Motion of a “small body” in non-metric gravity

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We describe “small bodies” in a non-metric gravity theory previously studied by this author. The main dynamical field of the theory is a certain triple of two-forms rather than the metric, with only the spacetime conformal structure, not metric, being canonically defined. The theory is obtained from general relativity (GR) in Plebanski formulation by adding to the action a certain potential. Importantly, the modification does not change the number of propagating degrees of freedom as compared to GR. We find that “small bodies” move along geodesics of a certain metric that is constructed with the help of a new potential function that appears in the matter sector. We then use the “small body” results to formulate a prescription for coupling the theory to general stress-energy tensor. In its final formulation the theory takes an entirely standard form, with matter propagating in a metric background and only the matter-gravity coupling and the gravitational dynamics being modified. This completes the construction of the theory and opens way to an analysis of its physical predictions.

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I. INTRODUCTION

In general relativity (GR) a test particle moves along a spacetime geodesic. The fact does not need to be added as a separate postulate of the theory. Indeed, general relativity possesses a well-defined initial value formulation, so to determine the evolution of a body one should just prescribe the initial data for its gravitational field and read off the trajectory from the resulting spacetime metric. For a “small body” this procedure gives the geodesic motion, and the Bianchi identities satisfied by the Einstein tensor are at the root of the derivation. A systematic procedure that allows to derive not only the geodesic motion, but also the corrections to it (the so-called gravitational self-force) has been recently described in [1].

In [2] the present author has described a large class of gravity theories that are based on two-forms rather than the metric. This class contains general relativity (in Plebanski formulation [3]) and can be arrived at rather simply, see [4], by dropping the simplicity constraints of Plebanski’s theory. A theory from the class is specified by a certain “potential” – a scalar function of certain components of the two-form field. Exactly as GR, the theory describes just two propagating degrees of freedom. The same class of gravity theories has appeared much earlier in works of Bengtsson and Peldan under the name of neighbours of GR, see e.g. [5]. These authors’ starting point (the pure connection formulation [6]) was, however, entirely different from that in [2], so the equivalence of models proposed in [5] to the theory described in [2] is not obvious and was pointed out in [7].

In the version proposed by this author, the basic dynamical field of the theory is an $su(2)$ Lie algebra-valued two-form (complexified in the Lorentzian signature case). With the Lie algebra being three-dimensional, the two-form field can be viewed as a triple of two-forms, and these can be declared to span the space of two-forms self-dual with respect to some metric. The knowledge of which two-forms are self-dual can be shown [8] to determine the conformal structure of the metric uniquely. Thus, any theory based on $su(2)$ Lie algebra-valued two-forms is naturally a theory of the conformal structure of spacetime. However, it is by no means obvious which metric in the conformal class so defined plays a physically distinguished role. Note that in Plebanski formulation of GR this problem does not arise as additional simplicity constraints that are imposed on the two-form field guarantee the existence of a preferred metric. It is not even clear that there is any physically distinguished metric in the theory. Indeed, this would be a metric in which test particles move along geodesics. To find whether there is such a metric, one would need to describe how the “usual” matter couples to the gravity theory in question. However, with the theory being that of two-forms rather than the metric, this is an unsolved problem. The only case that is currently understood is that of Yang-Mills fields which, being classically conformally-invariant, do not help.

The goal of this paper is to develop the physical interpretation of the theory [2] by studying the motion of a “small body”. This allows us to sidestep the unsolved problem of coupling to generic matter and develop the physical interpretation of the theory remaining entirely within its domain. We shall use the systematic procedure of [1] that needs only very little adaptation to the theory in question.

Our main result is that there is a physically distinguished metric in the theory [2] along whose geodesics test particles move. However, we find that it is the matter itself that supplies the conformal factor that determines this metric. Thus, we shall see that the coupling of matter to the gravity theory in question is characterized by a certain “mass” function, and the metric in which particles move along geodesics is obtained by choosing the conformal factor
so that this function is a constant. If the theory is to preserve the weak equivalence principle the mass function of all material bodies must be the same. This requirement introduces a certain universal potential function, see the main text. A very similar function appears on the gravitational side, and the theory is thus completely specified by prescribing the gravity and matter side potentials.

Having obtained an expression for the stress-energy “tensor” of a small body, it is not hard to extend it to a description of how the general stress-energy tensor of matter couples to our gravity theory. We give such a description, thus completing the construction of the theory and making the study of its physical predictions possible. We would like to emphasize at the outset that, in spite of the metric appearing in this theory only indirectly, the final formulation of the theory is entirely standard: one has usual matter fields moving in a metric background. Only the dynamics of gravity, as well as its coupling to matter are modified. However, unlike all previous modification schemes considered in the literature, the theory in question modifies both vacuum and non-vacuum GR without adding to it any new propagating degrees of freedom. We would like to stress that this feature of the theories considered here is quite striking, for it is a rather common belief that the only way to modify Einstein’s theory is to add to it new propagating modes.

The organization of this paper is as follows. In the next Section we review the Plebanski formalism for general relativity. We also describe how matter (e.g., ideal fluid) can be coupled to gravity in this formulation. In Section III we describe the modified gravity theories. We extend this description to the non-vacuum case in Section IV. In Section V we obtain an expression for the stress-energy-momentum two-form of a “small body” and use Bianchi identities to determine its motion in Section VI. An interpretation of the equations we obtain is contained in Section VII. We describe how a general stress-energy tensor is coupled to the gravity theory in question in section VII. Section IX gives a metric formulation of the theory that is most useful for practical applications. We conclude with a discussion.

II. PLEBANSKI FORMALISM

The aim of this Section is to review the Plebanski formulation of general relativity. As we have already mentioned, in this formulation Einstein’s gravity becomes a theory of two-forms rather than the metric. Plebanski’s formalism uses in a deep way the notions of self-duality on two forms (as determined by the spacetime metric) and it is the process of “abstracting” this notion from the underlying metric that allows for a deep reformulation of Einstein’s theory. The original Plebanski’s paper used spinor notations and is not very transparent for a reader who is not familiar with spinor techniques. An excellent exposition of the theory is also available in [9], where the problem of coupling to matter is discussed as well.

In this paper, to make it more easy to follow, we will try to avoid using spinors as hard as possible, only resorting to spinor techniques when they simplify computations. All such spinor calculations are banned to the Appendix. We have also decided to make the exposition of the Plebanski formalism as concrete as possible, so in this section we present it as a concrete recipe for deriving Einstein equations. However, before we give such an explicit description, let us state the main ideas abstractly.

Plebanski theory introduces a (complexified in Lorentzian signature) SO(3) vector bundle \( V \) over the spacetime \( M \), which can be referred to as the self-dual bundle, and a two-form field \( B^i, i = 1, 2, 3 \) taking values in \( V \). The triple of two forms \( B^i, i = 1, 2, 3 \) encodes information about the metric \( g \) on \( M \) via the requirement that \( B^i, i = 1, 2, 3 \) are self-dual two-forms with respect to \( g \). Indeed, the triple \( B^i \) spans a 3-dimensional subspace in the space of all two-forms, and declaring this to be the subspace of self-dual two forms defines the notion of Hodge duality on two forms, which, in turn, can be shown to uniquely determine the conformal class of the metric. However, a general triple \( B^i \) of two-forms contains too many components as compared to a metric. Indeed, it needs \( 3 \times 6 \) numbers to be specified, while a metric has only 10 components. To remedy this, Plebanski imposes the following “metricity” (or simplicity) conditions:

\[
B^i \wedge B^j \sim \delta^{ij},
\]

which give 5 equations on the two-form field (the trace of this equation gives the proportionality coefficient and is an identity). This brings the number of components in \( B^i \) down to 13, which is the required 10 components describing the metric, plus 3 gauge components related to availability of SO(3) gauge transformations. The volume form of this metric is then given by \((i/3)(B^i \wedge B^i)\). With the two-form field \( B^i \) being complex, one further needs 10 conditions that guarantee that the metric obtained is real Lorentzian. These conditions are given below in (7).

Thus, supplemented by the metricity conditions the two-form field contains just the right amount of information to describe a metric. One now needs a second order differential equation on \( B^i \). To obtain this one notices that there is a unique connection \( A^i \) satisfying:

\[
D_A B^i = 0,
\]
where $D_A B^i := d B^i + \epsilon^{ijk} A^j \wedge B^k$. Indeed, this gives $4 \times 3$ algebraic equations for $4 \times 3$ components of $A^i$, which fixes it uniquely, provided certain non-degeneracy conditions for $B^i$ are satisfied. We shall refer to this connection as $B$-compatible and denote it by $A_B$. When $B^i$ satisfies the metricity conditions (11) the $B$-compatible connection turns out to be equal to the self-dual part of the metric-compatible one. One can now compute the curvature symmetric and that its trace part must be a constant:

\[ R^i_{\ j} = \Phi^{ij} B^j, \]  

where the quantities $\Phi^{ij}$ are arbitrary at this stage. It can then be shown that the "internal" tensor $\Phi^{ij}$ must be symmetric and that its trace part must be a constant:

\[ \Phi^{ij} = \Psi^{ij} + \frac{1}{3} \delta^{ij} \Lambda, \quad \Psi^{ij} = \Psi^{(ij)}, \quad \Psi^{ij} \delta_{ij} = 0, \quad \Lambda = \text{const}. \]  

The relation (3) then becomes a set of 18 equations for 13 components of the $B$-field as well as for 5 (traceless symmetric) undetermined components $\Psi^{ij}$ of $\Phi^{ij}$. To see that (3) is equivalent to Einstein equations one notices that it states that the curvature of the self-dual part of the spin connection is self-dual as a two-form. This is known to be equivalent to the Einstein condition.

Finally, let us note that all equations of Plebanski theory can be obtained as Euler-Lagrange equations for the following action:

\[ S[B, A, \Psi] = \int B^i \wedge F^i(A) - \frac{1}{2} \left( \Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) B^i \wedge B^j. \]  

Indeed, the variation with respect to the traceless tensor $\Psi^{ij}$ gives (11), variation with respect to the connection gives (2), while variation with respect to the two-form field gives the main dynamical equation (3).

To make the above description little less abstract let us reformulate it as a concrete recipe for writing down Einstein equations once a spacetime metric is given. The starting point of Plebanski method of deriving Einstein equations is the same as in the tetrad method: one has to find a suitable tetrad. For a diagonal metric there is no ambiguity, but for a non-diagonal one it is possible to use the available freedom of Lorentz rotations to bring the tetrad to a convenient form. Thus, we assume that we have found a convenient collection of one-forms $e^I, i = 0, 1, 2, 3$ so that the metric is:

\[ ds^2 = e^I \otimes e^J \eta_{IJ}, \]  

where $\eta_{IJ}$ is the Minkowski metric.

The second step is to form a set of three two-forms $B^i, i = 1, 2, 3$ which are self-dual with respect to the given metric, and satisfy:

\[ B^i \wedge B^j \sim \delta^{ij}, \quad B^i \wedge (B^j)^* = 0, \quad \text{Re}(B^i \wedge B^j) = 0, \]  

where $(B^i)^*$ are the complex conjugate two-forms. This task is easy if one has a tetrad in one’s disposal, with a possible solution being:

\[ B^1 = i e^0 \wedge e^1 - e^2 \wedge e^3, \quad B^2 = i e^0 \wedge e^2 - e^3 \wedge e^1, \quad B^3 = i e^0 \wedge e^3 - e^1 \wedge e^2. \]  

It is easy to see that all the required conditions (7) are satisfied, and that the above two-forms are indeed self-dual with respect to (11). It can also be shown that the converse is true: given a triple of two-forms $B^i$ satisfying (7) there is a unique real metric with respect to which the two-forms $B^i$ are self-dual.

The third step is to find an $su(2)$ connection $A^i$ that is "compatible" with the above set $B^i$ of two-forms, in the sense that the covariant derivative of $B^i$ with respect to $A^i$ is zero: $D_A B^i = d B^i + \epsilon^{ijk} A^j \wedge B^k = 0$. To obtain such a connection one has to solve the following system of linear algebraic equations for the components of the connection:

\[ d B^1 + A^2 \wedge B^3 - A^3 \wedge B^2 = 0, \quad d B^2 + A^3 \wedge B^1 - A^1 \wedge B^3 = 0, \quad d B^3 + A^1 \wedge B^2 - A^2 \wedge B^1 = 0. \]  

In doing so we first have to take the exterior derivative of the three two forms $B^i$, and then write down the above system of $3 \times 4$ equations for the connection components $a^i : A^i = a^i_j \theta^j$. This is an exercise in algebra, of roughly the same degree of complexity as one that arises in the determination of the rotation coefficients in the tetrad-based approach. What simplifies the game somewhat is that in the tetrad-based approach, at least in principle, one has
4 × 6 equations to write, for the same number of the rotation coefficients, while in the case of Plebanski formulation there is just half that number. No information is lost, however, as all quantities in the Plebanski case are complex. In practice, for a given metric (usually possessing some symmetry properties), most of the connection coefficients are zero by symmetries both in the tetrad and the two-form cases, so the amount of work one has to do to find $A^i$ is only a little less than in the tetrad-based scheme. As in the tetrad-based scheme it is much easier to verify a conjectural solution than find one, as the latter involves some guesswork on which of the components are zero. Finding the connection is the most laborious part of the computation.

The fourth step is to compute the curvature two-form $F^i = dA^i + (1/2)e^i{}_{jk} A^j ∧ A^k$. In components:

$$F^1 = dA^1 + A^2 ∧ A^3, \quad F^2 = dA^2 + A^3 ∧ A^1, \quad F^3 = dA^3 + A^1 ∧ A^2.$$  \hfill (10)

This is a simple exercise in differentiation. The step of computing $F^i$ should be compared to the curvature computation in the tetrad-based approach. In that case one needs to compute six two-forms, as compared to only three in the Plebanski case.

The fifth step is to write all the two-forms that appear in $F^i$ in terms of the six basic two-forms that are used in the tetrad-based scheme, when one writes the curvature components in terms of the basic tetrad two-forms (before the Ricci tensor can be found by contracting a pair of indices).

The final step is to replace the tetrad two-forms appearing in the result for $F^i$ by their expressions in terms of $B^i$ and $(B^i)^*$. It turns out to be more convenient to use the two-forms $B^i = -(B^i)^*$. We have:

$$e^0 ∧ e^1 = \frac{1}{2i}(B^1 + B^1), \quad e^2 ∧ e^3 = \frac{1}{2}(B^1 - B^1),$$

$$e^0 ∧ e^2 = \frac{1}{2i}(B^2 + B^2), \quad e^3 ∧ e^1 = \frac{1}{2}(B^2 - B^2),$$

$$e^0 ∧ e^3 = \frac{1}{2i}(B^3 + B^3), \quad e^1 ∧ e^2 = \frac{1}{2}(B^3 - B^3).$$  \hfill (11)

This last step does not have a direct analog in the tetrad-based method.

This is it! Once an expression for the curvature $F^i$ in terms of the self- and anti-self dual two-forms is found, one can immediately write down the Einstein equations. Indeed, we have obtained the curvature in the form:

$$F^i = M^{ij} B^j + N^{ij} B^i,$$  \hfill (12)

where $M^{ij}, N^{ij}$ are some matrices built from the components of the metric and their first and second derivatives. The Plebanski field equations read:

$$F^i = \left( Ψ^{ij} + \frac{1}{3} δ^{ij} Λ \right) B^j + 4πGT^i,$$  \hfill (13)

where $Ψ^{ij}$ is a traceless matrix, $Λ$ is the cosmological constant and $T^i$ is the “stress-energy-momentum two-form”, see below, zero in vacuum. Thus, the vacuum Einstein equations are simply:

$\text{Tr} M = Λ, \quad N^{ij} = 0,$  \hfill (14)

which gives ten equations, as it should. As a bonus, we also obtain the components of the (self-dual part of the) Weyl curvature tensor. Indeed, we have $Ψ^{ij} = (M^{ij})_{(tf)}$, with $(tf)$ standing for the tracefree part.

Let us also write the general non-vacuum equations. To this end we need a general expression for the “stress-energy-momentum two-form” in terms of the quantities characterizing the gravitating matter. A general Lie-algebra valued two-form admits an expansion of the type \hfill (12), thus giving rise to $9 + 9 = 18$ components. However, in general relativity the “right-hand-side” of Einstein equations – the matter stress-energy-momentum tensor – has ten components. Thus, the stress-energy-momentum two form of Plebanski formalism cannot be a general Lie-algebra valued two-form. It needs to satisfy:

$$T^i ∧ B^j ∼ δ^{ij},$$  \hfill (15)

which should be compared to the first of the conditions in \hfill (7), plus certain reality conditions, see below. The conditions \hfill (15) imply that $T^i$ has the following structure:

$$T^i = \frac{1}{6} T B^i - \frac{1}{2} T^{ij} B^j,$$  \hfill (16)
Thus, we note that the components of

\[ M \]

identity:

This gives:

\[ B \]

system of 18 equations is thus that for 13 components of zero (no cosmological constant case) or constant (the cosmological constant), and is thus not an unknown field. The computes the

\[ B \]

quantities

is not complete. Note that in the GR case we have exactly the same system of 18 equations, but in that case for 18-5

quantities

is required to be real while its anti-symmetric part is purely imaginary.

For many applications one is interested in the simplest type of matter – that given by the perfect fluid. For the 18 components of the two-form field

\[ T^{ij}_\text{fluid} = \frac{1}{6}(\rho - 3P)B^i - \frac{1}{2}((\rho + P)\delta^{ij} + i\epsilon^{ijk}u^k)\overline{B}^j. \]

### III. MODIFIED GRAVITY: THE VACUUM CASE

We will start our presentation of the theory \cite{2} by describing the vacuum case. We follow closely a recent description \cite{1}.

The class of theories in question can be obtained by relaxing the simplicity conditions \cite{1} of the Plebanski theory. Thus, the main idea is to allow all the components of the two-form field

\[ B^i \]

to become dynamical. Recall that a general two-form field

\[ B^i \]

determines a conformal structure of spacetime by requiring that the triple

\[ B^i, B^j, u^i \]

are all required to be real. Thus, the symmetric part of

\[ T^{ij} \]

is required to be real while its anti-symmetric part is purely imaginary.

The non-vacuum Einstein equations then take the form:

\[ \text{Tr}M = \Lambda + 2\pi GT, \quad N^{ij} = -2\pi GT^{ij}. \]

How can one get natural field equations describing the dynamics of

\[ B^i \]

? As in Plebanski formulation of GR one first computes the

\[ B \]-compatible connection

\[ A_B \]

and then its curvature

\[ F(A_B) \]

. As before, it is natural and instructive to decompose the curvature into the basis of two-forms

\[ B^i \]

and

\[ \overline{B}^i \]

:

\[ F^i(A_B) = M^{ij}B^j + N^{ij}\overline{B}^j. \]

The most natural field equations are the same as in the Plebanski case \cite{3}. Thus, we require the curvature of the connection

\[ A_B \]

to be purely self-dual:

\[ F^i(A_B) = \Phi^{ij}B^j \iff M^{ij} = \Phi^{ij}, \quad N^{ij} = 0, \]

where

\[ \Phi^{ij} \]

is some purely gravitational tensor to be described below. The system of equations \cite{21} gives us 18 equations for the 18 components of the two-form field

\[ B^i \]

. However, it also contains the so-far unspecified functions

\[ \Phi^{ij} \]

and so is not complete. Note that in the GR case we have exactly the same system of 18 equations, but in that case for 18-5

quantities

\[ B^i \]

(the two-form field

\[ B^i \]

modulo the conditions

\[ B^i \wedge B^j \sim \delta^{ij} \]

). In addition the trace part of

\[ \Phi^{ij} \]

is either zero (no cosmological constant case) or constant (the cosmological constant), and is thus not an unknown field. The system of 18 equations is thus that for 13 components of

\[ B^i \]

and the remaining 5 components of

\[ \Phi^{ij} \]

.

In the general case the system of equations \cite{21} can be completed by considering analogs of “Bianchi” identities. Thus, we note that the components of

\[ M^{ij}, N^{ij} \]

in \cite{20} are not independent. Indeed, we have the following Bianchi identity:

\[ D_{A_B}F(A_B) = 0. \]

This gives:

\[ (D_{A_B}M^{ij}) \wedge B^j + (D_{A_B}N^{ij}\overline{B}^j) = 0. \]
Another important identity is obtained by using the compatibility equations $D_{A^i} B^j = 0$. Taking another covariant derivative and using the definition of the curvature we get:

$$\epsilon^{ijk} F^j (A_B) \wedge B^k = 0 \iff \epsilon^{ijk} M^{jl} B^l \wedge B^k = 0. \quad (24)$$

This last equation can be conveniently interpreted as follows. Let us define a conformal “internal” metric:

$$B^i \wedge B^j \sim h^{ij}. \quad (25)$$

Then (24) can be rewritten as:

$$\epsilon^{ijk} M^{jl} h^{lk} = 0. \quad (26)$$

Let us now also introduce an action principle that leads to (21) as Euler-Lagrange equations. This is easy to write, we have:

$$S[B, A, \Phi] = \int B^i \wedge F^i (A) - \frac{1}{2} \Phi_{ij} B^i \wedge B^j. \quad (27)$$

Varying this with respect to $A^i$ we get $D_A B^i = 0$, which allows to solve for $A$ in terms of $B$, varying the action with respect to $B^i$ we get (21). We also note that only the symmetric part of the field $\Phi^{ij}$ enters the action, so it is necessary to assume that $\Phi^{ij}$ in (21) is symmetric.

It remains to clarify the meaning of the variation with respect to $\Phi^{ij}$. To these end we shall use the Bianchi identities (23), (24). Using (23) and field equations (21) we see that we must have:

$$D_A B^i \Phi^{ij} \wedge B^j = 0. \quad (28)$$

Let us multiply this equation by the one-form $\iota_\xi B^i$ and sum over $i$. Here $\xi$ is an arbitrary vector field and $\iota_\xi B^i$ is one-form with components $(\iota_\xi B^i)_\mu := \xi^\alpha B_{\alpha \mu}$. However, for any vector field $\xi$ we have:

$$\iota_\xi B^i \wedge B^j = \frac{1}{2} \iota_\xi (B^i \wedge B^j) \sim h^{ij}, \quad (29)$$

where $h^{ij}$ is the internal metric introduced above. This gives us the following equation:

$$h^{ij} D_A \Phi^{ij} = 0. \quad (30)$$

Now, using the symmetry of $h^{ij}$ we can rewrite this equation as $h^{ij} (d\Phi^{ij} + 2 \epsilon^{ikl} A^k \Phi^{lj}) = 0$. However, the other Bianchi identity (24) together with the field equation $M^{ij} = \Phi^{ij}$ implies $\epsilon^{klm} \Phi^{li} h^{jm} = 0$ and so we must have:

$$h^{ij} d\Phi^{ij} = 0. \quad (31)$$

The identity (31) implies that the quantities $h^{ij}$ and $\Phi^{ij}$ are not independent. This can be seen quite clearly by considering the last term in the action (27). Using (29) and field equations (21) we see that we must have:

$$h^{ij} d\Phi^{ij} = 0. \quad (32)$$

The identity (31) implies that the quantities $h^{ij}$ and $\Phi^{ij}$ are not independent. This can be seen quite clearly by considering the last term in the action (27). Using (29) and field equations (21) we see that we must have:

$$h^{ij} d\Phi^{ij} = 0. \quad (33)$$

Below we shall characterize the “potential” $V(h)$ in more details. For now let us note that having expressed the unknown functions $\Phi^{ij}$ in terms of the the components of the two-form field $B^i$ we have closed the system of equations (21), as it is now a system of 18 equations for 18 unknowns - components of the $B^i$ field.

To understand the structure of the potential $V(h)$ it is convenient to parametrize the “internal” metric $h^{ij}$ by its trace and the traceless part:

$$h^{ij} = \frac{1}{3} \text{Tr}(h) \left( \delta^{ij} + H^{ij} \right), \quad (34)$$
where $H^{ij}$ is tracefree. It is then easy to see that for any function $f(h^{ij}) = f(\text{Tr}(h), H^{ij})$

$$\frac{\partial f}{\partial h^{ij}} = \left( \frac{\partial f}{\partial \text{Tr}(h)} - \frac{\partial f}{\partial H^{kl}} \frac{H^{kl}}{\text{Tr}(h)} \right) \delta^{ij} + 3 \frac{\partial f}{\text{Tr}(h) \partial H^{ij}}. \quad (35)$$

In particular, we have

$$\frac{\partial f}{\partial h^{ij}} h^{ij} = \frac{\partial f}{\partial \text{Tr}(h)} \text{Tr}(h). \quad (36)$$

Thus, one has:

$$V = \Phi^{ij} h^{ij} = \text{Tr}(h) \frac{\partial V}{\partial \text{Tr}(h)}, \quad (37)$$

where $\Phi^{ij}$ is given by (33). Thus, we learn that the potential must be a homogeneous function of order one in its argument $\text{Tr}(h)$:

$$V(h) = \Lambda \frac{\text{Tr}(h)}{3} U(H), \quad (38)$$

where $U$ is a dimensionless function that only depends on the tracefree part $H^{ij}$ of $h^{ij}$, and is normalized so that $U(0) = 0$, that is:

$$U(H) = 1 + \frac{\alpha}{2} \text{Tr}(H^2) + O(H^3), \quad (39)$$

where $\alpha$ is some dimensionless parameter. The quantity $\Lambda$ is a constant of dimensions $1/L^2$ that needs to be introduced to give $V(h)$ the correct dimensions. Below it will be identified with the cosmological constant. Using our definition of the “internal” metric $h^{ij}$ we can write:

$$B^i \wedge B^j = \frac{h^{ij}}{\text{Tr}(h)} (B^k \wedge B^k), \quad (40)$$

which of course defines $h^{ij}$ only up to a conformal factor. One can now write down the action (27) as a functional of only the two-form and the connection fields:

$$S[B, A] = \int B^i \wedge F^i(A) - \Lambda \frac{U(H)}{6} B^i \wedge B^i, \quad (41)$$

where $H^{ij}$ is defined as the traceless part of the internal metric $h^{ij}$, and is independent of the conformal freedom present in the definition of $h^{ij}$. Note that this action is an off-shell one, that is it can be varied with respect to the dynamical fields $B^i, A^i$ to obtain field equations. Alternatively, one can work with a version that uses extra “Lagrange multiplier” fields, see below.

Note that the dimensionfull constant $\Lambda$ here should be identified with the cosmological constant of our theory. Indeed, one can, e.g., consider the metric describing a homogeneous isotropic Universe. In such a Universe $H = 0$ by symmetries, and so the metric evolves exactly like in general relativity with the cosmological constant $\Lambda$. In other words, a solution of our theory describing a homogeneous isotropic Universe is the same as in GR with cosmological constant $\Lambda$. Thus, to define the modified gravity theory in question one only needs to specify a dimensionless function $U(H)$ of a dimensionless traceless tensor $H^{ij}$. All physical dimensionfull parameters present in the theory are as in general relativity. Let us remark that we could have included $\Lambda$ into the definition of the potential $U(H)$ and thus made it dimensionfull. However, as we shall see below from matter coupling considerations, it is more convenient to make the potential function dimensionless, for one can then use similar potential functions in both the gravity and the matter sectors.

In terms of the introduced potential $U(H)$ the main set (21) of field equations becomes:

$$F^i(A_B) = \Lambda \left( \frac{\partial U}{\partial H^{ij}} + \frac{1}{3} \delta^{ij} \tilde{U} \right) B^i, \quad (42)$$

where we have introduced the Legendre transform $\tilde{U}$ of the potential $U$:

$$\tilde{U} := U - \frac{\partial U}{\partial H^{kl}} H^{kl}. \quad (43)$$
The function $\tilde{U}$ can be viewed as either that of $H^{ij}$ or of the quantities:

$$\Psi^{ij} / \Lambda := \frac{\partial U}{\partial H^{ij}},$$

(44)

The theory with an arbitrary “cosmological function” $\Lambda(\Psi) := \Lambda \tilde{U}(\Psi / \Lambda)$ defined by the Lagrangian

$$S[B,A] = \int B^i \wedge F^i(A) - \frac{1}{2} \left( \Psi^{ij} - \frac{1}{3} \Lambda(\Psi) \delta^{ij} \right) B^i \wedge B^j,$$

(45)

is that of the original paper [2]. Field equations (41) are most easily obtained precisely in this “Lagrange multiplier” formulation. However, the viewpoint suggested by (41), namely that of the gravity theory being the BF theory (the first term in (41)) plus a potential term for the $H^{ij}$ components of the two-form field will be more convenient for our purposes here.

To summarize, we have seen that the condition $B^i \wedge B^j \sim \delta^{ij}$ of Plebanski formulation of GR can be relaxed and how the Bianchi identities still lead (in a unique way) to a consistent theory. Note that what one obtains is a class of gravity theories rather than one theory, for a theory is now specified by a choice of the dimensionless “potential” function $U(H)$ of the components $H^{ij}$ of the two-form field $B^i$. The potential can be completely arbitrary. One can obtain back general relativity (in Plebanski formulation) by making the potential function $U(H)$ infinitely steep so that the quantities $H^{ij}$ are set to zero. However, if one sets $U(H) = 1 \; \text{one obtains a topological theory} - \; \text{the so-called BF theory with a cosmological constant.}$

Having achieved a formulation of the theory in vacuum it is very important to continue to develop the theory and allow for a non-trivial right hand side of our equations - for matter to be present. Indeed, pure gravity is only of academic interest, and the real world gravity is both produced and felt by material bodies.

**IV. MODIFIED GRAVITY: NON-VACUUM CASE**

The vacuum field equations (21) of modified gravity were exactly the same as those (3) of the vacuum Plebanski theory. As in Plebanski case, it is natural to describe the effect of matter on the modified gravity “geometry” by allowing a non-zero $T^i$ to be present on the right hand side of (21). This is in the spirit of Einstein equations, where the stress-energy-momentum of matter appears on the right hand side of an equation for the curvature and thus affects the geometry. This in turn implies that the stress-energy-momentum of matter should satisfy some conservation laws which to a large effect determine its given in a certain background.

Thus, we shall keep the field equations (3) as our main dynamical equations even in the case of non-zero $T^i$. However, now that we have removed the condition $B^i \wedge B^j \sim \delta^{ij}$ it no longer consistent to impose the condition $T^i \wedge B^j \sim \delta^{ij}$ either. In general, the matter “stress-energy-momentum” two-form $T^i$ will have all components:

$$T^i = \frac{1}{2} Q^{ij} B^j - \frac{1}{2} T^{ij} \tilde{B}^j,$$

(46)

where the interpretation of components $T^{ij}$ is similar to that in GR, see (17), and $Q^{ij}$ are some new “internal” components of $T^i$. We note that, in the case $Q^{ij} \sim \delta^{ij}$, the trace $T$ of the usual GR stress-energy-momentum tensor is just a multiple of the trace $\text{Tr}(Q)$. The quantity $Q^{ij}$, by analogy with $T^{ij}$ can be referred to as the “internal” stress-momentum of matter. Let us decompose $Q^{ij}$ into its symmetric and anti-symmetric parts:

$$Q^{ij} = \chi^{ij} + i \epsilon^{ijk} \xi^k.$$

(47)

The quantities $\chi^{ij}, \xi^i$ then receive the interpretation of “internal” stress and momentum correspondingly. However, at this stage, there are no reasons to require $\chi^{ij}, \xi^i$ to be real, while the similar quantities in the “spacetime” stress-momentum $T^{ij}$ are observable quantities and thus must be real. We note that the various pieces in the decomposition (46) have the interpretations of stress and momentum (internal and “spacetime” ones), but there is no energy density anymore, the later appearing as the combination of the traces of the internal and spacetime stress tensors. For this reason, and also for brevity, we shall refer to $T^i$ as the stress-momentum two-form from now on.

Let us now rewrite our field equations as relations between the curvature and stress-momentum components:

$$F^i(A_B) = \Phi^{ij} B^j + 4 \pi G T^i \iff M^{ij} = \Phi^{ij} + 2 \pi G Q^{ij}, \; N^{ij} = -2 \pi G T^{ij},$$

(48)

where as before the “gravitational” quantity $\Phi^{ij}$ is at this stage arbitrary and is to be determined via the help of Bianchi identities. In the field equations (48) the quantity $G$ is the usual Newton’s constant. Note then that the
quantities $Q^{ij}, T^{ij}$ must have the dimensions of energy density, so the above interpretation of the components of $Q^{ij}$ as the “internal” stress-momentum is consistent with their dimensions.

Now, let us, as before, construct an action that leads to (48). We have:

$$S[B, A, \Phi, \phi_m] = \int B^i \wedge F^i(A) - \frac{1}{2} \Phi^{ij} B^i \wedge B^j - 4\pi G S_m[B, \phi_m], \quad (49)$$

where $\phi_m$ is a collective notation for all the matter fields, and $S_m[B, \phi_m]$ stands for the matter part of the action, which is assumed to depend on the “gravitational” background only via the two-form field $B^i$. In principle, one can also envisage the possibility of the matter fields (e.g. fermions) coupling directly to the connection $A^i$, but this does not bring anything conceptually new, only complicates the analysis, so we shall not consider this possibility any further. Defining:

$$T^i := \frac{\delta S_m}{\delta B^i}, \quad (50)$$

we get the equation (48) when varying the action with respect to the two-form field $B^i$, and the compatibility equations $D_A B^i = 0$ when varying the action with respect to the connection.

The next step in interpreting the above theory is to note that the stress-energy-momentum two-form $T^i$ satisfies some conservation laws. Indeed, since the matter part $S_m$ of the action must be diffeomorphism invariant, the following identity must hold:

$$0 = \delta \xi S_m = \int \frac{\delta S_m}{\delta \phi_m} \delta \xi \phi_m + \int T^i \wedge \delta \xi B^i, \quad (51)$$

where $\delta \xi$ is a variation of fields under an infinitesimal diffeomorphism generated by a vector field $\xi$. The first term in (51) vanishes by matter equations of motion, while the other term gives:

$$0 = \int T^i \wedge D_A \iota_\xi B^i, \quad (52)$$

where $\iota_\xi$ denotes contraction of a form with a vector field, so e.g. $\iota_\xi B^i$ is a one-form with components $(\iota_\xi B^i)_\mu := \xi^\alpha B^i_{\alpha \mu}$. This expression follows from the following formula for the action of diffeomorphisms on $B^i$:

$$\mathcal{L}_\xi B^i = \iota_\xi D_A B^i + D_A \iota_\xi B^i, \quad (53)$$

where $A$ can be taken to be arbitrary, and the fact that $D_A B^i = 0$ for $A = A_B$. Now integrating (52) by parts and taking into account that $\xi$ may be of compact support, we can conclude that the integrand must vanish:

$$D_A \iota_\xi T^i \wedge B^i = 0 \quad (54)$$

This should hold for any vector field $\xi$, so we get four “conservation” equations.

Another important identity that we can obtain for $T^i$ follows from gauge invariance of the action. Thus, we similarly write (51) but now consider the variations of the fields under a gauge transformation. For the $B^i$ field this is:

$$\delta \omega B^i = \omega^{ij} B^j, \quad (55)$$

where $\omega^{ij}$ is an infinitesimal anti-symmetric matrix - a Lie-algebra element of SO(3). We can therefore conclude that

$$T^i \wedge \omega^{ij} B^j = 0 \quad (56)$$

for any matrix $\omega$ and thus:

$$T^i \wedge B^j = 0 \quad (57)$$

In the language of the decomposition (46) this translates into:

$$Q^{[i} h^{j]} = 0 \iff \epsilon^{ijk} Q^{j} h^{lk} = 0. \quad (58)$$

This identity is satisfied by any $T^i$ that follows via (50) from a gauge-invariant action, and will be of importance below.
We can now derive a Bianchi identity for the gravitational quantities \( \Phi^{ij} \). Let us rewrite (23) as:

\[
D_{AB} \Phi^{ij} \wedge B^j + 4\pi G D_{AB} T^i = 0.
\]  

(59)

Let us now take the wedge product of this expression with the one-form \( \iota_\xi B^i \). Using (54) we see that the field \( \Phi^{ij} \) must satisfy:

\[
\iota_\xi B^i \wedge D_{AB} \Phi^{ij} \wedge B^j \sim h^{ij} D_{AB} \Phi^{ij} = 0,
\]

(60)

where we have used (29). We can now write down all the terms in the expression for the covariant derivative \( D_{AB} \Phi^{ij} \).

Similarly to what we had in the pure gravity case, there is a term containing the connection \( A^i_B \) and proportional to \( \varepsilon^{ijk} \Phi^{ijl} h^{lk} \). However, we can again conclude that this term is zero. Indeed, the second Bianchi identity (24) together with field equations (48) says:

\[
\varepsilon^{ijk} (\Phi^{jl} + 2\pi G Q^{jl}) h^{lk} = 0.
\]

(61)

However, we have seen above that the invariance of the material action under gauge transformations implies (48), and thus the second term here is zero, which implies that the first term is zero as well. Thus, from (60) we conclude that:

\[
h^{ij} d\Phi^{ij} = 0,
\]

(62)

which is exactly what we had in the pure gravity case. All the remaining steps from the previous Section go unchanged: we arrive at conclusion that the potential term in the action proportional to \( V = h^{ij} \Phi^{ij} \) is a homogeneous function of order one in \( \text{Tr}(h) \) times the cosmological constant \( \Lambda \) times an arbitrary dimensionless function \( U(H) \). Thus, the full theory is obtained as simply the gravitational plus matter parts:

\[
S[B, A, \phi_m] = \int B^i \wedge F^i(A) - \frac{\Lambda U(H)}{6} B^i \wedge B^i - 4\pi G S_m[B, \phi_m].
\]

(63)

This solves the problem of coupling of the class of generalized gravity theories we have been considering to generalized matter. It only remains to supplement the matter part of the action with some appropriate reality conditions, for it is in general complex. A way to do this is to require the components of \( \Phi \) to be “directly” observable to be real. However, to be able to do physics with our gravity theory we need to understand how material actions \( S_m[B, \phi_m] \) can be formed and which stress-momentum two-forms \( T^i \) can arise in our non-metric theory.

V. STRESS-MOMENTUM TWO-FORM OF A “SMALL” BODY

In the previous Section we have seen that the matter stress-momentum two-form \( T^i \) satisfies the following “conservation” equations:

\[
\iota_\xi B^i \wedge D_{AB} T^i = 0, \quad T^i \wedge B^j = 0.
\]

(64)

In GR the stress-momentum has the special form (15), and the second of these equations is automatically satisfied, while the first gives the usual conservation of energy equation when the two-form field \( B^i \) is metric \( B^i \wedge B^j \sim \delta_{ij} \). The conservation equation can then be used to conclude that “small bodies” move in GR along geodesics. A very important question for the theory just developed is what the notion of geodesic generalizes to in the case of an arbitrary two-form field background. To understand this we shall employ the same methods as are used in GR.

A particularly efficient method that allows to study this question has appeared recently in a paper by Gralla and Wald [1]. This work employs the machinery of asymptotic expansions to derive results on motion of “small” bodies in GR both in the leading approximation, which gives the result that bodies move along geodesics, as well as in the sub-leading one, which leads to results on gravitational self-force. For our purposes we only need the analog of the first of these. Thus, most of the machinery developed in this work is actually unnecessary here. However, some key ideas of Section IV of this paper will still be used.

As in [1], the first step is to derive, using the asymptotic expansion techniques, that the stress-energy-momentum tensor has a well-defined limit approaching a distribution. To this end, consider a family \( B^i(\lambda, x^\alpha) \) of two-form field backgrounds. Here \( \lambda \geq 0 \) parametrizes members of the family and \( x^\alpha \) are coordinates of some convenient coordinate system. We shall assume there exist coordinates such that \( B^i(\lambda, x^\alpha) \) are smooth in both \( \lambda \) and \( x^\alpha \) at least sufficiently far away \( r > \bar{R}\lambda, r^2 = \sum_{i=1}^{3}(x^i)^2 \) from the particle, where \( \bar{R} \) is some universal constant, and that for all \( \lambda, r > \bar{R}\lambda \)
the two-form $B^i(\lambda, x^\alpha)$ is a solution of vacuum field equations of our theory. Let us consider the following expansion for the two-form field as $\lambda \to 0$:

$$B^i(\lambda) = B^i + \lambda b^i + O(\lambda^2),$$

(65)

where for now both the background $B^i$ and the “perturbation” $b^i$ are functions of all coordinates $x^\alpha$, the background $B^i$ satisfies the vacuum field equations $F^i(A_B) = \Phi^{ij}B^j$, and the “perturbation” satisfies the linearized field equation, at least sufficiently far from the particle $r > \epsilon$.

Now given a background $B^i$ satisfying the field equations, let us define an operator $G^i_{\mu\nu}(b)$ via:

$$F^i(A_{B+\lambda b}) - \Phi^{ij}(B^j + \lambda b^j) = \lambda G^i(b) + O(\lambda^2).$$

(66)

In terms of $G^i$ the linearized field equations read $G^i(b) = 0$.

By definition, the stress-momentum two-form is defined as a distribution on spacetime whose action on an arbitrary smooth Lie-algebra valued anti-symmetric tensor $f^i_{\mu\nu}$ is given by integrating the right-hand-side of linearized field equations, i.e., $G^i(b)$ against $f^i$. Or, using the fact that $G^i$ is self-adjoint, we can define:

$$4\pi GT(f) = \int_M G^i(f) \wedge b^i,$$

(67)

where $b^i$ is as in (65).

The definition of the notion of self-adjoint that is used here is as follows. It can be shown that for arbitrary two-forms $b^i, f^i$ the four-form:

$$b^i \wedge G^i(f) - G^i(b) \wedge f^i = dX(b, f)$$

(68)

is a total derivative, where $X(b, f)$ is a certain 3-form depending on both $b$ and $f$, as well as the background $B^i$. We shall not attempt demonstrate this property in the present paper, as the computation is quite technical and it would take us too far.

If we now integrate the expression (68) over a region $r > \epsilon$ and use the fact that $b^i$ satisfies the linearized field equations $G^i(b) = 0, r > \epsilon$. We get:

$$\int_{r>\epsilon} b^i \wedge G^i(f) = \int_{r=\epsilon} X(b, f).$$

(69)

Taking the limit $\epsilon \to 0$ we get:

$$T(f) = \frac{1}{4\pi G} \lim_{\epsilon \to 0} \int_{r=\epsilon} X(b, f).$$

(70)

Similar to what happens in the GR case, it can be seen that this limit exists and is different from zero if $b^i \sim 1/r$ as $r \to 0$. Thus, when this is the case, the stress-momentum distribution can be written as:

$$T(f) = \int dt \epsilon^{\mu\nu\rho\sigma} T^i_{\mu\nu}(t) f^i_{\rho\sigma}(t).$$

(71)

Here $T^i(t)$ is a Lie-algebra valued two-form along the curve $\gamma$ (given in the chosen coordinate system by $x^i = 0$), and $f^i(t) := f^i(t, r = 0)$ is the value of the test two-form $f^i$ along the curve $\gamma$. We note that to write this expression we have chosen a particular background metric (see below on how this is done). Also note, for future use, that under conformal transformations of the background metric $dt \to \Omega dt, \epsilon^{\mu\nu\rho\sigma} \to \Omega^{-4} \epsilon^{\mu\nu\rho\sigma}$, and so for the above distribution to be invariant under conformal transformations of the auxiliary background metric the stress-momentum two-form must transform as $T^i_{\mu\nu}(t) \to \Omega^i T^i_{\mu\nu}(t)$. We shall use this fact in the next Section when we check the behaviour of the evolution equations under conformal transformations.

Now, using the Bianchi identity that holds in our theory, one concludes that the above distribution must vanish on test two-forms $f^i$ of the form:

$$D_A B^i,$$

(72)

where $\xi$ is an arbitrary vector field and $B^i$ is the background two-form field appearing in (65). As in [1], to derive consequences of the arising “conservation” equations, we shall first consider the case of special vector fields of the form:

$$\xi^\mu = x^\alpha F(x^1, x^2, x^3) c^\mu(t), \quad i = 1, 2, 3,$$

(73)
where $F(x^1, x^2, x^3)$ is an arbitrary function such that $F(r = 0) = 1$. As we have already mentioned the coordinates $x^i$ are chosen in such a way that the curve $\gamma$ along which the body is moving (i.e. in the neighbourhood of which vacuum field equations are assumed) corresponds to $r = 0$, where as usual $r^2 = \sum_{i=1}^3 (x^i)^2$. This still, however, leaves a considerable freedom in the choice of the coordinates. Let us use the background two-form field $B^i$ to help with this. Thus, recall that $B^i$ defines a conformal metric. Since any metric is locally flat, as in [1], we can always choose the coordinates locally so that this conformal metric is just the Minkowski metric. This means that, without loss of generality we can assume the two-form field $B^i$ in the small neighbourhood of $\gamma$ to be given by:

$$B^i = \Lambda^{i\underline{2}}B^{\underline{2}},$$  \hspace{1cm} (74)

where the two-forms $B^{\underline{2}}$ are those describing the Minkowski spacetime:

$$B^{\underline{2}} = idt \wedge dx^\underline{2} - \frac{1}{2}c^{ij\underline{2}}dx^i \wedge dx^j,$$  \hspace{1cm} (75)

and $\Lambda^{i\underline{2}}$ are arbitrary matrix-valued functions of spacetime coordinates. Note that we have introduced a new type of indices - underlined ones, to distinguish between the "internal" $SO(3)$ bundle where the original fields take values and the "metric" bundle where the metric two-form field $T\Lambda$ lives. The covariant derivative $D_A$ only acts on the original non-underlined indices. The matrix $\Lambda^{i\underline{2}}$ is defined modulo conformal rescalings of the metric introduced, which sends $B^{\underline{2}}_m \rightarrow \Omega^2 B^{\underline{2}}_m$, and, since the background two-form field is by itself independent of any choice of the metric, transforms $\Lambda^{i\underline{2}} \rightarrow \Omega^{-2}\Lambda^{i\underline{2}}$. One can also do a Lorentz rotation on the metric two-forms (75) that acts on the underlined indices. Thus, the quantities $\Lambda^{i\underline{2}}$ are only defined modulo such conformal rescalings and $SO(3)$ rotations.

Now using the special vector fields (73) in the test two-form (72), and taking into account that only the term in which the exterior derivative acts on the coordinate functions $x^i$ gives a non-zero contribution in the limit $r \rightarrow 0$, we get that the stress-momentum distribution must vanish on the following set of test two-forms:

$$\Lambda^{i\underline{2}}(t)dx^\underline{2} \wedge (ic^0(t)dx^\underline{2} - ic^{\underline{2}}(t)dt + e^{\underline{2}ik}c^{\underline{2}k}(t)dx^\underline{k}),$$  \hspace{1cm} (76)

for any choice of $\underline{l}$ and functions $c^0(t), c^{\underline{2}}(t)$. This gives us $3 \times 4$ conditions on $T^i(t)$ that we would like to exploit to deduce the form of the stress-momentum. Here and in what follows the notation $f(t)$ stands for the value of the function $f$ along the curve $r = 0$. Thus, $f(t) := f(t, r = 0)$.

It is now convenient to consider a related quantity: $T^{\underline{2}}(t) := T^i(t)\Lambda^{i\underline{2}}(t)$. The equation in question then becomes:

$$T^{\underline{2}}(t) \wedge dx^\underline{2} \wedge (ic^0(t)dx^\underline{2} - ic^{\underline{2}}(t)dt - e^{\underline{2}ik}c^{\underline{2}k}(t)dx^\underline{k}) = 0.$$  \hspace{1cm} (77)

Let us decompose:

$$T^{\underline{2}}(t) = iA^{\underline{2}l}(t)dt \wedge dx^\underline{2} - \frac{1}{2}B^{\underline{2}l}(t)e^{\underline{2}ik}c^{\underline{2}k}(t)dx^\underline{k} \wedge dx^\underline{l},$$  \hspace{1cm} (78)

where $A^{\underline{2}l}(t), B^{\underline{2}l}(t)$ are some unknown matrix-valued functions of time. Setting $c^{\underline{2}}(t) = 0$ and thus extracting the $c^0$ component of the conservation equations we immediately get:

$$A^{\underline{2}l} = 0.$$  \hspace{1cm} (79)

Setting $c^0(t) = 0$ and $c^{\underline{2}}(t) \sim \delta^{\underline{lm}}$ we get, after some algebra:

$$B^{\underline{2}l} + \text{Tr}(A)\delta^{\underline{lm}} - A^{\underline{2}m} = 0,$$  \hspace{1cm} (80)

where we have suppressed the dependence on $t$ for brevity. From this equation we immediately conclude that $B^{\underline{2}l}$ is also a symmetric matrix, and that $\text{Tr}(B) = -2\text{Tr}(A)$, while the traceless parts of $A^{\underline{2}l}, B^{\underline{2}l}$ are equal. Let us denote these traceless parts by $\chi^{\underline{2}l}(t)$, and (a multiple of) the trace of say $B^{\underline{2}l}$ by $m(t)$. Then we obtain the following form of the stress-momentum distribution:

$$T^{\underline{2}}(t) = \frac{1}{6}m(t)B^{\underline{2}} - \frac{1}{2}m(t)\bar{B}^{\underline{2}} + \chi^{\underline{2}l}(t)B^{\underline{2}l},$$  \hspace{1cm} (81)

where we have used the definition (75) of the Minkowski space two-forms, and $\bar{B}^{\underline{2}} := -(B^{\underline{2}})^*$ is the anti-self-dual two-forms. It is instructive to compare this result to the GR one. In that case $\Lambda^{i\underline{2}} = \delta^{i\underline{2}}$, and no non-trivial self-dual part of the stress-momentum tensor is possible, so $\chi^{\underline{2}l} = 0$. The remaining two-form is that corresponding to the ideal pressureless fluid, see [19], with coordinates chosen such that the momentum $u^i = 0$, as it should. We have thus
recovered the GR result, formula (45) of [1]. We see that the main modification arising in our case is the presence of an arbitrary traceless part \( \chi^{ij} \) in the self-dual part of the stress-momentum two-form.

The second “conservation” equation in (64) can also be exploited. After some simple algebra we find that along the curve \( r = 0 \) it is equivalent to the condition:

\[
e^{ijk}(\Lambda^{-1})^{ij}(t)\chi^{k}(t)\Lambda^{j}(t) = 0.
\]

Let us also note the transformation properties of the quantities that appeared in (81). Since \( T^{i} \to \Omega^{3}T^{i} \), we have \( m(t) \to \Omega^{-1}m(t), \chi^{ij}(t) \to \Omega^{-1}\chi^{ij}(t), \) which are the correct transformation properties for the quantities having the dimensions of mass.

VI. MOTION OF A “SMALL BODY”

Having extracted the form (81) of the stress-momentum two-form \( T^{i} \) of a “small body” we are ready to find equations that such a body must satisfy during its motion. Thus, we are looking for an analog of the GR statement that “small bodies” move along geodesics. To this end we once again use the fact that the stress-momentum distribution, whose form (81) we have determined above, must vanish on test two-forms of the form \( D_{A_{i}B_{j}}\xi B^{j} \), where \( \xi \) is an arbitrary vector field.

The computation one has to do is conceptually clear, but a bit involved. A particularly efficient way to do it is to use spinors. However, the resulting intermediate formulae are not particularly transparent for somebody not familiar with spinor techniques. For this reason we shall use a shortcut based on the fact that changes to the final result only come from the self-dual part of the stress-momentum two-form, and the contribution of the anti-self-dual sector is completely unchanged from the GR case. This has to be verified, and, as we have said, the easiest way to do this is to use spinors. We give a complete derivation in the Appendix. Here we only deal with (the most interesting) self-dual part, which can be easily done without spinors.

Thus, we decompose the stress-energy distribution two-form \( T^{i} \) into its self- and anti-self-dual parts and write:

\[
\int dt \left( T^{i}_{sd}(t) + T^{i}_{asd}(t) \right) \wedge D_{A_{i}B_{j}}\xi B^{j} = 0.
\]

This must hold for any vector field \( \xi \).

Let us analyze the \( T^{i}_{sd}(t) \) term of (83). In the previous subsection we have found that this part of \( T^{i} \) can be written as:

\[
T^{i}_{sd}(t) = \frac{1}{2}Q^{ij}B^{j},
\]

where \( B^{j} \) is the background two-form field and \( Q^{ij} \) is a tensor given by:

\[
Q^{ij} = (\Lambda^{-1})^{ij}(t) \left( \frac{1}{3}m(t)\delta^{ij} + \chi^{ij}(t) \right) (\Lambda^{-1})^{jk},
\]

where \( \chi^{ij}(t) \) is as introduced above in (81) and the matrix \( (\Lambda^{-1})^{jk} \) on the far right is not evaluated at \( r = 0 \). We will only be interested in this tensor along the curve \( r = 0 \). Then the quantity \( Q^{ij}(t) \) is symmetric \( Q^{ij}(t) = 0 \) because \( \chi^{ij}(t) \) is symmetric. We also get the following simple expression for the “mass” \( m(t) \):

\[
h^{ij}(t)Q^{ij}(t) = m(t),
\]

where \( h^{ij} \) is the “internal” metric now defined as:

\[
h^{ij} = (\Lambda^{-1})^{ij}.\]

We also see that the second “conservation” equation (82) becomes in terms of \( Q^{ij} \):

\[
e^{ijk}Q^{ij}(t)h^{jk}(t) = 0.
\]

Now using the fact that \( D_{A_{i}B_{j}}B^{j} = 0 \) we can take the two-forms \( B^{j} \) in (81) under the operator of covariant derivative. We then use the identity (29) to get for this first term:

\[
\frac{1}{4} \int dt Q^{ij}(t)D_{A_{i}B_{j}}(B^{j} \wedge B^{j}) \sim \int dt \left( \frac{1}{4}Q^{ij}(t)(\xi^{a}D_{a}h^{ij})(t) + \frac{m(t)}{4}(\nabla_{a}\xi^{a})(t) \right),
\]
where we have omitted unimportant numerical factors and used the background Minkowski metric $\mathbf{g}$ to write the result. As usual, the notation $f(t)$ stands for $f(t,r = 0)$ for any function on spacetime. In $\mathbf{89}$, the quantity $D_a$ stands for the components of the derivative operator that acts on spacetime indices as the metric compatible one and on the internal indices as the covariant derivative $D_{AB}$. To write the last term in $\mathbf{89}$ we have used the relation $\mathbf{80}$. Further, the covariant derivative $D$ was replaced in this term by the usual metric compatible one because there are no internal indices to act on. A detailed derivation of $\mathbf{89}$ is given in the Appendix. Now using the property $\mathbf{88}$ we see that the derivative operator $\nabla_a h^{ij}$ one first evaluates the derivative and only then evaluates the result at $r = 0$.

Let us now treat the second, $\mathbf{T}_{a\phi}$ term in $\mathbf{89}$. As we have already mentioned, this term is exactly the same as it is in GR. This is demonstrated in detail in the Appendix. So, we have for it:

$$\int dt \left( m(t) u_a u_b - \frac{1}{4} m(t) g_{ab} \right) (\nabla^a \xi^b)(t).$$

(91)

Here $u^a$ is the vector tangent to $\gamma$, and the quantity in the brackets is simply the traceless part of the standard particle stress-energy tensor $T_{ab} = m u_a u_b$.

Now adding $\mathbf{90}$ and $\mathbf{91}$ we get:

$$\int dt \left( \frac{1}{4} Q^{ij}(t)(\xi^a \nabla_a h^{ij})(t) + (m(t) u_a)(u^b \nabla_b \xi^a)(t) \right) = 0,$$

(92)

where we have rewritten the second term in a suggestive form. Integrating in this second term by parts, and using the fact that $\xi^a$ is an arbitrary vector field we get:

$$u^b \nabla_b (m(t) u_a) = \frac{1}{4} Q^{ij}(t)(\nabla_a h^{ij})(t).$$

(93)

Equation $\mathbf{93}$ is our main evolution equation for “small bodies”. It is instructive to see how the GR case gets reproduced. In that case there exists a unique background metric such that $h^{ij} = \delta^{ij}$. The right-hand-side of $\mathbf{93}$ then vanishes and we get the usual $u^b \nabla_b (m(t) u_a) = 0$, which implies that $m(t)$ is constant along the worldline of the “small” body, and that this worldline is a geodesic.

As a check of our result $\mathbf{93}$ we must make sure that it is invariant under conformal transformations of the metric used two write it. Indeed, in the theory under study only the conformal class of metrics is well-defined, not the metric itself. To see that the equation $\mathbf{93}$ is conformally invariant we recall the transformation properties of all the quantities: $m \to \Omega^{-1} m, Q^{ij} \to \Omega^2 Q^{ij}, h^{ij} \to \Omega^{-1} h^{ij}$ and finally $u_a \to \Omega u_a$, the last one following from the normalization condition $g^{ab} u_a u_b = 1$. Thus, the quantity $m u_a$ is conformally invariant, and we only need to worry about the change of the metric-compatible derivative operator. For brevity we drop the argument indicating time dependence in all the formulae and get:

$$\nabla_b (m u_a) = \nabla_b (m u_a) - m (u_b \nabla_a \ln \Omega + u_a \nabla_b \ln \Omega - g_{ab} u^c \nabla_c \ln \Omega).$$

(94)

This means that the left hand side of $\mathbf{93}$ transforms as:

$$u^b \nabla_b (m u_a) \to \Omega^{-1} u^b \nabla_b (m u_a) - \Omega^{-1} m \nabla_a \ln \Omega.$$

(95)

On the other hand, the right-hand-side of $\mathbf{93}$ transforms as:

$$- \frac{1}{2} Q^{ij} \nabla_a h^{ij} \to - \frac{1}{2} \Omega^2 Q^{ij} \nabla_a \Omega^{-1} h^{ij} =$$

$$\Omega^{-1} Q^{ij} \nabla_a h^{ij} + 2 \Omega^{-1} Q^{ij} h^{ij} \nabla_a \ln \Omega = - \frac{1}{2} \Omega^{-1} Q^{ij} \nabla_a h^{ij} - \Omega^{-1} m \nabla_a \ln \Omega.$$

(96)

which shows that the equation is indeed conformally invariant.
VII. INTERPRETATION

In the previous Section we have obtained the “small body” evolution equation for our gravity theory. The only difference with the GR case stems from the fact that a “small body” is allowed to have a non-trivial “internal stress” tensor $Q^{ij}(t)$ which then interacts non-trivially with the “non-metric” part of the background.

From (93) we see that, because of the non-zero right-hand-side, the motion does not seem to be geodesic. Interestingly, when the background is metric, i.e. there exists a metric in which $h^{ij} = \delta^{ij}$, the evolution is geodesic even in case the body has non-trivial “internal stress” $Q^{ij}(t)$. Thus, it is only when the two-form field in which the body moves is “non-metric” that we get (apparent, see below) deviations from geodesic motion.

To give an interpretation to (93), let us multiply this equation by $u^a$. Due to the normalization condition satisfied by this vector field, we have $u^a\nabla_a u_a = 0$, and so:

$$u^a\nabla_a m(t) = \frac{1}{4} Q^{ij}(t)(u^a\nabla_a h^{ij})(t).$$

(97)

Thus, the conservation equation no longer implies that the “mass” of the body is constant along the trajectory, but tells us something different. To see what, we will need to make an additional assumption about the nature of the “internal” stress-momentum of our body. Thus, we shall assume that the tensor $Q^{ij}(t)$ that appears in (93), (97) is independent of the direction of motion of the body. That is, we assume that the self-dual part of the body’s stress-momentum two-form is $(1/2)Q^{ij}B^j$ with the same $Q^{ij}$ no matter along which trajectory the body travels. Note that in general relativity this is true, with the self-dual part of $T^i$ being given by $T_{sd}^{ij} = (m/6)B^j$. Our assumption may be motivated by considering how the self-dual part of $T^i$ arises from some matter action via (50). Indeed, $T_{sd}^{ij}$ arises from a term of the form $(1/2)Q^{ij}B^i \wedge B^j$, where $Q$ is some matrix possibly depending on the “non-metric” components of $B^i$. This suggests that the self-dual, or “internal” part of the stress-energy two-form should only depend on the internal composition of the particle, and not on its motion.

If one makes this well-motivated assumption, then (97) must hold for any choice of the vector $u^a$. Thus, the following equation must hold:

$$dm = \frac{1}{4} Q^{ij}dh^{ij}.$$ 

(98)

Let us now recall that we have seen a similar equation before, equation (62), in Section III that dealt with the vacuum modified gravity. There it implied that the tensor $\Phi^{ij}$ that arises in the decomposition of the curvature into its self- and anti-self-dual parts must be given by a derivative of some potential function with respect to the internal metric. We see that a similar relation must be true here:

$$Q^{ij}(t) = 4 \left( \frac{\partial m(h^{ij})}{\partial h^{ij}} \right)(t).$$

(99)

Thus, the evolution equation implies that for each body there must exist a function $m(h^{ij})$ of the internal metric, such that the tensor $Q^{ij}$ describing the self-dual part of the stress-momentum of this body is given by the partial derivatives of the mass function with respect to the components of the internal metric. Because of this, we shall no longer write the argument indicating the time dependence next to $m$, as we now interpret the mass as a function of $h^{ij}$, which later must be evaluated on $h^{ij}(t)$.

Let us now consider the “mass” function $m(h^{ij})$ to be a function of the trace $\text{Tr}(h)$ of the internal metric and the traceless part $H^{ij}$. Then, using (35), as well as the fact that $m(h) = \text{Tr}(Qh)$ we see that the function $m(\text{Tr}(h), H)$ is a homogeneous function of degree 1/4 in $\text{Tr}(h)$. Therefore, we can write:

$$m(h) = \tilde{m} \frac{(\text{Tr}(h))^{1/4}}{3} W(H),$$

(100)

where $W(H) = W(H^{ij})$ is a dimensionless function of the traceless part $H^{ij}$ of the “internal” metric $h^{ij}$ normalized as $W(0) = 1$, and $\tilde{m}$ is a quantity with the dimensions of mass. Note that the formula (100) is consistent with the transformation properties $m \to \Omega^{-1}m, h \to \Omega^{-4}h$ of the quantities under the conformal transformations of the background metric. It should also be compared with an analogous formula (35) on the gravity side.

Let us now see what the fact (99) implies for the motion of the body. We can now replace the right-hand-side in (93) by $\nabla_a m$ to get:

$$u^b\nabla_b m(t)u_a = (\nabla_a m)(t).$$

(101)
Recall that this equation is conformally invariant, with the mass function transforming as \( m \to \Omega^{-1} m \). Note also that the mass function is now defined not only along the trajectory, but everywhere, and we can use the conformal freedom in choosing the background metric to select a special metric in which \( m = \bar{m} \) is a constant. Then in this metric, whose conformal factor is defined by the condition

\[
\text{Tr}(h)(W(H))^4 = 3
\]

the body moves along spacetime geodesics. Note that when the background is metric \( H = 0 \) the mass \( \bar{m} \) is the usual mass of the particle as we know it in general relativity. So, similar to what we saw happening in the case of pure gravity, the departure from the familiar behaviour is parametrized by a single dimensionless function \( W(H) \) of the traceless part of the internal metric.

It remains to discuss an interpretation of the function \( W(H) \). This function is set by the coupling of the matter component in question to the two-form field. In principle, it can be arbitrary, and moreover different for different matter components. In previous studies, see e.g. [10], of the theory it was shown that the non-metricity component in question to the two-form field.}

The “quantum average” of the matter action must be an integral of a four-form that can only be built from the 4-forms \( B_i \wedge B^i \). However, these 4-forms are proportional to the internal metric \( h^{ij} \) and the volume form \( B^i \wedge B_i \). Thus, the four-form in question must be proportional to the volume form. Further, since all the mass terms in \( S_m[B, \phi_m] \) depend non-trivially on the components \( H^{ij} \) of \( B \), the quantum average will depend on the function \( W(H) \). Thus, we see that the quantum average in \([103]\) must be equal to:

\[
S_{\text{eff}}[B, A] = \int B^i \wedge F^i - \frac{\Lambda_{\text{eff}}}{6} U_{\text{eff}}(H)(B^i \wedge B^i),
\]

where \( \Lambda_{\text{eff}} \) is the effective cosmological constant, and \( U_{\text{eff}}(H) \) is some effective potential normalized so that \( U_{\text{eff}}(0) = 1 \). Here both \( \Lambda_{\text{eff}} \) and \( U_{\text{eff}}(H) \) depend on details of the matter Lagrangian and the form of the coupling \( W(H) \). What we have reproduced via this heuristic argument is precisely the gravitational action \([11]\) with the potential \( U_{\text{eff}}(H) \) and the cosmological constant \( \Lambda_{\text{eff}} \). This strongly suggests that the potential function \( U(H) \) appearing in gravity must be related to the mass function \( W(H) \) appearing in the matter sector. However, to find this relation one must perform a complicated quantum computation. In this paper we shall treat both functions as phenomenological, but one should keep in mind that it should be possible to relate them in the final theory.

**VIII. COUPLING TO THE GENERAL STRESS-ENERGY TENSOR**

Considerations of previous Sections fixed the form of the stress-momentum two-form of a “small body”. However, in order to be able to develop physical consequences of our gravity theory it is necessary to describe how arbitrary types of matter couple to it. Fortunately, the above considerations on the form of the stress-momentum two-form of a “small body” allow us to describe coupling to generic matter.
Since our theory respects the weak equivalence principle and there is a preferred metric in which test bodies move along geodesics and any metric is locally flat, it is natural to postulate, as in general relativity, that all non-gravitational physics in our theory is the same as in flat space. In particular, it is natural to assume that the effect of any matter on gravity is still characterized just by the usual stress-energy tensor of matter. Of course, the coupling of this stress-energy tensor to gravity may and will be different in the theory under study.

The question thus reduces to that of describing how the usual stress-energy tensor of matter couples to our gravity theory. To answer it, the following formalism will be useful. As we have already done in Section [V], given a general two-form field background $B^i$, it will be convenient to introduce a set of “metric” two-forms. To this end, let us choose a representative in the conformal class defined by $B^i$, and choose a tetrad $\theta^I$, $I = 0, 1, 2, 3$. From the tetrad one can construct the two forms $B^{ij}: = \theta^i \wedge \theta^j$ and take the self-dual part with respect to the indices $IJ$. Let us refer to the so(3)-valued two-forms obtained this way as metric. They are the two-forms of Plebanski formulation of general relativity reviewed in Section [II].

As before we denote these metric forms by a bold letter. Thus, we get a two-form field $B^\mathbf{I}$ satisfying the metricity condition: $B^\mathbf{I} \wedge B^\mathbf{J} \sim \delta^\mathbf{IJ}$, as well as the reality conditions, see (7). As before, we shall continue to use the underlined indices to refer to quantities taking values in the “metric” SO(3) bundle that we have introduced via $B^\mathbf{I}$. Now, given the metric forms $B^\mathbf{I}$ defining the same notion of self-duality on two-forms as $B^i$, the original two-form field $B^i$ can be represented as a linear combination of the metric ones:

$$B^i = \Lambda^{i\mathbf{I}} B^\mathbf{I}.$$

(106)

The quantities $\Lambda^{i\mathbf{I}}$ are defined up to SO(3) rotations and rescalings of the metric two-forms $B^\mathbf{I}$. Thus, the invariant information contained in them is that in $9-4 = 5$ components, and we can parametrize a general two-form background $B^i$ by its metric two-forms $B^\mathbf{I}$ and by the quantities $\Lambda^{i\mathbf{I}}$, modulo conformal and SO(3) transformations. We shall see that this parametrization is very convenient for practical computations. In particular, the internal metric $h^{ij}$ is given in terms of the matrices $\Lambda^{i\mathbf{I}}$ by:

$$h^{ij} = \Lambda^{i\mathbf{I}} \Lambda^{\mathbf{I}j}.$$

(107)

Let us now consider the stress-energy two-form. The usual stress-energy tensor $T_{ab}$ can be decomposed into its trace $\text{Tr}(T)$ and traceless $T_{ab} - (1/4)g_{ab}\text{Tr}(T)$ parts. As we have explained above, we would like the stress-energy two-form $T^i$ of our theory to be constructed from the quantities $\text{Tr}(T)$ and $T_{ab} - (1/4)g_{ab}\text{Tr}(T)$. It can be expected that the traceless part will enter into the anti-self-dual part of the stress-momentum two-form to be constructed, and the trace part will enter into the self-dual part. Our experience with the stress-momentum two-form of a small body suggests that the anti-self-dual part of $T^i$ is essentially unchanged, and is given by:

$$T^{i}_{asd} = (\Lambda^{-1})^{i\mathbf{I}} \mathbf{T}^{\mathbf{I}}_{asd},$$

(108)

where $\Lambda^{\mathbf{I}i}$ is the matrix introduced in [105], and $\mathbf{T}^{\mathbf{I}}_{asd}$ is the “metric” anti-self-dual stress-momentum two-form, see Section [II]. Indeed, we have seen in [51] that for a small body, the anti-self-dual tensor $\Lambda^{\mathbf{I}i}T^{i}_{asd}$ has the usual form of one in the metric theory. We assume that this generalizes to arbitrary matter, and later check that this choice is consistent with energy conservation. This solves the problem of coupling the traceless part of $T_{ab}$ to our gravity theory.

The coupling of the trace part $\text{Tr}(T)$ is also suggested by what happens in the “small body” case. Thus, we saw that for a “small body”

$$T_{sd}^i = \frac{1}{2} Q^{ij} B^j,$$

(109)

and that the internal stress-momentum tensor $Q^{ij}$ is given by a derivative of a “mass function” [100] with respect to the internal metric. We shall keep the same relation in the general case and write:

$$Q^{ij} = \text{Tr}(T) \frac{\partial R_m(h)}{\partial h^{ij}},$$

(110)

where $\text{Tr}(T)$ is the trace of the stress-energy tensor (with dimensions of energy density), and the dimensionless function $R_m(h)$ is given by

$$R_m(h) = \frac{\text{Tr}(h)}{3} U_m(H),$$

(111)

where $U_m(H) := (W(H))^4$ is the matter sector potential. Unlike in [100], which uses $(\text{Tr}(h))^{1/4} W(H)$, we have now used the fourth power of this combination. This is necessary for the internal stress-momentum tensor $Q^{ij}$ to be
invariant under conformal transformations of the background metric. With this choice of $T(h)$ we get:

$$Q^{ij} = \text{Tr}(T) \left( \frac{\partial U_m}{\partial H^{ij}} + \frac{1}{3} \delta^{ij} U_m \right),$$  \hspace{1cm} (112)

where, as before, $\tilde{U}_m$ is the Legendre transform \cite{31} of the matter potential $U_m(H)$. Using the identity similar to \cite{31}, it is now easy to see that

$$h^{ij} dQ^{ij} = R_m(h) d\text{Tr}(T).$$  \hspace{1cm} (113)

We will need this identity below when we discuss the energy conservation.

The expressions \cite{108}, \cite{109} and \cite{112} determine the coupling of a general stress-energy tensor $T_{ab}$ to our gravity theory. The only extra input that needs to be specified on top of what is already present in general relativity is two dimensionless potentials $U(H), U_m(H)$ of the traceless part $H^{ij}$ of the internal metric. The potential $U(H)$ determines the dynamics of the vacuum gravity, and the material potential $U_m(H)$ is necessary to specify the coupling to matter.

It remains to check that the coupling specified is consistent with the standard energy conservation. The only thing that needs to be verified is that there are no changes in the self-dual part of the stress-energy two-form. The anti-self dual part does not change. A detailed argument involves spinors and is given in the Appendix. In our theory the conservation equation for $T^i$ is given by \cite{61}. Using \cite{109}, the self-dual part of this equation becomes:

$$\frac{1}{2} \xi B^i \wedge D_{Ab} Q^{ij} B^j = \frac{1}{4} \xi (B^i \wedge B^j) D_{Ab} Q^{ij},$$  \hspace{1cm} (114)

where we have again used the identity \cite{29}. Passing to the description of the two-form field $B^i$ in terms of the metric two-forms $B^\mu$, and taking note of the definition \cite{107} of the internal metric we can write this as:

$$\frac{1}{3} \xi (B^\mu \wedge B_\nu) \frac{1}{4} h^{ij} dQ^{ij} = \frac{1}{3} \xi (B^\mu \wedge B_\nu) R_m(h) d(\text{Tr}(T))/4.$$  \hspace{1cm} (115)

where we have used the second equation in \cite{64} to replace the covariant derivative $D_{Ab}$ by the usual one, and used \cite{113} to arrive at the final expression. The only modification here as compared to GR is the presence of the factor $R_m(h)$ in this expression. We see, therefore, that the conservation equation holds in the metric in which

$$\text{Tr}(h) U_m(H) = 3,$$  \hspace{1cm} (116)

in agreement with our finding \cite{102} in the previous Section. This establishes that the standard stress-energy tensor $T_{ab}$ conservation equation is consistent with the conservation equation \cite{64} for $T^i$ when the stress-momentum two-form is constructed from the components of $T_{ab}$ as specified in equations \cite{108}, \cite{109} and \cite{112}. A more thorough discussion of the energy conservation (including a treatment of the anti-self-dual part) may be found in the Appendix.

This finishes the formal development of our theory. We now have all the ingredients to study physics with it, as we know how to describe pure gravity, know how it is influenced by a general stress-energy tensor, and know how test matter moves in a given gravitational background. At the end, our gravity theory has all the standard ingredients of general relativity. The only new quantities that we have introduced are two dimensionless “potential” functions $U(H), U_m(H)$ depending on the components $H^{ij}$ of the two-form field. Choosing the metric conformal factor so that the condition \cite{110} is satisfied gives us completely standard physics for matter fields. The only thing that changes is the coupling of matter to gravity, as well as the gravitational dynamics.

**IX. RECIPE**

We finish our exposition of the new theory with an “explicitly metric” formulation that is useful for practical computations. The reader, however, will not see a metric below, only two-forms $B^\mu$ constructed from the metric, similar to what happens in Plebanski formulation of general relativity reviewed in Section \cite{11}. A more conventional formulation of the theory that uses the spacetime metric explicitly is also possible, see \cite{11} for a recent treatment.

As in GR in Plebanski formulation, see Section \cite{11} one starts with a metric and the corresponding set of self-dual metric two-forms, which we denote by $B^\mu$. One then forms a general linear combination of the metric forms:

$$B^i = \Lambda^i \Lambda^j,$$  \hspace{1cm} (117)

introduces the “internal” metric

$$h^{ij} = \Lambda^i \Lambda^j,$$  \hspace{1cm} (118)
and finds its traceless part:

\[ H^{ij} = \frac{3h^{ij}}{\text{Tr}(h)} - \delta^{ij}. \]

(119)

The theory is specified by two dimensionless potential functions normalized as:

\[ U(H) = 1 + \frac{\alpha}{2} \text{Tr}(H^2) + O(H^3), \quad U_m(H) = 1 + \frac{\beta}{2} \text{Tr}(H^2) + O(H^3), \]

(120)

where \( \alpha, \beta \) are dimensionless parameters. It can be seen that, apart from those already available in GR, these are the only parameters that are of relevance for the linearized theory. The matrices \( \Lambda^{ij} \) are required to satisfy:

\[ \text{Tr}(h)U_m(H) = 3. \]

(121)

After this is done, one finds the two-form field compatible connection \( A_B \) such that \( D_{AB}B^i = 0 \) (note that the derivative operator \( D_A \) only acts on the non-underlined indices). The field equations then read:

\[ \Lambda^{ij}F^i(A_{AB}) = \Lambda \left( \Lambda^{ij} \partial U \partial h^{ij} \Lambda^{ij} + \frac{1}{3} h^{ij} \bar{U} \right) B^i + 2\pi GT \left( \Lambda^{ij} \partial U_m \partial h^{ij} \Lambda^{ij} + \frac{1}{3} h^{ij} \bar{U}_m \right) B^i - 2\pi GT \bar{U} B^i, \]

(122)

where \( \bar{U}, \bar{U}_m \) are the Legendre transforms \(^\text{123}\) of the potentials \( U(H), U_m(H) \), \( \Lambda \) is the cosmological constant, \( T \) is the standard metric stress-energy tensor of matter, \( \bar{B}^i \) are the quantities constructed from the traceless part of the standard stress-energy tensor, and \( \bar{B}^i \) are the anti-self-dual metric two-forms. For example, for the ideal fluid:

\[ T = (\rho - 3P), \quad \bar{T}^i = (\rho + P)\delta^i + \epsilon^{ijk}u^j, \]

(123)

where \( \rho, P \) are the energy and pressure densities and \( u^i \) is (related to) the momentum vector. The limit to general relativity is obtained by making the gravitational potential infinitely steep, i.e. by sending \( \alpha \to \infty, H^{ij} \to 0 \) so that the product \( \alpha H^{ij} \) remains finite. When \( H^{ij} = 0 \) the matrix \( \Lambda^{ij} \) is an arbitrary \( \text{SO}(3) \) one, for example the identity matrix, and it is evident that \(^\text{122} \) reproduces Plebanski equations. The only new ingredient in \(^\text{122} \), apart from the usual metric and stress-energy tensors, are the quantities \( \Lambda^{ij} \) that change the gravitational dynamics and the coupling to matter. They are, however, non-dynamical, and their only job is to “twist” the theory as prescribed by two potentials \( U(H), U_m(H) \).

The discussion of the previous sections has demonstrated that the stress-energy tensor is still conserved in this theory in the usual way \( \nabla^aT_{ab} = 0 \), where \( \nabla^a \) is the metric-compatible derivative, and that test bodies move along geodesics. In both cases the relevant metric is the one that is used in the construction of the metric two-forms \( \bar{B}^i \). One can use the formulae given in this section as a definition of the theory. A reader who finds this definition a bit contrived should consult earlier sections for a simpler, but more abstract description. The physical exploration of this theory is left to future publications.

For applications it is sometimes more convenient to work not with the internal metric \( h^{ij} \), but with the matrices \( \Lambda^{ij} \) introduced above in \(^\text{117} \). Thus, let us describe an equivalent formulation of the theory in which the internal metric never appears and one works directly with the quantities \( \Lambda^{ij} \). Again, we start with a metric and the corresponding set of self-dual metric two-forms, which satisfy \( \bar{B}^i \wedge \bar{B}^i \sim \delta^{ij} \). As before, we form a general linear combination of the metric forms \(^\text{117} \). However, now instead of introducing two potentials \( U(H), U_m(H) \), let us work directly with the combinations: \( R(h) = (\text{Tr}(h)/3)U(H), R_m(h) = (\text{Tr}(h)/3)U_m(H) \), which are two (arbitrary) \( \text{SO}(3) \)-invariant functions of the internal metric \( h^{ij} \) normalized so that \( R(\delta) = R_m(\delta) = 1 \) and homogeneous of degree one in the quantity \( \text{Tr}(h) \). Let us view these functions as those of \( \Lambda^{ij} \). Then they are arbitrary (normalized) functions of matrices \( \Lambda^{ij} \) that are invariant under left and right action of \( \text{SO}(3) \), and transform as \( R \to \Omega^{-3}R, R_m \to \Omega^{-4}R_m \) when \( \Lambda^{ij} \to \Omega^{-2}\Lambda^{ij} \). In order for the metric used to construct \( \bar{B}^i \) to be the physical one, in which matter moves along geodesics, the quantities \( \Lambda^{ij} \) are required to satisfy the conditions:

\[ R_m(\Lambda) = 1. \]

(124)

As before, one now finds the \( B \)-compatible connection \( A_B \) such that \( D_{AB}B = 0 \) (note that the derivative operator \( D_A \) only acts on the non-underlined indices). After this is done, one computes the curvature of the connection \( A_B \). From \(^\text{122} \) we see that the right-hand side of the field equations contains matrices of the type \( \partial f(h)/\partial h^{ij} \). Let us note an identity:

\[ \Lambda^{ij} \partial f/\partial h^{ij} \Lambda^{ij} = \frac{1}{2} \partial f/\partial \Lambda^{ij} \Lambda^{ij}, \]

(125)
where $f$ on the right-hand-side is considered to be a function of the matrix $\Lambda^A_{\underline{\lambda}}$. Using this identity we can rewrite the field equations (122) in a way that uses directly the quantities $\Lambda^A_{\underline{\lambda}}$. One gets:

$$\Lambda^A_{\underline{\lambda}}F^i(A_{AB}) = \Lambda \left( \frac{1}{2} \frac{\partial R}{\partial \Lambda^A_{\underline{\lambda}}} \right) B^i_{\underline{\lambda}} + 2\pi G T^i_{\underline{\lambda}} \left( \frac{1}{2} \frac{\partial R_{\underline{\mu}}}{\partial \Lambda^A_{\underline{\lambda}}} \right) B^i_{\underline{\mu}} - 2\pi G T^i_{\underline{\lambda}} B^i_{\underline{\lambda}}. \quad (126)$$

The interpretation of the quantities $\Lambda$, $T$ and $T^i_{\underline{\lambda}}$ is as before. Thus, for the ideal fluid we have (128). The formulation that works directly with the internal metric “triads” $\Lambda^A_{\underline{\lambda}}$ thus leads to more compact field equations as compared to (122) and may be preferable for some purposes. The “twisting” role of the scalars $\Lambda^A_{\underline{\lambda}}$ is particularly clear in the formulation (126).

From the described Plebanski-like formulation (126) it may seem that the obtained field equations have little to do with the objects one usually works with in gravity, namely the spacetime metric and the stress-energy tensor of some matter that couples to this metric. However, we would like to stress that in the final formulation our theory is completely standard and works exactly with the same quantities. Thus, we have the physical metric and the matter couples to it in a completely standard way. The matter moves along geodesics of this physical metric and has the usual stress-energy tensor. What is non-standard is how field equations for this metric are obtained. To this end one introduces certain extra scalar fields $\Lambda^A_{\underline{\lambda}}$ and deforms Einstein equations in a way that does not generate any kinetic term for the scalar fields and is consistent with energy conservation.

Our theory in its final form may be compared to the “modified source gravity” of [12], where the authors, following [13], introduce a scalar field $\psi$ and consider a gravity theory described by the following simple Lagrangian:

$$\int_M \sqrt{-g}(R - U(\psi)) + S_m[e^{2\psi}g, \phi_m]. \quad (127)$$

Here $g$ is a dynamical metric, but note that the matter couples not to $g$ but to a conformally related metric $e^{2\psi}g$ instead. Importantly, the field $\psi$ is non-dynamical, i.e. does not have a kinetic term. In vacuum, the theory is just GR with a cosmological constant. But in general the theory for the physical metric $e^{2\psi}$ is different from GR and, in particular, the stress-energy tensor of matter sources the Einstein tensor of $e^{2\psi}g$ in a modified way.

The theory we have considered in this paper is similar to (127) in the sense that the coupling of the stress-energy tensor of matter to gravity is modified. The modification also arises from introducing non-dynamical scalar fields, even though there is now a multiplet $\Lambda^A_{\underline{\lambda}}$ instead of a single one in (127). However, unlike in the case of (127), the pure gravity theory is modified as well, with this modification being controlled by the potential $R(h)$. Another important difference is that, unlike in the theory (127) that modifies the homogeneous isotropic Universe solution, in our theory the scalars $\Lambda^A_{\underline{\lambda}}$ are set to SO(3) matrices in this case by symmetry, so the homogeneous isotropic cosmology is unmodified. But the principle according to which (127) is constructed is quite similar to that used in our theory. This is made especially clear by a recent reformulation [11] of the theory that works directly with the internal metric “triads” $\Lambda^A_{\underline{\lambda}}$.

Let us finish this section with two more remarks. As we see from the field equations (126), in general, the self- and anti-self-dual parts of the stress-momentum two-form, or, in other words, the trace and the tracefree parts of the stress-energy tensor of matter tensor appear on the right-hand side of field equations on a different footing. Indeed, there is an extra matrix multiplying the self-dual part proportional to $e^{2\psi}T$ such that both parts of its stress-energy tensor appear in (126) in the same way. Thus, we are looking for a function $R_m(\Lambda)$ such that $\partial R_m/\partial \Lambda \sim \Lambda^{-1}$, such that $R_m(\delta) = 1$ and which transforms under $\Lambda \to \Omega^{-2}\Lambda$ as $R_m \to \Omega^{-4}R_m$. This function is:

$$R_m(\Lambda) = (\text{det}(\Lambda))^{2/3}, \quad (128)$$

or, in terms of the internal metric $R_m(h) = (\text{det}(h))^{1/3}$. With this choice of the matter side potential the field equations take the form:

$$\Lambda^A_{\underline{\lambda}}F^i(A_{AB}) = \Lambda \left( \frac{1}{2} \frac{\partial R}{\partial \Lambda^A_{\underline{\lambda}}} \right) B^i_{\underline{\lambda}} + 4\pi G T^i_{\underline{\lambda}}, \quad (129)$$

with $T^i_{\underline{\lambda}}$ given by its usual expression in Plebanski theory, see e.g. [19] for the case of the ideal fluid. Let us finally note that the condition (124) in this case is simply $\text{det}(h) = 1$, which defines the so-called Urbantke metric [14]. Thus, we can rephrase the above discussion by saying that the Urbantke metric is distinguished in our theory by the fact that when matter couples to this metric the field equations take a particularly simple form (129). Urbantke metric has recently played a distinguished role in a reformulation of this theory proposed in [11]. However, unlike in this reference, instead of fixing the metric to which matter couples from the outset, we prefer to allow matter to couple to...
an arbitrary metric in the conformal class of $B^i$, and control this coupling by the matter side potential $R_m(\Lambda)$. It can be seen that a non-trivial coupling with $R_m(\Lambda)$ different from (128) leads to new interesting physical effects absent in the case (128), which in our opinion serves as a sufficient motivation to allow such a more general coupling.

It is also interesting to note that one can obtain a simple but still non-trivial theory with both potentials fixed by taking, in addition to (128), the gravitational potential to be given by the same expression $R(h) = (\det(h))^{1/3}$. The obtained theory has no adjustable parameters and its field equations read:

$$\Lambda^{ij}F^i(A_{AB}) = \frac{1}{3} \Lambda B^i + 4\pi G T^i. \quad (130)$$

If not for the presence of the quantities $\Lambda^{ij}$ the vacuum $T^i = 0$ version of these equations would be just the constant curvature condition. The presence of the extra scalars twisting this equation makes them more interesting, and, in particular, makes a non-trivial spherically-symmetric solution possible. However, in view of the fact that the modified gravity theory (130) does not have adjustable parameters it is likely to be in gross conflict with the standard gravity tests. Thus, the particularly simple version of the theory with both potentials fixes to be $R(h) = R_m(h) = (\det(h))^{1/3}$ is likely to be of only academic interest.

X. DISCUSSION

With most of the discussion being embedded in the main text we shall only make some “philosophical” remarks here. The role of the metric in general relativity is two-fold. First, a metric defines the spacetime causal structure (lightcones at every point). However, to define the causal structure one only needs the conformal metric, i.e. the metric modulo conformal rescalings. Second, every spacetime point in general relativity is effectively equipped with a set of rulers and a clock. It is for this purpose of measuring spacetime intervals that one needs a metric per se, not just a conformal metric. While the propagation of light is a very basic process that can arguably make sense to be built into the very definition of the spacetime structure, the availability of rulers and clocks at every point is on a very different footing. Indeed, a measurement of distances and time intervals is a complex physical process that requires in each case a macroscopic physical system - a solid body or a clock. The very availability of rulers and clocks at every spacetime point is quite striking, for even the empty space, which is by definition void of anything material, is endowed in GR with this structure. In our opinion this is a very anti-Machian feature of GR: according to Mach’s ideas only a relative description of material bodies in the Universe is possible, and an “empty” Universe filled with clocks should be approached with suspicion. One can argue that the notion of spacetime distance is a macroscopic one, and has no place in any reasonable microscopic description. The fact that macroscopic bodies behave as if to register spacetime intervals needs to be explained, not postulated. This is not so for the causal structure, as the tiny quanta of electromagnetic field constantly popping out of the vacuum can be argued to define the causal structure of spacetime by their very existence.

It thus seems reasonable to try to formulate a theory of the gravitational field which is based on the spacetime conformal structure, not on the spacetime metric. Experimentally we only know that the gravitational field is a universal long-range interaction. As such it should be possible to think about it as occurring due to exchange of some massless particles which can thus have only two possible polarizations. In GR these two polarizations arise from the gravitational field of a spacetime metric with its ten components and an additional 4-parameter group of gauge symmetries - diffeomorphisms. However, it seems impossible to build a similar description on just the conformal structure, as it is specified by 9 components, which is an odd number, and so no straightforward scheme with gauge symmetries (which reduces the number of DOF by an even number) can bring 9 total components down to two physical. A theory of gravity that is based on just the conformal structure would also have the problem that there would be no preferred scales in it, so the world described by it would not be realistic.

The theory that we have formulated in this paper describes spacetime geometry by specifying its conformal structure. The way this happens is that in addition to the conformal structure there are other fields in the theory - other components of the gravitational field. The total number of “components” of our basic two-form field is 18 - an even number and taking into account all the arising constraints it can be seen that the number of the arising polarizations of the graviton is still two. The vacuum theory is specified by one arbitrary dimensionless scalar function $U(H)$ of a traceless symmetric $3 \times 3$ matrix $H^{ij}$. When one couples the theory to matter one has to introduce yet another arbitrary function $U_m(H)$ with similar properties. It is this matter sector potential function $U_m(H)$ that can be shown to supply the conformal factor that defines the spacetime metric in which test bodies move along geodesics. The limit to general relativity is obtained by setting $H^{ij} = 0$.

As we have discussed, the gravity and material sector functions should be related, since at least in principle it should be possible to compute the gravity potential $U(H)$ as induced by quantum effects involving matter. Indeed, in Section VII we have seen how a version of “induced gravity” scenario is possible in our setting. With the current
state of the development of the theory such a quantum computation remains beyond our abilities and there is no choice but to treat the two functions as two independent phenomenological parameters (or rather two infinite set of parameters) of the theory. It should be noted, however, that the linearized theory is only sensitive to the two leading parameters from the arising infinite towers of coefficients. A work analyzing the effect of modification on the cosmological perturbation theory is currently in preparation. It should also be noted that the spherically-symmetric solution of the described gravity theory is known, see e.g. \[10\]. Implications of the modification for the motion of test bodies in the spherically-symmetric background need to be analyzed in light of findings of this paper.

Let us conclude by expressing our amazement at how far it was possible to develop our modified gravity theory just by following its internal logic. Indeed, the theory started its life in \[5\] as a rather complicated modification of the pure connection formulation of GR. However, as we have seen from the constructions presented in this paper, the theory turned out to be a very natural generalization of Plebanski gravity tightly constrained by “Bianchi” identities, both the vacuum theory and its coupling to matter. We have also seen that after specifying the matter sector potential \( U_m(H) \) (or, equivalently, \( R_m(h) \)) and thus specifying to which metric in the conformal class defined by \( B^i \) the matter couples, the theory takes entirely standard metric form.

It should also be emphasized how striking the results described in this paper are from the more familiar perspective of metric-based gravity theories. Indeed, it is commonly believed that in order to modify gravity one needs to quantization is attempted. We thus hope that the theory that started its life almost 20 years ago in \[5\] will continue this question to obtain physical predictions of the theory, but it will certainly be an essential question when the matter directly to the two-form field remains open. As we have seen in this paper, it is not necessary to answer this question to obtain physical predictions of the theory, but it will certainly be an essential question when the quantization is attempted. We thus hope that the theory that started its life almost 20 years ago in \[5\] will continue to be a source of interesting results for some more years to come.

Appendix: Spinor techniques and energy conservation

Let us, as before, select an arbitrary metric in the conformal class of metrics determined by \( B^i \), construct a tetrad, and then use it to identify the space of rank 2 mixed primed-unprimed spinors \( \lambda_{A'B'} \) with spacetime vectors \( \lambda^a \) (and also, using the metric, with spacetime one-forms \( \lambda_a \)). Thus, all spacetime indices are converted into spinor indices. In these notations our basic two-form field \( B^i \) is a self-dual two-form \( B_{AB}' \epsilon_{A'B'} \) (there is no component proportional to \( \epsilon_{AB} \) which would correspond to the anti-self-dual part). Thus, the two-form field \( B^i \) becomes described in this language by the metric that it defines, as well as by the quantity \( B_{AB}' \), symmetric in the unprimed spinor indices \( AB \). In the GR case \( B_{CD}' \sim \epsilon_{A(C'}\epsilon_{B|D)} \), but in general these quantities are arbitrary. The stress-energy two-form \( T^i \) is described by its self-dual \( T^i_{AB} \) and anti-self-dual \( T^i_{A'B'} \) components.

The conditions \( \text{[SS]} \) whose consequences we need to explore in spinor notations become:

\[
\int dt \left( -T^{i'AC}D_C^A \xi^B_B B_{AB}^i + T^{i'AC'}D_C^A \xi^B_B B_{AB}^i \right) = 0, \tag{131}
\]

where \( \xi^{AA'} \) is the spinorial representation of the vector field \( \xi^a \), and \( D_{AA'} \) is that of the covariant derivative operator \( D_{A'B'} \).

Let us now use the form \( \text{[SS]} \) of the stress-momentum two-form. The anti-self-dual component of \( T^i \) is given by:

\[
T^i_{A'B'} = (B^{-1})^i_{AB} T^A_{A'B'}, \tag{132}
\]

where \( T_{A'B'} \) has the same form as in GR \( T_{A'B'} = m u_{(A'} u_{B')} \), and is just the traceless part \( T_{ab} - (1/4) T \epsilon_{ab} \) of the stress-energy tensor \( T_{ab} \) of ideal pressureless fluid \( T_{ab} = m u_a u_b, u^a u_a = 1 \). The quantities \( B^i_{AB} \) what \( \Lambda_{ij} \) become in the spinor notations, and \( (B^{-1})^i_{AB} \) is the inverse matrix satisfying:

\[
(B^{-1})^i_{AB} B^j_{AB} = \delta^i_i, \quad (B^{-1})^i_{AB} B^j_{CD} = \epsilon_{A(C'} \epsilon_{B|D)} \tag{133}
\]

Let us thus look at the second term in \( \text{[SS]} \). The compatibility equation \( DB^i = 0 \) takes in spinor notations the following form:

\[
D_A^B B^i_{AB} = 0. \tag{134}
\]
This means that the quantity $B^i_{AB}$ can be taken out of the operator of the covariant derivative:

$$T^{iA'B'}D^A_C
A_B^i= T^{iA'B'}B^i_{AB}D^A_C
A_B^i= T^{iA'B'}B^i_{AB}
A_B^i,$$

where the last equality is due to the fact that there are no internal indices in the quantity that the covariant derivative operator acts on, and so the derivative operator can be replaced by the usual metric-compatible one. We now use the form (132) of the anti-self-dual part of the stress-momentum two form to conclude that the second term in (131) is given by:

$$T^{A'B'}C_A^B
A_B^i= mu_{A'B'}C_A^B
A_B^i,$$

where we have used the fact that $T_{A'B'A'B'}= mu_{A'B'}u_{B'}.$

We can now substitute (131) into the first term self-dual term in the conservation equation (130), and use the compatibility equation to take the quantity $B^i_{AB}$ under the operator of covariant derivative. The first term becomes:

$$-\frac{1}{2}Q^{ij}D^j_C
A_B^i,$$

We now use the identity (29), which in the spinor notations becomes:

$$B^i_{EF}B^j_{AB}= -\frac{1}{2}\epsilon_{EF}h^{ij},$$

to rewrite (137) as:

$$-\frac{1}{4}Q^{ij}D^j_C
A_B^i,$$

where we have used the relation (80). Now using the identity (88) we see that we can replace the covariant derivative in the first term here by the ordinary one. The covariant derivative in the second term can be replaced by the metric compatible one as the quantity it acts on does not have internal indices.

Combining it all together we get for the equation (131):

$$0 = \int dt \left( -\frac{1}{4}Q^{ij}h^{ij} + \frac{m}{4}\epsilon_{AB}u^i_{A'B'}C_A^{B'} + m u^i_{A'B'}C_A^{B'} \right),$$

where we have rewritten the second term in a suggestive way. We now use:

$$u^i_{A'B'} + \frac{1}{4}\epsilon_{AB}u^i_{A'B'} = u^i_{A'}u^i_{B'},$$

to get:

$$0 = \int dt \left( -\frac{1}{4}Q^{ij}h^{ij} + m u^i_{A'}u^i_{B'} \right),$$

or, in the usual tensorial notations:

$$0 = \int dt \left( -\frac{1}{4}Q^{ij}h^{ij} + m u^i_{A'}u^i_{B'} \right),$$

where to obtain the second equality we have integrated by parts in the second term. Since the vector field $\xi^a$ is arbitrary we can conclude that:

$$u^i_{A'}u^i_{B'} = \frac{1}{4}Q^{ij}h^{ij}. $$

This finishes our proof of (88).

Let us now consider a proof of energy conservation. For this we take the energy conservation equation in the form (134), which in spinor notations becomes:

$$\xi^B_{A'B'}B^i_{AB}(-D^C_{A'}T^i_{CA'} + D^A_{C'}T^i_{C'A'}) = 0.$$
Let us transform the anti-self-dual part first. We can use the compatibility equation \((134)\) to take the quantity \(B^i_{AB}\) under the operator of covariant derivative. We get for this term:

\[
\xi^B_A D^{AB'} B^{i}_{AB} T^{iA'} = \xi^B_A \nabla^{AB'} T^{A'B}, \tag{146}
\]

which coincides with the usual expression in the metric theory.

Let us now analyze the self-dual term. Here we can replace the self-dual stress-momentum two-form by its expression \((84)\) in terms of the tensor \(Q_{ij}\), and take the two-form \(B^i\) out of the operator of covariant derivative. We get for this first term:

\[
-\frac{1}{2} \xi^B_A B^{i}_{AB} B^C_\,^A D^{CA'} Q_{ij}, \tag{147}
\]

where \(Q_{ij}\) is now given by:

\[
Q_{ij} = T \frac{\partial R(h)}{\partial h^{ij}}, \tag{148}
\]

with \(T\) being the trace of the stress-energy tensor \(T_{ab}\). Now using \((138)\) we get:

\[
\frac{1}{4} \xi^B_A h^{ij} D^{A'}_B Q_{ij} = \xi^B_A R(h) \nabla^{A'}_B (T/4), \tag{149}
\]

where we have used the identity \((113)\). Combining the two terms above, and writing the result (up to an overall minus sign) in usual vector notations we get:

\[
R(h) \xi^a \nabla_a (T/4) + \xi^a \nabla^b (T_{ab} - (1/4) g_{ab} T) = 0. \tag{150}
\]

This coincides with the usual conservation equation \(\nabla^a T_{ab}\) when the metric is chosen so that \(R(h) = 1\). This finishes our demonstration of the fact that the usual energy conservation holds in our theory.

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