Dissipativity-based \( \mathcal{L}_2 \) gain-scheduled static output feedback design for rational LPV systems

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Abstract—This paper proposes the design of gain-scheduled static output feedback controllers for the stabilization of continuous-time linear parameter-varying systems with \( \mathcal{L}_2 \)-gain performance. The system is transformed into the form of a differential-algebraic representation which allows dealing with the broad class of systems whose matrices can present rational or polynomial dependence on the parameter. The proposed approach uses the definition of strict QSR-dissipativity, Finsler’s Lemma, and the notion of linear annihilators to formulate conditions expressed in the form of polytopic linear matrix inequalities for determining the gain-scheduled static output feedback control for system stabilization. One of the main advantages of the strategy is that it provides a simple design solution in a non-interactive manner. Furthermore, no restriction on the plant output matrix is imposed. Numerical examples highlight the effectiveness of the proposed method.

Index Terms—Linear parameter-varying systems, gain-scheduling, static output feedback, dissipativity, differential-algebraic representation.

I. INTRODUCTION

Static output feedback (SOF) design is a very important problem in control theory. In some cases, a feedback controller that uses all system state information can not be applied due to the impossibility of measuring all the states of the system. Then, an output feedback controller using only the available states has to be designed [1]. The SOF design is a challenging problem since its mathematical formulation leads to non-convex conditions which can not be solved by semidefinite programming (SDP) [2]. Even though there are many strategies that provide solutions to this problem, it is known that a definitive solution is yet to appear [1]. See [1]–[4] for an overview of the subject.

In the control of linear parameter-varying (LPV) systems, the gain-scheduling technique has received significant attention in the last decades [5], [6]. The gain-scheduling approach is based on the measurement of the time-varying parameter which adjusts the controller gains for the complete range of parameter variation [7], [8]. It is well known that a gain-scheduling approach provides less conservative results for LPV systems compared with others control methods [6], [7], [9].

In the literature, there are few works that use SOF gain-scheduling techniques for LPV systems. Methods for the gain-scheduled SOF (GS-SOF) design for continuous-time LPV systems have been published in [10], [11]. Recently, a GS-SOF design for LPV systems was developed in a two stage method, where it is necessary designing a non-scheduled static feedback in the first stage [12]. In [13], a GS-SOF non-iterative design procedure with \( H_2/H_\infty \) performance has been developed. However, as it is common in the field, these strategies consider the polytopic approach, then the LPV system can only be affine on the parameter. Few works consider a polynomial or rational dependence on the parameter. A paper considering this type of dependence is [14], where a state feedback design was developed for rational LPV systems. In [15], gain-scheduled dynamic output feedback design with \( H_2 \) performance has been proposed for rational LPV systems. In [16], a procedure for designing dynamic gain-scheduled controllers for rational LPV systems in the descriptor form was developed. Recently, [17] proposes a novel method to compute the \( \mathcal{L}_2 \)-gain for rational LPV systems, however no control law is designed. Many solutions to the gain-scheduled static output feedback design for discrete-time LPV systems have also been recently developed [18]–[21]. However, none of them consider rational dependence on the parameter.

Dissipativity theory was introduced some decades ago and has been extensively used in stability analysis and control systems design [22], [23]. Recently, [24] proved, under mild assumptions, that a specific case of dissipativity called strict QSR-dissipativity is a necessary and sufficient condition for SOF stabilizability of LTI systems. In [25], the same concept of dissipativity has been used to the linear SOF design for uncertain nonlinear systems.

In this paper, we develop a strategy based on some ideas presented in [24]. Our strategy provides sufficient conditions for the design of a gain-scheduled SOF that stabilizes continuous-time LPV systems with rational or polynomial dependence. Furthermore, we also consider the influence of external signals on the system and propose the stabilization with \( \mathcal{L}_2 \)-gain performance. The strategy here uses strict QSR-dissipativity, Finsler’s Lemma and a differential algebraic representation (DAR) for the LPV system to obtain polytopic LMI conditions for the SOF design. The main contribution of this paper is to consider others types of dependencies on the parameter, such as rational, or polynomial, since most works only consider an affine dependence. It is important to highlight that no restriction on the output plant matrix is imposed. Finally, differently from most papers dealing with SOF design, the proposed strategy is non-iterative and is solved in only one stage.
This paper is organized as follows. In Section 2, we present important theoretical preliminaries for the formulation of developed conditions. In Section 3, the proposed strategy for gain scheduling SOF stabilization is introduced. Extension of the proposed method to the $L_2$-gain performance case is realized within Section 4. In Section 5, some numerical examples are provided to illustrate the efficiency of the strategy. Finally, in Section 6, we have the conclusion of the paper.

**Notation.** For a matrix $H \in \mathbb{R}^{n \times m}$, $H^\top \in \mathbb{R}^{m \times n}$ means its transpose. Operators $H \succ 0$ and $H \succeq 0$ mean that the symmetric matrix $H$ is positive definite or positive semidefinite, respectively. $He\{A\}$ stands for $A + A^\top$. $I$, $0$ and $J$ denote all-ones, identity, null, and exchange matrices (i.e., anti-diagonal matrix with ones) of appropriate dimensions, which can be explicitly presented when relevant. For a symmetric block matrix, the symbol $*$ stands for the transpose of the blocks outside the main diagonal block. Additionally, for matrices $A$ and $H$, $\text{diag}(A, H)$ corresponds to the block-diagonal matrix. $\triangledown$ represents the gradient function. $\mathbb{R}^+$ denotes the set of elements $\beta \in \mathbb{R}$ such that $\beta \geq 0$. Finally, $\|f\|_2$ is used to denote the $l_2$ norm of $f(t) : \mathbb{R}^+ \to \mathbb{R}^n$, given by $\int_0^T f^\top(\tau)f(\tau)d\tau$.

II. Preliminaries

A. LPV systems

Consider an LPV system of the form

$$\begin{cases}
\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t),

y(t) = C(\rho)x(t),
\end{cases}$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measured output. Moreover, $\rho \in \Omega \subset \mathbb{R}^r$ is a vector of time-varying parameters and $A(\rho) \in \mathbb{R}^{n \times n}$, $B(\rho) \in \mathbb{R}^{n \times m}$, $C(\rho) \in \mathbb{R}^{p \times n}$ are polynomial or rational matrices on $\rho$.

**Assumption 1.** The elements of the parameters vector are bounded and the vector $\rho$ lies inside a polytope $\Omega$ of $N = 2^r$ vertices, where $r$ is the number of elements of $\rho$. The polytope $\Omega$ is given by

$$\Omega = \{\alpha(\rho(t)) \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0; i = 1, \ldots, N\},$$

(2)

where any point inside $\Omega$ can be represented by the convex combination of its vertices [26].

B. Differential-Algebraic Representation - DAR

The LPV system (1) can be described by a Differential Algebraic Representation (DAR), as presented in [27]. A Differential Algebraic Representation is given by

$$\begin{cases}
\dot{x} = A_1x + A_2\pi + A_3u,

y = C_1x + C_2\pi,

0 = Y_1(\rho)x + Y_2(\rho)\pi + Y_3(\rho)u,
\end{cases}$$

(3)

where $\pi(x, \rho, u) \in \mathbb{R}^{n_x}$ is an auxiliary vector that contains all nonlinear terms of (1) depending on $\rho$. $A_1 \in \mathbb{R}^{n \times n}$, $A_2 \in \mathbb{R}^{n \times n_x}$, $A_3 \in \mathbb{R}^{n \times m}$, $C_1 \in \mathbb{R}^{p \times n}$, $C_2 \in \mathbb{R}^{p \times n_x}$ are constant matrices and $Y_1(\rho) \in \mathbb{R}^{n_x \times n}$, $Y_2(\rho) \in \mathbb{R}^{n_x \times n}$, $Y_3(\rho) \in \mathbb{R}^{n_x \times m}$ are affine matrices of $\rho$.

The DAR of a system is not unique and a state-space representation (1) is well-posed in its DAR form if $Y_2(\rho)$ is invertible since from (3) we have

$$\begin{align}
\pi(x, u, \rho) &= Y_2^{-1}[\Sigma_1 x - \Sigma_3 u], \\
\dot{x} &= (A_1 - A_2 Y_2^{-1} Y_1)x + (A_3 - A_2 Y_2^{-1} Y_3)u.
\end{align}$$

(4)

**Remark 1.** The DAR [3] is an alternative and exact representation of system (1). It is important to highlight that it can model the whole class of LPV systems with rational dependence on the parameters without singularities at the origin [27]. A general procedure to obtain the DAR of the LPV system can be found in [27].

The motivation to represent the LPV system in a DAR form is that, in [3], the system matrices $A_i$ and $C_i$ are constant and the dependency on $\rho$ is transferred to the auxiliary matrices $Y_i(\rho)$. Moreover, the auxiliary matrices depend only affinely on $\rho$, which allows the use of techniques leading to convex design conditions expressed in the form of LMIs in this work.

C. Finsler’s Lemma

A version of Finsler’s Lemma from [28] is presented in the following Lemma.

**Lemma 1.** Consider $W \subseteq \mathbb{R}^{n_z}$ a given polytopic set, and let $Q_d : W \to \mathbb{R}^{n_w \times n_w}$ and $C_d : W \to \mathbb{R}^{n_z \times n_t}$ be given matrix functions, with $Q_d$ symmetric. Then, the following statements are equivalent

i) $\forall w \in W$ the condition that $z^\top Q_d(w)z > 0$ is satisfied $\forall z \in \mathbb{R}^{n_z} : C_d(w)z = 0$.

ii) $\forall w \in W$ there exists a certain matrix function $L : W \to \mathbb{R}^{n_w \times n_z}$ such that $Q_d(w) + L(w)C_d(w) + C_d(w)^\top L(w)^\top \succ 0$.

If $C_d$ and $Q_d$ are affine functions of $w$, and $L$ is a constant matrix to be determined, then ii) becomes a polytopic LMI condition which is sufficient for i). Lemma 1 also applies for testing negative definite functions. Clearly, $C_d$ is an annihilator of the vector $z$, which is not unique. Further details and a systematic procedure for determining linear annihilators are presented in [28] and [29].

D. Dissipativity

Consider an LTI system such as

$$\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t),

y(t) = Cx(t).
\end{cases}$$

(5)

System (5) is said to be dissipative if it is completely reachable and there exists a nonnegative storage function $V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$ and $V \in C^1$, and a locally integrable supply rate $r(u(t), y(t))$ such that $\dot{V} \leq r(u, y)$ [30]. Some definitions of dissipativity can be found in [23]. In this work, we use the definition of strict QSR-dissipativity given below.
Definition 1. A system is said to be strictly QSR-dissipative along all possible trajectories of (6) starting at $x(0)$, for all $t \geq 0$, if there exists $T(x) > 0$ such that

$$
\dot{V}(x) + T(x) \leq y^T Q y + 2y^T S u + u^T R u,
$$

where $S \in \mathbb{R}^{p \times m}$ is real and $Q \in \mathbb{R}^{p \times p}, R \in \mathbb{R}^{m \times m}$ are real and symmetric.

From a practical point of view, a dissipative system stores only a fraction of the energy supplied to it through $v(u, y)$ and only a fraction of its stored energy $V(x)$ can be delivered to its surroundings. Definition 1 can be related with Lyapunov stability. If a system is strictly QSR-dissipative with $V(x) > 0$ and $Q \preceq 0$, then the free system is asymptotically stable [30].

In this work, we consider quadratic Lyapunov functions

$$V(x) = x^T P x, \quad P > 0,$$

where $P \in \mathbb{R}^{n \times n}$, and a quadratic $\rho$-parameter dependent function $T(x, \rho)$ that can be defined in a polytopic domain

$$T(x, \rho) = x^T H(\rho) x, \quad H(\rho) = \sum_{i=1}^{N} \alpha_i H_i, \quad H_i > 0,$$

where $H_i \in \mathbb{R}^{n \times n}$. Also, we consider $Q$ and $S$ $\rho$-parameter dependent matrices in a polytopic domain

$$Q(\rho) = \sum_{i=1}^{N} \alpha_i Q_i, \quad S(\rho) = \sum_{i=1}^{N} \alpha_i S_i,$$

where $Q_i \in \mathbb{R}^{p \times p}$ and $S_i \in \mathbb{R}^{p \times m}$. Thus, considering (8)-(9)-(10), a version of the dissipativity condition [11] for the case of LPV systems [11] in a DAR form such as [11] is given by

$$t_d(x, u, \rho) = \nabla V^T [A_1 x + A_2 \pi x + A_3 u] + x^T H(\rho) x - y^T Q(\rho) y - 2y^T S(\rho) u - u^T R u \leq 0.$$

The system (11) is said to be robust strictly QSR-dissipative if (11) holds for all $\rho \in \Omega$.

III. GS-SOF STABILIZATION

The strategy proposed in this work consists in connecting Lemma 1 and dissipativity condition (11), assuming parameter dependent matrices on the supply rate and on function $T$. In order to apply Lemma 1 we consider the following notation

$$w = \rho(t), \quad W = \Omega, \quad n_x = r,$n_y = n_\pi, \quad n_u = n + n_\pi + m.$

Next, observe that $t_d(x, u, \rho)$ from (11) can be decomposed in the following manner

$$t_d(x, u, \rho) = \pi_d Y(\rho) \pi_d,$$

$$\pi_d = [x^T \pi^T u^T]^T, \quad Y(\rho) = \sum_{i=1}^{N} \alpha_i Y_i,$$

where $Y_i$ is a symmetric and linear matrix on all the unknown coefficients of $(Q_i, S_i, R, P)$. In addition, consider

$$C_d(\rho) = \begin{bmatrix} Y_1(\rho) & Y_2(\rho) & Y_3(\rho) \end{bmatrix} \quad (13)$$
as a linear annihilator of $\pi_d$. Since matrices $T(\rho)$ are affine on the parameter, these matrices can be represented in a polytopic domain, leading to the following representation of $C_d(\rho)$

$$C_d(\rho) = \sum_{i=1}^{N} \alpha_i C_{d_i} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} Y_{1_i} & Y_{2_i} & Y_{3_i} \end{bmatrix}.$$

The following theorem provides a solution for the design of a gain-scheduled SOF that stabilizes LPV systems.

Theorem 1. Let $\Omega$ be a polytope of $\rho(t)$ described by (2) and $C_d(\rho)$ a linear annihilator of $\pi_d$ described by (14). Given a scalar $\beta$, assume there exist symmetric matrices $P > 0, H_i > 0, R > 0, Q_i$, and matrices $S_i, L \in \mathbb{R}^{n \times n}$, such that

$$Y_i + LC_{d_i} + C_{d_i}^T L^T < 0,$$

$$X_{d_i} + L C_{s_i} + C_{s_i}^T L_s^T < 0,$$

for $i = 1, \ldots, N$, where $L_s = [\beta I - I]^T, C_s = [S_i^T R_i]$,

$$X_{d_i} = \begin{bmatrix} Q_i & S_i \end{bmatrix} R_i,$$

and

$$Y_i = \begin{bmatrix} PA_1 + A_1^T P - C_{d_i}^T Q_i C_{d_i} & * & * \\ (PA_2 - C_{d_i}^T Q_i C_{d_i})^T & -C_{d_i}^T Q_i C_{d_i} & * \\ (PA_3 - C_{d_i}^T S_i) & -S_i & -R_i \end{bmatrix}.$$

Then system (11) is robust strictly QSR-dissipative for all $\rho(t) \in \Omega$ and the gain-scheduled SOF

$$u = K(\rho) y, \quad K(\rho) = \sum_{i=1}^{N} \alpha_i K_i, \quad K_i = -R_i^{-1} S_i^T,$$

asymptotically stabilizes (11) for all $\rho \in \Omega$, around the origin.

Proof. First, consider the satisfaction of condition (15). Since $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$ for $i = 1, \ldots, N$, note that, by multiplying all the terms of (15) by $\alpha_i$ and summing them up from $i = 1$ to $i = N$, we obtain

$$\sum_{i=1}^{N} \alpha_i (Y_i + \text{He}\{LC_{d_i}\}) = Y(\rho) + \text{He}\{LC_{d}(\rho)\} < 0.$$

Since $C_d(\rho)$ is an annihilator of $\pi_d$, from Lemma 1 satisfaction of (18) implies that $\pi_d Y(\rho) \pi_d = t_d(x, u, \rho) < 0$ is also satisfied for all $\rho \in \Omega$ and $\forall \pi_d \neq 0$, where $t_d(x, u, \rho)$ was first defined in (11). Thus, system (11) is robust strictly QSR-dissipative for all $\rho \in \Omega$. In addition, note that as $H(\rho) > 0$, fulfilling

$$y^T Q(\rho) y + 2y^T S(\rho) u + u^T R u \leq 0$$

is sufficient to guarantee $\nabla V^T [A_1 x + A_2 \pi x + A_3 u] < 0$, which ensures the asymptotic stability of system (11) about the origin. Considering a vector $\zeta = [y^T u^T]^T$, condition (19) can be rewritten as $\zeta^T X_d(\rho) \zeta \leq 0$, where

$$X_d(\rho) = \begin{bmatrix} Q(\rho) & S(\rho) \\ S^T(\rho) & R \end{bmatrix}.$$
Let us recall that, we consider the gain-scheduled static output feedback given by

\[ u = K(\rho)y = -R^{-1} \sum_{i=1}^{N} \alpha_i S_i^T y = -R^{-1} S^T(\rho)y. \]  

(20)

By noting that \( C_s(\rho)\zeta = 0 \), with \( C_s(\rho) = [S^T(\rho) R] \), Lemma [1] can be applied. If there exists matrix \( L_s \) such that

\[ X_d(\rho) + \text{He}(L_s C_s(\rho)) < 0, \]  

(21)

then \( \zeta^T X_d(\rho) \zeta < 0 \) for all \( \rho \in \Omega \) and \( \zeta \neq 0 \), thus condition [19] is also satisfied, ensuring \( \nabla V^T [A_1 x + A_2 \pi + A_3 u] < 0 \). Note that by multiplying all the terms of (16) by \( \alpha_i \) and summing them up from \( i = 1 \) to \( i = N \), we obtain

\[ \sum_{i=1}^{N} \alpha_i (X_{d_i} + \text{He}(L_s C_{s_i})) = X_d(\rho) + \text{He}(L_s C_s(\rho)) < 0. \]  

(22)

Therefore, satisfaction of (16) implies fulfillment of (21) and that the system is stabilized by the SOF gain-scheduling given by (17), which completes the proof of Theorem [1].

IV. \( \mathcal{L}_2 \)-gain PERFORMANCE

In this section, we present an extension of the proposed GS-SOF design procedure to the case of \( \mathcal{L}_2 \)-gain performance when the system is affected by external disturbances. The approach presented here is inspired by the framework proposed in [31] for stabilization of an LTI system with \( \mathcal{L}_2 \)-gain performance.

First, consider an LPV system of the form

\[
\begin{align*}
\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + B_w(\rho)w(t), \\
z(t) &= A_z(\rho)x(t) + B_z(\rho)u(t) + D_z(\rho)w(t), \\
y(t) &= C(\rho)x(t) + D(\rho)w(t),
\end{align*}
\]  

(23)

that is the same system [1], with an additional external input \( w(t) \in \mathbb{R}^q \) and a controlled output \( z(t) \in \mathbb{R}^l \). As in [1], all system matrices can present rational or polynomial dependence on \( \rho \). This system in its DAR form is presented below

\[
\begin{align*}
\dot{x} &= A_1 x + A_2 \pi + A_3 u + A_4 w, \\
z &= B_1 x + B_2 \pi + B_3 u + B_4 w, \\
y &= C_1 x + C_2 \pi + C_3 w, \\
0 &= Y_1(\rho)x + Y_2(\rho)\pi + Y_3(\rho)u + Y_4(\rho)w,
\end{align*}
\]  

(24)

where matrices \( B_i \) are also constant matrices. Considering \( u = K(\rho)y \), the closed loop form of this system is given by

\[
\begin{align*}
\dot{x} &= \alpha_1 x + \alpha_2 \pi + \alpha_3 w, \\
z &= B_{\pi} x + B_{\pi} \pi + B_3 w, \\
0 &= \tilde{Y}_1(\rho)x + \tilde{Y}_2(\rho)\pi + \tilde{Y}_3(\rho)w,
\end{align*}
\]  

(25)

where

\[
\begin{align*}
\alpha_1 &= (A_1 + A_3 K(\rho)C_1), \\
\alpha_2 &= (A_2 + A_3 K(\rho)C_2), \\
\alpha_3 &= (A_4 + A_3 K(\rho)C_3), \\
\beta_1 &= (B_1 + B_3 K(\rho)C_1), \\
\beta_2 &= (B_2 + B_3 K(\rho)C_2), \\
\beta_3 &= (B_3 + B_3 K(\rho)C_3), \\
\tilde{Y}_1 &= (Y_1 + Y_3 K(\rho)C_1), \\
\tilde{Y}_2 &= (Y_2 + Y_3 K(\rho)C_2), \\
\tilde{Y}_3 &= (Y_4 + Y_3 K(\rho)C_3).
\end{align*}
\]  

(26)

The gain-scheduled static output feedback control problem with \( \mathcal{L}_2 \)-gain performance is equivalent to finding a control law \( u(t) = K(\rho(t))y(t) \) such that the closed loop (23) is asymptotically stable in the absence of disturbance \( w \) and the \( \mathcal{L}_2 \) norm of \( z \) is bounded such that

\[
||z||_2 \leq \gamma ||w||_2 + \theta.
\]  

(27)

with positive scalars \( \gamma \) and \( \theta \), where \( \theta \) is a bias term. When (27) is ensured, one can say that the system (23) is input to output stable with \( \mathcal{L}_2 \)-gain bounded by \( \gamma \). In order to guarantee asymptotic stability at the same time satisfying relation (27), we have the following sufficient condition [29]

\[
V + \gamma^{-1} z^T z - \gamma w^T w < 0,
\]  

(28)

with function \( V \) defined in (8). Note that by integrating both sides of (28), taking squares roots, and using the fact that \( \sqrt{a+b} \leq a+b \), for \( a, b \in \mathbb{R}^+ \), one arrives at \( ||z||_2 \leq ||w||_2 + \sqrt{V(x(0))} \), i.e., (27) with bias term \( \theta = \sqrt{V(x(0))} \).

Theorem 2. If there exists a scalar \( \gamma > 0 \) such that conditions (15) and (16) of Theorem 1 hold replacing matrices \( (P, A_1, A_2, A_3, C_1, C_2, T_1, T_2, T_3, L, H_1) \) by \( (\bar{P}, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{C}_1, \bar{C}_2, \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{L}, \bar{H}_1) \), respectively, where

\[
\bar{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & \bar{I}_q & 0 \\ 0 & 0 & \bar{I}_l \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & A_4 & 0_{n \times l} \\ 0_{l \times n} & -\bar{H}_{lxq} & 0_{l \times q} \end{bmatrix}, \\
\bar{A}_2 = \begin{bmatrix} A_2 \\ 0_{q \times nx} \end{bmatrix}, \quad \bar{T}_3 = \begin{bmatrix} 0_{q \times m} \\ 0_{l \times n} \end{bmatrix}, \quad \bar{T}_1 = \begin{bmatrix} T_1 \\ 0_{n \times l} \end{bmatrix}, \\
\bar{C}_2 = C_2, \quad \bar{C}_1 = \begin{bmatrix} C_1 \\ C_3 \end{bmatrix}, \quad \bar{T}_3 = \begin{bmatrix} T_2 \\ T_3 \end{bmatrix}, \\
\bar{T}_3 = \begin{bmatrix} T_3 \\ \bar{L} \end{bmatrix} \in \mathbb{R}^{(n+l) \times l}
\]  

(29)

with \( n_q = n + q + l \), then system (23) is robust strictly QSR-dissipative for all \( \rho(t) \in \Omega \), and the gain-scheduled SOF

\[ u = K(\rho)y, \quad K(\rho) = \sum_{i=1}^N \alpha_i K_i, \quad K_i = R^{-1} S_i^T, \]  

(30)

asymptotically stabilizes system (23) for all \( \rho(t) \in \Omega \) with \( \mathcal{L}_2 \)-gain bounded by \( \gamma \).

Proof. First, consider system (1) in its DAR form (3). Considering \( u = K(\rho)y \), we have the Lyapunov condition \( \dot{V}(x) < 0 \) that guarantees asymptotic stability for the closed-loop system (3), which is equivalently expressed as

\[
\begin{bmatrix} x \\ \pi \end{bmatrix}^T H e \begin{bmatrix} P A_1 + PA_3 K(\rho)C_1 \\ A_2 P + C_2 K^T(\rho)A_1 \end{bmatrix} \begin{bmatrix} x \\ \pi \end{bmatrix} < 0.
\]

(31)

Since \( u = K(\rho)y \) and also because of (3), matrix \( [T_1 + T_2 + T_3 K(\rho)C_2] \) is an annihilator of \( [\pi^T \pi]^T \). Thus Lemma [1] can be applied. If there exists matrix \( L_a \) such that

\[
\begin{align*}
He \begin{bmatrix} P A_1 & P A_2 \\ 0 & 0 \end{bmatrix} + & He \begin{bmatrix} P A_3 K C_1 & P A_3 K C_2 \\ 0 & 0 \end{bmatrix} \\
+ & He \begin{bmatrix} L_a [T_1 & T_2] \\ L_a T_3 K C_1 & C_2 \end{bmatrix} < 0,
\end{align*}
\]  

(32)
then \( \beta \) is satisfied for all \( \rho(t) \in \Omega \) and \( [x^T \pi^T]^T \neq 0 \).

On the other hand, asymptotic stability of system \( (25) \) with \( L_2 \)-gain performance is guaranteed fulfilling \( (28) \), which is equivalent to \( \pi_w^T Y_w \pi_w < 0 \), where \( \pi_w = [x^T \pi^T]^T \) and \( Y_w \) is given by

\[
\begin{bmatrix}
H \{ P A_1 \} + \gamma^{-1} B_1^T B_1 & \gamma^{-1} B_3^T B_3 - \gamma I \\
\gamma^{-1} B_3^T B_3 - \gamma I & \gamma^{-1} B_2^T B_3 \\
\gamma^{-1} B_3^T B_3 & \gamma^{-1} B_2^T B_2
\end{bmatrix}
\]

By noting from \( (25) \) that \( \hat{Y}_w = [\hat{Y}_1 \hat{Y}_3 \hat{Y}_2] \) is an annihilator of \( \pi_w \), Lemma \( \ref{lm2} \) can also be applied. If there exists a matrix \( L_w = [L_1^T L_2^T L_3^T]^T \) such that

\[
Y_w + L_w \hat{Y}_w + \hat{Y}_w^T L_w^T < 0,
\]

then \( \pi_w L_w \pi_w < 0 \) is satisfied for all \( \rho(t) \in \Omega \) and \( \pi_w \neq 0 \). Next, applying Schur complement in \( (33) \) followed by a congruence transformation with \( \text{diag}(I_2, I_2) \), we obtain

\[
\begin{bmatrix}
H \{ P A_1 \} + \gamma^{-1} B_1^T B_1 & \gamma^{-1} B_3^T B_3 - \gamma I \\
\gamma^{-1} B_3^T B_3 - \gamma I & \gamma^{-1} B_2^T B_2
\end{bmatrix}
\]

where \( L_w = [L_1^T L_2^T 0 L_3^T]^T \) and \( \hat{Y}_w = [\hat{Y}_1 \hat{Y}_3 \hat{Y}_2] \).

By taking into account the definitions of matrices \( A_1, A_2, B_1, B_2, B_3, \hat{X}_w, \hat{Y}_w \) in \( (26) \), the following equivalent expression for \( (34) \) is obtained in terms of the matrices \( \tilde{P}, \tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}_1, \tilde{C}_2, \tilde{Y}_w, \tilde{Y}_w \) given in \( (29) \)

\[
H \{ \tilde{P} \tilde{A}_1 \} + \gamma^{-1} \tilde{B}_1^T \tilde{B}_1 & \gamma^{-1} \tilde{B}_3^T \tilde{B}_3 - \gamma I \\
\gamma^{-1} \tilde{B}_3^T \tilde{B}_3 - \gamma I & \gamma^{-1} \tilde{B}_2^T \tilde{B}_3
\]

Note that condition \( (33) \) has the same form of condition \( (32) \).

Thus, by applying Theorem \( \ref{th1} \) with the bar matrices, one ensures satisfaction of \( \pi_w^T Y_w \pi_w < 0, \forall \pi_w \in \mathbb{R}^{n+q+1+n} : Y_w \pi_w = 0 \), and for all \( \rho(t) \in \Omega \) with the designed SOF gain-scheduled control \( (30) \), which in turn guarantees \( (28) \) along the trajectories of the closed-loop perturbed system \( (25) \) with \( L_2 \)-gain bounded by \( \gamma \).

\section*{A. Optimization Problem}

To design the GS-SOF that stabilizes system \( (25) \) while minimizing the \( L_2 \)-gain, the following optimization problem applies.

\[
\begin{align*}
\text{minimize} & \quad \gamma \\
\text{subject to} & \quad \gamma > 0, \quad P > 0, \quad \overline{P}_i > 0, \quad \text{for } i = 1, \ldots, N, \\
& \quad \text{conditions } (15) \text{ and } (16) \text{ are applied as in Theorem 2.}
\end{align*}
\]

\section*{V. NUMERICAL EXAMPLES}

This section presents numerical examples to illustrate the effectiveness of the proposed design method. For the implementation of the design conditions in the theorems, we use conventional SDP tools provided by \[32\] and \[33\].

\section*{B. Example 2}

Consider an LPV system as follows,

\[
\begin{bmatrix}
1 + \rho & -2 - 3 \rho \\
0 & 4 - \rho
\end{bmatrix} x + \begin{bmatrix} 1 \\ \rho \end{bmatrix} u + \begin{bmatrix} 2 - \rho \\ 1 \end{bmatrix} w, \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 - \rho \end{bmatrix} x + (1 + \rho) u, \\
y = \begin{bmatrix} 1 + \rho \\ \rho \end{bmatrix} x.
\]
with $\rho \in \{0, 1\}$. Replacing the limits of $\rho$ in the system, we obtain the same two vertices system of Example 2 from [13]. Consider a DAR of system (38) with

$$\pi = \begin{bmatrix} \rho x_1 & \rho x_2 & \rho u & \rho w \end{bmatrix}^T, \quad B_3 = 1, \quad B_4 = C_3 = 0,$$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -3 \, 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad Y_2^T = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}, \quad C_2^T = \begin{bmatrix} 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix},$$

$$\Upsilon_2 = -I_4, \quad \Upsilon_3 = \begin{bmatrix} 0 & 0 & \rho & 0 \end{bmatrix}, \quad \Upsilon_4 = \begin{bmatrix} 0 & 0 & 0 & \rho \end{bmatrix}.$$

By applying the optimization problem [36] with $\beta = -29.3$, we obtain the gain-scheduled SOF (30), with matrices $K_1 = -29.0522$, $K_2 = -29.2994$, that guarantees closed-loop stability with $L_2$-gain bounded by $\gamma = 5.2637$. Implementation of the control law is straightforward by following the same procedure illustrated in Example 1.

VI. CONCLUSION

This paper proposed a new strategy based on strict QSR-dissipativity for gain-scheduled SOF stabilization of LPV systems with $L_2$-gain performance. Finsler’s Lemma and linear annihilators have been applied to formulate polytopic LMI conditions for dissipativity analysis and gain-scheduled SOF design. We successfully applied the strategy in two numeral examples. The first presents rational system matrices and the second presents affine system matrices on the time-varying parameter, both being open-loop unstable. The main contribution of this paper consists that the system matrices can present polynomial or rational dependence, not only affine as it is common in the field, and no restriction on the output plant matrix is considered. In addition, differently from some strategies in the field, the formulated solution does not need to solve a static feedback problem as initial stage to design the gain-scheduled static output feedback.

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