THE PASSAGE FROM THE INTEGRAL TO THE RATIONAL GROUP RING IN ALGEBRAIC K-THEORY

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ABSTRACT. An open question is whether the map \( \widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G \) in reduced \( K \)-theory from the integral to the rational group ring is trivial for any group \( G \). We will show that this is false, with a counterexample given by the group \( \mathbb{Q}D_{32} \cdot \mathbb{Q}16 \cdot \mathbb{Q}D_{32} \). We will also show how to compute the image of the map \( \widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G \) using representation theoretic means, assuming \( G \) satisfies the Farrell-Jones conjecture.

1. Introduction

Let \( G \) be a group, \( R \) a ring and define \( RG \) to be the group algebra of \( G \) over \( R \). The algebraic \( K \)-theory group is defined as

\[
K_0RG := \mathbb{Z}\{\text{isomorphism classes of f.g. projective } RG\text{-modules}\} / \equiv
\]

where \( \equiv \) is the equivalence relation generated by \( [A \oplus B] \equiv [A] + [B] \).

For \( R \) being the ring of integers, the subgroup of \( K_0\mathbb{Z}G \) generated by the free modules is always a split summand isomorphic to \( \mathbb{Z} \). The reduced \( K \)-theory group \( \widetilde{K}_0\mathbb{Z}G \) is defined as the quotient of \( K_0\mathbb{Z}G \) by this summand.

The group \( \widetilde{K}_0\mathbb{Z}G \) is an important invariant of \( G \) appearing in a variety of geometric problems, the most notable as the group containing Wall’s finiteness obstruction. Wall \cite{Wal65} showed that for every finitely dominated CW-complex \( X \) with fundamental group \( G = \pi_1(X) \), there exists an element \( w(X) \) in \( \widetilde{K}_0\mathbb{Z}G \) which is trivial iff \( X \) is actually finite. Moreover, every element of \( \widetilde{K}_0\mathbb{Z}G \) is realized in this way from some finitely dominated CW-complex\[1\].

However, \( \widetilde{K}_0\mathbb{Z}G \) tends to be very hard to compute in general. One of the few structural things that can be said about \( \widetilde{K}_0\mathbb{Z}G \) is a theorem due to Swan that for \( G \) being a finite group \( \widetilde{K}_0\mathbb{Z}G \) is finite.

Changing the base ring to the rational numbers, we can define \( \widetilde{K}_0\mathbb{Q}G \) in a similar manner. As before, we obtain a splitting \( K_0\mathbb{Q}G \cong \mathbb{Z} \oplus \widetilde{K}_0\mathbb{Q}G \). For \( G \) a finite group, \( K_0\mathbb{Q}G \) is inherently easier to compute than its integral counterpart. Since in this case \( \mathbb{Q}G \) is a finite dimensional semisimple algebra over \( \mathbb{Q} \), it splits as a product of matrix algebras over division algebras over \( \mathbb{Q} \), one for each irreducible \( \mathbb{Q} \)-representation of \( G \). This means that \( K_0\mathbb{Q}G \cong \mathbb{Z}^{r_Q} \), where \( r_Q \) is the number of isomorphism classes of irreducible \( \mathbb{Q} \)-representations of \( G \). In particular, \( \widetilde{K}_0\mathbb{Q}G \) is a free abelian group of rank \( r_Q - 1 \).

\[1\] See also \cite{Var89}
We can thus see that for $G$ being a finite group, the map $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ defined via $[P] \mapsto [P \otimes \mathbb{Q}]$, for $P$ being a f.g. projective $\mathbb{Z}G$-module, is an isomorphism on the summands corresponding to free modules over $\mathbb{Z}G$ and $\mathbb{Q}G$ respectively, and trivial on the quotients $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$, since there it is a homomorphism from a finite to a free abelian group. Swan also showed the following slightly stronger result.

**Theorem 1.1** (Swan, [Swa60]). **Suppose** $G$ is a finite group and $P$ a finitely generated projective $\mathbb{Z}G$-module. Then $P \otimes \mathbb{Q}$ is free.

The statement that $\tilde{K}_0\mathbb{Q}G$ is free does not generalize to arbitrary groups. In fact, Kropholler, Moselle [KM91], and Leary [Lea00] constructed specific examples of groups which have 2-torsion elements in $K_0\mathbb{Q}G$. This means we cannot expect to find a straightforward generalization of Swans theorem to infinite groups.

Bass [Bas76] investigated this question and formulated what is now known as the strong Bass conjecture for $K_0\mathbb{Z}G$. For this, let $r : K_0\mathbb{Z}G \to HH_0(ZG)$ denote the Hattori-Stallings trace map and define $r_P(g)$ as the coefficient in the sum $r(P) = \sum_{[g] \in \text{Conj}(G)} r_P(g)$ under the isomorphism $HH_0(RG) = \bigoplus_{\text{Conj}(G)} R$.

**Conjecture 1.2** (Strong Bass Conjecture for $K_0\mathbb{Z}G$, [Bas76]). The function $r_P(g)$ is 0 for $g \neq 1$.

Lück, Reich [LR05], Section 3.1.3, show that the strong Bass conjecture for $K_0\mathbb{Z}G$ follows from the stronger claim that the map $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ vanishes rationally, and they also show that this holds true if $G$ satisfies the Farrell-Jones conjecture.

**Theorem 1.3** ([LR05], Proposition 3.11). **Assume** $G$ satisfies the Farrell-Jones conjecture. Then the map $\tilde{K}_0\mathbb{Z}G \otimes \mathbb{Q} \to \tilde{K}_0\mathbb{Q}G \otimes \mathbb{Q}$ is trivial.

We will give a definition of the Farrell-Jones conjecture below. In Remark 3.13 they ask whether this is true integrally.

**Conjecture 1.4** (Integral $\tilde{K}_0\mathbb{Z}G$-to-$\tilde{K}_0\mathbb{Q}G$-conjecture). The map $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ is trivial.

Part of this paper will show the following.

**Theorem 1.5** (See Section 9). The Integral $\tilde{K}_0\mathbb{Z}G$-to-$\tilde{K}_0\mathbb{Q}G$-Conjecture is false. A counterexample is given by the group $G := QD_{32} * Q_16 QD_{32}$, where $QD_{32}$ is the quasi-dihedral group of order 32, and $Q_16$ is the generalized quaternion group of order 16.

The other half of this paper is an investigation into how much the map $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ can fail to be trivial under the assumption that $G$ satisfies the Farrell-Jones conjecture. The Farrell-Jones conjecture states that the assembly map $\text{colim}_{(G/H) \in \text{OrG}_{\text{VCyc}}} KZH \to KZG$.
is a weak equivalence of spectra. Here $K_{\mathbb{Z}G}$ refers to the non-connective $K$-theory spectrum of $\mathbb{Z}G$, and the colimit in question is a homotopy colimit over the category $\text{Or}_{G, \text{Cyc}}$, which is the full subcategory of the orbit category of $G$ spanned by the objects $G/H$ with virtually cyclic isotropy group $H$. The Farrell-Jones conjecture has been shown to be true for a wide class of groups by the work of Bartels, Lück, Reich [BLR07], Bartels, Bestvina [BB18] Kammeyer, Lück, Rüping [KLR16], and Wegner [Weg15] among many others.

Now let $E\text{Fin}$ be a fixed model for the classifying space of finite subgroups together with a chosen CW-structure $(E\text{Fin}^{(k)})_{k \in \mathbb{N}}$. Write $(f, g): \bigsqcup_{i \in I} G/H_i \times S^0 \to \bigsqcup_{j \in J} G/K_j$ for the degree 0 attaching map of $E\text{Fin}$ with the $H_i$ and $K_j$ being finite subgroups of $G$. For a functor $F: \text{Or}_G \to \text{Ab}$ define

$$\ker^F := \ker (F(f) - F(g)): \bigoplus_{i \in I} F(G/H_i) \to \bigoplus_{j \in J} F(G/K_j).$$

**Theorem 1.6.** Suppose $G$ satisfies the Farrell-Jones conjecture. There is an exact sequence

$$0 \to \ker \widetilde{K}_0^{\mathbb{Q}G} \to \ker \text{SC} \to \ker \widetilde{K}_1^{\mathbb{Z}G} \to \Im(\widetilde{K}_0^{\mathbb{Z}G} \to \widetilde{K}_0^{\mathbb{Q}G}) \to 0.$$

This gives a certain limitation on the map $\widetilde{K}_0^{\mathbb{Z}G} \to \widetilde{K}_0^{\mathbb{Q}G}$. The terms $\ker \widetilde{K}_0^{\mathbb{Q}G}$, $\ker \text{SC}$ and $\ker \widetilde{K}_1^{\mathbb{Z}G}$ are computable by representation theoretic techniques and this is what allowed the computation of the above counterexample. The groups $\ker \widetilde{K}_0^{\mathbb{Q}G}$ and $\ker \text{SC}$ are always free, and $\ker \widetilde{K}_1^{\mathbb{Z}G}$ is free $p$-locally away from the prime 2. We will define the functor $\text{SC}$ in Section 6 and give a characterization in terms of $p$-adic characters for finite groups in Section 6.3.

One could still ask if the image of the map $\widetilde{K}_0^{\mathbb{Z}G} \to \widetilde{K}_0^{\mathbb{Q}G}$ is in fact only 2-torsion. The author at present believes this to be false. Examples of groups with odd torsion are however more challenging to construct and will be saved for a later publication.

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2. **Preliminaries**

Throughout, we will denote the *non-connective algebraic K-theory spectrum* of a non-commutative unital ring $R$ by $K_R$, see e.g. [Wei13], Chapter IV. Its homotopy groups $K_n R := \pi_n K_R$ are the algebraic $K$-theory groups of $R$.

Sections 3 to 5 are used to phrase and setup the Farrell-Jones conjecture and discuss how to deal with functors on the orbit category $\text{Or}_G$. Section 5 is concerned with primarily classical results about lower $K$-theory groups of finite groups $G$. The proofs of the main theorems will be found in section 7. The claimed counterexample to the integral $K_0^{\mathbb{Z}G}$-to-$K_0^{\mathbb{Q}G}$ conjecture is discussed in Section 9.
We will use the language of $\infty$-categories in our proofs. The author remarks that this choice is due to convenience, not necessity. The reader not familiar with the topic shall be assured that all arguments can be phrased using the notions of model categories and $t$-structures on a triangulated category, only that many formal arguments become harder to formulate (e.g. the existence of the object-wise $t$-structure on a functor category or exactness of many functors involved). A model for the notion of $\infty$-categories is given by the notion of quasi-categories developed by Joyal and Lurie, which are simplicial sets fulfilling a certain lifting property. The standard reference is [Lur12]. We further included some results used about $t$-structures on stable $\infty$-categories in Appendix A. We want to remark that most of our results will be phrased in a model independent way, treating the notion of $\infty$-categories as a black box. The terms limit and colimit will always be interpreted in an $\infty$-categorical way. In situations where our $\infty$-category $C$ arises from a model category $\mathcal{M}$, limits and colimits in $C$ are modelled by homotopy limits and homotopy colimits in $\mathcal{M}$. If $C$ is a 1-category, then the nerve $N(C)$ is an $\infty$-category in which limits model ordinary 1-categorical limits and similarly for colimits. We will often omit the notation for the nerve and just refer to the $\infty$-category $N(C)$ simply as $C$ when the context is clear. Given two $\infty$-categories $C$ and $D$, there is the $\infty$-category of functors $\text{Fun}(C, D)$ from $C$ to $D$. Functors $A: C \to D$ are sometimes written as $A(-)$ to highlight that the value of $A$ is dependent on the input. Natural transformations between functors will be depicted with a double arrow, like $A = \Rightarrow B$.

The $\infty$-category of spaces, sometimes also referred to as homotopy types or $\text{anima}$, will be denoted as $\text{Spc}$ and is characterized via a universal property as the free cocomplete $\infty$-category generated by a single object (the point) similar to how the category of sets is generated under coproducts by a single object (the set with a single element). It is modelled, for example, by the simplicial category of Kan complexes or the relative category of CW-complexes and weak equivalences being homotopy equivalences. The undercategory $\text{Spc}_{pt}$ is called the $\infty$-category of pointed spaces and will be denoted as $\text{Spc}_\ast$. We have a natural functor $(-)_+: \text{Spc} \to \text{Spc}_\ast$ that adds a disjoint basepoint. The $\infty$-category of spectra will be denoted as $\text{Sp}$ and is characterized again via a universal property as the stabilization of $\text{Spc}$, or equivalently, as the free cocomplete stable $\infty$-category generated by a single object (the sphere spectrum $S$). It is modelled, for example, by the relative category of $\Omega$-spectra and weak equivalences being maps that induce isomorphisms on all homotopy groups. Spectra will be denoted by bold-faced letters, e.g. $A, K, \underline{R}$ or $\text{Wh}(R; G)$. Since $\text{Sp}$ is stable, the suspension $A \mapsto \Sigma A$ defined as the pushout

produces an auto-equivalence of $\text{Sp}$ with itself. The suspension functor $\text{Spc}_\ast \to \text{Sp}$ will be denoted as $\Sigma^\infty$, the composition $\Sigma^\infty \circ (-)_+$ will be denoted as $\Sigma^\infty_\ast$. The homotopy groups of a spectrum $A$ are denoted as $\pi_n(A)$. Further, denote the 1-category of abelian groups by $\text{Ab}$. Taking homotopy groups yields functors $\pi_n: \text{Sp} \to \text{Ab}$. Spectra $A$ with the property that $\pi_n A = 0$ for $n < 0$ will be called connective, and spectra $A$ with the property that $\pi_n A = 0$ for $n > 0$ will.
be called cocommutative. The functor $\pi_0$ becomes an equivalence when restricted to the intersection of the full subcategories of connective and cocommutative spectra (essentially a consequence of the Brown representability theorem). Its inverse will be denoted by $\mathbf{H}$, or the Eilenberg-Maclane functor. The inclusion of the full subcategory of connective spectra into $\text{Sp}$ admits a right adjoint which will be called $\tau_{\geq 0}$, and we define for any $a \in \mathbb{Z}$ the functor $\tau_{\leq a}$ as $\Sigma^a \tau_{\geq 0} \Sigma^{-a}$. Similarly, the inclusion of the full subcategory of cocommutative spectra into $\text{Sp}$ admits a left adjoint which will be called $\tau_{\leq 0}$, and $\tau_{\leq a}$ is defined as $\Sigma^a \tau_{\leq 0} \Sigma^{-a}$ in the same way. For $a, b \in \mathbb{Z}$ the compositions $\tau_{\geq a} \tau_{\leq b}$ and $\tau_{\leq b} \tau_{\geq a}$ are naturally isomorphic and will denoted as $\mathbf{A} \mapsto \mathbf{A}[a, b]$. This type of structure defines a $t$-structure on $\text{Sp}$, more on this in Appendix A.

For a fixed $\infty$-category $\mathcal{C}$, for two given objects $c, c' \in \mathcal{C}$ the mapping space from $c$ to $c'$ will be denoted $\text{Map}_C(c, c')$. The subscript is omitted in the case of $\mathcal{C}$ being the $\infty$-category of spaces. $\text{Map}_C$ is a bi-functor into the category $\text{Spc}$, contravariant in the left and covariant in the right variable. We also use the notation $[c, c'] := \pi_0 \text{Map}_C(c, c')$. Note that $[c, c']$ is just the Hom-set of the 1-category given by the homotopy category of $\mathcal{C}$. If $\mathcal{C}$ is more a stable $\infty$-category, the space $\text{Map}_C(c, c')$ is naturally the zero-th space of a spectrum $\text{map}_C(c, c')$, which is called the mapping spectrum from $c$ to $c'$. Again $\text{map}_C$ is naturally a functor in both variables. We, similarly, omit the subscript in the case of $\mathcal{C}$ being the $\infty$-category of spectra. Since

$$[c, c'] = \pi_0 \text{Map}_C(c, c') \cong \pi_0 \text{map}_C(c, c')$$

is the zero-th homotopy group of a spectrum, the set $[c, c']$ comes naturally with the structure of an abelian group. The mapping space for two functors $F, G : D \to \mathcal{C}$ in the functor category will also be denoted as $\text{Nat}_\mathcal{C}(F, G)$ and is called the space of natural transformations from $F$ to $G$. If $\mathcal{C}$ is stable, the corresponding mapping spectrum is also written as $\text{nat}_\mathcal{C}(F, G)$.

If $G$ is a group, the category $\mathcal{M}$ of topological spaces with left $G$-action admits the structure of a closed simplicial model category [DK84]. The associated $\infty$-category is the $\infty$-category of $G$-spaces, $\text{G-Spc}$. Similarily, the category of pointed $G$-spaces, $\text{G-Spc}_p$ is defined as the over category $\text{G-Spc}_{pt}$, where $pt$ is the one-point space with trivial $G$-action. Let $H$ be a subgroup of $G$. We can think of the left $G$-set $G/H$ as a discrete $G$-space. We can also interpret the $n$-sphere $S^n$ as well as the $n$-disc $D^n$ as $G$-spaces by equipping them with the trivial $G$-action. If $X$ is an object of $\mathcal{M}$, i.e. a topological space with continuous left $G$-action, a $G$-CW-structure on $X$ refers to a sequence of $G$-spaces $(X^{(k)})_{k \geq 0}$, maps $i_k : X^{(k)} \to X^{(k+1)}$ such that there exist pushout squares in the category $\mathcal{M}$,

$$\prod_{i \in I_k} G/H_i \times S^n \xrightarrow{\phi_k} X^{(k)} \xrightarrow{i_k} \prod_{i \in I_k} G/H_i \times D^{n+1} \longrightarrow X^{(k+1)},$$
and an equivariant homeomorphism \( X \cong \text{colim}_k X^{(k)} \), where the colimit in question is over the tower given by the maps \( i_k : X^{(k)} \to X^{(k+1)} \). The maps

\[
\phi_k : \prod_{i \in I_k} G/H_i \times S^n \to X^{(k)}
\]

are called attaching maps. The indexing sets \( I_k \) can be arbitrary sets. The space \( X^{(k)} \) is also called the \( k \)-skeleton of \( X \). We also refer to \( X \) together with a fixed choice \( G \)-CW-structure as a \( G \)-CW-complex. By [DK84], Theorem 2.2, the cofibrant objects in \( \mathcal{M} \) are exactly retracts of \( G \)-CW-spaces. Moreover, every object of the \( \infty \)-category \( G\text{-Spc} \) can be represented by a \( G \)-CW-space. The maps \( G/H_i \times S^n \to G/H_i \times D^{n+1} \) are cofibrations in the model category \( \mathcal{M} \). This means that for a given \( G \)-CW-complex \( X \), the squares

\[
\prod_{i \in I_k} G/H_i \times S^n \xrightarrow{\phi_k} X^{(k)} \\
\prod_{i \in I_k} G/H_i \times D^{n+1} \xrightarrow{i_k} X^{(k+1)},
\]

are also pushout squares in the \( \infty \)-category \( G\text{-Spc} \). If \( X \) is an object of \( G\text{-Spc} \) we will define a \( G \)-CW-structure on \( X \) to be a \( G \)-CW-structure on any representing object of \( X \) in \( \mathcal{M} \). We will also refer to objects of \( G\text{-Spc} \) from now on as \( G \)-spaces. Note that often we will leave it implicit that a given object of \( G\text{-Spc} \), i.e. a \( G \)-space, is technically speaking only represented by a topological space with \( G \)-action up to weak equivalence.

3. \( G \)-HOMOLOGY THEORIES AND FUNCTORS ON THE ORBIT CATEGORY

Define the orbit category \( \text{Or}G \) as the full subcategory of the 1-category of \( G \)-sets spanned by the \( G \)-sets with transitive action. Equivalently, each object \( S \) of \( \text{Or}G \) is \( G \)-equivariantly isomorphic to a set of left cosets \( G/H \), acted on by \( H \). where \( H \) is the isotropy group of a chosen element \( s \) of \( S \). It is an elementary fact that each map in \( \text{Or}G \) can be decomposed into a composition of maps given by inclusions \( i : G/H \to G/H', kH \mapsto kH' \) for \( H \subset H' \) and conjugations \( c_g : G/H \to G/(g^{-1}Hg), kH \mapsto kHg = (kg)(g^{-1}Hg) \). If we have a \( G \)-space \( X \), we can think of the assignment \( G/H \mapsto X^H \) as a functor

\[
X^- : \text{Or}G^{\text{op}} \to \text{Spc}.
\]

Elmendorf’s Theorem states that mapping \( X \) to \( X^- \) produces an equivalence of \( \infty \)-categories \( G\text{-Spc} \cong \text{Fun}(\text{Or}G^{\text{op}}, \text{Spc}) \), see e.g. [DK84] Theorem 3.1. Note that it further refines to an equivalence \( G\text{-Spc}_\ast \cong \text{Fun}(\text{Or}G^{\text{op}}, \text{Spc}_\ast) \) for pointed \( G \)-spaces. Throughout this section, if \( A \) is a functor \( \text{Or}G \to \text{Sp} \), we will write \( A(H) \) for the value of \( A \) at the orbit \( G/H \).

**Definition 3.1** (Orbit smash product). Let \( X \) be a pointed \( G \)-space and \( A \) be a functor \( \text{Or}G \to \text{Sp} \). Define the orbit smash product as the coend (see Section A.3)

\[
X \wedge_{\text{Or}G} A := \int^{G/H} X^H \wedge A(H).
\]
It is straightforward to show that $X \wedge_{\text{Or} G} \mathbf{A} = \Sigma^\infty X \otimes_{\text{Or} G} \mathbf{A}$ where $\otimes_{\text{Or} G}$ is defined in section A.3. Since $\Sigma^\infty$ preserves colimits, it is straightforward to see that the functors $A_* := \pi_*(\wedge_{\text{Or} G} \mathbf{A})$ define a $G$-equivariant homology theory on CW-complexes in the sense of Lück [Lüc19, Definition 2.1]. Moreover, we can equip $\text{Fun}(\text{Or} G, \text{Sp})$ with the object-wise $t$-structure defined in Section A.8. Since $\Sigma^\infty$ also preserves connectivity, we get from Lemma A.17 that if $\mathbf{A}$ is object-wise connective, then $X \wedge_{\text{Or} G} \mathbf{A}$ is connective and further, if $X$ is $m$-connected, then so is $X \wedge_{\text{Or} G} \mathbf{A}$. As a special case we deduce the following useful lemma.

**Lemma 3.2.** Let $X$ be a pointed $G$-CW-complex and $\mathbf{A}$ a functor $\text{Or} G \to \text{Sp}_{\geq 0}$ with values in connective spectra. Denote by $X^{(k)}$ the $k$-skeleton of $X$. Then the homomorphism $\pi_n(X^{(k)} \wedge_{\text{Or} G} \mathbf{A}) \to \pi_n(X \wedge_{\text{Or} G} \mathbf{A})$ is an isomorphism for $n < k$ and an epimorphism for $n = k$.

**Proof.** Taking the cofiber of $X^{(k)} \to X^{(k+1)}$ gives a cofiber sequence

$$X^{(k)} \to X^{(k+1)} \to \bigvee_{i \in I} G/H_i \times S^{k+1}$$

Applying the exact functor $- \wedge_{\text{Or} G} \mathbf{A}$ thus gives the fiber sequence

$$X^{(k)} \wedge_{\text{Or} G} \mathbf{A} \to X^{(k+1)} \wedge_{\text{Or} G} \mathbf{A} \to \bigvee_{i \in I} (G/H_i \times S^{k+1}) \wedge_{\text{Or} G} \mathbf{A}.$$

Since $\mathbf{A}$ is object-wise connective, smashing preserves connectivity (see Lemma A.17), and thus $\pi_n((G/H_i \times S^{k+1}) \wedge_{\text{Or} G} \mathbf{A}) = 0$ for $n < k + 1$. The claimed statements now follow by induction for $X$ being finite dimensional $G$-CW from the long exact sequence in homotopy groups of the above fiber sequence. For general $X$ we have $X = \text{colim}_k X^{(k)}$. The functor $- \wedge_{\text{Or} G} \mathbf{A}$ preserves colimits thus $\pi_n(X \wedge_{\text{Or} G} \mathbf{A}) \cong \pi_n(\text{colim}_k (X^{(k)} \wedge_{\text{Or} G} \mathbf{A})) \cong \pi_n(X^{(k)} \wedge_{\text{Or} G} \mathbf{A})$ for $k > n$, thus reducing the lemma to the finite case. \hfill \Box

**Lemma 3.3.** Suppose $\mathbf{A}$ is a functor $\text{Or} G \to \text{Sp}$ and $X$ a $G$-CW-space that admits a 1-dimensional model of the form

$$\coprod_{i \in I} G/H_i \times S^0 \xrightarrow{f,g} \coprod_{j \in J} G/K_i$$

Then there is a fiber sequence

$$\bigvee_{i \in I} \mathbf{A}(H_i) \xrightarrow{f,g} \bigvee_{j \in J} \mathbf{A}(K_j) \to X_+ \wedge_{\text{Or} G} \mathbf{A}.$$

**Proof.** The functor $(-)_+ \wedge_{\text{Or} G} \mathbf{A}$ is an exact functor from the category of $G$-spaces to $\text{Sp}$. It thus sends the pushout square

$$\coprod_{i \in I} G/H_i \times S^0 \xrightarrow{f,g} \coprod_{j \in J} G/K_i$$

$$\coprod_{i \in I} G/H_i \times D^1 \to X$$
to a pushout square of spectra. We have equivalences \((G/H)_+ \wedge_{\text{Or}G} A \simeq A(H)\), \((G/H \times S^0)_+ \wedge_{\text{Or}G} A \simeq A(H) \vee A(H)\) and \((G/H \times D^1)_+ \wedge_{\text{Or}G} A \simeq A(H)\). This means we have the pushout square

\[
\bigvee_{i \in I} A(H_i) \vee A(H_i) \xrightarrow{(f,g)} \bigvee_{j \in J} A(K_j) \\
\bigvee_{i \in I} A(H_i) \xrightarrow{(\text{id},\text{id})} X_+ \wedge_{\text{Or}G} A.
\]

This is equivalent to the fiber sequence

\[
\bigvee_{i \in I} A(H_i) \vee A(H_i) \xrightarrow{\left(\begin{smallmatrix} \text{id} & \text{id} \\ -f & -g \end{smallmatrix}\right)} \bigvee_{i \in I} A(H_i) \vee \bigvee_{j \in J} A(K_j) \to X_+ \wedge_{\text{Or}G} A.
\]

Elementary row and column reduction now yields the desired fiber sequence. □

4. Assembly and the Farrell-Jones conjecture

**Definition 4.1.** Let \(G\) be a group. A family of subgroups \(\mathcal{F}\) is a set of subgroups that is closed under subgroups and conjugation.

**Example 4.2.** The following four examples of families will be relevant.
- The trivial family Triv consisting of only the subgroup \(\{1\}\).
- The family All consisting of all subgroups.
- The family Fin consisting of all finite subgroups.
- The family VCyc consisting of all virtually cyclic subgroups. A group \(H\) is virtually cyclic if it contains a cyclic subgroup of finite index.

If \(A\) is a functor \(\text{Or}G\) to \(\text{Sp}\) then

\[
\text{colim}_{\text{Or}G} A \simeq A(G/G)
\]

since \(G/G \cong \text{pt}\) is a terminal object in the category \(\text{Or}G\). The property of \(A\) satisfying assembly states that this still holds true when the domain, over which the colimit is taken, is suitably restricted.

**Definition 4.3.** Let \(\mathcal{F}\) be a family of subgroups for the group \(G\) and \(A\) a functor \(\text{Or}G\) to \(\text{Sp}\). Denote by \(\text{Or}G_{\mathcal{F}}\) the full subcategory of \(\text{Or}G\) spanned by the objects \(G/H\) with \(H \in \mathcal{F}\). Then the inclusion \(\text{Or}G_{\mathcal{F}} \subset \text{Or}G\) induces a natural map

\[
\text{colim}_{\text{Or}G_{\mathcal{F}}} A \to \text{colim}_{\text{Or}G} A \simeq A(G/G).
\]

We say \(A\) satisfies assembly for \(\mathcal{F}\) if this map is an equivalence.

**Lemma 4.4.** Assume we have a fiber sequence of functors \(\text{Or}G \to \text{Sp}\),

\[
A \implies B \implies C.
\]

If any two of them satisfy assembly for a family \(\mathcal{F}\), then so does the third.

**Proof.** Both \(\text{colim}_{\text{Or}G_{\mathcal{F}}} A\) as well as \(\text{colim}_{\text{Or}G} A\) are exact functors from the \(\infty\)-category \(\text{Fun}(\text{Or}G, \text{Sp})\) to \(\text{Sp}\) giving the diagram

\[
\begin{array}{ccc}
\text{colim}_{\text{Or}G_{\mathcal{F}}} A & \longrightarrow & \text{colim}_{\text{Or}G_{\mathcal{F}}} B & \longrightarrow & \text{colim}_{\text{Or}G_{\mathcal{F}}} C \\
\text{colim}_{\text{Or}G} A & \longrightarrow & \text{colim}_{\text{Or}G} B & \longrightarrow & \text{colim}_{\text{Or}G} C
\end{array}
\]
with rows being fiber sequences. The statement now follows from the 5-lemma. □

We are concerned with one particular type of functor on the orbit category - the functor that associates $G/H$ to the algebraic $K$-theory spectrum of its group algebra over a fixed base ring $R$, $KH$. However, note that algebraic $K$-theory is a priori only functorial in ring homomorphisms. This models the morphisms $KRH \rightarrow KRH'$ corresponding to inclusions $H \subset H'$. We also need functoriality with respect to conjugation morphisms $c_g : G/H \rightarrow G/(g^{-1}Hg)$. These can give a priori different ring homomorphisms $RH \rightarrow Rg^{-1}Hg$ depending on the choice of representative $g$, meaning there is no good functor $OrG \rightarrow$ Rings. This issue has been addressed by James Davis and Wolfgang Lück, and we will summarize the main results necessary for our work here.

**Lemma 4.5** (Davis, Lück [DL98] Chapter 2 and Lemma 2.4). Let $G$ be a group and $R$ be a ring. There exists a functor of 1-categories $K^\text{alg}R(-) : \text{Grpds} \rightarrow \Omega\text{-Sp}$ where Grpds is the category of groupoids and functors between them and $\Omega\text{-Sp}$ is the 1-category of $\Omega$-Spectra. If $G$ is a groupoid, the spectrum $K^\text{alg}R(G)$ is defined as the non-connective $K$-theory spectrum of the additive category $((P(RG_{\triangle})^\triangledown)^{opp})$

where $G_\triangle$ is the free symmetric monoidal category generated by $G$, $RC$ is the free $R$-linear category generated by $C$, $P(-)$ is idempotent completion, $(-)^\triangledown$ is the underlying groupoid, and $(-)^{opp}$ refers to the group completion of a symmetric monoidal $R$-category. It has the properties

1. If $F_i : G_0 \rightarrow G_1$ for $i = 0, 1$ are functors of groupoids and $T : F_0 \rightarrow F_1$ is a natural transformation between them, then the induced maps of spectra $K^\text{alg}R(F_i) : K^\text{alg}R(G_0) \rightarrow K^\text{alg}R(G_1)$ are homotopy equivalent.

2. Let $G$ be a groupoid. Suppose that $G$ is connected, i.e. there is a morphism between any two objects. For an object $x \in \text{Ob}(G)$, let $G_x$ be the full subgroupoid with precisely one object, namely $x$. Then the inclusion $i_x : G_x \rightarrow G$ induces a homotopy equivalence $K^\text{alg}R(i_x) : K^\text{alg}R(G_x) \rightarrow K^\text{alg}R(G)$ and $K^\text{alg}R(G_x)$ is isomorphic to the non-connective algebraic $K$-theory spectrum associated to the group ring $R\text{aut}G(x)$.

**Theorem 4.6.** Let $G$ be a group and $R$ be a ring. There exists a functor $KR(-) : OrG \rightarrow \text{Sp}$ with the properties

- $KR(G/H) \simeq KH$ where $KH$ is the non-connective algebraic $K$-theory spectrum of the group ring $RG$.
- If $H \subset H'$ giving the canonical map $G/H \rightarrow G/H'$, the induced map $KRH \rightarrow KRH'$ corresponds to the map induced by the ring homomorphism $RH \rightarrow RH'$.
- The action of the conjugation morphism $c_g : G/\{1\} \rightarrow G/\{1\}$ on $KR(G/\{1\}) \simeq KR$ is homotopic to the identity.
Let \( g \in G \). The action of the conjugation morphism \( c_g : G/H \to G/(g^{-1} H g) \) on the homotopy groups \( K_n R H \to K_n R(g^{-1} H g) \) is induced by the ring homomorphism \( RH \to R(g^{-1} H g), h \mapsto g^{-1} h g \) and independent of the chosen representative \( g \).

**Remark 4.7.** The action in the last point of \( c_g \) on \( K_0 \) can be understood in the following way. It sends a f.g. projective \( RH \)-module \( P \) to the \( R(g^{-1} H g) \)-module \( P^g \) with same underlying \( R \)-module as \( P \) and the scalar multiplication of an element \( h' \in g^{-1} H g \) on \( x \in P \) given by \( h' \cdot x := gh'g^{-1} \cdot x \).

**Proof.** The \( \infty \)-category of spectra is a localization of the nerve of the 1-category of \( \Omega \)-spectra. Let \( L : N(\Omega - \text{Sp}) \to \text{Sp} \) be the corresponding localization functor. There is also a natural functor \( G/(-) : \text{Or} G \to \text{Grpd} \) which sends the \( G \)-set \( G/H \) to the groupoid \( G/\int G/H \) with object set \( G/H \) and a morphism from \( gH \) to \( g'H \) for each element \( g'' \in G \) such that \( g''gH = g'H \). Define \( KR(-) \) as the composite

\[ KR(-) := L \circ K^\text{alg} R \circ G/(-) \]

with the functor \( K^\text{alg} R \) being given by Lemma 4.5.

For a subgroup \( H \) of \( G \) write \( BH \) for the groupoid with a single object and automorphism group \( H \). The statement that \( KR(G/H) \simeq KRH \) follows from Lemma 4.5(2), since the automorphism group of the object \( H \) in \( G/\int G/H \) is exactly \( H \). This means we have the inclusion functor \( BH \to G/\int G/H \), which produces the claimed equivalence in the first point.

If \( H \subset H' \) are two subgroups of \( G \), we have a functor \( BH \to BH' \), which fits into the commutative square

\[
\begin{array}{ccc}
BH & \rightarrow & G/\int G/H \\
\downarrow & & \downarrow \\
BH' & \rightarrow & G/\int G/H'.
\end{array}
\]

The functor \( K^\text{alg} R(BH) \to K^\text{alg} R(BH') \) is equivalent to the classical map \( KRH \to KRH' \) induced by the ring homomorphism \( RH \to RH' \), proving the second claim.

For the third claim we use Lemma 4.5(1). The groupoid \( G/\int G/\{1\} \) is contractible since every object is in fact both terminal and initial, which means that all endofunctors of \( G/\int G/\{1\} \) are connected via natural transformations. In particular the functor coming from the conjugation morphism \( c_g \) and the identity realize to homotopic maps in \( K \)-theory.

For the last point, we have a commutative square of groupoids

\[
\begin{array}{ccc}
BH & \xrightarrow{\text{inc}_H} & G/\int G/H \\
\downarrow g^{-1}(-)g & & \downarrow \text{inc}_{g^{-1}Hg} \circ \text{trans}_{g^{-1}} \\
B(g^{-1} H g) & \xrightarrow{\text{inc}_{g^{-1}Hg}} & G/\int (g^{-1} H g)
\end{array}
\]

where the functor \( g^{-1}(-)g : BH \to B(g^{-1} H g) \) is given via the group homomorphism that is the conjugation \( h \mapsto g^{-1} h g \) (it is trivial on objects as both groupoids have only a single object), and

\[ \text{trans}_{g^{-1}} : G/\int G/H \to G/\int G/H \]
is the functor that acts on objects as

\[ g' H \mapsto g^{-1} g' H, \]

and sends the morphism

\[ g' H \xrightarrow{k} k g' H \]

to

\[ g^{-1} g' H \xrightarrow{g^{-1}k g} g^{-1} k g' H. \]

Note that \( \text{trans}_{g^{-1}} \) is an auto-equivalence of \( G \sslash G/H \) that sends the object \( H \) to \( g^{-1} H \). The group homomorphism \( c_g : H \rightarrow g^{-1} H g \) induces the claimed map in \( K_n \), thus showing the last point. If \( g'' \) is another element in \( G \) such that \( c_g = c_{g''} : G/H \rightarrow G/g^{-1} H g \) represent the same map in \( \text{Or} G \), which is exactly the case when \( g(g'')^{-1} \in H \), it is elementary to show that \( \text{trans}_{g^{-1}} \) and \( \text{trans}_{g''}^{-1} \) are naturally equivalent functors, thus inducing homotopic maps on \( K \)-theory spectra and therefore the same map in \( K_n \). \( \square \)

**Definition 4.8** (Farrell-Jones conjecture). A group \( G \) has the Farrell-Jones property if the functor for non-connective algebraic \( K \)-theory \( K R : \text{Or} G \rightarrow \text{Sp} \) satisfies assembly for the family of virtually cyclic subgroups and any ring \( R \). We will also sometimes refer to this as saying that \( G \) satisfies the Farrell-Jones conjecture.

**Remark 4.9.** The Farrell-Jones conjecture is known to hold for a wide range of groups. A recent summary of results can be found in [RV18], Theorem 27.

In order to compute the colimits involved in the assembly maps, a useful tool for geometric arguments is the notion of classifying spaces for a family of subgroups \( F \) of \( G \).

**Definition 4.10.** Let \( F \) be a family of subgroups of \( G \). If \( X \) is a \( G \)-CW-space with

\[ X^H \simeq \begin{cases} \text{pt} & \text{if } H \in F \\ \emptyset & \text{if } H \notin F, \end{cases} \]

we call \( X \) a **classifying space for the family** \( F \). We will write \( E(G; F) \) or sometimes \( EF \) for such a \( G \)-CW-space \( X \).

**Remark 4.11.** Our choice of fixed notation for classifying spaces of families is justified, since they exist and are unique up to \( G \)-equivariant homotopy, see [Lüc05], Theorem 1.9.

**Example 4.12.** The universal cover \( EG \) of \( BG \) with its free \( G \)-action is a classifying space for the trivial family \( \text{Triv} \) consisting only of the single subgroup \( \{1\} \). The point with trivial \( G \)-action is a classifying space for the family of all subgroups.

**Theorem 4.13** (See also [MNN19], Proposition A.2). Let \( i_F \) be the inclusion of the category \( \text{Or} G_F \) into the category of \( G \)-spaces. Then

\[ \text{colim}(i_F) = E(G; F) \]

is a model for a classifying space for \( F \).

**Proof.** Let \( H \) be a subgroup of \( G \). Under the equivalence \( G \)-Spc \( \simeq \text{Fun}(\text{Or} G, \text{Sp}) \) given by Elmendorf’s theorem the operation of taking \( H \)-fixed points corresponds to
evaluation at $G/H$. Since colimits of functors are computed objectwise this means that taking $H$-fixed points commutes with colimits. Now, if $H \notin \mathcal{F}$, then

$$\text{colim}(i_{\mathcal{F}})^H \simeq \text{colim}_{G/K \in \text{Or}_G \mathcal{F}}((G/K)^H) = \emptyset.$$ 

Now suppose $H$ in $\mathcal{F}$. Then

$$\text{colim}(i_{\mathcal{F}})^H \simeq \text{colim}_{G/K \in \text{Or}_G \mathcal{F}}(\text{Map}_{\text{Or}_G \mathcal{F}}(G/H,G/K))$$

is the colimit over a corepresentable functor and thus contractible, see Example A.15.

The following lemma now explains why classifying spaces of families are such a useful tool for understanding assembly maps.

**Lemma 4.14.** Let $\mathcal{F}$ be a family of subgroups of $G$ and $A$ a functor $\text{Or}_G \to \text{Sp}$. There is a natural equivalence

$$\text{colim}_{\text{Or}_G \mathcal{F}} A \to E(G;\mathcal{F})_+ \wedge_{\text{Or}_G} A.$$ 

**Proof.** Since $\wedge_{\text{Or}_G}$ commutes with colimits we have

$$E(G;\mathcal{F})_+ \wedge_{\text{Or}_G} A \simeq (\text{colim}_{G/K \in \text{Or}_G \mathcal{F}}(G/K))_+ \wedge_{\text{Or}_G} A \simeq \text{colim}_{G/K \in \text{Or}_G \mathcal{F}}(G/K_+ \wedge_{\text{Or}_G} A) \simeq \text{colim}_{G/K \in \text{Or}_G \mathcal{F}}(A(K)).$$

We will also need the following two results on assembly in $K$-theory.

**Theorem 4.15** (See [Bar03]). The map

$$E\text{Fin}_+ \wedge_{\text{Or}_G} K R \to E\text{VCyc}_+ \wedge_{\text{Or}_G} K R$$

is split injective and is so naturally with respect to the ring $R$ and the group $G$.

**Lemma 4.16** (See [LR05], Proposition 2.14). The map

$$E\text{Fin}_+ \wedge_{\text{Or}_G} K Q \to E\text{VCyc}_+ \wedge_{\text{Or}_G} K Q$$

is an equivalence.

**Remark 4.17.** A consequence of Lemma 4.16 is that if $G$ satisfies the Farrell-Jones conjecture, the functor $KQ$— actually satisfies finite assembly. Since $KQ_H$ is connective for finite groups $H$, this implies that $KQ$ satisfies assembly too, in the sense that

$$KQG = \text{colim}_{G/H \in \text{Or}_G \text{Fin}} KQH$$

with the colimit in question being relative to the 1-category of abelian groups. This is because we have

$$H KQG \simeq \tau_{\leq 0} \text{colim}_{G/H \in \text{Or}_G \text{Fin}} KQH \simeq \text{colim}_{G/H \in \text{Or}_G \text{Fin}} \tau_{\leq 0} KQH \simeq \text{colim}_{G/H \in \text{Or}_G \text{Fin}} HKQH,$$

since the Postnikov truncation $\tau_{\leq 0}$ commutes with colimits.
5. The Whitehead spectrum $\text{Wh}(R; G)$ and the spectrum $\text{SC}(G)$

We remind the reader that for any ring $R$, $\widetilde{K}_0R$ is defined as the cokernel of the natural homomorphism $K_0\mathbb{Z} \to K_0R$. For a group ring $RG$ the group $K_0RG$ naturally has $K_0R$ as a split summand, with the split given via the augmentation map $RG \to R$ that sends all $g \in G$ to 1. If the base ring $R$ is such that every projective module is stably free, such as when $R$ is a PID or a local ring, it follows that $K_0R \cong K_0\mathbb{Z} \cong \mathbb{Z}$, and we have $K_0RG \cong \mathbb{Z} \oplus \widetilde{K}_0RG$.

We are ultimately interested in understanding the map $\widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G$.

Since our tool of choice, the Farrell-Jones conjecture, gives us a priori the full map on spectra $KZG \to K\mathbb{Q}G$, we would like to split off the superfluous data in a sensible way. This is where the Whitehead spectrum comes into play.

**Definition 5.1.** Given a group $G$ and a ring $R$, we define the Whitehead spectrum $\text{Wh}(R; G)$ to be the cofiber of the assembly map

$$BG_+ \wedge KR \to KR[G] \to \text{Wh}(R; G),$$

corresponding to the trivial family $\text{Triv}$.

**Example 5.2.** Let $R$ be a ring. Then

- $\pi_1 \text{Wh}(R; G) = K_1RG$ for $i > 0$, if $R$ is regular noetherian (see [Wei13], III, Definition 4.1),
- $\pi_0 \text{Wh}(R; G) = \widetilde{K}_0RG$, if $R$ is in addition a local ring or a PID (see [Wei13], II, §2),
- and furthermore, if $R$ is a field or the ring of integers, then

$$\pi_1 \text{Wh}(R; G) = K_1(RG)/\{rg|r \in R^\times, g \in G\}.$$ 

In the particular case of $R$ being the integers we have:

- $\pi_1 \text{Wh}(\mathbb{Z}; G) = K_1ZG$,
- $\pi_0 \text{Wh}(\mathbb{Z}; G) = \widetilde{K}_0ZG$,
- $\pi_1 \text{Wh}(\mathbb{Z}; G) = \text{Wh}(G) = K_1ZG/(\{\pm 1\}G)$, with $\text{Wh}(G)$ being the Whitehead group of $G$.

More generally, we can do the following construction. Let $EG$ be a universal cover of $BG$. Note that $EG$ is equivalently a classifying space for the trivial family. The functor

$$(B(-)_+ \wedge KR)(G/H) := (G/H \times EG)_+ \wedge_{\text{Or}G} KR$$

from $\text{Or}G$ to spectra comes with a natural transformation

$$\theta : B(-)_+ \wedge KR \to KR(-),$$

induced from the projection $G/H \times EG \to G/H$, to the functor $KR$. Let us explain why the notation $B(-)_+ \wedge KR$ makes sense. The value at $G/G$ is given as

$$EG_+ \wedge_{\text{Or}G} KR \simeq \text{colim}_{BG} KR \simeq BG_+ \wedge KR$$

where we used that the subcategory of $\text{Or}G$ generated by the single object $G/\{1\}$ is a $BG$ and that the action of $G$ on the value $KR(G/\{1\}) = KR$ is homotopically trivial. This means that the natural transformation $\theta$ becomes the assembly map

$$BG_+ \wedge KR \to KRG$$
when evaluated at $G/G$. If $G/H$ is an arbitrary object of $\text{Or}_G$, then $G/H \times E(G; \text{Triv}) \simeq \text{Ind}^G_H(E(H; \text{Triv}))$. The functor $\text{Ind}^G_H$ is the left Kan extension induced by the functor $\text{Or}^{op}_H \to \text{Or}^{op}_G$ under the equivalence of Elmendorf's theorem $G\text{-}\text{Spc} \simeq \text{Fun}(\text{Or}^{op}_G, \text{Spc})$. With this understood, we can use Theorem A.18 to get

$$(G/H \times EG)_+ \wedge_{\text{Or}_G} KR = \text{Ind}^G_H(E(H; \text{Triv})_+) \wedge_{\text{Or}_G} KR \simeq E(H; \text{Triv})_+ \wedge_{\text{Or}_H} KR \simeq BH_+ \wedge KR,$$

and see that $\theta$ has as component on the object $G/H$ the assembly map

$$BH_+ \wedge KR \to KRH.$$ 

This allows us to define the functor $\text{Wh}(R; -): \text{Or}_G \to \text{Sp}$ as the cofiber of this natural transformation.

**Remark 5.3.** More generally, if $\mathcal{F}$ is a family and $A: \text{Or}_G \to \text{Sp}$ is a functor, we can define

$$A_\mathcal{F}(G/H) := (G/H \times EF)_+ \wedge_{\text{Or}_G} A$$

and get via the projection $G/H \times EF \to G/H$ a natural transformation

$$A_\mathcal{F} \to A.$$

The functor $A_\mathcal{F}$ can be shown to satisfy $\mathcal{F}$-assembly, and we can think of it as a universal approximation of $A$ from the left by a functor that satisfies $\mathcal{F}$-assembly. This construction appears for example in [DQR11], Lemma 4.1.

The following is an essential lemma that is a consequence of Theorem 4.16 and Lemma A.18.

**Lemma 5.4.** The map

$$E\text{Fin}_+ \wedge_{\text{Or}_G} \text{Wh}(R; -) \to E\text{VCyc}_+ \wedge_{\text{Or}_G} \text{Wh}(R; -)$$

is split injective and is so naturally with respect to the ring $R$ and the group $G$.

**Proof.** We have the commutative diagram

$$
\begin{array}{ccc}
BG_+ \wedge KR & \xrightarrow{=} & BG_+ \wedge KR \\
\downarrow & & \downarrow \\
E\text{Fin}_+ \wedge_{\text{Or}_G} KR & \xrightarrow{} & E\text{VCyc}_+ \wedge_{\text{Or}_G} KR \\
\downarrow & & \downarrow \\
E\text{Fin}_+ \wedge_{\text{Or}_G} \text{Wh}(R; -) & \xrightarrow{} & E\text{VCyc}_+ \wedge_{\text{Or}_G} \text{Wh}(R; -)
\end{array}
$$

with the columns being fiber sequences, so a natural split in the middle map induces one on the bottom. Hence, the statement follows from theorem A.18.

An immediate consequence of Lemma A.4 is that if $G$ satisfies the Farrell-Jones conjecture, the functor $\text{Wh}(R; -)$ on $\text{Or}_G$ satisfies assembly for the family VCyc. moreover, Theorem A.18 implies that for $R = \mathbb{Q}$, the functor $\text{Wh}(\mathbb{Q}; -)$ then also satisfies assembly for the family Fin.
Corollary 5.5. If $G$ satisfies the Farrell-Jones conjecture, then the image of the map

$$\widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G$$

agrees with the image of the map

$$(\ast) \quad \pi_0(\text{Fin}_+ \wedge_{\mathbb{Z}} \text{Wh}(\mathbb{Z}, -)) \to \pi_0(\text{Fin}_+ \wedge_{\mathbb{Q}} \text{Wh}(\mathbb{Q}, -)) \cong \widetilde{K}_0\mathbb{Q}G.$$

In particular if $\ast$ vanishes $p$-locally for some prime $p$, then so does the map

$$\widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G.$$

Proof. By Lemma 5.4 the group $\pi_0(\text{Fin}_+ \wedge_{\mathbb{Z}} \text{Wh}(\mathbb{Z}, -))$ is a split summand of $\widetilde{K}_0\mathbb{Z}G$, similarly for $\mathbb{Q}$, so by naturality of the split with respect to ring homomorphisms, the map $\widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G$ splits as a sum of two maps, the second of which has to be trivial, since $\pi_0(\text{Fin}_+ \wedge_{\mathbb{Q}} \text{Wh}(\mathbb{Q}, -)) \cong \widetilde{K}_0\mathbb{Q}G$. To get the second statement, apply the functor $\mathbb{Z}(p) \otimes -$ and use exactness. □

Definition 5.6. Define the spectrum of singular characters $SC(G)$ as the cofiber

$$\text{Wh}(\mathbb{Z}; G) \to \text{Wh}(\mathbb{Q}; G) \to SC(G).$$

Write in short $SC(G) := \pi_0SC(G)$.

Note that we always have a long exact sequence

$$\cdots \to \text{Wh}(G) \to K_1(\mathbb{Q}G)/\{rg \in \mathbb{Q}^\times \cdot g \in G\} \to \pi_1SC(G) \to \widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G \to SC(G) \to \cdots$$

From this we see that the vanishing of the map $\widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G$ is equivalent to the injectivity of $\widetilde{K}_0\mathbb{Q}G \to SC(G)$. We will give a concrete description of the group SC$(G)$ for finite groups in the next section.

6. LOWER K-THEORY OF FINITE GROUPS

In this section assume $G$ is finite. We will write $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for the $p$-adic integers as well as $p$-adic rationals for a prime $p$. We will be concerned with the groups $K_0\mathbb{Z}_pG, K_0\mathbb{Q}G, K_0\mathbb{Q}_pG$, as well as $K_{-1}\mathbb{Z}G$.

Suppose in the following that $k$ is a subfield of $\mathbb{C}$. It is a standard fact in representation theory that the group ring $kG$ is semisimple. In particular, it decomposes uniquely as

$$kG \cong \prod_{I \in \text{Irr}_k(G)} M_{n_I \times n_I}(D_I)$$

where $\text{Irr}_k(G)$ is the set of isomorphism classes of irreducible $k$-representations of $G$ and $D_I$ are division algebras given by

$$D_I = \text{hom}_G(I, I),$$

and $n_I = \langle kG, I \rangle$ is the multiplicity of $I$ appearing in the regular representation $kG$. This is known as the Wedderburn decomposition of $kG$. The $K$-theory of a division algebra $D$ is $\mathbb{Z}$ in degree 0 since every left $D$-module is in fact a $D$-vectorspace.
and its negative $K$-theory vanishes since $D$ is regular noetherian. Using that $K$-theory commutes with products as well as invariance of $K$-theory under Morita-equivalence, we get the formula $K_0 kG \cong \mathcal{Z}(G)$ where $r_k(G)$ is the number of irreducible $k$-representations of $G$ and the irreducible representations form a set of generators for this group. We also get that $K_{-n}kG = 0$ for all $n > 0$.

**Definition 6.1** (Schur index, see also [Die06] 9.3. and [Ser77] 12.2.). Let $I$ be an irreducible $k$-representation of $G$. Then $D_I$ is a division algebra over its center $K_I$ of degree $m(I)$ with $m(I) = [D_I, E_I]$ for $E_I$ a maximal field contained in $D_I$. We call $m(I)$ the *Schur index* of $I$.

**Lemma 6.2** ([Ser77] 12.2., also [Isa76] Corollary 10.2). Let $\chi$ be an irreducible character. There exists an irreducible $k$-representation $I$, such that

$$\chi_I = m(I) \sum \rho(\chi),$$

with the $\rho(J)$ being all the distinct translates of $\chi$ under the action of the Galois group $\text{Gal}(\mathbb{C}/k)$, in other words the direct sum is over the orbit of the action of the Galois group $\text{Gal}(\mathbb{C}/k)$ on the set of characters. Conversely, if $I$ is an irreducible $k$-representation, its character splits as above.

**Corollary 6.3.** The map $K_0 \mathbb{Q}G \rightarrow K_0 \mathbb{Q}_pG$ is acting on the basis of irreducible $\mathbb{Q}$-representations by

$$[I] \mapsto \frac{m(I)}{m(K_i)} ([K_1] + \cdots + [K_{n_I}]).$$

where the $K_i$ are representatives of the irreducible $\mathbb{Q}_p$-representations that appear in $I \otimes \mathbb{Q}_p$ and $n_I$ depends on $I$. The Schur index $m(K_i)$ is independent of $i$. If $I$ and $J$ are distinct irreducible $\mathbb{Q}$-representations, the irreducible components appearing in their individual $p$-completions are pairwise non-isomorphic. In other words, the map

$$K_0 \mathbb{Q}G \rightarrow K_0 \mathbb{Q}_pG$$

splits as

$$\bigoplus_{I \in \text{Irr}_\mathbb{Q}(G)} \mathbb{Z} \rightarrow \bigoplus_{I \in \text{Irr}_\mathbb{Q}(G)} \mathbb{Z}\{K \in \text{Irr}_{\mathbb{Q}_p}(G)|K \text{ appears as a summand in } I \otimes \mathbb{Q}_p\}.$$

**Proof.** Split the character $\chi_I$ as in Lemma 6.2 for the case $k = \mathbb{Q}$, then apply the Lemma 6.2 to each of the irreducible constituents appearing, using the case $k = \mathbb{Q}_p$. \qed

**Definition 6.4** (Local Schur index). Let $I$ be an irreducible $\mathbb{Q}$-representation of $G$. The *local Schur index* $m_p(I)$ of $I$ at the prime $p$ is defined to be the Schur index of any of the irreducible components of $I \otimes \mathbb{Q}_p$ and is independent of this choice by the argument given above. The *local Schur index at infinity* $m_\infty(I)$ is similarly defined as the Schur index of any of the irreducible components of $I \otimes \mathbb{R}$. \footnote{As a good summary of what is currently known about rational and local Schur indices we recommend [Ung19].}
6.1. **Negative $K$-theory of finite groups.** The following result on negative $K$-theory of group rings is due to Carter.

**Theorem 6.5** (Carter, [Car80a]). Let $G$ be finite. The groups $K_{-i} ZG$ vanish for $i > 1$ and the group $K_{-1} ZG$ has the form

$$K_{-1} ZG = \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s$$

where

$$r = 1 - r_Q + \sum_{p \mid |G|} (r_{Q_p} - r_{F_p})$$

and $s$ is equal to the number of irreducible $\mathbb{Q}$-representations $I$ with even Schur index $m(I)$ but odd local Schur index $m_p(I)$ at every prime $p$ dividing the order of $G$.

We note that it can be shown that the smallest group $G$ such that $s > 0$ is the group $Q_{16}$. Its negative $K$-theory will be computed in Section 9.1.

**Remark 6.6.** If $G$ is a $p$-group, then by Magurn, [Mag13] Theorem 1, the rank $r = 0$, in other words $K_{-1} ZG$ only consists of 2-torsion.

6.2. **Localization squares for finite groups.** The negative $K$-theory groups of $ZG$ for $G$ a finite group have been first computed by Carter in [Car80a]. We will repeat the essential points. First, we need the following lemma, which is a consequence of a theorem due to Karoubi, see e.g. [Wei13], Prop. V.7.5.

**Lemma 6.7.** Let $G$ be a finite group and $P$ the set of primes dividing the order of $G$. Then the following square is a pullback square of spectra:

$$
\begin{array}{ccc}
KZG & \rightarrow & \bigvee_{p \in P} KZ_p G \\
\downarrow & & \downarrow \\
KZ[P^{-1}] G & \rightarrow & \bigvee_{p \in P} KQ_p G,
\end{array}
$$

where the maps appearing are induced by the corresponding inclusions of the involved rings.

**Corollary 6.8.** Fix the same assumptions as in the previous lemma. Then

$$
\begin{array}{ccc}
\text{Wh}(\mathbb{Z}; G) & \rightarrow & \bigvee_{p \in P} \text{Wh}(\mathbb{Z}_p; G) \\
\downarrow & & \downarrow \\
\text{Wh}(\mathbb{Z}[P^{-1}]; G) & \rightarrow & \bigvee_{p \in P} \text{Wh}(\mathbb{Q}_p; G)
\end{array}
$$

is a pullback square of spectra as well.

**Proof.** The square in question is given as the levelwise cofibers of the square

$$
\begin{array}{ccc}
BG_+ \wedge KZ & \rightarrow & \bigvee_{p \in P} BG_+ \wedge KZ_p \\
\downarrow & & \downarrow \\
BG_+ \wedge KZ[P^{-1}] & \rightarrow & \bigvee_{p \in P} BG_+ \wedge KQ_p
\end{array}
$$

and the square in Lemma 6.7. Both are pushouts via the previous lemma, hence the claim follows. \qed
The following is a straightforward consequence of the fact that any idempotent in the rational group algebra \( \mathbb{Q}G \) is already defined over \( \mathbb{Z}[P^{-1}]G \).

**Lemma 6.9.** The map \( K_0 \mathbb{Z}[P^{-1}]G \to K_0 \mathbb{Q}G \) is an isomorphism.

### 6.3. The singular character group \( SC(G) \) for finite \( G \)

Write \( \text{Conj}(G) \) for the set of conjugacy classes of \( G \). Assume \( I \) is a \( k \)-representation of \( G \) for \( G \) finite and \( k \) a field of characteristic 0. The *character* of \( I \) is defined as the function

\[
\chi_I : \text{Conj}(G) \to k \\
g \mapsto \text{tr}(g : I \to I).
\]

This is well-defined since the trace of an endomorphism is invariant under conjugation. Let \( \text{Cl}(G; k) := \text{Fun}(\text{Conj}(G), k) \) be the \( k \)-vector space of class functions of \( G \) with values in \( k \). It is a standard fact from representation theory that the association \( I \mapsto \chi_I \) gives an injection \( K_0 kG \to \text{Cl}(G; k) \). We call a class function in the image of this inclusion a *\( k \)-valued virtual character* of \( G \).

The character of the regular representation \( kG \) is given by

\[
\chi_{kG}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{else.} \end{cases}
\]

In general, for \( I \) any \( k \)-representation, the value of the character \( \chi_I \) at 1 is \( \chi_I(1) = \dim(I) \). It follows that we have a commutative square

\[
\begin{array}{ccc}
K_0 kG & \longrightarrow & K_0 k \\
\downarrow \chi & & \downarrow \dim \\
\text{Cl}(G; k) & \overset{\text{ev}_1}{\longrightarrow} & k.
\end{array}
\]

From this we can deduce that

\[
\tilde{K}_0 kG = \ker(K_0 kG \to K_0 k) \hookrightarrow \text{Fun}(\text{Conj}(G) \setminus \{[1]\}, k),
\]

in other words, we can interpret the reduced \( K \)-theory group as the set of \( k \)-valued virtual characters defined on non-trivial conjugacy classes.

Fix a prime \( p \). An element \( g \) of \( G \) is called singular with respect to \( p \) if \( p \) divides the order of \( g \). Write \( \text{Conj}_p(G) \) for the set of \( p \)-singular conjugacy classes of \( G \). The following theorem can be found in Serre [Ser77], Chapter 16, Theorem 34 and 36:

**Theorem 6.10.** The map \( K_0 \mathbb{Z}_pG \to K_0 \mathbb{Q}_pG \) is split injective, and the image consists of all virtual representations with characters vanishing on \( p \)-singular elements of \( G \).

**Definition 6.11.** As a consequence, we can identify the cokernel of \( K_0 \mathbb{Z}_pG \to K_0 \mathbb{Q}_pG \) with the set of virtual characters defined on \( \text{Conj}_p(G) \). Note that such a character always takes values in \( \mathbb{Q}(\zeta_n) \) where \( n \) is the order of the group \( G \) and \( \zeta_n \) is

---

3Berman’s theorem actually shows that the character \( \chi_I \) for a \( k \)-linear representation \( I \) is a well-defined function on \( k \)-conjugacy classes of \( G \) and the irreducible representations form an orthogonal basis of the space \( \text{Fun}(k\text{-Conj}(G), k) \) with respect to the scalar product given by

\[
\langle \chi_1, \chi_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1}).
\]

In particular the number \( r_k \) is equal to the number of \( k \)-conjugacy classes of \( G \). See [CR81], Theorem 21.5.
an \( n \)-th root of unity. We write \( \text{SC}_p(G) \) for the subgroup of \( \text{Fun}(\text{Conj}_p(G), \mathbb{Q}(\zeta_n)) \) spanned by those characters and we call them \( p \)-singular virtual characters of \( G \).

**Lemma 6.12.** Let \( G \) be finite and \( n \) be its order. The group \( \text{SC}(G) \) of singular characters of \( G \) from Definition 5.6 is isomorphic to the cokernel of the map

\[
\bigoplus_{p \text{ prime}, p\mid n} K_0\mathbb{Z}_pG \to \bigoplus_{p \text{ prime}, p\mid n} K_0\mathbb{Q}_pG.
\]

As a consequence of the previous remark, this can be identified with the subgroup of the group of functions

\[
\text{Fun}\left( \bigotimes_{p \text{ prime}, p\mid n} \text{Conj}_p(G), \mathbb{Q}(\zeta_n) \right)
\]

consisting of tuples \((\chi_p)\) where each \( \chi_p \) is the restriction of a \( \mathbb{Q}_p \)-valued virtual character of \( G \) to the set of \( p \)-singular elements of \( G \). In other words, we have an isomorphism

\[
\text{SC}(G) \cong \bigoplus_{p \text{ prime}, p\mid n} \text{SC}_p(G).
\]

**Remark 6.13.** The group \( K_0\mathbb{Z}_pG \) is free and of rank \( r_{\mathbb{F}_p} \), where \( r_{\mathbb{F}_p} \) is the number of irreducible \( \mathbb{F}_p \)-representations by Serre [Ser77], Chapter 14, Corollary 3 and Chapter 16, Corollary 1. From this it follows that \( \text{SC}(G) \) is finitely generated free of rank

\[
r_{\text{SC}} = \sum_{p \mid |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}),
\]

since it is isomorphic to the sum of the cokernels of the split injective maps

\[
K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG
\]

between free abelian groups of rank \( r_{\mathbb{F}_p} \) and \( r_{\mathbb{Q}_p} \), respectively.

**Remark 6.14.** If \( G \) is a finite \( p \)-group, by the above lemma we have \( \text{SC}(G) \cong \text{cok}(K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG) \). Moreover, since every non-trivial element of \( G \) is \( p \)-singular, by Theorem 6.10 the image of \( K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG \) is generated by those virtual representations \( I \) for which their character \( \chi_I \) vanishes away from 1, which means that \( K_0\mathbb{Z}_pG \) is generated by the free modules. Hence we have an isomorphism \( \text{SC}(G) \cong \tilde{K}_0\mathbb{Q}_pG \).

**Proof.** Recall the pullback square

\[
\begin{array}{ccc}
\text{Wh}(\mathbb{Z}; G) & \longrightarrow & \bigvee_{p \in P} \text{Wh}(\mathbb{Z}_p; G) \\
\downarrow & & \downarrow \\
\text{Wh}(\mathbb{Z}[P^{-1}]; G) & \longrightarrow & \bigvee_{p \in P} \text{Wh}(\mathbb{Q}_p; G)
\end{array}
\]

of Corollary 6.8. Denote by \( C \) the common vertical cofiber, i.e.

\[
C := \text{cof}\left( \text{Wh}(\mathbb{Z}; G) \to \text{Wh}(\mathbb{Z}[P^{-1}]; G) \right)
\]

\[
\simeq \text{cof}\left( \bigvee_{p \in P} \text{Wh}(\mathbb{Z}_p; G) \to \bigvee_{p \in P} \text{Wh}(\mathbb{Q}_p; G) \right).
\]
The groups $K_{-n}\mathbb{Z}_pG$ vanish for $n > 0$ (see [Car80b], Page 619), in other words $Wh(\mathbb{Z}_p; G)$ is connective. The spectrum $Wh(\mathbb{Q}_p; G)$ is connective as well, hence so is $C$, and we have

$$\pi_0 C = \text{cok} \left( \bigoplus_{p \text{ prime}, p|n} \widetilde{K}_0\mathbb{Z}_pG \to \bigoplus_{p \text{ prime}, p|n} \widetilde{K}_0\mathbb{Q}_pG \right).$$

Note that the summand of $\widetilde{K}_0\mathbb{Q}_pG$ corresponding to free $\mathbb{Q}_pG$-modules lies in the image of $K_0\mathbb{Z}_pG \to \widetilde{K}_0\mathbb{Q}_pG$ hence the cokernel does not change when going to unreduced $K$-theory, therefore

$$\pi_0 C \cong \text{cok} \left( \bigoplus_{p \text{ prime}, p|n} K_0\mathbb{Z}_pG \to \bigoplus_{p \text{ prime}, p|n} K_0\mathbb{Q}_pG \right).$$

There is a natural map $C \to SC(G)$ induced by the map $Wh(\mathbb{Z}[P^{-1}]; G) \to Wh(\mathbb{Q}; G)$, which by Lemma 6.9 is an isomorphism in $\pi_0$. The 5-lemma thus implies that $\pi_0 C \cong \pi_0 SC(G) = SC(G)$. □

**Remark 6.15.** If $G$ is a $p$-group then all non-trivial elements of $G$ are $p$-singular, hence $K_0\mathbb{Z}_pG \cong K_0\mathbb{Z} \cong \mathbb{Z}$. We can conclude that $\widetilde{K}_0\mathbb{Z}_pG = 0$. Since by Lemma 6.12 the group $SC(G)$ of singular characters is given by the cokernel of $K_0\mathbb{Z}_pG \to \widetilde{K}_0\mathbb{Q}_pG$ we have an isomorphism $SC(G) \cong \widetilde{K}_0\mathbb{Q}_pG$.

**Lemma 6.16.** Let $G$ be finite. There is a natural short exact sequence

$$0 \to \widetilde{K}_0\mathbb{Q}G \to SC(G) \to K_{-1}ZG \to 0,$$

which is a free resolution of the abelian group $K_{-1}ZG$, and the map $\widetilde{K}_0\mathbb{Q}G \to SC(G)$ simply sends a rational representation $I$ to the corresponding singular character $(\chi_p)_p \text{ prime}$ defined as $\chi_p(g) := \chi_I(g)$, where $\chi_I$ is the character of $I$.

**Proof.** The long exact sequence of the fiber sequence

$$Wh(\mathbb{Z}; G) \to Wh(\mathbb{Q}; G) \to SC(G)$$

gives the exact sequence

$$\widetilde{K}_0\mathbb{Z}G \to \widetilde{K}_0\mathbb{Q}G \to SC(G) \to K_{-1}ZG \to 0.$$

Theorem 1.1 implies that this sequence splits off to the left, giving the claimed short exact sequence. The group $\widetilde{K}_0\mathbb{Q}G$ is free since it is isomorphic to $\text{ker}(\widetilde{K}_0\mathbb{Q}G \to \widetilde{K}_0\mathbb{Q})$, which is, as a subgroup of the free abelian group $K_0\mathbb{Q}G$, again free and $SC(G)$ is free as discussed in Remark 6.13. Lastly, the claim that $\widetilde{K}_0\mathbb{Q}G \to SC(G)$ sends a representation to the singular character $(\chi_p)_p \text{ prime}$ follows from the fact that $\widetilde{K}_0\mathbb{Q}G \to SC(G)$ factors as

$$\widetilde{K}_0\mathbb{Q}G \to \bigoplus_{p \mid |G|} K_0\mathbb{Q}_pG \to SC(G),$$

where the first map is induced by the ring homomorphisms $\mathbb{Q} \to \mathbb{Q}_p$ and the second by Lemma 6.12. □
Define the Bockstein morphism \( \beta_n : \mathbb{H} \mathbb{Z}/n \to \Sigma \mathbb{H} \mathbb{Z} \) as the boundary morphism to the fiber sequence of spectra

\[
\mathbb{H} \mathbb{Z} \xrightarrow{n} \mathbb{H} \mathbb{Z} \to \mathbb{H} \mathbb{Z}/n.
\]

The following theorem is now a consequence of Lemma 6.16 and Theorem 6.5.

**Theorem 6.17.** Let \( G \) be finite and let \( s \) be the number of irreducible \( \mathbb{Q} \)-representations with even Schur index but odd local Schur index at every prime \( p \) dividing the order of \( G \). The map of spectra \( \text{Wh}(\mathbb{Z}; G)[-1, 0] \to \text{Wh}(\mathbb{Q}; G)[-1, 0] \) factorizes as

\[
\text{Wh}(\mathbb{Z}; G)[-1, 0] \xrightarrow{p} \Sigma^{-1} \mathbb{H}(\mathbb{Z}/2)^* \xrightarrow{(\beta_2)} \mathbb{H} \mathbb{Z}^* \xrightarrow{i} \mathbb{H} \tilde{K}_0 \mathbb{Q} G \cong \text{Wh}(\mathbb{Q}; G)[-1, 0]
\]

where the map \( p \) is given by the Postnikov truncation of \( \text{Wh}(\mathbb{Z}; G)[-1, 0] \) followed by the projection onto the torsion summand of \( K_{-1} \mathbb{Z} G \) and the map \( i : \mathbb{H} \mathbb{Z}^* \to \mathbb{H} \tilde{K}_0 \mathbb{Q} G \) is induced by the inclusion of all linear combinations of the irreducible \( \mathbb{Q} \)-representations that contribute to \( s \).

We want to stress the importance of this theorem to the reader in regard to the analysis of the map \( \tilde{K}_0 \mathbb{Z} G \to \tilde{K}_0 \mathbb{Q} G \). The first major obstruction for generalizing the triviality of \( \tilde{K}_0 \mathbb{Z} G \to \tilde{K}_0 \mathbb{Q} G \) from finite to arbitrary groups lies in the fact that while \( \text{Wh}(\mathbb{Z}; G)[-1, 0] \to \text{Wh}(\mathbb{Q}; G)[-1, 0] \) for finite groups \( G \) is trivial on homotopy groups, it is not the trivial map of spectra, unless \( s(G) = 0 \).

**Proof.** The map \( \tilde{K}_0 \mathbb{Z} G \to \tilde{K}_0 \mathbb{Q} G \) is zero for \( G \) being finite by Theorem 1.1 and \( \text{Wh}(\mathbb{Q}; G)[-1, 0] \) is actually concentrated in degree 0 since the negative \( K \)-theory of \( \mathbb{Q} G \) vanishes. This means we can apply Lemma A.4 to see that

\[
\text{Wh}(\mathbb{Z}; G)[-1, 0] \to \text{Wh}(\mathbb{Q}; G)[-1, 0]
\]

factors through a unique map

\[
\Sigma^{-1} \mathbb{H} K_{-1} \mathbb{Z} G \to \mathbb{H} \tilde{K}_0 \mathbb{Q} G.
\]

It corresponds under Lemma A.3 to the short exact sequence

\[
0 \to \tilde{K}_0 \mathbb{Q} G \to \text{SC}(G) \to K_{-1} \mathbb{Z} G \to 0
\]

from Lemma 6.16. Now \( K_{-1} \mathbb{Z} G = \mathbb{Z}^* \oplus (\mathbb{Z}/2)^* \) by Theorem 6.5. The abelian group \( \tilde{K}_0 \mathbb{Q} G \) is free and maps of degree 1 of the form \( \Sigma^{-1} \mathbb{H} \mathbb{Z} \to \mathbb{H} \mathbb{Z} \) are necessarily zero, since \( \text{Ext}^1_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}) = 0 \). This means the map \( \Sigma^{-1} \mathbb{H} K_{-1} \mathbb{Z} G \to \mathbb{H} \tilde{K}_0 \mathbb{Q} G \) further factors through the 2-torsion, i.e. as

\[
\Sigma^{-1} \mathbb{H}(\mathbb{Z}/2)^* \to \mathbb{H} \tilde{K}_0 \mathbb{Q} G.
\]

The generators of \( \tilde{K}_0 \mathbb{Q} G \) are given by the isomorphism classes of non-trivial irreducible \( \mathbb{Q} \)-representations of \( G \). By Theorem 6.5, each of these contributes to a single \( \mathbb{Z}/2 \)-summand in \( K_{-1} \mathbb{Z} G \) iff it has even global Schur index but odd local Schur index at every prime \( p \) dividing the order of \( G \), giving rise to a Bockstein morphism \( \beta_2 \). In other words, the map \( \Sigma^{-1} \mathbb{H}(\mathbb{Z}/2)^* \to \mathbb{H} \tilde{K}_0 \mathbb{Q} G \) factors further through \( i \),

\[
\Sigma^{-1} \mathbb{H}(\mathbb{Z}/2)^* \xrightarrow{(\beta_2)^*} \mathbb{H} \mathbb{Z}^* \xrightarrow{i} \mathbb{H} \tilde{K}_0 \mathbb{Q} G,
\]

with \( i \) being the inclusion of the subgroup of \( \tilde{K}_0 \mathbb{Q} G \) generated by all the irreducible \( \mathbb{Q} \)-representations that contribute to \( s \). \( \square \)
7. The map $K_0 \mathbb{Z}G \to K_0 \mathbb{Q}G$ for infinite groups

The following section is concerned with proving the main theorem. Throughout, assume that $G$ satisfies the Farrell-Jones conjecture and that $E\operatorname{Fin}$ is a fixed model for the classifying space of finite subgroups together with a chosen CW-structure $(E\operatorname{Fin}^{(k)})_{k \in \mathbb{N}}$. Write

$$(f, g): \prod_{i \in I} G/H_i \times S^0 \to \prod_{j \in J} G/K_j$$

for the degree 0 attaching map of $E\operatorname{Fin}$ with the $H_i$ and $K_j$ being finite subgroups of $G$. For a functor $F$: $OrG \to \operatorname{Ab}$ define

$$\ker^F := \ker(F(f) - F(g)) : \bigoplus_{i \in I} F(G/H_i) \to \bigoplus_{j \in J} F(G/K_j).$$

**Theorem 7.1.** There is an exact sequence

$$0 \to \ker \tilde{K}_0 \mathbb{Q} \to \ker \mathcal{S} \to \ker\tilde{K}_1 \mathbb{Z} \to \operatorname{im}(\tilde{K}_0 \mathbb{Z}G \to \tilde{K}_0 \mathbb{Q}G) \to 0$$

and the map $\ker\tilde{K}_1 \mathbb{Z} \to \operatorname{im}(\tilde{K}_0 \mathbb{Z}G \to \tilde{K}_0 \mathbb{Q}G)$ is the connecting map induced from the snake lemma applied to the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \bigoplus_{i \in I} \tilde{K}_0 \mathbb{Q}(H_i) \\
\downarrow & & \downarrow \quad f-g \\
0 & \longrightarrow & \bigoplus_{j \in J} \tilde{K}_0 \mathbb{Q}(K_j)
\end{array}
\begin{array}{ccc}
\longrightarrow & \bigoplus_{i \in I} SC(H_i) & \longrightarrow & \bigoplus_{i \in I} K_1 \mathbb{Z}(H_i) & \longrightarrow & 0 \\
\longrightarrow & \bigoplus_{j \in J} SC(K_j) & \longrightarrow & \bigoplus_{j \in J} K_1 \mathbb{Z}(K_j) & \longrightarrow & 0.
\end{array}
$$

Before we begin with the proof, we need a few more arguments. If $D$ is a 1-category, then the category of functors $D \to \operatorname{Ab} \subset \operatorname{Sp}$ with values in the heart $\operatorname{Ab}$ of $\operatorname{Sp}$ is again a 1-category and thus a natural transformation $\eta: \mathbf{A} \Longrightarrow \mathbf{B}$ between two functors $\mathbf{A}, \mathbf{B}: D \to \operatorname{Sp}$ with values in the heart is the zero map in the category $\operatorname{Fun}(D, \operatorname{Sp})$ if and only if its value on all the components $\eta_d: \mathbf{A}(d) \to \mathbf{B}(d)$ is the zero homomorphism for all $d \in D$. Note that here it is essential that the category of functors with values in the heart of $\operatorname{Sp}$ is again a 1-category, it is not true in general that a natural transformation between two functors with values in spectra is zero if all its components are zero maps.

Now, since the map $\tilde{K}_0 \mathbb{Z}H \to \tilde{K}_0 \mathbb{Q}H$ vanishes for all finite subgroups $H$, the natural transformation $H\tilde{K}_0 \mathbb{Z} \Longrightarrow H\tilde{K}_0 \mathbb{Q}$ becomes the zero map when restricted to the subcategory $OrG_{\mathbb{F}_\operatorname{in}}$. Furthermore, $\operatorname{Wh}(\mathbb{Q}; -)[-1, 0]$ is as a functor on $OrG_{\mathbb{F}_\operatorname{in}}$ concentrated in degree 0 since the negative $K$-theory of the group algebras $\mathbb{Q}H$ vanishes for $H$ being finite. By using the object-wise $t$-structure on $\operatorname{Fun}(OrG_{\mathbb{F}_\operatorname{in}}, \operatorname{Sp})$ (see Definition A.3), we are now in a position to apply Lemma A.4 with $\mathcal{C} = \operatorname{Fun}(OrG_{\mathbb{F}_\operatorname{in}}, \operatorname{Sp})$, and the map $f$ in question being the natural transformation $\operatorname{Wh}(\mathbb{Z}; -)[-1, 0] \Longrightarrow \operatorname{Wh}(\mathbb{Q}; -)[-1, 0]$. Lemma A.4 states that the natural transformation of functors

$$\operatorname{Wh}(\mathbb{Z}; -)[-1, 0] \Longrightarrow \operatorname{Wh}(\mathbb{Q}; -)[-1, 0]$$

descends to a unique natural transformation of functors $OrG_{\mathbb{F}_\operatorname{in}} \to \operatorname{Sp}$,

$$\Sigma^{-1}H\tilde{K}_1 \mathbb{Z} \Longrightarrow H\tilde{K}_0 \mathbb{Q}.$$
Proposition 7.2. If $G$ satisfies the Farrell-Jones conjecture, the image of the map $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ agrees with the image of
$$\pi_1E\text{Fin}_+ \wedge_{\text{Or}G} HK_{-1}\mathbb{Z} \to \tilde{K}_0\mathbb{Q}G,$$
as well as with the image of the map
$$\pi_1E\text{Fin}_+^{(1)} \wedge_{\text{Or}G} HK_{-1}\mathbb{Z} \to \tilde{K}_0\mathbb{Q}G$$induced by the inclusion $E\text{Fin}^{(1)} \subset E\text{Fin}$.

Proof. The fiber sequence
$$\text{H}_\sim \tilde{K}_0\mathbb{Z} \rightarrow \text{Wh}(\mathbb{Z}; G)[-1, 0] \rightarrow \Sigma^{-1}HK_{-1}\mathbb{Z} \rightarrow \ldots$$
of functors leads to the exact sequence
$$\cdots \rightarrow \pi_0E\text{Fin}_+ \wedge_{\text{Or}G} (\text{Wh}(\mathbb{Z}; -)[-1, 0]) \rightarrow \pi_0E\text{Fin}_+ \wedge_{\text{Or}G} \Sigma^{-1}HK_{-1}\mathbb{Z} \rightarrow \pi_1E\text{Fin}_+ \wedge_{\text{Or}G} \text{H}_\sim \tilde{K}_0\mathbb{Z} \rightarrow \ldots$$

Since $\text{H}_\sim \tilde{K}_0\mathbb{Z}$ is a connective functor and smashing with a $G$-space preserves connectivity, the group $\pi_{-1}E\text{Fin}_+ \wedge_{\text{Or}G} \text{H}_\sim \tilde{K}_0\mathbb{Z}$ vanishes, which means that the map
$$\pi_0E\text{Fin}_+ \wedge_{\text{Or}G} (\text{Wh}(\mathbb{Z}; -)[-1, 0]) \rightarrow \pi_0E\text{Fin}_+ \wedge_{\text{Or}G} \Sigma^{-1}HK_{-1}\mathbb{Z}$$is an epimorphism.

As discussed before, we have a commuting triangle of natural transformations of functors $\text{Or}G_{\text{Fin}} \rightarrow \text{Sp}$,

$$\text{Wh}(\mathbb{Z}; -)[-1, 0] \longrightarrow \text{H}_\sim \tilde{K}_0\mathbb{Z} \longrightarrow \Sigma^{-1}HK_{-1}\mathbb{Z}$$

Taking colimits over $\text{Or}G_{\text{Fin}}$, we get the triangle

$$\pi_0E\text{Fin}_+ \wedge_{\text{Or}G} (\text{Wh}(\mathbb{Z}; -)[-1, 0]) \longrightarrow \tilde{K}_0\mathbb{Q}G$$
$$\pi_1E\text{Fin}_+ \wedge_{\text{Or}G} HK_{-1}\mathbb{Z},$$

which together with Lemma 5.5 proves the first statement.

For the second statement we use Lemma 3.2 to get that
$$\pi_1E\text{Fin}_+^{(1)} \wedge_{\text{Or}G} HK_{-1}\mathbb{Z} \rightarrow \pi_1E\text{Fin}_+ \wedge_{\text{Or}G} HK_{-1}\mathbb{Z}$$is an epimorphism. This allows us to reduce further to the image of the composition
$$\pi_1E\text{Fin}_+^{(1)} \wedge_{\text{Or}G} HK_{-1}\mathbb{Z} \rightarrow \pi_1E\text{Fin}_+ \wedge_{\text{Or}G} HK_{-1}\mathbb{Z} \rightarrow \pi_0E\text{Fin}_+ \wedge_{\text{Or}G} \text{H}_\sim \tilde{K}_0\mathbb{Q} \equiv \tilde{K}_0\mathbb{Q}G.$$
Proof of Theorem 7.1. We can already reduce the image of
\[ \widetilde{K}_0 \mathbb{Z} \to \widetilde{K}_0 \mathbb{Q} \]
to that of the map
\[ \pi_1 E \text{Fin} \_1^{(1)} \wedge_{\text{Or} G} H K^{-1} \mathbb{Z} \to \widetilde{K}_0 \mathbb{Q} G \]
thanks to Corollary 7.2.

By Lemma 3.3, there is the following commutative diagram of spectra
\[
\begin{array}{ccc}
\Sigma^{-1} E \text{Fin} \_1^{(1)} \wedge_{\text{Or} G} H K^{-1} \mathbb{Z} & \longrightarrow & H \Sigma H \widetilde{K}_0 \mathbb{Q} G \\
(\ast) & & \\
H(\bigoplus_{i \in I} K^{-1} \mathbb{Z} H_i) & \longrightarrow & \Sigma H(\bigoplus_{i \in I} \widetilde{K}_0 \mathbb{Q} H_i) \\
& \downarrow f-g & \\
H(\bigoplus_{j \in J} K^{-1} \mathbb{Z} K_j) & \longrightarrow & \Sigma H(\bigoplus_{j \in J} \widetilde{K}_0 \mathbb{Q} K_j)
\end{array}
\]
with the columns being fiber sequences.

By Lemma A.5, the map induced on \( \pi_0 \) on the fibers is equivalent to the map induced by the snake lemma of the diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{i \in I} \widetilde{K}_0 \mathbb{Q} H_i & \longrightarrow & \bigoplus_{i \in I} \text{SC}(H_i) & \longrightarrow & \bigoplus_{i \in I} K^{-1} \mathbb{Z} H_i & \longrightarrow & 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \longrightarrow & \bigoplus_{i \in I} \widetilde{K}_0 \mathbb{Q} K_j & \longrightarrow & \bigoplus_{i \in I} \text{SC}(K_j) & \longrightarrow & \bigoplus_{i \in I} K^{-1} \mathbb{Z} K_j & \longrightarrow & 0
\end{array}
\]
with exact rows.

This means we get the claimed exact sequence
\[ 0 \to \ker \widetilde{K}_0 \mathbb{Q} \to \ker \text{SC} \to \ker K^{-1} \mathbb{Z} \to \text{im}(\widetilde{K}_0 \mathbb{Z} G \to \widetilde{K}_0 \mathbb{Q} G) \to 0. \]

\[ \square \]

8. Virtually cyclic groups

A group \( G \) is called virtually cyclic if it contains a cyclic subgroup of finite index. Virtually cyclic groups can be classified into three families of groups.

Lemma 8.1 (See Hem04, Lemma 11.4.) A group \( G \) is virtually cyclic if it is of one of the three forms

- \( G \) is finite.
- \( G \) is finite-by-infinite cyclic. This means that there is an exact sequence of groups
  \[ 1 \to H \to G \to C_\infty \to 1 \]
  with \( H \) being finite, and \( C_\infty \) an infinite cyclic group. We will call \( G \) of type VC1.
- \( G \) is finite-by-infinite dihedral. This means that there is an exact sequence of groups
  \[ 1 \to H \to G \to D_\infty \to 1 \]
  with \( H \) being finite, and \( D_\infty \) an infinite dihedral group. We will call \( G \) of type VC2.
8.1. Virtually cyclic groups of type 1. In the following fix a group \( G \) of type VC1 and write \( H \) for the unique maximal finite subgroup. Write \( \pi: G \to G/H \cong C_\infty \) for the canonical projection. Since the kernel \( H \) of \( G \to C_\infty \) is finite, the following is easy to show:

**Lemma 8.2.** Let \( G \) be of type VC1. Then a model of the classifying space \( E(G; \text{Fin}) \) is given by \( \mathbb{R} \) with the action lifted from the translation action of \( C_\infty = G/H \).

A \( C_\infty \)-CW-structure of \( \mathbb{R} \) with the translation action can be described with the following pushout square.

\[
\begin{array}{ccc}
C_\infty/1 \times S^0 & \xrightarrow{(id,t)} & C_\infty/1 \\
\downarrow & & \downarrow \\
C_\infty/1 \times D^1 & \longrightarrow & \mathbb{R}
\end{array}
\]

This generalizes for \( G \) being of type VC1 in the following way. Let \( \tilde{t} \in G \) be a choice of lift of the generator \( t \) in \( C_\infty \). Then the following is a pushout square of \( G \)-spaces.

\[
\begin{array}{ccc}
G/H \times S^0 & \xrightarrow{(id,\tilde{t})} & G/H \\
\downarrow & & \downarrow \\
G/H \times D^1 & \longrightarrow & \mathbb{R}
\end{array}
\]

Applying Lemma 8.3 now states that if \( F \) is any functor \( \text{Or}_G \to \text{Sp} \), then there is a fiber sequence

\[
F(H) \xrightarrow{1-\tilde{t}} F(H) \to E(G; \text{Fin})_+ \wedge F.
\]

The functors \( K_0 \mathbb{Q} \text{−} \) and \( K_{-1} \mathbb{Z} \text{−} \) satisfy finite assembly (see Remark 4.17 as well as Corollary 8.9). We thus have the exact sequences

\[
K_0 \mathbb{Q} H \xrightarrow{1-\tilde{t}} K_0 \mathbb{Q} H \to K_0 \mathbb{Q} G \to 0
\]

and

\[
K_{-1} \mathbb{Z} H \xrightarrow{1-\tilde{t}} K_{-1} \mathbb{Z} H \to K_{-1} \mathbb{Z} G \to 0.
\]

**Theorem 8.3.** Let \( G \) be a group of type VC1. Then \( K_0 \mathbb{Q} G \) is a finitely generated and free abelian group.

**Proof of Theorem 8.3.** Since \( K \mathbb{Q} \text{−} \) satisfies finite assembly, as remarked above, we have the exact sequence

\[
K_0 \mathbb{Q} H \xrightarrow{1-\tilde{t}} K_0 \mathbb{Q} H \to K_0 \mathbb{Q} G \to 0.
\]

What is left to understand is the action of \( \tilde{t} \) on \( K_0 \mathbb{Q} H \). The endomorphisms of the object \( G/H \) in the category \( \text{Or}_G \) are equal to \( N(H)/H = G/H = C_\infty = \langle t \rangle \). By Theorem 4.6, this action of \( \tilde{t} \) on \( K_0 \mathbb{Q} H \) sends a representation \( V = (V, \rho) \) to the representation \( V_t = (V, \rho((t^{-1})t-1)) \). Let \( \text{Irr}_\mathbb{Q}(G) \) be the set of isomorphism classes of irreducible representations of \( H \) over \( \mathbb{Q} \). If \( V \) is irreducible, then so is \( V_t \), hence

\[
K_0 \mathbb{Q} G \cong \text{cok}(1-t) \cong (K_0 \mathbb{Q} H)_{C_\infty} = \mathbb{Z}[\text{Irr}_\mathbb{Q}(G)]_{C_\infty} \cong \mathbb{Z}[\text{Irr}_\mathbb{Q}(G)/ \equiv]
\]

with \( \equiv \) being the equivalence relation generated by \( V \equiv V_t \). Hence \( K_0 \mathbb{Q} G \), is free generated by the finite set of \( C_\infty \)-equivalence classes of rational irreducible representations of \( H \). \( \square \)
Corollary 8.4. The map $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ is trivial for $G$ of type VC1.

Proof. A virtually cyclic group trivially satisfies the Farrell-Jones conjecture. Theorem 1.3 implies that the image of the map $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ is torsion, which has to be trivial, assuming Theorem 8.3. □

8.2. Virtually cyclic groups of type 2. In the following fix a group $G$ of type VC2. Write $\pi: G \to G/H \cong D_\infty$ for the canonical projection. Let $H$ be the kernel of $\pi$. It is not difficult to show that a group $G$ is of type VC2 iff $G/H \cong K_1 \ast_H K_2$, where $K_1$ and $K_2$ are two finite groups that both contain $H$ as an index 2 subgroup (see e.g. [LG13], Theorem 17). Waldhausen [Wal78] showed that in this case there is a fiber sequence

$$KRH \to KRK_1 \vee KRK_2 \to KRG/\text{Nil}^W_R,$$

where the spectrum $KRG/\text{Nil}^W_R$ is a natural split summand of $KRG$, i.e. we have

$$KRG \cong KRG/\text{Nil}^W_R \vee \text{Nil}^W_R.$$

Moreover, the spectrum $\text{Nil}^W_R$ is contractible if $RH$ is a regular coherent ring. We will get the same result using a geometric understanding of the classifying space $E(G;\text{Fin})$.

We can equip $\mathbb{R}$ with an action of $D_\infty = \langle a, b | a^2 = b^2 = 1 \rangle$ by sending $a$ to the reflection around 0 and $b$ to the reflection around 1/2. The $D_\infty$-space $\mathbb{R}$ is easily seen to be a model for $E(D_\infty;\text{Fin})$. If $G$ is any group of type VC2, we can equip $\mathbb{R}$ with a $G$-action via the projection $G \to D_\infty$. Since the kernel $H$ of $G \to D_\infty$ is finite, we conclude the following:

Lemma 8.5. Let $G$ be virtually cyclic of type 2. The $G$-space $\mathbb{R}$ with action lifted from the projection $G \to D_\infty$ is a model for $E(G;\text{Fin})$.

Consequently, we get a nice pushout description for the $G$-space $E(G;\text{Fin})$.

Lemma 8.6. Suppose $G = K_1 \ast_H K_2$ is of type VC2. Then there is a pushout square

$$
\begin{array}{ccc}
G/H \times S^1 & \longrightarrow & G/K_1 \sqcup G/K_2 \\
\downarrow & & \downarrow \\
G/H \times D^1 & \longrightarrow & E\text{Fin}
\end{array}
$$

of $G$-spaces, giving $E\text{Fin}$ a 1-dimensional $G$-CW-structure.

Proof. By Lemma 8.5 the space $\mathbb{R}$ with the action lifted from the projection $\pi: G \to D_\infty$ is a model for $E\text{Fin}$. This means we may as well assume that $G = D_\infty = \langle a, b | a^2 = b^2 = 1 \rangle$, i.e. $H = \{1\}$, $K_1 = \langle a \rangle$, $K_2 = \langle b \rangle$. Now it is an elementary exercise to see that $\mathbb{R}$ indeed fits into a pushout square of the shape

$$
\begin{array}{ccc}
D_\infty \times S^1 & \longrightarrow & D_\infty / \langle a \rangle \sqcup D_\infty / \langle b \rangle \\
\downarrow & & \downarrow \\
D_\infty \times D^1 & \longrightarrow & \mathbb{R}.
\end{array}
$$

□
Write $\iota_1$ for the inclusions $H \hookrightarrow K_1$. The functors $K_0 \mathbb{Q}^-$ and $K_{-1} \mathbb{Z}^-$ satisfy finite assembly (see Remark 4.17 as well as Corollary 8.9) therefore as a consequence of the pushout square from Lemma 8.6 we get the exact sequence

$$
\tilde{K}_0 \mathbb{Q} \mathbb{H} \xrightarrow{(\iota_{1,-1})} \tilde{K}_0 \mathbb{Q} K_1 \oplus \tilde{K}_0 \mathbb{Q} K_2 \to \tilde{K}_0 \mathbb{Q} G \to 0
$$

and we have a long exact sequence from Lemma 7.1

$$
0 \to \ker \tilde{K}_0 \mathbb{Q} \to \ker \tilde{K}_0 \mathbb{Q} \to \ker K_{-1} \mathbb{Z} \to \im(\tilde{K}_0 \mathbb{Z} G \to \tilde{K}_0 \mathbb{Q} G) \to 0
$$

with

$$
\ker \tilde{K}_0 \mathbb{Q} \cong \ker \left( \tilde{K}_0 \mathbb{Q} \mathbb{H} \xrightarrow{(\iota_{1,-1})} \tilde{K}_0 \mathbb{Q} K_1 \oplus \tilde{K}_0 \mathbb{Q} K_2 \right)
$$

$$
\ker \tilde{K}_0 \mathbb{Q} \cong \ker \left( \tilde{K}_0 \mathbb{Q} \mathbb{H} \xrightarrow{(\iota_{1,-1})} \tilde{K}_0 \mathbb{Q} K_1 \oplus \tilde{K}_0 \mathbb{Q} K_2 \right)
$$

$$
\ker K_{-1} \mathbb{Z} \cong \ker \left( K_{-1} \mathbb{Z} \mathbb{H} \xrightarrow{(\iota_{1,-1})} K_{-1} \mathbb{Z} K_1 \oplus K_{-1} \mathbb{Z} K_2 \right).
$$

**Remark 8.7.** We will construct an example of a group $G$ of type $VC_2$, for which the map $\tilde{K}_0 \mathbb{Z} G \to \tilde{K}_0 \mathbb{Q} G$ is non-trivial, in section 9.

8.3. **Negative $K$-theory of virtually cyclic groups.** The following theorem due to Farrell, Jones extends Carter’s results to virtually cyclic groups.

**Theorem 8.8** ([FJ93], Theorem 2.1.). Let $G$ be a virtually infinite cyclic group. Then

(a) $K_n \mathbb{Z} G = 0$ for all integers $n \leq -2$.

(b) $K_{-1} \mathbb{Z} G$ is generated by the images of $K_{-1} \mathbb{Z} F$ under the maps induced by the inclusions $F \subset G$ where $F$ varies over representatives of the conjugacy classes of finite subgroups of $G$.

(c) $K_{-1} \mathbb{Z} G$ is a finitely generated abelian group.

This has a few implications for groups that satisfy the Farrell Jones conjecture.

**Corollary 8.9.** Let $G$ be a group satisfying the Farrell Jones conjecture. Then

- $K_n \mathbb{Z} G = 0$ for all integers $n \leq -2$.
- The functor $K_{-1} \mathbb{Z}^-$ satisfies finite assembly in the sense that

$$
K_{-1} \mathbb{Z} G \cong \colim_{H \in \text{Or} \mathbb{G}_{\text{Fin}}} K_{-1} \mathbb{Z} H.
$$

**Proof.** The first statement is clear, since Theorem 8.8 (a) implies that the functor $K_{-1} \mathbb{Z}^-$ is ($-1$)-connective, when restricted to the category $\text{Or} \mathbb{G}_{VCyc}$ and thus

$$
K \mathbb{Z} G \cong \colim_{H \in \text{Or} \mathbb{G}_{VCyc}} K \mathbb{Z} H
$$

is ($-1$)-connective as well.

For the second statement, the Farrell Jones conjecture implies that

$$
\text{H} K_{-1} \mathbb{Z} G \simeq \tau_{\leq -1} \colim_{H \in \text{Or} \mathbb{G}_{VCyc}} K \mathbb{Z} H
$$

$$
\simeq \colim_{H \in \text{Or} \mathbb{G}_{VCyc}} \tau_{\leq -1} K \mathbb{Z} H \simeq \colim_{H \in \text{Or} \mathbb{G}_{VCyc}} \text{H} K_{-1} \mathbb{Z} H,
$$

since the Postnikov truncation $\tau_{\leq -1}$ commutes with colimits and the functor $K_{-1} \mathbb{Z}^-$ is $-1$-connective. Hence we need to show that

$$
\colim_{H \in \text{Or} \mathbb{G}_{\text{Fin}}} \text{H} K_{-1} \mathbb{Z} H \to \colim_{H \in \text{Or} \mathbb{G}_{VCyc}} \text{H} K_{-1} \mathbb{Z} H
$$

induced by the inclusion $\text{Or} G_{\text{Fin}} \subset \text{Or} G_{\text{VCyc}}$ is an isomorphism. Theorem 4.15 already states that it is injective. Surjectivity is implied by Theorem 8.8 (b). □

9. A COUNTEREXAMPLE TO THE INTEGRAL $\tilde{K}_0\mathbb{Z}G$-TO-$\tilde{K}_0\mathbb{Q}G$ CONJECTURE

The following section is concerned with an example of a group $G$ with the property that $\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0\mathbb{Q}G$ is non-trivial.

For the construction take $Q_{16}$ contained in the semidihedral group $QD_{32}$. We will show that the group $G = QD_{32} \ast Q_{16}$ has the property that $\tilde{K}_0\mathbb{Z}G$ maps onto a summand $\mathbb{Z}/2$ sitting inside $\tilde{K}_0\mathbb{Q}G$. The group $G$ is not special in this regard. All computations have been done using the computer algebra system GAP, [20].

9.1. The group $Q_{16}$. We let a presentation of $Q_{16}$ be given as

$$Q_{16} = \langle r, s \mid r^8 = 1, r^4 = s^2, srs^{-1} = r^7 \rangle.$$

It has the following conjugacy classes:

| Class | \{1\} | \{s^2\} | \{r^{2}, r^{4}\} | \{s, r^2s, r^4s, r^6s\} |
|-------|-------|-------|---------------|------------------|
| Order | 1     | 2     | 4             | 4                |
| Size  | 1     | 1     | 2             | 4                |

Since $Q_{16}$ is a 2-group, by Remark 6.6 the group $K^{-1}\mathbb{Z}Q_{16}$ must be torsion. Using the “wedderga” package in GAP, we can check the Schur indices appearing in the Wedderburn decomposition of $QQ_{16}$:

```gap
gap> G := QuaternionGroup(16);
gap> WedderburnDecompositionWithDivAlgParts( GroupRing( Rationals, G ) );
[[ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ], [ 1, rec( Center := NF(8,[ 1, 7 ]), DivAlg := true, LocalIndices := [ [ infinity, 2 ] ], SchurIndex := 2 ) ]]
```

A few comments on how to read this output are needed. As described in Section 6, the group algebra $\mathbb{Q}G$ splits as

$$\mathbb{Q}G \cong \prod_{I \in \text{Irrep}(G)} M_{n_I \times n_I}(D_I),$$

with the $D_I$ being finite dimensional division algebras over $\mathbb{Q}$. The function `WedderburnDecompositionWithDivAlgParts` returns a list containing information about each part $M_{n_I \times n_I}(D_I)$ appearing in the Wedderburn decomposition. First, we have 6 entries corresponding to the 6 irreducible representations of $Q_{16}$. The first number in each of the entries refers to the number $n_I$. Next to it is information about $D_I$. In our case the first 5 entries happen to have $D_I = \mathbb{Q}$. For the last entry, its division algebra $D$ is non-commutative, which is signalled by `DivAlg := true`. The center $A$ of $D$ is a finite field extension of $\mathbb{Q}$ and described as $\text{NF}(8,[1,7])$. This notation means that $A$ is a sub-field of the cyclotomic field.
extension $\mathbb{Q}(\zeta_8)$ being fixed by the subgroup $\{1, 7\}$ of the Galois group
$(\mathbb{Z}/8)\times = \{1, 3, 5, 7\}$. It is not difficult to see that $A = \mathbb{Q}(\sqrt{2})$, using the code:

```gap
gap> A := NF( 8, [ 1 , 7 ] );
NF(8,[1,7])
gap> Dimension(A);
2
gap> Sqrt(2) in A;
true
```

This means we have the decomposition

$$\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times M_{2\times 2}(\mathbb{Q}) \times D.$$  

The entry `SchurIndex` gives the global Schur index of the representation $I$ and is displayed only when it is bigger than 1. `LocalIndices` gives a list of all primes at which the local Schur index of $I$ is not equal to 1, together with the real Schur index for the value `infinity`.

In our case we can see that $\mathbb{Q}Q_{16}$ has a single irreducible rational representation $\alpha$ contributing to $s(\mathbb{Q}Q_{16})$, with endomorphism algebra $D$, hence $K_{-1}\mathbb{Q}Q_{16} = \mathbb{Z}/2$. This representation is concretely given by the action of $\mathbb{Q}Q_{16}$ on the quaternion algebra $H_{\mathbb{Q}(\sqrt{2})} := \mathbb{Q}(\sqrt{2}) \langle i, j | i^4 = j^4 = -1, ij = -ji \rangle$ over the field $\mathbb{Q}(\sqrt{2})$, realized by

$$
\begin{align*}
    r &\mapsto \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \\
    s &\mapsto j.
\end{align*}
$$

acting via left multiplication on $H_{\mathbb{Q}(\sqrt{2})}$. Note that $\alpha$ can also be characterized as the unique faithful irreducible $\mathbb{Q}$-representation of $\mathbb{Q}Q_{16}$.

### 9.2. The group $QD_{32}$

A presentation of $QD_{32}$ is given as

$$QD_{32} = \langle a, b | a^{16} = 1, b^2 = 1, bab = a^7 \rangle.$$  

It is easy to see that every element of $QD_{32}$ can be represented in the form $a^nb^i$ for $n = 0, \ldots, 15$ and $i = 0, 1$, from which it follows that $QD_{32}$ has in fact 32 elements. The inclusion $Q_{16} \to QD_{32}$ can be realized by sending $r \mapsto a^2, s \mapsto ab$ as seen by the calculations

$$(a^2)^4 = a^8 = a(a^7b)b = a(ba)b = (ab)^2$$

as well as

$$(ab)a^2 = a(a^{2+7})b = (a^2)^7(ab).$$

The image of this homomorphism consists of all $a^nb^i$ for which $n + i$ is even, of which there are exactly 16 elements from which it follows that it actually is an inclusion.

The conjugacy classes are given as follows

| Class     | $\{1\}$ | $\{a^8\}$ | $\{a^{27}b\}$ | $\{a^4, a^{10}\}$ | $\{a^{27}+1b\}$ | $\{a^4, a^{14}\}$ |
|-----------|----------|------------|----------------|-------------------|-----------------|-------------------|
| Order     | 1        | 2          | 2              | 4                 | 4               | 8                 |
| Size      | 1        | 8          | 2              | 2                 | 8               | 2                 |

| Class     | $\{a^8, a^{10}\}$ | $\{a, a^7\}$ | $\{a^4, a^9\}$ | $\{a^2, a^{15}\}$ | $\{a^{11}, a^{13}\}$ |
|-----------|---------------------|--------------|----------------|-------------------|----------------------|
| Order     | 8                   | 16           | 16             | 16                | 16                   |
| Size      | 2                   | 2            | 2              | 2                 | 2                    |
Similarly to before, $QD_{32}$ is a 2-group, so $K_{-1}QD_{32}$ is torsion. Doing the same computation of the Schur indices appearing in the Wedderburn decomposition of $QD_{32}$ we get:

gap> G := SmallGroup(32,19);
<pc group of size 32 with 5 generators>

```
gap> WedderburnDecompositionWithDivAlgParts( GroupRing( Rationals, G ) );
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ], [ 2, NF(8,[ 1, 7 ] ) ], [ 2, NF(16,[ 1, 7 ] ) ] ]
```

The values $(32,19)$ refer to the ID of $QD_{32}$ in the SmallGroups library of GAP.

Similarly to before, this means we have the Wedderburn decomposition

$$QD_{32} \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times M_{2 \times 2}(\mathbb{Q}) \times M_{2 \times 2}(A_1) \times M_{2 \times 2}(A_2),$$

with $A_1$ being the sub-field of $\mathbb{Q}(\zeta_8)$ fixed by $\{1, 7\} \subset \mathbb{Z}/8 \times$ and $A_2$ being the sub-field of $\mathbb{Q}(\zeta_{16})$ fixed by $\{1, 7\} \subset \mathbb{Z}/16 \times$. From this we can see that no irreducible rational representations contribute to $s$. Hence $K_{-1}QD_{32} = 0$.

9.3. The group $QD_{32} \ast_{Q_{16}} QD_{32}$. The group we want to consider is the group

$$G := QD_{32} \ast_{Q_{16}} QD_{32}.$$

A concrete presentation is given by

$$G = \left\langle a, b, a', b' \mid a^{16} = 1, b^2 = 1, aba^{-1} = a^7, a' = 1, b'^2 = 1, \right\rangle.$$

The group $G$ is virtually cyclic of type 2, which means that we can apply the formulas from Section 8.2. We have the long exact sequence

$$0 \to \ker \tilde{\text{Ker}}_{0Q} \to \ker \text{SC} \to \ker K_{-1Z} \to \text{im}(\tilde{\text{Ker}}_{0ZG} \to \tilde{\text{Ker}}_{0QG}) \to 0.$$

The previous calculations show that $K_{-1}Q_{16} = \mathbb{Z}/2$ and $K_{-1}QD_{32} = 0$, which gives $K_{-1Z} = \mathbb{Z}/2$. We claim that the map $\ker K_{-1Z} \to \text{im}(\tilde{\text{Ker}}_{0ZG} \to \tilde{\text{Ker}}_{0QG})$ is injective, which is equivalent to the map $\ker \text{SC} \to \ker K_{-1Z}$ being trivial. Since $Q_{16}$ and $QD_{32}$ are 2-groups, by Remark 6.13 we have isomorphisms $\text{SC}(Q_{16}) \cong \tilde{\text{Ker}}_{0Q_2}Q_{16}$ and similarly $\text{SC}(QD_{32}) \cong \tilde{\text{Ker}}_{0Q_2}QD_{32}$ by Lemma 6.12. This means that

$$\ker \text{SC} \cong \ker(\tilde{\text{Ker}}_{0Q_2}Q_{16} \ast_{Q_2} \tilde{\text{Ker}}_{0Q_2}QD_{32}) \cong \tilde{\text{Ker}}_{0Q_2}QD_{32} \ast_2 \tilde{\text{Ker}}_{0Q_2}QD_{32}.$$

By Corollary 6.3 the map $\tilde{\text{Ker}}_{0Q_2}Q_{16} \to \tilde{\text{Ker}}_{0Q_2}Q_{16}$ splits as

$$\bigoplus_{I \in \text{Irr}_2(G)} \mathbb{Z} \to \bigoplus_{I \in \text{Irr}_2(G)} \mathbb{Z} \{K \in \text{Irr}_2(Q_{16}) \mid K \text{ appears as a summand in } I \otimes \mathbb{Q}_2\}.$$

Now, since $K_{-1}Q_{16} = \mathbb{Z}/2$ there is a unique irreducible $\mathbb{Q}_2$-representation $\beta$ of $Q_{16}$ such that $\alpha \otimes_2 Q_2 = 2\beta$. The negative $K$-theory group $K_{-1}Q_{16}$ is generated by the image of $\beta$. Neither $\alpha$ nor $\beta$ can lie in the kernels of $\iota_1$ and $\iota_2$, respectively, since their inductions to $QD_{32}$ are neither the trivial nor regular representations (by looking at their dimensions), which shows the claim.

In summary, we have just shown that for the group $G$,

$$\text{im}(\tilde{\text{Ker}}_{0ZG} \to \tilde{\text{Ker}}_{0QG}) \cong \mathbb{Z}/2.$$
9.4. Other examples. The group $QD_{32}$ is not special beyond the property that the map $K_1 \mathbb{Z}Q_{16} \to K_1 \mathbb{Z}QD_{32}$ is not injective. In fact, the group $Q_{16}$ sits inside 5 different groups of order 32. This can be checked with the GAP code:

```
gap> for G in AllSmallGroups(32) do
>   if ForAny( NormalSubgroups(G), H -> IdSmallGroup(H) = [16, 9] ) then
>     Print(IdSmallGroup(G)); fi;
> od;
[32, 19], [32, 20], [32, 41], [32, 42], [32, 44]
```

Here the value (16, 9) refers to the ID of $Q_{16}$ in the SmallGroups library in GAP.

We will analyse them case by case:

1. ID $=[32, 19]$, also known as $QD_{32}$: As discussed $K_1 \mathbb{Z}QD_{32} = 0$.
2. ID $=[32, 20]$, also known as $Q_{32}$: Here we can show that $K_1 \mathbb{Z}Q_{16} \to K_1 \mathbb{Z}Q_{32}$ induces an isomorphism. (Note: This is not due to $Q_{32}$ but due to the fact that $Q_{16}$ is not special.)
3. ID $=[32, 41]$, also known as $Q_{16} \times C_2$: Here we can show that $K_1 \mathbb{Z}(Q_{16} \times C_2) = (\mathbb{Z}/2)^3$ and the map $K_1 \mathbb{Z}Q_{16} \to K_1 (Q_{16} \times C_2)$ corresponds to the diagonal $\mathbb{Z}/2 \to (\mathbb{Z}/2)^2$. In particular it is injective.
4. ID $=[32, 42]$, also known as $C_4 \circ D_8$: This group has $K_1 \mathbb{Z}C_4 \circ D_8 = 0$.
5. ID $=[32, 44]$, also known as $C_8.C_2^2$: Here we can show that the map $K_1 \mathbb{Z}Q_{16} \to K_1 \mathbb{Z}(C_8.C_2^2)$ induces an isomorphism.

In summary, the only virtually cyclic groups of type 2 that contain $Q_{16}$ as kernel for which $\text{im}(\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0QG)$ is non-trivial are the groups $QD_{32} \ast_{Q_{16}} QD_{32}, QD_{32} \ast_{Q_{16}} (C_4 \circ D_8)$, and $(C_4 \circ D_8) \ast_{Q_{16}} (C_4 \circ D_8)$.

In each of those cases we have

$$\text{im}(\tilde{K}_0\mathbb{Z}G \to \tilde{K}_0QG) = \mathbb{Z}/2.$$

10. Comparison to related functors

We can ask if there is a more general statement to the one considered in this paper on $K_0$ for higher $K$-groups. Here the most natural way to generalize to $\pi_1$ would be to understand the map

$$\text{Wh}_1(\mathbb{Z}; G) \to \text{Wh}_1(\mathbb{Q}; G).$$

For finite $G$, we have that

$$\text{Wh}_1(\mathbb{Z}; G) \cong \mathbb{Z}^{r_{\mathbb{R}}} \oplus SK_1(\mathbb{Z}G),$$

where $r_{\mathbb{R}}$ and $r_{\mathbb{Q}}$ are the number of real and rational representations, respectively, and $SK_1(\mathbb{Z}G)$ is a finite group, as well as

$$\text{Wh}_1(\mathbb{Q}; G) \cong \mathbb{Z}^{r_{\mathbb{Q}}},$$

and the map

$$\text{Wh}_1(\mathbb{Z}; G) \to \text{Wh}_1(\mathbb{Q}; G)$$

is an isomorphism on the free parts and has kernel $SK_1(\mathbb{Z}G)$.

Using the groups of type $G = H \times (C_\infty)^2$, the Bass-Heller-Swan decomposition already tells us that any defects of the maps in $K_1$, $\tilde{K}_0$ and $\text{Wh}_1$ for virtually cyclic groups $H$ will enter the picture. Thus we can easily find counterexamples to the possibility that for example the map $\text{Wh}_1(\mathbb{Z}; G) \to \text{Wh}_1(\mathbb{Q}; G)$ is an isomorphism.
any non-trivial group \( H \)) or that it kills all torsion (not true in \( \tilde{K}_0 \) by the results of this paper).

**Appendix A. Stable \( \infty \)-categories and \( t \)-structures**

In this appendix we develop some of the tools for dealing with \( t \)-structures on stable \( \infty \)-categories. The standard reference will be Section 1.2.1 in Lurie, [Lur17]. We note that if \( C \) is a stable \( \infty \)-category, its homotopy category is naturally a triangulated category. The notion of a \( t \)-structure has been first developed for triangulated categories in [BD82] and we will give a definition here.

**Definition A.1.** Let \( D \) be a triangulated category. A \( t \)-structure on \( D \) is a pair of full subcategories \( D_{\geq 0} \) and \( D_{\leq 0} \), both closed under isomorphisms, such that the following three conditions hold. Here \( D_{\geq n} := \Sigma^n D_{\geq 0} \) and \( D_{\leq n} := \Sigma^n D_{\leq 0} \) are defined as the essential images under the functors \( \Sigma^n \) for all \( n \in \mathbb{Z} \).

- If \( X \in D_{\geq 0}, Y \in D_{\leq -1} \) then \( \text{Hom}_D(X,Y) = 0 \).
- \( D_{\geq 1} \subset D_{\geq 0}, D_{\leq 1} \subset D_{\leq 0} \)
- For all objects \( X \) in \( D \) we have a distinguished triangle
  \[ X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X' \]
  with \( X' \in D_{\geq 0} \) and \( X'' \in D_{\leq -1} \).

A \( t \)-structure on a stable \( \infty \)-category is defined to be a \( t \)-structure on its homotopy category \( hC \). We define two types of subcategories of \( C \), namely \( C_{\geq n} \) and \( C_{\leq n} \), as the full subcategories of \( C \) corresponding to the subcategories \( hC_{\geq n} \) and \( hC_{\leq n} \) of \( hC \), respectively. The inclusions of the subcategories \( C_{\geq n} \) in \( C \) admit right adjoints denoted by \( \tau_{\geq n} \), which act as the identity when restricted to \( C_{\geq n} \). Consequently, \( C_{\geq n} \) is closed under colimits in \( C \). Dually, the subcategories \( C_{\leq n} \) in \( C \) admit left adjoints \( \tau_{\leq n} \), which act as the identity on \( C_{\leq n} \), and \( C_{\leq n} \) is closed under limits in \( C \). The compositions \( \tau_{\geq n} \circ \tau_{\leq m} \) and \( \tau_{\leq m} \circ \tau_{\geq n} \) are naturally equivalent and will be denoted as \( A \rightarrow A[m,n] \), or \( A \rightarrow A[n] \) in the case \( n = m \).

The intersection \( C^\heart := C_{\geq 0} \cap C_{\leq 0} \) is called the heart of \( C \) and is (equivalent to the nerve of) an abelian 1-category. There are functors \( \pi_n := (A \rightarrow (\Sigma^{-n} A)[0]) \), from \( C \) to \( C^\heart \) which will be called homotopy group functors.

**A.1. Homological algebra in the setting of \( t \)-structures.** The following section is concerned with the relationship between computations involving fiber sequences in \( C \) and homological algebra in the abelian category \( C^\heart \). The following theorem is the central part of this section: Fiber sequences in \( C \) give rise to long exact sequences in \( C^\heart \).

**Theorem A.2 (See [BD82], Theorem 1.3.6).** Let \( C \) be a stable \( \infty \)-category with a \( t \)-structure. Let
\[ X \xrightarrow{f} Y \xrightarrow{g} Z \]
be a fiber sequence. Then there is an induced long exact sequence
\[ \cdots \rightarrow \pi_{n+1}Z \rightarrow \pi_nX \xrightarrow{\pi_nf} \pi_nY \xrightarrow{\pi_ng} \pi_nZ \rightarrow \pi_{n-1}X \rightarrow \cdots \]
where the maps \( \pi_nZ \rightarrow \pi_{n-1}X \) come from \( \pi_n \) applied to the boundary map \( Z \rightarrow \Sigma X \) which realizes the cofiber of \( X \rightarrow Y \).

---

\(^4\)Our choice of notation clashes here with the one used in [BD82], where \( X[n] \) denotes the \( n \)-th suspension of \( X \).
Next, we are concerned with degree 1 maps between objects in the heart \( C \).

**Lemma A.3.** Let \( C \) be a stable \( \infty \)-category with a \( t \)-structure and let \( A, C \) be two objects in the heart \( C^\circ \). There is a natural isomorphism
\[
\phi: [C, \Sigma A] \cong \text{Ext}^1_{C^\circ}(C, A)
\]
where
\[
\phi(\beta: C \to \Sigma A) = (0 \to A \to \text{fib}(\beta) \to C \to 0).
\]

**Proof.** To show that \( \phi \) is well-defined, we still have to show that \( \text{fib}(\beta) \) lies in the heart of \( C \). To do so, note that by Theorem A.2 we have the long exact sequence in homotopy groups
\[
\cdots \to \pi_1(C) \to \pi_1(\Sigma A) \to \pi_0(\text{fib}(\beta)) \to \pi_0(C) \to \pi_0(\Sigma A) \to \cdots.
\]
Since \( A \) and \( C \) are in the heart, we have \( \pi_1(C) = 0, \pi_0(\Sigma A) = 0 \), which shows that \( \text{fib}(\beta) \) lies in the heart. Furthermore, \( \pi_0(C) = C \) and \( \pi_1(\Sigma A) = A \), which means we do, in fact, get the claimed exact sequence.

The inverse map \( \psi: \text{Ext}^1_{C^\circ}(C, A) \to [C, \Sigma A] \) is constructed as follows. A given exact sequence
\[
0 \to A \to B \to C \to 0
\]
in the heart produces a fiber sequence
\[
A \to B \to C
\]
in \( C \) which can be mapped to the boundary map \( \delta: C \to \Sigma A \). It is clear that that the two processes are mutually inverse. \( \square \)

The following lemma clarifies an argument used multiple times during the main part of this paper.

**Lemma A.4.** Let \( C \) be a stable \( \infty \)-category with a \( t \)-structure and let \( A \) be an object of \( C \) concentrated in degrees \(-1\) and 0 and \( B \) an object in the heart \( C^\circ \). Let \( f \) be map \( A \to B \) such that the composition \( A[0] \to A \xrightarrow{f} B \) is zero. Then \( f \) factorizes through an up to homotopy unique map \( \tilde{f}: A[-1] \to B \), i.e. we have a commutative triangle
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A[-1] & \xrightarrow{\tilde{f}} & B
\end{array}
\]

**Proof.** We have the fiber sequence
\[
A[0] \to A \to A[-1],
\]
which implies the long exact sequence
\[
[\Sigma A[0], B] \to [A[-1], B] \to [A, B] \to [A[0], B].
\]
The abelian group \( [\Sigma A[0], B] \) is zero since \( \Sigma A[0] \) is 1-connected and \( B \) was assumed to be concentrated in degree 0, so if \( f: A \to B \) is a map that becomes the zero map when precomposed with \( A[0] \to A \), it factors through a map \( \tilde{f}: A[-1] \to B \) which is unique up to homotopy. \( \square \)
The last part of this section is concerned with a technical lemma about the relationship between the induced map on fibers coming from a square involving degree 1 maps and the well known connecting map from the snake lemma in homological algebra.

**Lemma A.5.** Let $\mathcal{C}$ be a stable $\infty$-category with a $t$-structure and let $A_1, A_2, C_1$ and $C_2$ be objects in the heart $\mathcal{C}^\triangledown$ and suppose we have a commutative square

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\beta_1} & \Sigma A_1 \\
\downarrow f_C & & \downarrow \Sigma f_A \\
C_2 & \xrightarrow{\beta_2} & \Sigma A_2
\end{array}
$$

Write $\text{fib}_A := \text{fib}(f_A)$, $\text{cok}_A := \pi_\leq \text{fib}_A$, $\text{fib}_C := \text{fib}(f_C)$, and $\ker_C := \pi_0 \text{fib}_C$. Then the induced map $\pi_0 \text{fib}_C \to \pi_0 \Sigma \text{fib}_A$ agrees with the map $\delta: \ker_C \to \text{cok}_A$ induced by the snake lemma for the corresponding map of exact sequences

$$
0 \to A_1 \to B_1 \to C_1 \to 0 \\
0 \to A_2 \to B_2 \to C_2 \to 0
$$

in the heart $\mathcal{C}^\triangledown$.

Before we begin the proof of Lemma A.5, we want to establish some facts about the map induced by the well known snake lemma (see e.g. [Wei95], Lemma 1.3.2).

**Lemma A.6.** Suppose $\mathcal{A}$ is an abelian category and

$$
\begin{array}{ccc}
0 & \to & A_1 \\
\downarrow f_A & & \downarrow f_C \\
0 & \to & A_2 \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{\pi_1} & C_1
\end{array}
$$

$$
\begin{array}{ccc}
B_2 & \xrightarrow{\pi_2} & C_2 \\
\downarrow f_C & & \downarrow f_A \\
B_1 & \xrightarrow{\pi_1} & C_1
\end{array}
$$

is a diagram in $\mathcal{A}$ with exact rows. Write $\ker_C := \ker(f_C)$ and $\text{cok}_A := \text{cok}(f_A)$. Then:

1. There is an isomorphism

   $$\theta: \frac{\ker(B_1 \oplus A_2 \to (g_2, f_C) \to B_2)}{\text{im}(A_1 \to (g_1, f_A) \to B_1 \oplus A_2)} \cong \ker_C$$

   induced by the composition $B_1 \oplus A_2 \to B_1 \xrightarrow{\pi_1} C_1$, where the first map is the projection onto the first summand.

2. There is a natural map

   $$\phi: \frac{\ker(B_1 \oplus A_2 \to (g_2, f_C) \to B_2)}{\text{im}(A_1 \to (g_1, f_A) \to B_1 \oplus A_2)} \to \text{cok}_A$$

   induced by the composition $B_1 \oplus A_2 \to B_2 \to \text{cok}_A$, where the first map is projection onto the second summand.

3. The composition $\phi \theta^{-1}: \ker_C \to \text{cok}_A$ is the natural connecting map from the snake lemma.
(4) If \( \pi_1 \) and \( \pi_2 \) have sections \( s_1 : C_1 \to B_1, \ s_2 : C_2 \to B_2 \) such that

\[
\begin{array}{c}
B_1 \leftarrow s_1 C_1 \\
\downarrow fB \quad \downarrow fC \quad \downarrow \pi_1 \\
B_2 \leftarrow s_2 C_2
\end{array}
\]

commutes then \( \phi \theta^{-1} = 0. \)

**Proof.** To show that \( \theta \) is well-defined, we need to show:

- The composition
  \[
  A_1 \xrightarrow{(g_1,fA)} \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) \to B_1 \xrightarrow{\pi_1} C_1
  \]
  is the trivial map. This is simply because \( \pi_1 g_1 = 0. \) From this follows that the map
  \[
  \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) \to B_1 \xrightarrow{\pi_1} C_1
  \]
  factors through \( \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) / \text{im}(A_1 \xrightarrow{(g_1,fA)} B_1 \oplus A_2). \)

- The composition
  \[
  \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) \to B_1 \xrightarrow{\pi_1} C_1 \xrightarrow{fC} C_2
  \]
  is the trivial map. We have the following equalities of maps
  \[
  \begin{align*}
  \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) & \to B_1 \xrightarrow{\pi_1} C_1 \xrightarrow{fC} C_2 \\
  \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) & \to B_1 \xrightarrow{fB} B_2 \xrightarrow{\pi_2} C_2
  \end{align*}
  \]
  \[
  \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) \to A_2 \xrightarrow{g_2} B_2 \xrightarrow{\pi_2} C_2
  \]

The claim now follows since \( \pi_2 g_2 = 0. \) From this follows that the map

\[
\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2) \to B_1 \xrightarrow{\pi_1} C_1
\]

maps into \( \ker C. \)

To see that \( \theta \) is an isomorphism, we show two things:

- \( \ker(\theta) = 0. \) This is because
  \[
  \ker(B_1 \oplus A_2 \to B_1 \to C_1) = A_1 \oplus A_2
  \]
  which implies
  \[
  \ker(B_1 \oplus A_2 \to B_1 \to C_1) \cap \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-fB)} B_2)
  = \text{im}(A_1 \xrightarrow{(g_1,fA)} B_1 \oplus A_2)
  \]
  Hence, \( \ker(\theta) = 0. \)

- \( \text{cok}(\theta) = 0. \) A simple diagram chase using exactness at \( B_2 \) shows that
  \[
  \text{im}(\ker(B_1 \oplus A_2 \to B_2) \to B_1) = \ker(\pi_2 f_B)
  \]
  Hence, since \( B_1 \to C_1 \) is an epimorphism,
  \[
  \text{im}(\ker(B_1 \oplus A_2 \to B_2) \to B_1 \to C_1) = \ker C
  \]
  From this follows that \( \theta \) is an epimorphism.
We now need to show that $\phi$ is well-defined. To do so, we need to show that the composition

$$A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2 \rightarrow A_2 \rightarrow \text{cok} A$$

is trivial. This is clear, however, as the composition $A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2 \rightarrow A_2$ is just equal to $f_A$.

The next claim is that the map $\phi\theta^{-1}$ agrees with the map induced by the snake lemma. For simplicity, assume that $A$ is the category of abelian groups.

The traditional way of defining the boundary map $\delta$ goes as follows. Assume $c \in \ker C$. Using surjectivity of $B_1 \rightarrow C_1$, find a preimage $b_1 \in B$ of $c$. Since $\pi_2 f_B = f_C \pi_1$, the element $f(b_1)$ lies in the kernel of $\pi_2$; hence, there is a unique $a_2 \in A_2$ such that $g_2(a_2) = f(b_1)$. The image of $\delta$ of the element $c$ is defined as the class of $a_2$ in the cokernel $\text{cok} A$. The reason this agrees with $\phi\theta^{-1}$ is as follows. The class $[b_1,a_2]$ is just a preimage of $c$ under the map $\theta$ and the assignment $[b_1,a_2] \mapsto [a_2]$ is exactly what defines the map $\phi$.

Lastly, assume $\pi_1$ and $\pi_2$ have commuting sections $s_1$ and $s_2$ respectively. Then the map $C_1 \xrightarrow{(s_1,0)} B_1 \oplus A_2$ descends to the inverse of $\theta$. It is then clear that $\phi\theta^{-1} = 0$ since $\phi$ is induced by projection on the $A_2$ coordinate. □

We still need to introduce some new terminology. Suppose $D$ is a commutative square

$$\begin{array}{ccc}
X_1 & \xrightarrow{g_1} & Y_1 \\
\downarrow f_X & & \downarrow f_Y \\
X_2 & \xrightarrow{g_2} & Y_2
\end{array}$$

in a stable $\infty$-category $\mathcal{C}$. Define the \textit{iterated cofiber} of $D$ as

$$\text{cof}(D) := \text{cof}(\text{cof}(g_1) \rightarrow \text{cof}(g_2)) \simeq \text{cof}(\text{cof}(f_X) \rightarrow \text{cof}(f_Y))$$

Define $\square := \Delta^1 \times \Delta^1$. It is clear that taking iterated cofibers is functorial in the sense that it defines an exact functor

$$\text{cof}: \text{Fun}(\square, \mathcal{C}) \rightarrow \mathcal{C},$$

which is the left adjoint to the functor that sends an object $X \in \mathcal{C}$ to the square

$$\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & X.
\end{array}$$

The reason we are interested in this construction is that if we take the objects $X_i$ and $Y_i$ to be in the heart of a $t$-structure on $\mathcal{C}$, this allows us to model chain complexes of length $\leq 3$ in $\mathcal{C}$. The following lemma will make this precise.

---

5 An element-free proof can be done, of course. Our proof is sufficient by the Freyd-Mitchell embedding theorem.

6 This construction generalizes to arbitrary length by defining iterated cofibers of $n$-cubes in a similar manner.
Lemma A.7. Suppose $D$ is a commutative square

$$\begin{array}{c}
A_1 \xrightarrow{g_1} B_1 \\
\downarrow f_A \qquad \downarrow f_B \\
A_2 \xrightarrow{g_2} B_2
\end{array}$$

with values in $C^\otimes$. Then:

1. The iterated cofiber $\text{cof}(D)$ is concentrated in degrees 0, 1 and 2. Moreover, we have
   $$\pi_2\text{cof}(D) = \ker(A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2)$$
   $$\pi_1\text{cof}(D) = \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2)$$
   $$\pi_0\text{cof}(D) = \text{cok}(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2)$$

2. If the square $D$ has the form
   $$\begin{array}{c}
0 \xrightarrow{} B_1 \\
\downarrow f_B \\
0 \xrightarrow{} B_2
\end{array}$$
   then $\text{cof}(D) = \text{cof}(f_B)$.

3. If the square $D$ has the form
   $$\begin{array}{c}
A_1 \xrightarrow{} 0 \\
\downarrow f_A \\
A_2 \xrightarrow{} 0
\end{array}$$
   then $\text{cof}(D) = \Sigma\text{cof}(f_A)$.

4. If the square $D$ has the form
   $$\begin{array}{c}
0 \xrightarrow{} B_1 \\
\downarrow \\
A_2 \xrightarrow{} 0
\end{array}$$
   then $\text{cof}(D) = \Sigma(A_2 \oplus B_1)$.

Proof. Point (2) and (3) are trivial. For point (4), note that the space of morphisms $\text{Map}_C(0,0)$ is contractible, hence the square

$$\begin{array}{c}
0 \xrightarrow{} B_1 \\
\downarrow \\
A_2 \xrightarrow{} 0
\end{array}$$

is trivially commutative and taking vertical cofibers realizes to the zero map $A_2 \to \Sigma B_1$ which implies that $\text{cof}(D) = \text{cof}(A_2 \xrightarrow{0} \Sigma B_1) = \Sigma(A_2 \oplus B_1)$. 
Now assume $D$ is of the shape
\[
\begin{array}{c}
0 \\
\downarrow \\
A_2 \\
\downarrow \\
B_2.
\end{array}
\]

Then we have the following fiber sequence of square diagrams,
\[
\begin{array}{c}
0 \\
\downarrow \\
B_1 \\
\downarrow \\
A_2 \\
\downarrow \\
B_2 \\
\downarrow \\
\Sigma B_2.
\end{array}
\]

Taking vertical cofibers of the right hand cube results in the square
\[
\begin{array}{c}
A_2 \\
\downarrow \\
0 \\
\downarrow \\
\Sigma B_1 \\
\downarrow \\
\Sigma B_2.
\end{array}
\]

where the resulting square
\[
\begin{array}{c}
A_2 \\
\downarrow \\
0 \\
\downarrow \\
\Sigma B_2.
\end{array}
\]

classifies the map $g_2: \Sigma A_2 \to \Sigma B_2$. This means taking further cofibers results in the map
\[
\Sigma(A_2 \oplus B_1) \xrightarrow{(g_2, -f_B)} \Sigma B_2.
\]

This means we have a fiber sequence
\[
\text{cof}(D) \to \Sigma(A_2 \oplus B_1) \xrightarrow{(g_2, -f_B)} \Sigma B_2
\]
from which we can read off that $\text{cof}(D)$ is concentrated in degree 0 and 1 with the homotopy groups
\[
\pi_1 \text{cof}(D) = \ker(A_2 \oplus B_1 \xrightarrow{(g_2, -f_B)} B_2)
\]
\[
\pi_0 \text{cof}(D) = \text{cok}(A_2 \oplus B_1 \xrightarrow{(g_2, -f_B)} B_2).
\]

Now assume $D$ is a general commutative square of the form
\[
\begin{array}{c}
A_1 \\
\downarrow \\
A_2 \\
\downarrow \\
B_2.
\end{array}
\]

\[
\begin{array}{c}
A_1 \\
\downarrow \\
B_1 \\
\downarrow \\
B_2.
\end{array}
\]

\[
\begin{array}{c}
A_1 \\
\downarrow \\
A_2 \\
\downarrow \\
B_2.
\end{array}
\]
We have the following fiber sequence of square diagrams,

\[
\begin{array}{cccccc}
\Omega A_1 & \rightarrow & 0 & \rightarrow & A_1 & \rightarrow \text{g}_1 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & B_1 & \rightarrow & B_1 & \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & A_2 & \rightarrow & A_2 & \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & B_2 & \rightarrow & B_2 & \\
\end{array}
\]

which produces the following two exact sequences

\[
0 \rightarrow \pi_2 \text{cof}(D) \rightarrow A_1 \rightarrow \ker(A_2 \oplus B_1 \xrightarrow{(g_2, -f_B)} B_2) \rightarrow \pi_1 \text{cof}(D) \rightarrow 0
\]

as well as

\[
0 \rightarrow \text{cok}(A_2 \oplus B_1 \xrightarrow{(g_2, -f_B)} B_2) \rightarrow \pi_0 \text{cof}(D) \rightarrow 0,
\]

which proves point (1). □

We are now ready to prove Lemma A.5. Assume now that we have a diagram with short exact rows,

\[
\begin{array}{cccccc}
0 & \rightarrow & A_1 & \xrightarrow{g_1} & B_1 & \rightarrow & C_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A_2 & \xrightarrow{g_2} & B_2 & \rightarrow & C_2 & \rightarrow & 0 \\
\end{array}
\]

and write \(\text{cof}_A := \text{cof}(f_A), \text{cof}_B := \text{cof}(f_B), \text{cof}_C := \text{cof}(f_C), \ker_C := \pi_0 \text{fib}_C\) and \(\text{cok}_A := \pi_{-1} \text{fib}_A\).

Proof of Lemma A.5 We now come back to the claim that the map

\[\text{fib}_C \rightarrow \Sigma \text{fib}_A\]

induces the map described by the snake lemma in \(\pi_0\). Note that \(\text{cof}_A = \Sigma \text{fib}_A\), and similarly for \(C\), so to proof that \(\pi_0 \text{fib}_C \rightarrow \pi_0 \Sigma \text{fib}_A\) is the map induced by the snake lemma, it suffices to show the same thing for \(\pi_1\) on the cofibers.

Take the square \(D\)

\[
\begin{array}{ccc}
A_1 & \xrightarrow{g_1} & B_1 \\
\downarrow & & \downarrow \\
A_2 & \xrightarrow{g_2} & B_2.
\end{array}
\]
There is a commuting cube

\[
\begin{align*}
A_1 & \rightarrow 0 \\
\downarrow & \quad \downarrow & \quad \downarrow \\
B_1 & \rightarrow C_1 \\
\downarrow & \quad \downarrow & \quad \downarrow \\
A_2 & \rightarrow 0 \\
\downarrow & \quad \downarrow \\
B_2 & \rightarrow C_2.
\end{align*}
\]

Taking iterated cofibers yields a map \(\text{cof}(D) \rightarrow \text{cof}_C\). It is clear that the map induced on \(\pi_1\) of this is the map \(\theta\) described in lemma A.6. Moreover, the map in \(\pi_0\) is an isomorphism as well and \(\pi_2(D) = 0\) since \(A_1 \rightarrow B_1\) is injective, hence \(\text{cof}(D) \rightarrow \text{cof}_C\) is an equivalence.

Now take the fiber sequence of commutative squares

\[
\begin{align*}
0 & \rightarrow A_1 \rightarrow A_1 \\
\downarrow & \quad \downarrow & \quad \downarrow \\
B_1 & \rightarrow B_1 \rightarrow 0 \\
\downarrow & \quad \downarrow \\
0 & \rightarrow A_2 \rightarrow A_2 \\
\downarrow & \quad \downarrow \\
B_2 & \rightarrow B_2 \rightarrow 0.
\end{align*}
\]

Taking iterated cofibers gives the fiber sequence

\[
\text{cof}_B \rightarrow \text{cof}(D) \rightarrow \Sigma \text{cof}_A
\]

Here it is clear that the map \(\text{cof}(D) \rightarrow \Sigma \text{cof}_A\) on \(\pi_1\) becomes the map \(\phi\) in lemma A.6 Taking all things together we see that the map

\[
\text{cof}_C \rightarrow \Sigma \text{cof}_A \simeq \text{cof}_C \rightarrow \text{cof}(D) \rightarrow \Sigma \text{cof}_A
\]

gives the map \(\phi \theta^{-1}\) on \(\pi_1\), which by A.6 is the map induced by the snake lemma. □

A.2. **Functor categories and t-structures.** Given a stable \(\infty\)-category \(\mathcal{C}\) and a small \(\infty\)-category \(\mathcal{D}\), we know that \(\text{Fun}(\mathcal{D}, \mathcal{C})\) is again a stable \(\infty\)-category. If we have a \(t\)-structure on \(\mathcal{C}\), we can put a natural \(t\)-structure on \(\text{Fun}(\mathcal{D}, \mathcal{C})\):

**Definition A.8.** Suppose \(\mathcal{C}\) is a stable \(\infty\)-category with \(t\)-structure and \(\mathcal{D}\) a small \(\infty\)-category. The object-wise \(t\)-structure on \(\text{Fun}(\mathcal{D}, \mathcal{C})\) is defined via

\[
\begin{align*}
\text{Fun}(\mathcal{D}, \mathcal{C})_{\leq 0} & := \text{Fun}(\mathcal{D}, \mathcal{C}_{\leq 0}) \\
\text{Fun}(\mathcal{D}, \mathcal{C})_{\geq 0} & := \text{Fun}(\mathcal{D}, \mathcal{C}_{\geq 0}).
\end{align*}
\]

We view the category of functors \(\mathcal{D} \rightarrow \mathcal{C}_{\leq 0}\) as the full subcategory of functors \(\mathcal{D} \rightarrow \mathcal{C}\) with values in the subcategory \(\mathcal{C}_{\leq 0}\) and similarly for \(\geq 0\).

**Proof.** We have to check three things:

- \(\text{Fun}(\mathcal{D}, \mathcal{C})_{\leq 0}\) is closed under \(\Omega\). This is true since limits are computed object-wise and \(\mathcal{C}_{\leq 0}\) is closed under limits. Similarly, \(\text{Fun}(\mathcal{D}, \mathcal{C})_{\geq 0}\) is closed under \(\Sigma\).
• Given $X$ in $\text{Fun}(\mathcal{D},\mathcal{C})_{\geq 1}$ and $Y$ in $\text{Fun}(\mathcal{D},\mathcal{C})_{\leq 0}$ the abelian group $\pi_0\text{Nat}(X,Y)$ is zero. The space $\pi_0\text{Nat}(X,Y)$ is given as the end

$$\text{Nat}_C(X,Y) \simeq \int_{d \in \mathcal{D}} \text{Map}_C(X(d), Y(d))$$

by Lemma A.12. The spaces $\text{Map}_C(X(d), Y(d'))$ for $d, d' \in \mathcal{D}$ are all contractible by assumption, so the end in question is, as a limit of a functor into contractible objects, contractible as well, therefore $\text{Nat}_C(X,Y) \simeq \text{pt}$ and thus $[X,Y] = \pi_0\text{Nat}(X,Y) = 0$.

• For any $X$ in $\text{Fun}(\mathcal{D},\mathcal{C})$ there is a fiber sequence

$$X_1 \rightarrow X \rightarrow X_0$$

with $X_1$ in $\text{Fun}(\mathcal{D},\mathcal{C})_{\geq 1}$ and $X_0$ in $\text{Fun}(\mathcal{D},\mathcal{C})_{\leq 0}$. To see this, note that we have a fiber sequence $\tau_{\geq 1} \rightarrow \text{id}_\mathcal{C} \rightarrow \tau_{\leq 0}$ of functors $\mathcal{C} \rightarrow \mathcal{C}$. Precomposing with $X$ gives the fiber sequence

$$\tau_{\geq 1} X \rightarrow X \rightarrow \tau_{\leq 0} X$$

which is our desired fiber sequence.

Remark A.9. The heart of this $t$-structure is given as

$$\text{Fun}(\mathcal{D},\mathcal{C})^\diamond \simeq \text{Fun}(\mathcal{D},\mathcal{C}^\diamond) \simeq \text{Fun}(\mathcal{hD},\mathcal{C}^\diamond),$$

where right equivalence follows from $\mathcal{C}^\diamond$ being a 1-category.

A.3. Ends and Coends. The earliest account of ends and coends in the setting of $\infty$-categories goes back to [Cra10]. In the following section we always assume that $\mathcal{C}$ is a complete and cocomplete $\infty$-category and that $\mathcal{D}$ is an (essentially) small $\infty$-category. For an $\infty$-category $\mathcal{D}$ there always exists the twisted arrow category $\text{tw(}\mathcal{D})$, together with a natural functor $\text{tw(}\mathcal{D}) \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D}$, see [Bar14] Section 2 or [Gla15] Definition 2.1. Note that if $\mathcal{D} = N(\mathcal{D})$ for $\mathcal{D}$ a 1-category, then $\text{tw}(\mathcal{D}) \simeq N(\text{tw}(\mathcal{D}))$, where $\text{tw}(\mathcal{D})$ is the 1-categorical twisted arrow category.

Definition A.10. Let $F$ be a functor $\mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{C}$. Then the end of $F$, written as

$$\int_{d \in \mathcal{D}} F(d,d)$$

is defined as the limit over the composition

$$\text{tw}(\mathcal{D}) \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D} \xrightarrow{F} \mathcal{C}.$$ 

Dually, the coend

$$\int_{d \in \mathcal{D}} F(d,d)$$

is defined as the colimit of the composition

$$\text{tw}(\mathcal{D}^{\text{op}})^{\text{op}} \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D} \xrightarrow{F} \mathcal{C}.$$ 

---

7 The statement that a limit of a functor $F: \mathcal{C}' \rightarrow \mathcal{C}$ into contractible objects is contractible follows from the observation that $F$ is a terminal object in the $\infty$-category $\text{Fun}(\mathcal{C}',\mathcal{C})$ and the functor $\lim: \text{Fun}(\mathcal{C}',\mathcal{C}) \rightarrow \mathcal{C}$ preserves all limits and in particular terminal objects.
Example A.11. If $F : D^{op} \times D \to C$ is constant in the left variable, i.e. $F$ factors through the projection as $F : D^{op} \times D \to D \xrightarrow{F'} C$, then

$$\int_{d \in D} F(d, d) \simeq \lim F'.$$

Dually,

$$\int_{d \in D} F(d, d) \simeq \colim F'.$$

Similarly in the right variable.

Proposition A.12 ([GHN20], Prop. 5.1.). Let $A, B : D \to C$ be two functors. Then there is a natural equivalence

$$\Nat_C(A, B) \simeq \int_{d \in D} \Map_C(Ad, Bd),$$

where $\Nat_C(A, B)$ is the space of natural transformations from $A$ to $B$.

Theorem A.13 (Fubini’s Theorem, [AL18] Proposition 3.5.). Let $F : D^{op} \times D \times E^{op} \times E \to C$ be a functor. Then

$$\int_{d \in D} \int_{e \in E} F(d, d, e, e) \simeq \int_{e \in E} \int_{d \in D} F(d, d, e, e).$$

The assumption that $C$ is cocomplete implies that $C$ is tensored over spaces, i.e. for any space $X$ and object $c \in C$ there exists an object $X \otimes c$, which is defined as $\colim X \cdot c$, where $\cdot c$ is the constant functor $X \to C$ with value $c$. Dually, the assumption that $C$ is complete implies that $C$ is powered over spaces, i.e. there exists $c^X$ defined as $\lim X \cdot c$. Note that for fixed $X$, the functor $c \mapsto X \otimes c$ is left adjoint to the functor $c \mapsto c^X$.

Lemma A.14 (Yoneda Lemma, [AL18] Proposition 3.5.). Let $G$ be a functor $D \to C$. There exists a natural equivalence

$$G(d) \simeq \int_{d' \in D} \Map_D(d', d) \otimes G(d').$$

Dually, let $G$ be a functor $D^{op} \to C$. There exists a natural equivalence

$$G(d) \simeq \int_{d' \in D} G(d') \Map_D(d', d').$$

If $G : D^{op} \to \Spc$ the Yoneda lemma has the more familiar form

$$\Nat_{\Spc}(\Map_D(-, d), G) \simeq G(d).$$

Example A.15. Let $d \in D$. The colimit over a corepresentable functor $\Map_D(d, -) : D \to \Spc$ is contractible. This follows directly from the Yoneda Lemma by

$$\colim \Map_D(d, -) \simeq \int_{d' \in D^{op}} \Map_D(d, d') \otimes \pt \simeq \int_{d' \in D^{op}} \Map_{D^{op}}(d', d) \otimes \pt \simeq \pt,$$

where $\pt$ is the constant functor $D^{op} \to \Spc$ with value $\pt$. 
Lemma A.16. Let \( F : \mathcal{D} \to \mathcal{C} \) and \( p : \mathcal{D} \to \mathcal{D}' \) be functors. The left Kan extension functor \( \text{Lan}_p : \text{Fun}(\mathcal{D}, \mathcal{C}) \to \text{Fun}(\mathcal{D}', \mathcal{C}) \) is defined as the left adjoint to the precomposition functor \( \circ p : \text{Fun}(\mathcal{D}', \mathcal{C}) \to \text{Fun}(\mathcal{D}, \mathcal{C}) \).

Proof. Let \( G \) be a functor \( \mathcal{D}' \to \mathcal{C} \). There is the following sequence of natural equivalences

\[
\text{Nat}_\mathcal{C} \left( \int_{d \in \mathcal{D}} \text{Map}_{\mathcal{D}'}(p(d), -) \otimes F(d), G \right)
\]

\[\simeq \int_{d \in \mathcal{D}} \text{Nat}_\mathcal{C}(\text{Map}_{\mathcal{D}'}(p(d), -) \otimes F(d), G)\]

\[\simeq \int_{d \in \mathcal{D}} \int_{d' \in \mathcal{D}'} \text{Map}_\mathcal{C}(\text{Map}_{\mathcal{D}'}(p(d), d') \otimes F(d), G(d'))\]

\[\simeq \int_{d \in \mathcal{D}} \text{Map}_\mathcal{C}(F(d), \int_{d' \in \mathcal{D}'} G(d') \otimes \text{Map}_{\mathcal{D}'}(p(d), d'))\]

\[\simeq \int_{d \in \mathcal{D}} \text{Map}_\mathcal{C}(F(d), G(p(d)))\]

\[\simeq \text{Nat}_\mathcal{C}(F, G \circ p)\]

showing that the functor \( F \mapsto \int_{d \in \mathcal{D}} \text{Map}_{\mathcal{D}'}(p(d), -) \otimes F(d) \) is in fact the left adjoint to the precomposition by \( p \). \( \square \)

We will use coends in this paper for one particular major case. Let \( \mathcal{C} \) be a cocomplete symmetric monoidal \( \infty \)-category. If \( F : \mathcal{D}^{op} \to \mathcal{C} \) and \( G : \mathcal{D} \to \mathcal{C} \) then define

\[ F \otimes D G = \int_{d \in \mathcal{D}} F(d) \otimes G(d). \]

Lemma A.17. Suppose that \( \mathcal{C} \) is a symmetric monoidal \( \infty \)-category such that \( \otimes \) preserves colimits in both variables. Then the functor

\[ \otimes_D : \text{Fun}(\mathcal{D}^{op}, \mathcal{C}) \times \text{Fun}(\mathcal{D}, \mathcal{C}) \to \mathcal{C} \]

preserves colimits in both variables. If, furthermore, \( \mathcal{C} \) is equipped with a \( t \)-structure such that \( C_{\geq 0} \times C_{\geq 0} \) maps to \( C_{\geq 0} \) under \( \otimes \), then \( \text{Fun}(\mathcal{D}^{op}, C_{\geq 0}) \times \text{Fun}(\mathcal{D}, C_{\geq 0}) \) maps to \( C_{\geq 0} \). As a consequence, if \( F \in \text{Fun}(\mathcal{D}^{op}, \mathcal{C}) \) is object-wise \( m \)-connective and \( G \in \text{Fun}(\mathcal{D}, \mathcal{C}) \) is object-wise \( n \)-connective, then \( F \otimes_D G \) is \( m + n \)-connective in \( \mathcal{C} \).

Proof. Fix \( G : \mathcal{D} \to \mathcal{C} \) and let \( F = \text{colim}_{i \in I} F_i \) be the colimit of a system of functors \( \mathcal{D}^{op} \to \mathcal{C} \). Since colimits are computed object-wise, for each \( d \in \mathcal{D} \) we have

\[ F(d) \otimes G(d) \simeq (\text{colim}_{i \in I} F_i(d)) \otimes G(d) \simeq \text{colim}_{i \in I} (F_i(d) \otimes G(d)). \]
The coend is obtained by taking the colimit over $\text{tw}(D)$ of this functor, and by commutativity of colimits we get that $\otimes_D$ preserves colimits in the left variable. The proof for the right side is analogous.

Now assume that $C$ comes equipped with a $t$-structure and that $\otimes$ maps $C_{\geq 0} \times C_{\geq 0}$ to $C_{\geq 0}$. Assume that $G : D \to C$ and $F : D^{op} \to C$ are object-wise connective. This implies that the bifunctor $F \otimes G : D^{op} \times D \to C$ is object-wise connective. Since $C_{\geq 0}$ is closed under colimits, the colimit under the precomposition with $\text{tw}(D^{op})^{op} \to D^{op} \times D$ lies in $C_{\geq 0}$. □

The $D$-tensor product $\otimes_D$ comes with a projection formula, similar to one in the context of genuine equivariant spectra.

**Theorem A.18.** Suppose that $C$ is a symmetric monoidal $\infty$-category such that $\otimes$ preserves colimits in both variables. Let $p$ be a functor $D \to D'$, $F$ a functor $D^{op} \to C$, $G$ a functor $D' \to C$. There is a natural equivalence

$$\text{Lan}_p F \otimes_{D'} G \simeq F \otimes_D (G \circ p).$$

**Proof.** We have the following sequence of natural equivalences.

$$\text{Lan}_p F \otimes_{D'} G \simeq \int_{D'} \text{Lan}_p F(d') \otimes G(d')$$

$$\simeq \int_{D'} \left( \int_D \text{Map}_{D'}(p(d), d')F(d) \right) \otimes G(d')$$

$$\simeq \int_D \int_{D'} \left( \text{Map}_{D'}(p(d), d') \otimes G(d') \right)$$

$$\simeq \int_D \int_{D'} F(d) \otimes \left( \text{Map}_{D'}(p(d), d') \otimes G(d') \right)$$

$$\simeq \int_D F(d) \otimes \int_{D'} \text{Map}_{D'}(p(d), d') \otimes G(d')$$

$$\simeq \int_D F(d) \otimes G(p(d)) \simeq F \otimes_D (G \circ p).$$

□

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