CHOW-KÜNNETH DECOMPOSITION FOR UNIVERSAL FAMILIES OVER PICARD MODULAR SURFACES

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Dedicated to Jaap Murre

Abstract. We prove existence results for Chow–Künneth projectors on compactified universal families of Abelian threefolds with complex multiplication over a particular Picard modular surface studied by Holzapfel. Our method builds up on the approach of Gordon, Hanamura and Murre in the case of Hilbert modular varieties. In addition we use relatively complete models in the sense of Mumford, Faltings and Chai and prove vanishing results for $L^2$–Higgs cohomology groups of certain arithmetic subgroups in $SU(2,1)$ which are not cocompact.

1. INTRODUCTION

In this paper we discuss conditions for the existence of absolute Chow–Künneth decompositions for families over Picard modular surfaces and prove some partial existence results. In this way we show how the methods of Gordon, Hanamura and Murre [12] can be slightly extended to some cases but fail in some other interesting cases. Let us first introduce the circle of ideas which are behind Chow–Künneth decompositions. For a general reference we would like to encourage the reader to look into [26] which gives a beautiful introduction to the subject and explains all notions we are using.

Let $Y$ be a smooth, projective $k$–variety of dimension $d$ and $H^*$ a Weil cohomology theory. In this paper we will mainly be concerned with the case $k = \mathbb{C}$, where we choose singular cohomology with rational coefficients as Weil cohomology. Grothendieck’s Standard Conjecture $C$ asserts that the Künneth components of the diagonal $\Delta \subset Y \times Y$ in the cohomology $H^{2d}(Y \times Y, \mathbb{Q})$ are algebraic, i.e., cohomology classes of algebraic cycles. In the case $k = \mathbb{C}$ this follows from the Hodge conjecture. Since $\Delta$ is an element in the ring of correspondences, it is natural to ask whether these algebraic classes come from algebraic cycles $\pi_j$ which form a complete set of orthogonal idempotents

$$\Delta = \pi_0 + \pi_1 + \ldots + \pi_{2d} \in CH^d(Y \times Y)_\mathbb{Q}$$
summing up to $\Delta$. Such a decomposition is called a Chow–Künneth decomposition and it is conjectured to exist for every smooth, projective variety. One may view $\pi_j$ as a Chow motive representing the projection onto the $j$–the cohomology group in a universal way. There is also a corresponding notion for $k$–varieties which are relatively smooth over a base scheme $S$. See section 3, where also Murre’s refinement of this conjecture with regard to the Bloch–Beilinson filtration is discussed. Chow–Künneth decompositions for abelian varieties were first constructed by Shermenev in 1974. Fourier–Mukai transforms may be effectively used to write down the projectors, see [18, 26]. The cases of surfaces was treated by Murre [27], in particular he gave a general method to construct the projectors $\pi_1$ and $\pi_{2d-1}$, the so–called Picard and Albanese Motives. Aside from other special classes of 3–folds [1] not much evidence is known except for some classes of modular varieties. A fairly general method was introduced and exploited recently by Gordon, Hanamura and Murre, see [12], building up on previous work by Scholl and their own. It can be applied in the case where one has a modular parameter space $X$ together with a universal family $f : A \to X$ of abelian varieties with possibly some additional structure. Examples are given by elliptic and Hilbert modular varieties. The goal of this paper was to extend the range of examples to the case of Picard modular surfaces, which are uniformized by a ball, instead of a product of upper half planes. Let us now describe the general strategy of Gordon, Hanamura and Murre so that we can understand to what extent this approach differs and eventually fails for a general Picard modular surface with sufficiently high level structure.

Let us assume that we have a family $f : A \to X$ of abelian varieties over $X$. Since all fibers are abelian, we obtain a relative Chow–Künneth decomposition over $X$ in the sense of Deninger/Murre [6], i.e., algebraic cycles $\Pi_j$ in $A \times_X A$ which sum up to $\Delta \times_X \Delta$. One may view $\Pi_j$ as a projector related to $R^jf_*\mathbb{C}$. Now let $\overline{f} : \overline{A} \to \overline{X}$ be a compactification of the family. We will use the language of perverse sheaves from [8] in particular also the notion of a stratified map. In [11] Gordon, Hanamura and Murre have introduced the Motivic Decomposition Conjecture:

Conjecture 1.1. Let $\overline{A}$ and $\overline{X}$ be quasi–projective varieties over $\mathbb{C}$, $\overline{A}$ smooth, and $\overline{f} : \overline{A} \to \overline{X}$ a projective map. Let $\overline{X} = X_0 \supset X_1 \supset \ldots \supset X_{\dim(X)}$ be a stratification of $\overline{X}$ so that $\overline{f}$ is a stratified map. Then there are local systems $\mathcal{V}_\alpha^j$ on $X_\alpha = X_\alpha \setminus X_{\alpha-1}$, a complete set $\Pi_\alpha^j$ of orthogonal projectors and isomorphisms

$$\sum_{j, \alpha} \Psi^j_\alpha : R^j f_* \mathbb{Q} \xrightarrow{\cong} \bigoplus_{j, \alpha} IC_{X_\alpha} (\mathcal{V}_\alpha^j)[-j - \dim(X_\alpha)]$$

in the derived category.

This conjecture asserts of course more than a relative Chow–Künneth decomposition for the smooth part $f$ of the morphism $\overline{f}$. Due to the complicated structure of the strata in general its proof in general needs some
more information about the geometry of the stratified morphism $f$. In the course of their proof of the Chow–K"unneth decomposition for Hilbert modular varieties, see [12], Gordon, Hanamura and Murre have proved the motivic decomposition conjecture in the case of toroidal compactifications for the corresponding universal families. However to complete their argument they need the vanishing theorem of Matsushima–Shimura [21]. This theorem together with the decomposition theorem [3] implies that each relative projector $\Pi_j$ on the generic stratum $X_0$ only contributes to one cohomology group of $A$ and therefore, using further reasoning on boundary strata $X_\alpha$, relative projectors for the family $f$ already induce absolute projectors.

The plan of this paper is to extend this method to the situation of Picard modular surfaces. These were invented by Picard in his study of the family of curves (called Picard curves) with the affine equation

$$y^3 = x(x - 1)(x - s)(x - t).$$

The Jacobians of such curves of genus 3 have some additional $CM$–structure arising from the $\mathbb{Z}/3\mathbb{Z}$ deck transformation group. Picard modular surfaces are compactifications of two dimensional ball quotients $X = \mathbb{B}_2/\Gamma$ which parametrize such Jacobians and form a particular beautiful set of Shimura surfaces in the moduli space of abelian varieties of dimension 3. Many examples are known through the work of Holzapfel [15, 16]. Unfortunately the generalization of the vanishing theorem of Matsushina and Shimura does not hold for Picard modular surfaces and their compactifications. The reason is that $\mathbb{B}_2$ is a homogenous space for the Lie group $G = SU(2,1)$ and general vanishing theorems like Ragunathan’s theorem [4, pg. 225] do not hold. If $\mathbb{V}$ is an irreducible, non–trivial representation of any arithmetic subgroup $\Gamma$ of $G$, then the intersection cohomology group $H^1(X, \mathbb{V})$ is frequently non–zero, whereas in order to make the method of Gordon, Hanamura and Murre work, we would need its vanishing. This happens frequently for small $\Gamma$, i.e., high level. However if $\Gamma$ is sufficiently big, i.e., the level is small, we can sometimes expect some vanishing theorems to hold. This is the main reason why we concentrate our investigations on one particular example of a Picard modular surface $\overline{X}$ in section 4. The necessary vanishing theorems are proved by using Higgs bundles and their $L^2$–cohomology in section 6. Such techniques provide a new method to compute intersection cohomology in cases where the geometry is known. This methods uses a recent proof of the Simpson correspondence in the non–compact case by Jost, Yang and Zuo [17, Thm. A/B]. But even in the case of our chosen surface $\overline{X}$ we are not able to show the complete vanishing result which would be necessary to proceed with the argument of Gordon, Hanamura and Murre. We are however able to prove the existence of a partial set $\pi_0, \pi_1, \pi_2, \pi_3, \pi_7, \pi_8, \pi_9, \pi_{10}$ of orthogonal idempotents under the assumption of the motivic decomposition conjecture 1.1 on the universal family $\overline{A}$ over $\overline{X}$:
Theorem 1.2. Assume the motivic decomposition conjecture \( \mathcal{L} \) for \( \mathcal{A} \to \mathcal{X} \). Then \( \mathcal{A} \) supports a partial set of Chow–Künneth projectors \( \pi_i \) for \( i \neq 4, 5, 6 \).

Unfortunately we cannot prove the existence of the projectors \( \pi_4, \pi_5, \pi_6 \) due to the non–vanishing of a certain \( \mathbb{L}^2 \)–cohomology group, in our case \( H^1(X, S^2V_1) \), where \( V_1 \) is (half of) the standard representation. This is special to \( SU(2, 1) \) and therefore the proposed method has no chance to go through for other examples involving ball quotients. If \( H^1(X, S^2V_1) \) would vanish or consist out of algebraic Hodge \( (2, 2) \)–classes only, then we would obtain a complete Chow–Künneth decomposition. This is an interesting open question and follows from the Hodge conjecture, if all classes in \( H^1(X, S^2V_1) \) would have Hodge type \( (2, 2) \). We also sketch how to prove the motivic decomposition conjecture in this particular case, see section 7.2., however details will be published elsewhere. This idea generalizes the method from \([12]\), since the fibers over boundary points are not anymore toric varieties, but toric bundles over elliptic curves. We plan to publish the full details in a forthcoming publication and prefer to assume the motivic decomposition conjecture \( \mathcal{L} \) in this paper.

The logical structure of this paper is as follows:
In section 2 we present notations, definitions and known results concerning Picard Modular surfaces and the universal Abelian schemes above them. Section 3 first gives a short introduction to Chow Motives and the Murre Conjectures and then proceeds to our case in paragraph 3.2. The remainder of the paper will then be devoted to the proof of Theorem 1.2.

In section 4 we give a description of toroidal degenerations of families of Abelian threefolds with complex multiplication. In section 5 we describe the geometry of a class of Picard modular surfaces which have been studied by Holzapfel. In section 6 we prove vanishing results for intersection cohomology using the non–compact Simpson type correspondence between the \( \mathbb{L}^2 \)–Higgs cohomology of the underlying VHS and the \( \mathbb{L}^2 \)–de Rham cohomology resp. intersection cohomology of local systems. In section 7 everything is put together to prove the main theorem 1.2. The appendix (section 8) gives an explicit description of the \( \mathbb{L}^2 \)–Higgs complexes needed for the vanishing results of section 6.

2. The Picard modular surface

In this section we are going to introduce the (non–compact) Picard modular surfaces \( X = X_{\Gamma} \) and the universal abelian scheme \( \mathcal{A} \) of fibre dimension 3 over \( X \). For proofs and further references we refer to [9].

Let \( E \) be an imaginary quadratic field with ring of integers \( \mathcal{O}_E \). The Picard modular group is defined as follows. Let \( V \) be a 3-dimensional \( E \)–vector space and \( L \subset V \) be an \( \mathcal{O}_E \)–lattice. Let \( J : V \times V \to E \) be a nondegenerate Hermitian form of signature \( (2, 1) \) which takes values in \( \mathcal{O}_E \) if it is restricted to \( L \times L \). Now let \( G' = SU(J, V)/\mathbb{Q} \) be the special unitary group of \( (V, \phi) \).
This is a semisimple algebraic group over \( \mathbb{Q} \) and for any \( \mathbb{Q} \)-algebra \( A \) its group of \( A \)-rational points is

\[
G'(A) = \{ g \in \text{SL}(V \otimes_\mathbb{Q} A) \mid J(gu, gv) = J(u, v), \text{for all } u, v \in V \otimes_\mathbb{Q} A \}.
\]

In particular one has \( G'(\mathbb{R}) \simeq \text{SU}(2, 1) \). The symmetric domain \( \mathcal{H} \) associated to \( G'(\mathbb{R}) \) can be identified with the complex 2-ball as follows. Let us fix once and for all an embedding \( E \hookrightarrow \mathbb{C} \) and identify \( E \otimes_\mathbb{Q} \mathbb{R} \) with \( \mathbb{C} \). This gives \( V(\mathbb{R}) \) the structure of a 3-dimensional \( \mathbb{C} \)-vector space and one may choose a basis of \( V(\mathbb{R}) \) such that the form \( J \) is represented by the diagonal matrix \( [1, 1, -1] \). As \( \mathcal{H} \) can be identified with the (open) subset of the Grassmannian \( \text{Gr}_1(V(\mathbb{R})) \) of complex lines on which \( J \) is negative definite, one has

\[
\mathcal{H} \simeq \{(Z_1, Z_2, Z_3) \in \mathbb{C}^3 \mid |Z_1|^2 + |Z_2|^2 - |Z_3|^2 < 0\}/\mathbb{C}^*.
\]

This is contained in the subspace, where \( Z_3 \neq 0 \) and, switching to affine coordinates, can be identified with the complex 2-ball

\[
\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}.
\]

Using this description one sees that \( G'(\mathbb{R}) \) acts transitively on \( \mathbb{B} \).

The Picard modular group of \( E \) is defined to be \( G'(\mathbb{Z}) = \text{SU}(J, L) \), i.e. the elements \( g \in G'(\mathbb{Q}) \) with \( gL = L \). It is an arithmetic subgroup of \( G(\mathbb{R}) \) and acts properly discontinuously on \( \mathbb{B} \). The same holds for any commensurable subgroup \( \Gamma \subset G'(\mathbb{Z}) \), in particular if \( \Gamma \subset G'(\mathbb{Z}) \) is of finite index the quotient

\[
X_{\Gamma}(\mathbb{C}) = \mathbb{B}/\Gamma \text{ is a non-compact complex surface, the Picard modular surface. Moreover, for torsionfree } \Gamma \text{ it is smooth.}
\]

We want to describe \( X_{\Gamma}(\mathbb{C}) \) as moduli space for polarized abelian 3-folds with additional structure. For this we will give a description of \( X_{\Gamma}(\mathbb{C}) \) as the identity component of the Shimura variety \( S_K(G, \mathcal{H}) \).

Let \( G = \text{GU}(J, V)/\mathbb{Q} \) be the reductive algebraic group of unitary similitudes of \( J \), i.e. for any \( \mathbb{Q} \)-algebra \( A \)

\[
G'(A) = \{ g \in \text{GL}(V \otimes_\mathbb{Q} A) \mid \text{there exists } \mu(g) \in A^* \text{ such that } J(gu, gv) = \mu(g)J(u, v), \text{for all } u, v \in V \otimes_\mathbb{Q} A \}.
\]

As usual \( \mathbb{A} \) denotes the \( \mathbb{Q} \)-adeles and \( \mathbb{A}_f \) denotes the finite adeles. Let \( K \) be a compact open subgroup of \( G(\mathbb{A}_f) \), which is compatible with the integral structure defined by the lattice \( L \). I.e., \( K \) should be in addition a subgroup of finite index in \( G(\hat{\mathbb{Z}}) := \{ g \in G(\mathbb{A}_f) \mid g(L \otimes_\mathbb{Z} \hat{\mathbb{Z}}) = L \otimes_\mathbb{Z} \hat{\mathbb{Z}} \} \). Then one can define

\[
S_K(G, \mathcal{H})(\mathbb{C}) := G(\mathbb{Q})\backslash \mathcal{H} \times G(\mathbb{A}_f)/K.
\]

This can be decomposed as \( S_K(G, \mathcal{H})(\mathbb{C}) = \prod_{j=1}^{n_K} X_{\Gamma_j}(\mathbb{C}) \).

The variety \( S_K(G, \mathcal{H})(\mathbb{C}) \) has an interpretation as a moduli space for certain 3-dimensional abelian varieties. Recall that over \( \mathbb{C} \) an abelian variety \( A \) is determined by the following datum: a real vector space \( W(\mathbb{R}) \), a lattice \( W(\mathbb{Z}) \subset W(\mathbb{R}) \), and a complex structure \( j : \mathbb{C}^x \to \text{Aut}_\mathbb{R}(W(\mathbb{R})) \), for which there exists a nondegenerate \( \mathbb{R} \)-bilinear skew-symmetric form \( \psi : W(\mathbb{R}) \times W(\mathbb{R}) \to \mathbb{R} \) taking values in \( \mathbb{Z} \) on \( W(\mathbb{Z}) \) such that the form given
by \((w, w') \mapsto \psi(j(i)w, w')\) is symmetric and positive definite. The form \(\psi\) is called a Riemann form and two forms \(\psi_1, \psi_2\) are called equivalent if there exist \(n_1, n_2 \in \mathbb{N}_{>0}\) such that \(n_1 \psi_1 = n_2 \psi_2\). An equivalence class of Riemann forms is called a homogeneous polarization of \(A\).

An endomorphism of a complex abelian variety is an element of \(\text{End}(A)\) preserving \(W(\mathbb{Z})\) and commuting with \(j(z)\) for all \(z \in \mathbb{C}^\times\). A homogenously polarized abelian variety \((W(\mathbb{R}), W(\mathbb{Z}), j, \psi)\) is said to have complex multiplication by an order \(\mathcal{O}\) of \(E\) if and only if there is a homomorphism \(m : \mathcal{O} \rightarrow \text{End}(A)\) such that \(m(1) = 1\), and which is compatible with \(\psi\), i.e. \(\psi(m(\alpha)w, w') = \psi(w, m(\alpha)w')\) where \(\rho\) is the Galois automorphism of \(E\) induced by complex conjugation (via our fixed embedding \(E \hookrightarrow \mathbb{C}\)). We shall only consider the case \(\mathcal{O} = \mathcal{O}_E\) in the following.

One can define the signature of the complex multiplication \(m\), resp. the abelian variety \((W(\mathbb{R}), W(\mathbb{Z}), j, \psi, m)\) as the signature of the hermitian form \((w, w') \mapsto \psi(w, iw') + i\psi(w, w')\) on \(W(\mathbb{R})\) with respect to the complex structure imposed by \(m\) via \(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}\). We write \(m(s, t)\) if \(m\) has signature \((s, t)\).

Finally for any compact open subgroup \(K \subset G(\hat{\mathbb{Z}})\) as before one has the notion of a level-\(K\) structure on \(A\). For a positive integer \(n\) we denote by \(A_n(\mathbb{C})\) the group of points of order \(n\) in \(A(\mathbb{C})\). This group can be identified with \(W(\mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z}\) and taking the projective limit over the system \((A_n(\mathbb{C}))_{n \in \mathbb{N}_{>0}}\) defines the Tate module of \(A\):

\[
T(A) := \varprojlim A_n(\mathbb{C}) \simeq W(\mathbb{Z}) \otimes \hat{\mathbb{Z}}.
\]

Now two isomorphisms \(\varphi_1, \varphi_2 : W(\mathbb{Z}) \otimes \hat{\mathbb{Z}} \simeq L \otimes \hat{\mathbb{Z}}\) are called \(K\)-equivalent if there is a \(k \in K\) such that \(\varphi_1 = k \varphi_2\) and a \(K\)-level structure on \(A\) is just a \(K\)-equivalence class of these isomorphisms.

**Proposition 2.1.** For any compact open subgroup \(K \subset G(\hat{\mathbb{Z}})\) there is a one-to-one correspondence between

1. the set of points of \(S_K(G, \mathcal{H})(\mathbb{C})\) and
2. the set of isomorphism classes of \((W(\mathbb{R}), W(\mathbb{Z}), j, \psi, m_{(2,1), \varphi})\) as above.

**Proof.** [9, Prop.3.2]

**Remark 2.2.** If we take

\[
K_N := \{g \in G(A_f) \mid (g - 1)(L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \subset N \cdot (L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})\},
\]

then a level-\(K\) structure is just the usual level-\(N\) structure, namely an isomorphism \(A_N(\mathbb{C}) \rightarrow L \otimes \mathbb{Z}/N\mathbb{Z}\). Moreover \(K_N \subset G(\hat{\mathbb{Q}}) = \Gamma_N\), where \(\Gamma_N\) is the principal congruence subgroup of level \(N\), i.e. the kernel of the canonical map \(G'(\mathbb{Z}) \rightarrow G'(\mathbb{Z}/N\mathbb{Z})\). In this case the connected component of the identity of \(S_{K_N}(G, \mathcal{H})\) is exactly \(X_{\Gamma_N}(\mathbb{C})\).

We denote with \(\mathcal{A}_f\) the universal abelian scheme over \(X_{\Gamma}(\mathbb{C})\). In section [9] the compactifications of these varieties will be explained in detail. For the time being we denote them with \(\overline{X}_{\Gamma}\) and \(\overline{\mathcal{A}}_{\Gamma}\).
As the group $\Gamma$ will be fixed throughout the paper we will drop the index $\Gamma$ if no confusion is possible.

3. Chow motives and the conjectures of Murre

Let us briefly recall some definitions and results from the theory of Chow motives. We refer to [26] for details.

3.1. For a smooth projective variety $Y$ over a field $k$ let $\text{CH}^j(Y)$ denote the Chow group of algebraic cycles of codimension $j$ on $Y$ modulo rational equivalence, and let $\text{CH}^j(Y)_\mathbb{Q} := \text{CH}^j(Y) \otimes \mathbb{Q}$. For a cycle $Z$ on $Y$ we write $[Z]$ for its class in $\text{CH}^j(Y)$. We will be working with relative Chow motives as well, so let us fix a smooth connected, quasi-projective base scheme $S \to \text{Spec} k$. If $S = \text{Spec} k$ we will usually omit $S$ in the notation. Let $Y, Y'$ be smooth projective varieties over $S$, i.e., all fibers are smooth. For ease of notation (and as we will not consider more general cases) we may assume that $Y$ is irreducible and of relative dimension $g$ over $S$. The group of relative correspondences from $Y$ to $Y'$ of degree $r$ is defined as

$$\text{Corr}^r(Y \times_S Y') := \text{CH}^{r+g}(Y \times_S Y')_\mathbb{Q}.$$ 

Every $S$-morphism $Y' \to Y$ defines an element in $\text{Corr}^0(Y \times_S Y')$ via the class of the transpose of its graph. In particular one has the class $[\Delta_{Y/S}] \in \text{Corr}^0(Y \times_S Y')$ of the relative diagonal. The self correspondences of degree 0 form a ring, see [26, pg. 127]. Using the relative correspondences one proceeds as usual to define the category $M_S$ of (pure) Chow motives over $S$. The objects of this pseudoabelian $\mathbb{Q}$-linear tensor category are triples $(Y, p, n)$ where $Y$ is as above, $p$ is a projector, i.e. an idempotent element in $\text{Corr}^0(Y \times_S Y)$, and $n \in \mathbb{Z}$. The morphisms are

$$\text{Hom}_{M_S}((Y, p, n), (Y', p', n')) := p' \circ \text{Corr}^{n'-n}(Y \times_S Y') \circ p.$$ 

When $n = 0$ we write $(Y, p)$ instead of $(Y, p, 0)$, and $h(Y) := (Y, [\Delta_Y])$.

Definition 3.1. For a smooth projective variety $Y/k$ of dimension $d$ a Chow-Künneth-decomposition of $Y$ consists of a collection of pairwise orthogonal projectors $\pi_0, \ldots, \pi_{2d}$ in $\text{Corr}^0(Y \times Y)$ satisfying

1. $\pi_0 + \ldots + \pi_{2d} = [\Delta_Y]$ and
2. for some Weil cohomology theory $H^*$ one has $\pi_i(H^*(Y)) = H^i(Y)$.

If one has a Chow-Künneth decomposition for $Y$ one writes $h^i(Y) = (Y, \pi_i)$. A similar notion of a relative Chow-Künneth-decomposition over $S$ can be defined in a straightforward manner, see also introduction. Towards the existence of such decomposition one has the following conjecture of Murre:

Conjecture 3.2. Let $Y$ be a smooth projective variety of dimension $d$ over some field $k$.

1. There exists a Chow-Künneth decomposition for $Y$. 

For all $i < j$ and $i > 2j$ the action of $\pi_i$ on $\text{CH}^j(Y)_\mathbb{Q}$ is trivial, i.e. $\pi_i \cdot \text{CH}^j(Y)_\mathbb{Q} = 0$.

The induced step filtration on $F^n\text{CH}^j(Y)_\mathbb{Q} := \text{Ker} \pi_{2j} \cap \cdots \cap \text{Ker} \pi_{2j-n+1}$ is independent of the choice of the Chow–Künneth projectors, which are in general not canonical.

The first step of this filtration should give exactly the subgroup of homological trivial cycles $\text{CH}^j(Y)_\mathbb{Q}$.

There are not many examples for which these conjectures have been proved, but they are known to be true for surfaces [26], in particular we know that we have a Chow-Künneth decomposition for $X$.

In the following theorem we are assuming the motivic decomposition conjecture which was explained in the introduction. The main result we are going to prove in section 7 is:

**Theorem 3.3.** Under the assumption of the motivic decomposition conjecture A has a partial Chow–Künneth decomposition, including the projectors $\pi_i$ for $i \neq 4, 5, 6$ as in Part (1) of Murre’s conjecture.

Over the open smooth part $X \subset \overline{X}$ one has the relative projectors constructed by Deninger and Murre in [6], see also [18]: Let $S$ be a fixed base scheme as in section 3. We will now state some results on relative Chow motives in the case that $A$ is an abelian scheme of fibre dimension $g$ over $S$.

Firstly we have a functorial decomposition of the relative diagonal $\Delta_{A/S}$.

**Theorem 3.4.** There is a unique decomposition

$$\Delta_{A/S} = \sum_{s=0}^{2g} \Pi_i \text{ in } \text{CH}^g(A \times_S A)_\mathbb{Q}$$

such that $(\text{id}_A \times [n])^* \Pi_i = n^i \Pi_i$ for all $n \in \mathbb{Z}$. Moreover the $\Pi_i$ are mutually orthogonal idempotents, and $[\Gamma_{[n]}] \circ \Pi_i = n^i \Pi_i = \Pi_i \circ [\Gamma_{[n]}]$, where $[n]$ denotes the multiplication by $n$ on $A$.

**Proof.** [6, Thm. 3.1] \hfill \square

Putting $h^i(A/S) = (A/S, \Pi_i)$ one has a Poincaré-duality for these motives.

**Theorem 3.5.** (Poincaré-duality)

$$h^{2g-i}(A/S)^\vee \simeq h^i(A/S)(g)$$

**Proof.** [18, 3.1.2] \hfill \square

3.2. We now turn back to our specific situation. From Theorem 3.3 we have the decomposition $\Delta_{A/X} = \Pi_0 + \cdots + \Pi_6$.

We will have to extend these relative projectors to absolute projectors. In order to show the readers which of the methods of [11], where Hilbert modular varieties are considered, go through and which of them fail in our case, we recall the main theorem (Theorem 1.3) from [11]:

Putting $h^i(A/S) = (A/S, \Pi_i)$ one has a Poincaré-duality for these motives.
Theorem 3.6. Let \( p : A \to X \) as above satisfy the following conditions:

1. The irreducible components of \( X \setminus X \) are smooth toric projective varieties.
2. The irreducible components of \( A \setminus A \) are smooth projective toric varieties.
3. The variety \( A/X \) has a relative Chow-Künneth decomposition.
4. \( X \) has a Chow-Künneth decomposition over \( k \).
5. If \( x \) is a point of \( X \) the natural map
   \[
   CH^r(A) \to H^{2r}_{B}(A_z(\mathbb{C}), \mathbb{Q})^{\pi_1^{\text{top}}(X,x)}
   \]
   is surjective for \( 0 \leq r \leq d = \dim A - \dim X \).
6. For \( i \) odd, \( H^i_B(A_z(\mathbb{C}), \mathbb{Q})^{\pi_1^{\text{top}}(X,x)} = 0 \).
7. Let \( \rho \) be an irreducible, non-constant representation of \( \pi_1^{\text{top}}(X,x) \) and \( V \) the corresponding local system on \( X \). Assume that \( V \) is contained in the \( i \)-th exterior power \( R^i\rho_!\mathbb{Q} = \Lambda^i R^1\rho_!\mathbb{Q} \) of the monodromy representation for some \( 0 \leq i \leq 2d \). Then the intersection cohomology \( H^q(X,V) \) vanishes if \( q \neq \dim X \).

Under these assumptions \( A \) has a Chow-Künneth decomposition over \( k \).

As it stands we can only use conditions (3),(4) and (5) of this theorem, all the other conditions fail in our case. As for conditions (1) and (2) we will have to weaken them to torus fibrations over an elliptic curve. This will be done in section 4.

Condition (3) holds because of the work of Deninger and Murre ([6]) on Chow-Künneth decompositions of Abelian schemes.

Condition (4) holds in our case because of the existence of Chow-Künneth projectors for surfaces (see [26]).

In order to prove condition (5) and to replace conditions (6) and (7) we will from section 5 on use a non-compact Simpson type correspondence between the \( L^2 \)-Higgs cohomology of the underlying variation of Hodge structures and the \( L^2 \)-de Rham cohomology (respectively intersection cohomology) of local systems. This will show the vanishing of some of the cohomology groups mentioned in (6) of Theorem 3.6 and enable us to weaken condition (7).

4. The universal abelian scheme and its compactification

In this section we show that the two conditions (1) and (2) of Theorem 3.6 fail in our case. Instead of tori we get toric fibrations over an elliptic curve as fibers over boundary components. The main reference for this section is [23].

4.1. Toroidal compactifications of locally symmetric varieties. In this paragraph an introduction to the theory of toroidal compactifications of locally symmetric varieties as developed by Ash, Mumford, Rapoport and Tai in [2] is given. The main goal is to fix notation. All details can be found in [2], see the page references in this paragraph.
Let $D = G(\mathbb{R})/K$ be a bounded symmetric domain (or a finite number of bounded symmetric domains, for the following discussion we will assume $D$ to be just one bounded symmetric domain), where $G(\mathbb{R})$ denotes the $\mathbb{R}$-valued points of a semisimple group $G$ and $K \subset G(\mathbb{R})$ is a maximal compact subgroup. Let $\hat{D}$ be its compact dual. Then there is an embedding

$$D \hookrightarrow \hat{D}. \tag{1}$$

Note that $G^0(\mathbb{C})$ acts on $\hat{D}$. We pick a parabolic $P$ corresponding to a rational boundary component $F$, by $Z^0$ we denote the connected component of the centralizer $Z(F)$ of $F$, by $P^0$ the connected component of $P$ and by $\Gamma$ a (torsion free, see below for this restriction) congruence subgroup of $G$. We will explicitly be interested only in connected groups, so from now on we can assume that $G^0 = G$. Set

- $N \subset P^0$: the unipotent radical
- $U \subset N$: the center of the unipotent radical
- $U_{\mathbb{C}}$: its complexification
- $V = N/U$
- $\Gamma_0 = \Gamma \cap U$
- $\Gamma_1 = \Gamma \cap P^0$
- $T = \Gamma_0 \setminus U_{\mathbb{C}}$.

Note that $U$ is a real vector space and by construction, $T$ is an algebraic torus over $\mathbb{C}$. Set

$$D(F) := U_{\mathbb{C}} \cdot D,$$

where the dot denotes the action of $G^0(\mathbb{C})$ on $\hat{D}$. This is an open set in $\hat{D}$ and we have the inclusions

$$D \subset D(F) = U_{\mathbb{C}} \cdot D \subset \hat{D} \tag{2}$$

and furthermore a complex analytic isomorphism

$$U_{\mathbb{C}} \cdot D \simeq U_{\mathbb{C}} \times E_P \tag{3}$$

where $E_P$ is some complex vector bundle over the boundary component corresponding to $P$. We will not describe $E_P$ any further, the interested reader is referred to [2], chapter 3. The isomorphism in (3) is complex analytic and takes the $U_{\mathbb{C}}$-action on $\hat{D}$ from the left to the translation on $U_{\mathbb{C}}$ on the right. We once for all choose a boundary component $F$ and denote its stabilizer by $P$.

From (2) we get (see [2], chapter 3 for details, e.g. on the last isomorphism)

$$\Gamma_0 \setminus D \subset \Gamma_0 \setminus (U_{\mathbb{C}} \cdot D) \simeq \Gamma_0 \setminus (U_{\mathbb{C}} \times E_P) \simeq T \times \mathcal{E},$$

where $\mathcal{E} = \Gamma_0 \setminus E_P$. 
The torus $T$ is the one we use for a toroidal embedding. Furthermore $D$ can be realized as a Siegel domain of the third kind:

$$D \simeq \{(z,e) \in U_C \cdot D \simeq U_C \times \mathcal{E} \mid \text{Im}(z) \in C + h(e)\},$$

where

$$h : \mathcal{E} \to U$$

is a real analytic map and $C \subset U$ is an open cone in $U$.

A finer description of $C$ which is needed for the most general case can be found in [2].

We pick a cone decomposition $\{\sigma_\alpha\}$ of $C$ such that

$$(\Gamma_1/\Gamma_0) \cdot \{\sigma_\alpha\} = \{\sigma_\alpha\}$$

with finitely many orbits and $C \subset \bigcup \sigma_\alpha \subset \overline{C}$.

This yields a torus embedding

$$T \subset X_{\{\sigma_\alpha\}}.$$

We can thus partially compactify the open set $\Gamma_0 \setminus (U_C \cdot D)$:

$$\Gamma_0 \setminus (U_C \cdot D) \simeq \Gamma_0 \setminus U_C \times \mathcal{E} \hookrightarrow X_{\{\sigma_\alpha\}} \times \mathcal{E}.$$

The situation is now the following:

$$\Gamma_0 \setminus (U_C \cdot D) \simeq T \times \mathcal{E} \hookrightarrow X_{\{\sigma_\alpha\}} \times \mathcal{E}$$

$$\cup$$

$$\Gamma_0 \setminus D.$$

We proceed to give a description of the vector bundle $\mathcal{E}_P$ in order to describe the toroidal compactification geometrically.

Again from [2] (pp.233) we know that

$$D \cong F \times C \times N$$

as real manifolds and

$$D(F) \cong F \times V \times U_C,$$

where $V = N/U$ is the abelian part of $N$.

Now set

$$D(F)' := D(F) \mod U_C.$$

This yields the following fibration:

$$\begin{array}{c}
D(F) \\
\pi_1 \downarrow \text{fibres } U_C \\
\pi \downarrow \\
D(F)' \\
\pi_2 \downarrow \text{fibres } V \\
\rightarrow F.
\end{array}$$
Taking the quotient by $\Gamma_0$ yields a quotient bundle

$$ \Gamma_0 \backslash D(F) $$

$$ \pi_T \text{ fibres } T := \Gamma_0 \backslash U_C $$

$$ D(F)' $$

So, $T$ is an algebraic torus group with maximal compact subtorus

$$ T_{cp} := \Gamma_0 \backslash U. $$

Take the closure of $\Gamma_0 \backslash D$ in $X_{\{\sigma\}} \times \mathcal{E}$ and denote by $(\Gamma_0 \backslash D)\{\sigma\}$ its interior.

Factor $D \to \Gamma \backslash D$ by

$$ D \to \Gamma_0 \backslash D \to \Gamma_1 \backslash D \to \Gamma \backslash D. $$

It is the following situation we aim at obtaining:

$$ (\Gamma_0 \backslash D)\{\sigma\} \leftarrow \Gamma_0 \backslash D \to \Gamma_1 \backslash D \to \Gamma \backslash D $$

$$ (\Gamma_0 \backslash D(c))\{\sigma\} \leftarrow \Gamma_0 \backslash D(c) \to \Gamma_1 \backslash D(c) \to \Gamma \backslash D. $$

Here $D(c)$ is a neighborhood of our boundary component. More precisely for any compact subset $K$ of the boundary component and any $c \in C$ define

$$ D(c, K) = \Gamma_1 \cdot \left\{ (z, e) \in U_C \times \mathcal{E} \mid \text{Im } z \in C + h(e) + c \text{ and } e \text{ lies above } K \right\}. $$

Then by reduction theory for $c$ large enough, $\Gamma$-equivalence on $D(c, K)$ reduces to $\Gamma_1$-equivalence.

This means that we have an inclusion

$$ \Gamma_1 \backslash D(c) \hookrightarrow \Gamma \backslash D $$

$$ \simeq (\Gamma_1/\Gamma_0) \backslash (\Gamma_0 \backslash D(c)) $$

where the quotient by $\Gamma_1/\Gamma_0$ is defined in the obvious way.

Furthermore $\Gamma_0 \backslash D \hookrightarrow (\Gamma_0 \backslash D)\{\sigma\}$ directly induces

$$ \Gamma_0 \backslash D(c) \hookrightarrow (\Gamma_0 \backslash D(c))\{\sigma\}. $$

Having chosen $\{\sigma\}$ such that $(\Gamma_1/\Gamma_0) \cdot \{\sigma\} = \{\sigma\}$, we get

$$ \Gamma_1 \backslash D(c) \hookrightarrow (\Gamma_1/\Gamma_0) \\backslash (\Gamma_0 \backslash D(c))\{\sigma\} $$

which yields the partial compactification and establishes the diagram (7).

The following theorem is derived from the above.
Theorem 4.1. With the above notation and for a cone decomposition \( \{ \sigma_\alpha \} \) of \( C \) satisfying the condition (4), the diagram

\[
\begin{array}{c}
\Gamma_1 \setminus D(c) \\
\downarrow \\
(\Gamma_1/\Gamma_0) \setminus (\Gamma_0 \setminus D(c))_{\{\sigma_\alpha\}}
\end{array}
\]

yields a (smooth if \( \{ \sigma_\alpha \} \) is chosen appropriately) partial compactification of \( \Gamma \setminus D \) at \( F \).

4.2. Toroidal compactification of Picard modular surfaces. We will now apply the results of the last paragraph to the case of Picard modular surfaces and give a finer description of the fibres at the boundary.

Theorem 4.2. For each boundary component of a Picard modular surface the following holds. With the standard notations from \[2\] (see also the last paragraph and \[23\] for the specific choices of \( \Gamma_0, \Gamma_1 \) etc.)

\[
(\Gamma_1/\Gamma_0) \setminus (\Gamma_0 \setminus D)
\]

is isomorphic to a punctured disc bundle over a CM elliptic curve \( A \). A toroidal compactification

\[
(\Gamma_1/\Gamma_0) \setminus (\Gamma_0 \setminus D)_{\{\sigma_\alpha\}}
\]

is obtained by closing the disc with a copy of \( A \) (e.g. adding the zero section of the corresponding line bundle).

We now turn to the modification of condition (2). The notation we use is as introduced in chapter 3 of \[8\]. Let \( \tilde{P} \) be a relatively complete model of an ample degeneration datum associated to our moduli problem. As a general reference for degenerations see \[25\], see \[8\] for the notion of relatively complete model and \[23\] for the ample degeneration datum we need here. In \[23\] the following theorem is proved.

Theorem 4.3. (i) The generic fibre of \( \tilde{P} \) is given by a fibre-bundle over a CM elliptic curve \( E \), whose fibres are countably many irreducible components of the form \( \mathbb{P} \), where \( \mathbb{P} \) is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \).

(ii) The special fibre of \( \tilde{P} \) is given by a fibre-bundle over the CM elliptic curve \( E \), whose fibres consist of countably many irreducible components of the form \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Remark 4.4. In this paper we work with some very specific Picard modular surfaces and thus the generality of Theorem 4.3 is not needed. It will be needed though to extend our results to larger families of Picard modular surfaces, see section 7.2..
5. Higgs bundles on Picard modular surfaces

In this section we describe in detail the Picard modular surface of Holzapfel which is our main object. We follow Holzapfel [15, 16] very closely. In the remaining part of this section we explain the formalism of Higgs bundles which we will need later.

5.1. Holzapfel’s surface. We restrict our attention to the Picard modular surfaces with compactification $X$ and boundary divisor $D \subseteq X$ which were discussed by Picard [28], Hirzebruch [14] and Holzapfel [15, 16]. These surfaces are compactifications of ball quotients $X = \mathbb{B}/\Gamma$ where $\Gamma$ is a subgroup of $SU(2,1;\mathcal{O})$ with $\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\omega$, $\omega = \exp(2\pi i/3)$, i.e., $\mathcal{O}$ is the ring of Eisenstein numbers. In the case $\Gamma = SU(2,1;\mathcal{O})$, studied already by Picard, the quotient $\mathbb{B}/\Gamma$ is $\mathbb{P}^2 \setminus 4$ points, an open set of which is $U = \mathbb{P}^2 \setminus \Delta$ and $\Delta$ is a configuration of 6 lines (not a normal crossing divisor). $U$ is a natural parameter space for a family of Picard curves

$$y^3 = x(x-1)(x-s)(x-t)$$

of genus 3 branched over 5 (ordered) points $0, 1, s, t, \infty$ in $\mathbb{P}^1$. The parameters $s, t$ are coordinates in the affine set $U$. If one looks at the subgroup

$$\Gamma' = \Gamma \cap SL(3, \mathbb{C}),$$

then $X = \mathbb{B}/\Gamma'$ has a natural compactification $\overline{X}$ with a smooth boundary divisor $D$ consisting of 4 disjoint elliptic curves $E_0 + E_1 + E_2 + E_3$, see [15, 16]. This surface $\overline{X}$ is birational to a covering of $\mathbb{P}^2 \setminus \Delta$ and hence carries a family of curves over it. If we pass to yet another subgroup $\Gamma'' \subset \Gamma$ of finite index, then we obtain a Picard modular surface $\overline{X} = \overline{E \times E}$

with boundary $D$ a union of 6 elliptic curves which are the strict transforms of the following 6 curves

$$T_1, T_\omega, T_\omega^2, E \times \{Q_0\}, E \times \{Q_1\}, E \times \{Q_2\}$$

on $E \times E$ in the notation of [15, page 257]. This is the surface we will study in this paper. The properties of the modular group $\Gamma''$ are described in [15, remark V.5]. In particular it acts freely on the ball. $\overline{X}$ is the blowup of $E \times E$ in the three points $(Q_0, Q_0)$ (the origin), $(Q_1, Q_1)$ and $(Q_2, Q_2)$ of triple intersection. Note that $E$ has the equation $y^2z = x^3 - z^3$. On $E$ we have an action of $\omega$ via $(x : y : z) \mapsto (\omega x : y : z)$. $E$ maps to $\mathbb{P}^1$ using the projection

$$p : E \to \mathbb{P}^1, \quad (x : y : z) \mapsto (y : z).$$

This action has 3 fixpoints $Q_0 = (0 : 1 : 0)$ (the origin), $Q_1 = (0 : i : 1)$ and $Q_2 = (0 : -i : 1)$ which are triple ramification points of $p$. Therefore one has $3Q_0 = 3Q_1 = 3Q_2$ in $CH^1(E)$. In order to proceed, we need to know something about the Picard group of $\overline{X}$. 
Lemma 5.1. In $\text{NS}(E \times E)$ one has the relation
\[ T_1 + T_\omega + T_{\omega^2} = 3(0 \times E) + 3(E \times 0). \]

Proof. Since $E$ has complex multiplication by $\mathbb{Z}[\omega]$, the Néron–Severi group has rank 4 and divisors $T_1, T_\omega, 0 \times E$ and $E \times 0$ form a basis of $\text{NS}(E \times E)$. Using the intersection matrix of this basis, the claim follows. \( \square \)

The following statement is needed later:

Lemma 5.2. The log–canonical divisor is divisible by three:
\[ K_X + D = 3L \]

for some line bundle $L$.

Proof. If we denote by $\sigma : \overline{X} \to E \times E$ the blowup in the three points $(Q_0, Q_0), (Q_1, Q_1)$ and $(Q_2, Q_2)$, then we denote by $Z = Z_1 + Z_2 + Z_3$ the union of all exceptional divisors. We get:
\[ \sigma^*T = D_1 + Z, \sigma^*T_\omega = D_2 + Z, \sigma^*T_{\omega^2} = D_3 + Z, \]
and
\[ \sigma^*E \times Q_0 = D_4 + Z_1, \sigma^*E \times Q_1 = D_5 + Z_2, \sigma^*E \times Q_2 = D_6 + Z_3. \]

Now look at the line bundle $K_{\overline{X}} + D$. Since
\[ K_{\overline{X}} + D = \sigma^*K_{E \times E} + Z + D = Z + D, \]
we compute
\[ K_{\overline{X}} + D = \sum_{i=1}^{6} D_i + \sum_{j=1}^{3} Z_j. \]

The first sum,
\[ D_1 + D_2 + D_3 = -3Z + \sigma^*(T_1 + T_\omega + T_{\omega^2}) = -3Z + 3\sigma^*(0 \times E + E \times 0). \]
is divisible by 3. Using $3Q_0 = 3Q_1 = 3Q_2$, the rest can be computed in $\text{NS}(\overline{X})$ as
\[ D_4 + D_5 + D_6 + Z = \sigma^*(E \times 0 + E \times Q_1 + E \times Q_2) = 3\sigma^*(E \times 0). \]

Therefore the class of $K_{\overline{X}} + D$ in $\text{NS}(\overline{X})$ is given by
\[ K_{\overline{X}} + D = -3Z + 3\sigma^*(0 \times E) + 6\sigma^*(E \times 0) \]
and divisible by 3. Since $\text{Pic}^0(\overline{X})$ is a divisible group, $K_{\overline{X}} + D$ is divisible by 3 in $\text{Pic}(\overline{X})$ and we get a line bundle $L$ with $K_{\overline{X}} + D = 3L$ whose class in $\text{NS}(\overline{X})$ is given by
\[ L = \sigma^*(0 \times E) - Z + 2\sigma^*(E \times 0). \]

If we write
\[ \sigma^*(0 \times E) = D_0 + Z_1, \]
we obtain the equation
\[ L = D_0 + D_5 + D_6. \]
in NS($\mathbb{X}$). Note that $D_0$ intersects both $D_5$ and $D_6$ in one point. All $D_i$, $i = 1, \ldots, 6$ have negative selfintersection and are disjoint.

It is not difficult to see that $L$ is a nef and big line bundle since $\mathbb{X}$ has logarithmic Kodaira dimension $2$ [15]. $L$ is trivial on all components of $D$ by the adjunction formula, since they are smooth elliptic curves.

The rest of this section is about the rank 6 local system $\mathbb{V} = R^1p_*\mathbb{Z}$ on $X$. The following Lemma was known to Picard [28], he wrote down $3 \times 3$ monodromy matrices with values in the Eisenstein numbers:

**Lemma 5.3.** $\mathbb{V}$ is a direct sum of two local systems $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ of rank 3. The decomposition is defined over the Eisenstein numbers.

**Proof.** The cohomology $H^1(C)$ of any Picard curve $C$ has a natural $\mathbb{Z}/3\mathbb{Z}$ Galois action. Since the projective line has $H^1(\mathbb{P}^1, \mathbb{Z}) = 0$, the local system $\mathbb{V} \otimes \mathbb{C}$ decomposes into two $3$–dimensional local systems

$$\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$$

which are conjugate to each other and defined over the Eisenstein numbers. □

Both local systems $\mathbb{V}_1, \mathbb{V}_2$ are irreducible and non–constant.

5.2. **Higgs bundles on Holzapfel’s surface.** Now we will study the categorical correspondence between local systems and Higgs bundles. It turns out that it is sometimes easier to deal with one resp. the other.

**Definition 5.4.** A Higgs bundle on a smooth variety $Y$ is a holomorphic vector bundle $E$ together with a holomorphic map

$$\vartheta : E \to E \otimes \Omega^1_Y$$

which satisfies $\vartheta \wedge \vartheta = 0$, i.e., an $\text{End}(E)$ valued holomorphic 1–form on $Y$.

Each Higgs bundle induces a complex of vector bundles:

$$E \to E \otimes \Omega^1_Y \to E \otimes \Omega^2_Y \to \ldots \to E \otimes \Omega^d_Y.$$  

**Higgs cohomology** is the cohomology of this complex. The Simpson correspondence on a projective variety $\bar{Y}$ gives an equivalence of categories between polystable Higgs bundles with vanishing Chern classes and semisimple local systems $\mathbb{V}$ on $Y$ [29 Sect. 8]. This correspondence is very difficult to describe in general and uses a deep existence theorem for harmonic metrics. For quasi–projective $Y$ this may be generalized provided that the appropriate harmonic metrics exist, which is still not known until today. There is however the case of VHS (Variations of Hodge structures) where the harmonic metric is the Hodge metric and is canonically given. For example if we have a smooth, projective family $f : A \to X$ as in our example and $\mathbb{V} = R^m f_* \mathbb{C}$ is a direct image sheaf, then the corresponding Higgs bundle is

$$E = \bigoplus_{p+q=m} E^{p,q}.$$
where $E^{p,q}$ is the $p$–the graded piece of the Hodge filtration $F^\bullet$ on $\mathcal{H} = V \otimes \mathcal{O}_X$. The Higgs operator $\vartheta$ is then given by the graded part of the Gauß–Manin connection, i.e., the cup product with the Kodaira–Spencer class. In the non–compact case there is also a corresponding log–version for Higgs bundles, where $\Omega^1_Y$ is replaced by $\Omega^1_Y(\log D)$ for some normal crossing divisor $D \subset Y$ and $E$ by the Deligne extension. Therefore we have to assume that the monodromies around the divisors at infinity are unipotent and not only quasi–unipotent as in [17, Sect. 2]. This is the case in Holzapfel’s example, in fact above we have already checked that the log–canonical divisor $K_X + D$ is divisible by three. We refer to [29] and [17] for the general theory. In our case let $E = E^{1,0} \oplus E^{0,1}$ be the Higgs bundle corresponding to $V_1$ with Higgs field

$$\vartheta : E \to E \otimes \Omega^1_X(\log D).$$

This bundle is called the uniformizing bundle in [29, Sect. 9].

Let us return to Holzapfel’s example. We may assume that $E^{1,0}$ is 2–dimensional and $E^{0,1}$ is 1–dimensional, otherwise we permute $V_1$ and $V_2$.

**Lemma 5.5.** $\vartheta : E^{1,0} \to E^{0,1} \otimes \Omega^1_X(\log D)$ is an isomorphism.

*Proof.* For the generic fiber this is true for rank reasons. At the boundary $D$ this is a local computation using the definition of the Deligne extension. This has been shown in greater generality in [17, Sect. 2-4] (cf. also [20, Sect. 4]), therefore we do not give any more details. $\square$

Let us summarize what we have shown for Holzapfel’s surface $\overline{X}$:

**Corollary 5.6.** $K_{\overline{X}}(D)$ is nef and big and there is a nef and big line bundle $L$ with

$$L^{\otimes 3} \cong K_{\overline{X}}(D).$$

The uniformizing bundle $E$ has components

$$E^{1,0} = \Omega^1_X(\log D) \otimes L^{-1}, \quad E^{0,1} = L^{-1}.$$

The Higgs operator $\vartheta$ is the identity as a map $E^{1,0} \to E^{0,1} \otimes \Omega^1_X(\log D)$ and it is trivial on $E^{0,1}$.

### 6. Vanishing of intersection cohomology

Let $X$ be Holzapfel’s surface from the previous section. We now want to discuss the vanishing of intersection cohomology

$$H^1(X, \mathbb{W})$$

for irreducible, non–constant local systems $\mathbb{W} \subseteq R^i p_* \mathbb{Q}$. Let $V_1$ be as in the previous section. Denote by $(E, \vartheta)$ the corresponding Higgs bundle with

$$E = (\Omega^1_X(\log D) \otimes L^{-1}) \oplus L^{-1}$$

and Higgs field

$$\vartheta : E \to E \otimes \Omega^1_X(\log D).$$
Our goal is to compute the intersection cohomology of $V_1$. We use the isomorphism between $L^2$– and intersection cohomology for $C$–VHS, a theorem of Cattani, Kaplan and Schmid together with the isomorphism between $L^2$–cohomology and $L^2$–Higgs cohomology from [17] Thm. A/B. Therefore for computations of intersection cohomology we may use $L^2$–Higgs cohomology. We refer to [17] for a general introduction to all cohomology theories involved.

**Theorem 6.1.** The intersection cohomology $H^q(X, V_1)$ vanishes for $q \neq 2$. By conjugation the same holds for $V_2$.

**Proof.** We need only show this for $q = 1$, since $V_1$ has no invariant sections, hence $H^0(X, V_1) = 0$ and the other vanishing follow via duality

$$H^q(X, V_1) \cong H^{2\dim(X) - q}(X, V_2)$$

from the analogous statement for $V_2$. The following theorem provides the necessary technical tool.

**Theorem 6.2 ([17] Thm. B]).** The intersection cohomology $H^q(X, V_1)$ can be computed as the $q$–th hypercohomology of the complex

$$0 \to \Omega^0(E)_{(2)} \to \Omega^1(E)_{(2)} \to \Omega^2(E)_{(2)} \to 0$$

on $X$, where $E$ is as above. This is a subcomplex of

$$E \to E \otimes \Omega^1_X(\log D) \to E \otimes \Omega^2_X(\log D).$$

In the case where $D$ is smooth, this is a proper subcomplex with the property

$$\Omega^1(E)_{(2)} \subseteq \Omega^1_X \otimes E.$$

**Proof.** This is a special case of the results in [17]. The subcomplex is explicitly described in section 8 of our paper.

**Lemma 6.3.** Let $E$ be as above with $L$ nef and big. Then the vanishing

$$H^0(\Omega^1_X(\log D) \otimes \Omega^1_X \otimes L^{-1}) = 0$$

implies the statement of theorem [6.1].

**Proof.** We first compute the cohomology groups for the complex of vector bundles and discuss the $L^2$–conditions later. Any logarithmic Higgs bundle $E = \oplus E^{p,q}$ coming from a VHS has differential

$$\vartheta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X(\log D).$$

In our case $E = E^{1,0} \oplus E^{0,1}$ and the restriction of $\vartheta$ to $E^{0,1}$ is zero. The differential

$$\vartheta : E^{1,0} \to E^{0,1} \otimes \Omega^1_X(\log D)$$

is the identity. Therefore the complex

$$(E^\bullet, \vartheta) : E \to E \otimes \Omega^1_X(\log D) \to E \otimes \Omega^2_X(\log D)$$
looks like:

\[
\begin{align*}
&\left( \Omega^1_X(\log D) \otimes L^{-1} \right) \oplus \left( L^{-1} \otimes \Omega^1_X(\log D) \right) \\
&\downarrow \cong \downarrow \\
&\left( \Omega^1_X(\log D) \otimes L^{-1} \otimes \Omega^2_X(\log D) \right) \oplus \left( L^{-1} \otimes \Omega^1_X(\log D) \right).
\end{align*}
\]

Therefore it is quasi–isomorphic to a complex

\[
L^{-1} \xrightarrow{0} S^2 \Omega^1_X(\log D) \otimes L^{-1} \xrightarrow{0} \Omega^1_X(\log D) \otimes \Omega^2_X(\log D) \otimes L^{-1}
\]

with trivial differentials. As \( L \) is nef and big, we have

\[
H^0(L^{-1}) = H^1(L^{-1}) = 0.
\]

Hence we get

\[
H^1(X, (E^*, \vartheta)) \cong H^0(X, S^2 \Omega^1_X(\log D) \otimes L^{-1})
\]

and \( H^2(X, (E^*, \vartheta)) \) is equal to

\[
H^0(X, K_X \otimes L)^v \oplus H^0(X, \Omega^1_X(\log D) \otimes \Omega^2_X(\log D) \otimes L^{-1}) \oplus H^1(X, S^2 \Omega^1_X(\log D) \otimes L^{-1}).
\]

If we now impose the \( L^2 \)–conditions and use the complex \( \Omega^*_1(E) \) instead of \( (E^*, \vartheta) \), the resulting cohomology groups are subquotients of the groups described above. Since

\[
\Omega^1_1(E) \subseteq \Omega^1_X \otimes E
\]

we conclude that the vanishing

\[
H^0(X, \Omega^1_1(\log D) \otimes \Omega^1_X(\log D) \otimes L^{-1}) = 0
\]

is sufficient to deduce the vanishing of intersection cohomology. \( \square \)

Now we verify the vanishing statement.

**Lemma 6.4.** In the example above we have

\[
H^0(\Omega^1_1X(\log D) \otimes \Omega^1_X(\log D) \otimes L^{-1}) = 0.
\]

**Proof.** Let \( \sigma : X \to E \times E \) be the blow up of the 3 points of intersection. Then one has an exact sequence

\[
0 \to \sigma^* \Omega^1_{E \times E} \to \Omega^1_X \to i_* \Omega^1_Z \to 0,
\]

where \( Z \) is the union of all (disjoint) exceptional divisors. Now \( \Omega^1_{E \times E} \) is the trivial sheaf of rank 2. Therefore \( \Omega^1_X(\log D) \otimes \Omega^1_X(\log D) \otimes L^{-1} \) has as a subsheaf 2 copies of \( \Omega^1_X(\log D) \otimes L^{-1} \). The group

\[
H^0(X, \Omega^1_X(\log D) \otimes L^{-1})
\]

is zero by the Bogomolov–Sommese vanishing theorem (see [1], Cor. 6.9), since \( L \) is nef and big. In order to prove the assertion it is hence sufficient to show that

\[
H^0(Z, \Omega^1_1Z(\log D) \otimes \Omega^1_Z(\log D) \otimes L^{-1}) = 0.
\]
But $Z$ is a disjoint union of $\mathbb{P}^1$'s. In our example we have $K_X(D) \otimes \mathcal{O}_Z \cong \mathcal{O}_Z(3)$ since $(L.Z) = 1$ and therefore $\Omega^1_Z \otimes L^{-1} \cong \mathcal{O}_Z(-3)$. Now we use in addition the conormal sequence

$$0 \to N^*_Z \to \Omega^1_X(\log D)|_Z \to \Omega^1_Z(\log(D \cap Z)) \to 0.$$ 

Note that $N^*_Z = \mathcal{O}_Z(1)$. Twisting this with $\Omega^1_Z \otimes L^{-1} \cong \mathcal{O}_Z(-3)$ gives an exact sequence

$$0 \to \mathcal{O}_Z(-2) \to \Omega^1_X(\log D) \otimes \Omega^1_Z \otimes L^{-1} \to \mathcal{O}_Z(-1) \to 0.$$

On global sections this proves the assertion.

So far we have only shown the vanishing of $H^q(X, V_1)$ and hence of $H^q(X, V)$ for $q \neq 2$. In order to apply the method of Gordon, Hanamura and Murre, we also have to deal with the case $\Lambda^i V$.

**Theorem 6.5.** Let $\rho$ be an irreducible, non–constant representation of $\pi_1(X)$, which is a direct factor in $\Lambda^k(V_1 \oplus V_2)$ for $k \leq 2$. Then the intersection cohomology group

$$H^1(X, V_\rho)$$

is zero.

**Proof.** Let us first compute all such representations: if $k = 1$ we have only $V_1$ and its dual. If $k = 2$, we have the decomposition

$$\Lambda^2(V_1 \oplus V_2) = \Lambda^2 V_1 \oplus \Lambda^2 V_2 \oplus \text{End}(V_1).$$

Since $V_1$ is 3–dimensional, $\Lambda^2 V_1 \cong V_2$ and therefore the only irreducible, non–constant representation that is new here is $\text{End}^0(V_1)$, the trace–free endomorphisms of $V_1$. Since we have already shown the vanishing $H^1(X, V_{1,2})$, it remains to treat $H^1(X, \text{End}^0(V_1))$. The vanishing of $H^1(X, \text{End}^0(V_1))$ is a general and well–known statement: The representation $\text{End}^0(V_1)$ has regular highest weight and therefore contributes only to the middle dimension $H^2$. A reference for this is [19, Main Thm.], cf. [4, ch. VII] and [30].

The vanishing of $H^1(X, \text{End}^0(V_1))$ has the following amazing consequence, which does not seem easy to prove directly using purely algebraic methods. In the compact case this has been shown by Miyaoka, cf. [24].

**Lemma 6.6.** In our situation we have

$$H^0_{L^2}(X, S^3 \Omega^1_X(\log D)(-D) \otimes L^{-3}) = 0.$$

**Proof.** Write down the Higgs complex for $\text{End}^0(E)$. In degree one, a term which contains

$$S^3 \Omega^1_X(\log D)(-D) \otimes L^{-3}$$

occurs. Since $H^1$ vanishes, this cohomology group must vanish too.
Finally we want to discuss the case $k = 3$. Unfortunately here the vanishing techniques do not work in general. But we are able to at least give a bound for the dimension of the remaining cohomology group. Namely for $k = 3$, one has

$$\Lambda^3(\mathcal{V}_1 \oplus \mathcal{V}_2) = \Lambda^3\mathcal{V}_1 \oplus \Lambda^3\mathcal{V}_2 \oplus (\Lambda^2\mathcal{V}_1 \otimes \mathcal{V}_2) \oplus (\Lambda^2\mathcal{V}_2 \otimes \mathcal{V}_1).$$

Here the only new irreducible and non-constant representation is

$$S^2\mathcal{V}_1 \subseteq \mathcal{V}_1 \otimes \mathcal{V}_1$$

and its dual. We would like to compute $H^1(X, S^2\mathcal{V}_1)$ using a variant of the symmetric product of the $L^2$–complexes $\Omega^*(S)_{(2)}$ as described in the appendix. The Higgs complex without $L^2$–conditions looks as follows:

$$\begin{align*}
&S^2\Omega_X^1(\log D) \otimes L^{-2} \oplus (\Omega_X^1(\log D) \otimes L^{-2}) \oplus L^{-2} \\
&\downarrow \\
&S^2\Omega_X^1(\log D) \otimes L^{-2} \oplus \Omega_X^1(\log D) \otimes L^{-2} \oplus \Omega_X^1(\log D) \\
&\downarrow \\
&S^2\Omega_X^1(\log D) \otimes L^{-2} \oplus \Omega_X^1(\log D) \otimes L^{-2} \oplus \Omega_X^1(\log D) \oplus (L^{-2} \otimes \Omega_X^1(\log D))
\end{align*}$$

Again many pieces of differentials in this complex are isomorphisms or zero. For example the differential

$$S^2\Omega_X^1(\log D) \otimes L^{-2} \otimes \Omega_X^1(\log D) \rightarrow \Omega_X^1(\log D) \otimes L^{-2} \otimes \Omega_X^2(\log D)$$

is a projection map onto a direct summand, since for every vector space $W$ we have the identity

$$S^2W \otimes W = S^3W \oplus (W \otimes \Lambda^2W).$$

Therefore the Higgs complex for $S^2(E)$ is quasi–isomorphic to

$$\begin{array}{c}
L^{-2} \rightarrow S^2\Omega_X^1(\log D) \otimes L^{-2} \\
\sigma \rightarrow S^2\Omega_X^1(\log D) \otimes L^{-2} \otimes \Omega_X^1(\log D).
\end{array}$$

We conclude that the first cohomology is given by

$$H^0(X, S^2\Omega_X^1(\log D) \otimes L^{-2}).$$

If we additionally impose the $L^2$–conditions (see appendix), then we see that the first Higgs cohomology of $S^2(E, \theta)$ vanishes, provided that we have

$$H^0(X, S^2\Omega_X^1(\log D) \otimes \Omega_X^1 \otimes L^{-2}) = 0.$$ 

Using

$$0 \rightarrow \sigma^*\Omega^1_{E \times E} \rightarrow \Omega^1_X \rightarrow i_*\Omega^1_Z \rightarrow 0$$

we obtain an exact sequence

$$0 \rightarrow H^0(X, S^2\Omega_X^1(\log D) \otimes L^{-2}) \rightarrow H^0(X, S^2\Omega_X^1(\log D) \otimes \Omega_X^1 \otimes L^{-2}) \rightarrow H^0(Z, S^2\Omega_X^1(\log D) \otimes \Omega_Z^1 \otimes L^{-2}).$$

A generalization of [23] example 3 leads to the vanishing

$$H^0(X, S^2\Omega^1_X(\log D) \otimes L^{-2}) = 0.$$ 

Since $\Omega^1_X(\log D)|_Z = \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2)$, we get

$$H^0(Z, S^2\Omega^1_X(\log D) \otimes \Omega^1_Z \otimes L^{-2}) = \mathbb{C}^3,$$
because $\Omega_X(\log D) \otimes \Omega^1_Z \otimes L^{-2} = \mathcal{O}_Z \oplus \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-2)$. However we are not able to decide whether these 3 sections lift to $\overline{X}$. When we restrict to forms with fewer poles, then the vanishing will hold for a kind of cuspidal cohomology.

### 7. Proof of the Main Theorem

In paragraph 7.1 we prove our main theorem, in paragraph 7.2 we give some indication on the proof of the motivic decomposition conjecture in our case, however the details will be published in a forthcoming paper. We thus will drop the assumption on the motivic decomposition conjecture in Theorem 7.2.

#### 7.1. From Relative to Absolute.

We now state and prove our main theorem. Let $p: \overline{A} \rightarrow \overline{X}$ be the compactified family over Holzapfel’s surface. Assume the motivic decomposition conjecture 1.1 ([5], [10, Conj. 1.4]) for $A/X$. In the proof we will need an auxiliary statement which was implicitly proven in section 6:

**Lemma 7.1.** Let $x \in \overline{X}$ be a base point. Then $\pi^\text{top}_1(X, x)$ acts on the Betti cohomology group $H^2_j(A_x(\mathbb{C}), \mathbb{Q})$. Then, for $0 \leq j \leq d = 3$, the cycle class map $CH^j(A) \rightarrow H^{2j}(A_x(\mathbb{C}), \mathbb{Q})$ is surjective.

*Proof.* By Lemma 5.3 the sheaf $R^1p_\ast\mathbb{C}$ is a sum of two irreducible representations of $\pi^\text{top}_1(X, x)$. By the proof of Theorem 6.5., $R^2p_\ast\mathbb{C}$ decomposes into a one–dimensional constant representation and three irreducible ones. The constant part corresponds to the identity in $\text{End}(V_1)$ and therefore to the polarization class on the fibers, which is a Hodge class. Therefore the invariant classes in $H^2(\mathcal{A}_x(\mathbb{C}), \mathbb{Q})$, and by duality also in $H^4(\mathcal{A}_x(\mathbb{C}), \mathbb{Q})$, consist of Hodge classes and are hence in the image of the cycle class map by the Hodge conjecture for divisors (and curves).

Now we can prove our main theorem:

**Theorem 7.2.** Assuming the motivic decomposition conjecture 1.1 the total space of the family $p: \overline{A} \rightarrow \overline{X}$ supports a partial set of Chow–Küneth projectors $\pi_i$ for $i \neq 4, 5, 6$.

*Proof.* The motivic decomposition conjecture 1.1 states that we have a relative Chow–Küneth decomposition with projectors $\Pi^i_\alpha$ on strata $X_\alpha$ which is compatible with the topological decomposition theorem [3]

$$\sum_{j, \alpha} \Psi^j_\alpha: \mathbb{R}p_\ast\mathbb{Q}_A \xrightarrow{\cong} \bigoplus_{j, \alpha} IC_{X_\alpha}(\mathcal{V}^j_\alpha)[-j - \dim(X_\alpha)].$$

Now we want to pass from relative Chow–Küneth decompositions to absolute ones. We use the notation of [11] and for the reader’s convenience we recall everything. Let $P^i/X$ and $P^i_\alpha/X$ be the mutually orthogonal projectors adding up to the identity $\Delta(A/X) \in CH_{\dim(A)}(A \times_X A)$ such that

$$(P^i/X) \ast \mathbb{R}p_\ast\mathbb{Q}_A = IC_X(R^ip_\ast\mathbb{Q}_A)[-i], \ (P^i_\alpha/X) \ast \mathbb{R}p_\ast\mathbb{Q}_A = IC_{X_\alpha}(\mathcal{V}^i_\alpha)[-i - \dim(X_\alpha)],$$
where the sheaves $V^i_\alpha$ are local systems supported over the cusps. The projectors $P^i_\alpha/X$ on the boundary strata decompose further into Chow–Künneth components, since the boundary strata consist of smooth elliptic curves and the stratification has the product type fibers described in Theorem 4.1. Let us now summarize what we know about the local systems $R^i p_* C$ on the open stratum $X_0$ from section 6: $R^3 p_* C$ is a sum of two irreducible representations and has no cohomology except in degree 2 by Theorem 6.5. $R^2 p_* C$ contains a trivial subsystem and the remaining complement has no cohomology except in degree 2 again by Theorem 6.5. $R^3 p_* C$ also contains a trivial subsystem and its complement has cohomology possibly in degrees 1, 2, 3, see section 6. By duality similar properties hold for $R^i p_* C$ with $i = 4, 5, 6$. Using these properties together with Lemma 7.1 we can follow closely the proof of Thm. 1.3 in [11]: First construct projectors $\pi_{2r}$ which are constituents of $(P^2r/X)_{alg}$ for $0 \leq r \leq 3$. This follows directly from Lemma 7.1 as in Step II of [11, section 1.7.]. Step III from [11, section 1.7.] is valid by the vanishing observations above. As in Step IV of loc. cit. this implies that we have a decomposition into motives in $CHM(k)$

\[
M^{2r-1} = (A, P^{2r-1}, 0) \ (1 \leq r \leq d), \quad M^{2r}_{trans} = (A, P^{2r}_{trans}, 0) \ (0 \leq r \leq d), \\
M^{2r}_{alg} = (A, P^{2r}_{alg}, 0) \ (0 \leq r \leq d),
\]

plus additional boundary motives $M^2_\alpha$ for each stratum $X_\alpha$. As in Step V of [11] we can split $M^{2r}_{alg}$ further. The projectors constructed in this way define a set of Chow–Künneth projectors $\pi_i$ for $i \neq 4, 5, 6$, since the relative projectors which contribute to more than one cohomology only affect cohomological degrees 4, 5 and 6.

\[\square\]

**Remark 7.3.** If $H^1(X, S^2V_1)$ vanishes or consists of algebraic $(2, 2)$ Hodge classes only, then we even obtain a complete Chow–Künneth decomposition in the same way, since algebraic Hodge $(p,p)$–classes define Lefschetz motives $Z(-p)$ which can be split off by projectors in a canonical way. Therefore the Hodge conjecture on $A$ would imply a complete Chow–Künneth decomposition. However the Hodge conjecture is not very far from proving the total decomposition directly.

### 7.2. Motivic decomposition conjecture

The goal of this paragraph is to sketch the proof of the motivic decomposition conjecture [11] in the case we treat in this paper. The complete details for the following argument will be published in a future publication. First note that since $A$ is an abelian variety we can use the work of Deninger and Murre ([6]) on Chow–Künneth decompositions of Abelian schemes to obtain relative Chow–Künneth projectors for $A/X$. To actually get relative Chow–Künneth projectors for $\overline{A/X}$, we observe the following.

Recall our results in section 4. We showed that the special fibres over the smooth elliptic cusp curves $D_i$ are of the form $Y_s = E \times \mathbb{P}^1 \times \mathbb{P}^1$. We do not need the cycle class map

\[CH_s(Y_s \times Y_s) \to H_s(Y_s \times Y_s)\]
to be an isomorphism as in \cite{10} Thm. 1. Since the boundary strata on $\overline{X}$ are smooth elliptic curves it is sufficient to know the Hodge conjecture for the special fibres. But the special fibres are composed of elliptic curves and rational varieties by our results in section \[. Therefore the methods in \cite{10} can be refined to work also in this case and we can drop the assumption in theorem \[7.2\].

**Remarks 7.4.** We hope to come back to this problem later and prove the motivic decomposition conjecture for all Picard families. The existence of absolute Chow–Künneth decompositions however seems to be out of reach for other examples since vanishing results will hold only for large arithmetic subgroups, i.e., small level.

8. **Appendix: Algebraic $L^2$–sub complexes of symmetric powers of the uniformizing bundle of a two–dimensional complex ball quotient**

$\overline{X}$ a 2-dim projective variety with a normal crossing divisor $D$, $X = \overline{X} \setminus D$; assume that the coordinates near the divisor are $z_1, z_2$.

Consider the uniformizing bundle of a 2-ball quotient

$$E = \left( \Omega^1_X(\log D) \otimes \mathcal{K}_X^{-1/3}(\log D) \right) \oplus \mathcal{K}_X^{-1/3}(\log D)$$

We consider two cases: 1) $D$ is a smooth divisor (the case we need) and 2) $D$ is a normal crossing divisor.

**Case 1:** Assume that $D$ is defined by $z_1 = 0$. Taking $v$ as the generating section of $\mathcal{K}_X^{-1/3}(\log D)$, $\frac{dz_1}{z_1} \otimes v, dz_2 \otimes v$ as the generating sections of $\Omega^1_X(\log D) \otimes \mathcal{K}_X^{-1/3}(\log D)$, then the Higgs field

$$\vartheta : E \to E \otimes \Omega^1_X(\log D)$$

is defined by setting $\vartheta(\frac{dz_1}{z_1} \otimes v) = v \otimes \frac{dz_1}{z_1}$, $\vartheta(dz_2 \otimes v) = v \otimes dz_2$, and $\vartheta(v) = 0$.

Clearly, if $\vartheta$ is written as $N_1 \frac{dz_1}{z_1} + N_2 dz_2$, then $N_1(\frac{dz_1}{z_1} \otimes v) = v$, $N_1(dz_2 \otimes v) = 0$, $N_1(v) = 0$, $N_2(\frac{dz_1}{z_1} \otimes v) = 0$, $N_2(dz_2 \otimes v) = v$, $N_2(v) = 0$; the kernel of $N_1$ is the subsheaf generated by $dz_2 \otimes v$ and $v$. Using the usual notation, we then have

$$\text{Gr}_1 W(N_1) = \text{generated by } \frac{dz_1}{z_1} \otimes v$$
$$\text{Gr}_0 W(N_1) = \text{generated by } dz_2 \otimes v$$
$$\text{Gr}_{-1} W(N_1) = \text{generated by } v.$$
So, one has

$$\Omega^0(E)(2) = z_1\left\{ \frac{dz_1}{z_1} \otimes v \right\} + \left\{ dz_2 \otimes v \right\} + \left\{ v \right\}$$

$$= \text{Ker} N_1 + z_1 E;$$

$$\Omega^1(E)(2) = \frac{dz_1}{z_1} \otimes \left( z_1\left\{ \frac{dz_1}{z_1} \otimes v \right\} + z_1\left\{ dz_2 \otimes v \right\} + z_1 \left\{ v \right\} \right)$$

$$+ dz_2 \otimes \left( z_1\left\{ \frac{dz_1}{z_1} \otimes v \right\} + \left\{ dz_2 \otimes v \right\} + \left\{ v \right\} \right)$$

$$= \frac{dz_1}{z_1} \otimes z_1 E + dz_2 \otimes \left( \text{Ker} N_1 + z_1 E \right);$$

$$\Omega^2(E)(2) = \frac{dz_1}{z_1} \wedge dz_2 \otimes \left( z_1\left\{ \frac{dz_1}{z_1} \otimes v \right\} + z_1\left\{ dz_2 \otimes v \right\} + z_1 \left\{ v \right\} \right)$$

$$= \frac{dz_1}{z_1} \wedge dz_2 \otimes z_1 E,$$

where $\{x\}$ represents the line bundle generated by an element $x$.

**Case 2:** As before, taking $v$ as the generating section of $K^{-1/3}_X (\log D)$, $\frac{dz_1}{z_1} \otimes v$, $\frac{dz_2}{z_2} \otimes v$ as the generating sections of $\Omega^1_X (\log D) \otimes K^{-1/3}_X (\log D)$, then the Higgs field

$$\vartheta : E \rightarrow E \otimes \Omega^1_X (\log D)$$

is defined by setting $\vartheta(\frac{dz_1}{z_1} \otimes v) = v \otimes \frac{dz_1}{z_1}$, $\vartheta(\frac{dz_2}{z_2} \otimes v) = v \otimes \frac{dz_2}{z_2}$, and $\vartheta(v) = 0$.

Clearly, if $\vartheta$ is written as $N_1 \frac{dz_1}{z_1} + N_2 \frac{dz_2}{z_2}$, then $N_1(\frac{dz_1}{z_1} \otimes v) = v$, $N_1(\frac{dz_2}{z_2} \otimes v) = 0$, $N_1(v) = 0$, $N_2(\frac{dz_1}{z_1} \otimes v) = 0$, $N_2(\frac{dz_2}{z_2} \otimes v) = v$, $N_2(v) = 0$; the kernel of $N_1$ (resp. $N_2$) is the subsheaf generated by $\frac{dz_2}{z_2} \otimes v$ (resp. $\frac{dz_1}{z_1} \otimes v$) and $v$. We
then have

\[
\begin{align*}
\text{Gr}_1 W(N_1) &= \text{generated by } \frac{dz_1}{z_1} \otimes v \\
\text{Gr}_0 W(N_1) &= \text{generated by } \frac{dz_2}{z_2} \otimes v \\
\text{Gr}_{-1} W(N_1) &= \text{generated by } v \\
\text{Gr}_1 W(N_2) &= \text{generated by } \frac{dz_2}{z_2} \otimes v \\
\text{Gr}_0 W(N_2) &= \text{generated by } \frac{dz_1}{z_1} \otimes v \\
\text{Gr}_{-1} W(N_2) &= \text{generated by } v \\
\text{Gr}_1 W(N_1 + N_2) &= \text{generated by } \left(\frac{dz_1}{z_1} + \frac{dz_2}{z_2}\right) \otimes v \\
\text{Gr}_0 W(N_1 + N_2) &= \text{generated by } \left(\frac{dz_1}{z_1} - \frac{dz_2}{z_2}\right) \otimes v \\
\text{Gr}_{-1} W(N_1 + N_2) &= \text{generated by } v
\end{align*}
\]

So, one has

\[
\begin{align*}
\Omega^0(E)(2) &= z_1\left\{\frac{dz_1}{z_1} \otimes v\right\} + z_2\left\{\frac{dz_2}{z_2} \otimes v\right\} + \{v\} \\
&= \text{Ker}N_1 \cap \text{Ker}N_2 + z_2\text{Ker}N_1 + z_1\text{Ker}N_2; \\
\Omega^1(E)(2) &= \frac{dz_1}{z_1} \otimes (z_1\left\{\frac{dz_1}{z_1} \otimes v\right\} + z_1 z_2\left\{\frac{dz_2}{z_2} \otimes v\right\} + z_1\{v\}) \\
&\quad + \frac{dz_2}{z_2} \otimes (z_2\left\{\frac{dz_2}{z_2} \otimes v\right\} + z_1 z_2\left\{\frac{dz_1}{z_1} \otimes v\right\} + z_2\{v\}) \\
&= \frac{dz_1}{z_1} \otimes (z_1\text{Ker}N_2 + z_1 z_2\text{Ker}N_1) + \frac{dz_2}{z_2} \otimes (z_2\text{Ker}N_1 + z_1 z_2\text{Ker}N_2); \\
\Omega^2(E)(2) &= \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \otimes z_1 z_2 E,
\end{align*}
\]

For the above two cases, it is easy to check that \(\vartheta(\Omega^0(E)(2)) \subset \Omega^1(E)(2)\) and \(\vartheta(\Omega^1(E)(2)) \subset \Omega^2(E)(2)\). Thus, together \(\vartheta \wedge \vartheta = 0\), we have the complex

\[
0 \to \Omega^0(E)(2) \to \Omega^1(E)(2) \to \Omega^2(E)(2) \to 0
\]

with \(\vartheta\) as the boundary operator.
Now we take the 2nd-order symmetric power of \((E, \vartheta)\), we obtain a new Higgs bundle \(S^2(E, \vartheta)\) (briefly, the Higgs field is still denoted by \(\vartheta\)) as follows,

\[
S^2(E, \vartheta) = S^2\left(\Omega^1_X(\log D)\right) \otimes K^{-2/3}_X(\log D) \oplus \Omega^1_X(\log D) \otimes K^{-2/3}_X(\log D) \oplus K^{-2/3}_X(\log D).
\]

The Higgs field \(\vartheta\) maps \(S^2(E, \vartheta)\) into \(S^2(E, \vartheta) \otimes \Omega^1_X(\log D)\) and \(S^2(E, \vartheta) \otimes \Omega^1_X(\log D)\) into \(S^2(E, \vartheta) \otimes \Omega^2_X(\log D)\) so that one has a complex with the differentiation \(\vartheta\) as follows

\[
(*) \quad 0 \to S^2(E, \vartheta) \to S^2(E, \vartheta) \otimes \Omega^1_X(\log D) \to S^2(E, \vartheta) \otimes \Omega^2_X(\log D) \to 0;
\]

more precisely, one has

\[
\begin{align*}
\vartheta\left(S^2\left(\Omega^1_X(\log D)\right) \otimes K^{-2/3}_X(\log D)\right) & \subset \left(\Omega^1_X(\log D) \otimes K^{-2/3}_X(\log D)\right) \otimes \Omega^1_X(\log D) \\
\vartheta\left(\Omega^1_X(\log D) \otimes K^{-2/3}_X(\log D)\right) & \subset K^{-2/3}_X(\log D) \otimes \Omega^1_X(\log D) \\
\vartheta\left(K^{-2/3}_X(\log D)\right) & = 0
\end{align*}
\]

and

\[
\begin{align*}
\vartheta\left(\left(S^2\left(\Omega^1_X(\log D)\right) \otimes K^{-2/3}_X(\log D)\right) \otimes \Omega^1_X(\log D)\right) & \subset \left(\Omega^1_X(\log D) \otimes K^{-2/3}_X(\log D)\right) \otimes \Omega^2_X(\log D) \\
\vartheta\left(\Omega^1_X(\log D) \otimes K^{-2/3}_X(\log D) \otimes \Omega^1_X(\log D)\right) & \subset K^{-2/3}_X(\log D) \otimes \Omega^2_X(\log D) \\
\vartheta\left(K^{-2/3}_X(\log D) \otimes \Omega^1_X(\log D)\right) & = 0
\end{align*}
\]

Note: Let \(V\) be a \(SL(2)\)-module, then

\[
S^2 V \otimes V \simeq S^3 V \oplus V \otimes \wedge^2 V.
\]

In general, one needs to consider the representations of \(GL(2)\); in such a case, we can take the determinant of the representation in question, and then go back to a representation of \(SL(2)\).
\[ S^2(E, \vartheta) = S^2(\Omega_X^1(\log D)) \otimes K_X^{-2/3}(\log D) \]
\[ \oplus \Omega_X^1(\log D) \otimes K_X^{-2/3}(\log D) \]
\[ \oplus K_X^{-2/3}(\log D) \]

\[ S^2(E, \vartheta) \otimes \Omega_X^1(\log D) = (S^2(\Omega_X^1(\log D)) \otimes K_X^{-2/3}(\log D)) \otimes \Omega_X^1(\log D) \]
\[ \oplus (\Omega_X^1(\log D) \otimes K_X^{-2/3}(\log D)) \otimes \Omega_X^1(\log D) \]
\[ \oplus K_X^{-2/3}(\log D) \otimes \Omega_X^1(\log D) \]

Assuming that the divisor \( D \) is smooth, we next want to consider the \( L^2 \)-holomorphic Dolbeault sub-complex of the above complex (*):

\[ 0 \to (S^2(E, \vartheta)) \to (S^2(E, \vartheta) \otimes \Omega_X^1(\log D)) \to (S^2(E, \vartheta) \otimes \Omega_X^2(\log D)) \to 0, \]
and explicitly write down \((S^2(E, \vartheta) \otimes \Omega_X^i(\log D))(2)\).

Note that taking symmetric power for \( L^2 \)-complex does not have obvious functorial properties in general.

We will continue to use the previous notations. For simplicity, we will further set \( v_1 = d_{z_1} \otimes v \) and \( v_2 = d_{z_2} \otimes v \); we also denote \( e_1 \otimes e_2 + e_2 \otimes e_1 \) by \( e_1 \otimes e_2 \), the symmetric product of the vectors \( e_1 \) and \( e_2 \).

Thus, \( S^2(\Omega_X^1(\log D)) \otimes K_X^{-2/3}(\log D) \), as a sheaf, is generated by \( v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_2; \Omega_X^1(\log D) \otimes K_X^{-2/3}(\log D) \) is generated by \( v_1 \otimes v, v_2 \otimes v; \) and \( K_X^{-2/3}(\log D) \) is generated by \( v \otimes v \). Also, it is easy to check how \( N_1, N_2 \) act on these generators; as for \( N_1 \), we have (Note \( N_1 v_1 = v, N_1 v_2 = 0, N_1 v = 0 \).)

\[ N_1(v_1 \otimes v_1) = 2v_1 \otimes v \]
\[ N_1(v_1 \otimes v_2) = v_2 \otimes v \]
\[ N_1(v_2 \otimes v_2) = 0 \]
\[ N_1(v_1 \otimes v) = v \otimes v \]
\[ N_1(v_2 \otimes v) = 0 \]
\[ N_1(v \otimes v) = 0. \]

Clearly, \( N_1 \) maps \( S^2(\Omega_X^1(\log D)) \otimes K_X^{-2/3}(\log D) \) into \( \Omega_X^1(\log D) \otimes K_X^{-2/3}(\log D) \), \( \Omega_X^1(\log D) \otimes K_X^{-2/3}(\log D) \) into \( K_X^{-2/3}(\log D) \), and then \( K_X^{-2/3}(\log D) \) to 0. So, \( N_1 \) is of index 3 on the 2\(^{nd}\)-order symmetric power \( S^2 E \) (as is obvious from the abstract theory since \( N_1 \) is of index 2 on \( E \)); and we then have the following
gradings
\[ \text{Gr}_2W(N_1) = \text{generated by } v_1 \odot v_1 \]
\[ \text{Gr}_1W(N_1) = \text{generated by } v_1 \odot v_2 \]
\[ \text{Gr}_0W(N_1) = \text{generated by } v_1 \odot v, v_2 \odot v_2 \]
\[ \text{Gr}_{-1}W(N_1) = \text{generated by } v_2 \odot v \]
\[ \text{Gr}_{-2}W(N_1) = \text{generated by } v \odot v. \]

(Note that \( N_1 \), acting on \( E \), has two invariant (irreducible) components, one being generated by \( v_1, v \), the other by \( v_2 \), so that \( N_1 \) has three invariant components on \( S^2E \), as is explicitly showed in the above gradings.)

Now we can write down \( L^2 \)-holomorphic sections of \( S^2E \), namely the sections generated by \( v_1 \odot v, v_2 \odot v_2, v_2 \odot v, v \odot v \), and \( z_1S^2E \); in the invariant terms, they should be
\[ (S^2(E, \vartheta))(2) = E \odot \text{Im}N_1 + S^2(\text{Ker}N_1) + z_1S^2E. \]

Now it is easy to also write down \( (S^2(E, \vartheta) \otimes \Omega^1_{X}(\log D))(2) \) and \( (S^2(E, \vartheta) \otimes \Omega^2_{X}(\log D))(2) \):
\[ (S^2(E, \vartheta) \otimes \Omega^1_{X}(\log D))(2) = \frac{dz_1}{z_1} \otimes (S^2(\text{Im}N_1) + z_1S^2E) \]
\[ + dz_2 \otimes (E \odot \text{Im}N_1 + S^2(\text{Ker}N_1) + z_1S^2E); \]
\[ (S^2(E, \vartheta) \otimes \Omega^2_{X}(\log D))(2) = \frac{dz_1}{z_1} \wedge dz_2 \otimes (S^2(\text{Im}N_1) + z_1S^2E). \]

Similarly, one can determine the algebraic \( L^2 \)-sub complex of \( S^n(E, \vartheta) \) for any \( n \in \mathbb{N} \).

9. Acknowledgement

It is a pleasure to dedicate this work to Jaap Murre who has been so tremendously important for the mathematical community. In particular we want to thank him for his constant support during so many years.

We are grateful to Bas Edixhoven, Jan Nagel and Chris Peters for organizing such a wonderful meeting in Leiden. Many thanks go to F. Grunewald, R.-P. Holzapfel, J.-S. Li and J. Schwermer for very helpful discussions.

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