Buoyant convection in porous annulus with discrete sources-sink pairs and internal heat generation

M. Sankar¹,†, S. Kemparaju², B.M.R. Prasanna³ and S. Eswaramoorthi⁴

¹Department of Mathematics, School of Engineering, Presidency University, Bangalore, India
²Department of Mathematics, GFGC, Doddaballapur, Karnataka, India
³Department of Mathematics, Siddaganga Institute of Technology, Tumkur, India
⁴Department of Mathematics, Dr. N.G.P. Arts and Science College, Coimbatore India.

E-mail: †msankar@presidencyuniversity.in

Abstract. In the present analysis, numerical investigation of buoyancy-driven convective thermal transport in a vertical annulus has been performed when the interior and exterior cylinders are discretely heated and cooled by heat source-sink pair arrangements. The upper and lower boundaries and unheated parts of inner and outer cylinders are insulated. For the porous annulus, the Brinkman-extended Darcy formulation is adopted for modeling the fluid flow in the porous medium. Also, the effects of internal heat generation are investigated. The governing equations in terms of vorticity-stream function formulation are solved by an implicit finite difference technique based ADI and SLOR methods. In particular, the study is focused on the effects of different sources and sinks arrangements on the convective flow and associated thermal transport features.

1. Introduction

Buoyancy-driven convection in an annular geometry has been considered as an ideal heat transport tool in many industrial applications. Hence, many researchers investigated convective thermal transport in an upright annular space both experimentally and numerically. Initially, de Vahl Davis and Thomas [1] investigated convection in the vertical annulus where it is heated isothermally and suggested correlations for heat transport rates. Prasad and Kulacki [2] performed pioneering experiments in determining the effect of curvature ratio in an annular enclosure with uniform thermal conditions. Buoyancy convection in an upright annulus by maintaining constant heat flux at the inner wall is numerically analyzed by Keyhani et al. [3] and Khan and Kumar [4]. Kumar and Kalam [5] studied natural convection within an annular region and deliberated the inconsistencies existing among the numerical and experimental results. Oudina and Bessaih [6] performed magentoconvection of liquid metal in a vertical annulus by considering magnetic field in axial and radial directions. Sankar and Do [7] analyzed numerically the impact of localized heating on convective heat transport in an upright annular geometry with two heaters placed at inner cylinder. By considering different sizes and locations, Sankar et al. [8, 9] numerically analyzed convective heat transport due to dual heaters in a vertical annulus and the analysis carried out to know in detail the influence of localized heating on the streamlines, isotherms and global heat transport rates. Recently, in a vertical annulus, Oudina [10] analyzed the hydrodynamic stability of buoyant convection by considering heat sources of
various lengths. Natural convective heat transport in rectangular geometry containing three
discrete thermal sources is analyzed by Bae and Hyun [11]. The influence of different sources
and sinks combinations on convection thermal transport in a cavity is analyzed numerically by
Deng [12] and found that the number of sources and sinks pair has strong influence on the rate
of heat transfer.

In a porous annular geometry, Havstad and Burns [13] numerically investigated buoyant
thermal transport and presented thermal transport rates for moderate cylinder spacing and for
high temperature differences. Recently, Sankar et al. [14] analyzed the location and size effects of
discrete thermal source on buoyant convective thermal transport in a porous annular enclosure.
Using the Darcy model, Saied and Pop [15] performed numerical simulations on convection in
a porous geometry under the influence of an isoflux or isothermal discrete heat source. In a
rectangular porous cavity, Sivasankaran et al. [16] analyzed the influence of localized heating on
buoyant convection and found that the heat transport rates can be enhanced with the Ra and
Da numbers, but suppresses as aspect ratio increases. Convective thermal transport in a square
porous geometry which is driven by buoyancy force has been numerically analyzed by Sankar et
al. [17] to know the influence of discrete heating and cooling on free convection by considering
different locations. Prasad and Chui [18] numerically analyzed the buoyant convection
within a cylindrical porous enclosure and obtained several correlations for the maximum cavity
temperature and the overall Nusselt number. Reddy and Narasimham [19] numerically analyzed
the conjugate convective heat transfer in a vertical annulus containing a vertical heat generating
rod and presented thermal distributions and heat transport rates. Sankar et al. [20] studied
natural convection due to discrete heating in an upright concentric porous annular region with
internal heat generation.

Though the annular enclosure is common in practice, the earlier studies on buoyant convection
due to source-sink combinations are primarily devoted to rectangular enclosures. Therefore, in
this analysis, a sincere attempt has been made to numerically investigate the effects of source-
sink pair locations on natural convection in a vertical porous annulus in the presence of internal
heat generation.

2. Mathematical Formulation

For the present study, the physical geometry, displayed in Fig 1, is a vertical cylindrical annulus of
height $L$ and annular width $D$ is filled with fluid-saturated porous medium and an internally
heat generating quantity. The annular enclosure is designed from two vertical, co-axial cylindrical
tubes having internal radius $r_i$, and external radius $r_o$. Along the interior and exterior cylinders,
the discrete heat source(s) and sink(s), of length 0.2 units, are placed. The heat source is
kept at constant higher temperature $θ_h$ and the sink is supplied constant lower temperature
$θ_c$ and remaining portion of inner and outer walls, the upper and lower portions are insulated.
The fluid phase temperature is same as solid phase temperature everywhere is assumed in the
region of porous and the Local Thermal Equilibrium (LTE) assumption is applied in the current
investigation. The porous matrix is assumed to be rigid, and in local thermodynamic equilibrium
with the fluid. By applying the above mentioned approximations, the governing equations in
vorticity-stream function formulation can be written as [Sankar et al. [20]]

\[
\sigma \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial R} + W \frac{\partial T}{\partial Z} = \nabla^2 T + \left( \frac{Ra_I}{Ra} \right)
\]

\[
\frac{1}{\phi} \frac{\partial \zeta}{\partial t} + \frac{1}{\phi^2} \left[ U \frac{\partial \zeta}{\partial R} + W \frac{\partial \zeta}{\partial Z} - \frac{U \zeta}{R} \right] = \frac{Pr}{\phi} \left[ \nabla^2 \zeta - \frac{\zeta}{R^2} \right] - \frac{Pr Ra}{D_o} \zeta - Pr Ra \frac{\partial T}{\partial R}
\]

\[
\zeta = \frac{1}{R} \left[ \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} \right]
\]
\[ U = \frac{1}{R} \frac{\partial \psi}{\partial Z}, \quad W = \frac{1}{R} \frac{\partial \psi}{\partial R} \]  

Here, \( \nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} \). The dimensionless variables used in the above equations can be found in Sankar et al. [20]. Also, the non-dimensional parameters are: \( Ra = \frac{g \beta \Delta \theta D^3}{\nu \alpha} \), the external Rayleigh number, \( Ra_I = \frac{g \beta Q D^5}{\nu \alpha^2 \rho_0 C_p} \), the internal Rayleigh number, \( Pr = \frac{\nu}{\alpha} \), the Prandtl number, \( Da = \frac{K}{D^2} \), the Darcy number, \( A = \frac{H}{D} \) the aspect ratio and \( \lambda = \frac{r_o}{r_i} \) the radius ratio.

The dimensionless auxiliary conditions are

\[ t = 0: \quad U = W = T = 0, \quad \psi = \zeta = 0; \text{ at } 0 \leq Z \leq A \text{ and } \frac{1}{\lambda - 1} \leq R \leq \frac{\lambda}{\lambda - 1}. \]

\[ t > 0: \quad \psi = \frac{\partial \psi}{\partial R} = 0, \quad \frac{\partial T}{\partial R} = 0 \quad \text{at } R = \frac{1}{\lambda - 1} \text{ and unheated regions on inner wall} \]

\[ \psi = \frac{\partial \psi}{\partial R} = 0, \quad T = 1 \quad \text{at } R = \frac{1}{\lambda - 1} \text{ and at sources} \]

\[ \psi = \frac{\partial \psi}{\partial R} = 0, \quad \frac{\partial T}{\partial R} = 0 \quad \text{at } R = \frac{\lambda}{\lambda - 1} \text{ and unheated regions on outer wall} \]

\[ \psi = \frac{\partial \psi}{\partial R} = 0, \quad T = 0 \quad \text{at } R = \frac{\lambda}{\lambda - 1} \text{ and at sinks} \]

\[ \psi = \frac{\partial \psi}{\partial Z} = 0, \quad \frac{\partial T}{\partial Z} = 0 \quad \text{at } Z = 0 \text{ and } Z = A. \]

The rate of heat transport in the annulus is significant factor in heat transfer applications. Thus, the local Nusselt number is \( Nu = \frac{hl}{k} = \frac{q_h L}{k(\theta_h - \theta_c)} \) along the heater and this leads to \( Nu = \frac{\partial T}{\partial R} \). The global heat transport rate is represented by the average Nusselt numbers for two pair of source and sink are given by

\[ Nu_B = \frac{1}{0.2} \int_{0.2}^{0.4} Nu dZ \quad \text{and} \quad Nu_T = \frac{1}{0.2} \int_{0.6}^{0.8} Nu dZ. \]  

3. Solution Methodology

The governing partial differential equations together with initial and boundary conditions are numerically solved using the finite difference based ADI and SLOR methods. The average Nusselt number at each sources and sinks are calculated by employing Simpson’s rule. The details of method and grid independency are discussed elaborately in the previous works [7, 8, 9] and hence are not repeated here. A Fortran program is written and validated with the standard benchmark data.

4. Results and Discussion

Convective thermal transport in a cylindrical porous annular region containing source-sink pairs and internally generating heat substance are analyzed numerically. The pattern of fluid movement and heat transfer are illustrated with the help of streamlines, isotherms, local and global \( Nu \) profiles. The heat transport rate for various values of internal and external \( Ra \) and Darcy number in terms of the average thermal transport at source- sink pairs are presented.

The fluid flow and thermal distribution in the porous annulus for three Darcy number \( 10^{-5}, 10^{-3} \) and \( 10^{-1} \) are presented in Figs. 2 and 3 through streamlines and isotherms for
both first and second arrangements of source-sink pairs. The fluid flow depends on Darcy number as observed through the streamlines and isotherms. For lower Darcy number $10^{-5}$ in both the arrangements, a simple circulating flow pattern is observed and thermal lines are slightly parallel revealing the dominant conduction mode and weaker convective motion caused by the dense porous medium. However, for higher Darcy number $10^{-1}$, the resistance from the friction factor decreased due to absorbency in the porous medium, and the flow is caused solely by buoyancy forces. At higher Darcy number, flow velocity becomes strong and hence the main vortex of the streamlines moves towards the sinks at outer wall of the annulus. Also, at higher Darcy number it is noted that the convection mode is dominant and the flow enter to the core of porous material and is observed through the streamlines and isotherms. For the case of second arrangement, the sources and sinks are arranged alternately on inner and outer wall, a bi-cellular symmetric flow pattern is observed due to the decomposition of their buoyancy forces. A clockwise rotating eddy is located in lower portion and an anti-clockwise eddy is found in upper region of annulus as indicated by the streamlines in Fig. 3. It is observed through the isothermal lines that the fluid is thermally stratified for first arrangement. In second arrangement, at the center core of the annulus fluid maintains almost an average temperature due to the hot fluid moving upwards combined with the cold fluid moving downwards.

In this discussion, the combined influence of internal and external Rayleigh numbers on the flow pattern and temperature profiles in the annulus are presented. The effect of internal and external heating on the convective flow and isotherms are presented in Figs. 4 and 5 for both the arrangements. The $Da$ value is kept at $10^{-2}$ and the internal and external Rayleigh number values are varied to obtain the effects on natural convection due to source-sink pairs. A simple circular flow pattern in streamlines and less slanted isotherms can be observed in the annulus when internal Rayleigh number less than the external Rayleigh number. But, when internal Rayleigh number is enlarged to the higher value, the eddy at the center of streamline divided into two circular pattern and mainly moves towards the top sink with larger strength, due to strong internal heat generation. The isotherms tend to move towards the top sink compared to bottom sink as shown in Figs. 4(c) and 5(c), which shows that the dominance of internal heat generation effect. As the values of internal and external $Ra$ are equal, a simple circular flow pattern in streamlines and less slanted isotherms can be observed in the annulus. However, as external Rayleigh number is enhanced to the extreme value $10^7$, the convective flow as well as the thermal pattern shows that the dominance of external heat generation. In general, it is found that the internal heating is significant when its value is higher than the external Rayleigh number.

The discrete source-sink pair arrangement in finite enclosures plays an important role in cooling of electronic equipment. The heat transport rate in the porous annulus is demonstrated through average Nusselt number over different internal and external Rayleigh number and Darcy number. The impact of external Rayleigh and Darcy numbers on the average Nusselt number is measured from the sources for fixed value of $Ra_I = 10^5$ are illustrated in Fig. 6 for the first arrangement. In general, by increasing the value of external Rayleigh number the overall heat transfer rates can be enhanced, also it is observed that the heat transfer rate increases when the Darcy number increases. Further, it is observed that the heat transfer rates are more at bottom source compared to the top source on the inner wall. However, in the case of second arrangement the rate of heat transfer at the bottom source increases and the top source decreases as increase in external Rayleigh and Darcy number for fixed value of $Ra_I = 10^5$, but beyond the value of external Rayleigh number $Ra = 10^5$, for any value of Darcy number, the rate of heat transfer at bottom source is almost similar and it little rises as compared to the rate of heat transfer at the top source as shown in Fig. 7.

The effect of external Rayleigh and Darcy numbers on the average Nusselt number measured from the sinks at the outer wall for a fixed internal Rayleigh number $Ra_I = 10^5$, reveals a
higher heat transfer rate at the top sink compared to bottom sink as shown in Fig. 8 for the first arrangement. Figure 9 represents global Nusselt number variation for the case of alternate source and sink arrangements for different values of Rayleigh and Darcy numbers. A closer observation of the result exhibits that the heat transfer rate at the top sink increases and at the bottom sink decreases up to the value of $Ra = 10^5$, after increasing the value of $Ra$ the heat transfer rate is similar at both top and bottom sinks. The effect of internal Rayleigh and Darcy numbers on the average Nusselt number measured from the sinks at the outer wall for a fixed external Rayleigh number $Ra = 10^5$, reveals a higher heat transfer rate at the top sink compared to bottom sink. A closer observation of the figure also exhibits that the thermal transport rate at the top and bottom sinks are similar up to the value of $Ra_I = 10^5$ and beyond this value, the heat transfer rate at the top sink decreases and bottom sink increases.

5. Conclusions
The impact of discontinuous heating from the arrangement of source-sink pairs on the buoyancy-driven convective flow in a porous annular geometry containing internally heat generating substance is numerically investigated. From these simulations, the following observations are achieved: The arrangement of sources and sinks on the side walls produces unicellular or multicellular flow in the annulus. The impact of internal heating is superior in the porous annulus when the magnitude of internal $Ra$ is same or maximum as compared with the external Rayleigh number. The average Nusselt number is small at low Darcy number but, at higher value of Darcy number, the overall thermal transport increases. The global heat transport is found to be higher for bottom source as compared to top source. Also, the rate of thermal transport enhances as the value of external $Ra$ and $Da$ numbers increases in the case of first arrangement and for the second arrangement the heat transfer rate is similar beyond the value of $Ra = 10^5$. The overall thermal transport rate is observed to be maximum for lower source as compared to top source and the thermal transport rate decreases as the value of internal $Ra$ and $Da$ numbers increases in the case of first arrangement and for the second arrangement the heat transfer rate is similar up to the value of $Ra_I = 10^5$. The average heat absorption is found to be higher for top sink as compared to bottom sink and the thermal transport rate increases as the value of internal Rayleigh and Darcy number increases in the case of first arrangement and for the second arrangement, the heat transfer rate is similar up to the value of $Ra_I = 10^5$.

References
[1] De Vahl Davis G and Thomas R W 1969 High-speed Computing in Fluid Dynamics, Phys. Fluids Suppl. II 198
[2] Prasad V and Kulacki F V 1985 ASME. J. Heat Transfer. 107 (3) 596
[3] Keyhani M, Kulacki F V and Christensen R N 1983 ASME. J. Heat Transfer. 105 454
[4] Khan J A and Kumar R 1999 ASME. J. Heat Transfer. 111 909
[5] Kumar R and Kalam M A 1991 Int. J. Heat Mass Transfer. 34 (2) 513
[6] Oudina F M and Bessah R 2016 J. Applied Fluid Mechanics. 9(4) 1655
[7] Sankar M and Do Y 2010 Int. Communications in Heat Mass Transfer. 37 600
[8] Sankar M, Jang B and Do Y 2014 Journal of Porous Media. 17 (5) 373
[9] Sankar M, Do Y, Ryu S and Jang B 2015 Numerical Heat Transfer, Part A: Applications. 68(8) 847
[10] Oudina F M 2017 Engineering Science and Technology. 20 1324
[11] Bae J H and Hyun J M 2004 Int. J. Therm. Sci. 43 8
[12] Deng Q H 2008 Int. J. Heat Mass Transfer. 51 5949
[13] Havstad M A and Burns P J 1982 Int. J Heat and Mass Transfer. 25(11) 1755
[14] Sankar M, Park Y, Lopez J M and Do Y 2011 Int. J. Heat Mass Transfer. 54 (7-8) 1493
[15] Saeid N H and Pop I 2005 Journal of Porous Media. 8(1) 55
[16] Sivasankaran S, Do Y and Sankar M 2011 Transp. Porous Med. 86 291
[17] Sankar M, Park J, Kim D and Do Y 2013 Numerical Heat Transfer, Part A: Applications. 63 1
[18] Prasad V and Chui A 1989 ASME. J. Heat Transfer. 111 (4) 916
Acknowledgments
The authors Sankar, Kempuraju, Prasanna and Eswaramoorthi respectively acknowledges the Managements of Presidency University, Bengaluru, GFGC, Doddaballapur, SIT, Tumkur and Dr. N.G.P. Arts and Science College, Coimbatore for their support and encouragement. Also, M. Sankar’s work is financial supported by the Vision Group of Science and Technology, Government of Karnataka under Grant Number KSTePS/ VGST-KFIST (L1)/2017.

Figure 1. Physical configuration and coordinate system. First arrangement of sources and sinks (Left) and second arrangement of sources and sinks (Right)
Figure 2. Effect of Darcy number on streamlines and isotherms for fixed values of $Ra = 10^7$ and $Ra_I = 10^5$. (Left) $Da = 10^{-5}$ (Middle) $Da = 10^{-3}$ (Right) $Da = 10^{-1}$ (First arrangement)

Figure 3. Effect of Darcy number on streamlines and isotherms for fixed values of $Ra = 10^7$ and $Ra_I = 10^5$. (Left) $Da = 10^{-5}$ (Middle) $Da = 10^{-3}$ (Right) $Da = 10^{-1}$ (Second arrangement)
Figure 4. Effect of $Ra_I$ and $Ra$ on streamlines and isotherms for $Da = 10^{-2}$. (Left) $Ra_I = 10^3$ and $Ra = 10^5$ (Middle-1) $Ra_I = 10^7$ and $Ra = 10^5$ (Middle-2) $Ra_I = 10^5$ and $Ra = 10^5$ (Right) $Ra_I = 10^5$ and $Ra = 10^7$ (First arrangement)

Figure 5. Effect of $Ra_I$ and $Ra$ on streamlines and isotherms for $Da = 10^{-2}$. (Left) $Ra_I = 10^3$ and $Ra = 10^5$ (Middle-1) $Ra_I = 10^7$ and $Ra = 10^5$ (Middle-2) $Ra_I = 10^5$ and $Ra = 10^5$ (Right) $Ra_I = 10^5$ and $Ra = 10^7$ (Second arrangement)
Figure 6. Effect of Rayleigh and Darcy numbers on the average Nusselt number for $Ra_l = 10^5$. (Left) $Da = 10^{-5}$, (Right) $Da = 10^{-1}$ (First arrangement). Solid and dashed lines respectively represents bottom and top sources.

Figure 7. Effect of Rayleigh and Darcy numbers on the average Nusselt number for $Ra_l = 10^5$. (Left) $Da = 10^{-5}$, (Right) $Da = 10^{-1}$ (Second arrangement). Solid and dashed lines respectively represents bottom and top sources.
Figure 8. Effect of Internal Rayleigh and Darcy numbers on the average Nusselt number for $Ra = 10^5$ (Left) $Da = 10^{-5}$, (Right) $Da = 10^{-1}$ (First arrangement). Solid and dashed lines respectively represents bottom and top sources.

Figure 9. Effect of Internal Rayleigh and Darcy numbers on the average Nusselt number for $Ra = 10^5$ (Left) $Da = 10^{-5}$, (Right) $Da = 10^{-1}$ (Second arrangement). Solid and dashed lines respectively represents bottom and top sources.