Cluster Algebras and Scattering Diagrams

Part II
Cluster Patterns and Scattering Diagrams*

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Abstract. We review some important results by Gross, Hacking, Keel, and Kontsevich on cluster algebra theory, namely, the column sign-coherence of $C$-matrices and the Laurent positivity, both of which were conjectured by Fomin and Zelevinsky. We digest and reconstruct the proofs of these conjectures by Gross et al. still based on their scattering diagram method, however, without relying on toric/birational geometry. At the same time, we also give a detailed account of the correspondence between the notions of cluster patterns and scattering diagrams. Most of the results in this part are found in or translated from the known results in the literature. However, the approach, the construction of logic and proofs, and the overall presentation are new. Also, as an application, we show that there is a one-to-one correspondence between $g$-vectors and cluster variables in cluster patterns with arbitrary coefficients.

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## Contents

0 Introduction to Part II

1 Cluster patterns and separation formulas
   1.1 Seeds and mutations ........................................ 6
   1.2 Cluster patterns ............................................. 8
   1.3 $C$-matrices, $G$-matrices, and $F$-polynomials .......... 9
   1.4 Separation formulas and tropicalization .................. 14
   1.5 Sign-coherence of $C$-matrices and Laurent positivity .... 17

2 More about $C$- and $G$-matrices
   2.1 Mutations and second duality .............................. 19
   2.2 Dual mutations and third duality .......................... 21
   2.3 Principal extension of $B$-patterns ......................... 25
   2.4 $G$-cones and $G$-fans ..................................... 26
   2.5 Rank 2 examples of $G$-fans ............................... 29
   2.6 Piecewise linear isomorphisms between $G$-fans ......... 32
   2.7 Proof of Theorem 2.17 ...................................... 35
   2.8 Properties of $\hat{c}$-vectors .............................. 37

3 Scattering diagrams and $C$- and $G$-matrices
   3.1 Supports of scattering diagrams ............................ 39
   3.2 Proof of sign-coherence conjecture ......................... 40

4 More about $F$-polynomials
   4.1 Fock-Goncharov decomposition .............................. 43
   4.2 Nontropical parts and $F$-polynomials ...................... 46
   4.3 Detropicalization ............................................. 48

5 Scattering diagrams
   5.1 Fixed data and seed .......................................... 50
   5.2 Walls .................................................................. 52
   5.3 Scattering diagrams ........................................... 55
   5.4 Wall-crossing automorphisms ................................. 56
   5.5 Rank 2 examples of consistent scattering diagrams $D_s$ .. 59

6 Scattering diagrams and $F$-polynomials
   6.1 Mutations of scattering diagrams ............................ 62
   6.2 Relation between $\mathfrak{D}(\Sigma s_{i_0})$ and $\mathfrak{D}_s$ ........ 65
   6.3 $F$-polynomials and wall-crossing automorphisms .......... 68
   6.4 Theta functions and Laurent positivity .................... 70
   6.5 Linear independence of cluster monomials ................ 73
   6.6 Remarks on singular case .................................... 76

7 Some applications
   7.1 Detropicalization revisited .................................. 79
   7.2 Bijection between $g$-vectors and $x$-variables .......... 80

References

Index
0 Introduction to Part II

In this part we review some important results by Gross, Hacking, Keel, and Kontsevich \[\text{GHKK18}\] on cluster algebra theory, namely, the \textit{column sign-coherence of C-matrices} and the \textit{Laurent positivity}, both of which were conjectured by Fomin and Zelevinsky \[\text{FZ07, FZ02}\].

Let us start by recalling the background. In the paper \[\text{FZ07}\], Fomin and Zelevinsky developed a new method to study \textit{cluster patterns}, which are the underlying algebraic and combinatorial structure to define cluster algebras. The basic ingredients of the method are \textit{C-matrices}, \textit{G-matrices}, and \textit{F-polynomials} associated with a given cluster pattern. Through the \textit{separation formulas}, the cluster pattern is recovered from them. Moreover, one can regard \textit{C}- and \textit{G}-matrices as the \textit{tropical part} of the cluster pattern, while \textit{F}-polynomials as the \textit{nontropical part}, so that the separation formulas naturally separate these two parts. In many applications of cluster algebras or cluster patterns, this separation plays a central role.

In \[\text{FZ07}\] several conjectures on the properties of \textit{C-} and \textit{G-}matrices and \textit{F}-polynomials were also proposed. Among them, there are two prominent conjectures, namely,

- \textit{The sign-coherence conjecture}: Every \textit{C}-matrix is column sign-coherent.
- \textit{The Laurent positivity conjecture}: Every \textit{F}-polynomial has no negative coefficients. (It is equivalent to the earlier conjecture in \[\text{FZ02}\] on the positivity of the coefficients in the Laurent polynomial expressions for cluster variables.)

Due to their importance, these conjectures, together with other conjectures in \[\text{FZ07}\], have been intensively studied, and proved in some special cases by various methods. Most notably, in the skew-symmetric case, the sign-coherence conjecture was proved by \[\text{DWZ10, Pla11, Nag13}\], and the Laurent positivity conjecture was proved by \[\text{LS15, Dav18}\]. However, the extending these proofs to the most general case, i.e., the skew-symmetric case, seemed difficult and demanding some new ideas.

In \[\text{GHKK18}\] Gross, Hacking, Keel, and Kontsevich finally proved both conjectures together in full generality by the \textit{scattering diagram method}. The scattering diagrams were originally introduced in \[\text{KS06}\] to study the toric degenerations of Calabi-Yau manifolds in homological mirror symmetry in two dimensions, and they were generalized to higher dimensions by \[\text{GST11}\].

Roughly speaking, we have the correspondence between the notions in cluster patterns and scattering diagrams as in Table 1. Let us admit this correspondence. Then, by the definition/construction of scattering diagrams, any normal vector of a wall is either positive or negative. This proves the sign-coherence conjecture. On the other hand, any theta function has a
II.0. Introduction to Part II

Table 1: Correspondence between the notions of cluster patterns and scattering diagrams

| cluster patterns          | scattering diagrams          |
|---------------------------|-----------------------------|
| $G$-matrices              | chambers                     |
| $C$-matrices              | normal vectors of walls      |
| $F$-polynomials           | wall-crossing automorphisms  |
| cluster variables         | theta functions              |

manifestly positive combinatorial description by broken lines as shown in [GHKK18]. This proves the Laurent positivity conjecture. Therefore, establishing the above correspondence is the crucial step to prove the conjectures.

In this part we digest and reconstruct the results and the proofs of the above theorems in [GHKK18] still based on their scattering diagram method, however, without relying on toric/birational geometry therein. To be more specific, we set the following guidelines while preparing the manuscript:

- We assume that readers are familiar with basic concepts in cluster algebras to some extent, for example, seeds and mutations, cluster variables, coefficients, the Laurent phenomenon, the finite type classification, etc.
- The goal of the part is to give reasonably self-contained proofs of the sign-coherence and the Laurent positivity theorems based on the formulation of cluster patterns by [FZ07] with the help of some key properties of scattering diagrams from [GHKK18].
- At the same time, we also give a detailed account of the correspondence between the notions of cluster patterns and scattering diagrams in Table 1.
- We employ the Fock-Goncharov decomposition of mutations with tropical sign to work on $F$-polynomials. This substitutes and avoids the toric/birational geometrical setting in [GHKK18], including cluster varieties, which plays a key role at several points in their original proofs of the conjectures.
- We rely crucially on some basic properties of scattering diagrams in [GHKK18] that are independent of the geometrical setting therein. The details and the proofs of these results will be given in Part III of this monograph.
- Our approach is entirely based on the formulation of cluster patterns in [FZ07]. However, after the paper [FZ07], considerable progress has been made in the subject. So, we also take it into account and reorganize and/or streamline some part of the formulation. Most notably, we do not rely on the notion of principal coefficients explicitly except for the
polynomial property of $F$-polynomials.

Most of the results in this part are found in or translated from the known results in [GHKK18], and/or in other related literatures, especially, a series of papers by Reading [RS16, Rea14, Rea20b, Rea20a]. However, the approach, the construction of logic and proofs, and the overall presentation are new. For example, we prove the sign-coherence conjecture inductively along the $n$-regular tree graph together with establishing the correspondence in Table 1 for $C$- and $G$-matrices. This approach is completely different from the toric geometrical argument in [GHKK18]. On the other hand, it is close to the approach in [Rea14, Rea20b] using mutation fans therein. In fact, many of our results can be also obtained from the results on the mutations fans therein. A subtle but important difference is that we work on $C$- and $G$-matrices without referring to mutation fans at all. Therefore, our approach is more direct. To summarize, our presentation is complementary to the above references, and we believe it serves as an alternative guide for the scattering diagram formalism for cluster algebras.

Finally let us mention that there are already several excellent applications of the scattering diagram method to cluster algebra theory (e.g., Mul16, CGM+17, Bri17, CHL20, CL20, Zho20, Qin19, Mou19, DM21).
1 Cluster patterns and separation formulas

In this section we recall several basic notions and facts on cluster patterns by following the formulation of Fomin and Zelevinsky especially in \cite{FZ07}. Most of the materials are taken or adapted from \cite{FZ07}, and we assume that the readers are familiar with them to some extent. Thus, we skip the proofs of the results unless we think necessary.

1.1 Seeds and mutations

Below we fix a positive integer $n$, which is called the rank of the forthcoming seeds, cluster patterns, etc.

**Definition 1.1** (Skew-symmetrizable matrix). An $n \times n$ integer matrix $B = (b_{ij})_{i,j=1}^n$ is said to be skew-symmetrizable if there is a positive rational diagonal matrix $D = (d_i \delta_{ij})_{i,j=1}^n$ such that $DB$ is skew-symmetric, i.e,

$$d_ib_{ij} = -d_jb_{ji} \quad (1.1)$$

holds. The matrix $D$ is called a (left) skew-symmetrizer of $B$.

**Remark 1.2.** Often a skew-symmetrizer $D$ is assumed to be an integer matrix. However, it is natural to use a rational one in order to match with the convention in \cite{GHKK18}. See (2.44). So, we employ the condition here.

For a skew-symmetrizable matrix $B$, its diagonal entries $b_{ii}$ vanish due to (1.1). Also, the condition (1.1) is rephrased as

$$DBD^{-1} = -B^T, \quad (1.2)$$

where for any matrix $M$, $M^T$ stands for the transpose of $M$.

For our purpose, we especially employ the following (nonstandard) definition of seeds.

**Definition 1.3** (Seed). Let $\mathcal{F}_X$ and $\mathcal{F}_Y$ be a pair of fields both of which are isomorphic to the rational function field of $n$-variables with coefficients in $\mathbb{Q}$. A (labeled) seed in $(\mathcal{F}_X, \mathcal{F}_Y)$ is a triplet $\Sigma = (x, y, B)$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are $n$-tuples of free generating elements in $\mathcal{F}_X$ and $\mathcal{F}_Y$, respectively, and $B = (b_{ij})_{i,j=1}^n$ is an $n \times n$ skew-symmetrizable (integer) matrix. We call $x$, $y$, and $B$, respectively, the $x$-variables, the $y$-variables, and the exchange matrix of $\Sigma$. We also call the pair $(\mathcal{F}_X, \mathcal{F}_Y)$ the ambient fields of a seed $\Sigma$ and the forthcoming cluster patterns, etc.

**Remark 1.4.** In the standard terminology of \cite{FZ07}, the $x$-variables above are cluster variables without coefficients, while the $y$-variables above are coefficients in the universal semifield $\mathbb{Q}_{sf}(y) \subset \mathcal{F}_Y$. Here, $x$- and $y$-variables are on an equal footing and independent of each other. In particular, $y$-variables do not serve as coefficients for $x$-variables.
For any seed $\Sigma = (x, y, B)$, we attach an $n$-tuple $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)$ of elements in the ambient field $\mathcal{F}_X$ defined by
$$\hat{y}_i = \prod_{j=1}^{n} x_j^{b_{ji}}. \quad (1.3)$$

They play an important role in cluster algebra theory. We call them $\hat{y}$-variables. They are algebraically independent if and only if $B$ is nonsingular.

For any integer $a$, we define
$$[a]_+ := \max(a, 0). \quad (1.4)$$
We have a useful identity:
$$a = [a]_+ - [-a]_+. \quad (1.5)$$

**Definition 1.5** (Seed mutation). For any seed $\Sigma = (x, y, B)$ and $k \in \{1, \ldots, n\}$, we define a new seed $\Sigma' = (x', y', B')$ by the following rule:

$$x'_i = \begin{cases} x_k^{-1} \left( \prod_{j=1}^{n} x_j^{-\hat{y}_j} \right)(1 + \hat{y}_k) & i = k, \\ x_i & i \neq k, \end{cases} \quad (1.6)$$

$$y'_i = \begin{cases} y_k^{-1} & i = k, \\ y_i y_k^{\hat{b}_{ki}}(1 + y_k)^{-\hat{b}_{ki}} & i \neq k, \end{cases} \quad (1.7)$$

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} + b_{ik} b_{kj} + [-b_{ik}] + b_{kj} & i, j \neq k, \end{cases} \quad (1.8)$$

where $\hat{y}_k$ in (1.6) is the $\hat{y}$-variable in (1.3). The seed $\Sigma'$ is called the mutation of $\Sigma$ in direction $k$, and denoted by $\mu_k(\Sigma) = \mu_k(x, y, B)$.

The following facts ensure that $(x', y', B')$ is indeed a seed.

(a). If $D$ is a skew-symmetrizer of $B$, then it is also a skew-symmetrizer of $B'$. In particular, the matrix $B'$ is skew-symmetrizable.

(b). Regard the mutation $\mu_k$ as a map from the set of seeds to itself. Then, $\mu_k$ is an involution. In particular, $x'$ and $y'$ are free generating elements in $\mathcal{F}_X$ and $\mathcal{F}_Y$, respectively.

Also, one can easily verify that $|B'| = |B|$, so that $B'$ is nonsingular if and only if $B$ is nonsingular.

The following fact can be easily verified.

**Proposition 1.6** ([FZ07 Prop. 3.9]). The $\hat{y}$-variables mutate as

$$\hat{y}'_i = \begin{cases} \hat{y}_k^{-1} & i = k, \\ \hat{y}_i \hat{y}_k^{\hat{b}_{ki}}(1 + \hat{y}_k)^{-\hat{b}_{ki}} & i \neq k \end{cases} \quad (1.9)$$
in the ambient field $\mathcal{F}_X$ by the same rule as (1.7).

There is some flexibility to write the mutation formulas (1.6)–(1.8) as follows. (The choice $\varepsilon = 1$ reduces to the one in (1.6)–(1.8).)

**Proposition 1.7** ([Nak12, Eqs. (2.9), (2.10)], [BFZ05, Eq. (3.1)]). Let $\varepsilon \in \{1, -1\}$. Then, the right hand sides of the following equalities do not depend on the choice of $\varepsilon$:

$$
\begin{align*}
  x'_i &= \begin{cases} 
    x^{-1}_k \left( \prod_{j=1}^n x_{[\varepsilon b_{jk}]}^j \right) \left( 1 + \tilde{y}_k^\varepsilon \right) & i = k, \\
    x_i & i \neq k,
  \end{cases} \\
  y'_i &= \begin{cases} 
    y^{-1}_k \left[ \varepsilon b_{ki} \right]_+ (1 + \tilde{y}_k^\varepsilon)^{-b_{ki}} & i = k, \\
    y_i y_k^\varepsilon & i \neq k,
  \end{cases} \\
  b'_{ij} &= \begin{cases} 
    -b_{ij} & i = k \text{ or } j = k, \\
    b_{ij} + b_{ik} \varepsilon b_{kj} + [-\varepsilon b_{ik}]_+ b_{kj} & i, j \neq k.
  \end{cases}
\end{align*}
$$

(1.10) (1.11) (1.12)

The claim can be easily checked with the identity (1.5). We call the above formulas the $\varepsilon$-expressions of mutations.

### 1.2 Cluster patterns

Let $\mathbb{T}_n$ be the $n$-regular tree graph, that is, a graph without cycles such that exactly $n$-edges are attached to each vertex. We assume that the edges are labeled by $1, \ldots, n$ such that the edges attached to each vertex have different labels. By abusing the notation, the set of vertices of $\mathbb{T}_n$ is also denoted by $\mathbb{T}_n$. We say that a pair of vertices in $t$ and $t'$ in $\mathbb{T}_n$ are ($k$-)adjacent, or $t'$ is ($k$-)adjacent to $t$, if they are connected with an edge labeled by $k$.

**Definition 1.8** (Cluster pattern/B-pattern). A collection of seeds $\Sigma = \{\Sigma_t = (x_t, y_t, B_t)\}_{t \in \mathbb{T}_n}$ in $(\mathcal{F}_X, \mathcal{F}_Y)$ indexed by $\mathbb{T}_n$ is called a cluster pattern if, for any vertices $t, t' \in \mathbb{T}_n$ that are $k$-adjacent, the equality $\Sigma_{t'} = \mu_k(\Sigma_t)$ holds. The collection of exchange matrices $B = \{B_t\}_{t \in \mathbb{T}_n}$ extracted from a cluster pattern $\Sigma$ is called the B-pattern of $\Sigma$.

We arbitrary choose a distinguished vertex (the initial vertex) $t_0$ in $\mathbb{T}_n$. Any cluster pattern $\Sigma = \{\Sigma_t = (x_t, y_t, B_t)\}_{t \in \mathbb{T}_n}$ is uniquely determined from the initial seed $\Sigma_{t_0}$ at the initial vertex $t_0$ by repeating mutations.

Note that one can take a common skew-symmetrizer $D$ for all $B_t$ in $B$. By (1.2), we have

$$
DB_tD^{-1} = -B_t^T. 
$$

(1.13)

In particular, if $B_t$ is skew-symmetric for some $t$, it is skew-symmetric for any $t$. Such a cluster pattern or a $B$-pattern is said to be skew-symmetric.
1.3. C-matrices, G-matrices, and F-polynomials

Similarly, if \( B_t \) is nonsingular for some \( t \), it is nonsingular for any \( t \). Such a cluster pattern or a \( B \)-pattern is said to be nonsingular.

For a seed \( \Sigma_t = (x_t, y_t, B_t) (t \in \mathbb{T}_n) \) in a cluster pattern \( \Sigma \), we use the notation

\[
x_t = (x_{1;t}, \ldots, x_{n;t}), \quad y_t = (y_{1;t}, \ldots, y_{n;t}), \quad B_t = (b_{ij;t})_{i,j=1}^n.
\]  

(1.14)

For the initial seed \( \Sigma_{t_0} = (x_{t_0}, y_{t_0}, B_{t_0}) \), we often drop the indices \( t_0 \) as

\[
x_{t_0} = x = (x_1, \ldots, x_n), \quad y_{t_0} = y = (y_1, \ldots, y_n), \quad B_{t_0} = B = (b_{ij})_{i,j=1}^n.
\]  

(1.15)

if there is no confusion. We also use similar notations for \( \hat{y} \)-variables \( \hat{y}_t = (\hat{y}_{1;t}, \ldots, \hat{y}_{n;t}) \) and the initial \( \hat{y} \)-variables \( \hat{y}_{t_0} = \hat{y} = (\hat{y}_1, \ldots, \hat{y}_n) \).

We recall the definition of cluster algebras, though the focus of this part is to study the properties of cluster patterns rather than cluster algebras.

Definition 1.9 (Cluster algebra). For a cluster pattern \( \Sigma \), the associated cluster algebra \( A(\Sigma) \) is the \( \mathbb{Z} \)-subalgebra of the ambient field \( \mathcal{F}_X \) generated by all \( x \)-variables (cluster variables) \( x_{i,t} (i = 1, \ldots, n; t \in \mathbb{T}_n) \).

1.3 C-matrices, G-matrices, and F-polynomials

Following [FZ07], we introduce C-matrices, G-matrices, and F-polynomials. They play a crucial role in the study of cluster patterns.

1.3.1 Matrix notations

We start by introducing the following matrix notations (see, e.g., [NZ12]). For an \( n \times n \) matrix \( A \), let \([A]_+\) denote the matrix obtained from \( A \) by replacing the entry \( a_{ij} \) with \( [a_{ij}]_+ \). Let \( A^{\bullet k} \) denote the matrix obtained from \( A \) by replacing the entry \( a_{ij} \) (\( j \neq k \)) with 0. Similarly, let \( A^{k \bullet} \) denote the matrix obtained from \( A \) by replacing the entry \( a_{ij} \) (\( i \neq k \)) with 0.

For any \( n \times n \) matrices, the following facts are easily checked, where for the last two equalities we use \( (1.3) \).

\[
AB^{\bullet k} = (AB)^{\bullet k}, \quad A^{k \bullet} B = (AB)^{k \bullet},
\]  

(1.16)

\[
A^{\bullet k} B = A^{\bullet k} B^{k \bullet} = AB^{k \bullet},
\]  

(1.17)

\[
A[B]_+ + [-A]_+ B = A[-B]_+ + [A]_+ B,
\]  

(1.18)

If \( AB = CD \), then \( A[B]_+ - C[D]_+ = A[-B]_+ - C[-D]_+ \).

(1.19)

Let \( J_k \) be the \( n \times n \) diagonal matrix obtained from the identity matrix by replacing the \( k \)th diagonal entry with \(-1\). The following properties will be repeatedly used.
Lemma 1.10. Let $A$ be any $n \times n$ matrix whose diagonal entries are 0. Let $P = J_k + A^k \bullet$ or $J_k + A^k \bullet$. Then, we have
\[
A^k \bullet J_k = -A^k \bullet, \quad J_k A^k = A^k \bullet, \quad J_k A^k = -A^k \bullet, \quad P^2 = I, \quad |P| = -1.
\]

Proof. Let us prove (1.21) for $P = J_k + A^k \bullet$. By $a_{kk} = 0$, we have
\[
P^2 = J_k^2 + J_k A^k + A^k \bullet J_k + A^k \bullet A^k = I - A^k + A^k \bullet = I.
\]
Also, we have $|P| = |J_k| = -1$. \hfill \Box

Example 1.11. The first case of the matrix mutation (1.8) is written as
\[
(B')^k \bullet = -B^k \bullet, \quad (B')^k = -B^k \bullet.
\]

Also, the entire mutation (1.8) is written as [BFZ05, Eq. (3.2)]
\[
B' = (J_k + [-B]^k_+)B(J_k + [B]^k_+).
\]

By (1.21), we have the relation
\[
(J_k + [-B]^k_+)B' = B(J_k + [B]^k_+).
\]

1.3.2 $C$-matrices

Let $\Sigma$ be any cluster pattern, and let $B$ be the $B$-pattern of $\Sigma$. Let $t_0 \in \mathbb{T}_n$ be a given initial vertex.

We introduce a collection of matrices $C^{t_0} = C = \{C_t\}_{t \in \mathbb{T}_n}$ called a $C$-pattern, which is uniquely determined from $B$ and $t_0$, therefore, eventually only from $t_0$ and $B_{t_0}$.

Definition 1.12 (C-matrices). The $C$-matrices $C_t = (c_{ij;t})_{i,j=1}^n$ ($t \in \mathbb{T}_n$) of a cluster pattern $\Sigma$ (or a $B$-pattern $B$) with an initial vertex $t_0$ are $n \times n$ integer matrices that are uniquely determined by the following initial condition and the mutation rule:
\[
C_{t_0} = I, \quad C_{t'} = C_{t}J_k + C_{t}[B_t]^k_+ + [-C_{t}]^k_+ B_t,
\]
or more explicitly,
\[
c_{ij;t'} = \begin{cases} 
-c_{ik;t} & j = k, \\
c_{ij;t} + c_{ik;t}[b_{kj};t]_+ + [-c_{ik;t}]+b_{kj;t} & j \neq k,
\end{cases}
\]

where $t$ and $t'$ are $k$-adjacent. Each column vector $c_{i;t} = (c_{ij;t})_{j=1}^n$ of a matrix $C_t$ is called a $c$-vector.
Remark 1.13. The $C$-matrices appeared implicitly as the lower halves of the extended exchange matrices for the principal coefficients in [FZ07, Def. 3.1].

The mutation (1.28), in particular, implies
\[
C_{k; t'} = -C_{k; t}, \tag{1.30}
\]
or equivalently,
\[
C_t^{*k} = -C_t^{*k}. \tag{1.31}
\]

One can easily check that the mutation (1.29) is involutive using (1.3), (1.8), and (1.30). Alternatively, in the matrix form, by multiplying $J_k$ from the right to (1.28), we have
\[
C_{t'} J_k = C_t + C_t^{*k} [B_t]_+^{k*} + [-C_t]_+^{k} B_t^{*k}. \tag{1.32}
\]
By (1.24) and (1.31), this can be written as
\[
C_t = C_{t'} J_k + C_{t'} [-B_{t'}]_+^{k*} + [C_{t'}]_+^{k} B_{t'}. \tag{1.33}
\]
Then, by (1.18), we have
\[
C_t = C_{t'} J_k + C_{t'} [B_{t'}]_+^{k*} + [-C_{t'}]_+^{k} B_{t'}, \tag{1.34}
\]
which is the desired result. Moreover, by (1.18) again, we have the following $\varepsilon$-expression of (1.28):

**Proposition 1.14** ([NZ12, Eq. (2.4)]). Let $\varepsilon \in \{1, -1\}$. Then, the right hand side of the following equality does not depend on the choice of $\varepsilon$:
\[
C_{t'} = C_t J_k + C_t [\varepsilon B_t]_+^{k*} + [-\varepsilon C_t]_+^{k} B_t. \tag{1.35}
\]

### 1.3.3 G-matrices

Under the same assumption of $C$-matrices, we introduce another collection of matrices $G_{t_0} = G = \{G_t\}_{t \in \mathbb{T}_n}$ called a $G$-pattern, which is uniquely determined from $\mathcal{B}$, $t_0$, and $C_{t_0}$ defined above.

**Definition 1.15** (G-matrices). The $G$-matrices $G_t = (g_{ij;t})_{i,j=1}^n$ ($t \in \mathbb{T}_n$) of a cluster pattern $\Sigma$ (or a $B$-pattern $\mathcal{B}$) with an initial vertex $t_0$ are $n \times n$ integer matrices that are uniquely determined by the following initial condition and the mutation rule:
\[
G_{t_0} = I, \tag{1.36}
\]
\[
G_{t'} = G_t J_k + G_t [B_t]_+^{k*} - B_{t_0} [C_t]_+^{k}. \tag{1.37}
\]
or more explicitly,

\[
g_{ij:t'} = \begin{cases} 
-g_{ik:t} + \sum_{\ell=1}^n g_{i\ell:t} [-b_{\ell k:t}] + \sum_{\ell=1}^n b_{i\ell:t_0} [-c_{\ell k:t}] & j = k, \\
g_{ij:t} & j \neq k,
\end{cases}
\]  

(1.38)

where \(t\) and \(t'\) are \(k\)-adjacent. Each column vector \(g_{i:t} = (g_{ji:t})_{j=1}^n\) of a matrix \(G_t\) is called a \(g\)-vector.

**Remark 1.16.** The \(g\)-vectors were originally introduced as the degree vectors of cluster variables with principal coefficients in [FZ07].

The first nontrivial property so far is the following relation between \(C\)- and \(G\)-matrices.

**Proposition 1.17** ([FZ07, Eq. (6.14)]). The following equality holds for any \(t \in T_n\).

(First duality)

\[
G_t B_t = B_{t_0} C_t.
\]  

(1.39)

The proof in [FZ07, Eq. (6.14)] relied on the fact that \(g\)-vectors are the degree vectors of cluster variables with principal coefficients. We give an alternative proof based on the definitions of \(C\)- and \(G\)-matrices here. Let \(d(t, t')\) denote the distance in \(T_n\) between two vertices \(t, t' \in T_n\).

**Proof.** We prove it by the induction on \(t\) along \(T_n\) from \(t_0\). For \(t = t_0\), the equality (1.39) holds because \(C_{t_0} = G_{t_0} = I\). Suppose that the equality (1.39) holds for \(t \in T_n\) such that \(d(t_0, t) = d\). Let \(t'\) be the vertex that is \(k\)-adjacent to \(t\) such that \(d(t_0, t') = d + 1\). Then, we have

\[
G_t B_{t'} = G_t (J_k + [-B_t]_+^k) B_{t'} - B_{t_0} [-C_t]^k B_{t'}^k
= G_t B_t (J_k + [B_t]_+^k) + B_{t_0} [-C_t]^k B_{t'}^k
= B_{t_0} C_t (J_k + [B_t]_+^k) + B_{t_0} [-C_t]^k B_t
= B_{t_0} C_{t'},
\]  

where in the second equality we used (1.24) and (1.26).

**Remark 1.18.** There has been no name on this equality so far in the literature; however, due to its importance, we name it as above, where the duality means a relation between \(C\)- and \(G\)-matrices. (Later we will have the second and the third dualities, which are more nontrivial.)

Thanks to (1.19) and (1.39), one can easily check that the mutation (1.37) is involutive as before. Moreover, by (1.19) and (1.39) again, we have the following \(\varepsilon\)-expression of (1.37).
Proposition 1.19 ([FZ07] Eqs. (6.12), (6.13)). Let $\varepsilon \in \{1, -1\}$. Then, the right hand side of the following equality does not depend on the choice of $\varepsilon$:

$$G_{t'} = G_t J_k + G_t[-\varepsilon B_t]_+^k - B_{t_0}[-\varepsilon C_t]_+^k. \quad (1.41)$$

1.3.4 $F$-polynomials

Let $y = (y_1, \ldots, y_n)$ be an $n$-tuple of formal variables. Although it is not necessary at this moment, it is natural to identify them with the initial $y$-variables $y_{t_0}$ of $\Sigma$ as we will see soon. Under the same assumption of $C$-matrices, we introduce a collection of rational functions $F_{t_0} = \mathbf{F} = \{F_{i:t}(y)\}_{i=1,\ldots,n; t \in \mathbb{T}_n}$ in $y$ called a $F$-pattern, which is uniquely determined from $B$, $t_0$, and $C_{t_0}$ as follows.

Definition 1.20 ($F$-polynomials). The $F$-polynomials $F_{i:t}(y) (i = 1, \ldots, n; t \in \mathbb{T}_n)$ of a cluster pattern $\Sigma$ (or a $B$-pattern $B$) with an initial vertex $t_0$ are rational functions in $y$ with coefficients in $\mathbb{Q}$ that are uniquely determined by the following initial condition and the mutation rule:

$$F_{i:t_0}(y) = 1, \quad (1.42)$$

$$F_{i:t'}(y) = \begin{cases} M_{k:t}(y) \frac{F_{k:t}(y)}{F_{i:t}(y)} & i = k, \\ F_{i:t}(y) & i \neq k, \end{cases} \quad (1.43)$$

where $t$ and $t'$ are $k$-adjacent, and $M_{k:t}(y)$ is a rational function in $y$ defined as follows:

$$M_{k:t}(y) = \prod_{j=1}^{n} y_j^{[c_{jk:t}]} \prod_{j=1}^{n} F_{j:t}(y)^{[b_{jk:t}]}_+ + \prod_{j=1}^{n} y_j^{[-c_{jk:t}]} \prod_{j=1}^{n} F_{j:t}(y)^{-b_{jk:t}}_+. \quad (1.44)$$

In the situation in (1.43), we have

$$M_{k:t}(y) = M_{k:t'}(y). \quad (1.45)$$

Therefore, the mutation (1.43) is involutive. We do not discuss the $\varepsilon$-expression for (1.43), because we do not use it here.

The $F$-polynomials were originally introduced as the specialized $x$-variables with principal coefficients under the specialization $x_1 = \cdots = x_n = 1$ in [FZ07]. The Laurent phenomenon applied to them yields the following important fact, which justifies the name $F$-polynomials.

Theorem 1.21 ([FZ07] Prop. 3.6]). For any $i = 1, \ldots, n$ and $t \in \mathbb{T}_n$, the rational function $F_{i:t}(y)$ is a polynomial in $y$ with coefficients in $\mathbb{Z}$.

Remark 1.22. In case we need to clarify the choice of the initial vertex $t_0$, we write $C$-matrices and others as, $C_{t_0} = (c_{i,j,t_0}^0)_{i,j=1}^n, C_{t_0}' = (c_{i,j,t_0}')_{i,j=1}^n, g_{i,j,t_0}^0, F_{i,t_0}(y)$. Conversely, if there is no confusion, we suppress the superscript $t_0$ as above for simplicity.
1.4 Separation formulas and tropicalization

Let us present one of the most fundamental results on cluster patterns in \[\text{FZ07}.\]

**Theorem 1.23** (Separation Formulas \[\text{FZ07, Prop. 3.13, Cor. 6.3}\]). Let \(\Sigma\) be any cluster pattern, and let \(t_0 \in \mathbb{T}_n\) be a given initial vertex. Let \(C, G,\) and \(F\) be the \(C\)-, \(G\)-, and \(F\)-patterns of \(\Sigma\) with the initial vertex \(t_0\). Let \(x, y,\) and \(\hat{y}\) be the initial \(x\)-, \(y\)-variables, and \(\hat{y}\)-variables for the initial seed \(\Sigma_{t_0}\). Then, the following formulas hold.

\[
x_{i;t} = \left( \prod_{j=1}^{n} x_j^{g_{ji};t} \right) F_{i;t}(\hat{y}), \quad (1.46)
\]

\[
y_{i;t} = \left( \prod_{j=1}^{n} y_j^{c_{ji};t} \right) \prod_{j=1}^{n} F_j^{b_{ji};t} (y), \quad (1.47)
\]

**Proof.** Here is an alternative proof to the one in \[\text{FZ07}.\] Using the mutations of \(C\)- and \(G\)-matrices, and \(F\)-polynomials, one can easily verify (1.46) and (1.47) by the induction on \(t\) along \(\mathbb{T}_n\) from \(t_0\) with the help of (1.39). We leave the details as an exercise to the readers.

Since \(\hat{y}\)-variables mutate exactly in the same manner as \(y\)-variables, a parallel formula holds for them:

\[
\hat{y}_{i;t} = \left( \prod_{j=1}^{n} \hat{y}_j^{\hat{c}_{ji};t} \right) \prod_{j=1}^{n} F_j^{\hat{b}_{ji};t} (\hat{y}). \quad (1.48)
\]

Alternatively, one can derive it from (1.46) with the help of the duality (1.39).

Thank to the separation formulas, the study of cluster patterns reduces to the study of \(C\)- and \(G\)-matrices, and \(F\)-polynomials. Moreover, the formulas play a central role in the interplay between the cluster algebraic structure and various subjects.

The original meaning of the separation formula in \[\text{FZ07, Cor. 6.3}\] is that a general version of the formula (1.46) for \(x\)-variables with coefficients naturally “separates” the additions of \(x\)-variables and coefficients. Unfortunately, we cannot observe it in (1.46), because our \(x\)-variables are without coefficients. Below we attach an additional meaning of “separation” to the formulas in view of the tropicalization.

First, we briefly recall the notion of tropicalization in the context of semifields \[\text{FZ07}.\]
Definition 1.24. A semifield $P$ is a multiplicative abelian group equipped with a binary operation $\oplus$ on $P$ that is commutative, associative, and distributive, i.e., $(a \oplus b)c = ac \oplus bc$. The operation $\oplus$ is called the addition in $P$.

Here we exclusively use the following two semifields.

Example 1.25. (a). Universal semifield $Q_{sf}(u)$. Let $u = (u_1, \ldots, u_n)$ be an $n$-tuple of commutative variables. We say that a rational function $f(u) \in Q(u)$ in $u$ has a subtraction-free expression if it is expressed as $f(u) = p(u)/q(u)$, where $p(u)$ and $q(u)$ are nonzero polynomials in $u$ whose coefficients are nonnegative. Let $Q_{sf}(u)$ be the set of all rational functions in $u$ having subtraction-free expressions. Then, $Q_{sf}(u)$ is a semifield by the usual multiplication and addition in $Q(u)$.

(b). Tropical semifield $\text{Trop}(u)$. Let $u = (u_1, \ldots, u_n)$ be an $n$-tuple of commutative variables. Let $\text{Trop}(u)$ be the set of all Laurent monomials of $u$ with coefficient 1, which is a multiplicative abelian group by the usual multiplication. We define the addition $\oplus$ by

$$\prod_{i=1}^{n} u_i^{a_i} \oplus \prod_{i=1}^{n} u_i^{b_i} := \prod_{i=1}^{n} u_i^{\min(a_i, b_i)}.$$  \hspace{1cm} (1.49)

Then, $\text{Trop}(u)$ becomes a semifield. The addition $\oplus$ is called the tropical sum.

Consider $Q_{sf}(u)$ and $\text{Trop}(u)$ with common generating variables $u = (u_1, \ldots, u_n)$. Then, we have a unique semifield homomorphism

$$\pi_{\text{trop}} : Q_{sf}(u) \rightarrow \text{Trop}(u)$$  \hspace{1cm} (1.50)

such that $\pi_{\text{trop}}(u_i) = u_i$ for any $i = 1, \ldots, n$. We call it the tropicalization homomorphism. Roughly speaking, it extracts the “leading monomial” of a rational function $f(u)$ in $Q_{sf}(u)$.

Example 1.26. For $u = (u_1, u_2, u_3)$,

$$\pi_{\text{trop}} \left( \frac{3u_1u_2^3u_3^2 + 2u_1^2u_2u_3}{3u_2^2 + u_1^2u_2^2 + u_1u_3^2u_3} \right) = \frac{u_1u_2u_3}{u_2^2} = u_1u_2^{-1}u_3.$$  \hspace{1cm} (1.51)

Now let $\Sigma$ be any cluster pattern, and let $t_0 \in \mathbb{T}_n$ be a given initial vertex. Let $y_{t_0} = y$ be the initial $y$-variables in the ambient field $F_Y$. Then, the semifield generated by $y$ in $F_Y$ (by the usual multiplication and addition) is identified with the universal semifield $Q_{sf}(y)$. Moreover, any $y$-variable $y_{i;t}$ belongs to $Q_{sf}(y)$ because the mutation \hspace{1cm} (1.7) is a subtraction-free operation. Let us apply the tropicalization homomorphism $\pi_{\text{trop}}^{t_0} : Q_{sf}(y) \rightarrow \text{Trop}(y)$ to $y$-variables.
Definition 1.27 (Tropical $y$-variable). We call the image $\pi_{t_0}^{t_0}(y_{i;t})$ a tropical $y$-variable with respect to the initial vertex $t_0$.

The following fact can be easily deduced from (1.43) by the induction on $t$ along $T_n$ from $t_0$.

**Proposition 1.28** ([FZ07, Eq. (5.5)]).

\[
\pi_{t_0}^{t_0}(F_{i;t}(y)) = 1. \tag{1.52}
\]

**Remark 1.29.** The equality (1.52) does not mean the polynomial $F_{i;t}(y)$ has a nonzero constant term. For example, $\pi_{t_0}(y_1 + y_2) = 1$.

Tropical $y$-variables are naturally identified with $c$-vectors as follows:

**Proposition 1.30** ([FZ07, Eq. (3.14)]). We have

\[
\pi_{t_0}^{t_0}(y_{i;t}) = y^{c_{i;t}} := \prod_{j=1}^{n} y_{j}^{c_{ji;t}}. \tag{1.53}
\]

**Proof.** This follows from the separation formula (1.47) and (1.52).

**Remark 1.31.** In fact, (1.53) is the definition of $c$-vectors in [FZ07, Eq. (5.8)], where the tropical $y$-variables are identified with principal coefficients.

Now looking back the separation formula (1.47) for $y$-variables, we see that it naturally “separates” the tropical and the nontropical parts of $y$-variables.

On the other hand, the tropicalization of $x$-variables with respect to the initial $x$-variables $x$ does not yield $g$-vectors; rather, it yields the so-called $d$-vectors (denominator vectors in [FZ07]), which are the leading exponents of the Laurent polynomial expression of $x$-variables. However, if we regard the operation of setting all $F_i$-polynomials $F_{i;t}(y)$ to 1 also as “a kind of tropicalization” for $x$-variables, then the separation formula (1.46) for $x$-variables can be viewed again as separating the “tropical” and the “nontropical” parts of $x$-variables, where a $g$-vector $g_{i;t} = (g_{ji;t})_{j=1}^{n}$ is identified with a tropical $x$-variable defined by

\[
x^{g_{i;t}} := \prod_{j=1}^{n} x^{g_{ji;t}}. \tag{1.54}
\]

In the same token, we call

\[
y^{c_{i;t}} := \prod_{j=1}^{n} y^{c_{ji;t}}. \tag{1.55}
\]
a tropical $\hat{y}$-variable with respect to the initial vertex $t_0$. Thanks to the duality (1.39), we see that the notion is compatible with a tropical $x$-variable as

\[ \hat{y}^{c_{i,t}} = \prod_{j=1}^{n} (x^{g_{j,t}})^{b_{j,t}}. \quad (1.56) \]

Alternatively, based on the duality (1.39), we introduce a $\hat{c}$-vector $\hat{c}_{i,t}$ by

\[ \hat{c}_{i,t} = B_{t_0}c_{i,t} = G_{t}b_{i,t}, \quad (1.57) \]

where $b_{i,t}$ is the $i$th column vector of $B_{t}$. In other words, it is the $i$th column vector of a $\hat{C}$-matrix defined by

\[ \hat{C}_{t} := B_{t_0}C_{t} = G_{t}B_{t}. \quad (1.58) \]

Then, the tropical $\hat{y}$-variable in (1.56) is expressed also as

\[ \hat{y}^{c_{i,t}} = x^{\hat{c}_{i,t}}. \quad (1.59) \]

These expressions and notions play a major role in the scattering diagram method later.

1.5 Sign-coherence of $C$-matrices and Laurent positivity

In the paper [FZ07] several conjectures on the properties of cluster patterns were proposed. Here we recall especially two (three, precisely speaking) conjectures among them, which are the theme of this part.

Let $C$ and $F$ be the $C$- and $F$-patterns of any cluster pattern $\Sigma$ with a given initial point $t_0 \in \mathbb{T}_n$.

The first conjecture is the following:

**Conjecture 1.32** (Unit constant property [FZ07, Conj. 5.4]). Every $F$-polynomial $F_{i;t}(y)$ has constant term 1.

This seemingly innocent conjecture turned out to be difficult to prove simply by the induction on $t$ along $\mathbb{T}_n$. However, it was noticed by [FZ07] that Conjecture 1.32 is equivalent to another remarkable conjecture on $C$-matrices and $c$-vectors.

**Definition 1.33** (Row/Column sign-coherence). We say that a vector $v \in \mathbb{Z}^n$ is positive (resp. negative) if it is nonzero vector and all nonzero components are positive (resp. negative). We say that a matrix $M$ is column sign-coherent (resp. row sign-coherent) if each column vector (resp. row vector) of $M$ is either positive or negative.

**Conjecture 1.34** (Column sign-coherence of $C$-matrices [FZ07, Prop. 5.6]). Every $C$-matrix $C_{t}$ is column sign-coherent. Equivalently, every $c$-vector $c_{i;t}$ is either positive or negative.
The equivalence between Conjectures 1.32 and 1.34 easily follows from the mutation of $F$-polynomials [FZ07, Prop. 5.6].

The sign-coherence conjecture was proved in the skew-symmetric case by [DWZ10, Pla11, Nag13] with the representation/categorical methods, and in general by [GHKK18] with the scattering diagram method.

**Theorem 1.35 ([GHKK18, Cor. 5.5]).** Conjectures 1.32 and 1.34 hold for any $\Sigma$ and $t_0$.

The second conjecture we consider is the following:

**Conjecture 1.36** (Laurent positivity [FZ07, §3], [FZ02, §3]). Every $F$-polynomial $F_{i;t}(y) \in \mathbb{Z}[y]$ has no negative coefficients.

Through the separation formula, this is equivalent to the earlier conjecture in [FZ02] on the positivity of the coefficients in the Laurent polynomial expressions for $x$-variables. This conjecture was proved for surface type by [MSW11], for acyclic type by [KQ14], in the skew-symmetric case by [LS15, Dav18], with various methods, and in general by [GHKK18] with the scattering diagram method.

**Theorem 1.37 ([GHKK18, Theorem 4.10]).** Conjecture 1.36 holds for any $\Sigma$ and $t_0$.

From these conjectures (now theorems) many other conjectures in [FZ07] and new results follow (e.g., [NZ12, CHL20, CL20, Nak21]). Therefore, they are considered to be in the heart of cluster algebra theory.

As mentioned in Introduction, the purpose of the part is to digest and reconstruct the proofs of these conjectures in [GHKK18]. Therefore, we temporally forget Theorems 1.35 and 1.37 and keep Conjectures 1.32, 1.34 and 1.36 as conjectures until we prove them here.

**Remark 1.38.** Theorem 1.35 was proved in [GHKK18] with some toric geometrical setting. In contrast, we prove Conjecture 1.34 without any toric geometrical setting.
2 More about $C$- and $G$-matrices

In this section we study more about $C$- and $G$-matrices, namely, the tropical part of a cluster pattern. Here we only consider a $B$-pattern $B$, and we do not refer to a cluster pattern $\Sigma$. Therefore, we regard that $C$- and $G$-matrices are directly associated with a $B$-pattern $B$.

2.1 Mutations and second duality

Let $B$ be any $B$-pattern, and let $C_{t_0}^B = C$ and $G_{t_0}^B = G$ are the $C$- and $G$-patterns of $B$ with a given initial vertex $t_0$. In view of Proposition 1.30, we introduce the following notion.

Definition 2.1 (Tropical sign). Suppose that a $c$-vector $c_{i;t_0} = c_{i:t}$ is either positive or negative. Then, the common sign $\varepsilon_{i;t_0} = \varepsilon_{i;t} \in \{1, -1\}$ of all nonzero components of $c_{i;t}$ is called the tropical sign of $c_{i;t}$.

Example 2.2. (a). For the initial vertex $t_0$, we have $C_{t_0} = I$. Therefore, we have

$$\varepsilon_{i;t_0} = 1, \quad (i = 1, \ldots, n). \quad (2.1)$$

(b). Let $t, t' \in \mathbb{T}_n$ be vertices that are $k$-adjacent. Then, by (1.30), we have

$$\varepsilon_{k;t'} = -\varepsilon_{k;t}. \quad (2.2)$$

Proposition 2.3. Suppose that the sign-coherence conjecture holds. Then, the following facts hold.

(a). For any $t \in \mathbb{T}_n$, we have

(Unimodularity [NZ12, Prop. 4.2])

$$|C_t| = |G_t| \in \{1, -1\}, \quad (2.3)$$

(Second duality [NZ12, Eq. (3.11)])

$$D^{-1}G_t^T DC_t = I, \quad (2.4)$$

or, equivalently,

$$D^{-1}C_t^T DG_t = I, \quad (2.5)$$

where $D$ is a common skew-symmetrizer of $B$.

(b). For any $t, t' \in \mathbb{T}_n$ that are $k$-adjacent, we have

(Mutations [NZ12, Prop. 1.3])

$$C_{t'} = C_t(J_k + [\varepsilon_{k;t}B_t]^k_+), \quad (2.6)$$

$$G_{t'} = G_t(J_k + [-\varepsilon_{k;t}B_t]^k_+). \quad (2.7)$$
Since we need some details of the proof later, we present it below.

**Proof.** We first prove (b). By the column sign-coherence of $C_t$, the tropical sign $\varepsilon_{k,t}$ is defined for any $k$. Then, we have

$$[-\varepsilon_{k,t} C^\bullet_{t}]^k_+ = 0. \quad (2.8)$$

We set $\varepsilon = \varepsilon_{k,t}$ in the $\varepsilon$-expressions of mutations (1.35) and (1.41). The resulting relations are (2.6) and (2.7).

We prove (a) by the induction on $t$ along $T_n$ starting from $t_0$. Let us introduce the following statement:

(a) The claim (a) holds for any $t \in T_n$ such that $d(t_0, t) = d$.

First, $(a)_0$ holds because $C_{t_0} = G_{t_0} = I$. Next, suppose that $(a)_d$ holds for some $d$. Let $t, t' \in T_n$ be vertices that are $k$-adjacent such that $d(t_0, t) = d$ and $d(t_0, t') = d + 1$. Let us temporarily set the matrices in (2.6) and (2.7) as

$$P = J_k + [\varepsilon_{k,t} B_{t}]^k_+, \quad Q = J_k + [-\varepsilon_{k,t} B_{t}]^k_+.$$

Then, by Lemma 1.10 we have

$$P^2 = I, \quad |P| = |Q| = -1. \quad (2.9)$$

Thus, by the induction assumption $|C_t| = |G_t| = \pm 1$, we have $|C_{t'}| = |G_{t'}| = \mp 1$. Also, by (1.13), we have

$$DPD^{-1} = Q^T. \quad (2.10)$$

Using the induction assumption $D^{-1}(G_t)^T D C_t = I$, we have

$$D^{-1}G_{t'}^T D C_{t'} = D^{-1}(Q^T G_t^T) D (C_t P) = P (D^{-1}G_t^T D C_t) P = P^2 = I. \quad (2.11)$$

Therefore, $(a)_{d+1}$ holds.

**Remark 2.4.** For the later use, we record the following facts in the above proof.

- To prove (b), we only use the sign-coherence of $C_t$ for $t$ therein.
- To prove $(a)_d \implies (a)_{d+1}$, we only use the sign-coherence of $C_t$ such that $d(t_0, t) = d$.

The following expressions of formulas (2.6) and (2.7) by $c$- and $g$-vectors are also useful.

**Proposition 2.5.** Suppose that the sign-coherence conjecture holds. Then, we have

$$c_{i,t'} = \begin{cases} -c_{k,t} & i = k, \\ c_{i,t} + [\varepsilon_{k,t} b_{k,t}]_+ c_{k,t} & i \neq k, \end{cases} \quad (2.12)$$
2.2. Dual mutations and third duality

\[ g_{i:t'} = \begin{cases} -g_{k:t} + \sum_{j=1}^{n} [-\varepsilon_{k:t}b_{j:k:t}] + g_{j:t} & i = k, \\ g_{i:t} & i \neq k. \end{cases} \] \quad (2.13)

We note that the matrices \( C_t \) and \( B_{t_0} \) in turn determine \( B_t \).

**Proposition 2.6** ([NZ12, Eq. (2.9)]). Suppose that the sign-coherence conjecture holds. Then, the following equality holds:

\[ C_t^T (D B_{t_0}) C_t = D B_t. \] \quad (2.14)

**Proof.** By the dualities (1.39) and (2.5), we have

\[ C_t^T D B_{t_0} C_t = C_t^T D G_t B_t = D(D^{-1} C_t^T D G_t) B_t = D B_t. \] \quad (2.15)

2.2 Dual mutations and third duality

Let \( B = \{B_t\}_{t \in \mathbb{T}_n} \) be any \( B \)-pattern, and let \( t_0 \) be a given initial vertex. Since we now vary the initial vertex below, we write the corresponding \( C \)- and \( G \)-matrices as \( C_{t_0}^t \) and \( G_{t_0}^t \) as in Remark 1.22.

We note that the mutation (1.12) is compatible with the matrix transpose. Namely, if \( B' = \mu_k(B) \), then \( B'^T = \mu_k(B^T) \). Therefore, \( B^T := \{B_t^T\}_{t \in \mathbb{T}_n} \) is also a \( B \)-pattern, which is called the transpose of \( B \). We temporarily write the associated \( C \)- and \( G \)-matrices as \( \tilde{C}_{t_0}^t \), \( \tilde{G}_{t_0}^t \).

We borrow the following notation from [FG19]: For any matrix \( M \), \( \varepsilon_{\bullet \bullet}(M) \) (resp. \( \varepsilon_{\bullet \bullet}(M) \)) stands for the common sign of the \( k \)th row (resp. \( k \)th column) assuming that it is either positive or negative. For example, the tropical sign \( \varepsilon_{i:t} \) in Definition 2.1 is written as \( \varepsilon_{\bullet \bullet}(C_{t_0}^t) \).

**Proposition 2.7.** Suppose that the sign-coherence conjecture holds. Then, the following facts hold:

(a). For any \( t_0, t \in \mathbb{T}_n \), we have

(Third duality [NZ12, Eq. (1.13)])

\[ C_{t_0}^t = (\tilde{G}_{t_0}^t)^T, \] \quad (2.16)

\[ G_{t_0}^t = (\tilde{C}_{t_0}^t)^T. \] \quad (2.17)

In particular, each \( G \)-matrix \( G_{t_0}^t \) is row sign-coherent, and the equality

\[ \varepsilon_{\bullet \bullet}(G_{t_0}^t) = \varepsilon_{\bullet \bullet}(\tilde{C}_{t_0}^t) \] \quad (2.18)

holds.

(b). For any \( t_0, t_1, t \in \mathbb{T}_n \) such that \( t_0 \) and \( t_1 \) are \( k \)-adjacent, we have
Proof. We prove the claims by the induction on the distance \(d(t_0, t)\) in \(\mathbb{T}_n\), where we vary \(t_0\) for \((a)\) and \(t\) for \((b)\).

Let us introduce the following claims:

\((a)_d\). The claim \((a)\) holds for any \(t_0, t \in \mathbb{T}_n\) such that \(d(t_0, t) = d\).

\((b)_d\). The claim \((b)\) holds for any \(t_0, t_1, t \in \mathbb{T}_n\) such that \(d(t_0, t) = d\) and \(t_0\) and \(t_1\) are \(k\)-adjacent.

We prove the claims in the following order,

\[
(a)_0 \implies (b)_0 \implies (a)_1 \implies (b)_1 \implies (a)_2 \implies \ldots ,
\]

assuming all preceding claims.

First, we show that \((a)_0\) and \((b)_0\) hold. The claim \((a)_0\), where \(t_0 = t\), trivially holds, because all relevant matrices are the identity matrix \(I\). Let us prove \((b)_0\), where \(t_0 = t\). By assumption, \(t_0\) and \(t_1\) are \(k\)-adjacent. Then, by Proposition \(2.3\) \((b)\) and \((1.24)\), we have

\[
C_{t_0}^{t_1} = C_{t_0}^{t_1} (J_k + [-\varepsilon_{k^*} (G_{t_0}^{t_0}) B_{t_0}]) C_{t_0}^{t_0},
\]

\[
G_{t_0}^{t_1} = G_{t_0}^{t_1} (J_k + [-\varepsilon_{k^*} (G_{t_0}^{t_0}) B_{t_0}]) G_{t_0}^{t_0}.
\]

Thus, \((b)_0\) holds.

Next, assuming the claims in \((2.21)\) up to \((b)_d\), we show \((a)_{d+1}\). Assume \(d(t_0, t) = d\) and \(d(t_1, t) = d + 1\) in \((2.19)\). Take the transpose of \((2.19)\). Then, we apply \((2.16)\), which is valid thanks to \((a)_{d}\), to its right hand side. We have

\[
(C_{t_0}^{t_1})^T = G_{t_0}^{t_1} (J_k + [-\varepsilon_{k^*} (G_{t_0}^{t_0}) B_{t_0}])^T.
\]

The right hand side is \(\tilde{G}_{t_1}^{t_0}\) by \((2.7)\). Thus, we have \((2.19)\). The other case is similar.

Finally, assuming the claims in \((2.21)\) up to \((a)_{d+1}\), we show \((b)_{d+1}\). This is the nontrivial part. It is enough to concentrate on the following situation. Let \(t_0, t_1, t, t' \in \mathbb{T}_n\) be vertices such that \(d(t_0, t) = d\), \(d(t_0, t') = d + 1\), where
\( t_1 \) and \( t_0 \) are \( k \)-adjacent, while \( t \) and \( t' \) are \( \ell \)-adjacent. They are depicted as follows, where \( t_1 \) could be between \( t_0 \) and \( t \).

\[
\begin{array}{c}
  & k \\
\hline
 t_1 & t_0 & t & t' \\
\end{array}
\]

By (2.6) and (2.19) with \((b)\_d\), we obtain

\[
C^{t_1}_{t'} = C^{t_1}_t ( J_\ell + [\varepsilon_{\varepsilon_\ell}(C^{t_1}_t) B_1]_+^{\varepsilon_\ell}) \\
= (J_k + [-\varepsilon_{k\varepsilon}(G^{t_0}_t) B_0]_+^{k\varepsilon}) C^{t_0}_t (J_\ell + [\varepsilon_{\varepsilon_\ell}(C^{t_1}_t) B_1]_+^{\varepsilon_\ell}).
\]

By \((a)\_d+1\), the sign \( \varepsilon_{k\varepsilon}(G^{t_0}_t) \) is well-defined. Let us temporally assume the identity:

\[
(J_k + [-\varepsilon_{k\varepsilon}(G^{t_0}_t) B_0]_+^{k\varepsilon}) C^{t_0}_t (J_\ell + [\varepsilon_{\varepsilon_\ell}(C^{t_1}_t) B_1]_+^{\varepsilon_\ell}) \\
= (J_k + [-\varepsilon_{k\varepsilon}(G^{t_0}_t) B_0]_+^{k\varepsilon}) C^{t_0}_t (J_\ell + [\varepsilon_{\varepsilon_\ell}(C^{t_0}_t) B_1]_+^{\varepsilon_\ell}).
\]

Then, the right hand side of (2.25) reduces to \((J_k + [-\varepsilon_{k\varepsilon}(G^{t_0}_t) B_0]_+^{k\varepsilon}) C^{t_0}_t\), which is the desired result for (2.19) of \((b)\_d+1\). Similarly, from the identity

\[
(J_k + [\varepsilon_{k\varepsilon}(G^{t_0}_t) B_0]_+^{k\varepsilon}) C^{t_0}_t (J_\ell + [-\varepsilon_{\varepsilon_\ell}(C^{t_1}_t) B_1]_+^{\varepsilon_\ell}) \\
= (J_k + [\varepsilon_{k\varepsilon}(G^{t_0}_t) B_0]_+^{k\varepsilon}) C^{t_0}_t (J_\ell + [-\varepsilon_{\varepsilon_\ell}(C^{t_0}_t) B_1]_+^{\varepsilon_\ell}),
\]

we obtain (2.20) of \((b)\_d+1\). It remains to prove (2.20) and (2.27). To do it, we need to know a precise relation between signs \( \varepsilon_{k\varepsilon}(G^{t_0}_t) \) and \( \varepsilon_{k\varepsilon}(G^{t_0}_{t'}) \), and also \( \varepsilon_{\varepsilon_\ell}(C^{t_1}_t) \) and \( \varepsilon_{\varepsilon_\ell}(C^{t_0}_t) \).

The following facts hold:

(i). By (2.7), \( G^{t_0}_t \) and \( G^{t_0}_{t'} \) differ only in the \( \ell \)-th column.

(ii). By \((b)\_d\), \( C^{t_0}_t \) and \( C^{t_0}_{t'} \) differ only in the \( k \)-th row.

(iii). By the duality (2.4), the \( k \)-th row of \( G^{t_0}_t \) has only a unique component \( \varepsilon \) at \( \ell \)-th column if and only the \( \ell \)-th column of \( C^{t_0}_t \) has a unique nonzero component \( \varepsilon' \) at \( k \)-th row. Furthermore, if it happens, \( \varepsilon = \varepsilon' \in \{1, -1\} \), thanks to the unimodularity (2.3).

Now we consider two cases.

**Case 1.** Suppose that the \( k \)-th row of \( G^{t_0}_t \) has a nonzero component at some column other than the \( \ell \)-th column. Then, from Facts (i)–(iii) above and the row and column sign-coherence of \( G \)- and \( C \)-matrices in \((a)\_d+1\), we conclude that

\[
\varepsilon_{k\varepsilon}(G^{t_0}_t) = \varepsilon_{k\varepsilon}(G^{t_0}_{t'}), \quad \varepsilon_{\varepsilon_\ell}(C^{t_1}_t) = \varepsilon_{\varepsilon_\ell}(C^{t_0}_t).
\]

Therefore, (2.26) and (2.27) hold trivially.
Case 2. Suppose that the $k$th row of $C_{t_0}^{t}$ has only one nonzero component that is at the $\ell$th column. Let us write this component as $\varepsilon \in \{1, -1\}$. As mentioned in Fact (iii), $\varepsilon$ coincides with the only one nonzero component in the $\ell$th column of $C_{t_0}^{t}$ that is at the $k$th row. Then, by (2.26) and $(b)_d$, we have

$$\varepsilon_{k\bullet}(C_{t_0}^{t}) = -\varepsilon, \quad \varepsilon_{k\bullet}(C_{t_0}^{t}) = \varepsilon, \quad \varepsilon_{\bullet\ell}(C_{t_0}^{t}) = -\varepsilon, \quad \varepsilon_{\bullet\ell}(C_{t_0}^{t}) = \varepsilon. \quad (2.29)$$

Let us prove (2.26), which is now written as

$$(J_k + [-\varepsilon B_{t_0}]_+^{k\bullet}) C_{t_0}^{t} (J_\ell + [-\varepsilon B_{t_0}]_+^{\ell\bullet}) = (J_k + [\varepsilon B_{t_0}]_+^{k\bullet}) C_{t_0}^{t} (J_\ell + [\varepsilon B_{t_0}]_+^{\ell\bullet}). \quad (2.30)$$

By (1.21), it is rewritten as

$$(J_k + [\varepsilon B_{t_0}]_+^{k\bullet})(J_k + [-\varepsilon B_{t_0}]_+^{k\bullet}) C_{t_0}^{t} = C_{t_0}^{t} (J_\ell + [\varepsilon B_{t_0}]_+^{\ell\bullet})(J_\ell + [-\varepsilon B_{t_0}]_+^{\ell\bullet}). \quad (2.31)$$

After some manipulation, it reduces to

$$(B_{t_0} C_{t_0}^{t})^{k\bullet} = C_{t_0}^{t} (B_{t})^{k\bullet}. \quad (2.32)$$

By the assumption, we have $(C_{t_0}^{t})^{k\bullet} = (G_{t_0}^{t})^{k\bullet} = \varepsilon E_{k\ell}$, where $E_{k\ell}$ is the matrix whose entries are zero except for the $(k, \ell)$-entry that is 1. Then, the right hand side of (2.32) can be rewritten as

$$(C_{t_0}^{t})^{k\ell} B_t = \varepsilon E_{k\ell} B_t = (G_{t_0}^{t})^{k\bullet} B_t = (G_{t_0}^{t} B_{t})^{k\bullet}. \quad (2.33)$$

Thus, (2.32) reduces to the duality (1.39). Therefore, (2.26) is proved. Similarly, skipping details a little, (2.27) reduces to

$$(B_{t_0})^{k} C_{t_0}^{t} = (G_{t_0}^{t} B_{t})^{\ell}. \quad (2.34)$$

Then, by repeating a similar argument, again (2.34) reduces to the duality (1.39). Therefore, (2.27) is proved.

Remark 2.8. For the later use we record the following facts in the above proof.

- To prove $(a)_d$, $(b)_d \implies (a)_{d+1}$, we only use the sign-coherence of $\tilde{C}_{t_0}^{t}$ such that $d(t_0, t) = d$.
- To prove $(b)_d$, $(a)_{d+1} \implies (b)_{d+1}$, we only use the sign-coherence of $C_{t_0}^{t}$, $C_{t_1}^{t}$, $\tilde{C}_{t_0}^{t}$, $\tilde{C}_{t_1}^{t}$ such that $d(t_0, t) = d$, $d(t_0, t') = d + 1$, $d(t_1, t) = d \pm 1$.

Remark 2.9. Combining three dualities (1.39), (1.41), (2.16), we have various relations among $C$- and $G$-matrices therein. For example, by (1.39) and (2.16), we have

$$B_{t_0} C_{t_0}^{t} = (B_{t}^{T} \tilde{C}_{t_0}^{t})^{T}, \quad C_{t_0}^{t} B_{t_0} = (\tilde{G}_{t_0}^{t} B_{t}^{T})^{T}. \quad (2.35)$$
2.3. Principal extension of $B$-patterns

The $\hat{C}$-matrices in (1.58) mutate as a “hybrid” of $C$- and $G$-matrices in the following sense.

**Proposition 2.10.** Suppose that the sign-coherence conjecture holds. Let $\hat{C}_{t_0} = B_{t_0} C_{t_0}$ and $\hat{C}_{t_1} = B_{t_1} C_{t_1}$. Then, for any $t_0, t_1, t, t' \in T_n$ such that $t_0$ and $t_1$, and also, $t$ and $t'$ are $k$-adjacent, respectively, we have

$$\hat{C}_{t_0} = \hat{C}_{t_0} (J_k + [\varepsilon_{k}(C_{t_0}) B_{t_0}]_{k}^{k}),$$  \hspace{1cm} \text{(2.36)}$$

$$\hat{C}_{t_1} = (J_k + [\varepsilon_{k}(C_{t_0}) B_{t_0}]_{k}^{k}) \hat{C}_{t_0}.$$  \hspace{1cm} \text{(2.37)}$$

**Proof.** The first formula is obtained from (1.58) by multiplying $B_{t_0}$ from the left. The second one is obtained from (2.20) by multiplying $B_{t_0}$ from the right.

\[ \square \]

2.3 Principal extension of $B$-patterns

For any $B$-pattern $B$ of rank $n$ and a given initial vertex $t_0 \in T_n$, we introduce a $2n \times 2n$ skew-symmetrizable matrix

$$\overline{B}_{t_0} = \begin{pmatrix} B_{t_0} & -I \\ I & O \end{pmatrix},$$  \hspace{1cm} \text{(2.38)}$$

which is called the principal extension of $B_{t_0}$. It is important that $\overline{B}_{t_0}$ is nonsingular. Choose any point $\overline{t}_0 \in T_{2n}$. We naturally identify $T_n$ as a subtree of $T_{2n}$ by identifying $t_0$ with $\overline{t}_0$, and also the edges with labels $k = 1, \ldots, n$ in $T_n$ and $T_{2n}$. Consider a $B$-pattern $\overline{B} = \{ \overline{B}_{t} \}_{t \in T_{2n}}$ of rank $2n$ generated by $\overline{B}_{t_0}$, which is called the principal extension of $\overline{B}$ with the initial vertex $t_0$.

The following stability properties have repeatedly appeared in the literatures in various forms. Here, we quote from [FG19].

**Proposition 2.11** (e.g., [FG19], Theorem 4.2, Remark 4.4). Let $C_t, G_t, F_{i;t}(y)$ be $C$- and $G$-matrices, and $F$-polynomials of $B$ with the initial point $t_0 \in T_n$. Let $\overline{C}_t, \overline{G}_t, \overline{F}_{i;t}(\overline{y})$ be $C$- and $G$-matrices, and $F$-polynomials of $\overline{B}$ with the same initial point $t_0 \in T_n \subset T_{2n}$, where $\overline{y} = (y_1, \ldots, y_{2n})$. Then, the following relations hold for any $t \in T_n \subset T_{2n}$:

$$\overline{C}_t = \begin{pmatrix} C_t & Z_t \\ O & I \end{pmatrix},$$  \hspace{1cm} \text{(2.39)}$$

$$\overline{G}_t = \begin{pmatrix} G_t & 0 \\ O & I \end{pmatrix},$$  \hspace{1cm} \text{(2.40)}$$

$$\overline{F}_{i;t}(\overline{y}) = \begin{cases} F_{i;t}(y) & i = 1, \ldots, n, \\ 1 & i = n + 1, \ldots, 2n, \end{cases}$$  \hspace{1cm} \text{(2.41)}$$

where $Z_t$ is a certain matrix. Moreover, $Z_t = O$ if the sign-coherence of $C$-matrices for $B$ holds.
These equalities can be proved by the induction on \( t \) along \( \mathbb{T}_n \) from \( t_0 \) \cite[Theorem 4.2]{FG19}.

By Proposition 2.11 any information on \( C_t, G_t, F_{i;t}(y) \) can be extracted from the one on \( \overline{C}_t, \overline{G}_t, \overline{F}_{i;t}(y) \).

### 2.4 \( G \)-cones and \( G \)-fans

From now on, we study the geometric aspect of \( G \)-matrices.

We recall some notions from convex geometry.

**Definition 2.12 (Cone).** Let \( M \simeq \mathbb{Z}^n \) be a lattice of rank \( n \). Let \( M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n \) be the extension to an \( n \)-dimensional vector space over \( \mathbb{R} \). Let \( N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \) and \( N_\mathbb{R} = \text{Hom}_\mathbb{R}(M_\mathbb{R}, \mathbb{R}) \) be their duals. Let \( \langle , \rangle \) be the canonical paring \( N_\mathbb{R} \times M_\mathbb{R} \rightarrow \mathbb{R} \).

- For given elements \( a_1, \ldots, a_r \in M \subset M_\mathbb{R} \), the subset of \( M_\mathbb{R} \) defined by
  \[
  \sigma(a_1, \ldots, a_r) := \mathbb{R}_{\geq 0}a_1 + \cdots + \mathbb{R}_{\geq 0}a_r
  \]  
  (2.42)
  is called a convex rational polyhedral cone, or simply, a cone generated by \( a_1, \ldots, a_r \). The set \( \sigma(\emptyset) := \{0\} \) is also a cone.
- A cone \( \sigma \) is said to be strongly convex if \( \sigma \cap (-\sigma) = \{0\} \).
- A cone is simplicial if it is generated by \( \mathbb{Z} \)-linearly independent elements in \( M \). Thus, a simplicial cone is strongly convex.
- A cone is nonsingular if it is generated by a subset of a \( \mathbb{Z} \)-basis in \( M \). Thus, a nonsingular cone is simplicial.
- The dimension of a cone \( \sigma \) is the dimension of the subspace of \( M_\mathbb{R} \) spanned by \( \sigma \).
- The dual cone \( \sigma^\vee \) of a cone \( \sigma \) is defined by
  \[
  \sigma^\vee = \{ u \in N_\mathbb{R} \mid \langle u, a \rangle \geq 0 \text{ for any } a \in \sigma \},
  \]
  which is a cone in \( N_\mathbb{R} \).
- A subset \( \tau \) of a cone \( \sigma \) is called a face of \( \sigma \) if there is some \( u \in \sigma^\vee \) such that \( \tau = \sigma \cap u^\perp \), where \( u^\perp = \{ a \in M_\mathbb{R} \mid \langle u, a \rangle = 0 \} \). In particular, for a simplicial cone \( \sigma \) with a \( \mathbb{Z} \)-linearly independent generating set \( S \), any face of \( \sigma \) is a cone generated by a subset of \( S \).

**Example 2.13.** For \( M = \mathbb{Z}^2 \) and \( M_\mathbb{R} = \mathbb{R}^2 \), \( \sigma(e_1) \) and \( \sigma(e_1, e_2) \) are strongly convex cones, while \( \sigma(e_1, -e_1) = \mathbb{R}e_1 \) and \( \sigma(e_1, -e_1, e_2, -e_2) = \mathbb{R}^2 \) are cones that are not strongly convex. The cone \( \sigma = \sigma(e_1, e_2) \) has four faces \( \sigma, \sigma(e_1), \sigma(e_2), \{0\} \), while the cone \( \sigma = \sigma(e_1, -e_1) \) has the unique face, which is \( \sigma \).

**Definition 2.14 (Fan).**

- A fan \( \Delta \) in \( M_\mathbb{R} \simeq \mathbb{R}^n \) is a nonempty (possibly infinite) set of strongly convex cones in \( M_\mathbb{R} \) satisfying the following conditions:
2.4. G-cones and G-fans

(i). If \( \tau \) is a face of \( \sigma \in \Delta \), then \( \tau \in \Delta \).
(ii). If \( \sigma, \tau \in \Delta \), then \( \sigma \cap \tau \in \Delta \).

- A subset \( |\Delta| = \bigcup_{\sigma \in \Delta} \sigma \) of \( M_\mathbb{R} \) is called the support of a fan \( \Delta \).
- A fan \( \Delta \) is simplicial if all cones of \( \Delta \) are simplicial.
- A fan \( \Delta \) is nonsingular if all cones of \( \Delta \) are nonsingular. Thus, a nonsingular fan is simplicial.
- A fan \( \Delta \) is complete if \( |\Delta| = M_\mathbb{R} \).

We apply these geometric notions to \( G \)-matrices following \([RS16, Rea14]\). In the rest of this section we temporarily assume the sign-coherence of \( C \)-matrices and study some basic properties of \( G \)-cones.

Let \( B \) be any \( B \)-pattern, and let \( G_{t_0} = G \) be the \( G \)-pattern of \( B \) with a given initial point \( t_0 \).

**Definition 2.15 (G-cone).** For each \( G \)-matrix \( G_t \), the cone spanned by its \( g \)-vectors in \( \mathbb{R}^n \)

\[
\sigma(G_t) := \sigma(g_{1;t}, \ldots, g_{n;t}) \tag{2.43}
\]

is called a \( G \)-cone. The interior of \( \sigma(G_t) \) is denoted by \( \sigma^o(G_t) \). Each face of \( \sigma(G_t) \) spanned by \( g_j \)'s excluding the \( i \)th \( g \)-vector \( g_{i;t} \) is denoted by \( \sigma_i(G_t) \) for \( i = 1, \ldots, n \).

By the unimodularity \([2.3]\), every \( G \)-cone \( \sigma(G_t) \) is a nonsingular cone, and its faces \( \sigma_i(G_t) \) \( (i = 1, \ldots, n) \) are cones of codimension one.

From now on, let us especially take a common skew-symmetrizer \( D \) of the \( B \)-pattern \( B \) in the following form:

\[
D = \text{diag}(\delta_1^{-1}, \ldots, \delta_n^{-1}), \tag{2.44}
\]

where \( \delta_1, \ldots, \delta_n \) are positive integers. (This seemingly awkward choice is suitable to the formulation of scattering diagrams in \([CHKK18]\) we use later. See Example 5.4.) Accordingly, we introduce the following nonstandard inner product \( (\cdot, \cdot)_D \) in \( \mathbb{R}^n \):

\[
(u, v)_D = u^T D v \quad (u, v \in \mathbb{R}^n). \tag{2.45}
\]

Below, for a given \( B \)-pattern \( B \), we always assume this inner product \( (\cdot, \cdot)_D \) in \( \mathbb{R}^n \). For \( n \neq 0 \in \mathbb{R}^n \), \( n^\perp \) is a hyperplane defined by

\[
n^\perp := \{ v \in \mathbb{R}^n \mid (n, v)_D = 0 \}. \tag{2.46}
\]

We say that a normal vector \( n \) of a face \( \sigma_i(G_t) \) of a cone \( \sigma(G_t) \) is inward for \( \sigma(G_t) \) if \( (n, g_{i;t})_D > 0 \).

In this geometrical setting, the duality \([2.4]/(2.5)\) can be rephrased in the following manner.
Proposition 2.16. Suppose that the sign-coherence conjecture holds. Then, the following dualities of vectors in $\mathbb{R}^n$ hold:

\[
(\delta_{t} c_{i; t}, g_{j; t})_{D} = \delta_{ij}, \quad (2.47)
\]
\[
(c_{i; t}, \delta_{j} g_{j; t})_{D} = \delta_{ij}. \quad (2.48)
\]

In particular, each c-vector $c_{i; t}$ is a normal vector of the face $\sigma_{i}(G_{t})$ of the cone $\sigma(G_{t})$ with respect to the inner product $(\cdot, \cdot)_{D}$. Moreover, it is inward for $\sigma(G_{t})$.

Let $\Delta(G^{t_0}) = \Delta(G)$ be the set of all faces of all $G$-cones $\sigma(G_{t})$ ($t \in \mathbb{T}_n$) in the $G$-pattern $G^{t_0}$. Let $t, t' \in \mathbb{T}_n$ be vertices that are $k$-adjacent. Then, by (2.13), two cones $\sigma(G_{t})$ and $\sigma(G_{t'})$ intersect each other in their common face $\sigma_k(G_{t}) = \sigma_k(G_{t'})$ with codimension one. However, for a cone $\sigma(G_{t''})$ obtained from $\sigma(G_{t})$ after several mutations, it is not clear at all that $\sigma(G_{t''})$ intersects $\sigma(G_{t})$ only in their common face so that the set $\Delta(G)$ forms a fan. It was conjectured by [RS16, Conj. 3.14] that this always happens. Then, it was shown to be true by [Rea14] under the sign-coherence conjecture.

Theorem 2.17 ([Rea14, Theorem 8.7]). Suppose that the sign-coherence conjecture holds. Then, for any $B$-pattern $B$ and a given initial vertex $t_0$, the set of cones $\Delta(G^{t_0})$ is a fan in $\mathbb{R}^n$.

In [Rea14] this was proved in a more general setting of mutation fans therein. We provide a self-contained proof of Theorem 2.17 in Section 2.7 without referring to mutation fans, because the tools and techniques we use therein are also relevant to our main purpose.

The resulting fan $\Delta(G^{t_0})$, which is nonsingular by (2.3), is called the $g$-vector fan in [RS16]. To simplify a little, we call it the $G$-fan of $B$ (or $\Sigma$) with the initial vertex $t_0$.

Let $\#G^{t_0}$ be the number of distinct $G$-matrices in $G^{t_0}$, which can be finite or infinite.

Proposition 2.18. Suppose that the sign-coherence conjecture holds. Then, the $G$-fan $\Delta(G^{t_0})$ is complete if and only if $\#G^{t_0}$ is finite.

Proof. This is true by the following topological reason. For each subset $T$ of $\mathbb{T}_n$, we attach a subset $\Delta_{T}$ of $\Delta(G^{t_{0}})$ consisting of the faces of all $\sigma(G_{t})$ with $t \in T$. Let $|\Delta_{T}|^{c}$ be the complement of the support $|\Delta_{T}|$ in $\mathbb{R}^n$. Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Let $D_{T} = |\Delta_{T}|^{c} \cap S^{n-1}$. For $T = \{t_{0}\}$, $D_{T}$ is topologically a $(n-1)$-dimensional open disk. We consider a strongly increasing sequence of subsets of $\mathbb{T}_n$, $T_{1} = \{t_{0}\} \subset T_{2} \subset T_{3} \subset \cdots$ with $|T_{s}| = s$. Correspondingly, we have a weakly decreasing sequence of subsets of $S^{n-1}$, $D_{T_{1}} \supset D_{T_{2}} \supset D_{T_{3}} \supset \cdots$. Then, $D_{T_{s}}$ is a disjoint union of a finite number of open disks, such that at each process of increasing $T_{s}$ to $T_{s+1}$, one of the followings occurs (by adding a $G$-cone):
• $D_{T_{s+1}} = D_{T_s}$.
• An open disk shrinks.
• An open disk splits into two open disks.
• An open disk disappears.

If $\# G_{t_0}$ is finite, the decreasing process of $D_{T_s}$ should terminate at some $s$. This occurs only when $D_{T_s} = \emptyset$. Thus, $\Delta(G_{t_0})$ is complete. If $\# G_{t_0}$ is infinite, the process never terminates. Then, when $T_s$ converges to $T_n$, $D_{T_s}$ converges to the (possibly infinite) union of points and (not open, but not necessarily closed) disks. Therefore, $|\Delta(G_{t_0})|^{c}$ is nonempty.

Remark 2.19. We say that a cluster pattern $\Sigma$ is of finite type if there are only finitely many distinct seeds in $\Sigma$. It is known, for example, by [Nak21, Theorem 5.2], that a cluster pattern $\Sigma$ is of finite type if and only if $\# G_{t_0}$ is finite for $G_{t_0}$ of $\Sigma$. Then, combining it with Proposition 2.18 we have that a cluster pattern $\Sigma$ is of finite type if and only if the $G$-fan $\Delta(G_{t_0})$ is complete for $G_{t_0}$ of $\Sigma$. Also, we note that the proof of [Nak21, Theorem 5.2] is based on the sign-coherence of $C$-matrices and the Laurent positivity.

2.5 Rank 2 examples of $G$-fans

In the rank 2 case, one can confirm Theorem 2.17 and Proposition 2.18 by explicitly calculating $G$-matrices.

Let us arrange a $B$-pattern of rank 2 in the following way:

$$
\cdots \leftrightarrow B_{t_{-2}} \leftrightarrow B_{t_{-1}} \leftrightarrow B_{t_0} \leftrightarrow B_{t_1} \leftrightarrow B_{t_2} \leftrightarrow \cdots.
$$

(2.49)

We take $B_{t_0}$ as the initial exchange matrix. Let us present the associated $G$-fan $\Delta(G)$ in $\mathbb{R}^2$ explicitly. Below the type of a $B$-pattern refers to the type of the Cartan matrix $A(B) = (a_{ij})$ associated with $B = B_{t_0}$ in the convention of [Kac90], where

$$
a_{ij} = \begin{cases} 
2 & i = j, \\
-|b_{ij}| & i \neq j.
\end{cases}
$$

(2.50)

(I). Finite type. The $G$-fan $\Delta(G)$ is complete as stated in Proposition 2.18

(a). Type $A_2$: Let

$$
B_{t_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

(2.51)
Along the mutation sequence $\mu_1, \mu_2, \cdots$, the $G$-matrices are explicitly calculated as

\begin{align*}
G_{t_0} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & G_{t_1} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & G_{t_2} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\
G_{t_3} &= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, & G_{t_4} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, & G_{t_5} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}

Both $G_{t_5}$ and $G_{t_0}$ define the same cone, corresponding to the celebrated \textit{pentagon periodicity}. The $G$-fan $\Delta(G)$ is depicted in Figure 1 (a).

(b). Type $B_2$: Let

\begin{equation}
B_{t_0} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}.
\end{equation}

We start from the initial $G$-matrix $G_{t_0} = I$. Along the mutation sequence $\mu_1, \mu_2, \cdots$, the following $g$-vectors appear in this order, showing periodicity:

\begin{equation}
\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{equation}

The $G$-fan $\Delta(G)$ is depicted in Figure 1 (b).

(c). Type $G_2$: Let

\begin{equation}
B_{t_0} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}.
\end{equation}

We start from the initial $G$-matrix $G_{t_0} = I$. Along the mutation sequence $\mu_1, \mu_2, \cdots$, the following $g$-vectors appear in this order, showing periodicity:

\begin{equation}
\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{equation}

The $G$-fan $\Delta(G)$ is depicted in Figure 1 (c).

(II). Infinite type. The $G$-fan $\Delta(G)$ is not complete.

(d). Type $A_1^{(1)}$: Let

\begin{equation}
B_{t_0} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.
\end{equation}

We start from the initial $G$-matrix $G_{t_0} = I$. Along the mutation sequence $\mu_1, \mu_2, \cdots$, the following $g$-vectors appear in this order:

\begin{equation}
\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \cdots.
\end{equation}
2.5. Rank 2 examples of $G$-fans

On the other hand, along the mutation sequence $\mu_2, \mu_1, \cdots$, the following $g$-vectors appear in this order:

$$
\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \end{pmatrix}, \cdots.
$$

(2.59)

The $G$-fan $\Delta(G)$ is depicted in Figure 1 (d). It covers the region excluding the half line $\mathbb{R}_+(1, -1)$.

(e). Type $A_2^{(2)}$: Let

$$B_{t_0} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}.$$  

(2.60)

We start from the initial $G$-matrix $G_{t_0} = I$. Along the mutation sequence $\mu_1, \mu_2, \cdots$, the following $g$-vectors appear in this order:

$$
\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -8 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ -12 \end{pmatrix}, \begin{pmatrix} 3 \\ -7 \end{pmatrix}, \cdots.
$$

(2.61)

On the other hand, along the mutation sequence $\mu_2, \mu_1, \cdots$, the following $g$-vectors appear in this order:

$$
\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ -8 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 7 \\ -12 \end{pmatrix}, \cdots.
$$

(2.62)

The $G$-fan $\Delta(G)$ is depicted in Figure 1 (e). It covers the region excluding the half line $\mathbb{R}_+(1, -2)$.

(f). Non-affine type: Let

$$B_{t_0} = \begin{pmatrix} 0 & -c \\ b & 0 \end{pmatrix}, \quad (b, c > 0, \ bc \geq 5).$$

(2.63)
II.2. More about $C$- and $G$-matrices

We start from the initial $G$-matrix $G_{t_0} = I$. Along the mutation sequence $\mu_1, \mu_2, \cdots$, the following $g$-vectors appear in this order:

$$g_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad g_3, \quad g_4, \cdots,$$

where $g_i$'s obey the recursion,

$$g_{i+2} = \begin{cases} -g_i + bg_{i+1} & \text{if } i \text{ odd}, \\ -g_i + cg_{i+1} & \text{if } i \text{ even}. \end{cases}$$

On the other hand, along the mutation sequence $\mu_2, \mu_1, \cdots$ from the initial $G$-matrix, the following $g$-vectors appear in this order:

$$g'_1 = \begin{pmatrix} c \\ -1 \end{pmatrix}, \quad g'_2 = \begin{pmatrix} bc - 1 \\ -b \end{pmatrix}, \quad g'_3, \quad g'_4, \cdots,$$

where $g'_i$'s obey the recursion,

$$g'_{i+2} = \begin{cases} -g'_i + cg'_{i+1} & \text{if } i \text{ odd}, \\ -g'_i + bg'_{i+1} & \text{if } i \text{ even}. \end{cases}$$

It is known [Rea14, Eqs. (9.11), (9.12)] that, in the limit $i \to \infty$, the rays $\sigma(g_i)$ and $\sigma(g'_i)$ monotonically converge to $\sigma(v)$ and $\sigma(v')$, respectively, where

$$v = \begin{pmatrix} bc - \sqrt{bc(bc - 4)} \\ -2b \end{pmatrix}, \quad v' = \begin{pmatrix} bc + \sqrt{bc(bc - 4)} \\ -2b \end{pmatrix}.$$

The $G$-fan $\Delta(G)$ is depicted in Figure II (f). It covers the region excluding $\sigma(v, v') \setminus \{0\}$ (the hatched region in the figure).

2.6 Piecewise linear isomorphisms between $G$-fans

In this subsection we study a canonical bijection between the sets $\Delta(G_{t_0})$ and $\Delta(G_{t_1})$ for any $t_0, t_1 \in T_n$. The idea comes from [GHKK18], but this is independent of their results. Also, the result of this and the next subsections has considerable overlap with the results in [Rea14, Rea20b], in which $G$-fans were studied as a part of mutation fans.

Let $B$ be any $B$ pattern. Let us start with the following consequence of the sign-coherence.

**Proposition 2.20.** Suppose that the sign-coherence conjecture holds. Then, for any $t_0, t \in T_n$, the cone $\sigma(G_{t_0})$ intersects the hyperplanes $e_i^\perp$ ($i = 1, \ldots, n$) only in the boundary of $\sigma(G_{t_0})$. 

Theorem 2.21. The following relation holds:

$$\varphi_{t_0} \circ \varphi_{t_1} \circ \varphi_{t_0} = \text{id.}$$

Proof. Note that, for $v' = \varphi_{t_0}^i(v)$, we have $v'_k = -v_k$ due to $b_{k0}^t = 0$. We also recall that $(B_{t_1})^k = -(B_{t_0})^k$. Then, for $v \in \mathbb{R}^n_{k, +}$,

$$\varphi_{t_0} \circ \varphi_{t_0}^i(v) = (J_k + [-B_{t_0}]^k)(J_k + [B_{t_0}]^k)v = (J_k + [-B_{t_0}]^k)^2v.$$ 

Therefore, the correspondence $\varphi_{t_0}^i : g_{i; t}^{t_0} \rightarrow g_{i; t}^{t_1}$ gives a bijection between $\Delta(\mathbf{G}^{t_0})$ and $\Delta(\mathbf{G}^{t_1})$ preserving the intersection and the inclusion of cones.
II.2. More about $C$- and $G$-matrices

\[ \phi_{t_0}^{-1} \leftarrow \Delta(G_{t_0}^{t_0}) \rightarrow \Delta(G_{t_0}^{t_1}) \]
\[ \Delta(G_{t_0}^{t_1}) \rightarrow \Delta(G_{t_1}^{t_1}) \]

Figure 2: Bijections between $G$-fans for type $A_2$.

**Proof.** We only need to prove (2.75). By Proposition 2.20, each cone $\sigma(G_{t_0}^{t_0})$ belongs to either $\mathbb{R}_{k,+}^n$ or $\mathbb{R}_{k,-}^n$, so that it is linearly transformed under $\phi_{t_0}^{t_1}$. Therefore, by (2.72), its image is $\sigma(G_{t_1}^{t_1})$. □

**Remark 2.23.** For the later use, we record the following fact in the above proof.

- To prove (2.75), we only use the formula (2.69) for $t_0$, $t_1$ and $t$ therein, and the row sign-coherence of $G_{t_0}^{t_0}$ therein.

**Example 2.24.** Let us clarify Proposition 2.22 explicitly for type $A_2$ based on the convention in Section 2.5. The fans \( \Delta(G_{t_0}^{t_1}) \), \( \Delta(G_{t_0}^{t_0}) \), \( \Delta(G_{t_1}^{t_1}) \) are depicted in Figure 2. Also, the piecewise linear maps $\phi_{t_0}^{t_1}$ and $\phi_{t_0}^{t_1}$ are given by combining the following linear maps:

\[ J_1 + [B_{t_0}]_+^1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad J_1 + [-B_{t_0}]_+^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.76) \]
\[ J_2 + [B_{t_0}]_+^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_2 + [-B_{t_0}]_+^2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \quad (2.77) \]

Then, the correspondence in Proposition 2.22 can be easily confirmed in Figure 2.

By applying the above isomorphism repeatedly along the tree $T_n$, we obtain the following important fact.

**Proposition 2.25.** Suppose that the sign-coherence conjecture holds. Then, for any vertices $t_0, t_1 \in T_n$, which are not necessarily adjacent, the correspondence $g_{t_0}^{t_0} \mapsto g_{t_1}^{t_1}$ gives a bijection between $\Delta(G_{t_0}^{t_0})$ and $\Delta(G_{t_1}^{t_1})$ preserving the intersection and the inclusion of cones.

For the later use, we also describe the relation between $\phi_{t_0}^{t_1}$ here and a certain piecewise linear isomorphism in [GHKK18].

Again, let $t_0, t_1 \in T_n$ be vertices that are $k$-adjacent. First, for $B_{t_0}$ and
k, we introduce a basis \( a_1, \ldots, a_n \) of \( \mathbb{R}^n \) as

\[
a_i = \begin{cases} 
-e_i + \sum_{j=1}^{n} [-b_{jk} t_0] + e_j & i = k, \\
e_i & i \neq k.
\end{cases}
\]  

(2.78)

In fact, by specializing the formula in (2.13) with \( t = t_0 \), and applying the fact \( \varepsilon_{k; t_0} = 1 \) therein, we see that

\[
a_i = g_{t_0}^{t_1}. \tag{2.79}
\]

The change of a basis of \( \mathbb{R}^n \) from \( e_1, \ldots, e_n \) to \( a_1, \ldots, a_n \) induces the following linear isomorphism \( \eta_{t_0}^{t_1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \)

\[
v \mapsto (J_k + [-B_{t_0}^k] \cdot v)
\]

such that \( \eta_{t_0}^{t_1}(a_i) = e_i \).

Next, for the same \( B_{t_0} \) and \( k \), we define a piecewise linear map \( T_{k; t_0} \) on \( \mathbb{R}^n \) following [GHKK18] as

\[
T_{k; t_0} : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
v \mapsto \begin{cases} 
(I + (B_{t_0})^k) v & v \in \mathbb{R}^n_{k, +}, \\
(J_k + [-B_{t_0}^k]) v & v \in \mathbb{R}^n_{k, -}.
\end{cases}
\]  

(2.81)

The following fact appeared in [Mul16, Rea20b].

**Proposition 2.26** ([Mul16, Lemma 5.2.1], [Rea20b, Cor. 4.4]). The above maps are related as follows:

\[
\varphi_{t_0}^{t_1} = \eta_{t_0}^{t_1} \circ T_{k; t_0}. \tag{2.82}
\]

**Proof.** By the definitions of these maps,

\[
\eta_{t_0}^{t_1} \circ T_{k; t_0}(v) = \begin{cases} 
(J_k + [-B_{t_0}^k]) (I + (B_{t_0})^k) v & v \in \mathbb{R}^n_{k, +}, \\
(J_k + [-B_{t_0}^k]) v & v \in \mathbb{R}^n_{k, -}.
\end{cases}
\]  

(2.83)

Thanks to (1.55), we have

\[
(J_k + [-B_{t_0}^k])(I + (B_{t_0})^k) = J_k + (B_{t_0})^k + [-B_{t_0}^k] = J_k + [B_{t_0}]^k.
\]

\[\square\]

### 2.7 Proof of Theorem 2.17

Let us prove Theorem 2.17 under the assumption of the sign-coherence of \( C \)-matrices.

We recall that the condition for \( \Delta(G^{t_0}) \) to form a fan is stated as follows:
**Condition 2.27.** For any \( t, t' \in \mathbb{T}_n \), the cones \( \sigma(G_{t}^{t_0}) \) and \( \sigma(G_{t'}^{t_0}) \) intersect in their common face.

Thanks to Proposition 2.22, Condition 2.27 is rephrased as follows, where we used the bijection between \( \Delta(G_{t}^{t_0}) \) and \( \Delta(G_{t'}^{t_0}) \) therein.

**Condition 2.28.** For any \( t, t' \in \mathbb{T}_n \), the cones \( \sigma(G_{t}^{t_0}) \) and \( \sigma(I) \) intersect in their common face, where \( \sigma(I) = \sigma(e_1, \ldots, e_n) \).

By changing the notation, we may prove Condition 2.28 for \( \sigma(G_{t}^{t_0}) \) instead of \( \sigma(G_{t'}^{t_0}) \) therein. Then, the condition can be further rephrased as follows:

**Claim.** For any \( t_0, t \in \mathbb{T}_n \), if a \( g \)-vector \( g_{t;i;t}^{t_0} \) is contained in \( \sigma(I) \), then \( g_{t;i;t}^{t_0} = e_\ell \) for some \( \ell \).

Let us prove the claim. For \( J \subset \{1, \ldots, n\} \), let \( J^c = \{1, \ldots, n\} \setminus J \) be its compliment, and let \( \sigma_J(G_{t}^{t_0}) \) be the face of \( \sigma(G_{t}^{t_0}) \) generated by \( g_{j;i;t}^{t_0} \) is \( (j \in J^c) \). For example, we have \( \sigma_{\{1, \ldots, n\}}(G_{t}^{t_0}) = \{0\}, \sigma_{\{i\}}(G_{t}^{t_0}) = \sigma(g_{i;i;t}^{t_0}) \).

Thanks to Proposition 2.16, we have

\[
\sigma_J(G_{t}^{t_0}) = \sigma(G_{t}^{t_0}) \cap \left( \bigcap_{j \in J} (c_{j;i;t}^{t_0})^\perp \right). \tag{2.84}
\]

On the other hand, since \( c_{j;i;t}^{t_0} \) is either positive or negative, we have

\[
\sigma(I) \cap (c_{j;i;t}^{t_0})^\perp = \sigma(I) \cap \bigcap_{k \in K[j]} (e_k)^\perp = \sigma_{K[j]}(I), \tag{2.85}
\]

where \( K[j] \) is the set of \( k \) such that the \( k \)th component of \( c_{j;i;t}^{t_0} \) is nonzero. Therefore, we have

\[
\sigma_J(G_{t}^{t_0}) \cap \sigma(I) = \sigma(G_{t}^{t_0}) \cap \left( \bigcap_{j \in J} \sigma_{K[j]}(I) \right) = \sigma(G_{t}^{t_0}) \cap \sigma_{K[J]}(I), \tag{2.86}
\]

where \( K[J] \) is the set of \( k \) such that the \( k \)th component of \( c_{j;i;t}^{t_0} \) is nonzero for some \( j \in J \). Now we set \( J = \{i\}^c \). Then, we have \( \sigma_{\{i\}^c}(G_{t}^{t_0}) = \sigma(g_{i;i;t}^{t_0}) \) and \( K[J] = \{1, \ldots, n\} \) or \( \{\ell\}^c \) for some \( \ell \). In the former case, we see from (2.86) that \( \sigma(g_{i;i;t}^{t_0}) \) intersects \( \sigma(I) \) trivially. In the latter case, we have

\[
\sigma(g_{i;i;t}^{t_0}) \cap \sigma(I) = \sigma(G_{t}^{t_0}) \cap \sigma(e_\ell). \tag{2.87}
\]

Therefore, if \( \sigma(g_{i;i;t}^{t_0}) \cap \sigma(I) \neq \{0\} \), we have \( \sigma(g_{i;i;t}^{t_0}) = \sigma(e_\ell) \). Then, by the unimodularity (2.3), we have \( g_{i;i;t}^{t_0} = e_\ell \) as desired. This completes the proof of Theorem 2.17.
2.8 Properties of \( \hat{c} \)-vectors

In this subsection we observe some properties of \( \hat{c} \)-vectors that are relevant to the scattering diagram method.

The following properties do not depend on the sign-coherence conjecture.

**Proposition 2.29.** The following facts hold:

(a). Any \( \hat{c} \)-vector \( \hat{c}_{i;t} \) is orthogonal to the \( c \)-vector \( c_{i;t} \).

(b). Any \( \hat{c} \)-vector \( \hat{c}_{i;t} \) is in the subspace in \( \mathbb{R}^n \) spanned by \( \sigma_i(G_t) \).

**Proof.** (a). Since \( DB_{t_0} \) is skew-symmetric, we have

\[
(c_{i;t}, \hat{c}_{i;t})_D = (c_{i;t})^T DB_{t_0} c_{i;t} = 0.
\]

(2.88)

(b). By the second expression of \( \hat{c} \)-vectors in (1.57), we have

\[
\hat{c}_{i;t} = \sum_{j=1}^{n} b_{ji;t} g_{j;t},
\]

(2.89)

where we recall that \( b_{ii;t} = 0 \).

**Definition 2.30.** Suppose that the sign-coherence conjecture holds. Let \( c_{i;t} \) be a \( c \)-vector, and let \( \varepsilon_{i;t} \) be its tropical sign in Definition 2.1. We call the vectors \( c_{i;t}^+ = \varepsilon_{i;t} c_{i;t} \) and \( \hat{c}_{i;t}^+ = \varepsilon_{i;t} \hat{c}_{i;t} \) a \( c^+\)-vector and a \( \hat{c}^+\)-vector, respectively.

**Remark 2.31.** Any \( c^+\)-vector \( c_{i;t}^+ \) is positive due to the sign-coherence assumption, while a \( \hat{c}^+\)-vector \( \hat{c}_{i;t}^+ \) is not so at all in general.

The following fact is related to the incoming/outgoing property of walls appearing later in Definition 5.14.

**Proposition 2.32.** Suppose that the sign-coherence conjecture holds. Then, if

\[
\hat{c}_{i;t}^+ \in \sigma_i(G_t),
\]

(2.90) then \( c_{i;t}^+ = e_\ell \) for some \( \ell \).

**Proof.** In view of (2.89), the condition (2.90) is equivalent to the following condition:

The vector \( \varepsilon_{i;t} b_{i;t} \) is either positive or zero.

(2.91)

Suppose that (2.91) happens, where we replace \( i \) with \( k \) therein for the sake of convenience. Let \( t' \) be the vertex that is \( k \)-adjacent to \( t \). Then, the mutation (2.13) is simplified by (2.91) as

\[
g_{i;t'} = \begin{cases} 
-g_{k;t} & i = k, \\
g_{i;t} & i \neq k.
\end{cases}
\]

(2.92)
Now we recall the row sign-coherence of $G$-matrices in Proposition 2.7. Then, (2.92) happens only when the following condition is satisfied:

- For each $\ell = 1, \ldots, n$, if the $\ell$th element of $g_{k:t}$ is nonzero, then the $\ell$th element of other $g_{i:t}$ ($i \neq k$) vanishes.

Thanks to the unimodularity (2.3) of $G_t$, one can sharpen the condition as follows:

- There is some $\ell$ such that $g_{k:t} = \pm e_\ell$, and the $\ell$th element of other $g_{i:t}$ ($i \neq k$) vanishes.

Then, by the duality (2.4), $c_{k:t} = \pm e_\ell$. Thus, $c_{k:t}^+ = e_\ell$. \hfill $\square$
3 Scattering diagrams and $C$- and $G$-matrices

In this section we present a proof of the sign-coherence conjecture based on the results on scattering diagrams in [GHKK18]. At the same time we also establish the correspondence between a $G$-fan and the support of a scattering diagram.

3.1 Supports of scattering diagrams

To write down a precise definition of scattering diagrams in [GHKK18] requires considerable pages, and it will be postponed in Section 5. Here we summarize the minimal information and results [GHKK18] that are needed to prove the sign-coherence conjecture.

Informally speaking, for a certain vector space $M_{\mathbb{R}} \cong \mathbb{R}^n$, a scattering diagram $\mathcal{D} = \{ w_\lambda = (\vartheta_\lambda, f_\lambda) \}_{\lambda \in \Lambda}$ in $M_{\mathbb{R}}$ is a finite or countably infinite collection of walls, where $\Lambda$ is an index set, and each wall $w_\lambda = (\vartheta_\lambda, f_\lambda)$ is a pair such that $\vartheta_\lambda$ is a (not necessarily strongly convex) cone of codimension one in $M_{\mathbb{R}}$ called the support of $w_\lambda$, and $f_\lambda$ is a certain formal power series called the wall function of $w_\lambda$. The union of the supports of walls

$$\text{Supp}(\mathcal{D}) = \bigcup_{\lambda \in \Lambda} \vartheta_\lambda \subset M_{\mathbb{R}}$$

(3.1)

is called the support of a scattering diagram $\mathcal{D}$.

For any nonsingular skew-symmetrizable matrix $B$, one can construct a scattering diagram $\mathcal{D}_s$ called a cluster scattering diagram, where $s$ is a seed data in [GHKK18] corresponding to $B$. There is a canonical isomorphism $\phi_s : M_{\mathbb{R}} \overset{\sim}{\rightarrow} \mathbb{R}^n$ such that the image of $\text{Supp}(\mathcal{D}_s)$ in $\mathbb{R}^n$ only depends on $B$. We write this image as $\mathcal{S}(B)$, and we call it the support diagram for $B$, for simplicity. A connected component of $\mathbb{R}^n \setminus \mathcal{S}(B)$ is called a chamber of $\mathcal{S}(B)$.

From the viewpoint of cluster patterns, the supports and the wall functions of walls contain the information of the tropical and the nontropical part of a cluster pattern, respectively. Since at this moment we are interested only in the tropical part, i.e., $C$- and $G$-matrices, we may safely forget wall functions, and concentrate on the support diagram $\mathcal{S}(B)$.

Let $D$ be a skew-symmetrizer of $B$ in the form (2.44), and let $(\ , \ )_D$ be the inner product in $\mathbb{R}^n$ defined in (2.45). The following facts are direct consequences of the definition/theorem of the scattering diagram $\mathcal{D}_s$ in [GHKK18] Theorems 1.12 & 1.28, which will appear later as Theorem 5.27.

**Proposition 3.1** ([GHKK18] Theorems 1.12 & 1.28). For any nonsingular skew-symmetrizable matrix $B$, the support diagram $\mathcal{S}(B)$ has the following properties.
II.3. Scattering diagrams and $C$- and $G$-matrices

(a). The support $\mathcal{S}_\lambda$ of each wall of $\mathcal{G}(B)$ has a positive normal vector $n_\lambda$ with respect to the inner product $(\ , \)_D.

(b). Each hyperplane $e_i^\perp$ ($i = 1, \ldots, n$) with respect to the inner product $(\ , \)_D is a subset of $\mathcal{G}(B)$.

(c). The interior $\sigma^o(I)$ of the cone $\sigma(I) = (e_1, \ldots, e_n)$ is a chamber of $\mathcal{G}(B)$. In other words, $\sigma^o(I)$ does not intersect $\mathcal{G}(B)$, while its boundary belongs to $\mathcal{G}(B)$. (This is a consequence of (a) and (b).)

Also, the following property of the support diagram $\mathcal{G}(B)$ is obtained from [GHKK18, Theorem 1.24], which will appear later as Theorem 6.7, together with Proposition 2.26. See the explanation after Theorem 6.7 for details.

Proposition 3.2 ([GHKK18 Theorem 1.24, Mul16 Lemma 5.2.1, Rea20b, Cor. 4.4]). Let $B$ be any nonsingular $B$-pattern. Let $t_0, t_1 \in \mathbb{T}_n$ be vertices that are $k$-adjacent. Let $\varphi_{t_0}^{t_1}$ be the map defined in (2.71). Then, the support diagrams $\mathcal{G}(B_{t_0})$ and $\mathcal{G}(B_{t_1})$ are related by

$$\varphi_{t_0}^{t_1}(\mathcal{G}(B_{t_0})) = \mathcal{G}(B_{t_1}). \ (3.2)$$

This is clearly a parallel result for $G$-fans in Proposition 2.22, which was proved under the assumption of the sign-coherence of $C$-matrices.

3.2 Proof of sign-coherence conjecture

Let us present a proof of Conjecture 1.34 relying on Propositions 3.1 and 3.2. Our strategy is to establish both

- the identification of the interior $\sigma^o(G_{t_0}^t)$ of each $G$-cone as a chamber in $\mathcal{G}(B_{t_0})$,
- the column sign-coherence of $C$-matrices $C_{t_0}^t$,

simultaneously by the induction on the distance $d(t_0, t)$ in $\mathbb{T}_n$, where both $t_0$ and $t$ vary.

Now we prove the following theorem. See [Rea20b Theorem 4.10] for a closely related result for mutation fans.

Theorem 3.3. Let $B = \{B_t\}_{t \in \mathbb{T}_n}$ be any nonsingular $B$-pattern. Then, for any $t_0, t \in \mathbb{T}_n$, the following facts hold:

(a). The set $\sigma^o(G_{t_0}^t)$ is a chamber of $\mathcal{G}(B_{t_0})$.

(b). Each $c$-vector $c_{t_0}^t$ is a normal vector of the support of some wall of $\mathcal{G}(B_{t_0})$.

(c). The column sign-coherence holds for the $C$-matrix $C_{t_0}^t$.

Proof. As in Section 2.2, we introduce the transpose $B^T = \{(B_t)^T\}_{t \in \mathbb{T}_n}$ of $B$. Let $\tilde{C}_{t_0}^t, \tilde{C}_{t_0}^t$ be $C$- and $G$-matrices of $B^T$. 

Consider the following claims \((a)\), \((b)\), \((c)\) for \(d = 0, 1, 2, \ldots\).

\((a)\). For any \(t_0, t \in \mathbb{T}_n\) such that \(d(t_0, t) = d\), the following facts hold:

\[
|C_{t_0}^t| = |G_{t_0}^t| \in \{1, -1\}, \quad (3.3)
\]

\[
D^{-1}(G_{t_0}^t)^T DC_{t_0}^t = I, \quad (3.4)
\]

\[
C_{t_0}^t = (G_{t_0}^t)^T, \quad (3.5)
\]

\[
C_{t_0}^t = (\tilde{C}_{t_0}^t)^T. \quad (3.6)
\]

\((b)\). For any \(t_0, t \in \mathbb{T}_n\) such that \(d(t_0, t) = d\), the following facts hold:

(i). The set \(\sigma_c(G_{t_0}^t)\) is a chamber of \(\mathcal{G}(B_{t_0})\).

(ii). Each \(c\)-vector \(c_{t_0}^t\) is a normal vector of the support of a wall of \(\mathcal{G}(B_{t_0})\).

(iii). The column sign-coherence holds for \(C_{t_0}^t\).

(iv). The row sign-coherence holds for \(C_{t_0}^t\).

\((c)\). For any \(t_0, t_1, t, t' \in \mathbb{T}_n\) such that

- \(d(t_0, t) = d\),
- \(t_0\) and \(t_1\) are \(k\)-adjacent,
- \(t\) and \(t'\) are \(k\)-adjacent,

the following facts hold:

\[
C_{t_0}^{t_1} = C_{t_0}^t (J_k + [\varepsilon_k (C_{t_0}^t B_{t_1})^k]), \quad (3.7)
\]

\[
G_{t_0}^{t_1} = G_{t_0}^t (J_k + [-\varepsilon_k (G_{t_0}^t B_{t_1})^k]), \quad (3.8)
\]

\[
C_{t_0}^{t_1} = (J_k + [-\varepsilon_k (G_{t_0}^t B_{t_1})^k] C_{t_0}^t, \quad (3.9)
\]

\[
G_{t_0}^{t_1} = (J_k + [\varepsilon_k (G_{t_0}^t B_{t_1})^k]) G_{t_0}^t. \quad (3.10)
\]

We prove the claims in the following order,

\[
(a)_0 \implies (b)_0 \implies (c)_0 \implies (a)_1 \implies (b)_1 \implies (c)_1 \implies (a)_2 \implies \cdots, \quad (3.11)
\]

assuming all preceding claims. Moreover, we run the same induction procedure for \(C\)- and \(G\)-matrices \(\tilde{C}_{t_0}^t\), \(\tilde{G}_{t_0}^t\) of \(B^T\) in the background.

We first prove \((a)_0\), \((b)_0\), and \((c)_0\). The claims \((a)_0\) and \((c)_0\) were proved in Propositions \(2.3\) and \(2.7\). The claim \((b)_0\) holds because \(C_{t_0}^t = G_{t_0}^t = I\), and Proposition \(3.1\) (c), where we have already received an important input from scattering diagrams.

Now, assuming the claims in \((3.11)\) up to \((c)_d\), we prove \((a)_{d+1}\). This was already proved in Propositions \(2.3\) and \(2.7\) by taking account of Remarks \(2.4\) and \(2.8\).

Next, assuming the claims in \((3.11)\) up to \((a)_{d+1}\), we prove \((b)_{d+1}\). (This is the highlights of the proof of the sign-coherence conjecture.) Let \(t_0, t_1, t \in \mathbb{T}_n\)
be vertices such that \( d(t_0, t) = 42 \), \( d(t_1, t) = 42 + 1 \), and \( t_0 \) and \( t_1 \) are \( k \)-adjacent. By (b)\(_d\), \( \sigma^o(G_{t_0}^{t_1}) \) is a chamber in \( \mathcal{S}(B_{t_0}) \). By (a)\(_d\), (c)\(_d\), Proposition 2.22 and Remark 2.23, we have \( \sigma(G_{t_0}^{t_1}) = \varphi_{t_0}^{t_1}(\sigma(G_{t_0}^{t_0})) \). Meanwhile, by Proposition 3.2, the piecewise linear isomorphism \( \varphi_{t_0}^{t_1} \), which is homeomorphism, maps a chamber in \( \mathcal{S}(B_{t_0}) \) to a chamber in \( \mathcal{S}(B_{t_1}) \). Therefore, \( \sigma^o(G_{t_0}^{t_1}) \) is a chamber in \( \mathcal{S}(B_{t_1}) \). This proves (i). By (i), each face \( \sigma_i(G_{t_0}^{t_1}) \) \((i = 1, \ldots, n)\), which is a cone of codimension one by (a)\(_d\), is a subset of the union of the supports of some walls of \( \mathcal{S}(B_{t_1}) \). Meanwhile, by (a)\(_d\), each \( c \)-vector \( c_{t_0}^{t_1} ; t \) is a normal vector of the face \( \sigma_i(G_{t_0}^{t_1}) \) with respect to the inner product \((,)_D\). Therefore, each \( c \)-vector \( c_{t_0}^{t_1} ; t \) is a normal vector of the support of some wall of \( \mathcal{S}(B_{t_1}) \). See Proposition 2.16. This proves (ii). Then, by Proposition 3.1 (a), the \( c \)-vector \( c_{t_0}^{t_1} ; t \) is either positive or negative. This proves (iii), i.e., the column sign-coherence of the \( C \)-matrix \( C_{t_0}^{t_1} \). The same result is proved in the background for the \( C \)-matrix \( \tilde{C}_{t_0}^{t_1} \) of the \( B \)-pattern \( B^T \). Thus, by (a)\(_d\), the \( G \)-matrix \( G_{t_0}^{t_1} \) is row sign-coherent. This proves (iv).

Finally, assuming the claims in (3.11) up to (b)\(_d\), we prove (c)\(_d\). This was already proved in Propositions 2.3 and 2.7 by taking account of Remarks 2.4 and 2.8.

We may roughly describe the results in Theorem 3.3 (a) that the \( G \)-fan \( \Delta(G_{t_0}) \) is embedded in the corresponding cluster scattering diagram \( \mathcal{D}_a \).

We immediately extend Theorem 3.3 (c) for any (possibly singular) \( B \)-pattern by Proposition 2.11.

**Theorem 3.4.** Let \( B = \{B_t\}_{t \in \mathbb{T}_n} \) be any (possibly singular) \( B \)-pattern. Then, for any \( t_0, t \in \mathbb{T}_n \), the column sign-coherence for the \( C \)-matrix \( C_{t_0}^{t} \) holds.

**Proof.** Let \( \overline{C}_t \) and \( \overline{G}_t \) be \( C \)- and \( G \)-matrices of the principal extension \( \overline{B} \) of \( B \) with the initial vertex \( t_0 \) in Section 2.3. The matrix \( \overline{C}_t \) is column sign-coherent by Theorem 3.3 (c). Then, the claim follows from (2.39).

This completes the proof of Conjecture 1.34.

From now on, we use the column sign-coherence of \( C \)-matrices not as an assumption but as a theorem.

**Remark 3.5.** In Part III we show that Propositions 3.1 and 3.2 actually holds for any (possibly singular) \( B \)-pattern. Thus, Theorem 3.3 also holds for any (possibly singular) \( B \)-pattern by the same proof. This gives a direct proof of the sign-coherence without using Proposition 2.11.
4 More about $F$-polynomials

Let us turn to study $F$-polynomials, namely, the nontropical part of a cluster pattern.

4.1 Fock-Goncharov decomposition

Let $\Sigma$ be any cluster pattern. We come back to the mutations of $x$- and $y$-variables, especially in the $\varepsilon$-expression (1.10) and (1.11). Let $t, t' \in T_n$ be vertices that are $k$-adjacent. One can regard mutations (1.10) and (1.11) as isomorphisms between rational function fields as follows, where $x_t, x_{t'}, y_t, y_{t'}$ are $n$-tuple of formal variables:

$$
\mu_{k;t} : \mathbb{Q}(x_{t'}) \to \mathbb{Q}(x_t), \quad \mathbb{Q}(y_{t'}) \to \mathbb{Q}(y_t),
$$

(4.1)

$$
\mu_{k;t}(x_{i;t'}) = \begin{cases} 
 x_{k;t}^{-1} \left( \prod_{j=1}^{n} x_{j;t}^{[\varepsilon b_{j,k};t]} \right) (1 + \hat{y}_{k;t}) & i = k, \\
 x_{i;t} & i \neq k,
\end{cases}
$$

(4.2)

$$
\mu_{k;t}(y_{i;t'}) = \begin{cases} 
 y_{k;t}^{-1} y_{i;t}^{[\varepsilon b_{k,i};t]} (1 + y_{k;t})^{-b_{k,i}} & i = k, \\
 y_{i;t} & i \neq k,
\end{cases}
$$

(4.3)

Note that the map is in the opposite direction, namely, from $t'$ to $t$. Here and below, we use common symbols for maps for $x$- and $y$-variables as above in view of the parallelism between them.

To make use of the flexibility of the choice of sign $\varepsilon$, we first choose a given initial vertex $t_0$. Since we have already established the sign-coherence of $C$-matrices $C^t_{t_0} = C_t$, the tropical sign $\varepsilon_{i;t}^{t_0} = \varepsilon_{i;t}$ in Definition 2.1 is defined for any $t$ and $i$. Now we set $\varepsilon = \varepsilon_{k;t}$ in (4.2) and (4.3).

Then, modifying the idea of [FG09, §2.1] with tropical signs, we consider the following decompositions of the maps $\mu_{k;t}$:

$$
\mu_{k;t} = \rho_{k;t} \circ \tau_{k;t}.
$$

(4.4)

Here, $\tau_{k;t}^{t_0} = \tau_{k;t}$ are the following isomorphisms,

$$
\tau_{k;t} : \mathbb{Q}(x_{t'}) \to \mathbb{Q}(x_t), \quad \mathbb{Q}(y_{t'}) \to \mathbb{Q}(y_t),
$$

(4.5)

$$
\tau_{k;t}(x_{i;t'}) = \begin{cases} 
 x_{k;t}^{-1} \left( \prod_{j=1}^{n} x_{j;t}^{[\varepsilon b_{j,k};t]} \right) & i = k, \\
 x_{i;t} & i \neq k,
\end{cases}
$$

(4.6)

$$
\tau_{k;t}(y_{i;t'}) = \begin{cases} 
 y_{k;t}^{-1} y_{i;t}^{[\varepsilon b_{k,i};t]} & i = k, \\
 y_{i;t} & i \neq k,
\end{cases}
$$

(4.7)
while $\rho_{k;0}^t = \rho_{k;t}$ are the following automorphisms,
\begin{align}
\rho_{k;t} : \mathbb{Q}(x_t) &\to \mathbb{Q}(x_t), \ \mathbb{Q}(y_t) \to \mathbb{Q}(y_t), \ \quad (4.8) \\
\rho_{k;t}(x_{i;t}) &= x_{i;t}(1 + y_{k;t}^{\varepsilon_{k;i}})^{-\delta_{ik}}, \ \quad (4.9) \\
\rho_{k;t}(y_{i;t}) &= y_{i;t}(1 + y_{k;t}^{\varepsilon_{k;i}})^{-b_{k;i}}. \ \quad (4.10)
\end{align}

Note that the maps $\tau_{k;t}$ and $\rho_{k;t}$ depend on the choice of the initial vertex $t_0$ through tropical signs. Also, they are defined only after establishing the sign-coherence of $C$-matrices. We call the decomposition (4.4) the Fock-Goncharov decomposition of a mutation $\mu_{k;t}$ (with tropical sign) with respect to the initial vertex $t_0$. We also call $\tau_{k;t}$ and $\rho_{k;t}$ the tropical part and the nontropical part of $\mu_{k;t}$, respectively. Indeed, it is clear that (1.6) and (1.7) are the exponential form of the mutations of $g$-vectors (2.13) and $c$-vectors (2.12), respectively.

**Remark 4.1.** The decomposition (4.4) was introduced in [FG09] for $\varepsilon = 1$. The signed version here were used, for example, in the application to the Stokes automorphisms [LN13] and dilogarithm identities [GNR17].

**Proposition 4.2.** Let $t, t' \in \mathbb{T}_n$ be vertices that are $k$-adjacent. Then, the following relations hold.
\begin{align}
\mu_{k;t'} \circ \mu_{k;t} &= \text{id}, \quad (4.11) \\
\tau_{k;t'} \circ \tau_{k;t} &= \text{id}. \quad (4.12)
\end{align}

**Proof.** The equality (4.11) is nothing but the involution of the mutations of $x$- and $y$-variables. The equality (4.12) for both $x$- and $y$-variables can be verified by the fact,
\begin{align}
\varepsilon_{k;t'} = -\varepsilon_{k;t}; \quad b_{k;i;t'} = -b_{k;i;t}; \quad b_{jk;t'} = -b_{jk;t}. \quad (4.13)
\end{align}

For any $t \in \mathbb{T}_n$, consider a sequence of vertices $t_0, t_1, \ldots, t_{r+1} = t \in \mathbb{T}_n$ such that they are sequentially adjacent with edges labeled by $k_0, \ldots, k_r$. Then, we define isomorphisms
\begin{align}
\mu_t^{t_0} := \mu_{k_0;t_0} \circ \mu_{k_1;t_1} \circ \cdots \circ \mu_{k_r;t_r} : \mathbb{Q}(x_t) \to \mathbb{Q}(x), \ \mathbb{Q}(y_t) \to \mathbb{Q}(y), \quad (4.14) \\
\tau_t^{t_0} := \tau_{k_0;t_0} \circ \tau_{k_1;t_1} \circ \cdots \circ \tau_{k_r;t_r} : \mathbb{Q}(x_t) \to \mathbb{Q}(x), \ \mathbb{Q}(y_t) \to \mathbb{Q}(y), \quad (4.15)
\end{align}
where we set $x_{t_0} = x, y_{t_0} = y$. Thanks to Proposition 4.2, $\mu_t^{t_0}$ and $\tau_t^{t_0}$ depend only on $t_0$ and $t$. Namely, we do not have to care about the redundancy $k_{s+1} = k_s$ in the sequence $k_0, \ldots, k_r$.

Recall the notions of tropical $x$- and $y$-variables in (1.54) and (1.53). The following proposition tells that the tropical parts $\tau_t^{t_0}$ are nothing but the mutations of tropical $x$- and $y$-variables.
Proposition 4.3. The following formulas hold:

\[ \mu_{t_0}^{t_0}(x_{i:t}) = x^{\mathbb{E}_{i:t}} F_{i:t}(\mathbf{y}) , \quad (4.16) \]

\[ \mu_{t_0}^{t_0}(y_{i:t}) = y^{c_{i:t}} \prod_{j=1}^{n} F_{j:t}(y)^{b_{j;i:t}} , \quad (4.17) \]

\[ \tau_{t_0}^{t_0}(x_{i:t}) = x^{\mathbb{E}_{i:t}} , \quad (4.18) \]

\[ \tau_{t_0}^{t_0}(y_{i:t}) = y^{c_{i:t}} . \quad (4.19) \]

**Proof.** The first two equalities are the separation formulas in Theorem 1.23. Let us prove the equality (4.18) for \( x \)-variables by the induction on \( t \) along \( T_n \) starting from \( t_0 \). For \( t = t_0 \), \( x^{\mathbb{E}_{i:t_0}} = x^{e_i} = x_i \). Therefore, (4.18) holds.

Suppose that it is true for \( t \) with \( d(t_0, t) = d \). Let \( t' \) be the vertex that is \( k \)-adjacent to \( t \) such that \( d(t_0, t') = d + 1 \). Then, we have

\[
\tau_{t_0}^{t_0}(x_{i:t'}) = (\tau_{t_0}^{t_0} \circ \tau_{k:t})(x_{i:t'}) \\
= \begin{cases} 
\tau_{t_0}^{t_0}(x_{i:t})^{-1} \prod_{j=1}^{n} x_j^{[-\varepsilon_{k:i} b_{k:i}]} & i = k , \\
\tau_{t_0}^{t_0}(x_{i:t}) = x^{\mathbb{E}_{i:t}} & i \neq k 
\end{cases} \\
= x^{\mathbb{E}_{i:t'}} ,
\]

where we used (2.13) in the last equality. Therefore, (4.18) holds for \( t' \). Similarly, the equality (4.19) for \( y \)-variables follows from (2.12). \( \square \)

Under these maps, \( \mathbf{y} \)-variables transform in the same way as \( y \)-variables as expected.

Proposition 4.4. The following formulas hold:

\[ \tau_{k:t}(\mathbf{y}_{i:t'}) = \begin{cases} 
\hat{y}_{k:t}^{-1} & i = k , \\
\hat{y}_{i:t} \hat{y}_{k:t}^{[\varepsilon_{k:i} b_{k:i}]} & i \neq k , 
\end{cases} \quad (4.21) \]

\[ \rho_{k:t}(\mathbf{y}_{i:t}) = \hat{y}_{i:t}(1 + \hat{y}_{k:t}^{\varepsilon_{k:i}})^{-b_{k:i}} , \quad (4.22) \]

\[ \tau_{t_0}^{t_0}(\mathbf{y}_{i:t}) = \hat{y}^{c_{i:t}} = x^{\mathbb{E}_{i:t}} , \quad (4.23) \]

where \( \mathbf{c}_{i:t} = B_{t_0} \mathbf{c}_{i:t} \) is a \( \mathbf{c} \)-vector in (1.57).

**Proof.** The equality (4.21) follows from (4.6) and (1.12). The equality (4.22) follows from (4.9). The equality (4.23) follows from (4.18) and (1.56), \( \square \)
4.2 Nontropical parts and $F$-polynomials

Observing Proposition 4.3, it is clear that the nontropical parts $\rho_{k:t}$ are responsible to generate and mutate $F$-polynomials. Let us make this statement more manifest.

In addition to the inner product $(\cdot, \cdot)_D$ in (2.45), we introduce a skew-symmetric form

$$\{u, v\}_{DB} := u^T DB v = (u, Bv)_D,$$

where $B = B_{t_0}$. Then, we introduce the following automorphisms $q_{k:t}$ for the initial $x$- and $y$-variables,

$$q_{k:t} : \mathbb{Q}(x) \to \mathbb{Q}(x), \quad \mathbb{Q}(y) \to \mathbb{Q}(y),$$

$$q_{k:t}(x^m) = x^m (1 + y^{c^+_k:t})^{-\delta_k c_k:t, m}_D,$$

$$q_{k:t}(y^n) = y^n (1 + y^{c^+_k:t})^{-\delta_k c_k:t, n}_{DB},$$

where $m, n \in \mathbb{Z}^n$, and $c^+_k:t = \varepsilon_k c_k:t$ and $\hat{c}^+_k:t = \varepsilon_k \hat{c}_k:t$ are a $c^+$-vector and a $\hat{c}^+$-vector, respectively, in Definition 2.30.

The seemingly asymmetric definitions in (4.26) and (4.28) are justified by the following properties.

**Proposition 4.5.** The following facts hold:

(a). We have the formulas

$$q_{k:t}(x^{g_{k:t}}) = x^{g_{k:t}} (1 + y^{c^+_k:t})^{-\delta_k},$$

$$q_{k:t}(y^{c_{k:t}}) = y^{c_{k:t}} (1 + y^{c^+_k:t})^{-b_{k:t}},$$

(b). The following relation holds:

$$\tau_{t_0} \circ \rho_{k:t} = q_{k:t} \circ \tau_{t_0}.$$

(c). If $t'$ and $t$ are $k$-adjacent, we have

$$q_{k:t'} \circ q_{k:t} = \text{id}.$$

**Proof.** (a). The claim follows from the following equalities:

$$\delta_k c_k:t, g_{k:t})_D = \delta_k,$$

$$\{\delta_k c_k:t, c_{k:t}\}_{DB} = b_{k:t}.$$

The equality (4.33) is the duality in (2.47). The equality (4.34) follows from (2.14) as follows:

$$D^{-1}(C^T_t DB_{t_0} C_t) = D^{-1}(DB_t) = B_t.$$
(b). Thanks to (4.29) and Propositions 4.3 and 4.4 we have
\[
x_{i; t} \overset{\rho_{k; t}}{\mapsto} x_{i; t} (1 + y_{k; t}^{+})^{-\delta_{ik}} \overset{\tau_{t}^{0}}{\mapsto} x_{i; t}^{\tau_{t}^{0}} (1 + y_{k; t}^{+})^{-\delta_{ik}}, \tag{4.36}
\]
\[
x_{i; t} \overset{\tau_{t}^{0}}{\mapsto} x_{i; t}^{\tau_{t}^{0}} (1 + y_{k; t}^{+})^{-\delta_{ik}} \tag{4.37}
\]
Therefore, the equality (1.31) holds also for \( y \)-variables. Similarly, we have
\[
y_{i; t} \overset{\rho_{k; t}}{\mapsto} y_{i; t} (1 + y_{k; t}^{+})^{-b_{k; t}} \overset{\tau_{t}^{0}}{\mapsto} y_{i; t}^{\tau_{t}^{0}} (1 + y_{k; t}^{+})^{-b_{k; t}}, \tag{4.38}
\]
\[
y_{i; t} \overset{\tau_{t}^{0}}{\mapsto} y_{i; t}^{\tau_{t}^{0}} (1 + y_{k; t}^{+})^{-b_{k; t}} \tag{4.39}
\]
Therefore, the equality (1.31) holds also for \( y \)-variables.

(c). This follows from \( c_{k; t'} = -c_{k; t} \).

For the same \( t_0, t_1, \ldots, t_{r+1} = t \in \mathbb{T}_n \) for \( \mu_{t}^{t_0} \) and \( \tau_{t}^{t_0} \) in (4.14) and (4.15), we define the automorphism
\[
q_{t_0}^{t_0} := q_{k_0; t_0} \circ q_{k_1; t_1} \circ \cdots \circ q_{k_{r}; t_{r}} : \mathbb{Q}(x) \rightarrow \mathbb{Q}(x), \mathbb{Q}(y) \rightarrow \mathbb{Q}(y). \tag{4.40}
\]
Again, by Proposition 4.5 (c), it depends only on \( t_0 \) and \( t \).

One can separate the tropical and the nontropical parts of \( \mu_{t}^{t_0} \) as follows:

**Proposition 4.6.** The following decomposition holds:
\[
\mu_{t}^{t_0} = q_{t_0}^{t_0} \circ \tau_{t}^{t_0} : \mathbb{Q}(x_{t}) \rightarrow \mathbb{Q}(x), \mathbb{Q}(y_{t}) \rightarrow \mathbb{Q}(y). \tag{4.41}
\]

**Proof.** First, we note that
\[
\rho_{k_0; t_0}(x_i) = x_i (1 + y_{k_0})^{-\delta_{ik_0}} = q_{k_0; t_0}(x_i), \tag{4.42}
\]
\[
\rho_{k_0; t_0}(y_i) = y_i (1 + y_{k_0})^{-b_{k_0; t_0}} = q_{k_0; t_0}(y_i). \tag{4.43}
\]

For \( x \)-variables, for example, by Proposition 4.5 (b), we have the following commutative diagram:
\[
\begin{array}{cccccc}
\mathbb{Q}(x_t) & \xrightarrow{\tau_{k_0; t_0}} & \mathbb{Q}(x_{t_0}) & \xrightarrow{\tau_{t_0}} & \mathbb{Q}(x_{t}) & \xrightarrow{\mu_{k_0; t_0}} \\
\mu_{k_0; t_0} & \xrightarrow{\rho_{k_0; t_0}} & \mathbb{Q}(x_{t_{-1}}) & \xrightarrow{\tau_{t_{-1}}} & \mathbb{Q}(x_{t_0}) & \\
\vdots & & \vdots & & \vdots & \\
\mathbb{Q}(x_{t_2}) & \xrightarrow{\tau_{k_1; t_1}} & \mathbb{Q}(x_{t_1}) & \xrightarrow{\mu_{k_1; t_1}} & \mathbb{Q}(x_{t_0}) & \xrightarrow{\rho_{k_1; t_1}} \\
\mathbb{Q}(x_{t_1}) & \xrightarrow{\tau_{k_0; t_0}} & \mathbb{Q}(x_{t_0}) & & & \mathbb{Q}(x_{t_0}) & \xrightarrow{\rho_{k_0; t_0} = q_{k_0; t_0}} \\
\end{array}
\tag{4.44}
\]
Therefore, the claim holds.
We conclude that the automorphisms $q_{t_0}^t$ generate the nontropical parts of $x$- and $y$-variables in the following manner.

**Theorem 4.7.** The following formulas hold:

\begin{align*}
q_{t_0}^t (x^{g_{i;t}}) &= x^{g_{i;t}} F_{i;t} (\hat{y}), \quad (4.45) \\
q_{t_0}^t (y^{c_{i;t}}) &= y^{c_{i;t}} \prod_{j=1}^n F_{j;t} (y)^{b_{j;i;t}}, \quad (4.46) \\
q_{t_0}^t (\hat{y}^{c_{i;t}}) &= \hat{y}^{c_{i;t}} \prod_{j=1}^n F_{j;t} (\hat{y})^{b_{j;i;t}}. \quad (4.47)
\end{align*}

**Proof.** The equalities (4.45) and (4.46) follow from Propositions 4.3 and 4.6. For example, for $x$-variables,

\begin{align*}
q_{t_0}^t (x^{g_{i;t}}) &= q_{t_0}^t (\tau_{t_0}^t (x_{i;t})) = \mu_{t_0}^t (x_{i;t}) = x^{g_{i;t}} F_{i;t} (\hat{y}). \quad (4.48)
\end{align*}

Similarly, the equality (4.47) follows from Propositions 4.4 and 4.6. \hfill \Box

### 4.3 Detropicalization

Let $\Sigma$ be any cluster pattern of rank $n$, and let $t_0$ be a given initial vertex. Let $S_n$ be the symmetric group of degree $n$. We define the (left) action of a permutation $\nu \in S_n$ on $x$- and $y$-variables, etc., as follows.

\begin{align*}
x' &= \nu x_t, \quad x'_i = x_{\nu^{-1}(i);t}, \quad (4.49) \\
y' &= \nu y_t, \quad y'_i = y_{\nu^{-1}(i);t}, \quad (4.50) \\
B' &= \nu B_t, \quad b'_{ij} = b_{\nu^{-1}(i)\nu^{-1}(j);t}, \quad (4.51) \\
C' &= \nu C_t, \quad c'_{ij} = c_{i\nu^{-1}(j);t}, \quad (4.52) \\
G' &= \nu G_t, \quad g'_{ij} = g_{i\nu^{-1}(j);t}. \quad (4.53)
\end{align*}

Note that these actions are compatible with the separation formulas in Theorem 1.23.

For any $t, t' \in T_n$ and any permutation $\nu \in S_n$, we define the following isomorphisms.

\begin{align*}
\nu_t : \mathbb{Q}(x_t) \to \mathbb{Q}(x'_t), \quad \mathbb{Q}(y_t) \to \mathbb{Q}(y'_t), \quad (4.54) \\
\nu_t (x_{i;t}) &= x_{\nu^{-1}(i);t'}, \quad (4.55) \\
\nu_t (y_{i;t}) &= y_{\nu^{-1}(i);t'}. \quad (4.56)
\end{align*}

Then, by Proposition 4.3 one can rephrase the periodicity condition for $G$-matrices and $x$-variables as follows:

\begin{align*}
G_{t_0}^t = \nu G_{t'}^t &\iff \tau_{t_0}^t = \tau_{t'}^{t_0} \circ \nu_t'(\text{for} x\text{-variables}), \quad (4.57)
\end{align*}
Also, the parallel statement holds for $C$-matrices and $y$-variables.

**Proposition 4.8.** Let $t_0, t, t'$ be any vertices in $T_n$. Suppose that

$$G_{t_0}^{t} = \nu G_{t_0}^{t'}$$

(4.59) holds. Then, we have

$$x_t = \nu x_{t'} \iff q_{t_0}^t = q_{t_0}^{t'} \text{ (for } x\text{-variables}).$$

(4.60)

Also, the parallel statement holds for $C$-matrices and $y$-variables.

**Proof.** The claim follows from (4.57), (4.58), and Proposition 4.6. 

Note that the permutation $\nu_{t'}$ drops off in the condition for $q_{t_0}^{t_0}$ in (4.60).

The properties

$$G_{t_0}^{t} = \nu G_{t_0}^{t'} \implies x_t = \nu x_{t'},$$

(4.61)

$$C_{t_0}^{t} = \nu C_{t_0}^{t'} \implies y_t = \nu y_{t'},$$

(4.62)

are called the *detropicalization* of $x$- and $y$-variables, respectively. They are the core of the *synchronicity phenomenon* occurring in cluster patterns systematically studied in [Nak21].

**Remark 4.9.** The implications (4.61) and (4.62) were proved for $\nu = \text{id}$ by [CHL20, Lemma 2.4 & Theorem 2.5] and for general $\nu$ by [Nak21, Theorem 5.2]. However, the proofs therein depend on the Laurent positivity in Theorem [1.37] which we temporarily dismiss. We come back to the point later in Section 7.1.
5 Scattering diagrams

In this section we present a precise formulation of scattering diagrams following \cite{GHK15, GHKK18}. To do this, we set the following guideline.

(a). Since in our approach we do not need scattering diagrams with \textit{frozen variables} (corresponding to \textit{coefficients of geometrical type} in \cite{FZ07}), we skip the relevant formulation. This makes a considerable simplification of the formulation.

(b). We omit the description of the underlying Lie algebra $\mathfrak{g}$ and the group $G$ in \cite{GHKK18} for scattering diagrams. They are crucial for the construction of scattering diagrams, but we can skip them for our purpose.

(c). We clarify and establish the connection between cluster patterns and scattering diagrams in detail by step-by-step examples.

(d). We mostly follow the notations and the conventions in \cite{GHKK18}. However, we take the \textit{transpose} of the skew-symmetric forms therein to match the convention of the exchange matrices in \cite{FZ07}.

5.1 Fixed data and seed

\textbf{Definition 5.1 (Fixed data).} A \textit{fixed data} $\Gamma$ consists of the following:

- A lattice $N \simeq \mathbb{Z}^n$ of rank $n$ with a skew-symmetric bilinear form
  \[ \{ \cdot, \cdot \} : N \times N \to \mathbb{Q}. \]  \hfill (5.1)

- A sublattice $N^o \subset N$ of finite index (equivalently, of rank $n$) such that
  \[ \{N^o, N\} \subset \mathbb{Z}. \]  \hfill (5.2)

- Positive integers $\delta_1, \ldots, \delta_n$ such that there is a basis $(e_1, \ldots, e_n)$ of $N$, where $(\delta_1 e_1, \ldots, \delta_n e_n)$ is a basis of $N^o$.

- $M = \text{Hom}(N, \mathbb{Z})$, $M^o = \text{Hom}(N^o, \mathbb{Z})$, $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$, and we regard $M \subset M^o \subset M_\mathbb{R}$.  \hfill (5.3)

For a given $N^o \subset N$ of finite index, the above $\delta_1, \ldots, \delta_n$ always exist; for example, take the elementary divisors of the embedding $N^o \hookrightarrow N$. However, other choices are equally good.

Let $\langle n, m \rangle : N^o \times M^o \to \mathbb{Z}$ be the canonical paring. We also write its linear extension $N \times M_\mathbb{R} \to \mathbb{R}$ by the same symbol $\langle n, m \rangle$. (Though it is a little confusing, the symbol $n$ is used to denote both the rank and an element in $N$.)

\textbf{Remark 5.2.} (a). In \cite{GHKK18} we already took the transpose of the form in \cite{GHKK18} as mentioned in the above guideline (d).

(b). In \cite{GHKK18} it is assumed that the numbers $\delta_1, \ldots, \delta_n$ are coprime. Here we do not require it. See also Remark \cite{5.29}.
**Definition 5.3** (Seed). A seed $s = (e_1, \ldots, e_n)$ for a fixed data $\Gamma$ is a basis of $N$ such that $(\delta_1 e_1, \ldots, \delta_n e_n)$ is a basis of $N^\circ$.

Let $s = (e_1, \ldots, e_n)$ be a seed for a fixed data $\Gamma$, and let $(e_1^*, \ldots, e_n^*)$ be the dual basis of $M$. Let $f_i = \delta_i^{-1} e_i^*$ ($i = 1, \ldots, n$). Then, $(f_1, \ldots, f_n)$ is a basis of $M^\circ$.

Accordingly, we define isomorphisms, all of which are denoted by the same symbol for simplicity,

$$
\phi_s : N \rightarrow \mathbb{Z}^n, \quad e_i \mapsto e_i, \quad (5.4)
$$

$$
\phi_s : M^\circ \rightarrow \mathbb{Z}^n, \quad M_{\mathbb{R}} \rightarrow \mathbb{R}^n, \quad f_i \mapsto e_i. \quad (5.5)
$$

The identifications under the above isomorphisms are written as, $N \simeq_s \mathbb{Z}^n$, $M^\circ \simeq_s \mathbb{Z}^n$, $M_{\mathbb{R}} \simeq_s \mathbb{R}^n$, respectively.

Let us identify $N \simeq_s \mathbb{Z}^n$ and $M^\circ \simeq_s \mathbb{Z}^n$. For $n \in N^\circ \subset N$ and $m \in M^\circ$, let $n \in \mathbb{Z}^n$ and $m \in \mathbb{Z}^n$ be the corresponding vectors. Then, the canonical paring $\langle n, m \rangle$ for $N^\circ \times M^\circ$ is given by

$$
\langle n, m \rangle = (n, m)_D := n^T D m, \quad D = \text{diag}(\delta_1^{-1}, \ldots, \delta_n^{-1}). \quad (5.6)
$$

This symmetric bilinear form $(\cdot, \cdot)_D$ agrees with the one in (2.45). We also regard it as a pairing $N \times M_{\mathbb{R}} \rightarrow \mathbb{R}$, depending on the context.

**Example 5.4.** A seed $s$ for a fixed data $\Gamma$ is identified with a seed $(x, y, B)$ (in the sense of Definition 1.3) in the following way: Let $x^m$ ($m \in M^\circ$) and $y^n$ ($n \in N$) be the monomial expressions with formal symbols $x$ and $y$ such that $x^{m+m'} = x^m x^{m'}$, $y^{n+n'} = y^n y^{n'}$. Let $\mathcal{F}_X$ and $\mathcal{F}_Y$ be the rational function fields generated by all formal exponentials $x^m$ and $y^n$, respectively. Then, a seed $(x_s, y_s, B_s) = (x, y, B)$ in $(\mathcal{F}_X, \mathcal{F}_Y)$ is defined by

$$
x_i = x^{f_i}, \quad y_i = y^{e_i}, \quad b_{ij} = \{\delta_i e_i, e_j\}. \quad (5.7)
$$

In particular, the matrix $D$ in (5.6) is a (left) skew-symmetrizer of $B = (b_{ij})$. Moreover, under the identification $N \simeq_s \mathbb{Z}^n$, the skew-symmetric bilinear form in (5.1) is given by

$$
\{n, n'\} = (n, n')_{DB} := n^T D B n'. \quad (5.8)
$$

This skew-symmetric bilinear form agrees with the one in (4.24). This is the reason why we take $D$ as (2.44) throughout. For a given seed $s$ (the initial seed) for a fixed data $\Gamma$ and a given initial vertex $t_0 \in T_n$, we associate a cluster pattern $\Sigma_{s,t_0}$ whose initial seed $\Sigma_{t_0}$ is given by the seed $(x, y, B)$ in (5.7). Below we consider the $C$-, $G$-, and $F$-patterns of $\Sigma_{s,t_0}$ with the initial vertex $t_0$ unless otherwise mentioned.
Remark 5.5. In the scattering diagram formalism, it is natural to consider the diagonal matrix with positive integer diagonals $\Delta = \text{diag}(\delta_1, \ldots, \delta_n)$ and the skew-symmetric rational matrix $\Omega = (\omega_{ij})_{i,j=1}^n$, $\omega_{ij} = \{e_i, e_j\}$. Then, the matrices in (5.6)–(5.8) are given by

$$D = \Delta^{-1}, \quad B = \Delta \Omega, \quad DB = \Omega. \tag{5.9}$$

5.2 Walls

Let us introduce the homomorphism

$$p^*: \ N \to M^\circ, \quad n \mapsto \{., n\}. \tag{5.10}$$

Example 5.6 (continued). Using the matrix $B$ in (5.7), we have

$$p^*(e_j) = \sum_{i=1}^n b_{ij} f_i. \tag{5.11}$$

Therefore, under our identification $N \simeq_{\text{a}} \mathbb{Z}^n$ and $M^\circ \simeq_{\text{a}} \mathbb{Z}^n$, the matrix representation of $p^*$ is just given by $B$. Also, again by (5.7) and (5.11),

$$x^{p^*(e_i)} = \prod_{j=1}^n x_{ji}^{b_{ji}} = \hat{y}_i. \tag{5.12}$$

From now on we assume the following condition for a fixed data $\Gamma$:

Assumption 5.7 (Injectivity Assumption). The map $p^*$ is injective.

Example 5.8 (continued.). Injectivity Assumption is equivalent to assume that the skew-symmetric form $\{., .\}$ in (5.1) is nondegenerate. Thus, $B$ in (5.7) is nonsingular, and $\hat{y}_1, \ldots, \hat{y}_n$ in (5.12) are algebraically independent.

Define

$$N^+ = N^+_s := \left\{ \sum_{i=1}^n a_i e_i \right\| a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n a_i \neq 0 \right\}, \tag{5.13}$$

$$P = P_s := \left\{ \sum_{i=1}^n a_i p^*(e_i) \right\| a_i \in \mathbb{Z}_{\geq 0} \right\} = p^*(N^+) \cup \{0\} \subset M^\circ. \tag{5.14}$$

We view $P$ as a monoid.

Example 5.9 (continued.). Under the identification $M^\circ \simeq_{\text{a}} \mathbb{Z}^n$, we have

$$P = \left\{ \sum_{i=1}^n a_i b_i \right\| a_i \in \mathbb{Z}_{\geq 0} \right\}, \tag{5.15}$$

where $b_i$ is the $i$th column of $B$. 

Let \( \mathbb{k} \) be any field of characteristic zero, e.g., \( \mathbb{k} = \mathbb{Q} \). Let \( \mathbb{k}[P] \) be the monoid algebra of \( P \) over \( \mathbb{k} \). Let \( J \) be the maximal ideal of \( \mathbb{k}[P] \) generated by \( P \setminus \{0\} \), and let

\[
\mathbb{k}[[P]] = \lim_{\leftarrow} \mathbb{k}[P]/J^\ell (5.16)
\]

be the completion with respect to \( J \). Using the formal exponentials \( x^m (m \in P) \), an element of \( \mathbb{k}[P] \) (resp. \( \mathbb{k}[[P]] \)) is written as a polynomial (resp. a formal power series) in \( x \) as

\[
f = \sum_{m \in P} c_m x^m, \quad (c_m \in \mathbb{k}). (5.17)
\]

**Remark 5.10.** In [GHKK18] the notation \( z^m \) is used. We use \( x^m \) because of the forthcoming identification with \( x \)-variables. This notation is also compatible with the one in Example 5.4.

**Example 5.11** (continued.). Under the identification (5.12), \( \mathbb{k}[P] \) is the polynomial algebra \( \mathbb{k}[\hat{y}_1, \ldots, \hat{y}_n] \). Also, \( \mathbb{k}[[P]] \) is the algebra of formal power series \( \mathbb{k}[[\hat{y}_1, \ldots, \hat{y}_n]] \).

**Definition 5.12** (Normalized automorphism). For \( n \in \mathbb{N}^+ \) and \( f = 1 + \sum_{k=1}^{\infty} c_k x^{k^p(n) \in \mathbb{k}[[P]]} \), we define an automorphism \( p_{n,f} \) of \( \mathbb{k}[[P]] \) by

\[
p_{n,f}(x^m) = x^m f^{(\delta(n)n,m)}, \quad (m \in P), (5.18)
\]

where \( \delta(n) \) is the smallest positive rational number such that \( \delta(n)n \in \mathbb{N}^\circ \). We call \( \delta(n) \) the **normalization factor** of \( n \) and \( p_{n,f} \) the normalized automorphism by \( n \) and \( f \).

**Example 5.13** (continued.). Let us identify \( N \simeq_s \mathbb{Z}^n \) and \( M^\circ \simeq_s \mathbb{Z}^n \).

(a). Let \( n = e_i \), where \( p^*(n) = B e_i = b_i \). Then, we have \( \delta(e_i) = \delta_i \). Let \( f = 1 + x^{b_i} = 1 + \hat{y}_i \). Then, the automorphism \( p_{n,f} \) is given by

\[
p_{n,f}(x^m) = x^m (1 + \hat{y}_i)^{\delta(e_i,m)}. \quad (m \in P \subset M^\circ). (5.19)
\]

(b). Let \( n = c_{i,t}^+ \), where \( p^*(n) = B c_{i,t}^+ = \hat{c}_{i,t}^+ \). We first claim that

\[
\delta(c_{i,t}^+) = \delta_i. (5.20)
\]

We prove it by the induction on \( t \) along \( \mathbb{T}_n \) starting from \( t_0 \). For \( t = t_0 \), this is the case (a) above. Assume that (5.20) holds for some \( t \). Let \( t' \) be the vertex that is \( k \)-adjacent to \( t \). By (2.6), we have

\[
C_{t'} D^{-1} = C_t (J_k + [\varepsilon_{k,t} B_t^k]_+^k) D^{-1} = C_t D^{-1} (J_k + [-\varepsilon_{k,t} (B_t^k)]_+^k), (5.21)
\]
where (1.13) is used for the last equality. Since \(|J_k + [-\varepsilon_{k;i};(B_t)^T]_{k_\#}| = -1\) by Lemma 1.10, each column vector \(\delta_i c^+_n\) belongs to \(N_0\). Furthermore, \(\delta_i\) is the smallest positive rational number satisfying this property because \(\delta_1 c^+_n, \ldots, \delta_n c^+_n\) are a \(\mathbb{Z}\)-basis of \(N^\circ\). This proves (5.20). Now, let \(f = 1 + x^e_{i;i} = 1 + \bar{y}^c_{i;i}\). Then, we have

\[p_n,f(x^m) = x^m(1 + \bar{y}^c_{i;i})(\delta_i c^+_n, m)_D, \quad (m \in P \subset M^\circ). \tag{5.22}\]

Observe that

\[p_n^{-\varepsilon_{i;i}} = q_{i;i}, \tag{5.23}\]

where \(p_n^{-1}\) is the inverse of \(p_n\), and \(q_{i;i}\) is the one in (4.26) viewed as an automorphism of \(k[[P]]\). The factor \(-\varepsilon_{i;i}\) in (5.23) is important for the integrity of the whole picture.

For any \(n \in N^+\), we define a hyperplane in \(M_\mathbb{R}\)

\[n^\perp := \{m \in M_\mathbb{R} \mid (n, m) = 0\} \subset M_\mathbb{R}. \tag{5.24}\]

We say that \(n \in N^+\) is primitive if there is no pair \(j \in \mathbb{Z}_{>1}\) and \(n' \in N^+\) such that \(n = jn'\). Let \(N^+_\text{pr}\) be the set of all primitive elements in \(N\).

Now we introduce the fundamental ingredient of scattering diagrams.

**Definition 5.14** (Wall). A wall for a seed \(s\) is a triplet \(w = (\mathfrak{d}, f)_n\), where

- \(n \in N^+_\text{pr}\),
- \(\mathfrak{d} \subset n^\perp\) is a (not necessarily strongly convex) cone in \(M_\mathbb{R}\) of codimension one,
- \(f\) is an element of \(k[[x^{p^*(n)}]] \subset k[[P]]\) with constant term 1,

\[f = 1 + \sum_{k=1}^{\infty} c_k x^{kp^*(n)}. \tag{5.25}\]

The cone \(\mathfrak{d}\) is called the support of a wall. No names were explicitly given for \(f\) and \(n\) in [GHKK18]. Since this is inconvenient, here we call them the wall function and the normal vector of a wall. We say that a wall \(w = (\mathfrak{d}, f)_n\) is incoming if

\[p^*(n) \in \mathfrak{d}. \tag{5.26}\]

Otherwise, we say it is outgoing.

**Remark 5.15.** Since \(\{n, n\} = 0\) by skew-symmetry, we have

\[p^*(n) \in n^\perp. \tag{5.27}\]
5.3 Scattering diagrams

Example 5.16 (continued). Let us identify \( N \simeq_s \mathbb{Z}^n \) and \( M^o \simeq_s \mathbb{Z}^n \).

(a) Let \( n = e_i, \mathfrak{d} = e_i^+ \), and \( f = 1 + \hat{y}_i \). Then, \( b_i = B e_i \in \mathfrak{d} \), and \( w_{i; t_0} = (e_i^+, 1 + \hat{y}_i) e_i \), is an incoming wall.

(b) Let \( n = c_{i; t}^+ = \varepsilon_{i; t} c_{i; t}, \mathfrak{d} = \sigma_i(c_i) \), and \( f = 1 + \hat{y}_{c_{i; t}}^+ \). Firstly, \( c_{i; t}^+ \) is primitive due to the unimodularity (2.3) of \( \mathbb{C} \)-matrices. Secondly, \( c_{i; t}^+ \) is a normal vector of \( \sigma_i(G_i) \) thanks to the duality in Proposition (2.16). Therefore, \( w_{i; t} = (\sigma_i(G_i), 1 + \hat{y}_{c_{i; t}^+}) c_{i; t}^+ \) is a wall. Moreover, due to Proposition (2.32), if \( c_{i; t}^+ \neq e_t \) for any \( \ell \), it is an outgoing wall. Note that, if \( t, t' \in \mathbb{T}_n \) are \( k \)-adjacent, then two walls \( w_{k; t} \) and \( w_{k; t'} \) are identical.

5.3 Scattering diagrams

Let \( \hat{J} \) be the maximal ideal of \( \mathbb{k}[[P]] \) generated by \( P \setminus \{0\} \). We say that \( f = 1 + \sum_{m \in P, m \neq 0} c_m x^m \in \mathbb{k}[[P]] \) is trivial modulo \( \hat{J}^\ell \) if its image in \( \mathbb{k}[[P]]/\hat{J}^\ell \) is 1.

Example 5.17 (continued). Under the identification \( \mathbb{k}[[P]] = \mathbb{k}[[\hat{y}_1, \ldots, \hat{y}_n]] \), \( f \) is trivial modulo \( \hat{J}^\ell \) if and only if \( f - 1 \) contains no monomial in \( \hat{y}_1, \ldots, \hat{y}_n \) whose total degree is less than \( \ell \).

Now we can give the definition of a scattering diagram.

Definition 5.18 (Scattering diagram). A scattering diagram \( \mathcal{D} \) for a seed \( s \) is a collection of walls \( \{w_\lambda = (\mathfrak{d}_\lambda, f_\lambda)_{n_\lambda}\}_{\lambda \in \Lambda} \) with respect to a seed \( s \), where \( \Lambda \) is a finite or countably infinite index set, satisfying the following finiteness condition:

- For each positive integer \( \ell \), there are only finitely many walls whose wall functions are not trivial modulo \( \hat{J}^\ell \).

Definition 5.19 (Support and singular locus). For a scattering diagram \( \mathcal{D} = \{(\mathfrak{d}_\lambda, f_\lambda)_{n_\lambda}\}_{\lambda \in \Lambda} \), the support and the singular locus of \( \mathcal{D} \) are defined by

\[
\text{Supp}(\mathcal{D}) = \bigcup_{\lambda \in \Lambda} \mathfrak{d}_\lambda, \quad (5.28)
\]
\[
\text{Sing}(\mathcal{D}) = \bigcup_{\lambda \in \Lambda} \mathfrak{d}_\lambda \cup \bigcup_{\lambda, \lambda' \in \Lambda, \dim \mathfrak{d}_\lambda \cap \mathfrak{d}_{\lambda'} = n-2} \mathfrak{d}_\lambda \cap \mathfrak{d}_{\lambda'}. \quad (5.29)
\]

Example 5.20 (continued). Let us identify \( N \simeq_s \mathbb{Z}^n \) and \( M^o \simeq_s \mathbb{Z}^n \). For the cluster pattern \( \Sigma_{s; t_0} \), we define a set of walls

\[
\mathcal{D}(\Sigma_{s; t_0}) := \{w_{i; t} = (\sigma_i(G_i), 1 + \hat{y}_{c_{i; t}}^+) c_{i; t}^+ | i = 1, \ldots, n; t \in \mathbb{T}_n\}, \quad (5.30)
\]

where we discard all duplicate walls so that \( \mathcal{D}(\Sigma_{s; t_0}) \) consists of mutually distinct walls. We may take a different label \((i; t)\) freely for identical walls.
The collection of walls $\mathcal{D}(\Sigma_{s,t_0})$ is not necessarily a scattering diagram, because the finiteness condition is not guaranteed. In fact, there are examples where $\mathcal{D}(\Sigma_{s,t_0})$ does not satisfy the finiteness condition. See Remark 6.15.

### 5.4 Wall-crossing automorphisms

**Definition 5.21** (Admissible curve). A curve $\gamma : [0,1] \to M_\mathbb{R}$ is *admissible* for a scattering diagram $\mathcal{D}$ if it satisfies the following properties:

- $\gamma$ does not intersect $\text{Sing}(\mathcal{D})$.
- The end points of $\gamma$ are in $M_\mathbb{R} \setminus \text{Supp}(\mathcal{D})$.
- It is a smooth curve, and it intersects $\text{Supp}(\mathcal{D})$ transversally.

**Definition 5.22** (Wall-crossing automorphism). For any scattering diagram $\mathcal{D}$ and any admissible curve $\gamma$, we define the associated *wall-crossing automorphism along $\gamma$*, $p_{\gamma,\mathcal{D}} \in \text{Aut}(\mathbb{R}[[P]])$ as follows: For each degree $\ell$, take all walls whose wall functions are not trivial modulo $\hat{J}_\ell$. There are only finitely many such walls. Let $(d_s, f_s)_{n_s}$ ($s = 1, \ldots, r$) be the walls among them which $\gamma$ intersects. We may assume that the curve $\gamma(t)$ intersects $d_s$ at time $t = t_s$ such that $0 < t_1 \leq t_2 \leq \cdots \leq t_r < 1$. (5.31)

For each $s$, we define the *intersection sign*

$$\epsilon_s = \begin{cases} 
1 & \langle n_s, \gamma'(t_s) \rangle < 0, \\
-1 & \langle n_s, \gamma'(t_s) \rangle > 0,
\end{cases} \quad (5.32)$$

where $\gamma'(t)$ is the velocity vector of $\gamma(t)$ at $t$. Then, we define

$$p_{\gamma,\mathcal{D}}^\ell = p_{n_r,f_r}^\ell \circ \cdots \circ p_{n_2,f_2}^\ell \circ p_{n_1,f_1}^\ell,$$  
$$p_{\gamma,\mathcal{D}} = \lim_{\ell \to \infty} p_{\gamma,\mathcal{D}}^\ell. \quad (5.33)$$

Note that $p_{\gamma,\mathcal{D}}$ only depends on the homotopy class of $\gamma$ in $M_\mathbb{R} \setminus \text{Sing}(\mathcal{D})$.

**Remark 5.23.** In [GHKK18], $p_{\gamma,\mathcal{D}}$ is called a *path-ordered product* of automorphisms $p_{n_s,f_s}$.

**Example 5.24** (continued). Let $\mathcal{D}(\Sigma_{s,t_0})$ be the collection of walls in Example 5.20. Even though the finiteness condition does not hold in general, the wall-crossing automorphism is still well-defined for a curve crossing only finitely many walls. Let $t_0, t_1, \ldots, t_{r+1} = t \in \mathbb{T}_n$ be a sequence of vertices such that they are sequentially adjacent with edges labeled by $k_0, k_1, \ldots, k_r$. Accordingly, we consider an admissible curve $\gamma_{t_0}^{t_0}$ further satisfying the following conditions:

- It is in the interior of the support $|\Delta(G_{t_0})|$ of the $G$-fan $\Delta(G_{t_0})$. 


• It starts in $\sigma^\circ(G_{t_{r+1}})$.
• It sequentially intersects walls

$$
(\sigma_{k_r}(G_{t_r}), 1 + \hat{y}^+_{c_{kr,t_r}})_{c_{kr,t_r}^+}, \ldots, (\sigma_{k_0}(G_{t_0}), 1 + \hat{y}^+_{c_{k_0,t_0}})_{c_{k_0,t_0}^+}.
$$

\hfill (5.35)

• It ends in $\sigma^\circ(G_{t_0})$.

By Proposition 2.16 for each $s = 0, \ldots, r$, the $c$-vector $c_{k_s,t_s}$ is inward for the cone $\sigma_{k_s}(G_{t_s})$. Thus, when $\gamma_{t_0}$ crosses $\sigma_{k_s}(G_{t_s})$ to get into $\sigma(G_{t_s})$, the velocity vector $\gamma'$ and the $c$-vector $c_{k_s,t_s}$ are always in the same direction with respect to $\sigma_{k_s}(G_{t_s})$. Since the normal vector of the wall is $n_{s} = \varepsilon_{k_s,t_s}c_{k_s,t_s}$, the factor in (5.32) is given by $-\varepsilon_{k_s,t_s}$. Then, by (4.40) and (5.23), the wall-crossing automorphism along $\gamma_{t_0}$ is identified with $q_{t_0}$ in (4.40) as follows:

\[
\tilde{p}_{t_0} := p_{s,t_0,\mathcal{D}(\Sigma_{s,t_0})} = p_{-\varepsilon_{k_0,t_0}} \circ \cdots \circ p_{-\varepsilon_{k_{r},t_r}} = q_{k_0,t_0} \circ \cdots \circ q_{k_r,t_r} = q_{t_0}.
\]

\hfill (5.36)

Note that the factor $-\varepsilon_{i;t}$ in (5.23) is absorbed in coordination with the factor in (5.32).

**Definition 5.25 (Equivalence/Consistency).**

• Two scattering diagrams $\mathcal{D}$ and $\mathcal{D}'$ with a common initial seed $s$ are equivalent if, for any curve $\gamma$ that is admissible for both $\mathcal{D}$ and $\mathcal{D}'$, the equality $p_{\gamma,\mathcal{D}} = p_{\gamma,\mathcal{D}'}$ holds.
• A scattering diagram $\mathcal{D}$ is consistent if for any admissible curve $\gamma$, the associated wall-crossing automorphism $p_{\gamma,\mathcal{D}}$ depends only on the end points of $\gamma$.

**Remark 5.26.** For a given scattering diagram, one obtains equivalent scattering diagrams by repeating the following procedures:

• To join a pair of walls $(\mathcal{D}_1, f_1)_n$ and $(\mathcal{D}_2, f_2)_n$ intersecting in their common face $\mathcal{D}_1 \cap \mathcal{D}_2$ of codimension two into a wall $(\mathcal{D}_1 \cup \mathcal{D}_2, f)_n$, or to split it conversely.
• To join a pair of walls $(\mathcal{D}, f_1)_n$ and $(\mathcal{D}, f_2)_n$ into a wall $(\mathcal{D}, f_1 f_2)_n$, or to split it conversely. One may join infinitely many walls with a common support as well by taking the infinite product of wall functions, or split it conversely.
• To add or to remove a trivial wall $(\mathcal{D}, 1)$.

Applying these operations, and especially removing all trivial walls, one can obtain a scattering diagram whose support is minimal among other equivalent diagrams.
The following is the first fundamental theorem on scattering diagrams in view of the application to cluster algebra theory.

**Theorem 5.27** ([GHKK18, Theorems 1.12 & 1.28]). Let $\mathfrak{s}$ be any initial seed for $\Gamma$. Then, there exists a scattering diagram $\mathcal{D}_s$ satisfying the following properties:

(a). $\mathcal{D}_{\text{in},s} := \{(e_i^\perp, 1 + \hat{y}_i)_{e_i} \mid i = 1, \ldots, n \} \subset \mathcal{D}_s$.
(b). $\mathcal{D}_s \setminus \mathcal{D}_{\text{in},s}$ consists only of outgoing walls, and their normal vectors are not equal to $e_i$ for any $i$.
(c). $\mathcal{D}_s$ is consistent.

Moreover, such a scattering diagram is unique up to the equivalence.

We call $\mathcal{D}_s$ a cluster scattering diagram (CSD for short) for $\mathfrak{s}$.

**Remark 5.28.** The second statement of (b) is taken from [GHKK18, Theorem 1.28, Remark 1.29], while all other statements are from [GHKK18, Theorem 1.12].

From now on we assume that $\mathcal{D}_s$ has the minimal support among other equivalent diagrams. Under the identification $M_{\mathbb{R}} \simeq_\mathfrak{s} \mathbb{R}^n$, we define $\mathcal{S}(B_{t_0}) := \text{Supp}(\mathcal{D}_s) \subset \mathbb{R}^n$. Now we see that the properties of the support diagram $\mathcal{S}(B_{t_0})$ given in Proposition 3.1 follow from Definition 5.14 and Theorem 5.27.

**Remark 5.29.** There is some redundancy of fixed data, because we do not require the coprimeness of $\delta_1, \ldots, \delta_n$ here. Suppose that the exchange matrix $B$ in (5.7) is indecomposable. For a fixed data $\Gamma$, another fixed data $\Gamma'$ is called a rescaling of $\Gamma$ if there is some rational number $\lambda > 0$ such that the following relations hold:

\[
N' = N, \quad (N^\circ)' = \lambda^{-1}N^\circ, \quad \delta_i' = \lambda^{-1}\delta_i, \quad (M^\circ)' = \lambda M^\circ.
\]

For such a pair, a seed $\mathfrak{s}$ for $\Gamma$ is also regarded as a seed for $\Gamma'$. Let $\phi_\lambda : \mathbb{K}[[P]] \to \mathbb{K}[[P']]$, $x^m \mapsto x^{\lambda m}$ be the induced algebra isomorphism. Then, we have, for $m \in P$,

\[
(p'_{n,\phi_\lambda(f)} \circ \phi_\lambda)(x^m) = p'_{n,\phi_\lambda(f)}(x^{\lambda m}) = x^{\lambda m} \phi_\lambda(f)(\delta_0(n) n, \lambda m) = x^{\lambda m} \phi_\lambda(f)(\delta(n) n, m) = (\phi_\lambda \circ p_{n,f})(x^m).
\]

Thus, cluster scattering diagrams $\mathcal{D}_s$ and $\mathcal{D}'_s$ for $\Gamma$ and $\Gamma'$, respectively, are identified under the above correspondence. Moreover, for $x_i, y_i, b_{ij}$ and $x'_i$, 

5.5 Rank 2 examples of consistent scattering diagrams $\mathcal{D}_s$

Let us present the scattering diagrams $\mathcal{D}_s$ in Theorem 5.27 in the rank 2 case under the identification $N \cong_s \mathbb{Z}^n$ and $M_\mathbb{R} \cong_s \mathbb{R}^n$, following [GHKK18]. Using the equivalence in Remark 5.29, we may assume that \( \{e_2, e_1\} = 1 \) without losing generality. We use the same convention and notations in Section 2.5.

(I). Finite type. The scattering diagram $\mathcal{D}_s$ is equivalent to $\mathcal{D}(\Sigma_{s,t_0})$ constructed in Example 5.20. In other words, $\mathcal{D}(\Sigma_{s,t_0})$ itself is a consistent scattering diagram.

(a). Type $A_2$. Let

$$\delta_1 = \delta_2 = 1; \ \hat{y}_1 = x_2, \ \hat{y}_2 = x_1^{-1}. \quad (5.44)$$

The scattering diagram $\mathcal{D}(\Sigma_{s,t_0})$ consists of five distinct walls of the form in (5.30). They are rearranged up to the equivalence of scattering diagrams as two incoming walls and one outgoing walls as

$$(e_1^\perp, 1 + \hat{y}_1)e_1, \ (e_2^\perp, 1 + \hat{y}_2)e_2, \ (\sigma((1,-1)), 1 + \hat{y}_1\hat{y}_2)(1,1). \quad (5.45)$$

This is the scattering diagram $\mathcal{D}_s$, which is depicted in Figure 3. The only property to be checked is the following consistency condition

$$p_{\gamma_1, \mathcal{D}_s} = p_{\gamma_2, \mathcal{D}_s} \quad (5.46)$$

for the admissible curves $\gamma_1$ and $\gamma_2$ in Figure 3. One can check it easily by the direct calculation as follows:

\[
\begin{align*}
\delta_1 = \delta_2 = 1; \ \hat{y}_1 = x_2, \ \hat{y}_2 = x_1^{-1}. \\
(e_1^\perp, 1 + \hat{y}_1)e_1, \ (e_2^\perp, 1 + \hat{y}_2)e_2, \ (\sigma((1,-1)), 1 + \hat{y}_1\hat{y}_2)(1,1).
\end{align*}
\]
\[ p_{\gamma_1, \mathcal{D}_s}(x_1) = p_{e_2, 1+\hat{y}_2}(p_{e_1, 1+\hat{y}_1}(x_1)) \]
\[ = p_{e_2, 1+\hat{y}_2}(x_1(1 + \hat{y}_1)) \]
\[ = x_1(1 + \hat{y}_1(1 + \hat{y}_2)) \]
\[ = x_1(1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2). \]  
\[ (5.47) \]

\[ p_{\gamma_2, \mathcal{D}_s}(x_1) = p_{e_1, 1+\hat{y}_1}(p_{(1,1), 1+\hat{y}_1 \hat{y}_2}(p_{e_2, 1+\hat{y}_2}(x_1)))) \]
\[ = p_{e_1, 1+\hat{y}_1}(p_{(1,1), 1+\hat{y}_1 \hat{y}_2}(x_1)) \]
\[ = p_{e_1, 1+\hat{y}_1}(x_1(1 + \hat{y}_1 \hat{y}_2)) \]
\[ = x_1(1 + \hat{y}_1)(1 + \hat{y}_1 \hat{y}_2(1 + \hat{y}_1)^{-1}) \]
\[ = x_1(1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2). \]
\[ (5.48) \]

Similarly,

\[ p_{\gamma_1, \mathcal{D}_s}(x_2) = p_{e_2, 1+\hat{y}_2}(p_{e_1, 1+\hat{y}_1}(x_2)) \]
\[ = p_{e_2, 1+\hat{y}_2}(x_2) \]
\[ = x_2(1 + \hat{y}_2). \]
\[ (5.49) \]

\[ p_{\gamma_2, \mathcal{D}_s}(x_2) = p_{e_1, 1+\hat{y}_1}(p_{(1,1), 1+\hat{y}_1 \hat{y}_2}(p_{e_2, 1+\hat{y}_2}(x_2)))) \]
\[ = p_{e_1, 1+\hat{y}_1}(p_{(1,1), 1+\hat{y}_1 \hat{y}_2}(x_2(1 + \hat{y}_2))) \]
\[ = p_{e_1, 1+\hat{y}_1}(x_2(1 + \hat{y}_1 \hat{y}_2)(1 + \hat{y}_2(1 + \hat{y}_1 \hat{y}_2)^{-1})) \]
\[ = p_{e_1, 1+\hat{y}_1}(x_2(1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2)) \]
\[ = x_2(1 + \hat{y}_2(1 + \hat{y}_1)^{-1} + \hat{y}_1 \hat{y}_2(1 + \hat{y}_1)^{-1}) \]
\[ = x_2(1 + \hat{y}_2). \]
\[ (5.50) \]

It is also instructive to give an alternative derivation of (5.46) in view of Proposition 4.8. Recall the action of a permutation \( \sigma \in S_n \) in (4.49)–(4.53). We know from (2.52) that

\[ G_{t_2} = s_{12}G_{t_{-3}}, \]
\[ (5.51) \]

where \( s_{12} \) is the transposition of 1 and 2. Also, the following property (the \textit{pentagon periodicity}) is the first thing we learn in cluster algebra theory.

\[ x_{t_2} = s_{12}x_{t_{-3}}. \]
\[ (5.52) \]

Therefore, by Proposition 4.8 we have (for \( x \)-variables)

\[ q_{t_2}^{t_0} = q_{t_{-3}}^{t_0}. \]
\[ (5.53) \]

Thus, by (5.36),

\[ \tilde{p}_{t_2}^{t_0} = \tilde{p}_{t_{-3}}^{t_0}. \]
\[ (5.54) \]
5.5. Rank 2 examples of consistent scattering diagrams $\mathcal{D}_s$

Since $\mathfrak{p}_{\gamma_1, \mathcal{D}_s} = (\tilde{\mathfrak{p}}_{t_0})^{-1}$ and $\mathfrak{p}_{\gamma_2, \mathcal{D}_s} = (\tilde{\mathfrak{p}}_{t_{-3}})^{-1}$, we have (5.46). Moreover, it clarifies the meaning of the consistency condition (5.46); namely, it guarantees the detropicalization of $x$-variables.

(b). Type $B_2$. Let
\[
\delta_1 = 1, \delta_2 = 2; \hat{y}_1 = x_2^2, \hat{y}_2 = x_1^{-1}.
\] (5.55)

The situation is the same as type $A_2$, and the outgoing walls are
\[
(\sigma((1, -2)), 1 + \hat{y}_1 \hat{y}_2)_{(1, 1)}, (\sigma((1, -1)), 1 + \hat{y}_1 \hat{y}_2^2)_{(1, 2)}.
\] (5.56)

(c). Type $G_2$. Let
\[
\delta_1 = 1, \delta_2 = 3; \hat{y}_1 = x_2^3, \hat{y}_2 = x_1^{-1}.
\] (5.57)

The situation is the same as type $A_2$, and the outgoing walls are
\[
(\sigma((1, -3)), 1 + \hat{y}_1 \hat{y}_2)_{(1, 1)}, (\sigma((1, -2)), 1 + \hat{y}_1 \hat{y}_2^3)_{(2, 3)},
\]
\[
(\sigma((2, -3)), 1 + \hat{y}_1 \hat{y}_2^2)_{(1, 2)}, (\sigma((1, -1)), 1 + \hat{y}_1 \hat{y}_2^3)_{(1, 3)}.
\] (5.58)

(II). Infinite type. The consistent scattering diagram $\mathcal{D}_s$ is an extension of $\mathcal{D}(\Sigma_{s, t_0})$ such that the supports of additional walls are in the complement of $|\Delta(G_{t_0})|$. In particular, $\mathcal{D}(\Sigma_{s, t_0})$ is a scattering diagram but it is not consistent.

(d). Type $A_1^{(1)}$. Let
\[
\delta_1 = \delta_2 = 2; \hat{y}_1 = x_2^2, \hat{y}_2 = x_1^{-2}.
\] (5.59)

There is one additional wall to $\mathcal{D}(\Sigma_{s, t_0})$ [GHKK18, Example 1.15], which is
\[
\left(\sigma((1, -1)), \left(\sum_{k=0}^{\infty} \hat{y}_1^k \hat{y}_2^k\right)^2\right)_{(1, 1)}.
\] (5.60)

(e). Type $A_2^{(2)}$. Let
\[
\delta_1 = 1, \delta_2 = 4; \hat{y}_1 = x_2^4, \hat{y}_2 = x_1^{-1}.
\] (5.61)

There is one additional wall to $\mathcal{D}(\Sigma_{s, t_0})$ [Rea20a, Theorem 3.4.], which is
\[
\left(\sigma((1, -2)), (1 + \hat{y}_1 \hat{y}_2)\left(\sum_{k=0}^{\infty} \hat{y}_1^k \hat{y}_2^{2k}\right)^2\right)_{(1, 2)}.
\] (5.62)

(f). Non-affine type. There are infinitely many additional walls inside the irrational cone spanned by $\bfv$ and $\bfv'$ in (2.68). The region is informally called the Badlands. We may also call it the dark side from the viewpoint of cluster patterns. The explicit description of walls therein is not yet known. However, for the skew-symmetric case $\delta_1 = \delta_2$, the existence of walls for every rational slope was proved in [DM21, Example 7.10].
6 Scattering diagrams and \( F \)-polynomials

In this section we show how \( F \)-polynomials and \( x \)-variables are related with the consistent scattering diagram \( D_s \). The Laurent positivity follows from the positivity of theta functions in \([GHKK18]\).

6.1 Mutations of scattering diagrams

We continue to assume that a fixed data \( \Gamma \) satisfies Injectivity Assumption throughout the section.

The following definition originates in \([FG09, \S2.1]\).

\textbf{Definition 6.1 (Seed mutation).} Let \( s = (e_1, \ldots, e_k) \) be any seed for a fixed data \( \Gamma \). For any \( k = 1, \ldots, n \), we define a new seed \( \mu_k(s) = s' = (e'_1, \ldots, e'_n) \) by

\[
e'_i = \begin{cases} -e_k & i = k, \\ e_i + [b_{ki}]e_k & i \neq k, \end{cases}
\]

\( b_{ij} = \{\delta_i e_i, e_j\} \),

(6.1)

which is called the \textit{mutation} of \( s \) in direction \( k \).

The transformation (6.1) is written by a matrix form \( P \) in Lemma 1.10. Thus, \( e'_1, \ldots, e'_n \) are certainly a basis of \( N \). Also, under the mutation (6.1), the basis \( \delta_1 e_1, \ldots, \delta_n e_n \) of \( N^\circ \) mutates as

\[
\delta_i e'_i = \begin{cases} -\delta_k e_k & i = k, \\ \delta_i e_i + [-b_{ik}]e_k & i \neq k, \end{cases}
\]

(6.2)

where we used the skew-symmetry \( b_{ki}\delta_i = -b_{ik}\delta_k \). Thus, \( \delta_1 e'_1, \ldots, \delta_n e'_n \) are a basis of \( N^\circ \) by the same reason as above. Therefore, \( s' \) is certainly a seed in the sense of Definition 5.3.

On can easily confirm that the mutation (6.2) induces the following mutation for the basis \( f_1, \ldots, f_n \) of \( M^\circ \), which are dual to \( \delta_1 e_1, \ldots, \delta_n e_n \):

\[
f'_i = \begin{cases} -f_k + \sum_{j=1}^n [-b_{jk}]f_j & i = k, \\ f_i & i \neq k. \end{cases}
\]

(6.3)

Also, we define the matrix \( B' = (b'_{ij}) \), \( b'_{ij} = \{\delta_i e'_i, e'_j\} \) as in (5.7). Then, \( B' \) coincides with the matrix mutation of \( B \) in direction \( k \) by (1.12) with \( \varepsilon = 1 \).

\textbf{Remark 6.2.} The transformation (6.3) already appeared in (2.78) under the identification \( M_\mathbb{R} \simeq_s \mathbb{R}^n \).

\textbf{Remark 6.3.} For \( s' = \mu_k(s) \), \( \mu_k(s') = s \) does not hold, because \( b'_{ki} = -b_{ki} \).

Namely, \( \mu_k \) is \textit{not} an involution on the set of all seeds for a fixed data \( \Gamma \). This causes a technical problem, for example, in the proof of the forthcoming
Theorem 6.13. To remedy the situation, we also define another mutation of a seed $s' = \mu_k^- (s)$ by

$$e'_i = \begin{cases} -e_k & i = k, \\ e_i + [-b_{ki}] + e_k & i \neq k. \end{cases}$$  \hspace{1cm} (6.4)$$

Again, we define the matrix $B' = (b'_{ij}), b'_{ij} = \{\delta_i e'_i, e'_j\}$. Then, $B'$ coincides with the matrix mutation of $B$ in direction $k$ by (1.12) with $\varepsilon = -1$.

Moreover, we have

$$\mu_k^- (\mu_k^+ (s)) = \mu_k^+ (\mu_k^- (s)) = s,$$  \hspace{1cm} (6.5)$$

where $\mu_k^\pm = \mu_k$. We call $\mu_k^\pm$ the \textit{signed mutations} of $s$. They appeared, for example, in [IN14].

For any seed $s = (e_1, \ldots, e_n)$ and $k = 1, \ldots, n$, we define a linear map

$$S_{k:s} : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}$$

$$m \mapsto m + \langle \delta_k e_k, m \rangle p^*(e_k).$$ \hspace{1cm} (6.6)$$

and its dual

$$S_{k:s}^* : N \rightarrow N$$

$$n \mapsto n + \{\delta_k e_k, n\} e_k.$$ \hspace{1cm} (6.7)$$

The following equality can be easily verified:

$$\langle S_{k:s}^* (n), S_{k:s} (x) \rangle = \langle n, x \rangle.$$ \hspace{1cm} (6.8)$$

Let

$$\mathcal{H}_{k,+} = \{m \in M_{\mathbb{R}} \mid \langle e_k, m \rangle \geq 0\}, \quad \mathcal{H}_{k,-} = \{m \in M_{\mathbb{R}} \mid \langle e_k, m \rangle \leq 0\}.$$ \hspace{1cm} (6.9)$$

Then, following [GHKK18], we define a piecewise-linear map

$$T_{k:s} : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}$$

$$m \mapsto \begin{cases} S_{k:s} (m) & m \in \mathcal{H}_{k,+}, \\ m & m \in \mathcal{H}_{k,-}. \end{cases}$$ \hspace{1cm} (6.10)$$

Remark 6.4. Under the identification $M_{\mathbb{R}} \cong_s \mathbb{R}^n$, it coincides with the map $T_{k:t_0}$ in (2.81).

Definition 6.5 (Mutation of $\mathcal{D}_s$). For the consistent scattering diagram $\mathcal{D}_s$ in Theorem 5.27 with minimal support, we define a collection of walls $T_k (\mathcal{D}_s)$, called the \textit{mutation of $\mathcal{D}_s$ in direction $k$}, by collecting all walls obtained below:

(a). For each wall $(\partial, f)_n$ of $\mathcal{D}_s$ other than $(e_k^+, 1 + \hat{y}_k)e_k$, let $\partial_+ = \partial \cap \mathcal{H}_{k,+}$, and $\partial_- = \partial \cap \mathcal{H}_{k,-}$. We split the wall $(\partial, f)_n$ into two walls

$$(\partial_+, f)_n, \quad (\partial_-, f)_n.$$ \hspace{1cm} (6.11)$$
where, if one of $\mathfrak{d}_+$ and $\mathfrak{d}_-$ is codimension more than 1, we throw that part away. Then, we replace $(\mathfrak{d}_+, f)_n$ with a new wall $(S_{k; s}(\mathfrak{d}_+), S_{k; s}(f)) S_{k; s}(n)$, where $S_{k; s}(f)$ is the formal power series obtained from $f$ by replacing each term $x^m$ ($m \in P$) with $x^{S_{k; s}(m)}$, while we leave $(\mathfrak{d}_-, f)_n$ as it is.

(b) Replace $(e_k \perp, 1 + \hat{y}_k e_k)$ with $(e'_k \perp, 1 + \hat{y}'_k e'_k)$, where $e'_k \perp = e_k \perp$ and $\hat{y}'_k = \hat{y}_k - 1$ by (6.1).

To summarize, in both (a) and (b), each support of a wall (after splitting) is transformed by the piecewise-linear map $T_{k; s}$. On the other hand, each wall function in (a) obeys the same linear map applied to its support, while the case (b) is exceptional.

**Remark 6.6.** Here we encounter a temporal problem that the walls above are not walls for the initial seed $s$ by the following reasons:

- The normal vectors of walls $T_{k; s}(\mathfrak{d})$ may not belong to $N^+_s = N_s^+$ in general, because the coefficients of $e_k$ might be negative.
- Accordingly, the wall functions $T_{k; s}(f)$ may not belong to $\mathbb{k}[P]$ in general. In particular, the wall function $1 + \hat{y}'_k = 1 + \hat{y}_k^{-1}$ does not belong to $\mathbb{k}[P]$.

These problems will be harmonically solved in the next theorem.

Here is the second fundamental theorem on scattering diagrams in view of the application to cluster algebra theory.

**Theorem 6.7 ([GHKK18 Theorem 1.24]).** The collection of walls $T_k(\mathfrak{D}_s)$ is a scattering diagram for the seed $s' = \mu_k(s)$. Moreover, $T_k(\mathfrak{D}_s)$ is equivalent to the consistent scattering diagram $\mathfrak{D}_{s'}$.

Let us explain how Proposition 3.2 follows from this theorem.

We do the identification $M_\mathbb{R} \simeq_\mathbb{R} \mathbb{R}^n$ for $\mathfrak{D}_s$, and $M_\mathbb{R} \simeq_\mathbb{R} \mathbb{R}^n$ for $\mathfrak{D}_{s'}$, respectively. Under the change of a basis of $M_\mathbb{R}$ in (6.3), two identifications are related by the isomorphism $\eta^{t_1}_{t_0}$ in (2.80). All relevant maps are summarized in the following commutative diagram, where $\phi^{t_1}_{t_0} = \eta^{t_1}_{t_0} \circ T_{k; t_0}$ as in Proposition 2.26:

\[
\begin{array}{ccc}
M_\mathbb{R} & \xrightarrow{T_{k; s}} & M_\mathbb{R} \\
\mathbb{R}^n & \xrightarrow{T_{k; t_0}} & \mathbb{R}^n \\
\downarrow \simeq_\mathbb{R} & & \downarrow \simeq_\mathbb{R} \\
\mathbb{R}^n & \xrightarrow{\eta^{t_1}_{t_0}} & \mathbb{R}^n.
\end{array}
\]

Then, for the support $\mathfrak{d}$ of each wall of $\mathfrak{D}_s$, the map $\mathfrak{d} \mapsto T_{k; s}(\mathfrak{d})$ in $M_\mathbb{R}$ is identified with the map $\mathfrak{d} \mapsto \phi^{t_1}_{t_0}(\mathfrak{d})$ in $\mathbb{R}^n$. Thus, by forgetting the wall functions, we obtain Proposition 3.2.
6.2 Relation between $\mathcal{D}(\Sigma_{s,t_0})$ and $\mathcal{D}_s$

So far, what we know about the relation between $\mathcal{D}(\Sigma_{s,t_0})$ and $\mathcal{D}_s$ are the rank 2 examples and the relations of the supports in Theorem 3.3. Below we show that the walls of $\mathcal{D}(\Sigma_{s,t_0})$ are integrated into a part of the consistent scattering diagram $\mathcal{D}_s$ in an essential way. To do that, we apply a parallel construction of the mutation of $\mathcal{D}_s$ to $\mathcal{D}(\Sigma_{s,t_0})$.

Throughout this subsection, we omit the normal vector $n \in N^+$ of a wall $(d, f)_n$ for simplicity, because it is recovered from $d$ or $f$ without ambiguity.

For $t_0 \in T_n$ and $k = 1, \ldots, n$, we define a linear map

$$S_{k;t_0} : \mathbb{R}^n \to \mathbb{R}^n \quad \mathbf{v} \mapsto (I + (B_{t_0})^k)\mathbf{v},$$

which correspond to the linear map $S_{k;s}$ in (6.6) under the identification $M_R \simeq_s \mathbb{R}^n$. Let $\mathbb{R}^n_{k,+}$ and $\mathbb{R}^n_{k,-}$ be the ones in (2.70). Then, the piecewise-linear map (2.81), which corresponds to (6.10), is written as

$$T_{k;t_0} : \mathbb{R}^n \to \mathbb{R}^n \quad \mathbf{v} \mapsto \begin{cases} S_{k;t_0}(\mathbf{v}) & \mathbf{v} \in \mathbb{R}^n_{k,+}, \\ \mathbf{v} & \mathbf{v} \in \mathbb{R}^n_{k,-}. \end{cases}$$

Let us explicitly denote the dependence of $\hat{y}$-variables to the initial vertex $t_0$ as

$$\hat{y}_{t_0}^\mathbf{v} := x^{B_{t_0}}. \quad (6.15)$$

**Definition 6.8 (Mutation of $\mathcal{D}(\Sigma_{s,t_0})$).** For the collection of walls $\mathcal{D}(\Sigma_{s,t_0})$, let $T_k(\mathcal{D}(\Sigma_{s,t_0}))$ be the collection of walls $T_k(d, f)$ obtained from each wall $(d, f)$ of $\mathcal{D}(\Sigma_{s,t_0})$ as below.

(a). For each wall $(d, f) = (\sigma_i(G^t_{i,t}), 1 + \hat{y}_{t_0}^{c_{i,t}^{t_0} + })$ such that $c_{i,t}^{t_0} \neq e_k$, we define

$$T_k(d, f) = \begin{cases} (S_{k;t_0}(d), S_{k;t_0}(f)) & d \subset \mathbb{R}^n_{k,+}, \\ (d, f) & d \subset \mathbb{R}^n_{k,-}. \end{cases} \quad (6.16)$$

where

$$S_{k;t_0}(1 + \hat{y}_{t_0}^{c_{i,t}^{t_0} + }) = S_{k;t_0}(1 + x^{c_{i,t}^{t_0} + }) := 1 + x^{S_{k;t_0}(c_{i,t}^{t_0} + )}. \quad (6.17)$$

Note that by Proposition 2.20 either $d \subset \mathbb{R}^n_{k,+}$ or $d \subset \mathbb{R}^n_{k,-}$ occurs. Thus, we do not have to divide a wall.

(b). For each wall $(d, f) = (\sigma_i(G^t_{i,t}), 1 + \hat{y}_{t_0}^{e_k})$ we define

$$T_k(d, f) = (\sigma_i(G^t_{i,t}), 1 + \hat{y}_{t_0}^{e_k}). \quad (6.18)$$
To summarize, in both (a) and (b) each support \( \mathfrak{d} \) is transformed by the piecewise-linear map \( T_{k;t_0} \). On the other hand, each wall function \( f \) in (a) obeys the same linear map applied to its support, while the case (b) is exceptional.

**Remark 6.9.** The same remark as Remark 6.6 is applied to walls in the above.

Let \( t_1 \in \mathbb{T}_n \) be the vertex that is \( k \)-adjacent to the initial vertex \( t_0 \). Let \( \eta^{t_1}_{t_0} \) be the linear map in (2.80). For any wall \((\mathfrak{d}, f) \in T_k(\mathfrak{D}(\Sigma_{s,t_0}))\), we define a new wall by

\[
\eta^{t_1}_{t_0}(\mathfrak{d}, f) = 1 + x^\epsilon : = (\eta^{t_1}_{t_0}(\mathfrak{d}), 1 + x^{\eta^{t_1}_{t_0}(f)}). \quad (6.19)
\]

We have a parallel result to Theorem 6.7. Also, this is an upgrade of Proposition 6.10 including wall functions.

**Proposition 6.10.** The walls \( \eta^{t_1}_{t_0}(\mathfrak{d}, f) \) in (6.19) are walls of \( \mathfrak{D}(\Sigma'_{s',t_1}) \) with \( s' = \mu_k(s) \). Moreover, it induces the following bijections of walls:

\[
\begin{align*}
\mathfrak{D}(\Sigma_{s,t_0}) & \xrightarrow{T_k} T_k(\mathfrak{D}(\Sigma_{s,t_0})) \xrightarrow{\eta^{t_1}_{t_0}} \mathfrak{D}(\Sigma'_{s',t_1}) \\
(\sigma_i(G^0_t), 1 + y c^{i+0}_{t_0}) & \mapsto (\sigma_i(G^0_t), 1 + y c^{i+0}_{t_1}) \mapsto (\sigma_i(G^0_t), 1 + y c^{i+1}_{t_1}), \quad (6.20)
\end{align*}
\]

where

\[
y c^{i+0}_{t_0} = x c^{i+1}_{t_0}, \quad y c^{i+1}_{t_1} = x c^{i+1}_{t_1}. \quad (6.21)
\]

**Proof.** We only need to show the correspondence by \( \eta^{t_1}_{t_0} \) in (6.20). First we prove the claim for the supports. Under the map \( \eta^{t_1}_{t_0} \circ T_k \), the support \( \sigma_i(G^0_t) \) maps to \( \eta^{t_1}_{t_0}(T_k; t_0(\sigma_i(G^0_t))) \). By Propositions 2.22 and 2.26, it equals to \( \sigma_i(G^0_t) \).

Next we prove the claim for the wall functions. Let us consider the case (a) in Definition 6.8 where \( c^{i+0}_{t_0} \neq \pm e_k \) is assumed. Under the map \( \eta^{t_1}_{t_0} \circ T_k \), the \( \hat{c}^+ \)-vector \( c^{i+0}_{t_0} \) obeys the same linear transformations as its support \( \sigma_i(G^0_t) \). Therefore, by comparing (2.20) and (2.37), its image equals to \( \epsilon^{i+0}_{t_0} c^{i+1}_{t_0} \) \( c^{i+1}_{t_1} \). Next we need to compare \( \epsilon^{i+0}_{t_0} c^{i+1}_{t_0} \) and \( \epsilon^{i+1}_{t_1} \). Here we repeat a similar argument after (2.27) in the proof of Proposition 2.7. By (2.19), \( C^0_t \) and \( C^1_t \) differ only in their \( k \)-th row. Since we assume \( c^{i+0}_{t_0} \neq \pm e_k \), there is at least one nonzero element in the \( i \)-th column of \( C^0_t \) other than at the \( k \)-th row. Thus, we have \( \epsilon^{i+0}_{t_0} = \epsilon^{i+1}_{t_1} \), and we conclude that \( c^{i+0}_{t_0} \) maps to \( c^{i+1}_{t_1} \) as desired. Now we consider the case (b) in Definition 6.8 where \( c^{i+0}_{t_0} = \pm e_k \) is assumed. It is enough to prove the equality

\[
\eta^{t_1}_{t_0}(-c^{i+0}_{t_0}) = c^{i+1}_{t_1}. \quad (6.22)
\]
Again by (2.19), we have \( c_{i;t}^t = \mp e_k \). Therefore, \( c_{i;t}^{t_0} = c_{i;t}^t = e_k \). Thus, the equality (6.22) is equivalent to

\[
\eta_{t_0}^t(-b_{k;t_0}) = b_{k;t_1},
\]

which can be proved as

\[
\eta_{t_0}^t(-b_{k;t_0}) = -(J_k + [-B_{t_0}]^*) b_{k;t_0} = -b_{k;t_0} = b_{k;t_1}.
\]

(This proof also explains the necessity of the exception in the case (b) in view of the exceptional change of tropical signs \( \varepsilon_{i;t}^{t_0} = -\varepsilon_{i;t}^t \) therein.)

\[\square\]

**Example 6.11** (cf. Example 2.24). Let us clarify Definition 6.8 and Proposition 6.10 explicitly for type \( A_2 \) based on the convention in Section 2.5. The relevant data are as follows:

\[
B_{t_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_{t_1} = B_{t_{-1}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (6.25)
\]

\[
\hat{y}_{1;t_0} = x_2, \quad \hat{y}_{2;t_0} = x_1, \quad \hat{y}_{1;t_1} = \hat{y}_{1;t_{-1}} = x_2, \quad \hat{y}_{2;t_1} = \hat{y}_{2;t_{-1}} = x_1, \quad (6.26)
\]

\[
S_{1;t_0} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S_{2;t_0} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (6.27)
\]

\[
\eta_{t_0}^t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta_{t_{-1}} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \quad (6.28)
\]

The results are depicted in Figure 4, where \( G_{t_0}^t = T_{k;t_0}(G_{t_0}^t) \) therein. See also Figure 2.

To describe the relation between \( \mathcal{D}(\Sigma_{s,t_0}) \) and \( \mathcal{D}_s \), we introduce the following notion.

**Definition 6.12.** For any wall \((\mathfrak{d}, f)\) and a scattering diagram \( \mathcal{D} \) for a common seed \( s \), we say that \((\mathfrak{d}, f)\) is **essential to** \( \mathcal{D} \) if there is a scattering diagram \( \mathcal{D}' \) that is equivalent to \( \mathcal{D} \) satisfying the following condition:

- The wall \((\mathfrak{d}, f)\) is a wall of \( \mathcal{D}' \); moreover, for any wall \((\mathfrak{d}', f')\) of \( \mathcal{D}' \) other than \((\mathfrak{d}, f)\), it holds that \( \dim \mathfrak{d} \cap \mathfrak{d}' < n - 1 \).

Combining Theorem 6.7 and Proposition 6.10, we obtain the following relation between \( \mathcal{D}(\Sigma_{s,t_0}) \) and \( \mathcal{D}_s \). (The claim (b) was already given in Theorem 3.3 (a), but it is included here again for the summary.)

**Theorem 6.13.** Under the identification \( M_\mathcal{R} \simeq_\mathcal{R} \mathbb{R}^n \), the following facts hold:

(a). Every wall \((\sigma_i(G_{t_0}^t), 1 + \hat{y}_{t_0}^{t_0})\) of \( \mathcal{D}(\Sigma_{s,t_0}) \) is essential to \( \mathcal{D}_s \).

(b). For any \( t \in \mathbb{T}_n \), the set \( \sigma^\circ(G_{t_0}^t), \) which is a chamber of \( \text{Supp}(\mathcal{D}(\Sigma_{s,t_0})) \), is also a chamber of \( \text{Supp}(\mathcal{D}_s) \).
Claim 6.14. Let $t$ be as above. Then, for any $r = 0, \ldots, d$, each wall $(\sigma_i(G^d_t), 1 + \hat{y}_{t_r}^{e_{i,r}^+})$ of $\mathcal{D}(\Sigma_{s,r,t})$ is essential to $\mathcal{D}_{s_r}$.

For each $i = 1, \ldots, n$, we prove the claim by the finite induction on $r$ from $d$ to $0$.

First, consider the case $r = d$, namely, $t_d = t$. We split the support of the wall $(e_i^+, 1 + \hat{y}_{t_d}^{e_i})$ of $\mathcal{D}_{s_d}$ in Theorem 5.27 into the orthants

$$e_i^+ = \bigcup_{\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+1}, \ldots, \kappa_n = \pm 1} \sigma(\kappa_1 e_1, \ldots, \kappa_{i-1} e_{i-1}, \kappa_{i+1} e_{i+1}, \ldots, \kappa_n e_n),$$  \hspace{1cm} (6.29)

so that the one with $\kappa_1 = \cdots = \kappa_n = 1$ is $\sigma_i(G^d_t) = \sigma_i(I)$. By Theorem 5.27, there is no other wall of $\mathcal{D}_{s_d}$ whose normal vector is $e_i$. Thus, $(\sigma_i(G^d_t), 1 + \hat{y}_{t_d}^{e_i})$ is essential to $\mathcal{D}_{s_d}$.
Next, suppose that the claim is true for some \( r \). Namely, for the wall \((\mathfrak{d}, f) := (\sigma_i(G_t^r), 1 + \hat{y}_{t_{r-1}})\) of \( \mathcal{D}(\Sigma_{s_r, t_r}) \), there is a scattering diagram \( \mathcal{D}' \) that is equivalent to \( \mathcal{D}_{s_r} \) such that the following holds:

- Under the identification \( M_{\mathbb{R}} \cong_{s_r} \mathbb{R}^n \), the wall \((\mathfrak{d}, f)\) is a wall of \( \mathcal{D}' \); moreover, for any wall \((\mathfrak{d}', f')\) of \( \mathcal{D}' \) other than \((\mathfrak{d}, f)\), it holds that \( \dim \mathfrak{d} \cap \mathfrak{d}' < n - 1 \).

We apply the same construction of \( T_{k_r}(\mathcal{D}_{s_r}) \) to \( \mathcal{D}' \) and obtain \( T_{k_r}(\mathcal{D}') \). Since the construction of \( T_{k_r}(\mathcal{D}') \) is compatible with that of \( T_{k_r}(\mathcal{D}(\Sigma_{s_r, t_r})) \), we have the property:

- Under the identification \( M_{\mathbb{R}} \cong_{s_r} \mathbb{R}^n \), the wall \( T_{k_r}(\mathfrak{d}, f) \) is a wall of \( T_{k_r}(\mathcal{D}') \); moreover, for any wall \((\mathfrak{d}', f')\) of \( T_{k_r}(\mathcal{D}') \) other than \( T_{k_r}(\mathfrak{d}, f) \), it holds that \( \dim T_{k_r}(\mathfrak{d}) \cap \mathfrak{d}' < n - 1 \).

By construction, \( T_{k_r}(\mathcal{D}') \) is equivalent to \( T_{k_r}(\mathcal{D}_{s_r}) \). Also, by Theorem 6.7, \( T_{k_r}(\mathcal{D}_{s_r}) \) is equivalent to \( \mathcal{D}_{s_{r-1}} \), where we recall \( \mu_{k_r}(s_r) = s_{r-1} \). Thus, \( T_{k_r}(\mathcal{D}') \) is also equivalent to \( \mathcal{D}_{s_{r-1}} \). Therefore, the wall \( T_{k_r}(\mathfrak{d}, f) \) is essential to \( \mathcal{D}_{s_{r-1}} \). Now, we change the identification from \( M_{\mathbb{R}} \cong_{s_r} \mathbb{R}^n \) to \( M_{\mathbb{R}} \cong_{s_{r-1}} \mathbb{R}^n \) by the linear isomorphism \( \eta_{t_{r-1}} \). Then, by Proposition 6.10, the wall \( T_{k_r}(\mathfrak{d}, f) \) is identified with \((\sigma_i(G_t^{r-1}), 1 + \hat{y}_{t_{r-1}})\). Therefore, we conclude that the wall \((\sigma_i(G_t^{r-1}), 1 + \hat{y}_{t_{r-1}})\) is essential to \( \mathcal{D}_{s_{r-1}} \). This completes the proof of the claim.

**Remark 6.15.** The following remark is due to Nathan Reading. Theorem 6.13 does not mean that \( \mathcal{D}(\Sigma_{s,t_0}) \) satisfies the finiteness condition. For example, consider the initial exchange matrix

\[
B_{t_0} = \begin{pmatrix}
0 & -1 & -1 \\
1 & 0 & -1 \\
1 & 1 & 0
\end{pmatrix}
\]  

(6.30)

(This matrix is singular, but it is irrelevant to the point here.) Then, there are infinitely many distinct faces \( \sigma_i(G_t) \) sharing the common normal vector \((0, 1, 0)\), thus, also with the common wall function. Therefore, in this case \( \mathcal{D}(\Sigma_{s,t_0}) \) is not a scattering diagram in the sense of Definition 5.18. However, Theorem 6.13 guarantees that one can combine them into finitely many walls to satisfy the finiteness condition. After this, \( \mathcal{D}(\Sigma_{s,t_0}) \) becomes a scattering diagram.

### 6.3 F-polynomials and wall-crossing automorphisms

Based on Theorem 6.13, now we can identify \( F \)-polynomials with wall-crossing automorphisms of \( \mathcal{D}_s \).
So far, a wall-crossing automorphism $p_{\gamma,D}$ is defined to act on $k[[P]]$ by (5.18) as a ring automorphism. Now we consider a $k[[P]]$-module $x_m^\gamma$ for any $m \in M^0$ such that the action $p_{\gamma,D}(x_m^\gamma)$ is defined by the same formula (5.18).

Let us recall the situation in Example 5.24. Consider a sequence of vertices $t_0, t_1, \ldots, t_{r+1} = t \in T_n$ such that they are sequentially adjacent with edges labeled by $k_0, \ldots, k_r$. Let $\gamma^0_t$ be an admissible curve therein. Namely, $\gamma^0_t$ starts in $\sigma(G_t)$, passing through $G$-cones in $\mathbb{R}^n$, and it ends in $\sigma^0(G_{t_0})$. Under the identification $M_\mathbb{R} \simeq s_\mathbb{R}^n$, we define

$$p^0_t := p_{\gamma^0_t,D_s}. \tag{6.31}$$

The following theorem clarifies the relation between $F$-polynomials and wall-crossing automorphisms in the scattering diagram $D_s$.

**Theorem 6.16** ([Rea20b, Theorem 5.6], [Rea20a, Theorem 2.9]). Under the identification $M_\mathbb{R} \simeq s_\mathbb{R}^n$, the following facts hold.

(a). The wall-crossing automorphism $p^0_t$ in (6.31) for $D_s$ coincides with the wall-crossing automorphism $\tilde{p}^0_t$ in (5.36) for $D(\Sigma_{s,t_0})$.

(b). The following formula holds in $x^{\mathbb{R}^n} k[[P]]$:

$$p^0_t (x^{\mathbb{R}^n}) = x^{\mathbb{R}^n} F_{t,t} (\hat{y}). \tag{6.32}$$

**Proof.** (a). By Theorems 6.13, one may assume that, up to the equivalence, the walls of $\mathcal{D}_s$ passed by the above curve $\gamma^0_t$ all belong to $\mathcal{D}(\Sigma_{s,t_0})$. Thus, we have

$$\tilde{p}^0_t = p_{\gamma^0_t,D(\Sigma_{s,t_0})} = p_{\gamma^0_t,D_s} = p^0_t. \tag{6.33}$$

(b). Recall that $\tilde{p}^0_t = q^0_t$ by (5.36). Therefore, by (4.45), we have the equality. \qed

**Remark 6.17.** By Injectivity Assumption, the initial exchange matrix $B$ is nonsingular. Thus, $\hat{y}_1, \ldots, \hat{y}_n$ are algebraically independent. Therefore, (6.32) uniquely determines $F$-polynomials $F_{t,t}(y)$ themselves.

### 6.4 Theta functions and Laurent positivity

Let us present the outline of the proof of the Laurent positivity by [GHKK18].

The following object was introduced in [GHKK18] to provide a combinatorial description of wall-crossing automorphisms.

**Definition 6.18** (Broken line). Let $\mathcal{D}$ be any scattering diagram with respect to a given initial seed $s$. Let $m_0 \in M^0 \setminus \{0\}$, and let $Q \in M_\mathbb{R} \setminus \text{Supp}(\mathcal{D})$. A broken line for $m_0$ with endpoint $Q$ is a piecewise-linear curve $\gamma: (-\infty, 0] \to M_\mathbb{R} \setminus \text{Sing}(\mathcal{D})$ satisfying the following properties:
(1) The endpoint \( \gamma(0) \) is \( Q \).

(2) There are numbers \(-\infty = t_0 < t_1 < t_2 < \cdots < t_{r+1} = 0 \) \((r \geq 0)\) such that \( \gamma \) is linear in each interval \( I_j = (t_j, t_{j+1}) \), while it breaks at each \( t_j \) for \( j = 1, \ldots, r \). Moreover, \( \gamma \) passes some walls of \( \mathfrak{D} \) at each break point \( t_j \). (It is possible that \( \gamma \) passes some walls in an interval \( I_j \) without breaking.)

(3) To each \( I_j \), a monomial \( c_j x^{m_j} \in \mathbb{K}[M^\circ] \) is attached. In particular, to \( I_0 \), a monomial \( x^{m_0} \) is attached, where \( m_0 \) is the given data.

(4) The velocity \( \gamma' \) is \(-m_j \) in the interval \( I_j \).

(5) Let \( \gamma_j \) be a segment of \( \gamma \) for the interval \( (t_j - \delta, t_j + \delta) \) with sufficiently small \( \delta > 0 \) such that \( \gamma_j \) passes walls only at \( t_j \). Then, \( c_j x^{m_j} \) is a monomial in \( p_{\gamma_j, \mathfrak{D}}(c_{j-1} x^{m_{j-1}}) \).

Finally, we define

\[
\text{Mono}(\gamma) := c_r x^{m_r}. \tag{6.34}
\]

Below we implicitly assume that \( Q \) is in a general position so that it has the maximal set of broken lines in the neighborhood of \( Q \). Namely, no broken line for \( Q' \) near \( Q \) drops out accidentally by intersecting \( \text{Sing}(\mathfrak{D}) \) when \( Q' \) approaches to \( Q \).

**Remark 6.19.** At \( t_j \), the velocity changes from \(-m_j \) to \(-m_{j+1} = -m_j - k_j p^*(n_j) \), where \( n_j \) is the common normal vector of the walls at \( t_j \), and \( k_j \) is some positive integer. Since \( p^*(n_j) \in n_j^\perp \), the broken line always crosses walls, not reflects nor stops at them as stated in Condition (2).

**Remark 6.20** ([GHKK18, Remark 3.2]). In Condition (5) of Definition 6.18 suppose that \( \gamma_j \) crosses possibly multiple walls \((\mathfrak{D}_\lambda, f_\lambda)_{n_j} \) \((\lambda \in \Lambda_j) \) at \( t_j \). Since \( \gamma \) does not intersect \( \text{Sing}(\mathfrak{D}) \), these walls share the common normal vector \( n_j \in N^\perp_{\text{pr}} \). Note that \( \gamma' = -m_{j-1} \) just before \( \gamma \) crosses these walls. Then, the intersection signs in (5.32) are given by

\[
\epsilon_j = \begin{cases} 
1 & \langle n_j, m_{j-1} \rangle > 0, \\
-1 & \langle n_j, m_{j-1} \rangle < 0. 
\end{cases} \tag{6.35}
\]

Thus, \( \epsilon_j \langle n_j, m_{j-1} \rangle = |\langle n_j, m_{j-1} \rangle| \). Therefore, by (5.18), (5.33), and (5.34), we have

\[
p_{\gamma_j, \mathfrak{D}}(c_{j-1} x^{m_{j-1}}) = c_{j-1} x^{m_{j-1}} \prod_{\lambda \in \Lambda_j} f^{|\langle \delta(n_j), n_{j-1}, m_{j-1} \rangle|}. \tag{6.36}
\]

It is crucial that there is no division in this expression for the forthcoming positivity of theta functions.
Definition 6.21 (Theta function). Under the same assumption and notations in Definition 6.18 the theta function $\vartheta_{Q,m_0}$ for $m_0$ with endpoint $Q$ is defined by

$$\vartheta_{Q,m_0} = \sum_{\gamma} \operatorname{Mono}(\gamma),$$

(6.37)

where the sum is over all broken lines for $m_0$ with endpoint $Q$. We also define

$$\vartheta_{Q,0} = 1.$$  

(6.38)

Let us quote the basic properties of theta functions from [GHKK18].

Proposition 6.22 ([GHKK18 Theorem 3.4]). For any scattering diagram $\mathcal{D}$, we have

$$\vartheta_{Q,m_0} \in x^{m_0} k[[P]].$$

(6.39)

Proposition 6.23 ([CPS Lemma 4.8], [GHKK18 Theorem 3.5]). Let $\mathcal{D}$ be a consistent scattering diagram. Let $m_0 \in M^o$ and $Q, Q' \in M_R \setminus \text{Supp}(\mathcal{D})$. Then, for any admissible curve $\gamma$ from $Q$ to $Q'$, we have

$$\vartheta_{Q',m_0} = p_{\gamma,\mathcal{D}}(\vartheta_{Q,m_0}).$$

(6.40)

Now let us specialize to the scattering diagram $\mathcal{D}_s$ in Theorem 5.27.

Proposition 6.24 ([GHKK18 Corollary 3.9]). Let $\mathcal{D}_s$ be the scattering diagram in Theorem 5.27. Let us identify $M_R \simeq s \mathbb{R}^n$. For any $G$-cone $\sigma(G_t)$, let $m_0 \in \sigma(G_t) \cap \mathbb{Z}^n$ and $Q \in \sigma^o(G_t)$. Then, we have

$$\vartheta_{Q,m_0} = x^{m_0}.$$  

(6.41)

From Propositions 6.23 and 6.24 we have the third fundamental theorem on scattering diagrams in view of the application to cluster algebra theory.

Theorem 6.25. Let $\mathcal{D}_s$ be the scattering diagram in Theorem 5.27. Let us identify $M_R \simeq s \mathbb{R}^n$. For any $G$-cone $\sigma(G_t)$, let $m_0 \in \sigma(G_t) \cap \mathbb{Z}^n$ and $Q \in \sigma^o(G_t)$. Then, for any admissible curve $\gamma$ from any point in $\sigma^o(G_t)$ to $Q$, we have

$$\vartheta_{Q,m_0} = p_{\gamma,\mathcal{D}}(x^{m_0}).$$

(6.42)

From Theorems 6.16 and 6.25 we have the following identification of $x$-variables in $\Sigma_{s,t_0}$ with theta functions.

Theorem 6.26 ([GHKK18 Theorem 4.9]). Let $\mathcal{D}_s$ be the scattering diagram in Theorem 5.27. Let us identify $M_R \simeq s \mathbb{R}^n$. Then, for any $t \in \mathbb{T}_n$ and $Q \in \sigma^o(G_{t_0})$, we have

$$\vartheta_{Q,s,t} = x^{s,t} F_t(\hat{y}) = x_{i;t}.$$  

(6.43)
6.5. Linear independence of cluster monomials

Proof. We set \( m_0 = g_{i,t} \) in (6.42). Then, comparing it with (6.32), we obtain the equality.

The following positivity result holds.

**Proposition 6.27** ([GHKK18 Theorem 1.13]). The scattering diagram \( \mathcal{D}_s \) in Theorem 5.27 is equivalent to a scattering diagram such that every wall function has a form

\[
 f = (1 + x^{p^{(n)}})^c 
\]

for some (not necessarily primitive) \( n \in \mathbb{N}^+ \) and a positive integer \( c \).

**Remark 6.28.** By Theorem 6.13, the complexity of wall functions with non-primitive \( n \) or \( c \neq 1 \) occurs only outside \( |\Delta(G)| \), i.e., in the dark side.

**Example 6.29.** The wall function in (5.60) can be split into the form (6.44) by

\[
 \left( \sum_{k=0}^{\infty} \hat{y}_1^k \hat{y}_2^k \right)^2 = \prod_{j=0}^{\infty} \left( 1 + \hat{y}_1^{2^j} \hat{y}_2^{2^j} \right)^2. 
\]

By the definition of broken lines, Remark 6.20, and Proposition 6.27, we have the fourth fundamental theorem on scattering diagrams in view of the application to cluster algebra theory.

**Theorem 6.30** ([GHKK18 Theorem 1.13 & Remark 3.2]). For the scattering diagram \( \mathcal{D}_s \) in Theorem 5.27, every theta function \( \vartheta_{Q,m_0} \in x^{m_0} \mathbb{k}[[P]] \) has only nonnegative integer coefficients.

Finally, we obtain the Laurent positivity in Theorem 1.37.

**Theorem 6.31** (Laurent positivity, [GHKK18 Theorem 4.10]). For any cluster patterns \( \Sigma \) and a given initial vertex \( t_0 \), every \( F \)-polynomial \( F_{i,t}(\mathbf{y}) \in \mathbb{Z}[\mathbf{y}] \) has no negative coefficients.

Proof. Recall that we have assumed Injectivity Assumption for \( \Gamma \). For a nonsingular \( B \)-pattern, it is a corollary of Theorems 6.30, 6.26, and Remark 6.17. The singular case reduces to the nonsingular case by Proposition 2.11.

6.5 Linear independence of cluster monomials

Let us present another important consequence of Theorem 6.26.

Following [FZ03b], we introduce the notion of cluster monomials.

**Definition 6.32** (Cluster monomials). For a given cluster pattern \( \Sigma \), a cluster monomial at \( t \in \mathbb{T}_n \) is a monomial (including 1) in the cluster variables \( x_{1,t}, \ldots, x_{n,t} \) at \( t \).
Two cluster variables taken from different \( t \) and \( t' \) in \( \mathbb{T}_n \) may coincide. For example, if \( t \) and \( t' \) are \( k \)-adjacent, a cluster monomial at \( t \) that does not contain \( x_{k; t} \) as a factor is also a cluster monomial at \( t' \). We regard them as the same cluster monomial. We are interested in the collection of all distinct cluster monomials taken from all \( t \in \mathbb{T}_n \).

The following property was conjectured by [FZ03b, Conj. 4.16], then partially proved by [CK08, DWZ10, GLS12, Pla11, CILF11, CIKLFP13] for various classes, and proved by [GHKK18] in full generality. See also Corollary 6.37.

**Theorem 6.33 ([GHKK18] Theorem 7.20).** For a given cluster pattern \( \Sigma \), all distinct cluster monomials are linearly independent over \( \mathbb{Z} \).

Below we present a proof of Theorem 6.33 following [GHKK18].

First, we express cluster monomials with theta functions in the same way as cluster variables given in Theorem 6.26. A cluster monomial at \( t \) has the following form

\[
x_t^a = \prod_{i=1}^{n} x_{i; t}^{a_i} \quad (a = (a_i) \in \mathbb{Z}_n^{\geq 0}).
\]  

(6.46)

We define the \( g \)-vector \( g_t^a \) of \( x_t^a \) by

\[
g_t^a = \sum_{i=1}^{n} a_i g_{i; t}.
\]  

(6.47)

By definition, this is in the \( G \)-cone \( \sigma(G_t) \). Under the situation of Theorem 6.26, we have

\[
\vartheta_{Q, g_t^a} = p_{\gamma, \mathcal{D}}(x_{g_t^a}) = p_{\gamma, \mathcal{D}} \left( \prod_{i=1}^{n} (x_{g_t^a})^{a_i} \right) = \prod_{i=1}^{n} p_{\gamma, \mathcal{D}}(x_{g_t^a})^{a_i} = x_t^a,
\]  

(6.48)

where we used (6.43) in the last equality.

By the Laurent phenomenon, we know that \( \vartheta_{Q, g_t^a} = x_t^a \) has a Laurent polynomial expression in \( x \). We say that a Laurent monomial in \( x \) is proper if it has a negative exponent for some \( x_i \).

The following lemma is a key to prove Theorem 6.33.

**Lemma 6.34 (In the proof of [GHKK18] Theorem 7.20).** In the Laurent polynomial expression of the theta function \( \vartheta_{Q, g_t^a} \) in (6.48), every Laurent monomial in \( x \) is proper if \( g_t^a \notin \sigma(G_{t_0}) \cap \mathbb{Z}_n = \mathbb{Z}_n^{\geq 0} \).

**Proof.** We may assume that \( g_t^a \neq 0 \). Suppose that \( \vartheta_{Q, g_t^a} \) contains a non-proper Laurent monomial \( M \) in \( x \). Then, there is a broken line \( \gamma \) for \( g_t^a \) with endpoint \( Q \) such that the final velocity \( \gamma'(0) \) does not have any positive
component. Since \( \gamma'(0) \) is not zero, it is a negative vector. Such a broken line has no break point; therefore, it is a ray ending at \( Q \) with constant velocity \(-g^a_t\). Thus, \( g^a_t \) is a positive vector. \( \square \)

Let us rephrase the above result in terms of cluster monomials.

**Lemma 6.35.** Let \( \Sigma \) be a given cluster pattern. In the Laurent polynomial expression of a cluster monomial \( x^a_t \) for \( \Sigma \) at \( t \), every Laurent monomial in \( x \) is proper if \( x^a_t \) does not coincide with any cluster monomial at \( t_0 \).

**Proof.** First, assume that the underlying \( B \)-pattern \( B \) is nonsingular. Suppose that \( x^a_t \) contains a non-proper Laurent monomial \( M \) in \( x \). Then, by (6.48) and Lemma 6.34, \( x^a_t = \vartheta Q, g^a_t \) and \( g^a_t \) belongs to \( \sigma(G_{t_0}) \). Thus, we can write \( g^a_t = \sum_{i=1}^n b_i e_i \) (\( b_i \in \mathbb{Z}_{\geq 0} \)). Then, we obtain

\[
x^a_t = \vartheta Q, g^a_t = p, \gamma, D(x^a_t) = p, \gamma, D(x^a_t) = x^a_t.
\]

Thus, \( x^a_t \) coincides with a cluster monomial at \( t_0 \).

Next, assume that \( B \) is singular. Consider the principal extension \( \overline{B} \). By Proposition 2.11 and the separation formula (1.46), \( x^a_t \) for \( \overline{B} \) is given by

\[
x^a_t = \prod_{i=1}^n \prod_{j=1}^n x_j^{g_{ji},i} F_{i:t}(\hat{y})^{a_i} \quad (i = 1, \ldots, n),
\]

where \( G_t \) and \( F_{i:t} \) are a \( G \)-matrix and an \( F \)-polynomial for \( B \). Thus, this expression formally coincides with the one for \( B \). However, in the formula, for \( \overline{B} \), we have

\[
\hat{y}_i = \sum_{j=1}^{2n} x_j^{b_{ji}} = x_{n+i} \sum_{j=1}^n x_j^{b_{ji}} \quad (i = 1, \ldots, n),
\]

and, for \( B \), we have \( \hat{y}_i = \sum_{j=1}^n x_j^{b_{ji}} \). To extract the result for \( B \), we note the following facts:

(a) Any cluster monomial \( x^a_t \) for \( B \) is obtained from the corresponding \( x^a_t \) for \( \overline{B} \) by the specialization \( x_{n+1} = \cdots = x_{2n} = 1 \).

(b) Since \( F \)-polynomials are polynomials in \( \hat{y} \), the exponents of \( x_{n+i} \) \( (i = 1, \ldots, n) \) are nonnegative for every Laurent monomial in the RHS of (6.50) for \( \overline{B} \). Therefore, all the above proper Laurent monomials are still proper under the specialization \( x_{n+1} = \cdots = x_{2n} = 1 \).

Now, suppose that \( x^a_t \) for \( B \) contains a non-proper Laurent monomial in \( x = (x_1, \ldots, x_n) \). Then, by (b), the corresponding \( x^a_t \) for \( \overline{B} \) also contains a non-proper Laurent monomial in \( \hat{x} = (x_1, \ldots, x_{2n}) \). Thus, by the first part of
the proof, \( x_t^a \) for \( \overline{B} \) coincides with a cluster monomial for \( \overline{B} \) at \( t_0 \). Therefore, by (a), \( x_t^a \) for \( B \) coincides with a cluster monomial for \( B \) at \( t_0 \). \( \square \)

The property presented in Lemma 6.35 is called the *proper Laurent monomial property* in \[CILF11\]. It was shown in \[CILF11, Theorem 6.4\] that the linear independence in Theorem 6.33 immediately follows from this property. For the reader’s convenience, we reproduce the proof here.

**Proof of Theorem 6.33.** Consider a \( \mathbb{Z} \)-linear relation among distinct cluster monomials. One can organize it in the following form:

\[
\sum_i c_i x_{t_i}^a = \sum_i c'_i x_{t_i}^{a'},
\]

(6.52)

where each cluster monomial \( x_{t_i}^a \) in the LHS does not coincide with any cluster monomial at \( t_0 \). Also, \( t_i = t_j \) may happen for \( i \neq j \). Then, by Lemma 6.35, the LHS is a sum of proper Laurent monomials. Thus, to have the equality, the both sides of (6.52) vanish. Then, it follows that \( c'_i = 0 \) for any \( i \) by the linear independence of the cluster monomials at \( t_0 \), which is obvious. One can repeat this argument by changing the initial vertex \( t_0 \) to other vertices in \( T_n \) and show that \( c_i = 0 \) for any \( i \).

**Remark 6.36.** Using the same idea, the linear independence of a wider class of theta functions including cluster monomials (called the *theta basis*) was proved in \[GHKK18, Theorem 7.20\].

One can immediately extend Theorem 6.33 to a cluster pattern with coefficients in the sense of \[FZ07\]. This was the original conjecture given by \[FZ07\].

**Corollary 6.37.** For any cluster pattern \( \Sigma \) with coefficients in a given semi-field \( \mathbb{P} \), all distinct cluster monomials are linearly independent over \( \mathbb{Z}\mathbb{P} \).

**Proof.** Suppose that there is a linear relation among cluster monomials over \( \mathbb{Z}\mathbb{P} \) for some cluster pattern with coefficients. Then, by trivializing the coefficients, we obtain a linear relation over \( \mathbb{Z} \) for the cluster pattern without coefficients. This is a contradiction. \( \square \)

### 6.6 Remarks on singular case

One can actually remove Injectivity Assumption of \( \Gamma \), or equivalently, the assumption of the nonsingularity of the corresponding \( B \)-pattern, and still obtain an analogous result to Theorem 6.26 directly. Since this is beyond the scope of Part II, we only give a sketch of the outline here.

On the \( F \)-polynomial side, we need to modify the result (4.45). We consider a cluster pattern with *principal coefficients* at \( t_0 \) in \[FZ07\]. Let
\( \mathbb{P} = \text{Trop}(y) \) be the tropical semifield of \( y = y_{t_0} \). Let \( \mathbb{ZP} \) be the group ring of \( \mathbb{P} \), and let \( \mathbb{QP} \) be the fraction field of \( \mathbb{ZP} \). The \( \hat{y} \)-variables are defined by

\[
\hat{y}_{i; t} = y_{i; t} \prod_{j=1}^{n} x_{j; t}^{b_{ji; t}}. \tag{6.53}
\]

In particular, let \( \hat{y} = (\hat{y}_i) \) be the initial \( \hat{y} \)-variables,

\[
\hat{y}_i = \hat{y}_{i; t_0} = y_{i; t_0} \prod_{j=1}^{n} x_{j; t}^{b_{ji; t}}, \tag{6.54}
\]

which are algebraically independent in the ambient field \( \mathcal{F}_X = (\mathbb{QP})(x) \) whether \( B \) is nonsingular or singular. Let \( t, t' \in \mathbb{T}_n \) be vertices that are \( k \)-adjacent. The \( y \)-variables transform in \( \mathbb{P} \) by

\[
y_{i; t'} = \begin{cases} y_{k; t}^{-1} [\varepsilon b_{ki; t}]_+ (1 + \hat{y}_{k; t}^\varepsilon)^{-b_{ki; t}} & i = k, \\ y_{i; t} y_{k; t}^{-\varepsilon b_{ki; t}} (1 + \hat{y}_{k; t}^\varepsilon)^{b_{ki; t}} & i \neq k. \end{cases} \tag{6.55}
\]

Meanwhile, one can regard mutations of \( x \)-variables as isomorphisms of fields as follows, where \( x_{t}, x_{t'} \) are \( n \)-tuple of formal variables:

\[
\mu_{k; t} : (\mathbb{QP})(x_{t'}) \rightarrow (\mathbb{QP})(x_{t}), \tag{6.56}
\]

\[
\mu_{k; t}(x_{i; t'}) = \begin{cases} x_{k; t}^{-1} \left( \prod_{j=1}^{n} x_{j; t}^{-\varepsilon b_{jk; t}} \right) \frac{1 + \hat{y}_{k; t}^\varepsilon}{1 + \hat{y}_{k; t}^\varepsilon} & i = k, \\ x_{i; t} & i \neq k. \end{cases} \tag{6.57}
\]

Compare it with (4.12). Note that, for the tropical sign \( \varepsilon_{k; t} \), we have

\[
1 + \hat{y}_{k; t}^\varepsilon = 1, \tag{6.58}
\]

thus, the factor \( 1 + \hat{y}_{k; t}^\varepsilon \) in (6.55) and (6.57) disappears by setting \( \varepsilon = \varepsilon_{k; t} \). Then, we just repeat the same decomposition (4.4) and so on, and we obtain the same formula as (4.45),

\[
q_{t_0}^{t_0}(x^\mathbb{S}; t) = x^\mathbb{S}; t F_{i; t}(\hat{y}), \tag{6.59}
\]

but now \( \hat{y}_1, \ldots, \hat{y}_n \) are algebraically independent as mentioned.

Meanwhile, on the scattering diagram side, following [GHKK18 Appendix B], we consider the extension of the lattice \( M^\circ \) of a fixed data \( \Gamma \),

\[
\tilde{M}^\circ = M^\circ \oplus N. \tag{6.60}
\]

Then, we introduce a parallel map to \( p^* \) in (5.10),

\[
\tilde{p}^* : N \rightarrow \tilde{M}^\circ, \tag{6.61}
\]
**(II.6. Scattering diagrams and \(F\)-polynomials)**

\[
(\tilde{p}^*(n))(n' + m') = \{n', n\} + \langle n, m \rangle, \quad (n' \in N^\circ, m' \in M).
\]  
(6.62)

We have

\[
\tilde{p}^*(e_j) = \sum_{i=1}^n b_{ij} f_i + e_j,
\]

(6.63)

so that the map \(\tilde{p}^*\) is injective whether \(B\) is singular or not. Also, under the identification (5.7), we have

\[
x{\tilde{p}^*}(e_i) = y_i \prod_{j=1}^n x_{b_{ij}^*} =: \hat{y}_i,
\]

(6.64)

which coincides with (6.53). We also need to extend the monoid \(P\) in (5.14) with \(\tilde{P} \subset \tilde{M}^\circ\) as follows:

(i). \(\tilde{P} = \sigma \cap \tilde{M}^\circ\), where \(\sigma\) is a \(2r\)-dimensional strongly convex cone in \(\tilde{M}_\R^\circ\).

(ii). \(\tilde{p}^*(e_1), \ldots, \tilde{p}^*(e_r) \in \tilde{P}\).

Such \(\tilde{P}\) is not unique at all, and we choose one arbitrarily. The result does not depend on the choice of \(\tilde{P}\). Then, for a wall \((\varnothing, f)_n\), we replace the wall function \(f\) in (5.25) with

\[
f = 1 + \sum_{k=1}^\infty c_k x^{k\tilde{p}^*(n)} \in \mathbb{k}[[\tilde{P}]],
\]

(6.65)

while \(n\) and \(\varnothing\) remain the same. Then, under this modification, we have a parallel result for Theorems 6.25 and 6.26 without assuming Injectivity Assumption. More details will be found in Part III.
7 Some applications

Let us give some applications of the results and the techniques presented so far.

7.1 Detropicalization revisited

We return to the situation in Section 4.3.

As mentioned in Remark 4.9, the following result was given for $\sigma = \text{id}$ by [CHL20], and for general $\sigma$ by [Nak21], where both rely on the Laurent positivity. Here, we give an alternative proof based on Theorem 6.16 and the consistency of the scattering diagram $\mathcal{D}_s$, without relying on the Laurent positivity.

**Theorem 7.1** (Detropicalization [CHL20 Lemma 2.4 & Theorem 2.5], [Nak21 Theorem 5.2]). Let $\Sigma$ be any cluster pattern of rank $n$, and let $t_0$ be a given initial vertex. Then, for any $t, t' \in \mathbb{T}_n$ and any permutation $\nu \in S_n$, the following facts hold:

(a). $G_t = \nu G_{t'} \implies x_t = \nu x_{t'}$.

(b). $C_t = \nu C_{t'} \implies y_t = \nu y_{t'}$.

**Proof.** First, assume that the underlying $B$-pattern $B$ is nonsingular.

(a). Assume that $G_t = \nu G_{t'}$. Then, $\sigma(G_t) = \sigma(G_{t'})$. Therefore, for $p_t^{t_0}$ in (6.31), we have

$$p_t^{t_0} = p_{t'}^{t_0}$$

by the consistency of the scattering diagram $\mathcal{D}_s$. Then, we have $q_t^{t_0} = q_{t'}^{t_0}$ (for $x$-variables) by Theorem 6.16 (a) and (5.36). Therefore, $x_t = \nu x_{t'}$ by Proposition 4.8. Alternatively, by Theorem 6.16 (b) and (7.1), we have

$$p_t^{t_0}(x^{g;i}_t) = x^{g;i}_{t'} F_{i;t}^{t_0}(\hat{y}) = x_{i;t},$$

$$p_t^{t_0}(x^{g;i}_t) = p_{t'}^{t_0}(x^{g_{\nu^{-1}(i);t'}}) = x^{g_{\nu^{-1}(i);t'}} F_{\nu^{-1}(i);t'}^{t_0}(\hat{y}) = x_{\nu^{-1}(i);t'}.\quad (7.2)$$

Thus, we have $x_t = \nu x_{t'}$.

(b). Assume that $C_t = \nu C_{t'}$. Then, by the duality (2.4) and (2.14), we obtain

$$\delta_{\nu(i)} = \delta_i, \quad G_t = \nu G_{t'}, \quad B_t = \nu B_{t'}.$$

See [Nak21 Prop. 4.4 & Cor. 4.5] for details. Then, from (7.2) and (7.3), we have

$$F_{i;t}^{t_0}(\hat{y}) = F_{\nu^{-1}(i);t'}^{t_0}(\hat{y}).\quad (7.5)$$
By Remark 6.17 we have

$$F_{i;t}(y) = F_{\nu^{-1}(i);t'}(y).$$

(7.6)

Thus, by the separation formula (1.47), we obtain $y_t = \nu y_{t'}$.

Next, we consider the case when $B$ is singular. Thanks to Proposition 2.11, it reduces to the nonsingular case as follows:

$$G_t = \nu G_{t'} \implies \bar{C}_t = \nu \bar{C}_{t'} \implies \bar{x}_t = \nu \bar{x}_{t'},$$

(7.7)

$$C_t = \nu C_{t'} \implies \bar{C}_t = \nu \bar{C}_{t'} \implies \bar{y}_t = \nu \bar{y}_{t'},$$

(7.8)

where $\Sigma = \{\Sigma_t = (x_t, y_t, B_t)\}_{t \in \mathbb{T}_n}$ is a cluster pattern in Proposition 2.11.

7.2 Bijection between $g$-vectors and $x$-variables

Here we prove a statement, which sharpens Theorem 7.1 (a). This was proved by [CIKLFP13] in the skew-symmetric case with the representation/categorical method. Also, the implication $\implies$ was proved by [CL20, Theorem 3.2] in the skew-symmetrizable case with principal coefficients. In contrast to the proof of Theorem 7.1, here we rely on the Laurent positivity.

Theorem 7.2. Let $\Sigma$ be any cluster pattern, and let $t_0$ be a given initial vertex. Then, we have

$$g_{i;t}^{t_0} = g_{i';t'}^{t_0} \iff x_{i;t} = x_{i';t'}.$$  

(7.9)

Proof. ($\iff$) First we apply the separation formula (1.46) to $x_{i';t'}$ with respect to the initial vertex $t$. Then, by assumption, we have

$$x_{i';t'} = \left( \prod_{j=1}^{\ell} x_{i';t'} g_{i',t'}^j \right) F_{i';t'}(\hat{y}_t) = x_{i;t}.$$  

(7.10)

By the equivalent statement of the sign-coherence in Conjecture 1.32 which is now proved, the constant term of $F_{i';t'}(y_t)$ is 1. Then, by the Laurent positivity, we have $F_{i';t'}(y_t) = 1$ and also $g_{i';t'} = e_i = g_{i;t}$. Then, applying the bijection in Proposition 2.25 repeatedly, we obtain $g_{i';t'}^{t_0} = g_{i;t}^{t_0}$.

($\implies$) From the assumption, we obtain $g_{i';t'}^{t_0} = e_i$ by the opposite procedure to the above. Thus, we have

$$x_{i';t'} = x_{i;t} F_{i';t'}^{t_0}(\hat{y}_t).$$  

(7.11)

Changing the role of $t$ and $t'$, we also have

$$x_{i;t} = x_{i';t'} F_{i;t'}^{t_0}(\hat{y}_t').$$  

(7.12)
From two equalities, we obtain

$$F^t_{i',t'}(\hat{y}_i)F^t_{i't}(\hat{y}_i') = 1. \quad (7.13)$$

Again, by the constant term 1 of $F$-polynomials and the Laurent positivity, we have $F^t_{i',t'}(y_i) = F^t_{i't}(y_i) = 1$. Therefore, $x_{i;t} = x_{i't'}$.

$(\Rightarrow \Leftarrow)$ Let us give an alternative proof using the consistency of the scattering diagram $\mathcal{D}_s$ without using the Laurent positivity. Assume that $B$ is nonsingular. Let us identify $M_\mathbb{R} \simeq \mathbb{R}^n$. Then, we have

$$x_{i;t} = p^t_0(x^{g;i;t}), \quad x_{i't'} = p^t_0(x^{g;i';t'}) = p^t_0(x^{g;i}). \quad (7.14)$$

By assumption, two cones $\sigma(G_i), \sigma(G_{i'})$ have the ray $\sigma(g_{i;t}) = \sigma(g_{i';t'})$ as a common face. Let $\gamma^t_i$ be an admissible curve from $\sigma^0(G_i)$ to $\sigma^0(G_{i'})$ for $\mathcal{D}_s$. (It may not be completely contained in $|\Delta(G^t)|$.) For each degree $\ell$, by ignoring all walls of $\mathcal{D}_s$ whose wall functions are trivial modulo $\hat{J}^\ell$, one can take $\gamma^t_i$ close enough to the ray $\sigma(g_{i;t})$ so that $\gamma^t_i$ passes through only walls of $\mathcal{D}_s$ that contain $\sigma(g_{i;t})$. Since the normal vectors of these walls are orthogonal to $g_{i;t}$, the contribution to the wall-crossing automorphism $p_{\gamma^t_i, \mathcal{D}_s}$ from each wall is trivial. Therefore, we have

$$p_{\gamma^t_i, \mathcal{D}_s}(x^{g;i;t}) \equiv x^{g;i;t} \mod \hat{J}^\ell. \quad (7.15)$$

Since $\ell$ is arbitrary, this implies that

$$p_{\gamma^t_i, \mathcal{D}_s}(x^{g;i;t}) = x^{g;i;t}. \quad (7.16)$$

By the consistency of $\mathcal{D}_s$, we have

$$p^t_0 \circ p_{\gamma^t_i} = p^t_0. \quad (7.17)$$

By combining (7.14)-(7.17), we obtain $x_{i;t} = x_{i't'}$. Next, we consider the case when $B$ is singular. Thanks to Proposition 2.11, it reduces to the nonsingular case as follows:

$$g^{t_0}_{i;t} = g^{t_0}_{i't'} \Rightarrow \bar{g}^{t_0}_{i;t} = \bar{g}^{t_0}_{i't'} \Rightarrow \bar{t}_{i;t} = \bar{t}_{i't'} \Rightarrow x_{i;t} = x_{i't'}. \quad (7.18)$$

\[ \square \]

**Remark 7.3.** The first proof of the implication $\Rightarrow \Leftarrow$ is a little simplified version of the proof of [CL20, Theorem 3.2].

**Remark 7.4.** The above proof (as for the implication $\Rightarrow \Leftarrow$, the first one) is applicable to any cluster pattern with arbitrary coefficients in the sense of [FZ07], with a little care of coefficients. Originally in [FZ07], a $g$-vector is uniquely associated with each $x$-variable only for cluster patterns with geometric type satisfying a certain condition in [FZ07, Eq. (7.10)]. Now we can safely associate a $g$-vector with each $x$-variable for any cluster pattern with arbitrary coefficients.
Remark 7.5. As for $y$-variables and $c$-vectors, the implication
\[ c_{i:t}^{t_0} = c_{i':t'}^{t_0} \iff y_{i:t} = y_{i':t'} \quad (7.19) \]
holds by Proposition 1.30. However, the opposite implication does not hold. For example, for type $A_2$ case in Section 2.5, we have
\[ y_{2:t_0} = y_2, \quad y_{2:t_1} = y_2(1 + y_1), \quad (7.20) \]
which is already a counterexample.

One can rephrase Theorem 7.2 using the notion of the cluster complex in [FZ03a].

Definition 7.6 (Cluster complex). The cluster complex $\Delta(\Sigma)$ of a cluster pattern $\Sigma$ is the simplicial complex whose vertices are $x$-variables (cluster variables) and whose simplices are nonempty subsets of clusters.

We define a parallel notion for a $G$-pattern $G^{t_0}$ of $\Sigma$, where we encounter the conflict of notation with a $G$-fan $\Delta(G^{t_0})$. However, two notions are essentially equivalent, so that we abuse the notation.

Definition 7.7 (G-complex). The G-complex $\Delta(G^{t_0})$ of a G-pattern $G^{t_0}$ is the simplicial complex whose vertices are $g$-vectors in $G^{t_0}$ and whose simplices are nonempty sets of $g$-vectors belonging to a common $G$-matrix.

One can geometrically identify a simplex in the G-complex $\Delta(G^{t_0})$ with the intersection $\sigma \cap S^{n-1}$ of a cone $\sigma$ in the $G$-fan $\Delta(G^{t_0})$ and the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

We have a corollary of Theorem 7.2.

Corollary 7.8. For any cluster pattern $\Sigma$ and any vertex $t_0$, we have an isomorphism of simplicial complexes
\[ \Delta(\Sigma) \xrightarrow{\cong} \Delta(G^{t_0}) \quad (7.21) \]

Due to Remark 7.4, we can extend the result to any cluster pattern with arbitrary coefficients in the sense of [FZ07]. This extends the result of [CIKLFP13, Cor. 5.6] in the skew-symmetric case.

Corollary 7.9. For any cluster pattern $\Sigma$ with arbitrary coefficients in the sense of [FZ07] and any vertex $t_0$, we have the same isomorphism as $(7.21)$.

We have a further corollary of Corollary 7.9 and Theorem 2.17. The equivalence $(a) \iff (d)$ below is a part of a conjecture by [FZ03b, Conj. 4.14(b)], and it was proved by [GSV08] with algebraic arguments. Here we replace the proof with geometric arguments.
Corollary 7.10 (cf. [GSV08, Theorem 5]). For any cluster pattern $\Sigma$ with arbitrary coefficients in the sense of [FZ07] and any vertex $t_0$, the following conditions for $t, t' \in T_n$ are equivalent:

(a). $x_t$ and $x_{t'}$ contain exactly $n - 1$ common elements (as a set), where $n$ is the rank of $\Sigma$.

(b). $\sigma(G_t)$ and $\sigma(G_{t'})$ intersects in their common face of codimension one.

(c). There are some $t'' \in T_n$ that is adjacent to $t$ and some permutation $\nu \in S_n$ such that $G_{t''} = \nu G_{t'}$.

(d). There are some $t'' \in T_n$ that is adjacent to $t$ and some permutation $\nu \in S_n$ such that $x_{t''} = \nu x_{t'}$.

Proof. The equivalences $(a) \iff (b)$ and $(c) \iff (d)$ are due to Corollary 7.9 while $(b) \iff (c)$ is due to Theorem 2.17 \qed
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Index

$B$-pattern, 8
  nonsingular, 9
  principal extension, 25
  skew-symmetric, 9
  transpose of, 21
$C$-matrix, 10
$C$-pattern, 10
$F$-pattern, 13
$F$-polynomial, 13
$G$-complex, 82
$G$-cone, 27
$G$-fan, 28
$G$-matrix, 11
$G$-pattern, 11
$C$-matrix, 17
$\hat{C}$-matrix, 17
$\hat{c}$-vector, 17
$\hat{y}$-variable, 7
  tropical, 17
$\varepsilon$-expression, 8
  for $C$-matrices, 11
  for $G$-matrices, 13
c-vector, 10
d-vector, 16
g-vector, 12
g-vector fan, see $G$-fan
$k$-adjacent, 8
$x$-variable (cluster variable), 6
  tropical, 16
$y$-variable (coefficient), 6
  tropical, 16
ambient field, 6

Badlands, 61
broken line, 70

chamber, 39
cluster algebra, 9
cluster complex, 82
cluster monomial, 73
cluster pattern
  of finite type, 29
cluster scattering diagram (CSD), 58
cone, 26
  $G^-$, 27
  convex rational polyhedral, 26
  dual, 26
  nonsingular, 26
  simplicial, 26
  strongly convex, 26
CSD, see cluster scattering diagram
detropicalization, 49, 79
duality
  first, 12
  second, 19
  third, 21
face (of a cone), 26
fan, 26
  $G^-$, 28
  complete, 27
  nonsingular, 27
  simplicial, 27
finiteness condition, 55
fixed data, 50
Fock-Goncharov decomposition, 44
  nontropical part, 44
  tropical part, 44
initial vertex, 8
Injectivity Assumption, 52
intersection sign, 56
Laurent positivity, 18, 73

matrix
  $C^-$, 10
  $G^-$, 11
  $\hat{C}^-$, 17
  column/row sign-coherent, 17
  exchange, 6
  skew-symmetrizable, 6
mutation
  dual, 22
  of $C$- and $G$-matrices, 19
  of a cluster scattering diagram, 63
  seed, 7
  seed (for a fixed data), 62
  signed, 63
normal vector (of a wall), 54
normalization factor, 53
normalized automorphism, 53

pattern
  $B^-$, 8
  $C^-$, 10
  $F^-$, 13
