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VALUES OF HARMONIC WEAK MAASS FORMS ON HECKE ORBITS

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Abstract. Let \( q := e^{2\pi i z} \), where \( z \in \mathbb{H} \). For an even integer \( k \), let \( f(z) := q^k \prod_{m=1}^{\infty} (1 - q^m)^{c(m)} \) be a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \). For a positive integer \( m \), let \( T_m \) be the \( m \)th Hecke operator and \( D \) be a divisor of a modular curve with level \( N \). Both subjects, the exponents \( c(m) \) of a modular form and the distribution of the points in the support of \( T_m.D \), have been widely investigated.

When the level \( N \) is one, Bruinier, Kohnen, and Ono obtained, in terms of the values of \( j \)-invariant function, identities between the exponents \( c(m) \) of a modular form and the points in the support of \( T_m.D \). In this paper, we extend this result to general \( \Gamma_0(N) \) in terms of values of harmonic weak Maass forms of weight 0. By the distribution of Hecke points, this applies to obtain an asymptotic behaviour of convolutions of sums of divisors of an integer and sums of exponents of a modular form.

1. Introduction

Let \( \mathbb{H} \) be the complex upper half plane. For a positive integer \( N \), let \( Y_0(N) \) be the modular curve of level \( N \) defined by \( \Gamma_0(N) \backslash \mathbb{H} \), and \( X_0(N) \) denote the compactification of \( Y_0(N) \) by adjoining the cusps. Let \( J_0(N) \) be the jacobian of a modular curve \( X_0(N) \). We denote by \( \text{Div}(C) \) the divisor group of a curve \( C \). If \( f \) is a function on \( C \) and \( D = \sum_{P \in C} n_P P \) is a divisor of \( C \), we define

\[
    f(D) := \sum n_P f(P).
\]

The \( m \)th normalized Hecke operator \( T_m \) acts on \( \text{Div}(Y_0(N)) \), and it is denoted by \( T_m.D \) for \( D \in \text{Div}(Y_0(N)) \). We call \( T_m.D \) the \( m \)th Hecke orbit of \( D \). Especially, when \( D \) is a divisor corresponding to \( i \in \mathbb{H} \), a point in the support of \( T_m.D \) is called a Hecke point. Hecke points have been investigated from several perspectives such as their distribution on the fundamental domain for \( \Gamma_0(N) \) and the rank of a subgroup of \( J_0(N) \) generated by Hecke points, and so on. Let \( q := e^{2\pi i z} \), where \( z \in \mathbb{H} \). For an even integer \( k \), let \( f(z) := q^k \prod_{m=1}^{\infty} (1 - q^m)^{c(m)} \) be a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \). The exponents \( c(m) \) of a modular form were investigated in various works (for examples, see [4, 5, 23]). For example, Borcherds [5] proved that if \( f \) has a Heegner divisor, then the \( m \)th exponent \( c(m) \) is the \( m^2 \)th coefficient of a fixed modular form of half integral weight. Bruinier, Kohnen, and Ono [9] obtained a connection between these exponents of a modular form and the points in the support of \( T_n.D \).

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For the modular invariant $j$, let \( J(z) := j(z) - 744 \). For positive integers \( k \) and \( m \), let \( \sigma_k(m) := \sum_{d|m} d^k \), and \( \sigma_f(m) := \sum_{d|m} c(d) \). Bruinier, Kohnen, and Ono [9] proved the following identities between values \( J(T_m.D_f) \) and sum of exponents in the product expansion of \( f \):
\[
\sum_{d|m} c(d)d = 2k\sigma_1(m) + J(T_m.D_f)
\]
for every positive integer \( m \), where \( D_f \) denotes the divisor of \( f \) on \( X_0(N) \). In other words, the value \( J(T_m.D_f) \) can be expressed as the sum of the following values:

1. a multiple of the divisor function \( \sigma_1(m) \),
2. the convolution of \( \sigma_1(m) \) (sum of divisors) and \( \sigma_f(m) \) (sum of exponents).

They applied this result to prove the modularity of the generating series for \( \sigma_f(m) \) and to obtain several \( p \)-adic properties of \( J(T_m,D_f) \) and exponents of a meromorphic modular form \( f \). Based on the argument in [2], the result was extended to several cases such as \( \Gamma_0(N) \) with genus zero by Ahlgren [2], Jacobi forms by Choie and Kohnen [12], and higher levels by the first author [11].

For general positive integers \( N \), the first author studied in [11] the generalization of [9] to a harmonic weak Maass form \( J_{N,1} \) of weight 0 defined as a Poincaré series (instead of a weakly holomorphic modular form of weight 0). It was proved in [11] that the value \( J_{N,1}(T_m.D_f) \) can be expressed as the sum of the following values:

1. a linear combination of the divisor functions \( \sigma_1(nm) \) for \( n|N \),
2. the convolution of \( \sigma_1(m) \) (sum of divisors) and \( \sigma_f(m) \) (sum of exponents),
3. the regularized Petersson inner product \( R_{f,N}(m) \) of a meromorphic modular form and a cusp form.

In this paper, we show that \( R_{f,N}(m) \), the value of the regularized Petersson inner product in identities [11], is zero, and so we give explicit identities between values \( J_{N,1}(T_m.D_f) \) and sums of exponents in the product expansion of \( f \). As an application, we obtain an asymptotic behavior for the convolution of \( \sigma_1(m) \) (sum of divisors) and \( \sigma_f(m) \) (sum of exponents) as \( m \to \infty \).

Recently, Bringmann, Kane, Löbrich, Ono, and Rolen [7] showed that for any fixed \( N \) the generating series for \( J_{N,1}(T_m.D_f) \) is basically modular. Moreover, their result implies that there is a cusp form such that, for each \( m \), \( R_{f,N}(m) \), the value of regularized Petersson inner product, is given by the \( m \)th coefficient of a fixed cusp form.

Let \( \mathcal{F}_1 \) denote the usual fundamental domain for the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \) given by
\[
\mathcal{F}_1 := \left\{ z \in \mathbb{H} \middle| |z| > 1, \quad -\frac{1}{2} \leq \text{Re}(z) < \frac{1}{2} \right\} \cup \left\{ z \in \mathbb{H} \middle| |z| = 1, \quad \text{Re}(z) \leq 0 \right\}
\]
and
\[
\mathcal{F}_N := \bigcup_{\gamma \in \text{SL}_2(\mathbb{Z}) \setminus \Gamma_0(N)} \gamma \mathcal{F}_1.
\]
Here we choose coset representatives for \( \text{SL}_2(\mathbb{Z})/\Gamma_0(N) \) such that
\[
\mathcal{F}_N \subset \left\{ z \in \mathbb{H} \middle| |\text{Re}(z)| \leq \frac{1}{2} \right\}.
\]
Then, \( \mathcal{F}_N \) is a fundamental domain for the action of \( \Gamma_0(N) \) on \( \mathbb{H} \). Let \( \mathcal{C}_N \) be the set of inequivalent cusps of \( \Gamma_0(N) \). Let \( k \) be an even integer and \( f \) be a meromorphic modular form of weight \( k \) on
\( \Gamma_0(N) \). For \( \tau \in \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q} \), let \( Q_\tau \) be the image of \( \tau \) under the canonical map from \( \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q} \) to \( X_0(N) \). For \( \tau \in \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q} \), we denote by \( \nu_\tau^{(N)}(f) \) the order of zero of \( f \) at \( Q_\tau \) on \( X_0(N) \).

Let us note
\[
g(T_m, D_f) = \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) g(T_m, \tau).
\]

Moreover, for a divisor \( D = \sum n_{\gamma} Q_{\gamma} \) of \( X_0(N) \), we can give a more explicit expression of \( T_m, D \).

For a positive integer \( m \) prime to \( N \), let
\[
T(m) := \{ \gamma = (a \ b) \ 0 \ 0 \ d \mid a, b, d \in \mathbb{Z}, a > 0, ad = m, \text{ and } 0 \leq b < d \}.
\]

Then, we have
\[
T_m, D = \sum n_{\gamma} \sum_{\gamma \in T(m)} Q_{\gamma z}.
\]

Next, we define the Ramanujan theta-operator by
\[
\theta(f)(z) := \frac{1}{2\pi i d} f(z).
\]

Let
\[
f_{\theta}(z) := \frac{\theta f(z)}{f(z)} - \frac{k}{12} E_{2}(z),
\]

where \( E_2 \) is the usual normalized Eisenstein series of weight 2 for \( \text{SL}_2(\mathbb{Z}) \).

Let \( N > 1 \) and \( I_\nu \) be the usual modified Bessel functions as in \([\text{I}]\). For a positive integer \( n \), we define the Poincaré series of weight 0 and index \( n \) by
\[
F_{N,n}(z, s) := \sum_{\gamma \in \Gamma_0(N) \setminus \Gamma_0(N)} \pi \ |n \text{Im}(\gamma z)|^{1/2} I_{s-1/2}(2\pi n \text{Im}(\gamma z)) e(-n \text{Re}(\gamma z)),
\]

where \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \) and \( e(z) := e^{2\pi iz} \).

Let \( j_{N,n}(z) \) be the continuation of \( F_{N,n}(z, s) \) as \( s \to 1 \) from the right. Then, the function \( j_{N,n} \) is a harmonic weak Maass form of weight 0 on \( \Gamma_0(N) \) (see \([\text{II}]\) Section 2 for details). Let \( J_{N,n}(z) := j_{N,n}(z) - \beta_{N,n} \), where \( \beta_{N,n} \) is the constant term of the Fourier expansion of \( j_{N,n} \) at the cusp \( i\infty \).

For square-free \( N \), let \( D(N) \) be the number of divisors of \( N \), and \( \{d_1, d_2, \ldots, d_{D(N)-1}, N\} \) be the set of distinct divisors of \( N \) such that \( d_{i_1} < d_{i_2} \) if \( i_1 < i_2 \). Let \( A_N \) be the \( (D(N) - 1) \times (D(N) - 1) \) matrix whose \( ij \)-entry \( a_{ij} \) is defined by
\[
a_{ij} = \left( 1 - \frac{\gcd(d_i, d_j)^2}{d_j} \right).
\]

Let \( A_{f,j} \) be a matrix obtained from \( A_N \) by replacing the \( j \)th column of \( A_N \) with a column matrix whose \( i \)th component is \( \nu_{1/d_i}^{(N)}(f) - \frac{k}{12} \). With this notation, we state our main theorem.
Theorem 1.1. Let $k$ be an even integer and $N > 1$ be a positive integer. Suppose that
\begin{equation}
(1.1) \quad f(z) = q^{h_0} \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}
\end{equation}
is a meromorphic modular form of weight $k$ on $\Gamma_0(N)$. Then
\[- \sum_{\tau \in F_N} \nu_{\tau}^{(N)}(f) - \sum_{m=1}^{\infty} \left( \sum_{\tau \in F_N} \nu_{\tau}^{(N)}(f) \theta_{N,m}(\tau) \right) q^m = f_0(z) - \mathcal{E}_2(z),\]
where $\mathcal{E}_2$ is a modular form in the Eisenstein space of weight 2 on $\Gamma_0(N)$. Moreover, if $N$ is square free, then, for every positive integer $m$ prime to $N$,
\begin{equation}
(1.2) \quad -J_{N,1}(T_m, D_f) = \sum_{d|m} dc(d) + 24 \left( \sum_{1 \leq j \leq D(N)-1} \frac{\text{det}(A_{f,j})}{\text{det}(A_N)} + \frac{k}{12} \right) \sigma_1(m).
\end{equation}

Remark 1.2. The modular form $\mathcal{E}_2$ in Theorem (1.1) is determined by the order of zero or pole of $f$ at each cusp. In many cases, a modular form $\mathcal{E}_2$ can be expressed as a sum of explicit modular forms. For example, if $N$ is square free, then
\[\mathcal{E}_2(z) = \sum_{1 \leq j \leq D(N)-1} \frac{\text{det}(A_{f,j})}{\text{det}(A_N)} (E_2(z) - d_j E_2(d_jz)).\]

Let $D := \sum_{z \in S} n_z Q_z$ be a divisor of $Y_0(N)$, where $S$ is a finite set in $F_N$. For a positive real number $r \geq 1$, we define a divisor $D_{\geq r}$ by
\[D_{\geq r} = \sum_{\substack{z \in S \mid \text{im}(z) > r}} n_z Q_z.\]
Here, $\tilde{z}$ is a complex number in $F_N$, which is equivalent to $z$ under the action of $\Gamma_0(N)$. By the argument of Duke [15] and equidistribution of Hecke points ([17], [13] and [14]), Theorem 1.1 implies the following theorem.

Theorem 1.3. Let $k, N$, and $f$ be given as in Theorem (1.1). Assume that $N$ is square free. Let $m$ be a positive integer prime to $N$, and $h_f$ denote the sum of the orders of zero or pole of $f$ at $Q_\tau$ on $Y_0(N)$. Then
\[\lim_{m \to \infty} \frac{1}{\sigma_1(m)} \left( 24 \left( \sum_{1 \leq j \leq D(N)-1} \frac{\text{det}(A_{f,j})}{\text{det}(A_N)} + \frac{k}{12} \right) \sigma_1(m) - \sum_{d|m} dc(d) - e \left( - (T_m, D_f)_{\geq 1} \right) \right)
= \frac{3h_f}{\pi |\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)|} \lim_{\epsilon \to 0} \int_{\mathcal{F}_N(\epsilon)} J_{N,1}(z) \frac{dx dy}{y^2},\]
where $c(n)$ are complex numbers determined by (1.1). Here, $\mathcal{F}_N(\epsilon)$ is defined by $\mathcal{F}_N \cup \tau \in \mathcal{E}_N B_{\tau}(\epsilon)$, where $B_{\tau}(\epsilon)$ is given in (3.7).

Recently, Ali and Mani [3] proved an upper bound for exponents $c(m)$ in the product expansion of $f$. The sum $\sum_{d|m} dc(d)$ looks like a kind of convolution of $\sigma_1(m)$ (a sum of divisors) and $\sigma_f(m)$.
(a sum of exponents of \( f \)). The above inequality means that, as \( m \to \infty \), this convolution has a similar asymptotic behavior as that of the sum of divisors of \( m \) except its main term.

The remainder of the paper is organized as follows. In Section 2, we introduce some preliminaries for meromorphic 1-forms on \( X_0(N) \). In Section 3, we provide some basic facts on regularized Petersson inner product, and prove that \( f_\theta \) is orthogonal to every cusp form of weight 2 on \( \Gamma_0(N) \) with respect to regularized Petersson inner product if \( f \) is a meromorphic modular form on \( \Gamma_0(N) \). In Section 4, we recall some results related to the distribution of Hecke points for \( \Gamma_0(N) \). In Section 5, we prove our main theorems: Theorems 1.1 and 1.3.

2. RESIDUES OF A MEROMORPHIC 1-FORM ON \( X_0(N) \)

Let \( f \) be a meromorphic modular form of weight 2 on \( \Gamma_0(N) \). Assume that \( t \) is a cusp of \( \Gamma_0(N) \). Let \( \sigma_t \in \text{SL}_2(\mathbb{Z}) \) be a matrix such that \( \sigma_t(\infty) = t \), and \( \Gamma_0(N)_t \) denote the stabilizer of the cusp \( t \) in \( \Gamma_0(N) \). We define a positive integer \( \alpha_t \) by

\[
\sigma_t^{-1}\Gamma_0(N)\sigma_t = \{ \pm (\frac{1}{\ell}, 1) : \ell \in \mathbb{Z} \},
\]

and we call \( \alpha_t \) the width of \( \Gamma_0(N) \) at the cusp \( t \). The Fourier expansion of \( f \) at the cusp \( t \) is given by

\[
(f|_{2}\sigma_t)(z) = \sum a_t(n)q^{n/\alpha_t},
\]

where \(|_k\) denotes the usual weight \( k \) slash operator. If a cusp \( t \) is equivalent to \( \infty \), the Fourier coefficients \( a_t(n) \) of \( f \) at the cusp \( t \) are simply denoted by \( a(n) \).

For \( \tau \in \mathbb{H} \cup \{ \infty \} \cup \mathbb{Q} \), let \( Q_\tau \) be the image of \( \tau \) under the canonical map from \( \mathbb{H} \cup \{ \infty \} \cup \mathbb{Q} \) to \( X_0(N) \). Then, \( fdz \) can be considered as a meromorphic 1-form on \( X_0(N) \). Thus, we denote by \( \text{Res}_{Q_\tau}fdz \) the residue of \( f \) at \( Q_\tau \) on \( X_0(N) \). Let \( \text{Res}_\tau f \) be the residue of \( f \) at \( \tau \) on \( \mathbb{H} \). The description of \( \text{Res}_{Q_\tau}fdz \) is given in terms of \( \text{Res}_\tau f \). For \( \tau \in \mathbb{H} \), let \( e_\tau \) be the order of the isotropy subgroup of \( \Gamma_0(N) \) at \( \tau \). Then, we have

\[
(2.1) \quad \text{Res}_{Q_\tau}f \ dz = \begin{cases} 
\frac{1}{e_\tau} \text{Res}_\tau f, & \text{if } \tau \in \mathbb{H}, \\
\frac{1}{2\pi i} \alpha_\tau a_\tau (0), & \text{if } \tau \in \mathbb{C}_N.
\end{cases}
\]

Let us note that if \( k \) is an even integer and \( f \) is a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \), then \( f_\theta \) is a meromorphic modular form of weight 2 on \( \Gamma_0(N) \). The residue of \( f_\theta \) at each point on \( X_0(N) \) is determined by the order of its zero or pole of \( f \) at that point. Let \( \text{ord}_\tau(f) \) be the order of the zero or pole of \( f \) at \( \tau \) on \( \mathbb{H} \). Since we have

\[
(cz + d)^{-2}E_2 \left( \frac{az + b}{cz + d} \right) = E_2(z) + \frac{12}{2\pi i} \cdot \frac{c}{cz + d}
\]

for all \((a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z}) \), we obtain

\[
(f_\theta|_{2}\sigma_t)(z) = \frac{\theta(f|_{k}\sigma_t)(z)}{(f|_{k}\sigma_t)(z)} - k \frac{12}{12} E_2(z)
\]

for a cusp \( t \). Thus, we have

\[
(2.2) \quad \text{Res}_\tau f_\theta \ dz = \begin{cases} 
\frac{1}{2\pi i} \nu^{(N)}_\tau (f), & \text{if } \tau \in \mathbb{H}, \\
\frac{k}{2\pi i} \left( \nu^{(N)}_\tau (f) - \frac{k}{12} \right), & \text{if } \tau \in \mathbb{C}_N.
\end{cases}
\]
3. Regularized Petersson inner product

Petersson defined an inner product of two cusp forms with the same weight. The Petersson inner product was extended by Borcherds [6] to the case in which one of the two forms is a weakly holomorphic modular form. In this section, following [6] and [11], we define regularized Petersson inner product of a cusp form and a meromorphic modular form with the same weight. We prove that if $f$ is a meromorphic modular form on $\Gamma_0(N)$, then the regularized Petersson inner product of $f_\theta$ with any cusp form of weight 2 on $\Gamma_0(N)$ is zero.

Let $k$ be an even integer and $f$ be a meromorphic modular form of weight $k$ on $\Gamma_0(N)$. Let $\text{Sing}(f)$ be the set of singular points of $f$ on $\mathcal{F}_N$. For a positive real number $\varepsilon$, an $\varepsilon$-disk $B_\tau(\varepsilon)$ at $\tau$ is defined by

$$(3.1)\quad B_\tau(\varepsilon) := \begin{cases} 
\{ z \in \mathbb{H} : |z - \tau| < \varepsilon \}, & \text{if } \tau \in \mathbb{H}, \\
\{ z \in \mathcal{F}_N : \text{Im}(\sigma_\tau z) > 1/\varepsilon \}, & \text{if } \tau \in \{i\infty\} \cup \mathbb{Q}.
\end{cases}$$

Let $\mathcal{F}_N(f,\varepsilon)$ be a punctured fundamental domain for $\Gamma_0(N)$ defined by

$$\mathcal{F}_N(f,\varepsilon) := \mathcal{F}_N \setminus \bigcup_{\tau \in \text{Sing}(f) \cup \mathcal{C}_N} B_\tau(\varepsilon).$$

Let $g$ be a cusp form of weight $k$ on $\Gamma_0(N)$. The regularized Petersson inner product $(f,g)_{\text{reg}}$ of $f$ and $g$ is defined by

$$(f,g)_{\text{reg}} := \lim_{\varepsilon \to 0} \int_{\mathcal{F}_N(f,\varepsilon)} f(z)\overline{g(z)}\frac{dxdy}{y^{k-2}}.$$ 

Then, we have the following proposition.

**Proposition 3.1.** Let $k$ be an even integer, and $f$ be a meromorphic modular form of weight $k$ on $\Gamma_0(N)$. Then, for every cusp form $g$ of weight 2 on $\Gamma_0(N)$,

$$(f_\theta,g)_{\text{reg}} = 0.$$ 

**Proof.** Let $\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ be the unique normalized cusp form of weight 12 on $\text{SL}_2(\mathbb{Z})$. Let

$$F(z) := \frac{f(z)^{12}}{\Delta(z)^k}.$$ 

Then, we have

$$d((\log |F(z)|^2)g(z)d\bar{z}) = \frac{\partial_z F(z)\overline{F(z)}}{F(z)\overline{F(z)}}g(z)dzd\bar{z} = \frac{\partial_z F(z)}{F(z)}g(z)(-2i)dzd\bar{z}.$$ 

Let us note that $\Delta$ has no zeros and no poles on $\mathbb{H}$. Therefore, according to [9] Theorem 1], we have

$$\frac{\theta(\Delta)}{\Delta} = E_2.$$ 

The function $\partial_z F(z)/F(z)$ is given as

$$\frac{\partial_z F(z)}{F(z)} = 12\frac{\partial_z f(z)}{f(z)} - k\frac{\partial_z \Delta(z)}{\Delta(z)} = 12\frac{\partial_z f(z)}{f(z)} - k(2\pi i)E_2(z) = (24\pi i)f_\theta(z).$$
Thus, we have
\[
(3.2) \quad d((\log_e |F(z)|^2)g(z)d\zeta) = (48\pi)f_\theta(z)g(z)dx\,dy.
\]

In order to apply the Stokes theorem, we describe the boundary of $F_N$. For a positive real number $\varepsilon$, we define
\[
\gamma_\tau(\varepsilon) := \begin{cases} 
\{ z \in \mathbb{H} : |z - \tau| = \varepsilon \} & \text{if } \tau \in \mathbb{H}, \\
\{ z \in F_N : \text{Im}(\sigma_\tau z) = 1/\varepsilon \} & \text{if } \tau \in \{i\infty \} \cup \mathbb{Q}.
\end{cases}
\]
Assume that $\varepsilon$ is sufficiently small. If $\partial^*F_N(f,\varepsilon)$ denotes the closure of the set $\partial F_N(f,\varepsilon) - \partial F_N$ in $\mathbb{C}$, then
\[
(3.3) \quad \partial^*F_N(f,\varepsilon) = \bigcup_{\tau \in \text{Sing}(f) \cup C_N} \gamma_\tau(\varepsilon),
\]
where $\partial D$ denotes the boundary of $D$ for a subset $D$ of $\mathbb{C}$. From (3.2) and (3.3), the Stokes theorem implies
\[
\int_{F_N(f_\theta,\varepsilon)} f_\theta(z)g(z)dx\,dy = \int_{\partial^*F_N(f_\theta,\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)g(z)d\zeta = \sum_{\tau \in \text{Sing}(f_\theta) \cup C_N} \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)g(z)d\zeta.
\]

For each $\gamma \in \text{SL}_2(\mathbb{Z})$, the absolute value $|\gamma_\tau(z)|$ exponentially decays as $\text{Im}(z) \to \infty$, since $g$ is a cusp form. Thus, if $\tau \in C_N$, then $\lim_{\varepsilon \to 0} \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)g(z)d\zeta = 0$.

To complete the proof, we assume that $\tau \in \text{Sing}(f_\theta)$. Then
\[
\left| \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)g(z)d\zeta \right| \leq \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}|(\log_e |F(z)|^2)||g(z)||d\zeta| \\
\leq \max\{|(\log_e |F(z)|^2)| : z \in \gamma_\tau(\varepsilon)\}M_1 \int_{\gamma_\tau(\varepsilon)} |d\zeta| \quad (\text{some constant } M_1) \\
\leq \max\{|(\log_e |F(z)|^2)| : z \in \gamma_\tau(\varepsilon)\}M_1(2\pi\varepsilon).
\]
The function $F(z)$ can be expressed around $\tau$ as
\[
F(z) = (z - \tau)^{12\nu_f(N)}F_0(z),
\]
where $F_0(z)$ is a nowhere vanishing holomorphic function around $\tau$. If $\varepsilon$ is sufficiently small, then, for any $z \in \gamma_\tau(\varepsilon)$ we have
\[
|(\log_e |F(z)|^2)| \leq |(\log_e |(z - z_0)|^{24\nu_f(N)}(f))| + |(\log_e |F_0(z)|^2)| \\
\leq |(\log_e |(z - z_0)|^{24\nu_f(N)}(f))| + M_2 \quad (\text{some fixed constant } M_2) \\
= |24\nu_f(N)(f)\log_e \varepsilon| + M_2.
\]
Thus, for sufficiently small \( \varepsilon \), we have
\[
\left| \int_{\gamma_{\tau}(\varepsilon)} \frac{1}{48\pi} (\log_2 |F(z)|^2) dz \right| \leq (|24\nu_2(N)(f)\log_2 \varepsilon| + M_2)M_1(2\pi \varepsilon).
\]
This implies that, for \( \tau \in \text{Sing}(f_\theta) \),
\[
\lim_{\varepsilon \to 0} \int_{\gamma_{\tau}(\varepsilon)} \frac{1}{48\pi} (\log_2 |F(z)|^2) dz = 0.
\]
Thus, we complete the proof. \( \square \)

4. Equidistribution of Hecke points

Let \( \{u_j\}_{j \geq 0} \) be an orthonormal basis of the residual and cuspidal spaces of \( L^2(\Gamma_0(N) \setminus \mathbb{H}) \), i.e., \( u_0 \) is a constant with the eigenvalue \( \lambda_0 = 0 \) and \( u_j \) is a Maass form for \( \Gamma_0(N) \) with eigenvalue \( \lambda_j = s_j(1-s_j) \) for \( j \geq 1 \). Further, assume that \( \lambda_j \) are ordered so that \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \). For each cusp \( t \in \mathbb{Q} \cup \{\infty\} \), let \( E_t(z, s) \) be the Eisenstein series at \( t \) for \( \text{Re}(s) > 1 \), which is given by
\[
E_t(z, s) = \sum_{\gamma \in \Gamma_0(N) \setminus \Gamma_0(N)} (\text{Im}(\sigma_t^{-1}\gamma z))^s.
\]
Here, \( \Gamma_0(N)_t \subset \Gamma_0(N) \) is the stability group of \( t \). For the properties of \( E_t(z, s) \), see [19, §15].

According to [19, Theorem 15.5], any \( f \in L^2(\Gamma_0(N) \setminus \mathbb{H}) \) has the spectral decomposition
\[
f(z) = \sum_{j \geq 0} \langle f, u_j \rangle u_j(z) + \sum_{t \in \mathcal{C}_N} \frac{1}{4\pi} \int_{\mathbb{R}} \langle f, E_t(1/2 + ir) \rangle E_t(z, 1/2 + ir) dr
\]
(valid in \( L^2 \)-sense) and converges absolutely and uniformly on compact sets if \( f \) and \( \Delta f \) are smooth and bounded.

We now follow the proof of [17, Theorem 3.1]. Let
\[
f_C = \langle f, u_0 \rangle = \text{the projection of } f \text{ onto the constant subspace},
\]
\[
f_{\mathcal{M}}(z) = \sum_{j \geq 1} \langle f, u_j \rangle u_j(z),
\]
\[
f_E(z) = \sum_{t \in \mathcal{C}_N} \frac{1}{4\pi} \int_{\mathbb{R}} \langle f, E_t(1/2 + ir) \rangle E_t(z, 1/2 + ir) dr.
\]
Note that
\begin{equation}
(4.1) \quad f_C = \langle f, u_0 \rangle = \int_{\mathcal{F}_N} f(z) d\mu(z),
\end{equation}
where \( d\mu(z) := \frac{3}{\pi|S_{1/2}(2, \Gamma_0(N))|} \cdot \frac{dx dy}{y^2} \) is the normalized Haar measure; so, \( \int_{\mathcal{F}_N} d\mu(z) = 1 \).

Let \( \lambda_j(n) \) be the \( n \)th Fourier coefficient of \( u_j \). By the Ramanujan conjecture, there exists \( \theta \geq 0 \) such that \( |\lambda_j(n)| \leq cn^{\theta+\epsilon} \), for any \( \epsilon > 0 \). So, we get
\begin{equation}
(4.2) \quad \frac{1}{\sigma_1(n)} \| T_n f_{\mathcal{M}} \|_2 \leq cn^{-2+\theta+\epsilon} \| f_{\mathcal{M}} \|_2.
\end{equation}
Note that the value of \( \theta \) has been lowered to \( \frac{7}{64} \) by Kim and Sarnak [20, Appendix 2].
In \cite[§6, §7 and §8]{22}, an explicit change-of-basis formula between the Eisenstein series attached to cusps and newform Eisenstein series attached to pairs of primitive Dirichlet characters is described. The Eisenstein series attached to a Dirichlet character is an eigenfunction of Hecke operators $T_n$ for $\gcd(n, N) = 1$, and the absolute values of the corresponding eigenvalues are bounded above by $\sigma_0(n)n^{-\frac{1}{2}}$. So, we get

\begin{equation}
\frac{1}{\sigma_1(n)}\|T_n f_E\|_2 \leq c n^{-\frac{1}{2}+\epsilon}\|f_E\|_2.
\end{equation}

If we combine (4.1), (4.2), and (4.3), then we obtain the following theorem. For more general result, see \cite{13}.

**Theorem 4.1.** Let $f \in L^2(\Gamma_0(N)\backslash \mathbb{H})$. For a positive integer $n$ prime to $N$, we have

$$\left\| \frac{1}{\sigma_1(n)} T_n f - \int_{\mathcal{F}_N} f(z) \, d\mu(z) \right\|_2 \leq c n^{-\frac{1}{2}+\theta+\epsilon}\|f\|_2$$

for any $\epsilon > 0$. The constant $c_\epsilon$ depends on $\epsilon$.

The pointwise convergence can be derived from \cite[Proposition 8.2]{14}. Note that elliptic differential operators are differential operators that generalize the Laplace-Beltrami operator $\Delta$. For an integer $m \geq 2$, assume that $f, \Delta^m f \in L^2(\Gamma_0(N)\backslash \mathbb{H})$. Then, by \cite[Proposition 8.2]{14}, for a compact subset $\omega \subset \mathcal{F}_N$, there exist constants $C_1(\omega)$ and $C_2(\omega)$ such that, for any $z_0 \in \omega$

$$\left| \frac{1}{\sigma_1(n)} T_n f(z_0) - \int_{\mathcal{F}_N} f(z) \, d\mu(z) \right| \leq C_1(\omega) \left\| \frac{1}{\sigma_1(n)} T_n f - \int_{\mathcal{F}_N} f(z) \, d\mu(z) \right\|_2 + C_2(\omega) \left\| \frac{1}{\sigma_1(n)} T_n (\Delta^m f) - \int_{\mathcal{F}_N} (\Delta^m f)(z) \, d\mu(z) \right\|_2.$$  

So, we have the following corollary.

**Corollary 4.2.** Assume that $f, \Delta^2 f \in L^2(\Gamma_0(N)\backslash \mathbb{H})$. Take a compact $\omega \subset \Gamma_0(N)\backslash \mathbb{H}$ and a positive number $\epsilon$. Then, there exists a constant $C_{\omega, \epsilon}$ depending on $\omega$ and $\epsilon$, such that, for a positive integer $n$ prime to $N$, for any $z_0 \in \omega$,

$$\left| \frac{1}{\sigma_1(n)} T_n f(z_0) - \int_{\mathcal{F}_N} f(z) \, d\mu(z) \right| \leq C_{\omega, \epsilon} n^{-\frac{1}{2}+\theta+\epsilon} \max\{\|f\|_2, \|\Delta^2 f\|_2\}.$$  

5. Proofs

Let $M^Eis_k(\Gamma_0(N))$ be the space of modular forms orthogonal to all the cusp forms of weight $k$ on $\Gamma_0(N)$, which is called the Eisenstein space of weight $k$ on $\Gamma_0(N)$. In the following lemma, we prove that if $N$ is square-free, then, for a positive integer $n$ prime to $N$, the $n$th coefficient of a modular form in $M^Eis_k(\Gamma_0(N))$ is a multiple of $\sigma_1(n)$. Recall the notations $D(N)$, $d_j$, and $A_N$ from Section \cite{11}. Now, we prove the following lemma related to properties for modular forms in an Eisenstein space.

**Lemma 5.1.** Suppose that $\mathcal{E}_2(z) := \sum_{n=0}^{\infty} b(n)q^n$ is a modular form in $M^Eis_2(\Gamma_0(N))$, and that $N$ is square free. Then, the following statements are true.
Proof. (1) We claim that there is a basis of $dE\a$ where $\sum a$ have

$$
E(z) = \sum_{1 \leq j \leq D(N) - 1} \frac{\det(A_j)}{\det(A_N)}(E_2(z) - d_jE_2(d_jz)).
$$

Therefore, the modular forms $E(z)$ expressed as a linear combination of $E_2(z) - d_jE_2(d_jz)$ having the form

$$
E(z) = \sum_{1 \leq j \leq D(N) - 1} a_j(E_2(z) - d_jE_2(d_jz)).
$$

Recall that $E_2$ has the Fourier expansion of the form

$$(5.1) \quad E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n.$$ 

Then, the $n$th coefficient of $E_2(z)$ is given by

$$-24 \sum_{1 \leq j \leq D(N) - 1} a_j(\sigma_1(n) - d_j\sigma_1(n/d_j))$$

for $n > 0$, and $a_j$ does not depend on $n$. Here, $\sigma_1(n/d) = 0$ if $n$ is not divisible by $d$. Thus, we have the proof of the lemma.

Now, we prove the claim. Suppose that

$$\sum_{1 \leq j \leq D(N) - 1} a_j(E_2(z) - d_jE_2(d_jz)) = 0,$$

where $a_j$ are complex numbers. We assume that complex numbers $a_j$ are not all zero. Then, we have

$$\sum_{1 \leq j \leq D(N) - 1} a_jE_2(z) = \sum_{1 \leq j \leq D(N) - 1} a_jd_jE_2(d_jz).$$

Comparing the $n$th coefficients of the forms on both sides for $n$ prime to $N$, we have

$$\sum_{1 \leq j \leq D(N) - 1} a_jE_2(z) = \sum_{1 \leq j \leq D(N) - 1} a_jd_jE_2(d_jz) = 0.$$ 

Take the smallest positive integer $d_{j_0}|N$ such that $a_{j_0} \neq 0$. Then, we have

$$-a_{j_0}d_{j_0}E_2(d_{j_0}z) = \sum_{1 \leq j \leq D(N) - 1} a_jd_jE_2(d_jz) - a_{j_0}d_{j_0}E_2(d_{j_0}z).$$

Comparing the $d_{j_0}$th coefficients of the forms on both sides, we have $a_{j_0} = 0$. This is a contradiction. Therefore, the modular forms $E_2(z) - dE_2(dz)$, $d|N$ and $d \neq 1$, are linearly independent.

Let us note

$$\dim_\mathbb{C} M_2^{Eis}(\Gamma_0(N)) = D(N) - 1.$$
since $N$ is square free. Thus,

$$\{ (E_2(z) - dE_2(dz) : d \mid N \text{ and } d \neq 1 \}$$

is a basis of $M_2^{Eis}(\Gamma_0(N))$. This completes the proof of the claim.

(2) From the proof of (1), we may assume that

$$\mathcal{E}_2(z) = \sum_{1 \leq j \leq D(N) - 1} a_j(E_2(z) - d_jE_2(d_jz)).$$

Let us note that $E_2(z) - \frac{3}{\pi \Im(z)}$ is a non-holomorphic modular form of weight 2 on $SL_2(\mathbb{Z})$. By direct computation, there are $\gamma \in SL_2(\mathbb{Z})$ and $\mu_j \in \mathbb{Z}$ such that

$$\left( \begin{array}{cc} d_j & 0 \\ 0 & 1 \end{array} \right) \gamma \left( \begin{array}{cc} 1 & \mu_j \\ 0 & d_j \gcd(d_j, d_i) \end{array} \right) \left( \begin{array}{cc} 1 & \mu_j \\ 0 & d_j \gcd(d_j, d_i) \end{array} \right) = \gamma \left( \begin{array}{cc} 1 & \mu_j \\ 0 & d_j \gcd(d_j, d_i) \end{array} \right).$$

Thus,

$$(E_2(z) - d_jE_2(d_jz))|_2 \left( \frac{1}{d_j} \right) = E_2(z) - \frac{\gcd(d_j, d_i)^2}{d_j}E_2 \left( \frac{\gcd(d_j, d_i)^2}{d_j}z + \frac{\mu_j \gcd(d_j, d_i)}{d_j} \right).$$

This implies that $a_j$ are the solution of the system

$$c_{d_i} = \sum_{1 \leq j \leq D(N) - 1} \left( 1 - \frac{\gcd(d_j, d_i)^2}{d_j} \right) a_j$$

for $1 \leq i \leq D(N) - 1$. Thus, the Cramer’s rule completes the proof. \hfill \square

Now, we prove Theorem 1.1.

**Proof.** Note that

$$\{ E_2(z) - dE_2(dz) \mid d \mid N, \ d \neq 1 \}$$

forms a basis of $M_2^{Eis}(\Gamma_0(N))$ by the proof of Lemma 5.1. Therefore, we can take a modular form $\mathcal{E}_2 \in M_2^{Eis}(\Gamma_0(N))$ such that the constant term of $\mathcal{E}_2$ at each cusp except cusps equivalent to $i \infty$ is the same as that of $f_\theta$. Suppose that $\mathcal{E}_2$ has the Fourier expansion of the form

$$\mathcal{E}_2(z) = \sum_{n=0}^{\infty} b(n)q^n.$$

Note that, by (2.1) and (2.2), we have

$$\frac{2\pi i}{e_\tau} \text{Res}_\tau f_\theta = \nu^{(N)}(f)$$

for $\tau \in \mathcal{F}_N$. Thus, from [11, Lemma 3.1], we obtain

$$(5.2) \quad (f_\theta - \mathcal{E}_2, \xi_0(j_{N,m}))_{\text{reg}} = \beta_{N,m}(a_\theta(0) - b(0)) + a_\theta(m) - b(m) + \sum_{\tau \in \mathcal{F}_N} \nu^{(N)}(f)j_{N,m}(\tau),$$

where $a_\theta(m)$ is the $m$th Fourier coefficient of $f_\theta$ and $\xi_0$ is a differential operator defined by

$$\xi_0(f)(z) := 2i \left. \frac{\partial}{\partial z} \right| f(z).$$
By the same argument in the proof of Lemma 3.1, we have
\[
(f_\theta - \mathcal{E}_2, \xi_0(J_{N,m}))_{\text{reg}} = a_\theta(m) - b(m) + \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau).
\]

Note that \(\xi_0(j_{N,m}) = \xi_0(J_{N,m})\) since \(J_{N,m}(z) = j_{N,m}(z) - \beta_{N,m}\). Therefore, from (5.2) and (5.3), we have
\[
a_\theta(0) - b(0) = \frac{1}{\beta_{N,m}} \left( \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) - \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) j_{N,m}(\tau) \right) = - \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f).
\]

Proposition 3.1 implies that
\[
(f_\theta - \mathcal{E}_2, \xi_0(j_{N,m}))_{\text{reg}} = 0.
\]

Therefore, from (5.3), we have
\[
a_\theta(m) - b(m) = - \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau)
\]
for every positive integer \(m\). Thus, from (5.4) and (5.5), we obtain
\[
f_\theta(z) - \mathcal{E}_2(z) = - \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) - \sum_{m=1}^{\infty} \left( \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) \right) q^m.
\]

By (1.1) and the Fourier expansion of \(E_2\) given in (5.1), \(f_\theta\) has the Fourier expansion of the form
\[
f_\theta(z) = h_\infty + \sum_{n=1}^{\infty} \sum_{d|n} dc(d) q^n - \frac{k}{12} + 2k \sum_{n=1}^{\infty} \sigma_1(n) q^n.
\]

Let us note that the constant term of \(f_\theta\) at cusp \(c\) is \(\nu_1^{(N)}(f) - k/12\). Suppose that \(m\) is prime to \(N\). Then, Lemma 5.1 implies that
\[
b(m) = -24 \left( \sum_{1 \leq j \leq D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} \right) \sigma_1(m).
\]

Here, \(A_{f,j}\) is a matrix obtained from \(A_N\) by replacing the \(j\)th column of \(A_N\) with a column matrix whose \(i\)th component is \(\frac{\nu_1^{(N)}(f)}{\alpha_{1,d_i}} - \frac{k}{12}\). Let us note that if \(\gcd(m, N) = 1\), then \(J_{N,m} = J_{N,1}|T_m\). Therefore, by (5.5), (5.7), and (5.8), we have
\[
-J_{N,1}(T_m, D_f) = -J_{N,m}(D_f) = - \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) = a_\theta(m) - b(m)
\]
\[= \sum_{d|m} dc(d) + 24 \left( \sum_{1 \leq j \leq D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} + \frac{k}{12} \right) \sigma_1(m).
\]

□
To prove Theorem 1.3, we follow the argument of the proof of [15, Proposition 3]. We fix \( \epsilon > 0 \). Let \( \psi : \mathbb{R}_{>0} \to \mathbb{R} \) be a \( C^\infty \) function with \( 0 \leq \psi_\epsilon(y) \leq 1 \) for all \( y \in \mathbb{R}_{>0} \) and

\[
\psi_\epsilon(y) = \begin{cases} 
0, & \text{if } y \leq 1, \\
1, & \text{if } y > 1 + \epsilon.
\end{cases}
\]

For a positive integer \( n \), consider the Poincaré series defined by

\[
P_{n, \epsilon}(z) := \sum_{\gamma \in \Gamma_0(N) \setminus \Gamma_0(N)} \psi_\epsilon(\text{Im}(\gamma z)) e(-n(\gamma z)).
\]

From this, we obtain the following proposition.

**Proposition 5.2.** Let \( \theta \) be given as in Section 4. Fix \( n \in \mathbb{Z}_{\geq 1} \), \( \epsilon > 0 \), and \( z_0 \in \mathbb{H} \). For any positive integer \( m \) prime to \( N \) and any \( \epsilon' > 0 \), we have

\[
\left| \frac{1}{\sigma_1(m)} \left\{ \sum_{ad=m, \ b \ (\text{mod } d)} J_{N,n} \left( \frac{az_0 + b}{d} \right) - \sum_{ad=m, \ b \ (\text{mod } d)} P_{n, \epsilon} \left( \frac{az_0 + b}{d} \right) \right\} - \lim_{\epsilon'' \to 0} \int_{\mathcal{F}_N(\epsilon'')} J_{N,n}(z) \, d\mu(z) \right| \leq C_{n, \epsilon'} m^{-\frac{1}{2} + \epsilon'} \max\{\|F_{n, \epsilon}\|_2, \|\Delta^2 F_{n, \epsilon}\|_2\},
\]

where \( F_{n, \epsilon} := J_{N,n} - P_{n, \epsilon} \) and \( C_{n, \epsilon'} \) is the constant given as in Corollary 4.2.

**Proof.** For a positive integer \( n \), let \( P_{n, \epsilon} \) be the Poincaré series as in (5.9). From [11, Theorem 2.1], it follows that \( F_{n, \epsilon} \in L^2(\Gamma_0(N) \setminus \mathbb{H}) \) for a fixed \( n \geq 1 \).

Recall that for \( \phi \in L^2(\Gamma_0(N) \setminus \mathbb{H}) \) and \( m \geq 1 \) with \( \gcd(m, N) = 1 \), the normalized Hecke operator \( T_m \) can be represented by

\[
T_m \phi(z) = \sum_{\gamma \in T(m)} \phi(\gamma z).
\]

By Corollary 4.2 we find that for any \( \epsilon' > 0 \) and \( m \)

\[
\left| \frac{1}{\sigma_1(m)} (T_m F_{n, \epsilon})(z_0) - \int_{\mathcal{F}_N} F_{n, \epsilon}(z) \, d\mu(z) \right| \leq C_{n, \epsilon'} m^{-\frac{1}{2} + \epsilon'} \max\{\|F_{n, \epsilon}\|_2, \|\Delta^2 F_{n, \epsilon}\|_2\}.
\]

For \( z_0 \in \mathbb{H} \), we have

\[
\left( 1 - \frac{1}{\sigma_1(m)} \right) T_m F_{n, \epsilon}(z_0) = \frac{1}{\sigma_1(m)} \left\{ \sum_{ad=m, \ b \ (\text{mod } d)} J_{N,n} \left( \frac{az_0 + b}{d} \right) - \sum_{ad=m, \ b \ (\text{mod } d)} P_{n, \epsilon} \left( \frac{az_0 + b}{d} \right) \right\}.
\]

Note that

\[
\lim_{\epsilon'' \to 0} \int_{\mathcal{F}_N(\epsilon'')} F_{n, \epsilon}(z) \, \frac{dx \, dy}{y^2} = \int_{\Gamma_0(N) \setminus \mathbb{H}} \psi_\epsilon(y) e^{-2\pi i nz} \, d\mu(y) = \int_0^\infty \psi_\epsilon(y) y e^{2\pi ny} \, dy = \int_0^1 e^{2\pi inx} \, dx = 0
\]

for every positive integer \( n \). So, we get

\[
\int_{\mathcal{F}_N} F_{n, \epsilon}(z) \, d\mu(z) = \lim_{\epsilon'' \to 0} \int_{\mathcal{F}_N(\epsilon'')} J_{N,n}(z) \, d\mu(z).
\]
If we combine (5.10), (5.11), and (5.12), then we get the desired result. \qed

We define

\[ Q_{1,\epsilon}(z) := \psi_\epsilon(\text{Im}(\bar{z}))e(-\bar{z}) \]

for \( \epsilon > 0 \). Then, we obtain the following proposition.

**Proposition 5.3.** For any \( m \) with \( \gcd(m, N) = 1 \), we obtain

\[
(5.13) \quad \left| \frac{1}{\sigma_1(m)}(J_{N,1}(T_m.D_f) - Q_{1,\epsilon}(T_m.D_f)) - h_f \lim_{\epsilon' \to 0} \int_{F_N(\epsilon')} J_{N,1}(z) d\mu(z) \right| \leq H_f C(f, \epsilon') m^{\frac{1}{2} + \theta + \epsilon'} \max\{\|F_{1,\epsilon}\|_2, \|\Delta^2 F_{1,\epsilon}\|_2\},
\]

where \( h_f \) denotes the sum of the orders of zero or pole of \( f \) at \( Q_\tau \) on \( Y_0(N) \).

**Proof.** Let \( \epsilon > 0 \) be fixed. Note that

\[ J_{N,1}(T_m.D_f) = \sum_{\tau \in F_N} \nu_\tau^{(N)}(f) \sum_{b \equiv m (\text{mod } d)} J_{N,1}(a\tau + b) \]

and

\[ P_{1,\epsilon}(T_m.D_f) = \sum_{\tau \in F_N} \nu_\tau^{(N)}(f) \sum_{b \equiv m (\text{mod } d)} P_{1,\epsilon}(a\tau + b) . \]

Therefore, by Proposition 5.2 for any \( m \) with \( \gcd(m, N) = 1 \), we have

\[
(5.14) \quad \left| \frac{1}{\sigma_1(m)}(J_{N,1}(T_m.D_f) - P_{1,\epsilon}(T_m.D_f)) - h_f \lim_{\epsilon' \to 0} \int_{F_N(\epsilon')} J_{N,1}(z) d\mu(z) \right| \leq H_f C(f, \epsilon') m^{\frac{1}{2} + \theta + \epsilon'} \max\{\|F_{1,\epsilon}\|_2, \|\Delta^2 F_{1,\epsilon}\|_2\},
\]

for any \( \epsilon' > 0 \), where \( H_f := \sum_{\tau \in F_N} |\nu_\tau^{(N)}(f)| \) and \( C(f, \epsilon') := \max\{C_{\tau,\epsilon'} \mid \tau \in F_N, \nu_\tau^{(N)}(f) \neq 0\} \).

Recall that \( \bar{z} \) is a unique complex number in \( F_N \) which is equivalent to \( z \) under the action of \( \Gamma_0(N) \). If \( \text{Im}(\bar{z}) > 1 \), then for any \( \gamma \in \Gamma_0(N) \), \( \text{Im}(\gamma \bar{z}) \leq 1 \) unless \( \gamma \in \Gamma_0(N)_{\infty} \).

Suppose that \( \text{Im}(\bar{z}) \leq 1 \) and that there exists \( \gamma \in \Gamma_0(N) \) such that \( \text{Im}(\gamma \bar{z}) > 1 \). Then, there exists \( \ell \in \mathbb{Z} \) such that \(-\frac{1}{2} < \text{Re}(\gamma \bar{z}) + \ell \leq \frac{1}{2}\), and so

\[ \gamma \bar{z} + \ell = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \gamma \bar{z} \in F_N. \]

Since \( \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \gamma \in \Gamma_0(N) \), we have \( \gamma \bar{z} + \ell = \bar{z} \), so \( \text{Im}(\gamma \bar{z}) = \text{Im}(\bar{z}) \leq 1 \), which is a contradiction. Therefore, if \( \text{Im}(\bar{z}) \leq 1 \), then for any \( \gamma \in \Gamma_0(N) \), we get \( \text{Im}(\gamma \bar{z}) \leq 1 \).

Thus, we have

\[ P_{1,\epsilon}(z) = P_{1,\epsilon}(\bar{z}) = \sum_{\gamma \in \Gamma_0(N)_{\infty} \backslash \Gamma_0(N)} \psi_\epsilon(\text{Im}(\gamma \bar{z}))e(-\gamma \bar{z}) = Q_{1,\epsilon}(z). \]

Therefore, from (5.14), we obtain the desired result. \qed
From Proposition 5.2 and Proposition 5.3, we obtain the following theorem. This gives the distribution of values of $J_{N,1}$ on Hecke orbits.

**Theorem 5.4.** We have

$$
\lim_{m \to \infty} \frac{1}{\sigma_1(m)} \left( J_{N,1}(T_m, D_f) - e \left( (T_m, D_f) > 1 \right) \right) = \frac{3h_f}{\pi|\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)|} \lim_{\epsilon' \to 0} \int_{\mathcal{F}_N(\epsilon')} J_{N,1}(z) \frac{dx dy}{y^2}.
$$

**Proof.** Let $\epsilon > 0$ be fixed. For any positive integer $m$ which is prime to $N$, we have

$$
\left| \frac{1}{\sigma_1(m)} \left( J_{N,1}(T_m, D_f) - e \left( (T_m, D_f) > 1 \right) \right) - h_f \lim_{\epsilon' \to 0} \int_{\mathcal{F}_N(\epsilon')} J_{N,1}(z) d\mu(z) \right|
$$

$$
\leq \left| \frac{1}{\sigma_1(m)} \left( J_{N,1}(T_m, D_f) - Q_{1,\epsilon}(T_m, D_f) \right) - h_f \lim_{\epsilon' \to 0} \int_{\mathcal{F}_N(\epsilon')} J_{N,1}(z) d\mu(z) \right|
$$

$$
+ \frac{1}{\sigma_1(m)} |Q_{1,\epsilon}(T_m, D_f) - e \left( (T_m, D_f) > 1 \right) |.
$$

Note that

$$
|Q_{1,\epsilon}(T_m, D_f) - e(- (T_m, D_f))_{>1}| \leq \sum_{\tau \in \mathcal{F}_N} \left| \nu^{(N)}_{\tau}(f) \right| \sum_{\gamma \in T(m)} \left\{ |\psi_{\epsilon}(\text{Im}(\gamma \tau)) - 1| |e(- \gamma \tau)|, \text{ if } 1 < \text{Im}(\gamma \tau) \leq 1 + \epsilon, \right.
$$

$$
0, \text{ otherwise.}
$$

Now, we follow the proof of [15 Proposition 3]. Fix $0 < \epsilon < \frac{1}{4}$ and consider the incomplete Eisenstein series

$$
g_{\epsilon}(z) := \sum_{\gamma \in \Gamma_0(N) \setminus \Gamma_0(N)} \phi_{\epsilon}(\text{Im}z),
$$

where $\phi_{\epsilon} : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function supported in $(1 - \epsilon, 1 + 2\epsilon)$ with $0 \leq \phi_{\epsilon}(y) \leq 1$ for all $y \in \mathbb{R}_+$ and $\phi_{\epsilon}(y) = 1$ for $1 \leq y \leq 1 + \epsilon$. By Corollary 4.2, we see that for any $\epsilon' > 0$, $z_0 \in \mathbb{H}$, and $m$ with $\gcd(m, N) = 1$,

$$
\left| \frac{1}{\sigma_1(m)} T_m g_{\epsilon}(z_0) - \int_{\mathcal{F}_N} g_{\epsilon}(z) d\mu(z) \right| \leq C_{z_0, \epsilon'} m^{-\frac{1}{2} + \epsilon' + \theta} \max\{ \|g_{\epsilon}\|_2, \|\Delta^2 g_{\epsilon}\|_2 \}.
$$

Then, there exists a constant $D(f, \epsilon')$ such that

$$
\frac{1}{\sigma_1(m)} |Q_{1,\epsilon}(T_m, D_f) - e(- (T_m, D_f))_{>1}| \leq \frac{e^{2\pi(1+\epsilon)}}{\sigma_1(m)} \sum_{\tau \in \mathcal{F}_N} |\nu^{(N)}_{\tau}(f)| (T_m g_{\epsilon})(\tau)
$$

$$
\leq C_f e^{2\pi(1+\epsilon)} \left( \int_{\mathcal{F}_N} g_{\epsilon}(z) d\mu(z) + D(f, \epsilon') m^{-\frac{1}{2} + \theta + \epsilon'} \max\{ \|g_{\epsilon}\|_2, \|\Delta^2 g_{\epsilon}\|_2 \} \right),
$$

where $C_f := \# \left\{ \tau \in \mathcal{F}_N \mid \nu^{(N)}_{\tau}(f) \neq 0 \right\} \times \max\left\{ |\nu^{(N)}_{\tau}(f)| \mid \tau \in \mathcal{F}_N \right\}$. 

Therefore, from Proposition 5.3 and (5.15), we have
\begin{equation}
\left| \frac{1}{\sigma_1(m)} (J_{N,1}(T_m.D_f) - e ((T_m.D_f) > 1)) \right| \left| h_f \lim_{\varepsilon'' \to 0} \int_{F_N(\varepsilon'')} J_{N,1}(z) d\mu(z) \right| \\
\leq \left( H_f C(f, \varepsilon') \max\{\|F_{1,\varepsilon}\|_2, \|\Delta^2 F_{1,\varepsilon}\|_2\} + C_f \varepsilon^{2(1+\varepsilon)} D(f, \varepsilon') \max\{\|g_{\varepsilon}\|_2, \|\Delta^2 g_{\varepsilon}\|_2\} \right) m^{-\frac{1}{2} + \theta + \varepsilon} \\
+ C_f \varepsilon^{2(1+\varepsilon)} \int_{F_N} g_{\varepsilon}(z) d\mu(z).
\end{equation}

For a fixed $\varepsilon$, taking $m \to \infty$, we get
\begin{equation}
\lim_{m \to \infty} \frac{1}{\sigma_1(m)} (J_{N,1}(T_m.D_f) - e ((T_m.D_f) > 1)) = h_f \lim_{\varepsilon'' \to 0} \int_{F_N(\varepsilon'')} J_{N,1}(z) d\mu(z) + C_f \varepsilon^{2(1+\varepsilon)} \int_{F_N} g_{\varepsilon}(z) d\mu(z).
\end{equation}

Note that (5.19) holds for any fixed $0 < \varepsilon < \frac{1}{4}$. Since
\[
\int_{F_N} g_{\varepsilon}(z) d\mu(z) = \frac{3}{\pi [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_0^\infty \phi_{\varepsilon}(y) \frac{dy}{y^2} \to 0,
\]
as $\varepsilon \to 0$, we get
\[
\lim_{m \to \infty} \frac{1}{\sigma_1(m)} (J_{N,1}(T_m.D_f) - e ((T_m.D_f) > 1)) = h_f \lim_{\varepsilon'' \to 0} \int_{F_N(\varepsilon'')} J_{N,1}(z) d\mu(z).
\]

Finally, Theorem 1.3 comes from Theorem 5.4 and (1.2).

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