A weighted fractional problem involving a singular nonlinearity and an $L^1$ datum

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Abstract

In this article, we show the existence of a unique entropy solution to the following problem:

$$(-\Delta)^s_{p,\alpha} u = f(x)h(u) + g(x) \text{ in } \Omega,$$

$$u > 0 \text{ in } \Omega,$$

$$u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where the domain $\Omega \subset \mathbb{R}^N$ is bounded and contains the origin, $\alpha \in [0, \frac{N-ps}{2})$, $s \in (0, 1)$, $2 - \frac{s}{N} < p < \infty$, $sp < N$, $g \in L^1(\Omega)$, $f \in L^q(\Omega)$ for $q > 1$ and $h$ is a general singular function with singularity at 0. Further, the fractional $p$-Laplacian with weight $\alpha$ is given by

$$(-\Delta)^s_{p,\alpha} u(x) = \text{P. V. } \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}} \frac{dy}{|x|^\alpha |y|^\alpha}, \forall x \in \mathbb{R}^N.$$

Keywords: Weighted fractional Sobolev spaces, singular nonlinearity, entropy solution.

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1 Introduction

We consider the following problem

$$(-\Delta)^s_{p,\alpha} u = f(x)h(u) + g(x) \text{ in } \Omega,$$

$$u > 0 \text{ in } \Omega,$$

$$u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

(1.1)

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where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $s \in (0, 1)$, $2 - \frac{s}{p} < p < \infty$, $sp < N$, $0 \leq \alpha < \frac{N - sp}{2}$ with $g \in L^1(\Omega)$ and $f \in L^q(\Omega)$, for some $q > 1$ are two positive functions. The singularity $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-increasing and continuous function such that

$$\lim_{v \to 0^+} h(v) \in (0, \infty) \quad \text{and} \quad \lim_{v \to \infty} h(v) := h(\infty) < \infty. \quad (1.2)$$

Regarding the growth conditions on $h$ near zero and infinity it will be considered that there are constants $K_1, K_2, M, N > 0$ in such a way that

$$h(v) \leq \frac{K_1}{v^\gamma} \quad \text{if} \quad v < M, \quad \text{for} \quad 0 < \gamma < 1, \quad (1.3)$$

$$h(v) \leq \frac{K_2}{v^\theta} \quad \text{if} \quad v > N, \quad \text{for} \quad \theta > 0. \quad (1.4)$$

Note that in the case (1.4) we have $h(\infty) = 0$. The weighted fractional $p$-Laplacian is defined as

$$(-\Delta)^{s,\alpha}_{p,\alpha} u(x) = P \mathcal{V} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{N+sp}} \frac{dy}{|x|^{\alpha}|y|^{\alpha}}, \quad \forall x \in \mathbb{R}^N. \quad (1.5)$$

This operator $(-\Delta)^{s,\alpha}_{p,\alpha}$ is the nonlocal version of the local operator the operator $-\text{div}(|x|^{-\alpha} \nabla \cdot |x|^{-2p} \nabla u)$. The Caffarelli–Kohn–Nirenberg inequality, derived in [7], is strongly related to the problem

$$-\text{div}(|x|^{-\alpha} \nabla |u|^{p-2} \nabla u) = 0, \quad \text{in} \quad \mathbb{R}^N \quad (1.6)$$

and each weak supersolution to the above problem (1.6) satisfies the weak Harnack inequality, for every $\alpha < N - p$. Thus, in this sense, we say that $|x|^{-\alpha}$ is an admissible weight. A nonlocal version of the Caffarelli–Kohn–Nirenberg inequality, in a bounded domain, has been introduced by Abdellaoui & Bentifour in [2] with $\alpha = \frac{N - sp}{2}$. For the case $\Omega = \mathbb{R}^N$, the generalization of the classical Caffarelli–Kohn–Nirenberg inequality is given in [2] for $\alpha < \frac{N - sp}{2}$ and is known as the weighted fractional Sobolev inequality. This condition on $\alpha$ is related in some sense to the admissible weight described in [13]. The derivation of the weighted fractional Sobolev inequality is based on the following improved Hardy inequality, i.e. for every $u \in C_0^\infty(\Omega)$,

$$\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy - \Lambda_{N,s,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{sp}} \, dx \geq C \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \frac{dxdy}{|x|^{\frac{N}{2}}|y|^{\frac{N}{2}}}. \quad (1.7)$$

Here, $\Lambda_{N,s,p}$ is the best Hardy constant and $v(x) = |x|^{\frac{N-sp}{p}} u(x)$. Therefore, the class of operators like $(-\Delta)^{s,\alpha}_{p,\alpha}$ appear naturally when one deals with the inequality (1.7). We quote [2] [3] [11] for further references.

The main goal of this manuscript is to obtain the existence and uniqueness of an entropy solution to (1.1). The concept of entropy solution is very useful when technical difficulties arise in proving the uniqueness of the weak solution. Regarding such approach in problems involving an $L^1$ datum we point out [1] [5] [4] and its references. Such notion was considered
for the first time by Boccardo et al. [5] and Bénilan et al [4], where the authors considered the following problem with an $L^1$ datum $g$:

$$
A(u) = g(x) \text{ in } \Omega,
$$

$$
u = 0 \text{ in } \Omega, \tag{1.8}
$$

where $u \mapsto A(u)$ is a monotone operator acting on $W^{1,p}_0(\Omega)$ for $1 < p < N$. The above mentioned problem (1.8) is a local sub-case of the problem (1.1) with $s = 1$, $\alpha = 0$ and $f = 0$. Although there is a good amount of literature pertaining to the case $f \not\equiv 0$, the notion of entropy solution was not previously applied in the literature to guarantee the existence and uniqueness of solution.

The case of measure data was considered in [9], where the authors have shown the existence of a renormalized solution to the following problem:

$$
-\text{div}(a(x, \nabla u)) = \mu \text{ in } \Omega,
$$

$$
u = 0 \text{ in } \Omega. \tag{1.9}
$$

Here, $\mu$ is a bounded Radon measure and $u \mapsto -\text{div}(a(x, \nabla u))$ is a monotone operator acting on $W^{0,p}_0(\Omega)$. Kuusi et al. in [16], dealt with a Dirichlet problem involving an $L^1$ datum and the fractional $p$-Laplacian. The authors used the basic and optimal nonlinear Wolff potential estimates to obtain the existence of solution in an appropriate fractional Sobolev space. An important piece of work which is worth mentioning here is due to Abdellaoui et al. in [1]. The authors have considered the case $f \equiv 0$ of the problem (1.1) and obtained a unique entropy solution using some algebraic inequalities.

An improvement of the problems above was considered by Panda et al. [23], where the authors studied a local singular problem involving a measure datum. The problem (1.1) with $\alpha = 0$, $s = 1$, $p = 2$ and with $\alpha = 0$, $s \in (0,1)$, $p = 2$ has been analysed by Panda et al. [23] and by Ghosh et al. [12] respectively. The authors guaranteed the existence of a weak solution as well as a very weak solution to the problem using an approximation argument. The case of purely singular problem, i.e. (1.1) with $g \equiv 0$, involving the fractional $p$-Laplace operator was treated in [8] for $\gamma > 0$. Some equally important works in the literature on these type of problems that involves a singular term and a source term in the form of a measure can be found in [15, 20, 21, 22] and the references therein.

In this manuscript, by using adapting the ideas of [1, 23], we extend the work of Abdellaoui [1] by considering a singular function $h$, having a singularity at 0, in the right hand side of the problem. Due to the irregularity near the boundary, singular problems admits solutions in a weak distributional sense, for compactly supported test functions. See Section 2 for the notations.

**Theorem 1.1.** There exists a positive nontrivial weak solution $u$ to the problem (1.1) in $X^{t,m,\alpha}_0(\Omega)$, for every $1 \leq m < \frac{N(p-1)}{N-s}$ and $0 < t < s$, in the sense of Definition 2.8. More precisely,

$$
\int_Q \frac{|u(x) - u(y)|^m}{|x-y|^{N+tm}} \frac{dxdy}{|x|^\alpha |y|^\alpha} \leq C, \tag{1.10}
$$

for every $1 \leq m < \frac{N(p-1)}{N-s}$ and every $0 < t < s$. Moreover, $u \in T^{s,p,\alpha}_0(\Omega)$. 

The weak solution space do not lie in the natural energy space corresponding to the operator \((-\Delta)_{p,\alpha}\), i.e. \(X^{s,p,\alpha}_0(\Omega)\), but it has a lower degree of differentiability and integrability. The next theorem proves the uniqueness result.

**Theorem 1.2.** Let \(g \in L^1(\Omega)\). Then, the problem (1.1) admits a unique entropy solution in \(T_0^{s,p,\alpha}(\Omega)\) in the sense of Definition 2.9.

The paper is organized as follows: In Section 2, we introduce the weighted fractional Sobolev space which is the natural solution space to the given problem (1.1). Further, we also define some useful function spaces and provide some auxiliary results. The existence of a positive weak solution to the problem (1.1) is proved in Section 3 by an approximation argument and using some apriori estimates. Section 4 is all about showing the existence and uniqueness of a positive entropy solution to (1.1). In this process, we show the equivalence between the weak solution and the entropy solution to (1.1). Additionally, we prove the existence of an entropy solution to the problem (1.1) with \(h(u) = \frac{1}{u^\gamma}\) and with a general function \(g(x, w) = w^r + g(x)\) for \(r < p - 1\).

## 2 Properties of weighted fractional Sobolev space and some preliminary definitions & results

We now introduce the weighted fractional Sobolev space which is a natural solution space to the problem (1.1). Let us consider \(\Omega\) to be a bounded domain in \(\mathbb{R}^N\), \(s \in (0, 1)\), \(1 < p < \infty\) and \(\alpha \in [0, \frac{N-2p}{2})\). Denote \(d\nu = \frac{dx}{|x|^{2\alpha}},\ d\mu = \frac{dxdy}{|x-y|^{N+sp}|x|^{\alpha}|y|^{\alpha}}\).

The fractional Sobolev space with weight \(\alpha\) (refer [2]) is defined by

\[
W^{s,p,\alpha}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N, d\nu) : \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p d\mu < \infty \right\}.
\]

Further, denote \(Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)\) with \(\Omega^c = \mathbb{R}^N \setminus \Omega\). Then, the space \(X^{s,p,\alpha}(\Omega)\) is defined by

\[
X^{s,p,\alpha}(\Omega) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, } u|_{\Omega} \in L^p(\Omega, d\nu) \text{ and } \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}|x|^{\alpha}|y|^{\alpha}} < \infty \right\}
\]

and this space is a Banach space endowed with the norm:

\[
||u||_{X^{s,p,\alpha}(\Omega)} = ||u||_{L^p(\Omega,d\nu)} + \left( \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}|x|^{\alpha}|y|^{\alpha}} \right)^\frac{1}{p}.
\]

Define a space \(X^{s,p,\alpha}_0(\Omega)\) by

\[
X^{s,p,\alpha}_0(\Omega) = \{ u \in X^{s,p,\alpha}(\Omega) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \},
\]
which is a Banach space when endowed with the following norm:

\[ \|u\|_{X_0^{s,p,\alpha}(\Omega)}^p = \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp} |x|^{\alpha}|y|^{\alpha}} \, dx \, dy. \]

If \( \alpha = 0 \), denote \( W^{s,p,\alpha}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N), X^{s,p,\alpha}(\Omega) = X^{s,p}(\Omega), X_0^{s,p,\alpha}(\Omega) = X_0^{s,p}(\Omega) \) which are the usual well-known fractional Sobolev spaces. Since the domain \( \Omega \) is bounded, the embedding \( L^1(\Omega, d\nu) \hookrightarrow L^1(\Omega) \) is continuous and we will use this embedding throughout the article.

The following is the weighted fractional Sobolev inequality which is an improvement of the fractional Sobolev inequality and is obtained by the help of weighted Hardy inequality (see [17]).

**Theorem 2.1** ([2], Theorem 1.4, Theorem 1.5). Let \( s \in (0, 1) \) and \( 1 < p < \frac{N}{s} \). If \( \alpha < \frac{N - sp}{2} \), then there exists constant \( S = S(s, N, p, \alpha) > 0 \) such that

\[ \int_{\mathbb{R}^N} |u(x) - u(y)|^p d\mu \geq S \left( \int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_s}}{|x|^{2sp^*_s/p}} \right)^{p^*_s/p} \]

for every \( u \in C_c^\infty(\mathbb{R}^N) \). Here, the constant \( p^*_s = \frac{Np}{N - sp} \) is the fractional critical Sobolev exponent.

The next theorem is known as the fractional Caffarelli–Kohn–Nirenberg inequality in a bounded domain, proved in [2].

**Theorem 2.2.** If \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) and \( \alpha = \frac{N - sp}{2} \), then for every \( q < p \) there exists constant \( C = C(\Omega, s, q, N) > 0 \) such that

\[ \int_{\mathbb{R}^N} |u(x) - u(y)|^p d\mu \geq C \left( \int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_q,s}}{|x|^{2sp^*_q,s/p}} \right)^{p^*_q,s/p} \]

for every \( u \in C_c^\infty(\mathbb{R}^N) \), where \( p^*_q,s = \frac{Np}{N - sq} \).

The truncation function \( T_k : \mathbb{R} \to \mathbb{R} \), for a fixed \( k > 0 \), is defined by

\[ T_k(s) = \max\{-k, \min\{k, s\}\} \]

and define

\[ G_k(v) = (|v| - k)^+ \text{sign}(v). \]

Clearly, \( T_k(v) + G_k(v) = v \) for every \( v \in \mathbb{R} \).

**Definition 2.3.** We define \( T_0^{s,p,\alpha}(\Omega) \) to be the class of measurable functions \( u \) on \( \Omega \) such that \( T_k(u) \in X_0^{s,p,\alpha}(\Omega) \) for all \( k > 0 \).

The following well-known algebraic inequalities will be frequently used throughout this article.
Remark 2.4. Let $p \geq 1$, $\beta > 0$ and $a_1, a_2 \in [0, \infty]$. Then, there exist constants $c_1, c_2, c_3 > 0$ such that

1. $(a_1 + a_2)^\beta \leq c_1 a_1^\beta + c_2 a_2^\beta$,
2. $|a_1 - a_2|^{p-2}(a_1 - a_2)(a_1^\beta - a_2^\beta) \geq c_3 |a_1^\frac{\beta + p - 1}{p} - a_2^\frac{\beta + p - 1}{p}|^p$,
3. $|a_1 + a_2|^{\beta - 1}|a_1 - a_2|^p \leq c_3 |a_1^\frac{\beta + p - 1}{p} - a_2^\frac{\beta + p - 1}{p}|^p$ for $\beta \geq 1$,
4. for $a_1, a_2 \in \mathbb{R}$,
   $$|a_1 - a_2|^{p-2}(a_1 - a_2)(T_k(a_1) - T_k(a_2)) \geq |T_k(a_1) - T_k(a_2)|^p,$$
   $$|a_1 - a_2|^{p-2}(1 - 2^{-p})|a_1 - a_2|^p, \text{ when } p \geq 2,$$
5. $(a_1^{p-2}a_1 - a_2^{p-2}a_2) \cdot (a_1 - a_2) \geq 2^{2-p}|a_1 - a_2|^p$, when $p \geq 2$,
6. $(a_1^{p-2}a_1 - a_2^{p-2}a_2) \cdot (a_1 - a_2) \geq \frac{1}{2} (|a_1| + |a_2|)^{p-2} |a_1 - a_2|^2$, when $p < 2$.

Definition 2.5 (I). A measurable function $u : \Omega \to \mathbb{R}$ is said to be in the weighted Marcinkiewicz space, denoted by $M^q(\Omega, d\nu)$ for $0 < q < \infty$, if

$$\nu(\{x \in \Omega : |u(x)| > t\}) \leq \frac{C}{t^q}, \text{ for } t > 0 \text{ and } 0 < C < \infty.$$ 

Assume $\Omega \subset \mathbb{R}^N$ to be bounded, then

1. $M^{q_1}(\Omega, d\nu) \subset M^{q_2}(\Omega, d\nu)$, $\forall q_1 \geq q_2 > 0$,
2. for $1 \leq q < \infty$ and $0 < \epsilon < q - 1$, we have the following continuous embedding

$$L^q(\Omega, d\nu) \subset M^q(\Omega, d\nu) \subset L^{q-\epsilon}(\Omega, d\nu). \quad (2.11)$$

Abdellaoui & Bentifour [2] provided a useful comparison principle which is stated in the following lemma.

Lemma 2.6. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p \geq 1$, $H(\cdot, \cdot)$ is a positive continuous function and $\frac{H(x,v)}{v^{p-1}}$ is decreasing for $v > 0$. Furthermore, let $v, w \in X^{s,p,\alpha}_0(\Omega)$ satisfy

$$\begin{cases}
(-\Delta)^{s,p,\alpha}_\nu v \geq H(x,v) & \text{in } \Omega, \\
(-\Delta)^{s,p,\alpha}_\nu w \leq H(x,w) & \text{in } \Omega, \\
v, w > 0 & \text{in } \Omega.
\end{cases}$$

Then, $w \leq v$ in $\Omega$.

We now provide different notions of solutions to problem [11].

Definition 2.7. Let $u$ be a measurable function. Then $u \in T_0^{s,p,\alpha}(\Omega)$ if for every $k > 0$,

$$T_k(u) \in X^{s,p,\alpha}_0(\Omega).$$
Definition 2.8 (Weak solution). Let \( g \in L^1(\Omega) \). A function \( u \in X_0^{t,m,\alpha}(\Omega) \), for \( 0 < t < s \) and \( 1 \leq m < p \), is said to be a positive weak solution to (1.1) if
\[
\int_{Q} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))d\mu = \int_{\Omega} fh(u)\phi + \int_{\Omega} g\phi \quad (2.12)
\]
for every \( \phi \in C_c^\infty(\Omega) \) and for every \( \omega \subset \subset \Omega \), there exists \( C_\omega > 0 \) such that \( u \geq C_\omega > 0 \).

In many cases, it is very tough to guarantee the uniqueness of weak solutions and to overcome this difficulty, the concept of entropy solution comes into picture to prove the uniqueness of solutions.

Definition 2.9 (Entropy solution). Let \( g \in L^1(\Omega) \). A function \( u \in \mathcal{T}_0^{s,p,\alpha}(\Omega) \) is said to be an entropy solution to (1.1) if it satisfies the following:

1. \[
\int_{D_l} |u(x) - u(y)|^{p-1}d\mu \to 0 \quad \text{as } l \to \infty, \quad (2.13)
\]
where
\[
D_l = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : l + 1 \leq \max\{|u(x)|, |u(y)|\} \text{ with } \min\{|u(x)|, |u(y)|\} \leq l\},
\]
2. for every \( \varphi \in X_0^{s,p,\alpha}(\Omega) \cap L^\infty(\Omega) \), \( k > 0 \),
\[
\int_{Q} |u(x) - u(y)|^{p-2}(u(x) - u(y))[T_k(u(x) - \varphi(x)) - T_k(u(y) - \varphi(y))]d\mu \leq \int_{\Omega} fh(u)T_k(u - \varphi) + \int_{\Omega} gT_k(u - \varphi). \quad (2.14)
\]

The solution \( u \) is positive if for every \( \omega \subset \subset \Omega \), there exists \( C_\omega > 0 \) such that \( u \geq C_\omega > 0 \).

Remark 2.10. It is easy to show that every entropy solution of (1.1) is a weak solution of (1.1).

3 Existence of weak solution - Proof of Theorem 1.1

In this section, we prove the existence of a positive weak solution to the problem (1.1) in the sense of Definition 2.8. The proof is through approximation, since the problem involves an \( L^1 \) function \( g \). Let us consider the following approximating problem:
\[
(-\Delta)^s_{p,\alpha} u_n = f_nh_n(u_n + 1/n) + g_n \text{ in } \Omega,
\]
\[
u_n > 0 \text{ in } \Omega,
\]
\[
u_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]
(3.15)

Here, \( f_n = T_n(f) \), \( h_n = T_n(h) \) and \( \{g_n\} \subset L^\infty(\Omega) \) is an increasing sequence such that \( g_n \to g \) strongly in \( L^1(\Omega) \). We say \( u_n \in X_0^{s,p,\alpha}(\Omega) \) is a weak solution to (3.15), if \( u_n \) satisfies
\[
\int_{Q} |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))d\mu = \int_{\Omega} f_nh_n(u_n + 1/n)\phi + \int_{\Omega} g_n\phi, \quad (3.16)
\]
for every \( \phi \in C_c^\infty(\Omega) \).
Lemma 3.1. Let $h$ satisfies (1.3) and (1.4) with $0 < \gamma, \theta < \infty$. Then, for a fixed $n \in \mathbb{N}$, there exists a unique weak solution $u_n \in X_0^{s,p,\alpha}(\Omega) \cap L^\infty(\Omega)$ to (3.11).

Proof. The proof follows the ideas of Panda et al. [23] using Schauder’s fixed point argument. Let us define a map,

$$G : L^p(\Omega, d\nu) \rightarrow X_0^{s,p,\alpha}(\Omega) \subset L^p(\Omega, d\nu)$$

such that for any $w \in L^p(\Omega, d\nu)$ we obtain a unique weak solution $v \in X_0^{s,p,\alpha}(\Omega)$ to the following problem by Lax-Milgram theorem.

$$(-\Delta)^s_{p,\alpha} v = h_n \left( |w| + \frac{1}{n} \right) f_n + g_n \text{ in } \Omega,$$

$$v = 0 \text{ on } \partial \Omega.$$  \hspace{1cm} (3.17)

Further, $C_c^\infty(\Omega)$ is dense in $X_0^{s,p,\alpha}(\Omega)$ with respect to the norm topology of $X_0^{s,p,\alpha}(\Omega)$. Hence, we are allowed to choose $v$ as a test function in the weak formulation of (3.17). Thus, we get

$$\int_Q |v(x) - v(y)|^p d\mu = \int_\Omega h_n \left( |w| + \frac{1}{n} \right) f_n v + \int_\Omega g_nv \leq K_1 \int_{\{|w| + \frac{1}{n} < M\}} \frac{f_n v}{(|w| + \frac{1}{n})^\gamma} + \max_{|M,N|} \{h_n\} \int_{(M \leq (|w| + \frac{1}{n}) \leq N)} f_n v$$

$$+ K_2 \int_{\{|w| + \frac{1}{n} > N\}} \frac{f_n v}{(|w| + \frac{1}{n})^\gamma} + C(n) \int_\Omega |v| \leq K_1 n^{1+\gamma} \int_{\{|w| + \frac{1}{n} < M\}} |v| n \max_{|M,N|} \{h_n\} \int_{(M \leq (|w| + \frac{1}{n}) \leq N)} |v|$$

$$+ K_2 n^{1+\theta} \int_{\{|w| + \frac{1}{n} > N\}} |v| + C(n) \int_\Omega |v| \leq C'(n, \gamma, \theta) \int_\Omega |v| d\nu = C'(n, \gamma, \theta) \|v\|_{L^1(\Omega, d\nu)}.$$  \hspace{1cm} (3.18)

Now by applying the weighted Sobolev embedding theorem, i.e. Theorem 2.1, there exists a constant $C(n, \gamma, \theta) > 0$ such that

$$\|v\|_{X_0^{s,p,\alpha}(\Omega)} \leq C(n, \gamma, \theta),$$

where $C(n, \gamma, \theta)$ is independent of $w$. We will now show the continuity and compactness of the operator $G$ to apply the Schauder fixed point theorem.

Claim 1: The map $G$ is continuous over $X_0^{s,p,\alpha}(\Omega)$.

Let us consider a sequence $\{w_k\} \subset X_0^{s,p,\alpha}(\Omega)$ that converges strongly to $w$ with respect to the $X_0^{s,p,\alpha}$-norm. Thus, by the uniqueness of the weak solution, we denote $v_k = G(w_k)$, $v = G(w)$ and let $\bar{v}_k = v_k - v$. According to the Remark 2.4 (5) and (6),

$$|v_k(x) - v_k(y)| + |v(x) - v(y)| = |\bar{v}_k(x) - \bar{v}_k(y)|^p \leq 2 \int_\Omega \left( h_n \left( |w_k| + \frac{1}{n} \right) f_n - h_n \left( |w| + \frac{1}{n} \right) f_n \right) \bar{v}_k \hspace{1cm} (3.19)$$
and
\[
\text{if } p \geq 2, \int_Q |\vec{v}_k(x) - \vec{v}_k(y)|^p d\mu \leq 2^{p-2} \int_{\Omega} \left( h_n \left( |w_k| + \frac{1}{n} \right) f_n - h_n \left( |w| + \frac{1}{n} \right) f_n \right) \vec{v}_k. \tag{3.20}
\]

Hence, by Theorem 2.11 and Hölder’s inequality we establish the following for the case of \( p \geq 2 \).
\[
\|\vec{v}_k\|_{X_0^{s,p,\alpha}(\Omega)}^p \leq \left\| h_n \left( |w_k| + \frac{1}{n} \right) f_n - h_n \left( |w| + \frac{1}{n} \right) f_n \right\|_{L^{(p^*)'}(\Omega)} \|\vec{v}_k\|_{L^{p^*}(\Omega)}
\leq C_1(N, p, \alpha, \Omega) \|h_n \left( |w_k| + \frac{1}{n} \right) f_n - h_n \left( |w| + \frac{1}{n} \right) f_n \|_{L^{(p^*)'}(\Omega)} \left\| \frac{\vec{v}_k}{x^{2\alpha/p}} \right\|_{L^{p^*}(\Omega)}
\leq C_2 \left\| h_n \left( |w_k| + \frac{1}{n} \right) f_n - h_n \left( |w| + \frac{1}{n} \right) f_n \right\|_{L^{(p^*)'}(\Omega)} \|\vec{v}_k\|_{X_0^{s,p,\alpha}(\Omega)},
\]

where \((p_*)'\) is the Hölder conjugate of \( p_* \). This implies
\[
\|\vec{v}_k\|_{X_0^{s,p,\alpha}(\Omega)} \leq C \left\| h_n \left( |w_k| + \frac{1}{n} \right) f_n - h_n \left( |w| + \frac{1}{n} \right) f_n \right\|_{L^{(p^*)'}(\Omega)}^{1/p-1}.
\]

Now on applying the dominated convergence theorem we prove that \( \lim_{k \to \infty} \|\vec{v}_k\|_{X_0^{s,p,\alpha}(\Omega)} = 0 \).
This gives \( \lim_{k \to \infty} \|v_k - v\|_{X_0^{s,p,\alpha}(\Omega)} = 0 \). In this similar manner, for the case \( 1 < p < 2 \), we can prove the strong convergence \( v_k \to v \) in \( X_0^{s,p,\alpha}(\Omega) \). Thus, \( G \) is continuous from \( X_0^{s,p,\alpha}(\Omega) \) to \( X_0^{s,p,\alpha}(\Omega) \).

**Claim 2:** \( G(X_0^{s,p,\alpha}(\Omega)) \) is relatively compact in \( X_0^{s,p,\alpha}(\Omega) \).

Let \( \{w_k\} \) is a bounded sequence in \( X_0^{s,p,\alpha}(\Omega) \). Then, there exists \( w \in X_0^{s,p,\alpha}(\Omega) \) and a subsequence of \( \{w_k\} \) (still denoted as \( \{w_k\} \)) such that \( w_k \to w \) weakly in \( X_0^{s,p,\alpha}(\Omega) \). We have already proved in (3.18) that
\[
\|G(w_k)\|_{X_0^{s,p,\alpha}(\Omega)} \leq C,
\]
for \( C > 0 \) independent of \( k \). Thus, there exists \( v \in X_0^{s,p,\alpha}(\Omega) \) such that \( G(w_k) \to v \) weakly in \( X_0^{s,p,\alpha}(\Omega) \). Now by passing the limit \( k \to \infty \) in the weak formulation and using some basic computation we prove that \( G(w) = v \). Following the similar lines used in the proof of Claim 1, we finally show that \( \lim_{k \to \infty} \|G(w_k) - G(w)\|_{X_0^{s,p,\alpha}(\Omega)} = 0 \). Hence the claim.

Thus, on using the Schauder fixed point theorem to \( G \), we obtain a fixed point \( u_n \in X_0^{s,p,\alpha}(\Omega) \) that is also a weak solution to the problem (3.15) in \( X_0^{s,p,\alpha}(\Omega) \). Further, by the strong maximum principle [Lemma 2.3, [19]], \( u_n > 0 \) in \( \Omega \).

**Claim 3:** Uniqueness of weak solution.

To prove this claim, suppose the problem (3.15) admits two different weak solutions \( u_n \) and \( v_n \). Let us take \( \phi = (u_n - v_n)^+ \) as a test function in the weak formulation (3.16). Thus, for
\( p \geq 2 \), with the consideration of Lemma 9 in [18], we obtain the following.

\[
0 \leq \int_{\Omega} \left| (u_n(x) - v_n(x))^+ - (u_n(y) - v_n(y))^+ \right|^pd\mu \\
\leq \int_{\Omega} \left[ |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y)) \right] \\
\times \left[ (u_n(x) - v_n(x))^+ - (u_n(y) - v_n(y))^+ \right] d\mu \\
= \int_{\Omega} (f_n h_n(u_n + \frac{1}{n}) - f_n h_n(v_n + \frac{1}{n}))(u_n - v_n)^+ \\
\leq 0.
\]

This implies \((u_n - v_n)^+ = 0\) a.e in \(\Omega\), since \(u_n = v_n = 0\) in \(\mathbb{R}^N \setminus \Omega\) and thus \(u_n \leq v_n\) a.e. in \(\Omega\). The same proof holds for \(1 < p < 2\). Proceeding similarly with \(\phi = (v_n - u_n)^+\) as a test function, we can show that \(u_n \geq v_n\) a.e. in \(\Omega\). This proves the claim.

Since the right hand side of (3.15) belongs to \(L^\infty(\Omega)\), by virtue of Lemma 4.3 of [1], we conclude that \(u_n \in L^\infty(\Omega)\).

**Lemma 3.2.** Let \(u_n\) be a unique positive weak solution to (3.15). Then,

1. the sequence \(\{u_n\}\) is an increasing sequence w.r.t \(n\) and for every set \(\omega \subset \subset \Omega\) there exists constant \(C_\omega > 0\), independent of \(n\), such that

\[
u_n \geq C_\omega > 0, \text{ for every } n \in \mathbb{N} \tag{3.21}\]

2. \(h(u_n)f \in L^1(\Omega)\).

**Proof.** Let us consider the problems satisfied by \(u_n\) and \(u_{n+1}\). Then, subtracting these two problems and taking the test function \((u_n - u_{n+1})^+\) in its weak formulation we get

\[
0 \leq \int_{\Omega} \left| (u_n(x) - u_{n+1}(y))^+ - (u_n(x) - u_{n+1}(y))^+ \right|^pd\mu \\
\leq \int_{\Omega} \left( I_p(u_n) - I_p(u_{n+1}) \right) ((u_n(x) - u_{n+1}(x))^+ - (u_n(y) - u_{n+1}(y))^+) d\mu \\
= \int_{\Omega} \left( f_nh_n(u_n + \frac{1}{n}) - f_{n+1}h_{n+1}(u_{n+1} + \frac{1}{n+1}) \right) (u_n - u_{n+1})^+ + \int_{\Omega} (g_n - g_{n+1})(u_n - u_{n+1})^+ \\
\leq \int_{\Omega} \left( f_{n+1}h_{n+1}(u_n + \frac{1}{n+1}) - f_n h_n(u_{n+1} + \frac{1}{n+1}) \right) (u_n - u_{n+1})^+ \\
\leq 0,
\]

where \(I_p(v) = |v(x) - v(y)|^{p-2}(v(x) - v(y))\). This implies \((u_n - u_{n+1})^+ = 0\) a.e. in \(\Omega\) and hence \(u_n \leq u_{n+1}\) a.e. in \(\Omega\).

Further, from the definition of the problem (3.15), \(u_1 > 0\) in \(\Omega\) and belongs to \(L^\infty(\Omega)\), i.e. there exists \(M > 0\) such that \(\|u_1\|_{L^\infty(\Omega)} \leq M\). Since \(\{u_n\}\) is an increasing sequence, \(u_n\) verifies
for all $n \geq 1$. This proves (1).

Consider the following nonlinear eigenvalue problem:

\begin{equation}
(-\Delta)^s_{p,\alpha} v = \lambda |v|^{p-2} v \text{ in } \Omega,
\end{equation}

\[ v = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \]

Moreover, from the weighted fractional Sobolev embedding, we have the Rayleigh quotient as follows.

\[ \lambda_1 = \inf_{v \in X_{0}^{s,p,\alpha}(\Omega), v \neq 0} \left\{ \frac{\|v\|_{p,0}^p}{\|v\|_{p,\infty}^p} \right\} \]

and $\lambda_1 \in (0, \infty)$. Let $\phi_1$ be the eigenfunction corresponding to the eigenvalue $\lambda_1$. Then, it is not difficult to prove that either $\phi_1 > 0$ a.e. in $\Omega$ or $\phi_1 < 0$ a.e. in $\Omega$, refer Proposition 2.3 of [14]. Indeed, if $\psi = \phi_1$ is chosen as a test function to the following weak formulation

\[ \int_{\Omega} |\phi_1(x) - \phi_1(y)|^{p-2}(\phi_1(x) - \phi_1(y))(\psi(x) - \psi(y))d\mu = \lambda_1 \int_{\Omega} |\phi_1|^{p-2}\phi_1 \psi, \]

we get $\phi_1 \equiv 0$ a.e. in $\Omega$. So, $\phi_1 > 0$ a.e. in $\Omega$. We already know that $u_1 \in X_{0}^{s,p,\alpha}(\Omega)$ is a weak solution to (3.15) with $n = 1$ and $h$ is a nonincreasing function. Thus, we may choose $c > 0$ sufficiently small such that

\[ (-\Delta)^s_{p,\alpha}(c\phi_1) - f_1(h_1(c\phi_1 + 1) - g_1 < 0 \]

\[ = (-\Delta)^s_{p,\alpha}u_1 - f_1(h_1(u_1 + 1) - g_1, \text{ in } \Omega. \]

By using the comparison principle, stated in Lemma 2.6 we observe $c\phi_1 \leq u_1$ in $\Omega$. Following the arguments as in Lazer and McKenna [17], for $\gamma < 1$, we get $h(\phi_1) \in L^1(\Omega)$. This implies $h(u_1)f \leq h(c\phi_1)f \in L^1(\Omega)$. Therefore, $h(u_n)f \in L^1(\Omega)$ for each $n$ and this concludes the proof of (2).

In the proceeding lemma we find some apriori estimates so that we can pass the limit $n \to \infty$ in (3.16).

**Lemma 3.3.** Let $h$ satisfies the growth conditions given in (1.3), (1.4) for $0 < \gamma < 1$, $\theta > 0$ and let $u_n \in X_{0}^{s,p,\alpha}(\Omega)$ be a positive weak solution to (3.15). Then, the sequence $\{u_n\}$ is bounded in $L^{m_1}(\Omega, d\nu)$ for every $m_1 < \frac{N(p-1)}{N-2p}$. Furthermore,

\[ \int_Q \frac{\left| u_n(x) - u_n(y) \right|^m}{|x-y|^{N+tm}} \frac{dx dy}{|x|^\alpha |y|^\alpha} \leq C, \]

\[ (3.23) \]

for every $1 \leq m < \frac{N(p-1)}{N-s}$ and $0 < t < s$.

**Proof.** Let $u_n \in X_{0}^{s,p,\alpha}(\Omega)$ be a unique positive weak solution to (3.15). Then, for any $k \geq 1$,
using $\phi = T_k(u_n)$ as a test function in (3.16), we have
\[
\int_Q |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(T_k(u_n(x)) - T_k(u_n(y)))d\mu \\
= \int_\Omega f_n h_n(u_n + 1/n)T_k(u_n) + \int_\Omega g_n T_k(u_n) \\
\leq K_1 \int_{(u_n + \frac{1}{n} < M)} \frac{f_n T_k(u_n)}{(u_n + \frac{1}{n})} + \max\{h(s)\} \int_{(M \leq u_n + \frac{1}{n} \leq N)} f_n T_k(u_n) \\
+ K_2 \int_{(u_n + \frac{1}{n} > N)} \frac{f_n T_k(u_n)}{(u_n + \frac{1}{n})} + k \int_\Omega g_n \\
\leq K_1 M^{1-\gamma} \int_{(u_n + \frac{1}{n} < M)} f + k \max\{h(s)\} \int_{(M \leq u_n + \frac{1}{n} \leq N)} f \\
+ \frac{K_2 k}{N^\gamma} \int_{(u_n + \frac{1}{n} > N)} f + C_1 k \\
\leq C_1 k
\] (3.24)

On using Theorem 2.1, Remark 2.4 and the above inequality (3.24), we have
\[
S \left( \int_\Omega \left| \frac{T_k(u_n)(x)|^{p^*_s}}{|x|^{2\alpha p^*_s}} \right| dx \right)^{\frac{p}{p^*_s}} \leq \int_Q |T_k(u_n)(x) - T_k(u_n)(y)|^p d\mu \\
\leq \int_Q |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(T_k(u_n(x)) - T_k(u_n(y)))d\mu \\
\leq C_1 k, \ \forall k > 0.
\] (3.25)

Hence,
\[
\nu(\{x \in \Omega : |u_n(x)| \geq k\}) \leq \nu(\{x \in \Omega : |T_k(u_n(x))| = k\}) \\
\leq C_2 \int_\Omega \left| \frac{T_k(u_n)(x)|^{p^*_s}}{|x|^{2\alpha p^*_s}} \right| dx \\
\leq \frac{C}{k^\frac{N(p-1)}{N-sp}}.
\] (3.26)

Therefore, by Definition 2.5, $\{u_n\}$ is bounded in $M^\frac{N(p-1)}{N-sp}(\Omega, d\nu)$ and hence bounded in $L^{m_1}(\Omega, d\nu)$ for every $m_1 < \frac{N(p-1)}{N-sp}$. Consequently, the sequence $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{m_2}(\Omega, d\nu)$ for every $m_2 < \frac{N}{N-sp}$.

Let us choose $m < \frac{N(p-1)}{N-s}$ and $t < s$ very close to $s$ such that $\frac{mp(s-t)}{p-m} < \alpha$. Denote
\[
w_n(x) = T_1(u_n(x)) - \frac{1}{(u_n(x) + 1)^\beta}, \text{ for some } \beta > 0 \text{ which will be chosen later.}
\]
On using $w_n$ as a test function in (3.16), we establish the following.
\[
\int_{\Omega} |u_n(x) - u_n(y)|^{\beta-2}(u_n(x) - u_n(y))(T_1(u_n(x)) - T_1(u_n(y))) d\mu \\
+ \int_{\Omega} |u_n(x) - u_n(y)|^{\beta-2}(u_n(x) - u_n(y)) \left( \frac{1}{(u_n(y) + 1)^\beta} - \frac{1}{(u_n(x) + 1)^\beta} \right) \\
= \int_{\Omega} f_n h_n(u_n + 1/n) \left( T_1(u_n(x)) - \frac{1}{(u_n(x) + 1)^\beta} \right) + \int_{\Omega} g_n \left( T_1(u_n(x)) - \frac{1}{(u_n(x) + 1)^\beta} \right) \\
\leq K_1 \int_{(u_n + \frac{1}{n} < M)} \left( \frac{f_n T_1(u_n(x))}{(u_n + \frac{1}{n})^{\beta}} + \max_{[M,N]} \{h(s)\} \int_{(M \leq u_n + \frac{1}{n} \leq N)} f_n T_1(u_n(x)) + K_2 \int_{(u_n + \frac{1}{n} > N)} (u_n + \frac{1}{n})^\beta \right) + C_1 \\
\leq K_1 M^{1-\gamma} \int_{(u_n + \frac{1}{n} < M)} f + \max_{[M,N]} \{h(s)\} \int_{(M \leq u_n + \frac{1}{n} \leq N)} f + K_2 \frac{N^\theta}{N^\theta} \int_{(u_n + \frac{1}{n} > N)} f + C_1 \\
\leq C_2. \tag{3.27}
\]

On the other hand, the use of Remark 2.1 implies that
\[
\int_{\Omega} |(u_n + 1)^{\frac{\beta-1}{p}}(x) - (u_n + 1)^{\frac{\beta-1}{p}}(y)|^p d\mu \\
\leq \int_{\Omega} |u_n(x) - u_n(y)|^{\beta-2}(u_n(x) - u_n(y)) \left( \frac{1}{(u_n(y) + 1)^\beta} - \frac{1}{(u_n(x) + 1)^\beta} \right) \\
\leq \int_{\Omega} |T_1(u_n(x)) - T_1(u_n(y))|^p d\mu \\
+ \int_{\Omega} |u_n(x) - u_n(y)|^{\beta-2}(u_n(x) - u_n(y)) \left( \frac{1}{(u_n(y) + 1)^\beta} - \frac{1}{(u_n(x) + 1)^\beta} \right) \\
\leq \int_{\Omega} |u_n(x) - u_n(y)|^{\beta-2}(u_n(x) - u_n(y))(T_1(u_n(x)) - T_1(u_n(y))) d\mu \\
+ \int_{\Omega} |u_n(x) - u_n(y)|^{\beta-2}(u_n(x) - u_n(y)) \left( \frac{1}{(u_n(y) + 1)^\beta} - \frac{1}{(u_n(x) + 1)^\beta} \right). \tag{3.28}
\]

Thus, the inequalities (3.27) and (3.28) together yields
\[
\int_{\Omega} |(u_n + 1)^{\frac{\beta-1}{p}}(x) - (u_n + 1)^{\frac{\beta-1}{p}}(y)|^p d\mu \leq C. \tag{3.29}
\]

Abdellaoui et al. [Lemma 3.2, [1]] established the following inequality for certain range of $\beta > 0$.
\[
\int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^m}{|x-y|^{N+tm}} \frac{dxdy}{|x|^\alpha |y|^\alpha} \leq C_4 \left( \int_{\Omega} \frac{|(u_n + 1)^{\frac{\beta-1}{p}}(x) - (u_n + 1)^{\frac{\beta-1}{p}}(y)|^p}{(u_n(x) + 1)^\beta (u_n(y) + 1)^\beta} d\mu \right)^{\frac{m}{p}} \tag{3.30},
\]
for $m < \frac{N(p-1)}{N-s}$ and $0 < t < s$. Then, by combining (3.29) and (3.30), the following is obtained.
\[
\int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^m}{|x-y|^{N+tm}} \frac{dxdy}{|x|^\alpha |y|^\alpha} \leq C. \tag{3.31}
\]
We can always choose a $R > 0$ such that $|x - y| \sim |y|$ for $y \in \mathbb{R}^N \setminus B_R$, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Assume that $(x, y) \in \Omega \times (B_R \setminus \Omega)$, then

$$\sup_{(x,y) \in \Omega \times (B_R \setminus \Omega)} \left\{ \frac{1}{|x - y|^{N + mt}} \right\} \leq C < \infty.$$  

Since $\{u_n\}$ is bounded in $L^{m_1}(\Omega, d\nu)$ for every $m_1 < \frac{N(p - 1)}{N - sp}$, the following is obtained for $m < \frac{N(p - 1)}{N - s}$.

$$\int_{\Omega} \int_{B_R \setminus \Omega} \frac{|u_n(x) - u_n(y)|^m}{|x - y|^{N + mt}} \frac{dxdy}{|x|^\alpha |y|^\alpha} = \int_{\Omega} \int_{B_R \setminus \Omega} \frac{|u_n(x)|^m}{|x - y|^{N + mt}} \frac{dxdy}{|x|^\alpha |y|^\alpha}$$

$$\leq C \int_{\Omega} \int_{B_R \setminus \Omega} \frac{|u_n(x)|^m}{|x|^\alpha |y|^\alpha} \frac{dxdy}{|x|^\alpha |y|^\alpha}$$

$$\leq C_3 \quad (3.32)$$

and similarly

$$\int_{\Omega} \int_{\mathbb{R}^n \setminus B_R} \frac{|u_n(x) - u_n(y)|^m}{|x - y|^{N + mt}} \frac{dxdy}{|x|^\alpha |y|^\alpha} = \int_{\Omega} \int_{\mathbb{R}^n \setminus B_R} \frac{|u_n(x)|^m}{|x|^\alpha |y|^\alpha} \frac{dxdy}{|x|^\alpha |y|^\alpha}$$

$$\leq C_4 \int_{\mathbb{R}^n \setminus B_R} \frac{dy}{|y|^{N + mt + \alpha}}$$

$$\leq C_5 \quad (3.33)$$

On combining (3.32) and (3.33) we obtain

$$\int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{|u_n(x) - u_n(y)|^m}{|x - y|^{N + mt}} \frac{dxdy}{|x|^\alpha |y|^\alpha} \quad (3.34)$$

In a similar way we can prove

$$\int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^m}{|x - y|^{N + mt}} \frac{dxdy}{|x|^\alpha |y|^\alpha} \quad (3.35)$$

With the consideration of (3.31), (3.34) and (3.35), the claim given in (3.23) is proved which concludes the proof. 

We now prove our first main result, i.e. the existence of positive weak solution to (1.1) stated in Theorem 1.1.

**Proof of Theorem 1.1** According to Lemma 3.3 the sequence $\{T_k(u_n)\}$ is bounded in $X_0^{s,p,\alpha}(\Omega)$ and $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{m_2}(\Omega, d\nu)$, for every $m_2 < \frac{N}{N - sp}$. Thus, there exists a measurable function $u$ such that, up to a subsequential level, $T_k(u_n) \to T_k(u)$ weakly in $X_0^{s,p,\alpha}(\Omega)$, $|u|^{p-2}u \in L^{m_2}(\Omega, d\nu)$ and $T_k(u) \in X_0^{s,p,\alpha}(\Omega)$. Hence, $u_n \to u$ a.e. in $\Omega$ and $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

Let us fix the following notations:

$$\Phi(x, y) = \phi(x) - \phi(y), \quad J_n(x, y) = |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)),$$
\[ I(x, y) = |u(x) - u(y)|^{p-2}(u(x) - u(y)), \quad d\mu = \frac{1}{|x - y|^{N+sp}} \frac{dx\,dy}{|x|^\alpha |y|^\alpha}. \]

Then, from (3.16) we have
\[ \int_Q I_n(x, y)\Phi(x, y)d\mu = \int_\Omega f_n h_n(u_n + 1/n)\phi + \int_\Omega g_n\phi, \quad \forall \phi \in C_c^\infty(\Omega). \]

On rewriting the above equation, we obtain
\[ \int_Q I(x, y)\Phi(x, y)d\mu + \int_Q (I_n(x, y) - I(x, y))\Phi(x, y)d\mu = \int_\Omega f_n h_n(u_n + 1/n)\phi + \int_\Omega g_n\phi. \]

Since \( g_n \to g \) strongly in \( L^1(\Omega) \), we have
\[ \lim_{n \to \infty} \int_\Omega g_n\phi = \int_\Omega g\phi, \quad \forall \phi \in C_c^\infty(\Omega). \]

Now with the help of the dominated convergence theorem and (3.21), we are able to pass the limit \( n \to \infty \) in the following integral.
\[ \lim_{n \to \infty} \int_\Omega f_n h_n(u_n + 1/n)\phi = \int_\Omega h(u)f\phi. \]

We express
\[ \int_\Omega (I_n(x, y) - I(x, y))\Phi(x, y)d\mu = \int_{\Omega \times \Omega} (I_n(x, y) - I(x, y))\Phi(x, y)d\mu + \int_{(\mathbb{R}^N \setminus \Omega) \times \Omega} (I_n(x, y) - I(x, y))\Phi(x, y)d\mu + \int_{\Omega \times (\mathbb{R}^N \setminus \Omega)} (I_n(x, y) - I(x, y))\Phi(x, y)d\mu = T_{1,n} + T_{2,n} + T_{3,n}. \]

Clearly, \( I_n \to I \) a.e. in \( \mathbb{R}^N \times \mathbb{R}^N \). By the Vitali’s lemma and Lemma 3.2, \( I_n \to I \) strongly in \( L^1(\Omega \times \Omega, d\mu) \). Therefore, \( T_{1,n} \to 0 \) as \( n \to \infty \).

Consider \((x, y) \in (B_R \setminus \Omega) \times \Omega\), then
\[ \sup_{(x, y) \in (B_R \setminus \Omega) \times \Omega} \left\{ \frac{1}{|x - y|^{N+sp}} \right\} \leq C < \infty \]

and
\[ \frac{|(I_n(x, y) - I(x, y))\Phi(x, y)|}{|x - y|^{N+sp}|x|\alpha |y|\alpha} \leq C \frac{(|u_n(y)|^{p-1} + |u(y)|^{p-1})|\phi(y)|}{|x|\alpha |y|\alpha} = CJ_n(x, y). \]

Denote \( J(x, y) = \frac{2|u(y)|^{p-1}|\phi(y)|}{|x|\alpha |y|\alpha} \). Then, \( J_n \to J \) strongly in \( L^1((B_R \setminus \Omega) \times \Omega) \). Hence, by the dominated convergence theorem, \( T_{2,n} \to 0 \) as \( n \to \infty \) and similarly we can also prove that \( T_{3,n} \to 0 \) as \( n \to \infty \). Hence, on passing the limit \( n \to \infty \) in (3.36), we have
\[ \int_Q I(x, y)\Phi(x, y)d\mu = \int_\Omega h(u)f\phi + \int_\Omega g\phi, \quad \forall \phi \in C_c^\infty(\Omega). \]

(3.37)
Further, (1.10) follows from (3.25) and the Fatou’s lemma. Finally, with the consideration of Lemma 3.2 and (3.32), it is proved that \( u \) is a weak solution to (1.1) in the sense of Definition 2.8. Hence the proof.

Remark 3.4. If we assume \( g \) to be a bounded Radon measure, then the above existence result, i.e. Theorem 1.1 holds.

4 Existence of entropy solution - Proof of Theorem 1.2

The problem (1.1) admits a positive weak solution by Theorem 1.1. According to Remark 2.10, every entropy solution of (1.1) is also a weak solution to (1.1), i.e. the entropy solution satisfies (2.12). The following lemma is a consequence of Lemma 3.8 of [1], which says that every entropy solution to (1.1) satisfies (2.12) for a larger class of test function space.

Lemma 4.1 (Lemma 3.8, [1]). Suppose \( u \) is a positive entropy solution to (1.1). Then, \( u \) satisfies

\[
\int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))\psi(x) - \psi(y))d\mu = \int_\Omega fh(u)\psi + \int_\Omega g\psi,
\]

for every \( \psi \in X_0^{s,p,\alpha}(\Omega) \cap L^\infty(\Omega) \) s.t. \( \psi \equiv 0 \) in the set \( \{ u > l \} \) for some \( l > 0 \).

Let \( u_n \) be a weak solution to (3.15). Hence, the sequence \( \{ u_n \} \) is an increasing sequence w.r.t \( n \), by Lemma 3.2. Further, (3.25) implies that \( \{ T_k(u_n) \} \) is a uniformly bounded sequence in \( X_0^{s,p,\alpha}(\Omega) \) for every \( k > 0 \). Before proving the existence of an entropy solution, we provide the following compactness result.

Lemma 4.2 (Lemma 3.6, [1]). Let the sequence \( \{ u_n \} \subset X_0^{s,p,\alpha}(\Omega) \) be a positive increasing sequence w.r.t \( n \) and \( (-\Delta)^s_{p,\alpha} u_n \geq 0 \) in \( \Omega \). Furthermore, assume the sequence \( \{ T_k(u_n) \} \) is uniformly bounded in \( X_0^{s,p,\alpha}(\Omega) \) for every \( k > 0 \). Then, there exists \( u \) such that \( u_n \uparrow u \) a.e. in \( \Omega \), \( T_k(u) \in X_0^{s,p,\alpha}(\Omega) \) and

\[
T_k(u_n) \to T_k(u) \text{ strongly in } X_0^{s,p,\alpha}(\Omega).
\]

We now prove Theorem 1.2 and show the equivalence between the weak solution and the entropy solution to (1.1).

Proof of Theorem 1.2 The proof of this theorem is divided into the following two parts.

Part 1: There exists a positive entropy solution to (1.1) in the sense of Definition 2.9. By Theorem 1.1 we guarantee the existence of a weak solution \( u \) to (1.1). Then, (3.25) and Lemma 4.2 imply that \( T_k(u_n) \to T_k(u) \) strongly in \( X_0^{s,p,\alpha}(\Omega) \).

Let us take \( \phi = T_1(G_1(u_n)) \), for a fixed \( l > 0 \), in (3.16) and we get

\[
\int_Q |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(T_1(G_1(u_n))(x) - T_1(G_1(u_n))(y))d\mu
\]

\[
= \int_\Omega f_n h_n(u_n + 1/n)T_1(G_1(u_n)) + \int_\Omega g_n T_1(G_1(u_n))
\]

\[
\leq \int_{\{u_n \geq l\}} f_n h_n(l + 1/n) + \int_{\{u_n \geq l\}} g_n. \tag{4.38}
\]
Consider the set $D_l$ as given in Definition 2.39. Thus, for $(x, y) \in D_l$ it is easy to show that

$$|u_n(x) - u_n(y)|^{p-1} \leq |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(T_k(G_t(u_n))(x) - T_k(G_t(u_n))(y)).$$

(4.39)

By considering (4.38), (4.39) and using the Fatou’s lemma, we have

$$\int_{D_l} |u(x) - u(y)|^{p-1}d\mu \leq \lim_{n \to \infty} \inf \left\{ \int_{D_l} |u_n(x) - u_n(y)|^{p-1}d\mu \right\}
\leq \int_{\{u \geq l\}} fh(l) + \int_{\{u \geq l\}} g.
$$

In fact,

$$\int_{\{u \geq l\}} fh(l) + \int_{\{u \geq l\}} g \to 0 \text{ as } l \to \infty.
$$

Hence, we establish (2.13), i.e. we have

$$\int_{D_l} |u(x) - u(y)|^{p-1}d\mu \to 0 \text{ as } l \to \infty.
$$

Now it remains to prove (2.14). For this, we consider $\varphi \in X^{s,p,\alpha}_0(\Omega) \cap L^\infty(\Omega)$. Using $T_k(u_n - \varphi)$ as a test function in (3.16) and following the notations used in the proof of Theorem 1.1, we get

$$\int_Q I_n(x, y)(T_k(u_n - \varphi)(x) - T_k(u_n - \varphi)(y))d\mu = \int_{\Omega} f_n h_n(u_n + 1/n)T_k(u_n - \varphi) + \int_{\Omega} g_n T_k(u_n - \varphi).
$$

(4.40)

The integrand in the first term of the above equation (4.40) can be decomposed as

$$I_n(x, y)(T_k(u_n - \varphi)(x) - T_k(u_n - \varphi)(y)) = I_{1,n}(x, y) + I_{2,n}(x, y),
$$

(4.41)

where

$$I_{1,n}(x, y) = |(u_n(x) - \varphi(x)) - (u_n(y) - \varphi(y))|^{p-2}((u_n(x) - \varphi(x)) - (u_n(y) - \varphi(y)))
\times [T_k(u_n - \varphi)(x) - T_k(u_n - \varphi)(y)]$$

and

$$I_{2,n}(x, y) = [I_n(x, y) - |(u_n(x) - \varphi(x)) - (u_n(y) - \varphi(y))|^{p-2}((u_n(x) - \varphi(x)) - (u_n(y) - \varphi(y)))]
\times [T_k(u_n - \varphi)(x) - T_k(u_n - \varphi)(y)].
$$

Clearly, $I_{1,n}(x, y) \geq 0$ a.e. in $Q$ and

$$I_{1,n}(x, y) \to |(u(x) - \varphi(x)) - (u(y) - \varphi(y))|^{p-2}((u(x) - \varphi(x)) - (u(y) - \varphi(y)))
\times [T_k(u - \varphi)(x) - T_k(u - \varphi)(y)] \text{ a.e. in } Q.
Thus, on using the Fatou’s lemma we establish
\[
\lim_{n \to \infty} \inf \left\{ \int_Q I_{1,n}(x,y) \, d\mu \right\} \geq \int_Q \left| (u(x) - \varphi(x)) - (u(y) - \varphi(y)) \right|^{p-2} \\
(\varphi(x) - \varphi(y)) \times \left[ T_k(u - \varphi)(x) - T_k(u - \varphi)(y) \right] \, d\mu.
\]
(4.42)

According to Abdellaoui et al. [11],
\[
\lim_{n \to \infty} \int_Q I_{2,n}(x,y) \, d\mu \\
= \int_Q \left[ I(x,y) - \left| (u(x) - \varphi(x)) - (u(y) - \varphi(y)) \right|^{p-2} \right] \\
\times \left[ T_k(u - \varphi)(x) - T_k(u - \varphi)(y) \right] \, d\mu.
\]
(4.43)

On the other hand, by using the dominated convergence theorem, Lemma 3.2 and the strong convergence $T_k(u_n) \to T_k(u)$ in $X_0^{s,p,\alpha}(\Omega)$, we observe that
\[
\int_{\Omega} f_n h_n(u_n+1/n)T_k(u_n-\varphi) + \int_{\Omega} g T_k(u_n-\varphi) \to \int_{\Omega} f h(u)T_k(u-\varphi) + \int_{\Omega} g T_k(u-\varphi), \text{ as } n \to \infty.
\]
(4.44)

Therefore, on combining the results from (4.41) – (4.44) and then passing the limit $n \to \infty$ in (4.40), we conclude
\[
\int_Q I(x,y)(T_k(u - \varphi)(x) - T_k(u - \varphi)(y)) \, d\mu \leq \int_{\Omega} f h(u)T_k(u-\varphi) + \int_{\Omega} g T_k(u-\varphi),
\]
(4.45)

for every $\varphi \in X_0^{s,p,\alpha}(\Omega) \cap L^\infty(\Omega)$. Thus, (2.14) holds.

Part 2: Uniqueness of entropy solutions.

Let $u$ be the positive entropy solution obtained from Part 1 of this proof and $u_n$ be the unique solution to (3.15). Then, from Lemma 4.2 $u = \lim_{n \to \infty} \sup \{u_n\}$.

We prove this theorem by method of contradiction. For that, suppose $\bar{u}$ is another entropy solution to (1.1). Let us fix $n$ and define $\psi_n = (u_n - \bar{u})^+$. Thus, for $k \gg \|u_n\|_{L^\infty(\Omega)}$, $\psi_n = (u_n - T_k(\bar{u}))^+$. This implies $\psi_n \in X_0^{s,p,\alpha}(\Omega) \cap L^\infty(\Omega)$ and $\psi_n \equiv 0$ in the set $\{\bar{u} > \|u_n\|_{L^\infty(\Omega)}\}$. Hence, on choosing $\phi = \psi_n$ in (3.16), we have
\[
\int_Q |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_n(x) - \psi_n(y)) \, d\mu = \int_{\Omega} f_n h_n(u_n + 1/n) \psi_n + \int_{\Omega} g_n \psi_n \\
\leq \int_{\Omega} f h(u_n) \psi_n + \int_{\Omega} g \psi_n.
\]
(4.46)

Since $\bar{u}$ is an entropy solution to (1.1), by Lemma 4.1 we get
\[
\int_Q |\bar{u}(x) - \bar{u}(y)|^{p-2}(\bar{u}(x) - \bar{u}(y))(\psi_n(x) - \psi_n(y)) \, d\mu = \int_{\Omega} f h(\bar{u}) \psi_n + \int_{\Omega} g \psi_n.
\]
(4.47)
Subtracting (4.47) from (4.46) we reach that

\[ 0 \leq C \int_Q |\psi_n(x) - \psi_n(y)|^p \leq \int_Q \left( |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |\bar{u}(x) - \bar{u}(y)|^{p-2}(\bar{u}(x) - \bar{u}(y)) \right) (\psi_n(x) - \psi_n(y))d\mu \leq \int_\Omega f(h(u_n) - h(\bar{u}))\psi_n \leq 0. \]

Therefore, \( \psi_n = 0 \) a.e. in \( Q \) and \( u_n \leq \bar{u} \) a.e. for every \( n \). Use the fact \( u = \lim_{n \to \infty} \sup \{u_n\} \) to obtain

\[ u \leq \bar{u} \text{ a.e in } Q. \tag{4.48} \]

Further, choose \( h \gg k \). Then, the two entropy solutions \( u \) and \( \bar{u} \) satisfy

\[ \int_Q I(x, y)[T_k(u(x) - T_h(\bar{u}(x))) - T_k(u_n(y) - T_h(\bar{u}(y)))]d\mu \leq \int_\Omega (fh(u) + g)T_k(u - T_h(\bar{u})) \tag{4.49} \]

and

\[ \int_Q \bar{I}(x, y)[T_k(\bar{u}(x) - T_h(u(x))) - T_k(\bar{u}(y) - T_h(u(y)))]d\mu \leq \int_\Omega (f\bar{u}(\bar{u}) + g)T_k(\bar{u} - T_h(u)), \tag{4.50} \]

where

\[ I(x, y) = |\bar{u}(x) - \bar{u}(y)|^{p-2}(\bar{u}(x) - \bar{u}(y)) \text{ and } \bar{I}(x, y) = |u(x) - u(y)|^{p-2}(u(x) - u(y)). \]

Clearly, as \( h \to \infty \),

\[ \int_\Omega gT_k(u - T_h(\bar{u})) + \int_\Omega gT_k(\bar{u} - T_h(u)) \to 0 \tag{4.51} \]

and

\[ \int_\Omega fh(u)T_k(u - T_h(\bar{u})) + \int_\Omega fh(\bar{u})T_k(\bar{u} - T_h(u)) \to \int_\Omega fh(u)T_k(u - \bar{u}) + \int_\Omega fh(\bar{u})T_k(\bar{u} - u). \tag{4.52} \]

From (4.48), we already have \( u \leq \bar{u} \) a.e. in \( Q \). Thus, the right hand side of (4.52) becomes

\[ \int_\Omega fh(u)T_k(u - \bar{u}) + \int_\Omega fh(\bar{u})T_k(\bar{u} - u) = \int_\Omega \int fh(u) - h(\bar{u})T_k(u - \bar{u}) \leq 0. \tag{4.53} \]

Let us define the following two sets:

\[ D(h) = \{(x, y) \in Q : u(x) < h, u(y) < h\} \text{ and } \bar{D}(h) = \{(x, y) \in Q : \bar{u}(x) < h, \bar{u}(y) < h\}. \]
Since \( u \leq \bar{u} \), we get \( u < h \) in the set \( \{ \bar{u} < h \} \). On combining the inequalities (4.49) \(-\) (4.53), we obtain

\[
o(h) \geq \int_{Q} I(x, y)(T_k(u(x) - T_h(\bar{u}(x))) - T_k(u_n(y) - T_h(\bar{u}(y))))d\mu \\
+ \int_{Q} \bar{I}(x, y)(T_k(\bar{u}(x) - T_h(u(x))) - T_k(\bar{u}(y) - T_h(u(y))))d\mu \\
= \int_{D(h)} I(x, y)(T_k(u(x) - T_h(\bar{u}(x))) - T_k(u_n(y) - T_h(\bar{u}(y))))d\mu \\
+ \int_{Q \setminus D(h)} I(x, y)(T_k(u(x) - T_h(\bar{u}(x))) - T_k(u_n(y) - T_h(\bar{u}(y))))d\mu \\
+ \int_{D(h)} \bar{I}(x, y)(T_k(\bar{u}(x) - u(x)) - T_k(\bar{u}(y) - u(y)))d\mu \\
+ \int_{Q \setminus D(h)} \bar{I}(x, y)(T_k(\bar{u}(x) - T_h(u(x))) - T_k(\bar{u}(y) - T_h(u(y))))d\mu \\
= J_1(h) + J_2(h) + J_3(h) + J_4(h). \tag{4.54}
\]

It is easy to check that

\[
J_1(h) = \int_{D(h)} [\bar{I}(x, y) - I(x, y)](T_k(\bar{u}(x) - u(x)) - T_k(\bar{u}(y) - u(y)))d\mu \\
\geq C \int_{D(h)} |T_k(\bar{u}(x) - u(x)) - T_k(\bar{u}(y) - u(y))|^p d\mu. \tag{4.55}
\]

According to Abdellaoui et al. \[1\],

\[
J_2(h) + J_3(h) + J_4(h) \geq o(h). \tag{4.56}
\]

Hence, by the inequalities (4.51) \(-\) (4.56), we establish the following.

\[
o(h) + C \int_{D(h)} |T_k(\bar{u}(x) - u(x)) - T_k(\bar{u}(y) - u(y))|^p d\mu \leq o(h).
\]

Now on passing the limit \( h \rightarrow \infty \), we conclude that

\[
\int_{\mathbb{R}^N} |T_k(\bar{u}(x) - u(x)) - T_k(\bar{u}(y) - u(y))|^p d\mu = 0.
\]
This implies $T_k(\bar{u} - u)$ is a constant function. Since $u = \bar{u} = 0$ in $\mathbb{R}^N \setminus \Omega$, we reach the conclusion that $u = \bar{u}$ a.e. in $Q$.

Thus, from Part 1 and Part 2 of this proof we conclude that $u \in T_0^{s,p,\alpha}(\Omega)$ is a unique positive entropy solution to (1.1) in the sense of Definition 2.9 and is equivalent to the weak solution as obtained in Theorem 1.1.

\section{4.1 Problem (1.1) with a power nonlinearity.}

We consider the following problem with a singularity, power nonlinearity and an $L^1$ datum. We show that this problem possesses an entropy solution.

\[(−\Delta)^s_{p,\alpha} w = \frac{f(x)}{w^\gamma} + w^r + g(x) \text{ in } \Omega, \quad r < p - 1, \gamma < 1,\]
\[w > 0 \text{ in } \Omega,\]
\[w = 0 \text{ in } \mathbb{R}^N \setminus \Omega.\]  

\begin{equation}
(4.57)
\end{equation}

The following is the definition of entropy solution to the problem (4.57).

\textbf{Definition 4.3.} A function $w \in T_0^{s,p,\alpha}(\Omega)$ is said to be an entropy solution of (4.57) if it satisfies (2.13) and for every $\varphi \in W_0^{s,p,\alpha}(\Omega) \cap L^\infty(\Omega)$ and every $k > 0$,

\begin{equation}
\int_Q |w(x) - w(y)|^{p-2}(w(x) - w(y))[T_k(w(x) - \varphi(x)) - T_k(w(y) - \varphi(y))]d\mu \\
\leq \frac{1}{k} \int_\Omega \frac{f}{w^\gamma} T_k(w - \varphi) + \int_\Omega w^r T_k(w - \varphi) + \int_\Omega g T_k(w - \varphi). 
\end{equation}

\begin{equation}
(4.58)
\end{equation}

Further, for every $\omega \subset \subset \Omega$, there exists a $C_\omega > 0$ such that $w \geq C_\omega > 0$.

\textbf{Theorem 4.4.} Let $r < p - 1$ and $0 < \gamma < 1$. Then, the problem (4.57) admits a positive entropy solution in $T_0^{s,p,\alpha}(\Omega)$ in the sense of Definition 4.3.

\textbf{Proof.} Denote $f_n = T_n(f)$ and let $\{g_n\} \subset L^\infty(\Omega)$ be a nonnegative, increasing sequence converging strongly to $g$ in $L^1(\Omega)$. Consider the following approximating problem:

\[(−\Delta)^s_{p,\alpha} w_n = \frac{f_n(x)}{(w_n + 1/n)^\gamma} + w_n^r + g_n(x) \text{ in } \Omega,\]
\[w_n > 0 \text{ in } \Omega,\]
\[w_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega.\]  

\begin{equation}
(4.59)
\end{equation}

The corresponding energy functional is given by

\[\Psi(w_n) = \frac{1}{p} \int_Q |w_n(x) - w_n(y)|^p d\mu - \frac{1}{1 - \gamma} \int_\Omega f_n \left( (w_n^+ + 1/n)^{1-\gamma} - (1/n)^{1-\gamma} \right) \]
\[- \frac{1}{r + 1} \int_\Omega (w_n^+)^{r+1} - \int_\Omega g_n w_n^+.\]  

\begin{equation}
(4.60)
\end{equation}
The functional $\Psi$ is $C^1$, coercive and weakly lower semi-continuous in $X_0^{s,p,\alpha}(\Omega)$. Thus, there exists a critical point of $\Psi$ and hence a weak solution to the problem (4.57), denoted by $w_n \in X_0^{s,p,\alpha}(\Omega)$. The strong maximum principle [Lemma 2.3, [19]] and the comparison principle, i.e. Lemma 2.6, imply that the solution $w_n > 0$ is unique and the sequence $(w_n)$ is an increasing sequence with respect to $n$.

Clearly, by Lemma 2.6 we obtain $u_n \leq w_n$ in $\Omega$ for every $n$, where $u_n$ is the positive weak solution to (3.15) with $h(t) = t^{-\gamma}$. According to Lemma 3.2, $u_n^{-\gamma} f \in L^1(\Omega)$ and hence $w_n^{-\gamma} f \in L^1(\Omega)$.

**Claim:** the sequence $\{w_n\}$ is uniformly bounded in $L^{p-1}(\Omega)$.

We will use the method of contradiction to prove our claim. Assume that $\{w_n\}$ is an unbounded sequence in $L^{p-1}(\Omega)$. Hence, there exists a subsequence of $\{w_n\}$, still denoted as $\{w_n\}$, such that $M_n = \|w_n\|_{L^1(\Omega)} \to \infty$ as $n \to \infty$. Let us define

$$\tilde{w}_n = \frac{w_n}{M_n^{p-1}}.$$ 

This implies $\|\tilde{w}_n\|_{L^{p-1}(\Omega)} = 1$. Since $w_n$ is a solution to (4.57), $\tilde{w}_n$ satisfies

$$( -\Delta )^{s}_{p,\alpha} \tilde{w}_n = \frac{f_n(x)}{M_n(w_n + 1/n)^\gamma} + M_n^{r-1/p} \tilde{w}_n^r + M_n^{-1} g_n(x) \in \Omega,$$

$$\tilde{w}_n > 0 \text{ in } \Omega,$$

$$\tilde{w}_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

Denote $H_n = \frac{f_n}{M_n(w_n + 1/n)^\gamma} + M_n^{r-1/p} \tilde{w}_n^r + M_n^{-1} g_n$. Then, it is simple to prove that $\|H_n\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. Following the proof of Lemma 3.3, we establish that $\{\tilde{w}_n\}$ is bounded in $L^{m_1}(\Omega, d\nu)$ for every $m_1 < \frac{N(p-1)}{N-sp}$ and also bounded in $X_0^{t,m,\alpha}(\Omega)$ for every $1 \leq m < \frac{N(p-1)}{N-s}$, $0 < t < s$. Thus, there exists $\tilde{w} \in X_0^{t,m,\alpha}(\Omega) \cap L^{m_1}(\Omega, d\nu)$ such that $T_k(\tilde{w}) \in X_0^{s,p,\alpha}(\Omega)$ and $T_k(\tilde{w}_n) \to T_k(\tilde{w})$ weakly in $X_0^{s,p,\alpha}(\Omega)$. We now use the Vitali’s lemma to prove $\tilde{w}_n^{p-1} \to \tilde{w}^{p-1}$ strongly in $L^1(\Omega)$ and we obtain $\|\tilde{w}\|_{L^{p-1}(\Omega)} = 1$.

On readopting the proof of Lemma 3.3 with the test function $T_k(\tilde{w}_n)$ we obtain

$$\|T_k(\tilde{w}_n)\|_{X_0^{t,m,\alpha}(\Omega)} \to 0,$$ as $n \to \infty$.

Thus, $T_k(\tilde{w}) = 0$ for every $k$ and this contradicts to $\|\tilde{w}\|_{L^{p-1}(\Omega)} = 1$. This proves the claim and $\{w_n\}$ is uniformly bounded in $L^r(\Omega)$.

We can now guarantee the existence of a weak solution $w \in X_0^{t,m,\alpha}(\Omega)$ to the problem (4.57) for every $1 \leq m < \frac{N(p-1)}{N-s}$, $0 < t < s$. Further, $T_k(w_n) \to T_k(w)$ weakly in $X_0^{s,p,\alpha}(\Omega)$. The proof follows verbatim of the proofs in Lemma 3.3 and Theorem 1.1.

We have already shown that $\{w_n\}$ is an increasing sequence. Hence, by Lemma 4.2, $T_k(w_n)$ converges strongly to $T_k(w)$ in $X_0^{s,p,\alpha}(\Omega)$. Proceeding on the similar lines used in the proof of Theorem 1.2 we get the existence of an entropy solution $w$ to (4.57) in the sense of Definition 4.3. Hence the proof.
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