PROOF OF SERBAN'S CONJECTURE

M.C.Bergère*

CEA-Saclay, Service de Physique Théorique
F-91191 Gif sur Yvette Cedex, FRANCE

ABSTRACT

We prove Serban's conjecture which simplifies greatly the expression of the advanced single-particle Green function in the Calogero-Sutherland model. The importance of proving this conjecture is that it reorganizes the form factor in terms of two dimensional Coulomb gas correlators and confirms the possible existence of a bosonization procedure for this system.

Submitted to Journal of Mathematical Physics.
*Membre du C.N.R.S.
1 Introduction.

The Calogero-Sutherland model Ref.1, attracted much of attention during the last few years, due to its connection to random matrix theory, fractional statistics in one dimension, conformal field theory, etc... This model is exactly solvable and its solution is known to a large extent. The corresponding Hamiltonian

\[ H = -\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} + \beta (\beta - 1) \frac{\pi^2}{L^2} \sum_{i<j} \frac{1}{\sin^2 \left( \frac{\pi}{L} (x_i - x_j) \right)} \]  

(1.1)

describes the motion of \( N \) particles along a circle of length \( L \). The eigenstates of this Hamiltonian can be expressed Ref.2, in terms of the so-called Jack polynomials in the variables \( z_i = \exp(2i\pi x_i/L) \). For \( \beta = 0 \) or 1, the particles behave like free spinless bosons or fermions, and for rational \( \beta \) the model describes particles with a fractional statistic (excitations from the Fermi sea with \( kq \) quasi-particles and \( kp \) quasi-holes if \( k \) is a positive integer and \( \beta = p/q \)).

The two points correlation functions were calculated using two different methods. The first one generalizes Dyson’s work on the correlations of the eigenvalues of random matrices Ref.3, which led to explicit expressions for the static density-density correlations for \( \beta = 1/2, 1, 2 \). For these values of \( \beta \), Simons, Lee and Al’tshuler in Ref.3, using supersymmetric techniques initiated by Efetov Ref.3, calculated the dynamical density-density correlations; then Haldane and Zirnbauer obtained in Ref.4, the retarded (for \( \beta = 2 \)) and the advanced (for \( \beta = 1/2, 1, 2 \)) single-particle Green functions.

The other method works for any rational \( \beta \) and uses the properties of Jack polynomials. The dynamical density-density correlations and the retarded single-particle Green functions were obtained for integer \( \beta \) by Lesage, Pasquier and Serban in Ref.5 and for rational \( \beta \) by Ha in Ref.6. The correlations functions can be expressed in terms of ”form factors” for some operator \( A \):

\[ |F|^2 = |< v_1, ..., v_p; w_1, ..., w_q | A | 0 >|^2 \]  

(1.2)
where the $v$’s and the $w$’s are the rapidities of the quasi-holes and of the quasi-particles respectively. $F$ can be decomposed as a product

$$F = F^{(0)} \times F^{(1)}$$

The expression $F^{(0)}$ is a purely statistical contribution given by

$$\left| F^{(0)} \right|^2 = \frac{\prod_{i<j} |v_i - v_j|^{2/\beta} \prod_{k<l} |w_k - w_l|^{2\beta}}{\prod_i (v_F^2 - v_i^2)^{1-1/\beta} \prod_k (w_F^2 - w_k^2)^{1-\beta} \prod_{i,k} (v_i - w_k)^2}$$  \hspace{1cm} (1.3)$$

where $v_F$ is half the size (in rapidity) of the Fermi sea. In the case of the advanced Green function ($A = \Psi^+$) it was found in Ref.4 that there are two contributions: one with one quasi-particle (zero quasi-hole) and one with $q+1$ quasi-particles and $p$ quasi-holes (if $\beta=p/q$). For this second contribution, $F^{(1)}_\beta$ was found in Ref.4, for $\beta=2, 1, 1/2$, to be

$$F_2^{(1)} = C_2 \frac{d}{dz} \left( \frac{(z-v_1)(z-v_2)}{(z-w_1)^2} \right)_{z=w_0}$$

$$F_1^{(1)} = 0$$

$$F_{1/2}^{(1)} = C_{1/2} \int_{w_0}^\infty \frac{dz}{\sqrt{z-w_0}} \left( \frac{z-v_1}{\sqrt{(z-w_1)(z-w_2)}} - 1 \right)$$  \hspace{1cm} (1.4)$$

Moreover, the authors proposed the following conjecture: for any positive integer $\beta = p$,

$$F_p^{(1)} = C_p \left( \frac{d}{dz} \right)^{p-1} \left\{ \prod_{i=1}^p \frac{(z-v_i)}{(z-w_1)^p} \right\}_{z=w_0}$$  \hspace{1cm} (1.5)$$

Recently, the single particle advanced Green function have been calculated for all rational $\beta$ by Serban, Lesage and Pasquier Ref.7. Their method was to sum the corresponding combination of Jack polynomials over all Young tableau with $r$ ($w_0$ in the thermodynamic limit) columns of length $N-1$, $p$ legs
and q arms, and to take the thermodynamic limit, that is N going to infinity. They obtained (1.3) for $F^{(0)}$, and $F^{(1)}_{\beta}$ as

$$
F^{(1)}_{\beta} = \frac{\prod_{i=0}^{q} \prod_{j=1}^{p} (w_i - v_j)^{\beta}}{\prod_{i<j} (v_i - v_j) \prod_{0 \leq i \leq j \leq q} (w_i - w_j)^{2\beta-1}} \cdot \int_{w_q}^{w_{q-1}} d\xi_{q-1} \cdots \int_{w_2}^{w_1} d\xi_{1} \prod_{i<j} (\xi_i - \xi_j) \prod_{i=1}^{q-1} \prod_{j=0}^{q} (\xi_i - w_j)^{\beta-1}. 
$$

.$$
. \partial_{v_1} \cdots \partial_{v_p} \left[ \prod_{i<j} (v_i - v_j) \prod_{j=1}^{p} \left( \prod_{i=0}^{q} (w_i - v_j)^{1-\beta} \prod_{i=1}^{q-1} (\xi_i - v_j)^{-1} \right) \right]. 
$$(1.6)

Then, Serban conjectured that this expression reduces to

$$
F^{(1)}_{\beta} = (\beta - 1) \frac{[\Gamma (\beta)]^{q}}{2i\pi} \oint_{C_w} dz \frac{\prod_{i=1}^{p} (v_i - z)}{\prod_{j=0}^{q} (w_j - z)^{\beta}} 
$$

(1.7)

where the contour $C_w$ surrounds the points $w_1, \ldots, w_q$.

The conjecture was proven for $\beta$ integer (that is Haldane and Zirnbauer conjecture (1.5)), and the result (1.4) for $\beta = 1/2$ has been obtained from (1.7), as well as the numerical equality between (1.6) and (1.7) for $\beta = 1/2$ and $3/2$.

In this publication, we prove Serban’s conjecture, not only for $\beta = p/q$, but for a set of different $\beta_i$ attached to each rapidity $w_i$. The importance of proving this conjecture is due to the interpretation of the result in terms of basic conformal operators. It was already pointed out by Khveshchenko Ref.8, that the density-density correlation function and that the one particle retarded Green function could be simply reexpressed in terms of 2D Coulomb gaz correlators

$$
<V_{a_1}(z_1) \cdots V_{a_n}(z_n)> = \prod_{i<j} (z_i - z_j)^{a_i a_j} i f \sum_{i=1}^{n} a_i = 0
$$

(1.8)
and zero otherwise. Equation (1.7) proves, in the more complicated situation of the advanced Green function, that the same property is true:

\[ F = F^{(0)} \times F^{(1)} = \frac{(\beta - 1) [\Gamma(\beta)]^q}{\prod_i (v_i^2 - v_i^2)^{\frac{\beta}{2}} \prod_k (w_k^2 - v_k^2)^{\frac{1-\beta}{2}}} \]

\[
\frac{1}{2i\pi} \oint_{C_w} dz < V_{-\sqrt{\beta}}(z) \prod_{i=1}^{p} V_{-\frac{1}{\sqrt{\beta}}}(v_i) \prod_{k=0}^{q} V_{\sqrt{\beta}}(w_k) > \quad (1.9)
\]

where a charge \(\sqrt{\beta}\) is given to the quasi-particle operator, \(-\frac{1}{\sqrt{\beta}}\) to the quasi-hole operator, and \(-\sqrt{\beta}\) to the screening operator introduced in order to have conservation of the charge. At the present time, we do not know what has to be done with the terms coming from the Fermi sea \((v_F\) dependent) but this might be overcome naturally if we later develop a bosonisation procedure of the Calogero-Sutherland model. Several arguments plead in favour of such a construction and specially in Ref.9, it is shown that the Jack polynomials are the singular vectors of the Virasoro and of the \(W_N\) algebra. It is however surprising to see in (1.9) that the correlators \(V_a\) are not taken in complexified space-time points but in the rapidities. Let us finally mention that the relation between (1.6) and (1.7) can certainly be q-deformed if we work with the relativistic Ruijsenaars-Schneider model Ref.10, where the Jack polynomials are q-deformed into the Macdonald polynomials Ref.11.

## 2 Proof of the conjecture.

Given a set of \((q+1)\) variables \(\beta_i\) (large enough to make all integrals convergent), and two sets of variables, \((q+1)\) variables \(w_i\) and \(p\) variables \(v_i\), we define \(C\) as

\[
C = \int_{w_q}^{w_q-1} d\xi_1 \cdots \int_{w_2}^{w_1} d\xi_1 \prod_{i<j}^{q-1} (\xi_i - \xi_j) \prod_{i=1}^{q-1} \prod_{j=0}^{q} |\xi_i - w_j|^{\frac{\beta}{2} - 1},
\]

\[
\frac{\partial v_1 \cdots \partial v_p}{\prod_{j=0}^{q} \prod_{k=1}^{p} (w_j - v_k)^{1-\beta_j} \prod_{i=1}^{q-1} \prod_{k=1}^{p} (\xi_i - v_k)^{1-\beta_j} \prod_{k<l} (v_k - v_l)}
\]

\[
\prod_{j=0}^{q} \prod_{k=1}^{p} (w_j - v_k)^{1-\beta_j} \prod_{k<l} (v_k - v_l)
\]
Our result is

\[
C = \frac{\prod_{j=0}^q \Gamma \left( \beta_j \right)}{\Gamma \left( \sum_{j=0}^q \beta_j - p - 1 \right)} \frac{\prod_{0 \leq i < j \leq q} (w_i - w_j)^{\beta_i + \beta_j - 1}}{\prod_{j=0}^p \prod_{k=1}^q (w_j - v_k)} \cdot \frac{(-1)^{k_p}}{2\pi i} \oint_{C_0} \frac{dz}{(w_0 - z)^{\beta_0} \prod_{j=1}^q (z - w_j)^{\beta_j}}
\]

(2.2)

where \( w_0 > w_1 \) and \( C_0 \) surrounds the cut \([w_0, +\infty)\); we suppose that \( \sum_{j=0}^q \beta_j > p + 1 \) to ensure the convergence of the integral in (2.2) (otherwise, subtractions should have to be considered).

We note that if all the variables \( \beta_j \) are equal to \( p/q \), we have

\[
\frac{\prod_{j=0}^q \Gamma \left( \beta_j \right)}{\Gamma \left( \sum_{j=0}^q \beta_j - p - 1 \right)} = (\beta - 1) \left[ \Gamma (\beta) \right]^q
\]

(2.3)

The proof of this result requires several steps of different nature; in order to have an easier reading, the main part of the proof is written in this section while four appendices are devoted to technical details.

The expression \( C \) in (2.1) is made of two parts: on the first line, the integrals \( d\xi \); on the second line the derivatives \( \partial_v \), the factorization being prevented by the \( (\xi - v)^{-1} \) terms. A first appendix (A) generalizes Cauchy’s identity in order to eliminate the \( (\xi - v)^{-1} \) terms. \( C \) becomes a sum of terms, each of them being factorized into \( d\xi \) integrals and \( \partial_v \) derivatives without \( (\xi - v)^{-1} \) terms. The second appendix (B) generalizes a result by Dixon (1905) Ref.12, who performs a change of variables which makes obvious the calculation of the integrals \( d\xi \); the generalization consists in adding the variable \( w_0 \) to Dixon’s result which generates a contour integral \( \oint dz \). These two appendices are sufficient to prove the conjecture in the case \((q-1) \geq p\). However, the case \((q-1) \leq p\) is more difficult because we have extra \( \partial_v \) derivatives to perform over terms which do not contain \( (\xi - v)^{-1} \) anymore. Appendix C is devoted to the calculation and to the properties of these \( \partial_v \) derivatives.
Finally, appendix D, which also, is only needed in the case $q-1 \leq p$, develops the properties of the integrals

$$I_p = \frac{1}{2i\pi} \oint_{C_0} dz \frac{P(z)}{(w_0 - z)^{\beta_0} \prod_{j=1}^q (z - w_j)^{\beta_j}}$$

(2.4)

where $P(z)$ is a polynomial in $z$ and where $C_0$ surrounds the cut $[w_0, +\infty]$.

Let us mention that the proof is written for a set of generic values of the variables $v_i$ without care to the case where several $v$'s are equal; however, our result remain valid in that case since (2.1) has no pole singularity at $v_i = v_j$ (Appendix C (C2)).

First case: $q-1 \geq p$.

In expression (2.1), we use the generalized Cauchy’s identity (A3) to write

$$\prod_{i<j} (\xi_i - \xi_j) \prod_{k<l} (v_k - v_l) \prod_{i=1}^{q-1} \prod_{j=1}^p (\xi_i - v_j) = \sum (-)^{\delta_{l_p}} \det \frac{1}{\xi_{j_k} - v_j} \prod_{i,j \notin I_p} (\xi_i - \xi_j)$$

(2.5)

where $I_p = (j_1 < j_2 < \ldots < j_p) \subset (1, \ldots, q-1)$ and $\delta_{l_p} = (q-1)p - \sum_{i=1}^p j_i$.

Because of the absence of terms of the type $(v_i - v_j)^{-1}$ for different $j$’s, the action of the derivatives $\partial v_j$ in (2.1) factorizes. We get a product of the type

$$\prod_{j=1}^p \frac{\partial v_j}{\prod_{i=0}^q (w_i - v_j)^{1-\beta_i}} \left(\xi_{a_j} - v_j\right)^{-1}$$

$$= \prod_{j=1}^p \left\{ \left[ \sum_{i=0}^q \beta_i - \frac{\partial}{\partial \xi_{a_j}} \left(\xi_{a_j} - v_j\right)^{-1} \right] \left(\xi_{a_j} - v_j\right)^{-1} \right\}$$

(2.6)
where the $a_j$'s are a permutation of the $j_k$'s in $I_p$. The right hand side of (2.6) can now be taken inside the integrals $d\xi$ to be integrated by parts (we assume that the corresponding $\text{Re}\beta_j'$s $> 1$ to avoid the contribution of the end-points); the expression (2.6) becomes

$$
\prod_{j=1}^{p} \left[ \sum_{i=0}^{q} \frac{\beta_i - 1}{(w_i - v_j)(\xi_{a_j} - w_i)} \right]
$$

(2.7)

For any product of terms in the determinant (2.5), we have obtained a product of terms (2.7); consequently, the determinant in (2.5) can be replaced by the determinant

$$
\det \begin{vmatrix}
\frac{q}{\sum_{i=0}^{\beta_i - 1}} (\xi_j - v_i)(w_i - v_j)
\end{vmatrix}
$$

(2.8)

where $J_p = (l_1 < l_2 < .. < l_p) \subset (0, ..., q)$.

The situation now, is that the determinant $\left| (\xi_j - v_i)^{-1} \right|$ in (2.5) is replaced by the determinant $\left| (\xi_j - w_l)^{-1} \right|$ with the $l_j$’s in $J_p$. We may now use the generalized Cauchy’s identity (A3) backwards to sum over the different $I_p$’s and we obtain, inside the sum over $J_p$,

$$
\prod_{i<j}^{l_j} \frac{\xi_i - \xi_j}{\Pi_{j=k}^{l_j} (w_j - w_k)} = (-)^{\delta_{J_p - \eta}} \prod_{i<j}^{l_j} \frac{(\xi_i - \xi_j)}{\Pi_{i=1}^{l_j} \Pi_{k=1}^{l_j} (\xi_i - w_k)}
$$

(2.9)

where the $l_j$’s belong to $J_p$, and $\delta_{J_p} = pq - \sum_{k=1}^{1} l_k$ (here, $\eta = 1$ if $l_1 = 0$ and 0 otherwise).

We proved, up to now, that

$$
C = \prod_{k<l} \frac{1}{v_k - v_l} \sum_{J_p} (-)^{\delta_{J_p}} \prod_{i \in J_p} (\beta_i - 1) \prod_{j<k} (w_j - w_k) \det \left| \frac{1}{w_k - v_j} \right|
$$

$$
. \int_{w_q}^{w_q-1} d\xi_{q-1} \cdots \int_{w_2}^{w_1} d\xi_1 \prod_{i<j} (\xi_i - \xi_j) \prod_{q=1}^{q} \frac{1}{\Pi_{i=1}^{q} |\xi_i - w_j|^{\beta_j - 1}}
$$

(2.10)
where $\beta'_j = \beta_j$ if $j \notin J_p$ and $\beta'_j = \beta_j - 1$ if $j \in J_p$.

We are now in situation to use Dixon’s result with an extra variable $w_0$ (B4,B10) and to write the second line of (2.10) as

$$
\prod_{j=0}^{q} \Gamma \left( \beta_j' \right) \prod_{0 \leq i < j \leq q} (w_i - w_j)^{\beta_i' + \beta_j' - 1} 
\cdot \left( -1 \right) \frac{1}{2i\pi} \oint_{C_0} dz \frac{1}{(w_0 - z)^{\beta_0'} \prod_{j=1}^{q} (z - w_j)^{\beta_j'}}. 
$$

(2.11)

The expression $C$ becomes

$$
C = \frac{\prod_{j=0}^{q} \Gamma \left( \beta_j \right)}{\Gamma \left( \sum_{j=0}^{q} \beta_j - p - 1 \right)} \prod_{k<l} (v_k - v_l) \prod_{0 \leq i < j \leq q} (w_i - w_j)^{\beta_i + \beta_j - 2} 
\cdot \left( -1 \right) \frac{1}{2i\pi} \oint_{C_0} dz \frac{1}{(w_0 - z)^{\delta_{dp}} \prod_{j=1}^{q} (z - w_j)^{\beta_j'}} 
\cdot \sum_{J_p} (-)^{\delta_{dp}} \prod_{k=1}^{p} (z - w_k) \det \left| \frac{1}{w_i - w_j} \prod_{i<j \in J_p} (w_i - w_j) \right|. 
$$

(2.12)

Again, as a consequence of the generalized Cauchy’s identity, we may now sum over all $J_p$’s; from (A8) with $q+1 > p$; the third line of (2.12) is nothing but

$$
\prod_{k=1}^{p} (z - v_k) \frac{\prod_{0 \leq i < j \leq q} (w_i - w_j) \prod_{k<l} (v_k - v_l)}{\prod_{i=0}^{q} \prod_{j=1}^{p} (w_i - v_j)}. 
$$

(2.13)

The fact that the indices of the $w$’s run from 0 to $q$ (instead of 1 to $q+1$) makes in (A8), a shift by $p$ of $\delta_{J_p}$; this explains why we do not have a $(-)^p$ in (2.13).

This result ends the proof for the case $(q-1) \geq p$. 

9
Second case: \( q-1 \leq p \).

Again, we use the generalized Cauchy’s identity in order to calculate the left hand side of (2.5), but here we have more variables \( v_j \) than variables \( \xi_i \); using the relation (A2), we write

\[
\frac{\prod_{i<j} (\xi_i - \xi_j) \prod_{k<l} (v_k - v_l)}{\prod_{j=1}^{q} \prod_{j=1}^{q} (\xi_i - v_j)} = \sum_{I_{q-1}} (-)^{t_{q-1}} \det \left| \frac{1}{\xi_i - v_{ij}} \prod_{k<l} (v_k - v_l) \right|
\]

(2.14)

where \( I_{q-1} = (j_1 < j_2 < \ldots < j_{q-1}) \subset (1, \ldots, p) \) and \( \delta_{t_{q-1}} = \sum_{k=1}^{q-1} j_k - q + 1 \). Consequently, the derivatives \( \partial_{v_j} \) split into two parts depending whether \( j \) is in \( I_{q-1} \) or not. The second line of (2.1) becomes

\[
\frac{1}{\prod_{k<l} (v_k - v_l)} \sum_{I_{q-1}} (-)^{t_{q-1}} \cdot \frac{\prod_{j \notin I_{q-1}} (\partial_{v_j}) \left[ \prod_{j=0}^{q} \prod_{j \notin I_{q-1}} (w_i - v_j)^{1-\beta_i} \prod_{k<l} (v_k - v_l) \right]}{\prod_{j=0}^{q} \prod_{j \notin I_{q-1}} (w_i - v_j)^{1-\beta_i}} \cdot \frac{\prod_{j \in I_{q-1}} (\partial_{v_j}) \left[ \prod_{j=0}^{q} \prod_{j \in I_{q-1}} (w_i - v_j)^{1-\beta_i} \det \left| \frac{1}{\xi_i - v_{jk}} \right| \right]}{\prod_{j=0}^{q} \prod_{j \in I_{q-1}} (w_i - v_j)^{1-\beta_i}}
\]

(2.15)

In the third line of (2.15), the derivatives \( \partial_{v} \) factorize so that this term can be calculated in exactly the same way as for the case \( (q-1) \geq p \). The determinant \((\xi_i - v_{jk})^{-1}\) can be replaced in the same manner by

\[
\det \left| \sum_{r=0}^{q} \left( \frac{\beta_r - 1}{(\xi_i - w_r)(w_r - v_{jk})} \right) \right| = \sum_{J_{q-1}} \prod_{r \in J_{q-1}} (\beta_r - 1) \det \left| \frac{1}{\xi_i - w_r} \right| \det \left| \frac{1}{w_r - v_{jk}} \right|
\]

(2.16)

where \( J_{q-1} = (l_1 < l_2 < \ldots < l_{q-1}) \subset (0, \ldots, q) \). We now use Cauchy’s identity (A1), to write

\[
\det \left| \frac{1}{\xi_i - w_r} \right| = (-)^{\frac{q(q-1)}{2} + \delta_{J_{q-1}}} \frac{\prod_{i<j} (\xi_i - \xi_j) \prod_{j<k} (w_j - w_k)}{\prod_{r=1}^{q-1} \prod_{j=1}^{q-1} (\xi_i - w_{lj})}
\]

(2.17)
where $\delta_{J_{q-1}} = \sum_{j=1}^{q-1} l_j - q + 1$ and $\eta = 1$ if $l_1 = 0$ (and 0 otherwise).

Again, after inserting (2.17) into the integrals $d\xi$, we may integrate using Dixon’s change of variables (B4, B10) to get

$$
\left(-\right)^{\frac{q(q-1)}{2}} \frac{\prod_{j=0}^{q-1} \Gamma \left(\beta_j\right)}{\Gamma \left(\sum_{j=0}^{q} \beta_j - q\right)} \prod_{0 \leq i < j \leq q} \left(w_i - w_j\right)^{\beta_i + \beta_j - 2}.
$$

$$
\left(-\right)^{\frac{1}{2}} \frac{1}{2\pi i} \oint_{C_0} \frac{dz}{(w_0 - z)^{\beta_0} \prod_{j=1}^{q-1} \Gamma \left(\beta_j\right)} \prod_{0 \leq i < j \leq q} \left(w_i - w_j\right)^{\beta_i + \beta_j - 2}.
$$

Using a consequence of the generalized Cauchy’s identity for a given set of $q-1$ variables $v_{j_k}$’s in $I_{q-1}$, the third line of (2.18) can be summed over $J_{q-1}$; here again, because the $w$’s are labelled from 0 to $q+1$, $\delta_{J_{q-1}}$ has to be shifted by $q-1$ in order to apply (A8) with $q+1 > q-1$. We get for the third line of (2.15)

$$
\left(-\right)^{\frac{1}{2}} \prod_{j=0}^{q-1} \frac{\Gamma \left(\beta_j\right)}{\Gamma \left(\sum_{j=0}^{q} \beta_j - q\right)} \prod_{0 \leq i < j \leq q} \left(w_i - w_j\right)^{\beta_i + \beta_j - 1} \prod_{k \leq l \in I_{q-1}} \left(v_k - v_l\right) \prod_{i=0}^{q-1} \prod_{j \in I_{q-1}} \left(w_i - v_j\right) \prod_{k \leq l \in I_{q-1}} \left(v_k - v_l\right).
$$

At this point, we proved that, in the case $(q-1) \leq p$, the expression $C$ may be written as

$$
C = \left(-\right)^{\frac{1}{2}} \prod_{j=0}^{q-1} \frac{\Gamma \left(\beta_j\right)}{\Gamma \left(\sum_{j=0}^{q} \beta_j - q\right)} \prod_{0 \leq i < j \leq q} \left(w_i - w_j\right)^{\beta_i + \beta_j - 1} \prod_{k \leq l \in I_{q-1}} \left(v_k - v_l\right) \prod_{i=0}^{q-1} \prod_{j \in I_{q-1}} \left(w_i - v_j\right) \prod_{k \leq l \in I_{q-1}} \left(v_k - v_l\right).
$$
\[
\prod_{k<l \in I_{q-1}} (v_k - v_l) \frac{(-1)^{q-1}}{2i\pi} \oint_{C_0} \frac{dz}{(w_0 - z)^{\beta_0}} \prod_{j=1}^{q} (z - w_j)^{\beta_j} = \prod_{j=1}^{q} (z - v_j) \quad (2.20)
\]

The purpose of appendices C and D is to calculate the sum over \( I_{q-1} \) in (2.20). In (D23) we show that the second and third lines of (2.20) are equal to

\[
\left(-\frac{1}{2}\right)^{(q-1)(q-2)} \frac{\Gamma \left( \sum_{j=0}^{q} \beta_j - q \right)}{\Gamma \left( \sum_{j=0}^{q} \beta_j - p - 1 \right)} \prod_{k<l} (v_k - v_l),
\]

\[
\frac{(-1)}{2i\pi} \oint_{C_0} \frac{dz}{(w_0 - z)^{\beta_0}} \prod_{j=1}^{q} (z - w_j)^{\beta_j} = \prod_{k=1}^{p} (z - v_k) \quad (2.21)
\]

and this ends the proof of the conjecture in the case \((q-1)\leq p\).

ACKNOWLEDGMENTS

I thank D. Serban for her constant encouragement, for many discussions about the work in Ref. 7 and for her careful reading of the manuscript.
Given two sets of variables \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_m)\), if \(n=m\), Cauchy's identity is
\[
\prod_{i<j} (x_i - x_j) \prod_{k<l} (y_k - y_l) = (-)^{n(n-1)/2} \det \begin{vmatrix} \frac{1}{x_i - y_k} \end{vmatrix}_{i,j,k,l}^{n,n} \tag{A1}
\]
(apart from the sign, this result is relatively evident from the properties of the determinant, the homogeneity and the pole structure). Now, if \(n \leq m\), we may generalize this relation by systematically organizing the residues of the poles; we get:
\[
\prod_{i<j} (x_i - x_j) \prod_{k<l} (y_k - y_l) = \sum_{I_n} (-)^{\delta_{I_n}} \det \begin{vmatrix} \frac{1}{x_i - y_k} \end{vmatrix}_{k,l \notin I_n}^{m,m} \prod_{i<j} (x_i - y_j) \tag{A2}
\]
where \(I_n = (j_1 < j_2 < \ldots < j_n) \subset (1, \ldots, m)\) and \(\delta_{I_n} = \sum_{i=1}^{n} j_i - n\). Of course, if \(n \geq m\), the situation is symmetric and we have
\[
\prod_{i<j} (x_i - x_j) \prod_{k<l} (y_k - y_l) = \sum_{I_m} (-)^{\delta_{I_m}} \det \begin{vmatrix} \frac{1}{x_i - y_k} \end{vmatrix}_{i,j \notin I_m}^{m,m} \prod_{i<j} (x_i - x_j) \tag{A3}
\]
where \(I_m = (l_1 < l_2 < \ldots < l_m) \subset (1, \ldots, n)\) and \(\delta_{I_m} = mn - \sum_{k=1}^{m} l_k\). We call (A2-3) generalized Cauchy’s identities. In (A2-3), we adopt the convention that \(\prod \frac{1}{x_i - y_j} = 1\) if there is 0 or 1 variable \(z \notin I\).

Let us note the following property: for \(n < m\), if we let a given variable \(x_p \to \infty\), the left-hand side of (A2) behaves as \(x_p^{-(m-n+1)} \leq x_p^{-2}\) while the right-hand side behaves as \(x_p^{-1}\). Consequently, the coefficient of \(x_p^{-1}\) in the right-side of (A2) is zero. This coefficient can be calculated: we define the matrix \(\Delta_{I_n,p}\) such that
\[
(\Delta_{I_n,p})_{ik} = \begin{cases} 
1 & \text{for } i \neq p \\
\frac{1}{x_i - y_{j_k}} & \text{for } i = p
\end{cases} \tag{A4}
\]
then,
\[
\sum_{I_n} (-\delta_{I_n}) \det \Delta_{I_n,p} \prod_{k<l, k,l \notin I_n} (y_k - y_l) = 0 \quad \text{(A5)}
\]

We are now using (A5) to calculate
\[
\Phi(z) = \sum_{I_n} (-\delta_{I_n}) \prod_{k=1}^n (z - y_{j_k}) \det \left| \frac{1}{x_i - y_{j_k}} \right| \prod_{k<l, k,l \notin I_n} (y_k - y_l) \quad \text{(A6)}
\]

The determinant in (A6) is a sum of terms of the type \((x_1 - y_{\alpha_1})^{-1} (x_2 - y_{\alpha_2})^{-1} \ldots (x_n - y_{\alpha_n})^{-1}\) where \((\alpha_1, \ldots, \alpha_n)\) is a permutation of \((j_1, \ldots, j_n)\). For such a term, we write
\[
\prod_{k=1}^n (z - y_{j_k}) = \sum_{J} \prod_{k \notin J} (z - x_k) \prod_{j \in J} (x_j - y_{\alpha_j})
\]

where \(J\) is a set of \(|J|\) indices \(\subset (1, \ldots, n)\). Consequently, when summing over all terms of the determinant, we get
\[
\prod_{k=1}^n (z - y_{j_k}) \det \left| \frac{1}{x_i - y_{j_k}} \right| = \sum_{J} \prod_{k \notin J} (z - x_k) \det \left| \left( \frac{1}{x_i - y_{j_k}} \right)_{J} \right|
\]

where the matrices \(\left( (x_i - y_{j_k})^{-1} \right)_{J}\) have \(|J|\) lines of 1 corresponding to the indices \(i \in J\). Their determinants are trivially zero for \(|J| \geq 2\). Now, for \(|J| = 1\), the sum over \(I_n\) in (A6) gives zero because of (A5); the only non vanishing contribution comes from \(|J| = 0\). We just proved that
\[
\Phi(z) = \prod_{k=1}^n (z - x_k) \frac{\prod_{i<j} (x_i - x_j) \prod_{k<l} (y_k - y_l)}{\prod_{i=1}^n \prod_{j=1}^n (x_i - y_j)} \quad \text{(A7)}
\]

By symmetry, for \(n > m\), we have
\[
\sum_{I_m} (-\delta_{I_m}) \prod_{i=1}^m (z - x_i) \det \left| \frac{1}{x_i - y_k} \right| \prod_{i<j, i,j \notin I_m} (x_i - x_j)
\]
\[
= \prod_{i=1}^m (z - y_i) \frac{\prod_{i<j} (x_i - x_j) \prod_{k<l} (y_k - y_l)}{\prod_{i=1}^n \prod_{j=1}^m (x_i - y_j)} \quad \text{(A8)}
\]
where, this time, we used determinants of matrices with columns of (-1)'s. Let us mention that if the indices of the $x$'s run from 0 to $n-1$ instead of 1 to $n$ (as it is the case in eq.(2.12,2.18)), then, $\delta_{lm}$ has to be replaced in (A8) by $\delta_{lm} + m$.

4 Appendix B

In this appendix, we calculate the integrals $d\xi$ which appear in the first line of (2.1). We first calculate the integrals without the variable $w_0$; the end of the appendix is devoted to the introduction of $w_0$ generating a contour integral in the complex plane. Let us define

$$J_{q-1} = \int_{w_q}^{w_{q-1}} d\xi_{q-1} \cdots \int_{w_2}^{w_1} d\xi_1 \prod_{i<j} (\xi_i - \xi_j) \prod_{i=1}^{q} \prod_{j=1}^{q} |\xi_i - w_j|^{\beta_j-1}.$$  \hspace{1cm} (B1)

Clearly,

$$J_1 = \frac{\Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)} (w_1 - w_2)^{\beta_1 + \beta_2 - 1}$$

A naive calculation of $J_2$ gives three products of hypergeometric functions, namely

$$J_2 = \prod_{i<j} (w_i - w_j)^{\beta_i + \beta_j - 1} \frac{\Gamma(\beta_1) \Gamma(\beta_2) \Gamma(\beta_3)}{\Gamma(\beta_1 + \beta_2) \Gamma(\beta_2 + \beta_3)}.$$

\begin{equation}
\begin{bmatrix}
F(\beta_3, -\beta_1; \beta_2 + \beta_3; \alpha) & F(1 - \beta_3, \beta_1; \beta_1 + \beta_2; 1 - \alpha) \\
+ F(\beta_3, 1 - \beta_1; \beta_2 + \beta_3; \alpha) & F(-\beta_3, \beta_1; \beta_1 + \beta_2; 1 - \alpha) \\
- F(\beta_3, 1 - \beta_1; \beta_2 + \beta_3; \alpha) & F(1 - \beta_3, \beta_1; \beta_1 + \beta_2; 1 - \alpha)
\end{bmatrix}
\end{equation}

where $\alpha = (w_2 - w_3) / (w_1 - w_3)$. The remarkable fact is that the square bracket $[.]$ in (B2) is $\alpha$ independent and equal to

$$\frac{\Gamma(\beta_1 + \beta_2) \Gamma(\beta_2 + \beta_3)}{\Gamma(\beta_1 + \beta_2 + \beta_3) \Gamma(\beta_2)}.$$
This result is due to Elliot Ref.13, who generalizes Legendre’s relation from the theory of elliptic integrals. A generalization of Elliot’s result has been given by Dixon Ref.12, who proved that

\[
J_{q-1} = \frac{\prod_{j=1}^{q} \Gamma \left( \beta_j \right)}{\Gamma \left( \sum_{j=1}^{q} \beta_j \right)} \prod_{i<j} (w_i - w_j)^{\beta_i + \beta_j - 1} \]  \hspace{1cm} (B3)

by performing a clever change of variables: given the function

\[
f(\theta) = \prod_{j=1}^{q} (\theta - w_j)
\]

we define new variables

\[
x_j = (-)^{q-1} \frac{\prod_{i=1}^{q-1} (\xi_j - w_i)}{f'(w_j)} \text{ for } j = 1, ..., q
\]

It is easy to verify that all variables \(x_j\) are \(\geq 0\); moreover, we have

\[
\sum_{j=1}^{q} x_j = 1
\]

so that \(x_q\) can be considered as dependant of the other \(x’s.\) The jacobian of the transformation is

\[
\frac{d \left( \xi_1, \ldots, \xi_{q-1} \right)}{d \left( x_1, \ldots, x_{q-1} \right)} = \frac{\prod_{i<j} (w_i - w_j)}{\prod_{i<j} (\xi_i - \xi_j)}
\]

Consequently,

\[
J_{q-1} = \prod_{i<j} (w_i - w_j)^{\beta_i + \beta_j - 1} \int_0^1 \ldots \int_0^1 \prod_{j=1}^{q} \left[ dx_j x_j^{\beta_j - 1} \right] \delta \left( \sum_{j=1}^{q} x_j - 1 \right)
\]

which proves (B3).
Next, we wish to calculate (or to transform into a single contour integral in the complex plane) the integrals

\[ K_{q-1} = \int_{w_q}^{w_{q-1}} d\xi_{q-1} \cdots \int_{w_2}^{w_1} d\xi_1 \prod_{i<j} (\xi_i - \xi_j) \prod_{i=1}^{q-1} \prod_{j=0}^{q} |\xi_i - w_j|^{\beta_j-1} \]  \hspace{1cm} (B4)

In order to transform \( K_{q-1} \), we write

\[ \prod_{i=1}^{q-1} |\xi_i - w_0| = \sum_{j=1}^{q} A_j \prod_{i=1}^{q-1} |\xi_i - w_j| \]  \hspace{1cm} (B5)

with

\[ A_j = \frac{\prod_{k \neq j} (w_0 - w_k)}{\prod_{k \neq j} |w_j - w_k|} \]  \hspace{1cm} (B6)

Let us first consider the case where \( \beta_0 = n \) integer. Then,

\[ \prod_{i=1}^{q-1} |\xi_i - w_0|^{n-1} = (n-1)! \sum_{\{p_j\}}^{q} \prod_{j=1}^{q} \frac{A_j^{p_j}}{p_j!} \prod_{i=1}^{q-1} |\xi_i - w_j|^{p_j} \]  \hspace{1cm} (B7)

so that we can use (B1, B3) to integrate the \( d\xi \) integrals with \( \beta_j \) replaced by \( \beta_j + p_j \). We get

\[ K_{q-1} = \frac{(n-1)!}{\Gamma \left( \sum_{j=1}^{q} \beta_j + n - 1 \right)} \prod_{1 \leq i < j \leq q} (w_i - w_j)^{\beta_i + \beta_j - 1} \cdot \sum_{\{p_j\}}^{q} \prod_{j=1}^{q} \left\{ \frac{\Gamma (\beta_j + p_j)}{p_j!} (w_0 - w_j)^{n-1-p_j} \right\} \]  \hspace{1cm} (B8)

On the other hand, the contour integral around \( z = w_0 \)

\[ \frac{1}{2i\pi} \oint_{C_0} dz \frac{1}{(z-w_0)^n \prod_{j=1}^{q} (z-w_j)^{\beta_j}} \]

is a derivative

\[ \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial z^{n-1}} \left\{ \frac{1}{\prod_{j=1}^{q} (z-w_j)^{\beta_j}} \right\}_{z=w_0} \]
equal to

\[ \frac{(-1)^{n-1}}{\prod_{j=1}^{q} \left[ \Gamma \left( \beta_j \right) (w_0 - w_j)^{\beta_j + n-1} \right]} \times \sum_{\{p_j\}} \prod_{j=1}^{q} \left\{ \frac{\Gamma \left( \beta_j + p_j \right)}{p_j!} (w_0 - w_j)^{n-1-p_j} \right\} \]  

(B9)

If we compare (B8) and (B9), we get for \( \beta_0 = n \) integer

\[ K_{q-1} = \frac{\prod_{j=0}^{q} \Gamma \left( \beta_j \right)}{\Gamma \left( \sum_{j=0}^{q} \beta_j - 1 \right)} \prod_{0 \leq i < j \leq q} (w_i - w_j)^{\beta_i + \beta_j - 1} \times \frac{(-1)}{2i\pi} \int_{C_0} dz \frac{1}{(w_0 - z)^{\beta_0} \prod_{j=1}^{q} (z - w_j)^{\beta_j}}. \]  

(B10)

We now prove that (B10) is also true for any \( \beta \)'s such that the integral \( dz \) converges around a contour \( C_0 \) which surrounds the cut \([w_0, +\infty] \), namely \( \sum_{j=0}^{q} \beta_j > 1 \). Let us generalize the discrete sum (B7) using the integral representation

\[ (a + b)^\gamma = \frac{1}{\Gamma (-\gamma)} \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} dz \Gamma (-z) \Gamma (z - \gamma) a^z b^{\gamma-z} \]  

(B11)

where \( \text{Re} \gamma \leq \sigma \leq 0 \). Strictly speaking, this integral representation is a priori valid for \( \text{Re} \gamma < 0 \). However, we may, by deformation of the contour, generalize to \( \text{Re} \gamma > 0 \) provided that \( |\text{Im} \gamma| > 0 \) in order to avoid a possible pinch of the contour and the extraction of residues. Consequently, we assume that \( \beta_0 \) has a small imaginary part for the demonstration and we analytically continue the result to real \( \beta_0 \). We generalize (B11) and write (B7) as

\[ \prod_{i=1}^{q-1} |\xi_i - w_0|^{\beta_0-1} = \frac{1}{\Gamma (1-\beta_0)} \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} dz \prod_{j=1}^{q-1} \int dz_j. \]

\[ \times \prod_{j=1}^{q} \left\{ \Gamma (-z_j) A_j^{z_j} \prod_{i=1}^{q-1} |\xi_i - w_j|^{z_j} \right\} \delta_{Z,\beta_0-1} \]  

(B12)
where $\delta_{Z,\beta_0-1}$ is a symbol which simplifies the writing and which means that

$$z_q = \beta_0 - 1 - \sum_{j=1}^{q-1} z_j$$

We may now apply (B1, B3) in order to integrate the $d\xi$ integrals with $\beta_j$ replaced by $\beta_j + z_j$ and get

$$\frac{1}{\Gamma \left( \sum_{j=0}^{q} \beta_j - 1 \right)} \frac{1}{\Gamma (1 - \beta_0)} \prod_{1 \leq i < j \leq q} (w_i - w_j)^{\beta_i + \beta_j - 1}.$$  

$$\frac{1}{(2i\pi)^q-1} \int \ldots \int d\xi_j \prod_{i=1}^{q-1} \{ \Gamma (-z_j) \Gamma \left( \beta_j + z_j \right) (w_0 - w_j)^{\beta_0 - 1 - z_j} \} \delta_{Z,\beta_0-1} \tag{B13}$$

On the other hand, the integral

$$\frac{1}{2i\pi} \oint_{C_0} dz \frac{1}{(w_0 - z)^{\beta_0} \prod_{j=1}^{q} (z - w_j)^{\beta_j}}$$  

can be calculated from the integral of the discontinuity along the cut $[w_0, +\infty]$.

$$- \frac{\sin \pi \beta_0}{\pi} \int_0^\infty dx \ x^{-\beta_0} \prod_{j=1}^{q} (x + w_0 - w_j)^{-\beta_j} \tag{B15}$$

Next, we use the integral representation (B11), for $j=1$ to $q-1$, in order to write (B15) as

$$- \frac{\sin \pi \beta_0}{\pi} \frac{1}{\prod_{j=1}^{q-1} \Gamma \left( \beta_j \right)}$$

$$\cdot \frac{1}{(2i\pi)^q-1} \int \ldots \int d\xi_j \Gamma (-z_j) \Gamma \left( \beta_j + z_j \right) (w_0 - w_j)^{z_j - \beta_j}.$$  

$$\cdot \int_0^\infty dx \ x^{-\beta_0 + \sum_{j=1}^{q-1} z_j} (x + w_0 - w_q)^{-\beta_q}$$  

19
After integration of the dx integral, (B14) is equal to

\[ -\frac{\sin \pi \beta_0}{\pi} \frac{1}{\prod_{j=1}^q \Gamma (\beta_j)}. \]  

(B16)

\[ -\frac{1}{(2i\pi)^{q-1}} \int \cdots \int \prod_{j=1}^{q-1} dz_j \prod_{j=1}^q \left[ \Gamma (-z_j) \Gamma (\beta_j + z_j) (w_0 - w_j)^{-z_j - \beta_j} \right] \delta_{Z, \beta_0 - 1} \]

If we compare (B16) and (B13), we just proved (B10) for a non integer \( \beta_0 \).

This ends the calculation of \( K_{q-1} \) which represents the central part in the proof of the conjecture.

5 Appendix C

In this appendix we calculate and we give some properties of the expression

\[ A = \frac{\partial v_1 \cdots \partial v_p}{\prod_{i=0}^p \prod_{j=1}^q (w_i - v_j)^{1-\beta_i} \prod_{k<l} (v_k - v_l)} \left[ \prod_{j=1}^q \frac{\partial v_j}{\partial v_i} \left[ f (w, v) (v_i - v_j) \right] \right] \]  

(C1)

In order to calculate this expression, we simply calculate the residues of the poles in the variables \( v_j \); the first property which is clear is that there is no singularity when two \( v \)'s coincide. The only way to get a pole at \( v_i = v_j \) is when the derivatives \( \partial v_i \) or/and \( \partial v_j \) act upon the term \( (v_i - v_j) \) at the numerator of (C1). Clearly, for \( f (w, v) \) symmetric in \( v_i \) and \( v_j \),

\[ \frac{\partial v_i \partial v_j}{f (w, v) (v_i - v_j)} \left[ f (w, v) (v_i - v_j) \right] = f''_{ij} (w, v) - \frac{f' (w, v) - f'_j (w, v)}{f (w, v) (v_i - v_j)} \]  

(C2)

and the residue of the pole at \( v_i = v_j \) is zero (and this property remains evidently true when performing the other derivatives). Consequently, we are
going to extract the poles when \( v_i = w_k \) neglecting systematically all poles at \( v_i = v_j \) since their residues vanish. We write

\[
A = \prod_{i=0}^{q} \frac{R_i}{w_i - v_1} \tag{C3}
\]

and we find

\[
R_i = (\beta_i - 1) \frac{\partial v_3 \cdots \partial v_p \left[ f_2 \cdots f_p \prod_{j>1} (w_i - v_j) \Delta_1 \right]}{f_2 \cdots f_p \prod_{j>1} (w_i - v_j) \Delta_1} \tag{C4}
\]

where \( f_k = \prod_{i=0}^{q} (w_i - v_k)^{1-\beta_i} \) and \( \Delta_1 = \prod_{1<i<j} (v_i - v_j) \).

We now extract the poles in \( v_2 \) and write

\[
A = \sum_{\{i_1, i_2\} = 0}^{q} \frac{R_{i_1, i_2}}{(w_{i_1} - v_1)(w_{i_2} - v_2)} \tag{C5}
\]

We find

\[
R_{i_1, i_2} = (\beta_{i_1} - 1)(\beta_{i_2} - 1 - \delta_{i_1, i_2}).
\]

\[
\frac{\partial v_3 \cdots \partial v_p \left[ f_3 \cdots f_p \prod_{j>2} (w_{i_1} - v_j)(w_{i_2} - v_j) \Delta_2 \right]}{f_3 \cdots f_p \prod_{j>2} (w_{i_1} - v_j)(w_{i_2} - v_j) \Delta_2} \tag{C6}
\]

where \( \Delta_2 = \prod_{2<i<j} (v_i - v_j) \). We can proceed and extract successively all the poles in \( v_i \); the result can be written in the following way: given \((q+1)\) complementary (and possibly empty) sets \( I_0, I_1, \ldots, I_q \) such that

\[
I_r \cap I_s = \Phi
\]

\[
\bigcup_{r=0}^{q} I_r = (1, \ldots, p) \tag{C7}
\]

then

\[
A = \sum_{\{I_0, \ldots, I_q\}} \prod_{r=0}^{q} \left\{ \frac{\Gamma(\beta_r)}{\Gamma(\beta_r - |I_r|)} \prod_{j \in I_r} \frac{1}{w_r - v_j} \right\} \tag{C8}
\]

where \(|I_r|\) is the number of elements in \( I_r \).
Let us write, as an example, the case where \( \beta = p \) integer and where we have only two variables \( w_0 \) and \( w_1 \). Then,

\[
A = (p - 1) (p - 1)! \sum_{I \subset \{1, \ldots, p\}} C_{p-2}^{[I]-1} \prod_{i \in I} \frac{1}{w_0 - v_i} \prod_{i \not\in I} \frac{1}{w_1 - v_i} \tag{C9}
\]

In the rest of the appendix, we show that \( A \) is the ratio of two determinants, and we develop a recurrent construction satisfied by expressions of the same type of \( A \).

We define successively the following symbols: given the symbol \([u] = [u]_0\) attached to a given variable \( u \), and the symbols

\[
[u]_n = n u^{n-1} + u^n [u] \tag{C10}
\]

then, we define the completely symmetric symbols

\[
[v_1, \ldots, v_p] = \frac{1}{\Delta} \det \begin{bmatrix}
[v_1]_0 & [v_2]_0 & \cdots & [v_p]_0 \\
[v_1]_1 & [v_2]_1 & \cdots & [v_p]_1 \\
\vdots & \vdots & \ddots & \vdots \\
[v_1]_{p-1} & [v_2]_{p-1} & \cdots & [v_p]_{p-1}
\end{bmatrix} \tag{C11}
\]

where \( \Delta = \left(-\right)^{n(n-1)/2} \prod_{i<j} (v_i - v_j) \) is the Vandermonde determinant. Clearly, these symbols can also be defined recursively by

\[
[v, w] = \frac{(1 + v [v]) [w]}{v - w} + \frac{(1 + w [w]) [v]}{w - v}
\]

\[
[u, v, w] = \frac{(2u + u^2 [u]) [v, w]}{(u - v) (u - w)} + \text{circ.perm.}
\]

\[
\vdots
\]

\[
[v_1, \ldots, v_p] = \frac{((p - 1) v_1^{p-2} + v_1^{p-1} [v_1]) [v_2, \ldots, v_p]}{\Pi_{i=2}^{p} (v_1 - v_i)} + \text{circ.perm.} \tag{C12}
\]

To illustrate this construction, we give two examples. First, let us define

\[
[u] = \frac{\beta}{u} \tag{C13}
\]
then, it is easy to construct the different symbols; using the properties of determinants, we obtain

\[ [v_1, \ldots, v_p] = \Gamma(\beta + p) \frac{1}{v_1 \ldots v_p} \]  

(C14)

Moreover, since

\[ [v]_n = \frac{\partial v(\beta + n)}{v^\beta} \]

we have

\[ [v_1, \ldots, v_p] = \frac{\partial v_1 \ldots \partial v_p \left[ v_1^\beta \ldots v_p^\beta \prod_{i<j} (v_i - v_j) \right]}{v_1^\beta \ldots v_p^\beta \prod_{i<j} (v_i - v_j)} \]  

(C15)

As a second example, we define

\[ [u] = \sum_{i=0}^{q} \frac{\beta_i - 1}{w_i - u} \]  

(C16)

Then, from

\[ [v]_n = \frac{\partial v \left[ v^n \prod_{i=0}^{q} (w_i - v)^{1-\beta_i} \right]}{\prod_{i=0}^{q} (w_i - v)^{1-\beta_i}} \]  

(C17)

and the properties of determinants, we find that

\[ [v_1, \ldots, v_p] = A \]  

(C18)

as given in (C1) and (C8).

6 Appendix D

We are interested in the structure of the integrals

\[ I_{P(z)} = \frac{1}{2\pi i} \oint_{C_0} dz \frac{P(z)}{(w_0 - z)^{\beta_0} \prod_{j=1}^{q} (z - w_j)^{\beta_j}} \]  

(D1)
where $P(z)$ is a polynomial and where the sum over the variables $\beta$ is large enough to make the integral convergent along the cut $C_0 = [w_0, +\infty]$. To simplify the writing of this appendix, we keep abusively a $z$ dependence on the left hand side of (D1). From

$$
\frac{1}{2i\pi} \oint_{C_0} dz \frac{\partial}{\partial z} \left\{ \frac{z^n}{(w_0 - z)^{\beta_0 - 1} \prod_{j=1}^{q} (z - w_j)^{\beta_j - 1}} \right\} = 0 \quad (D2)
$$

we obtain the relation

$$
- n I_{z^{n-1}} \prod_{j=0}^{q} (z-w_j) + \sum_{j=0}^{q} (\beta_j - 1) I_{z^n \prod_{\beta_j} (z-w_i)} = 0 \quad (D3)
$$

In the special case where $n=0$, we have

$$
\sum_{j=0}^{q} (\beta_j - 1) I_{\prod_{\beta_j} (z-w_i)} = 0 \quad (D4)
$$

Equation (D4) shows that the function $I_{z^q}$ is linearly dependent of the functions $I_{z^p}$ for $0 \leq p < q$. Also, for $n > 0$, equation (D3) shows that the functions $I_{z^p}$ for $p > q$ can be decomposed on the basis of the $q$ functions $I_{z^p}$ ($0 \leq p < q$). We have constructed a $q$ dimensional vectorial space $E_q$.

The decomposition of the functions $I_{P(z)}$ on the basis of functions $I_{z^p}$ ($0 \leq p < q$) is particularly simple if the polynomial $P(z)$ is of degree $< q$: it becomes particularly difficult otherwise because of the relations (D3-4). We are going to introduce other basis for the vectorial space $E_q$ which are specially adapted when the polynomial $P(z)$ is known from its roots $P(z) = \prod_i (z-v_i)$. Let us give ourselves $q$ different values of the variable $v$ say $v_1, \ldots, v_q$; then, we define a basis of $E_q$ as a set of $q$ independent functions

$$
I_{(j)} = I_{\prod_{\beta_j} (z-w_i)} \text{ for } j = 1, \ldots, q \quad (D5)
$$

The transformation from the basis $I_{z^p}$ to the basis (D5) can be written as

$$
I_{z^p} = \sum_{j=1}^{q} \frac{v_j^p}{\prod_{i \neq j} (v_j - v_i)} I_{(j)} \quad (D6)
$$

and consequently, if $P_{q-1}(z)$ is a polynomial in $z$ of degree at most $q-1$, then

$$
I_{P_{q-1}(z)} = \sum_{j=1}^{q} \frac{P_{q-1}(v_j)}{\prod_{i \neq j} (v_j - v_i)} I_{(j)} \quad (D7)
$$
As we already mentioned, the decomposition of $I_{z^q}$, $I_{z^{q+1}}$...is more elaborate; let us give the example of $I_{z^q}$. From (D4), we have

$$D_{q+1}I_{z^q} + \sum_{j=0}^{q} (\beta_j - 1) I_{\prod_{i \neq j} (z-w_i)} = 0 \quad (D8)$$

where $D_p = \sum_{j=0}^{q} \beta_j - p$. Since the polynomial $\prod_{i \neq j} (z-w_i) - z^q$ is of degree q-1 in z, we may apply (D7); we obtain

$$I_{z^q} = \sum_{r=1}^{q} \left[ P_q(v_r) - \frac{1}{D_{q+1}} \sum_{j=0}^{q} (\beta_j - 1) \prod_{i \neq j} (v_r - w_i) \right] I_{(r)} \quad (D9)$$

Given a polynomial $P_q(z)$ of degree q in z (with the coefficient of $z^q$=1), we may write

$$I_{P_q(z)} = \sum_{r=1}^{q} \left[ P_q(v_r) - \frac{1}{D_{q+1}} \sum_{j=0}^{q} (\beta_j - 1) \prod_{i \neq j} (v_r - w_i) \right] I_{(r)} \quad (D10)$$

If moreover, the choice of the variables $v_1, ..., v_q$ which defines the basis $I_{(r)}$ coincide with the roots of the polynomial $P_q(z)$, that is $P_q(v_r) = 0$ for $r = 1, ..., q$, then

$$I_{\prod_{i=1}^{q} (z-v_i)} = \frac{(-1)^{q+1}}{D_{q+1}} \sum_{r=1}^{q} \prod_{i=0}^{q} \prod_{s \neq r} (v_r - v_s) \left[ v_r \right] I_{(r)} \quad (D11)$$

where the symbol $[v]$ is defined in Appendix C (C6).

We now extend the above construction to polynomials of degree larger than q. Given a set of $p > q$ values $v_1, ..., v_p$ of a variable $v$, we define a set of p functions as

$$I_{(r)} = I_{\prod_{i \neq r} (z-v_i)} \quad (D12)$$

Clearly these p functions are necessarily dependent in the vectorial space $E_q$. Nevertheless, the relation (D6) remains valid

$$I_{z^j} = \sum_{r=1}^{p} \frac{v_r^j}{\prod_{i \neq r} (v_r - v_i)} I_{(r)} \quad for \ j = 0, 1, ..., p-1 \quad (D13)$$
In order to express $I_{z^p}$, we use (D3) with $n=p-q$; we get

$$D_{p+1} I_{z^p} - (p - q) I_{z^p}^{-q-1} \left[ \prod_{j=0}^{q} (z - w_j) - z^{q+1} \right] + \sum_{j=0}^{q} \left( \beta_j - 1 \right) I_{z^p}^{-q} \left[ \prod_{i \neq j} (z - w_i) - z^q \right] = 0$$

so that, from (D13)

$$I_{z^p} = \sum_{r=1}^{p} \left\{ v_r^p + \frac{(-)^{q+1}}{D_{p+1}} \prod_{j=0}^{q} (w_j - v_r) \left[ (p - q) v_r^{p-q-1} + v_r^{p-q} [v_r] \right] \right\},$$

$$\prod_{i \neq r} (v_r - v_i)$$

(D14)

Now, given a polynomial $P(z) = \prod_{i=1}^{p} (z - v_i)$ where the $v$’s are supposed to be all different, we may write

$$I_{\prod_{i=1}^{p} (z-v_i)} = \frac{(-)^{q+1}}{D_{p+1}} \prod_{j=0}^{q} (w_j - v_r) \prod_{i \neq r} (v_r - v_i) \left[ v_r \right]_{p-q} \prod_{i \neq r} (v_r - v_i)$$

(D15)

where $[v_r]_n$ is defined in Appendix C (C10).

Let us illustrate this result on the example where $p=q+1$. In that case the $q+1$ functions $\prod_{r} (z - v_i)$ can themselves be decomposed according to (D11) generating a set of $q(q+1)/2$ functions

$$\prod_{i=1}^{q+1} (z - v_i)$$

(D16)

When transforming (D15) in terms of $\prod_{r,s} (z-v_i)$, we obtain two contributions depending whether we express $\prod_{r} (z-v_i)$ or $\prod_{s} (z-v_i)$ in terms of $\prod_{r,s} (z-v_i)$; these two contributions construct the combination

$$\frac{[v_r]_1 [v_s]_0}{v_r - v_s} + \frac{[v_s]_1 [v_r]_0}{v_s - v_r} = [v_r, v_s]$$

(D17)

where $[v_r, v_s]$ is defined in Appendix C (C12). The expression (D15) becomes

$$I_{\prod_{i=1}^{q+1} (z-v_i)} = \frac{(-)^{2q+2}}{D_{q+2} D_{q+1}} \sum_{r < s} \prod_{j=0}^{q} \left[ (w_j - v_r) (w_j - v_s) \right] \prod_{i \neq r,s} \left[ (v_r - v_i) (v_s - v_i) \right] [v_r, v_s] \prod_{r,s} (z-v_i)$$

(D18)
We insist again on the fact that this decomposition is not a decomposition over a basis of \( E_q \) but it is exactly the kind of decomposition we need to prove the conjecture in the case \( q-1 \leq p \).

We now proceed by recurrence for any \( p > q \). We suppose that for \( p-1 \), we have

\[
I_{\prod_{i=1}^{p-1} (z-v_i)} = \left( -\right)^{(p-q)(q+1)} \frac{\Gamma \left( \sum_{j=0}^q \beta_j - p \right)}{\Gamma \left( \sum_{j=0}^q \beta_j - q \right)} \sum_{J_{q-1}} \prod_{j=0}^q \prod_{k \notin J_{q-1}} (w_j - v_k) \left[ \bigcup_{k \notin J_{q-1}} (v_k) \right] \overline{I_{\prod_{k \notin J_{q-1}} (v_k)}} (D19)
\]

where \( J_{q-1} \) is a set of q-1 indices \((j_1, ..., j_{q-1}) \subset (1, ..., p-1) \) and \([\bigcup_{k \notin J_{q-1}} (v_k) \] is a notation for the symbol of Appendix C (C11) where the indices \( k \) belong to the complement of \( J_{q-1} \) in \( (1, ..., p-1) \). The notation \( \overline{I_{(J_{q-1})}} \) means \( \overline{I_{(J_{q-1})}} \).

Now, we consider (D15) and we replace \( I_{(r)} \) by its expression (D19); we get

\[
I_{\prod_{i=1}^{p} (z-v_i)} = \left( -\right)^{(p-q+1)(q+1)} \frac{\Gamma \left( \sum_{j=0}^q \beta_j - p - 1 \right)}{\Gamma \left( \sum_{j=0}^q \beta_j - q \right)} \sum_{r=1}^p \prod_{r=0}^q (w_j - v_r) \prod_{i \neq r} (v_r - v_i) \left[ v_r \right]_{p-q} \cdot \\
\cdot \sum_{J_{q-1}} \prod_{j=0}^q \prod_{k \notin J_{q-1}} (w_j - v_k) \prod_{s \notin k \neq r} (v_s - v_i) \left[ \bigcup_{k \notin J_{q-1}} (v_k) \right] \overline{I_{J_{q-1}}} (D20)
\]

where \( J_{q-1} = (j_1, ..., j_{q-1}) \subset \{(1, ..., p) \setminus \{r\}\} \). We now invert the sum over \( r \) and the sum over \( J_{q-1} \),

\[
I_{\prod_{i=1}^{p} (z-v_i)} = \left( -\right)^{(p-q+1)(q+1)} \frac{\Gamma \left( \sum_{j=0}^q \beta_j - p - 1 \right)}{\Gamma \left( \sum_{j=0}^q \beta_j - q \right)} \cdot \\
\cdot \sum_{J_{q-1}} \prod_{j=0}^q \prod_{k \notin J_{q-1}} (w_j - v_k) \prod_{s \notin k \neq r} (v_s - v_i) \cdot \sum_{r \notin J_{q-1}} \prod_{i \notin J_{q-1}} (v_r - v_i) \left[ v_r \right]_{p-q} \left[ \bigcup_{k \notin J_{q-1}} (v_k) \right] \overline{I_{J_{q-1}}} (D21)
\]
where \( J_{q-1} = (j_1, \ldots, j_{q-1}) \subset (1, \ldots, p) \). By Appendix C (C12), the sum over \( r \) in the second line of (D21) is nothing but \( \left[ \bigcup_{k \notin J_{q-1}} (v_k) \right] \). We have obtained (D19) where \( p \) has been replaced by \( p+1 \); this ends the proof of the recurrence.

We now rewrite (D21) in a way which is directly applicable to section 2 eq.(2.20-21). We have

\[
\frac{1}{\prod_{s \notin J_{q-1}} \prod_{i \in J_{q-1}} (v_s - v_i)} = (-)^{(2p-q)(q-1)/2 + \delta_{J_{q-1}}/2} \frac{\prod_{k<l} (v_k - v_l) \prod_{k,l \notin J_{q-1}} (v_k - v_l)}{\prod_{k<l} (v_k - v_l)} \tag{D22}
\]

where \( \delta_{J_{q-1}} = \sum_{k=1}^{q-1} j_k - q + 1 \). From the definition of the symbol \( \left[ \bigcup_{k \notin J_{q-1}} (v_k) \right] \) as given in Appendix C (C18,C1), we may write

\[
\sum_{J_{q-1}} (-)^{\delta_{J_{q-1}}} \left[ \prod_{j \notin J_{q-1}} (\partial_{v_j}) \right] \left[ \prod_{i=0}^{q} \prod_{j \notin J_{q-1}} (w_i - v_j)^{1 - \beta_i} \prod_{k<l} (v_k - v_l)^{-\beta_i} \right] \prod_{k,l \in J_{q-1}} (v_k - v_l) \frac{\prod_{j \in J_{q-1}} (z - v_j)}{2i\pi} \oint_{C_0} dz \prod_{j \in J_{q-1}} (w_0 - z)^{\beta_0} \prod_{j=1}^{q} (z - w_j)^{\beta_j} \nn
= (-)^{(q-1)(q-2)/2} \frac{\Gamma \left( \sum_{j=0}^{q} \beta_j - q \right)}{\Gamma \left( \sum_{j=0}^{q} \beta_j - p - 1 \right)} \prod_{k<l} (v_k - v_l) \frac{\prod_{j \in J_{q-1}} (z - v_j)}{2i\pi} \oint_{C_0} dz \prod_{j \in J_{q-1}} (w_0 - z)^{\beta_0} \prod_{j=1}^{q} (z - w_j)^{\beta_j} \tag{D23}
\]

which is the relation used at the end of section 2.
1 F. Calogero, J. Math. Phys. 10, 2191, 2197 (1969),
   B. Sutherland, J. Math. Phys. 12, 246 (1971); Phys. Rev. A4, 2019 (1971);
   5, 1372 (1972).
2 P. J. Forrester, Nucl. Phys. B388, 671 (1992)
3 F. J. Dyson, J. Math. Phys. 3, 140, 157 (1962),
   B. D. Simons, P. A. Lee, B. L. Al'tshuler, Nucl. Phys. B409, 487 (1993),
   K. B. Efetov, Adv. Phys. 32, 53 (1983).
4 F. D. M. Haldane, M. R. Zirnbauer, Phys. Rev. Lett. 71, 4055 (1993),
   M. R. Zirnbauer, F. D. M. Haldane, Phys. Rev. B52, 8729 (1995).
5 F. Lesage, V. Pasquier, D. Serban, Nucl. Phys. B435, 585 (1995).
6 Z. N. C. Ha, Phys. Rev. Lett. 73, 1574 (1994); Nucl. Phys. B435, 604 (1995).
7 D. Serban, F. Lesage, V. Pasquier, Nucl. Phys. B466, 499 (1996).
8 D. V. Khveshchenko, Int. J. Mod. Phys. B9, 1639 (1995).
9 K. Mimachi, Y. Yamada, Comm. Math. Phys. 174, 447 (1995),
   H. Awata, Y. Matsuo, S. Odake, J. Shiraishi, Nucl. Phys. B449, 347 (1995).
10 S. N. M. Ruijsenaars, H. Schneider, Ann. Phys. 170, 370 (1986),
   S. N. M. Ruijsenaars, Comm. Math. Phys. 110, 191 (1987),
   H. Konno, Nucl. Phys. B473, 579 (1996).
11 I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed. (Clarendon Press, 1995).
12 A. L. Dixon, Proc. London Math. Soc. (2), 3, 206 (1905)
13 H. Bateman, Higher Transcendental functions, Vol 1, p.85, eq.(13),
   E. B. Elliot, Messenger of Math, 33, 31 (1904).