Weierstraß type representation of timelike surfaces with constant mean curvature

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Abstract

We derive a correspondence between (Lorentzian) harmonic maps into the pseudosphere $S^2_1$, with appropriate regularity conditions, and certain connection 1-forms. To these harmonic maps, we associate a representation of type Weierstrass, and we apply it to construct timelike surfaces with constant mean curvature.

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Introduction

As is well known, every solution to the elliptic sinh-Gordon (or sinh-Laplace) equation:

\[ \omega_{zz} + \sinh \omega = 0 \]

describes a constant mean curvature (CMC) surface in Euclidean 3-space \( \mathbb{E}^3 \). The symmetric quadratic form \( I = e^\omega dzd\bar{z} \) is the induced metric of the CMC surface. With respect to the conformal structure determined by \( I \), the Gauß map is a harmonic map into \( S^2 \).

F. Pedit, H. Wu and the first named author established a loop group theoretic Weierstraß-type representation for harmonic maps of Riemann surfaces into compact Riemannian symmetric spaces \[13\]. This loop group theoretic Weierstraß-type representation is frequently referred to in the literature as the \textit{DPW-method}.

Furthermore G. Haak and the first named author used the DPW-method intensively for the investigation of CMC surfaces \[11\]. H. Wu gave a simple way for determining a normal form of the Weierstraß data for CMC surfaces in \[37\]. In \[12\], Pedit, the first and third named author gave a reinterpretation of the classical Weierstraß representation of minimal surfaces in terms of the DPW-method.

On the other hand, every solution \( \phi \) to the sine-Gordon equation:

\[ \phi_{xy} + \sin \phi = 0. \]

describes a (weakly regular) pseudospherical surface in Euclidean 3-space \( \mathbb{E}^3 \). The solution \( \phi \) is the angle of two asymptotic directions. The quadratic form \( II = \sin \phi dxdy \) is the second fundamental form of the surface. With respect to the Lorentzian conformal structure determined by \( II \), the Gauß map is a (Lorentzian) harmonic map into the unit 2-sphere \( S^2 \).

M. Melko and I. Sterling \[27\]–\[28\] presented a modern approach to pseudospherical surfaces via the theory of finite type (Lorentzian) harmonic maps into \( S^2 \).

Recently the third named author established a Weierstraß-type representation for pseudospherical surfaces in \( \mathbb{E}^3 \) in her thesis \[35\]. In particular, she showed that there exist certain normalized potentials for pseudospherical surfaces. Finally, in \[14\] it was shown that finite type pseudospherical surfaces can be constructed from particularly simple normalized potentials.
Both, the sine-Gordon and the elliptic sinh-Gordon equation are real forms of the complexified sine-Gordon equation. But the complexified sine-Gordon equation has still different real forms: the hyperbolic sinh-Gordon and the hyperbolic cosh-Gordon equations. Solutions to these equations do not describe CMC surfaces in $\mathbb{E}^3$ but in Minkowski 3-space $\mathbb{E}^3_1$.

In fact, let $M$ be a timelike CMC surface in $\mathbb{E}^3_1$ parametrized locally by null coordinates $(x, y)$. Denote by $D$ the discriminant of the characteristic equation for the shape operator. Then the Gauß-Codazzi equations of $M$ become

$$\omega_{xy} + H^2 \sinh \omega = 0, \text{ if } D > 0,$$
$$\omega_{xy} + \frac{H^2}{2} e^\omega = 0, \text{ if } D = 0,$$
$$\omega_{xy} + H^2 \cosh \omega = 0, \text{ if } D < 0.$$ 

At this point we would like to point out a certain similarity between timelike CMC surfaces and CMC surfaces in hyperbolic 3-space. Note that both $H^3$ and $\mathbb{E}^3_1$ are naturally imbedded in Minkowski 4-space $\mathbb{E}^4_1$.

Let $M$ be a CMC surface in hyperbolic 3-space $H^3$ parametrized by isothermic coordinates (isothermal-curvature line coordinates) $z = x + \sqrt{-1} y$, then its Gauß-Codazzi equations become

$$\omega_{z\bar{z}} + (H^2 - 1) \sinh \omega = 0, \text{ if } H^2 > 1,$$
$$\omega_{z\bar{z}} - \frac{1}{2} e^{-\omega} = 0, \text{ if } H^2 = 1,$$
$$\omega_{z\bar{z}} + (H^2 - 1) \cosh \omega = 0, \text{ if } H^2 < 1.$$ 

Therefore, at the level of Gauß-Codazzi equations, timelike CMC surface geometry can be considered as a “hyperbolic version” of CMC surface geometry in $H^3$. There is another similarity between timelike surface geometry in $\mathbb{E}^3_1$ and surface geometry in $H^3$. In fact, timelike HIMC surfaces, \textit{i.e.}, timelike surfaces with harmonic inverse mean curvature in $\mathbb{E}^3_1$ \[15\] and Bonnet surfaces in $H^3$, (which are not Willmore surfaces) \[7\] are described by the same integrable equation, namely the \textit{Painlevé equations of type V} and VI.

The hyperbolic sinh-Gordon equation and the Liouville equation have been studied extensively by the soliton theoretic approach. For instance, Babelon and Bernard \[1\] studied (the infinitesimal version of) the dressing transformations for the hyperbolic sinh-Gordon equation. L. McNertney
studied Bäcklund transformations for the hyperbolic sinh-Gordon and the
Liouville equation via the classical theory of line-congruences in her thesis
[26]. H.-S. Hu [22] and the second named author [24] gave a Darboux form of
Bäcklund transformations on the hyperbolic sinh-Gordon equation. Namely,
in [22] and [24], Bäcklund transformations are reformulated as transformations on extended framings.

However as far as the authors know, only few studies on the cosh-Gordon
equation are available. M. V. Babich obtained finite-gap solutions to the
elliptic cosh-Gordon equation [2]. Babich and A. I. Bobenko studied minimal
surfaces in $H^3$ in terms of finite-gap solutions of the elliptic cosh-Gordon
equation [3]. V. Y. Novokshenov studied radial-symmetric solutions to the
elliptic cosh-Gordon equation [31]. The radial-symmetry reduces the elliptic
cosh-Gordon equation to the third Painlevé equation. Moreover he studied
the minimal surfaces in $H^3$ corresponding to these solutions to the elliptic
cosh-Gordon equation [30].

In this paper we establish a Weierstraß-type representation for timelike
CMC surfaces in Minkowski 3-space. The Weierstraß-type representation
gives a unified theory of constructing solutions to the sinh-Gordon, the Liou-
ville and the cosh-Gordon equations. Moreover our Weierstraß-type represen-
tation is regarded as nonlinear d’Alembert formula for these three nonlinear
wave equations.

This paper is organized as follows:

After establishing the requisite facts on geometry of surfaces in Minkowski
3-space in Section 1, we devote Section 2 to prepare ingredients from loop
group theory.

In Section 3, we derive a correspondence between harmonic maps into the
pseudosphere $S^2_1$ with appropriate regularity and a certain kind of connection
one-forms.

The Weierstraß-type representation for (Lorentzian) harmonic maps into
$S^2_1$ is introduced in Section 4. We apply the Weierstraß-type representation
for constructing timelike CMC surfaces in Section 5.

In the final section, we discuss fundamental examples of timelike CMC
surfaces via the Weierstraß-type representation.
1 Timelike surfaces

1.1 We start with preliminaries on the geometry of timelike surfaces in Minkowski 3-space.

Let $\mathbb{E}_1^3$ be Minkowski 3-space with Lorentzian metric $\langle \cdot, \cdot \rangle$. The metric $\langle \cdot, \cdot \rangle$ is expressed as $\langle \cdot, \cdot \rangle = -du_1^2 + du_2^2 + du_3^2$ in terms of the natural coordinate system $(u_1, u_2, u_3)$ of the Cartesian 3-space $\mathbb{R}^3$.

Let $M$ be a connected orientable 2-manifold and $\varphi : M \to \mathbb{E}_1^3$ an immersion. The immersion $\varphi$ is said to be timelike if the induced metric $I$ of $M$ is Lorentzian. The induced Lorentzian metric $I$ determines a Lorentz conformal structure $\mathcal{C}_I$ on $M$. We treat $(M, \mathcal{C}_I)$ as a Lorentz surface and $\varphi$ as a conformal immersion. For the general theory of Lorentz surfaces, we refer to T. Weinstein [36].

Hereafter we will assume that $M$ is an orientable timelike surface in $\mathbb{E}_1^3$ (immersed by $\varphi$).

It is worthwhile to remark that there exists no compact timelike surface in $\mathbb{E}_1^3$. (See B. O’Neill [32], p. 125.)

Let $(x, y)$ be a null coordinate system with respect to the conformal structure $\mathcal{C}_I$. Then the first fundamental form $I$ is written in terms of $(x, y)$ as follows:

$$ I = e^{\omega} \, dx \, dy. $$

Now let $N$ be a unit normal vector field of $M$. Namely a vector field $N$ along $M$ satisfying

$$ \langle N, N \rangle = 1, \quad \langle \varphi_x, N \rangle = \langle \varphi_y, N \rangle = 0. $$

The second fundamental form $II$ of $M$ derived from $N$ is defined by

$$ II = -\langle d\varphi, \, dN \rangle. $$

The shape operator $S$ of $M$ derived from $N$ is

$$ S := -dN. $$

The shape operator $S$ is related to $II$ by

$$ II(X,Y) = \langle SX, Y \rangle $$
for all vector fields $X$, $Y$ on $M$. The mean curvature $H$ of $M$ is defined by

$$H = \frac{1}{2} \text{tr} S.$$ 

Note that $H$ is computed by the following formula:

$$H = \frac{1}{2} \text{tr}(II \cdot I^{-1}).$$ 

Note that the Gaußian curvature $K$ of $M$ is computed as

$$K = \det S = \det(II \cdot I^{-1}).$$

(See [32], p. 107.) The characteristic values of $S$, i.e., the (complex) solutions to

$$\det(tI - S) = 0, \quad I = \text{identity of } TM,$$

are called the principal curvatures. Since the metric $I$ is indefinite, both principal curvatures may be non real complex numbers. It is easy to check from the definitions that $H$ is the mean of the two principal curvatures and $K$ is the product of the two principal curvatures.

A point $p$ of $M$ is said to be an umbilic point if $II$ is proportional to $I$ at $p$. Equivalently, $p$ is an umbilic point if and only if the two principal curvatures at $p$ are the same real number and the corresponding eigenspace is 2-dimensional.

A timelike surface is said to be a totally umbilic surface if all the points are umbilical.

It is known that every totally umbilic timelike surface in $\mathbb{E}_1^3$ is congruent to an open portion of a pseudosphere

$$S^2_1(r) := \{ u \in \mathbb{E}_1^3 \mid \langle u, u \rangle = r^2 \}$$

of radius $r > 0$ or a timelike plane.

Let $\varphi : M \to \mathbb{E}_1^3$ be a timelike surface with unit normal vector field $N$ as before. Then, on a simply connected null coordinate region $\mathbb{D}$, we can define an orthonormal frame field $\mathcal{F}$ defined by

$$\mathcal{F} = (e^{-\omega/2}(-\varphi_x + \varphi_y), e^{-\omega/2}(\varphi_x + \varphi_y), N) : \mathbb{D} \to O_1^{++}(3),$$

where $O_1^{++}(3)$ denotes the identity component of the Lorentz group

$$O_1(3) = \{ A \in \text{GL}(3; \mathbb{R}) \mid \langle Au, Av \rangle = \langle u, v \rangle, \ u, v \in \mathbb{E}_1^3 \}.$$
Throughout this paper, we identify $E_3^1$ with the Lie algebra $g = \mathfrak{sl}(2; \mathbb{R})$. We take the following basis \(\{i, j', k'\}\) of $\mathfrak{sl}(2; \mathbb{R})$:

\[
i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The basis $\{i, j', k'\}$ satisfies the following relation:

\[
i^2 = -1, \quad j'^2 = k'^2 = 1, \quad ij' = -j'i = k', \quad j'k' = -k'j' = -i, \quad k'i = -ik' = j'.
\]

Here 1 denotes the identity matrix:

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Hereafter we identify $E_3^1$ with $g$ via this basis.

\[
(u_1, u_2, u_3) \rightarrow u_1 i + u_2 j' + u_3 k'.
\] (1.1)

The real algebra $\mathbb{H}'$ generated by $\{1, i, j', k'\}$ is called the algebra of *split quaternions*. The algebra $\mathbb{H}'$ is isomorphic to the algebra $M(2; \mathbb{R})$ of all 2 by 2 real matrices. The commutation relations of $g$ are given by

\[
[i, j'] = 2k', \quad [j', k'] = -2i, \quad [k', i] = 2j'.
\]

By the linear isomorphism (1.1) the Lorentz metric $\langle \cdot, \cdot \rangle$ corresponds to the scalar product:

\[
\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY), \quad X, Y \in g.
\] (1.2)

This scalar product induces a biinvariant Lorentz metric of constant curvature $-1$ on the special linear group $G = \text{SL}(2; \mathbb{R})$. Hence $G$ is identified with the anti-de Sitter 3-space $H_3^1$. (See [32].)

The special linear group $G$ acts isometrically on $g$ via the Ad-action:

\[
\text{Ad} : G \times g \rightarrow g; \quad \text{Ad}(g)X = gXg^{-1}.
\]
The Ad-action induces a double covering $G \to O_1^{++}(3)$ of the Lorentz group $O_1^{++}(3)$.

By using this double covering we can find a lift $\hat{\Phi}$ (called a coordinate frame) of $\mathcal{F}$ to $SL(2;\mathbb{R})$:

$$\text{Ad}(\hat{\Phi})(i,j',k') = \mathcal{F}.$$  \hspace{1cm} (1.3)

The coordinate frame $\hat{\Phi}$ satisfies the following Frenet (or Gauss-Weingarten) equations [23]:

$$\frac{\partial}{\partial x} \hat{\Phi} = \hat{U}, \quad \frac{\partial}{\partial y} \hat{\Phi} = \hat{V},$$  \hspace{1cm} (1.4)

where

$$\hat{U} = \begin{pmatrix} -\frac{1}{2} \omega_x & -Qe^{-\frac{\omega}{2}} \\ \frac{H}{2} e^{\frac{\omega}{2}} & \frac{1}{4} \omega_x \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} \frac{1}{4} \omega_y & -\frac{H}{2} e^{\frac{\omega}{2}} \\ Re^{\frac{\omega}{2}} & -\frac{1}{4} \omega_y \end{pmatrix},$$  \hspace{1cm} (1.5)

and $Q := \langle \varphi_{xx}, N \rangle$, $R := \langle \varphi_{yy}, N \rangle$, $H = 2e^{-\omega} \langle \varphi_{xy}, N \rangle$.

The function $H$ coincides with the mean curvature of $\varphi$. It is easy to see that $Q^\# := Qdx^2$ and $R^\# := Rd\Omega^2$ are globally defined on $M$. The quadratic differentials $Q^\#$ and $R^\#$ are called the Hopf differentials of $M$.

The second fundamental form $II$ of $M$ is related to $Q$ and $R$ by

$$II = Q^\# + R^\# + HI.$$  

This formula implies that the common zeros of $Q$ and $R$ coincide with the umbilic points of $M$.

The Gauss equation which describes a relation between $K$, $H$ and $Q$ takes the following form:

$$H^2 - K = 4e^{-2\omega}QR.$$  

Note that the condition $QR = 0$ does not imply the umbilicity of $M$. (See T. K. Milnor [29]).

The Gauss-Codazzi equation, i.e., the integrability condition of the Frenet equations,

$$\hat{V}_x - \hat{U}_y + [\hat{U}, \hat{V}] = 0$$

has the following form:

$$\omega_{xy} + \frac{1}{2} H^2 e^{\omega} - 2QRe^{-\omega} = 0,$$  \hspace{1cm} (G)
\[ H_x = 2e^{-ω}Q_y, \quad H_y = 2e^{-ω}R_x. \]  \hfill (C)

The Codazzi equations (C) show that the constancy of the mean curvature \( H \) is equivalent to the condition \( Q_y = R_x = 0 \), i.e., \( Q = Q(x), \quad R = R(y) \).

**Remark 1.1** Let \((M, C)\) be a Lorentz surface and \((x, y)\) a null coordinate system. Then the following differential operators are well defined:

\[
d' := \frac{∂}{∂x}dx, \quad d'' := \frac{∂}{∂y}dy. \quad (1.6)
\]

A function \( f : M \to \mathbb{R} \) is said to be a **Lorentz holomorphic function** [resp. **Lorentz anti-holomorphic function**] if \( d''f = 0 \) [resp. \( d'f = 0 \)].

Next a 1-form \( A = A_x dx + A_y dy \) is said to be a **Lorentz holomorphic 1-form** if \( A = A_x dx \) and \( A_x \) is a Lorentz holomorphic function. Similarly \( A \) is said to be a **Lorentz anti-holomorphic 1-form** if \( A = A_y dy \) and \( A_y \) is a Lorentz anti-holomorphic function.

According to these terminologies, the constancy of mean curvature is characterized as follows:

**Let** \( \varphi : M \to \mathbb{E}_1^3 \) **be a timelike surface. Then \((M, \varphi)\) is of constant mean curvature if and only if \( Q \) is a Lorentz holomorphic function and \( R \) is a Lorentz anti-holomorphic function.**

**Remark 1.2** Let \((M, C)\) be a Lorentz surface and \((x, y), \ (\tilde{x}, \tilde{y})\) null coordinate systems. Then these two coordinate systems are related by

\[
\frac{∂\tilde{x}}{∂y} = 0, \quad \frac{∂\tilde{y}}{∂x} = 0.
\]

Namely \( \tilde{x} \) and \( \tilde{y} \) depends only on \( x \) and \( y \) respectively.

On timelike surfaces of constant mean curvature \( H \geq 0 \), a special (local) coordinate system is available (\cite{15, 21, 29, 36}).

**Proposition 1.3** Let \( \varphi : M \to \mathbb{E}_1^3 \) **be a timelike surface of constant mean curvature \( H \neq 0 \). Assume that \((M, \varphi)\) has real distinct principal curvatures. Then there exists a local coordinate system \((x, y)\) such that

\[
I = e^ω dx dy, \quad II = \frac{H}{2} \left\{ dx^2 + 2e^ω dx dy + dy^2 \right\}. \quad (1.7)
\]
With respect to this local coordinate system, the Gauss-Codazzi equation become
\[ \omega_{xy} + H^2 \sinh \omega = 0. \]  

(shG)

The partial differential equation (shG) is called the hyperbolic sinh-Gordon equation or affine Toda field equation of type \( A^{(1)}_1 \).

**Remark 1.4** Let \((x, y)\) be the local coordinate system in the preceeding Proposition. Introduce a local coordinate system \((u, v)\) by \(x = u + v, \ y = -u + v\). Then \(I\) and \(II\) are represented as
\[
I = e^\omega (-du^2 + dv^2), \quad II = 2He^\omega \left( -\sinh \frac{\omega}{2} du^2 + \cosh \frac{\omega}{2} dv^2 \right).
\]

The local coordinate system \((u, v)\) is (Lorentz) isothermal and a curvature-line coordinate system. Such a coordinate system \((u, v)\) is called an isothermic coordinate system.

Timelike CMC surfaces with real distinct principal curvatures are called isothermic timelike CMC surfaces. Note that an isothermic coordinate system is characterized as a local null coordinate system \((x, y)\) such that \(Q = R \neq 0\). See [15].

A very different situation is discussed in the

**Proposition 1.5** Let \(\varphi : M \rightarrow \mathbb{E}^3_1\) be a timelike surface of constant mean curvature \(H \neq 0\). Assume that \((M, \varphi)\) has imaginary principal curvatures. Then there exists a local coordinate system \((x, y)\) such that
\[
I = e^\omega dxdy, \quad II = \frac{H}{2} \left\{ dx^2 + 2e^\omega dxdy - dy^2 \right\}.
\]

(1.8)

With respect to this local coordinate system, the Gauss-Codazzi equations become
\[ \omega_{xy} + H^2 \cosh \omega = 0. \]  

(chG)

The partial differential equation (chG) is called the hyperbolic cosh-Gordon equation.

The local coordinate system \((x, y)\) is called an anti-isothermic coordinate system. The anti-isothermic coordinate system is characterized as a local null coordinate system such that \(Q = -R \neq 0\).

Timelike CMC surfaces with imaginary principal curvatures are called anti-isothermic timelike CMC surfaces. The notion of “anti-isothermic coordinate” has been introduced by [15] (Definition 4.15).
Remark 1.6 Let \( \varphi : M \to \mathbb{E}^3_1 \) be a timelike surface of constant mean curvature \( H \). Assume that \((M, \varphi)\) has two equal and real principal curvatures. Then the Gauss equation of \((M, \varphi)\) becomes the Liouville equation:

\[
\omega_{xy} + \frac{H^2}{2} e^\omega = 0. \tag{L}
\]

1.3 Next, we shall define the Gauß map of a timelike surface. Let \( M \) be a timelike surface and \( N \) a unit normal vector field to \( M \). The Gauß map \( \psi \) of \( M \) is a smooth map of \( M \) into \( S^2_1 \), which assigns to each \( p \in M \), the point \( \psi(p) \in \mathbb{E}^3_1 \) obtained by parallel translation of the unit normal vector \( N_p \) of \( M \) at \( p \) to the origin of \( \mathbb{E}^3_1 \).

The constancy of the mean curvature is characterized by the harmonicity of the Gauß map (cf. [29]).

Proposition 1.7 The Gauss map of a timelike surface is harmonic if and only if the mean curvature is constant.

Remark 1.8 Let \((M_1, g_1)\) and \((M_1, g_2)\) be (semi-) Riemannian manifolds and \( \psi : M_1 \to M_2 \) be a smooth map. Then \( \psi \) is said to be a harmonic map if its tension field \( \tau(\psi) \):

\[
\tau(\psi) := \text{tr}(\nabla d\psi)
\]

vanishes. In case \( \dim M_1 = 2 \), the harmonicity of a smooth map \( \psi \) is invariant under the conformal transformation of \((M_1, g_1)\). In particular, when \((M_1, g_1)\) is a Lorentzian 2-manifold, a harmonic map \( \psi \) is often called a Lorentzian harmonic map.

Note that the constancy of the Gaußian curvature is characterized by the following.

Proposition 1.9 Let \( M \) be a timelike surface. Assume that the Gaußian curvature has a constant sign on \( M \). Then the second fundamental form \( II \) gives \( M \) another (semi-) Riemannian metric. With respect to the conformal structure determined by \( II \), the Gauß map of \( M \) is harmonic if and only if \( K \) is constant.

The Ad-action of \( G \) on \( S^2_1 \) is transitive and isometric. The isotropy subgroup \( K \) of \( G \) at \( k' \) is

\[
K = \{ u_0 \mathbf{1} + u_3 k' \mid u_0^2 - u_3^2 = -1 \}.
\]
The isotropy subgroup \(K\) is isomorphic to the multiplicative group \(\mathbb{R}^*\).

The natural projection \(\pi : G = H^3 \to S^2_1\), given by \(\pi(g) = \text{Ad}(g)k', g \in G\), defines a principal \(\mathbb{R}^*\)-bundle \(H^3_1\) over \(S^2_1\). The fibering \(\pi : H^3_1 \to S^2_1\) is called the Hopf-fibering of \(S^2_1\).

The Lie algebra \(k\) of \(K\) is given by \(k = \mathbb{R}k'\). The tangent space of \(S^2_1\) at the origin \(k'\) is \(m = \mathbb{R}i \oplus \mathbb{R}j'\). Let \(\sigma\) be the involution of \(g\) defined by \(\sigma = \text{Ad}(k') = \Pi_k - \Pi_m\), where \(\Pi_k\) and \(\Pi_m\) are the projections from \(g\) onto \(\mathfrak{k}\) and \(\mathfrak{m}\) respectively. Then the pair \((g, \sigma)\) is a symmetric Lie algebra data associated with the Lorentzian symmetric space \(S^2_1 = G/K\).

2 Loop groups

2.1 To study timelike CMC surfaces in the spirit of [13], we need to introduce some notation involving loop groups. Let us denote the polynomial loop algebra of \(g = \mathfrak{sl}(2; \mathbb{R})\) by \(\Lambda_{\text{pol}} g\):

\[
\Lambda_{\text{pol}} g = \left\{ \xi(\lambda) = \sum_{\text{finite}} \xi_j \lambda^j : S^1 \to g \right\},
\]

(2.1)

where \(S^1\) denotes the unit circle in \(\mathbb{C}\).

Let \(\sigma\) be the involution of \(g\) corresponding to the Lorentzian symmetric space \(S^2_1 = G/K\) defined above. Then the polynomial twisted loop algebra of \(g\) is defined by

\[
\Lambda_{\text{pol}} g_{\sigma} = \left\{ \xi(\lambda) \in \Lambda_{\text{pol}} g \mid \sigma(\xi(\lambda)) = \xi(-\lambda) \right\}.
\]

(2.2)

For the purposes of this paper we need a certain Banach-completion of \(\Lambda_{\text{pol}} g_{\sigma}\). To this end we introduce the norm \(|\cdot|_1\) for \(g\):

\[
|A|_1 := \max_j \left\{ \sum_{i=1}^2 |a_{ij}| \right\}, \quad A = (a_{ij}) \in g.
\]

We extend this norm to the polynomial loop algebra in the following way:

\[
||\xi|| = \sum |\xi_j|_1, \quad \xi = \sum \xi_j \lambda^j \in \Lambda_{\text{pol}} g.
\]

Denote the completion of \(\Lambda_{\text{pol}} g\) and \(\Lambda_{\text{pol}} g_{\sigma}\) with respect to the norm \(||\cdot||\) by \(\Lambda g\) and \(\Lambda g_{\sigma}\) respectively. Then the Lie algebras \(\Lambda g\) and \(\Lambda g_{\sigma}\) are Banach
Lie algebras. (cf. Proposition 4.2.1 in [35].) Actually, these are Banach algebras of continuous functions on $S^1$. Moreover since the involution $\sigma$ is inner, these Banach Lie algebras are isomorphic to each other.

Next we introduce the following Lie subalgebras of $\Lambda g$:

$$\Lambda^+ g = \left\{ \xi(\lambda) = \sum_{j \geq 0} \xi_j \lambda^j \in \Lambda g \right\}, \quad \Lambda^- g = \left\{ \xi(\lambda) = \sum_{j \leq 0} \xi_j \lambda^j \in \Lambda g \right\},$$

$$\Lambda^*_+ g = \left\{ \xi(\lambda) = \sum_{j > 0} \xi_j \lambda^j \in \Lambda g \right\}, \quad \Lambda^*_- g = \left\{ \xi(\lambda) = \sum_{j < 0} \xi_j \lambda^j \in \Lambda g \right\}.$$  (2.3)

Then we have the following decompositions as direct sums of linear spaces:

$$\Lambda g = \Lambda^+_g \oplus \Lambda^- g = \Lambda^*_- g \oplus \Lambda^*_+ g.$$  (2.5)

Similarly, we introduce the following Lie subalgebras of the twisted loop algebra:

$$\Lambda^+_g \sigma = \left\{ \xi(\lambda) = \sum_{j \geq 0} \xi_j \lambda^j \in \Lambda g \sigma \right\}, \quad \Lambda^-_g \sigma = \left\{ \xi(\lambda) = \sum_{j \leq 0} \xi_j \lambda^j \in \Lambda g \sigma \right\},$$

$$\Lambda^*_+ g \sigma = \left\{ \xi(\lambda) = \sum_{j > 0} \xi_j \lambda^j \in \Lambda g \sigma \right\}, \quad \Lambda^*_- g \sigma = \left\{ \xi(\lambda) = \sum_{j < 0} \xi_j \lambda^j \in \Lambda g \sigma \right\}.$$  (2.6)

Then we have the following decompositions as direct sums of linear spaces:

$$\Lambda g \sigma = \Lambda^*_+ g \sigma \oplus \Lambda^- g \sigma = \Lambda^*_- g \sigma \oplus \Lambda^*_+ g \sigma.$$  (2.8)

It is not difficult to see that one can analogously define connected Banach Lie groups: $\Lambda G$, $\Lambda^\pm G$ and $\Lambda^*_\pm G$, whose Lie algebras are $\Lambda g$, $\Lambda^\pm g$ and $\Lambda^*_\pm g$ respectively.

For the twisted case we have the following Banach Lie groups: $\Lambda G_\sigma$, $\Lambda^\pm G_\sigma$ and $\Lambda^*_\pm G_\sigma$, whose Lie algebras are $\Lambda g_\sigma$, $\Lambda^\pm g_\sigma$ and $\Lambda^*_\pm g_\sigma$ respectively.
2.2 In this subsection we recall the classically known Birkhoff decomposition theorem for loop groups of complex special linear group SL(2; C).

We use this result in order to prove that a similar factorization holds for the loop groups \( \tilde{\Lambda}G \), and \( \tilde{\Lambda}G_\sigma \) defined below.

**Theorem 2.1 (Birkhoff decomposition of \( \Lambda G^C \))**

\[
\Lambda G^C = \bigcup_{w \in \mathcal{T}} \Lambda^- G^C \cdot w \cdot \Lambda^+ G^C, \tag{2.9}
\]

Here \( \mathcal{T} \) denotes the group of homomorphisms from \( S^1 \) into the subgroup of diagonal matrices of \( SL(2; C) \), that is,

\[
\mathcal{T} = \left\{ \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^{-a} \end{pmatrix} \middle| a > 0 \right\}.
\]

Moreover, the multiplication maps

\[
\Lambda^- G^C \times \Lambda^+ G^C \to \Lambda G^C, \quad \Lambda^\pm G^C \times \Lambda^- G^C \to \Lambda G^C. \tag{2.10}
\]

are diffeomorphisms onto the open dense subsets \( \mathcal{B}_A^\pm (-, +) \) and \( \mathcal{B}_A^\pm (+, -) \) of \( \Lambda G^C \), called the big cells of \( \Lambda G^C \). In particular if \( \gamma \) is an element of \( \mathcal{B}_A = \mathcal{B}_A^\pm (-, +) \cap \mathcal{B}_A^\pm (+, -) \), then \( \gamma \) has unique decompositions:

\[
\gamma = \gamma_- \cdot \ell_+ = \gamma_+ \cdot \ell_-, \quad \gamma_\pm \in \Lambda^\pm_G^C, \quad \ell_\pm \in \Lambda^\pm_G^C.
\]

Here the subgroups \( \Lambda^\pm_G^C \) are defined by

\[
\Lambda^-_G^C = \left\{ \gamma \in \Lambda^- G^C \mid \gamma(\lambda) = 1 + \sum_{k \leq -1} \gamma_k \lambda^k \right\},
\]

\[
\Lambda^+_G^C = \left\{ \gamma \in \Lambda^+ G^C \mid \gamma(\lambda) = 1 + \sum_{k \geq 1} \gamma_k \lambda^k \right\}.
\]

Next let \( \tilde{\Lambda}G \) be the subset of \( \Lambda G \) whose elements, as maps defined on \( S^1 \), admit analytic continuations to \( \mathbb{C}^* \).

\[
\tilde{\Lambda}G = \left\{ \gamma \in \Lambda G \mid \gamma : S^1 \to G \text{ extends analytically to } \mathbb{C}^* \right\}. \tag{2.11}
\]
Similarly we define
\[ \tilde{\Lambda}G_{\sigma} = \{ \gamma \in \Lambda G_{\sigma} \mid \gamma \text{ extends analytically to } \mathbb{C}^* \}. \tag{2.12} \]

It is easy to check that \( \tilde{\Lambda}G \) is a subgroup of \( \Lambda G \). Similarly, \( \tilde{\Lambda}G_{\sigma} \) is a subgroup of \( \Lambda G_{\sigma} \).

For \( \tilde{\Lambda}G \) we will use the topology induced from \( \Lambda G \). Then we obtain the following theorem.

**Theorem 2.2 (Birkhoff decomposition of \( \tilde{\Lambda}G \))**

\[ \tilde{\Lambda}G = \bigsqcup_{w \in \mathcal{T}} \tilde{\Lambda}^- G \cdot w \cdot \tilde{\Lambda}^+ G. \]

Here \( \mathcal{T} \) denotes the group of homomorphisms from \( S^1 \) into the subgroup of diagonal matrices of \( SL(2; \mathbb{C}) \). Moreover, the multiplication maps

\[ \tilde{\Lambda}^+ G \times \tilde{\Lambda}^+ G \to \tilde{\Lambda}G, \quad \tilde{\Lambda}^- G \times \tilde{\Lambda}^- G \to \tilde{\Lambda}G \]

are diffeomorphism onto the open dense subsets \( \mathcal{B}^o(-, +) \) and \( \mathcal{B}^o(+, -) \) of \( \tilde{\Lambda}G \), called the big cells of \( \tilde{\Lambda}G \). In particular if \( \gamma \) is an element of \( \mathcal{B}^o = \mathcal{B}^o(-, +) \cap \mathcal{B}^o(+, -) \), then \( \gamma \) has unique decompositions:

\[ \gamma = \gamma_- \cdot \ell_+ = \gamma_+ \cdot \ell_-, \quad \gamma_\pm \in \tilde{\Lambda}_\pm G, \quad \ell_\pm \in \tilde{\Lambda}^\pm G. \]

**Proof** The idea of this proof is similar to the one presented in the Appendix of [35].

Let \( g \in \tilde{\Lambda}G \) with expansion \( g(\lambda) = \sum g_j \lambda^j \).

Note that the coefficients \( g_j \) in the expansion of \( g \) are real.

Over the unit circle \( S^1 \), we obtain a decomposition \( g = g_- \cdot w \cdot g_+ \) by the classical Birkhoff Decomposition Theorem 2.2. It remains to show that actually every factor of \( g \) defines an element in \( \tilde{\Lambda}G \).

First we show all factors of \( g \) are in \( \Lambda G \). To see this we introduce the automorphism \( \kappa \) of \( \Lambda G^\mathbb{C} \) which is defined by

\[ \kappa(\gamma)(\lambda) := \sum \tilde{\gamma}_j \lambda^j \]

for every

\[ \gamma(\lambda) = \sum \gamma_j \lambda^j \in \Lambda G^\mathbb{C}. \]

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It is obvious to verify that $\kappa$ leaves $\Lambda^+G^C$ and $\Lambda^-G^C$ invariant and fixes all $w \in \mathcal{T}$. Moreover, $\Lambda G$ is the fixed point set of $\kappa$. Thus $g = \kappa(g) = \kappa(g_-) \cdot w \cdot \kappa(g_+)$. The classical Birkhoff Decomposition Theorem 2.2 now implies $\kappa(g_-) = g_- \cdot v_-$ and $\kappa(g_+) = v_+ \cdot g_+$, where $v_+ \in \Lambda^+G^C$. Moreover, we have $v_- \cdot w = w \cdot v_+^{-1}$.

Applying $\kappa$ again to $\kappa(g_-) = g_- \cdot v_-$ and taking into account that $\kappa$ is an automorphism of order two, it follows that

$$g_- = \kappa(\kappa(g_-)) = \kappa(g_- \cdot v_-) = g_- \cdot v_- \cdot \kappa(v_-).$$

Thus we obtain $\kappa(v_-) = (v_-)^{-1}$.

Analogously, from $\kappa(g_+) = v_+ \cdot g_-$, we obtain $\kappa(v_+) = (v_+)^{-1}$.

If $w = 1$, then we are done. Assume now $w \neq 1$. Then $v_-$ and $v_+$ are lower triangular. Moreover, the equation relating $v_-$ and $v_+$ via $w$ shows that the diagonal part of $v_-$ is independent of $\lambda$. Thus $\kappa(v_-) = (v_-)^{-1}$ shows that the diagonal entries of $v_-$ have modulus 1. Let $d$ denote the diagonal part of $v_-$ and let $\sqrt{d}$ denote its square root. Then we set $g'_- := g_- \cdot (\sqrt{d})^{-1}$ and obtain $\kappa(g'_-) = g'_- \cdot v'_-$, where $v'_-$ is lower triangular with 1's on the diagonal. Writing now $g = g'_- \cdot w \cdot g'_+$, where $g'_+$ is defined by $g'_+ = \sqrt{d} \cdot g_+$, then we see that the corresponding $v'_+$ has 1's on the diagonal. Now we take the square root of $v'_-$ and multiply $g'_-$ by its inverse on the right, obtaining $g''_-$. A straightforward computation shows that $\kappa$ fixes $g''_-$, whence $g''_-$ is in $\Lambda G$. As a consequence, the corresponding $g''_+$ is also in $\Lambda G$.

Next we need to show that $g_-$ and $g_+$ are in the " $\sim$ "-group.

We know from the definition that $g_-$ has a holomorphic extension to the exterior of the unit disk and is finite at $\infty$. Thus $(g_-)^{-1} \cdot g = w \cdot g_+$ has a holomorphic extension to the exterior of the unit disk. Hence $g_+$ also has a holomorphic extension to the exterior of the unit disk. Altogether, $g_+$ has a holomorphic extension to $\mathbb{C}^*$. This proves the first part of the Theorem.

For the second part we note first that the big cells are indeed dense, since the cosets involving $w \neq 1$ are of nonzero codimension in in $\Lambda G$.

The rest of the claim follows by the fact the multiplication maps are induced by the diffeomorphisms $\Lambda_g^{-}G^C \times \Lambda^+G^C \rightarrow \Lambda G^C$ and $\Lambda^+G^C \times \Lambda^-G^C \rightarrow \Lambda G^C$ via the loop group correspondences those we presented in this section. $\square$

Finally, we consider the twisted loop groups defined earlier. We also define the twisted analytic loop groups derived from $\Lambda G_\sigma$ in the obvious way. Then we have
**Theorem 2.3** (Birkhoff decomposition of $\tilde{\Lambda}G_\sigma$) Theorem 2.2 also holds for the twisted groups.

**Proof** We note that the twisting involution $\sigma$ is given by an inner automorphism of $G$. Therefore, the twisted loop group and the untwisted loop group are isomorphic. Actually, the isomorphism from the untwisted loop group to the twisted loop group is given easily: powers $\lambda^k$ on the diagonal are doubled, powers $\lambda^k$ in the $(1,2)$-position are replaced by $\lambda^{2k+1}$ and in the $(2,1)$-position they are replaced by $\lambda^{2k-1}$. The claim now follows. $\blacksquare$

**Remark 2.4** The proof above actually also shows the Birkhoff Decomposition Theorem for $\Lambda G_\sigma$.

**2.3** We have the following fundamental decomposition theorem:

**Theorem 2.5** (Iwasawa decomposition of $\Lambda G_\sigma \times \Lambda G_\sigma$)

Let $\Delta(\Lambda G_\sigma \times \Lambda G_\sigma)$ denote the diagonal subgroup of $\Lambda G_\sigma \times \Lambda G_\sigma$. Then we have

$$\Lambda G_\sigma \times \Lambda G_\sigma = \bigsqcup \Delta(\Lambda G_\sigma \times \Lambda G_\sigma) \cdot (1, w) \cdot (\Lambda^- G_\sigma \times \Lambda^+ G_\sigma),$$

where $w \in T$ is as in Theorem 2.2.

Moreover, the multiplication maps

$$\Delta(\Lambda G_\sigma \times \Lambda G_\sigma) \times (\Lambda^- G_\sigma \times \Lambda^+ G_\sigma) \to \Lambda G_\sigma \times \Lambda G_\sigma,$$

$$\Delta(\Lambda G_\sigma \times \Lambda G_\sigma) \times (\Lambda^+ G_\sigma \times \Lambda^- G_\sigma) \to \Lambda G_\sigma \times \Lambda G_\sigma$$

are diffeomorphisms onto the open dense subsets $I^+ \Lambda_\Lambda(+-)$ and $I^- \Lambda_\Lambda(+-)$ of $\Lambda G_\sigma \times \Lambda G_\sigma$—called the big cells of $\Lambda G_\sigma \times \Lambda G_\sigma$.

**Proof** Take $(g,h) \in \Lambda G_\sigma \times \Lambda G_\sigma$. Decompose $g^{-1} \cdot h$ according to the Birkhoff decomposition of $\Lambda G_\sigma$ (Theorem 2.2):

$$g^{-1} \cdot h = u_-w u_+, \quad u_\pm \in \Lambda G_\sigma, \quad w \in T.$$

It is easy to verify that the splitting

$$(g,h) = (gu_-, gu_-)(1,w)(u_-^{-1},u_+)$$

gives the Iwasawa decomposition of $(g,h)$ in the untwisted loop group $\Lambda G \times \Lambda G$. (cf. Theorem 4.1 in [6].) Since the factors $gu_-$ and $u_\pm$ are $\sigma$-twisted,
this splitting is the required (Iwasawa) splitting in the twisted loop group $\Lambda G_\sigma \times \Lambda G_\sigma$.

For the second claim we note that our definitions imply

$$\Delta(\Lambda G_\sigma \times \Lambda G_\sigma) \cap (\Lambda^- G_\sigma \times \Lambda^+ G_\sigma) = \{1\}.$$  

Thus the splitting is unique, whence the map is a bijection onto its image. The proof that it is a diffeomorphism is almost verbatim the same as for the Birkhoff decomposition. In fact, the proof follows as in [10].

It remains to show that the big cell is dense. But if $(g, h)$ is given, then $g^{-1} \cdot h$ is in $\Lambda G_\sigma \cong \Lambda G$ and from the Birkhoff decomposition Theorem 2.3, we know that in every neighbourhood of this element there is an element in the big cell. Therefore, in every neighbourhood of $g$ and $h$ there exists some $g'$ and $h'$ such that $(g')^{-1} \cdot h'$ is in the big cell of $\Lambda G_\sigma$. But the proof above shows that then $(g', h')$ is in the big cell of $\Lambda G_\sigma \times \Lambda G_\sigma$. □

2.4 For twisted loop groups of elements with analytic extension, we have the following decomposition theorem:

**Theorem 2.6 (Iwasawa decomposition of $\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma$)**

Let $\Delta(\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma)$ denote the diagonal subgroup of $\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma$. Then we have

$$\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma = \bigsqcup \Delta(\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma) \cdot (1, w) \cdot (\tilde{\Lambda}^- G_\sigma \times \tilde{\Lambda}^+ G_\sigma),$$

where $w \in \mathcal{T}$ is as in Theorem 2.2.

Moreover, the multiplication maps

$$\Delta(\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma) \times (\tilde{\Lambda}^- G_\sigma \times \tilde{\Lambda}^+ G_\sigma) \to \tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma,$$

$$\Delta(\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma) \times (\tilde{\Lambda}^+_* G_\sigma \times \tilde{\Lambda}^- G_\sigma) \to \tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma$$

are diffeomorphisms onto the open dense subsets $\mathcal{I}(+, -)$ and $\mathcal{I}(-, +)$ of $\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma$—called the big cells of $\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma$.

**Proof** Take $(g, h) \in \tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma$. Decompose $g^{-1} \cdot h$ according to the Birkhoff decomposition of $\tilde{\Lambda}G_\sigma$ (Theorem 2.2):

$$g^{-1} \cdot h = u_- w u_+, \text{ } u_\pm \in \tilde{\Lambda}G_\sigma, \text{ } w \in \mathcal{T}.$$
As we showed in the proof of Theorem 2.4, the splitting
\[(g, h) = (gu_-, gu_-)(1, w)(u_-)^{-1}, u_+)\]
gives the Iwasawa decomposition of \((g, h)\) in the twisted loop group \(\tilde{\Lambda}G_\sigma \times \tilde{\Lambda}G_\sigma\).

We need to check that \(u_-\) and \(u_+\) are in the \(\sim\)-group. By definition, \(u_-\) has a holomorphic extension to the exterior of the unit disk and is finite at \(\infty\). Thus \(u_+ = (gu_-w)^{-1}h\) has a holomorphic extension to the exterior of the unit disk. Thus \(u_+\) has holomorphic extension to \(\mathbb{C}^*\). Similarly \(u_-\) has also a holomorphic extension to \(\mathbb{C}^*\). This proves the claim. The remaining assertions follow from the previous Theorem, since we use the induced topology. \(\square\)

**Remark 2.7** P. Kellersch [25] generalized the classical Iwasawa decomposition for untwisted loop groups of compact simple Lie groups to those for loop groups of general simple Lie groups. Moreover V. Balan and the first named author [4] generalized the splitting theorem due to Kellersch to those for general Lie groups.

**Remark 2.8** I. T. Gohberg and his collaborators investigated splittings for matrix valued functions over (separate) contours and more general Banach algebras of functions. For instance, the Birkhoff splitting for matrix valued functions with coefficients in the *Wiener algebra*:

\[\mathcal{A} = \left\{ f(\lambda) = \sum f_j \lambda^j : S^1 \to \mathbb{C} \mid \sum |f_j| < \infty \right\}\]

was proven by Gohberg in [17]. But also splittings for matrix valued functions with coefficients in the Wiener algebra on the *real line* were obtained [8]. For more information, we refer to [17] and [8] references therein.

For our geometric purposes—a Weierstraß type representation for timelike surfaces in Minkowski space—, we need *real* Banach algebras of functions. In fact, we need actually a *real* loop parameter \(\lambda\).

Fortunately all the loop group elements occurring in our geometric context (extended framings etc.) have analytic extensions to \(\mathbb{C}^*\), i.e. these geometric loop group elements are all contained in \(\tilde{\Lambda}G\). For this reason we have presented in this section the Birkhoff and Iwasawa decomposition theorems for the twisted analytic loop group \(\tilde{\Lambda}G_\sigma\).
3 Harmonic maps into $S_1^2$

3.1 In this section we shall derive a correspondence between harmonic maps from a simply connected Lorentz surface $D$ to $S_1^2$ and certain kind of flat connections. (so-called zero curvature representation).

We note that since the harmonic map equation is a local condition, it suffices to consider harmonic maps from simply connected Lorentz surfaces into $S_1^2$.

For the rest of this paper $D$ will always denote a simply connected region of the Minkowski plane $(\mathbb{R}^2(x,y), dx dy)$ containing the origin.

The following result is the starting point of our approach.

**Proposition 3.1** A smooth map $\psi : D \to S_1^2 \subset \mathbb{E}^3_1$ is harmonic if and only if

$$\frac{\partial^2 \psi}{\partial x \partial y} = \rho \psi$$

for some function $\rho$ on $D$.

3.2 Let $\psi : D \to S_1^2$ be a smooth map and $\pi : G \to S_1^2$ the Hopf fibration as before. Since $D$ is simply connected, $\psi$ has a smooth lift $\Psi : D \to G$ unique up to the right $K$-action. Such a lift $\Phi$ is called a framing of $\psi$. Note that a framing $\Psi$ is related to $\psi$ by

$$\psi = \text{Ad}(\Psi)k'.$$  \hspace{1cm} (3.2)

Since $D$ is simply connected, the pull-back bundle $\psi^* G$ is necessarily a trivial bundle $D \times K$. So the group $G$ of gauge transformations of $\psi^* G$ is identified with $C^\infty(D,K)$.

We wish to describe the harmonicity of $\psi$ in terms of a framing. Let $\mu_G$ be the Maurer-Cartan form of $G$. It is well-known that $\mu_G$ satisfies the Maurer-Cartan equation:

$$d\mu_G + \frac{1}{2}[\mu_G \wedge \mu_G] = 0.$$  \hspace{1cm} (3.3)

The pulled back 1-form $\alpha = \Psi^* \mu_G = \Psi^{-1}d\Psi$ of $\mu_G$ by $\Psi$ then satisfies

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$
The identity \((3.3)\) is equivalent with the integrability condition for the existence of a smooth map \(\Psi : \mathbb{D} \to G\) such that \(\alpha = \Psi^* \mu_G\). (Frobenius theorem). By definition, \(\alpha\) is a \(\mathfrak{g}\)-valued 1-form on \(\mathbb{D}\). The \(\mathfrak{g}\)-valued 1-form \(\alpha\) has a type decomposition along the decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m};\)

\[
\alpha = \alpha_0 + \alpha_1. \tag{3.4}
\]

Here \(\alpha_0\) and \(\alpha_1\) denote the \(\mathfrak{k}\)-valued part and \(\mathfrak{m}\)-valued part respectively. Write

\[
\alpha_0 = \alpha'_0 \, dx + \alpha''_0 \, dy, \quad \alpha_1 = \alpha'_1 \, dx + \alpha''_1 \, dy. \tag{3.5}
\]

Then \(\alpha_0\) and \(\alpha_1\) are decomposed with respect to the conformal structure of \(\mathbb{D}\) as follows:

\[
\alpha'_0 = \alpha'_0 + \alpha''_0, \quad \alpha'_1 = \alpha'_1 + \alpha''_1, \quad \alpha''_0 = \alpha''_0 + \alpha''_1. \tag{3.6}
\]

\[
\alpha'_0 = \partial_x \, dx, \quad \alpha''_0 = \partial_y \, dy, \quad \alpha'_1 = \alpha'_m \, dx, \quad \alpha''_1 = \alpha''_m \, dy. \tag{3.7}
\]

Define \(\alpha'\) and \(\alpha''\) by

\[
\alpha' := \alpha'_0 + \alpha'_1, \quad \alpha'' := \alpha''_0 + \alpha''_1. \tag{3.8}
\]

The 1-forms \(\alpha'\) and \(\alpha''\) are called the \((1,0)\)-part and \((0,1)\)-part of \(\alpha\) respectively.

With respect to the conformal structure of \(\mathbb{D}\), we decompose the exterior differential operator \(d\) (cf. (1.6)):

\[
d = d' + d'', \quad d' = \frac{\partial}{\partial x} \, dx, \quad d'' = \frac{\partial}{\partial y} \, dy. \tag{3.9}
\]

Then we have

\[
\alpha' = \Psi^{-1} d' \Psi, \quad \alpha'' = \Psi^{-1} d'' \Psi. \tag{3.10}
\]

By the usual computations we obtain the following (cf. \([20], [27]\))

**Proposition 3.2** Let \(\psi : \mathbb{D} \to S^2_1\) be a smooth map with framing \(\Psi\). Then \(\psi\) is harmonic if and only if

\[
d(* \alpha_1) + [\alpha_0 \wedge * \alpha_1] = 0 \tag{3.11}
\]

for \(\alpha = \Psi^{-1} d \Psi\).

Here \(*\) denotes the Hodge star operator acting on \((\mathfrak{g}\text{-valued})\) one forms on \(\mathbb{D}\) defined by

\[
* \, dx = dx, \quad * \, dy = -dy. \tag{3.12}
\]
3.3 Let $A^1(\mathbb{D}; g)$ be the space of all $g$-valued one-forms and $\mathcal{A}$ the affine space of all connection 1-forms on the product bundle $\mathbb{D} \times G$. The space $\mathcal{A}$ is an affine space associated to the linear space $A^1(\mathbb{D}; g)$. We shall choose the trivial flat connection as the origin of $\mathcal{A}$, then the space $\mathcal{A}$ is identified with $A^1(\mathbb{D}; g)$. Hereafter we shall identify $\mathcal{A}$ with $A^1(\mathbb{D}; g)$ in this way.

**Definition 3.3** A connection $\alpha \in \mathcal{A} = A^1(\mathbb{D}; g)$ is admissible provided that

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \quad d(*\alpha_1) + [\alpha_0 \wedge *\alpha_1] = 0. \quad (3.13)$$

The space of all admissible connections on $\mathbb{D}$ is denoted by $\mathcal{A}^*$. The Frobenius’ theorem implies the following:

**Lemma 3.4** Let $\alpha$ be an admissible connection. Then there exists a harmonic map $\psi : \mathbb{D} \to S^2_1$ such that for any framing $\Psi$ of $\psi$ we have $\Psi^{-1}d\Psi = \alpha$.

Recall that framings are unique up to multiplication on the left by matrices independent of $x$ and $y$. We will remove this freedom in the loop group formalism discussed below.

3.4 Furthermore for an admissible connection $\alpha$, we define the the scalar field $S(\alpha)$ by

$$S(\alpha)(x, y) := \langle \alpha'_m(x, y), \alpha'_m(x, y) \rangle \cdot \langle \alpha''_m(x, y), \alpha''_m(x, y) \rangle. \quad (3.14)$$

Let us define the linear subspaces $\mathcal{A}_+, \mathcal{A}_0, \mathcal{A}_-$ of $\mathcal{A}^*$ by

$$\mathcal{A}_+ := \{ \alpha \in \mathcal{A}^* \mid S(\alpha) > 0 \}, \quad \mathcal{A}_- := \{ \alpha \in \mathcal{A}^* \mid S(\alpha) < 0 \}; \quad (3.15)$$

$$\mathcal{A}_0 := \{ \alpha \in \mathcal{A}^* \mid S(\alpha) = 0 \}.$$  

It is easily checked that all the spaces; $\mathcal{A}^*$, $\mathcal{A}_\pm$ and $\mathcal{A}_0$ are invariant under the action of the gauge group as well as under conformal changes of $\mathbb{D}$. Note that the gauge group $\mathcal{G}$ acts on $\mathcal{A}$ as follows:

$$g^*\alpha = g^{-1} dg + \text{Ad}(g^{-1})\alpha \quad (3.16)$$

for $g \in \mathcal{G}, \alpha \in \mathcal{A}$. 

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Proposition 3.5 Let $\mathcal{H}^*$ be the space of all harmonic maps from $\mathbb{D}$ to $S^2_1$. Define the linear subspaces $\mathcal{H}_+, \mathcal{H}_-$ and $\mathcal{H}_0$ of $\mathcal{H}^*$ by

\begin{align*}
\mathcal{H}_+ &= \{ \psi \in \mathcal{H}^* \mid \langle \psi_x, \psi_x \rangle \langle \psi_y, \psi_y \rangle > 0 \text{ on } \mathbb{D} \}, \quad (3.17) \\
\mathcal{H}_- &= \{ \psi \in \mathcal{H}^* \mid \langle \psi_x, \psi_x \rangle \langle \psi_y, \psi_y \rangle < 0 \text{ on } \mathbb{D} \}, \quad (3.18) \\
\mathcal{H}_0 &= \{ \psi \in \mathcal{H}^* \mid \langle \psi_x, \psi_x \rangle \langle \psi_y, \psi_y \rangle = 0 \text{ on } \mathbb{D} \}. \quad (3.19)
\end{align*}

Then there are the following bijective correspondences:

\begin{align*}
\mathcal{H}^* &\longleftrightarrow \mathcal{A}^*/\mathcal{G}, \quad \mathcal{H}_\pm \longleftrightarrow \mathcal{A}_\pm/\mathcal{G}, \quad \mathcal{H}_0 \longleftrightarrow \mathcal{A}_0/\mathcal{G}. \quad (3.20)
\end{align*}

Here $\mathcal{G}$ denotes the gauge transformation group of the bundle $\psi^*G = \mathbb{D} \times K$. These correspondences are described by $\alpha = \Psi^{-1}d\Psi$ via a framing $\Psi$ of $\psi$.

Proof It suffices to show that a harmonic map $\psi \in \mathcal{H}_+$ [resp. $\mathcal{H}_0$] corresponds to a gauge class of an admissible connection with $\pm S(\alpha) > 0$ [resp. $S(\alpha) = 0$]. However this is apparent from the relations:

\begin{align*}
\frac{\partial}{\partial x}\psi &= \text{Ad}(\Psi)[\alpha'_m, k'], \\
\frac{\partial}{\partial y}\psi &= \text{Ad}(\Psi)[\alpha''_m, k'].
\end{align*}

Remark 3.6 Here is a differential geometric interpretation of the subspaces $\mathcal{H}_\pm, \mathcal{H}_0$.

We know that to any harmonic map $\psi \in \mathcal{H}_+$, there exists a timelike immersion $\varphi$. (See Section 4.) The condition $\psi \in \mathcal{H}_+$ [resp. $\psi \in \mathcal{H}_-$] is equivalent to the positivity [resp. negativity] of the discriminant for the characteristic equation of the shape operator of $\varphi$. Of course the condition $\psi \in \mathcal{H}_0$ corresponds to the property “$\varphi$ has real repeated principal curvatures”.

3.5 To close this section, we introduce the so-called spectral parameter.

Definition 3.7 Let $\alpha \in \mathcal{A}$ be a connection. A loop $\alpha_\lambda$ of connections through $\alpha$ is defined by the following rule:

\begin{align*}
\alpha_\lambda = \alpha_0 + \lambda \alpha'_1 + \lambda^{-1} \alpha''_1, \quad \lambda \in \mathbb{R}^+
\end{align*}
Note that $S(\alpha_\lambda) \equiv S(\alpha)$.

The following observation is fundamental for our approach.

**Proposition 3.8** A connection $\alpha \in \mathcal{A}$ is admissible if and only if the loop $\alpha_\lambda$ through $\alpha$ satisfies

$$d\alpha + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad (3.22)$$

for every $\lambda$.

It is clear that each $\alpha_\lambda$ is admissible whenever $\alpha$ is, and $\alpha_\lambda$ generates the same loop. For every $\alpha_\lambda$ satisfying $(3.12)$, there exists a one-parameter family of smooth maps $\Psi_\lambda : D \to G$ depending smoothly on $\lambda$ such that $\Psi_\lambda^{-1}d\Psi_\lambda = \alpha_\lambda$. In this paper, we shall normalize from here on $\Psi_\lambda$ by

$$\Psi_\lambda(0, 0) \equiv 1. \quad (3.23)$$

Such normalized one-parameter family of maps $\Psi_\lambda$ is called an extended framing of the harmonic map $\psi$. To every harmonic maps $\psi : D \to S^2_1$, there is a naturally associated one-parameter family of harmonic maps $\{\psi_\lambda\}$ such that $\psi_1 = \psi$ parametrized by $\lambda \in \mathbb{R}^+$. We shall therefore refer to $\psi_\lambda$ as a loop of harmonic maps (through $\psi$).

The extended framing $\Psi_\lambda$ of $\psi$ has its values in $\tilde{\Lambda}G_\sigma$ and can therefore be regarded as a mapping $\Psi = \Psi_\lambda : D \to \tilde{\Lambda}G_\sigma$ into the twisted loop group. More generally we have the following (cf. p. 116 in [19]).

**Proposition 3.9** (Harmonicity equation in terms of extended framings)

1. Let $\Psi = \Psi_\lambda : D \to \tilde{\Lambda}G_\sigma$ be a smooth map which satisfies the following equations:

$$\Psi(0,0;\lambda) \equiv 1,$$

$$\Psi_\lambda^{-1}d\Psi_\lambda = \text{linear in } \lambda \quad (= A + \lambda B \text{ for some } A, B), \quad (\Lambda_\sigma)$$

$$\Psi_\lambda^{-1}d''\Psi_\lambda = \text{linear in } \lambda^{-1} \quad (= C + \lambda^{-1}D \text{ for some } C, D)$$

where $A$ and $C$ are $\mathfrak{k}$-valued and $B$ and $D$ are $\mathfrak{m}$-valued 1-forms.

Then $\psi_\lambda := \text{Ad}(\Psi_\lambda)k'$ defines a loop of harmonic maps into $S^2_1$.

2. Let $\psi : D \to S^2_1$ be a harmonic map. Then there exists a solution $\Psi_\lambda$ of $(\Lambda_\sigma)$ such that $\psi = \text{Ad}(\Psi_1)k'$.

**Remark 3.10** C.-H. Gu [18] investigated the Cauchy problem for Lorentzian harmonic maps from $E^2_1$ into $S^2_1$. 

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4  Weierstraß-type representation for harmonic maps

4.1 In this section, we establish a Weierstraß-type representation for Lorentzian harmonic maps into $S^2_1 = G/K$.

We start by introducing the following linear spaces:

\[ \Lambda_{-\infty,1} := \left\{ \xi' \in \Lambda g_\sigma \mid \xi' = \sum_{k=-\infty}^{1} \xi'_k \lambda^k \right\}, \quad (4.1) \]
\[ \Lambda_{-1,\infty} := \left\{ \xi'' \in \Lambda g_\sigma \mid \xi'' = \sum_{k=-1}^{\infty} \xi''_k \lambda^k \right\}. \quad (4.2) \]

**Definition 4.1** Let us denote the space of all $\Lambda_{-\infty,1}$-valued Lorentz holomorphic 1-forms on $D$ by $\mathcal{P}'$. Similarly we denote by $\mathcal{P}''$ the space of $\Lambda_{-1,\infty}$-valued Lorentz anti-holomorphic 1-forms on $D$.

By definition the elements $\xi' \in \mathcal{P}'$ and $\xi'' \in \mathcal{P}''$ have the following form:

\[ \xi' = \left( \sum_{k=-\infty}^{1} \xi'_k(x) \lambda^k \right) dx, \quad \xi'' = \left( \sum_{k=-1}^{\infty} \xi''_k(y) \lambda^k \right) dy. \quad (4.3) \]

Here the coefficients $\xi'_k(x)$ [resp. $\xi''_k(y)$] are smooth functions of $x$ [resp. $y$]. We call elements of $\mathcal{P}'$ and $\mathcal{P}''$ by $(1,0)$-potentials and $(0,1)$-potentials for harmonic maps.

4.2 We will show below that the space $\mathcal{P}' \times \mathcal{P}''$ can serve as the spaces of Weierstraß data for the construction of harmonic maps from $D$ into $S^2_1$.

**Remark 4.2** In the case of CMC surfaces in Euclidean 3-space $\mathbb{E}^3$ the corresponding potentials are the holomorphic potentials for harmonic Gauß maps from Riemann surfaces into $S^2$. In particular for the description of finite type CMC surfaces in $\mathbb{E}^3$, it turns out to be very helpful to use these potentials. See [11].

**Theorem 4.3** (Weierstraß type representation) Let $\{\xi', \xi''\} \in \mathcal{P}' \times \mathcal{P}''$ be a potential.


Solve the two independent initial value problems:
\begin{align}
\frac{d}{dx} \Psi' &= \Psi' \xi', \\
\frac{d^2}{dx^2} \Psi'' &= \Psi'' \xi'', \\
\Psi'(x = 0) &= \Psi''(y = 0) = 1.
\end{align}

Then the Iwasawa decomposition
\[ (\Psi', \Psi'') = (\Psi, \Psi)(L^{-1}_-, L^{-1}_+) \in \Delta(\tilde{\Lambda} G_\sigma \times \tilde{\Lambda} G_\sigma) \cdot \tilde{\Lambda}^+ G_\sigma \times \tilde{\Lambda}^+ G_\sigma \]
gives an extended framing \( \Psi \). Hence \( \psi := \text{Ad}(\Psi)k' \) gives a loop of harmonic maps into \( S^2_1 \).

**Proof** From the Iwasawa splitting (Theorem 2.6) of \((\Psi', \Psi'')\) we know locally around \((0,0)\):
\[ \Delta(\tilde{\Lambda} G_\sigma \times \tilde{\Lambda} G_\sigma) \ni (\Psi, \Psi) = (\Psi', \Psi'')(L_-, L_+). \]

The Maurer-Cartan form \((\Psi, \Psi)^{-1} d(\Psi, \Psi)\) is computed as
\[ (\Psi, \Psi)^{-1} d(\Psi, \Psi) = (L^{-1}_- \xi' L_- + L^{-1}_- dL_-, L^{-1}_- \xi'' L_+ + L^{-1}_- dL_+), \]
where we have used
\[ (\Psi')^{-1} d\Psi' = \xi', \quad (\Psi'')^{-1} d\Psi'' = \xi''. \]

By the construction, the two components of (4.7) are equal. We call them \( \alpha \). One reads off that the dx-part of the first component of the right hand side of (4.7) only involves positive powers of \( \lambda \) with exponents equal to or smaller than 1, while the dx-part of the second component involves only exponents equal to or greater than 0. Thus in \( \alpha \) only the exponents 0 and 1 occur in the dx-part. A similar argument for the dy-part shows that it only involves the terms with exponents \(-1\) and 0.

Thus we obtain
\[ \alpha = \Psi^{-1} d\Psi = \lambda^{-1} (\xi'' dY) + (\xi'_0 dx + \xi''_0 dy) + \lambda (\xi'_1 dx). \]
Hence \( \Psi \) is an extended framing, since \( \xi'_1, \xi'' \in m, \ \xi'_0, \xi''_0 \in \mathfrak{k}. \)

**4.3** This section is in some sense the converse of the previous one. However, we shall compute potentials only in some normal form, which, in general, will
involve singularities. A relation between smooth surfaces and singularity free potentials of the form (4.3) has not been established yet.

Let $\psi : \mathbb{D} \to G/K$ be a harmonic map with an extended framing $\Psi$. Then we perform both type of Birkhoff decompositions (Theorem 2.3) for $\Psi$ as long as it is in both big cells. First we consider the Birkhoff decomposition of the type:

$$\tilde{\Lambda}^- G_\sigma \times \tilde{\Lambda}^+ G_\sigma \subset \tilde{\Lambda} G_\sigma. \quad (4.9)$$

Since the big cell $\mathcal{B}(-, +)$ of $\tilde{\Lambda} G_\sigma$ is open and $\Psi$ is continuous, the set

$$D_1 := \{(x, y) \in \mathbb{D} \mid \Psi(x, y) \text{ belongs to the big cell } \mathcal{B}(-, +) \} \quad (4.10)$$

is an open subset of $\mathbb{D}$. Note that $D_1$ contains $(0, 0)$. Set $S_1 := \mathbb{D} \setminus D_1$. Similarly, we have $D_2$ and $S_2$ for the splitting:

$$\tilde{\Lambda}^+ G_\sigma \times \tilde{\Lambda}^- G_\sigma \subset \tilde{\Lambda} G_\sigma. \quad (4.11)$$

We can perform the two Birkhoff splittings on the extended framing $\Psi$ over $\mathbb{D} \setminus S, \ S := S_1 \cup S_2$:

$$\Psi = \Psi_+ L_+, \ \Psi_- \in \tilde{\Lambda}^- G_\sigma, \ L_+ \in \tilde{\Lambda}^+ G_\sigma. \quad (4.12)$$

$$\Psi = \Psi_- L_, \ \Psi_+ \in \tilde{\Lambda}^+ G_\sigma, \ L_- \in \tilde{\Lambda}^- G_\sigma. \quad (4.13)$$

From the splitting (4.12), we obtain

$$\Psi_-^{-1} d\Psi_+ = L_+(\Psi_-^{-1} d\Psi)L_+^{-1} - dL_+ L_+^{-1}. \quad (4.14)$$

Since $\alpha_\lambda = \alpha_0 + \lambda \alpha_1' + \lambda^{-1} \alpha_1''$, we have

$$L_+(\Psi_-^{-1} d\Psi)L_+^{-1} = L_+(\alpha_0 + \lambda \alpha_1' + \lambda^{-1} \alpha_1'')L_+^{-1}. \quad (4.15)$$

Since $L_+ \in \tilde{\Lambda}^+ G_\sigma$, $L_+$ has the decomposition

$$L_+ = \sum_{k \geq 0} L_k^+(x, y) \lambda^k. \quad (4.16)$$

Compare the left and right hand sides of the $(1, 0)$-part of (4.14). The left hand side of (4.14) contains negative powers of $\lambda$. On the other hand, the right hand side contains nonnegative powers of $\lambda$ only. Thus we have

$$\Psi_-^{-1} \frac{\partial \Psi_-}{\partial x} = 0. \quad (4.17)$$
Therefore $\Psi_-$ depends only on $y$ and $\lambda$.

Next by comparing the left and right hand sides of the $(0,1)$-part of (4.14), we obtain:

$$\Psi^-_1 \, d'' \Psi_-= \lambda^{-1} \{ \text{Ad} \left( L_0^+ \right) \alpha'''_1 \} \, dy. \quad (4.18)$$

The left hand side of (4.14) depends only on $y$ (and $\lambda$). Hence

$$\eta'' := \text{Ad} \left( L_0^+ \right) \alpha'''_1 \, dy \quad (4.19)$$

is an $m$-valued anti-holomorphic 1-form on $\mathbb{D} \setminus \mathcal{S}$.

By similar arguments for $\Psi_+ = \Psi \, L_-^{-1}$, where $L_- = \sum_{k \leq 0} L_k^{-} \lambda^k$, we have

$$\Psi_+^{-1} \frac{\partial \Psi_+}{\partial y} = 0. \quad (4.20)$$

Thus $\Psi_+$ depends only on $x$ (and $\lambda$). Moreover the 1-form

$$\eta' := \text{Ad} \left( L_0^- \right) \alpha'_1 \, dx \quad (4.21)$$

is an $m$-valued holomorphic 1-form on $\mathbb{D} \setminus \mathcal{S}$. Obviously

$$\xi' := \lambda \eta' \in \mathcal{P}', \quad \xi'' := \lambda^{-1} \eta'' \in \mathcal{P}''. \quad (4.22)$$

One can check that the pair of potentials $\xi', \xi''$ reproduces the harmonic map $\psi$ via Weierstraß representation. (cf. Lemma 4.5 and Theorem 4.10 in [13] and Theorem 2.1 in [37].)

**Theorem 4.4 (The normalized potentials)**

Let $\psi : \mathbb{D} \to G/K$ be a harmonic map with $\psi(0,0) = K$ and $\Psi$ an extended framing of $\psi$. Then there exists an open subset $(0,0) \in \mathbb{D} \setminus \mathcal{S}$ on which $\Psi$ splits into

$$\Psi = \Psi_- L_+ = \Psi_+ L_-, \quad (4.23)$$

$$\Psi_+ \in \tilde{\Lambda}_+ G_\sigma, \quad L_- \in \tilde{\Lambda}_- G_\sigma.$$ 

The 1-forms $\eta', \eta''$ defined by

$$\eta'(x) = \Psi_+^{-1} d \Psi_+ \lambda, \quad \eta''(y) = \Psi_-^{-1} d \Psi_- \lambda^{-1} \quad (4.24)$$

are an $m$-valued holomorphic 1-form and an anti-holomorphic 1-form on $\mathbb{D} \setminus \mathcal{S}$ respectively.
Conversely, any harmonic map \( \psi : \mathbb{D} \rightarrow G/K \) with \( \psi(0,0) = K \) can be constructed from a pair of \( m \)-valued 1-forms \( \eta', \eta'' \) which are holomorphic and anti-holomorphic respectively. The harmonic map \( \psi \) is constructed via the Weierstraß representation with potentials

\[
\xi' := \lambda \eta', \quad \xi'' := \lambda^{-1} \eta''.
\] (4.25)

The pair of 1-forms \( \{ \eta', \eta'' \} \) is defined uniquely. Following Wu [37] and [35], we call the pair \( \{ \eta', \eta'' \} \) (or the pair \( \{ \xi', \xi'' \} \) defined by \( \xi' = \lambda \eta', \xi'' := \lambda^{-1} \eta'' \)) the normalized potentials for \( \psi \) with the origin as the reference point.

Up to now we have only very little information on the singular set. Thus in this paper, we restrict our attention to holomorphic and anti-holomorphic potentials \( \{ \xi', \xi'' \} \).

**Remark 4.5** In Euclidean CMC surface geometry, the holomorphic potentials can be extended meromorphically to the simply connected Riemann surface \( \mathbb{D} \). The poles of the potentials are in the singular set \( \mathcal{S} \).

**Remark 4.6** Balan and the first named author studied Weierstraß-type representation of harmonic maps from Riemann surfaces into noncompact Riemannian symmetric spaces [5].

## 5 Normalized potentials for timelike CMC surfaces

### 5.1 In this section, we shall apply the Weierstraß representation for harmonic maps into \( S^2_{1} \) to timelike CMC surfaces.

Let \( \varphi : \mathbb{D} \rightarrow \mathbb{E}^3_{1} \) be a timelike CMC surface with Gauss map \( \psi \) and coordinate frame \( \hat{\Phi} \) (See (1.3)). The Gauss-Codazzi equations (G) and (C) of \( \varphi \) are invariant under the deformation:

\[
Q \longmapsto Q_\lambda := \lambda Q, \quad R \longmapsto R_\lambda := \lambda^{-1} R, \quad \lambda \in \mathbb{R}^+. \quad (5.1)
\]

Integrating the Frenet equations (1.4) with \( Q_\lambda \) and \( R_\lambda \), one obtains a one-parameter family of timelike surfaces \( \{ \hat{\varphi}_\lambda \} \). This deformation does not effect the induced metric and the mean curvature. Hence all the surfaces \( \{ \hat{\varphi}_\lambda \} \) are
isometric and have the same constant mean curvature. The family \{\hat{\varphi}_\lambda\} is called the associated family of \varphi.

This one-parameter deformation of \hat{\varphi} satisfies the following Lax equations:

\[
\frac{\partial}{\partial x} \hat{\Phi}_\lambda = \hat{\Phi}_\lambda \hat{U}(\lambda), \quad \frac{\partial}{\partial y} \hat{\Phi}_\lambda = \hat{\Phi}_\lambda \hat{V}(\lambda),
\]

(5.2)

\[
\hat{U}(\lambda) = \begin{pmatrix}
-\frac{1}{4} \omega_x & -\lambda Q e^{-\frac{\varphi}{2}} \\
\frac{H}{2} e^{-\frac{\varphi}{2}} & \frac{1}{4} \omega_x
\end{pmatrix}, \quad \hat{V}(\lambda) = \begin{pmatrix}
\frac{1}{4} \omega_y & -\frac{H}{2} e^{-\frac{\varphi}{2}} \\
\lambda^{-1} R e^{-\frac{\varphi}{2}} & -\frac{1}{4} \omega_y
\end{pmatrix}.
\]

(5.3)

A solution \hat{\Phi}_\lambda to (5.2) is not an extended framing for the harmonic Gauß map \psi in the sense of the definition given in Section 3.5, since the \lambda-distribution does not fit.

To relate these two \(S^1\)-families to each other we perform the transformation

\[
\Phi := g(\lambda)^{-1} \hat{\Phi}_\lambda g(\lambda), \quad \lambda \in \mathbb{R}^+
\]

(5.4)

with

\[
g(\lambda) = \begin{pmatrix}
0 & -\sqrt{\lambda} \\
1/\sqrt{\lambda} & 0
\end{pmatrix}.
\]

Then the coefficient matrices of the Lax pair \{\hat{U}, \hat{V}\} are changing into

\[
U = g(\lambda)^{-1} \hat{U}(\lambda^2) g(\lambda) = \begin{pmatrix}
\frac{1}{4} \omega_x & -\lambda \frac{H}{2} e^{-\frac{\varphi}{2}} \\
\lambda Q e^{-\frac{\varphi}{2}} & -\frac{1}{4} \omega_x
\end{pmatrix},
\]

(5.5)

\[
V = g(\lambda)^{-1} \hat{V}(\lambda^2) g(\lambda) = \begin{pmatrix}
-\frac{1}{4} \omega_y & -\lambda^{-1} R e^{-\frac{\varphi}{2}} \\
\lambda^{-1} \frac{H}{2} e^{-\frac{\varphi}{2}} & \frac{1}{4} \omega_y
\end{pmatrix}.
\]

Comparing with [23] we conclude

**Proposition 5.1** *(Sym formula)*

Let \Phi be a solution to the Lax equations (5.4) and (5.5). Then

\[
\varphi_\lambda = -\frac{1}{H} \left\{ \frac{\partial}{\partial t} \Phi \cdot \Phi^{-1} + \frac{1}{2} \text{Ad}(\Phi) k' \right\}, \quad \lambda = e^t \in \mathbb{R}^+
\]

(S)

describes a real loop of timelike surfaces of constant mean curvature \(H\). The first fundamental form \(I\) and the Gauß map \(N_\lambda\) of \varphi_\lambda are given by

\[
I = e^{\omega} dx dy \quad \text{and} \quad N_\lambda = \text{Ad}(\Phi) k'.
\]

(5.6)
The logarithmic derivative part $\varphi^K_\lambda$ of $\varphi_\lambda$:

$$\varphi^K_\lambda = -\frac{1}{H} \frac{\partial}{\partial t} \Phi \cdot \Phi^{-1}$$

describes a real loop of timelike surfaces with constant Gaussian curvature $4H^2$.

5.2 Next we calculate the normalized potential $\{\xi', \xi''\}$ in terms of the fundamental quantities of the timelike CMC surface $\varphi$. More precisely we shall clarify the role of the normalized potentials in the construction of timelike CMC surfaces.

First decompose $\alpha := \Phi^{-1} d\Phi$ as

$$\Phi^{-1} d\Phi = \alpha_0 + \lambda \alpha'_1 + \lambda^{-1} \alpha''_1.$$ 

Next we perform the Birkhoff decompositions of $\Phi$ with regard to $\tilde{\Lambda}_+^+ G_\sigma \times \tilde{\Lambda}^- G_\sigma \subset \tilde{\Lambda} G_\sigma$, $\tilde{\Lambda}^- G_\sigma \times \tilde{\Lambda}^+ G_\sigma \subset \tilde{\Lambda} G_\sigma$, over the big cells $\mathcal{B}(-, +)$ and $\mathcal{B}(+, -)$ of $\tilde{\Lambda} G_\sigma$:

$$\Phi = \Phi_+ L_+ = \Phi_- L_+.$$ (5.7)

We have seen above that with

$$L_+ = \sum_{k \geq 0} L^+_k \lambda^k, \quad L_- = \sum_{k \leq 0} L^-_k \lambda^k,$$ (5.8)

the potentials

$$\xi' := \Phi^{-1}_+ d\Phi_+, \quad \xi'' := \Phi^{-1}_- d\Phi_-$$ (5.9)

have the form:

$$\xi' = \lambda \left\{ L_0^+(x, y) \alpha'_1 L_0^+(x, y)^{-1} \right\},$$ (5.10)

$$\xi'' = \lambda^{-1} \left\{ L_0^-(x, y) \alpha''_1 L_0^-(x, y)^{-1} \right\}.$$ (5.11)

More explicitly we have

$$\Phi^{-1}_+ \frac{\partial \Phi_+}{\partial x} = \lambda \left\{ L^-_0 \left( \begin{array}{cc} 0 & -\frac{H}{2}e^{\omega/2} \\ Q(x)e^{-\omega/2} & 0 \end{array} \right) (L^{-}_0)^{-1} \right\},$$ (5.12)
\[
\Phi^{-1} \frac{\partial \Phi}{\partial y} = \lambda^{-1} \left\{ \begin{array}{c}
L_0^+ \left( \begin{array}{cc}
0 & -R(y)e^{-\omega/2} \\
\frac{H}{2} e^{\omega/2} & 0
\end{array} \right) (L_0^+)^{-1}\end{array} \right\}. \tag{5.13}
\]

Comparing the two sides of (5.13) we see that the left hand side depends on \( y \) and \( \lambda \) only, while the right hand side depends on \( x, y \) and \( \lambda \). Recall from the Coddazi equation (C) that since the mean curvature \( H \) is constant, \( R \) depends only on \( y \). Hence we derive

\[
\Phi^{-1} \frac{\partial \Phi}{\partial y} = \lambda^{-1} \left\{ L_0^+(0, y) \left( \begin{array}{cc}
0 & -R(y)e^{-\omega(0,y)/2} \\
\frac{H}{2} e^{\omega(0,y)/2} & 0
\end{array} \right) L_0^+(0, y)^{-1}\end{array} \right\}.
\]

We abbreviate \( L_0^+(0, y) \) by \( W(y) \) and compute \( W(y) \) more explicitly. From (5.7) we obtain the Maurer-Cartan equation:

\[
L_+ \alpha L^{-1}_+ - dL_+ L^{-1}_+ = \Phi^{-1} d\Phi. \tag{5.14}
\]

Comparing the \( \lambda^0 \)-terms of both sides in (5.14) we obtain,

\[
0 = W_0 \alpha_0'' W_0^{-1} - dW_0 W_0^{-1}. \tag{5.15}
\]

Hence

\[
dW_0 = W_0 \alpha_0''. \tag{5.16}
\]

Namely we have

\[
\frac{dW_0}{dy} = W_0 \left\{ \frac{\omega_y(0, y)}{4} \right\} k'. \tag{5.17}
\]

Thus \( W_0(y) \) is given explicitly by

\[
W_0(y) = \left( \begin{array}{cc}
e^{-\omega(0,y)/4+c_1} & 0 \\
0 & e^{\omega(0,y)/4+c_2}
\end{array} \right), \quad c_1, c_2 \in \mathbb{R}. \tag{5.18}
\]

Since we require the initial condition:

\[
W_0(0) = 1,
\]

we see that the \( (0,1) \)-potential \( \xi'' \) is given by

\[
\xi'' = \lambda^{-1} \left( \begin{array}{cc}
0 & -R(y)e^{-\omega(0,y) + \omega(0,0)/2} \\
\frac{H}{2} e^{\omega(0,y) - \omega(0,0)/2} & 0
\end{array} \right) dy. \tag{5.19}
\]
By similar arguments for $\Phi_+ = \Phi L_+^{-1}$, we obtain
\[
\Phi_+^{-1} \frac{\partial \Phi_+}{\partial x} = \lambda \left\{ L_0(x, 0) \begin{pmatrix} 0 & -\frac{H}{2} e^{\omega(x, 0)/2} \\ Q(x) e^{-\omega(x, 0)/2} & 0 \end{pmatrix} L_0^{-1}(x, 0) \right\} .
\]

The function $\Gamma_0(x) := L_0^-(x, 0)$ is a solution to
\[
\frac{d\Gamma_0}{dx} = \Gamma_0 \left\{ -\frac{\omega_x(x, 0)}{4} \right\} k',
\]
(5.20)

Thus we have
\[
\Gamma_0(x) = \begin{pmatrix} e^{\omega(x, 0)/4 + c_3} & 0 \\ 0 & e^{-\omega(x, 0)/4 + c_4} \end{pmatrix}, \quad c_3, \ c_4 \in \mathbb{R}.
\]
(5.21)

Since we require the initial condition
\[
\Gamma_0(0, 0) = 1,
\]
the $(1, 0)$-potential $\xi'$ is given by
\[
\xi' = \lambda \left( \begin{array}{cc} 0 & -\frac{H}{2} e^{\omega(x, 0) - \omega(0, 0)/2} \\ Q(x) e^{-\omega(x, 0) + \omega(0, 0)/2} & 0 \end{array} \right) dx.
\]
(5.22)

**Theorem 5.2** The normalized potentials with the origin as a reference point are given by
\[
\xi' = \lambda \left( \begin{array}{cc} 0 & -\frac{H}{2} f(x) \\ Q(x) f(x) & 0 \end{array} \right) dx,
\]
\[
\xi'' = \lambda^{-1} \left( \begin{array}{cc} 0 & -R(y)/g(y) \\ \frac{H}{2} g(y) & 0 \end{array} \right) dy.
\]

Here the functions $f(x)$ and $g(y)$ are given by
\[
f(x) = \exp \{\omega(x, 0) - \omega(0, 0)/2\}, \quad g(y) = \exp \{\omega(0, y) - \omega(0, 0)/2\}.
\]

**Remark 5.3** From this result one sees that the normalized potentials are essentially the Hopf differentials together with the values of the conformal factor of the metric restricted to one pair of null coordinate lines.
Remark 5.4 (Nonlinear d’Alembert’s formulas)

It is known that every solution for the linear wave equation:

\[ \omega_{xy} = 0 \]

can be written as the sum of a Lorentz holomorphic function and a Lorentz anti-holomorphic function by the so called d’Alembert formula:

\[ \omega(x, y) = f(x) + g(y) \].

The Weierstraß-type representation (Theorem 4.3) together with Theorem 5.2 provide us with nonlinear analogues of d’Alembert’s formula for the hyperbolic sinh-Gordon equation, the Liouville equation and the hyperbolic cosh-Gordon equation. More precisely for any initial data \( f(x) \) and \( g(y) \), the normalized potentials with \( Q = H/2 \), \( R = \epsilon H/2 \), \( \epsilon = 0, \pm 1 \) produce solutions \( \omega \) to the hyperbolic sinh-Gordon equation (\( \epsilon = 1 \)), the Liouville equation (\( \epsilon = 0 \)) and the hyperbolic cosh-Gordon equation (\( \epsilon = -1 \)) via the Weierstraß-type representation (Theorem 4.3). (See Proposition 1.3, Proposition 1.5 and Remark 1.6.)

6 Examples

Even though the results of the previous sections give a one-to-one correspondence between special potentials and timelike CMC surfaces, making this correspondence explicit is a different matter.

To give examples we start with the easiest case from the potential point of view.

6.1 (Cylinders and pseudospheres)

Example 6.1 (hyperbolic cylinders) Let us take the following potentials:

\[ \xi' = \frac{\lambda}{4} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \, dx, \quad \xi'' = \frac{\lambda^{-1}}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, dy. \] (6.1)

Then the normalized potentials \( \{ \xi', \xi'' \} \) produce the associated family of timelike CMC surface \( \varphi : \mathbb{R}^2 \to \mathbb{E}^3 \) of mean curvature \( 1/2 \);

\[ \varphi(x, y) = (\sinh \frac{x - y}{2}, \frac{x + y}{2}, \cosh \frac{x - y}{2}). \] (6.2)
The image of $\varphi$ is a timelike hyperbolic cylinder. The induced metric of $\mathbb{R}^2$ is $I = dx dy$. Thus $\varphi$ is an isometric imbedding of the Minkowski plane $(\mathbb{R}^2, dx dy)$ into $\mathbb{E}^3_1$.

**Example 6.2** (Circular cylinders)

The normalized potentials $\{\xi', \xi''\}$ defined by

$$
\xi' = \frac{\lambda}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dx, \quad \xi'' = \frac{\lambda^{-1}}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dy
$$

(6.3)

produce the associated family of timelike CMC surface $\varphi : \mathbb{R}^2 \rightarrow \mathbb{E}^3_1$ of mean curvature $1/2$;

$$
\varphi(x,y) = \left( \frac{x-y}{2}, \sin \frac{x+y}{2}, -\cos \frac{x+y}{2} \right).
$$

(6.4)

The image of $\varphi$ is a timelike circular cylinder. The induced metric of $\mathbb{R}^2$ is $I = dx dy$. Thus $\varphi$ is an isometric immersion of the Minkowski plane $(\mathbb{R}^2, dx dy)$ into $\mathbb{E}^3_1$.

Note that both, timelike hyperbolic cylinder and timelike circular cylinder, correspond to the vacuum solution of the hyperbolic sinh-Gordon equation.

**Example 6.3** (Totally umbilical pseudosphere)

The normalized potentials $\{\xi', \xi''\}$ with $Q = R = 0$ produce totally umbilical pseudosphere in $\mathbb{E}^3_1$.

6.2 Next we shall give examples of timelike CMC surfaces with repeated real principal curvatures. Namely, timelike CMC surfaces with $QR = 0$. Such surfaces have no Euclidean counterparts.

As in the case of indefinite affine spheres with $AB = 0$ (See Section 9.2 in [9]), timelike CMC surfaces with $QR = 0$ have specific shapes. To investigate such surfaces, we recall the notion of a B-scroll introduced by L. Graves [16].

**Definition 6.4** Let $\gamma = \gamma(s)$ be a smooth curve in $\mathbb{E}^3_1$ defined on an interval $I \subset \mathbb{R}$. Then $\gamma$ is said to be a null Frenet curve if

1. $\langle \gamma', \gamma' \rangle = 0$,
2. there exist vector fields $A, B, C$ along $\gamma$ and two functions $\kappa$ and $\tau$ such that
   $$
   A = \gamma', \quad \langle A, B \rangle = 1,
   $$

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\[ (A, A) = (B, B) = 0, \quad (C, C) = 1, \]
\[ (A, C) = (B, C) = 0. \]  
\[ (6.5) \]

\[
\frac{d}{ds} (A, B, C) = (A, B, C) \begin{pmatrix}
0 & 0 & -\tau \\
0 & 0 & -\kappa \\
\kappa & \tau & 0
\end{pmatrix}.
\]

The frame field \( L = (A, B, C) \) along \( \gamma \) is called the \textit{null Frenet frame field} of \( \gamma \). The two functions \( \kappa \) and \( \tau \) are called the \textit{curvature} and \textit{torsion} of \( \gamma \) respectively.

**Definition 6.5** Let \( \gamma \) be a null Frenet curve in \( \mathbb{E}_1^3 \). The ruled surface

\[ \varphi(s, t) = \gamma(s) + tB(s) : I \times \mathbb{R}^* \to \mathbb{E}_1^3 \]

(6.6)

is called \textit{B-scroll} of \( \gamma \).

Since

\[ I = (t\tau)^2 ds^2 + 2ds dt, \]

every \( B \)-scroll is timelike. (In fact \( \det I = -1 \).) The mean curvature of \( \varphi \) is \( H(s, t) = \tau(s) \). Thus \( \varphi \) is of constant mean curvature if and only if \( \tau \) is constant.

A specific example of a null Frenet curve with constant torsion 1 is

\[ \gamma(s) = \left( \frac{\sinh(2s)}{2}, \frac{\cosh(2s)}{2}, s \right) \]

with null Frenet frame field:

\[ A(s) = (\cosh 2s, \sinh 2s, 1), \]

\[ B(s) = \frac{1}{2} (-\cosh(2s), -\sinh(2s), 1), \]

\[ C(s) = (-\cosh 2s, -\sinh 2s, 0). \]

The \( B \)-scroll \( \varphi \) of \( \gamma \):

\[ \varphi(s, t) = \left( \frac{\sinh(2s) - t \cosh(2s)}{2}, \frac{\cosh(2s) - t \sinh(2s)}{2}, s + \frac{t}{2} \right) \]

is a timelike surface with constant mean curvature 1. ([20], p. 33.)

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Now let \( \varphi \) be a constant mean curvature \( B \)-scroll of a null Frenet curve \( \gamma \). The local coordinate system \((x, y)\) defined by
\[
x := s - \frac{2}{\tau^2} t - \frac{1}{2}, \quad y := s + \frac{1}{2},
\]
(6.7)
is a null coordinate system of \( \varphi \). Then
\[
I = e^{\omega(x,y)} \, dx \, dy, \quad e^{\omega(x,y)} = \frac{4}{H^2((x - y) + 1)^2},
\]
(6.8)
\[
Q(x) = 0, \quad R(y) = H(y - \frac{1}{2}).
\]
(6.9)
Thus the normalized potentials are given by
\[
\xi' = \lambda \begin{pmatrix} 0 & (x + 1)^{-2} \\ 0 & 0 \end{pmatrix} \, dx,
\]
\[
\xi'' = \lambda^{-1} \begin{pmatrix} 0 & -R(y)H(1 - y)^2/2 \\ (1 - y)^{-2} & 0 \end{pmatrix} \, dy.
\]

Conversely, we shall prove that timelike CMC surfaces derived from normalized potentials of the form:
\[
\xi' = \lambda \begin{pmatrix} 0 & -\frac{H}{2} f(x) \\ 0 & 0 \end{pmatrix} \, dx, \quad \xi'' = \lambda^{-1} \begin{pmatrix} 0 & -R(y)/g(y) \\ \frac{H}{2} g(y) & 0 \end{pmatrix} \, dy.
\]
(6.10)
are \( B \)-scrolls of constant mean curvature \( H \).

To this end we consider the initial value problems:
\[
d'\Phi_+ = \Phi_+ \xi', \quad d''\Phi_- = \Phi_- \xi'',
\]
\[
\Phi_+(x = 0) = \Phi_-(y = 0) = 1
\]
with potential (6.10).

The solution \( \Phi_+ \) is easily obtained as
\[
\Phi_+ = \begin{pmatrix} 1 & -\lambda F(x) \\ 0 & 1 \end{pmatrix}, \quad F(x) = \frac{H}{2} \int_0^x f(x) \, dx.
\]
(6.11)
The inverse loop of \( \Phi_+ \) is
\[
\Phi_+^{-1} = \begin{pmatrix} 1 & \lambda F(x) \\ 0 & 1 \end{pmatrix}.
\]
(6.12)
We shall get the extended framing $\Phi$ derived from $\{\xi', \xi''\}$ by the technique used in [11]. (See section 3.6 of [11].) Express $\Phi_-$ as

$$\Phi_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 + \sum_{j=1}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \lambda^{-j}. \quad (6.13)$$

Take a $\tilde{\Lambda}G_\sigma^+$-valued map $\mathcal{G}$;

$$\mathcal{G}(x, y) = \begin{pmatrix} 1 & -\lambda F/(1 + c_1 F) \\ 0 & 1 \end{pmatrix}. \quad (6.14)$$

Then $\tilde{\Phi} := \Phi_+^{-1} \Phi_- \mathcal{G}$ is computed as

$$\tilde{\Phi} = \tilde{\Phi}_0 + \sum_{j \geq 1} \tilde{\Phi}_j \lambda^{-j} = \begin{pmatrix} 1 + c_1 F & 0 \\ 0 & 1/(1 + c_1 F) \end{pmatrix} + \text{negative powers.}$$

Put $\check{\Phi}_+ = \tilde{\Phi}_0 \mathcal{G}^{-1}$ and $\check{\Phi}_- := \tilde{\Phi}_0^{-1}$. Then $\check{\Phi}_- \in \tilde{\Lambda}^- G_\sigma$, $\check{\Phi}_+ \in \tilde{\Lambda}^+ G_\sigma$ and $\check{\Phi}_+^{-1} \check{\Phi}_- = \check{\Phi}_- \check{\Phi}_+$. Thus we obtain

$$\check{\Phi}_- = L_- \quad (6.15)$$

and the extended framing $\Phi$ is computed as follows:

$$\Phi = \Phi_+ L_- = \Phi_+ \check{\Phi}_- = \Phi_- \check{\Phi}_+^{-1}. \quad (6.16)$$

By the Sym formula, Proposition 5.1, we obtain the constant mean curvature immersion

$$\varphi(x, y) = \gamma(y) + q(x, y) B(y), \quad (6.18)$$

$$\gamma(y) := -\frac{1}{H} \left\{ \frac{\partial \Phi}{\partial t} \Phi_-^{-1} + \frac{1}{2} \text{Ad}(\Phi_-) k' \right\}, \quad (6.19)$$

$$B(y) := \frac{\lambda}{H^2} \text{Ad}(\Phi_-)(j' - i), \quad q(x, y) = \frac{F(x)}{H \{1 + c_1(y) F(x)\}}. \quad (6.20)$$

A direct computation shows that $\gamma(y)$ is a null Frenet curve and $\varphi$ is the $B$-scroll of $\gamma$. 

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References

[1] O. Babelon and D. Bernard, Affine solitons: a relation between tau functions, dressing and Bäcklund transformations, Inter. J. Modern Phys. A 8 (1993), 507-543.

[2] M. V. Babich, Real finite-gap solutions of the equation $\Delta u = \text{chu}$, Mat. Zametki 50 (1991), No. 1, 3–9, 158, English translation; Math. Notes 50 (1991), No. 1–2, 663–667.

[3] M. Babich and A. I. Bobenko, Willmore tori with umbilic lines and minimal surfaces in hyperbolic space, Duke Math. J. 72 (1993), 151–185.

[4] V. Balan and J. Dorfmeister, Birkhoff decomposition and Iwasawa decomposition for general untwisted loop groups, preprint.

[5] V. Balan and J. Dorfmeister, A Weierstrass-type representation for harmonic maps from Riemann surface into semisimple symmetric spaces, in preparation.

[6] M. J. Bergvelt and M. Guest, Actions of loop groups on harmonic maps, Trans. Amer. Math. Soc. 326 (1991) 861–886.

[7] A. I. Bobenko and U. Eitner, Painlevé Equations in Differential Geometry of Surfaces, Lecture Notes in Math., 1753, Springer Verlag, 2000.

[8] K. F. Clancey and I. Gohberg, Factorization of matrix functions and singular integral operators, Operator Theory: Advances and Applications, 3, Birkhäuser Verlag, Basel-Boston, Mass., 1981, ISBN 3-7643-1297-1.

[9] J. Dorfmeister and U. Eitner, Weierstrass-type representation of affine spheres, preprint (1999).

[10] J. Dorfmeister, H. Gradl and J. Szmigielski, Systems of PDEs obtained from factorization in loop groups, Acta Appl. Math. 53 (1998), 1-58.

[11] J. Dorfmeister and G. Haak Meromorphic potentials and smooth CMC surfaces, Math. Z., 224 (1997), 603–640.

[12] J. Dorfmeister, F. Pedit and M. Toda, Minimal surfaces via loop groups, Balkan J. Geom. Appl. 2 (1997), 25–40.

[13] J. Dorfmeister, F. Pedit and H. Wu, Weierstrass type representations of harmonic maps into symmetric spaces, Comm. Analysis and Geom. 6 (1998) 633-668.
[14] J. Dorfmeister and I. Sterling, Finite type Lorentz harmonic maps and the method of Symes, preprint (2000).

[15] A. Fujioka and J. Inoguchi, Timelike surfaces with harmonic inverse mean curvature, preprint (submitted to same proceeding).

[16] L. Graves, Codimension one isometric immersions between Lorentz spaces, Trans. Amer. Math. Soc. 252 (1979), 367-392.

[17] I. T. Gokhberg, A factorization problem in normed rings, functions of isometric and symmetric operators and singular integral equations, Russian Math. Surveys 19 (1964), 63–114.

[18] C.-H. Gu, On the harmonic maps from $\mathbb{R}^{1,1}$ to $S^{1,1}$, J. reine Angew. Math. 346 (1984), 101-109.

[19] M. A. Guest, Harmonic Maps, Loop Groups and Integrable Systems, London Math. Soc. Student Texts 38, Cambridge Univ. Press, Cambridge, 1997.

[20] M. A. Guest and Y. Ohnita, Actions of loop groups, deformations of harmonic maps and applications, Selected Papers on Harmonic Analysis, Groups and Invariants, Amer. Math. Soc. Translations, vol. 183, Providence, 1998, pp. 33–50.

[21] H. Hu, Sine-Laplace equations, Sinh-Laplace equations and harmonic maps, Manuscripta Math. 40 (1982), 205-216.

[22] H. Hu, On the geometry of sinh-Gordon equation, Qualitative Aspects and Applications of Nonlinear Evolution Equations (Trieste 1993) World Sci. Publishing, River Edge, NJ., 1994, pp. 35–47.

[23] J. Inoguchi, Timelike surfaces of constant mean curvature in Minkowski 3-space, Tokyo J. Math. 21 (1998), 141-152.

[24] J. Inoguchi, Darboux transformations on timelike constant mean curvature surfaces, J. Geom. Phys. 32 (1999), 57–78.

[25] P. Kellersch, Eine Verallgemeinerung der Iwasawa Zerlegung in Loop Gruppen, Ph. D. Thesis, Technische Universität München, 1999.

[26] L. McNerney, One-parameter families of surfaces with constant curvature in Lorentz 3-space, Ph. D. Thesis, Brown Univ., 1980.

[27] S. Melko and I. Sterling, Applications of soliton theory to the construction of pseudospherical surfaces in $\mathbb{R}^3$, Ann. of Global Anal. Geom. 11 (1993), 65–107.
[28] S. Melko and I. Sterling, Integrable systems, harmonic maps and the classical theory of surfaces, *Harmonic Maps and Integrable Systems* (A. P. Fordy and J. C. Wood eds.), Aspects of Math. E 23, Viewig, Braunschweig, (1994), 129–144.

[29] T. K. Milnor, Harmonic maps and classical surface theory in Minkowski 3-space, Trans. Amer. Math. Soc. 280 (1983), 161–185.

[30] V. Y. Novokshenov, Minimal surfaces in the hyperbolic space and radial-symmetric solution of the cosh-Laplace equation, *Algebraic and Geometric Methods in Mathematical Physics* (Kaciveli, 1993), Math. Phys. Stud. 19, Kluwer Acad. Publ., Dordrecht, 1996, pp. 357–370.

[31] V. Y. Novokshenov, Radial-symmetric solution of the cosh-Laplace equation and the distribution of its singularities, Russian J. Math. Phys. 5 (1997), 211–226.

[32] B. O’Neill, *Semi-Riemannian Geometry with Application to Relativity*, Pure and Applied Math., vol. 130, Academic Press, Orlando, 1983.

[33] D. Petersen and V. Kac, Infinite flag varieties and conjugacy theorem, Proc. Nat. Acad. Sci. U.S.A., 80 (1983), 1778–1782.

[34] A. Pressley and G. Segal, *Loop Groups*, Oxford Math. Monographs, Oxford University Press, 1986.

[35] M. Toda, *Pseudo spherical surfaces via moving frames and loop groups*, Ph. D. Thesis, University of Kansas (2000).

[36] T. Weinstein, *An Introduction to Lorentz Surfaces*, de Gruyter Exposition in Math. vol. 22, Walter de Gruyter, Berlin, 1996.

[37] H. Wu, A simple way for determining the normalized potentials for harmonic maps, Ann. Global Anal. Geom. 17 (1999), 189–199.

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