S¹-EQUIVARIANT LOCAL INDEX AND QUANTIZATION CONJECTURE FOR NON-COMPACT SYMPLECTIC MANIFOLDS

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Abstract. We define an $S¹$-equivariant index for non-compact symplectic manifolds with Hamiltonian $S¹$-action. We use the perturbation by Dirac-type operator along the $S¹$-orbits. We give a formulation and a proof of quantization conjecture for this $S¹$-equivariant index. We also give a comment on the relation between our $S¹$-equivariant index and the index of transverse elliptic operators.

1. Introduction

In [3] we gave a formulation of index theory of Dirac-type operator on open manifolds using torus fibration and the perturbation by Dirac-type operator along fibers. In [4] we gave a refinement of it for a family of torus bundles with some compatibility conditions. In [5] we used equivariant version of them to give a geometric proof of quantization conjecture for Hamiltonian torus action on closed symplectic manifolds. In this paper we give a formulation of $S¹$-equivariant index theory for non-compact symplectic manifold with Hamiltonian $S¹$-action based on the framework of [3]. The resulting index is a homomorphism from $R(S¹)$ to $\mathbb{Z}$, and if the manifold is closed, then the $S¹$-equivariant index coincides with the Riemann-Roch character as a functional on $R(S¹)$.

We use a perturbation by the Dirac-type operator along $S¹$-orbits. On the other hand Braverman[2] gave an index theory on open manifolds based on a perturbation by the vector field induced from certain equivariant map, e.g., momentum map. His index theory realizes the index of transverse elliptic operators developed by Atiyah[1] and Paradan-Vergne[11]. Both our construction and Braverman’s construction use perturbation by operators along the orbits, and hence, they have conceptual similarity. We show that they are equal for the proper moment map case. We also show that they have different nature. In fact we will give an example in this paper which shows the difference.

In our construction it is straightforward to give a formulation and a proof of quantization conjecture for non-compact symplectic manifolds with Hamiltonian $S¹$-action. Vergne[12] proposed a quantization conjecture for non-compact symplectic manifolds, which was proved by Ma-Zhang[7, 8] and Paradan gave a new proof in [10]. Vergne’s conjecture is based on the index theory of transverse elliptic operators. Ma and Zhang showed Vergne’s conjecture for the proper moment map case using Braverman’s index theory. We do not assume neither compactness of the fixed point set nor properness of the momentum map as in [12, 7, 8] and [10]. We only assume that the inverse image of each integer point is compact.

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This paper is organized as follows. In Section 2, we give a brief review of the construction in [3] to define an $S^1$-equivariant index $\text{ind}_{S^1}(X, V)$. In Section 3, we apply the construction in Section 2 to the symplectic geometry case. We define an $S^1$-equivariant local Riemann-Roch number $\text{RR}_{S^1}(M, L)$ for Hamiltonian $S^1$-action on non-compact symplectic manifold. In Section 4 we give a quantization conjecture for non-compact symplectic manifold with Hamiltonian $S^1$-action. In Section 5, we give comments on relation between our equivariant index and that developed by Braverman [2] and Ma-Zhang [7]. In Appendix A we give some details of the explicit computation of the kernel of a family of perturbed Dirac operators on the cylinder. The family contains the perturbations in this paper and that in [2], and it shows the difference between two equivariant indices.

1.1. Notations.

- For each $n \in \mathbb{Z}$ let $\mathcal{C}(n)$ be the complex line with the standard action of the circle group $S^1$ of weight $n$.
- Let $\rho$ be a representation space of $S^1$. For each $n \in \mathbb{Z}$ we denote by $\rho^{(n)}$ the multiplicity of the weight $n$ representation in $\rho$ i.e., we put

$$\rho^{(n)} := \dim \left( \text{Hom}_{S^1}(\mathcal{C}(n), \rho) \right).$$

We will also use the same notation for elements in the representation ring $R(S^1)$.

2. Definition of the $S^1$-equivariant index

In this section we give a brief review of the construction in [3] to define an index $\text{ind}_{S^1}(X, V)$.

2.1. Setting. Let $X$ be a non-compact Riemannian manifold. Let $W$ be a $\mathbb{Z}/2$-graded $\text{Cl}(TX)$-module bundle with the Clifford multiplication $c$. Suppose that the circle group $S^1$ acts on $X$ in an isometric way and the action lifts to $W$ so that it commutes with $c$. We assume that there exists an open subset $V$ of $X$ which satisfies the following assumption.

Assumption 2.1. (1) The complement $X \setminus V$ is compact.
(2) $S^1$ acts on $V$ without fixed points.
(3) There exists a formally self-adjoint operator $D_{S^1} : \Gamma(W|_V) \to \Gamma(W|_V)$ which satisfies the following conditions.
   (a) $D_{S^1}$ contains only the derivatives along the $S^1$-orbits, and its restriction to each orbit is a Dirac-type operator along the orbit.
   (b) For each tangent vector $u$ which is normal to the $S^1$-orbit, $D_{S^1}$ anti-commutes with the Clifford multiplication of $\tilde{u}$:

$$c(\tilde{u}) \circ D_{S^1} + D_{S^1} \circ c(\tilde{u}) = 0,$$

where $\tilde{u}$ is the vector field along the $S^1$-orbit which is obtained by $u$ and the $S^1$-action.
(4) For all $x \in V$ the kernel of the restriction of $D_{S^1}$ to the orbit $S^1 \cdot x$ is trivial i.e.,

$$\ker(D_{S^1}|_{S^1 \cdot x}) = 0.$$

If $(X, V, D_{S^1})$ satisfies these conditions, then we call $(X, V, D_{S^1})$ is acyclic.
2.2. Definition of $\text{ind}_{S^1}(X, V)$. Following the procedure as in [3, 5], we can define the $S^1$-equivariant index $\text{ind}_{S^1}(X, V, D_{S^1}, W) = \text{ind}_{S^1}(X, V, W) = \text{ind}_{S^1}(X, V) \in R(S^1)$. Let us recall the definition briefly. We first deform an end of $X$ into a complete Riemannian manifold $\hat{X}$ (e.g., cylindrical end) in an $S^1$-equivariant way. We also deform $W$ and $D_{S^1}$ into $\hat{W}$ and $\hat{D}_{S^1}$ (e.g., translationally invariant) on the end of $\hat{X}$. Let $D$ be an $S^1$-invariant Dirac-type operator on $\Gamma(\hat{W})$ (which is translationally invariant on the end of $\hat{X}$). Let $\rho_V$ be an $S^1$-invariant cut-off function such that $\rho_V = 0$ on $X \setminus V$ and $\rho_V = 1$ on the end of $\hat{X}$. For $t \gg 0$ a perturbation $D + t\rho_V \hat{D}_{S^1}$ gives a Fredholm operator on $L^2(\hat{W})$. We can show that the index does not depend on the choice of the completion $\hat{X}$, and we define $\text{ind}_{S^1}(X, V)$ as this index, which satisfies the excision formula, sum formula and product formula. See [3] for details.

3. $S^1$-EQUIVARIANT LOCAL INDEX FOR SYMPLECTIC MANIFOLDS WITH HAMILTONIAN $S^1$-ACTION

Let $(M, \omega)$ be a (possibly non-compact) symplectic manifold with a pre-quantizing line bundle $(L, \nabla)$. Suppose that the circle group $S^1$ acts on $(M, \omega)$ and the action lifts to $(L, \nabla)$. Let $\mu : M \to \mathbb{R}$ be the associated moment map. We assume the following compactness.

Assumption. For each $n \in \mathbb{Z}$, the inverse image $\mu^{-1}(n)$ is a compact subset.

Take and fix an $S^1$-invariant $\omega$-compatible almost complex structure $J$ on $M$ so that we have the associated metric $g^J$ and $\mathbb{Z}/2$-graded $\text{Cl}(TM)$-module bundle $W_L := \wedge^T M^01 \otimes L$. Using the orthogonal projection from $TM|\mu^{-1}M^{S^1}$ to the tangent bundle along the orbits we can define $D_{S^1}$, the twisted Dirac-type operator along $S^1$-orbits. Note that the restriction of $D_{S^1}$ to each orbit is the de Rham operator with coefficient in the flat bundle $(L, \nabla)|_{\text{orbit}}$. We first recall some basic properties.

Lemma 3.1. (1) For each $x \in M \setminus M^{S^1}$, the space of global parallel sections $H^0(S^1 \cdot x, (L, \nabla)|_{S^1 \cdot x})$ vanishes if and only if $\ker(D_{S^1}|_{S^1 \cdot x})$ vanishes.

(2) For each $x \in M \setminus M^{S^1}$ and $n \in \mathbb{Z}$, the multiplicity $H^0(S^1 \cdot x, (L, \nabla)|_{S^1 \cdot x})^{(n)}$ vanishes if and only if the multiplicity $\ker(D_{S^1}|_{S^1 \cdot x})^{(n)}$ vanishes.

(3) If $H^0(S^1 \cdot x, (L, \nabla)|_{S^1 \cdot x}) \neq 0$, then we have $\mu(x) \in \mathbb{Z}$. In particular we have $\mu(M^{S^1}) \subset \mathbb{Z}$.

(4) If $\mu(x) \in \mathbb{Z}$ and $H^0(S^1 \cdot x, (L, \nabla)|_{S^1 \cdot x}) \neq 0$, then we have $H^0(S^1 \cdot x, (L, \nabla)|_{S^1 \cdot x}) = \mathbb{C}_{\mu(x)}$. In particular if $x \in M^{S^1}$, then we have $L_x = \mathbb{C}_{\mu(x)}$.

We take an $S^1$-invariant relatively compact open neighborhood $X_{\mu, n}$ of the compact set $\mu^{-1}(n)$ as $\mu^{-1}(n) \subset X_{\mu, n} \subset \mu^{-1}([n - 1/2, n + 1/2])$. We put $V_{\mu, n} := X_{\mu, n} \setminus \mu^{-1}(n)$.

Proposition 3.2. $(X_{\mu, n}, V_{\mu, n}, D_{S^1}|_{V_{\mu, n}})$ is acyclic.

Proof. Since $\mu(V_{\mu, n}) \cap \mathbb{Z} = \emptyset$ we have $(V_{\mu, n})^{S^1} = \emptyset$ and $\ker(D_{S^1}|_{S^1 \cdot x}) = 0$ for all $x \in V_{\mu, n}$ by (1) and (3) in Lemma 3.1. For each tangent vector $u \in T_x V_{\mu, n}$ which is normal to the orbit $S^1 \cdot x$ let $\tilde{u}$ be the induced parallel vector field along the orbit. Since $D_{S^1}$ is the de Rham operator with coefficient in $(L, \nabla)|_{S^1 \cdot x}$, it anti-commutes with the Clifford multiplication of the parallel vector field $\tilde{u}$. \hfill $\Box$

We can define the index $\text{ind}_{S^1}(X_{\mu, n}, V_{\mu, n}) \in R(S^1)$ by applying the construction in Section 2.
Proposition 3.3. The index \( \text{ind}_{S^1}(X_{\mu,n}, V_{\mu,n}) \) does not depend on the choice of \( X_{\mu,n} \).

Proof. Suppose that we take two relatively compact neighborhoods \( X_{\mu,n} \) and \( X'_{\mu,n} \) of \( \mu^{-1}(n) \) with the required properties. It is enough to show that if \( X_{\mu,n} \subset X'_{\mu,n} \), then we have \( \text{ind}_{S^1}(X_{\mu,n}, V_{\mu,n}) = \text{ind}_{S^1}(X'_{\mu,n}, V'_{\mu,n}) \). The equality follows from the excision formula of \( \text{ind}_{S^1}(\cdot, \cdot) \). \( \square \)

Definition 3.4. For each \( n \in \mathbb{Z} \), we put \( \text{RR}^{(n)}_{S^1,\text{loc}}(M, L) := \text{ind}_{S^1}(X_{\mu,n}, V_{\mu,n})(n) \in \mathbb{Z} \) and define \( \text{RR}^{(n)}_{S^1,\text{loc}}(M, L) \in \text{Hom}(R(S^1), \mathbb{Z}) \) by putting

\[
\text{RR}^{(n)}_{S^1,\text{loc}}(M, L) : C(n) \mapsto \text{RR}^{(n)}_{S^1,\text{loc}}(M, L).
\]

We call \( \text{RR}^{(n)}_{S^1,\text{loc}}(M, L) \) the \( S^1 \)-equivariant local Riemann-Roch number.

Remark 3.5. The number \( \text{RR}^{(n)}_{S^1,\text{loc}}(M, L) \) may be non-zero for infinitely many \( n \). We do not know whether \( S^1 \)-equivariant local Riemann-Roch number \( \text{RR}^{(n)}_{S^1,\text{loc}}(M, L) \) has distributional nature or not.

Remark 3.6. Using the equivariant version of the acyclic compatible systems in [4] it would be possible to define the equivariant local index \( \text{RR}^{S^1}_{G,\text{loc}}(M, L) \) and \( \text{RR}^{S^1}_{G,\text{loc}}(M, L) \) for any compact torus \( G \) and \( \xi \) in the weight lattice of \( G \).

Suppose that \( M \) is a compact manifold without boundary. For the \( S^1 \)-equivariant data \( (M, \omega, L, \nabla) \), the \( S^1 \)-equivariant Riemann-Roch number \( \text{RR}^{S^1}(M, L) \) is defined as the index of the \( S^1 \)-equivariant spin\(^r\) Dirac operator twisted by \( L \).

Theorem 3.7. If \( M \) is a compact symplectic manifold without boundary, then we have

\[
\text{RR}^{S^1}_{S^1,\text{loc}}(M, L) = \text{RR}^{S^1}(M, L),
\]

where the right hand side is regarded as a functional on \( R^1(S^1) \).

Proof. We show \( \text{RR}^{(n)}_{S^1,\text{loc}}(M, L) = \text{RR}^{S^1}(M, L)(n) \) for each \( n \in \mathbb{Z} \). By (2) and (4) in Lemma 3.1 we have \( \text{ker}(D_{S^1}|_{S^1,x}) \in H^0(S^1, x, (L, \nabla)|_{S^1,x})(n) = 0 \), for each \( x \notin X_{\mu,n} \), and hence, by shifting trick and the localization theorem for \( S^1 \)-acyclic compatible system ([5 Theorem 2.41]), we have

\[
\text{RR}^{S^1}(M, L)(n) = \text{RR}^{S^1}(M, L \otimes C(-n))(0) = \text{ind}_{S^1}(X_{\mu,n,0}, V_{\mu,n,0}, W_L \otimes C(-n)|X_{\mu,n,0})(0)
+ \sum_{k \neq 0} \text{ind}_{S^1}(X'_{\mu-n,k}, V'_{\mu-n,k}, W_L \otimes C(-n)|X'_{\mu-n,k})(0),
\]

where \( X'_{\mu-n,k} \) is an \( S^1 \)-invariant relatively compact open neighborhood of \( (\mu - n)^{-1}(k) \cap M^{S^1} = \mu^{-1}(n+k) \cap M^{S^1} \) and we put \( V'_{k} := X'_{k} \setminus \mu^{-1}(k) \cap M^{S^1} \). On the other hand we have \( L_x = C(k) \) for each \( x \in \mu^{-1}(k) \cap M^{S^1} \) by (4) in Lemma 3.1. We can apply the vanishing theorem ([5 Theorem 4.1]) and we have \( \text{ind}_{S^1}(X'_{\mu-n,k}, V'_{\mu-n,k}, W_L \otimes C(-n)|X'_{\mu-n,k})(0) = 0 \) for all \( k \neq 0 \). Note that we may assume that \( X_{\mu-n,0} = X_{\mu,n} \) and \( V_{\mu-n,0} = V_{\mu,n} \). So we have \( \text{RR}^{S^1}(M, L)(n) = \text{RR}^{S^1}(X_{\mu,n}, V_{\mu,n}, W_L \otimes C(-n)|X_{\mu,n})(0) = \text{RR}^{S^1}_{S^1,\text{loc}}(M, L \otimes C(-n)) = \text{RR}^{(n)}_{S^1,\text{loc}}(M, L). \) \( \square \)
4. Quantization conjecture for $RR_{S^1, loc}$

Let $(M, \omega), (L, \nabla)$ and $\mu$ be the data as in Section 3. Namely $(M, \omega)$ is a symplectic manifold and $(L, \nabla)$ is a pre-quantizing line bundle with Hamiltonian $S^1$-action whose moment map is $\mu$. We assume that $\mu^{-1}(n)$ is a compact subset for each $n \in \mathbb{Z}$. Suppose that an integer $n$ is a regular value of $\mu$. Then we have a compact symplectic orbifold $M(n) := \mu^{-1}(n)/S^1$ with the prequantizing line bundle $L(n) := (L \otimes \mathbb{C}(-n), \nabla)|_{\mu^{-1}(n)}/S^1$. One can define the Riemann-Roch number $RR(M(n), L(n))$ of the prequantized symplectic orbifold $M(n)$ as the index of the spin$^c$ Dirac operator twisted by $L(n)$.

**Theorem 4.1.** If an integer $n \in \mathbb{Z}$ is a regular value of $\mu$, then we have

$$RR_{S^1, loc}^{(n)}(M, L) = RR(M(n), L(n)).$$

*Proof.* By the excision formula, the index $RR_{S^1, loc}^{(n)}(M, L) = ind_{S^1}(X_{\mu,n}, V_{\mu,n}, W_L|_{X_{\mu,n}})^{(n)}$ is localized at any neighborhood of $\mu^{-1}(n)$. On the other hand by the normal form theorem (e.g., [5, Proposition 5.11]) we may assume that the neighborhood has the form $\mu^{-1}(n) \times_{S^1} T^*S^1$. Since the $S^1$-invariant part of the index of $T^*S^1 = S^1 \times \mathbb{R}$ with the standard structure is equal to 1, we have $RR_{S^1}^{(n)}(M, L) = ind_{S^1}(X_{\mu,n}, V_{\mu,n}, W_L|_{X_{\mu,n}})^{(n)} = ind(\mu^{-1}(n)/S^1, W_{\mu,n}) = RR(M(n), L(n))$ by the product formula.

**Remark 4.2.** Kirwan [6] and Meinrenken-Sjamaar [9] gave definitions of $RR(M(n), L(n))$ for a critical value $n$ of $\mu$. We do not understand relation between them and $RR_{S^1, loc}^{(n)}(M, L)$.

5. Relation with the transverse index

Vergne [12] gave a formulation of quantization conjecture for non-compact symplectic manifolds with Hamiltonian action of a general compact Lie group $G$, in which the compactness of the zero set of the induced vector field (Kirwan vector filed) is assumed. Her conjecture concerns with the transverse index which was defined by Atiyah [1] and studied by Paradan-Vergne [11]. Her conjecture was proved by Ma-Zhang [7, 8] and Paradan gave a new proof in [10]. In [7] they defined an equivariant index $Q(L): R(G) \to \mathbb{Z}$ under the assumption of the properness of the moment map, and showed that the quantization conjecture for $Q(L)$. Namely for each irreducible representation $\rho$ of $G$, the number $Q(L)(\rho)$ is equal to the Riemann-Roch number of the symplectic quotient. They used the index theorem due to Braverman [2]. He showed that a perturbation of Dirac operator gives an analytic realization of the transverse index $\chi_G(M) = \chi_G(M, \mu)$. The perturbation term is the Clifford action of the induced vector field. If the induced vector field has compact zero set, then the equivariant index $Q(L)$ is equal to the transverse index $\chi_G(M)$. Since both equivariant indices $Q(L)$ and $RR_{S^1, loc}(M, L)$ satisfy the quantization conjecture, we have the following.

**Proposition 5.1.** Let $(M, \omega, L, \nabla)$ be a prequantized symplectic manifold equipped with a Hamiltonian $S^1$-action. If the moment map is proper and it does not have any critical points, then we have

$$Q(L) = \chi_{S^1}(M, \mu) = RR_{S^1, loc}(M, L).$$

We do not have direct proof of the second equality which does not use the quantization conjecture. On the other hand the following example implies that our equivariant index $RR_{S^1}(M, L)$ has different behaviour from the transverse index.
Example 5.2. Let $m$ be a non-zero integer and $M$ the product of the circle $S^1$ and a small interval centered at $m$. Consider the standard metric and the symplectic structure on $M$. Let $L$ be the trivial complex line bundle over $M$ which is equipped with a structure of prequantizing line bundle over $M$. Consider the natural $S^1$-action on $M$, and we take its lift to $L$ so that $S^1$ acts trivially on the fiber direction. One has the associated momentum map $\mu$ which is equal to the projection to the interval factor. Since $m$ is non-zero $\mu$ does not have neither critical points nor zeros, and hence, the associated vector field $\mu_M$ on $M$ does not vanish. Then [2, Lemma 3.12] implies that the associated transverse index $\chi_{S^1}(M, \mu)$ vanishes. (In fact one can check that the kernel of the perturbation of the Dirac operator by $\mu_M$ vanishes by the direct computation.) On the other hand one can check that the kernel of the perturbation by $D_{S^1}$ is one dimensional and it is isomorphic to $C(n)$, hence, we have $RR_{S^1}^{(n)}(M, L) = \delta_{mn}$. In particular we have $RR_{S^1}(M, L) \neq \chi_{S^1}(M, \mu)$. See Appendix A for details of the computation.

### Appendix A. Perturbations on the cylinder and some computations

In this appendix we give some details of the computations of the kernel of the perturbed Dirac operator on the cylinder. We consider a family of perturbations which come from [2], [8] and [3].

#### A.1. Settings.

1. $m$ : integer
2. $M = \mathbb{R} \times S^1$ with coordinate functions $(r, \theta)$
3. $g = dr^2 + d\theta^2$ : Riemannian metric
4. $\omega = dr \wedge d\theta$ : symplectic structure
5. $J : \partial_r \mapsto \partial_\theta, \partial_\theta \mapsto -\partial_r$ : almost complex structure
6. We use $\partial_\theta$ as a frame of $TM_{\mathbb{C}} = (TM, J)$.
7. $W^+ = M \times \mathbb{C}, W^- = TM_{\mathbb{C}}, W = W^+ \oplus W^-$
8. $c : T^*M \to \text{End}(W)$ : Clifford action defined by
   \[
   c(dr) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad c(d\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
   \]
9. $\rho : \mathbb{R} \to (m - 1/2, m + 1/2)$ : smooth non-decreasing function with
   \[
   \rho(r) = \begin{cases}
   r & (m - 1/4 < r < m + 1/4) \\
   m - 1/2 & (r < m - 1/2) \\
   m + 1/2 & (r > m + 1/2)
   \end{cases}
   \]
10. $\nabla = d - 2\pi \rho(r) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) d\theta$ : Clifford connection of $W$
11. $D : \Gamma(W) \to \Gamma(W)$ : Dirac operator,
   \[
   D = \begin{pmatrix} 0 & -\partial_\theta - \sqrt{-1}\partial_r + 2\pi\sqrt{-1}\rho \\ -\partial_\theta - \sqrt{-1}\partial_r - 2\pi\sqrt{-1}\rho & 0 \end{pmatrix}
   \]
12. Let $S^1$ acts on $M$ in the standard way, and we take a lift of the $S^1$-action on $W$ so that the action on the fiber direction is trivial. All the data are preserved by the $S^1$-action.
(13) $D_{S^1} : \Gamma(W) \to \Gamma(W)$: Dirac operator along the $S^1$-orbits:
$$D_{S^1} = \begin{pmatrix} 0 & -\partial_\theta + 2\pi \sqrt{-1} \rho \\ \partial_\theta - 2\pi \sqrt{-1} \rho & 0 \end{pmatrix}$$

(14) $\mu := -2\pi \rho : M \to \mathbb{R}$
(15) $\mu^M = -2\pi \rho \partial_\theta \in \Gamma(TM)$: induced vector field
(16) $f : M \to \mathbb{R}_+$: smooth positive function on $M$ such that $f(r) = |r|$ for $|r-m| > 1/2$

**Remark A.1.** This data is a completion of the standard Hamiltonian $S^1$-action on $(m-1/2, m+1/2) \times S^1$, which has a cylindrical end and translationally invariance on the end. The map $\mu$ gives the moment map of the $S^1$-action on $(m-1/4, m+1/4) \times S^1$, and $f^\varepsilon$ is an admissible function for $(W, \mu, \nabla)$ for any $\varepsilon > 0$ in the sense of [2].

A.2. **Perturbation of $D$.** For $s, t, \varepsilon_1, \varepsilon_2 \geq 0$ we consider the following perturbation of $D$:
$$D_{s,t,\varepsilon_1,\varepsilon_2} := D + \sqrt{-1}sf^{\varepsilon_1}c(\mu^M) + tf^{\varepsilon_2}D_{S^1} = \begin{pmatrix} 0 & D_{s,t,\varepsilon_1,\varepsilon_2}^- \\ D_{s,t,\varepsilon_1,\varepsilon_2}^+ & 0 \end{pmatrix},$$
where
$$D_{s,t,\varepsilon_1,\varepsilon_2}^+ = (1 + tf^{\varepsilon_2})(\partial_\theta - 2\pi \sqrt{-1} \rho) - \sqrt{-1}\partial_r - 2\pi \sqrt{-1}sf^{\varepsilon_1}\rho$$
and
$$D_{s,t,\varepsilon_1,\varepsilon_2}^- = -(1 + tf^{\varepsilon_2})(\partial_\theta - 2\pi \sqrt{-1} \rho) - \sqrt{-1}\partial_r + 2\pi \sqrt{-1}sf^{\varepsilon_1}\rho.$$

Note that $D_{1,0,\varepsilon_1,\varepsilon_2}$ ($\varepsilon_1 > 0$) is the perturbation considered in [2] and [3], and $D_{0,t,\varepsilon_1,0}$ is the one considered in [3].

A.3. **ker$_L^2(D_{s,t,\varepsilon_1,\varepsilon_2}^+)$.** For $\phi \in \Gamma(W^+)$ by taking the Fourier expansion we write
$$\phi(r, \theta) = \sum_{n \in \mathbb{Z}} a_n(r)e^{2\pi \sqrt{-1}n\theta}.$$

Then we have
$$D_{s,t,\varepsilon_1,\varepsilon_2}^+ \phi(n) = \sum_{n \in \mathbb{Z}} \sqrt{-1}(2\pi((1 + tf^{\varepsilon_2})(n-\rho) - sf^{\varepsilon_1}\rho)a_n(r) - a_n'(r))e^{2\pi \sqrt{-1}n\theta},$$
and hence,
$$D_{s,t,\varepsilon_1,\varepsilon_2}^+ \phi = 0 \iff 2\pi((1 + tf^{\varepsilon_2})(n-\rho) - sf^{\varepsilon_1}\rho)a_n(r) - a_n'(r) = 0 \iff a_n(r) = \alpha_n \exp \left(2\pi \int_0^r ((1 + tf^{\varepsilon_2})(n-\rho) - sf^{\varepsilon_1}\rho)dr \right) (\alpha_n \in \mathbb{C}).$$

Now we determine the condition to $\phi \in$ ker$_L^2(D_{s,t,\varepsilon_1,\varepsilon_2}^+)$. Since $\rho = m \pm 1/2$ and $f = |r|$ for $\pm r$ large enough we have
$$a_n(r) = \alpha_n \exp \left(2\pi \int_0^r ((1 + t|r|^{\varepsilon_2})(n-\rho) \mp 1/2) \pm (m \pm 1/2)s|r|^{\varepsilon_1}dr \right)$$
$$= \alpha_n \exp \left(2\pi(n-\rho) \mp 1/2 \left(r + \frac{tr|r|^{\varepsilon_2}}{\varepsilon_2 + 1} \right) - 2\pi(m \pm 1/2)\frac{sr|r|^{\varepsilon_1}}{\varepsilon_1 + 1} \right).$$

Suppose that $\int_{-\infty}^\infty |a_n(r)|^2dr < \infty$. 


(I) $\varepsilon_1 > \varepsilon_2$. In this case when we take $r \gg 0$ we have $m + 1/2 > 0$, and when we take $-r \gg 0$ we have $m - \frac{1}{2} < 0$. So we have $m = 0$, and hence, we have $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2}) \neq 0$ if and only if $m = 0$. If $m = 0$, then $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})$ is an infinite dimensional vector space generated by $\{a_n(r)e^{2\pi\sqrt{-1}n\theta} \mid n \in \mathbb{Z}\}$.

(II) $\varepsilon_1 < \varepsilon_2$. In this case as in the same way for (I) we have $m - 1/2 < n < m + 1/2$, and hence, $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2}) = \mathbb{C}\langle a_m(r)e^{2\pi\sqrt{-1}m\theta} \rangle$.

(III) $\varepsilon_1 = \varepsilon_2$. In this case when we take $r \gg 0$ and $-r \gg 0$ we have

$$\left(n - m - \frac{1}{2}\right)t - \left(m + \frac{1}{2}\right)s < 0 \text{ and } \left(n - m + \frac{1}{2}\right)t - \left(m - \frac{1}{2}\right)s > 0.$$  

So we have if $t = 0$ and $s > 0$, then $m = 0$, and if $t > 0$, then

$$\left(1 + \frac{s}{t}\right)\left(m - \frac{1}{2}\right) < n < \left(1 + \frac{s}{t}\right)\left(m + \frac{1}{2}\right).$$

In this case $\dim \ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})$ depends on $s/t$.

A.4. $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})$. For $\phi \partial_\theta \in \Gamma(W^-)$ by taking the Fourier expansion we write

$$\phi(r, \theta) = \sum_{n \in \mathbb{Z}} a_n(r)e^{2\pi\sqrt{-1}n\theta}.$$  

Then we have

$$D_{s,t,\varepsilon_1,\varepsilon_2}^\phi = -\sum_{n \in \mathbb{Z}} \sqrt{-1} \left(2\pi((1 + tf^{\varepsilon_2})(n - \rho) - sf^{\varepsilon_1}\rho)a_n(r) + a'_n(r)\right)e^{2\pi\sqrt{-1}n\theta},$$

and hence,

$$D_{s,t,\varepsilon_1,\varepsilon_2}^\phi = 0 \iff 2\pi((1 + tf^{\varepsilon_2})(n - \rho) - sf^{\varepsilon_1}\rho)a_n(r) + a'_n(r) = 0$$

$$\iff a_n(r) = \alpha_n \exp\left(-2\pi \int_{r}^{0}((1 + tf^{\varepsilon_2})(n - \rho) - sf^{\varepsilon_1}\rho)dr\right) \quad (\alpha_n \in \mathbb{C}).$$

As in the same way for $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})$ one can check that there are no $L^2$-solutions of $D_{s,t,\varepsilon_1,\varepsilon_2}^\phi = 0$.

A.5. Computation of indices. We specialize the parameters and have computations of two indices, the transverse index $\chi_{S^1}(M, \mu)$ in $[2]$ and the equivariant local Riemann-Roch number $RR^{S^1,\text{loc}}(M, L)$.

A.5.1. $\chi_{S^1}(M, \mu)$. When we take $s = 1$, $t = 0$ and $\varepsilon_1 > \varepsilon_2$ we have the following.

**Proposition A.2.**

$$\ker_{L^2}(D_{1,0,\varepsilon_1,\varepsilon_2}^+) = \mathbb{C}\langle\{\delta_{m0}a_n(r)e^{2\pi\sqrt{-1}n\theta} \mid n \in \mathbb{Z}\}\rangle, \quad \ker_{L^2}(D_{1,0,\varepsilon_1,\varepsilon_2}^-) = 0.$$  

In particular we have

$$\chi_{S^1}(M, \mu) = \mathbb{C}\langle\{\delta_{m0}a_n(r)e^{2\pi\sqrt{-1}n\theta} \mid n \in \mathbb{Z}\}\rangle = \bigoplus_{n \in \mathbb{Z}} \delta_{m0}\mathbb{C}(n).$$
A.5.2. $RR_{S^1, \text{loc}}(M, L)$. When we take $s = \varepsilon_2 = 0$ we have the following.

**Proposition A.3.**

$$\text{Ker}_{L^2}(D_{0, \varepsilon_1, 0}^+) = \mathbb{C}\langle a_m(r)e^{2\pi\sqrt{-1}m\theta}\rangle, \quad \text{Ker}_{L^2}(D_{0, \varepsilon_1, 0}^-) = 0.$$ 

In particular we have

$$RR_{S^1, \text{loc}}(M, L) = \mathbb{C}\langle a_m(r)e^{2\pi\sqrt{-1}m\theta}\rangle = \mathbb{C}(m).$$

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