SIZE MATTERS IN UNIVALENT FOUNDATIONS

TOM DE JONG AND MARTÍN HÖTZEL ESCARDÓ

School of Computer Science, University of Birmingham, Birmingham, B15 2TT, UK
e-mail address: t.dejong@pgr.bham.ac.uk
URL: https://www.cs.bham.ac.uk/~txd880

School of Computer Science, University of Birmingham, Birmingham, B15 2TT, UK
e-mail address: m.escardo@cs.bham.ac.uk
URL: https://www.cs.bham.ac.uk/~mhe

ABSTRACT. We investigate predicative aspects of constructive univalent foundations. By predicative and constructive, we respectively mean that we do not assume Voevodsky’s propositional resizing axioms or excluded middle. Our work complements existing work on predicative mathematics by exploring what cannot be done predicatively in univalent foundations. Our first main result is that nontrivial (directed or bounded) complete posets are necessarily large. That is, if such a nontrivial poset is small, then weak propositional resizing holds. It is possible to derive full propositional resizing if we strengthen nontriviality to positivity. The distinction between nontriviality and positivity is analogous to the distinction between nonemptiness and inhabitedness. Moreover, we prove that locally small, nontrivial (directed or bounded) complete posets necessarily lack decidable equality. We prove our results for a general class of posets, which includes directed complete posets, bounded complete posets and sup-lattices. Secondly, we show that each of Zorn’s lemma, Tarski’s greatest fixed point theorem and Pataraia’s lemma implies propositional resizing. Hence, these principles are inherently impredicative and a predicative development of order theory must therefore do without them. Thirdly, we clarify, in our predicative setting, the relation between the traditional definition of sup-lattice that requires suprema for all subsets and our definition that asks for suprema of all small families. Finally, we investigate the interdefinability and interaction of type universes of propositional truncations and set quotients in the absence of propositional resizing axioms.

1. INTRODUCTION

We investigate predicative aspects of constructive univalent foundations. By predicative and constructive, we respectively mean that we do not assume Voevodsky’s propositional resizing axioms [Voe11, Voe15] or excluded middle. Most of this paper work is situated in our larger programme of developing domain theory constructively and predicatively in univalent foundations. In previous work [dJE21], we showed how to give a constructive and

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predicative account of many familiar constructions and notions in domain theory, such as Scott’s $D_{\omega}$ model of untyped $\lambda$-calculus and the theory of continuous dcpos. The present work complements this and other existing work on predicative mathematics (e.g. [AR10, Sam87, CSSV03]) by exploring what cannot be done predicatively, as in [Cur10a, Cur10b, Cur15, Cur18, CR12]. We do so by showing that certain statements crucially rely on resizing axioms in the sense that they are equivalent to them. Such arguments are important in constructive mathematics. For example, the constructive failure of trichotomy on the real numbers is shown [BR87] by reducing it to a nonconstructive instance of excluded middle.

Our first main result is that nontrivial (directed or bounded) complete posets are necessarily large. In [dJE21] we observed that all our examples of directed complete posets have large carriers. We show here that this is no coincidence, but rather a necessity, in the sense that if such a nontrivial poset is small, then weak propositional resizing holds. It is possible to derive full propositional resizing if we strengthen nontriviality to positivity in the sense of [Joh84]. The distinction between nontriviality and positivity is analogous to the distinction between nonemptiness and inhabitedness. We prove our results for a general class of posets, which includes directed complete posets, bounded complete posets and sup-lattices, using a technical notion of a $\delta_V$-complete poset. We also show that nontrivial locally small $\delta_V$-complete posets necessarily lack decidable equality. Specifically, we can derive weak excluded middle from assuming the existence of a nontrivial locally small $\delta_V$-complete poset with decidable equality. Moreover, if we assume positivity instead of nontriviality, then we can derive full excluded middle.

Secondly, we prove that each of Zorn’s lemma, Tarski’s greatest fixed point theorem and Pataraia’s lemma implies propositional resizing. Hence, these principles are inherently impredicative and a predicative development of order theory in univalent foundations must thus forgo them.

Thirdly, we clarify, in our predicative setting, the relation between the traditional definition of sup-lattice that requires suprema for all subsets and our definition that asks for suprema of all small families. This is important in practice in order to obtain workable definitions of dcpo, sup-lattice, etc. in the context of predicative univalent mathematics.

Finally, we investigate the inter-definability and interaction of type universes of propositional truncations and set quotients in the absence of propositional resizing axioms. Following a construction due to Voevodsky, we construct set quotients from propositional truncations. However, while Voevodsky assumed propositional resizing rules in his construction, we show that, when propositional truncations are available, resizing is not needed to prove the universal property of the set quotient, even though the quotient will live in a higher type universe.

Our foundational setup is the same as in [dJE21], meaning that our work takes places in intensional Martin-Löf Type Theory and adopts the univalent point of view [Uni13]. This means that we work with the stratification of types as singletons, propositions (or sub-singletons or truth values), sets, 1-groupoids, etc., and that we work with univalence. At present, higher inductive types other than propositional truncation are not needed. Often the only consequences of univalence needed here are functional and propositional extensionality. An exception is Section 2.3. Full details of our univalent type theory are given at the start of Section 2.

Related work. Curi investigated the limits of predicative mathematics in CZF [AR10] in a series of papers [Cur10a, Cur10b, Cur15, Cur18, CR12]. In particular, Curi shows
(see [Cur10a, Theorem 4.4 and Corollary 4.11], [Cur10b, Lemma 1.1] and [Cur15, Theorem 2.5]) that CZF cannot prove that various nontrivial posets, including sup-lattices, dcpos and frames, are small. This result is obtained by exploiting that CZF is consistent with the anti-classical generalized uniformity principle GUP [vdB06, Theorem 4.3.5]. Our related Theorem 3.23 is of a different nature in two ways. Firstly, our theorem is in the spirit of reverse constructive mathematics [Ish06]: Instead of showing that GUP implies that there are no non-trivial small dcpos, we show that the existence of a non-trivial small dcpo is equivalent to weak propositional resizing, and that the existence of a positive small dcpo is equivalent to full propositional resizing. Thus, if we wish to work with small dcpos, we are forced to assume resizing axioms. Secondly, we work in univalent foundations rather than CZF. This may seem a superficial difference, but a number of arguments in Curi’s papers [Cur15, Cur18] crucially rely on set-theoretical notions and principles such as transitive set, set-induction, weak regular extension axiom wREA, which cannot even be formulated in the underlying type theory of univalent foundations. Moreover, although Curi claims that the arguments of [Cur10a, Cur10b] can be adapted to some version of Martin-Löf Type Theory, it is presently not known whether there is any model of univalent foundations which validates GUP.

Finally, the construction of set quotients using propositional truncations is due to Voevodsky and also appears in [Uni13, Section 6.10] and [RS15, Section 3.4]. While Voevodsky assumed resizing rules for his construction, we investigate the inter-definability of propositional truncations and set quotients in the absence of propositional resizing axioms.

**Organization.** Section 2: Foundations and size matters, including impredicativity, relation to excluded middle, univalence and closure under embedded retracts. Section 3: Nontrivial and positive $\delta_V$-complete posets and reductions to impredicativity and excluded middle. Section 4: Predicative invalidity of Zorn’s lemma, Tarski’s fixed point theorem and Pataraia’s lemma. Section 5: Comparison of completeness w.r.t. families and w.r.t. subsets. Section 6: Inter-definability of set quotients and propositional truncations. Section 7: Conclusion and future work.

2. Foundations and Size Matters

We work with a subset of the type theory described in [Uni13] and we mostly adopt the terminological and notational conventions of [Uni13]. We include $+$ (binary sum), $\Pi$ (dependent products), $\Sigma$ (dependent sum), $\text{Id}$ (identity type), and inductive types, including 0 (empty type), 1 (type with exactly one element $*$: 1), $\mathbb{N}$ (natural numbers). We assume a universe $U_0$ and two operations: for every universe $U$ a successor universe $U^+$ with $U : U^+$, and for every two universes $U$ and $V$ another universe $U \sqcup V$ such that for any universe $U$, we have $U_0 \sqcup U \equiv U$ and $U \sqcup U^+ \equiv U^+$. Moreover, $(-) \sqcup (-)$ is idempotent, commutative, associative, and $(-)^+$ distributes over $(-) \sqcup (-)$. We write $U_1 \equiv U_0^+$, $U_2 \equiv U_1^+$, $\ldots$, and so on. If $X : U$ and $Y : V$, then $X + Y : U \sqcup V$ and if $X : U$ and $Y : X \to V$, then the types $\Sigma_{x : X} Y(x)$ and $\Pi_{x : X} Y(x)$ live in the universe $U \sqcup V$; finally, if $X : U$ and $x, y : X$, then $\text{Id}_X(x, y) : U$. The type of natural numbers $\mathbb{N}$ is assumed to be in $U_0$ and we postulate that we have copies $0_U$ and $1_U$ in every universe $U$. We assume function extensionality and propositional extensionality tacitly, and univalence explicitly when needed. Finally, we use a single higher inductive type: the propositional truncation of a type $X$ is denoted by $\|X\|$ and we write $\exists_{x : X} Y(x)$ for $\|\Sigma_{x : X} Y(x)\|$.
2.1. The Notion of Size. We introduce the fundamental notion of a type having a certain size and specify the impredicativity axioms under consideration (Section 2.2). We also note the relation to excluded middle (Section 2.2) and univalence (Section 2.3). Finally in Section 2.4 we review embeddings and sections and establish our main technical result on size, namely that having a certain size is closed under retracts whose sections are embeddings.

Definition 2.1 (Size, UF-Slice.html in [E+21]). A type \( X \) in a universe \( U \) is said to have size \( V \) if it is equivalent to a type in the universe \( V \). That is, \( X \) has-size \( V \) if
\[
\sum_{Y:V} (Y \simeq X).
\]

2.2. Impredicativity and Excluded Middle. We consider various impredicativity axioms and their relation to (weak) excluded middle. The definitions and propositions below may be found in [Esc20, Section 3.36], so proofs are omitted here.

Definition 2.2 (Impredicativity axioms).
(i) By \( \text{Propositional-Resizing}_{U,V} \) we mean the assertion that every proposition \( P \) in a universe \( U \) has size \( V \).
(ii) The type of all propositions in a universe \( U \) is denoted by \( \Omega_U \). Observe that \( \Omega_U : U^+ \). We write \( \Omega\text{-Resizing}_{U,V} \) for the assertion that the type \( \Omega_U \) has size \( V \).
(iii) The type of all \( \neg\neg \)-stable propositions in a universe \( U \) is denoted by \( \Omega_{\neg\neg}^U \), where a proposition \( P \) is \( \neg\neg \)-stable if \( \neg\neg P \) implies \( P \). By \( \Omega_{\neg\neg}\text{-Resizing}_{U,V} \) we mean the assertion that the type \( \Omega_{\neg\neg}^U \) has size \( V \).
(iv) For the particular case of a single universe, we write \( \Omega\text{-Resizing}_{U} \) and \( \Omega_{\neg\neg}\text{-Resizing}_{U} \) for the respective assertions that \( \Omega_U \) has size \( U \) and \( \Omega_{\neg\neg}^U \) has size \( U \).

Proposition 2.3.
(i) The principle \( \Omega\text{-Resizing}_{U,V} \) implies \( \text{Propositional-Resizing}_{U,V} \) for every two universes \( U \) and \( V \).
(ii) The conjunction of \( \text{Propositional-Resizing}_{U,V} \) and \( \text{Propositional-Resizing}_{V,U} \) implies \( \Omega\text{-Resizing}_{U,V^+} \) for every two universes \( U \) and \( V \).

It is possible to define a weaker variation of propositional resizing for \( \neg\neg \)-stable propositions only (and derive similar connections), but we don’t have any use for it in this paper.

Definition 2.4 ((Weak) excluded middle).
(i) Excluded middle in a universe \( U \) asserts that for every proposition \( P \) in \( U \) either \( P \) or \( \neg P \) holds.
(ii) Weak excluded middle in a universe \( U \) asserts that for every proposition \( P \) in \( U \) either \( \neg P \) or \( \neg\neg P \) holds.

We note that weak excluded middle says precisely that \( \neg\neg \)-stable propositions are decidable and is equivalent to de Morgan’s Law.

Proposition 2.5. Excluded middle implies impredicativity. Specifically,
(i) Excluded middle in \( U \) implies \( \Omega\text{-Resizing}_{U,U_0} \).
(ii) Weak excluded middle in \( U \) implies \( \Omega_{\neg\neg}\text{-Resizing}_{U,U_0} \).
2.3. Size and Univalence. With univalence we can prove that Propositional-Resizing$_{U,V}$ and $\Omega$-Resizing$_{U,V}$ are subsingletons. More generally, univalence allows us to prove that the statement that $X$ has size $V$ is a proposition, which is needed at the end of Section 3.4.

**Proposition 2.6 (cf. has-size-is-subsingleton in [Esc20]).** If $V$ and $U \sqcup V$ are univalent universes, then $X$ has-size $V$ is a proposition for every $X : U$.

The converse also holds in the following form.

**Proposition 2.7.** The type $X$ has-size $U$ is a proposition for every $X : U$ if and only if $U$ is a univalent universe.

*Proof.* Since $X$ has-size $U$ is $\Sigma_{Y : U}(Y \simeq X)$, this follows from [Esc20, Section 3.14].

2.4. Size and Retracts. We show our main technical result on size here, namely that having a size is closed under retracts whose sections are embeddings.

**Definition 2.8 (Sections, retractions and embeddings).**

(i) A *section* is a map $s : X \to Y$ together with a left inverse $r : Y \to X$, i.e. the maps satisfy $r \circ s \sim \text{id}$. We call $r$ the *retraction* and say that $X$ is a *retract* of $Y$.

(ii) A function $f : X \to Y$ is an *embedding* if the map $\text{ap}_f : (x = y) \to (f(x) = f(y))$ is an equivalence for every $x, y : X$. (See [Uni13, Definition 4.6.1(ii)].)

(iii) A *section-embedding* is a section $s : X \to Y$ that moreover is an embedding. We also say that $X$ is an *embedded retract* of $Y$.

We recall the following facts about embeddings and sections.

**Lemma 2.9.**

(i) A function $f : X \to Y$ is an embedding if and only if all its fibres are subsingletons, i.e. $\Pi_{y : Y} \text{is-subsingleton}(\Sigma_{x : X}(f(x) = y))$. (See [Uni13, Proof of Theorem 4.6.3].)

(ii) If every section is an embedding, then every type is a set. (See [Shu16, Remark 3.11(2)].)

(iii) Sections to sets are embeddings. (See [Esc20, lc-maps-into-sets-are-embeddings].)

In phrasing our results it is helpful to extend the notion of size from types to functions.

**Definition 2.10 (Size (for functions), UF-Slice.html in [E+21]).** A function $f : X \to Y$ is said to have size $V$ if every fibre has size $V$.

**Lemma 2.11 (cf. UF-Slice.html in [E+21]).**

(i) A type $X$ has size $V$ if and only if the unique map $X \to 1_{U^0}$ has size $V$.

(ii) If $f : X \to Y$ has size $V$ and $Y$ has size $V$, then so does $X$.

(iii) If $s : X \to Y$ is a section-embedding and $Y$ has size $V$, then $s$ has size $V$ too, regardless of the size of $X$.

*Proof.* The first two claims follow from the fact that for any map $f : X \to Y$ we have an equivalence $X \simeq \Sigma_{y : Y} \text{fib}_f(y)$ (see [Uni13, Lemma 4.8.2]). For the third claim, suppose that $s : X \to Y$ an embedding with retraction $r : Y \to X$. By the second part of the proof of Theorem 3.10 in [Shu16], we have $\text{fib}_s(y) \simeq \| s(r(y)) = y \|$, from which the claim follows. 

**Lemma 2.12.**

(i) If $X$ is an embedded retract of $Y$ and $Y$ has size $V$, then so does $X$.

(ii) If $X$ is a retract of a set $Y$ and $Y$ has size $V$, then so does $X$.
Proof. The first statement follows from (ii) and (iii) of Lemma 2.11. The second follows from the first and item (iii) of Lemma 2.9.

3. Largeness of Complete Posets

A well-known result of Freyd in impredicative mathematics says that every complete small category is a poset [Fre64, Exercise D of Chapter 3]. In other words, complete categories are necessarily large and only complete posets can be small, at least impredicatively. Predicatively, by contrast, we show that many weakly complete posets (including directed complete posets, bounded complete posets and sup-lattices) are necessarily large. We capture these structures by a technical notion of a \( \delta_V \)-complete poset in Section 3.1. In Section 3.2 we define when such structures are nontrivial and introduce the constructively stronger notion of positivity. Section 3.3 and Section 3.4 contain the two fundamental technical lemmas and the main theorems, respectively. Finally, we consider alternative formulations of being nontrivial and positive that ensure that these notions are properties rather than data and shows how the main theorems remain valid, assuming univalence.

3.1. \( \delta_V \)-complete Posets. We start by introducing a class of weakly complete posets that we call \( \delta_V \)-complete posets. The notion of a \( \delta_V \)-complete poset is a technical and auxiliary notion sufficient to make our main theorems go through. The important point is that many familiar structures (dcpos, bounded complete posets, sup-lattices) are \( \delta_V \)-complete posets (see Examples 3.3).

Definition 3.1 (\( \delta_V \)-complete poset, \( \delta_{x,y,P} \), \( \bigvee \delta_{x,y,P} \)). A poset is a type \( X \) with a subsingleton-valued binary relation \( \sqsubseteq \) on \( X \) that is reflexive, transitive and antisymmetric. It is not necessary to require \( X \) to be a set, as this follows from the other requirements. A poset \( (X, \sqsubseteq) \) is \( \delta_V \)-complete for a universe \( V \) if for every pair of elements \( x \sqsubseteq y \) and every subsingleton \( P \) in \( V \), the family
\[
\delta_{x,y,P} : 1 + P \to X
\]
\[
\text{inl}(\star) \mapsto x;
\]
\[
\text{inr}(p) \mapsto y;
\]
has a supremum \( \bigvee \delta_{x,y,P} \) in \( X \).

Remark 3.2 (Every poset is \( \delta_V \)-complete, classically). Consider a poset \( (X, \sqsubseteq) \) and a pair of elements \( x \sqsubseteq y \). If \( P : V \) is a decidable proposition, then we can define the supremum of \( \delta_{x,y,P} \) by case analysis on whether \( P \) holds or not. For if it holds, then the supremum is \( y \), and if it does not, then the supremum is \( x \). Hence, if excluded middle holds in \( V \), then the family \( \delta_{x,y,P} \) has a supremum for every \( P : V \). Thus, if excluded middle holds in \( V \), then every poset (in any universe) is \( \delta_V \)-complete.

The above remark naturally leads us to ask whether the converse also holds, i.e. if every poset is \( \delta_V \)-complete, does excluded middle in \( V \) hold? As far as we know, we can only get weak excluded middle in \( V \), as we will later see in Proposition 3.6. This proposition also shows that in the absence of excluded middle, the notion of \( \delta_V \)-completeness isn’t trivial. For now, we focus on the fact that, also constructively and predicatively, there are many examples of \( \delta_V \)-complete posets.
Examples 3.3.

(i) Every \( \mathcal{V} \)-sup-lattices is \( \delta_\mathcal{V} \)-complete. That is, if a poset \( X \) has suprema for all families \( I \to X \) with \( I \) in the universe \( \mathcal{V} \), then \( X \) is \( \delta_\mathcal{V} \)-complete.

(ii) The \( \mathcal{V} \)-sup-lattice \( \Omega_\mathcal{V} \) is \( \delta_\mathcal{V} \)-complete. The type \( \Omega_\mathcal{V} \) of propositions in \( \mathcal{V} \) is a \( \mathcal{V} \)-sup-lattice with the order given by implication and suprema by existential quantification. Hence, \( \Omega_\mathcal{V} \) is \( \delta_\mathcal{V} \)-complete. Specifically, given propositions \( Q, R \) and \( P \), the supremum of \( \delta_{Q,R,P} \) is given by \( Q \lor (R \land P) \).

(iii) The \( \mathcal{V} \)-powerset \( \mathcal{P}_\mathcal{V}(X) \equiv \mathcal{V} \to \Omega_\mathcal{V} \) of a type \( X \) is \( \delta_\mathcal{V} \)-complete. Note that \( \mathcal{P}_\mathcal{V}(X) \) is another example of a \( \mathcal{V} \)-sup-lattice (ordered by subset inclusion and with suprema given by unions) and hence \( \delta_\mathcal{V} \)-complete.

(iv) Every \( \mathcal{V} \)-bounded complete posets is \( \delta_\mathcal{V} \)-complete. That is, if \( (X, \sqsubseteq) \) is a poset with suprema for all bounded families \( I \to X \) with \( I \) in the universe \( \mathcal{V} \), then \( (X, \sqsubseteq) \) is \( \delta_\mathcal{V} \)-complete. A family \( \alpha : I \to X \) is bounded if there exists some \( x : X \) with \( \alpha(i) \sqsubseteq x \) for every \( i : I \). For example, the family \( \delta_{x,y,P} \) is bounded by \( y \).

(v) Every \( \mathcal{V} \)-directed complete poset (dcpo) is \( \delta_\mathcal{V} \)-complete, since the family \( \delta_{x,y,P} \) is directed. We note that [dJE21] provides a host of examples of \( \mathcal{V} \)-dcpos.

3.2. Nontrivial and Positive Posets. In Remark 3.2 we saw that if we can decide a proposition \( P \), then we can define \( \lor x \lor y \equiv \delta_{x,y,P} \) by case analysis. What about the converse? That is, if \( \delta_{x,y,P} \) has a supremum and we know that it equals \( x \) or \( y \), can we then decide \( P \)? Of course, if \( x = y \), then \( \lor x \lor y = x = y \), so we don’t learn anything about \( P \). But what if we add the assumption that \( x \neq y \)? It turns out that constructively we can only expect to derive decidability of \( \lnot P \) in that case. This is due to the fact that \( x \neq y \) is a negated proposition, which is rather weak constructively, leading us to later define (see Definition 3.8) a constructively stronger notion for elements of \( \delta_\mathcal{V} \)-complete posets.

Definition 3.4 (Nontrivial). A poset \( (X, \sqsubseteq) \) is nontrivial if we have designated \( x, y : X \) with \( x \sqsubseteq y \) and \( x \neq y \).

Lemma 3.5. Let \( (X, \sqsubseteq, x, y) \) be a nontrivial poset. We have the following implications for every proposition \( P : \mathcal{V} \):

(i) if the supremum of \( \delta_{x,y,P} \) exists and \( x = \lor \delta_{x,y,P} \), then \( \lnot P \) is the case.

(ii) if the supremum of \( \delta_{x,y,P} \) exists and \( y = \lor \delta_{x,y,P} \), then \( \lnot \lnot P \) is the case.

Proof. Let \( P : \mathcal{V} \) be an arbitrary proposition. For (i), suppose that \( x = \lor \delta_{x,y,P} \) and assume for a contradiction that we have \( p : P \). Then \( y \equiv \delta_{x,y,P}(\text{inr}(p)) \sqsubseteq \lor \delta_{x,y,P} = x \), which is impossible by antisymmetry and our assumptions that \( x \sqsubseteq y \) and \( x \neq y \). For (ii), suppose that \( y = \lor \delta_{x,y,P} \) and assume for a contradiction that \( \lnot P \) holds. Then \( x = \lor \delta_{x,y,P} = y \), contradicting our assumption that \( x \neq y \).

Proposition 3.6 (cf. Section 4 of [dJE21]). Let \( 2 \) be the poset with exactly two elements \( 0 \sqsubseteq 1 \). If \( 2 \) is \( \delta_\mathcal{V} \)-complete, then weak excluded middle in \( \mathcal{V} \) holds.

Proof. Suppose that \( 2 \) were \( \delta_\mathcal{V} \)-complete and let \( P : \mathcal{V} \) be an arbitrary subsingleton. We must show that \( \lnot P \) is decidable. Since \( 2 \) has exactly two elements, the supremum \( \lor \delta_{0,1,P} \) must be 0 or 1. But then we apply Lemma 3.5 to get decidability of \( \lnot P \).

That the conclusion of the implication in Lemma 3.5(ii) cannot be strengthened to say that \( P \) is the case is shown by the following observation.
Proposition 3.7. Recall Examples 3.3, which show that \( \Omega_\mathcal{V} \) is \( \delta_\mathcal{V} \)-complete. If for every two propositions \( Q \) and \( R \) with \( Q \sqsubseteq R \) and \( Q \neq R \) we have that the equality \( R = \bigvee \delta_{Q,R,P} \) in \( \Omega_\mathcal{V} \) implies \( P \) for every proposition \( P : \mathcal{V} \), then excluded middle in \( \mathcal{V} \) follows.

Proof. Assume the hypothesis in the proposition. We are going to show that \( \lnot \lnot P \rightarrow P \) for every proposition \( P : \mathcal{V} \), from which excluded middle in \( \mathcal{V} \) holds. Let \( P \) be a proposition in \( \mathcal{V} \) and assume that \( \lnot \lnot P \). This yields \( 0 \neq P \), so by assumption the equality \( P = \bigvee \delta_{0,P,P} \) implies \( P \). But, recalling item (ii) of Examples 3.3, we have exactly this equality \( \bigvee \delta_{0,P,P} = P \).

We have seen that having a pair of elements \( x,y \) with \( x \sqsubseteq y \) and \( x \neq y \) is very weak constructively. As promised in the introduction of this section, we now introduce a constructively stronger notion.

Definition 3.8 (Strictly below, \( x \sqsubseteq y \)). Let \((X,\sqsubseteq)\) be a \( \delta_\mathcal{V} \)-complete poset and \( x,y : X \). We say that \( x \) is strictly below \( y \) if \( x \sqsubseteq y \) and, moreover, for every \( z \sqsupseteq y \) and every proposition \( P : \mathcal{V} \), the equality \( z = \bigvee \delta_{x,z,P} \) implies \( P \).

Note that with excluded middle, \( x \sqsubseteq y \) is equivalent to the conjunction of \( x \sqsubseteq y \) and \( x \neq y \). But constructively, the former is much stronger, as the following example and proposition illustrate.

Example 3.9 (Strictly below \( \Omega_\mathcal{V} \)). Recall from Examples 3.3 that \( \Omega_\mathcal{V} \) is \( \delta_\mathcal{V} \)-complete. Let \( P : \mathcal{V} \) be an arbitrary proposition. Observe that \( 0_\mathcal{V} \neq P \) precisely when \( \lnot \lnot P \) holds. However, \( 0_\mathcal{V} \) is strictly below \( P \) if and only if \( P \) holds.

Proposition 3.10. For a \( \delta_\mathcal{V} \)-complete poset \((X,\sqsubseteq)\) and \( x,y : X \), we have that \( x \sqsubseteq y \) implies both \( x \sqsubseteq y \) and \( x \neq y \). However, if the conjunction of \( x \sqsubseteq y \) and \( x \neq y \) implies \( x \sqsubseteq y \) for every \( x,y : \Omega_\mathcal{V} \), then excluded middle in \( \mathcal{V} \) holds.

Proof. Note that \( x \sqsubseteq y \) implies \( x \sqsubseteq y \) by definition. Now suppose that \( x \sqsubseteq y \) and assume \( x = y \) for a contradiction. Since we assumed \( x \sqsubseteq y \), the equality \( y = \bigvee \delta_{x,y,0} \), implies that \( 0_\mathcal{V} \) holds. But this equality holds since \( x = y \) by our other assumption, so \( x \neq y \), as desired.

For \( P : \Omega_\mathcal{V} \) we observed that \( 0_\mathcal{V} \neq P \) is equivalent to \( \lnot \lnot P \) and that \( 0_\mathcal{V} \sqsubseteq P \) is equivalent to \( P \), so if we had \( ((x \sqsubseteq y) \times (x \neq y)) \rightarrow x \sqsubseteq y \) in general, then we would have \( \lnot \lnot P \rightarrow P \) for every proposition \( P \) in \( \mathcal{V} \), which is equivalent to excluded middle in \( \mathcal{V} \).

Lemma 3.11. Let \((X,\sqsubseteq)\) be a \( \delta_\mathcal{V} \)-complete poset and \( x,y,z : X \). The following hold:

(i) If \( x \sqsubseteq y \sqsubseteq z \), then \( x \sqsubseteq z \).

(ii) If \( x \sqsubseteq y \sqsubseteq z \), then \( x \sqsubseteq z \).

Proof. For (i), assume \( x \sqsubseteq y \sqsubseteq z \), let \( P \) be an arbitrary proposition in \( \mathcal{V} \) and suppose that \( z \sqsubseteq w \). We must show that \( w = \bigvee \delta_{x,w,P} \) implies \( P \). But \( y \sqsubseteq z \), so we know that the equality \( w = \bigvee \delta_{y,w,P} \) implies \( P \). Now observe that \( \bigvee \delta_{x,w,P} \subseteq \bigvee \delta_{y,w,P} \), so if \( w = \bigvee \delta_{x,w,P} \), then \( w = \bigvee \delta_{y,w,P} \), finishing the proof. For (ii), assume \( x \sqsubseteq y \sqsubseteq z \), let \( P \) be an arbitrary proposition in \( \mathcal{V} \) and suppose that \( z \sqsubseteq w \). We must show that \( w = \bigvee \delta_{x,w,P} \) implies \( P \). But \( x \sqsubseteq y \) and \( y \sqsubseteq w \), so this follows immediately.

Proposition 3.12. Let \((X,\sqsubseteq)\) be a \( \mathcal{V} \)-sup-lattice and let \( y : X \). The following are equivalent:

(i) the least element of \( X \) is strictly below \( y \);

(ii) for every family \( \alpha : I \rightarrow X \) with \( I : \mathcal{V} \) and \( y \subseteq \bigvee \alpha \), there exists some element \( i : I \).

(iii) there exists some \( x : X \) with \( x \sqsubseteq y \).
Proof. Write \( \bot \) for the least element of \( X \). By Lemma 3.11 we have:

\[
\bot \sqsubseteq y \iff \exists x : X (\bot \sqsubseteq x \sqsubseteq y) \iff \exists x : X (x \sqsubseteq y),
\]

which proves the equivalence of (i) and (iii). It remains to prove that (i) and (ii) are equivalent. Suppose that \( \bot \sqsubseteq y \) and let \( \alpha : I \to X \) with \( y \sqsubseteq \bigvee \alpha \). Using \( \bot \sqsubseteq y \sqsubseteq \bigvee \alpha \) and Lemma 3.11, we have \( \bot \sqsubseteq \bigvee \alpha \). Hence, we only need to prove \( \bigvee \alpha \sqsubseteq \bigvee \delta_{\bot,\bigvee \alpha,\exists : I} \), but \( \alpha_j \sqsubseteq \bigvee \delta_{\bot,\bigvee \alpha,\exists : I} \) for every \( j : I \), so this is true indeed. For the converse, assume that \( y \) satisfies (ii), suppose \( z \sqsubseteq y \) and let \( P : \mathcal{V} \) be a proposition such that \( z = \bigvee \delta_{\bot, z, P} \). We must show that \( P \) holds. But notice that \( y \sqsubseteq z = \bigvee \delta_{\bot, z, P} = \bigvee (p : P \mapsto z) \), so \( P \) must be inhabited as \( y \) satisfies (ii).

Item (ii) in Proposition 3.12 says exactly that \( y \) is a positive element in the sense of [Joh84, p. 98]. We note that item (iii) in Proposition 3.12 makes sense even when \((X, \sqsubseteq)\) is not a \( \mathcal{V} \)-sup-lattice, but just a \( \delta_{\mathcal{V}} \)-complete poset. Accordingly, we make the following definition.

**Definition 3.13 (Positive element).** An element of a \( \delta_{\mathcal{V}} \)-complete poset is **positive** if it satisfies item (iii) in Proposition 3.12.

An element of a \( \mathcal{V} \)-dcpo is called **compact** if it is inaccessible by directed joins of families indexed by types in \( \mathcal{V} \) [dJE21, Definition 44].

**Proposition 3.14.** A compact element \( x \) of a \( \mathcal{V} \)-dcpo with least element \( \bot \) is positive if and only if \( x \neq \bot \).

Proof. One implication is taken care of by Proposition 3.10. For the converse, suppose that \( x \neq \bot \). We show that \( \bot \) is strictly below \( x \). For if \( x \sqsubseteq y = \bigvee \delta_{\bot, y, P} \), then by compactness of \( x \), there must exist \( i : 1 + P \) such that \( x \sqsubseteq \delta_{\bot, y, P}(i) \) already. But \( i \) can’t be equal to \( \text{inl}(*) \), since \( x \) is assumed to be different from \( \bot \). Hence, \( i = \text{inr}(p) \) and \( P \) must hold.

Looking to strengthen the notion of a nontrivial poset, we make the following definition, whose terminology is inspired by Definition 3.13.

**Definition 3.15 (Positive poset).** A \( \delta_{\mathcal{V}} \)-complete poset \( X \) is **positive** if we have designated \( x, y : X \) with \( x \) strictly below \( y \).

**Examples 3.16.**

(i) Consider an element \( P \) of the \( \delta_{\mathcal{V}} \)-complete poset \( \Omega_{\mathcal{V}} \). The pair \((0_{\mathcal{V}}, P)\) witnesses nontriviality of \( \Omega_{\mathcal{V}} \) if and only if \( \mapsto P \) holds, while it witnesses positivity if and only if \( P \) holds.

(ii) Consider the \( \mathcal{V} \)-powerset \( \mathcal{P}_{\mathcal{V}}(X) \) on a type \( X \) as a \( \delta_{\mathcal{V}} \)-complete poset (recall Examples 3.3). We write \( \emptyset : \mathcal{P}_{\mathcal{V}}(X) \) for the map \( x \mapsto \emptyset_{\mathcal{V}} \). Say that a subset \( A : \mathcal{P}_{\mathcal{V}}(X) \) is nonempty if \( A \neq \emptyset \) and inhabited if there exists some \( x : X \) such that \( A(x) \) holds. The pair \((\emptyset, A)\) witnesses nontriviality of \( \mathcal{P}_{\mathcal{V}}(X) \) if and only if \( A \) is nonempty, while it witnesses positivity if and only if \( A \) is inhabited. In particular, \( \mathcal{P}_{\mathcal{V}}(X) \) is positive if and only if \( X \) is an inhabited type.
3.3. Retract Lemmas. We show that the type of propositions in \( \mathcal{V} \) is a retract of any positive \( \delta_\mathcal{V} \)-complete poset and that the type of \( \neg\neg \)-stable propositions in \( \mathcal{V} \) is a retract of any nontrivial \( \delta_\mathcal{V} \)-complete poset.

**Definition 3.17** (\( \Delta_{x,y} : \Omega_\mathcal{V} \to X \)). Suppose that \( (X, \sqsubseteq, x, y) \) is a nontrivial \( \delta_\mathcal{V} \)-complete poset. We define \( \Delta_{x,y} : \Omega_\mathcal{V} \to X \) by the assignment \( P \mapsto \bigvee \delta_{x,y,P} \).

We will often omit the subscripts in \( \Delta_{x,y} \) when it is clear from the context.

**Definition 3.18** (Locally small). A \( \delta_\mathcal{V} \)-complete poset \( (X, \sqsubseteq) \) is locally small if its order has values of size \( \mathcal{V} \), i.e. we have \( \sqsubseteq_\mathcal{V} : X \to X \to \mathcal{V} \) with \( (x \sqsubseteq y) \preceq (x \sqsubseteq_\mathcal{V} y) \) for every \( x, y : X \).

**Examples 3.19.**

(i) The \( \mathcal{V} \)-sup-lattices \( \Omega_\mathcal{V} \) and \( \mathcal{P}_\mathcal{V}(X) \) (for \( X : \mathcal{V} \)) are locally small.

(ii) All examples of \( \mathcal{V} \)-depos in [dJE21] are locally small.

**Lemma 3.20.** A locally small \( \delta_\mathcal{V} \)-complete poset \( (X, \sqsubseteq) \) is nontrivial, witnessed by elements \( x \sqsubseteq y \), if and only if the composite \( \Omega_{\mathcal{V}} \twoheadrightarrow \Omega_{\mathcal{V}} \Delta_{x,y} \to X \) is a section.

**Proof.** Suppose first that \( (X, \sqsubseteq, x, y) \) is nontrivial and locally small. We define \( r : X \to \Omega_{\mathcal{V}} \)

\[
 r(z) = \begin{cases} \top_{\mathcal{V}} & \text{for } z \not\sqsubseteq x \\ \bot_{\mathcal{V}} & \text{for } z \sqsubseteq x \end{cases}
\]

Note that negated propositions are \( \neg\neg \)-stable, so \( r \) is well-defined. Let \( P : \mathcal{V} \) be an arbitrary \( \neg\neg \)-stable proposition. We want to show that \( r(\Delta_{x,y}(P)) = P \). By propositional extensionality, establishing logical equivalence suffices. Suppose first that \( P \) holds. Then \( \Delta_{x,y}(P) \equiv \bigvee \delta_{x,y,P} = y \), so \( r(\Delta_{x,y}(P)) = r(y) \equiv (y \not\sqsubseteq x) \) holds by antisymmetry and our assumptions that \( x \sqsubseteq y \) and \( x \not= y \). Conversely, assume that \( r(\Delta_{x,y}(P)) \) holds, i.e. that we have \( \bigvee \delta_{x,y,P} \not\sqsubseteq x \). Since \( P \) is \( \neg\neg \)-stable, it suffices to derive a contradiction from \( \neg P \). So assume \( \neg P \). Then \( x = \bigvee \delta_{x,y,P} \), so \( r(\Delta_{x,y}(P)) = r(x) \equiv x \not\sqsubseteq_\mathcal{V} x \), which is false by reflexivity.

For the converse, assume that \( \Omega_{\mathcal{V}} \twoheadrightarrow \Omega_{\mathcal{V}} \Delta_{x,y} \to X \) has a retraction \( r : \Omega_{\mathcal{V}} \to X \). Then \( 0_{\mathcal{V}} = r(\Delta_{x,y}(0_{\mathcal{V}})) = r(x) \) and \( 1_{\mathcal{V}} = r(\Delta_{x,y}(1_{\mathcal{V}})) = r(y) \), where we used that \( 0_{\mathcal{V}} \) and \( 1_{\mathcal{V}} \) are \( \neg\neg \)-stable. Since \( 0_{\mathcal{V}} \not= 1_{\mathcal{V}} \), we get \( x \not= y \), so \( (X, \sqsubseteq, x, y) \) is nontrivial, as desired. \( \square \)

The appearance of the double negation in the above lemma is due to the definition of nontriviality. If we instead assume a positive poset \( X \), then we can exhibit all of \( \Omega_{\mathcal{V}} \) as a retract of \( X \).

**Lemma 3.21.** A locally small \( \delta_\mathcal{V} \)-complete poset \( (X, \sqsubseteq) \) is positive, witnessed by elements \( x \sqsubseteq y \), if and only if for every \( z \sqsubseteq_\mathcal{V} y \), the map \( \Delta_{x,z} : \Omega_\mathcal{V} \to X \) is a section.

**Proof.** Suppose first that \( (X, \sqsubseteq, x, y) \) is positive and locally small and let \( z \sqsubseteq_\mathcal{V} x \) be arbitrary. We define \( r_z : X \to \Omega_{\mathcal{V}} \)

\[
 r_z(w) = \begin{cases} \top_{\mathcal{V}} & \text{for } w \not\sqsubseteq z \\ \bot_{\mathcal{V}} & \text{for } w \sqsubseteq z \end{cases}
\]

Let \( P : \mathcal{V} \) be arbitrary proposition. We want to show that \( r_z(\Delta_{x,z}(P)) = P \). Because of propositional extensionality, it suffices to establish a logical equivalence between \( P \) and \( r_z(\Delta_{x,z}(P)) \). Suppose first that \( P \) holds. Then \( \Delta_{x,z}(P) = z \), so \( r_z(\Delta_{x,z}(P)) = r_z(z) \equiv
\((z \subseteq_\mathcal{V} z)\) holds as well by reflexivity. Conversely, assume that \(r_z(\Delta_{x,z}(P))\) holds, i.e. that we have \(z \subseteq_\mathcal{V} \bigvee \delta_{x,z} P\). Since \(\bigvee \delta_{x,z} P \subseteq z\) always holds, we get \(z = \bigvee \delta_{x,z} P\) by antisymmetry. But by assumption and Lemma 3.11, the element \(x\) is strictly below \(z\), so \(P\) must hold.

For the converse, assume that for every \(z \equiv_\mathcal{V} y\), the map \(\Delta_{x,z} : \Omega_\mathcal{V} \to X\) has a retraction \(r_z : X \to \Omega_\mathcal{V}\). We must show that the equality \(z = \Delta_{x,z}(P)\) implies \(P\) for every \(z \equiv_\mathcal{V} y\) and proposition \(P : \mathcal{V}\). Assuming \(z = \Delta_{x,z}(P)\), we have \(1_\mathcal{V} = r_z(\Delta_{x,z}(1_\mathcal{V})) = r_z(z) = r_z(\Delta_{x,z}(P)) = P\), so \(P\) must hold indeed. Hence, \((X, \subseteq, x, y)\) is positive, as desired. 

3.4. Small Completeness with Resizing. We present our main theorems here, which show that, constructively and predicatively, nontrivial \(\delta_\mathcal{V}\)-complete posets are necessarily large and necessarily lack decidable equality.

**Lemma 3.22** (Small). A \(\delta_\mathcal{V}\)-complete poset is small if it is locally small and its carrier has size \(\mathcal{V}\).

**Theorem 3.23.**

(i) There is a nontrivial small \(\delta_\mathcal{V}\)-complete poset if and only if \(\Omega_{\neg\neg}\operatorname{-Resizing}_\mathcal{V}\) holds.

(ii) There is a positive small \(\delta_\mathcal{V}\)-complete poset if and only if \(\Omega\operatorname{-Resizing}_\mathcal{V}\) holds.

**Proof.** (i) Suppose that \((X, \subseteq, x, y)\) is a nontrivial small \(\delta_\mathcal{V}\)-complete poset. By Lemma 3.20, we can exhibit \(\Omega_{\mathcal{V}}^{-}\) as a retract of \(X\). But \(X\) has size \(\mathcal{V}\) by assumption, so by Lemma 2.12 and the fact that \(\Omega_{\mathcal{V}}^{-}\) is a set, the type \(\Omega_{\mathcal{V}}^{-}\) has size \(\mathcal{V}\) as well. For the converse, note that \((\Omega_{\mathcal{V}}^{-}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})\) is a nontrivial \(\mathcal{V}\)-sup-lattice with \(\bigvee \alpha\) given by \(\neg\neg \exists i. \alpha_i\). And if we assume \(\Omega_{\neg\neg}\operatorname{-Resizing}_\mathcal{V}\), then it is small.

(ii) Suppose that \((X, \subseteq, x, y)\) is a positive small poset. By Lemma 3.21, we can exhibit \(\Omega_{\mathcal{V}}\) as a retract of \(X\). But \(X\) has size \(\mathcal{V}\) by assumption, so by Lemma 2.12 and the fact that \(\Omega_{\mathcal{V}}\) is a set, the type \(\Omega_{\mathcal{V}}\) has size \(\mathcal{V}\) as well. For the converse, note that \((\Omega_{\mathcal{V}}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})\) is a positive \(\mathcal{V}\)-sup-lattice. And if we assume \(\Omega\operatorname{-Resizing}_\mathcal{V}\), then it is small. 

**Lemma 3.24** (retract-is-discrete and subtype-is-\(\neg\neg\)-separated in \([E^{+}21]\)).

(i) Types with decidable equality are closed under retracts.

(ii) Types with \(\neg\neg\)-stable equality are closed under retracts.

**Theorem 3.25.** There is a nontrivial locally small \(\delta_\mathcal{V}\)-complete poset with decidable equality if and only if weak excluded middle in \(\mathcal{V}\) holds.

**Proof.** Suppose that \((X, \subseteq, x, y)\) is a nontrivial locally small \(\delta_\mathcal{V}\)-complete poset with decidable equality. Then by Lemmas 3.20 and 3.24, the type \(\Omega_{\mathcal{V}}^{-}\) must have decidable equality too. But negated propositions are \(\neg\neg\)-stable, so this yields weak excluded middle in \(\mathcal{V}\). For the converse, note that \((\Omega_{\mathcal{V}}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})\) is a nontrivial \(\mathcal{V}\)-sup-lattice that has decidable equality if and only if weak excluded middle in \(\mathcal{V}\) holds.

**Theorem 3.26.** The following are equivalent:

(i) There is a positive locally small \(\delta_\mathcal{V}\)-complete poset with \(\neg\neg\)-stable equality.

(ii) There is a positive locally small \(\delta_\mathcal{V}\)-complete poset with decidable equality.

(iii) Excluded middle in \(\mathcal{V}\) holds.

**Proof.** Note that (ii) \(\Rightarrow\) (i), so we are left to show that (iii) \(\Rightarrow\) (ii) and that (i) \(\Rightarrow\) (iii). For the first implication, note that \((\Omega_{\mathcal{V}}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})\) is a positive \(\mathcal{V}\)-sup-lattice that has decidable equality if and only if excluded middle in \(\mathcal{V}\) holds. To see that (i) implies (iii), suppose that
$(X, \sqsubseteq, x, y)$ is a positive locally small $\delta_V$-complete poset with $\neg\neg$-stable equality. Then by Lemmas 3.21 and 3.24 the type $\Omega_V$ must have $\neg\neg$-stable equality. But this implies that $\neg\neg P \to P$ for every proposition $P$ in $V$ which is equivalent to excluded middle in $V$. \hfill \Box

Lattices, bounded complete posets and dcpos are necessarily large and necessarily lack decidable equality in our predicative constructive setting. More precisely,

**Corollary 3.27.**

(i) There is a nontrivial small $V$-sup-lattice (or $V$-bounded complete poset or $V$-dcpo) if and only if $\Omega_{\neg\neg}$-Resizing$_V$ holds.

(ii) There is a positive small $V$-sup-lattice (or $V$-bounded complete poset or $V$-dcpo) if and only if $\Omega$-Resizing$_V$ holds.

(iii) There is a nontrivial locally small $V$-sup-lattice (or $V$-bounded complete poset or $V$-dcpo) with decidable equality if and only if weak excluded middle in $V$ holds.

(iv) There is a positive locally small $V$-sup-lattice (or $V$-bounded complete poset or $V$-dcpo) with decidable equality if and only if excluded middle in $V$ holds.

The above notions of non-triviality and positivity are data rather than property. Indeed, a nontrivial poset $(X, \sqsubseteq)$ is (by definition) equipped with two designated points $x, y : X$ such that $x \sqsubseteq y$ and $x \neq y$. It is natural to wonder if the propositionally truncated versions of these two notions yield the same conclusions. We show that this is indeed the case if we assume univalence. The need for the univalence assumption comes from the fact that the notion of having a given size is property precisely if univalence holds, as shown in Propositions 2.6 and 2.7.

**Definition 3.28** (Nontrivial/positive in an unspecified way). A poset $(X, \sqsubseteq)$ is *nontrivial in an unspecified way* if there exist some elements $x, y : X$ such that $x \sqsubseteq y$ and $x \neq y$, i.e. $\exists x : X \exists y : X ((x \sqsubseteq y) \times (x \neq y))$. Similarly, we can define when a poset is *positive in an unspecified way* by truncating the notion of positivity.

**Theorem 3.29.** Suppose that the universes $V$ and $V^+$ are univalent.

(i) There is a small $\delta_V$-complete poset that is nontrivial in an unspecified way if and only if $\Omega_{\neg\neg}$-Resizing$_V$ holds.

(ii) There is a small $\delta_V$-complete poset that is positive in an unspecified way if and only if $\Omega$-Resizing$_V$ holds.

**Proof.** (i) Suppose that $(X, \sqsubseteq)$ is a $\delta_V$-complete poset that is nontrivial in an unspecified way. By Proposition 2.6 and univalence of $V$ and $V^+$, type $\Omega_{\neg\neg}$ has-size $V$ is a proposition. By the universal property of the propositional truncation, in proving that $\Omega_{\neg\neg}$ has-size $V$ we can therefore assume that are given points $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$. The result then follows from Theorem 3.23. (ii) By reduction to item (ii) of Theorem 3.23. \hfill \Box

Similarly, we can prove the following theorems by reduction to Theorems 3.25 and 3.26.

**Theorem 3.30.**

(i) There is a locally small $\delta_V$-complete poset with decidable equality that is nontrivial in an unspecified way if and only if weak excluded middle in $V$ holds.

(ii) There is a locally small $\delta_V$-complete poset with decidable equality that is positive in an unspecified way if and only if excluded middle in $V$ holds.
4. Maximal Points and Fixed Points

In this section we construct a particular example of a \( \mathcal{V} \)-sup-lattice that will prove very useful in studying the predicative validity of some well-known principles in order theory.

**Definition 4.1** (Lifting, cf. [EK17]). Fix a proposition \( P_U \) in a universe \( U \). Lifting \( P_U \) with respect to a universe \( V \) is defined by

\[
\mathcal{L}_V(P_U) \equiv \sum_{Q: \Omega_V} (Q \to P_U).
\]

This is a subtype of \( \Omega_V \) (the map \( \text{pr}_1: \mathcal{L}_V(P_U) \to \Omega_V \) is an embedding) and it is closed under \( \mathcal{V} \)-suprema (in particular, it contains the least element).

**Examples 4.2.**

(i) If \( P_U \equiv 0_U \), then \( \mathcal{L}_V(P_U) \simeq (\Sigma_Q: \Omega_V, \neg Q) \simeq (\Sigma_Q: \Omega_V (Q = 0_V)) \simeq 1 \).

(ii) If \( P_U \equiv 1_U \), then \( \mathcal{L}_V(P_U) \equiv (\Sigma_Q: \Omega_V (Q \to 1_U)) \simeq \Omega_V \).

What makes \( \mathcal{L}_V(P_U) \) useful is the following observation.

**Lemma 4.3.** Suppose that the poset \( \mathcal{L}_V(P_U) \) has a maximal element \( Q: \Omega_V \). Then \( P_U \) is equivalent to \( Q \), which is the greatest element of \( \mathcal{L}_V(P_U) \). In particular, \( P_U \) has size \( V \). Conversely, if \( P_U \) is equivalent to a proposition \( Q: \Omega_V \), then \( Q \) is the greatest element of \( \mathcal{L}_V(P_U) \).

**Proof.** Suppose that \( \mathcal{L}_V(P_U) \) has a maximal element \( Q: \Omega_V \). We wish to show that \( Q \simeq P_U \). By definition of \( \mathcal{L}_V(P_U) \), we already have that \( Q \to P_U \). So only the converse remains. Therefore suppose that \( P_U \) holds. Then, \( 1_V \) is an element of \( \mathcal{L}_V(P_U) \). Obviously \( Q \to 1_V \), but \( Q \) is maximal, so actually \( Q = 1_V \), that is, \( Q \) holds, as desired. Thus, \( Q \simeq P_U \). It is then straightforward to see that \( Q \) is actually the greatest element of \( \mathcal{L}_V(P_U) \), since \( \mathcal{L}_V(P_U) \simeq \Sigma_{Q': \Omega_V} (Q' \to Q) \). For the converse, assume that \( P_U \) is equivalent to a proposition \( Q: \Omega_V \). Then, as before, \( \mathcal{L}_V(P_U) \simeq \Sigma_{Q': \Omega_V} (Q' \to Q) \), which shows that \( Q \) is indeed the greatest element of \( \mathcal{L}_V(P_U) \).

**Corollary 4.4.** Let \( P_U \) be a proposition in \( U \). The \( \mathcal{V} \)-sup-lattice \( \mathcal{L}_V(P_U) \) has all \( \mathcal{V} \)-infima if and only if \( P_U \) has size \( V \).

**Proof.** Suppose first that \( \mathcal{L}_V(P_U) \) has all \( \mathcal{V} \)-infima. Then it must have an infimum for the empty family \( 0_V \to \mathcal{L}_V(P_U) \). But this infimum must be the greatest element of \( \mathcal{L}_V(P_U) \). So by Lemma 4.3 the proposition \( P_U \) must have size \( V \).

Conversely, suppose that \( P_U \) is equivalent to a proposition \( Q: \mathcal{V} \). Then the infimum of a family \( \alpha: I \to \mathcal{L}_V(P_U) \) with \( I: \mathcal{V} \) is given by \( (Q \times \Pi_i: \mathcal{V}_i) : \mathcal{V} \).

**Definition 4.5** (Zorn’s-Lemma_{\mathcal{V},U,T}). Let \( U, \mathcal{V} \) and \( T \) be universes. Zorn’s-Lemma_{\mathcal{V},U,T} asserts that every pointed \( \mathcal{V} \)-dcpo with carrier in \( U \) and order taking values in \( T \) (cf. [dJE21]) has a maximal element.

It is important to note that Zorn’s lemma does not imply the Axiom of Choice in the absence of excluded middle [Bel97]. If it did, then the following would be useless, since the Axiom of Choice implies excluded middle, which in turn implies propositional resizing.

**Theorem 4.6.** Zorn’s-Lemma_{\mathcal{V},U,V} implies Propositional-Resizing_{\mathcal{V},U,V}.

In particular, Zorn’s-Lemma_{\mathcal{V},U,V} implies Propositional-Resizing_{\mathcal{V}+',V}. 
Proof. Suppose that Zorn’s-Lemma for \( V \)-\( U \),\( V+ \) were true. Then \( \mathcal{L}_V(P) : V^+ \sqcup U \) has a maximal element for every \( P : \Omega_U \). Hence, by Lemma 4.3, every \( P : \Omega_U \) has size \( V \).

We can also use Lemma 4.3 to show that the following version of Tarski’s fixed point theorem [Tar55] is not available predicatively.

**Definition 4.7 (Tarski’s-Theorem for\( \mathcal{V} \).)** The assertion Tarski’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) says that every monotone endofunction on a\( \mathcal{V} \)-sup-lattice with carrier in a universe \( \mathcal{U} \) and order taking values in a universe \( \mathcal{T} \) has a greatest fixed point.

**Theorem 4.8.** Tarski’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) implies Propositional-Resizing for\( \mathcal{U}, \mathcal{V} \).

In particular, Tarski’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) implies Propositional-Resizing for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \).

Proof. Suppose that Tarski’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) were true and let \( P : \Omega_U \) be arbitrary. Consider the\( \mathcal{V} \)-sup-lattice \( \mathcal{L}_V(P) : V^+ \sqcup U \). By assumption, the identity map on this poset has a greatest fixed point, but this must be the greatest element of \( \mathcal{L}_V(P) \), which implies that \( P \) has size \( V \) by Lemma 4.3.

Another famous fixed point theorem, for dcpos this time, is due to Pataaraia [Pat97, Esc03] which says that every monotone endofunction on a pointed dcpo has a least fixed point. (A dcpo is called pointed if it has a least element.) A crucial step in proving Pataaraia’s theorem is the observation that every dcpo has a greatest monotone inflationary endofunction. (An endomap \( f : X \rightarrow X \) is inflationary when \( x \sqsubseteq f(x) \) for every \( x : X \).) We refer to this intermediate result as Pataaraia’s lemma.

**Definition 4.9 (Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \), Pataaraia’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \)).**

(i) Pataaraia’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) says that every monotone endofunction on a pointed\( \mathcal{V} \)-dcpo with carrier in a universe \( \mathcal{U} \) and order taking values in a universe \( \mathcal{T} \) has a least fixed point.

(ii) Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) says that every\( \mathcal{V} \)-dcpo with carrier in a universe \( \mathcal{U} \) and order taking values in a universe \( \mathcal{T} \) has a greatest monotone inflationary endofunction.

A careful analysis of the proof in [Esc03, Section 2] shows that in our predicative setting we can still prove that Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) implies Pataaraia’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \). However, Pataaraia’s lemma is not available predicatively.

**Theorem 4.10.** Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) implies Propositional-Resizing for\( \mathcal{U}, \mathcal{V} \).

In particular, Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) implies Propositional-Resizing for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \).

Proof. Suppose that Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) were true and let \( P : \Omega_U \) be arbitrary. Consider the\( \mathcal{V} \)-dcpo \( \mathcal{L}_V(P) : V^+ \sqcup U \). By assumption, it has a greatest monotone inflationary endomap \( g : \mathcal{L}_V(P) \rightarrow \mathcal{L}_V(P) \). We claim that \( g(0_V) \) is a maximal element of \( \mathcal{L}_V(P) \), which would finish the proof by Lemma 4.3. So suppose that we have \( Q : \mathcal{L}_V(P) \) with \( g(0_V) \sqsubseteq Q \). Then we must show that \( Q \sqsubseteq g(0_V) \). Define \( f_Q : \mathcal{L}_V(P) \rightarrow \mathcal{L}_V(P) \) by \( Q' \mapsto Q' \lor Q \). Note that \( f_Q \) is monotone and inflationary, so that \( f_Q \sqsubseteq g \). Hence, \( Q = f_Q(0_V) \sqsubseteq g(0_V) \).

**Remark 4.11.** For a single universe \( \mathcal{U} \), the usual proofs (see respectively [Tar55] and [Esc03, Section 2]) of Tarski’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) and Pataaraia’s-Lemma for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \) are also valid in our predicative setting, and hence, so is Pataaraia’s-Theorem for\( \mathcal{V}, \mathcal{U}, \mathcal{T} \). However, in light of Theorem 3.23, these statements are not useful predicatively, because one would never be able to find interesting examples of posets to apply the statements to.
Finally, we note that Zorn’s lemma implies Pataraia’s lemma with the following universe parameters. Together with Theorem 4.10 this yields another proof that Zorn’s-Lemma$_{\mathcal{V},\mathcal{V}^+}$ implies Propositional-Resizing$_{\mathcal{V}^+,\mathcal{V}}$.

**Lemma 4.12.** Zorn’s-Lemma$_{\mathcal{V},\mathcal{U} \sqcup \mathcal{T},\mathcal{U} \sqcup \mathcal{T}}$ implies Pataraia’s-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$.

*Proof.* Assume Zorn’s-Lemma$_{\mathcal{V},\mathcal{U} \sqcup \mathcal{T},\mathcal{U} \sqcup \mathcal{T}}$ and let $D : \mathcal{U}$ be $\mathcal{V}$-dcpo with order taking values in $\mathcal{T}$. Consider the type $\text{MI}_D$ of monotone and inflationary endomaps on $D$. We can order these maps pointwise to get a $\mathcal{V}$-dcpo with carrier and order taking values in $\mathcal{U} \sqcup \mathcal{T}$. Finally, $\text{MI}_D$ has a least element: the identity map. Hence, by our assumption, it has a maximal element $g : D \to D$. It remains to show that $g$ is in fact the greatest element. To this end, let $f : D \to D$ be an arbitrary monotone inflationary endomap on $D$. We must show that $f \sqsubseteq g$. Since $f$ is inflationary, we have $g \sqsubseteq f \circ g$. So by maximality of $g$, we get $g = f \circ g$. But $f$ is monotone and $g$ is inflationary, so $f \sqsubseteq f \circ g = g$, finishing the proof.$\square$

The answer to the question whether Pataraia’s theorem (or similarly, a least fixed point theorem version of Tarki’s theorem) is inherently impredicative or (by contrast) does admit a predicative proof has eluded us thus far.

## 5. Families and Subsets

In traditional impredicative foundations, completeness of posets is usually formulated using subsets. For instance, dcpo’s are defined as posets $D$ such that every directed subset $D$ has a supremum in $D$. Examples 3.3 are all formulated using small families instead of subsets. While subsets are primitive in set theory, families are primitive in type theory, so this could be an argument for using families above. However, that still leaves the natural question of how the family-based definitions compare to the usual subset-based definitions, especially in our predicative setting, unanswered. This section aims to answer this question. We first study the relation between subsets and families predicatively and then clarify our definitions in the presence of impredicativity. In our answers we will consider sup-lattices, but similar arguments could be made for posets with other sorts of completeness, such as dcpo’s.

**All Subsets.** We first show that simply asking for completeness w.r.t. all subsets is not satisfactory from a predicative viewpoint. In fact, we will now see that even asking for all subsets $X \to \Omega_\mathcal{T}$ for some fixed universe $\mathcal{T}$ is problematic from a predicative standpoint.

**Theorem 5.1.** Let $\mathcal{U}$ and $\mathcal{V}$ be universes and fix a proposition $P_\mathcal{U} : \mathcal{U}$. Recall $\mathcal{L}_\mathcal{V}(P_\mathcal{U})$ from Definition 4.1, which has $\mathcal{V}$-suprema. Let $\mathcal{T}$ be any type universe. If $\mathcal{L}_\mathcal{V}(P_\mathcal{U})$ has suprema for all subsets $\mathcal{L}_\mathcal{V}(P_\mathcal{U}) \to \Omega_\mathcal{T}$, then $P_\mathcal{U}$ has size $\mathcal{V}$ independently of $\mathcal{T}$.

*Proof.* Let $\mathcal{T}$ be a type universe and consider the subset $S$ of $\mathcal{L}_\mathcal{V}(P_\mathcal{U})$ given by $Q \mapsto 1_\mathcal{T}$. Note that $S$ has a supremum in $\mathcal{L}_\mathcal{V}(P_\mathcal{U})$ if and only if $\mathcal{L}_\mathcal{V}(P_\mathcal{U})$ has a greatest element, but by Lemma 4.3, the latter is equivalent to $P_\mathcal{U}$ having size $\mathcal{V}$.$\square$
All Subsets Whose Total Spaces Have Size $\mathcal{V}$. The proof above illustrates that if we have a subset $S : X \rightarrow \Omega_T$, then there is no reason why the total space $\Sigma_{x:X}(x \in S) :\equiv \Sigma_{x:X}(S(x) \text{ holds})$ should have size $\mathcal{T}$. In fact, for $S(x) :\equiv 1_\mathcal{T}$ as above, the latter is equivalent to asking that $X$ has size $\mathcal{T}$.

**Definition 5.2** (Total space of a subset, $\mathcal{T}$). Let $\mathcal{T}$ be a universe, $X$ a type and $S : X \rightarrow \Omega_\mathcal{T}$ a subset of $X$. The *total space* of $S$ is defined as $\mathcal{T}(S) :\equiv \Sigma_{x:X}(x \in S)$.

A naive attempt to solve the problem described in Theorem 5.1 would be to stipulate that a $\mathcal{V}$-sup-lattice $X$ should have suprema for all subsets $S : X \rightarrow \Omega_\mathcal{V}$ for which $\mathcal{T}(S)$ has size $\mathcal{V}$. Somewhat less naively, we might be more liberal and ask for suprema of subsets $S : X \rightarrow \Omega_{\mathcal{U},\mathcal{V}}$ for which $\mathcal{T}(S)$ has size $\mathcal{V}$. Here the carrier of $X$ is in a universe $\mathcal{U}$. Perhaps surprisingly, even this more liberal definition is too weak to be useful as the following example shows.

**Example 5.3** (Naturally occurring subsets whose total spaces are not necessarily small). Let $X$ be a poset with carrier in $\mathcal{U}$ and suppose that it has suprema for all (directed) subsets $S : X \rightarrow \Omega_{\mathcal{U},\mathcal{V}}$ for which $\mathcal{T}(S)$ has size $\mathcal{V}$. Now let $f : X \rightarrow X$ be a Scott continuous endofunction on $X$. We would want to construct the least fixed point of $f$ as the supremum of the directed subset $S :\equiv \{ \bot, f(\bot), f^2(\bot), \ldots \}$. Now, how do we show that its total space $\mathcal{T}(S) :\equiv \Sigma_{x:X}(\exists m : \mathbb{N} \mid x = f^m(\bot))$ has size $\mathcal{V}$? A first guess might be that $\mathbb{N} \simeq \mathcal{T}(S)$, which would do the job. However, it’s possible that $f^m(\bot) = f^{m+1}(\bot)$ for some natural number $m$, which would mean that $\mathcal{T}(S) \simeq \text{Fin}(m)$ for the least such $m$. The problem is that in the absence of decidable equality on $X$ we might not be able to decide which is the case. But $X$ seldom has decidable equality, as we saw in Theorems 3.25 and 3.26.

**Remark 5.4.** The example above also makes clear that it is undesirable to impose an injectivity condition on families, as the family $\mathbb{N} \rightarrow X, n \mapsto f^n(\bot)$ is not necessarily injective. In fact, for every type $X : \mathcal{U}$ there is an equivalence between embeddings $I \hookrightarrow X$ with $I : \mathcal{V}$ and subsets of $X$ whose total spaces have size $\mathcal{V}$, cf. [E+21, Slice.html].

All $\mathcal{V}$-covered Subsets. The point of Example 5.3 is analogous to the difference between Bishop finiteness and Kuratowski finiteness. Inspired by this, we make the following definition.

**Definition 5.5** ($\mathcal{V}$-covered subset). Let $X$ be a type, $\mathcal{T}$ a universe and $S : X \rightarrow \Omega_\mathcal{T}$ a subset of $X$. We say that $S$ is $\mathcal{V}$-covered for a universe $\mathcal{V}$ if we have a type $I : \mathcal{V}$ with a surjection $e : I \rightarrow \mathcal{T}(S)$.

In the example above, the subset $S :\equiv \{ \bot, f(\bot), f^2(\bot), \ldots \}$ is $\mathcal{U}_0$-covered, because $\mathbb{N} \rightarrow \mathcal{T}(S)$.

**Theorem 5.6.** For $X : \mathcal{U}$ and any universe $\mathcal{V}$ we have an equivalence between $\mathcal{V}$-covered subsets $X \rightarrow \Omega_{\mathcal{U},\mathcal{V}}$ and families $I \rightarrow X$ with $I : \mathcal{V}$.

**Proof.** The forward map $\varphi$ is given by $(S,I,e) \mapsto (I,\text{pr}_1 \circ e)$. In the other direction, we define $\psi$ by mapping $(I,\alpha)$ to the triple $(S,I,e)$ where $S$ is the subset of $X$ given by $S(x) :\equiv \exists_{i : I} x = \alpha(i)$ and $e : I \rightarrow \mathcal{T}(S)$ is defined as $e(i) :\equiv (\alpha(i), [[i, \text{refl}]]$. The composite $\varphi \circ \psi$ is easily seen to be equal to the identity. To show that $\psi \circ \varphi$ equals the identity, we need the following intermediate result, which is proved using function extensionality and path induction.
**Claim.** Let \( S, S' : X \to \Omega_\mathcal{U} \lor \mathcal{V} \), \( e : I \to \tau(S) \) and \( e' : I \to \tau(S') \). If \( S = S' \) and \( \text{pr}_1 \circ e \sim \text{pr}_1 \circ e' \), then \((S, e) = (S', e')\).

The result then follows from the claim using function extensionality and propositional extensionality.

**Corollary 5.7.** Let \( X \) be a poset with carrier in \( \mathcal{U} \) and let \( \mathcal{V} \) be any universe. Then \( X \) has suprema for all \( \mathcal{V} \)-covered subsets \( X \to \Omega_\mathcal{U} \lor \mathcal{V} \) if and only if \( X \) has suprema for all families \( I \to X \) with \( I : \mathcal{V} \).

**Proof.** This is because the supremum of a \( \mathcal{V} \)-covered subset is equal to the supremum of the corresponding family and vice versa by inspection of the proof of Theorem 5.6.

**Families and Subsets in the Presence of Impredicativity.** Finally, we compare our family-based approach to the subset-based approach in the presence of impredicativity.

**Theorem 5.8.** Assume \( \Omega \)-Resizing\(_{\mathcal{T}, \mathcal{U}_0}\) for every universe \( \mathcal{T} \). Then the following are equivalent for a poset \( X \) in a universe \( \mathcal{U} \):

(i) \( X \) has suprema for all subsets;

(ii) \( X \) has suprema for all \( \mathcal{U} \)-covered subsets;

(iii) \( X \) has suprema for all subsets whose total spaces have size \( \mathcal{U} \);

(iv) \( X \) has suprema for all families \( I \to X \) with \( I : \mathcal{U} \).

**Proof.** Clearly (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii). We show that (iii) implies (i), which proves the equivalence of (i)–(iii). Assume that \( X \) has suprema for all subsets whose total spaces have size \( \mathcal{U} \) and let \( S : X \to \Omega_\mathcal{T} \) be any subset of \( X \). Using \( \Omega \)-Resizing\(_{\mathcal{T}, \mathcal{U}_0}\), the total space \( \tau(S) \) has size \( \mathcal{U} \). So \( X \) has a supremum for \( S \) by assumption, as desired. Finally, (ii) and (iv) are equivalent by Corollary 5.7.

Notice that (iv) in Theorem 5.8 implies that \( X \) has suprema for all families \( I \to X \) with \( I : \mathcal{V} \) and \( \mathcal{V} \) such that \( \mathcal{V} \lor \mathcal{U} \equiv \mathcal{U} \). Typically, in the examples of \([dJE21]\) for instance, \( \mathcal{U} \equiv \mathcal{U}_1 \) and \( \mathcal{V} \equiv \mathcal{U}_0 \), so that \( \mathcal{V} \lor \mathcal{U} \equiv \mathcal{U} \) holds. Thus, our \( \mathcal{V} \)-families-based approach generalizes the traditional subset-based approach.

6. Set Quotients, Propositional Truncations and Propositional Resizing

We investigate the inter-definability and interaction of type universes of propositional truncations and set quotients in the absence of propositional resizing axioms. In particular, we will see that it not so important if the set quotient or propositional truncation lives in a higher universe. What is paramount instead is whether the universal property applies to types in arbitrary universes.

We start by recalling (the universal property of) the propositional truncation, which, borrowing terminology from category theory, we could also call the subsingleton reflection or propositional reflection.

**Definition 6.1** (Propositional truncation, \( \| - \| \)). A propositional truncation of a type \( X \), if it exists, is a proposition \( \| X \| \) with a map \( |-| : X \to \| X \| \) such that every function \( f : X \to P \) to any proposition factors through \( |-| \).
Some sources, e.g. [Uni13], also demand that the diagram above commutes definitionally: for every \(x : X\), we have \(f(x) \equiv f(|x|)\). Having definitional equalities has some interesting consequences, such as being able to prove function extensionality [KECA17, Section 8]. We do not require definitional equalities, but notice that we do have \(f(x) = \tilde{f}(|x|)\) (up to an identification) for every \(x : X\), as \(P\) is a subsingleton. In particular it follows using function extensionality that \(\tilde{f}\) is the unique factorization.

Notice that if a propositional truncation exists, then it is unique up to unique equivalence.

**Remark 6.2.** Some remarks regarding universes are in order:

(i) In Definition 6.1, the subsingleton \(P\) may live in an arbitrary universe, regardless of the universe in which \(X\) sits. The importance of this will be revisited throughout this section and in Example 6.4 in particular.

(ii) In Definition 6.1, we haven’t specified in what universe \(\|X\|\) should be. When adding propositional truncations as higher inductive types, one typically assumes that \(\|X\| : \mathcal{U}\) if \(X : \mathcal{U}\), and indeed this is what we did in the previous sections. In this section, however, we will be more general and instead assume that \(\|X\| : F(U)\) where \(F\) is a (meta)function on universes, so that the above case is obtained by taking \(F\) to be the identity. We will also consider \(F(U) = U_1 \sqcup U\) in the final subsection.

While in general propositional truncations may fail to exist in intensional Martin-Löf Type Theory, it is possible to construct a propositional truncation of some types in specific cases [EX15, Section 3.1]. A particular example [KECA17, Corollary 4.4] is for a type \(X\) with a weakly constant (viz. any of its values are equal) endo function \(f\): the propositional truncation of \(X\) can be constructed as \(\Sigma_{x : X} (x = f(x))\), the type of fixed points of \(f\).

We review an approach by Voevodsky, who used resizing rules, to constructing propositional truncations in general in the next section.

### 6.1. Propositional Truncations and Propositional Resizing

Voevodsky [Voe11] introduced propositional resizing rules in order to construct propositional truncations [PVW15, Section 2.4]. Here we review Voevodsky’s construction, paying special attention to the universes involved.

**NB. We do not assume the availability of propositional truncations in this section.**

**Definition 6.3** (Voevodsky propositional truncation, \(\|X\|_v\)). The Voevodsky propositional truncation \(\|X\|_v\) of a type \(X : \mathcal{U}\) is defined as

\[
\|X\|_v \equiv \prod_{\mathcal{U}^+} (\text{is-subsingleton}(P) \to (X \to P) \to P).
\]

Because of function extensionality, one can show that \(\|X\|_v\) is indeed a proposition for every type \(X\). Moreover, we have a map \(\mid - \mid_v : X \to \|X\|_v\) given by \(\mid x \mid_v \equiv (P, i, f) \mapsto f(x)\).

Observe that \(\|X\|_v : \mathcal{U}^+\), so using the notation from Remark 6.2, we have \(F(U) = \mathcal{U}^+\). This prompted Voevodsky to consider resizing \(\|X\|_v\) to a proposition in the same universe \(\mathcal{U}\).
However, as we will argue for set quotients, it does not matter so much where the truncated proposition lives; it is much more important that we can eliminate into subsingletons in arbitrary universes, i.e. that $|\neg|_v$ satisfies the right universal property. Given $X : \mathcal{U}$ and a map $f : X \to P$ to a proposition $P : \mathcal{U}$ with $i : \text{is-subsingleton}(P)$, we have a map $\|X\|_v \to P$ given as $\Phi \mapsto \Phi(P,i,f)$. However, if the proposition $P$ lives some other universe $\mathcal{V}$, then we seem to be completely stuck. To clarify this, we consider the example of functoriality.

**Example 6.4.** If we have a map $f : X \to Y$ and $X : \mathcal{U}$ and $Y : \mathcal{U}$, then we get a lifting simply by precomposition, i.e. we define $\langle f \rangle_v : \|X\|_v \to \|Y\|_v$ by $\langle f \rangle_v(\Phi) :\equiv (P,i,g) \mapsto \Phi(P,i,g \circ f)$. But obviously, we also want functoriality for maps $f : X \to Y$ with $X : \mathcal{U}$ and $Y : \mathcal{V}$, but this is impossible with the above definition of $\langle f \rangle_v$, because for $\|X\|_v$ we are considering propositions in $\mathcal{U}$, while for $\|Y\|_v$, we are considering propositions in $\mathcal{V}$.

In particular, even if the types $X : \mathcal{U}$ and $Y : \mathcal{V}$ are equivalent, then it does not seem possible to construct an equivalence between $\|X\|_v$ and $\|Y\|_v$. This issue also comes up if one tries to prove that the map $|\neg|_v : X \to \|X\|_v$ is a surjection [Esc20, Section 3.34.1].

**Proposition 6.5** (cf. Theorem 3.8 of [KECA17]). If our type theory has propositional truncations with $\|X\| : \mathcal{U}$ whenever $X : \mathcal{U}$, then $\|X\|_v$ has size $\mathcal{U}$.

**Proof.** We will show that $\|X\|$ and $\|X\|_v$ are logically equivalent (i.e. we have maps in both directions), which suffices, because both types are subsingletons. We obtain a map $\|X\| \to \|X\|_v$ by applying the universal property of $\|X\|$ to the map $|\neg|_v : X \to \|X\|_v$. Observe that it is essential that the universal property allows for elimination into subsingletons in universes other than $\mathcal{U}$, as $\|X\|_v : \mathcal{U}^+$. For the function in the other direction, simply note that $\|X\| : \mathcal{U}$, so that we can construct $\|X\|_v \to \|X\|$ as $\Phi \mapsto \Phi(\|X\|,i,|\neg|)$ where $i$ witnesses that $\|X\|$ is a subsingleton.

Thus, as is folklore in the univalent foundations community, we can view higher inductive types as specific resizing axioms. But note that the converse to the above proposition does not appear to hold, because even if $\|X\|_v$ had size $\mathcal{U}$, then it still wouldn’t have the appropriate universal property. This is because the definition of $\|X\|_v$ is a dependent product over propositions in $\mathcal{U}$ only, which now includes $\|X\|_v$, but still misses propositions in other universes. In the presence of resizing axioms, we could obtain the full universal property, because we would have (equivalent copies of) all propositions in a single universe.

### 6.2. Set Quotients from Propositional Truncations

In this section we assume to have propositional truncations with $\|X\| : F(\mathcal{U})$ when $X : \mathcal{U}$ for some (meta)function $F$ on universes. We will be mainly interested in $F(\mathcal{U}) = \mathcal{U}$ and $F(\mathcal{U}) = \mathcal{U}_i \sqcup \mathcal{U}$ for the reasons explained below. We prove that we can construct set quotients using propositional truncations. The construction is due to Voevodsky and also appears in [Uni13, Section 6.10] and [RS15, Section 3.4]. However, while Voevodsky assumed propositional resizing rules in his construction, the point of this section is to show that resizing is not needed to prove the universal property of the set quotient, provided propositional truncations are available. Our proof follows our earlier Agda developments [Esc18] (see also [Esc20, Section 3.37]) and is fully formalized: UF-Quotient-F.html in [E+21]
6.2.1. Images and Surjections. It will be convenient to first state and prove two lemmas on images and surjections.

**Definition 6.6** (Image, im(f), surjection, corestriction).

1. The *image* of a function \( f : X \to Y \) is defined as \( \text{im}(f) \equiv \Sigma_{y : Y} \exists_{x : X} (f(x) = y) \).
2. A function \( f : X \to Y \) is a *surjection* if for every \( y : Y \), there exists some \( x : X \) such that \( f(x) = y \).
3. The *corestriction* of a function \( f : X \to Y \) is the function \( f : X \to \text{im}(f) \) given by \( x \mapsto (f(x), [x, \text{refl}] ) \).

**Remark 6.7.** Note that if \( X : \mathcal{U} \) and \( Y : \mathcal{V} \) and \( f : X \to Y \), then \( \text{im}(f) : \mathcal{V} \uplus F(\mathcal{U} \uplus \mathcal{V}) \), because \( \Sigma_{x : X} (f(x) = y) : \mathcal{U} \uplus \mathcal{V} \) and \( \parallel - \parallel \) takes types in \( \mathcal{W} \) to subsingletons in \( F(\mathcal{W}) \). In case \( F \) is the identity, then we obtain the simpler \( \text{im}(f) : \mathcal{U} \uplus \mathcal{V} \).

**Lemma 6.8.** Every corestriction is surjective.

**Proof.** By definition of the corestriction. \(\square\)

**Lemma 6.9** (Image induction). If \( f : X \to Y \) is a surjection, then the following induction principle holds: for every subsingleton-valued \( P : Y \to \mathcal{W} \), with \( \mathcal{W} \) an arbitrary universe, if \( P(f(x)) \) holds for every \( x : X \), then \( P(y) \) holds for every \( y : Y \).

In the other direction, for any map \( f : X \to Y \), if the above induction principle holds for the specific family \( P(y) \equiv \exists_{x : X} (f(x) = y) \), then \( f \) is a surjection.

**Proof.** Suppose that \( f : X \to Y \) is a surjection, let \( P : Y \to \mathcal{W} \) be subsingleton-valued and assume that \( P(f(x)) \) holds for every \( x : X \). Now let \( y : Y \) be arbitrary. We are to prove that \( P(y) \) holds. Since \( f \) is a surjection, we have \( \exists_{x : X} (f(x) = y) \). But \( P(y) \) is a subsingleton, so we may assume that we have a specific \( x : X \) with \( f(x) = y \). But then \( P(y) \) must hold, because \( P(f(x)) \) does by assumption.

For the other direction, notice that if \( P(y) \equiv \exists_{x : X} (f(x) = y) \), then \( P(f(x)) \) clearly holds for every \( x : X \). So by assuming that the induction principle applies, we get that \( P(y) \) holds for every \( y : Y \), which says exactly that \( f \) is a surjection. \(\square\)

6.2.2. Set Quotients. We now construct set quotients using images and specialize image induction to the set quotient.

**Definition 6.10** (Equivalence relation). An *equivalence relation* on a type \( X \) is a binary type family \( \approx : X \to X \to \mathcal{V} \) such that it is

1. subsingleton-valued, i.e. \( x \approx y \) is a subsingleton for every \( x, y : X \);
2. reflexive, i.e. \( x \approx x \) for every \( x : X \);
3. symmetric, i.e. \( x \approx y \) implies \( y \approx x \) for every \( x, y : X \);
4. transitive, i.e. the conjunction of \( x \approx y \) and \( y \approx z \) implies \( x \approx z \) for every \( x, y, z : X \).

**Definition 6.11** (Set quotient, \( X/\approx \)). We define the *set quotient* of \( X \) by \( \approx \) to be the type \( X/\approx \equiv \text{im}(e_\approx) \) where

\[
e_\approx : X \to (X \to \Omega_Y)
\]
\[
x \mapsto (y \mapsto (x \approx y, p(x, y)))
\]

and \( p \) is the witness that \( \approx \) is subsingleton-valued.
Of course, we should prove that $X/\approx$ really is the quotient of $X$ by $\approx$ by proving a suitable universal property. The following definition and lemmas indeed build up to this. For the remainder of this section, we will fix a type $X : \mathcal{U}$ with an equivalence relation $\approx : X \to X \to \mathcal{V}$.

**Remark 6.12.** By Remark 6.7, and because $\Omega_V : \mathcal{V}^+$, we have $X/\approx : \mathcal{T} \sqcup F(\mathcal{T})$ with $\mathcal{T} :\equiv \mathcal{V}^+ \sqcup \mathcal{U}$. In the particular case that $F$ is the identity, then we obtain the simpler $X/\approx : \mathcal{V}^+ \sqcup \mathcal{U}$.

**Lemma 6.13.** The quotient $X/\approx$ is a set.

**Proof.** Observe that $(X/\approx) \equiv \text{im}(e_\approx)$ is a subtype of $X \to \Omega_V$ (as $\text{pr}_1 : X/\approx \to (X \to \Omega_V)$ is an embedding), that $X \to \Omega_V$ is a set (by function extensionality) and that subtypes of sets are sets.

**Definition 6.14 ($\eta$).** The map $\eta : X \to X/\approx$ is defined to be the corestriction of $e_\approx$.

Note that the quotient $X/\approx$ lives in the universe $\mathcal{U} \sqcup \mathcal{V}^+$, because $\Omega_V : \mathcal{V}^+$ and $X : \mathcal{U}$. But we can still prove the following induction principle for (subsingleton-valued) families into arbitrary universes.

**Lemma 6.15 (Set quotient induction).** For every subsingleton-valued $P : X/\approx \to \mathcal{W}$, with $\mathcal{W}$ any universe, if $P(\eta(x))$ holds for every $x : X$, then $P(x')$ holds for every $x' : X/\approx$.

**Proof.** The map $\eta$ is surjective by Lemma 6.8, so that Lemma 6.9 yields the desired result.

**Definition 6.16 (Respect equivalence relation).** A map $f : X \to A$ respects the equivalence relation $\approx$ if $x \approx y$ implies $f(x) = f(y)$ for every $x, y : X$.

Observe that respecting an equivalence relation is property rather than data, when the codomain $A$ of the map $f : X \to A$ is a set.

**Lemma 6.17.** The map $\eta : X \to X/\approx$ respects the equivalence relation $\approx$ and the set quotient is effective, i.e. for every $x, y : X$, we have $x \approx y$ if and only if $\eta(x) = \eta(y)$.

**Proof.** By definition of the image and function extensionality, we have for every $x, y : X$ that $\eta(x) = \eta(y)$ holds if and only if
\[
\forall z : X (x \approx z \iff y \approx z) \tag{*}
\]
holds. If $(*)$ holds, then so does $x \approx y$ by reflexivity and symmetry of the equivalence relation. Conversely, if $x \approx y$ and $z : X$ is such that $x \approx z$, then $y \approx z$ by symmetry and transitivity; and similarly if $z : X$ is such that $y \approx z$. Hence, $(*)$ holds if and only if $x \approx y$ holds. Thus, $\eta(x) = \eta(y)$ if and only if $x \approx y$, as desired.

The universal property of the set quotient states that the map $\eta : X \to X/\approx$ is the universal function to a set preserving the equivalence relation. We can prove it using only Lemma 6.15 and Lemma 6.17, without needing to inspect the definition of the quotient.

**Theorem 6.18 (Universal property of the set quotient).** For every set $A : \mathcal{W}$ in any universe $\mathcal{W}$ and function $f : X \to A$ respecting the equivalence relation, there is a unique function $\bar{f} : X/\approx \to A$ such that the diagram
commutes.

Proof. Let \( A : \mathcal{W} \) be a set and \( f : X \to A \) respect the equivalence relation. The following auxiliary type family will be at the heart of our proof:

\[
B : X/\approx \to \mathcal{V}^+ \sqcup U \sqcup \mathcal{W} \\
x' \mapsto \Sigma_{a:A} \exists_{x:X} ((\eta(x) = x') \times (f(x) = a)).
\]

Claim. The type \( B(x') \) is a subsingleton for every \( x' : X/\approx \).

Proof of claim. By function extensionality, the type expressing that \( B(x') \) is a subsingleton for every \( x' : X/\approx \) is itself a subsingleton. So by set quotient induction, it suffices to prove that \( B(\eta(x)) \) is a subsingleton for every \( x : X \). So assume that we have \( (a, p), (b, q) : B(\eta(x)) \).

It suffices to show that \( a = b \). The elements \( p \) and \( q \) witness

\[
\exists_{x_1 : X} ((\eta(x_1) = \eta(x)) \times (f(x_1) = a))
\]

and

\[
\exists_{x_2 : X} ((\eta(x_2) = \eta(x)) \times (f(x_2) = b))
\]

respectively. By Lemma 6.17 and the fact that \( f \) respects the equivalence relation, we obtain \( f(x) = a \) and \( f(x) = b \) and hence the desired \( a = b \).

Next, we define \( k : \Pi_{x:X} B(\eta(x)) \) by \( k(x) = (f(x), |x, \text{refl}, \text{refl}|) \). By set quotient induction and the claim, the function \( k \) induces a dependent map \( \tilde{k} : \Pi_{(x', X/\approx)} B(x') \).

We then define the (nondependent) function \( \bar{f} : X/\approx \to A \) as \( \text{pr}_1 \circ \tilde{k} \). We proceed by showing that \( \bar{f} \circ \eta = f \). By function extensionality, it suffices to prove that \( \bar{f}(\eta(x)) = f(x) \) for every \( x : X \). But notice that:

\[
\bar{f}(\eta(x)) = \text{pr}_1(k(\eta(x))) = \text{pr}_1(\tilde{k}(x)) = f(x).
\]

Since \( k(\eta(x)) = \tilde{k}(x) \) because of the claim.

Finally, we wish to show that \( \bar{f} \) is the unique such function, so suppose that \( g : X/\approx \to A \) is another function such that \( g \circ \eta = f \). By function extensionality, it suffices to prove that \( g(x') = \bar{f}(x') \) for every \( x' : X/\approx \), which is a subsingleton as \( A \) is a set. Hence, set quotient induction tells us that it is enough to show that \( g(\eta(x)) = \bar{f}(\eta(x)) \) for every \( x : X \), but this holds as both sides of the equation are equal to \( f(x) \).

Remark 6.19 (cf. Section 3.21 of [Esc20]). In univalent foundations, if we wish to express unique existence of an element \( x : X \) satisfying \( P(x) \) for some type family \( P : \mathcal{U} \to \mathcal{V} \), then we should phrase it as is-singleton(\( \Sigma_{x:X} P(x) \)), where is-singleton(\( Y \)) \( \equiv Y \times \text{is-subsingleton}(Y) \). That is, we require that there is a unique pair \( (x, p) : \Sigma_{x:X} P(x) \). This becomes important when the type family \( P \) is not subsingleton-valued. However, if \( P \) is subsingleton-valued, then it is equivalent to the traditional formulation of unique existence: i.e. that there is an \( x : X \) with \( P(x) \) such that every \( y : X \) with \( P(y) \) is equal to \( x \). This happens to be the situation in Theorem 6.18, because of function extensionality and the fact that \( A \) is a set.
We stress that although the set quotient increases universe levels, see Remark 6.12, it does satisfy the appropriate universal property, so that resizing is not needed.

Having small set quotients is closely related to propositional resizing, as we show now.

**Proposition 6.20.** Suppose that \( \| - \| \) does not increase universe levels, i.e. in the notation of Remark 6.2, the function \( F \) is the identity.

(i) If \( \Omega\)-Resizing\( _{\mathcal{V},\mathcal{U}} \) holds for universes \( \mathcal{U} \) and \( \mathcal{V} \), then the set quotient \( X/\approx \) has size \( \mathcal{U} \) for any type \( X : \mathcal{U} \) and any \( \mathcal{V} \)-valued equivalence relation.

(ii) Conversely, if the set quotient \( 2/\approx \) has size \( \mathcal{U}_0 \) for every \( \mathcal{V} \)-equivalence relation on \( 2 \), then Propositional-Resizing\( _{\mathcal{V},\mathcal{U}_0} \) holds.

**Proof.** (i): If we have \( \Omega\)-Resizing\( _{\mathcal{V},\mathcal{U}} \), then \( \Omega \mathcal{V} \) has size \( \mathcal{U} \), so that \( X/\approx \equiv \text{im}(e_\approx) \) has size \( \mathcal{U} \) too when \( X : \mathcal{U} \) and \( \approx \) is \( \mathcal{V} \)-valued. (ii): Let \( P : \mathcal{V} \) be any proposition and consider the \( \mathcal{V} \)-valued equivalence relation \( x \approx_P y \equiv (x = y) \lor P \) on \( 2 : \mathcal{U}_0 \). Notice that

\[
(2/\approx_P) \text{ is a subsingleton } \iff P \text{ holds,}
\]

so we see that if \( 2/\approx_P \) has size \( \mathcal{U}_0 \), then so does is-subsingleton \( (2/\approx_P) \) and therefore \( P \).

6.3. **Propositional Truncations from Set Quotients.** The converse, constructing propositional truncations from set quotients, is more straightforward, although we must pay some attention to the universes involved in order to get an exact match.

**Theorem 6.21.** If set quotients exist, then every type has a propositional truncation.

**Proof.** Let \( X : \mathcal{U} \) be any type and consider the \( \mathcal{U}_0 \)-valued equivalence relation equivalence relation \( x \approx_1 y \equiv 1 \). To see that \( X/\approx_1 \) is a subsingleton, note that by set quotient induction it suffices to prove \( \eta(x) = \eta(y) \) for every \( x, y : X \). But by Lemma 6.17 we have \( \eta(x) = \eta(y) \) if and only if \( x \approx_1 y \), and the latter holds for every \( x, y : X \), so \( X/\approx_1 \) is indeed a subsingleton. Now if \( P : \mathcal{V} \) is any subsingleton and \( f : X \rightarrow P \) is any map, then \( f \) respects the equivalence relation \( \approx_1 \) on \( X \), simply because \( P \) is a subsingleton. Thus, by the universal property of the quotient, we obtain the desired map \( \bar{f} : X/\approx_1 \rightarrow P \) and hence, \( X/\approx_1 \) has the universal property of the propositional truncation.

**Remark 6.22.** Because the set quotients constructed using the propositional truncation live in higher universes, we embark on a careful comparison of universes. Suppose that propositional truncations of types \( X : \mathcal{U} \) exist and that \( \| X \| : F(\mathcal{U}) \). Then by Remark 6.12, the set quotient \( X/\approx_1 \) in the proof above lives in the universe \( (\mathcal{U}_1 \sqcup \mathcal{U}) \sqcup F(\mathcal{U}_1 \sqcup \mathcal{U}) \).

In particular, if \( F \) is the identity and the propositional truncation of \( X : \mathcal{U} \) lives in \( \mathcal{U} \), then the quotient \( X/\approx_1 \) lives in \( \mathcal{U}_1 \sqcup \mathcal{U} \), which simplifies to \( \mathcal{U} \) whenever \( \mathcal{U} \) is at least \( \mathcal{U}_1 \). In other words, the universes of \( \| X \| \) and \( X/\approx_1 \) match up for types \( X \) in every universe, except the first universe \( \mathcal{U}_0 \).

If we always wish to have \( X/\approx_1 \) in the same universe as \( \| X \| \), then we can achieve this by assuming \( F(\mathcal{V}) \equiv \mathcal{U}_1 \sqcup \mathcal{V} \), which says that the propositional truncations stay in the same universe, except when the type is in the first universe \( \mathcal{U}_0 \) in which case the truncation will be in the second universe \( \mathcal{U}_1 \).
7. Conclusion

Firstly, we have shown, constructively and predicatively, that nontrivial dcpos, bounded complete posets and sup-lattices are all necessarily large and necessarily lack decidable equality. We did so by deriving a weak impredicativity axiom or weak excluded middle from the assumption that such nontrivial structures are small or have decidable equality, respectively. Strengthening nontriviality to the (classically equivalent) positivity condition, we derived a strong impredicativity axiom and full excluded middle.

Secondly, we proved that Zorn’s lemma, Tarski’s greatest fixed point theorem and Pataraia’s lemma all imply impredicativity axioms. Hence, these principles are inherently impredicative and a predicative development of order theory (in univalent foundations) must thus do without them.

Thirdly, we clarified, in our predicative setting, the relation between the traditional definition of a lattice that requires completeness with respect to subsets and our definition that asks for completeness with respect to small families.

Finally, we investigated the inter-definability and interaction of type universes of propositional truncations and set quotients in the absence of propositional resizing axioms. In particular, we show that in the presence of propositional truncations, but without assuming propositional resizing, it is possible to construct set quotients that happen to live in higher type universes but that do satisfy the appropriate universal properties with respect to sets in arbitrary type universes.

In future work, we wish to study the predicative validity of Pataraia’s theorem and Tarski’s least fixed point theorem. Curi [Cur15, Cur18] develops predicative versions of Tarki’s fixed point theorem in some extensions of CZF. It is not clear whether these arguments could be adapted to univalent foundations, because they rely on the set-theoretical principles discussed in the introduction. The availability of such fixed-point theorems might be useful for application to inductive sets [Acz77], which we might otherwise introduce in univalent foundations using higher inductive types [Uni13]. In another direction, we have developed a notion of apartness [BV11] for continuous dcpos [GHK03] that is related to the notion of being strictly below introduced in this paper. Namely, if \( x \sqsubseteq y \) are elements of a continuous dcpo, then \( x \) is strictly below \( y \) if \( x \) is apart from \( y \). In [dJ21], we give a constructive analysis of the Scott topology [GHK03] using this notion of apartness.

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