Abstract. Classification is at the heart of the scientific enterprise, from bacteria in biology to groups in mathematics. A central classification project in mathematical logic is motivated by Gödel’s incompleteness theorems. Indeed, logical systems are classified according to how hard it is to establish that no contradiction can be derived in these systems, yielding the Gödel hierarchy: a linear hierarchy claimed to encompass all natural/foundationally important systems. The medium range of this hierarchy is based on second-order arithmetic, a system with roots in Hilbert-Bernays’ Grundlagen der Mathematik.

The Gödel hierarchy exhibits remarkable robustness: enriching the language of the medium range to that of higher-order arithmetic does not change the picture, while switching to the inclusion ordering introduces only a few natural outliers, and also a parallel hierarchy for the medium range based on the axiom of determinacy from set theory. In this paper, we introduce numerous such parallel hierarchies for the inclusion-based and higher-order Gödel hierarchy, based on basic convergence theorems for nets. Among these higher-order hierarchies, we identify the Plato hierarchy which yields the (medium range of) the Gödel hierarchy under the canonical embedding of higher-order into second-order arithmetic. The Plato hierarchy can be defined in Hilbert-Bernays’ Grundlagen, while the erstwhile embedding preserves equivalences, translating e.g. an equivalence involving the monotone convergence theorem for nets to the well-known equivalence of arithmetical comprehension and the monotone convergence theorem for sequences.

1. Introduction

1.1. In a nutshell. An important and central aspect of mathematical logic is the study of hierarchies of logical systems. The Gödel hierarchy ([62]) is perhaps the most important such hierarchy, motivated by Gödel’s incompleteness theorems and going back to Hilbert’s second problem from his famous list of open problems ([22]). The aim of this paper is to exhibit numerous parallel hierarchies for the medium range of the Gödel hierarchy, based on convergence theorems for nets in basic spaces like Cantor space or the unit interval. Among those, we identify the Plato hierarchy which yields the (medium range of) the Gödel hierarchy under the canonical embedding of higher-order arithmetic into second-order arithmetic. The erstwhile embedding also preserves equivalences, in that it converts certain higher-order equivalences into the well-known equivalences from Reverse Mathematics (RM hereafter; see Section [24]). As an example, the equivalence involving the monotone convergence theorem for nets in Theorem [3.7] is converted to the well-known equivalence of
arithmetic comprehension and the monotone convergence theorem for sequences from [61, III.2]. We obtain many such examples in Section 3.5 including the Big Five of RM. We show in Remark 3.3 that, just like second-order arithmetic, the systems in the Plato hierarchy are definable in Hilbert-Bernays’ Grundlagen der Mathematik ([24][25]). We work with the Gödel hierarchy based on inclusion and involving higher-order arithmetic, as discussed in Section 1.2.

Now, as shown in [34], the axiom of determinacy from set theory, introduced in [41], gives rise to a hierarchy parallel to the medium range of the Gödel hierarchy, so one is led to wonder what is so special about the hierarchies in this paper. At the risk of impertinence, it is our opinion that convergence theorems for nets, being almost a century old, are simply more natural from the point of view of mainstream mathematics than the axiom of determinacy from set theory. Moreover, the former theorems do not involve formula classes, as opposed to the fragments of the axiom of determinacy in [34], which is a concrete/objective difference.

Lest the above be misinterpreted, we emphasise that the study of the axiom of determinacy in all its forms is an interesting and worthwhile topic in mathematical logic, in our opinion. However, we also believe that if one takes the qualifiers ‘natural’ and ‘mainstream’ seriously, then there is a fundamental/foundational difference between hierarchies based on axioms from set theory and hierarchies generated by theorems that originated outside of set theory. Moreover, the Plato hierarchy is ‘extra natural/special’ in that it gives rise to the medium range of the Gödel hierarchy under the canonical embedding of higher-order into second-order arithmetic. Equivalences involving the Plato hierarchy are also translated by this embedding to well-known equivalences involving the Big Five from RM, as shown in Section 3.5.

Finally, a brief introduction to the Gödel hierarchy may be found in Section 1.2 while we summarise our results in Section 1.3. As discussed in Section 2, we shall mainly study nets/Moore-Smith sequences with index sets that are subsets of Baire space \( \mathbb{N}^\mathbb{N} \). As such, the associated hierarchies are part of third-order arithmetic, i.e. only one step up from second-order arithmetic.

1.2. Hilbert, Gödel, and classification. During his invited lecture at the second International Congress of Mathematicians of 1900 in Paris, David Hilbert presented his famous list of 23 open problems ([22]) that would have a profound influence on modern mathematics. Hilbert’s list contains a number of foundational/logical problems. For instance, Problem 2 pertains to the consistency of mathematics, i.e. the fact that no contradiction can be proved in mathematics. Hilbert later elaborated on Problem 2 by formulating Hilbert’s program for the foundations of mathematics ([23]); this program calls for a proof of consistency of all of mathematics using only methods from so-called finitistic mathematics.

However, Gödel’s famous incompleteness theorems ([19]) are generally believed to show that Hilbert’s program is impossible: Gödel namely showed that any logical system rich enough to express arithmetic, cannot even prove its own consistency, let alone that of all of mathematics. Moreover, one can build stronger and stronger logical systems by consecutively appending the formula expressing the system’s consistency (or inconsistency). This proliferation of logical systems has not led to chaos, but to remarkable order and surprising regularity, as follows: as a positive
outcome of Gödel’s negative solution to Hilbert’s program, the notion of consistency gave rise to the Gödel hierarchy presented in Figure 1: a collection of logical systems linearly ordered via increasing consistency strength.

As to its import, the Gödel hierarchy is claimed to capture all systems that are natural or have an important foundational status. For instance, Simpson claims the following regarding the Gödel hierarchy and the associated consistency strength ordering ‘<’ in the Gödel Centennial volume:

It is striking that a great many foundational theories are linearly ordered by <. Of course it is possible to construct pairs of artificial theories which are incomparable under <. However, this is not the case for the “natural” or non-artificial theories which are usually regarded as significant in the foundations of mathematics. (62)

Burgess and Koellner corroborate Simpson’s claims in [9, §1.5] and [30, §1.1]; the former refers to the Gödel hierarchy as the Fundamental Series. Precursors to the Gödel hierarchy may be found in the work of Wang ([70]) and Bernays ([5, 7]). Friedman ([15]) has studied the linear nature of the Gödel hierarchy in great detail, including many more systems than present in Figure 1. The importance of the logical systems present in Figure 1 is discussed below the latter.

We now discuss the systems in Figure 1 and their role in mathematics and computer science. In this light, the Gödel hierarchy becomes a central object of study in logic to which all sub-fields contribute.

(i) Bounded arithmetic provides a logical framework for the study of polynomial time computation, and hence the famous ‘P versus NP’ problem ([10 Chapter I and II]).
(ii) The system $\text{RCA}_0$ is the ‘base theory’ of Reverse Mathematics (See Section 2.1) in which computable mathematics can be formalised ([60, 61, 64]).

(iii) The system $\text{WKL}_0$ provides a partial realisation of Hilbert’s program ([59, 62]). The ‘finitistic’ mathematics as in this program, is shown by Tait to be captured by the system $\text{PRA}$ ([65]).

(iv) The system $\text{ATR}_0$ is the upper limit of Weyl-Feferman predicative mathematics ([65, 62]).

(v) The system $\text{Z}_2$, called second-order arithmetic, originates from the logical system $H$ used by Hilbert-Bernays in Grundlagen der Mathematik ([24, 25]).

(vi) The system $\text{ZFC}$ is Zermelo-Fraenkel set theory with the axiom of choice, i.e. the standard/typical foundations of mathematics ([28]).

(vii) The systems above $\text{ZFC}$ consist of large cardinal axioms. The latter express regularities of the universe of sets and settle the truth of (certain) theorems independent of $\text{ZFC}$ ([28]).

Finally, the Gödel hierarchy exhibits some remarkable robustness: we can perform the following modifications and the hierarchy remains largely unchanged.

(I) Instead of the consistency strength ordering, we can order via inclusion: Simpson claims that inclusion and consistency strength yield the same Gödel hierarchy as depicted in [62, Table 1] and Figure 1. Some exceptional statements do fall outside of the inclusion-based Gödel hierarchy.

(II) We can replace the systems with their higher-order counterparts boasting a much richer language. These higher-order systems generally prove the same sentences as their second-order counterpart for (large parts of) the language of second-order arithmetic.

As suggested by item (I), there are some examples of theorems that fall outside of the Gödel hierarchy based on inclusion, like special cases of Ramsey’s theorem and the axiom of determinacy from set theory ([26, 34]). The latter axiom restricted to certain formula classes even yields a parallel hierarchy for the medium range of the Gödel hierarchy based on inclusion. It is now a natural question whether there are more examples of such parallel hierarchies. The aim of this paper is to provide a positive answer, assuming the aforementioned modifications (inclusion ordering and richer language) in items (I-II). Our new hierarchies are based on convergence theorems for nets, as discussed in the next section.

1.3. Nets and bootstraps. Abstraction is an integral part of mathematics, from Euclid’s Elements to the present day. In this spirit, E. H. Moore presented a framework called General Analysis at the 1908 ICM in Rome ([35]) that was to be a ‘unifying abstract theory’ for various parts of analysis. Indeed, Moore’s framework captures various limit notions in one abstract concept ([59]) and even includes a generalisation of the concept of sequence to possibly uncountable index sets (called directed sets), nowadays called nets or Moore-Smith sequences. These were first described in [37] and then formally introduced by Moore and Smith in [35]. They also established the generalisation from sequences to nets of various basic theorems
due to Bolzano-Weierstrass, Dini, and Arzelà ([38, §8-9]). More recently, nets are central to the development of domain theory (see [17,18,20]), including a definition of the Scott and Lawson topologies in terms of nets. Moreover, sequences cannot be used in this context, as expressed in a number of places:

[. . . ] clinging to ascending sequences would produce a mathematical theory that becomes rather bizarre, whence our move to directed families. ([20, p. 59])

Turning to foundations, we feel that the necessity to choose chains where directed subsets are naturally available (such as in function spaces) and thus to rely on the Axiom of Choice without need, is a serious stain on this approach. ([1, §2.2.4]).

Thus, nets enjoy a rich history, as well as a mainstream (and essential) status in mathematics and computer science. Motivated by the above, the study of nets in RM was undertaken in [53,55]. We continue the RM study of nets in this paper, and the truly novel result is that basic convergence theorems of nets give rise to parallel hierarchies for the medium range of the Gödel hierarchy, modulo the caveats from Section 1.2. In particular, we show that these theorems ‘bootstrap’ themselves to higher levels of the hierarchy when combined with higher-order comprehension axioms from the medium range. Moreover, the so-called Plato hierarchy stands out among these hierarchies, as discussed below.

Since uncountable index sets are first-class citizens in the theory of nets, we work in Kohlenbach’s higher-order RM (see Section 2.1). The exact formalisation of nets in higher-order RM is detailed in Definition 2.4 and Section 2.3. Now, in Sections 3.3 to 3.6 we restrict ourselves to nets indexed by subsets of Baire space, i.e. part of third-order arithmetic, as such nets are already general enough to obtain our main results. We study (slightly) more general index sets in Section 3.7 with significantly stronger results. Our results for the monotone convergence theorem \( \text{MCT}^C_{\text{net}} \) for nets in Cantor space indexed by subsets of Baire space, are neatly summarised by Figure 2; the associated logical systems are defined in Section 2.2.

Figure 2 only provides one example and we shall obtain a number of such parallel hierarchies in Section 3 based on the following theorems.

(i) The Bolzano-Weierstrass theorem for nets (Section 3.2).
(ii) The existence of moduli of convergence for nets (Section 3.3).
(iii) The Moore-Osgood theorem for nets (Section 3.4).
(iv) Numerous variations including the anti-Specker property and the Arzelà and Ascoli-Arzelà theorems (Section 3.2) and Cauchy nets (Section 3.3).

We refer to the hierarchy formed by \( \Pi^1_k-\text{CA}^0_\omega + \text{MCT}^C_{\text{net}} \) for \( k \geq 0 \) as the bootstrap hierarchy as the logical strength (at least \( \Pi^1_{k+1}-\text{CA}^0_\omega \)) is ‘bootstrapped’ from two essential parts, namely \( \Pi^1_k-\text{CA}^0_\omega \) and \( \text{MCT}^C_{\text{net}} \) that are weak(er) in isolation.

To obtain the aforementioned results, \( \text{MCT}^C_{\text{net}} \) is shown to be equivalent to a new comprehension principle \( \text{BOOT} \) from Section 3.1 and similar results for the other convergence theorems. We also show that \( \text{BOOT} \) implies the uncountable Heine-Borel theorem as in \( \text{HBU} \) from Section 2.2 and obtain a number of implications and equivalences for these and related principles introduced in Section 3.5, yielding:

\[
\text{BOOT}^k_{k+1} \rightarrow \text{BOOT}_2 \rightarrow \Sigma-\text{TR} \rightarrow \text{BOOT} \rightarrow \text{HBU} \rightarrow \Delta\text{-comprehension. (1.1)}
\]
Moreover, applying the ECF-translation (see Remark 2.5) to (1.1), one obtains the well-known picture involving the Big Five from RM as follows:

$$\Pi^1_k \text{-CA}_0 \rightarrow \Pi^1_{k+1} \text{-CA}_0 \rightarrow \text{ATR}_0 \rightarrow \text{ACA}_0 \rightarrow \text{WKL}_0 \rightarrow \text{RCA}_0.$$  

(1.2)

We stress that the ECF-translation is the canonical embedding of higher-order into second-order arithmetic, replacing as it does higher-order objects by the codes typical of the practise of RM. In a nutshell, a particular hierarchy populated by BOOT, MCT$^\text{net}$, and HBU becomes the medium range of the Gödel hierarchy when applying a natural and well-known lossy syntactical translation. For these reasons, the former hierarchy shall be baptised the *Plato hierarchy*, in honour of Plato’s famous writings on ideal objects and their importance in mathematics and its foundations.

**Figure 2.** The Gödel hierarchy (based on inclusion and higher types) with a parallel branch for the medium range

Finally, we study two ‘more general’ convergence theorems, respectively for nets in function spaces and for nets involving index sets beyond Baire space. The former theorem ‘bootstraps itself’, i.e. become stronger and stronger without the need for additional comprehension, as discussed in Section 3.6. The latter theorem carries us beyond second-order arithmetic, and shows that our proofs readily generalise to higher types. Nonetheless, results associated to index sets beyond Baire space are still mapped into the lower regions of second-order arithmetic by ECF, as discussed in Section 3.7.
2. Preliminaries

We introduce \textit{Reverse Mathematics} in Section 2.1 as well as its generalisation to \textit{higher-order arithmetic}, and the associated base theory $\text{RCA}_0^\omega$. We introduce some essential axioms in Section 2.2. We provide a brief introduction to nets and related concepts in Section 2.3. As noted in Section 1.2, we mostly study nets \textit{indexed by subsets of Baire space}, i.e. part of third-order arithmetic; the associated bit of set theory shall be represented in $\text{RCA}_0^\omega$ as in Definition 2.4.

2.1. \textbf{Reverse Mathematics}. Reverse Mathematics is a program in the foundations of mathematics initiated around 1975 by Friedman (13, 14) and developed extensively by Simpson (61). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics. We refer to [64] for a basic introduction to RM and to [60, 61] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach’s higher-order RM ([32]) essential to this paper, including the base theory $\text{RCA}_0^\omega$ (Definition 2.1). As will become clear, the latter is officially a type theory but can accommodate (enough) set theory via Definition 2.4.

First of all, in contrast to ‘classical’ RM based on second-order arithmetic $\mathbb{Z}_2$, higher-order RM uses $\mathbb{L}_\omega$, the richer language of \textit{higher-order arithmetic}. Indeed, while the latter is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, etcetera. To formalise this idea, we introduce the collection of all finite types $\mathbb{T}$, defined by the two clauses:

(i) $0 \in \mathbb{T}$ and (ii) If $\sigma, \tau \in \mathbb{T}$ then $(\sigma \to \tau) \in \mathbb{T},$

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and where $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that $\mathbb{Z}_2$ only includes objects of type 0 and 1.

Secondly, the language $\mathbb{L}_\omega$ includes variables $x^\rho, y^\rho, z^\rho, \ldots$ of any finite type $\rho \in \mathbb{T}$. Types may be omitted when they can be inferred from context. The constants of $\mathbb{L}_\omega$ includes the type 0 objects 0, 1 and $<, +, \times, =_0$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of ‘$=_0$’ as follows: for any objects $x^\tau, y^\tau$, we have

$$[x =_\tau y] \equiv (\forall z^1 \ldots z^n)(x z^1 \ldots z^n =_0 y z^1 \ldots z^n), \quad (2.1)$$

if the type $\tau$ is composed as $\tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0)$. Furthermore, $\mathbb{L}_\omega$ also includes the \textit{recursor constant} $R_\sigma$ for any $\sigma \in \mathbb{T}$, which allows for iteration on type $\sigma$-objects as in the special case (2.2). Formulas and terms are defined as usual. One obtains the sub-language $\mathbb{L}_{n+2}$ by restricting the above type formation rule to produce only type $n + 1$ objects (and related types of similar complexity).

\textbf{Definition 2.1.} The base theory $\text{RCA}_0^\omega$ consists of the following axioms.

(a) Basic axioms expressing that 0, 1, $<_0, +_0, \times_0$ form an ordered semi-ring with equality $=_0$.

(b) Basic axioms defining the well-known $\Pi$ and $\Sigma$ combinators (aka $K$ and $S$ in [2]), which allow for the definition of $\lambda$-abstraction.

(c) The defining axiom of the recursor constant $R_0$: For $m^0$ and $f^1$:

$$R_0(f, m, 0) := m \text{ and } R_0(f, m, n + 1) := f(n, R_0(f, m, n)). \quad (2.2)$$
(d) The **axiom of extensionality**: for all \( \rho, \tau \in \mathbf{T} \), we have:

\[
(\forall x^\rho, y^\rho, \varphi^\rho \tau \rightarrow \varphi(x) =_\tau \varphi(y)).
\]

(E_{\rho, \tau})

(e) The induction axiom for quantifier-free formulas of \( \mathcal{L}_\omega \).

(f) \( \text{QF-AC}^{1,0} \): The quantifier-free Axiom of Choice as in Definition 2.2.

**Definition 2.2.** The axiom \( \text{QF-AC} \) consists of the following for all \( \sigma, \tau \in \mathbf{T} \):

\[
(\forall x^\sigma)(\exists y^\tau) A(x, y) \rightarrow (\exists Y^\sigma)(\forall x^\sigma) A(x, Y(x)),
\]

(QF-AC\(^{\sigma, \tau}\))

for any quantifier-free formula \( A \) in the language of \( \mathcal{L}_\omega \).

We let \( \text{IND} \) be the induction axiom for all formulas in \( \mathcal{L}_\omega \). The system \( \text{RCA}_0 + \text{IND} \) has the same first-order strength as Peano arithmetic.

As discussed in [32, §2], \( \text{RCA}_0^n \) and \( \text{RCA}_n \) prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. Recursion as in (2.2) is called **primitive recursion**; the class of functionals obtained from \( \mathbf{R}_\sigma \) for all \( \rho \in \mathbf{T} \) is called **Gödel’s system** \( T \) of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in [32] p. 288-289).

**Definition 2.3** (Real numbers and related notions in \( \text{RCA}_0^n \)).

(a) Natural numbers correspond to type zero objects, and we use ‘\( n^0 \)’ and ‘\( n \in \mathbb{N} \)’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘\( q \in \mathbb{Q} \)’ and ‘\( q \in \mathbb{Q} \)' have their usual meaning.

(b) Real numbers are coded by fast-converging Cauchy sequences \( (q_n) : \mathbb{N} \rightarrow \mathbb{Q} \), i.e. such that \((\forall n^0, i^0)(|q_n - q_{n+i}| < q_1)\). We use Kohlenbach’s ‘hat function’ from [32] p. 289 to guarantee that every \( q^1 \) defines a real number.

(c) We write ‘\( x \in \mathbb{R} \)' to express that \( x^1 := (q^1) \) represents a real as in the previous item and write \( [x](k) := q_k \) for the \( k \)-th approximation of \( x \).

(d) Two reals \( x, y \) represented by \( (q^k) \) and \( (r^k) \) are **equal**, denoted \( x =_\mathbb{R} y \), if \((\forall n^0)(|q_n - r_n| \leq 2^{-n+1})\). Inequality ‘\( <_\mathbb{R} \)' is defined similarly. We sometimes omit the subscript ‘\( \mathbb{R} \)' if it is clear from context.

(e) Functions \( F : \mathbb{R} \rightarrow \mathbb{R} \) are represented by \( \Phi^{1 \rightarrow 1} \) mapping equal reals to equal reals, i.e. \((\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \rightarrow \Phi(x) =_\mathbb{R} \Phi(y))\).

(f) The relation ‘\( x \leq_\tau y \)' is defined as in (2.1) but with ‘\( \leq_0 \)' instead of ‘\( =_0 \)’. Binary sequences are denoted ‘\( f^1 \)’, ‘\( g^1 \)’, ‘\( f^1, g^1 \)’, but also ‘\( f, g \in \mathbb{C} \)' or ‘\( f, g \in \mathbb{H} \)’. Elements of Baire space are given by \( f^1 \), \( g^1 \), but also denoted ‘\( f, g \in \mathbb{N} \)'.

(g) For a binary sequence \( f^1 \), the associated real in \( [0, 1] \) is \( r(f) := \sum_{n=0}^{\infty} f(n) \).

(h) Sets of type \( \rho \) objects \( X^{\rho \rightarrow 0}, Y^{\rho \rightarrow 0}, \ldots \) are given by their characteristic functions \( F_X^{\rho \rightarrow 0} \leq_{\rho \rightarrow 0} 1 \), i.e. we write ‘\( x \in X \)' for \( F_X(x) =_0 1 \).

The following special case of item (h) is singled out, as it will be used frequently.

**Definition 2.4.** [\( \text{RCA}_0^n \)] A ‘subset \( D \) of \( \mathbb{N}^n \)' is given by its characteristic function \( F_D^{\rho \leq 1} \), i.e. we write ‘\( f \in D \)' for \( F_D(f) = 1 \) for any \( f \in \mathbb{N}^n \). A ‘binary relation \( \leq \) on a subset \( D \) of \( \mathbb{N}^n \)' is given by the associated characteristic function \( G^{\leq(1 \times 1) \rightarrow 0}_\rho \), i.e. we write ‘\( f \leq g \)' for \( G_{\leq}(f, g) = 1 \) and any \( f, g \in D \). Assuming extensionality on the reals as in item (e), we obtain characteristic functions that represent subsets of

\[\text{\footnotesize{\(\text{\scriptsize{To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language \( \mathcal{L}_\omega \): only quantifiers are banned.}}\)}}\]
proves \( A \) for sequences \( \langle \cdot \rangle \). Identifying codes with the objects being coded, it is no exaggeration to refer to these formulations using types, namely only using type zero and one objects. Similarely) called ‘associates’ or ‘RM-codes’ (see [31, §4]). The ECF-interpretation connects \( \text{RCA}_0 \) and \( \text{RCA}_0 \) (see [32, Prop. 3.1]) in that if \( \text{RCA}_0 \) proves \( A \), then \( \text{RCA}_0 \) proves \( [A]_{\text{ECF}} \), again ‘up to language’, as \( \text{RCA}_0 \) is formulated using sets, and \( [A]_{\text{ECF}} \) is formulated using types, namely only using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the canonical embedding of higher-order into second-order RM. For completeness, we also list the following notational convention for finite sequences.

**Notaion 2.6 (Finite sequences).** We assume a dedicated type for ‘finite sequences of objects of type \( \rho \)’, namely \( \rho^* \). Since the usual coding of pairs of numbers goes through in \( \text{RCA}_0 \), we shall not always distinguish between 0 and 0*. Similarly, we do not always distinguish between \( 's^\rho' \) and \( 's^\rho' \), where the former is ‘the object \( s \) of type \( \rho \)’, and the latter is ‘the sequence of type \( \rho^* \) with only element \( s^\rho \)’. The empty sequence for the type \( \rho^* \) is denoted by ‘\( (\cdot)_{\rho} \)’, usually with the typing omitted.

Furthermore, we denote by |\( [s] = n \)\| the length of the finite sequence \( s^\rho = \langle s^\rho_0, s^\rho_1, \ldots, s^\rho_{n-1} \rangle \), where |\( (\cdot) | = 0 \), i.e. the empty sequence has length zero. For sequences \( s^\rho, t^\rho \), we denote by \( 's\ast t' \) the concatenation of \( s \) and \( t \), i.e. \( (s\ast t)(i) = s(i) \) for \( i < |s| \) and \( (s\ast t)(j) = t(|s|−j) \) for \( |s| ≤ j < |s|+|t| \). For a sequence \( s^\rho \), we define \( \pi N := \langle s(0), s(1), \ldots, s(N−1) \rangle \) for \( N^0 < |s| \). For a sequence \( \alpha^0_{\rho} \), we also write \( \pi N = (\alpha(0), \alpha(1), \ldots, \alpha(N−1)) \) for any \( N^0 \). By way of shorthand, \( (\forall q^\rho \in Q^\rho)A(q) \) abbreviates \( (\forall^0 < |Q|)A(Q(i)) \), which is (equivalent to) quantifier-free if \( A \) is.

### 2.2. Some axioms of higher-order RM

We introduce some functionals which constitute the counterparts of second-order arithmetic \( Z_2 \), and some of the Big Five systems, in higher-order RM. We use the formulation from [32][41].

First of all, \( \text{ACA}_0 \) is readily derived from:

\[
(\exists \mu^2)(\forall f^1)[(\exists n)(f(n) = 0) \rightarrow [(f(\mu(f)) = 0) \land (\forall i < \mu(f))f(i) \neq 0] \quad (\mu^2)
\]

\[
\quad \land [(\forall n)(f(n) \neq 0) \rightarrow \mu(f) = 0]],
\]

and \( \text{ACA}^\omega_0 = \text{RCA}^\omega_0 + (\mu^2) \) proves the same sentences as \( \text{ACA}_0 \) by [27, Theorem 2.5]. The (unique) functional \( \mu^2 \) in \( (\mu^2) \) is also called Feferman’s \( \mu \) ([2]), and is clearly discontinuous at \( f = 11 \ldots \); in fact, \( (\mu^2) \) is equivalent to the existence of \( F : \mathbb{R} \rightarrow \mathbb{R} \) such that \( F(x) = 1 \) if \( x \geq 0 \), and 0 otherwise ([32, §3]), and to

\[ (\exists \varphi^2 \leq 2')(\forall f^1)[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0]. \quad (\exists^2) \]

Secondly, \( \Pi^1_1\text{-}\text{CA}_0 \) is readily derived from the following sentence:

\[ (3S^2 \leq 2')(\forall f^1)[[(\exists y^0)(\forall n^0)(f(\overline{\text{y}}n) = 0) \leftrightarrow S(f) = 0], \quad (S^2) \]

and \( \Pi^1_1\text{-}\text{CA}_0^\omega = \text{RCA}_0^\omega + (S^2) \) proves the same \( \Pi^1_1 \)-sentences as \( \Pi^1_1\text{-}\text{CA}_0 \) by [50, Theorem 2.2]. The (unique) functional \( S^2 \) in \( (S^2) \) is also called the Suslin functional.
By definition, the Suslin functional $S^2$ can decide whether a $\Sigma^1_3$-formula as in the left-hand side of $(S^2)$ is true or false. We similarly define the functional $S^2_3$ which decides the truth or falsity of $\Sigma^1_k$-formulas; we also define the system $\Pi^1_k$-$\text{CA}_0^\omega$ as $\text{RCA}_0^\omega + (S^2_3)$, where $(S^2_3)$ expresses that $S^2_3$ exists. Note that we allow formulas with function parameters, but not functionals here. In fact, Gandy’s Superjump \cite{33} constitutes a way of extending $\Pi^1_k$-$\text{CA}_0^\omega$ to parameters of type two. We identify the functionals $\exists^2$ and $S^2_3$ and the systems $\text{ACA}_0^\omega$ and $\Pi^1_k$-$\text{CA}_0^\omega$ for $k = 0$.

Thirdly, full second-order arithmetic $\mathbb{Z}_2$ is readily derived from $\cup_k \Pi^1_k$-$\text{CA}_0^\omega$, or from:

$$\exists^3 \leq 1 \forall E \exists Y \exists f : (\exists f^1)(\forall Y^2)[(\exists f^1)Y(f) = 0 \leftrightarrow E(Y) = 0]$$

and we therefore define $\mathbb{Z}_2^0 \equiv \text{ACA}_0^\omega + (\exists^3)$ and $\mathbb{Z}_2^\omega \equiv \cup_k \Pi^1_k$-$\text{CA}_0^\omega$, which are conservative over $\mathbb{Z}_2$ by \cite{27} Cor. 2.6. Despite this close connection, $\mathbb{Z}_2^\omega$ and $\mathbb{Z}_2^\alpha$ can behave quite differently, as discussed in e.g. \cite{44, 45} §2.2. The functional from $(\exists^3)$ is also called ‘\exists^3’, and we use the same convention for other functionals.

Finally, the Heine-Borel theorem states the existence of a finite sub-cover for an open cover of certain spaces. Now, a functional $\Psi : \mathbb{R} \to \mathbb{R}^+$ gives rise to the canonical cover $\cup_{x \in I^\Psi} I^\Psi_x$ for $I = [0,1]$, where $I^\Psi$ is the open interval $(x - \Psi(x), x + \Psi(x))$. Hence, the uncountable cover $\cup_{x \in I^\Psi} I^\Psi_x$ has a finite sub-cover by the Heine-Borel theorem; in symbols:

$$(\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists y_1, \ldots, y_k \in I)(\forall x \in I) (\exists i \leq k)(x \in I^\Psi_{y_i}).$$  \hspace{1cm} (HBU)

Note that HBU is almost verbatim Cousin’s lemma (see \cite{11} p. 22), i.e. the Heine-Borel theorem restricted to canonical covers. The latter restriction does not make much of a big difference, as studied in \cite{51}. By \cite{44, 45}, $\mathbb{Z}_2^0$ proves HBU but $\mathbb{Z}_2^\omega + \text{QF-AC}^{0,1}$ cannot, and many basic properties of the gauge integral \cite{10, 32} are equivalent to HBU. Although strictly speaking incorrect, we sometimes use set-theoretic notation, like reference to the cover $\cup_{x \in I^\Psi} I^\Psi_x$ inside $\text{RCA}_0^\omega$, to make proofs more understandable. Such reference can in principle be removed in favour of formulas of higher-order arithmetic.

2.3. Introducing nets. We introduce the notion of net and associated concepts. We first consider the following standard definition from \cite{29} Ch. 2.

**Definition 2.7.** [Nets] A set $D \neq \emptyset$ with a binary relation ‘$\preceq$’ is directed if

(a) The relation ‘$\preceq$’ is transitive, i.e. $(\forall x, y, z \in D)(x \preceq y \land y \preceq z \to x \preceq z)$.

(b) For $x, y \in D$, there is $z \in D$ such that $x \preceq z \land y \preceq z$.

(c) The relation ‘$\preceq$’ is reflexive, i.e. $(\forall x \in D)(x \preceq x)$.

For such $(D, \preceq)$ and topological space $X$, any mapping $x : D \to X$ is a net in $X$. We denote $\lambda d. x(d)$ as ’$x_d$’ or ’$x_d : D \to X$’ to suggest the connection to sequences. The directed set $(D, \preceq)$ is not always explicitly mentioned together with a net $x_d$.

Except for the final Section 3.7, we only use directed sets that are subsets of Baire space, i.e. as given by Definition 2.4. Similarly, we only study nets $x_d : D \to \mathbb{R}$ where $D$ is a subset of Baire space. Thus, a net $x_d$ in $\mathbb{R}$ is just a type $1 \to 1$ functional with extra structure on its domain $D$ provided by ‘$\preceq$’ as in Definition 2.4 i.e. part of third-order arithmetic.

The definitions of convergence and increasing net are of course familiar.
Definition 2.8. [Convergence of nets] A net \( x_d \) is a net in \( X \), we say that \( x_d \) converges to the limit \( \lim_{d} x_d = y \in X \) if for every neighbourhood \( U \) of \( y \), there is \( d_0 \in D \) such that for all \( e \geq d_0 \), \( x_e \in U \).

Definition 2.9. [Increasing nets] A net \( x_d : D \to \mathbb{R} \) is increasing if \( a \leq b \) implies \( x_a \leq_R x_b \) for all \( a, b \in D \).

Definition 2.10. A point \( x \in X \) is a cluster point for a net \( x_d \) in \( X \) if every neighbourhood \( U \) of \( x \) contains \( x_u \) for some \( u \in D \).

The previous definition yields the following nice equivalence: a topological space is compact if and only if every net therein has a cluster point (see Proposition 3.4). All the below results can be formulated using cluster points only, but such an approach does not address the question what the counterpart of 'sub-sequence' for nets is. Indeed, an obvious next step following Definition 2.10 is to take smaller and smaller points, i.e. convergence results about nets do apply to sequences. Of course, a sub-net of a sequence is not necessarily a sub-sequence, i.e. some care is advisable in these matters. Nonetheless, the Bolzano-Weierstrass theorem for nets does for instance imply the monotone convergence theorem for sequences (see §3.1.1 for details).

3. Main results

3.1. Introduction: the bootstrap hierarchy. The results in [53,55] establish that basic convergence theorems for nets are extremely hard to prove, while the limits therein are similarly hard to compute. In this paper, we show that the first-order strength of such theorems can also 'explode', i.e. increase dramatically when combined with certain comprehension axioms. These results in turn give rise to the hierarchies described in Section 1.3. To this end, we show in the next sections that various convergence theorems for nets imply, or are even equivalent to, the following higher-order comprehension axiom.

Definition 3.1. [BOOT] \((\forall Y^2)(\exists X^1)(\forall n^0)[n \in X \leftrightarrow (\exists f^1)(Y(f,n) = 0)]\).

The principle \(\text{BOOT}_{k+1} \) is simply \(\text{BOOT} \) involving \( k \) quantifier alternations in the right-hand side (see Definition 2.29 for full details). The name 'BOOT' derives from the word 'bootstrap'. We refer to the hierarchy formed by \(\Pi_k^1 \cdot \text{CA}_0^\omega + \text{BOOT} \) as the bootstrap hierarchy as the logical strength of the latter system (in casu at least \(\Pi_k^1 \cdot \text{CA}_0 \)) is 'bootstrapped' from two essential parts, namely \(\Pi_k^1 \cdot \text{CA}_0^\omega \) and \(\text{BOOT} \) that are weak(er) in isolation.

Theorem 3.2. The system \(\Pi_k^1 \cdot \text{CA}_0^\omega + \text{BOOT} \) proves \(\Pi_{k+1}^1 \cdot \text{CA}_0 \). The system \(\text{RCA}_0^\omega + \text{BOOT} \) proves the same second-order sentences as \(\text{ACA}_0 \).
Proof. For the first part, a $\Pi_{k+1}^1$-formula from $L_2$ is clearly equivalent to a formula of the form $(\forall f^1)(Y(f, n) = 0)$ given $S_2^k$. For the second part, $\text{RCA}_0 + \text{BOOT}$ readily proves $\text{ACA}_0$, while the ECF-translation establishes that $\text{BOOT}$ proves the same second-order sentences as $\text{ACA}_0$. Indeed, as discussed in Remark 2.5, the ECF-translation replaces the functional $Y^2$ in $\text{BOOT}$ by a total associate $\alpha^1$, i.e. the right-hand side of $[\text{BOOT}]_{\text{ECF}}$ is thus $(\exists f^1)(\exists m^0)(\alpha(f^1 m, n) = 1)$. Given $\text{ACA}_0$, there is clearly a set $X$ that collects all $n$ satisfying this formula. □

The previous theorem is hardly surprising given the form of $\text{BOOT}$. By contrast, the equivalence between $\text{BOOT}$ and the monotone convergence theorem $\text{MCT}_{\text{net}}$ for nets in Cantor space indexed by Baire space from Section 3.2 is rather surprising, in our opinion. Moreover, the addition of moduli of convergence for nets gives rise to an equivalence involving $\text{BOOT}$ and countable choice in Section 3.3. The Moore-Osgood theorem for nets is shown to exhibit similar behaviour in Section 3.4. By Theorem 3.2, these convergence theorems give rise to the ‘bootstrap hierarchy’ and variations. One of these hierarchies is also ‘special/unique’ in the following sense.

In Section 3.5, we establish a number of implications and equivalences involving $\text{BOOT}$, $\text{HBU}$, $\text{MCT}_{\text{net}}$ and related principles, including (3.1) as follows:

$$\text{BOOT}_{k+1} \rightarrow \text{BOOT}_2 \rightarrow \Sigma-\text{TR} \rightarrow \text{BOOT} \rightarrow \text{HBU} \rightarrow \Delta\text{-comprehension.} \quad (3.1)$$

We shall observe that applying the ECF-translation to (3.1), one obtains the well-known picture involving the Big Five as in (3.2), as discussed in detail in Section 3.2:

$$\Pi_{k}^1\text{-CA}_0 \rightarrow \Pi_{1}^1\text{-CA}_0 \rightarrow \text{ATR}_0 \rightarrow \text{ACA}_0 \rightarrow \text{WKL}_0 \rightarrow \text{RCA}_0. \quad (3.2)$$

Moreover, ECF maps equivalences involving principles from (3.1), like $\text{MCT}_{\text{net}}^{[0,1]} \leftrightarrow \text{BOOT}$, to well-known RM-equivalences, like the equivalence between arithmetical comprehension and the monotone convergence theorem for sequences ($[0,1]$ III.2]). We stress that the ECF-translation is the canonical embedding of higher-order into second-order arithmetic, replacing as it does higher-order objects by the codes typical of the practise of RM and second-order arithmetic. In the other direction, Theorems 3.26 and 3.38 show that certain second-order proofs, namely involving Specker sequences, almost verbatim translate to proofs of $\text{MCT}_{\text{net}}^{[0,1]} \rightarrow \text{BOOT}$ and generalisations.

In a nutshell, a particular hierarchy populated by $\text{BOOT}$, $\text{MCT}_{\text{net}}^C$, and $\text{HBU}$ as in (3.1) becomes the medium range of the Gödel hierarchy as in (3.2) when applying a natural and well-known lossy syntactical translation; this translation also maps equivalences involving the former to equivalences involving the latter. The Big Five and the medium range of the Gödel hierarchy therefore emerge as a special case of higher-order arithmetic, similar to the way quantum mechanics and general relativity have Newtonian mechanics as a special case. In this light, the former hierarchy is baptised the *Plato hierarchy*, inspired by Plato’s writings on ideal objects and their fundamental role in the foundations of mathematics.

Finally, we study two ‘more complicated’ convergence theorems: for nets in the function space $[0,1] \rightarrow [0,1]$ (Section 3.6) and for nets with index sets beyond Baire space (Section 3.7). On the hand, the latter section was added for completeness, i.e. to show that our proofs would easily generalise to higher types, while the general case is perhaps best treated in a set-theoretic framework. On the other hand,

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6The axioms $\text{BOOT}_k$ and $\Sigma\text{-TR}$ are introduced in Definitions 3.28 and 3.29.
Section 8.7 shows that results for such large index sets are still mapped to the lower regions of second-order arithmetic, namely $\Pi^1_2$-CA$_0$, by ECF. Section 3.3 is interesting as we obtain a convergence theorem for nets in functions spaces-still in the language third-order arithmetic-that ‘bootstraps itself’, i.e. does not need additional comprehension axioms (like $S^2_5$) to become stronger and stronger.

We finish this section with some historical remarks pertaining to BOOT.

Remark 3.3 (Historical notes). First of all, the bootstrap principle BOOT is definable in Hilbert-Bernays’ system $H$ from the Grundlagen der Mathematik (see [25, Supplement IV]). In particular, the functional $\nu$ from [25, p. 479] immediately yields the set $X$ from BOOT (and similarly for its generalisations), viewing the type two functional $Y^2$ as a parameter (which is done throughout [25, Supplement IV]). Thus, the Plato and Gödel hierarchies have the same historical roots.

Secondly, Feferman’s axiom (Proj1) from [12] is similar to BOOT. The former is however formulated using sets, which makes it more ‘explosive’ than BOOT in that full $\mathbb{Z}_2$ follows when combined with $(\mu^2)$, as noted in [12] I-12. The axiom (Proj1) only became known to us after the results in this paper were finished.

3.2. Convergence theorems for nets. We show that a number of convergence theorems for nets gives rise to $\Pi^1_{k+1}$-CA$_0$ in combination with $\Pi^1_k$-CA$_0$. This is done by establishing the connection between these theorems and BOOT from the previous section. We first study the Bolzano-Weierstrass theorem as follows.

Definition 3.4. [BW$_{\text{net}}^C$] A net in Cantor space indexed by a subset of Baire space has a convergent sub-net.

Theorem 3.5. The system $\text{ACA}_0^\omega + \text{BW}_{\text{net}}^C$ proves $\Pi^1_1$-CA$_0$.

Proof. A $\Sigma^1_1$-formula $\varphi(n) \in L_2$ is readily seen to be equivalent to a formula \((\exists g^1)(Y(g, n) = 0)\) for $Y^2$ defined in terms of $\exists^2$. Let $D$ be the set of finite sequences in Baire space and let $\preceq_D$ be the inclusion ordering, i.e. $w \preceq_D v$ if $(\forall i < |w|)(\exists j < |v|)(w(i) = v(j))$. Now define the net $f_w : D \to C$ as $f_w := \lambda k. F(w, k)$ where $F(w, k)$ is 1 if $(\exists i < |w|)(Y(w(i), k) = 0)$, and zero otherwise. Using $\text{BW}_{\text{net}}$, let $\varphi : B \to D$ be such that $\lim_b f_{\varphi(b)} = f$. We now establish this equivalence:

\[(\forall n^0)(\exists g^1)(Y(g, n) = 0) \iff f(n) = 1.\] (3.3)

For the reverse direction, note that for fixed $n_0$, if $Y(g, n_0) > 0$ for all $g^1$, then $f_w(n_0) = 0$ for any $w \in D$. The definition of limit then implies $f(n_0) = 0$, i.e. we have established (the contraposition of) the reverse direction. For the forward direction in (3.3), suppose there is some $n_0$ such that $(\exists g^1)(Y(g, n_0) = 0) \wedge f(n_0) = 0$. Now, $\lim_b f_{\varphi(b)} = f$ implies that there is $b_0 \in B$ such that for $b \succeq_B b_0$, we have $f_{\varphi(b)}(n_0) = 0$ for any $n_0$, i.e. $f_{\varphi(b)}(n_0) = 0$ for $b \succeq_B b_0$. Let $g_0^1$ be such that $Y(g_0, n_0) = 0$, and use the second item in Definition 2.11 for $d = \langle g_0 \rangle$, i.e. there is $b_1 \in B$ such that $\varphi(b) \succeq_D \langle g_0 \rangle$ for any $b \succeq_B b_1$. Now let $b_2 \in B$ be such that $b_2 \succeq_B b_0, b_1$ as provided by Definition 2.11. On one hand, $b_2 \succeq_B b_1$ implies that $\varphi(b_2) \succeq_D \langle g_0 \rangle$, and hence $f_{\varphi(b_2)}(n_0) = F(\varphi(b_2), n_0) = 1$, as $g_0$ is in the finite sequence $\varphi(b_2)$ by the definition of $\preceq_D$. On the other hand, $b_2 \succeq_B b_0$ implies that $f_{\varphi(b_2)}(n_0) = f(n_0) = 0$, a contradiction. Hence, (3.3) follows, yielding $\{ n : \varphi(n) \}$, as required by $\Pi^1_1$-CA$_0$. □

The functional $\nu$ from [25, p. 479] is such that if $(\exists f^1)A(f)$, the function $(\nu f)A(f)$ is the lexicographically least such $f^1$. The formula $A$ may contain type two parameters, as is clear from e.g. [25, p. 481] and other definitions.
The previous theorem is elegant, but hides an important result involving the monotone convergence theorem for nets. As to its provenance, the latter theorem can be found in e.g. [8, p. 103], but is also implicit in domain theory ([17, 18]). Indeed, the main objects of study of domain theory are dcpos, i.e. directed-complete posets, and an increasing net converges to its supremum in a dcpo.

**Definition 3.6.** [MCT$_\text{net}$] Any increasing net in $C$ indexed by a subset of $\mathbb{N}$ converges in $C$.

Note that we use the lexicographic ordering $\leq_{\text{lex}}$ on $C$ in the previous definition, i.e. $f \leq_{\text{lex}} g$ if either $f =_1 g$ or there is $n^0$ such that $\mathcal{F}n = \mathcal{F}n$ and $f(n+1) < g(n+1)$.

**Theorem 3.7.** The system $\text{RCA}_0^\omega$ proves that $\text{MCT}_\text{net} \leftrightarrow \text{BOOT}$.

**Proof.** We first prove the equivalence assuming $(\exists^2)$. For the forward direction, fix some $Y^2$ and consider $f_w$ from the proof of the theorem. Note that $v \leq_D w \rightarrow f_v \leq_{\text{lex}} f_w$, i.e. this net is indeed increasing. Let $f = \lim w f_w$ be the limit provided by $\text{MCT}_\text{net}$ and verify that $(\ref{eq:lemma5.2})$ also holds in this case. In this way, we obtain the equivalence required by $\text{BOOT}$. Note that $\exists^2$ is necessary for defining $\leq_D$.

For the reverse direction, let $x_d : D \rightarrow C$ be an increasing net in $C$ and consider the formula $(\exists d \in D)(x_d \geq_{\text{lex}} \sigma * 00 \ldots)$, where $\sigma^*$ is a finite binary sequence. The latter formula is equivalent to a formula of the form $(\exists^1 g^1)(Y(g,n) = 0)$ where $Y^2$ is defined in terms of $\exists^2$ and $n$ codes a finite binary sequence. To define the limit $f$ required by $\text{MCT}_\text{net}$, $f(0)$ is 1 if $(\exists d \in D)(x_d \geq_{\text{lex}} 100 \ldots)$ and zero otherwise. One then defines $f(n+1)$ in terms of $\mathcal{F}n$ in the same way, yielding the equivalence from the corollary given $(\exists^2)$.

Next, we establish the theorem assuming $\neg(\exists^2)$, which implies that all functionals on Baire space are continuous (see [32, 33]). In this light, $\text{BOOT}$ reduces to (essentially) $\text{ACA}_0$ by the proof of Theorem 5.2. Similarly, any formula involving a type one quantifier $(\exists d \in D)(\ldots x_d \ldots)$ may be equivalently replaced by $(\exists^0 \sigma^*)(\sigma * 00 \ldots \in D \land \ldots x_{\sigma^*00\ldots})$, which now involves a type zero quantifier (modulo coding). Thus, $\text{MCT}_\text{net}$ also reduces to (essentially) the monotone convergence theorem for sequences, and the latter is equivalent to $\text{ACA}_0$ by [61, III.2]. Hence, we have proved the theorem in both cases and the law of excluded middle $(\exists^2) \lor \neg(\exists^2)$ finishes the proof. \hfill \Box

**Corollary 3.8.** The systems $\Pi^1_k \cdot \text{CA}_0^\omega + \text{BW}^\text{net}_{\omega^0}$ and $\Pi^1_k \cdot \text{CA}_0^\omega + \text{MCT}_\text{net}$ prove $\Pi^1_{k+1} \cdot \text{CA}_0$ ($k \geq 0$). The system $Z^2_2$ proves $\text{MCT}_\text{net}$.

**Proof.** By Theorem 5.2 and the fact that $(\exists^3)$ trivially proves $\text{BOOT}$. \hfill \Box

By the second part of the corollary, the power, strength, and hardness of $\text{MCT}_\text{net}$ have nothing to do with the Axiom of Choice. We actually study the connection between the latter and the convergence of nets in Section 5.3.

Of course, there is nothing special about Cantor space in the previous results. Let $\text{BW}^\text{net}_{\omega^0}$ and $\text{MCT}^\text{net}_{\omega^0}$ be respectively the Bolzano-Weierstrass and monotone convergence theorem for nets in the unit interval indexed by subsets of Baire space.

**Corollary 3.9.** The system $\text{ACA}_0^\omega + \text{IND} + X$ proves $\text{BOOT}$, for $X$ equal to either $\text{BW}^\text{net}_{\omega^0}$ or $\text{MCT}^\text{net}_{\omega^0}$. 

Definition 3.10. [Cauchy sequence] A net \( \langle x_n \rangle \) in \([0,1]\) is Cauchy if \( \forall \varepsilon > 0 \exists d \in D \) \( \forall n,m \geq d (|x_n - x_m| < \varepsilon) \).

Definition 3.11. [Cauchy modulus] A net \( \langle x_n \rangle \) in \([0,1]\) is Cauchy with a modulus if there is \( \Phi : [0,1] \to D \) such that \( \forall \varepsilon > 0 \forall n \exists d \in D \) \( \Phi(\varepsilon)(|x_n - x_d| < \varepsilon) \).

On one hand, the convergence of Cauchy sequences in the unit interval is equivalent to ACA\(_0\) by [61 III.2.2], i.e. we expect the generalisation to Cauchy nets to exhibit similar behaviour to MCT\(_{\text{net}}^{[0,1]}\). One the other hand, MCT\(_{\text{net}}^{[0,1]}\) obviously follows from the two following facts:

(i) An increasing net in \([0,1]\) indexed by a subset of \(\mathbb{N}\) is Cauchy.

(ii) A Cauchy net in \([0,1]\) indexed by a subset of \(\mathbb{N}\) converges.
One readily shows that item (i) gives rise to hierarchies as in Corollary 3.9 while item (ii) is provable in $\text{RCA}_0 + \text{IND}$. Item (iii) is therefore quite weak and we shall enrich it with a Cauchy modulus, as follows.

**Definition 3.12.** $[\text{CAU}_{\text{mod}}]$ An increasing net in $[0,1]$ is Cauchy with a modulus.

Using the splitting of $(\exists^3)$ involving $(\exists^2)$ (see [52]), one readily proves that $\text{RCA}_0 + \text{CAU}_{\text{mod}}$ has the same first-order strength as $\text{RCA}_0^\omega$.

**Theorem 3.13.** The system $\text{ACA}_0^\omega + \text{CAU}_{\text{mod}}$ proves $\Pi^1_1 - \text{CA}_0$.

**Proof.** A $\Sigma^1_1$-formula $\varphi(n) \in L_2$ is readily seen to be equivalent to a formula $(\exists f^1)(Y(f,n) = 0)$ for $Y^2$ defined in terms of $\exists^2$. Let $D$ be the set of finite sequences in Baire space and let $\preceq_D$ be the inclusion ordering, i.e. $w \preceq_D v$ if $(\forall i < |w|)(\exists j < |v|)(w(i) = v(j))$. Now define the net $x_w : D \to \mathbb{R}$ as $x_w := \pi(\lambda k.F(w,k))$ where $F(w,k)$ is 1 if $(\exists i < |w|)(Y(w(i),k) = 0)$, and zero otherwise. Note that $x_w$ is increasing by definition. Let $\Phi : \mathbb{N} \to D$ be such that $(\forall k^n)(\forall v, v \succeq_D \Phi(k))(|x_w - x_v| < \frac{1}{2^k})$. We now establish this equivalence:

$$(\forall n^0)\left[(\exists f^1)(Y(f,n) = 0) \leftrightarrow (\exists g^1 \in \Phi(n))(Y(g,n) = 0)\right]. \quad (3.5)$$

The reverse direction in (3.5) is trivial. For the forward direction, suppose there is some $n_0$ such that $(\exists f^1)(Y(f,n_0) = 0) \land (\forall g^1 \in \Phi(n_0))(Y(g,n_0) > 0)$. Let $f_0$ be such that $Y(f_0,n_0) = 0$, implying $F(\Phi(n_0),n_0) = 0$ and $F(w_0,n_0) = 1$ for $w_0 := \Phi(n_0) \ast \langle f_0 \rangle$. Hence $|x_\Phi(n_0) - x_{w_0}| = \frac{1}{2^n}$ and $w_0 \succeq_D \Phi(n_0)$, a contradiction. Thus, (3.5) holds and yields the set $\{n : \varphi(n)\}$, as required by $\Pi^1_1 - \text{CA}_0$. \qed

The theorem also yields a nice splitting as follows.

**Corollary 3.14.** The system $\text{RCA}_0^\omega$ proves $[\text{CAU}_{\text{mod}} + \text{ACA}_0] \leftrightarrow [\text{BOOT} + \text{QF-AC}^{0,1}]$.

**Proof.** For the reverse implication, the proof of Theorem 3.7 yields $\text{BOOT} \to \text{MCT}_{\text{net}}^{[0,1]}$ with minimal adaptation. Let $x_d : D \to [0,1]$ be an increasing net and let $x \in [0,1]$ be the limit provided by $\text{MCT}_{\text{net}}^{[0,1]}$. Now apply $\text{QF-AC}^{0,1}$ to the formula $(\forall k^n)(\exists d \in D)(|x_d - x| < \frac{1}{2^k})$ and note that the resulting functional is a Cauchy modulus since $x_d$ is an increasing net.

For the forward implication, we again use $(\exists^2) \lor (\exists^2)$. In case $(\exists^2)$, all functions on Baire space are continuous by [52] [3]. In this case, $\text{QF-AC}^{0,1}$ is immediate from $\text{QF-AC}^{0,0}$ (included in $\text{RCA}_0^\omega$) and $\text{BOOT}$ reduces to $\text{ACA}_0$ as noted in the proof of Theorem 3.7. In case of $(\exists^2)$, the proof of the theorem yields (3.5); $\text{BOOT}$ and $\text{QF-AC}^{0,1}$ are now immediate as the right-hand side of (3.5) is decidable. \qed

The definition of a ‘modulus of net convergence’ is now obvious following Definition 3.11. Let $\text{MCT}_{\text{net}}^{[0,1]}$ and $\text{BW}_{\text{mod}}^{[0,1]}$ be resp. $\text{MCT}_{\text{net}}^{[0,1]}$ and $\text{BW}_{\text{net}}^{[0,1]}$ with the addition of a modulus of convergence.

**Corollary 3.15.** The system $\text{ACA}_0^\omega + \text{BW}_{\text{mod}}^{[0,1]}$ proves $\text{BOOT} + \text{QF-AC}^{0,1}$.

**Proof.** Immediate by the proof of the theorem and the observation that for an increasing net, a modulus of convergence of a sub-net is also a Cauchy modulus for the (original) net. \qed

**Corollary 3.16.** The system $\text{RCA}_0^\omega$ proves $[\text{MCT}_{\text{mod}}^{[0,1]} + \text{ACA}_0] \leftrightarrow [\text{BOOT} + \text{QF-AC}^{0,1}]$.

**Proof.** By Corollaries 3.7 and 3.14. \qed
A similar result can now be obtained for the Arzelà and Ascoli-Arzelà theorems for nets studied in [33, §3.2.2]. Moreover, to derive BW\([0,1]^{\mathbb{N}}\) from item (ii) at the beginning of this section, one requires COH\(_{net}\), i.e. the statement *any net in the unit interval contains a Cauchy sub-net*. The associated property for sequences is equivalent to COH from the RM zoo (see [33]). Clearly, COH\(_{net}\) upgraded with a modulus would also give rise to e.g. Corollary 3.15.

3.4. The Moore-Osgood theorem for nets. We study the *Moore-Osgood theorem* which provides a sufficient criterion for the existence of double limits. We show that this theorem for nets is explosive in the same way as in the previous sections. Our motivation is that the above proofs can be viewed as a kind of double limit construction involving nets and sequences.

As to history, E. H. Moore’s version of the Moore-Osgood theorem apparently goes back to 1900 (see [21], p. 100), while Osgood’s version goes back to 1907 (see [47]). As expected, Moore-Smith deal with double (net) limits in [38, §7]. We use the following version of the Moore-Osgood theorem, similar to [4, Lemma 2.3], where \(D\) is assumed to be a subset of Baire space.

**Definition 3.17. [MOT]** Let \((D, \preceq_D)\) be a directed set. For a sequence of nets \(x_{d,n} : (D \times \mathbb{N}) \to [0, 1]\), if \(\lim_{n \to \infty} x_{d,n} = y_d\) for some net \(y_d : D \to [0, 1]\) and if the net \(\lambda d.x_{d,n}\) is uniformly Cauchy, then \(\lim_{d} y_d = z\) for some \(z \in [0, 1]\).

A sequence of nets \(x_{d,n}\) is uniformly Cauchy if the \(d\) claimed to exist by Definition 3.10 does not depend on the sequence parameter \(n\). This definition is equivalent to uniform convergence in \(Z^2_2 + QF-AC^{0,1}\). We use uniform Cauchyness because one generally needs non-trivial comprehension and choice to obtain a *sequence* of limits from the existence of the individual limits \(\lim_d x_{d,n}\) for all \(n\).

**Theorem 3.18.** The system ACA\(^0\) + IND + MOT proves \(\Pi_1^1\)-CA\(_0\).

**Proof.** A \(\Sigma_1^1\)-formula \(\varphi(n) \in L_2\) is readily seen to be equivalent to a formula \((\exists f^1)(Y(f,n) = 0)\) for \(Y^2\) defined in terms of \(\exists^2\). Let \(D\) be the set of finite sequences in Baire space and let \(\preceq_D\) be the inclusion ordering, i.e. \(w \preceq_D v\) if \((\forall i < |w|)(\exists j < |v|) (w(i) = v(j))\). Now define \(F(w,k) = 1\) if \((\exists i < |w|)(Y(w(i), k) = 0)\), and zero otherwise, and define the sequence of nets \(x_{w,k} := \sum_{i=0}^{k} \frac{F(w,i)}{2^{i+1}}\). By definition, we have \(\lim_{k \to \infty} x_{w,k} = y_w\), where \(y_w := \sum_{i=0}^{\infty} \frac{F(w,i)}{2^{i+1}}\). To prove that \(x_{w,k}\) is uniformly Cauchy, use IND to establish that for every \(m^0 \geq 1\), there is \(w\) of length \(m\) such that \((\forall i < m)[(\exists g^1)(Y(g,i) = 0) \to Y(w(i),i) = 0]\). For \(m \geq 1\) and such \(w\), note that \(x_{w,k}\) is below \(x_{w,k+1} + \frac{1}{2^m}\) for any \(k\) and \(v \succeq_D w\), i.e. uniform Cauchyness.

Let \(z\) be the limit provided by MOT, i.e. \(\lim_w y_w = z\). One now readily establishes the following equivalence for \(\eta\) as in the proof of Corollary 3.9

\[
(\forall n^0)[(\exists g^1)(Y(g,n) = 0) \leftrightarrow \eta(z)(n) = 1].
\]

Clearly, 3.8 yields \(\{n : \varphi(n)\}\), as required by \(\Pi_1^1\)-CA\(_0\). \(\square\)

Finally, one can obtain BOOT from MOT in the same way as in the previous sections, while introducing moduli would similarly yield QF-AC\(^{0,1}\). To establish BOOT \(\to\) MOT, note that \(y_d\) is a Cauchy net due to the assumptions in MOT.
3.5. **Connecting higher-order and second-order arithmetic.** In the previous sections, we have established the results in Figure 2 by connecting convergence theorems for nets to the comprehension axiom \textsc{boot}. In this section, we connect this axiom to \textsc{hbu} and other higher-order axioms, and identify the *Plato hierarchy*. As summarised by Table 3 below, the \textsc{ecf}-translation converts higher-order results into familiar second-order results pertaining to the Big Five of RM and the medium range of the Gödel hierarchy. The exact nature of this connection is discussed in Remark 3.19. In light of the results in Table 3, it is no exaggeration to claim that the Big Five and the associated RM arise as special cases of the higher-order theorems in the first column of Table 3. For this reason, the hierarchy formed by \textsc{boot} and its ilk in (3.7) is referred to as the *Plato hierarchy*, inspired by Plato’s famous writings on ideal objects and their role in foundations of mathematics.

\[
\textsc{boot}_{k+1} \rightarrow \textsc{boot}_2 \rightarrow \Sigma\text{-}\textsc{tr} \rightarrow \textsc{boot} \rightarrow \textsc{hbu} \rightarrow \Delta\text{-}\text{comprehension} \quad (3.7)
\]

For reasons of space, a perfunctory treatment of \textsc{atr}_0/\Sigma\text{-}\textsc{tr} and beyond is given. Similarly, the Vitali covering theorem for (un)countable covers, denoted \textsc{whbu}, and \textsc{wwkl} (see [61, X.1]) are studied in [46], and the associated equivalences would fit between the third and fourth line of Table 3. Dini’s theorem for nets, denoted \textsc{din} \textsc{net}, is equivalent to \textsc{hbu} by [55, 3.2.1], which yields the equivalence between \textsc{wklt} and Dini’s theorem for sequences, denoted \textsc{din} \textsc{seq}, from [6] via \textsc{ecf}. Basic theorems pertaining to the gauge integral are equivalent to \textsc{hbu} by [44, §3.3], and \textsc{ecf} yields equivalences between \textsc{wklt}_0 and theorems pertaining to the Riemann integral (see [61, IV.2]), but including these would overstretch the following table.

| Higher-order | Second-order | Proof |
|--------------|--------------|-------|
| \Delta\text{-}\text{comprehension} | \Delta_1^1\text{-}\text{comprehension} | Theorem 3.25 |
| \textsc{whbu} | \textsc{wwkl}/\text{countable Vitali} | [40] |
| \textsc{hbuc} | \textsc{wklt}_0/\text{countable Heine-Borel} | [61, IV.1-2] |
| \textsc{hbuc} \rightarrow \Sigma\text{-}\text{sep} | \textsc{wklt}_0 \rightarrow \Sigma_1^1\text{-}\text{sep} | Theorem 3.23 |
| \textsc{din} \textsc{net} \leftrightarrow \textsc{hbuc} | \textsc{wklt}_0 \leftrightarrow \textsc{din} \textsc{seq} | [55, §3]. |
| \textsc{wklt}_0 \leftrightarrow \textsc{hbuc} | \textsc{wklt}_0 \leftrightarrow \text{countable Heine-Borel} | Remark 3.27 |
| \textsc{boot} | \textsc{aca}_0 | Theorem 3.29 |
| \textsc{boot} \leftrightarrow \textsc{hbuc} | \textsc{aca}_0 \leftrightarrow \textsc{wklt}_0 | Theorem 3.24 |
| \textsc{boot} \leftrightarrow \text{mct}_C | \textsc{aca}_0 \leftrightarrow \text{mct}_C \textsc{seq} | Theorem 3.7 |
| \textsc{boot} \leftrightarrow \text{range} | \textsc{aca}_0 \leftrightarrow \text{range} | Theorem 3.20 |
| \Sigma\text{-}\text{tr} | \textsc{atr}_0 | Definition 3.28 |
| \textsc{t-sep} \rightarrow \Sigma\text{-}\text{tr} | \Sigma_1^1\text{-}\text{separation} \rightarrow \textsc{atr}_0 | below Definition 3.29 |
| \textsc{boot}_2 | \Pi_1^1\text{-}\text{ca}_0 | Definition 3.29 |
| \textsc{boot}_{k+1} | \Pi_1^1\text{-}\text{ca}_0 | below Definition 3.29 |

Figure 3. The connection between the Plato and Gödel hierarchies: \textsc{ecf} converts the first to the second column, via the third.

By Table 3 (5.7) becomes the following sequence of implications under \textsc{ecf}:

\[
\Pi_1^1\text{-}\text{ca}_0 \rightarrow \Pi_1^1\text{-}\text{ca}_0 \rightarrow \textsc{atr}_0 \rightarrow \textsc{ca}_0 \rightarrow \textsc{wklt}_0 \rightarrow \textsc{rca}_0. \quad (3.8)
\]

---

8The definitions of \textsc{boot}_{k} and \Sigma\text{-}\text{tr} may be found in Definitions 3.28 and 3.29.
Moreover, results go in both directions: the proof of Theorem 3.20 establishes that
\( \text{MCT}^{[0,1]}_{\text{net}} \rightarrow \text{BOOT} \) using \( \Delta \)-comprehension. This proof is an almost verbatim copy of the associated second-order proof in [61, p. 107], i.e. there is also a connection at the level of proofs. This is not an isolated case: many so-called recursive counterexamples give rise to reversals in RM, and these results can be lifted to obtain higher-order results in many cases, as studied in detail in [56].

We however first need to discuss the nature of the translation described in Table 3.

Remark 3.19 (The nature of ECF). We discuss the meaning of the words ‘A is converted into B by the ECF-translation’. Such statement is obviously not to be taken literally, as e.g. \([\text{BOOT}]_{\text{ECF}}\) is not verbatim ACA\(_0\). Nonetheless, \([\text{BOOT}]_{\text{ECF}}\) follows from ACA\(_0\) by noting that \((\exists f^1)(Y(f, n) = 0) \leftrightarrow (\exists \sigma^0)(Y(\sigma * 00, n) = 0)\) for continuous \(Y^2\) (see Theorem 3.2). Similarly, \([HBU]_{\text{ECF}}\) is not verbatim the Heine-Borel theorem for countable covers, but the latter does imply the former by noting that for continuous functions, the associated canonical cover has a trivial countable sub-cover enumerated by the rationals in the unit interval. In general, that continuous objects have countable representations is the very foundation of the formalisation of mathematics in \(L_2\), and identifying continuous objects and their countable representations is routinely done. Thus, when we say that \(A\) is converted into \(B\) by the ECF-translation, we mean that \([A]_{\text{ECF}}\) is about a class of continuous objects to which \(B\) is immediately seen to apply, with a possible intermediate step involving representations. Since this kind of step forms the bedrock of classical RM, it would therefore appear harmless in this context.

For the rest of this section, we shall establish the results in Table 3.

First of all, let \(\text{MCT}^{C}_{\text{seq}}\) be the monotone convergence theorem for sequences in \(C\), which is equivalent to ACA\(_0\) by [61, III.2]. The ECF-translation converts \(\text{MCT}^{C}_{\text{seq}} \leftrightarrow \text{BOOT} \) into \(\text{MCT}^{C}_{\text{seq}} \leftrightarrow \text{ACA}_0\) following Remark 3.19. Indeed, if a net \(x_d\) is continuous in \(d\), then \((\exists d \in D)(x_d > y)\) is equivalent to a \(\Sigma^0_1\)-formula and the ‘usual’ interval halving proof goes through for \([\text{MCT}^{C}_{\text{seq}}]_{\text{ECF}}\) given ACA\(_0\).

Secondly, ACA\(_0\) is equivalent to range, i.e. the existence of the range of any one-to-one \(f : \mathbb{N} \rightarrow \mathbb{N}\), by [61, III.1.3]; \text{BOOT} satisfies a similar equivalence involving the existence of the range of any type two functional.

Theorem 3.20. The system \(\text{RCA}_0^2\) proves that \text{BOOT} is equivalent to

\[
(\forall G^2)(\exists X^1)(\forall n^0)[n \in X \leftrightarrow (\exists f^1)(G(f) = n)].
\]

(RANGE)

Proof. The forward direction is immediate. For the reverse direction, define \(G^2\) as follows for \(n^0\) and \(g^1\): put \(G((n^0) * g) = n + 1\) if \(Y(g, n) = 0\), and 0 otherwise. Let \(X \subseteq \mathbb{N}\) be as in RANGE and note that

\[
(\forall m^0 \geq 1)(m \in X \leftrightarrow (\exists f^1)(G(f) = m) \leftrightarrow (\exists g^1)(Y(g, m - 1) = 0)).
\]

which is as required for \text{BOOT} after trivial modification. \(\square\)

It goes without saying that \([\text{RANGE}]_{\text{ECF}}\) is merely range, i.e. the existence of the range of any one-to-one \(f : \mathbb{N} \rightarrow \mathbb{N}\), following Remark 3.19.

Thirdly, another neat result is that HBU follows from \text{BOOT}, in contrast to the known comprehension axioms of third-order arithmetic.

Theorem 3.21. The system \(\text{RCA}_0^2 + \text{IND} + \text{BOOT}\) proves HBU while \(\mathbb{Z}_2^2 + \text{QF-AC}^{0,1}\) does not prove \text{BOOT} or HBU.
Proof. The first negative result follows directly from Theorem 3.2, while $\mathbb{Z}^\omega + \text{QF-AC}^{0,1}$ $\not\vdash \text{HBU}$ has been established in [44,45]. For the positive result, we prove $\text{HBU}_c$, i.e. the Heine-Borel compactness of Cantor space, as follows

$$\forall \sigma \in \mathbb{N}, (\forall f \in \mathbb{C})(\exists g \in \mathbb{C})(\forall i \leq k)(f \in [\mathcal{T}_iG(f_i)]).$$  \hspace{1cm} (HBU$_c$)

Note that $\text{HBU} \leftrightarrow \text{HBU}_c$ over $\text{RCA}_0^\omega$ by the proof of [44] Theorem 3.3. Fix $G^2$ and let $A(\sigma)$ be the following formula

$$\exists g \in C)[G(g) \leq |\sigma| \wedge \sigma \ast 00 \cdots \in [\mathfrak{H}G(g)]],$$  \hspace{1cm} (3.9)

where $\sigma^0\ast$ is a finite sequence of natural numbers. Note that the formula in (3.9) in square brackets is quantifier-free. Thus, $\text{BOOT}$ provides a set $X \subseteq \mathbb{N}$ such that $(\forall \sigma^0\ast)(\sigma \in X \leftrightarrow A(\sigma))$, with minimal coding. Now, we have $(\forall f \in C)(\exists n^0)A(f,n)$ since we may take $g = f$ and $n = G(f)$. Hence, we have $(\forall f \in C)(\exists n_0^0)(\exists n \leq 0_n)A(f,n)$. Let $\sigma_1, \ldots, \sigma_{2n_0+1}$ enumerate all binary sequences of length $n_0 + 1$ and define $f_i := \sigma_i \ast 00 \ldots$ for $i \leq 2n_0+1$. Intuitively speaking, we now apply (3.9) for $f_i$ and obtain $g_i$ for each $i \leq 2n_0+1$. Then $(g_1, \ldots, g_{2n_0+1})$ provides the finite sub-cover for $G$. Formally, it is well-known that $\text{ZF}$ proves the ‘finite’ axiom of choice via mathematical induction (see e.g. [67, Ch. IV]). Similarly, one readily uses $\text{IND}$ to prove the existence of the aforementioned finite sequence based on (3.9).

We could replace $\text{IND}$ in the previous theorem by $\text{QF-AC}^{0,1}$, which would be applied to (3.9) to yield the required finite sub-cover. The final part of the proof was first used in [51] to prove without using the Axiom of Choice the equivalence between $\text{HBU}$ and a version involving more general covers. Note that $\text{BOOT} \rightarrow \text{HBU}$ becomes $\text{ACA}_0 \rightarrow \text{WKL}_0$ when applying the ECF-translation.

Fourth, $\text{WK}_0^L$ is equivalent to the separation axiom $\Sigma^0_1$-SEP, i.e. the schema (3.10) for $L_2$-formulas $\varphi_i \in \Sigma^0_1$, by [11, IV.4.4]. Now consider the following separation axiom $\Sigma$-SEP and note that $\text{HBU} \rightarrow \Sigma$-SEP becomes $\text{WK}_0^L \rightarrow \Sigma^0_1$-SEP under $\text{ECF}$.

**Definition 3.22.** $\Sigma$-SEP] For $\varphi_i(n) \equiv (\exists f_1^i)(Y_i(f_i,n) = 0)$, we have

$$\forall n^0(\neg \varphi_1(n) \lor \neg \varphi_2(n)) \rightarrow (\exists Z^1)(\forall n^0) [\varphi_1(n) \rightarrow n \in Z \wedge \varphi_2(n) \rightarrow n \notin Z].$$  \hspace{1cm} (3.10)

**Theorem 3.23.** The system $\text{RCA}_0^\omega + \text{IND} + \text{QF-AC}^{1,1}$ proves $\text{HBU} \rightarrow \Sigma$-SEP.

**Proof.** Suppose $\varphi_i$ is as in $\Sigma$-SEP and satisfies the antecedent of (3.10). Note that using $\text{IND}$, it is straightforward to prove that for every $m^0$, there is a finite binary sequence $\sigma^m\ast$ such that $|\sigma| = m$ and

$$\forall n < m)[\varphi_1(n) \rightarrow (\sigma(n) = 1) \wedge \varphi_2(n) \rightarrow (\sigma(n) = 0)].$$  \hspace{1cm} (3.11)

Now let $A(n,Z)$ be the formula in square brackets in (3.10) and suppose we have $(\forall Z^1)(\forall n^0)A(n,Z)$. Note that $\neg A(n,Z)$ hides two existential quantifiers involving $f_1, f_2$. Applying $\text{QF-AC}^{1,1}$, we obtain $G : C \rightarrow \mathbb{N}$ such that $(\forall Z^1)(\forall n < G(Z))\neg A(n,Z)$. Apply $\text{HBU}_c$ to the canonical cover $\bigcup_{f \in C}[\mathcal{T}_iG(f)]$ and obtain a finite sub-cover $f_0, \ldots, f_k$, i.e. $\bigcup_{i \leq k}[\mathcal{T}_iG(f_i)]$ also covers $C$. Let $k_0$ be $\max_{i \leq k} G(f_i)$ and consider binary $\sigma_0$ of length $k_0 + 2$ satisfying (3.11). Then $g_0 := \sigma_0 \ast 00 \ldots$
is in some neighbourhood of the finite sub-cover, say \( g_0 \in [f_j G(f_j)] \). By definition, \( k_0 \geq G(f_j) \), i.e. \( \overline{G}(f_j) = \overline{G}G(f_j) = f_j G(f_j) \). However, \((3.11)\) is false for \( m = G(f_j) \) and \( \sigma = \overline{f}G(f_j) \), a contradiction. \( \square \)

The usual ‘interval halving’ proof (going back to Cousin in [11]) establishes the reversal, also using countable choice. Variations of Corollary 3.24 are in [42–44], all involving quite different proofs.

**Corollary 3.24.** The system ACA\(_0\) + IND + QF-AC\(^{1,1}\) + HBU proves ATR\(_0\).

**Proof.** The schema \((3.10)\) for L\(_2\)-formulas \( \varphi_i \in \Sigma^1_1 \) is called \( \Sigma^1_1 \)-separation and equivalent to \( \text{ATR}_0 \) by [61] V.5.1. This separation axiom immediately follows from \((\exists^2)\) and \( \Sigma^0_1 \)-SEP, and hence the theorem finishes the proof. \( \square \)

Fifth, the crux of numerous reversals \( T \rightarrow \text{ACA}_0 \) is that the theorem \( T \) (somehow) allows for the reduction of \( \Sigma^0_1 \)-formulas to \( \Delta^0_1 \)-formulas, while \( \Delta^0_1 \)-comprehension (included in \( \text{RCA}_0 \)) then yields the required \( \Sigma^1_1 \)-comprehension, and \( \text{ACA}_0 \) follows immediately. We now show that this technique elegantly extends to \( \text{BOOT} \), which in turn allows us to lift proofs from the second-order to the higher-order framework.

First of all, our base theory (plus countable choice) proves the following higher-order version of \( \Delta^0_1 \)-comprehension.

\[
(\forall Y^2, Z^2)[(\forall n^0)((\exists f^1)(Y(f, n) = 0) \leftrightarrow (\exists g^1)(Z(g, n) = 0)) \quad (\Delta\text{-comprehension}) \rightarrow (\exists X^1)(\forall n^0)(n \in X \leftrightarrow (\exists f^1)(Y(f, n) = 0))]
\]

Note that the ECF-translation converts \( \Delta \)-comprehension into \( \Delta^0_1 \)-comprehension, while QF-AC\(^{0,1} \) becomes QF-AC\(^{0,0} \), following Remark 3.19.

**Theorem 3.25.** The system \( \text{RCA}_0^{\omega} \+ \text{QF-AC}^{0,1} \) proves \( \Delta \)-comprehension.

**Proof.** The antecedent of \( \Delta \)-comprehension implies the following

\[
(\forall n^0)(\exists g^1, f^1)(Z(g, n) = 0 \rightarrow Y(f, n) = 0). \tag{3.12}
\]

Applying QF-AC\(^{0,1} \) to \((3.12)\) yields \( \Phi^{0 \rightarrow 1} \) such that

\[
(\forall n^0)((\exists g^1)(Z(g, n) = 0) \rightarrow Y(\Phi(n), n) = 0), \tag{3.13}
\]

and by assumption an equivalence holds in \((3.13)\), and we are done. \( \square \)

The previous theorem demonstrates its importance in the following proof.

**Theorem 3.26.** The system \( \text{RCA}_0^{\omega} \+ \text{QF-AC}^{0,1} \) proves \( \text{MCT}^{[0,1]} \rightarrow \text{BOOT} \).

**Proof.** In case \( \neg(\exists^2) \), note that \( \text{MCT}^{[0,1]} \) also implies \( \text{MCT}^{[0,1]}_\text{seq} \) as sequences are nets with directed set \([n, \leq_k]\). By [61] III.2.1, \( \text{ACA}_0 \) is available, which readily implies \( \text{BOOT} \) for continuous \( Y^2 \), but all functions on Baire space are continuous by [22] §3.

In case \( (\exists^2) \), we shall establish RANGE and obtain \( \text{BOOT} \) by Theorem 3.20. We let \((D, \preceq_D) \) be a directed set with \( D \) consisting of the finite sequences in \( \mathbb{N}^N \) and \( v \preceq_D w \) if \((\forall i < |v|)(\exists j < |w|)(v(i) = 1 \leftrightarrow w(j)) \). Now fix some \( Y^2 \) and define the net \( c_w : D \rightarrow [0, 1] \) as \( c_w := \sum_{i=0}^{|w|} 2^{-Y(w(i))} \). Clearly, \( c_w \) is increasing and let \( c \) be the limit provided by \( \text{MCT}^{[0,1]} \). Now consider the following equivalence:

\[
(\exists f^1)(Y(f) = k) \leftrightarrow (\exists w^*), ([c_w - c] < 2^{-k} \rightarrow (\exists g \in w)(Y(g) = k)), \tag{3.14}
\]

for which the reverse direction is trivial thanks to \( \lim_{n} c_w = c \). For the forward direction in \((3.14)\), assume the left-hand side holds for \( f = f^1 \) and fix some \( w^*_0 \) such
that $|c - c_w| < \frac{1}{2^k}$. Since $c_w$ is increasing, we also have $|c - c_w| < \frac{1}{2^k}$ for $w \geq_D w_0$. Now there must be $f_0$ in $w_0$ such that $Y(f_0) = k$, as otherwise $w_1 = w_0 \ast \{f_1\}$ satisfies $w_1 \geq_D w_0$ but also $c_{w_1} > c$, which is impossible.

Note that (3.14) has the right form to apply $\Delta$-comprehension (modulo some coding), and the latter provides the set required by RANGE. □

The previous proof is exactly the final part of the proof of [61, III.2.2], save for the replacement of sequences by nets. In other words, proofs from classical RM can be ‘recycled’ as proofs related to the Plato hierarchy. The aforementioned ‘reuse’ comes at a cost however: the proof of $\text{MCT}^{\text{net}}_{\text{net}} \rightarrow \text{BOOT}$ in Corollary 3.9 does not make use of countable choice. The net $c_w$ from the proof should be called a Specker net, similar to Specker sequences, pioneered in [63]. The previous is not an isolated case: many so-called recursive counterexamples give rise to reversals in RM, and these results can often be lifted to obtain higher-order results, as studied in [56]. We list another example of the reuse of recursive counterexamples in Section 3.7.

We note in passing that the above ‘excluded middle’ trick yields the disjunction $\text{ACA}_0 \leftrightarrow [\text{BOOT} \lor (\exists \beta)]$, which is converted into a tautology by ECF.

Next, there is a straightforward generalisation of WKL that is equivalent to HBU.

**Remark 3.27** (Uniform theorems). Dag Normann and the author study the RM and computability theory of uniform theorems in [45]. A theorem is uniform if the objects claimed to exist by the theorem depend on few of its parameters. For instance, the contraposition of $\text{WKL}$ expresses that a binary tree with no paths must be finite. It is readily seen that the latter is equivalent to

$$\forall G^2)(\exists m^0)(\forall T \leq_1 1)(\forall \alpha \in C)(\exists G(\alpha) \notin T) \rightarrow (\forall \beta \in C)(\exists m \notin T).$$ (WKL$_u$)

Note that WKL$_u$ expresses that a binary tree $T$ is finite if it has no paths, and the upper bound $m$ only depends on a realiser $G$ of ‘$T$ has no paths’. For this reason, WKL$_u$ is called uniform weak König’s lemma. It is easy to show that WKL$_0 \leftrightarrow$ HBU by adapting the proof of [22, Theorem 4.6]. It goes without saying that most theorems from the RM of WKL$_0$ have uniform versions that are equivalent to HBU. For instance, uniform versions of the Pincherle, Heine, and Fejér theorems are studied in [45]. Moreover, as documented in [45, Appendix A], many proofs from the literature actually establish the uniform version of the theorem at hand, including the first proof of Heine’s theorem in Stillwell’s introduction to RM ([64]). Finally, the original König’s lemma (see e.g. [61, III.7]) can be given a similar ‘uniform’ treatment, something worthy of future study.

Finally, the Big Five systems ATR$_0$ and $\Pi^1_1$-CA$_0$ boast the following higher-order counterparts, which we mention for completeness only.

**Definition 3.28.** [\Sigma-TR] For $\theta(n, X) \equiv (\exists f^1)(Z(f, X, n) = 0)$ and $Z^2$, we have

$$(\forall X^1)(\text{WO}(X) \rightarrow (\exists Y^1)H_\theta(X, Y)).$$

**Definition 3.29.** [\text{BOOT}$_2$]

$$(\forall Y^2)(\exists X^1)(\forall n^0)(n \in X \leftrightarrow (\exists f^1)(\forall g^1)(Y(f, g, n) = 0)).$$

It is straightforward to show that the ECF-translation converts \Sigma-TR and \text{BOOT}$_2$ into ATR$_0$ and $\Pi^1_1$-CA$_0$. Moreover, let T-SEP be \Sigma-SEP generalised to formulas
Proof. First of all, we prove Theorem 3.31. We first prove the following theorem.

Due to the boundedness property of that it implies \( \Pi^0_1 \)-separation in [61, V.5.1], one readily obtains \( T\text{-SEP} \rightarrow \Sigma^0_3\text{-TR} \). The crucial part is that given countable choice as in QF-AC^{0,1}, \((\exists^0_1 Y)(\forall X)(X, Y)\) has the same form as the \( \varphi_i\) in T-SEP. In light of BOOT and its properties, the generalisation BOOT \( k+1 \) to systems at the level of \( \Pi^1_k\text{-CA}_0 \) is now obvious.

3.6. Convergence in function spaces. In the previous sections, we have obtained a number of convergence theorems for nets that give rise to parallel hierarchies as sketched in Figure [2]. Of course, these theorems do not involve formula classes, but the associated hierarchies are still based on formula classes via \( \Pi^1_k\text{-CA}_0 \).

In this section, we formulate MON, a (third-order) convergence theorem for nets that does not need \( \Pi^1_k\text{-CA}_0 \) to bootstrap to the next level, but rather ‘boots itself’, i.e. RCA \( \omega \) + MON can prove \( \Pi^1_k\text{-CA}_0 \) for any \( k \), via longer and longer proofs.

Now, we have previously considered nets in basic spaces like \( 2^\omega \) and \([0, 1]\). While Moore-Smith in [38] limited themselves to nets in \( \mathbb{R} \), Vietoris already studied nets in (much) more general spaces in [69], even in the early days of nets. Hence, it is a natural question how strong MCT \( \omega \) becomes for nets in e.g. function spaces. Note that this generalisation still is part of the language of third-order arithmetic.

In this section, we show that for nets in the function space \([0, 1]\) → \([0, 1]\), the associated monotone convergence theorem MON becomes extremely powerful, in that it implies \( \Pi^1_k\text{-CA}_0 \) for any \( k \) without additional axioms.

Definition 3.30. [MON] Let \((D, \leq_D)\) be a directed set where \( D \subseteq \mathbb{N}^\omega \). Any increasing net \( F_d : D \to (I \to I) \) converges to some \( H : I \to I \).

Recall that a net \( F_d : D \to (I \to I) \) is increasing if we have that:

\[
(\forall x \in I)(\forall d, e \in D)(d \leq_D e \to F_d(x) \leq_F F_e(x)).
\]

Due to the boundedness property of \( F_d \), for fixed \( x \in I \), the net \( F_d(x) \) converges to some limit, and the limit function from MON is obtained by putting all these individual limits together, as in Theorem 3.33 We first prove the following theorem.

Theorem 3.31. The system RCA \( \omega \) + MON proves (\( \exists^2 \)).

Proof. First of all, we prove MON → (\( \exists^2 \)). Let \( F_n \) be the piecewise linear function that is zero for \( x = 0 \) and 1 for \( x \geq \frac{1}{n} \). Consider the directed set \( (\mathbb{N}, \leq) \) and the net \( F_n \). The latter is increasing in that \((\forall n, m \in \mathbb{N})(\exists x \in [0, 1])(n \leq m \to F_n(x) \leq F_m(x))\), and hence \( F_n \) has a limit \( H : I \to I \) by MON. Clearly, \( H(0) = 0 \) and \( H(x) = 1 \) for \( x \in (0, 1] \), i.e. \( H \) is discontinuous, and [32] §3 yields (\( \exists^2 \)).

Secondly, note that the variable ‘\( f \)’ in the definition of the Suslin functional (\( \exists^2 \)) can be restricted to Cantor space without loss of generality. Moreover, if \( f \in C \) is eventually constant 0 (resp. constant 1), then \( (\exists^0_1 \eta)(\forall \eta)(f(\eta) = 0) \) clearly holds (resp. does not hold). Given \( \exists^2 \), we can decide whether \( f \in C \) is eventually constant, i.e. we may restrict ourselves to \( f \in C \) that are not eventually constant when defining the Suslin functional. Recall that \( \exists^2 \) defines a functional \( \eta^1_{\to 1} \) that converts real numbers in \([0, 1]\) into binary representation, choosing a tail of zeros whenever there are two possibilities.

Now, let \( D \) be the set of finite sequences in Baire space and let \( \leq_D \) be the inclusion ordering, i.e. \( w \leq_D v \) if \((\forall i < |w|)(\exists j < |v|)(w(i) = v(j))\). For \( w^T \in D \), define the net \( F_w(f) \) as 1 if \( (\exists^0_1 \eta)(\forall \eta)(f(\eta) = 0) \) and 0 otherwise. Define \( G_w : D \to (I \to I) \) as \( G_w(x) := F_w(\eta(x)) \). Note that for \( w \leq_D v \), we have
$G_w(x) \leq G_w(x)$ for all $x \in I$, i.e. $G_w$ is increasing in the sense of nets. Let $H : I \to I$ be the limit $\lim_w G_w$, and consider:

$$\forall f^1 \in C \exists H_0(f) = 1 \iff (\exists g^1)(\forall n^0)(f(n) = 0),$$

(3.15)

where $H_0(f)$ is $H(r(f))$ if $r(f)$ has a unique binary representation, and otherwise 0 or 1 depending on whether $f$ is eventually constant 0 or eventually constant 1. For any $f \in C$, (3.15) is immediate in the ‘otherwise’ case in $H_0(f)$, by the above. In the unique representation case, if $H_0(f) = H(r(f)) = 1$ then the definition of limit implies that there is $w \in D$ such that for all $v \geq_D w$, we have $G_v(r(f)) = F_v(f) = 1$, which immediately yields the right-hand side of (3.15). Now let $g_0^1$ be such that $(\forall n^0)(f(n) = 0)$ in the unique representation case and suppose $H_0(f) = H(r(f)) = 0$. Again by the definition of limit, there is $w \in D$ such that for all $v \geq_D w$, we have $G_v(r(f)) = F_v(f) = 0$. This yields a contradiction for $v = w * (g_0^1)$, and (3.15) follows. Clearly, the latter defines ($S^2$).

\[\square\]

**Corollary 3.32.** For any $k$, the system $\text{RCA}_0^\omega + \text{MON}$ proves ($S^2_k$) and BOOT.

**Proof.** To obtain BOOT, repeat the proof of the theorem with $(\forall n^0)(f[\overline{n}]) = 0)$ replaced by $Y(f, g)$. To obtain ($S^2_2$), $(\exists g^1)(\forall h^1)(\exists n)(g[\overline{n}]) = 0)$ is equivalent to the formula $(\exists g^1)(Y(f, g) = 0)$, where $Y^2$ is defined in terms of $S^2$. Now repeat the previous step until $S^2_k$ is obtained. \[\square\]

Finally, MON is not that much more ‘exotic’ than e.g. $\text{MCT}_{\text{net}}^{[0,1]}$ by the following.

**Theorem 3.33.** The system $\text{RCA}_0^\omega$ proves $[\text{MCT}_{\text{net}}^{[0,1]} + \text{QF-AC}^{1,1} + (\exists^2)] \to \text{MON}$.

**Proof.** Let $F_d$ be as in MON. By $\text{MCT}_{\text{net}}^{[0,1]}$, for fixed $x \in I$, the net $F_d(x)$ converges to some limit $y \in I$, implying the following formula:

$$(\forall x \in I)(\exists y \in I)(\forall k^0)(\exists d \in D)(|F_d(x) - y| < \frac{1}{2^k}).$$

Apply QF-AC$^{0,1}$ to the underlined formula to obtain

$$(\forall x \in I)(\exists y \in I)(\exists d^0 \in D)(\forall k^0)(|F_d(x) - y| < \frac{1}{2^k}),$$

which qualifies for QF-AC$^{1,1}$ in the presence of ($\exists^2$) and coding of the second existential quantifier as a type one object. The resulting functional is the limit as required for MON. \[\square\]

The previous proof actually provides a modulus of convergence for the limit process $\lim_d F_d = H$. Moreover, introducing a modulus of convergence in MON, one obtains mutatis mutandis that the enriched principle implies QF-AC$^{1,1}$, and hence an equivalence in the previous theorem. One can also prove that MON is equivalent to the following straightforward generalisation of BOOT:

$$\forall Y^2(\exists G^2)(\forall f^1)(G(f) = 0 \iff (\exists g^1)(Y(f, g) = 0)).$$

The proof is similar to that of Theorem 3.31 and we therefore omit it.
3.7. Index sets beyond Baire space. In this section, we study the Bolzano-Weierstrass theorem for nets with index sets beyond Baire space, namely subsets of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$. Such index sets are also studied in [55] Appendix A in the context of computability theory and RM, but we stress that these results are only given (here and in [55]) by way of illustration: the general study of nets is perhaps best undertaken in a suitable set theoretic framework. That is not to say this section should be dismissed as spilerei; our results come with conceptual motivation:

(i) Index sets beyond Baire space do occur ‘in the wild’, namely in e.g. fuzzy mathematics and the iterated limit theorem, by Remark 3.40.

(ii) It is a natural question whether the above proofs generalise to higher types.

(iii) In light of Corollary 3.8, it is a natural question whether nets with index sets beyond Baire space take us beyond second-order arithmetic.

(iv) It is a natural question whether $\text{ECF}$ maps results pertaining to index sets beyond Baire space into second-order arithmetic.

As we will see below, the answer is positive for each of the three questions. Thus, similar to Definition 2.4 we introduce the following.

Definition 3.34. [RCA$_0^w$] A ‘subset $E$ of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$’ is given by its characteristic function $F_E^v \leq 3$, i.e. we write ‘$Y \in E$’ for $F_E(Y) = 1$ for any $Y^2$. A ‘binary relation $\preceq$ on the subset $E$ of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$’ is given by the associated characteristic function $G^v_{\preceq} \rightarrow 0$, i.e. we write ‘$Y \preceq Z$’ for $G^0_{\preceq}(Y, Z) = 1$ and any $Y, Z \in E$.

Definition 3.35. [$\text{BW}_{\text{net}}^1$] Any net in Cantor space indexed by a subset of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ has a convergent sub-net.

Theorem 3.36. The system $Z_2^\Omega + \text{BW}_{\text{net}}^1$ proves $\Pi^2_1$-$\text{CA}_0$.

Proof. A $\Sigma^2_1$-formula $\varphi(n) \in L_3$ is readily seen to be equivalent to a formula $(\exists Y^2)(Z(Y, n) = 0)$ for $Z^3$ defined in terms of $\exists^3$. Let $E$ be the set of finite sequences in $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ and let $\preceq_E$ be the inclusion ordering, i.e. $w \preceq_E v$ if $(\forall v < |w|)(\exists j < |v|)(w(i) = v(j))$. Define the net $f_w : E \to C$ as $f_w := \lambda k; F(w, k)$ where $F(w, k) = 1$ if $(\exists i < |w|)(Z(w(i), k) = 0)$, and zero otherwise. Using $\text{BW}_{\text{net}}^1$, let $\phi : B \to E$ and $f^1$ be such that $\lim_0 \phi(b) = f$. We now establish that

\[
(\forall v^0)[(\exists Y^2)(Z(Y, n) = 0) \leftrightarrow f(n) = 1].
\]

(3.16)

For the reverse direction, note that for fixed $n_0$, if $Z(Y, n_0) = 0$ for all $Y^2$, then $f_w(n_0) = 0$ for any $w \in E$. The definition of limit then implies $f(n_0) = 0$, i.e. we have established (the contraposition of) the reverse direction. For the forward direction in (3.16), suppose there is some $n_0$ such that $(\exists Y^2)(Z(Y, n_0) = 0) \land f(n_0) = 0$. Now, $\lim_0 \phi(b) = f$ implies that there is $b_0 \in B$ such that for $b \preceq_B b_0$, we have $\overline{f_\phi(b)}(n_0) = \overline{f}(n_0)$, i.e. $f_\phi(b)(n_0) = 0$ for $b \preceq_B b_0$. Let $Y^0$ be such that $Z(Y_0, n_0) = 0$, and use the second item in Definition 2.11 for $d = (Y_0)$, i.e. there is $b_1 \in B$ such that $\phi(b_1) \preceq_E (Y_0)$ for any $b \preceq_B b_1$. Now let $b_2 \in B$ be such that $b_2 \preceq_B b_1$ as provided by Definition 2.14. On one hand, $b_2 \preceq_B b_1$ implies that $\phi(b_2) \preceq_E (Y_0)$, and hence $f_\phi(b_2)(n_0) = F(\phi(b_2), n_0) = 1$, as $Y_0$ is in the finite sequence $\phi(b)$ by the definition of $\preceq_E$. On the other hand, $b_2 \preceq_B b_0$ implies that $f_\phi(b_2)(n_0) = f(n_0) = 0$, a contradiction. Hence the forward direction follows and so does (3.16), yielding the set $\{n : \phi(n)\}$, as required by $\Pi^2_1$-$\text{CA}_0$. \hfill \Box

We could formulate a monotone convergence theorem similar to $\text{BW}_{\text{net}}^1$, and the former would be equivalent to $\text{BOOT}^1$ below in the same way as for Theorem 3.7.
We limit ourselves to the following results which establish that quite general convergence theorems are still mapped in the lower regions of second-order arithmetic.

**Corollary 3.37.** The system $Z_2^\Omega + \text{BW}^1_{\text{net}}$ proves the comprehension axiom:

$$\forall Z^3(\exists X^1)(\forall n^0)(n \in X \leftrightarrow (\exists Y^2)(Z(Y, n) = 0)).$$

(BOOT$^1$)

The $L_2$-sentence $[\text{BOOT}^1_{\text{ECF}}]$ is provable in $\Pi^1_2$-$\text{CA}_0$.

**Proof.** The first part is immediate by the proof of the theorem. For the second part, let $\gamma^1$ be a total associate for $Z^3$ in $\text{BOOT}^1$. The right-hand side of $[\text{BOOT}^1_{\text{ECF}}]$ is

$$(\exists \alpha^1)((\forall \beta^1)(\exists m^0)(\alpha(\beta m) > 0) \land (\exists k^0)(\gamma(\beta k, n) = 1)), \quad (3.17)$$

and the set consisting of such $n^0$ is clearly definable in $\Pi^1_2$-$\text{CA}_0$. \hfill \square

Next, Specker nets are used in the proof of Theorem 3.26 to establish

**Theorem 3.38.** The system $Z_2^\Omega + \text{QF-AC}^{0,2} + \text{BW}^1_{\text{net}}$ proves the following:

$$\forall G^3(\exists X^1)(\forall n^0)[n \in X \leftrightarrow (\exists Y^2)(G(Y) = n)]. \quad (\text{RANGE}^1)$$

**Proof.** A slight modification of the proof of Theorem 3.26 goes through as follows: let $E$ be the set of finite sequences in $\mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ and let $\leq_E$ be the inclusion relation, for which $\exists^3$ is needed (instead of $\exists^2$). The Specker net $c_{w^2} : E \rightarrow [0, 1]$ is defined in exactly the same way as in Theorem 3.26, namely as $c_{w^2} := \sum_{i=0}^{|w^2|-1} 2^{-Z(w^2(i))}$, where $Z^3$ is given. The associated version of (3.17) is:

$$\forall Y^2(Z(Y) = k) \leftrightarrow (\forall w^2)(|c_{w^2} - c| < 2^{-k} \rightarrow (\exists V \in w)(Z(V) = k)), \quad (3.18)$$

where $c = \lim_w c_{w^2}$ is provided by $\text{BW}^1_{\text{net}}$. Applying $\text{QF-AC}^{0,2}$ to (3.18) as in the proof of Theorem 3.26 yields the set $X \subset \mathbb{N}$ required for $\text{RANGE}^1$. \hfill \square

In light of the proofs of Theorem 3.30 and 3.38 it is now clear that the above proofs readily generalise to higher types. To avoid repetition, we do not study further generalisations of convergence theorems for nets in this paper. We do list some nice results: let $\text{BW}^\sigma_{\text{net}}$ be the obvious generalisation of $\text{BW}^1_{\text{net}}$ to index sets of type $\sigma + 1$ objects. A straightforward modification of Theorem 3.30 implies that $\text{RCA}^\sigma_0 + (\exists^k+2) + \text{BW}^k_{\text{net}}$ proves $\Pi^k+1$-comprehension for $k \geq 1$. Hence, the general Bolzano-Weierstrass theorem for nets is extremely hard to prove.

Recall Corollary 3.14 which implies $\text{CAU}_{\text{mod}} \leftrightarrow \text{QF-AC}^{0,1}$ over $Z_2^\Omega$. Let $\text{CAU}^2_{\text{mod}}$ be the generalisation of $\text{CAU}_{\text{mod}}$ to index sets that are subsets of $\mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$.

**Corollary 3.39.** The system $\text{RCA}^\sigma_0 + (\exists^4)$ proves $\text{QF-AC}^{0,2} \leftrightarrow \text{CAU}^2_{\text{mod}}$.

**Proof.** Generalise the proof of Theorem 3.13 in the same way as Theorem 3.36. \hfill \square

Let $\text{CAU}^\sigma_{\text{mod}}$ be the obvious generalisation of $\text{CAU}^2_{\text{mod}}$ to sets of type $\sigma + 1$ objects. One then readily proves $\text{QF-AC}^{0,k} \leftrightarrow \text{CAU}^k_{\text{net}}$ over $\text{RCA}^\sigma_0 + (\exists^k+2)$.

We finish this section with a conceptual remark on ‘large’ index sets and their occurrence in mathematics and logic.

**Remark 3.40** (Large index sets). Zadeh founded the field of fuzzy mathematics in [71]. The core notion of fuzzy set is a mapping that assigns values in $[0, 1]$, i.e. a ‘level’ of membership, rather than the binary relation from usual set theory. The first two chapters of Kelley’s General Topology (29) are generalised to the
setting of fuzzy mathematics in [48]. As an example, [48 Theorem 11.1] is the fuzzy generalisation of the classical statement that a point is in the closure of a set if and only if there is a net that converges to this point. However, as is clear from the proof of this theorem, to accommodate fuzzy points in $X$, the net is indexed by the space $X \to [0,1]$. Moreover, the iterated limit theorem (both the fuzzy and classical versions: [48 Theorem 12.2] and [29]) involves an index set $E_m$ indexed by $m \in D$, where $D$ is an index set. Thus, ‘large’ index sets are found in the wild.

We may also formulate two arguments in favour of ‘large’ index sets based on the results in [44, 55], as follows. First of all, by way of an exercise, the reader should generalise the well-known formulation of the Riemann integral in terms of nets (see e.g. [29, p. 79]) to the gauge integral as studied in [44, §3.3]. As will become clear, this generalisation involves nets indexed by $\mathbb{R} \to \mathbb{R}$-functions.

Secondly, the results in [55, §4.3-4.5] connect continuity and open sets to nets, all in $\mathbb{R}$, while avoiding the Axiom of Choice. As is clear from the proofs (esp. the use of the net $x_d := d$, replacing $\mathbb{R}$ by a larger space requires the introduction of nets with a similarly large index set. In particular, to show that a net-closed set $C$ is closed (see [55 Theorem 4.15] for $C \subseteq \mathbb{R}$), one seems to need nets with an index set the same cardinality as $C$.

Acknowledgement 3.41. Our research was supported by the John Templeton Foundation via the grant a new dawn of intuitionism with ID 60842. We express our gratitude towards this institution. We thank Adrian Mathias, Thomas Streicher, and Anil Nerode for their valuable advice. Opinions expressed in this paper do not necessarily reflect those of the John Templeton Foundation.

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