Observational constraints on spatial anisotropy of $G$ from orbital motions

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Abstract
A phenomenological anisotropic variation $\Delta G/G$ of the Newtonian gravitational coupling parameter $G$, if real, would affect the orbital dynamics of a two-body gravitationally bound system in a specific way. We analytically work out the long-term effects that such a putative modification of the usual Newtonian inverse-square law would induce on the trajectory of a test particle orbiting a central mass. Without making any \textit{a priori} simplifying assumptions concerning the orbital configuration of the test particle, it turns out that its osculating semi-major axis $a$, eccentricity $e$, pericenter $\varpi$ and mean anomaly $M$ undergo long-term temporal variations, while the inclination $I$ and the node $\Omega$ are left unaffected. Moreover, the radial and the transverse components of the position and the velocity vectors $r$ and $v$ of the test particle, experience non-vanishing changes per orbit, contrary to the out-of-plane ones. Then, we compute our theoretical predictions for some of the major bodies of the solar system by orienting the gradient of $G(r)$ toward the Galactic center and keeping it fixed over the characteristic timescales involved. By comparing our calculation to the latest observational determinations for the same bodies, we infer $\Delta G/G \lesssim 10^{-17}$ over about 1 AU. Finally, we consider also the supermassive black hole hosted by the Galactic center in Sgr A$^*$ and the main sequence star S2 orbiting it in about 16 years, obtaining just $\Delta G/G \lesssim 10^{-2}$ over 1 KAU.

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1. Introduction

The possibility that the Newtonian coupling parameter $G$ may experience macroscopic spacetime variations ranging from laboratory to cosmological scales has been investigated both theoretically (Sciama 1953, Brans andDicke 1961, Dicke 1962, Long 1974, Gershteyn and Gershteyn 1988, Linde 1990, de Sabbata et al 1992, Melnikov 1994a, 1994b, Capozziello et al 1996, Melnikov 1996, Capozziello and deRitis 1997, Capozziello et al 1998, Drinkwater et al 1998, Fischbach and Talmadge 1998, Barrow and O’Toole 2001, Danielsson 2001, Krause and Fischbach 2001, Murphy et al 2001, Uzan 2003, deSabbata et al 2004, Capozziello 2005, Clifton et al 2005, Hamber and Williams 2005, Bailey and Kostelecký 2006, Brownstein and Moffat 2006, García-Berro et al 2007, Bertolami et al 2008, Capozziello and Francaviglia 2008, Adelberger et al 2009, Uzan 2009, Ramí 2010, Kostelecký and Tasson 2011, Uzan 2011) and experimentally/observationally (Wagoner 1970, Vinti 1972, Ulrich 1974, Long 1976, Warburton and Goodkind 1976, Mikkelsen and Newman 1977, Anderson et al 1978, Blinnikov 1978, Hut 1981, Chan et al 1982, Kislik 1983, Chan and Paik 1984, Gillies 1987, Burgess and Cloutier 1988, Talmadge et al 1988, Krauss and White 1992, Izmailov et al 1993, Paik et al 1994, Bertolami and García-Bellido 1996, Gillies 1997, Gaztañaga et al 2002, Gershteyn et al 2002, 2004, Unnikrishnan and Gillies 2002a, 2002b, Adelberger et al 2003, Long 2003, Abramyan 2004, Barrow 2005, Boucher 2005, Kononogov and Mel’nikov 2005, García-Berro et al 2007, Iorio 2007, Bertolami and Santos 2009, Newman et al 2009, Li 2009, Lämmerzahl 2011, Piedipalumbo et al 2011, Uzan 2011) since the early insights by Milne (1935, 1937), Dirac (1937) andJordan (1937, 1939).

In this paper, we deal with possible smooth anisotropic spatial variations of $G$, i.e. we consider the case $G = G(r)$ from a purely phenomenological point of view. We stress that in our analysis we do not rely upon any specific theoretical scheme encompassing such a spatial variability of $G$: the interested reader may consult the previously cited specialized literature. Quite generally, we express a putative anisotropic dependence of $G$ on the spatial coordinates by parameterizing it with a gradient $\nabla G$ along a fixed direction in space $\xi$ as

$$G(r) \simeq G_0 + \nabla G_0 \cdot r,$$

with

$$\nabla G_0 = |\nabla G_0| \hat{\xi}.$$

The subscript ‘0’ in the two aforementioned equations refers to quantities evaluated at the origin of the spatial coordinates, which, in our case, coincides with a generic body of mass $M$ acting as the source of the gravitational field. By means of equation (1), we are assuming that the putative variations of $G$ are rather smooth over the spatial extensions considered. A change of $G$ like that of equation (1) is usually absent in the standard alternative metric theories of gravity treated within the parameterized post-Newtonian (PPN) framework (Will 1993) where $G$ may depend on the velocity $V$ of the frame in which the experiments are performed with respect to a preferred frame (Will 1971); see, e.g., Vinti (1972), Damour and Esposito-Farèse (1994) and Will (1993) for analyses of the orbital consequences of such kind of anisotropies. A spatial anisotropy of $G$ depending on the angle between the line of interaction of two gravitating bodies and a reference direction with respect to distant stars was experimentally investigated by Gershteyn et al (2002, 2004) in a series of Earth-based laboratory investigations, which were subsequently critically analyzed by Unnikrishnan and Gillies (2002a). Unnikrishnan and Gillies (2002a) remarked that, in general, spatial anisotropies of $G$ may occur depending on how nearby masses and their distribution can affect the gravitational interaction between two

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3 They claimed to have measured an anisotropy as large as $5.4 \times 10^{-4}$ with a torsion balance.
bodies, and also because of preferred frame effects. If the modification of the gravitational interaction depends on the gravitational potential generated by other masses (Brans and Dicke 1961, Sciama 1953) in a somewhat Machian fashion, then the most distant ones dominate and the spatial anisotropy is expected to be small (Unnikrishnan and Gillies 2002a). It is, then, reasonable to expect that the Galaxy yields the most important contribution to the anisotropy by assuming $\hat{\xi}$ directed toward the Galactic center (GC) (Unnikrishnan and Gillies 2002a).

In the framework of our parameterization of equations (1) and (2), such a scenario may offer, in principle, interesting observational perspectives if suitable astronomical bodies are chosen. Indeed, if we insert equation (1) in the usual expression of the Newtonian inverse-square law, a small modification of it occurs

$$A = -\frac{M(\nabla G_0 \cdot \mathbf{r})}{r^3} = -\frac{M|\nabla G_0| (\hat{\xi} \cdot \mathbf{r})}{r^3}. \quad (3)$$

Now, the ecliptic coordinates of the GC are (Reid and Brunthaler 2004)

$$\lambda_{GC} = 183.15^\circ,$$
$$\beta_{GC} = -5.61^\circ, \quad (4)$$

so that the angle between $\hat{\xi}$ and $\mathbf{r}$ for, say, a typical planet of the solar system is rather small. Thus,

$$\frac{\Delta G}{G} \sim \frac{|\nabla G_0|}{G_0} r. \quad (5)$$

Unnikrishnan and Gillies (2002a) argued that the order of magnitude of the Galactic-induced anisotropy is

$$\frac{\Delta G}{G} \sim \frac{GM_{Gal}}{c^2 d} \sim 10^{-6}, \quad (6)$$

where $c$ is the speed of light in vacuum, $M_{Gal} \sim 10^{12} M_\odot$ (Battaglia et al 2005) is the mass of the Galaxy, and $d = 8.28$ kpc (Gillessen et al 2009) is the distance from the GC. In our picture, it would naively be equivalent to perturbing accelerations

$$A \lesssim 10^{-6} A_N, \quad (7)$$

where the standard inverse-square Newtonian accelerations $A_N$ are

$$4 \times 10^{-2} \text{ m s}^{-2} \leq A_N \leq 4 \times 10^{-6} \text{ m s}^{-2} \quad (8)$$

for the major bodies of the solar system. In principle, perturbing accelerations as large as

$$A \sim 10^{-8} – 10^{-12} \text{ m s}^{-2} \quad (9)$$

may have interesting observational consequences. We will analytically work out them in detail. Indeed, relying upon simple order of magnitude evaluations may be misleading since important factors of the order of $O(e^j)$, $j = 1, 2, \ldots$, or $O(e^{-j})$, $j = 1, 2, \ldots$, in the usually small eccentricities $e$ of the bodies adopted as probes may be neglected. Unnikrishnan and Gillies (2002a) performed a preliminary calculation concerning the Earth–Moon system. They started from a certain value $|\dot{G}/G| \leq 4 \times 10^{-12}$ year$^{-1}$ (Dickey et al 1994) of the upper bound in the fractional time change of $G$ obtained with the lunar laser ranging (LLR) technique. Then, Unnikrishnan and Gillies (2002a) stated that the same analysis can be useful as far as the spatial anisotropy of $G$ is concerned as well. They noted that an anisotropic spatial variation of $G$ like that of equation (1) should exhibit a harmonic signal with the same approximate monthly periodicity of the orbital lunar motion as the line joining the Earth and the Moon sweeps out different directions with respect to the GC direction. Finally, since the accuracy
with which it is possible to measure a periodic signal may be of the same order of, or better than that for a secular trend, Unnikrishnan and Gillies (2002a) concluded by inferring

\[ \frac{\Delta G}{G} \leq 4 \times 10^{-12}. \]  

(10)

Repeating the same reasonings with the latest results from LLR (Williams et al. 2004, Müller and Biskupek 2007, Williams et al. 2009, Hofmann et al. 2010) would yield

\[ \frac{\Delta G}{G} \leq (1 - 0.4) \times 10^{-12}, \]  

(11)

in neat disagreement with the results by Gershteyn et al. (2002). Gershteyn et al. (2002) pointed out that such a disagreement exists if it is assumed that the \( G \) anisotropy depends neither on the magnitude of the interacting masses nor on the distance between them; Gershteyn et al. (2002) remarked that the masses and the distances involved in the analysis by Unnikrishnan and Gillies (2002a) drastically differ from those used by Gershteyn et al. (2002) in their experiment.

We propose to obtain much more accurate bounds than that in equation (10) by calculating in detail all the orbital effects of equation (3) which, actually, depends on the mutual distance between \( M \) and the test body: we will assume that it is independent of their masses. As far as the other performed and/or proposed astronomical tests of \( G(r) \) are concerned (Wagoner 1970, Vinti 1972, Ulrich 1974, Warburton and Goodkind 1976, Mikkelsen and Newman 1977, Anderson et al. 1978, Blinnikov 1978, Hut 1981, Kislik 1983, Burgess and Cloutier 1988, Talmadge et al. 1988, Abramyan 2004, Iorio 2007, Bertolami and Santos 2009, Li 2009), quite different explicit theoretical isotropic models and empirical approaches have been followed so far; in particular, an exponential Yukawa-type model and the third Kepler law have often been adopted.

This paper is organized as follows. In section 2, we analytically work out various orbital effects caused by a phenomenological dipole-type spatial variation of \( G \) like that of equation (1) averaged over one orbital revolution of a test particle. In section 3, a comparison with the latest solar system planetary observations is made; we also consider the stellar system around the Galactic black hole. In section 4, we summarize our findings and give the conclusions.

2. Calculation of the orbital effects

The orbital motions of test particles are, in principle, affected by equation (3) whose effects can be worked out with standard perturbative techniques (Brouwer and Clemence 1961, Bertotti et al. 2003). By defining

\[ \psi_0 = \frac{[\nabla G_0]}{G_0}, \quad \mu_0 = G_0M, \]  

(12)

the radial component \( A_R \) of equation (3), evaluated onto the unperturbed Keplerian ellipse, is

\[ A_R = -\frac{\psi_0 \mu_0 (1 + e \cos f) [\cos u(\xi_x \cos \Omega + \xi_y \sin \Omega) + \sin u(\xi_z \sin I + \cos I(\xi_y \cos \Omega - \xi_x \sin \Omega))]}{a(1-e^2)}, \]  

(13)

the transverse and out-of-plane components \( A_T, A_N \) vanish. In equation (13), \( a, I, \Omega, u = \omega + f, \omega, f \) are the osculating semi-major axis, the inclination of the orbit plane to the reference \( \{x, y\} \) plane, the longitude of the ascending node\(^6\), the argument of latitude,

\(^4\) It is so also because 1 year contains almost 12 lunar cycles.

\(^5\) Note that \( [\psi_0] = L^{-1}, \) while \( [G_0M] = L^3 T^{-2} \), as usual.

\(^6\) It is an angle in the reference \( \{xy\} \) plane measured from the reference \( x \) direction to the line of the nodes, which is the intersection of the orbital plane with the reference \( \{xy\} \) plane.
the argument of pericentre, and the true anomaly, respectively, of the test particle. Note that equation (13) clearly shows that the putative anisotropic \( G \) effect depends, among other things, on the distance between the two bodies as well. By assuming \( \dot{\xi} \) constant over one orbital revolution of the test particle, a straightforward first-order application of the Gauss perturbative equations (Brouwer and Clemence 1961, Bertotti et al 2003) yields for \( a, e, \) the longitude of the pericenter \( \varpi = \Omega + \omega, \) and \( M \) the following non-vanishing long-term rates of change:

\[
\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{2\psi_0 \mu_0}{\sqrt{a} e^3 n} \left\{ \sin \omega \left( \hat{\xi}_e \cos \Omega + \hat{\xi}_e \sin \Omega \right) + \cos \omega \left( \hat{\xi}_e \sin I + \cos I \left( \hat{\xi}_e \cos \Omega - \hat{\xi}_e \sin \Omega \right) \right\},
\]

\[
\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\psi_0 \mu_0 (1 - e^2)}{a^2 e^3 n} \left\{ \sin \omega \left( \hat{\xi}_e \cos \Omega + \hat{\xi}_e \sin \Omega \right) + \cos \omega \left( \hat{\xi}_e \sin I + \cos I \left( \hat{\xi}_e \cos \Omega - \hat{\xi}_e \sin \Omega \right) \right\},
\]

\[
\frac{\mathrm{d}\varpi}{\mathrm{d}t} = -\frac{\psi_0 \mu_0 (1 - e^2)}{a^2 e^3 n} \left\{ \cos \omega \left( \hat{\xi}_e \cos \Omega + \hat{\xi}_e \sin \Omega \right) + \sin \omega \left( \hat{\xi}_e \sin I + \cos I \left( \hat{\xi}_e \cos \Omega - \hat{\xi}_e \sin \Omega \right) \right\},
\]

\[
\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\psi_0 \mu_0 (1 - e^2)^{3/2}}{a^2 e^3 n} \left\{ \cos \omega \left( \hat{\xi}_e \cos \Omega + \hat{\xi}_e \sin \Omega \right) + \sin \omega \left( \hat{\xi}_e \sin I + \cos I \left( \hat{\xi}_e \cos \Omega - \hat{\xi}_e \sin \Omega \right) \right\}. \tag{14}
\]

The inclination \( I \) and the node \( \Omega \) are left unaffected. Note that the quantity \( n \) entering equation (14) is the unperturbed Keplerian mean motion, i.e. \( n = \sqrt{\mu_0/a^3} \). The long-term rates of change of equation (14) are exact in the sense that no \textit{a priori} assumptions concerning \( \hat{\xi} \) and the orbital configuration of the test particle were made. All the formulas of equation (14) are singular for \( e \rightarrow 0 \); however, it is just an unphysical artifact, which can be cured by adopting the well-known non-singular elements (Brouwer and Clemence 1961, Broucke and Cefola 1972)

\[
\begin{align*}
\hat{h} & \doteq e \sin \varpi, \\
\hat{k} & \doteq e \cos \varpi, \\
\hat{l} & \doteq \varpi + M.
\end{align*} \tag{15}
\]

It is also important to remark that the validity of equation (14) is not restricted to any specific reference frame, being, instead, quite general.

It is useful to work out the radial, transverse, and out-of-plane shifts over one orbital revolution of the position and velocity vectors \( \mathbf{r} \) and \( \mathbf{v} \) as well. They can be analytically worked out according to Casotto (1993). For the shifts \( \Delta R, \Delta T, \Delta N \) of \( \mathbf{r} \), we have

\[
\Delta R = \frac{2\pi \psi_0 a^2 (1 - e^2)^2}{e^2} \left\{ \sin \omega \left( \hat{\xi}_e \cos \Omega + \hat{\xi}_e \sin \Omega \right) + \cos \omega \left( \hat{\xi}_e \sin I + \cos I \left( \hat{\xi}_e \cos \Omega - \hat{\xi}_e \sin \Omega \right) \right\},
\]

\[
\Delta T = \frac{4\pi \psi_0 a^2 (1 - e^2)}{e^2} \left\{ \cos \omega \left( \hat{\xi}_e \cos \Omega + \hat{\xi}_e \sin \Omega \right) + \sin \omega \left( \hat{\xi}_e \sin I + \cos I \left( \hat{\xi}_e \cos \Omega - \hat{\xi}_e \sin \Omega \right) \right\},
\]

\[
\Delta N = 0. \tag{16}
\]

\[\text{It is an angle in the orbital plane reckoning the point of closest approach with respect to the line of the nodes.}\]

\[\text{It is a time-dependent angle in the orbital plane determining the instantaneous position of the test particle with respect to the pericentre.}\]

\[\text{It is a ‘dogleg’ angle.}\]
Table 1. Formal uncertainties in the secular variations of the semi-major axes \(a\) and the eccentricities \(e\) of the inner planets of the solar system. They were obtained by dividing the formal errors in \(a\) and \(e\) in table 3 of Pitjeva (2007) by the time interval \(\Delta t = 93\) years (1913–2006) of the data records used for the EPM2006 ephemerides (Pitjeva 2007). The errors in \(e\) were computed as \(\sigma_e = \sqrt{\left(\frac{\partial e}{\partial h}\right)^2 \sigma_h^2 + \left(\frac{\partial e}{\partial k}\right)^2 \sigma_k^2}\). The realistic uncertainties may be up to one order of magnitude larger.

| Planet  | \(\sigma_a\) (m year\(^{-1}\)) | \(\sigma_e\) (year\(^{-1}\)) |
|---------|-------------------------------|-----------------------------|
| Mercury | \(3 \times 10^{-3}\)          | \(4.43 \times 10^{-12}\)   |
| Venus   | \(2 \times 10^{-3}\)          | \(1.8 \times 10^{-13}\)    |
| Earth   | \(1 \times 10^{-3}\)          | \(5 \times 10^{-14}\)      |
| Mars    | \(3 \times 10^{-3}\)          | \(5 \times 10^{-14}\)      |

The shifts \(\Delta v_R, \Delta v_T, \Delta v_N\) of \(v\) are
\[
\Delta v_R = -\frac{2\pi \psi_0 a^2 n (1 + e)^{1/2}}{\epsilon^2 \sqrt{1 - e^2}} \left[ \cos \omega (\hat{\xi}_x \cos \Omega + \hat{\xi}_y \sin \Omega) \\
+ \sin \omega [\hat{\xi}_x \sin I + \cos I (\hat{\xi}_y \cos \Omega - \hat{\xi}_z \sin \Omega)] \right],
\]
\[
\Delta v_T = -\frac{2\pi \psi_0 a^2 n \sqrt{1 - e^2}}{\epsilon^2} \left[ -\sin \omega (\hat{\xi}_x \cos \Omega + \hat{\xi}_z \sin \Omega) \\
+ \cos \omega [\hat{\xi}_x \sin I + \cos I (\hat{\xi}_y \cos \Omega - \hat{\xi}_z \sin \Omega)] \right],
\]
\[
\Delta v_N = 0.
\]
Concerning the singularities occurring for \(e \to 0\), also for equations (16) and (17), the same considerations as for equation (14) hold.

3. Confrontation with the observations

Here, we compare our theoretical results of section 2 with the latest observationally determined quantities pertaining the orbital motions of some of the major bodies of the solar system obtained by processing long data records of various types with different ephemerides (Pitjeva 2007, Fienga et al 2011, Fienga et al 2010, Pitjeva 2010) in order to preliminarily infer upper bounds on \(\psi_0\). In principle, one should explicitly include equation (1) in the dynamical force models fit to the observations and re-process the entire data set with such an ad hoc modified theory by varying the parameters to be estimated, the data, etc. Otherwise, the putative signal may be partly or totally absorbed in the estimation of, say, the initial state vectors. However, this is beyond the scope of our work.

In computing the anomalous effects for different bodies, we refer their orbital configurations to a heliocentric frame with mean ecliptic and equinox at the epoch J2000. In this, the unit vector pointing to the GC is
\[
\hat{\xi}_x = -0.993, \\
\hat{\xi}_y = -0.054, \\
\hat{\xi}_z = -0.097.
\]
In table 1, we quote reasonable evaluations for the secular variations of the semi-major axes \(a\) and the eccentricities \(e\) of the inner planets of the solar system for which the most accurate data are currently available. They were computed by dividing the formal, statistical \(1 - \sigma\) errors in \(a, e\) of the EPM2006 ephemerides (Pitjeva 2007) by the time interval \(\Delta t = 93\) years covered by the observations used for constructing them. The realistic uncertainties may be up to one order of magnitude larger. Table 2 displays the latest determinations of the corrections \(\Delta \dot{\sigma}\) to
As a result, we are able to obtain reasonable values for it which make the putative anomalous effects of equation (14) not larger than 10. It is a plane tangent to the Celestial Sphere at the point where the object of interest is located.

Since the standard Newtonian/Einsteinian secular precessions of the longitudes of the perihelia $\psi$ of the eight planets of the solar system plus Pluto determined with the EPM2008 (Pitjeva 2010), the INPOP08 (Fienga et al 2010), and the INPOP10a (Fienga et al 2011) ephemerides. Only the usual Newtonian–Einsteinian dynamics was modeled so that, in principle, the corrections $\Delta \psi$ may account for any other unmodeled/mismodeled dynamical effect. Concerning the values quoted in the third column from the left, they correspond to the smallest uncertainties reported by Fienga et al (2010). Note the small uncertainty in the correction to the precession of the terrestrial perihelion, obtained by processing Jupiter VLBI data (Fienga et al 2010).

| Planet  | $\Delta \psi$ (Pitjeva 2010) | $\Delta \psi$ (Fienga et al 2010) | $\Delta \psi$ (Fienga et al 2011) |
|---------|-----------------------------|---------------------------------|----------------------------------|
| Mercury | $-4 \pm 5$                  | $-10 \pm 30$                    | $0.2 \pm 3$                     |
| Venus   | $24 \pm 33$                 | $-4 \pm 6$                      | $-$                              |
| Earth   | $6 \pm 7$                   | $0 \pm 0.016$                   | $-$                              |
| Mars    | $-7 \pm 7$                  | $0 \pm 0.2$                     | $-$                              |
| Jupiter | $67 \pm 93$                 | $142 \pm 156$                   | $-$                              |
| Saturn  | $-10 \pm 15$                | $-10 \pm 8$                     | $0 \pm 2$                       |
| Uranus  | $-3890 \pm 3900$            | $0 \pm 20000$                   | $-$                              |
| Neptune | $-4440 \pm 5400$            | $0 \pm 20000$                   | $-$                              |
| Pluto   | $2840 \pm 4510$             | $-$                             | $-$                              |

Table 3. Upper bounds on the anisotropic percent variation $\Delta G / G$ inferred from $a, e, \psi$ for the inner planets of the solar system. We posed $\Delta G \lesssim \sigma_{\psi_0}(r) = \sigma_{\psi_0} a (1 + e^2)/2$ for each planet, where $\sigma_{\psi_0}$ was obtained by comparing the theoretical predictions of equation (14) to the uncertainties listed in tables 1 and 2.

| Planet  | $\Delta G / G$ | $\Delta \psi$ | $\Delta \psi$ |
|---------|----------------|---------------|---------------|
| Mercury | $2 \times 10^{-16}$ | $8 \times 10^{-15}$ | $3 \times 10^{-15}$ |
| Venus   | $9 \times 10^{-18}$ | $1 \times 10^{-17}$ | $1 \times 10^{-17}$ |
| Earth   | $1 \times 10^{-17}$ | $2 \times 10^{-18}$ | $3 \times 10^{-18}$ |
| Mars    | $4 \times 10^{-16}$ | $3 \times 10^{-16}$ | $3 \times 10^{-15}$ |

The results are shown in table 3. Even by rescaling the bounds obtained from table 1 by one order of magnitude, the anisotropy of $G$ in the planetary regions of the solar system is very tightly constrained, being of the order of $10^{-15}$–$10^{-17}$.

It may be interesting to consider a completely different astronomical scenario, both from the point of view of its components and of the distance scales involved. In table 4, we quote the relevant physical and orbital parameters of the system constituted by the supermassive black hole (SBH) hosted by the GC in Sgr A* and the main sequence star S2 orbiting it in about 16 years at a distance of approximately 1 kau (Gillessen et al 2009). In this case, the angular elements refer to a coordinate system whose reference $z$ axis is directed along the line of sight: the reference $(x, y)$ plane coincides with the plane of the sky\(^{10}\), with the $x$ axis pointing toward the celestial North Pole. The results of equation (14) are applicable to S2 as well since its mass is about five orders of magnitude smaller than that of the SBH: clearly, in this case, it is $\hat{x} = -\hat{z}$. Concerning the use of equation (14) in the Sgr A* scenario, one may wonder why

\(^{10}\) It is a plane tangent to the Celestial Sphere at the point where the object of interest is located.
Relevant physical and orbital parameters of the SBH-S2 system in Sgr A*.

| $\mu_0$ (m$^3$ s$^{-2}$) | $a$ (m) | $e$ | $I$ (deg) | $\Omega$ (deg) | $\omega$ (deg) | $\sigma_\omega$ (arcsec year$^{-1}$) |
|-------------------------|---------|----|---------|---------------|----------------|----------------|
| $5.70 \times 10^{26}$  | $1.54 \times 10^{14}$ | $0.8831$ | $134.87$ | $226.53$       | $64.98$        | $182$          |

an essentially Newtonian scheme is adopted instead of a general relativistic one. In principle, one could assume as the reference path a fully post-Newtonian one\(^\text{11}\) (Calura et al 1997, 1998) and work out the effects of a given small extra acceleration like equation (3) with respect to it according to the perturbative scheme set up by Calura et al (1997, 1998), which is a general relativistic generalization of another standard perturbative approach based on the planetary Lagrange equations (Bertotti et al 2003). Actually, it is, in practice, useless since the only addition with respect to the orbital effects like, e.g., the precession of the pericenter resulting from the standard scenario would consist of further, small mixed GTR-perturbation orbital effects, completely irrelevant in strengthening the bounds inferred. Viewed from a different point of view, the ratio of the average distance $r_{S2}$ of S2 from the SBH to its Schwarzschild radius is, after all, as large as $1.7 \times 10^4$.

By using the expression for the putative precession of the stellar pericenter in equation (14), it can be compared to the present-day uncertainty in observationally determining its secular rate in table 4. In this case, the constraints on the $G$ anisotropy are very weak. Indeed, we have just

$$\psi_0^{-1} \geq 0.45 \text{ pc,}$$

(19)

corresponding to

$$\frac{\Delta G}{G} \leq 1.5 \times 10^{-2}$$

(20)

over about 1 kau. Note that the bound of equation (20) is tighter by one order of magnitude than that could be inferred by simply posing

$$\frac{\Delta G}{G} \lesssim \frac{\sigma_\mu}{\mu_0} = 1.2 \times 10^{-1},$$

(21)

from $\sigma_\mu = 0.50 \times 10^6 \mu_\odot$ (Gillessen et al 2009).

4. Summary and conclusions

We looked at phenomenological anisotropic spatial variations $\Delta G/G$ of the Newtonian gravitational coupling parameter $G$, in the form $\Delta G = V G \cdot r$, and analytically worked out the impact that they may have on the trajectory of a test particle orbiting a central body of mass $M$. More specifically, we focussed on the cumulative orbital changes obtained perturbatively by averaging over one period of revolution of the test particle the effects due to $\Delta G/G$ on

\(^\text{11}\) In doing so, it would be implicitly assumed that the effects due to a putative $G$ anisotropy are smaller than the post-Newtonian ones as well.
its path. As a result, the osculating semi-major axis $a$, the eccentricity $e$, the pericenter $\varpi$ and the mean anomaly $M$ of the orbiter experience non-vanishing long-term changes, which depend on the overall orbital geometry of the test particle and on the direction $\hat{\xi}$ of the putative gradient $\nabla G$. We analytically worked out the long-term variations per orbit $\Delta a$, $\Delta e$, $\Delta \varpi$ and $\Delta M$ of the position and velocity vectors $r$ and $v$ of the test particle as well. We found that both the radial and the transverse components of $r$ and $v$ are affected by long-term changes per orbit, while the out-of-plane ones are left unaffected. We kept $\hat{\xi}$ fixed during the integrations; moreover, no a priori simplifying assumptions on $e$ and $I$ were assumed so that our results are exact in this respect.

Then, we compared our theoretical predictions to the most recently determined observational quantities for some of the major bodies of the solar system. By assuming that the dominant contribution to the hypothetical anisotropy of $G$ is due to the Galaxy, we took $\hat{\xi}$ directed toward the GC, which has a small inclination with respect to the ecliptic. By using the heliocentric orbits of the inner planets, we were able to constrain $\Delta G / G$ to a level of about $10^{-17}$ over $\sim 1$ au, several orders of magnitude better than in previous analyses based on LLR only. We looked also at the star S2 orbiting the Galactic black hole at a distance of about 1 kau along a highly elliptical ellipse, but, in this case, we got just $\Delta G / G \lesssim 10^{-2}$.

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