Order reduction for critical traveling wave problems

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Abstract. The paper deals with the order reduction for critical traveling wave problems. The specificity of such traveling waves is that they separate waves with qualitatively different behaviors. We show how the application of the geometric theory of singular perturbations allows us to reduce the traveling wave problem for the original PDE system to the analysis the projection of the system onto its slow invariant manifold. We illustrate this approach to the problem of finding the point-to-periodic traveling wave for the reaction-diffusion model.

1. Introduction

The paper discusses how the geometric theory of singular perturbations and the method of invariant manifolds [1–10] can be used to the order reduction for traveling waves problems with singular perturbations [11–13]. This approach allows us to investigate critical traveling waves. The specificity of such traveling waves is that they separate waves with qualitatively different behaviors. We demonstrate this approach via the reaction–diffusion–advection system.

Consider the reaction–diffusion–advection system in one spatial dimension

\[
\frac{\partial \mathbf{u}}{\partial t} = \varepsilon D \frac{\partial^2 \mathbf{u}}{\partial s^2} - A \frac{\partial \mathbf{u}}{\partial s} + \mathbf{F}(\mathbf{u}),
\]

where \( \mathbf{u} = \mathbf{u}(s, t) \in \mathbb{R}^n \), \( s \in \mathbb{R} \) is the spatial coordinate, \( t > 0 \) is time, \( D \) is the real diagonal matrix of diffusion coefficients, \( A \) is the real diagonal matrixes of the velocity in the advection process and \( \mathbf{F} (\mathbf{u}) \) is the vector of reaction terms. The coefficient \( 0 < \varepsilon \ll 1 \) represents the relative sizes of the diffusion.

We are interested in traveling wave solutions to equation (1), which propagate at constant speed \( c > 0 \) preserving their spatial profile.

After transforming to a co-moving coordinate \( \xi = s - ct \), equation (1) becomes

\[
-c \frac{d\tilde{\mathbf{u}}}{d\xi} = \varepsilon D \frac{d^2\tilde{\mathbf{u}}}{d\xi^2} - A \frac{d\tilde{\mathbf{u}}}{d\xi} + \mathbf{F}(\tilde{\mathbf{u}}),
\]

where \( \tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\xi) = \mathbf{u}(s, t) \). Introducing \( \mathbf{v} = \frac{d\tilde{\mathbf{u}}}{d\xi} \), equation (2) yields

\[
\frac{d\tilde{\mathbf{u}}}{d\xi} = \mathbf{v},
\]

\[
\varepsilon D \frac{d\mathbf{v}}{d\xi} = (A - c)\mathbf{v} - \mathbf{F}(\tilde{\mathbf{u}}).
\]
Periodic solutions, as well as heteroclinic and homoclinic trajectories of the system (3), correspond to traveling waves of the original system (1). 

The application of the geometric theory of invariant manifolds allows us to replace the study of the full system (3) with the analysis of a system of lower dimension [10]. The order reduction occurs due to the decomposition of (3) in the vicinity of the attractive invariant surface into the fast subsystem and the independent slow subsystem. This independent slow subsystem is the projection of the original system onto its slow invariant manifold.

Recall that the degenerate system for (3) is

\[
\frac{d\tilde{u}}{d\xi} = v,
\]

\[
0 = (A - c)v - F(\tilde{u}).
\]

(4)

Note that (4) is equivalent to the equation

\[
(A - c)\frac{d\tilde{u}}{d\xi} = F(\tilde{u}),
\]

(5)

which follows from (2) as \(\varepsilon = 0\).

The second equation in (4) determines the slow surface of (3). The slow surface can be considered as a zero-order approximation \((\varepsilon = 0)\) of the slow invariant manifold of the system [8–10]: the slow invariant manifold of the system can be found in the form of asymptotic expansion

\[
v = v(\tilde{u}, \varepsilon) = v_0(\tilde{u}) + \varepsilon v_1(\tilde{u}) + O(\varepsilon^2),
\]

where

\[
v_0 = (A - c)^{-1}F(\tilde{u}).
\]

On this slow manifold, the flow of the system (3) is governed by

\[
\frac{d\tilde{u}}{d\xi} = v(\tilde{u}, \varepsilon).
\]

(6)

If the slow invariant manifold \(v = v(\tilde{u}, \varepsilon)\) is attractive, then the analysis of the system (3) can be replaced by the analysis of (6) with a high degree of accuracy. Hence, problems with traveling wave solutions of (1) can be reduced to the study of periodic solutions, heteroclinic and homoclinic trajectories of the system (6).

2. Order Reduction

As an example of the approach above we consider the reaction-diffusion model with two reacting components, i.e., when \(u = (x, y)^T\) and the advection effects are neglected. In dimensionless form, the model is described by the following equations:

\[
\frac{\partial x}{\partial t} = \varepsilon \frac{\partial^2 x}{\partial s^2} + f(x, y),
\]

\[
\frac{\partial y}{\partial t} = \varepsilon \frac{\partial^2 y}{\partial s^2} + g(x, y).
\]

(7)

where

\[
f(x, y) = \frac{\alpha(\nu_0 + x^\gamma)}{1 + x^\gamma} - x(1 + y),
\]

\[
g(x, y) = x(\beta + y) - \delta y,
\]
$x$ and $y$ are the dimensionless concentrations of the reacting components; $\alpha, \beta, \gamma, \delta,$ and $\nu_0$ are the dimensionless positive parameters, moreover, $\beta > 1$ and $\gamma > 1$ [14].

Following the algorithm described above, when searching for traveling wave solutions to (7), we look for solutions

$$
\begin{align*}
  x(s, t) &= \varphi(s - ct) = \varphi(\xi), \\
  y(s, t) &= \psi(s - ct) = \psi(\xi).
\end{align*}
$$

From (7) and (8) we get

$$
\begin{align*}
  -c \frac{d\varphi}{d\xi} &= \varepsilon \frac{d^2\varphi}{d\xi^2} + \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi), \\
  -c \frac{d\psi}{d\xi} &= \frac{\varepsilon d^2\psi}{k d\xi^2} + \varphi(\beta + \psi) - k\delta\psi,
\end{align*}
$$

or, in the form of (3),

$$
\begin{align*}
  \frac{d\varphi}{d\xi} &= p, \\
  \frac{d\psi}{d\xi} &= q, \\
  \varepsilon \frac{dp}{d\xi} &= -cp - \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi), \\
  \varepsilon \frac{dq}{d\xi} &= -ckq - k\varphi(\beta + \psi) + k\delta\psi.
\end{align*}
$$

The corresponding degenerate system is

$$
\begin{align*}
  \frac{d\varphi}{d\xi} &= p, \\
  \frac{d\psi}{d\xi} &= q, \\
  0 &= -cp - \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi) := h_1, \\
  0 &= -ckq - k\varphi(\beta + \psi) + k\delta\psi := h_2.
\end{align*}
$$

The last two equations in (10) give the unique solution

$$
\begin{align*}
  p &= P_0(\varphi, \psi) = -\frac{\alpha(\nu_0 + \varphi^\gamma)}{c(1 + \varphi^\gamma)} + \frac{1}{c}\varphi(1 + \psi), \\
  q &= Q_0(\varphi, \psi) = -\frac{1}{c}\varphi(\beta + \psi) + \frac{\delta}{c}\psi,
\end{align*}
$$

which determines the slow surface of system (9). The slow surface is attractive since [10]

$$
\text{tr } B(\varphi, \psi) < 0, \quad \text{det } B(\varphi, \psi) > 0,
$$

where

$$
B = \begin{pmatrix}
  \frac{\partial h_1}{\partial p} & \frac{\partial h_1}{\partial q} \\
  \frac{\partial h_2}{\partial p} & \frac{\partial h_2}{\partial q}
\end{pmatrix} = \begin{pmatrix}
  -c & 0 \\
  0 & -ck
\end{pmatrix}.
$$
According to the geometric theory of singular perturbations in an \( \varepsilon \)-neighborhood of the slow surface, there exists a stable (attractive) slow invariant manifold which can be represented in the form

\[
\begin{align*}
p &= P(\varphi, \psi, \varepsilon) = P_0(\varphi, \psi) + \varepsilon P_1(\varphi, \psi) + O(\varepsilon^2), \\
q &= Q(\varphi, \psi, \varepsilon) = Q_0(\varphi, \psi) + \varepsilon Q_1(\varphi, \psi) + O(\varepsilon^2).
\end{align*}
\]  

(12)

To calculate the first–order approximation to the slow invariant manifold we substitute (12) into the invariance equations

\[
\begin{align*}
\varepsilon \left( \frac{\partial P}{\partial \varphi} P + \frac{\partial P}{\partial \psi} \right) &= -cP - \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi), \\
\varepsilon \left( \frac{\partial Q}{\partial \varphi} P + \frac{\partial Q}{\partial \psi} \right) &= -ckQ - k\varphi(\beta + \psi) + k\delta \psi,
\end{align*}
\]  

which follows from (9). Hence, we obtain

\[
\begin{align*}
\varepsilon \left( \frac{\partial P_0}{\partial \varphi} P_0 + \frac{\partial P_0}{\partial \psi} Q_0 \right) + O(\varepsilon^2) &= -c \left( P_0 + \varepsilon P_1 + O(\varepsilon^2) \right) - \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} + \varphi(1 + \psi), \\
\varepsilon \left( \frac{\partial Q_0}{\partial \varphi} P_0 + \frac{\partial Q_0}{\partial \psi} Q_0 \right) + O(\varepsilon^2) &= -ck \left( Q_0 + \varepsilon Q_1 + O(\varepsilon^2) \right) - k\varphi(\beta + \psi) + k\delta \psi.
\end{align*}
\]

From these equations, equating the coefficients of the first power of \( \varepsilon \) and taking into account (11), we get

\[
\begin{align*}
P_1(\varphi, \psi) &= -\frac{1}{c^2} \left[ \alpha \gamma \varphi^{\gamma-1}(1 - \nu_0) \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) - \varphi^2(\beta + \psi) + \delta \varphi \psi \right], \\
Q_1(\varphi, \psi) &= -\frac{1}{c^2 k} \left[ (\beta + \psi) \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) + (\varphi - \delta)(\varphi(\beta + \psi) - \delta \varphi) \right].
\end{align*}
\]

Thus, the first–order approximation to the slow motion of (9) is described by

\[
\begin{align*}
\frac{d\varphi}{d\xi} &= -\frac{\alpha(\nu_0 + \varphi^\gamma)}{c(1 + \varphi^\gamma)} + \frac{1}{c^2} \varphi(1 + \psi) \\
&\quad - \varepsilon \left[ \frac{\alpha \gamma \varphi^{\gamma-1}(1 - \nu_0)(\alpha(\nu_0 + \varphi^\gamma)}{(1 + \varphi^\gamma)(1 + \varphi^\gamma)} - 1 - \psi \right] \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) - \varphi^2(\beta + \psi) + \delta \varphi \psi \right] + O(\varepsilon^2), \\
\frac{d\psi}{d\xi} &= -\frac{1}{c} \varphi(\beta + \psi) + \frac{\delta}{c} \psi \\
&\quad - \varepsilon \left[ (\beta + \psi) \left( \frac{\alpha(\nu_0 + \varphi^\gamma)}{1 + \varphi^\gamma} - \varphi(1 + \psi) \right) + (\varphi - \delta)(\varphi(\beta + \psi) - \delta \varphi) \right] + O(\varepsilon^2).
\end{align*}
\]  

(13)

The advantages of the system (13) over (9) are that it has no singular perturbations and its order is lower.

3. Point–to–periodic Traveling Waves
The wave properties of the original system (1) are determined by the properties of the system (3) under \( \varepsilon \to 0 \), i.e. by the properties of the system (4). For \( \varepsilon = 0 \), the singular points of (3) coincide with the equilibria of (5) and are determined by \( \mathbf{F}(\tilde{\mathbf{u}}) = 0 \).
Let us consider (9) for \( \alpha = 12 \), \( \beta = 1.5 \), \( \gamma = 3 \), \( \delta = 1.7 \), and \( \nu_0 = 0.01 \). In this case, there are three equilibria of the corresponding degenerate system (10): the unstable node \( P_1 \), the saddle \( P_2 \), and the unstable focus \( P_3 \), see Figure 1. Note, that these equilibria are the projections of the singular points \( \tilde{P}_1 \), \( \tilde{P}_2 \), and \( \tilde{P}_3 \) of (9) onto \((\varphi, \psi)\)-plane.

Adding the corresponding asymptotic boundary conditions to (10), we can obtain a trajectory connecting equilibrium \( P_1 \) and \( \omega \)-periodic orbit that arises via an Andronov-Hopf bifurcation near \( P_3 \), see Figures 2 and 3. Moreover, the system (9) has a solution tending to the unstable equilibrium point \( \tilde{P}_1 \) as \( \xi \to -\infty \) and to the stable \( \omega(\varepsilon) \)-periodic solution as \( \xi \to +\infty \), where \( \omega(\varepsilon) \to \omega \) as \( \varepsilon \to 0 \). This solution determines the profile of point-to-periodic traveling waves of the system (7). The proof of the existence of the \( \omega \)-periodic orbit near \( P_3 \), as well as, the point-to-periodic traveling waves is beyond the scope of this paper. Readers who have an interest in such problems are referred to [15–23].

It should be noted that the trajectory shown in Figure 2 is a canard [24–27]. Such trajectories are used to model critical phenomena, see, for example, [10, 12, 28, 29]. They play the role of intermediate forms between the trajectories corresponding to oscillations with negligibly small amplitudes and the trajectories with relaxation oscillations. Traveling waves with a canard profile are critical. They separate traveling waves with qualitatively different behavior.

Figure 4 shows the point-to-periodic trajectories of the systems (10) (see the blue line) and (13) (see the red line) as well as the \((\varphi, \psi)\)-projection of the corresponding trajectory of the system (9) (see the black line). All these trajectories are very close to each other, which means that the reduced systems preserve the essential properties of the qualitative behavior of the original system.

Note that in this case, for reduction, we can restrict ourselves to the zeroth approximation of the invariant manifold, and replace the study of the system (7) with the analysis of system (10). However, in many cases, to reflect the behavior of the original models it is necessary to use the first- or higher-order approximation of the invariant manifold [30].
4. Conclusions
In the paper, we discussed the approach of order reduction for critical traveling wave problems. The approach discussed is based on the use of the geometric theory of singular perturbations. By use of the reaction–diffusion system we show how the traveling wave problems for the original PDE system can be reduced to the study of the projection of this system onto its slow invariant manifold. Analysis of the reduced system made it possible to recognize critical traveling waves of the original system. It was shown that the critical waves play the role of intermediate forms between waves with different qualitative behavior.

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6. References
[1] Bogolyubov N N and Mitropolsky Yu A 1961 *Asymptotic Methods in the Theory of Nonlinear Oscillations* (New York: Gordon and Breach).
[2] Bogolyubov N N and Mitropolsky Yu A 1963 The Method of Integral Manifolds in Nonlinear Mechanics *Contributions to Differential Equations* 2 123-196.
[3] Hale J 1961 Integral manifolds of perturbed differential systems *Annals of Mathematics. Second Series* 73(3) 496-531.
[4] Fenichel N 1979 Geometric singular perturbation theory for ordinary differential equations *J. Differ. Equ.* 31 53-98.
[5] Henry D 1981 Geometrical Theory of Semilinear Parabolic Equations *Lect Notes Math* 804.
[6] Sobolev V A 1984 Integral manifolds and decomposition of singularly perturbed systems *System and Control Lett.* 5 169-179.
[7] Jones C K R T 1994 Geometric Singular Perturbation Theory (Dynamical Systems, Montecatini Terme) *Lect Notes Math* 1609 44-118.
[8] Mishchenko E F and Rozov N Kh 1980 Differential Equations with Small Parameters and Relaxation Oscillations (New York: Plenum Press).
[9] Mishchenko E F, Kolesov Yu S, Kolesov A Yu and Rozov N Kh 1995 Asymptotic Methods in Singularly Perturbed Systems (New York: Plenum Press).
[10] Shchepakina E, Sobolev V and Mortell M P 2014 Singular Perturbations. Introduction to system order reduction methods with applications Lect. Notes Math 2114.
[11] Schneider K, Shchepakina E and Sobolev V 2003 A new type of travelling wave Math. Method. Appl. Sci. 26 1349-1361.
[12] Shchepakina E and Sobolev V 2005 Black Swans and Canards in Laser and Combustion Models Singular Perturbation and Hysteresis 207-255.
[13] Mishchenko E F, Sadovnichii V A, Kolesov A Yu and Rozov N Kh 2005 Autowave Processes in Nonlinear Diffusive Media (Moscow: Fizmatlit).
[14] Sevčikova H, Kubiček M and Marek M 1984 Concentration waves — effects of an electric field Mathematical Modelling in Science and Technology 477-482.
[15] Dunbar S R 1986 Traveling wave in diffusive predator-prey equations: Periodic orbits and point–to–periodic heteroclinic orbits SIAM J. Appl. Math. 46 1057-1078.
[16] Huang W 2008 Traveling waves connecting equilibrium and periodic orbit for reaction-diffusion equations with time delay and nonlocal response J. Differ. Equ. 244 1230-1254.
[17] Huang Y and Weng P 2014 Periodic traveling wave train and point–to–periodic traveling wave for a diffusive predator-prey system with Ivlev–type functional response J. Math. Anal. Appl. 417 376-393.
[18] Liang D, Weng P X and Wu J H 2012 Travelling wave solutions in a delayed predator-prey diffusion PDE system: point–to–periodic and point–to–point waves IMAJ Appl. Math. 77 516-545.
[19] Duehring D and Huang W 2007 Periodic traveling waves for diffusion equations with time delayed and non–local responding reaction J. Dyn. Diff. Equat. 19 457-477.
[20] Hasik K and Trofimchuk S 2014 Slowly oscillating wavefronts of the KPP–Fisher delayed equation Discrete Contin. Dyn. Syst. 34 3511-3533.
[21] Hasik K, Kopfova J, Nabelkova P and Trofimchuk S 2016 Traveling waves in the nonlocal KPP–Fisher equation: Different roles of the right and the left interactions J. Differ. Equ. 260 6130-6175.
[22] Merkin J H, Poole A J and Scott S K 1997 Chemical wave responses to periodic stimuli in vulnerable excitable media J. Chem. Soc. Farad. Trans. 93(9) 1741-1745.
[23] Bordiougov G and Engel H 2006 From trigger to phase waves and back again Physica D 215 25-37.
[24] Diener M 1979 Nessie et Les Canards (Strasbourg: Publication IRMA).
[25] Benoît E, Callot J L, Diener F and Diener M 1981-1982 Chasse au canard Collect. Math. 31-32 37-119.
[26] Arnold V I, Afraimovich V S, Il’yashenko Yu S and Shil’nikov L P 1994 Theory of Bifurcations Dynamical Systems. Encyclopedia of Mathematical Sciences (New York: Springer-Verlag) 5.
[27] Eckhaus M W 1979 Asymptotic Analysis of Singular Perturbations Studies in Mathematics and Its Applications 9.
[28] Gorelov G N and Sobolev V A 1991 Mathematical modelling of critical phenomena in thermal explosion theory Combust. Flame 87 203-210.
[29] Gorelov G N and Sobolev V A 1992 Duck–trajectories in a thermal explosion problem Appl. Math. Lett. 5 3-6.
[30] Korobeinikov A, Shchepakina E and Sobolev V 2016 Paradox of enrichment and system order reduction: bacteriophages dynamics as case study Math. Med. Biol. 33(3) 359-369.