Smooth Lyapunov Functions for Multistable Differential Inclusions

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Abstract: We provide a converse Lyapunov theorem for differential inclusions with upper
semicontinuous right-hand side, admitting a finite number of compact, globally attractive,
weakly invariant sets, and evolving on Riemannian manifolds. Such properties entail multistable
behavior in differential inclusions and may gather interest in a number of applications where
uncertainty and discontinuities of the vector field play a major role.

Keywords: Discontinuous control, Multivalued mapping, Lyapunov function, Attractors.

1. INTRODUCTION

Available in the literature is a vast body of work regarding converse Lyapunov results for time-invariant nonlinear systems with a globally asymptotically stable equilibrium (Massera (1949)), time-varying ones (Kurzweil (1956)), set stability (Hoppensteadt (1966)), robust stability (Lin et al. (1996)), differential inclusions (Clarke et al. (1998); Teel and Praly (2000)), hybrid systems (Cai et al. (2007)).

While most contributions focus on Lyapunov stability of a single connected attractor, e.g. an equilibrium point, several applications in system biology, mechanics, and electronics, have called for a global analysis of the so-called “multistable” systems. The term encompasses a variety of non-trivial dynamical behaviors - almost global stability, multiple equilibria, periodicity, almost periodicity, chaos - and commonly refers to the existence of a compact invariant set which is simultaneously globally attractive and decomposable in a finite number of smaller compact invariant sets. Typically, such set is not Lyapunov stable and, for this reason, the standard aforementioned approaches fail to work in a multistable setting. However, several other approaches (Gelig et al. (1978); Nitecki and Shub (1975); Conley and Sciences (1978); Shub et al. (2013); Patrão (2011); Rantzer (2001); Angeli (2004); Efimov (2012); Angeli and Efimov (2015)) have success in the global analysis of multistable systems.

In this work we consider differential inclusions on Riemannian manifolds with upper semicontinuous right-hand side, and we study the properties which both entail multistable behavior and the existence of a smooth Lyapunov function. Our interest in multistable differential inclusions mainly lies in analyzing the robustness properties of systems with uncertain and discontinuous vector fields (for instance, mechanical systems with dry friction and impacts). Our main contribution consists in establishing the appropriate attractivity and acyclicity notions yielding local uniform asymptotic stability of some sectors partitioning the state manifold. The term “uniform” stands here for uniformity with respect to both initial conditions and solutions sets departing from the same initial condition.

2. PRELIMINARIES

2.1 Differential inclusions on Riemannian manifolds

We start our analysis by recalling standard notions about differential inclusions evolving on Riemannian manifolds (Abraham et al. (1988); Aubin and Cellina (1984); Clarke et al. (1998)). Let $M$ be an $n$-dimensional, connected, and geodesically complete Riemannian manifold without boundary and let $g$ be the Riemannian metric associated with $M$. The distance between points $x, y \in M$ according to $g$. Let $F$ be a multivalued mapping of the tangent bundle $TM$, that is a multivalued mapping $F: M \ni T_xM$ such that $F(x) \subset T_xM$ for all $x \in M$.

The object of our study will then be the differential inclusion:

\[
\frac{d}{dt}x(t) \in F(x(t)),
\]

with $x(t) \in M$. A solution of (1) on an interval $[a, b]$ is an absolutely continuous function $X : [a, b] \rightarrow M$ such that $\dot{X}(t) \in F(X(t))$ holds almost everywhere on $[a, b]$.

A manifold $M$ is said to be connected if it is not the union of two disjoint open sets, and is said to have no boundary if every point belonging to $M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$. The property of geodesical completeness is the property of every maximal geodesic $\gamma(t)$ on $M$ being extendible for all $t \in \mathbb{R}$. Geodesical completeness relates to the notion of completeness of $M$ as a metric space via the Hopf-Rinow theorem (do Carmo (1992)) and implies compactness of all closed and bounded sets of $M$. 

\[\text{ }\]
Definition 2. \( [a,b] \) will denote with \( S(x) \) the set of maximal solutions of (1) with initial condition \( x \in \mathcal{M} \), namely \( X(t,x) \in S(x) \) will satisfy \( X(0,x) = x \). We will denote the attainable set from \( x \in \mathcal{M} \) at \( t \in \mathbb{R} \) with \( \varphi_t(x) := \{ y \in \mathcal{M} : \exists X(t,x) \text{ and } y = X(t,x) \} \), and the attainable set from a set \( S \subseteq \mathcal{M} \) at time \( t \in \mathbb{R} \) with \( \varphi_t(S) := \bigcup_{x \in S} \varphi_t(x) \). We define the reachable set from a set \( S \) at time \( T \geq 0 \) as \( R^{ST}(S) := \bigcup_{t \in [0,T]} \varphi_t(S) \). The reachable set from \( S \) is defined as \( R(S) := \bigcup_{T \geq 0} R^{ST}(S) \).

Assumption 1. \( F(x) \) is a nonempty compact convex subset of \( T_x \mathcal{M} \) for every \( x \in \mathcal{M} \). Furthermore, \( F \) is upper semicontinuous, i.e., for any \( x \in \mathcal{M} \) and any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d(x_1, x_2) \leq \delta \) implies \( T_{x_1}, \gamma_\delta(F(x_2)) \subseteq F(x_1) + \varepsilon \mathbb{B} \) where \( T_{x_1}, \gamma_\delta \) denotes the parallel transport operator from \( x_2 \) to \( x_1 \) along their connecting geodesic \( \gamma_\delta \) and \( \mathbb{B} \) denotes the open unit ball.

2.2 Limit sets and invariance

Definition 2. The \( \omega \)-limit set (respectively, the \( \alpha \)-limit set) of a solution \( X \in S(x) \) of (1) is defined as the set of all \( y \in \mathcal{M} \) such that there exists a sequence \( \{ t_n \}_{n \in \mathbb{N}} > 0 \) satisfying \( t_n \to +\infty \) (respectively, \( t_n \to -\infty \)) as \( n \to +\infty \) and \( \lim_{n \to +\infty} d(X(t_n, x), y) = 0 \).

Definition 3. (Invariance). A set \( Z \subseteq \mathcal{M} \) is said to be weakly forward (respectively, backward) invariant if, for any \( x \in Z \), there exists a solution \( X \in S(x) \) such that \( X(t, x) \in Z \) for all \( t \geq 0 \) (respectively, \( t \leq 0 \)). A set \( Z \subseteq \mathcal{M} \) is weakly invariant if it is both weakly forward and backward invariant. Strong invariance requires the aforementioned conditions being satisfied for all solutions \( X \in S(x) \).

Here we recall a number of properties of \( \omega \)-limit sets (Filippov, 2013, Section 12), and similar ones will hold for \( \alpha \)-limit sets along solutions of the backward inclusion.

Lemma 4. For any \( x \in \mathcal{M} \), and any solution \( X \in S(x) \) bounded forward in time, it holds that \( \omega(X) \) is closed, weakly invariant, non-empty, connected, and satisfies \( \lim_{t \to +\infty} d(X(t), \omega(X)) = 0 \).

2.3 Decomposition

Instrumental in the proof of our main result will be the notions of decomposition and filtration ordering for a number of compact and weakly invariant sets. We will denote with \( |w| \) the set-point distance of point \( w \in \mathcal{M} \) from set \( \Xi \subseteq \mathcal{M} \), i.e., \( |w| := \inf_{x \in \Xi} d(x, w) \).

Definition 5. A decomposition for a compact and weakly invariant set \( \Lambda \) is a finite family of disjoint compact and weakly invariant sets \( \Lambda_1, \ldots, \Lambda_N \) - referred to as the atoms of the decomposition - such that \( \Lambda = \bigcup_{i=1}^N \Lambda_i \).

Definition 6. The basins of attraction and repulsion for a set \( \Xi \) are defined as:

\[ \alpha(\Xi) = \{ x \in \mathcal{M} : \exists X \in S(x) \text{ and } \lim_{t \to +\infty} |X(t,x)| = 0 \}, \]

\[ \varrho(\Xi) = \{ x \in \mathcal{M} : \exists X \in S(x) \text{ and } \lim_{t \to -\infty} |X(t,x)| = 0 \}. \]

Following from the definition, the basins of attraction and repulsion for \( \Xi \) are understood in a weak sense.

Definition 7. A connecting orbit between two disjoint sets \( \Xi_1, \Xi_2 \) exists if \( \alpha(\Xi_1) \cap \varrho(\Xi_2) \neq \emptyset \).

The existence of a connecting orbit between two disjoint sets \( \Xi_1, \Xi_2 \) will then be denoted as \( \Xi_1 \prec \Xi_2 \).

Definition 8. Let \( \Xi_1, \ldots, \Xi_N \) be a decomposition of a compact and weakly invariant set \( \Xi \).

1. A 1-cycle is an index \( i \in \{ 1, \ldots, N \} \) such that \( \alpha(\Xi_i) \cap \varrho(\Xi_j) \neq \emptyset \).
2. An \( r \)-cycle \( (r \geq 2) \) is an ordered \( r \)-tuple of distinct indices \( i_1, \ldots, i_r \in \{ 1, \ldots, N \} \) such that \( \Xi_{i_1} \prec \Xi_{i_2} \prec \ldots \prec \Xi_{i_r} \prec \Xi_{i_1} \).
3. A filtration ordering is a numbering of the \( \Xi_i \)s so that \( \Xi_i \prec \Xi_j \implies i < j \) with \( i, j \in \{ 1, \ldots, N \} \).

Note that the existence of a filtration ordering automatically rules out the existence of \( 1 \)- and \( r \)-cycles, namely it rules out the existence of homoclinic trajectories and heteroclinic cycles among the atoms of \( \Xi \).

In this paper, we focus our attention to a specific set \( \mathcal{W} \) which will satisfy a global attractivity property for all solutions of (1). Let \( \mathcal{W} \) be any closed and weakly invariant set such that

\[ \mathcal{W} \supseteq \bigcup_{x \in \mathcal{M}, X \in S(x)} \alpha(X) \cup \omega(X). \]

In particular, we make the following assumptions on \( \mathcal{W} \).

Assumption 9. \( \mathcal{W} \) is compact and admits a decomposition \( \mathcal{W}_1, \ldots, \mathcal{W}_N \).

Assumption 10. The decomposition \( \mathcal{W}_1, \ldots, \mathcal{W}_N \) admits a filtration ordering.

Remark 11. The rationale behind Assumption 10 is that the existence of \( 1 \)- or \( r \)-cycles among the atoms of \( \mathcal{W} \) prevents the construction of continuous Lyapunov functions which are constant on each atom of \( \mathcal{W} \) and dissipating outside \( \mathcal{W} \). In fact, a Lyapunov function dissipating along a \( 1 \)- or \( r \)-cycle to some atom \( \mathcal{W}_i \) would either be non-continuous or non-constant on \( \mathcal{W}_i \). Furthermore, compactness of \( \mathcal{W} \) automatically entails compactness of all \( \alpha \)- and \( \omega \)-limit sets of (1), which significantly simplifies the construction of Lyapunov functions.

3. MAIN RESULT

Definition 12. Differential inclusion (1) is said to satisfy the \textit{practical global attractivity} (pGATT) property with respect to \( \mathcal{W} \) if there exists \( q \geq 0 \) such that:

\[ \forall x \in \mathcal{M}, X \in S(x), \lim_{t \to +\infty} |X(t,x)|_W \leq q. \]

If \( q = 0 \), then we say that the global attractivity (GATT) property holds.

Definition 13. Differential inclusion (1) is said to satisfy the \textit{limit property} (LIM) property with respect to \( \mathcal{W} \) if:

\[ \forall x \in \mathcal{M}, X \in S(x), \inf_{t \geq 0} |X(t,x)|_W = 0. \]
Definition 14. Differential inclusion (1) is said to be practical global stability (pGS) property with respect to \( W \) if there exist \( q \geq 0 \) and a class-\( \mathcal{K}_\infty \) function \( \mu \) such that:

\[ \forall x \in M, \forall X \in S(x), \forall t \geq 0, \quad |X(t,x)|_W \leq \mu(|x|_W) + q. \quad (5) \]

Definition 15. Any function \( V : M \rightarrow \mathbb{R} \) is called a practical Lyapunov function for differential inclusion (1) if it satisfies the following two properties:

- there exists a constant \( c \geq 0 \), and class-\( \mathcal{K}_\infty \) functions \( \alpha_1, \alpha_2, \beta \) such that:
  \[ \alpha_1(|x|_W) \leq V(x) \leq \alpha_2(|x|_W + c) \quad \forall x \in M; \quad (6) \]
- \( V \) is differentiable along trajectories of (1), namely \( V(X(t,x)) \) is differentiable in \( t \in \mathbb{R} \) for all \( x \in M \) and all \( X \in S(x) \). Furthermore, there exist a class-\( \mathcal{K}_\infty \) function \( \alpha \) and a constant \( q \geq 0 \) such that:
  \[ \frac{d}{dt} V(X(t,x)) \bigg|_{t=0} \leq -\alpha(|x|_W) + q \quad (7) \]
  for all \( x \in M \), and all \( X \in S(x) \).

If (7) holds for \( q = 0 \), then \( V \) is called a Lyapunov function.

If, in addition, the set \( \bigcup_{x \in W} \{V(x)\} \) is a singleton for all \( i \in \{1, \ldots, N\} \), then \( V \) is called a Lyapunov function constant on invariant sets. Moreover, if \( dV(x) \) is defined and vanishes for all \( x \in W \), then \( V \) is called a Lyapunov function flat on attractors.

Remark 16. If \( q = 0 \) in (7) and the Lyapunov function is smooth on \( M \), then condition (7) is equivalent to the following dissipation inequality:

\[ \max_{v \in F(x)} \{dV(x) \cdot v\} \leq -\alpha(|x|_W), \quad (8) \]

for all \( x \in M \), where \( dV(x) \) denotes the gradient of \( V \) at \( x \), i.e., the covector whose action on \( v \) yields the Lie derivative of \( V \) along \( v \) at \( x \). If \( c = 0 \) in (6), it would be straightforward to prove global asymptotic stability of \( W \) via standard arguments, i.e., that \( |X(t,x)|_W \leq \beta(|x|_W, t) \) for some \( \mathcal{K}_\mathcal{L} \) function \( \beta \), and all \( x \in W \) and all \( t \geq 0 \).

However, we have mentioned that the invariant set of a multistable system typically lacks Lyapunov stability and also uniform attractivity, and this is due to the so-called stickiness effect in proximity of anti-stable manifolds\(^3\).

We are now ready to state our main result.

**Theorem 17.** Consider the differential inclusion (1). Let \( W \subseteq M \) be a compact and weakly invariant set containing all \( \alpha \) - and \( \omega \)-limit sets as in (2). Let Assumptions 1, 9, and 10 hold true. Then, the following properties are equivalent for (1):

1. GATT;
2. practical GATT;
3. LIMP and pGS;
4. existence of a practical Lyapunov function;
5. existence of a Lyapunov function;
6. existence of a smooth Lyapunov function flat on attractors.

In particular, under Assumptions 1 and 9, the conjunction of the GATT property and Assumption 10 implies the existence of a smooth Lyapunov function flat on attractors. Conversely, under Assumptions 1-9, the existence of a Lyapunov functions implies GATT and Assumption 10.

**Proof.** Implications 6 \( \Rightarrow \) 5, 5 \( \Rightarrow \) 4, and 1 \( \Rightarrow \) 2 are trivial. Implications 5 \( \Rightarrow \) 3, 3 \( \Rightarrow \) 2, and 2 \( \Rightarrow \) 1, follows by adapting the arguments of Claim 1 in Angeli and Efimov (2015), Lemma 3.4 in Angeli et al. (2004b), and Claim 4 in Angeli and Efimov (2015) (together with our Proposition 22) respectively. Implication 4 \( \Rightarrow \) 2 is proved by observing that \( V(x) \leq -\alpha(|x|_W) + q \leq -(\alpha \circ \alpha^{-1})(V(x)) + \alpha(c) + q \). Then there exists a \( \mathcal{K}_\mathcal{L} \) function \( \beta \) such that \( V(X(t,x)) \leq \beta(V(x), t) + \bar{q} \) for some \( \bar{q} > 0 \). Implication 5 \( \Rightarrow \) 1 follows by standard Lyapunov-LaSalle arguments. Implication 1 \( \Rightarrow \) 6, i.e., our converse Lyapunov result in Proposition 22, is proved in Section 4.

## 4. EXISTENCE OF A SMOOTH LYAPUNOV FUNCTION

This Section addresses the proof of our converse Lyapunov result, namely Proposition 22. Such proof involves novel results (e.g. Appendix A) and, in particular, a number of known results (for instance, construction and smoothing of a Lyapunov function in the standard Euclidean setting) which apply to the case of connected, Riemannian, without boundary, and geodesically complete manifolds in a straightforward fashion. Due to space constraints, we thoroughly report the former results and refer the reader to related references for the latter.

### 4.1 Sectors

A particular construction of sectors partitioning the manifold \( M \) will be instrumental in the proof of our main converse Lyapunov results, as follows. Consider the following sets:

\[ A_i := \bigcup_{j \leq i} R(W_j), \quad B_i := \bigcup_{j > i} \mathbb{R}(W_j). \]

**Lemma 18.** The following properties hold true for all \( i \in \{1, \ldots, K\} \):

1. **Closedness:** \( A_i \) and \( B_i \) are closed;
2. **Disjointness:** \( A_i \cap B_i = \emptyset \);
3. **Invariance:** \( A_i \) and \( M \setminus B_i \) (respectively \( B_i \) and \( M \setminus A_i \)) are strongly forward (backward) invariant;
4. **Attractivity:** if \( x \in M \setminus B_i \), then it holds that \( \lim_{t \to \infty} |X(t,x)|_{A_i} = 0 \) for all \( X \in S(x) \);
5. **Repulsion:** if \( x \in M \setminus A_i \), then it holds that \( \lim_{t \to \infty} |X(t,x)|_{B_i} = 0 \) or \( \lim_{t \to \infty} |X(t,x)| = +\infty \) for all \( X \in S(x) \);
6. **Closed neighborhoods:** there exist a closed neighborhood \( A_i \) of \( A_i \) and a closed neighborhood \( B_i \) of \( B_i \) such that \( A_i \cap \bar{B_i} = B_i \cap \bar{A_i} = \emptyset \) and \( A_i \cup \bar{B_i} = M \); in particular, we select \( A_N = M \) and \( B_N = \emptyset \).

**Proof.** (i) We omit the proof of closedness of \( A_i \) for space reasons. Closedness of \( B_i \) follows by observing that the definition of \( B_i \) for (1) is equivalent to the definition of set \( A_i \) for the backward inclusion \( \bar{x} \in -F(x) \).

(ii) Disjointness follows by Assumption 10 and closedness of \( A_i \) and \( B_i \). Indeed, if there exists \( x \in A_i \cap
Proof. Observe that the definition of set $B_1$ for the forward inclusion (1) is equivalent to the definition of set $A_1$ for the backward inclusion $\dot{x} \in -F(x)$. We can then invoke Appendix A to prove that $B_1$ is uniformly locally asymptotically stable on $B_1$ along the solutions of the backward inclusions, and thus invoke the main results in Clarke et al. (1998) or Teel and Praly (2000) to prove the Proposition.

4.3 Patching the Lyapunov functions

For any $i \in \{1, \ldots, N\}$, define $G_i := A_i \cup B_i$.

Proposition 21. For any $i \in \{1, \ldots, N\}$, there exists a smooth function $L_i : M \to [0,1]$ and class-$K$ functions $\alpha_i, \mu_i$ such that:

\[
\alpha_i(|x|_{A_i}) \leq L_i(x) \quad \forall x \in M, \tag{13}
\]
\[
\max_{x \in F(x)} dL_i(x) \cdot v \leq -\mu_i(|x|_{G_i}) \quad \forall x \in M. \tag{14}
\]

Moreover, the following properties are satisfied:

\[
L_i(A_i) = 0, \quad L_i(B_i) = 1, \tag{15}
\]
\[
dL_i(G_i) = 0 \quad . \tag{16}
\]

Proof. The proof makes use of Propositions 19 and 20, and it follows along the lines of (Angeli and Eftimov, 2015, Proposition 2, page 12, column 2).

Proposition 22. Let Assumptions 1, 9, 10, and GATT hold true. Then, there exists a smooth function $L : M \to \mathbb{R}_{\geq 0}$ and class-$K$ functions $\alpha_1, \alpha$ such that:

\[
\alpha_1(|x|_{W_i}) \leq L(x) \quad \forall x \in W_i, \tag{17}
\]
\[
\max_{x \in F(x)} dL(x) \cdot v \leq -\alpha(|x|_{W_i}) \quad \forall x \in W_i. \tag{18}
\]

Moreover, function $L$ is flat on attractors, i.e. $dL(W) = 0$.

Proof. The proof makes use of Proposition 21 and follows along the lines of (Angeli and Eftimov, 2015, Proposition 2, pp. 12-13).

5. EXAMPLES

Scalar example. Consider the differential inclusion:

\[
\dot{x} \in -(x + a_1)/(x + a_2 + a_3)(x + a_2 + a_3), \tag{19}
\]

with state $x \in \mathbb{R}$ and parameters $(a_1, a_2, a_3) \in [-1/2, 1/2]^3$, as depicted in Figure 1 A family of compact and weakly invariant sets $W = W_{i1} \cup W_{i2} \cup W_{i3} = [-5/2, -3/2] \cup [-1/2, 1/2] \cup [3/2, 5/2]$ is found by collecting any point $x^* \in \mathbb{R}$ satisfying $0 = -(x + a_1)/(x + a_2 + a_3)(x + a_2 + a_3)$ for some combination of $(a_1, a_2, a_3) \in [-1/2, 1/2]^3$. Consider now the following $C^1$ Lyapunov function:

\[
V(x) = \begin{cases} 
(x + 5/2)^2 & x \leq -5/2 \\
0 & -5/2 < x < -3/2 \\
-9x - 12x^2 - 4x^3 & -3/2 \leq x < -1/2 \\
2 & -1/2 \leq x < 1/2 \\
9x - 12x^2 + 4x^3 & 1/2 \leq x < 3/2 \\
0 & 3/2 \leq x < 5/2 \\
(x - 5/2)^2 & x \geq 5/2
\end{cases} \tag{20}
\]

It is possible to show (Figure 1) that $V$ satisfies $V(x) \geq \frac{1}{2}|x|^2$. 

\[
dV(x) \cdot (-1)(x + a_1)/(x + a_2 + a_3)(x + a_2 + a_3) \leq -|x|^4, \tag{21}
\]

for all $x \in \mathbb{R}$ and all $(a_1, a_2, a_3) \in [-1/2, 1/2]^3$, and thus qualifies as a Lyapunov function for (19). Since
Assumptions 1, 9, and properties (21) are satisfied, we conclude by virtue of Theorem 17 that (19) satisfies the GATT property with respect to \( W \) and, moreover, \( W \) satisfies Assumption 10, i.e., it admits a filtration ordering \((i_1, i_2, i_3) = (1,3,2)\) obtained by simply observing minima and maxima of \( V \).

Nonlinear pendulum with dry friction. A nonlinear pendulum with angle \( \theta \in S \), velocity \( \omega \in \mathbb{R} \), and length \( a \), whose motion is subject to dry friction

\[
\gamma(\omega) = \begin{cases} 
\{f_D \text{ sign } \omega \} & \omega \neq 0 \\
[-f_D, f_D] & \omega = 0,
\end{cases}
\]

for some \( f_D > 0 \), is described by the differential inclusion:

\[
\left( \begin{array}{c} \dot{\theta} \\ \dot{\omega} \end{array} \right) \in -a\sin \theta - \gamma(\omega).
\]  

(22)

By selecting any point \((\theta^*, \omega^*) \in S \times \mathbb{R}\) satisfying \(0 = \dot{\theta} = \omega^* \) and \(0 = \dot{\omega} = -a \sin \theta^* - \gamma(\omega^*)\), we obtain the family of compact and weakly invariant sets \( W = W_{i_1} \cup W_{i_2} \) where \( W_{i_1} := \{(\theta^*, 0) : \theta^* \in \text{arc sin}[-f_D, f_D]/\omega \} \) and \( W_{i_2} := \{(\theta^*, 0) : \theta^* \in \pi + \text{arc sin}[-f_D, f_D]/\omega \} \).

Taking the time derivative of the energy \( E(\theta, \omega) = a(1 - \cos(\theta)) + \omega^2/2 \) along the solutions of (22) yields

\[
\dot{E} = -f_D|\omega| \quad \forall \theta \in S, \forall \omega \neq 0.
\]

(23)

Since \( E \) is strictly decreasing whenever \( \omega \neq 0 \), and since the trajectories of the phase portrait of (22) are perpendicular to the \( \theta \) axis, i.e., \( \partial \theta / \partial \omega = 0 \) for all \( \theta \in S \), there is no homoclinic or heteroclinic cycle among the atoms \( W_{i_1} \) and \( W_{i_2} \). In particular, we can set \((i_1, i_2) = (1,2)\). Moreover, LaSalle’s invariance principle, properness of \( E \), and dissipation (23), imply that the set \( W \) is globally attractive on \( S \times \mathbb{R} \). Since Assumptions 1, 9, and the GATT property are satisfied, we can then conclude by virtue of Theorem 17 that the existence of a smooth Lyapunov function \( V \), flat on \( W \), which satisfies inequalities (6)-(8) for some class-\( \mathcal{K}_\infty \) function \( \alpha_1, \alpha_2, \alpha \) and \( c = 0 \) is necessary for system (22) and the compact weakly invariant set \( W \).

6. CONCLUSIONS

We have provided a converse Lyapunov theorem for differential inclusions with upper semicontinuous right-hand side, evolving on Riemannian manifolds, and characterized by multistable behavior. Multistability is entailed by the existence of a decomposable family of compact, globally attractive and weakly invariant sets. We have shown that a necessary condition for the existence of a Lyapunov function is the absence of homoclinic and heteroclinic cycles among the aforementioned weakly invariant sets.

Under this very assumption, a nontrivial result is the local uniform asymptotic stability of some sectors partitioning the state manifold, where “uniformity” is intended here with respect to both initial conditions and solutions sets departing from the same initial condition.

Appendix A. ASYMPTOTIC STABILITY IN SECTORS

In this Section, we let Assumptions 1, 9, and 10, and GATT hold true, and then we prove uniform local asymptotic stability of \( A_t \) on any closed subset of \( M \setminus B_t \) for any \( i \in \{1, \ldots, N\} \), namely Corollary 26.

We recall the following results for differential inclusions.

Lemma 23. \( A_t \) is bounded for any \( i \in \{1, \ldots, N\} \).

Proof. For a given compact neighborhood \( Q \) of \( W \), the reachable set \( \mathcal{R}^{T_0}(Q) \) is bounded for all \( k \in \mathbb{N} \) and all \( T \geq 0 \) (Corollary 4.7 in Goebel and Teel (2006)). By virtue of Angeli et al. (2004a) and due to the GATT property, there exists a time \( T_0 \geq 0 \) such that \( \mathcal{R}^{T_0}(Q) = \mathcal{R}(Q) \), thus \( \mathcal{R}(Q) \) is bounded. By observing that, for any point \( x \in \mathcal{R}(W_k) \), there exists a solution \( X \in S(x) \) with \( \lim_{t \rightarrow -\infty} |X(t, x)|_{W_k} = 0 \), which is thus captured by \( Q \) backward in time, we conclude that \( \mathcal{R}(W_j) \subseteq \mathcal{R}(Q) \) is bounded. Then, \( A_t = \bigcup_{j \leq t} \mathcal{R}(W_j) \) is bounded as well.

Lemma 24. (Angeli et al. (2004a)). Assume \( Q \) is a compact neighborhood of \( A_t \) such that \( Q \subseteq M \setminus B_t \). Then, there exists a time \( T_0 \geq 0 \) such that \( \mathcal{R}(Q) = \mathcal{R}^{T_0}(Q) \). In particular, \( \mathcal{R}(Q) \) is compact. Furthermore, \( \mathcal{R}(Q) \cap B_t = \emptyset \).

Lemma 25. Select a closed subset \( A_t \) of \( C_t \). Then, for any compact neighborhood \( Q \) of \( A_t \), there exists a compact weakly invariant set \( V \) of \( A_t \) such that \( V \subseteq Q \) and:

\[
\varphi_t(V) \subseteq Q \quad \forall t \geq 0.
\]

Proof. Let \( V_k := \text{cls} \mathbb{B} \{A_t, 1/k\} \) for all \( k \in \mathbb{N} \). Assume by contradiction that for some neighborhood \( Q \) of \( A_t \), there exist a sequence of non-negative times \{\( t_k \)\}_{k \in \mathbb{N}} and points \{\( x_k \)\}_{k \in \mathbb{N}} satisfying \( x_k \in \varphi^{-1}(V_k) \setminus Q \). By the definition of attainable set, there exist a sequence of points \{\( y_k \)\}_{k \in \mathbb{N}} and solutions \{\( Y_k \)\}_{k \in \mathbb{N}} satisfying \( y_k \in V_k, Y_k \in S(y_k) \), and \( Y_k(t, y) = x_k \). Due to Lemma 24, \( y_k \rightarrow \tilde{y} \) as \( k \rightarrow +\infty \) for some \( \tilde{y} \in A_t \). Corollary 4.7 in Goebel and Teel (2006) proves that \( \mathcal{R}^{T_0}(V_1) \) is bounded for all \( k \in \mathbb{N} \), and thus, by upper semicontinuity of \( F \), the solutions \( Y_k \) are equicontinuous and uniformly bounded. By the Arzelà-Ascoli theorem, for any \( h \in \mathbb{N} \) there exists a subsequence of the \( Y_k \)'s uniformly convergent to a solution \( Y \in S(\tilde{y}) \) on \([0, h]\) such that \( Y_{k+1}(t, \tilde{y}) = Y(t, \tilde{y}) \) for all \( t \in [0, h] \) and all \( h \in \mathbb{N} \). Therefore there exists a solution \( Y \in S(\tilde{y}) \),
defined for all $t \geq 0$, which coincides with $Y_k$ on any finite interval $[0, h]$ and for any $h \in \mathbb{N}$.

It holds that either $\sup_{k \in \mathbb{N}} t_k < +\infty$ or $\sup_{k \in \mathbb{N}} t_k = +\infty$. In the first case, w.l.o.g. we can consider the subsequence of the $k$s such that $\lim_{k \to +\infty} t_k = t < +\infty$. Due to the pointwise convergence, $Y(t, \dot{y}) = \lim_{k \to +\infty} Y_k(t, y_k)$. Then, due to $Y_k(t, y_k) \not\in Q$, it follows that

$$Y(t, \dot{y}) \not\in \text{int } Q. \quad (A.1)$$

However, observe that $\lim_{k \to +\infty} y_k = \bar{y} \in A_i$ for some subsequence of the $k$s, and thus $Y(t, \dot{y}) \in A_i$ by strong forward invariance of $A_i$ (Lemma 18), thus contradicting (A.1).

Consider now the case of $\sup_{k \in \mathbb{N}} t_k = +\infty$. W.l.o.g., since $|y_k|_{A_i} \leq 1/k$ for all $k \in \mathbb{N}$ and $Q$ is a neighborhood of $A_i$, we can select $t_k \geq 0$ such that $Y_k(t_k, y_k) \in \partial Q$ and $Y_k(t, y_k) \in \text{int } Q$ for all $0 \leq t < t_k$. For any $k \in \mathbb{N}$, we can then define $x_k \in \mathcal{M}$ and $X_k \in \mathcal{S}(x_k)$ by setting $X_k(t, x_k) := Y_k(t_k + t, y_k)$ for all $t \in \mathbb{R}$. It immediately follows that $X_k(-t_k, x_k) = y_k$, $X_k(0, x_k) \not\in \text{int } Q$, and $X_k(t, x_k) \in \text{int } Q$ for all $t \in [-t_k, 0)$ since the sequence of the $X_k$s is locally eventually bounded, it convergees pointwise to a solution $\bar{X} \in \mathcal{S}(x)$ where $\bar{x} := \lim_{k \to +\infty} x_k \not\in \text{int } Q$. We claim that $\bar{X} \in Q$ for all $t \leq 0$. Indeed, for any $t \leq 0$, the point $\bar{X}(t, \bar{x})$ results from the pointwise convergence of $X_k(t, x_k)$ as $k \to +\infty$. Since $X_k(t, x_k) \in \mathcal{Q}$ for all $t \in \mathcal{R}$, we have just found a 1-cycle to $A_i$, thus contradicting Assumption 10.

**Corollary 26.** Select a closed subset $A_i$ of $\mathcal{M} \setminus B_i$. Then, $A_i$ is uniformly locally asymptotically stable on $A_i$, i.e. there exists a class-$\mathcal{KL}$ function $\beta$ such that:

$$|X(t, x)|_{A_i} \leq \beta(|x|_{A_i}, t) \quad \forall t \geq 0 \quad \forall x \in A_i \quad \forall X \in \mathcal{S}(x).$$

**Proof.** By virtue of Lemmas 18 and 25, set $A_i$ satisfies both the uniform local Lyapunov stability property and the local weak attractiveness property

$$\inf_{t \geq 0} |X(t, x)|_{A_i} = 0 \quad \forall x \in A_i \quad \forall X \in \mathcal{S}(x).$$

The arguments of (Angeli et al., 2004a, Corollary 3.8 and Remark 3.6) can then be adapted to yield local uniform asymptotic stability of $A_i$ on $A_i$.

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