Statistics of speckle patterns

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We develop a general method for calculating statistical properties of the speckle pattern of coherent waves propagating in disordered media. In some aspects this method is similar to the Boltzmann-Langevin approach for the calculation of classical fluctuations. We apply the method to the case where the incident wave experiences many small angle scattering events during propagation, but the total angle change remains small. In many aspects our results for this case are different from results previously known in the literature. The correlation function of the wave intensity at two points separated by a distance \( r \), has a long range character. It decays as a power of \( r \) and changes sign. We also consider sensitivities of the speckles to changes of external parameters, such as the wave frequency and the incidence angle.

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In this article we consider statistical properties of waves propagating through a disordered medium and described by the stationary (scalar) wave equation,

\[
(\nabla^2 + k^2n^2(r))\psi(r) = 0,
\]

where \( n(r) \) is the index of refraction assumed to be a random Gaussian function. This problem is relevant for a variety of important physical situations, ranging from electromagnetic waves propagating through the interstellar space or the atmosphere, to electron transport in disordered conductors. The wave density, \( I(r) = |\psi(r)|^2 \), exhibits sample specific random fluctuations (speckles) due to interference of waves traveling along different paths. The statistical properties of speckles have been discussed in the past. The problem can be characterized by several characteristic lengths: the propagation distance, \( Z \), the elastic mean free path, \( \ell \) and the transport length \( \ell_tr \sim \ell/\theta_0 \), which is the typical distance for backscattering. Here \( \theta_0 \sim k\xi \) is the typical scattering angle on the distance \( \ell \), and \( \xi \) is the correlation length of \( n(r) \). In the diffusive regime \( Z \gg \ell_tr \), the problem has been studied in Refs. [1, 2, 3]. The limit of "directed waves" \( \ell_tr \gg Z \gg \ell \) has been studied in many papers, see for example [4, 5, 6, 7] and references therein. In the latter case the wave experiences many small angle scattering events, but the total change of its propagation angle \( \theta \) remains small.

In this Letter we develop a general method for calculating speckle correlations, which enables us to treat both diffusive and directed wave cases on equal footing. It is general characteristic lengths: the propagation distance, \( Z \), the elastic mean free path, \( \ell \) and the transport length \( \ell_tr \sim \ell/\theta_0 \), which is the typical distance for backscattering. Here \( \theta_0 \sim k\xi \) is the typical scattering angle on the distance \( \ell \), and \( \xi \) is the correlation length of \( n(r) \). In the diffusive regime \( Z \gg \ell_tr \), the problem has been studied in Refs. [1, 2, 3]. The limit of "directed waves" \( \ell_tr \gg Z \gg \ell \) has been studied in many papers, see for example [4, 5, 6, 7] and references therein. In the latter case the wave experiences many small angle scattering events, but the total change of its propagation angle \( \theta \) remains small.

In this Letter we develop a general method for calculating speckle correlations, which enables us to treat both diffusive and directed wave cases on equal footing. It is similar, in some of its aspects, to the Langevin scheme for the description of classical fluctuations [8, 9, 10]. We shall demonstrate the method for the case \( \ell_tr \gg Z \gg \ell \) by calculating the speckle correlations and their sensitivity to various perturbations, such as a change in the frequency of the wave, its incidence angle and a change of the refraction coefficient. In many aspects, our results differ from those obtained in the previous studies [4, 5, 6, 7]. Among the differences are the slow power law decay of the density correlator as a function of coordinates and its change of sign, see Fig. 1.

The central object of our approach is the ray distribution function \( f(r, s) \)

\[
f(r, s) = \frac{\rho^2 dp}{2\pi^2} \int dr' \psi \left( r - \frac{r'}{2} \right) \psi^* \left( r + \frac{r'}{2} \right) e^{is \cdot r'},
\]

which is the probability of finding a ray at point \( r \) pointing in the direction specified by the unit vector \( s \), in particular \( f(r) = \int ds f(r, s) \). The average distribution function \( \langle f(r, s) \rangle \) satisfies the Boltzmann kinetic equation,

\[
s \cdot \frac{\partial \langle f(r, s) \rangle}{\partial r} = I_{st}\{\langle f(r, s) \rangle\},
\]

\[
I_{st}\{\langle f \rangle\} = \int d^2s W(s - \bar{s}) |\langle f(r, s) \rangle - \langle f(r, s) \rangle|.
\]

Here \( \langle \ldots \rangle \) denotes averaging over the random realizations of \( n(r) \), the integral over the directions is normalized to unity, \( \int d^2s = 1 \), and \( W(\delta s) = \frac{\rho^2}{4} \int d^3r g(r)e^{i\delta s \cdot r} \) is the

FIG. 1: The asymptotic behavior of the density correlation function \( C(\rho) = (\delta I(\rho) \delta I(0)) \). \( \lambda \) is the light wavelength, \( \ell \) is the elastic mean free path, \( Z \) is the slab width, and \( \theta_0 \) and \( \theta \) are the typical scattering angle of a ray traveling a distance \( \ell \) and \( Z \) respectively.
angular diffusion equations, of directions, $s$ |

| $\ell$ f | where Eqs. (5,6) follow from diagrams b)-d) in Fig. 2. |

| One can prove Eqs. (3-6) using the standard impurity diagram technique. Equations (3-6) are obtained by summing the ladder diagrams in Fig. 2 a), whereas Eqs. (6) follow from diagrams b)-d) in Fig. 2. |

| Equations (3-6) are valid when the mean free path is sufficiently large, $\ell = (\int ds W(s))^{-1} \gg \xi^2/\lambda$, and $|r - r'| \gg \lambda$. Here $r$ and $r'$ are observation points, $\xi = (\int ds^2 g(r)/\int s^2 g(r))^1/2 > \lambda$ is the disorder correlation length, and $\lambda = 2\pi/k$ is the wave length. For $|r - r'| \ll \lambda$ Eqs. (3-6) are not valid and to evaluate the correlation function one has to calculate the diagram shown in Fig. 2 e). |

| In the regime of small angle scattering, $\theta_0 \ll 1$, and when $|r - r'| \gg \lambda$ the rays undergo diffusion in the space of directions, $s$. In this case Eqs. (3-6) reduce to a set of angular diffusion equations, |

| s · $\partial f(r,s) \over \partial r$ = $D_0 \nabla_s^2 f(r,s)$, (7) |

| s · $\partial \delta f(r,s) \over \partial r$ = $\delta r \nabla_s [D_0 \nabla_s \delta f(r,s) - j_L(r,s)]$, (8) |

| $\langle j_L^L(r,s) j_L^\perp(r,s) \rangle = 2\pi D_0 g^2 f^2 \delta_s \delta(r - \bar{r})$. (9) |

| Here $j_L^L(r,s)$ are the Langevin current sources, $D_0 = \frac{1}{2} \ell^2 r_{tr}^{-1} = \frac{1}{2} \int ds^2 s(1 - s \cdot s') W(s - s')$ is the diffusion constant in the space of angles $s$, and $\nabla_s = \hat{\phi} \partial / \partial \phi + \hat{\theta} \sin(\phi) \partial / \partial \theta$ is the gradient operator, with $\hat{\phi} = (- \sin \phi, \cos \phi, 0)$, and $\hat{\theta} = (\cos \phi \cos \theta, - \sin \phi \cos \theta, - \sin \phi)$. |

| Further simplification emerges at larger spatial scales, $|r - r'| \gg \ell_{tr}$. In this case Eqs. (3-6) can be reduced to diffusion equations, $\nabla^2 f = 0$, and $\nabla (-D \nabla \delta I + J) = 0$. Here $D = \ell_{tr}/3$ is the (real space) diffusion constant, and correlation function for Langevin currents has the form: $\langle J_\alpha(r) J_\beta(r') \rangle = \lambda^2 \ell_{tr} g \langle I(r) \rangle^2 \delta_{\alpha\beta} \delta(r - r')$. In this case, the correlation function $\langle \delta I(r) \delta I(r') \rangle$ decays as $1/r$ and as $1/r^2$ for $r \gg \ell_{tr}$ and $\lambda \ll \ell_{tr}$ respectively. |

| The scheme presented above can be generalized to treat speckle sensitivities to changes of external parameters, such as the wavenumber, $\Delta k$, or smooth changes of the refractive index, $\Delta n(r)$. We characterize the sensitivities by the correlation function $\langle \delta f(0) \delta f(\gamma) \rangle$, where $\gamma = \Delta k + \Delta n(r)$. In this case the Langevin source correlator is given by, |

| $\langle L(r,s;0)L(r',s';\gamma) \rangle = \frac{\pi}{k^2} \delta(r - r') \int d^2 s_1 W(s - s_1) f_\nu(r,s) f_\nu(r_1,s_1) - f_\nu(r,s) W(s - s') f_\nu(r,s') \right]$, (10) |

| The method based on Eqs. (3-11) is similar to the Langevin approach describing classical time and space fluctuations of a single particle distribution function.
The fundamental difference between our problem and that of classical fluctuations manifests itself in the form of the correlation function of the Langevin sources. In the classical problem the Langevin sources are $\delta$-correlated both in time and space and their variance is proportional to the average distribution function $\langle f \rangle$. In contrast, in Eqs. (9,11) the Langevin source variance is quadratic in $\langle f \rangle$ and $\delta$-correlated only in space.

To illustrate the use of Eqs. (8,11) we consider the case when a wave of intensity $I_b$ is incident on a disordered slab of thickness $Z$, such that $\ell \ll Z \ll \ell_{tr}$, as shown in the inset of Fig. 1. The results presented below are calculated to leading order in the small total scattering angle, $\theta^2 = \delta I_b Z$. In this case the correlation function $C(r-r') = \langle \delta I(r)\delta I(r') \rangle$ is strongly anisotropic (here $\delta I(r) = I(r) - \langle I(r) \rangle$). Therefore below we shall use the notation: $r = (z,\vec{\rho})$, where $\vec{\rho}$ denotes a two-component vector in the plane perpendicular to the $z$-axis and $z$ denotes the distance between observation points along the $z$ axis. When $\rho = 0$, i.e. the observation points are located along the $z$ axis, the correlation function for $z \ll Z$ is,

$$C(z) = \frac{I_b^2}{4k^2g^2z^2}. \quad (12)$$

Equation (12) matches the results for the diffusive case $Z \gg \ell_{tr}$ when $\theta$ is of order unity.

When $z < \rho/\theta$, i.e. the observation points are located essentially on a plane perpendicular to the $z$ axis, a general formula for $C(\rho)$, can be derived from Eqs. (7,11),

$$C(\rho) = \frac{I_b^2}{4D_b k^2} \int_0^{\rho/\ell} \frac{d\zeta}{\zeta} \int_0^\infty dq J_0(q \rho) \times \frac{d}{d\zeta} \exp \left[ -\frac{\ell}{2} \int_0^{\zeta} d\eta \left( 1 - \hat{g} \left( \frac{\eta}{k} \right) \right) \right], \quad (13)$$

where $\hat{g}(\rho) = \int dzg(\rho^2 + z^2) / \int dzg(z)$. The integral in Eq. (13) contains a term proportional to a $\delta$-function, $\delta(\rho)$. This term represents the rapidly decaying (at $\rho \sim \lambda/\theta$) part of the correlator and corresponds to the part of diagram Fig. 2(b) without the impurity ladders after the Hikami box, see diagram e). The $\delta$-function term results from the semiclassical approximation employed in the derivation of Eqs. (11,12), which limits the spatial resolution to $\delta \rho \gg \lambda$. In order to resolve the spatial structure of the short distance part of the correlator diagram e) needs to be evaluated more accurately. This gives the following asymptotic behavior,

$$C(\rho) \approx \begin{cases} e^{-2(\rho \theta^2)/\theta} & \text{if } \rho \sim \alpha \lambda/\theta, \\ \frac{\epsilon_4}{\epsilon_3} b_0^2 D_b^2 \frac{\theta}{k^2\theta^2 \rho^2} & \text{if } \ell \theta_0 \ll \rho \ll \ell \Theta Z, \\ \exp \left( -\frac{b_1^2}{\epsilon_3} Z \frac{\theta}{\theta_0} \right) & \text{if } \Theta Z \ll \rho \ll \frac{Z \theta^2}{\theta_0^2}, \end{cases} \quad (14)$$

where $\alpha^2 = \log(k\ell\theta^4/\theta_0)$, $b_1 = \int_0^\infty dxg(x)$ is a constant of order unity, $b_2 = 3^{1/3}\Gamma(5/3)/8 \approx 0.163$, and $b_3 = 27/128 \approx 0.21$. The tail of the correlation function (the regime $\rho > \theta Z^2/\theta_0$) is also described by Eq. (13) and depends on the precise form of the disorder correlation $q(r)$, since this limit is dominated by rare scattering events. The qualitative form of the function $C(\rho)$ is shown in Fig. 1.

Let us consider now the statistics of density, integrated over a disk of radius $R$, $P = \int_{\rho \ll R} d^2\rho I(\rho)$. Using Eqs. (13,14) we get,

$$\langle (\delta P)^2 \rangle \approx \frac{I_b^2}{2k^2 R^4} \left[ \frac{\pi}{8} + \frac{b_1^2}{2k^2 R^2} \frac{\alpha \theta}{\theta_0} \right], \quad (15)$$

where $b_1^2 = 2b_1/\pi, b_2^2 = 3^{1/3}\Gamma(5/6)\pi/2^{11/3}\Gamma(7/6)$, and $b_3^2 = 1/2\pi^2$.

Consider now the sensitivity of the integrated density $P(\omega)$ to a change in the wave frequency $\Delta \omega = c \Delta k$, where $c$ is the speed of the wave. It can be characterized by the experimentally accessible quantity,

$$\frac{\langle (P(\omega + \Delta \omega) - P(\omega))^2 \rangle}{\langle (\delta P)^2 \rangle} \approx \frac{(\Delta \omega/k)^2}{(\omega^*)^2}, \quad (16)$$

where

$$\omega^* = \sqrt{\frac{15}{2} \frac{c}{\theta^2 Z}}. \quad (17)$$

A qualitative explanation of the scale $\omega^*$ is similar to that given for the sensitivity of the conductance fluctuations $12.13$. Let us estimate the characteristic change in the phase of a typical orbit due to the frequency change $\Delta \omega = c \Delta k$. The typical length spread of the orbits is of order $\theta Z^2$. Therefore the phase difference is $\Delta k = \Delta \omega/c$ is the change in the wavenumber. Thus a complete change of the speckle pattern occurs when the phase, $\Delta \omega Z^2/c$ is of order one, namely $\Delta \omega \sim c/\theta^2 Z^2$, in agreement with Ref. 12.

As another application of our scheme let us consider the sensitivity of speckles to a change in the incidence angle, $\phi$, of the wave, see the inset in Fig. 1 (thus the incident wave function now has a form $\psi = \sqrt{15} \exp[i k z \cos \phi + i k \rho \sin \phi]$). One may characterize this sensitivity by the correlation function,

$$\frac{\langle \delta P(\phi)\delta P(0) \rangle}{\langle (\delta P)^2 \rangle} \approx e^{-3/2} \left[ 1 + \frac{\beta (e^{-\frac{3\pi^2}{2\sigma^2} - e^{-\frac{3\pi^2}{2\sigma^2}}})}{k^2 \ell^4 D_b^2} \right], \quad (18)$$

where $\phi^* = (\theta k Z)^{-1}$, $\beta$ is a factor of order unity, and it is assumed that $R \ll \theta Z$. The first term of this equation follows from formula 15 and the correlator 11, with initial conditions

$$f_{\pm}(\bar{\rho}, z = 0) = I_0 e^{\pm ik z} \bar{p} \delta(s - s_0), \quad (19)$$
where \( s_0 = (\cos \phi, s_\perp) \approx (1, s_\perp) \), with \(|s_\perp| = \sin \phi \approx \phi\), assuming \( \phi \ll 1 \). The second term in the right hand side of Eq. (13) is computed from diagrams of the type shown in Fig. 2(d), containing two Hikami boxes. It represents a small correction in the parameter \( \xi / \ell \theta_0 \), however, this term becomes the dominant contribution when \( \phi \gg \phi' \).

On a more general level Eqs. (11) provide a framework for calculation of time correlation of speckle patterns in the case where the refraction index changes in time, provided these changes are much slower than the time of propagation of the wave to the observation point.

Our results also may be easily extended to cases with light polarization, optically active media, Faraday effect, and coherent short wave pulses as long as their duration is longer than \( \tau = \ell / c \). These issues are left for future studies.

The problem considered here is similar to the problem of universal conductance fluctuations in metallic samples \[12, 14\], which are also of interference nature. Therefore we would like to discuss the relation between the two problems. In the single particle approximation the conductance of a metallic sample, \( G \sim \int d\phi T(\phi) \) can be expressed in terms of the electron transmission probability through the sample, \( T(\phi) \propto \int d\rho d^2 s \cdot (z \cdot s) f(\rho, s) \), integrated over the incidence angle of the incoming wave, \( \phi \).

Thus the variance of the conductance fluctuations, \( \delta G \) is proportional to a double integral of the correlation function \( \langle \delta T(\phi) \delta T(\phi') \rangle \). In principle, the latter can be calculated using Eqs. (11), or, equivalently, by calculating diagrams shown in Fig. 2(b)-d). However, as we explain below, this does not account for the conductance fluctuations. The latter arise from diagrams of the form shown in Fig. 2(f).

In the limit of directed waves, \( Z \ll \ell_{tr} \), there is no backscattering. Therefore the transmission probability does not fluctuate, \( \delta T(\phi) \sim \delta G = 0 \). Thus to compare the two problems we have to consider the diffusive case, \( Z \gg \ell_{tr} \), where \( \delta G \sim e^2 / h \). The correlation function \( \langle \delta T(\phi) \delta T(\phi') \rangle \), in the diffusive regime, still has a structure similar to the correlation function given by Eq. (13). Namely, it contains two contributions. The first contribution comes from diagrams \[2\]d), and describes relatively strong fluctuations of the transmission coefficient. However, it is very sensitive to the change of \( \phi \), and after the integration over \( \phi \) gives a small contribution to \( \delta G^2 \). The second contribution originates from diagrams of the type Fig. 2(f), and is analogous to the second term in Eq. (13). Although its amplitude is smaller than that of the first term, it is insensitive to the change of \( \phi \), and after the integration over \( \phi \) yields the dominant contribution to \( \delta G^2 \). Thus, conductance fluctuations are not described by Eqs. (11), and should be calculated from the diagrams of the type shown in Fig. 2(f) (see the corresponding discussion in Ref. \[3\]).

Finally we would like to mention that the results presented above substantially differ from those known in the literature (see for example Refs. \[1, 3, 4, 5\]). First, the correlation function \( 13 \) exhibits a universal long range power law behavior in a wide range of values of \( \rho \). The only non-universal regimes are at the tail, \( \rho \gg Z \theta^2 / \theta_0 \), and the short distance region, \( \rho \sim \xi \). In contrast, in the results presented in Refs. \[4, 5, 6, 7\], \( C(\rho) \) depends on the detailed form of \( g(r) \), and usually decays exponentially at \( \rho > \xi \). Second, in contrast with previous results, \( C(\rho) \) changes its sign as a function of \( \rho \), which is a consequence of the current conservation. This conservation law also implies, that the fluctuations of the integrated intensity over disks of radius \( R > Z \theta \) is proportional to \( R \), see Eq. (15), rather than \( R^2 \), as would follow from Refs. \[4, 5, 6, 7\].

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