On transfer inequalities in Diophantine Approximation

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Abstract – Let Θ be a point in \( \mathbb{R}^n \). We split the classical Khintchine’s Transference Principle into \( n - 1 \) intermediate estimates which connect exponents \( \omega_d(\Theta) \) measuring the sharpness of the approximation to \( \Theta \) by linear rational varieties of dimension \( d \), for \( 0 \leq d \leq n - 1 \). We also review old and recent results related to these \( n \) exponents.

1. Introduction and results.

We revisit in this note some results of homogeneous transfer in Diophantine Approximation. Let us first recall the classical Khintchine’s Transference Principle in its primary form [4, 12]. Let \( n \) be a positive integer and let \( \Theta = (\theta_1, \ldots, \theta_n) \) be a point in \( \mathbb{R}^n \). We shall assume in all the forthcoming statements that the real numbers 1, \( \theta_1, \ldots, \theta_n \) are linearly independent over the field \( \mathbb{Q} \) of rational numbers. Khintchine’s Transference Principle relates the sharpness of the rational simultaneous approximation to \( \theta_1, \ldots, \theta_n \) with the measure of linear independence over \( \mathbb{Q} \) of 1, \( \theta_1, \ldots, \theta_n \). Let us first quantify these notions.

Definition 1. We denote respectively by \( \omega_0(\Theta) \) and \( \omega_{n-1}(\Theta) \) (the meaning of the indices 0 and \( n - 1 \) will be explained afterwards) the supremum, possibly infinite, of the real numbers \( \omega \) for which there exist infinitely many integer \((n+1)\)-tuples \((x_0, \ldots, x_n)\) satisfying respectively the inequation

\[
\max_{1 \leq i \leq n} |x_0 \theta_i - x_i| \leq \left( \max_{0 \leq i \leq n} |x_i| \right)^{-\omega} \quad \text{or} \quad |x_0 + x_1 \theta_1 + \cdots + x_n \theta_n| \leq \left( \max_{0 \leq i \leq n} |x_i| \right)^{-\omega}.
\]

Now we can state Khintchine’s Transference Principle as follows:

Theorem 1. The inequalities

\[
\frac{\omega_{n-1}(\Theta)}{(n - 1) \omega_{n-1}(\Theta) + n} \leq \omega_0(\Theta) \leq \frac{\omega_{n-1}(\Theta) - n + 1}{n}
\]

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hold for any point \( \Theta \) in \( \mathbf{R}^n \) with \( 1, \theta_1, \ldots, \theta_n \) linearly independent over \( \mathbf{Q} \). Moreover, both inequalities are optimal.

It is convenient to view \( \mathbf{R}^n \) as a subset of \( \mathbf{P}^n(\mathbf{R}) \) via the usual embedding 
\((x_1, \ldots, x_n) \mapsto (1, x_1, \ldots, x_n)\). In the sequel, we shall identify \( \Theta \) with its image in \( \mathbf{P}^n(\mathbf{R}) \).

Let us introduce for each integer \( d \) with \( 0 \leq d \leq n - 1 \), an exponent \( \omega_d(\Theta) \) which measures the approximation to the point \( \Theta \in \mathbf{P}^n(\mathbf{R}) \) by rational linear projective subvarieties of dimension \( d \), in term of their height. Denote by \( d \) the projective distance on \( \mathbf{P}^n(\mathbf{R}) \) (it will be defined in Section 2 below; notice however that the use of any locally equivalent distance, such as the distance associated to the supremum norm on \( \mathbf{R}^n \), would lead to the same exponents). For any real linear subvariety \( L \) of \( \mathbf{P}^n(\mathbf{R}) \), we denote by
\[
\min_{P \in L} d(\Theta, P)
\]
the minimal distance between \( \Theta \) and the real points \( P \) of \( L \). When \( L \) is rational over \( \mathbf{Q} \), we indicate moreover by \( H(L) \) its height, that is the Weil height of any system of Plücker coordinates of \( L \). The Weil height is normalized by using the Euclidean norm at the Archimedean place of \( \mathbf{Q} \).

**Definition 2.** Let \( d \) be an integer with \( 0 \leq d \leq n - 1 \). We denote by \( \omega_d(\Theta) \) the supremum of the real numbers \( \omega \) for which there exist infinitely many rational linear projective subvarieties \( L \subset \mathbf{P}^n(\mathbf{R}) \) such that
\[
\dim(L) = d \quad \text{and} \quad d(\Theta, L) \leq H(L)^{-1-\omega}.
\]

Definitions 1 and 2 are consistent, since \( d(\Theta, L) \) compares respectively with
\[
\max_{1 \leq i \leq n} \left| \theta_i - \frac{x_i}{x_0} \right| \quad \text{and} \quad \max_{0 \leq i \leq n} \left| y_i + y_1 \theta_1 + \cdots + y_n \theta_n \right|
\]
when \( L \) is either the rational point (case \( d = 0 \)) with homogeneous coordinates \((1, x_1/x_0, \ldots, x_n/x_0)\), or the hyperplane (when \( d = n - 1 \)) with homogeneous equation \( y_0 X_0 + \cdots + y_n X_n = 0 \).

W. Schmidt was the first to investigate in [21] the properties of these exponents \( \omega_d(\Theta) \). In fact, he did not introduce them explicitly; however Theorems 9–18 of [21] provide us with various relations between more general families of exponents, which coincide with the exponents \( \omega_d(\Theta) \) in our more limited framework. Note that in the setting of [21], the point \( \Theta \), which belongs here to \( \mathbf{P}^n(\mathbf{R}) = \mathbf{P}(\mathbf{R}^{n+1}) \), has to be replaced by the associated line in \( \mathbf{R}^{n+1} \). The consideration of real linear subspaces \( \Theta \) of \( \mathbf{R}^{n+1} \) with arbitrary dimension, gives then rise to various angles and consequently to further exponents. See Section 4 of [2] for a projective formulation similar to the present one. Here is our main result which slightly improves on earlier inequalities due to W. Schmidt [21].
Theorem 2. For any point Θ with 1, θ_1, . . . , θ_n linearly independent over Q, and for any integer d with 1 ≤ d ≤ n − 1, we have the estimate

$$\frac{d \omega_d(\Theta)}{\omega_d(\Theta) + d + 1} \leq \omega_{d+1}(\Theta) \leq \frac{(n - d)\omega_d(\Theta) - 1}{n - d + 1}.$$

Following the terminology of [21], we split the above estimate into two parts; namely the Going-up transfer

$$\omega_{d+1}(\Theta) \geq \frac{(n - d)\omega_d(\Theta) + 1}{n - d - 1}, \quad 0 \leq d \leq n - 2,$$

and the Going-down transfer

$$\omega_{d-1}(\Theta) \geq \frac{d \omega_d(\Theta)}{\omega_d(\Theta) + d + 1}, \quad 1 \leq d \leq n - 1.$$

The Going-up transfer (2) refines Theorem 11 of [21] in our restricted situation. It will be established in Section 3. The Going-down transfer (3) is a very special case of Theorem 10 from [21]. We shall not prove it here again, although a similar argumentation using the inner product instead of the wedge product, could be employed. Note that exterior and inner products are algebraic translations of the geometrical operations of join and intersection for subvarieties in a projective space.

An interesting feature of Theorem 2 is that it covers Khintchine’s Transference Principle. Iterating respectively (2) and (3), we easily find by induction on the integer k the lower bounds

$$\omega_{d+k}(\Theta) \geq \frac{(n - d)\omega_d(\Theta) + k}{n - d - k}, \quad 0 \leq d \leq n - 2, \quad 1 \leq k \leq n - d - 1,$$

and

$$\omega_{d-k}(\Theta) \geq \frac{(d - k + 1)\omega_d(\Theta)}{k \omega_d(\Theta) + d + 1}, \quad 1 \leq d \leq n - 1, \quad 1 \leq k \leq d.$$

Thus, selecting respectively in (4) and (5) the pairs (d, k) = (0, n − 1) and (d, k) = (n − 1, n − 1), we obtain (1).

Jarník [8, 9] has established that Khintchine’s Transference Principle is best possible. Since it is a formal corollary of all the inequalities occurring in (2) and (3), it follows that any of these inequalities is optimal. Let us call spectrum of the function ω_d, the set of values taken by the exponents ω_d(Θ), when Θ ranges over R^n, with 1, θ_1, . . . , θ_n linearly independent over Q. Then, Theorem 2 implies the following
Corollary. For any integer \( d \) with \( 0 \leq d \leq n - 1 \), we have the lower bound

\[
\omega_d(\Theta) \geq \frac{d + 1}{n - d},
\]

and equality holds in (6) for almost all \( \Theta \) with respect to Lebesgue measure on \( \mathbb{R}^n \). The spectrum of the function \( \omega_d \) is equal to the whole interval \( [(d + 1)/(n - d), +\infty] \).

Proof. We deduce from (4) and (5) the lower bounds

\[
\omega_d(\Theta) \geq \frac{n \omega_0(\Theta) + d}{n - d} \quad \text{and} \quad \omega_d(\Theta) \geq \frac{(d + 1)\omega_{n-1}(\Theta)}{(n - d - 1)\omega_{n-1}(\Theta) + n}.
\]

By Dirichlet’s Box Principle, we know that \( \omega_0(\Theta) \geq 1/n \) and \( \omega_{n-1}(\Theta) \geq n \). Then, each of the two preceding inequalities imply (6). Moreover, (5) and (4) also provide us with the converse estimates

\[
\omega_0(\Theta) \geq \frac{\omega_d(\Theta)}{d \omega_d(\Theta) + d + 1} \quad \text{and} \quad \omega_{n-1}(\Theta) \geq (n - d)\omega_d(\Theta) + n - d - 1,
\]

so that \( \omega_0(\Theta) > 1/n \) and \( \omega_{n-1}(\Theta) > n \) whenever \( \omega_d(\Theta) > (d + 1)/(n - d) \). Now, the easy part of the classical Khintchine-Groshev Theorem [7, 11] shows that the set of points \( \Theta \) for which \( \omega_0(\Theta) > 1/n \), or equivalently \( \omega_{n-1}(\Theta) > n \), has Lebesgue measure 0 in \( \mathbb{R}^n \). Thus, equality holds in (6) for almost all \( \Theta \).

As for the spectrum of the functions \( \omega_d \), notice that all lower bounds (4) turn out to be equalities

\[
\omega_{d+k}(\Theta) = \frac{(n - d)\omega_d(\Theta) + k}{n - d - k}, \quad 0 \leq d \leq n - 2, 1 \leq k \leq n - d - 1,
\]

whenever \( \omega_{n-1}(\Theta) \) has minimal value \( n \omega_0(\Theta) + n - 1 \), with regard to Khintchine’s Transference Principle (1). Now, Jarník [9] has established that for any \( w \) with \( 1/n \leq w \leq +\infty \), there exists a point \( \Theta \in \mathbb{R}^n \) such that

\[
\omega_0(\Theta) = w \quad \text{and} \quad \omega_{n-1}(\Theta) = nw + n - 1.
\]

For such a point \( \Theta \), we then have

\[
\omega_d(\Theta) = \frac{nw + d}{n - d}, \quad 0 \leq d \leq n - 1.
\]

It follows that, for any \( 0 \leq d \leq n - 1 \), the spectrum of the function \( \omega_d \) coincides with the interval \( [(d + 1)/(n - d), +\infty] \).
Remark. Similarly, the inequalities (5) become equalities when

\[
\omega_0(\Theta) = \frac{\omega_{n-1}(\Theta)}{(n-1)\omega_{n-1}(\Theta) + n}.
\]

Using now the construction of [8], we obtain in this way a second family

\[
\omega_d(\Theta) = \frac{(d + 1)w}{(n - d - 1)w + n}, \quad 0 \leq d \leq n - 1,
\]

of \(n\)-tuples \((\omega_0(\Theta), \ldots, \omega_{n-1}(\Theta))\) indexed by the real parameter \(w \geq n\). We are thus lead to address the following

Problem. Find the spectrum in \((\mathbb{R} \cup \{+\infty\})^n\) of the \(n\)-tuples

\[
\left(\omega_0(\Theta), \ldots, \omega_{n-1}(\Theta)\right),
\]

when \(\Theta\) ranges over \(\mathbb{R}^n\), with \(1, \theta_1, \ldots, \theta_n\) linearly independent over \(\mathbb{Q}\).

Necessary assumptions are provided by the inequalities (4) and (5), together with the lower bounds (6). Are these conditions sufficient? It holds true for \(n = 2\), as follows from our work [14] whose content is described below.

Let us now display a different kind of refinement of Khintchine’s Transference Principle, that has recently been pointed out by Y. Bugeaud and M. Laurent in Theorem 8 of [2]. Following the “hat” notations of [1], introduce first uniform analogues of the above-mentioned exponents \(\omega_0(\Theta)\) and \(\omega_{n-1}(\Theta)\).

Definition 3. We denote respectively by \(\hat{\omega}_0(\Theta)\) and \(\hat{\omega}_{n-1}(\Theta)\) the supremum of the real numbers \(\omega\) such that for all sufficiently large real number \(X\), there exists a non-zero integer \((n+1)\)-tuple \((x_0, \ldots, x_n)\), with supremum norm \(\max_{0 \leq i \leq n} |x_i| \leq X\), which satisfies respectively the inequation

\[
\max_{1 \leq i \leq n} |x_0 \theta_i - x_i| \leq X^{-\omega} \quad \text{or} \quad |x_0 + x_1 \theta_1 + \cdots + x_n \theta_n| \leq X^{-\omega}.
\]

We can now refine Theorem 1 in the following way.

Theorem 3. The inequalities

\[
\frac{(\hat{\omega}_{n-1}(\Theta) - 1)\omega_{n-1}(\Theta)}{(n - 2)\hat{\omega}_{n-1}(\Theta) + 1)\omega_{n-1}(\Theta) + (n - 1)\hat{\omega}_{n-1}(\Theta)} \leq \omega_0(\Theta) \leq \frac{(1 - \hat{\omega}_0(\Theta))\omega_{n-1}(\Theta) - n + 2 - \hat{\omega}_0(\Theta)}{n - 1},
\]

5.
hold for any point $\Theta$ in $\mathbb{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbb{Q}$.

The above estimate is stronger than (1), since

$$\hat{\omega}_{n-1}(\Theta) \geq n \quad \text{and} \quad \hat{\omega}_0(\Theta) \geq \frac{1}{n}$$

by Dirichlet’s Box Principle.

Notice that Theorem 3 is optimal in dimension $n = 2$. In that case, the uniform exponents $\hat{\omega}_0(\Theta)$ and $\hat{\omega}_1(\Theta)$ are linked by Jarník’s equation [10]

$$\hat{\omega}_0(\Theta) = 1 - \frac{1}{\hat{\omega}_1(\Theta)},$$

so that Theorem 3 reads equivalently

$$\frac{(\hat{\omega}_1(\Theta) - 1)\omega_1(\Theta)}{\omega_1(\Theta) + \hat{\omega}_1(\Theta)} \leq \omega_0(\Theta) \leq \frac{\omega_1(\Theta) - \hat{\omega}_1(\Theta) + 1}{\hat{\omega}_1(\Theta)}.$$ \hspace{1cm} (8)

It is established in [14] that the subset of $(\mathbb{R}_{>0} \cup \{+\infty\})^4$ made up by all possible quadruples

$$\left(\omega_1(\Theta), \omega_0(\Theta), \hat{\omega}_1(\Theta), \hat{\omega}_0(\Theta)\right)$$

when $\Theta$ ranges over $\mathbb{R}^2$ with $1, \theta_1, \theta_2$ linearly independent over $\mathbb{Q}$, is essentially described by the relations (7-8), together with the obvious lower bound $\hat{\omega}_1(\Theta) \geq 2$. Thus, the transfer inequalities (8) cannot be sharpened for generic points $\Theta$.

To end this introduction, let us ask for various extensions of the above results. First, Khintchine’s Transference Principle has naturally been extended to any system of real linear forms [6, 22], and even more generally, to systems of linear inequalities in an adelic context [3]. We address the problem of splitting these further transfer relations through adequate intermediate exponents, on the model of Theorem 2. That may possibly be achieved by improving the transfer results of [21]. In an other direction, one should also ask for some refined version of Theorem 2, involving uniform exponents as in Theorem 3.

Transference principles play an important role in proving Schmidt’s Subspace Theorem (see e.g. [22]), which extends the classical Roth’s theorem on the rational approximations to an algebraic number [17]. They also occur in questions of approximation to a real number $\xi$ by algebraic numbers of bounded degree. With regard to that topic, it might be fruitful to investigate more specifically transfer inequalities between exponents of the form $\omega_d(\xi, \ldots, \xi^n)$, as well as their uniform analogues $\hat{\omega}_d(\xi, \ldots, \xi^n)$ introduced (over the notation $\hat{\omega}_{d,1}(\Theta)$ with $\Theta = (\xi, \ldots, \xi^n)$) in Definition 4 of [2]. See the articles [5, 13, 18, 19] and the reports [2, 20] for relating those exponents with various results on algebraic approximation.
2. Algebraic formulation

We reformulate in this section Definition 2 in term of wedge product. One relates the exponent $\omega_d(\Theta)$ to integer solutions of a system of linear inequations, as in Definition 1. This approach will enable us to employ standard arguments from the geometry of numbers.

First, we equip the real vector space $\mathbb{R}^{n+1}$ with the usual dot product and extend it naturally to the Grassmann algebra $\Lambda(\mathbb{R}^{n+1})$, by requiring that for any orthonormal basis $\{e_i\}_{0 \leq i \leq n}$ of $\mathbb{R}^{n+1}$, the family of wedge products $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $0 \leq i_1 < \ldots < i_k \leq n$, $0 \leq k \leq n + 1$, is an orthonormal basis of $\Lambda(\mathbb{R}^{n+1})$. For any $X \in \Lambda(\mathbb{R}^{n+1})$, we denote by $|X|$ its Euclidean norm. Now, let $P$ and $Q$ be points in $\mathbb{P}^n(\mathbb{R})$ with homogeneous coordinates $x$ and $y$. The projective distance between $P$ and $Q$ is measured by the ratio

$$d(P, Q) = \frac{|x \wedge y|}{|x||y|},$$

which does not depend on the choices of $x$ and $y$. Using Lagrange’s identity

$$|x \wedge y|^2 + (x \cdot y)^2 = |x|^2|y|^2,$$

we see that $d(P, Q)$ turns out to be the sine of the acute angle determined in $\mathbb{R}^{n+1}$ by the two lines $Rx$ and $Ry$. See [16, 20] for further properties of the projective distance.

Now, let $L$ be a $d$-dimensional linear subvariety in $\mathbb{P}^n(\mathbb{R})$. Write $L = \mathbb{P}(V)$, where $V$ is the $(d + 1)$-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by the homogeneous coordinates of the points of $L$. Select a basis $\{x_0, \ldots, x_d\}$ of $V$ and put

$$X = x_0 \wedge \ldots \wedge x_d.$$

Using the canonical basis $\{e_i\}_{0 \leq i \leq n}$ of $\mathbb{R}^{n+1}$, we may identify $\Lambda^{d+1}(\mathbb{R}^{n+1})$ with $\mathbb{R}^{n+1}_{d+1}$. The multivector $X$ is then called a system of Plücker coordinates of $L$ (or of $V$). Note that $X$ is determined up to multiplication by a non-zero real number. It is known that the correspondence $L \mapsto X$ establishes a bijection between the set of $d$-dimensional linear subvariety of $\mathbb{P}^n(\mathbb{R})$ and the set of non-zero decomposable multivectors in $\Lambda^{d+1}(\mathbb{R}^{n+1})$, up to an homothety.

**Lemma 1.** Let $\Theta$ be a point in $\mathbb{P}^n(\mathbb{R})$ with homogeneous coordinates $y$ and let $L$ be a linear subvariety of $\mathbb{P}^n(\mathbb{R})$ with Plücker coordinates $X$. Then

$$d(\Theta, L) = \frac{|y \wedge X|}{|y||X|}.$$
Proof. Write $L = \mathbf{P}(V)$ as above. If $y$ is orthogonal to $V$, we have

$$d(\Theta, L) = 1 = \frac{|y \wedge X|}{|y||X|}.$$ 

Otherwise, denote by $y'$ the orthogonal projection on $V$ of the vector $y$. The minimal angle $\langle R_y, Rx \rangle$, when $x$ ranges along $V \setminus \{0\}$, is clearly reached for $x = y'$. Therefore

$$d(\Theta, L) = \frac{|y \wedge y'|}{|y||y'|}.$$ 

We may assume without loss of generality that $y \notin V$. Let $\{e_i\}_{1 \leq i \leq d+1}$ be an orthonormal basis of $V$, which we complete by an orthogonal unitary vector $e_0$ to get an orthonormal basis $\{e_i\}_{0 \leq i \leq d+1}$ of $Ry \oplus V$. Choose $X = e_1 \wedge \ldots \wedge e_{d+1}$ as a system of Plücker coordinates of $V$. Then $|X| = 1$. Write now $y = ae_0 + y'$ for some $a \in R$. We have

$$|y \wedge y'| = |a||e_0 \wedge y'| = |a||y'| \quad \text{and} \quad |y \wedge X| = |a||e_0 \wedge e_1 \wedge \ldots \wedge e_{d+1}| = |a|,$$

so that

$$d(\Theta, L) = \frac{|y \wedge y'|}{|y||y'|} = \frac{|a|}{|y|} = \frac{|y \wedge X|}{|y||X|},$$

as required. ∎

Suppose now that $L$ is rational over $Q$, or equivalently that $V$ can be generated by points $x_0, \ldots, x_d$ belonging to $Q^{n+1}$. Then $X = x_0 \wedge \ldots \wedge x_d$ has rational coordinates in $R^{n+1}$. We define $H(L)$ as the Weil height of the $n+1$-tuple $X$. Let us indicate an useful interpretation of the height $H(L)$ in term of determinant of a lattice.

**Lemma 2.** The group $V \cap Z^{n+1}$ is a lattice in $V$. For any $Z$-basis $\{x_0, \ldots, x_d\}$ of $V \cap Z^{n+1}$, put $X = x_0 \wedge \ldots \wedge x_d$. Then

$$\det(V \cap Z^{n+1}) = |X| = H(L).$$

The notation $\det(V \cap Z^{n+1})$ indicates here the volume, with respect to the induced Euclidean norm on $V$, of the unit parallelepiped constructed on any such $Z$-basis $\{x_0, \ldots, x_d\}$.

Proof. The $Z$-module $V \cap Z^{n+1}$ has rank $d+1$, since $V$ is rational over $Q$. The multivector $X$ is clearly a system of Plücker coordinates of $L$, and $X$ is primitive in the group $\Lambda^d(Z^{n+1})$ since $\{x_0, \ldots, x_d\}$ is a $Z$-basis of $V \cap Z^{n+1}$. Therefore $|X| = H(L)$. Moreover

$$\det(V \cap Z^{n+1}) = \sqrt{\det(\det(x_i \cdot x_j)_{0 \leq i,j \leq d})} = |X|,$$

by Cauchy-Binet formula. We refer to Theorem 1 of [21] for more details and for an extension of Lemma 2 to number fields. ∎

Lemmas 1 and 2 enable us to handle more easily the quantities $\omega_d(\Theta)$ through the following alternative
Definition 4. Let $y$ be homogeneous coordinates of the point $\Theta$. For any $d$ with $0 \leq d \leq n - 1$, the exponent $\omega_d(\Theta)$ is the supremum of the real numbers $\omega$ for which there exist infinitely many integer decomposable multivectors $X \in \Lambda^{d+1}(\mathbb{Z}^{n+1})$ such that

$$|y \wedge X| \leq |X|^{-\omega}.$$  

For any $d$-dimensional rational linear subvariety $L$, select a system of Plücker coordinates $X$ as in Lemma 2. Then

$$H(L) = |X| \quad \text{and} \quad d(\Theta, L) = |y|^{-1}|X|^{-1}|y \wedge X|.$$  

Now, the equivalence of Definitions 2 and 4 is clear.

Remark. We should have obtained the same exponent $\omega_d(\Theta)$, when dropping in Definition 4 the assumption that $X$ is decomposable in $\Lambda^{d+1}(\mathbb{Z}^{n+1})$. We shall not use this property, which can be deduced from Mahler’s theory of compound convex bodies [15, 22]. To that purpose, observe that for fixed positive real numbers $Y < X$, the convex body defined as the set of $X$ in $\Lambda^{d+1}(\mathbb{R}^{n+1})$ satisfying the linear inequations

$$|X| \leq Y^d X \quad \text{and} \quad |y \wedge X| \leq Y^{d+1},$$

compares with the $(d+1)$-th compound of the convex body defined in $\mathbb{R}^{n+1}$ by

$$|x| \leq X \quad \text{and} \quad |y \wedge x| \leq Y.$$  

Further applications to transfer inequalities will be given elsewhere.

3. Proof of the Going-up inequality.

We are now able to prove inequality (2). Fix homogeneous coordinates $y$ of $\Theta$ and follow Definition 4. Let $X$ be a decomposable multivector in $\Lambda^{d+1}(\mathbb{Z}^{n+1})$ such that

$$|y \wedge X| \leq |X|^{-\omega}.$$  

We may suppose without loss of generality that $X$ is primitive in the group $\Lambda^{d+1}(\mathbb{Z}^{n+1})$. Then $X$ is a system of Plücker coordinates of some rational $(d+1)$-dimensional subspace $V \subset \mathbb{R}^{n+1}$. By Lemma 2, we may write $X = x_0 \wedge \ldots \wedge x_d$ for some $\mathbb{Z}$-basis $\{x_0, \ldots, x_d\}$ of the lattice $V \cap \mathbb{Z}^{n+1}$. Specifically, we know from Lemma 2 that

$$\det(V \cap \mathbb{Z}^{n+1}) = |X|.$$
Let $W$ be the orthogonal complement to $V$ in $\mathbb{R}^{n+1}$. Therefore, $W$ is a rational subspace of $\mathbb{R}^{n+1}$ with dimension $n - d$. Denote by $\Lambda$ the orthogonal projection on $W$ of the lattice $\mathbb{Z}^{n+1}$. Then, $\Lambda$ is a lattice in $W$ with determinant

$$\det(\Lambda) = \frac{\det(\mathbb{Z}^{n+1})}{\det(V \cap \mathbb{Z}^{n+1})} = \frac{1}{|X|}.$$ 

On the other hand, the Euclidean ball in $W$

$$\{z \in W; \ |z| \leq R\},$$

centered at the origin of $W$ with radius $R$, has volume $v_{n-d}R^{n-d}$, where $v_k = \pi^k/\Gamma(1+k/2)$ denotes the volume of the unit Euclidean ball in $\mathbb{R}^k$. Choosing now

$$R = 2v_{n-d}^{1/(n-d)}|X|^{-1/(n-d)},$$

Minkowski’s convex body theorem shows that there exists a non-zero element $\xi \in \Lambda$ with norm

$$|\xi| \leq R.$$ 

Lift up now $\xi$ into an element $x \in \mathbb{Z}^{n+1}$ whose orthogonal projection on $W$ is $\xi$, and put

$$X' = x \wedge X = \xi \wedge X.$$ 

Then $X'$ is a decomposable multivector in $\Lambda^{d+2}(\mathbb{Z}^{n+1})$. Making use of Hadamard’s inequality, we first bound from above its norm

$$|X'| = |\xi||X| \leq R|X| \ll |X|^{(n-d-1)/(n-d)}.$$ 

A second use of Hadamard’s inequality enables us to bound the wedge product

$$|y \wedge X'| = |y \wedge \xi \wedge X| \leq |\xi||y \wedge X| \leq R|X|^{-\omega} \ll |X|^{-\left((\omega+1)/(n-d)\right)}.$$ 

Combining the two last inequalities, we obtain

$$|y \wedge X'| \ll |X'|^{-(n-d-1)/(n-d-1)}.$$ 

Taking now $\omega$ arbitrarily close to $\omega_d(\Theta)$, we have established the lower bound (2).
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