A Polynomial-time Algorithm to Achieve Extended Justified Representation

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Abstract

We consider a committee voting setting in which each voter approves of a subset of candidates and based on the approvals, a target number of candidates are to be selected. In particular we focus on the axiomatic property called extended justified representation (EJR). Although a committee satisfying EJR is guaranteed to exist, the computational complexity of finding such a committee has been an open problem and explicitly mentioned in multiple recent papers. We settle the complexity of finding a committee satisfying EJR by presenting a polynomial-time algorithm for the problem. Our algorithmic approach may be useful for constructing other multi-winner voting rules.

Keywords: Social choice theory, committee voting, multi-winner voting, approval voting, computational complexity

JEL: C63, C70, C71, and C78

1. Introduction

The topic of multi-winner/committee voting has witnessed a renaissance with a number of new and interesting developments in the last few years (see [1,7] for recent surveys). We consider a committee voting setting in which each voter approves of a subset of candidates and based on the approvals, a target $k$ number of candidates are selected. The setting has been referred to as approval-based multi-winner voting or committee voting with approvals. The setting has inspired a number of natural voting rules [9, 3, 10, 4, 15]. Many of the voting rules attempt to satisfy some notion of representation. However it has been far from clear what axiom captures the representation requirements.
Aziz et al. [2, 3] proposed two compelling representation axioms called justified representation (JR) and extended justified representation (EJR). Interestingly, Sánchez-Fernández et al. [13] presented an intermediate property called proportional justified representation (PJR). The idea behind all the three properties is that a cohesive and large enough group deserves sufficient number of approved candidates in the winning set of candidates. Interestingly, it is known that there always exists a committee satisfying the strongest property EJR [3]. However to date, it has been unknown whether a committee satisfying EJR can be computed in polynomial time. For the two weaker representation notions, polynomial-time algorithms have been presented for finding a committee satisfying JR [2, 3] and PJR [6, 14]. On the other hand, the computational complexity of finding a committee satisfying EJR has been open. Aziz et al. [2, 3] mentioned the problem in their original paper. The problem has been reiterated in subsequent work. Brill et al. [6] state that

"it remains an open problem whether committees providing EJR can be computed efficiently."

Sánchez-Fernández et al. [14] mention the same problem:

"Whether a voting rule exists that satisfies the extended justified representation and can be computed in polynomial time remains an open issue."

In a different paper, Sánchez-Fernández et al. [12] state the following.

"In contrast, it is conjectured that finding committees that provide EJR is computationally hard."

Incidentally, there exists an interesting rule called PAV (Proportional Approval Voting) that satisfies EJR [3]. In PAV, each voter is viewed as getting an additional score of $1/j$ for getting the $j$-th approved candidate in the committee. The PAV rule returns a committee with the highest total PAV score for the voters. The PAV rule has a fascinating history as it was proposed by the Danish polymath Thorvald N. Thiele in the 19th century and then rediscovered by Forrest Simmons [8]. Finding a PAV outcome is NP-hard [4, 15] and W[1]-hard even if each voter approves of 2 candidates [4]. Thiele also presented a greedy sequential version of PAV. The rule that is referred to as SeqPAV (Sequential PAV) or RAV (reweighted approval voting) does not even satisfy JR [3].

1 The property PJR was independently proposed by Haris Aziz in October 2014 who referred to it as weak EJR.
2 For JR, a simple linear-time algorithm call GreedyAV finds a committee satisfying JR.
3 It has recently been shown that a committee satisfying PJR can be computed in polynomial time. Brill et al. [6] proved that SeqPhragmén (an algorithm proposed by Swedish mathematician Phragmén in the 19th century) is polynomial-time and returns a committee satisfying PJR. Independently and around the same time as the result by Brill et al. [6], Sánchez-Fernández et al. [13] presented a different algorithm that finds a PJR committee and also satisfies other desirable monotonicity axioms.
One natural approach to find a committee satisfying \textit{EJR} is to enumerate possible committees and then test them for \textit{EJR}. However the number of committees is exponential and even testing whether a committee satisfies \textit{EJR} is coNP-complete \cite{3}. Aziz et al. \cite{3} presented a result that implies that if \( k \) is a constant, then a committee satisfying \textit{EJR} can be computed in time \( \text{poly}(n \cdot |C|^k) \). The result does not show that finding a committee satisfying \textit{EJR} is polynomial-time solvable in general or whether it is fixed parametrized tractable.

\textit{Contributions.} We present the first polynomial-time algorithm to find a committee that satisfies \textit{EJR}. The result implies that there exists a polynomial-time algorithm to find a committee that satisfies the weaker property of \textit{PJR}. As mentioned earlier, it has only recently been proven in two independent papers that a committee satisfying \textit{PJR} can be computed in polynomial time \cite{6,14}. Both of the algorithms in \cite{6} and \cite{14} sequentially build a committee while optimizing some flow or load balancing objective. In contrast, our algorithm uses an approach based on swapping candidates from inside a committee with candidates from outside the committee. The correctness of our algorithm relies on a careful insight on the connection between \textit{EJR} and a property we refer to as \textit{PAV}-swap-freeness. We feel that this simple idea of allowing swaps may lead to other interesting algorithms for \textit{EJR} as well as other compelling properties in multi-winner voting problems.

\section{Approval-based Committee Voting and Representation Properties}

We consider a social choice setting with a set \( N = \{1, \ldots , n\} \) of voters and a set \( C \) of \( m \) candidates. Each voter \( i \in N \) submits an approval ballot \( A_i \subseteq C \), which represents the subset of candidates that she approves of. We refer to the list \( \vec{A} = (A_1, \ldots , A_n) \) of approval ballots as the \textit{ballot profile}. We will consider \textit{approval-based multi-winner voting rules} that take as input a quadruple \((N, C, \vec{A}, k)\), where \( k \) is a positive integer that satisfies \( k \leq m \), and return a subset \( W \subseteq C \) of size \( k \), which we call the \textit{winning set}, or \textit{committee}.

\textbf{Definition 1 (Justified representation (JR)).} Given a ballot profile \( \vec{A} = (A_1, \ldots , A_n) \) over a candidate set \( C \) and a target committee size \( k \), we say that a set of candidates \( W \) of size \( |W| = k \) satisfies justified representation for \((\vec{A}, k)\) if

\[ \forall X \subseteq N : |X| \geq \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq 1 \implies (|W \cap (\cup_{i \in X} A_i)| \geq 1) \]

The rationale behind this definition is that if \( k \) candidates are to be selected, then, intuitively, each group of \( \frac{n}{k} \) voters “deserves” a representative. Therefore, a set of \( \frac{n}{k} \) voters that have at least one candidate in common should not be completely unrepresented.
Definition 2 (Proportional Justified Representation (PJR)). Given a ballot profile \((A_1, \ldots, A_n)\) over a candidate set \(C\), a target committee size \(k\), \(k \leq m\), and integer \(\ell\) we say that a set of candidates \(W\), \(|W| = k\), satisfies \(\ell\)-proportional justified representation for \((\vec{A}, k)\) if
\[
\forall X \subseteq N : |X| \geq \ell \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq \ell \implies (|W \cap (\cup_{i \in X} A_i)| \geq \ell)
\]
We say that \(W\) satisfies proportional justified representation for \((\vec{A}, k)\) if it satisfies \(\ell\)-proportional justified representation for \((\vec{A}, k)\) and all integers \(\ell \leq k\).

Definition 3 (Extended justified representation (EJR)). Given a ballot profile \((A_1, \ldots, A_n)\) over a candidate set \(C\), a target committee size \(k\), \(k \leq m\), we say that a set of candidates \(W\), \(|W| = k\), satisfies \(\ell\)-extended justified representation for \((\vec{A}, k)\) and integer \(\ell\) if
\[
\forall X \subseteq N : |X| \geq \ell \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq \ell \implies (\exists i \in X : |W \cap A_i| \geq \ell).
\]
We say that \(W\) satisfies extended justified representation for \((\vec{A}, k)\) if it satisfies \(\ell\)-extended justified representation for \((\vec{A}, k)\) and all integers \(\ell \leq k\).

It is easy to observe that EJR implies PJR which implies JR. So any committee that satisfies EJR also satisfies the other two properties.

3. PAV-score and Swaps

The PAV-score of a voter \(i\) for a committee \(W\) is
\[
H(|W \cap A_i|)
\]
where
\[
H(p) = \begin{cases} 
0, & \text{for } p = 0 \\
\sum_{j=1}^{p} \frac{1}{j}, & \text{for } p > 0.
\end{cases}
\]
The PAV-score of a committee \(W \subseteq C\) is defined as
\[
\sum_{i \in N} H(|W \cap A_i|).
\]

The PAV rule that we discussed in the introduction outputs a set \(W \subseteq C\) of size \(k\) with the highest PAV-score.

We say that a committee \(W\) such that \(|W| = k\) satisfies PAV-swap-freeness if there exists no \(c' \in W, c \in C \setminus W\) s.t. PAV-score\((W \setminus \{c'\}) \cup \{c\}\) > PAV-score\((W)\). Note that if a committee \(W\) has the highest possible PAV-score, it satisfies PAV-swap-freeness.

We now define marginal contribution as used in \cite{3}. For each candidate \(w \in W\), we define \(MC(w, W)\) its marginal contribution as the difference between the PAV-score of \(W\) and that of \(W \setminus \{w\}\):
\[
MC(w, W) = PAV\text{-score}((W) - PAV\text{-score}((W \setminus \{w\})).
\]
Let $MC(W)$ denote the sum of marginal contributions of all candidates in $W$:

$$MC(W) = \sum_{w \in W} MC(w, W).$$

We now formally state as a lemma an observation that was already made in [3].

**Lemma 1.** For any committee $W$ such that $|W| = k$, $\sum_{c \in W} MC(c, W) \leq \{|i \in N : A_i \cap W \neq \emptyset\}$. Moreover there exists at least one $c \in W$ such that $MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}|/k \leq n/k$.

**Proof.** Pick a voter $i \in N$, and let $j = |A_i \cap W|$. If $j > 0$, this voter contributes exactly $\frac{1}{j}$ to the marginal contribution of each candidate in $A_i \cap W$, and hence her contribution to $MC(W)$ is exactly 1. If $j = 0$, this voter does not contribute to $MC(W)$ at all. Therefore, we have $MC(W) = \sum_{c \in W} MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}| \leq n$. Since there are exactly $k$ candidates, there exists some $c \in W$ such that $MC(c, W) \leq |\{i \in N : A_i \cap W \neq \emptyset\}|/k \leq n/k$. \(\square\)

We now prove that if a committee satisfies $PAV$-swap-freeness, then it satisfies $EJR$. The argument is almost identical to the argument that the outcome of $PAV$ satisfies $EJR$ [3]. However, we reproduce it just for the sake of completeness because we will further refine this argument.

**Lemma 2.** If a committee satisfies $PAV$-swap-freeness, then it satisfies $EJR$.

**Proof.** Suppose that there is a committee $W$ such that $|W| = k$ that satisfies $PAV$-swap-freeness but violates $EJR$. Since $W$ violates $EJR$, there is a value of $\ell \geq 1$ and a set of voters $N^*$, $|N^*| = s \geq \ell \cdot \frac{n}{k}$. We know that at least one of the $\ell$ candidates approved by all voters in $N^*$ is not elected; let $c$ be some such candidate. Each voter in $N^*$ has at most $\ell - 1$ representatives in $W$, so the marginal contribution of $c$ (if it were to be added to $W$) would be at least $s \cdot \frac{1}{\ell} \geq \frac{n}{k}$. On the other hand, by Lemma 1, we have $\sum_{c \in W} MC(c, W) \leq n$.

Now, consider some candidate $w \in W$ with the smallest marginal contribution; clearly, his marginal contribution is at most $\frac{n}{k}$. If it is strictly less than $\frac{n}{k}$, we are done, as we can improve the total $PAV$-score by swapping $w$ and $c$, a contradiction.

Therefore suppose it is exactly $\frac{n}{k}$, and therefore the marginal contribution of each candidate in $W$ is exactly $\frac{n}{k}$. We know that $A_i \cap W \neq \emptyset$ for each $i \in N^*$, because otherwise $\sum_{w \in W} MC(w', W) \leq n - 1$ (by Lemma 1) which implies that the marginal contribution of $w$ is less than $\frac{n}{k}$. Hence we know that $A_i \cap W \neq \emptyset$ for all $i \in N^*$. Pick some candidate $w' \in W \cap A_i$ for some $i \in N^*$, and set $W' = (W \setminus \{w'\}) \cup \{c\}$. Observe that after $w'$ is removed, adding $c$ increases the total $PAV$-score by at least

$$(s - 1) \cdot \frac{1}{\ell} + \frac{1}{\ell - 1} > \frac{s}{\ell} \geq n/k.$$

Thus, the $PAV$-score of $W'$ is higher than that of $W$, a contradiction again. \(\square\)
Although PAV-swap-freeness is a much weaker property than maximizing total PAV-score, it is surprising that it already implies EJR. In the next section, this insight helps us to come up with useful algorithms.

4. MaxSwapPAV

Based on PAV-score improving swaps, one can formulate the following algorithm called SwapPAV.

SwapPAV: Start from a random committee of size \( k \). Keep implementing swaps that increase the total PAV-score of the committee while such a swap is possible. Return the committee if no more improving swaps are possible.

Our first observation is that Swap-PAV always terminates. The reason is that each time we implement the swap, the PAV-score of the committee increases. This can only happen finitely often as \( PAV-score((W) \leq nH(k) \leq n(ln k + 1) \) for any committee \( W \) of size \( k \). In fact, one can easily prove that with each improving swap, the PAV-score increases by at least \( 1/k^3 \), so that the total number of swaps cannot exceed \( n(ln k + 1)k! \). This observation already gives us the first FPT algorithm for finding a committee satisfying EJR.

We now show how we can modify SwapPAV to find a committee satisfying EJR in polynomial time. We modify SwapPAV as follows. If \( W \) is not PAV-swap-free, then we look at all possible swaps and only implement the swap which makes biggest difference to the PAV-score. We impose an extra condition that we only swap if the improvement in the total PAV-score is at least \( 1/2k^3 \). The algorithm is specified as Algorithm 1 (MaxSwapPAV).

\[
\text{Algorithm 1 MaxSwapPAV}
\]
\[
\text{Require: } (N, \bar{A}, k).
\]
\[
\text{Ensure: } W
\]
\[
1: \quad W \leftarrow \text{any committee of size } k.
2: \quad \text{while } \exists \ c, c' \in C \setminus W \text{ s.t. } PAV-score((W \setminus \{c'\}) \cup \{c\}) - PAV-score(W) \geq 1/2k^3 \text{ do}
3: \quad \text{for each } c', c \in C \setminus W \text{ do}
4: \quad \text{diff}(c, c') = PAV-score((W \setminus \{c'\}) \cup \{c\}) - PAV-score(W)
5: \quad \text{end for}
6: \quad \text{Find } c' \in W, c \in C \setminus W \text{ with the maximum } \text{diff}(c, c').
7: \quad W \leftarrow (W \setminus \{c'\}) \cup \{c\}
8: \quad \text{end while}
9: \quad \text{return } W.
\]

We now argue why MaxSwapPAV returns a committee satisfying EJR and it terminates in polynomial time. The most crucial lemma for both statements is Lemma 3. The lemma is stronger than Lemma 2 and requires a more careful analysis.
Lemma 3. Suppose that $W$ does not satisfy EJR, then there exist $c' \in W, c \in C \setminus W$ s.t. $\text{PAV-score}(W \setminus \{c'\}) \cup \{c\}) - \text{PAV-score}(W) \geq 1/2k^3$.

Proof. Since $W$ violates EJR, there is a value of $\ell \geq 1$ and a set of voters $N^*$, $|N^*| = s \geq \ell \cdot \frac{n}{k}$. We know that at least one of the $\ell$ candidates approved by all voters in $N^*$ is not elected; let $c$ be some such candidate. Each voter in $N^*$ has at most $\ell - 1$ representatives in $W$, so the marginal contribution of $c$ (if it were to be added to $W$) would be at least $s \cdot \frac{1}{\ell} \geq \frac{n}{k}$.

Let $w \in W$ be the candidate with smallest marginal contribution. By Lemma 1, $\text{MC}(w, W) \leq \frac{n}{k}$. If $\text{MC}(w, W) \leq \frac{n}{k} - \frac{1}{2k^3}$, then replacing $w$ with $c$ results in a committee $W'$ with PAV-score increasing by at least $\frac{1}{2k^3}$, and so we are done.

So, we assume that $\text{MC}(w, W) > \frac{n}{k} - \frac{1}{2k^3}$. This implies that $\text{MC}(w', W) > \frac{n}{k} - \frac{1}{2k^3}$ for every $w' \in W$. Since $\sum_{w' \in W} \text{MC}(w', W) \leq n$, we use this fact to find the maximum possible marginal contribution among all candidates in $W$.

Let the maximum marginal contribution be $\text{MC}(b, W)$ of candidate $b$. In that case we know that

$$\sum_{w' \in W} \text{MC}(w', W) \leq n$$
$$\iff \sum_{w' \in W \setminus \{b\}} \text{MC}(w', W) + \text{MC}(b, W) \leq n$$
$$\iff \text{MC}(b, W) \leq n - \sum_{w' \in W \setminus \{b\}} \text{MC}(w', W)$$
$$\iff \text{MC}(b, W) < n - (k - 1)\left(\frac{n}{k} - \frac{1}{2k^3}\right)$$
$$\iff \text{MC}(b, W) < \left(\frac{2nk^3 - 2nk^3 + 2nk^2 + k - 1}{2k^3}\right)$$
$$\iff \text{MC}(b, W) < \frac{n}{k} - \frac{1}{2k^2} - \frac{1}{2k^3} < \frac{n}{k} + \frac{1}{2k^2}$$

Hence, it follows that $\text{MC}(w', W) < \frac{n}{k} + \frac{1}{2k^2}$ for every $w' \in W$.

We now claim that there is a candidate $w' \in W$ that is also in $\bigcup_{i \in N^*} A_i$. Suppose not. This means that no one in $N^*$ approves of anybody in $W$ and so by Lemma 1, $\sum_{w' \in W} \text{MC}(w', W) \leq |N \setminus N^*| \leq n - \frac{n}{k}$. Thus, $\text{MC}(w, W) \leq \frac{n}{k} - \frac{1}{2k^3} < \frac{n}{k} - \frac{1}{2k^3}$, contradicting our assumption that $\text{MC}(w, W) > \frac{n}{k} - \frac{1}{2k^3}$.

Now pick any $w' \in W \cap \bigcup_{i \in N^*} A_i$. As $w \in \bigcup_{i \in N^*} A_i$, this implies that $|A_i \cap (W \setminus \{w'\})| \leq \ell - 2$ for some $i \in N^*$. Hence, $\text{MC}(c, (W \setminus \{w'\}) \cup \{c\})$ would be at least $\frac{n}{k} + \frac{1}{2k} - \frac{1}{k} \geq \frac{n}{k} + \frac{1}{2k}$. Therefore, replacing $w'$ with $c$ results in a committee $W'$ with PAV-score increasing by at least $\frac{1}{2k^3} \geq \frac{1}{2k^3}$. This proves the lemma.

Lemma 3 is the foundation for proving the main properties of the MaxSwapPAV algorithm.

Proposition 1. MaxSwapPAV returns a committee that satisfies EJR.
Proof. MaxSwapPAV returns a committee $W$ such that there exist no $c' \in W$ and $c \in C \setminus W$ s.t. $PAV-score((W \setminus \{c'\}) \cup \{c\}) - PAV-score(W) \geq 1/2k^3$. By Lemma 3 such a committee satisfies EJR.

Proposition 2. MaxSwapPAV runs in polynomial time $O(n^2 mk^4 \ln k)$.

Proof. We first show that the total number of swaps in MaxSwapPAV cannot exceed $2n(ln k + 1)k^3$. In Lemma 4 we proved that each swap in the algorithm improves the $PAV$-score by at least $\frac{1}{2k^3}$. Since the $PAV$ score of any committee cannot exceed $n(ln k + 1)$, there can be at most $2n(ln k + 1)k^3$ swaps. Each swap requires examining $O(km)$ pairs of candidates. For each pair, we need to make $O(n)$ operations.

It follows from the two propositions above that MaxSwapPAV returns a committee satisfying EJR in polynomial time.

5. Discussion

To conclude, we presented the first polynomial-time algorithm for finding a committee that satisfies EJR. In Table 1 we summarize the justified representation related properties satisfied by different polynomial-time algorithms in the literature.

| Rules                   | $JR$ | $PJR$ | $EJR$ |
|-------------------------|------|-------|-------|
| MaxSwapPAV (this paper) | ✓    | ✓     | ✓     |
| SeqPhragmén [6, 11]     | ✓    | ✓     | ✗     |
| Open DHondt (ODH) [14]  | ✓    | ✓     | ✗     |
| GreedyAV [3, 16]        | ✓    | ✗     | ✓     |
| SeqPAV [16]             | ✗    | ✗     | ✗     |

Table 1: Related and known polynomial-time algorithms for approval-based committee voting.

Our result shows that EJR is as amenable to efficient computation as PJR. Depending on particular specifications, our algorithm to find a committee satisfying EJR can also be used to formulate particular voting rules.

Skowron et al. [15] mentioned that SeqPAV can be seen as a desirable approximation algorithm for PAV. Our alternative approach of allowing exchanges rather than sequentially building a committee seems to be closer to one of the defining features of PAV that it satisfies EJR.

The approach of implementing swaps of candidates also makes it possible to move towards fairer representation from a default committee without having to disband the whole committee. The swapping procedure can also be used as post-processing step after running any other committee rule. If the initial committee is the outcome of SeqPAV, then we know that the committee already
guarantees at least \( (1 - \frac{1}{e}) \) of the maximum possible \( PAV \)-score \(^{15}\). Hence it follows that subsequent \( PAV \)-score improving swaps can only further increase the score.

Our algorithmic result also adds a new talking point to the debate between the harmonic scoring approach of Thiele versus the load balancing approach of Phragmén that started over a hundred years ago \(^8\). Note that Brill et al. \(^6\) showed that SeqPhragmén—one of the efficient algorithms within Phragmén’s framework of multi-winner rules—satisfies \( PJR \). On the other hand, SeqPAV the well-known polynomial-time algorithm using Thiele’s approach of harmonic weights does not even satisfy \( JR \). However, we have shown that by allowing swaps of candidates, one can satisfy \( EJR \) which is stronger than \( PJR \).

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