On Controllable Abundance Of Saturated-input Linear Discrete Systems¹)

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Abstract

Several theorems on the volume computing of the polyhedron spanned by an n-dimensional vector set with the finite-interval parameters are presented and proved firstly, and then are used in the analysis of the controllable regions of the linear discrete time-invariant systems with saturated inputs. A new concept and continuous measure on the control ability, control efficiency of the input variables, and the diversity of the control laws, named as the controllable abundance, is proposed based on the volume computing of the regions and is applied to the actuator placing and configuring problems, the optimizing problems of dynamics and kinematics of the controlled plants, etc. The numerical experiments show the effectiveness of the new concept and methods for investigating and optimizing the control ability and efficiency.

Key words: linear systems; discrete-time systems; controllability; reachability; controllable region; polyhedron volume; actuator placement;

1. Introduction

The input variables of the most practical controlled plants are fallen into the finite variable intervals for the restricted actuators or input energy, and can be modeled as the saturated-input linear discrete time-invariant (SLDTI) systems as follows

$$x_{k+1} = Ax_k + Bu_k, \|u_k\| \leq 1, u_k \in R^n$$

(1)

where $u_k$ is the saturated input variables with the normalization of the variable intervals (see e.g., Hu & Qiu, 1998; Jamak, 2000 and the references therein), $x_k \in R^n$ is the state variables; $A \in R^{n \times n}$ and $B \in R^{n \times r}$ are the system and input matrix, respectively. For the practical systems, without loss of the generality, the matrix $A$ is assumed non-singular, otherwise, the dynamics of some sub-space of the systems is as

$$x^{(i)}_{k+1} = 0 \cdot x^{(i)}_{k+1} + B^{(i)}u_k$$

i.e., there is not exist any dynamics in the sub-space.

The state controllability and reachability are very important concepts in system analysis and synthesis, and are used widely in the various aspects of the control theory. Because of the saturated inputs, the state controllability and reachability become particularly important for the SLDTI systems. Therefore, many works are devoted to that and the controllable and reachable regions are two new concepts proposed for the controllability and reachability studies on the SLDTI systems. Considering that the reachable regions can be equal to the controllable regions of some kind of duality systems (see e.g., Hu, Lin, & Qiu, 2002), the controllable regions and other controllable properties mentioned later will include the reachable regions and the corresponding reachable properties, respectively.

For the SLDTI systems, the controllable regions are defined, and the boundaries of the regions, the extreme control law on the boundaries, and the control design with the saturated actuators are discussed in succession (see e.g., Fisher & Gayek, 1987; D’Alessandra & De Santis, 1992; Lasserre, 1993; Bernstein & Michel, 1995; Hu & Qiu, 1998; Hu, Lin, & Qiu, 2002). But the control ability and efficiency of the saturated inputs didn't been discussed with regret. In fact, excepting the two-value logic index of state controllability, whether there exist the more accurate measures on the control ability and efficiency for the SLDTI systems or other systems? In additional, the problems on the relations among the controllable regions, the control ability/efficiency, and the solution space of the input sequences for stabilizing the given states are very interesting problems. The studies and answers on these problems are helpful to many interesting works in the control engineering, such as,

1) optimizing the dynamics and kinematics of the open-loop controlled plants in their designing and manufacturing processes, e.g., the optimizing the technic parameters in...
the designing and manufacturing of the electric motors, robot, etc..

(2) determining the input variables from the possible input variables in the modeling process for more effective stabilizing the controlled systems, e.g., designing of the place of the fuel nozzles in the heating furnaces, the assignment of the leader or sub-leaders of the team control systems, etc..

(3) promoting the diversity of the control laws and improving the state trajectory and control performance of the closed-loop control systems, etc.

The above problems on the controllable abundance, proposed later as a synthesis concept on the control ability, control efficiency of the input variables, and the diversity of the control laws, are closely related to the shapes and sizes of the controllable regions. Here, the control ability is mainly refer to the size of the controllable states, i.e., the size of the controllable region, the control efficiency is mainly refer to the wasting time and energy in the controlling process, and the diversity of the control laws is mainly refer to the multiple choices of the control laws and state trajectories for controlling the state, i.e. the size of the solution space of the input sequence. This paper will be devoted to study the controllable abundance. For the general LTI systems with the unconstrained input variables, the state controllability means that the controllable regions and the solution space of the corresponding input sequence are infinite space, but, for the SLDTI systems, the controllable region and the solution space maybe are finite space. Therefore, their shapes and sizes are very important factors for the control problems.

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2. Shape And Volume Of High-dimensional Polyhedron

Before discussing on the controllable abundance, the shape and volume of a special polyhedron in n-dimensional (n-D) geometry space are discussed firstly and several theorems on the computation of these shape and volume are given and proven. The polyhedron volume will be used to measure the volume of the controllable region of the SLDTI systems

2.1. Definition Of A Special Polyhedron In n-D

In the algebra and geometry fields, some analysis and computing problems on polyhedrons, such as computing the boundary, shape, volume, etc., are dealt with usually (see e.g., Brøndsted, 1982; Ziegler, 1995). The controllable regions for the SLDTI systems are belong to a class of special polyhedrons spanned by a vector set with the finite-interval parameters in high-dimensional geometry space, and the analysis and computation of the special polyhedrons will be contributed to the system analysis and synthesis in the control theory field.

The special polyhedrons are defined as follows.

**Definition 1.** The polyhedrons spanned by the n-D vectors of matrix \( \mathbf{A}_m = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m] \in \mathbb{R}^{n \times m} \) and the parameter set with the finite intervals are defined as follows

\[
\begin{align*}
C_q (\mathbf{A}_m) & = \left\{ \sum_{i=1}^{m} c_i \mathbf{a}_i : \forall c_i \in [0,1], i = 1, m \right\} \\
D_q (\mathbf{A}_m) & = \left\{ \sum_{i=1}^{m} c_i \mathbf{a}_i : \forall c_i \in [-1,1], i = 1, m \right\}
\end{align*}
\]

where \( q = \text{rank}(\mathbf{A}_m) \). The polyhedrons are the q-D parallel polyhedrons in the n-D space and are convex bodies.

Because the polyhedrons \( C_q (\mathbf{A}_m) \) and \( D_q (\mathbf{A}_m) \) can be transformed each other via the parallel displacement and stretching transformation, for the convenience of discussion, when one of two polyhedrons is discussed, the obtained conclusions can be generalized to the other one.

In fact, the q-D parallel polyhedrons \( C_q (\mathbf{A}_m) \) are surrounded by a series of (q-1)-D parallel polyhedrons spanned and shifted by the q-1 vectors from the vector set \( \mathbf{A}_m \). Likewise, the (q-1)-D parallel polyhedrons are surrounded by a series of (q-2)-D parallel polyhedrons, and so on.

The polyhedron \( C_q (\mathbf{A}_m) \) in the 2-D can be illustrated as Fig.1. If vectors \( \mathbf{a}_i \in \mathbb{R}^2 (i = 1,3) \), the parallelogram ABCD and the parallel hexagon ABFEGC are the 2-D polyhedrons spanned by the set \( \{\mathbf{a}_1, \mathbf{a}_2\} \) and \( \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \) in the 2-D space, respectively. If vectors \( \mathbf{a}_i \in \mathbb{R}^3 (i = 1,3) \), these parallelograms and parallel hexagons are the 2-D polyhedrons in the 3-D space.

![Fig1 2-D parallel hexagon spanned by set \{a1, a2, a3\}](image)

**2.2. Volume Of The Parallel Polyhedrons**

By Definition 1, we have,

\[
V_q (D_q (\mathbf{A}_m)) = 2^q V_q (C_q (\mathbf{A}_m))
\]

The volume \( V_q (C_q (\mathbf{A}_m)) \) can be computed in three case, such as, \( m = n, m > n \) and \( m < n \), as follows.
Theorem 1. For any matrix \( A_n \in \mathbb{R}^{m \times n} \), the volume \( V_n(C_n(A_n)) \) of the \( n \)-D parallel polyhedrons \( C_n(A_n) \) is
\[
V_n(C_n(A_n)) = \left| \det(A_n) \right|
\] (4)

Proof. Two cases are discussed as follows.

1. If \( \text{rank}(A_n) = q < n \), only \( q \)-D polyhedron in \( n \)-D space, can be spanned by vector set \( A_n \). Therefore, the volume of \( C_q(A_n) \) in the \( n \)-D space is 0, i.e.,
\[
V_n(C_q(A_n)) = |\det(A_n)| = 0
\]
(2) If the matrix \( A_n \) is a non-singular identity matrix, we have,
\[
V_n(C_q(A_n)) = |\det(A_n)| = 1
\]
(3) If the matrix \( A_n \) is nonsingular but is not an identity matrix, it is surely transformed as an identity matrix via three kinds of the elementary column transformations. Because Eq. (4) holds for the identity matrix, only that these column transformations don't change the volume, of Eq. (4) is proven, and then Eq. (4) holds for any non-singular \( A_n \).

Three kinds of the column transformations are as follows.

Transformation 1. Multiplying a column by a non-zero constant \( c \). For the transformation, the two sides of Eq. (4) will be multiplied by \( |c| \), i.e., Transformation 1 doesn't changing the property of Eq. (4).

Transformation 2. Exchanging some two columns. For the transformation, \( \det(A_n) \) will reverse sign and the volume \( V_n(C_n(A_n)) \) doesn't change, and then Transformation 2 doesn't changing the property of Eq. (4).

Transformation 3. Summing one column to other. Without loss of generality, it is assumed that the 2nd column is summed to the 1st column. Firstly, for the determinant, we have,
\[
det(a_1 + a_2, a_2, \cdots, a_n) = det(a_1, a_2, \cdots, a_n) + det(a_2, a_2, \cdots, a_n)
\] (5)
Secondly, for the volume, we have,
\[
V_n(C_q(A_n)) = |a_1||h(a_2, a_1)| \times h(a_2, C_2(A_2)) \times \cdots
\] (6)
where \( h(a_1, C_{i-1}(A_{i-1})) \) is the vertical distance from the space point \( a_1 \) to the \( (i-1) \)-D polyhedron \( C_{i-1}(A_{i-1}) \) in the \( n \)-D space.

Considered that two parallelograms spanned by set \( \{a_1, a_2\} \) and \( \{a_1 + a_2, a_2\} \) in Fig. 2 are with the same area, we have,
\[
V_2(C_2(a_1, a_2)) = |a_1||h(a_2, a_1)| = |a_1 + a_2||h(a_2, a_1 + a_2)| = V_2(C_2(a_1 + a_2, a_2))
\] (7)

In addition, by the definition of the function \( h(a_1, C_{i-1}(A_{i-1})) \), we have,
\[
h(a_1, C_{i-1}(a_1, a_2, a_3, \cdots)) = h(a_1, C_{i-1}(a_1 + a_2, a_2, a_3, \cdots)
\] (8)
By Eq. (6), (7), and (8), we conclude that
\[
V_n(C_q(A_n)) = V_n(C_q(a_1 + a_2, a_2, a_3, \cdots))
\] (9)
And then, by Eq. (5) and (9), we have, then the Transformation 3 doesn't changing the property of Eq. (7).

Summing up the above analysis process, Eq. (7) is proven to be true.

Theorem 2. For any full row rank matrix \( A_m \in \mathbb{R}^{m \times n} \), we have,
\[
V_n(C_n(A_m)) = \sum_{(i,j,k,\cdots,\ell)} V_n(C_n(A_{i,j,k,\cdots,\ell}))
\] (10)
where \( A_{i,j,k,\cdots,\ell} = [a_{i_1}, a_{i_2}, \cdots, a_{i_n}] \), the column-label multi-tuple set \( Q^n_m \) is constituted by all possible multi-tuple \( (i_1, i_2, \cdots, i_n) \) which elements are picked from the set \( \{1, 2, \ldots, m\} \).

Proof. Because the exchanging place of the vectors in \( A_m \) doesn't change the volume \( V_n(C_n(A_m)) \), without lost of generality, it is assumed that the first \( n \) vectors of the matrix \( A_m \) are linearly independent. Next, the inductive method is used to prove Eq. (10).

1. When \( m = n \), by Theorem 1, Eq. (10) holds.
2. When \( m = k \) for some \( k > n \), it is assumed that Eq. (10) holds, i.e., we have,
\[
V_n(C_n(A_m)) = \sum_{(i,j,k,\cdots,\ell)} V_n(C_n(A_{i,j,k,\cdots,\ell}))
\] (11)
(3) Next, Eq. (10) for \( m = k + 1 \) will be proven true.
In fact, the \( n \)-D polyhedron \( C_n(A_{i,j,k,\cdots,\ell}) \) is surrounded by a series the \((n-1)\)-D polyhedrons described as
In all \((n-1)\)-D polyhedrons \(\tilde{C}_{n-1}(\mathbf{A}_{i_1,i_2\cdots,i_{n-1}})\) corresponding to \((i_1,i_2\cdots,i_{n-1})\), only two \((n-1)\)-D polyhedrons locate in the outer surfaces of the \(n\)-D polyhedrons \(C_n(A_i)\) and are parallel.

In fact, the \(n\)-D polyhedrons \(C_n(A_{i+1})\) can be generated by extending the \(n\)-D polyhedrons \(C_n(A_i)\) along with the direction \(a_{k+1}\), and the half of the \((n-1)\)-D polyhedrons at the outer surface will be shifted with the direction \(a_{k+1}\).

The volume of the extending part is the sum of the product of the extending outer surface polyhedrons and the direction \(a_{k+1}\). For the two \((n-1)\)-D polyhedrons \(\tilde{C}_{n-1}(\mathbf{A}_{i_1,i_2\cdots,i_{n-1}})|^*\) corresponding to \((i_1,i_2\cdots,i_{n})\), only one will be shifted outer and the extending volume is \(V_q(C_n(\mathbf{A}_{i_1,i_2\cdots,i_{n}},a_{k+1}))\).

Therefore, we have

\[
V_q(C_n(A_{k+1})) = V_q(C_n(A_k)) + \sum_{(i_1,i_2\cdots,i_{n}) \in \Omega_k} V_q(C_n(\mathbf{A}_{i_1,i_2\cdots,i_{n}},a_{k+1}))
\]

\[
= \sum_{(i_1,i_2\cdots,i_{n}) \in \Omega_k} V_q(C_n(\mathbf{A}_{i_1,i_2\cdots,i_{n}},a_{k+1}))
\]

\[
= \sum_{(i_1,i_2\cdots,i_{n}) \in \Omega_k} V_q(C_n(\mathbf{A}_{i_1,i_2\cdots,i_{n}},a_{k+1}))
\]

Hence, Eq. (10) holds for \(m = k+1\).

Summed up the above, Eq. (10) is proven true by the inductive method.

In fact, Theorem 1 can be regarded as a special case of Theorem 2 for \(n = m\). By Theorem 2, we have the following theorem on the volume \(V_q(C_q(A_m))\) for any matrix \(A_m\).

**Theorem 3.** For any matrix \(A_m \in \mathbb{R}^{m \times m}\), if \(\text{rank} A_m = q\), we have

\[
V_q(C_q(A_m)) = \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} V_q(C_q(\mathbf{A}_{i_1,i_2\cdots,i_q}))
\]

\[
= \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} \left| \det(\mathbf{A}_{i_1,i_2\cdots,i_q}) \right|^2
\]

**Proof.** (1) When \(q = n\), Eq. (13) is equal to Eq. (10) in Theorem 2.

(2) When \(q < n\), there must exist a unitary orthogonal matrix \(U \in \mathbb{R}^{m \times m}\) to make the following transformation

\[
U \mathbf{A}_m = \left[ \begin{array}{c} \mathbf{G}_m \\ \mathbf{O}_{n-q,m} \end{array} \right], \quad \mathbf{g}_i \in \mathbb{R}^q, i = 1, m
\]

where \(\text{rank} \mathbf{G}_m = q\). Because the unitary orthogonal matrix \(U\) doesn’t change the volume of the polyhedron, we have

\[
V_q(C_q(A_m)) = V_q(C_q(U \mathbf{A}_m)) = V_q(C_q(\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_m))
\]

So that, by Theorem 2 and Theorem 1, we have,

\[
V_q(C_q(A_m)) = \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} V_q(C_q(\mathbf{A}_{i_1,i_2\cdots,i_q}))
\]

\[
= \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} \left| \det(\mathbf{A}_{i_1,i_2\cdots,i_q}) \right|
\]

Because

\[
\det(A^*_{i_1,i_2\cdots,i_q} \mathbf{A}_{i_1,i_2\cdots,i_q} = \left| \det(A^*_{i_1,i_2\cdots,i_q}) \right|^2 \mathbf{U}^* \mathbf{U}
\]

\[
= \left[ \det(\mathbf{G}_{i_1,i_2\cdots,i_q}) \right]^2
\]

By Eq. (14) and (15), we know, Eq. (13) holds.

For Theorem 2, if \(q = m < n\), we have

\[
V_m(C_m(A_m)) = \left| \det(\mathbf{A}_m) \right|^2
\]

and if \(q = n\), Theorem 2 is only a special case of Theorem 3.

By Theorem 2, the theorem on the volume computation of the polyhedrons via the rotation and scale transformations can be got as follows.

**Theorem 4.** For any full row rank matrix \(A_m \in \mathbb{R}^{m \times m}\) and any reversible matrix \(P \in \mathbb{R}^{m \times m}\), we have,

\[
V_q(C_q(P \mathbf{A}_m)) = \left| \det(P) \right| V_q(C_q(A_m))
\]

**Proof.** By the Theorem 2, we have

\[
V_q(C_q(P \mathbf{A}_m)) = \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} V_q(C_q(P \mathbf{A}_{i_1,i_2\cdots,i_q}))
\]

\[
= \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} |\det(P \mathbf{A}_{i_1,i_2\cdots,i_q})|\]

\[
= |\det(P)| \sum_{(i_1,i_2\cdots,i_q) \in \Omega^q_k} |\det(\mathbf{A}_{i_1,i_2\cdots,i_q})|
\]

\[
= |\det(P)| V_q(C_q(A_m))
\]
By Theorem 1 to Theorem 4, the volume of the polyhedron spanned by the any finite vector set can be computed, and the simpler and rapider computing methods of the polyhedron volume will be studied in the future.

3. The Controllable Abundance

For the SLDTI systems, based on the computing of the dimensions and volumes of the controllable regions, a new concept on the control ability and efficiency, named as the controllable abundance, will be defined and discussed later.

3.1. The Controllable Regions

The controllable region is a very important concept for the analysis and synthesis for the SLDTI systems and can be defined as follows (see Hu, Lin, & Qiu, 2002).

**Definition 2.** The all controllable states in the N sample steps constitute the N-steps controllable regions of the SLDTI systems, described as follows.

\[
\begin{align*}
R_{c,N} & = \{ x^0 | x^0 = -P_{c,N}u_{0,N-1}, \forall u_{0,N-1} \in [-1, 1]^{c-N} \} \\
& = \{ x^0 | x^0 = P_{c,N}u_{0,N-1}, \forall u_{0,N-1} \in [-1, 1]^{c-N} \}
\end{align*}
\]  

where the N-steps controllability matrix \( P_{c,N} \) and the input sequence \( u_{0,N-1} \) are respectively as follows

\[
P_{c,N} = \begin{bmatrix} A^{-N}B & A^{-N+1}B & \cdots & A^{-1}B \end{bmatrix}
\]  

\[
u_{0,N-1} = \begin{bmatrix} u_{N-1}^T & u_{N-2}^T & \cdots & u_0^T \end{bmatrix}
\]

Similar to the above definition, the N-steps reachable regions \( R_{c,N} \) and the N-steps reachability matrix \( P_{r,N} \) can be defined as follows.

\[
R_{r,N} = \{ x^N | x^N = P_{r,N}u_{0,N-1}, \forall u_{0,N-1} \in [-1, 1]^{r-N} \}
\]

\[
P_{r,N} = \begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}
\]

By Eq. (18) to Eq. (21), we can see, the controllable region of the SLDTI system \( \Sigma(A, B) \) can be equal to the reachable region of the SLDTI system \( \Sigma(A^{-1}, -A^{-1}B) \) (see Hu, Lin, & Qiu, 2002).

By Eq. (18), we know, the control ability and efficiency are closed relations to the shape and the size of the controllable regions. Comparing Definition 1 with Definition 2, the regions \( R_{c,N} \) can be regarded as a parallel polyhedron spanned by the matrix \( P_{c,N} \), and can be denoted as \( D_q (P_{c,N}) \). Therefore, with the aid of the analysis of the parallel polyhedrons in Section 2, the control ability and efficiency can be discussed in detailed.

3.2. Dimension And Volume Of The Controllable Region

The indices on the shape and size of these regions mainly include the dimension and the volume.

As discussed in Section 1, the system matrix \( A \) of the SLDTI systems is nonsingular, and then, by Eq. (19) and (21), we have

\[
\text{rank } P_{c,N} = \text{rank } P_{r,N} = r_n \leq r_n \leq n, \quad \forall N
\]

Therefore,

\[
dim(R_{c,N}) = dim(R_{r,N}) = r_n
\]

For the SLDTI systems, with the increasing of the sampling times, we have

\[0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq n\]

The set \( \{r_1, r_2, \cdots, r_n\} \) composed by the dimension of the every N-steps controllable regions is called as the dimension set of the controllable regions. The dimension and the dimension set reflect the size of the \( r_n \)-D controllable regions, the direct and rapid respond ability to the input variables in the N-steps, in some extent.

With the aid of Theorem 1 to Theorem 4, the volumes of the regions \( R_{c,N} \) and \( R_{r,N} \) can be computed as follows.

\[
v_{c,N} = V_{c} (D_q (P_{c,N})) = 2^n V_{c} (C_{r_n} (P_{c,N}))
\]

(23)

\[
v_{r,N} = V_{r} (D_q (P_{r,N})) = 2^n V_{r} (C_{r_n} (P_{r,N}))
\]

(24)

By the definitions of the controllable and reachable regions, according to the Theorem 4, if \( r_n = n \), we have

\[
v_{c,N} = |\det(A)|^{-N} v_{c,N}
\]

(25)

The volume \( v_{c,N} \) is continuous varying and its size reflect the control ability of the input variables. The volume \( v_{c,N} \) associated the dimension \( r_n \) will reflect sufficiently the control ability and efficiency as the following analysis.

(1) The bigger the dimension \( r_n \) is, the bigger the volume \( v_{c,N} \) of the \( r_n \)-D controllable region \( R_{c,N} \) are.

(2) The bigger the volume \( v_{c,N} \) is, the more the controllable states in the state space are, i.e., the better the state stabilizing ability of the systems is. Otherwise, the smaller the volume is, the less controllable states are and the worse the state stabilizing ability is.

(3) That the volume of the controllable region is bigger will lead to that designing the input sequence \( u_{0,N-1} \) and the state trajectories for the given control problems are with more choices, and then, will lead to the closed-loop control systems can be designed with the better performance on the wasting time, the wasting energy, etc. So that, the better time-optimal, energy-optimal, or other optimal control systems can be got easily.
About that, we have the following theorem.

**Theorem 5.** For the SLDTI system $\Sigma(A(\alpha), B(\alpha))$ with parameter $\alpha \in \Lambda$, if the controllable region satisfies
\[ R_{c,i}(\alpha) \subset R_{c,i}(\alpha'), \quad \forall \alpha, \alpha' \in \Lambda; \alpha < \alpha'; i = 1,2,\ldots \]  
(26)
\[
\lim_{i \to \infty} R_{c,i}(\alpha) = R_c, \quad \forall \alpha \in \Lambda
\]  
(27)
the bigger the controllable region $R_{c,N}(\alpha)$ with the varying parameter $\alpha$ is, the faster the state response speeds of the systems are and the bigger the solution spaces of the input sequence for the given initial state $x_0 \in R_c$ are.

**Proof.** (1) Firstly, for any parameter $\alpha$, we have
\[ R_{c,i}(\alpha) \subset R_{c,i+1}(\alpha), \quad i = 1,2,\ldots \]  
(28)
So, for any initial state $x_0 \in R_c$, there must exist the number $N$, $\alpha_1$ and $\alpha_2$ satisfied
\[ x_0 \in R_{c,N}(\alpha_1) \text{ and } x_0 \notin R_{c,N+1}(\alpha_1) \]  
(29)
\[ x_0 \in R_{c,N}(\alpha_2), \quad \alpha_2 > \alpha_1 \]  
(30)
Therefore, for controlling the given $x_0$ to the origin of the state space, the sampling steps for systems $\Sigma(A(\alpha_1), B(\alpha))$ must be $N$, but the sampling steps for systems $\Sigma(A(\alpha_2), B(\alpha_2))$ can be $N-1$. So, the state response speed of the systems with $\alpha_2$ is faster than that of systems with $\alpha_1$, i.e., the bigger the controllable region $R_{c,N}(\alpha)$ with the varying parameter $\alpha$ is, the faster the state response speed is.

(2) Denoting the solution space of input sequence $u_{0,N-1}(\alpha, x_0)$ controlling the above $x_0$ for systems with $\alpha$ as $U_{N-1}(\alpha, x_0)$. As above analysis, the input sequence controlling the above $x_0$ is at least $N$ steps $u_{0,N-1}(\alpha_1)$ for systems with $\alpha_1$, but $(N-1)$ steps $u_{0,N-2}(\alpha_2)$ for systems with $\alpha_2$. Therefore, we have
\[
\dim U_{N-2}(\alpha_1, x_0) = 0
\]
\[
\dim U_{N-1}(\alpha_1, x_0) \in [1, r]
\]
\[
\dim U_{N-2}(\alpha_2, x_0) = [1, r]
\]
\[
\dim U_{N-1}(\alpha_2, x_0) = \dim U_{N-2}(\alpha_2, x_0) + r
\]
(i.e.,
\[
\dim U_{0,N-1}(\alpha_2, x_0) > \dim U_{0,N-1}(\alpha_1, x_0)
\]
Hence, the bigger the controllable region $R_{c,N}(\alpha)$ with the varying parameter $\alpha$ is, the bigger the solution space of the input sequence for the given initial state $x_0 \in R_c$ is.

3.3. The Controllable Abundances

By **Theorem 5**, we know, the sizes of the dimension and volume of the controllable region are not only related to the control ability (i.e., the amount of the controllable states), but also related to the control efficiency (the wasting control time) and the diversity of the control laws and the expected state trajectories for stabilizing the initial state (i.e., the size of the solution space of the input sequence). Therefore, to meter accurately the control ability and efficiency for the SLDTI systems, a measure on the dimension and the size of the controllable regions is introduced as follows.

**Definition 3.** For the SLDTI systems, the two-tuples $(r_N, v_{c,N})$ consisted of the dimension $r_N$ and the volume $v_{c,N}$ of the $N$-steps controllable regions are defined as a measure metering the control ability and efficiency, named as the controllable abundance.

Similar to the above definition of the controllable abundance, the definition of the reachable abundance can be got.

In **Theorem 5**, Eq. (26) requires that the controllable region $R_{c,N}$ is monotonically extending outer with the varying parameter. The following discussions can generalize the theorem conclusion.

1) In fact, if Eq. (26) is not always true but
\[ v_{c,i}(\alpha_1) < v_{c,i}(\alpha_2), \quad \forall \alpha_1, \alpha_2 \in \Lambda; \alpha_1 < \alpha_2; i = 1,2,\ldots \]  
(31)
the bigger the volume of the controllable region with the parameter $\alpha$ is, the more the controllable states are, and maybe the bigger the size of the solution space of the input sequence are. So that, maximizing the volume $v_{c,N}$, i.e., the controllable abundance, is helpful with improving the control ability and efficiency.

2) If the controllable region $R_{c,i}(\alpha)$ doesn’t change with parameter $\alpha (\alpha \in \Lambda)$ monotonically, the parameter space $\Lambda$ must be regarded as a union of several sub-space $\Lambda_i$ where the region $R_{c,i}(\alpha)$ changes with parameter $\alpha (\alpha \in \Lambda_i)$ monotonically. If so, the optimizing problem of the controllable region on the parameter space $\Lambda$ is a local optimizing problem.

By **Theorem 5** and above discussions, we know, the bigger the controllable abundance is, the bigger of the solution space of the input sequence, and then the more abundance the state trajectories are and the more the choices designing the expected state trajectories for stabilizing the initial state are. The size of the controllable region reflects the control ability and efficiency, and reflects the diversity choosing the control laws and the state trajectories. Therefore, optimizing the controllable region will lead to optimizing the control ability and efficiency, and optimizing the diversity choosing the control laws and the state trajectories.
trajectories. The volume \( v_{c,N} \) is an appropriate index measuring the size of the controllable region \( R_{c,N} \) and then maximizing the volume \( v_{c,N} \) will result in optimizing the controllable region, i.e., optimizing the controllable abundance.

4. Optimization of Controllable Abundance

4.1. Control problems and Controllable Abundance

The control problems discussed in convention control theory can be classified as the following three kinds of the basic control problems.

1) Stabilizing control problems. This kind of the basic control problems is stabilizing the states in the neighborhood of some equilibrium state to the equilibrium state. The conventional control theory was mainly developed on that, and the stability analysis and synthesis methods are prepared and used widely in all aspects of the control field. The state controllability is a good concept and tools for this kind of the problems.

2) Reaching control problems. This kind of the basic control problems are transferring the system state from the initial states to the given goal states, or transferring along the given expected state trajectories. It is worth noting that this kind of problems is only on how to reach and its focus is not on the stabilizing on the goal state. For example, the temperature of the steel ingot is heated uniformly up to the expected temperature and is transferred to the rolling mill immediately without keeping the temperature long time. The state reachability is a good concept and tools for the analysis and synthesis for the problems.

3) Synthetical problems of reaching and stabilizing control. Some control problems in practical control engineering can be regarded as the combination of the two above kinds of problems and then the state controllability and reachability are needed to analyzing and solving the problems.

For the above three kinds of control problems, the requires on the control ability and efficiency will lead to the differential dynamics and kinematics properties of the controlled plants, and will lead to the difference designing and manufacturing of the controlled plants.

4.2. Determining of Input Variables

As discussed above, the controllable abundances determine the control ability and efficiency in some extent, and can be used for choosing the input variables from the possible input variable with the input power and the executing devices, such as, choosing the voltage input of the excitation or the main circuit as the input variable for D-C motor, placing the fuel nozzles in the heating furnaces, etc.

Consider the following SLDTI systems with multiply input-variable sets.

\[
x_{k+i} = A_i x_i + B_i u_{i}^{(i)} \quad x_i \in \mathbb{R}^n, u_{i}^{(i)} \in [-1,1]^s, i = 1, \ldots, s
\]  

where \( u_{i}^{(i)} \) is the \( i \)-th set of the possible input variables, \( A_i \) and \( B_i \) are respectively the corresponding system matrix and input matrix, number \( s \) is the set number. The determining problem is to choose the input variable set from the possible sets based on the computation of the \( N \)-steps controllable and reachable abundances. The determining method is stated as follows.

Firstly, defining and computing the \( N \)-steps controllable and reachable abundances for all possible input-variable sets as follows

\[
\left( v_{c,N}^{(i)}, v_{r,N}^{(i)} (\alpha) \right) = \left( v_{c,N}^{(i)}, \alpha v_{c,N}^{(i)} + (1-\alpha) v_{r,N}^{(i)} \right) \quad i = 1, s
\]

where \( N \) is the given time length for investigating the control ability and effort of the systems, \( \alpha \in [0,1] \) is the weighting coefficients.

\[
r_{c,N}^{(i)} = \text{rank } P_{c,N}^{(i)} = \text{rank } P_{c,N}^{(i)}
\]

\[
V_{c,N}^{(i)} = V_{c,N}^{(i)} (D_{c,N} (P_{c,N}^{(i)})) = 2^{-N} V_{c,N}^{(i)} (C_{c,N} (P_{c,N}^{(i)}))
\]

\[
P_{c,N}^{(i)} = \begin{bmatrix} A_i^{-N} B_i & A_i^{-N+1} B_i & \cdots & A_i^{-1} B_i \end{bmatrix}
\]

Similarly, \( v_{r,N}^{(i)} \) and \( P_{r,N}^{(i)} \) can be defined.

When \( \alpha = 1 \) and \( \alpha = 0 \), the optimizing problem is suitable for the stabilizing control problems and reaching control problems, respectively. When \( \alpha \in (0,1) \), solving the optimizing problem is for finding the take-off between the controllable and reachable abundances for the 3-rd kind of problems.

Secondly, choosing the input variable set with the greatest \( N \)-steps controllable and reachable abundances and the corresponding input variable set, i.e.,

\[
r_{c,N}^{\max} = \max_{i} r_{c,N}^{(i)}, i = 1, s
\]

\[
\beta_{\text{max}} = \arg \max_{i} \left| \left. r_{c,N}^{(i)} (\alpha) \right|_{\alpha = r_{c,N}^{\max}}, i = 1, s \right|
\]

In general, for the more sufficient study on the control ability and efficiency, the sampling-step number \( N \) considered in determine the input variables is selected as \( N \geq n \), i.e., the control ability and efficiency of the input variables after \( n \) sampling steps will be discussed. Therefore, based on the conventional control theory, for the systems with the state controllable systems, \( r_{c,N}^{\max} = n \).

Solving the above optimization problem is made on a discrete number of \( \beta (i = 1, s) \) based on the volume computing of parallel polyhedrons, and the optimal solution is can be easily by using some simple comparison operations.
4.3. Optimization Problems of Dynamics and kinematics of controlled plants

Consider the SLDTI systems with some technic parameters to be determined in the designing and manufacturing the controlled plants as follows.

\[ x_{k+1} = A(\beta)x_k + B(\beta)u_k \quad x_k \in \mathbb{R}^n, u_k \in [-1, 1]^r \]  \hspace{1cm} (39)

where \( \beta \in \Phi \) is the technic parameter set to be determined and \( \Phi \) is the corresponding parameter space. The parameter determining problem for enhancing the control ability and efficiency is equal to solve the maximum problem of the \( N \)-steps controllable and reachable abundances for the given sample step \( N \). The determining method is stated as follows.

Firstly, solving the maximum rank \( r_{\text{max}} \) of the controllability matrix for the parameter space \( \Phi \) as follows

\[
\begin{align*}
    r_{\text{max}} = & \max_{\beta \in \Phi} \text{rank} \left( P_{r,N}(\beta) \right) \\
    = & \max_{\beta \in \Phi} \text{rank} \left( A^{-N} B \ A^{-N+1} \cdots A^{-1} B \right) \beta
\end{align*}
\]  \hspace{1cm} (40)

Secondly, solving the following optimizing problems

\[
\max_{\beta \in \Phi} \alpha(\beta, \beta) = \alpha V_{r,N}(\beta) + (1 - \alpha) v_{r,N}(\beta)
\]  \hspace{1cm} (41)

where the parameters \( \alpha \) and \( N \) are chosen as in the last sub-section;

\[
v_{r,N}(\beta) = V_{x_N}(D(\beta))
\]

similarly, \( P_{r,N}(\beta) \) and \( v_{r,N}(\beta) \) can be defined as \( P_{r,N}(\beta) \) and \( v_{r,N}(\beta) \).

Because the optimizing problem (41) is on the continuous parameter space \( \Phi \) and the solving the partial derivatives vector of the goal function to the variables \( \beta \) is very difficult, the derivative free optimization methods are needed to solve these optimizing problems, such as Powell method.

In fact, the controllable and reachable abundances are depend on the poles of the systems and matrix \( B \). For example, if the matrix \( A \) of the SISO controllable system can be diagonalized by the state space transformation \( x = T_d \tilde{x} \), i.e.,

\[
\begin{align*}
    A &= T_d^{-1} A T_d = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \\
    \tilde{B} &= T_d^{-1} B = \left[ \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n \right]^T
\end{align*}
\]

where \( \lambda_i (i = 1, n) \) is the poles of the systems, matrix \( T_d \) is composed of all right eigenvectors of matrix \( A \), i.e., matrix \( T_d^{-1} \) is composed of all left eigenvectors of matrix \( A \), where the norms of all right and left eigenvectors are normalized to 1. By the row transformation, we have

\[
\begin{align*}
    \det \begin{bmatrix}
        \lambda_1^{-1} & \lambda_2^{-1} & \cdots & \lambda_n^{-1} \\
        \lambda_2^{-1} & \lambda_3^{-1} & \cdots & \lambda_n^{-1} \\
        \vdots & \vdots & \ddots & \vdots \\
        \lambda_n^{-1} & \lambda_1^{-1} & \cdots & \lambda_{n-1}^{-1}
    \end{bmatrix} &= f_{i_1, i_2, \ldots, i_n} \prod_{1 \leq j < s_n} (\lambda_j - \lambda_i) \quad (i_1, i_2, \ldots, i_n) \subset \Omega_N^n
\end{align*}
\]  \hspace{1cm} (42)

where \( f_{i_1, i_2, \ldots, i_n} \) is a function about \( \lambda(i = 1, n) \). And then, by \textbf{Theorem 3} and Eq. (24), the reachable abundances is

\[
v_{r,N}(\beta) = f_{N} \prod_{1 \leq j < s_n} (\lambda_j - \lambda_i) \quad (i_1, i_2, \ldots, i_n) \subset \Omega_N^n
\]  \hspace{1cm} (43)

where \( f_N \) is \( \det(T_d^{-1}) \) for \( N = n \) and is a positive multivariate polynomials about \( \lambda(i = 1, n) \) for \( N > n \). Therefore, maximizing \( v_{r,N} \) is depend on the enhancing of three factors, such as, the input matrix factor \( \tilde{b}_j \), the size and the distribution of the poles \( \lambda_i \). These roles of three factors on the optimization problem are analyzed as follows.

1) Enhancing \( f_N \) and \( \prod(\lambda_j - \lambda_i) \) will lead to enlarging the size of the poles. In general, the bigger \( |\lambda_i| \) is, the more activity the discrete-time systems are, i.e., the less stable the systems are. Therefore, for the optimizing problem of the reachable abundances for the practical systems, the size of the poles \( \lambda_i \) must be subject to limitations.

2) In fact, enlarging \( \prod(\lambda_j - \lambda_i) \) will make that the distribution of the poles \( \lambda_i \) is more uniformity under the limitation of the size of \( \lambda_i \). And then, the characteristic sub-systems of the optimized systems will be with the more distinct dynamics and the state of each sub-system will be controlled easily, i.e., the state reachability will be enhanced.

3) Considered that \( \tilde{b}_i = [T_d^{-1}]_{\cdot \cdot i_{\cdot \cdot}} B \) and \( [T_d^{-1}]_{\cdot \cdot j_{\cdot \cdot}} \) is the left eigenvector corresponding to the pole \( \lambda_j \), optimizing \( \tilde{b}_j \) will make that the size of the elements of vector \( B \) and the correlation between the vector \( B \) and the left eigenvectors of matrix \( A \) are enhanced. The stronger the correlations are, the bigger the reachable region in the characteristic sub-systems is, and then, the better the state reachability is.

Hence, the optimization problem (41) for the reachable abundances for the reaching control problem can be redefined as

\[
\begin{align*}
    \max_{\beta \in \Phi} v_{r,N} \\
    \text{s.t. } \max_{i = 1, n} |\lambda_i| \leq \lambda_{\text{max}}
\end{align*}
\]  \hspace{1cm} (44)

where \( \lambda_{\text{max}} \) is the expected maximum modulus of poles.
Similar to the reachable abundances, the controllable abundances for the diagonalized systems is

\[ v_{r,N} = f_N \prod_{i < j \leq n} (\lambda_i - \lambda_j) \times \prod_{i=1}^{n} \tilde{b}_i \]

where \( f_N \) is \( \det(T_N) \), for \( N = n \) and is a positive multivariate polynomials about \( |\lambda_i|^n (i = 1, n) \) for \( N > n \). Similar to the above analysis for \( v_{r,N} \), maximizing \( v_{c,N} \) will lead to optimizing the input matrix factor \( \tilde{h}_i \), the size and the distribution of the poles. In general, the smaller \( |\lambda_i| \) is, the more stable the discrete-time systems are, i.e., the less activity the systems are. For the optimizing problem of the controllable abundances, the size of the poles \( \lambda_i \) must be subject to limitations. Hence, the optimization problem (41) for the controllable abundances for the stabilizing control problem can be redefined as

\[
\max_{\beta \in \Phi} v_{c,N} \\
\text{st. \ min} |\lambda_i|_{i=1,n} \geq \lambda_{\min}
\]

where \( \lambda_{\min} \) is the expected minimum modulus of poles.

According to the definitions of two kinds of optimization problems, the trade-off problem (41) of the control ability and efficiency between the stabilizing control problem and reaching control problem can be defined as

\[
\max_{\beta \in \Phi} v_N = \alpha v_{r,N} + (1 - \alpha)v_{c,N}
\]

\[
\text{st. \ max} |\lambda_i|_{i=1,n} \leq \lambda_{\max}
\]

\[
\min |\lambda_i|_{i=1,n} \geq \lambda_{\min}
\]

where

\[ v_N = f_N \prod_{i < j \leq n} (\lambda_i - \lambda_j) \times \prod_{i=1}^{n} \tilde{b}_i \]

\[ \tilde{f}_N = \alpha f_N + (1 - \alpha)f_N \]

The above three optimizing problems with the constraint of the pole size will be mainly on optimizing the distribution of the poles \( \lambda_i \), the size of the elements of the input matrix \( B \), and the correlation between the elements of matrix \( B \) and the left eigenvectors of matrix \( A \), and the differences among the optimizing problems are that the distribution regions of the poles are the inside of circle with radius \( \lambda_{\max} \), the outside of circle with radius \( \lambda_{\min} \), and the inside of ring with radius \( \lambda_{\max} \) and \( \lambda_{\min} \), respectively.

5. Numerical experiments

In this section, three numerical experiments for actuator placement, actuator configuration, and parameter optimization of the plant dynamics, respectively, are made and the numerical results analyzed as follows.

Example 1 (Placing problem of the gas nozzles in the Ingot heating furnace) It is assumed that the ingot heating problem with two schemes placing nozzles can be regarded as a distributing and heating problem in the 2-D plane shown in Fig. 3. Which scheme is more suitable for the ingot heating problem with the greatest ability and efficiency of the fuel nozzles?

| \( f_1 \) | nozzle 1 | nozzle 2 | \( f_2 \) |
| --- | --- | --- | --- |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 12 | 11 | 10 | 9 | 8 | 7 |

\( f_1 \) is the nozzle 1

(a) Scheme 1

| \( f_1 \) | nozzle 1 | nozzle 2 | \( f_2 \) |
| --- | --- | --- | --- |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 12 | 11 | 10 | 9 | 8 | 7 |

\( f_2 \) is the nozzle 3

(b) Scheme 2

Fig. 3 The schemes placing the nozzles

By discretizing the time and the 2-D plane (see Lu, Y.Z. & Williams1984), considered the heat exchange among the elements and the heat dissipate of the elements, the system models for two schemes are modeled as follows

\[ x_{k+1} = Ax_k + Bu_k \]

where

\[ A = [b_{ij}] \]

\[ B_1, B_2 = [b_{ij}] \]

\[ a_{ij} = 0.95 - 0.03g_{ik} - 0.04g_{ik} + \sum_{j=1}^{n} (0.15g_{ij} + 0.18g_{ij}) \]

\[ a_{ij} = 0.15g_{ij} + 0.18g_{ij} \quad i \neq j \]

\[ b_{ij} = \begin{cases} f & \text{if the unit is nozzle } s \\ 0 & \text{otherwise} \end{cases} \]

For elements \( a_{ij} \) and \( b_{ij} \), if the \( i \)-th unit is on the outside in \( X/Y \) direction or not, \( g_{ij}^x / g_{ij}^y \) is 1 or 0, and if the \( i \)-th and \( j \)-th units are adjacent in \( X/Y \) direction or not, \( g_{ij}^x / g_{ij}^y \) is 1 or 0.
Considered the temperatures of the ingot is up to the expected temperatures and then is transferred to the rolling mill immediately, the heating problem can be regarded as a reaching control problem. The better of the two schemes can be determined on computing of the reachable abundances.

The computing results of the reachable abundances as Table 1 and then the scheme 1 is with the better reachable abundance, i.e., the better reach ability and efficiency.

Example 2. If the three nozzles can be configured with different fuel maximum-flows in the Scheme 1 in Fig 3, what is the best configuration of the fuel maximum-flows for the ingot heating problems.

For the different fuel maximum-flow of the nozzles, the three non-zero elements in the matrix $B$ can be described as 

$$f_1, f_2, 30 - f_1 - f_2 \in [0, 30]$$

When the sampling steps $N$ is chosen as 4 to 9 for investigating the reachable abundance, the optimizing results by matlab function "fmincon" are shown in Table 2.

Table 2. optimal configuration of the fuel maximum-flow 

| steps (N) | $f_1$ | $f_2$ | $f_3$ | $v_{r,N}$ |
|-----------|-------|-------|-------|-----------|
| 4         | 10.00 | 10.00 | 10.00 | 0.1064    |
| 5         | 9.712 | 10.27 | 10.02 | 11.43     |
| 6         | 9.591 | 10.37 | 10.04 | 179.5     |
| 7         | 9.515 | 10.43 | 10.06 | 1126      |
| 8         | 9.463 | 10.47 | 10.06 | 4120      |
| 9         | 9.425 | 10.51 | 10.06 | 10663     |

Comparing Table 2 with Table 1, we can see, the reachable abundance $v_{r,N}$ can be improved by optimizing the configuration of the fuel maximum-flow.

Example 3 Some mechanical vibrating system can be modeled as follows

$$x_{k+1} = \begin{bmatrix} 1 & S_T & 0 \\ -kS_T & 1-fS_T & u_k \end{bmatrix} x_k + \begin{bmatrix} 0 \\ S_T \end{bmatrix} u_k \quad k \in [-1,1]$$  

(53)

where $S_T = 0.05$ is the sample step length, $k \in [0,1,0.3]$ and $f \in [1.2,3.0]$ are parameters to be determine based on the controllable and reachable abundance. For the system (53), the following 3 cases are computed and analysis.

1) Stabilizing control problem (44) with $\lambda_{\text{max}} = 0.99$.
2) Reaching control problem (47) with $\lambda_{\text{min}} = 0.9$.
3) trade-off control problem (49) with $\alpha = 0.5$, $\lambda_{\text{max}} = 0.99$ and $\lambda_{\text{min}} = 0.9$.

The computational results by matlab function "fmincon" are shown in Table 3, where $\lambda(A)$ is the pole of the system and $r_{ab}$ is the relativity between the right eigenvector $E_j(A)$ of matrix $A$ and the columns of matrix $B$ as follows

$$r_{ab}(i,j) = \frac{||E_j(A)B_{i-th column}||}{||E_j(A)|| \cdot ||B_{j-th column}||}$$

Table 3. Optimized results of Example 3

| Case | $N$ | $v_{r,N}$ | $f$ | $\lambda(A)$ | $r_{ab}$ |
|------|----|----------|----|-------------|---------|
| 1    | 2  | 5.00E-04 | 0.3000 | 1.5579      | 0.9887  | 0.6061 |
|      | 20 | 0.4098   | 0.1800 | 1.1000      | 0.9900  | 0.7433 |
|      |    |          |       |             | 0.9550  | 0.9806 |
| 2    | 2  | 6.27E-04 | 0.3000 | 2.1500      | 0.9925  | 0.4472 |
|      | 20 | 2.5661   | 0.3000 | 2.1500      | 0.9900  | 0.9889 |
| 3    | 2  | 5.48E-04 | 0.3000 | 1.8700      | 0.9900  | 0.5547 |
|      | 20 | 1.0897   | 0.3000 | 1.8700      | 0.9250  | 0.9806 |

From Table 3, we can see, by the optimizations, each pole the open-loop controlled plant is with remarkable distinct sub-space dynamics, include the pole position and the relativity between the right eigenvector $E_j(A)$ and the matrix $B$, and then the control ability and efficiency are improved.

6. Conclusions

We gave a clear understanding of the control ability and efficiency of the input variable of the SLDTI based on the volume computing of the controllable and reachable regions, and then a new concept about that, named as the controllable and reachable abundance, proposed firstly and is applied to some interesting control problems. The simpler and rapid computing methods on these region volumes will be made further and the new concept will be applied widely to more control problems, such as optimal control, self-tuning control, predictive control, and receding-horizon control, etc...
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