LIPSCHITZ P-SUMMING MULTILINEAR OPERATORS
CORRESPOND TO LIPSCHITZ P-SUMMING OPERATORS

MAITE FERNÁNDEZ-UNZUETA

Abstract. We give conditions that ensure that an operator satisfying a Pietsch
domination in a given setting also satisfies a Pietsch domination in a different
setting. From this we derive that a bounded multilinear operator $T$ is Lipschitz
$p$-summing if and only if the mapping $f_T(x_1 \otimes \cdots \otimes x_n) := T(x_1, \ldots, x_n)$ is
Lipschitz $p$-summing. The results are based on the projective tensor norm. An
example with the Hilbert tensor norm is provided to show that the statement
may not hold when a reasonable cross-norm other than the projective tensor
norm is considered.

1. Introduction

A bounded multilinear operator $T$ on the tensor product of Banach spaces
uniquely determines a Lipschitz mapping $f_T$ by the relation $f_T(x_1 \otimes \cdots \otimes x_n) :=
T(x_1, \ldots, x_n)$, as explained in [7]. This relation makes it possible to use the theory
of Lipschitz mappings to study multilinear operators on Banach spaces.

In [6] J. Farmer and W.B. Johnson introduced the class of Lipschitz $p$-summing
mappings defined on metric spaces. The above-mentioned relation between $T$ and
$f_T$ makes it natural to ask for the multilinear mappings $T$ such that $f_T$ is a Lipschitz
$p$-summing mapping.

In this paper we prove that such multilinear mappings are precisely the class
of Lipschitz $p$-summing multilinear operators introduced in [1]. The corresponding
Lipschitz $p$-summing norms satisfy $\pi^L_p(f_T) = \pi^L_p(T)$ (Theorem 2.1). This result
generalizes the case of linear operators proved in [6, Theorem 2]. A main part of
its proof consists of moving from a Pietsch-type domination of $f_T$ (as a Lipschitz
operator) to another Pietsch-type domination of $f_T$ (as a $\Sigma$-operator associated to the
multilinear mapping $T$). Theorem 2.3 provides conditions in a more general
setting, to guarantee that such a motion is possible.

We use Hilbert-Schmidt multilinear operators to show that Theorem 2.1 may
not hold when a reasonable cross-norm other than the projective tensor norm is
considered.

Throughout the paper $X_1, \ldots, X_n$ and $Y$ are real Banach spaces and $B_{X_i}$ is
the closed unit ball of a space $X_i$. The completed projective tensor product of
$X_1, \ldots, X_n$ is denoted by $\hat{\otimes}_\pi X_i$ and the space of multilinear bounded operators
from $X_1 \times \cdots \times X_n$ to $Y$, by $\mathcal{L}(X_1, \ldots, X_n; Y)$. The linear operator determined
by $T$ with domain the tensor product will be denoted by $\hat{T}$. The linear theory of

2020 Mathematics Subject Classification. Primary 47L22; 47H60; 46T99; 46B28.
Key words and phrases. Lipschitz $p$-summing, Segre cone of Banach spaces, multilinear oper-
ator, Pietsch Domination, Hilbert-Schmidt multilinear operators.

The author was partially supported by CONACyT project 284110.
Banach spaces that we will use can be found in [5], the theory of tensor norms in [4] and [9] and the Lipschitz theory in [2] and [12].

Each bounded multilinear operator $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ can be uniquely associated with a Lipschitz mapping $f_T : \Sigma X_1, \ldots, X_n \to Y$ where $\Sigma X_1, \ldots, X_n := \{ x_1 \otimes \cdots \otimes x_n; x_i \in X_i \}$ is endowed with the metric induced by $X_1 \otimes \cdots \otimes X_n$ [17 Theorem 3.2]. The mapping $f_T(x_1 \otimes \cdots \otimes x_n) := T(x_1, \ldots, x_n)$ and $\Sigma X_1, \ldots, X_n$ are called the $\Sigma$-operator associated to $T$ and the Segre cone of $X_1, \ldots, X_n$, respectively.

A norm $\gamma$ on the vector space $X_1 \otimes \cdots \otimes X_n$ is said to be a reasonable cross-norm if it has the following two properties: (i) $\gamma(x_1 \otimes \cdots \otimes x_n) \leq \| x_1 \| \cdots \| x_n \|$ for every $x_i \in X_i$, $i = 1, \ldots, n$, and (ii) For every $x_i^* \in X_i^*$, the linear functional $x_i^* \otimes \cdots \otimes x_n^*$ on $X_1 \otimes \cdots \otimes X_n$ is bounded, and $\| x_i^* \otimes \cdots \otimes x_n^* \| \leq \| x_1^* \| \cdots \| x_n^* \|$. We denote the completed space as $\hat{\otimes}_\gamma X_i$. The space of multilinear operators such that its associated linear operator $T$ is continuous on $\hat{\otimes}_\gamma X_i$ will be denoted $\mathcal{L}_\gamma (X_1, \ldots, X_n; Y)$ and $\mathcal{L}_\gamma (X_1, \ldots, X_n)$ when $Y$ is the scalar field.

For a fixed reasonable cross-norm $\gamma$ and $1 \leq p < \infty$ we say, as in [11 Definition 5.1], that $T \in \mathcal{L}_\gamma (X_1, \ldots, X_n; Y)$ is Lipschitz $p$-summing with respect to $\gamma$ (briefly, $\gamma$-Lipschitz $p$-summing) if there exists $c > 0$ such that for $k \in \mathbb{N}$, $i = 1, \ldots, k$ and every $u_i := (u_1^i, \ldots, u_n^i), v_i := (v_1^i, \ldots, v_n^i) \in X_1 \times \cdots \times X_n$,

\begin{equation}
\sum_{i=1}^{k} \| T(u_i) - T(v_i) \|^p \leq c^p \sup_{\varphi \in B_{\mathcal{L}_\gamma (X_1, \ldots, X_n)}} \sum_{i=1}^{k} | \varphi(u_i) - \varphi(v_i) |^p
\end{equation}

The best $c$ above is denoted $\pi_p^{Lip, \gamma}(T)$. Whenever $\gamma$ is the projective tensor norm, it will be omitted in the notation.

If $T$ is a linear operator (i.e., $n = 1$ and $\gamma$ is the norm of $X_1$) it holds that $\pi_p^{Lip, \gamma}(T)$ is the usual $p$-summing norm of $T$ [5 p.31].

2. Main results

As introduced in [3], the Lipschitz $p$-summing norm $(1 \leq p < \infty)$ $\pi_p^L(T)$ of a (possibly nonlinear) mapping $T : X \to Y$ between metric spaces is the smallest constant $C$ so that for all $(x_i), (y_i)$ in $X$

\begin{equation}
\sum_{i=1}^{k} d(T(x_i), T(y_i))^p \leq C^p \sup_{f \in B_X^p} \sum_{i=1}^{k} | f(x_i) - f(y_i) |^p
\end{equation}

where $d$ is the distance in $Y$ and $X^p$ is the Lipschitz dual of $X$, that is, the space of all real valued Lipschitz functions defined on $X$ that vanish at a specified point $0 \in X$. It is a Banach space with the Lipschitz norm and its unit ball $B_{X^p}$ is a compact Hausdorff space in the topology of pointwise convergence on $X$. In the case of vector valued functions, that is, when $Y$ is a Banach space, we will use the notation $Lip_0(X, Y)$ to designate the Banach space of Lipschitz functions $f : X \to Y$ such that $f(0) = 0$ with pointwise addition and the Lipschitz norm.

**Theorem 2.1.** A multilinear operator $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is Lipschitz $p$-summing if and only if its associated $\Sigma$-operator $f_T : \Sigma X_1, \ldots, X_n \to Y$ satisfies (2.1) for some $C > 0$. In this case $\pi_p^L(f_T) = \pi_p^{Lip}(T)$.

The notions of Lipschitz $p$-summability compared in this note are two among several existing generalizations of the linear absolutely $p$-summing operators. Each of them comes with a generalization of the so-called Pietsch domination theorem. We will call them Pietsch-type domination. To the best of our knowledge, all of
Theorem 2.3. Fix in other context. The following theorem provides conditions for this type domination of an operator in one context to the Pietsch-type domination of it.

Lemma 2.2. Let \( \varphi : K_1 \to K_2 \) be a continuous mapping between compact (Hausdorff) spaces. Consider \( \Phi : C(K_2) \to C(K_1) \) defined as \( \Phi(g) := g \circ \varphi \) and its adjoint operator \( \Phi^* : M(K_1) \to M(K_2) \). Then, for any \( \mu \in M(K_1) \) and every \( g \in C(K_2) \) it holds

\[
\int_{K_1} (g \circ \varphi) d\mu = \int_{K_2} gd(\Phi^* \mu).
\]

If \( \mu \) is a probability measure on \( K_1 \), then \( \Phi^*(\mu) \) is a probability measure on \( K_2 \). If \( \mu \in M(K_1) \) is such that \( \Phi^*(\mu) \) is positive, then for every \( h \in C(K_2) \), \( h \geq 0 \) it holds \( 0 \leq \int_{K_1} h \circ \varphi d\mu \).

Proof. All the facts used in this proof can be found in [11]. The mapping \( \Phi \) and its adjoint operator \( \Phi^* \) are bounded linear operators with \( \|\Phi\| = \|\Phi^*\| = 1 \). The duality \( M(K_2) = C(K_2)^* \) means that for any \( \mu \in M(K_1) \) and every \( g \in C(K_2) \) it holds.

Assume that \( \mu \) is a probability measure. For each positive \( g \in C(K_2) \), \( g \circ \varphi \in C(K_1) \) is positive. Then \( 0 \leq \int_{K_2} g \circ \varphi d\mu \). By (2.2), the measure \( \Phi^*(\mu) \) is positive, too. Even more, applying (2.2) to the constant function \( g(k_2) \equiv 1 \) we get \( \Phi^*(\mu)(K_2) = \mu(K_1) = 1 \). Consequently, \( \Phi^*(\mu) \) is a probability measure.

Now, assume that \( \Phi^*(\mu) \) is a positive measure. Using (2.2) again, it follows that for \( h, f \in C(K_2) \), \( 0 \leq h \leq f \)

\[
0 \leq \int_{K_2} h d\Phi^*(\mu) = \int_{K_1} h \circ \varphi d\mu \leq \int_{K_2} f d\Phi^*(\mu) = \int_{K_1} f \circ \varphi d\mu.
\]

Our proof of Theorem 2.1 relies in the possibility of moving from the Pietsch-type domination of an operator in one context to the Pietsch-type domination of it in other context. The following theorem provides conditions for this.

Theorem 2.3. Fix \( 1 \leq p < \infty \). Let \( \varphi : K_1 \to K_2 \) be a continuous mapping between compact (Hausdorff) spaces, a non-empty set \( Z \subset C(K_2) \), \( F : Z \to Y \) a mapping into a Banach space \( Y \) and \( C > 0 \). If there exists a probability measure \( \rho_1 \) on \( K_1 \) such that for every \( h \in Z \)

\[
\|F(h)\|^p \leq C^p \int_{K_1} |h \circ \varphi|^p d\rho_1
\]

then, there exists a probability measure \( \rho_2 \) in \( K_2 \), such that for every \( h \in Z \)

\[
\|F(h)\|^p \leq C^p \int_{K_2} |h|^p d\rho_2.
\]

If the continuous mapping \( \varphi \) is surjective, then if \( \rho_2 \) is a probability measure on \( K_2 \) such that (2.4) holds, then there exists a probability measure \( \rho_1 \) on \( K_1 \) such that (2.3) holds.
Proof. Let \( \rho_1 \) be a probability measure satisfying (2.3). By Lemma 2.2 we know that \( \rho_2 := \Phi^*(\rho_1) \) is a probability measure on \( K_2 \) and that they satisfy (2.2). It remains to check that (2.4) holds. For an \( \Phi \) in Lemma 2.2 is an isometry (see [11, Theorem 2.2]) and consequently, \( \Phi^* \) holds. Since \( \Phi^* \rho \) satisfies (2.3), it also satisfies (2.4).

To prove the remaining assertion, assume that \( \varphi \) is surjective. In this case the operator \( \Phi \) in Lemma 2.2 is an isometry (see [11, Theorem 2.2]) and consequently, \( \Phi^* \) is a quotient operator. Let \( \rho_2 \) be a probability measure on \( K_2 \) such that (2.2) holds. Since \( \Phi^* \) is a surjective mapping, there exists \( \rho \in \mathcal{M}(K_1) \) such that \( \Phi^*(\rho) = \rho_2 \). Being \( \rho_2 \) positive, by Lemma 2.2 we will have that for \( h_i \in Z \) and \( \lambda_i \geq 0 \)

\[
\int_{K_1} \sum_{i=1}^{k} \lambda_i |h_i \circ \varphi|^p d\rho \leq \int_{K_1} \left( \sup_{k_1 \in K_1} \sum_{i=1}^{k} \lambda_i |h_i \circ \varphi(k_1)|^p \right) d\rho = \int_{K_2} \left( \sup_{k_1 \in K_1} \sum_{i=1}^{k} \lambda_i |h_i \circ \varphi(k_1)|^p \right) d\rho_2 = \sup_{k_1 \in K_1} \sum_{i=1}^{k} \lambda_i |h_i \circ \varphi(k_1)|^p.
\]

The first equality above follows if we write for \( c := \sup_{k_1 \in K_1} \sum_{i=1}^{k} \lambda_i |h_i \circ \varphi(k_1)|^p \)

\[
\int_{K_1} c d\rho = \int_{K_1} c (1_{K_2} \circ \varphi) d\rho = \int_{K_2} c \cdot 1_{K_2} d\rho_2 = \int_{K_2} c d\rho_2.
\]

Condition (2.4) and equality (2.2) applied to \( g(k_2) = |h_1(k_2)|^p \) \( (i = 1, \ldots, k) \), along with the inequality above, imply

\[
(2.5) \quad \sum_{i=1}^{k} \lambda_i \|F(h_i)\|^p \leq C \|h_i\|^p \int_{K_2} |h_i|^p d\rho_2 = C \sum_{i=1}^{k} \lambda_i \int_{K_2} |h_i|^p d\rho_2 \leq C \sup_{k_1 \in K_1} \sum_{i=1}^{k} \lambda_i |h_i \circ \varphi(k_1)|^p.
\]

Let \( Q_1 \) be the cone consisting of all positive linear combinations of functions on \( K_1 \) of the form

\[
q_h(k_1) := \|F(h)\|^p - C \|h(\varphi(k_1))\|^p, \quad h \in Z.
\]

From (2.5) we have that \( Q_1 \) can be separated from the positive cone \( P_1 := \{ f \in \mathcal{C}(K_1); f > 0 \} \) by a linear functional \( \tilde{\rho}_1 \in \mathcal{M}(K_1) \). Choosing \( -\tilde{\rho}_1 \) if necessary, we know the existence of some \( c \in \mathbb{R} \) such that for every \( q \in Q_1 \), \( f \in P_1 \)

\[
\int_{K_1} q d\tilde{\rho}_1 \leq c < \int_{K_1} f d\tilde{\rho}_1.
\]

This inequality used with functions \( \lambda q \) and \( f \equiv \mu \) where \( q \) is a fixed function in \( Q_1 \) and any \( \lambda, \mu > 0 \), implies that \( c = 0 \). We get from this that \( \tilde{\rho}_1 \) is positive. Normalizing \( \tilde{\rho}_1 \) we get a probability measure \( \rho_1 \) satisfying (2.3).

\[ \square \]
Proof of Theorem 2.7. Let \( \mathcal{L}(\Sigma_{X_1,\ldots,X_n}) \) be the space of \( \Sigma \)-operators associated to the bounded multilinear forms \( \mathcal{L}(X_1,\ldots,X_n) \). They are isometric spaces with the Lipschitz norm and the multilinear operator norm, respectively [7, Proposition 3.3]. Using the isometric inclusion \( \varphi : B_{\mathcal{L}(\Sigma_{X_1,\ldots,X_n})} \hookrightarrow B_{\Sigma_{X_1,\ldots,X_n}} \), we derive that whenever \( T \) satisfies (2.1), with \( X := \Sigma_{X_1,\ldots,X_n} \) and the same \( c \). Thus, \( \pi_p^L(f_T) \leq \pi_p^{LIP}(T) \).

To prove the reverse implication, we will consider restrictions to finite dimensional subspaces. For fixed finite dimensional subspaces \( E_i \subset X_i, i = 1, \ldots, n \) let \( \gamma \) be the reasonable cross-norm on \( \otimes E_i \) induced by the inclusion in \( \otimes_{\gamma} X_i \). Then \( \Sigma_{\gamma} := (\Sigma_{E_1,\ldots,E_n}, \gamma) \) is a metric subspace of \( X \) and \( f_T|_{\Sigma_{\gamma}} \) is a Lipschitz mapping satisfying (2.1). Since \( \pi_p^L(f_T) \) is defined by means of evaluations on finite sets, it holds that

\[
\pi_p^L(f_T) = \sup_{E_i \subset X_i} \{ \pi_p^L(f_T|_{\Sigma_{\gamma}}) \}.
\]

On the other hand, as proved in [10, Example 3.2], the class of Lipschitz \( p \)-summing multilinear operators is maximal. This means in particular that

\[
\pi_p^{LIP}(T) = \sup_{E_i \subset X_i} \{ \pi_p^{LIP,\gamma}(T|_{E_1 \times \cdots \times E_n}) \}.
\]

From now on, we will omit the spaces \( E_i \) in the notation and will assume that \( f_T \) is defined on \( \Sigma_{\gamma} \). By the Pietsch-type domination theorem proved in [8, Theorem 1], there exists a probability measure \( \mu \) in \( B_{\Sigma_{\gamma}} \) such that for any \( x := x_1 \otimes \cdots \otimes x_n, y := y_1 \otimes \cdots \otimes y_n \) in \( \Sigma_{\gamma} \)

\[
\|f_T(x) - f_T(y)\| \leq \pi_p^L(f_T)^p \int_{B_{\Sigma_{\gamma}}} |\zeta(x) - \zeta(y)|^p d\mu(\zeta).
\]

First we construct an induced probability measure \( \mu_1 \) on \( B_{(\otimes, E_i)_\#} \) which satisfies a Pietsch-type domination. Consider the restriction mapping:

\[
\varphi : B_{(\otimes, E_i)_\#} \to B_{\Sigma_{\gamma}} \quad \zeta \mapsto \zeta_{|\Sigma_{\gamma}}.
\]

Now we identify \( \Sigma_{\gamma} \) with its isometric copy in \( \mathcal{C}(B_{\Sigma_{\gamma}}) \) by means of the natural isometric mapping \( x \mapsto \delta_x \) where \( \delta_x(\zeta) := \zeta(x) \) for each \( \zeta \in B_{\Sigma_{\gamma}} \). For each pair \( x := x_1 \otimes \cdots \otimes x_n, y := y_1 \otimes \cdots \otimes y_n \) in \( \Sigma_{\gamma} \), let \( h_{x,y}(\eta) := \eta(x) - \eta(y) \in \mathcal{C}(B_{\Sigma_{\gamma}}) \), \( Z := \{ h_{x,y} : x, y \in \Sigma_{\gamma} \} \) and \( F(h_{x,y}) := f_T(x) - f_T(y) \). Inequality (2.6) implies that \( F \) satisfies (2.3) in Theorem 2.3. Since McShane’s extension of a Lipschitz mapping guarantees that \( \varphi \) is surjective, \( F \) satisfies also (2.3). That is, there is a probability measure \( \mu_1 \) on \( B_{(\otimes, E_i)_\#} \) satisfying that for each pair \( x := x_1 \otimes \cdots \otimes x_n, y := y_1 \otimes \cdots \otimes y_n \) in \( \Sigma_{\gamma} \)

\[
\|F(h_{x,y})\| \leq \pi_p^L(f_T)^p \int_{B_{(\otimes, E_i)_\#}} |\zeta(x) - \zeta(y)|^p d\mu_1(\zeta).
\]

We are now in position to adapt the proof of the linear case [6, Theorem 2] to our setting. There we can find the justification for the following facts. They can also be tracked from [2, Proposition 6.41].

Without loss of generality, we can assume that \( \mu_1 \) is separable. Let \( \alpha : \hat{\otimes_{\gamma}} E_i \to L_{\infty}(\mu_1) \) be the natural isometric embedding into \( \mathcal{C}(B_{(\otimes, E_i)_\#}) \) composed with the
natural inclusion into \( L_\infty(\mu_1) \), \( \alpha(w) := [\delta_w] \) and let \( i_{\infty,p} \) be the natural inclusion from \( L_\infty(\mu_1) \) into \( L_p(\mu_1) \). The following properties hold:

1. The mapping \( \alpha \) is weak* differentiable almost everywhere. This means that for (Lebesgue) almost every \( w_0 \in \mathbb{R} E_i \), there is a linear operator \( D_{w_0}^\alpha(\alpha) : \mathbb{R} E_i \to L_\infty(\mu_1) \) such that for all \( f \in L_1(\mu_1) \) and for every \( w \in \mathbb{R} E_i \),

\[
\lim_{t \to 0} \left\{ \frac{\alpha(w_0 + tw) - \alpha(w_0)}{t}, f \right\} = \left\langle D_{w_0}^\alpha(\alpha)(w), f \right\rangle.
\]

2. The operator \( i_{\infty,p} \alpha \) is differentiable almost everywhere. This means that for (Lebesgue) almost every \( w_0 \in \mathbb{R} E_i \), there is a linear operator \( D_{w_0}(i_{\infty,p} \alpha) : \mathbb{R} E_i \to L_p(\mu_1) \) such that

\[
\sup_{\|w\| \leq 1} \left\| i_{\infty,p} \alpha(w_0 + tw) - i_{\infty,p} \alpha(w_0) - D_{w_0}(i_{\infty,p} \alpha)(w) \right\|_p \to 0 \quad \text{as} \quad t \to 0.
\]

Select \( w_0 \in \mathbb{R} E_i \) where both derivatives exist. Since \( i_{\infty,p} \) is a is \( w^* - w^* \) continuous mapping, equality \( D_{w_0}(i_{\infty,p} \alpha) = i_{\infty,p} D_{w_0}^\alpha(\alpha) \) holds. Let \( \alpha_1(\alpha) := \alpha(w_0 + w) - \alpha(w_0) \). Then \( D_{w_0}^\alpha(\alpha) = D_{w_0}^\alpha(\alpha_1) \), \( \alpha_1(0) = 0 \) and \( \|D_{w_0}^\alpha(\alpha_1)\| \leq \text{Lip}(\alpha_i) \).

Consider the mapping \( \phi : B(\mathbb{R} E_i)^\# \to B(\mathbb{R} E_i)^\# \) defined as \( \phi(\rho(w)) := \rho(w + w_0) - \rho(w_0) \). By the duality described in Lemma [2.2] and being \( \phi \) a bijective isometry, it induces the existence of a probability measure \( \mu_2 \in \mathcal{M}(\mathbb{R} E_i)^\# \) such that for every \( x, y \in \Sigma_\gamma \),

\[
\|f_T(x) - f_T(y)\|^p \leq \pi_p^\#(f_T)^p \int_{B(\mathbb{R} E_i)^\#} \kappa(x + w_0) - \kappa(y + w_0) |d\mu_2(\kappa)|.
\]

Let \( \bar{Y} \) be the finite dimensional subspace \( \bar{\mathbb{R}}(\mathbb{R} E_i) \) of \( Y \). For each \( \epsilon > 0 \), let \( J : \bar{Y} \to \ell_\infty^m \) be a linear embedding with \( \|J\| = 1, \|J^{-1}\| \leq 1 + \epsilon \). Let \( \alpha_2(w) := \|\delta_{w+w_0} - \delta_{w_0}\| \in L_\infty(\mu_2) \) for \( w \in \mathbb{R} E_i \). The previous inequality allows us to define \( \beta(i_{\infty,p} \alpha_2(x)) := Jf_T(x) \) for every \( x \in \Sigma_\gamma \), where \( i_{\infty,p} \) denotes now the natural inclusion from \( L_\infty(\mu_2) \) to \( L_p(\mu_2) \). Then, we have the factorization \( Jf_T = \beta i_{\infty,p} \alpha_2 \), with \( \text{Lip}(\alpha_2) \leq 1 \), \( \text{Lip}(\beta) \leq \pi_p^\#(f_T) \). By the injectivity of \( \ell_\infty^m \) we can extend \( \beta \) to \( L_p(\mu_2) \) preserving its norm to obtain

\[
\begin{array}{ccc}
\Sigma_\gamma & \xrightarrow{f_T} & \bar{Y} \\
\alpha_2|_{\Sigma_\gamma} & \xrightarrow{J} & \ell_\infty^m \\
L_\infty(\mu_2) & \xrightarrow{i_{\infty,p}} & L_p(\mu_2).
\end{array}
\]

The mapping \( \phi \), which determines \( \mu_2 \) as the pull-back of \( \mu_1 \), induces the isometric onto isomorphisms \( A_p : L_p(B(\mathbb{R} E_i)^\#, \mu_2) \to L_p(B(\mathbb{R} E_i)^\#, \mu_1) \) defined as \( A_p(f) := f \circ \phi \), for \( 1 \leq p \leq \infty \). If \( 1 \leq p < \infty \) and \( 1 = \frac{1}{p} + \frac{1}{p'} \) then \( A_p^{-1} = A_{p'}^* \). These relations along with the fact that \( A_{p'}^* \) is \( w^* - w^* \) continuous, guarantee that the following derivatives of \( \alpha_2 \) at 0 exist and satisfy \( D_{w_0}^\alpha(\alpha_2) = A_{p'}^* D_{w_0}^\alpha(\alpha_1) \) and \( i_{\infty,p} D_{w_0}^\alpha(\alpha_2) = D_0(i_{\infty,p} \alpha_2) \). Consequently, \( \|D_{w_0}^\alpha(\alpha_2)\| \leq \text{Lip}(\alpha_2) \leq 1 \).

The following calculations are the same that the ones in [3 Theorem 2], evaluated only at vectors in \( \Sigma_\gamma \).
Let $\bar{\beta}_n(w) := n\bar{\beta}(\frac{w}{n})$. Then $\text{Lip}(\bar{\beta}_n) = \text{Lip}(\bar{\beta})$. Using that $Jf_T$ is homogeneous, for each $x_1 \otimes \cdots \otimes x_n \in \Sigma_\gamma$ it holds that

$$
\|J \circ f_T(x_1 \otimes \cdots \otimes x_n) - \bar{\beta}_n i_{\infty,p} D_0^w \alpha_2(x_1 \otimes \cdots \otimes x_n)\| = \\
\|\bar{\beta}_n i_{\infty,p} \alpha_2(x_1 \otimes \cdots \otimes x_n) - \bar{\beta}_n i_{\infty,p} D_0^w \alpha_2(x_1 \otimes \cdots \otimes x_n)\| \leq \\
\text{Lip}(\bar{\beta}) i_{\infty,p} \alpha_2(x_1 \otimes \cdots \otimes x_n) - D_0(i_{\infty,p} \alpha_2(x_1 \otimes \cdots \otimes x_n)) \| \xrightarrow{n \to \infty} 0.
$$

Since $\{\bar{\beta}_n\}$ is a bounded set in $Lip_0(L_p(\mu_2), L_{\infty}^{\gamma})$, it has a cluster point $\beta_0$ in it. Let $f_D := D_0^w(\alpha_2)|_{\Sigma_\gamma}$ and $T_D$ be the $\Sigma$-operator and the multilinear mapping respectively, associated to $D_0^w(\alpha_2)$. Then we have the factorization $Jf_T = \beta_0 i_{\infty,p} f_D$ with $\text{Lip}(f_D) \leq \|D_0^w(\alpha_2)\| \leq \text{Lip}(\alpha_2)$ and the factorization of the multilinear mapping $JT = \beta_0 i_{\infty,p} T_D$ with $\|T_D\| = \|D_0^w(\alpha_2)\| \|_{\mathbb{S}_E, E_i \to L_\infty} \leq \|D_0^w(\alpha_2)\| \mathbb{S}_{E_i} \to L_\infty \leq \text{Lip}(\alpha_2)$. The result already follows from these computations, if we use an observation in [1] Section 5]. For the sake of completeness we write the argument. Let $k \in \mathbb{N}$, $i = 1, \ldots, k$ and $x_i := x_i \otimes \cdots \otimes x_i^i$, $y_i := y_i \otimes \cdots \otimes y_i^i \in \Sigma_\gamma$. Using that the $p$-summing norm $\Pi_p(\cdot, \gamma, p)$ of the linear inclusion is one, we have that for $x_i, y_i \in \Sigma_\gamma$

$$
\sum_{i=1}^k \|Jf_T(x_i) - Jf_T(y_i)\|^p = \sum_{i=1}^k \|\beta_0 i_{\infty,p} f_D(x_i) - \beta_0 i_{\infty,p} f_D(y_i)\|^p \\
\leq \text{Lip}(\beta_0)^p \Pi_p(i_{\infty,p}) \sup_{\varphi \in B_{L_\infty}} \sum_{i=1}^k \|\varphi D_0^w(\alpha_2)(x_i) - \varphi D_0^w(\alpha_2)(y_i)\|^p \\
\leq \text{Lip}(\beta_0)^p \|D_0^w(\gamma, p)\| \sup_{\varphi \in B_{\mathbb{S}_{E_i}}} \sum_{i=1}^k \|\varphi(x_i) - \varphi(y_i)\|^p.
$$

The last inequality holds because $\mathbb{S}_{E_i}$ is a closed subspace of $\mathbb{S}_{E_i} X_i$ and consequently each norm-one linear form defined on $\mathbb{S}_{E_i} X_i$ has a norm-one extension. Then $\pi_p^{Lip}(f_T|_{\Sigma_\gamma}) \leq (1 + \epsilon) \text{Lip}(\alpha_2) \text{Lip}(\beta_0)$. Since this is true for arbitrary $\epsilon$ and also for arbitrary finite dimensional spaces $E_i \subset X_i$, $i = 1, \ldots, n$, we have that $\pi_p^{Lip}(f_T) \leq \text{Lip}(\alpha_2) \text{Lip}(\beta_0)$ and, consequently, $\pi_p^{Lip}(T) \leq \pi_p^{Lip}(f_T)$.

This result answers affirmatively Question 7.1 in [1]. In that paper it was also introduced the Lipschitz $p$-summability of a $\Sigma$-operator. Theorem 2.1 clearly implies that for $\Sigma$-operators both notions coincide.

3. COMPARISON OF LIPSCHITZ $p$-SUMMABILITIES IN THE CASE OF OTHER REASONABLE CROSSNORMS

Here we consider the analogous question when a reasonable cross-norm $\gamma$, other than the projective norm, is defined on $X_1 \otimes \cdots \otimes X_n$. Namely, if it is true that a multilinear mapping $T \in L_\gamma(X_1, \ldots, X_n; Y)$ is $\gamma$-Lipschitz $p$-summing (1.1) if and
only if its associated $\Sigma$-operator $f_T : (\Sigma X_1, \ldots, x_n, \gamma) \to Y$ is a Lipschitz $p$-summing mapping (2.1).

Using the isometric inclusion $\varphi : B_{\ell_p}(\Sigma X_1, \ldots, x_n) \hookrightarrow B_{\ell_p^\infty}$ it is direct to prove that if $T$ is a $\gamma$-Lipschitz $p$-summing multilinear operator, then $f_T$ on $\Sigma_\gamma$ is a Lipschitz $p$-summing mapping and $\pi_p^\infty(f_T : \Sigma_\gamma \to Y) \leq \Pi_p^{\infty, \gamma}(T)$. We will see that the reciprocal statement does not always hold. Note that in this case we are assuming that $T$ is continuous on $\hat{\gamma}$, $X_i$.

Lemma 3.1. Let $\gamma$ be a reasonable cross-norm defined on $X_1 \otimes \cdots \otimes X_n$.

(1) If $\Sigma_\gamma := (\Sigma X_1, \ldots, x_n, \gamma)$, then the identity mappings $Id : \Sigma \to \Sigma_\gamma$ and $Id : B_{\Sigma_\gamma} \to B_{\Sigma}$ are bi-Lipschitz with $\text{Lip}(Id) = 1$ and $\text{Lip}(Id^{-1}) \leq 4^{n-1}$.

(2) For $T \in \mathcal{L}_\gamma(X_1, \ldots, x_n; Y)$ and $1 \leq p < \infty$, $f_T : \Sigma_\gamma \to Y$ is a Lipschitz $p$-summing mapping if and only if $f_T : \Sigma \to Y$ is a Lipschitz $p$-summing mapping. In this case $\pi_p^\infty(f_T) \leq 4^{n-1}\pi_p^\infty(f_T : \Sigma_\gamma \to Y)$.

Proof. Recall that $f_T(x_1 \otimes \cdots \otimes x_n) := T(x_1, \ldots, x_n)$. The first assertion in (1) is proved in [7, Theorem 2.1] and the second follows immediately from it. To prove (2) it is enough to observe that inequality (2.1) holds for the supremum on $B_{\Sigma_\gamma}$ if and only if it holds on $B_{\Sigma}$ with a constant at most $4^{n-1}C$. □

Example 3.2. Let $H$ be the Hilbert reasonable cross-norm on $\ell_2 \otimes \ell_2$ (see, e.g. [8, Definition 5.8]) and consider $T : \ell_2 \times \ell_2 \to \ell_2 \otimes_H \ell_2$ defined as $T((a_i)_i, (b_i)_j) = \sum_{i,j} a_i b_j e_i \otimes e_j$. Then, for every $1 \leq p < \infty$, $f_T$ is Lipschitz $p$-summing as a Lipschitz mapping (2.1) on $\Sigma_H$, $T \in \mathcal{L}_{H}(\ell_2, \ell_2; \ell_2 \otimes_H \ell_2)$, but $T$ is not a $H$-Lipschitz $p$-summing bilinear mapping.

To prove the assertions, recall first that the completed space $\ell_2 \otimes_H \ell_2$ is a Hilbert space and $\{e_i \otimes e_j\}_{i,j}$ is an orthonormal basis for it. The linear mapping associated with $T$ satisfies $\widehat{T} \in \mathcal{L}(\ell_2 \otimes_H \ell_2, \ell_2 \otimes_H \ell_2)$ which says that $T \in \mathcal{L}_H(\ell_2, \ell_2; \ell_2 \otimes_H \ell_2)$. $T$ can be factorized as $T = i \circ S$ where $S \in \mathcal{L}(\ell_2, \ell_2; \ell_1)$ is defined as $S((a_n)_n, (b_j)_j) = (a_i b_j)_j$ and $i : \ell_1 \to \ell_2 \leftrightarrow \ell_2 \otimes_H \ell_2$ is the natural inclusion. Since $i$ is absolutely summing, $T$ is a Lipschitz 1-summing bilinear operator. By Theorem 2.1 $f_T : \Sigma \to \ell_2 \otimes_H \ell_2$ is a Lipschitz 1-summing mapping. By Lemma 5.1 we also have that $f_T : \Sigma_H \to \ell_2 \otimes_H \ell_2$ is Lipschitz 1-summing.

Now we check that $T$ is not a $H$-Lipschitz 1-summing bilinear operator. By [1, Theorem 5.2], this is equivalent to prove that $T$ is not a Hilbert-Schmidt bilinear operator. Thus, it is equivalent to prove that $\widehat{T} \in \mathcal{L}(\ell_2 \otimes_H \ell_2, \ell_2 \otimes_H \ell_2)$ is not a Hilbert-Schmidt linear operator [8, Proposition 5.10]. But this is clear since $\{e_i \otimes e_j\}_{i,j}$ is an orthonormal basis of the space and $\sum_{i,j=1}^{\infty} \|\widehat{T}(e_i \otimes e_j)\|^2$ is not finite. The same example serves to prove the case for any $1 < p < \infty$.

Acknowledgement

The author wishes to thank Samuel García-Hernández for helpful discussions during the preparation of this manuscript and the anonymous referee whose suggestions helped improve and clarify it.

References

1. Angulo-López J.C.; Fernández-Unzueta M. Lipschitz $p$-summing multilinear operators. J. Funct. Anal. 279 (2020), no. 4. doi.org/10.1016/j.jfa.2020.108572
2. Benyamini, Y; Lindenstrauss, J. *Geometric Nonlinear Functional Analysis Volume 1*; American Mathematical Society Colloquium Publications Volume 48, 2000.

3. Botelho G.; Pellegrino D.; Rueda P. A unified Pietsch domination theorem. J. Math. Anal. Appl. 365 (2010), no. 1, 269-276.

4. Defant, A.; Floret, K. Tensor norms and operator ideals. *North-Holland Mathematics Studies*, 176, Amsterdam, 1993.

5. Diestel, Jarchow H.; Tonge A. *Absolutely Summing Operators*. Cambridge Univ. Press, 1995.

6. Farmer, J.D.; Johnson, W. B. Lipschitz p-summing operators. Proc. Amer. Math. Soc. 137 (2009), no. 9, 2989-2995.

7. Fernández-Unzueta, M. The Segre cone of Banach spaces and multilinear mappings. Linear Multilinear Algebra 68 (2020), no. 3, 575-593.

8. Matos. M.C. Fully absolutely summing mappings, Math. Nachr. 258 (2003), 71-89.

9. Ryan, R. Introduction to tensor products of Banach spaces. Springer Monographs in Mathematics. *Springer-Verlag London, Ltd., London*, 2002.

10. García-Hernández, S., The duality between ideals of multilinear operators and tensor norms. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 114 (2020), no. 2

11. Semadeni, Z. Spaces of continuous functions on compact sets. Advances in Math. 1 (1965), fasc. 3, 319-382.

12. Weaver, N. Lipschitz algebras. Second edition. World Sci. 2018.

---

Centro de Investigación en Matemáticas (CIMAT), A.P. 402 Guanajuato, Gto., C.P. 36000 México

*Email address: maite@cimat.mx*