Bose-Einstein Condensation beyond the Gross-Pitaevskii Regime

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Introduction

- consider $N$ non-interacting bosons in $\Lambda_L = [0; L]^3$ with Hamiltonian

$$H_{N,L}^{\text{free}} = \sum_{j=1}^{N} -\Delta_{x_j}$$

with periodic boundary conditions, acting in $L^2_s(\Lambda_L^N) = \bigotimes_{\text{sym}}^N L^2(\Lambda_L)$

- the spectrum of $H_{N,L}^{\text{free}}$ consists of sums $\sum_{p \in 2\pi \mathbb{Z}^3/L} n_p |p|^2$ for $n_p \in \mathbb{N}$ and the eigenbasis $\varphi_{p_1} \otimes \cdots \otimes \varphi_{p_N}$ consists of product states built up from $\{x \mapsto \varphi_p(x) = L^{-3/2} e^{ipx}, p \in 2\pi \mathbb{Z}^3/L\}$

- the unique, normalized ground state vector $\psi_N$ with energy $E_N = 0$ is

$$\psi_N = \varphi_0^\otimes N, \quad \varphi_0 = L^{-3/2} \in L^2(\Lambda_L)$$

- in particular, if we measure $A = B \otimes 1 \otimes \cdots \otimes 1$, for $B = B^* \in \mathcal{L}(L^2_s(\Lambda_L^k))$ and some fixed $1 \leq k \leq N$ in the ground state $\psi_N$, then

$$\langle \psi_N, A\psi_N \rangle = \langle \varphi_0^\otimes k, B\varphi_0^\otimes k \rangle$$

- expectations of observables are therefore determined by $\varphi_0 \in L^2(\Lambda)$: we say the system exhibits Bose-Einstein condensation (BEC)
BEC has been verified experimentally in 1995, leading to the Nobel Prize in Physics for Eric Cornell, Carl Wieman and Wolfgang Ketterle.

In typical experiments, dilute Bose gases are cooled down to very low temperatures and one measures the velocity distribution; although such systems are dilute, the particles are of course still interacting.

An important problem in mathematical physics is therefore to understand the spectrum and appearance of BEC for interacting models with

\[
H_{N,L} = \sum_{j=1}^{N} -\Delta x_j + \sum_{1 \leq i < j \leq N} V(x_i - x_j),
\]

where \( V \geq 0 \) is assumed to be a repulsive short-range potential.

Of particular interest is to understand this in the thermodynamic limit in which the density \( \rho = N/L^3 \) is fixed and the number of particles \( N \to \infty \).
if we view the interaction as a perturbation, we may naively expect that

\[
\frac{E_N}{N} \approx \frac{\langle \varphi_0 \otimes ^N, H_N, \varphi_0 \otimes ^N \rangle}{N} \approx \frac{1}{2} \rho \int \Lambda dx \ V(x)
\]

due to important particle correlations this turns out to be wrong! To take correlations into account we introduce the scattering length of \( V \) which is defined by the zero-energy scattering equation

\[
\left[ -\Delta + \frac{1}{2} V(x) \right] f(x) = 0, \quad \lim_{|x| \to \infty} f(x) = 1
\]

for large enough \(|x|\), \( f \) is given by

\[
f(x) = 1 - \frac{a_0}{|x|}, \quad a_0 = \text{scattering length of } V
\]

in particular, we have for \( V \neq 0 \) that

\[
8\pi a_0 = \int \Lambda dx \ V(x)f(x) < \int \Lambda dx \ V(x)
\]
heuristically, the gas consists of $N(N-1)/2$ pairs and for $\rho \ll 1$, the ground state energy should be equal to $N(N-1)/2$ times that of two particles in a large box $L \gg 1$

by the scattering equation, the ground state energy of two particles in a large box $L \gg 1$ is approximately given by

$$L^{-3} \int_{|x| \leq L} dx \left( 2|\nabla f(x)|^2 + V(x)|f(x)|^2 \right) \approx 8\pi a_0 L^{-3}$$

so that we expect that the ground state energy of the $N$-body system is

$$\frac{E_N}{N} \approx 4\pi a_0 \rho$$

this has been proved in [Dyson, '57], [Lieb, Yngvason '98], showing that

$$\lim_{N \to \infty, \rho = N/L^3 = \text{const.}} \frac{E_N}{N} = 4\pi a_0 \rho \left( 1 - C(\rho a_0^3)^{1/17} \right)$$
given $\psi_N \in L^2_s(\Lambda_L^N)$, define its one-particle reduced density matrix by
\[ \gamma^{(1)}_N = \text{tr}_{2,\ldots,N} |\psi_N\rangle\langle\psi_N|, \]
then $0 \leq \gamma^{(1)}_N \leq 1$ with $\text{tr} \gamma^{(1)}_N = 1$ and
\[ \gamma^{(1)}_N(x; y) = \int_{\Lambda_{L-1}^N} dx_2 \ldots dx_N \psi_N(x; x_2, \ldots, x_N) \overline{\psi}_N(y; x_2, \ldots, x_N) \]

note that for $A = B \otimes 1 \otimes \cdots \otimes 1$ with $B = B^* \in L(L^2(\Lambda_L))$, we have
\[ \langle \psi_N, A\psi_N \rangle = \text{tr}(\gamma^{(1)}_N B) \]

we say that $(\psi_N)_{N \in \mathbb{N}}$ exhibits complete BEC into $\varphi \in L^2(\Lambda_L)$ if
\[ \lim_{N \to \infty} \langle \varphi, \gamma^{(1)}_N \varphi \rangle = 1 \iff \lim_{N \to \infty} \text{tr} |\gamma^{(1)}_N - |\varphi\rangle\langle\varphi|| = 0. \]

proving that the ground state $\psi_N$ exhibits BEC is a fundamental and long-standing open problem in mathematical physics
since a rigorous proof of BEC in the thermodynamic limit is out of reach, it is useful to study **ULTRA-DILUTE REGIMES** where \( \rho \to 0 \) as \( N \to \infty \)

to this end, we set \( L = N^{1-\kappa} \) for some \( \kappa > 0 \) and we study w.l.o.g. the rescaled system on \( \Lambda = [0; 1]^3 \) with Hamiltonian

\[
H_N = \sum_{j=1}^{N} -\Delta x_j + \sum_{1 \leq i < j \leq N} N^{2-2\kappa} V(N^{1-\kappa}(x_i - x_j)),
\]

acting in \( L^2_s(\Lambda^N) \) (note that \( H_N \sim N^{2-2\kappa} H_{N,L} \) are unitarily equivalent)

in this picture, \( N^{2-2\kappa} V(N^{1-\kappa}. \) represents a very singular short range potential and, by scaling, its scattering length is given by \( a_0 / N^{1-\kappa} \)

the thermodynamic limit corresponds to the choice \( \kappa = 2/3 \)

\( \kappa = 0 \) corresponds to the well-known **GROSS-PITAEVSKII REGIME**; in this regime, the kinetic and potential energies are of the same order \( \mathcal{O}(N) \) and one may expect a non-trivial theory in the limit \( N \to \infty \)

this talk deals with results slightly beyond the Gross-Pitaevskii regime, \( 0 < \kappa < 1/43 \)
Main Results

- to formulate our main results, the following observation is useful: given \( \psi_N \in L^2_s(\Lambda^N) \), we can decompose it into a sum
  \[
  \psi_N = \alpha_0 \varphi_0 \otimes N + \alpha_1 \otimes_s \varphi_0 \otimes N^{-1} + \ldots + \alpha_N
  \]
  for a sequence \( \alpha_j \in \bigotimes_j^{\text{sym}} L^2_{\perp}(\Lambda) \), where \( L^2_{\perp}(\Lambda) = \{ \varphi_0 \}^{\perp} \)
- the map \( \psi_N \mapsto U_N \psi_N = (\alpha_0, \ldots, \alpha_N) \) is unitary and maps \( L^2_s(\Lambda^N) \) into the truncated excitation Fock space
  \[
  \mathcal{F}^\leq_N = \bigoplus_{j=0}^N \bigotimes_j^{\text{sym}} L^2_{\perp}(\Lambda)
  \]
- on \( \mathcal{F}^\leq_N \), we introduce the number of particles operator \( \mathcal{N}_+ \), defined by
  \[
  (\mathcal{N}_+ \xi)^{(n)} = n \xi^{(n)} \quad \text{for every } \xi = (\xi^{(0)}, \ldots, \xi^{(N)}) \in \mathcal{F}^\leq_N ; \text{ we observe}
  \]
  \[
  1 - \langle \varphi_0, \gamma^{(1)}_N \varphi_0 \rangle = N^{-1} \langle \psi_N, \sum_{j=1}^N (1 - |\varphi_0 \rangle \langle \varphi_0 |)_{x_j} \psi_N \rangle = N^{-1} \langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle,
  \]
  so that \( \psi_N \) exhibits complete BEC into \( \varphi_0 \) if and only if
  \[
  \lim_{N \to \infty} N^{-1} \langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle = 0
  \]
Theorem (Adhikari, B., Schlein ’20)

Let $\kappa \in (0; 1/43)$, then the ground state energy $E_N$ of $H_N$ satisfies

$$|E_N - 4\pi a_0 N^{1+\kappa}| \leq CN^{1-\delta}$$

for some constant $C > 0$ and some (small) $\delta > 0$.

Moreover, if $\psi_N \in L^2_s(\Lambda^N)$ is an approximate ground state vector such that

$$\langle \psi_N, (H_N - E_N)^2 \psi_N \rangle \leq \zeta^2$$

for some $\zeta > 0$, then there exists $C = C_\zeta > 0$ such that

$$\langle U_N \psi_N, N_+ U_N \psi_N \rangle \leq CN^{1-\delta}.$$}

Finally, if $\kappa \in (0; 1/44)$ and $\psi_N = \chi(H_N - E_N \leq \zeta)\psi_N$, then for every $k \in \mathbb{N}$, there exists $C = C_{k,\zeta}$ such that

$$\langle U_N \psi_N, N_+^k U_N \psi_N \rangle \leq CN^{k(1-\delta)}.$$
Previous and related works

- the first proof of BEC for the ground state vector (in the more general setting of trapped particles in $\mathbb{R}^3$) was given in [Lieb, Seiringer '02] in the Gross-Pitaevskii regime, $\kappa = 0$

- further results in GP regime obtained in [Lieb, Seiringer 06'], [Nam, Rougerie, Seiringer '16], [Boccato, B., Cenatiempo, Schlein '17, '18], [Nam, Napiórkowski, Ricaud, Triay '20], [Hainzl '20]

- the result of [Lieb, Seiringer '02] can be extended to $\kappa \in [0; 1/10)$

- recently, [Fournais '20] proves BEC up to $\kappa \in [0; 2/5)$, inspired by the works [Brietzke, Fournais, Solovej, '19], [Fournais, Solovej '19] which led to a proof of the Lee-Huang-Yang formula

$$\lim_{N \to \infty, \rho = N/L^3 = \text{const.}} \frac{E_N}{N} = 4\pi a_0 \rho \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a_0^3} + o(\sqrt{\rho a_0^3})\right)$$

- most of these works estimate the ground state energy and prove BEC by providing suitable bounds on $\mathcal{N}_+$; controlling higher powers $\mathcal{N}_+^k$ for $k > 1$ is crucial for us to resolve the excitation spectrum of $H_N$
Theorem (B., Caporaletti, Schlein - in preparation)

Let $\kappa > 0$ be sufficiently small. The ground state energy $E_N$ of $H_N$ is given by

\[
E_N = 4\pi N^\kappa (N - 1) a_0 + e_\Lambda (a_0 N^\kappa)^2 - \frac{1}{2} \sum_{p \in \Lambda^*_+} \left[ p^2 + 8\pi a_0 N^\kappa - \sqrt{p^4 + 16\pi a_0 N^\kappa p^2 - \frac{(8\pi a_0 N^\kappa)^2}{2p^2}} \right] + o(1)
\]

where $e_\Lambda$ is given by the limit

\[
e_\Lambda = 2 - \lim_{M \to \infty} \sum_{p \in \Lambda^*_+: \ |p_1|, |p_2|, |p_3| \leq M} \frac{\cos(|p|)}{p^2}
\]

Moreover, the spectrum of $H_N - E_N$ below $\zeta N^{\kappa/2}$ consists of eigenvalues

\[
\sum_{p \in \Lambda^*_+} n_p \sqrt{|p|^4 + 16\pi a_0 N^\kappa |p|^2} + o(1 + \zeta^6)
\]

where $n_p \in \mathbb{N}$ for all $p \in \Lambda^*_+ = 2\pi \mathbb{Z}^3 \setminus \{0\}$. 
the results in [B., Caporaletti, Schlein '20] generalize [Boccato, B., Cenatiempo, Schlein '18], where the corresponding results have been obtained for the Gross-Pitaevskii regime, $\kappa = 0$

it is worth to mention that, by scaling, the **Bogoliubov correction**

$$
\frac{1}{2} \sum_{p \in \mathbb{A}_+^*} \left[ p^2 + 8\pi a_0 N^\kappa - \sqrt{p^4 + 16\pi a_0 N^\kappa p^2 - \left(8\pi a_0 N^\kappa\right)^2} \right] = \mathcal{O}(N^{5/2} \kappa)
$$

and this term corresponds to the Lee-Huang-Yang correction for $\kappa = 2/3$

the recent works [Fournais, Solovej '19], [Fournais '20] resolve the energy precisely up to this order and therefore prove BEC on length scales up to $\kappa \in [0; 2/5 + \varepsilon)$
Ideas from the Proof

Instead of analyzing $H_N$ directly, we analyze the unitarily equivalent operator $\mathcal{L}_N = U_N H_N U_N^*$ which acts in $\mathcal{F}_{+}^{\leq N}$.

$L_N$ can be expressed conveniently in terms of standard creation and annihilation operators $a_p, a_q^*$ for $p, q \in \Lambda^*_+$, which satisfy

$$[a_p, a_q^*] = \delta_{p,q}, \quad [a_p, a_q] = [a_p^*, a_q^*] = 0$$

$a_p$ is the adjoint of $a_p^*$ which acts on $\xi \in \mathcal{F}_{+}^{\leq N}$ as

$$(a_p^* \xi)^{(n)}(x_1, \ldots, x_n) \approx (e^{ipx} \otimes_s \xi^{(n-1)}(x_1, \ldots, x_n))$$

In particular, the number of particles operator $\mathcal{N}_+$ reads

$$\mathcal{N}_+ = \sum_{p \in \Lambda^*_+} a_p^* a_p$$

so that $a_p^* a_p$ counts the number of particles with momentum $p \in \Lambda^*_+$. 
the operator $\mathcal{L}_N = U_N H_N U_N^*$ takes the form

$$\mathcal{L}_N \approx \frac{N^{1+\kappa}}{2} \hat{V}(0) - \frac{N^\kappa \hat{V}(0)}{2N} \mathcal{N}_+^2 + \sum_{p \in \Lambda_+^*} |p|^2 a_p^* a_p$$

$$+ \frac{1}{2} \sum_{p \in \Lambda_+^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p} \right]$$

$$+ \frac{1}{\sqrt{N}} \sum_{p, q \in \Lambda_+^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right]$$

$$+ \frac{1}{2N} \sum_{p, q, r \in \Lambda_+^*} N^\kappa \hat{V}(r/N^{1-\kappa}) a_{p+r}^* a_q^* a_p a_q r$$

with modified operators $b_p^* = a_p^* \sqrt{1 - \mathcal{N}_+/N}$ (whose adjoint is $b_p$)

the goal is to prove a coercivity bound of the form

$$\mathcal{L}_N \geq 4\pi a_0 N^{1+\kappa} + c \mathcal{N}_+ + o(N)$$

for some positive constant $c > 0$
\begin{itemize}
  \item **Problem:** notice that the leading constant \( \frac{N^{1+\kappa} \hat{V}(0)}{2} \gg 4\pi a_0 N^{1+\kappa}! \)
  \item this means that missing energies are still hidden in the remaining (quadratic, cubic and quartic) contributions to \( \mathcal{L}_N \); to extract these energies, we must take into account PARTICLE CORRELATIONS!
  \item recall that \( \mathcal{L}_N \) expands \( H_N \) around energy of \( \varphi_0^\otimes N \); we would rather like to expand \( H_N \) around the energy of states which have roughly the form
    \[ \psi_N \approx \varphi_0^\otimes N \prod_{i<j} f(N^{1-\kappa}(x_i - x_j)), \]
    where \( f(N^{1-\kappa}. \) solves the scattering equation for \( N^{2-2\kappa} V(N^{1-\kappa}. \)
  \item for suitable \( (\eta_p)_{p \in \Lambda^*_+} \in \ell^2(\Lambda^*_+) \) (related to \( f(N^{1-\kappa}. \)), we model such states through **generalized Bogoliubov transformations**
    \[ T = \exp \left[ \frac{1}{2} \sum_{|p| \geq N^\alpha} \eta_p \left( b_p^* b_{-p}^* - b_p b_{-p} \right) \right] \]
\end{itemize}
**Action of** $T$: we have for

$$
\mathcal{K} = \sum_{p \in \Lambda_+^*} |p|^2 a_p^* a_p, \quad \mathcal{V}_N = \frac{1}{2N} \sum_{p,q,r \in \Lambda_+^*} N^\kappa \hat{V}(r/N^{1-\kappa}) a_{p+r}^* a_q^* a_p a_{q+r}
$$

that

$$
T^* \mathcal{K} T \approx \mathcal{K} + \sum_{|p| \geq N^\alpha} p^2 \eta_p^2 + \sum_{|p| \geq N^\alpha} p^2 \eta_p \left[ b_p^* b_{-p}^* + b_{-p} b_p \right]
$$

$$
T^* \mathcal{V}_N T \approx \mathcal{V}_N + \frac{1}{2N} \sum_{p,r \in \Lambda_+^*} N^\kappa \hat{V}(r/N^{1-\kappa}) \eta_{p+r} \eta_p
$$

$$
+ \frac{1}{2N} \sum_{p,r \in \Lambda_+^*} N^\kappa \hat{V}(r/N^{1-\kappa}) \eta_{r+p} \left[ b_p^* b_{-p}^* + b_p b_{-p} \right]
$$

**combine with**

$$
T^* \frac{1}{2} \sum_{p \in \Lambda_+^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_p^* b_{-p}^* + b_p b_{-p} \right] T
$$

$$
\approx \sum_{|p| \geq N^\alpha} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_p + \frac{1}{2} \sum_{p \in \Lambda_+^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_p^* b_{-p}^* + b_p b_{-p} \right]
$$
first of all, this **renormalizes** the constant term

\[
\frac{N^{1+\kappa}}{2} \hat{V}(0) + \sum_{|p| \geq N^\alpha} \left( p^2 \eta_p + N^\kappa \hat{V}(p/N^{1-\kappa}) + \frac{1}{2N} (\hat{V}(\cdot/N^{1-\kappa}) * \eta)(p) \right) \eta_p
\]

\[
\approx 4\pi a_0 N^{1+\kappa}
\]

at the same time, it regularizes the off-diagonal pairing term to

\[
\frac{1}{2} \sum_{p \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_p^* b_{-p} + b_p b_{-p} \right]
\]

\[
\rightarrow \sum_{|p| \leq N^\alpha} 4\pi a_0 N^\kappa \left[ b_p^* b_{-p} + b_p b_{-p} \right]
\]

with a new, renormalized potential \(8\pi a_0 N^\kappa \mathbf{1}(\cdot \leq N^\alpha)\)

notice in particular that this removes the singular ultra-violet behavior of \(N^\kappa V(\cdot/N^{1-\kappa})\) (as long as \(\alpha \ll 1 - \kappa\))
although we extract the leading order contribution $4\pi a_0 N^{1+\kappa}$, the
renormalized Hamiltonian is still not regular enough

the reason consists of the singular cubic and quartic contributions

$$\frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b^*_{p+q} a^*_p a_q + a^*_q a_{-p} b_{p+q} \right]$$

and

$$\frac{1}{2N} \sum_{p,q,r \in \Lambda_+^*} N^\kappa \hat{V}(r/N^{1-\kappa}) a^*_{p+r} a^*_q a_p a_{q+r}$$

in order to be able to prove the desired coercivity bound, we need to
renormalize the singular potential $N^\kappa \hat{V}(./N^{1-\kappa})$ to the regularized
potential $8\pi a_0 N^\kappa 1(|p| \leq N^\alpha)$ also in (parts of) these terms

as a result, we obtain, roughly speaking, a new Hamiltonian $\mathcal{M}_N$ of the
same form as $\mathcal{L}_N$, but with the new potential $8\pi a_0 N^\kappa 1(|p| \leq N^\alpha)$

for $\alpha \ll 1 - \kappa$, $\mathcal{M}_N$ can be bounded from below by standard arguments
Appendix

- the renormalized excitation Hamiltonian $G_N = T^* L_N T$ takes the form

$$G_N \approx 4\pi a_0 N^{1+\kappa} - \frac{N^\kappa \hat{V}(0)}{2N} N_+^2 + \sum_{p \in \Lambda^*_+} |p|^2 a_p^* a_p$$
$$+ \frac{1}{N} \sum_{p,q \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_p^* b_{-p} + a_{-p}^* a_p b_{p+q} + a_{-p}^* a_{p+q} b_{-p+q} \right]$$
$$+ \frac{1}{2N} \sum_{p,q,r \in \Lambda^*_+} N^\kappa \hat{V}(r/N^{1-\kappa}) a_p^* a_q^* a_p a_{q+r}$$

- although we extract the correct leading order contribution $4\pi a_0 N^{1+\kappa}$ to the ground state energy, the renormalized Hamiltonian is still not regular enough to prove the desired coercivity bound
one of the reasons is the cubic term which can be absorbed through $\mathcal{V}_N$
only up to an error of the size $CN^\kappa \mathcal{N}_+:

$$\frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda^*_N} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \geq -\delta \mathcal{V}_N - \delta^{-1} CN^\kappa \mathcal{N}_+$$

in fact, the cubic term still contains crucial contributions of order $N^\kappa \mathcal{N}_+$
which are important to resolve the second order correction to the ground state energy; hence, we can not expect to improve this lower bound

this has already been observed in [Boccato, B., Cenatiempo, Schlein, ’18] and we adapt their strategy to extract these contributions and renormalize the cubic term through a unitary map of the form

$$S = \exp \left[ \frac{1}{\sqrt{N}} \sum_{|p| \geq N^\alpha, |q| \leq N^\beta} \eta_p \left( b_{p+q}^* a_{-p}^* a_q - h.c. \right) \right]$$

for some $\beta < \alpha$
the renormalized excitation Hamiltonian $J_N = S^* G_N S$ takes the form

$$J_N \approx 4\pi a_0 N^{1+\kappa} - 4\pi a_0 N^\kappa N_+^2 / N + \sum_{p \in \Lambda_+^*} |p|^2 a_p^* a_p$$

$$+ \sum_{|p| \leq N^\alpha} 8\pi a_0 N^\kappa \left[ b_p^* b_p + \frac{1}{2} b_p^* b_{-p} + \frac{1}{2} b_p b_{-p} \right]$$

$$+ \frac{8\pi a_0 N^\kappa}{\sqrt{N}} \sum_{|p| \leq N^\alpha, |q| \leq N^\beta} \left[ b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right]$$

$$+ \frac{1}{2N} \sum_{p, q, r \in \Lambda_+^*} N^\kappa \hat{V}(r / N^{1-\kappa}) a_{p+r}^* a_q^* a_p a_q a_{q+r}$$

**Problem:** although the negative term $-4\pi a_0 N^\kappa N_+^2 / N$ and the cubic term can be expected to be $o(N)$, we can not conclude this a priori
naively, one might hope to renormalize the yet un-renormalized quartic term $\mathcal{V}_N$ which would yield a completely renormalized potential energy

$$U_N^*(\frac{1}{N} \sum_{1 \leq i < j \leq N} \nu_{\text{ren}}(x_i - x_j)) U_N$$

for $\nu_{\text{ren}}(r) = 8\pi a_0 N^{\kappa} 1(|r| \leq N^{\alpha})$

technically, this is clearly out of reach with our current estimates

instead, we can however try to renormalize $\mathcal{V}_N$ partially, leaving a non-negative contribution un-renormalized (that can simply be dropped in a lower bound)

QUESTION: which parts of $\mathcal{V}_N$ should we try to renormalize?
Observation: let us consider the cubic term in $\mathcal{J}_N$ and introduce effective field operators

$$
c_r^* = \frac{1}{\sqrt{N}} \sum_{v \in \Lambda^+_+ : v \neq -r, \ v \in P_L, v+r \in P^c_L} a^*_{v+r} a_v, \quad e_r^* = \frac{1}{2\sqrt{N}} \sum_{v \in \Lambda^+_+ : v \neq -r, \ v \in P_L, v+r \in P_L} a^*_{v+r} a_v
$$

for $P_L = \{ p : |p| \leq N^\beta \}$

in terms of these operators, the cubic contribution to $\mathcal{J}_N$ reads

$$
\frac{8\pi a_0 N^\kappa}{\sqrt{N}} \sum_{p \in P^c_H, q \in P_L: p+q \neq 0} \left[ b^*_{p+q} a^*_{-p} a_q + \text{h.c.} \right]
$$

$$
= 8\pi a_0 N^\kappa \sum_{p \in P^c_H} \left[ b^*_{-p} e_{-p} + e^*_{-p} b_{-p} + b^*_{-p} e^*_{p} + e_p b_{-p} + b^*_{-p} c_p^* + c_p b_{-p} \right]
$$

for $P_H = \{ p : |p| \geq N^\alpha \}$

this suggests to bound $\mathcal{J}_N$ from below by completing the square in the effective creation and annihilation operators

$$
g_p^* = b_p^* + c_p^* + e_p^*, \quad g_p = b_p + c_p + e_p
$$
to extract the missing contributions needed to complete the square, we renormalize $\mathcal{J}_N$ through a final, quartic conjugation $W$, defined by

\[
W = \exp \left( \frac{1}{2N} \sum_{r \in P_H, p, q \in P_L} \eta_r a_{p+r}^* a_{q-r}^* a_p a_q - h.c. \right)
= \exp \left( \frac{1}{2N} \sum_{r \in P_H} \eta_r c_r^* c_{-r}^* - h.c. \right)
\]

the renormalized Hamiltonian $\mathcal{M}_N = W^* \mathcal{J}_N W$ then takes the form

\[
\mathcal{M}_N \approx 4\pi a_0 N^{1+\kappa} + \sum_{p \in \Lambda^*_+} |p|^2 a_p^* a_p + 8\pi a_0 N^\kappa \sum_{p \in P^c_H} \left[ g_p^* g_p + \frac{1}{2} g_p^* g_{-p} + \frac{1}{2} g_{-p} g_p \right] + \frac{1}{2N} \sum_{r \in \Lambda^*, v, w \in \Lambda^*_+: (v, w) \in (P^2_L)^c \text{ and } (v+r, w-r) \in (P^2_L)^c} \hat{V}(r/N^{1-\kappa}) a_{v+r}^* a_{w-r}^* a_v a_w
\]
the quartic term is non-negative, by the pointwise positivity $V \geq 0$ and by the symmetry of the momentum restrictions

for the effective quadratic Hamiltonian, we complete the square

$$\sum_{p \in P^c_H} \left[ g_p^* g_p + \frac{1}{2} g_p^* g_{-p}^* + \frac{1}{2} g_{-p} g_p \right]$$

$$= \frac{1}{2} \sum_{p \in P^c_H} \left( g_p^* + g_{-p}^* \right) \left( g_p + g_{-p}^* \right) - \frac{1}{2} \sum_{p \in P^c_H} [g_p, g_p^*],$$

and conclude that

$$\mathcal{M}_N \gtrsim 4\pi a_0 N^{1+\kappa} + \mathcal{C} - 4\pi a_0 N^{\kappa+3\alpha}$$

choosing $\alpha > 0$ such that $3\alpha + \kappa < 1$, we conclude complete BEC for all approximate ground states that satisfy $\langle \xi_N, \mathcal{M}_N \xi_N \rangle \leq 4\pi a_0 N^{1+\kappa} + o(N)$