Non-existence of stable solutions for weighted $p$-Laplace equation

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Abstract
We provide sufficient conditions on $w \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that the weighted $p$-Laplace equation
\[- \text{div} \left( w(x)|\nabla u|^{p-2} \nabla u \right) = f(u) \text{ in } \mathbb{R}^N\]
does not admit any stable $C^{1,\zeta}_{\text{loc}}$ solution in $\mathbb{R}^N$ where $f(x)$ is either $-x^{-\delta}$ or $e^x$ for any $0 < \zeta < 1$.

Keywords: $p$-Laplacian; Non-existence; Stable solution

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1 Introduction

In this paper we are interested in finding conditions under which the quasilinear degenerate equations given by
\[-\Delta_{p,w} u = -u^{-\delta} \text{ in } \mathbb{R}^N \text{ and } -\Delta_{p,w} u = e^u \text{ in } \mathbb{R}^N\]
doesn’t admit a stable solution in $\mathbb{R}^N$.

Here $-\Delta_{p,w} u := \text{div} \left( w(x)|\nabla u|^{p-2} \nabla u \right)$ where $w : \mathbb{R}^N \rightarrow \mathbb{R}$ is a positive, measurable function satisfies one of the following conditions:

• $w \in C^1(\mathbb{R}^N)$ such that $w(x) \geq \eta$ for some constant $\eta > 0$.

• There exists $M > 0$ such that $w(x) = w(|x|) = w(r) \leq M|r|^{\theta}$ for $\theta \in \mathbb{R}^N$ and $r = |x|$.

The study of stable solutions for elliptic equation has been a subject of interest for the last three decades. When $p = 2$ and $w \equiv 1$ the general problem of stable solutions of the equation given by
\[-\Delta u = f(u) \text{ in } \Omega\]
with $f$ locally lipschitz continuous in $\mathbb{R}$ has been a subject of considerable research. Obtaining $L^\infty$ estimates is one of the major concern in this case and we provide here a brief description of the available results.
• For the case $3 \leq N \leq 9$, Cabré and Capella \cite{2} settled the problem for the unit ball.

• a general smooth bounded domain the boundedness was derived for $N \leq 3$ and $N = 4$ respectively by Nedev \cite{6} and Cabré \cite{1} respectively

• When $N \geq 10$ the existence of unbounded solutions was shown in Cabré and Capella \cite{2}.

For more information on this field one can consult the survey of Cabré \cite{3}. In case of Laplacian in $\mathbb{R}^N$, classification results for $f(u) = |u|^{p-1}u$, $p > 1$ or when $f(u) = e^u$ are already available see Farina \cite{7, 8}. In Farina \cite{7}, non-existence of stable solutions for $2 \leq N \leq 9$ was obtained when $f(u) = e^u$. Other significant work in these topic can be found in \cite{14} and reference therein. For the equation

$$-\text{div} (w(x)|\nabla u|^{p-2} \nabla u) = g(x)f(u) \text{ in } \mathbb{R}^N \quad (1)$$

the following results were obtained recently

• When $w = g = 1$, Le \cite{12} showed non-existence result for $p > 2$ and $N < \frac{p(p+3)}{p-1}$ when $f(x) = e^x$.

• When $w = g = 1$, Guo-Mei \cite{11} showed non-existence for finite Morse index solutions for $2 \leq p < N < \frac{p(p+3)}{p-1}$ when $f(x) = -x^{-\delta}$, $\delta > q_c$ and $q_c$ is given by

$$q_c = \frac{(p-1)[(1-p)N^2 + (p^2 + 2p)N - p^2] - 2p^2((p-1)(N-1)^{1/2})}{(N-p)((p-1)N - p(p+3))}$$

• For $g \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $g(x) \geq C|x|^{\alpha}$ for $|x|$ large enough, Chen et al \cite{4} showed the non-existence results for the case $f(x) = -x^{-\delta}$, $\delta > q_c$ and $2 \leq p < N < \mu_0(p,a) := \frac{p(p+3)+4a}{p-1}$ and

$$q_c = \frac{2(N+a)(p+a) - (N-p)[(p-1)(N+a) - p-a] - \alpha}{(N-p)[N(p-1) - p(p+3)]}$$

where $\alpha = 2(p+a)\sqrt{(p+a)(N+a + \frac{N-p}{p+1})}$.

For $f(x) = e^x$ they obtained that for $1 \leq p < N < \frac{p(p+3)+4a}{p-1}$ the problem does not have a stable solution.

Our main aim in this note is to establish some non-existence results for stable solutions of the equation \cite{14} provided $g(x) = 1$. Before we begin with the main results let us define the notion of weak solution and stable solution for the problem \cite{14}.

**Definition 1.1.** We say that $u \in C^1_{\text{loc}}(\mathbb{R}^N)$ is a weak solution to the equation \cite{14} if for all $\varphi \in C^1_{\text{loc}}(\mathbb{R}^N)$ we have,

$$\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} \nabla u \nabla \varphi dx - \int_{\mathbb{R}^N} f(u) \varphi dx = 0. \quad (2)$$
Definition 1.2. A weak solution \( u \) of equation (1) is said to be stable if for all \( \varphi \in C^1_c(\mathbb{R}^N) \) we have,

\[
\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} |\nabla \varphi|^2 dx + (p-2) \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-4} (\nabla u, \nabla \varphi)^2 dx - \int_{\mathbb{R}^N} f'(u) \varphi^2 dx \geq 0.
\]

Therefore if \( u \) is a stable solution of equation (1) then we have,

\[
\int_{\mathbb{R}^N} f'(u) \varphi^2 dx \leq (p-1) \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} |\nabla \varphi|^2 dx.
\]

2 Main Results

We start by denoting the equations as \((n)_e\) and \((n)_s\) for \( f(x) = e^x \) and \( f(x) = -x^3 - \delta \) respectively with \( n = 1, 2, 3, 4 \) and

\[
A_r = \|w\|_{L^{p+3} \left( B_{2r}(0) \right)}^{\frac{p+3}{p-1}}
\]

where \( B_r(0) \) is the ball centered at 0 with radius \( r \geq 0 \). We will also assume \( C > 0 \) to be a constant for the rest of the paper which may vary depending on the situation.

Theorem 2.1. Let \( p > 2 \) and \( w \in C^1(\mathbb{R}^N) \) be such that \( w(x) > C \). If for every ball \( B_R(0) \) we have, \( A_R = O(R^\mu) \) where \( \mu < \frac{p(p+3)}{p-1} \), then there does not exist any stable solution of the problem \((n)_e\) in \( C^{1,\zeta}_{loc}(\mathbb{R}^N) \) for any \( 0 < \zeta < 1 \).

Theorem 2.2. Let \( p > 2 \) and \( w \in L^{p+3} \left( \mathbb{R}^N \right) \) be such that \( w(x) = w(|x|) \leq C|x|^\theta \), then the problem \((n)_e\) does not admit a stable solution in \( C^{1,\zeta}_{loc}(\mathbb{R}^N \setminus \{0\}) \) for any \( 0 < \zeta < 1 \) provided \( \theta \in \cup_\alpha I_\alpha \), where \( I_\alpha = (-\frac{N}{p\alpha+1},p-\frac{N}{p\alpha+1}) \) for \( \alpha \in (0, \frac{4}{p(p-1)}) \). Moreover

i) If \( N < \frac{p(p+3)}{p-1} \), there exists \( \alpha_0 \in (0, \frac{4}{p(p-1)}) \) such that \( 0 \in I_{\alpha_0} \).

ii) If \( N \geq \frac{p(p+3)}{p-1} \), then \( \cup_{I_\alpha} \subset (-\infty, 0) \).

Remark 2.1. For \( w(x) \equiv 1 \) the equation \((n)_e\) becomes the following equation

\[
-\Delta_p u = e^u \text{ in } \mathbb{R}^N
\]

and Theorem 2.2 says that there does not exists any stable solution to the equation (5) provided one has \( N < \frac{p(p+3)}{p-1}, p > 2 \).
For $\delta > 0$ denote,
\[ k = \frac{\delta + p - 1 + 2l}{\delta + p - 1} \]
where \( l = \frac{2\delta}{p - 1} - \frac{p - 1}{2} \) and \( Q_r = \|w\|^k_{L^k(B_r, (0))} \).

Also
\[ B := (0, l) \]

Note that for $\delta > 0$ and $p > 2$ we have $k > 1$.

**Theorem 2.3.** Suppose $w \in C^1(\mathbb{R}^N)$ be such that $w(x) > C$ and for every ball $B_{2R}(0)$ we have, $Q_R = O(R^\tau)$. If $0 < \tau < pk$ with $p > 2$ and $\delta \geq \max\{1, \frac{(p-1)^2}{4}\}$ then there is no stable, positive solution of equation (1) in $C^{1,\zeta}(\mathbb{R}^N)$ for any $0 < \zeta < 1$.

**Theorem 2.4.** Let $p > 2$ and $w \in L^k_{\text{loc}}(\mathbb{R}^N)$ be such that $w(x) = w(|x|) \leq C|x|^{\theta}$. If $\theta \in \cup_{\beta \in B} K_{\beta}$ where $K_{\beta} = (-\frac{N(\delta+p-1)}{2\delta+\delta+p-1}, -\frac{N(\delta+p-1)}{2\delta+\delta+p-1})$ for $\beta \in B$ then there does not exist any positive, stable solution of the problem (1) in $C^{1,\zeta}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ for any $0 < \zeta < 1$. Moreover,

i) If $N < kp$ then there exist $\beta_0 \in B$ such that $0 \in K_{\beta_0}$

ii) If $N \geq kp$ then $\cup_{\beta \in B} K_{\beta} \subset (-\infty, 0)$

**Remark 2.2.** For $w(x) \equiv 1$ the equation (1) becomes the following equation
\[ -\Delta_p u + u^{\delta} = 0 \text{ in } \mathbb{R}^N \] (6)
and Theorem 2.4 says that there does not exists any positive, stable solution to the equation (6) provided one has $N < kp$, $p > 2$ and $\delta \geq \max\{1, \frac{(p-1)^2}{4}\}$.

### 3 Proof of Main Theorems

**Proof of Theorem 2.1.** Suppose $u \in C^{1,\zeta}(\mathbb{R}^N)$, $0 < \zeta < 1$ be a stable solution of equation (1) for $\mu < \frac{p(p+3)}{p-1}$ and let $\psi \in C^1_c(\mathbb{R}^N)$.

**Step 1.** Choosing $\varphi = e^{\rho \alpha u} \psi^p$ ($\alpha > 0$ to be chosen later) in equation (2). Since,
\[ \nabla \varphi = p\alpha e^{\rho \alpha u} \psi^p \nabla u + p e^{\rho \alpha u} \psi^{p-1} \nabla \psi \]
we obtain
\[ p\alpha \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{\rho \alpha u} \psi^p \, dx + \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} e^{\rho \alpha u} \psi^{p-1} \nabla u, \nabla \psi > \, dx = \int_{\mathbb{R}^N} e^{(\rho \alpha + 1)u} \psi^p \, dx. \]
Therefore using Young’s inequality for any \( \epsilon \in (0, p\alpha) \), we obtain

\[
p\alpha \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{p\alpha u_\psi p} \, dx
\leq p \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1} e^{p\alpha u_\psi p-1} |\nabla \psi| \, dx + \int_{\mathbb{R}^N} e^{(p\alpha+1)u_\psi p} \, dx
\leq \int_{\mathbb{R}^N} \epsilon \left( w^{\frac{p-1}{p}} |\nabla u|^{p-1} e^{(p\alpha-1)u_\psi p-1} \right)^{\frac{p}{p-1}}
+ C_\epsilon \left( w^{\frac{1}{p}} e^{\alpha u_\psi} |\nabla \psi|^p \right) \, dx + \int_{\mathbb{R}^N} e^{(p\alpha+1)u_\psi p} \, dx
= \epsilon \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{p\alpha u_\psi p} \, dx + C_\epsilon \int_{\mathbb{R}^N} w(x)e^{p\alpha u_\psi} |\nabla \psi|^p \, dx
+ \int_{\mathbb{R}^N} e^{(p\alpha+1)u_\psi p} \, dx
\]

Therefore we get,

\[
(p\alpha - \epsilon) \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{p\alpha u_\psi p} \, dx \leq C_\epsilon \int_{\mathbb{R}^N} w(x)e^{p\alpha u_\psi}|\nabla \psi|^p \, dx
+ \int_{\mathbb{R}^N} e^{(p\alpha+1)u_\psi p} \, dx. \tag{7}
\]

**Step 2.** Choose \( \varphi = e^{\frac{p\alpha}{2} u_\psi} \). Therefore,

\[
\nabla \varphi = \frac{p\alpha}{2} e^{\frac{p\alpha}{2} u_\psi} \nabla u + \frac{p}{2} e^{\frac{p\alpha}{2} u_\psi} e^{\frac{p-2}{2} \nabla u} \nabla \psi
\]

Putting \( \varphi \) and \( \nabla \varphi \) in the stability equation (4), we obtain

\[
\int_{\mathbb{R}^N} e^{(p\alpha+1)u_\psi p} \, dx \leq (p-1) \int_{\mathbb{R}^N} w(x)|\nabla u|^p \left( \frac{p\alpha}{2} \right)^2 e^{p\alpha u_\psi p} \, dx
+ (p-1) \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1} \left( \frac{p\alpha}{2} \right)^2 e^{p\alpha u_\psi p-1} |\nabla \psi| \, dx
+ (p-1) \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} \left( \frac{p}{2} \right)^2 e^{p\alpha u_\psi p-2} |\nabla \psi|^2 \, dx.
\]

Using Young’s inequality we estimate the last two terms

\[
(p-1) \int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1} \left( \frac{p\alpha}{2} \right)^2 e^{p\alpha u_\psi p-1} |\nabla \psi| \, dx
\leq \int_{\mathbb{R}^N} \frac{\epsilon}{2} \left( w^{\frac{p-1}{p}} |\nabla u|^{p-1} e^{(p\alpha-1)u_\psi p-1} \right)^{\frac{p}{p-1}} + C_\epsilon \left( w^{\frac{1}{p}} e^{\alpha u_\psi} |\nabla \psi|^p \right) \, dx
= \frac{\epsilon}{2} \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{p\alpha u_\psi p} \, dx + C_\epsilon \int_{\mathbb{R}^N} w(x)e^{p\alpha u_\psi} |\nabla \psi|^p \, dx,
\]

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and
\[(p - 1) \int_{\mathbb{R}^N} w(x)|\nabla u|^{p - 2}\left(\frac{P}{2}\right)^2 e^{\alpha u}\psi^{p - 2}|\nabla \psi|^2 \, dx\]
\[\leq \int_{\mathbb{R}^N} \frac{\epsilon}{2} \left( w \frac{\psi}{\psi(u)} |\nabla u|^{p - 2} e^{(p - 2)\alpha u}\psi^{p - 2}\right) \frac{\psi}{\psi(u)} + C_{\epsilon} \left( w \frac{\psi}{\psi(u)} e^{2\alpha u}|\nabla \psi|^2\right) \frac{\psi}{\psi(u)} \, dx\]
\[= \frac{\epsilon}{2} \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{\alpha u}\psi^p \, dx + C_{\epsilon} \int_{\mathbb{R}^N} w(x)e^{\alpha u}|\nabla \psi|^p \, dx.\]

Using these two estimates in the previous one, we obtain after using (7)
\[\int_{\mathbb{R}^N} e^{(p\alpha + 1)u}\psi^p \, dx\]
\[\leq \left( \frac{(p - 1)p^2\alpha^2}{4} + \epsilon \right) \int_{\mathbb{R}^N} w(x)|\nabla u|^p e^{\alpha u}\psi^p \, dx + C_{\epsilon} \int_{\mathbb{R}^N} w(x)e^{\alpha u}|\nabla \psi|^p \, dx\]
\[\leq \left( \frac{(p - 1)p^2\alpha^2}{4} + \epsilon \right) \frac{1}{p\alpha - \epsilon} \int_{\mathbb{R}^N} e^{(p\alpha + 1)u}\psi^p \, dx + C_{\epsilon} \int_{\mathbb{R}^N} w(x)e^{\alpha u}|\nabla \psi|^p \, dx.\]

Define \(\gamma_\epsilon = 1 - \left( \frac{(p - 1)p^2\alpha^2}{4} + \epsilon \right) \frac{1}{p\alpha - \epsilon}\). Note that \(\lim_{\epsilon \to 0} \gamma_\epsilon = 1 - \frac{2p(p - 1)}{4} > 0\) for \(\alpha \in \left(0, \frac{4}{p(p - 1)}\right)\).
Hence we can choose some \(\epsilon \in (0, 1)\) depending on \(p\) and \(\alpha\) such that \(\gamma_\epsilon > 0\).
Hence we get,
\[\int_{\mathbb{R}^N} e^{(p\alpha + 1)u}\psi^p \, dx \leq C \int_{\mathbb{R}^N} w(x)e^{\alpha u}|\nabla \psi|^p \, dx. \tag{8}\]

where \(C\) is a constant depending on the chosen \(\epsilon \in (0, 1)\). Next we choose some \(m\) such that \(p\alpha + 1 = m\) and apply (3) for \(\psi = \eta^m\) to obtain
\[\int_{\mathbb{R}^N} e^{(p\alpha + 1)u}\eta^{pm} \, dx \leq C \int_{\mathbb{R}^N} w(x)e^{\alpha u}\eta^{p(m - 1)}|\nabla \eta|^p \, dx\]
\[\leq \int_{\mathbb{R}^N} \epsilon \left( e^{\alpha u}\eta^{p(m - 1)} \right)^{\frac{p + 1}{pm}} + C_{\epsilon} (w|\nabla \eta|^p)^{p + 1} \, dx\]
\[\leq \epsilon \int_{\mathbb{R}^N} e^{(p\alpha + 1)u}\eta^{pm} \, dx + C_{\epsilon} \int_{\mathbb{R}^N} w^{p + 1}|\nabla \eta|^{p + 1} \, dx.\]

Choosing \(\eta_R \in C^1_c(\mathbb{R}^N)\) satisfying \(0 \leq \eta_R \leq 1\) in \(\mathbb{R}^N\), \(\eta_R = 1\) in \(B_R(0)\) and \(\eta_R = 0\) in \(\mathbb{R}^N \setminus B_{2R}(0)\) with \(|\nabla \eta| \leq \frac{C}{R}\) for \(C > 1\) we obtain after using the assumption on \(w\)
\[\int_{B_R(0)} e^{(p\alpha + 1)u} \, dx \leq CR^{-p(p\alpha + 1)} \int_{B_{2R}(0)} w^{p\alpha + 1} \, dx \tag{9}\]

Now since, \(A_R = \mathcal{O}(R^m)\) we have for sufficiently large \(R\),
\[\int_{B_R(0)} e^{(p\alpha + 1)u} \, dx \leq CR^{\left[p\left(\frac{m - 1}{p - m}\right)\right]}(p\alpha + 1) \, dx\]
which tends to 0 as \(R \to \infty\). Hence arriving at a contradiction. \(\square\)
Proof of Theorem 2.2. Following the exact proof of Theorem 2.1 for $w(x) \leq C|x|^\theta$ we have from (11),

$$\int_{B_R(0)} e^{p(\alpha+1)u} dx \leq CR^{-p(\alpha+1)} \int_{B_{2R}(0)} (w(x))^{p(\alpha+1)} r^{N-1} dr$$

$$\leq CR^{-p(\alpha+1)} \int_0^{2R} r^{p(\alpha+1)+N} dr$$

which implies that $R^{(\theta-p)(\alpha+1)+N} \to 0$ as $R \to \infty$, given that

$$-\frac{N}{p+1} < \theta < p - \frac{N}{p\alpha+1}$$

for any $\alpha \in (0, \frac{4}{p(p-1)})$. This proves the first part of the Theorem.

Note that when $N < \frac{p(p+3)}{p-1}$, there exist an $\alpha_0 \in (0, \frac{4}{p(p-1)})$ such that $N - p(p\alpha_0 + 1) < 0$, which implies $p - \frac{N}{p\alpha_0+1} > 0$.

Hence the interval $I_{\alpha_0} := (-\frac{N}{p\alpha_0+1}, p - \frac{N}{p\alpha_0+1})$ contains both positive and negative values of $\theta$. This along with the fact that for any $\theta$ in $I_{\alpha}$ and $\alpha \in (0, \frac{4}{p(p-1)})$ we have,

$$R^{(\theta-p)(\alpha+1)+N} \to 0 \text{ as } N \to \infty$$

shows that there are $\theta$, both positive and negative for which the problem (1) does not admit a stable solution.

Again when $N \geq \frac{p(p+3)}{p-1}$, we have $p - \frac{N(p-1)}{p+3} \leq 0$. Note that for any $\alpha \in (0, \frac{4}{p(p-1)})$, we have the strict inequality

$$p - \frac{N}{p+1} < p - \frac{N(p-1)}{p+3}.$$ 

Hence $I_{\alpha} \subset (-\infty, 0)$ when $N \geq \frac{p(p+3)}{p-1}$ and so all exponent $\theta$ must be negative for the non-existence to hold.

Proof of Theorem 2.3. Suppose $u \in C^1(\mathbb{R}^N)$ be a positive stable solution of equation (1) for $f(x) = -x^{-\delta}$ and $\psi \in C^1_c(\mathbb{R}^N)$.

**Step 1.** Choosing $\varphi = u^{-\alpha} \psi^\delta$, ($\alpha > 0$ to be chosen later) as a test function in the weak form (2), since

$$\nabla \varphi = -\alpha u^{-\alpha-1} \psi^\delta \nabla u + p\psi^{\delta-1} u^{-\alpha} \nabla \psi$$

we obtain

$$\alpha \int_{\mathbb{R}^N} w(x)u^{-\alpha-1} \psi^\delta |\nabla u|^{p-1} dx \leq p \int_{\mathbb{R}^N} w(x)u^{-\alpha} \psi^{\delta-1} |\nabla u|^{p-1} |\nabla \psi| dx + \int_{\mathbb{R}^N} u^{-\delta-\alpha} \psi^{\delta} dx \quad (10)$$
Now using the Young’s inequality for $\varepsilon \in (0, \alpha)$, we obtain

\[
p \int_{\mathbb{R}^N} w(x) u^{-\alpha} \psi^{p-1} |\nabla u|^{p-1} |\nabla \psi| \, dx
= p \int_{\mathbb{R}^N} \left( w(x) u^{-\alpha} \psi^{p-1} |\nabla u|^{p-1} |\nabla \psi| \right) \, dx
\leq \{ \varepsilon \int_{\mathbb{R}^N} w(x) u^{-\alpha} |\nabla u|^{p} \psi^{p} \, dx
+ C_\varepsilon \int_{\mathbb{R}^N} w(x) u^{-\alpha} |\nabla \psi|^{p} \, dx \}
\]

Plugging this estimate in (10), we get

\[
(\alpha - \varepsilon) \int_{\mathbb{R}^N} w(x) u^{-\alpha-1} |\nabla u|^{p} \psi^{p} \, dx \leq C_\varepsilon \int_{\mathbb{R}^N} w(x) u^{-\alpha} |\nabla u|^{p} \psi^{p} \, dx
+ \int_{\mathbb{R}^N} u^{-\delta-\alpha} \psi^{p} \, dx \quad (11)
\]

**Step 2.** Choosing $\varphi = u^{-\beta-\frac{P}{2}+1} \psi$, ($\beta > 0$ to be chosen later) as a test function in the stability equation, since

\[
\nabla \varphi = (-\beta - \frac{P}{2} + 1) u^{-\beta-\frac{P}{2}} \psi \nabla u + \frac{P}{2} \psi^{-\frac{P}{2}} u^{-\beta-\frac{P}{2}+1} \nabla \psi
\]

we obtain

\[
\delta \int_{\mathbb{R}^N} u^{-2\beta-\delta-P+1} \psi^{p} \, dx \leq (p-1)(-\beta - \frac{P}{2} + 1)^2 \int_{\mathbb{R}^N} w(x) u^{-2\beta-p} |\nabla u|^{p} \psi^{p} \, dx
+ (p-1)\frac{P^2}{4} \int_{\mathbb{R}^N} w(x) u^{-2\beta-p+2} |\nabla u|^{p-2} |\nabla \psi|^{2} \, dx
+ p(p-1)(-\beta - \frac{P}{2} + 1) \int_{\mathbb{R}^N} w(x) u^{-2\beta-p+1} |\nabla u|^{p-1} |\nabla \psi| \, dx = A + B + C
\]

where

\[
A = (p-1)(-\beta - \frac{P}{2} + 1)^2 \int_{\mathbb{R}^N} w(x) u^{-2\beta-p} |\nabla u|^{p} \psi^{p} \, dx
\]

\[
B = (p-1)\frac{P^2}{4} \int_{\mathbb{R}^N} w(x) u^{-2\beta-p+2} |\nabla u|^{p-2} |\nabla \psi|^{2} \, dx
\]

and

\[
C = p(p-1)(-\beta - \frac{P}{2} + 1) \int_{\mathbb{R}^N} w(x) u^{-2\beta-p+1} |\nabla u|^{p-1} |\nabla \psi| \, dx.
\]

Therefore we have

\[
\delta \int_{\mathbb{R}^N} u^{-2\beta-\delta-p+1} \psi^{p} \, dx \leq A + B + C \quad (12)
\]
Now, using the exponents $\frac{p}{p-\beta}$ and $\frac{2}{p}$ in the Young’s inequality, we have

$$B = (p - 1)\frac{p^2}{4} \int_{\mathbb{R}^N} w(x)u^{-2\beta - p + 2}\psi^{p-2}|\nabla u|^{p-2}|
abla \psi|^2\,dx$$

$$= (p - 1)\frac{p^2}{4} \int_{\mathbb{R}^N} (w^{\frac{p-2}{p}}|\nabla u|^{p-2}\psi^{(\frac{2p-2-\beta(p-2)}{p})})(w^{\frac{2}{p}}\psi^{\frac{2(p-2-\beta)}{p}})|\nabla \psi|^2)\,dx$$

$$\leq \frac{c}{2} \int_{\mathbb{R}^N} w(x)|\nabla u|^p \psi^p u^{-2\beta-p} \,dx + C_\epsilon \int_{\mathbb{R}^N} w(x)u^{-2\beta}|
abla \psi|^p \,dx$$

Also using the exponents $p' = \frac{p}{p-1}$ and $p$ in the Young’s inequality, we obtain

$$C = p(p - 1)(-\beta - \frac{p}{2} + 1) \int_{\mathbb{R}^N} w(x)u^{-2\beta - p + 1}\psi^{p-1}|\nabla u|^{p-1}|
abla \psi|\,dx$$

$$= p(p - 1)(-\beta - \frac{p}{2} + 1) \int_{\mathbb{R}^N} (w^{p'}|\nabla u|^{p-1}\psi^{p-1}u^{-\frac{2(p-\beta)}{p'}})(w^p u^{-\frac{2(p-\beta)}{p'}})|\nabla \psi|)\,dx$$

$$\leq \frac{c}{2} \int_{\mathbb{R}^N} w(x)|\nabla u|^p \psi^p u^{-2\beta-p} \,dx + C_\epsilon \int_{\mathbb{R}^N} w(x)u^{-2\beta}|
abla \psi|^p \,dx$$

Choosing $\alpha = 2\beta + p - 1 > 0$ in equation (11) we get

$$\int_{\mathbb{R}^N} w(x)u^{-2\beta-p}|\nabla u|^p \psi^p \,dx \leq \frac{1}{2\beta + p - \epsilon - 1} \{ C_\epsilon \int_{\mathbb{R}^N} w(x)u^{-2\beta}|
abla \psi|^p \,dx \} \ \text{(13)}$$

Using the inequality (13) and the above estimates on $B$ and $C$ in (12), we get

$$\delta \int_{\mathbb{R}^N} u^{-2\beta - \delta - p+1}\psi^p \,dx \leq \frac{(p - 1)(-\beta - \frac{p}{2} + 1)^2 + \epsilon}{(2\beta + p - \epsilon - 1)} \int_{\mathbb{R}^N} u^{-2\beta - \delta - p+1}\psi^p \,dx +$$

$$C_\epsilon \frac{(p - 1)(-\beta - \frac{p}{2} + 1)^2 + \epsilon}{(2\beta + p - \epsilon - 1)} \int_{\mathbb{R}^N} w(x)u^{-2\beta}|
abla \psi|^p \,dx.$$
Choosing the exponents $\gamma = \frac{2\beta + \delta + p - 1}{2\beta}$ and $\gamma' = \frac{\tau}{\beta - 1} = \frac{2\beta + \delta + p - 1}{\beta + p - 1}$ in the Young's inequality we get,

$$
\int_{\mathbb{R}^N} \left( \frac{\psi}{u} \right)^{2\beta + \delta + p - 1} dx \leq \epsilon \int_{\mathbb{R}^N} \left( \frac{\psi}{u} \right)^{2\beta + \delta + p - 1} dx + C\epsilon \int_{\mathbb{R}^N} w^\gamma \psi^{(\delta - 1)\gamma} |\nabla \psi|^p \gamma' dx
$$

Therefore we get the inequality

$$
\int_{\mathbb{R}^N} \left( \frac{\psi}{u} \right)^{2\beta + \delta + p - 1} dx \leq C \int_{\mathbb{R}^N} w^{\frac{2\beta + \delta + p - 1}{\beta + p - 1}} (\psi \frac{\delta - 1}{\beta} |\nabla \psi|)^{\frac{p(2\beta + \delta + p - 1)}{\beta + p - 1}} dx.
$$

(15)

Choose $\psi_R \in C^1_c(\mathbb{R}^N)$ satisfying $0 \leq \psi_R \leq 1$ in $\mathbb{R}^N$, $\psi_R = 1$ in $B_R(0)$ and $\psi_R = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$ with $|\nabla \psi_R| \leq \frac{C}{R}$ for $C > 1$,

$$
\int_{B_R(0)} \left( \frac{1}{u} \right)^{2\beta + \delta + p - 1} dx \leq CR^{\frac{2\beta + \delta + p - 1}{\beta + p - 1}} \int_{B_{2R}(0)} w^{\frac{2\beta + \delta + p - 1}{\beta + p - 1}} dx.
$$

(16)

Since $Q_R = O(R^\tau)$, letting $R \to \infty$ we have

$$
\int_{B_R(0)} \left( \frac{1}{u} \right)^{2\beta + \delta + p - 1} dx \leq CR^{\frac{\beta + p - 1 + 2\beta}{\beta + p - 1}} \to 0
$$

as $R \to \infty$, where $l := \frac{2\delta}{p - 1} - \frac{p - 1}{2}$.

This implies $\int_{\mathbb{R}^N} \left( \frac{1}{u} \right)^{2\beta + \delta + p - 1} dx = 0$ and so we arrive at a contradiction. □

**Proof of Theorem 2.4** Following the proof of Theorem 2.3 for $w(x) \leq C|x|^\theta$ we have from (16),

$$
\int_{B_R(0)} \left( \frac{1}{u} \right)^{2\beta + \delta + p - 1} dx \leq CR^{\frac{\beta + p - 1 + 2\beta}{\beta + p - 1}} \int_{B_{2R}(0)} w^{\frac{2\beta + \delta + p - 1}{\beta + p - 1}} dx
$$

$$
\leq CR^{\frac{\beta + p - 1 + 2\beta}{\beta + p - 1}} \int_{B_{2R}(0)} r^{\frac{\beta + p - 1 + 2\beta}{\beta + p - 1}} r^{N - 1} dr
$$

$$
= CR^{\frac{(\theta - p)(2\beta + \delta + p - 1)}{\beta + p - 1} + N}
$$

(17)

Now given, $\theta \in (\frac{N(\beta + p - 1)}{2\beta + \delta + p - 1}, \frac{N(\delta + p - 1)}{2\beta + \delta + p - 1})$ one has, $\frac{(\theta - p)(2\beta + \delta + p - 1)}{\beta + p - 1} + N < 0$.

Hence we have from (17) that $\int_{\mathbb{R}^N} \left( \frac{1}{u} \right)^{2\beta + \delta + p - 1} dx = 0$ which is a contradiction to the fact that $u$ is a stable solution.

When $N \geq pk$ then one has,

$$
p - \frac{N(\delta + p - 1)}{2\beta + \delta + p - 1} < p - \frac{N(\delta + p - 1)}{2l + \delta + p - 1} \leq 0 \text{ for all } \beta \in \mathcal{B}
$$
where 

\[ l := \frac{3\delta}{p-1} - \frac{p-1}{2}. \]

Therefore one has \( K_\beta \subset (-\infty, 0) \) for all \( \beta \in \mathcal{B} \).

For \( N < pk \), there exist \( \beta_0 \in (0, l) \) such that

\[ N - \frac{p(2\beta_0 + \delta + p - 1)}{\delta + p - 1} < 0 \]

which implies,

\[ p - \frac{N(\delta + p - 1)}{2\beta + \delta + p - 1} > 0. \]

Hence \( 0 \in K_{\beta_0} \). \( \square \)

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