Generalised quantum scissors for noiseless linear amplification

Matthew Winnel,\textsuperscript{1,*} Nedasadat Hosseinidehaj,\textsuperscript{1} and Timothy C. Ralph\textsuperscript{1}

\textsuperscript{1}Centre for Quantum Computation and Communication Technology, School of Mathematics and Physics, University of Queensland, St Lucia, Queensland 4072, Australia

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We generalise the concept of optical state truncation and noiseless linear amplification to enable truncation of the Fock-state expansion of an optical state to higher order and to simultaneously amplify it using linear optics. The resulting generalised quantum scissors are more efficient for noiseless linear amplification than employing multiple scissors in parallel and are experimentally practical. As a particular example, we focus on a third-order scissor device and demonstrate advantages in terms of fidelity with the target state, probability of success, distillable entanglement, and the amount of non-Gaussianity introduced.

The no-cloning theorem [1] forbids the deterministic, linear (i.e. phase insensitive) amplification of a quantum state. Hence, all deterministic linear amplifiers must introduce noise [2]. Nevertheless, non-deterministic noiseless linear amplification is possible if the amplifier is allowed to operate in a probabilistic, but heralded way, and the alphabet of states has an energy bound [3, 4].

Noiseless linear amplification has proven a very useful technique in quantum optics with numerous experimental demonstrations [5] such as distillation of entanglement [4], purification of entanglement [6], amplification of qubits [7] and enhanced metrology [8]. Proposed applications include continuous variable error correction [9], quantum key distribution [10–12] and quantum repeaters [13], and discrete variable Bell inequalities [14].

The non-deterministic quantum scissor introduced by Pegg et al [15] truncates an input optical field to first order, retaining only the vacuum and one-photon components of the input state. In their original proposal, Ralph and Lund introduced a modified scissor device which truncates an input state to first order and simultaneously amplifies it by increasing the amplitude of the one-photon component relative to the vacuum component [3]. For input states of small amplitude, the modified scissor acts as an ideal noiseless linear amplifier (NLA).

In order to go beyond small input amplitudes, Ralph and Lund proposed employing multiple quantum scissors in parallel [3]. For a large finite number of scissors, the set-up acts as an ideal NLA, however with a vanishing probability of success. If a small number of scissors are used, the amplification is “distorted”, that is, the Fock coefficients of the state obtained are multiplied by different constants than those required for ideal linear amplification. Other methods for NLA do not truncate the state but still distort Fock components higher than one [6, 16].

To improve the one photon scissor-based NLA, there have been attempts to generalise the modified quantum scissor to two photons [17], however the resulting device imposes an undesired non-linear sign change to the two photon component of the amplified state. A generalisation of the original quantum scissor to higher order has also been made but this does not allow for amplification [18–20]. Alternatively, a protocol that could truncate and amplify without distortion was described in Ref. [21], however this solution is impractical as it requires a massive high order optical non-linearity. As a result, no demonstration of NLA without distortion of the higher order Fock state components has been achieved, thus seriously limiting future applications.

Here, we generalise the concept of optical state truncation and amplification and propose a practical device which can correctly amplify the input state up to higher order. Our generalised scissors function in a way that is analogous to the original modified quantum scissor and uses only linear optics. Furthermore, the device naturally performs noiseless linear amplification without distorting the amplified Fock coefficients.

**Generalised scissors**: Suppose the input state in the Fock basis is \( |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \). An ideal NLA performs the transformation \( \hat{T}_{\text{ideal}} |\psi\rangle \rightarrow \sum_{n=0}^{\infty} g^n c_n |n\rangle \), where \( g \) is the gain of the NLA. The success probability is zero for any device that achieves this transformation perfectly [22].

The modified quantum scissors [3, 15] truncate and amplify an optical state in Fock space and is shown in Fig. 1. It performs the transformation

\[
\hat{T}_1 |\psi\rangle = \sqrt{\frac{1}{2g^2 + 1}} (c_0 |0\rangle \pm gc_1 |1\rangle),
\]

where the gain is \( g = \sqrt{\eta/(1-\eta)} \). The plus sign corresponds to the measurement outcome shown in Fig. 1, i.e., detection of a single photon at the upper port and no photons at the port on the right. The minus sign corresponds to the reverse measurement outcome, i.e., no photon at the top and a single photon at the side.

This device is called a quantum scissor since all Fock components greater than one are truncated. For the device to operate as an ideal NLA, the two photon component must be negligible, that is \( |g^2 c_2| \ll |gc_1| \). The one photon scissor acts as an ideal NLA only for small input states and the effect of the truncation is severe for large input states.

Our generalised three photon quantum scissor is shown in Fig. 2. It can amplify input states of larger amplitude. It per-
forms the transformation
\[
\hat{T}_3|\psi\rangle = \frac{\sqrt{6}}{8} \left( \frac{1}{g^2 + 1} \right)^2 \times (c_0|0\rangle + g c_1|1\rangle + g^2 c_2|2\rangle + g^3 c_3|3\rangle),
\]
where the gain is \( g = \sqrt{\eta/(1-\eta)} \) and the success probability is
\[
P_3 = 4 \times \langle \psi | \hat{T}_3 \hat{T}_3^\dagger | \psi \rangle^2.
\]

Three photons enter the device as a resource, three single photon detectors register single photons, and one detector registers no photon. Specifically, we consider the measurement outcome shown in Fig. 2, i.e. detecting three single photons at the top of the device and no photons at the other mode. There are three other possible patterns for getting three “clicks” and one “no click.” These other click patterns lead to a heralded phase shift of the state, but the magnitudes of the Fock components are unchanged. Thus, the success probability is effectively increased by a factor of four.

The three photon scissor truncates and ideally amplifies the input state to third order. The device operates as an ideal NLA as long as \(|g^4 c_4| \ll |g^3 c_3|\), which is a major improvement from the single photon scissor, at the cost of a reduced probability of success.

We have found that our device generalises (at least) to \(2^S-1\), where \(S=1, 2, 3\) etc. The next quantum scissor we have found is a seventh order device. For the rest of the paper we will refer to the scissors as 1-scissor, 3-scissor, 7-scissor, etc. The prefix in our notation is the number of photons entering the device as a resource, the number of detectors that register single photons, and the Fock state truncation of the output state. See the Supplemental Material for derivations and further details.

**Amplification of coherent states:** The fidelity \( F \) of the output state with the ideally amplified state (i.e. the target state) is useful as a measure of how well the input state has been amplified. When considering the performance of probabilistic amplifiers, it is also important to consider the success probability.

In Fig. 3 we plot the success probability and infidelity \((1-F)\) of the output state with the target state \(|g\gamma\rangle\) as a function of the gain \( g \) for a fixed input coherent state with amplitude \( \gamma = 0.1 \), and compare generalised scissors with NLA based on multiple 1-scissors in parallel. Despite the trade-off between fidelity and success probability we see that our 3-scissor simultaneously achieves higher fidelity and success probability than the NLA based on four 1-scissors.

**Entanglement distillation with generalised quantum scissors:** A two mode EPR state with squeezing parameter \( r \) is
\[
|\chi\rangle = \sqrt{1 - \chi^2} \sum_{n=0}^{\infty} \chi^n |nn\rangle,
\]
where \( \chi = \tanh r \) and the mean photon number is \( \bar{n} = \sinh^2 r \). The notation \(|nn\rangle\) is shorthand for the two mode Fock state \(|n\rangle \otimes |n\rangle\).

Placing a 1-scissor on one arm transforms the EPR state according to
\[
|\chi\rangle \rightarrow \sqrt{\frac{1 - \chi^2}{2 (g^2 + 1)}} (|00\rangle + g \chi |11\rangle).
\]

Placing the 3-scissor on one arm performs the transformation
\[
|\chi\rangle \rightarrow \sqrt{\frac{6}{8}} \sqrt{\frac{1 - \chi^2}{(g^2 + 1)^3}} \sum_{n=0}^{3} (g \chi)^n |nn\rangle.
\]

Both scissors herald states that have the form of a truncated EPR state, but with an effective increase in \( \chi \rightarrow g \chi \). Therefore, the scissors are useful to distill entanglement. These protocols generalise to EPR states distributed through loss, allowing purification of entanglement distributed over long distances \([6, 9]\).
For such protocols, the 1-scissor usually works best when limited to small $\chi$ and large loss [13]. The 3-scissor allows distillation protocols to operate in regimes of higher squeezing and less loss, at the cost of a reduced probability of success, and it also introduces less non-Gaussianity. To demonstrate this, we calculate the entanglement of formation and reverse coherent information [23] in the following.

After transmission of one mode of an EPR state through a pure loss channel of transmissivity $T$ followed by either a 1-scissor or a 3-scissor, we calculate the Gaussian entanglement of formation (GEOF) as an entanglement measure to evaluate the performance of each scissor. The GEOF quantifies the amount of two-mode squeezing required to prepare an entangled state from a classical state [24], which is a lower bound on the entanglement of formation.

The GEOF is calculated following Ref. [24] and using results from Ref. [25, 26]. Figure 4 shows the GEOF as a function of $g$ for the 1-scissor and our 3-scissor given EPR parameter $\chi = 0.3$ and channel transmissivity $T = 0.1$. Also shown is the amount of entanglement for the same EPR state and loss channel but with no quantum scissor. The deterministic bound assumes an infinitely squeezed EPR state sent through the same loss channel and no quantum scissor. Crossing the deterministic bound is a necessary condition for the distillation to be useful in error correction or repeater protocols [27].

The 3-scissor has a higher GEOF than the 1-scissor for the same gain. In particular, for these parameters the 3-scissor crosses the deterministic bound whilst the 1-scissor is unable to cross this bound.

To demonstrate the non-Gaussian effect of the scissors, we calculate the total reverse coherent information (RCI) [23], and compare it with the the Gaussian RCI, i.e. the total RCI calculated for a Gaussian state with the same covariance matrix. The RCI gives a lower bound on the distillable entanglement [23].

We plot the total RCI and Gaussian RCI as a function of gain in Fig. 5 for EPR parameter $\chi = 0.3$ and channel transmissivity $T = 0.1$. This plot demonstrates that for these parameters, the non-Gaussian entangled state heralded after the 1-scissor suffers severely from unwanted non-Gaussianity, whereas, the non-Gaussianity introduced by the 3-scissor is not so harsh, especially for small gain. The 3-scissor will be useful for protocols in regimes of less loss, larger initial squeezing, and larger gain.

**Imperfect operations:** An important consideration is how the performance of the 3-scissor is affected by experimental imperfections. Single photon detectors with quantum efficiency $\tau_d$ can be modelled by a lossy channel with transmissivity $\tau_d$ followed by a perfect single photon detector. We find that non-perfect efficiency impacts the success probability but has a small impact on the fidelity. Typically it is more feasible in an experiment to use on-off photon detectors. On-off detectors cannot discriminate between different numbers of photons, but can only distinguish between vacuum and non-vacuum. Again, for input states with small amplitudes, the effect of on-off detection on the fidelity is small.

The reason that our generalised scissors are robust to these practical issues is due to the detection scheme. The detection scheme separates all the light into several modes and performs single photon detection on those modes. In the high fidelity regime, there is only a small number of photons in the device at any one time (if there was not, the device would not be working in a high fidelity regime), and so errors due to on-off detection or imperfect detectors are rare.

Another important practical aspect of the device is the resource mode. The efficiency of the resource is modelled by a lossy channel with transmissivity $\tau_s$ following preparation of the Fock state source, $|1\rangle$ for the 1-scissor and $|3\rangle$ for the 3-scissor. The success probability and the fidelity are both negatively impacted, but for coherent states, not severely. For coherent state inputs, the 3-scissor still performs ideal truncation and amplification up to two photons if two photons rather
FIG. 4. a) Gaussian Entanglement of Formation (GEOF) and b) probability of success $P$ as a function of gain $g$ for an EPR state, with one arm propagated through a lossy channel followed by a 1-scissor or a 3-scissor for perfect set-up, and for non-ideal realistic resource and detectors ($\tau_s = \tau_d = 0.7$). Channel transmissivity is $T = 0.1$ and EPR parameter is $\chi = 0.3$. The “loss channel” is GEOF calculated for direct transmission, i.e. no quantum scissor. The “deterministic bound” is the amount of entanglement given that an infinitely squeezed state has been sent through the channel [24]. This plot shows a situation for which the crossing point of the deterministic bound, the minimum requirement for error correction [13], can be reached by the 3-scissor, even for a realistic experimental set-up, but is unobtainable by the 1-scissor.

than three are injected into the device, of course with a reduced fidelity since the truncation is at second order not three. This surprising result allows the 3-scissor to keep performing well even with loss on the resource mode. This result generalises to all scissors, i.e., the 7-scissor ideally truncates and amplifies coherent states even if less than seven photons are injected into the device.

In Fig. 4 we also include the effect of inefficient detectors and an inefficient source ($\tau_s = \tau_d = 0.7$). We find that under realistic conditions the 3-scissor can still distill entanglement above the deterministic bound under conditions for which this is impossible for the 1-scissor (See Supplemental Material for derivations with inefficiencies and more plots).

Discussion and conclusion: We have shown our generalised scissors amplify and truncate arbitrary input states without distorting the Fock coefficients, assuming perfect implementation. Considering more realistic devices, we find that in the working regime of high fidelity, imperfect single photon detectors, or employing on-off detectors has little effect on the fidelity and only impacts the success probability. Of greater importance is the efficiency of the Fock state source. For coherent state inputs, we find that the 3-scissor device is naturally and surprisingly robust to non-ideal resource efficiency. Realistic devices perform well for entanglement distillation as well.

Another possible application of our scissors would be to engineer optical states [28]. For example, making a slight change to our 3-scissor device, in particular by accepting a different measurement outcome, a different state will be heralded. This heralded state may be potentially useful (for instance, we speculate that it would be possible to generate truncated cat-like states in this way).

Generalised scissors belong to a class of protocols known as tele-amplification [29]. Scissors are tele-amplification devices upon taking the amplitude of the entangled cat-state resource to zero. Since scissors herald states with a hard truncation in Fock space, but in general tele-amplification does not, we speculate that it may be beneficial for some protocols to use a general tele-amplification device rather than a scissor.

The laws of quantum physics puts absolute limits on the performance of probabilistic NLA [22]. A natural question to ask is how do the scissors compare against these ultimate bounds. Within the high-fidelity region NLAs have success probabilities that decrease exponentially with $N$ (the order of truncation), and this is an unavoidable consequence of attempting noiseless linear amplification [22]. The fidelity and success probability together determine the overall performance of NLA devices.

Our scissors do not obtain the ultimate bound on the success probability [22], however this is the price to pay for such sim-
ple linear devices, employing just beamsplitters and photon detectors. To approach the quantum limit, one would require more complicated (probably highly nonlinear) devices, such as the proposal in Ref. [21].

In conclusion, we have introduced new quantum scissors which truncate and ideally amplify optical states using linear optical components. Compared to use of multiple scissors in parallel we found that the new scissors are more efficient for noiseless linear amplification and more practical for experimental implementation. This device may be scaled-up to \(2^N-1\) numbers of photons at the cost of a diminishing probability of success. We expect that our generalised scissors will in some situations improve the performance of existing experiments in quantum communication and make theorized protocols realisable in the near future.

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