SYMMETRIC SOLUTIONS FOR A PARTIAL DIFFERENTIAL ELLIPTIC EQUATION THAT ARISES IN STOCHASTIC PRODUCTION PLANNING WITH PRODUCTION CONSTRAINTS

DRAGOS-PATRU COVEI
DEPARTMENT OF APPLIED MATHEMATICS, THE BUCHAREST UNIVERSITY OF ECONOMIC STUDIES
PIATA ROMANA, 1ST DISTRICT, POSTAL CODE: 010374, POSTAL OFFICE: 22, ROMANIA

Abstract. In this article we consider the question of the existence of positive symmetric solutions to the problems of the following type $\Delta u = a(|x|) h(u) + b(|x|) g(u)$ for $x \in \mathbb{R}^N$, which are called entire large solutions. Here $N \geq 3$, and we assume that $a$ and $b$ are nonnegative continuous spherically symmetric functions on $\mathbb{R}^N$. We extend results previously obtained for special cases of $h$ and $g$ and we will describe a real-world model in which such problems may arise.

1. Introduction

The problem of establishing the existence of spherically symmetric solutions (i.e. $u(x) = u(r)$ where $r := |x|$ is the Euclidean norm) for the partial differential equation

$$\Delta u = a(|x|) h(u) + b(|x|) g(u) \quad \text{for } x \in \mathbb{R}^N \quad (N \geq 3),$$

is extensively studied both from the theoretical point of view and from modeling of various phenomena in the real world. From a theoretical point of view, using conditions on the potential functions $a, b \in C^{0, \beta}_{\text{loc}}(\mathbb{R}^N, [0, \infty))$ for some $\beta \in (0, 1)$, and on the nonlinearities $h, g \in C((0, \infty), [0, \infty)) \cap C^1((0, \infty), [0, \infty))$, the study of the existence solutions for the problem (1.1) was well argued, for the case where $h$ and $g$ are nondecreasing with $h(0) = g(0) = 0$ and $h(s) g(s) > 0$ for all $s > 0$, in the article of Lair and Mohammed [6]. Inspired by the works of Alvarez [1], Bensoussan, Sethi, Vickson and Derzko [3], Du and Guo [5], Lair and Mohammed [6], Lasry and Lions [7] and Porretta [9] here we assume that the nonlinearities $h$ and $g$ belongs to a new class of functions:

h1) $h : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing with $h(0) = 0$ and $h(s) > 0$ for all $s \in (0, \infty)$;

g1) $g : (0, \infty) \to \mathbb{R}$ is continuous and there exists $s_0 \in (0, \infty)$ such that $g$ is non-decreasing for all $s \in (s_0, \infty)$, $\lim_{s \to 0+} g(s) = g(s_0) = 0$, $g(s) < 0$ for all $s \in (0, s_0)$ and $g(s) > 0$ for all $s \in (s_0, \infty)$;

O) for $s_0 \in (0, \infty)$, which exists from the condition g1), we have

$$H(\infty) = \infty \text{ for all } u_0 > s_0,$$

where

$$H(\infty) := \lim_{t \to \infty} H(t), \quad H(t) := \int_{u_0}^t \frac{1}{h(t) + g(t)} dt,$$

and we are interested in the following goals for the problem (1.1): establishing a result of existence of spherically symmetric solutions for (1.1), determining the asymptotic behavior of solutions for (1.1), and last but not least is to write a description of a model from real-world where such problems (1.1) might arise.

More exactly, that we aim to emphasize on problems (1.1) from the theoretical point of view, is synthesized in:

Key words and phrases. Symmetric solutions; partial differential elliptic equation; stochastic production planning. 2010 AMS Subject Classification: Primary: 35J25, 35J47 Secondary: 35J96.
Theorem 1. We assume $h_1), g_1)$ and $O)$ hold. If $a$ and $b$ are nonnegative continuous spherically symmetric functions on $\mathbb{R}^N$, then

$$
\begin{cases}
  u(0) = u_0 \geq s_0,
  u_n(r) = u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}(a(s)h(u_{n-1}(s)) + b(s)g(u_{n-1}(s)))dsdt, \ r \geq 0
\end{cases}
$$

converges locally uniformly to a positive spherically symmetric function $u \in C^2 [0, \infty)$ and $u$ is a solution of the equation (2.2) such that $u(0) = u_0$.

In addition, for any $c_0 > 0$ we have the estimates:

A1) there exists $C_1 > u_0$ such that $u_0 \leq u(r) \leq C_1$, for all $r \in [0, c_0]$;

A2) there exists $C_2 > 0$ such that $0 \leq u'(r) \leq C_2(r + 1)$, for all $r \in [0, c_0]$.

Moreover, with the following notations

$$
m(s) = \begin{cases}
  a(s)h(u_0) & \text{if } g(u_0) = 0,
  ((a(s) + b(s)) \min \{h(u_0), g(u_0)\}) & \text{if } g(u_0) \neq 0,
\end{cases}
$$

$$
P(r) = \int_0^r t^{1-N} \int_0^t s^{N-1}m(s)dsdt, \ P(\infty) := \lim_{r \to \infty} P(r),
$$

$$
\overline{P}(r) = \int_0^r t^{1-N} \int_0^t s^{N-1}(a(s) + b(s))dsdt, \ \overline{P}(\infty) := \lim_{r \to \infty} \overline{P}(r),
$$

if $P(\infty) = \infty$ then $\lim_{r \to \infty} u(r) = \infty$ (i.e. $u$ is large), and $\lim_{r \to \infty} u(r) < \infty$ provided $\overline{P}(\infty) < \infty$ (i.e. $u$ is bounded).

Remark 2. If $a$ and $b$ are nondecreasing functions then the solution $u$ obtained in the Theorem 1 is convex.

In the following we will structure the paper in two sections. In the Section 2 we will prove the Theorem 1 and Remark 2. For this, we use a different approach to that of Lasry and Lions [7] or Alvarez [11] at least from the following aspects: approximating the solution for the considered problem (1.1), the direct study of the solved problem without call to auxiliary results, imposing another conditions on the boundary and last but not least by establishing the monotony of the solution. The difficulty that arises from these aspects is given by the influence of the sign of the function $g$. In the Section 3 we will describe a real-world model in which such problems may arise. In addition, new directions of study are open to research.

2. The Proof of Theorem

The main objective, is to establish the existence of solutions for the second order differential equation

$$
\begin{cases}
  (r^{N-1}u'(r))' = r^{N-1}(a(r)h(u(r)) + b(r)g(u(r))), \ r > 0,
  u(0) = u_0 \in [s_0, \infty), \ u'(0) = 0.
\end{cases}
$$

A common argument is to rewrite (2.1) in the integral form

$$
u(r) = u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}(a(s)h(u(s)) + b(s)g(u(s)))dsdt.
$$

So, setting a solution for (2.2) we get the solution for the proposed problem (1.1). We define the sequence of functions $\{u_n(r)\}_{n \in \mathbb{N}}$ in $[0, \infty)$ iteratively, as follows:

$$
\begin{cases}
  u_0(r) := u_0 \geq s_0,
  u_n(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1}(a(s)h(u_{n-1}(s)) + b(s)g(u_{n-1}(s)))dsdt, \ r \geq 0.
\end{cases}
$$

We note that for any $r \in [0, \infty)$ the sequence of functions $\{u_n(r)\}_{n \in \mathbb{N}}$ it is monotonically increasing with respect to variable $r$ independent of the value of $n \in \mathbb{N}$, which is a useful information in the following. To prove the existence of the limit

$$
u(r) := \lim_{n \to \infty} u_n(r),$$
we will prove that \( \{u_n(r)\}_{n \in \mathbb{N}} \) is a nondecreasing sequence on \([0, \infty)\). We use the mathematical induction method. We note that the verification step takes place

\[
\begin{align*}
  u_1(r) &= u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} \left( a(s) h(u_0(s)) + b(s) g(u_0(s)) \right) ds dt \\
  &= u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} \left( a(s) h(u_0) + b(s) g(u_0) \right) ds dt \\
  &\leq u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} \left( a(s) h(u_1(s)) + b(s) g(u_1(s)) \right) ds dt \\
  &= u_2(r),
\end{align*}
\]

and so \( u_1(r) \leq u_2(r) \) for any \( r \in [0, \infty) \). We assume \( u_{n-1}(r) \leq u_n(r) \) for any \( r \in [0, \infty) \) and we prove that

\[
u_n(r) \leq u_{n+1}(r) \text{ for any } n \in \mathbb{N} \text{ and } r \in [0, \infty).
\]

Indeed,

\[
u_n(r) = u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} \left( a(s) h(u_{n-1}(s)) + b(s) g(u_{n-1}(s)) \right) ds dt \\
\leq u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} \left( a(s) h(u_n(s)) + b(s) g(u_n(s)) \right) ds dt \\
= u_{n+1}(r),
\]

which ends the proof.

In the following we will show that the sequence of functions \( \{u_n(r)\}_{n \in \mathbb{N}} \) is bounded in any compact interval \([0, c_0]\) with \( c_0 > 0 \) arbitrary and independent of \( n \), property that couples with the monotony of the sequence ensures the uniform convergence

\[
u_n(r) \xrightarrow{\text{uniformly } r \in [0, c_0]} u(r) \text{ as } n \to \infty.
\]

To prove this, we notice that

\[
\left[ r^{N-1} (u_n(r))' \right]' = r^{N-1} \left( a(s) h(u_{n-1}(r)) + b(s) g(u_{n-1}(r)) \right) \\
\leq r^{N-1} \left( a(s) h(u_n(r)) + b(s) g(u_n(r)) \right) \\
\leq r^{N-1} \left( a(r) + b(r) \right) \left( h(u_n(r)) + g(u_n(r)) \right),
\]

where we have used the monotony of the sequence \( \{u_n(r)\}_{n \in \mathbb{N}} \). Integrating between 0 and \( r \) the inequality \((2.4)\), we find

\[
u_n(r)' \leq r^{1-N} \int_0^r t^{N-1} \left( a(t) + b(t) \right) \left( h(u_n(t)) + g(u_n(t)) \right) dt \\
\leq r^{1-N} \left( h(u_n(r)) + g(u_n(r)) \right) \int_0^r t^{N-1} \left( a(t) + b(t) \right) dt,
\]

and after rearrangement

\[
\frac{\left( u_n(r) \right)'}{h(u_n(r)) + g(u_n(r))} \leq r^{1-N} \int_0^r t^{N-1} \left( a(t) + b(t) \right) dt.
\]

A new integration from 0 to \( r \), in the inequality \((2.5)\), leads to

\[
\int_{u_0}^{u_n(r)} \frac{1}{h(t) + g(t)} dt \leq \mathcal{P}(r), \quad \mathcal{P}(r) := r^{1-N} \int_0^r t^{N-1} \left( a(t) + b(t) \right) dt,
\]

expression that can be written such

\[
u_n(r) \leq \mathcal{P}(r), \quad H(s) := \int_{u_0}^s \frac{1}{h(t) + g(t)} dt, s > u_0.
\]
Obviously the function $H : [u_0, \infty) \to [0, \infty)$ is bijective and strictly increasing for any $s \in [u_0, \infty)$, properties that which are transmitted to the inverse function $H^{-1}$. Applying the inverse $H^{-1}$ to inequality (2.6) we obtain

$$u_n (r) \leq H^{-1} (\overline{P} (r)).$$

Recapitulate, it is noticed that

$$u_n (r) \leq u_{n+1} (r) \quad \text{for any } n \in \mathbb{N}, \ r \in [0, \infty),$$

and

$$u_n (r) \leq u_n (c_0) \leq \overline{C}_1 := H^{-1} (\overline{P} (c_0)) < \infty \ \forall n \in \mathbb{N}, \ \forall r \in [0, c_0],$$

properties justifying the applicability of Dini’s theorem and implicitly establishing that

$$u_n (r) \xrightarrow{\text{uniformly}} u (r) \quad \text{for } n \to \infty \text{ and } c_0 > 0 \text{ arbitrary.}$$

So, letting to the limit when $n \to \infty$ in (2.3) we obtain the existence of a $c_0 := c_0 (u_0) > 0$ (maximal extreme to the right) for the existence maximal interval $(0, c_0)$ of solutions for (2.1) and an $u (r) := u_{u_0} (r) \in C^2 (0, c_0) \cap C^1 ([0, c_0])$ solution for the problem (2.1) in $(0, c_0)$. We prove that $u (r)$ exists in $(0, \infty)$ which is reduced to showing that $c_0 = \infty$. Assume the contrary, that $c_0 \in (0, \infty)$. A simple argument proves that $\lim u (r) = \infty$. Appealing to the inequality

$$\int_{u_0}^{u(r)} \frac{1}{h (t) + g (t)} dt \leq \overline{P} (r),$$

by letting to the limit, we get

$$\infty = \lim_{r \to c_0} \int_{u_0}^{u(r)} \frac{1}{h (t) + g (t)} dt = \lim_{r \to c_0} \int_{u_0}^{\infty} \frac{1}{h (t) + g (t)} dt \leq \lim_{r \to c_0} \overline{P} (r) < \infty,$$

and then a contradiction. We have proved that

$$u_n (r) \quad \text{locally uniformly} \quad r \to [0, \infty) u (r) \quad \text{for } n \to \infty,$$

and as a consequence $u (r)$ satisfy

$$\left\{ \begin{array}{l}
  u (0) = u_0 \\
  u (r) = u_0 + \int_{0}^{r} y^{1-N} \int_{0}^{y} t^{N-1} (a (t) h (u (t)) + b (t) g (u (t))) dt dy, \quad r \geq 0.
\end{array} \right.$$  

The regularity of the solution is a classic process that can be consulted in the paper of [4].

We prove the announced estimates. Let $c_0 > 0$ arbitrary. The claim A1) it is evident. Then it remains to test the existence of a parameter $\overline{C}_2 > 0$ such that $u' (r) \leq \overline{C}_2 (r+1)$ for any $r \in [0, c_0]$. Indeed, for any $r \geq 0$,

$$u' (r) = r^{1-N} \int_{0}^{r} t^{N-1} (a (t) h (u (t)) + b (t) g (u (t))) dt \leq \langle h (u (r)) + g (u (r)) \rangle \int_{0}^{r} (a (t) + b (t)) dt \leq \|a + b\|_{\infty} (h (u (c_0)) + g (u (c_0))) \int_{0}^{r} dt \leq \|a + b\|_{\infty} (h (u (c_0)) + g (u (c_0))) r,$$  

(2.10)
the inequalities that take place for any \( r \in [0, c_0] \). As a consequence
\[
C_2 := \| a + b\|_\infty (h(u(c_0)) + g(u(c_0))) ,
\]
check the affirmations.

Next, we check the convexity of the solution \( u \). Indeed, it is clear that
\[
(2.11) \quad \left( r^{N-1}u'_r(r) \right)' = r^{N-1}(a(r)h(u(r)) + b(r)g(u(r))).
\]
Integrating the equation \((2.11)\) from 0 to \( r > 0 \) we obtain
\[
\begin{align*}
r^{N-1}u'_r(r) &= \int_0^r s^{N-1}(a(s)h(u(s)) + b(s)g(u(s)))ds \\
&\leq a(r)h(u(r)) \int_0^r s^{N-1}ds + b(r)g(u(r)) \int_0^r s^{N-1}ds \\
&= a(r)h(u(r)) \frac{r^N}{N} + b(r)g(u(r)) \\
&= \frac{r^N}{N} (a(r)h(u(r)) + b(r)g(u(r))),
\end{align*}
\]
and, as a consequence
\[
(2.12) \quad \frac{u'_r(r)}{r} \leq \frac{1}{N} (a(r)h(u(r)) + b(r)g(u(r))), \forall r > 0.
\]
On the other hand \((2.11)\) can be written in form
\[
(2.13) \quad u''(r) + (N-1)\frac{u'_r(r)}{r} = a(r)h(u(r)) + b(r)g(u(r)).
\]
Substituting \((2.12)\) into \((2.13)\) we obtain
\[
(2.14) \quad a(r)h(u(r)) + b(r)g(u(r)) \leq u''(r) + \frac{N-1}{N} (a(r)h(u(r)) + b(r)g(u(r))).
\]
Rearranging the inequality \((2.13)\), we get
\[
0 < (a(r)h(u(r)) + b(r)g(u(r))) \left(1 - \frac{N-1}{N}\right) \leq u''(r), \forall r > 0,
\]
relation that coupled with
\[
u''(0) = \frac{a(0)h(u(0)) + b(0)g(u(0))}{N} \geq 0,
\]
leads to \( u''(r) \geq 0 \) for any \( r \geq 0 \).

Finally, we prove the limit of the solution on the boundary. In the case \( \overline{\mathcal{P}}(\infty) = \infty \), we observe that
\[
u(r) = u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}(a(s)h(u(s)) + b(s)g(u(s)))dsdt \\
\geq u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}(a(s)h(u_0) + b(s)g(u_0(s)))dsdt \\
\geq u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}m(s)dsdt,
\]
or, considering the inequality in which are interested
\[
(2.15) \quad u(r) \geq u_0 + \mathcal{P}(r).
\]
Consequently, passing to the limit in \((2.15)\) we obtain \( \lim_{r \to \infty} u(r) = \infty \). In the case \( \overline{\mathcal{P}}(\infty) < \infty \), we see that
\[
u(r) \leq H^{-1}(\overline{\mathcal{P}}(r)) \leq H^{-1}(\overline{\mathcal{P}}(\infty)) < \infty.
\]
We also point that
\[
\lim_{r \to \infty} \int_{0}^{r} y^{1-N} \int_{0}^{y} t^{N-1} (a(t) + b(t)) \, dt \, dy = \frac{1}{N-2} \int_{0}^{\infty} r (a(r) + b(r)) \, dr.
\]

3. The model

Consider a factory producing $N$ homogeneous goods and having an inventory warehouse. Let $p(t) = (p_1(t), ..., p_N(t)) \geq 0$ represents the production rate at time $t$ (control variable) and $y(t) = (y_1(t), ..., y_N(t))$ denote the inventory level for production rate at time $t$ (state variable). We point that a negative value of $y_i(t)$ indicates a backlogged demand for part $i$ (for example, due to obsolescence or perishability), while a positive value is the size of the inventory stored in the buffers. We consider $w = (w_1, ..., w_N)$ a $N$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the natural completed filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$, where $T$ is the length of planning period, generated by the standard Wiener process $w$. We note that the process $dw_i(t), i = 1, 2, ..., N$, can be formally expressed as $z_i(t)dt$, where $z_i(t)$ is considered to be the white noise process [2]. For a fall in the Theorem hypotheses, we assume that all the constant demand rate at time $t$ are equal with 0 and we consider the inventory production control problem

\[(3.1)\]
\[J(p_1, ..., p_N) := E \int_{0}^{\infty} (f_1(p(t)) + f_2(y(t))) e^{-\alpha t} \, dt,\]

where $f_1(x) = f_2(x) = |x|^2$ is the quadratic loss function. The stochastic differential equations governing $y_i$ are

\[(3.2)\]
\[dy_i(t) = p_i dt + \sigma_i dw_i, \quad y_i(0) = y^0_i, \quad i = 1, ..., N,\]

where $\sigma = (\sigma_1, ..., \sigma_N)$ is the non-zero constant diffusion coefficient, $\alpha > 0$ is the constant discount rate and $y^0_i$ is the initial inventory level. The aim is to minimize the stochastic production planning problem

\[(3.3)\]
\[
\inf \{ J(p_1, ..., p_N) \mid p_i \geq 0 \; \forall i = 1, 2, ..., N \},
\]

subject to the Itô equation (3.2). Let $z(x) = z(x_1, ..., x_N)$ denote the expected current-valued value of the control problem (3.2)-(3.3) with initial value $x$ in (3.2) so that $z(y^0)$ represents to the state-equation (3.2). In order to achieve this we apply the martingale principle: we search for a function $U(x)$ such that the stochastic process $M^c(t)$ defined below

\[M^c(t) = e^{-\alpha t} U(y(t)) - E \int_{0}^{\infty} (f_1(p(t)) + f_2(y(t))) e^{-\alpha t} \, dt\]

is supermartingale for all $p_1(t) \geq 0$, ..., $p_N(t) \geq 0$ and martingale for the optimal control $p^*(t) = (p^*_1(t), ..., p^*_N(t))$. If, this is achieved and the following transversality condition holds true

\[(3.4)\]
\[
\lim_{t \to \infty} E[e^{-\alpha t} U(y(t))] = 0,
\]

then, it can be shown that $-U(x) = z(x)$ is $C^2[0, \infty)$ and satisfies the associated dynamic programming partial differential equation or Hamilton-Jacobi-Bellman equation formally associated to the problem (3.2)-(3.3)

\[(3.5)\]
\[\alpha z - \frac{\sigma^2}{2} \Delta z - |x|^2 = \inf \{ p \nabla z + |p|^2 \mid p_i \geq 0 \; \forall i = 1, 2, ..., N \},\]

where $z := z(x_1, ..., x_N)$ is the corresponding value function. We point that the solution of this HJB equation is used to test controller for optimality or perhaps to construct a feedback controller. In the next, we give some ideas to solve the problem (3.5). Firstly, if $\frac{\partial z}{\partial x_i}(x_1, ..., x_N) \leq 0$ for all $i = 1, ..., N$, since this is the case where we are interested, we observe that

\[p \nabla z + |p|^2 = \frac{1}{4} |\nabla z|^2.\]
Indeed, setting
\[ F(p_1, \ldots, p_N) = p \nabla z + |p|^2 \]
\[ = p_1 \frac{\partial z}{\partial x_1}(x_1, \ldots, x_N) + \ldots + p_N \frac{\partial z}{\partial x_N}(x_1, \ldots, x_N) + \sum_{i=1}^{N} p_i^2 \]
we have
\[ F_{p_i}(p_1, \ldots, p_N) = 0 \iff \frac{\partial z}{\partial x_i}(x_1, \ldots, x_N) + 2p_i = 0 \]
for all \( i = 1, 2, \ldots, N \). Then, the critical point of the function \( F \) is
\[ p_i^* = -\frac{1}{2} \frac{\partial z}{\partial x_i}(x_1, \ldots, x_N) \text{ for } i = 1, \ldots, n. \]

On the other hand the Hessian matrix is
\[ H_F(p_1, \ldots, p_N) = \begin{pmatrix} 2 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 2 \end{pmatrix} \]
which is positive definite and so
\[ (p_1^*, \ldots, p_N^*) = \left( -\frac{1}{2} \frac{\partial z}{\partial x_1}(x_1, \ldots, x_N), \ldots, -\frac{1}{2} \frac{\partial z}{\partial x_N}(x_1, \ldots, x_N) \right) \]
is a global minimum point for (3.6). Then, we have
\[
F(p_1^*, \ldots, p_N^*) = (p_1^*, \ldots, p_N^*) \nabla z + \sum_{i=1}^{N} (p_i^*)^2
\]
\[ = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial z}{\partial x_i}(x_1, \ldots, x_N) \right)^2 + \frac{1}{4} |\nabla z|^2 \]
\[ = -\frac{1}{2} |\nabla z|^2 + \frac{1}{4} |\nabla z|^2 = -\frac{1}{4} |\nabla z|^2, \]
so that equation (3.5) can be written as
\[ \alpha z - \frac{|\sigma|^2}{2} \Delta z - |x|^2 = -\frac{1}{4} |\nabla z|^2 \text{ for } x \in \mathbb{R}^N, \]
or, equivalently
\[ -2 |\sigma|^2 \Delta z + |\nabla z|^2 + 4\alpha z = 4 |x|^2 \text{ for } x \in \mathbb{R}^N. \]
and, finally with the change of variable
\[ z = -v \]
we obtain
\[ 2 |\sigma|^2 \Delta v + |\nabla v|^2 = 4 |x|^2 + 4\alpha v \text{ for } x \in \mathbb{R}^N, \]
or equivalently
\[ \Delta v = \frac{4 |x|^2 + 4\alpha v - |\nabla v|^2}{2 |\sigma|^2} \text{ for } x \in \mathbb{R}^N, \]
Then, after changing the variable \( u(x) = e^{\frac{\sigma}{2} |x|^2} \), the equation becomes
\[ \left\{ \begin{array}{ll}
\Delta u(x) = \frac{1}{|\sigma|^2} |x|^2 u(x) + \frac{2\alpha}{|\sigma|^2} u(x) \ln u(x) & \text{for } x \in \mathbb{R}^N, \\
u(x) > 0 & \text{for } x \in \mathbb{R}^N, 
\end{array} \right. \]
which is exactly the same with the equation (1.1), for
\[ a(x) = \frac{1}{|\sigma|^2} |x|^2, b(x) = \frac{2\alpha}{|\sigma|^2}, h(u) = u, g(u) = u \ln u. \]
and $s_0 = 1$. A recapitulation of the changes of variables and notations are

$$2|\sigma|^2 \ln u(x) = v(x) = -z(x),$$

$$H(s) = \ln (\ln s + 1) - \ln (\ln u_0 + 1) \implies H^{-1}(s) = e^{e^{\ln(\ln u_0 + 1) - 1}},$$

$$P(r) = \frac{r^4}{4(N+2)|\sigma|^4 + \frac{\alpha r^2}{N |\sigma|^2}};$$

$$P(r) = \begin{cases} 
\frac{4|\sigma|(N+2)}{r^4} & \text{if } u_0 = 1, \\
\min \{ 1, u_0 \ln u_0 \} \left( \frac{r^4}{4(N+2)|\sigma|^4} + \frac{2\alpha}{|\sigma|^2} \right) & \text{if } u_0 \neq 1.
\end{cases}$$

Then, from Theorem 1 we can see that

$$z(x) < 0 \text{ for } x \in \mathbb{R}^N \text{ and } z(x) \to -\infty \text{ as } |x| \to \infty.$$ 

Let us point that, since $u(x_1, \ldots, x_N) = u(|x|)$ is defined by

$$u(x) = u_0 + \int_0^{|x|} y^{1-N} \int_0^y t^{N-1} \left( \frac{1}{|\sigma|^4} t^2 u(t) + \frac{2\alpha}{|\sigma|^2} u(t) \ln u(t) \right) dt dy,$$

we have that

$$z_{x_i} \leq 0 \text{ for all } i = 1, \ldots, N.$$

for any $x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$. Since the production rate were restricted to be nonnegative, in this case

$$p_i^* = \max \left\{ 0, -\frac{z_{x_i}}{2} \right\} = -\frac{z_{x_i}}{2} \text{ for } i = 1, \ldots, N.$$

**Remark 3.** Using the change of variable $u(r) = e^{-\frac{z(r)}{2|\sigma|^2}}$ we can rewrite the affirmations A1)-A2) such:

A1)' there exists $\overline{C}_1 > u_0$ such that

$$2|\sigma|^2 \ln u_0 \leq U(r) = -z(r) \leq 2|\sigma|^2 \ln \overline{C}_1 \text{ for any } r \in [0, c_0],$$

A2)' there exists $\overline{C}_2 > 0$ such that

$$U'(r) = -z'(r) \leq \frac{2|\sigma|^2}{u_0 \overline{C}_2} (r+1) \text{ for any } r \in [0, c_0].$$

**Remark 4.** In the model problem the result of existence of solution, obtained in Theorem 1, holds and for the case $N \in \{1, 2\}$.

**Remark 5.** In the optimization criterion (3.11) the quadratic loss function, $f_1(x) = f_2(x) = |x|^2$, was considered. In a future work we plan to explore optimization criterions involving other loss functions as well.

**Open problem.** Under hypotheses of the form h1) and g1) and under some suitable conditions on $a$ and $b$ we think that there exists $l \in [0, s_0]$ such that the problem

$$\Delta u = a(x) h(u) + b(x) g(u) \text{ for } x \in \mathbb{R}^N \ (N \geq 3),$$

has a unique positive solution $u \in C^2([0, \infty))$ with $u(x) \to l$. Moreover, such a solution guarantees the existence of a unique strong solution for (3.2) which makes the candidate optimal control admissible. Moreover, we can prove some growth estimates as in (3.10), (3.11) and then it can be proved the transversality condition (3.3).

**Acknowledgement.** The author would like to thank Professor T. A. P. for valuable comments and suggestions which further improved this article.
REFERENCES

[1] O. Alvarez, A quasilinear elliptic equation in $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A, 126 (1996), 911-921.
[2] L. Arnold, Stochastic Differential Equations, Wiley, New York, 1974.
[3] A. Bensoussan, S.P. Sethi, R. Vickson and N. Derzko, Stochastic production planning with production constraints, Siam J. Control and optimization, 22 (1984) 920-935.
[4] D.-P. Covei, On the radial solutions of a system with weights under the Keller–Osserman condition, J. Math. Anal. Appl., 447 (2017) 167-180.
[5] Y. Du, Z. Guo, Symmetry for elliptic equations in a half-space without strong maximum principle, Proc. Roy. Soc. Edinburgh Sect. A, 134A (2004) 259-269.
[6] A.V. Lair, A. Mohammed, Entire large solutions of semilinear elliptic equations of mixed type, Commun. Pure Appl. Anal., 8 (2009) 1607-1618.
[7] J.M. Lasry, P. L. Lions, Nonlinear Elliptic Equations with Singular Boundary conditions and Stochastic Control with State Constraints, Math. Ann., 283 (1989) 583-630.
[8] T. Leonori, Large solutions for a class of nonlinear elliptic equations with gradient terms, Adv. Nonlinear Stud., 7 (2007) 237-269.
[9] A. Porretta, Some uniqueness results for elliptic equations without condition at infinity, Commun. Contemp. Math., 5 (2003) 705-717.

E-mail address: coveidragos@yahoo.com