Suppose that i.i.d. random variables $X_1, X_2, \ldots$ are chosen uniformly from $[0, 1]$, and let $f : [0, 1] \to [0, 1]$ be an increasing bijection. Define $\mu_f$ to be the expected value of $f(X_i)$ for each $i$. Define the random variable $K_f$ to be minimal so that $\sum_{i=1}^{K_f} f(X_i) > t$ and let $N_f(t)$ be the expected value of $K_f$. We prove that if $c_f = \int_0^1 \int_0^{f(x)-u} f^{-1}(u) dx du$, then $N_f(t) = t + c_f \mu_f + o(1)$. This generalizes a result of Curgus and Jewett (2007) on the case $f(x) = x$.

1 Introduction

Renewal theory is a branch of mathematics with applications to waiting time distributions in queueing theory, ruin probabilities in insurance risk theory, the development of the age distribution of a population, and debugging programs [6, 8]. In this paper, we compare renewal processes, which are simple point processes $0 = x_0 < x_1 < x_2 < \ldots$ for which the differences $x_{i+1} - x_i$ for each $i \geq 0$ form an independent identically distributed sequence.

A famous problem about renewal processes was actually a problem from the 1958 Putnam exam [1]: Select numbers randomly from the interval $[0,1]$ until the sum is greater than 1. What is the expected number of selections?

The answer is $e$ and solutions have appeared in several papers [1, 2, 3, 4, 5]. A more general problem is to find the expected number of selections until the sum is greater than $t$. Let $M(t)$ denote this expected number. In [7], Curgus and Jewett showed that $M(t) = 2t + \frac{2}{3} + o(1)$ and $M(t) = \sum_{k=0}^{[t]} \frac{(-1)^k (t-k)^k}{k!} e^{t-k}$ [7].
An analogous question for products was posed in [10]: Select numbers randomly from the interval \([1,e]\) until the product is greater than \(e\). What is the expected number of selections?

Vandervelde found that the answer is \(e^{-1} + \frac{1}{e-1}\) and posed the more general question of finding the number of selections until the product is greater than \(e^t\) [10]. Let \(N(t)\) denote this expected number. Vandervelde conjectured that \(N(t) \leq M(t)\) for all \(t \geq 0\).

We prove the conjecture in Section 4, as well as the fact that \(N(t) = (e^{-1} + \frac{1}{e-1}) + o(1)\). We use the same proof to obtain the following more general result.

**Theorem 1.** Suppose that i.i.d. random variables \(X_1, X_2, \ldots\) are chosen uniformly from \([0,1]\), and let \(f : [0,1] \to [0,1]\) be an increasing bijection. Define \(\mu_f\) to be the expected value of \(f(X_i)\) for each \(i\). Define the random variable \(K_f\) be to be minimal so that \(\sum_{i=1}^{K_f} f(X_i) > t\) and let \(N_f(t)\) be the expected value of \(K_f\). If \(c_f = \int_0^1 \int_0^1 f(u) du dx\), then \(N_f(t) = t + c_f \mu_f + o(1)\).

As a corollary, this gives an alternative proof of the main result in [7], which was proved in that paper using results about delay functions.

**Corollary 2.** \(M(t) = 2t + \frac{2}{3} + o(1)\)

In Section 2, we find that \(N(t) = \frac{e^{-1}}{e} + e^{t-1} + \frac{1}{e-1}\) for \(t \in [0,1]\). We prove in Section 3 that \(\frac{d}{dt} (N(t)e^{-e^{t-1}}) = -\frac{e^{-1}}{e} e^{-e^{t-1}} N(t-1) - e^{-e^{t-1}}\), and we use this equation to find \(N(t)\) for \(t \in [1,2]\).

2 \(t \in [0,1]\)

The proof for \(t \in [0,1]\) is like the proof for \(t = 1\) in [10].

Let \(q_n = q_n(t)\) be the probability that a product of \(n\) numbers chosen from \([1,e]\) is not greater than \(e^t\). Define \(q_0 = 1\). The probability that the product exceeds \(e^t\) for the first time at the \(n^{th}\) selection is \((1 - q_n) - (1 - q_{n-1}) = q_{n-1} - q_n\). \(N(t)\) is equal to \(\sum_{n=1}^{\infty} n(q_{n-1} - q_n) = \sum_{n=0}^{\infty} q_n\).

For \(t \in [0,1]\) the region \(R_n(t)\) within the \(n\)-cube \([1,e]^n\) consisting of points \((x_1, \ldots, x_n)\), the product of whose coordinates is at most \(e^t\), is described by \(1 \leq x_1 \leq e^t, 1 \leq x_2 \leq \frac{e^t}{x_1}, \ldots, 1 \leq x_n \leq \frac{e^t}{x_1 \ldots x_{n-1}}\).
Theorem 5. In this section, we show that $\Theta_n = \int_{R_n} dx_1 \ldots dx_n$. Note that $\Theta_{n+1} = \int_{R_n} (x_{n+1}^t - 1) dx_1 \ldots dx_n$. Therefore $\Theta_{n+1} + \Theta_n = \int_{R_n} x_{n+1}^t dx_1 \ldots dx_n$.

Lemma 3. $\Theta_n = (-1)^n(1 - b_n e^t)$, where $b_n = 1 - t + t^2/2 - \ldots + (-1)^{n-1} t^{n-1}/(n-1)!$.

Proof. Make a change of variables $y_k = \ln x_k$, so $\Theta_{n+1} + \Theta_n = \int_{R_n} e^t dy_1 \ldots dy_n$. Clearly $R_n$ consists of the points $(y_1, \ldots, y_n)$ satisfying $y_k \in [0, 1]$ and $y_1 + \ldots + y_n \leq t$. Therefore $\Theta_{n+1} + \Theta_n = \frac{dy}{n^n}$ for $t \in [0, 1]$.

For $n \geq 1$ let $b_n$ be the $n^{th}$ partial sum of the Taylor series for $e^{-t}$ centered at 0, i.e., $1 - \frac{t}{1} + \frac{t^2}{2} - \ldots + (-1)^{n-1} \frac{t^{n-1}}{(n-1)!}$. We show that $\Theta_n = (-1)^n(1 - b_n e^t)$.

The quantities agree for $n = 1$. For $n \geq 2$, $\Theta_{n+1} + \Theta_n = (-1)^n(1 - b_n e^t) + (-1)^{n+1}(1 - b_{n+1} e^t) = e^t(-1)^n(b_n + b_{n+1} - b_n) = e^t(-1)^n \frac{1}{n!} = \frac{n!}{n!}$.

Therefore for $t \in [0, 1]$, $q_n = \frac{(-1)^n(1 - b_n e^t)}{(e-1)^n}$. The final step is to calculate the sum of the $q_n$.

Theorem 4. For $t \in [0, 1]$, $N(t) = \frac{e^t}{e} + e^{t-1} - \frac{t}{e}$.

Proof. $\sum_{n=0}^\infty q_n = 1 + \sum_{n=1}^\infty \frac{(-1)^n}{(e-1)^n} - \sum_{n=1}^\infty \frac{(-1)^n}{(e-1)^n} = \frac{e-1}{e} - \sum_{n=1}^\infty \frac{(-1)^n}{(e-1)^n}$. We evaluate the remaining term by writing $b_n$ as a sum and interchanging the order of summation.

$-\sum_{n=1}^\infty \frac{(-1)^n}{(e-1)^n} = -e^t \sum_{n=0}^\infty \frac{(-1)^n}{(e-1)^n} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} = -e^t \sum_{n=0}^\infty \frac{(-1)^k}{(e-1)^k} = \frac{e^t}{e e^{-1}} = \frac{e^t}{e^{-1}}$.

$3 \quad t \geq 1$

In this section, we show that $\frac{d}{dt}(N(t) e^{-\frac{t}{e-1}}) = -\frac{e}{e-1} e^{-\frac{t}{e-1}} N(t - 1) - e^{-\frac{t}{e-1}}$ for $t \geq 1$ and calculate $N(t)$ for $t \in [1, 2]$.

Theorem 5. $\frac{d}{dt}(N(t) e^{-\frac{t}{e-1}}) = -\frac{e}{e-1} e^{-\frac{t}{e-1}} N(t - 1) - e^{-\frac{t}{e-1}}$

Proof. In the next section we show that $N(t) = 1 + \frac{1}{e-1} \int_1^t N(t - \ln u) du$. If $s = t - \ln u$, then $N(t) = 1 + \frac{1}{e-1} e^t \int_t^1 N(s) e^{-s} ds$. Therefore $N'(t) = \frac{e^t}{e-1} (N(t) - N(t-1)) - 1$, so $\frac{d}{dt}(N(t) e^{-\frac{t}{e-1}}) = -\frac{e}{e-1} e^{-\frac{t}{e-1}} N(t - 1) - e^{-\frac{t}{e-1}}$.

Theorem 6. $N(t) = e^{-\frac{t}{e-1}} \left(-\frac{e-1}{e^2} + \frac{1}{e} + \frac{e}{e-1} \right) + \frac{2(e-1)}{e} - \frac{e}{e-1} - e^{-\frac{t}{e-1}}$ for $t \in [1, 2]$.
Proof. By Theorems 4 and 5, \(\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -2e^{-\frac{e}{e-1}t} - \frac{e}{e-1}.\) Therefore for \(t \in [1, 2],\) \(N(t)e^{-\frac{e}{e-1}t} = C + \frac{2(e-1)}{e}e^{-\frac{e}{e-1}t} - \frac{e}{e-1}t\) for a constant \(C = -\frac{e-1}{e^2} + \frac{1}{e} + \frac{e}{e-1}.\) In other words, \(N(t) = e^{\frac{e}{e-1}t}\left(-\frac{e-1}{e^2} + \frac{1}{e} + \frac{e}{e-1}\right) + \frac{2(e-1)}{e}e^{-\frac{e}{e-1}t}.\)  

For each integer \(i \geq 2,\) \(N(t)\) can be calculated similarly for \(t \in [i, i+1]\) based on the values of \(N(t)\) for \(t \in [i-1, i]\) using the fact that \(\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -\frac{e}{e-1}e^{-\frac{e}{e-1}t}N(t-1) - e^{-\frac{e}{e-1}t}.\)

4 Bounds on \(N(t)\)

The results in this section use the fact that \(\ln(1 + (e-1)t) \geq t\) for \(t \in [0, 1].\)

Lemma 7. \(\ln(1 + (e-1)t) \geq t\) for \(t \in [0, 1]\)

Proof. Define \(f(t) = \ln(1 + (e-1)t) - t\) for \(t \in [0, 1].\) Then \(f'(t) = \frac{e-1}{1+(e-1)t}-1,\) so \(f'(t) = 0.\) Clearly \(f'(t) > 0\) for \(t \in [0, \frac{e-2}{e-1}], f'(t) < 0\) for \(t \in (\frac{e-2}{e-1}, 1],\) and \(f(0) = f(1) = 0.\) Therefore \(f(t) \geq 0\) for \(t \in [0, 1].\)

The proof of the \(N(t)\) recurrence is like the proof of the \(M(t)\) recurrence in [7]. Let \(I = [0, 1]\) and define \(B_{0,t} = I^{\mathbb{N}}.\) For each \(n \in \mathbb{N},\) define \(B_{n,t} = \{x \in I^{\mathbb{N}} : \ln(1 + (e-1)x_1) + \ldots + \ln(1 + (e-1)x_n) \leq t\}.\) Let \(B = \bigcup_{n=1}^{\infty} \cap_{t=1}^{\infty} B_{n,k}.\) Clearly the measure of \(B\) in \(I^{\mathbb{N}}\) is 0, since \(\ln(1 + (e-1)x_1) + \ldots + \ln(1 + (e-1)x_n) \geq x_1 + \ldots + x_n.\)

Theorem 8. \(N(t) = 1 + \frac{1}{e-1} \int_{1}^{t} N(t - \ln u)du\)

Proof. Let \(t \geq 0\) and define the random variable \(F_t : I^{\mathbb{N}} \to \mathbb{N} \cup \{\infty\}\) by \(F_t(x) = \min\{n \in \mathbb{N} : \ln(1 + (e-1)x_1) + \ldots + \ln(1 + (e-1)x_n) > t\},\) with \(\min\emptyset = \infty.\) Since \(B\) has measure 0 and \(F_t^{-1}(\{\infty\}) = \cap_{n=1}^{\infty} B_{n,t} \subset B, F_t\) is finite almost everywhere on \(I^{\mathbb{N}}.\)

For \(n \in \mathbb{N},\) \(F_t^{-1}\{\{n\}\} = B_{n-1,t} - B_{n,t}.\) Thus \(F_t\) is a Borel function and \(N(t) = \int_{I^{\mathbb{N}}} F_t(x)dx.\) If \(t \geq 1\) and \(x = (x_1, x_2, x_3, \ldots) = (w, v_1, v_2, \ldots) = (w; v) \in I^{\mathbb{N}},\) then \(2 \leq F_t(x) \leq \infty\) and \(F_t(x) = F_t(w; v) = 1 + F_t-Ln(1+(e-1)w)(v).\)

By Fubini’s theorem, \(N(t) = \int_{I^{\mathbb{N}}} F_t(x)dx = \int_{I}^{1} \int_{I^{\mathbb{N}}} F_t(w; v)dvdw = \int_{0}^{1} (1 + \int_{I^{\mathbb{N}}} F_t-Ln(1+(e-1)w)(v)dv)dw = 1 + \int_{0}^{1} N(t - \ln(1 + (e-1)v))dw.\) If \(u = 1 + (e-1)v,\) then \(N(t) = 1 + \frac{1}{e-1} \int_{1}^{u} N(t - \ln u)du.\)
Theorem 9. $M(t) \geq N(t)$ for all $t \geq 0$

Proof. As in the last proof, define $F_t : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ so that $F_t(x) = \min \{n \in \mathbb{N} : \ln(1 + (e - 1)x_1) + \ldots + \ln(1 + (e - 1)x_n) > t\}$. Moreover, define $G_t : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ by $G_t(x) = \min \{n \in \mathbb{N} : x_1 + \ldots + x_n > t\}$. Since $\ln(1 + (e - 1)t) \geq t$ for $t \in [0, 1]$, then $F_t(x) \leq G_t(x)$ for all $t \geq 0$ and $x \in \mathbb{N}^n$. Thus $N(t) \leq M(t)$ for all $t \geq 0$. \[\square\]

We use Wald’s equation to derive bounds on $N(t)$.

Theorem 10. (Wald’s equation) Let $X_1, X_2, \ldots$ be i.i.d. random variables with common finite mean, and let $\tau$ be a stopping time which is independent of $X_{\tau+1}, X_{\tau+2}, \ldots$ for which $E(\tau) < \infty$. Then $E(X_1 + \ldots + X_\tau) = E(\tau)E(X_1)$.

Lemma 11. For all $t \geq 0$, $(e - 1)t < N(t) \leq (e - 1)(t + 1)$.

Proof. Suppose that i.i.d. random variables $X_1, X_2, \ldots$ are chosen uniformly from $[0, 1]$. Define $\mu$ to be the expected value of $\ln(1 + (e - 1)X_i)$ for each $i$. Define the random variable $K$ to be minimal so that $\sum_{i=1}^K \ln(1 + (e - 1)X_i) > t$ and define $S(t)$ to be the expected value of $\sum_{i=1}^K \ln(1 + (e - 1)X_i)$. By definition, $N(t)$ is the expected value of $K$.

By Wald’s equation, $N(t) = S(t)/\mu$, so $N(t) = (e - 1)S(t)$. Since $t < S(t) \leq t + 1$, then $(e - 1)t < N(t) \leq (e - 1)(t + 1)$. \[\square\]

In order to prove that $N(t) = (e - 1)(t + \frac{e-2}{2}) + o(1)$, we use two more well-known results.

Theorem 12. (Chernoff’s bound) Suppose $X_1, X_2, X_3, \ldots$ are i.i.d. random variables such that $0 \leq X_i \leq 1$ for all $i$. Set $S_n = \sum_{i=1}^n X_i$ and $\mu = E(S_n)$. Then, for all $\delta > 0$, $Pr(|S_n - \mu| \geq \delta \mu) \leq 2e^{-\frac{\delta^2 \mu}{2(\mu + \delta \mu)}}$.

Theorem 13. (Local Limit Theorem [9]) Let $X_1, X_2, \ldots$ be i.i.d. copies of a real-valued random variable $X$ of mean $\mu$ and variance $\sigma^2$ with bounded density and a third moment. Set $S_n = \sum_{i=1}^n X_i$, let $f_n(y)$ be the probability density function of $\frac{S_n - \mu n}{\sqrt{n}}$, and let $\phi(y)$ be the probability density function of the Gaussian distribution $N(0, \sigma^2)$. Then $\sup_{y \in \mathbb{R}} |f_n(y) - \phi(y)| = O(\frac{1}{\sqrt{n}})$.

Theorem 14. $N(t) = (e - 1)(t + \frac{e-2}{2}) + o(1)$
Proof. As in the last proof, suppose that i.i.d. random variables \( X_1, X_2, \ldots \) are chosen uniformly from \([0, 1]\). Define \( \mu \) to be the expected value and \( \sigma^2 \) to be the variance of \( \ln(1 + (e \cdot i)X_i) \) for each \( i \). Define the random variable \( K \) to be minimal so that \( \sum_{i=1}^{K} \ln(1 + (e \cdot i)X_i) > t \) and define \( S(t) \) to be the expected value of \( \sum_{i=1}^{K} \ln(1 + (e \cdot i)X_i) \). By definition, \( N(t) \) is the expected value of \( K \).

By Wald’s equation, \( N(t) = S(t)/\mu \), so \( N(t) = (e \cdot i - 1)S(t) \). It remains to prove that \( S(t) - t = \frac{e^2 - 2}{2} + o_i(1) \).

For each integer \( i \geq 0 \), define the random variable \( Y_i = \sum_{j=1}^{i} \ln(1 + (e \cdot i)X_j) \) and let \( p_i(u) \) be the probability density function for the random variable \( U = (t-Y_i)^{t+1} > t \land Y_i \leq t \). We will show that \( p_i(u) = e - e^u + o_i(1) \) for \( i \in [(e \cdot i - 1)t - c\sqrt{t}, (e \cdot i - 1)t + c\sqrt{t}] \) for all constants \( c \geq 0 \).

Define \( q_i(y) \) to be the density function for \( Y_i \). By Bayes’ Theorem and the fact that \( Pr(\ln(1 + (e \cdot i)X_i) \geq u) = 1 - \frac{e^{-u}}{e^{-1}} \), \( p_i(u) = \frac{q_i(t-u)(1-e^{-u})}{\int_0^{t} q_i(t-y)(1-e^{-y})dy} \).

Let \( \phi(x) \) be the density function of the distribution \( \mathcal{N}(0, \sigma^2) \). By the local limit theorem, \( p_i(u) = \frac{\phi(t-u)(1-e^{-u})}{\int_0^{t} \phi(t-y)(1-e^{-y})dy} = (e \cdot i - 1)(1 - \frac{e^{-u}}{e^{-1}}) + o_i(1) = e - e^u + o_i(1) \) for \( i \in [(e \cdot i - 1)t - c\sqrt{t}, (e \cdot i - 1)t + c\sqrt{t}] \).

Now define the random variable \( O = -t + \sum_{i=1}^{K} \ln(1 + (e \cdot i)X_i) \), and let \( O_i(t) \) be the expected value of \( (O|K = i+1) \). Furthermore define \( V_{i,u}(t) \) to be the expected value of \( (O|U = u^iK = i+1) \). Then for \( i \in [(e \cdot i - 1)t - c\sqrt{t}, (e \cdot i - 1)t + c\sqrt{t}] \), \( O_i(t) = \int_0^t p_i(u)V_{i,u}(t)du = \int_0^t (e - e^u + o_i(1))(\frac{1}{e^u}) \int_{u+1}^{\infty} (\ln(1 + (e \cdot i)X_i - u)du)du = \frac{e^2 - 2}{2} + o_i(1) \).

For any \( \epsilon > 0 \), there is a constant \( c = c(\epsilon) > 0 \) such that \( Pr(|O|K = i+1|t > c\sqrt{t}) < \epsilon \) by Chernoff’s bound. Therefore, there is a sequence \( \epsilon_0 > \epsilon_1 > \epsilon_2 > \ldots \) converging to 0 such that \( |S(t) - t - \sum_{i=1}^{\lfloor (e \cdot i - 1)t + c(\epsilon_j)\sqrt{t} \rfloor} O_i(t)Pr(K = i+1)| < \epsilon_j \). Thus \( S(t) - t = \frac{e^2 - 2}{2} + o_i(1) \).

The proof above also generalizes to other functions \( f \) besides \( f(x) = \ln(1 + (e \cdot i)X_i) \). In particular, \( \ln(1 + (e \cdot i)X_i) \) can be replaced in the proof with an increasing bijection \( f : [0, 1] \to [0, 1] \), thus implying Theorem II.

Proof. Suppose that i.i.d. random variables \( X_1, X_2, \ldots \) are chosen uniformly from \([0, 1]\). Define \( \mu_f \) to be the expected value and \( \sigma_f^2 \) to be the variance of \( f(X_i) \) for each \( i \). Define the random variable \( K_f \) to be minimal so that
\(\sum_{i=1}^{K_f} f(X_i) > t\) and define \(S_f(t)\) to be the expected value of \(\sum_{i=1}^{K_f} f(X_i)\). By definition, \(N_f(t)\) is the expected value of \(K_f\).

By Wald’s equation, \(N_f(t) = S_f(t)/\mu_f\). It remains to prove that \(S_f(t) = t\).

For each integer \(i \geq 0\), define the random variable \(Y_i = \sum_{j=1}^i f(X_j)\) and let \(p_i(u)\) be the probability density function for the random variable \(U = (t - Y_i)Y_{i+1} > t \land Y_i \leq t\). We will show that \(p_i(u) = \frac{1 - f^{-1}(u)}{\mu_f} + o_t(1)\) for \(i \in \left[\frac{t}{\mu_f} - c\sqrt{t}, \frac{t}{\mu_f} + c\sqrt{t}\right]\) for all constants \(c \geq 0\).

Define \(q_i(y)\) to be the density function for \(Y_i\). By Bayes’ Theorem and the fact that \(Pr(f(X_i) \geq u) = 1 - f^{-1}(u), p_i(u) = \frac{q_i(t-u)(1-f^{-1}(u))}{\int_0^1 q_i(t-y)(1-f^{-1}(y))dy}\).

Let \(\phi(x)\) be the density function of the distribution \(\mathcal{N}(0, \sigma_f^2)\). By the local limit theorem, \(p_i(u) = \frac{(\frac{1}{\sqrt{2\pi}} \phi(\frac{t-u-\mu_f}{\sigma_f} \pm O(\frac{1}{\sqrt{t}}))(1-f^{-1}(u))}{\int_0^1 (\frac{1}{\sqrt{2\pi}} \phi(\frac{t-y-\mu_f}{\sigma_f} \pm O(\frac{1}{\sqrt{t}}))(1-f^{-1}(y))dy}\frac{1}{\mu_f}(1-f^{-1}(u)) + o_t(1) for \(i \in \left[\frac{t}{\mu_f} - c\sqrt{t}, \frac{t}{\mu_f} + c\sqrt{t}\right]\).

Now define the random variable \(O = -t + \sum_{i=1}^{K_f} f(X_i)\), and let \(O_i(t)\) be the expected value of \((O|K_f = i + 1)\). Furthermore define \(V_{i,u}(t)\) to be the expected value of \((O|U = u \land K_f = i + 1)\). Then for \(i \in \left[\frac{t}{\mu_f} - c\sqrt{t}, \frac{t}{\mu_f} + c\sqrt{t}\right]\), \(O_i(t) = \int_0^1 p_i(u) V_{i,u}(t) du = \int_0^1 \left(\frac{1}{\mu_f}(1-f^{-1}(u)) + o_t(1)\right) \left(\frac{1}{\mu_f} f^{-1}(f(x) - u) dx\right) du = \int_0^1 \int_0^1 \frac{f^{-1}(f(x) - u) dx}{\mu_f} + o_t(1)\).

For any \(\epsilon > 0\), there is a constant \(c = c(\epsilon) > 0\) such that \(Pr(|K_f - \frac{t}{\mu_f}| > c\sqrt{t}) < \epsilon\) by Chernoff’s bound. Therefore, there is a sequence \(\epsilon_0 > \epsilon_1 > \epsilon_2 > \ldots\) converging to 0 such that \(S_f(t) = t - \sum_{i=1}^{\left[\frac{t}{\mu_f} + c(\epsilon)\sqrt{t}\right]} O_i(t) Pr(K_f = i + 1) < \epsilon_j\). Thus \(S_f(t) - t = \frac{\int_0^1 f^{-1}(f(x) - u) dx}{\mu_f} + o_t(1)\).

**References**

[1] L. Bush. The William Putnam mathematical competition. Amer Math Monthly 68 (1961) 18-33.

[2] H. Shultz. An expected value problem. Two-Year College Math J 10 (1979) 179.
[3] N. MacKinnon. Another surprising appearence of e. Math Gazette 74 (1990) 167-9.

[4] S. Schwartzman. An unexpected expected value. Math Teacher (1993) 118-20.

[5] E. Weisstein. Uniform sum distribution. From MathWorld. [http://mathworld.wolfram.com/UniformSumDistribution.html](http://mathworld.wolfram.com/UniformSumDistribution.html)

[6] J. Blanchet and P. Glynn. Uniform Renewal Theory with Applications to Expansions of Random Geometric Sums. Advances in Applied Probability 39(4) (2007) 1070-1097.

[7] B. Čurgus and R. I. Jewett, An unexpected limit of expected values, Expo. Math. 25 (2007), 1-20.

[8] J. Doob. Renewal Theory from the Point of View of the Theory of Probability. Transactions of the AMS 63(3) (1947) 422-438.

[9] L. Hervé and J. Ledoux. Local limit theorem for densities of the additive component of a finite Markov Additive Process. Statistics and Probability Letters 83 (2013) 2119-2128.

[10] S. Vandervelde. Expected Value Road Trip, Mathematical Intelligencer, 30(2) (2008) 17-18.