Pohlmeyer invariants are expressible in terms of DDF invariants

Urs Schreiber
Universität Duisburg-Essen
Essen, 45117, Germany

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It is shown that the Pohlmeyer invariants of the classical bosonic string are a proper subset of the classical DDF invariants. This makes the quantization of the Pohlmeyer invariants particularly transparent and allows to generalize them to the superstring.

The Pohlmeyer program \[1,2,3,4,5,6\] is an attempt to quantize the bosonic string by finding a consistent quantum deformation of the Poisson-algebra of a certain set of classical invariants, the so-called Pohlmeyer invariants. So far this program has succeeded only up to the as yet unproven quadratic generation hypothesis \[2\]. The hope has been expressed that after completion the Pohlmeyer program would yield an alternative quantization of the string which does not feature the usual quantum effects like the critical dimension.

Here we show that the Pohlmeyer invariants can consistently be quantized by expressing them in terms of DDF invariants whose quantization as DDF operators is well known and leads to the standard theory. The result reported in this letter is discussed in \[7\].

Let \(X(\sigma)\) and \(P(\sigma)\) be canonical coordinates and momenta of the bosonic string with Poisson brackets

\[
[X^\mu(\sigma), P^\nu(\kappa)]_{PB} = \delta^\mu_\nu \delta(\sigma - \kappa)
\]

and define

\[
\mathcal{P}_\pm^\mu(\sigma) = \frac{1}{\sqrt{2T}} (P^\mu(\sigma) \pm TX^\mu(\sigma))
\]

(Here \(X' = \partial_\sigma X\), \(T = 1/2\pi\alpha'\) is the string tension and we assume a trivial Minkowski background and shift all spacetime indices with \(\eta_{\mu\nu} = \text{diag}(-1,1,\ldots,1)\).)

Using the usual oscillators

\[
\mathcal{P}^\mu_+(\sigma) := \frac{1}{\sqrt{2\pi}} \sum_m \hat{\alpha}^m e^{-im\sigma}
\]

\[
\mathcal{P}^\mu_-(\sigma) := \frac{1}{\sqrt{2\pi}} \sum_m \alpha^m e^{+im\sigma}
\]

and center of mass coordinates and momenta

\[
x^\mu := \frac{1}{2\pi} \int X^\mu(\sigma) \, d\sigma
\]

\[
p^\mu := \int P^\mu(\sigma) \, d\sigma = \frac{1}{\sqrt{4\pi T}} \tilde{\alpha} = \frac{1}{\sqrt{4\pi T}} \alpha_0
\]

we can write down the left- and right-moving fields of vanishing conformal weight

\[
X^\mu_-(\sigma) := x^\mu - \frac{\sigma}{4\pi T} p^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{m \neq 0} \frac{1}{m} \alpha^m e^{+im\sigma}
\]

and

\[
X^\mu_+(\sigma) := x^\mu + \frac{\sigma}{4\pi T} p^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{m \neq 0} \frac{1}{m} \tilde{\alpha}^m e^{-im\sigma}
\]

By fixing an arbitrary lightlike vector field \(k\) on target space we define the objects

\[
R_\pm(\sigma) := \pm \frac{4\pi T}{k \cdot p} k \cdot X_\pm(\sigma)
\]

which are normalized such that

\[
R_\pm(\sigma + 2\pi) = R_\pm(\sigma) + 2\pi
\]

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Their derivative is
\[ R'_\pm(\sigma) = \frac{2\pi\sqrt{2T}}{k} k : \mathcal{P}(\sigma) \] (3)
and can be seen to be non-vanishing for all \( \sigma \) on all of phase space except for a set of measure 0, as discussed in [7].

Now the classical DDF observables \( A^\mu_m \) and \( \tilde{A}^\mu_m \) of the closed bosonic string are defined by
\[ A^\mu_m := \frac{1}{\sqrt{2\pi}} \int d\sigma \mathcal{P}^\mu_-(\sigma) e^{-imR_-(\sigma)} \]
\[ \tilde{A}^\mu_m := \frac{1}{\sqrt{2\pi}} \int d\sigma \mathcal{P}^\mu_+(\sigma) e^{imR_+(\sigma)}. \] (4)
The coordinate 0-mode \( k \cdot x \) in \( R_\pm \) couples the left- and right-moving Virasoro algebras. Splitting off this factor yields the ‘truncated’ observables
\[ a^\mu_m := A^\mu_m e^{-\frac{2\pi i}{\sqrt{2}} kx} \]
\[ \tilde{a}^\mu_m := \tilde{A}^\mu_m e^{-\frac{2\pi i}{\sqrt{2}} kx}. \] (5)

It is now easy to see that the objects
\[ D\{m, \bar{m}_j\} := a_{m_1}^{\mu_1} \cdots a_{m_r}^{\nu_r} \tilde{a}_{\bar{m}_1}^{\nu_1} \cdots \tilde{a}_{\bar{m}_s}^{\nu_s} e^{iN \frac{2\pi}{\sqrt{2}} kx} \] (6)
when satisfying the level matching condition
\[ \sum_i m_i = N = \sum_j \bar{m}_j, \] (7)
Poisson-commute with all the Virasoro constraints, i.e. with \((\mathcal{P}^\pm)^2(\sigma), \forall \sigma \). This are the classical DDF invariants of the classical closed bosonic string in flat target space.

From the Fourier mode-like objects \( A^\mu_m \) and \( \tilde{A}^\mu_m \) one reobtains quasi-local fields \( \mathcal{P}^R_\pm \) by an inverse Fourier transformation:
\[ \mathcal{P}^R_-(\sigma) := \frac{1}{\sqrt{2\pi}} \sum_m A^\mu_m e^{im\sigma} \]
\[ \mathcal{P}^R_+(\sigma) := \frac{1}{\sqrt{2\pi}} \sum_m \tilde{A}^\mu_m e^{-im\sigma}. \] (8)

Since \( R_\pm \) is invertible almost everywhere on phase space this is equal to
\[ \mathcal{P}^R_\pm(\sigma) = \left( (R_\pm)^{-1} \right)'(\sigma) \mathcal{P}^\mu((R_\pm)^{-1}(\sigma)). \] (9)

But this is just the formula for the transformation of the unit weight object \( \mathcal{P}_\pm \) under the reparameterization \( \sigma \mapsto R_\pm^{-1}(\sigma) \). This means that any functional \( F(\mathcal{P}_\pm) \) of the original \( \mathcal{P}_\pm \) which is reparameterization invariant remains invariant after substitution of \( \mathcal{P}^R_\pm \) for \( \mathcal{P}_\pm \):
\[ F(\mathcal{P}_\pm) = F(\mathcal{P}^R_\pm) \] (10)
This way all such reparameterization invariant functionals can be re-expressed in terms of DDF invariants.

One particular reparameterization invariant functional is the Wilson line. Let \( A_\mu \) be any constant gauge connection on target space, then
\[ F(\mathcal{P}_\pm) := \text{Tr} \mathbf{P} \exp \left( \int_0^{2\pi} \mathcal{P}_\pm^\mu(\sigma) A_\mu \right) \]
\[ := \sum_{n=0}^{\infty} Z^{\mu_1 \cdots \mu_n}(\mathcal{P}_\pm) \text{Tr}(A_{\mu_1} \cdots A_{\mu_n}) \] (11)
(where Tr is the trace and \( \mathbf{P} \) indicates path-ordering) is such a reparameterization invariant functional and in fact all the Taylor coefficients \( Z^{\mu_1 \cdots \mu_n}(\mathcal{P}_\pm) \) are, too. These are the Pohlmeyer invariants. They Poisson-commute with all the \((\mathcal{P}_\pm)^2(\sigma), \forall \sigma \).

Using the result (10) the representation of the Pohlmeyer invariants in terms of DDF invariants is immediate, since
\[ Z^{\mu_1 \cdots \mu_n}(\mathcal{P}_\pm) = Z^{\mu_1 \cdots \mu_n}(\mathcal{P}^R_\pm). \] (12)
This says that the Pohlmeyer invariants remain unaffected under the substitution of ordinary oscillators with DDF invariants \( a^\mu_m \rightarrow A^\mu_m, \tilde{a}^\mu_m \rightarrow \tilde{A}^\mu_m \). The level matching condition is fulfilled since all Pohlmeyer invariants are necessarily of weight 0.

The result is therefore that the enveloping algebra of Pohlmeyer invariants is a proper subset of the enveloping algebra of DDF invariants.
The consistent quantization of the DDF invariants is well known and yields a quantum algebra which closes on DDF invariants. It therefore induces a consistent quantization of the algebra of Pohlmeyer invariants in the sense that the commutator of two quantized Pohlmeyer invariants is again an invariant (simply because it is again a polynomial in the DDF operators), though not necessarily a combination of (polynomials of) Pohlmeyer invariants.

This consistent quantization of the algebra of the Pohlmeyer invariants, just like that of the DDF invariants, exists for every number $D$ of spacetime dimensions. But, by the well known no-ghost theorem, a representation on a Hilbert space with positive norm and decoupled longitudinal excitations requires precisely $D = 26$ (for the bosonic string).

Since the DDF invariants generalize to the superstring, the prescription (12) also gives us a generalization of the Pohlmeyer invariants to the superstring.

**Note:** After this work was completed we learned of the old articles [8, 9] where essentially the same results as given here were already reported. Their relevance for the Pohlmeyer program and for attempts at “alternative” quantizations of the string seems not to have been widely familiar [10].

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