Performance analysis on free-piston Stirling cryocooler based on an idealized mathematical model

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Abstract. Free-piston Stirling cryocoolers have extensive applications for its simplicity in structure and decrease in mass. However, the elimination of the motor and the crankshaft has made its thermodynamic characteristic different from that of Stirling cryocoolers with displacer driving mechanism. Therefore, an idealized mathematical model has been established, and with this model, an attempt has been made to analyse the thermodynamic characteristic and the performance of free-piston Stirling cryocooler. To certify this mathematical model, a comparison has been made between the model and a numerical model. This study reveals that due to the displacer damping force necessary for the production of cooling capacity, the free-piston Stirling cryocooler is inherently less efficient than Stirling cryocooler with displacer driving mechanism. Viscous flow resistance and incomplete heat transfer in the regenerator are the two major causes of the discrepancy between the results of the idealized mathematical model and the numerical model.

1. Introduction

Since the invention of free-piston Stirling cryocooler, which eliminates the motor as well as the crankshaft and features compactness and long-life, it has been applied in various areas such as aerospace industry, infrared detection, superconductor technology, cryophysics, cryobiology as well as domestic refrigerator and heat pump [1].

However, such mechanical simplification comes with a price. Without the connecting rods and crankshafts, the movements of the driving piston and the displacer become less predictable because the pneumatic force they sustain is affected by various parameters such as the geometry of the components, charge pressure and spring stiffness. Some unexpected factors could result in substantial deviation of their movement amplitudes and phases, which could lead to significant decrease in cooling capacity.

Nevertheless, the temptation of such mechanical simplification is so huge that great effort has been made to model and predict the movement of the driving piston and displacer with various mathematical models. R.A. Ackermann applied phasor analysis and eigenvalue analysis to study the dynamics of the driving piston and displacer as well as the transient equation in matrix form [2]. B.J. Huang and C.W. Lu developed a linear network analysis to model the propagation of gas flow and displacer motion inside the free-piston Stirling cryocooler [3]. A.T.A.M. de Waele and W. Liang
derived basic relations for the free-piston Stirling cryocooler with harmonic approximation and the assumption of zero void volume in the regenerator [4].

In those studies, the motion of the moving parts and cooling capacity are revealed from different points of view, and some practical considerations such as flow resistance inside the regenerator and pressure fluctuation inside the bounce space are included to bring the mathematical models close to the real machine. However, such practical considerations have made the mathematical models so complicated that from time to time only numerical results are available, and the essential properties of free-piston Stirling cryocooler can hardly be understood. Therefore, an idealized mathematical model, almost as idealized as that of Carnot cycle, is established and some important parameters are discussed based on this model, which would shed light on the thermodynamic nature of free-piston Stirling cryocooler.

2. The mathematical model

2.1. Object and assumptions

The mathematical model is established upon a sketch of free-piston Stirling cryocooler shown in figure 1. This cryocooler is comprised of two major parts, the compressor and the expander, which are connected by a tube. In the expander, the displacer is connected with a rod that extrudes into the bounce space, which is separated with the compression space by an ideal seal. Spring force and damping force are exerted on the rod as well as the displacer to adjust their motion. \(A_1\) is greater than \(A_2\) due to the existence of the rod. Detailed explanations of the physics notations in figure 1 are listed in table 1.

![Figure 1. A sketch of the free-piston Stirling cryocooler.](image)

To establish the model, following assumptions are made.

- The pressures of the working fluid inside the expansion space, compression space and the regenerator are always equal and uniform;
- The temperatures of the working fluid inside the expansion space, compression space and the regenerator are uniform and equal to \(T_L\), \(T_H\), and \((T_L + T_H)/2\), respectively;
- The regenerator is ideal, which implies that no friction occurs inside the regenerator while enthalpy flow remains zero;
- Neither leakage nor friction occurs in the clearance seals;
- During operation, the pressure inside the bounce spaces remains constant and equals to \(p_m\);
- The driving piston and the displacer execute simple harmonic motion, while the pressure oscillates in a simple harmonic manner. The phase angle of pressure is set to be zero;
- The working fluid is ideal gas;
The driving force applied on the driving piston is given by

\[ F_p = \alpha I = |F_p| \cos(\omega t + \varphi_F) = \text{Re}[F_p e^{i\omega t}] \]  

(1)

Table 1. Explanations of physics notations in figure 1.

| Physics Notation | Explanation |
|------------------|-------------|
| \( m_e \)        | the total mass of the displacer and the rod |
| \( k_e \)        | spring stiffness of the spring that connects to the rod |
| \( c_e \)        | damping coefficient of the damper that connects to the rod |
| \( A_1 \)        | the bottom area of the displacer on the side of the expansion space |
| \( A_2 \)        | the bottom area of the displacer on the side of the compression space |
| \( m_c \)        | the mass of the driving piston |
| \( k_c \)        | spring stiffness of the spring that connects to the driving piston |
| \( c_c \)        | drag coefficient of the driving piston |
| \( A_c \)        | the bottom area of the driving piston on both sides |

2.2. The dynamics of driving piston and displacer

Applying Newton’s Second law on the displacer gives

\[ pA_1 - pA_2 - p_m(A_1 - A_2) - k_e x_e - \frac{c_e (dx_e)}{dt} = \frac{m_e d^2 x_e}{dt^2} \]  

(2)

where \( p, p_m \) and \( x_e \) represent the instantaneous pressure, the average pressure and the displacement of the displacer, respectively. With the simple harmonic assumption, equation (2) can be reduced to

\[ (A_1 - A_2)p - k_e x_e - j\omega c_e x_e = -\omega^2 m_e x_e \]  

(3)

where \( \omega \) represents the angular frequency. In equation (3), the boldface physics notation represents a phasor, such as

\[ X = |X| e^{j\varphi_X} \]  

(4)

Phasor notation is a powerful tool in the analysis of simple harmonic motion, which can significantly simplify the mathematics. Solving equation (3) gives

\[ x_e = \frac{A_1 - A_2}{k_e - \omega^2 m_e + j\omega c_e} p \]  

(5)

which yields the amplitude and the phase angle of the displacer displacement

\[ |x_e| = \frac{(A_1 - A_2)|p|}{\sqrt{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2}} \]  

(6)

\[ \varphi_e = -\arctan \frac{\omega c_e}{k_e - \omega^2 m_e} \]  

(7)

Identical process can be applied to the driving piston, which gives the dynamic equation

\[ F_p + A_c p - k_c x_c - j\omega c_c x_c = -\omega^2 m_c x_c \]  

(8)

and the driving piston displacement

\[ x_c = \frac{F_p + A_c p}{k_c - \omega^2 m_c + j\omega c_c} \]  

(9)

Since the space inside the entire cryocooler is a closed volume, the sum of the gas mass in the compression space and the expansion space remains constant and is equal to the initial charged gas mass. Based on the ideal gas assumption, the sum of the gas mass is

\[ m = \frac{p}{R_g} \left( \frac{V_H}{T_H} + \frac{2V_R}{T_H + T_L} + \frac{V_L}{T_L} \right) = \frac{p_m}{R_g} \left( \frac{V_{H0}}{T_H} + \frac{2V_R}{T_H + T_L} + \frac{V_{L0}}{T_L} \right) = \frac{p_0 (V_{H0} + V_R + V_{L0})}{R_g T_H} \]  

(10)
where \(V_H, V_R\) and \(V_L\) represent the instantaneous volume of compression space, void volume of the regenerator and the instantaneous volume of expansion space. \(p_0\) is the initial charged pressure when the entire machine is at ambient temperature. \(V_{H0}\) and \(V_{L0}\) represent the volume of compression space and the expansion space, respectively, at equilibrium state. \(p_m\) is therefore given by

\[
p_m = p_0 \frac{V_{H0} + V_R + V_{L0}}{(V_{H0}/T_H + 2V_R/(T_H + T_L) + V_{L0}/T_L)T_H} \tag{11}
\]

With the simple harmonic motion assumption, \(V_H\) and \(V_L\) vary with time in such a manner

\[
V_H = V_{H0} - A_2 x_e + A_c x_c = V_{H0} - A_2 |x_e| \cos(\omega t + \varphi_e) + A_c |x_c| \cos(\omega t + \varphi_c) \tag{12}
\]

\[
V_L = V_{L0} + A_1 x_e = V_{L0} + A_1 |x_e| \cos(\omega t + \varphi_e) \tag{13}
\]

Combining (10), (12) and (13) gives

\[
\frac{p_m + |p| \cos(\omega t)}{R_g} \left[ \frac{V_{H0} - A_2 |x_e| \cos(\omega t + \varphi_e) - A_c |x_c| \cos(\omega t + \varphi_c)}{T_H} + \frac{2V_R}{T_H + T_L} \right] + \frac{V_{L0} + A_1 |x_e| \cos(\omega t + \varphi_e)}{T_L} = \frac{p_m (V_{H0}/T_H + 2V_R/V_{L0}/T_L)}{R_g (T_H + T_L)} \tag{14}
\]

Excluding all the second-order terms and applying phasor notation gives

\[
B_1 p + B_2 x_e + B_3 x_c = 0 \tag{15}
\]

where

\[
B_1 = \frac{V_{H0}}{T_H} + \frac{2V_R}{T_H + T_L} + \frac{V_{L0}}{T_L} \tag{16}
\]

\[
B_2 = p_m \left( -\frac{A_2}{T_H} + \frac{A_1}{T_L} \right) \tag{17}
\]

\[
B_3 = p_m \frac{A_c}{T_H} \tag{18}
\]

Equations (5), (9) and (15) give a complete description of the dynamics of the driving piston and the displacer and are the basis of the following discussion. Figure 2 presents the phasor diagram of the free-piston Stirling cryocooler given by equations (5), (9) and (15).
2.3. Power and efficiency

2.3.1. Cooling capacity. The cooling capacity at the expansion space is given by

\[ \langle Q \rangle_L = \frac{\omega}{2\pi} \int p dV_L = -\frac{A_2}{2} \omega |p| |x_e| \sin \varphi_e = -\frac{A_2}{2} \omega |p| \cdot \Im[x_e] \]  
(19)

Combining (19) with (5) gives

\[ \langle Q \rangle_L = \frac{A_2}{2} |p|^2 \frac{(A_1 - A_2) \omega^2 c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \]  
(20)

Similar process can be applied to calculate the rejected heat at the compression space and the power consumed by the damping force on the displacer.

\[ \langle Q \rangle_H = \frac{\omega}{2\pi} \int p dV_H = -\frac{1}{2} |p|^2 \cdot T_H \left( \frac{A_1}{T_{L}} - \frac{A_2}{T_{H}} \right) \cdot \frac{(A_1 - A_2) \omega^2 c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \]  
(21)

\[ \langle W \rangle_{pe} = \frac{\omega}{2\pi} \int F_p v_c dt = \frac{A_1 - A_2}{2} |p|^2 \frac{(A_1 - A_2) \omega^2 c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \]  
(22)

2.3.2. Compression power During operation, the compressor transforms the electrical work into acoustic work, which drives the expander to produce cooling capacity. However, due to the resistance of the coil and the friction force, such transformation is less than 100% in efficiency. The acoustic work is given by

\[ \langle pv \rangle = \frac{\omega}{2\pi} \int p v_c dt = \frac{1}{2} |p||v_c| \cos \varphi_{vc} = \frac{1}{2} |p| \Re[v_c] \]  
(23)

in which \( v_c \) represents the velocity of the driving piston, and \( \varphi_{vc} \) represents the phase angle of \( v_c \). According to (5) and (15), the velocity of the driving piston is given by

\[ v_c = \omega x_c = -\omega \left( \frac{B_1}{B_3} p + \frac{B_2}{B_3} x_e \right) = -\omega \left[ \frac{B_3 + B_2 (A_1 - A_2)(k_e - \omega^2 m_e - j\omega c_e)}{B_3} \right] \]  
(24)

Therefore, combining (23) and (24) gives the expression of acoustic power, which is

\[ \langle pv \rangle = -\frac{1}{2} |p|^2 \cdot \frac{T_H}{A_e} \left( \frac{A_1}{T_L} - \frac{A_2}{T_H} \right) \cdot \frac{(A_1 - A_2) \omega^2 c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \]  
(25)

Due to the friction force applied on the driving piston, the acoustic power that the driving piston produces is less than the power that drives the driving piston, which is obtained by combining (5), (9) and (15)

\[ \langle W \rangle_p = \frac{\omega}{2\pi} \int F_p v_c dt = \frac{1}{2} \left\{ \Re[F_p] \Re[v_c] + \Im[F_p] \Im[v_c] \right\} \]  
(26)

\[ = \frac{1}{2} A_e |p|^2 \cdot \frac{B_2}{B_3} \cdot \frac{(A_1 - A_2) \omega^2 c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \]  

\[ + \frac{1}{2} c_e \omega^2 |p|^2 \left\{ \left[ \frac{B_1}{B_3} + \frac{B_2}{B_3} (A_1 - A_2)(k_e - \omega^2 m_e)^2 + (\omega c_e)^2 \right]^2 + \frac{B_2}{B_3} \frac{(A_1 - A_2) \omega c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \right\} \]

Similarly, the power consumed by the friction force on the driving piston is

\[ \langle W \rangle_{pe} = \frac{\omega}{2\pi} \int F_{pe} v_c dt = \frac{1}{2} |F_{pe}| |v_c| = \frac{\omega c_e |x_c|^2}{2} \]  
(27)

\[ = \frac{1}{2} c_e \omega^2 |p|^2 \left\{ \left[ \frac{B_1}{B_3} + \frac{B_2}{B_3} (A_1 - A_2)(k_e - \omega^2 m_e)^2 + (\omega c_e)^2 \right]^2 + \frac{B_2}{B_3} \frac{(A_1 - A_2) \omega c_e}{(k_e - \omega^2 m_e)^2 + (\omega c_e)^2} \right\} \]
Combining (25), (26) and (27) gives
\[ \langle W \rangle_p = A_c \langle p v \rangle | + \langle W \rangle_{f_e} \]  
which implies that the power that drives the driving piston is partially consumed by the friction force and remainder is transformed into the acoustic power that drives the expander. Combining (20), (21), (22), (25), (26) and (27) can reveal the relationship among the various powers concerning the expander and the compressor

\[ |\langle W \rangle_p| - |\langle W \rangle_{f_e}| - |\langle W \rangle_{f_e}| = A_c |\langle p v \rangle| - |\langle W \rangle_{f_e}| = |\langle Q \rangle| - |\langle Q \rangle| \]  

Equation (29) shows that the damping force on the displacer and the friction force on the driving piston result in greater power than necessary to drive the driving piston. However, the discussion in Section 3.1 will reveal that the damping force on the displacer is so important to the production of cooling capacity that it is unwise to minimize it. Therefore, it is more reasonable to take \( A_c \langle p v \rangle \) rather than \( A_c \langle p v \rangle - |\langle W \rangle_{f_e}| \) as the denominator in the calculation of the COP.

\[ \text{COP} = \frac{\langle Q \rangle}{-A_c \langle p v \rangle} = \frac{T_L}{B_2 T_2} \cdot \frac{A_1}{-A_1} \]  

Equation (30) reveals that because of the necessary damping force on the displacer, the COP of free-piston Stirling cryocooler, though less than the COP of reverse Carnot cycle, is actually the upper limit of the COP of free-piston Stirling cryocooler. Therefore, free-piston Stirling cryocooler is inherently of lower efficiency.

3. Discussion

3.1. The displacer damping force and the frequency

From equation (20), the damping force on the displacer plays a critical role in the production of cooling capacity, even though damping force is generally considered as a negative factor for the cooling capacity. Consideration of the displacer motion, however, would justify the need for this damping force.

In general, it is necessary for a free-piston Stirling cryocooler to have a phase difference between the displacer displacement and the pressure in the expansion space to produce cooling capacity. A glance at equation (3) shows that neither the spring force nor the inertial force but the damping force could give rise to such a phase difference. Therefore, the damping force on the displacer is necessary to the cooling capacity. However, a large damping force cannot guarantee a large cooling capacity because the amplitude of the displacer displacement, which is also necessary for the cooling capacity, decreases as the damping force increases, as indicated by equation (6). Consequently, there exists an optimum \( c_e \) that maximizes the cooling capacity. Unfortunately, such an optimum \( c_e \) cannot be obtained just with the cooling capacity equation (20) because there is a rather complicated relationship between \( c_e \) and the amplitude of pressure, which will be discussed in Section 3.2, and the amplitude of pressure also plays a critical role in the production of cooling capacity, as equation (20) indicates.

Similar dilemma exists in the analysis of the optimum frequency. As indicated by equation (20), the resonant frequency of the displacer, which can cancel the term \( (k_e - \omega^2 m_e) \), is not the optimum frequency that maximizes the cooling capacity. The complicated relationship between the frequency and the pressure amplitude, as discussed in Section 3.2, makes it impossible to solve this optimum frequency.

3.2. The pressure amplitude

Equation (20) shows that the pressure amplitude is of great importance to the cooling capacity, and increasing the pressure amplitude is an effective approach to increase the cooling capacity. Unfortunately, such an important parameter is not a given condition but is calculated with given parameters such as the driving force on the driving piston, the geometry of the components and the frequency.
\[ |p| = \frac{|F_p|}{\sqrt{C^2_1 + C^2_2}} \]  
\[ C_1 = A_c + \frac{V_{H0}}{p_{m}A_c} + 2V_R/(1 + T_L/T_H) + V_{L0}T_H/T_L (k_c - \omega^2 m_c) \]  
\[ + \frac{A_1}{A_c} \left( \frac{T_H}{T_L} - 1 \right) (A_1 - A_2) [(k_c - \omega^2 m_c) (k_e - \omega^2 m_e) + \omega^2 c_e c_c] \]  
\[ C_2 = \frac{V_{H0} + 2V_R/(1 + T_L/T_H) + V_{L0}T_H/T_L}{\omega c_c} \]  
\[ + \frac{A_1}{A_c} \left( \frac{T_H}{T_L} - 1 \right) \omega (A_1 - A_2) [c_c (k_e - \omega^2 m_e) - c_e (k_c - \omega^2 m_c)] \]  
\[ \left((k_c - \omega^2 m_c) \right) + (\omega c_e)^2 \]  
Equations (31) to (33) indicate an extremely complicated relationship between the pressure amplitude and the given parameters. The optimum combination of given parameters that maximizes the pressure amplitude can hardly be achieved by such a complicated relationship. Still, the equations have provided some useful information.

First, reduction in \(V_{H0}, V_{L0}\) and \(V_R\), or the void volumes, is not always advantageous to enlarge the pressure amplitude and the cooling capacity, especially when \((k_c - \omega^2 m_c)\) is negative. In practice, however, \(V_{H0}\) and \(V_{L0}\) should be large enough to avoid collision between the moving components and the walls. Second, suppose the driving piston and the displacer can simultaneously reach the resonant state with a certain frequency, such a resonance frequency is not necessarily the optimum frequency that maximizes the pressure amplitude as well as the cooling capacity.

Figure 3 has plotted the pressure amplitude against the frequency with different \(V_{H0}\) and \(V_{L0}\). The given parameters are carefully chosen so that the compressor and the displacer can simultaneously reach the resonant state with frequency equals to 56.2 Hz. The figure shows that the resonant frequency (marked by a star) is far from the optimum frequency for both cases. Also, the pressure amplitude of larger \(V_{H0}\) and \(V_{L0}\) is greater than that of smaller \(V_{H0}\) and \(V_{L0}\) in some range of frequency. It is interesting to notice that the pressure amplitudes of both cases are nearly equal at the resonant frequency because with a very small \(c_c\), both \(C_1\) and \(C_2\) are almost independent of \(V_{H0}\) and \(V_{L0}\) at resonant state.

**Figure 3.** Pressure amplitude versus frequency.

**Figure 4.** Comparison between the idealized mathematical model and a numerical model.
3.3. A comparison between the idealized mathematical model and a numerical model

Since the idealized mathematical model is established upon various assumptions, it is interesting to compare this model with a more practical numerical model which is established with a commercial software Sage. Figure 4 plots the cooling capacity against frequency calculated by the idealized mathematical model and the numerical model. In the numerical model, curves of different Fmult, or the multiplier for viscous pressure drop inside the regenerator, are plotted for comparison. Moreover, the gross cooling capacity $Q_{\text{gross}}$ of the numerical model is the sum of the net cooling capacity $Q_{\text{net}}$ and the enthalpy flow at the cold end of the regenerator, which is due to the incomplete heat transfer between the working fluid and the regenerator matrices.

Figure 4 shows that there is a large discrepancy of cooling capacity between the idealized mathematical model and the numerical model ($Q_{\text{net}}$ with Fmult = 1). However, the discrepancy decreases as viscous flow resistance decreases from the normal value (Fmult = 1) to zero (Fmult = 0). Moreover, the enthalpy flow should also account for part of the discrepancy. The $Q_{\text{gross}}$ curve with zero Fmult of the numerical model is quite close to the curve of the idealized mathematical model for a wide range of frequency.

4. Conclusion

This paper has established an idealized mathematical model based on a series of assumptions. The expressions of the cooling capacity, the power consumption, COP as well as the pressure amplitude and the motion parameters of the moving components are derived. A brief discussion is also made regarding to the displacer damping force, the frequency and the pressure amplitude. A comparison has also been made between the idealized mathematical model and a numerical model. In summary, following conclusions can be achieved.

- The damping force on the displacer is necessary for the production of cooling capacity, and makes free-piston Stirling cryocooler inherently less efficient;
- Reduction in the void volumes is not always advantageous to enlarge the pressure amplitude and the cooling capacity;
- Resonance frequency is not necessarily the optimum frequency that maximizes the pressure amplitude and the cooling capacity;
- The viscous flow resistance and the incomplete heat transfer in the regenerator are the two major causes of the discrepancy between the results of idealized mathematical model and the numerical model.

While numerical models such as Sage can provide rather precise numerical results applicable to actual design, mathematical models, though based upon more simplifying assumptions and less accurate, provide formulae which can reveal the nature of free-piston Stirling cryocooler and help us understand the mechanism behind the numerical results. Future work will be devoted in establishing an optimal design method based on the idealized model, which would work as a preliminary optimization before further numerical optimization.

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