Superconformal Tensor Calculus on an Orbifold in 5D

Taichiro Kugo* and Keisuke Ohashi**

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

Superconformal tensor calculus on an orbifold $S^1/Z_2$ is given in five-dimensional (5D) spacetime. The four-dimensional superconformal Weyl multiplet and various matter multiplets are induced on the boundary planes from the 5D supermultiplets in the bulk. We identify those induced 4D supermultiplets and clarify a general method for coupling the bulk fields to the matter fields on the boundaries in a superconformal invariant manner.

* E-mail: kugo@gauge.scphys.kyoto-u.ac.jp
** E-mail: keisuke@gauge.scphys.kyoto-u.ac.jp
§1. Introduction

It is a very interesting idea that our world may be a 3-brane embedded in a higher-dimensional space-time. To investigate seriously various possibilities and problems in this framework of the ‘brane world scenario’,\(^1\)\(^2\)\(^3\) we need an off-shell formulation of supergravity in higher dimensions. Having this in mind, some groups have been developing supergravity\(^4\)\(^5\)\(^6\) and superconformal\(^7\)\(^8\) tensor calculus in five-dimensional (5D) space-time.

In this paper we report on superconformal tensor calculus in 5D space-time in which the fourth spatial direction, \(x^4 \equiv y\), is compactified on an orbifold \(S^1/Z_2\). We clarify the 4D superconformal multiplets induced on the boundary planes from the 5D bulk fields. In particular, we show that the 4D superconformal Weyl multiplet is induced on the boundary planes from the 5D bulk Weyl multiplet. Similarly to the rigid supersymmetry case,\(^8\) 5D bulk Yang-Mills multiplets induce 4D gauge multiplets on the boundaries if the vector components are assigned even \(Z_2\) parity. A hypermultiplet in 5D bulk produces a 4D chiral multiplet on the boundary. A linear multiplet in 5D bulk can also yield a 4D chiral multiplet on the boundary for a certain \(Z_2\) parity assignment, while, for the opposite \(Z_2\) parity assignment, it does not give 4D linear multiplet but, rather, a general-type multiplet.

Once we can identify the 4D superconformal multiplets induced on the boundary planes, it becomes immediately clear how to couple the 4D matter fields on the boundary to the bulk supergravity, Yang-Mills and matter fields in a superconformal invariant manner. Since the 4D compensating multiplet is also induced from the 5D bulk compensating multiplet, we can write down any 4D invariant action on the boundary planes using the known invariant action formulas of the 4D superconformal tensor calculus.\(^9\)\(^10\)\(^11\)\(^12\)\(^13\)\(^14\)

Actually this type of tensor calculus on an orbifold was first studied by Zucker.\(^15\) However, his tensor calculus is not a superconformal but, rather, a supergravity one, in which dilatation \(D\) and \(S\)-supersymmetry are already gauge-fixed. This fact (together with his choice of the linear multiplet for the compensator in the 5D bulk) lead to the inconvenient situation that a quite unfamiliar non-minimal version\(^16\) of 4D Poincaré supergravity is induced on the boundary planes. In our case, 4D superconformal symmetry is fully realized on the boundary planes. Then, it is clearly seen that the simplest 4D Poincaré supergravity, ‘old minimal’ version,\(^17\) is induced on the boundary planes if the hypermultiplet is chosen as the compensator in the 5D bulk.

This paper is organized as follows. We first, in §§2 and 3, respectively, state the 5D and 4D superconformal transformation rules of the Weyl multiplets and some matter multiplets. Comparing these transformation rules of the 5D bulk multiplets with the 4D transformation rules, as done by Mirabelli and Peskin\(^18\) in the rigid supersymmetry case, we identify in §4
all the basic 4D supermultiplets induced from the bulk field on the boundary. Then, in §5 we identify the 4D compensator induced from the bulk hypermultiplet compensator and explain how the brane action can be written down generally. This action gives superconformal invariant couplings of the bulk supergravity, Yang-Mills and matter fields to an arbitrary set of 4D matter fields on the boundary. Finally, in §6 we illustrate the procedure for writing invariant actions by considering the simplest case, in which the bulk is a pure supergravity system with $U(1)_R$ gauged and the boundary planes contain only tension terms. For convenience and to facilitate the practical use of the present tensor calculus, we add three appendices. The notation and conventions are briefly explained in Appendix A. Explicit expressions for the curvatures both in 5D and 4D cases are given in Appendix B. These make manifest the structure functions of the superconformal algebras in both cases. Previous results for the embedding and invariant action formulas in 4D superconformal tensor calculus are given in Appendix C in our present notation.

2. 5D superconformal multiplets

Tensor calculus for 5D supergravity was first formulated by Zucker, and later by the present authors in a more complete form. These formulations, however, do not contain conformal $S$ supersymmetry (nor dilatation $D$ symmetry in the case of the former), which leads to inconvenience when considering a general matter-coupled system, because we must carry out very tedious field redefinitions in order to recover the canonical Einstein and Rarita-Schwinger terms. These tedious field redefinitions can simply be bypassed by choosing improved gauge fixing conditions of $S$ supersymmetry and dilatation $D$ symmetries in the superconformal framework. In view of this, Bergshoeff et al. have derived the Weyl multiplets in 5D superconformal tensor calculus, and almost simultaneously Fujita and Ohashi presented the full superconformal tensor calculus including matter multiplets and invariant action formulas in a paper that we refer to as I henceforth. We here restate the transformation laws of the Weyl and matter multiplets given in I. For the purpose of convenience, which emphasis on practical use, we here list the explicit expressions for the superconformal covariant derivatives and curvatures, which were omitted in I.

2.1. 5D Weyl multiplet

The 5D Weyl multiplet consists of 32 Bose plus 32 Fermi fields,

$$e_\mu^a, \, \psi_\mu^i, \, V_\mu^{ij}, \, b_\mu, \, v^{ab}, \, \chi^i, \, D,$$

(2.1)

* These are off-shell formulations of 5D supergravity. For on-shell formulations of 5D supergravity, which have been known for a long time, see Refs.18)
Table I. Weyl multiplet in 5D.

| field   | type            | remarks            | SU(2) | Weyl-weight |
|---------|-----------------|--------------------|-------|-------------|
| $e_{\mu}^a$ | boson          | f"unbein          | 1     | -1          |
| $\psi_i^\mu$ | fermion       | SU(2)-Majorana    | 2     | $\frac{1}{2}$ |
| $b_{\mu}$ | boson          | real              | 1     | 0           |
| $V_{ij}^\mu$ | boson          | $V_{ij}^\mu = (V_{ij})^*$ | 3     | 0           |
| $v_{ab}$ | boson          | real, antisymmetric | 1     | 1           |
| $\chi^i$ | fermion        | SU(2)-Majorana    | 2     | $\frac{3}{2}$ |
| $D$     | boson          | real              | 1     | 2           |

dependent gauge fields

| $\omega_{\mu}^{ab}$ | boson          | spin connection | 1     | 0           |
| $\phi_i^\mu$ | fermion        | SU(2)-Majorana | 2     | $\frac{1}{2}$ |
| $f_{\mu}^a$ | boson          | real            | 1     | 1           |

whose properties are summarized in table I. We use $a, b, \cdots$ for the local Lorentz indices, $\mu, \nu, \cdots$ for the world vector indices and $i, j = 1, 2$ for SU(2). The first four fields $e_{\mu}^a$, $\psi_i^\mu$, $V_{ij}^\mu$ and $b_{\mu}$ are the gauge fields for ‘translation’ $P_a$, supersymmetry $Q_i^\mu$, $SU(2)$ $U_{ij}$ and dilatation $D$ transformations, respectively. The other gauge fields, $\omega_{\mu}^{ab}$ for the local Lorentz $M_{ab}$, $\phi_i^\mu$ for the conformal supersymmetry $S_i^\mu$ and $f_{\mu}^a$ for the special conformal boost $K_a$, are dependent fields given by functions of the above independent gauge fields, as a result of the imposition of the following constraints on the $P_a$, $Q_i^\mu$ and $M_{ab}$ curvatures, respectively:

\[ \hat{R}_{\mu}^{\nu}a(P) = 0 \quad \Rightarrow \quad \omega_{\mu}^{ab} = \omega_{\mu}^{0ab} + i(2\bar{\psi}_\mu^a \gamma^b \psi^a + \bar{\psi}_\mu^a \gamma^b \psi^a) - 2e_{\mu}^{[a}b] \]

with $\omega_{\mu}^{0ab} \equiv -2e^{[a}e_{\mu}^{b]} + e^{[a}e_{\mu}^{b]}e^c_\mu e^c_\sigma$.

\[ \gamma^\nu \hat{R}_{\mu}^{\nu}(Q) = 0 \quad \Rightarrow \quad \phi_i^\mu = (\frac{1}{3} \gamma^\mu \gamma^b + \frac{1}{12} \gamma_i^{\mu\sigma} \gamma^{ab}) \hat{R}_{ij}(Q) \]

\[ \hat{R}_{\mu}^{\nu}(M) = 0 \quad \Rightarrow \quad f_{\mu}^a = (\frac{1}{6} \delta_{\mu}^a \delta_{\nu}^b - \frac{1}{48} e^{[a}_\mu e^{b]}_\nu) \hat{R}_{ij}(M) \]

Here $\hat{R}_{\mu}^{\nu}(M) \equiv \hat{R}_{\mu}^{\nu}(M)e_{\nu}^a$, and the primes on the curvatures indicate that $\hat{R}_{ij}(Q) = \hat{R}_{ij}^0(Q)|_{\phi = 0}$ and $\hat{R}_{\mu}^{\nu}(M) = \hat{R}_{\mu}^{\nu}(M)|_{f_{\nu} = 0}$. A constraint-independent treatment for these dependent gauge fields were given in I, but here we prefer to impose the constraints (2.2) explicitly, since this is simpler in practice.

The full $Q, S$ and $K$ transformation laws of the Weyl multiplet are given as follows. With $\delta \equiv \bar{\varepsilon}^i Q_i + \bar{\eta}^i S_i + \xi^a_K K_a \equiv \delta_Q(\bar{\varepsilon}) + \delta_S(\bar{\eta}) + \delta_K(\xi^a_K)$, we have

\[ \delta e_{\mu}^a = -2i\bar{\varepsilon}\gamma^a\psi^\mu, \]

\[ \delta \psi^\mu_i = D^\mu \bar{\varepsilon}^i + \frac{1}{2} \varepsilon^a \bar{\gamma}^{a\mu}_{\nu \lambda} \bar{\varepsilon}^\nu \bar{\gamma}^{i\nu}_{\lambda} \]

\[ - \gamma^\mu \bar{\psi}_\mu^a. \]
\[ \delta b_\mu = -2i \bar{\epsilon} \phi_\mu - 2i \bar{\eta} \psi_\mu - 2 \xi K_\mu, \]
\[ \delta V^{ij}_\mu = -6i \epsilon^{(i} \phi^{j)} - 4i \bar{\epsilon}^{(i} \gamma^j \psi^{j)} - \frac{1}{4} \bar{\epsilon}^{(i} \gamma^j \chi^{j)} + 6i \bar{\eta}^{(i} \psi^{j)}, \]
\[ \delta v_{ab} = -\frac{i}{\sqrt{2}} \bar{\gamma}^{\pm \mp} R_{ab}(Q), \]
\[ \delta \chi^i = D \bar{\epsilon}^i - 2 \gamma^c \bar{\epsilon}^a \bar{D}_a v_{bc} + \gamma \cdot \bar{R}(U) \bar{D}_i \bar{\epsilon}^i - 2 \gamma^a \bar{\epsilon}^i \epsilon_{abc} v^{bc} v^{de} + 4 \gamma \cdot \eta^i, \]
\[ \delta D = -i \bar{\epsilon} \bar{D} \chi - 8i \bar{\epsilon} \bar{R}_{ab}(Q) v^{ab} + i \eta \chi, \] (2.3)

where the fermion bilinears like \( \eta \psi_\mu, \bar{\chi} \gamma_\mu \chi, \) etc. with their \( SU(2) \) spinor indices suppressed, always represent the northwest-southeast contraction \( \eta^i \psi_{\mu i}, \bar{\gamma}_i \gamma_\mu \chi_i, \) etc. The dot product \( \gamma \cdot T \) for a tensor \( T_{ab...} \) generally represents the contraction \( \gamma^{ab...} T_{ab...} \). The transformation rules of dependent fields, of course, follow from those of independent fields and are found to be

\[ \delta \omega^{ab}_\mu = 2i \bar{\epsilon} \gamma^{ab} \phi_\mu - 2i \bar{\epsilon} \gamma^a \bar{R}_b \eta^{(i} \psi^{j)} - \frac{1}{3} \bar{\eta} \gamma^a \bar{R}_b(Q) - \frac{1}{3} \bar{\epsilon} \gamma^a \bar{R}^{ab}(Q), \]
\[ \delta \phi^{ij}_\mu = D_{\mu} \eta^i - \frac{1}{3} \gamma_{abc} \eta^{vbc} + \gamma^{a} \epsilon \psi_{\mu v} - \xi K[a \epsilon^b \bar{\gamma}_a \psi_\mu \]
\[ + \frac{1}{3} \bar{\gamma} \gamma^a \bar{R}_b(Q) \gamma^{ab} \bar{R}_c(Q) - \frac{1}{3} \bar{\epsilon} \gamma^a \bar{R}^{ab}(Q), \]
\[ \delta f^a_\mu = D_{\mu} \xi K^a - 2i \bar{\epsilon} \gamma^{a} \phi_\mu - \frac{1}{3} \bar{\eta} \gamma^{a} \bar{R}_b(Q) + \frac{1}{3} \bar{\epsilon} \gamma^{a} \bar{R}^{ab}(Q), \]
\[ - \frac{1}{3} \gamma_{abc} \phi_{\mu v} - \frac{1}{3} \bar{\epsilon} \gamma^a \phi_\mu \gamma^{vbc} + \frac{1}{3} \bar{\epsilon} \gamma^a \bar{R}_b(Q) \gamma^{vbc} + \frac{1}{3} \bar{\epsilon} \gamma^a \bar{R}^{vbc}(Q) \gamma^{ab}, \]
\[ \delta \epsilon_\mu = \frac{1}{6} \bar{\epsilon} \gamma^a \phi_\mu v_{bc} - \frac{1}{6} \bar{\epsilon} \gamma^a \phi_\mu v^{ab} - \frac{1}{6} \bar{\epsilon} \gamma^a \bar{R}_b(U) \gamma^{vbc} - \frac{1}{6} \bar{\epsilon} \gamma^a \bar{R}^{vbc}(Q) \gamma^{ab}, \]
\[ - \frac{1}{6} \bar{\epsilon} \gamma^a \bar{R}_b(Q) v^{ab} - \frac{1}{6} \bar{\epsilon} \gamma^a \bar{R}^{vbc}(Q) \gamma^{ab}, \]
\[ + \epsilon_\mu^b \left( \frac{1}{2} i \bar{\epsilon} \bar{\gamma}_c \bar{R}_b(Q) \frac{1}{2} i \bar{\epsilon} \gamma^a \bar{D} \bar{R}_c(Q) + \frac{1}{2} i \bar{\epsilon} \gamma^a \bar{D} \bar{R}_c(Q) v^{ab} \right) \] (2.4)

Here the (unhatted) derivative \( D_{\mu} \) is covariant only with respect to the homogeneous transformations \( M_{ab}, D \) and \( U^{ij} \) (and the \( G \) transformation for non-singlet fields under the Yang-Mills group \( G \)), while the hatted derivative \( \hat{D}_{\mu} \) denotes the fully superconformal covariant derivative; that is, with \( h^A_{\mu} \) denoting the gauge fields of the transformation \( X_A \), we have

\[ D_{\mu} \equiv \partial_{\mu} - \sum_{X_A = M_{ab}D, U^{ij} G} h^A_{\mu} X_A, \quad \hat{D}_{\mu} \equiv D_{\mu} = \sum_{X_A = Q, S, K} h^A_{\mu} X_A. \] (2.5)
Table II. Matter multiplets in 5D.

| field     | type               | remarks                          | SU(2) | Weyl-weight |
|-----------|--------------------|----------------------------------|-------|-------------|
| Vector multiplet                                      |                                   |       |             |
| $W_\mu$   | boson              | real gauge field                 | 1     | 0           |
| $M$       | boson              | real                            | 1     | 1           |
| $\Omega^i$| fermion            | $SU(2)$-Majorana                 | 2     | 3           |
| $Y_{ij}$  | boson              | $Y^{ij} = Y^{ji} = (Y_{ij})^*$  | 3     | 2           |
| Hypermultiplet                                      |                                   |       |             |
| $\mathcal{A}_a^i$ | boson        | $\mathcal{A}_a^i = \epsilon^{ij} \mathcal{A}_j^a \rho_{ja} = -(\mathcal{A}_a^i)^*$ | 2     | 3           |
| $\zeta^a$ | fermion            | $\zeta^a \equiv (\zeta_a)^T \gamma_0 = \zeta^a T C$ | 1     | 2           |
| $\mathcal{F}_i^\alpha$ | boson           | $\mathcal{F}_i^\alpha \equiv M^Z Z \mathcal{A}_a^i$, $\mathcal{F}_i^\alpha = -(\mathcal{F}_i^\alpha)^*$ | 2     | 5           |
| Linear multiplet                                     |                                   |       |             |
| $L^{ij}$  | boson              | $L^{ij} = L^{ji} = (L_{ij})^*$  | 3     | 3           |
| $\varphi^i$| fermion            | $SU(2)$-Majorana                 | 2     | 7           |
| $E_a$     | boson              | real, constrained                | 1     | 4           |
| $N$       | boson              | real                            | 1     | 4           |

The explicit forms of the curvatures $\hat{R}_{\mu \nu}^A = \epsilon^{b e c} [\hat{D}_a, \hat{D}_b] A$ are given in Appendix B. The covariant derivatives appearing in Eqs. (2.3) and (2.4) are given explicitly by

\[
\begin{align*}
D_\mu \varepsilon^i &= \left( \partial_\mu - \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} + \frac{1}{2} b_\mu \right) \varepsilon^i - V^i_{\mu j} \varepsilon^j, \\
D_\mu \eta^i &= \left( \partial_\mu - \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} - \frac{1}{2} b_\mu \right) \eta^i - V^i_{\mu j} \eta^j, \\
D_\mu \gamma^a &= \left( \partial_\mu - b_\mu \right) \gamma^a - \omega^{ab}_\mu \gamma^b, \\
\hat{D}_\mu v_{ab} &= \partial_\mu v_{ab} + 2 \omega^{[a}_\mu \varepsilon^{b]c} - b v_{ab} + \frac{i}{8} \tilde{\psi} \gamma_{ab} \chi + \frac{3}{2} i \tilde{\psi} \hat{R}_{ab}(Q), \\
\hat{D}_\mu \chi^i &= \partial_\mu \chi^i - \partial_\mu \chi^i + 2 \gamma^c \gamma^{ab} \psi^i D_a v_{bc} - \gamma \cdot \hat{R}(U) j \psi^j + 2 \gamma^a \psi^i \epsilon_{abcde} v^{bc} v^{de} - 4 \gamma \cdot v \phi^i, \\
\hat{D}_\mu \chi^i &= \left( \partial_\mu - \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} - \frac{3}{2} b_\mu \right) \chi^i - V^i_{\mu j} \chi^j.
\end{align*}
\]

2.2. Matter multiplets in 5D

We here give the transformation rules for three kinds of matter multiplets: vector multiplets, hypermultiplets and linear multiplets. The components of these multiplets and their properties are listed in Table II.
2.2.1. Vector multiplet

All the component fields of this multiplet are Lie-algebra valued. For example, the first component scalar \( M \) is the matrix \( M^\alpha_\beta = M^I(t_I)^\alpha_\beta \), where the \( t_I \) are (anti-hermitian) generators of the gauge group \( G \): \([t_I, t_J] = -f_{IJ}Kt_K\). The \( Q \) and \( S \) transformation laws of the vector multiplet are given by

\[
\begin{align*}
\delta W_\mu &= -2i\bar{\epsilon}\gamma_\mu \Omega + 2i\bar{\epsilon}\psi_\mu M, \\
\delta M &= 2i\bar{\epsilon}\Omega, \\
\delta \Omega^i &= -\frac{i}{4}\gamma^{\mu\nu}\hat{F}_{\mu\nu}(W)\bar{\epsilon}^i - \frac{i}{2}\hat{\nabla}M\bar{\epsilon}^i + Y^i j\bar{\epsilon}^j - M\eta^i, \\
\delta Y^{ij} &= 2i\bar{\epsilon}^i(\hat{\nabla}\Omega^j) - i\bar{\epsilon}^i\gamma^j\Omega - \frac{i}{4}\bar{\epsilon}^i(\chi^j)M - 2ig\bar{\epsilon}^i[M, \Omega^j] - 2i\bar{\eta}^i(\Omega^j), \quad (2.7)
\end{align*}
\]

where the full covariant field strength \( \hat{F}_{\mu\nu}(W) \) and covariant derivatives are given explicitly by

\[
\begin{align*}
\hat{F}_{\mu\nu}(W) &= 2\partial_{[\mu}W_{\nu]} - g[W_\mu, W_\nu] + 4i\bar{\psi}[\gamma_\mu\eta_\nu] - 2i\bar{\psi}_\mu\psi_\nu M, \\
\hat{\nabla}_\mu M &= (\partial_\mu - b_\mu) M - g[W_\mu, M] - 2i\bar{\psi}_\mu\Omega, \\
\hat{\nabla}_\mu \Omega^i &= \nabla_\mu \Omega^i + \frac{i}{4}\gamma^{\mu\nu}\hat{F}(W)\bar{\psi}^i + \frac{i}{2}\hat{\nabla}M\bar{\psi}^i - Y^i j\bar{\psi}^j + M\phi^i. \quad (2.8)
\end{align*}
\]

Note that the gauge coupling constant \( g \) used in this paper is a symbolic notation; it represents different values for different factor groups when \( G \) is not a simple group.

2.2.2. Hypermultiplet

The hypermultiplet in 5D consists of scalars \( A^\alpha_i \), spinors \( \zeta^\alpha \) and auxiliary fields \( F^\alpha_i \). They carry the index \( \alpha (= 1, 2, \cdots, 2r) \) corresponding to the representation of the gauge group \( G' \), which is lowered (or raised) with a \( G' \)-invariant tensor \( \rho_{\alpha\beta} \) (and \( \rho^{\alpha\beta} \) with \( \rho^{\alpha\beta}\rho_{\gamma\beta} = \delta^\gamma_\beta \)) as \( A_{i\alpha} = A^\alpha_i \rho_{\alpha\beta} \). This multiplet gives an infinite dimensional representation of the central charge gauge group \( U(1)_Z \), which we regard as a subgroup of the full gauge group \( G = G' \times U(1)_Z \).

The \( Q \) and \( S \) transformation rules of \( A^\alpha_i \) and \( \zeta^\alpha \) are given by

\[
\begin{align*}
\delta A^\alpha_i &= 2i\bar{\epsilon}_i \zeta^\alpha, \\
\delta \zeta^\alpha &= \hat{\nabla}A^\alpha_j \bar{\epsilon}^j - \gamma^j \bar{\psi}^j A^\alpha_j - M \times A^\alpha_j \bar{\epsilon}^j + 3A^\alpha_j \eta^j, \\
\delta F^\alpha_i &= 2i\bar{\epsilon}_i (\alpha Z \zeta^\alpha) + 2i\bar{\epsilon}_i \Omega^0 F^\alpha_i, \quad (2.9)
\end{align*}
\]

where \( \theta_* = M_*, \Omega_*, \cdots \) represent the transformations with the parameters \( \theta \) including the central charge transformation, \( \delta_G(\theta) = \delta_{G'}(\theta) + \delta_Z(\theta^0) \); more explicitly, e.g.,

\[
M_\star A^\alpha_i = \delta_{G'}(M) A^\alpha_i + \delta_{Z}(M^0 = \alpha) A^\alpha_i = \sum_{I=1}^{n} g M^I(t_I)^\alpha_\beta A^\beta_i + \alpha Z A^\alpha_i. \quad (2.10)
\]
Here, $Z$ is the generator of the $U(1)_Z$ transformation. The $U(1)_Z$ transformation of $A_i^a$ defines an auxiliary field $F_i^a \equiv \alpha Z A_i^a$, where $\alpha$ is the scalar component of the $U(1)_Z$ vector multiplet $V^0 = (M_0 \equiv \alpha, W_0^\mu, \Omega^0, Y^{0ij})$. The $U(1)_Z$ transformations of the other components, $Z\zeta_\alpha$ and $ZF_i^a$, are defined by requiring

$$
0 = \hat{\mathcal{D}}\zeta_\alpha + \frac{1}{2} \gamma \cdot \nu \zeta_\alpha - \frac{1}{8} \chi^i A_i^\alpha + M_\alpha \zeta_\alpha - 2 \Omega_i^a A_i^\alpha,
$$

$$
0 = -\hat{\mathcal{D}}^a \hat{D}_a A_i^\alpha + M_\alpha M_\mu A_i^\alpha, 
$$

$$
+4i\bar{\Omega}_i \zeta_\alpha - 2Y_{ij} A_{\alpha j} - \frac{i}{4} \bar{\zeta}_\alpha \chi + \frac{1}{8} (D - 2v^2) A_i^\alpha. 
$$

Note that $\hat{\mathcal{D}}_\mu \zeta_\alpha$ and $M_\alpha \zeta_\alpha$ contain the central charge transformation terms $-W_\mu^0 Z \zeta_\alpha$ and $\alpha Z \zeta_\alpha$, respectively, and that both $\hat{\mathcal{D}}^a \hat{D}_a A_i^\alpha$ and $M_\alpha M_\mu A_i^\alpha$ contain $ZF_i^a = \alpha Z A_i^a$. Hence these conditions (2.11) indeed determine the $Z\zeta_\alpha$ and $ZF_i^a$.

The explicit forms of the covariant derivatives $\hat{\mathcal{D}}_\mu A_i^\alpha$ and $\hat{\mathcal{D}}_\mu \zeta_\alpha$ are given by

$$
\hat{\mathcal{D}}_\mu A_i^\alpha = \left( \partial_\mu - \frac{3}{2} b_\mu \right) A_i^\alpha - V_{\mu j} A_{\alpha j} - gW_{\mu}^{\alpha \beta} A_j^\beta - W_0^{0} \frac{1}{\alpha} F_i^\alpha - 2i\bar{\psi}_\mu \zeta_\alpha,
$$

$$
\hat{\mathcal{D}}_\mu \zeta_\alpha = D_\mu \zeta_\alpha - \hat{\mathcal{D}}^a A_i^\alpha \psi_\mu^a + \gamma \cdot \nu \psi_\mu^a A_i^\alpha + M_\mu A_i^\alpha \psi_\mu^a - 3A_{\alpha j}^\beta \psi_\mu^a,
$$

with $D_\mu \zeta_\alpha = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{a b} \gamma_{a b} - 2b_\mu \right) \zeta_\alpha - gW_{\mu}^{\alpha \beta} \zeta_\beta - W_0^{0} Z \zeta_\alpha. \tag{2.12}$

For completeness, we also give an explicit form for $\hat{\mathcal{D}}^a \hat{D}_a A_i^\alpha$, which can be obtained by using the formula (2.31) in Ref. [4],

$$
\delta(\varepsilon) \hat{D}_a \phi = \varepsilon^A X_A (\hat{D}_a \phi) = \varepsilon^A \hat{D}_a (X_A \phi) - \varepsilon^A f_{a B} X_B \phi. \tag{2.13}
$$

If we note that $\varepsilon^A f_{a B}$ in the last term is equal to the terms containing no gauge fields in $e^a_i \delta(\varepsilon) h_\mu^B$ (i.e., terms proportional to the vielbein $e^a_i$ in $\delta(\varepsilon) h_\mu^B$), we easily find

$$
\hat{\mathcal{D}}^a \hat{D}_a A_i^\alpha = \left( \partial^a - \frac{5}{2} b^a \right) (\hat{D}_a A_i^\alpha) - e^{\mu a} \omega^a_{\mu b} (\hat{D}^b A_i^\alpha) - V_{\mu j} A_{\alpha j} - gW_{\mu}^{\alpha \beta} A_j^\beta - W_0^{0} \frac{1}{\alpha} F_i^\alpha - 2i\bar{\psi}_\mu \zeta_\alpha,
$$

$$
- 2i \bar{\psi}_\mu \gamma^a \Omega_{\mu \beta} A_i^\beta - 2i \bar{\psi}_\mu \gamma^\alpha \Omega_{\alpha \beta} A_i^\beta - i\bar{\psi}_\mu \gamma_{a b c} \zeta_\alpha \nu^{b c} - 2i \bar{\psi}_\mu \gamma_{a} \zeta_\alpha. \tag{2.14}
$$

2.2.3. **Linear multiplet**

The linear multiplet consists of the components listed in Table [4] and may generally carry a charge of the gauge group $G$.

The $Q$ and $S$ transformation laws of the linear multiplet are given by

$$
\delta L_{ij} = 2i \varepsilon^{(i}(\varphi^j),
$$

- We used the notation $A_\mu$ to denote the gravi-photon field $W_\mu^0$ in previous papers. However, we here use $W_\mu^0$ instead, since $A_\mu$ is used to denote the $U(1)$ gauge field of the 4D superconformal group.
\[
\delta \varphi^i = -\hat{D}L^i \varepsilon_j + \frac{1}{2} \gamma^a \varepsilon^i E_a + \frac{1}{2} \varepsilon^i N + 2 \gamma \cdot v \varepsilon_j L^j + M_s L^{ij} \varepsilon_j - 6 L^{ij} \eta_j,
\]
\[
\delta E^a = 2i \varepsilon^i \hat{D} \varphi - 2i \varepsilon^i \gamma_a \varphi \varepsilon_{bc} + 6i \varepsilon^i \gamma_b \varphi \varepsilon^{ab} + 2i \varepsilon^i \gamma^a M_s \varphi - 4i \varepsilon^i \gamma^a \Omega^i_{ij} L_{ij} - 8i \eta_\gamma \gamma_a \varphi,
\]
\[
\delta N = -2i \varepsilon^i \hat{D} \varphi - 3i \varepsilon^i \gamma \cdot v \varphi + \frac{1}{2} i \varepsilon^i \chi^j L_{ij} + 4i \varepsilon^i \Omega^i_{ij} L_{ij} - 6i \eta \varphi,
\]

with $\theta_*$ defined above in Eq. (2.10). The closure of the algebra demands that $E^a$ satisfy the following $Q$- and $S$-invariant constraint:
\[
\hat{D}_a E^a + M_s N + 4i \tilde{\Omega}_a \varphi + 2 \tilde{Y}^i_{ij} L_{ij} = 0.
\]

§3. 4D superconformal tensor calculus

The $N = 1$ 4D formulation of superconformal tensor calculus has been known for a long time. Here we cite the results, mainly following Kugo and Uehara in the present notation. However, strangely enough, the transformation rules for the multiplets that carry the gauge group charges have not previously been given in the literature. Our expressions given here are also valid for such cases.

3.1. Weyl multiplet

The 4D superconformal group consists of the usual bosonic conformal transformations, $P_a$, $M_{ab}$, $D$ and $K_a$, plus a bosonic $U(1)$ symmetry $A$ and fermionic Majorana $Q$ and $S$ supersymmetries. For simplicity of notation, we use the same symbols for the gauge fields, curvatures, etc., in 4D as in 5D, although they, of course, denote different quantities. From this point, the world vector indices $\mu, \nu, \cdots$ and Lorentz indices $a, b, \cdots$ are considered to run only over 0, 1, 2 and 3. We attach a superscript indicating the dimensions, “(4)” or “(5)”, when the distinction is relevant. The Weyl multiplet in 4D consists of 12 Bose plus 12 Fermi gauge fields and no ‘matter’ fields:

\[
e^a_\mu, \quad \psi_\mu, \quad A_\mu, \quad b_\mu,
\]

where $A_\mu$ is the gauge field for the $U(1)$ transformation $A$. In this 4D case, the $M_{ab}$, $S$ and $K_a$ gauge fields $\omega_{ab}^\mu$, $\phi_\mu$ and $f_{\mu}^a$ are also dependent fields, as stipulated by the usual constraints,

\[
\hat{R}_{ab}^c(P) = 0, \quad \gamma^b \hat{R}_{ab}(Q) = 0, \quad \hat{R}_{ab}(M) - \frac{1}{2} \hat{R}_{ab}(A) = 0,
\]

where the tilde denotes the dual tensor $\tilde{F}_{ab} \equiv \epsilon_{abcd} F^{cd}/2$. The solution for the spin connection $\omega_{ab}^\mu$ to the first constraint takes the same form as that in 5D given in Eq. (2.2). The solutions
for $\phi_\mu$ and $f_\mu^a$ to the latter two constraints have coefficients that differ slightly from those in 5D, given in Eq. (2.2), and are given by
\[
\phi_\mu = -\frac{i}{3} \gamma^a \hat{R}_\mu a(Q) + \frac{i}{12} \gamma_{\mu ab} \hat{R}'_{ab}(Q),
\]
\[
f_\mu^a = \frac{1}{4} \hat{R}'_{\mu}^a(M) - \frac{1}{8} \hat{R}_\mu^a(A) - \frac{1}{24} \epsilon_\mu^a \hat{R}'(M).
\]
(3.3)

The $Q$, $S$, $K_a$ and $A$ transformation laws of the gauge fields are given as follows. With $\delta = \delta_Q(\varepsilon) + \delta_S(\eta) + \delta_K(\xi^a_K) + \delta_A(\theta)$,
\[
\delta e_\mu^a = -2i\bar{\varepsilon} \gamma^a \psi_\mu,
\]
\[
\delta \psi_\mu = \mathcal{D}_\mu \varepsilon + i\gamma_\mu \eta + \frac{2}{3} \theta i\gamma_5 \psi_\mu,
\]
\[
\delta b_\mu = -2\bar{\varepsilon} \phi_\mu + 2\bar{\eta} \psi_\mu - 2\xi_{K\mu},
\]
\[
\delta A_\mu = 4i\bar{\varepsilon}\gamma_5 \phi_\mu - 4i\bar{\eta} \gamma_5 \psi_\mu + \partial_\mu \theta,
\]
\[
\delta \omega_{\mu}^{ab} = 2\bar{\varepsilon} \gamma^a \phi_\mu - 2i\bar{\varepsilon} \gamma_\mu \hat{R}^{ab}(Q) + 2\bar{\eta} \gamma^a \psi_\mu - 4\xi^a_K e_\mu^b,
\]
\[
\delta \phi_\mu = \mathcal{D}_\mu \eta + i\gamma_\mu f_\mu^a - i\xi_K^a \gamma_\mu \psi_\mu + \frac{i}{4} \theta i\gamma_5 \hat{R}_{\mu a}(A) - \frac{1}{4} \gamma^a \gamma_5 \varepsilon \hat{R}_{\mu a}(A) - \frac{3}{4} \theta i\gamma_5 \phi_\mu,
\]
\[
\delta f_\mu^a = \mathcal{D}_\mu \xi^a_K - 2i\bar{\eta} \gamma^a \phi_\mu - i\bar{\varepsilon} \xi_K \mathcal{D}_\mu \hat{R}^{ab}(Q),
\]
(3.4)

where the covariant derivatives of transformation parameters are defined by
\[
\mathcal{D}_\mu \varepsilon = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{i}{2} b_\mu - \frac{2}{3} i\gamma_5 A_\mu \right) \varepsilon,
\]
\[
\mathcal{D}_\mu \eta = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{1}{2} b_\mu + \frac{3}{2} i\gamma_5 A_\mu \right) \eta,
\]
\[
\mathcal{D}_\mu \xi^a_K = \left( \partial_\mu - b_\mu \right) \xi^a_K - \omega_\mu^{ab} \xi_K^b.
\]
(3.5)

3.2. Matter multiplets

3.2.1. Gauge multiplet

A multiplet that contains the gauge field of a gauge group $G$ is a gauge multiplet $[B_\mu^g, \lambda^g, D^g]$. The $Q$, $S$ and $K$ transformation laws are given by
\[
\delta B_\mu^g = -i\bar{\varepsilon} \gamma_\mu \lambda^g,
\]
\[
\delta \lambda^g = -\frac{1}{2} \gamma \cdot \hat{F}(B^g) \varepsilon + i\gamma_5 \varepsilon D^g + \frac{3}{4} \theta i\gamma_5 \lambda^g,
\]
\[
\delta D^g = \bar{\varepsilon} \gamma_5 \hat{D} \lambda^g,
\]
(3.6)

where $\hat{F}_{\mu\nu}(B^g) \left(= \hat{R}_{\mu\nu}(G) \right)$ is a super-covariantized field strength given by
\[
\hat{F}_{\mu\nu}(B^g) = 2\partial_\mu B_\nu^g - [B_\mu^g, B_\nu^g] + 2i\bar{\psi}_\mu [\gamma_\mu] \lambda^g,
\]
\[
\hat{D}_\mu \lambda^g = \mathcal{D}_\mu \lambda^g + \frac{1}{2} \gamma \cdot \hat{F}(B^g) \psi_\mu - i\gamma_5 \psi_\mu D^g,
\]
with
\[
\mathcal{D}_\mu \lambda^g = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{3}{2} b_\mu - \frac{3}{4} A_\mu \gamma_5 \right) \lambda^g.
\]
(3.7)
Table III. Weyl and Matter multiplets in 4D.

| field | type | remarks | Weyl-weight |
|-------|------|---------|-------------|
| $e_{\mu}^{a}$ | boson | real | $-1$ |
| $\psi_{\mu}$ | fermion | Majorana | $-\frac{1}{2}$ |
| $A_{\mu}$ | boson | real | $0$ |
| $b_{\mu}$ | boson | real | $0$ |

**Weyl multiplet**

**Complex (real) general multiplet**

| $C$ | boson | complex (real) | $w$ |
| $\zeta$ | fermion | Dirac (Majorana) | $w + \frac{1}{2}$ |
| $H, K, B_{a}$ | boson | complex (real) | $w + 1$ |
| $\lambda$ | fermion | Dirac (Majorana) | $w + \frac{3}{2}$ |
| $D$ | boson | complex (real) | $w + 2$ |

**Gauge multiplet ($w = n = 0$)**

| $B_{\mu}^{g}$ | boson | adjoint rep. | $0$ |
| $\lambda^{g}$ | fermion | Majorana, adjoint rep. | $\frac{3}{2}$ |
| $D^{g}$ | boson | adjoint rep. | $2$ |

**Chiral multiplet ($w = n$)**

| $A$ | boson | complex | $w$ |
| $\chi$ | fermion | Majorana | $w + \frac{1}{2}$ |
| $\mathcal{F}$ | boson | complex | $w + 1$ |

**Real linear multiplet ($w = 2, n = 0$)**

| $C^{L}$ | boson | real | $2$ |
| $\zeta^{L}$ | fermion | Majorana | $\frac{5}{2}$ |
| $B_{a}^{L}$ | boson | real, constrained | $3$ |

As is well known in the rigid supersymmetry case, this gauge multiplet is embedded into a superfield strength multiplet $W_{\alpha}$, a chiral multiplet with an external spinor index $\alpha$, whose first component is $\lambda_{\alpha}^{g}$. [See Ref. [12] for discussion of superconformal multiplets with external Lorentz indices.]

### 3.2.2. Complex (or real) general multiplet $\Phi$

A maximal unconstrained multiplet whose first component is a complex scalar $C$, is called a ‘complex general multiplet’. Its full components are listed in Table [III]. The dilatation and $U(1)$ transformations of the first component $C$ define the Weyl and chiral weights, $w$ and $n$, respectively.
and the multiplet is characterized by these weights \( w \) and \( n \). When the chiral weight vanishes \( (n = 0) \), the complex general multiplet decomposes into two irreducible real general multiplets, whose components are all real or Majorana fields. The transformation laws of the complex general multiplet are given as follows (with those of the real general multiplet obtained by simply setting \( n = 0 \)):

\[
\begin{align*}
\delta C &= i\bar{\varepsilon}\gamma_5\zeta + \frac{1}{2}in\theta C, \\
\delta \zeta &= \left(i\gamma_5 H - K + i\beta - \hat{\phi}C\gamma_5\right)\varepsilon \\
& \quad + 2i(n + w\gamma_5)C\eta + \left(\frac{1}{2}in - \frac{3}{4}i\gamma_5\right)\theta\zeta, \\
\delta H &= \bar{\varepsilon}\gamma_5\hat{\phi}\zeta + \bar{\varepsilon}i\gamma_5\lambda - \bar{\eta}\left((w - 2)i\gamma_5 + in\right)\zeta + \left(\frac{1}{2}inH + \frac{3}{2}K\right)\theta, \\
\delta K &= i\bar{\varepsilon}\partial\zeta - \bar{\varepsilon}\lambda - \bar{\eta}\left((w - 2) + n\gamma_5\right)\zeta + \left(\frac{1}{2}inK - \frac{3}{2}H\right)\theta, \\
\delta B_a &= -i\bar{\varepsilon}\bar{D}_a\zeta - i\bar{\varepsilon}\gamma_a\lambda \\
& \quad - i\bar{\eta}\left((w + 1) + n\gamma_5\right)\gamma_a\zeta + \frac{1}{2}in\theta B_a + 2ni\xi_K aC, \\
\delta \lambda &= -\frac{1}{2}\gamma\cdot\hat{F}\varepsilon + 2i\gamma_5\varepsilon D \\
& \quad + \left(i\gamma_5 H + K - i\beta - \hat{\phi}C\gamma_5\right)(w + n\gamma_5)\eta \\
& \quad + \left(\frac{1}{2}in + \frac{3}{4}i\gamma_5\right)\theta\lambda + i(w + n\gamma_5)\xi_K^a\gamma_a\zeta, \\
\delta D &= \bar{\varepsilon}\gamma_5\hat{\phi}\lambda + \frac{1}{2}in\theta D \\
& \quad - \bar{\eta}(w\gamma_5 + n)\hat{\phi}\zeta - 2i\bar{\eta}(w\gamma_5 + n)\lambda + 2w\xi_K^a\hat{D}_a C + 2ni\xi_K^a B_a, 
\end{align*}
\]

(3.9)

where \( \hat{F}_{ab} \) is a field-strength-like quantity given by

\[
\hat{F}_{ab} = 2\hat{D}_{[a}B_{b]} + \frac{1}{2}\varepsilon_{abcd}[\hat{D}^c, \hat{D}^d]C. 
\]

(3.10)

To this point, this general multiplet has tacitly been assumed to carry no extra charges. If it carries charges of the gauge group \( G \), the transformation rules are slightly modified. First, the \( G \)-covariantization term \(-\delta_G(B^a_\mu)\) should also be included in the full-covariant derivative \( \hat{D}_\mu \). Second, the following terms should be added to the above transformation laws (3.9):

\[
\begin{align*}
\Delta[\delta B_a] &= \bar{\varepsilon}\gamma_5\gamma_a\lambda^a C, \\
\Delta[\delta \lambda] &= -\varepsilon D^a_\mu C + \gamma^a\varepsilon\frac{1}{2}\lambda^a\gamma_a\zeta + \gamma^a\gamma_5\varepsilon\frac{1}{2}\lambda^a\gamma_a\gamma_5\zeta, \\
\Delta[\delta D] &= \bar{\varepsilon}\gamma_5\gamma^a\lambda^a B_a + i\varepsilon\hat{\phi}(\lambda^a C) + \varepsilon D^a_\mu \zeta, 
\end{align*}
\]

(3.11)

where \( \theta_* = (\lambda^a_*, D^a_\mu) \) denotes the \( G \) transformation, \( \theta_* \Phi = \delta_G(\theta)\Phi \).
3.2.3. **Chiral multiplet** \( \Sigma \)

The chiral multiplet \( \Sigma = [\mathcal{A}, \mathcal{P}_R \chi, \mathcal{F}] (\mathcal{P}_R \equiv \frac{1}{2}(1 + \gamma_5)) \) can exist when \( w = n \), and the anti-chiral multiplet \( \Sigma^* = [\mathcal{A}^*, \mathcal{P}_L \chi, \mathcal{F}^*] \), where \( \mathcal{P}_L \equiv \frac{1}{2}(1 - \gamma_5) \), when \( w = -n \). Their embedding into the complex general multiplet is given by

\[
\Phi(\Sigma) = [\mathcal{A}, -i\mathcal{P}_R \chi, -\mathcal{F}, i\hat{\mathcal{F}}_a \mathcal{A}, -2i\mathcal{P}_L \lambda^a \mathcal{A}, -i\hat{D}^a \lambda^a \mathcal{A}],
\]

\[
\Phi(\Sigma^*) = [\mathcal{A}^*, i\mathcal{P}_L \chi, -\mathcal{F}^*, -i\hat{\mathcal{F}}^*_a \mathcal{A}^*, 2i\mathcal{P}_R \lambda^a \mathcal{A}^*, i\hat{D}^a \mathcal{A}^*].
\] (3.12)

The transformation laws of these multiplets can be read from those of \( \Phi \) as follows:

\[
\delta \mathcal{A} = \varepsilon \mathcal{P}_R \chi + \frac{i}{2}w \theta \mathcal{A},
\]

\[
\delta (\mathcal{P}_R \chi) = \mathcal{P}_R \left(-2i\hat{\mathcal{P}} \mathcal{A} \varepsilon + 2\mathcal{F} \varepsilon - 4w \mathcal{A} \eta + \frac{i}{2}(w - \frac{3}{2})\theta \chi \right),
\]

\[
\delta \mathcal{F} = -i\varepsilon \hat{\mathcal{P}} (\mathcal{P}_R \chi) + 2\varepsilon \mathcal{P}_L \lambda^a \mathcal{A} + 2(w - 1)\eta \mathcal{P}_R \chi + \frac{i}{2}(w - 3)\theta \mathcal{F}.
\] (3.13)

3.2.4. **Real linear multiplet** \( \mathcal{L} \)

This multiplet, which is denoted by \( \mathcal{L} = [C^L, \zeta^L, B^L_a] \), can exist only in the case that the weight is \( w = 2, n = 0 \). The vector component \( B^L_a \) is subject to the constraint

\[
0 = \hat{D}^a B^L_a - D^g_C^L + \bar{\lambda}^g \zeta^L.
\] (3.14)

Interestingly, this constraint is solvable in the case that this multiplet that \( G \)-inert or the matrix \( D^g \) is invertible. This multiplet is also embedded into the real general multiplet in the form

\[
\Phi(\mathcal{L}) = [C^L, \zeta^L, 0, 0, B^L_a, i\hat{\mathcal{P}} \zeta^L, \hat{D}^a \hat{D}_a C^L - i\bar{\lambda}^g \gamma_5 \zeta^L].
\] (3.15)

The transformation laws of the components can also be read from those of \( \Phi \).

§4. **Identification of \( N = 1, d = 4 \) supermultiplets at the boundary**

We must treat 5D and 4D fields simultaneously from this point. We use the vector indices \( \mu, \nu, \cdots \) and \( a, b, \cdots \) as the 4D indices running over 0, 1, 2 and 3, and write the fifth component as \( y \) for world vector and as 4 for the Lorentz vector. For instance, a 5D vector is written \( (V^\mu, V_y) \) or \( (V_a, V_5) \).

From the viewpoint of the four-dimensional boundary plane, any supermultiplet in the 5D bulk is reducible to an infinite number of supermultiplets of 4D superconformal algebra. We here identify all the basic 4D supermultiplets that each contains at least one bulk field on the boundary without derivative \( \partial_y \) (with respect to the transverse direction \( y \)) as a member.
4.1. 4D Weyl multiplet from a 5D one

The fields $\phi$ are classified as even and odd fields under the $Z_2$ parity transformation $y \equiv x^4 \to -y$. The $Z_2$ parity eigenvalue $\Pi(\phi) = \pm 1$ is defined by

$$\phi(-y) = \Pi(\phi) \phi(y).$$

(4.1)

For $SU(2)$-Majorana spinor fermions $\psi^i$ ($i = 1, 2$), however, the $Z_2$ parity transformation mixes the two components $\psi^1$ and $\psi^2$ as

$$\psi^i(-y) = \Pi(\psi) \gamma_5 M^i_j \psi^j(y),$$

where $M^i_j$ is a $2 \times 2$ matrix satisfying $M^* = -\sigma_2 M \sigma_2$, where $M = \sigma_3$ in our convention. We therefore define the combinations of spinors $\psi^i$

$$\psi_+(y) \equiv \psi^1_R(y) + \psi^2_R(y),$$

$$\psi_-(y) \equiv i(\psi^1_L(y) + \psi^2_L(y)),

\left(\psi^i_R \equiv \frac{1 \pm \gamma_5}{2} \psi\right)$$

(4.3)

which give the $Z_2$ parity eigenstates

$$\psi_\pm(-y) = \pm \Pi(\psi) \psi_\pm(y),$$

(4.4)

also satisfying the 4D Majorana property $\bar{\psi}_\pm \equiv (\psi_\pm)^\dagger \gamma^0 = \psi^T_\pm C_4$.

The $Z_2$ parity eigenvalues are assigned to fields by demanding the invariance of the action and the consistency of both sides of the superconformal transformation rules. We list in Table [V] the $Z_2$ parity eigenvalues for the Weyl multiplet fields and $Q$- and $S$-transformation parameters $\varepsilon$ and $\eta$, where the 'isovector' notation $\vec{t} = (t^1, t^2, t^3)$ is used, which we generally define for any symmetric $SU(2)$ tensor $t^{ij}$ [satisfying hermiticity $t^{ij} = (t_{ij})^*$] according to

$$t^i_j = t^{ik} \epsilon_{kj} \equiv i\vec{t} \cdot \vec{\sigma}^i_j.$$  

(4.5)

The even parity fields are non-vanishing on the 4D boundary planes at $y = 0$ and $y = \tilde{y}$ and can form 4D superconformal multiplets there. In four dimensions, the parameters of the $Q$ and $S$ supersymmetry transformations are both 4-component Majoranas. In accordance with this, half of the parameters of 5D the $Q$ and $S$ supersymmetries vanish on the boundaries, with only $\varepsilon_+$ and $\eta_-$, respectively, remaining non-vanishing. These non-vanishing parameters are indeed the Majorana spinors.

First of all, the 4D superconformal Weyl multiplet is induced on the boundary planes from the 5D bulk Weyl multiplet, and the multiplet members can be identified as follows, comparing the superconformal transformation laws in 4D and 5D cases; that is, the following
Table IV. \( Z_2 \) parity eigenvalues

| \( \Pi = +1 \) | \( e_\mu^a, e_y^a, \psi_{\mu+}, \psi_{y-}, \varepsilon_+, \eta_-, b_\mu, V_{\mu}^3, V_{y}^{1,2}, v^{4a}, X_+, D \) |
| \( \Pi = -1 \) | \( e_\mu^a, e_y^a, \psi_{\mu-}, \psi_{y+}, \varepsilon_-, \eta_+, b_\mu, V_{\mu}^3, V_{y}^{1,2}, v^{4a}, X_- \) |

**Weyl multiplet**

| \( \Pi_V \) | \( M, W_y, Y^{1,2}, \Omega_- \) |
| \( -\Pi_V \) | \( W_\mu, Y^3, \Omega_+ \) |

**Vector multiplet** \( \mathbf{V} \)

| \( \Pi_L \) | \( L^{1,2}, N, E^4, \varphi_+ \) |
| \( -\Pi_L \) | \( L^3, E^a, \varphi_- \) |

**Linear multiplet** \( \mathbf{L} \)

| \( \Pi_\hat{\alpha} \) | \( A^{2\hat{\alpha} - 1}_{i=1}, A^{2\hat{\alpha}}_{i=2}, F^{2\hat{\alpha} - 1}_{i=2}, F^{2\hat{\alpha}}_{i=1}, \zeta^\hat{\alpha}_- \) |
| \( -\Pi_\hat{\alpha} \) | \( A^{2\hat{\alpha}}_{i=2}, A^{2\hat{\alpha}_-}_{i=1}, F^{2\hat{\alpha} - 1}_{i=1}, F^{2\hat{\alpha}}_{i=2}, \zeta^\hat{\alpha}_+ \) |

**Hypermultiplet** \( \mathbf{H} \)

5D fields on the right-hand sides can be seen to transform in exactly the same way as the 4D Weyl multiplet obeying the superconformal transformation rule (3.4). (Noting that the 5D fields are always understood to be those evaluated at the boundary, \( y = 0 \) or \( \tilde{y} \), in the relations between the 4D and 5D cases):

\[
\begin{align*}
\varepsilon^{(4)a}_\mu &= \varepsilon^a_\mu, & \psi^{(4)}_\mu &= \psi_{\mu+}, & b^{(4)}_\mu &= b_\mu, \\
\omega^{(4)ab}_\mu &= \omega^{ab}_\mu, & A^{(4)}_\mu &= \frac{4}{3} (V^3_{\mu} + v_{4\mu}), \\
\phi^{(4)}_\mu &= \phi_{\mu-} - \gamma_5 \gamma^a v_{4\mu} \psi_{\mu+} + \Delta \phi_\mu, \\
f^{(4)a}_\mu &= f^a_\mu - \tilde{\psi}_{\mu+} \Delta \phi^a + \Delta f^a_\mu
\end{align*}
\]

with \( \Delta \phi_\mu \) and \( \Delta f^a_\mu \) given by

\[
\begin{align*}
\Delta \phi_\mu &= \frac{1}{2} i \gamma_5 \hat{R}_{\mu 4} (Q_-), \\
\Delta f^a_\mu &= -\frac{1}{6} \epsilon^{abc}_{\mu} (\hat{D}_b v_{c 4} + \frac{1}{2} \hat{R}_{bc} 3 (V)) + \frac{1}{2} \hat{R}_{\mu 4} a_4 (M).
\end{align*}
\]

Note, however, that the 4D \( Q \) supersymmetry transformation \( \delta^{(4)}_Q (\varepsilon) \) here is identified with the linear combination of 5D transformations at the boundaries

\[
\delta^{(4)}_Q (\varepsilon = \varepsilon_+) = \delta_Q (\varepsilon_+) + \delta_S (\gamma_5 \gamma^a v_{4\mu} \varepsilon_+) + \delta_K (\varepsilon_+ \Delta \phi^a),
\]

and the other 4D superconformal transformations, the \( U(1) \) transformation \( \delta^{(4)}_A (\theta) \), \( S \) supersymmetry transformation \( \delta^{(4)}_S (\eta) \), etc., are identified as

\[
\delta^{(4)}_A (\theta = \frac{4}{3} \theta^3) = \delta_U (\theta^3), \quad \delta^{(4)}_S (\eta = \eta_-) = \delta_S (\eta_-),
\]
\[ \delta_D^{(4)}(\rho) = \delta_D(\rho), \quad \delta_M^{(4)}(\lambda^{ab}) = \delta_M(\lambda^{ab}), \quad \delta_K^{(4)}(\xi^a_K) = \delta_K(\xi^a_K). \quad (4.9) \]

With these identifications of the fields and superconformal transformations, we have the following relation between the superconformal covariant derivatives in 4D and 5D:

\[ \hat{D}_a^{(4)} = \hat{D}_a - \delta_{U_3}(v_{a4}) - \delta_S(\Delta \phi_a) - \delta_K(\Delta f^a_{\mu}). \quad (4.10) \]

If we regard this equation and \[[\hat{D}_a, \hat{D}_b] = -\hat{R}_{ab}^A X_A\] as holding in any dimensions, we can most easily find the expressions (4.7) for \(\Delta \phi_\mu\) and \(\Delta f^a_\mu\). Indeed, comparing the coefficients of \(X_A = Q^i, M_{ab}\) and \(U^3\) on both sides of the commutators of Eq. (4.10), we straightforwardly find the relations between the curvatures in the 4D and 5D cases:

\[ \hat{R}_{ab}^{(4)}(Q) = \hat{R}_{ab}(Q) + 2\gamma[\mu \Delta \phi_{\beta}], \]
\[ \hat{R}_{ab}^{cd}(M) = \hat{R}_{ab}^{cd}(M) + 8\Delta f_{\mu}^a[\delta^b_d], \]
\[ \frac{3}{4} \hat{R}_{ab}^{(4)}(A) = \hat{R}_{ab}^{3}(U) + 2\hat{D}_{[a}v_{b]\]4. \quad (4.11)\]

Applying to these relations the constraints on the \(Q\) and \(M_{ab}\) curvatures in both cases, we immediately find the above expressions (4.7) for \(\Delta \phi_\mu\) and \(\Delta f^a_\mu\).

In addition to this 4D Weyl multiplet, the 5D Weyl multiplet also induces a 4D ‘matter’ multiplet. Indeed, the extra dimensional component \(e^4_y\) of the fünfbein is also non-vanishing on the boundaries and is \(S\)- and \(K\)-inert, so that it can be the first component of a superconformal multiplet. It turns out to be a general multiplet \(W_y\) with Weyl and chiral weights \((w, n) = (-1, 0)\). The identification of the multiplet members is given by

\[ W_y \equiv (C, \zeta, H, K, B_4, \lambda, D) \]
\[ = \left( e^4_y, -2\psi_{y-}, -2V^2_y, 2V^1_y, -2v_{oy}, \frac{i}{4}\gamma_5\chi + e^4_y + 2\phi_y + 2\gamma_5\gamma^b v_{b4}\psi_y, \right. \]
\[ \left. \left( \frac{1}{4}D - (v_{a4})^2 \right)e^4_y - 2f^4_y + \frac{i}{4}\chi + \gamma_5\psi_y \right). \quad (4.12) \]

These fields transform according to the general multiplet transformation rule (3.9), provided that the covariant derivatives \(\hat{D}_\mu^{(4)}\) appearing there are understood to be given by

\[ \hat{D}_\mu^{(4)} e^4_y = e_{\mu\omega}^a e^a_y = D_\mu^{(4)} e^4_y + 2\gamma_x^5 \gamma^b v_{b4}\psi_y - \partial_y e^4_y, \]
\[ \hat{D}_\mu^{(4)} \psi_{y-} = D_\mu^{(4)} \psi_{y-} + [ -V^1_y - i\gamma_5 V^2_y - i\gamma^b v_{by} + \frac{1}{2}(\hat{D}_\mu^{(4)} e^4_y) \gamma_5 \psi_{y+} - i e^4_y \gamma_5 \phi_\mu - \partial_y \psi_{y-} = \hat{R}_{\mu y}(Q)_{\mu y} + i\gamma_5 \phi_{y+} + i\gamma_5 \gamma^a v_{a4}\gamma_5 \psi_{y-}. \quad (4.13) \]

with the ‘homogeneous covariant derivative’ \(D_\mu^{(4)} = \partial_\mu - \delta_M(\omega^{ab}_\mu) - \delta_D(b_\mu) - \delta_A(A_\mu)\) covariant only with respect to the homogeneous transformations \(M_{ab}, D\) and \(A\). The last terms, \(-\partial_y e^4_y\)
in $\mathcal{D}_\mu^{(4)} e_y^4$ and $-\partial_y \psi_{\mu-}$ in $\mathcal{D}_\mu^{(4)} \psi_{y-}$, are unusual and appear as a result of the fact that $e_y^4$ and $\psi_{y-}$ carry strange ‘new’ charges, as we now explain.

Generally, if a 5D local transformation parameter $\Lambda(x, y)$ is $Z_2$-odd, it vanishes on the boundary. However, its first derivative with respect to $y$, $\partial_y \Lambda(x, y)$, is $Z_2$-even and gives a non-vanishing 4D gauge transformation parameter $\partial_y \Lambda(x, 0) \equiv \Lambda^{(1)}(x)$ on the boundary. Therefore, for the $Z_2$-odd parts of the 5D superconformal transformation parameters, there exist the corresponding 4D gauge transformations with parameters given as follows:

$$
\begin{align*}
\xi^y &\quad \text{of GC transformation } P \quad \to \quad \xi^{(1)}(x) \equiv \partial_y \xi^y(x, 0), \\
\varepsilon_- &\quad \text{of } Q \text{ supersymmetry} \quad \to \quad \varepsilon^{(1)}(x) \equiv \partial_y \varepsilon_-(x, 0), \\
\theta_1, \theta_2 &\quad \text{of SU(2) transformation } U \quad \to \quad \theta^{(1)}_{1,2}(x) \equiv \partial_y \theta_{1,2}(x, 0), \\
\lambda_a^4 &\quad \text{of local Lorentz } M \quad \to \quad \lambda^{(1)}_a(x) \equiv \partial_y \lambda_a^4(x, 0), \\
\eta_+ &\quad \text{of } S \text{ supersymmetry} \quad \to \quad \eta^{(1)}(x) \equiv \partial_y \eta_+(x, 0), \\
\xi^4_K &\quad \text{of special conformal } K \quad \to \quad \xi^{(1)}_K(x) \equiv \partial_y \xi^4_K(x, 0).
\end{align*}
$$

The general multiplet $W_y$ in Eq. (4.12) transforms non-trivially under these transformations. Under the first $\xi^{(1)}$ transformation, every member of $W_y$ undergoes a common scale transformation,

$$
\delta W_y = \xi^{(1)} W_y, \tag{4.15}
$$

and many members of $W_y$ are shifted inhomogeneously as Nambu-Goldstone fields under other transformations:

$$
\delta W_y = (0, -2\varepsilon^{(1)}, -2\theta^{(1)}_2, 2\theta^{(1)}_1, 0, 2\eta^{(1)}, -2\xi^{(1)}_K + \frac{i}{4} \varepsilon^{(1)} \gamma_5 \chi_+). \tag{4.16}
$$

We find the gauge fields for these transformations to be

$$
\begin{align*}
E^{(1)}_\mu &\equiv (e_y^4)^{-1} \partial_y e^{(4)}_\mu \quad \text{for } \xi^{(1)}, \\
\psi^{(1)}_\mu &\equiv \partial_y \psi_{\mu-} - E^{(1)}_\mu \psi_{y-} \quad \text{for } \varepsilon^{(1)}, \\
V^{(1),2}_\mu &\equiv \partial_y V^{(1),2}_\mu - E^{(1)}_\mu V^{(1),2}_y \quad \text{for } \theta^{(1)}_{1,2}, \\
\vdots &\quad \vdots \quad \vdots
\end{align*} \tag{4.17}
$$

The last terms $-\partial_y e^{(4)}_\mu$ in $\mathcal{D}_\mu^{(4)} e^4_y$ and $-\partial_y \psi_{\mu-}$ in $\mathcal{D}_\mu^{(4)} \psi_{y-}$, in Eq. (4.13) can be understood to be identically the terms that appear as the gauge covariantization $-\delta_{\xi^{(1)}} (E^{(1)}_\mu) - \delta_{\varepsilon^{(1)}} (\psi^{(1)}_\mu)$ using these gauge fields $E^{(1)}_\mu$ and $\psi^{(1)}_\mu$.

Since the general multiplet $W_y$ transforms non-trivially under these gauge transformations (4.14), the utility of the multiplet $W_y$ is rather limited. If we wish to use it in constructing 4D invariant actions on the brane with the other multiplets, we must satisfy the gauge invariance also under the transformations (4.14), which seems a non-trivial task.
4.2. From a vector multiplet

We define the $Z_2$-parity $\Pi_V$ of the vector multiplet $V = (M, W_{\mu y}, \Omega^i, Y^{ij})$ to be that of the first scalar component $M$. The $Z_2$-parity quantum numbers of the other members are given in Table [IV].

If a vector multiplet $V$ is assigned odd $Z_2$-parity $\Pi_V = -1$, then the components $W_\mu$, $\Omega_+$ and $Y^3$ are even and non-vanishing on the brane, and it gives a 4D gauge multiplet $(B_\mu^g, \chi^g, D^g)$ defined in Eq. (3.6) with $(w, n) = (0, 0)$ with the following identification:

$$(B_\mu^g, \chi^g, D^g) = (W_\mu, 2\Omega_+, 2Y^3 - \hat{D}_4 M).$$ (4.18)

This implies that, if $\Pi_V = -1$, the bulk Yang-Mills multiplets can also couple to the matter multiplets on the brane as the 4D Yang-Mills multiplets.

If a vector multiplet $V$ has even $Z_2$-parity, $\Pi_V = +1$, then we can identify the following real general-type multiplet with weight $(w, n) = (1, 0)$, whose first component is $M$:

$$(C, \zeta, H, K, B_a, \lambda, D) = \left(M, -2i\gamma_5\Omega_-, 2Y^1, 2Y^2, \hat{F}_{a4}(W) + 2v_{a4} M, -2\hat{D}_4 \Omega_+ + 2i\gamma^a v_{a4} \Omega_- - \frac{i}{4}\gamma_5 \chi_+ M, \hat{D}_4^2 M - 2\hat{D}_4 Y^3 - \frac{1}{4}DM + v_{a4}(2\hat{F}_{a4}(W) + v_{a4} M) + \frac{1}{2}\tilde{\chi}_+ \Omega_- \right).$$ (4.19)

The field $D$ in the term $-\frac{1}{4}DM$ is the auxiliary field $D$ in the 5D Weyl multiplet.

In the latter case of $\Pi_V = +1$, we can construct a 4D chiral multiplet with weight $(w, n) = (0, 0)$:

$$\mathcal{A} = \frac{1}{2}(W_y + ie_y^4M),$$
$$\chi = 2\psi_\gamma M + 2ie_y^4\gamma_5\Omega_-, $$
$$\mathcal{F} = (V_y^1 + iV_y^2)M - ie_y^4(y^1 + iy^2) - \psi_y(1 + \gamma_5)\Omega_-.$$(4.20)

However, this multiplet is also of limited utility because of its non-trivial transformation properties under the gauge transformation $\Lambda^{(1)} = \partial_y \Lambda(x, 0)$ as well as the above $\zeta^{(1)}, \epsilon^{(1)}, \theta_1^{(1)}$ and $\theta_2^{(1)}$ transformations.

4.3. From a hypermultiplet

A hypermultiplet $H^\alpha = (A^\alpha_i, \zeta^\alpha, F^\alpha_i) \ (\alpha = 1, 2, \cdots, 2r)$ generally splits into $r$ pairs $(H^{2\tilde{\alpha}-1}, H^{2\tilde{\alpha}}) \ (\tilde{\alpha} = 1, 2, \cdots, r)$ in the standard representation, in which $\rho_{\alpha\beta} = \epsilon \otimes 1_r$. 

18
Then, the following $2 \times 2$ matrix of $(A_{i}^{2\hat{\alpha}-1}, A_{i}^{\hat{\alpha}})$ for each $\hat{\alpha}$ possesses the same real structure as a quaternion $q = q^{0} + iq^{1} + jq^{2} + kq^{3}$ mapped to a $2 \times 2$ matrix:

$$
\begin{pmatrix}
A_{i=1}^{2\hat{\alpha}-1} & A_{i=2}^{2\hat{\alpha}-1} \\
A_{i=1}^{\hat{\alpha}} & A_{i=2}^{\hat{\alpha}}
\end{pmatrix}
\quad \leftrightarrow \quad q^{0}1_{2} - i\vec{q} \cdot \vec{\sigma} = 
\begin{pmatrix}
q^{0} - iq^{3} & -iq^{1} - q^{2} \\
-iq^{1} + q^{2} & q^{0} + iq^{3}
\end{pmatrix}.
$$

The matrix element fields of this matrix and those of the similar matrix $(F_{i}^{2\hat{\alpha}-1}, F_{i}^{\hat{\alpha}})$ for the auxiliary fields give the $Z_{2}$ parity eigenstates. For spinors, the pair $(\zeta_{2\hat{\alpha}-1}, \zeta_{\hat{\alpha}})$ for each $\hat{\alpha}$ satisfies the same realness condition as the $SU(2)$-Majorana spinor $\psi^{i} = (\psi^{1}, \psi^{2})$, so that we can define the $Z_{2}$-parity eigenstate 4D-Majorana spinors $\zeta_{2\hat{\alpha}}^{\hat{\alpha}}$ in the same way as in Eq. (4.3):

$$
\zeta_{2\hat{\alpha}}^{\hat{\alpha}} \equiv \zeta_{R}^{2\hat{\alpha}-1} + \zeta_{L}^{2\hat{\alpha}} \quad \text{and} \quad \zeta_{\hat{\alpha}}^{\hat{\alpha}} \equiv i\left(\zeta_{L}^{2\hat{\alpha}-1} + \zeta_{R}^{2\hat{\alpha}}\right).
$$

Then, if the 1-1 component $A_{i=1}^{2\hat{\alpha}-1}$ has $Z_{2}$-parity $\Pi_{\hat{\alpha}}$, the $Z_{2}$-parity quantum numbers of the other hypermultiplet members are given as listed in Table IV. For either choice of the $Z_{2}$ parity assignment $\Pi_{\hat{\alpha}} = \pm 1$, we obtain the following 4D chiral multiplet with weight $(w, n) = (3/2, 3/2)$:

$$
(A, \chi_{R}, F) = \left(\begin{array}{c}
A_{i=2}^{2\hat{\alpha}}, \\
-2i\zeta_{R}^{2\hat{\alpha}}
\end{array}\right), \quad (iM_{a}A + \hat{D}_{a}A)_{i=1}^{2\hat{\alpha}},
$$

$$
(A, \chi_{R}, F) = \left(\begin{array}{c}
A_{i=2}^{\hat{\alpha}}, \\
-2i\zeta_{R}^{\hat{\alpha}-1}
\end{array}\right), \quad (iM_{a}A + \hat{D}_{a}A)_{i=1}^{\hat{\alpha}-1},
$$

for $\Pi_{\hat{\alpha}} = \pm 1$, respectively. Since $M_{a}A_{i}^{a} = gM^{i}(t_{i})^{\alpha}_{\beta}A_{i}^{\beta} + F_{i}^{\beta}$, the $F$-components of these chiral multiplets contain the $F_{i}^{\alpha}$ components of the hypermultiplet.

### 4.4. From a linear multiplet

The $Z_{2}$ parity quantum numbers for the linear multiplet $L = (L^{i}, \varphi^{i}, E^{a}, N)$, are listed in Table IV.

In the case $\Pi_{L} = +1$, we can identify the following 4D chiral multiplet with the weight $(w, n) = (3, 3)$ on the brane:

$$
(A, \chi, F) = \left(-L^{1} + iL^{2}, \ 2\varphi_{+}, \ \frac{1}{2}(N + iE_{a}) - \hat{D}_{a}L^{3} - iM_{a}L^{3}\right).
$$

In the case $\Pi_{L} = -1$, the scalar component $L_{3}$ is non-vanishing on the brane. Since it is $S$- and $K$-inert and carries Weyl and chiral weights $(w, n) = (3, 0)$, there is a 4D general-type real multiplet with weight $(w, n) = (3, 0)$ starting with $L_{3}$. We identify the components other than the last $D$ component as

$$
(C, \zeta, H, K, B_{a}, \lambda, D)
= \left(L^{3}, -\varphi_{-}, -M_{a}L^{2} + \hat{D}_{a}L^{1}, M_{a}L^{1} + \hat{D}_{a}L^{2}, -\frac{1}{2}E_{a} + 2v_{a4}L^{3}, \right.
- i\hat{\Phi}^{(4)}\varphi_{+} + i\gamma_{5}\hat{D}_{a}\varphi_{+} + M_{a}\varphi_{+} - \gamma_{5}\gamma^{a}v_{a4}\varphi_{-}
- 2\Omega_{-}L^{1} + 2i\gamma_{5}\Omega_{-}L^{2} - \frac{i}{4}\gamma_{5}\varphi_{-}L^{2}, \ 2\varphi_{+}L^{3}, \ \ldots\right).
$$

(4.24)
§5. Compensator and general brane action

As is well known, we need not only the Weyl multiplet but also a special matter field called a ‘compensator’ in order to derive the superconformal invariant supergravity actions. The compensator multiplet is used to fix the extraneous gauge freedoms of the superconformal symmetries, like $D$, $A$ and $S$, as well as to saturate the required Weyl and chiral weights when applying the invariant action formulas.

The most common formulation in the 4D case is called ‘old minimal’ supergravity,\[1\] where the compensator used is a chiral multiplet $\Sigma$ with weight $w = n = 1$. The general superconformal invariant action in 4D is given by

$$S_{\text{brane}} = \int d^5x [\delta(y) + \delta(y - \tilde{y})] \times \left( \left[ \Sigma \bar{\Sigma} e^{K(S, \bar{S})} \right]_D + [f_{IJ}(S)W^I\alpha W^J_\alpha]_F + \left[ \Sigma^3 W(S) \right]_F \right), \quad (5.1)$$

where $K(S, \bar{S})$ and $W(S)$ are Kähler potential and superpotential functions, respectively, and $[\cdots]_D$ and $[\cdots]_F$ represent the D-term and F-term invariant action formulas in the 4D superconformal tensor calculus,\[3\] explained in Appendix C. The quantities $S_i$ are the 4D chiral matter multiplets with weight $w = n = 0$, and $W^I_\alpha$ denotes the superfield strength of the 4D Yang-Mills multiplets $V^I$. Both of these chiral and gauge multiplets, $S$ and $V$, may be genuine 4D multiplets existing solely on the brane or induced multiplets on the boundary from the 5D bulk multiplets.

The 4D Weyl multiplet used in expressing the action formulas in Eq. (5.1) should, of course, be the induced Weyl multiplet found in the previous section. Since gravity is unique, we cannot add a genuine 4D Weyl multiplet on the brane in addition to the induced one. In the same sense, we cannot add a genuine 4D compensator on the brane in addition to that induced from the bulk compensator fields.

Let us now identify the 4D compensator induced from the 5D bulk compensator. The most useful and common choice of the 5D compensator is the hypermultiplet, which we discussed in Ref. [3]. Consider the simplest case of a single-quaternion compensator ($p = 1$) in which the hypermultiplet compensator is given by $H^a = (A^a_i, \zeta^a_i, F^a_i)$ ($a = 1, 2$). Then, as seen in the previous section, this hypermultiplet gives the 4D chiral multiplet $\Sigma_c = (A_c, \chi_c, F_c)$ with weight $(w, n) = (3/2, 3/2)$ on the brane, assuming the $Z_2$ parity assignment $\Pi_a = +1$:

$$\Sigma_c : \begin{cases} A_c &= A^a_i = 2 \\ \chi_c & = -2i\zeta^a_i = 2 \\ F_c &= iF^a_{i=1} + ig(MA)^a_{i=1} + \hat{D}_4A^a_{i=1} \end{cases} \quad (5.2)$$
Since this multiplet $\Sigma_c$ carries Weyl and chiral weights $w = n = 3/2$, we should identify $\Sigma_c^{2/3}$ with the 4D chiral compensator $\Sigma$, with $w = n = 1$ induced on the brane. Note that, if the 5D superconformal gauges are fixed, for example, by the conditions

$$
D, \quad U^{ij} : \quad A^a_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
S^i : \quad \zeta^a = 0
$$

in the bulk, then, these yield the conditions

$$
D, \quad A : \quad A = 1, \quad S : \quad \chi = 0,
$$

on $\Sigma \equiv \Sigma_c^{2/3} = (A, \chi, \mathcal{F})$ on the boundary. These are the same 4D superconformal gauge fixing conditions as imposed on the usual chiral compensator in the case of a pure supergravity system.\[11]

This brane action (5.1) gives superconformal invariant coupling between the 4D matters on the brane and the 5D bulk fields through the induced Weyl, Yang-Mills and compensating multiplets.

\[\section{6. Altendorfer-Bagger-Nemeschansky approach}

We next illustrate how the action is given for the bulk-plus-brane system by considering the simplest case, in which the bulk is a pure supergravity system with $U(1)_R$ gauged, and the brane contains only the tension term. This is the system that was first constructed by Altendorfer, Bagger and Nemeschansky\[23] in an attempt to supersymmetrize the Randall and Sundrum\[24] scenario.

In our off-shell formulation, the system contains a 5D Weyl multiplet, the hypermultiplet compensator $H^a = (A^a_i, \zeta^a, F^a_i)$ ($a = 1, 2$) and a vector multiplet $V_0 = (M^a = \alpha, W^0_k, \Omega^{0i}, Y^{0ij})$ coupling to the central charge $Z$ of the hypermultiplet. Note here that this central charge vector multiplet $V_0$ is simultaneously the $U(1)_R$ gauge multiplet coupling to the $U(1)_R$-charge which is in general represented by a $2 \times 2$ matrix $t^a_R$ acting on the group index $a = 1, 2$ of $H^a$:

$$
t_R = i\bar{Q} \cdot \sigma = i(Q^1\sigma_1 + Q^2\sigma_2 + Q^3\sigma_3), \quad |\bar{Q}| = 1.
$$

Here, $V_0$ can be made to play such a double role if we add a ‘mass term’ $m\eta^{ab}\mathcal{L}_{VL}(V_0 \cdot \mathcal{L}(H_a, H_b))$ to the Lagrangian with a symmetric tensor $\eta^{ab}$ related to the $U(1)_R$ generator $t_R$ by $g_R(t_R)^{ab} = m\eta^{ab}/2$. (Here, $\mathcal{L}_{VL}(V \cdot \mathcal{L})$ is the invariant $V$-$L$ action formula $[\text{Eq. (4.5)$}$]
in I], and \( L(H_a, H'_b) \) is the embedding formula [Eq. (4.3) in I] of two hypermultiplets \( H \) and \( H' \) into a linear multiplet. The bulk Lagrangian is given in the form

\[
-2\mathcal{L}_{VL}(V_0 \cdot L(H^a, ZH_b + \frac{1}{2}m\eta^c H_c)) + \mathcal{L}_{VL}(V_0 \cdot L(-cV^2_0)),
\]

(6.2)

where \( ZH^a \) is the central-charge transformed hypermultiplet whose first component is \( ZA^a_i = \mathcal{F}^a_i/\alpha \), and \( L(\frac{1}{2}f_{IJ}V^IV^J) \) denotes generally the embedding formula [Eq. (4.1) in I] of vector multiplets \( V^I \) into a linear multiplet based on the homogeneous quadratic function \( f(M) = \frac{1}{2}f_{IJ}M^IM^J \). The quantity \( \mathcal{L}_{VL}(V_0 \cdot L(-cV^2_0)) \) here thus corresponds to the choice of \( f(\alpha) = -\alpha^2 \) and to the action with ‘norm function’ \( \mathcal{N}(\alpha) = c\alpha^3 \), where \( c \) is a constant coefficient. The explicit component form of the action (6.2) can be read from the general expression given in Ref. [5]. The extraneous gauges are generally fixed by the gauge conditions

\[
D : \quad \mathcal{N} = 1, \quad S : \quad \Omega^i N_i = 0, \quad K : \quad \hat{D}_a N = 0.
\]

(6.3)

The first \( D \) gauge-fixing condition \( \mathcal{N} = c\alpha^3 = 1 \) may equivalently be replaced by \( \mathcal{A}^a_i \mathcal{A}^i_a = -2 \), thanks to the auxiliary field equation \( \delta\mathcal{S}/\delta\mathcal{A}^i_a = \mathcal{A}^a_i \mathcal{A}^i_a + 2\mathcal{N} = 0 \). Similarly, if we use the equation of motion \( \delta\mathcal{S}/\delta\chi^i \propto \mathcal{A}^a_i \zeta_a + N_i \Omega^i_i = 0 \), the \( S \)-gauge condition \( N_i \Omega^i_i = 0 \) is equivalent to \( \mathcal{A}^a_i \zeta_a = 0 \). Then, imposing also \( \mathcal{A}^a_i \propto \delta^a_i \) as the \( U^u \) gauge-fixing conditions, these conditions reproduce the previous gauge-fixing conditions (5.3). The kinetic term \( -(1/4)F^2_{\mu\nu}(W^0) \) of the gravi-photon field \( W^0_\mu \) has the coefficient \( -(1/2)\mathcal{N}(\partial^2 \ln \mathcal{N}/\partial\alpha^2) \) in the action, so that it is properly normalized if the constant \( c \) is chosen to satisfy \( c\alpha = 2/3 \). Together with the \( D \) gauge condition \( \mathcal{N} = c\alpha^3 = 1 \), this determines \( \alpha = \sqrt{3/2} \) and \( c = (\sqrt{2/3})^3 \). The cosmological constant in the bulk is found to be \( -(8/3)g^2_R/\alpha^2 = -4g^2_R \). \[ \]

Note here that the consistency of the \( U(1)_R \) symmetry with the \( Z_2 \) parity requires \( Q^3 = 0 \). \[ \]

Recall that this can be seen if we consider the two terms in the covariant derivative,

\[
\mathcal{D}_\mu \mathcal{A}^a_i = \partial_\mu \mathcal{A}^a_i - g_R W^0_\mu(t_R)^a_b \mathcal{A}^b_i + \cdots
\]

(6.4)

Recall that the scalar component \( M^0 = \alpha \) of the central charge vector multiplet is \( Z_2 \)-even, so that the vector component \( W^0_\mu \) is \( Z_2 \)-odd, as seen in Table [IV]. Since \( W^0_\mu \) is \( Z_2 \)-odd and \( \mathcal{A}^{a=1}_i \) and \( \mathcal{A}^{a=2}_i \) carry opposite \( Z_2 \)-parity, the first and the second terms in Eq. (6.4) can have the same \( Z_2 \) even-oddness if and only if \( t_R \) possesses only off-diagonal components. In other words, \( t_R = i(Q^1 \sigma_1 + Q^2 \sigma_2) \) with no \( \sigma_3 \) component. (If the coupling constant \( g_R \) were lifted to the \( Z_2 \)-odd field, as in the Gherghetta-Pomarol and Falkowski-Lalak-Pokorski (GPFLP) approach, then the \( g_R W^0_\mu \) part would become \( Z_2 \)-even and \( t_R \) would be diagonal so that \( t_R = i\sigma_3 \).) After the \( SU(2) \) gauge is fixed in the bulk by the condition \( \mathcal{A}^a_i \propto \delta^a_i \), the \( U(1)_R \) gauge transformation is modified into the combined (diagonal) \( U(1) \)
transformation of the original $U(1)_R$ and $SU(2)$; e.g., $\delta_{U(1)_R} (\theta) F^a_{i} = i \theta [Q^1 \sigma_1 + Q^2 \sigma_2, \ F]_{i}$. However, the $SU(2)$ symmetry is \textit{explicitly} broken by the $Z_2$ parity assignment down to $U(1)$ with the generator $\sigma_3$. This breaking is manifest only at the boundaries, since $SU(2)$ is a local symmetry, while the $Z_2$ parity transformation relates the fields at $y$ only with those at $-y$. Since the $U(1)$ transformation of the generator $Q^1 \sigma_1 + Q^2 \sigma_2$ in $SU(2)$ is broken, the $U(1)_R$ symmetry is \textit{explicitly} broken in this Altendorfer-Bagger-Nemeschansky approach.

The brane tension terms are given by

$$S_{\text{brane}} = \int d^5x \left( A_1 \delta (y) + A_2 \delta (y - \bar{y}) \right) \left[ \Sigma^3 = \Sigma^2 \right]_F.$$

(6.5)

Here $A_1$ and $A_2$ are assumed to be real for simplicity and the F-term action formula (C.3) reads, for our chiral compensator $\Sigma_c = \left( A_c = 1, \ \chi_c = 0, \ \mathcal{F}_c \right)$, as

$$\left[ \Sigma^2_c \right]_F = e_4 \left( 2 (\mathcal{F}_c + \mathcal{F}^a) - 2 \bar{\psi}_i \gamma^{\mu \nu} \psi_\nu \right),$$

(6.6)

where $e_4$ is the four dimensional determinant of the vierbein, $e_4 = e/e_y^4 = e \cdot e_y^b$. Note that now

$$\mathcal{D}_4 A^a_i = (\partial_4 - \frac{3}{2} b_4) A^a_i - V_{a ij} A^{a j} - \frac{W_4^0}{\alpha} F^a_i - 2 i \bar{\psi}_i \zeta^a$$

$$\to \mathcal{D}_4 A^{a=2}_{i 1} = - \frac{W_4^0}{\alpha} \bar{\mathcal{F}}^2_1 + (V_4)_1^2 - g_R W_4^0 (t_R)_1^2,$$

(6.7)

where we have used $A^a_i = \delta^a_i$ and $\zeta^a = 0$, by the superconformal gauge fixing (5.3), and $\bar{F}^a_i$ is defined by

$$\bar{F}^a_i \equiv F^a_i - \frac{i}{2} m a \eta^a_b A^b_i = \bar{F}^a_i - g_R \alpha (t_R)^a_b A^b_i.$$  

(6.8)

This field $\bar{F}^a_i$ vanishes in the absence of the brane term $A_1 = A_2 = 0$. Then we find the real part of the $F$-component $\mathcal{F}_c$ of $\Sigma_c$ in (5.2) as

$$\text{Re} \mathcal{F}_c = \text{Re} \left\{ i (1 + i \frac{W_4^0}{\alpha}) \bar{\mathcal{F}}^2_1 + i g_R \alpha (t_R)_1^2 + (V_4 - g_R W_4^0 t_R)_1^2 \right\}$$

$$= \bar{\mathcal{F}}^1 - \frac{W_4^0}{\alpha} \bar{\mathcal{F}}^2 - g_R \alpha Q^1 - \bar{V}_4^2.$$  

(6.9)

Here $Q^1$ is the first component of the direction vector $\bar{Q} \equiv (Q^1, Q^2, Q^3)$ of the $U(1)_R$ generator in $SU(2)$, $\mathcal{F}^1$ and $\mathcal{F}^2$ are the 1 and 2 components of the ‘quaternion’ $\mathcal{F}^a_i$ ($\mathcal{F}^a_i = \mathcal{F}^0 1_2 - i \mathcal{F}^1 \sigma_1 - i \mathcal{F}^2 \sigma_2 - i \mathcal{F}^3 \sigma_3$), and the quantities $\bar{V}_4^k$ are defined by

$$(\bar{V}_4)^i_j \equiv (V_4 - g_R W_4^0 t_R)^i_j = \sum_{k=1}^{3} i \bar{V}_4^k (\sigma_k)^i_j.$$  

(6.10)
Since the auxiliary fields $\tilde{F}^1$, $\tilde{F}^2$ and $\tilde{V}_4^2$ appear in the bulk action in the form
\[
e^{-2\left(1 + \frac{(W_0^0)^2 - W_0^0 W_0^0}{\alpha^2}\right)\left((\tilde{F}^1)^2 + (\tilde{F}^2)^2\right) + 2(\tilde{V}_4^2)^2}
\]
with opposite signs and $W_0^0$ vanishes on the brane ($\Pi(W_0^0) = -1$), the solution to these auxiliary field equations of motion are given by
\[
\begin{align*}
\tilde{F}^1 &= \left(1 + \frac{(W_0^0)^2}{\alpha^2}\right)^{-1} e_4^y(A_1 \delta(y) + A_2 \delta(y - \bar{y})), \\
\tilde{F}^2 &= \left(1 + \frac{(W_0^0)^2}{\alpha^2}\right)^{-1} (-\frac{W_0^0}{\alpha}) e_4^y(A_1 \delta(y) + A_2 \delta(y - \bar{y})), \\
\tilde{V}_4^2 &= e_4^y(A_1 \delta(y) + A_2 \delta(y - \bar{y})).
\end{align*}
\]
Elimination of these auxiliary fields by substituting these solutions back into the action (6.11) plus (6.12), could potentially yield singular terms in the form of the squares of delta functions. However, in fact, we see that the contributions from $\tilde{F}^1$ and $\tilde{F}^2$ and from $\tilde{V}_4^2$ exactly cancel each other.

After eliminating these auxiliary fields, the brane action becomes
\[
S_{\text{brane}} = \int d^5x \left(A_1 \delta(y) + A_2 \delta(y - \bar{y})\right) e_4 \left(-4g_R Q^1\alpha - 2\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu\right),
\]
(6.13)
The scalar $\alpha$ of the $U(1)_R$ gauge multiplet is nonvanishing. If the $U(1)_R$ gauging is done with $Q^1 = 1$, i.e., $t_R = i\sigma_1$, then this essentially reproduces the brane action given by Altendorfer, Bagger and Nemeschansky.\(^{[3]}\) The point here is, however, that the parameters $A_1$ and $A_2$ remain arbitrary and are not determined by the supersymmetry requirement at all. Therefore, despite the fact that the bulk cosmological constant is given by the parameter $g_R\alpha$, the brane tensions are $A_1$ or $A_2$ times $-4g_R\alpha$, and thus have no relation to the bulk cosmological constant $-(8/3)g_R^2\alpha^2 = -4g_R^2$.\(^{[1]}\) Zucker noted the same point in his off-shell Poincaré supergravity formulation based on a linear multiplet compensator.\(^{[2]}\)

Let us comment on the Killing spinor in the Randall-Sundrum background.\(^{[24]}\)
\[
ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2.
\]
(6.14)
The Killing spinor is found by demanding that the $Q$ and $S$ transformation $\delta = \delta_Q(\varepsilon) + \delta_S(\eta)$ of the gravitino $\psi^i_\mu$, $\psi^i_y$ and the fermion components $\Omega^{0i}$ of $V_0$ and $\zeta^a$ of $H^a$ vanish. The $S$-transformation parameter $\eta$ is given as a function of $\varepsilon$ by the condition $\delta \Omega^{0i} = 0$ (and then $\delta \zeta^a = 0$ is automatically satisfied). Assuming that the Killing spinor $\varepsilon(y)$ depends only on the extra dimension coordinate $y$, one can show that such a Killing spinor can exist only when $A_1 = -A_2 = 2$, $Q^1 = 1$ ($Q^2 = Q^3 = 0$) and $k = 2\alpha g_R/3$. This implies that the
brane tension $\pm \tau$ of the two boundary planes should be $\pm \tau \equiv \pm 4g_R \alpha Q^1 \Lambda_1 = \pm 12k$ while the bulk cosmological term is $-4g_R^2 = -6k^2$. However, this value of the brane tension is twice as large as the Randall-Sundrum value, $\pm \tau = \pm 6k^2$. Zucker also noted this fact in his formulation. Since the effective four-dimensional cosmological term vanishes only when the Randall-Sundrum case, this implies that there exists no Killing spinor, and therefore the (global) supersymmetry is *spontaneously broken* on the Randall-Sundrum background. Note that this conclusion is in the framework of Altendorfer-Bagger-Nemeschansky approach. In fact, in the GPFLP approach whose off-shell formulation is given in Ref. [21], the same Randall-Sundrum background is shown to allow the existence of a Killing spinor.

**Acknowledgements**

The authors owe a lot to Tomoyuki Fujita, who collaborated with them at the early stages of this work. The authors would like to thank Tony Gherghetta, David E. Kaplan, Tatsuo Kobayashi, Nobuhiro Maekawa, Hiroaki Nakano and Stefan Vandoren for their encouragement and interest in this work. They also appreciate the Summer Institute 2001 held at Fuji-Yoshida, the discussions at which were valuable. T. K. is supported in part by a Grant-in-Aid for Scientific Research, No. 13640279, from the Japan Society for the Promotion of Science and a Grant-in-Aid for Scientific Research on Priority Areas, No. 12047214, from the Ministry of Education, Science, Sports and Culture, Japan.

**Appendix A**

--- **Conventions** ---

The gamma matrices $\gamma^a$ ($a = 0, 1, 2, \ldots, d - 1$) in $d$ dimensions satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $(\gamma_a)^\dagger = \eta^{ab} \gamma_b$, where $\eta^{ab} = \text{diag}(+, -, -, \cdots)$. Here $\gamma^{a \cdots b}$ is the antisymmetrized product of gamma matrices

$$
\gamma^{a \cdots b} = \gamma^{[a \cdots b]}, \tag{A.1}
$$

where the square brackets $[\cdots]$ represent complete antisymmetrization of the indices with weight 1. Similarly, $(\cdots)$ represents complete symmetrization with weight 1.

In five dimensions, we choose the Dirac matrices to satisfy

$$
\gamma^{a_1 \cdots a_5} = \epsilon^{a_1 \cdots a_5}, \tag{A.2}
$$

where $\epsilon^{a_1 \cdots a_5}$ is a totally antisymmetric tensor with $\epsilon^{01234} = 1$. 

25
The SU(2) index \( i (i=1,2) \) is raised and lowered with the antisymmetric \( \epsilon_{ij} \) tensor \( (\epsilon_{12} = \epsilon^{12} = 1) \) according to the northwest-to-southeast (NW-SE) contraction convention:

\[
A^i = \epsilon^{ij} A_j, \quad A_i = A^j \epsilon_{ji}. \tag{A.3}
\]

The charge conjugation matrix \( C_5 \) in 5D has the properties

\[
C_5^T = -C_5, \quad C_5^T C_5 = 1, \quad C_5 \gamma_a C_5^{-1} = \gamma^a. \tag{A.4}
\]

Our five-dimensional spinors \( \psi^i \) satisfy the SU(2)-Majorana condition

\[
\bar{\psi}^i \equiv (\psi_i)^\dagger \gamma^0 = \psi^T C_5 \tag{A.5}
\]

where spinor indices are omitted. When the SU(2) indices are suppressed in bilinear terms of the spinors, the NW-SE contraction is understood, e.g. \( \bar{\psi} \gamma^{a_1 \cdots a_n} \lambda = \bar{\psi}^i \gamma^{a_1 \cdots a_n} \lambda_i \).

In four dimensions, the Dirac matrices satisfy

\[
\gamma^{a_1 \cdots a_4} = -i \epsilon^{a_1 \cdots a_4} \gamma_5, \quad \epsilon^{0123} = 1, \tag{A.6}
\]

with the chirality matrix \( \gamma_5 \). The fifth Dirac matrix in 5D, \( \gamma^4 \), is anti-hermitian and related with \( \gamma_5 \) as \( \gamma^4 = i \gamma_5 \). The Majorana-condition is defined by

\[
\bar{\psi} \equiv \psi^\dagger \gamma^0 = \psi^T C_4 \tag{A.7}
\]

where the charge conjugation matrix \( C_4 \) in 4D has the properties

\[
C_4^T = -C_4, \quad C_4^T C_4 = 1, \quad C_4 \gamma_a C_4^{-1} = -\gamma^a. \tag{A.8}
\]

In the main text, we take as our convention the relation \( C_5 = -C_4 \gamma_5 \) between the charge conjugation matrices in 5D and 4D.

**Appendix B**

----- Curvatures in 4D and 5D ----- 

The curvatures \( \hat{R}_{\mu \nu}^A \) are defined by \( \hat{R}_{ab}^A X_A \equiv -[\hat{D}_a, \hat{D}_b] \) in terms of the full superconformal covariant derivative \( \hat{D}_a \equiv \partial_a - \sum_{A \neq P} h_{a}^{A} X_{A} \) and are written using the structure function in the form

\[
\hat{R}_{\mu \nu}^A = e_{\mu}^{b} e_{\nu}^{a} f_{ab}^{A} = 2 \partial_{[\mu} h_{\nu]}^{A} + h_{\nu}^{C} h_{\mu}^{B} f_{B C}^{A}. \tag{B.1}
\]

Here \( X_A \) and \( h_{\mu}^{A} \) denote the transformation operators and the corresponding gauge fields, respectively, whose explicit contents in 5D and 4D are listed in Table [V]. The quantity \( f_{AB}^{C} \)
is a ‘structure function’, defined by $[X_A, X_B] = f_{AB} C X_C$, which generally depends on the fields. The primed $f'_{BC} A$ is zero when $B = P_b$ and $C = P_c$, and otherwise $f'_{BC} A = f_{BC} A$.

The explicit expression of (B.1) for the curvature $\hat{R}_{\mu\nu} A$ can most easily be obtained from the gauge field transformation law $\delta(\varepsilon) h^A_\mu$ of the same generator $X_A$:

$$\varepsilon^B X_B h^A_\mu \equiv \delta(\varepsilon) h^A_\mu = \partial_{\mu} \varepsilon^A + \varepsilon^C h^A_\mu f'_{BC} A;$$

that is, we can obtain $\hat{R}_{\mu\nu} A$ by simply replacing $\varepsilon^C$ by $h^C_\mu$ in $\delta(\varepsilon) h^A_\mu$.

The explicit forms of the curvatures $\hat{R}_{\mu\nu} A$ in 4D are given by [4]

$$\hat{R}_{\mu\nu}^a(P) = 2\partial_{[\mu} e^a_{\nu]} - 2\omega^{\alpha\beta}_{[\mu} e^\alpha_{\nu]} + 2b_{[\mu} e^a_{\nu]} + 2i\tilde{\psi}_{[\mu} \gamma^a \psi_{\nu]},
\hat{R}_{\mu\nu}^{ab}(M) = 2\partial_{[\mu} \omega^{\alpha\beta}_{\nu]} - 2\omega^{\alpha\beta}_{[\mu} \omega^\gamma_{\nu]} + 8f_{[\mu} ^a e^b_{\nu]} - 4\tilde{\psi}_{[\mu} \gamma^a \phi_{\nu]} + 4i\tilde{\psi}_{[\mu} \gamma_5 \psi_{\nu]} \hat{R}_{\mu\nu}^{ab}(Q),
\hat{R}_{\mu\nu}(D) = 2\partial_{[\mu} b_{\nu]} + 4f_{[\mu} ^a e^b_{\nu]} + 4\tilde{\psi}_{[\mu} \phi_{\nu]},
\hat{R}_{\mu\nu}(A) = 2\partial_{[\mu} A_{\nu]} - 8i\tilde{\psi}_{[\mu} \gamma_5 \phi_{\nu]},
\hat{R}_{\mu\nu}(K) = 2\partial_{[\mu} f^a_{\nu]} - 2\omega^{\alpha\beta}_{[\mu} f^a_{\nu]} - 2b_{[\mu} f^a_{\nu]} + 2i\tilde{\psi}_{[\mu} \gamma^a \phi_{\nu]} + 2i\tilde{\psi}_{[\mu} \gamma_5 \psi_{\nu]} \hat{D}_b \hat{R}_{\mu\nu}^{ab}(Q),
\hat{R}_{\mu\nu}(Q) = 2\partial_{[\mu} \psi_{\nu]} - \frac{1}{2} \omega^{\alpha\beta\gamma\delta}_{[\mu} \psi_{\nu]} + b_{[\mu} \psi_{\nu]} - \frac{3}{2} i A_{[\mu} \gamma_5 \psi_{\nu]} + 2i\tilde{\psi}_{[\mu} \gamma_5 \psi_{\nu]} \hat{D}_b \hat{R}_{\mu\nu}^{ab}(Q),
\hat{R}_{\mu\nu}(S) = 2\partial_{[\mu} \phi_{\nu]} - \frac{1}{2} \omega^{\alpha\beta\gamma\delta}_{[\mu} \phi_{\nu]} - b_{[\mu} \phi_{\nu]} + \frac{3}{2} i A_{[\mu} \gamma_5 \phi_{\nu]} + 2i f_{[\mu} \gamma a \phi_{\nu]} + \frac{i}{2} \gamma^a \hat{R}_{[\mu a} (A) + i \gamma_5 \hat{R}_{[\mu a} (A) \psi_{\nu]}.\quad\text{(B.3)}$$

With the help of Bianchi identities $[\hat{D}_{[a}, [\hat{D}_{b}, \hat{D}_{c]}] = 0$, one can show that the constraints (3.2) imply the useful equalities [3]

$$\hat{R}_{ab}(D) = -\hat{R}_{[ab]}(M) = -\frac{1}{2} \hat{R}_{ab}(A),
\hat{R}_{ab}(Q) \equiv \frac{1}{2} \epsilon_{abcd} \hat{R}^{cd}(Q) = -i \gamma_5 \hat{R}_{ab}(Q),
\hat{D}_{[a} \hat{R}_{bc]}(Q) = -i \gamma_{[a} \hat{R}_{bc]}(S),
\hat{D}^b \hat{R}_{ab}(Q) = -i \gamma^b \hat{R}_{ab}(S), \quad \gamma \cdot \hat{R}(S) = 0,
i \gamma_5 \hat{R}_{ab}(S) \equiv \frac{i}{2} \epsilon_{abcd} \gamma_5 \hat{R}^{cd}(S) = \hat{R}_{ab}(S) - i \hat{\mathcal{D}} \hat{R}_{ab}(Q),\quad\text{(B.4)}$$
whose first component is given by a general function $g_{ab}(M) \equiv \hat{R}_{ab}(M)$ and $\tilde{g}_{ab} \equiv (1/2)\epsilon_{abcd}\hat{R}^{cd}$.

The curvatures $\hat{R}_{\mu\nu}^A$ in 5D are given explicitly by

$$
\hat{R}_{\mu\nu}^a(P) = 2\partial_{[\mu}e_{\nu]}^a - 2\omega_{[\mu}^a e_{\nu]}^b + 2b_{[\mu}e_{\nu]}^a + 2i\bar{\psi}_{[\mu}^{\gamma} a_{\nu]} \psi,
$$

$$
\hat{R}_{\mu\nu}^i(Q) = 2\partial_{[\mu}\bar{\psi}_{\nu]}^i - \frac{1}{2}\omega_{[\mu}^a \gamma_{\nu]} \bar{\psi}_{[\mu}^i + b_{[\mu}^a \bar{\psi}_{\nu]}^i - 2V_{[\mu}^i \bar{\psi}_{\nu]}^j + \gamma_{ab[\mu} \psi_{\nu]} v^{ab} - 2\gamma_{[\mu} \phi_{\nu]}^i,
$$

$$
\hat{R}_{\mu\nu}^{ab}(M) = 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^a \omega_{\nu]}^{b} - 4i\bar{\psi}_{[\mu}^{\gamma} \phi_{\nu]} + 2i\bar{\psi}_{[\mu}^{\gamma} \omega_{\nu]} v_{cd} + 4i\bar{\psi}_{[\mu}^{\gamma} a_{\nu]} \psi_{\nu]} v_{cd},
$$

$$
\hat{R}_{\mu\nu}(D) = 2\partial_{[\mu}b_{\nu]} + 4i\bar{\psi}_{[\mu} \phi_{\nu]} + 4f_{[\mu\nu]},
$$

$$
\hat{R}_{ij}(U) = 2\partial_{[i}V_{j]} - 2V_{[i}^j V_{j]} + 12i\bar{\psi}_{[i}^j \phi_{j]} - 4i\bar{\psi}_{[i}^{\gamma} \gamma_{j]} \psi_{a} + \frac{1}{2}\bar{\psi}_{[i}^{j} \gamma_{\nu]} \chi_{j]},
$$

$$
\hat{R}_{\mu\nu}(S) = 2\partial_{[\mu}\phi_{\nu]} - \frac{1}{2}\omega_{[\mu}^a \gamma_{\nu]} \phi_{a} + b_{[\mu}^a \phi_{\nu]} - 2V_{[\mu}^i \phi_{\nu]}^j + 2f_{[\mu}^a \gamma_{\nu]} \phi_{a} + \cdots,
$$

$$
\hat{R}_{\mu\nu}(K) = 2\partial_{[\mu}f_{\nu]} - 2\omega_{[\mu}^a f_{\nu]} + 2b_{[\mu}f_{\nu]} + 2i\bar{\psi}_{[\mu}^{\gamma} a_{\nu]} \phi_{a} + \cdots. \quad (B.5)
$$

Here, the dots in the $S^i$ and $K^a$ curvature expressions denote terms containing the other curvatures. The constraints (2.2) in 5D also imply the equalities,

$$
\hat{R}_{ab}(D) = -\frac{2}{3}\hat{R}_{ab}(M) = 0, \quad \hat{R}_{afe}^a(M) = \hat{R}_{afe}^a(M) = 0,
$$

$$
\hat{R}_{ab}^i(S) = \hat{\Phi} \hat{R}_{ab}^i(Q) + \gamma_{ab} \hat{D}^i Q_{bc} + \hat{R}_{ab}^i(Q) \bar{Q}_{bc} + \frac{1}{4} \bar{Q} \gamma_{ab} \hat{D}_c Q_{bc} v_{cd},
$$

$$
\hat{R}_{ab}^c(K) = \frac{1}{4} \hat{D}_d \hat{R}_{ab}^c d(M) + \frac{1}{2} \hat{R}_{da}^c Q_{bc} + \frac{1}{2} \hat{R}_{da}^c Q_{bc} + \frac{1}{2} \hat{R}_{da}^c Q_{bc} + \hat{R}_{ab}^d(Q), \quad (B.6)
$$

and the $S^i$ and $K^a$ curvatures can be written in terms of the other curvatures.

### Appendix C

—— Embedding and Invariant Action Formulas in 4D ——

A product of chiral multiplets also forms a chiral multiplet. More generally, for an arbitrary set of chiral multiplets $\Sigma^I = [\mathcal{A}^I, \mathcal{P}_X^I, \mathcal{F}^I]$, we can have a new chiral multiplet $g(\Sigma)$, whose first component is given by a general function $g(\mathcal{A})$ of $\{\mathcal{A}^I\}$ carrying a homogeneous degree in the Weyl weight.

$$
g(\Sigma) = [g(\mathcal{A}), \mathcal{P}_X^I g_I(\mathcal{A}), \mathcal{F}^I g_I(\mathcal{A}) - \frac{i}{4} \chi^I \mathcal{P}_X^I g_{IJ}(\mathcal{A})], \quad (C.1)
$$

where $g_I(\mathcal{A}) \equiv \partial g(\mathcal{A})/\partial \mathcal{A}^I$ and $g_{IJ}(\mathcal{A}) \equiv \partial^2 g(\mathcal{A})/\partial \mathcal{A}^I \partial \mathcal{A}^J$.

Similarly, for an arbitrary set of general multiplets $\Phi^I = [C^I, \zeta^I, \cdots]$ and an arbitrary function $f(C)$ homogeneous in the Weyl weight, we can have a new general multiplet $\Phi' = \cdots$.
where $R$ is the invariant $D\Phi$ with $f$ divided that the spinor conjugate $\bar{\psi}$ covariant derivative with respect to the homogeneous transformations following superconformal-invariant $F(\Phi)$(\ref{eq:superconformal_action}) whose components are given by\cite{1}

\[
C' = f(C), \quad \zeta' = \zeta f, \\
H' = H f - \frac{1}{4}\bar{\xi}^i\xi^j f_{ij}, \quad K' = K f + \frac{1}{4}\bar{\xi}^i\bar{\gamma}_5\zeta J f_{ij}, \\
B'_a = B'_a f + \frac{1}{4}\bar{\xi}^i\gamma^a\gamma^b f_{ij}, \\
\lambda' = \lambda f - \frac{1}{2}i\gamma_5\left(i\gamma_5 H + K + iB + \tilde{\Phi} C\gamma_5\right)^J \zeta f_{ij} - \frac{1}{4}\bar{\xi}^i\bar{\xi}^j\zeta^k f_{ijk}, \\
D' = D f + \frac{1}{2}(H^I H^J + K^I K^J + B^i B^j + \bar{\Phi} a C^I \bar{C}^a C^J + i\bar{\xi}^i\tilde{\Phi} \zeta^j - 2\bar{\xi}^1 \lambda^j) f_{ij} \\
+ \frac{1}{4}\bar{\xi}^i\bar{\gamma}_5(i\gamma_5 H + K + iB)\zeta^k f_{ijk} + \frac{1}{16}\bar{\xi}^i\bar{\xi}^j\bar{\xi}^k\zeta^l f_{ijkl},
\]

(C.2)

with $f_i \equiv \partial f(C)/\partial C^i$, etc. This formula is also valid for complex general multiplets, provided that the spinor conjugate $\bar{\psi}$ is understood to be $\psi^T C_4$, not $\psi^T \gamma^0$.

For a chiral multiplet $\Sigma^{(w=3)} = [\mathcal{A}, \mathcal{P}_R X, \mathcal{F}]$ with weight $w = n = 3$, we have the following superconformal-invariant $F$-term action formula:\cite{1}

\[
I_F = \int d^4x \left[ \Sigma^{(w=3)} \right]_F = \int d^4x \left[ \mathcal{F} - i\bar{\psi} \cdot \gamma \mathcal{P}_R X - 2\bar{\psi} a \gamma^{ab} \mathcal{P}_L \psi_b \mathcal{A} + \text{h.c.} \right].
\]

(C.3)

For a real general multiplet $\Phi^{(w=2, n=0)} = [C, \zeta, H, K, B_a, \lambda, D]$ with weight $w = 2$, $n = 0$, the invariant $D$-term action formula is given by\cite{1}

\[
I_D = \int d^4x \left[ \Phi^{(w=2, n=0)} \right]_D \\
= \int d^4x \left[ D - \bar{\psi} \cdot \gamma_5 \lambda + i\epsilon^{abcd} \bar{\psi}_a \gamma_b \psi_c \left( B_d - \bar{\psi}_d \zeta \right) \\
+ \frac{1}{3}(R(\omega) + 4i\bar{\psi}_\mu \gamma^{\mu \lambda} D_\nu \psi_\lambda) C + \frac{2}{3}i\bar{\xi}^i \bar{\gamma}_5 \bar{\gamma}^{\mu \nu} D_\mu \psi_\nu \right],
\]

(C.4)

where $R(\omega)$ is the scalar curvature constructed from the spin connection $\omega^{ab}_\mu$, and $D_\mu$ is the covariant derivative with respect to the homogeneous transformations $M_{ab}$, $D$ and $A$.

References

[1] P. Hořava and E. Witten, Nucl. Phys. B460 (1996), 506, hep-th/9510209, Nucl. Phys. B475 (1996), 94, hep-th/9603142.
[2] I. Antoniadis, Phys. Lett. 246B (1990), 377.
J. D. Lykken, Phys. Rev. D54 (1996), 3693.
N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. 429B (1998), 263.
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. 436B (1998), 257.
[3] M. Zucker, Nucl. Phys. B570 (2000), 267, hep-th/9907082; JHEP 0008 (2000) 016, hep-th/9909144.
[4] T. Kugo, K. Ohashi, Prog. Theor. Phys. 104 (2000), 835, hep-ph/0006231.
[5] T. Kugo and K. Ohashi, Prog. Theor. Phys. 105 (2001), 323, hep-ph/0010288.
[6] E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, JHEP 0106 (2001), 051, hep-th/0104113.
[7] T. Fujita and K. Ohashi, Prog. Theor. Phys. 106 (2001), 221, hep-th/0104130.
[8] E. A. Mirabelli and M. E. Peskin, Phys. Rev. D58 (1998), 065002, hep-th/9712214.
[9] M. Kaku, P. K. Townsend and P. Van Nieuwenhuizen, Phys. Rev. Lett. 39 (1977) 1109; Phys. Lett. B 69 (1977) 304; Phys. Rev. D 17 (1978) 3179; ibid 19 (1979) 3166; ibid 19 (1979) 3592.
M. Kaku and P. K. Townsend, Phys. Lett. 76B (1978), 54.
M. Kaku, P. K. Townsend and P. van Niewenhuizen, Phys. Rev. D19 (1979), 3166, 3592.
[10] S. Ferrara, M. T. Grisaru and P. van Nieuwenhuizen, Nucl. Phys. B 138 (1978), 430.
[11] T. Kugo and S. Uehara, Nucl. Phys. B 226 (1983), 49.
[12] T. Kugo and S. Uehara, Prog. Theor. Phys. 73 (1985), 235.
[13] P. van Niewenhuizen, Phys. Rep. 68 (1981), 189.
[14] A. Van Proeyen, Lecture in the Proceedings of the Winter School in Karpacz, 1983, ed. B. Milewski (World Scientific Pub. Co.).
[15] M. Zucker, Phys. Rev. D 64 (2001), 024024, hep-th/0009083.
[16] M.F. Sohnius and P.C. West, Nucl. Phys. B216 (1983), 100.
[17] K.S. Stelle and P.C. West, Phys. Lett. 74B (1978), 330.
S. Ferrara and P. van Niewenhuizen, Phys. Lett. 74B (1978), 333.
[18] E. Cremmer, “Supergravities in 5 dimensions,” in Superspace and supergravity, eds. S. Hawking and M. M. Roˇ cek, Cambridge University Press, 1980.
A. H. Chamseddine and H. Nicolai, Phys.Lett. B96 (1980), 89.
M. Günaydin, G. Sierra and P. K. Townsend, Nucl.Phys. B242 (1984), 244; Nucl.Phys. B253 (1985), 573.
M. Günaydin and M. Zagermann, Nucl.Phys. B572 (2000), 131, hep-th/9912027.
A. Ceresole and G. Dall’Agata, Nucl. Phys. B 585 (2000), 143, hep-th/0004111.
[19] T. Kugo and S. Uehara, Nucl. Phys. B222 (1983), 125.
[20] E. Bergshoeff, R. Kallosh and A. Van Proeyen, JHEP 0010 (2000) 033, hep-th/0007044.
[21] T. Fujita, T. Kugo and K. Ohashi, Prog. Theor. Phys. 106 (2001), 671, hep-th/0104130.
[22] B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985), 569.
[23] R. Altendorfer, J. Bagger and D. Nemeschansky, Phys. Rev. D63 (2001), 125025,
[24] L. Randall and R. Sundrum, Phys.Rev.Lett. 83 (1999) 3370, hep-ph/9905221; Phys. Rev. Lett. 83 (1999), 4690, hep-th/9906064.

[25] M. Günyaydin, G. Sierra and P.K. Townsend, in Ref. 18.

[26] T. Gherghetta and A. Pomarol, Nucl. Phys. B586 (2000), 141, hep-ph/0003128.

[27] A. Falkowski, Z. Lalak and S. Pokorski, Phys. Lett. B491 (2000), 172, hep-th/0004093; Phys. Lett. B 509 (2001), 337, hep-th/0009167; Nucl. Phys. B 613 (2001), 189, hep-th/0102145.