REGULARITY CRITERIA FOR THE 3D NAVIER-STOKES AND MHD EQUATIONS

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Abstract. We prove that a solution to the 3D Navier-Stokes or MHD equations does not blow up at \( t = T \) provided
\[
\limsup_{q \to \infty} \int_T^T \| \Delta_q (\nabla \times u) \|_\infty \, dt
\]
is small enough, where \( u \) is the velocity, \( \Delta_q \) is the Littlewood-Paley projection, and \( T_q \) is a certain sequence such that \( T_q \to T \) as \( q \to \infty \). This improves many existing regularity criteria.

KEY WORDS: Navier-Stokes equations; magneto-hydrodynamics system; regularity; blow-up criteria.
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1. Introduction

The three-dimensional incompressible magneto-hydrodynamics (MHD) equations are given by
\[
\begin{align*}
  u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p &= \nu \Delta u, \\
  b_t + u \cdot \nabla b - b \cdot \nabla u &= \mu \Delta b, \\
  \nabla \cdot u &= 0, \\
  \nabla \cdot b &= 0.
\end{align*}
\]
(1.1)

Here \( x \in \mathbb{R}^3, t \geq 0 \), \( u \) is the fluid velocity, \( p \) is the fluid pressure and \( b \) is the magnetic field. The parameter \( \nu \) denotes the kinematic viscosity coefficient of the fluid, and \( \mu \) is the reciprocal of the magnetic Reynolds number. The MHD equations are supplemented with initial conditions
\[
\begin{align*}
  u(x, 0) &= u_0(x), \\
  b(x, 0) &= b_0(x),
\end{align*}
\]
(1.2)

where \( u_0 \) and \( b_0 \) are divergence free vector fields in \( L^2(\mathbb{R}^3) \). When the magnetic field \( b(x,t) \) vanishes, (1.1) reduce to the incompressible Navier-Stokes equations (NSE). Solutions to the MHD equations also share the same scaling property with solutions to the NSE, that is,
\[
\begin{align*}
  u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\
  b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t), \\
  p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)
\end{align*}
\]

solve (1.1) with the initial data
\[
\begin{align*}
  u_{0\lambda} = \lambda u_0(\lambda x), \\
  b_{0\lambda} = \lambda b_0(\lambda x),
\end{align*}
\]
provided \( u(x,t) \) and \( b(x,t) \) solve (1.1) with the initial data \( u_0(x) \) and \( b_0(x) \). Spaces that are invariant under the above scaling are called critical. Examples of critical

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spaces include the Prodi-Serrin spaces

\[ L^1(0, \infty; L^r), \quad \frac{2}{r} + \frac{3}{r} = 1, \]

or their Besov counterparts

\[ L^1\left(0, \infty; B^{r-1+\frac{2}{s}+\frac{2}{q}}_{r,\infty}\right). \]

The existence of global regular solutions for the NSE and the MHD system in 3D is an outstanding open problem. The study of these models in critical spaces has been one of the focuses of research activities since the initial work of Kato [25]. It is known that if a solution belongs to a certain critical space, then it is regular. For instance, a smooth solution to the NSE \((0, T)\) does not blow up at \(T\) if

\[
\int_0^T \|\nabla \times u\|_\infty \, dt < \infty,
\]

which is known as the Beale-Kato-Majda (BKM) condition [2]. In term of scaling, this is the borderline of Prodi-Serrin criteria when the time integrability index is one, which does not require any dissipation and also applies to the Euler equations.

For the Euler equations, the BKM condition [13] was weakened by Planchon [29] to

\[
\lim_{\varepsilon \to 0} \sup_{q} \int_{T-\varepsilon}^T \|\Delta_q(\nabla \times u)\|_\infty \, dt < c,
\]

for small enough \(c\), where \(\Delta_q\) is the Littlewood-Paley projection.

For the 3D NSE, Cheskidov and Shvydkoy [14] proved the following improvement of the Beale-Kato-Majda condition [13]:

\[
\int_0^T \|\nabla \times u \leq Q\|_{B^{0}_{\infty;\infty}} \, dt < \infty,
\]

where \(2Q(t) = \Lambda(t)\) and \(\Lambda(t)\) is a wavenumber constructed based on the structure of the NSE. This condition is also weaker than all the Prodi-Serrin type regularity criteria. Indeed,

\[
u_{\leq Q} \in L^r(0, T; B_{\infty;\infty}^{-1+\frac{2}{r}}) \quad \text{for some} \quad 1 \leq r < \infty
\]

implies [13] (see [14]). Note that \(r\) cannot be taken as \(\infty\) here. A regularity criterion in the limit case \(r = \infty\) has been established by Escuadra, Seregin, and Sverak [19] in \(L^\infty(0; L^3)\), and extended by Gallagher, Koch, and Planchon to \(L^\infty(0; B^{-1+\frac{2}{r}}_{\infty;\infty})\) with \(3 < s, q < \infty\) in [20].

For the MHD system, Caflisch, Klapper and Steele [3] obtained the BKM-type condition in the inviscid case, but in terms of both \(u\) and \(b\). Cannone, Chen, and Miao [4] proved the extended version of the BKM criterion, as [14], but also in terms of both \(u\) and \(b\). In the viscous case, Wu [35] obtained some Prodi-Serrin type criteria which impose conditions on both \(u\) and \(b\). However there have been various indications, both numerically [22, 30] and theoretically, suggesting that the velocity field \(u\) plays a dominant role in interactions between \(u\) and \(b\). This point is still not clear though. In fact, in [17], the authors showed that even when the initial velocity vanishes, one can observe norm inflation in velocity at a small time due to the interaction of the magnetic field and the velocity field (see also [10] for norm inflation results in a wider range of spaces). Nevertheless, a large amount of
work has been devoted to establishing criteria which impose conditions only on the
velocity, see [5, 8, 7, 23, 24, 27, 34, 37]. In particular, He, Xin, and Zhou [24] obtained the following Prodi-Serrin conditions for regularity:
\[
\begin{align*}
  u &\in L^1(0,T; L^r) \quad \text{with} \quad \frac{2}{l} + \frac{3}{r} = 1 \quad \text{for} \quad 3 < r \leq \infty; \\
  \nabla u &\in L^1(0,T; L^r) \quad \text{with} \quad \frac{2}{l} + \frac{3}{r} = 2 \quad \text{for} \quad 3 < r \leq \infty.
\end{align*}
\]
(1.6)

He and Xin [24] also obtained the following regularity condition:
\[
(1.7) \quad u \in C([0,T]; L^3).
\]

Finally, remarkable improvements were made by Chen, Miao and Zhang who ob-
tained the extended BKM criterion [14] in terms of the velocity only [7], as well as the Prodi-Serrin regularity criterion in terms of Besov spaces [6]:
\[
(1.8) \quad u \in L^1(0,T; B^s_{r,\infty}) \quad \text{with} \quad \frac{2}{l} + \frac{3}{r} = 1 + s,
\]
where \[\lambda = 2^q, \quad u_p = \Delta_p u\] is the Littlewood-Paley projection of \(u\), and \(c_r\) is an adimensional constant that depends only on \(r\). Here \(r \in [2,6)\) in the case of the the MHD system (c.f. proof of Proposition 5.1), and \(r \in [2,\infty]\) for the NSE (when \(b \equiv 0\)). Let \(Q_r(t) \in \mathbb{N}\) be such that \(\lambda_{Q_r(t)} = \Lambda_r(t)\).

The wavenumber \(\Lambda_r(t)\), defined for each individual solution (in terms of \(u\) only), separates the inertial range from the dissipation range where the viscous terms \(\Delta u\) and \(\Delta b\) dominate. The dissipation wavenumber is defined so that the critical \(B^{−1+\frac{2}{r}}\) norm of \(u\) is small above \(\Lambda_r\) for some \(2 \leq r < 6\). So it is larger than the dissipation wavenumber for the NSE defined with \(r = \infty\) in [14]. This is due to the fact that there are less cancellations and more terms to control for the MHD system. It is worth mentioning that this method is also applied to the supercritical quasi-geostrophic equation in [15] and to the Hall-magneto-hydrodynamics system in [16]. Regularity criteria obtained in these papers are weaker than all the corre-
sponding Prodi-Serrin type criteria. The dissipation wavenumber is also related to the determining wavenumber used in [9, 12] to estimate the number of determining modes for fluid flows.

Our main result states as follows.

**Theorem 1.1.** Let \((u, b)\) be a weak solution to (1.1) on \([0,T]\). Assume that \((u(t),b(t))\) is regular on \((0,T)\). If for any \(2 \leq r < 6\) (\(2 \leq r \leq \infty\) when \(b \equiv 0\)),
\[
\limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q_r(t)} \|\Delta_q(\nabla \times u)\|_\infty dt \leq c_r,
\]
(1.10)
with a small constant $c_r > 0$ depending on $r$, then $(u(t), b(t))$ is regular on $(0, T]$.

**Remark 1.2.** The choice of constant $c_r$ will be made specific later on, see (3.32).

We will show that (1.10) is weaker than various existing regularity criteria. Indeed, define

$$T_q = \sup \{ t \in (T/2, T) : Q_r(\tau) < q, \forall \tau \in (T/2, t) \}.$$  

Note that $T_q$ is increasing. In addition, we show that

$$\lim_{q \to \infty} T_q = T.$$  

**Theorem 1.3.** Regularity condition (1.10) is satisfied provided one of the following holds.

1. $\lim_{q \to \infty} \sup_{T_q} \int_T^T \| \Delta_q(\nabla \times u) \|_\infty \, dt \leq c_r$.

2. $\lim_{\epsilon \to 0} \limsup_{q \to \infty} \int_{T-\epsilon}^T T_q \int_0^T \| \Delta_q(\nabla \times u) \|_\infty \, dt \leq c_r$.

3. $\int_0^T \sup_{r \leq Q_r(\tau)} \| \Delta_q(\nabla \times u) \|_\infty \, dt \leq \infty$.

4. $\limsup_{q \to \infty} \int_0^T 1_{r \leq Q_r(\tau)} \left( \lambda_q^{1-\frac{3}{r}+\frac{2}{q}} \| u_q(t) \|_r \right) dt \leq c_r \min \{ \nu, \mu \}^{l-1}, 1 \leq l < \infty, 2 \leq r < 6 (2 \leq r \leq \infty when b = 0)$.

5. $\limsup_{q \to \infty} \int_{T_q} \lambda_q^{1-\frac{3}{r}+\frac{2}{q}} \| u_q(t) \|_r \right) dt \leq c_r \min \{ \nu, \mu \}^{l-1}, 1 \leq l < \infty, 2 \leq r < 6 (2 \leq r \leq \infty when b = 0)$.

6. $\limsup_{q \to \infty} \int_{T_q} \lambda_q^{1-\frac{3}{r}+\frac{2}{q}} \| u_q(t) \|_r \right) dt \leq c_r \min \{ \nu, \mu \}^{l-1}, 1 \leq l < \infty, 2 \leq r < 6 (2 \leq r \leq \infty when b = 0)$.

7. $\int_0^T \left( \| u \|_{Q_r(\tau)} \right) \frac{T}{B_{\infty, \infty}} \right) dt \leq \infty, 1 \leq l < \infty, 2 \leq r < 6 (2 \leq r \leq \infty when b = 0)$.

8. $\limsup_{t \to T^-} \left( \| u(t) - u(T) \|_{B_{\infty, \infty}} \right) \frac{T}{2} \min \{ \nu, \mu \}, 2 \leq r < 6 (2 \leq r \leq \infty when b = 0)$.

**Remark 1.4.** In the case of the 3D NSE ($b = 0$), the value $r = \infty$ in both Theorem 1.1 and Theorem 1.3 gives the best regularity criteria due to Bernstein’s inequality.

**Remark 1.5.** If $u \in C((0, T], L^3)$, then $Q_r \in L^\infty(T/2, T)$ for any $3 \leq r \leq \infty$, and every regularity condition in Theorems 1.1 and 1.3 is automatically satisfied. However, these regularity conditions exclude the $L^\infty(0; T, L^3)$ criterion by Escauriaza, Seregin, and Sverak [19]. Notably, recent Tao’s result for the 3D NSE [32] Theorem 5.1] implies that $Q_\infty \in L^\infty(T/2, T)$ provided $u \in L^\infty(0; T, L^3)$. What is even more remarkable, Tao’s result provides a triple exponential control of $\Lambda_\infty$ in terms of $\| u \|_3$, and gives quantitative (triple logarithmic) blow up rate for $\| u \|_3$. 

For the 3D NSE, conditions (6), (8), and (3), (7) were obtained by Cheskidov, Shvydkoy in [13] and [14] respectively. Regarding the MHD system, condition (7) (and hence (4), (5), and (6)) is weaker than (1.6), (1.8) for \( r < 6 \), and condition (8) is weaker than (1.7). Conditions (1) and (2) are weaker than (1.4) for both the 3D NSE and MHD system.

The rest of the paper is organized as follows. In Section 2 we introduce some notations, recall the Littlewood-Paley theory and definitions of weak and regular solutions. Section 3 is devoted to proving Theorem 1.1 and Theorem 1.3.

2. Preliminaries

2.1. Notation. We denote by \( A \lesssim B \) an estimate of the form \( A \leq CB \) with some absolute constant \( C \), and by \( A \sim B \) an estimate of the form \( C_1B \leq A \leq C_2B \) with some absolute constants \( C_1, C_2 \). We write \( \| \cdot \|_p = \| \cdot \|_{L^p} \). The symbol \( (\cdot, \cdot) \) stands for the \( L^2 \)-inner product.

2.2. Littlewood-Paley decomposition. Our method relies on the Littlewood-Paley decomposition, which we briefly recall here. We refer the readers to the books by Bahouri, Chemin and Danchin [1] and Grafakos [21] for a more detailed description on this theory.

Let \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transform and inverse Fourier transform, respectively. A nonnegative radial function \( \chi \in C_0^\infty(\mathbb{R}^n) \) is chosen such that

\[
\chi(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq \frac{3}{4} \\
0, & \text{for } |\xi| \geq 1.
\end{cases}
\]

Let

\[
\varphi(\xi) = \chi(\xi/2) - \chi(\xi),
\]

and

\[
\varphi_q(\xi) = \begin{cases} 
\varphi(\lambda_q^{-1} \xi) & \text{for } q \geq 0, \\
\chi(\xi) & \text{for } q = -1.
\end{cases}
\]

Note that the sequence of the smooth functions \( \varphi_q \) forms a dyadic partition of unity. For a tempered distribution vector field \( u \) we define the Littlewood-Paley decomposition

\[
\begin{cases} 
\bar{h} = \mathcal{F}^{-1} \varphi, & \bar{h} = \mathcal{F}^{-1} \chi, \\
u_q := \Delta_q u = \mathcal{F}^{-1} (\varphi(\lambda_q^{-1} \xi) \mathcal{F} u) = \lambda_q^n \int \bar{h}(\lambda_q y) u(x - y) dy, & \text{for } q \geq 0, \\
u_{-1} = \mathcal{F}^{-1} (\chi(\xi) \mathcal{F} u) = \int \bar{h}(y) u(x - y) dy.
\end{cases}
\]

Then

\[
u = \sum_{q=-1}^{\infty} u_q
\]

holds in the distributional sense. To simplify the notation, we denote

\[
\tilde{u}_q = u_{q-1} + u_q + u_{q+1}, \quad u_{\leq Q} = \sum_{q=-1}^{Q} u_q, \quad u_{(P,Q]} = \sum_{q=P+1}^{Q} u_q.
\]

The Besov space \( B^s_{p,\infty} \) is defined as follows.
**Definition 2.1.** Let \( s \in \mathbb{R} \), and \( 1 \leq p \leq \infty \). Let 
\[
\|u\|_{B^s_{p,\infty}} = \sup_{q \geq -1} \lambda^s_q \|u_q\|_p.
\]

The Besov space \( B^s_{p,\infty} \) is the space of tempered distributions \( u \) such that \( \|u\|_{B^s_{p,\infty}} \) is finite.

Note that 
\[
\|u\|_{H^s} \sim \left( \sum_{q=-1}^{\infty} \lambda^{2s}_q \|u_q\|^2 \right)^{1/2},
\]
for each \( u \in H^s \) and \( s \in \mathbb{R} \).

We recall Bernstein’s inequality as follows.

**Lemma 2.2.** \([28]\) Let \( n \) be the space dimension and \( r \geq s \geq 1 \). Then for all tempered distributions \( u \),
\[
\|u\|_r \lesssim \lambda^n_q \|u_q\|_s.
\]

Throughout the paper we will utilize the following sharp version of Bony’s para-product decomposition
\[
\Delta_q(u \cdot \nabla v) = \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot \nabla) v_p + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot \nabla) v_{\leq p-2} + \sum_{p \geq q-2} \Delta_q(u_p \cdot \nabla) \tilde{v}_p,
\]
which holds because we insist on a rather specific choice of the bump function \( \varphi \) via \((2.11)\):
\[
\varphi(\xi) = \begin{cases} 
1, & \text{for } 1 \leq |\xi| \leq 3/2 \\
0, & \text{for } |\xi| \leq 3/4 \text{ or } |\xi| \geq 2,
\end{cases}
\]
see Section 2.3 of \([11]\) for more details.

We will also employ the commutator notation 
\[
[\Delta_q, u_{\leq p-2} \cdot \nabla] v_p = \Delta_q(u_{\leq p-2} \cdot \nabla) v_p - (u_{\leq p-2} \cdot \nabla) \Delta_q v_p,
\]
which allows to essentially move the derivative from \( v_p \) to \( u_{\leq p-2} \).

**Lemma 2.3.** For any \( 1 < r_1 < \infty \) and \( r_1 \leq r_2, r_3 \leq \infty \) with \( \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1} \), the commutator satisfies
\[
(2.12) \quad \|[\Delta_q, u_{\leq p-2} \cdot \nabla] v_p\|_{r_1} \lesssim \|v_p\|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \|u_{p'}\|_{r_2}.
\]
**Proof:** We rewrite the commutator by applying the definition of $\Delta_q$, integration by parts, first order Taylor’s formula, and a change of variable

$$
[\Delta_q, u_{\leq p-2} \cdot \nabla] v_p
= \int_{\mathbb{R}^3} \lambda_3^q h(\lambda_1(x-y))(u_{\leq p-2}(y) - u_{\leq p-2}(x)) \cdot \nabla_y v_p(y) dy
$$

$$
= -\int_{\mathbb{R}^3} \lambda_3^q \nabla_y h(\lambda_1(x-y)) \left( (u_{\leq p-2}(y) - u_{\leq p-2}(x)) \otimes v_p(y) \right) dy
$$

$$
= -\int_{0}^{1} \int_{\mathbb{R}^3} \lambda_3^q \nabla_y h(\lambda_1(x-y)) \left( ((y-x) \cdot \nabla u_{\leq p-2}(x + \tau(y-x)) \otimes v_p(y) \right) dy d\tau
$$

$$
= \int_{0}^{1} \int_{\mathbb{R}^3} \lambda_3^q \nabla_z h(\lambda_1 z) \left( (z \cdot \nabla u_{\leq p-2}(x - \tau z)) \otimes v_p(x - z) \right) dz d\tau.
$$

To continue, we apply Minkowski’s integral inequality, Hölder’s inequality (with respect to $x$ variable), and the translation invariance of Lebesgue norm, and obtain

$$
\begin{align*}
\| [\Delta_q, u_{\leq p-2} \cdot \nabla] u_q \|_{r_1} & \lesssim \int_{0}^{1} \int_{\mathbb{R}^3} \lambda_3^q |z| \| \nabla_z h(\lambda_1 z) \| \| \nabla u_{\leq p-2}(\cdot - \tau z) v_p(\cdot - z) \|_{r_1} d\tau dz

& \lesssim \int_{0}^{1} \int_{\mathbb{R}^3} \lambda_3^q |z| \| \nabla_z h(\lambda_1 z) \| \| \nabla u_{\leq p-2}(\cdot - \tau z) \|_{r_2} \| v_p(\cdot - z) \|_{r_3} d\tau dz

& \lesssim \| v_p \|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \| u_{p'} \|_{r_2} \int_{0}^{1} \int_{\mathbb{R}^3} \lambda_3^q |z| \| \nabla_z h(\lambda_1 z) \| d\tau dz

& \lesssim \| v_p \|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \| u_{p'} \|_{r_2},
\end{align*}
$$

where we used the fact that $\nabla h \in L^1$ in the last step. \qed

### 2.3. Definition of solutions

We recall some classical definitions of weak and regular solutions to a differential equation system (see, e.g., [18, 31, 33]).

**Definition 2.4.** A weak solution of (1.1) on $[0, T]$ is a pair of divergence free functions $(u, b)$ in the class

$$
u, b \in C_w([0, T]; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)),
$$

satisfying

$$
u(t, \phi(t)) - (u_0, \phi(0))
= \int_{0}^{t} (u(s), \partial_s \phi(s)) + \nu(u(s), \Delta \phi(s)) + (u(s) \cdot \nabla \phi(s), u(s)) - (b(s) \cdot \nabla \phi(s), b(s)) ds,
$$

$$
u(t, \phi(t)) - (b_0, \phi(0))
= \int_{0}^{t} (b(s), \partial_s \phi(s)) + \mu(b(s), \Delta \phi(s)) + (u(s) \cdot \nabla \phi(s), b(s)) - (b(s) \cdot \nabla \phi(s), u(s)) ds,
$$

for all test functions $\phi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ with $\nabla_x \phi = 0$. 


It is known that space $\dot{H}^s$ is critical for the 3D NSE and $\dot{H}^{\frac{s}{2}} \times \dot{H}^{\frac{s}{2}}$ is critical for the 3D MHD under the aforementioned natural scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ and $b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t)$. The continuity of a subcritical norm on a time interval implies that the solution is smooth on this interval. Thus we adapt the following definition of a regular solution.

**Definition 2.5.** A weak solution $(u, b)$ of (1.1) is regular on a time interval $I$ if $\|u(t)\|_{H^s}$ and $\|b(t)\|_{H^s}$ are continuous on $I$ for some $s > \frac{1}{2}$.

**Remark 2.6.**

1. A weak solution $(u, b)$ of (1.1) is regular on $(0, T)$ if and only if $(u, b) \in C^\infty(\mathbb{R}^3 \times (0, T))$.
2. A weak solution $(u, b)$ of (1.1) is regular on $(0, T)$ if and only if $(u, b) \in C^\infty(\mathbb{R}^3 \times (0, T))$ and
   \[
   \limsup_{t \to T^-} (\|u(t)\|_{H^s} + \|b(t)\|_{H^s}) < \infty,
   \]
   for some $s > \frac{1}{2}$.
3. If a weak solution $(u, b)$ of (1.1) is regular on $(0, T]$, then it can be extended as a smooth solution to a longer interval $(0, T + \varepsilon)$, for some $\varepsilon > 0$.

3. Regularity Criterion

In this section we will establish the regularity criterion in Theorem 1.1. Let $(u(t), b(t))$ be a weak solution of (1.1) on $[0, T]$. Recall our definition of the dissipation wavenumber $\Lambda_r(t)$ as in (1.9). We will see that the constant $c_r$ in the definition needs to be chosen sufficiently small in the course of the proof of Theorem 1.1. Recall $Q_r(t) \in \mathbb{N}$ be such that $\lambda_{Q_r(t)} = \Lambda_r(t)$. It follows immediately that
   \[
   \|u_p(t)\|_{p} < \lambda_p^{1-\frac{s}{2}} c_r \min \{\nu, \mu\}, \quad \forall p > Q_r(t),
   \]
and
   \[
   \|u_{Q_r(t)}(t)\|_{r} \geq c_r \min \{\nu, \mu\} \lambda_r^{1-\frac{s}{2}}(t),
   \]
provided $1 < \Lambda_r(t) < \infty$. For simplicity, from now on we drop the subscript $r$ in $Q_r$ unless specified otherwise. We also denote
   \[
   f(t) = \sum_{\nu \leq Q(t)} \lambda_{\nu} \|u_{\nu}(t)\|_{\infty}.
   \]

3.1. **Proof of Theorem 1.1.** In order to prove that $(u, b)$ does not blow up at $T$, it is sufficient to show that $\|u(t)\|_{H^s} + \|b(t)\|_{H^s}$ is bounded on $[T/2, T)$ for some $s > \frac{1}{2}$. Before getting into lengthy computations and estimates, we first outline the scheme of the proof:

1. Establish the main energy estimate stated in the following proposition.

**Proposition 3.1.** For any $2 \leq r < 6$ and $\frac{2}{3} < s < \frac{3}{2}$ (any $2 \leq r \leq \infty$ and $s > \frac{1}{2}$ in case $b \equiv 0$), there exist adimensional constants $\tilde{c}$ and $C$, such that
   \[
   \frac{d}{dt} (\|u\|_{H^s}^2 + \|b\|_{H^s}^2) \leq C f(t) (\|u\|_{H^s}^2 + \|b\|_{H^s}^2),
   \]
provided $c_r \leq \hat{c}$.

(ii) Show that one can weaken the condition $f \in L^1$ to the assumption \(1.10\), which still allows us to apply Grönwall’s inequality to \(3.14\).

(iii) Complete the claim of Theorem \(1.1\).

**Step (i):** This step is devoted to the proof of Proposition \(3.1\), which consists of heavy energy estimates through standard frequency localization techniques and the wavenumber splitting approach developed in the first author’s previous work \([14]\).

Since \((u(t), b(t))\) is regular on \((0, T)\), multiplying equations \((1.1)\) with \(\Delta^2_q u\) and \(\Delta^2_q b\) respectively yields

$$
\frac{1}{2} \frac{d}{dt} \|u_q\|_2^2 \leq -\nu \|\nabla u_q\|_2^2 - \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx
$$

$$
+ \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla b) \cdot u_q \, dx,
$$

$$
\frac{1}{2} \frac{d}{dt} \|b_q\|_2^2 \leq -\mu \|\nabla b_q\|_2^2 - \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla b) \cdot b_q \, dx
$$

$$
+ \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla u) \cdot b_q \, dx.
$$

Multiplying the above two inequalities by \(\lambda^{2s}_q\), adding the resulted inequalities, and taking summation for all \(q \geq -1\), we obtain

$$
\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda^{2s}_q \left( \|u_q\|_2^2 + \|b_q\|_2^2 \right) \leq - \sum_{q \geq -1} \lambda^{2s}_q \left( \nu \|\nabla u_q\|_2^2 + \mu \|\nabla b_q\|_2^2 \right)
$$

$$
- (I + J + K + L),
$$

(3.15)

with

$$
I = \sum_{q \geq -1} \lambda^{2s}_q \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx, \quad J = - \sum_{q \geq -1} \lambda^{2s}_q \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla b) \cdot u_q \, dx,
$$

$$
K = \sum_{q \geq -1} \lambda^{2s}_q \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla b) \cdot b_q \, dx, \quad L = - \sum_{q \geq -1} \lambda^{2s}_q \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla u) \cdot b_q \, dx.
$$

In the rest of this step, we will establish estimates for the flux terms \(I, J, K,\) and \(L\), as respectively stated in Lemma \(5.2\), Lemma \(5.3\), and Lemma \(5.4\). Proposition \(3.1\) is then an immediate consequence of \(3.15\) and the estimates in Lemmas \(5.2\) \(5.3\) \(5.4\).

In order to estimate the flux terms \(I, J, K,\) and \(L\), we will employ extensively standard harmonic analysis techniques, Bony’s paraproduct and commutator estimates. The key ingredient is that we split each flux term into high frequency and low frequency parts via the dissipation wavenumber \(\Lambda_r(t)\), such that the high frequency part can be absorbed by the diffusion.

**Lemma 3.2.** The flux term \(I\) satisfies, for any \(s > 0\) and \(r \geq 2\),

$$
|I| \lesssim c_r \nu \sum_{q > Q-3} \lambda^{2s+2}_q \|u_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda^{2s}_q \|u_q\|_2^2.
$$

(3.16)
**Proof:** We first recall Bony’s paraproduct decomposition

\[
\Delta_q (u \cdot \nabla u) = \sum_{|q-p| \leq 2} \Delta_q (u_{\leq p-2} \cdot \nabla u_p) + \sum_{|q-p| \leq 2} \Delta_q (u_p \cdot \nabla u_{\leq p-2}) + \sum_{p \geq q-2} \Delta_q (u_p \cdot \nabla \tilde{u}_p).
\]

Thus, applying Bony’s paraproduct above, the flux term \(I\) can be decomposed as

\[
I = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla u_p) \cdot u_q \, dx
\]

\[
+ \sum_{q \geq -1} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla u_{\leq p-2}) \cdot u_q \, dx
\]

\[
+ \sum_{q \geq -1} \sum_{p \geq q-2} \sum_{p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \tilde{u}_p) \cdot u_q \, dx
\]

\[
= I_1 + I_2 + I_3.
\]

with

\[
I_1 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, u_{\leq p-2} \cdot \nabla] u_p \cdot u_q \, dx
\]

\[
+ \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_{\leq q-2} \cdot \nabla) u_q \cdot u_q \, dx
\]

\[
+ \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla) \Delta_q u_p \cdot u_q \, dx
\]

\[
= I_{11} + I_{12} + I_{13},
\]

where we used \(\sum_{|q-p| \leq 2} \Delta_q u_p = u_q\). One can see that \(I_{12}\) vanishes since \(\text{div} u_{\leq q-2} = 0\). We now estimate \(I_{11}\), where we used the commutator notation

\[
[\Delta_q, u_{\leq p-2} \cdot \nabla] u_p = \Delta_q (u_{\leq p-2} \cdot \nabla u_p) - (u_{\leq p-2} \cdot \nabla) \Delta_q u_p.
\]
Further splitting $I_{11}$ based on definition of $\Lambda_r$, we get

$$|I_{11}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2, p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} |[\Delta q, u_{\leq p-2} \cdot \nabla]u_p \cdot u_q| \, dx$$

$$\leq \sum_{-1 \leq p \leq Q+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |[\Delta q, u_{\leq p-2} \cdot \nabla]u_p \cdot u_q| \, dx$$

$$+ \sum_{p>Q+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |[\Delta q, u_{Q+p-2} \cdot \nabla]u_p \cdot u_q| \, dx$$

$$+ \sum_{p>Q+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |[\Delta q, u_{(Q,p-2)} \cdot \nabla]u_p \cdot u_q| \, dx$$

$$\equiv I_{111} + I_{112} + I_{113}.$$
Now using Hölder’s inequality, definition of $\Lambda_r$, Bernstein’s inequality, and Jensen’s inequality, we deduce

\[
I_{113} \lesssim \sum_{p > Q + 2} \sum_{q \geq -1} \lambda_q^{2s} \| \Delta_q u_p \|_2 \| \nabla \| u_q \|_2 \\
\lesssim \sum_{p > Q + 2} \sum_{q \geq -1} \lambda_p^{2s} \| u_p \|_2 \sum_{p' \leq p - 2} \lambda_{p'} \| u_{p'} \|_r \\
\lesssim \sum_{p > Q + 2} \lambda_p^{\frac{2s}{r}} \| u_p \|_2 \sum_{q \geq -1} \sum_{q \leq p} \lambda_q^{2s} \| u_q \|_2 \sum_{p' \leq p - 2} \lambda_{p'} \| u_{p'} \|_r \\
\lesssim c_r \nu \sum_{p > Q + 2} \lambda_p^{2s + \frac{2}{r}} \| u_p \|_2 \sum_{q \geq -1} \sum_{q \leq p} \lambda_q^{2s} \| u_q \|_2 \sum_{p' \leq p - 2} \lambda_{p'}^{\frac{2s}{r}} \\
\lesssim c_r \nu \sum_{p > Q + 2} \lambda_p^{2s + 2} \| u_p \|_2 \sum_{q \geq -1} \sum_{q \leq p} \lambda_q^{2s} \| u_q \|_2 \\
\lesssim c_r \nu \sum_{p > Q + 2} \lambda_p^{2s + 2} \| u_p \|_2^2,
\]

since $r \geq 2$. Combining the previous four inequalities lead to

\[
|I_{11}| \lesssim c_r \nu \sum_{p > Q + 2} \lambda_p^{2s + 2} \| u_p \|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \| u_q \|_2^2.
\]

To estimate $I_{13}$, we first split it as

\[
|I_{13}| \leq \sum_{q \geq -1} \sum_{p \geq -1} \lambda_q^{2s} \left| \int_{\mathbb{R}^3} ((u_{\leq p - 2} - u_{\leq q - 2}) \cdot \nabla) \Delta_q u_p \cdot u_q \, dx \right| \\
= \sum_{-1 \leq q \leq Q} \sum_{p \geq -1} \lambda_q^{2s} \left| \int_{\mathbb{R}^3} ((u_{\leq p - 2} - u_{\leq q - 2}) \cdot \nabla) \Delta_q u_p \cdot u_q \, dx \right| \\
+ \sum_{q > Q} \sum_{p \geq -1} \lambda_q^{2s} \left| \int_{\mathbb{R}^3} ((u_{\leq p - 2} - u_{\leq q - 2}) \cdot \nabla) \Delta_q u_p \cdot u_q \, dx \right| \\
\equiv I_{131} + I_{132}.
\]
Using Hölder’s inequality, integration by parts and the fact $\nabla \cdot (u_{p-2} - u_{q-2}) = 0$, and definition of $f(t)$, we obtain

$$|I_{131}| \lesssim \sum_{-1 \leq q \leq Q} \lambda_q^{2s+1} \|u_q\|_\infty \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2$$

$$\lesssim f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2$$

$$\lesssim f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \|u_q\|_2^2.$$ 

Also, Hölder’s inequality and definition of $\Lambda_r$ yield

$$|I_{132}| \lesssim \sum_{q > Q} \lambda_q^{2s+1} \|u_q\|_r \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2 \|u_r\|_p$$

$$\lesssim c_r \nu \sum_{q > Q} \lambda_q^{2s+2-\frac{2}{r}} \sum_{|q-p| \leq 2} \lambda_p^{\frac{2}{r}} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2$$

$$\lesssim c_r \nu \sum_{q > Q-3} \lambda_q^{2s+2} \|u_q\|_2^2$$

The last three inequalities together imply

$$|I_{13}| \lesssim c_r \nu \sum_{q > Q-3} \lambda_q^{2s+2} \|u_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \|u_q\|_2^2.$$ 

Therefore, we conclude from (3.18), (3.19), (3.20) and the fact $I_{12} = 0$ that

$$|I_1| \lesssim c_r \nu \sum_{q > Q-3} \lambda_q^{2s+2} \|u_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2.$$ 

Regarding $I_2$ in (3.14), we claim that it enjoys the same estimate as $I_{11}$ in (3.18). Indeed, the commutator estimate (2.12) indicates that the commutator $[\Delta_q, u_{p-2} \cdot \nabla]u_p$ appeared in $I_{11}$ has the effect of moving a derivative from $u_p$ to $u_{p-2}$. Hence, $I_{11}$ has the same essential structure as $I_2$. Therefore, by (3.19), we have

$$|I_2| \lesssim c_r \nu \sum_{p > Q+2} \lambda_p^{2s+2} \|u_p\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2,$$ 

for $r \geq 2$. 


In the end, we deal with $I_3$ by using Hölder’s inequality, definition of $\Lambda_r$ and $f(t)$, Bernstein’s inequality, and change order of the summations, to obtain

\[
|I_3| \lesssim \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q(u_p \otimes \tilde{u}_p)| \nabla u_q \, dx \\
\lesssim \sum_{q > Q} \lambda_q^{2s+1} \|u_q\|_\infty \sum_{p \geq q-3} \|u_p\|_2^2 + \sum_{-1 \leq q \leq Q} \lambda_q^{2s+1} \|u_q\|_\infty \sum_{p \geq q-3} \|u_p\|_2^2 \tag{3.23}
\]

\[
\lesssim \sum_{q > Q} \lambda_q^{2s+1+\frac{3}{r}} \|u_q\|_r \sum_{p \leq q-3} \|u_p\|_2^3 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{p \geq q-3} \|u_p\|_2^2 \\
\lesssim c_r \nu \sum_{p > Q} \lambda_p^{2s+2} \|u_p\|_2^2 \sum_{Q < p \leq p+3} \lambda_q^{2s+2} + f(t) \sum_{p \leq -1} \lambda_p^{2s} \|u_p\|_2^2 \sum_{q \leq p+3} \lambda_q^{2s} \\
\lesssim c_r \nu \sum_{q > Q} \lambda_q^{2s+2} \|u_q\|_2^3 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2.
\]

Finally, the estimate (3.21) is a consequence of (3.17), (3.21), (3.22), and (3.23).

The other flux terms $J$, $K$, and $L$ will be estimated in a similar fashion. However, to fully exploit the cancellations among the flux, we estimate $J + L$ instead of handling $J$ and $L$ separately.

**Lemma 3.3.** For any $r$ and $s$ satisfying $\frac{1}{2} < s < 1$ and $2 \leq r < \frac{3}{s}$, we have

\[
|J + L| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|h_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|h_q\|_2^2. \tag{3.24}
\]

**Proof:** Similarly, using Bony’s paraproduct decomposition and the commutator notation, $J$ can be decomposed as

\[
J = -\sum_{q \geq -1} \sum_{|p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_{\leq p-2} \cdot \nabla b_p) \cdot u_q \, dx \\
-\sum_{q \geq -1} \sum_{|p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla b_{\leq p-2}) \cdot u_q \, dx \\
-\sum_{q \geq -1} \sum_{|p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla \tilde{b}_p) \cdot u_q \, dx \\
= J_1 + J_2 + J_3, \tag{3.25}
\]
while the flux $L$ can be decomposed as

$$
L = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b \leq -2 \cdot \nabla u) \cdot b_q dx \\
- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b \leq -2 \cdot \nabla) \Delta_q b_p \cdot b_q dx \\
- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((b \leq -2 \cdot \nabla) \Delta_q b_p \cdot b_q dx
$$

(3.27)

with

$$
L_1 = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b \leq -2 \cdot \nabla] u_p \cdot b_q dx \\
- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b \leq -2 \cdot \nabla) \Delta_q u_p \cdot b_q dx \\
- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((b \leq -2 \cdot \nabla) \Delta_q u_p \cdot b_q dx
$$

(3.28)

We shall show that the term $J_{12}$ cancels $L_{12}$. Indeed, using the fact $\sum_{|q-p| \leq 2} \Delta_p b_q = b_q$ and integration by parts, we notice

$$
\sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b \leq -2 \cdot \nabla) \Delta_q b_p \cdot b_q dx = \lambda_q^{2s} \int_{\mathbb{R}^3} (b \leq -2 \cdot \nabla) b_q \cdot b_q dx = 0;
$$

and analogously,

$$
\sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b \leq -2 \cdot \nabla) \Delta_q u_p \cdot u_q dx = \lambda_q^{2s} \int_{\mathbb{R}^3} (b \leq -2 \cdot \nabla) u_q \cdot u_q dx = 0.
$$
Therefore, we deduce

\[
J_{12} + L_{12} = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_{q}^{2s} \int_{\mathbb{R}^3} (b_{q-2} \cdot \nabla) \Delta q b_p \cdot (u_q + b_q) \, dx
- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_{q}^{2s} \int_{\mathbb{R}^3} (b_{q-2} \cdot \nabla) \Delta q u_p \cdot (b_q + u_q) \, dx
= - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_{q}^{2s} \int_{\mathbb{R}^3} (b_{q-2} \cdot \nabla) (u_p + b_p) \cdot (u_q + b_q) \, dx
= 0.
\]

The other terms appeared in (3.25)-(3.28) are estimated as follows. First, we further split \(J_{11}\) as

\[
|J_{11}| \leq \sum_{q > Q} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_{q}^{2s} \int_{\mathbb{R}^3} |(\Delta q, b_{q-2} \cdot \nabla) b_p \cdot u_q| \, dx
+ \sum_{-1 \leq q \leq Q} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_{q}^{2s} \int_{\mathbb{R}^3} |(\Delta q, b_{q-2} \cdot \nabla) b_p \cdot u_q| \, dx
\equiv J_{111} + J_{112}.
\]

Then, by Hölder’s inequality, (2.12), definition of \(\Lambda_r\), and Jensen’s inequality, it follows that

\[
|J_{111}| \lesssim \sum_{q > Q} \lambda_{q}^{2s} ||u_q||_r \sum_{|q-p| \leq 2} \sum_{p \geq -2} \lambda_p \lambda'_p \lambda_{p'}^{2s+1-rac{3}{r}}
\leq C_r \mu \sum_{q > Q} \lambda_{q}^{2s+1-rac{3}{r}} \sum_{|q-p| \leq 2} \sum_{p \geq -1} \lambda_p^{1+s-rac{3}{r}} ||b_p||_2 \sum_{p' \leq q} \lambda_{p'}^{1+s-rac{3}{r}} ||b_{p'}||_2
\leq C_r \mu \sum_{q > Q-2} \lambda_{q}^{2s+1-rac{3}{r}} ||b_q||_2 \sum_{p' \leq q} \lambda_{p'}^{1+s-rac{3}{r}} ||b_{p'}||_2
\leq C_r \mu \sum_{q > Q-2} \lambda_{q}^{s+1} ||b_q||_2 \sum_{p' \leq q} \lambda_{p'}^{s+1} ||b_{p'}||_2
\leq C_r \mu \sum_{p' \leq q} \lambda_{p'}^{s+1} ||b_{p'}||_2 \sum_{q > Q-2} \lambda_{q}^{s+1} ||b_q||_2
\leq C_r \mu \sum_{q \geq -1} \lambda_{q}^{2s+1} ||b_{q}||_2^2.
\]
where we needed $2 \leq r < \frac{3}{2}$. Also, by Hölder’s inequality, definition of $f(t)$ and Jensen’s inequality,

\[
|J_{112}| \lesssim \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \|u_q\|_\infty \sum_{|q-p| \leq 2} \sum_{p' \leq p - 2} \lambda_{p'} \|b_{p'}\|_2 \\
\lesssim f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s-1} \sum_{|q-p| \leq 2} \|b_p\|_2 \sum_{p' \leq q} \lambda_{p'} \|b_{p'}\|_2 \\
\lesssim f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s-1} \|b_q\|_2 \sum_{p' \leq q} \lambda_{p'} \|b_{p'}\|_2 \\
\lesssim f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s} \|b_q\|_2^2, \\
\]

where we used $\frac{1}{2} < s < 1$. The last three inequalities together imply that for $\frac{1}{2} < s < 1$ and $2 \leq r < \frac{4}{s}$,

\[
(3.30) \quad |J_{11}| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s} \|b_q\|_2^2. \\
\]

Similar analysis of employing wavenumber splitting, Hölder’s inequality, Bernstein’s inequality, definition of $\Lambda_r$ and $f(t)$ yields, for $r \geq 2$ and $s \geq 0$

\[
|J_{13}| \leq \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \left| (b_{\leq p - 2} - b_{\leq q - 2}) \cdot \nabla \Delta_q b_p u_q \right| \, dx \\
\leq \sum_{q > Q} \sum_{|q-p| \leq 2} \lambda_q^{2s+1} \|u_q\|_\infty \|b_{\leq p - 2} - b_{\leq q - 2}\|_2 \|b_p\|_2 \\
+ \sum_{-1 \leq q \leq Q} \sum_{|q-p| \leq 2} \lambda_q^{2s+1} \|u_q\|_\infty \|b_{\leq p - 2} - b_{\leq q - 2}\|_2 \|b_p\|_2 \\
\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+2 - \frac{2}{r}} \sum_{|q-p| \leq 2} \lambda_p^{\frac{3}{r}} \|b_{p} - b_{q-2}\|_2 \|b_p\|_2 \\
+ f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{|q-p| \leq 2} \|b_{p} - b_{q-2}\|_2 \|b_p\|_2 \\
\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+2 - \frac{2}{r}} \sum_{q-3 \leq p \leq q+2} \lambda_p^{\frac{3}{r}} \|b_p\|_2^2 + f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s} \|b_q\|_2^2 \\
\lesssim c_r \mu \sum_{q > Q - 3} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s} \|b_q\|_2^2.
\]

Combining (3.20), (3.29) and (3.31), we obtain

\[
(3.32) \quad |J_1 - J_{12}| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s} \|b_q\|_2^2.
\]
Notice that $J_2$ enjoys the same estimate as $J_{11}$, due to the similar reason that $I_2$ enjoys the same estimate as $I_{11}$. Thus, thanks to the estimate (3.30), we have for $\frac{1}{2} < s < 1$ and $2 \leq r < \frac{4}{s}$,

\[
|J_2| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s} \|b_q\|_2^2.
\]

Applying Hölder’s inequality, definition of $\Lambda_r$ and $f(t)$, Bernstein’s inequality, and change order of the summations, the last term $J_3$ of flux $J$ has the estimate

\[
|J_3| \lesssim \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \left| \Delta_q (b_p \otimes \tilde{b}_p) \nabla u_q \right| \, dx
\]

\[
\lesssim \sum_{q > Q} \lambda_q^{2s+1} \|u_q\|_\infty \sum_{p \geq q-3} \|b_p\|_2^2 + \sum_{p \geq q-3} \lambda_q^{2s+1} \|b_p\|_\infty \sum_{p \geq q-3} \|b_p\|_2^2
\]

\[
\lesssim \sum_{q > Q} \lambda_q^{2s+1+\frac{1}{2}} \|u_q\|_r \sum_{p \geq q-3} \|b_p\|_2^2 + f(t) \sum_{-1 \leq q \leq Q} \lambda_q^{2s} \sum_{p \geq q-3} \|b_p\|_2^2
\]

\[
\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+2} \|b_p\|_2^2 \sum_{Q < q \leq p+3} \lambda_q^{2s+2} + f(t) \sum_{p \geq -1} \lambda_p^{2s} \|b_p\|_2^2 \sum_{-1 \leq q \leq p+3} \lambda_q^{2s}
\]

\[
\lesssim c_r \mu \sum_{q > Q} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\]

Therefore, the equation (3.26) together with estimates (3.32), (3.33) and (3.34) leads to

\[
|J - J_{12}| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\]

Now we turn to estimates of terms in $L$ appeared in (3.27) and (3.28). Notice that $L_{11}$ can be estimated as $J_{11}$. Thus, it follows from (3.30) that

\[
|L_{11}| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s} \|b_q\|_2^2.
\]

After using integration by parts, the term $L_{13}$ can be estimated similarly to $J_{13}$. Hence, we have from (3.31)

\[
|L_{13}| \lesssim c_r \mu \sum_{q > Q-3} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s} \|b_q\|_2^2.
\]
To estimate $L_2$, we split the summation first as

$$|L_2| \leq \sum_{-1 \leq p \leq Q+2} \sum_{p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q (b_p \cdot \nabla u_{\leq p-2}) \cdot b_q| \, dx$$

$$+ \sum_{p>Q+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q (b_p \cdot \nabla u_{\leq Q}) \cdot b_q| \, dx$$

$$+ \sum_{p>Q+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q (b_p \cdot \nabla u_{(Q,p-2)}) \cdot b_q| \, dx$$

$$\equiv L_{21} + L_{22} + L_{23}.$$ 

Then using Hölder’s inequality and definition of $f(t)$ we arrive at

$$L_{21} \lesssim \sum_{-1 \leq p \leq Q+2} \|\nabla u_{\leq p-2}\|_\infty \|b_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s} \|b_q\|_2$$

$$\lesssim f(t) \sum_{-1 \leq p \leq Q+2} \|b_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s} \|b_q\|_2$$

$$\lesssim f(t) \lambda_q^{2s} \|b_q\|_2^2,$$

and

$$L_{22} \lesssim \sum_{p>Q+2} \|\nabla u_{\leq Q}\|_\infty \|b_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s} \|b_q\|_2$$

$$\lesssim f(t) \sum_{p>Q+2} \|b_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s} \|b_q\|_2$$

$$\lesssim f(t) \lambda_q^{2s} \|b_q\|_2^2.$$ 

Using Hölder’s inequality, definition of $\Lambda_r$ and Jensen’s inequality, we deduce

$$L_{23} \lesssim \sum_{p>Q+2} \|\nabla u_{(Q,p-2)}\|_r \|b_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s} \|b_q\|_2$$

$$\lesssim \sum_{p>Q+2} \lambda_p^\frac{r}{2} \|b_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s} \|b_q\|_2 \sum_{Q<p'\leq p-2} \lambda_{p'} \|u_{p'}\|_r$$

$$\lesssim c_r \mu \sum_{p>Q} \lambda_p^{2s+\frac{3}{2}} \|b_p\|_2^2 \sum_{Q<p'\leq p-2} \lambda_{p'}^{2-\frac{2}{r}}$$

$$\lesssim c_r \mu \sum_{p>Q} \lambda_p^{2s+\frac{3}{2}} \|b_p\|_2^2 \sum_{Q<p'\leq p-2} \lambda_{p'}^{2-\frac{2}{r}}$$

$$\lesssim c_r \mu \sum_{p>Q} \lambda_p^{2s+\frac{3}{2}} \|b_p\|_2^2,$$
since \( r \geq 2 \). The last four inequalities together indicate

\[
(3.38) \quad |L_2| \lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+2} ||b_p||_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} ||b_q||_2^2.
\]

To estimate \( L_3 \), we use integration by parts first, and then split it as

\[
(3.39) \quad |L_3| \leq \sum_{p > Q} \lambda_q^{2s} \sum_{-1 \leq q \leq p + 2} \left| \int_{\mathbb{R}^3} \Delta_q \left( b_p \otimes u_p \right) \nabla b_q \, dx \right|
\]

\[
= \sum_{p > Q} \lambda_q^{2s} \sum_{-1 \leq q \leq p + 2} \left| \int_{\mathbb{R}^3} \Delta_q \left( \hat{b}_p \otimes u_p \right) \nabla b_q \, dx \right|
\]

\[
+ \sum_{-1 \leq p \leq Q} \lambda_q^{2s} \sum_{-1 \leq q \leq p + 2} \left| \int_{\mathbb{R}^3} \Delta_q \left( \hat{b}_p \otimes u_p \right) \nabla b_q \, dx \right|
\]

\[
\equiv L_{31} + L_{32}.
\]

Proceeding as in the case of \( I_3 \), we obtain

\[
(3.40) \quad |L_{31}| \lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+1} ||b_p||_2 \sum_{-1 \leq q \leq p + 2} \lambda_q^{2s+1} ||b_q||_2
\]

\[
\lesssim c_r \mu \sum_{p > Q} \lambda_p^{2s+1} ||b_p||_2 \sum_{-1 \leq q \leq p + 2} \lambda_q^{2s+1} ||b_q||_2
\]

\[
\lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+1} ||b_q||_2^2;
\]

and

\[
|L_{32}| \lesssim \sum_{-1 \leq p \leq Q} ||u_p||_\infty ||b_p||_2 \sum_{-1 \leq q \leq p + 2} \lambda_q^{2s+1} ||b_q||_2
\]

\[
\lesssim f(t) \sum_{-1 \leq q \leq p + 2} \lambda_q^{2s+1} ||b_q||_2
\]

\[
\lesssim f(t) \sum_{-1 \leq p \leq Q} \lambda_q^{2s} ||b_p||_2 \sum_{-1 \leq q \leq p + 2} \lambda_q^{2s} ||b_q||_2 \lambda_q^{s+1}
\]

\[
\lesssim f(t) \sum_{-1 \leq q \leq Q + 2} \lambda_q^{2s} ||b_q||_2^2.
\]

Putting together the last three inequalities, we obtain

\[
(3.41) \quad |L_3| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} ||b_q||_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} ||b_q||_2^2.
\]

The combination of equations (3.29), (3.28) and inequalities (3.36), (3.40) yields

\[
(3.41) \quad |L - L_{12}| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} ||b_q||_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} ||b_q||_2^2.
\]

Thus, the estimate (3.24) is a consequence of (3.29), (3.35) and (3.41). \( \square \)
Lemma 3.4. For $\frac{1}{2} < s < 1$ and $2 \leq r < \frac{3}{s}$, we have

\begin{equation}
|K| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\end{equation}

**Proof:** As before, we start with Bony's decomposition,

$$K = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla b_p) \cdot b_q \, dx$$

$$+ \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla b_{\leq p-2}) \cdot b_q \, dx$$

\begin{equation}
K = K_1 + K_2 + K_3,
\end{equation}

with

$$K_1 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla |b_p| \cdot b_q) \, dx$$

$$+ \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_{\leq q-2} \cdot \nabla) |b_q| \cdot b_q \, dx$$

\begin{equation}
K_2 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((u_{\leq p-2} - u_{\leq q-2} \cdot \nabla) \Delta_q |b_p| \cdot b_q) \, dx
\end{equation}

Here we used $\sum_{|q-p| \leq 2} \Delta_q b_q = b_q$ to obtain $K_{12}$. We further notice that $K_{12} = 0$ by applying integration by parts and the fact $\nabla \cdot u_{\leq q-2} = 0$.

Comparing $K_{11}$ in (3.44) and $I_{11}$ in (3.18), one can see that $K_{11}$ shares the same structure as $I_{11}$ (with $u_p$ and $u_q$ replaced by $b_p$ and $b_q$). Thus, in view of (3.19), we have for $r \geq 2$

$$|K_{11}| \lesssim c_r \mu \sum_{q > Q+2} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2.$$

Similarly, examining the equations (3.44) and (3.20), (3.33) and (3.25) respectively, we realize that $K_{13}$ and $J_{13}$ have essentially the same structure, and so do $K_2$ and $J_2$. Hence, it follows from (3.31) and (3.33) that for $\frac{1}{2} < s < 1$ and $2 \leq r < \frac{3}{s}$

$$|K_{13}| + |K_2| \lesssim c_r \mu \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s} \|b_q\|_2^2.$$

Regarding $K_3$, we first use integration by parts

$$|K_3| \leq \sum_{p \geq -1} \lambda_p^{2s} \sum_{-1 \leq q \leq p+2} \left| \int_{\mathbb{R}^3} \Delta_q (u_p \otimes b_p) \nabla b_q \, dx \right|$$
and then notice that it has the same structure as $L_3$ in (3.39). Thus, it follows from (3.40) that for \( r \geq 2 \) and \( s \geq 0 \)
\[
|K_3| \lesssim c_r \mu \sum_{q \geq 1} \lambda_q^{2s+2} \|b_q\|^2 + f(t) \sum_{-1 \leq q \leq Q+2} \lambda_q^{2s} \|b_q\|^2.
\]
Combining the estimates above and equations (3.43) and (3.44) leads to the conclusion of the lemma.

\[\square\]

In order to finish the proof of Proposition 3.1 we conclude from inequality (3.15) and estimates in Lemmas 3.2 through 3.4 that for any small enough constant \( \tilde{c} \) we have
\[
d \frac{dt}{dt} \sum_{q \geq 1} \lambda_q^{2s} (\|u_q\|^2 + \|b_q\|^2) \lesssim f(t) \sum_{q \geq 1} \lambda_q^{2s} (\|u_q\|^2 + \|b_q\|^2),
\]
provided \( c_r \leq \tilde{c} \). This means that there exists an adimensional constant \( C = C(s) \), such that
\[
d \frac{dt}{dt} (\|u\|_{H^s}^2 + \|b\|_{H^s}^2) \leq C f(t) (\|u\|_{H^s}^2 + \|b\|_{H^s}^2),
\]
which is the estimate (3.14). Notice that the requirements \( \frac{1}{2} < s < 1 \) and \( 2 \leq r < \frac{2}{s} \) in Lemmas 3.3 and 3.4 indicate that (3.14) holds for any \( 2 \leq r < 6 \). However, when \( b \equiv 0 \), we have \( J \equiv K \equiv L = 0 \), and the range for \( r \) extends to \([2, \infty)\).

Step (ii): First note that by Grönwall’s inequality \( \|u\|_{H^s}^2 + \|b\|_{H^s}^2 \) is bounded on \([T/2, T]\) provided \( f \in L^1(0, T) \). Our next goal is to weaken the condition \( f \in L^1 \) to (1.10). For this purpose, we will prove

**Proposition 3.5.** Let \( C \) be the constant in (3.14) and \( \tilde{c} = \frac{1}{2e} (s - \frac{1}{2}) \ln 2 \). Assume that

\[
\limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_{\infty} \text{d}t \leq \tilde{c}.
\]

Let \( q^* \) be an integer such that
\[
\int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_{\infty} \text{d}t \leq 2\tilde{c}, \quad \forall q > q^*.
\]

There exists a constant
\[
M_{q^*} \sim q^* \lambda_{q^*}^2 (\|u(0)\|_2 + \|b(0)\|_2),
\]
such that

\[
\exp \left( \int_{T/2}^t C f(\tau) \text{d} \tau \right) \leq e^{T \lambda m_{q^*} \min \{\nu, \mu\}^{-1}} \sup_{\tau \in [T/2, t]} \|u(\tau)\|_{H^s}.
\]

**Proof:** We first split \( f \) as
\[
f(t) = f_{\leq q^*} + f_{> q^*}(t),
\]
with
\[
f_{\leq q^*} = \sum_{q \leq q^*} \lambda_q \|u_q\|_{\infty}, \quad f_{> q^*}(t) = \sum_{q > q^*} \lambda_q \|u_q\|_{\infty}.
\]
Thanks to Bernstein’s and energy inequalities, we have
\[
f_{\leq q^*}(t) \lesssim q^* \lambda_{q^*}^2 (\|u(0)\|_2 + \|b(0)\|_2) \leq M_{q^*}, \quad t \in (0, T).
\]
It follows that

\[ \exp \left( \int_{T/2}^t C f_{ \geq q^*} (\tau) \, d\tau \right) \leq e^{TCM\epsilon^r}. \]  

By definition of \( \Lambda_r(t) \) (see (3.13)) and Bernstein’s inequality we have

\[ c_r \min \{ \nu, \mu \} \Lambda_r^{1-\frac{\phi}{2}} \leq \| u_Q \|_r \leq \Lambda_r^{\frac{\phi}{2}} \| u_Q \|_2, \]  

provided \( \Lambda_r > 1 \). Therefore

\[ c_r \min \{ \nu, \mu \} \Lambda_r^{\epsilon} \leq \| u_Q \|_r \approx \Lambda_r^{1-3r} \| u_Q \|_2, \]  

with \( \epsilon = s - \frac{1}{2} > 0 \), which leads to

\[ 2^\epsilon \bar{Q}(t) \lesssim \frac{1}{c_r \min \{ \nu, \mu \}} \sup_{\tau \in [T/2, t]} \| u(\tau) \|_{H^r}, \]  

where

\[ \bar{Q}(t) = \sup_{\tau \in (T/2, t]} Q(\tau). \]  

Then we can estimate the integral of \( f_{ > q^*} \) over \((T/2, T)\) as follows,

\[
\begin{align*}
\int_{T/2}^t f_{ > q^*} (\tau) \, d\tau &= \int_{T/2}^t 1_{q^* < Q(\tau)} \sum_{q^* < q \leq Q(\tau)} \lambda_q \| u_q \|_\infty \, d\tau \\
&= \int_{T/2}^t 1_{q^* < Q(\tau)} \sum_{q^* < q \leq Q(\tau)} \lambda_q \| u_q \|_\infty \, d\tau \\
&\leq \sum_{q^* < q \leq \bar{Q}(t)} \int_{T/2}^t 1_{q \leq Q(\tau)} \lambda_q \| u_q \|_\infty \, d\tau \\
&\leq \bar{Q}(t) \sup_{q > q^*} \int_{T/2}^T 1_{q \leq Q(\tau)} \lambda_q \| u_q \|_\infty \, d\tau \\
&\leq \bar{Q}(t) 2^\epsilon \\
&= \frac{\bar{Q}(t) \epsilon \ln 2}{C}.
\end{align*}
\]

Combining the last inequality with (3.48) yields

\[ \exp \left( \int_{T/2}^t C f_{ > q^*} (\tau) \, d\tau \right) \leq 2^{\epsilon} \bar{Q}(t) \lesssim c_r^{-1} \min \{ \nu, \mu \}^{-1} \sup_{\tau \in [T/2,t]} \| u(\tau) \|_{H^r}. \]  

Therefore, the estimate (3.46) is a consequence of (3.47) and (3.49).

**Step (iii):** With the preparations done in Step (i) and Step (ii), we are ready to finish the proof of Theorem 1.1.

First, notice that Lebesgue norms of Littlewood-Paley projections of \( \nabla \times u \) and \( \nabla u \) are equivalent. Indeed, since \( \nabla \cdot u_q = 0 \), there exists a vector potential \( A_q \) defined as \( A_q = (-\Delta)^{-1}(\nabla \times u_q) \); consequently, \( \nabla \times u_q = -\Delta A_q \) and \( \nabla \cdot A_q = 0 \).
Thus, we have
\[ ||\nabla u_q||_p \sim \lambda_q ||u_q||_p = \lambda_q ||\nabla A_q||_p \leq \lambda_q ||\Delta A_q||_p = ||\nabla \times u_q||_p \leq \lambda_q ||u_q||_p \sim ||\nabla u_q||_p,\]
for all $1 \leq p \leq \infty$. In particular, there exists $M > 0$, such that
\[ \lambda_q ||u_q||_p \leq M ||\nabla \times u_q||_p, \quad 1 \leq p \leq \infty. \]

Now given $2 \leq r < 6$ ($2 \leq r \leq \infty$ when $b \equiv 0$), let
\[ c_r = \min\{\tilde{c}, \bar{c}/M\}, \]
where $\tilde{c}$ and $\bar{c}$ are constants from Propositions 3.1 and 3.5 respectively. We point out that the choice of $c_r$ in (3.52) is consistent with the condition $c_r \leq \tilde{c}$ in Proposition 3.1.

Now we fix $s \in \left(\frac{1}{2}, \min\{1, \frac{3}{2}\}\right)$.

Recall assumption (3.10) in Theorem 1.1
\[ \lim sup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(\tau)} ||\Delta_q (\nabla \times u)||_\infty d\tau \leq c_r. \]

Since $c_r \leq \tilde{c}/M$, assumption (3.44) of Proposition 3.5 holds thanks to (3.51). Applying Grönwall’s inequality to (3.14) and employing (3.46) from Proposition 3.5, we infer
\[ \|u(t)\|^2_{H^s} + \|b(t)\|^2_{H^s} \leq e^{TCM^\nu} c_r^{-1} \min\{\nu, \mu\}^{-1} \sup_{\tau \in [T/2, t]} ||u(\tau)||_{H^s} \left( \|u(T/2)\|^2_{H^s} + \|b(T/2)\|^2_{H^s} \right). \]

Taking sup in time in (3.53) yields
\[ \sup_{\tau \in [T/2, t]} ||u(\tau)||_{H^s} \leq e^{TCM^\nu} c_r^{-1} \min\{\nu, \mu\}^{-1} \left( \|u(T/2)\|^2_{H^s} + \|b(T/2)\|^2_{H^s} \right). \]

Inserting the bound (3.54) on the right hand side of (3.53), we obtain
\[ \|u(t)\|^2_{H^s} + \|b(t)\|^2_{H^s} \leq e^{2TCM^\nu} c_r^{-1} \min\{\nu, \mu\}^{-2} \left( \|u(T/2)\|^2_{H^s} + \|b(T/2)\|^2_{H^s} \right)^2, \]
for all $t \in (T/2, T)$, which implies that $(u, b)$ is regular on $(0, T]$.

### 3.2. Alternative versions of the criterion.

In this subsection we show that regularity conditions stated in Theorem 1.3 are stronger than condition (1.10), i.e., Theorem 1.3 is a corollary of Theorem 1.1.

**Lemma 3.6.** Let
\[ \mathcal{T}_q = \sup\{t \in (T/2, T) : Q(\tau) < q, \forall \tau \in (T/2, t)\}. \]

Then $\mathcal{T}_q$ is increasing and
\[ \lim_{q \to \infty} \mathcal{T}_q = T. \]
Moreover, either condition (1), (2), (3), or (8) in Theorem 1.3 implies regularity criterion (1.10).

\[
\limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(\tau)} \| \Delta_q (\nabla \times u) \|_{\infty} \, d\tau \leq c_r,
\]
in Theorem 1.1.

**Proof:** Recall that \( \tilde{Q}(t) = \sup_{\tau \in (T/2, t]} Q(\tau) \), and hence

\[
T_q = \sup\{ t \in (T/2, T) : Q(\tau) < q, \forall \tau \in (T/2, t) \} = \sup\{ t \in (T/2, T) : \tilde{Q}(t) \leq q \}.
\]

Note that \( T_q \) and \( \tilde{Q}(t) \) are increasing by definition. In addition, (3.48) and the fact that \((u, b)\) is regular on \((0, T)\) imply that \( \tilde{Q}(t) \) is bounded on \([T/2, T']\) for every \( \tau \in (T/2, T) \), and, consequently, (3.55) holds. Indeed, note that \( T_q \) is the time when \( \tilde{Q}(t) \) crosses \( q \) due to (3.56). So if

\[
\lim_{q \to \infty} T_q = T' < T,
\]

then \( \lim_{t \to T'} \tilde{Q}(t) = \infty \), thanks to the definition of \( Q(t) \) and the fact that solution is regular on \([T/2, T']\). That is, \( \tilde{Q}(t) \) is unbounded on \([T/2, T']\), which leads to a contradiction.

**Condition (1) implies regularity.** To show that condition (1) in Theorem 1.3 implies inequality (1.10) in Theorem 1.1, we observe that

\[
\limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(\tau)} \| \Delta_q (\nabla \times u) \|_{\infty} \, dt \leq \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(\tau)} \| \Delta_q (\nabla \times u) \|_{\infty} \, dt
\]

\[= \limsup_{q \to \infty} \int_{T_q}^{T} \| \Delta_q (\nabla \times u) \|_{\infty} \, dt,\]

where we used (3.56) to infer the last inequality.

**Condition (2) implies regularity.** This follows from the fact that for every \( \varepsilon > 0 \),

\[
\limsup_{q \to \infty} \int_{T_q}^{T} \| \Delta_q (\nabla \times u) \|_{\infty} \, dt \leq \limsup_{q \to \infty} \int_{T-\varepsilon}^{T} \| \Delta_q (\nabla \times u) \|_{\infty} \, dt,
\]
since \( T_q \to T \) as \( q \to \infty \).

**Condition (3) implies regularity.** Now suppose that

\[
\int_{T/2}^{T} \sup_{q \leq Q(\tau)} \| \Delta_q (\nabla \times u) \|_{\infty} \, dt < \infty,
\]

which implies that

\[
\limsup_{q \to \infty} \int_{T_q}^{T} \sup_{q_1 \leq Q(\tau)} \| \Delta_{q_1} (\nabla \times u) \|_{\infty} \, dt = 0,
\]
because $T_q \to T$ as $q \to \infty$. Therefore, employing the fact that $Q(t) \leq \bar{Q}(t)$ and (3.56), we have
\[
\limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(t)} \| \Delta_q(\nabla \times u) \|_\infty \, dt
\leq \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq \bar{Q}(t)} \sup_{q_i \leq Q(t)} \| \Delta_{q_i}(\nabla \times u) \|_\infty \, dt
\leq \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq \bar{Q}(t)} \sup_{q_i \leq Q(t)} \| \Delta_{q_i}(\nabla \times u) \|_\infty \, dt
= \limsup_{q \to \infty} \int_{T_q}^{T} \sup_{q_i \leq Q(t)} \| \Delta_{q_i}(\nabla \times u) \|_\infty \, dt
= 0,
\]
where the fact that $T_q$ is the time when $\bar{Q}(t)$ crosses $q$ is used again to obtain the last second equality in the estimate. As a consequence, inequality (1.10) in Theorem 1.1 is satisfied.

**Condition (8) implies regularity.** Now we show that (8) in Theorem 1.3 implies (1.10). First, the regularity of $u(t)$ on $(0, T)$ yields that $u(t)$ is smooth for every $t \in (0, T)$, and hence
\[
\limsup_{q \to \infty} \lambda_q^{-1+\frac{2}{r}} \| u_q(t) \|_r = 0, \quad \text{for all} \quad t \in (0, T).
\]

Then due to the weak continuity of $u(t)$ in $L^2(\mathbb{R}^3)$, reflected in the continuity of $\| u_q(t) \|_2$ on $[0, T]$, the smallness of the jump (8) implies that
\[
\limsup_{q \to \infty} \lambda_q^{-1+\frac{2}{r}} \| u_q(T) \|_r
\leq \limsup_{q \to \infty} \lambda_q^{-1+\frac{2}{r}} \| u_q(T) - u_q(t) \|_r + \limsup_{q \to \infty} \lambda_q^{-1+\frac{2}{r}} \| u_q(t) \|_r
\leq \frac{c_r}{2} \min\{\nu, \mu\},
\]
provided $t$ is close enough to $T$. Using the smallness of the jump (8) again and (3.57), we can infer that there exists $\varepsilon > 0$, such that
\[
\limsup_{q \to \infty} \sup_{t \in [T-\varepsilon, T]} \lambda_q^{-1+\frac{2}{r}} \| u_q(t) \|_r
\leq \sup_{t \in [T-\varepsilon, T]} \lambda_q^{-1+\frac{2}{r}} \| u_q(t) - u_q(T) \|_r + \limsup_{q \to \infty} \lambda_q^{-1+\frac{2}{r}} \| u_q(T) \|_r
\leq \frac{c_r}{2} \min\{\nu, \mu\} + \frac{c_r}{2} \min\{\nu, \mu\} = c_r \min\{\nu, \mu\}.
\]
Therefore there exists $q^*$, such that
\[
\lambda_q^{-1+\frac{2}{r}} \| u_q(t) \|_r < c_r \min\{\nu, \mu\}, \quad \forall q > q^*, t \in [T-\varepsilon, T],
\]
which implies that $\Lambda_r(t) \leq 2q^*$ for all $t \in [T-\varepsilon, T]$ (see (1.9)), and in fact on $[T/2, T]$ due to the regularity of $u(t)$ on $(0, T)$. Recall that $\Lambda_r(t) = 2^Q(t)$ and hence $Q(t)$ is bounded on $[T/2, T]$ as well. On the other hand, it follows from Bernstein’s
inequality and the fact that \( u \in L^\infty(T/2, T; L^2) \),
\[
\|u_q\|_\infty \lesssim 2^{3q/2}\|u_q\|_2 \in L^\infty(T/2, T) \text{ for every fixed } q \geq -1.
\]
Therefore, we have
\[
\sup_{q \leq Q(t)} \lambda_q \|u_q\|_\infty \in L^\infty(T/2, T).
\]
Consequently, in view of (3.50), the fact that \( Q(t) \leq 1 \) and (3.56), we infer
\[
\limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \|\Delta_q(\nabla \times u)\|_\infty dt \lesssim \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_\infty dt \lesssim \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_\infty dt \]
\[
= \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_\infty dt \lesssim \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_\infty dt \]
\[
= \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_\infty dt \lesssim \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q\|_\infty dt \]
\[
= 0,
\]
because \( T_q \to T \) as \( q \to \infty \). Hence (1.10) holds. \(\square\)

Finally, to establish that either of the conditions (4)–(7) in Theorem 1.3 implies inequality (1.10) in Theorem 1.1 and hence the regularity on \((0, T]\), we need the following.

**Lemma 3.7.** For all \( 1 \leq l < \infty \) and \( 1 \leq r \leq \infty \),
\[
\limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \lambda_q \|u_q(t)\|_\infty dt \leq (c_r \min\{\nu, \mu\})^{1-l} \limsup_{q \to \infty} \int_{T/2}^T 1_{q \leq Q(t)} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{2}} \|u_q(t)\|_r \right)^l dt \leq (c_r \min\{\nu, \mu\})^{1-l} \limsup_{q \to \infty} \int_{T_q}^T \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{2}} \|u_q(t)\|_r \right)^l dt.
\]

**Proof:** First, we notice that \( \lambda_q \leq \Lambda_r(t) = 2^Q(t) \) for \( q \leq Q(t) \) and hence
\[
1 \leq \left( \Lambda_r(t)/\lambda_q \right)^{2-\frac{2}{r}} \text{ for } l \geq 1 \text{ and } q \leq Q(t).
\]
Concurrently, it follows from (3.13) that
\[
1 \leq (c_r \min\{\nu, \mu\})^{-1} \Lambda_r^{-1}\frac{2}{r}(t) \|u_Q(t)\|_r
\]
and hence for \( l \geq 1 \)
\[
1 \leq (c_r \min\{\nu, \mu\})^{-l} \Lambda_r^{-\frac{2}{r}}(t) \|u_Q(t)\|_r^{l-1}.
\]
Using Bernstein’s inequality, inserting \([3.59]-[3.60]\) to the resulted integral, and applying Hölder’s inequality, we deduce

\[
\limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(t)} \lambda_q \|u_q(t)\|_\infty \, dt
\]

\[
\leq \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(t)} \lambda_q^{1+\frac{2}{r}} \|u_q(t)\|_r \, dt
\]

\[
\leq \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(t)} \Lambda_q^{2-\frac{2}{r}} \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \, dt
\]

\[
\leq (c_r \min\{\nu, \mu\})^{1-l} \limsup_{q \to \infty} \int_{T/2}^{T} \left( 1_{q \leq Q(t)} \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt
\]

\[
\leq (c_r \min\{\nu, \mu\})^{1-l} \limsup_{q \to \infty} \left( \int_{T/2}^{T} 1_{q \leq Q(t)} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt \right)^{\frac{1}{l}}
\]

\[
\leq (c_r \min\{\nu, \mu\})^{1-l} \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(t)} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt.
\]

Thus, the first inequality in \([3.58]\) is justified. In order to see the second inequality in \([3.58]\), we again use the fact that \(T_q\) is the first time when \(\hat{Q}(t)\) reaches \(q\):

\[
\limsup_{q \to \infty} \int_{T_q}^{T} 1_{q \leq Q(t)} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt
\]

\[
\leq \limsup_{q \to \infty} \int_{T/2}^{T} 1_{q \leq Q(t)} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt
\]

\[
= \limsup_{q \to \infty} \int_{T_q}^{T} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt.
\]

As a consequence of Lemma \(3.7\), one can see that condition \([1.10]\) is satisfied if (4) or (5) in Theorem \(1.3\) holds. On the other hand, we have for any \(\varepsilon > 0\)

\[
\limsup_{q \to \infty} \int_{T_q}^{T} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt \leq \limsup_{q \to \infty} \int_{T-\varepsilon}^{T} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt,
\]

since \(T_q \to T\) as \(q \to \infty\), and hence

\[
\limsup_{q \to \infty} \int_{T_q}^{T} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt \leq \limsup_{q \to \infty} \int_{T-\varepsilon}^{T} \left( \lambda_q^{-1+\frac{2}{r}+\frac{2}{r}} \|u_q(t)\|_r \right)^l \, dt.
\]

Therefore, condition \([1.10]\) is also satisfied if (6) in Theorem \(1.3\) holds.

In the end, we see that (7) implies (4) in Theorem \(1.3\) and hence it implies \([1.10]\).
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