On finite-dimensional representations of finite $W$-superalgebras

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Received September 12, 2021; accepted October 15, 2022; published online April 4, 2023

Abstract We first formulate and prove a version of Premet’s conjecture for finite $W$-superalgebras associated with basic Lie superalgebras. As in the case of $W$-algebras, Premet’s conjecture is very close to giving a classification of finite-dimensional simple modules of finite $W$-superalgebras. In the case of basic type I Lie superalgebras, we classify the finite-dimensional simple supermodules with the integral central character and give an algorithm to compute their characters based on the $\mathfrak{g}_0$-rough structure of $\mathfrak{g}$-modules.

Keywords $W$-superalgebras, Premet’s conjecture, finite-dimensional representations

MSC(2020) 17B10, 17B63, 17B69

Citation: Xiao H. On finite-dimensional representations of finite $W$-superalgebras. Sci China Math, 2023, 66: 1737–1750, https://doi.org/10.1007/s11425-021-2048-x

1 Introduction

Finite $W$-superalgebras are the Zhu algebras of affine $W$-superalgebras in the sense of [9]. The latter includes the well-known $N (N = 1, 2, 3, 4)$ superconformal algebras and plays a very important role in the supersymmetric quantum field theory. The affine $W$-superalgebras were constructed in [11] by the quantum Hamiltonian reduction in the general setting. However, the finite $W$-superalgebras appear in mathematics more indirectly. Generalizing the groundbreaking work [27], Wang and Zhao [29] and Zhao [31] first studied finite $W$-superalgebras from the viewpoint of modular Lie superalgebras. Here, the term modular means that the ground algebraically closed field has a positive characteristic.

Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a basic Lie superalgebra, and $\mathcal{W}_0$ (resp. $\mathcal{W}$) be the finite $W$- (resp. super-) algebra constructed from a fixed nilpotent element $e \in \mathfrak{g}_0$. Based on a relation between the finite $W$-algebra $\mathcal{W}_0$ and the $W$-superalgebra $\mathcal{W}$ found recently by Shu and Xiao [28], we study the finite-dimensional irreducible representations of finite $W$-superalgebras in this paper. Let $\text{Irr}^{\text{fin}}(\mathcal{W})$ stand for the set of isomorphism classes of irreducible modules.

Brown et al. [2] and Brundan and Goodwin [3] gave a Yangian presentation of $W$-superalgebras corresponding to principal nilpotent elements in the general linear Lie superalgebras. Relying on this explicit presentation, they gave a description of $\text{Irr}^{\text{fin}}(\mathcal{W})$ and further detailed information on their highest weight structures. Poletaeva and Serganova [26] proved an Amitsur-Levitzki identity for the $W$-superalgebras associated with principal nilpotent elements in the queer Lie superalgebras. They
obtained that any irreducible representation is finite-dimensional. These results indicate that the representation theory of finite $W$-superalgebras is quite different from that of finite $W$-algebras. By giving an explicit description of the structure of $W$-superalgebras associated with minimal nilpotent elements, Zeng and Shu [30] constructed their irreducible representations with dimension 1 or 2. Recently, Chen [4] investigated the Whittaker category $\mathcal{N}$ for basic Lie superalgebras. Through Skryabin’s equivalence, the category $\mathcal{N}$ is equivalent to $W$-Mod when $W$ is associated with a principal nilpotent $e$.

However, unlike the case of finite $W$-algebras, some fundamental problems in the representation theory of finite $W$-superalgebras are still open in general. In [28], Shu and Xiao generalized Losev’s Poisson geometric approach to the super case and made a step to give a classification of finite-dimensional irreducible representations of finite $W$-superalgebras in a more general setting. In this article, we make some progress to this problem by proving Premet’s conjecture for the $W$-superalgebras of basic Lie superalgebras. In particular, we classify the finite-dimensional simple $W$-supermodules with the integral central character and obtain an algorithm to compute their characters in the basic type I case.

We hope that the readers could be convinced that the difference between finite $W$-algebras and $W$-superalgebras probably not exceeds that between Lie algebras and Lie superalgebras.

1.1 Premet’s conjecture for finite $W$-superalgebras

Let $g = g_0 \oplus g_1$ be a basic Lie superalgebra over an algebraically closed field $\mathbb{K}$ with $\text{Char}(\mathbb{K}) = 0$, and $U$ and $U_0$ be the enveloping algebras of $g$ and $g_0$, respectively. Denote by $(\cdot, \cdot)$ the Killing form on $g$. Fix a nilpotent $e \in g_0$ and let $\chi \in g_0^*$ be the corresponding element to $e$ via the Killing form. Pick an $sl_2$-triple $(f, h, e) \subset g_0$ and let $g = \bigoplus_i g(i)$ (resp. $g_0 = \bigoplus_i (g(i) \cap g_0)$) be the $\mathbb{Z}$-grading given by the adjoint action of $h$. Denote by $W$ and $W_0$ the $W$-algebras associated with the pairs $(g, e)$ and $(g_0, e)$, respectively. Let $\bar{W}$ be the extended $W$-superalgebra $A_1$ defined in [28, Section 3] (note that it is denoted by $A_1$ in [18, Section 6]). The following relation among the three kinds of $W$-algebras was found in [28]: (1) we have an embedding $W_0 \hookrightarrow \bar{W}$ and the latter is generated over the former by $\dim(g_1)$ odd elements; (2) we have an isomorphism $\bar{W} \cong Cl(V_1) \otimes W$ of associative algebras, where $Cl(V_1)$ is the Clifford algebra over a vector space $V_1$ with a non-degenerate symmetric bilinear form (see Theorem 2.3 for the details). Essentially, as mentioned in [28], this makes $W_0$ to play a role in the representation theory of $W$ as $U_0$ does in that of $U$. The representation theories of $W$ and $\bar{W}$ are equivalent (see Proposition 2.5). However, as we will see in the present work, a significant advantage to consider $\bar{W}$ instead of $W$ is that it is easy to relate $\bar{W}$ with $W_0$. This enables us to use results on $W_0$.

Given an associative algebra $A$, we denote by $\text{id}(A)$ the set of two-sided ideals of $A$ and by $\text{Prim}^{\text{fin}}(A)$ the set of primitive ideals of $A$ with finite codimension. It is well known that $\text{Prim}^{\text{fin}}(A)$ is bijective with the set $\text{Irr}^{\text{fin}}(A)$ of isomorphism classes of finite-dimensional irreducible $A$-modules. Losev [14] constructed an ascending map $\bullet : \text{id}(W_0) \to \text{id}(U_0)$ and a descending map $\bullet^\dagger : \text{id}(U_0) \to \text{id}(W_0)$. These two maps are crucial to his study on the representations of $W_0$. The ascending map $\bullet$ sends $\text{Prim}^{\text{fin}}(W_0)$ to the set $\text{Prim}_{\mathcal{O}}(U_0)$ of primitive ideals of $U_0$ supported on the Zariski closure of the adjoint orbit $\mathcal{O} = G_0 \cdot \chi$. Denote by $Q = Z_{G_0}(e, h, f)$ the stabilizer of the triple $(e, h, f)$ in $G_0$ under the adjoint action. Let $C_c = Q/Q^e$, where $Q^e$ is the identity component of $Q$. Premet’s conjecture which was proved in [16] states that for any $J \in \text{Prim}_{\mathcal{O}}(U_0)$, the set $\{I \mid I \in \text{Prim}^{\text{fin}}(W), I^\dagger = J\}$ is a single $C_c$-orbit. This indicates to us an almost complete classification of $\text{Irr}^{\text{fin}}(W_0)$.

In this paper, we generalize the above fact to the super case. Recall that the super analog of the maps $\bullet$ and $\bullet^\dagger$ was established in [28]. By abuse of notation, we also denote them by $\bullet$ and $\bullet^\dagger$ from now on. Denote by $\text{Prim}_{\mathcal{O}}(U)$ the set of primitive ideals of $U$ supported on the Zariski closure of $\mathcal{O}$ (see Section 2 for the definition of ‘supported’ in the super context). In Section 2, we construct an action of $Q$ on $\bar{W}$ with a property that $Q^e$ leaves any two-sided ideal of $\bar{W}$ stable (see Proposition 2.1). This yields an action of $C_c$ on $\text{id}(\bar{W})$.

We also consider the $\mathbb{Z}_2$-graded version of the above setting. For a superalgebra $A = A_0 + A_1$, the $\mathbb{Z}_2$-graded $A$-modules will be called $A$-supermodules. An ideal $I$ of $A$ is said to be graded primitive if it is the annihilator of a simple object in the category of $A$-supermodules. Denote by $\text{gr} \text{Prim}(A)$ the set
of graded primitive ideals of \( \mathcal{A} \). For a notation \( \bullet \) used in the ungraded case, we always use \( \text{gr} \cdot \bullet \) in the \( \mathbb{Z}_2 \)-graded case by the same way as above. Since the action of \( Q \) on \( W \) is \( \mathbb{Z}_2 \)-homogeneous, we also have an action of \( C_e \) on \( \text{gr} \cdot \hat{\mathfrak{D}}(W) \). Our first main result reads as follows.

**Theorem 1.1.** For any \( J \in \text{Prim}\_0(\mathcal{U}) \), the set \( \{ \text{Cl}(V_j) \otimes I \mid I \in \text{Prim}\_0^\text{fin}(W), I^\dagger = J \} \) consisting of the primitive ideals of \( W \) lying over \( J \), is a single \( C_e \)-orbit. For any \( J \in \text{gr} \cdot \text{Prim}\_0(\mathcal{U}) \), the set consisting of the graded primitive ideals of \( W \) lying over \( J \) is also a single \( C_e \)-orbit.

We also have maps \( \bullet^\dagger : \hat{\mathfrak{D}}^b(\hat{W}) \to \hat{\mathfrak{D}}(\mathcal{U}) \) and \( \bullet : \hat{\mathfrak{D}}(\mathcal{U}) \to \hat{\mathfrak{D}}(\hat{W}) \), which can be defined similarly to \( \bullet^\dagger \) and \( \bullet \) (see Lemma 2.6). Theorem 1.1 is equivalent to saying that the set \( \{ \hat{I} \mid \hat{I} \in \text{Prim}\_0^\text{fin}(\hat{W}), \hat{I}^\dagger = J \} \) of primitive ideals lying over \( J \), is a single \( C_e \)-orbit.

The strategy of our proof is applying [16, Theorem 4.1.1] to the Harish-Chandra bimodule \( \mathcal{U} \) over \( \mathcal{U}_0 \) and the relation among \( W, W_0 \) and \( \hat{W} \) introduced previously. This is highly inspired by [18, Section 6].

We can recover \( \mathcal{I} \) from \( \text{Cl}(V_j) \otimes \mathcal{I} \) by Corollary 2.4. It is known that the map \( \bullet^\dagger \) sends \( \text{Prim}\_0^\text{fin}(W) \) to \( \text{Prim}\_0(\mathcal{U}) \) (see [29, Theorem 4.8]). So Theorem 1.1 almost completely reduces the problem of classifying \( \text{Prim}\_0^\text{fin}(W) = \text{Irr}\_0^\text{fin}(W) \) to that of \( \text{Prim}(\mathcal{U}) \). If we know \( \text{Prim}(\mathcal{U}) \) and \( C_e \) are trivial, Theorem 1.1 gives a description of \( \text{Irr}\_0^\text{fin}(W) \) (see Subsection 2.6). For the recent progress on the primitive ideals of Lie superalgebras, see for examples [8, 21, 23].

We say that \( M \in \text{Irr}\_0^\text{fin}(\hat{W}) \) (or \( M' \in \text{Irr}\_0^\text{fin}(W) \)) lies over a primitive ideal \( J \) of \( \mathcal{U} \) if so do their annihilators. It is well known that for the basic classical Lie superalgebras \( g \), any primitive ideal of \( \mathcal{U} \) is the annihilator \( \hat{J}(\lambda) \) of a highest weight simple module \( \hat{L}(\lambda) \) for some \( \lambda \in \mathfrak{h}^* \). We say that a finite-dimensional simple \( \hat{W} \)-module has the center character \( \lambda \) if it lies over \( \hat{J}(\lambda) \). Let \( \text{Irr}_\lambda(\hat{W}) \) stand for the set of isomorphism classes of \( \hat{W} \)-supermodules with the center character \( \lambda \). Define \( \text{Irr}_\lambda(W_0) \), \( \text{Irr}_\lambda^\text{fin}(W_0) \) and \( \text{Irr}_\lambda^\text{fin}(W) \) similarly. Theorem 1.1 gives us an action of \( C_e \) on \( \text{Irr}_\lambda^\text{fin}(W) \) and \( \text{Irr}_\lambda^\text{fin}(W) \) (see Subsection 2.5).

### 1.2 Finite-dimensional representations of basic type I \( W \)-superalgebras

In the remaining part of this section, let \( g = g_0 + g_1 \) be a basic type I simple Lie superalgebra. Namely, \( g \) is one of the following list:

- **Type(\( A \)) : \text{gl}(m \mid n), \text{sl}(m \mid n), \text{so}(n \mid n)/\mathbb{C}I_{n \mid n};**  
  **Type(\( C \)) : \text{osp}(2 \mid 2n).**

A classification of simple \( g \)-supermodules was obtained in [7]. It was proved that there is a one-to-one correspondence between the set of isomorphism classes of simple \( g \)-supermodules and that of the simple \( g_0 \)-modules. Given a simple \( g_0 \)-module \( V \), we denote by \( \hat{V} \) the simple \( g \)-supermodules under this correspondence, which is the unique simple quotient of the Kac module \( K(V) \). This result is fundamental to the present work. Using Skryabin’s equivalence, we decent this result to the context of \( W \)-algebras. More precisely, we prove that the sets \( \text{Irr}(W_0) \), \( \text{gr}\cdot\text{Irr}(\hat{W}) \) and \( \text{gr}\cdot\text{Irr}(W) \) are bijective with each other. By abuse of notation, for a simple \( W_0 \)-module \( N \), we also denote by \( \hat{N} \) the unique simple \( \hat{W} \)-supermodule under this correspondence. However, this classification of \( \text{Irr}(W) \) is not well organized. For example, it is difficult to see the behavior of the action of \( C_e \) under the correspondence. To fix up this problem, we give another better classification of \( \text{gr}\cdot\text{Irr}(\hat{W}) \). To that end, we present a triangular decomposition \( \hat{W} = W_+ \otimes \mathbb{K} W_0 \otimes \mathbb{K} W_- \) for \( \hat{W} \). This can be compared with the decomposition \( g = g_- + g_0 + g_+ \) of the type I simple Lie superalgebras. A crucial point is that \( W_0 \) is the ordinary finite \( W \)-algebra from \( (g_0, e) \).

Using this decomposition, for any finite-dimensional simple \( W_0 \)-module \( N \), we define the ‘Verma’ module \( \Delta^\text{fin}_N(N) \otimes \hat{W} \) and prove that it has a unique simple \( \mathbb{Z}_2 \)-graded quotient \( L^\text{fin}_N(N) \). We point out that it is easy to obtain a triangular decomposition \( W = W_- \otimes \mathbb{K} W_0 \otimes \mathbb{K} W_+ \) for the usual finite \( W \)-superalgebra \( W \) by a similar method used here. A triangular decomposition has already been obtained for \( W \) arising from the general Lie superalgebras by using a super Yangian presentation (see [2] for the principal nilpotent element \( e \) and [25] for the general case). Compared with the one for \( \hat{W} \), a disadvantage of the latter is that it is highly non-trivial to relate \( W_0 \) and \( W_0 \) for general \( e \), although the two algebras coincide when \( e \) is a principal nilpotent element.

Our main tool used to compute the character of simple \( \hat{W} \)-modules with the integral center character is the generalized Soergel functor \( \mathcal{V} \) for \( W_0 \) constructed in [18]. Let \( P \subset G_0 \) (resp. \( \mathfrak{p} = \text{Lie}(P) \)) be the
suitable parabolic subgroup (resp. subalgebra) constructed from an $\mathfrak{sl}_2$-triple in [18]. Denote by $O^\mathcal{P}$ the corresponding parabolic category $\mathcal{O}$ and by $\Lambda_\mathcal{P}$ the set consisting of the integral $\lambda \in \mathfrak{h}^*$ such that the highest weight simple module $L(\lambda)$ lies in $O^\mathcal{P}$. Let $V : O^\mathcal{P} \to O^\mathcal{P}(\mathfrak{g}_0, e)$ be the generalized Soergel functor for $\mathcal{W}_0$ defined in [18]. The notation will be recalled in Section 4. Let $\lambda \in \Lambda_\mathcal{P}$ with $V(L(\lambda)) \neq 0$. Describing $V(L(\lambda))$, Losev [18] established a character formula for the modules in $\text{Irr}^{\text{fin}}_\lambda(\mathcal{W}_0)$ with the integral $\lambda$. His character formula is based on the parabolic Kazhdan-Lusztig theory for $O^\mathcal{P}$. We give a description of $V(\tilde{L}(\lambda))$ for the simple $\mathfrak{g}$-supermodules $\tilde{L}(\lambda) \in O^\mathcal{P}$. Relying on the $\mathfrak{g}_0$-rough structure of simple $\mathfrak{g}$-supermodules, we compute the characters of modules in $\text{Irr}^{\text{fin}}_\lambda(\mathcal{W})$ for the integral $\lambda$. Note that, just like the even case, the set $\text{Irr}^{\text{fin}}_\lambda(\mathcal{W})$ is non-empty only if $\lambda \in \Lambda_\mathcal{P}$.

In summary, for $\mathfrak{g}$ being a basic type I Lie superalgebra, our main results are as follows.

(1) We obtain a triangular decomposition for $\mathcal{W}$ and some standard properties of Verma modules defined by it. We prove that the map

$$\text{Irr}^{\text{fin}}(\mathcal{W}_0) \to \text{gr. Irr}^{\text{fin}}(\mathcal{W}) : N \mapsto L^K_N(\mathcal{W})$$

is bijective and $C_\mathcal{P}$-equivariant (see Proposition 4.1). As an application, we also prove that $\text{gr. Prim}(\mathcal{W})$ is bijective with $\text{Prim}(\mathcal{W}_0)$ (see Corollary 4.2).

(2) For $\lambda \in \Lambda_\mathcal{P}$, let $V(L(\lambda)) = \bigoplus_{i \in I_\lambda} N_i$ be the description of $V(L(\lambda))$ obtained in [18]. Here, $I_\lambda$ is a finite set and $N_i \in \text{Irr}^{\text{fin}}_\lambda(\mathcal{W}_0)$ for $i \in I_\lambda$. Then we have

$$V(\tilde{L}(\lambda)) = \bigoplus_{i \in I_\lambda} L^K_{\tilde{\mathcal{W}}}(N_i)$$

(see Theorem 4.5).

(3) For the integral $\lambda$, we present an algorithm to compute the characters of modules in $\text{Irr}^{\text{fin}}_\lambda(\mathcal{W})$ (see Subsection 4.5).

Finally, we point out that the powerful tools about $W$-algebras developed by Losev [18] will be used in the whole paper. However, they are very technical and rely heavily on the geometry of nilpotent orbits. In the super case, we overcome these difficulties (see Proposition 2.1 and 4.1).

2 A super version of Premet’s conjecture

We first recall the definition of finite $W$- (super-) algebras in the sense of Premet. We continue with the notation from Subsection 1.1. Let $l = l_0 + l_1$ be a Lagrangian subspace of $\mathfrak{g}(-1)$ with respect to the super symplectic form $\chi([\cdot, \cdot])$. Thus, $l_0$ is automatically a Lagrangian subspace of $\mathfrak{g}_0(-1)$. Set

$$m = \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus l, \quad m_0 = \bigoplus_{i \leq -2} \mathfrak{g}_0(i) \oplus l_0$$

and

$$m_\chi = \{ x - \chi(x) \mid x \in m \}, \quad m_{0, \chi} = \{ x - \chi(x) \mid x \in m_0 \}.$$ 

The finite $W$-algebra $\mathcal{W}_0$ and the $W$-superalgebra $\mathcal{W}$ are defined as follows:

$$\mathcal{W}_0 = (U_0/U_0m_0)_\chi^{\text{ad}(m_0)} \quad \text{and} \quad \mathcal{W} = (U/Um_\chi)^{\text{ad}(m)}.$$

(2.1)

Let us recall the Poisson geometric realization of finite $W$- (super-) algebras in the sense of Losev [14]. Denote by $A_0$ (resp. $A$) the Poisson (resp. super-) algebra $S[\mathfrak{g}_0]$ (resp. $S[\mathfrak{g}]$) with the standard bracket $\{\cdot, \cdot\}$ given by $\{x, y\} = [x, y]$ for all $x, y \in \mathfrak{g}_0$ (resp. $\mathfrak{g}$). Let $\tilde{A}_0$ (resp. $\tilde{A}$) be the completion of $A_0$ (resp. $A$) with respect to the point $\chi \in \mathfrak{g}_0^*$ (resp. $\mathfrak{g}$) and $U_0$ (resp. $U$) be the formal quantization of $\tilde{A}_0$ (resp. $\tilde{A}$) given by $x \cdot y - y \cdot x = \hbar^2 [x, y]$ for all $x, y \in \mathfrak{g}_0$. Equip all the above algebras with the Kazhdan $K^*$-action arising from the $Z$-grading on $\mathfrak{g}$ and $t \cdot h = th$ for all $t \in K^*$.

Denote by $\omega$ the even symplectic form on $[f, \mathfrak{g}]$ given by $\omega(x, y) = \chi([x, y])$. Here, $f$ is the one in the $\mathfrak{sl}_2$-triple $\{e, f, h\}$ chosen in Subsection 1.1. Let $V = V_0 \oplus V_1$ be the superspace $[f, \mathfrak{g}]$ if $\dim(\mathfrak{g}(-1))$ is
even. Otherwise, let $V \subset [f, g]$ be a superspace which has the standard basis $v_i$ such that $\omega(v_i, v_j) = \delta_{i,-j}$ for $i, j \in \{1, \ldots, \pm(\dim([f, g]) - 1)/2\}$. We choose such a $V$ in the present paper following the definition of $W$-superalgebras given in [31].

For a superspace $V$ with an even symplectic form, $A_h(V)$ denotes the corresponding Weyl superalgebra (see [28, Example 1.5] for the definition). Specially, if $V$ is pure odd, then the Weyl superalgebra $A_h(V)$ corresponds to the Clifford algebra $Cl_h(V)$.

It is known in [16, Subsection 2.3] that there is a $Q \times K^*$-equivariant isomorphism of associative algebras. Here, for a vector space $V$ with an even symplectic form, $\bar{W}^\lambda$ is defined as the commutator of $\tilde{\Phi}_h(V_0)$ in $\bar{U}^\lambda_h$, and $\bar{W}^\lambda$ is defined similarly.

(2) There are isomorphisms

$$\tilde{W}_h^\lambda|_{K\cdot l.f.} = \bar{W}_h^\lambda \quad \text{and} \quad (\tilde{W}_h^\lambda)|_{K\cdot l.f.} = \bar{W}_h^\lambda$$

of associative algebras. Here, for a vector space $V$ with a $K^*$-action, $(V)|_{K\cdot l.f.}$ denotes the sum of all the finite-dimensional $K^*$-stable subspace of $V$.

(3) There is an embedding $\mathfrak{q} := \text{Lie}(Q) \hookrightarrow \bar{W}$ of Lie algebras such that the adjoint action of $\mathfrak{q}$ coincides with the differential of the $Q$-action.

Proof. (1) Suppose that $V_0$ has a basis $\{v_i\}_{1 \leq i \leq l}$ with $\omega(v_i, v_j) = \delta_{i,-j}$. The isomorphism $\Phi_{0, h}$ gives us a $Q$-equivariant embedding $\tilde{\Phi}_h : V_0 \hookrightarrow \bar{U}^\lambda_h$ with $[\tilde{\Phi}_h(v_i), \tilde{\Phi}_h(v_j)] = \delta_{i,-j}h$. Now the isomorphism $\tilde{\Phi}_h$ can be constructed as in the proof of [28, Theorem 1.6]. For the construction of $\Phi_{1, h}$, see also Case 1 in the proof of [28, Theorem 1.6]. The isomorphism $\Phi_h$ can be constructed from the embedding $\Phi_h : V \hookrightarrow \bar{U}^\lambda_h$ given by $\Phi_h|_{V_0} = \tilde{\Phi}_h$ and $\Phi_h|_{V_1} = \Phi_{1, h}$.

(2) The second isomorphism was proved in [16]. The remaining statements follow by the similar arguments as in the proof of [28, Theorem 3.8].

(3) View $\bar{U}$ as a Harish-Chandra $\mathfrak{u}_0$-bimodule and use [16, Subsection 2.5].

Remark 2.2. (1) In Proposition 2.1 above, we do not claim that $\Phi_h$ is $Q$-equivariant, although this is probably true.

(2) Note that (2.1) can be interpreted as the Hamiltonian reduction of the adjoint action of $\mathfrak{m}$ (resp. $\mathfrak{m}_0$) on $\bar{U}$ (resp. $\bar{U}_0$). Similarly, $\bar{W}$ can be viewed as the Hamiltonian reduction of the $\mathfrak{m}_0$-action on $\bar{U}$, namely, $\bar{W} = ([\bar{U}/[\bar{U}(\omega)])^{ad(\mathfrak{m}_0)}$. Moreover, there exists an odd commuting Lie superalgebra $\mathfrak{n} \subset \bar{W}$ such that $\bar{W} = (\bar{W}/[\bar{W}(\mathfrak{n}])^{ad(\mathfrak{n})}$. Thus we may divide the reduction $([\bar{U}/[\bar{U}])^{ad(\mathfrak{m})}$ into two steps. The algebra $\bar{W}$ is obtained from the first one. Our setting can be viewed as an example of quantum super versions of the reduction by stages in the classical symplectic geometry (see [20]).

Proposition 2.1 gives us the following $Q \times K^*$-equivariant version of [28, Theorem 4.1].

1) Here and in Proposition 2.1, the tensor product is taken in the category of complete, super $K[[\hbar]]$-algebras. For simplicity of notation, the similar abbreviations are used frequently in the present paper. It is not hard to see their exact meaning from the context.
Theorem 2.3. (1) We have a \( Q \times \mathbb{K}^* \)-equivariant embedding \( \mathcal{W}_0 \hookrightarrow \tilde{\mathcal{W}} \) of associative algebras. The latter is generated over the former by \( \dim(g_1) \) odd elements.

(2) Moreover, we have an isomorphism

\[
\Phi_1 : \tilde{\mathcal{W}} \rightarrow \text{Cl}(V_1) \otimes_{\mathbb{K}} \mathcal{W}
\]

of associative algebras. Here, \( \text{Cl}(V_1) \) is the Clifford algebra on the vector space \( V_1 \) with the symmetric bilinear form \( \chi([\cdot, \cdot]) \).

**Proof.** The proof is similar to the proof of [28, Theorem 4.1] and hence is omitted. \( \square \)

Since it will be frequently used later, it is helpful to recall the construction of \( \Phi_1 \) in the following slightly general setting.

**Proposition 2.4.** For a two-sided ideal \( \mathcal{I} \) of \( \tilde{\mathcal{W}} \), we have \( \tilde{\mathcal{I}} = \text{Cl}(V_1) \otimes_{\mathbb{K}} \mathcal{I} \). Here, \( \mathcal{I} \) is the two-sided ideal of \( \mathcal{W} \) consisting of elements anti-commuting with \( \text{Cl}(V_1) \).

**Proof.** By Theorem 2.3(2), there exist \( x_1, \ldots, x_{\dim(V_1)} \in \tilde{\mathcal{W}} \) with

\[
x_i^2 = 1 \quad \text{and} \quad x_i x_j = -x_j x_i \quad \text{for all distinct} \ i, j \in \{1, \ldots, \dim(V_1)\}.
\]

By a quantum analog of [28, Lemma 2.2(2)], we have \( \tilde{\mathcal{I}} = \text{Cl}(\mathbb{K}(x_1)) \otimes_{\mathbb{K}} \tilde{\mathcal{I}}_1 \) as associative algebras. Here, \( \tilde{\mathcal{I}}_1 \) denotes the space anti-commuting with \( x_1 \). Now the proposition follows by induction on \( \dim(V_1) \). \( \square \)

### 2.1 Equivalence of \( \mathcal{W}\text{-Mod} \) and \( \tilde{\mathcal{W}}\text{-Mod} \)

Let \( u_1 \) be a Lagrangian of \( V_1 \) and \( u_1^* \) be its dual (given by the non-degenerate symmetric two form). Note that \( V_1 = u_1 \oplus u_1^* \). View the exterior algebra \( \bigwedge (u_1^*) \) as a \( \text{Cl}(V_1) \)-module by

\[
u \cdot x = ux \quad \text{and} \quad v \cdot x = \omega(v, x) \quad \text{for all} \ u, x \in u_1^* \quad \text{and} \ v \in u_1.
\]

The following proposition establishes an explicit relation between the categories \( \mathcal{W}\text{-Mod} \) and \( \tilde{\mathcal{W}}\text{-Mod} \). It relates to Proposition 2.4 via the bijective map \( \text{In}^{\text{fin}}(\tilde{\mathcal{W}}) \rightarrow \text{Prim}^{\text{fin}}(\tilde{\mathcal{W}}) \).

**Proposition 2.5.** For any \( M \in \tilde{\mathcal{W}}\text{-Mod} \), we have an isomorphism

\[
\bigwedge (u_1^*) \otimes_{\mathbb{K}} M' \rightarrow M : x \otimes m \mapsto x \cdot m
\]

of \( \tilde{\mathcal{W}} \)-modules. Here, \( M' \) is the annihilator of \( u_1 \), which is naturally a \( \mathcal{W} \)-module and we view \( \bigwedge (u_1^*) \otimes_{\mathbb{K}} M' \) as a \( \tilde{\mathcal{W}} \)-module by the isomorphism \( \Phi_1 \) in Theorem 2.3. The functor \( \tilde{\mathcal{W}}\text{-Mod} \rightarrow \mathcal{W}\text{-Mod} : M \mapsto M' \) is an equivalence of categories with the inverse \( N \mapsto \bigwedge (u_1^*) \otimes_{\mathbb{K}} N \).

The proof is very similar to the proof of Proposition 2.4 and [28, Lemma 2.2(2)].

**Proof of Proposition 2.5.** Let \( x_1, \ldots, x_{\dim(u_1)} \) be a basis of \( u_1 \) and \( x_1^*, \ldots, x_{\dim(u_1)}^* \) be the dual basis of \( u_1^* \) with \( \omega(x_i, x_j^*) = \delta_{i,j} \). We claim that there is an isomorphism

\[
\Psi_1 : \text{Cl}(\mathbb{C}(x_1, x_1^*)) \otimes_{\mathbb{K}} \mathcal{W}_1 \rightarrow \tilde{\mathcal{W}}
\]

of associative algebras. Here, \( \mathcal{W}_1 \) is the super-commutator of \( x_1 \) and \( x_1^* \) in \( \tilde{\mathcal{W}} \) and the isomorphism is given by the multiplication in \( \tilde{\mathcal{W}} \). For any \( y \in \tilde{\mathcal{W}} \), we have

\[
y = y - x_1[x_1^*, y] - x_1^*[x_1, y] + x_1(x_1^*, y) - x_1^*[x_1, [x_1, y]] + x_1 x_1^*[x_1, [x_1, y]] + x_1^*[x_1, y] - x_1[x_1^*, [x_1, y]] + x_1 x_1^*[x_1, [x_1, y]].
\]

Therefore, \( \Psi_1 \) is surjective. Suppose that

\[
w_0 + x_1 w_1 + x_1^* w_2 + x_1 x_1^* w_3 = 0
\]
for some $w_i \in \tilde{W}_1$ $(i = 0, 1, 2, 3)$. Applying the operator $[x_1, [x_1^*, \bullet]]$ on both sides, we have $w_3 = 0$. By the same token, we have $w_i = 0$ for $i = 0, 1, 2$. So $\Phi_1$ is also injective. Thus the claim follows. Now we prove the proposition for the pair $(\tilde{W}_1, \tilde{W})$, namely, there is an isomorphism
\[
\Psi_{1,M} : \bigwedge (x_1^*) \otimes_k M_1' \to M
\]
of $\tilde{W}$-modules. Here, the notation has a similar meaning as in Proposition 2.5. Indeed, for any $m \in M$, we have
\[m = m - x_1^* (x_1 \cdot m) + x_1^* x_1 \cdot m.\]
Since $x_1 \cdot m$ and $m - x_1^* (x_1 \cdot m) \in M_1'$, $\Psi_{1,M}$ is surjective. Similarly, we can check that $\Psi_{1,M}$ is injective. Now the first statement follows by repeating the above procedure $\dim(u_1)$ times. The second statement is a direct consequence of the first one. 

2.2 Maps $\bullet^\dagger$ and $\bullet_\dagger$

We recall the constructions of maps $\bullet^\dagger$ and $\bullet_\dagger$ between $\mathfrak{i}\mathfrak{o}(W)$ and $\mathfrak{i}\mathfrak{o}(U)$ in [28] at first. For $I \in \mathfrak{i}\mathfrak{o}(W)$, we denote by $R_h(I) \subset W_h$ the Rees algebra associated with $I$ and by $R_h(I) \cap W_h^\wedge$ the completion of $R_h(I)$ at 0. Let $A(I)^\wedge_h = A_h(V)^\wedge \otimes R_h(I) \cap W_h^\wedge$ and set $\tilde{I}^\dagger = (\mathcal{U}_0 \cap \Phi_h(A(I)^\wedge_h) / (h - 1)$ for an ideal $J \subset U$. Define $J_0$ to be the unique (by [28, Proposition 3.4(3)]) ideal in $W$ such that $R_{\bar{h}}(J_0) = \Phi_h^{-1}(J_h) \cap R_{\bar{h}}(W)$. A $\mathfrak{g}_0$-bimodule $M$ is said to be the Harish-Chandra (HC) bimodule, if $M$ is finitely generated and the adjoint action of $\mathfrak{g}$ on $M$ is locally finite. For any two-sided ideal $J \subset U$ (resp. $I \subset W$), $J_0$ (resp. $I_0$) denotes the image of $J$ under the functor $\bullet^\dagger$ (resp. $\bullet_\dagger$) in [16, Section 3]. Here, we view $J$ and $I$ as HC-bimodules over $\mathfrak{g}_0$ and $W_0$, respectively.

**Lemma 2.6.** We have $(\text{Cl}(V_1) \otimes_k I)^\dagger = I_0$ and $I_0 = \text{Cl}(V_1) \otimes_k I_1$.

**Proof.** The $K^*$-action (see the paragraph before [16, Lemma 3.3.3]) defining the HC $U_0$-bimodule $\tilde{I}^\dagger$ is given by $t \cdot x = t^{-2} x$ for all $x \in \mathfrak{g}$ and $t \in K^*$. So $U_0 \cap \Phi_h(A(I)^\wedge_h) \cap R_{\bar{h}}(W)$ coincides with the $K^*$-local finite part of $\Phi_h(A(I)^\wedge_h)$. Thus the lemma follows. For a similar fact in the even case, see [16, Remark 3.4.4].

2.3 Properties of $\bullet^\dagger$ and $\bullet_\dagger$

For an associative algebra $A$, $\text{GKdim}(A)$ denotes the Gelfand-Kirillov dimension of $A$ (see, for the definition, [12]). The associated variety $V(J)$ of a two-sided ideal $J \in \mathfrak{i}\mathfrak{o}(U)$ is defined to be the associated variety $V(J_0) = J_0 = J \cap U_0$. We say that $J$ is supported on $V(J)$ in this case.

**Lemma 2.7.** For any two-sided ideal of $J \subset U$, we have
\[
\text{GKdim}(U/J) = \text{GKdim}(U_0/J_0) = \dim(V(J)).
\]

**Proof.** Note that we have the natural embedding $U_0/J_0 \hookrightarrow U/J$. The first equality follows from the definition of the Gelfand-Kirillov dimension (see [12, p.14, Definition and Remark]) and the Poincaré-Birkhoff-Witt (PBW) basis theorem. The second equality follows from [1, Corollary 5.4].

The following proposition is a super generalization of [14, Theorem 1.2.2(vii)] in a special case.

**Proposition 2.8.** For any $J \in \text{Prim}(U)$, $\{I \in \mathfrak{i}\mathfrak{o}(W) \mid I$ is prime, $I^\dagger = J\}$ is exactly the set consisting of the minimal prime ideals containing $J_1$. 

**Proof.** Suppose that $I$ is a prime ideal of $W$ with $I^\dagger = J$. [28, Proposition 4.5] implies $J_1 \subset I$. So $I$ has finite codimension in $W$. Hence, $I$ is minimal by [1, Corollary 3.6]. Let $I \subset W$ be a minimal prime ideal with $J_1 \subset I$. According to [28, Proposition 4.6], $J_1$ has finite codimension in $I$. Thus we can see that $I = \text{Cl}(V_1) \otimes_k I$ has finite codimension in $W$. Hence, $I_0 = W_0 \cap I$ has finite codimension in $W_0$. Since $I^\dagger \cap U_0 = (I_0)^\dagger$, we obtain that $I^\dagger$ is supported on $G_0 \cdot \chi$ by the proof of [14, Theorem 1.2.2(vii)]. Thus Lemma 2.7 in conjunction with [1, Corollary 3.6] yields $I^\dagger = J$. 

\(\square\)
Let $\sigma$ be the automorphism of the superalgebra $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ given by $\sigma(x) = x_0 - x_1$ for any $x = x_0 + x_1$ in $\mathcal{A}$. An ideal of $\mathcal{A}$ is $\mathbb{Z}_2$-graded if and only if it is invariant under $\sigma$. We have the following relation between primitive and graded primitive ideals of $\mathcal{A}$.

**Lemma 2.9** (See [24, Lemma 7.6.3]). For any graded primitive ideal $\mathcal{I}'$ of $\mathcal{A}$, there exists a primitive ideal $\mathcal{I} \subset \mathcal{A}$ such that $\mathcal{I}' = \mathcal{I} \cap \sigma(\mathcal{I})$.

### 2.4 Proof of the main result Theorem 1.1

We prove the theorem by a similar argument as in the proof of [16, Conjecture 1.2.1]. Indeed, given $J \in \text{Prim} \mathcal{A}$, let $\mathcal{I}_1, \ldots, \mathcal{I}_l$ be the minimal prime ideals containing $J$. Since $\text{Cl}(V_i) \otimes \mathbb{K} I_1$ is stable under $Q^e$, $\bigcap_{i \in C_e} \gamma(\text{Cl}(V_i) \otimes \mathbb{K} I_1)$ is $Q$-stable. Set $J^1 = \bigcap_{i \in C_e} \gamma(\text{Cl}(V_i) \otimes \mathbb{K} I_1)$, and then by [16, Theorem 4.1.1], we have $(J^1)_{\gamma(1)} = \bigcap_{i \in C_e} \gamma(\text{Cl}(V_i) \otimes \mathbb{K} I_1)$. Thus $J = (I_1)^{\gamma(1)} \supset J^1 \supset J$ (the first equality follows from Lemma 2.7 and [1, Corollary 3.6]). Hence, $J = \bigcap_{i \in C_e} \mathcal{I}(\gamma(1))$ by [10, Proposition 3.1.10] and Corollary 2.4. Thus we have $\mathcal{I} = \bigcap_{i \in C_e} \mathcal{I}(\gamma(1))$ by [10, Proposition 3.1.10] and Lemma 2.6. Now the proof is completed by Proposition 2.8.

In the $\mathbb{Z}_2$-graded case, the automorphism given by $g \in Q$ commutes with $\sigma$. Thus the second statement follows from the first one and Lemma 2.9.

### 2.5 Finite-dimensional representations of $\bar{\mathcal{W}}$

Now we point out the role of Theorem 1.1 in describing $\text{Irr}^{\text{fin}}(\bar{\mathcal{W}})$. We mentioned earlier that the map

$$\text{Irr}^{\text{fin}}(\bar{\mathcal{W}}) \rightarrow \text{Prim}^{\text{fin}}(\bar{\mathcal{W}}) : M \mapsto \text{Ann}(M)$$

is bijective. Given $\mathcal{I} \in \text{Prim}^{\text{fin}}(\bar{\mathcal{W}})$, $\bar{\mathcal{W}}/\mathcal{I}$ is isomorphic to $\text{End}(M)$ for some finite-dimensional vector space $M$ over $\mathbb{K}$ by a well-known fact for general finite-dimensional simple algebras. The inverse of the above map is given by $\mathcal{I} \mapsto M$. By Lemma 2.9, there is a similar bijection in the $\mathbb{Z}_2$-graded case.

Now let $M \in \text{Irr}^{\text{fin}}(\bar{\mathcal{W}})$ and $\mathcal{I} = \text{Ann}(M)$. If $g \in C_e = Q/Q^e$ and $g' \in Q$ is a representative of $g$, $^gM$ denotes the twist of $M$ by the algebra automorphism $g'$ of $\bar{\mathcal{W}}$. Obviously, the annihilator of $^gM$ is $g' \cdot \mathcal{I}$. Thus Theorem 1.1 is equivalent to saying that $\{^gM \mid g \in C_e\}$ equals the set of modules in $\text{Irr}^{\text{fin}}(\bar{\mathcal{W}})$ which are annihilated by $(\mathcal{I})_{i_{\gamma}}^1$.

### 2.6 The special case: $C_e = 1$

For a basic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ of type I, Letzter [13] established a bijection $\nu : \text{Prim}(\mathcal{U}) \rightarrow \text{Prim}(\mathcal{U})$. It follows from the construction that $\nu$ restricts to a bijection between $\text{Prim} \mathcal{U}$ and $\text{Prim} \mathcal{U}$. So we can describe $\text{Irr}^{\text{fin}}(\bar{\mathcal{W}})$ when $C_e$ is trivial. We know that the finite group $C_e$ is trivial when $\mathfrak{g}$ is of type $A(m \mid n)$ or $e$ is a principal nilpotent element in the type $C(n)$ Lie superalgebras. In the case of $\mathfrak{g} = \mathfrak{osp}(1, 2n)$, a description of $\text{Prim}(\mathcal{U})$ is given in [23, Theorems A and B]. The poset structure describing $\text{Prim}(\mathcal{U})$ is exactly the same as that of $\text{Prim}(\mathcal{U})$. It is straightforward to check that $\hat{L}(\lambda)$ is supported on $\hat{\mathcal{O}}$ if and only if so is $L(\lambda)$. Thus we show that Theorem 1.1 gives a description of $\text{Irr}^{\text{fin}}(\bar{\mathcal{W}})$ provided $C_e = 1$.

### 3 Graded irreducible representations

From now on, let $\mathfrak{g}$ be a basic Lie superalgebra of type I. The most essential feature is that they admit a $\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

of Lie superalgebras. Here, the term $\mathbb{Z}_2$-compatible means $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1 = \mathfrak{g}_1$ and $\mathfrak{g}_0 = \mathfrak{g}_0$. For a $\mathfrak{g}_0$-module $V$, view it as a $\mathfrak{g}_0 + \mathfrak{g}_1$-module with the trivial $\mathfrak{g}_1$ action and define $K(V) = \text{ind}^{\mathfrak{g}_0+\mathfrak{g}_1}_{\mathfrak{g}_0} V$. We refer to $K(\bullet)$ as the Kac functor from the category of $\mathfrak{g}_0$-modules to that of $\mathfrak{g}$-supermodules. The main result of [7] states...
that for any simple $\mathfrak{g}_0$-supermodule $V$, the Kac module $K(V)$ has a unique simple $\mathbb{Z}_2$-graded quotient $\hat{V}$, and the map $V \mapsto \hat{V}$ induces a bijection between the set of isomorphism classes of simple $\mathfrak{g}_0$-modules and of simple $\mathfrak{g}$-supermodules. It is well known that the above map sends the highest weight simple $U_0$-module $L(\lambda)$ to the highest weight simple $U$-module $\hat{L}(\lambda)$. Now we can give a classification of simple $\hat{W}$-supermodules (hence of $W$-supermodules) via the Kac equivalence and Skryabin’s equivalence.

A $\mathfrak{g}$-supermodule $M$ is called Whittaker (see [26]) if $m_{0,\chi}$ acts on it as a locally nilpotent endomorphism. It is easy to check that $\WrhatM := M_{m_{0,\chi}}$ is a $\hat{W}$-supermodule. Let $\hat{Q}_X$ be the left $U$-module $U/Um_{0,\chi}$. It also has a right $\hat{W}$-supermodule structure. For any $\hat{W}$-supermodule $N$, $\hat{Q}_X \otimes \hat{W} N$ is a left $U$-supermodule. Let $Q_{0,X} = U_0U_0m_{0,\chi}$ be the $(U_0, W_0)$-bimodule defined similarly. We have the following Skryabin’s equivalence for $\hat{W}$.

**Theorem 3.1.** The functor $\Wrhat$ and $Q_X \otimes_{W_0} \cdot$ are mutual quasi-equivalences between the categories of $\hat{W}$-supermodules and of Whittaker $U$-supermodules. For any $\hat{W}$-supermodule $N$, $Q_{0,X} \otimes_{W_0} N$ also has a $U$-supermodule structure, which is isomorphic to $\hat{Q}_X \otimes \hat{W} N$.

The second statement is very useful in our study of representations of $\hat{W}$. It enables us to use results on $W_0$. We may prove the theorem by a similar argument in the $W$-algebra cases (see [16] or [28] in $W$-superalgebra cases). Here, we provide a sketch to prove it.

**Proof of Theorem 3.1.** Let $A_{V_0}(\hat{W}) = A(V_0) \otimes_k W$. We claim that there is an isomorphism

$$U_{m_{0,\chi}}^\wedge \rightarrow (A(\hat{W}))_{m_{0,\chi}}$$

of topological algebras, where $U_{m_{0,\chi}}$ (resp. $A_{V_0}(\hat{W})_{m_{0,\chi}}$) is the completion of $U$ (resp. $A_{V_0}(\hat{W})$) with respect to the nilpotent Lie subalgebra $m_{0,\chi} \subset U$ (resp. commutative subalgebra $m_{0,\chi}$). This is an analog of [14, Theorem 1.2.1] for $W_0$, which states that $(U_0)_{m_{0,\chi}}$ is isomorphic to $(A(V_0) \otimes_k W_0)^{\wedge}_{m_{0,\chi}}$ as topological algebras. Our claim can be proved by the similar arguments therein.

View $Q_{0,X}$ as an $A(V_0) \otimes_k W_0$-module via the above second isomorphism, and then we have $Q_{0,X} = \mathbb{K}[m_{0,\chi}] \otimes_{W_0} A(V_0) \otimes_{W_0} W_0$-bimodules (see [14, p. 52]). Similarly, we have $\hat{Q}_X = \mathbb{K}[m_{0,\chi}] \otimes_k \hat{W}$ as $(A(V_0)(\hat{W}))$-bimodules. Therefore,

$$Q_{0,X} \otimes_{W_0} N = (\mathbb{K}[m_{0,\chi}] \otimes_k W_0) \otimes_{W_0} N = \mathbb{K}[m_{0,\chi}] \otimes_k N$$

has an $A_{V_0}(W)$-supermodule structure. Hence, it is a Whittaker $U$-supermodule via the homomorphism $U \hookrightarrow U_{m_{0,\chi}} \rightarrow A(V_0)(\hat{W})_{m_{0,\chi}}$. Repeating the argument of [28, Theorem 4.1], we see that the theorem follows.

**Theorem 3.2.** The sets $\text{Irr}(W_0)$, $\text{gr.Irr}(\hat{W})$ and $\text{gr.Irr}(W)$ are bijective with each other. Any simple $\hat{W}$-supermodule or equivalently simple $W$-supermodule is $\mathbb{Z}$-graded.

**Proof.** Obviously, the Kac functor maps the Whittaker $\mathfrak{g}_0$-modules to the Whittaker $\mathfrak{g}$-supermodules. According to Theorem 3.1, we have that the map $N \mapsto \hat{N} := \WrhatM \otimes_{W_0} N$ is a bijection between $\text{Irr}(W_0)$ and $\text{gr.Irr}(\hat{W})$. Since $m_{0,\chi} \subset U$ is $\mathbb{Z}$-homogeneous, the second statement follows from the fact that any simple $\mathfrak{g}$-supermodule is $\mathbb{Z}$-gradable (see the proof of [7, Theorem 4.1]).

## 4 Character formula

### 4.1 Triangular decomposition for $\hat{W}$

Let $U_+$ (resp. $U_-$) be the universal enveloping algebra of $\mathfrak{g}_0 + \mathfrak{g}_1$ (resp. $\mathfrak{g}_0 + \mathfrak{g}_{-1}$). Define their completion $(U_+)^\wedge$ and $(U_-)^\wedge$ similarly to $U_0^\wedge$. The restrictions of $\Phi_\wedge$ to $(U_+)^\wedge$ and $(U_-)^\wedge$ give the following isomorphisms:

$$\Phi_\wedge^+ : A^\wedge(V_0) \otimes_W (U_-)^\wedge \rightarrow (U_+)^\wedge$$

and

$$\Phi_\wedge^- : A^\wedge(V_0) \otimes_W (U_+)^\wedge \rightarrow (U_-)^\wedge$$

of associative algebras. Here, $W_-^\wedge$ and $W_+^\wedge$ are defined similarly to $W^\wedge$ in Proposition 2.1. Define

$$\hat{W}_- := (\hat{W}_-^\wedge)_{\mathbb{Z}^* - I, f}/(h - 1)$$

and

$$\hat{W}_+ := (\hat{W}_+^\wedge)_{\mathbb{Z}^* - I, f}/(h - 1).$$
They can be viewed as the $W$-superalgebras from $(g_{-1} + g_0, e)$ and $(g_0 + g_1, e)$.

Equip $U_h$ a $Z$-grading such that the subspace $U$ has the natural grading from $g$ and $h$ has the grading 0. The isomorphism $\Phi_h$ preserves the $Z$-grading by construction. Hence, there is a $Z$-grading $W = \bigoplus_{z \in \mathbb{Z}} W_z$ inherited from the one on $U$, and $\hat{W}_-$ and $\hat{W}_+$ are $Z$-graded subalgebras of $\hat{W}$.

**Proposition 4.1.** (1) There exist $Z$-homogeneous odd elements $x_1, \ldots, x_k \in \hat{W}_-$, $x_1^\perp, \ldots, x_k^\perp \in \hat{W}_+$ and $x_1, \ldots, x_l \in W_0$ such that they form a PBW basis of $\hat{W}$ in the super sense, where $k = \dim(g_{-1}) - \dim(g_0) = \dim(g_0)$ and $l = \dim(g_0)$. We emphasize that $W_0$ is the ordinary finite $W$-algebra from $(0_0, e)$.

(2) Let $\hat{W}^\#_e$ (resp. $\hat{W}^\#_l$) be the vector space of the exterior algebra generated by $x_1, \ldots, x_k$ (resp. $x_1^\perp, \ldots, x_k^\perp$). There are isomorphisms of vector spaces

\[
\hat{W} \cong \hat{W}^\#_e \otimes_K W_0 \otimes_K \hat{W}^\#_l, \quad \hat{W}_+ \cong W_0 \otimes_K \hat{W}^\#_+, \quad \hat{W}_- \cong W_0 \otimes_K \hat{W}^\#_-
\]

given by the multiplication of $\hat{W}$.

(3) For any irreducible $W_0$-module $N$, view it as a $\hat{W}_+$-module via the quotient $\hat{W}_+ \rightarrow W_0$ modulo the two-sided ideal generated by elements with positive $Z$-grading (or by the image of $\hat{W}^\#$ equivalently). Then the Verma module $\Delta^K_{\hat{W}}(N) := \hat{W} \otimes \hat{W}_+^{-1} N$ has a unique simple quotient $L^K_{\hat{W}}(N)$. The map

\[\text{Irr}^\text{fin}(W_0) \rightarrow \text{gr.Irr}^\text{fin}(\hat{W}) : N \mapsto L^K_{\hat{W}}(N)\]

is bijective and $C_e$-equivariant.

**Proof.** Statement (1) follows from a similar argument to the proof of existence of the PBW basis for $W_0$ in [14] or for $W$ in [28]. Let $\tilde{S}_e = (g_0)_{\mathfrak{e}} \oplus g_1$ and choose odd elements $x_1^\perp, \ldots, x_k^\perp \in g_{-1}$, $x_1^\perp, \ldots, x_l \in g_0$ such that they form a basis of the vector space $\tilde{S}_e$. The procedure in [28, (2.3)] shows that $(x_1^\perp)_h := x_1^\perp + h$ is in $\Phi_h(\hat{W}^\#_l)$. Here, $h$ denotes the higher order correcting term obtained in there. We can construct $(x_i^\perp)_h$ for $i = 2, \ldots, k$ and $(x_i)_h$ for $i = 1, \ldots, l$ similarly. Since $\hat{W}^\#_l / (h) \cong S[[\tilde{S}_e]]$, these elements generate $\hat{W}^\#_l$ as a $K[[h]]$-algebra. They also lie in $(\hat{W}^\#_l)_{K^{C_e}}$, since they are $K^{C_e}$-homogeneous. We can take the PBW basis as their image under the quotient map $(\hat{W}^\#_l)_{K^{C_e}} \rightarrow \hat{W}$ given by specializing $h$ to 1.

Claim (2) follows directly from (1).

Let $M$ be a $Z_2$-graded simple quotient of $\Delta^K_{\hat{W}}(N)$ and $\pi$ be the quotient homomorphism. By Theorem 3.2, we may assume that $M$ has a $Z$-grading with top degree 0. We claim that $\pi$ has to be a $Z$-graded homomorphism. Otherwise, for a non-zero $x \in N$, we may write $\pi(x) = \sum_{i=1}^n y_i$ for $Z$-homogeneous $y_i \in M$ and $i = 1, 2, \ldots, n > 1$. Suppose $\text{gr}(y_i) = d < 0$. Since $\hat{W}^\#_l \cdot y_l$ has top degree $d < 0$, it is a proper super-submodule of the simple supermodule $M$, which leads to a contradiction. Thus we have that any maximal super-submodule of $\Delta^K_{\hat{W}}(N)$ is a $Z$-graded submodule. Consequently, the sum of all the proper maximal super-submodules of $\Delta^K_{\hat{W}}(N)$ is the unique proper maximal sub-superscript module. For any $g \in C_e$, it is clear that $L^K_{\hat{W}}(N) = L^K_{\hat{W}}(gN)$. The claim (3) follows.

The following corollary combined with the main result of [17] gives us a complete classification of $\text{gr. Prim}(\hat{W})$ in the type $A$ case.

**Corollary 4.2.** For a basic type $I$ Lie superalgebra $g$, the sets $\text{Prim}(W_0)$, $\text{gr. Prim}(\hat{W})$ and $\text{gr. Prim}(W)$ are bijective with each other.

**Proof.** We decent Letzter’s bijection $\nu : \text{Prim}(U_0) \rightarrow \text{Prim}(U)$ to

\[\nu_{\hat{W}} : \text{Prim}(W_0) \rightarrow \text{gr. Prim}(\hat{W}).\]

For any $I \in \text{gr. Prim}(\hat{W})$, let $\hat{I}$ be the preimage of $I$ under the quotient $\hat{W}_+ \rightarrow W_0$. We claim that there is a unique primitive ideal of $\hat{W}$ lying over $\hat{I}$. We define $\nu_{\hat{W}}(I)$ to be this primitive ideal. The claim, and hence the corollary, can be proved by repeating the proof of Letzter’s theorem [24, Theorem 15.2.5] almost word by word. In fact, we only need to replace $R$ and $Q$ therein by $\hat{W}_+$ and $\hat{W}$, respectively. Proposition 4.1 is used to verify the conditions of [24, Lemma 7.6.12].

\[\square\]
4.2 Recalling the generalized Soergel functor $V$ in the even theory

The present subsection is devoted to recalling the results on the category $O$ and the generalized Soergel functor $V$ in [15, 18]. Choose a Levi subalgebra $(g_0)_0 \subset g_0$, an sl$_2$-triple $(e, f, h) \subset (g_0)_0$ and an integral element $\theta \in \mathfrak{z}((g_0)_0)$ as in [18, Subsection 2.6.1]. Recall that we have used the grading $g_0 = \bigoplus_{i \in \mathbb{Z}} g_0(i)$ with respect to $\text{ad}(h)$ to define the $W$-algebra $W_0$. We also need the grading $g_0 = \bigoplus_{i \in \mathbb{Z}} g_0(i)$ with respect to $\text{ad}(\theta)$, where $(g_0)_0$ is exactly the Levi subalgebra introduced previously. Let $p$ be the parabolic subalgebra $p = (g_0)_0 (\geq 0) + (g_0)_0 \cdot 0$. Here, $(g_0)_0 (\geq 0)$ (resp. $(g_0)_0 \cdot 0$) stands for the subalgebra of $(g_0)_0$ (resp. $g_0$) generated by elements with non-negative (resp. positive) grading from $\text{ad}(h)$ (resp. $\text{ad}(\theta)$).

Let $P$ be the corresponding parabolic subgroup and $O^P_\theta$ be the parabolic category $O$ generated by finitely generated $(P, \nu)$-equivariant modules, where $\nu$ is an $R$-equivariant embedding of the torus. For the reader's convenience, we give a brief explanation in [18, Subsection 6.3.1], which are given in [18, Subsection 6.3.1]. This Whittaker category is similar to the one considered in Theorem 3.1. It is described in [18, Subsection 4.1.2].

Let $u := p \cap [f, g_0]$, which is a Lagrangian subspace of $V_0$. Choose an $R \times K^*$-equivariant embedding $\iota : V_0 \hookrightarrow U_{\nu, h}$ as in [18, Subsection 4.1.2]. We have an isomorphism

$$\Phi_{0, h} : A_h^0(V_0) \otimes W_{0, h}^\wedge \rightarrow U_{\nu, h}$$

of quantum algebras from $\iota$ and $(W_{0, h})_{K^* - l.f}/(h-1) = W_0$.

The generalized Soergel functor $V : O_\theta^P \rightarrow O_\theta(g_0, e)^P_\theta$ is defined by three different but equivalent ways in [18]. We recall the first one. For $M \in O_\theta^P$, let $M_h^\wedge$ denote the completion of the Rees module $M_h$ with respect to the inverse image of the maximal ideal of $\mathfrak{z}$ under the homomorphism $(U_h) \rightarrow S[g_0]$ given by $h = 0$. Let $M'_h \subset M_h^\wedge$ be the annihilator of $\Phi_{0, h}(u)$. Then $M'_h$ is $\Phi_{0, h}(W_{0, h})$-stable, because $\Phi_{0, h}(W_{0, h})$ commutes with $\Phi_{0, h}(A_h^0(V_0)) \supset \Phi_{0, h}(u)$. The generalized Soergel functor $V$ is defined as follows:

$$V(M) := (M'_h)_{K^* - l.f}/(h-1).$$

There is a rational action of $R$ on $V(M)$ by the construction. For the simple module $L(\lambda) \in O_\theta^P$, we have

$$V(L(\lambda)) = \bigoplus_{i \in I_\lambda} L_{W_0}^\wedge(N^0_i).$$

(4.3)

Here, $L_{00}(\lambda)$ stands for the finite-dimensional $(g_0)_0$-module with the highest weight $\lambda$ and $N^0_i \subset L_{00}(\lambda)$ run over the finite-dimensional simple modules of $(W_0)^0$ lying over $J_0(\lambda) = \text{Ann}(L_{00}(\lambda))$. For simplicity, we write $L_{W_0}^\wedge(N^0_i)$ instead of $N_i$ ($i \in I_\lambda$).

4.3 Description of $V(\check{L}(\lambda))$ for $\lambda \in A_p$

Denote by $O^P_\nu(U)$ the category of $g$-supermodules lying in parabolic category $O^P_\nu$ for $g_0$. Similarly, let $O_\nu(g_0, e)^P_\theta(W)$ be the category of $W$-modules lying in $O_\nu(g_0, e)^P_\theta$.

The forthcoming Lemma 4.3 and Theorem 4.4 follow from [18, Subsection 6.3.1], which are given in the more general setting of Dixmier algebras. For the reader’s convenience, we give a brief explanation in our special case.

Let $\text{Wh}(g_0, e)^P_\theta$ be the category of $R$-equivariant generalized Whittaker modules defined in [18, Subsection 3.2.3]. This Whittaker category is similar to the one considered in Theorem 3.1. It is.
defined by a nilpotent Lie subalgebra of $\mathfrak{g}_0$ different from $\mathfrak{m}_0$. Let $\text{Wh}(\mathfrak{g}_0,e)_ω^R(\mathcal{U})$ stand for the category of $\mathfrak{g}$-supermodules lying in $\text{Wh}(\mathfrak{g}_0,e)_ω^R$. There is a generalized Skryabin’s equivalence $\mathcal{K} : \text{Wh}(\mathfrak{g}_0,e)_ω^R \rightarrow \text{O}_θ(\mathfrak{g}_0,e)_ω^R$ with inverse $\mathcal{K}^{-1}$ (see [18, Section 4] for the definition). It is easy to know that $\mathcal{K}$ sends $\mathcal{U}$-supermodules in $\text{Wh}(\mathfrak{g}_0,e)_ω^R$ to $\mathcal{W}$-supermodules in $\text{O}_θ(\mathfrak{g}_0,e)_ω^R$. The following lemma is an analog of Theorem 3.1 and can be proved similarly.

**Lemma 4.3.** There is a natural functor from $\text{O}_θ(\mathfrak{g}_0,e)_ω^R(\hat{\mathcal{W}})$ to $\text{Wh}(\mathfrak{g}_0,e)_ω^R(\mathcal{U})$ induced by $\mathcal{K}^{-1}$.

The following result is crucial to describe the image of simple objects in $\text{O}_θ^R(\mathcal{U})$ under $\mathcal{V}$.

**Theorem 4.4.** The functor $\mathcal{V} : \text{O}_θ^R \rightarrow \text{O}_θ(\mathfrak{g}_0,e)_ω^R$ sends simple $\mathcal{U}$-supermodules to simple objects in $\text{O}_θ^R(\mathcal{U})$.

**Proof.** By construction, we see that $\mathcal{V}$ restricts to a functor from $\text{O}_θ^R(\mathcal{U})$ to $\text{O}_θ(\mathfrak{g}_0,e)_ω^R(\hat{\mathcal{W}})$. Let $\mathcal{V}^* : (\mathfrak{g}_0,e)_ω^R \rightarrow \text{O}_θ^R$ be the right adjoint functor of $\mathcal{V}$ defined in [18, Proposition 4.4]. The construction (precisely the last paragraph of [18, p. 898]) of $\mathcal{V}^*$ conjunction with Lemma 4.3 implies that $\mathcal{V}^*$ sends $\mathcal{W}$-supermodules to $\mathcal{U}$-supermodules. Furthermore, $\mathcal{V}^*$ is restricted to a functor $\text{O}_θ(\mathfrak{g}_0,e)_ω^R(\hat{\mathcal{W}}) \rightarrow \text{O}_θ^R(\mathcal{U})$, which is the right adjoint functor to the restriction of $\mathcal{V}$. □

**Theorem 4.5.** For $\lambda \in \Lambda_p$, recall that $N_i \ (i \in I_\lambda)$ stand for the simple $\mathcal{W}_0$-modules appearing in (4.3). Then we have

$$\mathcal{V}(\hat{L}(\lambda)) = \bigoplus_{i \in I_\lambda} L^K_{\mathcal{W}_i}(N_i).$$

**Proof.** Since $L(\lambda) \subset \hat{L}(\lambda)$, it follows that

$$\bigoplus_{i} N_i \subset \mathcal{V}(\hat{L}(\lambda)).$$

Note that the action of $\mathcal{W}^#$ on $N_i$ for $i \in I_\lambda$ is trivial. Now the theorem follows from Proposition 4.1(3) and Theorem 4.4. □

The following result implies that the action of $C_e$ on $\text{gr.Ir}_{\lambda}^{\text{fin}}(\hat{\mathcal{W}})$ is transitive.

**Corollary 4.6.** For $\lambda \in \Lambda_p$, the map $\text{Ir}_{\lambda}^{\text{fin}}(W_0) \rightarrow \text{gr.Ir}_{\lambda}^{\text{fin}}(\hat{\mathcal{W}}) : N \mapsto L^K_{\mathcal{W}_i}(N)$ is bijective and $C_e$-equivariant.

**Proof.** The main result of [19] states that

$$\text{Ir}_{\lambda}^{\text{fin}}(W_0) = \{gN \mid g \in C_e, N = N_i \text{ for some } i \in I_\lambda\}.$$ 

Theorem 4.5 implies that $L^K_{\mathcal{W}_i}(N) \in \text{gr.Ir}_{\lambda}^{\text{fin}}(\hat{\mathcal{W}})$. Hence, we have $\text{Ir}_{\lambda}^{\text{fin}}(\hat{\mathcal{W}}) = \{gL^K_{\mathcal{W}_i}(N) \mid g \in C_e\}$ by Subsection 2.5 and Proposition 4.1(3). □

### 4.4 On $\mathfrak{g}_0$-rough structure of $\mathfrak{g}$-supermodules

To compute the characters of $\mathcal{W}$-supermodules, we need the expression

$$\hat{L}(\lambda) = \sum_{i \in S_\lambda} c_{i\lambda} \Delta_\mathcal{P}(\lambda_i) \quad (4.4)$$

in the Grothendieck group $K(\text{O}_θ^R)$ of the equivariant parabolic category $\text{O}_θ^R$ for $\mathfrak{g}_0$. Here, $\Delta_\mathcal{P}(\lambda_i)$ stands for the Verma module in $\text{O}_θ^R$ with the highest weight $\lambda_i$. The coefficients $c_{i\lambda}$ can be obtained from the $\mathfrak{g}_0$-rough structure of simple $\mathfrak{g}$-modules by the following two ways.

We may view $\hat{L}(\lambda)$ as a $\mathfrak{g}_0$-module and assume that

$$\hat{L}(\lambda) = \sum d_{\lambda_i} L(\mu_i)$$

in $K(\text{O}_θ^R)$. Here, the coefficients $d_{\lambda_i}$ are the multiplicities of $L(\mu_i)$ in $\hat{L}(\lambda)$. However in general, it is still open to determine $d_{\lambda_i}$. It can be computed by the Kazhdan–Lusztig theory of Lie algebras when
\(g = \mathfrak{g}(m | n)\) and \(\lambda\) is typical (see [7]). For the recent progress on the rough structures for type I Lie superalgebras and their applications, see also [4–6]. Thus, we can determine the coefficients \(c_{i\lambda}\) by the Kazhdan-Lusztig theory of \(O_{\nu}^P\).

The coefficients \(c_{i\lambda}\) may also be determined by the super parabolic Kazhdan-Lusztig theory which is still open in general presently. Let \(\hat{\mu}\) be the parabolic sub-superalgebra \(p + \mathfrak{g}_1\) (\(\geq 0\)), where \(\mathfrak{g}_1\) (\(\geq 0\)) is defined by the similar way as above. Suppose that

\[
\hat{L}(\lambda) = \sum_{\mu \in S_{\lambda}} c_{i\lambda}(\hat{\mu})(\lambda)
\]

in the Grothendieck group of the super parabolic category \(O_{\hat{\mu}}^P\) for \(\hat{\mu}\), where \(\hat{\Delta}(\lambda)\) is the parabolic Verma module in \(O_{\hat{\mu}}^P\). A filtration of Verma modules of Lie superalgebras by that of Lie algebras was given in [22, Theorem 3.2]. Generalizing this result to the parabolic case, we may find the coefficients \(c_{i\lambda}\) in (4.4).

4.5 Algorithm for character formulas

Now we present our algorithm to compute the characters of modules in \(\text{Irr}_{\lambda}^{\text{fin}}(W)\) for \(\lambda \in \Lambda_P\). It was obtained in [18, Theorem 4.8(iv)] that

\[
\text{Ch}(\mathcal{V}(\Delta_{\hat{P}}(\mu))) = \dim(L_{\mu 0}(\mu)) e^{\mu_0 - \rho} \prod_{i=1}^{k} (1 - e^{\mu_i})^{-1} \tag{4.5}
\]

Here, \((i = 1, 2, \ldots, k)\) are the weights of \(t\) in \((\mathfrak{g}_0)_{<0} \cap \mathfrak{z}_{\mathfrak{g}}(e)\), and \(\rho\) is the half of sum of all the positive roots of \(\mathfrak{g}_0\). Applying \(\mathcal{V}\) to both sides of (4.4) and by [18, Theorem 4.8], we have

\[
\text{Ch}(\mathcal{V}(\hat{L}(\lambda))) = \sum_{\mu \in S_{\lambda}} c_{i\lambda}(\dim(L_{\mu 0}(\lambda_i)) e^{\lambda_i - \rho} \prod_{i=1}^{k} (1 - e^{\mu_i})^{-1} \tag{4.6}
\]

Thanks to Theorem 4.5, we have that \(\mathcal{V}(\hat{L}(\lambda))\) is the direct sum of \(|I_{\lambda}|\) simple \(W\)-supermodules. These supermodules are transitive under the twist action of \(Q_0/Q_0^P\), where \(Q_0\) is the centralizer of \(\mathfrak{sl}_2\)-triple \((e, h, f)\) in \((G_0)_{0}\). Note that we consider the character with respect to the torus \(t = 3((\mathfrak{g}_0)_{0})\). Therefore, they have the same characters. Thus

\[
\text{Ch}(L^K_W(N_i)) = \dim(L_{\mu 0}(\lambda_i)) e^{\lambda_i - \rho} \prod_{i=1}^{k} (1 - e^{\mu_i})^{-1} \tag{4.7}
\]

Now by Subsection 2.5 and Corollary 4.6, we obtain a character formula for all \(M \in \text{gr.Irr}_{\lambda}^{\text{fin}}(W)\). Note that by definition there is an embedding \(t \hookrightarrow W \hookrightarrow \hat{W}\). Proposition 2.5 now yields

\[
\text{Ch}(L^K_{\mathcal{V}}(N_i)) = \dim(L^K_{\mathcal{V}}(N_i)) \prod_{i=1}^{l} (1 + e^{\mu_i})^{-1}.
\]

Here, \((L^K_W(N_i))'\) is the simple \(W\)-supermodule obtained from \(L^K_W(N_i)\) (see Proposition 2.5) and \(\mu_i\) (\(i = 1, 2, \ldots, l\)) are the weights of the Lagrangian \(u_i^0\).

By Proposition 4.1 and [18], in order to compute the characters of modules in \(\text{gr.Irr}_{\lambda}^{\text{fin}}(W)\), we only need to determine the coefficients \(c_{i\lambda}\) in (4.4). This is a fundamental problem in the representation theory of Lie superalgebras.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11801113) and Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center Located in Kyoto University. This work was motivated by communications with Tomoyuki Arakawa and a part of it was written during the author’s visit to Tomoyuki Arakawa at Research Institute for Mathematical Sciences. The author is indebted much to him for fruitful and helpful discussion. He thanks Hao Chang for the tremendous help in improving the language. He also thanks for the helpful communications from Bin Shu and comments from Yang Zeng. Finally, the author thanks the referees for their numerous helpful comments.
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