Three-dimensional model of a decision-making problem under multicriteria conditions

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Abstract. The article proposed and reviewed a 3D model of the utility matrix, showing its analysis and proposing a generalized algorithm for solving a multi-critical decision-making problem under conditions of uncertainty based on this model. The essence of the technique is that the matrix-cube is divided into sections parallel to any axis (the axis of alternative solutions, the axis of undefined external situations, or the axis of criteria) and as a result we get several two-dimensional utility matrices to which we can apply any of the standard methods for solving decision-making problems. The application of the described method to finding the optimal solution in the conditions of uncertainty is shown by example.

1. Introduction
In the decision-making process, the leader should be guided by many different conditions, criteria and random factors. However, most of the methods do not allow the manager to look at the task from the side of not one criterion, but several. After all, maximizing profits may not be the only priority, there are also approaches aimed at, for example, minimizing costs or minimizing the time spent.

Problems of choosing the optimal solution in the economy, production, social spheres are considered from different positions [1, 2]. One of the directions in research is the description of such tasks with a model of a game where the strategies of one’s opponent are considered external factors [3]. Under this approach, alternative solutions are evaluated by only one criterion. When multiple criteria are used, the task is usually brought to a single-criterion task by applying criteria importance assessment models [4, 5]. Such an assessment is subjective and requires enormous experience and intuition from the decision maker.

There was no doubt that the manager could take advantage of existing techniques to address multicriterial tasks. But these methods do not allow to consider each alternative solution taking into account random factors affecting the evaluation of the alternative on separate criterion.

The purpose of this article is to develop and investigate the possibility of using a three-dimensional utility matrix to solve multi-criteria problems in conditions of uncertainty.

Tasks:
1. Define the concept of a 3D utility matrix model;
2. Formulate the task of decision-making in conditions of uncertainty;
3. Define criteria and generate a utility matrix for each of them;
4. Generate a 3D utility matrix model;
5. Justify the rationality and optimality of using such a model.

The relevance of the article is due to the need for the manager to make a decision according to several
criteria if there are random external factors and there is insufficient information about the impact of these factors on the chosen options.

2. Define the concept of a 3D utility matrix model

The utility matrix is constructed by determining the expected revenues \( a_{ij} \) for the cases when the decision \( X_i \) will be made from the set of analyzed alternatives \( \{X_i, i = 1, m\} \), and the external situation not dependent of the decision maker will be such that it corresponds to the event \( \theta_i \) (from the set of events of the complete group \( \{\theta_j, j = 1, n\} \) affecting the economic result). As a result, we obtain a matrix

\[
A = (a_{ij})
\]

which is called a utility matrix [6, 7]. The structure of the utility matrix is as follows:

\[
A = X_1 \begin{pmatrix}
\theta_1 & \theta_2 & \ldots & \theta_n \\
\end{pmatrix}
X_2 \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\]

(as you can see, the element is on the intersection of the \( i \)-th row that relates to the solution and the \( j \)-th column that relates to the external random situation).

Let us present this matrix graphically (Figure 1), where the x-axis will be the events from the multiple events of the complete group, and the y-axis will be the alternatives from the multiple alternatives analyzed.

![Figure 1. Graphical representation of the utility matrix.](image)

This utility matrix is constructed for one criterion. To construct a utility matrix for multiple criteria at the same time, add a z-axis to which we attach a set of all the necessary criteria \( = \{C_k, k = 1, p\} \). In this way we obtain a three-dimensional utility matrix, the graphical display of which is shown in Figure 2.

This representation of the utility matrix enables you to perform the following actions quickly and without much computation:

- breaking a cube into vertical cuts it is perpendicular to an axis of criteria, to receive \( p \) of utility matrixes of one-criteria tasks in the conditions of uncertainty (1);
- breaking a cube into horizontal cuts it is perpendicular to an axis of alternatives, can obtain information on each of uncertain situations on separate criterion. Therefore, it becomes possible to define which situations will be the most favorable and which ones, on the contrary, the most adverse,
for specifically taken alternative both for separate, and for set (selection) of any criteria of a great number of C:

- breaking a cube into vertical cuts it is perpendicular to an axis of uncertain situations, to receive m of two-dimensional utility matrixes describing m of multicriterial tasks [8], the decision which will give information on that what alternative for specifically taken situation (event) will be the best and what - the worst, both for any separate criterion, and for their set.

By selecting a cut from the cube, we obtain a two-dimensional utility matrix to which any method of choosing the best alternative can be applied [9, 10]. In addition, any of the classical or derived methods can be applied to a three-dimensional utility matrix by the following algorithm:

1. The cube is sliced perpendicular to any of the axes, depending on the result the decision maker wants. If the decision maker needs to choose the best alternative for all previously defined criteria (e.g. maximization of profit and minimization of costs), regardless of which of the undefined events of the complete group will occur, the slicing is done perpendicular to the criteria axis (Figure 3.a). If the decision maker wants to know which of the undefined events of the complete group is most favorable for most criteria, regardless of the choice of a specific alternative, slicing should be done perpendicular to the axis of the criteria (Figure 3.b). In the case where the decision maker interest which of the criteria will be most accurately met and when choosing which of the alternatives this will occur, regardless of uncertain situations, the cut is made perpendicular to the axis of the alternatives (Figure 3.c).

2. Previously selected method applies to the two-dimensional matrices obtained from the slices to find the optimal solution, resulting in a column of values.

3. The method of finding the best solution (either the same or any other) is applied again to the resulting column. Next, a slice or slices to which this value corresponds can be isolated for further more detailed studies.

As a result, the solution to the problem of deciding by means of a three-dimensional utility matrix becomes the alternative/event/criterion that corresponds to the selected cut.

**Figure 2.** Graphic display of 3D utility matrix.
Figure 3. Slicing: (a) to select the best alternative; (b) to select the most favorable event; (c) for selection of criterion.
3. Application of a technique

Application of the above-described technique of finding the best solution under uncertainty conditions is shown in the following example.

The head of the company "PARUS Corporation" needs to make a decision on distribution of software maintenance work in several organizations. Service times must be distributed to maximize revenue, minimize costs, and employ as few employees as possible.

Many uncertain situations \( \{X_i, i = 1, m\} \) and many alternatives \( \{\theta_j, j = 1, n\} \) are described in article [7]. Let us define a variety of task criteria \( \mathcal{C} = \{\mathcal{C}_k, k = 1, p\} \):

- \( \mathcal{C}_1 \) - maximizing profit (2);
- \( \mathcal{C}_2 \) - minimization of expenses (3);
- \( \mathcal{C}_3 \) - minimization of the number used labor resources (employees of the company) (4).

Thus, we have a set of criteria \( \{\mathcal{C}_k, k = 1,3\} \).

The task of the manager is to find the best alternative that meets the largest number of criteria, regardless of the situation that may occur.

Form a utility matrix for each criterion \( \mathcal{C}_k \) from a set \( \mathcal{C} = \{\mathcal{C}_k, k = 1,3\} \).

Form matrices for each criterion:

\[
X_{11} = \begin{pmatrix}
\theta_{11} & \theta_{21} & \theta_{31} & \theta_{41} & \theta_{51} \\
\theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} \\
\theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} \\
\theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} & \theta_{35} \\
\theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} & \theta_{45} \\
\end{pmatrix}
\]

\( X_{12} = \begin{pmatrix}
\theta_{11} & \theta_{21} & \theta_{31} & \theta_{41} & \theta_{51} \\
\theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} \\
\theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} \\
\theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} & \theta_{35} \\
\theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} & \theta_{45} \\
\end{pmatrix}
\]

\( X_{13} = \begin{pmatrix}
\theta_{11} & \theta_{21} & \theta_{31} & \theta_{41} & \theta_{51} \\
\theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} \\
\theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} \\
\theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} & \theta_{35} \\
\theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} & \theta_{45} \\
\end{pmatrix}
\]

We will form a three-dimensional model of the utility matrix (Figure 4).

To determine the best alternative, we will highlight four cuts (by number of alternatives) and record their utility matrices (5, 6, 7, 8).

\[
X_1 = \begin{pmatrix}
\theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 \\
\theta_1 & \theta_1 & \theta_1 & \theta_1 & \theta_1 \\
\theta_2 & \theta_2 & \theta_2 & \theta_2 & \theta_2 \\
\theta_3 & \theta_3 & \theta_3 & \theta_3 & \theta_3 \\
\theta_4 & \theta_4 & \theta_4 & \theta_4 & \theta_4 \\
\end{pmatrix}
\]
Applying any classical or derived method to each matrix results in four optimal values $a_{1ij}$; $a_{2ij}$; $a_{3ij}$; $a_{4ij}$. When you write them as a column and apply the same or different method, you get the value $a_{k'ij}$, hence the optimal solution is an alternative $X_{k'}$.

4. Conclusion
In the course of the work, the methodology of solving multicriteria problems [11] of decision-making in conditions of uncertainty by drawing up a three-dimensional utility matrix and working directly with it was studied. Advantages of this technique are:
- an opportunity to look at a task along with several positions;
- possibility of simultaneous use of various criteria of search of an optimal solution;
- the technique considers several goals, set by the decision maker (maximizing profit, cost minimization, minimization of an expense of human resources, etc.), and allows to choose an optimal solution for all criteria (restrictions) at once.

Among the disadvantages is a perceptional complexity of representation, however, this is not too significant minus relative to the advantages of this method of finding solutions.

The three-dimensional utility matrix makes it possible to perform a comparative analysis of alternative solutions in order to determine suboptimal but appropriate solutions for the decision manager.

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