Control Synthesis for an Underactuated Cable Suspended System Using Dynamic Decoupling

Siddharth H. Nair\textsuperscript{1}, Ravi N. Banavar\textsuperscript{2} and D.H.S Maithripala\textsuperscript{3}

Abstract—This article studies the dynamics and control of a novel underactuated system, wherein a plate suspended by cables and with a freely moving mass on top, whose other ends are attached to three quadrotors, is sought to be stabilized at a specified attitude and a certain height, with the ball positioned at the center of mass of the plate. The freely moving mass introduces a 2-degree of underactuation into the system. The design proceeds through a decoupling of the quadrotors and the plate dynamics. Through a partial feedback linearization approach, the attitude of the plate and the translational height of the plate is initially controlled, while maintaining a bounded velocity along the $y$ and $z$ directions. These inputs are then synthesized through the quadrotors with a backstepping and timescale separation argument based on Tikhonov’s theorem.

I. INTRODUCTION

Quadrotor drones are increasingly gaining popularity in non-military security applications like surveillance, communication relays and civil applications like environmental monitoring, traffic control, disaster relief and construction [1]. Trajectory tracking controllers for quadrotors have been successfully studied in [2], [3], [4], [5] while control of quadrotor formations has been studied in [6], [7], [8]. Recent endeavours involve synthesizing control laws wherein quadrotors are required to transport loads from one point to another. In [9], a system consisting of a quadrotor and a flexible cable treated as serially-connected links is modelled in a coordinate-free form where the equations of motion are obtained using infinitesimal variations of elements belonging to a Lie group. First the desired forces on the links are derived so that the payload tracks a desired trajectory and next the thrust and moments acting on the quadrotor are derived so that these forces are in turn, generated by the quadrotor. In [10], multiple quadrotors carrying a point payload via rigid, massless links is considered while in [11], the suspended payload is a rigid body. The design philosophy is similar to that used in [9]- where they first design the desired forces in the links and then use the quadrotors to generate them. This work is extended in [12] to incorporate flexible cables. The development of such systems can find applications in real life situations like transportation and search and rescue operations.

Much of the previous effort in the field of cable suspended systems focusses on fully actuated systems. Here, we introduce an element of underactuation into the system in the form of a freely moving ball on a plate, thereby increasing the control complexity of the problem. The work [13] considers 1-D version of the problem of balancing a ball on a rod connected to two quadrotors via rigid links. The quadrotors are restricted to move only vertically and the stabilization of the ball is achieved by employing a model predictive controller for the linearized model of the system. In this article, we consider the full 3-D problem. The control approach we adopt can be summarized as follows: First, the quadrotors are decoupled from the ball-and-plate system and then the desired forces in the tethers are synthesized such that the control objectives are met. The forces in the tethers create both the force and the torque to position and orient the plate. These forces are then generated by the respective quadrotors using a backstepping-like strategy seen in [11]. The response of the quadrotors are assumed significantly faster than the ball-and-plate dynamics. The underactuated subsystem leads us to employ partial feedback linearization into our control design.

The remaining article is organized as follows. Section II formally sets up the problem by describing the system of interest, fixing up naming conventions and deriving a coordinate-form of the equations of motion using Lagrangian mechanics. In section III, the control systems are constructed followed by numerical validation via simulations in section IV.

II. PROBLEM FORMULATION

Consider three quadrotors with masses $m_1$, $m_2$ and $m_3$ and inertias $J_1$, $J_2$ and $J_3$ respectively, carrying a thin plate of mass $m_p$ and inertia $I_p$ via three inextensible cables of lengths $l_1$, $l_2$ and $l_3$ respectively. The plate also carries a ball of mass $m_b$. The inertial coordinate system is set up as shown in figure 1.
The location of the centre of mass of the plate in the inertial frame is denoted by \( o_p \in \mathbb{R}^3 \) and its attitude is denoted by \( R_p \in SO(3) \). Let \( x_i \in \mathbb{R}^3 \) be the vector from the centre of mass of plate to the point where the \( i \)-th cable is attached to the plate. These vectors are constant and lie in the coordinate system attached to the plate at its centre of mass. The vector \( r_b \in \mathbb{R}^2 \) represents the position of the ball on the plate. To represent \( r_b \) as a 3 dimensional vector, we define the matrix \( E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), which when multiplied by \( r_b \) yields the ball’s position in the plate’s reference frame.

In the inertial frame, the position of the ball is given by \( o_b = o_p + R_p Er_b \).

Since the tethers are inextensible, the locus of the positions of the \( i \)-th quadrotor is a sphere of radius \( l_i \) and centered at \( x_i \) in the plate’s coordinate system. Thus, its position in the inertial coordinate system is given by \( o_i = o_p + R_p x_i + l_i q_i \), where \( q_i \in S^2 \) is a unit vector aligned along the \( i \)-th cable. Assuming that the tethers aren’t hinged to the quadrotors rigidly, the attitude of the \( i \)-th quadrotor is denoted by \( R_i \in SO(3) \). To consolidate, the states of the system as a whole evolve over the configuration manifold \( Q \) given by

\[
Q = SO(3) \times \mathbb{R}^3 \times \mathbb{R}^2 \times (S^2 \times SO(3))^3
\]

The coordinates \( (o_p, R_p), (r_b), ((q_1, R_1), (q_2, R_2), (q_3, R_3)) \) describe the state of the system.

As the name suggests, each quadrotor is propelled using 4 motors which generate a net thrust along the yaw axis and a moment. Let the thrust force and moment generated by the \( i \)-th quadrotor in its coordinate system be given by \( f_i e_3 \) and \( M_i \) respectively where \( f_i \) is the magnitude of the thrust and \( e_3 \) is the unit vector aligned along the quadrotor’s yaw axis. Thus \( \{f_i, M_i\}_{i=1,2,3} \) are the control inputs to the system.

The objective is to design \( \{f_i, M_i\}_{i=1,2,3} \) for the quadrotor so as to stabilize the height of the attitude and the size of the plate’s centre of mass of the plate while simultaneously stabilizing the ball at the plate’s centre of mass. It must be noted that the 2-degree underactuation in the system precludes complete positional stabilization, and thereby a steady translational velocity in the \( y \) and \( x \) directions are the best achievable goals.

### A. Equations of Motion

The rotational kinematics of the \( i \)-th link, quadrotor and plate, respectively, are given by

\[
\begin{align*}
\dot{q}_i &= \omega_i \times q_i = \hat{\omega}_i q_i \\
R_i &= R_i \hat{\Omega}_i \\
\dot{R}_p &= \hat{\Omega}_p
\end{align*}
\]

where \( \omega_i \) is the velocity of rotation of \( q_i \) in the inertial frame satisfying \( \omega_i \cdot q_i = 0 \) and \( \Omega_i \) are the angular velocities of the plate and \( i \)-th quadrotor respectively in their respective coordinate systems, and the operator \( \hat{\cdot} \) is the map from \( \mathbb{R}^3 \) to the space of skew-symmetric matrices as defined by

\[
\hat{\chi} = \begin{bmatrix} 0 & -\chi_3 & \chi_2 \\ \chi_3 & 0 & -\chi_1 \\ -\chi_2 & \chi_1 & 0 \end{bmatrix}
\]

for \( x = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3 \).

The equations of motion are derived using a variational approach. The kinetic energy and potential energy of the system are given by

\[
\mathcal{T} = \frac{1}{2} m_p ||\dot{o}_p||^2 + \frac{1}{2} \Omega_i^T f_i \Omega_i
\]

\[
+ \sum_{i=1}^3 \frac{1}{2} m_i ||\dot{o}_p + R_p \hat{\Omega}_p x_i + l_i \dot{\omega}_i q_i||^2 + \frac{1}{2} \Omega_i^T f_i \Omega_i
\]

\[
+ \frac{1}{2} m_b ||\dot{o}_p + R_p \hat{\Omega}_p Er_b + R_p \dot{E}r_b||^2
\]

\[
\mathcal{U} = m_p g e_3^T \dot{o}_p + \sum_{i=1}^3 m_i g e_3^T (\dot{o}_p + R_p x_i + l_i q_i)
\]

\[
+ m_b g e_3^T (\dot{o}_p + R_p Er_b)
\]

The Lagrangian \( \mathcal{L} : Q \rightarrow \mathbb{R} \) of the system is obtained as the difference between the kinetic and potential energies, i.e., \( \mathcal{L} = \mathcal{T} - \mathcal{U} \). The action integral is given by \( \mathcal{A} = \int_{t_0}^{t_f} \mathcal{L} \, dt \). Further, we define the differential operator \( \mathbf{D}_u \) as the partial derivative of the operator at \( x \in Q \) with respect to the subscripted configuration variable \( u \) to yield a vector in the cotangent space \( T_x^* Q \). The variation of the action integral is expressed by the following equation

\[
\delta \mathcal{A} = \delta \int_{t_0}^{t_f} \mathcal{L} \, dt = \int_{t_0}^{t_f} \delta \mathcal{L} \, dt
\]

\[
= \int_{t_0}^{t_f} \left[ \mathbf{D}_{\dot{o}_p} \mathcal{L} \cdot \delta \dot{o}_p + \mathbf{D}_{\omega} \mathcal{L} \cdot \delta \omega + \mathbf{D}_{R_p} \mathcal{L} \cdot \delta R_p + \mathbf{D}_{\dot{E}r_b} \mathcal{L} \cdot \delta \dot{E}r_b + \mathbf{D}_{\hat{\Omega}_i} \mathcal{L} \cdot \delta \hat{\Omega}_i + \mathbf{D}_{\hat{\omega}_i} \mathcal{L} \cdot \delta \hat{\omega}_i + \sum_{i=1}^3 \mathbf{D}_{\dot{q}_i} \mathcal{L} \cdot \delta \dot{q}_i + \mathbf{D}_{\omega_i} \mathcal{L} \cdot \delta \omega_i + \mathbf{D}_{\Omega_i} \mathcal{L} \cdot \delta \Omega_i \right] \, dt
\]
To derive coordinate-free equations of motion, we use the exponential map to express infinitesimal variations of elements belonging to a Lie group as follows

\[
\delta g = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \exp(\varepsilon \chi) g
\]

where \( g \) is an element of a Lie group \( G \) and \( \chi \) is an element of the Lie algebra \( \mathfrak{g} \) of \( G \) ([14]). The rotation matrices and unit vectors belong to the Lie groups \( SO(3) \) and \( S^2 \) respectively, with their Lie algebras being the space of skew-symmetric matrices. Thus, their variations can be expressed as

\[
\delta R = R \hat{\eta}
\]

\[
\delta q = \hat{\xi} q = \xi \times q
\]

where \( \eta, \xi \in \mathbb{R}^3 \) are mapped to the Lie algebra elements via the \( \hat{\cdot} \) map. Using the rotational kinematic equations and the fact that the time derivative and variational operators commute, the variations of \( q \) and \( \Omega \) are obtained as

\[
\delta q = \hat{\xi} \times q + \xi \times \dot{q}
\]

\[
\delta \Omega = \hat{\eta} + \Omega \times \eta
\]

If \( u_i = f_i R e_3 \) is the thrust acting on the \( i \)th quadrotor, the virtual work done by the external forces (the forces and torques acting on each quadrotor) is given by

\[
\delta \mathcal{W} = \int_{t_0}^{t_f} \sum_{i=1}^{3} (u_i \cdot (\delta \dot{p}_i + \delta \dot{R}_p x_i + l_i \delta q_i)) + M_i \cdot \delta \eta_i) dt
\]

For variations of trajectories with fixed end points, the Langrange- D’Alembert principle gives us

\[
\delta \mathcal{A} = -\delta \mathcal{W}
\]

Substituting the expressions of the variations in (4), using integration by parts and the fact that the variations are arbitrary, the following Euler-Lagrange equations are obtained

\[
\frac{d}{dt} D_{\dot{p}} \mathcal{L} - D_{\dot{p}} \mathcal{L} = \sum_{i=1}^{3} u_i
\]

\[
\frac{d}{dt} D_{\dot{R}} \mathcal{L} - D_{\dot{R}} \mathcal{L} = 0
\]

\[
\frac{d}{dt} D_{\dot{\Omega}} \mathcal{L} + \Omega_p \times D_{\dot{\Omega}} \mathcal{L} - D_{\dot{\Omega}} \mathcal{L} = \sum_{i=1}^{3} \hat{\xi}_i R_p^{T} u_i
\]

\[
\dot{q_i} \frac{d}{dt} D_{\dot{q_i}} \mathcal{L} - \dot{q_i} D_{\dot{q_i}} \mathcal{L} = l_i \dot{q_i} u_i
\]

\[
\frac{d}{dt} D_{\dot{\Omega}} \mathcal{L} + \Omega \times D_{\dot{\Omega}} \mathcal{L} - D_{\dot{\Omega}} \mathcal{L} = M_i
\]

The partial derivatives involved in the above computations are given by

\[
D_{\dot{p}} \mathcal{L} = m_p \ddot{p} + \sum_{i=1}^{3} m_i(o_p + R_p \dot{\Omega}_p x_i + l_i \dot{q}_i)
\]

\[
+ m_b(\dot{\delta}_p + R_p \dot{\Omega}_p E r_b + R_p E \dot{r}_b)
\]

\[
D_{\dot{p}} \mathcal{L} = -m_p g e_3 - \sum_{i=1}^{3} m_i g e_3 - m_b g e_3
\]

\[
D_{\dot{R}} \mathcal{L} = m_b E^T R_p^T (\dot{\delta}_p + R_p \dot{\Omega}_p E r_b + R_p E \dot{r}_b)
\]

\[
D_{\dot{\Omega}} \mathcal{L} = -m_b E^T \dot{\Omega}_p R_p^T (\dot{\delta}_p + R_p \dot{\Omega}_p E r_b + R_p E \dot{r}_b) - m_b g E^T R_p^T e_3
\]

\[
D_{\delta_p} \mathcal{L} = (J_p - m_b [E r_b]^2 - \sum_{i=1}^{3} m_i e^2_i) \Omega_p + \sum_{i=1}^{3} m_i \dot{\xi}_i R_p^T (\dot{\delta}_p + l_i \dot{q}_i)
\]

\[
+ m_p [E r_b] R_p^T (\dot{\delta}_p + R_p E \dot{r}_b)
\]

\[
D_{\delta_R} \mathcal{L} = \sum_{i=1}^{3} m_i ([E r_b] R_p^T (\dot{\delta}_p + l_i \dot{q}_i) - g \hat{\xi}_i R_p^T e_3)
\]

\[
+ m_b ([E r_b] R_p^T (\dot{\delta}_p + R_p E \dot{r}_b) - g [E r_b] R_p^T e_3)
\]

\[
+ m_b [E r_b] R_p^T (\dot{\delta}_p + R_p \dot{\Omega}_p E r_b)
\]

\[
D_{\delta_\Omega} \mathcal{L} = m_b (l_i \dot{q}_i + l_i \dot{\Omega}_p E r_b)
\]

\[
D_{\delta_\Omega} \mathcal{L} = -m_b g e_3
\]

\[
D_{\delta_\Omega} \mathcal{L} = J_\Omega
\]

\[
D_{\delta_\Omega} \mathcal{L} = 0
\]

On substituting the above derivatives into the Euler-Lagrange equations, we obtain

\[
(m_p + \sum_{i=1}^{3} m_i + m_b) \ddot{\delta}_p + \sum_{i=1}^{3} m_i(R_p \dot{\Omega}_p^2 x_i - R_p \dot{\Omega}_p \xi_i \dot{q}_i - l_i (\dot{q}_i \dot{\Theta}_h - \dot{\Theta}_h^2 \dot{q}_i))
\]

\[
+ m_b(R_p \dot{\Omega}_p^2 E_r b - R_p [E r_b] \dot{\Omega}_p + 2 R_p \dot{\Omega}_p E r_b + R_p E \dot{r}_b)
\]

\[
+ (m_p + \sum_{i=1}^{3} m_i + m_b) g e_3 = \sum_{i=1}^{3} u_i
\]

(5)

\[
(m_p + \sum_{i=1}^{3} m_i + m_b)^2 \ddot{\Omega}_p + \dot{\Omega}_p (J_p - m_p [E r_b]^2 - \sum_{i=1}^{3} m_i e^2_i) \Omega_p
\]

\[
+ \sum_{i=1}^{3} m_i \dot{\xi}_i R_p^T (-l_i (\dot{q}_i \dot{\Theta}_h - \dot{\Theta}_h^2 \dot{q}_i)) + m_b [E r_b] [R_p^{T} \dot{\delta}_p + 2 \dot{\Omega}_p E r_b + E \dot{r}_b]
\]

\[
+ \sum_{i=1}^{3} m_i \dot{\xi}_i R_p^T \dot{p}_i = \sum_{i=1}^{3} \dot{\xi}_i R_p^T (u_i - m_i g e_3) - m_b g [E r_b] R_p^T e_3
\]

(7)

\[
m_i (\dot{q}_i \dot{p}_i + l_i \dot{q}_i + \dot{q}_i R_p \dot{\Omega}_p^2 x_i - \dot{q}_i R_p \dot{\Omega}_p x_i) = \dot{q}_i (u_i - m_i g e_3)
\]

(8)

\[
J_\dot{\Omega}_p + \dot{\Omega}_p J_\dot{\Omega}_p = M_i
\]

(9)

Replacing the expressions for \( \dot{\Theta}_h \) from equation (8) into (5)
and (7), the equations become

\[ (m_p + \sum_{i=1}^{3} m_i q_i q_i^T)(\dot{q}_i + q_i \omega_i) + \sum_{i=1}^{3} m_i q_i q_i^T (R_p \hat{\Omega}_p^2 \xi_i - R_p \dot{\xi}_p \hat{\Omega}_p) \\
+ m_b (\dot{q}_p + R_p \hat{\Omega}_p^2 \xi_i - R_p \dot{\xi}_p \hat{\Omega}_p) + \sum_{i=1}^{3} m_i q_i q_i^T \dot{u}_i \]

\[ - m_i \omega_i \dot{\omega}_i = \sum_{i=1}^{3} m_i q_i q_i^T \dot{u}_i \]  

\[ (m_p + \sum_{i=1}^{3} m_i q_i q_i^T)(\dot{q}_i + q_i \omega_i) + \sum_{i=1}^{3} m_i q_i q_i^T (R_p \hat{\Omega}_p^2 \xi_i - R_p \dot{\xi}_p \hat{\Omega}_p) \\
+ m_b (\dot{q}_p + R_p \hat{\Omega}_p^2 \xi_i - R_p \dot{\xi}_p \hat{\Omega}_p) + \sum_{i=1}^{3} m_i q_i q_i^T \dot{u}_i \]

(10)

Before moving on to the next section, it is worth noting that \( q_i q_i^T (\cdot) \) behaves like the projection operator \( q_i \langle \cdot, \cdot \rangle \) where \( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathbb{R}^3 \).

### III. Control Design

Observe that the inputs controlling the translational dynamics of the system and rotational dynamics of the plate appear as \( q_i q_i^T \dot{u}_i \) which is essentially the component of force \( u_i \) along the \( i \)-th tether. For convenience, \( q_i q_i^T \dot{u}_i \) is denoted as \( u_i^\parallel \) and the orthogonal component \( u_i^\perp \) is defined such that

\[ u_i = u_i^\parallel + u_i^\perp \]

It can be seen that equations (10) and (12), which describe the rotational and translational dynamics of the plate, are solely affected by the \( u_i^\parallel \)'s whereas equation (13) which describes the dynamics of the tethers is solely affected by the \( u_i^\perp \)'s. Thus, we adopt a procedure similar to that used in [11] where the controls \( u_i \) and \( M_i \) are designed in two steps: First, the quadrotors are replaced by fully actuated point masses and \( u_i^\parallel \)'s and \( u_i^\perp \)'s are designed independently to meet the control objectives. Then the \( M_i \)'s and \( f_i \)'s are designed for the quadrotors such that the thrust \( f_i R_i e_3 \) equals \( u_i = u_i^\parallel + u_i^\perp \).

#### A. Design of Parallel components

Before designing the parallel component of control \( u_i^\parallel \), we decouple the ball and plate system from the quadrotors by making the following observation. We define \( \mu_i \) to be the tension in the tether and then apply Newton’s second law for the \( i \)-th quadrotor along \( q_i \) to obtain the following equation

\[ m_i \dot{\mu}_i = q_i^T \dot{u}_i + q_i^T \dot{\mu}_i + \mu_i \omega_i \cdot (\dot{\omega}_i + \hat{\Omega}_p \times \omega_i) - l_i (\langle \dot{\omega}_i, \omega_i \rangle) \]

(15)

where \( \dot{\mu}_i = q_i^T \dot{u}_i + q_i^T \dot{\mu}_i + \mu_i \omega_i \cdot (\dot{\omega}_i + \hat{\Omega}_p \times \omega_i) \) is the acceleration of the \( i \)-th quadrotor, \( u_i^\parallel \) is the external force being applied parallel to \( q_i \) and \(- q_i q_i^T \dot{\mu}_i \) is the gravitational force acting along \( q_i \).

Expressing the dynamics of the ball and that of the plate (equations (10), (11) and (12)) in terms of these tensions \( \mu_i \)'s, we obtain the dynamics of the ball and plate system completely decoupled from the quadrotors as

\[ m_p \dot{\mu}_p = q_p^T \dot{u}_p + q_p^T \dot{\mu}_p + \mu_p \omega_p \cdot (\dot{\omega}_p + \hat{\Omega}_p \times \omega_p) - l_p (\langle \dot{\omega}_p, \omega_p \rangle) \]

(16)

\[ m_b (\dot{q}_b + R_p \hat{\Omega}_p^2 \xi_b - R_p \dot{\xi}_p \hat{\Omega}_p) + \sum_{i=1}^{3} \dot{\mu}_i (\langle \dot{\omega}_i, \omega_i \rangle) = 0 \]

(17)

We proceed to design \( \mu_i \)'s to stabilize the attitude and position of the plate and simultaneously stabilize the ball at the centre of the plate. Once suitable \( \mu_i \)'s have been chosen, the controls \( u_i^\parallel \) can be implemented by substituting for the accelerations from (16), (17) and (18) into (15).

#### Partial Feedback Linearization

The ball and plate system is described by the configuration manifold \( Q_{ball-plate} = \mathbb{R}^2 \times \mathbb{R}^3 \times SO(3) \). This system is acted upon by a force \( F = \sum_{i=1}^{3} \mu_i \) and a torque \( \tau = \sum_{i=1}^{3} \hat{\omega}_i \times R_i^T \mu_i \). Thus the ball and plate system is underactuated by \( 2 + 3 - 6 = 2 \) degrees of freedom.

We attempt to simplify the procedure to design the \( \mu_i \)'s by employing the technique of Partial Feedback Linearization (PFL) to linearize the translational and rotational dynamics of the plate. PFL is a standard technique for addressing underactuated systems wherein a partially linearizing feedback is implemented to simplify the system dynamics before control design for meeting the desired specifications.
is considered ([15]).

We proceed to rewrite the system dynamics by identifying invertible blocks in the Riemannian metric $M_{bp}$ of the ball and plate system.

\[
M_{bp} = \begin{bmatrix}
    m_b \Omega & m_b \Omega^T R_{p}^T & -m_b \Omega^T E_b
    \end{bmatrix}
\begin{bmatrix}
    m_b R_{bp} E
    & (m_p + m_b) I_3
    & -m_b R_{bp} E
    \end{bmatrix}
\begin{bmatrix}
    m_b [E_r b] E
    & m_b [E_r b] R_{p}^T
    & J_p - m_b [E_r b]
    \end{bmatrix}
\]

Defining

\[ M_{11} = m_b \Omega \]
\[ M_{12} = [m_b \Omega^T R_{p}^T - m_b \Omega^T E_b] \]
\[ M_{22} = \left[m_p + m_b\right] I_3 - m_b R_{bp} E [m_b R_{bp} E]^T J_p - m_b [E_r b] \]

we rewrite equations (16), (17) and (18) as

\[ M_{12} \dot{b} + M_{22} \dot{\phi} \Omega_{\phi} + N_2 = [F^T \tau]^T \]

where

\[ N_1 = m_b \Omega \]
\[ N_2 = \left[m_b \Omega [E_r b] + 2 \Omega_p E_r b + 2 \Omega R_p E_{rb} + m_b \Omega \right] \]

Noting that $M_{11}$ is invertible, $\dot{b}$ is substituted from (19) into (20) and the linearizing feedback given by

\[ [F^T \tau]^T = N_2 - M_{12}^{-1} M_{11} N_1 + (M_{22} - M_{12}^{-1} M_{11} M_{12}) [U_1 \ U_2]^T \]

where $U_1$ and $U_2$ are the new inputs to the system, is substituted into (19) and (20) to obtain the partially linearized equations

\[
\dot{b} = -M_{11}^{-1} N_1 - M_{12}^{-1} M_{12} [U_1 \ U_2]^T
\]
\[
\dot{\phi} = U_1
\]
\[
\Omega_p = U_2
\]

The inputs $U_1$ and $U_2$ are coupled into the dynamics of the ball via the matrix $M_{11} M_{12} = [E^T R_{p}^T - E^T E_b]$. $E^T R_{p}^T$ has a constant rank of 2, $R_{p} E_{3}$ spans its null space and $E^T R_{p}^T R_{p} E = U_2$. Using these facts, $U_1$ is chosen as

\[ U_1 = R_{p} E_{3} \left( -k_b \dot{\phi} - k_b \dot{\alpha} \right) + \frac{1}{m_b} R_{p} E ( -N_1 + M_{11} (k_4 r_{b} + k_3 t_{b}) ) \]

To stabilize the attitude of the plate, $U_2$ is chosen as

\[ U_2 = -k_\eta \eta - k_1 \Omega_{p} \]

where $\eta$ is the gradient of $\Psi(R) = \frac{1}{4} \text{trace}(I_{x \times 3} - R)$. The closed loop dynamics are described by the following equations

\[
\dot{b} = -k_4 \dot{t}_{b} - k_3 \dot{t}_{b} + E^T [E_r b] U_2
\]
\[
\dot{\phi} = R_{p} E_{3} \left( -k_5 \dot{\phi} - k_6 \dot{\alpha} \right) + \frac{1}{m_b} R_{p} E ( -N_1 + M_{11} (k_4 r_{b} + k_3 t_{b}) ) \]
\[
\Omega_{p} = -k_2 \eta - k_1 \Omega_{p} \]

Examine the first and the third equation. Choosing appropriate gains $k_2$ and $k_1$ ensures that the attitude gets stabilized "quickly", which in turn ensures that $U_2$ approaches 0. This leads to the last term in the first equation going to zero. An appropriate choice of $k_4$ and $k_3$ ensures that $\dot{t}_{b}$ asymptotically goes to zero. When both $\dot{t}_{b}$ and the attitude have been stabilized, the dynamics of the $z$ coordinate of the CoM of the plate are given by

\[
e_3^T \dot{\alpha}_p = e_3^T ( -k_8 \dot{\phi}_p - k_9 \dot{\alpha}_p )
\]

which, for appropriate choices of $k_5$ and $k_6$, stabilizes the $z$ coordinate of the CoM of the plate as well.

**Theorem 1**: Consider the closed loop system described by equations (23), (24) and (25). Then there exist positive scalars $k_1$, $k_2$, $k_3$, $k_4$, $k_5$, $k_6$ such that the state $(r_b, \dot{r}_b, e^T \dot{\phi}_p, e^T \dot{\alpha}_p, R_p, \Omega_p) = ([0 \ 0]^T, [0 \ 0]^T, 0, 0, I_3, [0 \ 0 \ 0]^T)$ is asymptotically stable and the velocities $\dot{\phi}_p$ and $\dot{\alpha}_p$ remain bounded where $\dot{\phi}_p = \dot{\phi}_p + \dot{\alpha}_p = \dot{\phi}_p + \dot{\alpha}_p$.

**Proof**: To prove the stability of the state $(r_b, \dot{r}_b, e^T \dot{\phi}_p, e^T \dot{\alpha}_p, R_p, \Omega_p) = ([0 \ 0]^T, [0 \ 0]^T, 0, 0, I_3, [0 \ 0 \ 0]^T)$ of the system described by equations (23), (24) and (25), consider the following Lyapunov function candidate

\[
V = \frac{k_4 + c_1 k_3}{2} [\dot{r}_b]^2 + c_1 e^T \dot{\phi}_p + \frac{1}{2} [\dot{\alpha}_p]^2 + (k_2 + c_2 k_1) \Psi + c_1 \eta^T \Omega_p + \frac{1}{2} [\Omega_p]^2 + k_6 [\dot{\alpha}_p e_3]^2 + \frac{1}{2} [\dot{\alpha}_p e_3]^2
\]

which is positive definite for small values of $c_1$ and $c_2$. The time derivative of the candidate function along the system trajectories is given by

\[
\dot{V} = k_4 \dot{r}_b + c_1 \dot{r}_b + c_2 \dot{\phi}_p + k_5 \eta^T \Omega_p + \dot{\alpha}_p e_3 + c_1 \eta^T \Omega_p + k_6 [\dot{\alpha}_p e_3]^2 + (k_2 + c_2 k_1) \Psi + k_6 [\dot{\alpha}_p e_3]^2 + \dot{\alpha}_p e_3 e_3^T (R_p - I) e_3 e_3^T (k_8 \dot{\phi}_p - k_9 \dot{\alpha}_p) + \frac{1}{m_b} R_{p} E ( -N_1 + M_{11} (k_4 r_{b} + k_3 t_{b}) ) \]

The cubic terms in the derivative can be bounded as follows

\[ i^T E^T [E_r b] ( -k_2 \eta - k_1 \Omega_{p} ) \leq || \dot{r}_b || || \dot{\phi}_p || \leq || k_4 || || \dot{r}_b || + k_3 || \dot{\phi}_p ||\]
\[ \dot{\alpha}_p e_3 e_3^T (R_p - I) e_3 e_3^T ( -k_5 \dot{\phi}_p - k_6 \dot{\alpha}_p ) \leq 2 || \dot{\alpha}_p e_3 || ( || k_5 || || \dot{\phi}_p e_3 || + k_6 || \dot{\alpha}_p e_3 ||)
\[ i^T E^T [E_r b] ( -k_2 \eta - k_1 \Omega_{p} ) \leq || \dot{r}_b || || \dot{\phi}_p || \leq || k_4 || || \dot{r}_b || + k_3 || \dot{\phi}_p || + g || \dot{\alpha}_p e_3 || ( || k_5 || || \dot{\phi}_p e_3 || + k_6 || \dot{\alpha}_p e_3 ||)
\]

Substituting the above bounds into (26), the derivative of the candidate function is bounded above by

\[
\dot{V} \leq z^T \mathbf{W} z + \delta_0
\]
where 

\[ z = [||r_b|| \ ||r_b|| \ ||\eta|| \ ||\Omega_p|| \ ||\dot{\Omega}_e^T e_3|| \ ||\dot{\Omega}_p^T e_3||]^T \]

\[ \mathbf{w} = \begin{bmatrix} -c_1 k_4 & 0 & 0 & 0 & k_4 \\ 0 & -k_3 + c_1 & 0 & 0 & 0 \\ 0 & 0 & -c_2 k_2 & 0 & 0 \\ 0 & 0 & 0 & -k_1 + c_2 & 0 \\ k_2 & \frac{k_3}{2} & \frac{k_2}{2} & 0 & 0 \end{bmatrix} \]

\[ \mathbf{o}^T = [||r_b|| ||r_b|| ||\eta|| ||\dot{\Omega}_p|| ||\dot{\Omega}_e^T e_3|| ||\dot{\Omega}_p^T e_3||]^T \]

To bound the cubic and quartic terms in \( \mathbf{o} \), we obtain bounds on \( \eta \) and \( \Omega_p \) as follows.

Consider the differentiable function

\[ V_2 = \frac{1}{2} ||\Omega_p||^2 + c_0 \eta^T \Omega_p + (k_2 + c_0 k_1) \Psi \]

defined on the state space describing the plate’s attitude, \((R_p, \Omega_p)\). This function can be shown to be positive definite on this state space if \( c_0 \) is small. The time derivative of the function is given by

\[ \dot{V}_2 = -k_1 ||\Omega_p||^2 - c_0 k_2 ||\eta||^2 + c_0 \dot{\eta}^T \Omega_p \leq (-k_1 + c_0) ||\Omega_p||^2 - c_0 k_2 ||\eta||^2 \]

where the second inequality follows from that fact that \( ||\eta|| \leq ||\Omega_p|| \). For \( k_1 > c_0 \), the time derivative of \( V_2 \) is bounded above by a negative definite quantity on the state space \((R_p, \Omega_p)\) and is thus, negative definite on this space as well. This implies that \( V_2 \) decays to zero exponentially and is bounded above by its initial value \( V_2(0) \). Since \( V_2 \) is positive definite on the considered space, we have that

\[ ||\Omega_p|| \leq C_1(V_2(0)) \quad ||\eta|| \leq C_2(V_2(0)) \]

where \( C_1 \) and \( C_2 \) are some constants in terms of \( V_2(0) \). These bounds can be used to express the cubic and quartic terms as quadratic terms to bound the derivative of \( V \) in (27) as

\[ \dot{V} \leq \mathbf{w}^T \mathbf{z} + ||\dot{\Omega}_p^T e_3|| |g|(-1 + ||\dot{e}_1^T R_p^T e_3||^2) \]  

(28)

and the desired state is asymptotically stable using Lyapunov’s direct stability theorem (116). To investigate the behaviour of the unstabilized states (the internal dynamics), we analyze the translational dynamics of the plate in the x-y plane

\[ \dot{e}_1^T \dot{\omega}_p = e_1^T U_1 \]  

(29)

\[ \dot{e}_1^T \dot{\omega}_p = e_1^T U_1 \]  

(30)

When \( r_b \), the attitude and z coordinate of the plate get stabilized, we have \( U_1 \rightarrow 0 \) and thus translational accelerations in the x-y plane die off. As a consequence, the translational velocities of the plate in the x and y directions, \( \dot{\omega}_p \) and \( \dot{\omega}_p \) become constant eventually and are thus bounded above by \( sup_{p \rightarrow \infty} \dot{\omega}_p \) and \( sup_{p \rightarrow \infty} \dot{\omega}_p \) (these limits exist owing to the continuity of the velocities).

Note that the Riemannian metric \( M_{\mathbb{R}} \) and potential energy of the ball and plate system, are invariant to flows along the vector fields \( \partial_{\dot{\omega}_p} \partial_{\dot{\omega}_p} \mathbb{T}_{Ball-Plate} \). These vector fields are in fact infinitesimal symmetries (17).

The designed \( U_1 \) and \( U_2 \) are mapped to \( F \) and \( \tau \) (the net force and torque acting on the plate) by the transformation

\[ \begin{bmatrix} F \\ \tau \end{bmatrix} = N_2 - M_{12} M_{11}^{-1} N_1 + (M_{22} - M_{12} M_{11}^{-1} M_{12}) \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \]

The \( \mu \)s are obtained from \( F \) and \( \tau \) by solving the following set of linear equations

\[ \begin{bmatrix} I & I \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \end{bmatrix} \begin{bmatrix} R_p^T \mu_1 \\ R_p^T \mu_2 \\ R_p^T \mu_3 \end{bmatrix} = \begin{bmatrix} R_p^T F \\ \tau \end{bmatrix} \]

Matrix \( A \) has a full row rank if the vectors \( x_1, x_2 \) and \( x_3 \) are coplanar but not collinear. Thus, the minimum norm solution for the \( \mu \)s is given by

\[ \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \text{diag}(R_p, R_p, R_p) A^T (AA^T)^{-1} \begin{bmatrix} R_p^T F \\ \tau \end{bmatrix} \]

The \( u_i \)'s are obtained from equation (17) (restated here for convenience) by substituting \( \dot{\omega}_p = U_1 \) and \( \dot{\Omega}_p = U_2 \)

\[ u_i\| = q_i Q_i^T m_i(\dot{\omega}_p + (R_p \Omega_p^T x_i - R_p x_i \dot{\Omega}_p)) + q_3 - l_i ||\omega||^2 q_i = \mu \]

B. Design of Perpendicular components

The perpendicular component \( u_i^\perp \) is chosen such that the tether is aligned in the direction \( q_{i\perp} = \frac{\mu}{||\mu||} \). Equation (13) describes the tether dynamics and is rewritten here for convenience

\[ m_i(\dot{q}_i \dot{\omega}_p + l_i \omega_i + \dot{\omega}_p R_p \dot{\Omega}_p^T x_i - \dot{q}_i R_p x_i \dot{\Omega}_p) = \dot{q}_i (u_i^\perp - m_i g e_3) \]

Grouping the coupled acceleration terms, we define

\[ a_i = \dot{\omega}_p + R_p \dot{\Omega}_p^T x_i - R_p x_i \dot{\Omega}_p + q_3 \]

and rewrite equation (13) as

\[ \frac{1}{l_i} \dot{q}_i a_i + \omega_i = \frac{1}{m_i} \dot{q}_i u_i^\perp \]
We refer to [11] to solve the tracking problem by choosing the control
\[ u^i = -m_i \hat{a}_i + m_i \dot{q}_i (g \hat{e}_{\omega_i} + (q_i^T \hat{\omega}_a) \dot{q}_i + \dot{q}_i^T \dot{\hat{\omega}}_a) \]  
(31)
where \( \hat{\omega}_a = \dot{q}_i q_i^T \hat{a}_i \) and \( e_{\omega_i} = \hat{\omega}_i - \omega_a \). Note that the acceleration terms in \( \dot{q}_i \) are substituted in terms of the \( u_i \)s for implementation. The closed loop dynamics is given by
\[ \dot{\hat{\omega}}_a = -k_1 e_{\omega_i} - k_a e_{\omega_i} - (q_i^T \hat{\omega}_a) \dot{q}_i + \dot{q}_i^T \dot{\hat{\omega}}_a \]  
(32)

C. Design of Quadrotor inputs

The total control \( f_i(\hat{r}_i) = u_i = u_i^T + u_i^\perp \) is to be generated by the \( i \)th quadrotor using the inputs \( f_i \) and \( M_i \). Again, this problem has been solved in [11] and we refer to their approach. A backstepping-like controller is used so that \( M_i \) orients the yaw axis of the quadrotor along that of \( u_i \), i.e., \( \hat{r}_i \). The first two rows of the desired \( \hat{r}_i \) are obtained by considering some smooth \( b_{i1}(t) \in S^2 \) which is used to form a right-handed coordinate system along with \( b_{i3} = \frac{u_i}{|u_i|} \) by defining the following desired attitude of the quadrotor:
\[ R_{i\hat{d}} = \begin{bmatrix} -b_{i1} b_{i2} & b_{i2} b_{i3} & b_{i1} b_{i3} \end{bmatrix} b_{i3} \]

Tracking errors for the attitude and angular velocity of the \( i \)th quadrotor are defined as
\[ e_{\hat{r}_i} = \frac{1}{2} (R_{i\hat{r}_i} - R_{i\hat{t}_i} R_{\hat{r}_i}) \]  
(33)
\[ e_{\Omega_i} = \hat{\Omega}_i - R_{i\hat{r}_i} \Omega_{\hat{r}_i} \]
where the map \( \hat{\cdot} \) is the inverse of the \( \cdot \) map and the desired angular velocity is obtained from the attitude kinematics as \( \hat{\Omega}_{\hat{r}_i} = (R_{i\hat{r}_i})^{\perp} \). With this, the thrust and moment of the quadrotor are chosen as
\[ f_i = ||u_i|| \]  
(34)
\[ M_i = -k_{r\hat{r}_i} \hat{\epsilon}_{\hat{r}_i} - \epsilon_{\hat{\omega}_a} + \Omega_i \times J_i \Omega_i \]
\[ -J_i(\hat{\Omega}_i R_{i\hat{r}_i} \Omega_{\hat{r}_i} - R_{i\hat{r}_i} \Omega_{\hat{r}_i}) \]  
(35)
for some positive constants \( \epsilon, k_r \) and \( k_{\hat{\omega}} \).

Theorem 2: Consider the full dynamic model given by (10)-(14). For a desired direction of the first body-fixed axes \( b_{i1} \), \( i = 1, 2, 3 \), control inputs (34) and (35), there exists some \( \epsilon^* > 0 \), such that for all \( \epsilon < \epsilon^* \), the zero equilibrium of the tracking errors of the quadrotors \( (e_{\hat{r}_i}, e_{\Omega_i}) = (0, 0) \) is exponentially stable and the state \( (r_i, \dot{r}_i, e_{\sigma_{\hat{r}_i}}, e_{\sigma_{\hat{\omega}_a}}, \dot{\sigma}_{\hat{\omega}_a}, \hat{\epsilon}_{\hat{\sigma}_{\hat{\omega}_a}}, \hat{\omega}_{\hat{\sigma}_{\hat{\omega}_a}}) = ([0, 0, 0], [0, 0], 0, 0, 0, 1, [0, 0, 0]) \) is asymptotically stable and the velocities \( \dot{\sigma}_{\hat{p}_1} \) and \( \dot{\sigma}_{\hat{p}_2} \) remain bounded. Then according to Tikhonov’s theorem ([16]), there exists some \( \epsilon^* > 0 \), such that for all \( \epsilon < \epsilon^* \), the hypothesis holds true.

IV. SIMULATIONS

For simulating the behaviour of the system under the action of the proposed control laws, the following system parameters are chosen:
\[ m_p = 0.75 \]
\[ m_b = 0.1 \]
\[ J_p = \begin{bmatrix} 0.006 & 0 & 0 \\ 0 & 0.008 & 0 \\ 0 & 0 & 0.012 \end{bmatrix} \]

Furthermore, prior to employing a numerical technique to obtain the trajectories described by equations (10)-(13), the
system states are initialised to the following values.

\begin{align*}
    r_b(0) &= [1, 1]^T, \quad \dot{r}_b(0) = [0.5, 0.5]^T \\
    q_1(0) &= [0, 0, 1]^T, \quad \omega_1(0) = [0, 0, 0]^T \\
    q_2(0) &= [-0.5126, 0.0854, 0.8544]^T, \quad \omega_2(0) = [0, 0, 0]^T \\
    q_3(0) &= [-0.5126, 0.0854, 0.8544]^T, \quad \omega_3(0) = [0, 0, 0]^T \\
    R_p(0) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Omega_p(0) = [1, 1, 2]^T
\end{align*}

All quantities are expressed in their respective SI units. The system trajectory subject to these simulation conditions, are depicted by the following plots.

Fig. 3: Quaternion describing the plate's attitude

Fig. 4: Angular velocity of the plate in its frame

Fig. 5: Position of the plate in the inertial frame

Fig. 6: Translational velocity of the plate in the inertial frame

Fig. 7: Position of the ball in the plate’s frame

Fig. 8: Velocity of the ball in the plate’s frame

Fig. 9: Direction of the first tether in the inertial frame

Figures 5 and 6 show that the position of the plate in the x-y plane becomes unbounded while the velocity attains a constant value. This behaviour is attributed to the underactuated nature of the system.
An animated depiction of the system with the proposed control strategy in effect is available at: [https://youtu.be/nGNS-eZxbVM](https://youtu.be/nGNS-eZxbVM)

**REFERENCES**

[1] R. W. Beard and T. W. McLain, *Small unmanned aircraft: Theory and practice*. Princeton university press, 2012.

[2] T. Lee, M. Leok, and N. H. McClamroch, “Geometric tracking control of a quadrotor uav on se (3),” in *Decision and Control (CDC), 2010 49th IEEE Conference on*. IEEE, 2010, pp. 5420–5425.

[3] G. Hoffmann, S. Waslander, and C. Tomlin, “Quadrotor helicopter trajectory tracking control,” in *AIAA guidance, navigation and control conference and exhibit*, 2008, p. 7410.

[4] F. Goodarzi, D. Lee, and T. Lee, “Geometric nonlinear pid control of a quadrotor uav on se (3),” in *European Control Conference (ECC), 2013*. IEEE, 2013, pp. 3845–3850.

[5] D. Maithripala and J. Berg, “Robust tracking control for underactuated autonomous vehicles using feedback linearization,” in *Advanced Intelligent Mechatronics (AIM), 2014 IEEE/ASME International Conference on*. IEEE, 2014, pp. 446–451.

[6] M. Turpin, N. Michael, and V. Kumar, “Trajectory design and control for aggressive formation flight with quadrotors,” *Autonomous Robots*, vol. 33, no. 1-2, pp. 143–156, 2012.

[7] ——, “Decentralized formation control with variable shapes for aerial robots,” in *Robotics and Automation (ICRA), 2012 IEEE International Conference on*. IEEE, 2012, pp. 23–30.

[8] D. S. Maithripala, J. M. Berg, D. Maithripala, and S. Jayasuriya, “A geometric virtual structure approach to decentralized formation control,” in *American Control Conference (ACC), 2014*. IEEE, 2014, pp. 5736–5741.

[9] F. A. Goodarzi, D. Lee, and T. Lee, “Geometric stabilization of a quadrotor uav with a payload connected by flexible cable,” in *American Control Conference (ACC), 2014*. IEEE, 2014, pp. 4925–4930.

[10] T. Lee, K. Sreenath, and V. Kumar, “Geometric control of cooperating multiple quadrotor uavs with a suspended payload,” in *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*. IEEE, 2013, pp. 5510–5515.

[11] T. Lee, “Geometric control of multiple quadrotor uavs transporting a cable-suspended rigid body,” in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*. IEEE, 2014, pp. 6155–6160.