Holistic Robust Data-Driven Decisions

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Abstract

The design of data-driven formulations for machine learning and decision-making with good out-of-sample performance is a key challenge. The observation that good in-sample performance does not guarantee good out-of-sample performance is generally known as overfitting. Practical overfitting can typically not be attributed to a single cause but instead is caused by several factors all at once. We consider here three overfitting sources: (i) statistical error as a result of working with finite sample data, (ii) data noise which occurs when the data points are measured only with finite precision, and finally (iii) data misspecification in which a small fraction of all data may be wholly corrupted. We argue that although existing data-driven formulations may be robust against one of these three sources in isolation they do not provide holistic protection against all overfitting sources simultaneously. We design a novel data-driven formulation which does guarantee such holistic protection and is furthermore computationally viable. Our distributionally robust optimization formulation can be interpreted as a novel combination of a Kullback-Leibler and Lévy-Prokhorov robust optimization formulation which is novel in its own right. However, we show how in the context of classification and regression problems that several popular regularized and robust formulations reduce to a particular case of our proposed novel formulation. Finally, we apply the proposed HR formulation on a portfolio selection problem with real stock data, and analyze its risk/return tradeoff against several benchmarks formulations. Our experiments show that our novel ambiguity set provides a significantly better risk/return trade-off.

Keywords: Data-driven Decision-making, Machine Learning, Robustness, Generalization, Distributionally Robust Optimization, Kullback-Leibler Divergence, Lévy-Prokhorov Metric.

1 Introduction

In this paper, we study data-driven decision-making in the context of stochastic optimization. Let \( \mathcal{X} \) be a set of decisions and \( \xi \) a random variable representing an uncertain scenario realizing in a set \( \Sigma \). For a given scenario \( \xi \in \Sigma \) of the uncertainty, and a decision \( x \in \mathcal{X} \), the loss incurred for decision \( x \) in scenario
\( \xi \) is denoted here as \( \ell(x, \xi) \in \mathbb{R} \). The random variable \( \tilde{\xi} \) is distributed according to a probability measure \( \mathbb{P} \) in the set of probability measures \( \mathcal{P} \) over \( \Sigma \). A decision with minimal expected loss can be found as the solution to the stochastic optimization problem

\[
\min_{x \in \mathcal{X}} \mathbb{E}_{\mathbb{P}}[\ell(x, \tilde{\xi})].
\]  

(1)

Stochastic optimization covers a wide variety of problems, such as decision-making under uncertainty and regression in machine learning. We will assume here that the set \( \mathcal{X} \) and \( \Sigma \) are compact and the loss function \( \ell \) is continuous which guarantees that the minimum in Equation (1) indeed exists.

**Example 1.1** (Machine Learning). Consider covariates \( (X, Y) \in \mathbb{R}^n \times \mathcal{Y} \) for a set of possible outputs \( \mathcal{Y} \) and \( n \in \mathbb{N} \), a set of parameters \( \Theta \), and a loss function \( L: \Theta \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \). Assume the data \((\tilde{X}, \tilde{Y})\) follows an out-of-sample probability distribution \( \mathbb{P} \). A myriad of popular machine learning problems attempt to find parameters as minimizers in

\[
\min_{\theta \in \Theta} \mathbb{E}_{\mathbb{P}}[L(\theta, \tilde{X}, \tilde{Y})].
\]

The objective function is denoted as the out-of-sample error of a regressor associated with the parameter \( \theta \in \Theta \). For example, least squares linear regression corresponds to the loss \( L(\theta, X, Y) = (Y - \theta^T X)^2 \) with \( \theta \) the coefficient of the considered linear models. When training a neural network, \( \theta \) corresponds to the weights of the network, and \( L(\theta, X, Y) \) the loss incurred when evaluating a network with weights \( \theta \) on data point \((X, Y)\).

**Example 1.2** (Newsvendor problem). Consider a newsvendor problem in which the decision maker decides on an inventory \( x \in \mathcal{X} \subset \mathbb{R}_+ \) in the face of uncertain upcoming demand \( \tilde{d} \in \mathbb{R}_+ \) following a distribution \( \mathbb{P} \). Any excess inventory leads to a cost \( h > 0 \) per unit, and any demand that is not satisfied leads to a cost \( b > 0 \) per unit. The newsvendor problem can be formulated as the stochastic optimization problem

\[
\min_{x \in \mathcal{X}} \mathbb{E}_{\mathbb{P}}[b(\tilde{d} - x)^+ + h(x - \tilde{d})^+].
\]

The distribution \( \mathbb{P} \) of the random variable \( \tilde{\xi} \) is in many practical situations unknown. Rather, the decision-maker can only observe historical data points \( \xi_1, \ldots, \xi_T \in \Sigma^T \). The goal is to construct a solution approximating the optimal solution of Problem (1) using only the data samples and without access to the data distribution. As we assume the order of the data points to be irrelevant here, the observed data can be compactly represented through its empirical distribution

\[
\hat{\mathbb{P}}_T := \frac{1}{T} \sum_{t \in [T]} \delta_{\xi_t},
\]  

(2)

where \( \delta_{\xi_t} \) is the point mass distribution at \( \xi_t \). Typically, one constructs an approximation or proxy for the unknown expectation of the loss (or the out-of-sample loss) \( \mathbb{E}_{\mathbb{P}}[\ell(x, \tilde{\xi})] \) for each decision \( x \) as a function of the
empirical distribution; say \( \hat{c}(x, \hat{P}_T) \). We denote such an approximation or proxy \( \hat{c} \) here as a cost predictor. The cost \( \hat{c}(x, \hat{P}_T) \) is then called the in-sample loss of the decision \( x \). The resulting data-driven counterpart to Problem (1) is then

\[
\min_{x \in \mathcal{X}} \hat{c}(x, \hat{P}_T). \tag{3}
\]

Perhaps the most natural such approximation involves substituting the out-of-sample loss for the empirical loss \( \hat{c}(x, \hat{P}_T) = \mathbb{E}_{\hat{P}_T}[\ell(x, \hat{\xi})] = \frac{1}{T} \sum_{t \in [T]} \ell(x, \xi_t) \). This yields the so called sample average approximations (SAA) in the context of decision-making and empirical risk minimization (ERM) in the context of machine learning.

However, substitution of a naive cost predictor for the unknown out-of-sample cost often results in decisions exhibiting poor generalization or out-of-sample performance: the in-sample cost \( \hat{c}(x, \hat{P}_T) \) may considerably underestimate the actual out-of-sample cost \( \mathbb{E}_P[\ell(x, \xi)] \) and lead to poor decisions [Smith and Winkler 2006]. This phenomenon is broadly known as overfitting. Constructing predictors which safeguard against overfitting is one of the fundamental challenges of learning and decision-making with data.

1.1 Robustness

To safeguard against overfitting, robust predictors resulting in reliable decisions are desirable. A robust predictor \( \hat{c} \) is here a predictor which verifies an out-of-sample guarantee of the form

\[
\hat{c}(x, \hat{P}_T) \geq \mathbb{E}_P[\ell(x, \xi)] \tag{4}
\]

with “high probability” on the data generation, for all \( x \in \mathcal{X} \). This property guarantees that the estimated in-sample cost \( \hat{c}(x, \hat{P}_T) \), estimated with data, is an upper bound on the out-of-sample cost \( \mathbb{E}_P[\ell(x, \xi)] \) with high probability. Hence, the predictor provides a conservative—rather than an overly optimistic—estimation of the out-of-sample cost. In other words, the out-of-sample cost is at least as good as what was estimated in-sample. As the predictor is intended to be used for optimization and solving the stochastic optimization (1), then the guarantee (4) must also be verified in its prescribed optimal solution \( \hat{x}^* \in \arg\min \hat{c}(x, \hat{P}_T) \). This is the case when the guarantee holds uniform in the decision, that is, \( \hat{c}(x, \hat{P}_T) \geq \mathbb{E}_P[\ell(x, \xi)], \forall x \in \mathcal{X} \) with high probability. Uniform guarantees ensure then when subsequently minimizing the estimated loss indeed results in minimizing an upper bound on the out-of-sample loss.

It is easy to construct overly conservative upper bounds on the out-of-sample cost. Indeed, a trivial predictor predicting large or even infinite cost would trivially verify the guarantee (4). However, such predictor overly conservative predictors would likely lead poor decisions. Hence, merely verifying an out-of-sample guarantee does not ensure good decisions. We will seek therefore efficient predictors which constitute in a sense we will make more precise the smallest upper bound on the out-of-sample loss.
1.2 Sources of Overfitting

The first important step toward constructing formulations which are robust against overfitting and generalize well is to understand precisely what we seek to robustify against (“robustness to what?”). In other words, what are the potential sources of overfitting which can cause data-driven formulations to generalize poorly? We discuss here what we believe to be the three most common sources which can cause overfitting and challenge generalization. We will point out later that most existing robust predictors in the literature protect against some but not (at least not efficiently) all sources simultaneously.

1.2.1 Statistical Error

Perhaps the most prevalent and fundamental source of overfitting comes from the fact that as the data set is finite, our knowledge of the true data distribution is necessarily limited. The cost prediction \( \hat{c}(x, \hat{P}_T) \) is therefore inevitably only an approximation of the true out-of-sample loss \( E_P[\ell(x, \xi)] \) and fluctuates with the randomness of the finite sample. When the randomness of the sample—and therefore of the empirical distribution \( \hat{P}_T \)—causes the prediction \( \hat{c}(x, \hat{P}_T) \) to underestimate the out-of-sample loss, the minimization problem \( \text{3} \) might indeed suggest seemingly good solutions \( x \) in-sample, i.e., with low predicted cost \( \hat{c}(x, \hat{P}_T) \), while performing poorly out-of-sample. For example, one source of statistical error in classification tasks is data imbalance. Even when the true out-of-sample distribution of the data is balanced, by the random nature of the sample one class may count considerably more data points than the others. This results in classifiers based on ERM to mistakenly reduce the in-sample classification error on this majority class at the expense of the others.

There is considerable work on how to design robust formulations against overfitting due to statistical error \cite{bertsimas2018,yrieux2019}. One particular example of such robust predictors is the Kullback-Leibler divergence distributionally robust optimization predictor (KL-DRO)

\[
\hat{c}_{KL}(x, \hat{P}_T) := \max \left\{ E_{P'}[\ell(x, \xi)] : P' \in P, \text{KL}(\hat{P}_T||P') \leq r \right\} \quad \forall x \in \mathcal{X},
\]

for \( r > 0 \), where \( \text{KL}(\mu||\nu) := \int \log \left( \frac{d\mu}{d\nu}(\xi) \right) d\mu(\xi) \) for all \( \mu \ll \nu \in P \), and \( +\infty \) otherwise, is the Kullback–Leibler divergence. Informally, when the observed samples are independent realizations of the out-of-sample distribution \( P \), then \( P \in \{ P' \in P : \text{KL}(\hat{P}_T||P') \leq r \} \) with high probability\footnote{This is only true when the distributions are of finite support. In the general case however, it still holds that \( E_P[\ell(x, \xi)] \leq \hat{c}_{KL}(x, \hat{P}_T) \) with high probability \cite{van2021}.}. Hence, the KL-DRO predictor minimizes a worst-case expectation over a set of distributions which includes the out-of-sample distribution \( P \) with high probability and hence is unlikely to underestimate the out-of-sample cost. \cite{van2021} prove that the KL-DRO predictor efficiently guards against overfitting caused by statistical error and in fact guarantees that the probability that the out-of-sample cost is underestimated decays exponen-
tially fast in the number of collected data points $T$. Furthermore, the KL-DRO predictor can be evaluated as a convex minimization problem (Love and Bayraksan 2015, Van Parys et al. 2021).

Lam (2019), Duchi et al. (2021) finally show that when this probability is desired to decay subexponentially, robustness against statistical error can be obtained by considering the sample variance penalization predictor (SVP)

$$\hat{c}_{\text{SVP}}(x, \hat{P}_T) := E_{\hat{P}_T} \left[ \ell(x, \hat{\xi}) \right] + \lambda \sqrt{\text{Var}_{\hat{P}_T} \left[ \ell(x, \hat{\xi}) \right]} \quad \forall x \in \mathcal{X},$$

for $\lambda \geq 0$, where $\text{Var}_{\hat{P}_T} \left[ \ell(x, \hat{\xi}) \right]$ is the empirical variance of the loss. In fact, Van Parys et al. (2021), Bennouna and Van Parys (2021) prove in a precise sense that the KL-DRO and SVP predictors are efficient predictors for robustness against statistical error. In some sense, both predictors optimally balance efficiency while guaranteeing a certain level of robustness against statistical error.

However, overfitting in practice is not caused by statistical error alone. In fact, statistical error is perhaps the most mild source of overfitting among those we will discuss as it diminishes with the samples size at rate $O(1/\sqrt{T})$. That is, for any decision $x$ and bounded and continuous loss function $\xi \mapsto \ell(x, \xi)$ we have:

$$E_{\hat{P}_T} \left[ \ell(x, \hat{\xi}) \right] - E_P \left[ \ell(x, \xi) \right] = O \left( \sqrt{\text{Var}[\ell(x, \xi)]}/T \right)$$

when the data points are independently sampled from the out-of-sample distribution $P$. Hence, statistical error is in general more prominent in settings with scarce data or high variance of the loss. We next discuss two sources of overfitting which require robust predictors even in a large data regime where the amount of statistical error is negligible.

### 1.2.2 Noise

In practice the observed data sample $\{\xi_1, \ldots, \xi_T\}$ may often not be a realization of $\hat{\xi}$ following the out-of-sample distribution $P$, but rather a realization of a noisy random variable $\hat{\xi} + \hat{n} \in \Sigma$. As the distribution of the noise $\hat{n}$ is not known, collecting more training data does not eliminate this potential source of overfitting.

Clearly, when no assumptions are made regarding the size of the noise $\hat{n}$ then it may completely swamp the signal $\hat{\xi}$ and nothing interesting can be said further. Assume however that the noise realizes in a known bounded set $0 \in \mathcal{N}$. An example of such set is the epsilon norm ball $B(0, \epsilon) = \{n : \|n\| \leq \epsilon\}$, which models norm bounded noise up to size $\epsilon \geq 0$. The cost predictors and their associate decisions may be guarded against such adversarial noise by considering an inflated loss function $\ell^N : \mathcal{X} \times \Sigma \rightarrow \mathbb{R}$, $(x, \xi) \mapsto \max \{\ell(x, \xi - n) : n \in \mathcal{N}, \xi - n \in \Sigma\}$ rather than the loss function $\ell$ directly. Recent works by Madry et al.\footnote{Note that this bound can be extended to $\min_{x \in \mathcal{X}} E_{\hat{P}_T} \left[ \ell(x, \hat{\xi}) \right] - \min_{x \in \mathcal{X}} E_P \left[ \ell(x, \xi) \right]$ under careful complexity assumptions on the decision set $\mathcal{X}$, such as finite VC dimension (Vapnik 1999, Maurer and Pontil 2009).}
Indeed consider robust predictors of the form

\[
\hat{c}_R(x, \hat{P}_T) := \max \left\{ \frac{1}{T} \sum_{t \in [T]} \ell(x, \xi_t - n_t) : n_t \in \mathcal{N}, \xi_t - n_t \in \Sigma \forall t \in [T] \right\} = \frac{1}{T} \sum_{t \in [T]} \ell^N(x, \xi_t).
\] (6)

The underlying intuition is that when the statistical error is negligible (i.e., the out-of-sample distribution is the empirical distribution of the noiseless sample), we are guaranteed that \(P \in \{ P_t \in [T]: n_t \in \mathcal{N}, \xi_t - n_t \in \Sigma \forall t \in [T]\}\). We remark hence that the robust predictor differs from the ERM predictor only in that an inflated loss function \(\ell^N\) is considered instead of the loss function \(\ell\) itself. The robust formulation is practical to the extent we have access to an oracle with which we can (approximately) evaluate the inflated loss function \(\ell^N\). Sometimes this inflated loss can be computed explicitly for commonly considered loss functions and norm bounded noise as we will illustrate in Section 4. In some settings evaluating the inflated loss function \(\ell^N\) can be hard as discussed by Sinha et al. (2017). Efficient approximation methods however may still be available as discussed by Madry et al. (2018), Bertsimas et al. (2021). In the remainder of the paper we will assume that an oracle is available which can evaluate the inflated loss function \(\ell^N\).

Perhaps the most well known instance of the robust approach discussed here is the popular LASSO predictor for linear regression (see Example 1.1) which can be written as

\[
\sqrt{\frac{1}{T} \sum_{t \in [T]} (Y_t - \theta^T X_t)^2} + \frac{\lambda}{\sqrt{T}} \| \theta \|_1 = \max \left\{ \sqrt{\frac{1}{T} \sum_{t \in [T]} (Y_t - \theta^T (X_t - n_t))^2} : \| n_t \|_2 \leq \lambda \forall t \in [T] \right\},
\]

for all \(\theta \in \Theta\) and \(\lambda \geq 0\) (Xu et al. 2009, Theorem 1). Hence, in this perspective, LASSO can be interpreted as to protect precisely against particular norm bounded noise.

It is important to note that protection against one form of overfitting does not necessarily translate to protection against all others. As an extreme case, consider the KL-DRO formulation (5) which as we pointed out in the previous section protects efficiently against statistical error. Its associated ambiguity set will in the presence of any amount of noise fail to contain the distribution \(P\) when it is a continuous distribution; see also Figure 1. Hence, the KL-DRO predictor can severely underestimate the out-of-sample cost under the presence of noise, even though it verifies out-of-sample guarantees in the clean data setting. This somewhat surprising observation perhaps helps explain why the Kullback-Leibler divergence is not as popular in practice in favor of other divergence metrics such as the Wasserstein distance

\[
W(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \int \| \xi - \xi' \| \, d\gamma(\xi, \xi') \quad \forall \mu, \nu \in \mathcal{P},
\] (7)

for a given norm \(\| \cdot \|\) and where the set \(\Gamma(\mu, \nu)\) denotes the collection of all joint measures on \(\Sigma \times \Sigma\) with marginals \(\mu\) and \(\nu\) (Ambrosio 2003) (which we also call the set of couplings between \(\mu\) and \(\nu\)). However, the W-DRO robust predictor \(\hat{c}_W(x, \hat{P}_T) := \sup \{ \mathbb{E}_{P}[\ell(x, \tilde{\xi})] : P' \in \mathcal{P}, W(\hat{P}_T || P') \leq \epsilon \}\) for \(\epsilon > 0\) and all \(x \in \mathcal{X}\), in turn, does not protect against both statistical error and noise, but rather protects precisely against certain types of noisy data. Indeed, Gao and Kleywegt (2023), Wang et al. (2022) show that when the event set \(\Sigma\)
Figure 1: Two distributions $\mu$ and $\nu$ which are equivalent up to a small distributional shift $\epsilon > 0$ are very dissimilar in terms of their entropic divergence, i.e., $\text{KL}(\mu, \nu) = \infty$.

is convex and the loss function $\ell$ is concave in its second argument, then

$$\hat{c}_W(x, \hat{P}_T) := \max \left\{ \frac{1}{T} \sum_{t \in [T]} \ell(x, \xi_t - n_t) : \frac{1}{T} \sum_{t \in [T]} \|n_t\| \leq \epsilon, \xi_t - n_t \in \Sigma \quad \forall t \in [T] \right\} \quad \forall x \in \mathcal{X}.$$ 

Hence, W-DRO protects here precisely against all noise which averaged over the samples remains bounded in norm. This observation indicates that even though W-DRO verifies out-of-sample guarantees (Mohajerin Esfahani and Kuhn 2018), it might not efficiently protect against statistical error, and the required ambiguity set size for such guarantees can be too large. Indeed, it has been observed empirically in deep learning that while type-$\infty$ W-DRO (adversarial training) provides robustness to noisy inputs, it leads worse overfitting gap than ERM (Rice et al. 2020).

The main characteristic of the noise considered here is that, in some sense, it remains small as it realizes in a compact set $\mathcal{N}$. The last source of data corruption we identify is perhaps the most corrupting as it will neither be small nor diminish with increasing sample size.

1.2.3 Misspecification

In most typical data sets, a hopefully small fraction of all data may be wholly corrupted. This can be the case when errors in the data collection occurred or when fake data made its way into the data set. We remark that this source of overfitting is quite distinct from the previously discussed overfitting source. In the previous setting, the noise $n \in \mathcal{N}$ remains bounded and consequently all samples carry at least some amount of information concerning the out-of-sample distribution. Here, however, we will assume that a small fraction $\alpha \ll 1$ of all data points $\xi_t \in \Sigma$ for $t \in [T]$ may carry no information at all. This specific cause of overfitting has been studied since the pioneering work of Tukey (1958) and Huber (1981) in the context of robust statistics. More recent work by Diakonikolas et al. (2019) has renewed interest in this source of overfitting in particular in the context of high-dimensional learning problems (Diakonikolas and Kane 2019) and adversarial machine learning (Goldblum et al. 2022). In particular, in the context of adversarial machine learning the noise described in Section 1.2.2 as well as the misspecification described here are also known as data poisoning.
In this context, the robust predictor

\[
\hat{c}_M(x, \hat{P}_T) := \max \left\{ \frac{1}{T} \sum_{t \in [T]} \ell(x, \xi_t - n'_t) : \sum_{t \in [T]} 1 \{ n'_t \neq 0 \} / T \leq \alpha, \quad \xi_t - n'_t \in \Sigma \quad \forall t \in [T] \right\}
\]

seems appropriate. The underlying intuition is that as it is assumed here that there is no statistical error (informally we again assume here that \( \hat{P}_T \) becomes the out-of-sample distribution of the corrupted samples) we are guaranteed that \( P \in \{ P_t \in \mathcal{T} : \sum_{t \in [T]} 1 \{ n'_t \neq 0 \} / T \leq \alpha, \quad \xi_t - n'_t \in \Sigma \quad \forall t \in [T] \} \). Therefore, we have \( E_\mathbb{E}[\ell(\hat{x}_M(\hat{P}_T), \xi)] \leq \hat{c}(\hat{x}_M(\hat{P}_T), \hat{P}_T) \) where \( \hat{x}_M(\hat{P}_T) \in \arg \min_{x \in \mathcal{X}} \hat{c}_M(x, \hat{P}_T) \), and this upper bound can be shown to be tight. Hence, the predicted cost of the optimal decision \( \hat{x}_M(\hat{P}_T) \) exceeds its unknown out-of-sample cost and hence no overfitting has taken place.

Each of these three discussed sources of overfitting may have a certain varying importance in practice. For example, noise will be the main source of overfitting in settings where abundant but highly noisy data is available. Statistical error will come to dominate in settings where scarce but high quality data is available. It is therefore instrumental to adapt the amount of protection offered against each of these three distinct sources to the appropriate level in a particular application. Existing robust formulations typically protect specifically against only one of these aspects. As already pointed out before, KL-DRO and SVP for instance protects efficiently against statistical error while LASSO and W-DRO protects efficiently against noise. An important question that arises is therefore

**Can we construct a predictor robust simultaneously against statistical error, noise and misspecification?**

This is precisely the objective of this paper. We seek to construct a holistic robust predictor which protects against all sources of overfitting to any desirable degree. In addition to (i) robustness, we seek robust formulations which (ii) are tractable (iii) and are efficient in that they are not overly conservative.

### 1.3 Contributions

To the best of our knowledge, no current robust data-driven decision formulation protects efficiently against noise and misspecification simultaneously. We hence first consider in Section 2 in the absence of statistical error, the problem of protecting against both noise and misspecification. This constitutes our first building block towards holistic robustness. For parameters \( \mathcal{N} \) and \( \alpha \in [0, 1] \), we introduce the Lévy-Prokhorov DRO predictor (LP-DRO)

\[
\hat{c}_{\mathcal{L}P}^{\mathcal{N}, \alpha}(x, \hat{P}_T) := \max \{ E_{P'}[\ell(x, \tilde{\xi})] : P' \in \mathcal{P}, \quad \text{LP}_{\mathcal{N}}(\hat{P}_T, P') \leq \alpha \}
\]

associated with the convex pseudo divergence metric \( \text{LP}_{\mathcal{N}}(\hat{P}_T, P') = \inf_{\gamma \in \Gamma(\hat{P}_T, P')} \int_\mathcal{N} (\xi - \xi' \notin \mathcal{N}) d\gamma(\xi, \xi') \). The pseudo divergence \( \text{LP}_{\mathcal{N}} \) can be interpreted as a variation of the well-studied Lévy-Prokhorov (LP) metric (Prokhorov 1956); see also Section 2.2. We show that, given the observed empirical distribution
of the corrupted samples \( \hat{P}_T \), the ambiguity set \( \{ P' \in \mathcal{P}, \text{LP}_N(\hat{P}_T, P') \leq \alpha \} \) captures precisely the set of possible out-of-sample distributions when the noise realizes in the set \( \mathcal{N} \) and less than \( \alpha T \) data points are misspecified. In particular, this implies that the LP-DRO predictor provides the robustness out-of-sample guarantee \( \hat{c}^{N,\alpha}(x, \hat{P}_T) \geq \mathbb{E}_{P}[\ell(x, \hat{\xi})] \) for all \( x \in \mathcal{X} \), where \( \hat{P} \) is the observed corrupted distribution, and \( P \) is the true out-of-sample distribution. Furthermore, we show that the LP-DRO predictor provides efficient robustness; any other predictor \( \hat{c} \) that satisfies the out-of-sample guarantee \( \hat{c}(x, \hat{P}_T) \geq \mathbb{E}_{P}[\ell(x, \hat{\xi})] \) for all \( x \in \mathcal{X} \) and \( \hat{P} \in \mathcal{P} \).

Furthermore, the LP-DRO formulation is tractable and in fact combines the \( \mathcal{N} \)-inflated loss considered to protect against noise with the conditional value-at-risk which is appropriate in the context of misspecification. Indeed, in Theorem 2.4 we show that the LP-DRO predictor is precisely characterized as a convex combination between a conditional value-at-risk (CVaR) and a worst-case loss, i.e.,

\[
\hat{c}^{N,\alpha}(x, \hat{P}_T) = (1 - \alpha) \text{CVaR}_{\hat{P}_T}(\ell^N(x, \hat{\xi})) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi).
\]

In particular, we point out that this result provides a first exact tractable reformulation for the robust optimization problems associated with the Lévy-Prokhorov metric studied by Erdogan and Iyengar (2006), a special case of our DRO predictor.

Next, we address the problem of protecting against statistical error on top of noise and misspecification. By combining our LP-DRO formulation which protects efficiently against noise and misspecification and an KL-DRO formulation which protects efficiently against statistical error, we construct a tractable robust predictor safeguarded against all three overfitting sources simultaneously.

We consider the setting where data is corrupted by adversarial noise and misspecification after being sampled from the true distribution. We introduce the holistic robust DRO predictor (HR)

\[
\hat{c}^{N,\alpha,r}(x, \hat{P}_T) := \max\{ \mathbb{E}_{P}[\ell(x, \hat{\xi})] : P' \in \mathcal{P}, \ Q' \in \mathcal{P}, \ \text{LP}_N(\hat{P}_T, Q') \leq \alpha, \ KL(Q'||P') \leq r \},
\]

with parameters \( \mathcal{N}, \alpha \in [0,1] \) and \( r \geq 0 \). These parameters set the desired robustness of the predictor against each source of overfitting separately. We prove that when noise realizes in a set \( \mathcal{N} \) and less than a fraction \( \alpha \) of data points are misspecified, the HR predictor provides uniform robustness out-of-sample guarantees: with probability larger than \( 1 - e^{-rT+O(1)} \), \( \hat{c}^{N,\alpha,r}(x, \hat{P}_T) \geq \mathbb{E}_{P}[\ell(x, \hat{\xi})] \) uniformly in \( x \in \mathcal{X} \), where \( \hat{P}_T \) is the empirical distribution of the corrupted samples (Theorem 3.1). In particular, our finite sample uniform guarantees does not depend on the complexity of the decision set \( \mathcal{X} \), which is crucial for large scale problems in which the set \( \mathcal{X} \) can be high-dimensional. We further show that HR provides efficient robustness: any other predictor that verifies such robustness out-of-sample guarantee is necessarily uniformly more conservative than the HR predictor (Theorem 3.2). In particular, this establishes theoretically the superiority of combining KL and the optimal transport metric compared to approaches discussed in Section
Computing the HR predictor is a challenging problem involving an optimization over distributions. We prove that for any bounded continuous loss function, the supremum in the HR-DRO formulation is attained at distributions of finite known support. Subsequently, the HR predictor can be computed as a finite convex optimization problem with only conic and linear constraints (Theorem 3.3). Moreover, using a dual formation of the predictor as a minimization problem (Theorem 3.4), we show that the predictor can be optimized efficiently and the problem \( \min_{x \in X} \mathcal{N}_{\alpha, r}(x, \tilde{P}_T) \) is equally tractable or hard as optimizing the inflated loss function \( \min_{x \in X} \mathbb{E}_{\tilde{P}_T}[\ell_N(x, \tilde{\xi})] \). This is in stark contrast to popular DRO approaches, such as Wasserstein DRO, where tractability typically requires somewhat stringent conditions on the loss function (Wang et al. 2021, Shafieezadeh-Abadeh et al. 2023).

In Section 4, we illustrate our approach on two examples and a portfolio selection application. We first illustrate our approach in the context of linear regression and classification examples. In particular, we show that our general approach recovers many popular robustification and regularization schemes from the literature. For instance, when applied to linear regression with \( L_1 \) loss, HR recovers Ridge regularization or the sample variance penalization of (Maurer and Pontil 2009). For linear classification with hinge loss, HR classifiers are nearly identical to the soft-margin classifiers of Cortes and Vapnik (1995). We then visualize the effect of the robustness parameters in linear classification through numerical illustrations. Finally, we apply our HR formulation in a portfolio selection problem with real stock data, and analyze its risk/return tradeoff against four benchmarks formulations (Wasserstein DRO, KL DRO, Markowitz and a Mean-CVaR formulation). Our experiments show that our HR formulation provides a significantly better risk/return trade-off. Furthermore, we show that in practice, HR ’s robustness parameters can be selected appropriately via validation, as HR out-perform benchmarks given the same budget of parameters. We also point out that after the releases of the first version of this paper, our HR formulation has been successfully used to protect neural networks training against adversarial attacks (noise) and data poisoning (misspecification) while providing stronger generalization (statistical error) in Bennouna et al. (2023).

Beyond the introduced predictors, we seek to highlight through this work two important points. First, the importance of determining precisely robustness objectives in designing robust approach (Robustness to what?). Second, the power of combining \( f \)-divergence and optimal transport, rather than using one or the other. \( f \)-divergences captures stochasticity aspects while optimal transport metrics capture corruption aspects. Indeed, \( f \)-divergences are by nature a statistically motivated objects and provide efficient robustness to statistical error (Bennouna and Van Parys 2021), while optimal transport metrics measure tampering of data points and provides efficient robustness to corruption. Combining both is of great merit. More generally, our HR approach suggests a theoretically disciplined approach to combining an \( f \)-divergence measure capturing the desired strength of statistical guarantees (KL for exponential guarantees, Pearson for sub-exponential guarantees) and an optimal transport metric capturing the desired corruption (LP for noise
and misspecification, Wasserstein for average bounded noise...).

**Notations** Random variable are denoted with a tilde in this paper. For example $\tilde{\xi}$ denotes the random variable with realizations $\xi \in \Sigma$. We denote with $\mathcal{B}(\Sigma)$ the set of all Borel measurable subsets of the topological space $\Sigma$. $A^o$ (or $\text{int}(A)$) and $\bar{A}$ (or $\text{cl}(A)$) denote respectively the interior and the closure of set $A$. For a given measure $\mu \in \mathcal{P}$, we denote its support by $\text{supp}(\mu) := \{A = \bar{A} \in \mathcal{B}(\Sigma) : \mu(A^c) = 0\} = \{\xi \in \Sigma : \exists U \in \mathcal{B}(\Sigma), \xi \in U^o, \mu(U^o) > 0\}$ for all measure $\mu$. We abuse the notation of measures by denoting $\mu(\xi) := \mu(\{\xi\})$ for all measure $\mu$ and $\xi \in \Sigma$. For all $K \in \mathbb{N}$, we denote $[K] = \{1, \ldots, K\}$.

## 2 Robustness Against Noise and Misspecification

Our focus in this section is robustness against noise and misspecification only. The underlying assumption is that the number $T$ of available data points $\{\xi_1, \ldots, \xi_T\}$ is sufficiently large so that the effect of statistical error is negligible. This assumption, made by a plethora of work on constructing data-driven formulations robust against noise [Madry et al. 2018], will be relaxed in Section 3. In what follows, we propose a novel distributionally robust predictor which we prove protects efficiently against noise and misspecification. This predictor will then serve as a first building block towards holistic robustness.

### 2.1 The LP-DRO predictor

To construct our robust predictor, we seek to build an ambiguity set capturing precisely the considered overfitting sources. We associate with the set $\mathcal{N}$ the convex pseudo divergence metric $\text{LP}_{\mathcal{N}} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}_+$ defined as

$$\text{LP}_{\mathcal{N}}(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \int \mathbb{1}(\xi - \xi' \notin \mathcal{N}) \, d\gamma(\xi, \xi') \quad \forall \mu, \nu \in \mathcal{P}. \quad (9)$$

The considered metric is hence a generalized optimal transport metric associated with a particular transport cost function which is only sensitive to whether or not the transport distance remains bounded in the set $\mathcal{N}$ or not. We consider the associated LP-DRO predictor

$$c_{\text{LP}}^{\mathcal{N}, \alpha}(x, \hat{\mathcal{P}}_T) := \max\{\mathbb{E}_{\mathcal{P}}[\ell(x, \tilde{\xi})] : \mathcal{P}' \in \mathcal{P}, \ \text{LP}_{\mathcal{N}}(\hat{\mathcal{P}}_T, \mathcal{P}') \leq \alpha\} \quad \forall x \in \mathcal{X}, \quad (10)$$

with parameters $\mathcal{N}$ and $\alpha \in [0, 1]$. As visualized in Figure 2, the LP-DRO predictor takes the supremum over all distributions close in probability.

Intuitively, the ball $\{\mathcal{P}' \in \mathcal{P} : \text{LP}_{\mathcal{N}}(\hat{\mathcal{P}}_T, \mathcal{P}') \leq \alpha\}$ contains all the distributions that can be obtained from shifting $\hat{\mathcal{P}}_T$ by subtracting noise bounded in the set $\mathcal{N}$ and misspecifying a random arbitrary fraction $\alpha$ of all data points. In fact, the distribution $\gamma$ in Equation (9) can be interpreted as a coupling moving the
distribution $\mu$ into the distribution $\nu$ ($\hat{P}_T$ into $P'$ in (10)) and the objective function as the probability that a sample has been moved beyond the set $N$. Hence, $\text{LP}_N(\hat{P}_T, P') \leq \alpha$ ensures that there exists a coupling that moves $\hat{P}_T$ into $P'$ by perturbing less than a fraction $\alpha$ of the data by beyond the set $N$. This is the misspecified part of the data. The remaining $1 - \alpha$ percent (or more) is perturbed by noise bounded in the set $N$. This is the noisy part of the data. The LP ambiguity set is therefore by construction expected to capture precisely our considered corruption—noise and misspecification. We formalize this in the following theorem. Proofs of this section are deferred to Appendix A.

Let $\tilde{\xi}$ be distributed as $P$, the true out-of-sample distribution. Define the corrupted random observation as

$$\tilde{\xi}_c = (\tilde{\xi} + \tilde{n})1(\tilde{c} = 0) + \tilde{\xi}_0 1(\tilde{c} = 1) \in \Sigma,$$

where $\tilde{n}$ is a noise random variable, $\tilde{c}$ is a binary random variable indicating whether the sample is corrupted or not, and $\tilde{\xi}_0$ is a corrupted observation following an arbitrary distribution. Note that $\tilde{\xi}$, $\tilde{n}$, $\tilde{c}$ and $\tilde{\xi}_0$ can all be correlated and adversarial—we in particular do not impose any independence structure.

**Theorem 2.1** (Corruption set characterization). Assume the noise $\tilde{n}$ realizes in the set $N$ and the misspecification probability is less than $\alpha$, i.e., $\text{Prob}(\tilde{c} = 1) < \alpha$. For a given distribution $P^c \in P$ of the corrupted sample $\tilde{\xi}_c \in \Sigma$, the set of possible out-of-sample distributions $P$ of $\tilde{\xi}$ is

$$\{P' \in P : \text{LP}_N(P^c, P') < \alpha\}.$$

**Corollary 2.2** (Robustness against noise and misspecification). Assume the noise $\tilde{n}$ realizes in the set $N$ and the misspecification probability is less than $\alpha$, i.e., $\text{Prob}(\tilde{c} = 1) < \alpha$. Let $P$ be the out-of-sample distribution of clean observations and $P^c$ the distribution of the corrupted observations $\tilde{\xi}$. We have

$$\ell_{\text{LP}}^{N, \alpha}(x, P^c) \geq \mathbb{E}_P[\ell(x, \tilde{\xi})], \forall x \in \mathcal{X}.$$

Corollary 2.2 ensures that the LP-DRO predictor provides a robustness guarantee against noise and mis-
Total loss = \( (1 - \alpha) \text{CVaR}^{\alpha} \)

Replace with \( \alpha \max_{\xi \in \Sigma} \ell(x, \xi) \)

Higher loss \( \ell^N \)

Figure 3: Illustration of the LP-DRO expression of Theorem 2.4. The circles represent the observed corrupted samples ordered by increasing inflated loss \( \ell^N(x, \xi[1]) \leq \ldots \leq \ell^N(x, \xi[T]) \), with \( p = [\alpha T] \). The filled part represent the \( 1 - \alpha \) fraction with highest inflated loss \( \ell^N \) (which is \( (1 - \alpha) \text{CVaR} \)) plus \( \alpha \) times the worst-case scenario. The adversary replaces the \( \alpha \) fraction of the samples with lowest inflated loss \( \ell^N \) with the worst-case loss.

The LP-DRO predictor characterized in Equation (10) involves maximizing over probability measures which may in generally be computational hard. The following result indicates however that the supremum defining our LP-DRO predictor can be computed almost in closed form.

**Theorem 2.4.** Let \( x \in X \), \( \alpha \in [0, 1] \) and a noise set \( \mathcal{N} \) be given. Denote with \( \{\xi_1, \ldots, \xi_K\} \) the support of \( \hat{P}_T \), and consider an ordering such that \( \ell^N(x, \xi[1]) \leq \ldots \leq \ell^N(x, \xi[K]) \). Introduce for all \( k \in [K] \) the auxiliary points \( \xi_k' \in \xi_k - \arg \max_{n \in \mathbb{N}, \xi_k - n} \ell(x, \xi_k - n) \) and \( \xi_\infty \in \arg \max_{\xi \in \Sigma} \ell(x, \xi) \). We have

\[
\hat{c}_{\text{LP}}^{N, \alpha}(x, \hat{P}_T) = (1 - \alpha) \text{CVaR}^{\alpha}_{\hat{P}_T} (\ell^N(x, \hat{\xi}^N)) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi) = \mathbb{E}_{\hat{P}_T^{N, \alpha}}[\ell(x, \hat{\xi}^N)]
\]

where a worst-case distribution \( \hat{P}_T^{N, \alpha} \in \mathcal{P}, \text{LP}_{N}(\hat{P}_T, \hat{P}_T^{N, \alpha}) \leq \alpha \) can be found as

\[
\hat{P}_T^{N, \alpha} = \sum_{k=p+1}^T \delta_{\xi_k' \hat{P}_T} + (1 - \alpha - \sum_{k=p+1}^T \hat{P}_T(\xi_k)) \delta_{\xi_\infty} + \alpha \delta_{\xi_\infty}
\]

where \( p \) the smallest index so that \( 1 - \alpha - \sum_{k=p+1}^T \hat{P}_T(\xi_k) \) is strictly positive.

We prove a slightly more general version of Theorem 2.4 in Appendix A.3 where \( \hat{P}_T \) is not required to be an empirical distribution. Theorem 2.4 shows that we can compute the LP-DRO predictor explicitly and
provides an intuitive interpretation on how noise and misspecification interact when both are considered simultaneously, which we illustrate in Figure 3. First, the loss function considered here is the $\mathcal{N}$-inflated counterpart $\ell^N$ of the loss function $\ell$ as is the case for predictors robust against noise $\hat{c}_R(x, \hat{P}_T)$ defined in (6). Second, the conditional value-at-risk of the inflated loss function above the $1 - \alpha$ quantile is considered, which corresponds to the predictor $\hat{c}_M(x, \hat{P}_T)$ robust against misspecification stated in Equation (8). Finally, the LP-DRO predictor considers the $1 - \alpha$ percent worst $\mathcal{N}$-inflated loss scenarios and substitutes the ignored $\alpha$ percent lowest loss scenarios with the worst-case loss $\max_{\xi \in \Sigma} \ell(x, \xi)$. This result is rather intuitive, as the misspecification is adversarial it will tend to target an $\alpha$ fraction of low loss scenarios and corrupt them into worst-case scenarios. The remaining $1 - \alpha$ fraction of samples is then perturbed adversarially by bounded noise resulting in the $\mathcal{N}$-inflated loss; see also Figure 3.

Remark 2.5 (Robust Statistics and Outliers). From Figure 3 and the previous discussion it can be remarked that the LP-DRO formulation assigns more influence to those noise realizations associated with a large inflated loss while at the same time suppresses the influence of those realizations with a small inflated loss. This runs counter to the general strategy of robust methods introduced by Tukey [1958], Huber [1981] which try to identify the $\alpha$ fraction of the data points which are misspecified outliers and consequently suppress their influence. We remark however that outlier identification is only then a sound strategy if structural conditions on the unknown distribution such as normality [Huber 1981] or bounded variance [Diakonikolas and Kane 2019] are imposed. Critically, here we make no such assumptions and hence there is also no concept of statistical outlier as opposed to recent work by Nietert et al. [2023].

Theorem 2.4 implies that the LP-DRO formulation can be tractably computed provided we can evaluate the inflated loss function $\ell^N$. Hence, our novel LP-DRO formulation is computationally not harder to solve than the robust formulations proposed by Madry et al. [2018]. In particular, when the loss function $\ell(x, \xi)$ is convex in the decision $x$ for all $\xi$, the LP-DRO predictor is convex in $x$ as well. Finally, our LP-DRO robust predictor can be interpreted as a generalization of the predictors $\hat{c}_R(x, \hat{P}_T)$ and $\hat{c}_M(x, \hat{P}_T)$ which protect against both noise and misspecification, simultaneously.

Corollary 2.6. When $\alpha T$ is integer, any $x \in \mathcal{X}$ and noise set $\mathcal{N}$, we have

$$\hat{c}^{\mathcal{N}, \alpha}_{LP}(x, \hat{P}_T) = \max \left\{ \frac{\sum_{t \in [T]} \ell(x, \xi_t - n_t - n'_t)}{T} : \sum_{t \in [T]} \frac{\mathbb{1}_{\{n'_t \neq 0\}}}{T} \leq \alpha, \ n_t \in \mathcal{N}, \ \xi_t - n_t - n'_t \in \Sigma \ \forall t \in [T] \right\}.$$
Lévy-Prokhorov Metric  The pseudo-metric $\pi_{LP}$ is intimately related to the classical Lévy-Prokhorov metric

$$\pi_{LP}(\mu, \nu) = \inf \{ \epsilon > 0 : \nu(A) \leq \mu(A') + \epsilon \quad \forall A \in \mathcal{B}(\Sigma) \}$$

for $\epsilon > 0$, for all $\mu$ and $\nu$ where $A' := \{ \xi \in A \mid \exists \xi' \in A, \| \xi' - \xi \| < \epsilon \}$. The Lévy-Prokhorov metric is perhaps best known for its topological properties. Indeed, weak convergence of probability distributions is equivalent to convergence in the Lévy-Prokhorov metric. Following Strassen (1965), we have that $\{ \nu \in \mathcal{P} : \pi_{LP}(\mu, \nu) \leq \epsilon \} = \{ \nu \in \mathcal{P} : \nu(A) \leq \mu(A') + \epsilon \forall A \in \mathcal{B}(\Sigma) \}$. The LP-DRO predictor hence comprises (for $N = \mathcal{B}(0, \epsilon)$ and $\alpha = \epsilon$) the Lévy-Prokhorov robust cost predictor $\sup \{ \mathbb{E}_{P'}[\ell(x, \tilde{\xi})] : \mathcal{P}' \in \mathcal{P}, \pi_{LP}(\hat{P}_T, \mathcal{P}') \leq \epsilon \}$ studied by Erdogan and Iyengar (2006) who propose an approximate evaluation. Hence, even in the restricted context of Lévy-Prokhorov robust optimization the result stated in Theorem 2.4 appears to be novel.

Type-$\infty$ Wasserstein Metric  The type-$p$ Wasserstein distance family for $p \geq 1$ between two distributions $\mu$ and $\nu$ is defined as $W_p(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int \| \xi - \xi' \|^p d\gamma(\xi, \xi') \right\}^{1/p}$. Givens and Shortt (1984, Proposition 3, Proposition 5) show that we have in the limit $\lim_{p \to \infty} W_p(\mu, \nu) = W_\infty(\mu, \nu) := \inf \{ \epsilon > 0 : \nu(A) \leq \mu(A') \forall A \in \mathcal{B}(\Sigma) \}$. Following again Strassen (1965), we have that $\{ \nu \in \mathcal{P} : \pi_{LP}(\mu, \nu) \leq \epsilon \} = \{ \nu \in \mathcal{P} : W_\infty(\mu, \nu) \leq \epsilon \}$. The LP-DRO predictor hence comprises (for $N = \{0\}$ and $\alpha = \epsilon$) the type-$\infty$ Wasserstein robust cost predictor $\sup \{ \mathbb{E}_{P'}[\ell(x, \tilde{\xi})] : \mathcal{P}' \in \mathcal{P}, W_\infty(\hat{P}_T, \mathcal{P}') \leq \epsilon \}$ studied recently by Bertsimas et al. (2022), Xie (2019) in the context of two-stage robust optimization and by Nguyen et al. (2020) in the context of conditional estimation.

Total Variation  The total variation distance between any distributions $\mu$ and $\nu$ is defined as

$$TV(\mu, \nu) = \inf \{ \epsilon > 0 : \nu(A) \leq \mu(A') + \epsilon \quad \forall A \in \mathcal{B}(\Sigma) \}.$$ 

Following again Strassen (1965), we have that $\{ \nu \in \mathcal{P} : \pi_{LP}(\mu, \nu) \leq \epsilon \} = \{ \nu \in \mathcal{P} : TV(\mu, \nu) \leq \epsilon \}$. The LP-DRO predictor hence comprises (for $N = \{0\}$ and $\alpha = \epsilon$) the total variation robust cost predictor $\sup \{ \mathbb{E}_{P'}[\ell(x, \tilde{\xi})] : \mathcal{P}' \in \mathcal{P}, TV(\hat{P}_T, \mathcal{P}') \leq \epsilon \}$. Total variation robust formulations were proven effective by Rahimian et al. (2019) to identify critical scenarios in stochastic optimization models. With this perspective, Theorem 2.4 can be seen as an extension of prior reformulations of TV-DRO (Shapiro 2017, Jiang and Guan 2018).
3 Holistic Robustness

The previous section shows that the LP-DRO predictor defined in Equation (10) provides robustness efficiently against noise and misspecification. On the other hand, as we mentioned in Section 1, the KL-DRO predictor provides efficient robustness against statistical error as pointed out recently by Duchi et al. (2021), Van Parys et al. (2021), Bennouna and Van Parys (2021). However, none of the discussed robust predictors protects against all sources of overfitting simultaneously.

The question hence becomes whether a predictor can be constructed that is simultaneously robust against all the three sources of overfitting. A natural idea is to combine the KL and LP ambiguity sets which separately capture robustness against statistical error, noise and misspecification, respectively.

3.1 A Holistic Robust Predictor

Consider the following data generation process in which sample data is generated by independent draws from the true out-of-sample distribution $P$. However, this sample data is not observed directly by the decision-maker. Rather, each sample is first corrupted by noise realizing in $N$. Finally, the decision maker only gets to observe data in which up to a fraction $\alpha$ of these noisy samples have been substituted by arbitrary fake data. Remark that as opposed to Section 2, the statistical error is here not negligible. We do not observe the full distribution of the corrupted uncertainty but rather must learn its distribution, and infer the clean distribution from samples.

We construct a hollistically robust predictor against statistical error, noise and misspecification. Let us first provide intuition on the construction. The goal is to construct an ambiguity set containing the true out-of-sample distribution $P$ with high probability, given the observed empirical distribution $\hat{P}_T$ of the corrupted sample. Let $\hat{Q}$ be the empirical distribution of the unobserved sample, from the true out-of-sample distribution $P$. Sanov’s theorem (Dembo and Zeitouni 2009, Theorem 6.2.10) indicates that this empirical distribution (intuitively) verifies

$$\text{KL}(\hat{Q}||P) \leq r$$

with high probability $1 - \exp(-rT + o(T))$. As samples from of the empirical distribution $\hat{Q}$ are perturbed into $\hat{P}_T$ with noise in $N$ and a fraction up to $\alpha$ is misspecified, Theorem 2.1 ensures that we must have $\text{LP}_N(\hat{P}_T, \hat{Q}) \leq \alpha$. Hence, the out-of-sample distribution $P$ is in the set

$$\left\{ P' \in \mathcal{P} : \exists \hat{Q} \in \mathcal{P}, \text{LP}_N(\hat{P}_T, \hat{Q}) \leq \alpha, \text{KL}(\hat{Q}||P') \leq r \right\}.$$  

Figure 4 illustrates the hypothesized data generation process visually.

Following this intuition, for a given noise support $N$, a misspecification fraction $\alpha \in [0,1]$ and a desired statistical robustness parameter $r$, we consider the holistic robust predictor

$$c^N_{\alpha,r}(x, \hat{P}_T) = \sup \left\{ E_{P'}[\ell(x, \xi)] : P' \in \mathcal{P}, \exists \hat{Q} \in \mathcal{P}, \text{LP}_N(\hat{P}_T, \hat{Q}) \leq \alpha, \text{KL}(\hat{Q}||P') \leq r \right\} \quad \forall x \in \mathcal{X}. \quad (12)$$

---

3 This is not true for a continuous distribution $P$. We state this here only for the purpose of intuition. The proof requires more elaborate arguments, see Appendix B.
Figure 4: When learning with corrupted data, data points are first sampled independently from $P$. Sampling from $P$ itself results in statistical errors which however remain bounded with high probability $1 - \exp(-rT + o(T))$ by $r$ as measured in by the KL divergence between $P$ and $Q$. Subsequently, an adversary corrupts these samples with noise in $N$ and misspecification at frequency at most $\alpha$. Theorem 2.1 guarantees that the distance between $\hat{Q}$ and $\hat{P}_T$ is at most $\alpha$ as measured by the pseudometric $\text{LP}_N$.

We formally prove that the HR predictor enjoys robustness in the presence of noise and misspecification. Proofs of this section are in Appendix B.

**Theorem 3.1 (Uniform Robustness Out-of-sample Guarantee).** Let $\hat{P}_T$ be the observed empirical distribution of corrupted samples. Assume that the noise realizes in a compact set in the interior of $\mathcal{N}$ and the corrupted portion of the data is less than $\alpha T$. We have

$$\Pr(\exists x \in \mathcal{X}, \hat{c}^{\mathcal{N},\alpha,r}_{\text{HR}}(x, \hat{P}_T) < \mathbb{E}_P[\ell(x, \tilde{\xi})]) \leq \exp(-rT + O(1)).$$

The previous result establishes that the HR predictor enjoys a uniform asymptotic out-of-sample guarantee and hence safeguards against the three sources of overfitting simultaneously. In particular, the robustness out-of-sample guarantee is uniform in the decision set $\mathcal{X}$, an important property for optimization as it implies $\Pr(\hat{c}^{\mathcal{N},\alpha,r}_{\text{HR}}(\hat{x}^{\text{HR}}, \hat{P}_T) \geq \mathbb{E}_P[\ell(\hat{x}^{\text{HR}}, \tilde{\xi})]) \geq 1 - \exp(-rT + O(1))$, where $\hat{x}^{\text{HR}}$ minimizes the HR predictor. In classical statistical learning literature, such uniform upper bounds are typically obtained through restricting the complexity decision set (or hypothesis class) $\mathcal{X}$, such as its VC dimension (Vapnik 1999).

The then obtained bounds depend on such complexity measures, which may render them impractical in high-dimensional applications. In the proof of Theorem 3.1, we in fact exhibit a finite sample result as well in which we show that the $O(1)$ term is independent of the complexity of the set $\mathcal{X}$. We in fact show that for all $T \geq 1$, we have $\Pr(\exists x \in \mathcal{X}, \hat{c}^{\mathcal{N},\alpha,r}_{\text{HR}}(x, \hat{P}_T) < \mathbb{E}_P[\ell(x, \tilde{\xi})]) \leq \exp(-rT + m(\Sigma, \delta) \log (4/\delta))$ for any $\delta > 0$, where $\mathcal{N}^\delta = \mathcal{N} + B(0, \delta)$ and $m(\Sigma, \delta)$ denotes the internal covering number of the support set $\Sigma$.

In the proof of Theorem 3.1, we also show that the ambiguity set of the HR predictor contains the out-of-sample distribution with high probability. This shows that the LP metric smooths the KL ambiguity sets overcoming some of its limitations (Liu et al. 2023). Indeed, while KL-DRO is proven to be optimal in a certain sense when data is clean (Van Parys et al. 2021), the KL ambiguity set never contains the out-of-sample distribution when this distribution is continuous. The HR ambiguity sets circumvents this issue by use of the optimal transport metric, and contains the out-of-sample distribution with high probability in the more general setting of corrupted observed data.
Finally, the following result shows that the HR predictor provides efficient robustness. That is, out of all predictors enjoying its out-of-sample guarantee it predicts the least conservative costs.

**Theorem 3.2 (Efficient Robustness).** Suppose \( 0 \in \text{int}(\mathcal{N}), \text{cl}(\text{int}(\mathcal{N})) = \mathcal{N}, \alpha > 0, r > 0 \) and \( \xi \to \ell(x,\xi) \) continuous and bounded for all \( x \in \mathcal{X} \). Let \( \hat{c} \) be a predictor that verifies the out-of-sample guarantee

\[
\limsup_{T \to \infty} \frac{1}{T} \log \text{Prob} \left( \exists x \in \mathcal{X}, \hat{c}(x,\hat{P}_T) < E_{\hat{P}}[\ell(x,\hat{\xi})] \right) \leq -r,
\]

for all out-of-sample distributions \( \hat{P} \) and adversary which misspecifies less than a fraction \( \alpha \) of training data points and perturbs the testing data with noise bounded in a compact set in the interior of \( \mathcal{N} \). Then, \( \hat{c} \) is necessarily uniformly more conservative than \( c_{HR}^{N,\alpha,r} \), that is, we have \( \hat{c}(x,\hat{P}) \geq c_{HR}^{N,\alpha,r}(x,\hat{P}) \), for all \( x \in \mathcal{X} \) and observed empirical distribution \( \hat{P} \).

The previous results show that the HR predictor provides precisely the desired robustness, and is in fact an efficient robust predictor. However, it is not clear whether the HR predictor is tractable as it is characterized as a maximum over probability measures. In the following result, we show that computing the predictor can be reduced to solving a tractable optimization problem. Moreover, we show that the supremum in Equation (12) admits an optimal solution with finite known support, which characterizes the worst-case adversary. This is rather a surprising result, as the predictor characterized in Equation (12) considers the worst-case distribution among all potentially continuous distributions on \( \Sigma \).

**Theorem 3.3 (Finite Primal Reduction).** Let \( \{\xi_1, \ldots, \xi_K\} \) be the support of \( \hat{P}_T \). For all \( x \in \mathcal{X} \), the HR predictor (12) admits the representation

\[
\hat{c}_{HR}^{N,\alpha,r}(x,\hat{P}_T) = \max \left\{ \sum_{k \in [K]} p'_k \ell^N(x,\xi_k) + p'_{K+1} \max_{\xi \in \Sigma} \ell(x,\xi) \right\}
\]

s.t. \( p' \in \mathbb{R}^{K+1}_+, \hat{q'} \in \mathbb{R}^{K+1}_+, s \in \mathbb{R}^K \),

\[
\sum_{k \in [K]} p'_k = 1, \quad \sum_{k \in [K+1]} \hat{q}_k = 1
\]

\[
\sum_{k \in [K+1]} \hat{q}_k \log \left( \frac{\hat{q}_k}{p'_k} \right) \leq r,
\]

\[
\sum_{k \in [K]} s_k \leq \alpha,
\]

\[
\hat{q}_k + s_k = \hat{P}_T(\xi_k) \quad \forall k \in [K].
\]

Notice that the nonlinear constraint in the maximization problem (13) can be represented as an exponential cone constraint by introducing the auxiliary variable \( t \in \mathbb{R}^K \) with \( \sum_{k \in [K]} t_k \leq r \) and the auxiliary constraints \((-t_k, \hat{P}_T(\xi_k), q_k) \in K_{exp} \) for all \( k \in [K] \) where \( K_{exp} \) is the exponential cone. Hence, problem (13) can be solved using off-the-shelf exponential cone optimization solvers [Dahl and Andersen 2021].

Given an optimal solution in the optimization problem (13) we can explicitly construct an optimal solution in the optimization problem (12) as well. To that end recall that for all \( k \in [K] \) the points \( \xi'_k \in \xi_k - \)
arg max_{\nu \in \mathcal{N}, \xi_k-n \in \Sigma} \ell(x, \xi_k - n) and \xi_{\infty} \in \arg \max_{\xi \in \Sigma} \ell(x, \xi). We can associate with any feasible (\hat{q}, s, p') in (13) the distributions \hat{\mathcal{Q}} = \sum_{k \in [K]} \hat{q}_k \delta_{\xi_k}' + \hat{q}_{K+1} \delta_{\xi_{\infty}}', p' = \sum_{k \in [K]} p'_k \delta_{\xi_k}' + p'_{K+1} \delta_{\xi_{\infty}} and \gamma = \sum_{k \in [K]} \hat{q}_k \delta_{\xi_k}, \xi_{\infty}. The constraints \hat{q}_k + s_k = \hat{\mathcal{P}}_T(\xi_k) for all k \in [K], \sum_{k \in [K+1]} p'_k = 1 and \sum_{k \in [K+1]} \hat{q}_k = 1 guarantee that \gamma \in \Gamma(\hat{\mathcal{P}}_T, \hat{\mathcal{Q}}) while \int 1(\xi - \xi' \notin \mathcal{N}) d\gamma(\xi, \xi') \leq \sum_{k \in [K]} s_k \leq \alpha guarantees \text{LP}_\mathcal{N}(\hat{\mathcal{P}}_T, \hat{\mathcal{Q}}) \leq \alpha. Furthermore, we have \text{KL}(\hat{\mathcal{Q}} || \mathcal{P}^r) = \sum_{k \in [K]} \hat{q}_k \log(\hat{q}_k/p'_k) \leq r and \text{E}_{\mathcal{P}^r}[\ell(x, \xi)] = \sum_{k \in [K]} p'_k \ell(x, \xi_k) + p'_{K+1} \ell(x, \xi_{\infty}) = \sum_{k \in [K]} p'_k \ell^N(x, \xi_k) + p'_{K+1} \max_{\xi \in \Sigma} \ell(x, \xi). Hence, for all feasible solutions in (13) we can associate finitely supported distributions feasible in the maximization problem (12) characterizing the HR predictor attaining the same objective value. A feasible solution (p', q', s) in problem (13) corresponds to the solution

\[ p' = \sum_{k \in [K]} p'_k \delta_{\xi_k}' + p'_{K+1} \delta_{\xi_{\infty}} \quad \text{and} \quad \hat{\mathcal{Q}} = \sum_{k \in [K]} \hat{q}_k \delta_{\xi_k}' + \hat{q}_{K+1} \delta_{\xi_{\infty}}. \]

feasible in the maximization problem (12) characterizing the HR predictor; see the proof of Theorem 3.3.

Although Theorem 3.3 provides a tractable ways to evaluate the HR predictor, finding an optimal decision in min_{x \in \mathcal{X}} \hat{c}_{\text{HR}}^{N, \alpha, r}(x; \hat{\mathcal{P}}_T) still requires solving a saddle point problem which may yet again be slightly awkward to solve in practice. In the following, we provide an equivalent tractable dual minimization representation of the HR predictor.

**Theorem 3.4 (Dual Formulation).** Let \{\xi_1, \ldots, \xi_K\} be the support of \hat{\mathcal{P}}_T. For all x \in \mathcal{X}, the HR predictor (12) admits for all r > 0 the dual representation

\[
\hat{c}_{\text{HR}}^{N, \alpha, r}(x; \hat{\mathcal{P}}_T) = \inf \left\{ \sum_{k \in [K]} w_k \hat{\mathcal{P}}_T(\xi_k) + \lambda (r - 1) + \beta \alpha + \eta \right. \\
\left. \text{s.t. } w \in \mathbb{R}^K, \lambda \geq 0, \beta \geq 0, \eta \in \mathbb{R}, \right.
\]

\[
w_k \geq \lambda \log \left( \frac{\lambda}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} \right); \quad w_k \geq \lambda \log \left( \frac{\lambda}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} \right) - \beta \quad \forall k \in [K],
\]

\[
\eta \geq \max_{\xi \in \Sigma} \ell(x, \xi). \]

In Appendix B.5, we prove this dual formulation in the more general case when \hat{\mathcal{P}}_T may be any possibly continuous distribution. Hence, minimizing the HR predictor min_{x \in \mathcal{X}} \hat{c}_{\text{HR}}^{N, \alpha, r}(x; \hat{\mathcal{P}}_T) using the dual formulation is tractable whenever the inflated loss function \hat{\ell}^N(x, \xi) can be evaluated efficiently and hence the HR formulation is not harder to solve than the robust formulation min_{x \in \mathcal{X}} \text{E}_{\hat{\mathcal{P}}_T}[\hat{\ell}^N(x, \xi)].

Finally, we prove that the HR predictor can be interpreted as a KL-DRO predictor (protecting against statistical error) applied to a perturbed distribution (guarding against noise and misspecification).

**Lemma 3.5.** Consider the worst-case distribution \hat{\mathcal{P}}_T^\alpha \in \arg \max \{\text{E}_{\mathcal{P}}[\ell(x, \xi)] : \mathcal{P} \in \mathcal{P}, \text{LP}_\mathcal{N}(\hat{\mathcal{P}}_T, \mathcal{P}) \leq \alpha\} defined in the context of Theorem 2.4. For all x \in \mathcal{X}, the HR predictor (12) admits the decomposition

\[
\hat{c}_{\text{HR}}^{N, \alpha, r}(x; \hat{\mathcal{P}}_T) = \hat{c}_{\text{KL}}(x, \hat{\mathcal{P}}_T^\alpha).
\]
This results implies an alternative way of computing the HR predictor. In fact, from Lemma 3.5 and Van Parys et al. (2021, Proposition 5) the following even more compact reduced formulation follows.

**Lemma 3.6.** Consider the worst-case distribution $\hat{P}_{T,N}^{\alpha} \in \arg \max \{E_{P'}[\ell(x, \xi)] : P' \in P, LP_{N}(\hat{P}_{T,N}, P') \leq \alpha\}$ defined in the context of Theorem 2.4. For all $x \in X$ and $r > 0$, the HR predictor \((12)\) admits the dual representation

$$\hat{c}_{HR}^{N,\alpha,r}(x, \hat{P}_{T}) = \min \left\{ \alpha - \exp(-r) \exp \left( \int \log (\alpha - \ell(x, \xi)) d\hat{P}_{T,N}^{\alpha}(\xi) \right) : \alpha \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}. \tag{15}$$

Furthermore, the minimization problem \((15)\) admits an minimizer $\alpha^*$ which is bounded as

$$\max_{\xi \in \Sigma} \ell(x, \xi) \leq \alpha^* \leq \frac{\max_{\xi \in \Sigma} \ell(x, \xi) - e^{-r} \int \ell(x, \xi) d\hat{P}_{T,N}^{\alpha}(\xi)}{1 - e^{-r}}. \tag{16}$$

We remark that the HR-DRO predictor is characterized in Lemma 3.6 as the solution to a univariate convex optimization problem which can be efficiently solved using a simple bisection search starting from Equation \((16)\) whereas a worst-case distribution $\hat{P}_{T,N}^{\alpha}$ can be trivially found based on Equation \((11)\) after having simply sorted the values $(\ell_{N}(x, \xi_1), \ldots, \ell_{N}(x, \xi_K))$ where here $(\xi_1, \ldots, \xi_K)$ denotes again the support of the empirical distribution $\hat{P}_{T,N}$.

### 3.2 Oblivious Adversaries

The adversaries considered in the previous section are called *adaptive* by Ben-David et al. (1994); the data is first sampled from the out-of-sample distribution and then corrupted. An alternative corruption model can be considered by reversing the order of sampling and corruption; the out-of-sample distribution is first corrupted by an adversary, and then data is sampled from the corrupted distribution. This corruption model is associated with oblivious adversaries (Ben-David et al. 1994). More formally, let $\xi$ be a random variable following the true out-of-sample distribution of the uncertainty $P$. Each observed data point will be a sample, not from $\xi \in \Sigma$, but from a corrupted data source $\xi^c = (\xi + \hat{n})1(\hat{c} = 0) + \xi_01(\hat{c} = 1) \in \Sigma$, where $\hat{n}$ is a noisy random variable, $\hat{c}$ is a binary random variable modeling whether the sample is misspecified, and $\xi_0$ is an arbitrary random variable representing the misspecified observation. We observe a finite data sample from this corrupted distribution. These adversaries are generally weaker than the previously considered adaptive adversaries (Ben-David et al. 1994, Zhu et al. 2022, Blanc et al. 2022). Although the focus of this paper will be on adaptive adversaries, we briefly indicate in this section that inverting the order of LP and KL in the HR predictor’s ambiguity set naturally leads to robustness to oblivious adversaries.

Let us first provide intuition on our construction. The goal is to construct an ambiguity set containing the true out-of-sample distribution $P$ with high probability, given the observed empirical distribution $\hat{P}_{T}$ of the corrupted sample. Assume that the noise $\hat{n}$ realizes in $\mathcal{N}$ and the probability of a sample being corrupted is no
Figure 5: In the data generation process with oblivious adversaries, an adversary can corrupt the data generation distribution away from $\mathcal{P}$ towards any $\mathcal{Q}$ within distance $\alpha$ as measured by our pseudo metric $\text{LP}_N$. Sampling from $\mathcal{Q}$ itself results in statistical errors which however remain bounded with high probability $1 - \exp(-rT)$ by $r$ as measured in by the KL divergence between $\hat{\mathcal{P}}_T$ and $\mathcal{Q}$.

more than the fraction $\alpha$ (i.e., $\text{Prob}(\hat{c} = 1) \leq \alpha$). Following Theorem 2.1, this means that the corrupted data could in effect have been sampled by the adversary from any distribution $\mathcal{Q}$ verifying $\text{LP}_N(\mathcal{Q}, \mathcal{P}) \leq \alpha$. Now $\hat{\mathcal{P}}_T$ is an empirical distribution sampled from the distribution $\mathcal{Q}$. Following Sanov’s theorem (Dembo and Zeitouni 2009, Theorem 6.2.10), we have $\text{KL}(\hat{\mathcal{P}}_T|| \mathcal{Q}) \leq r$ with probability $1 - \exp(-rT + o(T))$. Hence the out-of-sample distribution $\mathcal{P}$ is expected to be in the set $\{\mathcal{P}' \in \mathcal{P} : \exists \mathcal{Q}' \in \mathcal{P}, \text{KL}(\hat{\mathcal{P}}_T|| \mathcal{Q}') \leq r, \text{LP}_N(\mathcal{Q}', \mathcal{P}') \leq \alpha\}$ with high probability $1 - \exp(-rT + o(T))$. Figure 5 illustrates this data generation process visually. Following this intuition, we consider the holistic robust predictor for oblivious adversaries (HRo)

$$\hat{c}_{\text{HRo}}^{N, \alpha, r}(x, \hat{\mathcal{P}}_T) := \max \{\mathbb{E}_{\mathcal{P}'}[\ell(x, \hat{\xi})] : \mathcal{P}' \in \mathcal{P}, \mathcal{Q}' \in \mathcal{P}, \text{KL}(\hat{\mathcal{P}}_T|| \mathcal{Q}') \leq r, \text{LP}_N(\mathcal{Q}', \mathcal{P}') \leq \alpha\} \quad \forall x \in \mathcal{X}. \quad (17)$$

We prove next prove that the HRo predictor indeed provides robustness in the oblivious adversary setting.

**Theorem 3.7** (Robustness Out-of-sample Guarantee). Let $\hat{\mathcal{P}}_T$ be the observed empirical distribution of corrupted samples $\hat{\xi}$ by an oblivious adversary. Assume that the noise $\tilde{n}$ realizes in $\mathcal{N}$ and the probability of a sample being corrupted is no more than $\alpha$, i.e., $\text{Prob}(\tilde{c} = 1) \leq \alpha$. For all $x \in \mathcal{X}$, we are disappointed, i.e., the event $\mathbb{E}_\mathcal{P}[\ell(x, \hat{\xi})] > \hat{c}_{\text{HRo}}^{N, \alpha, r}(x, \hat{\mathcal{P}}_T)$ occurs, with probability at most $\exp(-rT + o(T))$.

The proof of the previous result is in Appendix C. We remark that Theorem 3.7 is weaker than its counterpart for adaptive adversaries found in Theorem 3.1. Indeed, the result here is holds only asymptotically as $T$ tends to infinity whereas the guarantee in Theorem 3.1 holds for finite samples as well. Furthermore, the guarantee in Theorem 3.1 is pointwise rather than uniform in $x \in \mathcal{X}$. Technically, this is due to a lack of a finite sample counterpart of the result by Dembo and Zeitouni 2009 (Exercise 4.5.5) to the oblivious adversaries. Uniform guarantees can hence be only guaranteed when $\mathcal{X}$ is finite by a union bound argument or $\mathcal{X}$ compact by way of an appropriate discretization method.

We will also not derive here an efficiency guarantee as done in Theorem 3.2. Rather, we do show here that a worst case distribution is attained in a fine support which we characterize. We prove a more general result for general distributions in Appendix C.

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4Again, this is only for the purposes of intuition, but does not hold theoretically.
Theorem 3.8 (Finite Primal Reduction). Let \( \{\xi_1, \ldots, \xi_K\} \) be the support of \( \hat{P}_T \in \mathcal{P} \). For all \( x \in \mathcal{X} \), the HR predictor admits for all \( r > 0 \) the finite primal representation

\[
\hat{c}^{N,\alpha,r}_{\text{HRo}}(x, \hat{P}_T) = \begin{cases} 
\max & \sum_{k \in [K]} p'_k \ell'(x, \xi_k) + p'_{K+1} \max_{\xi \in \Sigma} \ell(x, \xi) \\
\text{s.t.} & p' \in \mathbb{R}^{K+1}_+, \quad q'_p \in \mathbb{R}^{K+1}_+, \quad s \in \mathbb{R}^K, \\
& \sum_{k \in [K+1]} p'_k = 1, \quad \sum_{k \in [K+1]} q'_k = 1, \\
& \sum_{k \in [K]} \hat{p}_T(\xi_k) \log \left( \frac{\hat{p}_T(\xi_k)}{q'_k} \right) \leq r, \\
& \sum_{k \in [K]} s_k \leq \alpha, \\
& q'_k = p'_k + s_k \quad \forall k \in [K].
\end{cases}
\]

Given an optimal solution in the optimization problem (18) we can explicitly construct an optimal solution in the optimization problem (17) as well. To that end consider again for all \( k \in [K] \) the points \( \xi'_k \in \xi_k - \arg \max_{n \in \mathcal{N}, \xi_k - n} \ell(x, \xi_k - n) \) and let \( \xi_\infty \in \arg \max_{\xi \in \Sigma} \ell(x, \xi) \). We can associate with any feasible \((q', s, p')\) in (18) the distributions \( Q' = \sum_{k \in [K]} q'_k \delta_{\xi'_k} + q'_K \delta_{\xi_\infty} \), \( P' = \sum_{k \in [K]} p'_k \delta_{\xi'_k} + p'_K \delta_{\xi_\infty} \) and \( \gamma = \sum_{k \in [K]} p'_k \delta_{\xi'_k} + s_k \delta_{\xi_\infty} + q'_K \delta_{\xi_\infty} \). The constraints \( q'_k = p'_k + s_k \) for all \( k \in [K] \), \( \sum_{k \in [K+1]} p'_k = 1 \) and \( \sum_{k \in [K+1]} q'_k = 1 \) guarantee that \( \gamma \in \Gamma(Q', P') \) while \( \int \mathbb{P}((\xi - \xi') \notin \mathcal{N}) d\gamma(\xi, \xi') \leq \sum_{k \in [K]} s_k \leq \alpha \) guarantees LP\(_N\)(Q', P') \leq \alpha. Furthermore, we have KL(\( \hat{P}_T || Q' \)) = \( \sum_{k \in [K]} \hat{p}_T(\xi_k) \log(\hat{p}_T(\xi_k)/q'_k) \leq r \) and \( \mathbb{E}_{Q'}[\ell(x, \tilde{\xi})] = \sum_{k \in [K]} p'_k \ell(x, \xi'_k) + p'_{K+1} \ell(x, \xi_\infty) = \sum_{k \in [K]} \hat{p}_T(\xi_k) \log(\hat{p}_T(\xi_k)/q'_k) \leq r \) for all feasible solutions in (18) we can associate finitely supported distributions feasible in the maximization problem (17) characterizing the HRo predictor attaining the same objective value.

A similar dual formulation is also verified for the HRo predictor, useful for optimization.

Theorem 3.9 (Dual Formulation). Let \( \{\xi_1, \ldots, \xi_K\} \) be the support of \( \hat{P}_T \in \mathcal{P} \). For all \( x \in \mathcal{X} \), the HRo predictor (17) admits for all \( r > 0 \) the dual representation

\[
\hat{c}^{N,\alpha,r}_{\text{HRo}}(x, \hat{P}_T) = \begin{cases} 
\inf & \sum_{k \in [K]} w_k \hat{p}_T(\xi_k) + \lambda(r - 1) + \beta \alpha + \eta \\
\text{s.t.} & w \in \mathbb{R}^K, \quad \lambda \geq 0, \quad \beta \geq 0, \quad \eta \in \mathbb{R}, \\
& w_k \geq \lambda \log \left( \frac{\lambda}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} \right), \quad \forall k \in [K], \\
& \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi).
\end{cases}
\]

In Appendix C.3 we prove this dual formulation in a slightly more general case when \( \hat{P}_T \) may be any possibly continuous distribution. Hence, minimizing the HRo predictor \( \min_{x \in \mathcal{X}} \hat{c}^{N,\alpha,r}_{\text{HRo}}(x, \hat{P}_T) \) using the dual formulation is also tractable whenever the inflated loss function \( \ell^N(x, \xi) \) can be evaluated efficiently and hence the HRo formulation is not harder to solve than the robust formulation \( \min_{x \in \mathcal{X}} \mathbb{E}_{\hat{p}_T}[\ell^N(x, \tilde{\xi})] \) by Madry et al. (2018).
3.3 Special Cases

In this section, we will point out that for several special parameter settings, the HR formulation is equivalent and collapses to well-known robust formulations.

Perhaps the simplest setting is the case when no robustness against statistical error is desired ($r = 0$). From the characterizations given in Equations (12) it follows

$$
\hat{c}_{HR}^{N,\alpha,0}(x, \hat{P}_T) = \hat{c}_{LP}^{N,\alpha}(x, \hat{P}_T) = (1 - \alpha)\text{CVaR}^\alpha_{\hat{P}_T}(\ell^N(x, \xi)) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi).
$$

That is, the HR predictor naturally reduce to the LP-DRO predictor which was the topic of Section 2. When additionally no robustness against misspecification is desired ($\alpha = 0$), the HR predictor is equivalent to

$$
\hat{c}_{HR}^{N,0,0}(x, \hat{P}_T) = \hat{c}_{R}(x, \hat{P}_T) = \max_{k \in [K]} \hat{P}_T(\xi_k) \ell^N(x, \xi_k).
$$

This is the robust formulation advanced by Madry et al. (2018) which we discussed in Section 1.2.2. As remarked, this formulation provides robustness to noise bounded in the noise set $N$ but none of the other overfitting sources discussed in Section 1.2.

When no robustness against mispecification is desired ($\alpha = 0$), the HR predictor is equivalent to

$$
\hat{c}_{HR}^{N,0,r}(x, \hat{P}_T) = \left\{ \begin{array}{l}
\sup \sum_{k \in [K]} p_k \ell^N(x, \xi_k) + p_{K+1} \max_{\xi \in \Sigma} \ell(x, \xi) \\
\text{s.t. } p \in \mathbb{R}^{K+1}_+, \sum_{k \in [K]} \hat{P}_T(\xi_k) \log(\hat{P}_T(\xi_k)/p_k) \leq r.
\end{array} \right.
$$

In particular, if additionally no robustness against noise is desired ($N = \{0\}$), the HR formulation is recognized as the classical KL-DRO [5]. Instead of using the empirical loss associated with the observed data points, HR predictors consider the worst-case inflated loss $\ell^N$ and reweigh each sample according to the variable $p$ subject to a likelihood constraint. Additionally, a safeguard against unobserved events in $\Sigma$ is provided by assigning a weight $p_{K+1}$ to outcomes with worst-case loss $\max_{\xi \in \Sigma} \ell(x, \xi)$. Once a worst-case event has been observed, i.e., $\max_{k \in [K]} \ell^N(x, \xi_k) = \max_{\xi \in \Sigma} \ell(x, \xi)$, then it follows immediately from the monotonicity of the logarithm that without loss of optimality we may assume $p_{K+1} = 0$ and hence

$$
\hat{c}_{HR}^{N,0,r}(x, \hat{P}_T) = \left\{ \begin{array}{l}
\sup \sum_{k \in [K]} p_k \ell^N(x, \xi_k) \\
\text{s.t. } p \in \mathbb{R}^K_+, \sum_{k \in [K]} \hat{P}_T(\xi_k) \log(\hat{P}_T(\xi_k)/p_k) \leq r.
\end{array} \right.
$$

This final formulation is recognized as the robust likelihood formulation advanced by Wang et al. (2016).
4 Two Examples and an Application

Our proposed HR formulation is a general approach that may provide robustness to any nominal data-driven optimization problem. In this section, we illustrate this robust formulation in the particular context of linear regression and classification examples. We point out through these examples that several popular robustness and regularization schemes can be identified as special instances of the HR formulations advanced here. We also visualize the effect of the robustness parameters characterizing HR formulations, and the resulting worse case distribution, in the context of the linear classification. Finally, we experiment the efficacy of the proposed HR formulation in the context of a portfolio optimization with real data along with benchmarks. We also point out that after the releases of the first version of this paper, our HR formulation has been applied to the training of neural networks, and has been shown to protect neural networks against adversarial attacks (noise) and data poisoning (misspecification) while providing stronger generalization (statistical error) in Bennouna et al. (2023).

4.1 Linear Regression Example

We consider here a linear regression problem based on historically observed data \((X_t, Y_t)_{t \in [T]}\) and \(L_1\) loss function \(\ell(\theta, (X, Y)) = |\theta^T X - Y|\). Given a particular covariate and response variable \((X, Y) \in \mathbb{R}^d \times \mathbb{R}\), \(d > 1\), this loss function is convex in the regression vector \(\theta\) which we assume is constrained in the convex parameter set \(\Theta \subseteq \mathbb{R}^d\). Ordinary \(L_1\) regression considers minimizing the empirical risk

\[
\min_{\theta \in \Theta} \frac{1}{T} \sum_{t \in [T]} |\theta^T X_t - Y_t|.
\]

The resulting \(L_1\) regressor is however well documented to be susceptible to overfitting effects.

In what follows we first point out that several classical regression regularization schemes studied in the literature reduce to a particular case of the HR linear regression formulation introduced here. We do remark though that none of these regularization schemes protect against statistical error, noise and misspecification simultaneously as does HR linear regression. Typically, these regularization schemes are designed to protect against only one particular source of overfitting instead.

Robustness Against Statistical Error \((N = \{0\}, r > 0, \alpha = 0)\) Suppose our objective is robustness against statistical error only. Assume here that a worst case has been observed and \(\Sigma = \{(X_t, Y_t)\}_{t \in [T]}\). Hence, we have the identities \(\max_{k \in [K]} \ell^N(x, \xi_k) = \max_{\xi \in \Sigma} \ell(x, \xi) = \max_{k \in [K]} \ell(x, \xi_k)\). HR linear regression
reduces following Equation (22) to
\[
\min_{\theta \in \Theta} \frac{1}{T} \sum_{t \in [T]} p_t |\theta^T X_t - Y_t| \\
\text{s.t. } p \in \mathbb{R}_+^T, \quad \sum_{t \in [T]} \frac{1}{T} \log(1/(T p_t)) \leq r
\]
which is precisely the likelihood robust approach proposed in [Wang et al. (2016)]. We remark again that this approach protects against statistical error but offers no safeguards against either noise or misspecification.

Let us investigate the regime in which the robustness radius decreases as \( r(T) = r'/T \) with \( T \to \infty \). From [Namkoong and Duchi (2017) Theorem 2] for \( \Theta \) compact it follows that
\[
\left\{ \sup_{t \in [T]} \frac{1}{T} \sum_{t \in [T]} p_t |\theta^T X_t - Y_t| \\
\text{s.t. } p \in \mathbb{R}_+^T, \quad \sum_{t \in [T]} \log(1/(T p_t)) \leq r'
\right\}
= \frac{1}{T} \sum_{t \in [T]} |\theta^T X_t - Y_t| + \sqrt{\frac{2r'}{T} \text{Var}_{\hat{p}_T}(|\theta^T \hat{X} - \hat{Y}|)} + \Delta_T(\theta)
\]
with \( \lim_{T \to \infty} \sup_{\theta \in \Theta} \Delta_T(\theta)\sqrt{T} = 0 \) in probability which we recognize as the sample variance penalization approach of [Maurer and Pontil (2009)]. Hence, in the large data regime in the absence of noise and misspecification the HR regression formulations offer a variance regularization interpretation. As observed though by [Namkoong and Duchi (2017)] it should be remarked that the variance regularized formulation is not convex and hence it is often preferable to use convex likelihood robust formulation instead.

**Robustness Against Noise** \( (\mathcal{N} = B_2(0, \epsilon) \times \{0\}, r = 0, \alpha = 0) \) Suppose now our objective to protect again noise in the data input. As in [Xu et al. (2009)] we will assume that the covariate data may be noisy and subject to misspecification while the response is not. We assume more precisely that the covariate data is subjected to noise bounded in Euclidean norm while the response variable is noise free and hence take \( \mathcal{N} = B_2(0, \epsilon) \times \{0\} \) for \( \epsilon \geq 0 \). In this regression context we have that the inflated loss function can be evaluated efficiently and in fact admits the analytic expression \( \ell^N(\theta, (X, Y)) = |\theta^T X - Y| + \epsilon \|\theta\|_2 \).

As here only robustness against norm bounded noise is desired, HR linear regression reduces following Equation (20) to
\[
\min_{\theta \in \Theta} \frac{1}{T} \sum_{t \in [T]} \sup_{\|n_t\|_2 \leq \epsilon} |\theta^T (X_t - n_t) - Y_t| = \min_{\theta \in \Theta} \frac{1}{T} \sum_{t \in [T]} |\theta^T X_t - Y_t| + \epsilon \|\theta\|_2
\]
which is readily identified as classical Tikhonov or Ridge regularization. This robust interpretation of classical Ridge regularization was pointed out by [Xu et al. (2009)] already. We do indicate though that classical Ridge regularization does not guard against either statistical noise or misspecification. In fact, as we assumed here that the noise set is of the form \( \mathcal{N} = B_2(0, \epsilon) \times \{0\} \) it can be remarked that Ridge regularization only protects against noisy covariate data.
Holistic Robustness ($\mathcal{N} = B_2(0, \epsilon) \times \{0\}, r > 0, \alpha > 0$) While several classical “robust” formulations exist for linear regression, a natural question is how to combine the benefits of each formulation and provide efficient robustness. The previous sections in this paper suggest that Ridge regression protects against noise while SVP and the likelihood approach protect against statistical error. A natural combination of both, with additional protection against misspecification, naturally spurs from the HR predictor.

Holistic robust linear regression can be defined following the discussion in Section 3 for the particular loss function and noise sets considered here. We assume here the existence of a compact set $\Sigma$ so that (Holistic robust linear regression can be defined following the discussion in Section 3 for the particular loss function and noise sets considered here. We assume here the existence of a compact set $\Sigma$ so that the HR predictor.

Theorem 3.4 suggests the following HR linear regression formulation

$$\inf_{\theta \in \Theta, w \in \mathbb{R}^T} \frac{1}{T} \sum_{t \in [T]} w_t + \lambda (r - 1) + \beta \alpha + \eta$$

s.t. \( \theta \in \Theta, w \in \mathbb{R}^T, \lambda \geq 0, \beta \geq 0, \eta \in \mathbb{R}, \)

\[
w_t \geq \lambda \log \left( \frac{\|\theta^\top X_t - Y_t\|}{\eta - \max_{\nu \in [T]} \|\theta^\top X_t - Y_t\|} \right), \quad w_t \geq \lambda \log \left( \frac{\|\theta^\top X_t - Y_t\|}{\theta^\top X_t - Y_t + \epsilon'} \right) - \beta \quad \forall t \in [T],
\]

$$\eta \geq \max_{t \in [T]} |\theta^\top X_t - Y_t| + \epsilon \|\theta\|_2.$$ 

As we have pointed out in Section 3, this HR linear regression formulation is tractable convex minimization problems which can be solved using off-the-shelf optimization routines. Interestingly, this formulation does not appear to be a straightforward combination of the previous robust formulations at a first glance. However, it appears naturally from our HR-DRO results of Section 3 and indeed offers efficient simultaneous robustness to statistical error, noise and misspecification.

4.2 Linear Classification Example

A linear classifier characterized by a coefficient vector $\theta \in \Theta \subseteq \mathbb{R}^d$ and bias term $b \in \mathbb{R}$ predicts class labels $Y \in \{-1, 1\}$ as $\theta^\top X - b$ when given access to covariate data $X \in \mathbb{R}^d$. Given historical data $(X_t, Y_t)_{t \in [T]}$, empirical risk minimization following [Bertsimas et al., 2019] would suggest to use a classifier associated with a minimizer in

$$\min_{\theta \in \Theta, b \in \mathbb{R}} \frac{1}{T} \sum_{t \in [T]} \max\{1 - Y_t(\theta^\top X_t - b), 0\} = \min_{\theta \in \Theta, b \in \mathbb{R}} \frac{1}{T} \sum_{t \in [T]} \nu_t$$

s.t. \( \theta \in \Theta, b \in \mathbb{R}, \nu \in \mathbb{R}^T_+, \)

$$Y_t(\theta^\top X_t - b) \geq 1 - \nu_t \quad \forall t \in [T].$$

A data set is denoted as separable if there exists a separating classifier with coefficient vector $\hat{\theta} \in \Theta$ and bias term $\hat{b} \in \mathbb{R}$ so that $Y_t(\hat{\theta}^\top X_t - \hat{b}) \geq 1$ for all $t \in [T]$. Clearly, when the data points are separable the minimum in Equation (23) will be zero. The hinge loss function $\ell((\theta, b), (X, Y)) = \max\{1 - Y(\theta^\top X - b), 0\}$ considered here however does not require the data to be separable. As in the previous section we will assume that the covariate data may be noisy or may be misspecified while the labels data are correct. We assume
more precisely that the historical covariate data is subjected to noise bounded in Euclidean norm while the dependent data is noise free and hence take \( \mathcal{N} = \mathcal{B}_2(0, \epsilon) \times \{0\} \) for \( \epsilon \geq 0 \).

HR linear classification can now be defined following the discussion in Section 3.1 for the hinge loss function and noise set considered here. Assume here that \((X_t, Y_t) \in \Sigma = \{(X_t, Y_t) : t \in [T]\} + \mathcal{B}_2(0, \epsilon') \times \{0\}\) for some sufficiently large \(\epsilon' \geq \epsilon\). In this context the inflated loss function can be characterized as \(\ell^N((\theta, b), (X, Y)) = \max(1 - Y(\theta^T X - b) + \epsilon \|\theta\|_2, 0)\). Theorem 3.4 suggests the following holistic robust classifier

\[
\begin{align*}
\inf \quad & \frac{1}{T} \sum_{t \in [T]} w_t + \lambda (r - 1) + \beta \alpha + \eta \\
\text{s.t.} \quad & \theta \in \Theta, \ b \in \mathbb{R}, \ w \in \mathbb{R}^T, \ \lambda \geq 0, \ \beta \geq 0, \ \eta \in \mathbb{R}_+, \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta} \right) \quad \forall t \in [T], \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta - 1 + Y_t(\theta^T X_t - b) - \epsilon \|\theta\|_2} \right) \quad \forall t \in [T], \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta - 1 + \min_{t' \in [T]} Y_{t'}(\theta^T X_{t'} - b) - \epsilon' \|\theta\|_2} \right) - \beta \quad \forall t \in [T], \\
& \eta \geq \max_{t \in [T]} 1 - Y_t(\theta^T X_t - b) + \epsilon \|\theta\|_2
\end{align*}
\]

Theorem 3.4 suggests the following holistic robust classifier

\[
\begin{align*}
\inf \quad & \frac{1}{T} \sum_{t \in [T]} w_t + \lambda (r - 1) + \beta \alpha + \eta \\
\text{s.t.} \quad & \theta \in \Theta, \ b \in \mathbb{R}, \ w \in \mathbb{R}^T, \ \lambda \geq 0, \ \beta \geq 0, \ \eta \in \mathbb{R}_+, \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta} \right) \quad \forall t \in [T], \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta - 1 + Y_t(\theta^T X_t - b) - \epsilon \|\theta\|_2} \right) \quad \forall t \in [T], \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta - 1 + \min_{t' \in [T]} Y_{t'}(\theta^T X_{t'} - b) - \epsilon' \|\theta\|_2} \right) - \beta \quad \forall t \in [T], \\
& \eta \geq \max_{t \in [T]} 1 - Y_t(\theta^T X_t - b) + \epsilon \|\theta\|_2
\end{align*}
\]

to safeguard against statistical error, noise and misspecification. As we have pointed out in Section 3 the HR linear classification formulations is a tractable convex minimization problems which can be solved using off-the-shelf optimization routines.

In the remainder of this section we shall visualize the effect each of the robustness parameters \((\mathcal{N}, r, \alpha)\) has on the HR linear classifiers by way of a small example. We consider data points \((X_t, Y_t)_{t \in [T]}\) distributed in the two dimensional square \([0, 1]^2\). The data points of the two classes are distributed following uniform distributions on rectangles on the two sides of the hyperplane with coefficient vector \(\theta = (-1, 1)\) and bias term \(b = 0\). Figure 6 illustrates the data distribution along with the true classifier, and an ERM classifier run on smaller data sample. In the following experiments, we do not observe data points from this distribution directly but rather from a corrupted distributions and study the effect of robustness provided by our HR approach.

**Robustness Against Noise** \((\epsilon > 0, r = 0, \alpha = 0)\) Consider the HR robust classifier safeguarded against noise but not against either statistical error nor misspecification \((\epsilon > 0, r = 0, \alpha = 0)\). The HR robust
Figure 6: (a) Data points of class $-1$ (in red) are distributed following a uniform distribution on a rectangle of center $(0.3, 0.7)$, length 0.52 and width 0.46. Data points of class $1$ (in blue) are distributed following the symmetric of class $-1$ with respect to the true classifier $\theta = (-1, 1), b = 0$. (b) The ribbon around an ERM classifier represents its margin $\{X \in \mathbb{R}^d : |\theta^T X - b| \leq 1\}$ with width $1/\|\theta\|_2$. Here, the ERM loss is 0 as the data is separable.

can here be reduced to

$$
\min_{\theta, b \in \mathbb{R}} \frac{1}{T} \sum_{t \in [T]} \max \left\{ 1 - Y_t (\theta^T X_t - b) + \epsilon \|\theta\|_2, 0 \right\} = \begin{cases} \\
\min & \epsilon \|\theta\|_2 + \frac{1}{T} \sum_{t \in [T]} \nu_t' \\
\text{s.t.} & \theta \in \Theta, \ b \in \mathbb{R}, \ \nu \in \mathbb{R}^T, \\
& Y_t (\theta^T X_t - b) \geq 1 - \nu_t \ \forall t \in [T], \\
& \nu_t \geq -\epsilon \|\theta\|_2 \ \forall t \in [T] 
\end{cases}
$$

which is nearly identical to the popular soft-margin SVM of Cortes and Vapnik (1995) with the exception of the ultimate constraint; see also Bertsimas et al. (2019, Appendix A). The soft-margin SVM classifier balances between minimizing the total hinge loss and the maximization of the classifier margin $1/\|\theta\|_2$. The HR support vector machine formulation enjoys the alternative interpretation that soft-margin SVM minimizes the maximum hinge loss with covariate data subjected to norm bounded noise; see also Figure 8(a).

The equivalence between regularization and noise robustness for classification was noted already by Xu et al. (2009), Bertsimas et al. (2019). Data distributions where soft-margin SVM is known to outperform ERM are normal distributions for each class. We indicate that we can plausibly attribute this superior performance to the robustness against noise interpretation of soft-margin SVM. Observe that normally distributed points can be seen as uniformly distributed points perturbed by (mostly) small noise; see Figure 7.

When run on the uncorrupted uniformly distributed data points, ERM and HR robust classification perform similarly as shown in Figure 8(a). However, when run on the normally distributed noisy data points, HR robust classification out-performs ERM (see Figure 8(b)). One intuitive explanations is that the normally
Figure 7: Let $P^c_\epsilon$ denote a corrupted data distribution in which the uniform distributions characterizing $P$ for both label classes have been replaced by an equivalent normal distribution sharing the same mean and variance. (a) True distribution $P$ versus corrupted distribution $P^c$ for a the points associated with the label $1$. (b) Equivalent noise perturbation corrupting each of the uniform samples.

distributed cloud of points have a circular shape, hence, very few points actually are near the separating hyperplane $\{X \in \mathbb{R}^d : \theta^T X = b\}$ and shape the classifier making it sensitive to noise in a few individual data points. The soft-margin classifier, however, by encouraging a larger margin $1/\|\theta\|_2$ is associated with a separating hyperplane shaped by many more points decreasing sensitivity to the noise corrupting an individual data point. The soft-margin classifier viewed as an HR robust classifier offers an alternative perspective on its superior performance. The HR classifier considers the worst-case scenario of each data point perturbed by Euclidean bounded noise and hence it perturbs all data points in the direction of the hyperplane. Naturally, this leads to many more points in the vicinity of the hyperplane all of which shape the classifier thereby decreasing its noise sensitivity.

Figure 8: (a) ERM and HR robust classification run on uncorrupted data points. The circles around each point represent the Euclidean noise ball of radius $\epsilon = 0.08$. (b) ERM and HR robust classification with $\epsilon = 0.08$ run on the normally distributed noisy data points.
Robustness Against Noise and Statistical Error ($\epsilon = \epsilon' > 0, r > 0, \alpha = 0$) We consider now HR robust classification with $\epsilon > 0$ and $r > 0$. That is, we desire robustness against noise and statistical error simultaneously but not to misspecification as here $\alpha = 0$. The classifier following Equation (22) can be characterized here as the minimizer to

$$\min_{\theta \in \Theta, b \in \mathbb{R}} \sup_{t \in [T]} \max \left\{ 1 - Y_t (\theta^T X_t - b) + \epsilon \|\theta\|_2, 0 \right\}$$

s.t. $p \in \mathbb{R}_+^T$, $\sum_{t \in [T]} p_t = 1$, $\sum_{t \in [T]} \frac{1}{T} \log(1/(T p_t)) \leq r$.

The HR robust classifier combines soft-margin classification which provides robustness against noise with weighting all data points according to their likelihood as advocated by Wang et al. (2016). Instead of taking the empirical average of the loss of the data points, HR-SVM indeed redistributes the weights of the data points to provide robustness against statistical error.

Statistical error can play a prominent role in classification when the data is imbalanced. Even when the out-of-sample distribution gives an equal chance of observing both label classes, by the luck of the draw there may be an imbalance between the number of observed data points in each label class. Data imbalance becomes even more problematic if the out-of-sample distribution only gives a small chance of observing data associated with the minority label. Both ERM or soft-margin SVM will on imbalanced data sets favor accuracy in labeling the majority class at the expense of the minority class. Two common strategies to combat such data imbalance are artificially weighing the samples or by directly removing or adding samples (Chawla et al. 2002). Our HR robust classifier can be interpreted to reweigh the observed samples to avoid overfitting to data imbalance as well as safeguard against statistical error in general. The weighing here is based on the statistical likelihood of observing the sample and is closely related to empirical likelihood method by Owen (2001). It is important to note that this weighing is a function of the particular considered loss function and even adapts to the particular classifier considered. This in contrast to popular weighing methods (Chawla et al. 2002) which are generic and independent of the loss function.

Figure 9 illustrate the HR robust classifier on an imbalanced data set. Figure 9(a) shows in particular that increasing the parameter $r$ weighs the data points in a way as to favor the minority class. Figure 9(b) exhibits these weights visually for $r = 0.5$.

4.2.1 Robustness Against Noise and Misspecification ($\epsilon = \epsilon' > 0, r = 0, \alpha > 0$)

We remark that for the special case of $r = 0$ HR SVMs reduce to another well known robust SVM approach, namely the $\nu$-SVMs of Schölkopf et al. (2000). We consider here that the worst-case cost is independent of the classifier, and hence does not affect the optimization problem. We can therefore write the HR-SVM
Figure 9: (a) The HR robust classifier with $r \in \{0, 0.1, 0.5\}$ on 25 data points with ratio 1/4 imbalance between classes. (b) Weights characterizing the HR robust classifier with $r = 0.5$ for each data point.

The following formulation Equation (19) as

$$\min_{\theta \in \Theta, b \in B} \left(1 - \alpha\right) \text{CVaR}_{\mathbb{P}_T}^{\alpha} \left(\max \left\{1 - \hat{Y}^\top \hat{X} - b + \epsilon \|\theta\|_2, 0\right\}\right)$$

This formulation results in the well known $\nu$-SVMs classifiers by Akansu et al. (2016, Section 10).

4.3 Portfolio Selection Application

In this section, we illustrate the efficacy of the HR predictor in the context of a portfolio selection problem with real stock data. We are provided here with a set of assets $A$ and observe the returns $(\xi_{a,i})_{i \in \mathbb{T}}$ of each asset $a \in A$ for $T$ times steps. The goal is to design a portfolio $x \in \mathcal{X} := \{x \in \mathbb{R}^A_{+} : \sum_{a \in A} x_a = 1\}$ enjoying maximum expected return $\mathbb{E}[(x, \xi)]$ given unknown future random returns $\xi$. The loss function can be therefore be taken here as $\ell(x, \xi) = -(x, \xi)$ for all $\xi \in \mathbb{R}^A$ and $x \in \mathcal{X}$.

We select 100 arbitrary stocks from the S&P 500 (listed in Appendix D.1) and study the problem of designing the optimal portfolio selection among the considered assets. Our data consists of the historical rolling 30 days returns of each asset. For instance $\xi_{t,a}$ is the return of asset $a$ when bought 30 trading days prior to time $t$ and sold at time $t$.

While the nature and amount of data corruption and distribution shift were assumed known in the previous section, when working with real life data, it is unclear what type and how much robustness is needed. Here, we seek to analyze the capacity of our proposed ambiguity set in capturing the required robustness, and in adapting to the provided problem data. Even though any distribution shift in the stock data is likely not exactly in the considered noise and misspecification form studied here, the hope is that the proposed HR formulation is flexible enough to still provide practical robustness. This is in contrast to more rigid robust formulations which only provide efficient protection against at most a single overfitting source.
We compare our HR formulation to two DRO formulations and two classical portfolio selection formulations:

- **Type-1 Wasserstein DRO** where we solve the problem \( \inf_{x \in \mathcal{X}} \sup_{P'} \{ E_{P'}[\ell(x, \tilde{\xi})] : P' \in \mathcal{P}, W(\hat{P}^T, P') \leq \epsilon \} \), with parameter \( \epsilon \geq 0 \), using the 1-Wasserstein distance \( W_1 \) suggested in Mohajerin Esfahani and Kuhn (2018). In this particular setting, W-DRO reduces to a linear optimization problem (Mohajerin Esfahani and Kuhn 2018, eq 27).

- **KL DRO** where we solve the problem \( \inf_{x \in \mathcal{X}} \sup_{P'} \{ E_{P'}[\ell(x, \tilde{\xi})] : P' \in \mathcal{P}, KL(\hat{P}^T || P') \leq r \} \), with parameter \( r \geq 0 \). The problem here reduces to an exponential cone optimization problem (Van Parys et al. 2021).

- **Mean-CVaR** where we solve the problem \( \inf_{x \in \mathcal{X}} E_{\hat{P}^T}[\ell(x, \tilde{\xi})] + \rho CVaR_\gamma \hat{P}^T[\ell(x, \tilde{\xi})] \), with parameter \( \rho \geq 0 \). The Mean-CVaR model can be reduced to a linear optimization problem (Rockafellar et al. 2000). Here, we fix \( \gamma = 20\% \) which will serve then as a measure of risk of the portfolios.

- **Markowitz** where we solve the problem \( \inf_{x \in \mathcal{X}} E_{\hat{P}^T}[\ell(x, \tilde{\xi})] + \rho Var_{\hat{P}^T}[\ell(x, \tilde{\xi})] \) (Markowitz 1952). This Mean-Variance problem reduces to a second-order optimization problem.

To formulate the HR-DRO model, we take the noise set to be the 1-norm ball \( \mathcal{N} = B_1(0, \epsilon) \), \( \epsilon \geq 0 \). The associated inflated loss is given explicitly as \( \ell^N(x, \xi) = \ell(x, \xi) - \langle x, \xi \rangle + \epsilon \|x\|_\infty \). We let \( \Sigma = \{ \xi_t : t \in [T] \} + B_1(0, \epsilon) \) so that \( \max_{\xi \in \Sigma} \ell^N(x, \xi) = \max_{t \in [T]} \ell^N(x, \xi_t) \). Hence, following Theorem 3.4, the HR portfolio optimization problem reduces to

\[
\begin{align*}
\inf \quad & \frac{1}{T} \sum_{t \in [T]} w_t + \lambda (r - 1) + \beta \alpha + \eta \\
\text{s.t.} \quad & x \in \mathbb{R}^A_+, \ w \in \mathbb{R}^T, \ \lambda \geq 0, \ \beta \geq 0, \ \eta \in \mathbb{R}, \\
& \sum_{a \in [A]} x_a = 1, \ \ell_t \geq -\langle x, \xi_t \rangle + \epsilon \|x\|_\infty, \ \forall t \in [T], \\
& w_t \geq \lambda \log \left( \frac{\lambda}{\eta - \ell_t} \right), \ w_t \geq \lambda \log \left( \frac{\lambda}{\eta - \max_{t \in [T]} |\ell_t|} \right) - \beta \ \forall t \in [T], \\
& \eta \geq \max_{t \in [T]} \ell_t
\end{align*}
\]

and hence requires the solution of an exponential cone problem (Dahl and Andersen 2021).

**Risk/Return Trade-off** As is typical in portfolio selection, overall performance of a given portfolio is a balance between average expect return and portfolio risk. In this experiment, we measure risk by either the CVaR or variance of the random return. The former measures the average return of a fraction \( \gamma = 0.2 \) of the worst performing assets. Higher CVaR of the returns is more desirable. Except for the Mean-CVaR model when the risk measure is CVaR and Markowitz when the risk measure is variance, none of the other models is directly designed for the these targeted risk measures. We analyse here how each considered portfolio selection formulation performs in terms of the risk/return trade-off.
As outlined in Appendix D.1 for each of the four models, we consider 2000 distinct hyperparameters and train each of the resulting models on historical returns of 90 trading days from May 26, 2022, to October 10, 2022 and report their average test return and test risk measurement determined using the following 60 trading days from October 10, 2022 to December 30, 2022. Figure 10 shows the risk/return trade-off at each the 2000 × 5 formulations. The optimal frontier of each of the four models considered models is emphasized.

Figure 10 shows that HR offers considerably more flexibility in the trade-off risk-return. Moreover, the Pareto frontier of HR essentially dominates the other formulations in terms of Pareto frontier. This is expected for KL, as KL is indeed a special case of HR. However, it does show that HR is not a meritless extension of KL, but rather offers significantly more flexibility and performance gain. HR also significantly dominates W-DRO. This illustrates the benefit of combining optimal transport and KL divergence, each providing robustness against different kind of adversaries. Perhaps surprisingly, HR dominates Mean-CVaR even when the evaluated risk measure is CVaR. One explanation for this is that Mean-CVaR evaluates both the return and risk measure empirically which can lead to overfitting. HR on the other hand takes into consideration generalization issues with the help of the KL divergence. The same observation holds when comparing HR to classical Markowitz portfolio selection.

**Performance with validation** While the previous set of experiments show that HR can reach a better return-risk trade-off, a key question which remains is how to choose those hyper parameters resulting on models on the Pareto frontier. Indeed, while HR dominates the other formulations in terms of Pareto
frontier, the flexibility of HR makes it also reaches several values that under-perform compared to other models. Hence, a poor selection of the parameters can lead to poor performance. We next show how validation selection of the hyperparameters allows to effectively select the hyperparameters based on data.

Each of the five portfolio selection formulation is given the same budget of 2000 parameters (see Appendix D.1). In each trial, each method is trained on the first 80% of days of the training data, and the parameter with the highest expected return in the next 20% of training days is selected. The formulation is then retrained with the full training days and the selected parameter. We run 10 trials each with training data of 90 trading days and with the following 60 trading days as test data. The last testing day of the final trial is December 30th, 2022, and each trial uses return data 30 trading days apart. In total, the 10 trials use return data for roughly a year and half, between May 3rd, 2021 and December 30th, 2022.

Table 1 presents the expected return, CVaR and standard deviation of each formulation with parameters selected with validation averaged across the trials. The HR formulations significantly outperform the other formulations in term of expected return while the W-DRO formulations have better risk measures. Nevertheless, the risk of the HR formulations remain comparable to the W-DRO models, indicating an overall superior returns/risk trade-off for the HR formulations. These results indicate that validation can be used to select the HR robustness parameters appropriately.

| Formulation         | Mean-CVaR | Wasserstein | HR   | KL  | Markowitz |
|---------------------|-----------|-------------|------|-----|-----------|
| Average Return      | +0.15%    | -0.08%      | +0.95%| -1.09%| -0.77%    |
| Average CVaR        | -11.53%   | -11.26%     | -11.66%| -14.02%| -12.57%   |
| Average Std         | 9.53%     | 9.25%       | 10.41%| 10.40%| 9.56%     |

Table 1: Average performance through 10 trials.

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A Omitted proofs of Section 2

A.1 Definitions and results on the conditional value-at-risk

We define the scaled conditional value-at-risk of a random variable \( C(\tilde{\xi}) \) at a quantile level \( \alpha \) as

\[
(1 - \alpha)\text{CVaR}_P^\alpha(C(\tilde{\xi})) := \sup \left\{ E_P[C(\tilde{\xi})R(\tilde{\xi})] : E_P[R(\tilde{\xi})] = 1 - \alpha, \ R : \Sigma \to [0,1] \right\}
\]

\[= \min \left\{ \beta(1 - \alpha) + E_P[\max(C(\tilde{\xi}) - \beta, 0)] : \beta \in \mathbb{R} \right\}; \tag{25} \]

see for instance Eichhorn and Römisch (2005). Denote the quantile of a random variable \( P \) at level \( \alpha \) as \( \tau_\alpha := \inf \{ \tau : P(C(\tilde{\xi}) \leq \tau) \geq \alpha \} \). We have from Rockafellar et al. (2000, Theorem 1) that \( \tau_\alpha \) is a minimizer in (25) and hence

\[
(1 - \alpha)\text{CVaR}_P^\alpha(C(\tilde{\xi})) = \tau_\alpha(1 - \alpha) + E_P[\max(C(\tilde{\xi}) - \tau_\alpha, 0)]
\]

\[= E_P[C(\tilde{\xi})\mathbb{I}(C(\tilde{\xi}) > \tau_\alpha)] + \tau_\alpha(P(C(\tilde{\xi}) \leq \tau_\alpha) - \alpha). \tag{26} \]

A.2 Proof of Corollary 2.2, Theorem 2.3, and Theorem 2.1

We first prove Theorem 2.1.

**Proof of Theorem 2.1.** Denote with \( \mathcal{C} \) the set of possible distributions of the random variable \( \tilde{\xi} \) and let \( \mathcal{C}' = \{ P' \in \mathcal{P} : \text{LP}_{\mathcal{N}}(P^c, P') < \alpha \} \). We first show that \( \mathcal{C} \subseteq \mathcal{C}' \). Consider any \( P \in \mathcal{C} \). We have

\[
\text{LP}_{\mathcal{N}}(P^c, P) = \inf_{\gamma \in \Gamma(P^c, P)} \int \mathbb{1}(\xi - \xi' \notin \mathcal{N}) \, d\gamma(\xi, \xi')
\]

\[
\leq \inf \{ \text{Prob}(\tilde{\xi}_1 - \tilde{\xi}_2 \notin \mathcal{N}) : \tilde{\xi}_1 \text{ and } \tilde{\xi}_2 \text{ r.v. with respective distributions } P^c \text{ and } P \}
\]

\[
\leq \text{Prob}(\tilde{\xi} + \tilde{n})\mathbb{1}(\tilde{c} = 0) + \tilde{\xi}_0\mathbb{1}(\tilde{c} = 1) - \tilde{\xi} \notin \mathcal{N}
\]

\[
\leq \text{Prob}(\tilde{\xi} + \tilde{n})\mathbb{1}(\tilde{c} = 0) + \text{Prob}(\tilde{c} = 0) + \text{Prob}(\tilde{c} = 1)
\]

\[
< \text{Prob}(\tilde{n} \notin \mathcal{N})\text{Prob}(\tilde{c} = 0) + \alpha = \alpha.
\]

and hence \( P \in \mathcal{C}' \).

We now show that \( \mathcal{C}' \subseteq \mathcal{C} \) as well. Consider any \( P \in \mathcal{C}' \) and hence \( \text{LP}_{\mathcal{N}}(P^c, P) < \alpha \). Consider random variables \( \tilde{\xi}^c \sim P^c \) and \( \tilde{\xi} \sim P \) with a joint distribution \( \tilde{\gamma} \) such that \( \int \mathbb{1}(\xi - \xi' \notin \mathcal{N}) \, d\tilde{\gamma}(\xi, \xi') < \alpha \). This joint distribution exists as \( \{ \int \mathbb{1}(\xi - \xi' \notin \mathcal{N}) \, d\gamma(\xi, \xi') : \gamma \in \Gamma(P^c, P) \} = \text{LP}_{\mathcal{N}}(P^c, P) < \alpha \). Set \( \tilde{c} := \mathbb{1}(\tilde{\xi}^c - \tilde{\xi} \notin \mathcal{N}) \) and \( \tilde{n} := \mathbb{1}(\tilde{c} = 0) \cdot (\tilde{\xi}^c - \tilde{\xi}) + \mathbb{1}(\tilde{c} = 1) n \), for some \( n \in \mathcal{N} \neq \emptyset \), and \( \tilde{\xi}_0 := \tilde{\xi}^c \). It follows that \( \tilde{\xi} = (\tilde{\xi} + \tilde{n}) \cdot \mathbb{1}(\tilde{c} = 0) + \tilde{\xi}_0\cdot \mathbb{1}(\tilde{c} = 1) \). Furthermore, we have \( \tilde{n} \in \mathcal{N} \) almost surely and \( \text{Prob}(\tilde{c} = 1) = \int \mathbb{1}(\xi - \xi' \notin \mathcal{N}) \, d\tilde{\gamma}(\xi, \xi') < \alpha \). Hence, we have that \( P^c \in \mathcal{C} \) as well.
Corollary 2.2 follows immediately. Let us now prove Theorem 2.3.

**Proof of Theorem 2.3.** Consider an arbitrary $\mathbb{P}^c \in \mathcal{P}$, $x \in \mathcal{X}$ and $\mathbb{P} \in \{ \mathbb{P}^\prime \in \mathcal{P} : \text{LP}_N(\mathbb{P}^c, \mathbb{P}^\prime) < \alpha \}$. Then Theorem 2.1 ensures that there exists an adversary, with the prescribed power corrupting $\mathbb{P}$ into $\mathbb{P}^c$. Suppose $\hat{c}$ verifies the out-of-sample guarantee. Hence, we must have $\hat{c}(x, \mathbb{P}^c) \geq \mathbb{E}[\ell(x, \hat{\xi})]$. As $\mathbb{P} \in \{ \mathbb{P}^\prime \in \mathcal{P} : \text{LP}_N(\mathbb{P}^c, \mathbb{P}^\prime) < \alpha \}$ is arbitrary we must have in fact that $\hat{c}(x, \mathbb{P}^c) \geq \sup \{ \mathbb{E}[\ell(x, \hat{\xi})] : \mathbb{P}^\prime \in \mathcal{P}, \text{LP}_N(\mathbb{P}^c, \mathbb{P}^\prime) \leq \alpha \} = \mathbb{E}^{\mathbb{P}^c}(\ell(x, \mathbb{P}^c))$.

### A.3 Generalization and proof of Theorem 2.4

**Theorem A.1.** For all $x \in \mathcal{X}$, $Q \in \mathcal{P}$, noise set $\mathcal{N}$ and $\alpha \in [0, 1]$, the following equality holds

$$
\max \left\{ \mathbb{E}_P[\ell(x, \xi)] : \mathbb{P}^\prime \in \mathcal{P}, \text{LP}_N(Q, \mathbb{P}^\prime) \leq \alpha \right\} = (1 - \alpha) CVaR^\beta_N(x) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi). \quad (27)
$$

Let $\xi_\infty \in \arg \max_{\xi \in \Sigma} \ell(x, \xi)$ and define a selection $S(\xi) \in \xi - \arg \max_{n \in \mathcal{N}, \xi - n \in \Sigma} \ell(x, \xi - n)$. There exists an optimal $\mathbb{P}^* \in (27)$ and a coupling $\gamma^* \in \Gamma(Q, \mathbb{P}^\prime)$ with $\int 1(\xi - \xi' \notin \mathcal{N}) \, d\gamma^*(\xi, \xi') \leq \alpha$ for which we have

$$
\mathbb{P}^*(\{S(\xi) : \xi \in \text{supp} Q \cup \{\xi_\infty\}) = 1 \quad \text{and \quad} \gamma^*(\{(\xi, S(\xi)) : \xi \in \text{supp} Q \cup \{\xi, \xi_\infty\} : \xi \in \text{supp} Q\}) = 1.
$$

**Proof.** We can rewrite the LHS of Equation (27) as

$$
\sup \left\{ \mathbb{E}_P[\ell(x, \xi)] : \mathbb{P}^\prime \in \mathcal{P}, \gamma \in \Gamma(Q, \mathbb{P}^\prime), \int 1(\xi - \xi' \notin \mathcal{N}) \, d\gamma(\xi, \xi') \leq \alpha \right\} \quad (28)
$$

We will prove the equality by first constructing a feasible solution to this problem that attains the RHS term in Equation (27). Subsequently, we then show than any other feasible solution attains an objective smaller than the RHS.

**Constructing a feasible solution attaining the RHS.** We start by providing some intuition on the construction that will follow. A feasible solution consists of a coupling $\gamma$ that moves mass from the distribution $Q$ into the distribution $\mathbb{P}^\prime$. For $(\xi, \xi') \in \Sigma \times \Sigma$, the quantity $\gamma(\xi, \xi')$ represents the amount of mass moved from $\xi$, in the distribution $Q$, to $\xi'$ in the distribution $\mathbb{P}^\prime$. The maximization problem seeks to move this mass in a way so to maximize the expected loss for the new mass distribution $\mathbb{P}^\prime$ under the constraint that at most $\alpha$ mass is moved by more than accounted for in $\mathcal{N}$. Hence, we are allowed to move $\alpha$ mass of points arbitrary far and $1 - \alpha$ with distance bounded in $\mathcal{N}$. In order to maximize the expectation, we naturally move the $\alpha$ mass of points (that can be moved arbitrarily) into the worst-case event in $\Sigma$ (that maximizes the loss $\ell$), which we denote later by $\xi_\infty$. The remaining mass is moved to the worst-case scenario within the noise set $\mathcal{N}$.
Let $\xi_\infty \in \text{arg max}\{\ell(x, \xi) : \xi \in \Sigma\}$. Consider a Bernoulli random variable $\hat{\zeta} \in \{0, 1\}$ independent of $\tilde{\zeta}$ with distribution $Z$ such that $Z(\hat{\zeta} = 1) = (Q(\ell^N(x, \hat{\zeta}) \leq \tau_\alpha) - \alpha)/Q(\ell^N(x, \tilde{\zeta}) = \tau_\alpha)$ and $Z(\hat{\zeta} = 0) = 1 - Z(\hat{\zeta} = 1)$ for $\tau_\alpha = \inf\{\tau \in \mathbb{R} : Q(\ell^N(x, \tilde{\zeta}) \leq \tau) \geq \alpha\}$ the $\alpha$-quantile of the inflated loss function. We denote by $\tilde{Q}$ the product distribution $Q \otimes Z$. Define the mapping $S_{N, \alpha} : \Sigma \times \{0, 1\} \to \Sigma$ as

$$S_{N, \alpha}(\xi, \zeta) = \begin{cases} 
\xi - \text{arg max}\{\ell(x, \xi - n) : n \in N, \xi - n \in \Sigma\} & \text{if } \ell^N(x, \xi) > \tau_\alpha, \\
\xi - \text{arg max}\{\ell(x, \xi - n) : n \in N, \xi - n \in \Sigma\} & \text{if } \ell^N(x, \xi) = \tau_\alpha \text{ and } \zeta = 1, \\
\xi_\infty & \text{if } \ell^N(x, \xi) = \tau_\alpha \text{ and } \zeta = 0, \\
\xi_\infty & \text{if } \ell^N(x, \xi) < \tau_\alpha.
\end{cases}$$

Define now a point $(P', \gamma)$ through

$$P'(A) := \tilde{Q}((\xi, \zeta) \in \Sigma \times \{0, 1\} : S_{N, \alpha}(\xi, \zeta) \in A) \quad \forall A \in B(\Sigma),$$

and

$$\gamma(\Pi) := \tilde{Q}((\xi, \zeta) \in \Sigma \times \{0, 1\} : (\xi, S_{N, \alpha}(\xi, \zeta)) \in \Pi) \quad \forall \Pi \in B(\Sigma \times \Sigma).$$

By definition of $P'$, it is clear that $\text{supp} P' \subseteq \{\xi - \text{arg max}_{n \in N, \xi - n \in \Sigma} \ell(x, \xi - n) : \xi \in \text{supp} Q\} \cup \{\xi_\infty\}$. We next show that the solution is feasible and its cost is exactly the RHS.

To show feasibility we need to have that $\gamma \in \Gamma(Q, P')$. For all event $A \in B(\Sigma)$, we have $\gamma(A \times \Sigma) = \tilde{Q}((\xi, \zeta) \in \Sigma \times \{0, 1\} : (\xi, S_{N, \alpha}(\xi, \zeta)) \in A \times \Sigma)) = Q(A)$. Furthermore $\gamma(\Sigma \times A) = \tilde{Q}((\xi, \zeta) \in \Sigma \times \{0, 1\} : S_{N, \alpha}(\xi, \zeta) \in A) = P'(A)$. Hence $\gamma \in \Gamma(Q, P')$. It also remains to verify the last constraint. We have

$$\int 1(\xi - \xi' \notin N) \, d\gamma(\xi, \xi') = \gamma((\xi, \xi') \in \Sigma \times \Sigma : \xi - \xi' \notin N))$$

$$= \tilde{Q}((\xi, \zeta) \in \Sigma \times \{0, 1\} : \xi - S_{N, \alpha}(\xi, \zeta) \notin N)$$

$$\leq \tilde{Q}(\ell^N(x, \hat{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 0 \text{ or } \ell^N(x, \hat{\xi}) < \tau_\alpha) \quad (29)$$

$$= \tilde{Q}(\ell^N(x, \hat{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 0) + \tilde{Q}(\ell^N(x, \hat{\xi}) < \tau_\alpha)$$

$$= Q(\ell^N(x, \hat{\xi}) = \tau_\alpha)Z(\hat{\zeta} = 0) + Q(\ell^N(x, \hat{\xi}) < \tau_\alpha) \quad (30)$$

where $\text{[29]}$ is by definition of $S_{N, \alpha}$ and $\text{[30]}$ is justified by the independence between $\hat{\zeta}$ and $\tilde{\zeta}$. The ultimate equality is by definition of $Z(\hat{\zeta} = 0)$. 

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Let us now compute the cost of this solution. We can decompose this cost as

\[ \mathbb{E}_{P'}[\ell(x, \tilde{\xi})] = \mathbb{E}_\gamma[\ell(x, \tilde{\xi})] = \mathbb{E}_{\tilde{\gamma}}[\ell(x, S_{N, \alpha}(\tilde{\xi}, \tilde{\zeta}))] \]

\[ = \mathbb{E}_{\tilde{\gamma}}[\ell(x, S_{N, \alpha}(\tilde{\xi}, \tilde{\zeta}))1(\ell^N(x, \tilde{\xi}) > \tau_\alpha)] + \mathbb{E}_{\tilde{\gamma}}[\ell(x, S_{N, \alpha}(\tilde{\xi}, \tilde{\zeta}))1(\ell^N(x, \tilde{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 1)] \]

\[ + \mathbb{E}_{\tilde{\gamma}}[\ell(x, S_{N, \alpha}(\tilde{\xi}, \tilde{\zeta}))1(\ell^N(x, \tilde{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 0, \text{ or } \ell^N(x, \tilde{\xi}) < \tau_\alpha)] \]

(31)

Let us now examine the second and third terms. By definition of \(S_{N, \alpha}\) the second term can be written as

\[ \mathbb{E}_{\tilde{\gamma}}[\ell(x, S_{N, \alpha}(\tilde{\xi}, \tilde{\zeta}))1(\ell^N(x, \tilde{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 1)] = \mathbb{E}_{\tilde{\gamma}}[\ell^N(x, \tilde{\xi})1(\ell^N(x, \tilde{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 1)] \]

\[ = \tau_\alpha \mathbb{E}_{\tilde{\gamma}}[1(\ell^N(x, \tilde{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 1)] \]

\[ = \tau_\alpha \tilde{Q}(\ell^N(x, \tilde{\xi}) = \tau_\alpha)Z(\tilde{\zeta} = 1) \]

\[ = \tau_\alpha (Q(\ell^N(x, \tilde{\xi}) = \tau_\alpha) - \tau_\alpha) \]

where the last two equalities are by the independence between \(\tilde{\xi}\) and \(\tilde{\zeta}\) and by the definition of \(Z(\tilde{\zeta} = 1)\).

We now examine the third term. By definition of \(S_{N, \alpha}\), we have

\[ \mathbb{E}_{\tilde{\gamma}}[\ell(x, S_{N, \alpha}(\tilde{\xi}, \tilde{\zeta}))1(\ell^N(x, \tilde{\xi}) = \tau_\alpha \text{ and } \tilde{\zeta} = 0, \text{ or } \ell^N(x, \tilde{\xi}) < \tau_\alpha)] \]

\[ = \max_{\xi \in \Sigma} \ell(x, \xi) \mathbb{E}_{\tilde{\gamma}}[1(\ell^N(x, \xi) = \tau_\alpha \text{ and } \xi = 0, \text{ or } \ell^N(x, \tilde{\xi}) < \tau_\alpha)] \]

\[ = \max_{\xi \in \Sigma} \ell(x, \xi) \left[ \tilde{Q}(\ell^N(x, \tilde{\xi}) = \tau_\alpha)Z(\xi = 0) + \tilde{Q}(\ell^N(x, \tilde{\xi}) < \tau_\alpha) \right] \]

\[ = \max_{\xi \in \Sigma} \ell(x, \xi) \left[ Q(\ell^N(x, \tilde{\xi}) = \tau_\alpha) - Q(\ell^N(x, \tilde{\xi}) < \tau_\alpha) + \alpha + Q(\ell^N(x, \tilde{\xi}) < \tau_\alpha) \right] \]

\[ = \alpha \max_{\xi \in \Sigma} \ell(x, \xi) \]

where the two last equalities are by independence of \(\tilde{\xi}\) and \(\tilde{\zeta}\) and definition of \(Z(\tilde{\zeta} = 0)\). Plugging the values of the second and third term in (31), we get exactly the RHS after observing Equation (26).

**Proving LHS\leq\text{RHS}**. We show that the cost of every feasible solution of the supremum problem (28) is smaller than the RHS. Let \(P' \in \mathcal{P}\) and \(\gamma \in \Gamma(Q, P')\) be a feasible solution to the supremum problem (28). Let \(\gamma(\{(\xi, \xi') \in \Sigma \times \Sigma : \xi - \xi' \not\in \mathcal{N}\}) = \alpha' \leq \alpha\). As \(\gamma\) has marginals \(Q\) and \(P'\) we have

\[ \mathbb{E}_{P'}[\ell(x, \tilde{\xi'})] = \mathbb{E}_\gamma[\ell(x, \tilde{\xi'})] = \mathbb{E}_\gamma[\ell(x, \tilde{\xi'})1(\tilde{\xi} - \tilde{\xi'} \in \mathcal{N})] + \mathbb{E}_\gamma[\ell(x, \tilde{\xi'})1(\tilde{\xi} - \tilde{\xi'} \not\in \mathcal{N})] \]

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Let us bound each of the two terms. We start by the second term. We have

$$
E_{\gamma} [\ell(x, \xi') \mathbb{1}(\xi - \xi' \notin \mathcal{N})] \leq \gamma((\xi, \xi') \in \Sigma \times \Sigma : \xi' - \xi \notin \mathcal{N}) \sup_{\xi \in \Sigma} \ell(x, \xi) = \max_{\xi \in \Sigma} \ell(x, \xi).
$$

We now turn to the first term. When $\xi - \xi' \in \mathcal{N}$, we have $\ell^N(x, \xi) = \max_{n \in \mathcal{N}} \ell(x, \xi - n) \geq \ell(x, \tilde{\xi})$. Hence,

$$
E_{\gamma} [\ell(x, \xi') \mathbb{1}(\tilde{\xi} - \xi' \in \mathcal{N})] \leq E_{\gamma} [\ell^N(x, \tilde{\xi}) \mathbb{1}(\tilde{\xi} - \xi' \notin \mathcal{N})] \leq \sup_{R: \Sigma \times \Sigma \to [0, 1]} \{E_{\gamma} [\ell^N(x, \tilde{\xi}) R(\tilde{\xi}, \xi')] : E_{\gamma} [R(\tilde{\xi}, \xi')] = 1 - \alpha\}
$$

where the sup is taken over measurable functions. The second inequality follows from $E_{\gamma} [\mathbb{1}(\tilde{\xi} - \xi' \notin \mathcal{N})] = 1 - \gamma((\xi, \xi') \in \Sigma \times \Sigma : \xi - \xi' \notin \mathcal{N})) = 1 - \alpha$. Furthermore,

$$
E_{\gamma} [\ell(x, \xi') \mathbb{1}(\tilde{\xi} - \xi' \in \mathcal{N})] \leq \sup_{R: \Sigma \times \Sigma \to [0, 1]} \{E_{\gamma} [\ell^N(x, \tilde{\xi}) R(\tilde{\xi}, \xi')] : E_{\gamma} [R(\tilde{\xi}, \xi')] = 1 - \alpha\}
$$

Here the first equality holds by the law of total expectation and the fact that $\ell^N(x, \tilde{\xi})$ does not depend on $\xi'$. The second inequality follows from the change of variables $R'(\tilde{\xi}) = E_{\gamma}(R(\tilde{\xi}, \xi'|\tilde{\xi}))$. That is, the random variable $R'$ denotes the conditional expectation of the random variable $R$ given $\tilde{\xi}$. The last equality is by definition of the conditional value-at-risk. Combining the bound on the first and second term, we get

$$
E_{\gamma} [\ell(x, \tilde{\xi})] \leq (1 - \alpha) \text{CVaR}_{\Sigma}^R (\ell^N(x, \tilde{\xi})) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi)
$$

which is exactly the RHS. \qed
A.4 Proof of Corollary 2.6

Proof of Corollary 2.6. Consider an ordering of the observed samples $\xi_1, \ldots, \xi_T$ by increasing inflated loss so that $\ell^N(x, \xi_1) \leq \ell^N(x, \xi_2) \leq \cdots \leq \ell^N(x, \xi_T)$. We have

\[
\max \left\{ \frac{1}{T} \sum_{t \in [T]} \ell(x, \xi_t - n_t - n_t') : \sum_{t \in [T]} \mathbb{1} \{n_t' \neq 0\} / T \leq \alpha, \ n_t \in \mathcal{N}, \ \xi_t - n_t - n_t' \in \Sigma \ \forall t \in [T] \right\}
\]

\[
= \max \left\{ \frac{1}{T} \sum_{t \in [T]} \ell^N(x, \xi_t - n_t') : \sum_{t \in [T]} \mathbb{1} \{n_t' \neq 0\} / T \leq \alpha, \ n_t \in \mathcal{N}, \ \xi_t - n_t' \in \Sigma \ \forall t \in [T] \right\}
\]

\[
= \sum_{t=0}^{T} \ell^N(x, \xi_t) / T + \alpha \max_{\xi \in \Sigma} \ell(x, \xi)
\]

\[
= (1 - \alpha) \text{CVaR}_{P_T}^\alpha(\ell^N(x, \xi)) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi) = \tilde{c}^N_{\text{LP}}(x, \hat{P}_T).
\]

The first equality follows from the definition of the inflated loss function as $\ell^N(x, \xi) = \max_{n \in \mathcal{N}, \xi - n \in \Sigma} \ell(x, \xi - n)$. The second equality use the fact that $\alpha T$ is integer. The penultimate equality follows from Equation (26). The final equality follow from Theorem 2.4.

B Omitted proofs of Section 3.1

B.1 Proof of Theorem 3.1

We first prove the following key lemma.

Lemma B.1. Let $\hat{P}_T$ be the empirical distribution of $T$ independent samples with distribution $P$ on a compact set $\Sigma$. Then for all $\delta > 0$ we have

\[
\text{Prob} \left( \exists \mathcal{P}' \in \mathcal{P}, \ \text{LP}_{B(0, \delta)}(\hat{P}_T, \mathcal{P}') \leq \delta, \ \text{KL}(\mathcal{P}', \mathcal{P}) \leq r \right) \geq 1 - \left( \frac{4}{\delta} \right)^{m(\Sigma, \delta)} \exp(-rT)
\]

where $m(\Sigma, \delta) := \min \{ k \geq 0 : \exists k_1, \ldots, k_k \in \Sigma \text{ s.t. } \cup_{i=1}^{k} B(\xi_i, \delta) \supseteq \Sigma \}$ denotes the internal covering number of the support set $\Sigma$.

Proof of Theorem 3.1. We have

\[
\text{Prob}(\exists \mathcal{P}' \in \mathcal{P} \text{ s.t. } \text{LP}_{B(0, \delta)}(\hat{P}_T, \mathcal{P}') \leq \delta, \ \text{KL}(\mathcal{P}', \mathcal{P}) \leq r)
\]

\[
= \text{Prob}(\exists \mathcal{P}' \in \mathcal{P} \text{ s.t. } \pi_{\text{LP}}(\hat{P}_T, \mathcal{P}') \leq \delta, \ \text{KL}(\mathcal{P}', \mathcal{P}) \leq r))
\]

\[
= \text{Prob}(\hat{P}_T \in \{ \hat{P} \in \mathcal{P} : \exists \mathcal{P}' \in \mathcal{P} \text{ s.t. } \pi_{\text{LP}}(\hat{P}, \mathcal{P}') \leq \delta, \ \text{KL}(\mathcal{P}', \mathcal{P}) \leq r \})
\]

\[
= 1 - \text{Prob}(\hat{P}_T \in \mathcal{A})
\]

where $\mathcal{A}$ is defined as $\mathcal{A}^c = \{ \hat{P} \in \mathcal{P} : \exists \mathcal{P}' \in \mathcal{P} \text{ s.t. } \pi_{\text{LP}}(\hat{P}, \mathcal{P}') \leq \delta, \ \text{KL}(\mathcal{P}', \mathcal{P}) \leq r \}$. We will show using results from Dembo and Zeitouni (2009) that $\text{Prob}(\hat{P}_T \in \mathcal{A}) \leq (4/\delta)^{m(\Sigma, \delta)} \exp(-T \inf_{\mathcal{P}' \in \mathcal{A}} \text{KL}(\mathcal{P}', \mathcal{P}))$.  

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where \( A^\delta = \{ P' \in P : \pi_{LP}(P', P'') \leq \delta, P'' \in A \} \) is the \( \delta \)-inflation of the set \( A \), by the LP distance \( \pi_{LP} \).

This last inequality immediately leads to the conclusion by remarking that \( P' \in A^\delta \implies KL(P', P) > r \).

Indeed, suppose that there exists \( P'' \in A^\delta \) with \( KL(P'', P) \leq r \). By definition of \( A^\delta \), there exists \( P' \in A \) such that \( \pi_{LP}(P'', P') = \pi_{LP}(P', P'') \leq \delta \). As we have both \( \pi_{LP}(P', P'') \leq \delta \) and \( KL(P'', P) \leq r \) which implies that \( P' \in A^\delta \), a contradiction.

Let us now show that \( \text{Prob}(\hat{P}_T \in A) \leq (4/\delta)^{m(\Sigma, \delta)} \exp(-T \inf_{P' \in A^\delta} KL(P', P)) \). Denote \( B_{LP}^A(P') := \{ P'' \in P : \pi_{LP}(P'', P') \leq \delta \} \) the LP ball inside \( A \), which is compact as \( \pi_{LP} \) is continuous in the weak topology and \( \Sigma \) is compact [Prokhorov (1956), Dembo and Zeitouni (2009, Exercise 4.5.5)] established that for any set \( A \) and \( \delta > 0 \), we have for all \( n \geq 1 \) the upper bound

\[
\text{Prob}(\hat{P}_T \in A) \leq m_{LP}(A, \delta) \exp \left(-T \inf_{P' \in A^\delta} KL(P', P)\right)
\]

where \( m_{LP}(A, \delta) = \min\{ k \geq 0 : \exists P_1, \ldots, P_k \in A \text{ s.t. } U_{i=1}^k B_{LP}(P_i, \delta) \supseteq A \} \) denotes the internal covering number of the set \( A \) of interest with LP balls of radius \( \delta \). [Dembo and Zeitouni (2009, Exercise 6.2.19)] upper bounds the covering number of any set \( A \) in terms of the covering number of the event set \( \Sigma \) as \( m_{LP}(A, \delta) \leq m_{LP}(P, \delta) \leq (4/\delta)^{m(\Sigma, \delta)} \), for all \( \delta > 0 \).

**Proof of Theorem 3.1.** Denote with \( N'' \) the compact set in the interior of \( N \) where the noise realizes and \( \alpha' < \alpha \) the fraction of data points misspecified. There exists \( \delta > 0 \) such that \( N'' + B(0, \delta) \subset N \) and \( \alpha' + \delta \leq \alpha \). Denote with \( \hat{Q}_T \) the empirical distribution of clean samples from \( \hat{P} \) before corruption. Theorem 2.1 ensures that \( LP_{N''}((\hat{P}_T, \hat{Q}_T) \leq \alpha' \). Furthermore, Lemma B.1 ensures that

\[
\text{Prob}(P \in \{ P'' \in P : \exists P' \in P, \hat{Q} \in P \text{ s.t. } LP_{N''}((\hat{P}_T, \hat{Q}) \leq \alpha', LP_B((0, \delta)(\hat{P}, P') \leq \delta, KL(P', P'') \leq r) \}) = 1 - (4/\delta)^{m(\Sigma, \delta)} \exp(-rT).
\]

Combining these two results yields

\[
1 - (4/\delta)^{m(\Sigma, \delta)} \exp(-rT) \leq \text{Prob}(P \in \{ P'' \in P : \exists P' \in P, \hat{Q} \in P \text{ s.t. } LP_{N''}((\hat{P}_T, \hat{Q}) \leq \alpha', LP_B((0, \delta)(\hat{P}, P') \leq \delta, KL(P', P'') \leq r) \}) = \text{Prob}(P \in \{ P'' \in P : \exists P' \in P \text{ s.t. } LP_{N''}((\hat{P}_T, P') \leq \alpha + \delta, KL(P', P'') \leq r) \}).
\]

The final equality follows from \( U' := \{ P'' \in P : \exists P' \in P, \hat{Q} \in P \text{ s.t. } LP_{N''}((\hat{P}_T, \hat{Q}) \leq \alpha', LP_B((0, \delta)(\hat{P}, P') \leq \delta, KL(P', P'') \leq r) \} = \{ P'' \in P : \exists P' \in P \text{ s.t. } LP_{N''}((\hat{P}_T, P') \leq \alpha + \delta, KL(P', P'') \leq r) \}. \) Hence it follows that

\[
\text{Prob} \left( \sup_{P' \in U'} E_{\hat{P}}[\ell(x, \hat{\xi})] \geq E_{\hat{P}}[\ell(x, \hat{\xi}), \forall x \in X] \right) \geq 1 - (4/\delta)^{m(\Sigma, \delta)} \exp(-rT)
\]

we get the desired result by noticing that \( \hat{c}_{HR}^{N, \alpha, \tau}(x, \hat{P}_T) \geq \sup_{P' \in U'} E_{\hat{P}}[\ell(x, \hat{\xi})] \) as \( U' \) is a subset of the
ambiguity set of $\hat{c}_{HR}^{N,\alpha,r}$. □

B.2 Proof of Theorem 3.2

Proof of Theorem 3.2 Suppose for the sake of contradiction that there exists a predictor $\hat{c}$ verifying the out-of-sample guarantee and such that there exists $x_0$ and $\hat{P}_T \in \mathcal{P}$ for some $T$ such that

$$\hat{c}(x_0, \hat{P}_T) < \hat{c}_{HR}^{N,\alpha,r}(x_0, \hat{P}_T). \tag{32}$$

Denote $D = \{\xi_1, \ldots, \xi_T\}$ the data points which constitute the empirical distribution $\hat{P}_T$. Without loss of generality, we may assume $\ell_N(x_0, \xi_1) \leq \ldots \leq \ell_N(x_0, \xi_T)$. Denote $\epsilon := \hat{c}_{HR}^{N,\alpha,r}(x_0, \hat{P}_T) - \hat{c}(x_0, \hat{P}_T) > 0$.

The goal is to construct a data generation process and a malicious adversary (with the prescribed power) for which $\hat{c}$ does not verify the out-of-sample guarantee thereby reaching a contradiction. We will first show that the inequality \((32)\) is still verified with slightly perturbed parameters $\alpha$ and $N$. These perturbed parameters will be convenient to construct our malicious adversary.

Recall that

$$\hat{c}_{HR}^{N,\alpha,r}(x_0, \hat{P}_T) = \sup\{\mathbb{E}_{P'}[\ell(x_0, \hat{\xi})] : P', \hat{Q} \in \mathcal{P}, \operatorname{LP}_N(\hat{P}_T, \hat{Q}) \leq \alpha, \operatorname{KL}(\hat{Q}||P') \leq r\} \quad \forall x \in \mathcal{X}.$$ 

Remark that the function $\hat{c}_{HR}^{N,\alpha,r}(x_0, \hat{P}_T)$ is continuous in the parameters $\alpha$ and $r$. Indeed, both the pseudo divergence $\operatorname{LP}_\eta$ and the KL divergence are jointly convex in the optimization variables $P'$ and $\hat{Q}$ and the objective function $\mathbb{E}_{P'}[\ell(x_0, \hat{\xi})]$ is linear, which implies that $\hat{c}_{HR}^{N,\alpha,r}(x_0, \hat{P}_T)$ is concave in $(\alpha, r) \in \mathbb{R}_+^2$. As concave functions are continuous in the interior of their domains, $\hat{c}_{HR}^{N,\alpha,r}$ is continuous at $r > 0$ and $\alpha > 0$.

Hence, we can consider $0 < r' < r$ and $0 < \alpha' < \alpha$, with $\alpha'$ rational so that $\hat{c}_{HR}^{N,\alpha',r'}(x_0, \hat{P}_T) \geq \hat{c}_{HR}^{N,\alpha,r}(x_0, \hat{P}_T) - \epsilon/4$. Consider an integer $k \geq 1$ such that $Tk\alpha'$ is integer and denote $T' = kT$. Note that $\hat{P}_T$ can also be interpreted as an empirical distribution of $T'$ data points (by duplicating all data points $k$ times) and our adversary can misspecify up to exactly $T'\alpha' \in \mathbb{N}$ data points, which will be useful later on.

We now build our data generating distribution $P'$ and our adversary. Let us consider a worst-case corrupted version of our new dataset defined as

$$D' = \{\xi_i' := \xi_i + \arg\max_{\delta \in \mathcal{N}, \xi_i + \delta \in \Sigma} \ell(x_0, \xi_i + \delta)\}_{i=1}^T \cup \{\xi'_{\infty} \in \arg\max_{\xi \in \Sigma} \ell(x_0, \xi)\}$$

which is well defined as $\xi \rightarrow \ell(x_0, \xi)$ is continuous and $\mathcal{N}$ is compact. Each of the points $\xi_i'$ is a perturbed version of $\xi_i$ by adversarial noise, and $\xi'_{\infty}$ is a worst-case data point maximizing the loss. As the empirical distribution $\hat{P}_T$ is finitely supported, from Lemma 3.5 and Corollary 2.6 it follows that we can consider the
maximizers in the HD problem, which write as

\[ \hat{p}^{N, \alpha'}_T = \sum_{i=\tau+1}^{T} \delta_{\xi_i} \hat{p}_T(\xi_i) + (1 - \alpha') \sum_{i=\tau+1}^{T} \hat{p}_T(\xi_i) \delta_{\xi_i} + \alpha' \delta_{\xi_\infty} \]

with \( \tau \) the smallest integer such that \( 1 - \alpha' - \sum_{i=\tau+1}^{T} \hat{p}_T(\xi_i) > 0 \), and \( P^*\) verifying \( \text{supp}(P^*) \subseteq \mathcal{D}' \) such that

\[ \mathcal{L}_{N}(\hat{p}_T, \hat{p}^{N, \alpha'}_T) \leq \alpha', \quad \text{KL}(\hat{p}^{N, \alpha', \delta}_{HR}(x_0, \hat{p}_T)) = \mathbb{E}_{P^*}[\ell(x_0, \hat{\xi})]. \]

We have \( \text{supp}(\hat{p}^{N, \alpha'}_T) \subseteq \mathcal{D}' \), and as \( \alpha' \tau' \) is integer, it follows that \( \hat{p}^{N, \alpha'}_T(\xi') \in \frac{1}{\tau'} \mathbf{N} \) for all \( \xi' \in \mathcal{D}' \). This implies that \( \hat{p}^{N, \alpha'}_T \) can also be seen as the empirical distribution of \( T' \) points in \( \Sigma \).

Finally, to complete building our adversary, we perturb slightly the support of \( P^* \) and \( \text{supp}(\hat{p}^{N, \alpha'}_T) \). As \( \xi \to \ell(x_0, \xi) \) is continuous and \( \text{cl}(\text{int}(N)) = N \), we can find points \( \xi'' \) such that \( \xi'' - \xi \in \text{int}(N) \) so that for \( \hat{p}^* \) defined as \( \hat{p}^*(\xi'') = P^*(\xi') \) for all \( i \) and \( \hat{p}^*(\xi'') = P^*(\xi'') \), we have \( \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})] + \epsilon/4 \geq \mathbb{E}_{P^*}[\ell(x_0, \hat{\xi})] = \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})] = \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})]. \)

Here, we move each point \( \xi'' \) of \( \mathcal{D}' \) (the support of \( P^* \)) into a close point \( \xi''' \). Define \( N' = \{0\} \cup \{\xi'' - \xi \} \subset \text{int}(N) \). Define similarly the distribution \( \hat{p}^{N', \alpha'}_T \) through \( \hat{p}^{N', \alpha'}_T(\xi''') = \hat{p}^{N, \alpha'}_T(\xi') \) for all \( i \) and \( \hat{p}^{N', \alpha'}_T(\xi'') = \hat{p}^{N, \alpha'}_T(\xi'') \). The distributions \( \hat{p}^{N', \alpha'}_T \) and \( P^* \) preserve the properties of \( \hat{p}^{N, \alpha'}_T \) and \( P^* \); as \( \mathcal{N} \subset N \), we have \( \mathcal{L}_{N}(\hat{p}_T, \hat{p}^{N', \alpha'}_T) \leq \mathcal{L}_{N}(\hat{p}_T, \hat{p}^{N, \alpha'}_T) \leq \alpha' \) and as the support of \( \hat{p}^{N', \alpha'}_T \) and \( P^* \) moved in exactly the same way, we have \( \text{KL}(\hat{p}^{N', \alpha'}_T, \hat{p}^*) = \text{KL}(\hat{p}^{N, \alpha'}_T, \hat{p}) \leq r \). Hence, our constructed distributions verify

\[ \mathcal{L}_{N}(\hat{p}_T, \hat{p}^{N', \alpha'}_T) \leq \alpha', \quad \text{KL}(\hat{p}^{N', \alpha', \delta}_{HR}(x_0, \hat{p}_T)) \leq \alpha', \quad \mathbb{E}_{P^*}[\ell(x_0, \hat{\xi})] \geq \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})] - \epsilon/4 \geq \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})] - \epsilon/2 > \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})]. \]

We are now ready to construct our data generation process and our adversary. Consider the data generation process of out-of-sample distribution \( \hat{p}^* \) and denote \( \hat{Q}^*_t \) its empirical distribution for all \( t \in N \). In particular, \( \hat{Q}^*_t \) is a random variable. As \( \hat{p}^{N', \alpha'}_T \) shares the same support as \( \hat{p}^* \), and \( \hat{p}^{N', \alpha'}(\xi) \subset \frac{1}{\tau'} \mathbf{N} \), the distribution \( \hat{p}^{N', \alpha'}_T \) can be seen as a potential realization of the empirical distribution of \( \hat{Q}^*_t \). As \( \mathcal{L}_{N}(\hat{p}_T, \hat{p}^{N', \alpha'}_T) \leq \alpha' \), there exists an adversary following Corollary that can perturb \( \hat{p}^{N', \alpha'}_T \) into \( \hat{p}_T \) by noise limited to the set \( N' \) and misspecification of less than \( \alpha' \). For this adversary, denote \( \hat{p}^*_T \) the corrupted distribution from the clean samples \( Q^*_t \) for all \( t \). We have

\[ \text{Prob} \left( \hat{c}(x_0, \hat{p}^*_T) < \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})] \right) \geq \text{Prob} \left( \hat{p}^*_mT = \hat{p}_T \right) \geq \text{Prob} \left( \hat{q}^*_mT = \hat{p}^{N', \alpha'}_T \right), \quad \forall m \geq 1 \]

Indeed, for any data size \( mT' \), if the empirical distribution from \( \hat{p}^* \) realizes as \( \hat{p}^{N', \alpha'}_T \), then the considered adversary can corrupt \( \hat{q}^*_mT = \hat{p}^{N', \alpha'}_T \) into \( \hat{p}^*_mT = \hat{p}_T \). Then, inequality (33) ensures that \( \mathbb{E}_{\hat{p}_T}[\ell(x_0, \hat{\xi})] > \hat{c}(x_0, \hat{p}_T) = \hat{c}(x_0, \hat{p}^*_mT) \).
If suffices now to lower bound the probability of the event \( \hat{Q}_{mT'}^* = \hat{P}_{T'}^{N',\alpha'} \). As \( \bar{P}^* \) is a discrete distribution supported on less than \( T + 1 \) points, and as \( \hat{P}_{T'}^{N',\alpha'}(\xi) \subset \frac{1}{T'}N \), Theorem 11.4 in [Cover and Thomas (1991)] implies

\[
\text{Prob}(\hat{Q}_{mT'}^* = \hat{P}_{T'}^{N',\alpha'}) \geq \frac{1}{(mT' + 1)^{T+1}} \exp \left( -mT' \text{KL}(\hat{P}_{T'}^{N',\alpha'}, \bar{P}^*) \right) \geq \frac{1}{(mT' + 1)^{T+1}} \exp \left( -mT'r' \right), \ \forall m \geq 1.
\]

Taking \( m \to \infty \) leads then to

\[
\limsup_{t \to \infty} \frac{1}{t} \log \text{Prob}(\hat{c}(x_0, \hat{P}_{T'}^*) < \mathbb{E}_P[\ell(x_0, \xi)]) \geq -r' > -r,
\]

a contradiction with \( \hat{c} \) verifying the out-of-sample guarantee for out-of-sample distribution \( \bar{P}^* \) and the considered adversary. \( \square \)

### B.3 Proof of Theorem 3.3

**Proof of Theorem 3.3.** Following Theorem 3.2 and the change of variables \( w'_k = w_k - \lambda \) we have

\[
\hat{c}_{\text{HR}}^{N,\alpha,r}(x, \hat{P}_T) = \begin{cases} 
\inf_{w \in \mathbb{R}^K, \lambda \geq 0, \beta \geq 0, \eta \in \mathbb{R}} \sum_{k \in [K]} w'_k \hat{P}_T(\xi_k) + \lambda r + \beta \alpha + \eta \\
\text{s.t. } w'_k \geq \lambda \left( \log \left( \frac{\lambda}{\eta - \max_{\xi \in \mathbb{X}_E} \ell(x, \xi)} \right) - 1 \right) \quad \forall k \in [K], \\
\quad w'_k \geq \lambda \left( \log \left( \frac{\lambda}{\eta - \max_{\xi \in \mathbb{X}_E} \ell(x, \xi)} \right) - 1 \right) - \beta \quad \forall k \in [K], \\
\quad \eta \geq \max_{\xi \in \mathbb{X}_E} \ell(x, \xi).
\end{cases}
\]

(34)

Introduce now the associated Lagrangian function

\[
L(w', \lambda, \beta, \eta; \hat{q}, s) := \sum_{k \in [K]} w'_k \hat{P}_T(\xi_k) + \lambda r + \beta \alpha + \eta + \sum_{k \in [K]} \hat{q}_k \left( \lambda \left( \log \left( \frac{\lambda}{\eta - \max_{\xi \in \mathbb{X}_E} \ell(x, \xi)} \right) - 1 \right) - w'_k \right) \\
+ \sum_{k \in [K]} s_k \left( \lambda \left( \log \left( \frac{\lambda}{\eta - \max_{\xi \in \mathbb{X}_E} \ell(x, \xi)} \right) - 1 \right) - \beta - w'_k \right) \\
= \sum_{k \in [K]} w'_k \left( \hat{P}_T(\xi_k) - \hat{q}_k - s_k \right) + \beta \left( \alpha - \sum_{k \in [K]} \hat{s}_k \right) + \lambda r + \eta \\
+ \sum_{k \in [K]} \hat{q}_k \lambda \left( \log \left( \frac{\lambda}{\eta - \max_{\xi \in \mathbb{X}_E} \ell(x, \xi)} \right) - 1 \right) + \sum_{k \in [K]} s_k \lambda \left( \log \left( \frac{\lambda}{\eta - \max_{\xi \in \mathbb{X}_E} \ell(x, \xi)} \right) - 1 \right)
\]
and dual function

\[
g(\hat{q}, s) := \inf \left\{ L(w, \lambda, \eta ; \hat{q}, s) : w' \in \mathbb{R}^K, \lambda \in \mathbb{R}_+^K, s \in \mathbb{R}_+^K, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}
\]

\[
= \sum_{k \in [K]} \chi_{-\infty}(\hat{p}'(x_k)) = \hat{q} + s_k + \chi_{-\infty}(\sum_{k \in [K]} s_k \leq \alpha)
\]

\[
= \left\{ \begin{array}{l}
\inf \lambda r + \eta + \sum_{k \in [K]} \hat{q}_k \left( \log \frac{\lambda}{\eta - \ell(x, \xi_k)} - 1 \right) + \sum_{k \in [K]} s_k \left( \log \frac{\lambda}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} - 1 \right) \\
\text{s.t. } \lambda \in \mathbb{R}_+, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi)
\end{array} \right.
\]

where \( \chi_{-\infty}(S) = 0 \) if \( S \) is true and \( -\infty \) otherwise. We remark that the minimization problem in Equation (34) satisfies the Slater constraint qualification condition. As Slater’s constraint qualification condition is met we have strong duality and the dual optimal value is attained (Bertsekas 2009, Proposition 5.3.1). Hence,

\[
\hat{c}^{N, \alpha, T}(x, \hat{p}) = \max \left\{ g(\hat{q}, s), \hat{q} \in \mathbb{R}^K_+, s \in \mathbb{R}^K_+ \right\}
\]

\[
= \max \left\{ \begin{array}{l}
\max \inf \left\{ \lambda r + \eta + \sum_{k \in [K]} \hat{q}_k \left( \log \frac{\lambda}{\eta - \ell(x, \xi_k)} - 1 \right) + \sum_{k \in [K]} s_k \left( \log \frac{\lambda}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} - 1 \right) : \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi), \lambda \in \mathbb{R}_+ \right\} \\
\text{s.t. } \hat{q} \in \mathbb{R}^K_+, s \in \mathbb{R}^K_+, \hat{p}_T(x_k) = \hat{q}_k + s_k \quad \forall k \in [K], \\
\sum_{k \in [K]} s_k \leq \alpha
\end{array} \right.
\]

\[
= \max \left\{ \begin{array}{l}
\max \inf \left\{ \lambda r + \eta + \sum_{k \in [K]} \hat{q}_k \sup_{p' \geq 0} \left( \ell'(x, \xi_k) - \eta \right) \frac{\hat{p}'_k}{\hat{q}_k} + \lambda \log \frac{\hat{p}'_k}{\hat{q}_k} + \left( \sum_{k \in [K]} s_k \right) \max_{\xi \in \Sigma} \ell(x, \xi) - \eta \right\} \\
\text{s.t. } \hat{q} \in \mathbb{R}^K_+, s \in \mathbb{R}^K_+, \hat{p}_T(x_k) = \hat{q}_k + s_k \quad \forall k \in [K], \\
\sum_{k \in [K]} s_k \leq \alpha
\end{array} \right.
\]

\[
= \max \left\{ \begin{array}{l}
\max \inf \left\{ \lambda r + \eta + \sum_{k \in [K]} \hat{q}_k \left[ \ell'(x, \xi_k) - \eta \right] \frac{\hat{p}'_k}{\hat{q}_k} + \lambda \log \frac{\hat{p}'_k}{\hat{q}_k} + \left( \sum_{k \in [K]} s_k \right) \max_{\xi \in \Sigma} \ell(x, \xi) - \eta \right\} \\
\text{s.t. } \hat{q} \in \mathbb{R}^K_+, s \in \mathbb{R}^K_+, \hat{p}' \in \mathbb{R}^{K+1}, \hat{p}_T(x_k) = \hat{q}_k + s_k \quad \forall k \in [K], \\
\sum_{k \in [K]} s_k \leq \alpha
\end{array} \right.
\]

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Proposition 5.5.7) with respect to the saddle point function from which the second equality follows where we choose $r$

Here the third equality follows from the standard convex conjugate identifies. First, observe that $\sup_{r \geq 0} a r + \log(r) = -(1 + \log(-a))$ for $a \leq 0$. Hence, for every $k \in [K]$ we have

$$-\lambda \left( 1 + \log \left( -\frac{\eta - \ell^N(x, \xi_k)}{\lambda} \right) \right) = \sup_{r_k \geq 0} (\eta - \ell^N(x, \xi_k)) r_k + \lambda \log(r)$$

and

$$-\lambda \left( 1 + \log \left( -\frac{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)}{\lambda} \right) \right) = \sup_{r_{K+1} \geq 0} (\eta - \max_{\xi \in \Sigma} \ell(x, \xi)) r_{K+1} + \lambda \log(r)$$

from which the second equality follows where we choose $r_k = p_k'/q_k$ and $r_{K+1} = p_{K+1}'/(\sum_{k \in [K]} s_k)$ for some vector $p \in R_{+}^{K+1}$. The fourth equality follows readily from a standard minimax theorem of Bertekas (2009, Proposition 5.5.7) with respect to the saddle point function

$$L(\eta, \lambda, p') := \lambda r + \eta + \sum_{k \in [K]} \hat{q}_k \left( (\ell^N(x, \xi_k) - \eta) \frac{p_{k'}}{q_k} + \lambda \log \left( \frac{p_{k'}}{q_k} \right) \right)$$

$$= \lambda \left( r - \sum_{k \in [K]} \hat{q}_k \log \frac{q_k}{p_{k'}} - (\sum_{k \in [K]} s_k) \log \left( \frac{p_{K+1}}{q_{K+1}} \right) \right) + \eta(1 - \sum_{k \in [K+1]} p_k')$$

$$+ \sum_{k \in [K]} p_k \ell^N(x, \xi_k) + p_{K+1} \max_{\xi \in \Sigma} \ell(x, \xi)$$

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where we may exploit \( r > 0 \). The penultimate inequality follows from the fact that we may choose without loss of optimality \( p_{K+1} = 1 - \sum_{k \in [K]} p_k \) as the logarithm is an increasing function. The final equality follows similarly from defining \( q_{K+1} = 1 - \sum_{k \in [K]} q_k \) and observing that we have \( \sum_{k \in K} \hat{p}_T(\xi_k) = 1 = \sum_{k \in K} q_k + \sum_{k \in K} s_k \). □

### B.4 Proof of Lemma 3.5

**Proof of Lemma 3.5** First, observe that

\[
\hat{c}_{HR}^{N,\alpha,r}(x, \hat{P}_T) := \sup \left\{ \mathbb{E}_{P'}[\ell(x, \bar{\xi})] : P' \in \mathcal{P}, \bar{\xi} \in \mathcal{P}, \mathbb{P}_N(\hat{p}_T, \bar{\xi}) \leq \alpha, \KL(\hat{q}\|P') \leq r \right\}
\]

\[
= \sup \left\{ \sup \left\{ \mathbb{E}_{P'}[\ell(x, \bar{\xi})] : P' \in \mathcal{P}, \KL(\hat{q}\|P') \leq r \right\} : \bar{\xi} \in \mathcal{P}, \mathbb{P}_N(\hat{p}_T, \bar{\xi}) \leq \alpha \right\}
\]

\[
= \sup \left\{ \inf \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) d\tilde{q}(\xi) + (r-1)\lambda + \eta : \lambda \geq 0, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\} : \tilde{q} \in \mathcal{P}, \mathbb{P}_N(\hat{p}_T, \tilde{q}) \leq \alpha \right\}
\]

\[
\leq \inf \left\{ \sup \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) d\tilde{q}(\xi) + (r-1)\lambda + \eta : \tilde{q} \in \mathcal{P}, \mathbb{P}_N(\hat{p}_T, \tilde{q}) \leq \alpha \right\} : \lambda \geq 0, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}.
\]

Here, the second equality follows from Lemma C.1. Second, we remark that for any \( \lambda \geq 0 \) and \( \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \) we have that \( a \mapsto \lambda \log \left( \frac{\lambda}{\eta - a} \right) \) is an increasing function. In particular, denote with \( \{\xi_1, \ldots, \xi_K\} \) the support of \( \hat{p}_T \in \mathcal{P} \) ordered to that \( \ell^N(x, \xi_1) \leq \cdots \leq \ell^N(x, \xi_K) \) then clearly we also have \( \log \left( \frac{\lambda}{\eta - \ell^N(x, \xi_1)} \right) \leq \cdots \leq \log \left( \frac{\lambda}{\eta - \ell^N(x, \xi_K)} \right) \). Following now Theorem 2.4 we have for all \( \lambda \geq 0 \) and \( \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \) that

\[
\sup \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) d\tilde{q}(\xi) : \tilde{q} \in \mathcal{P}, \mathbb{P}_N(\hat{p}_T, \tilde{q}) \leq \alpha \right\} = \mathbb{E}_{\hat{P}_T^N} \left[ \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \right].
\]

with worst-case distribution \( \hat{P}_T^{N,\alpha} \in \mathcal{P} \) satisfying \( \mathbb{P}_N(\hat{p}_T, \hat{P}_T^{N,\alpha}) \leq \alpha \) and given explicitly in Equation (11). We observe that the worst-case distribution \( \hat{P}_T^{N,\alpha} \in \mathcal{P} \) is not a function of \( \eta \) nor \( \lambda \). Van Parys et al. (2021, Proposition 5) now implies that

\[
\hat{c}_{HR}^{N,\alpha,r}(x, \hat{P}_T) \leq \inf \left\{ \mathbb{E}_{\hat{P}_T^{N,\alpha}} \left[ \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \right] + (r-1)\lambda + \eta : \lambda \geq 0, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}
\]

\[
\leq \hat{c}_{KL}(x, \hat{P}_T^{N,\alpha}) = \mathbb{E}_{\hat{P}_T^{N,\alpha}} \left[ \ell(x, \hat{\xi}) \right].
\]
for some worst-case distribution \( \hat{p}_T^{N,\alpha,r} \in \mathcal{P} \) satisfying \( \text{KL}(\hat{p}_T^{N,\alpha,r} || \hat{p}_T^{N,\alpha,r}) \leq r \). Clearly, as our defined worst-case distributions \( \hat{p}_T^{N,\alpha} \in \mathcal{P} \) and \( \hat{p}_T^{N,\alpha,r} \in \mathcal{P} \) satisfy \( \text{LP}_{\mathcal{X}}(\hat{p}_T, \hat{p}_T^{N,\alpha}) \leq \alpha \) and \( \text{KL}(\hat{p}_T^{N,\alpha,r} || \hat{p}_T^{N,\alpha,r}) \leq r \), respectively. Hence, the distributions \( \hat{Q} = \hat{p}_T^{N,\alpha} \) and \( \hat{P}' = \hat{p}_T^{N,\alpha,r} \) are feasible in the maximization problem characterizing the HD predictor in Equation (12). Hence, we have

\[
\hat{c}_{\mathcal{HR}}^{N,\alpha,r}(x, \hat{P}_T) \geq \mathbb{E}_{\hat{P}_T^{N,\alpha,r}}[\ell(x, \xi)] = \hat{c}_{\text{KL}}(x, \hat{P}_T^{N,\alpha})
\]

from which the claim follows immediately. \( \square \)

### B.5 Generalization and Proof of Theorem 3.4

**Theorem B.2 (General Dual Formulation).** Let \( \hat{P} \in \mathcal{P} \). For all \( x \in \mathcal{X} \), the HD robust predictor (12) admits for all \( r > 0 \) the dual representation

\[
\hat{c}_{\mathcal{HR}}^{N,\alpha,r}(x, \hat{P}) = \inf \left\{ \int w(\xi) \, d\hat{P}(\xi) + \lambda(r-1) + \beta \alpha + \eta \right\}
\]

\[\text{s.t.} \quad w : \Sigma \rightarrow \mathbb{R}, \; \lambda \geq 0, \; \beta \geq 0, \; \eta \in \mathbb{R}, \]

\[w(\xi) \geq \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right), \quad w(\xi) \geq \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) - \beta \quad \forall \xi \in \Sigma.\]

**Proof.** We have that

\[
\hat{c}_{\mathcal{HR}}^{N,\alpha,r}(x, \hat{P}) := \sup \left\{ \mathbb{E}_{\hat{P}'}[\ell(x, \xi)] : \hat{P}' \in \mathcal{P}, \; \hat{Q} \in \mathcal{P}, \; \text{LP}_{\mathcal{X}}(\hat{P}, \hat{Q}) \leq \alpha, \; \text{KL}(\hat{Q} || \hat{P}') \leq r \right\}
\]

\[= \sup \left\{ \sup \left\{ \mathbb{E}_{\hat{P}'}[\ell(x, \xi)] : \hat{P}' \in \mathcal{P}, \; \text{KL}(\hat{Q} || \hat{P}') \leq r \right\} : \hat{Q} \in \mathcal{P}, \; \text{LP}_{\mathcal{X}}(\hat{P}, \hat{Q}) \leq \alpha \right\}
\]

\[= \sup \left\{ \inf \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r-1)\lambda + \eta : \lambda \geq 0, \; \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\} \right\}
\]

\[: \hat{Q} \in \mathcal{P}, \; \text{LP}_{\mathcal{X}}(\hat{P}, \hat{Q}) \leq \alpha \]

\[= \lim_{\epsilon \downarrow 0} \sup \left\{ \inf \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r-1)\lambda + \eta + \epsilon : \lambda \geq 0, \; \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon \right\} \right\}
\]

\[: \hat{Q} \in \mathcal{P}, \; \text{LP}_{\mathcal{X}}(\hat{P}, \hat{Q}) \leq \alpha \]

\[= \lim_{\epsilon \downarrow 0} \inf \left\{ \sup \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r-1)\lambda + \eta : \hat{Q} \in \mathcal{P}, \; \text{LP}_{\mathcal{X}}(\hat{P}, \hat{Q}) \leq \alpha \right\} \right\}
\]

\[: \lambda \geq 0, \; \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon \]
Here, the second equality follows from Lemma C.1. The third inequality follows from the fact that
\[
\inf \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r - 1)\lambda + \eta : \lambda \geq 0, \, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}
\leq \inf \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r - 1)\lambda + \eta : \lambda \geq 0, \, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon \right\}

\leq \inf \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r - 1)\lambda + \eta : \lambda \geq 0, \, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon \right\}
\]
for any \( \epsilon > 0 \). We will now show that the fourth equality follows from the minimax theorem of Sion (1958). Let
\[
L(\lambda, \eta, \hat{Q}) = \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) \, d\hat{Q}(\xi) + (r - 1)\lambda + \eta
\]
be our saddle point function and remark that this function is convex in \((\lambda, \eta)\) for any fixed \(\hat{Q}\) and concave (linear) in \(\hat{Q}\) for any fixed \((\lambda, \eta)\). As shown in the proof of Van Parys et al. (2021) Proposition 5) the variables \((\lambda, \eta)\) can be restricted to the compact set
\[
\eta \leq M_3 := \frac{\max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon - \exp(-r) \min_{\xi \in \Sigma} \ell(x, \xi)}{1 - \exp(-r)} \quad \text{and} \quad \lambda \leq M_r := \exp(\log(M_3 - \min_{\xi \in \Sigma} \ell(x, \xi) - r)
\]
without loss of optimality. Furthermore, we have the dual representation
\[
L(\lambda, \eta, \hat{Q}) = \max_{p^r \in P_+} \int \ell(x, \xi) \, dp^r + \eta(1 - \int \, dp^r) + \lambda(r - \text{KL}(\hat{Q}||P^r)) \quad \text{(Van Parys et al. 2021)}
\]
where \(P_+\) denotes here all nonnegative Borel measures on \(\Sigma\) establishing the lower semicontinuity of the function \(L(\lambda, \eta, \hat{Q})\) in \((\lambda, \beta)\) for any \(\hat{Q}\). The minimax theorem of Sion (1958) applies as \(L(\lambda, \eta, \hat{Q})\) is continuous in \(\hat{Q}\) as the integrant
\[
\xi \mapsto \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right)
\]
is continuous and bounded for any \(\lambda\) and \(\eta\) such that \(\eta - \ell(x, \xi) \geq \epsilon > 0\) for all \(\xi \in \Sigma\).

Finally,
\[
\mathcal{E}^{N, \alpha, r}_{HR}(x, \hat{P}) = \lim_{\epsilon \downarrow 0} \left\{ \begin{array}{l}
\inf \left\{ \int (1 - \alpha)\beta + \int \max \left( \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) - \beta, 0 \right) \, dp^r(\xi) + \alpha \lambda \log (\lambda/(\eta - \max_{\xi \in \Sigma} \ell(x, \xi)))
\right. \\
\left. \quad + (r - 1)\lambda + \eta \right. \\
\text{s.t.} \quad \beta \leq \lambda \log (\lambda/(\eta - \max_{\xi \in \Sigma} \ell(x, \xi))) \quad \lambda \geq 0, \, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon.
\end{array} \right.
\]

\[
= \lim_{\epsilon \downarrow 0} \left\{ \begin{array}{l}
\inf \left\{ \int \max \left( \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right), \beta \right) \, dp^r(\xi) + (r - 1)\lambda - \alpha \beta + \eta + \alpha \lambda \log (\lambda/(\eta - \max_{\xi \in \Sigma} \ell(x, \xi)))
\right. \\
\left. \quad \text{s.t.} \quad \beta \leq \lambda \log (\lambda/(\eta - \max_{\xi \in \Sigma} \ell(x, \xi))) \quad \lambda \geq 0, \, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon.
\end{array} \right.
\]

\[
= \lim_{\epsilon \downarrow 0} \left\{ \begin{array}{l}
\inf \left\{ \int \max \left( \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right), \lambda \log \left( \frac{\lambda}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} \right) - \beta' \right) \, dp^r(\xi) + (r - 1)\lambda + \alpha \beta' + \eta
\right. \\
\left. \quad \text{s.t.} \quad \beta' \geq 0, \, \lambda \geq 0, \, \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) + \epsilon.
\end{array} \right.
\]


\[ \inf \int \max \left( \lambda \log \left( \frac{\eta - \ell(x, \xi)}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} \right), \sigma \log \left( \frac{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)}{\eta - \max_{\xi \in \Sigma} \ell(x, \xi)} \right) \right) \, dP' (\xi) + (r - 1) \lambda + \alpha \beta' + \eta \]

\text{s.t.} \quad \beta' \geq 0, \quad \lambda \geq 0, \quad \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi).

The first equality follows from Theorem A.1 where we remark that

\[ \max_{n \in N, \xi \notin n \in \Sigma} \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi - n)} \right) = \lambda \log \left( \lambda/\eta - \ell^{N}(x, \xi) \right) \]

and that we can restrict \( \beta \leq \max_{\xi \in \Sigma} \lambda \log \left( \lambda/\eta - \ell^{N}(x, \xi) \right) = \lambda \log \left( \lambda/\eta - \max_{\xi \in \Sigma} \ell(x, \xi) \right) \). The second equality follows from \( \beta + \max \left( \lambda \log \left( \lambda/\eta - \ell^{N}(x, \xi) \right) - \beta, 0 \right) = \max \left( \lambda \log \left( \lambda/\eta - \ell^{N}(x, \xi) \right), \beta \right) \). The third equality follows from the change of variable \( \beta' = \lambda \log \left( \lambda/\eta - \max_{\xi \in \Sigma} \ell(x, \xi) \right) - \beta \). The final inequality follows from the lower semicontinuity of the objective function due to the representation

\[ \int \max \left( \lambda \log \left( \frac{\lambda}{\eta - \ell^{N}(x, \xi)} \right), \beta \right) \, dP' (\xi) + (r - 1) \lambda + \alpha \beta' + \eta \]

\[ = \sup \left\{ L(\lambda, \eta, \hat{\xi}) : \hat{\xi} \in \mathcal{P}, \quad L_{P}(\hat{\xi}) \leq \alpha \right\} \]

where the lower semicontinuity of \( L(\lambda, \eta, \hat{\xi}) \) in \( (\lambda, \eta) \) was established before.

\[ \square \]

\section{Omitted proofs of Section 3.2}

\subsection{Proof of Theorem 3.7}

\textbf{Proof of Theorem 3.7.} We distinguish two cases.

\textbf{Case I:} \( \mathbb{E}[\ell(x, \hat{\xi})] = \max_{\xi \in \Sigma} \ell(x, \xi) \). In this case, the distribution \( \mathbb{P} \) is supported on the set \( \Sigma_{\infty} = \arg \max_{\xi \in \Sigma} \ell(x, \xi) \). Let \( \Sigma_{N} = (\Sigma_{\infty} + N) \cap \Sigma \). As before, we denote with \( \hat{\mathbb{P}}_{T} \) the empirical distribution of the data and we define \( \hat{\alpha}_{\infty}^{T} := \hat{\mathbb{P}}_{T}(\Sigma_{\infty}), \hat{\alpha}_{N}^{T} := \hat{\mathbb{P}}_{T}(\Sigma_{N}) \) and \( \hat{\alpha}_{\Sigma}^{T} := \hat{\mathbb{P}}_{T}(\Sigma \setminus (\Sigma_{\infty} \cup \Sigma_{N})) \). Intuitively, \( \hat{\alpha}_{\infty}^{T} \) denotes the fraction of all observed data points in the set \( \Sigma_{\infty} \), \( \hat{\alpha}_{N}^{T} \) the fraction of all observed data points close but not in the set \( \Sigma_{\infty} \) and \( \hat{\alpha}_{\Sigma}^{T} \) the remaining fraction of samples far from \( \Sigma_{\infty} \). Similarly, we let \( \mathbb{Q} \) denote the distribution of the noisy data \( \hat{\xi} \) and define \( \alpha_{\infty} = \mathbb{Q}(\Sigma_{\infty}), \alpha_{N} = \mathbb{Q}(\Sigma_{N}) \) and \( \alpha_{\Sigma} = \mathbb{Q}(\Sigma \setminus (\Sigma_{\infty} \cup \Sigma_{N})) \). We remark that the random variable \( T(\hat{\alpha}_{\infty}^{T}, \hat{\alpha}_{N}^{T}, \hat{\alpha}_{\Sigma}^{T}) \) is distributed as a multinomial distribution with parameter \( (\alpha_{\infty}, \alpha_{N}, \alpha_{\Sigma}) \). From the fact that \( L_{P}(\mathbb{Q}, \mathbb{P}) \leq \alpha \) it follows that we must have that \( \alpha_{\Sigma} \leq \alpha \).

Define first a distribution \( \bar{\mathbb{Q}}_{T} \in \mathcal{P} \) through the characterization

\[ \bar{\mathbb{Q}}_{T}(B) := \int_B \frac{1}{\mathbb{P}(\Sigma_{\infty})} \alpha_{\infty} d\mathbb{P}_{T}(\xi) + \int_B \frac{1}{\mathbb{P}(\Sigma_{N})} \alpha_{N} d\mathbb{P}_{T}(\xi) + \int_B \frac{1}{\mathbb{P}(\Sigma \setminus (\Sigma_{\infty} \cup \Sigma_{N}))} \alpha_{\Sigma} d\mathbb{P}_{T}(\xi) \]

for any measurable set \( B \) in \( \Xi \). Notice that by construction we have that \( \bar{\mathbb{Q}}_{T}(\Sigma_{\infty}) = \alpha_{\infty}, \bar{\mathbb{Q}}_{T}(\Sigma_{N}) = \alpha_{N} \) and \( \bar{\mathbb{Q}}_{T}(\Sigma \setminus (\Sigma_{\infty} \cup \Sigma_{N})) = \alpha_{\Sigma} \).
and \( \bar{Q}_T(\Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega)) = \alpha^\Sigma \) with Radon-Nikodym derivative

\[
\frac{d\bar{P}_T}{d\bar{Q}_T}(\xi) = \mathbb{1}\{\xi \in \Sigma_\infty\} \frac{\hat{\alpha}_T^\infty}{\alpha^\infty} + \mathbb{1}\{\xi \in \Sigma_N^\omega\} \frac{\hat{\alpha}_T^N}{\alpha^N} + \mathbb{1}\{\xi \in \Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega)\} \frac{\hat{\alpha}_T^\Sigma}{\alpha^\Sigma}
\]

for all \( \xi \in \Sigma \). Intuitively, the distribution \( \bar{Q}_T \) is a simple rescaling of the distribution \( \hat{P}_T \) in each of the regions of interest \( \Sigma_\infty, \Sigma_N^\omega \) and \( \Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega) \) so that the total mass in those regions coincides with the total probability mass assigned to those regions by the distribution \( Q \). Hence, we have

\[
\text{KL}(\hat{P}_T||\bar{Q}_T) = \int \log \left( \frac{d\hat{P}_T}{d\bar{Q}_T}(\xi) \right) d\hat{P}_T(\xi) = \int_{\Sigma_\infty} \log \left( \frac{d\hat{P}_T}{d\bar{Q}_T}(\xi) \right) d\hat{P}_T(\xi) + \int_{\Sigma_N^\omega} \log \left( \frac{d\hat{P}_T}{d\bar{Q}_T}(\xi) \right) d\hat{P}_T(\xi) + \int_{\Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega)} \log \left( \frac{d\hat{P}_T}{d\bar{Q}_T}(\xi) \right) d\hat{P}_T(\xi) = \hat{\alpha}_T^\infty \log \left( \frac{\hat{\alpha}_T^\infty}{\alpha^\infty} \right) + \hat{\alpha}_T^N \log \left( \frac{\hat{\alpha}_T^N}{\alpha^N} \right) + \hat{\alpha}_T^\Sigma \log \left( \frac{\hat{\alpha}_T^\Sigma}{\alpha^\Sigma} \right) = D((\hat{\alpha}_T^\infty, \hat{\alpha}_T^N, \hat{\alpha}_T^\Sigma), (\alpha^\infty, \alpha^N, \alpha^\Sigma))
\]

It can be shown that asymptotically in \( T, Q^\infty(D((\hat{\alpha}_T^\infty, \hat{\alpha}_T^N, \hat{\alpha}_T^\Sigma), (\alpha^\infty, \alpha^N, \alpha^\Sigma)) > r) \leq \exp(-rT+o(T)) \) for all \( r > 0 \) \cite{Csiszar1998, Agrawal2020} exploiting that \( T(\hat{\alpha}_T^\infty, \hat{\alpha}_T^N, \hat{\alpha}_T^\Sigma) \) is distributed as a multinomial distribution with parameter \((\alpha^\infty, \alpha^N, \alpha^\Sigma)\). Hence, \( Q^\infty(\text{KL}(\hat{P}_T||\bar{Q}_T) \leq r) = 1 - \text{Prob}(D((\hat{\alpha}_T^\infty, \hat{\alpha}_T^N, \hat{\alpha}_T^\Sigma), (\alpha^\infty, \alpha^N, \alpha^\Sigma)) > r) \geq 1 - \exp(-rT+o(T)) \).

Remark that we can associate with any point \( \xi \in \Sigma_\infty \cup \Sigma_N^\omega \) a point \( \Pi_{\Sigma_\infty}(\xi) \in \Sigma_\infty \) so that \( \xi - \Pi_{\Sigma_\infty} \in \mathcal{N} \). Furthermore, denote with \( \xi_\infty \) an arbitrary point in \( \Sigma_\infty \). Define the random variable \( \bar{P}_T \) through

\[
\bar{P}_T(B) = \bar{Q}_T(\{\xi \in \Sigma_\infty \cup \Sigma_N^\omega : \Pi_{\Sigma_\infty}(\xi) \in B\}) + \bar{Q}_T(\Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega)) \cdot \mathbb{1}\{\xi_\infty \in B\}
\]

for all measurable sets \( B \) in \( \Sigma \). It can be remarked that \( \bar{P}_T \in \mathcal{P}(\Sigma^\infty) \) as indeed we have \( \bar{P}_T(\Sigma^\infty) = \bar{Q}_T(\{\xi \in \Sigma_\infty \cup \Sigma_N^\omega : \Pi_{\Sigma_\infty}(\xi) \in \Sigma_\infty\}) + \bar{Q}_T(\Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega)) = \bar{Q}_T(\Sigma) = 1 \). We can interpret \( \bar{P}_T \) as the resulting measure after transporting the probability measure \( \bar{Q}_T \) by moving all mass at any location \( \xi \in \Sigma_\infty \cup \Sigma_N^\omega \) to their associated element in \( \Sigma_\infty \) over a distance bounded in \( \mathcal{N} \) and the remaining mass at location \( \xi \not\in \Sigma_\infty \cup \Sigma_N^\omega \) is moved to the point \( \xi_\infty \in \Sigma_\infty \). That is, according to a coupling \( \gamma_T(\xi, \Pi_{\Sigma_\infty}(\xi)) = \bar{Q}_T(\xi) \) for all \( \xi \in \Sigma_\infty \cup \Sigma_N^\omega \) and \( \gamma_T(\xi, \xi_\infty) = \bar{Q}_T(\xi) \) for all \( \xi \in \Sigma \setminus (\Sigma_\infty \cup \Sigma_N^\omega) \). Note that indeed \( \gamma_T \in \Gamma(\bar{Q}_T, \bar{P}_T) \). We
have thus

$$\text{LP}_\mathcal{N}(\tilde{Q}_T, \tilde{P}_T) = \inf_{\gamma \in \Gamma(\tilde{Q}_T, \tilde{P}_T)} \int 1 \{ \xi - \xi' \notin \mathcal{N} \} \ d\gamma(\xi, \xi')$$

$$\leq \int 1 \{ \xi - \xi' \notin \mathcal{N} \} \ d\gamma_T(\xi, \xi')$$

$$= \int_{\Sigma_\infty \cup \Sigma_\infty'} 1 \{ \xi - \Pi_{\Sigma_\infty}(\xi) \notin \mathcal{N} \} \ d\tilde{Q}_T(\xi) + \int_{\Sigma \setminus (\Sigma_\infty \cup \Sigma_\infty')} 1 \{ \xi - \xi_\infty \notin \mathcal{N} \} \ d\bar{Q}_T(\xi)$$

$$= \int_{\Sigma \setminus (\Sigma_\infty \cup \Sigma_\infty')} d\tilde{Q}_T(\xi)$$

$$= \alpha^\infty \leq \alpha.$$

Here the second equality follows from the fact that by construction $\xi - \Pi_{\Sigma_\infty}(\xi) \in \mathcal{N}$ when $\xi \in \Sigma_\infty \cup \Sigma_\infty'$.

Hence,

$$Q^\infty \left( \mathbb{E}_F[\ell(x, \tilde{\xi})] \right) \leq \hat{c}^{N, \alpha, r}_\text{HRo}(x, \hat{P}_T)$$

$$= Q^\infty \left( \max_{\xi \in \Sigma} \ell(x, \xi) = \hat{c}^{N, \alpha, r}_\text{HRo}(x, \hat{P}_T) \right)$$

$$= Q^\infty \left( \exists \mathbb{P}' \in \mathcal{P}(\Sigma_\infty), \exists \mathbb{Q}' \in \mathcal{P}, \text{KL}(\hat{P}_T || \mathbb{Q}') \leq r, \text{LP}_\mathcal{N}(\mathbb{Q}', \mathbb{P}') \leq \alpha \right)$$

$$\geq Q^\infty \left( \text{KL}(\hat{P}_T || \tilde{Q}_T) \leq r, \text{LP}_\mathcal{N}(\tilde{Q}_T, \tilde{P}_T) \leq \alpha \right)$$

$$= Q^\infty \left( \text{KL}(\hat{P}_T || \tilde{Q}_T) \leq r \right)$$

$$\geq 1 - \exp(-rT + o(T)).$$

**Case II:** $\mathbb{E}_F[\ell(x, \tilde{\xi})] < \max_{\xi \in \Sigma} \ell(x, \xi)$. Let $\tilde{\xi} \sim \mathbb{P}$ be the random variable of the true uncertainty before noise and corruption. Let $\tilde{\xi}^c$ be the random noisy and corrupted observation obtained from $\tilde{\xi}$ and denote with $\mathbb{Q} \in \mathcal{P}$ its distribution. Note that $\tilde{\xi}$ and $\tilde{\xi}^c$ may indeed be correlated and let $\hat{\gamma} \in \Gamma(\mathbb{Q}, \mathbb{P})$ be their joint distribution. We have

$$\text{LP}_\mathcal{N}(\mathbb{Q}, \mathbb{P}) = \inf_{\gamma \in \Gamma(\mathbb{Q}, \mathbb{P})} \int 1 \{ \xi - \xi' \notin \mathcal{N} \} \ d\hat{\gamma}(\xi, \xi')$$

$$\leq \int 1 \{ \xi - \xi' \notin \mathcal{N} \} \ d\hat{\gamma}(\xi, \xi')$$

$$= \text{Prob} \left( \tilde{\xi}^c - \tilde{\xi} \notin \mathcal{N} \right).$$

As the noise $\hat{n}$ realizes in $\mathcal{N}$, the event $\tilde{\xi}^c - \tilde{\xi} \notin \mathcal{N}$ occurs only in the case of misspecification of the sample. Hence, $\text{LP}_\mathcal{N}(\mathbb{Q}, \mathbb{P}) \leq \text{Prob} (\hat{c} = 1) \leq \alpha$.

Let $\mathcal{D} := \{ \hat{P} \in \mathcal{P} : \hat{c}^{N, \alpha, r}_\text{HRo}(x, \hat{P}) < \mathbb{E}_F[\ell(x, \tilde{\xi})] \}$. We have

$$Q^\infty \left( \hat{c}^{N, \alpha, r}_\text{HRo}(x, \hat{P}_T) \geq \mathbb{E}_F[\ell(x, \tilde{\xi})] \right) = 1 - Q^\infty \left( \hat{P}_T \in \mathcal{D} \right).$$
If $\mathcal{D}$ is empty, the result is trivial. We suppose now it is not. As $\hat{P}_T$ is an empirical distribution sampled from the corrupted distribution $Q$, Sanov’s theorem ensures that
\[
Q^\infty \left( \hat{c}_{HRo}^{N,\alpha,r}(x, \hat{P}_T) \geq \mathbb{E}_P[\ell(x, \hat{\xi})] \right) \geq 1 - \exp \left( -T \inf_{\hat{P} \in \mathcal{D}} \text{KL}(\hat{P}||Q) + o(T) \right).
\]

It suffices to show that $\inf_{\hat{P} \in P} \text{KL}(\hat{P}||Q) \geq r$. For this purpose, we show that for all $\hat{P} \in \tilde{D}$, $\text{KL}(\hat{P}||Q) \geq r$. By the lower semicontinuity of $\hat{c}_{HRo}^{N,\alpha,r}(x, \cdot)$ (see Lemma C.4), we have $\tilde{D} \subseteq \{ \hat{P} \in P : \hat{c}_{HRo}^{N,\alpha,r}(x, \hat{P}) \leq \mathbb{E}_P[\ell(x, \hat{\xi})]\}$.

Let $\hat{P} \in \tilde{D}$ and suppose for the sake of contradiction that $\text{KL}(\hat{P}||Q) < r$. We will show that $\hat{c}_{HRo}^{N,\alpha,r}(x, \hat{P}) > \mathbb{E}_P[\ell(x, \hat{\xi})]$ contradicting therefore that $\hat{P} \in \tilde{D}$.

Recall that $Q$ is the distribution of the corrupted observations and $P$ is the distribution before corruption. We have $\text{LP}_N(Q, P) \leq \alpha$ and $\text{KL}(\hat{P}||Q) < r$. Therefore, $P \in \{ P' \in P : \forall Q' \in \mathcal{P}, \text{KL}(\hat{P}||Q') \leq r, \text{LP}_N(Q', P') \leq \alpha \}$ which implies that $\hat{c}_{HRo}^{N,\alpha,r}(x, \hat{P}) \geq \mathbb{E}_P[\ell(x, \hat{\xi})]$ by definition of $\hat{c}_{HRo}^{N,\alpha,r}$ stated in Equation (17).

Let $\xi_\infty \in \arg \max_{\xi \in \Sigma} \ell(x, \xi)$ and let denote $\delta_{\xi_\infty}$ the a Dirac distribution on $\xi_\infty$. The function $\lambda \in [0, 1] \rightarrow \text{KL}(\hat{P}||Q + \lambda \delta_{\xi_\infty})$ is continuous in $\lambda$ as the KL divergence is lower semicontinuous and any convex function is upper semicontinuous on a locally simplicial set (Rockafellar [2015], Theorem 10.2). Therefore, as $\text{KL}(\hat{P}||Q) < r$, we can chose $\lambda > 0$ small enough such that $\text{KL}(\hat{P}||Q + \lambda \delta_{\xi_\infty}) \leq r$. We show next that this implies that $(1 - \lambda)\mathbb{P} + \lambda \delta_{\xi_\infty} \in \{ P' \in P : \exists Q' \in \mathcal{P}, \text{KL}(\hat{P}||Q') \leq r, \text{LP}_N(Q', P') \leq \alpha \}$. Denote $Q(\lambda) = (1 - \lambda)Q + \lambda \delta_{\xi_\infty}$ and $P(\lambda) = (1 - \lambda)P + \lambda \delta_{\xi_\infty}$. It suffices to show that $\text{LP}_N(Q(\lambda), P(\lambda)) \leq \alpha$, as then, we have the inclusion in the set by choosing $Q' = Q(\lambda)$.

We have
\[
\text{LP}_N(Q(\lambda), P(\lambda)) = \inf \left\{ \int 1(\xi - \xi' \not\in \mathcal{N}) d\gamma(\xi, \xi') : \gamma \in \Gamma(Q(\lambda), P(\lambda)) \right\}
\leq \inf \left\{ \int 1(\xi - \xi' \not\in \mathcal{N}) d\gamma(\xi, \xi') : \gamma = (1 - \lambda)\gamma' + \lambda \delta_{\xi_\infty}, \gamma' \in \Gamma(Q, P) \right\}
= \inf \left\{ (1 - \lambda) \int 1(\xi - \xi' \not\in \mathcal{N}) d\gamma'(\xi, \xi') : \gamma' \in \Gamma(Q, P) \right\} + \lambda \| 0 \not\in \mathcal{N} \|
= (1 - \lambda)\text{LP}_N(Q, P) = (1 - \lambda)\alpha \leq \alpha.
\]

Hence, we have $P(\lambda) \in \{ P' \in P : \exists Q' \in \mathcal{P}, \text{KL}(\hat{P}||Q') \leq r, \text{LP}_N(Q', P') \leq \alpha \}$ which implies by definition of $\hat{c}_{HRo}^{N,\alpha,r}$ that $\hat{c}_{HRo}^{N,\alpha,r}(x, \hat{P}) \geq \mathbb{E}_P[\ell(x, \hat{\xi})] = (1 - \lambda)\mathbb{E}_P[\ell(x, \hat{\xi})] + \lambda \max_{\xi \in \Sigma} \ell(x, \xi) > \mathbb{E}_P[\ell(x, \hat{\xi})]$.

\section{C.2 Proof of Theorem 3.8}

\textit{Proof of Theorem 3.8.} We can write the supremum in Equation (17) as
\[
\sup \left\{ \mathbb{E}_P[\ell(x, \hat{\xi})] : P' \in \mathcal{P}, Q' \in \mathcal{P}, \gamma \in \Gamma(Q', P'), \int \log \left( \frac{d\hat{P}_T}{dQ'}(\xi) \right) d\hat{P}_T(\xi) \leq r, \int 1(\xi - \xi' \not\in \mathcal{N}) d\gamma(\xi, \xi') \leq \alpha \right\}.
\]
For all $k \in [K]$, let $\xi_k' \in \xi_k^* - \arg\max_{n \in \mathcal{N}} \ell(x, \xi_k - n)$ and let $\xi_{\infty} \in \arg\max_{\xi \in \Sigma} \ell(x, \xi)$. We will show that there exists an optimal solution $\bar{\mathcal{P}}' \in \mathcal{P}, \bar{\mathcal{Q}}' \in \mathcal{P}$ and $\bar{\gamma} \in \Gamma(\bar{\mathcal{Q}}', \bar{\mathcal{P}}')$ verifying: (i) $\supp \bar{\mathcal{P}}' \subseteq \{\xi_k' : k \in [K]\} \cup \{\xi_{\infty}\}$, (ii) $\supp \bar{\mathcal{Q}}' \subseteq \{\xi_k : k \in [K]\} \cup \{\xi_{\infty}\}$, (iii) $\supp \bar{\gamma} \subseteq \{(\xi_k, \xi_k') : k \in [K]\} \cup \{(\xi_k, \xi_{\infty}) : k \in [K]\} \cup \{\xi_{\infty}, \xi_{\infty}\}$.

This result implies the stated finite formulation, by choosing $p_k' = \bar{\mathcal{P}}'(\xi_k')$, $q_k' = \bar{\mathcal{Q}}'(\xi_k)$ and $s_k = \gamma(\xi_k, \xi_{\infty})$ for all $k \in [K]$ as well as $q_{K+1} = \bar{\mathcal{Q}}'(\xi_{\infty})$ and $p_{K+1}' = \bar{\mathcal{P}}'(\xi_{\infty})$.

Let $\mathcal{P}^\star \in \mathcal{P}, \mathcal{Q}^\star \in \mathcal{P}$ be an optimal solution in Equation (17). Denote $\Xi := \supp(\mathcal{Q}^\star) \setminus \{\xi_k : k \in [K]\} \cup \{\xi_{\infty}\}$.

We start by constructing $\bar{\mathcal{Q}}'$. Consider $\bar{\mathcal{Q}}' \in \mathcal{P}$ defined as $\bar{\mathcal{Q}}'(A) = \mathcal{Q}^\star(A \setminus \Xi) + \mathcal{Q}^\star(\Xi \cup (\xi_{\infty} \in A))$ for all events $A \in \mathcal{B}(\Sigma)$. Notice that indeed we have $\supp(\bar{\mathcal{Q}}') \subseteq \{\xi_k : k \in [K]\} \cup \{\xi_{\infty}\}$. We first verify that the new solution still verifies $\text{KL}(\bar{\mathcal{P}}'|\bar{\mathcal{Q}}') \leq r$. We have

$$
\int \log \left( \frac{d\bar{\mathcal{P}}'}{d\mathcal{Q}'}(\xi) \right) d\bar{\mathcal{P}}'(\xi) = \sum_{k \in [K]} \log \left( \frac{\bar{\mathcal{P}}'(\xi_k)}{\mathcal{Q}^\star(\xi_k)} \right) \bar{\mathcal{P}}'(\xi_k)
\leq \sum_{k \in [K]} \log \left( \frac{\bar{\mathcal{P}}'(\xi_k)}{\mathcal{Q}^\star(\xi_k)} \right) \bar{\mathcal{P}}'(\xi_k) \leq r
$$

where the first inequality uses the fact that the logarithm is increasing and the second inequality is by feasibility of $\mathcal{Q}^\star$.

Theorem A.1 ensures that there exists $\bar{\mathcal{P}}' \in \arg\max \{\mathcal{E}_{\mathcal{P}'}[\ell(x, \xi)] : \mathcal{P}' \in \mathcal{P}, \text{LP}_{\mathcal{N}}(\mathcal{Q}', \mathcal{P}') \leq \alpha\}$ such that $\supp \bar{\mathcal{P}}' \subseteq \{\xi_k' : k \in [K]\} \cup \{\xi_{\infty}\}$. Consider the solution $\bar{\mathcal{P}}' \in \mathcal{P}, \bar{\mathcal{Q}}' \in \mathcal{P}$. We have

$$
\max \{\mathcal{E}_{\mathcal{P}'}[\ell(x, \xi)] : \mathcal{P}' \in \mathcal{P}, \mathcal{Q}' \in \mathcal{P}, \text{KL}(\bar{\mathcal{P}}'|\mathcal{Q}') \leq r, \text{LP}_{\mathcal{N}}(\mathcal{Q}', \mathcal{P}') \leq \alpha\}
= \max \{\max \{\mathcal{E}_{\mathcal{P}'}[\ell(x, \xi)] : \mathcal{P}' \in \mathcal{P}, \text{LP}_{\mathcal{N}}(\mathcal{Q}', \mathcal{P}') \leq \alpha\} : \mathcal{Q}' \in \mathcal{P}, \text{KL}(\bar{\mathcal{P}}'|\mathcal{Q}') \leq r\}
$$

Hence, the solution $\bar{\mathcal{P}}', \bar{\mathcal{Q}}'$ is feasible, as $\bar{\mathcal{Q}}'$ is feasible in the outer maximum and $\bar{\mathcal{P}}'$ is feasible in the inner by construction. Finally, Theorem A.1 ensures that there exists a coupling $\bar{\gamma} \in \Gamma(\bar{\mathcal{Q}}', \bar{\mathcal{P}}')$ of support $\supp \bar{\gamma} \subseteq \{(\xi_k, \xi_k') : k \in [K]\} \cup \{(\xi_k, \xi_{\infty}) : k \in [K]\} \cup \{\xi_{\infty}, \xi_{\infty}\}$ corresponding to this solution.

Let us show that $\bar{\mathcal{Q}}'$ also attains the optimal cost. Using Theorem A.1 and the CVaR formula (24), we have

$$
\mathcal{E}_{\mathcal{Q}^\star}[\ell(x, \xi)] = \max \{\mathcal{E}_{\mathcal{P}'}[\ell(x, \xi)] : \mathcal{P}' \in \mathcal{P}, \text{LP}_{\mathcal{N}}(\mathcal{Q}^\star, \mathcal{P}') \leq \alpha\}
= \sup \{\mathcal{E}_{\mathcal{Q}^\star}[\ell^N(x, \xi)R(\xi)] : \mathcal{E}_{\mathcal{Q}^\star}(R(\xi)) = 1 - \alpha, R : \Sigma \rightarrow [0, 1]\} + \alpha \max_{\xi \in \Sigma} \ell(x, \xi)
$$

(36)
Consider the mapping \( S : \Sigma \rightarrow \Sigma \) defined as

\[
S(\xi) = \begin{cases} 
\xi & \text{if } \xi \in \{\xi_k : k \in [K]\}, \\
\xi_\infty & \text{otherwise.}
\end{cases}
\]

Informally, this mapping moves the mass distribution \( Q'^* \) into the mass distribution \( \bar{Q}' \). Recall that \( \ell(x, \xi_\infty) = \max_{\xi \in \Sigma} \ell(x, \xi) \). Hence, for all \( \xi \in \Sigma \), we have \( \ell^N(x, \xi) \leq \ell^N(x, S(\xi)) \). Hence,

\[
\sup\{\mathbb{E}_{Q'^*}(\ell^N(x, \xi)R(\xi)) : \mathbb{E}_{Q'^*}(R(\xi)) = 1 - \alpha, \ R : \Sigma \rightarrow [0, 1]\} \\
\leq \sup\{\mathbb{E}_{Q'^*}(\ell^N(x, S(\xi))R(\xi)) : \mathbb{E}_{Q'^*}(R(\xi)) = 1 - \alpha, \ R : \Sigma \rightarrow [0, 1]\} \\
= \sup\{\sum_{k \in [K]} \int_{S(\xi) = \xi_k} \ell^N(x, \xi_k)R(\xi)dQ'^*(\xi) + \int_{S(\xi) = \xi_\infty} \ell^N(x, \xi_\infty)R(\xi)dQ'^*(\xi) : \mathbb{E}_{Q'^*}(R(\xi)) = 1 - \alpha, \ R : \Sigma \rightarrow [0, 1]\} \\
= \sup\{\sum_{k \in [K]} \bar{R}(\xi_k)Q'^*(S(\xi) = \xi_k) + \bar{R}(\xi_\infty)Q'^*(S(\xi) = \xi_\infty) = 1 - \alpha, \ \bar{R} : \Sigma \rightarrow [0, 1]\} \\
= \sup\{\mathbb{E}_{\bar{Q}'}(\ell^N(x, \xi)\bar{R}(\xi)) : \mathbb{E}_{\bar{Q}'}(\bar{R}(\xi)) = 1 - \alpha, \ \bar{R} : \Sigma \rightarrow [0, 1]\}.
\]

Plugging this inequality in (36), and using Theorem A.1 we get

\[
\mathbb{E}_{P'}(\ell(x, \hat{\xi})) \leq \sup\{\mathbb{E}_{\bar{Q}'}(\ell^N(x, \hat{\xi})\bar{R}(\hat{\xi})) : \mathbb{E}_{\bar{Q}'}(\bar{R}(\hat{\xi})) = 1 - \alpha, \ \bar{R} : \Sigma \rightarrow [0, 1]\} + \alpha \max_{\xi \in \Sigma} \ell(x, \xi) \\
= \sup\{\mathbb{E}_{\bar{Q}'}(\ell(x, \hat{\xi})) : P' \in \mathcal{P}, \ LP_\lambda(\bar{Q}', P') \leq \alpha\} \\
= \mathbb{E}_{\bar{Q}'}(\ell(x, \hat{\xi})).
\]

Hence, the distributions \( \bar{P}', \bar{Q}' \) are also optimal solutions. \( \square \)

**Lemma C.1 (Love and Bayraksan (2015), Van Parys et al. (2021)).** We have

\[
\hat{c}_{KL}(x, P') = \min \left\{ \int \lambda \log \left( \frac{\lambda}{\eta - \ell(x, \xi)} \right) dP'(\xi) + (r - 1)\lambda + \eta : \lambda \geq 0, \ \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}
\]

for all \( P' \in \mathcal{P} \) and \( r > 0 \).

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C.3 Generalization and Proof of Theorem 3.9

**Theorem C.2** (General Dual Formulation). Let \( \hat{P} \in \mathcal{P} \). For all \( x \in \mathcal{X} \), the HRo predictor \( \beta \) admits for all \( r > 0 \) the dual representation

\[
\hat{c}_{HRo}^{N, \alpha, r}(x, \hat{P}) = \begin{cases} 
\inf_{w : \Sigma \rightarrow \mathbb{R}} \int w(\xi) \, d\hat{P}(\xi) + \lambda(r - 1) + \beta \alpha + \eta \\
\text{s.t. } \lambda \geq 0, \beta \in \mathbb{R}, \eta \in \mathbb{R}, \quad w(\xi) \geq \lambda \log \left( \frac{\lambda}{\eta - \xi} \right), \quad w(\xi) \geq \lambda \log \left( \frac{\lambda}{\eta - \max_{\xi' \in \Sigma} \ell(x, \xi')} \right) \quad \forall \xi \in \Sigma, \\
\eta \geq \max_{\xi \in \Sigma} \ell(x, \xi).
\end{cases}
\]

**Proof.** We have that

\[
\hat{c}_{HRo}^{N, \alpha, r}(x, \hat{P}) := \sup \left\{ \mathbb{E}_{P'}[\ell(x, \hat{\xi})] : P' \in \mathcal{P}, Q' \in \mathcal{P}, \text{KL}(\hat{P}|\|Q') \leq r, \text{LP}_N(Q', P') \leq \alpha \right\}
\]

\[
= \sup \left\{ \mathbb{E}_{P'}[\ell(x, \hat{\xi})] : P' \in \mathcal{P}, \text{LP}_N(Q', P') \leq \alpha \right\} \quad \forall Q' \in \mathcal{P}, \text{KL}(\hat{P}|\|Q') \leq r
\]

\[
= \sup \left\{ (1 - \alpha)\text{CVaR}_N(\ell_N(x, \hat{\xi})) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi) : Q' \in \mathcal{P}, \text{KL}(\hat{P}|\|Q') \leq r \right\}
\]

\[
= \sup \left\{ \beta(1 - \alpha) + \mathbb{E}_{Q'}[\max(\ell_N(x, \hat{\xi}) - \beta, 0)] : \beta \leq \max_{\xi \in \Sigma} \ell(x, \xi) \right\} \quad \forall Q' \in \mathcal{P}, \text{KL}(\hat{P}|\|Q') \leq r
\]

\[
\quad + \alpha \max_{\xi \in \Sigma} \ell(x, \xi)
\]

\[
= \inf \left\{ \beta(1 - \alpha) + \sup \left\{ \mathbb{E}_{Q'}[\max(\ell_N(x, \hat{\xi}) - \beta, 0)] : Q' \in \mathcal{P}, \text{KL}(\hat{P}|\|Q') \leq r \right\} : \beta \leq \max_{\xi \in \Sigma} \ell(x, \xi) \right\}
\]

\[
\quad + \alpha \max_{\xi \in \Sigma} \ell(x, \xi)
\]

Here, the second equality follows from Theorem A.1. The third equality follows from the dual characterization of the conditional value-at-risk stated in Equation (25) where we remark that we can restrict \( \beta \) to be satisfy \( \beta \leq \max_{\xi \in \Sigma} \ell_N(x, \xi) = \max_{\xi \in \Sigma} \ell(x, \xi) \) as \( 0 \in \mathcal{N} \). We will now show that the fourth equality follows from the minimax theorem of Sion (1958). Let

\[
L(\beta, Q') = \beta(1 - \alpha) + \mathbb{E}_{Q'}[\max(\ell_N(x, \tilde{\xi}) - \beta, 0)]
\]

be our saddle point function and remark that this function is convex in \( \beta \) for any fixed \( Q' \) and concave (linear) in \( Q' \) for any fixed \( \beta \). As for any \( Q' \) the function \( L(\beta, Q') \) is convex in \( \beta \in \mathbb{R} \) it is also continuous. We also need that \( L(\beta, Q') \) is upper semicontinuous in \( Q' \) for any fixed \( \beta \). From the maximum theorem of Berge (1997 p. 116) and the compactness of the noise set \( \mathcal{N} \) it follows that the inflated loss function is continuous. Clearly, the function \( \max(\ell_N(x, \tilde{\xi}) - \beta, 0) \) is continuous as well and is bounded by \( \max_{\xi \in \Sigma} \ell_N(x, \xi) = \max_{\xi \in \Sigma} \ell(x, \xi) < \infty \). Hence, the function \( L(\beta, Q') \) is continuous in \( Q' \) for any \( \beta \) as we have indeed for any sequence \( Q'_n \in \mathcal{P} \).
Lemma C.3 (General Dual Formulation II). Let \( \hat{\phi} \in \mathcal{P} \). For all \( x \in \mathcal{X} \), the HRo predictor \( \hat{\mathcal{V}}_{HRo}^{N,\alpha,r}(x, \hat{\phi}) \) admits for all \( r > 0 \) the dual representation

\[
\hat{\mathcal{V}}_{HRo}^{N,\alpha,r}(x, \hat{\phi}) = \begin{cases} 
\inf \limits_{\omega : \Sigma \to \mathbf{R}^r, \beta \in \mathbf{R}^+, \eta \in \mathbf{R}} & \eta - \exp(-r) \exp \left( \int \log (\eta - w(\xi)) \, d\hat{\phi}(\xi) \right) + \beta \alpha \\
\text{s.t.} & w(\xi) \geq \ell^N(x, \xi), \ w(\xi) \geq \max_{\xi \in \Sigma} \ell(x, \xi) - \beta \ \forall \xi \in \Sigma, \\
& \eta \geq \max_{\xi \in \Sigma} \ell(x, \xi). 
\end{cases}
\]
Proof. We shown in the Proof of Theorem C.2 that
\[
\hat{c}_{\text{HRO}}^{\alpha, r}(x, \hat{P}) = \inf \left\{ \beta (1 - \alpha) + \sup \left\{ \mathbb{E}_{Q'}[\max(\ell^N(x, \hat{\xi}) - \beta, 0)] : Q' \in \mathcal{P}, \text{KL}(\hat{P}||Q') \leq r \right\} : \beta \leq \max_{\xi \in \Sigma} \ell(x, \xi) \right\} + \alpha \max_{\xi \in \Sigma} \ell(x, \xi).
\]

Using Van Parys et al. (2021, Proposition 5), we have
\[
\hat{c}_{\text{HRO}}^{\alpha, r}(x, \hat{P}) = \begin{cases} 
\inf \eta - \exp(-r) \exp \left( \int \log (\eta - \max(\ell^N(x, \xi) - \beta, 0)) \, d\mathbb{P}(\xi) \right) + \beta (1 - \alpha) + \alpha \max_{\xi \in \Sigma} \ell(x, \xi) \\
\text{s.t. } \beta \leq \max_{\xi \in \Sigma} \ell(x, \xi), \eta \geq \max_{\xi \in \Sigma} \max(\ell^N(x, \xi) - \beta, 0) 
\end{cases}
\]
\[
= \begin{cases} 
\inf \eta' - \exp(-r) \exp \left( \int \log (\eta' - \max(\ell^N(x, \xi), \beta)) \, d\mathbb{P}(\xi) \right) - \beta \alpha + \alpha \max_{\xi \in \Sigma} \ell(x, \xi) \\
\text{s.t. } \beta \leq \max_{\xi \in \Sigma} \ell(x, \xi), \eta' \geq \max_{\xi \in \Sigma} \max(\ell^N(x, \xi), \beta) 
\end{cases}
\]
\[
= \begin{cases} 
\inf \eta' - \exp(-r) \exp \left( \int \log (\eta' - \max(\ell^N(x, \xi), \max_{\xi \in \Sigma} \ell(x, \xi) - \beta')) \, d\mathbb{P}(\xi) \right) + \beta' \alpha \\
\text{s.t. } \beta' \geq 0, \eta' \geq \max_{\xi \in \Sigma} \ell(x, \xi) 
\end{cases}
\]
The second equality follows from the fact that the logarithm function is an increasing function. The third and fourth inequalities follow from the change of variables \(\eta' = \eta + \beta\) and \(\beta' = \max_{\xi \in \Sigma} \ell(x, \xi) - \beta\). Finally, remark that \(\max_{\xi \in \Sigma} \ell^N(x, \xi) = \max_{\xi \in \Sigma} \ell(x, \xi)\) as \(0 \in \mathcal{N}\). \qed

C.4 Continuity of the HR predictor

Lemma C.4. For \(\alpha > 0\), \(r > 0\) and \(x \in \mathcal{X}\), the cost predictor \(\hat{P} \mapsto \hat{c}_{\text{HRO}}^{\alpha, r}(x, \hat{P})\) is weakly continuous.

Proof. Let again \(\ell^\infty(x) = \max_{\xi \in \Sigma} \ell(x, \xi)\) and define
\[
\hat{c}_{\epsilon}(\hat{P}) := \inf f(\beta, \eta) := \eta - \exp(-r) \exp \left( \int \log (\eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta)) \, d\mathbb{P}(\xi) \right) + \beta \alpha \\
\text{s.t. } \beta \geq 0, \eta \geq \ell^\infty(x) + \epsilon
\]
for all \(\hat{P} \in \mathcal{P}\) and \(\epsilon > 0\). Let us first show that the feasible set of the optimization problem in Equation (38) can be restricted to a compact set independent of \(\hat{P}\) without loss of optimality. For any \(\beta > M_1 := \ell^\infty(x) - \min_{\xi \in \Sigma} \ell^N(x, \xi)\) we have that the objective function simplifies to
\[
f(\beta, \eta) = \eta - \exp(-r) \exp \left( \int \log (\eta - \ell^N(x, \xi)) \, d\mathbb{P}(\xi) \right) + \beta \alpha
\]
which is strictly increasing in \(\beta\) as \(\alpha > 0\). Therefore, we may impose \(\beta \leq M_1\) without loss of optimality.
Van Parys et al. (2021, Proposition 5) ensures that for any fixed $\beta$ the minimum in $\eta$ is attained at

$$\ell^\infty(x) \leq \eta \leq M_2 = \frac{\ell^\infty(x) + \epsilon - \exp(-r) \min_{\xi \in \Sigma} \ell^N(x, \xi)}{1 - \exp(-r)}.$$ 

Let us show the continuity of the function $c_\epsilon : \mathcal{P} \to \mathbb{R}$, defined as

$$c_\epsilon(\hat{P}) = \begin{cases} \inf & \eta - \exp(-r) \exp \left( \int \log \left( \eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta) \right) \, d\hat{P}(\xi) \right) + \beta \alpha \\ \text{s.t.} & M_1 \geq \lambda \geq 0, \ M_1 \geq \beta \geq 0, \ M_2 \geq \eta \geq \ell^\infty(x) + \epsilon. \end{cases}$$

(39)

for all $\hat{P} \in \mathcal{P}$. We will use the maximum theorem of (Berge 1997, p. 116) to show that the function $c_\epsilon : \mathcal{P} \to \mathbb{R}$ is continuous. It suffices to show that the set defined by the constraints is compact and that the objective function is continuous. First, the feasible region is clearly compact and nonempty. We need to show that the objective function is continuous in $(\beta, \eta, \hat{P})$. Consider an arbitrary sequence $(\beta_i, \eta_i, \hat{P}_i)$ for $i \geq 1$ with limit $(\beta, \eta, \hat{P})$. Remark that we have

$$\lim_{i \to \infty} \int \log \left( \eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta) \right) \, d\hat{P}_i(\xi) = \int \log \left( \eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta) \right) \, d\hat{P}(\xi)$$

as the integrant is bounded in $[\log(\epsilon), \log(M_2 - \min_{\xi \in \Sigma} \ell^N(x, \xi))]$ and continuous. It is immediate that we have also

$$|\log \left( \eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta) \right) - \log \left( \eta' - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta') \right)| \leq (|\eta - \eta'| + |\beta - \beta'|) / \epsilon.$$ 

for all $(\beta, \eta)$ and $(\beta', \eta')$ feasible in (38). Hence,

$$\lim_{i \to \infty} \left| \int \log \left( \eta_i - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta_i) \right) \, d\hat{P}_i(\xi) - \int \log \left( \eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta) \right) \, d\hat{P}(\xi) \right| \leq \lim_{i \to \infty} \left| \int \log \left( \eta_i - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta_i) \right) \, d\hat{P}_i(\xi) - \int \log \left( \eta - \max(\ell^N(x, \xi), \ell^\infty(x) - \beta) \right) \, d\hat{P}(\xi) \right| \leq \lim_{i \to \infty} \left( |\eta_i - \eta| + |\beta_i - \beta| \right) / \epsilon \, d\hat{P}_i(\xi) = 0.$$

As the exponential function is continuous, we can apply Berge’s maximum theorem and conclude that $c_\epsilon$ is continuous.

Following Lemma C.3 it follows that $c_\epsilon(\hat{P}) \geq c^{N, \alpha, r}_{\text{HFC}}(x, \hat{P})$ for all $\hat{P}$, as any feasible dual solution $(\beta, \eta)$ in the problem (39) is also feasible in problem (37). Furthermore, for a given $\hat{P} \in \mathcal{P}$, if $(\beta, \eta)$ is a feasible solution of problem (37) with cost $c$, then the dual point $(\lambda, \beta, \eta + \epsilon)$ is feasible in problem (38) and its
cost is less than \( c + \epsilon \). Hence, \( c_\epsilon(\hat{P}) \leq c_{\text{HRo}}^{N,\alpha,r}(x,\hat{P}) + \epsilon \). We conclude that for all \( \hat{P} \in \mathcal{P} \) and \( \epsilon > 0 \),
\[
c_\epsilon(\hat{P}) \geq c_{\text{HRo}}^{N,\alpha,r}(x,\hat{P}) \geq c_\epsilon(\hat{P}) - \epsilon.
\]
Furthermore, as \( c_\epsilon \) is continuous for all \( \epsilon > 0 \), we have that \( c_{\text{HRo}}^{N,\alpha,r}(x,\cdot) \) is continuous as uniform limits of continuous functions are continuous.

\[\square\]

### D  Further details on experiments

#### D.1 Portfolio optimization experiments

**Hyperparameters** Denote by \( \mathcal{U}_n([a,b]) \) a uniform discretization of \([a,b]\) with \( n \) points, starting at \( a \) included. The parameters of each model are chosen as:

- **HR-DRO**: \( \epsilon \in \{0\} \cup \{10^k : k \in \mathcal{U}_{19}([-1,3])\} \), \( \alpha \in \{0, 0.01, 0.03, 0.05, 0.1\} \), \( r \in \{0\} \cup \{10^k : k \in \mathcal{U}_{19}([-1,3])\} \).
- **W-DRO**: \( \epsilon \in \{10^k : k \in \mathcal{U}_{2000}([-1,3])\} \).
- **KL-DRO**: \( r \in \{10^k : k \in \mathcal{U}_{2000}([-1,3])\} \).
- **Mean-CVaR**: \( \rho \in \mathcal{U}_{2000}([0,100]) \).
- **Markowitz**: \( \rho \in \mathcal{U}_{2000}([0,100]) \).

**Selected stocks tickers** SYF, FDX, PSA, TFC, GEN, LNT, CTVA, GWW, IVZ, IDXX, FTV, VRSK, RSG, SO, D, HBAN, ETSY, MRO, TYL, URI, CINF, IPG, BK, DE, WMT, PEG, ANET, MCHP, RCL, CEG, UNH, K, PCAR, STX, MCD, UNP, BXP, DOW, AIZ, PM, AME, ULTA, TMUS, NVR, EA, HWM, LVS, T, ESS, GILD, DD, META, MCO, SCHW, LHX, WST, HAL, TFX, JKH, CSGP, AXP, SBAC, KHC, ADBE, ACGL, PYPL, STE, FMC, PANW, SNA, WY, NDSN, MRNA, MMM, MET, IR, SEDG, ELV, CAG, UPS, ECL, OTIS, AIG, CDW, LIN, CSCK, CMCSA, CRM, BA, SEE, KMI, HCA, APTV, ANSS, EMN, ISRG, ILMN, AVB, DLTR, TM0, FANG, WMR, BDX, TGT, LUV, KMB, TJX, VTR, REGN, MKTX, DNV, HIG, SPGI, MA, ZION, DXC, L, CMA, DXCM, CAT, CBRE, CBOE, ALB, STT, NEM, TSLA, HUM, AMP, AMAT, NRG, BBY, BSX, ALLE, VTRS, ORCL, O, NVDA, FE, ZTS, NJJ, DHR, J, CTSH, CI, SWK, MAR, GEHC, LYB, DHI, PEAK, SBUX, NWS, MS, EL, AON, SNPS, PPG, BALL, ITW, DGX, MOH, LLY, SPG, WRK, ADP, CMI, NOC, CTAS, EXR, CPB, REG, GE, TDY, ZBRA, BWA, NWL, GOOG, LMT, MLDZ, KDP, RJF, UAL, APD, RF, PG, MDT, VMC, BBWI, PH, QRVO, EXPD, POOL, ALGN, AFL, CNC, PGR, MAA, HII, CMG, MSI, MCK, TEL, GD, MOS, PNC, PNW, NOW, ABC, RHI, EQIX, GL, CARR, MLM, IP, HRL, XYL, MPC, MTD, CF, WFC, WELL, HST, CPT, RTX, AAL, BF.B, BG, EOG, JNPR, VFC, BAC, TXN, CAH, ADM, TSN, FOX, ROP, DTE, PAYX, FITB, TAP, AVGO, JCI, HPE, ADI, NI, MNST, DVA, AWK, MFM, FAST, TXT, PRU, KMX, DPZ, BKNG, ERM, CCL, DFS, HPQ, HAS, PTC, MSCI, VRTX, AMT, RVTY, BR, SYK, XOM, CVX, GRMN, TTWO, LDOS, STLD,
