QUENCHED ASYMPTOTICS FOR SYMMETRIC LÉVY PROCESSES INTERACTING WITH POISSONIAN FIELDS

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Abstract. We establish explicit quenched asymptotics for pure-jump symmetric Lévy processes in general Poissonian potentials, which is closely related to large time asymptotic behavior of the solution to nonlocal parabolic Anderson problem with Poissonian interaction. In particular, when the density function with respect to the Lebesgue measure of the associated Lévy measure is given by
\[
\rho(z) = \frac{1}{|z|^{d+\alpha}} \mathbf{1}_{(|z| \leq 1)} + e^{-|z|^\theta} \mathbf{1}_{(|z| > 1)}
\]
for some \( \alpha \in (0, 2), \theta \in (0, \infty) \) and \( c > 0 \), exact quenched asymptotics are derived for potentials with the shape function given by \( \varphi(x) = 1 \wedge |x|^{-d-\beta} \) for \( \beta \in (0, \infty) \) with \( \beta \neq 2 \). We also discuss quenched asymptotics in the critical case (e.g., \( \beta = 2 \) in the example mentioned above).

Keywords: symmetric Lévy process; Poissonian potential; quenched asymptotic; non-local parabolic Anderson problem

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1. Background and main results

This paper is devoted to the analysis of large time asymptotic behavior of the solution to nonlocal parabolic Anderson problem with Poissonian interaction:
\[
\frac{\partial u}{\partial t} = Lu - V^\omega u
\]
on \([0, \infty) \times \mathbb{R}^d\) with the initial condition \( u(0, x) = 1 \). Here, \( L \) is the infinitesimal generator of pure-jump symmetric Lévy process \( Z := (Z_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^d} \) with the characteristic exponent
\[
\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\langle \xi, z \rangle)) \nu(dz)
\]
for some symmetric Lévy measure \( \nu \) (i.e., \( \nu \) is a Radon measure on \( \mathbb{R}^d \setminus \{0\} \) that satisfies \( \int_{\mathbb{R}^d \setminus \{0\}}(|z|^2 \wedge 1) \nu(dz) < \infty \) and \( \nu(A) = \nu(-A) \) for any \( A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \)); the potential
\[
V^\omega(x) = \int_{\mathbb{R}^d} \varphi(x - y) \mu^\omega(dy),
\]
where \( \mu^\omega \) is a Poissonian random measure on \( \mathbb{R}^d \) with density \( \rho \, dx, \rho > 0 \), on a given probability space \( (\Omega, \mathbb{Q}) \), and \( \varphi \) is a non-negative profile function on \( \mathbb{R}^d \). We refer to the monograph [17] for background on this topic. Throughout this paper, \( \mathbb{Q} \) and \( \mathbb{E}_\mathbb{Q} \) denote the probability and the expectation, respectively, generated by the Poissonian field; while \( \mathbb{P}_x \) and \( \mathbb{E}_x \) denote the probability and the expectation, respectively, corresponding to the Lévy process \( Z \) with the starting point \( x \in \mathbb{R}^d \).

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Under mild assumptions (see Subsection 2.2), the solution to the problem (1.1) enjoys the Feynman-Kac representation
\begin{equation}
(1.4) \quad u^\omega(t, x) = E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) \right].
\end{equation}
Thus, the analysis of the properties of the solution to (1.1) can be done via (1.4) by estimating $u^\omega(t, x)$. There are a number of works on the large time behavior of $u^\omega(t, x)$ in both the annealed sense (averaged with respect to $Q$) and the quenched sense (almost sure with respect to $Q$). In this paper we will mainly analyse the quenched behavior of $u^\omega(t, x)$ for pure-jump symmetric Lévy processes in Poissonian potentials with more general profile function $\varphi$. Let us begin with recalling the history on related topics. The annealed asymptotics of $u^\omega(t, x)$ was first established by Donsker-Varadhan [6] for symmetric (but not necessarily isotropic) non-degenerate $\alpha$-stable processes (including Brownian motion). They proved in [6, Theorem 3] that, when the profile $\varphi(x)$ is of order $o(1/|x|^{d+\alpha})$ as $|x| \to \infty$, which is referred to the light tailed case, Theorem 6.3') that
\begin{equation}
(1.5) \quad \lim_{t \to \infty} \frac{\log E_Q[u^\omega(t, x)]}{td/(d+\alpha)} = -\rho^\alpha/(d+\alpha) \left( \frac{d + \alpha}{\alpha} \right) \left( \frac{\alpha \lambda_1(B(0, 1))}{d} \right)^{d/(d+\alpha)},
\end{equation}
where
\begin{equation}
\lambda_1(B(0, 1)) = \inf_{\text{open } U, |U| = w_d} \lambda_{1}(U),
\end{equation}

where $w_d$ is the volume of the unit ball and $\lambda_{1}(U)$ is the principal Dirichlet eigenvalue for symmetric $\alpha$-stable process killed on exiting $U$. In particular, when the symmetric $\alpha$-stable process is isotropic, it follows from the Faber-Krahn isoperimetric inequality that the infimum in the definition of $\lambda_{1}(B(0, 1))$ above is attained on the ball of radius $r_d = w_d^{-1/d}$ and so $\lambda_{1}(B(0, 1)) = w_d^{\alpha/d} \lambda_{1}(B(0, 1))$. Then, in this case (1.5) is reduced into
\begin{equation}
(1.6) \quad \lim_{t \to \infty} \frac{\log E_Q[u^\omega(t, x)]}{td/(d+\alpha)} = -\rho w_d^\alpha/(d+\alpha) \left( \frac{d + \alpha}{\alpha} \right) \left( \frac{\alpha \lambda_1(B(0, 1))}{d} \right)^{d/(d+\alpha)}.
\end{equation}
Later Ōkura [12] extended [6, Theorem 3] to a large class of symmetric Lévy processes whose exponent $\psi \exp(-t \psi(\cdot)^{1/2}) \in L^1(\mathbb{R}^d; dx)$ for all $t > 0$ and can be written as
\begin{equation}
(1.7) \quad \psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha), \quad |\xi| \to 0
\end{equation}
for some $\alpha \in (0, 2]$. Here, $\psi^{(\alpha)}(\xi)$ is the characteristic exponent of a symmetric non-degenerate $\alpha$-stable process $Z^{(\alpha)}$ (see (2.2) below) satisfying some kind of summability condition on $\psi^{(\alpha)}(\xi) := \inf_{t \geq 1} t^\alpha \psi^{(\alpha)}(t^{-\alpha})$; see Subsection 2.1 for more details. More explicitly, it was shown in [12, Theorem 4.1] that (1.5) still holds for symmetric Lévy processes above with $\lambda_{1}(B(0, 1))$ defined via the principle Dirichlet eigenvalue for the killed symmetric $\alpha$-stable process $Z^{(\alpha)}$ with exponent $\psi^{(\alpha)}$ given in (1.7).

When the characteristic exponent of the Lévy process $Z$ further satisfies
\begin{equation}
\psi(\xi) = O(|\xi|^\alpha), \quad |\xi| \to 0,
\end{equation}
and the shape function $\varphi$ fulfills $K := \lim_{|x| \to \infty} \varphi(x)|x|^{d+\beta} \in (0, \infty)$ for some $0 < \beta < \alpha$ (which is referred to the heavy tailed case), Ōkura proved in [13, Theorem 6.3'] that
\begin{equation}
(1.8) \quad \lim_{t \to \infty} \frac{\log E_Q[u^\omega(t, x)]}{td/(d+\beta)} = -\rho w_d^\beta \left( \frac{\beta}{d + \beta} \right) K^{d/(d+\beta)}
\end{equation}
holds for symmetric Lévy processes satisfying (1.7). See Pastur [15] for the first result on this direction when $Z$ is Brownian motion. The reader also can be referred to [13, Theorem 6.1] and [14, Theorem 1 and Remarks] for the study in the critical case, i.e., $K := \lim_{|x| \to \infty} \phi(x)|x|^{d+\alpha} \in (0, \infty)$; see the appendix for details. In particular, according to all the conclusions above, the annealed asymptotics of $u^\omega(t,x)$ is of order $t^{-d/(d+\beta)}$ when $\phi(x) = K (1 \wedge |x|^{\beta})$. However, in the light tailed case, the right hand side of (1.5) for the annealed asymptotics of $u^\omega(t,x)$ is independent of $K$ and $\beta$, while in the heavy tailed cases that of (1.8) only depends on the constants $K$ and $\beta$.

Compared with the annealed asymptotics, the study of the quenched asymptotics of $u^\omega(t,x)$ is relatively limit. The first result for the quenched asymptotics of $u^\omega(t,x)$ for Brownian motion moving in a Poissonian potential was established by Sznitman in [16, Theorem], which showed that when $\phi$ is compactly supported (which in particular corresponds to the shape function $\phi(x) = K (1 \wedge |x|^{\beta})$ with $\beta = \infty$, and so it belongs to the special light tailed case), $\mathbb{Q}$-almost surely for all $x \in \mathbb{R}^d$,

$$
\lim_{t \to \infty} \frac{\log u^\omega(t,x)}{t/(\log t)^{\beta/2}} = - \left( \frac{\rho_{\omega d}}{d} \right)^{2/d} \lambda_{\text{BM}}(B(0,1)),
$$

(1.9)

where $\lambda_{\text{BM}}(B(0,1))$ is the principle Dirichlet eigenvalue for the Brownian motion killed on exiting $B(0,1)$. More recently, the quenched asymptotics of $u^\omega(t,x)$ for symmetric Lévy processes satisfying (1.7) have been extensively studied in [9]; see [9, Table 1 in p. 165] for results concerning explicit Lévy processes.

Concerning Brownian motions in a heavy tailed Poissonian potential, for example, $\phi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0, 2)$, it was shown in [8, Theorem 2] that $\mathbb{Q}$-almost surely for any $x \in \mathbb{R}^d$,

$$
\lim_{t \to \infty} \frac{\log u^\omega(t,x)}{t/(\log t)^{\beta/d}} = - \frac{d}{d + \beta} \left( \frac{\beta}{d(d + \beta)} \right)^{\beta/d} \left( \frac{\rho_{\omega d}}{d} \Gamma \left( \frac{\beta}{d + \beta} \right) \right)^{(d+\beta)/d}.
$$

(1.10)

(Indeed, the second order asymptotics were proved in [8].)

However, the quenched asymptotics of $u^\omega(t,x)$ for symmetric Lévy processes in heavy tailed cases, as well as in the light tailed cases when $\phi$ does not have compact support, are still unknown. The goal of this paper is to fill up these gaps. To state our main contribution, in the following two results we are restricted ourselves on the special but typical shape function $\phi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0, \infty]$.

**Theorem 1.1.** Let $Z$ be a rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ with $\alpha \in (0, 2)$. Then,

(i) When $\beta \in (\alpha, \infty]$, $\mathbb{Q}$-almost surely for all $x \in \mathbb{R}^d$,

$$
-(d + \alpha)^{\alpha/(d+\alpha)} \left[ \left( \frac{\alpha}{d} \right)^{d/(d+\alpha)} + \left( \frac{d}{\alpha} \right)^{\alpha/(d+\alpha)} \right] A_1 \leq \liminf_{t \to \infty} \frac{u^\omega(t,x)}{t^{d/(d+\alpha)}} \\
A_1 \leq \limsup_{t \to \infty} \frac{u^\omega(t,x)}{t^{d/(d+\alpha)}} \\
A_1 \leq - \alpha(\alpha + d/2)^{-d/(\alpha+d)} A_1,
$$

where

$$
A_1 = \left( \frac{\rho_{\omega d}}{d} \right)^{\alpha/(d+\alpha)} [\lambda_1^{(\alpha)}(B(0,1))]^{d/(d+\alpha)}.
$$

(ii) When $\beta \in (0, \alpha)$, $\mathbb{Q}$-almost surely for all $x \in \mathbb{R}^d$,

$$
-(d + \alpha)^{\beta/(d+\beta)} \left[ \left( \frac{\beta}{d} \right)^{d/(d+\beta)} + \left( \frac{d}{\beta} \right)^{\beta/(d+\beta)} \right] A_1 \leq \liminf_{t \to \infty} \frac{u^\omega(t,x)}{t^{d/(d+\beta)}} \\
A_1 \leq \limsup_{t \to \infty} \frac{u^\omega(t,x)}{t^{d/(d+\beta)}}.
$$
For some $\alpha \in (0, 2)$, $\theta \in (0, \infty]$ and $c > 0$, where $f \asymp g$ means that there is a constant $c_0 \geq 1$ such that $c_0^{-1}g \leq f \leq c_0f$. Then,

(i) When $\beta \in (2, \infty]$, $Q$-almost surely for all $x \in \mathbb{R}^d$,

$$\lim_{t \to \infty} \frac{u^\omega(t, x)}{t^{d/(d + \beta)}} = - \left( \frac{\rho_d}{d} \right)^{2/d} \lambda_1^{(2)}(B(0, 1)),$$

where $\lambda_1^{(2)}(B(0, 1))$ is the first Dirichlet eigenvalue for the killed Brownian motion when exiting the ball $B(0, 1)$ and with the covariance matrix $(a_{ij})_{1 \leq i, j \leq d}$ as follows

$$a_{ij} = \int_{\mathbb{R}^d \setminus \{0\}} z_iz_j \nu(dz), \quad 1 \leq i, j \leq d.$$ 

(ii) When $\beta \in (0, 2)$, $Q$-almost surely for all $x \in \mathbb{R}^d$,

$$\lim_{t \to \infty} \frac{u^\omega(t, x)}{t^{d/(d + \beta)}} = - \left( \frac{\rho_d}{d + \beta} \right)^{\gamma(d + \beta)/d} \left[ \frac{\rho_d}{d + \beta} \right]^{(d+\beta)/d}.$$ 

Theorems 1.1 and 1.2 show that the quenched asymptotics of $u^\omega(t, x)$ for pure-jump symmetric Lévy process in Poissonian potentials depend not only on the shape function $\varphi$ in the potential, but also the properties of the Lévy measure (for large jumps) of the Lévy process $Z$. This phenomenon happens for the annealed asymptotics of $u^\omega(t, x)$, but there is much more involved in the quenched asymptotics. For instance, considering the example with $\theta \in (0, 1)$ in Theorem 1.2 which satisfies (1.7) with $\alpha = 2$, in the light tailed case the precise value of the annealed asymptotics of $u^\omega(t, x)$ is independent of $\theta$ by (1.6), but that of the quenched asymptotics of $u^\omega(t, x)$ do depend on $\theta$ by Theorem 1.2(i). The same occurs for the heavy tailed case. On the other hand, in both light tailed and heavy tailed cases, for rotationally symmetric $\alpha$-stable processes, by Theorem 1.1 the correct order of the quenched asymptotics of $u^\omega(t, x)$ is the same as that of the annealed asymptotics; however, according to Theorem 1.2, it is not true for symmetric Lévy processes with exponential decay for large jumps; see [9, Section 1] for more discussions on this point in the light tailed setting.

Next, we briefly make comments on in our proofs for the quenched asymptotics of $u^\omega(t, x)$ for pure-jump symmetric Lévy process in general Poissonian potentials.

(i) Compared with [9], the crucial ingredient to handle general light tailed cases is the observation that, due to the light tail of the potential, $V^\omega(x)$ is compared with $\tilde{V}^\omega(x)$ whose shape function $\varphi$ is compactly supported. This enables us to use the classical approach in [16, 9]; that is, when the shape function $\varphi$ has a compact support, $Q$-almost surely there exists a large area where the potential is zero and so the principle Dirichlet eigenvalue of the process $Z$ is naturally involved in the quenched asymptotics of $u^\omega(t, x)$. When $\beta = \infty$ (this is just the case that the shape function $\varphi$ has a compact support), Theorems 1.1(i) and 1.2(i) have been
proven in [9]; see [9, Table 1 in p. 165] for more details. Based on this and the strategy of the approach mentioned above, we believe that assertions of [9] should hold true for all light tailed cases.

(ii) In heavy tailed cases, the potential $V^\omega(x)$ will play a dominated role in the quenched asymptotics of $u^\omega(t, x)$. Similar to the Brownian motion case studied in [8], it is natural to expect that the main contribution of $u^\omega(t, x)$ defined by (1.4) comes from the process $Z$ which spends most of the time in the area where $V^\omega(x)$ takes small value. Motivated by the fact, we partly adopt the argument in [8] to treat upper bounds of the principle Dirichlet eigenvalue for the random Schrödinger operator associated with the equation (1.1), which in turn yield explicit quenched asymptotics of $u^\omega(t, x)$ in general heavy tailed setting.

(iii) To consider quenched asymptotics for pure-jump symmetric Lévy processes in both light tailed and heavy tailed potentials at the same time, we give an unified approach which is inspired by [2] (which studied quenched asymptotics for Brownian motion in renormalized Poissonian potentials) and based on recent development on (Dirichlet) heat kernel estimates for symmetric jump processes. We emphasize that the argument of lower bounds for quenched asymptotics of $u^\omega(t, x)$ here is different from that in [9]. In particular, the lower bound for the quenched asymptotics of $u^\omega(t, x)$ in Theorem 1.1(1) for symmetric rotationally $\alpha$-stable process slightly improves that in [9]; see [9, Remark 5.1(4)].

We further mention that our main results for quenched estimates of $u^\omega(t, x)$ hold (see Theorems 3.7 and 3.8) for pure-jump symmetric Lévy process in general Poissonian potentials, so the results should apply various examples discussed in [9, Section 5]. It is also possible to extend them to symmetric Lévy processes with non-degenerate Brownian motion as did in [9], and the details are left to interested readers. Instead, to highlight the power of our approaches, we will present the quenched estimates of $u^\omega(t, x)$ with the critical potential (for example, $\varphi(x) = 1 \wedge |x|^{-d-\alpha}$ with $\alpha$ being in (1.7)) in the appendix. Specially, we can prove that

**Proposition 1.3.**

(i) Let $Z$ be a rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ with $\alpha \in (0, 2)$, and $\varphi(x) = 1 \wedge |x|^{-d-\alpha}$. Then, $Q$-almost surely for all $x \in \mathbb{R}^d$,

$$-\infty < \liminf_{t \to \infty} \frac{u^\omega(t, x)}{t^{d/(d+\alpha)}} \leq \limsup_{t \to \infty} \frac{u^\omega(t, x)}{t^{d/(d+\alpha)}} < 0.$$

(ii) Let $Z$ be a pure-jump rotationally symmetric Lévy process given in Theorem 1.2, and $\varphi(x) = 1 \wedge |x|^{-d-2}$. Then, $Q$-almost surely for all $x \in \mathbb{R}^d$,

$$-\infty < \liminf_{t \to \infty} \frac{u^\omega(t, x)}{t/(\log t)^{2/d}} \leq \limsup_{t \to \infty} \frac{u^\omega(t, x)}{t/(\log t)^{2/d}} < 0.$$

The rest of the paper is arranged as follows. In Section 2, we give some preliminaries and main assumptions of our paper. Section 3 is the main part of our paper, and it is split into three subsections. In particular, after establishing quenched bounds for $u^\omega(t, x)$ and estimates for the principle Dirichlet eigenvalue, we will derive general quenched estimates of $u^\omega(t, x)$ in here. Section 4 is devoted to proofs of Theorems 1.1 and 1.2. Finally, in the appendix we present upper quenched bounds for the principle Dirichlet eigenvalue in the heavy tailed case, and quenched estimates of $u^\omega(t, x)$ with the critical potential.

2. Preliminaries and Assumptions
2.1. Lévy processes. Let $Z := (Z_t, P_x)_{t \geq 0, x \in \mathbb{R}^d}$ be a pure-jump symmetric Lévy process on $\mathbb{R}^d$ with the characteristic exponent $\psi$ given by (1.2). Throughout the paper, we will assume the following two conditions on the exponent $\psi$:

(i) $e^{-t \psi^{1/2}(\cdot)} \in L^1(\mathbb{R}^d, dx)$ for all $t > 0$;

(ii) $\psi(\xi) = \psi^{(a)}(\xi) + o(|\xi|^\alpha), \quad |\xi| \to 0$ for some $a \in (0, 2]$, where

$$
\psi^{(a)}(\xi) = \left\{ \begin{array}{ll}
\int_0^\infty \int_{S^{d-1}} \frac{1 - \cos(\xi \cdot r z)}{r^{1+a}} \mu(dz) \, dr, & \alpha \in (0, 2), \\
\xi : A_{1+}, & \alpha = 2.
\end{array} \right.
$$

with $\mu$ being a symmetric finite measure on unit sphere $S^{d-1}$ and $A = (a_{ij})_{1 \leq i, j \leq d}$ being a symmetric nonnegative definite matrix. Moreover, $\inf_{|\xi|=1} \psi^{(a)}(\xi) > 0$, and, for each $\delta, r > 0$,

$$
\sum_{\xi \in \mathbb{R}^d} \exp(-\delta \psi^{(a)}(\xi)) < \infty,
$$

where $\psi^{(a)}(\xi) = \inf_{t \geq 1} t^a \psi^{(a)}(t^{-1} \xi)$.

It is clear that under (i) the process $Z$ has a transition density function $p(t, x - y) = p(t, x, y)$ with respect to the Lebesgue measure such that $p(t, 0) = \sup_{x \in \mathbb{R}^d} p(t, x) < \infty$ for all $t > 0$. We further suppose that $p(t, x)$ is strictly positive for all $t > 0$ and $x \in \mathbb{R}^d$.

Note that, the asymptotic condition (2.1) on $\psi(\xi)$ for $|\xi| \to 0$ in (ii) is essentially based on the property of the Lévy measure $\nu$ on $\{z \in \mathbb{R}^d : |z| > 1\}$. For example, according to [9, Proposition 5.2(i)], if $\nu$ has finite second moment, i.e., $\int_{\{|z| > 1\}} |z|^2 \nu(dz) < \infty$, then (2.1) holds with $\alpha = 2$ and $\psi^{(2)}(\xi) = \xi : A \xi$, where $A = (\tilde{a}_{ij})_{1 \leq i, j \leq d}$ with $\tilde{a}_{ij} = \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} z_i z_j \nu(dz)$.

In this paper, we always let $D$ be a bounded domain (i.e., connected open set) of $\mathbb{R}^d$. Let $Z^D := (Z^D_t, P^x)_{t \geq 0, x \in D}$ be the subprocess of $Z$ killed on exiting $D$. Then, $Z^D$ has transition density function $p^D(t, x, y) = p(t, x, y) - E_x \left( p(t - \tau_D, X_{\tau_D}, y) I_{\{\tau_D < t\}} \right), \quad t > 0, x, y \in D,$

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Denote by $(P^D_t)_{t \geq 0}$ the Dirichlet semigroup associated with the process $X^D$. Since $D$ is bounded and $p^D(t, x, y) \leq p(t, x, y) = p(t, x - y) \leq p(t, 0) < \infty$ for all $t > 0$ and $x, y \in D$, the operators $P^D_t$ are compact and admit a sequence of positive eigenvalues

$$
0 < \lambda_1(U) < \lambda_2(U) \leq \lambda_3(U) \leq \cdots \to \infty.
$$

When $Z$ is a symmetric $\alpha$-stable process with $\alpha \in (0, 2]$, the eigenvalues will be denoted by $\lambda_i^{(a)}(U)$ for $i \geq 1$.

2.2. Random potential. Consider the random potential $V^\omega$ given by (1.3), which can be written as

$$
V^\omega(x) = \sum_i \varphi(x - \omega_i), \quad x \in \mathbb{R}^d,
$$

and the points $\omega_i$ are from a realization of a homogeneous Poisson point process in $\mathbb{R}^d$ with parameter $\rho > 0$. In this paper, we assume that the nonnegative shape function $\varphi$ is continuous, and satisfies

$$
\int_{\mathbb{R}^d} (e^{\varphi(x)} - 1) \, dx < \infty,
$$
where \( \bar{\phi}(x) = \sup_{z \in B(x,1)} \phi(z) \). Then, following the proof of [8, Lemma 5], we know that \( Q \)-almost surely there is \( r(\omega) > 0 \) such that for all \( r \geq r(\omega) \),

\[
(2.4) \quad \sup_{x \in B(0,r)} V^\omega(x) \leq 3d \log r.
\]

A typical example that satisfies (2.3) is the function \( \phi(x) = K(C \wedge |x|^{-d - \theta}) \) for some positive constants \( K, C, \theta \). Indeed, if there are constants \( c_0, \theta > 0 \) such that

\[
\phi(x) \leq \frac{c_0}{(1 + |x|)^{d+\theta}}, \quad x \in \mathbb{R}^d,
\]

then, according to [1, Lemma 2.1], we even have that \( Q \)-almost surely there is \( r(\omega) > 0 \) so that for all \( r \geq r(\omega) \),

\[
\sup_{x \in B(0,r)} V^\omega(x) \leq c \left( 1 + \frac{\log r}{\log \log r} \right),
\]

where \( c > 0 \) is independent of \( r(\omega) \) and \( r \). In particular, (2.4) yields that for \( Q \)-almost surely, \( V^\omega \) belongs to the local Kato class relative to the process \( Z \), i.e., \( Q \)-almost surely,

\[
\limsup_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t \mathbb{E}_x \{ V^\omega(X_s) \mathbb{1}_{\{ X_s \in B(0,R) \}} \} \, ds = 0
\]

for all \( R > 0 \).

2.3. **Feynman-Kac semigroup.** Since, \( Q \)-almost surely, \( V^\omega \) belongs to the local Kato class relative to the process \( Z \), we can well define the random Feynman-Kac semigroups \( (T_t^{V^\omega})_{t \geq 0} \) and \( (T_t^{V^\omega,D})_{t \geq 0} \) as follows

\[
T_t^{V^\omega} f(x) = \mathbb{E}_x \left[ f(Z_t) e^{-\int_0^t V^\omega(Z_s) \, ds} \right], \quad f \in L^2(\mathbb{R}^d; dx), t > 0,
\]

\[
T_t^{V^\omega,D} f(x) = \mathbb{E}_x \left[ f(Z_t) e^{-\int_0^t V^\omega(Z_s) \, ds} \mathbb{1}_{\{ \tau_D > t \}} \right], \quad f \in L^2(D; dx), t > 0.
\]

Under our setting, both \( (T_t^{V^\omega})_{t \geq 0} \) and \( (T_t^{V^\omega,D})_{t \geq 0} \) admit strictly positive and bounded symmetric kernels \( p^{V^\omega}(t, x, y) \) and \( p^{V^\omega,D}(t, x, y) \) with respect to the Lebesgue measure respectively, such that

\[
p^{V^\omega}(t, x, y) \leq p(t, x, y) = p(t, x - y), \quad x, y \in \mathbb{R}^d, t > 0,
\]

and

\[
p^{V^\omega,D}(t, x, y) \leq p^D(t, x, y) \leq p(t, x - y), \quad x, y \in D, t > 0.
\]

On the other hand, it is known that \( (T_t^{V^\omega})_{t \geq 0} \) can be generated by the random non-local Schrödinger operator \( -H^\omega \) with \( H^\omega := -L + V^\omega \), where \( L \) is the infinitesimal generator of the Lévy process \( Z \). Hence, the semigroup \( (T_t^{V^\omega,D})_{t \geq 0} \) corresponds to the Schrödinger operator \( -H^\omega \) with the Dirichlet conditions on \( D^c \). In particular, the operators \( T_t^{V^\omega,D} \) are compact, so that \( Q \)-almost surely the spectrum of the operator \( -H^\omega \) with the Dirichlet conditions on \( D^c \) is discrete:

\[
0 < \lambda_1^{V^\omega,D} < \lambda_2^{V^\omega,D} \leq \lambda_3^{V^\omega,D} \leq \cdots \to \infty.
\]

For simplicity, below we write \( \lambda_1^{V^\omega,D} \) as \( \lambda_{V^\omega,D} \), which will play an important role in our paper. It further follows that

\[
(2.5) \quad \| T_t^{V^\omega,D} \|_{L^2(D; dx) \to L^2(D; dx)} \leq e^{-\lambda_1^{V^\omega,D} t} = e^{-\lambda_{V^\omega,D} t}, \quad t > 0
\]

and that for any \( x \in D \),

\[
(2.6) \quad \mathbb{E}_x \left[ \exp \left( -\int_0^t V^\omega(Z_s) \, ds \right) \delta_x(Z_t) : \tau_D > t \right] = \sum_{k=1}^\infty e^{-\lambda_k^{V^\omega,D} t} e_k(x)^2, \quad t > 0,
\]
where $\|\cdot\|_{L^2(D;dx)} \rightarrow L^2(D;dx)$ is denoted by the operator norm from $L^2(D;dx)$ and $L^2(D;dx)$, and \( \{e_k(x)\}_{k \geq 1} \) are the eigenfunctions corresponding to $\{\lambda_k^{V_0}D\}_{k \geq 1}$ respectively with $\|e_k\|_{L^2(D;dx)} = 1$ for all $k \geq 1$. According to (2.5), it then holds that
\[
(2.7) \quad \int_D E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \, dx \leq |D| e^{-t \lambda V_0.0}, \quad t > 0.
\]
Thanks to (2.6), we also have
\[
(2.8) \quad e^{-t \lambda V_0.0} \leq \int_D E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) \delta_x(Z_t) : \tau_D > t \right] \, dx, \quad t > 0.
\]

3. General bounds for quenched asymptotics of $u^\omega(t, x)$

In this section, we establish general bounds for quenched asymptotics of $u^\omega(t, x)$. Let $Z$ be a pure-jump symmetric Lévy process on $\mathbb{R}^d$ and $V^\omega$ be the random potential given by (1.3), both of which satisfy all the assumptions in the previous section. For the index $\alpha \in (0, 2]$ given in (2.1), we will consider the following two cases.

- **Light tailed case (L)** The shape function $\varphi$ in the random potential $V^\omega(x)$ satisfies that
  \[
  \lim_{|x| \to \infty} \varphi(x)|x|^{d+\alpha} = 0.
  \]
- **Heavy tailed case (H)** The characteristic exponent $\psi(\xi)$ of the process $Z$ fulfills that $\psi(\xi) = O(|\xi|^\alpha)$ as $|\xi| \to 0$, and there are constants $\beta \in (0, \alpha)$ and $K > 0$ such that, for the shape function $\varphi$ in the random potential $V^\omega(x)$, it holds that
  \[
  \lim_{|x| \to \infty} \varphi(x)|x|^{d+\beta} = K.
  \]

The section is split into three parts. We first show quenched bounds for $u^\omega(t, x)$, and then present estimates for the principle Dirichlet eigenvalue $\lambda_{V_0.0}$. General explicit results for quenched estimates of $u^\omega(t, x)$ are given in Subsection 3.3.

3.1. Quenched bounds for $u^\omega(t, x)$. Because of the homogeneities of the Lévy process $Z$ and the potential $V^\omega$, the distribution of the quenched bounds in this section does not depend on the starting point, and so we can take $x = 0$ in the proof. In this part, we derive some pointwise quenched bounds for $u^\omega(t, 0)$. Some of arguments below are motivated by the arguments in [2, Section 4].

3.1.1. Upper bounds.

**Proposition 3.1.** For any $t > 0$, $0 < \delta < t$, $R > 0$ and $a > 1$, and for all $\omega \in \Omega$, $u^\omega(t, 0) \leq P_0(\tau_{B(0,R)} \leq t)$
\[
+ \min \left\{ p(\delta, 0)^{1/2} |B(0, R)|^{1/2} \exp(-t^{1/2} \lambda_{V_0.0,B(0,R)}),
\quad p(\delta, 0)^{1/a} |B(0, R)|^{1/a} \exp(-a^{-1}(t - \delta) \lambda_{V_0.0,B(0,R)}) \right\}.
\]

**Proof.** Write $B(0, R)$ as $B_R$ for simplicity. We mainly follow the idea of [7, Lemma 2.1]. For any $t > 0$, $R > 0$ and $\omega \in \Omega$,
\[
u^\omega(t, 0) = E_0 \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) \right]
\leq E_0 \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_{B_R} > t \right] + P_0(\tau_{B_R} \leq t)
\]
equality follows from (2.5).

Next, we will estimate $I_1$ in two different ways.

First, we repeat the proof of [9, Lemma 3.1] as follows. For any $0 < \delta < t$,

$$ I_1 = T_t^{V_\omega, B_R} 1_{B_R}(0) = T_{t/2}^{V_\omega, B_R} T_{t-\delta/2}^{V_\omega, B_R} 1_{B_R}(0) $$

$$ = \langle p^{V_\omega, B_R}(\delta/2, 0, \cdot), T_{t-\delta/2}^{V_\omega, B_R} 1_{B_R} \rangle_{L^2(B_R; dx)} $$

$$ \leq \|p^{V_\omega, B_R}(\delta/2, 0, \cdot)\|_{L^2(B_R; dx)} \|T_{t-\delta/2}^{V_\omega, B_R} 1_{B_R}\|_{L^2(B_R; dx)} $$

$$ \leq \|p(\delta/2, 0, \cdot)\|_{L^2(\mathbb{R}^d; dx)} e^{-(t-\delta/2)\lambda_{V_\omega, B_R}} \|1_{B_R}\|_{L^2(B_R; dx)} $$

$$ = p(\delta, 0)^{1/2} |B_R|^{1/2} \exp(-(t-\delta/2)\lambda_{V_\omega, B_R}), $$

where in the first inequality we used the Cauchy-Schwarz inequality and the second inequality follows from (2.5).

Second, for any $0 < \delta < t$, by the Hölder inequality with $a, b > 1$ satisfying $1/a + 1/b = 1$,

$$ I_1 \leq \left( E_0 \left[ \exp \left( -b \int_0^\delta V_\omega(Z_s) \, ds \right) \right] \right)^{1/b} \left( E_0 \left[ \exp \left( -a \int_0^\delta V_\omega(Z_s) \, ds \right) : \tau_{B_R} > t \right] \right)^{1/a} $$

$$ \leq \left( \int_{B_R} p^{B_R}(\delta, 0, x) E_x \left[ \exp \left( -a \int_0^{t-\delta} V_\omega(Z_s) \, ds \right) : \tau_{B_R} > t - \delta \right] dx \right)^{1/a} $$

$$ \leq p(\delta, 0)^{1/a} \left( \int_{B_R} E_x \left[ \exp \left( -a \int_0^{t-\delta} V_\omega(Z_s) \, ds \right) : \tau_{B_R} > t - \delta \right] dx \right)^{1/a} $$

$$ \leq p(\delta, 0)^{1/a} |B_R|^{1/a} \exp(-a^{-1}(t-\delta)\lambda_{a^{-1}V_\omega, B_R}), $$

where in the last inequality we used (2.7).

Therefore, the assertion follows from all the estimates above.

\[ \square \]

3.1.2. Lower bounds.

**Lemma 3.2.** For any bounded domain $D \subset \mathbb{R}^d$, $0 < \delta < t$ and $a, b > 1$ with $1/a + 1/b = 1$, and for any $\omega \in \Omega$,

$$ \int_D E_x \left[ \exp \left( - \int_0^t V_\omega(Z_s) \, ds \right) : \tau_D > t \right] dx $$

$$ \geq p(\delta, 0)^{-1} p(t, 0)^{-ab^{-1}} |D|^{-2ab^{-1}} \exp(-a(t+\delta)\lambda_{a^{-1}V_\omega, D}). $$

**Proof.** We start from (2.8), i.e.,

$$ e^{-t\lambda_{V_\omega, D}} \leq \int_D E_x \left[ \exp \left( - \int_0^t V_\omega(Z_s) \, ds \right) \delta_x(Z_t) : \tau_D > t \right] dx, \quad t > 0. $$

Replacing $t$ and $V_\omega$ by $t + \delta$ and $a^{-1}V_\omega$ respectively in the inequality above, we get by the Hölder inequality that for all $a, b > 1$ with $1/a + 1/b = 1$,

$$ e^{-(t+\delta)\lambda_{a^{-1}V_\omega, D}} \leq \int_D E_x \left[ \exp \left( -a^{-1} \int_0^{t+\delta} V_\omega(Z_s) \, ds \right) \delta_x(Z_{t+\delta}) : \tau_D > t + \delta \right] dx $$

$$ \leq \left( \int_D E_x \left[ \exp \left( - \int_0^t V_\omega(Z_s) \, ds \right) \delta_x(Z_{t+\delta}) : \tau_D > t + \delta \right] dx \right)^{1/a} $$

$$ \times \left( \int_D E_x \left[ \exp \left( - \frac{b}{a} \int_{t}^{t+\delta} V_\omega(Z_s) \, ds \right) : \tau_D > t + \delta \right] dx \right)^{1/b} $$

$$ =: I_1 \times I_2.$$
On the one hand, by the Markov property,

\[ I_1 \leq \left( \int_D E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) \delta_x(Z_{t+\delta}) : \tau_D > t \right] \, dx \right)^{1/a} \]

\[ \leq \left( \int_D E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) p(\delta, x - Z_t) : \tau_D > t \right] \, dx \right)^{1/a} \]

\[ \leq p(\delta, 0)^{1/a} \left( \int_D E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \, dx \right)^{1/a}. \]

On the other hand, also due to the Markov property,

\[ I_2 = \left( \int_D \int_D p^D(t, y - x) E_y \left[ \exp \left( - \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \, dy \, dx \right)^{1/b} \]

\[ \leq p(t, 0)^{1/b} |D|^{1/b} \left( \int_D E_y \left[ \exp \left( - \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \, dy \right)^{1/b} \]

\[ \leq p(t, 0)^{1/b} |D|^{2/b}. \]

Combining with both estimates above, we find that

\[ \int_D E_x \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \, dt \]

\[ \geq p(\delta, 0)^{-1} p(t, 0)^{-a/b} |D|^{-2a/b} \exp \left( -a(t + \delta) \lambda_{a^{-1} V^\omega, D} \right). \]

The proof is completed. \(\square\)

**Proposition 3.3.** For any bounded domain \(D \subset \mathbb{R}^d\) with \(0 \in D\), subdomain \(D_1 \subset D\), \(0 < \delta < t\), \(a, b > 1\) with \(1/a + 1/b = 1\) and \(\omega \in \Omega\),

\[ u^\omega(t, 0) \geq E_0 \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \]

\[ \geq p(\delta, 0)^{-a} p(t - \delta, 0)^{-a/2} |D_1|^{-2a/2} \left( E_0 \left[ \exp \left( \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \right)^{-a/b} \]

\[ \times \left( \inf_{x \in D_1} p^D(\delta, 0, x) \right)^a \left[ \exp \left( a^2 t \lambda_{a^{-2} V^\omega, D_1} \right) \right]. \]

**Proof.** For \(0 < \delta < t\), by the Hölder inequality, for any \(a, b > 1\) with \(1/a + 1/b = 1\),

\[ E_0 \left[ \exp \left( -a^{-1} \int_\delta^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \]

\[ \leq \left( E_0 \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \right)^{1/a} \]

\[ \times \left( E_0 \left[ \exp \left( \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \right)^{1/b}. \]

Note that, according to the Markov property,

\[ E_0 \left[ \exp \left( -a^{-1} \int_\delta^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \]

\[ = \int_D p^D(\delta, 0, x) E_x \left[ \exp \left( -a^{-1} \int_0^{t-\delta} V^\omega(Z_s) \, ds \right) : \tau_D > t - \delta \right] \, dx. \]
\[ \geq \left( \inf_{x \in D_1} p^D(\delta, 0, x) \right) \int_{D_1} E_x \left[ \exp \left( -a^{-1} \int_0^{t-\delta} V^\omega(Z_s) \, ds \right) : \tau_{D_1} > t - \delta \right] \, dx. \]

Hence,
\[
E_0 \left[ \exp \left( - \int_0^t V^\omega(Z_s) \, ds \right) : \tau_D > t \right] \\
\geq \left( \inf_{x \in D_1} p^D(\delta, 0, x) \right)^a \left( \int_{D_1} E_x \left[ \exp \left( -a^{-1} \int_0^{t-\delta} V^\omega(Z_s) \, ds \right) : \tau_{D_1} > t - \delta \right] \, dx \right)^a \\
\times \left( E_0 \left[ \exp \left( \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \right)^{-a/b} \\
\geq \left( \inf_{x \in D_1} p^D(\delta, 0, x) \right)^a \left( E_0 \left[ \exp \left( \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \right)^{-a/b} \\
\times \exp \left( -a^2 t \lambda_{a-2V^\omega,D_1} \right) \left( E_0 \left[ \exp \left( \frac{b}{a} \int_0^\delta V^\omega(Z_s) \, ds \right) : \tau_D > \delta \right] \right)^{-a/b},
\]

where in the last inequality we used Lemma 3.2. The proof is finished. \( \square \)

### 3.2. Estimates for the principle Dirichlet eigenvalue

In order to apply Propositions 3.1 and 3.3 to obtain explicit quenched asymptotics for \( u^\omega(t, x) \), we need to estimate the principle Dirichlet eigenvalue \( \lambda_{V^\omega,D} \).

It was known that the large time asymptotic behavior of the solution to (1.1) is closely connected to the integrated density of states of the random Schrödinger operator \( H^\omega = -L + V^\omega \), which is defined by

\[ (3.2) \quad N(\lambda) = \lim_{R \to \infty} \frac{1}{(2R)^d} E_Q \left[ \sharp \{ k \in \mathbb{N} : \lambda^\omega_k, B(0,R) \leq \lambda \} \right], \]

with \( \lambda^\omega_k, B(0,R) \) being the \( k \)-th smallest eigenvalue of \( H^\omega \) with the Dirichlet conditions on \( B(0,R)^c \). See [13, Section 5] for the existence of the limit above. Indeed, the existence of the limit in (3.2) was proved by using that spatial superadditivity property of \( E_Q \left[ \sharp \{ k \in \mathbb{N} : \lambda^\omega_k, B(0,R) \leq \lambda \} \right] \), and so it is in fact the supremum over \( R > 0 \). Furthermore, it was observed that \( N(\lambda) \) is the Laplace transform of the expectation of \( u^\omega(t, x) \) given by (1.4) on \( (\Omega, Q) \). Then, an appropriate Tauberian theorem can be used to derive the information on the tail of \( N(\lambda) \) as \( \lambda \to 0 \) from the large time behavior of \( E_Q[u^\omega(t, x)] \).

Due to the corresponding Abelian theorem, the converse is also true. We note that the study of \( N(\lambda) \) requires the use of the associated pinned process rather than the symmetric Lévy process \( Z \) itself.

#### 3.2.1. Lower bounds of \( \lambda^\omega, B(0,R) \) for \( R \) large enough

To estimate lower bounds of \( \lambda^\omega, B(0,R) \), we now recall some known results about the integral density \( N(\lambda) \) of states of the random Schrödinger operator \( H^\omega = -L + V^\omega \) defined by (3.2). It has been proved in [13, Theorems 6.2 and 6.3] that

\[ \lim_{\lambda \to 0} \lambda^{d/(\beta \wedge \alpha)} \log N(\lambda) = -k_0, \]

where

\[ k_0 := \begin{cases} \\
\rho \lambda_{(a)}(B(0,1))^{d/\alpha}, & \text{case (L)}, \\
\frac{1}{\beta} \left( \frac{d}{d+\beta} \right)^{d/\beta} \left( \Gamma \left( \frac{\beta}{d+\beta} \right) \rho w_{d}^{(d+\beta)/\beta} K^{d/\beta} \right), & \text{case (H)},
\end{cases} \]

are the parameters.
where
\[ \lambda_{(a)}(B(0,1)) = \inf_{U,|U|=w_d} \lambda_{1}^{(a)}(U), \]

\( w_d \) is the volume of the unit ball, and \( \lambda_{1}^{(a)}(U) \) is the principle Dirichlet eigenvalue for the symmetric \( \alpha \)-stable process killed on exiting \( U \) and with the exponent \( \psi^{(a)}(\xi) \) given in (2.1). In particular, when this symmetric \( \alpha \)-stable process is isotropic, \( \lambda_{(a)}(B(0,1)) = w_d^{\alpha/d} \lambda_{1}^{(a)}(B(0,1)) \). With this at hand, we can see from the arguments of (2.3)–(2.6) in [7, Section 2] that for any \( \varepsilon \in (0,1) \), \( Q \)-almost surely there is \( R_{\varepsilon}(\omega) > 0 \) such that for every \( R \geq R_{\varepsilon}(\omega) \),
\[ \lambda_{V^\omega,B(0,R)} \geq (1-\varepsilon) \left( \frac{k_0}{d \log R} \right)^{(\alpha\wedge\beta)/d}. \]

3.2.2. Upper bounds of \( \lambda_{V^\omega,B(z,r)} \) for \( r \) large enough with some \( z \). The following proposition is crucial for lower bounds of quenched asymptotic of \( u^\omega(t, x) \).

**Proposition 3.4.** The following two statements hold.

(i) In light tailed case (L), for any \( \kappa > 1 \) and \( \eta, \zeta \in (0,1) \), \( Q \)-almost surely there exists \( r_{\kappa,\eta,\zeta}(\omega) > 0 \) such that for all \( r \geq r_{\kappa,\eta,\zeta}(\omega) \), there is \( z := z(r, \omega) \in \mathbb{R}^d \) with \( |z| \leq M_{\kappa,\eta}(r) \),
\[ \lambda_{V^\omega,B(z,r)} \leq (1+\zeta)\lambda_{1}^{(a)}(B(0,1))r^{-\alpha}, \]
where
\[ M_{\kappa,\eta}(r) = r^{-\kappa} \exp \left( \frac{w_d}{d} (1 + 2\eta)r^d \right), \]
and \( \lambda_{1}^{(a)}(B(0,1)) \) is the principle eigenvalue for the symmetric \( \alpha \)-stable process killed on exiting \( B(0,1) \) and with the exponent \( \psi^{(a)}(\xi) \) given in (2.1).

(ii) In heavy tailed case (H), for any \( l > 1 \) large enough, \( \kappa > 1 \) and \( \zeta \in (0,1) \), \( Q \)-almost surely there exists \( r_{l,\kappa,\zeta}(\omega) > 0 \) such that for all \( r \geq r_{l,\kappa,\zeta}(\omega) \), there is \( z := z(r, \omega) \in \mathbb{R}^d \) with \( |z| \leq M_{\kappa}(r) \),
\[ \lambda_{V^\omega,B(z,r^{\beta/\alpha})} \leq (1+\zeta)q_1 r^{-\beta}, \]
where
\[ M_{\kappa}(r) = r^{-\kappa} e^{rd} \]
and
\[ q_1 = \frac{d}{d+\beta} \left( \frac{\beta}{d(d+\beta)} \right)^{\beta/d} \left[ \rho w_d \Gamma \left( \frac{\beta}{d+\beta} \right) \right]^{(d+\beta)/d} K. \]

**Proof.** The proof of the assertion (ii) is a little more delicate, and so we postpone it into the appendix. Here we only give the proof of the assertion (i). Note that the argument for the assertion (i) with some modifications works for the critical case; see Proposition 5.7. Fix \( \kappa > 1 \) and \( \eta \in (0,1) \), and set \( I_r := ((2(1+\eta)r)^d) \cap \{z \in \mathbb{R}^d : |z| \leq M_{\kappa,\eta}(r)\} \) for any \( r > 0 \). Define \( \varphi_0(r) = \sup_{|x| \geq r} \varphi(x) \) and \( \varphi_0(x) = \varphi_0(|x|) \). It is clear that \( \varphi(x) \leq \varphi_0(x) \), and \( \varphi_0(r) \) is a decreasing function on \([0, \infty)\) such that
\[ \lim_{r \to \infty} \varphi_0(r)r^{d+\alpha} = 0. \]

For any \( z \in I_r \) and \( \varepsilon > 0 \), define
\[ F_\varepsilon(z) = \{ \text{the ball } B(z, (1+\eta)r) \text{ at least contains one Poisson point}\}, \]
\[ G_r(z) = \left\{ \sup_{y \in \mathcal{B}(z,r)} \sum_{\omega_i \notin \mathcal{B}(z,(1+\eta)r)} \varphi_0(y - \omega_i) \geq 2\varepsilon r^{-\alpha} \right\}. \]

We will estimate \( Q(\cap_{z \in I_r} (F_r(z) \cup G_r(z))) \).

Note that \( \{ F_r(z) \}_{z \in I_r} \) are i.i.d., and that \( Q(F_r(0)) = 1 - e^{-w_0d((1+\eta)r)^d} \). Hence, there is \( r_0(\kappa, \eta) > 0 \) such that for all \( r \geq r_0(\kappa, \eta) \),

\[
Q(\cap_{z \in I_r} F_r(z)) \leq (1 - e^{-w_0d((1+\eta)r)^d})^{\frac{1}{2} \left( \frac{M_{\kappa,\eta}(r)}{(1+\eta)r} \right)^{d}} \leq \exp \left( \frac{-1}{2} e^{-w_0d((1+\eta)r)^d} \left( \frac{M_{\kappa,\eta}(r)}{(1+\eta)r} \right)^{d} \right) \leq \exp \left( -2^{-1}(1+\eta)^{-d}r^{-d(1+\kappa)}e^{-w_0d((1+\eta)r)^d} \right) \leq \exp(-r^d),
\]

where in the second inequality we used the fact that \( 1 - x \leq e^{-x} \) for all \( x > 0 \).

On the other hand, for \( y \in \mathcal{B}(0, r) \) and \( \omega_i \notin \mathcal{B}(0, (1+\eta)r) \), \( |y - \omega_i| \geq \eta|\omega_i|/(1+\eta) \). By the fact that \( \varphi_0(x) = \varphi_0(|x|) \) and the deceasing property of \( \varphi_0(r) \),

\[
\sup_{y \in \mathcal{B}(0, r)} \sum_{\omega_i \notin \mathcal{B}(0, (1+\eta)r)} \varphi_0(y - \omega_i) \leq \sum_{\omega_i \notin \mathcal{B}(0, (1+\eta)r)} \varphi_0(\eta|\omega_i|/(1+\eta)).
\]

Hence,

\[
Q \left[ \exp \left( \frac{1}{\varphi_0(\eta r)} \sup_{y \in \mathcal{B}(0, r)} \sum_{\omega_i \notin \mathcal{B}(0, (1+\eta)r)} \varphi_0(y - \omega_i) \right) \right] \leq Q \left[ \exp \left( \frac{1}{\varphi_0(\eta r)} \sum_{\omega_i \notin \mathcal{B}(0, (1+\eta)r)} \varphi_0(\eta|\omega_i|/(1+\eta)) \right) \right] = \exp \left( \rho \int_{\mathbb{R}^d \setminus \mathcal{B}(0, (1+\eta)r)} \left( e^{\varphi_0(\eta r)^{-1} \varphi_0(\eta|z|/(1+\eta))} - 1 \right) dz \right) \leq \exp \left( e^{\rho(1+\eta)^d} \int_{\mathbb{R}^d \setminus \mathcal{B}(0, r)} \varphi_0(\eta r) \varphi_0(\eta z) dz \right),
\]

where in the last inequality we used the fact that \( e^x - 1 \leq ex \) for all \( x \in (0, 1] \). By (3.9), for any \( \varepsilon > 0 \) there is a constant \( r_1(\eta, \varepsilon) \geq r_0(\kappa, \eta) \) such that for all \( r \geq r_1(\eta, \varepsilon) \),

\[
Q \left[ \exp \left( \frac{1}{\varphi_0(\eta r)} \sup_{y \in \mathcal{B}(0, r)} \sum_{\omega_i \notin \mathcal{B}(0, (1+\eta)r)} \varphi_0(y - \omega_i) \right) \right] \leq \exp \left[ \frac{\varepsilon r^{-\alpha}}{\varphi_0(\eta r)} \right].
\]

Therefore, according to the Markov inequality,

\[
Q(G_r(0)) \leq Q \left[ \exp \left( \frac{1}{\varphi_0(\eta r)} \sup_{y \in \mathcal{B}(0, r)} \sum_{\omega_i \notin \mathcal{B}(0, (1+\eta)r)} \varphi_0(y - \omega_i) \right) \geq \exp \left( \frac{2\varepsilon r^{-\alpha}}{\varphi_0(\eta r)} \right) \right] \leq \exp \left[ -\frac{\varepsilon r^{-\alpha}}{\varphi_0(\eta r)} \right] \leq \exp(-c_1 r^d),
\]
where $c_1 > 0$ is independent of $r$ (and may depend on $\eta$ and $\varepsilon$). Since $\{G_r(z)\}_{z \in I_r}$ have the same distribution (but are not independent with each other), we find that

$$ Q(\cup_{z \in I_r} G_r(z)) \leq 2 \left( \frac{M_{\kappa,\eta}(r)}{(1+\eta)r} \right)^d \exp(-c_1 r^d). $$

Combining with both estimates above, we find that for $r$ large enough

$$ Q(\cap_{z \in I_r} (F_r(z) \cup G_r(z)) \leq Q(\cap_{z \in I_r} F_r(z)) + Q(\cup_{z \in I_r} G_r(z)) $$

$$ \leq \exp(-r^d) + 2 \left( \frac{M_{\kappa,\eta}(r)}{(1+\eta)r} \right)^d \exp(-c_1 r^d) $$

$$ \leq \exp(-r^d) + c_2 \exp(-c_3 r^d), $$

where $c_2, c_3 > 0$ are independent of $r$. The Borel-Cantelli lemma tells us that $Q$-almost surely there exists $r_{\kappa,\eta,\varepsilon}(\omega)$ such that for all $r \geq r_{\kappa,\eta,\varepsilon}(\omega)$, there is $z := z(r, \omega) \in \mathbb{R}^d$ with $|z| \leq M_{\kappa,\eta}(r)$ so that both $F_r(z)$ and $G_r(z)$ fail to hold.

Below, we fix this $z$ for all $r \geq r_{\kappa,\eta,\varepsilon}(\omega)$. Since $G_r(z)$ fails to occur,

$$ \sup_{y \in B(z,r)} \sum_{\omega_i \in B(z,(1+\eta)r)} \varphi_0(y - \omega_i) \leq 2\varepsilon r^{-\alpha} $$

and so, also thanks to $\varphi(x) \leq \varphi_0(x)$,

$$ \lambda_{V^\omega, B(z,r)} \leq \lambda_{\tilde{V}^\omega, B(z,r)} + 2\varepsilon r^{-\alpha}, $$

where

$$ \tilde{V}^\omega(x) = \sum_{\omega_i \in B(z,(1+\eta)r)} \varphi_0(x - \omega_i). $$

On the other hand, because $F_r(z)$ does not happen, $\tilde{V}^\omega(x) = 0$ for all $x \in \mathbb{R}^d$, and so

$$ \lambda_{\tilde{V}^\omega, B(z,r)} = \lambda_1(B(z, r)). $$

Therefore, for any $\zeta \in (0, 1)$ and $r \geq r_{\kappa,\eta,\varepsilon}(\omega)$ large enough,

$$ \lambda_{V^\omega, B(z,r)} \leq \lambda_1(B(z, r)) + 2\varepsilon r^{-\alpha} $$

$$ = \lambda_1(B(0, r)) + 2\varepsilon r^{-\alpha} \leq (1 + \zeta/2)r^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)) + 2\varepsilon r^{-\alpha}. $$

where in the last inequality we used Lemma 3.5 below. The proof is completed by taking $\varepsilon$ small enough. \qed

The following was proved in [9, Proposition 5.1].

**Lemma 3.5.** Let $Z$ be a symmetric Lévy process satisfying (1.7), and $\lambda_1(B)$ be the principle Dirichlet eigenvalue of the process $Z$ killed when exiting $B$. Then, for any fixed $\zeta > 0$, there is $r_0 := r_0(\zeta) > 0$ so that for all $r \geq r_0$,

$$ \lambda_1(B(0, r)) \leq (1 + \zeta)r^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)), $$

where $\lambda_1^{(\alpha)}(B(0, 1))$ is the principle Dirichlet eigenvalue of the symmetric $\alpha$-stable process $Z^{(\alpha)}$ with characteristic exponent $\psi^{(\alpha)}(\xi)$ given in (2.1) and killed when exiting $B(0, 1)$.

**Remark 3.6.** For our use later, we will apply $a^{-2}V^\omega$ instead of $V^\omega$. Here, we note that the following conclusions for the potential $a^{-2}V^\omega(x)$, which immediately follow from the proof of Proposition 3.4.

(i) In the light tailed case (L), for any $a > 1$, (5.5) holds for $\lambda_{a^{-2}V^\omega, B(z,r)}$ in place of $\lambda_{V^\omega, B(z,r)}$ with some $\kappa > 1$ and $\eta, \zeta \in (0, 1)$ (independent of $a$) and for all $r \geq r_{\kappa,\eta,\alpha}(\omega)$, which depends on $a$;
(ii) In the heavy tailed case (H), for any $a > 1$, (3.7) holds for $\lambda_{a \cdot V, \omega, B(z, r)}$ in place of $\lambda_{V, B(z, r)}$ with $\kappa, l > 1$ and $\zeta \in (0, 1)$ (all are independent of $a$),

$$q_1^* = a^{-2} q_1 = \frac{d}{d + \beta} \left( \frac{\beta}{d (d + \beta)} \right)^{\beta/d} \left[ \exp \left( \frac{\beta}{d} \right) \right]^{(d + \beta)/d} a^{-2} K$$

(in place of $q_1$), and for all $r \geq r_{1, \kappa, a}(\omega)$, which depends on $a$ too.

3.3. Refinement of quenched estimates of $u^{\omega}(t, x)$.  

**Theorem 3.7.** Assume that for any $t, R \geq 1$ with $R \geq \phi(t)$,

$$P_0(t_{B(0, R)} \leq t) \leq \Phi(t, R),$$

where $\phi(t)$ is an increasing function on $[1, \infty)$, and $\Phi(v_1, v_2)$ is a nonnegative function defined on $[1, \infty)^2$ such that $v_1 \mapsto \Phi(v_1, v_2)$ is increasing for fixed $v_2$ and $v_2 \mapsto \Phi(v_1, v_2)$ is decreasing for fixed $v_1$. Let $k_0$ be the constant defined in (3.3). Then,

(i) In light tailed case (L), for any $\varepsilon > 0$, $Q$-almost surely there is $R_{\varepsilon}(\omega) \geq 1$ so that for any $R \geq \max\{R_{\varepsilon}(\omega), \phi(t)\}$ and $t \geq 1$,

$$u^{\omega}(t, 0) \leq \Phi(t, R) + C(\varepsilon) R^{d/2} \exp \left(-t(1 - 2\varepsilon) \left( \frac{k_0}{d \log R} \right)^{\alpha/d} \right),$$

where $C(\varepsilon)$ is independent of $R$ and $t$.

(ii) In heavy tailed case (H), for any $\varepsilon > 0$ and $a > 1$, $Q$-almost surely there is $R_{\varepsilon, a}(\omega) \geq 1$ so that for any $R \geq \max\{R_{\varepsilon, a}(\omega), \phi(t)\}$ and $t \geq 1$,

$$u^{\omega}(t, 0) \leq \Phi(t, R) + C(\varepsilon, a) R^{d/a} \exp \left(-t(1 - 2\varepsilon) \left( \frac{k_0}{d \log R} \right)^{\beta/d} \right),$$

where $C(\varepsilon, a) > 0$ is independent of $R$ and $t$.

**Proof.** We first consider the light tailed case. According to Proposition 3.1 with $\delta$ small enough and (3.5), for any $\varepsilon > 0$, $Q$-almost surely there is $R_{\varepsilon}(\omega) \geq 1$ so that for any $t \geq 1$ and $R \geq \max\{R_{\varepsilon}(\omega), \phi(t)\}$,

$$u^{\omega}(t, 0) \leq \Phi(t, R) + C_1(\varepsilon) R^{d/2} \exp \left(-t(1 - 2\varepsilon) \left( \frac{k_0}{d \log R} \right)^{\alpha/d} \right).$$

In the heavy tailed case, we note that, from the argument for (3.5), for all $\varepsilon \in (0, 1)$ and $a > 1$, $Q$-almost surely there is $R_{\varepsilon, a}(\omega) \geq 1$ such that for every $R \geq R_{\varepsilon, a}(\omega)$,

$$a^{-1} \lambda_{a \cdot V, \omega, B(0, R)} \geq (1 - \varepsilon) \left( \frac{k_0}{d \log R} \right)^{\beta/d},$$

where the right hand side of the inequality above is independent of $a$. With this, we can obtain the desired assertion by following the arguments in the light tailed case.

**Theorem 3.8.** Assume that for any $\delta \in (0, 1/2)$ and $r \geq 1$

$$\inf_{z \in B(0, r)} P_{B(0, 2r)}(\delta, z) \geq \Psi_{\delta}(r),$$

where $\Psi_{\delta}(r)$ is a non-negative decreasing function on $[1, \infty)$. Then,

(i) In light tailed case (L), for any $\delta \in (0, 1/2)$, $\kappa > 1$, $a > 1$, $\eta, \zeta \in (0, 1)$, $Q$-almost surely there is $R_{\kappa, a, \eta, \zeta}(\omega) \geq 1$ so that for any $R \geq R_{\kappa, a, \eta, \zeta}(\omega)$ and $t \geq 1$,

$$u^{\omega}(t, 0) \geq C(\kappa, \delta, \eta, a) M_{\kappa, a}(R)^{-4\delta d} \left[ \psi_{\delta}(2 M_{\kappa, a}(R)) \right]^a \exp \left(-a^2(1 + \zeta) \lambda_{(a)}(B(0, 1)) t R^{-a} \right),$$

where $\psi_{\delta}(c) = \int_0^\infty \exp(-c z^{-\delta}) \, dz$.
where
\[ M_{\kappa,\eta}(R) = R^{-\kappa} \exp \left( \frac{\omega_d p}{d} ((1 + 2\eta)R)^d \right), \]
and \( \lambda^{(\alpha)}_1(B(0,1)) \) is the principle Dirichlet eigenvalue for the symmetric \( \alpha \)-stable process killed on exiting \( B(0,1) \) and with the exponent \( \psi^{(\alpha)}(\xi) \) given in (2.1).

(ii) In heavy tailed case \((H)\), for any \( \delta \in (0,1/2), \kappa > 1 \) large enough, \( a > 1 \) and \( \varsigma \in (0,1), \) \( Q \)-almost surely there is \( R_{\kappa,a,\varsigma}(\omega) \geq 1 \) so that for any \( R \geq R_{\kappa,a,\varsigma}(\omega) \) and \( t \geq 1, \)
\[ w^{\omega}(t,x) \geq C(\kappa,\delta,\varsigma,a)M_{\kappa}(R)^{-4\delta d}[\Psi^{(\alpha)}(2M_{\kappa,\eta}(R))]^a \exp \left( -(1 + \varsigma)q_1 tR^{-\beta} \right), \]
where
\[ M_{\kappa}(R) = R^{-\kappa} \exp(R^d) \]
and \( q_1 \) is given by (3.8).

Proof. We only prove the assertion (i), since the assertion (ii) can be verified similarly by applying Proposition 3.4(ii) and Remark 3.6(ii) instead of Proposition 3.4(i) and Remark 3.6(i), respectively.

For any \( a > 1, \kappa > 1 \) and \( \eta, \varsigma \in (0,1), \) let \( D = B(0,2M_{\kappa,\eta}(r)) \) and \( D_1 = B(z, (1 + \eta)r) \) for \( r \geq r_{\kappa,\eta,\varsigma,a}(\omega), \) where \( r_{\kappa,\eta,\varsigma,a}(\omega) \), \( M_{\kappa,\eta}(r) \) and \( z := z(r, \omega) \) are given in Proposition 3.4(i) and Remark 3.6(i). Since \( |z| \leq M_{\kappa,\eta}(r), \) \( D_1 \subset D. \) Then, according to Propositions 3.3 and 3.4(i) as well as Remark 3.6(i), for any \( \delta \in (0,1/2) \) and \( t \geq 1, \)
\[ w^{\omega}(t,0) \geq p(\delta,0)^a p(t - \delta,0)^{-a^2/b}(w_d((1 + \eta)r)^d)^{-2a^2/b} \exp(-3\delta a \log(M_{\kappa,\eta}(r))) \]
\[ \times [\Psi^{(\alpha)}(2M_{\kappa,\eta}(r))]^a \exp \left( -a^2(1 + \varsigma)tr^{-\alpha}\lambda_1^{(\alpha)}(B(0,1)) \right) \]
\[ \geq C_1(\delta,\eta,a)r^{-2a^2d/b}M_{\kappa,\eta}(r)^{-3\delta d} \]
\[ \times [\Psi^{(\alpha)}(2M_{\kappa,\eta}(r))]^a \exp \left( -a^2(1 + \varsigma)tr^{-\alpha}\lambda_1^{(\alpha)}(B(0,1)) \right) \]
\[ \geq C_2(\kappa,\delta,\eta,a)M_{\kappa,\eta}(r)^{-4\delta d}[\Psi^{(\alpha)}(2M_{\kappa,\eta}(r))]^a \exp \left( -a^2(1 + \varsigma)tr^{-\alpha}\lambda_1^{(\alpha)}(B(0,1)) \right), \]
where in the first inequality \( b > 1 \) such that \( 1/a + 1/b = 1 \) and we used (2.4) and (3.11), and the second inequality follows from the fact that for all \( t \geq 1 \) and \( \delta \in (0,1/2), \)
\( p(t - \delta,0) \leq p(\delta,0), \) due to the deceasing property of the function \( t \mapsto p(t,0). \) The proof is finished. \( \square \)

4. Examples

In this section, we will present the proofs of Theorems 1.1 and 1.2. Note that, both symmetric Lévy processes in these two examples are rotationally invariant and satisfy the assumptions in Subsection 2.1. Meanwhile, the shape function \( \varphi(x) = 1 \land |x|^{-d-\beta} \) fulfills the assumptions in Subsection 2.2 as well.

4.1. Rotationally invariant symmetric \( \alpha \)-stable processes.

Proof of Theorem 1.1. For rotationally symmetric \( \alpha \)-stable process \( Z \) with \( \alpha \in (0,2), \)
\( \psi(\xi) = c_0|\xi|^\alpha \) for some \( c_0 > 0, \) and so (1.7) holds with \( \psi^{(\alpha)}(\xi) = \psi(\xi). \) Thus, for given shape function \( \varphi(x) = 1 \land |x|^{-d-\beta} \) with \( \beta \in (0,\infty], \) the light tailed case (resp. the heavy tailed case) corresponds to \( \beta > \alpha \) (resp. \( \beta \in (0,\alpha) \)). Furthermore, it is well known that, for symmetric \( \alpha \)-stable process \( Z, \) (3.10) holds with
\[ \Phi(t,r) = C^*t r^{-\alpha} \]
and \( \phi(t) = t^{1/\alpha} \), and (3.11) holds with
\[
\Psi_\delta(r) \geq \frac{C_\delta \delta}{r^{d+\alpha}};
\]
see [3, 5].

(i) We first consider \( \beta > \alpha \), which is referred to the light tailed case. According to Theorem 3.7(i), for any \( \varepsilon > 0 \), \( Q \)-almost surely there is \( R_\varepsilon(\omega) \geq 1 \) so that for any \( t \geq 1 \) and \( R \geq \max\{R_\varepsilon(\omega), t^{1/\alpha}\} \),
\[
u^0(t, 0) \leq \frac{C_2 t}{R^\alpha} + C_2(\varepsilon) R^{d/2} \exp\left(-t(1-2\varepsilon) \left(\frac{k_0}{d \log R}\right)^{\alpha/d}\right),
\]
where \( C_2(\varepsilon) > 0 \) is a constant independent of \( R \) and \( t \), and \( k_0 = \rho w_d \lambda_1^{(\alpha)}(B(0, 1))^{d/\alpha} \) with \( \lambda_1^{(\alpha)}(B(0, 1)) \) being the principle Dirichlet eigenvalue for the rotationally symmetric \( \alpha \)-stable process \( Z \) killed on exiting \( B(0, 1) \). Letting
\[
R = \exp\left((1-2\varepsilon)d/\alpha + \varepsilon(d/2) - d/(\alpha + \varepsilon) \left(\frac{k_0}{d}\right)^{\alpha/(d+\alpha)} t^{d/(d+\alpha)}\right)
\]
for \( t \) large enough, we arrive at the desired upper bound by letting \( \varepsilon \to 0 \).

On the other hand, by Theorem 3.8(i), for any \( \kappa, \alpha > 1, \delta \in (0, 1/2) \) and \( \eta, \varsigma \in (0, 1) \), \( Q \)-almost surely there is \( R_{\kappa,\alpha,\eta,\varsigma}(\omega) \geq 1 \) so that for any \( R \geq R_{\kappa,\alpha,\eta,\varsigma}(\omega) \) and \( t \geq 1 \),
\[
u^0(t, 0) \geq CR^{(d\delta + (d+\alpha)\kappa)} \exp(-AR^d - BtR^{-\alpha}),
\]
where
\[
A = \frac{\rho_d \rho}{d}(1+2\eta)^d[a(d+\alpha) + 4\delta d], \quad B = a^2(1+\varsigma)\lambda_1^{(\alpha)}(B(0, 1))
\]
and \( C > 0 \) is a constant independent of \( t \) and \( R \). Letting
\[
R = \left(\frac{\alpha B}{dA}\right)^{1/(d+\alpha)} t^{1/(d+\alpha)}
\]
for \( t \) large enough, we prove the lower bound by taking \( \delta, \eta, \varsigma \to 0 \) and \( a \to 1 \).

(ii) For heavy tailed case (i.e., \( \beta \in (0, \alpha) \)), it follows from Theorem 3.7(ii) that for all \( a > 1 \) and \( \varepsilon \in (0, 1) \), \( Q \)-almost surely there is \( R_{a,\varepsilon}(\omega) \geq 1 \) such that for every for any \( t \geq 1 \) and \( R \geq \max\{R_{a,\varepsilon}(\omega), t^{1/\alpha}\} \),
\[
u^0(t, 0) \leq \frac{C_1 t}{R^\alpha} + C_2(a, \varepsilon) R^{d/\alpha} \exp\left(-t(1-2\varepsilon) \left(\frac{k_0}{d \log R}\right)^{\beta/d}\right),
\]
where \( k_0 \) is given by (3.3). Then, choosing
\[
R = \exp\left((1-2\varepsilon)d/\alpha + \varepsilon(d/\alpha) - d/(\beta + \varepsilon) \left(\frac{k_0}{d}\right)^{\beta/(\beta + \varepsilon)} t^{d/(\beta + \varepsilon)}\right)
\]
for \( t \) large enough, we arrive at the upper bound by letting \( \varepsilon \to 0 \) and \( a \to \infty \).

Due to Theorem 3.8(ii), for any \( \kappa, a > 1, \delta \in (0, 1/2) \) and \( \varsigma \in (0, 1) \), \( Q \)-almost surely there is \( R_{\kappa,a,\delta}(\omega) \geq 1 \) so that for any \( R \geq R_{\kappa,a,\delta}(\omega) \) and \( t \geq 1 \),
\[
u^0(t, 0) \geq CR^{(d\delta + (d+\alpha)\kappa)} \exp(-AR^d - BtR^{-\beta}),
\]
where
\[
A = a(d+\alpha) + 4\delta d, \quad B = (1+\varsigma)q_1.
\]
Letting
\[
r = \left(\frac{B}{dA}\right)^{1/(d+\beta)} t^{1/(d+\beta)}
\]
for $t$ large enough, we prove the lower bound by taking $\varsigma, \delta \to 0$ and $a \to 1$. 

Remark 4.1. We used two different ways to estimate $I_t$ in the proof of Proposition 3.1, which yield two different quenched upper bounds for $u^\omega(t,0)$ in Theorem 3.7. For this example, if we follow the argument for light tailed case (i.e., $\beta \in (\alpha, \infty]$) to deal with the heavy tailed case (i.e., $\beta \in (0, \alpha)$), then we can only obtain that when $\beta \in (0, \alpha)$, $Q$-almost surely for all $x \in \mathbb{R}^d$, 

$$
\limsup_{t \to \infty} \frac{u^\omega(t, x)}{\operatorname{ld}(d+\beta)} \leq -\frac{\alpha}{(\alpha + d/2)^{d/(d+\beta)}} A_2,
$$

which is weaker than the desired assertion for the upper bound in Theorem 1.1(ii).

4.2. Rotationally symmetric processes with large jumps of exponential decay.

Proof of Theorem 1.2. For symmetric pure jump Lévy process $Z$ with Lévy measure $\nu$ given in Theorem 1.2, by [9, Proposition 5.2(i)], (1.7) holds with $\alpha = 2$ and $\psi(\alpha)(\xi) = \xi \cdot A \xi$, where $A = (a_{ij})_{1 \leq i,j \leq d}$ with 

$$
a_{ij} = \int_{\mathbb{R}^d \setminus \{0\}} z_i z_j \nu(dz), \quad 1 \leq i,j \leq d.
$$

Thus, for given shape function $\varphi(x) = 1 \wedge |x|^{-d-\beta}$ with $\beta \in (0, \infty]$, the light tailed case (resp. the heavy tailed case) corresponds to $\beta > 2$ (resp. $\beta \in (0, 2)$).

Furthermore, according to [4, Theorems 1.2 and 1.4], for any $t \geq 1$ and $x \in \mathbb{R}^d$ with $|x| \geq 2t^{1/((2-\theta)/\gamma)}$, 

$$
p(t, x) \leq c_1 \exp \left( -c_2 |x|^\theta \left( \log \left( \frac{|x|}{t} \right) \right)^{(\theta-1)/\theta} \right).
$$

This along with Lemma 4.2 below yields that (3.10) holds with 

$$
\Phi(t, r) = c_3 \exp(-c_4 t^\theta),
$$

and $\phi(t) = 2t$. On the other hand, by [11, Theorem 1.1], we know that (3.11) holds with 

$$
\Psi_0(r) \geq c_5 \exp(-c_6 r^\theta \log(r)^{(\theta-1)/\theta}.
$$

For simplicity, we only prove the light tailed case (i.e., $\beta \in (2, \infty)$), since the heavy tailed case can be treated similarly. First, by Theorem 3.7(i), for any $\varepsilon > 0$, $Q$-almost surely there is $R_\varepsilon(\omega) \geq 1$ so that for any $t \geq 1$ and $R \geq \max\{R_\varepsilon(\omega), 2t\}$, 

$$
C_1(\varepsilon) = 0 \text{ is a constant independent of } R \text{ and } t, \quad k_0 = \rho \omega_{d}[\lambda_1^{(1)}(B(0, 1))]^{d/2} \text{ with } 

\lambda_1^{(1)}(B(0, 1)) \text{ being the principle Dirichlet eigenvalue for the Brownian motion killed on exiting } B(0, 1) \text{ and with the covariance matrix } A \text{ above. Letting } R = C t^{1/(1+\delta)} \text{ for large } C \text{ and } t, \text{ we prove the desired upper bound by taking } \varepsilon \to 0.
$$

On the other hand, according to Theorem 3.8(i), for any $\kappa, a > 1$ and $\eta, \varsigma \in (0, 1)$, $Q$-almost surely there is $R_{\kappa,a,\eta,\varsigma}(\omega) \geq 1$ so that for any $R \geq R_{\kappa,a,\eta,\varsigma}(\omega)$ and $t \geq 1$, 

$$
uu^\omega(t,0) \geq C_2 \exp \left( -c_3 (M_{\kappa,a}(R))^{a(\theta \wedge 1)} (\log M_{\kappa,a}(R))^{(\theta-1)/\theta} \right)
$$

$$
\times \exp \left( -a^2 (1+\varsigma) \lambda_1^{(2)}(B(0, 1)) t R^{-2} \right)
$$

$$
\geq C_4 \exp \left[ -C_5 R^{-\alpha a(\theta \wedge 1)+d(\theta-1)/\theta} \exp \left( a(\theta \wedge 1) \frac{w d \theta}{d} ((1+2\eta) R)^d \right) \right].
$$
\[ \times \exp\left( -a^2(1 + \zeta)\lambda_1^{(2)}(B(0,1))tR^{-2} \right). \]

Choosing \( \kappa \) large enough and
\[ R = \frac{1}{1 + 2\eta} \left( \frac{d}{a(1 + \theta)w_d \rho} \right)^{1/d} (\log t)^{1/d} \]
for \( t \) large enough, we prove the lower bound by taking \( \eta, \zeta \to 0 \) and \( a \to 1 \).

**Lemma 4.2.** For any Lévy process \( Z \), it holds for all \( t, R > 0 \) that
\[ \mathbb{P}_0(\tau_{B(0,R)} \leq t) \leq 2 \sup_{s \in [t,2t]} \mathbb{P}_0(|Z_s| \geq R/2). \]

**Proof.** For any \( t, R > 0 \),
\[ \mathbb{P}_0(\tau_{B(0,R)} \leq t) = \mathbb{P}_0(\max_{s \in [0,t]} |Z_s| \geq R) \]
\[ = \mathbb{P}_0(\max_{s \in [0,t]} |Z_s| \geq R, |Z_{2t}| \geq R/2) + \mathbb{P}_0(\max_{s \in [0,t]} |Z_s| \geq R, |Z_{2t}| \leq R/2) \]
\[ \leq \mathbb{P}_0(|Z_{2t}| \geq R/2) + \mathbb{E}_0(1_{\{\tau_{B(0,R)} \leq t\}} \mathbb{P}_{Z_{\tau_{B(0,R)}}}(|Z_{2t} - Z_{\tau_{B(0,R)}}| \geq R/2)) \]
\[ \leq 2 \sup_{s \in [t,2t]} \mathbb{P}_0(|Z_s| \geq R/2). \]

The proof is complete.

\[ \square \]

5. **Appendix**

5.1. **Proof of Proposition 3.4(ii).** In this part, we will present the proof of Proposition 3.4(ii). The proof mainly follows from the argument in [8, Section 4.1]. Note that since the paper [8] studied second order asymptotics for Brownian motion in a heavy tailed Poissonian potential, the proof is much more involved. In particular, the argument in [8, Section 4.1] only works for part of heavy tailed potentials (i.e., for the shape function \( \varphi(x) = 1 + |x|^{-(d+\beta)} \) with \( \beta \in (0,2) \) and \( d + \beta > 2 \)). Now, in our setting we can prove Proposition 3.4(ii) holds for all heavy tailed potentials, because only the first order asymptotics for the first Dirichlet eigenvalue is concerned here.

To highlight differences from the argument in [8, Section 4.1], we rewrite Proposition 3.4(ii) as follows, where notations are the same as those in [8].

**Proposition 5.1.** In heavy tailed case (H), for \( M > 1 \) large enough, any \( \kappa > 1 \) and \( \varepsilon > 0 \), \( \mathbb{Q} \) almost surely there exists \( t_{M,\kappa,\varepsilon}(\omega) > 0 \) such that, for all \( t \geq t_{M,\kappa,\varepsilon}(\omega) \) there is \( z := z(t,\omega) \in \mathbb{R}^d \) so that \( |z| \leq t(\log t)^{-\kappa} \) and
\[ \lambda_{B(z,M(\log t)^{\beta/(\alpha d)}} \leq (1 + \varepsilon)\lambda(t), \]
where \( \lambda(t) = q_1(\log t)^{-\beta/d} \), and \( q_1 \) is given by (3.8).

In heavy tailed case, by the continuity of \( \varphi \) and (3.1), for any \( \theta > 0 \), there exists a constant \( C(\theta) > 0 \) such that for all \( x \in \mathbb{R}^d \),
\[ \varphi(x) \leq \varphi_0(x) := (K + \theta)(C(\theta) \wedge |x|^{-d-\beta}). \]

Thus, to consider upper bounds for the first eigenvalue corresponding to the shape function \( \varphi \), it suffices to study that associated with the function \( \varphi_0(x) \). For simplicity, in the proof below we just take
\[ \varphi_0(x) := 1 \wedge |x|^{-d-\beta}, \quad x \in \mathbb{R}^d, \]
since the argument goes through for \( \varphi_0(x) = (K + \theta)(C(\theta) \wedge |x|^{-d-\beta}) \) and then the desired assertion follows by letting \( \theta \) small enough.

Let \( N > 1/d \), and \( M > 1 \) large enough. Define \( \Lambda_N(t) = [-(\log t)^N, (\log t)^N] \) and \( B_M(t) = B(0, M(\log t)^{\beta/(\alpha d)}) \). First, we have
Lemma 5.2. For any $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that for all $t$ large enough,

$$Q \left( \sup_{y \in B_M(t)} \sup_{\omega \notin \Lambda_N(t)} |y - \omega|^d > \varepsilon (\log t)^{-\beta} \right) \leq \exp \left( -c(\varepsilon) (\log t)^{dN+\beta(N-1/d)} \right).$$

Proof. For $t$ large enough, and for $\omega_i \notin \Lambda_N(t)$ and $y \in B_M(t)$, by $N > 1/d$ and $\beta \in (0, \alpha)$, we have $|\omega_i - y| \geq |\omega_i|/2$, and so

$$\sup_{y \in B_M(t)} \omega \notin \Lambda_N(t) |y - \omega|^{d-\beta} \leq 2^{d+\beta} \sum_{\omega \notin \Lambda_N(t)} |\omega|^{d-\beta}.$$  

Note that, since $\{\omega_i\}$ are from a realization of a homogeneous Poisson point process on $\mathbb{R}^d$ with parameter $\rho$, for $t$ large enough,

$$E_Q \exp \left\{ (\log t)^{d+\beta} N \sum_{\omega \notin \Lambda_N(t)} |\omega|^{d-\beta} \right\} = \exp \left( \rho \int_{\mathbb{R}^d \setminus \Lambda_N(t)} \left( e^{(\log t)^{d+\beta} N |z|^{-(d+\beta)}} - 1 \right) \, dz \right) \leq \exp \left( c_1 (\log t)^{dN} \right),$$

where in the first inequality we used the fact that $e^x - 1 \leq ex$ for all $x \in (0, 1]$. Therefore, by the Markov inequality, for $t$ large enough,

$$Q \left( \sup_{y \in B_M(t)} \sup_{\omega \notin \Lambda_N(t)} |y - \omega|^{d-\beta} > \varepsilon (\log t)^{-\beta/d} \right) \leq \exp \left( c_1 (\log t)^{dN} - \varepsilon 2^{-d-\beta} (\log t)^{d+\beta N - \beta/d} \right) \leq \exp \left( -c_2 (\log t)^{dN+\beta(N-1/d)} \right),$$

where in the last inequality we used the fact that $N > 1/d$. The proof is complete.

For any $t > 0$, define

$$H(t) = \log E_Q[\exp(-tV^\omega(0))], \quad \rho_0(t) = \left( \frac{(d + \beta) t}{a_1} \right)^{-(d+\beta)/\beta} \frac{a_1}{\rho_0(t)} \gamma\left( \frac{\beta}{d + \beta} \right),$$

where

$$a_1 = \rho w_d \Gamma\left( \frac{\beta}{d + \beta} \right).$$

In particular,

$$\rho_0(\lambda(t)) = \left( \frac{a_1 \beta}{d(d + \beta)} \right)^{-(d+\beta)/d} (\log t)^{d(d+\beta)/d}$$

and

$$H(\rho_0(\lambda(t))) + \lambda(t) \rho_0(\lambda(t)) = -d \log t + o(1),$$

where in the latter equality we used the fact that

$$H(t) = -a_1 t^{d/(d+\beta)} + O(e^{-t}), \quad t \to \infty;$$
see [8, Lemma 1]. Next, we introduce a transformed measure defined by
\[
\tilde{Q}_t(d\omega) = \left[e^{-H(\rho_0(\lambda(t))) - \rho_0(\lambda(t))V^\omega(0)}}\right] Q(d\omega), \quad t > 0.
\]
Then, it follows from [8, Lemma 7(1)] that \((\omega, \tilde{Q}_t)\) is a Poisson point process on \(\mathbb{R}^d\) with intensity \(\rho e^{-\rho_0(\lambda(t))\varphi_0(z)}dz\). Furthermore, we have

**Lemma 5.3.** For \(t\) large enough,
\[
\|E_{\tilde{Q}_t}[V^\omega(x)] - \lambda(t)\| \leq o((\log t)^{-\beta/d}).
\]

**Proof.** For any \(x \in B_M(t)\),
\[
E_{\tilde{Q}_t}[V^\omega(x)] = \rho \int_{\mathbb{R}^d} \varphi_0(x - z)e^{-\rho_0(\lambda(t))\varphi_0(z)}dz
\]
\[
= \rho \int_{B_{2M}(t)} \varphi_0(x - z)e^{-\rho_0(\lambda(t))\varphi_0(z)}dz + \rho \int_{\mathbb{R}^d \setminus B_{2M}(t)} \varphi_0(x - z)e^{-\rho_0(\lambda(t))\varphi_0(z)}dz.
\]
It is easy to see that for \(t\) large enough
\[
(5.2) \quad \rho \sup_{x \in B_M(t)} \int_{B_{2M}(t)} \varphi_0(x - z)e^{-\rho_0(\lambda(t))\varphi_0(z)}dz \leq \rho \int_{B_{2M}(t)} e^{-\rho_0(\lambda(t))\varphi_0(z)}dz
\]
\[
\leq c_1 \exp(-c_2(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}),
\]
where \(c_1, c_2 > 0\) is independent of \(t\) (but depending on \(M\)). Thus, for \(x \in B_M(t)\) and for \(t\) large enough,
\[
E_{\tilde{Q}_t}[V^\omega(x)] \leq \rho \int_{\mathbb{R}^d \setminus B_{2M}(t)} |x - z|^{-d-\beta} e^{-\rho_0(\lambda(t))|z|^{-d-\beta}}dz + c_1 \exp(-c_2(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}).
\]
On the other hand, we can check that
\[
\rho \int_{B_{2M}(t)} |z|^{-d-\beta} e^{-\rho_0(\lambda(t))|z|^{-d-\beta}}dz \leq c_3 \exp(-c_4(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}).
\]
Then, for \(t > 0\) large enough,
\[
E_{\tilde{Q}_t}[V^\omega(0)] = \rho \int_{\mathbb{R}^d} |z|^{-d-\beta} e^{-\rho_0(\lambda(t))|z|^{-d-\beta}}dz + c_5 \exp(-c_6(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)})
\]
\[
= \lambda(t) + O(\exp(-c(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}))
\]
for some constant \(c > 0\).

Next, for any \(x \in B_M(t)\) and \(t\) large enough, by the fact that \(\varphi_0(z) = 1 \wedge |z|^{-d-\beta}\) and the mean value theorem,
\[
|E_{\tilde{Q}_t}(V^\omega(x) - V^\omega(0))| \leq \rho \int_{\mathbb{R}^d \setminus B_{2M}(t)} \varphi_0(x - z) - \varphi_0(z)|e^{-\rho_0(\lambda(t))\varphi_0(z)}dz
\]
\[
+ O(\exp(-c(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}))
\]
\[
\leq c_7(\log t)^{\beta/(d\alpha)} \int_{\mathbb{R}^d \setminus B_{2M}(t)} |z|^{-d-\beta-1} e^{-c_8(\log t)^{(d+\beta)/d}|z|^{-d-\beta}}dz
\]
\[
+ O(\exp(-c(\log t)^{(d+\beta)(\alpha-\beta)/(d\alpha)}))
\]
\[
\leq c_9(\log t)^{-\beta/d(1-\beta/\alpha)/d},
\]
thanks to \(\beta \in (0, \alpha)\) again. This proves the desired assertion. \(\square\)

Now, we are back to the probability estimate for \(V^\omega(x)\) under the measure \(Q\).
Lemma 5.4. There is a constant $\delta \in (0, 1/2)$ such that for any $\varepsilon, M > 0$ there is a constant $t_{\delta, \varepsilon, M} > 0$ such that for all $t \geq t_{\delta, \varepsilon, M}$,
\[
Q \left( \sup_{x \in B_M(t)} |V^\omega(x) - \lambda(t)| \leq \varepsilon (\log t)^{-\beta/d} \right) \geq c(\delta, \varepsilon, M) t^{-d} \exp((\log t)^{\delta}).
\]

Proof. For any given $\gamma > 0$, by Lemma 5.3, for $t$ large enough,
\[
\sup_{x \in B_M(t)} |E_{\tilde{Q}_t}(V^\omega(x) - V^\omega(0))| \leq \frac{\varepsilon}{4} (\log t)^{-\beta/d}.
\]

For any $\gamma \in (1/2, 1)$, we further define
\[
E_1 = \left\{ V^\omega(0) - \lambda(t) \in \left[ (\log t)^{-\beta/d-\gamma}, \frac{\varepsilon}{4} (\log t)^{-\beta/d} \right] \right\}
\]
and
\[
E_2 = \left\{ \sup_{x \in B_M(t)} |V^\omega(x) - V^\omega(0) - E_{\tilde{Q}_t}(V^\omega(x) - V^\omega(0))| \geq \frac{\varepsilon}{2} (\log t)^{-\beta/d} \right\}.
\]

Then, for $t$ large enough,
\[
E_1 E^c_2 \subset \left\{ \sup_{x \in B_M(t)} |V^\omega(x) - \lambda(t)| \leq \varepsilon (\log t)^{-\beta/d} \right\}.
\]

Hence,
\[
Q \left( \sup_{x \in B_M(t)} |V^\omega(x) - \lambda(t)| \leq \varepsilon (\log t)^{-\beta/d} \right)
\geq e^{H(\rho_0(\lambda(t)))} E_{\tilde{Q}_t}(e^{\rho_0(\lambda(t))} V^\omega(0) \mathbb{1}_{E_1 \setminus E_2})
\geq \exp \left( -d \log t - \rho_0(\lambda(t))(\lambda(t) + (\log t)^{-\beta/d-\gamma}) \right) \tilde{Q}_t(E_1 \setminus E_2)
\geq \exp \left( -d \log t - \rho_0(\lambda(t))(\log t)^{-\beta/d-\gamma} + o(1) \right) \tilde{Q}_t(E_2)
\geq c_1 t^{-d} \exp(c_2(\log t)^{1-\gamma})(\tilde{Q}_t(E_1) - \tilde{Q}_t(E_2)),
\]
where in the third inequality we used (5.1).

As shown in [8, Lemma 7(iii)],
\[
(\log t)^{(d+2\beta)/(2d)} (V^\omega(0) - \lambda(t))
\]
under $\tilde{Q}_t$ converges in law to a non-degenerate Gaussian random variable. Then,
\[
\tilde{Q}_t(E_1) = \tilde{Q}_t \left( (\log t)^{(d+2\beta)/(2d)} (V^\omega(0) - \lambda(t)) \in \left[ (\log t)^{1/2-\gamma}, \frac{\varepsilon}{4} (\log t)^{1/2} \right] \right)
\]
is bounded from below by a positive constant for $t$ large enough, thanks to $\gamma \in (1/2, 1)$.

On the other hand, defining
\[
\tilde{\mu}^\omega_t(dz) := \mu^\omega(dz) - \rho e^{-\rho_0(\lambda(t))} \varphi_0(z) dz,
\]
we write
\[
V^\omega(x) - V^\omega(0) - E_{\tilde{Q}_t}(V^\omega(x) - V^\omega(0))
= \int_{\mathbb{R}^d} (\varphi_0(x-z) - \varphi_0(-z)) \tilde{\mu}^\omega_t(dz)
= \int_{B_{2M(t)}} (\varphi_0(x-z) - \varphi_0(-z)) \tilde{\mu}^\omega_t(dz) + \int_{\mathbb{R}^d \setminus B_{2M(t)}} (\varphi_0(x-z) - \varphi_0(-z)) \tilde{\mu}^\omega_t(dz).
\]
Note that, by the fact that \( \varphi_0(x) = 1 \wedge |x|^{-\beta} \),

\[
\sup_{x \in B_{2M}(t)} \left| \int_{B_{2M}(t)} (\varphi_0(x - z) - \varphi_0(-z)) \bar{\mu}^\omega_t(dz) \right|
\leq \sup_{x \in B_{2M}(t)} \int_{B_{2M}(t)} |\varphi_0(x - z) - \varphi_0(-z)| \mu^\omega_t(dz)
+ \rho \sup_{x \in B_{2M}(t)} \int_{B_{2M}(t)} |\varphi_0(x - z) - \varphi_0(-z)| e^{-\rho_0(\lambda(t))\varphi_0(z)} dz
\leq \int_{B_{2M}(t)} \bar{\mu}^\omega_t(dz) + 2\rho \int_{B_{2M}(t)} e^{-\rho_0(\lambda(t))\varphi_0(z)} dz.
\]

Hence, according to the second inequality in (5.2), for \( t \) large enough,

\[
\hat{Q}_t \left( \sup_{x \in B_{2M}(t)} \left| \int_{B_{2M}(t)} (\varphi_0(x - z) - \varphi_0(-z)) \bar{\mu}^\omega_t(dz) \right| \geq \frac{\varepsilon}{4} (\log t)^{-\beta/d} \right)
\leq \hat{Q}_t \left( \int_{B_{2M}(t)} \bar{\mu}^\omega_t(dz) \geq \frac{\varepsilon}{8} (\log t)^{-\beta/d} \right)
\leq \left[ \frac{\varepsilon}{8} (\log t)^{-\beta/d} \right]^{-2} E_{\hat{Q}_t} \left[ \int_{B_{2M}(t)} \bar{\mu}^\omega_t(dz) \right]^2
= \left[ \frac{\varepsilon}{8} (\log t)^{-\beta/d} \right]^{-2} \rho \int_{B_{2M}(t)} e^{-\rho_0(\lambda(t))\varphi_0(z)} dz
\leq c_3 \exp(-c_4 (\log t)^{(d+\beta)(\alpha-\beta)/(\alpha d)}),
\]

where in the second inequality we used the Markov inequality and the equality above follows from the fact that the \( Q_t \)-mean of \( \int_{B_{2M}(t)} \bar{\mu}^\omega_t(dz) \) is zero.

Furthermore, according to the mean value theorem, for \( t \) large enough,

\[
\sup_{x \in B_{2M}(t)} \left| \int_{R^d \setminus B_{2M}(t)} (\varphi_0(x - z) - \varphi_0(-z)) \bar{\mu}^\omega_t(dz) \right|
= \sup_{x \in B_{2M}(t)} \left| \int_{R^d \setminus B_{2M}(t)} \int_0^1 \frac{d}{d\theta} \varphi_0(\theta x - z) d\theta \bar{\mu}^\omega_t(dz) \right|
\leq \int_{R^d \setminus B_{2M}(t)} \sup_{x \in B_{2M}(t), \theta \in (0,1)} \left| \frac{d}{d\theta} \varphi_0(\theta x - z) \right| \bar{\mu}^\omega_t(dz)
+ \rho \int_{R^d \setminus B_{2M}(t)} \sup_{x \in B_{2M}(t), \theta \in (0,1)} \left| \frac{d}{d\theta} \varphi_0(\theta x - z) \right| e^{-\rho_0(\lambda(t))\varphi_0(z)} dz
= \int_{R^d \setminus B_{2M}(t)} \sup_{x \in B_{2M}(t), \theta \in (0,1)} \left| \frac{d}{d\theta} \varphi_0(\theta x - z) \right| \bar{\mu}^\omega_t(dz)
+ 2\rho \int_{R^d \setminus B_{2M}(t)} \sup_{x \in B_{2M}(t), \theta \in (0,1)} \left| \frac{d}{d\theta} \varphi_0(\theta x - z) \right| e^{-\rho_0(\lambda(t))\varphi_0(z)} dz
\leq c_5 (\log t)^{\beta/(\alpha d)} \int_{R^d \setminus B_{2M}(t)} |z|^{-d-\beta-1} \bar{\mu}^\omega_t(dz)
+ c_5 (\log t)^{\beta/(\alpha d)} \int_{R^d \setminus B_{2M}(t)} |z|^{-d-\beta-1} e^{-\rho_0(\lambda(t))\varphi_0(z)} dz.
\]
Note that
\[
(\log t)^{\beta/(ad)} \int_{\mathbb{R}^d \setminus B_M(t)} |z|^{-d-\beta-1} e^{-\rho_0(\lambda(t))\varphi_0(z)} \, dz \leq c_6 (\log t)^{-\beta/d - (1-\beta/a)/d}
\]
for \( t \) large enough, and that the \( \tilde{Q}_t \)-mean of \( \int_{\mathbb{R}^d \setminus B_M(t)} |z|^{-d-\beta-1} \tilde{\mu}_t^\omega(dz) \) is zero and the variance of it is bounded above by \( c_7 (\log t)^{-(d+2\beta+2)/d} \). Hence, for \( t \) large enough, by the Markov inequality,
\[
\tilde{Q}_t \left( \sup_{z \in B_M(t)} \left| \int_{\mathbb{R}^d \setminus B_M(t)} (\varphi_0(x-z) - \varphi_0(-z)) \tilde{\mu}_t^\omega(dz) \right| \geq \frac{\varepsilon}{4} (\log t)^{-\beta/d} \right) \\
\leq \tilde{Q}_t \left( c_8 (\log t)^{2(\beta+\beta/a)/d} E_{\tilde{Q}_t} \left[ \int_{\mathbb{R}^d \setminus B_M(t)} |z|^{-d-\beta-1} \tilde{\mu}_t^\omega(dz) \right]^2 \right) \\
\leq c_9 (\log t)^{-1-2(1-\beta/a)/d}.
\]

Combining all the estimates above, we arrive at that \( \tilde{Q}_t(E_2) \) tends to zero when \( t \to \infty \), and so \( \tilde{Q}_t(E_1) - \tilde{Q}_t(E_2) \) is bounded below by a positive constant for \( t \) large enough. This along with (5.3) yields the desired assertion. \( \square \)

Now, we can present the

Proof of Proposition 5.1. Fix \( \kappa > 1 \), and set \( I_t := ((2(\log t)^N)Z^d) \cap \{ z \in \mathbb{R}^d : |z| \leq t(\log t)^{-\kappa} \} \) for any \( t > 0 \). For any \( z \in I_t \) and \( \varepsilon > 0 \), define
\[
F_r(z) = \left\{ \sup_{x \in z + B_M(t)} |\tilde{V}_r^\omega(x) - \lambda(t)| \geq \frac{\varepsilon}{2} (\log t)^{-\beta/d} \right\},
\]
\[
G_r(z) = \left\{ \sup_{x \in z + B_M(t)} \sum_{\omega_i \notin z + \lambda_N(t)} |z - \omega_i|^{-d-\beta} \geq \frac{\varepsilon}{4} (\log t)^{-\beta/d} \right\}.
\]
where
\[
\tilde{V}_r^\omega(x) = \sum_{\omega_i \in z + \Lambda_N(t)} |x - \omega_i|^{-d-\beta}.
\]
We will estimate \( Q(\cap_{z \in I_t} (F_t(z) \cup G_t(z))) \).

Note that \( \{ G_t(z) \}_{z \in I_t} \) have the same distribution such that for any \( z \in I_t \) and \( t \) large enough
\[
Q(G_t(z)) = Q(G_t(0)) \leq (-c_1(\varepsilon)(\log t)^{dN + \beta(N-1/d)}) \]
thanks to Lemma 5.2. On the other hand, \( \{ F_t(z) \}_{z \in I_t} \) are i.i.d., and, according to Lemmas 5.4 and 5.2, for any \( z \in I_t \) and \( t \) large enough,
\[
Q(F_t(z)) = Q(F_t(0)) \leq Q(F_t(0) \setminus G_t(0)) + Q(G_t(0)) \\
= Q \left( \sup_{x \in B_M(t)} |V_r^\omega(x) - \lambda(t)| \geq \frac{\varepsilon}{4} (\log t)^{-\beta/d} \right) + Q(G_t(0)) \\
\leq 1 - c_2(\varepsilon)t^{-d}\exp((\log t)^{\delta}) + \exp \left( -c_1(\varepsilon)(\log t)^{dN + \beta(N-1/d)} \right) \\
\leq 1 - c_3(\varepsilon)t^{-d}\exp((\log t)^{\delta}).
\]
Hence,
\[
Q(\cap_{z \in I_t}(F_t(z) \cup G_t(z))) \leq Q(\cap_{z \in I_t} F_t(z)) + Q(\cup_{z \in I_t} G_t(z)) \\
\leq \left[ 1 - c_3(\varepsilon) t^{-d} \exp((\log t)^{\delta}) \right] c_d (\log t)^{-d(\varepsilon + N) d} \\
+ \exp \left(-c_5(\varepsilon)(\log t)^{dN + \beta(N-1/d)} \right) \\
\leq \exp \left(-c_6(\varepsilon)(\log t)^{dN + \beta(N-1/d)} \right) \\
+ \exp \left(-c_5(\varepsilon)(\log t)^{dN + \beta(N-1/d)} \right) \\
\leq \exp \left(-c_7(\varepsilon)(\log t)^{dN + \beta(N-1/d)} \right),
\]
where in the third inequality we used the fact that \(1 - x \leq e^{-x}\) for all \(x > 0\). The Borel-Cantelli lemma yields that \(Q\)-almost surely for all \(t\) large enough there exists \(z := z(t, \omega) \in I_t\) for which both \(F_t(z)\) and \(G_t(z)\) fail to happen.

Below, we will fix this \(z \in I_t\) for all \(t\) large enough. Then, it holds that
\[
\lambda_{V^\omega, B(z, B_M(t))} \leq \lambda_1(B(z, B_M(t))) + \sup_{x \in B(z, B_M(t))} V^\omega(x) \\
\leq \lambda_1(B(0, B_M(t))) + \sup_{x \in B(z, B_M(t))} \tilde{V}^\omega(x) + \sup_{x \in B(z, B_M(t))} \sum_{\omega, |z - \omega_i|^{-d - \beta}} |z - \omega_i|^{-d - \beta} \\
\leq 2M^{-\alpha}(\log t)^{-\beta/d} \lambda_1^0(B(0, 1)) + \lambda(t) + \frac{3\varepsilon}{4} (\log t)^{-\beta/d},
\]
where in the last inequality we used Lemma 3.5. Letting \(\varepsilon\) small enough and \(M\) large enough in the inequality above, we then prove the desired assertion. \(\Box\)

5.2. Quenched estimates of \(u^\omega(t, x)\): critical case. In this part, we will briefly show that the arguments of Theorems 3.7 and 3.8 with some modifications still work for the following

- **Critical case (C)** The characteristic exponent \(\psi(\xi)\) of the pure-jump symmetric Lévy process \(Z\) fulfills that \(\psi(\xi) = O(|\xi|^\alpha)\) as \(|\xi| \to 0\), and the shape function \(\varphi\) in the random potential \(V^\omega(x)\) satisfies

\[
0 < \liminf_{|x| \to \infty} \varphi(x)|x|^{d+\alpha} \leq \limsup_{|x| \to \infty} \varphi(x)|x|^{d+\alpha} < \infty.
\]

In the critical case, it was shown in [13, Theorem 6.4] that the integrated density \(N(\lambda)\) of states of the random Schrödinger operator \(H\) defined by (3.2) satisfies that
\[
-\infty < \liminf_{\lambda \to 0} \lambda^{d/\alpha} \log N(\lambda) \leq \limsup_{\lambda \to 0} \lambda^{d/\alpha} \log N(\lambda) < 0.
\]

Then, according to the arguments in Subsection 3.2.1 and the proof of Theorem 3.7, we have

**Theorem 5.5.** In the critical case (C), assume that (3.10) holds. Then, there is a constant \(k_0 > 0\) such that for any \(\varepsilon > 0\), \(Q\)-almost surely there is \(R_\varepsilon(\omega) \geq 1\) so that for any \(R \geq \max\{R_\varepsilon(\omega), \phi(t)\}\) and \(t \geq 1\),
\[
u^\omega(t, 0) \leq \Phi(t, R) + C(\varepsilon) R^{d/2} \exp \left(-t(1 - 2\varepsilon) \left(\frac{k_0}{d \log R}\right)^{\alpha/d}\right),
\]
where \(C(\varepsilon)\) is independent of \(R\) and \(t\).
When $Z$ is a symmetric $\alpha$-stable process with the exponent $\psi^{(\alpha)}(\xi)$ given in (2.2) for some $\alpha \in (0, 2]$, and $K := \lim_{|x| \to \infty} \varphi(x)|x|^{d+\alpha} \in (0, \infty)$, Okura proved precise annealed asymptotics of $u^\omega(t, x)$ in [14, Theorem and Remark ii]; that is, for all $x \in \mathbb{R}^d$, 
\[ \lim_{t \to \infty} \log \mathbb{E}_0[u^\omega(t, x)] \frac{t^d}{t^{d/(d+\alpha)}} = -C(\rho, K), \]
where 
\[ C(\rho, K) = \inf_{f \in L^2(\mathbb{R}^d, dx) \cap B_c(\mathbb{R}^d); \|f\|_{L^2(\mathbb{R}^d, dx)} = 1} \{ D(f, f) + W(f^2) \} \]
with $B_c(\mathbb{R}^d)$ being the set of measurable functions with compact support, $D(f, f)$ being the Dirichlet form associated with the symmetric $\alpha$-stable process $Z$, and 
\[ W(f^2) = \rho \int_{\mathbb{R}^d} \left[ 1 - \exp \left( -K \int_{\mathbb{R}^d} \frac{f(y)^2}{|x-y|^{d+\alpha}} dy \right) \right] dx. \]
Then, by the Tauberian theorem of exponential type (see [10, Theorem 3]), we have 
\[ \lim_{\lambda \to 0^+} \lambda^{d/\alpha} \log N(\lambda) = -k_0 := -\frac{\alpha}{d+\alpha} \left( \frac{d}{d+\alpha} \right)^{d/\alpha} C(\rho, K). \]
So, in this case we have a precise expression for the constant $k_0$ in Theorem 5.5.

For quenched lower bounds of $u^\omega(t, x)$, we have the following statement.

**Theorem 5.6.** In the critical case (C), assume that (3.11) holds. Then, there is a constant $A > 0$ such that for any $\delta \in (0, 1/2)$, $\kappa > 1$, $a > 1$, $\eta, \varsigma \in (0, 1)$, $Q$-almost surely there is $R_{\kappa, \alpha, \eta, \varsigma}(\omega) \geq 1$ so that for any $R \geq R_{\kappa, \alpha, \eta, \varsigma}(\omega)$ and $t \geq 1$, 
\[ u^\omega(t, 0) \geq C(\kappa, \delta, \eta, a) M_{\kappa, \eta}(R)^{-4d\delta} \exp \left( -a^2(1 + \varsigma) A \rho R^{-\alpha} \right), \]
where 
\[ M_{\kappa, \eta}(R) = R^{-\kappa} \exp \left( \frac{w_{d+\rho}}{d} ((1 + 2\eta) R)^d \right). \]

To prove Theorem 5.6, we need the following proposition, which is analogous to Proposition 3.4.

**Proposition 5.7.** In the critical case (C), for any $\kappa > 1$ and $\eta, \varsigma \in (0, 1)$, $Q$-almost surely there exists $r_{\kappa, \alpha, \eta, \varsigma}(\omega) > 0$ such that for all $r \geq r_{\kappa, \alpha, \eta, \varsigma}(\omega)$, there is $z := z(r, \omega) \in \mathbb{R}^d$ with $|z| \leq M_{\kappa, \eta}(r)$, 
\[ \lambda_{V^\omega, B(z, r)} \leq (1 + \varsigma) \left( \lambda_1^{(\alpha)}(B(0, 1)) + C_0 \right) r^{-\alpha}, \]
where 
\[ M_{\kappa, \eta}(r) = r^{-\kappa} \exp \left( \frac{w_{d+\rho}}{d} ((1 + 2\eta) r)^d \right), \]
\[ C_0 = K \left( 1 + \frac{c_\alpha}{d+\alpha} \right), \quad c_\alpha = \int_0^1 (e^u - 1) u^{-2+\alpha/(d+\alpha)} du, \]
and $\lambda_1^{(\alpha)}(B(0, 1))$ is the principle Dirichlet eigenvalue for the symmetric $\alpha$-stable process killed on exiting $B(0, 1)$ and with the exponent $\psi^{(\alpha)}(\xi)$ given in (2.1).

**Proof.** We use some notations from the proof of Proposition 3.4(i). For any $\kappa > 1$ and $\eta \in (0, 1)$, let $I_r := ((2(1 + \eta)r)^d \cap \{ z \in \mathbb{R}^d : |z| \leq M_{\kappa, \eta}(r) \}$ for any $r > 0$. Still define $\varphi_0(r) := \sup_{|z| \geq r} \varphi(x)$ and $\varphi_0(x) = \varphi_0(|x|)$. It is clear that $\varphi(x) \leq \varphi_0(x)$, and $\varphi_0(r)$ is a decreasing function on $[0, \infty)$ such that for $r$ large enough 
\[ (1 + \eta)^{-1} r^{-d-\alpha} \leq \varphi_0(r) \leq (1 + \eta)r^{-d-\alpha}, \]
thanks to (5.4).
Now, for any \( z \in I_r \), define \( F_r(z) \) as in the proof of Proposition 3.4(i), and
\[
G_r(z) = \left\{ \sup_{y \in B(z,r)} \sum_{\omega_i \notin B(z,(1+\eta)r)} \varphi_0(y - \omega_i) \geq C_\ast r^{-\alpha} \right\}
\]
for some constant \( C_\ast > 0 \) which is chosen later. As shown in the proof of Proposition 3.4(i), for \( r > 1 \) large enough,
\[
Q(\cap_{z \in I_r} F_r(z)) \leq \exp(-r^d/2).
\]

On the other hand, by the deceasing property of \( \varphi_0(r) \) and (5.6), for all \( r \) large enough,
\[
Q \left[ \exp \left( \frac{1}{\varphi_0(\eta r)} \sup_{y \in B(0,r)} \sum_{\omega_i \notin B(0,(1+\eta)r)} \varphi_0(y - \omega_i) \right) \right] \leq \exp \left( \int_{\mathbb{R}^d \setminus B(0,(1+\eta)r)} e^{\varphi_0(\eta r) - \varphi_0(\eta |\omega_i|/(1+\eta))} - 1 \right) dz \leq \exp \left( \frac{\rho w_d (1 + \eta) 3^d c_\alpha r^d}{d + \alpha} \right).
\]

This along with the Markov inequality yields that
\[
Q(G_r(0)) \leq \exp \left( \frac{1}{\varphi_0(\eta r)} \sup_{y \in B(0,r)} \sum_{\omega_i \notin B(0,(1+\eta)r)} \varphi_0(y - \omega_i) \right) \geq \exp \left( \frac{C_\ast r^{-\alpha}}{\varphi_0(\eta r)} \right) \leq \exp \left( \frac{\rho w_d (1 + \eta) 3^d c_\alpha r^d}{d + \alpha} - C_\ast K^{-1}(1 + \eta)^{-1} r^d \right).
\]

Since \( \{G_r(z)\}_{z \in I_r} \) have the same distribution (but are not independent with each other), we find that
\[
Q(\cup_{z \in I_r} G_r(z)) \leq 2 \left( \frac{M_{\kappa,\eta}(r)}{(1 + \eta)r} \right)^d \exp \left[ \frac{\rho w_d (1 + \eta) 3^d c_\alpha}{d + \alpha} r^d - C_\ast K^{-1}(1 + \eta)^{-1} r^d \right] = 2(1 + \eta)^{-d} r^{-(\kappa + 1)d} \exp \left[ \rho w_d \left( \frac{(1 + \eta) 3^d c_\alpha}{d + \alpha} + (1 + 2\eta)^d \right) r^d - K^{-1} C_\ast (1 + \eta)^{-1} r^d \right].
\]

Now, we take
\[
C_\ast = K(1 + 2\eta)^d \left( 1 + \frac{c_\alpha}{d + \alpha} \right)
\]
and so \( Q(\cup_{z \in I_r} G_r(z)) < \infty \).

Therefore,
\[
Q(\cap_{z \in I_r} (F_r(z) \cup G_r(z))) \leq Q(\cap_{z \in I_r} F_r(z)) + Q(\cup_{z \in I_r} G_r(z)) < \infty.
\]

Hence, by the Borel-Cantelli lemma, \( Q \)-almost surely there exists \( z := z(r, \omega) \in \mathbb{R}^d \) such that \( |z| \leq M_{\kappa,\eta}(r) \), and both \( F_r(z) \) and \( G_r(z) \) fail to hold.

With this at hand, one can follow the proof of Proposition 3.4(i) to get the desired assertion. \( \square \)
According to Proposition 5.7, one can further repeat the argument for Theorem 3.8 to prove Theorem 5.6. The details are omitted here. We note that, also thanks to Proposition 5.7, the constant $A$ in Theorem 5.6 can be taken

$$A_0 = \lambda_1^{(\alpha)}(B(0, 1)) + C_0,$$

where $C_0$ is given in Proposition 5.7. Furthermore, as an application of Theorems 5.5 and 5.7, we can obtain Proposition 1.3.

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