Diffusion laws, path information and action principle

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Abstract

This is an attempt to address diffusion phenomena from the point of view of information theory. We imagine a regular hamiltonian system under the random perturbation of thermal (molecular) noise and chaotic instability. The irregularity of the random process produced in this way is taken into account via the dynamic uncertainty measured by a path information associated with different transition paths between two points in phase space. According to the result of our previous work, this dynamic system maximizes this uncertainty in order to follow the action principle of mechanics. In this work, this methodology is applied to particle diffusion in external potential field. By using the exponential probability distribution of action (least action distribution) yielded by maximum path information, a derivation of Fokker-Planck equation, Fick’s laws and Ohm’s law for normal diffusion is given without additional assumptions about the nature of the process. This result suggests that, for irregular dynamics, the method of maximum path information, instead of the least action principle for regular dynamics, should be used in order to obtain the correct occurring probability of different paths of transport. Nevertheless, the action principle is present in this formalism of stochastic mechanics because the average action has a stationary associated with the dynamic uncertainty. The limits of validity of this work is discussed.

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1 Introduction

Diffusion is a mechanism by which components of a mixture are transported around the mixture by means of random molecular motion. Over 200 years ago, Bertholot postulated[1] that the flow of mass by diffusion across a plane was proportional to the concentration gradient of the diffusant across that plane. About 50 years later, Fick introduced[2] two differential equations that quantified the above statement for the case of transport through thin membranes. Fick’s First Law states that the flux $J$ of a component of concentration $n$ across a membrane is proportional to the concentration gradient in the membrane:

$$J(x) = -D \frac{\partial n(x)}{\partial x}$$  \hspace{1cm} (1)

where $x$ is the position variable (for 3 dimensional space, $\frac{\partial}{\partial x}$ is replaced by the gradient $\nabla$). Fick’s Second Law states that the rate of time change of concentration of diffusant at a point is proportional to the rate of spacial change of concentration gradient at that point within the mixture

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x} \left[ D \frac{\partial n(x)}{\partial x} \right].$$  \hspace{1cm} (2)

If $D$ is constant everywhere in the mixture, the above equation becomes

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n(x)}{\partial x^2}.$$

Above normal diffusion laws are very precisely tested in experiments in most of solids, liquids and gases and are widely studied in nonequilibrium thermostatistics together with the Fokker-Planck equation of diffusion probability, Fourier law of heat conduction and Ohm’s law of electrical charge conduction[3, 4, 5, 6]. Many efforts to derive theoretically Fick’s laws were concentrated on special models of solids in which particles are transported. Other phenomenological derivations are also possible if one supposes Brownian motion and Markovian process[7], or Kolmogorov conditions[8]. Although diffusion is associated with stochastic process with certain dynamic uncertainty, theoretical study and derivation of these laws using information methods, as far as we know, is still an open question.

In this paper, we describe an attempt to investigate irregular dynamics by an information method. We imagine a hamiltonian systems under random perturbation of thermal noise and chaotic motion. This perturbation can be either internal or external. A isolated system containing a large number of
particles in random motion is an example of internal perturbation. A Brownian particle is an example of external perturbation due to random molecular motion around the particle. In these cases, the energy of the system and other mechanical quantities can fluctuate. The geodesics of regular dynamics will be deformed in such a stochastic way that following exactly the evolution of each mechanical quantity (e.g., action) is practically inconceivable. Hence in this approach, we use the action of the unperturbed hamiltonian system. The stochastic effect of random perturbation is taken into account through the dynamic uncertainty or information. Based on maximum information principle in connection with the unperturbed action, we developed a formalism of probabilistic dynamics within which the diffusion laws are a natural result of the differential equations of the probability distribution of action. The information addressed in this work is a quantification of the uncertainty of irregular dynamic process. Here we consider the following two dynamic uncertainties:

1. Between any two fixed cells a and b in phase space, there may be different possible paths (labelled by \( k=1,2,\ldots,w \)) each having a probability \( p_k(b|a) \) to be followed by the system. This is the uncertainty considered by Feynman in his formulation of quantum mechanics\[13\]. Here we introduce it for randomly perturbed dynamic systems.

2. There are different possible paths leaving the cell \( a \) and leading to different final cells \( b \) in a final phase volume \( B \), each having a probability \( p_k(x|a) \) to be followed by the system, where \( x \) is the position of arbitrary \( b \). This uncertainty is the basic consideration for the definition of Kolmogorov-Sinai entropy\[10\] used to describe chaotic systems.

These dynamic uncertainties were proved\[9,11\] to take their maximum when the most probable paths are just the paths of least action. In what follows, we briefly present the method of maximum uncertainty or path information for irregular dynamic systems and the concomitant “least action distributions” of transition probability. Then we focus on the particles diffusing in external field with potential energy \( U(x) \). The Hamiltonian of the particles is given by \( H = E + U \) where \( E \) is the kinetic energy. The above mentioned diffusion laws will be derived directly from the so called least action distribution.
2 Maximum path information

We suppose that the uncertainty concerning the choice of paths by the systems between two fixed cells is measured with the following path information

\[ H_{ab} = - \sum_{k=1}^{w} p_k(b|a) \ln p_k(b|a). \]  

(3)

where \( p_k(b|a) \) is the transition probability between the cell \( a \) in the initial phase volume \( A \) and a cell \( b \) in the final phase volume \( B \) via a path \( k \) \((k = 1, 2, \ldots, w)\). We have the following normalization

\[ \sum_{k=1}^{w} p_k(b|a) = 1. \]  

(4)

It is also supposed that each path is characterized by its action \( A_{ab}(k) \) defined in the same way as in regular dynamics without random forces by

\[ A_{ab}(k) = \int_{t_{ab}(k)}^{t} L_k(t) dt \]  

(5)

where \( L_k(t) = E - U \) is the Lagrangian of the system at time \( t \) along the path \( k \). The average action is given by

\[ A_{ab} = \sum_{k=1}^{w} p_k(b|a) A_{ab}(k). \]  

(6)

It is known that, for a regular process, the trajectories of the system should have a stationary action \((\delta A_{ab}(k) = 0)\). For irregular dynamic process, this least action principle does not apply since there are many possible paths with different actions, not only the minimum action. For this kind of process, we suppose the stable probability distribution corresponds to a stationary uncertainty or path information \( H_{ab} \) under the constraint associated with action. This means the following operation:

\[ \delta[H_{ab} + \alpha \sum_{k=1}^{w} p_k(b|a) - \eta \sum_{k=1}^{w} p_k(b|a) A_{ab}(k)] = 0 \]  

(7)

leading to

\[ p_k(b|a) = \frac{1}{Z} \exp[-\eta A_{ab}(k)], \]  

(8)
where the partition function

\[ Z = \sum_k \exp[-\eta A_{ab}(k)]. \]  

(9)

The physical meaning of \( \eta \) will be discussed in the following section.

It is proved that the distribution Eq. (8) is stable with respect to the fluctuation of action. It is also proved that Eq. (8) is a least (stationary) action distribution, i.e., the most probable paths are just the paths of least action. We can write

\[ \delta p_k(b|a) = -\eta p_k(b|a) \delta A_{ab}(k) = 0, \]

which means \( \delta A_{ab}(k) = 0 \) leading to Euler-Lagrange equation

\[ \frac{\partial}{\partial t} \frac{\partial L_k(t)}{\partial \dot{x}} - \frac{\partial L_k(t)}{\partial x} = 0 \]  

(10)

and to Hamiltonian equations via the Legendre transformation \( H = P\dot{x} - L_k(t) \):

\[ \dot{x} = \frac{\partial H}{\partial P} \text{ and } \dot{P} = -\frac{\partial H}{\partial x}, \]  

(11)

where \( P = m\dot{x} \) is the momenta of the system. These equations are satisfied by the most probable paths but not the other paths. In general, the paths have neither \( \delta A_{ab}(k) = 0 \) nor \( \delta A_{ab} = 0 \), although the average action \( A_{ab} \) is convex (concave) for \( \eta > 0 \) (\( \eta < 0 \)). Nevertheless, the average action does have a stationary associated with the stationary of the information \( H_{ab} \) in Eq. (7), i.e.,

\[ -\eta \delta A_{ab} + \delta H_{ab} = 0. \]  

(12)

Here we used \( \sum_{k=1}^{w} \delta p_k(b|a) = 0 \). Eq. (12) implies that, although the least action principle of classical mechanics cannot apply when the dynamics is perturbed by random and unstable noise, the maximum path information introduced above underlies in fact the same physics in which the action principle is present as a average effect in association with the stationary dynamic uncertainty. Eq. (12) will be used below in order to derive the averaged Euler-Lagrange equation for irregular dynamics.

### 3 A calculation of transition probability of diffusion

In what follows, the action is calculated and analyzed with the Euler method. Let us look at a particle of mass \( m \) diffusing along a given path \( k \) from a
cell $a$ to a cell $b$ of its phase space. The path is cut into $N$ infinitesimally
small segments each having a spatial length $\Delta x_i = x_i - x_{i-1}$ with $i = 1,...,N$
($x_0 = x_a$ and $x_N = x_b$). $t = t_i - t_{i-1}$ is the time interval spent by the system
on every segment. The Lagrangian on the segment $i$ is given by

$$L(x_i, \dot{x}_i, t) = \frac{m(x_i - x_{i-1})^2}{2(t_i - t_{i-1})^2} - \left( \frac{\partial U}{\partial x} \right)_i \frac{(x_i - x_{i-1})}{2} - U(x_{i-1})$$  \tag{13}$$

where the first term on the right hand side is the kinetic energy of the particle,
the second and the third are the average potential energy on the segment $i$. The action of segment $i$ is given by

$$A_i = \frac{m(\Delta x_i)^2}{2t} + F_i \frac{\Delta x_i}{2} t - U(x_{i-1}) t,$$  \tag{14}$$

where $F_i = -\left( \frac{\partial U}{\partial x} \right)_i$ is the force on the segment $i$. According to Eq.(8), the
transition probability $p_{i/i-1}$ from $x_{i-1}$ to $x_i$ on the path $k$ is given by

$$p_{i,k} = \frac{1}{Z_i} \exp \left( -\eta \left[ \frac{m}{2t} \Delta x_i^2 + F_i \frac{t}{2} \Delta x_i \right]_k \right)$$  \tag{15}$$

where $Z_i$ is calculated as follows

$$Z_i = \int_{-\infty}^{\infty} dx_i \exp \left( -\eta \left[ \frac{m}{2t} \Delta x_i^2 + F_i \frac{t}{2} \Delta x_i \right]_k \right)$$  \tag{16}$$

= \exp \left[ F_i^2 \frac{\eta^3}{8m} \right] \sqrt{\frac{2\pi t}{mn}}.$$

The potential energy of the point $x_{i-1}$ disappears in the expression of $p_{i,k}$
because it does not depend on $x_i$.

At this stage, the physical meaning of $\eta$ can be shown with a general
relationship given by the following calculation of the variance :

$$\overline{\Delta x_i^2} = \frac{1}{Z_i} \int_{-\infty}^{\infty} dx_i \Delta x_i^2 \exp \left( -\eta \left[ \frac{m}{2t} \Delta x_i^2 + F_i \frac{t}{2} \Delta x_i \right]_k \right)$$  \tag{17}$$

= \frac{t}{m\eta}.$$

This result can be compared to Brownian motion having $\overline{\Delta x_i^2} = 4Dt$. In this case we have $\eta = \frac{1}{2mD}$. If we still consider detailed balance, we
get \( D = \mu k_B T \) where \( \mu \) is the mobility of the diffusing particles, \( k_B \) the Boltzmann constant and \( T \) the temperature. This means

\[
\eta = \frac{1}{2m\mu k_B T} = \frac{\gamma}{2k_B T}
\]

(18)

where \( m\gamma = \frac{1}{\mu} \) is the friction constant of the particles in the diffusion mixture.

The total action given by

\[
A_{ab}(k) = \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} \left[ \frac{m(\Delta x_i)^2}{2t} + F_i \frac{t}{2} \Delta x_i - U(x_{i-1}) \right]_k
\]

(19)

so the transition probability from \( a \) to \( b \) via the path \( k \) is the following:

\[
p_k(b|a) = \frac{1}{Z} \exp \left(-\eta \sum_{i=1}^{N} \left[ \frac{m(\Delta x_i)^2}{2t} + F_i \frac{t}{2} \Delta x_i \right]_k \right)
\]

(20)

\[
= p(b|a)^{-1} \prod_{i=1}^{N} p_{k,i/i-1}
\]

where

\[
Z = \sum_{k=1}^{w} \exp \left(-\eta \sum_{i=1}^{N} \left[ \frac{m(\Delta x_i)^2}{2t} + F_i \frac{t}{2} \Delta x_i \right]_k \right)
\]

(21)

\[
= \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_{N-1} \exp \left(-\eta \sum_{i=1}^{N} \left[ \frac{m(x_i - x_{i-1})^2}{2t} + F_i \frac{t}{2} (x_i - x_{i-1}) \right]_k \right)
\]

\[
= \left( \exp \left[ F_i^2 \eta t^3 \right] \frac{2\pi t}{8m} \eta \right)^N p(b|a) = Z_i^N p(b|a)
\]

and

\[
p(b|a) = \exp \left[ F_i^2 \eta (t_b - t_a)^3 \frac{\sqrt{m\eta}}{8\pi (t_b - t_a)} \right] \prod_{i=1}^{N} \exp \left(-\eta \left[ \frac{m(x_i - x_{i-1})^2}{2(t_b - t_a)} + F_i \frac{t}{2} (x_i - x_{i-1}) \right]_k \right)
\]

(22)

In the above calculation, we have fixed the initial point \( x_0 = x_a \) and the final point \( x_N = x_b \).

Now in order to see the behavior of transition probability with respect to final point, we have to relax \( x_b = x \) and let it vary arbitrarily as other
intermediate points. This implies we take into account the second uncertainty due to chaotic motion mentioned in the introduction. The corresponding transition probability \( p_k(x|a) \) from \( a \) to arbitrary \( x \) via the path \( k \) has been derived with the maximum path information combined with action\[11\]. Here we only put it as follows:

\[
p_k(x|a) = p(b|a)p_k(b|a) = \prod_{i=1}^{N} p_{k,i|i-1},
\]

which is normalized by

\[
\sum_b \sum_{k=1}^{w} p_k(x|a) = \int dx_1 dx_2 ... dx_{N-1} dx p_k(x|a) = 1.
\] (24)

4 A derivation of the Fokker-Planck equation

The Fokker-Planck equation describes the time evolution of the probability density function of position and velocity of a particle. This equation can be derived if we suppose that the diffusion particles follow Brownian motion and Markovian process\[7\], or Kolmogorov conditions\[8\]. In what follows, it is shown that this equation can be derived in a quite general way without above assumptions. It is just the differential equation satisfied by the least action distributions given by Eq.(15) and Eq.(23).

The calculation of the derivatives \( \frac{\partial p_{i|i-1}}{\partial t} \), \( \frac{\partial F_i p_{i|i-1}}{\partial x_i} \), and \( \frac{\partial^2 p_{i|i-1}}{\partial x_i^2} \) straightforwardly leads to

\[
\frac{\partial p_{i|i-1}}{\partial t} = -\frac{\tau}{m} \frac{\partial (F_i p_{i|i-1})}{\partial x_i} + \frac{1}{2m\eta} \frac{\partial^2 p_{i|i-1}}{\partial x_i^2}.
\] (25)

This is the Fokker-Planck equation, where \( \tau \) is the mean free time supposed to be the time interval \( t \) of the particle on each segment of its path. In view of the Eq.(23), it is easy to show that this equation is also satisfied by \( p_k(x|a) \) if \( x_i \) is replaced by \( x \), the final position. Now let \( n_a \) and \( n_b \) be the particle density at \( a \) and \( b \), respectively. The following relationship holds

\[
n_b = \sum_k n_a p_k(x|a)
\] (26)
which is valid for any \( n_a \). This means \((n = n_b \text{ and } x = x_b)\):

\[
\frac{\partial n}{\partial t} = -\frac{\tau}{m} \frac{\partial (F n)}{\partial x} + \frac{1}{2m\eta} \frac{\partial^2 n}{\partial x^2},
\]

(27)

which describes the time evolution of the particle density.

5 \textbf{Fick’s laws of diffusion}

If the external force \( F \) is zero, we get

\[
\frac{\partial n}{\partial t} = \frac{1}{2m\eta} \frac{\partial^2 n}{\partial x^2}
\]

(28)

This is the second Fick’s law of diffusion. The first Fick’s law Eq.(11) can be easily derived if we consider matter conservation \( \frac{\partial n(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \) where \( J(x,t) \) is the flux of the particle flow.

6 \textbf{Ohm’s law of charge conduction}

Considering the charge conservation \( \frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} \), where \( \rho(x,t) = qn(x,t) \) is the charge density, \( j(x,t) = qJ(x,t) \) is the flux of electrical currant and \( q \) is the charge of the currant carriers, we have, from Eq.(27),

\[
j = \frac{\tau}{m} F \rho - \frac{1}{2m\eta} \frac{\partial \rho}{\partial x}
\]

(29)

where \( F = qE \) is the electrostatic force on the carriers and \( E \) is the electric field. If the carrier density is uniform everywhere, i.e., \( \frac{\partial n}{\partial x} = 0 \), we get the Ohm’s law

\[
j = \frac{\tau}{m} qE \rho = \sigma E,
\]

(30)

where \( \sigma = \frac{q\rho \tau}{m} = \frac{m \rho^2 \tau}{m} \) is the formula of electrical conductivity widely used for metals.
7 Passage from stochastic to regular dynamics

The present probabilistic representation of dynamics using least action distribution underlies a biased formalism of classical mechanics defined with the deterministic Hamiltonian functional $H$. It is clear that Eqs. (10) and (11) do not exist for the paths other than the unperturbed geodesics (least action ones). We have in general $\delta A_{ab}(k) \neq 0$ which is either positive or negative, i.e.,

$$\delta A_{ab}(k) = \int_a^b \left[ \frac{\partial}{\partial t} \frac{\partial L_k(t)}{\partial \dot{x}} - \frac{\partial L_k(t)}{\partial x} \right] \varepsilon dt \neq 0$$ (31)

where $\varepsilon$ is an arbitrary variation of $x$ ($\varepsilon$ is zero at $a$ and $b$). Using the same mathematics as in [12], we can prove that, if $dA_{ab}(k) = \frac{\partial A_{ab}(k)}{\partial x} dx - \frac{\partial A_{ab}(k)}{\partial \dot{x}} d\dot{x} >$ (or $< 0$, a deformed Euler-Lagrange equation follows

$$\frac{\partial}{\partial t} \frac{\partial L_k(t)}{\partial \dot{x}} - \frac{\partial L_k(t)}{\partial x} > 0 \ (or < 0).$$ (32)

Then with the help of the Legendre transformation $H = P\dot{x} - L_k$ [12], we can derive

$$\dot{x} = \frac{\partial H}{\partial P} \ \text{and} \ \dot{P} > \ (or <) - \frac{\partial H}{\partial x},$$ (33)

which are the Hamiltonian equations for irregular dynamic systems along the paths with non stationary action. The second equation above violates the Newton’s second law.

However, as mentioned above, the average action has a stationary associated with stationary path information as shown in Eq. (12). This “least average action” can be used in the following calculations to derive an averaged version of the equations of motion of classical mechanics.

$$\delta A_{ab} = \sum p_k(b|a) \delta A_{ab}(k) + \sum \delta p_k(b|a) A_{ab}(k)$$

$$= \int_a^b \left[ \left< \frac{\partial}{\partial t} \frac{\partial L_k(t)}{\partial \dot{x}} \right> - \left< \frac{\partial L_k(t)}{\partial x} \right> \right] \varepsilon dt + \frac{\delta H_{ab}}{\eta}.$$ (34)
Now taking into account Eq. (12), and using the usual mathematics, we obtain an extended version of the Euler-Lagrange equations:

\[
\left\langle \frac{\partial}{\partial t} \frac{\partial L_{kab}(t)}{\partial \dot{x}} \right\rangle - \left\langle \frac{\partial L_{kab}(t)}{\partial x} \right\rangle = 0
\]  

(36)

and of the Hamiltonian equations

\[
\left\langle \dot{x} \right\rangle = \left\langle \frac{\partial H}{\partial P} \right\rangle, \quad \left\langle \dot{P} \right\rangle = -\left\langle \frac{\partial H}{\partial x} \right\rangle
\]  

(37)

where the average \( \langle \cdot \rangle \) is taken over all the possible path between two points.

At first glance, it may seems surprising that the non least action paths have non zero probability of occurring and the fundamental equations such as Eqs. (10) and (11) be violated within classical mechanics. It should be noticed that the probabilistic approach of this work addresses the Hamiltonian systems subject to thermal noise and chaotic motion. So strictly speaking, the Hamiltonian of the systems does not exist. It exists only in an average manner as shown by Eq. (36) and Eq. (37). The random perturbation is of course responsible for the violation of the fundamental equations of classical mechanics. In fact, we can introduce a random force \( R \) into the classical equations. For a path whose action is not at stationary, Eq. (37) implies a Langevin equation:

\[
\dot{P} = -\frac{\partial H}{\partial x} + R
\]  

(38)

where \( R \) is positive if \( \delta A_{ab}(k) > 0 \) and negative if \( \delta A_{ab}(k) < 0 \). From Eq. (37), we must have \( \langle R \rangle = 0 \).

Logically, this probabilistic dynamics should recover the regular dynamics when the uncertainty tends to zero. In fact, the dispersion of action or the width of the least action distribution can be measured by the variance \( \sigma^2 = \overline{\mathcal{A}^2} - \overline{\mathcal{A}}^2 = \overline{\mathcal{A}^2} - A_{ab}^2 \). From Eqs. (3), (6), (8) and (9), we get

\[
\sigma^2 = -\frac{\partial A_{ab}}{\partial \eta}, \quad (39)
\]

\[
A_{ab} = -\frac{\partial}{\partial \eta} \ln Z, \quad (40)
\]
and

\[ H(a, b) = \ln Z + \eta A_{ab} = \ln Z - \eta \frac{\partial}{\partial \eta} \ln Z \]  

(41)

When the system becomes less and less irregular, \( \sigma^2 \) diminishes and the paths become closer and closer to the least action ones having stationary action \( A_{ab}^{\text{stat}} \). When \( \sigma^2 \to 0 \), the significant contribution to the partition function \( Z \) comes from the geodesics having \( A_{ab}^{\text{stat}} \), i.e., \( Z \to \exp[-\eta A_{ab}^{\text{stat}}] \). From Eq. (40), \( A_{ab} \to A_{ab}^{\text{stat}} \). Then considering Eq. (41), it is clear that the path information \( H(a, b) \to 0 \). In this case, the stationary average action Eq. (12) becomes \( \delta A_{ab}^{\text{stat}} = 0 \), the usual action principle, and Eqs. (36) and (37) will recover Eqs. (10) and (11). At the same time, diffusion phenomena completely vanish and the diffusion laws are replaced by the laws of regular mechanics.

8 Concluding remarks

We have presented an attempt to investigate diffusion phenomena with a method which consists in placing hamiltonian systems under the random perturbation of thermal noise and chaotic instability and addressing the irregular dynamics produced in this way with a information approach. A path information is used to measure the dynamic uncertainty associated with different possible transport paths in phase space between two points. On the basis of the assumption that the paths can be physically characterized by their action (defined by using the unperturbed Hamiltonian), we show that the maximum path information yields a path probability distribution in exponential of action.\(^1\) This least action distribution provides a simple and general derivation of the Fokker-Planck equation, the Fick’s laws, and the Ohm’s law. The mathematics of this derivation is not new. It can be found in many text books treating diffusion on the basis of Brownian motion and Markovian process. The new point of this work is using the information

\(^1\)We would like to note here that this exponential transition probability distribution of action has nothing to do with the Feynman’s postulated quantum propagator proportional to the path integrals of an exponential of action multiplied by the imaginary number \( i \). Although the mathematics of this work is similar to that of the path integrals, the physics is totally different. In path integral quantum theory, the transition probability between two points is the absolute square of the propagator, hence it is not necessarily exponential of action.
method combined with the action principle underlying a transition probability in exponential of action and a probabilistic formalism of dynamics for the hamiltonian systems under random perturbation. The reasoning is simple and the mathematics is straightforward. The usual assumptions for the derivation of diffusion laws, e.g., Brownian motion, Markovian process, Gaussian random forces, or Kolmogorov conditions, are unnecessary in this approach.

Although in this approach the action of individual paths is not at stationary, the stationary path information assures that the action of the most probable paths has a stationary and that the average action over all possible paths have also a stationary under the constraint associated with dynamic uncertainty. This result suggests the following analogy: as the Newton’s law is the consequence of action principle of regular hamiltonian systems, the diffusion laws can also be considered as the consequences of this same principle which presents itself through the stationary average action associated with irregular dynamic uncertainty.

The validity of the present work is of course limited by the mathematical tools we use in this work. As far as we can see, the results of this work probably does not hold if, 1) the dynamic uncertainty cannot be measured by Shannon information; 2) the paths of the diffusants is not sufficiently smooth for the application of the Euler method; 3) the paths do not allow continuous variational method. The latter two cases may occur when, for example, the phase space is porously occupied such as in fractal phase space.

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