Finite-time consensus protocols for multi-dimensional multi-agent systems

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Abstract
A finite-time consensus protocol is proposed for multi-dimensional multi-agent systems, using direction-preserving signum controls. Filippov solutions and nonsmooth analysis techniques are adopted to handle discontinuities. Sufficient and necessary conditions are provided to guarantee infinite-time convergence and boundedness of the solutions. It turns out that the number of agents which have continuous control law plays an essential role in finite-time convergence. In addition, it is shown that the unit balls introduced by $\ell_p$ norms, where $p \in [1, \infty]$, are invariant for the closed loop.

Keywords Multi-agent systems · Network consensus · Finite-time convergence · Nonsmooth analysis

1 Introduction

Multi-agent systems (MAS) have a broad spectrum of applications in both military and civilian environments and have been a hot area of research for decades. The essential goal of the control of multi-agent systems is to let the agents achieve some states cooperatively with only local information exchange, e.g., [1, 2]. Among the vast applications of MAS, the consensus problem, where the primary objective is to drive agent states to an agreement, is a benchmark problem [3]. The current paper considers finite-time consensus protocols which, first, have the advantage of providing a guarantee on the time it takes to reach consensus and, second, provide better robustness and disturbance rejection properties than the asymptotic case [4].

Existing results on finite-time consensus can roughly be divided into two groups: namely those exploiting continuous or discontinuous control protocols. These groups have in common that the nonlinear control laws are not Lipschitz continuous at the desired consensus space. For example, continuous strategies are usually based on applying a nonlinear state feedback strategy that includes fractional powers, e.g., [5–8], or on a high gain converging to infinity as time approaches the converging time, e.g., [9]. The discontinuous strategies utilize nonsmooth control tools, e.g., [10–16], typically exploiting signum functions which require only coarse measurements. In [11], the authors construct a finite-time consensus law using binary information, namely the sign of state differences of each pair of agents. A major motivation of the current paper is [12] where finite-time consensus is proved for signum-based discontinuous control protocols (using Filippov solutions). However, this work does not guarantee that once consensus is achieved, the agent states remain constant/static, which is referred as static consensus. Instead, the group as a whole could move and trajectories...
could grow unbounded. This is not desired in many applications. Sufficient conditions for boundedness of Filippov solutions are given in [17] for a general class of nonlinear multi-agent systems that include the dynamics in [12] as special cases. However, in [17], only asymptotic convergence properties are considered. In the current paper, conditions guaranteeing finite-time convergence properties and boundedness of the trajectories are derived.

The contributions of this paper are highlighted as follows. First, necessary and sufficient conditions to achieve the finite-time static consensus, for a control strategy using direction-preserving signum function, are presented. This builds on and extends the results in [12, 18]. These conditions hinge upon the number of the agents with continuous dynamics. More precisely, for the case with more than two agents, finite-time static consensus is achieved if and only one of the agents can have a continuous control strategy; for the case with only two agents, the same is achieved except both of the agents have continuous dynamics. Second, the considered dynamics of the agents are in arbitrarily finite dimension which differs significantly from the scalar case. For any \( p \in [1, \infty] \), a corresponding finite-time consensus protocol is proposed such that the unit ball in the \( \ell_p \) norm is invariant under the dynamics of the closed loop. The results developed in this paper have potential applications in, for example, the attitude control [19], and circumnavigation [20].

The structure of the paper is as follows. In Sect. 2, we introduce terminologies and notations on graph theory and stability analysis of discontinuous dynamical systems. Section 3 presents the problem formulation of finite-time consensus. The main result is presented in Sect. 4, which includes illustrative examples. Conclusions are stated in Sect. 5.

**Notation** With \( \mathbb{R}_- \), \( \mathbb{R}_+ \), \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{\leq 0} \), we denote the sets of negative, positive, nonnegative and nonpositive real numbers, respectively. A positive semidefinite (symmetric) matrix \( M \) is denoted as \( M \succeq 0 \). The \( i \)th row of a matrix \( M \) is given by \( M_i \). The vectors \( e_1, e_2, \ldots, e_n \) denote the canonical basis of \( \mathbb{R}^n \), whereas the vectors \( 1_n \) and \( 0_n \) represent a \( n \)-dimensional column vector with each entry being 1 and 0, respectively. We will omit the subscript \( n \) when no confusion arises. The \( \ell_p \) norm with \( p \in [1, \infty] \) is denoted as \( \| \cdot \|_p \). The notation \( B(x, \delta) \) represents the open ball centered at \( x \) with radius \( \delta > 0 \) with the \( \ell_2 \) norm. For a multi-agent system with each agent state \( x_i \in \mathbb{R}^k \), define the consensus space as

\[
 C = \{ x \in \mathbb{R}^k | \exists \bar{x} \in \mathbb{R}^k \text{ such that } x = 1_n \otimes \bar{x} \}. \tag{1}
\]

where \( \otimes \) denotes the Kronecker product. Finally, we denote the direction-preserving signum function as

\[
 \text{sgn}(w) = \begin{cases} 
 \frac{w}{\|w\|_p}, & \text{if } w \neq 0, \\
 0, & \text{if } w = 0, 
\end{cases} \tag{2}
\]

for \( w \in \mathbb{R}^k \) and any \( p \in [1, \infty] \).

**2 Preliminaries**

In this section, we briefly review some essentials from graph theory [21, 22] and give some results on Filippov solutions [23] of differential equations with discontinuous vector fields.

An undirected graph \( G = (\mathcal{I}, \mathcal{E}) \) consists of a finite set of nodes \( \mathcal{I} = \{1, 2, \ldots, n\} \) and a set of edges \( \mathcal{E} \subset \mathcal{I} \times \mathcal{I} \) of unordered pairs of elements of \( \mathcal{I} \). To any edge \((i, j) \in \mathcal{E}\), we associate a weight \( w_{ij} > 0 \). The weighted adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is defined by \( a_{ij} = w_{ij} \) if \((i, j) \in \mathcal{E}\) and \( a_{ij} = 0 \) otherwise. Note that \( A = A^T \) and that \( a_{ij} = 0 \) as no self-loops are allowed. For each node \( i \in \mathcal{I} \), its degree \( d_i \) is defined as \( d_i = \sum_{j=1}^{n} a_{ij} \). The graph Laplacian \( L \) is defined as \( L = A - \Delta \) with \( \Delta \) a diagonal matrix such that \( \Delta_{ii} = d_i \). As a result, \( LI = 0 \). Finally, we say that a graph \( G \) is connected if, for any two nodes \( i \) and \( j \), there exists a sequence of edges that connects them. In order to simplify the notation in the proofs, we set the weights \( w_{ij} \) to be one. All the results in this paper hold for general positive nonzero \( w_{ij} \).

The following result essentially states that the Schur complement of a graph Laplacian is itself a graph Laplacian.

**Lemma 1** [24] Consider a connected undirected graph \( G \) with Laplacian matrix \( L \), then all Schur complements of \( L \) are well defined, symmetric, positive semi-definite, with diagonal elements \( > 0 \), off-diagonal elements \( \leq 0 \), and with zero row and column sums.

In the remainder of this section, we discuss Filippov solutions. Let \( f \) be a map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), and let \( 2^\mathbb{R}^n \) denote the collection of all subsets of \( \mathbb{R}^n \). Then, the Filippov set-valued map of \( f \), denoted \( \mathcal{F}[f] : \mathbb{R}^n \to 2^{\mathbb{R}^n} \), is defined as

\[
 \mathcal{F}[f](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \text{co} \{ f(B(x, \delta) \setminus S) \}, \quad \tag{3}
\]

where \( S \) is a subset of \( \mathbb{R}^n \), \( \mu \) denotes the Lebesgue measure and \( \text{co}(A) \) denotes the convex closure of a set \( A \). If \( f \) is continuous at \( x \), then \( \mathcal{F}[f](x) \) contains only the point \( f(x) \).

A Filippov solution of the differential equation \( \dot{x} = f(x) \) on \([0, T] \subset \mathbb{R}\) is an absolutely continuous function \( x : [0, T] \to \mathbb{R}^n \) that satisfies the differential inclusion

\[
 \dot{x}(t) \in \mathcal{F}[f](x(t)) \tag{4}
\]
for almost all \( t \in [0, T] \). A Filippov solution \( t \mapsto x(t) \) is maximal if it cannot be extended forward in time, that is, if \( t \mapsto x(t) \) is not the result of the truncation of another solution with a larger interval of definition. Since Filippov solutions are not necessarily unique, we need to specify two types of invariant sets. A set \( \mathcal{R} \subset \mathbb{R}^n \) is called weakly invariant if, for each \( x_0 \in \mathcal{R} \), at least one maximal solution of (4) with initial condition \( x_0 \) is contained in \( \mathcal{R} \). Similarly, \( \mathcal{R} \subset \mathbb{R}^n \) is called strongly invariant if, for each \( x_0 \in \mathcal{R} \), every maximal solution of (4) with initial condition \( x_0 \) is contained in \( \mathcal{R} \). For more details, see [23, 25].

If \( V : \mathbb{R}^n \to \mathbb{R} \) is locally Lipschitz, then its generalized gradient \( \partial V : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) is defined by

\[
\partial V(x) := \text{co} \{ \lim_{i \to \infty} \nabla V(x_i) : x_i \to x, x_i \not\in S \cup \Omega_V \},
\]

(5)

where \( \text{co} \{ \lambda \} \) denotes the convex hull of a set \( \lambda \), \( \nabla V \) denotes the gradient operator, \( \Omega_V \subset \mathbb{R}^n \) denotes the set of points where \( V \) fails to be differentiable and \( S \subset \mathbb{R}^n \) is a set of measure zero that can be arbitrarily chosen to simplify the computation. Namely, the resulting set \( \partial V(x) \) is independent of the choice of \( S \) [26].

Given a set-valued map \( T : \mathbb{R}^n \to \mathbb{R}^n \), the set-valued Lie derivative \( \mathcal{L}_T V : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) of a locally Lipschitz function \( V : \mathbb{R}^n \to \mathbb{R} \) with respect to \( T \) at \( x \) is defined as

\[
\mathcal{L}_T V(x) := \{ a \in \mathbb{R} | \exists \gamma \in T(x) \text{ such that } \gamma^T v = a, \forall \zeta \in \partial V(x) \}.
\]

(6)

If \( T(x) \) is convex and compact \( \forall x \in \mathbb{R}^n \), then \( \mathcal{L}_T V(x) \) is a closed and bounded interval in \( \mathbb{R} \), possibly empty, for each \( x \).

A LaSalle’s invariance principle for discontinuous differential equations (4) with nonsmooth Lyapunov functions and a result on finite-time convergence for (4) can be found in [12].

3 Problem formulation

Consider the nonlinear multi-agent system

\[
x_i = u_i, \quad i \in \mathcal{I} = \{1, 2, \ldots, n\},
\]

(7)

defined on a connected network \( \mathcal{G} \) with \( n \) agents and \( m \) edges, i.e., \( |\mathcal{E}| = m \), where \( x_i(t), u_i(t) \in \mathbb{R}^k \) are the state and the input of agent \( i \) at time \( t \), respectively. In this paper, we say that the states \( x = [x_1^T \cdots x_n^T]^T \) of a multi-agent system converge to static consensus in finite time if, for any initial condition, there exists a time \( \tau^* > 0 \) such that \( x(t) \) converges to a static vector in \( \mathcal{C} \) as \( t \to \tau^* \), i.e., there exists a vector \( \bar{x} \) such that \( x(t) = \mathbf{1}_n \otimes \bar{x} \) for all \( t \geq \tau^* \).

There are two motivations for the aim in this paper. First, for scalar version of (7), i.e., \( x_i \in \mathbb{R} \), one finite-time consensus protocol stated in [12] is

\[
x = \text{sgn}(-Lx),
\]

(8)

where \( x \in \mathbb{R}^n \), \( L \in \mathbb{R}^{n \times n} \) is the Laplacian of a connected undirected graph. It was presented that all the Filippov solutions of (8) converge to the average-max-min-consensus, i.e.,

\[
\frac{1}{2} \min_{i=1}^n x_i(0) + \max_{i=1}^n x_i(0),
\]

in finite time. However, this is not precise as the trajectories could go unbounded in \( \mathcal{C} \), see [17]. We explain this phenomenon by recalling an example from [17], which also serves as a counterexample to the result in Sect. 4 [12].

Example [17]

Consider

\[
x_1 = \text{sgn}(x_2 + x_3 - 2x_1), \]

\[
x_2 = \text{sgn}(x_1 + x_3 - 2x_2),
\]

\[
x_3 = \text{sgn}(x_1 + x_2 - 2x_3),
\]

defined on a circular graph with three nodes and all edges with unit weight, and \( x_i \in \mathbb{R} \). In this case, \( \mathcal{C} = \text{span} \{ \mathbf{1}_n \} \). We can show that for any initial condition \( x(0) \in \mathcal{C} \), all Filippov solutions converge to \( \mathcal{C} \) in finite time. However, once they enter \( \mathcal{C} \), the solutions can be unbounded. Indeed, suppose that at time \( t_0 \) we have \( x(t_0) \in \mathcal{C} \), then

\[
\mathcal{F}[h](x(t_0)) = \text{co} \{ v_1, v_2, v_3, -v_1, -v_2, -v_3 \},
\]

(9)

where \( v_1 = [1, 1, -1]^T \), \( v_2 = [1, -1, 1]^T \), and \( v_3 = [-1, 1, 1]^T \). Since \( \sum_{i=1}^3 v_i = \frac{1}{3} \mathbf{1} \), we have that

\[
\left\{ \eta \mathbf{1} | \eta \in \left[ -\frac{1}{3}, \frac{1}{3} \right] \right\} \subset \mathcal{F}[h](x(t_0)).
\]

(10)

Hence, any function \( x(t) = \eta(t) \mathbf{1} \) with \( \eta(t) \) differentiable almost everywhere and satisfying \( \eta(t) \in \left[ -\frac{1}{3}, \frac{1}{3} \right] \) for all \( t > t_0 \) is a Filippov solution for this system. Consequently, not all Filippov solutions converge to a static vector in \( \mathcal{C} \).

For system (8), Theorem 7 in [17] provides a guarantee for asymptotic convergence to static consensus. The main result in [17] relies on replacing at least one sgn function with a function that is continuous at the origin. However, this result only guarantees asymptotic convergence.

Second, different from the scalar version of (7), in high-dimensional multi-agent system, it is often desired to keep the states within the \( \mathcal{C} \), ball spanned by the initial condition, for \( p \in [1, \infty] \). One motivating example is the attitude consensus problem using angle-axis representation, where the state of each agent is \( x_i \in \mathbb{R}^3 \). We refer to [19] for details on attitude control. For any control law using axis-angle representations of absolute rotations, i.e., the principle logarithmic of rotation matrices, a singularity occurs at \( \|x\|_2 = \pi \).

Then, by keeping the \( \mathcal{C} \)-ball \( B(\mathbf{0}, \pi - \epsilon) \) invariant for the dynamics of \( x_i, i \in \mathcal{I} \), the singularity is avoided when it is avoided initially.
Motivated by the above, the aim of the paper is formulated as follows.

**Aim** Design control protocols

\[ u_i = f_1 \left( \sum_{j=1}^{n} a_{ij} (x_j - x_i) \right), \quad i \in \mathcal{I}, \]  
(11)

where \( f_i : \mathbb{R}^k \to \mathbb{R}^k \) and \( a_{ij} \) is the \( ij \)-th element of the adjacency matrix \( A \), to system (7) such that all solutions converge to a static vector in \( C \) in finite time with the \( \ell_p \) ball spanned by the initial condition being invariant.

## 4 Main results

Combining (7) and (11), the closed-loop system is given as

\[ x_i = f_i \left( \sum_{j=1}^{n} a_{ij} (x_j - x_i) \right), \quad i \in \mathcal{I}. \]  
(12)

Denoting \( \bar{L} = L \otimes I_k \) and \( \bar{L}_i = L_i \otimes I_k \), (12) can be further written in a compact form as

\[ x = f(-\bar{L} x) \]  
(13)

in which \( x = [x_1^T \ x_2^T \ \cdots \ x_n^T]^T \in \mathbb{R}^{kn} \) collects the states of all agents and \( f(y) = [f_1^T(y_1) \ \cdots \ f_n^T(y_n)]^T \).

In light of the success of scalar binary control protocols to achieve finite-time consensus [11, 12, 14], we shall design control protocols based on the signum function for multidimensional multi-agent systems.

In this paper, we consider the nonlinear controller \( u_i \) as in (11) with some \( f_i = \text{sgn} \) as in (2). Motivated by Theorem 7 in [17], one sufficient condition to guarantee the boundedness of solutions is to have at least one \( f_i \) being Lipschitz-continuous at the origin, which prompts us to the following assumption on \( f_i, i \in \mathcal{I} \).

**Assumption 1** For some set \( \mathcal{I}_c \subset \mathcal{I} \), the function \( f \) in (13) satisfies the following conditions:

i) For \( i \in \mathcal{I}_c \), the function \( f_i : \mathbb{R}^k \to \mathbb{R}^k \) is locally Lipschitz continuous and satisfies \( f_i(0) = 0 \) and \( f_i(y)^T y = \|f_i(y)\| \|y\| > 0 \) for all \( y \neq 0 \) (i.e., the functions \( f_i \) are direction preserving).

ii) For \( i \in \mathcal{I} \setminus \mathcal{I}_c \), the function \( f_i \) is the direction-preserving signum, i.e., \( f_i = \text{sgn} \).

**Remark 2** For the scalar case, direction preserving in Assumption 1 is simply sign preserving, i.e., \( f_i(0) = 0 \) and \( f_i(y_i)y_i > 0 \) for all \( y_i \neq 0 \).

Notice that the \( \text{sgn} \) function in (2) is locally Lipschitz and direction preserving on \( \mathbb{R}^k \setminus \{0\} \), and for any \( w \neq 0 \), we have \( \|\text{sgn}(w)\|_p = 1 \).

To handle the discontinuities in \( f \) that arise from the signum function in Assumption 1, we understand the solution of (13) in the sense of Filippov, i.e., we consider the differential inclusion

\[ \dot{x} \in \mathcal{F}[h](x) \]  
(14)

where \( h = f(-\bar{L} x) \).

So far, one could expect that the finite-time convergence of system (14) to consensus depends on the conditions of \( \mathcal{I} \) and \( \mathcal{I}_c \). We start the analysis with the special case \( \mathcal{I}_c = \mathcal{I} \).

It is well known that Lipschitz continuous vector fields give mere asymptotic convergence [27], but this result is stated for completeness.

**Proposition 1** Consider the nonlinear consensus protocol (13) satisfying Assumption 1 with \( \mathcal{I}_c = \mathcal{I} \). Then, the (unique) solution of (13) converges to consensus only asymptotically, i.e., finite-time consensus is not achieved.

One key property of system (7) with controller (11) satisfying Assumption 1 is that any bounded ball in the \( \ell_p \)-norm is strongly invariant. This is formulated in the following lemma.

**Lemma 2** Consider the differential inclusion (14) satisfying Assumption 1 for some \( p \in [1, \infty) \). If one of the following two conditions is satisfied

i) \( |\mathcal{I}| = 2 \) and \( |\mathcal{I}_c| = 0 \);

ii) \( |\mathcal{I}| \geq 2 \) and \( |\mathcal{I}_c| \geq 1 \),

then the set \( S_p(C) = \{x \in \mathbb{R}^{nk} \|x\|_p \leq C, i \in \mathcal{I} \} \), for any \( C > 0 \) is a constant, is strongly invariant.

The proof is given in the appendix.

Now we are in the position to state the main result of this paper.

**Theorem 3** Consider the nonlinear consensus protocol (13) satisfying Assumption 1 and the corresponding differential inclusion (14) for some \( p \in [1, \infty) \). Then, the following statements hold:

i) If \( |\mathcal{I}| > 2 \), all Filippov solutions of (14) converge to static consensus in finite time if and only if \( |\mathcal{I}_c| = 1 \).

ii) If \( |\mathcal{I}| = 2 \), all Filippov solutions of (14) converge to static consensus in finite time if and only if \( |\mathcal{I}_c| \leq 1 \).
Proof The proofs of sufficiency and necessity of the two statements are considered separately.
(Sufficiency) The Lyapunov function candidate
\[ V(x) = \sqrt{x^T L x} \]  
(15)
is introduced. Note that \( V(x) = 0 \) for all \( x \in \mathbb{C} \) and that \( V \) is convex, hence, regular\(^1\). The set-valued Lie derivative of \( V \) in (15) will be considered for \( x \notin \mathbb{C} \), hereby evaluating the two cases in the statement of Theorem 3 separately. In both cases, it will be shown that
\[
\max \mathcal{L}_{\mathcal{F}(i)} V(x) \leq -c
\]  
(16)
for some \( c > 0 \) that is independent of \( x \notin \mathbb{C} \). Then, finite-time consensus follows from Proposition 4 in [12], hereby exploiting a strongly invariant set from Lemma 2.

1) The case \(|\mathcal{I}| = 1\) is considered first, and we assume that \(|\mathcal{I}_c| = 1\). As in the proof of Lemma 2, we use the extended differential inclusion \( \mathcal{F} \) defined in (54), which satisfies property (55). Consequently, \( \mathcal{L}_{\mathcal{F}(i)} V(x) \subseteq \mathcal{L}_+ V(x) \) with the latter given by (6) as
\[
\mathcal{L}_+ V(x) = \left\{ \frac{1}{\sqrt{x^T L x}} x^T L v \mid v \in \mathcal{F}(x) \right\},
\]  
(17)
which follows from the observation that the generalized gradient of (15) reduces to the regular gradient for \( x \notin \mathbb{C} \).

As the case \(|\mathcal{I}| > 2\) is considered, there exists exactly one agent with locally Lipschitz continuous dynamics. Without loss of generality, let this agent have index 1. Then, it follows from the property of \( f_i \) in Assumption 1 that
\[
x^T L_1 f_1(-L_1 x) \leq 0.
\]  
(18)
Next, for \( i \in \{2, \ldots, n\} \), it holds that
\[
\mathcal{F}[\text{sgn}]|-L_i x| = \left\{ \frac{-L_i x}{\|L_i x\|_p}, \|L_i x\| \neq 0, \|L_i x\|_p \leq 1 \right\}.
\]  
(19)
such that, for all \( v_i \in \mathcal{F}[\text{sgn}]|-L_i x|,
\[
x^T L_i v_i = \begin{cases} 
\frac{-\|L_i x\|^2}{\|L_i x\|_p}, & \|L_i x\| \neq 0, \\
0, & \|L_i x\| = 0.
\end{cases}
\]  
(20)
Due to the equivalence of norms on finite-dimensional vector spaces, there exists \( d_1 > 0 \) such that \( d_1 \|L_i x\|_p \leq \|L_i x\|_2 \) for all \( x \in \mathbb{R}^n \). Applying this to (20) yields
\[
x^T L_i^2 v_i \leq -d_1 \|L_i x\|_2
\]  
(21)
for \( i \in \{2, \ldots, n\} \), and note that this indeed holds for both cases in (20). Now, after recalling the definition of \( \mathcal{L}_+ V(x) \) in (17), the combination of (18) and (21) shows that, for any \( a \in \mathcal{L}_+ V(x) \),
\[
a \leq -\frac{d_1}{\sqrt{x^T L x}} \left( \sum_{i=2}^n \|L_i x\|_2 \right).
\]  
(22)
Note that, as \( \bar{L} = L \otimes I_k \) with \( L \) a graph Laplacian satisfying \( 1^T L = 0^T \), it holds that
\[
\|\bar{L}_i x\| = \|\sum_{i=2}^n \bar{L}_i x\| \leq \sum_{i=2}^n \|\bar{L}_i x\|,
\]  
(23)
where the triangle inequality is used to obtain the inequality. Then, the use of (23) in (22) yields
\[
a \leq -\frac{1}{2} \frac{d_1}{\sqrt{x^T L x}} \left( \sum_{i=2}^n \|L_i x\|_2 \right).
\]  
(24)
By further exploiting that \( L \) is a graph Laplacian, it holds that \( L \) and \( L^2 \) can be written as
\[
L = U^T A U, \quad L^2 = U^T A^2 U,
\]  
(25)
where \( A = \text{diag}(0, \lambda_2, \ldots, \lambda_n) \) is a diagonal matrix with Laplacian real-valued eigenvalues satisfying \( 0 < \lambda_2 \) and \( \lambda_i \leq \lambda_{i+1} \), and the matrix \( U \) collects the corresponding eigenvectors. From (25), it can be seen that
\[
L^2 L - c_1 L \geq 0
\]  
(26)
for any \( c_1 \in (0, \lambda_2] \). Consequently, using \( L = L \otimes I_k \), it follows that
\[
\left( \sum_{i=1}^n \|\bar{L}_i x\| \right)^2 \geq x^T \bar{L}^T \bar{L} x \geq c_1 x^T \bar{L} x.
\]  
(27)
After taking the square root (note that \( x^T \bar{L} x > 0 \) for all \( x \notin \mathbb{C} \) in (27) and using (24), the result
\[
a \leq -\frac{\sqrt{c_1}}{2} \frac{d_1}{\sqrt{x^T L x}} \leq -\frac{d_1 \sqrt{c_1}}{2} < 0
\]  
(28)
follows.

ii) The proof for the case \(|\mathcal{I}| = 2\) and \(|\mathcal{I}_c| \leq 1\) follows similarly.

Next, by Proposition 4 in [12], we have that the trajectories converge to \( \mathcal{Z}_{\mathcal{F}, \Lambda} \) in finite time. The remaining task is to

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\(^1\) The definition of a regular function can be found in [26]
characterize the set $\mathcal{Z}_{\mathcal{F}, \mathcal{V}}$. So far we have shown that $x \notin \mathcal{Z}_{\mathcal{F}, \mathcal{V}}$ for $\forall x \notin \mathcal{C}$ which implies that $\mathcal{Z}_{\mathcal{F}, \mathcal{V}} \subset \mathcal{C}$. By the fact that $\mathcal{C}$ is closed, we have $\mathcal{Z}_{\mathcal{F}, \mathcal{V}} \subset \mathcal{C}$. Moreover, when $x \in \mathcal{C}$, $x_i = 0$ where $\{i\} = \mathcal{L}_e$ which implies $x_i$ remains constant. In conclusion, the finite-time convergence to static consensus is guaranteed.

(Necessity) The necessity of the conditions in i) and ii) can be proven by showing the following equivalent formulation: For any $|\mathcal{I}| \geq 2$ and $|\mathcal{I}_e| \geq 2$, there exists (at least one) Filippov solution of (14) that does not converge to (static) consensus in finite time. Note that the conditions in both statements are now considered simultaneously.

For the case $\mathcal{I} = \mathcal{I}_e$, the desired result immediately follows from Proposition 1. Therefore, in the remainder of this proof, we consider the case in which $\mathcal{L}_e$ is a strict subset of $\mathcal{I}$ and we restrict analysis to the case $|\mathcal{I}| \geq 3$.

Consider the function

$$V_i(x) = \|\bar{L}_ix\|_p$$

for $i \in \mathcal{I} \setminus \mathcal{I}_e$ and the set

$$S(\delta) = \{x \in \mathbb{R}^nx | \sqrt{x^T\bar{L}_ix} \leq \delta\}.$$  

Note that $\mathcal{C} \subset S(\delta)$ for any $\delta \leq 0$. By the sufficiency proof of Theorem 3, we have that $S(\delta)$ is strongly invariant. More precisely, by the Lipschitz continuity of $f_i$ for $i \in \mathcal{I}_e$, there exists $\epsilon$ such that $\|f_i(-L_i)x\|_2 \leq 1$ for any $\|L_ix\|_2 \leq \epsilon$ and $i \in \mathcal{I}_e$. Furthermore,

$$\|\sum_{j \in \mathcal{N}_i} (x_j - x_i)\|_2 \leq \sum_{j \in \mathcal{N}_i} \|x_j - x_i\|_2$$

$$\leq (m(\sum_{i \in \mathcal{E}} \|x_j - x_i\|_{2}^{2}))^{\frac{1}{2}},$$

where we recall that $m = |\mathcal{E}|$. Then, by choosing $\delta_e < \frac{\epsilon}{\sqrt{m}}$, we have $\|\bar{L}_ix\|_2 \leq \epsilon$ for any $i \in \mathcal{I}_e$ and $x \in S(\delta_e)$. By the equivalence of the $\ell_p$-norms on finite dimensional space, we assume without loss of generality that we can choose proper $\delta_i$ such that $\|L_ix\|_p \leq \epsilon$ for any $i \in \mathcal{I}_e$ and $x \in S(\delta_i)$.

We consider the evolution of $V_i$ along trajectories $x$ in $S(\delta_i)$. To this end, note that $V_i(x)$ in (29) is locally Lipschitz and convex and, as a result, regular. In the evaluation of the set-valued Lie derivative $\mathcal{L}_{\mathcal{F}[h]} V_i(x)$ as in (6), we will only consider the subset of time for which $\mathcal{L}_{\mathcal{F}[h]} V_i(x(t))$ is non-empty. Moreover, as before, the cases $p \in [1, \infty]$ and $p = \infty$ will be considered separately. We denote $\zeta = [\zeta_1 \cdots \zeta_n]^T$ and $v = [v_1 \cdots v_n]^T$, where $\zeta_j, v_j \in \mathbb{R}^r, j = 1, \ldots, n$, for any $\zeta \in dV_i(x)$ and $v \in \mathcal{F}[h](x)$, respectively.

1) Let $p \in [1, \infty)$. If $V_i(x) \neq 0$, then we have that

$$\zeta_j = 0, \ j \notin \mathcal{N}_i, \quad (33)$$

$$\zeta_i \in L_{\mathcal{I}_i} \|\bar{L}_ix\|_p^{1-p} \psi(\bar{L}_ix), \quad (34)$$

$$\zeta_j \in \|\bar{L}_ix\|_p^{1-p} \psi(\bar{L}_ix), \ j \in \mathcal{N}_i, \quad (35)$$

where $\psi$ is given as in (48). If $V_i(x) = 0$, we have that $\zeta$ satisfies $\zeta_j = 0$ for $j \notin \mathcal{N}_i$, and the vector $\omega$, whose components are $\frac{1}{L_{\mathcal{I}_i}} \zeta_i$ and $\zeta_j$ for all $j \in \mathcal{N}_i$, satisfies $\|\omega\|_q \leq 1$ with $q = \frac{p}{p-1}$ for $p > 1$ and $q = \infty$ for $p = 1$.

If $V_i(x) = 0$, then we have that $\zeta_j = 0, j \notin \mathcal{N}_i$, and $\zeta_i \in L_{\mathcal{I}_i} \|\bar{L}_ix\|_p^{1-p} \psi(\bar{L}_ix), j \in \mathcal{N}_i$. Indeed, if there exists $a \in \mathcal{L}_{\mathcal{F}[h]} V_i(x)$, there exists $\nu \in \mathcal{F}[h](x)$ such that $a = v^T \zeta_i + \sum_{j \in \mathcal{N}_i} v^T \zeta_j$ for any $\omega$, where the components of $\omega$ are $\frac{1}{L_{\mathcal{I}_i}} \zeta_i$ and $\zeta_j$ for all $j \in \mathcal{N}_i$, satisfying $\|\omega\|_q \leq 1$. Hence, such $a$ can only be equal to 0 and max $\mathcal{L}_{\mathcal{F}[h]} V_i(x) \leq 0$.

If $V_i(x) \neq 0$, for any $a \in \mathcal{L}_{\mathcal{F}[h]} V_i(x)$, there exists a $\nu \in \mathcal{F}[h](x)$ such that

$$a = \sum_{j=1}^{n} \|L_ix\|_p^{1-p} L_{ij} v_j \psi(\bar{L}_ix)$$

$$\leq \sum_{j=1, j \neq i}^{n} |L_{ij}| - L_{ii}$$

$$= 0,$$

where the inequality is implied by Hölder’s inequality and the fact that $\|v_j\|_p \leq 1, j \neq i$, $v_i = -\frac{L_{ix}}{\|\bar{L}_ix\|_p}$ and $\|\psi(\bar{L}_ix)\|_q = \|\bar{L}_ix\|_p^{q-1}$.

2) Let $p = \infty$. The generalized gradient of $V_i$ is given as

$$\partial V_i(x) = \co\{\zeta \mid \zeta_j = e_{\mathcal{F}} \mathcal{F} \sgn((\bar{L}_ix)_j)L_{ii} \}$$. 

$$\zeta_j = -e_{\mathcal{F}} \mathcal{F} \sgn((\bar{L}_ix)_j), \quad (37)$$

$$e_{\mathcal{F}} \in \mathbb{R}^r, \ j \in \mathcal{N}_i, \ (\bar{L}_ix)_j = V_i(x).$$

Then, for any $a \in \mathcal{L}_{\mathcal{F}[h]} V_i(x)$, there exists a $\nu \in \mathcal{F}[h](x)$ such that

$$a = v_1 \zeta_1 + \sum_{j \in \mathcal{N}_i} v_j \zeta_j$$

$$= \mathcal{F} \sgn((\bar{L}_ix)_j)L_{ii} e_{\mathcal{F}} + \sum_{j \in \mathcal{N}_i} \mathcal{F} \sgn((\bar{L}_ix)_j)e_{\mathcal{F}} v_j$$

$$\leq \sum_{j=1, j \neq i}^{n} |L_{ij}| - L_{ii}$$

$$= 0.$$

Combining the above two cases, we have shown that
max $\mathcal{L}_{\mathcal{I}_c} V_i(x(t)) \leq 0$ \hspace{1cm} (39)

for any $i \in \mathcal{I} \setminus \mathcal{I}_c$ and for $x(t) \in S(\delta_i)$.

In the remainder of this proof, we will construct a Filippov solution of (14) in $S(\delta_i)$ which does not achieve static consensus in finite time. Here, we assume without loss of generality that the nodes are ordered such that $\mathcal{I}_c = \{1, \ldots, |\mathcal{I}_c|\}$ and $\mathcal{I} \setminus \mathcal{I}_c = \{|\mathcal{I}_c| + 1, \ldots, n\}$ and partition the Laplacian $L$ accordingly as

$L = \begin{bmatrix} L_{cc} & L_{cd} \\ L_{dc} & L_{dd} \end{bmatrix}$. \hspace{1cm} (40)

Then, consider the solutions for an initial condition $x_0 = [x_0^c \ x_0^d]^T \in S(\delta)$ that satisfies the set of equations

$L_{cc}x_0^c + L_{cd}x_0^d = 0$, \hspace{1cm} (41)

$L_{dc}x_0^c + L_{dd}x_0^d = 0$, \hspace{1cm} (42)

and note that such solution exists as $L_{dd}$ is invertible. (This follows from the standing assumption that the graph $G$ is connected and Lemma 1). Recall that the left-hand side of (42) can be written as $L_i x_0 = 0$ for $i \in \mathcal{I} \setminus \mathcal{I}_c$ and, thus, as $V_i(x_0) = 0$ with $V_i$ as in (29). Furthermore, since $|\mathcal{I}_c| \geq 2$, the solution $x_0$ can be chosen such that $x_0^d \notin \mathcal{C}$. Then, by the result (39), it follows that $V_i$ is nonincreasing along trajectories, such that $V_i(x(t)) = 0$ for all $i \in \mathcal{I} \setminus \mathcal{I}_c$ and all $t \geq 0$. Consequently, any trajectory with initial condition $x_0 \in S(\delta)$ satisfying (41) and (42) satisfies $x^d(t) = -L_{dd}^{-1}L_{dc}x^c(t)$ for all $t \geq 0$.

In this case, the dynamics of the nodes with continuous dynamics can be expressed as

$\dot{x}^c = f^c(-(L_{cc} - L_{cd}L_{dc})^{-1}L_{dd})x^c$, \hspace{1cm} (43)

where $f^c$ collects the Lipschitz continuous functions $f_i$ in Assumption 1 for $i \in \mathcal{I}_c$. As a result of Lemma 1, the matrix $L_{cc} - L_{cd}L_{dc}^{-1}L_{cc}$ is itself a graph Laplacian, such that dynamics (43) can be regarded as a special case of (13) in which all nodes have continuous dynamics. As such, the result follows from Proposition 1, finalizing the proof of necessity. \hspace{1cm} \blacksquare

Remark 4 In Lemma 2 and Theorem 3, we set the edge weights $w_{ij}$ of the graph $G$ to one to simplify the notation in the proofs. However, all results in this paper hold for general positive $w_{ij}$. For example, it can be verified that the calculations in (57)–(58), (66)–(67) and (38) hold for the case with general edge weights.

Remark 5 In [12], the stated final consensus value for the system $\dot{x} = \text{sgn}(-Lx)$, where $x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n$ and $L$ is the Laplacian matrix of any connected undirected graph, is the average-max-min-consensus, i.e.,

$\min_{i \in \mathcal{I}} x_i(0) + \max_{i \in \mathcal{I}} x_i(0)$ \hspace{1cm} (44)

However, this result is not precise as the solutions of $\dot{x} = \text{sgn}(-Lx)$ are not bounded, see Example 1 and [17]. For system (14) satisfying Assumption 1, the final consensus vector can be formulated as follows. When $|\mathcal{I}| = 2$ and $|\mathcal{I}_c| = 0$, it is straightforward to see that the final consensus value is the average of the initial condition $x(0) = [x_1(0) + x_2(0)]/2$. For other cases, i.e., $|\mathcal{I}_c| = 1$, the final consensus value depends on the position of the vertices in $\mathcal{I}_c$ in the graph as well as the associated nonlinear functions $f_i$. However, for both cases, the trajectories are in a compact set $S_p = \{x \in \mathbb{R}^d ||x||_p \leq C, i \in \mathcal{I}\}$ where $C = \max_{i \in \mathcal{I}} ||x_i(0)||_p$. Furthermore, the upper bound on $a$ in (28) can be used to derive an estimation of the settling time.

Remark 6 The notion of Filippov solutions, which provides a dedicated tool to analyze systems with discontinuities, captures infinitely fast chattering by allowing for solutions that slide along the switching manifold [23]. Furthermore, since $\text{sgn}(0) = 0$, infinitely fast output chattering of controllers $u_i$ does not occur on the switching manifold, i.e., the consensus space $\mathcal{C}$. However, the chattering phenomenon can be the result of the implementation of the discontinuous control strategies, especially if the controller suffers from delay. For some techniques to eliminate the chattering, see, e.g., [28].

Example 2 Consider consensus on SO(3) [19] again, where $x_i \in \mathbb{R}^3$ is the axis-angle representation of a rotation matrix $R_i \in \text{SO}(3)$. The kinematics of $x_i$ is given as $\dot{x}_i = L_{x_i}u_i, i \in \mathcal{I}$ where $L_{x_i} \in \mathbb{R}^{3x3}$ satisfies $x_i^T L_{x_i} x_i = x_i^T$ and $u_i$ is the control input. Thus,

$\frac{d}{dt} \frac{1}{2} ||x_i||_2^2 = x_i^T u_i$. \hspace{1cm} (44)

The singularity of the axis-angle representation, which occurs at $||x_i||_2 = \pi$, is avoided if $S_2(\pi - \epsilon)$ is strongly invariant for $\epsilon > 0$. By the proof of Lemma 2, which deals with the time variation of the maximal norm of the states, this is guaranteed if $u_i, i \in \mathcal{I}$ satisfy Assumption 1. Hence, Assumption 1 and Lemma 2 can be used as criteria for design of finite-time consensus protocols on SO(3).

We close this subsection with demonstrating the result in Theorem 3 by an example.
Example 3  Consider system (14) with \( x_i \in \mathbb{R}^2 \) defined on the graph given in Fig. 1 and satisfying Assumption 1. Let \( p = 2 \) (i.e., the \( \ell_2 \)-norm is considered) in \( \text{sgn} \) defined by (2).

First, consider \( f_i = \text{sgn} , i = 1, \ldots, 4 \) and \( f_5 \) is the identity function. Hence, condition (i) in Theorem 3 is satisfied. The trajectory of this system with randomly generated initial conditions is depicted in Fig. 2. Here, we can see that finite-time consensus is achieved.

Next, using the same initial conditions, set \( f_i = \text{sgn} , i = 1, \ldots, 3 \) and \( f_4, f_5 \) as the identity function. Then, both conditions (i) and (ii) are violated, so finite-time consensus is not expected. Indeed, in this case we can only have asymptotic convergence to consensus as shown in Fig. 3.

Remark 7  For vectors, the component-wise signum function, denoted as \( \text{sgn}_c \), is also commonly used in the literature. If we assume that, for \( i \in I_c \subseteq I \), each component of the function \( f_i : \mathbb{R}^k \rightarrow \mathbb{R}^k \) is Lipschitz continuous and
sign-preserving and that \( f_i = \text{sgn}_\varepsilon \) for \( i \in \mathcal{I}\setminus \mathcal{I}_c \), then the same conclusions hold for system \((14)\) as in Theorem 3.

5 Conclusions

In this paper, we considered the finite-time consensus problem for high-dimensional multi-agent systems. A finite-time consensus control protocol is proposed, using direction-preserving signum functions. The protocol can be designed such that the \( \ell_p \)-norm unit ball is strongly invariant for any \( p \in [1, \infty) \). For this control law, sufficient and necessary conditions were presented to guarantee finite-time convergence and boundedness of the solutions. Future work will focus on finite-time tracking problems as well as the practical implementation of the proposed control protocol, where care must be taken to avoid chattering.

Appendix

Proof of Lemma 2 We divide the proof into two parts, discussing the cases \( p \in [1, \infty) \) and \( p = \infty \) separately.

1) Let \( p \in [1, \infty) \). We introduce a Lyapunov function candidate

\[
V(x) = \max_{i \in \mathcal{I}} \frac{1}{p} \|x_i\|_p^p
\]

and note that \( V(x) \leq \frac{1}{p} C^p \) implies that \( x \in S_p(C) \). Since the function \((\cdot)^p \) is convex on \( \mathbb{R}_{\geq 0} \), it can be observed that \( V \) is convex and, hence, regular. In the remainder of the proof, we will show that \( V(x(t)) \) is nonincreasing along all Filippov solutions of \((14)\), implying strong invariance of the set \( S_p(C) \) for any \( C > 0 \).

Let \( a(x) \) denote the set of indices that achieve the maximum in \((45)\) as

\[
a(x) = \{ i \in \mathcal{I} \mid \|x_i\|_p^p = pV(x) \}.
\]

Then, the generalized gradient of \( V \) in \((45)\) is given by

\[
\partial V(x) = \text{co}\{ e_i \otimes \psi(x_i) \mid i \in a(x) \}
\]

where

\[
\psi(x_i) = \begin{bmatrix}
|x_{i,1}|^{p-1} I_{\|x_i\|_p} \text{sgn}(x_{i,1}) \\
\vdots \\
|x_{i,k}|^{p-1} I_{\|x_i\|_p} \text{sgn}(x_{i,k})
\end{bmatrix}
\]

and \( x_i = [x_{i,1} \ldots x_{i,k}]^T \in \mathbb{R}^k \).

Next, let \( \Psi \) be defined as

\[
\Psi = \left\{ t \geq 0 \mid x(t) \text{ and } \frac{d}{dt} V(x(t)) \text{ exist} \right\}.
\]

Since \( x \) is absolutely continuous (by definition of Filippov solutions) and \( V \) is locally Lipschitz, by Lemma 1 in [29] it follows that \( \Psi = \mathbb{R}_{\geq 0} \setminus \Psi \) for a set \( \Psi \) of measure zero and

\[
\frac{d}{dt} V(x(t)) \in \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t))
\]

for all \( t \in \Psi \), such that the set \( \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) \) is nonempty for all \( t \in \Psi \). For \( t \in \Psi \), we have that \( \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) \) is empty, and hence \( \max \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) = -\infty < 0 \) by definition. Therefore, we only consider \( t \in \Psi \) in the rest of the proof.

Next, we will consider the cases \( x \in C \) and \( x \notin C \) separately.

First, for \( x \in C \), it can be observed that \( a(x) = \mathcal{I} \). Then, the following two cases can be distinguished.

i) \( |\mathcal{I}| \geq 2 \) and \( |\mathcal{I}_c| \geq 1 \). As there is at least one agent with continuous vector field, there exists \( i \in \mathcal{I} \) such that \( f_i \) is locally Lipschitz and direction preserving. Then, by definition of the Filippov set-valued map, it follows that \( v_i = 0 \) for all \( v \in \mathcal{F}_h[x] \) (recall \( x \in C \)). As \( \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) \) is nonempty (by considering \( t \in \Psi \)), there exists \( a \in \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) \) such that \( a = \zeta^TV \) for all \( \zeta \in \partial V(x(t)) \), see definition \((6)\). Choosing \( \zeta = e_i \otimes \psi(x_i(t)) \), it follows that \( a = (e_i \otimes \psi(x_i(t)))^T v = 0 \), which implies that \( \max \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) = 0 \leq 0 \), i.e., \( V(x) \) is nonincreasing for any \( x \in C \).

ii) \( |\mathcal{I}| = 2 \) and \( |\mathcal{I}_c| = 0 \). In this case, system \((13)\) can be written as

\[
\begin{cases}
x_1 = \frac{x_2 - x_1}{\|x_2 - x_1\|_p}, \\
x_2 = \frac{x_1 - x_2}{\|x_1 - x_2\|_p}.
\end{cases}
\]

Then, by using definition \((3)\), it can be shown that, for \( x_1 = x_2 \) (i.e., \( x \in C \)), any element \( v \) in the Filippov set-valued map of \((51)\) satisfies \( v_1 = -v_2 \). Stated differently, the following implication holds with \( v = [v_1^T \ v_2^T]^T \):

\[
v \in \mathcal{F}_h[x], \ x \in C \Rightarrow v_1 = -v_2.
\]

Next, by recalling that \( a(x) = \mathcal{I} \) (see \((46)\)), it follows from \((47)\) that

\[
\partial V(x) = \text{co}\{ e_1 \otimes \psi(x_1(t)), e_2 \otimes \psi(x_2(t)) \}
\]

with \( x_1 = x_2 \). Now, following a similar reasoning as in item (i) on the basis of the definition of the set-valued Lie derivative in \((6)\), it can be concluded that \( a = \zeta^TV \) is necessarily 0, such that \( \max \mathcal{L}_{\mathbb{R}_{\geq 0}} V(x(t)) = 0 \leq 0 \) for all \( x \in C \).
Second, the case \( x \not\in C \) is considered. For this case, Theorem 1 in [30] is applied to obtain
\[
\mathcal{F}[h](x) \subseteq \bigcap_{i=1}^N \mathcal{F}[fi](\mathcal{L}_j x) =: \mathcal{F}(x),
\]
(A10)
after which it follows from the definition of the set-valued Lie derivative (6) that
\[
\mathcal{L}_x V(x) \subseteq \mathcal{L}_x V(x).
\]
(A11)
Therefore, in the remainder of the proof for the case \( x \not\in C \), we will show that \( \max \mathcal{L}_j[\mathcal{F}] V(x(t)) \leq 0 \), which implies the desired result by (55). As before, it is sufficient to consider the set \( \Psi \) such that \( \mathcal{L}_j[\mathcal{F}] V(x(t)) \) is nonempty for all \( t \in \Psi \).

Now, take an index \( i \in a(x) \) such that \( \mathcal{L}_j x \neq 0 \). Note that such \( i \) indeed exists. Namely, assume in order to establish a contradiction that \( \mathcal{L}_j x = 0 \) for all \( i \in a(x) \). If \( \mathcal{L}_j x = \mathcal{I} \), then there exists \( \ell \in [1, \ldots, k] \) such that
\[
\beta(\ell) := \arg \max_{j \in I} x_{j, \ell} \subseteq \mathcal{I},
\]
(A12)
i.e., there exists a state component \( \ell \) that does not have the same value for all agents. Otherwise, \( x \in C \), which is a contradiction. Then, for any \( i \in \beta(\ell) \) with \( j \in N_i \setminus \beta(\ell) \), we have \( \mathcal{L}_j x \neq 0 \), where \( N_i \) is the set of neighbors of agent \( i \). If \( a(x) \subseteq \mathcal{I} \) and \( \mathcal{L}_j x = 0 \) for all \( i \in a(x) \), then for any \( i \in a(x) \) with \( j \in N_i \setminus a(x) \), we have
\[
0 = \psi^T(x_i) \mathcal{L}_j x
\]
(A13)
where inequality (58) is based on Hölder’s inequality, and the last inequality is implied by \( \|x_i\| > \|x_j\| \) for any \( j \in N_i \setminus a(x) \). This is a contradiction.

For the index \( i \in a(x) \) satisfying \( \mathcal{L}_j x \neq 0 \), it follows from Assumption 1 that there exists \( \gamma > 0 \) such that
\[
\mathcal{F}[fi](\mathcal{L}_j x) = (-\gamma \mathcal{L}_j x),
\]
(A15)
i.e., for any \( \nu \in \mathcal{F}(x) \) it holds that \( \nu_i = -\gamma \mathcal{L}_j x \). Note that this is a result of the direction-preserving property of either the direction-preserving signum (for a nonzero argument, then \( \gamma = \frac{1}{\|\mathcal{L}_j x\|} \)) or the Lipschitz continuous function (by Assumption 1). Then, choosing \( \zeta \in \partial V(x) \) as \( \zeta = e_i \otimes \psi(x_i) \) (recall that \( i \in a(x) \)), it follows from (6) that
\[
\mathcal{L}_j[\mathcal{F}] V(x) = (-\gamma \psi^T(x_i) \mathcal{L}_j x).
\]
(A16)
Next, by observing (58), we have
\[
\psi^T(x_i) \mathcal{L}_j x \geq 0,
\]
(A17)
which implies \( \mathcal{L}_j[\mathcal{F}] V(x) \subseteq \mathcal{R}_x^{\geq 0} \).

Summarizing the results of the two cases leads to the condition
\[
\max \mathcal{L}_x V(x) \leq 0
\]
(A18)
for all \( x \in \mathcal{R}_x^{\geq 0} \), which proves strong invariance of \( \mathcal{S}_p(C) \) for all \( C > 0 \).

2) Let \( p = \infty \). Consider
\[
V(x) = \max_{i \in I} \|x_i\|_{\infty}
\]
(A19)
as a Lyapunov function candidate. Since the proof shares the same structure and reasoning as the case \( p \in [1, \infty) \), we only provide a sketch of the proof.

In this case, the set \( a(x) \) in (46) is
\[
a(x) = \{ i \in \mathcal{I} \mid \|x_i\|_{\infty} = V(x) \},
\]
(A20)
whereas the generalized gradient of \( V \) reads
\[
\partial V(x) = \text{co}(e_i \otimes (\mathcal{F} \text{ sgn} [x_{i, \ell} e_{\ell}]) \mid e_i \in \mathcal{R}^n)
\]
(A21)
\[
0 = \mathcal{F} \text{ sgn} [x_{i, \ell} e_{\ell}]^T \mathcal{L}_j x
\]
(A22)
where \( e_i \otimes (\mathcal{F} \text{ sgn} [x_{i, \ell} e_{\ell}]) \in \partial V(x) \) and the last inequality is implied by \( \|x_i\| > \|x_j\| \) for any \( j \in N_i \setminus a(x) \). This is a contradiction.

For the index \( i \in a(x) \) satisfying \( \mathcal{L}_j x \neq 0 \), it follows from Assumption 1 that there exists \( \gamma > 0 \) such that
\[
\mathcal{F}[fi](\mathcal{L}_j x) = (-\gamma \mathcal{L}_j x),
\]
(A15)
i.e., for any \( \nu \in \mathcal{F}(x) \) it holds that \( \nu_i = -\gamma \mathcal{L}_j x \). Note that this is a result of the direction-preserving property of either the direction-preserving signum (for a nonzero argument, then \( \gamma = \frac{1}{\|\mathcal{L}_j x\|} \)) or the Lipschitz continuous function (by Assumption 1). Then, choosing \( \zeta \in \partial V(x) \) as \( \zeta = e_i \otimes (\mathcal{F} \text{ sgn} [x_{i, \ell} e_{\ell}]) \) (recall that \( i \in a(x) \)), it follows from (6) that
\[ L_{[\mathcal{F}]n}(x) = \{-\gamma(\mathcal{F}) \text{sgn} (x_{i,r}) e_r^T \dot{L} x\}. \] (A24)

Next, by observing (67), we have \((\mathcal{F}) \text{sgn} (x_{i,r}) e_r^T L x \geq 0\), which implies \(L_{[\mathcal{F}]n}(x) \subset \mathbb{R}^+_0\).

In summary, we have

\[
\max L_{[\mathcal{F}]}(x) \leq 0
\]

for all \(x \in \mathbb{R}^m\), which proves strong invariance of \(\mathcal{S}_p(C)\) for all \(C > 0\).

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