On the possibility of the Fourier method using for finding of a solution of the sixth-order mathematical model with derivatives with respect to measure

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Abstract. In the present paper the separation variables method is applied to finding an exact solution of the sixth-order mathematical model with nonsmooth solutions. Analyzing arising difficulties, in particular the spectral problem, we use a pointwise method of interpretation of solutions proposed by Yu.V. Pokornyi. This method showed its effectiveness in constructing of an exact parallel to the classical theory of differential equations, including oscillation theorems, both second and fourth orders.

1. Introduction
In this paper we investigate a mathematical model represented in the form of a mixed problem

\[
\begin{align*}
M(x) \frac{d^2 u}{dx^2} &= (pu''''_{xx})'' - (ru''''_{xx})_{x} + (qu'_x)_x - uQ'_x; \\
- (pu''''_{xx}) (0, t) + \gamma_1 u''_{xx}(0, t) &= 0; \\
(pu''''_{xx})'_{x} (0, t) - ru''''_{xx}(0, t) + \gamma_2 u'_x(0, t) &= 0; \\
- (pu''''_{xx})_{xx} (0, t) + (ru''''_{xx})'_{x} (0, t) - gu'_x(0, t) + \gamma_3 u(0, t) &= 0; \\
(pu''''_{xx}) (\ell, t) + \gamma_4 u''_{xx}(\ell, t) &= 0; \\
- (pu''''_{xx})'_{x} (\ell, t) + ru''''_{xx}(\ell, t) + \gamma_5 u'_x(\ell, t) &= 0; \\
(pu''''_{xx})_{xx} (\ell, t) - (ru''''_{xx})'_{x} (\ell, t) + gu'_x(\ell, t) + \gamma_6 u(\ell, t) &= 0; \\
u(x, 0) &= \psi_0(x); \\
u'_x(x, 0) &= \psi_1(x).
\end{align*}
\]

It arises when we describe small transverse free vibrations of a rod system with internal singularities and placed on a double elastic foundation with localized singularities that lead to a loss of smoothness of the solution. We are looking for a solution of the equation in (1) in the class \( E \) of functions \( u(x, t) \), each of which is continuously differentiable with respect to \( x \in [0; \ell] \) for each fixed \( t \), the derivative \( u'_x(x, t) \) is absolutely continuous on \([0; \ell]\), the second derivative \( u''''_{xx}(x, t) \) is \( \mu \)-absolutely continuous on \([0; \ell]\); the third derivative \( u'''_{xx}(x, t) \) has bounded variation on \([0; \ell]\); the quasi-derivative \( (pu''''_{xx}) (x, t) \) is continuously differentiable on \([0; \ell]\); \( (pu''''_{xx})'_{x} (x, t) \) is absolutely continuous on \([0; \ell]\); \( (pu''''_{xx})_{xx} (x, t) \) is \( \sigma \)-absolutely continuous on \([0; \ell]\); for each
The equation
\[ M_\sigma(x) \frac{\partial^2 u}{\partial t^2} = (pu_{xx}^m)_{x\sigma} - (ru_x^m)_{x\sigma} + (q u_x^m)_{x\sigma} - u'Q_\sigma \]

is given almost everywhere (with respect to measure \( \sigma \)) on the Cartesian product of the extension \([0; \ell]_\sigma \) of the segment \([0; \ell] \) and the time interval \([0; T] \). The set \([0; \ell]_\sigma \) is constructed as follows. Let \( S(\sigma) \) be a set of discontinuity points of the function \( \sigma(x) \). We introduce on \( J_\sigma = [0; \ell] \setminus S(\sigma) \) the metric \( \varrho(x; y) = |\sigma(x) - \sigma(y)| \). The resulting metric space \((J_\sigma; \varrho)\) is not complete. The standard completion leads (up to isomorphism) to the set \([0; \ell]_S \), in which each point \( \xi \in S(\sigma) \) is replaced by a pair of points \( \xi - \xi, \xi + \xi \), which were previously limiting. Inducing ordering from the original set, we have the inequalities \( x < \xi - \xi < \xi + \xi < y \) for all \( x, y \) for which the inequalities \( x < \xi < y \) are satisfied in the original segment.

The function \( v(x) \) at the points \( \xi - \xi \) and \( \xi + \xi \) of the set \([0; \ell]_S \) is defined by limiting values. For the function defined in this way, we will retain the previous designation. The function defined on this set becomes continuous in the sense of the metric \( \varrho(x; y) \).

The union \([0; \ell]_S \) and \( S(\sigma) \) gives us the set \([0; \ell]_\sigma \), in which each point \( \xi \in S(\sigma) \) is replaced by the triple \( \{\xi - \xi; \xi; \xi + \xi\} \). We assume that the equation is defined precisely on this set, and at the points \( \xi \in S(\sigma) \) the equation itself has the form

\[ \Delta M(\xi) \frac{\partial^2 u}{\partial t^2}(\xi, t) = \Delta (pu_{xx}^m)(\xi, t) - \Delta (ru_x^m)_{x\xi}(\xi, t) + \Delta (g(x)u_x^m)(\xi, t) - u(\xi, t)\Delta Q(\xi), \]

where \( \Delta v(\xi) = v(\xi + \xi) - v(\xi - \xi) \) is a complete jump of the function \( v(x) \) at the point \( \xi \).

Notice that a qualitative theory equations with nonsmooth solutions began to develop rapidly after the publication in 1999 of the work of Yu. V. Pokorny [1]. We emphasize especially the monographs [2–5], works [6–21]. This efficiency is explained quite simply: when we apply derivatives with respect to the measures, the equation, in contrast to theory of generalized functions, becomes defined in each point and makes it possible to use qualitative methods for analyzing of solutions. Indeed, when using the theory of Schwartz-Sobolev distributions according difficult problems are emerged. The first problem is that only weak solvability can be established. Hence equations are not suitable for applications. The second problem, which has not yet been solved, arises when the generalized function multiply by the discontinuous one. The third problem is that equations in generalized functions are the equalities of two functionals defined on the space of basic functions. Hence it is extremely difficult to apply methods of qualitative analysis to such equations.

When applying the Fourier method to find a solution of (1), the spectral problem arises

\[
\begin{align*}
LX &\equiv - (pu_{xx}^m)_{x\sigma} + (ru_x^m)_{x\sigma} - (q u_x^m)_{x\sigma} + XQ_\sigma = \lambda M_\sigma X(x); \\
\varphi_1 X &\equiv - (pu_{xx}^m)(0) + \gamma_1 u_x^m(0) = 0;
\end{align*}
\]

\[
\begin{align*}
\varphi_2 X &\equiv (pu_{xx}^m)'(0) - ru_x^m(0) + \gamma_2 u_x^m(0) = 0;
\end{align*}
\]

\[
\begin{align*}
\varphi_3 X &\equiv - (pu_{xx}^m)(\ell) + (ru_x^m)'(\ell) - \gamma_3 u_x^m(\ell) = 0;
\end{align*}
\]

\[
\begin{align*}
\varphi_4 X &\equiv (pu_{xx}^m)'(\ell) + \gamma_4 u_x^m(\ell) = 0;
\end{align*}
\]

\[
\begin{align*}
\varphi_5 X &\equiv - (pu_{xx}^m)'(\ell) + ru_x^m(\ell) + \gamma_5 u_x^m(\ell) = 0;
\end{align*}
\]

\[
\begin{align*}
\varphi_6 X &\equiv (pu_{xx}^m)'(\ell) - (ru_x^m)'(\ell) + \gamma_6 u_x^m(\ell) = 0.
\end{align*}
\]

Here \( \lambda \) is a spectral parameter.

By writing \( \gamma_1 = \infty \), we mean the boundary condition \( u_x^m(0, t) = 0 \); similarly, \( \gamma_i = \infty \) means that the derivative or the function in front of \( \gamma_i \) is equal to zero.

Let the following conditions be satisfied:
(i) \( p(x), r(x), g(x), Q(x) \) and \( F(x) \) are functions of bounded variation on \([0; \ell]\);
(ii) \( \inf_{x \in [0;\ell]} p(x) > 0; \)
(iii) \( r(x) \) and \( g(x) \) are absolutely continuous on \([0; \ell]\);
(iv) \( Q(x) \) does not decrease by \([0; \ell]\);
(v) function \( \mu(x) \) generating on \([0; \ell]\) measure, strictly increases on \([0; \ell]\).
(vi) all \( \gamma_i \) are non-negative.
(vii) one of the following conditions holds: all \( \gamma_i = \infty \) or all \( \gamma_i \) are positive.

In view of these conditions, one can prove that spectrum problem (2) is real, consists of eigenvalues, the geometric multiplicity of each of them is finite, and the algebraic multiplicity is equal to one. The methods of proof are identical to those described in [5], which were applied to the justification of the oscillatory theory for the fourth-order boundary value problem with derivatives with respect to measure.

**Theorem 1.** Let conditions i–vii be satisfied; \( \{\lambda_n\} \) be eigenvalues of the problem (2), and each of them is written as much as is their geometric multiplicity. Then the series

\[
\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{2/3+\delta}}
\]

converges for any \( \delta > 0 \).

The proof of the theorem is carried out by the methods described in [21] for estimating the growth rate of the fourth order boundary value problem with derivatives with respect to measure. To formulate the main result of the work, we introduce the following class of functions.

The set \( E_\sigma \) consists of functions \( X(x) \), each of which is continuously differentiable on \([0; \ell]\), the derivative \( X'_x(x) \) is absolutely continuous on \([0; \ell]\), the second derivative \( X''_x(x) \) \( \mu \) is absolutely continuous on \([0; \ell]\); the third derivative \( X'''_{xx}(x) \) is function of bounded variation on \([0; \ell]\); the quasi-derivative \( (pX'''_{xx})_x(x) \) is continuously differentiable on \([0; \ell]\); \( (pX'''_{xx})_x(x) \) is absolutely continuous on \([0; \ell]\); \( (pX'''_{xx})_xx(x) \) \( \sigma \) is absolutely continuous on \([0; \ell]\).

2. The main results

The main result of the work is the theorem.

**Theorem 2.** Let conditions i–vii be satisfied; in addition, \( \psi_i(x) \in E_\sigma \), functions \( \frac{L\psi_i(x)}{M_\sigma(x)} \) are continuously differentiable on \([0; \ell]\), \( \left( \frac{L\psi_i(x)}{M_\sigma(x)} \right)'_x \) are absolutely continuous on \([0; \ell]\), and \( \left( \frac{L\psi_i(x)}{M_\sigma(x)} \right)''_xx \) are functions of bounded variation on \([0; \ell]\) \((i = 1, 2); \varphi_j\psi_0 = \varphi_j(L\psi_0) = \varphi_j\psi_1 = 0 \((j = 1, 2, \ldots, 6)\). Then the function

\[
u(x, t) = \sum_{k=1}^{\infty} \varphi_k(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

where \( \varphi_k(x) \) is the normalized amplitude function corresponding to the eigenvalue \( \lambda_k \),

\[
A_k = \int_0^\ell M'_{\ell}(x)\varphi_k(x)\psi_0(x)d\sigma, \quad B_k = \int_0^\ell M'_{\ell}(x)\varphi_k(x)\psi_1(x)d\sigma,
\]

is a solution of mathematical model (1), and series (3) can be differentiated by \( t \) twice, and by \([0; \ell]\) six times: first twice by \( x \), then by \( \mu \), again twice by \( x \), and then by \( \sigma \); series (3) converges absolutely and uniformly on the rectangle \([0; \ell] \times [0; T]\).
Proof. Let us apply the classical proof scheme. Let us estimate the coefficients of the Fourier series of the function \( \psi_0(x) \). We have

\[
A_k = \int_0^\ell \psi_0(x) M'_\ell(x) \varphi_k(x) d\sigma = 
\]

\[
= \int_0^\ell \psi_0(x) \frac{1}{\lambda_k} \left( - (p \varphi_{k,xx}^m)'_x + (r \varphi_{k,xx}^m)'_x - (g \varphi_{k,xx}^m)'_x + Q'_\sigma \varphi_k \right) d\sigma = 
\]

\[
= - \int_0^\ell \psi_0(x) \frac{1}{\lambda_k} (p \varphi_{k,xx}^m)'_x (x) d\sigma + \int_0^\ell \psi_0(x) \frac{1}{\lambda_k} (r \varphi_{k,xx}^m)'_x (x) d\sigma - \int_0^\ell \psi_0(x) \frac{1}{\lambda_k} (g \varphi_{k,xx}^m)'_x (x) d\sigma + \int_0^\ell \psi_0(x) \frac{1}{\lambda_k} Q'_\sigma \varphi_k (x) d\sigma.
\]

We integrate the first integral of the last equality six times, the second four times, the third two, and also use the properties of the function \( \varphi_k(x) \). Integration by parts for a more complex object (a graph) is justified in [22]. We have

\[
A_k = \frac{1}{\lambda_k} \left( - \psi_0 \left( p \varphi_{k,xx}^m \right)'_x \right) + \psi_0 \left( p \varphi_{k,xx}^m \right)'_x - \psi_0'' \left( p \varphi_{k,xx}^m \right)'_x + \varphi''_k \left( p \psi_{0,xx}^m \right)'_x \\
- \varphi''_k \left( p \psi_{0,xx}^m \right)'_x - \varphi''_k \left( r \varphi_{k,xx}^m \right)'_x + \varphi''_k \left( r \varphi_{k,xx}^m \right)'_x - \int \varphi_k \left( p \psi_{0,xx}^m \right)'_x d\sigma + \\
+ \psi_0 \left( r \varphi_{k,xx}^m \right)'_x - \psi_0 \left( r \varphi_{k,xx}^m \right)'_x + \varphi' \left( r \varphi_{k,xx}^m \right)'_x + \varphi_0 \left( r \varphi_{k,xx}^m \right)'_x - \int \varphi_k \left( r \psi_{0,xx}^m \right)'_x d\sigma - \\
- \psi_0 \left( g \varphi_k \right)' - g \psi_0 \varphi_k - \int \varphi_k \left( g \psi_0 \right)'_x d\sigma + \int \varphi_k \left( Q'_\sigma \right) \psi_0 (x) d\sigma = \\
= \frac{1}{\lambda_k} \int \varphi_k \psi_0 d\sigma = \frac{1}{\lambda_k} \int \varphi_k \left( M'_\ell \psi_0 \right) (x) \left( \frac{L \psi_0}{M'_\ell} \right) (x) d\sigma.
\]

The last equality means that the numbers \( \lambda_k A_k \) are the coefficients of the Fourier series of the function \( \left( \frac{L \psi_0}{M'_\ell} \right) (x) \). Therefore, the series \( \sum_{k=1}^{\infty} |\lambda_k^3 A_k| \) converges.

Similarly, estimating \( B_k \), we find that \( \lambda_k B_k \) are the coefficients of the Fourier series of a continuous function \( \left( \frac{L \psi_0}{M'_\ell} \right) (x) \) on \([0, \ell]\). The convergence of the series \( \sum_{k=1}^{\infty} (\lambda_k B_k)^2 \) follows from the analogue of Bessel’s inequality

\[
\sum_{k=1}^{\infty} (\lambda_k B_k)^2 \leq \int_0^\ell \left( M'_\ell \psi_0 (x) \left( \frac{L \psi_0}{M'_\ell} \right) (x) \right)^2 d\sigma,
\]
The theorem is proved. The series obtained by formal differentiation have the form

\[
u_x'(x, t) = \sum_{k=1}^{\infty} \varphi_{kx}'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
u_{xx}'(x, t) = \sum_{k=1}^{\infty} \varphi_{kxx}'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
u_{xxx}'(x, t) = \sum_{k=1}^{\infty} \varphi_{kxxx}'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
(pu_{xxx})_x'(x, t) = \sum_{k=1}^{\infty} (p \varphi_{kxxx})_x'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
(pu_{xxx})_{xx}'(x, t) = \sum_{k=1}^{\infty} (p \varphi_{kxxx})_{xx}'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
(pu_{xxx})_{xxx}'(x, t) = \sum_{k=1}^{\infty} (p \varphi_{kxxx})_{xxx}'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
 ru_{xx}'(x, t) = \sum_{k=1}^{\infty} (r \varphi_{kxx})_x'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
 ru_{xx}'(x, t) = \sum_{k=1}^{\infty} (r \varphi_{kxx})_{xx}'(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
(gv_x)'_\sigma(x, t) = \sum_{k=1}^{\infty} (g \varphi_{kx})'_\sigma(x) \left( A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right),
\]

\[
u_x(t) = \sum_{k=1}^{\infty} \varphi_k(x) \left( -A_k \sqrt{\lambda_k} \sin \sqrt{\lambda_k} t + B_k \cos \sqrt{\lambda_k} t \right),
\]

\[
u_{tt}(x, t) = \sum_{k=1}^{\infty} \varphi_k(x) \left( -A_k \lambda_k \sin \sqrt{\lambda_k} t + B_k \sqrt{\lambda_k} \cos \sqrt{\lambda_k} t \right),
\]

All series above are estimated

\[
K \sum_{k=1}^{\infty} \left( |A_k \lambda_k| + |B_k \sqrt{\lambda_k}| \right),
\]

which converges. This implies the uniform and absolute convergence of all series obtained from (3) by term-by-term differentiation. And since the function \(u(x, t)\), defined by equality (3), satisfies the boundary and initial conditions, the function \(u(x, t)\) is indeed a solution of problem (1). The theorem is proved.

In addition to the conditions stated above, we require that all arising mixed derivatives do not depend on the order of differentiation.

**Theorem 3.** Mathematical model (1) has a unique solution.
By Fubini’s theorem, we obtain

\[ M_t'(x) \frac{\partial^2 u}{\partial x^2} = (pu''_{xx\mu})_{xx\sigma} - (ru'_{xx})_{xx\sigma} + (gu'_x)_\sigma - uQ'_\sigma; \]

\[ -(pu''_{xx\mu})(0, t) + \gamma_1 u''_{xx}(0, t) = 0; \]

\[ (pu''_{xx\mu})_{x}(0, t) - ru''_{xx}(0, t) + \gamma_2 u'_x(0, t) = 0; \]

\[ -(pu''_{xx\mu})_{xx}(0, t) + (ru''_{xx})_{x}(0, \ell) - gu'_x(0, t) + \gamma_3 u(0, t) = 0; \]

\[ u_x(0, 0) = 0; \]

\[ u'_x(0, 0) = 0. \]

Let \( (x^*, T^*) \) be the point at which the solutions are different.

Consider the integral

\[ \int \int_0^T \int_0^\ell \frac{\partial u}{\partial t} \left( M_t'(x)u''_{tt} - (pu''_{xx\mu})_{xx\sigma} + (ru''_{xx})_{xx\sigma} - (gu'_x)_\sigma + uQ'_\sigma \right) d\sigma dt. \]  

(16)

We split the integral into two groups. Consider the first group consisting of a single term. By Fubini’s theorem, we obtain

\[ \int \int_0^T \int_0^\ell \frac{\partial u}{\partial t} M_t'(x)u''_{tt} d\sigma dt = \int \int_0^\ell \frac{1}{2} \frac{\partial}{\partial t} \left( \left( \frac{\partial u}{\partial t} \right)^2 \right) dM dt = \]

\[ = \frac{1}{2} \int_0^\ell \left( \left( \frac{\partial u}{\partial t} \right)^2 (s, T^*) - \left( \frac{\partial u}{\partial t} \right)^2 (s, 0) \right) dM(s) = \frac{1}{2} \int \left( \frac{\partial u}{\partial t} \right)^2 (s, T^*) dM(s) \]

(17)

as \( u'_x(0, 0) = 0. \)

For the second group, we successively find

\[ \int \int_0^T \int_0^\ell \frac{\partial u}{\partial t} \left( - (pu''_{xx\mu})_{xx\sigma} + (ru''_{xx})_{xx\sigma} - (gu'_x)_\sigma + uQ'_\sigma \right) d\sigma dt = \]

\[ = \int_0^T \left( - u'_x (pu''_{xx\mu})_{xx} \right|_0^\ell + \int_0^\ell ru''_{xx} u''_{xx} dx - u'_x (gu'_x)_x \right|_0^\ell + \int_0^T (ru''_{xx})_{xx} u''_{xx} \ d\mu + \]

\[ + u'_x (ru'_{xx}) \right|_0^\ell - u''_{xx} (ru''_{xx}) \right|_0^\ell + \gamma_1 \int_0^T u''_{xx}(0, t) dt + \gamma_2 \int_0^T u'_x(0, t) dt + \gamma_3 \int_0^T u'_x(0, t) u(0, t) dt + \]

\[ = \gamma_1 \int_0^T u''_{xx}(0, t) dt + \gamma_2 \int_0^T u'_x(0, t) dt + \gamma_3 \int_0^T u'_x(0, t) u(0, t) dt + \]

\[ = \gamma_1 \int_0^T u''_{xx}(0, t) dt + \gamma_2 \int_0^T u'_x(0, t) dt + \gamma_3 \int_0^T u'_x(0, t) u(0, t) dt + \]

\[ + \gamma_4 \int_0^T u''_{xx}(0, t) u(0, t) dt + \gamma_5 \int_0^T u'_x(0, t) u(0, t) dt + \gamma_6 u(0, t) = 0; \]

\[ u(x, 0) = 0; \]

\[ u'_x(x, 0) = 0. \]
of non-negative terms hold. If all
\( \gamma_i > 0 \), it follows that the equalities
\[ u''_{xx}(0, T^*) = u'_x(0, T^*) = u(0, T^*) = u''_{xx}(\ell, T^*) = u'_x(\ell, T^*) = u(\ell, T^*) = 0 \] (18)
hold. If all \( \gamma_i = \infty \), then the equalities (18) will be fulfilled automatically.

From equality
\[ \int_0^\ell p(x) \left( u'''_{xx}(x, T^*) \right)^2 \, dx = 0 \]
it follows that $p(x)u'''_{xx\mu}(x,T^*) = 0$ almost everywhere with respect to the measure $\mu$. Then, $u(x,T^*)$ is a square trinomial, which, together with the conditions (18), means the validity of the identity $u(x,T^*) \equiv 0$, which contradicts our assumption.

The theorem is proved.

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