A Characterization of Polynomially Convex Sets in Banach Spaces

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Abstract

Let $E$ be a Banach space and $\Delta^*$ be the closed unit ball of the dual space $E^*$. For a compact set $K$ in $E$, we prove that $K$ is polynomially convex in $E$ if and only if there exist a unital commutative Banach algebra $A$ and a continuous function $f : \Delta^* \to A$ such that (i) $A$ is generated by $f(\Delta^*)$, (ii) the character space of $A$ is homeomorphic to $K$, and (iii) $K = \overrightarrow{sp}(f)$ the joint spectrum of $f$. In case $E = C(X)$, where $X$ is a compact Hausdorff space, we will see that $\Delta^*$ can be replaced by $X$.

1 Introduction

Let $K$ be a compact set in the $n$-dimensional complex space $\mathbb{C}^n$. It is said that $K$ is polynomially convex if for every $\lambda \in \mathbb{C}^n \setminus K$ there exists a polynomial $p$ such that $p(\lambda) = 1$ and $|p(w)| < 1$ for all $w \in K$. It is known that if $K \subset \mathbb{C}$ then $K$ is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected (e.g. [6, Theorem 2.3.7]). A characterization of polynomially convex sets in $\mathbb{C}^n$...
is given as follows. Suppose that $A$ is a commutative Banach algebra with unit element $1$ and character space $\mathcal{M}(A)$. For an $n$-tuple $a = (a_1, \ldots, a_n)$ of elements of $A$, the joint spectrum $\tilde{s}\bar{P}(a)$, defined by [6, Definition 2.3.2]

$$\tilde{s}\bar{P}(a) = \{(\phi(a_1), \ldots, \phi(a_n)) : \phi \in \mathcal{M}(A)\},$$

(1.1)

is a compact set in $\mathbb{C}^n$. In case $a_1, \ldots, a_n$ generate $A$, the space $\mathcal{M}(A)$ is homeomorphic to $\tilde{s}\bar{P}(a)$ through the homeomorphism $\phi \mapsto (\phi(a_1), \ldots, \phi(a_n))$.

The following theorem provides a characterization of those compact subsets of $\mathbb{C}^n$ which arise in this way as character spaces of commutative Banach algebras generated by $n$ elements.

**Theorem 1.1** (e.g. [6, Theorem 2.3.6]). A compact set $K$ in $\mathbb{C}^n$ is polynomially convex if and only if there exists a unital commutative Banach algebra $A$ which is generated by $n$ elements $a_1, \ldots, a_n$ with $K = s\bar{P}(a_1, \ldots, a_n)$.

The main purpose of this paper is to establish analogous results for compact sets in infinite dimensional Banach spaces. Throughout the paper, the space of all continuous complex valued functions on a compact Hausdorff space $X$ is denoted by $C(X)$. For every $f \in C(X)$, the uniform norm of $f$ is defined by $\|f\|_X = \sup\{|f(x)| : x \in X\}$. Given a Banach space $(E, \| \cdot \|)$, the closed unit ball of the topological dual $E^*$ is denoted by $\Delta^*$, which is a compact Hausdorff space for the weak* topology. For every $x \in E$, the mapping $\tilde{x} : \Delta^* \to \mathbb{C}$, $\phi \mapsto \phi(x)$, belongs to $C(\Delta^*)$ with $\|\tilde{x}\|_{\Delta^*} = \|x\|$. Hence, the mapping $x \mapsto \tilde{x}$ is an isometric isomorphism of $E$ onto a closed subspace of $C(\Delta^*)$. For this reason, we may consider $E$ as a closed subspace of $C(\Delta^*)$.

The paper is outlined as follows. In Section 2, we briefly recall some properties of polynomially convex sets in a Banach space $E$ and, for a compact set $K$ in $E$, we study the algebra $\mathcal{P}(K)$ of all functions in $C(K)$ that can be approximated uniformly on $K$ by polynomials. The main results are included in Section 3. First, we briefly recall some properties of the joint spectrum $s\bar{P}(f)$ of an infinite system of elements $f : X \to A$ of a Banach algebra $A$. Next, we consider the Banach space $C(X)$ and present a characterization of polynomially convex sets. Finally, we consider arbitrary Banach spaces and prove that a compact set $K$ in a Banach space $E$ is polynomially convex if
and only if there exist a unital commutative Banach algebra $A$ and a continuous function $f : \Delta^* \to A$ such that (i) $A$ is generated by $f(\Delta^*)$, (ii) the character space of $A$ is homeomorphic to $K$, and (iii) $K = \text{sP}(f)$.

2 Algebras of Polynomials on Banach Spaces

Let $E$ and $F$ be complex Banach spaces. A mapping $P : E \to F$ is called an $n$-homogeneous polynomial if there exists a symmetric continuous $n$-linear map $A : E^n \to F$ such that $P(x) = A(x, \ldots, x)$, for every $x \in E$. Any finite sum of homogeneous polynomials is called a polynomial from $E$ into $F$.

The simplest and, perhaps, the most useful polynomials are obtained by multiplying linear functionals together. For a finite set $\{\phi_1, \ldots, \phi_n\}$ of functionals in $E^*$ and a vector $y \in F$, the mapping $P : x \mapsto \phi_1(x) \cdots \phi_n(x)y$ defines an $n$-homogeneous polynomial of finite type from $E$ into $F$. By the polarization formula [10, Theorem 1.10], every $n$-homogeneous polynomial of finite type can be expressed in the form

$$P(x) = \phi_1^n(x)y_1 + \phi_2^n(x)y_2 + \cdots + \phi_m^n(x)y_m, \ x \in E, \quad (2.1)$$

where $\phi_1, \ldots, \phi_m \in E^*$ and $y_1, \ldots, y_m \in F$. Any finite sum of homogeneous polynomials of finite type is called a polynomial of finite type. Since, in this paper, we consider only polynomials of finite type, we reserve the notation $\mathcal{P}(E, F)$ for the space of such polynomials from $E$ into $F$, and we write $\mathcal{P}(E)$ for $\mathcal{P}(E, \mathbb{C})$. For more information on polynomials in Banach spaces see, for example, [4], [10], or [11].

We declare that proofs of some results presented in this section may be found in the literature. The proofs, however, are included for the reader’s convenience.

Definition 2.1. Let $K$ be a compact set in $E$. Denoted by $\hat{K}_E$, the polynomially convex hull of $K$ in $E$ is defined by

$$\hat{K}_E = \{x \in E : |P(x)| \leq \|P\|_K, \ P \in \mathcal{P}(E)\}, \quad (2.2)$$

where $\| \cdot \|_K$ is the uniform norm over $K$. The set $K$ is polynomially convex if $K = \hat{K}_E$.  

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We will often write \( \hat{K} \) instead of \( \hat{K}_E \) when the underlying space \( E \) is tacitly understood. First of all, let us show that polynomially convexity is not altered when passing from a subspace to the space.

**Proposition 2.2.** Let \( E \) be a closed subspace of a Banach space \( F \), and \( K \) be a compact set in \( E \). Then the polynomially convex hull of \( K \) in \( E \) and the polynomially convex hull of \( K \) in \( F \) coincide; that is, \( \hat{K}_E = \hat{K}_F \).

**Proof.** Since the restriction to \( E \) of every polynomial in \( P(F) \) is a polynomial in \( P(E) \), we have \( \hat{K}_E \subset \hat{K}_F \). To prove the reverse inclusion, take an element \( x \in \hat{K}_F \). First, we show that \( x \in E \). If \( \phi \in F^* \) and \( \phi|_E = 0 \) then \( \|\phi\|_K = 0 \). Regarding \( \phi \) as a polynomial in \( P(F) \), we get \( \phi(x) = 0 \). Since \( E \) is closed in \( F \), the Hahn-Banach Theorem implies that \( x \in E \). Another use of the Hahn-Banach Theorem shows that every polynomial \( P \in P(E) \) extends to a polynomial \( \bar{P} \in P(F) \). Clearly, \( \|P\|_K = \|\bar{P}\|_K \) and thus, for \( x \in \hat{K}_F \),

\[
|P(x)| = |\bar{P}(x)| \leq \|P\|_K = \|\bar{P}\|_K.
\]

Since \( x \in E \), we obtain \( x \in \hat{K}_E \). Therefore, \( \hat{K}_F \subset \hat{K}_E \).

We remark that, by [9, Corollary 2.4], if \( K \) is compact then \( \hat{K} \) is also compact.

**Definition 2.3.** Let \( K \) be a compact set in a Banach space \( E \). We define \( \mathcal{P}_0(K) \) to be the space of the restriction to \( K \) of all polynomials \( P \in \mathcal{P}(E) \). We define \( \mathcal{P}(K) \) as the closure of \( \mathcal{P}_0(K) \) in \( C(K) \).

Clearly, \( \mathcal{P}(K) \) is a uniform algebra on \( K \).

**Proposition 2.4.** The algebra \( \mathcal{P}(K) \) is isometrically isomorphic to \( \mathcal{P}(\hat{K}) \).

**Proof.** Clear. 

Next, we identify the character space of \( \mathcal{P}(K) \) as the polynomially convex hull \( \hat{K} \). Before that, the following lemma is in order.

**Lemma 2.5.** Let \( \{\psi_\alpha : \alpha \in I\} \) be a bounded net in \( E^* \) that converges, in the weak* topology, to some \( \psi \in E^* \). Then \( \psi_\alpha \to \psi \), uniformly on compact sets in \( E \).
Proof. Assume that \( \{\psi_\alpha\} \) is bounded by \( M \) and that \( K \) is a compact set in \( E \). Given \( \varepsilon > 0 \), there exist \( x_1, \ldots, x_n \in K \) such that \( K \subset \bigcup_{i=1}^n B(x_i, \varepsilon) \), where \( B(x, \varepsilon) \) is the open ball with center \( x \) and radius \( \varepsilon \). Set

\[
U_0 = \{ \psi \in E^* : |\psi(x_i) - \psi_0(x_i)| < \varepsilon, i = 1, \ldots, n \}.
\]

Then \( U_0 \) is a neighbourhood of \( \psi_0 \) in the weak* topology of \( E^* \). There is \( \alpha_0 \in I \) such that \( \psi_\alpha \in U_0 \), for \( \alpha \geq \alpha_0 \). For every \( x \in K \), there exists \( i \in \{1, \ldots, n\} \) such that \( \|x - x_i\| < \varepsilon \). Now, for \( \alpha \geq \alpha_0 \),

\[
|\psi_\alpha(x) - \psi_0(x)| \leq |\psi_\alpha(x) - \psi_\alpha(x_i)| + |\psi_\alpha(x_i) - \psi_0(x_i)| + |\psi_0(x_i) - \psi_0(x)| \\
\leq M \varepsilon + \varepsilon + \|\psi_0\| \varepsilon \leq (2M + 1) \varepsilon.
\]

Since \( x \in K \) is arbitrary, we get \( \|\psi_\alpha - \psi_0\|_K \leq (2M + 1) \varepsilon \), for \( \alpha \geq \alpha_0 \). This shows that \( \psi_\alpha \to \psi_0 \) uniformly on \( K \).

Theorem 2.6. The character space of \( \mathcal{P}(K) \) is homeomorphic to \( \hat{K} \).

Proof. Using Proposition 2.4 we assume that \( K \) is polynomially convex. Define a mapping \( J : K \to \mathfrak{M}(\mathcal{P}(K)) \) by \( J(x) = \delta_x \), where \( \delta_x : f \mapsto f(x) \) is the evaluation homomorphism of \( \mathcal{P}(K) \). The mapping \( J \) imbeds \( K \) homeomorphically as a compact subset of \( \mathfrak{M}(\mathcal{P}(K)) \); see [3, Chapter 4]. We just need to show that \( J \) is surjective. Take a character \( \phi \) of \( \mathcal{P}(K) \). Since \( E^* \subset \mathcal{P}(E) \), we may consider \( \phi \) as a linear functional on \( E^* \). We show that \( \phi \) is weak* continuous on \( E^* \). Using a result of Banach (e.g. [8, Corollary 4, p 250]), it suffices to show that \( \phi \) is weak* continuous on bounded subsets of \( E^* \). Suppose that \( \{\psi_\alpha\} \) is a bounded net in \( E^* \) converging to \( \psi_0 \) in the weak* topology. By Lemma 2.5, \( \psi_\alpha \to \psi_0 \) uniformly on \( \hat{K} \) which, in turn, implies that \( \phi(\psi_\alpha) \to \phi(\psi_0) \). Hence, \( \phi \) is a weak* continuous functional on \( E^* \). By [12, Theorem 3.10], there is \( x \in E \) such that \( \phi(\psi) = \psi(x) \) for all \( \psi \in E^* \). Therefore, \( \phi(P) = P(x) \) for every polynomial \( P \in \mathcal{P}(E) \). Moreover, since \( \phi \) is a character of \( \mathcal{P}(K) \), we have \( |P(x)| = |\phi(P)| \leq \|P\|_K \), for every polynomial \( P \in \mathcal{P}(E) \). This shows that \( x \in \hat{K} = K \). Since \( \phi = \delta_x \) on \( \mathcal{P}(E) \), a dense subset of \( \mathcal{P}(K) \), we get \( \phi = \delta_x \) on \( \mathcal{P}(K) \). \( \square \)
3 A Characterization of Polynomially Convex Sets

In this section, we present the main results of the paper characterizing polynomially convex compact sets in Banach spaces. To start we briefly recall some properties of the joint spectrum of an infinite system of elements of a Banach algebra.

Suppose that \( A \) is a unital commutative Banach algebra, that \( X \) is a nonempty set, and that \( f : X \to A \) is a function. The joint spectrum of \( f \) is defined to be the set of all functions \( \lambda : X \to \mathbb{C} \) such that the ideal of \( A \) generated by \( \{ \lambda(x)1 - f(x) : x \in X \} \) is proper. To distinguish between the usual spectrum and the joint spectrum, the former is denoted by \( \text{sp}(f) \) and the latter is denoted by \( \vec{\text{sp}}(f) \). It is proved (e.g. see [2]) that

\[
\vec{\text{sp}}(f) = \{ \phi \circ f : \phi \in \mathcal{M}(A) \}. \tag{3.1}
\]

It is worth mentioning that when \( X \) is enriched with some structure (topological, algebraical, etc.), and \( f : X \to A \) is an appropriate morphism, many structural properties of \( f \) are inherited by every \( \lambda \in \vec{\text{sp}}(f) \). For more on joint spectrum, see [5] and [7], for example.

**Proposition 3.1.** Suppose that \( A \) is a unital commutative Banach algebra, that \( X \) is a compact Hausdorff space, and that \( f : X \to A \) is a continuous function. If \( f(X) \) generates \( A \), then the mapping \( \Phi : \mathcal{M}(A) \to \vec{\text{sp}}(f) \), \( \phi \mapsto \phi \circ f \), is a homeomorphism.

**Proof.** Since \( f : X \to A \) is continuous, the joint spectrum \( \vec{\text{sp}}(f) \) is a compact set in \( \mathcal{C}(X) \); see e.g. [2, Theorem 3.3]. We just need to prove that \( \Phi \) is a continuous bijection. Obviously \( \Phi \) is surjective. We show that \( \Phi \) is injective. Assume that \( \phi_1 \circ f = \phi_2 \circ f \), for \( \phi_1, \phi_2 \in \mathcal{M}(A) \). Then \( \phi_1 = \phi_2 \) on the range \( f(X) \). Since \( f(X) \) generates \( A \), we get \( \phi_1 = \phi_2 \) and thus \( \Phi \) is injective.

The continuity of \( \Phi \) follows from [1, Proposition 3.5]. It is proved that, with respect to the weak* topology of \( A^* \), the mapping \( A^* \to \mathcal{C}(X) \), \( \phi \mapsto \phi \circ f \), is continuous on bounded subsets of \( A^* \). Since \( \mathcal{M}(A) \) is bounded, we conclude that \( \Phi \) is continuous. \( \square \)
Before presenting our main results, the following lemma is in order.

**Lemma 3.2.** Let $A$ be a unital commutative Banach algebra, $X$ be a compact Hausdorff space, and $f : X \to A$ be a continuous function.

(i) With each polynomial $P \in \mathcal{P}(\mathcal{C}(X))$ is associated an element $Pf \in A$ such that $\phi(Pf) = P(\phi \circ f)$, for all $\phi \in \mathfrak{M}(A)$.

(ii) The set $\{Pf : P \in \mathcal{P}(\mathcal{C}(X))\}$ is an algebra by itself and is dense in the subalgebra of $A$ generated by $f(X)$.

**Proof.**

(i) We start with the simplest polynomials in $\mathcal{P}(\mathcal{C}(X))$, that is functionals $\psi \in \mathcal{C}(X)^*$. By the Riesz Representation Theorem, there exists a regular complex Borel measure $\mu$ such that $\psi(g) = \int_X gd\mu$, ($g \in \mathcal{C}(X)$).

By [12, Theorem 3.27], the vector-valued integral $\int fd\mu$ is defined as an element of $A$ such that $\phi(\int fd\mu) = \int \phi \circ f d\mu$, for all $\phi \in A^*$. Define $\psi f = \int fd\mu$ and $\psi^m f = (\psi f)^m$, for $m \geq 0$. Since each polynomial $P \in \mathcal{P}(\mathcal{C}(X))$ is a linear combination of $\{\psi^m : \psi \in \mathcal{C}(X)^*, m \geq 0\}$, we see that $Pf$, as an element of $A$, is defined with no ambiguity. Moreover, $\phi(Pf) = P(\phi \circ f)$, for all $\phi \in \mathfrak{M}(A)$. Note, however, that the latter equality may fail to hold for $\phi \in A^*$.

(ii) Without loss of generality, assume that $A$ is generated by $f(X)$. Since $\mathcal{P}(\mathcal{C}(X))$ is an algebra, the set $A_0 = \{Pf : P \in \mathcal{P}(\mathcal{C}(X))\}$ is an algebra. Since $f(x) = \delta_x f$ and $\delta_x \in \mathcal{P}(\mathcal{C}(X))$, for every $x \in X$, we get $f(X) \subset A_0$. Therefore, $A_0$ is dense in $A$.

Extending Theorem 1.1, we now give a characterization of polynomially convex sets in $\mathcal{C}(X)$, where $X$ is a compact Hausdorff space.

**Theorem 3.3.** For a compact set $K$ in $\mathcal{C}(X)$, the following statements are equivalent.

(i) The set $K$ is polynomially convex.
There is a unital commutative Banach algebra $A$ and a continuous function $f : X \to A$ such that $f(X)$ generates $A$ and $K = \mathfrak{sP}(f)$.

**Proof.**

(i) $\Rightarrow$ (ii) Assume that $K$ is polynomially convex. For every $x \in X$, consider the evaluation functional $\delta_x : g \mapsto g(x)$ of $\mathcal{C}(X)$. Define a function $f : X \to \mathcal{C}(K)$ by $f(x) = \delta_x$. We show that $f$ is continuous. Since $K$ is compact, as a subset of $\mathcal{C}(X)$, it is equicontinuous. This means that, given $\varepsilon > 0$, every $x_0 \in X$ has a neighborhood $U_{x_0}$ such that $|g(x) - g(x_0)| \leq \varepsilon$, $(x \in U_{x_0}, g \in K)$.

This implies that $\|f(x) - f(x_0)\|_K < \varepsilon$ for $x \in U_{x_0}$. Therefore, $f$ is continuous.

Take $A$ as the closed subalgebra of $\mathcal{C}(K)$ generated by $f(X)$. We claim that $A = \mathcal{P}(K)$. Since every element of $f(X)$ is a continuous functional of $\mathcal{C}(X)$, we have $f(X) \subset \mathcal{P}(K)$ and thus $A \subset \mathcal{P}(K)$. To see the reverse inclusion, we just need to show that the restriction to $K$ of every polynomial $P \in \mathcal{P}(\mathcal{C}(X))$ belongs to $A$. Since every such polynomial is a finite linear combination of $\{\varphi^m : \varphi \in \mathcal{C}(X)^*, m \geq 0\}$, it suffices to show that the restriction to $K$ of every functional $\varphi \in \mathcal{C}(X)^*$ belongs to $A$. This can be seen by the Krein-Millman Theorem. It is known that the set of extreme points of the closed unit ball of $\mathcal{C}(X)^*$ is exactly the set $f(X) = \{\delta_x : x \in X\}$ of unit point masses. Hence, by the Krein-Millman Theorem, every functional $\phi \in \mathcal{C}(X)^*$ is a weak* limit of finite linear combinations of elements of $f(X)$. By Lemma 2.5, convergence with respect to the weak* topology yields uniform convergence on compact sets. Hence, every functional $\phi \in \mathcal{C}(X)^*$ is a uniform limit, on $K$, of elements of $A$. We conclude that $\mathcal{P}(K) \subset A$.

By Theorem 2.6, since $K$ is polynomially convex, we get $\mathfrak{m}(A) = K$. Also, for every $g \in K$, we have $\phi_g \circ f = g$, for

$$(\phi_g \circ f)(x) = \phi_g(f(x)) = \phi_g(\delta_x) = \delta_x(g) = g(x), \quad (x \in X).$$

Therefore,

$$\mathfrak{sP}(f) = \{\phi \circ f : \phi \in \mathfrak{m}(A)\} = \{\phi_g \circ f : g \in K\} = \{g : g \in K\} = K.$$

(ii) $\Rightarrow$ (i) Assume that $K = \mathfrak{sP}(f)$, where $f : X \to A$ is a continuous function with $A$ a unital commutative Banach algebra generated by $f(X)$.
Take a function $g \in \hat{K}$. To verify that $g \in K$, we show that $g = \phi \circ f$, for some character $\phi \in \mathcal{M}(A)$. Let $A_0 = \{Pf : P \in \mathcal{P}(\mathcal{C}(X))\}$ be the subalgebra of $A$ defined in Lemma 3.2 Define a mapping $\phi_g : A_0 \to \mathbb{C}$ by $\phi_g(Pf) = Pg$. We show that $\phi_g$ is a well-defined continuous homomorphism. Since $g \in \hat{K}$, we have, for every $P \in \mathcal{P}(\mathcal{C}(X))$,

$$|\phi_g(Pf)| = |Pg| \leq \|P\|_K$$

$$= \sup\{|P(\phi \circ f)| : \phi \in \mathcal{M}(A)\}$$

$$= \sup\{|\phi(Pf)| : \phi \in \mathcal{M}(A)\} \leq \|Pf\|.$$ (3.2)

Therefore, $Pf = 0$ implies $Pg = 0$ which shows that $\phi_g$ is well-defined. Obviously, $\phi_g$ is a homomorphism. Also (3.2) shows that $\phi_g$ is continuous on $A_0$. Since $A$ is generated by $f(X)$, by Lemma 3.2, $A_0$ is dense in $A$. Hence, $\phi_g$ extends to a character of $A$, still denoted by $\phi_g$. We have

$$(\phi_g \circ f)(x) = \phi_g(f(x)) = \phi_g(\delta_x f) = \delta_x(g) = g(x), \quad (x \in X).$$

Therefore, $g = \phi_g \circ f$ and thus $g \in K$. \qed

We conclude the paper with our final result on characterising polynomially convex sets in arbitrary Banach spaces.

**Theorem 3.4.** For a compact set $K$ in a Banach space $E$, the following are equivalent.

(i) The set $K$ is polynomially convex in $E$.

(ii) There is a unital commutative Banach algebra $A$ and a continuous function $f : \Delta^* \to A$ such that $A$ is generated by $f(\Delta^*)$ and $K = \delta \overline{\mathcal{P}}(f)$. Moreover, $K$ is homeomorphic to $\mathcal{M}(A)$.

**Proof.** It follows from the fact that $E$ is isometrically isomorphic to a closed subspace of $\mathcal{C}(\Delta^*)$, along with propositions 2.2 and 3.1 and Theorem 3.3. \qed

As an illustration of Theorem 3.4, the following example is given.
Example

Let $E$ be an arbitrary Banach space and let $K = \{e_0, e_1, e_2, \ldots \}$ be a set in $E$ with $e_n \to e_0$. We show that $K$ is polynomially convex. Take $A = \mathcal{C}(K)$, which is, in fact, the space of all convergent complex sequences. Define a function $f : \Delta^* \to A$ by $f(\psi)(x) = \psi(x), \; x \in K$. Since $K$ is compact, by Lemma 2.5 $f$ is continuous. Let $s_j = (\delta_{ij})_{i=1}^\infty$, where $\delta_{ij}$ is the Kronecker delta. Then $S = \{s_1, s_2, \ldots \}$ generates $A$ and $S \subset f(\Delta^*)$. Hence, $f(\Delta^*)$ generates $A$. Finally, since $\mathfrak{M}(A) = \{\delta_x : x \in K\}$ and

$$\delta_x \circ f(\psi) = \delta_x(f(\psi)) = f(\psi)(x) = \psi(x) = \delta_x(\psi) \quad (x \in K)$$

we get

$$\tilde{S}\tilde{P}(f) = \{\delta_x \circ f : x \in K\} = \{\delta_x : x \in K\} = K.$$  

To be more precise, note that the last equality is, in fact, a homeomorphism. All requirements in Theorem 3.4 are fulfilled and $K$ is polynomially convex.

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