Coordinated Output Regulation of Heterogeneous Linear Systems under Switching Topologies

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Abstract

This paper constructs a framework to describe and study the coordinated output regulation problem for multiple heterogeneous linear systems. Each agent is modeled as a general linear multiple-input multiple-output system with an autonomous exosystem which represents the individual offset from the group reference for the agent. The multi-agent system as a whole has a group exogenous state which represents the tracking reference for the whole group. Under the constraints that the group exogenous output is only locally available to each agent and that the agents have only access to their neighbors’ information, we propose observer-based feedback controllers to solve the coordinated output regulation problem using output feedback information. A high-gain approach is used and the information interactions are allowed to be switched over a finite set of fixed networks containing both graphs that have a directed spanning tree and graphs that do not. The fundamental relationship between the information interactions, the dwell time, the non-identical dynamics of different agents, and the high-gain parameters is given. Simulations are shown to validate the theoretical results.

Key words: Heterogeneous linear dynamic systems; Coordinated output regulation; Switching communication topology

1 Introduction

Coordinated control of multi-agent systems has recently drawn large attention due to its broad applications in physical, biological, social, and mechanical systems [2–5]. The key idea of “coordination” algorithm is to realize a global emergence using only local information interactions [6, 7]. The coordination problem of a single-integrator network is fully studied with an emphasis on the system robustness to the input time delays and switching communication topologies [6–9], discrete-time dynamical models [10, 11], nonlinear couplings [12], the convergence speed evaluation [13], the effects of quantization [14], and the leader-follower tracking [15].

Following these ideas, the study of coordination of multiple linear dynamic systems becomes an attractive and fruitful research direction for the control community recently. For example, the authors of [16] generalize the existing works on coordination of multiple single-integrator systems to the case of multiple linear time-invariant single-input systems. For a network of neutrally stable systems and polynomially unstable systems, the author of [17] proposes a design scheme for achieving synchronization. The case of switching communication topologies is considered in [18] and a so-called consensus-based observer is proposed to guarantee leaderless synchronization of multiple identical linear dynamic systems under a jointly connected communication topology. Similar problems are also considered in [19] and [20], where a frequently connected communication topology is studied in [19] and an assumption on the neutral stability is imposed in [20]. The authors of [21] propose a neighbor-based observer to solve the synchronization problem for general linear time-invariant systems. An individual-based observer and a low-gain technique are used in [22] to synchronize a group of linear systems with open-loop poles at most polynomially unstable. In addition, the classical Laplacian matrix is generalized in [23] to a so-called interaction matrix. A D-scaling approach is then used to stabilize this interaction matrix under both fixed and switching communication topologies. Synchronization of multiple heterogeneous linear systems has been investigated under both fixed and switching communication topologies [24–26]. A similar problem is studied in [27, 28], where a high-gain approach is proposed to dominate the non-identical dynamics of the agents. The cases of frequently connected and jointly connected communication topologies are studied in [29] and [30], respectively, where a slow switching condition and a fast switching condition are presented. Recently, the generalizations of
In this paper, we generalize the classical output regulation problem of an individual linear dynamic system to the coordinated output regulation problem of multiple heterogeneous linear dynamic systems. We consider the case where each agent has an individual offset and simultaneously there is a group tracking reference. The individual offset and the group reference are generated by autonomous systems (i.e., systems without inputs). Each individual offset is available only through constrained communication among the agents, i.e., the group reference trajectory is available to only a subset of the agents. Our goal is to find an observer-based feedback controller for each agent such that the output of each agent converges to a given trajectory determined by the combination of the individual offset and the group reference. Motivated by the approach proposed in [27], we propose a unified observer to solve the coordinated output regulation problem of multiple heterogeneous general linear dynamics, where the open-loop poles of the agents can be exponentially unstable and the dynamics are allowed to be different both with respect to dimensions and parameters. This relaxes the common assumption of identical dynamics [17, 18, 20, 21, 29] or open-loop poles at most polynomially unstable [18, 20, 26]. The main contribution of this work is that the information interaction is allowed to be switching from a graph set containing both a directed spanning tree set and a disconnected graph set for the case of heterogeneous [18, 20, 26]. The high-gain technique is used and the relationships between the dwell time [38], the non-identical dynamics among different agents and the high-gain parameters are also given.

The remainder of the paper is organized as follows. In Section 2, we give some basic definitions on network model. In Section 3, we formulate the problem of coordinated output regulation of multiple heterogeneous linear systems. We then propose the state feedback control law with a unified observer design in Section 4. Two case studies are given in Section 5. Numerical studies are carried out in Section 6 to validate our designs of observer-based controllers and a brief concluding remark is drawn in Section 7.

2 Network Model

We use graph theory to model the communication topology among agents. A directed graph $G$ consists of a pair $(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a finite, nonempty set of nodes and $E \subseteq V \times V$ is a set of ordered pairs of nodes. An edge $(v_i, v_j)$ denotes that node $v_i$ can obtain information from node $v_j$. All neighbors of node $v_i$ are denoted as $N_i := \{v_j | (v_j, v_i) \in E\}$. For an edge $(v_i, v_j)$ in a directed graph, $v_i$ is the parent node and $v_j$ is the child node. A directed path in a directed graph is a sequence of edges of the form $(v_i, v_j), (v_j, v_k), \ldots$. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all other nodes.

For a leader-follower graph $\mathcal{G} := (V, \mathcal{E})$, we have $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where $v_0$ is the leader and $v_1, v_2, \ldots, v_n$ denote the followers. The leader-follower adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{(n+1) \times (n+1)}$ is defined such that $a_{ij}$ is positive if $(v_j, v_i) \in \mathcal{E}$ while $a_{ij} = 0$ otherwise. Here we assume that $a_{ii} = 0$, $i = 0, 1, \ldots, n$, and the leader has no parent, i.e., $a_{0j} = 0$, $j = 0, 1, \ldots, n$. The leader-follower “grounded” Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ associated with $A$ is defined as $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$, where $i \neq j$.

In this paper, we assume that the leader-follower communication topology $G_{(t)}$ is time-varying and switching from a finite set $\{\mathcal{G}_k\}_{k \in \Gamma}$, where $\Gamma = \{1, 2, \ldots, \delta\}$ is an index set and $\delta \in \mathbb{N}$ indicates its cardinality. We impose the technical condition that $G_{(t)}$ is right continuous, where $\sigma : [t_0, \infty) \rightarrow \Gamma$ is a piecewise constant function of time. That is to say, $G_{(t)}$ remains constant for $t \in [t_\ell, t_{\ell+1})$, $\ell = 0, 1, \ldots$ and switches at $t = t_\ell$, $\ell = 1, 2, \ldots$. In addition, we assume that $\inf \limits_{t \rightarrow \infty} t_\ell > \tau_2 > 0$, $\ell = 0, 1, \ldots$. Thus, $\tau_2$ is a constant known as the dwell time [38].

Let the sets $\{\mathcal{A}_k\}_{k \in \Gamma}$ and $\{L_k\}_{k \in \Gamma}$ be the leader-follower adjacency matrices and leader-follower grounded Laplacian matrices associated with $\{\mathcal{G}_k\}_{k \in \Gamma}$, respectively. Consequently, the time-varying leader-follower adjacency matrix and time-varying leader-follower grounded Laplacian matrix are defined as $\mathcal{A}_{(t)} = [a_{ij}(t)]$ and $L_{(t)} = [l_{ij}(t)]$.

Other notation in this paper: $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote, respectively, the minimum and maximum eigenvalues of a real symmetric matrix $P$, $P^T$ denotes the transpose of $P$, and $I_n$ denotes the $n \times n$ identity matrix.

3 Problem Formulation

3.1 Agent Dynamics

Suppose that we have $n$ agents modeled by the linear MIMO systems:

$$\dot{x}_i = A_i x_i + B_i u_i,$$  

(1)

where $x_i \in \mathbb{R}^{n_i}$ is the agent state, $u_i \in \mathbb{R}^{m_i}$ is the control input, $A_i \in \mathbb{R}^{n_i \times n_i}$, and $B_i \in \mathbb{R}^{n_i \times m_i}$.

Also suppose that there is an individual autonomous exosystem for each $v_i \in V$,

$$\omega_i = S_i \omega_i,$$  

(2)
relative estimation information is available using the same
where \( \omega_i \in \mathbb{R}^{q_i} \) and \( S_i \in \mathbb{R}^{q_i \times q_i} \).

In addition, there is a group autonomous exosystem for the
multi-agent system as a whole:
\[
\dot{x}_0 = A_0 x_0,
\]
where \( x_0 \in \mathbb{R}^{n_0} \) and \( A_0 \in \mathbb{R}^{n_0 \times n_0} \).

### 3.2 Control Architecture

The control of each agent is supposed to have the structure
shown in Fig. 1. More specifically, for the individual
autonomous exosystem tracking, available output information
for agent \( v_i \in \mathcal{V} \) is
\[
y_{s_i} = C_{si} x_i + C_{wi} \omega_i,
\]
where \( C_{si} \in \mathbb{R}^{p_1 \times n_i} \) and \( C_{wi} \in \mathbb{R}^{p_1 \times q_i} \).

For the group autonomous exosystem tracking, only
neighbor-based output information is available due to the
constrained communication. This means that not all the
agents have access to \( y_0 \). The available information is the
neighbor-based sum of each agent’s own output relative to
that of its’ neighbors, i.e.,
\[
\xi_i = \sum_{j=0}^{n} a_{ij}(t)(y_{di} - y_{dj})
\]
is available for each agent \( v_i \in \mathcal{V} \), where \( a_{ij}(t), i = 0, 1, \ldots, n, j = 0, 1, \ldots, n \), is entry \((i, j)\) of the adjacency matrix \( A_{\sigma(t)} \) associated with \( \mathcal{G}_{\sigma(t)} \) defined in Section 2 at time \( t \), \( y_{di} \) can be represented by \( y_{di} = C_{di} x_i, i = 1, 2, \ldots, n \) and \( y_{d0} = C_0 x_0 \), where \( C_{di} \in \mathbb{R}^{p_2 \times n_i}, i = 1, 2, \ldots, n \) and \( C_0 \in \mathbb{R}^{p_2 \times n_0} \). Also, the
relative estimation information is available using the same
communication topologies, i.e.,
\[
\hat{y}_i = \sum_{j=0}^{n} a_{ij}(t)(\hat{y}_j - \hat{y}_j)
\]
is available for each agent \( v_i \in \mathcal{V} \), where \( \hat{y}_i \) is an estimation
produced internally by each agent \( v_i \in \mathcal{V} \).

Fig. 2 gives an example of information flow among the
agents and the group autonomous exosystem \( v_0 \) for \( n = 3 \) agents.

#### 3.3 Switching Topologies

For the communication topology set \( \{ \mathcal{G}_k \}_{k \in \Gamma} \), we assume
that \( \mathcal{G}_k, \forall k \in \Gamma \) is a graph containing a directed spanning
tree with \( v_0 \) rooted. Without loss of generality, we relabel \( \Gamma_c := \{ 1, 2, \ldots, 3 \} \) \( (1 \leq \delta_1 \leq \delta_0) \), where \( \delta_1 \in \mathbb{N} \).

The remaining graphs are labeled as \( \mathcal{G}_{\delta_1} \), \( \forall \delta_1 \in \Gamma_d \), where \( \Gamma_d := \{ \delta_1 + 1, \delta_1 + 2, \ldots, \delta_0 \} \). Denote the graph set \( \mathcal{G}_{\delta_0} = \{ \mathcal{G}_k \}_{k \in \Gamma_c} \) and the graph set \( \mathcal{G}_{\delta_d} = \{ \mathcal{G}_{\delta_1} \}_{\delta_1 \in \Gamma_d} \), respectively. We
also denote \( T_0^d(t) \) and \( T_0^c(t) \) the total activation time when
\( \mathcal{G}_{\delta(c)} \in \mathcal{G}_{\delta_d} \) and total activation time when \( \mathcal{G}_{\delta(c)} \in \mathcal{G}_{\delta_c} \) during
\( \zeta \in [t_0, t) \) for \( t_0 \geq t_0 \).

**Assumption 1** The dwell time \( \tau_{\delta} \) is a positive constant.

**Assumption 2** Given a positive constant \( \kappa \), there exists a
\( T_0 \geq t_0 \) such that \( T_0^c(t) \geq \kappa T_0^d(t) \) for all \( t \geq T_0 \).

**Remark 1** Note that a sufficient condition satisfying Assumption 2 is that \( \mathcal{G}_{\delta} \) is non-empty and given a \( T > 0 \) and \( \tau_{\delta} > 0 \), for any \( t \geq t_0 \), the switching signal \( \sigma(t) \) satisfies
\( \{ t | (\mathcal{G}_{\sigma(t)} \in \mathcal{G}_{\delta_0}) \cap [t, t + T) \neq \emptyset \} \). Such a condition is also referred as “frequently connected” condition (i.e., the communication topology that contains a directed spanning tree is active frequently enough \([19, 23]\)). Note that this condition implies that there exists a time sequence \( 0 = t_0 < t_1 < \ldots < t_{\ell} \ldots \) such that \( \{ t | (\mathcal{G}_{\sigma(t)} \in \mathcal{G}_{\delta_0}) \cap [t, t + T) \neq \emptyset \} \) for all \( \ell = 0, 1, \ldots, T_{\ell + 1} - T_{\ell} \leq 2T \). Therefore, there exists a
\( t_0 \in [t_0, t_0 + 2T] \) such that \( T_0^c(t) \geq \kappa T_0^d(t) \) for all \( t \geq T_0 \).

### 3.4 Control Objective

The control objective of each agent is to track a given trajectory
determined by the combination of the group reference \( x_0 \) and the individual offset \( \omega_i, i = 1, 2, \ldots, n \). Such a combination is captured by the coordinated output regulation
tracking error (i.e., the total tracking error representing the
combination of both individual tracking and group tracking of each agent):
\[
e_i = D_{ai} x_i + D_{wi} \omega_i + D_0 x_0.
\]
Given that Assumption 3 is satisfied, we can perform the state transformation given in Step 1 of [27] by considering \( y_0 \) and \( x_0 \) together. We can construct a new state \( \tilde{\mathbf{x}}_i = W_i \begin{bmatrix} x_i \\ \omega_i \end{bmatrix} \) with the dynamics

\[
\dot{\mathbf{\tilde{x}}}_i = \mathbf{\tilde{A}}_i \mathbf{\tilde{x}}_i + \mathbf{\tilde{B}}_iu_i = \begin{bmatrix} A_i & \mathbf{\tilde{A}}_{i12} \\ 0 & \mathbf{\tilde{A}}_{i22} \end{bmatrix} \mathbf{\tilde{x}}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{6a}
\]

\[
\mathbf{e}_{di} = \mathbf{\tilde{c}}_i \mathbf{\tilde{x}}_i = \begin{bmatrix} C_i & \mathbf{\tilde{c}}_{i1} \\ \mathbf{\tilde{c}}_{i2} \\ \mathbf{\tilde{c}}_{i22} \end{bmatrix} \mathbf{\tilde{x}}_i, \tag{6b}
\]

where \( \mathbf{e}_{di} = y_{di} - y_{d0} \), and the details designs on \( W_i, \mathbf{\tilde{A}}_i, \mathbf{\tilde{B}}_i, \mathbf{\tilde{C}}_i \) are given in [27]. It was shown that pair \((\mathbf{\tilde{A}}_i, \mathbf{\tilde{C}}_i)\) is observable and the eigenvalues of \( \mathbf{\tilde{A}}_{i22} \) are a subset of the eigenvalues of \( S_i \) and \( A_0 \), \( i = 1, 2, \ldots, n \).

### 4.2 Regulated State feedback Control Law

We now design a controller to regulate \( e_i \) to zero for each agent based on the state information \( \mathbf{\tilde{x}}_i = \begin{bmatrix} \mathbf{\tilde{x}}_{i1} \\ \mathbf{\tilde{x}}_{i2} \end{bmatrix} \), where \( \mathbf{\tilde{x}}_{i1} \in \mathbb{R}^{n_i} \).

We impose the following assumptions on the structure of the systems.

**Assumption 4**

- \((A_i, B_i)\) is stabilizable, \( i = 1, \ldots, n \).
- \((A_i, B_i, D_{si})\) is right-invertible, \( i = 1, \ldots, n \).
- \((A_i, B_i, D_{si})\) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of \( S_i \) or \( A_0 \), \( i = 1, \ldots, n \).

**Lemma 1** Let Assumption 4 hold. Then, the regulator equations (7) are solvable and the state-feedback controller \( u_i = F_i(\mathbf{\tilde{x}}_{i1} - \Pi_i \mathbf{\tilde{x}}_{i2}) + \Gamma_i \mathbf{\tilde{x}}_{i2} \) ensures that \( \lim_{t \to \infty} e_i(t) = 0 \), \( i = 1, 2, \ldots, n \), where \( \Pi_i, \Gamma_i \) are the solutions of the following regulator equations

\[
\Pi_i \mathbf{\tilde{A}}_{i22} = A_i \Pi_i + \mathbf{\tilde{A}}_{i12} + B_i \Gamma_i, \tag{7a}
\]

\[
0 = D_{si} \Pi_i + \begin{bmatrix} D_{si} & D_{so} \end{bmatrix}, \quad i = 1, 2, \ldots, n, \tag{7b}
\]

and \( F_i \) is chosen such that \( A_i + B_i F_i \) is Hurwitz.

**Proof:** It follows from [39] and the similar analysis of proof of Lemma 3 in [27], we can show that the regulator equations

\[
\begin{bmatrix} y_{si} \\ y_{di} - y_{d0} \end{bmatrix} = \begin{bmatrix} C_{si} & C_{wi} & 0 \\ C_{di} & 0 & -C_0 \end{bmatrix} \begin{bmatrix} x_i \\ \omega_i \\ x_0 \end{bmatrix}.
\]

Fig. 2. Information flow associated with three agents \( v_1, v_2, v_3 \), the individual autonomous exosystems \( \omega_1, \omega_2, \omega_3 \), and the group autonomous exosystem \( v_0 \).

Thus, our objective is to guarantee that \( \lim_{t \to \infty} e_i(t) = 0 \).

We design an observer-based controller with available individual output information and neighbor-based group output information to solve this problem.

For the system shown in Fig. 2, the overall control can correspond to a formation control problem, where \( \omega_i \) encodes the relative position between each agent and the leader while the leader \( x_0 \) defines the overall motion of the group.
(7) are solvable given that Assumption 4 is satisfied. Then, by considering \( \dot{\xi}_i = A_i \hat{x}_i + B_i u_i \) as the system to be regulated for the classic output regulation result \([40]\), we know that \( u_i = P_i (\tau_i - \Pi_i \xi_i) + \Gamma_i \xi_i \) ensures that \( \lim_{t \to \infty} e_i(t) = 0, i = 1, \ldots, n \), where \( \Pi_i \) and \( \Gamma_i \) are the solutions of the regulator equations (7).

We next design observers to estimate \( \tau_i \) based on output information \( y_{si} \) and \( \zeta_i \) for each agent.

### 4.3 Pseudo-identical Linear Transformation

Note that the individual offset \( \omega_i \) can be estimated by \( y_{si} \) and the group reference \( x_0 \) can be estimated by \( \hat{\zeta}_i \). In contrast, the internal state information \( \tau_i \) for each agent can be obtained by either \( y_{si} \) or \( \hat{\zeta}_i \). In this section, we use the combination of \( y_{si} \) and \( \hat{\zeta}_i \) to give a unified observer design.

We define \( \chi_i = T_i \tau_i \in \mathbb{R}^{m_i}, i = 1, 2, \ldots, n \), where \( m = m_0 + \max_{i=1,2,\ldots,n} (n_0 + q_i) \), \( p = p_1 + p_2 \), and

\[
T_i = \begin{bmatrix}
\mathcal{C}_i \\
\vdots \\
\mathcal{C}_i A_i^{-1}
\end{bmatrix}.
\]

Note that \( T_i \) is full column rank since the pair \((\mathcal{A}_i, \mathcal{C}_i)\), \( i = 1, 2, \ldots, n \) is observable. This implies that \( T_i^T T_i \) is nonsingular. Therefore, it follows that

\[
\dot{\chi}_i = (\mathcal{A}_i + \mathcal{L}_i) \chi_i + \mathcal{B}_i u_i, \quad i = 1, 2, \ldots, n, \tag{8a}
\]

\[
y_{si} = \mathcal{C}_i \chi_i, \quad i = 1, 2, \ldots, n, \tag{8b}
\]

where \( \mathcal{A}_i = \begin{bmatrix} 0 & I_p (\sigma - 1) \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(m+\overline{m}, m)}, \mathcal{C}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix} \), \( \mathcal{B}_i = T_i \mathcal{B}_i \), \( \mathcal{C} = \begin{bmatrix} I_p & 0 \end{bmatrix} \in \mathbb{R}^{(p+\overline{m}, \overline{m})} \) for some matrix \( L_i \in \mathbb{R}^{p \times \overline{m}} \).

### 4.4 Unified Observer Design

Motivated by \([27]\), based on the available output information \( y_{si} \) and the neighbor-based group output information \( \zeta_i \), the distributed observer is proposed for (8) as

\[
\dot{\hat{\chi}}_i = (\mathcal{A}_i + \mathcal{L}_i) \hat{\chi}_i + \mathcal{B}_i u_i + S(\varepsilon) \mathcal{P} \mathcal{C}^T \times \left( \begin{bmatrix} y_{si} \\ \sum_{j=0}^n a_{ij}(t)(y_{di} - y_{dj}) \end{bmatrix} - \begin{bmatrix} \hat{y}_{si} \\ \sum_{j=0}^n a_{ij}(t)(\hat{y}_i - \hat{y}_j) \end{bmatrix} \right), \tag{9}
\]

where \( a_{ij}(t), i = 0, 1, \ldots, n, j = 0, 1, \ldots, n \), is entry \((i, j)\) of the adjacency matrix \( \mathcal{A}_\sigma(i) \) associated with \( \mathcal{G}_\sigma(i) \) defined in Section 2 at time \( t \), \( \hat{y}_{si} = \mathcal{C}_i \hat{x}_i, \hat{y}_i = \mathcal{C}_i \hat{x}_i, i = 1, \ldots, n \), \( \mathcal{G}_1 \) is first \( p_1 \) rows of \( \mathcal{G} \), \( \mathcal{G}_2 \) is the remaining \( p_2 \) rows of \( \mathcal{G} \), and \( \hat{y}_0 = 0 \). In addition, \( S(\varepsilon) = \text{diag}(I_p \varepsilon^{-1}, I_p \varepsilon^{-2}, \ldots, I_p \varepsilon^{-\overline{m}}) \), where \( \varepsilon \in (0, 1] \) is a positive constant to be determined, and \( \mathcal{P} = \mathcal{P}^T \) is a positive definite matrix satisfying

\[
\mathcal{A} \mathcal{P} + 2 \mathcal{P} \mathcal{C}^T - 2 \mathcal{P} \mathcal{C}^T \begin{bmatrix} I_{p_1} & 0 \\ 0 & \theta I_{p_2} \end{bmatrix} \mathcal{P} + l \mathcal{C} = 0, \tag{10}
\]

where \( \theta = \min_{k \in \Gamma} \beta_k \) and \( \beta_k \) will be determined later. Note that the existence of \( \mathcal{P} \) is due to the fact that \( (\mathcal{A} \mathcal{P} + 2 \mathcal{P} \mathcal{C}^T, \mathcal{C}) \) is observable.

**Lemma 2** • All the eigenvalues of \( L_k \) are in the closed right-half plane with those on the imaginary axis being simple, where \( L_k \) is associated with \( \mathcal{G}_k \) defined in Section 2, and some \( \mathcal{G}_k \in \{ \mathcal{G}_k \}_{k \in \Gamma} \).

• Furthermore, all the eigenvalues of \( L_k \) are in the open right-half plane for \( \mathcal{G}_k \in \{ \mathcal{G}_k \}_{k \in \Gamma} \).

**Proof:** See Theorem 4.29 in \([41]\) and Lemma 1.6 in \([42]\).

**Lemma 3** Let Assumptions 1, 2, 4, and 3 hold and assume that \( \kappa > \frac{\alpha + 4 \max_{k \in \Gamma} (\beta_k^2 \max(\mathcal{P}))}{1 - \alpha} \), where \( \alpha \in (0, 1) \). Then, there exists an \( \varepsilon^* \in (0, 1] \) such that, if \( \varepsilon \in (0, \varepsilon^*], \lim_{t \to \infty} (\chi_i(t) - \hat{\chi}_i(t)) = 0, i = 1, 2, \ldots, n \), for systems (9).

**Proof:** Note that for all \( i = 1, 2, \ldots, n \), \( \sum_{j=0}^n a_{ij}(t)(y_{di} - y_{dj}) = \sum_{j=0}^n a_{ij}(t)(y_{dj} - y_{di}) = \sum_{j=0}^n l_j(t)(e_{dj} - e_{jd}) \). Define \( \hat{\chi}_i = \chi_i - \hat{\chi}_i \). It then follows from (8) and (9) that

\[
\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i - S(\varepsilon) \mathcal{P} \mathcal{C}^T) \begin{bmatrix} y_{si} - \hat{y}_{si} \\ \sum_{j=0}^n a_{ij}(t)(\hat{y}_i - \hat{y}_j) \end{bmatrix}, \quad i = 1, 2, \ldots, n,
\]

where \( l_j(t), i = 1, \ldots, n, j = 1, \ldots, n \), is the \((i, j)\)th entry of the adjacency matrix \( L_\sigma(i) \) associated with \( \mathcal{G}_\sigma(i) \) defined in Section 2 at time \( t \). It follows that

\[
\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i - S(\varepsilon) \mathcal{P} \mathcal{C}^T) \begin{bmatrix} \mathcal{G}_1 \hat{\chi}_i \\ \mathcal{G}_2 \sum_{j=0}^n a_{ij}(t)(\hat{\chi}_j) \end{bmatrix}, \quad i = 1, 2, \ldots, n.
\]

1. The upper bound of the high-gain parameter may be conservative. We can use an empirical approach to derive a feasible \( \varepsilon^* \) in the practical applications.
By introducing $\xi_i = \varepsilon^{-1} S^{-1}(\varepsilon) \xi_i$, and after some manipulation, we have that

$$
\varepsilon \dot{\xi}_i = \langle \mathcal{A} + L \xi \rangle \xi_i - \mathcal{P} \mathcal{C}^T \left( \begin{bmatrix} G_i \xi_i \\ \varepsilon^2 \sum_{j=1}^n \xi_j \end{bmatrix} \right),
$$

where

$$
L_{i\varepsilon} = \begin{bmatrix} 0 \\ \varepsilon^{n+1} L_\varepsilon S(\varepsilon) \end{bmatrix} = O(\varepsilon).
$$

Note that

$$
\begin{bmatrix} G_i \\ \varepsilon \end{bmatrix} \xi_i = \mathcal{C} \xi_i, \quad \text{for all } i = 1, 2, \ldots, n.
$$

The overall dynamics can be written as

$$
\varepsilon \dot{\xi} = (I_n \otimes \mathcal{A} + L \varepsilon - (I_n \otimes \mathcal{P} \mathcal{C}^T) \cdot \begin{bmatrix} I_n \\ 0 \\
0 \\ 0 \end{bmatrix} (I_n \otimes \mathcal{C}) \varepsilon \xi,
$$

where $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_n^T]^T$ and $L \varepsilon = \text{diag}(L_{1\varepsilon}, L_{2\varepsilon}, \ldots, L_{n\varepsilon})$.

Note that $-L_k, k \in \Gamma_c$ is a Hurwitz stable matrix according to Lemma 2. Therefore, we can always guarantee that $-L_k + \beta_k I_n$ is also a Hurwitz stable matrix by choosing $\beta_k$ sufficiently small. In particular, we choose $\beta_k$ as a positive constant satisfying $\beta_k < \min \Re \{ \lambda(L_k) \}$, $k \in \Gamma_c$, where $\min \Re \{ \lambda(L_k) \}$ denote the minimum value of all the real parts of the eigenvalues of $L_k$. Then, we define piecewise Lyapunov function candidate $V_k = \varepsilon \xi^T (P_k \otimes (\mathcal{P}^{-1})) \xi$, where $P_k$ is a positive definite matrix satisfying

$$
P_k (-L_k + \beta_k I_n) + (-L_k + \beta_k I_n)^T P_k = -L_k < 0, \quad k \in \Gamma_c,
$$

$$
P_k (-L_k) + (-L_k)^T P_k \leq 0, \quad k \in \Gamma_d,
$$

where the second inequality is due to Lemma 2.

It then follows that for all $k \in \Gamma_c$,

$$
\dot{V}_k \leq -\xi^T \left( P_k \otimes (\mathcal{P}^{-1}) \right) \xi + \xi^T \left( P_k \otimes (\mathcal{P}^{-1}) \right) L \varepsilon \xi
$$

$$
-2 \xi^T \left( P_k \otimes \left( \mathcal{C}^T \begin{bmatrix} I_p \\ 0 \\
0 \\ 0 \end{bmatrix} \mathcal{C} \right) \right) \xi
$$

$$
-2 \xi^T \left( P_k L_k \otimes \left( \mathcal{C}^T \begin{bmatrix} 0 \\ 0 \\
0 \\ 0 \end{bmatrix} \mathcal{C} \right) \right) \xi
$$

$$
\leq \xi^T \left( P_k \otimes (\mathcal{P}^{-1} - \mathcal{P}^{-1} \mathcal{P} - 2 \theta \mathcal{C}^T \begin{bmatrix} 0 \\ 0 \\
0 \\ 0 \end{bmatrix} \mathcal{C}) \right) \xi + 2 \xi^T (P_k \otimes (\mathcal{P}^{-1}) L \varepsilon \xi
$$

$$
-2 \xi^T \left( P_k L_k \otimes \left( \mathcal{C}^T \begin{bmatrix} 0 \\ 0 \\
0 \\ 0 \end{bmatrix} \mathcal{C} \right) \right) \xi
$$

where we have used (10). Note that $\lambda_{\max} \left( \mathcal{C}^T \begin{bmatrix} I_p \\ 0 \\
0 \\ 0 \end{bmatrix} \mathcal{C} \right) = \max \{ \theta, 1 \}$. It follows that $\dot{V}_k \leq \frac{1}{\lambda_k} \lambda_k V_k$, $\forall k \in \Gamma_d$, if

$$
\| L \varepsilon \| < \frac{\lambda_{\min}(P_k) \lambda_{\max}(\mathcal{P})}{\lambda_{\max}(P_k) \lambda_{\min}(\mathcal{P})},
$$

$\forall k \in \Gamma_d$. Therefore, $V_k \leq \frac{1}{\lambda_k} \lambda_k V_k$, $\forall k \in \Gamma_d$, and the overall system is exponentially stable.
Following the similar analysis of \[38, 43\], we let \(\sigma = p_j\) on \([t_{j-1}, t_j)\) for \(p_j \in \Gamma\). Then, for any \(t\) satisfying \(t_0 < t_1 < \cdots < t_l < t < t_{l+1}\), define \(V = e^{\zeta T}(P_{\zeta(T)} \odot \mathcal{B}^{-1})\zeta\) for (11). We have that, \(V \zeta \in [t_{j-1}, t_j)\),

\[
V(\zeta) \leq e^{-\frac{1}{2}\lambda \theta}(\zeta - t_{j-1})V(t_{j-1}) \\
\leq e^{-\frac{1}{2}\lambda \theta}(\zeta - t_{j-1})V(t_{j-1}), \quad p_j \in \Gamma_c,
\]

\[
V(\zeta) \leq e^{\frac{1}{2}\lambda \theta}(\zeta - t_{j-1})V(t_{j-1}) \\
\leq e^{\frac{1}{2}\lambda \theta}(\zeta - t_{j-1})V(t_{j-1}), \quad p_j \in \Gamma_d,
\]

where \(\lambda^C = \min_{k \in \Gamma} \lambda_k = \frac{1}{\lambda_{\min}(\mathcal{B})}, \lambda^d = \max_{k \in \Gamma_d} \lambda_k = 2\max\{\theta, 1\} \lambda_{\max}(\mathcal{B})\). Define \(a = \frac{\lambda}{\lambda_{\min}(\mathcal{B})}, \lambda^C = \lambda^d = \lambda_{\max}(\mathcal{B})\).

We then know that \(V(t_j) \leq \lim_{t \to \infty} V(t)\). Thus, it follows that

\[
V(t) \leq a^p e^{-\frac{1}{2}\lambda \theta}(t - t_0)\zeta V(\zeta(t_0)) \\
\leq e^{-\frac{1}{2}\lambda \theta}(t - t_0)\zeta V(\zeta(t_0)) \\
= e^{-\frac{1}{2}\lambda \theta}(t - t_0)\zeta V(\zeta(t_0)).
\]

Furthermore, set \(\lambda = a^p \lambda^C\), where some \(a \in (0, 1)\). We then have that \(\lambda^C = \frac{\alpha + 4\max(\mathcal{B})\lambda_{\min}(\mathcal{B})}{1 - a}\), and

\[
V(t) \leq e^{-\frac{1}{2}\lambda \theta}(t - t_0)\zeta V(\zeta(t_0)).
\]

It follows that if \(e < \frac{\alpha t_0}{\lambda_{\min}(\mathcal{B}) \lambda_{\max}(\mathcal{B})}\), we have for (11) that

\[
\| \zeta(t) \| \leq c \epsilon e^{\frac{1}{2}\lambda \theta}(t - t_0)\| \zeta(t_0) \|,
\]

where \(c = \sqrt{\lambda_{\max}(\mathcal{B}) \lambda_{\min}(\mathcal{B}) / \lambda_{\max}(\mathcal{B}) \lambda_{\min}(\mathcal{B})}\).

Therefore, we choose \(\epsilon^*\) satisfying \(\epsilon^* < \frac{\alpha t_0}{\lambda_{\min}(\mathcal{B}) \lambda_{\max}(\mathcal{B})}\) and \(\| \mathcal{Z} \| < \min_{t \to \infty} \epsilon^* \frac{\lambda_{\min}(\mathcal{B}) \lambda_{\max}(\mathcal{B})}{\lambda_{\max}(\mathcal{B}) \lambda_{\min}(\mathcal{B})}\). It then follows that

\[
\lim_{t \to \infty} (\zeta(t) - \tilde{\zeta}(t)) = 0, \quad i = 1, 2, \ldots, n.
\]

**Remark 2** Note that the condition of \(\kappa\) is necessary when the communication topology is switching. Roughly speaking, we need to guarantee that the influence of “the good topology” beats that of “the bad topology” since the states of open-loop systems might diverge very fast due to the existence of unstable modes. The parameter \(\kappa\) is used to describe the relationship between \(T_{\zeta(T)}\) and \(T_{\zeta(T)}\), i.e., the remaining times of “good topology” and “bad topology”, respectively. The derived upper bound on \(\kappa\) might not be tight. However, we would like to emphasize that the significance is on the qualitative effects instead of quantitative effects. In practical applications, we can use an empirical approach to derive a feasible \(\kappa\), as illustrated in Section 6.

From the unified observer design, we then have that

\[
\tilde{x}_i = (T_{\zeta(T)} - 1)^{-1} T_{\zeta(T)} \hat{x}_i = [\xi_{1,1}, \xi_{1,2}]^T, \quad i = 1, 2, \ldots, n,
\]

which will be used in the control input design.

**4.5 Main Results**

In this section, we show that the observer architecture introduced in the previous sections provide an asymptotically stable closed-loop system, as presented in Theorems 1 below. The observer-based controller is proposed as

\[
u_i = F^T \hat{x}_1 + (\Gamma_l - F \Pi)^T \hat{x}_2,
\]

where \(\Pi_l\) and \(\Gamma_l\) are the solutions of the regulator equation (7), and \(\hat{x}_1\) and \(\hat{x}_2\) can be obtained from (9).

**Theorem 1** Let Assumptions 1, 2, 3 and 4 hold and assume that \(\kappa \geq \frac{\alpha + \max(\mathcal{B})\lambda_{\min}(\mathcal{B})}{1 - a}\), where \(\alpha \in (0, 1), \mathcal{B}\) and \(\mathcal{P}\) are given by (10). Then, there exists \(\epsilon^* \in (0, 1]\) such that, if \(\epsilon \in (0, \epsilon^*]\), (15) ensures that \(\lim_{t \to \infty} e_i(t) = 0, \quad i = 1, 2, \ldots, n\), for the multi-agent system (1)-(4).

**Proof:** Follows from Lemmas 1 and 3, and the separation principle. \(\blacksquare\)

**Remark 3** If the leader-follower communication topology \(\mathcal{G}\) is time-invariant, Assumptions 1 and 2 are not required, and therefore the high-gain parameter only depends on the non-identifier dynamics of the agents.

**5 Case Studies**

We notice that (9) give a unified way using \(y_i\) and \(\zeta\) to estimate \(x_i\), \(\phi_0\), and \(x_0\). One drawback of such a general approach is that the dimension of the observer \(\hat{x}_1\) may be unnecessarily large for some cases with special structures. We next give particular structural designs on two special cases, i.e., the case when \((A_i, C_{di})\) is observable and the case when \((A_i, C_{di})\) is observable.\(^2\)

\(^2\) These two cases are special cases of the first item of Assumption 3.
Based on the information of the individual output information \( y_{si} \), the following individual observer for each agent \( v_i \) is proposed

\[
\hat{x}_i = \bar{A}_i \hat{x}_i + \bar{B}_i u_i + K_{ai}(\bar{C}_i \hat{x}_i - y_{si}), \quad (18a)
\]

\[
[\xi_i^T, \omega_i^T]^T = W_i^{-1} \hat{x}_i, \quad i = 1, 2, \ldots, n, \quad (18b)
\]

where \( K_{ai} \) is chosen such that \( \bar{A}_i + K_{ai} \bar{C}_i \) is Hurwitz stable, \( i = 1, 2, \ldots, n \).

**Step III: group observer**

We transform (3) into its canonical form. Define \( \chi_0 = T_0 x_0 \in \mathbb{R}^{p_m} \), where

\[
T_0 = \begin{bmatrix}
C_0 \\
\vdots \\
C_0 A_0^{n_0-1}
\end{bmatrix}.
\]

Then, based on the neighbor-based group output information \( \zeta \), the distributed observer is proposed

\[
\hat{x}_{0i} = (\mathcal{A}_0 + \mathcal{L}_0) \hat{x}_{0i} - S(\hat{e}) \mathcal{P}_0 \hat{e}_0^T \left( \sum_{j=0}^{n} a_{ij}(t) (y_{di} - y_{dj}) - \sum_{j=0}^{n} a_{ij}(t) (\hat{y}_i - \hat{y}_j) \right), \quad (20a)
\]

\[
\hat{x}_{0i} = (T_0^T T_0)^{-1} T_0^T \hat{x}_{0i}, \quad i = 1, 2, \ldots, n, \quad (20b)
\]

where \( a_{ij}(t), \ i = 0, 1, \ldots, n, \ j = 0, 1, \ldots, n, \) is entry \((i,j)\) of the adjacency matrix \( \mathcal{A}_{ij} \) associated with \( \mathcal{G}_{ij} \) defined in Section 2 at time \( t \), the relative estimation information \( \sum_{j=0}^{n} a_{ij}(t) (y_{di} - y_{dj}) \) is obtained using the communication infrastructure with \( \hat{y}_i = C_{0i} \hat{x}_i - C_{0i} \hat{x}_{0i}, \ i = 1, 2, \ldots, n \) and \( \hat{y}_0 = 0 \). In addition, \( S(\hat{e}) = \text{diag}(I_p e^{-1}, I_p e^{-2}, \ldots, I_p e^{-p_m}) \), where \( e \in [0,1] \) is a positive constant, and \( \mathcal{P} = \mathcal{P}^T \) is a positive definite matrix satisfying

\[
\mathcal{A}_0 \mathcal{P} + \mathcal{P} \mathcal{A}_0^T - 2\mathcal{P} \mathcal{C}_0 \mathcal{C}_0^T \mathcal{P} + I_{p_m} = 0, \quad (21)
\]
and θ is a positive constant satisfying $\theta < \frac{1}{2} \min_{k=1}^{n} \pi_k \min \Re \{\lambda(I_k)\}$.

**Step IV: controller design**

The observer-based controller is proposed as

$$ u_i = F_i \tilde{x}_i + (\Gamma_{ii} - F_i \Pi_{ii}) \hat{\omega}_i + (\Gamma_{2i} - F_i \Pi_{2i}) \hat{x}_{0i}, \quad (22) $$

where $\Pi_{1i}, \Gamma_{1i}, \Pi_{2i},$ and $\Gamma_{2i}$ are the solutions of the following regulator equations

$$\begin{align}
\Pi_{1i} \dot{S}_i &= A_i \Pi_{1i} + B_i \Gamma_{1i}, \\
0 &= D \Pi_{1i} + D w_i, \\
\Pi_{2i} A_0 &= A_i \Pi_{2i} + B_i \Gamma_{2i}, \\
0 &= D \Pi_{2i} + D_0, \quad i = 1, 2 \ldots, n, 
\end{align}$$

(23a)-(23d)

and $F_i$ is chosen such that $A_i + B_i F_i$ is Hurwitz.

**Corollary 2** Let Assumptions 1, 2, 3 (the first item is replaced by that $(A_i, C_{di})$ is observable), and 4 hold and assume that $\kappa \geq \frac{\alpha + \sqrt{\alpha^2 + 4 \Theta}}{2 \Theta}$, where $\alpha \in (0, 1)$, $\Theta$ and $\mathcal{P}$ are given by (21). Also, let $\tilde{x}_i$ and $\hat{\omega}_i$ be obtained in (18), and $\hat{x}_{0i}$ be obtained in (20). Then, there exists $\epsilon_i^* \in (0, 1]$ such that, if $\epsilon \in (0, \epsilon_i^*)$, (22) ensures that $\lim_{t \to \infty} e_i(t) = 0, i = 1, 2 \ldots, n$, for the multi-agent system (1)-(4).

**Proof:** The proof is straightforward following the similar analysis given in Lemmas 1 and 3. \qed

5.2 **Case II:** $(A_i, C_{di})$ is observable

In this section, we use $y_{di}$ to estimate $\omega_i$ and use $\xi_i$ to estimate both $x_i$ and $x_0$. The control of each agent is supposed to have the structure shown in Fig. 4.

We also replace the first item of Assumption 3 with that $(A_i, C_{di})$, for all $i = 1, 2 \ldots, n$ is observable.

**Step I: redundant mode remove**

We first write the state and output of $x_i$ and $x_0$ for each agent in the compact form

$$\begin{bmatrix}
\dot{x}_i \\
\dot{x}_0
\end{bmatrix} =
\begin{bmatrix}
A_i & 0 \\
0 & A_0
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_0
\end{bmatrix} +
\begin{bmatrix}
B_i \\
0
\end{bmatrix} u_i,$n

$$e_{di} = y_{di} - y_{d0} =
\begin{bmatrix}
C_{di} & -C_0
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_0
\end{bmatrix}.$$

We can then construct a new state $\chi_i = W_i \begin{bmatrix} x_i \\ x_0 \end{bmatrix}$ and perform the state transformation such that

$$\dot{\chi}_i = (\mathcal{A} + \mathcal{L}_i) \chi_i + \mathcal{B}_i u_i,$n

$$e_{di} = C \chi_i,$n

(25a)-(25b)

where $\mathcal{A} = \begin{bmatrix} 0 & I_{p(\pi - 1)} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{p \times p \pi}$, $\mathcal{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}$, $\mathcal{B}_i = T_i \mathcal{B}$, $C = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in \mathbb{R}^{p \times p \pi}$ for some matrix $L_i \in \mathbb{R}^{p \times p \pi}$.

Therefore, it follows that

$$\dot{\chi}_i = (\mathcal{A} + \mathcal{L}_i) \chi_i + \mathcal{B}_i u_i,$n

$$e_{di} = C \chi_i,$n

(26a)-(26b)

Based on the neighbor-based group output information $\zeta_i$, the distributed observer is proposed for (26) as

$$\hat{\chi}_i = (\mathcal{A} + \mathcal{L}_i) \hat{\chi}_i + \mathcal{B}_i \hat{u}_i + S(\xi) \mathcal{P} \mathcal{G}_i^T$$

---

**Fig. 4. Control architecture for agent $v_i$**
\begin{align}
\times \left(\sum_{i=0}^{n} a_{ij}(t)(y_{di} - y_{dj}) - \sum_{j=0}^{n} a_{ij}(t)(\hat{y}_i - \hat{y}_j)\right), \quad (27a)
\end{align}

\begin{align}
[x_{i}^T, \hat{x}_{i0}]^T = W_{ij}^{-1}(T_{ij}^{T}T_{ij})^{-1}T_{ij}^{T}\tilde{x}_{i}, \quad i = 1, 2, \ldots, n, \quad (27b)
\end{align}

where $a_{ij}(t), i = 0, 1, \ldots, n, j = 0, 1, \ldots, n,$ is an entry $(i, j)$ of the adjacency matrix $\tilde{A}_{\sigma(i)}$ associated with $\tilde{G}_{\sigma(i)}$ defined in Section 2 at time $t$, $\hat{x}_i = \tilde{G}_{\sigma(i)} \hat{y}_i, i = 1, 2, \ldots, n$, $\hat{y}_0 = 0$. In addition, $S(\omega) = \text{diag}(I_p \epsilon^{-1}, I_p \epsilon^{-2}, \ldots, I_p \epsilon^{-\bar{n}})$, where $\epsilon \in (0, 1]$ is a positive constant, and $P = P^T$ is a positive definite matrix satisfying

$$A^TP + P^T(2\theta P - \epsilon) \leq 0, \quad (28)$$

where $\theta$ is a positive constant satisfying $\theta < \frac{1}{2} \min_{i \in \mathbb{Z}_+} \min \Re\{\lambda_{i}(L_k)\}.$

**Step III: individual observer**

Based on the information of $\hat{x}_i$ and the individual output information $y_{si}$, the following individual observer for each agent is proposed

$$\hat{\dot{x}}_i = S_i \hat{x}_i + K_{si}(C_{si} \hat{x}_i + C_{wi} \hat{\omega}_i - y_{si}), \quad i = 1, 2, \ldots, n, \quad (29)$$

where $K_{si}$ is chosen such that $S_i + K_{si}C_{wi}$ is Hurwitz stable.

**Step IV: controller design**

The observer-based controller is proposed as

$$u_i = F_i \hat{x}_i + (\Gamma_{1i} - F_i \Pi_{1i}) \hat{\omega}_i + (\Gamma_{2i} - F_i \Pi_{2i}) \hat{\omega}_0, \quad (30)$$

where $\Pi_{1i}, \Gamma_{1i}, \Pi_{2i},$ and $\Gamma_{2i}$ are the solutions of the following regulator equations

\begin{align}
\Pi_{1i}S_i &= A_i \Pi_{1i} + B_i \Gamma_{1i}, \quad (31a) \\
0 &= D_i \Pi_{1i} + D_{wi}, \quad (31b) \\
\Pi_{2i}A_0 &= A_i \Pi_{2i} + B_i \Gamma_{2i}, \quad (31c) \\
0 &= D_i \Pi_{2i} + D_0, \quad i = 1, 2, \ldots, n. \quad (31d)
\end{align}

and $F_i$ is chosen such that $A_i + B_i F_i$ is Hurwitz.

**Corollary 3** Let Assumptions 1, 2, 3 (the first item is replaced by that $(A_1, C_{si})$ is observable), and 4 hold and assume that $\kappa > \alpha + 4\theta \max_{i \in \mathbb{Z}_+} \epsilon_i$, where $\alpha \in (0, 1)$, $\theta$ and $\epsilon$ are given by (28). Also, let $\hat{x}_i$ and $\hat{\omega}_0$ be obtained in (27), and $\hat{\omega}_i$ be obtained in (29). Then, there exists $\epsilon_i^* \in (0, 1)$ such that, if $\epsilon \in (0, \epsilon_i^*)$, (30) ensures that $\lim_{t \to \infty} \epsilon_i(t) = 0, \ i = 1, 2, \ldots, n,$ for the multi-agent system (1)-(4).

**Proof:** See [1].

---

**6 Simulation Results**

In this section, we illustrate the theoretical results. Consider a network of three agents as shown in Fig. 2. We assume that the adjacency matrix $\tilde{A}_{\sigma(1)}$ associated with $\tilde{G}_{\sigma(1)}$ is switching periodically. Denote $\ell = 0, 20, 40, \ldots$

\begin{align}
&= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{when } t \in [\ell, \ell + 6), \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{when } t \in [\ell + 2, \ell + 8),
\end{align}

Also, let $\hat{x}_i$ and $\hat{\omega}_0$ be obtained in (27), and $\hat{\omega}_i$ be obtained in (29). Then, there exists $\epsilon_i^* \in (0, 1)$ such that, if $\epsilon \in (0, \epsilon_i^*)$, (30) ensures that $\lim_{t \to \infty} \epsilon_i(t) = 0, \ i = 1, 2, \ldots, n,$ for the multi-agent system (1)-(4).

**Example 1**

We give an example to validate Theorem 1, the dynamics of the agents are described as $A_1 = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$C_{s1} = C_{d1} = D_{s1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_{s2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_{d2} = \begin{bmatrix} 0 \end{bmatrix}, \quad D_{s2} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$C_{s3} = C_{d3} = D_{s3} = \begin{bmatrix} 0 \end{bmatrix}$. The dynamics of the individual autonomous exosystems are described as $S_i = 0, \quad C_{wi} = D_{wi} = -1, \ i = 1, 2, 3, \quad \omega_1(0) = -2, \quad \omega_2(0) = -4, \quad \omega_3(0) = -6$. The dynamics of the group autonomous exosystem are described as $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

$D_0 = -C_0$.

Following the design scheme proposed in Section 4, for the solutions of the regulator equations (7), we have that $F_1 = \begin{bmatrix} 1 & -4.5 & -6 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 1.0345 & -0.4138 \\ 0.1379 & 0.3448 \\ -0.1724 & 0.0690 \end{bmatrix}$.
Fig. 5. Output convergence of system (1), (2), and (3) under the observer-based controller (15) for Theorem 1

Fig. 6. Error convergence of system (1), (2), and (3) under the observer-based controller (15) for Theorem 1

Example 2

We next give an example to validate Corollary 2. In this section, the dynamics of the agents are described as

\[
A_1 = \begin{bmatrix}
0 & 3 & 0 \\
0 & 0 & 2 \\
0 & -1 & 0 \\
\end{bmatrix},
B_1 = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix},
C_{d1} = D_{d1} = \begin{bmatrix}
1 & 1 & 1 \\
\end{bmatrix}.
\]

\[
A_2 = \begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix},
B_2 = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix},
C_{d2} = D_{d2} = \begin{bmatrix}
1 & 0 \\
\end{bmatrix}.
\]

\[
A_3 = \begin{bmatrix}
0 & 1 \\
-2 & -2 \\
\end{bmatrix},
B_3 = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix},
C_{d3} = D_{d3} = \begin{bmatrix}
1 & 0 \\
\end{bmatrix}.
\]

The dynamics of the individual autonomous exosystem are described as \( \theta = 0 \) for (10). We also have \( \Pi = \begin{bmatrix}
0.0690 & 0.1724 \\
1.0345 & -0.4138 \\
0.1379 & 0.3448 \\
-0.1724 & 0.0690 \\
\end{bmatrix} \)
for agent \( v_1 \), \( F_1 = \begin{bmatrix}
-1 & -4.5 & -6 \\
\end{bmatrix} \).

\( \Gamma_1 = \begin{bmatrix}
0.0690 & 0.1724 \\
-1 & -4.5 & -6 \\
0 & 1.0345 & -0.4138 \\
0 & 0.1379 & 0.3448 \\
0 & -0.1724 & 0.0690 \\
\end{bmatrix} \)
for agent \( v_1 \), \( F_2 = \begin{bmatrix}
-2 & -3 \\
\end{bmatrix} \).

\( \Pi_2 = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \Gamma_2 = \begin{bmatrix}
-1 & 0 \\
\end{bmatrix} \)
for agent \( v_2 \), \( F_3 = \begin{bmatrix}
0 & 1 \\
\end{bmatrix} \).

\( \Pi_3 = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \Gamma_3 = \begin{bmatrix}
2 & 1 \\
\end{bmatrix} \)
for agent \( v_3 \). We also have \( \epsilon = 0.2 \) for (9) and \( \theta = 0.1 \) for (10).

Figs. 7 and 8 show, respectively, the state convergence and the error convergence of system (1), (2), and (3) under the observer-based controller (22). We see that coordinated output regulation is realized even when there exists multiple heterogenous dynamics and the information interactions are switching. This agrees with Corollary 2.

Example 3

We give an example to validate Corollary 3. The dynamics of the agents are described as

\[
A_1 = \begin{bmatrix}
0 & 3 & 0 \\
0 & 0 & 2 \\
0 & -1 & 0 \\
\end{bmatrix},
B_1 = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix},
C_{d1} = D_{d1} = \begin{bmatrix}
1 & 1 & 1 \\
\end{bmatrix}.
\]

\[
A_2 = \begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix},
B_2 = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix},
C_{d2} = D_{d2} = \begin{bmatrix}
1 & 0 \\
\end{bmatrix}.
\]

\[
A_3 = \begin{bmatrix}
0 & 1 \\
-2 & -2 \\
\end{bmatrix},
B_3 = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix},
C_{d3} = D_{d3} = \begin{bmatrix}
1 & 0 \\
\end{bmatrix}.
\]

Figs. 7 and 8 show, respectively, the state convergence and the error convergence of system (1), (2), and (3) under the observer-based controller (22). We see that coordinated output regulation is realized even when there exists multiple heterogenous dynamics and the information interactions are switching. This agrees with Corollary 2.
Following the design scheme proposed in Section 5.2, for the solutions of regulator equations (31), we have that $F_i = \begin{bmatrix} -1 & -4.5 & -6 \end{bmatrix}$, $\Pi_{i1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\Gamma_{i1} = 0$, $\Pi_{2i} = \begin{bmatrix} 1.0345 & -0.4138 \\ 0.1379 & 0.3448 \\ -0.1724 & 0.0690 \end{bmatrix}$, $\Gamma_{21} = \begin{bmatrix} 0.0690 & 0.1724 \end{bmatrix}$ for agent $v_1$, $F_3 = \begin{bmatrix} -2 & -3 \end{bmatrix}$, $\Pi_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\Gamma_{12} = 0$, $\Pi_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma_{22} = \begin{bmatrix} -1 & 0 \end{bmatrix}$ for agent $v_2$, $F_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}$, $\Pi_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\Gamma_{13} = -2$, $\Pi_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma_{23} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ for agent $v_3$. We also have $\epsilon = 0.2$ for (27), $\theta = 0.1$ for (28), and $K_{si} = 1$, $i = 1, 2, 3$ for (29).

Figs. 9 and 10 show, respectively, the state convergence and the error convergence of system (1), (2), and (3) under the observer-based controller (30). We see that coordinated output regulation is realized even when there exists multiple heterogeneous dynamics and the information interactions are switching. This agrees with Corollary 3.

7 Conclusions

This paper studied the coordinated output regulation problem of multiple heterogeneous linear systems. We first formulated the coordinated output regulation problem and specified the information that is available for each agent. A high-gain based distributed observer and an individual observer were introduced for each agent and observer-based controllers were designed to solve the problem. The information interactions among the agents and the group autonomous exosystem were allowed to be switching over a finite set of fixed networks containing both the graph having a spanning tree and the graph having not. The relationship of the information interactions, the dwell time, the non-identical dynamics and the information interactions are switching. This agrees with Corollary 3.

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Fig. 9. Output convergence of system (1), (2), and (3) under the observer-based controller (30) for Corollary 3

Fig. 10. Error convergence of system (1), (2), and (3) under the observer-based controller (30) for Corollary 3

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