Snell's Law for Shear Zone Refraction in Granular Materials

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We present experiments on slow shear flow in a split-bottom linear shear cell, filled with layered granular materials. Shearing through two different materials separated by a flat material boundary is shown to give narrow shear zones, which refract at the material boundary in accordance with Snell’s law in optics. The shear zone is the one that minimizes the dissipation rate upon shearing, i.e. a manifestation of the principle of least dissipation. We have prepared the materials as to form a granular lens. Shearing through the lens is shown to give a very broad shear zone, which corresponds to fulfilling Snell’s law for a continuous range of paths through the cell.

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Shearing, deformation or processing of granular materials are of practical concern on many scales: faults and land slides in geological systems, industrial powder processing, or mixing of materials. Flows occur in loose to dense granular systems; fast or slow\(^1\). Flow in dense systems tend to be localized to narrow regions, shear bands, whereas most of the material is moved as solid blocks. Shear bands may be down to a few particles\(^2\), but wide zones and bulk flow also occur, very much depending on the boundary conditions of the flows\(^3, 4, 5\).

From a physical viewpoint, in sheared or processed granular materials energy is dissipated. They will come to rest unless driven, be it by gravity, by moving walls or by any other means. In many cases a steady-state is reached, where the dissipated energy balances the input energy. In classical mechanics the time evolution of any system may be deduced from the principle of least action, and in optics the corresponding principle of least time implies Snell’s refraction law for light. A corresponding extremum principle for dissipative processes has been sought after and the principle of least dissipation\(^6\) can account for the dynamics of some systems, but not for others\(^6, 7\). Shearing of granular materials is a process for which the principle is proposed to be valid\(^8\). In a modified split-bottom Couette-cell setup the granular material was forced into a steady-state with a localized shear zone in the bulk of the material\(^3, 4, 5\). This was done with a single material, and by applying a minimum dissipation principle the shape of the shear zone could be explained\(^8\). This principle must in principle take the form of a volume integral, namely that the motion is such that the dissipation rate integral is minimized. We consider the system to be invariant in the z-direction, in which case we propose the following form of the integral

\[
\int p\mu |\nabla v(x, y)| \, dx \, dy \, dz = \min ,
\]

where \(v(x, y)\) is the z-direction velocity field. The shear zone is here considered a continuous sequence of thin surfaces with face normal \(\nabla v\). The dissipation rate is found by multiplying with the shear force between two sliding planes: the overall pressure \(p\) times the effective friction coefficient \(\mu\). For now, we make the simplification that shearing is localized to narrow shear bands, in which most of the sliding and thus energy dissipation take place. This allows the reduction of the integral to be over the surface of the shear zone, and we recover the form used in\(^8, 9\)

\[
\int \Delta v p\mu dS = \min .
\]

Here \(\Delta v\) is the velocity difference between the two sides. A particular manifestation of the minimization principle is that a shear zone going through a layer of one material into a layer of a different material will be refracted at the material boundary. Numerical work with idealized particles and periodic boundary conditions suggested that the effective friction \(\mu\) may be understood as a material index\(^9\), from which it followed that the refraction of the shear zone follows Snell’s refraction law.

We test this experimentally by the setup sketched in Figure\(^1\). A purpose-built split-bottom shear cell is filled with two materials, “coarse” sand and “smooth” glass beads. The material boundary can be prepared at different angles with respect to the shear cell. The shear cell is a Plexiglas tube of length 37 cm, outer radius 2 cm, and side thickness 2 mm. It is cut in two parts which slide with respect to each-other; mounted vertically through holes in two wooden boards. The inside walls are roughened by horizontal stripes of 2 mm glass beads glued on. The cell is filled vertically half and half with sand and glass beads. Upon filling a thin plastic divider is placed within the tube at an angle \(\phi\) with respect to the direct cut through the cell. After every 2 cm of filling it is raised correspondingly in order to allow the material to settle and create contact over the material boundary. The “coarse” material is sand; between 50 \(\mu m\) - 1 mm, highly non-spherical. Pinkish sand is obtained by adding a tiny amount of color powder. The “smooth” material is fabricated glass beads in two colors; between 50 to 200 \(\mu m\), fairly round. From the bottom every other layer of 2 cm is filled with pink sand/yellow glass and
FIG. 1: Sketch of the experimental setup. To the left: the geometry of the shear cell. Upper right: top view of a cut through the cell. Lower right: the filling of material makes color layers.

gray sand/black glass until layer 3. The cell is sheared exactly 2 cm, which means that opposite colors tell the shear zone’s position. The material is vacuumed away and photos made layer by layer. At layer boundaries the colors change, in-between the contrast is good for at least 1.5 cm.

Except at the ends, the system is invariant in the vertical direction, reducing the two-dimensional problem of Eq. (2) to the one of finding the easiest path for the shear zone between its two end points. This line is not the shortest path but the path that minimizes dissipation. The problem is exactly the same as in geometrical optics, where Fermat’s principle of least time implies Snell’s refraction law. This gives

$$\mu_1 \sin \theta_1 = \mu_2 \sin \theta_2$$  \hspace{1cm} (3)

at the material boundary, where the index of refraction from optics is replaced by an effective material index.

We have performed the experiment with five different angles of $\phi$: 22°, 36°, 45°, 54°, and 67°; shown as (a) to (e) in Figure 2. The shear zone is the line between pink and gray sand and the line between yellow and black glass beads. In each of the materials we observe that the shear zone is a straight line, supporting the assumption that one may assign an effective friction and a material index to each material. Second the two line segments meet in a point at the material boundary, and is thus refracted as predicted by Snell’s law.

Qualitatively the effect is very clear. However, one should keep in mind that the material index can not be expected to be history independent. The preparation involves some gentle tapping to let the material settle; the settling or density will vary to some degree between and within experiments. Furthermore, the system is subjected to gravity, which may give different packing as a function of depth in the shear cell. For the experiment in Figure 2(c) ($\phi=45^\circ$) we show the measured angles of incidence as a function of depth, see Figure 3. There are some fluctuations due to inhomogeneity in the preparation. We observed that small slips could occur at the material boundary, which locally perturb the shear zone. At the top, a few layers are influenced by the loose mass sliding over, and at the bottom a straight shear zone is imposed by the setup. Both effects are end effects, and in a large portion in the middle of the cell the shear zone is stable.

FIG. 2: Refraction at the material boundary. Its orientation $\phi$ is: (a) 22°, (b) 36°, (c) 45°, (d) 54°, and (e) 67°. The images are from layer 7 in (b-d) and layer 5 in (a) and (e). The shear zone clearly deviates from the shortest path in all cases.
From each of the five experiments the material index ratio is determined, see Figure 4. For each angle an average is made over the available levels, that is to say excluding levels where slip or other irregular behavior was seen. In numerical simulations where the properties of each material are perfectly controllable, the index ratio was tuned and in that way the measured index ratio ($\mu_1/\mu_2 = \sin \theta_2/\sin \theta_1$) was checked against an expected value [9]. In these experiments the expected index ratio is an unknown, not only dependent on the material, but also on the preparation – the history of the system. We measure the index ratio to be between 1.5 to 2.6. The shear resistance of the sand and the glass was tested separately, to get an order of magnitude estimate of their index ratio. A flat disk on a rod was submerged in a beaker with the material, and upon pulling it out the required force was measured. Loosely packed, the sand to glass friction ratio was 1.5, whereas material compacted by tapping gave the ratio 2.5. Firstly, this tells that the index ratios found from Snell’s law are reasonable. Secondly, the index ratio depends on the preparation to a large degree. A natural interpretation is that the irregular shape of sand makes its index increase when the material is compacted. As a consequence one should understand the $\mu_{1,2}$ in Eq. (3) to include the preparation history in order for the principle to hold. Nevertheless, Snell’s law as a manifestation of the minimum dissipation principle for a well prepared layered sample is shown to be valid. As such it confirms the validity of the principle itself. It is clear from the experiments that increasing inhomogeneity of the packing soon ruins the applicability of Snell’s law, however, this does not invalidate the general principle.

Moving on from the single refraction setup, we demonstrate the effect of refraction at two boundaries by looking at a wedge geometry, see Figure 5(a-b). The shear zone is refracted twice as Snell’s law predicts. We observed that this shear zone is narrow and invariant with depth in the tube. This is as expected when there is a clear minimum of the functional in Eq. (2). Consider instead of the wedge shape a granular lens centered in the cell. A convex lens of sand between glass beads will, when properly shaped according to the material index ratio, make a continuous range of shear zones which fulfill Snell’s law. They have equal dissipation rate, thus, the minimum dissipation principle as formulated in Eq. (2) fails to select one of the shear zones.

We have done the shear experiment with a granular lens that, save for the difficulties in preparation, is designed after an index ratio of 2.0. The result is shown in Figure 5(c-f). The shown images are all from within a single layer. As opposed to a narrow zone that makes all cuts within a layer look identical, these images show a strong shift of the apparent position of the shear zone. That is the signature of a very wide shear zone and bulk flow. The fact that the bulk flow did not spread out over the entire lens may be because of imperfections in the lens shape. Note that a successful experiment requires both the shape of the lens and the material index ratio to be accurately prepared. We found that small variations in lens thickness or filling could ruin the lens effect.

The granular lens experiment shows that in that case the narrow shear band assumption that allowed us to go from the volume integral in Eq. (1) to the surface integral of Eq. (2) is invalid. Snell’s law is not directly invalidated in the case of the lens in the sense that the shear to some approximation is distributed over a range of allowed shear zones. However, it does not provide a way to tell how the shear would be distributed among
FIG. 5: Refraction at two material boundaries. (a) A wedge of glass beads in sand and (b) a wedge of sand between glass beads. For both wedges the shear zone was sharply defined. (c-f) A “granular lens” of sand showed a very wide shear zone. The pictures are all from within layer ten at the positions (measured from the top of the layer): (c) 5 mm, (d) 9 mm, (e) 13 mm, and (f) 18 mm. This apparent shift of the shear zone within a layer actually means that there is bulk flow over nearly half the tube width.

In conclusion we present shear experiments with two materials in contact, separated by a plane material boundary. Shearing through both materials we obtain a narrow and well-defined shear zone. The shear zone is shown to refract at the material boundary. By assigning a material index to each material, the refraction is found to obey Snell’s law. Thereby we confirm the minimum dissipation principle in Eq. (2).

We seek to further investigate the range of validity of the extremum principle. Inspired by optics we design a granular lens. When properly shaped, a continuous range of shear zones obey Snell’s law and have the same total dissipation rate. Shearing through the lens is shown to produce a very wide shear zone. The minimum principle in the form of Eq. (1) or (2) does not provide a way to tell how shear will be distributed between possible shear zones. If investigated with sufficient accuracy, the granular lens may be used to investigate the validity of the minimum principle or corrections to the dissipation integral. We believe that a numeric approach to this will be very valuable.

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[10] The cited angles are nominal and we attempted to make these angles. Small deviations are inevitable, but all measurements of angles are made relative to the material boundary so that these nominal angles do not enter explicitly in the results.