Some remarks on the first Hardy-Littlewood conjecture

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Abstract

Starting from the first Hardy-Littlewood conjecture some topics will be covered: an empirical approach to the distribution of the twin primes in classes mod(10) and a simplified proof of the Bruns theorem.

Finally, it will be explored an approach based on numerical analysis: Monte Carlo Method and Low discrepancy Sequences will be used to prove the convergence of the conjecture to the expected values.

\textbf{Keywords} — win prime numbers, theories, conjectures, Monte Carlo methods.

1 Introduction

The twin prime conjecture also known as Polignac’s conjecture is one of the oldest and best-known unsolved problems in number theory and in all of mathematics: it states that for every positive even natural number $k$, there are infinitely many consecutive prime pairs $p$ and $p’$ such that $p’ - p = k$. The case $k = 2$ is the twin prime conjecture.

Even if the conjecture has not been proved, in spite of many challenges, most mathematicians believe it is true.

Recently Arenstorf, in 2004 \cite{1}, proposed a proof of the conjecture but an error was found after its publication, leaving the conjecture open to
What we know for sure, from empirical analysis, is that as numbers get larger, twin primes become increasingly rare.

A second twin prime conjecture, called the strong twin prime conjecture or first Hardy-Littlewood conjecture (1923) [3], states that the number $\pi_2(n)$ of twin primes less than or equal to $n$ is asymptotically equal to

$$\pi_2(n) \sim 2C_2 \int_2^n \frac{dx}{(\ln(x))^2}$$

where $C_2$ is the so-called twin primes constant [5].

Even if both conjectures have not been proved, models for the primes, based on some statistical distribution, can provide the asymptotic value of various statistics about the primes: the “naïve” Cramér random model (1936) which models the set of prime numbers by a random set, is too simplified to give accurate results, but tends to give predictions of the right order of magnitude [13, 14].

The starting point of the model is that for every natural number at random in $x, x + \varepsilon x$, for any fixed $\varepsilon > 0$ and large $x$, the probability that it will be prime is about

$$\frac{1}{\ln(x)} \approx \frac{1}{\ln(n)}$$

and each natural number has an independent probability of lying in the model set of the primes. This leads to a conjecture of the form:

$$\pi_2(n) \sim \frac{x}{(\ln(\ln(x)))^2}$$

and in particular that there are infinitely many twin primes.

It is worth noting that in 1996 Ribenboim proved that:

$$\pi_2(n) \leq c \Pi_2 \frac{x}{(\ln(x))^2} \left[1 + O\left(\frac{\ln_2(x)}{\ln(x)}\right)\right]$$

where $\Pi_2$ is the twin primes constant and $c$ is another constant, that according to Hardy-Littlewood conjecture is 2 and that in 1999 has been reduced to 6.8325 [4] down from previous values [15].

2 Hardy-Littlewood conjecture: an asymptotic distribution of twin primes

The distribution of pairs of primes has been studied with the Chi-square $\chi^2$ statistic approach [2], in order to compare experimental data to the

\[\text{Notation. The use of the asymptotic notations } \mathcal{O}, o, \sim \text{ is standard, as well as the symbol } \approx \text{ used to denote rough, conjectural or heuristic approximations.}\]

\[\text{but it's quite obvious that 'p is prime' and 'p+2 is prime' are not independent events, because p+2 is automatically odd and more likely to be prime.}\]
expected values. On the basis of this analysis, it has been possible to verify the hypothesis that twin primes thin out in classes with the same cardinality.

Let:

\[ P = \text{the set of primes} \]

\[ X_i(2, m) := \#\{ (p_i, p_{i+2}) : p_i, p_{i+2} \in P \text{ and } \frac{p_i}{10} \equiv m \mod 10 \}, \text{ (i.e. } m = p_i \mod 10) \] (5)

With \( i = 1, 2, 3 \) i.e. \( X_1(2, 1), X_3(2, 7), X_3(2, 9) \) and \( X_3 \).
\( \pi_i^2(n) \) be the twin prime counting function of class \( X_i(2, m) \)

Numerical analysis provides the following result:

![Image](image.png)

Figure 1: Twin prime counting function, \( \pi_i^2(n) \), of class \( X_1(2, 1) \)

And similar results for the classes \( X_2(2, 7) \) and \( X_3(2, 9) \). It may be clearly seen that the three classes converge toward the same value: 33.3% and the Chi-square \( \chi^2 \) statistic approach justifies a random distribution of the twin primes in the three classes.

Hence it may be conjectured, under empirical evidence, as follows:

\[ \pi_i^2(n) \sim \frac{2}{3} C_2 \int_2^n \frac{dn}{(\ln \ln(n))^2}, i = 1, 2, 3 \] (6)

\( S = \# \bigcup X_i(2, m) \) differs from \( 2s \) because classes of only one element (pairs \( (3, 5)(5, 7) \)) have not been considered in the numerical model
In other words, the asymptotic distribution of pairs of twin primes \((p_i, p_i + 2)\) in the three classes \(X_i(2, m)\), \(m = p_i \mod(10), m = 1, 7, 9\) may be described as statistically random: no strong empirical evidence appears to the contrary.

The fact that twin primes behave more randomly than primes, is also supported by the works by Kelly and Pilling [8], [9] pointing out that the occurrences of twin primes in any sequence of primes are like fixed probability random events.

3 From Hardy-Littlewood conjecture to the Bruns theorem

V. Brun wanted to analyze the sum

\[
\sum_{p, p+2 \text{ primes}} \frac{1}{p} + \frac{1}{p+2}
\]

hoping that the sum would be infinite and thus giving a solution to the twin primes conjecture. Instead what he proved in 1919, by means of a specific sieve, is that the sum of reciprocals of the twin primes converges to a finite value [6].

\[
\sum_{p, p+2 \text{ primes}} \frac{1}{p} + \frac{1}{p+2} \approx 1.9 < +\infty
\]

If the series had diverged, it would have indicated that there are an infinite number of twin primes but the proof that it converges does not provide more information about Polignac’s conjecture. The original proof of the convergence was based on the Bruns simple pure sieve (principle of Inclusion-Exclusion) but it is possible to provide a simplified demonstration starting from the Hardy-Littlewood conjecture.

**Proof**

First of all it’s easy to observe that:

\[
\int_{2}^{n} \frac{dx}{(\ln \ln(x))^2} \sim \frac{n}{(\ln \ln(n))^2}
\]

(9)

In fact, let:

\[
f(n) = \int_{2}^{n} \frac{dx}{(\ln \ln(x))^2}
\]

(10)

and

\[
g(n) = \frac{n}{(\ln \ln(x))^2}
\]

(11)

\[\text{It’s worth noting that the numerical analysis leads to a similar result also in case of cousin primes, sexy primes and Sophie Germain primes}\]

\[\text{even if some small differences appear in the speed of convergence rate}\]
Some remarks on the first Hardy-Littlewood conjecture

then
\[
\frac{f(n)}{g(n)} = \frac{f'(n)}{g'(n)} = \frac{1}{1 - 2/\ln(n)} = 1
\]

Unfortunately, the asymptotical equivalence doesn't provide any information about the behavior of the ratio:
\[
\int_2^n \frac{dx}{(\ln(\ln(x))} \in [2, +\infty[ (13)
\]

In order to bound the integral with a degree of approximation, in the set \([2, +\infty[\) we proceed as follows:
\[
\int_2^n \frac{dx}{(\ln(\ln(x))} = \int_2^n \frac{dx}{\ln(x)} - \int_2^n \frac{x}{\ln(x)} = li(n) - li(2) - \frac{n}{\ln(n)} + \frac{2}{\ln(2)} (14)
\]

with
\[
li(n) = \int_0^n \frac{dn}{\ln(n)} (15)
\]

The asymptotic expansion (Poincaré expansion) of \(li(n)\) for \(x \to \infty\) gives:
\[
li(n) \sim \frac{n}{\ln(n)} \sum_{k=0}^{\infty} \frac{k!}{(\ln(n))^k} (16)
\]

i.e. \(\sum li(n)\)

\[
li(n) \sim \frac{n}{\ln(n)} + \frac{n}{\ln^2(n)} + \frac{2n}{\ln^3(n)} + \cdots (17)
\]

Hence assuming Eq.??:
\[
\pi_2(n) \sim 2C_2 \cdot \left(-li(2) + \frac{2}{\ln(2)} + \frac{n}{\ln^2(n)} + \frac{2n}{\ln^3(n)} + \frac{6n}{\ln^4(n)} + \cdots\right) (18)
\]

Where \(li(2) = 1.045163 \cdots \) The series is not convergent and it is reasonable an approximation where the series is truncated at a finite number of terms with an error roughly of the same size as the next term.

In fact the problem associated to divergence is that for a fixed \(\varepsilon\), the error in a divergent series will reach to an \(\varepsilon\)-dependent minimum, but as more terms are added the error then increases without bound and tends to infinity.

Since for every \(n \in N, n \geq 10^{12}\) we have:
\[
\frac{1}{\ln^3(n)} \geq \frac{6}{\ln^4(n)} (19)
\]

Hence we can write for every \(n \in N, n \geq 10^{12}\) i.e. in the set \([n, +\infty[\)
\[
1 < \frac{\pi_2(n)}{2C_2 \frac{n}{\ln(n)}} \leq 1 + \frac{2}{\ln(n)} + \frac{7}{\ln^2(n)} (20)
\]

\(\ast\)This implies also: \(li(n) - n \ln(n) = O(n \ln 2n)\)
Some remarks on the first Hardy-Littlewood conjecture

\[ 1 < \frac{\pi_2(n)}{n^2(n)} \leq \approx 1.4277 \]  

(21)

Hence if we assume the Hardy-Littlewood conjecture we can say it exists a number \( \bar{n} \in \mathbb{N} \) such that for every \( n \geq \bar{n} \):

\[ \pi_2(n) \leq K \frac{n}{\ln^2(n)} \]  

(22)

It is worth noting that the ratio

\[ \frac{\pi_2(n)}{n^2(n)} \]  

(23)

has been studied by many authors under the general condition:

\[ \frac{\pi_2(n)}{n^2(n)} < 2C_2 + \varepsilon \]  

(24)

Recently Jie Wu [16] proved that for sufficient large \( n \):

\[ \frac{\pi_2(n)}{n^2(n)} < 4.5 \]  

(25)

Now let’s consider the sum in Eq.7

\[ \sum_{p, p+2 \text{ primes}} \frac{1}{p} + \frac{1}{p+2} \]  

(26)

Since

\[ \frac{1}{p} + \frac{1}{p+2} \leq \frac{2}{p} \]  

(27)

the convergence of Eq.7 is equivalent to the convergence of

\[ \sum_{p, p+2 \text{ primes}} \frac{1}{p} \]  

(28)

there are two possibilities:

a) Twin primes are finite in number (in this case the sum of the series is finite and the convergence is proved)

b) Twin primes are not finite in number, in this case:

Let \( r \) be the \( r^{th} \) twin prime \( q_r \):

\[ r = \pi_2(q_r) \leq K \frac{q_r}{\ln^2(q_r)} \leq K \frac{q_r}{\ln^2(r+1)}, \text{ since } q_r > r + 1, \forall r \in \mathbb{N} \]  

(29)

Hence

\[ \frac{1}{q_r} \leq K \frac{1}{r \ln^2(r+1)} \]  

(30)

And:

\[ \sum_{p, p+2 \text{ primes}} \frac{1}{p} = \sum_{1}^{\infty} \frac{1}{q_r} \leq K \sum_{1}^{\infty} \frac{1}{r \ln^2(r+1)} \]  

(31)

\( ^{\tau} \)the proof is the same as in [10]
Some remarks on the first Hardy-Littlewood conjecture

For the comparison test also the series

\[ \sum_{p, \ p+2 \text{ primes}} \frac{1}{p} \]  (32)

converges.

4 Calculation of the integral \(2C_2 \int_2^\infty \frac{dx}{(\ln \ln x)^2}\) using MonteCarlo approach

Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods are widely used in numerical analysis, especially in physics and finance. Consider an integral of the form: \(I = \int_{\Omega} f(x)dx\). Where \(\Omega\) is the domain of integration and \(f(x)\) a bounded real function.

Most direct quadrature methods are based on the Riemann definition of an integral (a finite sum of ordered 'areas' under the curve \(y = f(x)\)): MC and QMC methods are explained by Lebesgue integration: the finite sum do not depend on the order, it is enough that the function can be somehow 'measured'.

By the strong law of large numbers, if \(U\) is a uniformly distributed random variable on \(\Omega\) then the average of the sum of \(f(U_i)\ i \in [1, N]\) converges to \(I\) almost surely when \(n\) tends to infinity, i.e.:

\[ \int_{\Omega} f(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} f(U_i) \]  (33)

Hence, while conventional numerical methods calculate the integrand at regularly spaced points, MC method samples the integrand at random points \(U_i\), \(i \in [1, N]\) (\(N\) is the number of samples).

The critical issue with this points, is that they may not be equally distributed in the domain and this leads to the need to increase the number of samples, and consequently the run-times.

This problem can be solved with QMC methods, making use of quasi-random numbers that are more well-distributed [11]. Although quasi-random numbers come from a deterministic algorithm, they pass a statistical test of randomness.

Among these methods those which make use of low discrepancy sequences (LDS)[12] are based on the property of lack of apparent pattern in the distance\(^8\) between couples of primes and for this reason conforming a set of quasi-random numbers.

\(^8\)6 is the most common separation distance up to about \(n \approx 1.74 \times 10^{35}\)

It has been explored the application of MC and QMC methods to the Hardy-Littlewood integral calculation:
Some remarks on the first Hardy-Littlewood conjecture

\[ 2C_2 \int_2^n \frac{dx}{(\ln \ln(x))^2} \]

using low discrepancy sequences (LDS) and Mathematica software \(^9\) (Annex I).

The following TABLE \(\text{34}\) provides the results of MC and LDS methods:

| powers  | \(\pi_2(n)\) | \(2C_2 \int_2^n \frac{dx}{(\ln \ln(x))^2}\) | % error | MC H-L | % error | LDS H-L | % error |
|---------|---------------|---------------------------------|---------|--------|---------|---------|---------|
| 10^2   | 2             | 4.84                            | 1.42 \times 10^2 | 3.03   | 51.458 | 6.2     | 210.18  |
| 10^3   | 8             | \(1.35 \times 10^3\)            | 6.92 \times 10^4 | 6.92   | 13.510 | 1.35 \times 10^6 | 68.1979 |
| 10^4   | 35            | \(4.58 \times 10^4\)            | 3.08 \times 10^6 | 2.3 \times 10^8 | 34.412 | 4.64 \times 10^10 | 32.6141 |
| 10^5   | 205           | \(2.14 \times 10^7\)            | 4.49     | 1.08 \times 10^2 | 47.247 | 2.28 \times 10^10 | 11.3977 |
| 10^6   | 1224          | \(1.25 \times 10^7\)            | 2.02     | 6.34 \times 10^2 | 48.243 | 1.37 \times 10^10 | 12.0495 |
| 10^7   | 8169          | \(8.25 \times 10^7\)            | 9.67 \times 10^3 | 4.21 \times 10^3 | 48.462 | 9.2 \times 10^10 | 12.5951 |
| 10^8   | 58980         | \(5.88 \times 10^8\)            | 3.83 \times 10^4 | 3.09 \times 10^4 | 47.530 | 6.61 \times 10^10 | 12.1327 |
| 10^9   | 440312        | \(4.4 \times 10^9\)             | 1.27 \times 10^5 | 2.29 \times 10^5 | 48.021 | 4.99 \times 10^10 | 13.394 |
| 10^10  | 3424506       | \(3.43 \times 10^{10}\)         | 1.73 \times 10^6 | 9.69 \times 10^6 | 49.351 | 3.91 \times 10^{10} | 14.1356 |
| 10^11  | 27412679      | \(2.74 \times 10^{11}\)         | 4.15 \times 10^7 | 1.51 \times 10^7 | 44.766 | 3.15 \times 10^10 | 14.8062 |
| 10^11  | 2244376048    | \(2.24 \times 10^{11}\)         | 9.00 \times 10^7 | 1.22 \times 10^8 | 45.36  | 2.59 \times 10^10 | 15.512  |

Since the convergence rate of Monte Carlo method is close\(^10\) to \(O\left(\frac{1}{\sqrt{N}}\right)\), the error rate decreases as the value of \(N\) increases (i.e. as a function \(\pi_2(n)\) increases) as described in literature.

In the following example, the convergence is not proved due to the low number of points \(N\) considered in the calculation \((n = 10^{11}, \ N = 17548)\) but it can be seen the advantage of using LDS due to a faster rate of convergence.

Finally, the following table (Table 2) provides the results of Monte Carlo method with \(n = 10^{11}, \ N = 17548\)

5 CONCLUSIONS

In spite of many challenges and improvements due to numerical analysis, twin primes are still an unsolved problem in number theory: the first Hardy-Littlewood conjecture can be described as a milestone in this field.

This paper has proposed an empirical analysis of the twin primes distribution that leads to write the conjecture in terms of classes \(\text{mod}(10)\) marked by the same cardinality, according to a statistically random system.

\(^9\)It is worth notice that a compensating constant \(a \times 7.39\) has been used, depending on the limits of integration, the minimal and maximal values of the set of samples, and the dimensions of the integrand \(^12\).

\(^10\)and it is rather slow: quadrupling the number of sampled points it will halve the error.
Furthermore starting from the conjecture it has been provided an elementary demonstration of the Bruns theorem about the convergence of the sum of the reciprocal of the twin primes.

Finally it has been explored a less conventional method of calculation of the Hardy-Littlewood integral based on Monte Carlo and LDS approach. The result of the calculation with a sufficient number of samples is compelling and provides (for any given \( n \) larger than \( n = 10^6 \) say) a small relative error, for the MC case.

6 Annex: MonteCarlo code (Mathematica)

\[
c2 = 0.66016181584686957392781211001455577843262360284733413319448
powers = \{10, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}, 10^{-10},
\]
REFERENCES

10^{11};

nooftwillp = {2, 8, 35, 205, 1224, 8169, 58980, 440312, 3424506, 27412679, 224376048};

HLconjecture = {4.8361883278, 13.5354875604, 45.7955004115,
214.210938311, 1248.7087356371, 8248.026893808,
8753.814979424, 440367.7942273770,
3425308.1557430851, 27411416.5321785837,
224368864.81819439};

mcHLintegrand =
Table[
   Table[
      1/(Log[x])^2,
      {x, 2, powers[[k]], (powers[[k]] - 2)/17547}],
   {k, 1, Length[powers]}];

mcHLsummatories =
Table[2 c2*
   Sum[powers[[k]]* mcHLintegrand[[k, i]]*
      RandomReal[]/Length[mcHLintegrand[[k]]],
   {i, 1, Length[mcHLintegrand[[k]]] - 1}],
   {k, 1, Length[mcHLintegrand]}];

{3.00751, 6.88785, 23.121, 106.42, 631.225,
4226.86, 30603.9, 234703., 1.79265*10^{-6},
1.41899*10^{-7}, 1.22599*10^{-8}}

comparisons =
Table[{pow2[[k]], pi2n[[k]], ScientificForm[HLconjecture[[k]], 3],
   ScientificForm((Abs[pi2n[[k]]] - HLconjecture[[k]])*100/pi2n[[k]],
   3), ScientificForm[mcHLsummatories[[k]], 3],
   (Abs[pi2n[[k]]] - mcHLsummatories[[k]])*100/pi2n[[k]],
   ScientificForm[summatories[[k]], 3],
   N[(Abs[pi2n[[k]]] - summatories[[k]])*100/pi2n[[k]], 3]},
   {k, 1, Length[summatories]}];
PrependTo[comparisons, {"powers",
   "\![\*SubscriptBox[\[Pi\], \(2\)](n)\]", "HL conj.", "% error",
   "mc HL", "% error", "LDS HL", "% error"}]; MatrixForm[comparisons]

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| $\pi_2 (n)$ | $2C_2 \int_{2}^{n} \frac{dx}{(\ln \ln(x))^2}$ | $\Delta$ | $\text{MonteCarlo}$ | $2C_2 \int_{2}^{n} \frac{dx}{(\ln \ln(x))^2}$ | $\Delta$ |
|-------------|----------------------------------|--------|---------------------|----------------------------------|--------|
| 10          | 2.0 4,8361883278                  | 141,809416  | 4.85351% | 142,675500% |
| $10^2$      | 8.0 13,5354875604                | 69,193595% | 13.35    | 66.875000% |
| $10^3$      | 35.0 45,7955004115               | 30,844287% | 45,5171  | 30.048857% |
| $10^4$      | 205.0 214,2109398311             | 4,493141%  | 212,089  | 3,458049% |
| $10^5$      | 1224.0 1248,7087356371           | 2,018688%  | 1237,49  | 1,102124% |
| $10^6$      | 8169.0 8248,0296898308           | 0,967434%  | 8290,81  | 1,491125% |
| $10^7$      | 58980.0 58753,816497934          | -0,383492% | 59792,1  | 1,376907% |
| $10^8$      | 440312.0 440367,79422737         | 0,012672%  | 431697  | -1,956567% |
| $10^9$      | 3424506.0 3425308,1557430        | 0,023424%  | 3,33627E+06 | -2,576605% |
| $10^{10}$   | 27412679.0 27411416,5322786      | -0,004605% | 2,76981E+07 | 1,041201% |
| $10^{11}$   | 22437604.0 224368864,681182      | -0,003201% | 2,23159E+08 | -0,542414% |