Explicit Form of Evolution Operator of Three Atoms Tavis–Cummings Model

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Abstract

In this letter the explicit form of evolution operator of three atoms Tavis–Cummings model is given, which is a generalization of the paper quant-ph/0403008.

The purpose of this letter is to give an explicit form to the evolution operator of Tavis–Cummings model (II) with some atoms. This model is a very important one in Quantum Optics and has been studied widely, see [2] as general textbooks in quantum optics.

We are studying a quantum computation and therefore want to study the model from this point of view, namely the quantum computation based on atoms of laser–cooled and trapped linearly in a cavity. We must in this model construct a controlled NOT gate or

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other controlled unitary gates to perform a quantum computation, see [3] as a general introduction to this subject.

For that aim we need the explicit form of evolution operator of the models with one, two and three atoms (at least). As to the model of one atom or two atoms it is more or less known (see [4]), while as to the case of three atoms it has not been given as far as we know. Since we succeeded in finding the explicit form for three atoms case we report it \(^1\).

The Tavis–Cummings model (with \(n\)-atoms) that we will treat in this paper can be written as follows (we set \(\hbar = 1\) for simplicity).

\[
H = \omega 1_L \otimes a^\dagger a + \frac{\Delta}{2} \sum_{i=1}^{n} \sigma_i^{(3)} \otimes 1 + g \sum_{i=1}^{n} \left( \sigma_i^{(+)} \otimes a + \sigma_i^{(-)} \otimes a^\dagger \right),
\]

where \(\omega\) is the frequency of radiation field, \(\Delta\) the energy difference of two level atoms, \(a\) and \(a^\dagger\) are annihilation and creation operators of the field, and \(g\) a coupling constant, and \(L = 2^n\). Here \(\sigma_i^{(+)}\), \(\sigma_i^{(-)}\) and \(\sigma_i^{(3)}\) are given as

\[
\sigma_i^{(s)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_s \otimes 1_2 \otimes \cdots \otimes 1_2 \text{ (i – position)} \in M(L, \mathbb{C})
\]

where \(s\) is +, – and 3 respectively and

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Here let us rewrite the hamiltonian (1). If we set

\[
S_+ = \sum_{i=1}^{n} \sigma_i^{(+)} \quad S_- = \sum_{i=1}^{n} \sigma_i^{(-)} \quad S_3 = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^{(3)},
\]

then (1) can be written as

\[
H = \omega 1_L \otimes a^\dagger a + \Delta S_3 \otimes 1 + g \left( S_+ \otimes a + S_- \otimes a^\dagger \right) \equiv H_0 + V,
\]

which is very clear. We note that \(\{S_+, S_-, S_3\}\) satisfy the \(su(2)\)-relation

\[
[S_3, S_+] = S_+, \quad [S_3, S_-] = -S_-, \quad [S_+, S_-] = 2S_3.
\]

\(^1\)J.C.Retamal et al might have obtained the same result by another method [5]
However, the representation $\rho$ defined by
$$
\rho(\sigma_+) = S_+, \quad \rho(\sigma_-) = S_-, \quad \rho(\sigma_3/2) = S_3
$$
is a reducible representation of $su(2)$.

We would like to solve the Schrödinger equation

$$
i \frac{d}{dt} U = HU = (H_0 + V) U,
$$

where $U$ is a unitary operator (called the evolution operator). We can solve this equation by using the method of constant variation. The result is well-known to be

$$
U(t) = (e^{-it\omega S_3} \otimes e^{-it\omega a a^\dagger}) e^{-itg(S_+ \otimes a + S_- \otimes a^\dagger)}
$$

under the resonance condition $\Delta = \omega$, where we have dropped the constant unitary operator for simplicity. Therefore we have only to calculate the term (8) explicitly, which is however a very hard task $^2$. In the following we set

$$
A = S_+ \otimes a + S_- \otimes a^\dagger
$$

for simplicity. We can determine $e^{-itgA}$ for $n = 1$ (one atom case), $n = 2$ (two atoms case) and $n = 3$ (three atoms case) completely.

**One Atom Case** In this case $A$ in (9) is written as

$$
A_1 = \begin{pmatrix}
0 & a \\
a^\dagger & 0
\end{pmatrix}.
$$

By making use of the relation

$$
A_1^2 = \begin{pmatrix}
a a^\dagger & 0 \\
0 & a^\dagger a
\end{pmatrix} = \begin{pmatrix}
N + 1 & 0 \\
0 & N
\end{pmatrix}
$$

with the number operator $N$ we have

$$
e^{-itgA_1} = \begin{pmatrix}
\cos(tg\sqrt{N+1}) & -i \frac{\sin(tg\sqrt{N+1})}{\sqrt{N+1}} a \\
-i \frac{\sin(tg\sqrt{N})}{\sqrt{N}} a^\dagger & \cos(tg\sqrt{N})
\end{pmatrix}.
$$

$^2$the situation is very similar to that of the paper quant-ph/0312060 in [7]
We obtained the explicit form of solution. However, this form is more or less well–known, see for example the second book in [2]. We note that (12) can be decomposed as

$$\begin{pmatrix}
\cos\left(\tan^{-1}\sqrt{N} + 1\right) & -i\frac{\sin\left(\tan^{-1}\sqrt{N} + 1\right)}{\sqrt{N+1}} a \\
-i\frac{\sin\left(\tan^{-1}\sqrt{N}\right)}{\sqrt{N}} a^\dagger & \cos\left(\tan^{-1}\sqrt{N}\right)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
-i\frac{\tan\left(\tan^{-1}\sqrt{N} + 1\right)}{\sqrt{N+1}} a^\dagger & 1
\end{pmatrix}
\begin{pmatrix}
\cos\left(\tan^{-1}\sqrt{N} + 1\right) & 0 \\
0 & \cos\left(\tan^{-1}\sqrt{N}\right)
\end{pmatrix}
\begin{pmatrix}
1 & -i\frac{\tan\left(\tan^{-1}\sqrt{N} + 1\right)}{\sqrt{N+1}} a \\
0 & 1
\end{pmatrix}.$$  

(13)

This is a Gauss decomposition of unitary operator. This may be used to construct a theory of “quantum” representation of a non–commutative group, which is now under consideration.

**Two Atoms Case** In this case $A$ in (9) is written as

$$A_2 = \begin{pmatrix}
0 & a & a & 0 \\
0 & 0 & a \\
a^\dagger & 0 & 0 & a \\
0 & a^\dagger & a^\dagger & 0
\end{pmatrix}.$$  

(14)

Our method is to reduce the $4 \times 4$–matrix $A_2$ in (14) to a $3 \times 3$–matrix $B_1$ in the following to make our calculation easier. For that aim we prepare the following matrix

$$T = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

then it is easy to see

$$T^\dagger A_2 T = \begin{pmatrix}
0 & 0 & \sqrt{2}a & 0 \\
0 & \sqrt{2}a^\dagger & 0 & \sqrt{2}a \\
\sqrt{2}a^\dagger & 0 & \sqrt{2}a & 0 \\
0 & \sqrt{2}a^\dagger & 0 & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 \\
0 \\
B_1
\end{pmatrix}.$$
where $B_1 = J_+ \otimes a + J_- \otimes a^\dagger$ and $\{J_+, J_-\}$ are just generators of (spin one) irreducible representation of (3). We note that this means a well–known decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

Therefore to calculate $e^{-itgA_2}$ we have only to do $e^{-itgB_1}$. Noting the relation

$$B_1^3 = \begin{pmatrix} 2(2N + 3) & 2(2N + 1) & 2(2N - 1) \\ 2(2N + 1) & 2(2N - 1) & 2(2N + 3) \\ 2(2N - 1) & 2(2N + 3) & 2(2N + 1) \end{pmatrix} B \equiv DB_1,$$

we obtain

$$e^{-itgB_1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

(15)

where

$$b_{11} = \frac{N + 2 + (N + 1)\cos\left(tg\sqrt{2(2N + 3)}\right)}{2N + 3}, \quad b_{12} = -i \frac{\sin\left(tg\sqrt{2(2N + 3)}\right)}{\sqrt{2N + 3}} a,
$$

$$b_{13} = \frac{-1 + \cos\left(tg\sqrt{2(2N + 3)}\right)}{2N + 3} a^2, \quad b_{21} = -i \frac{\sin\left(tg\sqrt{2(2N + 1)}\right)}{\sqrt{2N + 1}} a^\dagger,
$$

$$b_{22} = \cos\left(tg\sqrt{2(2N + 1)}\right), \quad b_{23} = -i \frac{\sin\left(tg\sqrt{2(2N + 1)}\right)}{\sqrt{2N + 1}} a,
$$

$$b_{31} = \frac{-1 + \cos\left(tg\sqrt{2(2N - 1)}\right)}{2N - 1} (a^\dagger)^2, \quad b_{32} = -i \frac{\sin\left(tg\sqrt{2(2N - 1)}\right)}{\sqrt{2N - 1}} a^\dagger,
$$

$$b_{33} = \frac{N - 1 + N\cos\left(tg\sqrt{2(2N - 1)}\right)}{2N - 1}.\]
Three Atoms Case  In this case \( A \) in (9) is written as

\[
A_3 = \begin{pmatrix}
0 & a & a & 0 & a & 0 & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & a & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & 0 & a & 0 \\
0 & a^\dagger & a^\dagger & 0 & 0 & 0 & 0 & a \\
a^\dagger & 0 & 0 & 0 & 0 & a & a & 0 \\
0 & a^\dagger & 0 & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & a^\dagger & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & 0 & a^\dagger & 0 & a^\dagger & a^\dagger & 0
\end{pmatrix}.
\]  \tag{16}

We would like to look for the explicit form of solution like (12) or (15). If we set

\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & 2 \sqrt{3} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 2 \sqrt{3} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

then it is not difficult to see

\[
T^\dagger A_3 T = \begin{pmatrix}
0 & a \\
a^\dagger & 0
\end{pmatrix}
\begin{pmatrix}
0 & a \\
a^\dagger & 0
\end{pmatrix}^\ast
\begin{pmatrix}
0 & \sqrt{3} a & 0 & 0 \\
\sqrt{3} a^\dagger & 0 & 2 a & 0 \\
0 & 2 a^\dagger & 0 & \sqrt{3} a \\
0 & 0 & \sqrt{3} a^\dagger & 0
\end{pmatrix}
\equiv \begin{pmatrix}
A_1 \\
A_1 \\
B_2
\end{pmatrix}.
This means a decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$. Therefore we have only to calculate $e^{-itgB_2}$, which is however not easy. The result is

$$
e^{-itgB_2} = \begin{pmatrix}
    f_1(N + 2) & -\sqrt{3}ih_2(N + 2)a & 2\sqrt{3}f_3(N + 2)a^2 & -6ih_3(N + 2)a^3 \\
    -\sqrt{3}ih_2(N + 1)a^\dagger & f_2(N + 1) & -2ih_3(N + 1)a & 2\sqrt{3}f_3(N + 1)a^2 \\
    2\sqrt{3}f_3(N)(a^\dagger)^2 & -2ih_3(N)a^\dagger & f_4(N) & -\sqrt{3}ih_4(N)a \\
    -6ih_3(N - 1)(a^\dagger)^3 & 2\sqrt{3}f_3(N - 1)(a^\dagger)^2 & -\sqrt{3}ih_4(N - 1)a^\dagger & f_5(N - 1)
  \end{pmatrix}
$$

(17)

where

$$f_1(N) = \left\{ w_+(N)\cos(tg\sqrt{\lambda_+(N)}) - w_-(N)\cos(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_2(N) = \left\{ w_+(N)\cos(tg\sqrt{\lambda_+(N)}) - w_-(N)\cos(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$h_2(N) = \left\{ \frac{-w_+(N)}{\sqrt{\lambda_+(N)}}\sin(tg\sqrt{\lambda_+(N)}) - \frac{-w_-(N)}{\sqrt{\lambda_-(N)}}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_3(N) = \left\{ \cos(tg\sqrt{\lambda_+(N)}) - \cos(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$h_3(N) = \left\{ \frac{1}{\sqrt{\lambda_+(N)}}\sin(tg\sqrt{\lambda_+(N)}) - \frac{1}{\sqrt{\lambda_-(N)}}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_4(N) = \left\{ w_+(N)\cos(tg\sqrt{\lambda_-(N)}) - w_-(N)\cos(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}),$$

$$h_4(N) = \left\{ \frac{-w_+(N)}{\sqrt{\lambda_-(N)}}\sin(tg\sqrt{\lambda_-(N)}) - \frac{-w_-(N)}{\sqrt{\lambda_+(N)}}\sin(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_5(N) = \left\{ w_+(N)\cos(tg\sqrt{\lambda_-(N)}) - w_-(N)\cos(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}),$$

$$\tilde{h}_3(N) = \left\{ \sqrt{\lambda_+(N)}\sin(tg\sqrt{\lambda_-(N)}) - \sqrt{\lambda_-(N)}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)})$$

and

$$d(N) = 16N^2 + 9, \lambda_\pm = 5N \pm \sqrt{d(N)}, v_\pm = -2N - 3 \pm \sqrt{d(N)}, w_\pm = 2N - 3 \pm \sqrt{d(N)}.$$

We obtained the explicit form of evolution operator of the Tavis–Cummings model for three atoms case, so there are many applications to quantum optics or mathematical
physics, see for example [4]. In the near future we will apply the result to a quantum computation based on atoms of laser–cooled and trapped linearly in a cavity [6].

We conclude this paper by making a comment. The Tavis–Cummings model is based on (only) two energy levels of atoms. However, an atom has in general infinitely many energy levels, so it is natural to use this possibility. We are also studying a quantum computation based on multi–level systems of atoms (a qudit theory) [7]. Therefore we would like to extend the Tavis–Cummings model based on two–levels to a model based on multi–levels. This is a very challenging task.

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