GEOMETRIC OPTICS FOR RAYLEIGH WAVETRAINS IN D-DIMENSIONAL NONLINEAR ELASTICITY

ARIC WHEELER AND MARK WILLIAMS

Abstract. A Rayleigh wave is a type of surface wave that propagates in the boundary of an elastic solid with traction (or Neumann) boundary conditions. Since the 1980s much work has been done on the problem of constructing a leading term in an approximate solution to the rather complicated second-order quasilinear hyperbolic boundary value problem with fully nonlinear Neumann boundary conditions that governs the propagation of Rayleigh waves. The question has remained open whether or not this leading term approximate solution is really close in a precise sense to the exact solution of the governing equations. We prove a positive answer to this question for the case of Rayleigh wavetrains in any space dimension \(d \geq 2\). The case of Rayleigh pulses in dimension \(d = 2\) has already been treated by Coulombel and Williams. For highly oscillatory Rayleigh wavetrains we are able to construct high-order approximate solutions consisting of the leading term plus an arbitrary number of correctors. Using those high-order solutions we then perform an error analysis which shows (among other things) that for small wavelengths the leading term is close to the exact solution in \(L^\infty\) on a fixed time interval independent of the wavelength. The error analysis is carried out in a somewhat general setting and is applicable to other types of waves for which high order approximate solutions can be constructed.

Contents

Part 1. General introduction and main results

Part 2. Exact solutions near approximate ones

1. Introduction
2. Norms and basic estimates
3. A priori estimates and local existence.
4. Main result
5. Uniform estimates for the coupled nonlinear systems

Part 3. Construction of approximate solutions for the traction problem

6. Introduction
7. Spaces of profiles
8. Cascade of profile equations
9. Solvability conditions for \(L_{ff}(U) = F, l_f(U) = G\)
10. Order of construction
11. Amplitude equations
12. Final steps in the construction of the \(U_k\)

Part 4. Tame estimate for the amplitude equation

References
Part 1. General introduction and main results

The problem of constructing oscillatory, approximate solutions to the traction problem in nonlinear elasticity (0.1) has been considered by a number of authors including [Lar83, Hun06, BGC12, BGC16, CW16]. These papers construct 2-scale, WKB-type, approximate solutions consisting of a leading term and, in the case of [CW16], a first corrector as well. This paper provides a “rigorous justification” (explained below) of such approximate solutions for wavetrains in any number of space dimensions $d \geq 2$.

The traction problem is a Neumann-type boundary value problem for a quasilinear second-order hyperbolic system, and like most Neumann-type problems the boundary conditions exhibit a certain degeneracy - the uniform Lopatinski condition fails. The specific nature of the failure in this case is manifested by the presence of surface waves, or Rayleigh waves, that propagate in the boundary along characteristics of the Lopatinski determinant. Approximate surface wave solutions of both pulse-type and wavetrain-type (defined below) have been constructed.

The task of rigorously justifying these approximate solutions, that is, showing that they are close in a precise sense to true exact solutions of the traction problem was begun in [CW16], which provided a justification in the case of pulses in two space dimensions; see Remark 0.2. In this paper we treat the case of wavetrains in any space dimension $d \geq 2$ by entirely different methods.

We now describe the form of the traction problem to be considered here. Let the unknown $\phi = (\phi_1, \ldots, \phi_d)(t, x)$ represent the deformation of an isotropic, hyperelastic, Saint Venant-Kirchhoff (SVK) material whose reference (or undeformed) configuration is $\omega = \{x = (x_1, \ldots, x_d) : x_d > 0\}$. We will study the case where the deformation results from the application of a surface force $g(t, x)$, but our methods extend easily to the case where forcing is applied in the interior as well. Here $\phi(t, \cdot) : \omega \to \mathbb{R}^d$ and $g(t, \cdot) : \partial \omega \to \mathbb{R}^d$. The equations are a second-order, nonlinear $d \times d$ system

$$\begin{align*}
\partial_t^2 \phi - \text{Div}(\nabla \phi \sigma(\nabla \phi)) &= 0 \text{ in } x_d > 0 \\
\nabla \phi \sigma(\nabla \phi)n &= g \text{ on } x_d = 0, \\
\phi(t, x) &= x \text{ and } g = 0 \text{ in } t \leq 0,
\end{align*}$$

(0.1)

where $n = (0, \ldots, 0, -1)$ is the outer unit normal to the boundary of $\omega$, $\nabla \phi = (\partial_x \phi_i)_{i=1,\ldots,d}$ is the spatial gradient matrix, $\sigma$ is the stress $\sigma(\nabla \phi) = \lambda \text{tr} E \cdot I + 2\mu E$ with Lamé constants $\lambda$ and $\mu$ satisfying $\mu > 0$, $\lambda + \mu > 0$, and $E$ is the strain $E(\nabla \phi) = \frac{1}{2} (\nabla \phi \cdot \nabla \phi - I)$. Here $\text{tr} E$ denotes the transpose of $\nabla \phi$, $\text{tr} E$ is the trace of the matrix $E$, and

$$\text{Div} M = \sum_{j=1}^d \partial_j m_{i,j} \text{ for a matrix } M = (m_{i,j})_{i,j=1,\ldots,d}.$$  

(0.2)

The system (0.1) has the form of a second-order quasilinear system with fully nonlinear Neumann boundary conditions:

$$\begin{align*}
\partial_t^2 \phi - \sum_{i=1}^d \partial_x_i (A_i(\nabla \phi)) &= 0 \text{ in } x_d > 0 \\
\sum_{i=1}^d n_i A_i(\nabla \phi) &= g \text{ on } x_d = 0 \\
\phi(t, x) &= x \text{ and } g = 0 \text{ in } t \leq 0,
\end{align*}$$

(0.3)

where the real functions $A_i(\cdot)$ are $C^\infty$ (in fact, polynomial) in their arguments.
Defining the displacement $U(t, x) = \phi(t, x) - x$, we rewrite (0.3) as

$$\partial_t^2 U - \sum_{i=1}^{d} \partial_{x_i} A_i(\nabla U) = 0 \text{ in } x_d > 0$$

(0.4)

$$\sum_{i=1}^{d} n_i A_i(\nabla U) = g \text{ on } x_d = 0$$

$$U(t, x) = 0 \text{ and } g = 0 \text{ in } t \leq 0,$$

where the functions $A_i$ are related to $A_i$ in the obvious way. Writing $x' = (x_1, \ldots, x_{d-1})$ for the tangential spatial variables, we take highly oscillatory wavetrain boundary data with the \textit{weakly nonlinear} scaling (defined below):

$$g = g^\varepsilon(t, x') = \varepsilon^2 G\left(t, x', \frac{\beta \cdot (t, x')}{\varepsilon}\right), \quad \varepsilon \in (0, 1],$$

(0.5)

where $G(t, x', \theta) \in H^\infty(\mathbb{R}_t \times \mathbb{R}_x^{d-1} \times \mathbb{T}_y)$. Here $\beta \in \mathbb{R}^d \setminus 0$ is a frequency in the elliptic region of (the linearization at $\nabla U = 0$ of) (0.4), chosen so that the uniform Lopatinskii condition fails at $\beta$. \footnote{See section 3 for definitions and more detail on the choice of $\beta$. The existence of such special frequencies $\beta$ was predicted by Lord Rayleigh $[\text{LP83}]$.} We will refer to $\beta$ as a Rayleigh \textit{frequency}. We expect the response $U^\varepsilon(t, x)$ to be a Rayleigh wavetrain propagating in the boundary.

We note that for a fixed $\varepsilon$ the existence of an exact solution $U^\varepsilon$ of (0.3) on a time interval $(-\infty, T_\varepsilon]$ follows from the main result of $[\text{SN89}]$. Since Sobolev norms of $g^\varepsilon(t, x)$ clearly blow up as $\varepsilon \to 0$, the times of existence $T_\varepsilon$ provided by $[\text{SN89}]$ converge to zero as $\varepsilon \to 0$. One of the goals of this paper is to show that solutions actually exist on a fixed time interval independent of $\varepsilon$.

The strategy, which is an example of a general method going back to $[\text{Gu93}]$, is to first construct high-order approximate solutions on a fixed time interval independent of $\varepsilon$, and then to construct exact solutions (which are known to be unique) that are “close” to the approximate solutions on a time interval that is possibly shorter but still independent of $\varepsilon$. Carrying this out will at the same time achieve the second main goal of the paper, which is to show that the (first-order) approximate solutions constructed by earlier authors are indeed close to the exact solutions for $\varepsilon$ small.

In part 3 (Theorem 12.5) we construct high-order approximate solutions $U^\varepsilon_a$ of (0.3) on a time interval $(\infty, T]$, with $T > 0$ independent of $\varepsilon$, of the form

$$U^\varepsilon_a(t, x) = \sum_{k=2}^{N} \varepsilon^{k} U_k(t, x, \beta \cdot (t, x')/\varepsilon, x_d),$$

(0.6)

where the “profiles” $U_k$ are \textit{bounded} and can be written

$$U_k(t, x, \theta, Y) = U_k^\ast(t, x, \theta, Y),$$

with $U_k^\ast$ periodic in $\theta$ and exponentially decaying in $Y$.\footnote{More precisely, $U_k$ lies in the space $S$ of Definition (0.4), the sense in which $U_k^\ast$ is an \textit{approximate} solution is specified in Theorem 12.5.}

We stated above that the scaling in the choice of boundary data (0.5) is the \textit{weakly nonlinear} scaling. For the wavelength $\varepsilon$, the weakly nonlinear scaling of $g^\varepsilon(t, x')$ is by definition the smallest amplitude, in this case $\varepsilon^2$, for which the equations for the leading order profile $U_2$ are nonlinear. A higher power of $\varepsilon$ would lead to a linear problem for $U_2$. The weakly nonlinear scaling was identified by Lardner $[\text{Lar83}]$, who made a first attempt to formulate the profile equation for the leading term $U_2$ in (0.6). When the Fourier mean of $G, G(t, x')$, is zero, it turns out that $U_2 = U_2^\ast$ and is completely determined by its trace on the boundary. The equation for that trace, which is
usually referred to as the amplitude equation, is a nonlinear, Burgers-type equation (11.6) that is nonlocal in \( \theta \).

The following theorem summarizes our main results, Theorems 1.4 and 12.6 as applied to nonlinear elasticity in the case where the boundary forcing \( G \in H^\infty \). Those results are more precise than Theorem 0.1 for example, they allow boundary data of finite smoothness and give precise information on the regularity and “size” of both the exact solution \( U^\varepsilon(t,x) \) and the profiles \( U_k(t,x,\theta,Y) \).

**Theorem 0.1.** Consider the SVK problem (0.4) with boundary data (0.5), where \( G(t,x',\theta) \in H^\infty(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{T}_\theta) \). There exist constants \( \varepsilon_0 > 0 \) and \( T > 0 \) such that the exact solution \( U^\varepsilon(t,x) \) of (0.4) exists and is \( C^\infty \) on \( \Omega := (-\infty,T) \times \mathbb{R}_x^d \times \mathbb{R}_Y^+ \) and satisfies for any \( p \geq 2 \)

\[
(0.8) \quad \left| U^\varepsilon - \left( \varepsilon^2 U_2 + \cdots + \varepsilon^p U_p \right) \right|_{\theta = \theta(t,x') \in \mathbb{R}, Y = \frac{d}{\varepsilon}} \in L^\infty(\Omega), \quad \varepsilon \in (0, \varepsilon_0],
\]

where \( \varepsilon^2 U_2 + \cdots + \varepsilon^p U_p \) is the approximate solution constructed in Theorem 12.5.

For any given choice of \( p \) in (0.8), the proof of the theorem depends on being able to construct a number of bounded profiles beyond \( U_p \). This theorem implies that the explicit qualitative information contained in the approximate solution really does apply to the exact solution. This includes information about amplitude, group velocity (Remark 11.1), and internal rectification (Remark 12.3). Moreover, the estimate shows (0.8) shows that even higher profiles contain information about the exact solution.

**Remark 0.2** (Pulses versus wavetrains). When the function \( G(t,x',\theta) \) in (0.5) belongs to \( H^\infty(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{T}_\theta) \) instead of \( H^\infty(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{T}_\theta) \), we obtain Rayleigh pulses instead of Rayleigh wavetrains. The Fourier spectrum of \( G \), that is, the \( k \)-support of \( \hat{G}(t,x',k) \), is now a subset of \( \mathbb{R} \) rather than \( \mathbb{Z} \) and may include a full neighborhood of \( k = 0 \). A consequence of this is that the decay of pulses in the interior (the \( Y \) decay) is much weaker than for wavetrains, generally only polynomial rather than exponential decay. Since the construction of successive correctors \( U_k \), \( k \geq 3 \) requires successive integrations over the unbounded domain \( \mathbb{R}_Y \times \mathbb{R}_Y \), starting at \( k = 4 \) the profile \( U_k \) becomes too large for \( \varepsilon^k U_k \) to be considered a “corrector”. Thus, in the pulse case it is impossible to construct higher-order approximate solutions, and the method of this paper cannot be used to justify approximate solutions.

In the paper [CW16] approximate, leading-term pulse solutions \( \varepsilon^2 U_2 \) were justified in the case of two space dimensions by a different method that relied on precise microlocal energy estimates for certain “singular systems” associated to (0.1). These estimates were proved using Kreiss symmetrizers [Kre70] for the linearized SVK problem whose construction relied on the fact that in 2D, the linearized problem is strictly hyperbolic. For space dimensions three and higher, the linearized SVK problem fails to be strictly hyperbolic, and in fact exhibits complicated characteristics of variable multiplicity. Although Kreiss symmetrizers are now available in a number of situations where strict hyperbolicity fails, including some cases of variable multiplicity [Met00, MZ05], the linearized SVK problem in higher dimensions lies beyond the reach of current Kreiss-symmetrizer technology. The method of [CW16] can also be used to justify approximate wavetrain solutions \( \varepsilon^2 U_2 \) in 2D, but the lack of Kreiss symmetrizers prevents us from using that method for either pulses or wavetrains in space dimensions \( \geq 3 \).

**0.1. Organization of the paper.** In part 2 we consider a class of second-order, quasilinear hyperbolic problems with Neumann boundary conditions (1.1) that includes the SVK system (0.4). The boundary data can be taken to be oscillatory wavetrain data of the form (0.5), but it could have other forms. We assume that we are given an approximate solution \( u^\varepsilon \) of “high-order” in the sense of Assumption 1.1 on some fixed time interval independent of \( \varepsilon \), and we seek a nearby exact solution of the form \( u^\varepsilon = u^\varepsilon_{\text{a}} + v^\varepsilon \). The essential step is to study the “error problem” satisfied by \( v^\varepsilon \) in order to show that \( v^\varepsilon \) exists on a fixed time interval independent of \( \varepsilon \in (0, \varepsilon_0] \) for some \( \varepsilon_0 > 0 \),
and that \( v^\varepsilon \) is “small” compared to \( u^\varepsilon_0 \). This is accomplished in the proof of the main result of this part, Theorem 4.6.

The proof of Theorem 4.6 has two main components. The first is the well-posedness theory of \[\text{Shi88, SN89}\] for problems of the form \((1.1)\). The needed results are summarized in section 3 and we apply the local existence and continuation results, Propositions 3.8 and 3.9, to the problem \((1.1)\) with \( \varepsilon \) fixed. As already noted this yields solutions on time intervals \( T^\varepsilon_\alpha \) that apparently converge to zero as \( \varepsilon \to 0 \). These well-posedness results are based on estimates for linearized hyperbolic and elliptic problems associated to \((1.1)\) that are stated in Propositions 3.3 and 3.5. These estimates were proved by taking advantage of the special structure of the Neumann problem \((1.1)\). Although they are not as precise as the microlocal estimates that can be proved with Kreiss symmetrizers in the case of two space dimensions, they are sufficiently precise for our purposes here. Roughly speaking, the relative lack of precision is compensated for by the fact that we now have a high-order approximate solution to work with.

The second main component is the proof of simultaneous a priori estimates for a trio of coupled nonlinear problems consisting of \((1.1)\) and two other problems derived from it - the problems \((1.6)\) (which is equivalent to \((1.1)\)), \((1.7)\), and \((1.9)\). These estimates, which are summarized in Proposition 4.7, are carried out in section 5. The estimate of Proposition 4.7 is given in terms of an \( \varepsilon \)-dependent energy \( E^\varepsilon_s(t) \) (Definition 4.2) for the coupled systems. An essential feature of this estimate for the purpose of obtaining a time of existence independent of \( \varepsilon \) is that it is \textit{uniform}: the constants that appear in it are either independent of \( \varepsilon \) or converge to zero as \( \varepsilon \to 0 \). The two components are put together in section 4 in a continuous induction argument proving Theorem 4.6.

The main novelties of part 2 lie in the choice of an energy \( E^\varepsilon_s(t) \) that can be estimated “without loss” in terms of itself as in Proposition 4.7, and in the proof of the uniform estimates of section 5.

Part 3 is devoted to the construction of high-order approximate solutions \( U^\varepsilon_\alpha \) to the SVK system \((0.4)\). These solutions satisfy Assumption 4.1 so Theorem 4.6 applies to show that they are close to exact solutions. This result is formulated in Theorem 12.6 which is a more precise version of Theorem 0.1. The writing of part 3 was strongly influenced by \[\text{Mar10}\], which treats surface waves for first-order conservation laws with linear, homogeneous boundary conditions \( Cu = 0 \), and the second Chapter of \[\text{Mar11}\], which constructs approximate solutions consisting of a leading term \( U_2 \) and (part of a) first corrector \( U_3 \) for a simplified version of the SVK model. We give more detail later about our debt to these works; here we just note that the main novelty in our construction of approximate solutions lies in our construction of arbitrarily high order profiles.

In part 4 we provide a new proof of a tame estimate for the amplitude equation \((11.5)\). The solution of this equation determines the leading term in the approximate solution of SVK \((0.1)\). A tame estimate is needed to obtain a time of existence that depends on a fixed low order of regularity of the solution, rather than a time that decreases to zero as higher regularity is considered. Such an estimate (without slow spatial variables) was already proved in \[\text{Hun06}\] for a modified form of the amplitude equation; however, he did not use it to obtain a time of existence independent of high order regularity. The proof we give here applies directly to the form of the amplitude equation given by \((11.5)\) and incorporates the slow spatial variables.

Parts 2, 3, and 4 are written so that they can be read independently of one another. We include additional introductory material in the introductions to parts 2 and 3 sections 1 and 6.

**Remark 0.3.** Constants \( C, C_j, A_j, B_j \), etc. appearing in the various estimates are always independent of \( \varepsilon \) unless explicit \( \varepsilon \)-dependence is indicated. Also, we occasionally write \( |f| \lesssim |g| \) as shorthand for \( |f| \leq C|g| \) for some \( C > 0 \).
Part 2. Exact solutions near approximate ones

1. Introduction

Letting \((t, x) = (t, x', x_d) \in \mathbb{R}^{1+d}_+\) we consider the following \(N \times N\) nonlinear hyperbolic problem with Neumann boundary conditions on a half-space:

\[
P(u) = \partial_t^2 u - \sum_{i=1}^d \partial_i (A_i (D_x u)) = 0 \text{ in } x_d \geq 0
\]

\[
Q(u) = \sum_{i=1}^d n_i A_i (D_x u) = g(t, x') \text{ on } x_d = 0
\]

\[
u = 0 \text{ in } t \leq 0,
\]

where \(g = 0\) in \(t \leq 0\) and \(n = (0, \ldots, 0, -1) \in \mathbb{R}^d\) is the outward unit normal. Our main interest is in highly oscillatory wavetrain boundary data

\[
g = g^\varepsilon(t, x') = \varepsilon^2 G \left( t, x', \frac{\beta \cdot (t, x')}{\varepsilon} \right),
\]

where \(G(t, x', \theta) \in H^1(\mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{T})\) for some large \(t\), but the results of this part depend on only Assumptions 3.6 and 4.1 and thus apply to other types of boundary data.

The traction problem in nonlinear elasticity takes the form (1.1), and as we will show in part 3, the factor \(\varepsilon^2\) in (1.2) gives the weakly nonlinear scaling.

We assume that a “high-order” approximate solution \(u^\varepsilon_a(t, x)\) has been constructed. This is by definition a function with the regularity and growth properties listed in Assumption 4.1 which satisfies

\[
P(u^\varepsilon_a) = -\varepsilon M R^\varepsilon := -F^\varepsilon(t, x)
\]

\[
Q(u^\varepsilon_a) = g^\varepsilon - \varepsilon M r^\varepsilon := g^\varepsilon(t, x') - G^\varepsilon(t, x'),
\]

where the functions \(R^\varepsilon, r^\varepsilon\) have the properties given in Assumption 4.1 and \(M > 0\) is an integer that will be specified later. In the case of the traction problem such an approximate solution is constructed in part 3.

We look for an exact solution of (1.1) in the form

\[
u^\varepsilon = u_a + v.
\]

Let \(v = \varepsilon^M \nu\) and \(w = \partial_t v = \varepsilon^M \omega\), where \(\omega = \partial_t \nu\). The hyperbolic error problem that \(v\) must satisfy is:

\[
(a) \partial_t^2 v - \sum_{i=1}^d \partial_i [A_i (D_x (u_a + v)) - A_i (D_x u_a)] = F(t, x)
\]

\[
(b) \sum_{i=1}^d n_i [A_i (D_x (u_a + v)) - A_i (D_x u_a)] = G(t, x') \text{ on } x_d = 0.
\]

\[\text{We write } A_i (D_x u) = A_i (\partial_i u, \ldots, \partial_i u), \text{ where } u \text{ takes values in } \mathbb{R}^N \text{ and } A_i \text{ has } N \text{ real components.}\]

\[\text{The functions } u_a \text{ and } v \text{ clearly depend on } \varepsilon, u_a = u^\varepsilon_a \text{ and } v = v^\varepsilon, \text{ but we shall often suppress the superscripts } \varepsilon \text{ on these and other obviously } \varepsilon \text{-dependent functions.}\]
Below we set $A_{ij}(U) := \partial A_i / \partial U_j$, where $U = (U_1, \ldots , U_d)$ with $U_j \in \mathbb{R}^N$ a placeholder for $\partial_j u$. The problem (1.5) is equivalent to the following hyperbolic problem for $\nu$:

$$
(a) \partial_t^2 \nu - \sum_{i,j=1}^{d} \partial_i [A_{ij}(D_x(u_a + v))\partial_j \nu] = \\
\varepsilon^{-M} \left[ \mathcal{F} + \sum_{i=1}^{d} \partial_i \left( A_i(D_x(u_a + v)) - A_i(D_x u_a) - \sum_{j=1}^{d} A_{ij}(D_x(u_a + v))\partial_j v \right) \right] := \mathcal{F}_1
$$

(1.6)

(b) $\sum_{i=1}^{d} n_i [A_{ij}(D_x(u_a + v)) \partial_j \nu] = \\
\varepsilon^{-M} \left[ \mathcal{G} - \left( \sum_i n_i [A_i(D_x(u_a + v)) - A_i(D_x u_a) - \sum_j A_{ij}(D_x(u_a + v))\partial_j v] \right) \right] := \mathcal{G}_1.$

The function $\omega$ satisfies the hyperbolic problem obtained by differentiating (1.6) with respect to $t$:

$$
(a) \partial_t^2 \omega - \sum_{i,j=1}^{d} \partial_i [A_{ij}(D_x(u_a + v))\partial_j \omega] = \varepsilon^{-M} [\partial_t \mathcal{F} - H(D_x^2 v)] := \mathcal{F}_2
$$

(1.7)

(b) $\sum_{i=1}^{d} n_i [A_{ij}(D_x(u_a + v)) \partial_j \omega] = \varepsilon^{-M} [\partial_t \mathcal{G} - H_b(D_x v)] := \mathcal{G}_2,$

where

$$
H(D_x^2 v) := - \sum_{i,j=1}^{d} \partial_i [(A_{ij}(D_x(u_a + v)) - A_{ij}(D_x u_a))\partial_j \partial_t u_a]
$$

(1.8)

$$
H_b(D_x v) := - \sum_{i,j=1}^{d} n_i (A_{ij}(D_x(u_a + v)) - A_{ij}(D_x u_a)) \partial_j \partial_t u_a.
$$

The problem (1.6) for $\nu$ can also be written as the following elliptic problem for $\nu$:

$$
(a) \sum_{i,j=1}^{d} \partial_i [A_{ij}(D_x u_a)\partial_j \nu] + \lambda \nu = \\
- \partial_t \nu + \lambda \int_0^t \omega(s,x) ds + \varepsilon^{-M} [\mathcal{F}(t,x) + E(D_x^2 v)] := \mathcal{F}_3
$$

(1.9)

(b) $\sum_{i,j=1}^{d} n_i [A_{ij}(D_x u_a) \partial_j \nu] = \varepsilon^{-M} [\mathcal{G}(t,x^\prime) + E_b(D_x v)] := \mathcal{G}_3,$

The form of the problems (1.7) and (1.9) is similar to problems that appear in the induction scheme of [SN89].
where $\lambda > 0$ is large enough and
\[
E(D^2_x v) := -\sum_{i=1}^{d} \partial_i \left( A_i(D_x(u + v)) - A_i(D_x u_a) - \sum_{j=1}^{n} A_{ij}(D_x u_a) \partial_j v \right) = \\
-\sum_{i=1}^{d} \partial_i \left[ \int_{0}^{1} (1 - \theta)(d^2 A_i(D_x(u + \theta v))(D_x v, D_x v) d\theta \right], \\
E_0(D_x v) = \sum_{i=1}^{d} n_i \left[ \int_{0}^{1} (1 - \theta)(d^2 A_i(D_x(u + \theta v))(D_x v, D_x v) d\theta \right].
\]

(1.10)

Remark 1.1. Conversely, one can check that sufficiently regular solutions $(\omega^\varepsilon, \nu^\varepsilon)$ of the coupled systems (1.7), (1.9) actually satisfy $\omega = \partial_t \nu$, and that $\nu$ then satisfies (1.6). This can be shown by differentiating (1.9) with respect to $t$, and using the result of that together with (1.7) to show that $\partial_t \nu - \omega$ satisfies an elliptic problem with vanishing interior and boundary data. Substituting $\partial_t \nu = \omega$ into (1.9), we then see that $\nu$ satisfies (1.6). This argument, which is not needed in this paper, was used in [SNS79] to prove local existence for (1.1) for data with no $\varepsilon$-dependence, Proposition 3.8 below.

Proposition 3.8 gives for each fixed $\varepsilon$ a solution $(\omega^\varepsilon, \nu^\varepsilon)$ of (1.4), (1.7), and (1.9) on a time interval $T_\varepsilon$ that depends on $\varepsilon$. Our main task in the error analysis is to prove that solutions of (1.6) exist on a fixed time interval independent of $\varepsilon$. This requires estimates that are uniform with respect to $\varepsilon$. We will show that one can prove such estimates for an appropriate $\varepsilon$-dependent “energy”, $\mathcal{E}_\varepsilon(t)$ (Definition 1.2) defined in terms of $\nu$ and $\omega$, by estimating solutions of the systems (1.6), (1.7), and (1.9) using the linear a priori estimates for hyperbolic and elliptic boundary problems given in section 3.

The hyperbolic estimate of Proposition 3.3 exhibits a loss of one-half spatial derivative on the boundary, corresponding to the degeneracy in the Neumann boundary condition (or more precisely, to the failure of the uniform Lopatinski condition in the elliptic region). By estimating the systems (1.6), (1.7), and (1.9) simultaneously, we are able, roughly speaking, to use the elliptic estimate for (1.9) to gain back what is lost in the hyperbolic estimates. This process is carried out in section 5 where we obtain estimates uniform with respect to $\varepsilon$ for an appropriate $\varepsilon$-dependent “energy” $\mathcal{E}_\varepsilon(t)$, whose definition (Definition 1.2) involves both $\nu$ as in (1.6), (1.7) and $\omega$ as in (1.7). This estimate of $\mathcal{E}_\varepsilon(t)$ is stated in Proposition 4.7. In the remainder of section 4 we show how to use this estimate to prove Theorem 4.6 and thereby complete the error analysis.

2. Norms and basic estimates

In this section we define the norms and spaces needed to state and prove the main results of part 2. We also prove basic properties of the norms that will be used repeatedly in section 5.

Notation 2.1. For any nonnegative integers $s$, $k$ let
\[
E^{s,k}[0, T] = \bigcap_{i=0}^{\infty} C^i([0, T] : H^{s+k-i}({\mathbb R}^d)) \quad \text{and} \quad E_b^{s,k}[0, T] = \bigcap_{i=0}^{\infty} C^i([0, T] : H^{s+k-i}({\mathbb R}^{d-1})).
\]

We sometimes write $E^{s,0}[0, T] = E^s[0, T]$ and $E_b^{s,0}[0, T] = E_b^s[0, T]$. 2. For $u \in E^{s,k}[0, T], \ t \in [0, T], \ and \ \varepsilon \in (0, 1]$ we define the $(t, \varepsilon)$-dependent norm $|u(t)|_{s,k,\varepsilon}$ by
\[
|u(t)|_{s,k,\varepsilon} := \sup_{|\alpha| \leq s, |\beta| \leq k} \varepsilon^{|\alpha|+|\beta|} \partial_t^\alpha \partial_x^\beta u(t, \cdot)|_{L^2(x)}.
\]

For $f = f(t, x') \in E_b^{s,k}[0, T]$ we similarly define boundary norms
\[
|f(t)|_{s,k,\varepsilon} := \sup_{|\alpha| \leq s, |\beta| \leq k} \varepsilon^{|\alpha|+|\beta|} \partial_t^\alpha \partial_{x'}^\beta f(t, \cdot)|_{L^2(x')}.
\]
We will sometimes write \(|u(t)|_{s,0,\varepsilon} = |u(t)|_{s,\varepsilon}\) and do similarly for the boundary norms.

3. For \(u \in E^{s,k}[0,T]\) we set \(|u|_{s,k,\varepsilon,T} := \sup_{t \in [0,T]} |u(t)|_{s,k,\varepsilon}\) and for \(f \in E^{s,k}_b[0,T]\) we set \(\langle f \rangle_{s,k,\varepsilon,T} := \sup_{t \in [0,T]} \langle f(t) \rangle_{s,k,\varepsilon}\).

4. If \(T' < 0\) one defines \(E^{s,k}[T',T]\) and the corresponding norm \(|u|_{s,k,\varepsilon,T',T}\) as the obvious analogues of \(E^{s,k}[0,T]\) and \(|u|_{s,k,\varepsilon,T}\). When \(u \in E^{s,k}[T',T]\) and vanishes in \(t < 0\), we write \(|u|_{s,k,\varepsilon,T',T}\) instead as simply \(|u|_{s,k,\varepsilon,T}\).

5. For a nonnegative integer \(k\) and \(f = f(t,x)\) we set

\[
\begin{align*}
D^k f &= (\partial^k_t f, |\alpha| = k) \quad \text{and} \quad \overline{D}^k f = (\partial^k_t f, |\alpha| \leq k), \\
\overline{D}^k f &= (\partial^k_x f, |\beta| = k) \quad \text{and} \quad \overline{D}^k_x f = (\partial^k_x f, |\beta| \leq k).
\end{align*}
\]

We normally write \(D\) or \(\overline{D}\) in place of \(D^1\) or \(\overline{D}^1\).

6. All scalar functions appearing in part 2 are real-valued. This applies to components of vectors and entries of matrices.

The next proposition is an immediate consequence of the definitions.

**Proposition 2.2.** Let \(C\) denote a constant independent of \(\varepsilon\) and let \(\varepsilon_0 > 0\). If \(\varepsilon^2 \overline{D}^2_x \nu |_{s,0,\varepsilon} \leq C\) for \(\varepsilon \in (0,\varepsilon_0]\), then

\[
|\varepsilon^2 D x \nu|_{s,1,\varepsilon} \leq C \quad \text{and} \quad |\varepsilon^2 \nu|_{s,2,\varepsilon} \leq C \quad \text{for} \quad \varepsilon \in (0,\varepsilon_0].
\]

**Remark 2.3.** Below for \(\nu\) as in (1.6), (1.7), (1.9) we will need \(\nu\) to satisfy both \(\varepsilon^2 \overline{D}^2_x \nu |_{s,0,\varepsilon} \leq C\) and \(\varepsilon \overline{D}_x \nu |_{s,0,\varepsilon} \leq C\) for an appropriate choice of \(s\) and range of (small) \(\varepsilon\). Observe that the second of these two properties is not a consequence of the first.

When proving properties of the \(|\cdot|_{s,k,\varepsilon}\) norms the following simple observation (used already for the \(|\cdot|_{s,0,\varepsilon}\) norm in [Mar10]) is quite useful.

**Lemma 2.4.** Given \(u = u(t,x)\) and \(\varepsilon > 0\) define the rescaled function \(\tilde{u}(\tilde{t},\tilde{x})\) by \(\tilde{u}(\frac{t}{\varepsilon},\frac{x}{\varepsilon}) = u(t,x)\). Then

\[
|u(t)|_{s,k,\varepsilon} = \varepsilon^{d/2}|\tilde{u}(t/\varepsilon)|_{s,k,1}.
\]

Here \(\tilde{u}(t/\varepsilon)|_{s,k,1}\) means \(\tilde{u}(\tilde{t})|_{s,k,1}\) evaluated at \(\tilde{t} = t/\varepsilon\).

**Proof.** This follows immediately by change of variables from

\[
\varepsilon^{\alpha+\beta} \partial_t^\alpha \partial_x^\beta u(t,x) = \partial_{\tilde{t}}^\alpha \partial_{\tilde{x}}^\beta \tilde{u}(\frac{t}{\varepsilon},\frac{x}{\varepsilon}).
\]

**Proposition 2.5** (Sobolev estimate). Let \(s > \frac{d}{2}\). Then

\[
\begin{align*}
(a) \quad |u(t)|_{L^\infty(x)} &\leq \varepsilon^{-\frac{d}{4}} |u(t)|_{s,\varepsilon} \\
(b) \quad |\partial_t^\alpha u(t)|_{L^\infty(x)} &\leq \varepsilon^{-\frac{d}{4} - |\alpha|} |u(t)|_{s+|\alpha|,\varepsilon}
\end{align*}
\]

**Proof.** We have

\[
|u(t)|_{L^\infty(x)} = |\tilde{u}(\frac{t}{\varepsilon})|_{L^\infty(\tilde{x})} \leq |\tilde{u}(\frac{t}{\varepsilon})|_{H^s(\tilde{x})} \leq |\tilde{u}(\frac{t}{\varepsilon})|_{s,0,1} = \varepsilon^{-\frac{d}{4}} |u(t)|_{s,0,\varepsilon}
\]

by Lemma 2.5. The estimate (b) follows directly from (a).

---

\(\text{The functions } u_{\alpha}, \nu, \text{ and } \omega \text{ in (1.6), (1.7), and (1.9) all vanish in } t < 0.\)
The following product estimate, used in the treatment of first order conservation laws in [Mar10], will be useful to us here.

**Proposition 2.6 (Product estimate).** Suppose \( a \) and \( b \) are \( \geq 0 \), \( a + b \leq s \), and \( s > \frac{d}{2} \). Then

\[
(2.8) \quad |(uv)(t)|_{s-a-b,\varepsilon} \leq C \varepsilon^{-\frac{d}{2}} |u(t)|_{s-a,\varepsilon} |v(t)|_{s-b,\varepsilon}
\]

**Proof.** Following [Mar10] we begin with the well-known estimate

\[
(2.9) \quad |u(t)v(t)|_{H^{s-a-b}(x)} \leq C |u(t)|_{H^{s-a}(x)} |v(t)|_{H^{s-b}(x)}.
\]

Thus, for \( k \leq s - a - b \) we have

\[
(2.10) \quad |\partial^k_t (u(t)v(t))|_{H^{s-a-b-k}(x)} \leq C \sup_{k_1+k_2=k} |\partial^{k_1}_t u(t)\partial^{k_2}_t v(t)|_{H^{s-a-k_1-b-k_2}(x)} \leq
\]

\[
\sup_{k_1+k_2=k} |\partial^{k_1}_t u(t)|_{H^{s-a-k_1}(x)} |\partial^{k_2}_t v(t)|_{H^{s-b-k_2}(x)} \leq C |u(t)|_{s-a,0,1} |v(t)|_{s-b,0,1}.
\]

This implies \( |u(t)v(t)|_{s-a-b,0,1} \leq C |u(t)|_{s-a,0,1} |v(t)|_{s-b,0,1} \), so an application of Lemma 2.4 finishes the proof.

\[
\square
\]

**Proposition 2.7 (Trace estimate).** For \( g = g(t,x',x_d) \) and any nonnegative integer \( s \) we have\(^7\)

\[
(2.11) \quad \varepsilon \langle \Lambda^\frac{1}{2}_x g(t) \rangle_{s,\varepsilon} \lesssim |g(t)|_{s,1,\varepsilon}.
\]

**Proof.** We have

\[
(2.12) \quad \varepsilon \langle \Lambda^\frac{1}{2}_x g(t) \rangle_{s,\varepsilon} = \sup_{|\alpha| \leq s} \varepsilon^{\alpha+|\alpha'|+1} \left| \Lambda^\frac{1}{2}_x \partial^{\alpha}_t \partial^{\alpha'}_x g(t,x',0) \right|_{L^2(x')} \lesssim
\]

\[
\sup_{|\alpha| \leq s} \varepsilon^{\alpha+|\alpha'|+1} \left( |\partial^{\alpha}_t \partial^{\alpha'}_x g(t,x',x_d)|_{L^2(x)} + |\partial^{\alpha}_t \partial^{\alpha'}_x g(t,x',x_d)|_{L^2(x)} \right) \lesssim |g(t)|_{s,1,\varepsilon},
\]

where we have used \( \langle \Lambda^\frac{1}{2}_x h(x',0) \rangle_{L^2(x')} \lesssim |h(x',x_d)|_{H^1(x)} \) to get the first inequality.

\[
\square
\]

3. A PRIORI ESTIMATES AND LOCAL EXISTENCE.

In this section we state (with one essential modification in Proposition 3.3) the results from [Shi88, SN89] that will be used in this paper.

For some \( T > 0 \) consider the following linear hyperbolic problem:

\[
(3.1) \quad \sum_{i,j=1}^d n_i A_{ij}(t,x) \partial^j u := f(t,x) \text{ in } [-\infty, T] \times \mathbb{R}^d_+ \]

\[
\sum_{i,j=1}^d n_i A_{ij}(t,x) \partial^j u := g(t,x') \text{ on } [-\infty, T] \times \mathbb{R}^{d-1} \]

\[
u = 0 \text{ in } t \leq 0.
\]

**Definition 3.1.** 1. For any nonnegative integer \( L \) let \( \mathcal{B}^L_T \) denote the set of real-valued functions \( f(t,x) \) on \( [0,T] \times \mathbb{R}^d_+ \) with \( \|f\|_{C^L([0,T] \times \mathbb{R}^d_+)} < \infty \).

2. Set \( \|u(t,\cdot)\| := \|u(t,x)|_{L^2(x)} \) and \( \langle g(t,\cdot) \rangle := |g(t,x')|_{L^2(x')} \).

---

\(^7\)Here \( \Lambda^{1/2}_x \) is the operator defined by \( \mathcal{F}(\Lambda^{1/2}_x h) = \sqrt{1 + |\xi|^2} \mathcal{F}(h) \), where \( \mathcal{F} \) denotes the Fourier transform with respect to \( x' \).
Assumption 3.2. a) The entries of $A_{ij}$ belong to $B^2_d$.
b) For any $(t, x) \in [0, T] \times \mathbb{R}^d_+$ we have $A_{ij}^t = A_{ji}$ ($M^t$ denotes the transpose of the matrix $M$).
c) There exist positive constants $\delta_1$ and $\delta_2$ such that for $u \in H^2(\mathbb{R}^d_+)$ and $t \in [0, T]$

$$
\sum_{i,j=1}^d (A_{ij}(t, \cdot)\partial_j u, \partial_i u)_{L^2(x)} \geq \delta_1 \|D_x u\|^2 - \delta_2 \|u\|^2.
$$

The next proposition gives an improved version of an estimate in [Shi88].

**Proposition 3.3** (Hyperbolic estimate). Let $T > 0$ and consider the problem (3.1) where the coefficients $A_{ij}$ satisfy Assumption 3.2 and $u \in E^2[0, T]$ with $u(0, x) = \partial_t u(0, x) = 0$. There exists $K_1 = K_1(T, \delta_1, \delta_2, \max|A_{ij}|W^{1, \infty})$ such that $u$ satisfies

$$
\|\mathcal{D}(t, \cdot)\|^2 \leq K_1 \int_0^t (\|f(s, \cdot)\|^2 + \langle \Lambda_{\delta_2}^1 g(s, \cdot) \rangle^2) ds \text{ for } t \in [0, T],
$$

Proof. The proof is identical to the proof of Theorem 6.8 in [Shi88], except for one essential change.

The proof there uses (for example on p. 181) a commutator estimate (Theorem Ap. 5)

$$
\|e^{-\gamma t}[a(x), \Phi]u\|_{L^2} \leq C|a|_{C^{1, \mu}} e^{-\gamma t} \|u\|_{L^2},
$$

where $\Phi$ is a classical pseudodifferential operator of order one (depending on a parameter $\gamma \geq 1$) and $\mu$ is any small positive number. In place of (3.3) one can use an improved estimate where the Hölder norm $|a|_{C^{1, \mu}}$ is replaced by $|a|_{W^{1, \infty}}$, the Lipschitz norm of $a$. Thus, the constant $K_1$ here depends on $\max|A_{ij}|W^{1, \infty}$ rather than $\max|A_{ij}|_{C^{1, \mu}}$ as in [Shi88].

**Remark 3.4.** Proposition 3.3 will be applied in the error analysis of section 3 to the problems (1.6) and (1.7). The fact that $K_1$ depends just on the Lipschitz norm of the $A_{ij}$ is crucial for the application we give here, since the coefficients $A_{ij}(D_x u_a + D_x v)$ in (1.6), (1.7), although quite regular for each fixed $\varepsilon$, are no better than $W^{1, \infty}$ uniformly with respect to small $\varepsilon$.

Next we give an estimate for the linear elliptic problem:

$$
- \sum_{i,j=1}^d \partial_i (A_{ij}(t, x)\partial_j u) + \lambda u := f(t, x) \text{ in } [0, T] \times \mathbb{R}^d_+
$$

$$
\sum_{i,j=1}^d \partial_i A_{ij}(t, x)\partial_j u := g(t, x') \text{ on } [0, T] \times \mathbb{R}^{d-1},
$$

where $t$ is treated as a parameter.

**Proposition 3.5** (Elliptic estimate, [SN89]). Consider the problem (3.5) where the coefficients $A_{ij}$ satisfy Assumption 3.2 and $u \in E^2[0, T]$. There exist positive constants $\lambda_0(T, \delta_1, \delta_2, \max|A_{ij}|W^{1, \infty})$ and $K_2(T, \delta_1, \delta_2, \max|A_{ij}|W^{1, \infty})$ such that for $\lambda \geq \lambda_0$ the function $u$ satisfies

$$
\|\mathcal{D}^2(t, \cdot)\| \leq K_2 \left(\|f(t, \cdot)\| + \langle \Lambda_{\delta_2}^1 g(t, \cdot) \rangle\right) \text{ for } t \in [0, T].
$$

This estimate is part of Theorem 4.4 of [SN89]. We will apply it to the problem (1.9) for $\nu$ to gain spatial derivatives.

The following is our main structural assumption on the nonlinear problem (1.1) (and (5.8)).
\section*{Assumption 3.6.} a) Let $R > 0$ and let $U \in \mathbb{R}^{dN}$; here $dN$ is the number of scalar arguments of the $\mathbb{R}^{N}$-valued functions $A_i$ in (1.1). The $A_i$ are $C^\infty$ functions of $U$ and satisfy $A_i(0) = 0$.

b) The $N \times N$ matrices $A_{ij} := \partial A_i/\partial U_j$ satisfy $A_{ij}^T = A_{ji}$ ($M^T$ denotes the transpose of the matrix $M$).

c) There exist positive constants $\delta_1$ and $\delta_2$ such that for $u \in H^2(\mathbb{R}^d_+)$ and $|U| \leq 4R$

(3.7)

$$\sum_{i,j=1}^{d} (A_{ij}(U) \partial_j u, \partial_i u)_{L^2(x)} \geq \delta_1 \|u\|^2_{H^1} - \delta_2 \|u\|^2.$$

\section*{Remark 3.7.} The equations of nonlinear elasticity for three-dimensional, isotropic, hyperelastic materials are shown in \cite{SN89}, pages 8-10, to satisfy Assumption 3.6 for small enough $R > 0$. The Saint Venant-Kirchhoff model (0.1) for which we construct approximate solutions $v_n^0$ in part 3 belongs to this class of models.

Next we state a local existence theorem for the nonlinear initial boundary value problem

$$P(u) = \partial_t^2 u - \sum_{i=1}^{d} \partial_i (A_i(D_x u)) = f(t, x) \text{ in } x_d \geq 0$$

(3.8)

$$Q(u) = \sum_{i=1}^{d} n_i A_i(D_x u) = g(t, x') \text{ on } x_d = 0$$

$$u(0, x) = v_0(x), \quad u_t(0, x) = v_1(x).$$

\section*{Proposition 3.8} (Local existence and uniqueness: \cite{SN89}, Theorem 2.4). Suppose that Assumption 3.6 holds, let $L$ be an integer $\geq \lceil d/2 \rceil + 8$ (here $\lceil r \rceil$ is the greatest integer $\leq r$), and let $T_0$ and $\mathbb{B}$ be positive constants. Assume that the data in (3.8) satisfy

(3.9)

(a) $v_0 \in H^L(\mathbb{R}^d_+)$, $v_1 \in H^{L-1}(\mathbb{R}^d_+)$, $f \in C^{L-1}(0, T_0; L^2(\mathbb{R}^d_+)) \cap E^{L-2}(0, T_0)$,

$$g \in C^{L-1}(0, T_0; H^{1/2}(\mathbb{R}^d_+)) \cap E^{L-2,1/2}(0, T_0),$$

(b) $v_0$, $v_1$, $f$, and $g$ satisfy corner compatibility conditions of order $L - 2$ and $|v_1|_{L^\infty} + |D_x v_0|_{L^\infty} \leq R$,

(c) $|v_0|_{H^{[d/2] + 8}} + |v_1|_{H^{[d/2] + 7}} + |f|_{H^{[d/2] + 6,0,1,T_0}} + \langle \Lambda_{\frac{1}{2},g} \rangle_{[d/2] + 6,0,1,T_0} \leq \mathbb{B}.$

Then there exists $T_1 = T_1(\mathbb{B}) \in (0, T_0)$ such that (3.3) has a unique solution $u \in E^L[0, T_1]$ satisfying $|u|_{W^{1, \infty}} \leq 3R$.

\section*{Proposition 3.9} (Continuation). Suppose that Assumption 3.6 holds, let $L$ be an integer $\geq \lceil d/2 \rceil + 8$, and let $T_0$ and $\mathbb{B}$ be positive constants. Suppose that $f$ and $g$ have the regularity in (3.9) (a) and that for some $T \in (0, T_0]$ we are given a solution of (3.8) such that $u \in E^L[0, T]$ with

$$|u|_{L,0,1,T} \leq \mathbb{B}/2 \quad \text{and} \quad |u|_{W^{1,\infty}(0, T) \times \mathbb{R}^d} \leq R.$$ 

Then there is a time step $\Delta T$ depending on $\mathbb{B}$, $f$, and $g$, but not on $T$, such that $u$ extends to a solution on $[0, \min(T + \Delta T, T_0)]$ and satisfies $|u|_{L,0,1,\min(T + \Delta T, T_0)} \leq \mathbb{B}$.

\section*{Remark 3.10.} 1. The compatibility conditions referred to in (3.9) (b) are rather complicated to state and their precise form is not needed in this paper; they are stated on p. 6 of \cite{SN89}.

2. Proposition 3.9 is not stated explicitly in \cite{SN89}, but it is a Corollary of Proposition 3.8 and its proof.

3. Theorem 2.4 of \cite{SN89} deals with a wider class of systems than (3.8).

\footnote{Here $g \in E_b^{L-2,1/2}(0, T_0) \Leftrightarrow \Lambda_{1/2,g} \in E_b^{L-2}(0, T_0)$.}

\footnote{The norms on $f$ and $g$ are defined in Notation 2.1.}
In this section we describe the main result relating approximate and exact solutions, Theorem 4.6, and give an outline of the main argument, deferring some of the proofs until section 5.

Throughout the remainder of Part 2 we make the following assumption about the objects appearing in the problem (1.3) satisfied by the approximate solution $u_\varepsilon$.

**Assumption 4.1.** Suppose $M > \frac{d}{2} + 2$, let $T$ be fixed once and for all as the time of existence of the approximate solution $u_\varepsilon$ for $\varepsilon \in (0, 1]$, and set $\Omega := (-\infty, T] \times \mathbb{R}_+^d$. We suppose $s \geq \left\lfloor \frac{d}{2} \right\rfloor + 6$ (here $[r]$ is the greatest integer $\leq r$), that $u_\varepsilon \in E^{s+2}(-\infty, T]$ and vanishes in $t < 0$, and that for some positive constants $A_1, A_2$ we have

\begin{align}
|\varepsilon^\alpha \partial^\alpha_{t,x} u_\varepsilon|_{L^\infty(\Omega)} &\leq \varepsilon^2 A_1 \text{ for all } |\alpha| \leq s + 2, \\
|R_\varepsilon(t)|_{s+1,\varepsilon} + |r_\varepsilon(t)|_{s+2,\varepsilon} &\leq A_2 \text{ for all } t \in [0, T],
\end{align}

where $R_\varepsilon, r_\varepsilon$ are as in (1.3).

Next we define an $\varepsilon$-dependent “energy” $\mathcal{E}_\varepsilon(t)$ for which an estimate uniform with respect to $\varepsilon$ is stated in Proposition 4.7.

**Definition 4.2** (The energy $\mathcal{E}_\varepsilon(t)$). For $\nu^\varepsilon, \omega^\varepsilon$ as in (1.6), (1.7), (1.9) and $t \in \mathbb{R}$, let $\mathcal{E}_\varepsilon(t) = |\varepsilon^2 \mathcal{D}_x^2 \nu(t)|_{s,\varepsilon} + |\varepsilon \mathcal{D}_t \nu(t)|_{s,\varepsilon} + |\varepsilon^2 \mathcal{D}_t \omega(t)|_{s,\varepsilon}$. Here and below $s \geq \left\lfloor \frac{d}{2} \right\rfloor + 6$ as in Assumption 4.1.

**Remark 4.3.** Propositions 3.8 or 3.9 are stated for the system (3.8) and thus apply to (1.1). We want to apply these propositions to the systems (1.6), (1.7), (1.9) for fixed $\varepsilon$. To do this, we use the simple observation that if $u$ is a solution of (1.1), then the functions $\nu, \omega$ defined by

\begin{equation}
\nu = \varepsilon^{-M}(u - u_0), \quad \omega = \partial_t \nu
\end{equation}

satisfy (1.6), (1.7), and (1.9). Moreover, since $u_0 \in E^{s+2}(-\infty, T]$ it is clear that

\begin{equation}
u \in E^{s+2}[0, T] \Rightarrow (\mathcal{D}_x^2 \nu, \mathcal{D}_t \nu, \mathcal{D}_t \omega) \in E^s[0, T],
\end{equation}

and it follows directly from the definition of the spaces $E^s[0, T]$ that the converse of (1.3) is also true when (4.2) holds.

The following lemma is needed among other things for our applications of Propositions 3.8 and 3.9 as well as the linear estimates of Propositions 3.8 and 3.9.

**Lemma 4.4.** Suppose $T_\varepsilon \in (0, T]$ and that for some $C_1 > 0$ we have $\mathcal{E}_\varepsilon(t) \leq C_1$ for all $t \in [0, T_\varepsilon]$. Then$^{11}$

\begin{align}
(a) \quad |u_\varepsilon|_{W^1,\infty(\Omega)} &\leq \varepsilon A_1, \\
(b) \quad |v|_{W^1,\infty([0, T_\varepsilon] \times \mathbb{R}_+^d)} &\leq C_1 \varepsilon^{M-\frac{d}{2}-1}.
\end{align}

**Remark 4.5.** Observe that if $\mathcal{E}_\varepsilon(t) \leq C_1$ for all $t \in [0, T_\varepsilon]$ and $\varepsilon \in (0, \varepsilon_1]$, we can use Lemma 4.4 to insure, by reducing $\varepsilon_1$ if necessary, that for $\varepsilon \in (0, \varepsilon_1)$:

\begin{align}
|u_\varepsilon|_{W^1,\infty(\Omega)} &< R/2 \quad \text{and} \quad |v|_{W^1,\infty([0, T_\varepsilon] \times \mathbb{R}_+^d)} < R/2,
\end{align}

where $R$ is the constant in Assumption 3.6(a).

The following theorem is the main result of this part.

$^{11}$The proof of (4.4)(b) is contained in step 2 of the proof of Lemma 5.1 Part (a) is immediate from (1.1).
Theorem 4.6. Consider the nonlinear hyperbolic Neumann problem \((1.1)\) under Assumption \((3.6)\) and suppose that \(u_0^\varepsilon\) is an approximate solution satisfying Assumption \((4.1)\).

(a) There exist constants \(\varepsilon_3 > 0\) and \(C_2 > 0\) such that for \(\varepsilon \in [0, \varepsilon_3]\) the coupled systems \((1.6), (1.7), (1.9)\) have a unique solution on the time interval \([0, T]\) which satisfies

\[\mathcal{E}_\varepsilon^s(t) \leq C_2^s\text{ for all } t \in [0, T].\]

(b) For \(\varepsilon \in [0, \varepsilon_3]\) the problem \((1.1)\) has a unique exact solution \(u^\varepsilon \in E^{s+2}(-\infty, T]\) given by

\[u^\varepsilon = u_0^\varepsilon + v^\varepsilon = u_0^\varepsilon + \varepsilon^M \nu^\varepsilon,\]

where

\[|\varepsilon^2 D_x^2 v^\varepsilon(t)|_{s,\varepsilon} + |\varepsilon^2 \partial_t v^\varepsilon(t)|_{s,\varepsilon} + |\varepsilon^2 \partial_t \nu^\varepsilon(t)|_{s,\varepsilon} \leq \varepsilon^M C_2^s \text{ for } t \in [0, T] \text{ and } \varepsilon \in (0, \varepsilon_3].\]

In particular this implies \(|v^\varepsilon|_{W^{1,\infty}(\Omega)} \leq C_2^\varepsilon M^{-\frac{d}{2} - 1}.

The proof of Theorem 4.6 is based on the following a priori estimate for the coupled systems, whose proof is carried out in section 5.

Proposition 4.7. Suppose \(C_1\) and \(\varepsilon_1\) are positive constants and that for \(T_\varepsilon \in (0, T]\) and \(\varepsilon \in (0, \varepsilon_1]\), we are given a solution \((\nu^\varepsilon, \omega^\varepsilon)\) of the coupled systems on \([0, T_\varepsilon]\) which satisfies \(\mathcal{E}_\varepsilon^s(t) \leq C_1\) for all \(t \in [0, T_\varepsilon]\). Then there exist positive constants \(B_1 = B_1(T, A_1, K_1, K_2, \lambda)^{12}, B_2 = B_2(A_1, A_2, K_2, \lambda)\) such that for \(\varepsilon \in (0, \varepsilon_2]\) and \(t \in [0, T_\varepsilon]\)

\[\mathcal{E}_\varepsilon^s(t)^2 \leq B_1 \int_0^t \left( (\mathcal{E}_\varepsilon^s(\sigma))^2 + \varepsilon^2 |R^\varepsilon(\sigma)|^2_{s+1,\varepsilon} + |\nu^\varepsilon(\sigma)|^2_{s+2,\varepsilon} \right) d\sigma + \varepsilon^2 B_2.

This proposition will allow us to apply a continuous induction argument to prove Theorem 4.6.

Proposition 4.8. There exist constants \(C_2 > 0\) and \(\varepsilon_3 > 0\) such that for \(\varepsilon \in (0, \varepsilon_3]\) and \(T_\varepsilon \in (0, T]\), if \((\nu^\varepsilon, \omega^\varepsilon)\) is a solution of the three coupled nonlinear systems \((1.6), (1.7), (1.9)\) which satisfies \(\mathcal{E}_\varepsilon^s(t) \leq C_2\) for all \(t \in [0, T_\varepsilon]\), then in fact \(\mathcal{E}_\varepsilon^s(t) \leq C_2/2\) for all \(t \in [0, T_\varepsilon]\).

Assuming Proposition 4.7 we prove Proposition 4.8. In order to apply Proposition 4.7, we note that for any positive \(C_1\) and \(\varepsilon_1\) we take \(C_2\) and \(\varepsilon_3\) with the desired properties. We choose \(C_2^{\varepsilon} \geq 4(A_2^{\varepsilon} + 1) e^{B_1^{\varepsilon}/B_1}\). Taking this \(C_2\) as the choice of “\(C_1\)” in Proposition 4.7, we get \(\varepsilon_3\) to be a corresponding choice of “\(\varepsilon_2\)”. If \(\varepsilon \in (0, \varepsilon_3]\) and \((\nu^\varepsilon, \omega^\varepsilon)\) satisfies \(\mathcal{E}_\varepsilon^s(t) \leq C_2\) for all \(t \in [0, T_\varepsilon]\), then \((4.7)\) and Gronwall’s inequality imply

\[\mathcal{E}_\varepsilon^s(t)^2 \leq \int_0^t e^{B_1^{\varepsilon}(t-\sigma)} \left( (\varepsilon^2 |R^\varepsilon(\sigma)|^2_{s+1,\varepsilon} + |\nu^\varepsilon(\sigma)|^2_{s+2,\varepsilon} \right) d\sigma + \varepsilon^2 B_2 e^{B_1^{\varepsilon}t}/B_1 \text{ for all } t \in [0, T_\varepsilon].

Reducing \(\varepsilon_3\) if necessary so that \(\varepsilon^2 B_2 \leq 1\) for \(\varepsilon \in (0, \varepsilon_3]\), we obtain for all \(t \in [0, T_\varepsilon]\) and \(0 < \varepsilon \leq \varepsilon_3(C_2, A_1, A_2):\)

\[\mathcal{E}_\varepsilon^s(t)^2 \leq A_2^{\varepsilon} \int_0^t e^{B_1^{\varepsilon}(t-\sigma)} d\sigma + \varepsilon^2 B_2 e^{B_1^{\varepsilon}t}/B_1 \leq (A_2^{\varepsilon} + 1) e^{B_1^{\varepsilon}T}/B_1 \leq C_2^{\varepsilon}/4.

Next we show that Proposition 4.8 implies Theorem 4.6.

\[\square\]

\[\square\]The constants \(K_1, K_2,\) and \(\lambda\) appear in the linear estimates \((4.3), (4.6).\)

\[\square\]Since \(\nu\) and \(\omega\) vanish in \(t < 0\), we have \(\mathcal{E}_\varepsilon^s(t) = 0 < \frac{C_1}{B_1}\) for \(t < 0\). We also use Remark 4.3 here.

\[\square\]Here we see why it is essential that the constant \(B_1\) in Proposition 4.7 not depend on \(C_1\).
Proof of Theorem 4.6. 1. We choose \( \varepsilon_3 \) and \( C_2 \) to be the same as in Proposition 4.8. For each \( \varepsilon \in (0, \varepsilon_3] \) we will use a continuous induction argument to show that the solution of the coupled systems exists and satisfies (4.5) on all of \([0, T]\).

2. Suppose that for a given \( \varepsilon \in (0, \varepsilon_3] \) and some \( T_1 < T \), we have \( \mathcal{E}_\varepsilon(t) \leq C_2/2 \) for all \( t \in [0, T_1] \). Then the continuation result, Proposition 3.9, implies that there exists a time step \( \Delta T_\varepsilon > 0 \) (which depends on \( \varepsilon \) and \( C_2 \) but not on \( T_1 \)) such that \((\nu^\varepsilon, \omega^\varepsilon)\) extends to the time interval \([0, T + \Delta T_\varepsilon] \) and satisfies \( \mathcal{E}_\varepsilon(t) \leq C_2 \) for all \( t \in [0, T_1 + \Delta T_\varepsilon] \).

3. Fix \( \varepsilon \in (0, \varepsilon_3] \) and let \( T_\varepsilon^* = \text{sup}\{T' \in [0, T] : \mathcal{E}_\varepsilon(t) \leq C_2 \text{ for all } t \in [0, T']\} \). Observe that since \( \mathcal{E}_\varepsilon(t) = 0 \) for all \( t \leq 0 \), step 2 implies that \( T_\varepsilon^* \geq \Delta T_\varepsilon \). We now prove by contradiction that \( T_\varepsilon^* = T \), so suppose \( T_\varepsilon^* < T \). Then \( \mathcal{E}_\varepsilon(t) \leq C_2 \) for all \( t \in [0, T_\varepsilon^* - \frac{\Delta T_\varepsilon}{2}] \), and hence Proposition 4.8 implies \( \mathcal{E}_\varepsilon(t) \leq C_2 \) for all \( t \in [0, T_\varepsilon^* - \frac{\Delta T_\varepsilon}{2}] \). By step 2 we have \( \mathcal{E}_\varepsilon'(t) \leq C_2 \) for all \( t \in [0, T_\varepsilon^* + \frac{\Delta T_\varepsilon}{2}] \). Contradiction. This proves part (a).

4. Using Remark 4.3 and Lemma 4.4 we see that part (b) follows from part (a).

Thus it remains only to prove Proposition 4.7.

5. Uniform Estimates for the Coupled Nonlinear Systems

This section is devoted to proving Proposition 4.7. Recall that

\[
\mathcal{E}_\varepsilon(t) = |\varepsilon^2 \partial_x^2 \nu(t)|_{s, \varepsilon} + |\varepsilon \partial_x \nu(t)|_{s, \varepsilon} + |\varepsilon^2 \partial_x \omega(t)|_{s, \varepsilon}.
\]

Under the assumptions of that Proposition, the strategy will be to apply the elliptic estimate (3.6) to the problem (1.9) to estimate the first term in \( \mathcal{E}_\varepsilon(t) \), to apply the hyperbolic estimate (3.3) to the problem (1.6) to estimate the second term, and to apply the same hyperbolic estimate to the problem (1.7) to estimate the third term. We will use Remark 4.3 to insure that these systems satisfy the hypotheses of Propositions 3.3 and 3.5.

Throughout this section the constants \( \varepsilon_1, C_1 \), and \( T_\varepsilon \) are as given in the statement of Proposition 4.7.

5.1. Tangential derivative estimates. The first step is to estimate tangential \( \partial_{t,x}^\alpha := \partial_{t,x}^\alpha t \), derivatives. We define

\[
|u(t)|_{\mathcal{E}_\varepsilon, t,x, \alpha} = \sup_{|\alpha| \leq s} \varepsilon^{|\alpha|} |\partial_{t,x}^\alpha u(t, \cdot)|_{L^2(x)}
\]

(5.1)

\[
\mathcal{E}_{\varepsilon, t,x, \alpha} = |\varepsilon^2 \partial_x^2 \nu(t)|_{\mathcal{E}_\varepsilon, t,x, \alpha} + |\varepsilon \partial_x \nu(t)|_{\mathcal{E}_\varepsilon, t,x, \alpha} + |\varepsilon^2 \partial_x \omega(t)|_{\mathcal{E}_\varepsilon, t,x, \alpha}.
\]

and proceed to show for \( \varepsilon, B_1, B_2 \) as described in Proposition 4.7

\[
|\mathcal{E}_{\varepsilon, t,x, \alpha}(t)|^2 \leq B_1 \int_0^t \left( (\mathcal{E}_\varepsilon(\sigma))^2 + \varepsilon^2 |\nu^\varepsilon(\sigma)|_{s+1, \varepsilon}^2 + |r^\varepsilon(\sigma)|_{s+2, \varepsilon}^2 \right) d\sigma + \varepsilon^2 (B_1 [\mathcal{E}_\varepsilon(t)]^2 + B_2).
\]

We begin by estimating \( |\varepsilon \partial_x \nu(t)|_{\mathcal{E}_\varepsilon, t,x, \alpha} \) by applying the estimate (3.3) to \( \varepsilon^\alpha \partial^\alpha (\varepsilon \partial_{t,x}^\alpha) \) for \( |\alpha| \leq s \).

Lemma 5.1 (Interior commutator for (1.6)). Let \( \varepsilon_1 \) and \( C_1 \) be as in Proposition 4.7. There exist positive constants \( \varepsilon_2(C_1) \leq \varepsilon_1 \) and \( B_1 = B_1(A_1) \) such that for \( \varepsilon \in [0, \varepsilon_2] \) and all \( t \in [0, T_\varepsilon] \)

\[
|\partial_t [\varepsilon^\alpha \partial^\alpha, A_{ij}(D_x(vu_a + v))] \partial_t (\varepsilon \nu)|_{L^2(x)} \leq B_1 \mathcal{E}_\varepsilon(t) \text{ for all } |\alpha| \leq s.
\]

Here (and below) \( i, j \in \{1, \ldots, d\} \).

\(^{15}\)Here “\( \varepsilon^\alpha \partial^\alpha (\varepsilon \partial_{t,x}^\alpha) \)” denotes the result of multiplying the problem (1.6) by \( \varepsilon \) and then applying the operator \( \varepsilon^\alpha \partial^\alpha \).
Proof. 1. We note first that each component of \([\varepsilon^\alpha \partial^\alpha, A_{ij}(D_x(u_a + v))]\partial_j(\varepsilon \nu)\) is a finite sum of terms of the form \(16\)
\[ (5.4) \quad \varepsilon^{1+|\alpha|} H(D_x u_a, D_x v)(\partial^{\beta_1} D_x u_a) \ldots (\partial^{\beta_q} D_x u_a)(\partial^{\gamma_1} D_x v) \ldots (\partial^{\gamma_p} D_x v) \partial^\delta \partial_j \nu, \]
where \(H\) is a \(C^\infty\) function of its arguments and
\[ (5.5) \quad |\beta_k| \leq |\alpha|, |\gamma| \leq |\alpha|, |\xi| \leq |\alpha| - 1, \text{ and } |\beta| + |\gamma| + |\xi| = |\alpha| \leq s. \]
Either \(p\) or \(q\) can be zero, but not both \(17\). When the \(\partial_t\) derivative in \((5.3)\) is taken, four cases arise depending on whether \(\partial_t\) hits \(H\), one of the \(\partial^{\beta_k} D_x u_a\), one of the \(\partial^{\gamma_k} D_x v\), or \(\partial^\delta \partial_j \nu\).

2. In treating these cases we will use the following estimates, where \(m = 1\) or \(2\):
\[ (5.6) \quad |\partial^{\beta_k} D_x^m u_a(t)|_{L^\infty} \leq A_1 \varepsilon^{2-|\beta_k|-m} \]
\[ |\partial^{\gamma_k} D_x^m v(t)|_{s-|\gamma_k|, \varepsilon} \leq C_1 \varepsilon^{M-|\gamma_k|-m} \]
\[ |\partial^\delta D_x^m \nu(t)|_{s-|\delta|, \varepsilon} \leq \mathcal{E}_\varepsilon^s(t) \varepsilon^{-|\delta|-m}. \]
In this section we take, e.g., \(\partial^{\beta_k} := \partial_{t,x}^k\), but the above estimates also hold for \(\partial^{\beta_k} := \partial_{t,x}^{2\beta_k}\). These estimates follow directly from assumption \(14\) the assumption \(\mathcal{E}_\varepsilon^s(t) \leq C_1\), and the definition of \(\mathcal{E}_\varepsilon^s(t)\). For example, since \(\mathcal{E}_\varepsilon^s(t) \leq C_1\), we have
\[ (5.7) \quad \varepsilon^{|Dv|}_{s, \varepsilon} \leq C_1 \varepsilon^M \Rightarrow |v, Dv|_{s, \varepsilon} \leq C_1 \varepsilon^{M-1} \Rightarrow |\partial^{\gamma_k} (v, Dv)|_{s-|\gamma_k|, \varepsilon} \leq C_1 \varepsilon^{M-|\gamma_k|-1}. \]

We note also that the Sobolev estimate of Proposition \(2.5\) implies
\[ (5.8) \quad |(v, Dv)(t)|_{L^\infty} \lesssim \varepsilon^{-\frac{d}{2}} |(v, Dv)(t)|_{s, \varepsilon} \lesssim C_1 \varepsilon^{M-\frac{d}{2}-1} \]
\[ |D(v, Dv)(t)|_{L^\infty} \lesssim \varepsilon^{-\frac{d}{2}} |D(v, Dv)(t)|_{s-1, \varepsilon} \lesssim C_1 \varepsilon^{M-\frac{d}{2}-2}. \]
In the second estimate we have used \((5.7)\) with \(|\gamma_k| = 1\).\[^{18}\]

Remark 5.2. Since \(M > \frac{d}{2} + 2\) we see that the Lipschitz norm of \(D_x v^\varepsilon\) is bounded (\(\lesssim C_1\)) on \([0, T_\varepsilon] \times \mathbb{R}^d_+\) when \(\mathcal{E}_\varepsilon^s(t) \leq C_1\) on \([0, T_\varepsilon]\). Moreover, we can (and do) choose \(\varepsilon_2\) so that
\[ (5.9) \quad |D_x v|_{W^{1, \infty}(0, T_\varepsilon) \times \mathbb{R}^d_+} \leq 1 \text{ for } \varepsilon \in (0, \varepsilon_2]. \]
A bound like \((5.9)\) independent of \(C_1\) is needed to carry out the argument of Proposition \(4.8\). This is because the constants \(K_1\), \(K_2\), and \(\lambda\) in the linear estimates \((3.3)\) and \((3.6)\) depend on \(|D_x v|_{W^{1, \infty}}\), while \(B_1\) in Proposition \(4.7\) in turn depends on \(K_1\), \(K_2\), and \(\lambda\). For the argument of Proposition \(4.8\) to work, the constant \(B_1\) must not depend on \(C_1\).

3. Consider the case where \(\partial_t\) hits one of the components \(\partial^{\beta_k} D_x u_a\), say the first, producing a term we denote as \(\partial^{\beta_1} D_x^2 u_a\). When \(\gamma \geq 1\), setting \(|\gamma| := \sum_{k=1}^p |\gamma_k|\) and using \(-s - |\gamma| - |\xi| \geq 0\), we apply \((5.6)\) and the product estimate of Proposition \(2.6\) \(p\) times to obtain \(19\)
\[ (5.10) \quad |(\partial^{\gamma_1} D_x v) \ldots (\partial^{\gamma_p} D_x v) \partial^\delta \partial_j \nu|_{L^2} \lesssim \varepsilon^{-pd/2} |(\partial^{\gamma_1} D_x v)|_{s-|\gamma_1|, \varepsilon} \ldots |(\partial^{\gamma_p} D_x v)|_{s-|\gamma_p|, \varepsilon} |\partial^\delta \partial_j \nu|_{s-|\xi|, \varepsilon} \lesssim \varepsilon^{-pd/2} C_1 \varepsilon^{M|\gamma| - p} \mathcal{E}_\varepsilon^s(t) \varepsilon^{-|\delta|-1}. \]

Using \((5.6)\) and \((5.8)\) we also have
\[ (5.11) \quad |\varepsilon^{1+|\alpha|} H(D_x u_a, D_x v)(\partial^{\beta_1} D_x^2 u_a)(\partial^{\beta_2} D_x u_a) \ldots (\partial^{\beta_q} D_x u_a)|_{L^\infty} \lesssim \varepsilon^{1+|\alpha|} A_1^p \varepsilon^{-|\beta|-1}, \]
\(^{16}\)In \((5.4)\) as well as in similar expressions below, we suppress the indices that label components of \(D_x u_a, D_x v,\) or \(\nu\). Note also that here the \(\beta_j, \gamma_k\) are multi-indices.\(^{17}\)When, for example, \(q = 0\), this means that none of the terms \(\partial^{\beta_k} D_x u_a\) appears in \((5.4)\).\(^{18}\)We considered \((v, Dv)\) instead of \(D_x v\) here in order to include a proof of Lemma \(4.1(b)\).\(^{19}\)A similar use of the product estimate is made in section 1.10 of [Mar10] in her study of first-order hyperbolic conservation laws.
so by combining these two estimates we find

\[(5.12)\]
\[|\varepsilon^{1+|\alpha|} H(D_x u_0, D_x v)(\partial^{\beta_1} D_x^2 u_0)(\partial^{\beta_2} D_x u_0) \ldots (\partial^{\beta_p} D_x u_0)(\partial^{\gamma_1} D_x v) \ldots (\partial^{\gamma_p} D_x v) \partial^\zeta \partial_j \nu|_{L^2(x)} \leq C_1^p A_1^q E_\varepsilon^r(t) \varepsilon^{p(M - \frac{d}{2} - 1) - |\alpha| - |\gamma| - |\zeta|} \varepsilon^{q-1} \leq C_1^p A_1^q E_\varepsilon^r(t) \varepsilon^\mu \quad \text{for} \ t \in [0, T_\varepsilon].\]

We see that if \( \mu > 0 \) using (5.13), \( q \geq 0, p \geq 1, \) and \( M - \frac{d}{2} - 1 > 1, \) so there exists \( \varepsilon_2(C_1) \) such that the right side of (5.12) is \( A_1^q E_\varepsilon^r(t) \) for \( \varepsilon \in (0, \varepsilon_2) \).

When \( p = 0 \) we must have \( q \geq 1, \) and instead of (5.12) we obtain

\[(5.13)\]
\[|\varepsilon^{1+|\alpha|} H(D_x u_0, D_x v)(\partial^{\beta_1} D_x^2 u_0)(\partial^{\beta_2} D_x u_0) \ldots (\partial^{\beta_p} D_x u_0)(\partial^{\gamma_1} D_x v) \ldots (\partial^{\gamma_p} D_x v) \partial^\zeta \partial_j \nu|_{L^2(x)} \leq A_1^q E_\varepsilon^r(t) \varepsilon^{q-1} \leq A_1^q E_\varepsilon^r(t).\]

4. Next consider the case when \( \partial_1 \) hits \( \partial^\zeta \partial_j \nu \) producing \( \partial^\zeta D_x^2 \nu \). When \( p \geq 1 \) instead of (5.12) we find by arguing as above.

\[(5.14)\]
\[|\varepsilon^{1+|\alpha|} H(D_x u_a, D_x v)(\partial^{\beta_1} D_x^2 u_a)(\partial^{\beta_2} D_x u_a) \ldots (\partial^{\beta_p} D_x u_a)(\partial^{\gamma_1} D_x v) \ldots (\partial^{\gamma_p} D_x v) \partial^\zeta D_x^2 \nu|_{L^2(x)} \leq C_1^p A_1^q E_\varepsilon^r(t) \varepsilon^{-1} \varepsilon^{p(M - \frac{d}{2} - 1) - |\alpha| - |\gamma| - |\zeta|} \varepsilon^q \leq C_1^p A_1^q E_\varepsilon^r(t) \varepsilon^\mu \quad \text{for} \ t \in [0, T_\varepsilon].\]

Here \( \mu > 0 \) has the same value as in (5.13); the “extra” factor of \( \varepsilon^{-1} \) that was previously contributed by \( D_x^2 u_a \) is now contributed by \( D_x^2 \nu \). When \( p = 0 \) we clearly obtain an estimate just like (5.13).

5. The case when \( \partial_1 \) hits one of the \( \partial^{\alpha} D_x v \) in (5.14) clearly yields exactly the same right hand side as in (5.14). Finally, when \( \partial_1 \) hits \( H \) in (5.1) one obtains an expression that can be estimated using the other cases.

Next we estimate the interior forcing term in \( \varepsilon^\alpha \partial^\alpha (\varepsilon F) \), namely \( \varepsilon^{1+|\alpha|} \partial^\alpha F_1 \).

**Lemma 5.3** (Interior forcing for (1.16)). Let \( \varepsilon_1 \) and \( C_1 \) be as in Proposition 4.7. There exist positive constants \( \varepsilon_2(C_1) \leq \varepsilon_1 \) and \( B_1 = B_1(A_1) \) such that for \( \varepsilon \in (0, \varepsilon_2) \) and all \( t \in [0, T_\varepsilon] \)

\[(5.15)\]
\[|\varepsilon^{1+|\alpha|} \partial^\alpha F_1(t)|_{L^2(x)} \leq B_1 E_\varepsilon^r(t) + \varepsilon |R^\varepsilon(t)|_{s, \varepsilon} \quad \text{for all} \ |\alpha| \leq s.\]

**Proof.** 1. The first term of \( F_1 \) is \( \varepsilon^{-M} F = R^\varepsilon \), which obviously gives rise to the term \( \varepsilon |R^\varepsilon(t)|_{s, \varepsilon} \) in (5.15).

2. The remaining part of \( F_1 \) can be written in the form

\[(5.16)\]
\[\sum_{i=1}^d \partial_i [H_i(D_x u_a, D_x v)(D_x v, D_x v)],\]

where \( H_i \) is a \( C^\infty \) function of its arguments. First expanding out

\[(5.17)\]
\[\varepsilon^{1+|\alpha|} \partial^\alpha [H_i(D_x u_a, D_x v)(D_x v, D_x v)]\]

we obtain a sum of terms of the form

\[(5.18)\]
\[\varepsilon^{1+|\alpha|} H(D_x u_a, D_x v)(\partial^{\beta_1} D_x u_a) \ldots (\partial^{\beta_p} D_x u_a)(\partial^{\gamma_1} D_x v) \ldots (\partial^{\gamma_p} D_x v)(\partial^\zeta D_x v) \partial^\zeta D_x v,\]

where

\[(5.19)\]
\[|\beta| + |\gamma| + |\zeta| = |\alpha| \leq s.\]

\[20\text{The constant} \ \varepsilon_2 \text{will be decreased a finite number of times before we arrive at the final choice of} \ \varepsilon_2 \text{that appears in the statement of Proposition 4.7.}\]

\[21\text{The constant} \ B_1 \text{in (5.15) is, of course, not necessarily the same as the} \ B_1 \text{in Lemma 5.1. We are only interested in keeping track of what such constants depend on, not their particular values. We will continue to redefine certain constants in this way in order to reduce the number of distinct labels needed for constants.}\]
and $q$ or $p$ (possibly both) can be zero. Applying $\partial_t$ to (5.18), one again obtains four cases depending on whether $\partial_t$ hits $H$, one of the $\partial^{x_k} D_x u_a$, one of the $\partial^\nu D_x v$, or one of the final two factors in (5.18).

3. For example, in the fourth case we obtain by arguing as in the proof of Lemma 5.1

$$\epsilon^{1+|\alpha|} H(D_x u_a, D_x v)(\partial^{x_k} D_x u_a)(\partial^{x_k} D_x u_a) \ldots (\partial^{x_k} D_x u_a)(\partial^{x_k} D_x v) \ldots (\partial^\nu D_x v)(\partial^\nu D_x v) \partial^\kappa D_x^2 \nu|_{L^2(x)}$$

$$\leq C_{1}^{p+1} A_1^o \epsilon^s(t) \epsilon^{-1} \epsilon^{p(\frac{M-4}{t})} \epsilon^{(|\alpha|-|\beta|)-|\gamma|-|\kappa|} \epsilon^q \leq C_{1}^{p+1} A_1^o \epsilon^s(t) \epsilon^\mu \text{ for } t \in [0, T]$$

where $\mu > 0$. In (5.20) we have exhibited the “extra” factor of $\epsilon^{-1}$ contributed by $\partial^\kappa D_x^2 \nu$. The other factors of $C_1$, $A_1$, and $\epsilon$ arise just as in (5.10), (5.7).

4. The second and third cases (described in step 2) yield exactly the same estimate. As before the first case can be treated using the second and third cases.

It remains to estimate the boundary commutator and boundary forcing for $\epsilon^\alpha \partial^\alpha (\epsilon^{1.6})$.

**Lemma 5.4** (Boundary commutator for (1.6)). Let $\epsilon_1$ and $C_1$ be as in Proposition 4.7. There exist positive constants $\epsilon_2(C_1) \leq \epsilon_1$ and $B_1 = B_1(A_1)$ such that for $\epsilon \in [0, \epsilon_2]$ and all $t \in [0, T]$

$$\epsilon |A^\frac{1}{2}_x \epsilon^{[\alpha]} \partial^\alpha, A_{ij}(D_x(u_a + v))|_{L^2(x)} \leq B_1 \epsilon^s(t) \text{ for all } |\alpha| \leq s.$$ 

**Proof.** 1. Denote the commutator $[\epsilon^{[\alpha]} \partial^\alpha, A_{ij}(D_x(u_a + v))]|_{0, 1, \epsilon}$ by $C$ and observe that $\epsilon C$ is a sum of terms of the form (5.3). By the trace estimate of Proposition 2.7 we have

$$\epsilon |A^\frac{1}{2}_x \epsilon^{[\alpha]} \partial^\alpha, C(t)|_{0, 1, \epsilon} \leq |C(t)|_{0, 1, \epsilon} \leq |C(t)|_{L^2(x)} + |\epsilon D_x C(t)|_{L^2(x)}.$$ 

Since $\epsilon D_x C(t)$ is a sum of terms of the form $D_x (5.4)$, the second term on the right in (5.22) has already been estimated in Lemma 5.1.

2. To estimate $|C(t)|_{L^2(x)}$ we must estimate terms like $|\epsilon^{-1} (5.3)|_{L^2(x)}$. In the cases $p \geq 1$, $p = 0$ we obtain respectively

$$|\epsilon^{-1} (5.3)|_{L^2(x)} \leq C_1 A_1^o \epsilon^s(t) \epsilon^{p(\frac{M-4}{t})} \epsilon^{(|\alpha|-|\beta|)-|\gamma|-|\kappa|} \epsilon^q \leq C_1 A_1^o \epsilon^s(t) \epsilon^\mu$$

$$|\epsilon^{-1} (5.3)|_{L^2(x)} \leq A_1^o \epsilon^s(t) \epsilon^{(|\alpha|-|\beta|)-|\kappa|} \epsilon^q \leq A_1^o \epsilon^s(t),$$

where $\mu > 0$ in the first case, and $q \geq 1$ in the second case.

**Lemma 5.5** (Boundary forcing for (1.6)). Let $\epsilon_1$ and $C_1$ be as in Proposition 4.7. There exist positive constants $\epsilon_2(C_1) \leq \epsilon_1$ and $B_1 = B_1(A_1)$ such that for $\epsilon \in [0, \epsilon_2]$ and all $t \in [0, T]$

$$|\epsilon^\alpha \partial^\alpha G_1(t)|_{L^2(x)} \leq B_1 \epsilon^s(t) + \langle r^\epsilon(t) \rangle_{s+1, \epsilon} \text{ for all } |\alpha| \leq s.$$ 

**Proof.** 1. The first term of $G_1$ is $\epsilon^{-M} G = r^\epsilon$, which clearly gives rise to the term $\langle r^\epsilon(t) \rangle_{s+1, \epsilon}$ in (5.15).

2. The remaining part of $G_1$ is a sum of terms like

$$H_i(D_x u_a, D_x v)(D_x u_a, D_x v),$$

where $H_i$ is a $C^\infty$ function of its arguments. Denote $\epsilon^\alpha \partial^\alpha (5.25)$ by $K$ and observe that $\epsilon K$ is a sum of terms of the form (5.18). By the trace estimate of Proposition 2.7 we have

$$\epsilon |A^\frac{1}{2}_x \epsilon^{[\alpha]} \partial^\alpha, K(t)|_{0, 1, \epsilon} \leq |K(t)|_{0, 1, \epsilon} \leq |K(t)|_{L^2(x)} + |\epsilon D_x K(t)|_{L^2(x)}.$$ 

Since $\epsilon D_x K(t)$ is a sum of terms of the form $D_x (5.18)$, the second term on the right in (5.26) has already been estimated in Lemma 5.3.
3. To estimate $|K(t)|_{L^2(x)}$ we must estimate terms like $|\varepsilon^{-1} E_x(t)\varepsilon_{\varepsilon}^2|_{L^2(x)}$. For any integer $p \geq 0$ we obtain

$$
|\varepsilon^{-1} E_x(t)\varepsilon_{\varepsilon}^2|_{L^2(x)} \leq C_1^{p+1} A^{p+1}_1 E_x(t) \varepsilon^{(p+1)(M-\frac{d}{2}-1)} \varepsilon^{o_{|\alpha|-|\beta|-|\gamma|} \varepsilon^{q-1}} \leq C_1^{p+1} A^{p+1}_1 E_x(t) \varepsilon^p
$$

where $\mu > 0$.

The next proposition summarizes what we have shown so far.

**Proposition 5.6.** Suppose $\varepsilon_1$ and $C_1$ are positive constants and that for $\varepsilon \in (0, \varepsilon_1)$ and $t \in [0, T]$, we are given a solution $(u^\varepsilon, \omega^\varepsilon)$ of the three coupled systems on $[0, T]$ which satisfies $E_x^\varepsilon(t) \leq C_1$ for all $t \in [0, T]$. Then there exist positive constants $\varepsilon_2(C_1) \leq \varepsilon_1$ and $B_1(A_1, K_1)$ such that for $\varepsilon \in [0, \varepsilon_2]$ and all $t \in [0, T]$

$$
|\varepsilon \Delta \nu(t)|_{E,\varepsilon_{\varepsilon}^2} \leq B_1 \int_0^t (|E_x^\varepsilon(\sigma)|^2 + \varepsilon^2 |R^\varepsilon(\sigma)|^2 |s, \varepsilon| + |r^\varepsilon(\sigma)|^2 |s+1, \varepsilon|) \, d\sigma.
$$

**Proof.** The proposition is obtained by applying the hyperbolic estimate (3.3) to $\varepsilon^\alpha \partial^\alpha (\varepsilon^2 (1.7))$ for $|\alpha| \leq s$. The function that plays the role of "$f" in (3.3) is a sum of terms estimated in Lemmas 5.1 and 5.3, while the function that plays the role of "$g" is a sum of terms estimated in Lemmas 5.4 and 5.5.

We now estimate $|\varepsilon^2 \Delta \omega(t)|_{E,\varepsilon_{\varepsilon}^2}$ by applying the estimate (3.3) to $\varepsilon^\alpha \partial^\alpha (\varepsilon^2 (1.7))$ for $|\alpha| \leq s$.

**Lemma 5.7 (Interior commutator for (1.7)).** Let $\varepsilon_1$ and $C_1$ be as in Proposition 4.7. There exist positive constants $\varepsilon_2(C_1) \leq \varepsilon_1$ and $B_1 = B_1(A_1)$ such that for $\varepsilon \in [0, \varepsilon_2]$ and all $t \in [0, T]$

$$
|\partial_i [\varepsilon^\alpha \partial^\alpha, A_{ij}(D_x(u_a + v))]|_{E,\varepsilon_{\varepsilon}^2} \leq B_1 |E_x^\varepsilon(t)|_{L^2(x)}
$$

for all $|\alpha| \leq s$.

Here (and below) $i, j \in [1, \ldots, d]$.

**Proof.** The proof can be obtained by repeating verbatim the proof of Lemma 5.1 and replacing $\nu$ with $\varepsilon \omega$ wherever $\nu$ occurs in that proof.

**Lemma 5.8 (Interior forcing for (1.7)).** Let $\varepsilon_1$ and $C_1$ be as in Proposition 4.7. There exist positive constants $\varepsilon_2(C_1) \leq \varepsilon_1$ and $B_1 = B_1(A_1)$ such that for $\varepsilon \in [0, \varepsilon_2]$ and all $t \in [0, T]$

$$
|\varepsilon^{2+|\alpha|} \partial^\alpha \mathcal{F}_2(t)|_{L^2(x)} \leq B_1 |E_x^\varepsilon(t)| + |\varepsilon |R^\varepsilon(t)|_{s+1, \varepsilon}
$$

for all $|\alpha| \leq s$.

**Proof.** 1. The first term of $\mathcal{F}_2$ is $\varepsilon^{-M} \partial_i \mathcal{F} = \partial_i R^\varepsilon$, which obviously gives rise to the term $|\varepsilon^{s+1, \varepsilon}$ in (5.30).

2. The remaining part of $\mathcal{F}_2$ can be written in the form

$$
\sum_{i=1}^d \partial_i [H_i(D_x u_a, D_x v) D_x \nu \partial_i \partial_j u_a]
$$

where $H_i$ is a $C^\infty$ function of its arguments. First expanding out

$$
|\varepsilon^{2+|\alpha|} \partial^\alpha [H_i(D_x u_a, D_x v) D_x \nu \partial_i \partial_j u_a]
$$

we obtain a sum of terms of the form

$$
\varepsilon^{2+|\alpha|} H(D_x u_a, D_x v)(\partial^{\beta_1} D_x u_a) \cdots (\partial^{\beta_n} D_x u_a)(\partial^{\gamma_1} D_x v) \cdots (\partial^{\gamma_p} D_x v)(\partial^\kappa D_x \nu)(\partial^\kappa \partial_i u_a),
$$

where

$$
|\beta| + |\gamma| + |\zeta| + |\kappa| = |\alpha| \leq s,
$$

and any of the summands in (5.31) can be zero. Applying $\partial_i$ to (5.33), one again obtains four cases depending on whether $\partial_i$ hits $H$, one of the $\partial^{\beta_1} D_x u_a$, one of the $\partial^{\gamma_1} D_x v$, $\partial^\kappa D_x \nu$, or $\partial^\kappa \partial_i u_a$.

---

Here $"\varepsilon^\alpha \partial^\alpha (\varepsilon^2 (1.7))"$ denotes the result of multiplying the problem (1.6) by $\varepsilon$ and then applying the operator $\varepsilon^\alpha \partial^\alpha$. 
3. Consider for example the third case. When \( p \geq 1 \), letting “case 3” denote the typical term in that case, we obtain by arguing as in the proof of Lemma 5.1

\[
| \text{case 3} \mid \lesssim \varepsilon^{2+|\alpha|}(A_1^p\varepsilon^{-|\beta|})(C_1^{p\varepsilon^{-|\gamma|}}e^{-(\gamma-|\gamma|\varepsilon)})^2(A_1^{-|\gamma|}) = C_1^{p\varepsilon^{-|\gamma|}}A_1^{p+1}E_\varepsilon(t)e^{(M-\frac{d}{4}-1)}\varepsilon^{4|\alpha|-|\beta|}e^{\varepsilon|\gamma|} \lesssim C_1^{p\varepsilon^{-|\gamma|}}A_1^{p+1}E_\varepsilon(t)e^{\mu} \quad \text{for} \quad t \in [0,T],
\]

where \( \mu > 1 \).

When \( p = 0 \) we simply obtain

\[
| \text{case 3} \mid \lesssim A_1^{q+1}E_\varepsilon(t)e^{(M-\frac{d}{4}-1)}\varepsilon^{4|\alpha|-|\beta|}e^{\varepsilon|\gamma|} \lesssim A_1^{q+1}E_\varepsilon(t). \]

4. The estimate in case 3 for \( p \geq 1 \) readily implies the same estimate for case 2, and it is clear that the remaining two cases give the same estimates as in case 3.

\[\square\]

Lemma 5.9 (Boundary commutator for (1.7)). Let \( \varepsilon_1 \) and \( C_1 \) be as in Proposition 4.7. There exist positive constants \( \varepsilon_2(C_1) \leq \varepsilon_1 \) and \( B_1 = B_1(A_1) \) such that for \( \varepsilon \in [0,\varepsilon_2] \) and all \( t \in [0,T] \)

\[
\varepsilon^2|A_1^x\varepsilon^{|\alpha|}\partial^\alpha G(t,\varepsilon)|_{L^2(x')} \leq B_1E_\varepsilon^s(t) \quad \text{for all} \quad |\alpha| \leq s.
\]

Proof.

The proof can be obtained by repeating verbatim the proof of Lemma 5.4 and replacing \( \nu \) with \( \varepsilon\omega \) wherever \( \nu \) occurs in that proof.

\[\square\]

Lemma 5.10 (Boundary forcing for (1.7)). Let \( \varepsilon_1 \) and \( C_1 \) be as in Proposition 4.7. There exist positive constants \( \varepsilon_2(C_1) \leq \varepsilon_1 \) and \( B_1 = B_1(A_1) \) such that for \( \varepsilon \in [0,\varepsilon_2] \) and all \( t \in [0,T] \)

\[
|\varepsilon^2A_1^x\varepsilon^{|\alpha|}\partial^\alpha G(t,\varepsilon)|_{L^2(x')} \leq B_1E_\varepsilon^s(t) + |(r^\varepsilon(t))_{s+2,\varepsilon}| \quad \text{for all} \quad |\alpha| \leq s.
\]

Proof.

1. The first term of \( G_2 \) is \( \varepsilon^{-M}\partial_t G = \partial_t \varepsilon^\alpha \), which clearly gives rise to the term \( |(r^\varepsilon(t))_{s+2,\varepsilon}| \) in (5.15).

2. The remaining part of \( G_2 \) is a sum of terms like

\[
H_i(D_xu_\varepsilon,D_xv)D_x\nu \partial_t \partial_j u_\varepsilon
\]

where \( H_i \) is a \( C^\infty \) function of its arguments. Denote \( \varepsilon^\alpha\partial^\alpha(5.39) \) by \( K \) and observe that \( \varepsilon^2K \) is a sum of terms of the form (5.33). By the trace estimate of Proposition 2.7 we have

\[
\varepsilon^2(A_1^xK(t))_{0,\varepsilon} \leq \varepsilon(|K(t)|_{0,1,\varepsilon} + |\varepsilon^2D_xK(t)|_{L^2(x)}).
\]

Since \( \varepsilon^2D_xK(t) \) is a sum of terms of the form \( D_x(5.33) \), the second term on the right in (5.40) has already been estimated in Lemma 5.8.

3. To estimate \( |\varepsilon^2K(t)|_{L^2(x)} \) we must estimate terms like \( |\varepsilon^{-1}(5.33)|_{L^2(x)} \). For the cases \( p \geq 1 \), \( p = 0 \) we obtain by the usual procedure, respectively,

\[
|\varepsilon^{-1}(5.33)|_{L^2(x)} \lesssim C_1^{p\varepsilon^{-|\gamma|}}A_1^{q+1}E_\varepsilon^s(t)e^{\mu}, \quad \text{where} \quad \mu > 0
\]

\[\square\]

We now update Proposition 5.6 to reflect the results of the previous four lemmas.

Proposition 5.11. Suppose \( \varepsilon_1 \) and \( C_1 \) are positive constants and that for \( \varepsilon \in (0,\varepsilon_1] \) and \( T_\varepsilon \in [0,T] \), we are given a solution \( (r^\varepsilon,\omega^\varepsilon) \) of the three coupled systems on \( [0,T_\varepsilon] \) which satisfies \( E_\varepsilon^s(t) \leq C_1 \) for all \( t \in [0,T_\varepsilon] \). Then there exist positive constants \( \varepsilon_2(C_1) \leq \varepsilon_1 \) and \( B_1(A_1,K_1) \) such that for \( \varepsilon \in [0,\varepsilon_2] \) and all \( t \in [0,T_\varepsilon] \)

\[
|\varepsilon D\nu(t)|_{E^{\varepsilon,t}_{s\tan}}^2 + |\varepsilon^2 D\omega(t)|_{E^{\varepsilon,t}_{s\tan}}^2 \leq B_1 \int_0^t \left( |E_\varepsilon^s(\sigma)|^2 + \varepsilon^2 |R^\varepsilon(\sigma)|_{s+1,\varepsilon}^2 + |(r^\varepsilon(\sigma))_{s+2,\varepsilon}^2 \right) d\sigma.
\]
Proof. The proposition is obtained by applying the hyperbolic estimate (3.3) to \(\varepsilon^0 \partial^\alpha (\varepsilon^2 (1.7))\) for \(|\alpha| \leq s\), and combining the result with that of Proposition 5.8.

It remains to estimate \(|\varepsilon^2 D_x^2 \nu |_{E_s, \text{tan}}\) by applying the elliptic estimate (3.3) to \(\varepsilon^0 \partial^\alpha (\varepsilon^2 (1.9))\) for \(|\alpha| \leq s\).

**Lemma 5.12** (Interior commutator for (1.9)). Let \(\varepsilon_1\) and \(C_1\) be as in Proposition 4.4. There exist positive constants \(\varepsilon_2(C_1) \leq \varepsilon_1\) and \(B_1 = B_1(A_1)\) such that for \(\varepsilon \in [0, \varepsilon_2]\) and all \(t \in [0, T_\varepsilon]\)

\[
|\partial_t [\varepsilon^0 \partial^\alpha, A_{ij}(D_x u_a)] \partial_j (\varepsilon^2 \nu)|_{L^2(\alpha)} \leq \varepsilon B_1 E_\varepsilon^s (t) \quad \text{for all } |\alpha| \leq s.
\]

**Proof.** We must estimate a sum of the terms of the form \(\varepsilon \partial \alpha(\nu)\), except that now \(H = H(D_x u_a)\) and \(p = 0\). Thus, the work is already done in the proof of Lemma 5.1.

**Lemma 5.13** (Interior forcing for (1.9)). Let \(\varepsilon_1\) and \(C_1\) be as in Proposition 4.4. There exist positive constants \(\varepsilon_2(C_1) \leq \varepsilon_1\) and \(B_1 = B_1(A_1)\) such that for \(\varepsilon \in [0, \varepsilon_2]\) and all \(t \in [0, T_\varepsilon]\)

\[
|\varepsilon^2 D^2 \omega(t)|_{E^s, \text{tan}} + \left( \lambda \int_0^t \mathcal{E}^s_{\text{tan}} (\sigma) d\sigma + \varepsilon \lambda \mathcal{E}^s_{\text{tan}} (t) \right) + \varepsilon B_1 \mathcal{E}^s (t) + \varepsilon^2 A_2 \quad \text{for all } |\alpha| \leq s.
\]

**Proof.** From equation (1.9)(a) we see that there are four terms to estimate.
1. The term \(\varepsilon^{-M} F = R^\varepsilon\) in \(F_3\) clearly gives rise to the term \(\varepsilon^2 A_2\) in (5.17).
2. The term \(\varepsilon^{-M} E(D_x^2 v)\) can be written in the form

\[
\sum_{i=1}^d \partial_i [H_1(D_x u_a, D_x v)(D_x v, D_x v)],
\]

just like (5.16). The estimates of Lemma 5.3 thus show

\[
\varepsilon^2 + |\alpha| |\partial^\alpha | \varepsilon^2 A_2 \quad \text{in } (5.17).
\]

3. We have \(\varepsilon^2 + |\alpha| |\partial^\alpha | \varepsilon^2 A_2 \quad \text{in } (5.17),\) a term estimated in (5.42).

4. With \(\partial^\alpha = \partial^\alpha_{L_x^2, x}\), when \(\alpha_0 \geq 1\) or \(\alpha_0 = 0\) we have respectively

\[
\partial^\alpha \left( \lambda \int_0^t \omega(\sigma, x) d\sigma \right) = \lambda \partial^\beta \omega, \quad \text{where } |\beta| = |\alpha| - 1,
\]

\[
\partial^\alpha \left( \lambda \int_0^t \omega(\sigma, x) d\sigma \right) = \lambda \int_0^t \partial^\gamma \omega(\sigma, x) d\sigma, \quad \text{where } |\gamma| = |\alpha|,
\]

and corresponding estimates

\[
\varepsilon^2 + |\alpha| |\partial^\alpha | \varepsilon^2 + |\alpha| |\partial^\alpha | \varepsilon^2 A_2 \quad \text{in } (5.47).
\]

**Lemma 5.14** (Boundary commutator for (1.9)). Let \(\varepsilon_1\) and \(C_1\) be as in Proposition 4.4. There exist positive constants \(\varepsilon_2(C_1) \leq \varepsilon_1\) and \(B_1 = B_1(A_1)\) such that for \(\varepsilon \in [0, \varepsilon_2]\) and all \(t \in [0, T_\varepsilon]\)

\[
|\varepsilon^2 \Lambda F^2 [\varepsilon^0 \partial^\alpha, A_{ij}(D_x u_a)] \partial_j \nu|_{L^2(\alpha)} \leq \varepsilon B_1 \mathcal{E}^s (t) \quad \text{for all } |\alpha| \leq s.
\]

**Proof.** The estimate is immediate from the argument in the \(p = 0\) case of the proof of Lemma 5.4.
Lemma 5.15 (Boundary forcing for (1.9)). Let \( \varepsilon_1 \) and \( C_1 \) be as in Proposition 4.7. There exist positive constants \( \varepsilon_2(C_1) \leq \varepsilon_1 \) and \( B_1 = B_1(A_1) \) such that for \( \varepsilon \in [0, \varepsilon_2] \) and all \( t \in [0, T_\varepsilon] \)
\[
|\varepsilon^2 A_{t,x}^1 e^{[\alpha]} \partial^\alpha G_3(t)|_{L^2(x')} \leq \varepsilon B_1 E^{s}(t) + \varepsilon A_2 \text{ for all } |\alpha| \leq s.
\]

Proof. We have \( G_3 = \varepsilon^{-M} G + \varepsilon^{-M} E_0(D_{x', v}) \), where the second term is a sum of terms of the form (5.25). So this lemma follows from the proof of Lemma 5.5.

Combining the results of the previous four lemmas, we obtain the following estimate for \( |\varepsilon^2 D^2_{x'} \nu(t)|_{E_{s, \tan}} \) by applying the elliptic estimate (3.3) to \( \varepsilon^\alpha \partial^\alpha (\varepsilon^2 (1.9)) \) for \( |\alpha| \leq s \).

Proposition 5.16. Suppose \( \varepsilon_1 \) and \( C_1 \) are positive constants and that for \( \varepsilon \in (0, \varepsilon_1] \) and \( T_\varepsilon \leq [0, T] \), we are given a solution \( \langle \nu', \omega' \rangle \) of the three coupled systems on \( [0, T_\varepsilon] \) which satisfies \( \partial^\alpha E_{s}(t) \leq C_1 \) for all \( t \in [0, T_\varepsilon] \). Then there exist positive constants \( B_1(A_1) \) and \( \varepsilon_2(C_1) \leq \varepsilon_1 \) such that for \( \varepsilon \in [0, \varepsilon_2] \) and all \( t \in [0, T_\varepsilon] \)
\[
(5.50) \quad |\varepsilon^2 D^2_{x'} \nu(t)|_{E_{s, \tan}} \leq K_2 \left( \varepsilon B_1 E^{s}(t) + |\varepsilon^2 D\omega|_{E_{s, \tan}} + \lambda \int_0^t E^{s}_{\varepsilon, \tan}(\sigma) d\sigma + \varepsilon \lambda E^{s}_{\varepsilon, \tan}(t) + \varepsilon A_2 \right).
\]

Putting together Propositions 5.11 and 5.16 we obtain

Proposition 5.17. Under the assumptions of Proposition 5.16 there exist positive constants \( B_1(T, A_1, K_1, K_2, \lambda) \), \( B_2(A_2, K_2) \), and \( \varepsilon_2(C_1, A_1, K_1, K_2, \lambda) \leq \varepsilon_1 \) such that for \( \varepsilon \in [0, \varepsilon_2] \) and all \( t \in [0, T_\varepsilon] \)
\[
(5.51) \quad |E^{s}_{\varepsilon, \tan}(t)|^2 \leq B_1 \int_0^t \left( |E^{s}_{\varepsilon}(|\sigma|)|^2 + \varepsilon^2 |R^{\varepsilon}(|\sigma|)|^2_{s+1, \varepsilon} + \|r^{\varepsilon}(\sigma)\|_{s+2, \varepsilon}^2 \right) d\sigma + \varepsilon^2 (B_1 |E^{s}_{\varepsilon}(t)|^2 + B_2).
\]

Proof. Take the square of (5.50), use (5.42) to estimate the term \( |\varepsilon^2 D\omega|_{E_{s, \tan}}^2 \) that appears on the right, and add the resulting estimate of \( |\varepsilon^2 D^2_{x'} \nu(t)|_{E_{s, \tan}}^2 \) to the estimate (5.42). The left side now equals \( |E^{s}_{\varepsilon, \tan}(t)|^2 \) and can be used to absorb the term \( K_2^2 \varepsilon^2 \lambda^2 |E^{s}_{\varepsilon, \tan}(t)|^2 \) from the right, provided \( \varepsilon_2 \) is small enough. Finally, to obtain (5.51) we have also used
\[
(5.52) \quad K_2^2 \lambda^2 \left( \int_0^t |E^{s}_{\varepsilon, \tan}(\sigma)|^2 d\sigma \right)^2 \leq K_2^2 \lambda^2 T^2 \int_0^t |E^{s}_{\varepsilon, \tan}(\sigma)|^2 d\sigma \leq K_2^2 \lambda^2 T^2 \int_0^t |E^{s}_{\varepsilon}(\sigma)|^2 d\sigma.
\]

5.2. Normal derivative estimates. In order to complete the proof of Proposition 4.7 we must estimate
\[
(5.53) \quad |\varepsilon^{|\alpha|} \partial^\alpha_{x,x'} (\varepsilon D\nu, \varepsilon^2 D\omega, \varepsilon^2 D^2_{x'} \nu)|_{L^2(x')} \text{ for } |\alpha| \leq s
\]
when \( \partial \) derivatives are present in \( \partial^\alpha_{x,x'} \). In this section \( \partial^\beta \) will always be taken to mean \( \partial^\beta_{x, x', x_d} \), where \( \beta_d \leq |\beta| \). We have a noncharacteristic boundary \( (A_d \text{ is nonsingular}) \), so we can “use the equation” to control normal derivatives in an inductive argument starting with the control we now have over tangential derivatives (the case \( \alpha_d = 0 \) in (5.53)). Although this type of argument is standard, we have three equations here and a rather complicated object \( E^{s}_{\varepsilon} \) to estimate, so some care is needed both to formulate the induction assumption concisely and to avoid unnecessary work. Thus, we shall provide some details.

First we define for \( s_0 \leq s \):
\[
(5.54) \quad E^{s}_{\varepsilon, s_0}(t) = |\varepsilon^2 D^2_{x'} \nu(t)|_{E_{s, s_0}} + |\varepsilon^2 D\nu(t)|_{E_{s, s_0}} + |\varepsilon^2 D\omega(t)|_{E_{s, s_0}}.
\]
By induction on \( s_0 \) we will prove:

**Proposition 5.18.** Let \( s_0 \in \{0, \ldots, s\} \). Suppose \( \varepsilon_1 \) and \( C_1 \) are positive constants and that for \( \varepsilon \in (0, \varepsilon_1] \) and \( T \in [0, T_c] \), we are given a solution \((\nu^\varepsilon, \omega^\varepsilon)\) of the three coupled systems on \([0, T_c]\) which satisfies \( \mathcal{E}_s^\varepsilon(t) \leq C_1 \) for all \( t \in [0, T_c] \). Then there exist positive constants \( B_1 = B_1(T, A_1, K_1, K_2, \lambda), B_2 = B_2(A_1, A_2, K_2) \), and \( \varepsilon_2 = \varepsilon_2(C_1, A_1, K_1, K_2, \lambda) \) such that for \( \varepsilon \in [0, \varepsilon_2] \) and all \( t \in [0, T_c] \)

\[
(5.55) \quad \left[ \mathcal{E}_s^\varepsilon(t) \right]^2 \leq B_1 \int_0^t \left[ (\mathcal{E}_s^\varepsilon(\sigma))^2 + \varepsilon^2 |R^\varepsilon(\sigma)|^2_{s+1,\varepsilon} + (r^\varepsilon(\sigma))^2_{s+2,\varepsilon} \right] d\sigma + \varepsilon^2 (B_1[\mathcal{E}_s^\varepsilon(t)]^2 + B_2).
\]

Proof. 1. The case \( s_0 = s \) is the same as the estimate asserted in Proposition 4.17; the case \( s_0 = 0 \) is treated in Proposition 5.17. 

2. **Induction assumption.** Let \( s_0 < s \) and assume that (5.55) holds for this \( s_0 \). It remains to show that (5.55) holds for \( s_0 + 1 \). It is perhaps surprising that the terms \( \varepsilon^2 \mathcal{D}_s^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \) and \( \varepsilon \mathcal{D}_s^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \) can be estimated *without* having to “use the equation.” Also, in estimating the remaining term \( \varepsilon^2 \mathcal{D}_2^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \) we will only need to use one equation, namely (1.10)(a).

3. **The term** \( \varepsilon^2 \mathcal{D}_s^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \). With \( \alpha = (\alpha_0, \alpha', s_0 + 1) \) satisfying \( |\alpha| \leq s \) (here and in the remaining steps), we first estimate

\[
(5.56) \quad \varepsilon^2 \mathcal{D}_s^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon = \varepsilon^{|\alpha|} |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_d^{s_0+1} (\varepsilon^2 \mathcal{D}_s \nu)|_{L^2(x)}
\]

If \( D \) is replaced by \( \partial_t \) in (5.56), we can swap this \( \partial_t \) with one of the \( \partial_d \) derivatives to obtain

\[
|\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_d^{s_0+1} (\varepsilon^2 \mathcal{D}_s \nu)|_{L^2(x)} = \varepsilon^{|\alpha|} |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_d^{s_0+1} (\varepsilon^2 \mathcal{D}_s \nu)|_{L^2(x)} \leq \mathcal{E}_{s_0+1}^\varepsilon(t).
\]

When \( D \) is absent in (5.56), the desired estimate is immediate.

4. **The term** \( \varepsilon \mathcal{D}_s^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \). We consider only

\[
(5.57) \quad \varepsilon^{|\alpha|} |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_d^{s_0+1} (\varepsilon \mathcal{D}_s \nu)|_{L^2(x)}. \]

When \( D \) is replaced by \( \partial_t \) or \( \partial_{x'} \), we can swap that derivative with one of the \( \partial_d \) derivatives in \( \partial_d^{s_0+1} \) to obtain (5.58) ≤ \( \mathcal{E}_{s_0+1}^\varepsilon(t) \). When \( D \) is replaced by \( \partial_d \), we can rewrite (5.58) as

\[
(5.58) \quad \varepsilon^{|\alpha|-1} |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_d^{s_0+1} (\varepsilon^2 \partial_d^2 \nu)|_{L^2(x)} \leq \mathcal{E}_{s_0+1}^\varepsilon(t).
\]

5. **The term** \( \varepsilon^2 \mathcal{D}_2^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \). We will show

\[
(5.59) \quad |\varepsilon^2 \mathcal{D}_2^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \leq C(A_1) \mathcal{E}_{s_0}^\varepsilon + \varepsilon C(A_2) \mathcal{E}_{s_0}^\varepsilon + \varepsilon C(A_1, A_2).
\]

To estimate \( |\varepsilon^2 \mathcal{D}_2^\varepsilon \nu(t) |E_{s_0+1}^\varepsilon \) we consider

\[
(5.60) \quad \varepsilon^{|\alpha|} |\partial_t^{\alpha_0} \partial_{x'}^{\alpha'} \partial_d^{s_0+1} (\varepsilon^2 \mathcal{D}_2 \nu)|_{L^2(x)}. \]

If exactly one \( \partial_d \) appears in \( \mathcal{D}_2 \), for example, if \( \mathcal{D}_2 = \partial_t \partial_d \) with \( i < d \), we can swap the \( \partial_t \) with one of the \( \partial_d \) derivatives in \( \partial_d^{s_0+1} \) to obtain (5.60) ≤ \( \mathcal{E}_{s_0}^\varepsilon(t) \). We get the same estimate, of course, if no \( \partial_d \) appears in \( \mathcal{D}_2 \).

To treat the remaining case where \( \mathcal{D}_2 = \partial_d^2 \), we first use equation (1.10)(a) to write

\[
(5.61) \quad \partial_d^2 \nu = -A_{dd}^{-1} \left[ -\partial_t^2 \nu + \partial_d(A_{dd}) \partial_d \nu + \sum_{i \text{ or } j \neq d} \partial_i(A_{ij} \partial_j \nu) + R^\varepsilon + \sum_i \partial_i[H_i(D_x u, D_x v)(D_x v, D_x v)] \right].
\]

Here \( i, j \in \{1, \ldots, d\} \), the coefficients \( A_{ij} = A_{ij}(D_x u, D_x v) \), and we have used (5.10).
Replacing $D^2_t\nu$ in (5.60) by the right side of (5.58), we examine first the contribution to (5.60) from $-A_{dd}^{-1} \left( \sum_i \partial_i [H_i(D_{xu_a}, D_{xv})(D_{xu_a}, D_{xv})] \right)$, which is a sum of terms of the form $H_{ij}(D_{xu_a}, D_{xv})\partial_{ij}^2 \nu$ with $i$ or $j \neq d$. Thus, we must estimate terms like

$$\varepsilon^{2+|\alpha|} H \cdot (\partial^{\beta_1} D_{xu_a}) (\partial^{\beta_2} D_{xu_a}) \ldots (\partial^{\beta_q} D_{xu_a}) (\partial^{\gamma_1} D_{xv}) \ldots (\partial^{\gamma_p} D_{xv}) \partial^i \partial_j \nu |_{L^2(x)},$$

where $|\beta| + |\gamma| + |\zeta| = |\alpha|$, at least one of $i, j$ (say $i$) is $\neq d$, and the total number of $\partial_d$ derivatives appearing in $(\partial^{\beta_1}, \partial^{\gamma_1}, \partial^{\zeta})$ is $s_0 + 1$.

First consider the “worst” case, that is, when $p = 0$. We obtain

$$\varepsilon^{2+|\alpha|} e^{-|\beta|} A^q [\partial^i \partial_j \nu |_{L^2(x)} \leq \varepsilon^{2+|\alpha|} e^{-|\beta|} A^q [\partial^i \partial_j \nu |_{L^2(x)} \leq \varepsilon A^q [\partial^i \partial_j \nu |_{L^2(x)}.$$

Here the second inequality is immediate when $\zeta_d \leq s_0$; if $\zeta_d = s_0 + 1$ we swap $\partial_i$ with one of the $\partial_d$ derivatives in $\partial^i$.

When $p \geq 1$ we obtain using the product estimate of Proposition 2.6 $p$ times:

$$\varepsilon^{2+|\alpha|} e^{-|\beta|} A^q [\partial^i \partial_j \nu |_{L^2(x)} \leq \varepsilon A^q [\partial^i \partial_j \nu |_{L^2(x)}$$

for $\varepsilon \in (0, \varepsilon_2)$ if $\varepsilon_2$ small enough (use $p(M - \frac{d}{2} - 1) > 1$).

**Remark 5.19.** The estimate (5.64) illustrates that when factors like $\partial^{\gamma} D_{xv}$ are present, it is not necessary to use the induction assumption to obtain an estimate consistent with (5.50). Every term in the contribution to (5.60) when $D^2_t\nu$ is replaced by

$$-A_{dd}^{-1} \left( \sum_i \partial_i [H_i(D_{xu_a}, D_{xv})(D_{xu_a}, D_{xv})] \right)$$

includes at least one such factor and is again dominated by $\varepsilon C(A_1) \mathcal{E}_\varepsilon^s$.

Next consider the contribution to (5.60) when $D^2_t\nu$ is replaced by $-A_{dd}^{-1} \partial^2_\nu = -A_{dd}^{-1} \partial_i \partial_j \nu$. Using step 3 we easily obtain

$$\varepsilon^{2+|\alpha|} |(A_{dd}^{-1} \partial^2_\nu) |_{L^2(x)} \leq C(A_1) \mathcal{E}_\varepsilon^s + \varepsilon C(A_1) \mathcal{E}_\varepsilon^s.$$

The second term here arises when $\partial^\alpha$ hits $A_{dd}$.

By entirely similar or easier estimates we find

$$\varepsilon^{2+|\alpha|} |(A_{dd}^{-1} \partial_i \partial_j \nu) |_{L^2(x)} \leq C(A_1) \mathcal{E}_\varepsilon^s \quad (5.66)$$

This completes the proof of (5.55).

**6.** The results of steps 3, 4, 5 imply

$$\mathcal{E}_{\varepsilon,s+1}^s (t) \leq C(A_1) \mathcal{E}_{\varepsilon,s}^s + \varepsilon C(A_1) \mathcal{E}_\varepsilon^s + \varepsilon C(A_1, A_2),$$

which in turn implies that (5.55) holds for $s_0 + 1$. This completes the induction step. 

□

We can now finish the proof of Proposition 4.7

**End of the proof of Proposition 4.7** The case $s_0 = s$ in Proposition 5.18 gives

$$[\mathcal{E}_\varepsilon^s (t)]^2 \leq B_1 \int_0^t \left( [\mathcal{E}_\varepsilon^s (\sigma)]^2 + \varepsilon^2 [R^s (\sigma)]^2 \right) d\sigma + \varepsilon^2 (B_1 [\mathcal{E}_\varepsilon^s (t)]^2 + B_2).$$

For $\varepsilon_2$ small enough the term $\varepsilon^2 B_1 [\mathcal{E}_\varepsilon^s (t)]^2$ can be absorbed into the left side, yielding the estimate (4.7) of Proposition 4.7.
As explained in section 4, this completes the proof of Theorem 4.6.

Part 3. Construction of approximate solutions for the traction problem

In this part we construct high order approximate solutions to the equations of the Saint Venant-Kirchhoff model of nonlinear elasticity with traction boundary conditions. It will simplify the exposition and greatly lighten the notation to carry out the construction in two space dimensions, but the construction in higher dimensions goes through with only obvious (and almost exclusively notational) changes. We change notation slightly from part 2 and denote the normal variable $x_d$ here by $y$ and the single tangential spatial variable by $x$ in place of the earlier $x'$.

6. Introduction

We consider the equations of the Saint Venant-Kirchhoff model (0.1) in two space dimensions:

$$\partial_t^2 \phi - \nabla \cdot (\nabla \phi \sigma(\nabla \phi)) = 0 \text{ on } y > 0$$

(6.1)$$\nabla \phi \sigma(\nabla \phi)n = \varepsilon^2 G(t, x, \frac{\beta \cdot (t, x)}{\varepsilon}) := \varepsilon^2 \begin{bmatrix} f \\ g \end{bmatrix} \text{ on } y = 0$$

$$\phi(t, x, y) = (x, y) \text{ and } G = 0 \text{ in } t \leq 0,$$

where $\phi = (\phi_1, \phi_2)$ is the deformation, the $2 \times 2$ matrix $\sigma$ is the stress (defined below (0.1)), $n = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and the boundary forcing is given by $G(t, x, \theta) \in H^\infty([0, T_0] \times \mathbb{R}^x \times T_\theta)$ for some $T_0 > 0$.\footnote{Whenever we use an expression like $G(t, x, \theta) \in H^\infty([0, T_0] \times \mathbb{R}^x \times T_\theta)$, where $G$ is a function that vanishes in $t < 0$, it is to be understood that $G$ vanishes to infinite order at $t = 0$.}

We take $\beta$ of the form

$$\beta = (-c, 1) \in \mathbb{R}^2$$

(6.2)

for a $c$ whose choice is discussed below. The case where $G$ has finite regularity can easily be treated, but at the cost of much additional bookkeeping. In order to highlight the phenomenon of internal rectification we assume that the the Fourier mean (or zero-th Fourier mode $G^0(t, x)$) of $G$ is zero.\footnote{The construction goes through just as well if $G^0$ is not zero. See Remark 12.3 for more on internal rectification.}

We now write the system (0.4) for the displacement $U(t, x, y) = \phi(t, x, y) - (x, y)$ as

(a) $\partial_t^2 U + \nabla \cdot (L(\nabla U) + Q(\nabla U) + C(\nabla U)) = 0 \text{ on } y > 0$

(6.3)$$\begin{align*}
(b) \quad & L_2(\nabla U) + Q_2(\nabla U) + C_2(\nabla U) = \varepsilon^2 \begin{bmatrix} f \\ g \end{bmatrix} \text{ on } y = 0 \\
& U = 0 \text{ in } t \leq 0,
\end{align*}$$

where $L = (L_1, L_2)$, $Q = (Q_1, Q_2)$, and $C = (C_1, C_2)$ are respectively linear, quadratic, and cubic functions of $\nabla U$.\footnote{The construction goes through just as well if $G^0$ is not zero. See Remark 12.3 for more on internal rectification.}
6.1. **Choice of \( c \).** Letting \((\tau, \xi, \omega)\) denote variables dual to \((t, x, y)\), we can write the principal symbol of the operator obtained by linearizing the left side of (6.3) at \( \nabla U = 0 \) as

\[
L(\tau, \xi, \omega) = \begin{pmatrix}
\tau^2 - (r - 1)|\xi|^2 - |\xi, \omega|^2 & -(r - 1)\xi \omega \\
-(r - 1)\xi \eta & \tau^2 - (r - 1)|\xi|^2 - |\xi, \omega|^2
\end{pmatrix}
\]

The constant \( r > 1 \) is the ratio of the squares of pressure \( c_d \) and shear \( c_s \) velocities.\(^{25}\) The matrix \( L(\beta, \omega) \) has characteristic roots \( \omega_j \) and vectors \( r_j \) satisfying

\[
det L(\beta, \omega_j) = 0 \text{ and } L(\beta, \omega_j)r_j = 0, j = 1, \ldots, 4.
\]

The boundary frequency \( \beta \) is said to lie in the elliptic region when \( c^2 - 1 \) and \( \frac{c^2}{r} - 1 \) are negative. The \( \omega_j \) are then purely imaginary, and we take \( \omega_1, \omega_2 \) to have positive imaginary part. Thus, we have \( \omega_3 = \bar{\omega}_1, \omega_4 = \bar{\omega}_2 \) and we take

\[
r_1 = \begin{pmatrix}
-\omega_1 \\
1
\end{pmatrix}, \quad r_2 = \begin{pmatrix}
1 \\
\omega_2
\end{pmatrix}, \quad r_3 = \bar{r}_1, r_4 = \bar{r}_2.
\]

If we define \( q = q(c) > 0 \) by

\[
q^2 = -\omega_1 \omega_2,
\]

then the condition for \( \beta = (-c, 1) \) to be a Rayleigh frequency is that

\[
(2 - c^2)^2 = 4q^2(c), \quad \text{or equivalently } 2 - c^2 = 2q.
\]

This equation is equivalent to the statement that

\[
det B_{Lop} = 0,
\]

where \( B_{Lop} \) is the Lopatinski matrix derived in section 9. For the existence of \( 0 < c < 1 \) satisfying (6.9) we refer for example to [Tay77], and we fix \( \beta = (-c, 1) \) for this choice of \( c \). We will see below that it is the vanishing of the determinant (6.10) that gives rise to Rayleigh waves.

In the remainder of this part we will construct approximate solutions of (6.3) of the form

\[
U_a^\varepsilon(t, x, y) = \sum_{n=2}^N \varepsilon^k U_k(t, x, y, \theta, Y)|_{\theta = \frac{x - ct}{\varepsilon}, \frac{y}{\varepsilon}},
\]

where the profiles \( U_k \) belong to the space \( S \) of Definition 7.1. The function \( U_a^\varepsilon(t, x, y) \) is constructed (Theorem 12.5) to satisfy

\[
\begin{align*}
\partial_t^2 U_a^\varepsilon + \nabla \cdot (L(U_a^\varepsilon) + Q(\nabla U_a^\varepsilon) + C(\nabla U_a^\varepsilon)) &= \varepsilon^{N-1} E_N(t, x, y, \frac{x - ct}{\varepsilon}, \frac{y}{\varepsilon}) \quad \text{on } y > 0 \\
L_2(\nabla U_a^\varepsilon) + Q_2(\nabla U_a^\varepsilon) + C_2(\nabla U_a^\varepsilon) - \varepsilon^2 \int f^g = \varepsilon^N e_N(t, x, 0, \frac{x - ct}{\varepsilon}, 0) \text{ on } y = 0,
\end{align*}
\]

where \( E_N, e_N \) lie in the space \( S^\varepsilon \) of Definition 7.2.

In Theorem 12.6 we combine the results of theorems 11.6 and 12.5 to state our main result for nonlinear elasticity, which makes precise the sense in which the approximate solution is close to exact solution.

In Chapter 2 of [Mar11] A. Marcou constructed \( U_2 \) and part of \( U_3 \) for a simplified SVK-type model: there were no cubic terms \( C(\nabla U) \) and a number of the quadratic terms in \( Q(\nabla U) \) were dropped. It will be clear from the exposition below that her analysis of that model was helpful to us in constructing the approximate solution.

\(^{25}\)We have \( c_s^2 = \mu \) and \( c_d^2 = (\lambda + 2\mu) \), where \( \lambda, \mu \) are the Lamé constants. The form (6.1) is obtained by taking units of time so that \( c_s = 1 \). Observe \( r = c_d^2/c_s^2 > 1 \) since \( \lambda + \mu > 0 \).
7. Spaces of profiles

In this section we define spaces that contain all the kinds of functions that will arise in the construction of profiles. The first two definitions concern functions defined on the “interior”, that is, the set where \( y > 0, Y > 0 \).

**Definition 7.1.** Let the space \( S \) be given by \( S = S \oplus S^* \) where \( S = H^\infty([0, T] \times \mathbb{R}_x \times [0, \infty)_y) \) is the usual Sobolev space and \( S^* \) consists of functions \( u^*(t, x, \theta, Y) \in H^\infty([0, T] \times \mathbb{R}_x \times \mathbb{T}_\theta \times [0, \infty)_Y) \) satisfying the additional restriction that:

\[
|\partial_{t,x,\theta,Y} u^*(t, x, \theta, Y)|_{L^2(\mathbb{R}_s)} \leq C_\alpha e^{-\delta Y}
\]

where \( \alpha \) is a multi-index, and \( C_\alpha \) and \( \delta \) are positive constants. Note that \( \delta \) is independent of \( \alpha \) but not independent of \( u^* \).

This space was used, for example, by Marcou [Mar10] in her study of first-order hyperbolic conservation laws, and also in Chapter 2 of [Mar11] in her study of an SVK-type model.

The intervals \( y \in [0, \infty) \) in \( S \) and \( Y \in [0, \infty) \) in \( S^* \) contain different variables. A given element \( u \in S \) can be written as \( u(t, x, y, \theta, Y) = u(t, x, y) + u^*(t, x, \theta, Y) \), where \( u \in S \) and \( u^* \in S^* \). Moreover, since each \( u \in S \) is periodic with respect to \( \theta \), we can further decompose \( u \) as

\[
u(t, x, y, \theta, Y) = \overline{u}(t, x, y) + u^0(t, x, y, \theta) + \sum_{n \neq 0} u^n(t, x, y, \theta) e^{in\theta} = u^0(t, x, y, \theta) + u^{os}(t, x, \theta, Y),
\]

where

\[
u^0(t, x, y, \theta) := \overline{u}(t, x, y) + u^0(t, x, y) \] \( u^{os}(t, x, \theta, Y) := \sum_{n \neq 0} u^n(t, x, y, \theta) e^{in\theta} \).

The spaces \( S \) and \( S^* \) are each closed under multiplication, but \( S \) is not closed under multiplication, since it does not contain products \( uv^* \), where \( u \in S \) \( v^* \in S^* \). This forces us to introduce the extended space \( S^e \).

**Definition 7.2** (The space \( S^e \)).

1) A function \( u(t, x, y, \theta, Y) \) is called mixed if it is a finite linear combination of functions of the form \( \overline{a}(t, x, y)b^*(t, x, \theta, Y) \) where \( \overline{a} \in S \) and \( b^* \in S^* \). Let \( S^m \) be the space of all such linear combinations.

2) The extended space \( S^e := S \oplus S^m \).

A function \( u(t, x, y, \theta, Y) \in S^e \) is periodic in \( \theta \) so we can write

\[
u = u^0(t, x, y, \theta) + \sum_{n \neq 0} u^n(t, x, y, \theta) e^{in\theta} = u^0(t, x, y, \theta) + u^{os}(t, x, \theta, Y)
\]

where

\[
u^0(t, x, y, \theta) = \overline{u}(t, x, y) + u^0(t, x, y, \theta) + u^{0,m}(t, x, y, \theta) \] \( u^{os}(t, x, \theta, Y) = u^{os,*}(t, x, \theta, Y) + u^{os,m}(t, x, \theta, Y) \) with \( u^{0,*} \in S^* \), \( u^{0,m} \in S^m \)

\( u^{os,*}(t, x, \theta, Y) = u^{os,*}(t, x, \theta, Y) + u^{os,m}(t, x, \theta, Y) \) with \( u^{os,*} \in S^* \), \( u^{os,m} \in S^m \).

On the “boundary”, that is, the set where \( y = 0, Y = 0 \) we have

**Definition 7.3.** Let \( S^b = H^\infty([0, T] \times \mathbb{R}_x \times \mathbb{T}_\theta) \).

Functions in \( S^b \) can be written \( f(t, x, \theta) = \overline{f}(t, x) + f^{os}(t, x, \theta) \), where the Fourier mean of \( f^{os} \) is zero. We note also that

\[
u \in S^e \Rightarrow \nu|_{y=0,Y=0} \in S^b
\]

The following proposition, whose proof is immediate from (7.4) and the definitions, records several of the properties of \( S^e \).
Proposition 7.4. For elements \( u \in S^e \) we refer here to the pieces defined in (7.4) and (7.5).

1) The space \( S^e \) is closed under multiplication.
2) For \( u(t, x, y, \theta, Y) \in S^e \) we have \( \lim_{Y \to \infty} u = u(t, x, Y) \).
3) Any piece of \( u \) whose superscripts include one or more of \( * \), \( \text{osc} \), or \( m \) is exponentially decaying in \( Y \).

In the nonlinearity of the Saint Venant-Kirchhoff model, we have products of elements of \( S \). In order to deal with the fact that such products lie in \( S^e \) but not necessarily in \( S \), it will be useful to Taylor expand functions \( u^m \in S^m \) in the \( y \) variable as follows.

\[
(8.1) \quad u^m(t, x, y, \theta, Y) = u^m(t, x, 0, \theta, Y) + \varepsilon \frac{y}{e} \partial_y u^m(t, x, 0, \theta, Y) + \cdots + \varepsilon^k \frac{y^k}{k!} \partial_y^k u^m(t, x, 0, \theta, Y) + \varepsilon^{k+1} \frac{y^{k+1}}{e^{k+1}} r_{k+1}(t, x, y, \theta, Y),
\]

where \( r_{k+1} \in S^m \). Next define a modification of \( u^m \), \( u^m_{k+1, \text{mod}} \in S^m \), by

\[
(8.2) \quad u^m_{k+1, \text{mod}}(t, x, y, \theta, Y) = u^m(t, x, 0, \theta, Y) + \varepsilon \frac{y}{e} \partial_y u^m(t, x, 0, \theta, Y) + \cdots + \varepsilon^k \frac{y^k}{k!} \partial_y^k u^m(t, x, 0, \theta, Y) + \varepsilon^{k+1} R_{k+1}(t, x, y, \theta, Y),
\]

where \( R_{k+1} = Y^{k+1} r_{k+1}(t, x, y, \theta, Y) \in S^m \). This turns out to be useful because of the following two properties:

\[
(7.9) \quad u^m_{k+1, \text{mod}} - \varepsilon^{k+1} R_{k+1} \in S^e
\]

\[
u^m(t, x, y, \theta, Y)|_{Y = \frac{y}{e}} = u^m_{k+1, \text{mod}}(t, x, y, \theta, Y)|_{Y = \frac{y}{e}}.
\]

Roughly speaking, the properties (7.9) will allow us to replace elements of \( S^m \) by elements of \( S^e \) at the price of an error \( \varepsilon^{k+1} R_{k+1} \in \varepsilon^{k+1} S^m \) which is harmless for the purposes of constructing approximate solutions.

Remark 7.5. The piece \( u^0(t, x, y, Y) \) is the “Fourier mean” of \( u \in S^e \). Since it is common in geometric optics to use \( \underline{u} \) to denote the Fourier mean, in order to avoid confusion we will not refer to either \( u \) or \( u^0 \) as the “mean” of \( u \).

8. Cascade of profile equations

We look for an approximate solution of (6.3) given by the following ansatz:

\[
(8.1) \quad U^\varepsilon_a(t, x, y) = \sum_{n=2}^{N} \varepsilon^n U^a_n(t, x, y, \theta, Y)|_{\theta = \frac{x-ct}{\varepsilon}, Y = \frac{y}{e}}.
\]

Plugging in \( U^\varepsilon_a \) into (6.3) and grouping terms according to powers of \( \varepsilon \) gives:

\[
\left[ \sum_{k=2}^{N} \varepsilon^{k-2} \left( L_{ff}(U_k) - \left( \frac{H_{k-1}}{R_{k-1}} \right) \right) + \varepsilon^{N-1} E^\varepsilon_N \right]|_{\theta = \frac{x-ct}{\varepsilon}, Y = \frac{y}{e}} \text{ on } y > 0
\]

\[
L_2(\nabla U^\varepsilon_a) + Q_2(\nabla U^\varepsilon_a) + C_2(\nabla U^\varepsilon_a) - \varepsilon^2 \left( \begin{array}{c} f \varepsilon^2 \\ g \end{array} \right) \bigg|_{\theta = \frac{x-ct}{\varepsilon}, Y = \frac{y}{e}} \text{ on } y = 0
\]

\[
(8.3) \quad \left[ \sum_{k=2}^{N} \varepsilon^{k-1} \left( l_f(U_k) - \left( \frac{h_{k-1}}{R_{k-1}} \right) \right) + \varepsilon^{N} e^\varepsilon_N \right]|_{\theta = \frac{x-ct}{\varepsilon}, Y = \frac{y}{e}} \text{ on } y = 0.
\]
The operators $L_{ff}$ and $l_f$ are defined below in (8.5). The functions $H_{k-1}$, $K_{k-1}$, $h_{k-1}$, $k_{k-1}$ as well as $E_N^a$ and $e_N^a$ are determined by (8.2) and (8.3) as nonlinear functions of the profiles $U_2, \ldots, U_{k-1}$ and belong to $S^e$. Formulas for $H_{k-1}, \ldots, k_{k-1}$ that are as explicit as we need for the profile construction are given below.

Clearly, in order to obtain high order approximate solutions we would like to choose the $U_k \in S$ so that the following equations hold:

\[
L_{ff}(U_k) = \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}, \quad \text{on } y > 0, Y > 0
\]

(8.4)

\[
l_f(U_k) = \begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix}, \quad \text{on } y = 0, Y = 0.
\]

To specify the objects appearing in (8.2) and (8.3) we first define linear operators involving derivatives with respect to fast variables $\theta, Y$ and slow variables $t, x, y$. The constant $r > 1$ in the formulas below is same as in (8.4).

\[
L_{ff} := \begin{pmatrix} c^2 - r & 0 \\ 0 & c^2 - 1 \end{pmatrix} \partial_{\theta\theta} - \begin{pmatrix} 0 & r - 1 \\ r - 1 & 0 \end{pmatrix} \partial_{\theta Y} - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{YY}
\]

(8.5)

\[
L_{fs} := -2c\partial_{\theta\theta} - \begin{pmatrix} 2r & 0 \\ 0 & 2 \end{pmatrix} \partial_{x\theta} - \begin{pmatrix} 0 & r - 1 \\ r - 1 & 0 \end{pmatrix} [\partial_{xY} + \partial_{\theta Y}] - \begin{pmatrix} 2 & 0 \\ 0 & 2r \end{pmatrix} \partial_{YY}
\]

\[
L_{ss} := \partial_{tt} - \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \partial_{xx} - \begin{pmatrix} 0 & r - 1 \\ r - 1 & 0 \end{pmatrix} \partial_{xy} - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{yy}
\]

(8.6)

\[
l_f := \begin{pmatrix} 0 & 1 \\ r - 2 & 0 \end{pmatrix} \partial_{\theta} + \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{Y}
\]

\[
l_s := \begin{pmatrix} 0 & 1 \\ r - 2 & 0 \end{pmatrix} \partial_{x} + \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{y}.
\]

Next we give formulas for the terms $(H_{k-1}, K_{k-1})$ and $(h_{k-1}, k_{k-1})$ in (8.4). Profiles $U_j$ with $j < 2$ are defined to be zero. The various operators $A_{..}, B_{..}, Q_j(\ldots), C_j(\ldots)$ that appear are defined further below.

\[
\begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix} = -L_{fs}(U_{k-1}) - L_{ss}(U_{k-2}) + \sum_{i+j=k-2} A_{ss}(U_i, U_j) + \sum_{i+j=k-1} A_{fs}(U_i, U_j) + \sum_{i+j=k} A_{ffs}(U_i, U_j)
\]

\[
+ \sum_{i+j=k+1} A_{fff}(U_i, U_j) + \sum_{l+m+n=k-2} B_{ssss}(U_l, U_m, U_n) + \sum_{l+m+n=k-1} B_{fsss}(U_l, U_m, U_n)
\]

\[
+ \sum_{l+m+n=k} B_{ffss}(U_l, U_m, U_n) + \sum_{l+m+n=k+1} B_{fffs}(U_l, U_m, U_n) + \sum_{l+m+n=k+2} B_{ffff}(U_l, U_m, U_n)
\]

\[\text{It will turn out that these equations can be solved with } U_k \in S \text{ for } k = 2, \ldots, 5. \text{ For } k \geq 6 \text{ we will need to modify the right side of the interior equation. See Remark 8.7.}\]
and \((h_{k-1}, k_{k-1})\), as in (8.3), is given by the expression:

\[
\begin{align*}
(h_{k-1} & \\
k_{k-1}) = - l_5(U_{k-1}) - \sum_{i+j=k-1} Q_2(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j) - \sum_{i+j=k} [Q_2(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j) + Q_2(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j)] \\
&- \sum_{i+j=k} C_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_m, U_n) - \sum_{i+j=k} C_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_m, U_n) \\
&+ C_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_m, U_n) + C_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_m, U_n) \\
&- \sum_{i+j=k} C_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_m, U_n) + \left(\frac{f_{k-1}}{g_{k-1}}\right),
\end{align*}
\]

where \(\left(\frac{f_{k-1}}{g_{k-1}}\right) = \left(\frac{f}{g}\right)\) as in (6.3) if \(k = 3\) and is zero otherwise.

The notation here is an extension of that used by Marcou in Chapter 2 of [Mar11]. Recall that in (6.3) we had \(\nabla \cdot Q(\nabla U) = \partial_x Q_1(\nabla U) + \partial_y Q_2(\nabla U)\) with \(Q_1\) the first column of \(Q\) and \(Q_2\) the second. The \(Q_j\) are quadratic in \(\nabla U\), so with some abuse we can write

\[
Q_j(\nabla U) = Q_j(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j),
\]

where the first pair of derivatives act on the first profile in the argument, and the second pair acts on the second argument. Thus, the expression on the right denotes a column vector whose entries are linear combinations of terms of the form \(\partial_x u \partial_y v\), \(\partial_y u \partial_y v\), etc., where \(U = \begin{pmatrix} u \\ v \end{pmatrix}\).

Similarly, each entry of \(Q_2(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j)\) is a linear combination of terms like \(\partial_y u \partial_y v\), \(\partial_x v \partial_y u\), \(\partial_x v \partial_y u\) for \(U_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}\). The trilinear functions \(C_1, C_2\) are defined in an analogous manner. Thus, each entry of \(Q_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_m, U_n)\) is a linear combination of terms of the form \(\partial_y u \partial_y v \partial_y u\), \(\partial_y v \partial_y u \partial_y v\), etc..

The \(A\) and \(B\) functions are related to the \(Q\) and \(C\) functions by the following relations:

\[
A_{fff}(U_i, U_j) := \partial_0 Q_1(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j) + \partial_Y [Q_2(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j)]
\]

\[
A_{fss} := \partial_0 Q_1(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y}) + \partial_Y Q_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y}) + \partial_0 Q_1(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y}) + \partial_Y Q_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})
\]

\[
A_{ffs} := \partial_0 Q_1(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y}) + \partial_Y Q_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y}) + \partial_0 Q_1(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y}) + \partial_Y Q_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})
\]

\[
A_{sfs}(U_i, U_j) := \partial_0 Q_1(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j) + \partial_Y Q_2(\partial_{\partial Y}; \partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j)
\]

For example, each entry of the \(\partial_x Q_1(\partial_{\partial Y}; \partial_{\partial Y})(U_i, U_j)\) term of \(A_{fss}(U_i, U_j)\) is a linear combination of terms of the form \(\partial_x (\partial_y u \partial_y u)\), \(\partial_x (\partial_y v \partial_y v)\), etc.

---

27The exact coefficients in the linear combinations could be determined, but they are not needed for the profile construction.

28Here and in (8.12) we suppress some of the \((U_i, U_j)\) or \((U_i, U_m, U_n)\) arguments.
It turns out that this implies that the expressions for profiles and the coefficients of \( \varepsilon^k \), the following inequalities hold:

\[
A_{ff}(U_i, U_j) \implies i + j = k + 3 \implies 2 \leq i, j \leq k + 1
\]

\[
B_{fff} (U_i, U_m, U_n) \implies l + m + n = k + 4 \implies 2 \leq l, m, n \leq k.
\]

This implies that \( \binom{H_{k-1}}{K_{k-1}} \), which is part of the coefficient of \( \varepsilon^{k-2} \) in (8.2), depends only on the profiles \( U_2, \ldots, U_{k-1} \). Moreover, from these bounds we observe that cubic terms appear first in the expressions for \( H_3, K_3 \).

**Remark 8.1.** A necessary condition for obtaining a solution \( U_k \in S \) to the equation \( L_{ff}(U_k) = \binom{H_{k-1}}{K_{k-1}} \) is that the right side belong to \( S^* \). Inspection of (8.7) already shows that \( (H_2, K_2) \in S^* \). It turns out that \( U_2(t, x, y) = 0 \), which will imply that \( (H_3, K_3) \in S^* \) and \( (H_4, K_4) \in S \), but we will find that \( (H_j, K_j) \in S^m \) not \( S \) for \( j \geq 5 \). Thus, in section 12 we will modify the \( (H_j, K_j) \), \( j \geq 5 \), to elements \( (H^*_j, K^*_j) \) in \( S \), using (7.7) and (7.8). By a careful choice of \( U_{j-1} \), we will eventually arrange \( (H^*_j, K^*_j) \in S^* \).

Observe that there is no constraint of the form \( h_{k-1}, k_{k-1} \in S^* \). This is because the boundary condition in (8.4) involves only the traces of these functions on the boundary \( y = Y = 0 \).

**9. Solvability conditions for** \( L_{ff}(U) = F, l_{ff}(U) = G \)

Motivated by the form of the profile equations (8.4), we consider in this section the general question of finding real-valued solutions \( U = \left( \begin{array}{c} u \\ v \end{array} \right) \in S \) of systems of the form

\[
L_{ff}(U) = F \text{ on } y, Y > 0
\]

\[
l_{ff}(U) = G \text{ on } y, Y = 0,
\]

where \( F \in S^e, G \in S^b \).

\[^{29}\text{More precisely, there is no reason for these function to lie in } S; \text{ barring unlikely cancellations, they lie in } S^m \text{ not } S.\]
A necessary condition for the existence of a real-valued solution \( U \in S \) is that \( F \) and \( G \) be real-valued with \( F \in S^* \), \( G \in S^b \), and so we assume that. We will see that the existence of a solution depends on certain additional solvability conditions being satisfied. The operators \( L_{ff} \) and \( l_f \) both annihilate elements of \( S \), so we will look for \( U = U^{0s} + U^{osc} \in S^* \). Writing \( F = F^{0s} + F^{osc} \) and \( G = G^{0s} + G^{osc} \), we look for \( U \) of the form

\[
U = U^{0s} + U^{osc} = U_p + U_h,
\]

where we try to make these pieces satisfy

\[
\begin{align*}
(a) \quad & L_{ff} U^{0s} = F^{0s}, \quad l_f (U^{0s}) = \mathcal{G} \\
(b) \quad & L_{ff} U_p = F^{osc} \\
(c) \quad & L_{ff} (U_h) = 0, \quad l_f (U_h) = G^{osc} - l_f (U_p).
\end{align*}
\]

Here and below equations with \( L_{ff} \) or \( l_f \) on the left hold respectively on \( y, Y > 0 \) or \( y, Y = 0 \).

We will use Fourier series to analyze these equations. For \( F^{osc} \) and \( F^{0s} \) we write

\[
F^n(t, x, Y) = \begin{pmatrix} f^n_1 \\ f^n_2 \end{pmatrix}, \quad n \neq 0, \quad F^{0s}(t, x, Y) = \begin{pmatrix} f^{0s}_1 \\ f^{0s}_2 \end{pmatrix}.
\]

Consider first (9.3)(a). Since \( U^{0s} = U^{0s}(t, x, Y) \) is independent of \( \theta \), the interior equation simplifies to:

\[
- \left( \frac{\partial^2}{\partial y^2} u^{0s} \right) = \begin{pmatrix} f^{0s}_1 \\ f^{0s}_2 \end{pmatrix},
\]

which has the unique solution in \( S^* \) given by

\[
U^{0s} = \begin{pmatrix} u^{0s} \\ v^{0s} \end{pmatrix} = \left( \begin{array}{c} -
\int^\infty_y f^{0s}_1(t, z, z)dzds \\
\frac{1}{r} \int^\infty_y f^{0s}_2(t, z, z)dzds \end{array} \right).
\]

Observing that \( l_f (U^{0s}) = \left( \int^\infty_y f^{0s}_1(t, z, z)dz, \int^\infty_y f^{0s}_2(t, z, z)dz \right) \), we see that the boundary condition in (9.3)(a) can hold only if we impose the solvability condition

\[
\begin{pmatrix} 1 \\ \frac{1}{r} \end{pmatrix} \int^\infty_y f^n_1(t, z, z)dz = \mathcal{G}(t, x).
\]

Next consider the interior equations in (9.3)(b)(c), which we will solve by diagonalization after rewriting them as first-order systems. The equation for the \( n \)th Fourier mode in (9.3)(b) is

\[
- \partial_y u^n - in(r - 1) \partial_y v^n - n^2(c^2 - r)u^n = f^n_1 \\
- r \partial_y v^n - in(r - 1) \partial_y u^n - n^2(c^2 - 1)v^n = f^n_2,
\]

where \( (u^n, v^n) = U^n(t, x, Y), n \neq 0 \). Introducing \( \tilde{U} = (U, \partial_Y U) \) and \( \tilde{F} = (0, F) \), we rewrite this as the \( 4 \times 4 \) first order system

\[
(\partial_Y - G(\beta, n)) \tilde{U}^n = \left( \begin{array}{c} \partial_Y - \begin{pmatrix} 0 \\ D(\beta, n) \end{pmatrix} \\ B(n) \end{pmatrix} \right) \begin{pmatrix} U^n \\ \partial_Y U^n \end{pmatrix} = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1/r \end{array} \right) \tilde{F}^n := \tilde{F}^n,
\]

where the matrices \( B(n) \) and \( D(\beta, n) \) are given by:

\[
B(n) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad D(\beta, n) = n^2 \begin{pmatrix} r - c^2 & 0 \\ 0 & 1/r \end{pmatrix}
\]

and \( \beta = (-c, 1) \) as before. The matrix \( G(\beta, n) \) has eigenvalues \( in\omega_j, j = 1, \ldots, 4 \), where

\[
\omega_1 = c^2 - 1, \quad \omega_2 = \frac{c^2}{r} - 1, \quad \omega_3 = \omega_1 \quad \text{and} \quad \omega_4 = \omega_2
\]
(ω₁, ω₂ are purely imaginary with positive imaginary part), and corresponding right eigenvectors:

\[
R_1(n) = \begin{pmatrix}
-ω₁ \\
ω₂ \\
inω₁
\end{pmatrix} \quad R_2(n) = \begin{pmatrix}
1 \\
ω₂ \\
inω₁
\end{pmatrix} \quad R_3(n) = \overline{R_1(-n)} \quad R_4(n) = \overline{R_2(-n)}.
\]

We also choose corresponding left (row) eigenvectors \(L_i(n), i = 1, \ldots, 4\) satisfying \(L_iR_j = δ_{ij}\).

In the case when \(F^{osc} = 0\) (that is, \(Lff(U_h) = 0\)) we use the \(L_i(n), R_j(n)\) to diagonalize the left side of (9.8) and find easily that real, decaying solutions of \(Lff(U_h) = 0\) must have the form

\[
U_h(t, x, θ, Y) = \sum_{n≠0} U^n_h(t, x, Y)e^{inθ}
\]

where the \(U^n_h\) are given by the formulas:

\[
U^n_h = \begin{cases}
σ_1(t, x, n)e^{inω_{1}Y}r_1 + σ_2(t, x, n)e^{inω_{2}Y}r_2 & \text{for } n > 0 \\
σ_3(t, x, n)e^{inω_{3}Y}r_3 + σ_4(t, x, n)e^{inω_{4}Y}r_4 & \text{for } n < 0.
\end{cases}
\]

Here \(r_j\) is a column vector consisting of the first two components of \(R_j(n)\), and the \(σ_j\) are undetermined scalar functions satisfying \(σ_3(t, x, n) = \overline{σ_1(t, x, −n)}\) and \(σ_4(t, x, n) = \overline{σ_2(t, x, −n)}\) for \(n < 0\).

Similarly, diagonalization yields particular real, decaying solutions of \(Lff(U_p) = F^{osc}\) with Fourier modes of the form

\[
U^n_p(t, x, Y) = \sum_{j=1}^4 τ^n_j(t, x, Y)r_j,
\]

where

\[
τ^n_j(t, x, Y) := \begin{cases}
\int_0^Y e^{inω_j(Y−s)}F^n_j(t, x, s)ds & \text{for } j = 1, 2 \text{ and } n > 0 \\
\int_0^Y e^{inω_j(Y−s)}F^n_j(t, x, s)ds & \text{for } j = 3, 4 \text{ and } n > 0 \\
\int_0^∞ e^{inω_j(Y−s)}F^n_j(t, x, s)ds & \text{for } j = 1, 2 \text{ and } n < 0 \\
\int_0^∞ e^{inω_j(Y−s)}F^n_j(t, x, s)ds & \text{for } j = 3, 4 \text{ and } n < 0
\end{cases}
\]

and \(F^n_j = L_j(n)\bar{F}^n\).

Finally, consider the boundary equation in (9.3) (c), where the right side is now determined. We have

\[
lf(U^n_h) = \left( \begin{array}{cc}
∂Yu^n + inv^n \\
r∂Yv^n + (r−2)inv^n
\end{array} \right) = \left( \begin{array}{ccc}
0 & 1 & 0 \\
(r−2)in & 0 & r
\end{array} \right) \bar{U}^n_h := C(β, n)\bar{U}^n_h, \ n ≠ 0.
\]

Using (9.9) for \(n > 0\) we write

\[
C(β, n)\bar{U}^n_h = [C(β, n)R_1, C(β, n)R_2] \begin{pmatrix}
σ_1(t, x, n) \\
σ_2(t, x, n)
\end{pmatrix} = \begin{pmatrix}
2 − c^2 & 2ω_2 \\
2ω_1 & c^2 − 2
\end{pmatrix} \begin{pmatrix}
σ_1 \\
σ_2
\end{pmatrix} := \text{inB}_{Lop} \begin{pmatrix}
σ_1 \\
σ_2
\end{pmatrix}
\]

Recall that the boundary frequency \(β = (−c, 1)\) was chosen so that \(B_{Lop}\) is singular. Clearly

\[
\ker B_{Lop} = \text{span} \left( \begin{pmatrix}
ω_2 \\
−q
\end{pmatrix} \right), \ \text{coker } B_{Lop} = \text{span}(q, \omega_2), \ \text{where } q^2 = −ω_1ω_2 \text{ and } q > 0.
\]

---

30 The functions \(\bar{U}_h = (U_h, ∂YU_h)\) have the same form with \(R_j(n)\) in place of \(r_j\).

31 The \(ω_j\) and \(r_j\) here are the same as in (9.9), (9.7).

32 Here we use \(2 − c^2 = 2q\).
We obtain a solvability condition for \( l_f(U_h) = G^{osc} - l_f(U_p) \) by considering

\[
(9.15) \quad l_f(U^n_h) = \text{inB}_{Lop} \left( \frac{\sigma_1(t, x, n)}{\sigma_2(t, x, n)} \right) = G^n - C(\beta, n)\bar{U}_p^n, \text{ for, say, } n > 0.
\]

With (9.14) we see that

\[
(9.16) \quad (q \, \omega_2) \left( G^n - C(\beta, n)\bar{U}_p^n \right) = 0, \quad n \neq 0, \text{ or equivalently}
\]

\[
(q \, \omega_2) (G^{osc} - l_f(U_p)) = 0
\]

is a necessary and sufficient condition for the existence of a solution in \( S^* \) of \( (9.3) \) (c).

Assuming that (9.16) holds we now complete the construction of \( U_h \). For \( n > 0 \) we want to choose \( \sigma_1, \sigma_2 \) so that (9.15) holds. Although \( B_{Lop} \) has a one dimensional kernel \( K := \text{span} \left( \frac{\omega_2}{-q} \right) \), we can fix a solution of (9.15) by taking

\[
(9.17) \quad \left( \frac{\sigma_1(t, x, n)}{\sigma_2(t, x, n)} \right) = \frac{1}{\text{in}} B_{Lop}^{-1} \left( G^n - C(\beta, n)\bar{U}_{k,p}^n \right) \text{ for } n > 0,
\]

where \( B_{Lop}^{-1} \) is the inverse of \( B_{Lop} : K^\perp \to \text{Im} B_{Lop} \).

**Remark 9.1** (Solvability conditions). Summarizing, we have found the following solvability conditions for obtaining a solution \( U \in S \) to (9.1) when \( F = F + F^* + F^m \in S^c, G \in S^b \):

\[a) \quad F = 0, \quad F^m = 0 \quad \text{(equivalently, } F \in S^s)\]

\[b) \quad \left( \int_0^\infty \frac{\rho_0^e(t, x, z)dz}{\int_0^\infty \rho_0^e(t, x, z)dz} \right) = 0 \quad \text{for } n > 0 \]

\[c) \quad (q \, \omega_2) (G^{osc} - l_f(U_p)) = 0.
\]

Given any such solution \( U \), we can obtain another solution in \( S \) by adding any \( V(t, x, y) \in S \).

We close this section by observing that since \( B_{Lop} \) is singular, there are nontrivial decaying solutions \( U \in S \) to

\[
(9.19) \quad L_{ff}(U) = 0, \quad l_f(U) = 0.
\]

The kernel of \( B_{Lop} \) is spanned by \( \left( \frac{\omega_2}{-q} \right) \), so using (9.19) we see that for \( n > 0 \)

\[
(9.20) \quad l_f(U^n) = \text{inB}_{Lop} \left( \frac{\sigma_1(t, x, n)}{\sigma_2(t, x, n)} \right) = 0 \Leftrightarrow \left( \frac{\sigma_1(t, x, n)}{\sigma_2(t, x, n)} \right) = \alpha(t, x, n) \left( \frac{\omega_2}{-q} \right)
\]

for some scalar function \( \alpha \) to be determined. Thus, we obtain nontrivial, real decaying solutions \( U_\alpha(t, x, \theta, Y) \in S^* \) of (9.19) defined by

\[
(9.21) \quad U^n_\alpha(t, x, Y) = \alpha(t, x, n) \left( \omega_2 e^{i\omega_2 Y} r_1 - q e^{i\omega_2 Y} r_2 \right) := \alpha(t, x, n) \bar{r}(n, Y), \text{ for } n > 0
\]

\[
U^n_\alpha(t, x, Y) = \alpha(t, x, n) \bar{r}(n, Y), \text{ for } n < 0,
\]

where

\[
(9.22) \quad \alpha(t, x, n) = \overline{\alpha}(t, x, -n) \text{ and } \bar{r}(n, Y) = \overline{\bar{r}}(t, x, -n).
\]

We will see below that solutions like \( U_\alpha \) are used in the analysis of the profile equations to insure that the third solvability condition (9.18) (c) holds.
10. Order of construction

Consider again the cascade of profile equations

\begin{equation}
L_{ff}(U_k) = \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}, \quad l_f(U_k) = \begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix}, \quad k = 2, 3, \ldots,
\end{equation}

where the $H_j, \ldots, k_j$ are given by (8.7), (8.8), $U_j = 0$ for $j \leq 1$, and we seek $U_k \in S$. In general when the expressions (8.7), (8.8) are evaluated using profiles $U_j \in S$, one obtains elements of $S^e$ not $S^*$ (as is required by Remark 9.1). Thus we need to work with modifications of the $H_j, K_j$ which we denote by $H'_j, K'_j$. These have to be defined inductively; for example, we will see that the choice of $U_{k-1}$ is made so that $H'_k = 0, K'_k = 0$. The definition of the $H'_j, K'_j$ is given in (12.15); in this section, whose purpose is mainly to lay out the order of construction of the pieces of $U_k$, we only need to know that $H'_j, K'_j$ depend only on $U_k$ for $k \leq j$ and belong to $S$.

We split the $U_k$'s into five pieces,

\begin{equation}
U_k(t, x, y, \theta, Y) = U_{k,0}(t, x, y) + U_{k,0}^{osc}(t, x, Y) + U_{k,h}(t, x, \theta, Y) + U_{k,p}(t, x, \theta, Y) + U_{k,0}(t, x, \theta, Y),
\end{equation}

so now $U_k(t, x, y, \theta, Y) = U_k + U_{k,0}^{osc}$, $U_k^{osc} = U_{k,h} + U_{k,p} + U_{k,0}$.

The terms $U_k, U_{k,0}$ are without analogues in (9.2): we will see that they are chosen to arrange solvability conditions for $U_{k+1}$.

The pieces of $U_k$ are constructed to satisfy

\begin{itemize}
  \item[(a)] $L_{ff}(U_{k,0}^{osc}) = \begin{pmatrix} H_{k-1}^{0} \\ K_{k-1}^{0} \end{pmatrix}$
  \item[(b)] $L_{ff}(U_{k,p}) = \begin{pmatrix} H_{k-1}^{osc} \\ K_{k-1}^{osc} \end{pmatrix}$
  \item[(c)] $L_{ff}(U_{k,h}) = 0$ \quad $l_f(U_{k,h}) = \begin{pmatrix} h_{k-1}^{osc} \\ k_{k-1} \end{pmatrix}$
  \item[(d)] $L_{ff}(U_{k,0}, \alpha) = 0$ \quad $l_f(U_{k,0}, \alpha) = 0$ \quad $(q \omega_2 \left(\begin{pmatrix} h_{k}^{osc} \\ k_{k} \end{pmatrix} - l_f(U_{k+1,0}, \alpha) = 0
  \item[(e)] $\left(\begin{pmatrix} H'_{k-1}^{0} \\ K'_{k-1}^{0} \end{pmatrix} = 0$ \quad $\int_0^\infty \left(\begin{pmatrix} H_{k-1}^{0} \\ K_{k-1}^{0} \end{pmatrix} dY = \begin{pmatrix} h_{k}^{0} \\ k_{k} \end{pmatrix}$
\end{itemize}

where the first equation in each line is on $y, Y > 0$ and the second or third, when present, is on $y = Y = 0$. The equations in (10.3) have obvious counterparts in (9.3), (9.18), (9.19). Those in line (e) are used to determine $U_k$. We sometimes refer to (10.3) as (10.3)$_k$.

The construction is done inductively. When solving for $U_k$ we assume that real profiles $U_2, \ldots, U_{k-1}$ in $S$ have been found satisfying (10.3)$_j$, for $j \leq k - 1$. We also suppose that the $U_j, j \geq k$, although undetermined, lie in $S$.

The first elements to determine are $U_{k,p}$ and $U_{k,0}^{osc}$, which depend only on profiles $U_j, j \leq k - 1$. From (9.5) we obtain

\begin{equation}
U_{k,0}^{osc} = -\int_Y^\infty \int_s^\infty \left(\begin{pmatrix} H'_{k-1}^{0} \\ K'_{k-1}^{0} \end{pmatrix} dY = \begin{pmatrix} h_{k}^{0} \\ k_{k} \end{pmatrix}
\end{equation}
From (9.10) we see that \( U_{k,p}^n(t, x, Y) \) is given for \( n > 0 \) by\(^{\text{33}}\)

\[
U_{k,p}^n = -\int_0^Y \left[ \frac{\omega_1 H_{k-1}^n - K_{k-1}^n}{-2i\omega_1 c_n^2} r_1 + \frac{\omega_2 K_{k-1}^n}{2i\omega_2 c_n^2} r_2 ds \right] + \frac{\omega_1 H_{k-1}^n - K_{k-1}^n}{-2i\omega_1 c_n^2} r_3 + \frac{\omega_2 K_{k-1}^n}{2i\omega_2 c_n^2} r_4 ds,
\]

(10.5)

and for \( n < 0 \) we have \( U_{k,p}^n = \bar{U}_{k,p}^n \). By definition of \( S^* \) (Definition 7.1) Sobolev norms \( H^s(t, x) \), \( s \in \mathbb{N} \) of \( H_{k-1}^n(t, x, Y) \) and \( K_{k-1}^n(t, x, Y) \) are rapidly decaying with respect to \( n \) and exponentially decaying with respect to \( Y \); so (10.5) implies \( U_{k,p} \in S^* \).

Knowing \( U_{k,p} \) we can now construct \( U_{k,h} \). By the induction assumption, line (c) of (10.3)\(^{-1} \) shows that the solvability condition (recall (9.16)) for \( l_f(U_{k,h}) \) does hold. Using (9.9) and (9.17) we obtain

\[
U_{k,h}^n = \sigma_{1,k}(t, x, n)e^{i\omega_1 Y} r_1 + \sigma_{2,k}(t, x, n)e^{i\omega_2 Y} r_2 \quad \text{for} \quad n > 0,
\]

(10.6)

where

\[
\left( \frac{\sigma_{1,k}(t, x, n)}{\sigma_{2,k}(t, x, n)} \right) = \frac{1}{i\omega_1 B_{Lop}} \left( \frac{\omega_1}{\omega_2} - C(\beta, n) \tilde{U}_{k,h}^n \right).
\]

(10.7)

Moving next to \( U_{k,\alpha} \), we see that the first two equations of line (d) of (10.3)\(_k\), together with (9.21), (9.22), imply that

\[
U_{k,\alpha}^n(t, x, Y) = \alpha_k(t, x, n) \tilde{r}(n, Y), \quad n \neq 0
\]

(10.8)

for some function \( \alpha_k \) to be determined. This function is determined in section 11 so that the third equation in line (d) of (10.3)\(_k\) holds. It turns out to depend only on the pieces of \( U_k \) that are already known and previous profiles.

The last piece to construct is \( U_k \). This function is determined in section 12 so that the equations in line (e) of (10.3)\(_k\) hold. It turns out to depend only on \( U_k^0, U_{k,p}, U_{k,h}, U_{k,\alpha} \) and previous profiles.

**Remark 10.1 (Order of construction).** To summarize, the order of construction is

\[
U_{k,p} \text{ or } U_{k,p}^0, U_{k,h}, U_{k,\alpha}, \tilde{U}_k.
\]

The first two pieces depend only on previous profiles, \( U_{k,h} \) depends also on \( U_{k,p} \), while \( U_{k,\alpha} \) depends also on \( U_k^0, U_{k,p} \) and \( U_{k,h} \). Finally, \( \tilde{U}_k \) depends also on all four other pieces of \( U_k \).

11. Amplitude equations

We now discuss the construction of \( U_{k,\alpha} \), which is chosen so that the third equation in line (d) of (10.3)\(_k\), the solvability condition for \( U_{k,h+1} \), holds. Recall from (9.21) that the Fourier modes of \( U_{k,\alpha} \) have the form

\[
U_{k,\alpha}^n(t, x, Y) = \alpha_k(t, x, n) \tilde{r}(n, Y), \quad n \neq 0
\]

(11.1)

for an “amplitude” \( \alpha_k \) to be determined. The solvability condition can be written

\[
(q - \omega_2) \left( \frac{\omega_1}{k_k^0} - C(\beta, n) \tilde{U}_{k+1,p}^n \right) = 0, \quad n \neq 0.
\]

(11.2)

To see first the dependence of the terms in (11.2) on \( U_k \), we can use the formula (10.5)\(_{k+1}\) for \( U_{k+1,p}^n \), together with the formulas for \( H_k^n \), \( K_k^n \), \( h_k^n \), \( k_k^n \) given by (8.7), (8.8). The functions \( H_k, K_k \), defined in (12.15) and appearing in (10.5)\(_{k+1}\), are modifications of \( H_k, K_k \) belonging to \( S^* \). Like

\[^{\text{33}}\text{Here we use } L_1(n) = (-inr - \omega^2), -in\omega_1, \omega_1, -r)/(-2i\omega_1 c_n^2), \quad L_2(n) = (in\omega_2, ik(\epsilon^2 - 1), 1, r\omega_2)/(2i\omega_2 c_n^2).\]
Since \( H_k, K_k \) these functions depend just on the \( U_j \) for \( j \leq k \). For the purposes of this section the only other information we need is that the terms of \( H'_k, K'_k \) in which \( U_k \) appears are exactly the same as the terms of \( H_k, K_k \) in which \( U_k \) appears. This observation allows us to use (5.7). Thus, we can write

\[
(a) \left( \frac{H_k^n}{K_k^n} \right) = -[L_{fs}(U_k)]^n + [A_{ff}(U_2, U_k)]^n + [A_{ff}(U_k, U_2)]^n + N_1^n(U_2, \ldots, U_{k-1})
\]

(11.3) \( (b) \left( \frac{h_k^n}{k_k^n} \right) = -[l_s(U_k)]^n - [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_k)]^n - [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_k, U_2)]^n + N_2^n(U_2, \ldots, U_{k-1}) + \left( \frac{f_k^n}{g_k^n} \right)
\]

where the \( N_j \)'s are (known) nonlinear functions depending only on the profiles \( U_2, \ldots, U_{k-1} \), and the corresponding expression for \( \left( \frac{H_k^n}{K_k^n} \right) \) differs from (11.3)(a) only in the function \( N_1 \).

Writing \( U_k = U_k + U_k^{0*} + U_{k,\alpha_k} + U_{k,h} + U_{k,p} \) and using the bilinearity of the operators with arguments \( (U_2, U_k) \) or \( (U_k, U_2) \), we claim we can modify the above to:

\[
\left( \frac{H_k^n}{K_k^n} \right) = -[L_{fs}(U_{k,\alpha})]^n + [A_{ff}(U_2, U_{k,\alpha})]^n + [A_{ff}(U_{k,\alpha}, U_2)]^n + N_3^n(U_2, \ldots, U_{k-1}, U_k^{0*}, U_{k,h}, U_{k,p})
\]

\[
\left( \frac{h_k^n}{k_k^n} \right) = -[l_s(U_{k,\alpha})]^n - [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_{k,\alpha})]^n - [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_{k,\alpha}, U_2)]^n + N_4^n(U_2, \ldots, U_{k-1}, U_k^{0*}, U_{k,h}, U_{k,p}) + \left( \frac{f_k^n}{g_k^n} \right).
\]

This follows just from the fact that the right side of (11.3) does not depend on \( U_k^{0*} \). At this point in the construction the arguments of \( N_3 \) and \( N_4 \) are known. Thus, when (11.4) is plugged into (11.2) we obtain an equation in which the only unknown is \( \alpha_k \). We refer to this as the amplitude equation for \( \alpha_k \).

Although it is a lot of work to unravel the explicit form of the equation for \( \alpha_k \), it is already clear from a glance at (11.2), (10.5), and (11.4) that the equation for \( \alpha_2 \) has a quadratic nonlinearity with a forcing term depending only on \( (f_2, g_2) \), and that the equation for \( \alpha_k \) is a linearized form of the equation for \( \alpha_2 \).

Nonlocal amplitude equations involving bilinear Fourier multipliers arising in nonlinear elasticity and other areas have been studied by a number of authors including [Lar83, Hun06, BG09, Mar10, BGC12, Sec15, CW16]. The next proposition describes the form of the equation that arises in isotropic hyperelastic nonlinear elasticity.

**Proposition 11.1.** a) The amplitude equation for \( \alpha_2 \) has the form

\[
\partial_t \alpha_2 + c\partial_t \alpha_2 + \mathcal{H}(B(\alpha_2, \alpha_2)) = G_2(f_2, g_2)
\]

where \((-c, 1) = \beta\), \( \mathcal{H} \) denotes the Hilbert transform with respect to \( \theta \) \( (\mathcal{H} f(k) := -\text{sgn}(k) f(k)) \), and \( B \) is the bilinear Fourier multiplier given by:

\[
B(\alpha_2, \alpha_2)(n) := \frac{-1}{4\pi c_0} \sum_{n' \neq 0} b(-n, n - n', n') \alpha_2(t, x, n - n') \alpha_2(t, x, n').
\]

The kernel \( b(n_1, n_2, n_3) \), determined in [Hun06, CW16], is symmetric in its arguments and homogeneous of degree two. The constant \( c_0 \) is determined in [CW16].

---

\(^{34}\) The term \([l_s(U_k)]^n \) is independent of \( U_k \) since \( l_s \) is linear and \( n \neq 0 \).
b) For $k \geq 3$ the amplitude equation has the form

\begin{equation}
\partial_t \alpha_k + c \partial_x \alpha_k + 2\mathcal{H}(B(\alpha_2, \alpha_k)) = G_k(U_2, \ldots, U_{k-1}).
\end{equation}

**Remark 11.1.** The analogue of (11.5) for space dimensions $d \geq 3$ is given in [CW16]. The only change is that the $c \partial_x$ operator is replaced in higher dimensions by $c \nabla_x \cdot \nabla_x$, where the Rayleigh frequency is now $(-c\eta, \eta)$, $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$, and $c$ is as before. It is shown in chapter 2 of [CW16] that the vector field $\partial_t + c \frac{\eta}{|\eta|} \cdot \nabla_x$, which governs the speed and direction of Rayleigh waves along the boundary, is a characteristic vector field of the Lopatinski determinant.

The well-posedness of the amplitude equation (11.5) has been studied in [Hun06,CW16]. Here we need a version of the result in which the time of existence of $\alpha_2$, even for $H^\infty$ solutions, depends only on a fixed low order of regularity. We state the result for the following problem in $d \geq 2$ space dimensions (so $x' \in \mathbb{R}^{d-1}$):

\begin{equation}
\partial_t \alpha + v \cdot \nabla_x \alpha + \mathcal{H}(B(\alpha, \alpha)) = G(t, x', \theta)
\end{equation}

\begin{equation}
\alpha(t, x', \theta) = 0 \text{ in } t < 0,
\end{equation}

where $\mathcal{H}$ and $B$ are as in Proposition 11.1, and $v \in \mathbb{R}^{d-1}$ is a fixed vector.

**Proposition 11.2.** Let $m \geq m_1 > \frac{d}{2} + 2$ and suppose $G(t, x', \theta) \in C([0, T_0]; H^m(\mathbb{R}^{d-1} \times \mathbb{T}))$ for some $T_0 > 0$. For every $R > 0$ there exists a $T = T(m_1, R) \leq T_0$ such that if $|G| \in C([0, T_0]; H^{m_1}(\mathbb{R}^{d-1} \times \mathbb{T})) < R$, then there exists a unique solution $\alpha \in C([0, T]; H^m(\mathbb{R}^{d-1} \times \mathbb{T})) \cap C^1([0, T]; H^{m-1}(\mathbb{R}^{d-1} \times \mathbb{T}))$ to the problem (11.8).

**Remark 11.2.** 1) This proposition follows directly from the main well-posedness result of [Hun06]. The essential step for obtaining a time of existence depending on a fixed low order of regularity $H^{m_1}$ is to obtain a tame estimate of the form

\begin{equation}
\frac{d}{dt} \|\alpha(t)\|^2_{H^m} \leq C \|\alpha(t)\|^2_{H^{m_1}} \|\alpha(t)\|_{H^{m_1}}
\end{equation}

for solutions to the Cauchy problem with zero interior forcing and nonzero initial data. Although Hunter only uses a weaker nontame estimate, namely,

\begin{equation}
\frac{d}{dt} \|u(t)\|^2_{H^m} \leq C \|u(t)\|^3_{H^m},
\end{equation}

to obtain a time of existence that shrinks with increasing $H^m$ regularity, he actually proves the better estimate (11.9). The arguments using the tame estimate to get a time of existence depending just on $H^{m_1}$ regularity are standard and given, for example, in [Tay17], Chapter 16.

2) Given $\alpha_2$ and $G$ as in Proposition 11.2, one readily obtains (again using estimates contained in the proofs of [Hun06]) a solution with the same regularity on the same time interval to the linear problem

\begin{equation}
\partial_t \alpha + v \cdot \nabla_x \alpha + 2\mathcal{H}(B(\alpha_2, \alpha)) = G(t, x', \theta)
\end{equation}

\begin{equation}
\alpha(t, x', \theta) = 0 \text{ in } t < 0,
\end{equation}

corresponding to (11.4).

3) If $G \in H^\infty$ in (11.5) then one can show $\alpha \in H^\infty$ by using the equation to deduce increased regularity with respect to $t$. If one has both $G$ and $\alpha_2$ in $H^\infty$, the same remark applies to the solution of (11.11).

4) In part 4 we give a new proof of the tame estimate (11.9) that applies directly to the form of the amplitude equation given in (11.5) and also incorporates the slow tangential space variables.

---

35 J. Hunter in [Hun06] studies a different, but equivalent, form of the equation which shares the essential feature that his kernel “$b$” is unbounded with positive homogeneity. [Sec13] studies a related equation with unbounded, positively homogeneous kernel on a plasma-surface interface. The kernels studied in [BG09, Mar10] are bounded.
12. Final steps in the construction of the $U_k$

We now use the results of the previous two sections to complete the construction of the profiles. In the construction of $U_2$ and $U_3$, we ask to reader to accept for a moment that the definition of $(H_k', K_k')$ \[(12.15)\] implies that

\[(12.1) \quad (H_j, K_j) \in S^* \text{ for } j \leq k = (H_k', K_k') \in S^*.
\]

This reflects the fact that in such cases there is no need to modify $(H_k, K_k)$.

Recall that all profiles are required to vanish in $t \leq 0$.

12.1. The profile $U_2$. The pieces of the leading profile $U_2$ are determined by the equations \[(10.3)\] 2. The only nonvanishing piece turns out to be $U_{2,\alpha}$.

**Proposition 12.1.** The leading order profile $U_2$ is given by $U_2 = U_{2,\alpha}(t, x, \theta, Y) \in S^*$.

**Proof.** 1. We follow the procedure outlined in section 10. Since the profiles $U_j$, $j \leq 1$ are zero, we have $(H_j, K_j) = 0$ for $j \leq 1$, so \[(12.1)\] and the formulas \[(10.4), (10.5), \text{ and } (10.6), (10.7)\] imply immediately that $U_{2,0}^*$, $U_{2,p}$, and $U_{2,h}$ are zero. The term $U_{2,\alpha}$ is given by \[(10.8), \text{ where } \alpha_2 \text{ is provided by Proposition 11.2}\]

2. The only remaining component to find is $U_2$. This piece is determined by line (e) of \[(10.3)\]. We now use the fact, which will be clear from the definition \[(12.15)\], that

\[(12.2) \quad \left(\frac{H_k'}{K_k'}\right) = \left(\frac{H_k}{K_k}\right) \text{ for all } k,
\]

so we can use \[(8.7)\] to write the interior equation as

\[(H_3, K_3) = A_{fff}(U_2, U_2) + A_{fff}(U_3, U_3) + B_{fff}(U_2, U_2, U_2) - L_{ss}(U_2) - L_{ss}(U_3) = 0.
\]

Using Proposition \[(7.4)\] it is easy to see that the terms involving fast derivatives vanish, so this equation reduces to

\[(12.3) \quad L_{ss}(U_2) = 0 \text{ on } y > 0.
\]

The boundary conditions for $u_2, v_2$ come from

\[(12.4) \quad \int_0^\infty \left(\frac{H_2^0}{K_2^0}\right) dY = \left(\frac{h_2^0}{k_2^0}\right) \text{ on } y = Y = 0.
\]

Inspection of \[(8.7)\] shows that $(H_2, K_2) \in S^*$, so $(H_2, K_2) = (H_2', K_2')$\[36\] Thus \[(12.4)\] is

\[\int_0^\infty -[L_{fs}(U_2)]^0 + \partial_Y [Q_2(\partial_\theta Y; \partial_\theta Y)(U_2, U_2)]^0 dY = -[l_s(U_2)]^0 - [Q_2(\partial_\theta Y; \partial_\theta Y)(U_2, U_2)]^0.
\]

Since $U_{2,0}^* = 0$ we have $[L_{fs}(U_2)]^0 = 0$. Computing the integral gives

\[-[Q_2(\partial_\theta Y; \partial_\theta Y)(U_2, U_2)]^0 = -[l_s(U_2)]^0 - [Q_2(\partial_\theta Y; \partial_\theta Y)(U_2, U_2)]^0,
\]

which reduces to $l_s(U_2) = 0$ on $y = 0$\[37\] With \[(12.3)\] this gives $U_{2,\alpha} = 0$. \[\square\]

---

\[36\] We use \[(12.1)\] here.

\[37\] We use here the fact that $G(t, x') = 0$. 

---

GEOMETRIC OPTICS FOR RAYLEIGH WAVETRAINS IN D-DIMENSIONAL NONLINEAR ELASTICITY 39
12.2. The profile $U_3$. The profile $U_3$ is determined by the equations $[10.3]_3$. Knowing that $U_2 = 0$, we see from the formulas $[8.7]$ that

$$
(12.5) \quad \left( \frac{H_3}{K_3} \right) \in S^* \text{ and } \left( \frac{H_4}{K_4} \right) \in S.
$$

Proposition 12.2. There exists a profile $U_3 \in S$ satisfying the equations $[10.3]_3$.

Proof. 1. By $[12.1]$ we have $(H_2, K_2) = (H'_2, K'_2)$, so we can use $[8.7]$ in the formulas $[10.4]$, $[10.5]$, and $[10.6]$, $[10.7]$ to construct the pieces $U^0_3$, $U_{3,p}$, and $U_{3,h}$. The term $U_{3,\alpha}$ is given by $[10.8]$, where $\alpha_3$ is provided by Proposition $11.2$.

2. Finally, we construct $U_3$, which is determined by line (e) of $[10.3]_3$. Using $[12.2]$ and the definitions of $H_4, K_4$ provided in $[8.7]$, we obtain for the interior equation

$$
(12.6) \quad \left( \frac{H_4}{K_4} \right) = -L_{fs}(U_4) - L_{ss}(U_3) + A_{fs}(U_2, U_2) + A_{fs}(U_2, U_3) + A_{ff}(U_3, U_3) + B_{fff}(U_3, U_2) + B_{fff}(U_2, U_2) = 0.
$$

This simplifies to

$$
(12.7) \quad L_{ss}(U_3) = 0
$$

by an argument similar to that which gave $[12.3]$.

The boundary conditions for $U_3$ come from the formula

$$
(12.8) \quad \int_0^\infty \left( \frac{H_3^0}{K_3^0} \right) dY = \left( \frac{h_3^0}{k_3^0} \right) \text{ on } y = Y = 0.
$$

Applying $[12.1]$ again, we can use the definitions of $H_3, K_3, h_3$ and $k_3$ in $[12.8]$ to obtain

$$
(12.9) \quad \int_0^\infty [-L_{fs}(U_3) + A_{ff}(U_2, U_3) + A_{ff}(U_3, U_2) + N_1(U_2)^0] dY = \int_0^\infty [-l_s(U_3) - Q_2(\partial_{\theta_1}; \partial_{\theta_1})(U_2, U_3) - Q_2(\partial_{\theta_2}; \partial_{\theta_2})(U_3, U_2) + N_2(U_2)^0]
$$

To simplify this expression recall that the $A$'s are related to the $Q$'s as described in the formulas $[8.10]$. The integral of the term $[A_{ff}(U_3, U_2)]^0$ can be expanded

$$
(12.10) \quad \int_0^\infty [A_{ff}(U_3, U_2)]^0 dY = \int_0^\infty \partial_{\theta_1} Q_1(\partial_{\theta_1}; \partial_{\theta_1})(U_3, U_2) + \partial_{\theta_2} Q_2(\partial_{\theta_2}; \partial_{\theta_2})(U_3, U_2)]^0 dY = -[Q_2(\partial_{\theta_2}; \partial_{\theta_2})(U_3, U_2)]^0 |_{Y=0},
$$

since $[\partial_{\theta_2} Q_1(\partial_{\theta_1}; \partial_{\theta_1})(U_3, U_2)]^0$ vanishes. Notice that the right hand side is a term appearing in $(h_3^0, k_3^0)^*$. Doing the same for the other $A_{ff}$ term, we reduce $[12.9]$ to the following:

$$
(12.11) \quad \int_0^\infty [-L_{fs}(U_3) + N_1(U_2, U_2)]^0 dY = [-l_s(U_3) + N_2(U_2)]^0.
$$

Using $U_2^0 = 0$, which implies $[L_{ss}(U_2)]^0 = 0$, and $L_{fs}(U_3) = L_{fs}(U_3^*)$, we get the final form of the boundary conditions:

$$
(12.12) \quad l_s(U_3) = -l_s(U_3^*) + \int_0^\infty [L_{fs}(U_3^*) - \partial_{\theta} Q_1(\partial_{\theta}; \partial_{\theta})(U_2, U_2)]^0 dY.
$$

In view of $[12.7]$ and $[12.12]$ we obtain a unique solution $U_3 \in S$. This completes the construction of $U_3$. \qed
Remark 12.3. One of the goals of Chapter 2 of [Mar11] is to show that even though $G(t,x) = 0$ and $U_2 \in S^*$, it can happen that $U_3 \neq 0$. This conclusion is reached by showing that the right side of (12.12), or rather its analogue in her simplified model, is not 0 and hence neither is $U_3$. This is an example of “internal rectification”. The computation of [Mar11] shows there is every reason to expect that the right side of (12.12) is nonzero in the Saint Venant-Kirchhoff model as well, except for rare accidents. When that happens, the error analysis of part 2 shows that internal rectification is truly present in the exact solution.

12.3. The profiles $U_k$, $k \geq 4$. It remains to construct $U_k \in S$ satisfying (10.3)$_k$, assuming that profiles $U_2, \ldots, U_{k-1}$ in $S$ satisfying (10.3)$_j$, $j \leq k - 1$ have already been constructed. By the construction of $U_2$ and $U_3$ we see that $(H_j, K_j) \in S^*$ for $j \leq 4$, so there was no need to modify $(H_j, K_j)$ for these $j$.

But for $j \geq 5$ we must expect $(H_j, K_j)$ to contain terms in $S^m$. For instance, the term $\partial_x(\partial_y u_1 \partial_y v_2)$ from $A_{fss}(U_3, U_2) \in S^m$, since $u_1 \neq 0$ (normally), which implies that $H_5, K_5 \notin S^*$. In order to construct the higher profiles we must now define the $(H'_j, K'_j)$, $j \geq 5$. We define the $(H'_j, K'_j)$ as elements of $S$, even though they will turn out by the choice of $U_{j-1}$ to lie in $S^*$.

For functions $f = f + f^* + f^m \in S^c$, we can define a modification $f^{mod} = f + f^* + f^{m,mod}$, where $f^{m,mod}$ is as in 3). Applying this to the $H_{k-1}, K_{k-1}$ for $6 \leq k \leq N$ we obtain a preliminary modification

$$
(H_{k-1}^{mod})_{K_{k-1}} = \frac{(H_{k-1})_{K_{k-1}}}{(K_{k-1})_{K_{k-1}}} + \frac{(H_{k-1}^*)_{K_{k-1}}}{(K_{k-1})_{K_{k-1}}} + M_{k-1,0} + \epsilon Y M_{k-1,1} + \cdots + \varepsilon N - (k - 2) - 2 \gamma N - (k - 2) - 2 M_{k-1,1} - N - (k - 2) - 1 R_{k-1,1} - N - (k - 2) - 1,
$$

where the $M_{k-1,j} \in S^*$ are defined by

$$
M_{p,j} := \partial_x \left( \frac{H_p}{K_p} \right) (t, x, 0, \theta, Y)/j! \text{ for } p \geq 5, \ M_{p,j} := 0 \text{ for } p \leq 4.
$$

Noting that $L_{ff} U_k$ is part of the coefficient of $\epsilon^k$ in (8.2), we define

$$
(H'_k)_{K'_k} := \frac{(H_{k-1})_{K_{k-1}}}{(K_{k-1})_{K_{k-1}}} + \frac{(H_{k-1}^*)_{K_{k-1}}}{(K_{k-1})_{K_{k-1}}} + M_{k-1,0} + Y M_{k-2,1} + Y^2 M_{k-3,2} \ldots + Y^{k-6} M_{5,k-6}.
$$

Proposition 12.4. For each $2 \leq k \leq N$, there exists a profile $U_k \in S$ satisfying the equations (10.3)$_k$.

Proof. 1. The statement has been proved for $k = 2, 3$. The $U_j, j \leq k - 1$ are assumed to be known and to satisfy (10.3)$.j$, $j \leq k - 1$, so the $H'_j, K'_j, h_j, k_j, j \leq k - 1$ are known, and we can use the formulas (10.4), (10.5), and (10.6), (10.7) to construct the pieces $U^0_k, U_{k,p}$, and $U_{k,h}$. The term $U_{k,\alpha}$ is given by (10.8), where $\alpha_k$ is provided by Proposition 11.2.

2. Finally, we construct $U_k$, which is determined by line (e) of (10.3)$_k$. Using (12.2) we can use the definitions of $H_{k+1}, K_{k+1}$ to write the interior equation as

$$
\frac{(H'_{k+1})_{K'_{k+1}}}{(K'_{k+1})_{K'_{k+1}}} = -L_{ss}(U_k) - L_{fs}(U_{k+1}) + A_{fss}(U_k, U_2) + A_{fss}(U_k, U_2) + B_{fff}(U_k, U_2, U_2) + B_{fff}(U_k, U_2, U_2) + B_{fff}(U_k, U_2, U_2) + N_{k+1}(U_2, \ldots, U_{k-1}) = 0.
$$
where \( N_{k+1} \) is a known nonlinear function. As in earlier arguments this easily reduces to
\[
L_{ss}(U_k) = \frac{N_{k+1}(U_2, ..., U_{k-1})}{U_k}.
\]

Starting at \( H'_k, K'_k \) (or \( k = 6 \)), the function \( N \) is expected to be nonzero since \( A_{ss}(U_3, U_3) \) contains terms like \( \partial_x [\partial_y \partial_x U_{k+1}] \neq 0 \), which are normally nonzero. Observe that the function \( N_{k+1} \) is different from the corresponding function in the expression for \((H_{k+1}, K_{k+1})\), but \( N_{k+1} \) is not different.

Next, we look at the boundary conditions given by:
\[
\int_0^\infty \left( \frac{H'_k}{K_k} \right) \, dy = \left( \frac{h_0^0}{k_0^0} \right) \text{ on } y = Y = 0.
\]

Notice that here the distinction between \( H_k, K_k \) and \( H'_k, K'_k \) has an effect, because the terms \( M_{k,j} \) in (12.15) do not integrate to zero. Observing (again) that the terms of \( H'_k, K'_k \) in which \( U_k \) appears are exactly the same as the terms of \( H_k, K_k \) in which \( U_k \) appears, we can use (8.7) to write
\[
\int_0^\infty \left[ -L_{fs}(U_k) + A_{ff}(U_k, U_2) + A_{ff}(U_2, U_k) + N_1(U_2, ..., U_{k-1}) \right] \, dy =
\]
\[
- \left[ l_s(U_k) - Q_2(\partial_{\theta Y}; \partial_{\theta Y})(U_k, U_2) - Q_2(\partial_{\theta Y}; \partial_{\theta Y})(U_2, U_k) + N_2(U_2, ..., U_{k-1}) \right] \, dy
\]
where the \( N_j \)'s are known nonlinear functions of the lower order profiles. An argument similar to the one in section [12.2] allows us to simplify this to
\[
l_s(U_k) = -l_s(U_{k+1}) + \int_0^\infty L_{fs}(U_{k+1}) \, dy + [N(U_2, ..., U_{k-1})] \, dy \text{ on } y = Y = 0.
\]

The equations (12.17) and (12.20) together with the initial condition \( U_k = 0 \) in \( t \leq 0 \) uniquely determine \( U_k \). This completes the construction of \( U_k \) and the inductive step.

**Theorem 12.5.** (a) Assume \( d = 2 \). Let \( U_k, k = 2, ..., N \) be given by proposition [12.4]. Then the approximate solution \( U^\varepsilon(t, x, y) = \sum_{k=2}^N \varepsilon^k U_k(t, x, y, \frac{x-c t}{\varepsilon}, \frac{y}{\varepsilon}) \) satisfies
\[
\partial_t^2 U^\varepsilon + \nabla \cdot (L(\nabla U^\varepsilon)) + Q(\nabla U^\varepsilon) + C(\nabla U^\varepsilon) = \varepsilon^{N-1} E_N(t, x, y, \frac{x-c t}{\varepsilon}, \frac{y}{\varepsilon}) \text{ on } y > 0
\]
\[
L_2(\nabla U^\varepsilon) + Q_2(\nabla U^\varepsilon) + C_2(\nabla U^\varepsilon) = \varepsilon^2 \left[ f \right] = \varepsilon^N \epsilon_N(t, x, 0, \frac{x-c t}{\varepsilon}, 0) \text{ on } y = 0,
\]
where \( E_N, \epsilon_N \in S^\varepsilon \).

(b) Assume \( d \geq 3 \). The same result holds, where now \( U_k = U_k(t, x', x_d, \theta, Y) \) and
\[
U^\varepsilon = \left( \varepsilon^2 U_2 + \cdots + \varepsilon^n U_p \right)_{\theta = \frac{x-c t}{\varepsilon}, Y = \frac{y}{\varepsilon}}.
\]

Here \( \beta = (-c|\eta|, \eta) \) is a Rayleigh frequency as described in Remark [11.1].

**Proof.** 1. First consider the interior equation. Using [8.2] we obtain \[12\]
\[
\partial_t^2 U^\varepsilon + \nabla \cdot (L(\nabla U^\varepsilon)) + Q(\nabla U^\varepsilon) + C(\nabla U^\varepsilon) = \]
\[
\sum_{k=2}^N \varepsilon^{k-2} (L_{ff}(U_k) - \left( \frac{H'_k}{K_k} \right)) + \varepsilon^{N-1} E_N = \sum_{k=2}^N \varepsilon^{k-2} \left( \frac{H'_k}{K_k} \right) - \left( \frac{H_k}{K_k} \right) + \varepsilon^{N-1} E_N = \]
\[
\sum_{k=6}^N \varepsilon^{k-2} \left( \frac{H'_{k-1}}{K_{k-1}} \right) - \left( \frac{H_{k-1}}{K_{k-1}} \right) + \varepsilon^{N-1} E_N,
\]

\[42\] Here and below we suppress the evaluations \( \theta = \frac{x-c t}{\varepsilon}, Y = \frac{y}{\varepsilon} \).
where for the last equality we used (12.23) and the fact that the $H_j, K_j$ are modified only for $j \geq 5$. Substituting into the right side from (12.13), (12.15) yields

\begin{equation}
(12.25)
\end{equation}

Proof. which gives (12.27).

2. On the boundary we have directly from (8.3)

\begin{equation}
(12.26)
\end{equation}

3. As already noted, the proof for $d \geq 3$ is just a repetition of that for $d = 2$ with mainly notational changes. For example, in solving for $U_2$ one now uses the form of the amplitude equation given in Remark 11.1.

Combining the results of Theorems 12.6 and 12.5 we obtain our main result for the SVK system:

\textbf{Theorem 12.6.} Consider the traction problem in nonlinear elasticity (1.1), where $G(t, x', \theta) \in H^\infty([0, T] \times \mathbb{R}^{d-1} \times \mathbb{T}, d \geq 2$. With $\Omega := (-\infty, T] \times \mathbb{T}_+$ let $M > \frac{d}{2} + 2$ and let

\begin{equation}
(12.25)
\end{equation}

be the approximate solution constructed in Theorem 12.5 for $\varepsilon \in (0, 1)$ and some positive $T \in T_0$.

(a) Suppose $s \geq \left\lfloor \frac{d}{2} \right\rfloor + 6$. There exist constants $C > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ the problem (1.1) has a unique solution $u_\varepsilon = u_\varepsilon^a + v \in E^{s+2}(-\infty, T]$ such that $v_\varepsilon$ satisfies the estimate

\begin{equation}
(12.26)
\end{equation}

In particular this implies $|v_\varepsilon|_{W^{s+2}(-\infty, T]} \leq C \varepsilon \varepsilon^{M+1-1} - 1$.

(b) With $s$ as in part (a), let $p$ be an integer $\geq 2$, choose $M$ such that $p < M - \frac{d}{2} - 1$, let $u_\varepsilon^a$ be as in (12.25), and let $u_\varepsilon = u_\varepsilon^a + v$ be the exact solution as in part (a). Then we have

\begin{equation}
(12.27)
\end{equation}

\textbf{Proof.} Using the definitions of the spaces $S$ and $S^\varepsilon$ (section 7) and taking $R_\varepsilon, r_\varepsilon$ to be given by

\begin{equation}
(12.28)
\end{equation}

where $E_{M+1}^\varepsilon, e_{M+1}$ are as in (12.21), it is straightforward to check that Assumption 3.7 is satisfied by $u_\varepsilon^a, R_\varepsilon, r_\varepsilon$ for an appropriate choice of $A_1, A_2$. Thus, Remark 3.7 allows us to apply Theorem 4.6 to prove part (a). We also have

\begin{equation}
(12.29)
\end{equation}

which gives (12.27).

\hfill \Box
Part 4. Tame estimate for the amplitude equation

In this section we give a new proof of a tame a priori estimate for the amplitude equation \((11.8)\). This is the main step in obtaining a time of existence for very regular solutions that depends only on a fixed low order of regularity. The same proof provides a tame estimate for the pulse analogue of \((11.8)\) considered by [CW16].

**Proposition 12.1.** Let \(v \in \mathbb{R}^{d-1}\) be a fixed velocity vector, and assume that the kernel \(b\) in \((11.6)\) is symmetric with respect to its arguments and that there exists a constant \(C > 0\) such that
\[
\forall (n_1, n_2, n_3) \in \mathbb{Z}^3, \quad |b(n_1, n_2, n_3)| \leq C \min(|n_1n_2|, |n_2n_3|, |n_1n_3|).
\]

Let \(m \geq m_1 > \frac{d}{2} + 2\). Then sufficiently smooth solutions of the Cauchy problem
\[
(12.31) \quad \partial_t u + \sum_{j=1}^{d-1} v_j \partial_j u + \mathcal{H}(B(u, u)) = 0, \quad u|_{t=0} = u_0.
\]
satisfy the estimate
\[
(12.32) \quad \left| \frac{d}{dt} \|u(t)\|_{H^m}^2 \right| \leq C \|u(t)\|_{H^m}^2 \|u(t)\|_{H^{m+1}}.
\]

In isotropic elastodynamics, the kernel \(b\) that appears in \((11.6)\) satisfies the bound \((12.30)\) as can be seen immediately by inspection of the basic kernels written down in formulas (2.56)-(2.58) of [CW16] (or formula (3.20) of [Hum06]).

The proof of proposition 12.1 uses the following lemma from [RR82]. Here and below, integration with respect to \(k\) or \(l\) is summation over \(\mathbb{Z}^d\).

**Lemma 12.2.** Suppose \(G : (\mathbb{R}^{d-1} \times \mathbb{Z}) \times (\mathbb{R}^{d-1} \times \mathbb{Z}) \to \mathbb{C}\) is a locally integrable measurable function that can be decomposed into a finite sum
\[
(12.33) \quad G(\xi, k, \eta, l) = \sum_{j=1}^{K} G_j(\xi, k, \eta, l)
\]
such that for each \(j\) we have either
\[
(12.34) \quad \sup_{\xi, k} \int |G_j(\xi, k, \eta, l)|^2 d(\eta, l) < C \quad \text{or} \quad \sup_{\eta, l} \int |G_j(\xi, k, \eta, l)|^2 d(\xi, k) < C.
\]

Then
\[
(12.35) \quad (f, g) \to \int G(\xi, k, \eta, l) f(\xi - \eta, k - l)g(\eta, l)d(\eta, l)
\]
defines a continuous bilinear map of \(L^2 \times L^2 \to L^2\), and
\[
(12.36) \quad \left| \int G(\xi, k, \eta, l) f(\xi - \eta, k - l)g(\eta, l)d(\eta, l) \right|_{L^2} \leq C \|f\|_{L^2} \|g\|_{L^2}.
\]

**Proof of Proposition 12.1.** 1. We consider a solution \(u\) to the Cauchy problem \((12.31)\) that is sufficiently smooth for all manipulations below to be rigorous. Using the Fourier expression of the \(H^m\) norm, we see that is enough to estimate just the \(L^2\) norms of the functions \(u, \partial_\alpha^m u, \partial_\alpha^p u\) with \(|\alpha| = m (\alpha \in \mathbb{N}^{d-1})\). All other partial derivatives of \(u\) can be dealt with by interpolating between such ‘extreme’ cases. Let us first prove the following bounds on the operator \(B^{\mathbb{R}}\).

\[\text{Lemma 12.30 was proved in [CW16] for the case of pulses. The remainder of this proof differs from the argument in [CW16]; the argument in [CW16] gave only a nontame estimate for } \frac{d}{dt} \|u\|_{H^m}^2.\]
Lemma 12.3. Under the assumptions of Proposition [12.7], the bilinear operator $B$ is symmetric. It satisfies the Leibniz rule

$$\partial_\theta B(u, v) = B(\partial_\theta u, v) + B(u, \partial_\theta v),$$

and more generally the Leibniz rule at any order of differentiation in $\theta$, as well as the bound\(^{44}\)

$$\forall u, v \in \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{T}; \mathbb{R}), \quad \left| \int_{\mathbb{R}^{d-1} \times \mathbb{T}} u \mathcal{H}(B(u, v)) \, dy \, d\theta \right| \leq C \|v\|_{H^{m_1}} \|u\|^2_{L^2},$$

for a suitable constant $C$ and any integer $m_1$ satisfying $m_1 > \frac{d}{2} + 2$. (The Sobolev norms refer to the space domain $\mathbb{R}^{d-1} \times \mathbb{T}_\theta$.)

The fact that $B$ is symmetric comes from the symmetry of the kernel $b$ with respect to its three arguments. We now consider functions $u, v$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{T}; \mathbb{R})$. We will take advantage of some cancelation arising from the skew-symmetric operator $\mathcal{H}$. We compute

$$\int_{\mathbb{R}^{d-1} \times \mathbb{T}} u \mathcal{H}(B(u, v)) \, dy \, d\theta = i \int_{\mathbb{R}^{d-1} \times \mathbb{Z} \times \mathbb{Z}} \bar{u}(y, k) \tilde{u}(y, k - l) \tilde{v}(y, l) \text{sgn}(-k) b(-k, k - l, l) \, dy \, dk \, dl$$

$$= i \int_{\mathbb{R}^{d-1} \times \mathbb{Z} \times \mathbb{Z}} \bar{u}(y, -k) \tilde{u}(y, k - l) \tilde{v}(y, l) \text{sgn}(-k) b(-k, k - l, l) \, dy \, dk \, dl$$

$$= \frac{i}{2} \int_{\mathbb{R}^{d-1} \times \mathbb{Z} \times \mathbb{Z}} \bar{u}(y, -k) \tilde{u}(y, k - l) \tilde{v}(y, l) \left( \text{sgn}(-k) + \text{sgn}(k - l) \right) b(-k, k - l, l) \, dy \, dk \, dl,$$

where we have used the fact that $u$ is real valued and the symmetry of $b$. Let us observe that if $-k$ and $k - l$ have opposite signs, then the quantity $\text{sgn}(-k) + \text{sgn}(k - l)$ vanishes. If $-k$ and $k - l$ have the same sign, then the sum of signs is either $2$ or $-2$, and there holds

$$|k| \leq |l|, \quad \text{and} \quad |k - l| \leq |l|.$$

With (12.30) this yields

$$\left| (\text{sgn}(-k) + \text{sgn}(k - l)) b(-k, k - l, l) \right| \leq C|\text{sgn}(-k) + \text{sgn}(k - l)||k - l||l| \leq C|l|^2.$$

Using the Cauchy-Schwarz and $L^2 - L^1$ convolution inequalities, we derive the bound

$$\left| \int_{\mathbb{R}^{d-1} \times \mathbb{T}} u \mathcal{H}(B(u, v)) \, dy \, d\theta \right| \leq C \int_{\mathbb{R}^{d-1} \times \mathbb{Z} \times \mathbb{Z}} \bar{u}(y, -k) ||\tilde{u}(y, k - l)||^2 |\tilde{v}(y, l)| \, dy \, dk \, dl$$

$$\leq C \int_{\mathbb{R}^{d-1}} ||\bar{u}(y, \cdot)||^2_{L^2} ||\tilde{u}(y, \cdot)||^2_{L^2} ||\tilde{v}(y, \cdot)||^2_{L^1} \, dy$$

$$\leq C \|u\|^2_{L^2} \sup_{y \in \mathbb{R}^{d-1}} \|v(y, \cdot)\|_{H^q(\mathbb{T})} \text{ for any } q > \frac{5}{2}.$$

Applying the Sobolev imbedding Theorem completes the proof of Lemma 12.3.

2. Let us consider an integer $m \geq m_1$ with $m_1$ as in Lemma 12.3. We consider a sufficiently smooth solution $u$ to (12.31) and compute (the transport terms with respect to the variables $y$ can be removed by a change of $(t, y)$ variables):

$$\frac{d}{dt} \|u(t)\|^2_{L^2} = -2 \int_{\mathbb{R}^{d-1} \times \mathbb{R}} u \mathcal{H}(B(u, u)) \, dy \, d\theta.$$

Applying the bound in Lemma 12.3 we get

$$\frac{d}{dt} \|u(t)\|^2_{L^2} \leq C \|u(t)\|^2_{L^2} ||u(t)||_{H^{m_1}} \leq C \|u(t)\|^2_{H^m} \|u(t)\|_{H^{m_1}}$$

\(^{44}\)The bounds obviously extend by continuity to functions in appropriate Sobolev spaces and are not restricted to functions in the Schwartz class.
3. Let us now differentiate \([12.31]\) \(m\) times with respect to \(\theta\), and get
\[
\partial_t \partial_y^m u + \sum_{j=1}^{d-1} v_j \partial_j \partial_y^m u + 2 \mathcal{H} (B(\partial_y^m u, u)) = - \sum_{m'=1}^{m-1} \binom{m}{m'} \mathcal{H} (B(\partial_y^{m'} u, \partial_y^{m-m'} u)).
\]
Taking the \(L^2\) scalar product with \(\partial_y^m u\), we get
\[
\frac{d}{dt} \| \partial_y^m u(t) \|_{L^2}^2 = - 4 \int_{\mathbb{R}^{d-1} \times \mathbb{T}} \partial_y^m u \mathcal{H} (B(\partial_y^m u, u)) \, dy \, d\theta
\]
\[
- 2 \sum_{m'=1}^{m-1} \binom{m}{m'} \int_{\mathbb{R}^{d-1} \times \mathbb{T}} \partial_y^m u \mathcal{H} (B(\partial_y^{m'} u, \partial_y^{m-m'} u)) \, dy \, d\theta.
\]
For the first integral we apply the estimate of Lemma \([12.3]\) and get
\[
\left| \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \partial_y^m u \mathcal{H} (B(\partial_y^m u, u)) \, dy \, d\theta \right| \leq C \| u(t) \|_{H^{m+1}} \| u(t) \|_{H^m}^2.
\]

4. Now we consider the remaining terms in \([12.38]\). We let \(\xi\) or \(\eta\) denote Fourier variables dual to \(\xi\). Assuming without loss of generality \(m - 1 \geq m' \geq m - m' \geq 1\), and letting \(\chi_A(k, l), \chi_B(k, l)\) be the characteristic functions of \(\{ |k| \leq |l| \}\), \(\{ |l| < |k| \}\) respectively, we consider one of the remaining terms
\[
\int_{\mathbb{R}^{d-1} \times \mathbb{R}} k^m \hat{u}(\xi, k)(k-l)^{m'} \hat{u}(\xi-\eta, k-l)^{m-m'} \hat{u}(\eta, l)b(-k, k-l, l) \chi_A(k, l) + \chi_B(k, l) \, d\xi(k) \, d\eta \leq |F_A(\xi, k, \eta, l) + F_B(\xi, k, \eta, l)| := A + B,
\]
where \(F_A, F_B\) have the obvious definitions.\(^{45}\)

5. We can estimate \(A\) by Cauchy-Schwarz after estimating the \(L^2(\xi, k)\) norm of
\[
H_A(\xi, k) := \int (k-l)^{m'} \hat{u}(\xi-\eta, k-l)^{m-m'} \hat{u}(\eta, l)b(-k, k-l, l) \chi_A(k, l) \, d\eta(k).
\]
For this we apply Lemma \([12.2]\) to the kernel
\[
G_A(\xi, k, \eta, l) := \frac{|k-l|^{m'} |l|^{m-m'} |l||k-l| \chi_A}{\langle \xi, \eta, k-l \rangle} \lesssim \frac{1}{\langle \xi-\eta, k-l \rangle^{m-2}}.
\]
Here we have used \([12.30]\), the fact that \(m' \geq 1\), and the fact that on supp \(\chi_A\) we have \(|k-l| \leq 2|l|\). Observe that this estimate does not work if \(m' = 0\). This gives
\[
A \leq \| u(t) \|_{H^{m+1}}^2 \| u(t) \|_{H^m}.
\]

6. To estimate \(B\) we use \(|k|^m \lesssim |l|^m + |k-l|^m\) to write
\[
B \leq \int (k-l)^{m} \hat{u}(\xi, k)(k-l)^{m'} \hat{u}(\xi-\eta, k-l)^{m-m'} \hat{u}(\eta, l)b(-k, k-l, l) \chi_B(k, l) \, d\xi(k) \, d\eta \leq |F_1 + F_2|.
\]
\(^{45}\)We replace \(\text{sgn}(-k)\) by one in these estimates.
7. Estimate of $B_1$. Pairing $l^m$ with $\hat{u}(\eta, l)$, we can use Cauchy-Schwarz to estimate $B_1$ after estimating the $L^2(\eta, l)$ norm of

$$
H_{B_1}(\eta, l) := \int (k - l)^{m'} \hat{u}(\xi - \eta, k - l)l^{m-m'} \hat{u}(\xi, k)b(-k, k - l, l)\chi_B(k, l)d(\xi, k).
$$

To do this we apply lemma $[12.2]$ to the kernel

$$
G_{B_1}(\xi, k, \eta, l) := \frac{|k - l|^{m'} |l^{m-m'}| |k - l| \chi_B}{(\xi, k)^m(\xi - \eta, k - l)^{m-1}} \lesssim \frac{1}{(\xi - \eta, k - l)^{m-1/2}}.
$$

Here we have used $[12.30]$, the fact that $m' \geq 1$, and the fact that on supp $\chi_B$ we have $|k - l| \leq 2|k|$, $|l| \leq |k|$. This gives

$$
B_1 \lesssim \|u(t)\|_{H^m}^2 \|u(t)\|_{H^{m-1}}.
$$

8. Estimate of $B_2$. In the integral that defines $B_2$ make the change of variables

$$
(\xi, k, \eta, l) \rightarrow (\xi, k, \alpha, p)
$$

to obtain

$$
\int p^m \hat{u}(\xi, k)p^{m'} \hat{u}(\alpha, p)(k - p)^{m-m'} \hat{u}(\xi - \alpha, k - p)b(-k, p, k - p)\chi_B(k, k - p)d(\xi, k)d(\alpha, p)
$$

Pairing $p^m$ with $\hat{u}(\alpha, p)$, we can use Cauchy-Schwarz to estimate $B_2$ after estimating the $L^2(\alpha, p)$ norm of

$$
H_{B_2}(\alpha, p) := \int p^m \hat{u}(\xi - \alpha, k - p)(k - p)^{m-m'} \hat{u}(\xi, k)b(-k, p, k - p)\chi_B(k, k - p)d(\xi, k).
$$

For this we apply lemma $[12.2]$ to the kernel

$$
G_{B_2}(\xi, k, \alpha, p) := \frac{|k|^m |k - p|^{m-m'} |k - p| \chi_B}{(\xi, k)^m(\xi - \alpha, k - p)^{m-1}} \lesssim \frac{1}{(\xi - \alpha, k - p)^{m-1/2}}.
$$

Here we have used $[12.30]$, the fact that $m - m' \geq 1$, and the fact that on supp $\chi_B$ we have $|k - p| \leq |k|$, $|p| \leq 2|k|$. This gives

$$
B_2 \lesssim \|u(t)\|_{H^m}^2 \|u(t)\|_{H^{m-1}}.
$$

9. These estimates go through unchanged if factors like $(k - l)^{m'}$, $l^{m-m'}$ are replaced by $\langle k - l \rangle^{m'}$, $\langle l \rangle^{m-m'}$.

The $y$-partial derivatives of $u$ are estimated in an entirely similar way. The factors $k^m$, $(k - l)^{m'}$, and $l^{m-m'}$ in $[12.40]$ are now replaced by $\langle k \rangle^m$, $\langle \xi - \eta \rangle^{m'}$, $\langle \eta \rangle^{m-m'}$. The dichotomy $\{k \leq |l|\}$, $\{|l| < |k|\}$ is replaced by $\{\xi \leq |\eta|\}$, $\{|\eta| < |\xi|\}$ with respective characteristic functions $\chi_A(\xi, \eta)$, $\chi_B(\xi, \eta)$.

Putting these estimates together gives the estimate $[12.32]$ for $m \geq m_1 > \frac{d}{2} + 2$.

---

References

[BG09] S. Benzoni-Gavage. Local well-posedness of nonlocal Burgers equations. *Differential and Integral Equations*, 22(3-4):303–320, 2009.

[BGC12] S. Benzoni-Gavage and Jean-François Coulombel. On the amplitude equations for weakly nonlinear surface waves. *Arch. Ration. Mech. Anal.*, 205(3):871–925, 2012.

[BGC16] S. Benzoni-Gavage and J.-F. Coulombel. Amplitude equations for weakly nonlinear surface waves in variational problems. In *Shocks, Singularities and Oscillations in Nonlinear Optics and Fluid Mechanics*, pages 1–32. Springer, INdAM Series, 2016.

---

46This estimate does not work if $m' = m$. 
[CM78] R. Coifman and Y. Meyer. Commutateurs operateurs d’integrales singulieres et operateurs multilineaires. *Ann. Instit. Fourier*, 28(3):177–202, 1978.

[CW16] J.-F. Coulombel and M. Williams. Geometric optics for surface waves in nonlinear elasticity. *To appear in Memoirs of the AMS*, 2016.

[Guë93] O. Guès. Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires. *Asymptotic Anal.*, 6(3):241–269, 1993.

[Hun06] J. Hunter. Short-time existence for scale-invariant Hamiltonian waves. *J. Hyperbolic Differ. Equ.*, 3(2):247–267, 2006.

[Kre70] H. O. Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.

[Lar83] R. W. Lardner. Nonlinear surface waves on an elastic solid. *Internat. J. Engrg. Sci.*, 21(11):1331–1342, 1983.

[Mar10] A. Marcou. Rigorous weakly nonlinear geometric optics for surface waves. *Asymptot. Anal.*, 69(3-4):125–174, 2010.

[Mar11] A. Marcou. *Interactions d’ondes et de bord*. These, Universite de Bordeaux, 2011.

[Mét00] G. Métivier. The block structure condition for symmetric hyperbolic systems. *Bull. London Math. Soc.*, 32(6):689–702, 2000.

[MZ05] G. Métivier and K. Zumbrun. Hyperbolic boundary value problems for symmetric systems with variable multiplicities. *J. Differential Equations*, 211(1):61–134, 2005.

[RR82] J. Rauch and M. Reed. Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension. *Duke Math. J.*, 49(2):397–475, 1982.

[Sec15] P. Secchi. Nonlinear surface waves on the plasma-vacuum interface. *Quart. Appl. Math.*, 73(4):711–737, 2015.

[Shi88] Y. Shibata. On the Neumann problem for some linear hyperbolic systems of second order. *Tsukuba J. Math.*, 12(1):149–209, 1988.

[SN89] Y. Shibata and G. Nakamura. On a local existence theorem of Neumann problem for some quasilinear hyperbolic systems of 2nd order. *Math. Zeitschrift*, 202(1):1–64, 1989.

[Str85] J.W. Strutt. (Lord Rayleigh.) On waves propagated along the plane surface of an elastic solid. *Proc. Lond. Math. Soc.*, 17:4–11, 1885.

[Tay77] M. Taylor. Rayleigh waves in linear elasticity as a propagation of singularities phenomenon. In *Partial Differential Equations and Geometry, Proc. Conf., Park City, Utah, 1977*, pages 273–291. Dekker, New York, 1977.

[Tay11] M. E. Taylor. *Partial differential equations III. Nonlinear equations*, volume 117 of *Applied Mathematical Sciences*. Springer, 2011.