BOSONISATION AND PARASTATISTICS: AN EXAMPLE AND AN ALTERNATIVE APPROACH

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ABSTRACT. Definitions of the parastatistics algebras and known results on their Lie (super)algebraic structure are reviewed. The notion of super-Hopf algebra is discussed. The bosonisation technique for switching a Hopf algebra in a braided category \(H_M\) (\(H\): a quasitriangular Hopf algebra) into an ordinary Hopf algebra is presented and it is applied in the case of the parabosonic algebra. A bosonisation-like construction is also introduced for the same algebra and the differences are discussed.

1. INTRODUCTION AND DEFINITIONS

Throughout this paper we are going to use the following notation conventions:
If \(x\) and \(y\) are any monomials of the tensor algebra of some \(k\)-vector space, we are going to call commutator the following expression:

\[ [x, y] = x \otimes y - y \otimes x \equiv xy - yx \]

and anticommutator the following expression:

\[ \{x, y\} = x \otimes y + y \otimes x \equiv xy + yx \]

By the field \(k\) we shall always mean \(\mathbb{C}\), and all tensor products will be considered over \(k\) unless stated so. Finally we freely use Sweedler’s notation for the comultiplication throughout the paper.

Parafermionic and parabosonic algebras first appeared in the physics literature by means of generators and relations, in the pioneering works of Green [6] and Greenberg and Messiah [5]. Their purpose was to introduce generalizations of the usual bosonic and fermionic algebras of quantum mechanics, capable of leading to generalized versions of the Bose-Einstein and Fermi-Dirac statistics (see: [15]). We start with the definitions of these algebras:

Let us consider the \(k\)-vector space \(V_B\) freely generated by the elements: \(b^i_+, b^-_j, \ i, j = 1, ..., n\). Let \(T(V_B)\) denote the tensor algebra of \(V_B\) (i.e.: the free algebra generated by the elements of the basis). In \(T(V_B)\) we consider the two-sided ideal \(I_B\) generated by the following
elements:
\[
\{b_\xi^i, b_\eta^j, b_\epsilon^k\} - (\epsilon - \eta)\delta_{jk}b_\xi^i = - (\epsilon - \xi)\delta_{ik}b_\eta^j \tag{1}
\]
for all values of \(\xi, \eta, \epsilon = \pm 1\) and \(i, j, k = 1, \ldots, n\).

We now have the following:

**Definition 1.1.** The parabosonic algebra in 2\(n\) generators \(P_B^{(n)}\) (\(n\) parabosons) is the quotient algebra of the tensor algebra of \(V_B\) with the ideal \(I_B\):

\[
P_B^{(n)} = T(V_B)/I_B
\]

In a similar way we may describe the parafermionic algebra in 2\(n\) generators (\(n\) parafermions): Let us consider the \(k\)-vector space \(V_F\) freely generated by the elements: \(f_\xi^+, f_\eta^-, i, j = 1, \ldots, n\). Let \(T(V_F)\) denote the tensor algebra of \(V_F\) (i.e.: the free algebra generated by the elements of the basis). In \(T(V_F)\) we consider the two-sided ideal \(I_F\) generated by the following elements:

\[
\{[f_\xi^i, f_\eta^j, f_\epsilon^k] - \frac{1}{2}(\epsilon - \eta)^2\delta_{jk}f_\xi^i + \frac{1}{2}(\epsilon - \xi)^2\delta_{ik}f_\eta^j \}
\]

for all values of \(\xi, \eta, \epsilon = \pm 1\) and \(i, j, k = 1, \ldots, n\).

We get the following definition:

**Definition 1.2.** The parafermionic algebra in 2\(n\) generators \(P_F^{(n)}\) (\(n\) parafermions) is the quotient algebra of the tensor algebra of \(V_F\) with the ideal \(I_F\):

\[
P_F^{(n)} = T(V_F)/I_F
\]

2. (Super-)Lie and (Super-)Hopf Algebraic Structure of \(P_B^{(n)}\) and \(P_F^{(n)}\)

Due to its simpler nature, parafermionic algebras were the first to be identified as the universal enveloping algebras (UEA) of simple Lie algebras. This was done almost at the same time by S.Kamefuchi, Y.Takahashi in [9] and by C. Ryan, E.C.G. Sudarshan in [21]. In fact the following stem from the above mentioned references (see also [16]):

**Lemma 2.1.** In the \(k\)-vector space \(P_F^{(n)}\) we consider the \(k\)-subspace generated by the set of elements:

\[
\{([f_\xi^i, f_\eta^j], f_k^\epsilon) | \xi, \eta, \epsilon = \pm, i, j, k = 1, \ldots, n\}
\]

The above subspace endowed with a bilinear multiplication \(\{\ldots\}\) whose values are determined by the values of the commutator in \(P_F^{(n)}\), i.e:

\[
\{f_\xi^i, f_\eta^j\} = [f_\xi^i, f_\eta^j]
\]
and:

$$\langle [f^\xi_i, f^\eta_j], f^\epsilon_k \rangle = [[f^\xi_i, f^\eta_j], f^\epsilon_k] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{jk} f^\xi_i - \frac{1}{2}(\epsilon - \xi)^2 \delta_{ik} f^\eta_j$$

is a simple complex Lie algebra isomorphic to $B_n = \text{so}(2n+1)$. The basis in the Cartan subalgebra of $B_n$ can be chosen in such a way that the elements $f^+$ (respectively: $f^-$) are negative (respectively: positive) root vectors.

Based on the above observations, the following is finally proved:

**Proposition 2.2.** The parafermionic algebra in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $B_n = \text{so}(2n+1)$ (according to the well known classification of the simple complex Lie algebras), i.e:

$$P_F^{(n)} \cong U(B_n)$$

An immediate consequence of the above identification is that parafermionic algebras are ordinary Hopf algebras, with the generators $f^\pm_i, i = 1, \ldots, n$ being primitive elements. The Hopf algebraic structure of $P_F^{(n)}$ is completely determined by the well known Hopf algebraic structure of the Lie algebras, due to the above isomorphism. For convenience we quote the relations explicitly:

$$\Delta(f^\pm_i) = f^\pm_i \otimes 1 + 1 \otimes f^\pm_i$$

$$\varepsilon(f^\pm_i) = 0$$

$$S(f^\pm_i) = -f^\pm_i$$

The algebraic structure of parabosons seemed to be somewhat more complicated. The presence of anticommutators among the trilinear relations defining $P_B^{(n)}$ “breaks” the usual (Lie) antisymmetry and makes impossible the identification of the parabosons with the UEA of any Lie algebra. It was in the early ’80’s that was conjectured [15], that due to the mixing of commutators and anticommutators in $P_B^{(n)}$ the proper mathematical “playground” should be some kind of Lie superalgebra (or: $\mathbb{Z}_2$-graded Lie algebra). Starting in the early ’80’s, and using the recent (by that time) results in the classification of the finite dimensional simple complex Lie superalgebras which was obtained by Kac (see: [7, 8]), T.D.Palev managed to identify the parabosonic algebra with the UEA of a certain simple complex Lie superalgebra. In [18, 19] (see also [17]), T.D.Palev shows the following:

**Lemma 2.3.** In the $k$-vector space $P_B^{(n)}$ we consider the $k$-subspace generated by the set of elements:

$$\left\{ b_i^\pm b_j^\eta b_k^\xi | \xi, \eta, \epsilon = \pm, i, j, k = 1, \ldots, n \right\}$$
This vector space is turned into a superspace (\(\mathbb{Z}_2\)-graded vector space) by the requirement that \(b_1^\xi\) span the odd subspace and \(\{b_1^\xi, b_2^\eta\}\) span the even subspace.

The above vector space endowed with a bilinear multiplication \(\langle...,\rangle\) whose values are determined by the values of the anticommutator and the commutator in \(P_B^{(n)}\), i.e.:

\[
\langle b_1^\xi, b_2^\eta \rangle = \{b_1^\xi, b_2^\eta\}
\]

and:

\[
\{\{b_1^\xi, b_2^\eta\}, b_3^\phi\} = \{b_1^\xi, \{b_2^\eta, b_3^\phi\}\} = (\epsilon - \eta)\delta_{jk}b_1^\xi + (\epsilon - \xi)\delta_{ik}b_2^\eta
\]

respectively, according to the above mentioned gradation, is a simple, complex super-Lie algebra (or: \(\mathbb{Z}_2\)-graded Lie algebra) isomorphic to \(B(0, n) = osp(1, 2n)\). The basis in the Cartan subalgebra of \(B(0, n)\) can be chosen in such a way that the elements \(b^+\) (respectively: \(b^-\)) are negative (respectively: positive) root vectors.

Note that, according to the above lemma, the even part of \(B(0, n)\) is spanned by the elements \(\{\{b_1^\xi, b_2^\eta\} | i, j = 1, \ldots, n\}\) and is a subalgebra of \(B(0, n)\) isomorphic to the Lie algebra \(sp(2n)\). It’s Lie multiplication can be readily deduced from the above given commutators and reads:

\[
\langle \{b_1^\xi, b_2^\eta\}, \{b_3^\phi, b_4^\psi\} \rangle = \{\{b_1^\xi, b_2^\eta\}, \{b_3^\phi, b_4^\psi\}\} =
\]

\[
(\epsilon - \eta)\delta_{jk}\{b_1^\xi, b_2^\eta\} + (\epsilon - \xi)\delta_{ik}\{b_2^\eta, b_3^\phi\} + (\phi - \eta)\delta_{jl}\{b_3^\phi, b_4^\psi\} + (\phi - \xi)\delta_{il}\{b_4^\psi, b_1^\xi\}
\]

On the other hand the elements \(\{b_k^\epsilon | \epsilon = \pm, k = 1, \ldots, n\}\) constitute a basis of the odd part of \(B(0, n)\).

Note also, that \(B(0, n)\) in Kac’s notation, is the classical simple complex orthosymplectic Lie superalgebra denoted \(osp(1, 2n)\) in the notation traditionally used by physicists until then.

Based on the above observations, Palev finally proves (in the above mentioned references):

**Proposition 2.4.** The parabosonic algebra in \(2n\) generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra \(B(0, n)\) (according to the classification of the simple complex Lie superalgebras given by Kac), i.e.:

\[
P_B^{(n)} \cong U(B(0, n))
\]

The universal enveloping algebra \(U(L)\) of a Lie superalgebra \(L\) is not a Hopf algebra, at least in the ordinary sense. \(U(L)\) is a \(\mathbb{Z}_2\)-graded associative algebra (or: superalgebra) and it is a super-Hopf algebra in
a sense that we briefly describe: First we consider the braided tensor product algebra $U(L) \otimes U(L)$, which means the vector space $U(L) \otimes U(L)$ equipped with the associative multiplication:
\[(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} ac \otimes bd\]
for $b, c$ homogeneous elements of $U(L)$, and $|.|$ denotes the degree of an homogeneous element (i.e.: $|b| = 0$ if $b$ is an even element and $|b| = 1$ if $b$ is an odd element). Note that $U(L) \otimes U(L)$ is also a superalgebra or: $\mathbb{Z}_2$-graded associative algebra. Then $U(L)$ is equipped with a coproduct
\[\Delta : U(L) \rightarrow U(L) \otimes U(L)\]
which is an superalgebra homomorphism from $U(L)$ to the braided tensor product algebra $U(L) \otimes U(L)$ :
\[\Delta(ab) = \sum (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2 = \Delta(a) \cdot \Delta(b)\]
for any $a, b$ in $U(L)$, with $\Delta(a) = \sum a_1 \otimes a_2$, $\Delta(b) = \sum b_1 \otimes b_2$, and $a_2, b_1$ homogeneous. $\Delta$ is uniquely determined by it’s value on the generators of $U(L)$ (i.e.: the basis elements of $L$):
\[\Delta(x) = 1 \otimes x + x \otimes 1\]
Similarly, $U(L)$ is equipped with an antipode $S : U(L) \rightarrow U(L)$ which is not an algebra anti-homomorphism (as in ordinary Hopf algebras) but a braided algebra anti-homomorphism (or: “twisted” anti-homomorphism) in the following sense:
\[S(ab) = (-1)^{|a||b|} S(b) S(a)\]
for any homogeneous $a, b \in U(L)$.

All the above description is equivalent to saying that $U(L)$ is a Hopf algebra in the braided category of $\mathbb{C}\mathbb{Z}_2$-modules $\mathbb{C}\mathbb{Z}_2\mathcal{M}$ or: a braided group where the braiding is induced by the non-trivial quasitriangular structure of the $\mathbb{C}\mathbb{Z}_2$ Hopf algebra i.e. by the non-trivial $R$-matrix:
\[R_g = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)\]  
(4)
where $1, g$ are the elements of the $\mathbb{Z}_2$ group which is now written multiplicatively.

We recall here (see [13]) that if $(H, R_H)$ is a quasitriangular Hopf algebra, then the category of modules $H\mathcal{M}$ is a braided monoidal category, where the braiding is given by a natural family of isomorphisms $\Psi_{V,W} : V \otimes W \cong W \otimes V$, given explicitly by:
\[\Psi_{V,W}(v \otimes w) = \sum (R^2_H \triangleright w) \otimes (R^1_H \triangleright v)\]  
(5)
for any $V, W \in \text{obj}(\mathcal{HM})$.

Combining eq. (4) and (5) we immediately get the braiding in the $\text{CZ}_2\mathcal{M}$ category:

$$\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$$  

This is obviously a symmetric braiding, so we actually have a symmetric monoidal category $\text{CZ}_2\mathcal{M}$, rather than a truly braided one.

In view of the above description, an immediate consequence of proposition 2.4, is that the parabosonic algebras $P^{(n)}_B$ are super-Hopf algebras, with the generators $b^\pm_i$, $i = 1, ..., n$ being primitive elements. It’s super-Hopf algebraic structure is completely determined by the super-Hopf algebraic structure of Lie superalgebras, due to the above mentioned isomorphism. Namely the following relations determine completely the super-Hopf algebraic structure of $P^{(n)}_B$:

$$\Delta(b^\pm_i) = 1 \otimes b^\pm_i + b^\pm_i \otimes 1 \quad \varepsilon(b^\pm_i) = 0 \quad S(b^\pm_i) = -b^\pm_i$$  

### 3. Bosonisation as a Technique of Reducing Supersymmetry

A general scheme for “transforming” a Hopf algebra $B$ in the braided category $\mathcal{HM}$ ($H$: some quasitriangular Hopf algebra) into an ordinary one, namely the smash product Hopf algebra: $B \star H$, such that the two algebras have equivalent module categories, has been developed during ’90 ’s. The original reference is [10] (see also [11, 12]). The technique is called bosonisation, the term coming from physics. This technique uses ideas developed in [20], [13]. It is also presented and applied in [3], [4], [1]. We review the main points of the above method:

In general, $B$ being a Hopf algebra in a category, means that its structure maps are morphisms in the category. In particular, if $H$ is some quasitriangular Hopf algebra, $B$ being a Hopf algebra in the braided category $\mathcal{HM}$, means that $B$ is an algebra in $\mathcal{HM}$ (or: $H$-module algebra) and a coalgebra in $\mathcal{HM}$ (or: $H$-module coalgebra) and at the same time $\Delta_B$ and $\varepsilon_B$ are algebra morphisms in the category $\mathcal{HM}$. (For more details on the above definitions one may consult for example [14]).

Since $B$ is an $H$-module algebra we can form the cross product algebra $B \rtimes H$ (also called: smash product algebra) which as a k-vector space is $B \otimes H$ (i.e. we write: $b \rtimes h \equiv b \otimes h$ for every $b \in B$, $h \in H$), with multiplication given by:

$$(b \otimes h)(c \otimes g) = \sum b(h_1 \triangleright c) \otimes h_2 g$$  

(8)
∀ b, c ∈ B and h, g ∈ H, and the usual tensor product unit.
On the other hand B is a (left) H-module coalgebra with H: quasi-
triangular through the R-matrix: \( R_H = \sum R_H^{(1)} \otimes R_H^{(2)} \). Quasitriangularity “switches” the (left) action of H on B into a (left) coaction \( \rho : B \rightarrow H \otimes B \) through:

\[
\rho(b) = \sum R_H^{(2)} \otimes (R_H^{(1)} \triangleright b)
\]

and B endowed with this coaction becomes (see [11, 12]) a (left) H-
comodule coalgebra or equivalently a coalgebra in \( H \mathcal{M} \) (meaning that \( \Delta_B \) and \( \varepsilon_B \) are (left) H-comodule morphisms, see [14]).

We recall here (see: [11, 12]) that when H is a Hopf algebra and B
is a (left) H-comodule coalgebra with the (left) H-coaction given by:
\( \rho(b) = \sum b^{(i)} \otimes b^{(0)} \), one may form the cross coproduct coalgebra \( B \rtimes H \), which as a k-vector space is \( B \otimes H \) (i.e. we write: \( b \rtimes h \equiv b \otimes h \) for every \( b \in B, h \in H \)), with comultiplication given by:

\[
\Delta(b \otimes h) = \sum b_1 \otimes b_2^{(1)} h_1 \otimes b_2^{(0)} \otimes h_2
\]

and counit: \( \varepsilon(b \otimes h) = \varepsilon_B(b) \varepsilon_H(h) \). (In the above we use in the elements of B upper indices included in parenthesis to denote the components of the coaction according to the Sweedler notation, with the convention that \( b^{(i)} \in H \) for \( i \neq 0 \)).

Now we proceed by applying the above described construction of the
cross coproduct coalgebra \( B \rtimes H \), with the special form of the (left) coaction given by eq. (9). Replacing thus eq. (9) into eq. (10) we get for the special case of the quasitriangular Hopf algebra \( H \) the cross coproduct comultiplication:

\[
\Delta(b \otimes h) = \sum b_1 \otimes R_H^{(2)} h_1 \otimes (R_H^{(1)} \triangleright b_2) \otimes h_2
\]

Finally we can show that the cross product algebra (with multiplication
given by (8)) and the cross coproduct coalgebra (with comultiplication
given by (11)) fit together and form a bialgebra (see: [11, 12, 13, 20]).

This bialgebra, furnished with an antipode:

\[
S(b \otimes h) = (S_H(h_2)) u(R^{(1)} \triangleright S_B(b)) \otimes S(R^{(2)} h_1)
\]

where \( u = \sum S_H(R^{(2)}) R^{(1)} \), and \( S_B \) the (braided) antipode of B, be-
comes (see [11]) an ordinary Hopf algebra. This is the smash product
Hopf algebra denoted \( B \ast H \). In [10] it is further proved that the cate-
gory of the braided modules of B (B-modules in \( H \mathcal{M} \)) is equivalent to the category of the (ordinary) modules of \( B \ast H \).
3.1. **An example of Bosonisation.** In the special case that $B$ is some super-Hopf algebra, then: $H = \mathbb{C}Z_2$, equipped with it’s non-trivial quasitriangular structure, formerly mentioned. In this case, the technique simplifies and the ordinary Hopf algebra produced is the smash product Hopf algebra $B \star \mathbb{C}Z_2$. The grading in $B$ is induced by the $\mathbb{C}Z_2$-action on $B$:

$$g \triangleright b = (-1)^{|b|}b$$

for $b$ homogeneous in $B$. Utilizing the non-trivial $R$-matrix $R_g$ and using eq. (13) and eq. (14) we can readily deduce the form of the induced $\mathbb{C}Z_2$-coaction on $B$:

$$\rho(b) = \begin{cases} 1 \otimes b, & b: \text{even} \\ g \otimes b, & b: \text{odd} \end{cases}$$

The above mentioned action and coaction enable us to form the cross product algebra and the cross coproduct coalgebra according to the preceding discussion which finally form the smash product Hopf algebra $B \star \mathbb{C}Z_2$. The grading of $B$, is “absorbed” in $B \star \mathbb{C}Z_2$, and becomes an inner automorphism:

$$gbg = (-1)^{|b|}b$$

where we have identified: $b \star 1 \equiv b$ and $1 \star g \equiv g$ in $B \star \mathbb{C}Z_2$ and $b$ homogeneous element in $B$. This inner automorphism is exactly the adjoint action of $g$ on $B \star \mathbb{C}Z_2$ (as an ordinary Hopf algebra).

The following proposition is proved -as an example of the bosonisation technique- in [11]:

**Proposition 3.1.** Corresponding to every super-Hopf algebra $B$ there is an ordinary Hopf algebra $B \star \mathbb{C}Z_2$, its bosonisation, consisting of $B$ extended by adjoining an element $g$ with relations, coproduct, counit and antipode:

$$
\begin{align*}
g^2 &= 1 \\
gb &= (-1)^{|b|}bg \\
\Delta(g) &= g \otimes g \\
\Delta(b) &= \sum b_1 g^{[b_2]} \otimes b_2 \\
S(g) &= g \\
S(b) &= g^{-[b]}S(b) \\
\varepsilon(g) &= 1 \\
\varepsilon(b) &= \varepsilon(b)
\end{align*}
$$

where $S$ and $\varepsilon$ denote the original maps of the super-Hopf algebra $B$. Moreover, the representations of the bosonised Hopf algebra $B \star \mathbb{C}Z_2$ are precisely the super-representations of the original superalgebra $B$.

The application of the above proposition in the case of the para-bosonic algebra $P_{B}^{(n)} \cong U(B(0,n))$ is straightforward: we immediately get it’s bosonised form $P_{B(g)}^{(n)}$ which by definition is:

$$P_{B(g)}^{(n)} \equiv P_{B}^{(n)} \star \mathbb{C}Z_2 \cong U(B(0,n)) \star \mathbb{C}Z_2$$
Utilizing equations (7) which describe the super-Hopf algebraic structure of the parabosonic algebra $P_B^{(n)}$, and replacing them into equations (15) which describe the ordinary Hopf algebra structure of the bosonised superalgebra, we immediately get the explicit form of the (ordinary) Hopf algebra structure of $P_B^{(n)} \cong P_B^{(n)} \ast \mathbb{C}Z_2$ which reads:

$$
\Delta(b_i^\pm) = b_i^\pm \otimes 1 + g \otimes b_i^\pm \\
\Delta(g) = g \otimes g \\
\varepsilon(b_i^\pm) = 0 \\
\varepsilon(g) = 1 \\
S(b_i^\pm) = b_i^\pm g = -g b_i^\pm \\
S(g) = g \\
g^2 = 1 \\
\{g, b_i^\pm\} = 0 
$$

(16)

where we have again identified $b_i^\pm \ast 1 \equiv b_i^\pm$ and $1 \ast g \equiv g$ in $P_B^{(n)} \ast \mathbb{C}Z_2$.

3.2. **An alternative approach.** Let us describe now a slightly different construction (see: [2]), which achieves the same object: the determination of an ordinary Hopf structure for the parabosonic algebra $P_B^{(n)}$.

Defining:

$$
N_{lm} = \frac{1}{2} \{ b_l^+, b_m^- \}
$$

we notice that these are the generators of the Lie algebra $u(n)$:

$$
[N_{kl}, N_{mn}] = \delta_{lm} N_{kn} - \delta_{kn} N_{ml}
$$

We introduce now the elements:

$$
\mathcal{N} = \sum_{i=1}^{n} N_{ii} = \frac{1}{2} \sum_{i=1}^{n} \{ b_i^+, b_i^- \}
$$

which are exactly the linear Casimirs of $u(n)$.

We can easily find that they satisfy:

$$
[N, b_i^\pm] = \pm b_i^\pm
$$

Based on the above we inductively prove:

$$
[N^m, b_i^+] = b_i^+ ((\mathcal{N} + 1)^m - \mathcal{N}^m)
$$

(17)

We now introduce the following elements:

$$
K^+ = \exp(i \pi \mathcal{N}) \equiv \sum_{m=0}^{\infty} \frac{(i \pi \mathcal{N})^m}{m!}
$$
and:
\[ K^- = \exp(-i\pi N) \equiv \sum_{m=0}^{\infty} \frac{(-i\pi N)^m}{m!} \]

Utilizing the above power series expressions and equation (17) we get
\[ \{K^+, b_i^\pm\} = 0 \quad \{K^-, b_i^\pm\} = 0 \quad (18) \]

A direct application of the Baker-Campbell-Hausdorff formula leads also to:
\[ K^+ K^- = K^- K^+ = 1 \quad (19) \]

We finally have the following proposition:

**Proposition 3.2.** Corresponding to the super-Hopf algebra \( P_B^{(n)} \) there is an ordinary Hopf algebra \( P_B^{(n)}(K^\pm) \), consisting of \( P_B^{(n)} \) extended by adjoining two elements \( K^+, K^- \) with relations, coproduct, counit and antipode:

\[ \Delta(b_i^\pm) = b_i^\pm \otimes 1 + K^\pm \otimes b_i^\pm \]
\[ \Delta(K^\pm) = K^\pm \otimes K^\pm \]
\[ \varepsilon(b_i^\pm) = 0 \quad \varepsilon(K^\pm) = 1 \]
\[ S(b_i^\pm) = b_i^\pm K^\mp \quad S(K^\pm) = K^\mp \]
\[ K^+ K^- = K^- K^+ = 1 \quad \{K^+, b_i^\pm\} = 0 = \{K^-, b_i^\pm\} \quad (20) \]

**Proof.** Consider the k-vector space \( k\langle b_i^+, b_j^-, K^\pm \rangle \) freely generated by the elements \( b_i^+, b_j^-, K^+, K^- \). Denote \( T(b_i^+, b_j^-, K^\pm) \) its tensor algebra. In the tensor algebra we denote \( I_{BK} \) the ideal generated by all the elements of the forms (11), (18), (19). We define:

\[ P_B^{(n)}(K^\pm) = T(b_i^+, b_j^-, K^\pm)/I_{BK} \]

Consider the k-linear map
\[ \Delta : k\langle b_i^+, b_j^-, K^\pm \rangle \rightarrow P_B^{(n)}(K^\pm) \otimes P_B^{(n)}(K^\pm) \]
determined by its values on the basis elements, specified in equation (20). By the universality property of the tensor algebra this map extends to an algebra homomorphism:

\[ \Delta : T(b_i^+, b_j^-, K^\pm) \rightarrow P_B^{(n)}(K^\pm) \otimes P_B^{(n)}(K^\pm) \]

Now we can trivially verify that:
\[ \Delta(\{K^\pm, b_i^\pm\}) = \Delta(K^+ K^- - 1) = \Delta(K^- K^+ - 1) = 0 \quad (21) \]
Considering the usual tensor product algebra $P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$ with multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ for any $a, b, c, d \in P_{B(K^\pm)}^{(n)}$ we also compute:

$$\Delta([b_i^j, b_j^k]) = (\epsilon - \eta) \delta_{jk} b_i^k - (\epsilon - \xi) \delta_{ik} b_j^k = 0$$  \hspace{0.5cm} (22)

Relations (21), and (22), mean that $I_{B K} \subseteq ker \Delta$ which in turn implies that $\Delta$ is uniquely extended as an algebra homomorphism from $P_{B(K^\pm)}^{(n)}$ to the usual tensor product algebra $P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$ according to the diagram:

\[ T(b_i^+, b_j^-, K^\pm) \xrightarrow{\Delta} P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)} \]

\[ \xrightarrow{\pi} P_{B(K^\pm)}^{(n)} \]

Following the same procedure we construct an algebra homomorphism $\varepsilon : P_{B(K^\pm)}^{(n)} \rightarrow P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$ and an algebra antihomomorphism $S : P_{B(K^\pm)}^{(n)} \rightarrow P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$ which are completely determined by their values on the generators of $P_{B(K^\pm)}^{(n)}$ (i.e.: the basis elements of $k\langle b_i^+, b_j^-, K^\pm \rangle$). Note that in the case of the antipode we start by defining a linear map $S$ from $k\langle b_i^+, b_j^-, K^\pm \rangle$ to the opposite algebra $(P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)})^{op}$, with values determined by equation (20) and following the above described procedure we end up with an algebra antihomomorphism $S : P_{B(K^\pm)}^{(n)} \rightarrow P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$.

Now it is sufficient to verify the rest of the Hopf algebra axioms (i.e.: coassociativity of $\Delta$, counity property for $\varepsilon$, and the compatibility condition which ensures us that $S$ is an antipode) on the generators of $P_{B(K^\pm)}^{(n)}$. This can be done with straightforward computations (see [2]).

The above constructed algebra $P_{B(K^\pm)}^{(n)}$, is an ordinary Hopf algebra in the sense that the comultiplication is extended to the whole of $P_{B(K^\pm)}^{(n)}$ as an algebra homomorphism:

$$\Delta : P_{B(K^\pm)}^{(n)} \rightarrow P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$$

where $P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$ is considered as the tensor product algebra with the usual product:

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$
for any \(a, b, c, d \in P_{B(K^\pm)}^{(n)}\) and the antipode extends as usual as an algebra anti-homomorphism.

4. Discussion

It is interesting to see the relation between the above constructed Hopf algebras \(P_{B(g)}^{(n)}\) and \(P_{B(K^\pm)}^{(n)}\.

From the point of view of the structure, an obvious question arises: While \(P_{B(g)}^{(n)}\) is a quasitriangular Hopf algebra through the \(R\)-matrix: \(R_g\) given in eq. (4), there is yet no suitable \(R\)-matrix for the Hopf algebra \(P_{B(K^\pm)}^{(n)}\). Thus the question of the quasitriangular structure of \(P_{B(K^\pm)}^{(n)}\) is open.

Another interesting point, concerns the representations of \(P_{B(K^\pm)}^{(n)}\) versus the representations of \(P_{B(g)}^{(n)}\). The difference in the comultiplication between the above mentioned Hopf algebras, leads us to the question of whether the tensor product of representations of \(P_{B(g)}^{(n)}\) behave differently from the tensor product of representations of \(P_{B(K^\pm)}^{(n)}\).

Finally another open problem which arises from the above mentioned approach, is whether the above construction of \(P_{B(K^\pm)}^{(n)}\) can be extended for the universal enveloping algebra of an arbitrary Lie superalgebra, using power series of suitably chosen Casimirs.

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