The principle of invariance in the Donsker form to the partial sum processes of finite order moving averages

N. S. Arkashov *, a

aNovosibirsk State Technical University, Karl Marx Ave., 20, Novosibirsk 630073, Russia.

UDC 519.214

Abstract

We consider the process of partial sums of moving averages of finite order with a regular varying memory function, constructed from a stationary sequence, variance of the sum of which is a regularly varying function. We study the Gaussian approximation of this process of partial sums with the aid of a certain class of Gaussian processes, and obtain sufficient conditions for the $C$-convergence in the invariance principle in the Donsker form.

keywords: invariance principle, fractional Brownian motion, moving average, Gaussian process, memory function, regular varying function

1 Introduction and Statement of Main Results

In the present paper, sufficient conditions are obtained for $C$-convergence in the metric space $D[0,1]$ of a normalized process of partial sums of moving averages of finite order to a Gaussian process constructed from a fractional Brownian motion. We will use constructions, and also adhere to the [1] presentation scheme. A significant drawback of the estimates of the proximity of processes obtained in [1] in the principle of invariance in the Strassen form is the “local conditions” for each element of a non-random sequence that forms two-sided moving averages. Note that, in most cases, it is not possible to restore information about the behavior of this non-random sequence during statistical data analysis. This shortcoming was the motivation for writing this article, in which the mentioned “local conditions” are replaced by an “integral condition” for the regular behavior of the
variance of the process of partial sums of two-sided moving averages. Note that from the
assumption that this “integral condition” is satisfied, one can obtain estimates for the
scale parameter and the Hurst parameter (see [2]).

Let \{X_j; j \in \mathbb{Z}\} be a stationary (in the narrow sense) sequence of random variables
represented as a two-sided moving average:

\[ X_j = \sum_{k=-\infty}^{\infty} a_{j-k} \xi_k, \quad (1) \]

where \{\xi_k; k \in \mathbb{Z}\} is a sequence of i. i. d. random variables with zero means and unit
variances, \{a_k; k \in \mathbb{Z}\} is a nonrandom square summable sequence of real numbers.

We introduce the notation:

\[ S_n = \sum_{i=1}^{n} X_i \text{ for } n \geq 1, \quad S_0 = 0. \]

In what follows, we assume that \( \text{Var}(S_n) \) is a regularly varying function

\[ \text{Var}(S_n) = h^2(n)n^{2H}, \quad (2) \]

where \( h \) is a slowly varying function at \(+\infty\) (e.g., see [3]) and \( H \in (0, 1) \).

Denote by \( \mathcal{R}_\nu \) the class of regularly varying functions with exponent \( \nu \geq 0 \), defined
on the interval \([0, +\infty)\), different from constants having the following representation:
\( l(t)t^\nu \), where \( l(t) \) is a slowly varying at \(+\infty\), non-negative and non-decreasing function on
\([0, +\infty)\) (we assume that \( 0^0 = 1 \)).

We construct a sequence of moving average of finite order

\[ v_0 = 0, \quad v_k = \sum_{i=0}^{k-1} X_{k-i} \Delta M(i), \quad k \geq 1, \quad (3) \]

where \( M \in \mathcal{R}_\nu \) and \( \Delta M(i) = M(i+1) - M(i) \). Following [4], [5], the function \( M \) will be
called the memory function. We define the partial sums process of the sequence (3)

\[ R_n = \sum_{k=0}^{n} v_k, \quad n = 0, 1, \ldots. \quad (4) \]

Note that a possible “physical meaning” for the \( \{R_n\} \) random walk is presented in [6].

By \( B_H(t) \), we denote the fractional Brownian motion, i.e., centered Gaussian process
with covariance function

\[ R(t, s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad (5) \]

where \( H \in (0, 1) \).

Before stating the main result of the paper, we present the assertion underlying its
proof.
**Proposition 1.** Let condition (2) and $E|\xi_1|^\alpha < \infty$ be satisfied for some $\alpha$ such that $\alpha \geq 2$ and $\alpha H > 1$. Then, as $n \to \infty$, the processes $S_{[nt]} / \sqrt{\text{Var}(S_n)}$ $C$-converge in $D[0, 1]$ to $B_H(t)$.

In Proposition 1 the expression $[x]$ denotes the largest integer not exceeding the real number $x$.

Recall that by $C$-convergence in $D[0, 1]$ we mean the weak convergence of distributions of measurable (in Skorokhod’s topology) functionals on $D[0, 1]$ that are continuous in uniform topology at the points of the space $C[0, 1]$ (e.g., see [7]).

Note that the specified $C$-convergence is established in [8] under more stringent moment constraints than those in Proposition 1. We also note that in the case $H > 1/2$ the assertion of Proposition 1 was proved in [9, Theorem 2].

Let us define the Gaussian process $Z_{\nu, H}$:

$$Z_{\nu, H}(t) = \nu \int_0^t B_H(t-s)s^{\nu-1}ds.$$ 

We introduce the notation

$$r_n(t) = \frac{R_{[nt]}}{h(n)n^H M(n)}, \quad t \in [0, 1], \quad n = 1, 2, \ldots.$$ 

**Theorem 1.** Let the condition of Proposition 1 be fulfilled. Then

1) if $M \in \mathcal{R}_\nu$, $\nu > 0$, then the processes $r_n(t)$ $C$-converge in $D[0, 1]$ to $Z_{\nu, H}(t)$ as $n \to \infty$;

2) if $M \in \mathcal{R}_0$, then the processes $r_n(t)$ $C$-converge in $D[0, 1]$ to $\left(1 - \frac{M(0)}{M(+\infty)}\right) B_H(t)$ as $n \to \infty$.

**Corollary 1.** Let condition (2) be fulfilled. Then, as $n \to \infty$,

1) for the case of $M \in \mathcal{R}_\nu$, $\nu > 0$ the following equivalence is valid

$$\text{Var}(R_n) \sim \text{Var}(Z_{\nu, H}(1))M^2(n)h^2(n)n^{2H};$$ 

2) for the case of $M \in \mathcal{R}_0$ it holds that

$$\text{Var}(R_n) \sim \left(1 - \frac{M(0)}{M(+\infty)}\right)^2 M^2(n)h^2(n)n^{2H}.$$ 

**2  Proofs.**

Let us divide the proof of Proposition 1 into lemmas 2–9. Note that the proof of this proposition follows the scheme of the proof of Theorem 2 in [10]. Introduce the following notations:

$$s_n(t) = S_{[nt]} / \sqrt{\text{Var}(S_n)}, \quad t \in [0, 1], \quad n = 1, 2, \ldots.$$
\[ A_{k,n}(t) = (n^H h(n))^{-1} \sum_{j=-k+1}^{-k+[nt]} a_j. \]

Using the above notation, we immediately obtain (see [9])

\[ s_n(t) = \sum_{k \in \mathbb{Z}} A_{k,n}(t) \xi_k. \]

**Lemma 1.** For all \( 1 \geq t \geq \tau \geq 0 \) it holds that

\[ \sum_{k \in \mathbb{Z}} (A_{k,n}(t) - A_{k,n}(\tau))^2 = \frac{([nt] - [n\tau])2^H h^2([nt] - [n\tau])}{n^{2H} h^2(n)}. \]

**Proof.** We have

\[ \sum_{k \in \mathbb{Z}} (A_{k,n}(t) - A_{k,n}(\tau))^2 = n^{-2H} h^{-2}(n) \sum_{i \in \mathbb{Z}} \left( \sum_{j=-i+1}^{i} a_j \right)^2 = \frac{\text{Var}(S_{[nt]-[n\tau]})}{n^{2H} h^2(n)}. \]

Using (2), we obtain the assertion of the lemma. \( \square \)

**Lemma 2.** For all \( 1 \geq t \geq \tau \geq 0 \) the following convergence is valid as \( n \to \infty \)

\[ \mathbb{E}s_n(t)s_n(\tau) \to \mathbb{E}B_H(t)B_H(\tau). \]

**Proof.** We have

\[ \mathbb{E}(s_n(t) - s_n(\tau))^2 = \sum_{k \in \mathbb{Z}} (A_{k,n}(t) - A_{k,n}(\tau))^2 = \frac{([nt] - [n\tau])2^H h^2([nt] - [n\tau])}{n^{2H} h^2(n)}, \]

where the last equality follows from Lemma 1. From here we get

\[ \mathbb{E}(s_n(t) - s_n(\tau))^2 \to (t - \tau)^{2H} = \mathbb{E}(B_H(t) - B_H(\tau))^2. \]

Using the convergence of the second moments of the one-dimensional projections of the random processes \( s_n(t) \) we deduce the equality

\[ 2\mathbb{E}s_n(t)s_n(\tau) = \mathbb{E}(s_n(t))^2 + \mathbb{E}(s_n(\tau))^2 - \mathbb{E}(s_n(t) - s_n(\tau))^2 \to 2\mathbb{E}B_H(t)B_H(\tau). \]
Lemma 3 \((\text{III})\). Let \(\{X_k\}_{k=1,...,n}\) be independent centered random variables and \(E|X_k|^\alpha < +\infty\) for all \(k\) and some \(\alpha \geq 2\). Put

\[
S_n = \sum_{k=1}^{n} X_k, \quad M_{\alpha,n} = \sum_{k=1}^{n} E|X_k|^\alpha, \quad B_n = \sum_{k=1}^{n} E X_k^2.
\]

Then

\[
E|S_n|^\alpha \leq c(\alpha)(M_{\alpha,n} + B_n^{\alpha/2}),
\]

where \(c(\alpha)\) is a positive constant depending only on \(\alpha\).

Lemma 4. The following inequality is valid:

\[
E|s_n(t) - s_n(\tau)|^\alpha \leq C \left( \left\lfloor nt \right\rfloor - \left\lfloor n\tau \right\rfloor \right)^{\alpha H} \left( \frac{h([nt] - [n\tau])}{h(n)} \right)^\alpha,
\]

where \(C\) is a constant depending on the distributions of \(\xi_0\), \(\alpha\) and \(\{a_i\}\).

**Proof.** By Lemma 3 and the Fatou theorem we obtain

\[
E\left| \sum_{k \in \mathbb{Z}} (A_{k,n}(t) - A_{k,n}(\tau)) \xi_k \right|^\alpha \\
\leq c(\alpha) \left( \sum_{k \in \mathbb{Z}} |A_{k,n}(t) - A_{k,n}(\tau)|^\alpha E|\xi_0|^\alpha + \left( \sum_{k \in \mathbb{Z}} (A_{k,n}(t) - A_{k,n}(\tau))^2 \right)^{\alpha/2} \right) \\
\leq c(\alpha)(1 + E|\xi_0|^\alpha) \left( \sum_{k \in \mathbb{Z}} (A_{k,n}(t) - A_{k,n}(\tau))^2 \right)^{\alpha/2}.
\]

Note that we applied here the following elementary inequality:

\[
\sum_{k \in \mathbb{Z}} |b_k|^\gamma \leq \left( \sum_{k \in \mathbb{Z}} |b_k| \right)^\gamma, \quad \gamma \geq 1.
\]

Next, applying Lemma 3 to the right-hand side of the last inequality in (6), we obtain the assertion of the lemma.

\[\square\]

Lemma 5. Let \(g\) be a slowly varying function defined on \([1, +\infty)\) and \(\varepsilon\) be an arbitrary positive number. Then the sequence

\[
\left\{ \frac{\max_{1 \leq k \leq n} g(k) k^\varepsilon}{g(n)n^\varepsilon} \right\}
\]

is bounded for all \(n \geq 1\).
Proof. Using Statement 4 from [12, section 1.5], we find that there exists $B$ such that \( \sup_{g(x)g(x) > e^x} \g(n) g(x) \rightarrow 1 \) holds as $n \to +\infty$, which immediately implies the assertion of the lemma.

From Lemma 4 and 5 we immediately obtain the following assertion.

**Lemma 6.** Let $\alpha H > 1$. Then the following inequality is valid

\[
E|s_n(t) - s_n(\tau)|^\alpha \leq C \left( \frac{[nt] - [nt]}{n} \right)^{\frac{\alpha H + 1}{2}},
\]

where $C$ is a constant depending on the distribution of $\xi_0$, $\alpha$ and \( \{a_i\} \).

From Lemma 6, applying Theorem 3 from [7, p. 52] (where it is necessary to take $[nt]/n$ as the step function $F_n(t)$), we deduce that for any $\varepsilon > 0$ it holds

\[
\lim_{\delta \to 0} \limsup_{n \to +\infty} \mathbb{P} \left( \sup_{s|\delta} |s_n(t) - s_n(s)| > \varepsilon \right) = 0. \tag{7}
\]

**Lemma 7 (\[13\]).** For all $k \in \mathbb{Z}$ the following is valid

\[
|a_{k+1} + \ldots + a_{k+n}| \leq \left( 4\sqrt{\text{Var}(S_n)} \sum_{k \in \mathbb{Z}} |a_k|^2 \left( 1 + \frac{1}{2\sqrt{\text{Var}(S_n)}} \right) \right)^{1/2}.
\]

**Lemma 8 (\[9\] Lemma 7).** Let \( \{b_{ni}; \ n \geq 1, \ i \in \mathbb{Z}\} \) be an array of real numbers, and let \( \{\zeta_{ni}; \ n \geq 1, \ i \in \mathbb{Z}\} \) be an array of random variables satisfying the following conditions:

- **L1.** $\lim_{n \to \infty} \sum_{i \in \mathbb{Z}} b_{ni}^2 = 1$;
- **L2.** $\lim_{n \to \infty} \sup_{i \in \mathbb{Z}} |b_{ni}| = 0$;
- **L3.** For every $n \geq 1$ the sequence $\{\zeta_{ni}, \ i \in \mathbb{Z}\}$ consists of i.i.d. random variables with mean zero and variance 1;
- **L4.** $\lim_{K \to \infty} \sup_{n \geq 1} \mathbb{E}\zeta_{n0}^2 I(|\zeta_{n0}| > K) = 0$.

Then the sums $\sum_{i \in \mathbb{Z}} b_{ni}\zeta_{ni}$ converge in distribution to a standard Gaussian random variable as $n \to +\infty$.

**Lemma 9.** The finite-dimensional distributions of random processes $s_n(t)$ converge to the corresponding finite-dimensional distributions of random process $B_H(t)$ as $n \to \infty$.

**Proof.** It suffices to prove that $\sum_{i=1}^l c_i s_n(t_i)$ converges in distribution to $\sum_{i=1}^l c_i B_H(t_i)$ for every finite set of numbers $\{c_i; \ i = 1, \ldots, l\}$. So, we observe first that $z_n \equiv \sum_{i=1}^l c_i s_n(t_i) = \sum_{k \in \mathbb{Z}} \sum_{i=1}^l c_i A_{k,n}(t_i) \xi_k$. Further (see Lemma 2)
\[ E_{z_n}^2 = \sum_{i,j=1}^{l} c_i c_j E(s_n(t_i)s_n(t_j)) \rightarrow \sum_{i,j=1}^{l} c_i c_j E(B_H(t_i)B_H(t_j)) \]

\[ = E \left( \sum_{i=1}^{l} c_i B_H(t_i) \right)^2 , \quad n \to \infty. \]

Use the notation

\[ \delta^2 = E \left( \sum_{i=1}^{l} c_i B_H(t_i) \right)^2 , \quad z_n = z_n/\delta. \]

Thus

\[ E z_n^2 = \sum_{k \in \mathbb{Z}} \left( \sum_{i=1}^{l} c_i/\delta A_{k,n}(t_i) \right)^2 \rightarrow 1, \quad n \to \infty. \]

Now, under the conditions of Lemma 8 put \( b_{nk} = \sum_{i=1}^{l} c_i/\delta A_{k,n}(t_i), \) \( \zeta_{nk} = \zeta_k. \) It is easy to see that, in this case, condition L1 is valid. Using Lemma 7 we find, as \( n \to +\infty, \) that

\[ \sup_k \left| \sum_{i=1}^{l} (c_i A_{k,n}(t_i)) \right| = O\left(n^{-H/2}h^{-1/2}(n)\right), \quad n \to \infty. \]

Hence, condition L2 is valid. It is easy to verify that conditions L3 and L4 hold as well. So, the random variables \( z_n \) converge in distribution to a normal random variable with mean zero and variance \( \delta^2. \)

Proof of Proposition 1. Lemmas 6, 9 and relation (7) immediately imply the assertion of the proposition.

Proof of Theorem 1. By changing the order of summation in (4), represent \( R_n \) as

\[ \sum_{n=0}^{\infty} S_n - \int_{[0,1]} s_n dM(n). \]

Further, since \( r_n(t) = \frac{R_n(t/h[n])}{h[n]M(n)} \) we get the representation of \( r_n(t) \) as:

\[ \int_0^{[nt]/n} s_n([nt] - [nu]/n) \frac{dM(un)}{M(n)}. \quad (8) \]

Lemma 10. Let \( M \in \mathcal{R}_\nu, \) \( \nu \geq 0. \) Then, the family of distributions of the stochastic processes \( \{r_n(t)\} \) is dense in \( D[0,1] \) with respect to the uniform metric.

Proof. With the representation (8) in mind, we find an upper bound for

\[ \sup_{|t-s| < \delta} \left| \int_0^{[nt]/n} s_n([nt] - [nu]/n) \frac{dM(un)}{M(n)} \right. \]

\[ \left. - \int_0^{[ns]/n} s_n([ns] - [nu]/n) \frac{dM(un)}{M(n)} \right|. \quad (9) \]
Let, for definiteness, \( s < t \). The obvious inequalities hold
\[
\left| \int_0^{[nt]/n} s_n \left( \left[ nt \right] - \left[ nu \right] \right) / n \frac{dM(\nu)}{M(n)} - \int_0^{[ns]/n} s_n \left( \left[ ns \right] - \left[ nu \right] \right) / n \frac{dM(\nu)}{M(n)} \right|
\leq \int_0^{[ns]/n} \left| s_n \left( \left[ nt \right] - \left[ nu \right] \right) / n - s_n \left( \left[ ns \right] - \left[ nu \right] \right) / n \right| \frac{dM(\nu)}{M(n)}
+ \int_{[ns]/n}^{[nt]/n} \left| s_n \left( \left[ nt \right] - \left[ nu \right] \right) / n \right| \frac{dM(\nu)}{M(n)}.
\]

If \(|t - s| < \delta\), then, for all sufficiently large \( n \), it holds that
\[
\left| \frac{nt - nu}{n} - \frac{ns - nu}{n} \right| < 2\delta.
\]

Since \( u \geq \frac{[ns]}{n} \), we have
\[
\frac{[nt] - [nu]}{n} \leq \frac{[nt] - [ns]}{n} < 2\delta.
\]

Finally, we obtain the following upper bound on (9):
\[
\sup_{|t-s|<\delta, \ t,s \in [0,1]} \left| \int_0^{[nt]/n} s_n \left( \left[ nt \right] - \left[ nu \right] \right) / n \frac{dM(\nu)}{M(n)} - \int_0^{[ns]/n} s_n \left( \left[ ns \right] - \left[ nu \right] \right) / n \frac{dM(\nu)}{M(n)} \right|
\leq 2 \sup_{|t-s|<2\delta, \ t,s \in [0,1]} |s_n(t) - s_n(s)|.
\]

Whence, using the relation (7), we obtain the assertion of the lemma.

To prove Theorem 1 (taking into account Lemma 10), it is sufficient to prove the weak convergence of the finite-dimensional distributions of the processes \( r_n(t) \).

We decompose the proof of convergence of the finite-dimensional distributions in the case \( \nu > 0 \) into Lemmas 11–15 and, respectively, in the case \( \nu = 0 \) into Lemmas 16–18.

In the proofs of the following propositions, we will use the following simple fact. Since \( l \) is a slowly varying at \(+\infty\) function, there exists an infinitely small sequence \( \{\varepsilon_n\} \) such that
\[
n\varepsilon_n \to +\infty, \ l(n\varepsilon_n)/l(n) \to 1.
\]

Recall that \( M \in \mathcal{R}_\nu \) for \( \nu > 0 \) can be represented as \( M(t) = t^\nu l(t) \), where \( l \) is a function that is nondecreasing on the interval \([0, +\infty)\) and slowly varying at \(+\infty\).

**Lemma 11.** As \( n \to +\infty \), for every \( t \in [0, 1] \) it holds that:
\[
r_n(t) - \int_0^{[nt]/n} s_n \left( \left[ nt \right] - \left[ nu \right] \right) / n \ d\nu \to 0.
\]
Proof. Let $t \in (0, 1]$. The following inequalities are valid

$$
\left| \int_0^{[nt]/n} s_n([nt] - [nu])/n u^\nu \frac{dl(un)}{l(n)} \right| \\
\leq \int_{\varepsilon_0}^{[nt]/n} |s_n([nt] - [nu])/n| u^\nu \frac{dl(un)}{l(n)} \\
+ \int_{\varepsilon_0}^{[nt]/n} |s_n([nt] - [nu])/n| u^\nu \frac{dl(un)}{l(n)} \\
\leq \sup_{t \in [0, 1]} |s_n(t)| \left( \frac{l([nt])}{l(n)} - \frac{l(n\varepsilon_n)}{l(n)} \right) + \sup_{t \in [0, 1]} |s_n(t)| \varepsilon_n \to 0,
$$

(10)

where the convergence to 0 is implied by the fact that

$$
\sup_{t \in [0, 1]} |s_n(t)| \xrightarrow{d} \sup_{t \in [0, 1]} |B_H(t)|
$$

and

$$
\frac{l([nt])}{l(n)} \to \frac{l(n\varepsilon_n)}{l(n)} \to 0.
$$

We have

$$
\int_0^{[nt]/n} s_n([nt] - [nu])/n \frac{l(un)}{l(n)} du^\nu \\
= \int_0^{[nt]/n} s_n([nt] - [nu])/n \left( \frac{l(un)}{l(n)} - 1 \right) du^\nu \\
+ \int_0^{[nt]/n} s_n([nt] - [nu])/n du^\nu.
$$

(11)

Let us estimate the first term on the right side (11):

$$
\left| \int_0^{[nt]/n} s_n([nt] - [nu])/n \left( \frac{l(un)}{l(n)} - 1 \right) du^\nu \right| \\
\leq \int_0^{\varepsilon_0} |s_n([nt] - [nu])/n| \left( 1 - \frac{l(un)}{l(n)} \right) du^\nu \\
+ \int_0^{[nt]/n} |s_n([nt] - [nu])/n| \left( 1 - \frac{l(un)}{l(n)} \right) du^\nu \\
\leq \sup_{t \in [0, 1]} |s_n(t)| \varepsilon_n \sup_{t \in [0, 1]} |s_n(t)| \left( 1 - \frac{l(n\varepsilon_n)}{l(n)} \right) \to 0.
$$

(12)
Next, we note that
\[
\left| \int_0^{[nt]/n} s_n(\lfloor nt \rfloor - [nu]/n) \frac{dM(un)}{M(n)} - \int_0^{[nt]/n} s_n(\lfloor nt \rfloor - [nu]/n) \, du' \right| \\
\leq \left| \int_0^{[nt]/n} s_n(\lfloor nt \rfloor - [nu]/n) u' \frac{dl(un)}{l(n)} \right| + \int_0^{[nt]/n} s_n(\lfloor nt \rfloor - [nu]/n) \left( \frac{l(un)}{l(n)} - 1 \right) \, du'.
\]
Whence, using the relations (10) and (12), we obtain the assertion of the lemma.

Introduce the notation:
\[
s_n^{(1)}(t) = S_{[nt]+1}/\sqrt{\text{Var}(S_n)}, \quad t \in [0, 1], \quad n = 1, 2, \ldots.
\]

**Lemma 12.** Let \( E|\xi_1|^{\alpha} < +\infty \) for some \( \alpha \) such that \( \alpha \geq 2 \) and \( \alpha H > 1 \). Then, as \( n \to +\infty \), it holds that:
\[
\sup_{t \in [0,1]} |s_n^{(1)}(t) - s_n(t)| \overset{P}{\to} 0.
\]

**Proof.** Consider arbitrary \( \varepsilon > 0 \). The following obvious equality holds:
\[
P(\sup_{t \in [0,1]} |s_n^{(1)}(t) - s_n(t)| > \varepsilon) \leq nP(|X_1| > \varepsilon h(n)n^H).
\]
Applying Chebyshev’s inequality to the right side (13), we derive
\[
P(\sup_{t \in [0,1]} |s_n^{(1)}(t) - s_n(t)| > \varepsilon) \leq \frac{n^{1-\alpha H}}{\varepsilon^\alpha h^\alpha(n)} E|X_1|^{\alpha} \to 0.
\]
Note that the finiteness of \( E|X_1|^{\alpha} \) follows from Lemma 4.

**Lemma 13.** As \( n \to +\infty \), for every \( t \in [0, 1] \) it holds that:
\[
\left| \int_0^{[nt]/n} s_n(\lfloor nt \rfloor - [nu]/n) \, du' - \int_0^t (-s_n^{(1)}(u)) \, d(t - u)^{\nu} \right| \overset{P}{\to} 0.
\]

**Proof.** Let \( t \in (0, 1] \). Let us represent \( \int_0^t s_n^{(1)}(u) \, d(t - u)^{\nu} \) as a sum:
\[
\int_0^t s_n^{(1)}(u) \, d(t - u)^{\nu} = \int_0^{[nt]/n} s_n^{(1)}(u) \, d(t - u)^{\nu} \\
+ \int_{[nt]/n}^t s_n^{(1)}(u) \, d(t - u)^{\nu}.
\]
We estimate the second term on the right-hand side of (14). Using Lemma 12 and Proposition 11, we have
\[
\left| \int_{[nt]/n}^t s_n^{(1)}(u) \, d(t - u)^{\nu} \right| \leq |s_n^{(1)}(t)| |(t - [nt]/n)^{\nu} | \overset{P}{\to} 0.
\]
Further, we note that the following equality is valid
\[
\int_0^{[nt]/n} s_n((nt - [nu])/n) \, du^\nu = -\int_0^{[nt]/n} s_n^{(1)}(u) \, d([nt]/n - u)^\nu. \tag{16}
\]

Using (16), we conclude that
\[
\left| \int_0^{[nt]/n} s_n((nt - [nu])/n) \, du^\nu - \int_0^{[nt]/n} (-s_n^{(1)}(u)) \, d(t - u)^\nu \right|
\leq \sup_{t \in [0,1]} |s_n^{(1)}(t)| \left| (t - [nt]/n)^\nu - (t^\nu - ([nt]/n)^\nu) \right| \xrightarrow{P} 0. \tag{17}
\]

To estimate the third integral in (17) we use the fact that the function 
\[f(u) = (t - u)^\nu - ([nt]/n - u)^\nu\] is monotone on the interval \([0, [nt]/n]\).

It’s obvious that
\[
\left| \int_0^{[nt]/n} s_n((nt - [nu])/n) \, du^\nu - \int_0^t (-s_n^{(1)}(u)) \, d(t - u)^\nu \right|
\leq \left| \int_0^t s_n^{(1)}(u) \, d(t - u)^\nu \right|
+ \left| \int_0^{[nt]/n} s_n((nt - [nu])/n) \, du^\nu - \int_0^{[nt]/n} (-s_n^{(1)}(u)) \, d(t - u)^\nu \right|. \tag{18}
\]

Applying (15) and (17) to the right-hand side of (18), we obtain the assertion of the lemma.

\[\square\]

**Lemma 14.** Let \(\{c_i; i = 1, \ldots, l\}\) be an arbitrary set of real numbers and \(\{t_i; i = 1, \ldots, l\}\) be an arbitrary set in the interval \([0, 1]\). Then, as \(n \to +\infty\), it holds that
\[
\sum_{i=1}^l c_i \int_0^{t_i} s_n^{(1)}(u) \, d(t_i - u)^\nu \overset{d}{\to} \sum_{i=1}^l c_i \int_0^{t_i} B_H(u) \, d(t_i - u)^\nu.
\]

**Proof.** Define the functional \(F : D[0, 1] \to \mathbb{R}\) as follows:
\[
F(f) = \sum_{i=1}^l c_i \int_0^{t_i} f(u) \, d(t_i - u)^\nu.
\]

It is clear that \(F\) is continuous in the uniform topology at the points of \(C[0, 1]\) and measurable on \(D[0, 1]\) in Skorokhod’s topology. Therefore, taking into account Lemma 12 and the conditions of Proposition 11 we conclude that \(F(s_n^{(1)})\) converges in distribution to \(F(B_H)\) (e.g., see [7]), and this immediately implies the assertion of the lemma.

\[\square\]
Lemma 15. As $n \to \infty$, the finite-dimensional distributions of the stochastic processes $r_n(t)$ converge to the finite-dimensional distributions of process $Z_{\nu,H}(t)$.

Proof. Let $\{c_i; \, i = 1, \ldots, l\}$ be an arbitrary set of real numbers and $\{t_i; \, i = 1, \ldots, l\} \subseteq [0, 1]$. Lemmas (11) and (13) imply that

$$\left| \sum_{i=1}^l c_i r_n(t_i) - \sum_{i=1}^l c_i \int_0^{t_i} (-s_n^{(1)}(u)) \, d(t_i - u)^\nu \right| \to 0.$$ 

Using this fact, Lemma 14, and the well-known Cramer–Wold device (e.g., see [14]), we obtain the assertion of the lemma.

Now we proceed to the proof of Theorem 1 for the case $\nu = 0$. Recall that in this case the function $M$ is non-negative, non-decreasing, non-constant on the interval $[0; +\infty)$ and slowly varying at $+\infty$.

Lemma 16. There is a positive constant $C$ such that for any $k, l \in [0, n]$ the following inequality is valid:

$$E|s_n(k/n) - s_n(l/n)|^2 \leq C\frac{|k - l|^H}{n^H}.$$ 

Proof. First of all, we have the equality (see Lemma 1)

$$E|s_n(k/n) - s_n(l/n)|^2 = \frac{|k - l|^{2H}k^2(|k - l|)}{n^{2H}k^2(n)}.$$ 

We represent this equality in the form

$$E|s_n(k/n) - s_n(l/n)|^2 = \frac{|k - l|^H |k - l|^H H^2(|k - l|)}{n^H H^2(n)}. \tag{19}$$

Applying Lemma 5 to the second factor on the right-hand side (19), we obtain the assertion of the lemma.

Lemma 17. For every $t \in [0, 1]$, as $n \to +\infty$, it holds that:

$$r_n(t) - \left( 1 - \frac{M(0)}{M(+\infty)} \right) s_n(t) \to 0.$$ 

Proof. Let $t \in (0, 1]$. Using the representation (8) for $r_n(t)$, we have

$$r_n(t) = \int_{n\varepsilon_n}^{[nt]/n} s_n(([nt] - [nu])/n) \frac{dM(un)}{M(n)} + \int_0^{\varepsilon_n} s_n(([nt] - [nu])/n) \frac{dM(un)}{M(n)}. \tag{20}$$

12
Let us estimate the first term in (20):

\[
\left| \int_{\varepsilon_n}^{\lfloor nt/n \rfloor} s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) \frac{dM(un)}{M(n)} \right| \leq \sup_{t \in [0,1]} |s_n(t)| \left( \frac{M(\lfloor nt \rfloor)}{M(n)} - \frac{M(\varepsilon_n n)}{M(n)} \right) \overset{P}{\to} 0. \quad (21)
\]

We represent the second term in (20) as the sum:

\[
\int_{0}^{\varepsilon_n} s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) \frac{dM(un)}{M(n)} = \int_{0}^{\varepsilon_n} (s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) - s_n(t)) \frac{dM(un)}{M(n)} + \int_{0}^{\varepsilon_n} s_n(t) \frac{dM(un)}{M(n)}. \quad (22)
\]

Consider the first term on the right-hand side of (22).

\[
\mathbb{E} \left( \int_{0}^{\varepsilon_n} (s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) - s_n(t)) \frac{dM(un)}{M(n)} \right)^2 = \int_{0}^{\varepsilon_n} \int_{0}^{\varepsilon_n} \mathbb{E}s_n(t,u)s_n(t,v) \frac{dM(un)}{M(n)} \frac{dM(vn)}{M(n)}. \quad (23)
\]

where \( s_n(t,u) = s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) - s_n(t) \). Further, using the obvious inequality: \( \mathbb{E}s_n(t,u)s_n(t,v) \leq \frac{1}{2}(\mathbb{E}s_n^2(t,u) + \mathbb{E}s_n^2(t,v)) \), as well as Lemma 16 we obtain \( \mathbb{E}s_n(t,u)s_n(t,v) \leq C\varepsilon_n^2(u^H + v^H) \). Applying the last inequality to the right side (23), we get

\[
\mathbb{E} \left( \int_{0}^{\varepsilon_n} (s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) - s_n(t)) \frac{dM(un)}{M(n)} \right)^2 \leq \frac{C}{2} \int_{0}^{\varepsilon_n} \int_{0}^{\varepsilon_n} (u^H + v^H) \frac{dM(un)}{M(n)} \frac{dM(vn)}{M(n)} \overset{P}{\to} 0, \quad n \to +\infty. \quad (24)
\]

From (24) it follows that as \( n \to +\infty \)

\[
\int_{0}^{\varepsilon_n} (s_n((\lfloor nt \rfloor - \lfloor nu \rfloor)/n) - s_n(t)) \frac{dM(un)}{M(n)} \overset{P}{\to} 0. \quad (25)
\]

Consider the third term on the right side (22). It’s obvious that

\[
\int_{0}^{\varepsilon_n} s_n(t) \frac{dM(un)}{M(n)} - s_n(t) \left( 1 - \frac{M(0)}{M(+\infty)} \right) = s_n(t) \left( \frac{M(\varepsilon_n n)}{M(n)} - \frac{M(0)}{M(n)} \right) - s_n(t) \left( 1 - \frac{M(0)}{M(+\infty)} \right) \overset{P}{\to} 0. \quad (26)
\]
Applying the relations (25), (26) to (22) we derive
\[ \int_0^{c_n} s_n((nt - [nu])/n) \frac{dM(un)}{M(n)} - s_n(t) \left( 1 - \frac{M(0)}{M(+\infty)} \right) \xrightarrow{P} 0. \] (27)

Next, applying the relations (21), (27) to (20), we obtain the assertion of the lemma.

Lemma 18. The finite-dimensional distributions of the stochastic processes \( r_n(t) \) converge to the finite-dimensional distributions of the process \( \left( 1 - \frac{M(0)}{M(+\infty)} \right) B_H(t) \) as \( n \to +\infty \).

Proof. Let \( \{c_i; i = 1, \ldots, l\} \) be an arbitrary set of real numbers and \( \{t_i; i = 1, \ldots, l\} \) be an arbitrary set in the interval \([0, 1]\). Lemma 17 implies that
\[ \sum_{i=1}^{l} c_i r_n(t_i) - \sum_{i=1}^{l} c_i s_n(t_i) \left( 1 - \frac{M(0)}{M(+\infty)} \right) \xrightarrow{P} 0. \]

Whence and from Lemma 9 as well as the Cramer–Wold device, the assertion of the lemma follows.

Proof of Corollary. First, we note that the assertion of Theorem 1 holds for Gaussian analogs of the processes \( r_n(t) \) when the random variables \( \xi_k \) in (1) are the standard Gaussian; in this case, \( \text{Var}(R_n) \) does not change.

If \( M \in \mathcal{R}_\nu, \nu > 0 \), then, as \( n \to \infty \), we have convergence in distribution of the sequence \( r_n(1) \) to \( Z_{\nu,H}(1) \). Moreover, if \( M \in \mathcal{R}_0 \), then the sequence \( r_n(1) \) converge in distribution to \( \left( 1 - \frac{M(0)}{M(+\infty)} \right) B_H(1) \) (recall that \( r_n(1) = \frac{R_n}{h(n)n^H M(n)} \)).

Next, we note that the random variable \( \frac{R_n}{\sqrt{\text{Var}(R_n)}} \sim \mathcal{N}(0,1) \) for all \( n \). Finally, we obtain the equivalence of the normalizing coefficients \( R_n \):
\[ \text{Var}(R_n) \sim \sigma^2 h^2(n)n^{2H} M^2(n), \ n \to \infty, \]
where
\[ \sigma^2 = \begin{cases} \text{Var}(Z_{\nu,H}(1)), & \text{if } M \in \mathcal{R}_\nu, \nu > 0, \\ (1 - M(0)/M(+\infty))^2, & \text{if } M \in \mathcal{R}_0. \end{cases} \]

References

[1] N. S. Arkashov, *The principle of invariance in the Strassen form to the partial sum processes of moving averages of finite order*, Sib. Elektron. Mat. Izv., 15 (2018), 1292–1300.
[2] N. S. Arkashov, *On a Method for the Probability and Statistical Analysis of the Density of Low Frequency Turbulent Plasma*, Computational Mathematics and Mathematical Physics, 59:3 (2019), 402–413.

[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, v. II, New York: John Wiley & Sons, 1971.

[4] A. I. Olemskoi and A. Ya. Flat, *Application of fractals in condensed-matter physics*, Phys. Usp., 36 (1993), 1087–1128.

[5] R. R. Nigmatullin, *Fractional integral and its physical interpretation*, Theor. Math. Phys., 90:3 (1992), 242–251.

[6] N. S. Arkashov and V. A. Seleznev, *Formation of a relation of nonlocalities in the anomalous diffusion model*, Theoret. and Math. Phys., 193:1 (2017), 1508–1523.

[7] A. A. Borovkov, A. A. Mogulskii and A. I. Sakhanenko, *Limit theorems for random processes*, in: Probability Theory 7, VINITI, v. 82, Moscow, 1995.

[8] Yu. A. Davydov, *The invariance principle for stationary processes*, Theory of Probability & Its Applications, 15:3 (1970), 487–498.

[9] T. Konstantopoulos, A. Sakhanenko, *Convergence and convergence rate to fractional Brownian motion for weighted random sums*, Sib. Elektron. Mat. Izv., 1 (2004), 47–63.

[10] N. S. Arkashov and I. S. Borisov, *Gaussian approximation to the partial sum processes of moving averages*, Siberian Math. J., 45:6 (2004), 1000–1030.

[11] V. V. Petrov, *Limit Theorems for the Sums of Independent Random Variables*, Nauka, Moscow, 1987.

[12] E. Seneta, *Regularly Varying Functions*, Nauka, Moscow, 1985.

[13] I. A. Ibragimov and Yu. V. Linnik, *Independent and Stationarily Connected Variables*, Wolters-Noordhoff Publishing Company, Groningen, The Netherlands, 1971.

[14] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.