DEGENERATE" 3-DIMENSIONAL SKLYANIN ALGEBRAS ARE MONOMIAL ALGEBRAS

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Abstract. The 3-dimensional Sklyanin algebras, $S_{a,b,c}$, form a flat family parametrized by points $(a, b, c) \in \mathbb{P}^2 - \mathcal{D}$ where $\mathcal{D}$ is a set of 12 points. When $(a, b, c) \in \mathcal{D}$, the algebras having the same defining relations as the 3-dimensional Sklyanin algebras are called "degenerate Sklyanin algebras". C. Walton showed they do not have the same properties as the non-degenerate ones. Here we prove that a degenerate Sklyanin algebra is isomorphic to the free algebra on $u, v,$ and $w$, modulo either the relations $u^2 = v^2 = w^2 = 0$ or the relations $uw = vw = wa = 0$. These monomial algebras are Zhang twists of each other. Therefore all degenerate Sklyanin algebras have the same category of graded modules. A number of properties of the degenerate Sklyanin algebras follow from this observation. We exhibit a quiver $Q$ and an ultramatricial algebra $R$ such that if $S$ is a degenerate Sklyanin algebra then the categories $Q\text{Gr} S$, $Q\text{Gr} k Q$, and $\text{Mod} R$, are equivalent. Here $Q\text{Gr}(\cdot)$ denotes the category of graded right modules modulo the full subcategory of graded modules that are the sum of their finite dimensional submodules. The group of cube roots of unity, $\mu_3$, acts as automorphisms of the free algebra on two variables, $F$, in such a way that $Q\text{Gr} S$ is equivalent to $Q\text{Gr}(F \rtimes \mu_3)$.

1. Introduction

Let $k$ be a field having a primitive cube root of unity $\omega$.

1.1. Let $\mathcal{D}$ be the subset of the projective plane $\mathbb{P}^2$ consisting of the 12 points:

$$\mathcal{D} := \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \} \cup \{ (a, b, c) \mid a^3 = b^3 = c^3 \}.$$  

The points $(a, b, c) \in \mathbb{P}^2 - \mathcal{D}$ parametrize the 3-dimensional Sklyanin algebras,

$$S_{a,b,c} = \frac{k(x,y,z)}{(f_1, f_2, f_3)},$$

where

$$f_1 = ayz + bzy + cx^2$$
$$f_2 = axz + bxz + cy^2$$
$$f_3 = axy + byx + cz^2.$$  

In the late 1980s, Artin, Tate, and Van den Bergh, [1] and [2], showed that $S_{a,b,c}$ behaves much like the commutative polynomial ring on 3 indeterminates. In many ways it is more interesting because its fine structure is governed by an elliptic curve endowed with a translation automorphism.
1.2. When \((a, b, c) \in \mathcal{D}\) we continue to write \(S_{a,b,c}\) for the algebra with the same generators and relations and call it a degenerate 3-dimensional Sklyanin algebra. This paper concerns their structure.

Walton [8] has shown that the degenerate Sklyanin algebras are nothing like the others: if \((a, b, c) \in \mathcal{D}\), then \(S_{a,b,c}\) has infinite global dimension, is not noetherian, has exponential growth, and has zero divisors, none of which happens when \((a, b, c) \notin \mathcal{D}\).

Unexplained terminology in this paper can be found in Walton’s paper [8].

1.3. Results. Our results show that the degenerate Sklyanin algebras are rather well-behaved, albeit on their own terms.

**Theorem 1.1.** Let \(S = S_{a,b,c}\) be a degenerate Sklyanin algebra.

1. If \(a = b\), then \(S\) is isomorphic to \(k\langle u, v, w \rangle\) modulo \(u^2 = v^2 = w^2 = 0\).

2. If \(a \neq b\), then \(S\) is isomorphic to \(k\langle u, v, w \rangle\) modulo \(uv = vw = wu = 0\).

**Corollary 1.2.** If \((a, b, c)\) and \((a', b', c')\) belong to \(\mathcal{D}\), then \(S_{a,b,c}\) and \(S_{a',b',c'}\) are Zhang twists of one another, and their categories of graded right modules are equivalent,

\[\text{Gr} S_{a,b,c} \equiv \text{Gr} S_{a',b',c'}\.

Of more interest to us is a quotient category, \(\text{QGr} S\), of the category of graded modules. This has surprisingly good properties. We now state the result and then define \(\text{QGr} S\).

**Theorem 1.3.** Let \(S = S_{a,b,c}\) be a degenerate Sklyanin algebra. There is a quiver \(Q\) and an ultramatricial algebra \(R\), both independent of \((a, b, c) \in \mathcal{D}\), such that

\[\text{QGr} S_{a,b,c} \equiv \text{QGr} kQ \equiv \text{Mod} R\]

where the algebra \(R\) is a direct limit of algebras \(R_n\), each of which is a product of three matrix algebras over \(k\), and hence a von Neumann regular ring.

Furthermore, there is an action of \(\mu_3\), the cube roots of unity in \(k\), as automorphisms of the free algebra \(F = k\langle X, Y \rangle\) such that

\[\text{QGr} S_{a,b,c} \equiv \text{QGr} (F \times \mu_3)\.

1.4. We work with right modules throughout this paper.

Let \(A\) be a connected graded \(k\)-algebra. If \(\text{Gr} A\) denotes the category of \(\mathbb{Z}\)-graded right \(A\)-modules and \(\text{Fdim} A\) is the full subcategory of modules that are direct limits of their finite dimensional submodules, then the quotient category

\[\text{QGr} A := \frac{\text{Gr} A}{\text{Fdim} A}\]

plays the role of the quasi-coherent sheaves on a “non-commutative scheme” that we call \(\text{Proj}_{nc} A\).

Artin, Tate, and Van den Bergh, showed that if \((a, b, c) \in \mathbb{P}^2 - \mathcal{D}\), then \(\text{QGr} S\) has “all” the properties enjoyed by \(\text{Qcoh} \mathbb{P}^2\), the category of quasi-coherent sheaves on the projective plane [1], [2].
1.5. **Consequences.** The ring $R$ in Theorem 1.3 is von Neumann regular because each $R_n$ is. The global dimension of $R$ is therefore equal to 1.

Let $S = S_{a,b,c}$ be a degenerate Sklyanin algebra. Finitely presented monomial algebras are coherent so $S$ is coherent. The full subcategory $\text{gr} S$ of $\text{Gr} S$ consisting of the finitely presented graded modules is therefore abelian. We write $\text{fdim} S$ for the full subcategory $\text{Gr} S$ consisting of the finite dimensional modules; we have $\text{fdim} S = (\text{gr} S) \cap (\text{Fdim} S)$. Since $\text{fdim} S$ is a Serre subcategory of $\text{gr} S$ we may form the quotient category

$$qgr S := \frac{\text{gr} S}{\text{fdim} S}.$$ 

The equivalence $Q\text{Gr} S \cong \text{Mod} R$ restricts to an equivalence

$$qgr S \cong \text{mod} R$$

where $\text{mod} R$ consists of the finitely presented $R$-modules.

Modules over von Neumann regular rings are flat so finitely presented $R$-modules are projective, whence the next result.

**Corollary 1.4.** Every object in $qgr S$ is projective in $Q\text{Gr} S$ and every short exact sequence in $qgr S$ splits.

The corollary illustrates, again, that the degenerate Sklyanin algebras are unlike the non-degenerate ones but still “nice”.

Since $R$ is only determined up to Morita equivalence there are different Bratteli diagrams corresponding to different algebras in the Morita equivalence class of $R$. There are, however, two “simplest” ones, namely the stationary Bratteli diagrams that begin

```
| 1 | 1 | 1 |
|---|---|---|
| 2 | 2 | 2 |
```

and

```
| 1 | 1 | 1 |
|---|---|---|
| 2 | 2 | 2 |
```

These diagrams correspond to the quivers $Q$ and $Q'$ that appear in section 1.6.

1.6. **The Grothendieck group** $K_0(qgr S)$. Let $S$ be a degenerate Sklyanin algebra and $K_0(qgr S)$ the Grothendieck group of finitely generated projectives in $qgr S$.

We write $O$ for $S$ as an object in $qgr S$. The left ideal of $A$ generated by $u - v$ and $v - w$ is free and $A/A(u - v) + A/(v - w)$ is spanned by the images of 1 and $u$, so $O \cong O(-1) \oplus O(-1)$. We show that

$$K_0(qgr S) = \mathbb{Z} \left[ \begin{matrix} 1 \\ 8 \end{matrix} \right] \oplus \mathbb{Z} \oplus \mathbb{Z}$$

with $[O] = (1, 0, 0)$. 

1.7. The word “degenerate”. The algebras $S_{a,b,c}$ form a flat family on $\mathbb{P}^2 - \mathfrak{D}$. The family does not extend to a flat family on $\mathbb{P}^2$ because the Hilbert series for a degenerate Sklyanin algebra is different from the Hilbert series of the non-degenerate ones. This is the reason for the quotation marks around “degenerate”. It is not unreasonable to think that there might be another compactification of $\mathbb{P}^2 - \mathfrak{D}$ that parametrizes a flat family of algebras that are the Sklyanin algebras on $\mathbb{P}^2 - \mathfrak{D}$. The obvious candidate to consider is $\mathbb{P}^2$ blown up at $\mathfrak{D}$.

1.8. Acknowledgements. I thank Chelsea Walton for carefully reading this paper and for several useful conversations about its contents.

2. The degenerate Sklyanin algebras are monomial algebras

Let

$$A = \frac{k\langle u, v, w \rangle}{(u^2, v^2, w^2)} \quad \text{and} \quad A' = \frac{k\langle u, v, w \rangle}{(uv, vw, wu)}.$$  

The next result uses the blanket hypothesis that the base field $k$ contains a primitive cube root of unity which implies that $\text{char } k \neq 3$.

**Theorem 2.1.** Suppose $(a, b, c) \in \mathfrak{D}$. Then

$$S_{a,b,c} \cong \begin{cases} A & \text{if } a = b \\ A' & \text{if } a \neq b. \end{cases}$$

**Proof.** The result is a triviality when $(a, b, c) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ so we assume that $a^3 = b^3 = c^3$ for the rest of the proof.

If $\lambda$ is a non-zero scalar, then $S_{a,b,c} \cong S_{\lambda a, \lambda b, \lambda c}$ so it suffices to prove the theorem when

1. $(a, b, c) = (1, 1, 1)$,
2. $(a, b, c) = (1, 1, c)$ with $c^3 = 1$ but $c \neq 1$, and
3. $a \neq b$ and $abc \neq 0$.

We consider the three cases separately.

1. Suppose $(a, b, c) = (1, 1, 1)$. Let $\omega$ be a primitive cube root of unity. Then

$$x^2 + y^2 = f_1 + f_2 + f_3,$$

$$x \omega y + \omega y^2 = f_1 + \omega^2 f_2 + \omega f_3,$$

$$x + \omega^2 y + \omega z = f_1 + \omega f_2 + \omega^2 f_3.$$  

Since $\{(1, 1, 1), (1, \omega, \omega^2), (1, \omega^2, \omega)\}$ is linearly independent,

$$\text{span}\{x + y + z, x + \omega y + \omega^2 z, x + \omega^2 y + \omega z\} = kx + ky + kz.$$  

Therefore $S_{1,1,1} \cong A$.

2. Suppose $c \neq 1$ and $(a, b, c) = (1, 1, c)$. Then

$$x + c^{-1} y + z = c^{-1} f_1 + c^{-1} f_2 + f_3,$$

$$x + c^{-1} y + z = c^{-1} f_1 + f_2 + c^{-1} f_3,$$

$$c^{-1} x + y + z = f_1 + c^{-1} f_2 + c^{-1} f_3.$$  

Since

$$\det \begin{pmatrix} 1 & 1 & c^{-1} \\ c^{-1} & 1 & 1 \\ 1 & c^{-1} & 1 \end{pmatrix} = (c^{-1} - 1)^2(c^{-1} + 2) \neq 0$$

Therefore $S_{1,1,c} \cong A$.
\{x + y + c^{-1}z, x + c^{-1}y + z, c^{-1}x + y + z\} \text{ is linearly independent. Hence } S_{1,1,c} \cong A.

(3) Suppose \( a \neq b \). Let
\[
\begin{align*}
u &= a^{-1}x + b^{-1}y + c^{-1}z, \\
v &= b^{-1}x + a^{-1}y + c^{-1}z, \\
w &= abc(x + y) + z.
\end{align*}
\]
Because \( \text{char } k \neq 3 \), the hypothesis that \( a \neq b \), implies \( \{u, v, w\} \) is linearly independent. Furthermore,
\[
\begin{align*}
u w &= (abc)^{-1}(f_1 + f_2) + f_3 \\
v w &= af_1 + bf_2 + cf_3, \\
w u &= bf_1 + af_2 + cf_3.
\end{align*}
\]
Hence \( S_{a,b,c} \cong A' \).

**Proposition 2.2.** The algebras \( A \) and \( A' \) are Zhang twists of each other.

**Proof.** The defining relations of \( A' \) are \( uv = vw = wu = 0 \) so the map \( \tau(u) = v, \tau(v) = w, \tau(w) = u \) extends to an algebra automorphism of \( A' \). The Zhang twist of \( A' \) by \( \tau \) is therefore the algebra generated by \( u, v, w \) with defining relations
\[
\begin{align*}
u u &= w \tau(u) = w = 0, \\
v v &= v \tau(v) = vw = 0, \\
w w &= w \tau(w) = wu = 0.
\end{align*}
\]
The Zhang twist of \( A' \) by \( \tau \) is therefore isomorphic to \( A \).

The defining relations of \( A \) are \( u^2 = v^2 = w^2 = 0 \) so the map \( \theta(u) = w, \theta(v) = u, \theta(w) = v \) extends to an algebra automorphism of \( A \). The Zhang twist of \( A \) by \( \theta \) is therefore the algebra generated by \( u, v, w \) with defining relations
\[
\begin{align*}
u u &= u \theta(v) = u^2 = 0, \\
v v &= v \theta(w) = v^2 = 0, \\
w w &= w \theta(u) = w^2 = 0.
\end{align*}
\]
The Zhang twist of \( A \) by \( \theta \) is therefore isomorphic to \( A' \).

Zhang [10] proved that if a connected graded algebra generated in degree one is a Zhang twist of another one, then their graded module categories are equivalent. This, and Proposition 2.2 implies the next result.

**Corollary 2.3.** Let \( S \) be a degenerate Sklyanin algebra. There are category equivalences
\[
\text{Gr } S \equiv \text{Gr } A \equiv \text{Gr } A'
\]
and
\[
\text{QGr } S \equiv \text{QGr } A \equiv \text{QGr } A'.
\]
In particular, \( \text{Gr } S_{a,b,c} \), and hence \( \text{QGr } S_{a,b,c} \), is the same for all \( (a, b, c) \in D \).

Walton [8] determined the Hilbert series of \( S \) by exhibiting \( S \) as a free module of rank 2 over a free subalgebra. Here is an alternative derivation of the Hilbert series using the description of \( S \) as a monomial algebra.

**Corollary 2.4.** Let \( (a, b, c) \in D \). The Hilbert series of \( S_{a,b,c} \) is \( (1 + t)(1 - 2t)^{-1} \).
Proof. Let $S$ be a degenerate Sklyanin algebra. Since $A$ is a Zhang twist of $A'$ the Hilbert series of $S$ is the same as that of

$$\frac{k(u, v, w)}{(u^2, v^2, w^2)} \cong \frac{k[u]}{(u^2)} * \frac{k[v]}{(v^2)} * \frac{k[w]}{(w^2)}$$

where the right-hand side of this isomorphism is the free product of three copies of $k[\varepsilon]$, the ring of dual numbers. Therefore

$$H_{S}(t) - 1 = 3H_{k[\varepsilon]}(t) - 2.$$ 

Since $H_{k[\varepsilon]}(t) = 1 + t$ it follows that $H_{S}(t)$ is as claimed. 

Here is another simple way to compute $H_{S}(t)$. A basis for $k\langle u, v, w \rangle/(u^2, v^2, w^2)$ is given by all words in $u, v, \text{and } w$, that do not have $uu, vv, \text{or } ww$, as a subword. The number of such words of length $n \geq 1$ is easily seen to be $3 \cdot 2^{n-1}$. Since $3 \cdot 2^{n-1} = 2^n + 2^{n-1}$ the Hilbert series of this algebra is $(1 + t)(1 - 2t)^{-1}$. The same argument applies to the algebra with relations $uv = vw = wu = 0$.

3. Two Quivers

3.1. The main result in [4] is the following.

Theorem 3.1. If $A$ is a finitely presented monomial algebra there is a finite quiver $Q$ and a homomorphism $f : A \to kQ$ such that $\sim \otimes_A kQ$ induces an equivalence

$$QGr A \equiv QGr kQ.$$ 

One can take $Q$ to be the Ufnarovskii graph of $A$.

A $k$-algebra is said to be matricial if it is a finite product of matrix algebras over $k$. A $k$-algebra is ultramatricial if it is a direct limit, equivalently a union, of matricial $k$-algebras.

Theorem 3.2. Let $(a, b, c) \in D$ and let $S_{a,b,c}$ be the associated degenerate Sklyanin algebra. There is a quiver $Q$ and an ultramatricial algebra, $R$, both independent of $(a, b, c)$, for which there is an equivalence of categories

$$QGr S_{a,b,c} \equiv QGr kQ \equiv ModR.$$ 

Proof. Since $S_{a,b,c}$ is a monomial algebra Theorem 3.1 implies that $QGr S_{a,b,c} \equiv QGr kQ$ for some quiver $Q$. By [6, Thm. 1.2], for every quiver $Q$ there is an ultramatricial $k$-algebra $R(Q)$ such that $QGr kQ \equiv ModR(Q)$. Because $QGr S_{a,b,c}$ is the same for all $(a, b, c) \in D$, $Q$ and $R(Q)$ can be taken the same for all such $(a, b, c)$.

3.2. The Ufnarovskii graph for $k\langle u, v, w \rangle/(u^2, v^2, w^2)$ is the quiver

(3-1)
The equivalence $\text{QGr } A \equiv \text{QGr } kQ$ is induced by the $k$-algebra homomorphism $f : A \to kQ$ defined by

$$f(u) = u_1 + u_2,$$
$$f(v) = v_1 + v_2,$$
$$f(w) = w_1 + w_2.$$

3.3. The Ufnarovskii graph for $k\langle u, v, w \rangle/(uv, vw, wu)$ is the quiver

![Quiver Diagram](image)

The equivalence $\text{QGr } A' \equiv \text{QGr } kQ'$ is induced by the $k$-algebra homomorphism $f' : A' \to kQ'$ defined by

$$f'(u) = u_1 + u_2,$$
$$f'(v) = v_1 + v_2,$$
$$f'(w) = w_1 + w_2.$$

3.4. Direct proof that $\text{QGr } kQ$ is equivalent to $\text{QGr } kQ'$. Since $\text{QGr } A \equiv \text{QGr } A'$, $\text{QGr } kQ$ is equivalent to $\text{QGr } kQ'$. This equivalence is not obvious but follows directly from [6, Thm. 1.8] as we will now explain.

Given a quiver $Q$ its $n$-Veronese is the quiver $Q^{(n)}$ that has the same vertices as $Q$ but the arrows in $Q^{(n)}$ are the paths in $Q$ of length $n$. By [6, Thm. 1.8], $\text{QGr } kQ \equiv \text{QGr } kQ^{(n)}$, this being a consequence of the fact that $kQ^{(n)}$ is isomorphic to the $n$-Veronese subalgebra $(kQ)^{(n)}$ of $kQ$.

Hence, if the $3$-Veronese quivers of $Q$ and $Q'$ are the same, then $\text{QGr } kQ$ is equivalent to $\text{QGr } kQ'$.

The incidence matrix of the $n$th Veronese quiver is the $n$th power of the incidence matrix of the original quiver. The incidence matrices of $Q$ and $Q'$ are

$$\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}$$

and the third power of each is

$$\begin{pmatrix}
2 & 3 & 3 \\
3 & 2 & 3 \\
3 & 3 & 2
\end{pmatrix}$$

so $\text{QGr } kQ \equiv \text{QGr } kQ'$. 
3.5. **The ultramatricial algebra $R$ in Theorem 3.2**
The ring $R$ in Theorem 3.2 is only determined up to Morita equivalence but we can take it to be the algebra that is associated to $Q$ in [6, Thm. 1.2]. That algebra has a stationary Bratteli diagram that begins

\[
\begin{array}{cccc}
& 1 & 1 & 1 \\
2 & & & \\
4 & & & \\
\end{array}
\]

and so on. More explicitly, $R = \lim\limits_{\rightarrow} R_n$ where

\[R_n = M_{2n}(k) \oplus M_{2n}(k) \oplus M_{2n}(k)\]

and the map $R_n \to R_{n+1}$ is given by

\[(r, s, t) \mapsto \left( \begin{array}{ccc}
s & 0 & t \\
0 & t & 0 \\
r & 0 & s \\
\end{array} \right).\]

**Proposition 3.3.** The ring $R$ in Theorem 3.2 is left and right coherent, non-noetherian, simple, and von Neumann regular.

**Proof.** It is well-known that ultramatricial algebras are von Neumann regular and left and right coherent.

The simplicity of $R$ can be read off from the shape of its Bratteli diagram. Let $x$ be a non-zero element of $R$. There is an integer $n$ such that $x \in R_n$ so $x = (x_1, x_2, x_3)$ where each $x_i$ belongs to one of the matrix factors of $R_n$. Some $x_i$ is non-zero so, as can be seen from the Bratteli diagram, the image of $x$ in $R_{n+2}$ has a non-zero component in each matrix factor of $R_{n+2}$. The ideal of $R_{n+2}$ generated by the image of $x$ is therefore $R_{n+2}$. Hence $RxR = R$.

The ring $R$ is not noetherian because its identity element can be written as a sum of arbitrarily many mutually orthogonal idempotents. \qed

Because $R$ is ultramatricial it is unit regular [3, Ch. 4] hence directly finite [3, Ch. 5], and it satisfies the comparability axiom [3, Ch. 8].

3.6. **$Q$ and $Q'$ are McKay quivers.** Section 2 of [7] describes how to associate a McKay quiver to the action of a finite abelian group acting semisimply on a $k$-algebra. Both $Q$ and $Q'$ can be obtained in this way.

**Lemma 3.4.** Let $F = k\langle X, Y \rangle$ be the free algebra with $\deg X = \deg Y = 1$. Let $\mu_3$ be the group of 3rd roots of unity in $k$.

1. If $\alpha : \mu_3 \to \text{Aut}_{\text{gr.alg}} F$ is the homomorphism such that $\xi \cdot X = \xi X$ and $\xi \cdot Y = \xi^2 Y$, then $kQ \cong F \rtimes_\alpha \mu_3$.
2. If $\beta : \mu_3 \to \text{Aut}_{\text{gr.alg}} F$ is the homomorphism such that $\xi \cdot X = \xi X$ and $\xi \cdot Y = Y$, then $kQ' \cong F \rtimes_\beta \mu_3$.

**Proof.** This is a special case of [7, Prop. 2.1]. \qed

Even without Lemma 3.4 one can see that $kQ$ is Morita equivalent to $F \rtimes_\alpha \mu_3$ and $kQ'$ is Morita equivalent to $F \rtimes_\beta \mu_3$ because, as we will explain in the next
Given a $\mu_3$-equivariant $F$-module $M$

\[ M_i = \{ m \in M \mid \xi \cdot m = \xi^i m \text{ for all } \xi \in \mu_3 \} \]

and place $M_i$ at the vertex labelled $i$ in $Q$ or $Q'$. In Lemma 3.4(1), the clockwise arrows give the action of $X$ on each $M_i$ and the counter-clockwise arrows give the action of $Y$ on each $M_i$. (In other words, there is a homomorphism $F \to kQ$ given by $X \mapsto u_1 + v_1 + w_1$ and $Y \mapsto u_2 + v_2 + w_2$.) This functor from $\mu_3$-equivariant $F$-modules to $\text{Mod} kQ$ is an equivalence of categories.

The foregoing is “well known” to those to whom it is common knowledge.

**Proposition 3.5.** Let $k$ be a field such that $\mu_3$, its 3rd roots of unity, has order 3. Let $F$ be the free $k$-algebra on two generators placed in degree one. Let $\alpha : \mu_3 \to \text{Aut}_{\text{gr.alg}} F$ be a homomorphism such that $F_1$ is a sum of two non-isomorphic representations of $\mu_3$. If $S$ is a degenerate Sklyanin algebra, then

$$Q\text{Gr} S \equiv Q\text{Gr}(F \rtimes_{\alpha} \mu_3).$$

3.6.1. **Remark.** In [5], it is shown that $Q\text{Gr} F$ is equivalent to $\text{Mod} T$ where $T$ is the ultramatricial algebra with Bratteli diagram

\[
\begin{array}{c}
1 \xrightarrow{1} 2 \xrightarrow{M_2} 4 \xrightarrow{M_4} 8 \xrightarrow{M_8} \cdots
\end{array}
\]

We do not know an argument that explains the relation between this diagram and that in (3-3) which arises in connection with $Q\text{Gr}(F \rtimes \mu_3)$.

3.7. **The Grothendieck group of qgr $S$.**

**Proposition 3.6.** Let $S$ be a degenerate Sklyanin algebra. Then

$$K_0(\text{qgr} S) \cong \mathbb{Z} \left[ \begin{array}{c} 1 \\ \frac{1}{8} \end{array} \right] \oplus \mathbb{Z} \oplus \mathbb{Z}$$

with $[\mathcal{O}] = (1, 0, 0)$.

**Proof.** Since $\text{qgr} S \equiv \text{mod} R$,

$$K_0(\text{qgr} S) \cong K_0(R) \cong \varinjlim K_0(R_n)$$

where the second isomorphism holds because $K_0(\mathcal{O})$ commutes with direct limits.

Since $R_n$ is a product of three matrix algebras, $K_0(R_n) \cong \mathbb{Z}^3$. We can pick these isomorphisms in such a way that the map $K_0(R_n) \to K_0(R_{n+1})$ induced by the inclusion $\phi_n : R_n \to R_{n+1}$ in the Bratteli diagram is left multiplication by

\[
(3-4) \quad M := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
\]

The directed system $\cdots \to K_0(R_n) \to K_0(R_{n+1}) \to \cdots$ is therefore isomorphic to $\cdots \to \mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^3 \xrightarrow{M} \cdots$.

As noted above,

$$M^3 := \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$
We have
\[ M^3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad M^3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad M^3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}. \]
Since \( \lim_{\mathbb{N}} K_0(R_n) \) is also the direct limit of the directed system
\[ \cdots \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \cdots, \]
\( K_0(R) \) is isomorphic to \( \mathbb{Z}[\frac{1}{8}] \oplus \mathbb{Z} \oplus \mathbb{Z} \). Together with the observation that \( \mathcal{O} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) this completes the proof. \( \square \)

4. Point modules for \( S_{a,b,c} \)

4.1. A point module over a connected graded \( k \)-algebra \( A \) is a graded \( A \)-module \( M = M_0 \oplus M_1 \oplus \cdots \) such that \( \dim_k M_n = 1 \) for all \( n \geq 0 \) and \( M = M_0 A \).

4.2. If one connected graded algebra is a Zhang twist of another their categories of graded modules are equivalent via a functor that sends point modules to point modules. Thus, to determine the point modules over a degenerate Sklyanin algebra it suffices to determine the point modules over \( A \), the algebra with relations \( u^2 = v^2 = w^2 = 0 \).

4.3. The letters \( u, v, w \) will now serve double duty: they are elements in \( A \) and are also an ordered set of homogeneous coordinates on \( \mathbb{P}^2 = \mathbb{P}(A_1^*) \), the lines in \( A_1^* \).

If \( M = ke_0 \oplus ke_1 \oplus \cdots \) is a point module with \( \deg e_n = n \), we define points \( p_n \in \mathbb{P}^2, n \geq 0 \), by
\[ p_n = (\alpha_n, \beta_n, \gamma_n) \]
where \( e_n \cdot u = \alpha_n e_{n+1}, e_n \cdot v = \beta_n e_{n+1}, \) and \( e_n \cdot w = \gamma_n e_{n+1} \). The \( p_n \)s do not depend on the choice of homogeneous basis for \( M \). We call \( p_0, p_1, \ldots \) the point sequence associated to \( M \).

4.4. Let \( E \) be the three lines in \( \mathbb{P}^2 \) where \(uvw = 0 \). We call the points that lie on two of those lines intersection points. If \( p \) is an intersection point the component of \( E \) that does not pass through \( p \) is called the line opposite \( p \). If \( L \) is a component of \( E \) we call the intersection point that does not lie on \( L \) the point opposite \( L \).

Lemma 4.1. Let \( (p_0, p_1, \ldots) \) be the point sequence associated to a point module \( M \).

1. Every \( p_n \) lies on \( E \).
2. If \( p_n \) is an intersection point then \( p_{n+1} \) lies on the line opposite \( p_n \) and can be any point on that line.
3. If \( p_n \) is not an intersection point and lies on \( L \), then \( p_{n+1} \) is the point opposite \( L \).

Proof. (1) If \( p_n \notin E \), then \( e_n \cdot u, e_n \cdot v, \) and \( e_n \cdot w \) are non-zero scalar multiples of \( e_{n+1} \). But \( u^2, v^2, \) and \( w^2 \), are zero in \( A \) so \( e_n \cdot u^2 = e_n \cdot v^2 = e_n \cdot w^2 = 0 \) which implies that \( e_{n+1} A = ke_{n+1} \) in contradiction of the fact that \( M \) is a point module.

(2) Suppose \( p_n \) lies on the lines \( u = 0 \) and \( v = 0 \). The line opposite \( p_n \) is the line \( w = 0 \). Since \( e_n \cdot u = e_n \cdot v = 0, e_n \cdot w \) must be a non-zero multiple of \( e_{n+1} \) so, since \( w^2 = 0, e_{n+1} \cdot w = 0; \) i.e., \( w(p_{n+1}) \neq 0 \). The other cases are similar.
Proposition 4.4. In particular, this is a special sequence and is the point opposite the line \( u = 0 \). However, Lemma 4.1 shows that [8, Thm. 1.7] is not correct. Proposition 4.3. If \( s = s_0 \ldots s_n \) is a non-zero word of length \( n + 1 \), there is a special truncated point module \( N \) of length \( n + 2 \) such that \( N \cdot s \neq 0 \) and \( N \cdot t = 0 \) for every \( t \in s^± \).

Proof. We make \( N := ke_0 \oplus \ldots \oplus ke_n + 1 \) into a right \( A \)-module by having \( a \in \{ u, v, w \} \) act as follows:

\[
e_i \cdot a = \begin{cases} 
e_{i+1} & \text{if } a = s_i \\ 0 & \text{if } a \neq s_i \end{cases}
\]

for \( 0 \leq i \leq n \) and \( e_n \cdot u = e_n \cdot v = e_n \cdot w = 0 \). Because \( s_i \neq s_{i+1} \), \( u^2 \), \( v^2 \), and \( w^2 \), act as zero on \( N \). By construction, \( e_0 \cdot s = e_n + 1 \neq 0 \) so \( N \) is a truncated point module. If \( t = t_0 \ldots t_n \in s^± \), then \( t_i \neq s_i \) for some \( i \), whence \( e_0 \cdot t = 0 \). Of course, \( e_i \cdot t = 0 \) for \( i \geq 1 \) so \( N \cdot t = 0 \).

The point sequence associated to \( N, p_0, \ldots, p_n \), is given by

\[
p_i = \begin{cases} (1, 0, 0) & \text{if } s_i = u \\ (0, 1, 0) & \text{if } s_i = v \\ (0, 0, 1) & \text{if } s_i = w. \end{cases}
\]

In particular, this is a special sequence and \( N \) is therefore a special truncated point module.

Proposition 4.4. Let \( a \in A_n \setminus \{0\} \). There is a truncated special point module \( N \) of length \( n + 1 \) such that \( N \cdot a \neq 0 \).

Proof. Let

\[
I = \{ b \in A \mid Nb = 0 \text{ for all truncated point modules } N \text{ of length } n + 1 \}.
\]

To prove the proposition we must show that \( I_n = 0 \).

Let \( s \) be a non-zero word of length \( n \). By Proposition 4.3, there is a truncated point module \( N \) of length \( n + 1 \) such that \( (\text{Ann } N)_n \subset s^± \). Hence \( I_n \subset \text{Span}(s^±) \). However, \( L_n \) is a basis for \( A_n \) so the intersection of \( \text{Span}(s^±) \) as \( s \) ranges over \( L_n \) is zero. Hence \( I_n = 0 \).

Corollary 4.5. If \( a \) is a non-zero element in \( A \) there is a special point module \( M \) such that \( M \cdot a \neq 0 \).

Proposition 4.4 implies that the natural map from \( A \) to its point parameter ring (see [8 Sect. 1]) is injective. Hence [8, Thm. 1.9] and [8, Cor. 1.10] are incorrect.
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