A variational approach for the quantum inverse scattering method

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Abstract
We introduce a variational approach for the quantum inverse scattering method to exactly solve a class of Hamiltonians via Bethe ansatz methods. We undertake this in a manner which does not rely on any prior knowledge of integrability through the existence of a set of conserved operators. The procedure is conducted in the framework of Hamiltonians describing the crossover between the low-temperature phenomena of superconductivity, in the Bardeen–Cooper–Schrieffer theory, and Bose–Einstein condensation. The Hamiltonians considered describe systems with interacting Cooper pairs and a bosonic degree of freedom. We obtain general exact solvability requirements which include seven subcases that have previously appeared in the literature.

1. Introduction

The quantum inverse scattering method (QISM) [1–3] is used for constructing quantum Hamiltonians with multiple conserved operators, and in turn obtaining their exact solutions by Bethe ansatz methods. A solution of the Yang–Baxter equation [4–6] may be used to construct a transfer matrix which generates the conserved operators of the Hamiltonian. The Bethe ansatz used to obtain the exact solution can assume many forms. The original approach due to Bethe [7] is commonly termed the co-ordinate Bethe ansatz, whereas a more modern approach in the framework of the QISM (subject to a suitable reference state) is the algebraic Bethe ansatz [1–3]. However there are several variants between these two formats, which rely on functional relations and assumed analyticity properties of the eigenvalue spectrum to obtain the exact solution [8–12]. We note that in principle the implementation of the co-ordinate Bethe ansatz is neither dependent on any prior knowledge of an associated solution of the Yang–Baxter equation nor on the conserved operators of the Hamiltonian that it generates.

Progress in cold atom physics has yielded many studies concerning the nature of the crossover between the low-temperature phenomena of superconductivity, from the Bardeen–Cooper–Schrieffer (BCS) theory, and Bose–Einstein condensation (BEC) [13–16]. Early theoretical accounts emphasized the need to study Hamiltonians which explicitly incorporate coupling between Cooper pairs of atoms and bosonic molecular modes [17, 18]. The goal
of the work presented here is to implement a strategy motivated by both the co-ordinate and algebraic Bethe ansatz approaches to obtain a general class of exactly solvable Hamiltonians, with both a bosonic and Cooper pairing degrees of freedom, such that they model BCS-BEC crossover behaviour. As we will show, this approach gives a unified construction for classes of exactly solvable Hamiltonians with multiple free coupling parameters. It reproduces some seven exactly solvable subcases which have previously appeared in the literature (including some for which the bosonic degree of freedom is decoupled from the system, leaving a BCS Hamiltonian) [19–25]. The availability of such exact results makes amenable the computation of correlation functions [23, 24, 26] which potentially can be compared to experimental results.

To highlight the mathematical motivation for our work, it is instructive to examine the evolution of integrable models of correlated electrons in one dimension. Here, there are many examples whereby models were first introduced and solved exactly via the co-ordinate Bethe ansatz approach, with the rederivation of the Hamiltonian through the Yang–Baxter equation coming as a later development. The first of these is the Hubbard model which was exactly solved by Lieb and Wu in 1968 [27]. It was not until 1986 that Shastry [28] first made connection to the Yang–Baxter equation for this model, with subsequent understanding of the algebraic structure developing through the 1990s (e.g. [29, 30]). The strong coupling limit of the Hubbard model leads to the $t$-$J$ model, which was shown to be exactly solvable for particular choices of the coupling parameters through the works of Schlottmann [31] and Sarkar [32]. In this case, it was not so long before the conserved operators were constructed through the Yang–Baxter equation [33, 34]. Following this there was a flourish of activity throughout the 1990s in the study of exactly solvable models for correlated electrons via co-ordinate Bethe ansatz, which included the Bariev model [35], the $q$-deformation of the supersymmetric $U$ model [36] and the Alcaraz–Bariev model which ultimately included both Hubbard and a solvable $t$-$J$ models as particular subcases [37]. All three Hamiltonians were later shown to be derivable from a solution of the Yang–Baxter equation. Some works showing the connection to the Yang–Baxter equation, viz [38, 39] for the Bariev model, and [40] for the $q$-deformed supersymmetric $U$ model, were reasonably rapid developments. The algebraic formulations for the Bethe ansatz solution were given in [41–43]. Recently, the origin of the solution of the Yang–Baxter equation for the Alcaraz–Bariev model has started to become clear [44].

There are two key differences between the construction of correlated electrons referred to above and our approach below. The first is that in the co-ordinate Bethe ansatz approach, the ansatz wavefunctions for the correlated electron models are taken to be superpositions of plane waves. We will instead formulate the wavefunctions as a factorizable operator acting on a reference state, which follows the spirit of the algebraic Bethe ansatz [1–3] and relates back to Richardson’s original calculations for pairing Hamiltonians [19]. However, we differ from the algebraic Bethe ansatz approach through a second key point in that we will work directly with a variational Hamiltonian. We follow the spirit of the co-ordinate Bethe ansatz approach [7, 27, 31, 32, 35–37] in that we do not resort to any knowledge of a transfer matrix or a set of conserved operators. By combining these two aspects of the co-ordinate and algebraic Bethe ansatz methods, we are able to formulate a powerful technique for constructing exactly solvable models in a very general fashion.

The structure of the manuscript is as follows. We begin section 2 by introducing a general Hamiltonian describing a reduced BCS model coupled to a bosonic degree of freedom. In section 2.1, we apply the Bethe ansatz to obtain constraints on the Hamiltonian’s coupling parameters which are sufficient for exact solvability of the Hamiltonian. There are two cases which are dealt with separately in sections 2.1.1 and 2.1.2. In section 3, we discuss the
connection between seven known exactly solvable subcases that are limits of the general model we consider.

2. Variational Hamiltonian

We consider the general family of pairing Hamiltonians coupled to a bosonic degree of freedom

\[ H = H_0 - H_1, \]

where

\[ H_0 = \alpha N_0 + \kappa N_0^2 + \sum_{k=1}^{L} f(z_k) N_k, \]

\[ H_1 = \beta \sum_{k=1}^{L} g(z_k) b_0 b_k^\dagger + \beta \sum_{k=1}^{L} \overline{g(z_k)} b_0^\dagger b_k + \sigma \sum_{k,s} g(z_k) \overline{g(z_s)} b_k^\dagger b_s, \]

for some real-valued function \( f(z) \) and complex-valued function \( g(z) \), and real-valued \( \alpha, \kappa, \beta \) and \( \sigma \), which will be subject to certain solvability constraints yet to be determined. The overline notation denotes complex conjugation, which imposes that the above Hamiltonian is Hermitian. The operators \( b_k^\dagger = c_k^\dagger - k c_k^\dagger k \) and \( N_k = b_k^\dagger b_k \) for \( k > 0 \) are hard-core Cooper pair creation and number operators, where \( c_k^\dagger \) are fermion creation operators and Cooper pairs are assumed to consist of paired fermions of zero total momentum. Spin labels have been suppressed, so we do not imply anything about the spin properties (e.g. singlet, triplet) of the pairing. We also have a single bosonic mode with operators \( b_0^\dagger \) and \( N_0 \). The particle operators satisfy the following commutation relations:

\[ [b_j, b_k] = 0 = [b_j^\dagger, b_k^\dagger] \quad \forall j, k \geq 0, \]

\[ [b_j, b_k^\dagger] = 0 \quad \forall j, k \geq 0, j \neq k, \]

\[ [b_0, b_0^\dagger] = I, \quad [b_k, b_k^\dagger] = I - 2N_k \quad \forall k > 0. \]

The Hamiltonian commutes with the total number operator \( N = N_0 + \sum_k N_k \). Hamiltonians in this family describe a system of bosonic molecules, condensed into a single bosonic degree of freedom, coupled to \( L \) Cooper pairs which are bosonic-like but must observe an exclusion principle. The Hamiltonians consist of two parts. The diagonal part \( H_0 \), given in equation (2), describes the bosonic mode and allowed Cooper pair energy levels. The self-interaction term \( \kappa N_0^2 \) describes a shift in the frequency of the bosonic mode as it is populated. The cross-interaction part \( H_1 \), given in equation (3), describes level-dependent molecule–pair coupling and pair–pair coupling when either \( \beta \) or \( \sigma \) is non-zero. For the special case \( \alpha = \beta = \kappa = 0 \), which suppresses any action of the Hamiltonian on the bosonic part of the underlying Hilbert space, the Hamiltonian (1) reduces to a general form of the BCS Hamiltonian. It is for this reason we can refer to the general pairing Hamiltonian as a BEC-BCS crossover Hamiltonian.

We do not expect that the Hamiltonian (1) is exactly solvable in general. However, we have found solvability conditions for various sub-classes of the Hamiltonian, in particular, the cases (i) \( \kappa = 0 \) of no self-interaction term and (ii) \( \sigma = 0 \) of no BCS pair–pair scattering term. Taking appropriate limits, these two cases are shown to reduce consistently to the same Hamiltonian when \( \kappa = 0 = \sigma \).
2.1. Exact solvability constraints

We would like to understand the extent to which an exact solution can be found for pairing Hamiltonians of the form given in equation (1) for the yet to be determined functions \( f(z) \) and \( g(z) \). To begin, motivated by the approach of Richardson [19], we assume the ansatz\(^1\)

\[
|\Psi\rangle = \prod_{j=1}^{M} C(y_j)|0\rangle
\]

for the eigenstates of (1), where \(|0\rangle\) denotes the vacuum state,

\[
C(y) = \gamma(y) b_0^\dagger + \sum_{k=1}^{L} h(y, z_k) b_k^\dagger, \quad y \in \mathbb{C},
\]

and \( h(y, z) \) is yet to be determined. We introduce the notation

\[
|\Psi_j\rangle = \prod_{l \neq j}^{M} C(y_l)|0\rangle \quad \text{and} \quad |\Psi_{ij}\rangle = \prod_{l \neq i, j}^{M} C(y_l)|0\rangle.
\]

Using the identities in (4), the following commutation relations are found:

\[
\begin{align*}
[b_0, C(y)] &= \gamma(y) I, \\
[b_k, C(y)] &= h(y, z_k)(I - 2N_k), \\
[N_0, C(y)] &= \gamma(y) b_0^\dagger, \\
[N_k, C(y)] &= h(y, z_k) b_k^\dagger, \\
[N_0^2, C(y)] &= \gamma(y) b_0^\dagger (I + 2N_0), \\
[H_0, C(y)] &= \alpha \gamma(y) b_0^\dagger + \sum_{k=1}^{L} f(z_k) h(y, z_k) b_k^\dagger + \kappa \gamma(y) b_0^\dagger (I + 2N_0).
\end{align*}
\]

We note that for an operator \( \hat{O} \),

\[
\left[ \hat{O}, \prod_{j=m}^{M} C(y_j) \right] = \sum_{l=m}^{M} \left( \prod_{r=m}^{l-1} C(y_r) \right) [\hat{O}, C(y_l)] \prod_{j=l+1}^{M} C(y_j).
\]

Using this identity, we can show

\[
\begin{align*}
b_0|\Psi\rangle &= \left[ b_0, \prod_{j=1}^{M} C(y_j) \right]|0\rangle \\
&= \sum_{j=1}^{M} \gamma(y_j)|\Psi_j\rangle, \\
b_k|\Psi\rangle &= \sum_{j=1}^{M} \left( \prod_{r=1}^{l-1} C(y_r) \right) [b_k, C(y_j)] \prod_{l=j+1}^{M} C(y_l)|0\rangle \\
&= \sum_{j=1}^{M} h(y_j, z_k)|\Psi_j\rangle - \sum_{j \neq i}^{M} h(y_j, z_k) h(y_i, z_k) b_k^\dagger |\Psi_{ij}\rangle.
\end{align*}
\]

\(^1\) For simplicity, throughout the paper we only consider eigenstates which do not involve blocked levels. For a review of the blocking effect we refer to [45]. There is no technical impediment to extend results to accommodate the general case with blocked levels, but we omit instances with blocked states for the sake of readability.
We may then calculate the following:

\[ H_0 |\Psi\rangle = (\alpha + \kappa) \sum_{j=1}^M \gamma(y_j)b_0^j|\Psi\rangle + \sum_{j=1}^M \sum_{k=1}^L f(z_k)h(y_j, z_k)b_1^j|\Psi\rangle \\
+ \kappa \sum_{j \neq k}^M \gamma(y_j)\gamma(y_i)b_0^j b_0^k|\Psi\rangle, \]

\[ H_1 |\Psi\rangle = \beta \sum_{j=1}^M \sum_{k=1}^L \gamma(y_j)g(z_k)b_1^j|\Psi\rangle + \beta \sum_{j=1}^M \sum_{k=1}^L g(z_k)h(y_j, z_k)b_0^j|\Psi\rangle \\
- \beta \sum_{j \neq k}^M \sum_{k=1}^L g(z_k)h(y_j, z_k)h(y_i, z_i)b_1^j b_1^k|\Psi\rangle \\
+ \sigma \sum_{j=1}^M \sum_{k, l=1}^L g(z_k)g(z_l)h(y_j, z_k)b_1^j b_1^l|\Psi\rangle \\
- \sigma \sum_{j \neq k}^M \sum_{k, l=1}^L g(z_k)g(z_l)h(y_j, z_k)b_1^j b_1^k|\Psi\rangle. \]

These equations give the action of the Hamiltonian \( H \) on the state \( |\Psi\rangle \). In order to determine the exact solution, we require that \( |\Psi\rangle \) be an eigenstate of \( H \). Thus we look to solve the eigenvalue problem

\[ (H_0 - H_1)|\Psi\rangle = E|\Psi\rangle \tag{6} \]

for some scalar \( E \). In general, this will not be possible. The set of constraints required to find a solution to equation (6) are called solvability constraints and define the manifold in the coupling parameter space along which the Hamiltonian \( (1) \) sustains an exact solution.

In order to find an exact solution, we look to write the sums involving any of the vectors \( b_1^j b_1^k|\Psi\rangle \), \( b_0^j b_1^k|\Psi\rangle \), \( b_1^j b_0^k|\Psi\rangle \), \( b_1^j|\Psi\rangle \), or \( b_0^k|\Psi\rangle \) as a combination of the vectors \( |\Psi\rangle \) and \( |\Psi\rangle \). As a first step, we write the sum over \( b_1^j b_1^k|\Psi\rangle \) terms as a sum over a \( b_1^j b_0^k|\Psi\rangle \) part and a \( b_0^k|\Psi\rangle \) part. However, comparing the \( b_1^j b_1^k|\Psi\rangle \) and \( b_0^j b_1^k|\Psi\rangle \) terms we see that the particular constraint chosen to reduce the \( b_1^j b_1^k|\Psi\rangle \) terms must be compatible with one chosen to reduce the \( b_1^j b_0^k|\Psi\rangle \) terms. For compatibility we introduce the constraint

\[ \overline{g(z_k)}h(y_j, z_k)h(y_i, z_i) = k(y_j, y_i)h(y_j, z_k) + k(y_i, y_j)h(y_i, z_k), \quad \forall y_j, y_i, z_k \in \mathbb{C}, \]

where \( k(y_j, y_i) \) is to be chosen later. This allows us to write

\[ H_1 |\Psi\rangle = \beta \sum_{j=1}^M \sum_{k=1}^L \gamma(y_j)g(z_k)b_1^j|\Psi\rangle + \beta \sum_{j=1}^M \sum_{k=1}^L g(z_k)h(y_j, z_k)b_0^j|\Psi\rangle \\
- 2\beta \sum_{j \neq k}^M \sum_{k=1}^L k(y_j, y_i)h(y_j, z_k)b_1^j b_1^k|\Psi\rangle \\
+ \sigma \sum_{j=1}^M \sum_{k, l=1}^L g(z_k)g(z_l)h(y_j, z_k)b_1^j b_1^l|\Psi\rangle \\
- 2\sigma \sum_{j \neq k}^M \sum_{k, l=1}^L g(z_k)k(y_j, y_i)h(y_j, z_k)b_1^j b_1^k|\Psi\rangle. \]
We then utilize definition (5) for $|\Psi\rangle$ to express the sums in terms of the desired vectors

$$H_1|\Psi\rangle = \beta \sum_{j=1}^{M} \sum_{k=1}^{L} \gamma(y_j)g(z_k)\beta_k^j|\Psi_j\rangle + \beta \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)h(y_j, z_k)\beta_k^j|\Psi_j\rangle$$

$$- 2\beta \sum_{j,l \neq j}^{M} k(y_j, y_l)\beta_j^l|\Psi_j\rangle + 2\beta \sum_{j,l \neq j}^{M} k(y_j, y_l)\gamma(y_j)\beta_j^l|\Psi_j\rangle$$

$$+ \sigma \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)g(z_k)h(y_j, z_k)\beta_k^j|\Psi_j\rangle$$

$$- 2\sigma \sum_{j,l \neq j}^{M} \sum_{k=1}^{L} g(z_k)k(y_j, y_l)\beta_j^l|\Psi_j\rangle + 2\sigma \sum_{j,l \neq j}^{M} \sum_{k=1}^{L} g(z_k)k(y_j, y_l)\gamma(y_j)\beta_j^l|\Psi_j\rangle.$$ 

Consider the full expression

$$H|\Psi\rangle = (\alpha + \kappa) \sum_{j=1}^{M} \gamma(y_j)\beta_k^j|\Psi_j\rangle - \beta \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)h(y_j, z_k)\beta_k^j|\Psi_j\rangle + 2\beta \sum_{j,l \neq j}^{M} k(y_j, y_l)\beta_j^l|\Psi_j\rangle$$

$$+ \sum_{j=1}^{M} \sum_{k=1}^{L} f(z_k)h(y_j, z_k)\beta_k^j|\Psi_j\rangle - \beta \sum_{j=1}^{M} \sum_{k=1}^{L} \gamma(y_j)g(z_k)\beta_k^j|\Psi_j\rangle$$

$$- \sigma \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)g(z_k)h(y_j, z_k)\beta_k^j|\Psi_j\rangle$$

$$+ 2\sigma \sum_{j,l \neq j}^{M} \sum_{k=1}^{L} g(z_k)k(y_j, y_l)\beta_j^l|\Psi_j\rangle - 2\sigma \sum_{j,l \neq j}^{M} \sum_{k=1}^{L} g(z_k)k(y_j, y_l)\gamma(y_j)\beta_j^l|\Psi_j\rangle$$

$$- 2\beta \sum_{j,l \neq j}^{M} k(y_j, y_l)\gamma(y_j)\beta_j^l|\Psi_j\rangle + \kappa \sum_{j=1}^{M} \sum_{l \neq j}^{M} \gamma(y_j)\beta_j^l|\Psi_j\rangle.$$ 

At this point, we observe the appearance of identical coefficients of the vectors $\beta\beta_0^j|\Psi_j\rangle$ and $\sigma \sum_{k=1}^{L} g(z_k)\beta_k^j|\Psi_j\rangle$ for $\sigma \neq 0 \neq \beta$. In finding the exact solution, we must choose constraints on the coefficients that are compatible with this. Thus, the next constraint must involve the terms that are not yet related. We seek to express the remaining sums over vectors $\beta_k^j|\Psi_j\rangle$ as a combination of the vectors $|\Psi_j\rangle, \beta\beta_0^j|\Psi_j\rangle$ and $\sigma \sum_{k=1}^{L} g(z_k)\beta_k^j|\Psi_j\rangle$ such that the latter two have the same coefficients. We choose the following constraint to reduce the terms to the desired vectors:

$$f(z_k)h(y_j, z_k) = y_jh(y_j, z_k) + g(z_k)r(y_j)$$

$$\Rightarrow h(y_j, z_k) = \frac{g(z_k)r(y_j)}{f(z_k) - y_j} \forall y_j, z_k \in \mathbb{C},$$

where the $r(y_j)$ will be determined by compatibility requirements. With these choices we find

$$\sum_{j=1}^{M} \sum_{k=1}^{L} f(z_k)h(y_j, z_k)\beta_k^j|\Psi_j\rangle = \sum_{j=1}^{M} \sum_{k=1}^{L} y_jh(y_j, z_k)\beta_k^j|\Psi_j\rangle + \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)r(y_j)\beta_k^j|\Psi_j\rangle$$

$$= \sum_{j=1}^{M} y_j|\Psi\rangle - \sum_{j=1}^{M} y_j\gamma(y_j)\beta_j^0|\Psi_j\rangle + \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)r(y_j)\beta_k^j|\Psi_j\rangle.$$
Use of this relation leads to
\[
H|\Psi\rangle = \sum_{j=1}^{M} y_j|\Psi\rangle + \sum_{j=1}^{M} (\sigma + \kappa - y_j)\gamma(y_j)b_j^\dagger|\Psi\rangle - \beta \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)h(y_j, z_k)b_k^\dagger|\Psi_j\rangle \\
+ 2\beta \sum_{j,l \neq j} M L k(y_j, y_l)b_l^\dagger|\Psi_j\rangle + \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)[r(y_j) - \beta \gamma(y_j)]b_k^\dagger|\Psi_j\rangle \\
- \sigma \sum_{j=1}^{M} \sum_{k=1}^{L} g(z_k)g^*(z_k)h(y_j, z_k)b_k^\dagger|\Psi_j\rangle + 2\sigma \sum_{j,l \neq j} M L g(z_k)k(y_j, y_l)b_l^\dagger|\Psi_j\rangle \\
- 2\beta \sum_{j,l \neq j} M L k(y_j, y_l)\gamma(y_l)b_l^\dagger b_j^\dagger|\Psi_j\rangle + \kappa \sum_{j=1,l \neq j} M L \gamma(y_j)\gamma(y_l)b_l^\dagger b_j^\dagger|\Psi_j\rangle \\
- 2\sigma \sum_{j,l \neq j} M L g(z_k)k(y_j, y_l)\gamma(y_l)b_l^\dagger b_j^\dagger|\Psi_j\rangle.
\]

At this stage we remark that the choice of constraint (7) sets the energy eigenvalues to a standardized form \(E = \sum_{j=1}^{M} y_j\).

Exact solvability is achieved by requiring that the coefficients of all terms other than \(|\Psi\rangle\) cancel. To meet this requirement, we choose (for \(g(z_k) \neq 0\))
\[
(y_j - \sigma)\gamma(y_j) + \beta \sum_{k=1}^{L} g(z_k)h(y_j, z_k) = 2\beta \sum_{l \neq j} M L k(y_j, y_l) \\
\beta \gamma(y_j) - r(y_j) + \sigma \sum_{k=1}^{L} g(z_k)h(y_j, z_k) = 2\sigma \sum_{l \neq j} M L k(y_j, y_l) \\
\beta(k(y_j, y_l)\gamma(y_l) + k(y_l, y_j)\gamma(y_j)) = \kappa \gamma(y_j)\gamma(y_l) \\
(8) \\
\sigma(k(y_j, y_l)\gamma(y_l) + k(y_l, y_j)\gamma(y_j)) = 0. \hspace{1cm} (9)
\]

However, we note that for \(\kappa \neq 0 \neq \sigma\) the constraints (8) and (9) are incompatible, and we have at least two separate cases. We must now derive two sets of solvability conditions, one for each case; however, they should agree in the appropriate limit \(\kappa = 0 = \sigma\) when they describe the same family of Hamiltonians. Some care will also be needed in the limit \(\beta \rightarrow 0\).

2.1.1. Case 1. No pair–pair interaction (\(\sigma = 0\)). The first case we look at is \(\sigma = 0\). Here, the solvability conditions reduce to
\[
\overline{g(z_k)}h(y_j, z_k)h(y_l, z_k) = k(y_j, y_l)h(y_l, z_k) + k(y_l, y_j)h(y_j, z_k) \hspace{1cm} (10) \\
(y_j - \sigma - \kappa)\gamma(y_j) + \beta \sum_{k=1}^{L} g(z_k)h(y_j, z_k) = 2\beta \sum_{l \neq j} M L k(y_j, y_l) \\
\beta \gamma(y_j) - r(y_j) = 0 \hspace{1cm} (12) \\
\beta(k(y_j, y_l)\gamma(y_l) + k(y_l, y_j)\gamma(y_j)) = \kappa \gamma(y_j)\gamma(y_l) \hspace{1cm} (13) \\
h(y_j, z_k) = \frac{g(z_k)r(y_j)}{\overline{f(z_k)}} - y_j \hspace{1cm} (14)
\]

Firstly, (12) is satisfied for non-trivial \(\gamma(y_j)\) only if \(\beta \gamma(y_j) = r(y_j)\).
Substituting (12) into (14) gives

\[ h(y_j, z_k) = \frac{\beta g(z_k) g'(y_j)}{f(z_k) - y_j} \]

and substituting this into (10) gives

\[ g(z_k) \frac{\beta g(z_k) g'(y_i)}{f(z_k) - y_j} = k(y_j, y_i) \frac{\beta g(z_k) g'(y_i)}{f(z_k) - y_j} + k(y_i, y_j) \frac{\beta g(z_k) g'(y_j)}{f(z_k) - y_j} \]

Rearranging,

\[ f(z_k) \frac{\beta k(y_j, y_i) g'(y_i) + k(y_i, y_j) g'(y_j)}{g'(y_i) g'(y_j)} = \frac{\beta^2 g(z_k) g'(z_k)}{1} - \beta^2 g(z_k) g'(z_k) = c_1. \]

Since the right-hand side does not depend on the parameter \( z_k \) we require that, for some constants \( c_1 \) and \( c_2 \),

\[ \beta k(y_j, y_i) g'(y_i) + k(y_i, y_j) g'(y_j) = c_1 g'(y_i) g'(y_j), \]

\[ \beta k(y_j, y_i) g'(y_i) + k(y_i, y_j) g'(y_j) = c_2 g'(y_i) g'(y_j), \]

\[ c_2 f(z_k) - \beta^2 g(z_k) g'(z_k) = c_1. \]

However, compatibility with constraint (13) requires

\[ c_2 = \kappa. \]

The solution is thus

\[ k(y_j, y_i) = c_1 - c_2 y_j \frac{g'(y_j)}{g'(y_i)} \]

\[ f(z_k) = c_2^{-1} (\beta^2 g(z_k) g'(z_k) + c_1), \]

\[ c_2 = \kappa, \]

for some constant \( c_1 \).

Substituting this into the constraint (11) completes the compatibility of constraints yielding

\[ \frac{1}{2 y_j} (\alpha + \kappa) + \sum_{k=1}^{L} c_2 f(z_k) - c_1 - 2 \sum_{l \neq j} c_1 - c_2 y_l \]

\[ c_2 = \kappa. \]

We refer to (15) as the Bethe ansatz equations for which the sub-family of Hamiltonians of (1) corresponding to \( \sigma = 0 \) is solvable. An equivalent expression is obtained by manipulating the terms

\[ \frac{1}{2 y_j} (\alpha + \kappa + c_2 (2M - 2 - L)) - \frac{1}{2} + \frac{1}{4} \sum_{k=1}^{L} \frac{2}{y_j / c_2 - f(z_k) / c_2} \]

\[ + \frac{c_1}{2 y_j} \left( \sum_{k=1}^{L} \frac{1}{f(z_k) - y_j} + \sum_{l \neq j} \frac{2}{y_j - y_l} \right) = \sum_{l \neq j} \frac{c_2}{y_j - y_l} \]

\[ c_2 = \kappa. \]

In the special case of \( \kappa = 0 = \sigma \), we find \( c_1 = -\beta^2 |g(z_k)|^2 = -G^2 \), \( c_2 = 0 \) and the Bethe ansatz equations reduce to

\[ \frac{\alpha}{2G} - \frac{y_j / G}{2} - \frac{1}{4} \sum_{k=1}^{L} \frac{2}{f(z_k) / G - y_j / G} = \frac{1}{2} \sum_{l \neq j} \frac{2}{y_j / G - y_l / G}. \]
We summarize by listing the constraining relations for exact solvability in the case $\sigma = 0$ here:
\[
    h(y_j, z_k) = \frac{\beta g(z_k) y'(y_j)}{f(z_k) - y_j}
\]
\[
    f(z_k) = \kappa^{-1} \beta^2 g(z_k) g(z_k) + \kappa^{-1} c_1
\]
\[
    y_j - (\alpha + \kappa) + \sum_{k=1}^{L} c_2 f(z_k) - c_1 = 2 \sum_{l \neq j}^{M} \frac{c_1 - c_2 y_j}{y_j - y_l}
\]
\[
    c_2 = \kappa.
\]
for constants $\beta$ and $c_1$. These constraints define the manifold in the coupling parameters of the pairing Hamiltonian (1) for which it is exactly solvable with eigenstates of the form (5) in the case $\sigma = 0$. In the above derivation, we find that $\gamma(y_j)$ is not fixed; however, it will be fixed by normalization of the eigenstate.

2.1.2. Case 2. No self-interaction term ($\kappa = 0$). Setting $\kappa = 0$ in the solvability conditions leads to the following set of constraints:
\[
    g(z_k) h(y_j, z_k) h(y_i, z_k) = k(y_j, y_i) h(y_j, z_k) + k(y_i, y_j) h(y_j, z_k) \quad (17)
\]
\[
    (y_j - \alpha) \gamma(y_j) + \beta \sum_{k=1}^{L} g(z_k) h(y_j, z_k) = 2 \beta \sum_{l \neq j}^{M} k(y_j, y_l) \quad (18)
\]
\[
    \beta \gamma(y_j) - r(y_j) + \sigma \sum_{k=1}^{L} g(z_k) h(y_j, z_k) = 2 \sigma \sum_{l \neq j}^{M} k(y_j, y_l) \quad (19)
\]
\[
    k(y_j, y_i) \gamma(y_i) + k(y_i, y_j) \gamma(y_j) = 0 \quad (20)
\]
\[
    h(y_j, z_k) = \frac{g(z_k) r(y_j)}{f(z_k) - y_j} \quad (21)
\]

Constraints (18) and (19) are compatible when
\[
    \beta r(y_j) = [\sigma (\alpha - y_j) + \beta^2] \gamma(y_j)
\]
which along with condition (20) result in the constraint
\[
    \beta [\sigma (\alpha - y_j) + \beta^2] k(y_j, y_i) r(y_i) + \beta [\sigma (\alpha - y_i) + \beta^2] k(y_i, y_j) r(y_j) = 0, \quad (22)
\]
and along with (21) result in
\[
    \beta h(y_j, z_k) = [\sigma (\alpha - y_j) + \beta^2] \frac{g(z_k) \gamma(y_j)}{f(z_k) - y_j}.
\]
Substituting (21) into (17) gives
\[
    \frac{g(z_k) g(z_k) r(y_j)}{f(z_k) - y_j} \frac{g(z_k) r(y_i)}{f(z_k) - y_i} = k(y_j, y_i) \frac{g(z_k) r(y_i)}{f(z_k) - y_i} + k(y_i, y_j) \frac{g(z_k) r(y_j)}{f(z_k) - y_j}
\]
which we rearrange to
\[
    f(z_k) \frac{k(y_j, y_i) r(y_i) + k(y_i, y_j) r(y_j)}{r(y_j) r(y_i)} - \frac{g(z_k) g(z_k)}{r(y_j) r(y_i)} = \frac{k(y_j, y_i) r(y_i) y_j + k(y_i, y_j) r(y_j) y_i}{r(y_j) r(y_i)}.
\]
Since the right-hand side of the equation does not depend on the parameter $z_k$ we must satisfy, for some constants $c_2$ and $c_1$, the following set of relations:

\[
\begin{align*}
k(y_j, y_i) r(y_j) y_j + k(y_i, y_j) r(y_i) y_i &= c_1 r(y_j) r(y_i), \\
k(y_j, y_i) r(y_j) + k(y_i, y_j) r(y_i) &= c_2 r(y_j) r(y_i), \\
c_2 f(z_k) - g(z_k) g(z_k) &= c_1.
\end{align*}
\]

For compatibility of the first two equations, we find that the solution must be of the form

\[k(y_j, y_i) = \frac{c_1 - c_2 y_j}{y_j - y_i} r(y_i).\]

However, for compatibility with constraint (22) we then require

\[\beta [\sigma (\alpha - y_j) + \beta^2] \frac{c_1 - c_2 y_j}{y_j - y_i} r(y_j) - \beta [\sigma (\alpha - y_i) + \beta^2] \frac{c_1 - c_2 y_i}{y_i - y_j} r(y_i) = 0\]

which we can rearrange to obtain

\[\beta [c_2 (\sigma \alpha + \beta^2) - c_1 \sigma] r(y_j) r(y_i) = 0\]

and we must have

\[c_2 (\sigma \alpha + \beta^2) \beta = c_1 \sigma \beta. \tag{23}\]

At this point we have the conditions

\[
\begin{align*}
c_1 \sigma \beta &= c_2 (\sigma \alpha + \beta^2) \beta, \\
k(y_j, y_i) &= \frac{c_1 - c_2 y_j}{y_j - y_i} r(y_j), \\
c_1 &= c_2 f(z_k) - g(z_k) g(z_k), \\
h(y_j, z_k) &= g(z_k) r(y_j) / f(z_k) - y_j.
\end{align*}
\]

For $\beta \neq 0$, these equations, along with equation (18), or equivalently (19), complete the compatibility of constraints yielding

\[
\frac{y_j - \alpha}{[\sigma (\alpha - y_j) + \beta^2]} + \sum_{k=1}^{L} \frac{c_2 f(z_k) - c_1}{f(z_k) - y_j} = 2 \sum_{l \neq j}^{M} \frac{c_1 - c_2 y_l}{y_l - y_i},
\]

\[c_1 \sigma = c_2 (\sigma \alpha + \beta^2). \tag{24}\]

For the case $\beta = 0$, the constraint (24) is no longer necessary. In this instance the bosonic degree of freedom decouples, and we may project onto a BCS system which will be discussed in the following subsection.

An equivalent expression for the Bethe ansatz equations is obtained by manipulating the terms

\[
\begin{align*}
\frac{1}{2y_j} \left( \frac{\alpha}{\sigma (c_1/c_2 - y_j)} + c_2 (2M - 2 - L) \right) - \frac{1}{2\sigma (c_1/c_2 - y_j)} + \frac{1}{4} \sum_{k=1}^{L} \frac{2}{y_j/c_2 - f(z_k)/c_2} \\
+ \frac{c_1}{2y_j} \left( \sum_{k=1}^{L} \frac{1}{f(z_k) - y_j} + \sum_{l \neq j}^{M} \frac{2}{y_j - y_l} \right) = \sum_{l \neq j}^{M} \frac{c_2}{y_j - y_l}
\end{align*}
\]

\[c_2 (\sigma \alpha + \beta^2) \beta = c_1 \sigma \beta.\]
In the special case of $\sigma = 0 = \kappa$, we find $c_2 = 0$, $\beta^2 c_1 = -\beta^2 g(z_k)g(z_l) = -G^2$ and the Bethe ansatz equations reduce to

$$\frac{\alpha}{2G} - \frac{y_j/G}{2} = \frac{1}{4} \sum_{k=1}^{L} \frac{2}{f(z_k)/G - y_j/G} = \frac{1}{2} \sum_{\ell \neq j}^{M} \frac{2}{y_j/G - y_l/G}.$$  

This result is in complete agreement with that of the first case, where we derived the Bethe ansatz equations for $\kappa \neq 0$ and $\sigma = 0$ and then set $\kappa = 0$ at the end to obtain the limiting case.

2.2. Recovering known exactly solvable subcases

In the above subsections we determined manifolds in the coupling parameters of (1) for which an exact solution exists. Taking appropriate limits of the general exactly solvable models yields eight subcases which we have presented in figure 1. Seven of these subcases are known [19–25].

The three tiers apparent in the graph in figure 1 correspond to the number of free parameters in the solvable models which determine the coupling interaction strengths and the functional relationship between $f(z)$ and $g(z)$. The top tier consists of the most general exactly solvable

2 While we have tried to be general in deriving this manifold, it is potentially possible to relax some of the assumptions made on the functions introduced in section 2.1. This will be considered further in future work.

3 In some of the subcases, a change of variable is required to bring the Hamiltonian into the form cited.
These eigenstates are obtained by renormalizing the eigenstate (5) to accommodate for the factor models which were derived in sections 2.1.1 and 2.1.2. These models each have three free parameters up to an arbitrary energy rescaling. Models in the middle tier have two free parameters and models in the lowest tier are described by only one free parameter.

For the $\sigma = 0$ case, the Hamiltonian is of the form

$$H = \alpha N_0 + \kappa N_0^2 + \sum_{k=1}^{L} f(z_k) N_k - \beta \sum_{k=1}^{L} (g(z_k) b_k^\dagger b_k + \overline{g(z_k)} b_k^\dagger b_k),$$

and is exactly solvable with eigenstates

$$|\Psi\rangle = \prod_{j=1}^{M} y_j(y_j) \left( b_0^\dagger + \sum_{k=1}^{L} \frac{\beta g(z_k)}{f(z_k) - y_j} b_k^\dagger \right) |0\rangle$$

when

$$\beta^2 |g(z_k)|^2 + c_1 = c_2 f(z_k), \quad c_2 = \kappa \quad (25)$$

Equation (25) affects the pairing symmetry in the BCS part of the model. The three independent parameters in this model are $\beta$, $\kappa$ and $c_1$.

For the $\kappa = 0$ case, the Hamiltonian is of the form

$$H = \alpha N_0 + \sum_{k=1}^{L} f(z_k) N_k - \beta \sum_{k=1}^{L} (g(z_k) b_k b_k^\dagger + \overline{g(z_k)} b_k^\dagger b_k) - \sigma \sum_{k,s} g(z_k) \overline{g(z_s)} b_k^\dagger b_s,$$

and is exactly solvable with eigenstates

$$|\Psi\rangle = \prod_{j=1}^{M} y_j(y_j) \left( b_0^\dagger + \sum_{k=1}^{L} \frac{(c_2 - c_1 y_j) g(z_k)}{c_1 \beta (f(z_k) - y_j)} b_k^\dagger \right) |0\rangle$$

when

$$c_2 (\sigma \alpha + \beta^2) = c_1 \sigma, \quad |g(z_k)|^2 + c_1 = c_2 f(z_k), \quad (26)$$

Equation (26) affects the pairing symmetry in the BCS part of the model. The $\alpha = 0$ case in that equation (26) affects the pairing symmetry in the BCS part of the model.

In the limit of $\beta = 0$, the $\kappa = 0$ model reduces to a model with no coupling between the bosonic degree of freedom and the Cooper pairs. Here we can simply project out the bosonic degree of freedom from the Hilbert space of states and consider the Hamiltonian as BCS type only. The BCS model in this case has neither (p + ip)-wave nor s-wave symmetry, but a generalization of both. The Hamiltonian is of the form

$$H_{\text{BCS}} = \sum_{k=1}^{L} f(z_k) N_k - \sigma \sum_{k,s} g(z_k) \overline{g(z_s)} b_k^\dagger b_s, \quad (27)$$

and is exactly solvable with eigenstates

$$|\Psi\rangle = \prod_{j=1}^{M} \left( \sum_{k=1}^{L} \frac{(c_2 - c_1 y_j) g(z_k)}{f(z_k) - y_j} b_k^\dagger \right) |0\rangle,$$

These eigenstates are obtained by renormalizing the eigenstate (5) to accommodate for the factor $\prod_{j=1}^{M} y_j(y_j)$ which approaches zero in the limit $\beta \to 0$. 

1. Inverse Problems 28 (2012) 035008
where
\[ |g(z_k)|^2 + c_1 = c_2 f(z_k), \]  
\[ -\frac{1}{\sigma} + \sum_{k=1}^{L} \frac{c_2 f(z_k) - c_1}{f(z_k) - y_j} = 2 \sum_{l \neq j}^{M} \frac{c_1 - c_2 y_l}{y_j - y_l}. \]

Here \( c_1 = 0 \) results in \((p + ip)\)-wave symmetry, while \( c_2 = 0 \) results in \(s\)-wave symmetry\(^5\).

We take this moment to reflect on some of the historical passage of events in integrable pairing Hamiltonians. Although the construction of exact eigenstates in the \(s\)-wave model traces back to Richardson’s work of 1963, it was only in 1997 that the conserved operators were constructed [46]. It was soon realized that these operators could be reproduced following the approach of Gaudin [20] by considering the rational class of Gaudin magnets. By extending this to the trigonometric and hyperbolic classes, more general integrable Hamiltonians were constructed in 2001 [47, 48]. It turns out that the conserved operators in the hyperbolic case are those of the \((p + ip)\)-wave Hamiltonian, a model which came to prominence through the work of Read and Green in 2000 [49]. However, it was not immediately apparent that the hyperbolic class of conserved operators could be combined to produce the \((p + ip)\)-wave Hamiltonian, and it was only in 2010 that the explicit relationship was made [26].

It is somewhat surprising that the above results show that the rational and hyperbolic cases can be seen as two limits of a more general pairing Hamiltonian (27) with the constraint (28) and the Bethe ansatz equations (29). The number of free parameters is the same in both limiting Hamiltonians. This is in some contrast to the case of the \(XXX\) spin chain, which is the rational limit of the \(XXZ\) spin chain obtained by a one-variable reduction in the coupling co-efficient of the \(S_i^z S_{i+1}^z\) interactions.

3. Conclusion

We introduced a variational approach for the QISM to exactly solve a class of Hamiltonians via Bethe ansatz methods. The procedure was conducted in the framework of variational Hamiltonians describing BCS-BEC crossover physics through interacting Cooper pairs and a bosonic degree of freedom. We obtained general exact solvability requirements which included seven subcases which have previously appeared in the literature. An initial question to consider is whether the general forms of exactly solvable Hamiltonians, in subsections 2.1.1 and 2.1.2 respectively, do admit a set of conserved operators. We have found that such a set is indeed constructible, either through the Gaudin algebra approach of [50] (by relaxing constraints imposed in subsection 3.1 of their work), or by a suitable adaptation of the classical Yang–Baxter equation approach of [51]. However, we emphasize that our motivation was to undertake calculations to obtain exactly solvable Hamiltonians in a manner which did not rely on any prior knowledge of integrability through the existence of a set of conserved operators.

Finally, we remark that there remains scope to further extend this approach to a more general level than that which we have considered here. For example the exactly solvable Russian doll BCS model [52, 53] does not fit into the above scheme, both in that the pair–pair interaction is not factorizable, and the wavefunction ansatz is of a different type. Furthermore, pairing Hamiltonians such as those in [47, 48, 51] contain interaction terms which are not present in our starting Hamiltonian (1), and similar extended exactly solvable models can also be constructed within the above approach.

\(^5\) Strictly speaking we also have to impose that the pairing is triplet type in the \((p + ip)\)-wave case and singlet type in the \(s\)-wave case.
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