Modification of ARMA (1, 1) model to simulate strictly stationary random series with uniform distribution

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Abstract. We consider a problem of simulating strictly stationary random series. A modification of an autoregressive–moving-average model of orders 1, 1 is proposed. Proposed algorithm allows to simulate stationary random series with uniform distribution. Properties of resulting random series are examined.

1. Introduction
Random series generation is used in modeling objects, situations and phenomena of different nature. Simulating stationary random series is an important task as stationarity is an assumption underlying many statistical procedures. Such series are considered in radio engineering, communication theory, fluid and gas mechanics, meteorology, oceanology etc. In many cases density of series members is ignored, but taking it in consideration and therefore simulating strictly stationary models can improve quality of models [1].

Below we consider a problem of simulating strictly stationary random series \{Y(t)\} where each element of the series is a standard uniformly distributed random variable, i.e. the probability density function (pdf) of each element is

\[ g(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } y \in [0,1], \\ 0, & \text{if } y > 1. \end{cases} \]

Uniform distribution is important to consider as it can be used as a basis for further transformations. For instance, Smirnov’s inverse transformation method can be used. Therefore, if we have a correlated series of uniformly distributed random values, we can get series with different distributions.

General case would require to know joint pdf of all of series’ values. Then a vector of corresponding dimension can be simulated, as pdf gives comprehensive information about process characteristics, including correlation properties [2]. Obvious downsize of this approach is its complexity. On the other hand joint pdf is rarely known, and researchers usually consider properties like covariation or correlation.

We only consider random series where correlation coefficient between adjacent elements if preassigned, i.e. autocovariance function \(R(\tau)\) is defined at point 1. Further we assume that autocovariance function is normalized.

There are different approaches for solving the given problem.

In [3] it is proposed to transform series with normal distribution. In [4] author proposes to use autoregressive model with binary orthogonal stochastic coefficients, while [5] uses the same approach
to simulate random vectors. In [6] some cases are considered when autoregressive model with deterministic coefficients can be used for modeling strictly stationary random series.

Overall, the algorithms that are used are often based on linear models that transform white noise to produce series with necessary properties. In general case such transformations change the pdf of the next series term, so the resulting series are wide sense, but not strictly stationary [7]. Obvious exception is normal distribution, as the linear combination of normally distributed random variables is also normally distributed. Examples of linear models include autoregressive (AR), moving average (MA) and autoregressive–moving-average (ARMA) models.

Works [8, 9] describe and examine an approach of modifying autoregressive model of order 1 (AR (1)) and moving average model of order 1 (MA (1)) that allows to conserve uniformity of distribution of terms of generated sequence. That is if original white noise random sequence has uniformly distributed terms, sequence after transformation will maintain that distribution, but will also have specified correlation properties. So the methods for obtaining strictly stationary random series with uniform distribution were described.

The task of the following article is to combine the methods described in works [8, 9] to obtain a method based on autoregressive–moving-average model of orders 1, 1 (ARMA (1,1)) and its examination. Approach for that combination is usage of random coefficients similar to the one described in [4, 5].

2. ARMA (1,1) model

Autoregressive model of order 1 is considered as follows:

\[ Y(t) = aY(t-1) + bX(t) \]

where \( a \geq 0, b \geq 0, a + b = 1 \). It transforms white noise sequence \( \{X(t)\} \) into a sequence \( \{Y(t)\} \) with autocovariance function \( R(\tau) = a^\tau \).

Value \( Y(t) \) at the moment \( t \) is a combination of its value at the moment \( (t-1) \) and some independent random variable \( X(t) \). Resulting sequence \( \{Y(t)\} \) is wide-sense stationary. If distribution of \( X(t) \) is normal, then \( \{Y(t)\} \) is also strictly stationary, as performed linear transformation does not affect “normality” of distribution.

Moving average model of order 1 (MA (1)) is written

\[ Y(t) = aX(t-1) + bX(t) \]

where \( a \geq 0, b \geq 0, a + b = 1 \) transforms white noise sequence \( \{X(t)\} \) into a sequence \( \{Y(t)\} \) with autocovariance function \( R(\tau) = 0 \) (when \( |\tau| > 1 \)).

Therefore autoregressive–moving-average model of orders 1, 1 (ARMA (1,1)) is defined by equation

\[ Y(t) = a_{AR}Y(t-1) + a_{MA}X(t-1) + bX(t) \]

where \( a_{AR} \geq 0, a_{MA} \geq 0, b \geq 0, a_{AR} + a_{MA} + b = 1 \). It transforms white noise sequence \( \{X(t)\} \) into sequence \( \{Y(t)\} \) with autocovariance function \( R(1) = R(-1) = a_{AR} + a_{MA}, \) and \( R(\tau) = a_{AR}^\tau \) when \( |\tau| \geq 2 \).

Value \( Y(t) \) at the moment 1 is a combination of its value at the moment \( (t-1) \) and two independent random variables \( X(t) \) and \( X(t-1) \). Resulting sequence \( \{Y(t)\} \) is wide-sense stationary.

3. Manipulating intervals of probability density function

This method of transforming the next value of the sequence generated by autoregressive model was discussed in [8], its application for the moving average model is described in [9].

Linear combination \( Y'(t) = Z(t-1) + b_m \cdot X(t) \) of two standard uniformly distributed random variables \( Z(t-1) \) and \( X(t) \) has a density function
if we assume that $b_m \geq 1$.

If $Z(t - 1) = Y(t - 1)$, then series $\{Y'(t)\}$ can be described as autoregressive. In case when $Z(t - 1) = X(t - 1)$ the series can be described with the moving average model.

We propose the following transformation of random variable $Y'(t)$: if its value $y' \in [0,0.5]$, then we change it to $1 - y'$. If $y' \in [b_m + 0.5, b_m + 1]$, then we change it to $2b_m - y' + 1$. We call this transformation manipulating intervals of probability density function. Result of that transformation is changing the pdf on intervals $[0,1]$ and $[b_m, b_m + 1]$. Let us call the obtained random variable $Y''(t)$, and let us denote its pdf $g''(y)$.

If $y' \in [0,0.5]$ or $y' \in [b_m + 0.5, b_m + 1]$ then $g''(y) = 0$.

If $y' \in [0.5,1]$ then

$$g''(y) = g'(y) + g'(1 - y) = \frac{y}{b_m} + \frac{1 - y}{b_m} = \frac{1}{b_m}.$$

Similarly, if $y' \in [b_m; b_m + 0.5]$ then

$$g''(y) = g'(y) + g'(2b_m - y + 1) = \frac{1}{b_m}.$$

That way pdf of $Y''(t)$ is

$$g''(y) = \begin{cases} 0, & \text{if } y < 0.5, \\ \frac{1}{b_m}, & \text{if } y \in [0.5, b_m + 0.5], \\ 0, & \text{if } y > b_m + 0.5, \end{cases}$$

which is the pdf of random variable that is uniformly distributed on the interval $[0.5, b_m + 0.5]$. Further transformation $y = (y'' - 0.5)/b_m$ gives us a realization of random variable $Y(t)$ with standard uniform distribution.

As the pdf of uniform distribution is symmetrical in respect to point 0.5, we can additionally transform the result by changing it to $(1 - y)$. That way we keep the pdf but invert the correlation coefficient between the adjacent terms of the series.

Obtaining the predefined value for $R(1)$ requires finding the correspondent value of $b_m$. That subsequently requires that we know the joint pdf between the adjacent terms of the series, which is the task that must be done separately for autoregressive model and moving average model.

Thus, each consecutive term of the series is either a linear combination of two independent random variables with uniform distribution or a linear function of such a linear combination.

**Manipulating intervals of pdf in AR model**

Algorithm for generating the sequence in that case consists of the following steps:

1. Find the value of coefficient $b_m$ that will give the desired value for $R(1)$.
2. If $R(1) > 0$ then obtain a temporary value $y' \leftarrow y(t - 1) + b_m x(t)$.
   
   Else $y' \leftarrow 1 - y(t - 1) + b_m x(t)$.
3. If $y' \in [0,0.5]$ then find the new value $y'' \leftarrow 1 - y'$.
   
   If $y' \in [b_m + 0.5, b_m + 1]$ then $y'' \leftarrow 2b_m - y' + 1$.
   
   If $y' \in (0.5, b_m + 0.5)$ then $y'' = y'$.
4. Transform $y''$ to get the new value $Y(t)$: $y \leftarrow (y'' - 0.5)/b_m$.
5. Repeat steps 2 to 4 as many times as necessary.

Joint probability density function of adjacent terms $Y(t)$ and $Y(t + 1)$ of the obtained sequence is
\[ g_{AR}(y_1, y_2) = \begin{cases} 
1, & \text{if } (y_1, y_2) \in B_1 \\
2, & \text{if } (y_1, y_2) \in B_2 \cup B_3 \\
0, & \text{if } (y_1, y_2) \notin B_1 \cup B_2 \cup B_3 
\end{cases} \]

for sets \(B_1, B_2, B_3\) depicted in figure 1. Detailed descriptions of those sets are not present in this work due to their bulkiness.

![Diagram of sets B1, B2, B3](image)

**Figure 1.** Sets \(B_1, B_2, B_3\).

As we now know the joint pdf, we can find the correlation coefficient between the adjacent terms of the generated sequence (see figure 2):

\[ R_{AR}(1) = \frac{1}{b_m^2} - \frac{3}{8b_m^2} \]

As we choose \(b_m \geq 1\), the maximal value of \(R_{AR}(1)\) that can be obtained is 0.625. Therefore, proposed method allows to simulate random series with \(R_{AR}(1) \in [-0.625, 0.625]\).

![Graph of R_AR(1) vs b_m](image)

**Figure 2.** Dependence of \(R_{AR}(1)\) on \(b_m\).

*Manipulating intervals of pdf in MA model*

Algorithm for generating the sequence consists of these steps:

1. Find the value of coefficient \(b_m\) that will give the desired value for \(R(1)\).
2. If $R(1) > 0$ then obtain a temporary value $y^* = x(t - 1) + b_m x(t)$.
   Else $y^* = 1 - x(t - 1) + b_m x(t)$.
3. If $y^* \in [0,0.5]$ then find the new value $y^{**} = 1 - y^*$.
   If $y^* \in [b_m + 0.5, b_m + 1]$ then $y^{**} = 2b_m - y^* + 1$.
   If $y^* \in (0.5, b_m + 0.5)$ then $y^{**} = y^*$.
4. Transform $y^{**}$ to get the new value $Y(t) : y = (y^{**} - 0.5)/b_m$.
5. Repeat steps 2 to 4 as many times as necessary.

Joint probability density function of adjacent terms $Y(t)$ and $Y(t + 1)$ of the obtained sequence is

$$g(y_1, y_2) = \begin{cases} 
1, & \text{if } (y_1, y_2) \in A_0, \\
-2b_m^2 y_2 + 2b_m^2 - b_m + 1, & \text{if } (y_1, y_2) \in A_1, \\
b_m y_1 - b_m y_2 + b_m^2 + 0.5b_m + 0.5, & \text{if } (y_1, y_2) \in A_2, \\
2b_m y_1 - b_m + 1, & \text{if } (y_1, y_2) \in A_3, \\
b_m y_1 + b_m y_2 - b_m^2 - 0.5b_m + 1, & \text{if } (y_1, y_2) \in A_4 \cup A_{10}, \\
2b_m^2 y_2 - 2b_m^2 + b_m + 1, & \text{if } (y_1, y_2) \in A_5, \\
2b_m^2 y_2 - b_m + 1, & \text{if } (y_1, y_2) \in A_6, \\
-b_m y_1 + b_m y_2 + 0.5b_m + 0.5, & \text{if } (y_1, y_2) \in A_7, \\
-2b_m y_1 + b_m + 1, & \text{if } (y_1, y_2) \in A_8, \\
-b_m y_1 - b_m + 0.5b_m + 1.5, & \text{if } (y_1, y_2) \in A_9, \\
-2b_m^2 + b_m + 1, & \text{if } (y_1, y_2) \in A_{11}, \\
0, & \text{if } (y_1, y_2) \in \bigcup_{i=0}^{12} A_i 
\end{cases}$$

for sets $A_0, A_1, ..., A_{12}$ depicted in figure 3. Detailed descriptions of those sets are also not present in this work.

![Figure 3. Sets $A_0, A_1, ..., A_{12}$.](image-url)
The correlation coefficient between the adjacent terms of the generated sequence is (see figure 4)

\[ R(1) = \frac{40b_m^5 - 15b_m^4 - 20b_m^3 + 15b_m^2 - 5b_m + 1}{40b_m^6} \]

\[ \text{Figure 4.} \text{ Dependence of } R_{MA}(1) \text{ on } b_m. \]

Maximal value of correlation coefficient between adjacent terms of the sequence that can be obtained if \( R(1) \approx 0.423194 \), which corresponds to \( b_m \approx 1.24855 \).

Therefore this method allows to obtain a MA (1) series with uniform distribution and \( R(1) \in [-0.423, 0.423] \).

*Manipulating intervals of probability density function in ARMA model*

We write out the equation for ARMA (1,1) model as follows:

\[ Y(t) = a_AR Y(t-1) + a_{MA} X(t-1) + b_{m,AR} X_{AR}(t) + b_{m,MA} X_{MA}(t) \]

where \( a_{AR} + a_{MA} + b_{m,AR} + b_{m,MA} = 1, a_{AR}, a_{MA}, b_{m,AR}, b_{m,MA} > 0 \). If all four random variables \( Y(t-1), X(t-1), X_{AR}(t), X_{MA}(t) \) are distributed uniformly and are pairwise independent, then resulting random variable \( Y(t) \) will have a distribution that is not uniform. Manipulating intervals of pdf also cannot be applied as we find a linear combination of four random variables. To use this approach we need to have linear combination of two random variables and yet take account of both \( Y(t-1) \) and \( X(t-1) \).

We propose to randomly decide for each term if it will be generated as a term of autoregressive model (with predefined probability \( p_{AR} \)) or of moving average model (with probability \( p_{MA} = 1 - p_{AR} \)).

For obtaining a combined method based on autoregressive–moving-average model we propose to use a random coefficient \( A \) and the equation for the model is written out as follows:

\[ Y(t) = A(Y(t) + b_{m,AR} X(t-1)) + (1 - A) (X(t) + b_{m,MA} X(t-1)). \]

Coefficient \( A \) is independent of all the random variables, and \( P(A = 1) = p_{AR}, P(A = 0) = 1 - p_{AR} \), therefore it determines either autoregressive or moving average model will be chosen for the next step.

We do not use different random variables \( X_{AR}(t), X_{MA}(t) \) as only one of them would be chosen on each step anyway.

As coefficient \( A \) is independent from terms of generated sequence, joint pdf of its adjacent term is

\[ g_{ARMA}(y_1, y_2) = p_{AR} g_{AR}(y_1, y_2) + (1 - p_{AR}) g_{MA}(y_1, y_2). \]
Correlation coefficient between adjacent terms of the generated sequence is

\[ R_{ARMA}(1) = p_{AR}R_{AR}(1) + (1 - p_{AR})R_{MA}(1). \]

For the terms of the sequence further distanced from each other covariation function is

\[ R(\tau) = p_{AR}^{[\tau]} \left( \frac{1}{b_{m,AR}} - \frac{3}{8b^2_{m,AR}} \right)^{[\tau]} \text{ when } |\tau| > 1. \]

Resulting algorithm can be written as follows:

1. Find the value of coefficients \( b_{m,AR}, b_{m,MA} \) that will give the desired value for \( R(1) \).
2. With probability \( p_{AR} \) go to step 3, else go to step 7.
3. If \( R(1) > 0 \) then obtain a temporary value \( y^* \leftarrow y(t - 1) + b_{m,AR}x(t) \).
   Else \( y^* \leftarrow 1 - y(t - 1) + b_{m,AR}x(t) \).
4. If \( y^* \in [0,0.5] \) then find the new value \( y^{**} \leftarrow 1 - y^* \).
   If \( y^* \in [b_{m,AR} + 0.5,b_{m,AR} + 1] \) then \( y^{**} \leftarrow 2b_{m,AR} - y^* + 1 \).
   If \( y^* \in [0.5,b_{m,AR} + 0.5] \) then \( y^{**} = y^* \).
5. Transform \( y^{**} \) to get the new value \( Y(t) : y \leftarrow (y^{**} - 0.5)/b_{m,AR} \).
6. Go to step 10.
7. If \( R(1) > 0 \) then obtain a temporary value \( y^* \leftarrow x(t - 1) + b_{m,MA}x(t) \).
   Else \( y^* \leftarrow 1 - x(t - 1) + b_{m,MA}x(t) \).
8. If \( y^* \in [0,0.5] \) then find the new value \( y^{**} \leftarrow 1 - y^* \).
   If \( y^* \in [b_{m,MA} + 0.5,b_{m,MA} + 1] \) then \( y^{**} \leftarrow 2b_{m,MA} - y^* + 1 \).
   If \( y^* \in [0.5,b_{m,MA} + 0.5] \) then \( y^{**} = y^* \).
9. Transform \( y^{**} \) to get the new value \( Y(t) : y \leftarrow (y^{**} - 0.5)/b_{m,MA} \).
10. Repeat steps 2 to 9 as many times as necessary.

If \( p_{AR} = 0 \) or \( p_{AR} = 1 \) then we get a “pure” moving average model or autoregressive model correspondingly.

If we choose \( b_{m,AR} = b_{m,MA} = \beta_m \) then the joint pdf is (see figure 5)

\[ R_{ARMA}(1) = \frac{1}{b_m} - \frac{3}{8b^2_m} + (1 - p_{AR}) \frac{-20b^3_m + 15b^2_m - 5b_m + 1}{40b^5_m}. \]

![Figure 5. Dependence of \( R_{ARMA}(1) \) on \( b_{m,AR} = b_{m,MA} = \beta_m \) for different values of \( p_{AR} \).](image-url)
4. Conclusion
Suggested method is a modification of an algorithm based on ARMA (1, 1) model. It allows to obtain strictly stationary random series with standard uniform distribution. As the uniform distribution is a base for inverse transform sampling method, the proposed approach may be used to simulate random series with wide range of probability distributions.

Computer modeling confirms the theoretical calculations and predictions of this paper.

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