**Abstract**

Relative entropy of coherence can be written as an entropy difference of the original state and the incoherent state closest to it when measured by relative entropy. The natural question is, if we generalize this situation to Tsallis or Rényi entropies, would it define good coherence measures? We define a Tsallis coherence monotone as a difference between Tsallis entropies of the original state and the incoherent state closest to it when measured by Tsallis relative entropy. Taking Rényi entropy instead of the Tsallis entropy, leads to the well-known distance-based Rényi coherence. We investigate the properties of Tsallis coherence, such as monotonicity and strong monotonicity, and provide continuity estimates for both Tsallis and Rényi coherences. Moreover, we look back at the generalized $f$-coherence, defined as a difference between quasi-entropies of the dephased and the original states, prove its continuity, and deduce that it is not possible to distill a higher $f$-coherence state from a lower coherence state via genuine incoherent operations.

**1 Introduction**

Quantum coherence describes the existence of quantum interference, and it is often used in thermodynamics [1, 6, 14], transport theory [22, 33], and quantum optics [10, 24], among few applications. Recently, problems involving coherence included quantification of coherence [2, 17, 20, 21, 25, 35], distribution [19], entanglement [5, 28], operational resource theory [3, 5, 9, 32], correlations [13, 15, 29], with only a few references mentioned in each. See [27] for a more detailed review.

The golden standard for any “good” coherence measure is for it to satisfy four criteria presented in [2]: vanishing on incoherent states; monotonicity under incoherent operations; strong monotonicity under incoherent operations, and convexity. Alternatively, the last two properties can be substituted by an additivity for subspace independent states, which was shown in [35]. See Preliminaries for more details.

A number of ways has been proposed as a coherence measure, but only a few satisfy all necessary criteria [2, 36, 37]. A broad class of coherence measures are defined as the minimal distance $D$ to the set of incoherent states $I$, as

$$C_D(\rho) = \min_{\delta \in I} D(\rho, \delta).$$

It was shown in [2] that for a relative entropy there is a closed expression of a distance-based coherence:

$$\min_{\delta \in I} S(\rho|\delta) = S(\rho||\Delta(\rho)) = S(\Delta(\rho)) - S(\rho),$$

(1.1)
here $\Delta(\rho)$ is the dephased state in a pre-fixed basis, see Notation 2.2.

Another approach to generate physically relevant coherence measures is to consider different incoherent operations. Normally, one considers incoherent operations (IO) [2], which have Krauss operators that each preserve the set of incoherent states (see Definition 2.3). A smaller set is called genuine incoherent operations (GIO) [8], which act trivially on incoherent states, see Definition 2.4. See [4] for a larger list of incoherent operations, and their comparison. For these types of incoherent operations one may look at similar properties as the ones presented in [2]. Restricted to GIO, one would obtain a measure of genuine coherence when it is non-negative and monotone, or a coherence monotone when it is also strongly monotone under GIO.

In [30] the following generalized genuine coherence measure was proposed:

$$C_f(\rho) = S_f(\Delta(\rho)) - S_f(\rho),$$

here $S_f(\rho)$ is a quasi entropy, which could be defined in two ways, one of which is $S_f(\rho) = -S_f(\rho\|I)$.

Here we show the operational meaning of this $f$-coherence, by showing that it is not possible to distill a higher coherence states from a lower coherence state via GIO, Theorem 3.4. To prove this result, we first show the continuity of $f$-coherence, Theorem 3.2.

Next, we investigate the properties of the following Tsallis coherence

$$CT_\alpha(\rho) := S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho),$$

here $\Delta_\alpha(\rho)$ is the closest incoherent state to $\rho$ when measured by Tsallis relative entropy, i.e.

$$S^T_\alpha(\rho\|\Delta_\alpha(\rho)) := \min_{\delta \in I} S^T_\alpha(\rho\|\delta).$$

The explicit form of $\Delta_\alpha$ is given in [21], and it is the same for Rényi and Tsallis relative entropies. This coherence was motivated by one of the expressions for the relative entropy of coherence (1.1), since $\Delta(\rho)$ is the closest incoherent state to $\rho$ when measured by the relative entropy.

Surprisingly, taking Rényi entropy above leads to the well-known distance-based Rényi coherence:

$$CR_\alpha(\rho) = \min_{\delta \in I} S^R_\alpha(\rho\|\delta) = S^R_\alpha(\rho\|\Delta_\alpha(\rho)) = S^R_\alpha(\Delta_\alpha(\rho)) - S^R_\alpha(\rho).$$

Besides investigating properties of new Tsallis coherence, we also prove continuity estimates for both Tsallis and Rényi coherences, Theorems 5.11 and 5.12.

2 Preliminaries

2.1 Coherence

Let $\mathcal{H}$ be a $d$-dimensional Hilbert space. Let us fix a basis $\mathcal{E} = \{|j\rangle\}_{j=1}^d$ of vectors in $\mathcal{H}$.  

2.1 Definition. A state $\delta$ is called incoherent if it can be represented as follows $\delta = \sum_j \delta_j |j\rangle \langle j|$.

2.2 Notation. Denote the set of incoherent states for a fixed basis $\mathcal{E} = \{|j\rangle\}_{j}$ as $\mathcal{I} = \{\rho = \sum_j p_j |j\rangle \langle j|\}$. A dephasing operation in $\mathcal{E}$ basis is the following map:

$$\Delta(\rho) = \sum_j \langle j| \rho |j\rangle |j\rangle \langle j|.$$
2.3 Definition. A CPTP map $\Phi$ with the following Kraus operators

$$\Phi(\rho) = \sum_n K_n \rho K_n^* ,$$

is called the incoherent operation (IO) or incoherent CPTP (ICPTP), when the Kraus operators satisfy

$$K_n I K_n^* \subset I , \text{ for all } n ,$$

besides the regular completeness relation $\sum_n K_n^* K_n = I$.

Any reasonable measure of coherence $C(\rho)$ should satisfy the following conditions

- (C1) $C(\rho) \geq 0$, and $C(\rho) = 0$ if and only if $\rho \in I$;
- (C2) Non-selective monotonicity under IO maps (monotonicity)

$$C(\rho) \geq C(\Phi(\rho)) ;$$

- (C3) Selective monotonicity under IO maps (strong monotonicity)

$$C(\rho) \geq \sum_n p_n C(\rho_n) ,$$

where $p_n$ and $\rho_n$ are the outcomes and post-measurement states

$$\rho_n = \frac{K_n \rho K_n^*}{p_n}, \quad p_n = \text{Tr} K_n \rho K_n^* .$$

- (C4) Convexity,

$$\sum_n p_n C(\rho_n) \geq C \left( \sum_n p_n \rho_n \right) ,$$

for any sets of states $\{\rho_n\}$ and any probability distribution $\{p_n\}$.

Alternatively, instead of the last two conditions, one can impose the following one

- (C5) Additivity for subspace-independent states: For $p_1 + p_2 = 1$, $p_1, p_2 \geq 0$, and any two states $\rho_1$ and $\rho_2$,

$$C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2) .$$

In [35] it was shown that (C3) and (C4) are equivalent to (C5) condition.

These properties are parallel with the entanglement measure theory, where the average entanglement is not increased under the local operations and classical communication (LOCC). Notice that coherence measures that satisfy conditions (C3) and (C4) also satisfies condition (C2).

In [8] a class of incoherence operations was defined, called genuinely incoherent operations (GIO) as quantum operations that preserve all incoherent states.

2.4 Definition. An IO map $\Lambda$ is called a genuinely incoherent operation (GIO) is for any incoherent state $\delta \in I$,

$$\Lambda(\delta) = \delta .$$
An operation $\Lambda$ is GIO if and only if all Kraus representations of $\Lambda$ has all Kraus operators diagonal in a pre-fixed basis \[8\].

Conditions (C2), (C3) and (C4) can be restricted to GIO maps to obtain different classes of coherence measures.

2.5 Definition. In this case, a measure of genuine coherence satisfies at least (C1) and (C2). And if a coherence measure fulfills conditions (C1), (C2), (C3) it is called genuine coherence monotone.

A larger class than GIO was defined in \[32, 34\].

2.6 Definition. An IO map $\Lambda$ is called strictly incoherent operation (SIO) if its Kraus representation operator commute with dephasing, i.e. for $\Lambda(\rho) = \sum_j K_j \rho K_j^*$, we have for any $j$,

$$K_j \Delta(\rho) K_j^* = \Delta(K_j \rho K_j^*).$$

Since Kraus operators of GIO maps are diagonal in $E$ basis, any GIO map is SIO as well, i.e. $GIO \subset SIO$, \[8\].

A class of operators generalizing SIO was introduced in \[3\].

2.7 Definition. An IO map $\Lambda$ is called dephasing-incoherent operation (DIO) if it itself commute with dephasing operator, i.e.

$$\Lambda(\Delta(\rho)) = \Delta(\Lambda(\rho)).$$

Thus, we have $GIO \subset SIO \subset DIO$.

One may consider an additional property, closely related to the entanglement theory:

- (C6) Uniqueness for pure states: for any pure state $|\psi\rangle$ coherence takes the form:

$$C(\psi) = S(\Delta(|\psi\rangle)),$$

where $S$ is the von Neumann entropy and $\Delta$ is the dephasing operation defined as

$$\Delta(\rho) = \sum_j \langle j | \rho | j \rangle | j \rangle \langle j |.$$

2.2 Renyi and Tsallis coherences

Relative entropy of coherence

$$C(\rho) = \min_{\delta \in \mathcal{Z}} S(\rho\|\delta) = S(\rho\|\Delta(\rho)) = S(\Delta(\rho)) - S(\rho). \tag{2.1}$$

The fact that $\Delta(\rho)$ is the closest incoherent state to $\rho$ when measured by relative entropy was shown in \[].

Having three forms of relative entropy of coherence, Renyi and Tsallis coherences have been defined multiple ways. Recall, that Tsallis entropy is defined as for $\alpha \in (0, 2]$

$$S^T_\alpha(\rho) = \frac{1}{1-\alpha} \left[ \text{Tr} \rho^\alpha - 1 \right],$$

Tsallis relative entropy is defined as

$$S^T_\alpha(\rho\|\delta) = \frac{1}{\alpha-1} \left[ \text{Tr} (\rho^\alpha \delta^{1-\alpha}) - 1 \right].$$
Rényi entropy is defined as for $\alpha \in (0, \infty)$

$$S^R_\alpha (\rho) = \frac{1}{1 - \alpha} \log \text{Tr} \rho^\alpha ,$$

and Renyi relative entropy is defined as

$$S^R_\alpha (\rho\|\delta) = \frac{1}{\alpha - 1} \log \text{Tr} \left( \rho^\alpha \delta^{1-\alpha} \right).$$

Motivated by different forms involved in the definition of relative entropy of coherence\[2.1\] Renyi coherence can be defined as

\begin{align*}
CR^1_\alpha (\rho) &= \min_{\delta \in \mathcal{I}} S^R_\alpha (\rho\|\delta) , \quad (2.2) \\
CR^2_\alpha (\rho) &= S^R_\alpha (\Delta(\rho)) - S^R_\alpha (\rho) , \quad (2.3) \\
CR^3_\alpha (\rho) &= S^R_\alpha (\rho\|\Delta(\rho)) . \quad (2.4)
\end{align*}

The first definition $CR^1_\alpha$ is a particular case of any distance-based coherence \[\], and was separately discussed in\[26\]. The second definition $CR^2_\alpha$ was introduced in\[7\]. The third definition $CR^3_\alpha$ was introduced in\[4\].

Similarly, Tsallis coherence can be defined as

\begin{align*}
CT^1_\alpha (\rho) &= \min_{\delta \in \mathcal{I}} S^T_\alpha (\rho\|\delta) , \quad (2.5) \\
CT^2_\alpha (\rho) &= S^T_\alpha (\Delta(\rho)) - S^T_\alpha (\rho) . \quad (2.6)
\end{align*}

These definitions are all different, in particular, due to the fact that the closest incoherent state to a state $\rho$ when measured by either Renyi or Tsallis relative entropy is not a state $\Delta(\rho)$. From\[4\,21\] the closest incoherent state to a state $\rho$ for either Rényi or Tsallis relative entropies is

$$\Delta_\alpha (\rho) = \frac{1}{N(\rho)} \sum_j \langle j | \rho^\alpha | j \rangle^{1/\alpha} | j \rangle \langle j | \in \mathcal{I} , \quad (2.7)$$

where $N(\rho) = \sum_j \langle j | \rho^\alpha | j \rangle^{1/\alpha}$. The corresponding relative entropy becomes

$$CT^1_\alpha (\rho) = S^T_\alpha (\rho\|\Delta_\alpha (\rho)) = \frac{1}{\alpha - 1} \left[ N(\rho)^\alpha - 1 \right] , \quad (2.8)$$

and

$$CR^1_\alpha (\rho) = S^R_\alpha (\rho\|\Delta_\alpha (\rho)) = \frac{\alpha}{\alpha - 1} \log N(\rho) . \quad (2.9)$$

Interestingly enough difference-based Tsallis coherence when $\alpha = 2$ is related to the distance-based coherence induced by the Hilbert-Schmidt distance\[8\]

$$C^H_2 (\rho) := \min_{\delta \in \mathcal{I}} \| \rho - \delta \|^2_2 = S^T_2 (\Delta(\rho)) - S^T_2 (\rho) ,$$

where $\| \rho - \delta \|^2_2 = \text{Tr} (\rho - \sigma)^2$. 
2.3 Generalized coherences

Any proper distance $D(\rho, \sigma)$ between two quantum states, can induce a potential candidate for coherence. The distance-based coherence measure is defined as follows \[2\].

2.8 Definition.

$$CD(\rho) := \min_{\delta \in \mathcal{I}} D(\rho, \delta) ,$$

i.e. the minimal distance between the state $\rho$ and the set of incoherent states $\mathcal{I}$ measured by the distance $D$.

- (C1) is satisfied whenever $D(\rho, \delta) = 0$ iff $\rho = \delta$.
- (C2) is satisfied whenever $D$ is contracting under CPTP maps, i.e. $D(\rho, \sigma) \geq D(\Phi(\rho), \Phi(\sigma))$.
- (C4) is satisfied whenever $D$ is jointly convex.

Since the relative entropy, Renyi and Tsallis relative entropies satisfy all three above conditions for $\alpha \in [0, 1)$, (C1), (C2), and (C4) are satisfied for $C(\rho), CR_1^\alpha, CT_1^\alpha$.

Another generalization was considered in \[30\], which is based on quasi-relative entropy.

2.9 Definition. For strictly positive bounded operators $A$ and $B$ acting on a finite-dimensional Hilbert space $\mathcal{H}$, and for any continuous function $f : (0, \infty) \to \mathbb{R}$, the quasi-relative entropy (or sometimes referred to as the $f$-divergence) is defined as

$$S_f(A\|B) = \text{Tr}(f(L_B R_A^{-1}) A) ,$$

where left and right multiplication operators are defined as $L_B(X) = BX$ and $R_A(X) =XA$.

Having the spectral decomposition of operators one can calculate the quasi-relative entropy explicitly \[12, 31\]. Let $A$ and $B$ have the following spectral decomposition

$$A = \sum_j \lambda_j \langle \phi_j | \langle \phi_j | , \quad B = \sum_k \mu_k \langle \psi_k | \langle \psi_k | .$$

Here the sets \{|$\phi_j$\rangle\langle$\phi_j$|\}$j,k$, \{|$\psi_k$\rangle\langle$\psi_k$|\}$j,k$ form orthonormal bases of $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators. By \[31\], the quasi-relative entropy is calculated as follows

$$S_f(A\|B) = \sum_{j,k} \lambda_j f \left( \frac{\mu_k}{\lambda_j} \right) |\langle \psi_k | \langle \phi_j ||^2 .$$

2.10 Assumption. To define $f$-coherence, we assume that the function $f$ is operator convex and operator monotone decreasing and $f(1) = 0$.

$f$-entropy was defined in two ways in \[30\]

$$S_f^1(\rho) := -S_f(\rho\|I) = -\sum_j \lambda_j f \left( \frac{1}{\lambda_j} \right) ,$$

$$S_f^2(\rho) := f(1/d) - S_f(\rho\|I/d) = f(1/d) - \sum_j \lambda_j f \left( \frac{1}{d\lambda_j} \right) ,$$

where \{$\lambda_j$\}$j$ are the eigenvalues of $\rho$. 
2.11 Definition. For either $f$-entropy, the $f$-coherence is then defined as

$$ C_f(\rho) := S_f(\Delta(\rho)) - S_f(\rho). $$

If $\{\lambda_j\}$ are the eigenvalues of $\rho$, and the diagonal elements of $\rho$ in $\mathcal{E}$ basis are $\chi_j = \langle j | \rho | j \rangle$, then from (2.12), we have

$$ C_f(\rho) = \sum_j \lambda_j f(\frac{1}{\lambda_j}) - \sum_j \chi_j f(\frac{1}{\chi_j}). $$

Since $f(x) = -\log(x)$ is operator convex, coherence measure defined above coincides with the relative entropy of coherence (2.1) [2]:

$$ C_{\log}(\rho) = S_{\log}(\Delta(\rho)) - S_{\log}(\rho) = S(\Delta(\rho)) - S(\rho) = C(\rho). $$

The function $f(x) = \frac{1}{1-\alpha} (1-x^{1-\alpha})$ is operator convex for $\alpha \in (0, 2)$. The coherence monotone then becomes the Tsallis relative entropy of coherence

$$ C^{1}_\alpha(\rho) = \frac{1}{1-\alpha} \left[ \sum_j \chi_j^{\alpha} - \sum_j \lambda_j^{\alpha} \right] = C_{T\alpha}^2(\rho). $$

2.4 Properties

Here we list which properties (C1)-(C5) are satisfied by which coherences and under which conditions. For Renyi and Tsallis entropies we do not consider a case when $\alpha = 1$ and the entropies reduce to the relative entropy of coherence.

|          | (C1) | (C2) | (C3) | (C4) | (C5) |
|----------|------|------|------|------|------|
| CD       | ✓    | ✓    |      | ✓    |
| $CR^1_{\alpha}$ $\alpha \in [0, 1]$ | ✓    | ✓    |      |      |
| $CR^2_{\alpha}$ $\alpha \in (0, 2]$ | ✓    | ✓    |      |      |
| $CT^1_{\alpha}$ $\alpha \in [0, 1]$ | ✓    | ✓    |      |      |
| $CT^2_{\alpha}$ $\alpha \in (0, 2]$ | ✓    | ✓    |      |      |
| $C_f$    | ✓    | ✓    |      |      |

The fact that $CT^2_{\alpha}$ and $CR^2_{\alpha}$ are monotone under GIO can be derived from GIO monotonicity of $C_f$ [30], or it was shown separately in [7]. There are examples when the monotonicity of both are violated under a larger class of operators when $\alpha > 1$.

$CT^2_{\alpha}$ satisfies a modified version of additivity (C5), which $CR^2_{\alpha}$ also violates [7],

$$ CT^2_{\alpha}(p_1 \rho_1 \oplus p_2 \rho_2) = p_1^\alpha CT^2_{\alpha}(\rho_1) + p_2^\alpha CT^2_{\alpha}. $$

(a) In [30] it was shown that $C_f$, and in particular $CR^2_{\alpha}$ and $CT^2_{\alpha}$, reach equality in the strong monotonicity under a convex mixture of diagonal unitaries in any dimension, which implies these coherences reach equality.
in strong monotonicity under GIO in 2- and 3- dimensions. Moreover, these coherences are strongly monotone under GIO on pure states in any dimension.

\[ CR^1_\alpha(\rho), CT^1_\alpha(\rho) \text{ violate strong monotonicity } \] 21 26. In 21 it was shown that \[ CT^1_\alpha(\rho) \text{ satisfies a modified version of the strong monotonicity: for } \alpha \in (0, 2] \]
\[
\sum p_n q_n^{1-\alpha} CT^1_\alpha(\rho_n) \leq CT^1_\alpha(\rho),
\]
where \( p_n = \text{Tr}(K_n \rho K_n^*) \), \( q_n = \text{Tr}(K_n \Delta_\alpha(\rho) K_n^*) \) and \( \rho_n \) is a post-measurement state.

Clearly, (C6) is not satisfied for any Renyi or Tsallis coherences in its original form, therefore it was not included in the list. However, the values of coherences on pure states can be easily calculated in some cases.

3 \text{ \textbf{f-coherence distillation}}

3.1 Continuity of \textit{f}-entropy and \textit{f}-coherence

In addition to the above list of properties of the \textit{f}-coherence, one can add its continuity in the following form.

3.1 Lemma. Let \( \rho \) and \( \sigma \) be two states such that \( \epsilon := \frac{1}{2} \| \rho - \sigma \|_1 \). Then
\[
|S^1_\text{f}(\rho) - S^1_\text{f}(\sigma)| \leq - (1 - \epsilon) f \left( \frac{1}{1 - \epsilon} \right) - \epsilon f \left( \frac{d - 1}{\epsilon} \right)
\]
\[
|S^2_\text{f}(\rho) - S^2_\text{f}(\sigma)| \leq f \left( \frac{1}{d} \right) - (1 - \epsilon) f \left( \frac{1}{d(1 - \epsilon)} \right) - \epsilon f \left( \frac{d - 1}{d \epsilon} \right).
\]

Denote either of the right hand-sides as \( H(\epsilon) \), and note that \( H \) is continuous in \( \epsilon \), and goes to zero when \( \epsilon \to 0 \).

Proof. Recall that for any convex function \( f \), the transpose of it \( \tilde{f}(x) = xf(1/x) \) is also convex. We adopt a convention \( 0 \cdot \infty = 0 \), so for a convex function \( f \) such that \( f(1) = 0 \), we have \( \tilde{f}(0) = \tilde{f}(1) = 0 \). Then \textit{f-} entropy (2.12) can be written using a transpose function as
\[ S^1_\text{f}(\rho) = -S_f(\rho\|I) = -\text{Tr}(\rho f(\rho^{-1})) = -\text{Tr}(\tilde{f}(\rho)), \]
and
\[ S^2_\text{f}(\rho) = -S_f(\rho\|I/d) = f(1/d) - \text{Tr}(\rho f(\{d\rho\}^{-1})) = f(1/d) - \frac{1}{d} \text{Tr}(\tilde{f}(d\rho)). \]

In 18 Theorem 1, it was proved that for \( S_f(\rho) = -\text{Tr}g(\rho) \) and any convex function \( g \) the following holds
\[ |S_g(\rho) - S_g(\sigma)| \leq g(1) - g(1 - \epsilon) - (d - 1) \left( g \left( \frac{\epsilon}{d - 1} \right) - g(0) \right), \]
when \( \epsilon = \frac{1}{2} \| \rho - \sigma \|_1 \). And in Corollary 3, the result was generalized for non-unit trace density matrices: let \( \rho \) and \( \sigma \) be two states of the same trace \( t \), and let \( \epsilon = \frac{1}{2} \| \rho - \sigma \|_1 \in [0, t] \), then
\[ |S_g(\rho) - S_g(\sigma)| \leq g(t) - g(t - \epsilon) - (d - 1) \left( g \left( \frac{\epsilon}{d - 1} \right) - g(0) \right). \]
Adapting this result to our situation, it holds that
\[ |S_f^R(\rho) - S_f^R(\sigma)| \leq -(1 - \epsilon)f \left( \frac{1}{1 - \epsilon} \right) - \epsilon f \left( \frac{d - 1}{\epsilon} \right). \]
And similarly, for \( \tilde{\epsilon} := \frac{d}{2} \| d\rho - d\sigma \|_1 \in [0, d] \)
\[ |S_f^R(\rho) - S_f^R(\sigma)| \]
\[ = \frac{1}{d} \left| \text{Tr}(\tilde{f}(d\rho)) - \text{Tr}(\tilde{f}(d\sigma)) \right| \]
\[ \leq \frac{1}{d} \left[ \tilde{f}(d) - \tilde{f}(d - \tilde{\epsilon}) - (d - 1) \left( \tilde{f}(\tilde{\epsilon}/(d - 1)) - \tilde{f}(0) \right) \right] \]
\[ = f \left( \frac{1}{d} \right) - (1 - \epsilon)f \left( \frac{1}{d(1 - \epsilon)} \right) - \epsilon f \left( \frac{d - 1}{de} \right). \]

From this continuity result, one can obtain continuity of the \( f \)-coherence.

**3.2 Theorem.** Let \( \rho \) and \( \sigma \) be two states such that \( \epsilon := \frac{1}{2} \| \rho - \sigma \|_1 \). Let \( H(\epsilon) \) be as in the previous theorem for the corresponding \( f \)-entropy. Then for \( f \)-coherences we obtain
\[ |C_f(\rho) - C_f(\sigma)| \leq 2H(\epsilon). \]

**Proof.** Let \( \rho \) and \( \sigma \) be two states with \( \epsilon = \frac{1}{2} \| \rho - \sigma \|_1 \). Since \( \| \cdot \|_1 \) is monotone under CPTP maps, in particular, under dephasing operation, it follows that
\[ \| \Delta(\rho) - \Delta(\sigma) \|_1 \leq \| \rho - \sigma \|_1 \leq 2\epsilon. \]
Therefore, from continuity results above Theorem 3.1 for either \( f \)-coherence and the corresponding \( f \)-entropy, we obtain
\[ |C_f(\rho) - C_f(\sigma)| \]
\[ \leq |S_f(\Delta(\rho)) - S_f(\Delta(\sigma))| + |S_f(\rho) - S_f(\sigma)| \]
\[ \leq 2H(\epsilon). \]

**3.2 Coherence distillation**

In [8] it was shown that it is not possible to distill a higher coherence state \( \sigma \) from a lower coherence state \( \rho \) via GI operations when coherence is measured by a relative entropy of coherence (which equal to the distillable coherence). The same result holds for \( f \)-coherences as well, which relies on the continuity property of coherence above, and the GIO monotonicity of \( f \)-coherence [30]. For completeness sake, we present the adapted proof from [8] below.

**3.3 Definition.** A state \( \sigma \) can be distilled from the state \( \rho \) at rate \( 0 < R \leq 1 \) if there exists an operation \( \rho^{} \otimes n \rightarrow \tau \) such that \( \| \text{Tr}_{ref} \tau - \sigma^{} \otimes nR \|_1 \leq \epsilon \) and \( \epsilon \rightarrow 0 \) as \( n \rightarrow \infty \). The optimal rate at which distillation is possible is the supremum of \( R \) over all protocols fulfilling the aforementioned conditions.
3.4 Theorem. Given two states $\rho$ and $\sigma$ such that 

$$C_f(\rho) < C_f(\sigma),$$

it is not possible to distill $\sigma$ from $\rho$ at any rate $R > 0$ via GIO operations.

Proof. Suppose the contradiction holds, assume that there are two states $\rho$ and $\sigma$ such that $C_f(\rho) < C_f(\sigma)$, and that the distillation is possible. In particular, for large enough $n$, it is possible to approximate one copy of $\sigma$. In other words, for any $\epsilon > 0$, there is a GIO $\Lambda$ such that 

$$\|\text{Tr}_{n-1}\Lambda(\rho^\otimes n) - \sigma\|_1 \leq \epsilon.$$ 

By Lemma 12 in [8], there exists a GIO $\tilde{\Lambda}$ acting only on one copy of $\rho$, such that

$$\text{Tr}_{n-1}\Lambda(\rho^\otimes n) = \tilde{\Lambda}(\rho).$$

Thus, for any $\epsilon > 0$, there is a GIO $\tilde{\Lambda}$ such that

$$\|\tilde{\Lambda}(\rho) - \sigma\|_1 \leq \epsilon.$$ 

Using the asymptotic continuity of $f$-coherence, Theorem 3.2 for these two $\epsilon$-close states, we obtain

$$\left|C_f(\tilde{\Lambda}(\rho)) - C_f(\sigma)\right| \leq H(\epsilon).$$

Recall that $H(\epsilon)$ for either $f$-coherence is continuous in $\epsilon \in (0,1)$ and it goes to zero when $\epsilon \to 0$. Therefore, summarizing from the beginning, for any $\delta > 0$, there is GIO $\tilde{\Lambda}$ such that

$$\left|C_f(\tilde{\Lambda}(\rho)) - C_f(\sigma)\right| < \delta. \quad (3.1)$$

Take $\delta := \frac{1}{2}(C_f(\sigma) - C_f(\rho)) > 0$. Since $C_f$ is GIO monotone, for any GIO $\Lambda$, we have

$$C_f(\tilde{\Lambda}(\rho)) \leq C_f(\rho).$$

Therefore,

$$\delta \leq \frac{1}{2}(C_f(\sigma) - C_f(\tilde{\Lambda}(\rho)) < C_f(\sigma) - C_f(\tilde{\Lambda}(\rho)) .$$

This is a contradiction to (3.1).

\[ \square \]

4 New Renyi and Tsallis coherences

Playing off the last expression in the definition of the relative entropy of coherence 2.1 we define coherence measure as follows:

$$CT_\alpha(\rho) := S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho),$$

for Tsallis entropy, and

$$CR_\alpha(\rho) := S^R_\alpha(\Delta_\alpha(\rho)) - S^R_\alpha(\rho),$$
for Renyi entropy. Recall that here $\Delta_\alpha(\rho)$ is the closest incoherent state to $\rho$ when measured by the Rényi or Tsallis relative entropy, i.e.

$$S_\alpha(\rho\|\Delta_\alpha(\rho)) := \min_{\delta \in I} S_\alpha(\rho\|\delta) .$$

Recall from [2.7] that

$$\Delta_\alpha(\rho) = \frac{1}{N(\rho)} \sum_j \langle j | \rho^\alpha | j \rangle^{1/\alpha} | j \rangle \langle j | ,$$

where $N(\rho) = \sum_j \langle j | \rho^\alpha | j \rangle^{1/\alpha}$. Having this explicit form of $\Delta_\alpha(\rho)$, both coherences can be explicitly calculated

$$CT_\alpha(\rho) = \frac{1}{1 - \alpha} [\text{Tr} (\Delta_\alpha(\rho)^\alpha) - \text{Tr} \rho^\alpha]$$

$$= \frac{1}{1 - \alpha} \left[ \frac{1}{N(\rho)\alpha} - 1 \right] \text{Tr} \rho^\alpha$$

$$= \frac{N(\rho)^\alpha - 1}{\alpha - 1} \frac{\text{Tr} \rho^\alpha}{N(\rho)^\alpha}$$

$$= S^T_\alpha(\rho\|\Delta_\alpha(\rho)) \frac{\text{Tr} \rho^\alpha}{N(\rho)^\alpha}$$

$$= CT^1_\alpha(\rho) \frac{\text{Tr} \rho^\alpha}{N(\rho)^\alpha} \geq 0 .$$

The last two equalities come from (2.8). Similarly, from (2.9) for the Renyi coherence

$$CR_\alpha(\rho) = \frac{1}{1 - \alpha} [\log \text{Tr} (\Delta_\alpha(\rho)^\alpha) - \log \text{Tr} \rho^\alpha]$$

$$= \frac{1}{1 - \alpha} \left[ \log \left( \frac{1}{N(\rho)\alpha} \text{Tr} \rho^\alpha \right) - \log \text{Tr} \rho^\alpha \right]$$

$$= \frac{\alpha}{\alpha - 1} \log N(\rho)$$

$$= S^R_\alpha(\rho\|\Delta_\alpha(\rho))$$

$$= CR^1_\alpha(\rho) .$$

This means that for Renyi entropy of coherence we have a similar expressions to the relative entropy of coherence [2.1]

$$CR^1_\alpha(\rho) = \min_{\delta \in I} S^R_\alpha(\rho\|\delta) = S^R(\rho\|\Delta_\alpha(\rho)) = S^R_\alpha(\Delta_\alpha(\rho)) - S^R_\alpha(\rho) .$$

Therefore the distance-based Rényi coherence $CR^1_\alpha(\rho)$ coincides with the new definition $CR_\alpha(\rho)$, and therefore we will mostly focus on the new definition of Tsallis coherence $CT_\alpha(\rho)$.

5 Tsallis coherence

5.1 Positivity

As we noted above, the Tsallis coherence is non-negative. Note that this is a non-trivial statement, that cannot be directly observed by the monotonicity of entropy under linear CPTP maps, as it was done for $CT^2_\alpha, CR^2_\alpha, C_f$, since the map $\rho \rightarrow \Delta_\alpha(\rho)$ is non-linear.
5.2 Vanishing only on incoherent states

5.1 Proposition. $CT_\alpha(\rho) = 0$ if and only if $\rho \in \mathcal{I}$ is incoherent.

Proof. First, suppose that the state $\rho \in \mathcal{I}$ is incoherent, then $\Delta_\alpha(\rho) = \rho$. Therefore, $CT_\alpha(\rho) = S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho) = 0$.

Now, suppose that $C_\alpha(\rho) = 0$. From calculations above, since Tr$\rho^\alpha > 0$ for a non-zero state, this means that $S^T_\alpha(\rho\|\Delta_\alpha(\rho)) = 0$, which happens only when $\rho = \Delta_\alpha(\rho) \in \mathcal{I}$. Therefore, $\rho \in \mathcal{I}$ is incoherent. \qed

5.3 Value on pure states

Let $\rho = |\psi\rangle \langle \psi|$ be a pure state. Since $\rho^\alpha = \rho$, then

$$CT_\alpha(\rho) = \frac{1}{1-\alpha} [\text{Tr}(\Delta_\alpha(\rho)^\alpha) - \text{Tr}\rho^\alpha] = S^T_\alpha(\Delta_\alpha(\rho)) .$$

To calculate this Tsallis entropy explicitly, we note that Tr$(\Delta_\alpha(|\psi\rangle \langle \psi|)^\alpha) = N(|\psi\rangle \langle \psi|)^{-\alpha}$, where $N(|\psi\rangle \langle \psi|) = \sum_j |\langle \psi|j\rangle|^2/\alpha$. Thus,

$$CT_\alpha(\rho) = \frac{1}{1-\alpha} \left[ \left( \sum_j |\langle \psi|j\rangle|^{2/\alpha} \right)^{-\alpha} - 1 \right] .$$

5.4 Comparison with $CT^1_\alpha$

Recall that from our previous calculations,

$$CT_\alpha(\rho) = CT^1_\alpha(\rho) \frac{\text{Tr}\rho^\alpha}{N(\rho)^\alpha} .$$

Let us denote as $\lambda_j := \langle j | \rho^\alpha | j \rangle$. Then

$$\text{Tr}(\rho^\alpha) = \sum_j \langle j | \rho^\alpha | j \rangle = \|\lambda\|_1 .$$

And

$$N(\rho)^\alpha = \left( \sum_j \langle j | \rho^\alpha | j \rangle^{1/\alpha} \right)^\alpha = \|\lambda\|^{1/\alpha} .$$

Here $\| \cdot \|_p$ denotes the Schatten $p$-norm. Since Schatten $p$-norms are monotone decreasing in $p$, we have that

$CT_\alpha(\rho) \geq CT^1_\alpha(\rho) , \text{ for } 0 < \alpha < 1 ,$

and

$CT_\alpha(\rho) \leq CT^1_\alpha(\rho) , \text{ for } 1 < \alpha < 2 .$

5.5 Monotonicity

5.2 Theorem. $CT_\alpha(\rho)$ is invariant under diagonal unitaries.
Proof. Let $U = \sum_n e^{i\phi_n} |n\rangle \langle n|$ be a unitary diagonal in $\mathcal{E}$ basis. Then

$$\Delta_\alpha(U\rho U^*) = \frac{1}{\sum \langle j| U\rho^\alpha U^* |j\rangle^{1/\alpha}} \sum \langle j| U\rho^\alpha U^* |j\rangle^{1/\alpha} |j\rangle \langle j|$$

$$= \frac{1}{\sum \langle j| e^{i\phi_j} \rho^{\alpha} e^{-i\phi_j} |j\rangle^{1/\alpha}} \sum \langle j| e^{i\phi_j} \rho^{\alpha} e^{-i\phi_j} |j\rangle^{1/\alpha} |j\rangle \langle j|$$

$$= \Delta_\alpha(\rho)$$

Since the Tsallis entropy is invariant under unitaries itself, we have

$$CT_\alpha(U\rho U^*) = CT_\alpha(\rho) .$$

5.3 Lemma. If $CT_\alpha$ is monotone under any unitary (not necessarily diagonal in $\mathcal{E}$) for any state $\rho$, then $CT_\alpha$ is invariant under all unitaries for all states.

Proof. Suppose that for any unitary $U$ and any state $\rho$, Tsallis coherence satisfies

$$CT_\alpha(U\rho U^*) \leq CT_\alpha(\rho) . \quad (5.1)$$

Since Tsallis entropy is invariant under unitaries, this means that

$$S_\alpha^T(\Delta_\alpha(U\rho U^*)) \leq S_\alpha^T(\Delta_\alpha(\rho)) .$$

By definition this implies

$$\text{Tr}\{\Delta_\alpha(U\rho U^*)^\alpha\} \leq \text{Tr}\{\Delta_\alpha(\rho)^\alpha\} ,$$

or equivalently

$$[\text{Tr}\rho^\alpha]\left[\sum \langle j| U\rho^\alpha U^* |j\rangle^{1/\alpha}\right]^{-\alpha} \leq [\text{Tr}\rho^\alpha]\left[\sum \langle j| \rho^\alpha |j\rangle^{1/\alpha}\right]^{-\alpha} .$$

Since $x^\alpha$ is a monotone function for $\alpha > 0$, we have

$$\sum \langle j| \rho^\alpha |j\rangle^{1/\alpha} \leq \sum \langle j| U\rho^\alpha U^* |j\rangle^{1/\alpha} . \quad (5.2)$$

In the equation above take a unitary $V = U^*$ and a state $\omega = U\rho U^*$, so $\rho = V\omega V^*$. Then the following holds

$$\sum \langle j| V\omega^\alpha V^* |j\rangle^{1/\alpha} \leq \sum \langle j| \omega^\alpha |j\rangle^{1/\alpha} . \quad (5.3)$$

Since both equations $[5.2]$ and $[5.3]$ hold for all states and all unitaries, there must be an equality in both of them. This implies the equality in the monotonicity inequality $[5.1]$.

5.4 Theorem. $CT_\alpha$ is not monotone under all unitaries for all states.

Proof. From previous Lemma $5.3$, if we find a unitary and a state such that $CT_\alpha$ is not invariant under, then $CT_\alpha$ is not monotone under all unitaries.

Restrict the dimension to two, and denote the fixed basis as $\mathcal{E} = \{|0\rangle, |1\rangle\}$. Then any unitary can be written as

$$U = \begin{pmatrix} a & b \\ -e^{i\phi}b^* & e^{i\phi}a^* \end{pmatrix}$$
where \( |a|^2 + |b|^2 = 1 \) and some \( \phi \). Take \( \rho = |\psi\rangle \langle \psi| \) a pure normalized state:

\[
|\psi\rangle = \frac{1 - \sqrt{3}}{2\sqrt{2}} |0\rangle + \frac{1 + \sqrt{3}}{2\sqrt{2}} |1\rangle .
\]

Recall that for any pure state \( |\Psi\rangle \), Tsallis coherence is

\[
CT_\alpha(|\Psi\rangle \langle \Psi|) = S^T_\alpha(\Delta_\alpha(|\Psi\rangle \langle \Psi|)) ,
\]

and

\[
\text{Tr}(\Delta_\alpha(|\Psi\rangle \langle \Psi|)^\alpha) = N(|\Psi\rangle \langle \Psi|)^{-\alpha} .
\]

Therefore, to show that \( CT_\alpha \) is not invariant, we need to show that \( N(\rho) \) is not invariant. For \( \rho \) above, we have

\[
N(\rho) = |\langle 0|\psi\rangle|^{2/\alpha} + |\langle 1|\psi\rangle|^{2/\alpha} = 8^{-1/\alpha} \left[ (1 - \sqrt{3})^{2/\alpha} + (1 + \sqrt{3})^{2/\alpha} \right] .
\]

And calculating the action of the unitary on the state, we obtain

\[
U |\psi\rangle = \left[ a \frac{1 - \sqrt{3}}{2\sqrt{2}} + b \frac{1 + \sqrt{3}}{2\sqrt{2}} \right] |0\rangle + e^{i\phi} \left[ a^* \frac{1 + \sqrt{3}}{2\sqrt{2}} - b^* \frac{1 - \sqrt{3}}{2\sqrt{2}} \right] |1\rangle .
\]

Therefore,

\[
N(U \rho U^*) = \left| a \frac{1 - \sqrt{3}}{2\sqrt{2}} + b \frac{1 + \sqrt{3}}{2\sqrt{2}} \right|^{2/\alpha} + \left| a^* \frac{1 + \sqrt{3}}{2\sqrt{2}} - b^* \frac{1 - \sqrt{3}}{2\sqrt{2}} \right|^{2/\alpha}.
\]

There are multitude options to choose to make (5.4) and (5.5) not equal to each other for \( \alpha \neq 1 \) (e.g. \( a = b = 1/\sqrt{2} \)), proving that \( CT_\alpha \) is not invariant, and therefore not monotone under all unitaries.

5.5 **Theorem.** Tsallis coherence is not monotone under GIO.

**Proof.** Let us fix the basis \( E = \{|0\rangle, |1\rangle\} \) Let \( \rho = |\psi\rangle \langle \psi| \) be a pure state with \( |\langle 0|\psi\rangle|^2 = \chi = 3/4 \) and \( |\langle 1|\psi\rangle|^2 = 1 - \chi = 1/4 \).

For a pure state \( \rho \) the entropy is zero, and therefore

\[
CT_\alpha(\rho) = S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho)
= S^T_\alpha(\Delta_\alpha(\rho))
= \frac{1}{1 - \alpha} \left[ \text{Tr} \{ \Delta_\alpha(\rho)^\alpha \} - 1 \right]
= \frac{1}{1 - \alpha} \left[ \left( \sum_j x_j^{1/\alpha} \right)^\alpha - 1 \right]
= \frac{1}{1 - \alpha} \left[ \frac{4}{(3^{1/\alpha} + 1)^\alpha} - 1 \right].
\]

Let \( \Lambda \) be GIO, with Kraus operators \( \Lambda(\rho) = K_1 \rho K_1^* + K_2 \rho K_2^* \) where Kraus operators are diagonal in \( E \) basis

\[
K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad K_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.
\]
Figure 1: Comparison between $CT_\alpha(\rho)$ in green and $CT_\alpha(\Lambda(\rho))$ in blue as functions of $\alpha$.

Clearly $\sum_n K_n^* K_n = I$. Then

$$\Lambda(\rho) = \begin{pmatrix} \frac{3}{4} & a \\ a & \frac{1}{4} \end{pmatrix},$$

where $a = \frac{3 + \sqrt{3}}{8\sqrt{2}}$. The eigenvalues of this matrix are $\beta_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{\frac{1}{4} + 4a^2} \right)$. And the normalized eigenvector corresponding to $\beta_{1,2}$ are

$$|\psi_{1,2}\rangle = \frac{1}{\sqrt{a^2 + (\beta_{1,2} - \frac{3}{4})^2}} \begin{pmatrix} a \\ \beta_{1,2} - \frac{3}{4} \end{pmatrix}.$$

Therefore, $\text{Tr}(\lambda(\rho)^\alpha) = \beta_1^\alpha + \beta_2^\alpha$, and

$$N(\Lambda(\rho)) = \sum_j \beta_1 |\langle j|\psi_1\rangle|^{2/\alpha} + \beta_2 |\langle j|\psi_2\rangle|^{2/\alpha}.$$

And the Tsallis coherence is then

$$CT_\alpha(\Lambda(\rho)) = \frac{1}{1 - \alpha} \left[ \frac{1}{N(\Lambda(\rho))^{\alpha}} - 1 \right] \text{Tr}(\Lambda(\rho)^\alpha).$$

From Figure 1 we see that, for example, for $\alpha = 0.2$, monotonicity has failed

$$CT_\alpha(\Lambda(\rho)) > 0.5 > CT_\alpha(\rho).$$

5.6 Definition. A GIO map $\Lambda$ that commutes with $\Delta_\alpha$ is called $\alpha$-GIO.

A unitary diagonal under a fixed basis $\mathcal{E}$ is an $\alpha$-GIO for any $\alpha$. For $\alpha = 0$, $\Delta_\alpha(\rho) = \frac{1}{d} I$, which commutes with any GIO.

5.7 Theorem. Tsallis coherence is monotone under $\alpha$-GIO.

Proof. By definition

$$CT_\alpha(\rho) - CT_\alpha(\Lambda(\rho)) = S^T_\alpha(\Lambda(\rho)) - S^T_\alpha(\rho) + S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\Delta_\alpha(\Lambda(\rho))).$$

Since Tsallis entropy is monotone under CPTP maps, $S^T_\alpha(\Lambda(\rho)) - S^T_\alpha(\rho) \geq 0$. $\Lambda$ commutes with $\Delta_\alpha$, and $\Lambda$ is GIO, so it leaves the incoherent states, such as $\Delta_\alpha(\rho)$, invariant, therefore

$$S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\Delta_\alpha(\Lambda(\rho))) = S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\Lambda(\Delta_\alpha(\rho))) = 0.$$
5.6 Strong monotonicity.

5.8 Theorem. Tsallis coherence $CT_\alpha(\rho)$ reaches equality in strong monotonicity for convex mixtures of diagonal unitaries. Therefore, $CT_\alpha(\rho)$ reaches equality in strong monotonicity under GIO in two- and three-dimensions, when Kraus operators are proportional to diagonal unitaries.

Proof. Consider a GIO $\Lambda$ that is a probabilistic mixture of diagonal unitaries, i.e., let

$$\Lambda(\rho) = \sum_k \alpha_k U_k \rho U_k^*,$$

where $\alpha_j \in [0, 1]$ with $\sum \alpha_k = 1$, and for some $\phi_{kn}$, the unitaries $U_k$ are diagonal in $\mathcal{E}$. Then from Theorem 5.2 since $CT_\alpha$ is invariant under diagonal unitaries, we have

$$\sum_k \alpha_k CT_\alpha(U_k \rho U_k^*) = \left(\sum_k \alpha_k\right) CT_\alpha(\rho) = CT_\alpha(\rho).$$

In general, $CT_\alpha$ fails strong monotonicity for IO maps.

5.9 Theorem. Tsallis coherence $CT_\alpha(\rho)$ fails strong monotonicity under IO maps.

Proof. We use example from [26], which was used to show that $CR^1_\alpha$ fails strong monotonicity under IO maps. Consider a three-dimensional space spanned by standard orthonormal basis $\mathcal{E} = \{|0\rangle, |1\rangle, |2\rangle\}$. Let the density matrix be

$$\rho = \frac{1}{4} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let the Kraus operators of the IO map be

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $|a|^2 + |b|^2 = 1$ to satisfy the condition $K_1^* K_1 + K_2^* K_2 = I$. It is straightforward to check that these Kraus operators leave the space of incoherent states $\mathcal{I}$ invariant. The output states are

$$\rho_1 = \frac{1}{p_1} K_1 \rho K_1^* = \frac{1}{2 + |a|^2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |a|^2 \end{pmatrix}, \quad \rho_2 = \frac{1}{p_2} K_2 \rho K_2^* = \frac{1}{1 + |b|^2} \begin{pmatrix} 1 & b^* & 0 \\ b & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $p_1 = \frac{2 + |a|^2}{4}$ and $p_2 = \frac{1 + |b|^2}{4}$. Notice that $\rho_1 \in \mathcal{I}$ is diagonal and therefore incoherent, and $\rho_2 = |\psi\rangle \langle \psi|$ is the pure state with $|\psi\rangle = \frac{1}{\sqrt{1 + |b|^2}} (|0\rangle + b |1\rangle)$.

The $\alpha$ power of $\rho$ is the state

$$\rho^\alpha = \frac{1}{2^{1+\alpha}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$
Figure 2: Comparison between $CT_\alpha(\rho)$ in blue and $p_2CT_\alpha(\rho_2)$ in green as functions of $\alpha$ for $b = 0.9$.

And therefore the Tsallis coherence is

$$CT_\alpha(\rho) = S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho) = \frac{4}{1-\alpha} \left[ (2 + 2^{4/\alpha})^{1-\alpha} - 2^{-(1+\alpha)} \right].$$

Since $\rho \in \mathcal{I}$ is incoherent, $CT_\alpha(\rho_1) = 0$. And since $\rho_2$ is a pure state, the Tsallis coherence is

$$p_2CT_\alpha(\rho_2) = p_2S^T_\alpha(\Delta_\alpha(\rho_2)) = \frac{1+|b|^2}{4} \left[ (1 + |b|^2)(1 + |b|^{2/\alpha})^{1-\alpha} - 1 \right].$$

From Figure 2 we have, for example, for $b = 0.9$ and $\alpha = 0.2$, we have

$$CT_\alpha(\rho) < 0.4 < p_2CT_\alpha(\rho_2) = \sum_j p_jCT_\alpha(\rho_j).$$

For strong monotonicity property it is important how the quantum channel is written in terms of its Kraus operators. We just showed that in 2- or 3-dimensions, if GIO is written as a convex mixture of diagonal unitaries, then Tsallis coherence reaches equality. However, if GIO is written in some other way, we show that Tsallis coherence may fail strong monotonicity.

5.10 Theorem. Tsallis coherence fails strong monotonicity under GIO, even on pure states, if Kraus operators are not proportional to unitaries.

Proof. We are going to use the same example as in Theorem 5.5. Let us fix the basis $\mathcal{E} = \{|0\rangle, |1\rangle\}$. Let $\rho = |\psi\rangle \langle \psi|$ be a pure state with $|\langle \psi|0\rangle|^2 = \chi = 3/4$ and $|\langle \psi|1\rangle|^2 = 1 - \chi = 1/4$.

For a pure state $\rho$ the entropy is zero, and therefore

$$CT_\alpha(\rho) = S^T_\alpha(\Delta_\alpha(\rho))$$

$$= \frac{1}{1-\alpha} \left[ \text{Tr} \{\Delta_\alpha(\rho)\}^{\alpha} \right] - 1$$

$$= \frac{1}{1-\alpha} \left[ \left( \sum_j \chi_j^{1/\alpha} \right)^\alpha - 1 \right]$$

$$= \frac{1}{1-\alpha} \left[ \frac{4}{(3^{1/\alpha} + 1)^\alpha} - 1 \right].$$
Let $\Lambda$ be GIO, with Kraus operators $\Lambda(\rho) = K_1 \rho K_1^* + K_2 \rho K_2^*$ where Kraus operators are diagonal in $E$ basis

$$K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad K_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$ 

Clearly $\sum_n K_n^* K_n = I$. Then the post-measurement states $\rho_n = \frac{1}{p_n} K_n \rho K_n^*$ are also pure, where $|\psi_n\rangle = \sqrt{p_n} K_n |\psi\rangle$ and $p_n = \langle \psi | K_n^* K_n |\psi\rangle$. Let us denote $|\langle \psi_n | j \rangle| = \xi_{nj} = \frac{1}{p_n} |\langle j | K_n |\psi\rangle|^2 = \frac{1}{p_n} |k_{nj}|^2 \chi_j$, and $p_n = \sum_j |k_{nj}|^2 \chi_j$. Then $p_1 = \frac{9}{16}$ and $p_2 = \frac{7}{16}$, and

$\xi_{11} = \frac{2}{3}, \xi_{12} = \frac{1}{3}, \xi_{21} = \frac{6}{7}, \xi_{22} = \frac{1}{7}.$

Therefore,

$$CT_{\alpha}(\rho_1) = S_T^T(\Delta_{\alpha}(\rho_1))
= \frac{1}{1 - \alpha} \left[ \text{Tr} \left\{ (\Delta_{\alpha}(\rho_1))^\alpha \right\} - 1 \right]
= \frac{1}{1 - \alpha} \left[ \frac{1}{(\sum_j \xi_{1j}^{1/\alpha})^{\alpha}} - 1 \right]
= \frac{1}{1 - \alpha} \left[ \frac{3}{(2^{1/\alpha} + 1)^\alpha} - 1 \right].$$

Similarly,

$$CT_{\alpha}(\rho_2) = S_T^T(\Delta_{\alpha}(\rho_2))
= \frac{1}{1 - \alpha} \left[ \text{Tr} \left\{ (\Delta_{\alpha}(\rho_2))^\alpha \right\} - 1 \right]
= \frac{1}{1 - \alpha} \left[ \frac{1}{(\sum_j \xi_{2j}^{1/\alpha})^{\alpha}} - 1 \right]
= \frac{1}{1 - \alpha} \left[ \frac{7}{(6^{1/\alpha} + 1)^\alpha} - 1 \right].$$

From Figure 3 we have, for example, for $\alpha = 0.2$, strong monotonicity fails since $p_1 CT_{\alpha}(\rho_1) + p_2 CT_{\alpha}(\rho_2) > 0.42 > CT_{\alpha}(\rho)$.

5.7 Continuity for both Tsallis and Rényi coherences

5.11 Theorem. Let $\rho = |\psi\rangle \langle \psi|$ and $\sigma = |\phi\rangle \langle \phi|$ be pure states on $\mathbb{C}^d$ such that $\frac{1}{2} \|\rho - \sigma\|_1 = \epsilon$. Then, we obtain

$$|CT_{\alpha}(\rho) - CT_{\alpha}(\sigma)| \leq \frac{1}{1 - \alpha} \left( d^{1-\alpha} - \left( d^{1-\frac{1}{2}} + H(\epsilon) \right)^{-\alpha} \right), \text{ for } 0 < \alpha < 1,$$
Figure 3: Comparison between $CT_\alpha(\rho)$ in red and $p_1CT_\alpha(\rho_1) + p_2CT_\alpha(\rho_2)$ in blue as functions of $\alpha$.

and

$$|CT_\alpha(\rho) - CT_\alpha(\sigma)| \leq \frac{1}{\alpha - 1} \left[ 1 - (1 - H(\epsilon) )^{-\alpha} \right] , \text{ for } 1 < \alpha < 2 ,$$

where $H(\epsilon) = 1 - (1 - \epsilon)^{1/\alpha} - \epsilon^{1/\alpha}(d - 1)^{1/\alpha}$. Both right hand-sides converge to zero when $\epsilon$ goes to zero.

Proof. Denote $\chi_j = |\langle \psi | j \rangle|^2$ and $\xi_j = |\langle \phi | j \rangle|^2$. Then,

$$|CT_\alpha(\rho) - CT_\alpha(\sigma)| = \frac{1}{|1 - \alpha|} \left| \left( \sum_j \chi_j^{1/\alpha} \right)^{-\alpha} - \left( \sum_j \xi_j^{1/\alpha} \right)^{-\alpha} \right|$$

$$= \frac{1}{|1 - \alpha|} \left| [\text{Tr} f(\Delta(\rho))]^{-\alpha} - [\text{Tr} f(\Delta(\sigma))]^{-\alpha} \right| ,$$

where $f(x) = x^{1/\alpha}$ is convex function for $0 < \alpha < 1$ and $-f$ is convex for $\alpha > 1$, and recall that $\Delta(\rho) = \sum_j \chi_j |j\rangle \langle j|$ and $\Delta(\sigma) = \sum_j \xi_j |j\rangle \langle j|$.

Since trace-norm is monotone under CPTP maps, and $\Delta$ is a CPTP map, we obtain

$$\frac{1}{2} \|\Delta(\rho) - \Delta(\sigma)\|_1 \leq \frac{1}{2} \|\rho - \sigma\|_1 = \epsilon .$$

By continuity of $f$-entropy [18], the difference for $0 < \alpha < 1$ is bounded by

$$|\text{Tr} f(\Delta(\rho)) - \text{Tr} f(\Delta(\sigma))| \leq H(\epsilon) ,$$

where $H(\epsilon)$ is calculated for $f(x) = x^{1/\alpha}$, and therefore has expression as in the theorem statement. For $\alpha > 1$, $-f$ is convex, and therefore,

$$|\text{Tr} f(\Delta(\rho)) - \text{Tr} f(\Delta(\sigma))| \leq -H(\epsilon) ,$$
where the right-hand side is positive for $\alpha > 1$.

For $0 < \alpha < 1$, notice that the constant sequence is majorized by both $(\tfrac{1}{\alpha})_j < (\chi)_j$ and $(\tfrac{1}{\alpha})_j < (\xi)_j$, therefore, since $f(x) = x^{1/\alpha}$ is a convex function, by results on Schur-concavity [11, 16, 23], we have $\sum_j \chi_j^{1/\alpha}, \sum_j \xi_j^{1/\alpha} > d^{1-\frac{1}{\alpha}}$. For $\alpha > 1$, since $x < x^{1/\alpha}$, then $\sum_j \chi_j^{1/\alpha}, \sum_j \xi_j^{1/\alpha} > 1$.

For the function $g(x) = x^{-\alpha}$ and any $0 < s < c$, we have $|g'(s)| > |g'(c)|$, and therefore by the Mean Value Theorem, there exist $s, c \in (0, 1]$, such that $d^{1-\frac{1}{\alpha}} \leq s \leq c$, and

$$
\left| |\text{Tr}(\Delta(\rho))|^{-\alpha} - |\text{Tr}(\Delta(\sigma))|^{-\alpha} \right| = |\text{Tr}(\Delta(\rho)) - \text{Tr}(\Delta(\sigma))| \cdot |g'(c)| \\
\leq H(\epsilon) |g'(s)| \\
= \left( d^{1-\alpha} - \left( d^{1-\frac{1}{\alpha}} + H(\epsilon) \right)^{-\alpha} \right).
$$

Similarly, by the Mean Value Theorem, there exists $s, c \geq 1$, such that

$$
\left| |\text{Tr}(\Delta(\rho))|^{-\alpha} - |\text{Tr}(\Delta(\sigma))|^{-\alpha} \right| = |\text{Tr}(\Delta(\rho)) - \text{Tr}(\Delta(\sigma))| \cdot |g'(c)| \\
\leq -H(\epsilon) |g'(s)| \\
= \left( 1 - (1 - H(\epsilon))^{-\alpha} \right).
$$

Thus, we obtain the statement of the theorem.

Similarly to the above, we can show the continuity for Renyi coherence for pure states

**5.12 Theorem.** Let $\rho = |\psi\rangle \langle \psi|$ and $\sigma = |\phi\rangle \langle \phi|$ be pure states on $\mathbb{C}^d$ such that $\frac{1}{2} \|ho - \sigma\|_1 = \epsilon$. Then, we obtain

$$
|CR^1_\alpha(\rho) - CR^1_\alpha(\sigma)| \leq \frac{\alpha}{1 - \alpha} \log \left( d^{1-\frac{1}{\alpha}} + H(\epsilon) \right) + \log d, \text{ for } 0 < \alpha < 1,
$$

and

$$
|CR^1_\alpha(\rho) - CR^1_\alpha(\sigma)| \leq \frac{\alpha}{\alpha - 1} \log (1 - H(\epsilon)), \text{ for } 1 < \alpha < 2,
$$

where $H(\epsilon) = 1 - \epsilon^{1/\alpha} - \epsilon^{1/\alpha} (d - 1)^{1-\frac{1}{\alpha}}$. Both right hand-sides converge to zero when $\epsilon$ goes to zero.

**Proof.** Denote $\chi_j = |\langle \psi | j \rangle|^2$ and $\xi_j = |\langle \phi | j \rangle|^2$. Then,

$$
|CR^1_\alpha(\rho) - CR^1_\alpha(\sigma)| = \frac{\alpha}{1 - \alpha} \left| \log \left( \sum_j \chi_j^{1/\alpha} \right) - \log \left( \sum_j \xi_j^{1/\alpha} \right) \right| \\
= \frac{1}{1 - \alpha} \left| \log \text{Tr}(\Delta(\rho)) - \log \text{Tr}(\Delta(\sigma)) \right|,
$$

where $f(x) = x^{1/\alpha}$ is convex function for $0 < \alpha < 1$ and $-f$ is convex for $\alpha > 1$, and recall that $\Delta(\rho) = \sum_j \chi_j |j\rangle \langle j|$ and $\Delta(\sigma) = \sum_j \xi_j |j\rangle \langle j|$.

Since trace-norm is monotone under CPTP maps, and $\Delta$ is a CPTP map, we obtain

$$
\frac{1}{2} \|\Delta(\rho) - \Delta(\sigma)\|_1 \leq \frac{1}{2} \|\rho - \sigma\|_1 = \epsilon.
$$

By continuity of $f$-entropy [18], the difference for $0 < \alpha < 1$ is bounded by

$$
|\text{Tr}(\Delta(\rho)) - \text{Tr}(\Delta(\sigma))| \leq H(\epsilon),
$$
where $H(\epsilon)$ is calculated for $f(x) = x^{1/\alpha}$, and therefore has expression as in the theorem statement. For $\alpha > 1$, $-f$ is convex, and therefore,

$$|\text{Tr} f(\Delta(\rho)) - \text{Tr} f(\Delta(\sigma))| \leq -H(\epsilon),$$

where the right-hand side is positive for $\alpha > 1$.

For $0 < \alpha < 1$, notice that the constant sequence is majorized by both $(\frac{1}{\alpha})_j < (\chi)_j$ and $(\frac{1}{\alpha})_j < (\xi)_j$, therefore, since $f(x) = x^{1/\alpha}$ is a convex function, by results on Schur-concavity [11, 16, 23], we have $\sum_j \chi_j^{1/\alpha}, \sum_j \xi_j^{1/\alpha} > d^{-\frac{1}{\alpha}}$. For $\alpha > 1$, since $x < x^{1/\alpha}/\alpha$, then $\sum_j \chi_j^{1/\alpha}, \sum_j \xi_j^{1/\alpha} > 1$.

For the function $g(x) = \log x$ and any $0 < s < c < 1$, we have $|g'(s)| > |g'(c)|$, and therefore by the Mean Value Theorem, there exist $s, c \in (0, 1]$, such that $d^{1-\frac{1}{\alpha}} \leq s \leq c$, and

$$|\left[\text{Tr} f(\Delta(\rho))\right]^{-\alpha} - \left[\text{Tr} f(\Delta(\sigma))\right]^{-\alpha}| = \left|\text{Tr} f(\Delta(\rho)) - \text{Tr} f(\Delta(\sigma))\right| |g'(c)|$$

$$\leq H(\epsilon)|g'(s)|$$

$$= \left|\log d^{1-\frac{1}{\alpha}} - \log \left(d^{1-\frac{1}{\alpha}} + H(\epsilon)\right)\right|. $$

Similarly, by the Mean Value Theorem, there exists $s, c \geq 1$, such that

$$|\left[\text{Tr} f(\Delta(\rho))\right]^{-\alpha} - \left[\text{Tr} f(\Delta(\sigma))\right]^{-\alpha}| = \left|\text{Tr} f(\Delta(\rho)) - \text{Tr} f(\Delta(\sigma))\right| |g'(c)|$$

$$\leq -H(\epsilon)|g'(s)|$$

$$= \log(1 - H(\epsilon)) .$$

Thus, we obtain the statement of the theorem. \(\square\)

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