Long time stability of KAM tori for nonlinear wave equation *

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Abstract: It is proved that the KAM tori (thus quasi-periodic solutions) are long time stable for infinite dimensional Hamiltonian systems generated by nonlinear wave equation, by constructing a partial normal form of higher order around the KAM torus and showing \( p \)-tame property persists under KAM iterative procedure and normal form iterative procedure.

Key words: KAM tori, Normal form, Stability, \( p \)-tame property, KAM technique.

1 Introduction and main results

Since the initial work \cite{13, 14, 15, 18} of infinite dimensional KAM theory by Kuksin and Wayne, there has been a lot of work about the existence of KAM tori for the nonlinear wave equation (NLW)

\[
u_{tt} - u_{xx} + V(x)u + u^3 + h.o.t. = 0
\]  

subject to Dirichlet boundary conditions \( u(t, 0) = u(t, \pi) = 0 \). For examples, Wayne \cite{18} obtains the existence when \( V(x) \) does not belong to some set of “bad” potentials; Kuksin \cite{15} considers parameterized potentials \( V(x, \xi) \) and shows that there are many quasi-periodic solutions for “most” parameters \( \xi \)'s; Pöschel \cite{17} proves that the potential \( V = V(x) \) can be replaced by a fixed constant potential \( V \equiv m \); Yuan \cite{19} shows the existence of KAM tori for any prescribed non-constant potential.

All of these results are obtained by the classic KAM iteration which involves the so-called second Melnikov conditions. Thus, every KAM torus is linearly stable, and around it a normal form of order 2 is obtained. Based on the normal form, one can directly obtain \( \delta^{-1} \) long time stability of KAM tori, where \( \delta \) is the distance between initial data of the solutions and KAM tori.

A natural question is whether the KAM tori are stable in a longer time such as \( \delta^{-\mathcal{M}} \) for any given \( \mathcal{M} \geq 0 \). There has been a lot of work on the long time stability of the equilibrium point \( u = 0 \) and some approximate invariant tori for partial differential equations. For examples, see \cite{1, 2, 4, 5, 6, 8, 10, 12}. It is proposed by Eliasson \cite{11} that whether such stability results can be proven in a neighborhood of a given KAM torus. Also see \cite{7} by Berti and Biasco. Recently, the long time stability of KAM tori for nonlinear Schrödinger equation was obtained in \cite{9}. The basic

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idea is that, first, define a suitable $p$-tame property which generalizes the idea in [5]; second, prove the $p$-tame property persists under KAM iterative procedure and normal form iterative procedure; third, based on some reasonable non-resonant conditions, one can construct a partial normal form of high order around a KAM torus; finally, combining the $p$-tame property with the partial normal form of high order, one can deduce that the solutions starting in the neighborhood of a KAM torus still stay in the neighborhood of the KAM torus for a polynomial long time, that is, the KAM tori are stable in a long time.

In this paper, we will prove the long time stability of KAM tori for nonlinear wave equation. The method follows a parallel course as in [9] except two essential differences: (1) more regularity is needed for the Hamiltonian vector field generated by nonlinear wave equation to guarantee that the KAM iterative procedure works (it is necessary for measure estimate and see [10] for the details); (2) since the frequencies of nonlinear wave equation has worse approximations than nonlinear Schrödinger equation, whether the non-resonant conditions hold when constructing the partial normal form of higher order? We will deal with the first problem by modifying the definition of $p$-tame norm (see (2.3) in Definition 2.4) and estimate the measure of non-resonant set by the method as in [3] (see Section 5 for the details). The following is our main result:

**Theorem 1.1.** Consider the nonlinear wave equation

$$u_{tt} = u_{xx} - (m + M_\xi)u + \varepsilon u^3,$$

subject to Dirichlet boundary conditions $u(t, 0) = u(t, \pi) = 0$, where $m$ is a non-negative constant and $M_\xi$ is a real Fourier multiplier,

$$M_\xi \sin jx = \xi_j \sin jx,$$  (1.3)

Given an integer $n \geq 1$ and a real number $p \geq 1$, for any sufficiently small $\varepsilon > 0$, there exists a large subset $\Pi \subset \Pi$ such that for each $\xi \in \Pi$ equation (1.2) possesses a linearly stable $n$-dimensional KAM torus $\mathcal{T}_\xi$ in Sobolev space $H^p_0([0, \pi])$. Moreover, for arbitrarily given $\mathcal{M}$ with $0 \leq \mathcal{M} \leq C(\varepsilon)$ (where $C(\varepsilon)$ is a constant depending on $\varepsilon$ and $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$) and $p \geq 24(\mathcal{M} + 7)^3 + 1$, there exists a small positive $\delta_0$ depending on $n, p$ and $\mathcal{M}$, and for any $0 < \delta < \delta_0$, any solution $u(t, x)$ of equation (1.2) with the initial datum satisfying

$$d_{H^p_0[0, \pi]}(u(0, x), \mathcal{T}_\xi) := \inf_{w \in \mathcal{T}_\xi} ||u(0, x) - w||_{H^p_0[0, \pi]} \leq \delta,$$

then

$$d_{H^p_0[0, \pi]}(u(t, x), \mathcal{T}_\xi) := \inf_{w \in \mathcal{T}_\xi} ||u(t, x) - w||_{H^p_0[0, \pi]} \leq 2\delta, \quad \text{for all } t \leq \delta^{-\mathcal{M}}.$$

**Remark 1.** Instead of equation (1.2), we can also prove the long time stability of KAM tori for general nonlinear wave equations, such as

$$u_{tt} = u_{xx} - V(x)u + \varepsilon g(x, u),$$  (1.4)

where $V(x)$ is a smooth and $2\pi$ periodic potential, and $g(x, u)$ is a smooth function on the domain $\mathbb{T} \times \mathscr{U}$, $\mathscr{U}$ being a neighborhood of the origin in $\mathbb{R}$. Equation (1.4) was discussed in [5] and shown that the origin is stable in long time by the infinite dimensional Birkhoff normal form theorem.
The rest of the present paper is organized as follows. In §2, we give some basic notations and the definition of $p$-tame norm for a Hamiltonian vector field. In §3, we construct a normal form of order 2, which satisfies $p$-tame property, around the KAM tori based on the standard KAM method (see Theorem 3.1) and a partial normal form of order $\mathcal{M} + 2$ in the $\delta$-neighborhood of the KAM tori (see Theorem 3.2). Since the iterative procedure is parallel to [9], we only prove the measure estimate in detail. Finally, due to the partial normal form of order $\mathcal{M} + 2$ and $p$-tame property, we show that the KAM tori are stable in a long time (see Theorem 3.3). In §4, we finish the proof of Theorem 1.1.

In §5, we list some properties of $p$-tame norm. These properties are used in the proof of Theorem 3.1 and Theorem 3.2 to ensure the $p$-tame property surviving under KAM iterative procedure and normal form iterative procedure.

2 Definition of $p$-tame norm for a Hamiltonian vector field

We will define $p$-tame norm for a Hamiltonian vector field in this section. First we introduce the functional setting and the main notations concerning infinite dimensional Hamiltonian systems. Consider the Hilbert space of complex-valued sequences $\ell^2_p := \left\{ q = (q_1, q_2, \ldots) : ||q||_p := \sum_{j \geq 1} |q_j|^j^{2p} < +\infty \right\}$ with $p > 1/2$, and the symplectic phase space

$$(x,y,z) \in D(s) \times \mathbb{C}^n \times \ell^2_{b,p} := \mathcal{S}^p, \quad z := (q, \bar{q}) \in \ell^2_{b,p} := \ell^2_p \times \ell^2_p,$$

where $D(s) := \{ x \in \mathbb{C}^n / (2\pi \mathbb{Z})^n \mid ||\text{Im } x|| < s \}$ is the complex open $s$-neighborhood of the $n$-torus $\mathbb{T}^n := \mathbb{R}^n / (2\pi \mathbb{Z})^n$, equipped with the canonical symplectic structure:

$$\sum_{j=1}^n dy_j \wedge dx_j + \sqrt{-1} \sum_{j \geq 1} dq_j \wedge d\bar{q}_j.$$

Let

$$D(s, r_1, r_2) = \left\{ (x,y,z) \in \mathcal{S}^p \mid ||\text{Im } x|| < s, ||y|| < r_1, ||z||_p < r_2 \right\},$$

where $|| \cdot ||$ denote the sup-norm for complex vectors and

$$||z||_p = ||q||_p + ||\bar{q}||_p \quad \text{with } z = (q, \bar{q}).$$

Let $(x,y,z) \in D(s, r_1, r_2)$, any analytic function $W : D(s, r_1, r_2) \to \mathbb{C}$ can be developed in a totally convergent power series

$$W(x,y,z) = \sum_{\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^2} W^{\alpha \beta}(x)^\alpha z^\beta, \quad \bar{Z} = \mathbb{Z} \setminus \{0\}.$$

Note that there is a multilinear, symmetric, bounded map

$$W^{\alpha \beta}(x) \in \mathcal{L}\left( \mathbb{C}^n \times \cdots \times \mathbb{C}^n \times \ell^2_{b,p} \times \cdots \times \ell^2_{b,p}, \mathbb{C} \right)$$

with $|\alpha| - \text{times} \mathbb{C}^n \times \cdots \times \mathbb{C}^n$ and $|\beta| - \text{times} \ell^2_{b,p} \times \cdots \ell^2_{b,p}$. 

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such that
\[ \tilde{W}^{\alpha\beta}(x)(y_1, \ldots, y_n; z_1, \ldots, z_n) = W^{\alpha\beta}(x)y^\alpha z^\beta, \]
where \(|\alpha| = \sum_{j=1}^n |\alpha_j|, |\beta| = \sum_{j \in \mathbb{Z}} |\beta_j|, \) and \(|\cdot|\) denotes the 1-norm here and below.

We will study the Hamiltonian system
\[ (\dot{x}, \dot{y}, \dot{z}) = X_W(x, y, z), \]
where \(X_W\) is the Hamiltonian vector field of \(W\),
\[ X_W = (W_y, -W_x, \sqrt{-1}JW_z), \]
and
\[ J := \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right). \]

**Definition 2.1.** Consider a function \(W(x; \xi) : D(s) \times \Pi \to \mathbb{C}\) is analytic in the variable \(x \in D(s)\) and \(C^1\)-smooth in the parameter \(\xi \in \Pi\) in the Whitney's sense\(^1\) and the Fourier series of \(W(x; \xi)\) is given by
\[ W(x; \xi) = \sum_{k \in \mathbb{Z}^n} \hat{W}(k; \xi)e^{\sqrt{-1}k \cdot x}, \]
where
\[ \hat{W}(k; \xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} W(x; \xi)e^{-\sqrt{-1}k \cdot x}dx \]
is the \(k\)-th Fourier coefficient of \(W(x; \xi)\), and \(\langle \cdot, \cdot \rangle\) denotes the usual inner product, i.e.
\[ \langle k, x \rangle = \sum_{j=1}^n k_jx_j. \]
Then define the norm \(|| \cdot ||_{D(s) \times \Pi}\) of \(W(x; \xi)\) by
\[ ||W||_{D(s) \times \Pi} = \sup_{\xi \in \Pi, \|y\| \leq r^2} \left( \sum_{j \geq 1} \|\hat{W}(k; \xi)\|^2 \right)^{1/2}. \] (2.1)

**Definition 2.2.** Let
\[ D(s, r) = \{(x, y) \in D(s) \times \mathbb{C}^n \mid \|Im x\| < s, \|y\| < r^2 \}. \]
Consider a function \(W(x, y; \xi) : D(s, r) \times \Pi \to \mathbb{C}\) is analytic in the variable \((x, y) \in D(s, r)\) and \(C^1\)-smooth in the parameter \(\xi \in \Pi\) with the following form
\[ W(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} W^\alpha(x; \xi)y^\alpha. \]
Then define the norm \(|| \cdot \||_{D(s, r) \times \Pi}\) of \(W(x, y; \xi)\) by
\[ ||W||_{D(s, r) \times \Pi} = \sum_{\alpha \in \mathbb{N}^n} ||\hat{W}^\alpha||_2^{2|\alpha|}. \] (2.2)

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\(^1\)In the whole of this paper, the derivatives with respect to the parameter \(\xi \in \Pi\) are understood in the sense of Whitney.
where $\mathcal{H}^\alpha = \left\|W^\alpha(x,\xi)\right\|_{D(x)\times\Pi}^{\text{\tiny \text{\alpha-times}}}$, $\mathcal{H}^\alpha \in \mathcal{L}(\mathbb{C}^n \times \cdots \times \mathbb{C}^n, \mathbb{C})$ is an $|\alpha|$-linear symmetric bounded map such that

$$\mathcal{H}^\alpha(y,\cdots,y) = \mathcal{H}^\alpha y^\alpha,$$

and $\left\|\cdot\right\|$ is the operator norm of multilinear symmetric bounded maps.

**Definition 2.3.** Consider a function $W(x,y,z;\xi) : D(s,r) \times \Pi \rightarrow \mathbb{C}$ is analytic in the variable $(x,y,z) \in D(s,r)$ and $C^1$-smooth in the parameter $\xi \in \Pi$ with the following form

$$W(x,y,z;\xi) = \sum_{\beta \in \mathbb{N}^2} W^\beta(x,y;\xi)z^\beta.$$

Define the modulus $|W|_{D(s,r)\times\Pi}(z)$ of $W(x,y,z;\xi)$ by

$$|W|_{D(s,r)\times\Pi}(z) := \sum_{\beta \in \mathbb{N}^2} |W^\beta|_{D(s,r)\times\Pi} z^\beta. \quad (2.3)$$

**Definition 2.4.** Let

$$W(x,y,z;\xi) := W_h(x,y,z;\xi) = \sum_{\beta \in \mathbb{N}^2} W^\beta_h(x,y;\xi)z^\beta$$

be a function is analytic in the variable $(x,y,z) \in D(s,r)$ and $C^1$-smooth in the parameter $\xi \in \Pi$, and let

$$\left\|(z^\beta)\right\|_{p,1} := \frac{1}{h} \sum_{j=1}^{h} \left\|z^{(1)}\right\|_1 \cdots \left\|z^{(j-1)}\right\|_1 \left\|z^{(j)}\right\|_p \left\|z^{(j+1)}\right\|_1 \cdots \left\|z^{(h)}\right\|_1. \quad (2.4)$$

Define the $p$-tame operator norm for $W_z$ by

$$\left\|\left\|W_z\right\|\right\|_{p,D(s,r)\times\Pi} := \sup_{0 \neq z^\beta \in \mathbb{C}^2, 1 \leq j \leq h-1} \frac{\left\||W_z|_{D(s,r)\times\Pi}(z^{(1)},\ldots,z^{(h-1)})\right\|_{p+1}}{\left\|(z^{h-1})\right\|_{p,1}}, \quad (2.5)$$

define the 1-operator norm for $W_z$ by

$$\left\|\left\|W_z\right\|\right\|_{1,D(s,r)\times\Pi} := \sup_{0 \neq z^\beta \in \mathbb{C}^2, 1 \leq j \leq h-1} \frac{\left\||W_z|_{D(s,r)\times\Pi}(z^{(1)},\ldots,z^{(h-1)})\right\|_1}{\left\|(z^{h-1})\right\|_{1,1}}, \quad (2.6)$$

and define the operator norm for $W_v$ ($v = x$ or $y$) by

$$\left\|\left\|W_v\right\|\right\|_{D(s,r)\times\Pi} := \sup_{0 \neq z^\beta \in \mathbb{C}^2, 1 \leq j \leq h} \frac{\left\|W_v|_{D(s,r)\times\Pi}(z^{(1)},\ldots,z^{(h)})\right\|}{\left\|(z^{h})\right\|_{1,1}}, \quad (2.7)$$

Finally define the $p$-tame norm of the Hamiltonian vector field $Xw$ as follows,

$$\left\|\left\|X_w\right\|\right\|_{p,D(s,r)\times\Pi} = \left\|\left\|W_x\right\|\right\|_{D(s,r)\times\Pi} + \frac{1}{r} \left\|\left\|W_w\right\|\right\|_{D(s,r)\times\Pi} + \frac{1}{r} \left\|\left\|W_z\right\|\right\|_{p,D(s,r)\times\Pi}. \quad (2.8)$$
Remark 3. Based on (2.5) and (2.7) in Definition 2.4, for each of a bounded map form

\[ \text{property on the domain } D \]

Remark 2. Then define the p-tame norm of the Hamiltonian vector field \( X \)

\[ \text{as in [16], define} \]

Moreover, we say that a Hamiltonian vector field \( X \)

\[ \text{following estimates hold} \]

\[ \text{Theorem 3.1. (Normal form of order 2) Consider a perturbation of the integrable Hamiltonian} \]

\[ H(x, y, q, \xi) = N(y, q, \xi) + R(x, y, q, \xi) \]

\[ \text{where} \]

\[ \|W_v\|_{D(s,r,r) \times \Pi} := \|W_v\|_{D(s,r,r) \times \Pi}^h, \quad v = x \text{ or } y, \quad (2.9) \]

and

\[ \|W_z\|_{p,D(s,r,r) \times \Pi}^T = \max \left\{ \|W_z\|_{p,D(s,r,r) \times \Pi}^T, \|W_z\|_{1,D(s,r,r) \times \Pi} \right\}^h. \quad (2.10) \]

Remark 2. In view of (2.5), \( |JW_z|_{D(s,r,r) \times \Pi} \) is required as a bounded map form \( q^2 \) to \( q^2 + 1 \) instead of a bounded map form \( q^2 \) to \( q^2 + 1 \) as in [17].

Remark 3. Based on (2.5) and (2.7) in Definition 2.4, for each \( (x, y, z) \in \mathcal{P}^P \) and \( \xi \in \Pi \), the following estimates hold

\[ \|(W_h)_z(x, y, z; \xi)\|_p \leq \|(W_h)_z(x, y, z; \xi)\|_{p+1} \leq \|W_h\|_{p,D(s,r,r) \times \Pi}^T \|z\|_p \leq \|W_h\|_{1,D(s,r,r) \times \Pi} \max \{h-2, 0\}, \quad (2.11) \]

\[ \|(W_h)_x(x, y, z; \xi)\| \leq \|W_h\|_{p,D(s,r,r) \times \Pi}^T \|z\|_1^h, \quad (2.12) \]

and

\[ \|(W_h)_y(x, y, z; \xi)\| \leq \|W_h\|_{p,D(s,r,r) \times \Pi} \|z\|_1^h. \quad (2.13) \]

Definition 2.5. Let \( W(x, y, z; \xi) = \sum_{h \geq 0} W_h(x, y, z; \xi) \) be a Hamiltonian analytic in the variable \( (x, y, z) \in D(s, r, r) \) and \( C^1 \)-smooth in the parameter \( \xi \in \Pi \), where

\[ W_h(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^3, |\beta| = h} W^\beta_h (x, y; \xi) z^\beta. \]

Then define the p-tame norm of the Hamiltonian vector field \( X_W \) by

\[ \|X_W\|_{p,D(s,r,r) \times \Pi}^T := \sum_{h \geq 0} \|X_W\|_{p,D(s,r,r) \times \Pi}^T. \quad (2.14) \]

Moreover, we say that a Hamiltonian vector field \( X_W \) (or a Hamiltonian \( W(x, y, z; \xi) \)) has p-tame property on the domain \( D(s, r, r) \times \Pi \) if and only if \( \|X_W\|_{p,D(s,r,r) \times \Pi} < \infty \).

3 The abstract results

As in [16], define

\[ \|w\|_{\mathcal{P}^P, D(s,r,r)} = \|x\| + \frac{1}{r} \|y\| + \frac{1}{r} \|z\|_p \quad (3.1) \]

for each \( w = (x, y, z) \in D(s, r, r) \), and define the usual weighted norm of Hamiltonian vector field \( X_U \) on the domain \( D(s, r, r) \times \Pi \) by

\[ \|X_U\|_{\mathcal{P}^P, D(s,r,r) \times \Pi} = \sup_{(x,y,z;\xi) \in D(s,r,r) \times \Pi} \left( \|U_x\| + \frac{1}{r} \|U_y\| + \frac{1}{r} \|U_z\|_{p+1} \right). \quad (3.2) \]

Theorem 3.1. (Normal form of order 2) Consider a perturbation of the integrable Hamiltonian

\[ H(x, y, q, \tilde{q}; \tilde{\xi}) = N(y, q, \tilde{\xi}) + R(x, y, q, \tilde{q}; \tilde{\xi}) \]

\[ \text{where} \]

\[ \|W_v\|_{D(s,r,r) \times \Pi} := \|W_v\|_{D(s,r,r) \times \Pi}^h, \quad v = x \text{ or } y, \quad (2.9) \]

and

\[ \|W_z\|_{p,D(s,r,r) \times \Pi}^T = \max \left\{ \|W_z\|_{p,D(s,r,r) \times \Pi}^T, \|W_z\|_{1,D(s,r,r) \times \Pi} \right\}^h. \quad (2.10) \]
defined on the domain $D(s_0, r_0, r_0) \times \Pi$ with $s_0, r_0 \in (0, 1]$, where

$$N(y, q, \bar{q}; \bar{\xi}) = \sum_{j=1}^{n} \omega_j(\bar{\xi})y_j + \sum_{j \geq 1} \Omega_j(\bar{\xi})q_j \bar{q}_j$$

is a family of parameter dependent integrable Hamiltonian and

$$R(x, y, q, \bar{q}; \bar{\xi}) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}} R^{\alpha \beta \gamma}(x; \bar{\xi})y^\alpha q^\beta \bar{q}^\gamma$$

is the perturbation. Suppose the tangent frequency and normal frequency satisfy the following assumption:

**Frequency Asymptotics.**

$$\omega_j(\bar{\xi}) = \sqrt{j^2 + m + \xi_j}, \quad \text{for } 1 \leq j \leq n$$

and

$$\Omega_j(\bar{\xi}) = \sqrt{(j+n)^2 + m + \xi_{j+n}}, \quad \text{for } j \geq 1,$$

where $\xi = (\xi_j)_{j \geq 1} \in \Pi$. The perturbation $R(x, y, q, \bar{q}; \bar{\xi})$ has $p$-tame property on the domain $D(s_0, r_0, r_0) \times \Pi$ and satisfies the small assumption:

$$\varepsilon := |||X_0|||_{p, D(s_0, r_0, r_0)} \leq \eta^{1/2} \varepsilon, \quad \text{for some } \eta \in (0, 1),$$

where $\varepsilon$ is a positive constant depending on $s_0, r_0$ and $n$. Then there exists a subset $\Pi_\eta \subset \Pi$ with the estimate

$$\text{Meas } \Pi_\eta \geq (\text{Meas } \Pi)(1 - O(\eta^{1/2})).$$

For each $\xi \in \Pi_\eta$, there is a symplectic map

$$\Psi : D(s_0/2, r_0/2, r_0/2) \to D(s_0, r_0, r_0),$$

such that

$$\tilde{H}(x, y, q, \bar{q}; \bar{\xi}) := H \circ \Psi = \tilde{N}(x, y, q, \bar{q}; \bar{\xi}) + \tilde{R}(x, y, q, \bar{q}; \bar{\xi}),$$

where

$$\tilde{N}(x, y, q, \bar{q}; \bar{\xi}) = \sum_{j=1}^{n} \omega_j(\bar{\xi})y_j + \sum_{j \geq 1} \Omega_j(\bar{\xi})q_j \bar{q}_j$$

and

$$\tilde{R}(x, y, q, \bar{q}; \bar{\xi}) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}, 2|\alpha| + |\beta| + |\gamma| \geq 3} \tilde{R}^{\alpha \beta \gamma}(x; \bar{\xi})y^\alpha q^\beta \bar{q}^\gamma.$$  

Moreover, the following estimates hold:

1. (for each $\xi \in \Pi_\eta$, the symplectic map $\Psi : D(s_0/2, r_0/2, r_0/2) \to D(s_0, r_0, r_0)$ satisfies

$$||\Psi - id||_{p, D(s_0/2, r_0/2, r_0/2)} \leq c\eta^6 \varepsilon$$

and

$$|||D\Psi - Id|||_{p, D(s_0/2, r_0/2, r_0/2)} \leq c\eta^6 \varepsilon,$$
where on the left-hand side hand we use the operator norm\(^2\)

\[
|||D\Psi - 1d|||_{p,D(s_0/2,r_0/2)} = \sup_{0 \neq w \in D(s_0/2,r_0/2)} \frac{||(D\Psi - 1d)w||_{p,D(s_0,r_0)}}{||w||_{p,D(s_0,r_0)}};
\]

(2) the frequencies \(\tilde{\omega}(\xi)\) and \(\tilde{\Omega}(\xi)\) satisfy

\[
||\tilde{\omega}(\xi) - \omega(\xi)|| + \sup_{j \geq 1} ||\partial_{\xi_j}(\tilde{\omega}(\xi) - \omega(\xi))|| \leq c\eta^8 \varepsilon \tag{3.11}
\]

and

\[
||\tilde{\Omega}(\xi) - \Omega(\xi)||_{-1} + \sup_{j \geq 1} ||\partial_{\xi_j}(\tilde{\Omega}(\xi) - \Omega(\xi))||_{-1} \leq c\eta^8 \varepsilon, \tag{3.12}
\]

where

\[
||w = (w_i)_{i \geq 1}||_{-1} := \sup_{i \geq 1} |w_i(\xi)|; \tag{3.13}
\]

(3) the Hamiltonian vector field \(X_R\) of the new perturbed Hamiltonian \(\bar{R}(x,y,q,\bar{q};\xi)\) satisfies

\[
|||X_R|||_{p,D(s_0/2,r_0/2)} \leq \varepsilon(1 + c\eta^8 \varepsilon), \tag{3.14}
\]

where \(c > 0\) is a constant depending on \(s_0, r_0\) and \(n\).

**Remark 4.** This theorem is parallel to Theorem 2.9 in [9] and is essentially due to a standard KAM proof. The same as in [9], the tame property (3.14) of \(X_R\) can be verified explicitly in view of Lemmas 5.1, 5.5. Moreover, as a corollary of this theorem, the existence and time \(\delta^{-1}\) stability can be obtained directly (\(p\)-tame property is not necessary here).

Starting from the normal form of order 2 obtained in Theorem [3.1] we will further construct a partial normal form of order \(\mathcal{M} + 2\) through \(\mathcal{M}\)-times symplectic transformations under some non-resonant conditions. To this end, some notations are given first. Given a large \(\mathcal{N} \in \mathbb{N}\), split the normal frequency \(\Omega(\xi)\) and normal variable \((q,\bar{q})\) into two parts respectively, i.e.

\[
\Omega(\xi) = (\hat{\Omega}(\xi), \hat{\Omega}(\xi)), \quad q = (\bar{q}, \hat{q}), \quad \bar{q} = (\bar{q}, \hat{q}),
\]

where

\[
\hat{\Omega}(\xi) = (\Omega_1(\xi), \ldots, \Omega_{\mathcal{N}}(\xi)), \quad \hat{q} = (q_1, \ldots, q_{\mathcal{N}}), \quad \hat{\bar{q}} = (\bar{q}_1, \ldots, \bar{q}_{\mathcal{N}})
\]

are the low frequencies and

\[
\hat{\Omega}(\xi) = (\Omega_{\mathcal{N} + 1}(\xi), \ldots, \Omega_{\mathcal{N} + 2}(\xi), \ldots), \quad \hat{q} = (q_{\mathcal{N} + 1}, q_{\mathcal{N} + 2}, \ldots), \quad \hat{\bar{q}} = (\bar{q}_{\mathcal{N} + 1}, \bar{q}_{\mathcal{N} + 2}, \ldots)
\]

are the high frequencies. Given \(0 < \hat{\eta} < 1\), and \(\tau > 2n + 5\), define the resonant sets \(\mathcal{R}_{\mathcal{M}, \hat{\eta}}\) by

\[
\mathcal{R}_{\mathcal{M}, \hat{\eta}} = \left\{ \xi \in \Pi_\eta : |(k, \tilde{\omega}(\xi)) + (\tilde{t}, \hat{\Omega}(\xi)) + (\tilde{t}, \hat{\Omega}(\xi))| \leq \frac{\hat{\eta}}{4^\mathcal{M}(\tau + 1)C(\mathcal{N}, \mathcal{M})} \right\}. \tag{3.15}
\]

\(^2\)where \(id\) denotes the identity map from \(\mathcal{P}^p \to \mathcal{P}^p\) and \(Id\) denotes its tangent map.
Let
\[ \mathcal{R} = \bigcup_{|k|+|l|+|\ell| \neq 0, |l|+|\ell| \leq \mathcal{M}+2, |l| \leq 2} \mathcal{R}_{k\ell} \]  
(3.16)
and
\[ \tilde{\Pi}_\eta = \Pi_\eta \setminus \mathcal{R}, \]  
(3.17)
where \( \Pi_\eta \) is defined in Theorem 3.1.

**Theorem 3.2.** (Partial normal form of order \( \mathcal{M} + 2 \)) Consider the normal form of order 2
\[ \hat{H}(x,y,q,\tilde{q};\tilde{\xi}) = \tilde{N}(y,q,\tilde{q};\tilde{\xi}) + \tilde{R}(x,y,q,\tilde{q};\tilde{\xi}) \]
obtained in Theorem 3.1. Given any positive integer \( \mathcal{M} \) and 0 < \( \tilde{\eta} \) < 1, there exist a small \( \rho_0 > 0 \) and a large positive integer \( \mathcal{N}_0 \) depending on \( s_0, r_0, n \) and \( \mathcal{M} \), such that for each 0 < \( \rho < \rho_0 \), any integer \( \mathcal{N} \) satisfying
\[ \mathcal{N}_0 < \mathcal{N} < \left( \frac{\tilde{\eta}}{2\rho} \right)^{\frac{1}{2s_0, \mathcal{M}+1}}, \]  
(3.18)
the non-resonant set \( \tilde{\Pi}_\eta \) fulfills the estimate
\[ \text{Meas } \tilde{\Pi}_\eta \geq (\text{Meas } \Pi_\eta)(1 - c\tilde{\eta}^{1/2}), \]  
(3.19)
and for any \( \xi \in \tilde{\Pi}_\eta \), there is a symplectic map
\[ \Phi : D(s_0/4, 4\rho, 4\rho) \to D(s_0/2, 5\rho, 5\rho), \]
such that
\[ \hat{H}(x,y,q,\tilde{q};\tilde{\xi}) := \hat{H} \circ \Phi = \tilde{N}(y,q,\tilde{q};\tilde{\xi}) + Z(y,q,\tilde{q};\tilde{\xi}) + P(x,y,q,\tilde{q};\tilde{\xi}) + Q(x,y,q,\tilde{q};\tilde{\xi}) \]  
(3.20)
is a partial normal form of order \( \mathcal{M} + 2 \), where
\[ Z(y,q,\tilde{q};\tilde{\xi}) = \sum_{4 \leq |\alpha|+|\beta|+2|\mu| \leq \mathcal{M}+2, |\mu| \leq 1} Z^{\alpha\beta\mu}(\tilde{\xi}) y^{\alpha} \tilde{q}^{\beta} \tilde{q}^{\mu} \tilde{q}^{\nu} \]
is the integrable term depending only on \( y \) and \( I_j = |q_j|^2, j \geq 1 \), and where
\[ P(x,y,q,\tilde{q};\tilde{\xi}) = \sum_{2|\alpha|+|\beta|+|\mu|+|v| \geq \mathcal{M}+3, |\mu| + |v| \leq 2} P^{\alpha\beta\gamma\mu}(x,\tilde{\xi}) y^{\alpha} \tilde{q}^{\beta} \tilde{q}^{\gamma} \tilde{q}^{\mu} \tilde{q}^{v} \]
and
\[ Q(x,y,q,\tilde{q};\tilde{\xi}) = \sum_{|\mu| + |v| \geq 3} Q^{\alpha\beta\gamma\mu}(x,\tilde{\xi}) y^{\alpha} \tilde{q}^{\beta} \tilde{q}^{\gamma} \tilde{q}^{\mu} \tilde{q}^{v}. \]
Moreover, we have the following estimates:
(1) the symplectic map \( \Phi \) satisfies
\[ ||\Phi - \text{id}||_{p,D(s_0/4, 4\rho, 4\rho)} \leq \frac{c\mathcal{M}^{294}}{\tilde{\eta}^2} \rho \]  
(3.21)
and
\[ |||D\Phi - Id|||_{p,D(s_0/4,4\rho,4\rho)} \leq \frac{cN^294}{\eta^2}; \]  
(3.22)

(2) the Hamiltonian vector fields \(X_Z, X_P\) and \(X_Q\) satisfy
\[ |||X_Z|||_{T\pi, p, D(s_0/4,4\rho,4\rho)} \leq c \rho \left( \frac{1}{\eta^2} \mathcal{N}^6(\mathcal{M}+7)^2 \rho \right)^{M}, \]  
(3.23)
and
\[ |||X_Q|||_{T\pi, p, D(s_0/4,4\rho,4\rho)} \leq c \rho, \]

where \(c > 0\) is a constant depending on \(s_0, r_0, n\) and \(\mathcal{M}\).

Proof. In this proof, Lemmas 5.1-5.5 are used, and the normal form iterative procedure is the same as Theorem 5.1 in [9]. Thus, we prove the measure estimate (3.19) in detail while omit the other parts of proof. Firstly, we will estimate the measure of the resonant sets \(R_{k\tilde{l}\hat{l}}\).

Case 1.
For \(|k| \neq 0\), without loss of generality, we assume
\[ |k_1| = \max_{1 \leq i \leq n} \{|k_1|, \ldots, |k_n|\}. \]  
(3.24)

Then
\[ |\partial_{k_1} (\langle k, \tilde{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle)| \]
\[ \geq |k_1||\partial_{k_1} \tilde{\omega}(\xi)| - |\partial_{k_1} (\sum_{i=2}^{n} k_i \tilde{\omega}(\xi) + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle)| \]
\[ \geq |k_1|(1 - c\eta^8 \varepsilon) - \left( \sum_{i=2}^{n} |k_i| + |\tilde{l}| + |\hat{l}| \right) c\eta^8 \varepsilon \quad \text{in view of (3.4), (3.5), (3.11) and (3.12)} \]
\[ \geq |k_1| - (|k| + \mathcal{M} + 2)c\eta^8 \varepsilon \quad \text{in view of } |\tilde{l}| + |\hat{l}| \leq \mathcal{M} + 2 \]
\[ \geq \frac{1}{4} |k_1| \quad \text{(by (3.24) and } \mathcal{M} \leq (2c\eta^8 \varepsilon)^{-1}) \]
\[ \geq \frac{1}{4}. \]

Hence,
\[ \text{Meas } \mathcal{R}_{k\tilde{l}\hat{l}} \leq \frac{4\eta}{4^3 \mathcal{M} (|k|+1)^2 C(\mathcal{N}, \tilde{l}) \cdot \text{Meas } \Pi_\eta}. \]  
(3.25)

Case 2.
If \(|k| = 0\) and \(|\tilde{l}| \neq 0\), without loss of generality, we assume
\[ |\tilde{l}_j| \neq 0 \]
and
\[ \tilde{l}_i = 0, \quad 1 \leq i \leq j - 1. \]
Then
\[ |\partial_{s_{m+1}}((k, \hat{\omega}(\xi)) + (\vec{l}, \hat{\Omega}(\xi)) + \langle \vec{l}, \hat{\Omega}(\xi) \rangle)| \]
\[ \geq |\vec{l}_j| |\partial_{s_{m+1}}\hat{\Omega}_j(\xi)| - |\partial_{s_{m+1}}((\vec{l}, \hat{\Omega}(\xi)) + \langle \vec{l}, \hat{\Omega}(\xi) \rangle - \vec{l}_j\hat{\Omega}_j(\xi))| \]
\[ \geq \left( \sqrt{(j+n)^2 + m + \xi_j} \right)^{-1} \left( |\vec{l}_j| (1 - c \eta^8 \varepsilon) - \left( \sum_{i=j+1}^{N} |\vec{l}_i| + |\vec{l}| \right) c \eta^8 \varepsilon \right) \] (by (3.5) and (3.12))
\[ \geq \left( \sqrt{(j+n)^2 + m + \xi_j} \right)^{-1} (|\vec{l}_j| - (\mathcal{M} + 2) c \eta^8 \varepsilon) \quad \text{(in view of } |\vec{l}| + |\vec{l}| \leq \mathcal{M} + 2\text{)}
\[ \geq \frac{1}{\mathcal{N}} |\vec{l}_j| \quad \text{(in view of } \mathcal{M} \leq (2c \eta^8 \varepsilon)^{-1} \text{ and } j \leq \mathcal{N})
\[ \geq \frac{1}{\mathcal{N}}. \]

Hence,
\[ \text{Meas } \mathcal{R}_{0\vec{l}} \leq \frac{4\hat{\eta} \mathcal{N}}{4^3.3 \mathcal{C}(\mathcal{N}, \vec{l})} \cdot \text{Meas } \Pi_\eta. \quad (3.26) \]

**Case 3.**
If \(|k| = 0, |\vec{l}| = 0 \text{ and } 1 \leq |\vec{l}| \leq 2\), then it is easy to see that \(|\vec{l}, \hat{\Omega}(\xi))| \text{ is not small, i.e.}
\[ \text{the sets } \mathcal{R}_{0\vec{l}} \text{ are empty for } |k| = 0, |\vec{l}| = 0 \text{ and } 1 \leq |\vec{l}| \leq 2. \quad (3.27) \]

Now we would like to estimate the measure of \( \mathcal{R} \) (see (3.16)). Following the notations in [3], we define the set
\[ \mathcal{L}_{n,\mathcal{N}} := \left\{ (k, \vec{l}, \hat{l}) \in \mathbb{Z}^n \times \mathbb{Z}^{\mathcal{N}} \times \mathbb{Z}^N \setminus \{0,0,0\} : |\vec{l}| \leq 2 \right\}, \]
and we split
\[ \mathcal{L} := \left\{ \vec{l} \in \mathbb{Z}^N : |\vec{l}| \leq 2 \right\} \]
as the union of the following four disjoint sets
\[ \mathcal{L}_0 = \{ \hat{l} = 0 \}, \]
\[ \mathcal{L}_1 = \{ \hat{l} = e_j \}, \]
\[ \mathcal{L}_{2+} = \{ \hat{l} = e_i + e_j \}, \]
\[ \mathcal{L}_{2-} = \{ \hat{l} = e_i - e_j, i \neq j \}, \]
where
\[ e_j = (0, \ldots, 0, 1, 0, \ldots) \]
and \( i, j \geq n + \mathcal{N} + 1 \).

Let \(|\vec{l}| = 2 \text{ and } \hat{l} = e_i + e_j \in \mathcal{L}_{2+} \text{ for some } i, j \geq n + \mathcal{N} + 1\). If
\[ \min\{i, j\} \geq |k| \cdot ||\mathcal{O}(\xi)|| + 2(\mathcal{M} + 2) \mathcal{N} + 1, \]
then it is easy to see that
\[ \left| (k, \hat{\omega}(\xi)) + (\vec{l}, \hat{\Omega}(\xi)) + \langle \vec{l}, \hat{\Omega}(\xi) \rangle \right| \geq 1, \]
which is not small. Namely, the resonant sets $\mathcal{R}_{k,l}$ is empty. So it is sufficient to consider
\[
\max\{i, j\} < |k| \cdot ||\tilde{\phi}(\xi)|| + 2(\mathcal{M} + 2)\mathcal{N} + 1,
\]
when the estimate (3.28) is given below. In fact, we obtain
\[
\begin{align*}
\text{Meas} \bigcup_{(k,i,l) \in X_n \cap L_2} \mathcal{R}_{k,l} & \leq \sum_{k \neq 0, (k,i,l) \in X_n \cap L_2} \frac{4\tilde{\eta}}{4\mathcal{M}(|k| + 1)^\tau C(\mathcal{N}, l)} \cdot \text{Meas} \Pi_\eta \\
& + \sum_{k = 0, (k,i,l) \in X_n \cap L_2} \frac{4\tilde{\eta}\mathcal{N}}{4\mathcal{M} C(\mathcal{N}, l)} \cdot \text{Meas} \Pi_\eta \\
& \leq c\tilde{\eta} \cdot \text{Meas} \Pi_\eta, \quad (3.28)
\end{align*}
\]
where $c_1 > 0$ is a constant depending on $n$ and $\tau$.

Similarly we obtain
\[
\begin{align*}
\text{Meas} \bigcup_{(k,i,l) \in X_n \cap L_0} \mathcal{R}_{k,l} & \leq c_2\tilde{\eta} \cdot \text{Meas} \Pi_\eta \quad (3.29)
\end{align*}
\]
and
\[
\begin{align*}
\text{Meas} \bigcup_{(k,i,l) \in X_n \cap L_1} \mathcal{R}_{k,l} & \leq c_2\tilde{\eta} \cdot \text{Meas} \Pi_\eta, \quad (3.30)
\end{align*}
\]
where $c_2 > 0$ is a constant depending on $n$ and $\tau$. Now let
\[(k, i, l) \in X_n \cap L_2,
\]
and assume $i > j$ without loss generality. In view of (3.5) and (3.12), there is a constant $C > 0$ such that
\[
\frac{\tilde{\Omega}_i(\xi) - \tilde{\Omega}_j(\xi)}{i - j} - 1 \leq \frac{C}{j}.
\]
Hence,
\[
\langle \tilde{i}, \tilde{\Omega}(\xi) \rangle = \tilde{\Omega}_i(\xi) - \tilde{\Omega}_j(\xi) = i - j + r_{ij},
\]
with
\[
|r_{ij}| \leq \frac{Cm}{j}
\]
and $m = i - j$. Then we have
\[
|\langle k, \tilde{\phi}(\xi) \rangle + \langle \tilde{i}, \tilde{\Omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle| \geq |\langle k, \tilde{\phi}(\xi) \rangle + \langle \tilde{i}, \tilde{\Omega}(\xi) \rangle + m| - |r_{ij}|.
\]
Therefore,
\[
\mathcal{R}_{k,l} \subset \mathcal{D}_{kimj} := \left\{ \langle k, \tilde{\phi}(\xi) \rangle + \langle \tilde{i}, \tilde{\Omega}(\xi) \rangle + m \leq \frac{\tilde{\eta}}{4\mathcal{M}(|k| + 1)^\tau C(\mathcal{N}, l)} + \frac{Cm}{j} \right\}.
\]
For $j \geq j_0$, we have
\[
\mathcal{D}_{kimj} \subset \mathcal{D}_{kimj_0}.
\]
Then it is sufficient to consider

\[ m \leq |k| \cdot |\Theta(\xi)| + 2(M + 2)N + 1, \]

and let

\[ j_0 = \tilde{\eta}^{-1/2}M(|k| + 1)^{\tau/2}C(N, \tilde{\eta})^{1/2}. \]

Then following the proof of Lemma 5 in [3], we obtain

\[
\text{Meas} \bigcup_{(k,l,l)\in \mathcal{X}_{n-1} \cap \mathcal{Z}_2} \mathcal{R}_{kll} \leq c_3 \tilde{\eta}^{1/2} \cdot \text{Meas} \Pi_\eta,
\]

where \( c_3 > 0 \) is a constant depending on \( n \) and \( \tau \). Finally, in view of (3.28)-(3.31) and (3.16), we obtain

\[
\text{Meas} \mathcal{R} \leq c \tilde{\eta}^{1/2} \cdot \text{Meas} \Pi_\eta,
\]

where \( c \) is a constant depending on \( n \) and \( \tau \). Then combining (3.17) with (3.32), we finish the proof of (3.19).

Based on the partial normal form of order \( M + 2 \) and \( p \)-tame property, we obtain the long time stability of KAM tori as follows.

**Theorem 3.3.** (The long time stability of KAM tori) Based on the partial normal form (3.20), for any \( p \geq 24(M + 7)^4 + 1 \) and \( 0 < \delta < \rho \), the KAM tori \( \mathcal{F} \) are stable in long time, i.e. if \( w(t) \) is a solution of Hamiltonian vector field \( X_H \) with the initial datum \( w(0) = (w_x(0), w_y(0), w_q(0), w_{\bar{q}}(0)) \) satisfying

\[ d_p(w(0), \mathcal{F}) \leq \delta, \]

then

\[ d_p(w(t), \mathcal{F}) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-M}. \]

**Proof.** Take \( \delta < \rho \) and \( \delta^{1/10} < \tilde{\eta} < 1 \). Let

\[ \delta^{-\frac{M+1}{p-1}} \leq N + 1 < \delta^{-\frac{M+1}{p-1}} + 1. \]

Then based on Theorem 3.2, we obtain a partial normal form of order \( M + 2 \)

\[ \tilde{H}(x,y,q,\bar{q};\xi) := H \circ \Phi = \tilde{N}(y,q,\bar{q};\xi) + Z(y,q,\bar{q};\xi) + P(x,y,q,\bar{q};\xi) + Q(x,y,q,\bar{q};\xi). \]

To obtain the estimate (3.33), it is sufficient to prove that

\[ |||X_P|||_{\mathcal{P}^p, D(s_0/4,4\delta,4\delta) \times \Pi_\eta} \leq \delta^{M+\frac{1}{2}} \]

and

\[ |||X_Q|||_{\mathcal{P}^p, D(s_0/4,4\delta,4\delta) \times \Pi_\eta} \leq \delta^{M+\frac{1}{2}}. \]
By a direct calculation,
\[
\delta^{\mathcal{N}+1} \left( \frac{c_{\mathcal{N}} \delta^6 \mathcal{M}^2 + 1}{\bar{\eta}^2} \right) \leq \delta^{\mathcal{N}+1} \left( \frac{c_{\mathcal{N}} \delta^6 \mathcal{M}^2 + 1}{\bar{\eta}^2} \right) \quad \text{(in view of the inequality (3.34))}
\]
\[
\leq \delta^{\mathcal{N}+1} \left( \frac{c_{\mathcal{N}} \delta^6 \mathcal{M}^2 + 1}{\bar{\eta}^2} \right) \quad \text{(in view of } p \geq 24(\mathcal{M} + 7)^4 + 1) \]
\[
\leq \delta^{\mathcal{N}+1} \quad \text{(by assuming } \delta \text{ is very small).} \tag{3.37}
\]

In view of the inequalities (3.23) and (3.37), we have
\[
\|X_p\|_{\mathcal{L}^1(T, \mathcal{D}(s_0/4, 4\delta, 4\delta))} \times \Pi \leq \delta^{\mathcal{N}+1/2}. \tag{3.38}
\]

Moreover, by the inequalities (5.4) and (3.38), we finish the proof of (3.35).

On the other hand,
\[
\|\hat{z}\|_1 = \sqrt{\sum_{|j| \geq \mathcal{N}+1} |z_j|^2 j^2}
\]
\[
= \sqrt{\sum_{|j| \geq \mathcal{N}+1} |z_j|^2 j^2 p/j^{(p-1)}}
\]
\[
\leq \|\hat{z}\|_p
\]
\[
\leq \delta^{\mathcal{N}+1} \|\hat{z}\|_p \quad \text{(by (3.34))}. \tag{3.39}
\]

Note that \(Q(x, y, q, \bar{q}, \xi) = O(||\hat{q}||_p^3)\). Then in view of (2.11)-(2.13) and (3.39), we finish the proof of (3.36). \(\square\)

4 Proof of Theorem 1.1

Step 1. Write equation (1.2) as an infinite dimensional Hamiltonian system.

Introducing coordinates \(q, \bar{q} \in \ell_p^2\) by
\[
u := u = \sum_{j \geq 1} \sqrt{\lambda_j} \phi_j, \quad \lambda_j(\bar{\xi}) = \sqrt{\mu_j + \bar{\xi}_j},
\]
where \(\mu_j = j^2 + m\) and \(\phi_j(x) = \sqrt{2/\pi} \sin jx\) are respectively the simple eigenvalues and eigenvectors of \(-\partial_{xx} + m\) with Dirichlet boundary conditions. The Hamiltonian of (1.2) is
\[
H_{NLW} = \int_0^\pi \left( \frac{\nu^2}{2} + \frac{1}{2} (\bar{u}_x^2 + (m + M_\xi)u_x) + \frac{1}{4} u^4 \right) dx
\]
\[
= \frac{1}{2} \sum_{j \geq 1} \lambda_j (\phi_j^2 + \bar{\phi}_j^2) + G(q),
\]
where
\[ G(q) = \frac{1}{4}\sum_{i\neq j, k\neq l=0}^{n} G_{ijkl} q_i q_j q_k q_l \]
with
\[ G_{ijkl} = \frac{1}{\sqrt{\lambda_j \lambda_k}} \int_0^\pi \phi_1 \phi_2 \phi_3 dx. \]

Introduce complex coordinates
\[ w_j := \frac{1}{\sqrt{2}} (q_j + \sqrt{-1} \dot{q}_j), \quad \tilde{w}_j := \frac{1}{\sqrt{2}} (q_j - \sqrt{-1} \dot{q}_j), \]
and let \( z = ((w_j)_{j\geq 1}, (\tilde{w}_j)_{j\geq 1}) \). Following Example 3.2 in [5], it is proven that there exists a constant \( c_p > 0 \) such that
\[ ||\tilde{X}_G(z^{(1)}, z^{(2)}, z^{(3)})||_{p+1} \leq c_p ||z^3||_{p, 1}. \]
(4.1)

In particular, when \( p = 1 \), the inequality (4.1) reads
\[ ||\tilde{X}_G(z^{(1)}, z^{(2)}, z^{(3)})||_1 \leq ||X_G(z^{(1)}, z^{(2)}, z^{(3)})||_1 \leq c_1 ||z^3||_{1, 1}. \]
(4.2)
The inequalities (4.1) and (4.2) show that the Hamiltonian vector field \( X_G(z) \) has \( p \)-tame property.

**Step 2. Introduce the action-angle variables.**

Without loss of generality, we choose \( \phi_1, \phi_2, \ldots, \phi_n \) as tangent direction and the other as normal direction. Let
\[ \tilde{w} = (w_1, \ldots, w_n) \quad \text{and} \quad \tilde{w} = (w_j)_{j\geq n+1} \]
be the tangent variable and normal variable, respectively. Then rewrite \( G(w, \tilde{w}) \) in the multiple-index as
\[ G(w, \tilde{w}) = \sum_{|\mu|+|v|+|\beta|=4} G^{\mu\nu\beta} w^\mu \tilde{w}^v z^\beta, \quad \mu, v \in \mathbb{N}^n, \beta \in \mathbb{N}^4, \]
(4.4)

where \( z = (\tilde{w}, \tilde{\tilde{w}}) \) and \( G^{\mu\nu\beta} = G_{ijkl} \) for some corresponding \( i, j, k, l \).

Introduce action-coordinates on the first \( n \) modes
\[ w_j := \sqrt{I_j + y_j e^{\sqrt{-1} T_j}}, \quad 1 \leq j \leq n \]
with
\[ I_j \in \left( \frac{\rho^2}{2}, \rho^2 \theta \right], \quad \theta \in (0, 1). \]

Then the symplectic structure is \( dy \wedge dx + \sqrt{-1} d\tilde{w} \wedge d\tilde{\tilde{w}} \), where \( y = (y_1, \ldots, y_n) \) and \( x = (x_1, \ldots, x_n) \).
Hence, (4.4) is changed into
\[ G(w, \tilde{w}) = G(x, y, z) \]
\[ = \sum_{|\mu|+|v|+|\beta|=4} 2^{\frac{1}{2}} G^{\mu\nu\beta} \sqrt{(\xi+y)^{\mu} e^{\sqrt{-1} T_{\mu, x}}} \sqrt{(\xi+y)^{v} e^{\sqrt{-1} T_{v, x}}} z^\beta \]
\[ = \sum_{|\mu|+|v|+|\beta|=4} (2^{\frac{1}{2}} G^{\mu\nu\beta} e^{\sqrt{-1} T_{\mu, x}}) \sqrt{(\xi+y)^{\mu} e^{\sqrt{-1} T_{v, x}}} z^\beta \]
\[ = \sum_{|\beta|\leq 4} G^{\beta}(x, y) z^\beta, \]
(4.5)
where
\[ \zeta = (\xi_1, \ldots, \xi_n), \quad G^{\beta}(x, y) = \sum_{|\mu| + |\nu| = 4 - |\beta|} 2^{\frac{1}{2}(|\mu| + |\nu|)} G^{\mu\nu\beta} e^{\sqrt{-1}(\mu - \nu, x)} \sqrt{(\zeta + y)^{\mu + \nu}}. \]

**Step 3. Show that the Hamiltonian vector field \(X_G\) has \(p\)-tame property.**

Note \(G\) has \(p\)-tame property in the coordinate \(\zeta\), and introducing action-angle variables is a coordinate transformation of finite variables, so \(G\) has \(p\)-tame property in the coordinate \((x, y, z)\). More precisely, assume \(G(x, y, z) = \sum_{|\beta| \leq 4} G^\beta(x, y) z^\beta\) is defined on the domain \((x, y, z) \in D(s, r, r)\) for some \(0 < s, r \leq 1\). In this step, we will show that the Hamiltonian vector field \(X_G(x, y, z)\) has \(p\)-tame property on the domain \(D(s, r, r)\).

Firstly, we would like to show
\[ |||G_\zeta|||^T_{p, D(s, r, r) \times \Pi} < \infty. \]

In view of (4.4),
\[ G_\zeta(x, y, z) = G_\zeta(w, \tilde{w}) = \sum_{|\mu| + |\nu| + |\beta| = 4} \beta G^{\mu\nu\beta} \tilde{w}^{\mu} w^{\nu} z^{-1}. \]

Let
\[ G^{\beta-1}_\zeta(x, y) = G^{\beta-1}_\zeta(w, \tilde{w}) = \sum_{|\mu| + |\nu| = 4 - |\beta|} \beta G^{\mu\nu\beta} \tilde{w}^{\mu} w^{\nu}, \]

and then
\[ G_\zeta(x, y, z) = \sum_{|\beta| \leq 4} G^{\beta-1}_\zeta(x, y) z^{\beta}. \]

Hence, we obtain
\[ |||\tilde{G}_\zeta|||_{D(s, r, r) \times \Pi, p+1} \leq c \left( \sum_{1 \leq i \leq 3} |||z^{(i)}|||_{p} + \sum_{1 \leq i \neq j \leq 3} \sum_{1 \leq i \leq 3} |||z^{(i)}|||_{1} |||z^{(j)}|||_{p} + \sum_{1 \leq i \leq 3} \sum_{1 \leq i \neq j \leq 3} |||z^{(i)}|||_{1} |||z^{(j)}|||_{1} + |||(z^{3})|||_{1, 1} \right), \quad (4.6) \]

where \(c > 0\) is a constant depending on \(s, r, n, p\), and the above inequality is based on the inequality (4.1) and the definition of \(|\cdot|_{D(s, r, r) \times \Pi}\) (see (2.2) for the details), and noting that \(\zeta = (\hat{w}, \hat{w}, \tilde{w}, \tilde{w})\) and \(z = (\hat{w}, \tilde{w})\). In particular, when \(p = 1\), the inequality (4.6) reads
\[ |||\tilde{G}_\zeta|||_{D(s, r, r) \times \Pi, 1} \leq \tilde{c} \left( \sum_{1 \leq i \leq 3} |||z^{(i)}|||_{1} + \sum_{1 \leq i \neq j \leq 3} |||z^{(i)}|||_{1} |||z^{(j)}|||_{1} + |||(z^{3})|||_{1, 1} \right), \quad (4.7) \]

where \(\tilde{c} > 0\) is a constant depending on \(s, r, n\). Based on the inequalities (4.6) and (4.7), we obtain
\[ |||G_\zeta|||^T_{p, D(s, r, r) \times \Pi} < \infty. \quad (4.8) \]

Similarly, we obtain
\[ |||G_x|||_{D(s, r, r) \times \Pi} < \infty \quad (4.9) \]

and
\[ |||G_y|||_{D(s, r, r) \times \Pi} < \infty. \quad (4.10) \]
In view of the inequalities (4.8), (4.9) and (4.10), we get
\[ ||X_G||_{p,D(x,r)} \times \Pi < \infty. \] (4.11)

Finally, we obtain a Hamiltonian \( H(x,y,q,\bar{q};\xi) \) having the following form
\[ H(x,y,q,\bar{q};\xi) = N(x,y,q,\bar{q};\xi) + R(x,y,q,\bar{q};\xi), \] (4.12)

where
\[ N(x,y,q,\bar{q};\xi) = H_0(w,\bar{w}) = \sum_{j=1}^{n} \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) q_j \bar{q}_j, \] (4.13)

with the tangent frequency
\[ \omega(\xi) = (\omega_j(\xi))_{1 \leq j \leq n}, \quad \omega_j = \sqrt{j^2 + m + \xi_j}, \] (4.14)

the normal frequency
\[ \Omega(\xi) = (\Omega_1(\xi),\Omega_2(\xi), \ldots), \quad \Omega_j(\xi) = \sqrt{(j+n)^2 + m + \xi_{j+n}}, \quad j \geq 1 \] (4.15)

and the perturbation
\[ R(x,y,q,\bar{q};\xi) = \varepsilon G(x,y,q,\bar{q};\xi). \] (4.16)

In view of (4.11), (4.14)-(4.16), all assumptions in Theorem 3.1 hold. According to Theorem 3.1, we obtain a KAM normal form of order 2 where the nonlinear terms satisfy \( p \)-tame property.

Furthermore, let \( \delta \) be given in the statement of Theorem 3.2 and \( \mathcal{N} \) be given in (3.34). Take \( \eta \) satisfying \( \delta^{1/6} < \eta < \mathcal{N}^{-3} \). Then we obtain a KAM partial normal form of order \( \mathcal{M} + 2 \) where the nonlinear terms satisfy \( p \)-tame property.

Finally, based on Corollary 3.3, for each \( \xi \in \Pi_\eta \subset \Pi_\infty \), the KAM torus \( \mathcal{T}_\xi \) for equation (1.2) is stable in a long time, i.e. for any solution \( u(t,x) \) of equation (1.2) with the initial datum satisfying
\[ d_{H_0^p(0,\pi)}(u(0,x),\mathcal{T}_\xi) \leq \delta, \]
then
\[ d_{H_0^p(0,\pi)}(u(t,x),\mathcal{T}_\xi) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}. \]

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5 Appendix: Properties of the Hamiltonian with $p$-tame property

The following lemma shows that $p$-tame property persists under Poisson brackets, which can be parallel proved following the proof of Theorem 3.1 in [9]:

**Lemma 5.1.** Suppose that both Hamiltonian functions

$$U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2} U^\beta(x, y; \xi)z^\beta$$

and

$$V(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2} V^\beta(x, y; \xi)z^\beta,$$

satisfy $p$-tame property on the domain $D(s, r, r) \times \Pi$, where

$$U^\beta(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} U^{\alpha \beta}(x, \xi)y^\alpha$$

and

$$V^\beta(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} V^{\alpha \beta}(x, \xi)y^\alpha.$$

Then the Poisson bracket $\{U, V\}(x, y, z; \xi)$ of $U(x, y, z; \xi)$ and $V(x, y, z; \xi)$ with respect to the symplectic structure $\sum_{1 \leq j \leq n}dy_j \wedge dx_j + \sqrt{-1}\sum_{j=1}^\infty dz_j \wedge dz_j$ has $p$-tame property on the domain $D(s - \sigma, r - \sigma', r - \sigma') \times \Pi$ for $0 < \sigma < s, 0 < \sigma' < r/2$. Moreover, the following inequality holds

$$|||X_U|||_{p, D(s, r, r) \times \Pi}^T \leq \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} |||X_V|||_{p, D(s, r, r) \times \Pi}^T |||X_V|||_{p, D(s, r, r) \times \Pi}^T,$$  

(5.1)

where $C > 0$ is a constant depending on $n$.

Denote $X_U^\epsilon$ by the flow of the Hamiltonian vector field of $U(x, y, z; \xi)$. It follows from Taylors formula that

$$V \circ X_U^\epsilon(x, y, z; \xi) = \sum_{j=0}^\infty \frac{\epsilon^j}{j!} V^{(j)}(x, y, z; \xi),$$  

(5.2)

where

$$V^{(0)}(x, y, z; \xi) := V(x, y, z; \xi), \quad V^{(j)}(x, y, z; \xi) := \{V^{(j-1)}, U\}(x, y, z; \xi).$$

Then based on (5.1) in Lemma 5.1 and (5.2), we have the following estimate of symplectic transformation, which can be parallel proved following the proof of Theorem 3.3 in [9]:

**Lemma 5.2.** Consider two Hamiltonians $U(x, y, z; \xi)$ and $V(x, y, z; \xi)$ satisfying $p$-tame property on the domain $D(s, r, r) \times \Pi$ for some $0 < s, r \leq 1$. Given $0 < \sigma < s, 0 < \sigma' < r/2$, suppose

$$|||X_U|||_{p, D(s, r, r) \times \Pi}^T \leq \frac{1}{2A},$$

where
Lemma 5.3. Consider two Hamiltonians
\[ U(x, y, z; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^2} U^{\alpha \beta}(x; \xi)^{\alpha \beta} \]
and
\[ V(x, y, z; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^2} V^{\alpha \beta}(x; \xi)^{\alpha \beta}. \]

Suppose \( V(x, y, z; \xi) \) has \( p \)-tame property on the domain \( D(s, r) \times \Pi \), i.e.
\[ |||X_V|||_{p, D(s, r) \times \Pi} < \infty. \]

For each \( \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^2, k \in \mathbb{Z}^n, j \geq 1 \) and some fixed constant \( \tau > 0 \), assume the following inequality holds
\[ |U^{\alpha \beta}(k; \xi)| + |\partial_j U^{\alpha \beta}(k; \xi)| \leq (|k| + 1)^\tau (|V^{\alpha \beta}(k; \xi)| + |\partial_j V^{\alpha \beta}(k; \xi)|), \]
where \( U^{\alpha \beta}(k; \xi) \) and \( V^{\alpha \beta}(k; \xi) \) are the \( k \)-th Fourier coefficients of \( U^{\alpha \beta}(x; \xi) \) and \( V^{\alpha \beta}(x; \xi) \), respectively. Then, \( U(x, y, z; \xi) \) has \( p \)-tame property on the domain \( D(s - \sigma, r, r) \times \Pi \) for \( 0 < \sigma < s \). Moreover, we have
\[ \frac{c}{\sigma^\tau} \leq |||X_V|||_{p, D(s, r) \times \Pi} \leq \frac{c}{\sigma^\tau} \leq |||X_V|||_{p, D(s, r) \times \Pi}. \quad (5.3) \]

where \( c > 0 \) is a constant depending on \( s \) and \( \tau \).

The following lemma compares \( p \)-tame norm with the usual weighted norm for Hamiltonian vector field, which can be parallel proved following the proof of Theorem 3.5 in [9]:

Lemma 5.4. Given a Hamiltonian
\[ U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^2} U^{\beta}(x, y; \xi)^{\beta} \]
satisfying \( p \)-tame property on the domain \( D(s, r) \times \Pi \) for some \( 0 < s, r \leq 1 \). Then we have
\[ |||X_U|||_{p, D(s, r) \times \Pi} \leq |||X_U|||_{p, D(s, r) \times \Pi} \leq |||X_U|||_{p, D(s, r) \times \Pi}. \quad (5.4) \]

Then based on (5.4) in Lemma 5.4 and the proof of Lemma A.4 in [16], we have the following lemma:
Lemma 5.5. Suppose the Hamiltonian
\[ U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}} U^\beta(x, y; \xi) z^\beta \]
has p-tame property on the domain \( D(s, r, r) \times \Pi \) for some \( 0 < s, r \leq 1 \). Let \( X_U \) be the phase flow generalized by the Hamiltonian vector field \( X_U \). Given \( 0 < \sigma < s \) and \( 0 < \sigma' < r/2 \), assume
\[ \|X_U\|_{T, p, D(s, r, r) \times \Pi} < \min\{\sigma, \sigma'\}. \]
Then, for each \( \xi \in \Pi \) and each \( |t| \leq 1 \), one has
\[ \|X_U - id\|_{p, D(s-\sigma, r-\sigma', r-\sigma')} \leq \|X_U\|_{T, p, D(s, r, r) \times \Pi}, \]  
(5.5)
where
\[ \|X_U - id\|_{p, D(s-\sigma, r-\sigma', r-\sigma')} = \sup_{w \in D(s-\sigma, r-\sigma', r-\sigma')} \|X_U w - idw\|_{p, D(s, r, r)}, \]  
(5.6)

References

[1] D. Bambusi, Birkhoff normal form for some nonlinear PDEs., Comm. Math. Phys., 234, 253–285 (2003)

[2] D. Bambusi, A Birkhoff normal form theorem for some semilinear PDEs., Hamiltonian dynamical systems and applications, 213–247 (2008)

[3] D. Bambusi, M. Berti and E. Magistrelli, Degenerate KAM theory for partial differential equations, J. Differential Equations, 250, 3379–3397 (2011)

[4] D. Bambusi, J. M. Delort, B. Grèbert and J. Szeftel, Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds, Comm. Pure Appl. Math., 60, 1665–1690 (2007)

[5] D. Bambusi and B. Grèbert, Birkhoff normal form for partial differential equations with tame modulus, Duke Math. J., 135, 507–567 (2006)

[6] D. Bambusi and N. N. Nekhoroshev, A property of exponential stability in nonlinear wave equations near the fundamental linear mode, Phys. D, 122, 73–104 (1998)

[7] M. Berti and L. Biasco, Branching of Cantor manifolds of elliptic tori and application to PDEs, Comm. Math. Phys. 305, 741–796 (2011)

[8] J. Bourgain, On diffusion in high-dimensional Hamiltonian systems and PDE, J. Anal. Math., 80, 1–35 (2000)

[9] H. Cong, J. Liu and X. Yuan, Stability of KAM tori for nonlinear Schrödinger equation, to appear Mem. Amer. Math. Soc.

[10] J. M. Delort and J. Szeftel, Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres, Int. Math. Res. Not., 37, 1897–1966 (2004)
[11] L. H. Eliasson, A talk in Fudan University, (2007)

[12] B. Grébert, R. Imekraz and É. Paturel, Normal forms for semilinear quantum harmonic oscillators. Commun. Math. Phys. 291, 763–798 (2009)

[13] S. B. Kuksin, Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum, Funct. Anal. Appl., 21, 192–205 (1987)

[14] S. B. Kuksin, Perturbations of quasiperiodic solutions of infinite-dimensional Hamiltonian systems, Math. USSR Izv., 32, 39-62 (1989)

[15] S. B. Kuksin, Nearly integrable infinite-dimensional Hamiltonian systems, Springer-Verlag, Berlin (1993)

[16] J. Pöschel, A KAM-theorem for some nonlinear partial differential equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 23, 119–148 (1996)

[17] J. Pöschel, Quasi-periodic solutions for a nonlinear wave equation, Comment. Math. Helv. 71(2), 269–296 (1996)

[18] C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, Comm. Math. Phys., 127, 479–528 (1990)

[19] X. Yuan, Invariant tori of nonlinear wave equations with a given potential, Discrete Contin. Dyn. Syst., 16, 615-634 (2006)