Exact Three-Point Functions of Determinant Operators in Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We introduce a nonperturbative approach to correlation functions of two determinant operators and one non-protected single-trace operator in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Based on the gauge/string duality, we propose that they correspond to overlaps on the string worldsheet between an integrable boundary state and a state dual to the single-trace operator. We determine the boundary state using symmetry and integrability of the dual superstring sigma model, and write down expressions for the correlators at finite coupling, which we conjecture to be valid for operators of arbitrary size. The proposal is put to test at weak coupling.

I. INTRODUCTION

To advance our understanding of nonperturbative dynamics in gauge theories, it is useful to study simple models with rich enough structures. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions ($\mathcal{N} = 4$ SYM) is one of the leading candidates for the following reasons: First it admits the planar large $N_c$ limit, which makes it amenable to analytical studies. Second it is a conformal field theory (CFT) and all the correlation functions are exactly using integrability. The application of integrability led to a complete determination of two-point functions of local operators [1]. It was applied also to the three-point function [2], but the result is still unsatisfactory since it is given by a series expansion which one needs to resum.

In this letter, we present the first fully nonperturbative result for the three-point function valid for a large class of operators [3]. Specifically, we study the correlator of two determinant operators and one non-protected single-trace operator. By interpreting this correlator as an overlap on the string worldsheet between a boundary state and a state dual to the single-trace operator, we write down nonperturbative expressions using the framework of thermodynamic Bethe ansatz (TBA) [4].

II. SET UP AND BASIC STRATEGY

The main subject is the three-point function of a non-protected single-trace operator $\mathcal{O}$ and two determinant operators $\mathcal{D}_{1,2} \equiv \det \mathfrak{Z}(a_{1,2})$ [5] with

$$\mathfrak{Z}(a) \equiv \frac{(1 + a^2)\Phi^1 + i(1 - a^2)\Phi^2 + 2ia\Phi^4}{\sqrt{2}},$$

where $\Phi^{1,2,4}$ are real scalar fields in $\mathcal{N} = 4$ SYM. Owing to the superconformal symmetry, its spacetime dependence is fixed to be [2, 6]

$$\langle \mathcal{D}_1 \mathcal{D}_2 \mathcal{O}(0) \rangle = \left(\frac{a_1 - a_2}{a_1a_2}\right)^{\Delta - J} \mathfrak{D}_\mathcal{O},$$

where $\mathfrak{D}_\mathcal{O}$ is the structure constant while $\Delta$ and $J$ are the conformal dimension and the $R$-charge of $\mathcal{O}$.

The goal of this letter is to compute $\mathfrak{D}_\mathcal{O}$ nonperturbatively using the gauge/string duality. As discussed in [7–9], the duality maps (2) to a closed string in AdS$_5 \times$S$^5$ which ends on a geodesic of a D-brane dual to determinant operators. On the worldsheet, it corresponds to an overlap of two states (see Figure 1). On the string worldsheet, this corresponds to an overlap between a boundary state $\langle \mathcal{G} \rangle$ and a state dual to $\mathcal{O}$. To evaluate such an overlap, we first consider a partition function $Z(J,R)$ of a cylinder worldsheet whose ends are capped off by the boundary states (see Figure 2). In the limit $R \to \infty$, the expansion of $Z(J,R)$ in the closed string channel is dominated by the ground state $|\Omega\rangle$

$$Z(J,R) = \sum_{\psi_{\mathfrak{c}}} \langle \mathcal{G}|\psi_{\mathfrak{c}}\rangle e^{-E_{\mathfrak{c}} R} \langle \psi_{\mathfrak{c}}|\mathcal{G}\rangle$$

$$R \to \infty \quad \langle \mathcal{G}|\Omega\rangle^2 e^{-E_\Omega R}.$$

By contrast, in the open string channel, $Z(J,R)$ can be viewed as the thermal free energy, and the limit corre-
sponds to the thermodynamic limit in which the volume of the space becomes infinite. This allows us to compute $\langle G|\Omega \rangle$ using TBA. The result for excited states can be obtained from $\langle G|\Omega \rangle$ by analytic continuation [10].

**III. CONSTRAINTS ON BOUNDARY STATES**

To apply the aforementioned strategy, we first determine the boundary state $\langle G \rangle$ in the infinite volume ($J \to \infty$) limit. For this, we assume that $\langle G \rangle$ is an integrable boundary state, namely a state corresponding to a boundary condition which preserves infinite many conserved charges [11]. The assumption is justified a posteriori by agreement with weak-coupling computations as we see later. For integrable boundary states, the overlap in the $J \to \infty$ limit can be factorized into two-particle overlaps

$$F_{AB}(u) \equiv \langle G|A'_A(u)X_B(\bar{u})\rangle_{J \to \infty}, \quad (4)$$

where $A'$s are magnons in the $N = 4$ SYM spin chain, and $A = AA$ and $B = BB$ are in the bifundamental representation of the $su(2|2)^2$ symmetry [12]. The rapidities $u$ and $\bar{u}$ are parity-conjugate to each other, and satisfy $x^+(\bar{u}) = -x^+(u)$ where $f^+(u) \equiv f(u + \frac{i}{2})$ and the Zhukovsky variable $x(u)$ is defined by $x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g}$ with $g = \frac{N}{4}$ and $\lambda$ being the t'Hooft coupling constant.

**a. Boundary Yang-Baxter equation** The integrable boundary state satisfies the so-called boundary Yang-Baxter equation (bYBE), which reads (see Figure 3)

$$\langle G|S_{24}S_{34}|X_1(u)X_2(v)X_3(\bar{v})X_4(\bar{u})\rangle_{J \to \infty} = \langle G|S_{13}S_{12}|X_1(u)X_2(v)X_3(\bar{v})X_4(\bar{u})\rangle_{J \to \infty}, \quad (5)$$

where $S_{kl}$ is the bulk $S$-matrix [12] between $X_k$ and $X_l$.

**b. Watson’s equation** The second constraint is Watson’s equation, which states that an exchange of particles is equivalent to a multiplication of the $S$-matrix. Explicitly, it reads (see Figure 4)

$$F_{AB}(u) = S_{AB}^{CD}(u, \bar{u})F_{CD}(\bar{u}). \quad (6)$$

**c. Decoupling equation** The last condition is the decoupling condition, which is equivalent to the boundary unitarity in [11]. It states that a pair of particle-antiparticle pairs must decouple from the rest of the overlap (see Figure 5) and reads

$$F_{AB}(u)\mathcal{C}^{BB'}F_{B'C}C^{C'C} = \delta_{A}^{C} \quad (7)$$

where $\mathcal{C}$ is the charge conjugation matrix [13], and $u^{3\gamma}$ is the crossing transformation defined by $x^\pm(u^{3\gamma}) = \frac{1}{x^\pm(u)}$.

**d. Solution** Solving these constraints, the two-particle overlap is fixed to be

$$F_{AB}(u) = \frac{x^+ - \frac{i}{4}}{x^-} \frac{\sigma_B(u)}{\sigma(u, \bar{u})}(-1)^{|A||B|}M_{A,B}, \quad (8)$$

where $| \bullet |$ denotes the grading of the index $\bullet$ and $\sigma(u, v)$ is the bulk dressing phase [14]. As shown in Appendix A, there are two choices for the matrix part $M_{A,B}$ and we conjecture that the true boundary state is given by a sum of the two. See also discussions below (16). $\sigma_B(u)$ is the boundary dressing phase satisfying

$$\sigma_B(\bar{u}) = \sigma_B(u), \quad \sigma_B(u)\sigma_B(u^{3\gamma}) = \frac{u^2}{u^2 + \frac{1}{4}}. \quad (9)$$

A solution is given by $\sigma_B(u) = 4^{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}G(x^+ - \frac{i}{4})$ with

$$\log G(x) = \int \frac{dz}{2\pi i} \log \Theta(g(z + z^{-1})) = \frac{\Gamma(\frac{1}{2} + iz)}{\Gamma(\frac{1}{2} + iz)} \frac{\Gamma(1 + iz)}{\Gamma(1 + iz)} \quad (10)$$

and $\Theta(u) = \Gamma(\frac{1}{2} - iz)\Gamma(1 + iz)$.
FIG. 6. The bABA and its unfolding interpretation. The structure of the reflection matrix allows us to map the bABA to an ABA of a closed chain with a single $psu(2|2)$ symmetry.

IV. g-FUNCTION FOR GROUND STATE

We now discuss the ground-state overlap $\langle G|\Omega \rangle$ for finite $J$. For this we consider $Z(J, R)$ in the open string channel (also known as the mirror channel) and take the limit $R \to \infty$:

$$Z(J, R) = \sum_{\psi_a} e^{-\widehat{E}_{\psi_a} J} \chi_R^{\infty} \mathcal{N} \int \mathcal{D} \rho e^{-\rho \mathcal{S}_{\text{eff}}[\rho]}.$$  \hspace{1cm} (11)

As shown above, in the limit $R \to \infty$ one can replace the sum over $\psi_a$ with a path integral of densities $\rho$.

a. Bethe equation in the mirror channel

The crucial input for writing down $\mathcal{S}_{\text{eff}}$ is the boundary asymptotic Bethe equation (bABA), which constrains the rapidities of magnons. Schematically, it reads (see Figure 6)

$$1 = \mathbb{R}_L(u_j) \prod_{k \neq j} \mathcal{S}(u_j, u_k) \mathbb{R}_R(u_j) \prod_{k \neq j} \mathcal{S}(u_j, \bar{u}_k),$$ \hspace{1cm} (12)

where $\mathbb{R}_{L,R}$ are the left/right reflection matrices. The reflection matrices are related to the infinite-volume overlap (8) by $[\mathbb{R}_L]^{\text{BB}}(u) = [\mathbb{R}_R]^{\text{BB}}(\bar{u}) = F_{\text{AC}}^{\text{CB}}(u^\dagger) e^{\Theta_{\text{CB}}} \mathcal{S}_{\text{BB}}$ with $u^\dagger$ being the mirror transformation defined by $x^+(u^\dagger) = 1/x^-(u)$ and $x^-(u^\dagger) = x^-(u)$. As a result, we find

$$[\mathbb{R}_L]^{\text{BB}}(u) = \frac{u-i}{u} \sigma B(u^\dagger) \mathcal{S}_{\text{BB}}^{\text{AA}}(\bar{u}^\gamma, u^\gamma),$$ \hspace{1cm} (13)

where $\mathcal{S}$ is a single copy of the $psu(2|2)$ S-matrix [12]. The structure of $\mathbb{R}_{L,R}$ allows for the unfolding interpretation; the bABA (12) can be mapped to an ABA of a closed string with a single $psu(2|2)$ symmetry (see Figure 6).

b. TBA equation

$\mathcal{S}_{\text{eff}}$ can be derived from ABA following the standard derivation of TBA. In the $R \to \infty$ limit, (11) can be approximated by the saddle point $\delta \mathcal{S}_{\text{eff}} / \delta \rho = 0$. Due to the unfolding structure, the saddle-point equations coincide with the standard TBA for the spectrum [15-17] with the identification $Y_{a,s}(\bar{u}) = Y_{a,-s}(u)$. They take a form of $\log Y_{a,s} = \varphi_{a,s}^\dagger$. For instance, $\varphi_{a,0}^\dagger$ reads (the full equations are given in [18])

$$\varphi_{a,0}^\dagger = -\hat{J} \hat{E}_a + \log(1 + Y_{0,0}) \ast (K_{\pm})_{b,a}^{\dagger}$$
$$+ \log(1 + \delta_{m,1}) \ast (K_{\pm})_{m-1,a}^{\dagger} + \log(1 + 1/Y_{2,2}) \ast (K_{\pm})_{\psi a}^{\dagger}$$
$$+ \log(1 + Y_{1,1}) \ast (K_{\pm})_{\psi a}^{\dagger}.$$ \hspace{1cm} (14)

Here we follow the notations in [19], and $\ast$ and $\hat{\ast}$ denote the convolutions along $[-\infty, \infty]$, $[0, \infty]$ and $[-2g, 2g]$ respectively. $K_{\pm}$ is a symmetrized kernel defined by $K(u, v) = K(u, \delta) + K(\delta, v)$ with $K$ being the standard TBA kernel.

c. g-function

The saddle point value of $\mathcal{S}_{\text{eff}}$ only gives the exponentially decaying piece in (3), $e^{-\hat{E}_R}$. To read off the overlap, we need to consider the one-loop fluctuation around the saddle point and the $O(1)$ normalization factor $\mathcal{N}$ in (11). Such analysis was performed in the literature [20-22], and the application to our problem leads to an expression for the ground states, which corresponds to BPS operators in $\mathcal{N} = 4$ SYM.

$$\langle G|\Omega \rangle = e^{\Sigma_a J_0 \ln \frac{\delta \Theta_{a,s}^\dagger}{\delta \log Y_{a,s}(v)}} \sqrt{\text{Det}(1 - \hat{G})} \left| \frac{\text{Det}(1 - \hat{G}^\ast)}{\text{Det}(1 - \hat{G}_+)} \right|.$$ \hspace{1cm} (15)

Here $a$ runs from 1 to $\infty$ and $\Theta_{a,s}(u) = i\delta_u \log r_{a}(\bar{u}) - \pi \delta(u) + \frac{1}{2} \partial_u \log S_{a,s}^\dagger(u, \bar{u})$ where $S_{a,s}^\dagger$ is the bound-state S-matrix and $r_a$ is the bound-state reflection factor given in Appendix A. $\hat{G}$ denotes the Fredholm determinant [23] and $\hat{G}_+$ is an integral kernel defined by $[\hat{G}_+](a,b)(u, v) = \frac{\delta \varphi_{a,s}^\dagger}{\delta \log Y_{a,s}(v)}$. Similarly, $\hat{G}$ is given by $\frac{\delta \varphi_{a,s}^\dagger}{\delta \log Y_{a,s}(v)}$ where $\varphi_{a,s}$ are the “right hand sides” of the standard TBA, $\log Y_{a,s} = \varphi_{a,s}$, without identification of $Y$-functions.

V. CONJECTURE FOR SL(2) SECTOR

We now generalize (15) to excited states in the SL(2) sector using the analytic continuation trick [10] following the standard TBA analysis.

a. g-function for excited states

After the analytic continuation, poles of $1/(1 + 1/Y_{1,0})$ cross the integration contour and modify the overlap (15). As a result, we find that the structure constant $\mathcal{D}_\sigma$ is given by

$$\mathcal{D}_\sigma = -\frac{J_0 + (-i)^J}{\sqrt{J}} e^{\Sigma_a J_0 \ln \frac{\delta \Theta_{a,s}^\dagger}{\delta \log Y_{a,s}(v)}} \left( \prod_{1 \leq s \leq 4} \frac{u_s^2 + \frac{1}{2} \sigma^2 B(u_s)}{u_s^2} \right) \sqrt{\text{Det}(1 - \hat{G}^\ast) \left| \frac{\text{Det}(1 - \hat{G}_+)}{\text{Det}(1 - \hat{G}^\ast)} \right|}.$$ \hspace{1cm} (16)

This is the main result of this letter which we conjecture to be valid for any length $J$ and at finite $\lambda$. Here the $J$-dependent prefactor reflects the fact that the true boundary state is a sum of two boundary states as mentioned below (8). $\hat{G}^\ast$ and $\hat{G}_+$ are given by [24]

$$\hat{G}^\ast \cdot f = \sum_{k=1}^{M} \frac{i K_{\pm}(u_k, u)}{\partial_u \log Y_{1,0}(u_k)} f(u_k) + \hat{G}_+ \cdot f,$$ \hspace{1cm} (17)

$$\hat{G}_+ \cdot f = \sum_{k=1}^{M/2} \frac{i (K_{\pm})_{\ast}(u_k, u)}{\partial_u \log Y_{1,0}(u_k)} f(u_k) + \hat{G}_+ \cdot f.$$
Here and below, $x$ and $X$ take various indices and symbols which represent different bound states. The sum in (17) come from the poles crossing the contours, and $u_k$’s are the magnon rapidities satisfying the parity condition,

$$u_{M+k} = \bar{u}_k \quad (1 \leq k \leq M/2). \quad (18)$$

They are the solutions to the exact Bethe equations [15],

$$\phi(u_j) = 2\pi i \left(n_j + \frac{1}{2}\right), \quad n_j \in \mathbb{Z}, \quad (19)$$

with

$$\phi(u) \equiv -J\tilde{E}_1 + \sum_{k=1}^{M} \log S_{11}^\bullet(u, u_k) + \sum_{x,X} \log(1 + Y_{X,x}) \ast K_{x,1}^{X\bullet}. \quad (20)$$

### b. Exact Gaudin determinants

The result (17) can be rewritten into a form similar to the so-called Gaudin determinants. For this, we first split the kernel $G^\bullet$ into a sum $\hat{S}$ and an integral $I$ (see (17)), and rewrite $\text{Det}(1-\hat{G}^\bullet)$ as $\text{Det}(1-\hat{S} - I) = \text{Det}(1-\hat{S}) \times \text{Det}(1-I)$ with $\hat{S} \equiv S/(1-I)$. Similarly, $\text{Det}(1-\tilde{G}^\bullet)$ can be split into a sum $\tilde{S}_+$ and an integral $I_+$, and can be re-expressed as $\text{Det}(1-\tilde{G}^\bullet_+) = \text{Det}(1-\tilde{S}_+) \times \text{Det}(1-I)_+$ with $\tilde{S}_+ \equiv S_+/I_+$. Next we consider $\partial_{u_k} \phi(u_j)$ ($j, k = 1, \ldots, M$). The derivative $\partial_{u_k}$ can act either on $p(u_k)$, log $S_{11}^\bullet$ or $Y_{X,x}$ in (20). We then eliminate $\partial_{u_k} Y_{X,x}$ by considering the excited state TBA (see [15] for the full set of equations)

$$\log Y_{a,0} = -J\tilde{E}_a + \sum_{k=1}^{M} \log S_{a1}^\bullet(u, u_k) + \sum_{x,X} \log(1 + Y_{X,x}) \ast K_{x,a}^{X\bullet}, \quad (21)$$

taking a derivative $\partial_{u_k}$ of both sides, and solving for $\partial_{u_k} Y_{X,x}$. The parity condition (18) is imposed only at the end of the computation. As a result of these manipulations, we find that $\text{det}(\partial_{u_k} \phi(u_j)) \propto \text{Det}(1-\hat{S})$ up to some constant of proportionality. The relation in particular shows that $\text{Det}(1-\hat{S})$ is actually a finite-dimensional determinant although it was initially defined as the Fredholm determinant. Details of the rewriting are explained in a toy example in Appendix B.

On the other hand, if we first impose the parity condition (18) and compute the derivatives $\partial_{u_k} \phi(u_j)$, we find that $\text{det}(\partial_{u_k} \phi(u_j)) \propto \text{Det}(1-\tilde{S}_+)$. Upon taking the ratio, the constants of proportionality cancel out and we obtain

$$\frac{\sqrt{\text{Det}(1-\hat{S})}}{\text{Det}(1-\tilde{S}_+)} = \frac{\sqrt{\text{det}(\partial_{u_k} \phi(u_j))}}{\text{det}(\partial_{u_k} \phi^+(u_j))}. \quad (22)$$

where $\phi^+$ denotes that we are imposing the parity condition before computing derivatives. These determinants can be viewed as the finite-volume version of the Gaudin determinants for the norm of the spin chain. They also resemble the finite-volume one-point functions in sin(h)-Gordon model [25–27].

Using this rewriting, we obtain an alternative representation for the Fredholm determinants in (16),

$$\frac{\sqrt{\text{Det}(1-\hat{G}^\bullet)}}{\text{Det}(1-\tilde{G}^\bullet_+)} = \frac{\sqrt{\text{det}(\partial_{u_k} \phi(u_j))}}{\text{det}(\partial_{u_k} \phi^+(u_j))} \frac{\sqrt{\text{Det}(1-I)}}{\text{Det}(1-I)_+}. \quad (23)$$

### c. Asymptotic formula

Using the representation (23), one can take the asymptotic limit of (16), in which the size of the operator becomes large $J \gg 1$. In this limit, the middle-node Y-functions are exponentially suppressed, $Y_{a,0} \rightarrow 0$, and one can show that both

$$\rho_{\lambda} \rightarrow \frac{\sqrt{\text{Det}(1-\hat{S})}}{\text{Det}(1-\tilde{S}_+)} \times \frac{\sqrt{\text{det}(\partial_{u_k} \phi(u_j))}}{\text{det}(\partial_{u_k} \phi^+(u_j))}. \quad (24)$$

Note that the determinants on the second line are the standard Gaudin-like determinants since all the finite-size corrections can be dropped. For generalization of (24) to operators outside the SL(2) sector, see [18]. A similar formula was found at weak coupling for the defect one-point functions [28].

### VI. Weak coupling test

To test our formula (24), we computed the four-point function of $D_{1,2}$ and two $20'$ operators $O_{20'}$ up to $O(\lambda^2)$. We then performed the operator product expansion to read off the conformal data of the spin-$S$ twist-2 operators $O_S$. The details are given in [18].

The results of the computation are summarized in Table I. We compared them against the integrability prediction (24) and observed a perfect match. This is quite a nontrivial test of our formalism since the results contain the transcendental number $\zeta_3$ and include the contributions from the boundary dressing phase $\sigma_B(u)$. Further tests at weak and strong couplings are provided in [18].

We also found that the structure constants exhibit a simple large spin behavior up to two loops,

$$\log \left[ \frac{\mathcal{D}_{O_S}}{\mathcal{D}_{O_S} \text{[tree level]}} \right] = f_1 \log S' + f_2 + O(1/S') \quad (25)$$
with $\log S' = \log S + \gamma_E$, where $\gamma_E$ is the Euler-Mascheroni constant, and

\begin{align*}
  f_1 &= -4g^2 \log 2 + 8g^4 \left[ \zeta_2 \log 2 + \frac{9}{2} \zeta_3 \right] + O(g^6), \\
  f_2 &= -2g^2 \zeta_2 + 8g^4 \left[ \frac{4}{5} (\zeta_2)^2 + \frac{3}{2} \zeta_3 \log 2 \right] + O(g^6).
\end{align*}

VII. CONCLUSION

In this letter, we applied the TBA formalism to write down a nonperturbative expression for the structure constant of two determinant operators and a single-trace operator in the $\text{SL}(2)$ sector of arbitrary size. Our result would provide a foundation for future developments, such as the reformulation in terms of the Quantum Spectral Curve [29], as was the case with the TBA for the spectrum. It would also be worth trying to extract various interesting physics from our formula. We also hope that our approach gives useful insights into the three-point functions of single-trace operators [2].

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[55] The kernels in this letter are related to the ones in [18] by the transposition.
Appendix A: Overlap and Reflection Matrix

The matrix part of the two-particle overlap is given by
\[
M_{a\alpha,bb} = \frac{1 + B}{2} \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\beta} + \frac{1 - B}{2} \epsilon_{\alpha}^{\alpha} \epsilon_{\beta}^{\beta},
\]
\[
M_{a\alpha,\beta\beta} = \frac{1 + E}{2} \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\beta} - \frac{1 + E}{2} \epsilon_{\alpha}^{\alpha} \epsilon_{\beta}^{\beta},
\]
\[
M_{a\alpha,\beta\beta} = G \epsilon_{\alpha}^{\alpha} \epsilon_{\beta}^{\beta}, \quad M_{a\alpha,bb} = G \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\beta},
\]
\[
M_{a\alpha,\beta\beta} = C \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\alpha}, \quad M_{a\alpha,\beta\beta} = H \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\alpha},
\]
\[
M_{a\alpha,\beta\beta} = H \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\alpha}, \quad M_{a\alpha,\beta\beta} = C \epsilon_{\alpha}^{\beta} \epsilon_{\beta}^{\alpha},
\]
where
\[
B = -\frac{x^- + z^2(x^+)^3}{z^2 x^+(1 + x^+ x^-)}, \quad E = \frac{x^4 x^4 + (x^-)^3}{z^2 x^-(1 + x^+ x^-)}.
\]
\[
C = -\frac{i [z^4 (x^+)^2 - (x^-)^2]}{2z^2 \sqrt{x^+ x^-} (1 + x^+ x^-)},
\]
\[
G = \frac{x^- + z^2 x^+}{2z \sqrt{x^+ x^-}}, \quad H = \frac{x^- - z^2 x^+}{2z \sqrt{x^+ x^-}}.
\]

There are two allowed solutions depending on the value of \( z, z = \pm i \).

The bound-state reflection factor \( r_a \) is given by
\[
r_a(u) = \left( \prod_{k=-1}^{a} \frac{u + ik + i}{u + ik} \right) \frac{\sigma_{B,a}(u^\gamma)}{\sigma_{a,a}(u^\gamma)},
\]
\[
\sigma_{B,a}(u) \equiv 2^{-E_a(u)} \frac{G(x^+ a)}{G(x^- a)},
\]
with \( f^{[+a]} \equiv f(u \pm \frac{a}{2}) \) and
\[
E_a(u) = \frac{a}{2} + \frac{q}{4} \left( \frac{1}{x^+ a} - \frac{1}{x^- a} \right).
\]

Appendix B: Fredholm Determinant in Toy Model

Here we write down an explicit expression for the Fredholm determinant in a simple toy model which only contains a single species of particles without bound states. We also elucidate the relation between the Fredholm determinants and the exact Gaudin determinants (22).

The excited state TBA is given by log \( Y = \phi \) with
\[
\phi(u) \equiv -J \hat{E} + \sum_{k=1}^{M} \log S(u, u_k) + \log(1 + Y) \ast K,
\]
where \( S \) is the S-matrix and \( K(u, v) \equiv -i \partial_a \log S(u, v) \). The rapidities \( u_k \)'s satisfy the exact Bethe equation
\[
\phi(u_j) = 2\pi i \left( n_j + \frac{1}{2} \right), \quad n_j \in \mathbb{Z}.
\]

Following the discussion in the main text, one obtains the deformed Fredholm kernel \( G^\bullet \cdot f = S \cdot f + 1 \cdot f \), with
\[
S : f \equiv \sum_{k=1}^{M} iK(u_k, u) f(u_k), \quad 1 \cdot f \equiv \frac{f}{1 + 1/Y} \ast K.
\]

Let us now consider the exact Gaudin norm \( \det(\partial_{u_k} \phi(u_j)) \). Using (B1), we obtain
\[
\partial_{u_k} \phi(u_j) = \partial_u \phi(u_k) [iK(u_k, u_j)] + 1 \cdot (\partial_{u_k} \epsilon(u_j)),
\]
with \( \epsilon(u) \equiv \log Y(u) \). To eliminate \( \partial_{u_k} \epsilon \), we differentiate the excited state TBA,
\[
\partial_{u_k} \epsilon(u) = -iK(u_k, u) + 1 \cdot (\partial_{u_k} \epsilon(u)),
\]
and invert it to get
\[
\partial_{u_k} \epsilon(u) = \frac{1}{1 - 1} \cdot [-iK(u_k, u)].
\]

Plugging it back to (B4), we obtain
\[
\partial_{u_k} \phi(u_j) = \partial_u \phi(u_k) \delta_{jk} - \frac{1}{1 - 1} \cdot [iK(u_k, u)] \bigg|_{u=u_j}.
\]

Comparing (B3) and (B7), we find
\[
\det(\partial_{u_k} \phi(u_j)) = \left( \prod_{k=1}^{M} \partial_u \phi(u_k) \right) \text{Det}(1 - \hat{S}),
\]
with \( \hat{S} \equiv S/(1 - 1) \). This relation allows us to relate the Fredholm determinant and the exact Gaudin determinant. The derivation can be readily generalized to \( G^\bullet \) and to theories with bound states such as \( N = 4 \) SYM, leading to the identity (22).