Transitions Between Chiral Spin Liquids and $Z_2$ Spin Liquids

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The Kalmeyer-Laughlin chiral spin liquids (CSL) and the $Z_2$ spin liquids are two of the simplest topologically ordered states. Here I develop a theory of a direct quantum phase transition between them. Each CSL is characterized by an integer $n$ and is topologically equivalent to the $1/2n$ Laughlin fractional quantum Hall (FQH) state. Depending on the parity of $n$, the transition from the CSL is either to a “twisted” version of the $Z_2$ spin liquid, the “doubled semion” model, or the conventional $Z_2$ spin liquid (“toric code”). In the presence of $SU(2)$ spin symmetry, the triplet gap remains open through the transition and only singlet operators acquire algebraic correlations. An essential observation is that the CSL/Laughlin FQH states can be understood in terms of bosonic integer quantum Hall (BIQH) states of Schwinger bosons or vortices, respectively. I propose several novel many-body wave functions that can interpolate through the transition.

Two of the simplest quantum spin liquids in two spatial dimensions that exhibit topological order \[1, 2\] are the Kalmeyer-Laughlin (KL) chiral spin liquid (CSL) \[3, 4\] and the $Z_2$ quantum spin liquid \[5\]. These states played significant roles in the development of the theory of topological order and in attempts to formulate a theory of high $T_c$ superconductivity \[6\]. It has been established that these states can be stabilized with local Hamiltonians \[7\] and there is increasingly strong evidence from numerical simulations that physically realistic Hamiltonians for frustrated magnets \[7–11\] or bosons in partially filled Chern bands \[12\] can also stabilize them \[13\]. However it has not been clear whether a direct, continuous quantum phase transition is possible between the Laughlin and $Z_2$ states. Aside from being of intrinsic interest, understanding such questions may be helpful for the challenge of identifying the nature of topological spin liquids; as \[2\] spin liquids are two of the simplest examples of gapless spin liquids \[13\].

The CSL was argued to be realized in frustrated spin-$1/2$ systems, which can mapped to a problem of bosons in a magnetic field via a Holstein-Primakoff transformation \[3\]. The bosons in turn form a $1/2n$ Laughlin fractional quantum Hall (FQH) state. Later it was shown \[4\] that the KL-CSL, for $n = 1$, can also be understood in terms of a projective construction where fermionic spinons form a fermionic integer quantum Hall state. In contrast, the $Z_2$ spin liquid is described at low energies by $Z_2$ gauge theory, due to pair condensation of spinons. There are two topologically distinct classes of $Z_2$ gauge theory \[14\] and thus two corresponding classes of $Z_2$ spin liquids; as will be reviewed below, these describe the “toric code” and “doubled-semion” models \[3, 6\].

In this paper, I develop a theory of a direct continuous transition between the CSL and the $Z_2$ states. For odd $n$, the transition is to the doubled-semion model, and for even $n$, it is to the toric code model. In the Appendix I also provide a theory of the transition between the doubled-semion and toric code models, which is shown to be described by QED$_3$. A key insight is that, in contrast to previous constructions, the CSL can be understood as a state where Schwinger bosons form a bosonic integer quantum Hall (BIQH) state with Hall conductance $\sigma_{xy} = 2n$. The BIQH state possesses no fractionalized excitations, and is topologically non-trivial only in the presence of a conserved $U(1)$ charge \[15, 16\]. It forms a simple example of a bosonic symmetry-protected topological (SPT) phase \[15–21\]. The bosonic Laughlin FQH state can be understood as a state where the vortices of a bosonic superfluid collectively form a BIQH state. Using these observations, I show that the transitions between the Laughlin and $Z_2$ states can be understood in terms of transitions between BIQH states and charge-$2$ superconductors (SC) (see Fig. [1]). The latter transitions can be shown to be in the XY universality class. Depending on the parity of $n$, the charge-$2$ SC can itself be mapped to a topologically non-trivial Ising paramagnet, of the type discussed in \[19\].

**BIQH states and transitions** – A bosonic system can form an IQH state with no fractionalized excitations, as long as the electrical Hall conductance is quantized as $\sigma_{xy}/\sigma_0 = 2n \equiv \bar{\sigma}_{xy}$, where $n$ is an integer and $\sigma_0 = Q^2/h$ is the appropriate conductance quantum for bosons of charge $Q$ (In the following we set $\sigma_0 = 1/2\pi$). This can be described by the following effective $U(1) \times U(1)$ Chern-Simons (CS) gauge theory:

$$L = \frac{1}{4\pi} K_{IJ} a^I \partial a^J + \frac{1}{2\pi} A_e \partial(a^1 + a^2),$$

(1)

with $K = \begin{pmatrix} 0 & 1 \\ 1 & 2(1-n) \end{pmatrix}$, where $A_e$ is the external elec-
TABLE I: Chern number assignments for the mean-field insulating states of partons $f_1$ and $f_2$, and resulting bosonic states, where $b = f_1 f_2$. The charge-$k$ SC breaks $U(1)$ charge conservation and does not have a well-defined quantized DC Hall conductivity; if its Goldstone mode is gapped by, e.g. a long-ranged interaction, then it describes a $Z_k$ SPT.

| $(C_1, C_2)$ | Resulting state | Hall conductance, $\sigma_{xy}/\sigma_0$ |
|-------------|----------------|---------------------------------|
| $(-1, -2)$  | BIQH            | 2                              |
| $(-k, k)$   | Charge-$k$ SC (Z$_k$ SPT) | -                              |
| $(-1, 0)$   | Trivial Mott insulator | 0                              |
| $(-1, -1)$  | 1/2-Laughlin     | 1/2                            |

tromagnetic gauge field, and where $a^\dagger \partial a^\dagger = \epsilon^{\mu\nu\lambda} a^\dagger_\mu a^\dagger_\nu a^\dagger_\lambda$. This theory has counterpropagating edge states and therefore no net thermal Hall effect, and no topologically non-trivial excitations (since $|\text{Det } K | = 1$). However the charge is carried by one of the chiral edge modes, leading to the quantized $\sigma_{xy}$: $U(1)$ charge conservation prohibits backscattering and thus protects the counterpropagating edge modes [13].

Let us focus on the case $n = 1$. This state can be understood in terms of composite fermions and reverse flux attachment [22], applied to bosons at filling fraction $\nu = 2$. To each boson we attach $-2\pi$ flux of a statistical gauge field, leading to composite fermions at filling factor $\nu_{c.f} = -2$, which then form the $\nu_{c.f} = -2$ IQH state. This composite fermion construction can be alternatively understood by writing the hard-core bosons in terms of two fermionic “partons”: $b = f_1 f_2$, which introduces a compact $U(1)$ gauge field $a$ associated with the gauge redundancy $f_1 \rightarrow e^{i\theta} f_1$, $f_2 \rightarrow e^{-i\theta} f_2$. We consider a mean-field state of the partons where $f_1$, $f_2$ each form a quantized Hall insulator with $C_1 = -1$ and $C_2 = 2$, respectively. In this language, $f_2$ plays the role of the “composite fermion,” while $f_1$ plays the role of the emergent statistical gauge field. When the gauge fluctuations are properly included (see Appendix B), this mean-field ansatz yields a BIQH state with no fractionalization, described by the field theory [1].

The BIQH wave function suggested by this construction is given by

$$\Psi_{\text{BIQH}}(\{t_i\}) = \Phi_{-1}(\{t_i\}) \Phi_{2}(\{t_i\}),$$

where $\Phi_C(\{t_i\})$ is a Slater determinant wave function for fermions forming a quantized Hall insulator with total Chern number $C$. A simple example of (2) for a continuum system at filling fraction $\nu = 2$ is given by

$$\Psi_{\text{BIQH}}(\{z_i\}) = \prod_{i < j} (z_i - z_j) \Phi_{-2}(\{z_i^*\}),$$

where now $\Phi_{-2}$ is a free fermion wave function for two filled Landau levels at $\nu = -2$.

The advantage of the above parton construction is that other, nearby collective states of the bosons can also be understood by considering mean-field states associated with other values of the Chern numbers (see Table I). The continuous transitions between these various states can therefore be described by the Chern number changing transition for $f_2$ or $f_1$, coupled to the emergent $U(1)$ gauge field $a$ [24,25].

For the current discussion, consider the case $(C_1, C_2) = (-k, k)$. We will show that the resulting state of the bosons is a charge-$k$ superconductor and, moreover, is closely related to a topologically non-trivial $Z_k$ SPT state (the case $k = 1$ has been described previously [15,20]). To see this, let $f_1$ and $f_2$ have charges 0 and 1, respectively, under an external gauge field $A_e$, such that $b$ has charge 1. Integrating out $f_1$, $f_2$ yields the following Lagrangian density

$$\mathcal{L} = \frac{k}{2\pi} A_e \partial a + \frac{k}{4\pi} A_e \partial A_e - \frac{1}{2} (\epsilon_{\mu\nu\lambda} \partial_\mu a_\lambda)^2 + \cdots,$$

where we have included a higher order term for $a$; in general, all higher order terms compatible with gauge invariance of $a$ and $A_e$ will be allowed. Remarkably, (3) implies that instantons of $a$ carry charge $k$, and therefore are prohibited from proliferating at low energies due to charge conservation. Consequently, $a$ is effectively a massless noncompact gauge field. (3) shows that the fluctuations of $a$ are coupled to $A_e$, and therefore correspond to physical density/current fluctuations. Thus this theory must describe a superfluid state, and $a$ is the dual of the Goldstone mode. Since we consider the bosons to carry charge, we will use the terms superconductor and superfluid interchangeably.

In this theory, the vortices of the superfluid correspond to the gapped fermionic excitations of the Chern bands, as these are minimally coupled to the dual Goldstone mode $a$. To see that this describes a charge-$k$ superconductor, suppose $A_e$ is dynamical, so that the dual Goldstone mode $a$ is gapped by the Anderson-Higgs mechanism. The low energy theory is $\mathcal{L} = \frac{k}{2\pi} a \partial a + a (j_1 - j_2) + \cdots$, where we have included the currents $j_1$ and $j_2$ associated with the partons $f_1$ and $f_2$, and the $\cdots$ indicate higher derivative terms. The equation of motion for $a$ is now a constraint: $\epsilon_{\mu\nu\lambda} \partial_\mu a_{\nu} = \frac{2k}{\pi} (j_2 - j_1)_{\mu}$. This directly shows that the gapped excitations associated with $f_1$ and $f_2$ each carry $2\pi/k$ units of flux, as expected for the vortices of a charge-$k$ superconductor. Another way to see this is to introduce a field $\xi_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$, and a Lagrange multiplier field $\varphi$ to enforce $\partial_\mu \xi_\mu = 0$. Integrating out $\xi_\mu$ then gives the Lagrangian of a charge-$k$ superconductor: $\mathcal{L} \propto (\partial_\varphi - k A_e)^2$.

In the case where $A_e$ is dynamical so that $a$ is gapped by the Anderson-Higgs mechanism, $b$ therefore forms a charge-$k$ gapped superconductor with $Z_k$ symmetry. Remarkably, this state is a topologically non-trivial $Z_k$ SPT phase. From the group cohomology classification [17], there are $k$ topologically distinct bosonic SPT states with $Z_k$ symmetry. In a topologically trivial $Z_k$ state, gauging the $Z_k$ symmetry will yield ordinary $Z_k$ gauge theory, the topologically properties of which can be described by a $U(1) \times U(1)$ CS theory with $K$-matrix $K = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$. This
is sometimes also referred to as the $Z_k$ toric code model: there are $k^2$ topologically distinct quasiparticles $(a, b)$, for $a, b = 0, \ldots, k - 1$, with statistics $\theta_{(a,b)} = 2\pi ab/k$. However, if we consider the response field $A_k$ to be a dynamical gauge field in $[3]$, we obtain a $U(1) \times U(1)$ CS theory with $K = \begin{pmatrix} 0 & k \\ k & j \end{pmatrix}$. The topological properties of such a theory are distinct from those of $Z_k$ toric code. $[37]$. The case $k = 2$ describes the so-called doubled semion model, which consists of four particles, $1, \bar{s}, s, s \times \bar{s}$, with statistics $\theta_1 = \theta_{\bar{s}} = \pi/2$, $\theta_s s \bar{s} = \pi$, and with $s^2 = \bar{s}^2 = 1$. Therefore the bosons $b$ must be forming a non-trivial $Z_k$ SPT state $[13]$.

The transition between the BIQH state with $\sigma_{xy} = 2$ and the charge-2 SC can now be understood as a transition where $f_1$ changes Chern number from $C_1 = -1 \rightarrow -2$, while $f_2$ remains inert. This suggests a wave function that can continuously interpolate between the two phases:

$$\Psi_{bgh-Z_2}(m, \{r_i\}) = \Phi_{f_1}(m, \{r_i\}) \Phi_2(\{r_i\}), \quad (4)$$

where $\Phi_{f_1}(m, \{r_i\})$ is a Slater determinant wave function for the $f_1$ state, which tunes from a quantized Hall insulator with $C_1 = -1$ to $-2$ as the tuning parameter $m$ is tuned through 0.

The field theory for this transition is given by

$$\mathcal{L} = \frac{1}{2} \frac{1}{4\pi} a \phi + \bar{\psi} \gamma^\mu (\partial_\mu - ia) \psi + m \bar{\psi} \psi + \frac{2}{2\pi} A_x \partial_x + \frac{2}{4\pi} A_y \partial_y A_x, \quad (5)$$

where $\psi$ is a two-component Dirac fermion associated with the Chern number changing transition of $f_1$, $\bar{\psi} \equiv \psi^\dagger \gamma^0$, and $\gamma^\mu$ are the Pauli matrices. When $m < 0$, integrating out the Dirac fermion $\psi$ cancels the CS term for $a$, giving $[3]$. When $m > 0$, integrating out $\psi$ yields the BIQH state. The action above can be recognized as the fermionization of the 3D XY transition $[24, 27]$ (see Appendix $[4]$, and therefore is dual to the following Chern-Simons-Higgs theory:

$$\mathcal{L} = |(\partial - ik A_c) \Phi|^2 + m|\Phi|^2 + \lambda|\Phi|^2 - \frac{2m}{4\pi} A_{xy} \partial_y A_x, \quad (6)$$

with $(k, n) = (2, 1)$, $\lambda > 0$, and $\Phi$ a complex scalar field. For general $(k, n)$, it is clear that $[6]$ describes a transition between a bosonic state with $\sigma_{xy} = 2n$ and a charge-$k$ superconductor. The critical theory $[6]$ is expected on general grounds: By definition, the bulk of an SPT phase is trivial, and so any XY ordering transition in the bulk of a BIQH state should be conventional. The above derivation using the parton construction provides both a highly non-trivial check and projected many-body wave functions. When $A_{ak}$ in $[3]$ is interpreted as an internal dynamical gauge field, the $\Phi$-condensed phase describes the doubled semion theory (see Appendix $[4]$), agreeing with the parton construction analysis.

The $n = 1$ BIQH state can also be understood in terms of two-component bosons, $b_\alpha$, where the spin index $\alpha = \uparrow, \downarrow$. In the parton construction we set $b_\alpha = f_1a_\alpha$, and consider $f_\alpha$ to form a spin singlet $\nu = -2$ IQH state (ie $f_\alpha$ form Chern insulators with Chern number $C_\alpha = 1$), while again $f_1$ forms a quantized Hall insulator with $C_1 = -1$. This construction was used in $[12]$ to describe the BIQH state, and also earlier in $[28]$ to study topological phase transitions. This suggests the following wave function for a continuum system: $\Psi((z_1^\uparrow, z_1^\downarrow) = \prod_{i<j}[z_i^\uparrow - z_j^\uparrow]^2 [z_i^\downarrow - z_j^\downarrow]^2 \prod_{\lambda \neq \mu} (z_i^\alpha - z_j^\mu)e^{-\sum_i |z_i^\mu|^2/4l_B^2}$ where $l_B$ is the magnetic length. The generalization to a lattice system is straightforward. Note that this is a spin singlet wave function. In the two-component case, the transition to the charge-2 SC is again understood as a transition where $C_1$ changes from $-1$ to $-2$. Since the spinful fermions $f_\alpha$ are not modified, $SU(2)$ spin symmetry can be preserved and the state can remain a spin singlet throughout the transition.

CSL/Laughlin FQH states in terms of BIQH states – Consider an $SU(2)$ invariant spin-1/2 system, and write the spin-1/2 operator in terms of Schwinger bosons as $\hat{\sigma} = \hat{b}^\dagger \sigma \hat{b}$, where $\hat{b}^\dagger = (\hat{b}^\dagger_\beta, \hat{b}^\dagger_\alpha) \equiv \hat{b}^\dagger_{\alpha \beta}$ is a two-component complex scalar boson. This description introduces a $U(1)$ gauge field $a$ associated with the gauge transformation $b \rightarrow e^{i\theta}\hat{b}$, which keeps physical operators invariant and implements the constraint $\hat{b}^\dagger \hat{b} = 1$. The description of the conventional magnetically ordered or valence bond solid (VBS) states in terms of Schwinger bosons (Fig. $[1]$) follows from previous work $[6]$. Here we consider a mean-field state where $b$ forms the spin singlet BIQH state with $\sigma_{xy} = 2$. Using the effective field theory $[11]$ for the BIQH state, it follows that the effective theory for the CSL is given by $[11]$, but with $A_x$ replaced by the dynamical gauge field $a$, and where the current of $b_\alpha$ is given by $j_{\beta \mu} = \frac{1}{\epsilon} \epsilon_{\alpha \beta \mu \nu} \partial_\nu a^\nu$. Integrating out $a^\nu$ then gives the known theory for the KL-CSL $[1, 3]$: $\mathcal{L} = -\frac{2}{4\pi} \sigma a \partial \sigma + a \cdot (\hat{j}_\uparrow + \hat{j}_\downarrow)$, where the currents $\hat{j}_\beta$ for the gapped bosons $b_\alpha$ are now included in the low energy theory as “test charges.”

If instead the mean-field band dispersion of $a_\alpha$ has $n$ minima at momenta $Q_{\beta} = (\beta, \ldots, n)$, then we can approximate $b_\alpha(\tau) \approx \sum_{\beta=1}^n e^{iQ_{\beta} \tau} a_\beta(\tau)$, leading to $n$ flavors of two-component bosons $b_{\alpha \beta}$. A mean-field state where each of $n$ flavors forms a spin singlet $\sigma_{xy} = 2$ BIQH state then leads, at long wavelengths, to the effective theory $\mathcal{L} = -\frac{2\pi}{2\pi} a \partial a + a \cdot (\sum_{\alpha \beta} j_{\alpha \beta})$, where $j_{\alpha \beta}$ is the current of $b_{\alpha \beta}$. This theory has $2n$ topologically distinct quasiparticle excitations with fractional statistics $\theta_{\alpha \beta} = \pi k^2/n \mod 2\pi$, and therefore is topologically equivalent to the original KL-CSL. In contrast, the fermionic spinon construction of $[4]$, where the fractional statistics are given by $\theta_{\alpha \beta} = \pi (k^2/n + k) \mod 2\pi$, has a different topological order when $n > 1$. Thus for the $n$ construction we have presented is in the same universality class as the state originally proposed by KL, but not to the state proposed in $[4]$. The reason is that $[4]$ starts
with a construction of fermionic spinons, \( \vec{S} = \frac{1}{2} e^i \vec{\sigma} c \), and considers a mean-field state where the fermionic spinons \( c \) form a \( \nu = 2n \) fermionic IQH state, leading to the extra shifts of \( \pi \) in \( \theta_v \). Within this mean-field theory it is not possible to understand the continuous transition to the \( Z_2 \) spin liquid, which necessitates the construction of this paper (see Appendix [B]).

The above construction suggests that the wave function for the CSL can be obtained from the Gutzwiller projection of the BIQH state of \( b \) onto the physical Hilbert space of one particle per site:

\[
|\Psi_{\text{phys}}\rangle = \mathcal{P}_G |\Phi_{\text{BIQH}}\rangle,
\]

where \( |\Phi_{\text{BIQH}}\rangle \) is the mean-field state describing the BIQH state of \( b \), and \( \mathcal{P}_G \) is the Gutzwiller projection. \( |\Psi_{\text{phys}}\rangle \) is in general different from the wave functions proposed originally [3, 4].

Now consider bosons with a conserved \( U(1) \) charge in the FQH regime, instead of \( SU(2) \) invariant spin-1/2 systems. The vortices of a Bose superfluid can be described in terms of a bosonic field \( \phi_v \), coupled to a non-compact gauge field \( A \), where the particle current is \( j_\nu = \frac{1}{2\pi} e^{\nu A} \partial_t A \). This implies that the vortices see the original particles as a magnetic field. When bosons are at filling fraction \( \nu = 1/2n \), the vortices are at an effective filling fraction \( \nu = n \), and therefore are poised to form a BIQH state. The effective field theory of the vortices is given by \( \mathcal{L} = \frac{1}{2} A \partial_t A + A \cdot j + \mathcal{L}_v(\phi_v) \), where \( \mathcal{L}_v(\phi_v) \) is the effective field theory of the vortices and \( j_v \) is the vortex current. When the vortices form the BIQH state, described by the effective theory [1], we obtain a \( U(1)^3 \) CS gauge theory described by a \( K \)-matrix

\[
K = \begin{pmatrix}
0 & 1 & 1 \\
1 & 2(n-1) & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

and the fractional statistics of the quasiparticles coincide with those of the \( 1/2n \) Laughlin state. This can be seen more directly by integrating out \( \phi_v \) and \( \phi_v^2 \) to obtain \( \mathcal{L} = \frac{\lambda}{2\pi} A \partial_t A + \frac{1}{2\pi} A \partial_t A \), which is the more conventional theory of the \( 1/2n \) Laughlin FQH state [1].

It follows that when the vortices form either the BIQH state, a trivial Mott insulator, or a superfluid, then the original bosons form either the Laughlin FQH state, the superfluid, or the trivial Mott insulator, respectively. Therefore the former transitions, studied in [23, 24], when interpreted in terms of vortices, describe the latter transitions, studied in [24].

Let us turn to wave functions. A simple trial wave function for a single vortex in a continuum bosonic superfluid takes the form [23]:

\[
|\Psi_v(\eta; \{ \xi_l \})\rangle = \prod_l (z_l - \eta) g(z_l - \eta) |\Psi_0(\{ \xi_l \})\rangle,
\]

where \( z_l = x_{l,z} + i x_{l,y} \) is the complex coordinate of the \( l \)th boson, \( \eta \) is the complex coordinate of the vortex, and \( g(r) \propto 1/r \) is a real function that allows for the relaxation of the particle density profile of the vortex in the interacting superfluid. The ground state wave function \( \Psi_0 \) of the interacting superfluid is usually considered to take the Jastrow form

\[
\Psi_0(\{ \xi_l \}) = e^{\sum_{i<j} u(r_i - r_j)},
\]

for a suitable real function \( u(r) \). This suggests that the many-body boson wave function for a state where the vortices have formed a BIQH state may be described by the following ansatz:

\[
\Psi_{\text{fgh}}(\{ \eta_l \}) = \int \prod_i d^2 \xi |\Psi_v(\{ \eta_l \}, \{ \xi_l \})\rangle |\Phi_{\text{BIQH}}(\{ \xi_l \})\rangle
\]

where \( |\Phi_{\text{BIQH}}(\{ \xi_l \})\rangle \) is the BIQH wave function for particles with complex coordinates \( \{ \xi_l \} \), and \( |\Psi_v(\{ \eta_l \}, \{ \xi_l \})\rangle \) is the natural generalization of \( |\Psi_v(\{ \xi_l \}\rangle \) to a state with many well-separated vortices. \( (S) \) and its connection to the effective field theory studied here is reminiscent of the standard FQH hierarchy construction [1].

Continuous transitions between CSL/Laughlin FQH states and \( Z_2 \) fractionalized states – Let us begin with the case of the \( SU(2) \) invariant spin liquid. When the Schwinger bosons form the BIQH state with \( \sigma_{xy} = 2 \), the spin-1/2 system forms the \( n = 1 \) KL-CSL state. When the Schwinger bosons form the charge-2 SC described above, then the arguments given previously imply that the resulting spin liquid is topologically equivalent to the “doubled semion” model. The critical theory is given by \( (\Phi) \), with \( A_\nu \) instead replaced by the emergent dynamical gauge field \( A \). A wave function that interpolates through the transition can be written by using \( (\Phi) \), and with \( |\Phi_{\text{BIQH}}\rangle \) replaced by the spinful wave function of the BIQH to charge-2 SC transition described above. The generalization to \( n > 1 \) is given in Appendix [D]. Note that in the above construction, the spins remain gapped through the transition: only spin singlet operators appear in the critical theory. More details about the physical operators at the critical point are discussed in Appendix [C].

A possible microscopic Hamiltonian that may tune through this transition is the following \( SU(2) \) symmetric spin-1/2 model on the Kagome lattice:

\[
H = \lambda H_{\text{csl}} + H_{\text{z2}},
\]

where \( H_{\text{csl}} = J_\nu \sum_{i,j,k \in \Delta} \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k) \) with the sum over the triangles of the Kagome lattice, with \( i, j, k \) ordered clockwise around the vertices of the triangles, and \( H_{\text{z2}} = J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle \langle ij \rangle \rangle} \vec{S}_i \cdot \vec{S}_j \). \( H_{\text{csl}} \) has recently been shown to lead to a gapped \( n = 1 \) CSL [3] and \( H_{\text{z2}} \) has recently been shown to lead to a topological spin liquid either in the toric code or doubled semion universality classes [3, 10, 11] when \( J_2/J_1 \) is close to zero. The considerations of this paper suggest that a natural transition out of the \( n = 1 \) CSL is to the doubled semion model; the simplest possible phase diagram of [3] is therefore that when \( \lambda \ll 1 \), \( H \) realizes the doubled semion model, and when \( \lambda \gg 1 \) it realizes the CSL, and there is a direct continuous transition between them when \( \lambda \sim 1 \). More complicated phase diagrams are also possible; in Appendix [A] we will provide a theory of a transition between the doubled semion and toric code theories, which may also appear in the global phase diagram (Fig. [F]).
In the case of bosons with a conserved $U(1)$ charge, the transition between the Laughlin FQH state and the $Z_2$ states can be understood as a transition where the vortices undergo a BIQH to charge-2 SC transition, which is again given by (A3), with $A_{\nu}$ instead replaced by the noncompact gauge field $A$ to which the vortices are coupled. Now, a wave function that interpolates through the transition is given by (5), with $\Phi_{\mathrm{bqh}}(\{\eta_i\})$ replaced by the BIQH charge-2 SC transition, (4).

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Appendix A: Continuous transition between doubled semion and toric code models

Here we note that it is possible to understand a continuous transition between two kinds of $SU(2)$ invariant $Z_2$ spin liquids, corresponding to the doubled semion and toric code models, respectively. Specifically, the doubled semion model can continuously transition to the toric code model in the presence of $SU(2)$ spin symmetry. The Lagrangian at long wavelengths is described by $QED_3$ with two flavors of Dirac fermions ($N_f = 2$). Remarkably, in this case the gaps of both triplet and singlet excitations approach zero at the quantum phase transition. In the absence of $SU(2)$ spin symmetry, there can be generically an intervening topologically trivial gapped insulator unless other symmetries are present to stabilize the direct transition.

To see this, first consider the BIQH phase diagram studied in [23, 24]. This can be understood using the following field theory:

$$\mathcal{L} = \sum_{k = \uparrow, \downarrow} \bar{\psi}_k [\gamma^\mu (\partial_\mu - i\alpha_\mu) + M_k] \psi_k - \frac{1}{g^2} (\epsilon_{\mu\nu\lambda} \partial_\lambda \alpha_\mu)^2 - \frac{1}{4\pi} A_e \partial A_e - \frac{1}{2\pi} A_e \partial \alpha,$$  \hspace{1cm} (A1)

where here $\alpha$ is an emergent $U(1)$ gauge field, $\psi_k$ for $k = \uparrow, \downarrow$ are each two-component Dirac fermions, and $k$ is a physical $SU(2)$ spin index. $\gamma^\mu$ are the Pauli matrices, and $\psi_k = \psi_{k\gamma}^\dagger$. $SU(2)$ symmetry implies that $M_\uparrow = M_\downarrow$. When $M_1 = M_2 > 0$, the theory is in the BIQH state; when $M_1 = M_2 < 0$, the theory is in a topologically trivial insulating state with $\bar{\sigma}_{xy} = 0$. When $M_1$ and $M_2$ have opposite signs, which breaks the $SU(2)$ spin symmetry, then the theory can be shown to be in a superfluid state. (A1) can be understood as arising from the following parton construction

$$b_k = f_0 f_k,$$

where $k = \uparrow, \downarrow$ and $b_k$ have charge 1 under the external gauge field $A_e$. Next, we assume a mean-field ansatz where $f_0$ forms a quantized Hall insulator with Chern number $C_0 = -1$, while each $f_k$ is undergoing a Chern-number changing transition from Chern numbers $C_k = 1$ to $C_k = 0$. (A1) is simply the field theory for this transition. Since the fermions are undergoing a Chern-number changing transition, the gap to spinful excitations closes at this transition. Note that the direct transition involving two Dirac fermions is protected by the $SU(2)$ spin symmetry; when it is broken, and there is no additional symmetry protecting $M_\uparrow = M_\downarrow$, the direct transition will split into two transitions. The charge-1 superfluid phase of $b_k$ requires $(C_\uparrow, C_\downarrow) = (1, 0)$ or $(0, 1)$, and therefore requires the $SU(2)$ to be broken. Note that in this theory, it is important that there is no chemical potential term for the fermions, which requires that the density of $f_\alpha$, and correspondingly $b_\alpha$, each be constant through the transition.

Now consider the above theory, but with the global $U(1)$ symmetry associated with $A_e$ explicitly broken to $Z_2$ by a charge-2 Higgs scalar:

$$\mathcal{L} = (|\partial - i 2 A_e|)^2 - m|\Phi|^2 + \lambda |\Phi|^2 - \frac{1}{4\pi} A_e \partial A_e$$

$$+ \sum_{k = \uparrow, \downarrow} \bar{\psi}_k [\gamma^\mu (\partial_\mu - i\alpha_\mu) + M_k] \psi_k - \frac{1}{2\pi} A_e \partial \alpha,$$  \hspace{1cm} (A3)

with $m > 0$ so that $\Phi$ is condensed: $\langle \Phi \rangle \neq 0$. The insulating states are now replaced by gapped states with a $Z_2$ global symmetry. Remarkably, the BIQH state descends into the topologically non-trivial $Z_2$ SPT state, while the trivial insulator descends into the topologically trivial $Z_2$ symmetric gapped state. The superfluid descends into a $Z_2$ symmetry breaking state. This can be seen as follows. The BIQH state has two counterpropagating edge modes, described by two chiral boson fields $\phi_L$ and $\phi_R$. The backscattering term $\cos(a(\phi_L + \phi_R))$, for integer $a$, is...
prohibited by the $U(1)$ charge conservation, because only $\phi_L$ transforms under the action of the $U(1)$ symmetry. If $U(1)$ is broken to $Z_2$, terms of the form $\cos(2\alpha(\phi_L + \phi_R))$ are now allowed in the Hamiltonian as they preserve the $Z_2$ symmetry. However if they generate an energy gap, then $(e^{i(\phi_L + \phi_R)}) \neq 0$, which breaks the $Z_2$ symmetry. Therefore the edge states still cannot be gapped without breaking the $Z_2$ symmetry. This is the signature of the topologically non-trivial $Z_2$ SPT state. In contrast, the trivial Mott insulator has no edge states in the presence of either the $U(1)$ or $Z_2$ global symmetries.

The critical theory between the $Z_2$ trivial and non-trivial SPT states becomes a critical theory between the doubled semion and toric code models when the global $Z_2$ symmetry is gauged. This can be done in the above theory by replacing $A_e$ with a dynamical $U(1)$ gauge field $a$:

$$\mathcal{L} = (|\partial - i2\alpha\Phi|^2 - m|\Phi|^2 + \lambda|\Phi|^2 - \frac{1}{4\pi} a \partial a)
+ \sum_{k=\uparrow,\downarrow} \tilde{\psi}_k [\gamma^\mu(\partial_\mu - i\alpha_\mu) + M_k] \psi_k - \frac{1}{2\pi} a \partial a \alpha.$$ (A4)

This occurs when we interpret $b_k$ as Schwinger bosons (which are related to the $SU(2)$ spins by $\vec{S} = \frac{1}{2}\vec{b}$). When $\Phi$ is condensed, at long wavelengths the effective theory can be written in terms of the phase $\theta$ of $\Phi$:

$$\mathcal{L} = (|\partial - 2ia|^2 + \sum_{k=\uparrow,\downarrow} \tilde{\psi}_k [\gamma^\mu(\partial_\mu - i\alpha_\mu) + M_k] \psi_k
- \frac{1}{4\pi} a \partial a - \frac{1}{2\pi} a \partial a \alpha)$$ (A5)

Picking the gauge $\theta = 0$, $a_0 = 0$ and integrating out $a$ then just gives the action of QED$_3$, with two flavors of Dirac fermions ($N_f = 2$):

$$\mathcal{L} = \sum_{k=\uparrow,\downarrow} \tilde{\psi}_k [\gamma^\mu(\partial_\mu - i\alpha_\mu) + M_k] \psi_k - \frac{1}{g^2} (\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda)^2$$ (A6)

In the large $N_f$ limit, it is known that a critical fixed point exists, although this has not been fully established when $N_f = 2$.

Appendix B: Projective construction of BIQH state

Here we fill in some details about why the construction presented in the main text describes the BIQH state. Recall that we set

$$b = f_1 f_2,$$ (B1)

and we consider a mean-field ansatz where $f_1$, $f_2$ form quantized Hall insulators with Chern numbers $(C_1, C_2) = (-1, 2)$. The emergent $U(1)$ gauge field $a$ is associated with the gauge redundancy $f_1 \rightarrow e^{i\beta} f_1$ and $f_2 \rightarrow e^{-i\beta} f_2$. Without loss of generality, suppose $f_1$, $f_2$ carry charges $1$ and $0$, respectively, under the external gauge field $A_e$. Integrating out $f_1$ and $f_2$ then yields the following effective theory for the gauge fields:

$$\mathcal{L} = -\frac{1}{4\pi} (A_e + a) \partial (A_e + a) + \frac{2}{4\pi} a \partial a$$

$$= -\frac{1}{4\pi} A_e \partial A_e - \frac{1}{2\pi} A_e \partial a + \frac{1}{4\pi} a \partial a.$$ (B2)

Since CS term for $a$ has unit coefficient, clearly this describes a gapped state with unique ground state degeneracy on all closed manifolds. The elementary excitations are particles/holes in the $f_1$ and $f_2$ states, which, after being dressed by a gauge flux due to the CS term, become bosonic excitations. Furthermore, integrating out $a$ yields the response theory

$$\mathcal{L} = -\frac{2}{4\pi} A_e \partial A_e,$$ (B3)

which shows that the state has Hall conductance $\sigma_{xy} = 2$. Therefore this state describes a BIQH state, with no intrinsic topological order.

An alternative way of seeing this result, and directly deriving the effective theory $\mathcal{L}$, is as follows. We introduce three gauge fields, $a^1$, $a^2$, and $a^3$, and their associated conserved currents $j_{\mu}^I = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_{\lambda}^I$, for $I = 1, ..., 3$. $j^1$ is taken to describe the current of $f_1$. The Chern number $2$ state of $f_2$ is then assumed to consist of two filled bands, each with Chern number 1, so that the current of $f_2$ can be described by two conserved currents, $j^2$ and $j^3$, each of which describes the dynamics of a Chern number 1 band. Now, since $j^I$ each describe the dynamics of a band with unit Chern number, the effective theory is:

$$\mathcal{L} = \frac{1}{4\pi} (a^2 \partial a^2 + a^3 \partial a^3 - a^1 \partial a^1) + \frac{1}{2\pi} a \partial(a^2 + a^3 - a^1) + \frac{1}{2\pi} A_e \partial a^1.$$ (B4)

Integrating out $a$ then gives $\mathcal{L}$, for $n = 1$.

Finally, note that the case where $(C_1, C_2) = (-k, k+1)$ also describes a BIQH state, with Hall conductance $\sigma_{xy} = k(k+1)$.

Appendix C: Physical operators at the CSL - $Z_2$ critical point

Recall the critical theory between the $1/2n$ Laughlin FQH state and the $Z_2$ fractionalized states is described by

$$\mathcal{L} = (|\partial - i2nA\Phi|^2 + m|\Phi|^2 + \lambda|\Phi|^2
- \frac{2n}{4\pi} A \partial A + \frac{1}{2\pi} A_e \partial A.$$ (C1)

In the case of the FQH transition, where the physical degrees of freedom are bosons with a conserved $U(1)$ charge, the boson destruction operator at the critical point is
$$b = \hat{M}\Phi,$$

where \(\hat{M}\) removes 2\(\pi\) units of flux of \(A\). Due to the CS term, 2\(\pi\) flux of \(A\) carries 2 units of \(A\) charge, and therefore the physical gauge-invariant operator must also remove a quanta of \(\Phi\). From the coupling of the external field, we see that a 2\(\pi\) flux of \(A\) will carry unit charge under \(A_e\) and therefore corresponds to the physical boson. \(\Phi\) describes double vortices, as it carries charge 2 under \(A\). Single vortices remain gapped through the transition and do not appear in the low energy theory at the critical point.

Now let us consider the \(SU(2)\) invariant spin-1/2 system. Here, the critical theory is described by \((C_1, C_2)\), but without the external gauge field \(A_e\) and with \(A\) replaced by the gauge field \(a\) to which the Schwinger bosons couple. The Higgs field \(\Phi\) itself can be physically understood as a spin singlet pair of the Schwinger bosons. The single spin operators do not appear as scaling operators in the critical theory. The fact that the partons \(f_\alpha\) remain in a gapped Chern insulator indicates that the triplet gap remains finite through the transition. The only scaling operators in the theory are therefore spin singlet operators. These include the gauge flux:

$$j_\mu = \frac{1}{2\pi} \epsilon_{\mu \nu \lambda} \partial_\lambda \times (\hat{S} \times \triangledown f) = \frac{1}{2} \pi f_{\mu} \partial_\lambda \times (\hat{S} \times \triangledown f).$$

This flux is proportional to the skyrmion number of \(\hat{S}\). The other basic gauge invariant operator is \(M\Phi\), which physically corresponds to the operator that changes the skyrmion number by 1. The scaling functions of these operators can be readily obtained through various large \(N\) approximations. These were performed for \(U(1)\) CS-Higgs theories in [22], where it was shown that the transitions are indeed continuous in the large \(N\) limit.

**Appendix D: CSL - \(Z_2\) transition for general \(n\)**

In the main text we discussed the transition between the \(n = 1\) CSL and the \(Z_2\) spin liquid (doubled semion). To describe the case with general \(n\), we start with the \(n\) flavors of two-component Schwinger bosons:

$$b_\alpha(\hat{\tau}) \approx \sum_{\beta=1}^{n} e^{-iQ_\beta \cdot \hat{\tau}} b_{\alpha \beta}(\hat{\tau})$$

for \(\beta = 1, ..., n\) and \(\alpha = \uparrow, \downarrow\), and we consider a state where one of the \(n\) flavors transitions from the \(\bar{\sigma}_{xy} = 2\) BIQH to the charge-2 SC while the other \(n - 1\) flavors stay in the \(\bar{\sigma}_{xy} = 2\) BIQH state. The resulting state is described by a \(U(1) \times U(1)\) CS theory with \(K = \left( \begin{array}{cc} 0 & 2 \\ 2 & 2n \end{array} \right)\), which is topologically equivalent to the toric code/doubled semion models when \(n\) is even/odd, respectively. The wave function that interpolates through this transition, for general \(n\), is given by [17], with \(|\Phi_{BIQH}|\) replaced by \(\sum_{\beta} e^{-iQ_\beta \cdot \hat{\tau}} (|0\rangle |f\rangle |f^\dagger \rangle |\Phi_{MF}|\).

The mean-field state of the fermionic partons \(f_{1,\beta}\) and \(f_{\beta,3}\) where \(f_{\alpha,3}\) for \(\beta = 1, ..., n\) form spin singlet Chern insulators with Chern number 2, and \(f_{1,\beta}\) form a Chern insulator with Chern number \(-1\), while one of the \(f_{1,\beta}\) undergoes a Chern number changing transition from Chern number \(-1\) to \(-2\). \(|0\rangle\) denotes the \(f\)-vacuum.

**Appendix E: Review of fermionization of 3D XY transition**

Here we will briefly review the fermionization of the 3D XY transition, which was proposed in [27]. It will be helpful to consider Table I of the main text. Notice that the Chern number assignment \((C_1, C_2) = (1, 0)\) for the mean-field states of \(f_1\) and \(f_2\) leads to a description of a topologically trivial Bose Mott insulator. In contrast, the case \((C_1, C_2) = (1, -1)\) describes a Chern number changing transition for \(f_2\). Such a critical theory is described by the following action:

$$\mathcal{L}_{\text{ferm}} = \frac{1}{4\pi^2} a \partial a + \hat{\psi} \gamma^\mu (\partial_\mu - ia_\mu) + m \hat{\psi}\psi$$

$$+ \frac{1}{4\pi} A_\mu \partial A_\mu + \frac{1}{2\pi} A_\mu \partial a, \quad (E1)$$

where \(m > 0\) describes the Mott insulator and \(m < 0\) describes the superfluid. Since this theory describes the Mott insulator - superfluid transition in the presence of particle-hole symmetry, it is conjectured to be equivalent to the conventional 3D XY critical point:

$$\mathcal{L}_{xy} = |(\partial - iA_e)\Phi|^2 + m |\Phi|^2 + \lambda |\Phi|^4 \quad (E2)$$

Rescaling \(A_e \to A_e' = A_e/2\) and subtracting both (E1) and (E2) by \(\frac{1}{2\pi} A_e \partial A_e\) gives the duality used in the main text.

**Appendix F: Insufficiency of slave fermion construction**

In this section, we discuss in some more detail the necessity of the BIQH construction of this paper, as compared with the fermionic IQH construction of [4], for understanding the transitions between the CSL and the \(Z_2\) spin liquids. The construction of [4] starts by writing the spin-1/2 operator in terms of slave fermions:

$$\hat{S} = \frac{1}{2} c^\dagger \bar{c} c, \quad (F1)$$

c is a two-component fermion. This leads to an emergent \(U(1)\) gauge field \(a\) associated with the gauge transformation \(c \to e^{ia} c\). The CSL corresponds to a state where the fermions \(c\) form a spin singlet Chern insulator with Chern number \(C_c = -2n\). For \(n = 1\), this leads to a state that is equivalent to the KL-CSL, while for \(n > 1\), the fractional statistics of the quasiparticles are slightly different from the KL-CSL, as summarized in the main text.

In order to describe a transition to a \(Z_2\) state, one possibility is to consider a transition where the pair field of the fermions, \(\Phi = f_1 f_2\), condenses, thus breaking the \(U(1)\) gauge symmetry to \(Z_2\). In contrast to the BIQH
state, in the fermionic case there is no theory of a transition from a fermion IQH state that simultaneously condenses the pair field and completely destroys the edge modes. Instead we can consider a scenario where the pair field condenses, but the edge modes are not destroyed. To understand the nature of the resulting state, let us focus on the case $n = 1$. The IQH states of the fermions can be described by two $U(1)$ CS gauge fields $a^\uparrow$, $a^\downarrow$, such that $j_{\alpha\mu} = \frac{1}{2\pi} c_{\mu\lambda} \partial_\lambda a^\alpha$ describes the current of $e_\alpha$. The effective theory is therefore:

$$\mathcal{L} = \frac{1}{4\pi} (a^\uparrow \partial a^\uparrow + a^\downarrow \partial a^\downarrow) + \frac{1}{2\pi} a \partial (a^\uparrow + a^\downarrow) + (\partial - 2i\alpha)\Phi \Phi^* + m|\Phi|^2 + \lambda|\Phi|^4.$$  

(F2)

Here, $\Phi$ represents the pair field $f_1 f_1$; its condensation breaks the $U(1)$ gauge symmetry to $Z_2$. In order to understand the topological properties of the $\Phi$-condensed phase, it is helpful to perform a particle-vortex duality on $\Phi$:

$$\mathcal{L} = \frac{1}{4\pi} (a^\uparrow \partial a^\uparrow + a^\downarrow \partial a^\downarrow) + \frac{1}{2\pi} a \partial (a^\uparrow + a^\downarrow + 2\alpha) + (\partial - i\alpha)\Phi \Phi^* + m'|\Phi|^2 + \lambda'|\Phi|^4.$$  

(F3)

Here, $j_{\Phi} = \frac{1}{2\pi} c_{\mu\lambda} \partial_\lambda \alpha$ is the current of the original $\Phi$, while $\Phi_v$ describes the vortices of $\Phi$. The phase where $\Phi$ is condensed corresponds to the case where $\Phi_v$ is uncondensed, and vice versa. Therefore, when $\Phi$ is condensed, $\Phi_v$ is uncondensed and can be integrated out, leaving us with a $U(1)^4$ CS theory with $K$-matrix

$$K = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$  

(F4)

This satisfies $|\text{Det } K| = 4$. However it has three positive eigenvalues and one negative eigenvalue, implying that the system has a net chiral central charge $c = 2$, and therefore has topologically protected gapless edge states. Such a state is topologically distinct from both the toric code and doubled semion models. It can be thought of as a chiral spin liquid with a discrete gauge structure, $eg$ a chiral topological superconductor with 4 chiral edge Majorana modes coupled to a fluctuating $Z_2$ gauge field. Such a state can be constructed using the honeycomb Kitaev model $[34, 35]$. While it is interesting that this “$Z_2$ CSL” also neighbors the KL-CSL, it does not correspond to the two different $Z_2$ spin liquids considered in the main text. It appears that the construction in terms of fermionic spinons is indeed insufficient to describe the transition between the CSL and the $Z_2$ spin liquids, although it does allow access to the transition to a $Z_2$ CSL.

**Appendix G: Topological properties of Abelian Chern-Simons-Higgs theories**

Here we will briefly review the topological properties of Abelian CS-Higgs theories $[14]$ (see also $[21]$ for a recent discussion). Specifically, consider a $U(1)_n$ CS term, coupled to a charge-$k$ Higgs field:

$$\mathcal{L} = (|\partial - ikA|\Phi|^2 + m|\Phi|^2 + \lambda|\Phi|^4 + \frac{n}{4\pi} A\partial A,$$  

(G1)

where $\lambda > 0$. When $\Phi$ is uncondensed this describes $U(1)_n$ CS theory, the topological properties of which are well known $[14]$; there are $n$ topologically distinct quasiparticles, with fractional statistics $\theta_a = \pi a^2/n$.

In order to understand the topological properties of the Higgs phase, where $\Phi$ is condensed and the gauge group $U(1)$ is broken to $Z_k$, it is helpful to perform a duality transformation on $\Phi$ and to consider the theory in terms of the vortices of $\Phi$. This is described by the theory

$$\mathcal{L} = (|\partial - i\alpha|\Phi_v|^2 + m|\Phi_v|^2 + \lambda|\Phi_v|^2$$

$$+ \frac{k}{2\pi} A\partial A + \frac{n}{4\pi} A\partial A,$$  

(G2)

where $\Phi_v$ is a complex scalar describing the vortices of $\Phi$. The condensed phase of $\Phi$ is therefore the uncondensed phase of $\Phi_v$. Considering the case where $\Phi_v$ is uncondensed (Higgs phase of (G1)), we can integrate it out to obtain a $U(1) \times U(1)$ CS theory:

$$\mathcal{L} = \frac{n}{4\pi} A\partial A + \frac{k}{2\pi} A\partial A.$$  

(G3)

The topological properties of such a theory can be directly read off from the $K$-matrix:

$$K = \begin{pmatrix} n & k \\ k & 0 \end{pmatrix}.$$  

(G4)

It has $k^2$ distinct quasiparticles. For $n$ even, this describes a topological phase where the low degrees of freedom are all bosons, and otherwise the low degrees of freedom contain fermions. In general, we can perform an $SL(2; Z)$ transformation $K \rightarrow K' = W^T K W$, with $W \in SL(2; Z)$, which keeps the topological properties of $K$ invariant, such that

$$K' = \begin{pmatrix} n - 2k & k \\ k & 0 \end{pmatrix}.$$  

(G5)

Therefore, the case where $n$ is a multiple of $2k$ leads to $K' = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$, which encodes the topological properties of ordinary $Z_k$ gauge theory.

When $n$ is even, the above can be viewed as a generalized version of $Z_k$ gauge theory that corresponds to one of the $k$ different kinds of $Z_k$ gauge theory found in $[14]$.

The case $n = k = 2$ describes the doubled-semission model discussed in the main text.
[1] X.-G. Wen, Quantum Field Theory of Many-Body Systems (Oxford Univ. Press, Oxford, 2004).

[2] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Rev. Mod. Phys. 80, 1083 (2008).

[3] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987); V. Kalmeyer and R. B. Laughlin, Phys. Rev. B 39, 11879 (1989).

[4] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B 39, 11413 (1989).

[5] P. A. Lee, N. Nagaosa, and X.-G. Wen, Rev. Mod. Phys. 78, 17 (2006); P. Anderson, Mater. Res. Bull. 8, 153 (1973); P. Anderson, Science 235, 1196 (1987); G. Baskaran, Z. Zhou, and P. Anderson, Solid State Commun. 63, 973 (1987); S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B 35, 8865 (1987); D. J. Thouless, Phys. Rev. B 36, 7187 (1987); N. Read and B. Chakraborty, Phys. Rev. B 40, 7333 (1989); N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991); X. G. Wen, Phys. Rev. B 44, 2664 (1991); T. Senthil and M. P. A. Fisher, Phys. Rev. B 62, 7850 (2000).

[6] A. Kitaev, Annals Phys. 303, 2 (2003), arXiv:quant-ph/9707021; R. Moessner and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 (2001); X.-G. Wen, Phys. Rev. Lett. 90, 016803 (2003); O. I. Motrunich and T. Senthil, Phys. Rev. Lett. 89, 277004 (2002); M. A. Levin and X.-G. Wen, Phys. Rev. B 71, 045110 (2005).

[7] H.-C. Jiang, Z. Wang, and L. Balents, Nature Phys. 8, 902 (2012).

[8] H.-C. Jiang, H. Yao, and L. Balents, Phys. Rev. B 86, 024424 (2012).

[9] B. Bauer, B. P. Keller, M. Dolfi, S. Trebst, and A. W. Ludwig (2013), arXiv:1303.6963v1.

[10] S. Yan, D. A. Huse, and S. R. White, Science 332, 1173 (2011).

[11] S. Depenbrock, I. P. McCulloch, and U. Schollwöck, Phys. Rev. Lett. 109, 067201 (2012).

[12] A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. 94, 086803 (2005); E. H. Rezayi, N. Read, and N. R. Cooper, Phys. Rev. Lett. 95, 160404 (2005); M. Hafezi, A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. A 76, 023613 (2007); N. Cooper, Advances in Physics 57, 539 (2008); G. Möller and N. R. Cooper, Phys. Rev. Lett. 103, 105303 (2009); N. Y. Yao, A. V. Gorshkov, C. R. Laumann, A. M. Läuchli, J. Ye, and M. D. Lukin, Phys. Rev. Lett. 110, 185302 (2013).

[13] P. A. Lee, Science 321, 1306 (2008); L. Balents, Nature 464, 199 (2010).

[14] R. Dijkgraaf and E. Witten, Commun. Math. Phys. 129, 393 (1990); F. A. Bais, P. van Driel, and M. de Wild Propitius, Nucl. Phys. B 393, 547 (1993).

[15] T. Senthil and M. Levin, Phys. Rev. Lett. 110, 046801 (2013).

[16] Y.-M. Lu and A. Vishwanath, Phys. Rev. B 86, 125119 (2012).

[17] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013); X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science 338, 1604 (2012).

[18] A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016 (2013).

[19] M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012).

[20] B. Swingle (2012), arXiv:1209.0776.

[21] M. Cheng and Z.-C. Gu (2013), arXiv:1302.4803.

[22] J. K. Jain, Composite Fermions (Cambridge University Press, 2007).

[23] R. B. Laughlin, Phys. Rev. Lett. 60, 2677 (1988).

[24] M. Barkeshli and J. McGreevy (2012), arXiv:1201.4393.

[25] T. Grover and A. Vishwanath, Phys. Rev. B 87, 045129 (2013).

[26] Y.-M. Lu and D.-H. Lee (2013), arXiv:1210.0990v1.

[27] W. Chen, M. P. A. Fisher, and Y.-S. Wu, Phys. Rev. B 48, 13749 (1993).

[28] M. Barkeshli and X.-G. Wen, Phys. Rev. B 86, 085114 (2012), arXiv:1012.2417.

[29] L. Onsager, Nuovo Cimento 6, 249 (1949); R. Feynman, in Progress in Low-Temperature Physics, edited by C. Gorter (North-holland, Amsterdam, 1955), vol. 1, p. 17; A. Fetter, Phys. Rev. Lett 27, 986 (1971); A. Fetter, Ann. Phys. (N.Y.) 70, 67 (1966).

[30] A similar Hamiltonian is being studied in L. Cincio, B. Bauer, G. Vidal, S. Trebst, A.W.W. Ludwig, in preparation (2013).

[31] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989).

[32] X.-G. Wen and Y.-S. Wu, Phys. Rev. Lett. 70, 1501 (1993).

[33] D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, Phys. Rev. Lett. 99, 097202 (2007); R. Thomale, E. Kapit, D. F. Schroeter, and M. Greiter, Phys. Rev. B 80, 104406 (2009).

[34] A. Kitaev, Annals of Physics 321, 2 (2006).

[35] H. Yao and S. A. Kivelson, Phys. Rev. Lett. 99, 247203 (2007).

[36] Hamiltonians with longer ranged interactions have also been found to stabilize the KL-CSLs [33]. Other CSLs have also been found as ground states of exactly solvable models [34, 35], although these are in a different universality class as compared with the KL-CSL studied in this paper.

[37] It is crucial for the discussion that we assigned integer charges to $f_1$ and $f_2$. This is required in order to have the conventional periodicities for the large gauge transformations of the $U(1) \times U(1)$ CS gauge fields, $f_C a^i \cdot dl \sim f_C a^i \cdot dl + 2\pi$ for non-contractible loops $C$.  