A New Family of Continuous Probability Distributions

M. El-Morshedy 1,2,*, Fahad Sameer Alshammari 1, Yasser S. Hamed 3, Mohammed S. Eliwa 2, and Haitham M. Yousof 4

1 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia; f.alshammari@psau.edu.sa
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; mseliwa@mans.edu.eg
3 Department of Mathematics and Statistics, College of Science, Taif University, Taif 21944, Saudi Arabia; yasersalah@tu.edu.sa
4 Department of Statistics, Mathematics and Insurance, Benha University, Benha 13518, Egypt; haitham.yousof@fcom.bu.edu.eg
* Correspondence: m.elmorshedy@psau.edu.sa

Abstract: In this paper, a new parametric compound G family of continuous probability distributions called the Poisson generalized exponential G (PGEG) family is derived and studied. Relevant mathematical properties are derived. Some new bivariate G families using the theorems of “Farlie-Gumbel-Morgenstern copula”, “the modified Farlie-Gumbel-Morgenstern copula”, “the Clayton copula”, and “the Renyi’s entropy copula” are presented. Many special members are derived, and a special attention is devoted to the exponential and the one parameter Pareto type II model. The maximum likelihood method is used to estimate the model parameters. A graphical simulation is performed to assess the finite sample behavior of the estimators of the maximum likelihood method. Two real-life data applications are proposed to illustrate the importance of the new family.

Keywords: poisson distribution; generalized exponential distribution; compounding; Farlie-Gumbel-Morgenstern; clayton copula; Ali-Mikhail-Haq copula; modeling; Lomax distribution; kernel density estimation

1. Introduction and Genesis

In statistical literature, we always assume that every real phenomenon can be modeled by some lifetime distributions. If we know this distribution(s), we can then analyze our phenomenon, as many lifetime distributions have been developed in this regard. The well-known Poisson distribution is one of the famous distributions that was also defined and studied as a new family of continuous distribution in the concept of compounding. Using the Poisson G family, several compound lifetime G families have been proposed and studied. In the compounding method, there are two different approaches available; one is by using zero truncated power series (ZTPS) distribution and the other by using zero truncated Poisson (ZTP) distribution directly with other continuous distributions. A comprehensive survey regarding the Poisson G models was recently proposed by [1].

In this paper, we propose and study a new family of distributions using ZTP distribution with a strong physical motivation. Suppose that a system has $N$ (a discrete random variable) subsystems functioning independently at a given time, where $N$ has a ZTP distribution with parameter $\lambda$ and the failure time of $i$th component $Y_i$ ($i = 1, 2, \ldots$) is independent of $N$. It is the conditional probability distribution of a Poisson-distributed random variable (RV), given that the value of the RV is not zero. The probability mass function (PMF) of $N$ is given by

$$P\lambda(N = n) = \frac{\lambda^n \exp(-\lambda)}{\Gamma(1 + n) C\lambda} |(n = N) and C\lambda = 1 - \exp(-\lambda)$$
Note that for ZTP RV, the expected value $E(N|\lambda)$ and variance $V(N|\lambda)$ are, respectively, given by $E(N|\lambda) = \lambda/C_\lambda$ and $V(N|\lambda) = \frac{\lambda^2}{C_\lambda}$. Suppose that for each subsystem, the failure time (i.e., $i$th component) has the generalized exponential generator (GE-G) defined by the cumulative distribution function (CDF)

$$
\mathcal{H}_{\theta,\beta,\xi}(x) = \left[1 - St_{\beta,\xi}(x)\right]^{\theta}, \theta, \beta > 0 \text{ and } x \in \mathbb{R},
$$

(1)

where the function $St_{\beta,\xi}(x) = \exp[-\beta \Delta_{\xi}(x)]$, $\Delta_{\xi}(x) = G_{\xi}(x)/\mathcal{G}_{\xi}(x)$ refers to the odd ratio function (ORF), $G_{\xi}(x)$ refers to the base-line CDF with parameters vector $\xi$, $\mathcal{G}_{\xi}(x) = 1 - G_{\xi}(x)$ refers to the base-line survival function (SF) of the base-line model and $\beta > 0$ is a shape parameter, $G_{\xi}(x)$ is the CDF of the base-line model, and $\frac{d}{dx} G_{\xi}(x) = g_{\xi}(x)$ is the probability density function (PDF) of the base-line model. Staying in (1) and for $\beta = 1$, the GE-G reduces to exponential G (E-G) (see [2]). Let $Y_i$ denote the failure time of the $i$th subsystem, and let

$$
X = \min\{Y_1, Y_2, \ldots, Y_N\},
$$

Then, the conditional CDF of $X$ given $N$ is

$$
F(x|N) = 1 - \Pr(X > x|N) = 1 - \left[1 - \mathcal{H}_{\theta,\beta,\xi}(x)\right]^N = 1 - \left\{1 - [1 - St_{\beta,\xi}(x)]^\theta\right\}^N.
$$

Therefore, the unconditional CDF of $X$, as described in [3–9], can be expressed as

$$
F_{X}(x) = C_{\lambda}^{-1}\left(1 - \exp\left\{-\lambda\left[1 - St_{\beta,\xi}(x)\right]^\theta\right\}\right), x \in \mathbb{R}
$$

(2)

The CDF in (2) is called the Poisson generalized exponential G (PGEG) family, $\mathcal{V} = (\lambda, \theta, \beta, \xi)$ is the parameter vector of the PGEG family. The corresponding PDF can be derived as

$$
f(x) = \lambda C_{\lambda}^{-1} h_{\theta,\beta,\xi}(x) \exp\left\{-\lambda\left[1 - St_{\beta,\xi}(x)\right]^\theta\right\}, x, \lambda > 0,
$$

(3)

where the function $h_{\theta,\beta,\xi}(x) = d\mathcal{H}_{\theta,\beta,\xi}(x)/dx$. Or, the PDF due to (3) can be re-expressed as

$$
f_{X}(x) = \lambda \beta C_{\lambda}^{-1} \frac{G_{\xi}(x)St_{\beta,\xi}(x)}{\mathcal{G}_{\xi}(x)^2}\left[1 - St_{\beta,\xi}(x)\right]^{1-\theta} \exp\left\{-\lambda\left[1 - St_{\beta,\xi}(x)\right]^\theta\right\}, x \in \mathbb{R}.
$$

(4)

A RV $X$ having PDF (4) is denoted by $X \sim$ PGE-G ($\mathcal{V}$). Some special cases of the PGE-G family are listed in Table 1.

| No. | Baseline Model | $\Delta_{\xi}(x)$ | $\mathcal{V} = (\lambda,\theta,\beta,\xi)$ | New Model |
|-----|----------------|-------------------|----------------------------------------|-----------|
| 1   | Exponential (E) | exp(cx) − 1       | $\lambda \in \mathbb{R} - \{0\}, \theta > 0, \beta > 0, \xi = (c > 0)$ | PGEE |
| 2   | Log-logistic (LL) | $(\xi)^c$ | $\lambda \in \mathbb{R} - \{0\}, \theta > 0, \beta > 0, \xi = (a, c > 0)$ | PGELL |
| 3   | Weibull (W) | exp(ax)$^\xi$ − 1 | $\lambda \in \mathbb{R} - \{0\}, \theta > 0, \beta > 0, \xi = (a, c > 0)$ | PGEW |
Table 1. Cont.

| No. | Baseline Model             | $\Delta_2(x)$                  | $Y = (\lambda, \phi, \xi, \delta)$ | New Model          |
|-----|---------------------------|--------------------------------|------------------------------------|-------------------|
| 4   | Fréchet (F)               | $\exp(x^{-c}) - 1$             | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (c > 0)$ | PGEF              |
| 5   | Rayleigh (R)              | $\exp(cx)^2 - 1$              | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (c > 0)$ | PGER              |
| 6   | Dagum (D)                 | $\left[\left(1 + \left(\frac{x}{\xi}\right)^{-a}\right)^c - 1\right]^{-1}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, b, c > 0)$ | PGED              |
| 7   | Burr type XII (BXII) (PII)| $(1 + x/a)^c - 1$             | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGEBXII           |
| 8   | Lindley (Li)              | $\exp\left[\frac{cx}{\beta}\right] - 1$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (c > 0)$ | PGE Li            |
| 9   | Inverse Rayleigh (IR)     | $\exp(ax^{-2}) - 1$           | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (c > 0)$ | PGEIR             |
| 10  | Half-logistic (HL)        | $\left\{\frac{1 - \exp(-cx)}{1 + \exp(-cx)}\right\}^{-1}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (c > 0)$ | PGEHL             |
| 11  | Inverse Exponential (IE)  | $\exp(ax^{-1}) - 1$           | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (c > 0)$ | PGEIE             |
| 12  | Inverse PII               | $\left\{\left(1 + ax^{-1}\right)^c - 1\right\}^{-1}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGEPII            |
| 13  | Gumbel (Gu)               | $(\exp\left[\frac{-\xi}{\lambda}\right] - 1) - 1$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a > c > 0)$ | PGE Gu            |
| 14  | Burr type XII (BXII)      | $\left\{\left(1 + (x/\lambda)^b\right)^c - 1\right\}^{-1}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGBXII            |
| 15  | Fréchet (F)               | $\exp(\phi x - c) - 1$        | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGEF              |
| 16  | Burr type X (BX)          | $\left\{\left(1 - \exp[-(ax)^2]\right)^c - 1\right\}^{-1}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGBX              |
| 17  | Standard Gumbel (Gu)      | $(\exp[\exp(-ax)] - 1) - 1$  | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a > 0)$ | PGE Gu            |
| 18  | Nadarajah-Haghighi (NH)   | $\exp[\{1 + ax\}^c - 1] - 1$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGENH             |
| 19  | Gompertz                  | $\exp[a (\exp(cx) - 1)] - 1$  | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGER              |
| 20  | Inverse Flexible Weibull  | $(\exp[\exp(ax - cx)] - 1) - 1$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGEIFW            |
| 21  | Inverse Gompertz (IGz)    | $\\left\{\exp\left[-\exp\left(\frac{cx}{\phi}\right)\right] - 1\right\}^{-1}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGE Igz           |
| 22  | Normal (N)                | $\frac{\phi (x/\phi)^c}{\Gamma(1 + c)}$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a > 0, c \in R)$ | PGE N             |
| 23  | Gamma (Ga)                | $\frac{1}{\Gamma(\frac{\phi}{2})} \gamma^{-1}(a, \frac{\phi}{2}) - 1$ | $\lambda \in \mathbb{R} - (0)$, $\phi > 0$, $\xi = (a, c > 0)$ | PGE Ga            |

Note that $\Gamma(.)$ refers to the gamma function and $\gamma(.,.)$ refers to the incomplete gamma function. Figure 1 gives some plots of the Poisson generalized exponential-Pareto type II (PGEPII) PDF (a) and PGEPII hazard rate function (HRF) (b) for some carefully selected
parameters value. Figure 2 presents some plots of the Poisson generalized exponential-exponential (PGEE) PDF (a) and PGEE HRF (b) for some carefully selected parameters value. Based on Figure 1a, it is noted that the PDF of the PGEPII can be “asymmetric right-skewed function” and “symmetric” with different shapes. Based on Figure 2a, it is seen that the PDF of the PGEE can be “asymmetric right-skewed function”, “asymmetric left-skewed function”, “bimodal”, and “symmetric” with different shapes. Based on Figure 1b, it is noted that the HRF of the PGEPII can be “upside down bathtub (λ = 4, θ = 2, β = 1, c = 1.5)”, “bathtub (λ = 6, θ = 1, β = 1, c = 1.75)”, “decreasing-constant (λ = θ = β = c = 1)”, “increasing-constant (λ = −1, θ = β = c = 1)”, and “increasing (λ = −1, θ = 2, β = 1, c = 1.5)”. Based on Figure 2b, it is noted that the HRF of the PGEE can be “bathtub (λ = 0.5, θ = 0.5, β = 0.75, c = 0.25)”, “decreasing-constant (λ = 5, θ = 0.25, β = c = 0.1)”, “upside down bathtub (λ = 10, θ = 1, β = c = 0.25)”, “constant (λ = 10, θ = 2, β = c = 0.05)”, and “increasing (J-shape) (λ = 0.5, θ = 0.5, β = 0.0007, c = 1)”.

![Figure 1. Plots of the PGEPII probability density function (PDF) (a) and PGEPII hazard rate function (HRF) (b).](image1)

![Figure 2. Plots of the PGEE PDF (a) and PGEE HRF (b).](image2)
The new family could be useful in modeling
1-The real datasets with “asymmetric monotonically increasing HRF”, as illustrated in Section 6.
2-The real datasets that have no extreme values, as shown in Section 6.
3-The real datasets whose nonparametric Kernel density is symmetric, as given in Section 6 (Figure 11).

The PGE-G family proved its superiority against many well-known families as shown below:

1-In modeling the failure times of the aircraft windshield, the PGE-G family is better than the special generalized mixture G family, the odd log-logistic G family, the Burr-Hatke G family, the transmuted Topp-Leone G family, the Gamma G family, the Kumaraswamy G family, the McDonald G family, the exponentiated G family, and the proportional reversed hazard rate G family under the Akaike information criteria, consistent information criteria, Bayesian information criteria, and Hannan-Quinn information criteria.

2-In modeling the service times of the aircraft windshield, the PGE-G family is better than the special generalized mixture G family, the odd log-logistic G family, the Burr-Hatke G family, the transmuted Topp-Leone G family, the Gamma G family, the Kumaraswamy G family, the McDonald G family, the exponentiated G family, and the proportional reversed hazard rate G family under the Akaike information criteria, consistent information criteria, Bayesian information criteria, and Hannan-Quinn information criteria.

2. Copula

For facilitating the mathematical modeling of the bivariate RVs, we derived some new bivariate PGE-G (Bv-PGE-G)-type systems of distributions using “Farlie-Gumbel-Morgenstern copula” (FGMCp for short) copula ([10–15]), modified FGMCp (see [16] for details), “Clayton copula (CCp)” (see [17] for details), “Renyi’s entropy copula (RECp)” [18], and Ali-Mikhail-Haq copula [19]. The multivariate PGE-G (Mv PGE-G)-type can be easily derived based on the Clayton copula. However, future works may be allocated to study these new models.

2.1. BvPGE-G Type via CCp

Let us assume that \( X_1 \sim \text{PGE-G}(\mathbf{Y}_1) \) and \( X_2 \sim \text{PGE-G}(\mathbf{Y}_2) \). The CCp depending on the continuous marginal functions \( \pi = 1 - w \) and \( \omega = 1 - \overline{w} \) can be considered as

\[
C_{\Omega}(\pi, \omega) = \left[ \max \left( \pi^{-\Omega} + \omega^{-\Omega} - 1 \right), 0 \right]^{-\frac{1}{\Omega}}, \Omega \in (-1, \infty) - \{0\}, \pi \in (0, 1) \text{ and } \omega \in (0, 1) \tag{5}
\]

Let \( \overline{w} = 1 - F_{\mathbf{Y}_1}(x_1) |_{\mathbf{Y}_2}, \overline{\omega} = 1 - F_{\mathbf{Y}_2}(x_2) |_{\mathbf{Y}_1} \) and

\[
F_{\mathbf{Y}_i}(x_i) |_{i=1,2} = C_{\lambda_i}^{-1} \left( 1 - \exp \left\{ -\lambda_i \left[ 1 - St_{\beta_i,\xi_i}(x_i) \right]^{\theta_i} \right\} \right).
\]

Then, the BvPGE-G-type distribution can be obtained from (5). A straightforward multivariate PGE-G (m-dimensional extension) via CCp can be easily derived analogously. The m-dimensional extension via CCp is a function operating in \([0, 1]^m\), and in that case, \( x_i \) is not a value in \([0, 1]\) necessarily.

2.2. BvPGE-G Type via RECp

Following [18], the RECp can be derived as \( C(w, \overline{w}) = x_2 w + x_1 \overline{w} - x_1 x_2 \), with the continuous marginal functions \( w = 1 - \overline{w} = F_{\mathbf{Y}_1}(x_1) \in (0, 1) \) and \( \overline{w} = 1 - \overline{w} = F_{\mathbf{Y}_2}(x_2) \in (0, 1) \), where the values \( x_1 \) and \( x_2 \) are in order to guarantee that \( C(w, \overline{w}) \) is of a copula.

Then, the associated CDF of the BvPGE-G will be

\[
F(x_1, x_2) = C(F_{\mathbf{Y}_1}(x_1), F_{\mathbf{Y}_2}(x_2)).
\]
where $F_X(x_i)$ is defined above. It is worth mentioning that in [18], the authors emphasize that this copula does not show a closed shape and numerical approaches become necessary.

2.3. BuPGE-G Type via FGMCp

Considering the FGMCp (see [10–15]), the joint CDF can be written as

$$C_\Omega(w, \bar{\omega}) = w\bar{\omega}\left(1 + \Omega\bar{w}\bar{\omega}\right),$$

where the continuous marginal function is $w \in (0, 1)$, $\bar{\omega} \in (0, 1)$ and $\Omega \in [-1, 1]$ where $C_\Omega(w, 0) = C_\Omega(0, \bar{\omega}) = 0$, which is “grounded minimum condition” and $C_\Delta(w, 1) = w$ and $C_\Delta(1, \bar{\omega}) = \bar{\omega}$, which is “grounded maximum condition”. The grounded minimum/maximum conditions are valid for any copula. Setting $\bar{\omega} = \bar{\omega}_1|_{\bar{\omega}_1 > 0}$ and $\bar{\omega} = \bar{\omega}_2|_{\bar{\omega}_2 > 0}$, then we have

$$F(x_1, x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right) = w\bar{\omega}\left(1 + \Omega\bar{w}\bar{\omega}\right).$$

The joint PDF can be derived from

$$c_\Omega(w, \bar{\omega}) = 1 + \Omega w^*\bar{\omega}^*, \left(w^* = 1 - 2w \text{ and } \bar{\omega}^* = 1 - 2\bar{\omega}\right)$$

or from

$$f_\Omega(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)c\left(F_{X_1}(x_1), F_{X_2}(x_2)\right),$$

where the two function $c_\Omega(w, \bar{\omega})$ and $f_\Omega(x_1, x_2)$ are densities corresponding to the joint CDFs $C_\Omega(w, \bar{\omega})$ and $F_\Omega(x_1, x_2)$.

2.4. BuPGE-G Type via Modified FGMCp

The modified formula of the modified FGMCp due to [17] can written as

$$C_\Omega(w, \bar{\omega}) = w\bar{\omega} + \Omega\text{O}(w)^*\Psi(\bar{\omega})^*,$$

with $\text{O}(w)^* = w\Omega(w)$ and $\Psi(\bar{\omega})^* = \bar{\omega}\Psi(\bar{\omega})$, where $\text{O}(w) \in (0, 1)$ and $\Psi(\bar{\omega}) \in (0, 1)$ are two continuous functions where $\text{O}(w = 0) = \text{O}(w = 1) = \Psi(\bar{\omega} = 0) = \Psi(\bar{\omega} = 1) = 0$.

Let

$$a = \inf\{\text{O}(w)^* : \partial\text{O}(w)^*/\partial w, \forall A_1(w)/\partial w\} < 0, b = \sup\{\text{O}(w)^* : \partial\text{O}(w)^*/\partial w, \forall A_1(w)/\partial w\} < 0, c = \inf\{\Psi(\bar{\omega})^* : \partial\Psi(\bar{\omega})^*/\partial \bar{\omega}, \forall A_2(\bar{\omega})/\partial \bar{\omega}\} > 0, d = \sup\{\Psi(\bar{\omega})^* : \partial\Psi(\bar{\omega})^*/\partial \bar{\omega}, \forall A_2(\bar{\omega})/\partial \bar{\omega}\} > 0.$$  

Then, for $1 \leq \min(ba, \eta \xi)$, we have

$$0 = \frac{\partial}{\partial w}\text{O}(w)^* - \frac{w}{\partial w}\text{O}(w) - \text{O}(w),$$

where

$$\Delta_1(w) = \left\{\frac{\partial}{\partial w}\text{O}(w)^* \text{ exists} \right\},$$

and

$$\Delta_2(\bar{\omega}) = \left\{\frac{\partial}{\partial \bar{\omega}}\Psi(\bar{\omega})^* \text{ exists} \right\}.$$  

The following four types can be derived and considered:

- **Type I**
Let \( \mathcal{H}_1(w) = \lambda_1 H_{\theta_1, \beta_1, \xi_1}(w) \) and \( \mathcal{H}_2(\omega) = \lambda_2 H_{\theta_2, \beta_2, \xi_2}(\omega) \). Then, the new bivariate version via modified FGMCp type I can be written as

\[
C_\Omega(w, \omega) = \Omega[O(w)^* \psi(\omega)^*] + \left( \begin{array}{c} C_{\lambda_1}^{-1} - C_{\lambda_1}^{-1} \exp[-\mathcal{H}_1(w)] \\ \times C_{\lambda_2}^{-1} - C_{\lambda_2}^{-1} \exp[-\mathcal{H}_2(\omega)] \end{array} \right),
\]

where

\[
O(w)^* = w\{1 - C_{\lambda_1}^{-1}[1 - \exp(-\mathcal{H}_1(w))])\}_{Y_1 > 0}
\]

and

\[
\psi(\omega)^* = \omega\{1 - C_{\lambda_2}^{-1}[1 - \exp(-\mathcal{H}_2(\omega))])\}_{Y_2 > 0}.
\]

- **Type II**
  Consider \( A(w; \Omega_1) \) and \( Z(\omega; \Omega_2) \) that satisfy the above conditions where \( A(w; \Omega_1)|_{(\Omega_1 > 0)} = w^{\Omega_1}(1 - w)^{1-\Omega_1} \) and \( Z(\omega; \Omega_2)|_{(\Omega_2 > 0)} = \omega^{\Omega_2}(1 - \omega)^{1-\Omega_2} \). Then, the corresponding bivariate version (modified FGMCp Type II) can be derived from

\[
C_{\Omega_0, \Omega_1, \Omega_2}(w, \omega) = w\omega[1 + \Omega_0 A(w; \Omega_1) Z(\omega; \Omega_2)].
\]

- **Type III**
  Let \( \tilde{A}(\omega) = w[\log(1 + \omega)]|_{(\omega = 1 - \omega)} \) and \( \tilde{Z}(\omega) = \omega[\log(1 + \omega)]|_{(\omega = 1 - \omega)} \). Then, the associated CDF of the BvPGE-G-FGM (modified FGMCp Type III) can be written as

\[
C_\Omega(w, \omega) = w\omega[1 + \Omega \tilde{A}(\omega) \tilde{Z}(\omega)].
\]

- **Type IV**
  Using the quantile concept, the CDF of the BvPGE-G-FGM (modified FGMCp Type IV) model can be obtained using

\[
C(w, \omega) = wF^{-1}(w) + \omega F^{-1}(\omega) - F^{-1}(w)F^{-1}(\omega)
\]

where \( F^{-1}(w) = Q(w) \) and \( F^{-1}(\omega) = Q(\omega) \).

2.5. BvPGE-G Type via Ali-Mikhail-Haq Copula

Under the “stronger Lipschitz condition” and following [19], the joint CDF of the Archimedean Ali-Mikhail-Haq copula can be written as

\[
C_\Omega(\nu, \nu) = \frac{\nu}{1 - \Omega \nu}, \Omega \in (-1,1),
\]

and the corresponding joint PDF of the Archimedean Ali-Mikhail-Haq copula can be expressed as

\[
c_\Omega(\nu, \nu) = \frac{1 - \Omega + 2\Omega \nu}{[1 - \Omega \nu]^2}, \Omega \in (-1,1).
\]

Then, for any \( \nu = 1 - F_{Y_1}(x_1) \) and \( \nu = 1 - F_{Y_2}(x_2) \), we have

\[
C_\Omega(x_1, x_2) = \frac{F_{Y_1}(x_1)F_{Y_2}(x_2)}{1 - \Omega [1 - F_{Y_1}(x_1)] [1 - F_{Y_2}(x_2)]} |_{\Omega \in (-1,1)}
\]

\[
c_\Omega(x_1, x_2) = \frac{1 - \Omega + 2\Omega }{[1 - \Omega [1 - F_{Y_1}(x_1)] [1 - F_{Y_2}(x_2)]^2} |_{\Omega \in (-1,1)}
\]
3. Properties

3.1. Expanding the Univariate PDF

In this subsection, we present a useful representation for the new PDF in (4). Based on the new representation, we can easily and directly derive the main statistical properties of the new family due to the exponentiated G (exp-G) family. Using the power series, we expand the quantity \( A_{\lambda, \theta, \kappa}(x) \). Then, the PDF in (4) can be expressed as

\[
f_Y(x) = C_\lambda^{-1} \beta \theta \sum_{\mathbf{A}, \mathbf{G}, \kappa} (-1)^{\mathbf{A} + \mathbf{G}} \frac{\lambda^{1+\mathbf{A}} G^\mathbf{G}(x) \left( 1 - St_{\beta \mathbf{G}}(x) \right)^{\mathbf{G}(\mathbf{A}+1)-1}}{B_{\beta(\mathbf{A}+1), \beta}(x)}.
\]

Considering the power series

\[
\left( 1 - \frac{z_1}{z_2} \right)^{z_2+1} = \sum_{\mathbf{A}, r=0}^{+\infty} (-1)^{\mathbf{A} + r} \frac{\Gamma(z_3 + 2)}{\mathbf{A}! r! \Gamma(z_3 - \mathbf{A} + 2 - r)} \left( \frac{z_1}{z_2} \right)^r (1 \text{ and } z_3) 0,
\]

and applying (7) to the quantity \( B_{\beta(\mathbf{A}+1), \beta}(x) \) in (6), we get

\[
f_Y(x) = C_\lambda^{-1} \beta \theta \sum_{\mathbf{A}, \mathbf{G}, \kappa} (-1)^{\mathbf{A} + \mathbf{G}} \frac{\lambda^{1+\mathbf{A}} \left( -1 \right)^{\mathbf{A} + \mathbf{G}} \Gamma(\theta(\mathbf{A} + 1)) \exp \left( - \left( \mathbf{A} + 1 \right) \beta \Delta \mathbf{G}(x) \right)}{C_{\beta(\mathbf{A}+1)}(x)}.
\]

Expanding \( C_{\beta(\mathbf{A}+1)}(x) \), we can write

\[
C_{\beta(\mathbf{A}+1)}(x) = \sum_{s=0}^{+\infty} (-1)^{s} (\mathbf{A} + 1)^{s} \frac{G^\mathbf{G}(x)^{s}}{\Gamma(s+1) G^\mathbf{G}(x)^{s}}.
\]

Inserting the above expression of \( C_{\beta(\mathbf{A}+1)}(x) \) in (8), the PGE-G density reduces to

\[
f_Y(x) = \theta \beta C_\lambda^{-1} \sum_{\mathbf{A}, \mathbf{G}, \kappa} \lambda^{1+\mathbf{A}} \left( -1 \right)^{\mathbf{A} + \mathbf{G}} \frac{\Gamma(\theta(\mathbf{A} + 1)) \left( \mathbf{A} + 1 \right)^{\kappa} G^\mathbf{G}(x) G^\mathbf{G}(x)^{\kappa}}{\mathbf{A}! \mathbf{G}! \Gamma(\theta(\mathbf{A} + 1) - \mathbf{A})}.
\]

Expanding \( \left( 1 - G^\mathbf{G}(x) \right)^{\kappa-2} \) via generalized binomial expansion, we get

\[
\left( 1 - G^\mathbf{G}(x) \right)^{\kappa-2} = \sum_{j=0}^{+\infty} \frac{\Gamma(1+\mathbf{A}^+)}{j! \Gamma(\mathbf{A} + 2)} G^\mathbf{G}(x)^{j}, \mathbf{A}^+ = \mathbf{A} + j + 1.
\]

Inserting (10) in (9), the PGE-G density can be expressed as

\[
f_Y(x) = \sum_{\mathbf{A}, \mathbf{G}, \kappa} v_{\mathbf{A}, \mathbf{G}, \kappa} \mathbf{G^\mathbf{G}}(x),
\]

which is an infinite linear combination of exp-G PDFs where \( \mathbf{G^\mathbf{G}}(x) = d \mathbf{G^\mathbf{G}}(x)/dx = \mathbf{A}^+ \pi(x) \mathbf{G^\mathbf{G}}(x)^{\mathbf{A}^+} \) is the PDF of the exp-G family with power \( \mathbf{A}^+ \) and \( v_{\mathbf{A}, \mathbf{G}, \kappa} \) is a constant where

\[
v_{\mathbf{A}, \mathbf{G}, \kappa} = \sum_{\mathbf{A}, \mathbf{G}, \kappa} \lambda^{1+\mathbf{A}} \theta \beta C_\lambda^{-1} \left( -1 \right)^{\mathbf{A} + \mathbf{G}} \frac{\Gamma(1+\mathbf{A}^+) \left( \mathbf{A} + 1 \right)^{\kappa} \Gamma(\theta(\mathbf{A} + 1)) \Gamma(1+\mathbf{A}^+) \mathbf{G^\mathbf{G}}(x) G^\mathbf{G}(x)^{\kappa}}{\mathbf{A}! \mathbf{G}! \Gamma(\theta(\mathbf{A} + 1) - \mathbf{A})}.
\]

Similarly, the CDF of the PGE-G family can also be expressed as

\[
F_Y(x) = \sum_{\mathbf{A}, \mathbf{G}, \kappa} v_{\mathbf{A}, \mathbf{G}, \kappa} \mathbf{G^\mathbf{G}}(x),
\]
where $G_{\kappa}(x)$ is the CDF of the exp-G family with power $k^*$. 

### 3.2. Convex-Concave Analysis

Convex PDFs play a very important role in many areas of mathematics. They are important especially in study of the “optimization problems” where they are distinguished by several convenient properties. In mathematical analysis, a certain PDF defined on a certain n-dimensional interval is called “convex PDF” if the line between any two points on the graph of the PDF lies above the graph between the two points.

The PDF in (4) and based on any base-line model (see Table 1) is said to be “concave PDF” if for any $X_1 \sim \text{PGE} - G(\mathbf{V}_1)$ and $X_2 \sim \text{PGE} - G(\mathbf{V}_2)$ the PDF satisfies

$$f(\Delta x_1 + \Delta x_2) \geq \Delta f_{\mathbf{V}_1}(x_1) + \Delta f_{\mathbf{V}_2}(x_2)_{|_{\Delta \leq 1}} \text{ and } \Delta = 1 - \Delta.$$ 

If the function $f(\Delta x_1 + \Delta x_2)$ is twice differentiable, then if $f''(\Delta x_1 + \Delta x_2) < 0, \forall x \in \mathbb{R}, f(\Delta x_1 + \Delta x_2)$ is “strictly convex”. If $f''(\Delta x_1 + \Delta x_2) \leq 0, \forall x \in \mathbb{R},$ then $f(\Delta x_1 + \Delta x_2)$ is “convex”.

The PDF in (4) is said to be “convex PDF” if for any $X_1 \sim \text{PGE} - G(\mathbf{V}_1)$ and $X_2 \sim \text{PGE} - G(\mathbf{V}_2)$ the PDF satisfies

$$f(\Delta x_1 + \Delta x_2) \leq \Delta f_{\mathbf{V}_1}(x_1) + \Delta f_{\mathbf{V}_2}(x_2)_{|_{\Delta \leq 1}} \text{ and } \Delta = 1 - \Delta.$$ 

If the function $f(\Delta x_1 + \Delta x_2)$ is twice differentiable, then if $f''(\Delta x_1 + \Delta x_2) > 0, \forall x \in \mathbb{R}, f(\Delta x_1 + \Delta x_2)$ is “strictly convex”. If $f''(\Delta x_1 + \Delta x_2) \geq 0, \forall x \in \mathbb{R},$ then $f(\Delta x_1 + \Delta x_2)$ is “convex”. If $f(\Delta x_1 + \Delta x_2)$ is “convex” and $c$ is a constant, then the function $c f(\Delta x_1 + \Delta x_2)$ is “convex”. If $f(\Delta x_1 + \Delta x_2)$ is “convex PDF”, then $[c f(\Delta x_1 + \Delta x_2)]$ is convex for every $c > 0$. If $f(\Delta x_1 + \Delta x_2)$ and $g(\Delta x_1 + \Delta x_2)$ are “convex PDF”, then $[f(\Delta x_1 + \Delta x_2) + g(\Delta x_1 + \Delta x_2)]$ is also “convex PDF”. If $f(\Delta x_1 + \Delta x_2)$ and $g(\Delta x_1 + \Delta x_2)$ are “convex PDF”, then $[f(\Delta x_1 + \Delta x_2), g(\Delta x_1 + \Delta x_2)]$ is also “convex PDF”. 

If the function $f^{-1}(\Delta x_1 + \Delta x_2)$ is “convex PDF”, then the function $f(\Delta x_1 + \Delta x_2)$ is “concave PDF”. If $f(\Delta x_1 + \Delta x_2)$ is “concave PDF”, then $f^{-1}(\Delta x_1 + \Delta x_2)$ is “convex PDF”.

### 3.3. Moments

Let $Y_{\kappa}$ be an RV having the exp-G family power with $k^*$ and $X$ be an RV having the PGE-G family. Then, the $r$th moment of the RV $X$ is $\mu_r = \mathbb{E}(X^r) = \sum_{k,j=0}^{n \rightarrow +\infty} v_{k,j} \mathbb{E}(Y_{\kappa^*}).$

Analogously, the $n$th moment around the arithmetic mean ($\mu'_{\kappa^*}$) of $X$ is

$$M_n = \mathbb{E}(X - \mu'_{\kappa^*})^n = \sum_{k,j=0}^{n \rightarrow +\infty} v_{k,j} \left( \frac{n}{r} \right) (-\mu'_{\kappa^*})^{n-r} \mathbb{E}(Y_{\kappa^*}).$$

### 3.4. Moment-Generating Function (MGF)

We present two formulas for obtaining the MGF. Clearly, the first formula can be derived from Equation (11) as

$$M_{X}(t) = \sum_{k,j=0}^{n \rightarrow +\infty} v_{k,j} M_{\kappa^*}(t),$$
where \( M_{\kappa^*}(t) \) is the MGF of the RV \( Y_{\kappa^*} \). However, the second formula is based on the concept of the quantile function (QF) as

\[
M_X(t) = \sum_{\kappa,j=0}^{+\infty} v_{\kappa,j} \tau(t, 1 + \kappa^*),
\]

where the integral

\[
\tau(t, \tau) = \int_{t}^{+\infty} \exp\left[t\exp(G(u)) \right] du
\]

can be numerically evaluated using the baseline QF, i.e., \( Q_G(u) = G^{-1}(u) \).

### 3.5. Incomplete Moments (IM)

The \( s \)th IM, say \( \Phi_{s,X}(t) \), of the RV \( X \) can be derived from (11) as \( \Phi_{s,X}(t) = \sum_{\kappa,j=0}^{n} v_{\kappa,j} I_{s,\kappa^*}(t) \) where \( I_{s,\kappa^*}(t) = \int_{-\infty}^{t} x^s g_{\kappa^*}(x) dx \). One of the main mathematical end economical applications of the first IM concerns “mean deviations (MD)” and “Bonferroni and Lorenz curves”, which are very useful in economics, insurance, demography, reliability, and medicine. The MD about the \( \mu'_1 \) of \( E([X - \mu'_1]) = a_1 \), and the MD about the median (M) of \( E([X - M]) = a_2 \) of the RV \( X \) are given by \( a_1 = 2\mu'_1 F(\mu'_1) - 2\Phi_{1,X}(\mu'_1) \) and \( a_2 = \mu'_1 - 2\Phi_{1,X}(M) \), respectively, where \( \mu'_1 = E(X) \) is the arithmetic mean of the RV \( X \), \( M = Q(0.5) \) is the median of the RV \( X \), and \( \Phi_{1,X}(t) \) is the first IM given by \( \Phi_{1,X}(t) \).

Now, we provide two ways to determine \( a_1 \) and \( a_2 \). First, \( \Phi_{1,X}(t) = \sum_{\kappa,j=0}^{n} v_{\kappa,j} I_{1,\kappa^*}(t) \) where \( I_{1,\kappa^*}(t) \) is the first IM of the exp-G family. Second, \( \Phi_{1,X}(t) = \sum_{\kappa,j=0}^{n} v_{\kappa,j} \omega_{\kappa^*}(t) \)

where \( \omega_{\kappa^*}(t) = \kappa^* \int_{t}^{G(t)} Q_G(u) du \) can be evaluated numerically. These results for \( \Phi_{1,X}(t) \) can be directly applied for calculating the Bonferroni and Lorenz curves defined, for a certain given probability \( P \), by \( B(P) = \Phi_{1,X}(Q(P))/\mu'_1 \) and \( L(P) = \Phi_{1,X}(Q(P))/\mu'_1 \), respectively.

### 3.6. Residual Life (RL) and Reversed Residual Life (RRL)

The \( q \)th moment of the RL of the RV \( X \) can be obtained from \( m_{q,X}(t) = E[(X - t)^q] \bigg|_{X > t} \) or from

\[
m_{q,X}(t) = \frac{1}{1 - F_{\mathcal{L}}(t)} \int_{t}^{\infty} (-t + x)^q f_{\mathcal{L}}(x) dx,
\]

which can also be written as

\[
m_{q,X}(t) = \frac{1}{1 - F_{\mathcal{L}}(t)} \sum_{\kappa,j=0}^{\infty} \sum_{h=0}^{q} v_{\kappa,j} \left( \begin{array}{c} q \\ h \end{array} \right) (-t)^{q-h} I_{q,\kappa^*}(t).
\]

For \( q = 1 \), we obtain the mean of the residual life (MRL) also called the life expectation (LE), which can be derived from \( m_{1,X}(t) = E[(X - t)] \bigg|_{X > t} \) and \( q \in N \) and represents the additional expected life for a certain system or component that is already alive at the age \( t \). On the other hand, the \( q \)th moment of the RRL is \( M_{q,X}(t) = E[(X - t)^q] \bigg|_{X \leq t} \) or

\[
M_{q,X}(t) = \frac{1}{F_{\mathcal{L}}(t)} \int_{0}^{t} (-x + t)^q f_{\mathcal{L}}(x) dx,
\]

which can also be expressed as

\[
M_{q,X}(t) = \frac{1}{F_{\mathcal{L}}(t)} \sum_{\kappa,j=0}^{\infty} \sum_{h=0}^{q} v_{\kappa,j} (-1)^h \left( \begin{array}{c} q \\ h \end{array} \right) (-t)^{q-h} I_{q,\kappa^*}(t).
\]
For $q = 1$, we obtain the mean waiting time (MWT), which is also called the mean inactivity time (MIT), which can be derived from $M_{1,X}(t) = E[(t - X)]|_{t \geq 0}$ and $q = 1$.

### 3.7. Mathematical Results and Numerical Analysis for Two Special Models

We present some mathematical results for two special models chosen from Table 1. All results listed in Table 2 were derived based on the mathematical results previously obtained in Sections 1–6. Table 2 (part I) gives mathematical results for the PGEPII model. The calculations of this Subsection involve several special functions, including the complete beta function

$$B(v_1, v_2) = \int_0^1 u^{v_1 - 1}(1 - u)^{v_2 - 1} du;$$

the incomplete beta function

$$B_q(v_1, v_2) = \int_0^q u^{v_1 - 1}(1 - u)^{v_2 - 1} du,$$

the complete gamma function

$$\Gamma(1 + v_1) = \int_0^{+\infty} t^{v_1} \exp(-t) dt = v_1! = \prod_{m=0}^{v_1-1} (v_1 - m),$$

the lower incomplete gamma function

$$\gamma(v_1, v_2)|_{(v_1 \neq 0, -1, -2, \ldots)} = \int_0^{v_2} t^{v_1-1} \exp(-t) dt = \sum_{v_3=0}^{v_2} \frac{(-1)^{v_3}}{v_3!} v_1^{v_2+v_3},$$

and the upper incomplete gamma function

$$\Gamma(v_1, v_2) = \Gamma(v_1) - \gamma(v_1, v_2).$$

| Property | Result | Support |
|----------|--------|---------|
| $E(X')$ | $\sum_{k,j=0}^{+\infty} \sum_{r,v=0}^{r} v_{k,j}k^r a^v (-1)^v \left( \frac{r}{v} \right) B(k^r, \frac{v-r}{r} + 1)$ | $c > r$ |
| $M_X(t)$ | $\sum_{k,j,r=0}^{+\infty} \sum_{v=0}^{r} v_{k,j}k^r a^v (-1)^v \left( \frac{r}{v} \right) B(k^r, \frac{v-r}{r} + 1)$ | $c > r$ |
| $\Phi_{k,X}(t)$ | $\sum_{k,j=0}^{+\infty} \sum_{v=0}^{s} v_{k,j}k^r a^v (-1)^v \left( \frac{s}{v} \right) B_1(k^r, \frac{v-s}{r} + 1)$ | $c > s$ |
| $\Phi_{1,X}(t)$ | $\sum_{k,j=0}^{+\infty} \sum_{v=0}^{1} v_{k,j}k^r a^v (-1)^v \left( \frac{1}{v} \right) B_1(k^r, \frac{v-1}{r} + 1)$ | $c > 1$ |
| $m_q,X(t)$ | $\sum_{k,j=0}^{+\infty} \sum_{v=0}^{t} v_{k,j}(m,q) k^r a^v (-1)^v \left( \frac{q}{v} \right) B_1(k^r, \frac{v-q}{r} + 1)$, \quad where \quad $v_{k,j}(m,q) = v_{k,j} \sum_{h=0}^{q} \left( \frac{q}{h} \right) (-1)^{q-h}$ | $t > 0$, \quad $q \in \mathbb{N}$, \quad $c > q$ |
### Table 2. Cont.

| Property | Result |
|----------|--------|
| $m_{1,X}(t)$ | \[ \frac{1}{t} \Gamma(t) \sum_{k,j=0}^{+\infty} \sum_{v=0}^{1} v_{k,j,v}(m,1) k^s a(-1)^v \left( \frac{1}{v} \right) B_t \left( k^s, \frac{v-1}{t} + 1 \right) \] |
| $M_{q,X}(t)$ | \[ \frac{1}{t} \sum_{k,j=0}^{+\infty} \sum_{q=1}^{q} v_{k,j,v}(M,q) k^s a(-1)^v \left( \frac{q}{v} \right) B_t \left( k^s, \frac{v-q}{t} + 1 \right) \] |
| $M_{1,X}(t)$ | \[ \frac{1}{t} \sum_{k,j=0}^{+\infty} \sum_{q=1}^{q} v_{k,j,v}(M,1) k^s a(-1)^v \left( \frac{1}{v} \right) B_t \left( k^s, \frac{v-1}{t} + 1 \right) \] |

| Property | Result |
|----------|--------|
| $E(X)$ | \[ \frac{1}{r} \Gamma(r + 1) \sum_{k,j=0}^{+\infty} v_{k,j}(h+1)^{1+k} \left( \frac{k^s - 1}{h} \right) \] |
| $M_X(t)$ | \[ \frac{1}{r} \Gamma(r + 1) \sum_{k,j=0}^{+\infty} v_{k,j}(h+1)^{1+k} \left( \frac{k^s - 1}{h} \right) \] |
| $\Phi_{1,X}(t)$ | \[ \frac{1}{r} \Gamma(r + 1) \sum_{k,j=0}^{+\infty} v_{k,j}(h+1)^{1+k} \left( \frac{k^s - 1}{h} \right) \] |
| $\Phi_{2,X}(t)$ | \[ \frac{1}{r} \Gamma(r + 1) \sum_{k,j=0}^{+\infty} v_{k,j}(h+1)^{1+k} \left( \frac{k^s - 1}{h} \right) \] |

4. Numerical Analysis for Some Measures

Table 3 below gives numerical analysis for the mean ($E(X)$), variance ($V(X)$), skewness ($S(X)$), and kurtosis ($K(X)$) for FGEP II model based on special case number 7 of Table 1 with $a = 1$. Based on results listed in Table 3, it is noted that $E(X)$ decreases as $\lambda$ increases, $S(X) \in (0.647392, \infty)$ and $K(X)$ ranging from 5.07 to $\infty$. 
Table 3. E(X), V(X), S(X), and kurtosis K(X) for PGEII model.

| λ   | θ   | β  | c   | E(X)       | V(X)       | S(X)       | K(X)       |
|-----|-----|----|-----|------------|------------|------------|------------|
| −100| 10  | 10 | 0.5 | 2.072196   | 0.2201758  | 1.479884   | 7.298747   |
| −50 | 10  | 10 | 0.5 | 1.833215   | 0.2047501  | 1.485328   | 7.352612   |
| 1   | 0.602749 | 0.0926237 | 1.947101 | 10.23900   |
| 10  | 0.320146 | 0.0886203 | 0.922245 | 6.964258   |
| 20  | 4.5 × 10⁻⁷ | 4.9 × 10⁻⁷ | 1557.789 | 2427588    |
| 50  | 3 × 10⁻¹⁸ | 3.2 × 10⁻¹⁸ | ∞        | ∞          |
| 1   | 0.00001 | 1.5 | 1.5 | 3.8 × 10⁻⁶ | 1.9 × 10⁻⁶ | 617.3573   | 518800.1   |

5. Estimation Method and Assessment

5.1. The Maximum Likelihood Estimation (MLE) Method

Let \( x_1, \ldots, x_n \) be an observed random sample (RS) from the PGE-G family with \( V = \left( \lambda, \theta, \beta, \xi^T \right)^T \). The function of the log-likelihood \( L(V) = \log \prod_{i=1}^{n} f_V(x_i) \) can be obtained and maximized directly using the \( R \) software (the “optim function”) or the program of \( 0 \)x (sub-routine of MaxBFGS) or \( MATH-CAD \) software or by solving the nonlinear equations of the likelihood derived from differentiating \( L(V) \). The score vector components \( U_{\lambda} = \frac{\partial}{\partial \lambda} \ell_V, U_{\theta} = \frac{\partial}{\partial \theta} \ell_V, U_{\beta} = \frac{\partial}{\partial \beta} \ell_V, \) and \( U_{\xi} = \frac{\partial}{\partial \xi} \ell_V \) can be easily derived from obtaining the nonlinear system \( U_{\lambda} = U_{\theta} = U_{\beta} = U_{\xi} = 0 \) and then simultaneously solving them for getting the MLE of \( V \). This system could be solved numerically for the complicated models using iterative algorithms such as the “Newton–Raphson” algorithms. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub models. Hypothesis tests of the type \( H_0 : \Omega = \Omega_0 \) versus \( H_1 : \Omega \neq \Omega_0 \), where \( \Omega \) is a vector formed with some components of \( V \) and \( \Omega_0 \) is a specified vector, can be performed using LR statistics. For example, the test of \( H_0 : \lambda = \theta = \beta = 1 \) versus \( H_1 : H_0 \) is not true and is equivalent to comparing the PGE-G and G distributions, and the LR statistic is given by \( \chi^2_{LR} = 2 \left[ \ell(V(\hat{\lambda}, \hat{\theta}, \hat{\beta}, \hat{\xi}^T)) - \ell(1, 1, 1, \hat{\xi}^T) \right] \), where \( \hat{\lambda}, \hat{\theta}, \hat{\beta} \) and \( \hat{\xi}^T \) are the MLEs under \( H \) and \( \hat{\xi}^T \) is the estimate under \( H_0 \).
5.2. Graphical Assessment

We present a graphical simulation for assessing the behavior of the finite sample of the MLEs for the PGEPII distribution. We maximized the log-likelihood function using a wide range of starting initial values. The starting initial values were taken in a fine scale. For the PGEPII model, for example, they were taken corresponding to all possible combinations of \( \lambda = 1, 2, \ldots, 100, \theta = 1, 2, \ldots, 100, \beta = 1, 2, \ldots, 100, \) and \( c = 1, 2, \ldots, 100. \) The proposed assessment is performed depending on the following assessing algorithm:

Using the QF of the PGEPII distribution, we generate 1000 samples of size \( n \) from the PGEPII and PGEE models where

\[
Q_U = \left( 1 - \frac{1}{\beta} \ln \left( 1 - \left[ 1 - \frac{1}{\lambda} \ln(1 - U_C^{\lambda}) \right]^\frac{1}{\theta} \right) \right)^{\frac{1}{\beta}} - 1
\]

Computing the standard errors (SEs) of the MLEs for the \( N = 1000 \) samples, SEs are obtained via inverting the “observed information matrix”.

Computing the corresponding biases and mean squared errors (MSEs) given for \( \mathbf{V} = (\lambda, \theta, \beta, c) \), we repeated these steps for \( n = 100, 200, \ldots, 500 \).

For PGEPII model, Figures 3–6 (left panels) show how the four biases vary with respect to \( n \). Figures 3–6 (right panels) show how the four MSEs vary with respect to \( n \). From Figures 3–6, the biases for each parameter are generally negative and increase to zero as \( n \to \infty \), and the MSEs for each parameter decrease to zero as \( n \to \infty \).

![Figure 3. Biases (left plot) and mean squared errors (MSEs) (right plot) for parameter \( \lambda \) (PGEPII model).](image-url)
Figure 3. Biases (left plot) and mean squared errors (MSEs) (right plot) for parameter $\lambda$ (PGEPII model).

Figure 4. Biases (left plot) and MSEs (right plot) for parameter $\theta$ (PGEPII model).

Figure 5. Biases (left plot) and MSEs (right plot) for parameter $\beta$ (PGEPII model).

Figure 6. Biases (left plot) and MSEs (right plot) for parameter $c$ (PGEPII model).

6. Modeling Failure and Service Times

Two real-life data applications to illustrate the importance and flexibility of the family are presented under the PII case. The fits of the PGEPII are compared with other PII models shown in Table 4.

The first dataset (aircraft windshield, $n = 84$):
The first real-life dataset represents the data on failure times of 84 aircraft windshield.

The second dataset (aircraft windshield, $n = 63$):
The second real-life dataset represents the data on service times of 63 aircraft windshield. The two real-life datasets were chosen based on matching their characteristics and the plots of the PDF in Figure 1 (the right panel). By examining Figure 1 (the right panel), it is noted that the new PDF can be asymmetric right-skewed function and symmetric with different shapes. On the other hand, by exploring the two real datasets, it is noted that densities are nearly symmetric functions. Additionally, the HRF of the new family includes the asymmetric monotonically increasing shape, and the HRF of the two real datasets are asymmetric monotonically increasing (see Figure 1 (left panel)). The two real datasets were reported by [20]. Many other symmetric and asymmetric useful real-life datasets can be found in [21–28]. Initial density shape is explored using the nonparametric “Kernel density estimation (KDE)” approach in Figure 7. The “normality” condition is checked via the “quantile–quantile (Q-Q) plot” in Figure 8. The initial shape of the empirical HRFs is discovered from the “total time in test (TTT)” plot in Figure 9. The extremes are spotted from the “box plot” in Figure 10. Based on Figure 7, it is noted that the densities are nearly symmetric functions. Based on Figure 8, we see that the “normality” nearly exists. Based on Figure 9, it is noted that the HRF is “asymmetric monotonically increasing shaped” for the two datasets. Based on Figure 10, it is showed that no extreme observations were founded. The goodness-of-fit (GOF) statistic called “Akaike information” (AICr), consistent-AIC (CAICr), Bayesian-IC (BICr), and Hannan–Quinn-IC (HQICr) were analyzed by comparing the competitive PII models.

| N.   | Model Abbreviation | Author |
|------|--------------------|--------|
| 1    | Special generalized mixture-PII SGMPII | [29]   |
| 2    | Odd log-logistic-PII OLLPII | [30]   |
| 3    | Reduced OLL-PII ROLLPII | [30]   |
6. Modeling Failure and Service Times

Two real-life data applications to illustrate the importance and flexibility of the family are presented under the PII case. The fits of the PGEPII are compared with other PII models shown in Table 4.

The first dataset (aircraft windshield, n = 84): The first real-life dataset represents the data on failure times of 84 aircraft windshield. The second dataset (aircraft windshield, n = 63): The second real-life dataset represents the data on service times of 63 aircraft windshield. The two real-life datasets were chosen based on matching their characteristics and the plots of the PDF in Figure 1 (the right panel). By examining Figure 1 (the right panel), it is noted that the new PDF can be asymmetric right-skewed function” and “symmetric” with different shapes. On the other hand, by exploring the two real datasets, it is noted that densities are nearly symmetric functions. Additionally, the HRF of the new family includes the asymmetric monotonically increasing shape, and the HRF of the two real datasets are asymmetric monotonically increasing (see Figure 1(left panel)). The two real datasets were reported by [20]. Many other symmetric and asymmetric useful real-life datasets can be found in [21–28]. Initial density shape is explored using the nonparametric “Kernel density estimation (KDE)” approach in Figure 7. The “normality” condition is checked via the “quantile–quantile (Q-Q) plot” in Figure 8. The initial shape of the empirical HRFs is discovered from the “total time in test (TTT)” plot in Figure 9. The extremes are spotted from the “box plot” in Figure 10. Based on Figure 7, it is noted that the densities are nearly symmetric functions. Based on Figure 8, we see that the “normality” nearly exists. Based on Figure 9, it is noted that the HRF is “asymmetric monotonically increasing shaped” for the two datasets. Based on Figure 10, it is showed that no extreme observations were founded. The goodness-of-fit (GOF) statistic called “Akaike information” (AICr), consistent-AIC (CAICr), Bayesian-IC (BICr), and Hannan–Quinn-IC (HQICr) were analyzed by comparing the competitive PII models.

Table 4. The competitive models.

| N. | Model                           | Abbreviation | Author |
|----|---------------------------------|--------------|--------|
| 1  | Special generalized mixture-PII | SGMPII       | [29]   |
| 2  | Odd log-logistic-PII            | OLLPII       | [30]   |
| 3  | Reduced OLL-PII                 | ROLLPII      | [30]   |
| 4  | Reduced Burr–Hatke-PII          | RBHPII       | [31]   |
| 5  | Transmuted Topp–Leone-PII       | TTLPII       | [32]   |
| 6  | Reduced TTL-PII                 | RTTLPII      | [32]   |
| 7  | Gamma-PII                       | GamPII       | [33]   |
| 8  | Kumaraswamy-PII                 | KumPII       | [34]   |
| 9  | McDonald-PII                    | McPII        | [34]   |
| 10 | Beta-PII                        | BPII         | [34]   |
| 11 | Exponentiated-PII               | EPII         | [35]   |
| 12 | PII                             | PII          | [36]   |
| 13 | Proportional reversed hazard rate PII | PRHRPII | New    |

However, many other PII extensions could be considered in comparisons [37–45]. For failure times real-life data, relevant numerical results are shown in Tables 5 and 6. Precisely, Table 5 gives the MLEs and SEs. Table 6 gives the four GOF test statistics. For service times real-life data, the results are presented in Tables 7 and 8. Precisely, Table 5 gives the MLEs and SEs, whereas Table 8 gives the four GOFs test statistics. Figures 11 and 12 give the probability-probability (P-P) plot, estimated PDF (EPDF), Kaplan-Meier survival (KMS) plot and estimated HRF (EHRF) plot for the two datasets, respectively. Based on Tables 6 and 8, it is noted that the PGEPII model gives the lowest values for all test statistics, where AICr = 264.231, CAICr = 264.737, BICr = 273.954, and HQICr = 268.139 for the first dataset, and AICr = 205.252, CAICr = 205.941, BICr = 213.824, and HQICr = 208.623 for the second
dataset among all fitted models. Hence, it could be chosen as the best model under these criteria.

Figure 7. Nonparametric Kernel density estimation (KDE) (1st and 2nd datasets).

Figure 8. Normal quantile-quantile (Q-Q) plots (1st and 2nd datasets, respectively).

Figure 9. Total time in test (TTT) plots (1st and 2nd datasets, respectively).
Figure 8. Normal quantile-quantile (Q-Q) plots (1st and 2nd datasets, respectively).

Figure 9. Total time in test (TTT) plots (1st and 2nd datasets, respectively).

Figure 10. Box plots (1st and 2nd datasets, respectively).

Table 5. Maximum Likelihood Estimation (MLEs) and standard errors (SEs) for 1st dataset.

| Model             | Estimates                  |
|-------------------|----------------------------|
| PGEPII ($\lambda, \theta, \beta, c$) | 2.82464 (7.4304) 1.03661 (0.07303) 0.002702 (0.00046) 3.69627 (0.0004) |
| KPII ($\theta, \beta, c, \alpha$)    | 2.61502 (0.3822) 100.276 (120.49) 5.27710 (9.8116) 78.6774 (186.01) |
| TTLPII ($\theta, \beta, c, \alpha$)  | -0.80751 (0.1396) 2.47663 (0.5418) 15,608 (1602.4) 38,628 (123.94) |
| BPII ($\theta, \beta, c, \alpha$)    | 3.60360 (0.6187) 33,6387 (63.715) 4.85070 (9.382) 118,837 (123.94) |
| PRHRPII ($\beta, c, \alpha$)        | $3.73 \times 10^6$ (0.1396) $4.17 \times 10^{-1}$ (0.00001) $4.51 \times 10^6$ (37.146) |
| SGMPPII ($\theta, c, \alpha$)       | $-1.04 \times 10^{-1}$ (0.1223) $9.83 \times 10^6$ (4843.3) $1.18 \times 10^7$ (501.04) |
| RTTLPII ($\theta, \beta, c$)        | $-0.84732$ (0.10011) $5.52057$ (1.1848) $1.5678$ (0.0959) |
| OLLPII ($\theta, c, \alpha$)        | $2.32636$ (2.14 $\times 10^{-1}$) $7.17 \times 10^5$ (1.19 $\times 10^4$) $2.3 \times 10^6$ (2.6 $\times 10^3$) |
| EPII ($\theta, c, \alpha$)          | 3.62610 (0.6236) 20,074.5 (2041.8) 26,257.7 (99.744) |
| GamPII ($\theta, c, \alpha$)        | 3.58760 (0.5133) 52,001.4 (7955.0) 37,029.7 (81.163) |
| ROLLPII ($\theta, c$)                | 3.89056 (0.3652) 0.57316 (0.0195) |
| RBHPII ($c, \alpha$)                 | 1,080,175 (983,309) 513,672 (23,231) |
| PII ($c, \alpha$)                    | 341.208 (5933.5) 346.070 (296.12) |
| GamPII ($\theta, c, \alpha$)        | 3.58760 (0.5133) 52,001.4 (7955.0) 37,029.7 (81.163) |
| ROLLPII ($\theta, c$)                | 3.89056 (0.3652) 0.57316 (0.0195) |
| RBHPII ($c, \alpha$)                 | 1,080,175 (983,309) 513,672 (23,231) |
| PII ($c, \alpha$)                    | 341.208 (5933.5) 346.070 (296.12) |

Table 6. Goodness-of-fit (GOF) statistics for 1st dataset.

| Model | AICr | BICr | CAICr | HQICr |
|-------|------|------|-------|-------|
| PGEPII| 264.231| 273.954| 264.737| 268.139 |
| OLLPII| 274.847| 282.139| 275.147| 277.779 |
| TTLPII| 279.140| 288.863| 279.646| 283.049 |
| GamPII| 282.808| 290.136| 283.105| 285.756 |
| BPII  | 285.435| 295.206| 289.935| 289.365 |
| EPII  | 288.799| 296.127| 289.096| 291.747 |
| ROLLPII| 289.690| 294.552| 289.839| 291.645 |
| SGMPPII| 292.175| 299.467| 292.475| 295.106 |
| RTTLPII| 313.962| 321.254| 314.262| 316.893 |
| PRHRPII| 321.754| 339.046| 322.054| 334.686 |
| PII    | 333.977| 338.862| 334.123| 335.942 |
| RBHPII| 341.208| 346.070| 341.356| 343.162 |
Figure 11. Estimated PDF (EPDF), estimated HRF (EHRF), probability–probability (P-P), and Kaplan-Meier survival (KMS) plots for the 1st dataset.

Table 7. MLEs and SEs for 2nd dataset.

| Model Estimates | Model Estimates | Model Estimates | Model Estimates |
|-----------------|-----------------|-----------------|-----------------|
| PGEPII (λ, β, c) | -4.38494 (10.4313) | 0.34355 (0.0009) | 0.10422 (0.1068) | 2.11596 (0.0017) |
| BPII (θ, β, c, a) | 1.921842 (0.3184) | 31.2594 (316.84) | 4.9684 (30.528) | 169.572 (339.21) |
| KPII (θ, β, c, a) | 1.66912 (0.2571) | 60.5673 (86.013) | 2.56490 (4.7589) | 65.0640 (177.59) |
| TTLPII (θ, β, c, a) | -0.607 (0.18746) | 1.78578 (0.4152) | 2123.39 (163.92) | 4822.79 (200.01) |
| RTTLPII (θ, β, c) | -0.67151 (0.18746) | 2.74496 (0.6696) | 1.01238 (0.1141) |  |
| PRHRPII (θ, β, c, a) | 1.59 × 10^6 (4.1 × 10^{-10}) | 3.93 × 10^{-1} (3.21 × 10^9) | 1.30 × 10^6 (3.8573) |  |
| SGMPPII (θ, c, a) | -1.04 × 10^{-1} (1.8 × 10^{-1}) | 6.45 × 10^{-6} (1.68 × 10^{10}) | 6.33 × 10^6 (3.8573) |  |
| GamPII (θ, c, a) | 1.9073232 (0.32132) | 35.842.433 (6945.074) | 39.197.37 (151.653) |  |
| OLLPII (θ, c, a) | 1.66419 (1.84 × 10^6) | 6.340 × 10^{10} (1.68 × 10^{10}) | 7.22 × 10^6 (162.22) |  |
| EPII (θ, c, a) | 1.914532 (0.34801) | 22.971.15 (3209.53) | 32.882.0 (162.22) |  |
| RBHPII (c, a) | 14.055.522 (422.01) | 53.203.423 (26.5232) | 39.197.37 (301.237) |  |
| ROLLPII (θ, c) | 2.372331 (0.26834) | 0.69109 (0.0449) | 32.882.0 (301.237) |  |
| PII (c, a) | 99.269.83 (11864.3) | 207.019.4 (301.237) | 32.882.0 (301.237) |  |
Table 8. GOF statistics for 2nd dataset.

| Model    | AICr   | BICr   | CAICr  | HQICr  |
|----------|--------|--------|--------|--------|
| PGEPII   | 205.252| 213.824| 205.941| 208.623|
| KPII     | 209.735| 218.308| 210.425| 213.107|
| TTLPII   | 212.900| 221.472| 213.589| 216.271|
| GamPII   | 211.666| 218.096| 212.073| 214.195|
| SGMPII   | 211.788| 218.218| 212.195| 214.317|
| BPII     | 213.922| 222.495| 214.612| 217.294|
| EPII     | 213.099| 219.529| 213.506| 215.628|
| OLLPII   | 215.808| 222.238| 216.215| 218.337|
| PRHRPII  | 224.597| 231.027| 225.004| 227.126|
| PII      | 222.598| 226.884| 222.798| 224.283|
| ROLLPII  | 225.457| 229.744| 225.657| 227.143|
| RTTLPII  | 230.371| 236.800| 230.778| 232.900|
| RBHPII   | 229.201| 233.487| 229.401| 230.887|

Further, the results of the LR statistics of the PGEPII model against the quasi-Poisson generalized exponential Pareto type II (QPGEPII), Poisson exponential Pareto type II (PEPII), and quasi-Poisson Pareto type II (QPPII) models under the first dataset are in Table 9. Based on the results of this table,

I-We reject the null hypotheses of the LR tests in favor of the PGEPII model.
II-We can confirm the significance of the parameters $\lambda$ and $\theta$ with $W_{LR} = 17.09761$, $W_{LR} = 14.27654$, and $W_{LR} = 9.00651$, respectively.

Table 9. The likelihood ratio (LR) statistics for the 1st dataset.

| Model           | Hypothesis                   | $W_{LR}$ | p-Value |
|-----------------|------------------------------|----------|---------|
| PGEPII vs. QPGEPII | $H_0$: $\lambda = 1$, $H_1$: $H_0$ false | 17.09761 | 0.0015  |
| PGEPII vs. PEPII   | $H_0$: $\theta = 1$, $H_1$: $H_0$ false | 14.27654 | 0.0122  |
| PGEPII vs. QPPII   | $H_0$: $\lambda = \theta = 1$, $H_1$: $H_0$ false | 9.00651  | 0.0953  |

The results of the LR statistics of the PGEPII model against the QPGEPII, PEPII, and QPPII models under the second dataset are in Table 10. Based on the results of this table,

I-We reject the null hypotheses of the LR tests in favor of the PGEPII model.
II-We can confirm the significance of the parameters $\lambda$ and $\theta$ with $W_{LR} = 33.01982$, $W_{LR} = 4.710811$, and $W_{LR} = 3.476109$, respectively.

Table 10. The LR statistics for the 2nd data.

| Model           | Hypothesis                   | $W_{LR}$ | p-Value |
|-----------------|------------------------------|----------|---------|
| PGEPII vs. QPGEPII | $H_0$: $\lambda = 1$, $H_1$: $H_0$ false | 33.01982 | 0.0011  |
| PGEPII vs. PEPII   | $H_0$: $\theta = 1$, $H_1$: $H_0$ false | 4.710811 | 0.0033  |
| PGEPII vs. QPPII   | $H_0$: $\lambda = \theta = 1$, $H_1$: $H_0$ false | 3.476109 | 0.07782 |
Table 10. The LR statistics for the 2nd data.

| Model          | Hypothesis                  | 𝓗₀         | 𝓗₁         | 𝓗₀ false | 𝓗₀ false p-value |
|----------------|-----------------------------|------------|------------|----------|-----------------|
| PGEPII vs. QPGEPII | 𝐻₀: 𝜆 = 1, 𝐻₁: 𝐻₀ false    | 33.01982   | 0.0011     |          |                 |
| PGEPII vs. PEPII    | 𝐻₀: 𝜃 = 1, 𝐻₁: 𝐻₀ false    | 4.710811   | 0.0033     |          |                 |
| PGEPII vs. QPPII    | 𝐻₀: 𝜆 = 𝜃 = 1, 𝐻₁: 𝐻₀ false | 3.476109   | 0.07782    |          |                 |

Figure 12. EPDF, EHRF, P-P, and KMS plots for the 2nd dataset.

7. Conclusions

In this article, a new parametric lifetime compound G family of continuous probability distributions called the Poisson generalized exponential G (PGEG) family is derived and studied. The PGEG family is defined based on the Poisson and the generalized exponential G families’ concept of compounding. The new density can be “asymmetric right-skewed function”, “asymmetric left-skewed”, “bimodal”, and “symmetric” with different shapes. The new HRF can be “upside down bathtub”, “bathtub”, “decreasing-constant”, “increasing-constant”, “increasing”, “constant”, and “increasing”. Relevant mathematical properties including moments, incomplete moments, and mean deviation are derived. Some new bivariate-type PGEG families using the “copula of Farlie-Gumbel-Morgenstern”, “copula of the modified Farlie-Gumbel-Morgenstern”, “the Clayton copula”, and “copula Renyi’s entropy” are presented. Many special members are derived, and special attention is devoted to the exponential (E) and the one parameter Pareto type II (PII) model. A simulation study is presented to assess the finite sample behavior of the estimators. The simulations are based on a certain given algorithm under the baseline PII model. Finally, two different real-life applications are proposed to illustrate the importance of the PGEG family. For all real data, for exploring the “initial shape”, the nonparametric Kernel density estimation is presented. For checking the “normality” condition, the “Quantile–Quantile plot” is presented. For discovering the shape of the HRFs, the “total time in test” plot is provided. To explore the extremes, the “box plot” is sketched. Based on PII base-line model, the PEWPII model gives the lowest values for all test statistics, where AICr = 
264.231, CAICr = 264.737, BICr = 273.954, and HQICr = 268.139 for the failure times data; AICr = 205.252, CAICr = 205.941, BICr = 213.824, and HQICr = 208.623 for the service times data.

Author Contributions: These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received financial support from Taif University Researchers Supporting Project Number (TURSP-2020/155), Taif University, Taif, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The two real datasets were reported by [20].

Acknowledgments: This Research was supported by Taif University Researchers Supporting Project Number (TURSP-2020/155), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Maurya, S.K.; Nadarajah, S. Poisson Generated Family of Distributions: A Review. Sankhya 2020, 1–57. [CrossRef]
2. Bourguignon, M.; Silva, R.B.; Cordeiro, G.M. The Weibull-G family of probability distributions. J. Data Sci. 2014, 12, 53–68.
3. Ramos, M.W.A.; Marinho, P.R.D.; Cordeiro, G.M.; da Silva, R.V.; Hamedani, G.G. The Kumaraswamy-G Poisson family of distributions. J. Stat. Theory Appl. 2015, 14, 222–239.
4. Aryal, G.R.; Yousof, H.M. The exponentiated generalized-G Poisson family of distributions. Econ. Qual. Control 2017, 32, 1–17. [CrossRef]
5. Abouelmagd, T.H.M.; Hamed, M.S.; Handique, L.; Goual, H.; Ali, M.M.; Yousof, H.M.; Korkmaz, M.C. A new class of distributions based on the zero truncated Poisson distribution with properties and applications. J. Nonlinear Sci. Appl. 2019, 12, 152–164. [CrossRef]
6. Alizadeh, M.; Yousof, H.M.; Rasekhi, M.; Altun, E. The odd log-logistic Poisson-G Family of distributions. J. Math. Ext. 2019, 12, 81–104.
7. Korkmaz, M.C.; Yousof, H.M.; Hamedani, G.G.; Ali, M.M. The Marshall–Olkin generalized G Poisson family of distributions. Pak. J. Stat. 2018, 34, 251–267.
8. Yousof, H.M.; Afify, A.Z.; Alizadeh, M.; Hamedani, G.G.; Jahanshahi, S.M.A.; Ghosh, I. The generalized transmuted Poisson-G family of Distributions. Pak. J. Stat. Oper. Res. 2018, 14, 759–779. [CrossRef]
9. Yousof, H.M.; Mansoor, M.; Alizadeh, M.; Afify, A.Z.; Ghosh, I. The Weibull-G Poisson family for analyzing lifetime data. Pak. J. Stat. Oper. Res. 2020, 16, 131–148. [CrossRef]
10. Farlie, D.J.G. The performance of some correlation coefficients for a general bivariate distribution. Biometrika 1960, 47, 307–323. [CrossRef]
11. Morgenstern, D. Einfache beispiele zweidimensionaler verteilungen. Mitteilungsbl. Math. Stat. 1956, 8, 234–235.
12. Gumbel, E.J. Bivariate exponential distributions. J. Am. Stat. Assoc. 1960, 55, 698–707. [CrossRef]
13. Gumbel, E.J. Bivariate logistic distributions. J. Am. Stat. Assoc. 1961, 56, 335–349. [CrossRef]
14. Johnson, N.L.; Kotz, S. On some generalized Farlie-Gumbel-Morgenstern distributions. Commun. Stat. Theory Appl. 1975, 4, 415–427. [CrossRef]
15. Johnson, N.L.; Kotz, S. On some generalized Farlie-Gumbel-Morgenstern distributions-II: Regression, correlation and further generalizations. Commun. Stat. Theory Appl. 1977, 6, 485–496. [CrossRef]
16. Balakrishnan, N.; Lai, C.D. Continuous Bivariate Distributions; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2009.
17. Nelsen, R.B. An Introduction to Copulas; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2007.
18. Pougaza, D.B.; Djafari, M.A. Maximum entropies copulas. In Proceedings of the 30th international workshop on Bayesian inference and maximum Entropy methods in Science and Engineering, Chamonix, France, 4–9 July 2010; pp. 329–336.
19. Ali, M.M.; Mikhail, N.N.; Haq, M.S. A class of bivariate distributions including the bivariate logistic. J. Multivar. Anal. 1978, 8, 405–412. [CrossRef]
20. Murthy, D.N.P.; Xie, M.; Jiang, R. Weibull Models; John Wiley & Sons: Hoboken, NJ, USA, 2004.
21. Yousof, H.M.; Afify, A.Z.; Abd El Hadi, N.E.; Hamedani, G.G.; Butt, N.S. On six-parameter Fréchet distribution: Properties and applications. Pak. J. Stat. Oper. Res. 2016, 281–299. [CrossRef]
22. Yousof, H.M.; Afify, A.Z.; Nadarajah, S.; Hamedani, G.; Aryal, G.R. The Marshall-Olkin generalized-G family of distributions with Applications. Statistica 2018, 78, 273–295.
23. Aryal, G.R.; Ortega, E.M.; Hamedani, G.G.; Yousof, H.M. The Topp-Leone generated Weibull distribution: Regression model, characterizations and applications. Int. J. Stat. Probab. 2017, 6, 126–141. [CrossRef]
24. Ibrahim, M. The compound Poisson Rayleigh Burr XII distribution: Properties and applications. *J. Appl. Probab. Stat.* 2020, 15, 73–97.

25. Altun, E.; Yousof, H.M.; Hamedani, G.G. A new log-location regression model with influence diagnostics and residual analysis. *Facta Unive. Ser.* 2018, 33, 417–449.

26. Altun, E.; Yousof, H.M.; Chakraborty, S.; Handique, L. Zografos-Balakrishnan. Burr XII distribution: Regression modeling and applications. *Int. J. Math. Stat.* 2018, 19, 46–70.

27. Elgohari, H.; Yousof, H.M. New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling. *Stat. Optim. Inf. Comput.* 2020, 8, 972–993. [CrossRef]

28. Ibrahim, M.; Altun, E.; Yousof, H.M. A new distribution for modeling lifetime data with different methods of estimation and censored regression modeling. *Stat. Optim. Inf. Comput.* 2020, 8, 610–630. [CrossRef]

29. Chesneau, C.; Yousof, H.M. On a special generalized mixture class of probabilistic models. *J. Nonlinear Model. Anal.* 2021, forthcoming.

30. Elgohari, H.; Yousof, H.M. A Generalization of Lomax Distribution with Properties, Copula and Real Data Applications. *Pak. J. Stat. Oper. Res.* 2020, 16, 972–993. [CrossRef]

31. Yousof, H.M.; Altun, E.; Ramires, T.G.; Alizadeh, M.; Rasekhi, M. A new family of distributions with properties, regression models and applications. *J. Stat. Manag. Syst.* 2018, 21, 163–188. [CrossRef]

32. Yousof, H.M.; Alizadeh, M.; Jahanshahi, S.M.A.; Ramires, T.G.; Ghosh, I.; Hamedani, G.G. The transmuted Topp-Leone G family of distributions: Theory, characterizations and applications. *J. Data Sci.* 2017, 15, 723–740.

33. El-Morshedy, M.; Eliwa, M.S. The odd flexible Weibull-H family of distributions: Properties and estimation with applications to complete and upper record data. *Filomat* 2019, 33, 2635–2652. [CrossRef]

34. Eliwa, M.S.; El-Morshedy, M.; Ali, S. Exponentiated odd Chen-G family of distributions: Statistical properties, Bayesian and non-Bayesian estimation with applications. *J. Stat. Appl. Stat.* 2020, 1–27. [CrossRef]

35. Tahir, M.H.; Hussain, M.A.; Cordeiro, G.M.; El-Morshedy, M.; Eliwa, M.S. A New Kumaraswamy Generalized Family of Distributions with Properties, Applications, and Bivariate Extension. *Mathematics* 2020, 8, 1989. [CrossRef]