A computational study of sliding blocks on inclined surfaces is presented. Assuming that the friction coefficient $\mu$ is a function of position, the probability $P(\lambda)$ for the block to slide down over a length $\lambda$ is numerically calculated. Our results are consistent with recent experimental data suggesting a power-law distribution of events over a wide range of displacements when the chute angle is close to the critical one, and suggest that the variation of $\mu$ along the surface is responsible for this.

1. Introduction

The dynamics of rigid bodies sliding on inclined planes under the action of gravity and possibly friction forces is a very old subject, present in most basic physics courses. However these systems are still being studied in order to better understand some of the intriguing properties of granular materials. Several recent papers\cite{1,2,3,4,5} present experimental and computational results that demonstrate the complexity of friction-related phenomena in specific cases. A recent study\cite{6} of cylinders sliding on a rigid aluminum bar by Brito and Gomes shows that even this simple system may present unusual features. Measuring the number $N(\lambda)$ of slidings with length larger than $\lambda$ (see figure (1)) the authors find broad regions that can be very well fitted by a power law $N(\lambda)/N(0) \sim \lambda^{-B}$, where the exponent $B$ does not seem to depend on the material or the inclination $\theta$ of the chute, when $\theta$ is close to the critical angle $\theta_c = \arctan \mu$. Their data are in agreement with the Gutenberg-Richter law\cite{7} for the distribution of earthquakes and with numerical simulations by Chen, Bak and Obukhov\cite{8} exhibiting self organized criticality. However, no theoretical explanation of this behavior was advanced.

In this paper we present a simple model in order to explain the behavior observed in the above mentioned experiments. We consider the problem of a block that slides down, under the effect of gravity and friction forces, on an inclined surface at an angle $\theta$ with the horizontal. We assume that friction is due to the existence of many uncorrelated contact points between the surfaces, and therefore becomes a rapidly...
varying function $\mu(x)$ of the block position $x$ on the chute. At $t = 0$ the block is set in motion with velocity $v_0$. If $\theta < \theta_c$, where $\theta_c = \arctan \mu$ is the “critical angle”, the block will stop with probability one after some finite displacement $\lambda$. We have in this case an ‘avalanche’ of size $\lambda$. Our aim is to obtain the probability $P(\lambda)$ that the block stops at position $\lambda$.

The total energy variation after the slide is

$$\Delta E = mg \Delta h - \frac{1}{2} m v_0^2,$$

where $\Delta h = h_f - h_0$ is the vertical displacement, $v_0$ the initial velocity and $m$ the mass of the block. Since energy is dissipated by friction forces, we also have

$$\Delta E = \int_0^\lambda F_{\mu}(x) \, dx = - \int_0^\lambda m g \cos(\theta) \mu(x) \, dx$$

where $\theta$ is the inclination of the plane, $x$ the block position along the plane, $\lambda(= \Delta x)$ the total displacement and $\mu(x)$ the local friction coefficient. Equations (1) and (2) admit analytical treatment. The results of our theoretical approach will be published elsewhere.

2. The Computational Model

In order to simulate the position-dependent behavior of $\mu$ we introduce a very simple computational model. This model is based on the following assumptions: We assume that friction occurs due to randomly scattered contact points between the two surfaces, and that these contact points are separated by a characteristic length $a$. The displacement of the block is discretized in small steps of length $a$, and we represent the “rugosity” of both surfaces by means of two binary strings of
0s and 1s. Each bit can be thought of as representing the average properties of the surface over a length $a$. If for instance a certain region is more prominent than the average, the corresponding bit is set to 1, and to 0 in the opposite case. Thus when the two surfaces are put in contact, only those regions will contribute to friction which both have a 1 on the corresponding location of their bit string.

We model the disorder by assigning random strings of bits to both the plane and the block before each experiment. For computational convenience we set the block-string length to be 32 bits. Typical plane lengths are on the other hand $10^5$ bits. The concentration of 1s on the plane- and block-string are $C_p$ and $C_b$ respectively.

In Fig. (2) we show a schematic diagram of a block of length 14 bits with concentration $C_b = 0.5$ of ones, sliding over a plane with length $L = 39$ bits, with $C_p = 0.5$.

The local coefficient of friction $\mu(x)$ is defined as

$$\mu(x) = b \frac{N(x)}{N_{max}}$$

where $N(x)$ is the number of coincident 1s and depends on both strings, $N_{max}$ is the block length in bits and $b$ is a constant that can be associated to the strength of each individual contact.

The block motion can be efficiently simulated in the following way:

1. The block starts at $x = 0$ with total energy $E_0 = \frac{1}{2}mv_0^2$.
2. We let the block slide over a distance $a$ corresponding to one bit, and its energy variation is calculated as
\[ \Delta E = aF(x) - mga \sin(\theta) = mga \left( \frac{bN(x)}{N_{\text{max}}} \cos(\theta) - \sin(\theta) \right) \] (4)

3. If the total energy after this change turns out to be zero or negative, the block has stopped. Otherwise we set \( E \rightarrow E + \Delta E \) and go to 2.

The critical angle \( \theta_c \) can be obtained by setting \( < \Delta E > = 0 \) in (4), and satisfies

\[ \tan \theta_c = \frac{b}{\mu} = bC_pC_b \] (5)

Figure 3: The accumulated distribution \( N(\lambda) \) of slidings larger than \( \lambda \) for several values of the inclination angle \( \theta \). Averages were done over \( 10^7 \) slidings. The critical angle \( \theta_c \) is \( \pi/4 \). The straight line \( N(\lambda) \sim \lambda^{-1/2} \) shows the expected behavior for \( \theta = \theta_c \).

3. Results

We measure the number \( N(\lambda) \) of slidings with size larger than \( \lambda \). We fix \( C_p = C_b = 0.5 \) and \( b = 4 \) so that \( \theta_c = \pi/4 \). In Fig. (3) average results are shown for \( 10^7 \) slidings on a plane of maximum length \( 10^5 \) bits and several values of \( \theta \leq \theta_c \). The straight line is shown for reference and corresponds to \( N(\lambda)/N(0) = \lambda^{-1/2} \). Our numerical results indicate that, when \( \theta \) is close to \( \theta_c \), the distribution of avalanches...
shows a power-law behavior with exponent $1/2$ over a wide range of sizes. This exponent has the same value as obtained in experiments. As expected on simple grounds, below $\theta_c$ the distribution becomes exponentially decreasing for large sizes and therefore there is a finite average size. As already mentioned, this problem admits analytical treatment as well. In Fig. (3) we show a preliminary theoretical result corresponding to $(\theta_c - \theta) = 10^{-4}$.

The average sliding size $\lambda$ was also measured, and the results are presented in Fig. (4). A power-law $\lambda \sim (\theta_c - \theta)^{-\nu}$ is obtained with $\nu = 1.00 \pm 0.02$. This value is consistent with $\nu = 1$, which is obtained if $\mu$ does not depend on position. In this case $\lambda = \frac{v_0 \cos \theta_c}{2g \sin (\theta_c - \theta)} \sim (\theta_c - \theta)^{-1}$, for $\theta$ close to $\theta_c$.

Figure 4: Mean sliding size as a function of inclination angle $\theta$. We see that $\lambda \sim (\theta_c - \theta)^{-\nu}$ holds for $\theta \to \theta_c$. 
4. Conclusions

A very simple model in which the coefficient of friction changes from point to point on the surface is able to reproduce a power-law behavior in the distribution of slidings of a block on an inclined chute, as recently observed in experiments. This holds for values of $\theta$, the inclination angle, smaller than but very close to $\theta_c$. In this limit, the average sliding size diverges as $(\theta_c - \theta)^{-1}$.

It is possible to do a theoretical study of this problem considering the random variation of the coefficient of friction. We show a preliminary result, and a complete study will be published elsewhere.

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