Polychronakos fractional statistics with a complex-valued parameter

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Abstract. A generalization of quantum statistics is proposed in a fashion similar to the suggestion of Polychronakos [Phys. Lett. B 365, 202 (1996)] with the parameter $\alpha$ varying between $-1$ (fermionic case) and $+1$ (bosonic case). However, unlike the original formulation, it is suggested that intermediate values are located on the unit circle in the complex plane. In doing so one can avoid the case $\alpha = 0$ corresponding to the Boltzmann statistics, which is not a quantum one. The limits of $\alpha \to +1$ and $\alpha \to -1$ reproducing small deviations from the Bose and Fermi statistics, respectively, are studied in detail. The equivalence between the statistics parameter and a possible dissipative part of the excitation spectrum is established. The case of a non-conserving number of excitations is analyzed. It is defined from the condition that the real part of the chemical potential equals zero. Thermodynamic quantities of a model system of two-dimensional harmonic oscillators are calculated.

1. Introduction

Quantum systems obeying the so-called fractional statistics different from the Bose or Fermi statistics have been studied broadly in the recent decades. Several approaches to the definition of fractional statistics are known [1]. This concept appears mainly in studies of low-dimensional systems and the realization in physical systems can occur effectively, for instance, the connection of the exclusion (anyon) statistics parameter and the interaction in one-dimensional systems was studied in [2, 3, 4]. The anyon–fermion mapping is also known in the application to ultracold gases [5]. A fractional (intermediate, or para-)statistics known as the Gentile statistics [1, 6] is obtained if the maximum occupation of a particular energy level is limited by some integer greater than unity. It can be applied in particular to the study of finite bosonic systems [7].

A simple generalization of the statistics was suggested by Polychronakos [8], see also [9]. In this work, a modified approach is proposed with a parameter of statistics being a complex number. Such a defined statistics has a seeming drawback as it requires that some physical quantities, like energy or chemical potential, are complex. This is however not a problem as, for instance, complex-valued energy is usually connected to some dissipative processes [10]. Moreover, approaches involving complex chemical potential have a vast application domain, ranging from quantum chromodynamics [11] to the physics of semiconductors [12].

The paper is organized as follows. Section 2 contains the description of statistics and details on the calculation scheme used in further analysis. In Section 3 and the limiting cases reproducing small deviations from the Bose and Fermi statistics are considered in detail. The case of a non-conserving number of excitations is studied in Section 4, where the energy of a two-dimensional oscillator system is calculated. A brief discussion is given in Section 5.
2. Definitions and calculation scheme

A generalization of quantum statistics is proposed in a fashion similar to the suggestion of Polychronakos [8], with the distribution function given by

$$ f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/T} - \alpha}, $$

(1)

where $T$ is temperature, $\mu$ is the chemical potential, and $\varepsilon$ is the energy of the respective level.

The parameter $\alpha$ varies between $-1$ (fermionic case) and $+1$ (bosonic case). However, unlike the original formulation, it is suggested that intermediate values are located on the unit circle, $\alpha = e^{i\pi\nu}$, but not on the real axis (see image on the left). In doing so, in particular, one can avoid the case $\alpha = 0$ corresponding to the Boltzmann statistics, which is not a quantum one.

For a system with the excitation spectrum $\varepsilon = \varepsilon_j$ the number of particles $N$ is given by

$$ N = \sum_j g_j f(\varepsilon_j) = \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu)/T} - \alpha}, $$

(2)

where the summation runs over all the available levels and $g_j$ is the degeneracy of the $j$th level.

To calculate thermodynamic properties one can proceed from the expression for energy $E$ in the following form:

$$ E = \sum_j \varepsilon_j g_j f(\varepsilon_j) = \sum_j \frac{\varepsilon_j g_j}{e^{(\varepsilon_j - \mu)/T} - \alpha} $$

(3)

provided that the chemical potential $\mu$ as a function of $T$ and $N$ is obtained from Eq. (2).

As the above relations involve complex numbers, in further considerations we will demand that the imaginary part of the number of particles is zero: $\text{Im} N = 0$.

3. Bose and Fermi limits

The bosonic limit is given by $\nu \to 0$, and thus $\alpha = e^{i\pi\nu} = 1 + i\pi\nu + \ldots$. Suppose for definiteness that the spectrum $\varepsilon_j$ is real and the chemical potential $\mu = \mu' + i\mu''$, with $\mu', \mu'' \in \mathbb{R}$. In the considered limit, the imaginary part $\mu''$ is a small number.

Expanding expression (2) for the number of particles $N$ into series with respect to small quantities $\nu$ and $\mu''$ we obtain in the first order:

$$ N = \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu)/T} - 1} + i \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu)/T} - 1}^2 \left( \pi \nu + \frac{\mu''}{T} e^{(\varepsilon_j - \mu')/T} \right). $$

(4)

Dropping the imaginary part one can see that $\mu'$ is nothing but the chemical potential of $N$ bosons with spectrum $\varepsilon_j$, $\mu' = \mu_{\text{Bose}}$, where

$$ N = \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu_{\text{Bose}})/T} - 1}. $$

(5)

Performing simple calculations we can obtain from the condition $\text{Im} N = 0$ the following expression for $\mu'$:

$$ \mu' = -\pi \nu T \frac{p}{N + p}, \quad \text{where} \quad p = \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu)/T} - 1}^2. $$

(6)
Now, let the spectrum is complex, $\varepsilon_j = \epsilon_j + i\gamma_j$, and the particles obey ordinary bosonic statistics. A small dissipative correction can be obtained using the calculations similar to the above analysis. Equaling the energy of the bosonic system with the complex spectrum and that of the system with the defined fractional statistics, the following relation can be obtained for $j$ being large enough:

$$\gamma_j = \pi \nu T \varepsilon_j.$$  \hfill (7)

The fermionic limit is given by $\nu \to 1$. Let $\nu = 1 - \bar{\nu}$ and thus $\alpha = e^{i\pi(1-\nu)} = -1 + i\pi\bar{\nu} + \ldots$. From this point, one can perform the same derivations as for the bosonic limit using $\bar{\nu}$ instead of $\nu$. The only exception is that for the real part of the chemical potential the Fermi-distribution naturally occurs, $\mu' = \mu_{\text{Fermi}}$, where

$$N = \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu_{\text{Fermi}})/T} + 1}.$$  \hfill (8)

and the sign in the definition of $p$ is changed accordingly: $p = \sum_j \frac{g_j}{(e^{(\varepsilon_j - \mu')/T} + 1)^2}$.

It is worth mentioning that the transition between the Bose and Fermi limits might be done also via the lower half-plane $\text{Im}\alpha < 0$, so that the fermionic limit is $\nu \to -1$. For definiteness, however, all the further considerations will be limited to $\text{Im}\alpha \geq 0$.

### 4. Results for two-dimensional oscillators

If the number of excitations in a system is not conserved, so that they can be freely created and annihilated, the chemical potential is zero. Let us modify this condition for a complex-valued chemical potential as follows: $\text{Re}\mu = 0$. With such a simplification in the problem one can proceed to the calculations using the scheme described in Section 2. Namely, from the equation linking the number of particles $N$ and chemical potential $\mu = i\mu''$

$$N = \text{Re}\ N + i \text{Im}\ N = \sum_j \frac{g_j}{e^{(\varepsilon_j - \mu'')/T} - (\cos \pi \nu + i \sin \pi \nu)},$$  \hfill (9)

where the statistics parameter is explicitly written as $\alpha = \alpha' + i\alpha'' = \cos \pi \nu + i \sin \pi \nu$. Demanding, as before, that $\text{Im}\ N = 0$ we obtain the relation between the imaginary part of the chemical potential $\mu''$ and the parameter of statistics $\nu$.

So far all the expressions are obtained in general, without reference to any special physical system. To demonstrate the results of calculations the system of two-dimensional harmonic oscillators is considered. Without loosing generality one can neglect the discreteness of energy levels and change the summation over energy levels into integration with some density of states. For the system under consideration, density of states $g(\varepsilon) \propto \varepsilon$. After simple transformations the condition $\text{Im}\ N = 0$ becomes:

$$\int_0^\infty \frac{\left( e^{\varepsilon/T} \sin \frac{\mu''}{T} + \sin \pi \nu \right) \varepsilon d\varepsilon}{e^{2\varepsilon/T} + 2e^{\varepsilon/T} \left( \sin \pi \nu \sin \frac{\mu''}{T} - \cos \pi \nu \cos \frac{\mu''}{T} \right) + 1} = 0.$$  \hfill (10)

It is convenient to demonstrate the solution of the above equation in a graphic form, see Fig. 1.

The real part $\mathcal{E}$ of energy $E = \mathcal{E} + i\Gamma$ given in a general form by Eq. (3) becomes in our case as follows:

$$\mathcal{E} = \int_0^\infty \frac{\left( e^{\varepsilon/T} \cos \frac{\mu''}{T} - \cos \pi \nu \right) \varepsilon^2 d\varepsilon}{e^{2\varepsilon/T} + 2e^{\varepsilon/T} \left( \sin \pi \nu \sin \frac{\mu''}{T} - \cos \pi \nu \cos \frac{\mu''}{T} \right) + 1}.$$  \hfill (11)
Figure 1. Solving Eq. (10). The function $\text{Im} N$ (vertical axis) is plotted versus $\sin \frac{\mu''}{T}$ (horizontal axis) for $\nu = 0.02$. Two solutions correspond to different possible signs of $\cos \frac{\mu''}{T}$ depending on the value of temperature $T$.

For simplicity, the units of energy and temperature are $\hbar \omega$, where $\omega$ is the oscillator frequency, and $\hbar \omega$ is put equal unity.

The temperature dependence of energy $E$ is shown in Fig. 2 in comparison with Fermi and Bose systems with the same parameters.

Figure 2. Energy (11) (vertical axis) versus temperature $T$ (horizontal axis) for different values of the statistics parameter $\nu$. The line correspondence is as follows (top to bottom): Bose-system (black dashed, thin); $\nu = 0.2$ (blue solid, thick); $\nu = 0.5$ (black solid, thick); $\nu = 0.7$ (green dashed, thick); Fermi-system (red solid, thin). Units are arbitrary.

5. Discussion

A generalization of the fractional statistics of Polychronakos is proposed using a complex-valued parameter. The energy of the model system of two-dimensional oscillators is calculated to confirm that the suggested generalization interpolates smoothly between the Bose- and Fermi-cases. It is shown that the limits of the statistics parameter reproducing the Bose and Fermi statistics correspond to a small dissipative correction to the energy spectrum. Further studies on the suggested fractional statistics are required to reveal its possible application as a tool for the analysis of bosonic and fermionic systems under special conditions, when the complex-valued quantities can occur effectively.

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