Direct limits in the heart of a t-structure: the case of a torsion pair

Carlos E. Parra *
Departamento de Matemáticas
Universidad de los Andes
(5101) Mérida
VENEZUELA
carlosparra@ula.ve

Manuel Saorín *
Departamento de Matemáticas
Universidad de Murcia, Aptdo. 4021
30100 Espinardo, Murcia
SPAIN
msaorinc@um.es

Abstract

We study the behavior of direct limits in the heart of a t-structure. We prove that, for any compactly generated t-structure in a triangulated category with coproducts, countable direct limits are exact in its heart. Then, for a given Grothendieck category $G$ and a torsion pair $t = (T, F)$ in $G$, we show that the heart $H_t$ of the associated t-structure in the derived category $D(G)$ is AB5 if, and only if, it is a Grothendieck category. If this is the case, then $F$ is closed under taking direct limits. The reverse implication is true for a wide class of torsion pairs which include the hereditary ones, those for which $T$ is a cogenerating class and those for which $F$ is a generating class. The results allow to extend results by Buan-Krause and Colpi-Gregorio to the general context of Grothendieck categories and to improve some results of (co)tilting theory of modules.

Mathematics Subjects Classification: 18E30, 18E15, 18E40, 16E05, 16E30.

1 Introduction

Beilinson, Bernstein and Deligne [BBD] introduced the notion of a t-structure in a triangulated category in their study of perverse sheaves on an algebraic or analytic variety. If $D$ is such a triangulated category, a t-structure is a pair of full subcategories satisfying suitable axioms (see the precise definition in next section) which guarantee that their intersection is an abelian category $H$, called the heart of the t-structure. This category comes with a cohomological functor $D \to H$. Roughly speaking, a t-structure allows to develop an intrinsic (co)homology theory, where the homology 'spaces' are again objects of $D$ itself.

Since their introduction t-structures have been used in many branches of Mathematics, with special impact in Algebraic Geometry and Representation Theory of Algebras. One line of research in the topic has been the explicit construction, for concrete triangulated categories, of wide classes of t-structures. This

*The authors thank Silvana Bazzoni and Jan Trlifaj for their quick answer to our questions, and Riccardo Colpi for sending us the manuscript [CG].

1Parra is supported by a grant from the Universidad de los Andes (Venezuela) and Saorín is supported by research projects from the Spanish Ministry of Education (MTM2010-20940-C02-02) and from the Fundación ‘Séneca’ of Murcia (04555/GERM/06), with a part of FEDER funds. The authors thank these institutions for their help.
approach has led to classification results in many cases (see, e.g., [HRS], [CG], [AJS], [SS], [Kn]...). A second line of research consists in starting with a well-behaved class of t-structures and trying to find necessary and sufficient conditions on a t-structure in the class so that the heart is a ‘nice’ abelian category. For instance, that the heart is a Grothendieck or even a module category. All the work in this direction which we know of has concentrated on a particular class of t-structures. Namely, in the context of bounded derived categories, Happel, Reiten and Smalø [HRS] associated to each torsion pair in an abelian category A, a t-structure in the bounded derived category $D(A)$. This t-structure is actually the restriction of a t-structure in $D(A)$ and several authors (see [CGM], [CMT], [MT], [CG]) have dealt with the problem of deciding when its heart $H_t$ is a Grothendieck or module category, in case $A$ is the module category $R$–Mod, for some (always associative unital) ring $R$. Concretely, in [CGM] the authors proved that if $t$ is faithful in $R$–Mod, with $F$ closed under taking direct limits, and $H_t$ is a Grothendieck category, then $t$ is a cotilting torsion pair. Later, in [CG] (see also [M]), it was proved that the converse is also true. The study of the case when $H_t$ is a module category was also initiated in [CGM] and continued in [CMT], where the authors gave necessary and sufficient conditions, when $t$ is faithful, for $H_t$ to be a module category.

It is clear from the work on the second line of research mentioned above that the main difficulty in understanding when $H_t$ is a Grothendieck category comes from the AB5 condition. In fact, the understanding of direct limits in $H_t$ or, more generally, in the heart of any t-structure is far from complete. The present paper is our first attempt to understand the AB5 condition on the heart of commonly used t-structures. We first give some general results for any t-structure in an arbitrary triangulated category and, later, we concentrate on the case of a torsion pair $t$ in any Grothendieck (not necessarily a module) category. In a forthcoming paper [PS], we will study the problem in the case of a compactly generated t-structure in the derived category $D(R)$ of a commutative Noetherian ring, using the classification results of [AJS].

We summarize in the following list the main results of the paper, all of which, except the first, are given for a torsion pair $t = (T, F)$ in a Grothendieck category $G$ and the heart $H_t$ of the associated t-structure in $D(G)$. The reader is referred to next section for the pertinent definitions.

1. (Theorem 3.7) If $D$ is a triangulated category with coproducts, then, for any compactly generated t-structure in $D$, countable direct limits are exact in the heart.

2. (Part of theorem 4.8) The heart $H_t$ is a Grothendieck category if, and only if, it is AB5 if, and only if, the canonical morphism $\lim_{\to} H^{-1}(M_i) \to H^{-1}(\lim_{\to} M_i)$ is a monomorphism, for each direct system $(M_i)_{i \in I}$ in $H_t$. In this case $F$ is closed under taking direct limits in $G$.

3. (Part of theorem 4.9) Suppose that $t$ satisfies one of the following conditions: i) $t$ is hereditary; ii) $F$ is a generating class; or iii) $T$ is a cogenerated class. The heart $H_t$ is a Grothendieck category if, and only if, $F$ is closed under taking direct limits in $G$.

4. (Part of proposition 5.3) The torsion class $T$ is cogenerated and $H_t$ is a Grothendieck category with a projective generator if, and only if, $t$ is a tilting torsion pair such that $F$ is closed under taking direct limits in $G$.

5. (Part of proposition 5.7) When $G$ is locally finitely presented and $F$ is a generating class, the class $F$ is closed under taking direct limits in $G$ if, and only if, $t$ is a cotilting torsion pair.

Let us point out that, as a consequence of our findings, some results in the second line of research mentioned above, as well as classical results on tilting and cotilting theory of module categories are extended or improved to more general Grothendieck categories. For example, Buan-Krause classification of torsion pairs in the category of finitely generated modules over a Noetherian ring ([BK]) is extended to any locally noetherian Grothendieck category (corollary 5.13). Similarly, result 5 of the list above is an extension of the result, essentially due to Bazzoni, that a faithful torsion pair in $R$–Mod is cotilting if, and only if, its torsion-free class is closed under direct limits.

The organization of the paper goes as follows. In section 2 we give all the preliminaries and terminology needed in the rest of the paper. In section 3 we study Grothendieck properties AB3, AB4, AB5 and their duals, for the heart of a t-structure in any ambient triangulated category. In particular, we prove result 1 in the list above. In Section 4 we give results 2 and 3 in the list above and their proofs. In the final section 5, we see that the results of the previous section naturally lead to tilting and cotilting theory in a general...
Grothendieck category. Results 4 and 5 in the list above are proved in this final section, as well as their already mentioned consequences.

2 Preliminaries and terminology

Recall that a category $I$ is (skeletally) small when its objects form a set (up to isomorphism). If $C$ and $I$ are an arbitrary and a small category, respectively, then a functor $I \to C$ will be called an $I$-diagram on $C$, or simply a diagram on $C$ when $I$ is understood. The category of $I$-diagrams on $C$ will be denoted by $[I, C]$. We will frequently write an $I$-diagram $X = (X_i)_{i \in I}$, where $X_i := X(i)$ for each $i \in \text{Ob}(I)$, whenever the images by $X$ of the arrows in $I$ are understood. When each $I$-diagram has a limit (resp. colimit), we say that $C$ has $I$-limits (resp. $I$-colimits). In such case, $\lim : [I, C] \to C$ (resp. $\text{colim} : [I, C] \to C$) will denote the ($I$-)limit (resp. ($I$-)colimit) functor and it is right (resp. left) adjoin to the constant diagram functor $\kappa : C \to [I, C]$. If $C$ and $D$ are categories which have $I$-limits (resp. $I$-colimits), we will say that a functor $F : C \to D$ preserves $I$-limits (resp. $I$-colimits) when the induced morphism $F(\lim X_i) \to \lim F(X_i)$ (resp. $\text{colim}F(X_i) \to F(\text{colim}X_i)$) is an isomorphism, for each $I$-diagram $(X_i)_{i \in I}$. The category $C$ is called complete (resp. cocomplete) when $I$-limits (resp. $I$-colimits) exist in $C$, for any small category $I$.

Recall that a particular case of limit functor (resp. colimit functor) is the ($I$-)product functor $\prod : [I, C] \to C$ (resp. ($I$-)coproduct functor $\coprod : [I, C] \to C$), when $I$ is just a set, viewed as a small category on which the identities are its only morphisms. Another particular case comes when $I$ is a directed set, viewed as a small category on which there is a unique morphism $i \to j$ exactly when $i \leq j$. The corresponding colimit functor is the $I$-direct limit functor $\text{lim} : [I, C] \to C$. The $I$-diagrams on $C$ are usually called $I$-directed systems in $C$. Dually, one has $I$-inverse systems and the ($I$-)inverse limit functor $\text{lim} : [I^{\text{op}}, C] \to C$.

When $A$ is an additive category and $S \subset \text{Ob}(A)$ is any class of objects, we shall denote by $\text{add}_{A}(S)$ (resp. $\text{Add}_{A}(S)$), or simply $\text{add}(S)$ (resp. $\text{Add}(S)$) if no confusion appears, the class of objects which are direct summands of finite (resp. arbitrary) coproducts of objects in $S$. Also, we will denote by $\text{Prod}_{A}(S)$ or $\text{Prod}(S)$ the class of objects which are direct summands of arbitrary products of objects in $S$. When $S = \{V\}$, for some object $V$, we will simply write $\text{add}_{A}(V)$ (resp. $\text{Add}_{A}(V)$) or $\text{add}(V)$ (resp. $\text{Add}(V)$) and $\text{Prod}_{A}(V)$ or $\text{Prod}(V)$. If $S$ is any set of objects, we will say that it is a set of generators when the functor $\prod_{s \in S} \text{Hom}_{A}(s, ?) : A \to \text{Ab}$ is a faithful functor. An object $G$ is a generator of $A$, when $\{G\}$ is a set of generators. We will employ a stronger version of these concepts, for a class $R \subset \text{Ob}(A)$. The class $R$ will be called a generating (resp. cogenerated) class of $A$ when, for each object $X$ of $R$, there is an epimorphism $R \to X$ (resp. monomorphism $X \to R$), for some $R \in R$. When, in addition, $A$ has coproducts, we shall say that an object $X$ is a compact object when the functor $\text{Hom}_{A}(X, ?) : A \to \text{Ab}$ preserves coproducts.

Recall the following hierarchy among abelian categories introduced by Grothendieck ([Gr]). Let $A$ be an abelian category.

- $A$ is $AB3$ (resp. $AB3^*$) when it has coproducts (resp. products);
- $A$ is $AB4$ (resp. $AB4^*$) when it is $AB3$ (resp. $AB3^*$) and the coproduct functor $\coprod : [I, A] \to A$ (resp. product functor $\prod : [I, A] \to A$) is exact, for each set $I$;
- $A$ is $AB5$ (resp. $AB5^*$) when it is $AB3$ (resp. $AB3^*$) and the direct limit functor $\text{lim} : [I^{\text{op}}, A] \to A$ (resp. inverse limit functor $\text{lim} : [I^{\text{op}}, A] \to A$) is exact, for each directed set $I$.

Note that the $AB3$ (resp. $AB3^*$) condition is equivalent to the fact that $A$ be cocomplete (resp. complete). Note that if $A$ is $AB3$ (resp. $AB3^*$) then, for each small category $I$, the corresponding limit (resp. colimit) functor is left (resp. right) exact, because it is a right (resp. left) adjoint functor.

An $AB5$ abelian category $G$ having a set of generators (equivalently, a generator), is called a Grothendieck category. Such a category always has enough injectives, and every object in it has an injective envelope (see [Gr]). Moreover, it is always a complete (and cocomplete) category (see [S] Corollary X.4.4). Given an object $V$ of $G$, another object $X$ is called $V$-generated (resp. $V$-presented) when there is an epimorphism $V(?) \to X$ (resp. an exact sequence $V(?) \to V(?) \to X \to 0$), for some $I$ and $J$. We will say that $X$ is $V$-cogenerated (resp. $V$-copresented) when there is a monomorphism $X \to V(?)$ (resp. an exact sequence $0 \to X \to V(?) \to V(?)$), for some sets $I$ and $J$. As it is customary, we will denote by $\text{Gen}(V)$, $\text{Pres}(V)$,
Cogen(V) and Copres(V) the classes of V-generated, V-presented, V-cogenerated and V-copresented objects, respectively. An object \( X \) of \( \mathcal{G} \) is called \emph{finitely presented} when \( \text{Hom}_\mathcal{G}(X, ?) : \mathcal{G} \rightarrow \text{Ab} \) preserves direct limits. When \( \mathcal{G} \) has a set of finitely presented generators and each object of \( \mathcal{G} \) is a direct limit of finitely presented objects, we say that \( \mathcal{G} \) is \emph{locally finitely presented}.

A \emph{torsion pair} in an abelian category \( \mathcal{A} \) is a pair \( t = (T, F) \) of full subcategories satisfying the following two conditions:

1. \( \text{Hom}_\mathcal{A}(T, F) = 0 \), for all \( T \in T \) and \( F \in F \);
2. For each object \( X \) of \( \mathcal{A} \) there is an exact sequence \( 0 \to T_X \to X \to F_X \to 0 \), where \( T_X \in T \) and \( F_X \in F \).

In such case the objects \( T_X \) and \( F_X \) are uniquely determined, up to isomorphism, and the assignment \( X \mapsto T_X \) (resp. \( X \mapsto F_X \)) underlies a functor \( t : \mathcal{A} \to T \) (resp. \( (1 : t) : \mathcal{A} \to F \)) which is right (resp. left) adjoint to the inclusion functor \( T \hookrightarrow \mathcal{A} \) (resp. \( F \hookrightarrow \mathcal{A} \)). We will frequently write \( X/t(X) \) to denote \((1 : t)(X)\). The torsion pair \( t \) is called \emph{hereditary} when \( T \) is closed under taking subobjects in \( \mathcal{A} \).

Slightly modifying [C] Definitions 2.3 and 2.6, when \( \mathcal{A} \) is AB3 (resp. AB3*), an object \( V \) (resp. \( Q \)) of \( \mathcal{A} \) is called \( 1\)-\emph{tilting} (resp. \( 1\)-\emph{cotilting}) when \( \text{Gen}(V) = \text{Ker} (\text{Ext}_\mathcal{A}^1(V, ?)) \) (resp. \( \text{Cogen}(Q) = \text{Ker} (\text{Ext}_\mathcal{A}^1(? , Q)) \)). In that case, one has \( \text{Gen}(V) = \text{Pres}(V) \) (resp. \( \text{Cogen}(Q) = \text{Copres}(Q) \)) and \((\text{Gen}(V), \text{Ker}(\text{Hom}_\mathcal{A}(?, Q), \text{Cogen}(Q)))\) is a torsion pair in \( \mathcal{A} \) called the \emph{tilting} (resp. \emph{cotilting}) \emph{torsion pair} associated to \( V \) (resp. \( Q \)).

We refer the reader to [N] for the precise definition of \emph{triangulated category}, but, diverting from the terminology in that book, for a given triangulated category \( \mathcal{D} \), we will denote by \(?[1] : \mathcal{D} \to \mathcal{D} \) its suspension functor. We will then put \(?[0] = 1_{\mathcal{D}} \) and \(?[k] \) will denote the \( k \)-th power of \(?[1] \), for each integer \( k \). (Distinguished) triangles in \( \mathcal{D} \) will be denoted \( X \to Y \to Z \xrightarrow{+} \), or also \( X \to Y \to Z \xrightarrow{w} X[1] \) when the connected morphism \( w \) need be emphasized. A \emph{triangulated functor} between triangulated categories is one which preserves triangles. Unlike the terminology used in the abstract setting of additive categories, in the context of triangulated categories a weaker version of the term ‘set of generators’ is commonly used. Namely, a set \( S \subset \text{Ob}(\mathcal{D}) \) is called a \emph{set of generators of} \( \mathcal{D} \) if an object \( X \) of \( \mathcal{D} \) is zero whenever \( \text{Hom}_\mathcal{D}(S[k], X) = 0 \), for all \( S \in S \) and \( k \in \mathbb{Z} \). In case \( \mathcal{D} \) has coproducts, we will say that \( \mathcal{D} \) is \emph{compactly generated} when it has a set of compact generators.

Recall that if \( \mathcal{D} \) and \( \mathcal{A} \) are a triangulated and an abelian category, respectively, then an additive functor \( H : \mathcal{D} \to \mathcal{A} \) is a cohomological functor when, given any triangle \( X \to Y \to Z \xrightarrow{+} \), one gets an induced long exact sequence in \( \mathcal{A} \):

\[
\cdots \to H^{n-1}(Z) \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots,
\]

where \( H^n := H \circ (?[n]) \), for each \( n \in \mathbb{Z} \).

Given a Grothendieck category \( \mathcal{G} \), we will denote by \( \mathcal{C}(\mathcal{G}), \mathcal{K}(\mathcal{G}) \) and \( \mathcal{D}(\mathcal{G}) \) the category of chain complexes of objects of \( \mathcal{G} \), the homotopy category of \( \mathcal{G} \) and the derived category of \( \mathcal{G} \), respectively (see [V], [Ke2]).

Let \((\mathcal{D}, ?[1])\) be a triangulated category. A \emph{t-structure} in \( \mathcal{D} \) is a pair \((\mathcal{U}, \mathcal{W})\) of full subcategories, closed under taking direct summands in \( \mathcal{D} \), which satisfy the following properties:

i) \( \text{Hom}_\mathcal{D}(U, W[-1]) = 0 \), for all \( U \in \mathcal{U} \) and \( W \in \mathcal{W} \);

ii) \( \mathcal{U}[1] \subseteq \mathcal{U} \);

iii) For each \( X \in \text{Ob}(\mathcal{D}) \), there is a triangle \( U \to X \to V \xrightarrow{+} \) in \( \mathcal{D} \), where \( U \in \mathcal{U} \) and \( V \in \mathcal{W}[-1] \).

It is easy to see that in such case \( \mathcal{W} = \mathcal{U}^\perp[-1] \) and \( \mathcal{U} = \perp(\mathcal{W}[-1]) = \perp(\mathcal{U}^\perp) \). For this reason, we will write a t-structure as \((\mathcal{U}, \mathcal{U}^\perp[1])\). We will call \( \mathcal{U} \) and \( \mathcal{U}^\perp \) the \emph{aisle} and the \emph{co-aisle} of the t-structure, respectively. The objects \( U \) and \( V \) in the above triangle are uniquely determined by \( X \), up to isomorphism, and define functors \( \tau_U : \mathcal{D} \to \mathcal{U} \) and \( \tau_U^\perp : \mathcal{D} \to \mathcal{U}^\perp \) which are right and left adjoints to the respective inclusion functors. We call them the \emph{left and right truncation functors} with respect to the given t-structure. Note that \((\mathcal{U}[k], \mathcal{U}^\perp[-k])\) is also a t-structure in \( \mathcal{D} \), for each \( k \in \mathbb{Z} \). The full subcategory \( \mathcal{H} = \mathcal{U} \cap \mathcal{W} = \mathcal{U} \cap \mathcal{U}^\perp[1] \) is called the \emph{heart} of the t-structure and it is an abelian category, where the short exact sequences ‘are’ the
triangles in \( \mathcal{D} \) with its three terms in \( \mathcal{H} \). Moreover, with the obvious abuse of notation, the assignments \( X \mapsto (\tau_U \circ \tau^{[1]}_{\mathcal{H}})(X) \) and \( X \mapsto (\tau^{[1]}_{\mathcal{H}} \circ \tau_U)(X) \) define naturally isomorphic functors \( \mathcal{D} \to \mathcal{H} \) which are cohomological (see [BD]). When \( \mathcal{D} \) has coproducts, the t-structure \( (\mathcal{U}, \mathcal{U}^{\perp}[1]) \) is called compactly generated when there is a set \( S \subseteq \mathcal{U} \), consisting of compact objects, such that \( \mathcal{U}^{\perp} \) consists of the \( Y \in \mathcal{D} \) such that \( \text{Hom}_{\mathcal{D}}(S[n], Y) = 0 \), for all \( S \in S \) and integers \( n \geq 0 \). In such case, we say that \( S \) is a set of compact generators of the aisle \( \mathcal{U} \).

**Examples 2.1.** The following are typical examples t-structures:

1. Let \( \mathcal{G} \) be a Grothendieck category and, for each \( k \in \mathbb{Z} \), denote by \( \mathcal{D}^{\leq k}(\mathcal{G}) \) (resp. \( \mathcal{D}^{\geq k}(\mathcal{G}) \)) the full subcategory of \( \mathcal{D}(\mathcal{G}) \) consisting of the complexes \( X \) such that \( H^j(X) = 0 \), for all \( j > k \) (resp. \( j < k \)). The pair \( (\mathcal{D}^{\leq k}(\mathcal{G}), \mathcal{D}^{\geq k}(\mathcal{G})) \) is a t-structure in \( \mathcal{D}(\mathcal{G}) \) whose heart is equivalent to \( \mathcal{G} \). Its left and right truncation functors will be denoted by \( \tau^{\leq k} : \mathcal{D}(\mathcal{G}) \to \mathcal{D}^{\leq k}(\mathcal{G}) \) and \( \tau^{> k} : \mathcal{D}(\mathcal{G}) \to \mathcal{D}^{> k}(\mathcal{G}) := \mathcal{D}^{\geq k}(\mathcal{G})[-1] \). For \( k = 0 \), the given t-structure is known as the canonical t-structure in \( \mathcal{D}(\mathcal{G}) \).

2. (Happel-Reiten-Smalø) Let \( \mathcal{G} \) be any Grothendieck category and \( t = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{G} \). One gets a t-structure \( (\mathcal{U}_t, \mathcal{U}_t^{\perp}[1]) = (\mathcal{U}_t, \mathcal{W}_t) \) in \( \mathcal{D}(\mathcal{G}) \), where:

\[
\mathcal{U}_t = \{ X \in \mathcal{D}^{\leq 0}(\mathcal{G}) : H^0(X) \in \mathcal{T} \}
\]

\[
\mathcal{W}_t = \{ Y \in \mathcal{D}^{\geq -1}(\mathcal{G}) : H^{-1}(Y) \in \mathcal{F} \}.
\]

In this case, the heart \( \mathcal{H}_t \) consists of the complexes \( M \) such that \( H^{-1}(M) \in \mathcal{F} \), \( H^0(M) \in \mathcal{T} \) and \( H^k(M) \neq 0 \), for all \( k \neq -1, 0 \). Each such complex is isomorphic in \( \mathcal{D}(\mathcal{G}) \) to a complex \( \cdots \to 0 \to X \xrightarrow{d} Y \to 0 \to \cdots \), concentrated in degrees \(-1\) and \( 0 \), such that \( \text{Ker}(d) \in \mathcal{F} \) and \( \text{Coker}(d) \in \mathcal{T} \).

3. Let \( \mathcal{D} \) be a triangulated category which has coproducts. An object \( X \) of \( \mathcal{D} \) will be called a tilting complex when \( \{ X \} \) is a set of compact generators of \( \mathcal{D} \) such that \( \text{Hom}_{\mathcal{D}}(X, X[n]) = 0 \), for all \( n \in \mathbb{Z} \setminus \{ 0 \} \). When \( \mathcal{D} \) is the base of a derivator (see [KN, Appendix 1]), in particular when \( \mathcal{D} = \mathcal{D}(\mathcal{G}) \) for a Grothendieck category \( \mathcal{G} \), the smallest full subcategory \( \mathcal{U}_X \) of \( \mathcal{D} \) which contains \( X \) and is closed under coproducts, extensions and application of the functor \( ?[1] \) is the aisle of a t-structure whose co-aisle is \( \mathcal{U}_X^C = \{ Y \in \text{Ob}(\mathcal{D}) : \text{Hom}_{\mathcal{D}}(X[n], Y) = 0 \}, \text{for all } n \geq 0 \} \) (see also [AJSE]). The corresponding heart \( \mathcal{H}_X \) is equivalent to the module category over the ring \( R := \text{End}_{\mathcal{D}}(X) \) and the equivalence of categories \( \text{Hom}_{\mathcal{D}}(X, ?) : \mathcal{H}_X \xrightarrow{\cong} R - \text{Mod} \) extends to a triangulated equivalence \( \mathcal{D} \xrightarrow{\cong} \mathcal{D}(R) \).

3 Colimits in the heart of a t-structure

In the sequel \( (\mathcal{D}, ?[1]) \) is a triangulated category. All throughout this section, we will fix a t-structure \( (\mathcal{U}, \mathcal{U}^{\perp}[1]) \) in \( \mathcal{D} \) and \( \mathcal{H} = \mathcal{U} \cap \mathcal{U}^{\perp}[1] \) will be its heart. We will denote by \( \tilde{H} : \mathcal{D} \to \mathcal{H} \) either of the naturally isomorphic functors \( \tau_U \circ \tau^{[1]}_{\mathcal{H}} \) or \( \tau^{[1]}_{\mathcal{H}} \circ \tau_U \), in order to avoid confusion, in case \( \mathcal{D} = \mathcal{D}(\mathcal{G}) \), with the classical cohomological functor \( H = H^0 : \mathcal{D}(\mathcal{G}) \to \mathcal{G} \).

**Lemma 3.1.** The following assertions hold:

1. If \( X \) is an object of \( \mathcal{U} \), then \( \tilde{H}(X) \cong \tau^{[1]}_{\mathcal{H}}(X[-1])[1] \) and the assignment \( X \mapsto \tilde{H}(X) \) defines an additive functor \( L : \mathcal{U} \to \mathcal{H} \) which is left adjoint to the inclusion \( j : \mathcal{H} \to \mathcal{U} \).

2. If \( Y \) is an object of \( \mathcal{U}^{\perp}[1] \), then \( \tilde{H}(Y) \cong \tau_U(Y) \) and the assignment \( Y \mapsto \tilde{H}(Y) \) defines an additive functor \( R : \mathcal{U}^{\perp}[1] \to \mathcal{H} \) which is right adjoint to the inclusion \( j : \mathcal{H} \to \mathcal{U}^{\perp}[1] \).

**Proof.** Suppose that \( X, U \in \mathcal{U} \) and that \( V \in \mathcal{U}^{\perp} \). A sequence of morphisms

\[
U \to X[-1] \xrightarrow{g} V \xrightarrow{h} U[1]
\]

is a distinguished triangle if, and only if, the sequence
\[ U[1] \to X \xrightarrow{d[1]} V[1] \xrightarrow{h[1]} U[2] \]

is so. It follows from this that \( \tau^{H[1]}(X) = \tau^{H[1]}(X[-1])[1] \), and then the isomorphism \( \tilde{H}(X) \cong \tau^{H[1]}(X[-1])[1] \) follows from the definition of \( \tilde{H} \). On the other hand, if \( Y \in \mathcal{U}^+[1] \), then, by the definition of \( \tilde{H} \), we get \( \tilde{H}(Y) \cong \tau_U(Y) \).

The part of assertion 1 relative to the adjunction is dual to that of assertion 2 since \((\mathcal{U}, \mathcal{U}^+[1])\) is a t-structure in \( D \) exactly when \((\mathcal{U}^+[1], \mathcal{U})\) is a t-structure in \( D^{op} \). We then prove the adjunction of assertion 2, which follows directly from the following chain of isomorphisms, using the fact that \( \tau_U : D \to \mathcal{U} \) is right adjoint to the inclusion \( j_U : \mathcal{U} \to D \):

\[
\text{Hom}_{\mathcal{U}^+[1]}(j(Z), Y) = \text{Hom}_{D}(j(Z), Y) \cong \text{Hom}_{\mathcal{U}}(j(Z), \tau_U(Y)) \cong \text{Hom}_{\mathcal{H}}(Z, \tilde{H}(Y)) = \text{Hom}_{\mathcal{H}}(Z, R(Y)).
\]

\[ \square \]

**Proposition 3.2.** Let \( D \) be a triangulated category which has coproducts (resp. products) and let \((\mathcal{U}, \mathcal{U}^+[1])\) be a t-structure in \( D \). The heart \( \mathcal{H} \) is an \( AB3 \) (resp. \( AB3^* \)) abelian category.

**Proof.** It is a known fact and very easy to prove that if \( L : C \to C' \) is a left adjoint functor and \( C \) has coproducts, then, for each family \((C_i)_{i \in I}\) of objects of \( C \), the family \((L(C_i))_{i \in I}\) has a coproduct in \( C' \) and one has an isomorphism \( \coprod_{i \in I} L(C_i) \cong L(\coprod_{i \in I} C_i) \).

Let now \((Z_i)_{i \in I}\) be a family of objects of \( \mathcal{H} \). Since the counit \( L \circ j \to 1_\mathcal{H} \) is an isomorphism, we have that \( (L \circ j)(Z_i) \cong Z_i \) and the fact that the family has a coproduct in \( \mathcal{H} \) follows directly from the previous paragraph.

The statement about products is dual to the one for coproducts. \[ \square \]

The following is an interesting type of t-structures.

**Definition 1.** Let us assume that \( D \) has coproducts (resp. products). The t-structure \((\mathcal{U}, \mathcal{U}^+[1])\) is called smashing (resp. co-smashing) when \( \mathcal{U}^+ \) (resp. \( \mathcal{U} \)) is closed under taking coproducts (resp. products) in \( D \). Bearing in mind that coproducts (resp. products) of triangles are triangles (see [N, Proposition 1.2.1]), this is equivalent to saying that the left (resp. right) truncation functor \( \tau_U : D \to \mathcal{U} \) (resp. \( \tau^{H[1]} : D \to \mathcal{U}^+ \)) preserves coproducts (resp. products).

**Proposition 3.3.** Let \( D \) be a triangulated category that has coproducts (resp. products). If \((\mathcal{U}, \mathcal{U}^+[1])\) is a t-structure whose heart \( \mathcal{H} \) is closed under taking coproducts (resp. products) in \( D \), then \( \mathcal{H} \) is an \( AB4 \) (resp. \( AB4^* \)) abelian category. In particular, that happens when \((\mathcal{U}, \mathcal{U}^+[1])\) is a smashing (resp. co-smashing) t-structure.

**Proof.** Note that \( \mathcal{U} \) (resp. \( \mathcal{U}^+ \)) is closed under taking coproducts (resp. products) in \( D \). Then the final assertion follows automatically from the first part of the proposition and from the definition of smashing (resp. co-smashing) t-structure.

We just do the \( AB4^* \) case since the \( AB4^* \) one is dual. Let \((0 \to X_i \to Y_i \to Z_i \to 0)_{i \in I}\) be a family of short exact sequences in \( \mathcal{H} \). According to [BBD], they come from triangles in \( D \). By [N, Proposition 1.2.2], we get a triangle in \( D \)

\[
\coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i \to \coprod_{i \in I} Z_i^{+},
\]

where the three terms are in \( \mathcal{H} \) since \( \mathcal{H} \) is closed under taking coproducts in \( D \). We then get the desired short exact sequence \( 0 \to \coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i \to \coprod_{i \in I} Z_i \to 0 \) in \( \mathcal{H} \). \[ \square \]

**Definition 2.** Let \( \mathcal{X} \) be any full subcategory of \( D \). A cohomological datum in \( D \) with respect to \( \mathcal{X} \) is a pair \((H, r)\) consisting of a cohomological functor \( H : D \to \mathcal{A} \), where \( \mathcal{A} \) is an abelian category, and \( r \) is an element of \( \mathbb{Z} \cup \{+\infty\} \) such that the family of functors \( (H^k)_{X_\mathcal{X}} : \mathcal{X} \to \mathcal{A} \xrightarrow{H^k} \mathcal{A}^{k<r} \) is conservative. That is, if \( X \in \mathcal{X} \) and \( H^k(X) = 0 \), for all \( k < r \), then \( X = 0 \).

The following is a useful result inspired by [CGM, Theorem 3.7]:
Proposition 3.4. Suppose that $D$ has coproducts and let $(H : D \to A, r)$ be a cohomological datum in $D$ with respect to the heart $H = U \cap U^+[1]$. Suppose that $I$ is a small category such that $I$-colimits exist and are exact in $A$. If, for each diagram $X : I \to H$ and each integer $k < r$, the canonical morphism $\text{colim}H^k(X_i) \to H^k(\text{colim}_H(X_i))$ is an isomorphism, then $I$-colimits are exact in $H$.

Proof. By Proposition 3.2, we know that $H$ is AB3 or, equivalently, cocomplete. Let us consider an $I$-diagram $0 \to X_i \overset{f_i}{\to} Y_i \overset{g_i}{\to} Z_i \to 0$ of short exact sequences in $H$. Formally speaking, this is just a functor $I \to H_{\text{sec}}$, where $H_{\text{sec}}$ denotes the category of short exact sequences in $H$. By right exactness of colimits, we then get an exact sequence in $H$: 

$$\text{colim}_H(X_i) \overset{f}{\to} \text{colim}_H(Y_i) \overset{g}{\to} \text{colim}_H(Z_i) \to 0.$$

We put $L := \text{Im}(f)$ and then consider the two induced short exact sequences in $H$:

$$0 \to L \to \text{colim}_H(Y_i) \overset{g}{\to} \text{colim}_H(Z_i) \to 0$$

$$0 \to W \to \text{colim}_H(X_i) \overset{p}{\to} L \to 0.$$

We view all given short exact sequences in $H$ as triangles in $D$ and use the cohomological condition of $H$ and the fact that $I$-colimits are exact in $A$, and get the following commutative diagram in $A$ with exact rows:

$$\begin{array}{cccccc}
\text{colim}^{k-1}(Y_i) & \text{colim}^{k-1}(Z_i) & \text{colim}H^k(X_i) & \text{colim}H^k(Y_i) & \text{colim}H^k(Z_i) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
H^{k-1}(\text{colim}_H(Y_i)) & H^{k-1}(\text{colim}_H(Z_i)) & H^k(L) & H^k(\text{colim}_H(Y_i)) & H^k(\text{colim}_H(Z_i))
\end{array}$$

where the vertical arrow $\text{colim}H^k(X_i) \to H^k(L)$ is the composition $\text{colim}H^k(X_i) \to H^k(\text{colim}_H(X_i)) \overset{H^k(p)}{\to} H^k(L)$, for each $k \in \mathbb{Z}$. For $k < r$, in principle, all the vertical arrows except the central one are isomorphisms. Then also the central one is an isomorphism, which implies that $H^k(p)$ is an isomorphism since, by hypothesis, the canonical morphism $H^k(X_i) \to H^k(\text{colim}_H(X_i))$ is an isomorphism. We then get that $H^k(W) = 0$, for all $k < r$, which implies that $W = 0$ due to definition 2. Therefore $p$ is an isomorphism.

Example 3.5. If $(U, U^+[1])$ be a compactly generated t-structure and let $S$ be a set of compact generators of its aisle. Then $H := \coprod_{S \in S} \text{Hom}_D(S, ?) : D \to A$ is a cohomological functor. Moreover, the pair $(H, 1)$ is a cohomological datum with respect to the heart $H = U \cap U^+[1]$.

Given a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} \cdots$$

of morphisms in $D$, we will call Milnor colimit of the sequence, denoted $\text{Mcolim} X_n$, what is called homotopy colimit in [N].

Lemma 3.6. Suppose that $D$ has coproducts and that $(U, U^+[1])$ is a compactly generated t-structure in $D$. Then $U^+$ is closed under taking Milnor colimits.

Proof. Let $D^c$ be the full subcategory of compact objects and take $C \in D^c$ arbitrary. We claim that, for any diagram in $D$ of the form:

$$(*) \quad X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots,$$

we have an isomorphism $\text{Hom}_D(C, \text{Mcolim}(X_n)) \cong \lim \text{Hom}_D(C, X_n)$. To see that, let us consider the canonical triangle

$$\coprod_{n \geq 0} X_n \overset{1-f}{\to} \coprod_{n \geq 0} X_n \to \text{Mcolim}(X_n) \to$$
If $C \in \mathcal{D}^c$ then we have that the following diagram is commutative:

$$
\begin{array}{c}
\prod_{n \geq 0} \text{Hom}_\mathcal{D}(C, X_n) \\
\downarrow f \\
\text{Hom}_\mathcal{D}(C, \prod_{n \geq 0} X_n)
\end{array}
\begin{array}{c}
\prod_{n \geq 0} \text{Hom}_\mathcal{D}(C, X_n) \\
\downarrow f' \\
\text{Hom}_\mathcal{D}(C, \prod_{n \geq 0} X_n)
\end{array}

Note that $1 - f_*$ is a monomorphism in $\text{Ab}$. Then we get an exact sequence in $\text{Ab}$ of the form:

$$
\cdots \rightarrow 0 \rightarrow \text{Hom}_\mathcal{D}(C, \prod_{n \geq 0} X_n) \rightarrow \text{Hom}_\mathcal{D}(C, \prod_{n \geq 0} X_n) \rightarrow \text{Hom}_\mathcal{D}(C, \text{Mcolim}(X_n)) \rightarrow 0
$$

This proves the claim since $\text{Coker}(1 - f_*) = \lim \text{Hom}_\mathcal{D}(C, X_n)$. Now if all the $X_n$ are in $\mathcal{U}^\perp$, then for each $C \in \mathcal{U} \cap \mathcal{D}^c$ and each $k \geq 0$, we have $\text{Hom}_\mathcal{D}(\mathcal{C}[k], \text{Mcolim}(X_n)) \cong \lim \text{Hom}_\mathcal{D}(C[k], X_n) = 0$. This shows that $\text{Mcolim}(X_n) \in \mathcal{U}^\perp$ since the t-structure is compactly generated.

**Theorem 3.7.** Suppose that $\mathcal{D}$ has coproducts and let $(\mathcal{U}, \mathcal{U}^\perp[1])$ be a compactly generated t-structure in $\mathcal{D}$. Countable direct limits are exact in $\mathcal{H} = \mathcal{U} \cup \mathcal{U}^\perp[1]$.

**Proof.** Let $I$ be a countable directed set. Then there is an ascending chain of finite directed subposets $I_0 \subset I_1 \subset I_2 \cdots$ such that $I = \cup_{n \in \mathbb{N}} I_n$ (see [AR Lemma 1.6]). If we put $i_n = \text{max}\{I_n\}$, for all $n \in \mathbb{N}$, we clearly have $i_n \leq i_{n+1}$, for all $n$ and $J = \{i_0, i_1, \ldots, i_n, \ldots\}$ is a cofinal subset of $I$ which isomorphic to $\mathbb{N}$ as an ordered set. Then we know that, for any category $\mathcal{C}$ with direct limits, the diagram

$$
\begin{array}{ccc}
[I, \mathcal{C}] & \xrightarrow{\text{colim}_j} & \text{colim}_j \mathcal{C} \\
\text{restriction} \downarrow & & \downarrow \text{colim}_j \\
[I, \mathcal{C}] & \rightarrow & \text{colim}_j \mathcal{C}
\end{array}
$$

is commutative.

We can then assume that $I = \mathbb{N}$. But the previous lemma tells us that if $X : \mathbb{N} \rightarrow \mathcal{H}$ ($n \mapsto X_n$) is any diagram, then the triangle

$$
\begin{array}{c}
\prod_{n \geq 0} X_n \\
\downarrow 1-f \\
\prod_{n \geq 0} X_n \\
\downarrow 1-f' \\
\text{Mcolim}(X_n)
\end{array}
\begin{array}{c}
\text{Mcolim}(X_n) \\
\downarrow 1-f \\
\text{Mcolim}(X_n) \\
\downarrow 1-f' \\
\text{Mcolim}(X_n)
\end{array}
$$

lives in $\mathcal{H}$ and, hence, it is an exact sequence in this abelian category. Therefore $\text{Mcolim}(X_n) \cong \lim_{\mathcal{H}} X_n$.

If now $\mathcal{S}$ is any set of compact generators of the aisle $\mathcal{U}$ and we take the cohomological functor $\hat{H} := \prod_{S \in \mathcal{S}} \text{Hom}_\mathcal{D}(S, ?) : \mathcal{D} \rightarrow \text{Ab}$, then, by the proof of the previous lemma, the induced map $\lim \hat{H}^k(X_n) \rightarrow \hat{H}^k(\lim \mathcal{H} X_n)$ is an isomorphism, for each $k \in \mathbb{Z}$. The result now follows from proposition 3.4 and example 3.5.

In view of last result, the following is a natural question:

**Question 3.8.** Let $\mathcal{D}$ be a triangulated category and $(\mathcal{U}, \mathcal{U}^\perp[1])$ be a smashing t-structure in $\mathcal{D}$. Is its heart $\mathcal{H}$ an AB5 abelian category? Is it so when the t-structure is compactly generated?

In next section we tackle this question for the (smashing) t-structure in $\mathcal{D}(\mathcal{G})$ defined by a torsion pair in the Grothendieck category $\mathcal{G}$. In a forthcoming paper [PS], we will settle it for essentially all the compactly generated t-structures in $\mathcal{D}(R)$, when $R$ is a commutative Noetherian ring.
4 When is the heart of a torsion pair a Grothendieck category?

All throughout this section $\mathcal{G}$ is a Grothendieck category and $t = (\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{G}$. We will follow the terminology and notation introduced in example 2.12. Note that $\mathcal{T}$ is closed under taking direct limits in $\mathcal{G}$, while $\mathcal{F}$ need not be so. Note also that $(U_k, W_k) = (U_k, U_k^[-1][1])$ is a smashing $t$-structure in $\mathcal{D}(\mathcal{G})$, so that, by proposition 3.3, the heart $\mathcal{H}_t$ is always an AB4 abelian category.

Lemma 4.1. Let $t = (\mathcal{T}, \mathcal{F})$ be a torsion pair in the Grothendieck category $\mathcal{G}$, let $(U_k, U_k^[-1][1])$ be its associated $t$-structure (see example 2.12) and let $\mathcal{H}_t$ be its heart. The functor $H^0 : \mathcal{H}_t \to \mathcal{G}$ is right exact while the functor $H^{-1} : \mathcal{H}_t \to \mathcal{G}$ is left exact. Both of them preserves coproducts.

Proof. The functor $H^k$ vanishes on $\mathcal{H}_t$, for each $k \neq -1, 0$. By applying now the long exact sequence of homologies to any short exact sequence in $\mathcal{H}_t$ the right (resp. left) exactness of $H^0$ (resp. $H^{-1}$) follows immediately. Since coproducts in $\mathcal{H}_t$ are calculated as in $\mathcal{D}(\mathcal{G})$, the fact that $H^0$ and $H^{-1}$ preserve coproducts is clear.

Definition 3. Let $I$ be a directed set and $\mathcal{C}$ be any cocomplete category. Given an $I$-directed system $\{(X_i)_{i \in I}, (u_{ij})_{i \leq j}\}$, we put $X_{ij} = X_i$, for all $i \leq j$. The colimit-defining morphism associated to the direct system is the unique morphism $f : \bigsqcup_{i \leq j} X_{ij} \longrightarrow \bigsqcup_{i \in I} X_i$ such that if $\lambda_i : X_i \longrightarrow \bigsqcup_{i \leq j} X_{ij}$ and $\lambda_j : X_j \longrightarrow \bigsqcup_{i \in I} X_i$ are the canonical morphisms into the coproducts, then $f \circ \lambda_{ij} = \lambda_i - \lambda_j \circ u_{ij}$ for all $i \leq j$.

By classical category theory (see, e.g. [P, Proposition II.6.2]), in the situation of last definition, we have that $L := \lim_{\to \mathcal{H}_t} X_i \cong \text{Coker}(f)$.

The following is the crucial result for our purposes. In its statements and all throughout the rest of the paper, unadorned direct limits are considered in $\mathcal{G}$, while we will denote by $\lim_{\to \mathcal{H}_t}$ the direct limit in $\mathcal{H}_t$.

Proposition 4.2. Let $t = (\mathcal{T}, \mathcal{F})$ be a torsion pair in the Grothendieck category $\mathcal{G}$ and let $\mathcal{H}_t$ be the heart of the associated $t$-structure in $\mathcal{D}(\mathcal{G})$. The following assertions hold:

1. If $(M_i)_{i \in I}$ is a direct system in $\mathcal{H}_t$, then the induced morphism $\lim_{\to \mathcal{H}_t} H^k(M_i) \longrightarrow H^k(\lim_{\to \mathcal{H}_t} M_i)$ is an epimorphism, for $k = -1$, and an isomorphism, for $k \neq -1$.

2. If $(F_i)_{i \in I}$ is a direct system in $\mathcal{F}$, there is an isomorphism $(\lim_{\to \mathcal{H}_t} F_i) [1] \cong \lim_{\to \mathcal{H}_t} (F_i [1])$ in $\mathcal{H}_t$.

3. If $(T_i)_{i \in I}$ is a direct system in $\mathcal{T}$, then the following conditions hold true:

(a) The induced morphism $\lim_{\to \mathcal{H}_t} (T_i[0]) \longrightarrow (\lim_{\to \mathcal{H}_t} T_i)[0]$ is an isomorphism in $\mathcal{H}_t$;

(b) The kernel of the colimit-defining morphism $f : \bigsqcup_{i \leq j} T_{ij} \longrightarrow \bigsqcup_{i \in I} T_i$ is in $\mathcal{T}$.

Proof. 1) It essentially follows from [CGM] Corollary 3.6], but, for the sake of completeness, we give a short proof. By lemma 4.1 we know that $H^0 : \mathcal{H}_t \to \mathcal{G}$ preserves colimits, in particular direct limits. Let $f : \bigsqcup_{i \leq j} M_{ij} \longrightarrow \bigsqcup_{i \in I} M_i$ be the associated colimit-defining morphism and denote by $W$ the image of $f$ in $\mathcal{H}_t$. Applying the exact sequence of homology to the exact sequence $0 \to W \longrightarrow \bigsqcup_{i \in I} M_i \longrightarrow L \to 0$, where $L = \lim_{\to \mathcal{H}_t} M_i$, and using [CGM] Lemma 3.5], we readily see that we have a short exact sequence $0 \to H^{-1}(W) \longrightarrow \bigsqcup_{i \in I} H^{-1}(M_i) \longrightarrow H^{-1}(L) \to 0$. But the last arrow in this sequence is the composition $\bigsqcup_{i \in I} H^{-1}(M_i) \longrightarrow \lim_{\to \mathcal{H}_t} H^{-1}(M_i) \overset{\text{can}}{\longrightarrow} H^{-1}(\lim_{\to \mathcal{H}_t} M_i)$, whose second arrow is then an epimorphism.

In order to prove the remaining assertions, we first consider any direct system $(M_i)_{i \in I}$ in $\mathcal{H}_t$ and the associated triangle given by the colimit-defining morphism:

$$\bigsqcup_{i \leq j} M_{ij} \overset{f}{\longrightarrow} \bigsqcup_{i \in I} M_i \overset{q}{\longrightarrow} \mathbb{Z} \overset{+}{\longrightarrow}$$
in $\mathcal{D}(\mathcal{G})$. We claim that if $v : Z \rightarrow L := \lim_{\rightarrow H_4}(M_i)$ is a morphism fitting in a triangle $N[1] \rightarrow Z \rightarrow L \rightarrow$, with $N \in H_4$, then $H^{-1}(v)$ induces an isomorphism $H^{-1}(Z)/t(H^{-1}(Z)) \cong H^{-1}(L)$.

By [BBD], we have a diagram:

\[
\begin{array}{ccc}
N[1] & \xrightarrow{f} & \prod_{i \leq j} M_{ij} \\
\downarrow & & \downarrow \\
Z & \xrightarrow{+} & L \\
\downarrow & & \\
\prod_{i \in I} M_i & \xrightarrow{=} & \lim_{\rightarrow H_4} M_i
\end{array}
\]

where the row and the column are triangles in $\mathcal{D}(\mathcal{G})$. Then we have that $N \cong \operatorname{Ker}_{H_4}(f)$ and $L \cong \operatorname{Coker}_{H_4}(f) = \lim_{\rightarrow H_4} M_i$. From the sequence of homologies applied to the column, we get an exact sequence

\[0 \rightarrow H^0(N) \rightarrow H^{-1}(Z) \rightarrow H^{-1}(L) \rightarrow 0.
\]

It then follows that $H^0(N) \cong t(H^{-1}(Z))$ and $H^{-1}(Z)/t(H^{-1}(Z)) \cong H^{-1}(L)$.

We pass to prove the remaining assertions.

2) Note that, by assertion 1, we have $H^0(\lim_{\rightarrow H_4}(F_i[1])) = 0$. On the other hand, when taking $M_i = F_i[1]$ in the last paragraph, the complex $Z$ can be identified with $\operatorname{cone}(f)[1]$, where $f : \prod_{i \leq j} F_{ij} \rightarrow \prod_{i \in I} F_i$ is the colimit-defining morphism. Then we have $H^{-1}(Z) \cong H^0(\operatorname{cone}(f)) = \operatorname{Coker}(f) \cong \lim_{\rightarrow H_4} F_i$, which, by the last paragraph, implies that $\lim_{\rightarrow H_4} (F_i[1]) \cong (\lim_{\rightarrow H_4} F_i)[1]$.

3) a) From assertion 1) we get that $0 = \lim_{\rightarrow H_4} H^{-1}(T_i[0]) \rightarrow H^{-1}(\lim_{\rightarrow H_4} (T_i[0]))$ is an epimorphism. In particular we have an isomorphism $\lim_{\rightarrow H_4} (T_i[0]) \cong H^0(\lim_{\rightarrow H_4}(T_i[0]))[0]$. But, by lemma 4.1 $H^0$ preserves direct limits and then the right term of this isomorphism is $\lim_{\rightarrow H_4}(T_i)[0]$.

b) Let us consider the induced triangle $\prod_{i \leq j} T_{ij} \rightarrow \prod_{i \in I} T_i \xrightarrow{q} Z \rightarrow$ in $\mathcal{D}(\mathcal{G})$. Without loss of generality, we identify $Z$ with the cone of $f$ in $\mathcal{C}(\mathcal{G})$, so that $H^{-1}(Z) = \operatorname{Ker}(f)$. But then, by the proved claim that we made after proving assertion 1 and by the previous paragraph, we get an isomorphism $\frac{\operatorname{Ker}(f)}{t(\operatorname{Ker}(f))} \cong H^{-1}(\lim_{\rightarrow H_4} T_i[0]) = 0$. \qed

**Lemma 4.3.** Let us assume that $H_4$ is an AB5 category and let

\[0 \rightarrow F_i \rightarrow F'_i \xrightarrow{w_i} T_i \rightarrow T'_i \rightarrow 0,
\]

be a direct system of exact sequences in $\mathcal{G}$, with $F_i, F'_i \in \mathcal{F}$ and $T_i, T'_i \in \mathcal{T}$ for all $i \in I$. Then the induced morphism $w := \lim_{\rightarrow H_4} w_i : \lim_{\rightarrow H_4} F'_i \rightarrow \lim_{\rightarrow H_4} T_i$ vanishes on $t(\lim_{\rightarrow H_4} F'_i)$.

**Proof.** Put $K_i := \operatorname{Im}(w_i)$ and consider the following pullback diagram and pushout diagram, respectively:

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & F_i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & F'_i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{K_i} & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & \bar{F}_i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{t(K_i)} & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & F_i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{K_i} & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & \bar{F}_i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{t(K_i)} & 0
\end{array}
\]
of viewing monomorphisms as inclusions, we get that the top row and the bottom row of the second diagram give exact sequences in \( \mathcal{H}_t \):

\[
\begin{array}{ccc}
0 & \to & t(K_i)[0] \\
& & \to F_i[1] \\
& & \to \tilde{F}_i[1] \\
0 & \to & \tilde{T}_i[0] \\
& & \to T_i'[0] \\
& & \to K_i/t(K_i)[1] \to 0
\end{array}
\]

which give rise to direct systems of short exact sequences in \( \mathcal{H}_t \). Using now the AB5 condition of \( \mathcal{H}_t \) and proposition \( \ref{prop:AB5} \), we get exact sequences in \( \mathcal{H}_t \):

\[
\begin{array}{ccc}
0 & \to & \lim_{\mathcal{T}} t(K_i)[0] \\
& & \to \lim_{\mathcal{T}} F_i[1] \\
& & \to \lim_{\mathcal{T}} \tilde{F}_i[1] \to 0 \\
0 & \to & \lim_{\mathcal{T}} \tilde{T}_i[0] \\
& & \to \lim_{\mathcal{T}} T_i'[0] \\
& & \to \lim_{\mathcal{T}} K_i/t(K_i)[1] \to 0
\end{array}
\]

which necessarily come from exact sequences in \( \mathcal{G} \):

\[
\begin{array}{ccc}
0 & \to & \lim_{\mathcal{F}} F_i[0] \\
& & \to \lim_{\mathcal{F}} \tilde{F}_i[1] \\
& & \to \lim_{\mathcal{F}} t(K_i) \to 0 \\
0 & \to & \lim_{\mathcal{F}} (K_i/t(K_i)) \\
& & \to \lim_{\mathcal{F}} \tilde{T}_i \\
& & \to \lim_{\mathcal{F}} T_i' \to 0.
\end{array}
\]

These sequences are obviously induced from applying the direct limit functor to the top row of the first diagram and the bottom row of the second diagram above, respectively. With the obvious abuse of notation of viewing monomorphisms as inclusions, we get that \( w := \lim_{\mathcal{T}} w_i \) vanishes on \( \lim_{\mathcal{T}} \tilde{F}_i \), that \( \lim_{\mathcal{F}} F_i = (\lim_{\mathcal{F}} F_i) \cap t(\lim_{\mathcal{T}} \tilde{F}_i) \) and that \( t(\lim_{\mathcal{F}} (K_i/t(K_i))) = 0 \). This last condition in turn implies that \( \lim_{\mathcal{T}} t(K_i) = t(\lim_{\mathcal{F}} K_i) \) since we have an exact sequence \( 0 \to \lim_{\mathcal{T}} t(K_i) \to \lim_{\mathcal{T}} K_i \to \lim_{\mathcal{T}} (K_i/t(K_i)) \to 0 \).

We finally prove that \( w(\lim_{\mathcal{F}} F_i') = 0 \). Note that \( \ker(w) = \ker(p) \), where \( p : \lim_{\mathcal{F}} F_i' \to \lim_{\mathcal{T}} K_i \) is the induced map. Consider now the following bicartesian square:

\[
\begin{array}{ccc}
\lim_{\mathcal{F}} \tilde{F}_i & \to & \lim_{\mathcal{T}} t(K_i) \\
\downarrow & & \downarrow \lambda \\
\lim_{\mathcal{F}} F_i' & \to & \lim_{\mathcal{T}} K_i
\end{array}
\]

Due to the fact that \( t(\lim_{\mathcal{T}} K_i) = \lim_{\mathcal{T}} t(K_i) \), we have a unique morphism \( \alpha : t(\lim_{\mathcal{F}} F_i') \to t(\lim_{\mathcal{T}} K_i) \) such that \( \lambda \circ \alpha = p(\lim_{\mathcal{F}} F_i') \). By the universal property of cartesian squares, we get a morphism \( u : t(\lim_{\mathcal{F}} F_i') \to \lim_{\mathcal{T}} \tilde{F}_i \) such that the composition \( t(\lim_{\mathcal{F}} F_i') \to \lim_{\mathcal{F}} \tilde{F}_i \to \lim_{\mathcal{F}} F_i' \) is the canonical inclusion. It follows that \( u \) is a monomorphism. Viewing \( u \) as in inclusion, we then have that \( t(\lim_{\mathcal{T}} F_i') \subseteq t(\lim_{\mathcal{T}} \tilde{F}_i) \), and we have already seen that \( w \) (and hence \( p \)) vanishes on \( t(\lim_{\mathcal{T}} \tilde{F}_i) \).

Last lemma will be fundamental to prove that the closure of \( \mathcal{F} \) under taking direct limits is a necessary condition for \( \mathcal{H}_t \) to be AB5. We now want to know the information that that closure property gives about the existence of a generator in \( \mathcal{H}_t \). This requires a few preliminary lemmas.

**Lemma 4.4.** Suppose that \( \mathcal{F} \) is closed under taking direct limits in \( \mathcal{G} \). Let \( (M_i)_{i \in I} \) be a direct system in \( \mathcal{C}(\mathcal{G}) \), where \( M_i \) is a complex concentrated in degrees \(-1, 0\), for all \( i \in I \). If \( M_i \in \mathcal{H}_t \), for all \( i \in I \), then the canonical morphism

\[
\lim_{\mathcal{H}_t} M_i \to \lim_{\mathcal{C}(\mathcal{G})} M_i
\]

is an isomorphism in \( \mathcal{D}(\mathcal{G}) \).
Proof. Note that $\lim_{\longleftarrow \mathcal{C}(\mathcal{G})} M_i$ is a complex of $\mathcal{H}_t$ and, hence, there is a canonical morphism $g : \lim_{\longleftarrow \mathcal{C}(\mathcal{G})} M_i \to \lim_{\longleftarrow \mathcal{C}(\mathcal{G})} M_i$. We then get a composition of morphisms in $\mathcal{G}$:

$$\lim_{\longleftarrow \mathcal{H}_t} H^k(M_i) \to H^k(\lim_{\longleftarrow \mathcal{H}_t} M_i) \xrightarrow{H^k(g)} H^k(\lim_{\longleftarrow \mathcal{C}(\mathcal{G})} M_i),$$

for each $k \in \mathbb{Z}$. But all complexes involved have homology concentrated in degrees $-1, 0$ and, by exactness of $\lim$ in $\mathcal{G}$, we know that the last composition of morphisms is an isomorphism. By proposition 4.2, we know that the first arrow in the composition is an isomorphism, for $k = 0$, and an epimorphism, for $k = 1$. Then both arrows in the composition are isomorphisms, for all $k \in \mathbb{Z}$, and so $g$ is an isomorphism in $D(\mathcal{G})$. 

Lemma 4.5. Let $p : M \to N$ be a morphism in $\mathcal{H}_t$ such that $H^0(p)$ and $H^{-1}(p)$ are epimorphisms in $\mathcal{G}$ and $\text{Ker}(H^0(p))$ is in $T$. Then $p$ is an epimorphism in $\mathcal{H}_t$.

Proof. Let us consider the induced triangle $M \xrightarrow{p} N \to W \xrightarrow{\iota} \text{ in } D(\mathcal{G})$. The long exact sequence of homologies and the hypotheses give that $H^{-2}(W) \cong \text{Ker}(H^{-1}(p))$ is in $\mathcal{F}$, that $H^{-1}(W) \cong \text{Ker}(H^0(p))$ is in $T$ and that $H^i(W) = 0$, for all $i \neq -2, -1$. It follows that $W[-1] \in \mathcal{H}_t$, so that we get a short exact sequence $0 \to W[-1] \to M \xrightarrow{p} N \to 0$ in $\mathcal{H}_t$.

Lemma 4.6. Let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair such that $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$. Then there is an object $V$ such that $\mathcal{T} = \text{Pres}(V)$. Moreover, the torsion pairs such that $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$ form a set.

Proof. Let us fix a generator $G$ of $\mathcal{G}$. Then each object of $\mathcal{G}$ is a directed union of those of its subobjects which are isomorphic to quotients $G^{(n)}/X$, where $n$ is a natural number and $X$ is a subobject of $G^{(n)}$. Let now take $T \in \mathcal{T}$ and express it as a directed union $T = \bigcup_{i \in I} U_i$, where $U_i \cong \frac{G^{(n_i)}_{X_i}}{X_i}$, for some integer $n_i > 0$ and some subobject $X_i$ of $G^{(n_i)}$. We now get an $I$-directed system of exact sequences

$$0 \to t(U_i) \to U_i \to U_i/t(U_i) \to 0.$$

Due to the AB5 condition of $\mathcal{G}$, after taking direct limits, we get an exact sequence

$$0 \to \lim_{\longleftarrow} t(U_i) \to T \to \lim_{\longleftarrow} \frac{U_i}{t(U_i)} \to 0.$$

It follows that $\lim_{\longleftarrow} \frac{U_i}{t(U_i)} \in T \cap \mathcal{F} = 0$ since $\mathcal{F}$ is closed under taking direct limits in $\mathcal{F}$. Then we have $T = \bigcup_{i \in I} t(U_i)$.

The objects $T' \in T$ which are isomorphic to subobjects of quotients $G^n/X$ form a skeletally small subcategory. We take a set $\mathcal{S}$ of its representatives, up to isomorphism, and put $V = \bigsqcup_{S \in \mathcal{S}} S$. The previous paragraph shows that each $T \in \mathcal{T}$ is isomorphic to a direct limit of objects in $\mathcal{S}$, from which we get that $\mathcal{T} \subseteq \text{Pres}(V)$. The converse inclusion is clear.

For the final statement, note that the last paragraph shows that the assignment $\mathfrak{t} \mapsto \mathcal{S}$ gives an injective map from the class of torsion pairs $\mathfrak{t}$ such that $\mathcal{F} = \lim_{\longleftarrow} \mathcal{F}$ to the set of subsets of $\bigcup_{n \in \mathbb{N}, X < G^n} \mathcal{S}(G^n/X)$, where $\mathcal{S}(M)$ denotes the set of subobjects of $M$, for each object $M$.

We can now give the desired information on the existence of a generator in $\mathcal{H}_t$. Recall that a subquotient of an object $X$ of $\mathcal{G}$ is a quotient $Y/Z$, where $Z \subseteq Y$ are subobjects of $X$. Note that these subquotients form a set, for each object $X$ in $\mathcal{G}$ (see [S, Proposition IV.6.6]).

Proposition 4.7. Let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$ such that $\mathcal{F}$ is closed under taking direct limits. Then the heart $\mathcal{H}_t$ has a generator.

Proof. We fix a generator $G$ of $\mathcal{G}$ and, using lemma 4.6, we fix an object $V$ such that $\text{Pres}(V) = \mathcal{T}$. We consider the class $\mathcal{N}$ consisting of the objects $N \in \mathcal{H}_t$ such that $H^{-1}(N)$ is isomorphic to a subquotient of $G^{(m)}$ and $H^0(N) \cong V^{(n)}$, for some natural numbers $m$ and $n$. We claim that this class is skeletally small. To see this, consider an object $F \in \mathcal{F}$ which is a subquotient of $G^{(m)}$, for some $m \in \mathbb{N}$, and put $\mathcal{N}_{F,n} = \{ N \in \mathcal{N} : H^{-1}(N) \cong F \text{ and } H^0(N) \cong V^{(n)} \}$, for each $n \in \mathbb{N}$. Bearing in mind that the subquotients of each $G^{(m)}$ form a set, it is enough to prove that we have an injective map $\Psi : \mathcal{N}_{F,n}/ \cong \to \text{Ext}^2_{\mathcal{G}}(V^{(n)}, F)$.
since the codomain of this map is a set. Indeed, we represent any object of \( \mathcal{N}_{F,n} \) as a complex concentrated in degrees \(-1, 0\). If \( \cdots \to 0 \to N^{-1} \xrightarrow{d} N^0 \to 0 \to \cdots \) is such a complex, then \( \Psi(N) \) will be the element of \( \text{Ext}^2_{\mathcal{G}}(V^{(n)}, F) \) given by the exact sequence\[ 0 \to F \to N^{-1} \xrightarrow{d} N^0 \to V^{(n)} \to 0.\]

To see that \( \Psi \) is well defined, we need to check that if \( N \cong N' \) then \( \Psi(N) = \Psi(N') \). An isomorphism \( f : N \xrightarrow{\sim} N' \) is represented by two quasi-isomorphisms \( N \xrightarrow{s} Y \xleftarrow{s'} N' \), where \( Y \) is also a complex concentrated in degrees \(-1\) and \( 0 \) (see, e.g., the proof of [CGM, Theorem 4.2]). Then, there is no loss of generality in assuming that \( f \) is a quasi-isomorphism, in which case we have a commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & F & \to & N^{-1} & \to & N^0 & \to & V^{(n)} & \to & 0 \\
0 & \to & F & \to & N^{-1} & \to & N^0 & \to & V^{(n)} & \to & 0
\end{array}
\]

Then the upper and lower rows of this diagram represent the same element of \( \text{Ext}^2_{\mathcal{G}}(V^{(n)}, F) \) (see [ML, Chapter III]). That is, we have \( \Psi(M) = \Psi(N) \), so that \( \Psi : \mathcal{N}_{F,n} / \cong \to \text{Ext}^2_{\mathcal{G}}(V^{(n)}, F) \) is well-defined. The injectivity of \( \Psi \) follows from the definition of \( \text{Ext}^2_{\mathcal{G}}(V^{(n)}, F) \) via congruencies (see [ML, Chapter III, Section 5]) and the fact that if we have a commutative diagram as the last one, then the induced chain map \( f : N \to N' \) is a quasi-isomorphism and, hence, an isomorphism in \( \mathcal{H}_k \).

We shall prove that any object of \( \mathcal{H}_k \) is an epimorphic image of a coproduct of objects of \( \mathcal{N} \), which will end the proof. Let \( M \in \mathcal{H}_k \) be any object, which we represent by a complex \( \cdots \to 0 \to M^{-1} \xrightarrow{d} M^0 \to 0 \to \cdots \), concentrated in degrees \(-1, 0\). We fix an epimorphism \( p : G^{(J)} \to M^{-1} \) in \( \mathcal{G} \), for some set \( J \). Then, for each finite subset \( F \subseteq J \), we have the following commutative diagram with exact rows, where the bottom and pre-bottom left and the upper right squares are cartesian:

\[
\begin{array}{ccccccc}
0 & \to & U_F & \to & G^{(F)} & \to & M_F & \to & t(X_F) & \to & 0 \\
0 & \to & U_F & \to & G^{(F)} & \to & M^0 \downarrow & \to & X_F & \to & 0 \\
0 & \to & U_J & \to & G^{(J)} & \to & M^0 \downarrow & \to & H^0(M) & \to & 0 \\
0 & \to & H^{-1}(M) & \to & M^{-1} \downarrow & \to & M^0 \downarrow & \to & H^0(M) & \to & 0
\end{array}
\]

The AB5 condition of \( \mathcal{G} \) implies that \( H^0(M) = \text{lim}_{F \subseteq J}(t(X_F)) \), and then the fact that \( \mathcal{F} \) is closed under taking direct limits implies that \( H^0(M) = \text{lim}_{F \subseteq J}(t(X_F)) \). This in turn implies that \( M^0 = \text{lim}_{F \subseteq J}(M^0_F) \). We denote by \( K_J \) the complex \( \cdots \to 0 \to G^{(J)} \xrightarrow{d} M^0 \to 0 \to \cdots \), concentrated in degrees \(-1, 0\). Similarly, for each finite subset \( F \subseteq J \), we denote by \( K_F \) the complex \( \cdots \to 0 \to G^{(F)} \to M^0_F \to 0 \to \cdots \) given by the upper row of last diagram. Note that \( K_J \) and \( K_F \) are in \( \mathcal{U}_k \). By explicit construction of the functor \( L : \mathcal{U}_k \to \mathcal{H}_k \) (see lemma 3.4), we see that \( L(K_J) \) and \( L(K_F) \) can be represented by the complexes obtained from \( K_J \) and \( K_F \) by replacing \( G^{(J)} \) and \( G^{(F)} \) by \( G^{(J)}/t(U_J) \) and \( G^{(F)}/t(U_F) \), respectively, in degree \(-1\). This allows us to interpret \( (L(X_F))_{F \subseteq J, F \text{ finite}} \) as a direct system in \( \mathcal{C}(\mathcal{G}) \) of complexes concentrated in degrees \(-1, 0\). But the fact that \( \mathcal{F} \) is closed under taking direct limits in \( \mathcal{G} \) and \( \text{lim}_{U_F} = U_J \) implies that \( t(U_J) = \text{lim}_{F \subseteq J} t(U_F) \), from which we easily get that \( L(K_J) = \text{lim}_{F \subseteq J} L(K_F) \). From lemma 3.4 we deduce that \( L(K_J) \cong \text{lim}_{F \subseteq J} L(K_F) \), so that \( L(K_J) \) is a quotient in \( \mathcal{H}_k \) of a coproduct of complexes \( N' \in \mathcal{H}_k \) such that \( H^{-1}(N') \) is a subquotient of \( G^m \), for some \( m \in \mathbb{N} \).
Note that the chain map $K_f \to M$ obtained from the diagram above induces a chain map $q : L(K_f) \to M$, because $p(t(U_f)) \subseteq t(H^{-1}(M)) = 0$. Moreover, by construction, we have that $H^{-1}(q)$ is an epimorphism and $H^0(q) = 1_{H^0(M)}$ is an isomorphism. By lemma 4.4 we get that $q$ is an epimorphism in $\mathcal{H}_k$. This, together with the previous paragraph, shows that each $M$ in $\mathcal{H}_k$ is a quotient of a coproduct of objects $N'_i$ of $\mathcal{H}_k$ as above. Replacing $M$ by one such $N'$, we can and shall restrict ourselves to the case when $M$ is a complex concentrated in degrees $-1, 0$, which is in $\mathcal{H}_k$ and satisfies that $H^{-1}(M)$ is a subquotient of $G^m$, for some $m \in \mathbb{N}$.

Suppose that $M$ is such a complex in the rest of the proof. By fixing an epimorphism $v : V(S) \to H^0(M)$ such that $\text{Ker}(v) \in \mathcal{T}$ and pulling it back along the canonical epimorphism $M^0 \to H^0(M)$, we obtain a complex $M : \cdots 0 \to M^{-1} \to M^0 \to 0 \to \cdots$, concentrated in degrees $-1, 0$, such that $H^{-1}(M) = H^{-1}(M)$, $H^0(M) = V(S)$ and it comes with an induced exact sequence $0 \to \text{Ker}(v)[0] \to M \to 0$ in $\mathcal{C}(\mathcal{G})$. Since the three terms of this sequence are in $\mathcal{H}_k$, we get that $\hat{v}$ is an epimorphism in $\mathcal{H}_k$. Note that $v$ exists because $\mathcal{T} = \text{Pres}(V)$.

Replacing now $M$ by $M$, we can assume without loss of generality that $H^0(M) = V(S)$, for some set $S$. Now for each finite subset $F \subset S$, we consider the following commutative diagram with exact rows, whose right square is cartesian.

$$
\begin{array}{cccccc}
0 & \to & H^1(M) & \to & M^{-1} & \to & N^0_F & \to & V(F) & \to & 0 \\
\vert & & \vert & & \vert & & \vert & & \vert & & \\
0 & \to & H^1(M) & \to & M^{-1} & \to & M^0 & \to & V(S) & \to & 0 \\
\end{array}
$$

We denote by $N_F$ the complex $\cdots \to 0 \to M^{-1} \to N^0_F \to 0 \to \cdots$, concentrated in degrees $-1, 0$, given by the upper row of the last diagram. Note that $N_F \in \mathcal{N}$, for each $F \subset S$ finite. Moreover, $(N_F)_{F \subset S, F \text{ finite}}$ is a direct system in $\mathcal{C}(\mathcal{G})$ such that $\lim_{\mathcal{C}(\mathcal{G})} (N_F) = M$. By lemma 4.4 we get that $\lim_{\mathcal{H}_k} (N_F) = M$, so that $M$ is a quotient in $\mathcal{H}_k$ of a coproduct of objects of $\mathcal{N}$.

We are now ready to give the two main results of the paper.

**Theorem 4.8.** Let $\mathcal{G}$ be a Grothendieck category, let $t = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$, let $(\mathcal{U}_k, \mathcal{U}_k^1[1])$ be its associated $t$-structure in $\mathcal{D}(\mathcal{G})$ and let $\mathcal{H}_k = \mathcal{U}_k \cap \mathcal{U}_k^1[1]$ be the heart. The following assertions are equivalent:

0. $\mathcal{H}_k$ is a Grothendieck category.

1. $\mathcal{H}_k$ is an $\text{AB}5$ abelian category.

2. If $(M_i)_{i \in I}$ is a direct system in $\mathcal{H}_k$, with

$$\prod_{i \leq j} M_{ij} \xrightarrow{f} \prod_{i \in I} M_i \to Z \xrightarrow{+}$$

the triangle in $\mathcal{D}(\mathcal{G})$ afforded by the associated colimit-defining morphism, then the composition $\lim_{\mathcal{H}_k} H^{-1}(M_i) \to H^{-1}(Z) \xrightarrow{\text{can}} H^{-1}(Z)/t(H^{-1}(Z))$ is a monomorphism.

3. For each direct system $(M_i)_{i \in I}$ in $\mathcal{H}_k$, the canonical morphism $\lim_{\mathcal{H}_k} H^{-1}(M_i) \to H^{-1}(\lim_{\mathcal{H}_k} M_i)$ is a monomorphism.

4. For each direct system $(M_i)_{i \in I}$ in $\mathcal{H}_k$, the canonical morphism $\lim_{\mathcal{H}_k} H^k(M_i) \to H^k(\lim_{\mathcal{H}_k} M_i)$ is an isomorphism, for all $k \in \mathbb{Z}$.

In that case, the class $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$.

**Proof.** 0) $\implies$ 1) is clear.

1) $\implies$ 0) By proposition 4.7, we just need to prove that $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$. Let $(M_i)_{i \in I}$ be any direct system in $\mathcal{F}$. For each $j \in I$, let us denote by $\gamma_j$ the composition $F_j \xrightarrow{i_j} \lim_{\mathcal{F}} F_i \xrightarrow{p}$
Theorem 4.9. \( \frac{\lim_{\mathcal{F}} F_i}{t(\lim_{\mathcal{F}} F_i)} \) where the morphisms are the obvious ones. Using the exactness of direct limits in \( \mathcal{G} \), it follows that \((\text{Ker}(\gamma_i))_{i \in I}\) is a direct system in \( \mathcal{F} \) such that \( \lim_{\mathcal{F}} \text{Ker}(\gamma_i) = t(\lim_{\mathcal{F}} F_i) \) is in \( \mathcal{T} \). Put \( T := t(\lim_{\mathcal{F}} F_i) \) and, for each \( i \in I \), consider the canonical map \( u_i : \text{Ker}(\gamma_i) \rightarrow T \) into the direct limit. We then get a direct system of exact sequences in \( \mathcal{G} \)

\[ 0 \rightarrow \text{Ker}(u_i) \rightarrow \text{Ker}(\gamma_i) \xrightarrow{u_i} T \rightarrow \text{Coker}(u_i) \rightarrow 0. \]

From lemma 4.3 we then get that the map \( u := \lim_{\mathcal{F}} u_i : \lim_{\mathcal{F}} \text{Ker}(\gamma_i) \rightarrow T \) vanishes on \( t(\lim_{\mathcal{F}} \text{Ker}(\gamma_i)) \). This implies that \( u = 0 \) since \( \lim_{\mathcal{F}} \text{Ker}(\gamma_i) = T \) is in \( \mathcal{T} \). But \( u \) is an isomorphism by definition of the direct limit. It then follows that \( T = 0 \), so that \( \lim_{\mathcal{F}} F_i \in \mathcal{F} \) as desired.

1) \( \implies \) 2) Note that, by the proof of 1) \( \implies \) 0), we know that \( \mathcal{F} \) is closed under taking direct limits in \( \mathcal{G} \). In particular, if \((M_i)_{i \in I}\) be a direct system in \( \mathcal{H}_t \) then \( t(\lim_{\mathcal{F}} H^{-1}(M_i)) = 0 \). For such a direct system, we get an induced direct system of short exact sequences in \( \mathcal{H}_t \)

\[ 0 \rightarrow H^{-1}(M_i)[1] \rightarrow M_i \rightarrow H^0(M_i)[0] \rightarrow 0. \]

From the AB5 condition of \( \mathcal{H}_t \) and proposition 4.2, we get an exact sequence in \( \mathcal{H}_t \)

\[ 0 \rightarrow \lim_{\mathcal{H}_t} H^{-1}(M_i)[1] \rightarrow \lim_{\mathcal{H}_t} M_i \rightarrow (\lim_{\mathcal{H}_t} H^0(M_i))[0] \rightarrow 0. \]

By taking homologies, we get that the canonical morphism \( \lim_{\mathcal{H}_t} H^{-1}(M_i) \rightarrow H^{-1}(\lim_{\mathcal{H}_t} M_i) \) is a monomorphism in \( \mathcal{G} \) and, by the proof of proposition 4.2, we know that \( H^{-1}(\lim_{\mathcal{H}_t} M_i) \cong H^{-1}(Z)/t(H^{-1}(Z)) \).

2) \( \iff \) 3) \( \iff \) 4) follow directly from proposition 4.2 its proof and the fact that all complexes in \( \mathcal{H}_t \) have homology concentrated in degrees \(-1\) and 0.

4) \( \implies \) 1) \( H^0 : \mathcal{D} (\mathcal{G}) \rightarrow \mathcal{G} \) is a cohomological functor, and then \((H^0, 1)\) is a cohomological datum for \( \mathcal{H}_t \). Then the implication follows from proposition 3.3.

From last theorem we get that the Grothendieck condition of the heart implies the closure of \( \mathcal{F} \) under taking direct limits. One can naturally asks if the converse is also true. Our second main result in the paper shows that this is the case for some familiar torsion pairs.

**Theorem 4.9.** Let \( \mathcal{G} \) be a Grothendieck category and let \( t = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{G} \) satisfying, at least, one the following conditions:

a) \( t \) is hereditary.

b) Each object of \( \mathcal{H}_t \) is isomorphic in \( \mathcal{D} (\mathcal{G}) \) to a complex \( F \) such that \( F^k = 0 \), for \( k \neq -1, 0 \), and \( F^k \in \mathcal{F} \), for \( k = -1, 0 \).

c) Each object of \( \mathcal{H}_t \) is isomorphic in \( \mathcal{D} (\mathcal{G}) \) to a complex \( T \) such that \( T^k = 0 \), for \( k \neq -1, 0 \), and \( T^k \in \mathcal{T} \), for \( k = -1, 0 \).

The following assertions are equivalent:

1. The heart \( \mathcal{H}_t \) is a Grothendieck category;

2. \( \mathcal{F} \) is closed under taking direct limits in \( \mathcal{G} \).

**Proof.** 1) \( \implies \) 2) follows from theorem 4.8

2) \( \implies \) 1) Let \( t \) be hereditary in this first paragraph. Let \((M_i)_{i \in I}\) be a direct system in \( \mathcal{H}_t \). With the terminology theorem 4.8 note that the first arrow of the composition \( \lim_{\mathcal{F}} H^{-1}(M_i) \rightarrow H^{-1}(Z) \xrightarrow{\text{can}} H^{-1}(Z)/t(H^{-1}(Z)) \) is always a monomorphism. As a consequence, when \( t \) is hereditary, the composition is automatically a monomorphism since its kernel is in \( \mathcal{T} \cap \mathcal{F} = 0 \). By the mentioned theorem, we get that \( \mathcal{H}_t \) is a Grothendieck category.

Suppose next that condition b holds. We claim that, in that case, each object of \( \mathcal{H}_t \) is isomorphic to a subobject of an object in \( \mathcal{F}[1] \). Indeed, if \( M \) is isomorphic to the mentioned complex \( F \), then its differential \( d : F^{-1} \rightarrow F^0 \) gives a triangle
in \( D(G) \) and, hence, it also gives an exact sequence in \( H_\lambda \)

\[ 0 \to M \to F^{-1}[1] \to F^0[1] \to 0. \]

We want to check that assertion 3 of theorem 4.8 holds, for which we will use the fact that \( F[1] \) is closed under taking quotients in \( H_\lambda \). By traditional arguments (see, e.g. [AR] Corollary 1.7), it is not restrictive to assume that the directed set \( I \) is an ordinal and that the given direct system in \( H_\lambda \) is continuous (smooth in the terminology of [AR]). So we start with a direct system \( (M_\alpha)_{\alpha < \lambda} \) in \( H_\lambda \), where \( \lambda \) is a limit ordinal and \( M_\beta = \lim_{\gamma \to \beta} M_\gamma \), whenever \( \beta \) is a limit ordinal such that \( \beta < \lambda \). Now, by transfinite induction, we can define a \( \lambda \)-direct system of short exact sequences in \( H_\lambda \)

\[ 0 \to M_\alpha \to F_\alpha[1] \to F'_\alpha[1] \to 0, \]

with \( F_\alpha, F'_\alpha \in F \) for all \( \alpha < \lambda \). Suppose that \( \beta = \alpha + 1 \) is nonlimit and that the sequence has been defined for \( \alpha \). Then the sequence for \( \beta \) is the bottom one of the following commutative diagram, where the upper left and lower right squares are bicartesian and \( F \) and \( F' \) denote objects of \( F \), and \( u_{\alpha+1} \) any monomorphism into an object of \( F[1] \):

\[
\begin{array}{ccccccccc}
0 & \to & M_\alpha & \to & F_\alpha[1] & \to & F'_\alpha[1] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M_{\alpha+1} & \to & N_\alpha & \to & F'_\alpha[1] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M_{\alpha+1} & \to & F_{\alpha+1}[1] & \to & F'_{\alpha+1}[1] & \to & 0
\end{array}
\]

Suppose now that \( \beta \) is a limit ordinal and that the sequence has been defined for all ordinals \( \alpha < \beta \). Using proposition 4.2 and the fact that \( F \) is closed under taking direct limits, we then get an exact sequence in \( H_\lambda \)

\[ \lim_{\gamma \to \alpha} M_\alpha \xrightarrow{g} (\lim_{\gamma \to \beta} F_\gamma)[1] \to (\lim_{\gamma \to \beta} F'_\gamma)[1] \to 0. \]

Recall that, by the continuity of the direct system, we have \( M_\beta = \lim_{\gamma \to \beta} M_\gamma \). We denote by \( W \) the image of \( g \) in \( H_\lambda \). We then get the following commutative diagram with exact rows.

\[
\begin{array}{ccccccccc}
0 & \to & \lim_{\gamma \to \beta} H^{-1}(M_\gamma) & \to & \lim F_\alpha & \to & \lim F'_\alpha & \to & \lim H^0(M_\alpha) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^{-1}(W) & \to & H^{-1}(\lim_{\gamma \to H_\lambda} F_\gamma[1]) & \to & H^{-1}(\lim_{\gamma \to H_\lambda} F'_\gamma[1]) & \to & H^0(W) & \to & 0
\end{array}
\]

All the vertical arrows are then isomorphisms since so are the two central ones. But the left vertical arrow is the composition \( \lim_{\gamma \to \beta} H^{-1}(M_\gamma) \to H^{-1}(\lim_{\gamma \to \beta} M_\gamma) \xrightarrow{H^{-1}(p)} H^{-1}(W) \), where \( p : \lim_{\gamma \to \beta} M_\gamma \to W \) is the obvious epimorphism in \( H_\lambda \). It follows that the canonical map \( \lim_{\gamma \to \beta} H^{-1}(M_\gamma) \to H^{-1}(\lim_{\gamma \to \beta} M_\gamma) \) is a monomorphism. Then it is an isomorphism due to proposition 4.2(1) and, by this same proposition and the isomorphic condition of the right vertical arrow in the above diagram, we conclude that \( H^k(p) \) is an isomorphism, for all \( k \in \mathbb{Z} \). Then \( p \) is an isomorphism in \( H_\lambda \) and the desired short exact sequence for \( \beta \) is defined.

The argument of the previous paragraph, when applied to \( \lambda \) instead of \( \beta \), shows that the induced morphism \( \lim_{\gamma \to \beta} H^{-1}(M_\gamma) \to H^{-1}(\lim_{\gamma \to \lambda} M_\lambda) \) is a monomorphism, and then assertion 3 of theorem 4.8 holds. It follows that \( H_\lambda \) is a Grothendieck category.
Finally, suppose that condition c holds. An argument dual to the one used for condition b, shows that, $T[0]$ generates $H_4$. On the other hand, by lemma 4.9, we have an object $V \in T$ such that $T = \text{Pres}(V)$. We easily derive that, for each $T \in T$, the kernel of the canonical epimorphism $V(\text{Hom}_G(V,T)) \rightarrow T$ is in $T$, so that the induced morphism $V[0](\text{Hom}_G(V,T)) \rightarrow T[0]$ is an epimorphism in $H_4$. Therefore $V[0]$ is a generator of $H_4$.

Note that the assignment $M \rightarrow \Psi(M) := V[0](\text{Hom}_{H_4}(V[0],M))$ is functorial. Indeed if $f : M \rightarrow N$ is a morphism in $H_4$, we define $\Psi(f) : V[0](\text{Hom}_{H_4}(V[0],M)) \rightarrow V[0](\text{Hom}_{H_4}(V[0],N))$ using the universal property of the coproduct in $H_4$. By definition, $\Psi(f)$ the unique morphism in $H_4$ such that $\Psi(f) \circ \iota^M = \iota^N$, where $\iota^M : V[0] \rightarrow V[0](\text{Hom}_{H_4}(V[0],M))$ is the $\alpha$-injection into the coproduct, where $\alpha \in \text{Hom}_{H_4}(V[0],M)$, and similarly for $\iota^N$.

The functor $\Psi : H_4 \rightarrow H_4$ comes with natural transformation $p : \Psi \rightarrow id_{H_4}$ which is epimorphic. Note that $\xi(M) := \text{Ker}(p_M) = T_M[0]$, for some $T_M \in T$, since $T[0]$ is closed under taking subobjects in $H_4$. We then get functors $\Psi, \xi : H_4 \rightarrow T \cong T[0] \cong H_4$, together with an exact sequence of functors $0 \rightarrow \xi \rightarrow \Psi \rightarrow id_{H_4} \rightarrow 0$. In particular, if $f : L \rightarrow M$ are morphisms in $H_4$, we have a commutative diagram in $T \cong T[0]$

$$
\begin{array}{ccc}
\xi(L) & \xrightarrow{\mu_L} & \Psi(L) \\
\downarrow{\xi(f)} & & \downarrow{\Psi(f)} \\
\xi(M) & \xrightarrow{\mu_M} & \Psi(M)
\end{array}
$$

But note that if $M$ is an object of $H_4$, then, viewing $\xi(M)$ and $\Psi(M)$ as objects of $T$, the complex $C(M) : \cdots \rightarrow 0 \rightarrow \xi(M) \rightarrow \Psi(M) \rightarrow 0 \rightarrow \cdots$ ($\Psi(M)$ in degree 0) is isomorphic to $M$ in $D(G)$. The diagram above tells us that the assignment $M \rightarrow C(M)$ gives a functor $C : H_4 \rightarrow C(G)$ such that if $q : C(G) \rightarrow D(G)$ is the canonical functor, then the composition $H_4 \xrightarrow{C} C(G) \xrightarrow{q} D(G)$ is naturally isomorphic to the inclusion $H_4 \hookrightarrow D(G)$.

Suppose that $(M_i)_{i \in I}$ is a direct system in $H_4$. Then $(C(M_i))_{i \in I}$ is a direct system in $C(G)$. By lemma 4.4, we know that $\lim_{\rightarrow C(G)} C(M_i) \cong \lim_{\rightarrow H_4} C(M_i) \cong \lim_{\rightarrow H_4} M_i$. Then we get an isomorphism $\lim_{\rightarrow H_4} C(M_i) \cong \lim_{\rightarrow H_4} C(M_i)$. Then assertion 3 of theorem 4.8 holds.

**Corollary 4.10.** Let $t = (\mathcal{T}, \mathcal{F})$ be a torsion pair such that either $\mathcal{F}$ is generating or $\mathcal{T}$ is cogenerating. The heart $H_t$ is a Grothendieck category if, and only if, $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$.

**Proof.** We assume that $\mathcal{F}$ is closed under taking direct limits, because, by theorem 4.8, we only need to prove the 'if' part of the statement.

Suppose first that $\mathcal{F}$ is a generating class and let $M \in H_t$ be any object, which we represent by a complex $\cdots \rightarrow 0 \rightarrow M^{-1} \rightarrow M^0 \rightarrow 0 \rightarrow \cdots$. By fixing an epimorphism $p : F^0 \rightarrow M^0$ and taking the pullback of this morphism along $d$, we may and shall assume that $M^0 = F^0 \in \mathcal{F}$. But then $\text{Im}(d)$ is in $\mathcal{F}$, which implies that $M^{-1} \in \mathcal{F}$ since we have an exact sequence $0 \rightarrow H^{-1}(M) \rightarrow M^{-1} \rightarrow \text{Im}(d) \rightarrow 0$, where the outer nonzero terms are in $\mathcal{F}$. Then condition b of theorem 4.9 holds.

Suppose that $\mathcal{T}$ is a cogenerating class. Then the injective objects of $\mathcal{G}$ are in $\mathcal{T}$. By an argument dual to the one followed in the previous paragraph, we see that each object $M \in H_t$ is isomorphic in $D(G)$ to a complex $\cdots \rightarrow 0 \rightarrow T^{-1} \rightarrow T^0 \rightarrow 0 \rightarrow \cdots$, where $T^{-1}$ is injective and $T^0 \in \mathcal{T}$. Then condition c of theorem 4.9 holds. $\square$

The following is now a natural question that remains open.

**Question 4.11.** Let $t = (\mathcal{T}, \mathcal{F})$ be a torsion pair in the Grothendieck category $\mathcal{G}$ such that $\mathcal{F}$ is closed under taking direct limits. Is the heart $H_t$ a Grothendieck (equivalently, AB5) category?
5 Tilting and cotilting torsion pairs revisited

All throughout this section, the letter $G$ denotes a Grothendieck category. We refer the reader to section 2 for the definition of (co)tilting object in an abelian category.

**Definition 4.** Let $A$ be an AB3 (resp. AB3*) abelian category. Two 1-tilting (resp. 1-cotilting) objects of $A$ are said to be equivalent when their associated torsion pairs coincide.

**Remark 5.1.** Recall that idempotents split in any abelian category. As a consequence, two 1-tilting objects $V$ and $V'$ are equivalent if, and only if, $\text{Add}(V) = \text{Add}(V')$. Similarly, two 1-cotilting objects $Q$ and $Q'$ are equivalent if, and only if, $\text{Prod}(Q) = \text{Prod}(Q')$.

With some additional hypotheses, one obtains the following more familiar characterization of 1-tilting objects. The dual result characterizes 1-cotilting objects in AB4* categories with an injective cogenerator.

**Proposition 5.2.** Let $A$ be an AB4 abelian category with a projective generator and let $V$ be any object of $A$. Consider the following assertions:

1. $V$ is a 1-tilting object.

2. The following conditions hold:
   
   (a) There exists an exact sequence $0 \to P^{-1} \to P^0 \to V \to 0$ in $A$, where the $P^k$ are projective;
   
   (b) $\text{Ext}_A^1(V, V^{(I)}) = 0$, for all sets $I$;
   
   (c) for some (resp. every) projective generator $P$ of $A$, there is an exact sequence
   
   $$0 \to P \to V^0 \to V^1 \to 0,$$

   where $V^i \in \text{Add}(V)$ for $i = 0, 1$.

The implications 2) $\Rightarrow$ 1), 1) $\Rightarrow$ 2.b and 1) $\Rightarrow$ 2.c hold. When $A$ has enough injectives, assertions 1 and 2 are equivalent.

**Proof.** As in module categories (see [CT, Proposition 1.3], and also [C, Section 2]).

Recall that an object $X \in \text{Ob}(G)$ is called self-small when the canonical morphism $\text{Hom}_G(X, X^{(I)}) \to \text{Hom}_G(X, X^{(I)})$ is an isomorphism, for each set $I$. A consequence of the results in the previous section is the following:

**Proposition 5.3.** Let $G$ be a Grothendieck category and let $t = (T, F)$ be a torsion pair in $G$. Consider the following assertions:

1. $t$ is a tilting torsion pair induced by a self-small 1-tilting object;

2. $T$ is a cogenerating class and the heart $\mathcal{H}_t$ is a module category;

3. $T$ is a cogenerating class and $\mathcal{H}_t$ is a Grothendieck category with a projective generator;

4. $t$ is a tilting torsion pair such that $\mathcal{H}_t$ is an AB5 abelian category;

5. $t$ is a tilting torsion pair such that $F$ is closed under taking direct limits in $G$.

Then the implications 1) $\iff$ 2) $\implies$ 3) $\iff$ 4) $\iff$ 5) hold.

**Proof.** Let $V$ be any 1-tilting object and $t = (T, F)$ its associated torsion pair. If $F \in F$ is any object, then its injective envelope $E(F)$ and its first cosyzygy $\Omega^{-1}(F) = \frac{E(F)}{\text{Gen}(V)}$ are in $T := \text{Gen}(V) = \text{Ker}(\text{Ext}_G^1(V, ?))$. It follows that $\text{Ext}_{\mathcal{H}_t}^1(V[0], F[1]) = \text{Ext}_G^2(V, F) = \text{Ext}_G^1(V, \Omega^{-1}(F)) = 0$. On the other hand, we have $\text{Ext}_{\mathcal{H}_t}^1(V[0], T[0]) \cong \text{Ext}_G^2(V, T) = 0$, for all $T \in T$. It follows that $F[0]$ is a projective object of $\mathcal{H}_t$. On the other hand, by the proof of theorem 4.9 under its condition c, we know that $V[0]$ is a generator of $\mathcal{H}_t$.

1) $\implies$ 2) Let the 1-tilting object $V$ be self-small. We then have an isomorphism
$\text{Hom}_{\mathcal{H}_t}(V[0], V[0])^\mathcal{Z} \cong \text{Hom}_{\mathcal{G}}(V, V)^\mathcal{Z} \cong \text{Hom}_{\mathcal{G}}(V, V^\mathcal{Z}) \cong \text{Hom}_{\mathcal{H}_t}(V[0], V[0])^\mathcal{Z}$, \quad (*)

for each set $I$. That is, $V[0]$ is a self-small object of $\mathcal{H}_t$ and, since it is a projective generator, it is easily seen that $V[0]$ is a compact object of $\mathcal{H}_t$. Then $V[0]$ is a projective generator of $\mathcal{H}_t$ and $\mathcal{H}_t$ is a module category (see [Po Corollary 3.6.4]).

3) $\implies 4)$ By the proofs of corollary 4.10 and theorem 4.9 we know that $\mathcal{T}[0]$ generates $\mathcal{H}_t$. If $G$ is a projective generator of $\mathcal{H}_t$, then it is necessarily of the form $G = V[0]$, where $V \in \mathcal{T}$. It easily follows from this that $\mathcal{T} = \text{Pres}(V) = \text{Gen}(V)$ and, hence, that $\mathcal{F} = \text{Ker}(\text{Hom}_{\mathcal{G}}(V, ?))$.

From the projectivity of $V[0]$ in $\mathcal{H}_t$ we get that $0 = \text{Ext}_{\mathcal{H}_t}^1(V[0], T[0]) = \text{Ext}_{\mathcal{G}}^1(V, T)$, for each $T \in \mathcal{T}$. Therefore we get that $\text{Gen}(V) \subseteq \text{Ker}(\text{Ext}_{\mathcal{G}}^1(V, ?))$. The proof of the converse inclusion is entirely dual to the corresponding one for cotilting objects, which is done in the implication 1) $\implies 3)$ of proposition 5.7.

2) $\implies 1)$ By the argument in the implication 3) $\implies 4)$, we can assume that $t$ is a tilting torsion pair induced by a 1-tilting object $V$ such that $V[0]$ is a projective generator of $\mathcal{H}_t$. From the fact that $V[0]$ is compact in $\mathcal{H}_t$ we derive that the isomorphism (*) above still holds. Then $V$ is self-small. The implication 2) $\implies 3)$ is clear and 4) $\implies 5)$ follows from theorem 3.5.

5) $\implies 3)$ That $\mathcal{T} = \text{Ker}(\text{Ext}_{\mathcal{G}}^1(V, ?))$ is cogenerated is clear since it contains all injective objects. The fact that $\mathcal{H}_t$ is a Grothendieck category follows then from corollary 4.10. Finally, by the first paragraph of this proof, we know that $V[0]$ is a projective generator of $\mathcal{H}_t$.

Example 5.4. A projective generator $P$ of $\mathcal{G}$ is always a 1-tilting object. However $P$ is self-small if, and only if, it is compact. Therefore if $\mathcal{G}$ has a projective generator but is not a module category, then the trivial torsion pair $t = (\mathcal{G}, 0)$ satisfies assertion 5, but not assertion 2 of last proposition.

However, the following is a natural question whose answer seems to be unknown.

Question 5.5. Let $R$ be a ring and $V$ be a 1-tilting $R$-module such that $\text{Ker}(\text{Hom}_R(V, ?))$ is closed under taking direct limits in $R - \text{Mod}$. Is $V$ equivalent to a self-small 1-tilting module $?$. Note that a 1-tilting $R$-module is self-small if, and only if, it is finitely presented (cf. [G1 Proposition 1.3]).

If $I$ is any set, then the product functor $\prod : \mathcal{G}^I = [I, \mathcal{G}] \to \mathcal{G}$ is left exact, but need not be right exact. We shall denote by $\prod^1 : \prod^I [I, \mathcal{G}] \to \mathcal{G}$ its first right derived functor. Given a family $(X_i)_{i \in I}$, we have that $\prod^1 i \in I X_i$ is the cokernel of the canonical morphism $\prod_{i \in I} E(X_i) \to \prod_{i \in I} E(X_i)_{-n}$.

Definition 5. An object $Q$ of the Grothendieck category $\mathcal{G}$ will be called strong 1-cotilting when it is 1-cotilting and $\prod_{i \in I} Q$ is in $\mathcal{F} := \text{Cogen}(Q)$, for each set $I$. The corresponding torsion pair is called a strong cotilting torsion pair.

Let $\mathcal{G}$ be a locally finitely presented Grothendieck category in this paragraph. An exact sequence $0 \to X \xrightarrow{u} Y \xrightarrow{p} Z \to 0$ is called pure-exact when it is kept exact when applying the functor $\text{Hom}_{\mathcal{G}}(U, ?)$, for every finitely presented object $U$. An object $E$ of $\mathcal{G}$ is pure-injective when $\text{Hom}_{\mathcal{G}}(?, E)$ preserves the exactness of all pure-exact sequences (see, e.g. [CB] or [FT] for details).

Lemma 5.6. Let $Q$ be a 1-cotilting object of $\mathcal{G}$. The following assertions hold:

1. If $\mathcal{G}$ is AB4* then $Q$ is strong 1-cotilting and the class $\mathcal{F} := \text{Cogen}(Q)$ is generating;

2. If $\mathcal{G}$ is locally finitely presented, then $Q$ is a pure-injective object and $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$. In particular, the equivalence classes of 1-cotilting objects form a set.

3. If there exists a strong 1-cotilting object $Q'$ which is equivalent to $Q$, then $Q$ is itself strong 1-cotilting.

Proof. 1) That $Q$ is strong 1-cotilting is straightforward since $\prod^1$ vanishes when $\mathcal{G}$ is AB4*. In order to prove that $\mathcal{F}$ is generating, it is enough to prove that all injective objects of $\mathcal{G}$ are homomorphic image of objects in $\mathcal{F}$. Indeed, if that is the case and $U$ is any object of $\mathcal{G}$, then fixing an epimorphism $p : F \to E(U)$, with $F \in \mathcal{F}$, and pulling it back along the inclusion $U \hookrightarrow E(U)$, we obtain an epimorphism $F' \to U$, for some $F' \in \mathcal{F}$. But, by the dual of proposition 5.2, we get that each injective cogenerator $E$ is an homomorphic image of an object in $\mathcal{F} = \text{Cogen}(Q)$. The
2) We follow Bazzoni’s argument (see [B]) and see that it also works in our context. First of all, note that an object $Y$ of $\mathcal{G}$ is pure-injective if, and only if, for every set $S$, each morphism $f : Y^S \to Y$ extends to $Y^S$ (cf. [CB] Theorem 1), [FT] Theorem 5.4). Then lemmas 2.1, 2.3 and 2.4, together with corollary 2.2 of [loc. cit] are valid here. We next consider proposition 2.5 in that paper. For it to work in our situation, we just need to check that if $\lambda$ is an infinite cardinal and $(A_\beta)_{\beta \in \lambda^{<\lambda}}$ is a family of $\lambda^{<\lambda}$ subsets of $\lambda$ such that $A_\alpha \cap A_\beta$ is finite, for all $\alpha \neq \beta$, then the images of the compositions $M^{A_\beta} \to M^A \to M^A$ form a family $(f_\beta)_{\beta \in \lambda^{<\lambda}}$ of subobjects of $\prod_{\alpha \in \lambda^{<\lambda}} M^{A_\alpha}$ which have direct sum. Note that this amounts to prove that, for each $\beta \in \lambda^{<\lambda}$, we have $(M^{(\lambda)} + M^{A_\beta}) \cap (M^{(\lambda)} + \sum_{\gamma \neq \beta} M^{A_\gamma}) = M^{(\lambda)}$. By the modular law, which is a consequence of the AB5 condition, we need to prove that $M^{(\lambda)} = [[M^{(\lambda)} + M^{A_\beta}) \cap (\sum_{\gamma \neq \beta} M^{A_\gamma})] = M^{(\lambda)}$. That is, we need to prove that $[(M^{(\lambda)} + M^{A_\beta}) \cap (\sum_{\gamma \neq \beta} M^{A_\gamma})] \subseteq M^{(\lambda)}$.

For simplicity, call an object $X$ of $\mathcal{G}$ finitely generated when it is homomorphic image of a finitely presented one. Clearly, Hom$_G(X, ?)$ preserves direct union of subobjects in that case. Due to the locally finite presented condition of $\mathcal{G}$, each object of this category is a directed union of finitely generated subobjects. Our task reduces to prove that if $X$ is a finitely generated subobject of $[M^{(\lambda)} + M^{A_\beta}) \cap (\sum_{\gamma \neq \beta} M^{A_\gamma})]$, then $X \subseteq M^{(\lambda)}$. To do that, we denote by Supp$(X)$ the set of $\alpha \in \lambda$ such that the composition $X \to M^\lambda \xrightarrow{\pi_\alpha} M$ is nonzero, where $\pi_\alpha : M^\lambda \to M$ is the $\alpha$-projection, for each $\alpha \in \lambda$. Bearing in mind that $\sum_{\gamma \neq \beta} M^{A_\gamma} = \bigcup_{\gamma \in \mathcal{F}} (\sum_{\gamma \in \mathcal{F}} M^{A_\gamma})$, with $\mathcal{F}$ varying on the set of finite subsets of $\lambda \setminus \{\beta\}$; the AB5 condition (see [S] V.1)) gives:

$$X = X \cap (\sum_{\gamma \neq \beta} M^{A_\gamma}) = X \cap \bigcup_{\mathcal{F}} (\sum_{\gamma \in \mathcal{F}} M^{A_\gamma}) = \bigcup_{\mathcal{F}} [X \cap (\sum_{\gamma \in \mathcal{F}} M^{A_\gamma})],$$

and the finitely generated condition of $X$ implies that $X = X \cap (\sum_{\gamma \in \mathcal{F}} M^{A_\gamma}) \subseteq \sum_{\gamma \in \mathcal{F}} M^{A_\gamma}$, for some $\mathcal{F} \subseteq \lambda \setminus \{\beta\}$ finite. As a consequence, we have Supp$(X) \subseteq \bigcup_{\gamma \in \mathcal{F}} A_\gamma$.

On the other hand, exactness of direct limits gives that $M^{(\lambda)} + M^{A_\beta} = \bigcup_{\mathcal{F} \subseteq \lambda, \mathcal{F} \text{ finite}} [M^{(\mathcal{F})} + M^{A_\beta}]$ and, again by the AB5 condition and the finitely generated condition of $X$, we get that $X \subseteq M^{(\mathcal{F})} + M^{A_\beta}$, for some finite subset $\mathcal{F} \subseteq \lambda$. This implies that Supp$(X) \subseteq \mathcal{F} \cup A_\beta$. Together with the conclusion of the previous paragraph, we then get that Supp$(X) \subseteq (\mathcal{F} \cup A_\beta) \cap (\bigcup_{\gamma \in \mathcal{F}} A_\gamma)$, and so Supp$(X)$ is a finite set and $X \subseteq M^{(\lambda)}$.

The previous two paragraphs show that proposition 2.5 and corollary 2.6 of [B] go on in our context. To complete Bazzoni’s argument in our situation, it remains to check the truth of her lemma 2.7. This amount to prove that if $0 \neq X \subseteq M^{(\lambda)}$ is a finitely generated subobject, then there exists a morphism $f : M^{A_\lambda} \to Q$ such that $f(X) \neq 0$. Indeed, take the subobject $\bar{X}$ of $M^\lambda$ such that $X = \frac{\bar{X}}{\text{Ann}(\bar{X})}$. Then Supp$(\bar{X})$ is an infinite subset of $\lambda$, and this allows us to fix a subset $A \subseteq \text{Supp}(\bar{X})$ such that $|A| = n_0$. If now $p : M^\lambda \to M^A$ is the canonical projection, then we get an induced morphism $\bar{p} : M^{A_\lambda} \to M^A$ such that $\bar{p}(X) \neq 0$. Since $\frac{M^A}{M^{(\lambda)}} \in \text{Cogen}(Q)$ we get a morphism $h : M^{A_\lambda} \to Q$ such that $h(\bar{p}(X)) \neq 0$. We take $f = h \circ \bar{p}$ and have $f(X) \neq 0$, as desired. Therefore $Q$ is pure-injective.

Finally, if $(E_i)_{i \in I}$ is a direct system in $\mathcal{F} = \text{Ker}(\text{Ext}^1_Y(?, Q))$ then the induced sequence $0 \to K \to \prod_{i \in I} F_i \xrightarrow{\prod} \lim_{i \in I} F_i \to 0$ is pure-exact. The fact that $\mathcal{F} = \lim \mathcal{F}$ follows, as in module categories, by applying to this sequence the long exact sequence of Ext$(?, Q)$. Moreover, equivalence classes of 1-cotilting objects are in bijection with the cotilting torsion pairs. Then lemma 1.5.2 applies.

3) For any set $J$, the functor $\prod_{i \in I} : [I, \mathcal{G}] \to \mathcal{G}$ is additive. This and the fact that $\mathcal{F} = \text{Cogen}(Q)$ is closed under direct summands imply that the class of objects $X$ such that $\prod_{i \in I} F_i \to \lim_{i \in I} F_i \to 0$ is pure-exact. The fact that $\mathcal{F} = \lim \mathcal{F}$ follows, as in module categories, by applying to this sequence the long exact sequence of Ext$(?, Q)$. Moreover, equivalence classes of 1-cotilting objects are in bijection with the cotilting torsion pairs. Then lemma 1.5.2 applies.

Finally, if $(E_i)_{i \in I}$ is a direct system in $\mathcal{F} = \text{Ker}(\text{Ext}^1_Y(?, Q))$ then the induced sequence $0 \to K \to \prod_{i \in I} F_i \xrightarrow{\prod} \lim_{i \in I} F_i \to 0$ is pure-exact. The fact that $\mathcal{F} = \lim \mathcal{F}$ follows, as in module categories, by applying to this sequence the long exact sequence of Ext$(?, Q)$. Moreover, equivalence classes of 1-cotilting objects are in bijection with the cotilting torsion pairs. Then lemma 1.5.2 applies.
For each set $I$, the product functor $\prod : [I, \mathcal{G}] \to \mathcal{G}$ preserves pullbacks since it is left exact. It also preserves split short exact sequences. It follows that the central square of the following commutative diagram is bicartesian since the cokernels of its two vertical arrows are isomorphic:

\[
\begin{array}{c}
0 \longrightarrow Q^J \longrightarrow E(Q^J) \longrightarrow \frac{E(Q^J)}{Q^J} \longrightarrow 0 \\
0 \longrightarrow Q^J \longrightarrow E(Q^J) \longrightarrow \frac{E(Q^J)}{Q^J} \longrightarrow 0 \\
\end{array}
\]

Due to the left exactness of the product functor, the second vertical arrow from right to left is a monomorphism. \(\therefore\) $T \subseteq \{0\}$ is a generating class. Consider the following assertions:

1. $T \subseteq \{0\}$ is a generating class.
2. $\mathcal{F}$ is closed under taking direct limits in \(\mathcal{G}\);
3. $t$ is a (strong) cotilting torsion pair.

Then the implications $1) \iff 2) \implies 3)$ hold. When \(\mathcal{G}\) is locally finitely presented, all assertions are equivalent.

**Proof.** Let $(F_i)_{i \in I}$ be a family in $\mathcal{F}$. Note that, in order to calculate the product $\prod_{D(\mathcal{G})} F_i[1]$ in $D(\mathcal{G})$, we first replace each $F_i$ by an injective resolution, which we assume to be the minimal one, and then take products in $\mathcal{C}(\mathcal{G})$. When $\mathcal{G}$ is not AB4*, the resulting complex can have nonzero homology in degrees $> 0$. However $\prod_{D(\mathcal{G})} F_i[1]$ is $U_t^+$, and using lemma 3.1, we easily see that $P := \prod_{\mathcal{H}_t} F_i[1] = \tau_t(\prod_{D(\mathcal{G})} F_i[1])$. Due to the fact that $D^{>0}(\mathcal{G}) \subseteq U_t^+$, we have a canonical isomorphism $P \cong \tau_t(\tau_t^{\leq 0}(\prod_{D(\mathcal{G})} F_i[1]))$, where $\tau_t^{\leq 0}$ denotes the left truncation with respect to the canonical t-structure $(D^{\leq 0}(\mathcal{G}); D^{>0}(\mathcal{G}))$. But $\tau_t^{\leq 0}(\prod_{D(\mathcal{G})} F_i[1])$ is quasi-isomorphic to the complex

\[
\cdots \longrightarrow 0 \longrightarrow \prod_{i \in I} E(F_i) \xrightarrow{\text{can}} \prod_{i \in I} \frac{E(F_i)}{F_i} \longrightarrow 0 \longrightarrow \cdots
\]
concentrated in degrees $-1$ and $0$. It follows easily that $H^{-1}(P) \cong \prod_{i \in I} F_i$ and $H^0(P) \cong t(\prod_{i \in I} F_i)$.

1) $\iff$ 2) is a direct consequence of corollary 4.10.

1), 2) $\implies$ 3) By the proofs of corollary 4.10 and theorem 4.9 we know that $\mathcal{F}[1]$ cogenerates $\mathcal{H}_t$. Then any injective cogenerator of $\mathcal{H}_t$ is of the form $Q[1]$, for some $Q \in \mathcal{F}$. Fixing such a $Q$, we get that $Q[1]^S$ is an injective cogenerator of $\mathcal{H}_t$, for each set $S$. This in turn implies that $Q[1]^{\mathcal{H}_t} \in \mathcal{F}[1]$. By the initial paragraph of this proof, we get that $t(\prod_{i \in S} Q) = 0$ which implies that $Q[1]^S \cong Q^S[1]$. From this we immediately derive that $\mathcal{F} = \text{Copres}(Q) = \text{Cogen}(Q)$.

We fix an object $Q \in \mathcal{F}$ such that $Q[1]$ is an injective cogenerator of $\mathcal{H}_t$ and pass to prove that $Q$ is 1-cotilting. The equality $t(\prod_{i \in S} Q) = 0$ proved above will give that $Q$ is strong 1-cotilting and the proof of this implication will be finished. First, the injectivity of $Q[1]$ in $\mathcal{H}_t$ implies that $\text{Ext}_1^Q(F, Q) \cong \text{Ext}_{\mathcal{H}_t}^1(F[1], Q[1]) = 0$. From this equality we derive that $\text{Cogen}(Q) = \mathcal{F} \subseteq \text{Ker}(\text{Ext}_1^Q(?, Q))$.

Let now $Z$ be any object in $\text{Ker}(\text{Ext}_1^Q(?, Q))$. The generating condition of $\mathcal{F}$ gives us an epimorphism $p : F \to Z$, with $F \in \mathcal{F}$. Putting $F' := \text{Ker}(p)$, we then get the following commutative diagram, where the upper right square is bicartesian:

If we apply the long exact sequence of $\text{Ext}(?, Q)$ to the central row and the central column of the last diagram, we get the following commutative diagram with exact rows:

It follows that $\alpha : \text{Ext}_2^0(Z, Q) \to \text{Ext}_2^0(Z, Q)$ is a monomorphism. If we now apply the long exact sequence of $\text{Ext}$ to the right column of the diagram $(*)$ above, we obtain that the canonical morphism $\text{Ext}_1^0(Z, Q) \to \text{Ext}_1^0(t(Z), Q)$ is an epimorphism, which implies that $\text{Ext}_1^0(t(Z), Q) = 0$ due to the choice of $Z$. It follows from this that $\text{Hom}_{\mathcal{H}_t}(t(Z)[0], Q[1]) = 0$, which implies that $t(Z) = 0$ since $Q[1]$ is a cogenerator of $\mathcal{H}_t$. We then get $\text{Ker}(\text{Ext}_1^0(?, Q)) \subseteq \mathcal{F} = \text{Cogen}(Q)$ and, hence, this last inclusion is an equality.

3) $\implies$ 2) (assuming that $\mathcal{G}$ is locally finitely presented) follows directly from lemma 5.6(2).

**Remarks 5.8.**

1. When $\mathcal{G}$ is locally finitely presented, by proposition 5.7 and lemma 5.6(3), we know that if $Q$ is a 1-cotilting object such that $\mathcal{F} = \text{Cogen}(Q)$ is a generating class of $\mathcal{G}$, then $Q$ is strong 1-cotilting.

2. When $\mathcal{G}$ is $\text{AB}^3*$, it follows from lemma 5.2 and proposition 7.7 that the following assertions are equivalent for a torsion pair $t = (T, \mathcal{F})$:

   (a) $\mathcal{F}$ is generating and closed under taking direct limits in $\mathcal{G}$

   (b) $t$ is a strong cotilting torsion pair such that $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$.

The following direct consequence of proposition 5.7 extends [CMT] Corollary 6.3.

**Corollary 5.9.** Let $V$ be a 1-tilting object such that $\mathcal{F} = \text{Ker}(\text{Hom}_\mathcal{G}(V, ?))$ is closed under taking direct limits in $\mathcal{G}$ (e.g., when $V$ is self-small). If $\mathcal{F}$ is a generating class, then the torsion pair $t = (\text{Gen}(V), \text{Ker}(\text{Hom}_\mathcal{G}(V, ?)))$ is strong cotilting.
We now make explicit what proposition 5.7 says in case $G$ is locally finitely presented and AB4* (see lemma 5.6):

**Corollary 5.10.** Let $G$ be locally finitely presented and AB4* and let $t = (T,F)$ be a torsion pair in $G$. The following assertions are equivalent:

1. $F$ is a generating class and the heart $H_t$ is a Grothendieck category;
2. $F$ is a generating class closed under taking direct limits in $G$;
3. $t$ is a cotilting torsion pair.

**Remarks 5.11.**
1. The last corollary extends the main result of [CG] (see [A] Theorem 6.2]).
2. All Grothendieck categories with enough projectives are $AB4^*$, but the converse is not true and there even exist $AB4^*$ Grothendieck categories with no nonzero projective objects (see [Roo] Theorem 4.1]).

In the particular case when $G = R - \text{Mod}$, for a ring $R$, a torsion pair $t$ has the property that $F$ is generating if, and only if, $t$ is faithful. That is, if and only if, $R \in F$. In a sense, Bazzoni’s result (see [B] Theorem 2.8]) states that if $t$ is a cotilting torsion pair in $R - \text{Mod}$ then its torsionfree class is closed under taking direct limits (and $t$ is faithful). By corollary 5.10 we also have the converse, which, as Silvana Bazzoni pointed out to us, can be also deduced from [GT] Corollary 8.1.10:  

**Corollary 5.12.** Let $R$ be a ring. A torsion pair $t = (T,F)$ in $R - \text{Mod}$ is cotilting if, and only if, it is is faithful and $F$ is closed under taking direct limits.

Recall that a Grothendieck category is called locally noetherian when it has a set of noetherian generators. The following result extends [BK] Theorem 1] (see lemma 5.6):

**Corollary 5.13.** Let $G$ be a locally finitely presented Grothendieck category which is locally noetherian and denote by $fp(G)$ its full subcategory of finitely presented (=noetherian) objects. There is a one-to-one correspondence between:

1. The torsion pairs $(X,Y)$ of $fp(G)$ such that $Y$ contains a set of generators;
2. The equivalence classes of 1-cotilting objects $Q$ of $G$ such that $\text{Cogen}(Q)$ is a generating class. When, in addition, $G$ is an $AB4^*$ category, they are also in bijection with
3. The equivalence classes of 1-cotilting objects of $G$.

The map from 1 to 2 takes $(X,Y)$ to the equivalence class $[Q]$, where $Q$ is a 1-cotilting object such that $\text{Cogen}(Q) = \{ F \in \text{Ob}(G) : \text{Hom}_G(X,F) = 0, \text{ for all } X \in X \}$. The map from 2 to 1 takes $[Q]$ to $(\text{Ker}(\text{Hom}_Q(?,Q)) \cap fp(G), \text{Cogen}(Q) \cap fp(G))$.

**Proof.** By lemma 5.6 when $G$ is $AB4^*$, the classes in 2) and 3) are the same. We then prove the bijection between 1) and 2). Given a torsion pair $(X,Y)$ in $fp(G)$ as in 1), by [CB] Lemma 4.4, we know that the torsion pair in $G$ generated by $X$ is $t = (T,F) = (\lim X, \lim Y)$. It follows from proposition 5.7 that $t$ is a cotilting torsion pair. We then get a 1-cotilting object $Q$, uniquely determined up to equivalence, such that $\text{Cogen}(Q) = \lim Y = \{ F \in G : \text{Hom}_G(X,F) = 0, \text{ for all } X \in X \}$.

Suppose now that $Q$ is any 1-cotilting object and its associated torsion pair $t = (T,F)$ has the property that $F$ is a generating class. Then $(X,Y) := (T \cap \text{fp}(G), F \cap \text{fp}(G))$ is a torsion pair in $\text{fp}(G)$. We claim that $Y$ contains a set of generators. Indeed, by hypothesis $F$ contains a generator $G$ of $G$. By the locally noetherian condition of $G$, we know that $G$ is the direct union of its noetherian (=finitely presented) subobjects. Then the finitely presented subobjects of $G$ form a set of generators of $G$ which is in $Y$, thus settling our claim.

On the other hand, in the situation of last paragraph, we have that $(\lim X, \lim Y)$ is a torsion pair in $G$ such that $\lim X \subseteq T$ and $\lim Y \subseteq F$. Then these inclusions are equalities and, hence, $t$ is the image of $(X,Y)$ by the map from 1 to 2 defined in the first paragraph of this proof. That the two maps, from 1 to 2 and from 2 to 1, are mutually inverse is then a straightforward consequence of this. 


Examples 5.14. The following are examples of locally finitely presented Grothendieck categories. So proposition [5.7] and corollary [5.10] apply to them.

1. Each category of additive functors $A \to Ab$, for every skeletally small additive category $A$. Equivalently (see [G, Proposition II.2]), each category $R-\text{Mod}$ of unitary modules over a ring $R$ with enough idempotents.

2. The category $\text{Qcoh}(X)$ of quasi-coherent sheaves over any quasi-compact and quasi-separated algebraic scheme ([GD, I.6.9.12]). When, in addition, $X$ is locally Noetherian, corollary [5.13] also applies to $\mathcal{G} = \text{Qcoh}(X)$.

3. Each quotient category $\mathcal{G}/T$, where $\mathcal{G}$ is a locally finitely presented Grothendieck category and $T$ is a hereditary torsion class in $\mathcal{G}$ generated by finitely presented objects ([ES, Proposition 2.4]).

We end the paper with the following question:

Question 5.15. Let $\mathcal{G}$ be a locally finitely presented Grothendieck category and $Q$ be a 1-cotilting object. Is $\mathcal{F} = \text{Cogen}(Q)$ a generating class of $\mathcal{G}$?

References

[AJS] L. Alonso, A. Jeremías, M. Saorín, Compactly generated $t$-structures on the derived category of a Noetherian ring. J. Algebra 324(4) (2010), 313-346.

[AJSO] L. Alonso, A. Jeremías, M.J. Souto, Construction of $t$-structures and equivalences of derived categories. Trans. AMS 355 (2003), 2523-2543.

[AR] J. Adámek, J. Rosický, Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, 189, Cambridge University Press (1994).

[B] S. Bazzoni, Cotilting modules are pure-injective. Proceed. AMS 131(12), 3665-3672.

[BBD] A. Beilinson, J. Bernstein, P. Deligne, “Faisceaux Pervers”. Analysis and topology on singular spaces, I, Luminy 1981. Astérisque 100, Soc. Math. France, Paris, (1982), 5-171.

[Br] T. Bridgeland, Stability conditions on triangulated categories. Annals of Math. 166 (2007), 317-345.

[BK] A.B. Buan, H. Krause, Cotilting modules over tame hereditary algebras. Pacific J. Math 211(1) (2003), 41-59.

[C] R. Colpi, Tilting in Grothendieck categories. Forum Math. 11 (1999), 735-759.

[CG] R. Colpi, E. Gregorio, The Heart of cotilting theory pair is a Grothendieck category. Preprint.

[CGM] R. Colpi, E. Gregorio, F. Mantese, On the Heart of a faithful torsion theory. J. Algebra 307 (2007), 841-863.

[CMT] R. Colpi, F. Mantese, A. Tonolo, When the heart of a faithful torsion pair is a module category. J. Pure and Appl. Algebra 215 (2011) 2923-2936.

[CT] R. Colpi, J. Trlifaj, Tilting modules and tilting torsion pairs. J. Algebra 178(2) (1995), 614-634.

[CBO] W. Crawley-Boevey, Locally finitely presented additive categories. Comm. Algebra 22(5) (1994), 1641-1674.

[ES] S. Estrada, M. Saorín, Locally finitely presented categories with no flat objects. Forum Math. DOI:10.1515/forum-2012-0054 (2012).

[G] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323-448.
[GT] R. Göbel, J. Trlifaj, *Approximations and endomorphism algebras of modules*. De Gruyter Expos. Maths **41** (2006).

[GKR] A.L. Gorodentsev, S.A. Kuleshov, A.N. Rudakov, *t-stabilities and t-structures on triangulated categories*. Izvest. RAN Ser. Math. **68**(4) (2004), 117-150.

[Gr] A. Grothendieck, *Sur quelques points d’Algèbre Homologique*. Tohoku Math. J. **9**(2) (1957), 119-221.

[GD] A. Grothendieck, J.A. Dieudonné, *Elements de Géometrie Algébrique I*. Grundlehren Math. Wiss **166**, 1971.

[HRS] D. Happel, I. Reiten, S.O. SmallØ, *Tilting in abelian categories and quasitilted algebras*. Memoirs AMS **120** (1996).

[Ke2] B. Keller, *Introduction to abelian and derived categories*, in 'Representations of reductive groups', edts W. Carter and M. Geck. Cambridge Univ. Press (1998), 41-62.

[KN] B. Keller, P. Nicolás, *Weight structures and simple dg modules for positive dg algebras*. Int. Math. Res. Notes **5** (2013), 1028-1078.

[ML] S. Mac Lane, *Homology*, 3rd edition. Springer-Verlag (1975).

[M] F. Mattiello, *On the heart associated to a faithful torsion pair*. Available on www.algant.eu/documents/theses/mattiello.pdf, (2011).

[MT] F. Mantese, A. Tonolo, *On the heart associated with a torsion pair*. Topology and Appl., **159** (2012), 2483-2489.

[N] A. Neeman, *Triangulated categories*. Ann. Math. Studies. Princeton University Press, **148** (2001).

[P] B. Pareigis, *Categories and functors*. Academic Press (1970).

[PS] C. Parra, M. Saorín, *Hearts of t-structures which are Grothendieck categories*. In preparation.

[Po] N. Popescu, *Abelian categories with applications to rings and modules*. London Math. Soc. Monogr **3**, Academic Press (1973).

[Pr] M. Prest, *Definable additive categories: purity and model theory*. Memoirs AMS **987** (2011).

[Roo] J.E. Roos, *Derived functors of inverse limits revisited*. J. London Math. Soc. **73**(2) (2006), 65-83.

[SS] M.J. Souto, S. Trepode, *t-structures on the bounded derived category of the Kronecker algebra*. Appl. Categ. Struct. **20**(5), 513-529.

[S] B. Stenström, *Rings of quotients*. Grundlehren der math. Wissensch., **217**, Springer-Verlag, (1975).

[V] L. Verdier, *Des catégories dérivées des catégories abéliennes*. Asterisque **239**, Soc. Math. France (1996).