On asymptotic optimality of Merton’s myopic portfolio strategies for discrete time market

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Abstract

The paper studies the properties of discrete time stochastic optimal control problems associated with the portfolio selection problem and related continuous time portfolio selection problems. We found that Merton’s strategy that is optimal for continuous time model can be used effectively for the discrete market model that has sufficiently small time steps and approximate the continuous time model. After natural discretization, the Merton’s strategy approximates the performance of the optimal strategy in discrete time model.

Keywords: optimal portfolio, utility, discretization, discrete time Itô formula

MSC: 93E20, 91G10.

1 Introduction

The paper studies discrete time stochastic optimal control problems and their relationships with continuous time optimal control problems. More precisely, we study optimal investment problems where $EU(X_T)$ is to be maximized. Here $X_T$ represents the total wealth at final time $T$, and $U(\cdot)$ is a utility function. We consider the case where $U(x) = x^\alpha$, $\alpha \in (0, 1)$. For continuous time market models, these utilities have a special significance, in particular, because the optimal strategies for them are known explicitly (the so-called Merton’s strategies). These strategies are myopic; they depend only on the current observations of the market parameters, including the risk free rate, the appreciation rate, and the volatility matrix, even for the case of unknown prior distributions and evolution law. The optimality of Merton’s strategies still holds when the market parameters are random and independent of the driving Brownian motion, i.e., in the case of ”totally unhedgeable” coefficients, according to Karatzas and Shreve [19], Chapter

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6. The solution that leads to myopic strategies goes back to Merton [23]; the case of random coefficients was discussed in Karatzas and Shreve [19] and Dokuchaev and Haussmann [8].

The real stock prices are presented as time series, so the discrete time (multi-period) models are more natural than continuous time models. On the other hand, continuous-time models give a good description of distributions and often allows explicit solutions of optimal investment problems.

For the real market, a formula for an optimal strategy derived for a continuous-time model can often be effectively used after the natural discretization. However, this strategy will not be optimal for time series observed in the real market. Therefore, it is important to extend the class of discrete time models that allow myopic and explicit optimal portfolio strategies. The problem of discrete-time portfolio selection has been studied in the literature, such as in Smith [29], Leland [21], Mossin [24], Merton [23], Samuelson [28], Fama [11], Hakansson [15], Hakansson [16], Elton and Gruber [10], Francis [12], Dumas and Liucinao [9], Östermark [25], Grauer and Hakansson [14], Pliska [26], Li and Ng [22], Xu et al [31], Çanakoğlu and Özekici [3], Zhang and Li [32].

If a discrete time market model is complete, then the martingale method can be used (see, e.g., Pliska [26]). Unfortunately, a discrete time market model can be complete only under very restrictive assumptions. For incomplete discrete time markets, the main tool is dynamic programming that requires solution of Bellman equation starting at terminal time. For the general case, this procedure requires to calculate the conditional densities at any step (see, e.g., Pliska [26] or Gikhman and Skorohod [13]). This is why the optimal investment problems for discrete time can be more difficult than for continuous time setting that often allows explicit solutions.

There are several special cases when investment problem allows explicit solution for discrete time, and, for some cases, optimal strategies are myopic (see Leland [21], Mossin [24], Hakansson [15]). However, the optimal strategy is not myopic and it cannot be presented explicitly for power utilities in general case. Hakansson [15] showed that the optimal strategy is not myopic for $U(x) = \sqrt{x}$ if returns have serial correlation and evolve as a Markov process.

In mean-variance discrete time multi-period setting, the optimal strategies represent some analog of Merton’s strategies. These strategies are myopic for mean-variance goal achieving problems and non-myopic if these goals have to be selected to solve a problem with constraints; see Li and Ng [22], Dokuchaev [4], Zhang and Li [32]. It appears that the problems with utility functions $U(x) = x^\alpha, \alpha < 1$, have different properties with respect to time discretization.
In particular, Dokuchaev [5] demonstrated that the direct discretization of continuous time optimal Merton’s strategies does not approximate the optimal strategy for the discrete time market for concave utility functions $U(x) = x^\alpha$ such that $U(x) = -\infty$ for $x < 0$. More precisely, the difference between the optimal expected utilities for discrete time and continuous time models does not disappear if the number of periods (or frequency of adjustments) grows. As the result, the optimal expected utility calculated for continuous time market cannot be approximated by piecewise constant strategies with possible jumps at given times $\{t_k\}_{k=1}^N$, even if $N \to +\infty$ and $t_k - t_{k+1} \to 0$.

In the present paper, we reconsider the discrete time optimal portfolio selection problems. We suggest a solution based on myopic Merton’s strategies that are optimal for related continuous time portfolio selection problems. We investigated the limit properties of the discrete time optimal portfolio selection problem when the step of the discretization converges to zero. We found that the performance of the discrete time strategy obtained directly from Merton’s strategy approximates of the optimal strategy. This suboptimal discrete time strategy is myopic. We consider the case of Gaussian noise in the discrete time equation for the price. This means that the stock price can be negative with non-zero probability; this feature does not affect the validity of the model since this probability converges to zero as the step of discretization converges to zero; see Appleby et al [2]. The proof is based on the application of the variant of the discrete Itô formula first introduced by Appleby et al [1]. It can be noted that the proof does not use the dynamic programming principle.

These results lead to a conclusion that the Merton’s strategies can be used effectively for the discrete time multi-period market models with power utilities $U(x) = x^\alpha$, $\alpha < 1$ that has sufficiently small time steps and approximate the continuous time model. This seems to contradict to the result from Dokuchaev [5]. However, there is not a contradiction. In the present paper, we assumed that $U(x) = Lx$ for $x < 0$, where $L > 0$ can be selected to be arbitrarily large. On the other hand, Dokuchaev [5] assumed that $U(x) = -\infty$ for $x < 0$. This difference in the problem setting leaded to different conclusions. Note that the utility function considered in the present paper is not concave; however, its shape is becoming “more concave” as $L \to +\infty$. Moreover, the impact of non-concavity of $U$ for any given $L$ disappears since this probability converges to zero as the step of discretization converges to zero.
2 Problem setting

In this paper we consider the following controlled stochastic difference equation

\[ x_{n+1} = x_n \left( 1 + hu_n a_n + \sqrt{hu_n b_n} \xi_{n+1} \right), \quad n = 0, 1, \ldots, N - 1, \]
\[ x(0) = x_0 > 0, \]  

(2.1)

where \( x_0 > 0 \) is nonrandom, \( \xi_n \) are random variables, \( a_n, b_n \) are nonrandom coefficients, \( u_n \) is a nonrandom control (strategy), \( n = 0, 1, \ldots, N \), \( N \in \mathbb{N}, h > 0 \) is a small parameter, calculated by formula

\[ h := \frac{T}{N}. \]  

(2.2)

The value \( T > 0 \) is fixed throughout all paper, but \( N \) can increase, which makes \( h \) decrease.

We can either consider equation (2.1) independently, or think about it as the Euler-Maruyama discretization of the following Itô stochastic equation

\[ dX_t = X_t u_t (a_t dt + b_t dW_t), \quad t \in [0, T], \quad x(0) = x_0, \]

(2.3)

where \( W \) is a standard Wiener process, \( b, a, u : [0, T] \to \mathbb{R} \) are continuous nonrandom functions. In this setting \( h \) is a step size of discretization of the interval \( [0, T] \) and \( N \) is a number of corresponding mesh points.

We recall that the Euler-Maruyama numerical method for equation (2.3) computes approximations \( X_n(h) \approx x(nh) \) by

\[ x_{n+1}(h) = x_n(h) \left( 1 - hu_n a(nh) + u(nh)b(nh)\Delta W_{n+1} \right), \]

(2.4)

where \( h > 0 \) is the constant step size and \( \Delta W_{n+1} = W((n + 1)h) - W(nh) \). We see that when

\[ \xi_{n+1} = \frac{W((n + 1)h) - W(nh)}{\sqrt{h}}, \quad a_n = a(nh), \quad b_n = b(nh), \quad u_n = u(nh), \]

(2.4) coincides with (2.1) and \( \xi_{n+1} \) is a standardized normal random variable. More information about Euler-Maruyama discretization and stochastic difference equations could be found, e.g., in Higham et al [17], Kloeden and Platen [20], Appleby et al [21].

Fix \( T > 0 \) and \( \alpha \in (0, 1) \). Assume that the following assumptions hold.

**Assumption 2.1.** Functions \( b, a : [0, T] \to \mathbb{R} \) are nonrandom and there exist \( \hat{a}, \hat{b}, \underline{a}, \underline{b} > 0 \) such that

\[ \underline{a} \leq |a_t| \leq \hat{a}, \quad \underline{b} \leq |b_t| \leq \hat{b}, \quad \forall t \in [0, T]. \]  

(2.5)

**Assumption 2.2.** The strategy is a nonrandom sequence \( (u_n)_{n=1,\ldots,N} \) such that

\[ \underline{u} \leq |u_n| \leq \hat{u}, \quad \text{where} \quad \underline{u} := \frac{\underline{a}}{2(1 - \alpha)\underline{b}^2}, \quad \hat{u} := \frac{2\hat{a}}{(1 - \alpha)\hat{b}^2}. \]  

(2.6)
Assumption 2.3. \((\xi_n)_{n\in\mathbb{N}}\) is a sequence of independent and \(\mathcal{N}(0,1)\) distributed random variables.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})\) be a complete filtered probability space. We assume that the filtration \(\{\mathcal{F}_n\}_{n\in\mathbb{N}}\) is naturally generated: \(\mathcal{F}_{n+1} = \sigma\{\xi_i + 1 : i = 0, 1, \ldots, n\}\), where the sequence \((\xi_n)_{n\in\mathbb{N}}\) satisfies Assumption 2.3.

We use the standard abbreviation “a.s.” for the wordings “almost sure” or “almost surely” throughout the text.

Among all the sequences \(\{x_n\}_{n\in\mathbb{N}}\) of the random variables we distinguish those for which \(X_n\) are \(\mathcal{F}_n\)-measurable for all \(n\in\mathbb{N}\). A detailed exposition of the definitions and facts of the theory of random processes can be found in, for example, Shiryaev [30].

Define for some \(\alpha \in (0,1)\) and \(L > 0\),

\[
U(x) = x^\alpha, \quad x \geq 0, \quad U(x) = Lx, \quad x < 0. \tag{2.7}
\]

Up to the end of the paper, we consider the following optimal control problem:

Maximize \(\mathbb{E}[U(x_N)]\) over \(u\), \(\tag{2.8}\)

where \(x\) is a solution to (2.1) with \(h = \frac{T}{N} \leq \frac{T}{N_0}\) and \(u = (u_i)_{i=1,\ldots,N}\), \(N \geq N_0\), the set of admissible strategies satisfying Assumption 2.2.

3 Optimal portfolio selection and the main result

Problem (2.8) has applications for optimal portfolio selection. It appears that (2.1) describes the dynamic of the total wealth \(x_n\) of an investor at time period \(n\) for a single stock discrete time market model with a risk-free investment with zero return. The dynamic of the stock price is described by the equation

\[
s_{n+1} = s_{n} \left(1 + ha_{n} + \sqrt{hb_{n}\xi_{n+1}}\right), \quad n = 0, 1, \ldots, N - 1, \quad s_0 = 1. \tag{3.1}
\]

It is assumed that the portfolio is distributed among the shares of this stock and the risk-free investment with zero return. A strategy \(\{u_n\}\) represents a dynamically selected ratio of investment in stock. More precisely, let \(\gamma_n\) be the quantity of stock shares in the portfolio at time \(n\), then \(u_n = \gamma_n s_n / x_n\), where \(\gamma_n s_n\) is the current value of the stock portfolio, \(x_n\) is the current total value of the portfolio. We select the strategy \(\{u_n\}\) in the class of admissible processes described above and calculate the quantity of shares \(\gamma_n = u_n x_n / s_n\); effectively, we select closed-loop strategies. It is assumed that the strategy is self-financing, i.e.,

\[
x_{n+1} - x_n = \gamma_n(s_{n+1} - s_n), \quad n = 0, 1, 2, \ldots
\]
where $\gamma_n = u_n x_n / s_n$ is the quantity of stock shares in the portfolio at time $n$.

In fact, the case of non-zero return for the risk free asset is also covered by this model, if one interprets $x_n$ as the discounted wealth and $s_n$ as the discounted stock price (discounted with respect to the risk-free asset). More detailed market model description can be found, e.g., in Pliska [26], Dokuchaev [3].

For this discrete time market model, a standard problem of optimal portfolio selection is to maximize the expectation of the utility function $U(x_N)$ of the terminal wealth $x_N$, i.e., to find a strategy $u^* = (u^*_i)_{i=1}^{N_0}, N \geq N_0$, satisfying Assumption 2.2 which solves optimal control problem

$$\text{Maximize } E[ U(x_N) ] \text{ over } u,$$  \hspace{1cm} (3.2)

where $U$ is some given concave utility function, $x$ is a solution to (2.1) with $h = T / N \leq T / N_0$.

Further, Itô equation (2.3) describes the evolution of the total wealth $X_t$ for a single stock continuous market model with zero risk-free interest rate where the stock price evolution is described by the Itô equation

$$dS_t = S_t (a_t dt + b_t dW_t), \quad t \in [0, T],$$

$$S(0) = 1.$$  \hspace{1cm} (3.3)

For this continuous time market model, a standard optimal portfolio selection problem is to maximize the expectation of the utility function $U(X_T)$ of the terminal wealth $X_T$, i.e., to find a strategy $u : [0, T] \times \Omega \to \mathbb{R}$ in a certain class of admissible strategies that solves optimal control problem

$$\text{Maximize } E[ U(X_T) ] \text{ over } u_t,$$  \hspace{1cm} (3.4)

where $X_t$ is a solution to (2.3). For the case when $U(x) = x^\alpha$, $\alpha \in (0, 1)$, the following so-called Merton’s strategy

$$u^*_t = \frac{a_t}{(1 - \alpha)b_t^2}, \quad t \in [0, T],$$  \hspace{1cm} (3.5)

is optimal in the continuous time setting where in the class of admissible strategies that include all bounded random processes adapted to the filtration generated by $S_t$; see, e.g., Karatzas and Shreve [19], Chapter 6, and Merton [23]. In fact, this strategy is optimal in an even wider class of random and adapted processes $u_t$, as well as in the setting with random $a_t$ and $b_t$ that are independent from $W_t$.

It can be seen that problem (2.8) is in fact a modification of problem (3.2). Note that the “utility function” $U(x)$ in (2.8) is not concave in $x \in \mathbb{R}$; however, its shape is becoming “more concave” as $L \to +\infty$.

Consider strategy $u^* = (u^*_i)_{i=1}^{N_0}$

$$u^*_n = \frac{a_n}{(1 - \alpha)b_n^2}, \quad n = 1, 2, \ldots, N.$$  \hspace{1cm} (3.6)
Assumption 2.2 is satisfied for this strategy. Notice that this strategy represents a direct discretization of Merton’s strategy (3.5). It can be also noted that strategy (3.6) does not depend on the choice of $L$.

Our main result can be formulated as the following.

**Theorem 3.1.** The strategy $u^*$ defined by (3.6) maximizes $EU(x_N)$ approximately for small enough $h = \frac{T}{N}$, meaning that

$$\sup_u EU(x_N) = EU(x^*_N) + O(h) \quad \text{as} \quad h \to 0,$$

where $x^*_N$ is the terminal wealth for strategy (3.6) and $O(h) \to 0$ as $h \to 0$, independently on $N$.

We show that the error of this approximation tends to zero as step size of discretization $h \to 0$ (which is equivalent that number of mesh points $N \to \infty$). The proof is heavily dependent on the application of the variant of the discrete Itô formula first introduced in Appleby et al [1], as well as on the fact that solution $x^*_n$ of (2.1) for strategy (3.6) is positive for all $n = 1, \ldots, N$ with probability which tends to zero when $h \to 0$ (or $N \to \infty$); see Appleby et al [2].

In Dokuchaev [3], it was shown that the direct discretization of continuous time optimal Merton’s strategies does not approximate the optimal strategy for the discrete time market if the utility function $U(x) = x^\alpha$ is extended as $U(x) = -\infty$ for $x < 0$. We found that this can be overcome using the functions $U$ with non-concavity that can be made arbitrarily small with selection of a large $L > 0$. Moreover, we show that the probability that this non-concavity will ever have any impact vanishes as $h \to 0$, since the probability that the wealth ever achieves zero vanishes as $h \to 0$.

Let us review the main steps of the proofs.

Let

$$\phi(x) = |x|^\alpha, \quad x \in \mathbb{R}. \quad (3.7)$$

First, we observe that the solution $x_n$ of (2.1) can be represented as

$$x_n = x_0 \prod_{i=0}^{n-1} (1 + hu_i a_i + \sqrt{hu_i b_i} \xi_{i+1}), \quad x_0 > 0, \quad n = 1, \ldots, N. \quad (3.8)$$

Hence

$$E\phi(x_n) = \phi(x_0) \prod_{i=0}^{n-1} E\phi(1 + hu_i a_i + \sqrt{hu_i b_i} \xi_{i+1}), \quad n = 1, \ldots, N. $$

Application of the discrete Itô formula to each $E\phi(1 + hu_i a_i + \sqrt{hu_i b_i} \xi_{i+1})$ gives that

$$\sup_u E\phi(x_N) = x_0^\alpha \prod_{n=0}^{N-1} \left[ 1 + \alpha h \frac{a_n^2}{2(1 - \alpha)b_n^2} \right] + O(h) \quad \text{as} \quad h \to 0.$$
Then we show that the probability
\[ P\{\omega : U(x_N(\omega)) \neq \phi(x_N(\omega))\} \]
can be made however small when \( N = T/h \) is big enough. Finally we prove that
\[
\sup_u E[U(x_N)] = x_0 \alpha \prod_{n=0}^{N-1} \left[ 1 + \alpha h \frac{a_n^2}{2(1 - \alpha)b_n^2} \right] + O(h) \quad \text{as} \quad h \to 0.
\]

The remaining part of the paper devoted to the proof of Theorem 3.1 accordingly to the outline given above.

**Appendix: proofs**

### A.1 Discrete Itô formula.

Discrete Itô formula which we use in this paper is similar to the formula first introduced by Appleby, Berkolaiko & Rodkina [1]. The main purpose of this formula is to mimic the classical Itô formula for continues processes when we deal with the discrete process described by the equation with small parameter \( h \), similar (2.1). Theorem A.1 below deals with the case which is slightly different than the one considered in Appleby et al. [1]. Theorem A.1 can also be obtain as a partial case of Lemma 5.1 from Rodkina and Dokuchaev [27], where the Itô formula was proved for the diagonal system of stochastic difference equations. However it is much easier to give a sketch of the proof here than to adapt Lemma 5.1 for (2.1).

**Theorem A.1.** Let Assumptions 2.1, 2.2, and 2.3 hold. Let for function \( \phi : \mathbb{R} \to \mathbb{R} \) there exist a number \( \delta \in (0, 1) \) and a function \( \phi_\delta : \mathbb{R} \to \mathbb{R} \) such that

(i) \( \phi_\delta \) has a bounded third derivative on \( \mathbb{R} \),

(ii) \( \phi_\delta(s) = \phi(s) \) for \( s \notin (-\delta, \delta) \),

(iii) \( |\phi_\delta(s) - \phi(s)| < K \) for some \( K > 0 \) and all \( s \in (-\delta, \delta) \).

Let \( x_n \) be a solution to equation (2.1) with \( h \) defined by (2.2).

Then there exists \( h_0 \) such that, for all \( h \leq h_0 \) and \( n = 0, 1, \ldots, N \),

\[
E \left( \phi(1 + hu_na_n + \sqrt{hu_nb_n\xi_{n+1}}) | \mathcal{F}_n \right) = \phi(1) + h\phi'(1)u_na_n + h^2 \frac{\phi''(1)}{2} u_n^2 b_n^2 + o(h), \quad (A.1)
\]

where

\[
|o(h)| \leq h^{3/2}K u_n^2 b_n^2,
\]

and \( K > 0 \) does not depend on \( N \).
Proof. Fix \( n = 1, \ldots, N \) and define

\[
\zeta_{n+1} := 1 + h u_n a_n + \sqrt{h u_n b_n} \xi_{n+1}, \quad u_{n+1} = h u_n a_n + \sqrt{h u_n b_n} \xi_{n+1},
\]
and, for all \( v \in \mathbb{R} \),

\[
\eta(v) := 1 + h u_n a_n + \sqrt{h u_n b_n} v.
\]

We expand \( \phi_\delta(\zeta_{n+1}) \) by Taylor formula and apply mathematical expectation

\[
E \phi_\delta(\zeta_{n+1}) = \phi_\delta(1) + \phi'_\delta(1) E u_{n+1} + \frac{\phi''_\delta(1)}{2} E u_{n+1}^2 + \frac{\phi'''_\delta(\theta)}{6} E u_{n+1}^3,
\]
where \( \theta \) is situated between 1 and \( 1 + h u_n a_n + \sqrt{h u_n b_n} \xi_{n+1} \). Applying (2.5) we arrive at an estimate

\[
\left| \frac{\phi'''_\delta(\theta)}{6} E u_{n+1}^3 \right| \leq K_1 E \left| h u_n a_n + \sqrt{h u_n b_n} \xi_{n+1} \right|^3 \leq K_2 u_n^3 h^3/2 |a_n^3 + 3a_n b_n^2| \leq K_3 u_n^2 b_n^2 h^{3/2},
\]
where \( K_i, i = 1, 2, 3 \), does not depend on \( n \). Note also that

\[
\phi_\delta(1) = \phi(1), \quad \phi'_\delta(1) = \phi'(1), \quad \phi''_\delta(1) = \phi''(1).
\]

So the only thing which needs to be done is to estimate

\[
\Delta_2 := E \left| \phi(\zeta_{n+1}) - \phi_\delta(\zeta_{n+1}) \right| = \frac{1}{\sqrt{2\pi}} \int_{|\eta(v)| \leq \delta} |\phi(\eta(v)) - \phi_\delta(\eta(v))| e^{-v^2/2} dv.
\]

Change the variables by

\[
s = 1 + h u_n a_n + \sqrt{h u_n b_n} v, \quad v = \frac{s - 1 - h u_n a_n}{\sqrt{h u_n b_n}}, \quad dv = \frac{ds}{\sqrt{h u_n b_n}}.
\]

Assume that \( \delta \) and \( h_0 > 0 \) are small enough and \(|s| \leq \delta, h \leq h_0 \). Then, for \( u_n b_n > 0 \) we have

\[
v = \frac{s - 1 - h u_n a_n}{\sqrt{h u_n b_n}} \leq \delta - 1 - h u_n a_n \leq -\frac{1}{2\sqrt{h u_n b_n}} \leq \frac{1}{2\sqrt{h u b}},
\]

while \( u_n b_n < 0 \) we have

\[
v = \frac{1 - \delta - h |u_n a_n|}{\sqrt{h}|u_n b_n|} \geq \frac{1}{\sqrt{2h}|u_n b_n|} \geq \frac{1}{2\sqrt{h u b}},
\]
so

\[
|v| \geq \frac{1}{2\sqrt{h u b}},
\]

which implies that

\[
e^{-v^2/2} \leq K_4 v^{-4} \leq K_5 h^2 |\hat{u}\hat{b}|^4 \leq K_6 h^2 u_n^2 b_n^2.
\]

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Note that $h_0 > 0$ chosen here does not depend on $n$, but only on bounds for $a, b$, i.e. $\hat{a}, \hat{b}, a, b > 0$ (see (2.6)).

This gives us

$$\Delta_2 = \frac{1}{\sqrt{2\pi}} \int_{|\eta(v)| \leq \delta} |\phi(\eta(v)) - \phi_0(\eta(v))| e^{-v^2/2} dv$$

$$\leq \frac{K_4 h^2 u_n^2 b_n^2}{\sqrt{2\pi}} \int_{|\eta(v)| \leq \delta} |\phi(\eta(v)) - \phi_0(\eta(v))| dv$$

$$= \frac{K_4 h^2 u_n^2 b_n^2}{\sqrt{2\pi} \sqrt{hu_n b_n}} \int_{|s| \leq \delta} |\phi(s) - \phi_0(s)| ds \leq K_5 h^{3/2} u_n^2 b_n^2,$$

where $K_i, i = 4, 5$, does not depend on $n$. This completes the proof.

\[\square\]

**Corollary 1.1.** For $\phi$, defined by (3.7), formula (A.1) takes the form

$$E \left( \phi(1 + hu_n a_n + \sqrt{hu_n b_n} \xi_{n+1}) | F_n \right)$$

$$= 1 + h\alpha u_n a_n + h^2 \frac{\alpha(\alpha - 1)}{2} u_n^2 b_n^2 + o(h),$$

where

$$|o(h)| \leq h^{3/2} K u_n^2 b_n^2,$$

and $K > 0$ does not depend on $n = 1, 2, \ldots, N$. Hence (A.2) can be written as

$$E \left( \phi(1 + hu_n a_n + \sqrt{hu_n b_n} \xi_{n+1}) | F_n \right)$$

$$= 1 + h\alpha u_n a_n + h^2 \frac{\alpha(\alpha - 1)}{2} u_n^2 b_n^2 [1 + h^{1/2} O_n(1)],$$

where $|O_n(1)| \leq K$ for all $n = 1, 2, \ldots, N$ and $h = \frac{T}{N} \leq h_0 = \frac{T}{N_0}$.

**A.2 Positivity of $x_n$ with probability close to 1**

In this section we follow ideas from Appleby et al [2], showing that even though a.s. positivity is impossible to achieve for solution of (2.1), positivity with arbitrarily high probability is observed as the number of mesh points $N$ grows large. Again, we are giving the sketch of the proof instead of adopting result from Appleby et al [2].

Let $x_n$ be a solution to (2.1) with positive initial value $x_0 > 0$ and $h = \frac{T}{N}$. Define

$$\Omega_N := \mathbb{P}\{\omega \in \Omega : x_n(\omega) > 0, \quad n = 1, \ldots, N\}. \quad (A.4)$$

**Lemma A.1.** Let Assumptions 2.1, 2.2, and 2.3 hold. Let $x_n$ be a solution to (2.1) with parameter $h = \frac{T}{N}$ and $x_0 > 0$. Let $\Omega_N$ be defined as in (A.4). Then, for each $\gamma \in (0, 1)$, we can find $N(\gamma)$ such that for all $N \geq N(\gamma)$

$$\mathbb{P}[\Omega_N] \geq 1 - \gamma.$$
Proof. Note that \( x_n \) is \( \mathcal{F}_n \)-measurable and is independent of \( \xi_{n+1} \). Let \( u_n b_n > 0 \). Then we have, for \( n = 0, 1, \ldots, N-1 \),

\[
\mathbb{P}\{ x_{n+1} > 0 | x_n > 0 \} = \mathbb{P}\{ x_n \left( 1 + h u_n a_n + \sqrt{h u_n b_n} \xi_{n+1} \right) > 0 | x_n > 0 \} \\
= \mathbb{P}\{ 1 + h u_n a_n + \sqrt{h u_n b_n} \xi_{n+1} > 0 | x_n > 0 \} = \mathbb{P}\{ \xi_{n+1} > -\frac{1 + h u_n a_n}{\sqrt{h u_n b_n}} | x_n > 0 \}
\]

(A.5)

where \( \Phi \) is a normal probability distribution function.

If \( u_n b_n < 0 \) we consider \( \bar{\xi}_{n+1} = -\xi_{n+1} \) and note that \( \bar{\xi}_{n+1} \) is also standard normal variable. So calculations (A.5) holds true in this case again.

Applying (A.5), the Mill’s estimate (see Karatzas and Shreve [18])

\[
\frac{x}{(1 + x^2) \sqrt{2\pi}} e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{x \sqrt{2\pi}} e^{-x^2/2}, \quad x > 0,
\]

and the inequality

\[
\frac{1 + h u_n a_n}{\sqrt{h u_n b_n}} \geq \frac{1}{2 \sqrt{h u b}},
\]

we conclude that for some \( h_1 > 0 \) and all \( h < h_1 \), we have

\[
\mathbb{P}\{ x_{n+1} > 0 | x_n > 0 \} \geq \Phi \left( \frac{1 + h u_n a_n}{\sqrt{h u_n b_n}} \right) \geq 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{1 + h u_n a_n}{\sqrt{h u_n b_n}} \right)^2} \geq 1 - K_1 \left( \frac{2 \sqrt{h b}}{1 + h u_n a_n} \right)^{-3} = 1 - K_1 \left( 2 \sqrt{h b} \right)^4 = 1 - K_2 h^2,
\]

where \( K_1, K_2 > 0 \) do not depend on \( n \). Then,

\[
\mathbb{P}[\Omega_N] := \prod_{n=0,1,\ldots,N-1} \mathbb{P}\{ x_{n+1} > 0 | x_n > 0 \} \geq \prod_{n=0,1,\ldots,N-1} \left( 1 - K_2 h^2 \right) = (1 - K_2 h^2)^N = \left( 1 - \frac{K_2 T^2}{N^2} \right)^N.
\]

Fix now \( \gamma \in (0,1) \) and find \( N(\gamma) \) such that for all \( N \geq N(\gamma) \)

\[
1 - \left( 1 - \frac{K_2 T^2}{N^2} \right)^N < \gamma.
\]

This implies that for all \( N \geq N(\gamma) \)

\[
\mathbb{P}[\Omega_N] \geq 1 - \gamma,
\]

which completes the proof. \( \square \)
A.3 Estimation of maximum $E\phi(x_N)$.

Calculation of strategy $\bar{u}$ to maximise $E\phi(x_N)$.

Let $x_n$ be a solution to (2.1) corresponding to a strategy $u := (u_1, \ldots, u_N) \in \mathbb{R}^N$ and let $h_0 = \frac{T}{N_0}$ and $O_n(1)$ be from formula (A.3), $|O_n(1)| < K$. Moreover, we assume that $h_0 = \frac{T}{N_0}$ is so small that, for $h = \frac{T}{N} \leq h_0$, $n = 1, \ldots, N$,

$$1 - h^{1/2}K > 0, \quad 1 + h\alpha a_n u_n + h\frac{\alpha(\alpha - 1)}{2} u_n^2 b_n^2 [1 - h^{1/2}] > 0.$$  \hfill (A.6)

Applying formula (A.3) we define, for $h = \frac{T}{N} \leq h_0 = \frac{T}{N_0}$ and $N \geq N_0$,

$$G(u) := E\phi(x_N) = \phi(x_0) \prod_{n=0}^{N-1} \mathbb{E} \left( \phi(1 + hu_n a_n + \sqrt{hu_n b_n} \xi_{n+1}) \bigg\vert \mathcal{F}_n \right)$$

$$= x_0^\alpha \prod_{n=0}^{N-1} \left[ 1 + \alpha h u_n a_n + h\frac{\alpha(\alpha - 1)}{2} u_n^2 b_n^2 [1 + h^{1/2}O_n(1)] \right],$$

$$\bar{u}_k = \frac{a_k}{(1 - \alpha)b_n^2 [1 + h^{1/2}O_k(1)]}, \quad k = 1, \ldots, N, \quad (A.8)$$

$$G(u) := x_0^\alpha \prod_{n=0}^{N-1} \left[ 1 + \alpha h u_n a_n + h\frac{\alpha(\alpha - 1)}{2} u_n^2 b_n^2 \right]. \quad (A.9)$$

Lemma A.2. Let Assumptions 2.1, 2.2, and 2.3 hold. Let $x_n$ be a solution to (2.1) with parameter $h$ and a strategy $u$. Then there exists $h_0 > 0$ such that for all $h \leq h_0$

(i) strategy $\bar{u}$, defined by (A.8), maximizes $G(u) = E\phi(x_N)$;

(ii) strategy $u^*$, defined by (3.6), maximizes $G(u)$.

Proof. Let $h_0$ be such that the Itô formula (A.3) as well as (A.6) hold.

To find maximum of $G$ we will calculate its partial derivatives. We have

$$\frac{\partial G}{\partial u_k} = x_0^2 \alpha h \left( a_k + (\alpha - 1) u_k b_k^2 [1 + h^{1/2}O_n(1)] \right)$$

$$\times \prod_{n=0, n \neq k}^{N-1} \left[ 1 + h\alpha a_n u_n + h\frac{\alpha(\alpha - 1)}{2} u_n^2 b_n^2 [1 + h^{1/2}O_n(1)] \right].$$

Then, by (A.6), solving the system

$$\frac{\partial G}{\partial u_k} = 0, \quad k = 1, \ldots, N,$$

is equivalent to solving the system

$$a_k + (\alpha - 1) u_k b_k^2 [1 + h^{1/2}O_k(1)] = 0, \quad k = 1, \ldots, N. \quad (A.10)$$
Solution to (A.10) is given by (A.8).

Let us prove now that \( \bar{u} := (\bar{u}_1, \ldots, \bar{u}_N) \), where \( \bar{u}_k \) are defined by (A.8), is a point of maximum for function \( G \). In order to do it, we find second partial derivatives of \( G \) at \( \bar{u} \). We have, for \( k = 1, \ldots, N \),

\[
\frac{\partial^2 G}{\partial u_k^2} = x_0^2 \alpha h (\alpha - 1)b_k^2[1 + h^{1/2}O_k(1)]
\]

\[
\times \prod_{n=0, n \neq k}^{N-1} \left[ 1 + \alpha ha_nu_n + h\frac{\alpha(\alpha - 1)}{2}u_n^2b_n^2[1 + h^{1/2}O_k(1)] \right],
\]

and

\[
\frac{\partial^2 G}{\partial u_k \partial u_j} =
\]

\[
x_0^2 \alpha^2 h^2 \left( a_k + (\alpha - 1)u_kb_k^2[1 + h^{1/2}O_k(1)] \right) \left( a_j + (\alpha - 1)u_jb_j^2[1 + h^{1/2}O_j(1)] \right)
\]

\[
\times \prod_{n=0, n \neq k,j}^{N-1} \left[ 1 + \alpha ha_nu_n + h\frac{\alpha(\alpha - 1)}{2}u_n^2b_n^2[1 + h^{1/2}O_k(1)] \right].
\]

Let \( y = (y_1, \ldots, y_N) \). Consider the following quadratic form

\[
Q(y) = \sum_{k,j=0}^{N-1} \frac{\partial^2 G}{\partial u_k \partial u_j} \bigg|_{u=\bar{u}} y_ky_j. \tag{A.11}
\]

Since

\[
\left( a_k + (\alpha - 1)\bar{u}_k b_k^2[1 + h^{1/2}O_k(1)] \right) \left( a_j + (\alpha - 1)\bar{u}_j b_j^2[1 + h^{1/2}O_j(1)] \right) = 0,
\]

(A.11) takes the form

\[
Q(y) = \sum_{k=0}^{N-1} \frac{\partial^2 G}{\partial u_k^2} \bigg|_{u=\bar{u}} y_k^2
\]

\[
= x_0^2 \alpha h (\alpha - 1) \sum_k b_k^2[1 + h^{1/2}O_k(1)] \prod_{n=0, n \neq k}^{N-1} \left[ 1 + \frac{\alpha a_n^2}{2(1 - \alpha)b_n^2[1 + h^{1/2}O(1)]} \right] y_k^2.
\]

Since \( \alpha - 1 < 0 \), but

\[
\alpha h > 0, \quad b_k^2[1 + h^{1/2}O_k(1)] > 0, \quad 1 + \frac{\alpha a_n^2}{2(1 - \alpha)b_n^2[1 + h^{1/2}O(1)]} > 0,
\]

the quadratic form \( Q(y) \) is negatively defined. So \( \bar{u} \) given by (A.8) is a point of maximum for \( G \).

By calculations similar to above we can prove that \( u^* \) defined by (3.6) is a point of maximum for \( G \) and

\[
\sup_u G = G(u^*) = x_0^2 \prod_{n=0}^{N-1} \left[ 1 + \alpha h a_n^2 \frac{2(1 - \alpha)}{b_n^2[1 + h^{1/2}O(1)]} \right]. \tag{A.12}
\]

\(\square\)
Estimation of the difference $G(\bar{u}) - G(u^*)$.

Substituting the value $u^*$ from (3.6) into (A.9) we arrive at

$$G(u^*) = x_0^N \prod_{n=0}^{N-1} \left[ 1 + \alpha h \frac{a_n^2}{2(1-\alpha)b_n^2} \right], \quad (A.13)$$

and substituting the value $\bar{u}$ from (A.8) into (A.7) we arrive at

$$G(\bar{u}) = x_0^N \prod_{n=0}^{N-1} \frac{1 + \alpha h}{2(1-\alpha)b_n^2} \left[ 1 + \frac{h^2}{2} O_n(1) \right], \quad (A.14)$$

where $O_n(1)$ is as in (A.3).

**Lemma A.3.** Let Assumptions 2.1, 2.2, and 2.3 hold. Let $G(u^*)$ and $G(\bar{u})$ are given as in (A.13) and (A.14), respectively. Then, for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $N > N(\varepsilon)$, $h \leq h_0 = \frac{T}{N(\varepsilon)}$, we have

$$|G(\bar{u}) - G(u^*)| \leq \varepsilon. \quad (A.15)$$

**Proof.** Choosing $h_0$ as in Corollary 1.1 and applying the estimate $|O_n(1)| \leq K$, we have, for all $h < h_0$,

$$\alpha \frac{a_n^2}{2(1-\alpha)b_n^2} \left[ 1 + \frac{h^2}{2} \right] \geq \alpha \frac{a_n^2}{2(1-\alpha)b_n^2} \left[ 1 - \frac{h^2}{2} \right], \quad (A.16)$$

Denoting

$$\tilde{\zeta}_{n,1} := \frac{\alpha K a_n^2}{2(1-h^{1/2}K)(1-h^{1/2}K)}, \quad \tilde{\zeta}_{n,2} := \frac{\alpha K a_n^2}{2(1-h^{1/2}K)(1-h^{1/2}K)} \times \frac{1}{2(1-h^{1/2}K)}, \quad \hat{A}_n := \frac{\alpha a_n^2}{2(1-\alpha)b_n^2}, \quad (A.17)$$

we can obtain from (A.14) and (A.16) that

$$x_0^N \prod_{n=0}^{N-1} \left[ 1 + h\hat{A}_n - h^{3/2} \tilde{\zeta}_{n,1} \right] \leq G(\bar{u}) \leq x_0^N \prod_{n=0}^{N-1} \left[ 1 + h\hat{A}_n + h^{3/2} \tilde{\zeta}_{n,2} \right]. \quad (A.18)$$

We note that

$$\hat{A}_n = \frac{\alpha a_n^2}{2(1-\alpha)b_n^2} \leq \frac{\alpha \hat{a}^2}{2(1-\alpha)b^2}. \quad (A.19)$$

Assuming $h_0$ so small that there exists $\tilde{K}_1, \tilde{K}_2 > 0$, which do not depend on $n$, such that

$$\tilde{\zeta}_{n,1} \in (0, \tilde{K}_1), \quad \tilde{\zeta}_{n,2} \in (0, \tilde{K}_2),$$

we have

$$|G(\bar{u}) - G(u^*)| \leq \varepsilon. \quad (A.20)$$
and
\[ \frac{\tilde{\zeta}_{n,1}}{1 + h\tilde{A}_n - h^{3/2}\tilde{\zeta}_{n,1}} \leq 2\tilde{K}_1, \quad \frac{\tilde{\zeta}_{n,2}}{1 + h\tilde{A}_n} \leq 2\tilde{K}_2, \]
we estimate
\[ \frac{1 + h\tilde{A}_n}{1 + h\tilde{A}_n - h^{3/2}\tilde{\zeta}_{n,1}} = 1 + \frac{h^{3/2}\tilde{\zeta}_{n,1}}{1 + h\tilde{A}_n - h^{3/2}\tilde{\zeta}_{n,1}} \leq \exp \left\{ \frac{h^{3/2}\tilde{\zeta}_{n,1}}{1 + h\tilde{A}_n - h^{3/2}\tilde{\zeta}_{n,1}} \right\} \leq e^{h^{3/2}\tilde{K}_1}. \]
So
\[ \frac{1 + h\tilde{A}_n - h^{3/2}\tilde{\zeta}_{n,1}}{1 + h\tilde{A}_n} \geq e^{-h^{3/2}\tilde{K}_1}. \]
Also,
\[ \frac{1 + h\tilde{A}_n + h^{3/2}\tilde{\zeta}_{n,2}}{1 + h\tilde{A}_n} \leq \exp \left\{ h^{3/2} \frac{\tilde{\zeta}_{n,2}}{1 + h\tilde{A}_n} \right\} \leq \exp \left\{ h^{3/2}2\tilde{K}_2 \right\}. \]
Denoting
\[ \tilde{K}_1 = 2\tilde{K}_1 T, \quad \tilde{K}_2 = 2\tilde{K}_2 T, \]
and recalling that
\[ \mathcal{G}(u^*) = x_0^\alpha \prod_{n=0}^{N-1} \left[ 1 + h\tilde{A}_n \right], \]
we obtain from (A.18) that
\[ \mathcal{G}(u^*) e^{-h^{1/2}\tilde{K}_1} \leq \mathcal{G}(\bar{u}) \leq \mathcal{G}(u^*) e^{h^{1/2}\tilde{K}_2}. \] (A.19)

Then,
\[ \mathcal{G}(u^*) \left[ e^{-h^{1/2}\tilde{K}_1} - 1 \right] \leq \mathcal{G}(\bar{u}) - \mathcal{G}(u^*) \leq \mathcal{G}(u^*) \left[ e^{h^{1/2}\tilde{K}_2} - 1 \right], \]
which implies that
\[ |\mathcal{G}(\bar{u}) - \mathcal{G}(u^*)| \leq \mathcal{G}(u^*) \max \left\{ e^{h^{1/2}\tilde{K}_2} - 1, 1 - e^{-h^{1/2}\tilde{K}_1} \right\}. \]

Also,
\[ |\mathcal{G}(u^*)| = x_0^\alpha \prod_{n=0}^{N-1} \left[ 1 + \frac{\alpha h}{2(1 - \alpha)b_n^2} \right] \leq x_0^\alpha e^{\sum_{n=0}^{N-1} \alpha h \frac{a_n^2}{2(1 - \alpha)b_n^2}} \leq x_0^\alpha e^{\frac{\alpha N a_2^2}{2(1 - \alpha)b_2^2}} = C_1, \]
where
\[ C_1 := x_0^\alpha e^{\frac{\alpha T a_2^2}{2(1 - \alpha)b_2^2}} > 0 \]
does not depend on \( n \). Then,
\[ |\mathcal{G}(\bar{u}) - \mathcal{G}(u^*)| \leq C_1 \max \left\{ e^{h^{1/2}\tilde{K}_2} - 1, 1 - e^{-h^{1/2}\tilde{K}_1} \right\}. \]

Now, fix \( \varepsilon > 0 \) and find \( N = N(\varepsilon) \) such that
\[ \max \left\{ e^{h^{1/2}\tilde{K}_2} - 1, 1 - e^{-h^{1/2}\tilde{K}_1} \right\} < \frac{\varepsilon}{C_1} \quad \text{such that} \quad h < \frac{T}{N(\varepsilon)}. \]
Then, for \( N > N(\varepsilon) \), inequality (A.15) holds.
A.4 Estimation of \( \text{max} \, \mathbf{E}U(x_N) \).

Estimation of \( \mathbf{E}|x_N|^2 \)

From (3.8) we obtain, for \( n = 1, 2, \ldots, N \):

\[
\mathbf{E}|x_n|^2 = |x_0|^2 \prod_{i=1}^{n-1} \left[ 1 + hu_i a_i + \sqrt{h} u_i b_i \xi_{i+1} \right]^2
\]

\[
\leq x_0^2 \prod_{i=1}^{n-1} \left[ 1 + h (2u_i a_i + hu_i^2 a_i^2 + u_i^2 b_i^2) + 2\sqrt{h} (1 + hu_i a_i) u_i b_i \xi_{i+1} + hu_i^2 b_i^2 (\xi^2 - 1) \right]
\]

\[
= x_0^2 \prod_{i=1}^{n-1} \left[ 1 + h (2u_i a_i + hu_i^2 a_i^2 + u_i^2 b_i^2) + 2\sqrt{h} (1 + hu_i a_i) \right]
\]

\[
\leq x_0^2 \prod_{i=1}^{n-1} [1 + hK_3] \leq |x_0|^2 [1 + hK_3]^n,
\]

so

\[
\mathbf{E}|x_n|^2 \leq x_0^2 [1 + hK_3] = |x_0|^2 e^{NhK_3} = x_0^2 e^{K_3T}.
\]

(A.20)

Estimation of \( \text{max} \, \mathbf{E}U(x_N) \).

Lemma A.4. Let Assumptions 2.1, 2.2 and 2.3 hold. Let \( x_n \) be a solution to (2.1) corresponding to some strategy \( u \) and with parameter \( h = \frac{T}{N} \). Let \( \mathcal{G}(u^*) \) be defined as in (A.13) and \( U \) be defined as in (2.7). Then, for each \( \varepsilon > 0 \) there exists \( N(\varepsilon) \in \mathbb{N} \) such that for all \( N > N(\varepsilon) \), \( h \leq h_0 = \frac{T}{N(\varepsilon)} \), we have

\[
|\sup_u \mathbf{E}U(x_N) - \mathcal{G}(u^*)| \leq \varepsilon.
\]

(A.21)

Proof. Fix \( \gamma \in (0, 1) \) and find \( N(\gamma) \) by Lemma A.1 Then, by definition of function \( U \), see (2.7), for \( \Omega_N \), defined by (A.4) with \( N \geq N(\gamma) \), we have

\[
U(x_N(\omega)) = \phi(x_N(\omega)) = |x_N(\omega)|^\alpha, \quad \omega \in \Omega_N,
\]

so

\[
\mathbf{P}\{\omega : U(x_N(\omega)) \neq \phi(x_N(\omega))\} \leq \mathbb{P}[\Omega \setminus \Omega_N] \leq \gamma.
\]

We have that

\[
\mathbf{E} |\phi(x_N) - U(x_N)| = \int_{\Omega} |\phi(x_N(\omega)) - U(x_N(\omega))| \, dP(\omega)
\]

\[
= \int_{\Omega \setminus \Omega_N} |x_N(\omega)|^\alpha dP(\omega) \leq \left( \int_{\Omega \setminus \Omega_N} |x_N(\omega)|^2 dP(\omega) \right)^{\frac{\alpha}{2}} \times \left( \int_{\Omega \setminus \Omega_N} dP(\omega) \right)^{\frac{2-\alpha}{2}}
\]

\[
\leq \left( \int_{\Omega} |x_N(\omega)|^2 dP(\omega) \right)^{\frac{\alpha}{2}} \times \left( \mathbb{P}[\Omega \setminus \Omega_N] \right)^{\frac{2-\alpha}{2}} \leq \left( \mathbf{E}|x_N|^2 \right)^{\frac{\alpha}{2}} \gamma^{\frac{2-\alpha}{2}}.
\]

(A.22)
Due to (A.20) we have
\[(E|x_N|^2)^{\frac{1}{2}} \leq K_4,\]
where \(K_4 > 0\) does not depend on \(n\). Let \(G(u)\) be defined by (A.7). Then
\[|EU(x_N) - G(u)| = |EU(x_N) - E\phi(x_N)| \leq E|\phi(x_N) - U(x_N)| \leq K_4^{\frac{2-\alpha}{2}}.\]

By section A.3, Lemma A.2, the maximum of \(E\phi(x_N)\) is reached when strategy \(u = \bar{u}\), where \(\bar{u}\) is calculated by formula (A.8). So,
\[
sup_u E\phi(x_N) = E\phi(x_N(\bar{u})) = G(\bar{u}).\]

Lemma A.3 gives the estimation of the difference between value \(G(\bar{u})\) and \(G(u^*)\), where strategy \(u^*\) is given by the explicit formula (3.6).

Now, fix \(\varepsilon > 0\), and choose
\[
\gamma < \left(\frac{\varepsilon}{2K_4}\right)^{\frac{2}{\alpha}}. \tag{A.23}
\]
Using Lemma A.1 find \(N(\gamma)\). Then, find \(N(\varepsilon/2) \geq N(\gamma)\) such that,
\[
|G(\bar{u}) - G(u^*)| \leq \varepsilon/2, \quad h \leq \frac{T}{N(\varepsilon/2)}, \quad N \geq N(\varepsilon/2). \tag{A.24}
\]
Since
\[EU(x_N) = E\phi(x_N) + O(\gamma^{\frac{2-\alpha}{2}}),\]
where \(|O(s)| \leq K_4|s|\), we have
\[
sup_u EU(x_N) = sup_u E\phi(x_N) + O(\gamma^{\frac{2-\alpha}{2}}).\]
Then, by (A.23) and (A.24) we have, for \(h \leq \frac{T}{N(\varepsilon/2)}, \quad N \geq N(\varepsilon/2),\)
\[
|sup_u EU(x_N) - G(u^*)| \leq |sup_u EU(x_N) - sup_u E\phi(x_N)| + |sup_u E\phi(x_N) - G(u^*)|
= |sup_u EU(x_N) - sup_u E\phi(x_N)| + |G(\bar{u}) - G(u^*)| \leq K_4^{\gamma^{\frac{2-\alpha}{2}}} + \frac{\varepsilon}{2} \leq \varepsilon. \tag{A.25}\]

We now in the position to complete the proof of Theorem 3.1. For small enough \(h = \frac{T}{N}\), the strategy \(u^*\) defined by (3.6) maximize \(EU(x_N)\) approximately, meaning that
\[
\sup_u EU(x_N) = x_0^2 \prod_{n=0}^{N-1} \left[1 + \alpha a_n \frac{a_n^2}{2(1-\alpha)b_n^3}\right] + \rho(N)
\]
where \(\rho(N) \to 0\) as \(N \to \infty\). Then the proof of Theorem 3.1 follows.
Conclusions

We investigated the discrete time optimal portfolio selection problems. We suggest a solution based on known solutions of related continuous time portfolio selection problems such that the so-called myopic Merton’s strategies are optimal. For this, we investigated the limit properties of the discrete time optimal portfolio selection problem when the step of the discretization converges to zero. We found that the performance of the discrete time strategy obtained directly from Merton’s strategy approximates the optimal strategy. This suboptimal discrete time strategy is myopic. The proof is based the application of the discrete Itô formula. The results of this paper lead to a conclusion that the Merton’s strategies can be used effectively for the discrete time multi-period market models.

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