Integration Approach to Ising Models

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Abstract

An integral representation of the partition function for general $n$-dimensional Ising models with nearest or non-nearest neighbours interactions is given. The representation is used to derive some properties of the partition function. An exact solution is given in a particular case.

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As a simple prototype of a statistical mechanical system that undergoes a phase transition for spatial dimensionality \( n > 1 \), the Ising(-Lenz) model \([1]\) has been extensively studied in various ways. Since the exact solution of the free energy \( F \) and the spontaneous magnetization for the two dimensional zero-field Ising model (on a square lattice) were obtained more than fifty years ago \([2, 3]\), many efforts have been made towards a detailed study of properties and the possible finding of exact solutions for the higher dimensional Ising model or for a two-dimensional Ising model with nonzero magnetic field, for reviews see e.g., \([4-7]\). In this paper we study the Ising models by an “integration approach”. We present an integral representation of the partition function for general \( n \)-dimensional Ising models with either nearest or non-nearest neighbours interactions. From this representation we prove that the partition function of \( n \)-dimensional homogeneous (i.e. translation invariant) Ising systems on a square lattice is uniquely given by the eigenvalues of the related interaction coupling matrix. For one-dimensional homogeneous ferromagnetic systems with positive coupling coefficients the partition function satisfies a special equality which means that the variation of the interaction couplings of the system is related to the variations of the external magnetic field and the temperature. For some special cases of interaction couplings, in all dimensions, we get an exact representation of the partition function in terms of a Bessel function. We also give an approximate representation of the general partition function for the homogeneous Ising model, which shows that its leading term is given by the largest positive eigenvalue (resp. the largest absolute value of the negative eigenvalues) of the related interaction coupling matrix in the ferromagnetic (resp. antiferromagnetic) case. In the case of nearest neighbours interactions in two dimensions our integral representation is connected with the study of the Ising limit of the \( \phi_4^2 \) double-well models, see e.g. \([8]\).

The partition function of the Ising models with the usual nearest neighbours interactions on a \( d \)-dimensional lattice with \( \Lambda \) lattice sites is given by, see e.g. \([4]\),

\[
Z^0_\Lambda = \sum_{\{\sigma\}} \exp \left\{ K \sum_{<ij>_0} \sigma_i \sigma_j + K' \sum_{i=1}^\Lambda \sigma_i \right\},
\]

(1)

where \( \sigma_i, i = 1, 2, ..., \Lambda \), takes values +1 and -1, \(<ij>_0\) denotes the nearest-neighbour pairs on the lattice and \( K = J/kT, K' = H/kT \) with \( T \) the temperature, \( k \) the Boltzmann
constant, $H$ the external magnetic field and $J$ the coupling constant. $J$ is positive for ferromagnetic systems and negative for antiferromagnetic systems.

We consider the partition function (1) in a more general form including both nearest and non-nearest neighbours interactions,

$$Z_{\Lambda} = \sum_{\{\sigma_i\}} \exp \left\{ K \sum_{i,j=1}^{\Lambda} C^0_{ij} \sigma_i \sigma_j + K' \sum_{i=1}^{\Lambda} \sigma_i \right\},$$

(2)

where $C^0 = (C^0_{ij})$ is a symmetric $\Lambda \times \Lambda$ matrix,

$$C^0_{ii} = 0, \quad C^0_{ij} = C^0_{ji}, \quad i \neq j, \quad i, j = 1, 2, \ldots, \Lambda.$$

(3)

$C^0_{ij}$ stands for the interaction coupling between $\sigma_i$ and $\sigma_j$. Clearly if $C^0_{<ij>0} = 1$ and $C^0_{ij} = 0$ for all non-nearest neighbours pairs, the partition function (2) is reduced to (1).

In the following we call $C^0$ the interaction coupling matrix.

Let $<ij>_1$, $<ij>_2$ and $<ij>_{\alpha}$ denote the next-to-nearest neighbours, next to next-to-nearest neighbours and $\alpha$-th next-to-nearest neighbours interactions, respectively. In the following we consider systems such that all interaction coupling constants for any given $\alpha$ are the same, i.e., $C_{<ij>_{\alpha}} = c_{\alpha}, \forall i, j = 1, 2, \ldots, \Lambda, \alpha = 0, 1, 2, \ldots, \Lambda - 1$. We also assume that the number of sites interacting with an arbitrary given site $i$ is the same for all $i = 1, \ldots, \Lambda$. We call these systems homogeneous. In particular this implies that we impose the periodic boundary condition on the boundary of the lattice if the lattice is a bounded portion of $\mathbb{Z}^d$ (or equivalently the lattice is identified with a $d$-dimensional torus).

Let $l_i$ be the number of lines connected to a lattice site $i$. $l_i$ is then a topological invariant of the lattice and is called the topological link number associated with the lattice site $i$. For homogeneous systems with only nearest-neighbour interactions, the link number $l_i$ is exactly equal to the number of the sites interacting with the lattice site $i$, and by homogeneity we have that $l_i$ is independent of $i$, i.e., $l_i = l$ for all $i = 1, \ldots, \Lambda$. For instance, for a periodic one-dimensional chain with nearest-neighbour interactions we have $l = 2$. For a two (resp. three) dimensional square lattice with nearest-neighbour interactions and periodic boundary conditions we have $l = 4$ (resp. $l = 6$). Therefore
for a homogeneous system in \( d \) dimensions with only nearest-neighbour interactions (i.e., such that \( C_{<ij>_0}^0 = 1 \) and \( C_{ij}^0 = 0 \) for all non-nearest neighbours pairs), the matrix \( C^0 \) has the following properties:

\[
\sum_{i=1}^{\Lambda} C_{ij}^0 = \sum_{i=1}^{\Lambda} C_{ji}^0 = l, \quad \forall j = 1, 2, ..., \Lambda; \tag{4}
\]

\[
\sum_{j=1}^{\Lambda} C_{ij}^0 = \Lambda l; \quad Tr(C^0) = 0. \tag{5}
\]

In fact, a higher dimensional system with nearest-neighbour interactions can always be viewed as a lower dimensional system with non-nearest-neighbour interactions, and vice versa. For example, for a one dimensional chain with nearest and next-to-nearest neighbours interactions,

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots
\end{array}
\]

we have \( C_{i,i+1}^0 \neq 0, C_{i,i+2}^0 \neq 0 \), i.e., a lattice site \( i \) has interactions with the lattice sites \( i \pm 1 \) and \( i \pm 2 \). This system is equivalent to a two dimensional band with nearest-neighbour interactions,

\[
\begin{array}{cccccccc}
2 & 4 & 6 & 8 & 10 & \ldots
\end{array}
\]

where the double lines stand for nearest-neighbour interactions in the one dimensional chain and the single lines stand for next-to-nearest neighbours interactions in the one dimensional chain. If different interaction couplings in different directions are assumed then we have \( C_{i,i+1}^0 \neq C_{i,i+2}^0 \).

To describe the interaction properties for both systems with nearest and non-nearest-neighbour interactions, we define a generalized link number \( L \) of the lattice by,

\[
L \equiv \sum_{i=1}^{\Lambda} C_{ij}^0 = \sum_{i=1}^{\Lambda} C_{ji}^0, \quad \forall j = 1, 2, ..., \Lambda. \tag{6}
\]
When the systems have only nearest-neighbour interactions $L$ is equal to the topological link number $l$ of the lattice. Instead of (5) we have, for homogeneous systems with either nearest-neighbour or non-nearest-neighbour interactions (in arbitrary dimension),

$$
\sum_{i,j=1}^{\Lambda} C_{ij}^0 = \Lambda L; \quad Tr(C^0) = 0. 
$$

(7)

Any real symmetric matrix $M$ (over the complex numbers $\mathbb{C}$) can be diagonalized by an orthogonal similarity transformation. The matrix used to diagonalize $M$ has as its columns an orthonormal set of eigenvectors for $M$. The resulting diagonal matrix has as its diagonal elements the eigenvalues of $M$. Let $u_i$ be an orthonormal basis in $\mathfrak{g}^\Lambda$ (as a $\Lambda$-dimensional Hilbert space, with scalar product $\cdot$) of the (column) eigenvectors of $C^0$, with eigenvalues $\lambda_i^0$, $i = 1, 2, ..., \Lambda$,

$$
C^0 u_i = \lambda_i^0 u_i, \quad \tilde{u}_i \cdot u_j = \delta_{ij}, \quad i, j = 1, 2, ..., \Lambda, 
$$

(8)

(with $\tilde{u}_i$ the adjoint vector to $u_i$). Let $A$ be the orthogonal matrix that diagonalizes $C^0$. Then

$$
A = (u_1, u_2, ..., u_\Lambda), 
$$

(9)

$$
\tilde{A}C^0 A = \text{diag}\{\lambda_1^0, \lambda_2^0, ..., \lambda_\Lambda^0\},
$$

(10)

where $\text{diag}\{\lambda_1^0, \lambda_2^0, ..., \lambda_\Lambda^0\}$ denotes the $\Lambda \times \Lambda$ matrix having $\lambda_1^0, \lambda_2^0, ..., \lambda_\Lambda^0$ on the diagonal and 0 elsewhere, and $\tilde{A}$ is the adjoint of $A$. Moreover we have

$$
\tilde{A} A = 1, \quad detA = det\tilde{A} = 1.
$$

(11)

From (8) we also have

$$
\sum_{j=1}^{\Lambda} (C^0)_{ij} A_{jk} = \lambda_k^0 A_{ik}. 
$$

(12)

Summing over the index $i$ on both sides of the equation (12) and using (8) we get

$$
(L - \lambda_k^0) \sum_{i=1}^{\Lambda} A_{ik} = 0.
$$

Therefore if $\lambda_k^0 \neq L$ for some $k = 1, ..., \Lambda$, then:

$$
\sum_{i=1}^{\Lambda} A_{ik} = 0. 
$$

(13)
Let us now consider the possibility that \( \lambda_k^0 = L \) for some \( k \), i.e., \( L \) is an eigenvalue of \( C^0 \). We study the eigenvector \( x = \text{column}(x_1, x_2, ..., x_\Lambda) \in \mathbb{C}^\Lambda \) to the eigenvalue \( L \) (if it exists) of \( C^0 \). We have \( \sum_{j=1}^\Lambda C_{ij}^0 x_j = L x_i \), \( i = 1, 2, ..., \Lambda \). From (3) we see that

\[
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

(14)
is an eigenvector of \( C^0 \) with eigenvalue \( L \) and with unit length (up to a global phase factor). By the orthonormality of the \( u_i \) we have \( \bar{u}_i u_i = 0 \), \( i = 2, 3, ..., \Lambda \). Hence from (14) the matrix that diagonalizes \( C^0 \) has the properties

\[
\sum_{i=1}^\Lambda A_{ik} = \begin{cases} 
0, & k \neq 1 \\
\sqrt{\Lambda}, & k = 1.
\end{cases}
\]

(15)

Generally the matrix \( C^0 \) has both positive and negative eigenvalues. Let \( \lambda_{\text{max}}^+ \) be the largest positive eigenvalue and \( \lambda_{\text{max}}^- \) be the largest absolute value of the negative eigenvalues. Set \( \lambda^\pm = \pm \lambda_{\text{max}}^\pm \pm \epsilon \) for any small constant \( \epsilon > 0 \). We set

\[
C^+ \equiv \lambda^+ I - C^0
\]

(16)
and

\[
C^- \equiv C^0 - \lambda^- I,
\]

(17)
where \( I \) is the \( \Lambda \times \Lambda \) identity matrix. \( C^\pm \) are then positive definite matrices with the properties:

\[
C^\pm_{ii} = \pm \lambda^\pm, \quad C^\pm_{ij} = C^\pm_{ji}, \quad i \neq j, \quad i, j = 1, 2, ..., \Lambda, \quad \text{Tr}(C^\pm) = \pm \lambda^\pm \Lambda \]

(18)
and

\[
\sum_{i=1}^\Lambda C_{ij}^\pm = \sum_{i=1}^\Lambda C_{ji}^\pm = \pm \lambda^\pm \mp L, \quad \forall j = 1, 2, ..., \Lambda,
\]

(19)

\[
\sum_{i,j=1}^\Lambda C_{ij}^\pm = (\pm \lambda^\pm \mp L)\Lambda.
\]

(20)
The eigenvectors of \( C^0 \) are also eigenvectors of \( C^\pm \),

\[
C^\pm u_i = \lambda_i^\pm u_i, \quad \text{with} \quad \lambda_i^\pm \equiv \pm \lambda^\pm \mp \lambda_i^0, \quad i = 1, 2, ..., \Lambda.
\]

(21)
We have (with $A$ as in (9)):

$$\tilde{A}C^{\pm}A = \text{diag}\{\lambda_1^{\pm}, \lambda_2^{\pm}, \ldots, \lambda_{\Lambda}^{\pm}\}. \quad (22)$$

Corresponding to (12) we have

$$(C^{\pm}A)_{ij} = \lambda_j^{\pm}A_{ij}. \quad (23)$$

Set $K^+ \equiv J^+/kT$, with $J^+ \equiv J$ (resp. $K^- \equiv J^-/kT$, with $J^- \equiv -J$) for ferromagnetic, i.e., with $J > 0$ (resp. antiferromagnetic, i.e., with $J < 0$) systems. From (16), (17) and $\sum_{i=1}^{\Lambda} \sigma_i^2 = \Lambda$, we see that the partition function (2) can be rewritten as

$$Z_{\pm}^{\pm} = \exp\left\{\pm\lambda^{\pm} K^{\pm} \Lambda\right\} \sum_{\{\sigma_i\}} \exp\left\{-K^{\pm} \sum_{i,j=1}^{\Lambda} C_{ij}^{\pm} \sigma_i \sigma_j + K'^{\pm} \sum_{i=1}^{\Lambda} \sigma_i\right\}, \quad (24)$$

where $Z_{\Lambda}^{+}$ (resp. $Z_{\Lambda}^{-}$) represents the partition function for ferromagnetic (resp. antiferromagnetic) systems. In formula (24) all the interaction couplings could be included in the elements of the matrix $C^{\pm}$. The parameter $J^\pm$ is no longer independent and could be scaled to be 1. $Z_{\Lambda}^{+}$ is obviously independent of $\epsilon$ by the definition of $\lambda^{\pm}$, (10) and (17).

Formula (24) can be transformed into an integration by using the following lemma, see e.g. [9]:

[Lemma]. Let $f$ and $S$ be real-valued measurable functions on $\mathbb{R}^d$ such that $F(\lambda) \equiv \int_{\mathbb{R}^d} f(x) \exp\{\lambda S(x)\} \, dx$ exists and $S \in C^2(\mathbb{R}^d)$. For $0 < \lambda \to \infty$, $F(\lambda)$ is equal to

$$F(\lambda) = \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}} \sum_{x^0} |\text{det}S''(x^0)|^{-\frac{1}{2}} \left[f(x^0) + O\left(\frac{1}{\lambda}\right)\right] \exp\left\{\lambda S(x^0)\right\}, \quad (25)$$

provided $S(x)$ has a finite number of non-degenerate maxima $x^0$ such that

$$S'(x)|_{x^0} = \nabla S(x)|_{x^0} = 0;$$

$$\text{det}S''(x)|_{x^0} = \text{det} \left(\frac{\partial^2 S(x)}{\partial x_i \partial x_j}\right)|_{x^0} \neq 0, \quad i, j = 1, 2, \ldots, d$$

and $\max_{x \in \mathbb{R}^d} S(x) = S(x^0)$. The sum in (25) goes over all such maxima $x^0$.

Now let

$$S(x) = -\sum_{i=1}^{d} (x_i^2 - 1)^2, \quad x_i \in \mathbb{R}, \quad x = (x_1, \ldots x_d) \in \mathbb{R}^d. \quad (26)$$
Then the critical points $x^0$ of $S$ are given by

$$
\left. \frac{\partial S(x)}{\partial x_i} \right|_{x^0} = -4(x_i^2 - 1)x_i |_{x^0} = 0, \quad i = 1, ..., d. \tag{27}
$$

Here we have

$$
det S'' = \det \left( \frac{\partial^2 S(x)}{\partial x_i \partial x_j} \right) = \det(-12x_i^2 - 4)\delta_{ij}). \tag{28}
$$

From (27) and (28) we see that all critical points (maxima) $x^0 \in \mathbb{R}^d$ satisfying $det S'' \neq 0$ are given by $x^0 = (x_i^0), i = 1, ..., d$, for all possible combinations of values $x_i^0 = \pm 1$. We have $\max_{x \in \mathbb{R}^d} S(x) = S(x^0) = 0$.

Let $\mathcal{C}$ denote the set of all critical points. By the Lemma we get asymptotically for $\lambda \to \infty$,

$$
\int_{\mathbb{R}^d} f(x)exp \left\{-\lambda \sum_{i=1}^{d} (x_i^2 - 1)^2 \right\} dx = \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}}8^{-\frac{d}{2}} \left[ \sum_{y \in \mathcal{C}} f(y) + O\left(\frac{1}{\lambda}\right) \right]
$$

for any $f$ such that the integration exists.

In particular we have

$$
\lim_{\lambda \to \infty} \lambda^\frac{d}{2} \int_{\mathbb{R}^d} f(x)exp \left\{-\lambda \sum_{i=1}^{d} (x_i^2 - 1)^2 \right\} dx = \left(2\pi\right)^{\frac{d}{2}}8^{-\frac{d}{2}} \sum_{y \in \mathcal{C}} f(y). \tag{29}
$$

We shall now apply this result to the case $d = \Lambda$, observing that $\sum_{y \in \mathcal{C}} f(y)$ amounts to sum over $y$ with $y_i = \pm 1$ (independently, for each $i = 1, ..., \Lambda$) and identifying then $y_i$ with the $\sigma_i$ occurring in (24) we see that the partition function (24) can be written as

$$
Z_{\Lambda}^\pm = \exp \left\{ \pm \lambda^\pm K^\pm \Lambda \right\} \left(\frac{\Lambda}{2\pi}\right)^{\frac{d}{2}}8^{\frac{d}{2}}.
$$

$$
\int_{\mathbb{R}^\Lambda} \exp \left\{-K^\pm \sum_{i,j=1}^{\Lambda} C_{ij}^\pm y_i y_j + K' \sum_{i=1}^{\Lambda} y_i - \lambda \sum_{i=1}^{\Lambda} (y_i^2 - 1)^2 \right\} dy|_{\lambda \to \infty}, \tag{30}
$$

with the notation $F(\lambda)|_{\lambda \to \infty} \equiv \lim_{\lambda \to \infty} F(\lambda)$.

We call (30) the integral representation of the partition functions $Z_{\Lambda}^\pm$. Now we shall study its properties. First however we make a remark:

**Remark 1.** By combining the first two terms in the exponential of the integrand in (30) into a quadratic form and making a translation of the integration variable $y_i \to y_i - a^\pm$ in (30), with

$$
a^\pm \equiv \frac{-K'}{2K^\pm \sum_{i=1}^{\Lambda} C_{ij}^\pm}, \tag{31}
$$

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we get

\[ Z_\Lambda^\pm = \exp \{ \pm \lambda^\pm K^\pm \Lambda \} \left( \frac{\lambda}{2\pi} \right)^\frac{\Lambda}{2} \frac{1}{2} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} (y_i + a^\pm)(y_j + a^\pm) 
+ K^\pm (a^\pm)^2 \sum_{i,j=1}^\Lambda C^\pm_{ij} - \lambda \sum_{i=1}^\Lambda y_i^2 \right\} dy \mid_{\lambda \to \infty} \]

\[ = \exp \{ \pm \lambda^\pm K^\pm \Lambda \} \left( \frac{\lambda}{2\pi} \right)^\frac{\Lambda}{2} \frac{1}{2} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} y_i y_j 
+ K^\pm (a^\pm)^2 \sum_{i,j=1}^\Lambda C^\pm_{ij} - \lambda \sum_{i=1}^\Lambda ((y_i - a^\pm)^2 - 1)^2 \right\} dy \mid_{\lambda \to \infty}. \]

Therefore it is clear that the external magnetic field \( H = K'kT = -2a^\pm J^\pm \sum_{i=1}^\Lambda C^\pm_{ij} \) contributes a global translation \( a^\pm \) of the critical points of the function (26) together with a constant factor \( \exp \left\{ K^\pm (a^\pm)^2 \sum_{i,j=1}^\Lambda C^\pm_{ij} \right\} \) to the zero-field partition functions \( Z_\Lambda^\pm (H = 0) \):

\[ Z_\Lambda^\pm (H = 0) = \exp \{ \pm \lambda^\pm K^\pm \Lambda \} \left( \frac{\lambda}{2\pi} \right)^\frac{\Lambda}{2} \frac{1}{2} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} y_i y_j 
- \lambda \sum_{i=1}^\Lambda ((y_i - a^\pm)^2 - 1)^2 \right\} dy \mid_{\lambda \to \infty}. \]  

(32)

Conversely, a global translation of the critical points in the zero-field partition function is equivalent with the introduction of an external magnetic field.

For convenience in Theorem 1 below we shall denote by \( \Lambda \) the number of points in a one dimensional sublattice in an \( n \)-dimensional square lattice, so that \( \Lambda^n \) is the total number of points of the given \( n \)-dimensional lattice. From (2) it is obvious that for given \( K, K' \) and lattice number \( \Lambda \), the partition function \( Z_\Lambda \) is uniquely determined by the interaction coupling matrix \( C^0 \). Further we can prove the following:

[Theorem 1]. For given \( K, K' \), the partition function \( Z_\Lambda^n \) of \( n \)-dimensional homogeneous Ising systems (described by (3)) on a square lattice with \( \Lambda^n \) lattice sites is uniquely given by the eigenvalues of the corresponding interaction coupling matrix.

[Proof]. We first consider the one-dimensional homogeneous case. In this case the
interaction coupling matrix $C^0$ has the form

$$C^0 = \begin{pmatrix}
  a_1 & a_2 & \cdots & a_\Lambda \\
  a_\Lambda & a_1 & \cdots & a_{\Lambda-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_2 & a_3 & \cdots & a_1
\end{pmatrix},$$

(33)

where $a_\alpha$, $\alpha = 1, 2, \ldots, \Lambda$ are real constants representing the $(\alpha - 1)$-th order nearest neighbours interactions. The diagonal element $a_1$ is in fact zero, $a_1 = 0$. Since the system is homogeneous we have

$$a_\alpha = a_{\Lambda-\alpha+2} \begin{cases} 
\alpha = 2, 3, \ldots, \frac{\Lambda}{2}, & \Lambda \text{ even} \\
\alpha = 2, 3, \ldots, \frac{\Lambda+1}{2}, & \Lambda \text{ odd}
\end{cases}$$

(34)

This implies that $C^0$ is symmetric.

The matrix (33) is generated by cyclically permuting the elements of the former row to the right. We call $C^0$ a right cyclic matrix. By using the $\Lambda \times \Lambda$ permutation matrix $P$,

$$P = \begin{pmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \\
  1 & 0 & 0 & \cdots & 0
\end{pmatrix},$$

(35)

the right cyclic matrix $C^0$ can be expressed as

$$C^0 = a_1 + a_2 P + a_3 P^2 + \cdots + a_\Lambda P^{\Lambda-1}.$$

(36)

The $k$-th component of the $\alpha$-th eigenvector of the $\Lambda \times \Lambda$ permutation matrix $P$ acting in $C^\Lambda$ is simply given by

$$\exp \left\{ \frac{2\pi i}{\Lambda} (\alpha - 1)(k - 1) \right\}, \quad k, \alpha = 1, 2, \ldots, \Lambda,$$

(37)

see e.g., [10, 11]. As the eigenvalues of a real symmetric matrix are real, both the real part and the imaginary part of the eigenvectors (37) are also eigenvectors for the matrix $P$. 

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We take the $k$-th component of an eigenvector $u_\alpha$, $\alpha = 1, 2, \ldots, \Lambda$, of $P$ to be normalized by the factor $1/\sqrt{\Lambda}$ as follows,

$$\begin{equation}
(u_\alpha)_k = \begin{cases} 
\frac{1}{\sqrt{\Lambda}} \cos \left( \frac{2\pi}{\Lambda} (k-1)(\alpha - 1) \right), & 1 \leq \alpha \leq \beta, \\
\frac{1}{\sqrt{\Lambda}} \sin \left( \frac{2\pi}{\Lambda} (k-1)(\alpha - 1) \right), & \Lambda \geq \alpha > \beta,
\end{cases}
\end{equation}
$$

(38)

where $k = 1, 2, \ldots, \Lambda$, $\beta = (\Lambda + 1)/2$ for $\Lambda$ odd and $\beta = (\Lambda + 2)/2$ for $\Lambda$ even. From (38) we see that $u_1$ is just the eigenvector (14).

The orthogonal matrix that diagonalizes $P$ is then given by

$$A = (u_1, u_2, \ldots, u_\Lambda).$$

(39)

It is direct to check that the matrix $A$ obtained in this way satisfies the equations (11) and (13). From (36), (17) and (16), the matrix $A$ also diagonalizes $C^\pm = \pm \lambda^\pm I \mp C^0$, in the way given by formula (22).

Changing the integration variable $y_i$ to be $\sum_{j=1}^{\Lambda} A_{ij} y_j$ in the integration (30) and using the relations (11) and (22) we get,

$$Z^\pm_\Lambda = \exp \left\{ \pm \lambda^\pm K^\pm \Lambda \right\} \left( \frac{\lambda}{2\pi} \right)^{\frac{\Lambda}{2}} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i=1}^{\Lambda} \lambda^\pm_i y_i^2 \right. \right.
$$

$$+K' \sum_{i,j=1}^{\Lambda} A_{ij} y_j - \lambda \sum_{i,j,k=1}^{\Lambda} \left( A_{ij} A_{ik} y_j y_k - 1 \right)^2 \right\} dy_i \bigg|_{\lambda \to \infty}.$$ 

(40)

From (38) and (39), we see that $A = (A_{ij})$ is independent of the elements of $C^\pm$. Therefore for one dimensional homogeneous systems the partition function (40) only depends the eigenvalues $\lambda^\pm_i$, i.e., the eigenvalues of the interaction coupling matrix $C^0$.

For two dimensional homogeneous square lattice systems with $\Lambda^2$ lattice sites, the interaction coupling matrix $C^0$, denoted here by $C^0_2$, may have various forms according to the ways of numbering the lattice sites. We number the lattice sites from left to right and from the first line to the last line of the lattice. The matrix $C^0_2$ is then of the form,

$$C^0_2 = \begin{pmatrix} 
C^0 & a_2 I & \cdots & a_{\Lambda} I \\
a_2 I & C^0 & \cdots & a_{\Lambda-1} I \\
\vdots & \vdots & \ddots & \vdots \\
a_{\Lambda} I & a_{\Lambda+1} I & \cdots & C^0 
\end{pmatrix} = C^0 \otimes I + I \otimes C^0,
$$

(41)
where $C^0$ and $a_\alpha$, $\alpha = 2, ... , \Lambda$, as in (33), and $I$ is the $\Lambda \times \Lambda$ identity matrix.

$C^0_2$ is again a right cyclic matrix generated by permuting the matrix blocks $(C^0, a_2 I, a_3 I, ..., a_\Lambda I)$. The eigenvectors of the interaction coupling matrix $C^0_2$ have the form,

$$v_{\alpha \beta} = u_\alpha \otimes u_\beta, \quad \alpha, \beta = 1, 2, ..., \Lambda,$$

with $u_\alpha, u_\beta$ as in (38). The matrix that diagonalizes $C^0_2$ is then given by

$$A = (v_{11}, v_{12}, ..., v_{21}, v_{22}, ..., v_{\Lambda \Lambda}).$$

$A$ is then a $\Lambda^2 \times \Lambda^2$ orthonormal matrix which is independent of $a_2, ..., a_\Lambda$. Obviously the matrix (43) also diagonalizes the positive definite matrix $C^0_2 = \pm \lambda^\Lambda I \otimes I \mp C^0_2$ in the integral representation of the partition functions. Therefore for two-dimensional homogeneous systems the partition functions are also only determined by the eigenvalues of the interaction coupling matrix for given $K$, $K'$ and $\Lambda$.

Similarly, for an $n (> 2)$-dimensional homogeneous system, the interaction coupling matrix $C^0_n$ has the form,

$$C^0_n = \begin{pmatrix} C^0_{n-1} & a_2 I & \cdots & a_\Lambda I \\ a_2 I & C^0_{n-1} & \cdots & a_{\Lambda-1} I \\ \vdots & \vdots & \ddots & \vdots \\ a_2 I & a_3 I & \cdots & C^0_{n-1} \end{pmatrix} = C^0_{n-1} \otimes I + I \otimes C^0_{n-1},$$

where $C^0_{n-1}$ is the interaction coupling matrix of an $(n - 1)$-dimensional homogeneous system.

The matrix that diagonalizes $C^0_n$ and $C^\pm_n = \pm \lambda^\Lambda I_n \mp C^0_n$, with $I_n$ the $\Lambda^n \times \Lambda^n$ identity matrix, is given by

$$A = (u_{a_1} \otimes u_{a_2} \otimes ... \otimes u_{a_\Lambda}), \quad \alpha_i = 1, 2, ..., \Lambda, \quad \forall i = 1, 2, ..., n,$$

where $u_{a_i}$ is given by (38). This proves the theorem.

[Theorem 2]. For one-dimensional homogeneous ferromagnetic Ising systems with interaction coupling matrix $C^0$ given by (33) and coupling coefficients $a_i > 0$, $i = 2, 3, ..., \Lambda$, the partition function $Z^+_\Lambda$ satisfies

$$\sum_{i=2}^\Lambda a_i \frac{\partial Z^+_\Lambda}{\partial a_i} + T \frac{\partial Z^+_\Lambda}{\partial T} + H \frac{\partial Z^+_\Lambda}{\partial H} + L \sum_{i=2}^\Lambda \frac{\partial Z^+_\Lambda}{\partial a_i} = K^+ \Lambda(\Lambda - 1)LZ^+_\Lambda.$$

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[Proof]. From (30) the partition function of one-dimensional ferromagnetic homogeneous systems takes the form,

$$Z^+ = \exp \left\{ \frac{\lambda}{2\pi} \frac{2}{\sqrt{8\pi}} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^+ \sum_{i=1}^\Lambda \lambda_i^+ y_i^2 \right\} \right\}$$

$$+ K' \sum_{ij=1}^\Lambda A_{ij} y_j - \lambda \sum_{ij=1}^\Lambda (A_{ij} A_{ik} y_j y_k - 1)^2 \right\} dy|_{\lambda \to \infty}$$

$$\equiv \int_{\mathbb{R}^\Lambda} \exp \left\{ \Gamma^+ \right\} dy|_{\lambda \to \infty},$$

where $A_{ij}, i, j = 1, 2, ..., \Lambda$ are the elements of the matrix (39), $\lambda_i^+ = \lambda^+ - \lambda_i^0 = \lambda_{\text{max}}^+ + \epsilon - \lambda_i^0$, $\lambda_i^0$ is an eigenvalue of $C^0$. For simplicity we have set in (47),

$$B^+ \equiv \exp \left\{ \frac{\lambda}{2\pi} \frac{2}{\sqrt{8\pi}} \right\},$$

$$\Gamma^+ \equiv -K^+ \sum_{i=1}^\Lambda \lambda_i^+ y_i^2 + K' \sum_{ij=1}^\Lambda A_{ij} y_j - \lambda \sum_{ij=1}^\Lambda (A_{ij} A_{ik} y_j y_k - 1)^2.$$

By (24) and the definitions (16), (17) of $C^+$, as we stated before, the partition function $Z^+_\Lambda$ is independent of the positive real number $\epsilon$. Hence we have

$$0 = \frac{\partial Z^+_\Lambda}{\partial \epsilon}.$$ 

On the other hand using Lebesgue dominated convergence one shows that the application of $\partial/\partial \epsilon$ to the right hand side of (47) yields

$$K^+ \Lambda Z^+_\Lambda + \exp \left\{ \frac{\lambda}{2\pi} \frac{2}{\sqrt{8\pi}} \right\}$$

$$\int_{\mathbb{R}^\Lambda} \exp \left\{ -K^+ \sum_{i=1}^\Lambda \lambda_i^+ y_i^2 + K' \sum_{ij=1}^\Lambda A_{ij} y_j - \lambda \sum_{ij=1}^\Lambda (A_{ij} A_{ik} y_j y_k - 1)^2 \right\}$$

$$\cdot \left[ -K^+ \sum_{i=1}^\Lambda y_i^2 \right] dy|_{\lambda \to \infty}.$$ 

(In fact the derivatives of $Z^+_\Lambda$ with respective to $\epsilon$ and, in the following, to $T$, $H$ and $a_i$, can be exchanged with the integration and the limitation in (17) because from (24) these derivatives only change the form of the function $f$ in (24)). Equating (51) with (50) we get

$$\Lambda Z^+_\Lambda - B^+ \int_{\mathbb{R}^\Lambda} \exp \left\{ \Gamma^+ \right\} \sum_{i=1}^\Lambda y_i^2 dy|_{\lambda \to \infty} = 0.$$ 

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In a similar way we get from (47)
\[
\frac{\partial Z^+}{\partial H} = B^+ \int_{\mathfrak{U}^+} \exp \left\{ \Gamma^+ \right\} \left[ \frac{1}{kT} \sum_{ij=1}^{\Lambda} A_{ij} y_j \right] \, dy|_{\lambda \to \infty}.
\] (53)
and
\[
\frac{\partial Z^+}{\partial T} = \lambda^+ \Lambda \left( -\frac{J^+}{kT^2} \right) Z^+ \\
+ B^+ \int_{\mathfrak{U}^+} \exp \left\{ \Gamma^+ \right\} \left[ \frac{J^+}{kT^2} \sum_{i=1}^{\Lambda} y_i^2 - \frac{H}{kT^2} \sum_{ij=1}^{\Lambda} A_{ij} y_j \right] \, dy|_{\lambda \to \infty}.
\] (54)
From (53) and (54) we deduce that
\[
\frac{\partial Z^+}{\partial T} + H \frac{\partial Z^+}{\partial H} = \lambda^+ \Lambda \left( -\frac{K^+}{T} \right) Z^+ \\
+ B^+ \int_{\mathfrak{U}^+} \exp \left\{ \Gamma^+ \right\} \left[ \frac{K^+}{T} \sum_{i=1}^{\Lambda} \lambda_i^+ y_i^2 \right] \, dy|_{\lambda \to \infty}.
\] (55)
Using (52) one gets
\[
T \frac{\partial Z^+}{\partial T} + H \frac{\partial Z^+}{\partial H} = B^+ \int_{\mathfrak{U}^+} \exp \left\{ \Gamma^+ \right\} \left[ -K^+ \sum_{i=1}^{\Lambda} \lambda_i^0 y_i^2 \right] \, dy|_{\lambda \to \infty}.
\] (56)
Now we discuss the properties of the eigenvalues of the matrix $C^0$ given by (33). We first recall a general result of matrix theory. Let $g(x)$ be polynomial in $x$ of degree $l$ with roots $(\rho_1, \rho_2, \ldots, \rho_l)$,
\[
g(x) = d_0 x^l + d_1 x^{l-1} + \ldots + d_{l-1} x + d_l = (-1)^l d_0 (\rho_1 - x)(\rho_2 - x)\ldots(\rho_l - x).
\] (57)
Let $\lambda_i$, $i = 1, 2, \ldots, m$, be the eigenvalues of a given $m \times m$ matrix $B$ over $\mathfrak{U}$, acting in $\mathfrak{U}^m$. Then
\[
|\rho I - B| = (\rho - \lambda_1)(\rho - \lambda_2)\ldots(\rho - \lambda_m).
\]
Replacing $x$ by $B$ in (57) we get
\[
g(B) = (-1)^l d_0 (\rho_1 I - B)(\rho_2 I - B)\ldots(\rho_l I - B).
\]
Hence
\[
|g(B)| = (-1)^m d_0^m |(\rho_1 I - B)||(\rho_2 I - B)|\ldots|(\rho_l I - B)|
\]
\[
= (-1)^m d_0^m \prod_{k=1}^{l} \prod_{i=1}^{m} (\rho_k - \lambda_i) = \prod_{i=1}^{m} g(\lambda_i),
\] (58)
From (36) we see that $C_0$ is a polynomial in the $\Lambda \times \Lambda$ permutation matrix $P$ acting in $\mathbb{C}^\Lambda$. As $|\rho I - P| = \rho^\Lambda - 1$, the eigenvalues of $P$ are

$$\zeta_\alpha = exp \left\{ \frac{2\pi i (\alpha - 1)}{\Lambda} \right\}, \quad \alpha = 1, 2, ..., \Lambda.$$

(59)

Noting that all diagonal elements of $C_0$ are equal to $a_1$, from (58) we have

$$|C_0 - \lambda^0 I| = |a_1 - \lambda^0 + a_2 P + a_3 P^2 + ... + a_\Lambda P^{\Lambda - 1}|$$

$$= \prod_{\alpha=1}^{\Lambda} \left[ a_1 - \lambda^0 + a_2 \zeta^\alpha + a_3 \zeta^{2\alpha} + ... + a_\Lambda \zeta^{(\Lambda - 1)\alpha} \right].$$

(60)

The eigenvalues of $C_0$ are then given by

$$\lambda^0_\alpha = a_2 \zeta^\alpha + a_3 \zeta^{2\alpha} + ... + a_\Lambda \zeta^{(\Lambda - 1)\alpha}, \quad \alpha = 1, 2, ..., \Lambda.$$

(61)

By the definition (6) we see that $\lambda^0_1 = \sum_{\alpha=2}^{\Lambda} a_\alpha$ is of special significance. It is just the generalized link number of the lattice, $L = \lambda^0_1$. For our case, $a_\alpha > 0$, $\alpha = 2, 3, ..., \Lambda$, and the link number $L$ is also the largest eigenvalue of the interaction coupling matrix $C_0$,

$$\lambda^+_{max} = \lambda^0_1 = L,$$

(62)

which means $\lambda^+_1 = \lambda^+_{max} + \epsilon - \lambda^0_1 = \epsilon$ and hence

$$\frac{\partial \lambda^+_1}{\partial \lambda^0_i} = 0, \quad i = 1, 2, ..., \Lambda.$$

(63)

From (61) $\lambda^0_\alpha$ satisfies

$$\sum_{\gamma=2}^{\Lambda} a_\gamma \frac{\partial \lambda^0_\alpha}{\partial a_\gamma} = \lambda^0_\alpha.$$  

(64)

$$\Theta_\alpha \equiv \sum_{\gamma=2}^{\Lambda} \frac{\partial \lambda^0_\alpha}{\partial a_\gamma} = \begin{cases} \Lambda - 1, & \alpha = 1, \\ -1, & \alpha \neq 1. \end{cases}$$

(65)

The eigenvector corresponding to $\lambda^0_\alpha$ is $u_\alpha$ given by (38),

$$C_0 u_\alpha = \lambda^0_\alpha u_\alpha.$$  

(66)

From (47), (64) and (33) we have,

$$\sum_{i=2}^{n} a_i \frac{\partial Z^+_\Lambda}{\partial a_i} = \sum_{i=2}^{n} \sum_{\mu=1}^{\Lambda} a_i \frac{\partial Z^+_\Lambda}{\partial \lambda^0_\mu} \frac{\partial \lambda^0_\mu}{\partial a_i} = \sum_{\mu=1}^{\Lambda} \frac{\partial Z^+_\Lambda}{\partial \lambda^0_\mu} \lambda^0_\mu$$

$$= LK^+ \Lambda Z^+_\Lambda + B^+ \int_{\mathbb{R}^\Lambda} exp \left\{ \Gamma^+ \right\} \left[ K^+ \sum_{i=2}^{\Lambda} \lambda^0_i y_i^2 \right] dy|_{\lambda \to \infty},$$

(67)
where the relation (62) has been used.

By (47), (65) and (63) we also have

\[ \sum_{i=2}^{n} \partial Z_{\lambda}^{+} \partial a_{i} = \sum_{\mu=1}^{\Lambda} \partial \lambda_{\mu}^{0} \Theta_{\mu} \]

\[ = K^{+} \Lambda (\Lambda - 1) Z_{\lambda}^{+} - B^{+} \int_{\mathbb{R}^{\Lambda}} \text{exp} \left\{ \Gamma^{+} \right\} \left[ K^{+} \sum_{i=2}^{n} y_{i}^{2} \right] dy|_{\lambda \to \infty}. \]  

(68)

From (56), (67) one has

\[ \sum_{i=1}^{\Lambda} a_{i} \partial Z_{\lambda}^{+} \partial a_{i} + T \partial Z_{\lambda}^{+} \partial T + H \partial Z_{\lambda}^{+} \partial H = L K^{+} \Lambda Z_{\lambda}^{+} \]

\[ + B^{+} \int_{\mathbb{R}^{\Lambda}} \text{exp} \left\{ \Gamma^{+} \right\} ( - K^{+} L y_{1}^{2} ) dy|_{\lambda \to \infty}. \]

(69)

By (52) and (68) one gets

\[ \sum_{i=2}^{\Lambda} \partial Z_{\lambda}^{+} \partial a_{i} = K^{+} \Lambda (\Lambda - 2) Z_{\lambda}^{+} + K^{+} B^{+} \int_{\mathbb{R}^{\Lambda}} \text{exp} \left\{ \Gamma^{+} \right\} y_{1}^{2} dy|_{\lambda \to \infty}. \]

(70)

Equations (69) and (70) then give

\[ \sum_{i=1}^{\Lambda} a_{i} \partial Z_{\lambda}^{+} \partial a_{i} + T \partial Z_{\lambda}^{+} \partial T + H \partial Z_{\lambda}^{+} \partial H + L \sum_{i=2}^{\Lambda} \partial Z_{\lambda}^{+} \partial a_{i} \]

\[ = L K^{+} \Lambda Z_{\lambda}^{+} + L K^{+} \Lambda (\Lambda - 2) Z_{\lambda}^{+} = L K^{+} \Lambda Z_{\lambda}^{+} (\Lambda - 1), \]

which is just the equality (66).

\[ \text{Remark 2. Equality (66) means that the variation of the interaction couplings of a system is related to the variations of the external magnetic field and the temperature.} \]

As the free energy \( F^{+} \), internal energy \( u^{+} \) and magnetization \( M^{+} \) per site are given respectively by

\[ F^{+} = - \frac{1}{\Lambda} kT \log Z^{+}, \]

(71)

\[ u^{+} = - T^{2} \frac{\partial}{\partial T} \left( - \frac{k}{\Lambda} \log Z^{+} \right) = \frac{T^{2} k}{\Lambda} \frac{\partial}{\partial T} \log Z^{+}, \]

(72)

\[ M^{+} = - \frac{\partial F^{+}}{\partial H} = \frac{T k}{\Lambda} \frac{\partial}{\partial H} \log Z^{+}, \]

(73)

(74) can be rewritten as

\[ u^{+} + M^{+} H - \sum_{i=2}^{\Lambda} (a_{i} + L) \frac{\partial F^{+}}{\partial a_{i}} = J^{+} (\Lambda - 1) L. \]
(74) gives the relation among the internal energy, magnetization, external field, free energy and interaction couplings of one-dimensional homogeneous Ising ferromagnetic systems.

**[Theorem 3]**. For one-dimensional homogeneous Ising ferromagnetic systems with interaction coupling matrix $C^0$ given by (33) and coupling coefficients $a_i > 0$, $i = 2, 3, ..., \Lambda$, the correlation function $g_{ij}$ between spins $i$ and $j$ satisfies

$$\sum_{i=2}^{\Lambda} a_i \frac{\partial g_{ij}}{\partial a_i} + T \frac{\partial g_{ij}}{\partial T} + H \frac{\partial g_{ij}}{\partial H} + L \sum_{i=2}^{\Lambda} \frac{\partial g_{ij}}{\partial a_i} = 0. \quad (75)$$

**[Proof]**. Let $f(\sigma)$ be an entire function of $\sigma_i$, $i = 1, 2, ..., \Lambda$, independent of $a_i$, $T$ and $H$. The average $< f(\sigma) >$ of $f(\sigma)$ is by definition given by

$$< f(\sigma) > = \frac{1}{Z_{\Lambda}} \sum_{\{\sigma_i\}} f(\sigma) \exp \left\{ K \sum_{i,j=1}^{\Lambda} C^0_{ij} \sigma_i \sigma_j + K' \sum_{i=1}^{\Lambda} \sigma_i \right\}, \quad (76)$$

where $Z_{\Lambda}$ is given by (2).

For ferromagnetic systems, using our integral representation we see that we can write (76) in the form:

$$< f(\sigma) > = \frac{1}{Z_{\Lambda}^+} B^+ \int_{\mathbb{R}^\Lambda} f(\sigma) \exp \left\{ \Gamma^+ \right\} dy |_{\lambda \to \infty} \equiv \frac{Z_{\Lambda}^+(f)}{Z_{\Lambda}^+}, \quad (77)$$

where $Z_{\Lambda}^+$, $B^+$ and $\Gamma^+$ are given by (47), (48) and (49) respectively, and

$$Z_{\Lambda}^+(f) \equiv B^+ \int_{\mathbb{R}^\Lambda} f(\sigma) \exp \left\{ \Gamma^+ \right\} dy |_{\lambda \to \infty}. \quad (78)$$

Let $U$ be a linear operator defined on smooth functions of $a_i$, $T$ and $H$ by

$$U = \sum_{i=2}^{\Lambda} a_i \frac{\partial}{\partial a_i} + T \frac{\partial}{\partial T} + H \frac{\partial}{\partial H} + L \sum_{i=2}^{\Lambda} \frac{\partial}{\partial a_i}. \quad (79)$$

From Theorem 2 we see that

$$U Z_{\Lambda}^+ = K^+ \Lambda (\Lambda - 1) L Z_{\Lambda}^+. \quad (80)$$

It is also straightforward to check that

$$U Z_{\Lambda}^+(f) = K^+ \Lambda (\Lambda - 1) L Z_{\Lambda}^+(f). \quad (81)$$
By (77), (80) and (81) we get

\[ U < f(\sigma) > = \frac{1}{Z_A^+} U Z_A^+(f) - \frac{Z_A^+(f)}{(Z_A^+)^2} U Z_A^+ \]

\[ = \frac{1}{Z_A^+} K^+ \Lambda (\Lambda - 1) L Z_A^+(f) - \frac{Z_A^+(f)}{(Z_A^+)^2} K^+ \Lambda (\Lambda - 1) L Z_A^+ = 0. \] (82)

For the correlation function \( g_{ij} \equiv < \sigma_i \sigma_j > - < \sigma_i > < \sigma_j > \), we get using the result (82):

\[ U g_{ij} = U < \sigma_i \sigma_j > - U < \sigma_i > < \sigma_j > - < \sigma_i > U < \sigma_j > = 0, \]

which is just the formula (75).

By using our integral approach to the Ising model, we have discussed some properties of the partition function for a one-dimensional homogeneous case. We now remark that for some zero magnetic field cases, in arbitrary dimensions, the partition function can be given exactly in terms of special functions.

Let \( A \) be the matrix that diagonalizes \( C^\pm \) as in (22). Let us consider the Ising system described by (24).

[Theorem 4]. If \( C^\pm \) is such that \( A \) satisfies

\[ \sum_{i=1}^{\Lambda} A_{ij} A_{ik} A_{im} A_{in} = \delta_{jk} \delta_{km} \delta_{mn} b_n \] (83)

for some constants \( b_n, \text{Re} b_n > 0, n = 1, 2, ..., \Lambda, \) then

\[ Z_A^+(H = 0) = \exp \{ (\pm \lambda^\pm K^+ - \lambda) \Lambda \} \frac{\Lambda}{\pi} \exp \left\{ \sum_{i=1}^{\Lambda} \frac{(K^\pm)^2 (\lambda^\pm - 2\lambda)^2}{8\lambda b_i} \right\} \]

\[ \prod_{i=1}^{\Lambda} \sqrt{\frac{K(\lambda^\pm - 2\lambda)}{\lambda b_i}} K_1^\pm \left( \frac{K^2 (\lambda^\pm - 2\lambda)^2}{8\lambda b_i} \right) \bigg|_{\lambda \to +\infty}, \] (84)

where \( K_1^\pm \) is the Bessel function of imaginary argument equal to \( \frac{1}{4} \).

[Proof]. The partition function of zero-field is given by (32). Set \( y_i = \sum_{j=1}^{\Lambda} A_{ij} p_j \) in (32), \( A = (A_{ij}) \) as in (3). By using (22), (11) and (83) we have

\[ \sum_{i,j=1}^{\Lambda} C_{ij}^\pm y_i y_j = \sum_{i=1}^{\Lambda} \lambda_i^\pm p_i^2, \] (85)

\[ \sum_{i=1}^{\Lambda} y_i^2 = \sum_{i=1}^{\Lambda} (\sum_{j=1}^{\Lambda} A_{ij} p_j)^2 = \sum_{j,k=1}^{\Lambda} \sum_{i=1}^{\Lambda} A_{ij} A_{ik} p_j p_k = \sum_{i=1}^{\Lambda} p_i^2 \] (86)
and
\[ \sum_{i=1}^{\Lambda} y_i^4 \sum_{j,k,m,n=1} A_{ij} A_{ik} A_{im} p_j p_k p_m p_n = \sum_{i=1}^{\Lambda} b_i p_i^4. \] (87)

Substituting \( y_i = \sum_{j=1}^{\Lambda} A_{ij} p_j \), (85), (86) and (87) into (32) we get
\[ Z_{\Lambda}^\pm (H = 0) = \exp \left\{ (\pm \lambda^\pm K^\pm - \lambda) \Lambda \left( \frac{\lambda}{2\pi} \right)^{\frac{4}{\Lambda}} \right\} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i=1}^{\Lambda} (\lambda_i^\pm - 2\lambda) p_i^2 - \lambda \sum_{i=1}^{\Lambda} b_i p_i^4 \right\} dp |_{\lambda \to \infty}. \]

From the formula (see e.g. [12])
\[ \int_{\mathbb{R}} \exp \left\{ -\mu x^4 - 2\nu x^2 \right\} dx = \frac{1}{2} \sqrt{\frac{2\nu}{\mu}} \exp \left\{ \frac{\nu^2}{2\mu} \right\} K_{\frac{1}{4}} \left( \frac{\nu^2}{2\mu} \right), \quad Re \mu > 0, \] (88)
we obtain (84).

In the following we present an asymptotic formula for the partition function (2) in the general case. We first make a Laplace transformation to the integral representation of the partition function (30).

From the formula
\[ \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^{\Lambda} B_{ij} p_i p_j \right\} dp = \left( \frac{\pi}{K^\pm} \right)^{\frac{\Lambda}{2}} \frac{1}{\sqrt{\text{det} B}} \] (89)
for any symmetric strictly positive definite matrix \( B \), (30) can be reexpressed as
\[ Z_{\Lambda}^\pm = c^\pm \lambda^\frac{\Lambda}{2} \int_{\mathbb{R}^\Lambda} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^{\Lambda} C_{ij}^\pm p_i p_j + 2i K^\pm \sum_{i,j=1}^{\Lambda} C_{ij}^\pm (y_j + a^\pm) \right. \]
\[ \left. -\lambda \sum_{i=1}^{\Lambda} (y_i^2 - 1)^2 \right\} dp dy |_{\lambda \to \infty}, \]
where \( a^\pm \) as in (31) and
\[ c^\pm \equiv \exp \left\{ \pm \lambda^\pm K^\pm \Lambda \right\} \left( \frac{1}{2\pi} \right)^{\frac{4}{\Lambda}} \exp \left\{ K^\pm (a^\pm)^2 \sum_{i,j=1}^{\Lambda} C_{ij}^\pm \right\} \left( \frac{K^\pm}{\pi} \right)^{\frac{4}{\Lambda}} \sqrt{\text{det} C^\pm}. \] (90)
By a translation $y_i \rightarrow y_i + 1$ and a rescaling $y_i \rightarrow y_i / \sqrt{\lambda}$ we get

$$Z^\pm_\Lambda = c^\pm \lambda^\frac{\Lambda}{2} \int_{\mathbb{R}^\Lambda} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^{\Lambda} C^\pm_{ij} p_i p_j + 2iK^\pm \sum_{i,j=1}^{\Lambda} (a^\pm + 1)C^\pm_{ij} p_i 
+ 2iK^\pm \sum_{i,j=1}^{\Lambda} C^\pm_{ij} y_i y_j - \lambda \sum_{i=1}^{\Lambda} (y_i^4 + 4y_i^3 + 4y_i^2) \right\} dp dy |_{\lambda \to \infty}$$

(91)

$$= c^\pm \int_{\mathbb{R}^\Lambda} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^{\Lambda} C^\pm_{ij} p_i p_j + 2iK^\pm \sum_{i,j=1}^{\Lambda} (a^\pm + 1)C^\pm_{ij} p_i 
+ \frac{2iK^\pm}{\sqrt{\lambda}} \sum_{i,j=1}^{\Lambda} C^\pm_{ij} y_i y_j - \sum_{i=1}^{\Lambda} \left( \frac{y_i^4}{\lambda} + \frac{4y_i^3}{\sqrt{\lambda}} + 4y_i^2 \right) \right\} dp dy |_{\lambda \to \infty}. \quad (91)$$

[Theorem 5]. For given $K$, $K'$, up to order $O(1/\lambda^2)$, the partition function of any $n$-dimensional homogeneous Ising system is given in terms of the link number $L$ and the largest positive eigenvalue $\lambda^+$ (resp. the largest absolute value $\lambda^-$ of the negative eigenvalues) of the related interaction coupling matrix for ferromagnetic (resp. antiferromagnetic) systems, according to the formula

$$Z^\pm_\Lambda = \exp \{ (\pm K^\pm L + K')\Lambda \} \left[ 1 + \frac{\Lambda}{\lambda} \left( \frac{3}{16} + \frac{3}{16} (2K^\pm (\pm \lambda^\pm \mp L) - K') \right) + \frac{1}{16} (2K^\pm (\pm \lambda^\pm \mp L) - K')^2 \pm \frac{K^\pm}{8} \lambda^\pm \right] + O\left( \frac{1}{\lambda^2} \right). \quad (92)$$

[Proof]. By expanding the integrand in (91) to order $1/\lambda$, before taking the limit $\lambda \to \infty$, and using the integration formulae

$$\int_{-\infty}^{\infty} x^{2n} \exp\{-px^2\} dx = \frac{(2n-1)!!}{(2p)^n} \sqrt{\frac{\pi}{p}}, \quad p > 0, \quad (93)$$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$, and

$$\int_{-\infty}^{\infty} x^{2n+1} \exp\{-px^2\} dx = 0, \quad p > 0, \quad (94)$$
we have

\[ Z_\Lambda^\pm = c^\pm \int_{\mathbb{R}^\Lambda} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i p_j + 2iK^\pm \sum_{i,j=1}^\Lambda (a^\pm + 1) C^\pm_{ij} p_i \right\} \]

\[ \exp \left\{ -4y_i^2 \right\} \left[ 1 + \frac{2iK^\pm}{\sqrt{\lambda}} \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i p_j - \sum_{i=1}^\Lambda \left( \frac{y_i^4}{\lambda} + \frac{4y_i^3}{\sqrt{\lambda}} \right) \right. \]

\[ + \frac{1}{2\lambda} \left( -4(K^\pm)^2 \sum_{i,j,k,l=1}^\Lambda C^\pm_{ij} C^\pm_{kl} p_i p_k y_j y_l + 16 \sum_{i,j=1}^\Lambda y_i^3 y_j^3 \right. \]

\[ -16iK^\pm \sum_{i,j,k=1}^\Lambda C^\pm_{ij} p_i y_j y_k^3 \right] dpdy + O\left( \frac{1}{\lambda^2} \right) \]

\[ = c^\pm \int_{\mathbb{R}^\Lambda} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i p_j + 2iK^\pm \sum_{i,j=1}^\Lambda (a^\pm + 1) C^\pm_{ij} p_i \right\} \exp \left\{ -4y_i^2 \right\} \]

\[ \left[ 1 - \sum_{i=1}^\Lambda \frac{y_i^4}{\lambda} + \frac{1}{2\lambda} \left( -4(K^\pm)^2 \sum_{i,j,k=1}^\Lambda C^\pm_{ij} C^\pm_{ij} p_i p_j \right) \right. \]

\[ + 16 \sum_{i,j=1}^\Lambda y_i^6 - 16iK^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i y_j^4 \right] dpdy + O\left( \frac{1}{\lambda^2} \right) \]

\[ = c^\pm \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i p_j + 2iK^\pm \sum_{i,j=1}^\Lambda (a^\pm + 1) C^\pm_{ij} p_i \right\} \]

\[ \left( \frac{\sqrt{\pi}}{2} \right)^\Lambda \left[ 1 + \frac{1}{\lambda} \left( \frac{3}{16} \Lambda - \frac{3iK^\pm}{8} \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i \right. \right. \]

\[ \left. \left. - \frac{(K^\pm)^2}{4} \sum_{i,j,k=1}^\Lambda C^\pm_{ij} C^\pm_{ij} p_k \right) \right] dp + O\left( \frac{1}{\lambda^2} \right) \]

\[ = c^\pm \left( \frac{\sqrt{\pi}}{2} \right)^\Lambda \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} (p_i - i(a^\pm + 1)) (p_j - i(a^\pm + 1)) \right\} \]

\[ -K^\pm (a^\pm + 1)^2 \sum_{i,j=1}^\Lambda C^\pm_{ij} \left[ 1 + \frac{1}{\lambda} \left( \frac{3}{16} \Lambda - \frac{3iK^\pm}{8} \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i \right. \right. \]

\[ \left. \left. - \frac{(K^\pm)^2}{4} \sum_{i,j,k=1}^\Lambda C^\pm_{ij} C^\pm_{ij} p_k \right) \right] dp + O\left( \frac{1}{\lambda^2} \right) \],

where it has been taken into account that

\[ \int_{\mathbb{R}^\Lambda} \int_{\mathbb{R}^\Lambda} O_{p,y}(\frac{1}{\lambda^2}) dpdy = O\left( \frac{1}{\lambda^2} \right), \]

\[ O_{p,y}(\frac{1}{\lambda^2}) \] standing for the remainder in the expansion of the integrand in (31) up to order \( \frac{1}{\lambda^2} \).
Let $p_i \rightarrow p_i + i(a^\pm + 1)$.

\[
Z^\pm_{\Lambda} = c^\pm \left(\frac{\sqrt{\pi}}{2}\right)^\Lambda \exp \left\{ -K^\pm (a^\pm + 1)^2 \sum_{i,j=1}^\Lambda C^\pm_{ij} \right\} \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i p_j \right\} \\
\left[ 1 + \frac{1}{\Lambda} \left( \frac{3}{16} \Lambda - \frac{3iK^\pm}{8} \sum_{i,j=1}^\Lambda C^\pm_{ij} (p_i + i(a^\pm + 1)) \right) \right. \\
\left. \frac{(K^\pm)^2}{4} \sum_{i,j,k=1}^\Lambda C^\pm_{ij} C^\pm_{kj} (p_i + i(a^\pm + 1))(p_k + i(a^\pm + 1)) \right] dp + O\left(\frac{1}{\lambda^2}\right)
\]

where

\[
\int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i,j=1}^\Lambda C^\pm_{ij} p_i p_j \right\} \sum_{i,j,k=1}^\Lambda C^\pm_{ij} C^\pm_{kj} p_i p_k dp \\
= \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i=1}^\Lambda \lambda^i q_i^2 \right\} \sum_{i,j,m,n=1}^\Lambda ((C^\pm)^2)_{ij} A_{im} A_{jn} q_m q_n dq \\
= \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i=1}^\Lambda \lambda^i q_i^2 \right\} \sum_{i=1}^\Lambda (\bar{A}(C^\pm)^2 A)_{ii} q_i^2 dq \\
= \int_{\mathbb{R}^\Lambda} \exp \left\{ -K^\pm \sum_{i=1}^\Lambda \lambda^i q_i^2 \right\} \sum_{i=1}^\Lambda (\lambda^i) q_i^2 dq = \frac{1}{2K^\pm} \left(\frac{\pi}{K^\pm}\right)^\frac{\Lambda}{2} \frac{1}{\sqrt{detC^\pm}} TrC^\pm.
\]

Therefore we get

\[
Z^\pm_{\Lambda} = \exp \left\{ \pm \lambda^\pm K^\pm \Lambda - K^\pm (2a^\pm + 1) \sum_{i,j=1}^\Lambda C^\pm_{ij} \right\} \left[ 1 + \frac{1}{\Lambda} \left( \frac{3}{16} \Lambda + \frac{3K^\pm}{8} \sum_{i,j=1}^\Lambda C^\pm_{ij} (a^\pm + 1) \right) \right. \\
\left. \frac{(K^\pm)^2(a^\pm + 1)^2}{4} \sum_{i,j,k=1}^\Lambda C^\pm_{ij} C^\pm_{kj} - \frac{K^\pm}{8} TrC^\pm \right] + O\left(\frac{1}{\lambda^2}\right).
\]

(95)
Using (19) and (20) we obtain
\[
Z^\pm_{\Lambda} = \exp \left\{ \left( \pm K^\pm L + K' \right) \Lambda \right\} \left[ 1 + \frac{\Lambda}{\lambda} \left( \frac{3}{16} + \frac{3}{16} \left( 2K^\pm \pm \lambda \pm L \right) - K' \right) \right.
\]
\[
+ \frac{1}{16} \left( 2K^\pm \pm \lambda \pm L \right)^2 = \frac{K^\pm}{\lambda^2} \right] + O\left( \frac{1}{\lambda^2} \right). \tag{96}
\]

From (96) one gets a corresponding asymptotic representation of free energy of the system,
\[
F^\pm_{\Lambda} = -kT \frac{1}{\Lambda} \log Z_{\Lambda} = -kT \left( \pm K^\pm L + K' \right)
\]
\[
= -kT \frac{1}{\Lambda} \log \left[ 1 + \frac{\Lambda}{\lambda} \left( \frac{3}{16} + \frac{3}{16} \left( 2K^\pm \pm \lambda \pm L \right) - K' \right) \right.
\]
\[
+ \frac{1}{16} \left( 2K^\pm \pm \lambda \pm L \right)^2 = \frac{K^\pm}{\lambda^2} \right] + O\left( \frac{1}{\lambda^2} \right). \tag{97}
\]

**Conclusions** From the integral representation of the partition function for general \(n\)-dimensional Ising models with both nearest and non-nearest neighbours interactions we have proved that for given \(K, K'\), the partition function of \(n\)-dimensional homogeneous systems on a square lattice is uniquely given by the eigenvalues of the related interaction coupling matrix. For one-dimensional homogeneous ferromagnetic systems with positive coupling coefficients, the partition function satisfies a special equality which means that the variation of the interaction couplings of a system is related to the variations of the external magnetic field and the temperature. For some special cases of interaction coupling, we obtained, for a Ising model in \(n\)-dimensions, an exact solution for the partition function in terms of Bessel functions. We also calculated the partition function of the \(n\)-dimensional Ising model to order \(O(1/\lambda^2)\). It turned out that the leading terms of the partition function for any homogeneous systems in arbitrary dimensions are given by the largest positive eigenvalue (resp. the largest absolute value of the negative eigenvalues) of the related interaction coupling matrix for ferromagnetic (resp. antiferromagnetic) systems.

The advantage of our integration approach is that one can analyse the partition function of the Ising model for various interaction couplings on lattices of arbitrary dimensions in terms of integrals. More results could be obtained by studying the properties of other
interaction coupling matrices. It would also be interesting to study the relation between the largest eigenvalues of the interaction coupling matrices in our integration approach and the largest eigenvalues of the usual transfer matrix approach to Ising models, see e.g., \[4, 5\]. Moreover the integral representation formula can also serve as an alternative way to study the correlation functions, and as a mean to provide numerical approximations to the thermodynamic functions of the Ising model.

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