SINGULARITIES OF LAGRANGIAN MEAN CURVATURE FLOW: ZERO-MASLOV CLASS CASE

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Abstract. We study singularities of Lagrangian mean curvature flow in $\mathbb{C}^n$ when the initial condition is a zero-Maslov class Lagrangian. We start by showing that, in this setting, singularities are unavoidable. More precisely, we construct Lagrangians with arbitrarily small Lagrangian angle and Lagrangians which are Hamiltonian isotopic to a plane that, nevertheless, develop finite time singularities under mean curvature flow.

We then prove two theorems regarding the tangent flow at a singularity when the initial condition is a zero-Maslov class Lagrangian. The first one (Theorem A) states that the rescaled flow at a singularity converges weakly to a finite union of area-minimizing Lagrangian cones. The second theorem (Theorem B) states that, under the additional assumptions that the initial condition is an almost-calibrated and rational Lagrangian, connected components of the rescaled flow converges to a single area-minimizing Lagrangian cone, as opposed to a possible non-area-minimizing union of area-minimizing Lagrangian cones. The latter condition is dense for Lagrangians with finitely generated $H_1(L,\mathbb{Z})$.

1. Introduction

In the last few years, mean curvature flow of higher codimension submanifolds has attracted some attention. Most of the work done has focused on finding initial conditions that assure the flow will exist for all time. For instance, under some natural convexity assumptions on the image of the Gauss map, long time existence and convergence results have been proved by J. Chen, J. Li, and Tian [4], Smoczyk [14, 15], Smoczyk and M.-T. Wang [16], M.-P. Tsui and M.-T. Wang [17], and M.-T. Wang [18, 19, 20]. On the other hand, finite time singularities for mean curvature flow in the higher codimension case are not so well understood and, reasoning in analogy with minimal surfaces, they are expected to exhibit a far more complicated behavior than in the codimension one case.

There is, therefore, interest in identifying initial conditions for the flow that are broad enough to admit singularities, but restrictive enough so that the singularities are, so to speak, “well-behaved”. A natural candidate for such an initial condition is Lagrangian because, when the ambient manifold is Kähler-Einstein, the Lagrangian condition is preserved by mean curvature flow (see [12]). Mu-Tao Wang observed in [18] that, when the ambient manifold is Calabi-Yau, almost-calibrated Lagrangians (see next section for the definition) cannot develop type I singularities, i.e., no sequence of rescaled...
flows at a singularity can converge strongly to a homothetically shrinking solution. Later, Jingyi Chen and Jiayu Li [3] showed that in this setting the sequence of rescaled flows converges weakly to an integer rectifiable stationary Lagrangian varifold which is also a cone.

In this paper we study finite time singularities for zero-Maslov class Lagrangians in $\mathbb{C}^n$, a more general condition than being almost-calibrated. The first result, Theorem A, states that the tangent flow at a singularity can be decomposed into a finite union of area-minimizing Lagrangian cones. Theorem B is a more interesting result because, assuming the initial condition is an almost-calibrated and rational Lagrangian, it states that the Lagrangian angle converges to a single constant on each connected component of the rescaled flow. In particular, this implies that connected components of the rescaled flow converge weakly to a single area-minimizing Lagrangian cone, instead of a possible non-area-minimizing union of area-minimizing Lagrangian cones. Heuristically speaking, such property qualifies the formation of singularities as being, so to speak, “well behaved”. Without such behavior, it would be hopeless to expect Lagrangian mean curvature flow to be more tractable than general higher codimension mean curvature flow. We remark that any Lagrangian $M$ with $H_1(M, \mathbb{Z})$ finitely generated can always be perturbed in order to become rational.

Assuming some rotational symmetry, we also construct zero-Maslov class exact Lagrangians that develop finite time singularities under Lagrangian mean curvature flow. These examples include Lagrangians with arbitrarily small oscillation of the Lagrangian angle and Lagrangians which are Hamiltonian isotopic to a plane.

The paper is organized as follows. In Section 2 we recall some standard definitions and results that will be useful throughout the rest of the paper. The main two results are discussed and stated in Section 3. Examples of finite time singularities for Lagrangian mean curvature flow are given in Section 4. The first result, Theorem A, is proven in Section 5. In Section 6 we derive evolution equations of some geometric quantities that will be needed in Section 7. In this section we prove Theorem B.

The author would like to express his gratitude to Richard Schoen for all of his guidance and insight. He would also like to thank Leon Simon and Brian White for enlightening discussions and constant availability.

2. Preliminaries

Let $J$ and $\omega$ denote, respectively, the standard complex structure on $\mathbb{C}^n$ and the standard symplectic form on $\mathbb{C}^n$. We consider also the closed complex-valued $n$-form given by

$$\Omega \equiv dz_1 \wedge \ldots \wedge dz_n$$
and the Liouville form given by
\[ \lambda \equiv \sum_{i=1}^{n} x_{i} dy_{i} - y_{i} dx_{i}, \quad d\lambda = 2\omega, \]
where \( z_{j} = x_{j} + iy_{j} \) are complex coordinates of \( \mathbb{C}^{n} \).

A smooth \( n \)-dimensional submanifold \( L \) in \( \mathbb{C}^{n} \) is said to be Lagrangian if \( \omega_{L} = 0 \) and this implies that (see [7])
\[ \Omega_{L} = e^{i\theta} \text{vol}_{L}, \]
where \( \text{vol}_{L} \) denotes the volume form of \( L \) and \( \theta \) is some multivalued function called the Lagrangian angle. When the Lagrangian angle is a single valued function the Lagrangian is called zero-Maslov class and if 
\[ \cos \theta \geq \varepsilon \]
for some positive \( \varepsilon \), then \( L \) is said to be almost-calibrated. Furthermore, if \( \theta \equiv \theta_{0} \), then \( L \) is calibrated by 
\[ \text{Re} \left( e^{-i\theta_{0}} \Omega \right) \]
and hence area-minimizing. In this case, \( L \) is referred as being Special Lagrangian.

Likewise, we define an integral \( n \)-varifold \( L_{1} \) and an integral \( n \)-current \( L_{2} \) to be Lagrangian if
\[ \int_{L_{1}} \phi|\omega \wedge \eta| dH^{n} = 0 \quad \text{for all } n-2 \text{ form } \eta \text{ and all smooth } \phi \in C_{C}^{\infty}(\mathbb{C}^{n}) \]
and
\[ \int_{L_{2}} \phi \omega \wedge \eta dH^{n} = 0 \quad \text{for all } n-2 \text{ form } \eta \text{ and all } \phi \in C_{C}^{\infty}(\mathbb{C}^{n}) \]
respectively. The concept of being Special Lagrangian can be easily extended to the case when \( L \) is an integral current.

For a smooth Lagrangian, the relation between the Lagrangian angle and the mean curvature is given by the following remarkable property (see for instance [11])
\[ H = J \nabla \theta. \]

Let \( L_{0} \) be a smooth Lagrangian in \( \mathbb{C}^{n} \) such that, for some constant \( C_{0} \), we have
\[ \mathcal{H}^{n}_{\text{loc}} \left( L_{0} \cap B_{R}(0) \right) \leq C_{0} R^{n} \]
for all \( R \) sufficiently large and assume that we have a solution \( (L_{t})_{0 \leq t < T} \) to mean curvature flow for which the second fundamental form of \( L_{t} \) is bounded by a time dependent constant. The same argument used in [12] and the maximum principle for noncompact manifolds proved by Ecker and Huisken in [6] imply that the Lagrangian condition is preserved. In this case, we say that we have a solution to Lagrangian mean curvature flow. Moreover, if \( L_{0} \) is also zero-Maslov class, then this condition is preserved by
the flow and, according to [13], the Lagrangian angles $\theta_t$ can be chosen so that
\[
\frac{d\theta_t}{dt} = \Delta \theta_t.
\]
An immediate application of the parabolic maximum principle shows that the almost-calibrated condition is preserved by Lagrangian mean curvature flow.

A Lagrangian $L_0$ is said to be rational if for some real number $a$
\[
\lambda(H_1(L_0, Z)) = \{a2k\pi | k \in \mathbb{Z}\}.
\]
Any Lagrangian having $H_1(L_0, Z)$ finitely generated can be perturbed in order to become rational. When $a = 0$ the Lagrangian is called exact. Furthermore, if $L_0$ is also zero-Maslov class, we will see in Section 4 that the rational condition is preserved by Lagrangian mean curvature flow, i.e.,
\[
\lambda(H_1(L_t, Z)) = \{a2k\pi | k \in \mathbb{Z}\}
\]
while the solution exists smoothly.

Assume now that the solution to mean curvature flow develops a singularity at the point $(x_0, T)$ in space-time. Then
\[
L_s^\sigma := \sigma(L_{T+s/\sigma^2 - x_0}) \quad \text{for} \quad -\sigma^2T < s < 0
\]
is also a solution to Lagrangian mean curvature flow and it is called a rescaled flow. It follows from [9, Lemma 8] that for every sequence $(\sigma_i)$ going to infinity there is a subsequence for which the mean curvature flow
\[
(L_s^{\sigma_i})_{-\sigma_i^2T < s < 0}
\]
converges weakly to a homothetically shrinking weak solution of mean curvature flow (Brakke flow). This solution is called tangent flow and depends on the sequence $(\sigma_i)$ taken.

3. Statement of results

Let $(L_t)_{0 \leq t < T}$ be a smooth solution to Lagrangian mean curvature flow in $\mathbb{C}^n$ satisfying, for some constant $C_0$, the area bounds
\[
\mathcal{H}^n(L_0 \cap B_R(0)) \leq C_0 R^n
\]
for all $R$ sufficiently large. Furthermore, assume that the flow develops a finite time singularity at time $T$ and that $L_0$ is zero-Maslov class with bounded Lagrangian angle. We denote the Lagrangian angle of a rescaled flow $(L_s^{\sigma_i})_{s < 0}$ by $\theta_{L,s}$. Arguing informally, the following theorem states that a sequence of rescaled flows at a singularity converges weakly to a finite union of integral Special Lagrangian cones.

**Theorem A.** If $L_0$ is zero-Maslov class with bounded Lagrangian angle, then for any sequence of rescaled flows $(L_s^{\sigma_i})_{s < 0}$ at a singularity, there exist a finite set $\{\theta_1, \ldots, \theta_N\}$ and integral Special Lagrangian cones $L_1, \ldots, L_N$
such that, after passing to a subsequence, we have for every smooth function $\phi$ compactly supported, every $f$ in $C^2(\mathbb{R})$, and every $s < 0$

$$\lim_{i \to \infty} \int_{L^i_s} f(\theta_{i,s}) \phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\bar{\theta}_j) \mu_j(\phi),$$

where $\mu_j$ and $m_j$ denote the Radon measure of the support of $L_j$ and its multiplicity respectively.

Furthermore, the set $\{ar{\theta}_1, \ldots, \bar{\theta}_N\}$ does not depend on the sequence of rescalings chosen.

**Remark 3.1.**

1) It is possible and expected that, for instance, $\{\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3\} = \{0, \pi, 2\pi\}$ but the supports of $L_1$, $L_2$, and $L_3$ are all the same.

2) In case $n = 2$, it is well known that the support of area-minimizing cones are planes intersecting transversely.

Theorem A follows from combining standard ideas from geometric measure theory with the evolution equation

$$\frac{d \theta_{i,s}^2}{dt} = \Delta \theta_{i,s}^2 - 2|H|^2.$$

We will show that, after using Huisken monotonicity formula [8], such equation implies that for all $t < 0$ and all positive $R$

$$\lim_{i \to \infty} \int_{-1}^{t} \int_{L^i_s \cap B_R(0)} |H|^2 + |x|^2 \, d\mathcal{H}^n \, ds = 0,$$

where $x$ denotes the vector determined by the point $x$ in $\mathbb{C}^n$ and $x^\perp$ denotes the projection of the vector $x$ onto the orthogonal complement of $T_x L^i_s$.

Hence, for almost all $s < 0$ we get that for all positive $R$

$$\lim_{i \to \infty} \int_{L^i_s \cap B_R(0)} |H|^2 + |x|^2 \, d\mathcal{H}^n = 0$$

and this implies that, after passing to a subsequence, $L^i_s$ converges weakly to a stationary integral varifold $L$ which is also a cone. Note that so far $L$ could be a union of three Lagrangian half-planes meeting at angles of $2\pi/3$ along a common boundary. We now sketch briefly why such configuration cannot occur because the proof of Theorem A consists essentially in exploiting this argument. Suppose that each of the half-planes have Lagrangian angles $\theta_1, \theta_2,$ and $\theta_3$. Then, for all sufficiently small $\varepsilon$, $\{|\theta_{i,s} - \theta_1| \leq \varepsilon\}$ converges to a half-plane and so

$$\lim_{i \to \infty} \mathcal{H}(\{\theta_{i,s} = \theta_1 + \varepsilon\} \cap B_R(0)) > 0$$

This is impossible because, using the coarea formula and Hölder’s inequality, we have
\[
\lim_{i \to \infty} \int_{-\infty}^{\infty} H^{n-1} (\{ \theta_{i,s} = u \} \cap B_R(0)) \, du = \lim_{i \to \infty} \int_{L_i \cap B_R(0)} \left| H \right| \, dH^n = 0.
\]

Theorem A raises the following question: Given \( \Sigma \) a sequence of connected components of \( L_i \cap B_R(0) \) that converges weakly to \( \Sigma \), does \( \Sigma \) need to be a SLag cone? In other words, does \( \theta_{i,s} \) need to converge to a constant? According to Theorem A we only know that \( \Sigma \) is a finite union of SLag cones which might have different Lagrangian angles and hence not necessarily area-minimizing. An affirmative answer to this question is necessary if one wants to make reasonable the possibility of developing a regularity theory for the flow.

Technically, the difficulty comes from the fact that because the sequence of smooth manifolds \( L_i \) are becoming singular when \( i \) goes to infinity, no Poincaré inequality holds with a constant independent of \( i \) and therefore we cannot conclude that, on each connected component of \( L_i \), the Lagrangian angles \( \theta_{i,s} \) converge to a constant. As a matter of fact, for the sequence of smooth surfaces

\[ L_\varepsilon \equiv \{ (z, w) \in \mathbb{C}^2 \mid zw = \varepsilon \}, \]

one can easily construct bounded functions \( f_\varepsilon \) for which the \( L^2 \) norm of its gradient goes to zero when \( \varepsilon \) goes to zero but nevertheless \( f_\varepsilon \) converges to a distinct constant on each complex plane. The question raised in the previous paragraph was addressed in [3, Theorem 5.1] but unfortunately this technical aspect was overlooked.

In order to deal with this difficulty, we require \( L_0 \) to satisfy two additional conditions, namely that it is an almost-calibrated and rational Lagrangian (see Section 2 for the definitions). We argued in Section 2 that these conditions are preserved by Lagrangian mean curvature flow.

**Theorem B.** If \( L_0 \) is almost-calibrated and rational, then, after passing to a subsequence of \( (L_i^s)_{s<0} \), the following property holds for all \( R > 0 \) and almost all \( s < 0 \).

For any convergent subsequence (in the Radon measure sense) \( \Sigma^i \) of connected components of \( B_{4R}(0) \cap L_i^s \) intersecting \( B_R(0) \), there exists a Special Lagrangian cone \( L \) in \( B_{2R}(0) \) with Lagrangian angle \( \bar{\theta} \) such that

\[
\lim_{i \to \infty} \int_{\Sigma^i} f(\theta_{i,s}) \phi \, dH^n = mf(\bar{\theta}) \mu(\phi),
\]

for every \( f \) in \( C(\mathbb{R}) \) and every smooth \( \phi \) compactly supported in \( B_{2R}(0) \), where \( \mu \) and \( m \) denote the Radon measure of the support of \( L \) and its multiplicity respectively.

Next, we give a heuristic argument explaining why the rational condition should play a role. From the pioneering work of Richard Hamilton both on Ricci flow and on mean curvature flow we know that it is helpful to find quantities that are constant on self-similar solutions. For that matter, let us consider

\[ L_s \equiv \sqrt{s}L_1 \]
to be a solution to Lagrangian mean curvature flow where $L_0$ is zero-Maslov class. A simple computation reveals that for all $s > 0$
\[
H(L_s) = x^\perp/(2s) \iff 2s \nabla \theta_s = -(Jx)^\top \iff 2s \, d\theta_s + \lambda = 0.
\]
Thus, we conclude that $L_s$ is exact and that if we denote by $\beta_s$ the primitive for the Liouville form $\lambda$, then $\beta_s + 2s \theta_s$ is constant in space for all $s$. Arguing informally, this suggests that showing convergence of the Lagrangian angle to a single constant should be equivalent to showing that the primitive for the Liouville form converges to a single constant. The advantage of doing so is that the gradient of $\beta_s$ is a first order quantity and thus easier to control than the gradient of $\theta_s$ which is a second order quantity.

We now sketch the main idea behind the proof of Theorem B. Assume, for the sake of simplicity, that $L_0$ is exact which implies that for each $i$ there is a family of smooth functions $\beta_{i,s}$ defined on $L_{i,s}$ such that $d\beta_{i,s} = \lambda$, or equivalently,
\[
J \nabla \beta_{i,s}(x) = -x^\perp \text{ for all } x \in M_{i,s}.
\]
Moreover, as it will be shown in Section 6, the functions $\beta_{i,s}$ can be chosen so that
\[
\frac{d}{ds}(\beta_{i,s} + 2s \theta_{i,s}) = \Delta(\beta_{i,s} + 2s \theta_{i,s}).
\]
This evolution equation combined with identity (11) implies that, after passing to a subsequence, $\beta_{i,s} + 2s \theta_{i,s}$ has a limit which is independent of $s$ and so it must converge to some constant $c_j$ on each Special Lagrangian cone $L_j$, with $j = 1, \ldots, N$. Hence, we obtain from Theorem A that $\beta_{i,s}$ converges to $c_j - 2s \bar{\theta}_j$ on each $L_j$. Moreover, we can assume without loss of generality that the set $\Lambda \equiv \{\bar{\beta}_1 - 2s \bar{\theta}_1, \ldots, \bar{\beta}_N - 2s \bar{\theta}_N\}$ has $N$ distinct values.

Let $\Sigma_i$ be a convergent sequence of connected components of $L_{i,s} \cap B_R(0)$. Because the gradient of $\beta_{i,s}$ is pointwise bounded and its $L^2$-norm converges to zero, we can show that the sequence of functions $\beta_{i,s}$ converges to a single constant when restricted to $\Sigma_i$ (see Proposition 3.1). Thus, the Lagrangian angle of $\Sigma_i$ must converge to a constant because otherwise two numbers in the set $\Lambda$ would be equal.

4. Examples of Finite Time Singularities

We construct examples of finite time singularities for mean curvature flow where the initial condition is a zero-Maslov class and exact Lagrangian.

For simplicity, we restrict ourselves to $\mathbb{C}^2$ but we note that the phenomena observed also occur in $\mathbb{C}^n$. Given a curve $\gamma$ in the complex plane, it is easy to see that
\[
L = \{ (\gamma \cos \alpha, \gamma \sin \alpha) \mid \alpha \in \mathbb{R}/2\pi \mathbb{Z} \}
\]
is a Lagrangian surface in $\mathbb{C}^2$. A choice of orientation for the curve $\gamma$ induces an orientation on $L$ and if $\gamma(s)$ denotes a parametrization of $\gamma$, then
\[
\Omega_L = \frac{\gamma \cdot \gamma'}{|\gamma| |\gamma'|} \text{vol}_L \quad \text{and} \quad \lambda_L = \langle i\gamma, \gamma' \rangle ds.
\]
Hence, we get that $L$ is exact and zero-Maslov class whenever $\gamma$ is diffeomorphic to a line.

If we evolve $L$ by mean curvature flow, the rotational symmetries are preserved and the corresponding $\gamma_t$ evolve according to
\[
\frac{dz}{dt} = k - \frac{z^\perp}{|z|^2},
\]
where $k$ is the curvature of $\gamma$ and $z^\perp$ denotes the projection of the position vector $z$ on the orthogonal complement of $T_x \gamma$.

For any $0 < \beta \leq \pi$, consider the following initial condition for the equivariant mean curvature flow
\[
\gamma_0(s) = (\sin(\pi s/\beta))^{-\beta/\pi} e^{is} \equiv r_0(s) e^{is}, \quad 0 < s < \beta.
\]
The corresponding Lagrangian surface $L_0$ is asymptotic to two oriented planes with Lagrangian angles $\pi$ and $2\beta$ and, when $\beta > \pi/2$, its intersection with $\mathbb{C} \times \{0\}$ can be seen in Figure 1. In order to compute the Lagrangian angle of $L_0$, we use the formula
\[
\theta_0(s) = \arg(\gamma_t \gamma'_t) = 2s + \arg(r'_0 + ir_0)
\]
and obtain that
\[
\theta_0(s) = (2 - \pi/\beta)s + \pi, \quad 0 < s < \beta.
\]
Note that the oscillation of the Lagrangian angle can be made arbitrarily small by choosing $\beta$ close to $\pi/2$.

We now sketch briefly three distinct behaviors for the equivariant mean curvature flow. When $0 < \beta < \pi/2$, the curve will expand indefinitely because the curvature term on the right hand side of (2) points outward and dominates the first-order term that points inward. As a matter of fact, Anciaux [1] found a self-expander with the same asymptotics at infinity as $\gamma_0$. When $\beta = \pi/2$, the Lagrangian surface is one of the Special Lagrangians studied in [7]. Thus, the curvature term equals the first order term on (2) because the curve is a fixed point for the flow. Finally, when $\pi/2 < \beta \leq \pi$, the first order term will be pointing inward and bigger than the curvature term, thus forcing the solution to have a finite time singularity at the origin (see Figure 2). This is the content of the next theorem.

**Theorem 4.1.** When $\pi/2 < \beta \leq \pi$, the Lagrangian mean curvature flow starting at $L_0$ develops a finite time singularity at the origin. The tangent flow is a union of two planes intersecting at a single point, both with Lagrangian angle $\beta/2$.

**Proof.** We start by proving short-time existence for the equivariant mean curvature flow. The procedure is well-known among the specialists but we include it here for the sake of completeness.

After rotating the coordinate axis by $(\pi - \beta)/2$, the curve $\gamma_0$ can be written as the graph of a function $u_0$ over the real axis. A straightforward computation shows the existence of some constant $C$ such that

\begin{equation}
|u'_0|_{C^2} + |xu'_0 - u_0|_{C^0} \leq C.
\end{equation}
For each fixed $n \in \mathbb{N}$, consider graphical solutions $\gamma^n_t \equiv (x, u^n_t(x))$ for the equivariant mean curvature flow with boundary conditions

$$u^n_0(x) = u_0(x) \text{ for } |x| \leq n, \quad u^n_0(\pm n) = u_0(\pm n) = u_0(n).$$

We will show uniform apriori $C^{2,\alpha}$-estimates for the sequence of functions $(u^n_t)$.

A simple computation reveals that $u^n_t$ solves the quasilinear equation

$$\frac{du}{dt} = \frac{u''}{1 + (u')^2} + \frac{xu' - u}{x^2 + u^2}.$$

**Lemma 4.2.** There exists positive $s_0$ and $\varepsilon$ so that $u^n_t(0) \geq \varepsilon$ for all $t \leq s_0$ and all $n \in \mathbb{N}$. Moreover, we have for all $t \leq s_0$ that $u_0(n)|x|/n \leq u^n_t(x) \leq u_0(x)$.

**Proof.** Consider a solution $(C_t)_{t \geq 0}$ to (2) having initial condition a circle of small radius centered at the origin that does not intersect $\gamma_0$. The maximum principle implies that the graph of $u^n_t$ cannot intersect $C_t$ and so the first assertion follows. The second assertion also follows from the maximum principle because $v^n(x) \equiv u_0(n)|x|/n$ and $u_0$ are a solution and supersolution for (4) respectively.

The function $v^n_t \equiv u^n_t - u_0$ satisfies the equation

$$\frac{dv}{dt} = \frac{v''}{1 + (u' + u_0')^2} + \frac{xv' - v}{x^2 + (v + u_0)^2} + F_t$$

where, due to (3),

$$F_t = \frac{u_0''}{1 + (u' + u_0')^2} + \frac{xu_0' - u_0}{x^2 + (v + u_0)^2}$$

is pointwise bounded. Hence, the maximum principle implies that $v^n_t$ is uniformly bounded for all $t \leq s_0$. Moreover, we obtain from Lemma 4.2 that

$$-u_0(n)/n \leq u^n_t(-n) \leq u_0'(-n) \quad \text{and} \quad u_0(n) \leq u^n_t(n) \leq u_0(n)/n$$

and so, it follows from (3) that $v^n_t(\pm n)$ converges to zero as $n$ goes to infinity.

Because $\phi^n_t \equiv v^n_t$ satisfies an evolution equation of the form

$$\frac{d\phi}{dt} = a(x, \phi')\phi'' + b(x, \phi, \phi')\phi' + c(x, \phi, \phi')\phi + G_t,$$

where $a > 0$ and $c, G_t$ are uniformly bounded functions, we obtain from the maximum principle that $v^n_t$ is uniformly bounded. Standard theory for quasilinear parabolic equations implies the existence of some constant $M$ for
which $|u^n_t - u_0|_{C^2,\alpha} < M$ for all $t \leq s_0$. Therefore, we can let $n$ go to infinity and obtain a solution $\gamma_t(x) \equiv (x, u_t(x))$ for the equivariant mean curvature flow.

Next, we argue that the flow $(\gamma_t)$ develops a finite time singularity. We need the following lemma.

**Lemma 4.3.** While the solution exists smoothly, the curve $\gamma_t$ can be parametrized by

$$\gamma_t(s) = r_t(s)e^{is} \quad \text{with } r_t(s) > 0, \quad 0 < s < \beta.$$ 

**Proof.** For any $0 < \alpha < \beta$, denote by $C_\alpha$ the line

$$C_\alpha = \{ r \exp^{i\alpha} | r \in \mathbb{R} \}.$$ 

Initially, we have that $C_\alpha$ and $\gamma_0$ intersect only once. Furthermore, it follows from the short-time existence estimates that $\gamma_t$ remains in the region below $\gamma_0$ and above the $x$-axis. Hence, the Sturmian Theorem proved by Angenent [2, Proposition 1.2.] implies that $C_\alpha$ and $\gamma_t$ must intersect exactly once while the solution exists smoothly. $\Box$

For the rest of this proof we parameterize the curves $\gamma_t$ as described in the previous lemma. The equation satisfied by $r_t$ becomes

**Lemma 4.4.**

$$\frac{dr}{dt} = -\frac{\theta'_t}{r} = \frac{rr'' - 2r^2 - 3(r')^2}{r(r')^2 + r^3},$$

**Proof.** Denote by $\partial_s$ the tangent vector

$$\partial_s = r'e^{is} + ire^{is}.$$ 

Then,

$$\langle d(re^{is})/ds, i\partial_s \rangle = dr/dt \langle e^{is}, i\partial_s \rangle = -rdr/dt.$$ 

On the other hand,

$$\langle d(re^{is})/ds, i\partial_s \rangle = \langle H, i\partial_s \rangle = \langle \nabla \theta_t, \partial_s \rangle = \theta'_t$$

and so the first identity follows. The second identity can be checked using

$$\theta_t(s) = 2s + \arg(r_t' + ir_t).$$ 

$\Box$

Let $A_t(\varepsilon)$ denote the area of the triangular-shaped region

$$\{ ue^{is} | \varepsilon \leq s \leq \beta - \varepsilon, \quad 0 \leq u \leq r_t(s) \}.$$ 

Note that

$$2A_t(\varepsilon) = \int_\varepsilon^{\beta-\varepsilon} r_t^2(s) \, ds$$ 

and that

$$2s < \theta_t(s) < 2s + \pi$$
because $\theta_t = 2s + \arg(r_t + ir_t)$. Therefore, 
\[
\frac{d}{dt}A_t(\varepsilon) = -\int_\varepsilon^{\beta-\varepsilon} \theta_t'(s) ds = (\theta_t(\varepsilon) - \theta_t(\beta - \varepsilon)) < \pi + 2\varepsilon - 2\beta.
\]

Because $\varepsilon$ can be chosen arbitrarily small, the flow must develop a finite time singularity if $\pi/2 < \beta \leq \pi$.

Denote by $T$ the instant of the first time singularity. We need to show that the singularity occurs at the origin. The key idea consists in showing that if that is not the case, then the tangent flow cannot be a union of Lagrangian planes, which is a contradiction to Theorem A. In order to do so, we need some preliminary lemmas.

**Lemma 4.5.** For all $t < T$
\[
\lim_{s \to 0} \theta_0(s) = 0 \quad \text{and} \quad \lim_{s \to \beta} \theta_t(s) = 2\beta.
\]

**Proof.** The maximum principle applied to $\theta_t$ implies that $\pi \leq \theta_t \leq 2\beta$ for all $t < T$. Suppose that there is $t_1 < T$, a sequence $(s_i)$ converging to zero, and $\varepsilon > 0$ so that 
\[
\lim_{i \to \infty} \theta_t(s_i) = \pi + 2\varepsilon.
\]

Recall that $L_t$ denotes the Lagrangian surfaces corresponding to $\gamma_t$ and consider the function 
\[
\phi_{t,\varepsilon} \equiv (\theta_t - \pi - \varepsilon)_+^3
\]
which is supported on $\{p \in L_t \mid \theta_t \geq \pi + \varepsilon\}$. Furthermore, 
\[
\frac{d\phi_{t,\varepsilon}}{dt} \leq \Delta \phi_{t,\varepsilon}.
\]

Huisken’s monotonicity formula [8] implies that for all $i \in \mathbb{N}$
\[
8\varepsilon^3 \leq \int_{L_0} \frac{\phi_{t,\varepsilon} \exp(-|x - x_i|^2/4t_1)}{4\pi t_1} d\mathcal{H}^2
\]
\[
= \int_{\{\theta_t \geq \pi + \varepsilon\}} \phi_{t,\varepsilon} \frac{\exp(-|x - x_i|^2/4t_1)}{4\pi t_1} d\mathcal{H}^2,
\]

where $x_i$ is the point $(\gamma_t(s_i), 0)$ in $\mathbb{C}^2$. For every $R > 0$, we have for all $i$ sufficiently large that 
\[
\{\theta_0 \geq \pi + \varepsilon\} \cap B_R(x_1) = \emptyset.
\]

Thus 
\[
\lim_{i \to \infty} \int_{\{\theta_0 \geq \pi + \varepsilon\}} \frac{\phi_{t,\varepsilon} \exp(-|x - x_i|^2/4t_1)}{4\pi t_1} d\mathcal{H}^2 = 0
\]
and this gives us a contradiction. \qed

This lemma is used to prove

**Lemma 4.6.** For all $t < T$ 
\[
\frac{dr}{dt} \leq 0.
\]
Proof. Taking into account that the parameterization described in Lemma 4.3 creates a tangential component on the deformation vector, we get that
\[
\frac{d\theta}{dt} = \Delta_{L_t} \theta + \left\langle \frac{dx}{dt}, \nabla \theta_t \right\rangle = \frac{\theta''}{|\gamma'|^2} + \theta' \left( \frac{1}{r|\gamma'|} \left( \frac{r}{|\gamma'|} \right)' + \frac{dr}{dt} \frac{r'}{|\gamma'|^2} \right).
\]
While the solution exists smoothly, we have that
\[
\lim_{s \to 0} \theta_t(s) = \pi \quad \text{and} \quad \lim_{s \to \beta} \theta_t(s) = 2\beta
\]
and thus, the Sturmian property [2, Proposition 1.2.] implies that the cardinality
\[
\# \{ s \mid \theta_t(s) = y \}
\]
is one if \( \pi < y < 2\beta \) and zero if \( y < \pi \) or \( y > 2\beta \). Hence
\[
\frac{dr}{dt} = -\frac{\theta_t'}{r} \leq 0
\]
for all \( t < T \). □

The curves \( \gamma_t \) are symmetric under reflection over a line with slope \( \tan(\beta/2) \) and so
\[
(5) \quad r_t(\beta/2 + s) = r_t(\beta/2 - s)
\]
for all \( t < T \). This implies that
\[
r_t'(\beta/2) = 0 \quad \text{for all} \quad t < T.
\]

**Lemma 4.7.** For any \( t < T \), \( r_t(s) \) is decreasing when \( s < \beta/2 \) and increasing when \( s > \beta/2 \).

*Proof.* Direct computation shows that \( \beta/2 \) is the only critical point of \( r_0 \) and that, denoting \( r_t' \) by \( u_t \),
\[
\frac{du_t}{dt} = \frac{u_{tt}}{(r')^2 + r^2} + u_t b(r_t, u_t, u_t') + u_t c(r_t, u_t, u_t'),
\]
where the functions \( b \) and \( c \) are bounded for each \( t < T \). Moreover,
\[
\lim_{s \to 0} u_t(s) = \infty \quad \text{and} \quad \lim_{s \to \beta} u_t(s) = -\infty
\]
and thus, the Sturmian property [2, Proposition 1.2.] implies that \( \beta/2 \) is the only critical point of \( r_t \). □

Suppose now that the singularity happens at a point \( x_0 \equiv ae^{i\alpha} \), with \( 0 < a \leq r_0(\alpha) \) and \( 0 < \alpha < \beta \). From Theorem A, we know that the tangent flow at the singularity is a union of planes and so, by White’s regularity Theorem [22],
\[
\limsup_{\delta \to 0} \frac{\mathcal{H}^1(\gamma_{T-\delta^2} \cap B_\delta(x_0))}{2\delta} \geq 2.
\]
We show next that this is impossible because for all \( \delta \) sufficiently small and all \( t < T \)
\[
\frac{\mathcal{H}^1(\gamma_t \cap B_\delta(x_0))}{2\delta} \leq 3/2.
\]
Without loss of generality we assume that $\alpha = \beta/2$ (the remaining cases are treated similarly). For any $\delta < a$, Lemma 4.6 and Lemma 4.7 imply that

$$\gamma_t \cap B_\delta(x_0)$$

is either empty or a connected curve. If the latter occurs, there is $\varepsilon(t) < \arcsin(\delta/a)$ for which

$$\gamma_t \cap B_\delta(x_0) = \{ \gamma_t(s) \mid |s - \beta/2| < \varepsilon \}.$$

Note that

$$(r_t(\beta/2 + \varepsilon) \cos(\varepsilon) - a)^2 + (r_t(\beta/2 + \varepsilon) \sin(\varepsilon))^2 = \delta^2$$

and so

$$|r_t(\beta/2 + \varepsilon) \cos(\varepsilon) - a| + |r_t(\beta/2 + \varepsilon) \sin(\varepsilon)| \leq \sqrt{2}\delta < 3/2\delta.$$

Combining this inequality with Lemma 4.6, Lemma 4.7, and (5), we obtain

$$\frac{\mathcal{H}_1(\gamma_t \cap B_\delta(x_0))}{2\delta} = \frac{1}{\delta} \int_{\beta/2}^{\beta/2+\varepsilon} \left( (r'_t)^2 + r_t^2 \right)^{1/2} ds$$

$$\leq \frac{r_t(\beta/2 + \varepsilon) - r_t(\beta/2)}{\delta} + \varepsilon \frac{r_t(\beta/2 + \varepsilon)}{\delta}$$

$$\leq \frac{r_t(\beta/2 + \varepsilon) - a}{\delta} + \varepsilon \frac{r_t(\beta/2 + \varepsilon)}{\delta}$$

$$\leq 3/2$$

for all $\delta$ sufficiently small.

Finally, we argue next that the tangent flow at the singularity is a union of two planes with Lagrangian angle $\pi/2 + \beta$. From (6) it follows that

$$\theta'_t(\beta/2 + s) = \theta'_t(\beta/2 - s)$$

and therefore, because the solution remains asymptotic to two planes with Lagrangian angles $\pi$ and $2\beta$, we obtain after integration that $\theta_t(\beta/2) = \pi/2 + \beta$. From Lemma 4.7 we know that $\gamma_t(\beta/2)$ is the closest point of $\gamma_t$ to the origin and so Theorem B implies the desired result. \hfill \Box

We can now use Theorem 4.4 to construct an exact and zero-Maslov Lagrangian class which is Hamiltonian isotopic to a Lagrangian plane that, nevertheless, develops a finite time singularity. Denote by $L_0$ the compact perturbation of a Lagrangian plane which is associated with the curve described in Figure 3. The dashed noncompact curve represents one of the curves described in Theorem 4.1 (slightly rotated so that it is not asymptotic to $L_0$) and has a finite time singularity at the origin at time $T$. The dashed circles shown in Figure 3 correspond to a Lagrangian torus, which will have a finite time singularity at time $T_1$. All these curves can be arranged so that $T < T_1$ and an explicit expression for such curves could be easily found. The short-time existence for the flow with initial condition $L_0$ follows from the same arguments used in the proof of Theorem 4.4. Because the two noncompact solutions we consider have different asymptotics, the
maximum principle implies that they can never intersect. Hence, the flow $(L_t)_{t \geq 0}$ must develop a finite-time singularity.

We end this section with a brief heuristic discussion of how could the flow $(L_t)_{t \geq 0}$ be continued after its finite-time singularity. It is expected that in the setting described above, the singularity occurs at the origin. In this situation, the Lagrangian surface at the time of the singularity decomposes into a union of an immersed 2-sphere (the immersion point being at the origin) and a Lagrangian surface diffeomorphic to the Lagrangian plane. As it was pointed out by Tom Ilmanen, there are two possible different evolutions for the Lagrangian surface after the singularity occurs: the immersed 2-sphere that has formed can evolve as an immersed 2-sphere or it can become an embedded torus which then evolves smoothly by mean curvature flow. In either case, the other connected piece will evolve smoothly to a Lagrangian plane.

5. Proof of Compactness Theorem A

The next proposition will be essential to prove Theorem A. As a mean of motivation, it could be easier to read first the proof of Theorem A and come back to the proposition when necessary.

Proposition 5.1. Let $(L^i)$ be a sequence of smooth zero-Maslov class Lagrangians in $\mathbb{C}^n$ such that, for some fixed $R > 0$, the following properties hold:

(a) There exists a constant $D_0$ for which
\[ \mathcal{H}^n(L^i \cap B_{2R}(0)) \leq D_0 R^n \quad \text{and} \quad \sup_{L^i \cap B_{2R}(0)} |\theta^i| \leq D_0 \]
for all $i \in \mathbb{N}$.

(b) \[
\lim_{i \to \infty} \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)) = 0
\]
and
\[
\lim_{i \to \infty} \int_{L^i \cap B_{2R}(0)} |H|^2 \, d\mathcal{H}^n = 0.
\]

Then, there exist a finite set $\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}$ and integral Special Lagrangians $L_1, \ldots, L_N$ such that, after passing to a subsequence, we have for every smooth function $\phi$ compactly supported in $B_R(0)$ and every $f$ in $C(\mathbb{R})$
\[
\lim_{i \to \infty} \int_{L^i} f(\theta_i) \phi \, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\bar{\theta}_j) \mu_j(\phi),
\]
where $\mu_j$ and $m_j$ denote, respectively, the Radon measure of the support of $L_j$ and its multiplicity.

Proof. From Allard compactness theorem for varifolds [10, Theorem 42.7] we obtain the existence of a subsequence, still denoted by $(L^i)$, converging in $B_{2R}(0)$ to a stationary integer rectifiable varifold $L$. Moreover,
\[
\int_{L} \phi |\omega \wedge \eta| \, d\mathcal{H}^n = 0
\]
for every $n-2$ form $\eta$ and all smooth $\phi \in C^\infty_0(B_{2R}(0))$, and this implies that $L$ is Lagrangian. It suffices to find integral Special Lagrangians $L_1, \ldots, L_N$, a finite set $\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}$, and some positive $\varepsilon_0$ such that, after passing to a subsequence of $(L^i)$, we have for all smooth $\phi$ compactly supported in $B_R(0)$, all $0 < \varepsilon < \varepsilon_0$, and all $j = 1, \ldots, N$,
\[
\lim_{i \to \infty} \int_{\{|\theta_i - \bar{\theta}_j| \leq \varepsilon\}} \phi \, d\mathcal{H}^n = m_j \mu_j(\phi)
\]
and
\[
\mu_L(\phi) = \sum_{j=1}^N m_j \mu_j(\phi),
\]
where $\mu_L$ and $\mu_j$ denote the Radon measure of $L$ and of the support of $L_j$ respectively, and $m_j$ denotes the multiplicity of $L_j$.

The idea for the proof is as follows. The regular points of $L$ form a dense open set and therefore we can pick $p$ in $L \cap B_R(0)$ such that, for some positive $\rho$, $B_\rho(p)$ is contained in $B_R(0)$ and the support of $L \cap B_\rho(p)$ is a smooth Special Lagrangian with angle $\bar{\theta}_1$. After adding some multiple of $\pi$ to $\bar{\theta}_1$ if necessary, we will show the existence of an integral Special Lagrangian $L_1$
and of $\varepsilon_1 > 0$ such that, for all smooth $\phi$ with compact support in $B_R(0)$ and all $0 < \varepsilon \leq \varepsilon_1$, we have

\[
\lim_{i \to \infty} \int_{\{\theta_i - \bar{\theta}_1 \leq \varepsilon\}} \phi \, d\mathcal{H}^n = m_1 \mu_1(\phi),
\]

where $\mu_1$ is the Radon measure of the support of $L_1$ and $m_1$ its multiplicity. Because the support of $L_1$ is stationary, the monotonicity formula implies that

\[
\mu_1(B_{2\rho}(0)) R^{-n} \geq \mu_1(B_R(p)) R^{-n} \geq \mu_1(B_{\rho}(p)) \rho^{-n} \geq \gamma_n
\]

for some universal constant $\gamma_n$.

In order to find $\bar{\theta}_2$ and the integral Special Lagrangian $L_2$, we repeat this process but this time applied to the sequence

\[
P_i \equiv \{ |\theta_i - \bar{\theta}_1| \geq \varepsilon_1 \},
\]

where the boundary will cause no difficulty because, as it will be seen in the proof of Lemma 5.2, we can assume that

\[
\lim_{i \to \infty} \mathcal{H}^{n-1}(\{\theta_i = \bar{\theta}_1 \pm \varepsilon_1\} \cap B_{2R}(0)) = 0
\]

and hence,

\[
\lim_{i \to \infty} \mathcal{H}^{n-1}(\partial P_i \cap B_{2R}(0)) \leq \lim_{i \to \infty} (\mathcal{H}^{n-1}(\partial L_i \cap B_{2R}(0)) + \mathcal{H}^{n-1}(\{\theta_i = \bar{\theta}_1 \pm \varepsilon_1\} \cap B_{2R}(0))) = 0.
\]

Condition (a) and (7) ensures that this will be done only finitely many times and hence the proposition will be proven as soon as we show (6).

The next lemma will be quite useful throughout the rest of the proof.

**Lemma 5.2.** For almost all endpoints $a$ and $b$, the sequence

\[
N^i \equiv \{ a \leq \theta_i \leq b\}
\]

contains a subsequence converging, in $B_{2R}(0)$, to a stationary integer rectifiable varifold $N$ in the varifold sense and to an integral current $\hat{N}$ with $\partial \hat{N} = 0$ in the current sense.

**Proof.** For almost all endpoints $a$ and $b$ we have

\[
\lim_{i \to \infty} \mathcal{H}^{n-1}(\{\theta_i = a\} \cup \{\theta_i = b\}) \cap B_{2R}(0)) = 0
\]

because, by the coarea formula,

\[
\int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{\theta_i = s\} \cap B_{2R}(0)) \, ds = \int_{L^i \cap B_{2R}(0)} |H| \, d\mathcal{H}^n
\]

\[
\leq \sqrt{D_0} R^n \left( \int_{L^i \cap B_{2R}(0)} |H|^2 \, d\mathcal{H}^n \right)^{1/2}.
\]
The first variation formula yields, for any vector field $Y$ supported in $B_{2R}(0)$

$$
\delta N^i(Y) = -\int_{\overline{\Omega} \cap B_{2R}(0)} \langle H, Y \rangle d\mathcal{H}^n + \int_{\partial \overline{\Omega} \cap B_{2R}(0)} \langle Y, \nu \rangle d\mathcal{H}^{n-1},
$$

where $\nu$ denotes the exterior unit normal. Hence, whenever the sup norm of $Y$ satisfies $|Y|_{\infty, B_{2R}(0)} \leq 1$, we get

$$
|\delta N^i(Y)| \leq \sqrt{C_0 R^n} \left( \int_{\overline{\Omega} \cap B_{2R}(0)} |H|^2 d\mathcal{H}^n \right)^{1/2}
+ \mathcal{H}^{n-1}((\{\theta_i = a\} \cup \{\theta_i = b\}) \cap B_{2R}(0))
+ \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)).
$$

Furthermore, if $\vartheta$ is any $n-1$ form compactly supported in $B_{2R}(0)$ with $|\vartheta| \leq 1$, then

$$
|\partial N^i(\vartheta)| \leq \mathcal{H}^{n-1}((\{\theta_i = a\} \cup \{\theta_i = b\}) \cap B_{2R}(0))
+ \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)).
$$

We can now apply Allard compactness theorem for varifolds and Federer and Fleming compactness theorem for currents (see [10, Theorem 27.3]) in order to complete the proof of the lemma.

Condition (a) implies the existence of a finite set $F \subset \mathbb{N}$ such that, whenever $l \notin F$, we have for all $i \in \mathbb{N}$

$$
\{ |\theta_i - (\bar{\theta}_1 + l\pi)| \leq \pi \} \cap B_{2R}(0) = \emptyset.
$$

**Lemma 5.3.** There is a universal constant $\gamma_n$ so that, for all $\varepsilon < \pi/2$,

$$
\lim_{i \to \infty} \sum_{l \in F} \mathcal{H}^n(\{ |\theta_i - (\bar{\theta}_1 + l\pi)| \leq \varepsilon \} \cap B_\rho(p)) = \mathcal{H}^n(L \cap B_\rho(p)) \geq \gamma_n \rho^n.
$$

**Proof.** The first equality is true because for almost all intervals $[a, b]$ such that

$$
[a, b] \cap \{ \bar{\theta}_1 + l\pi : l \in \mathbb{Z} \} = \emptyset,
$$

we have

$$
\limsup_{i \to \infty} \mathcal{H}^n(\{ a \leq \theta_i \leq b \} \cap B_\rho(p)) = 0.
$$

Otherwise we could, by Lemma 5.2, extract a subsequence converging to a integer rectifiable varifold $N$ with support in $L$ and such that

$$
\mu(B_\rho(p)) > 0,
$$

where $\mu$ is the Radon measure associated to $N$. This is impossible because for some positive $\delta$ we have

$$
\sup_{\{a \leq \theta_i \leq b\}} |\cos(\theta_i - \bar{\theta}_1)| \leq 1 - \delta,
$$
and so varifold convergence implies that
\[(1 - \delta)\mu(B_{\rho}(p)) \geq \lim_{i \to \infty} \int_{\{a \leq \theta_i \leq b\} \cap B_{\rho}(p)} \left| \text{Re} \left( e^{-i\bar{\theta}_1} \Omega \right) \right| d\mathcal{H}^n = \mu(B_{\rho}(p)).\]

\[\square\]

Renaming $\bar{\theta}_1$ to be $\bar{\theta}_1 + l\pi$ for some $l$ in $F$, we can find a sequence $(\varepsilon_k)$ converging to zero and a constant $K = K(D_0)$ such that
\[\limsup_{i \to \infty} \mathcal{H}^n\left( \{\theta_i - \bar{\theta}_1 \leq \varepsilon_k \} \cap B_{\rho}(p) \right) \geq K\rho^n\]
for all $k \in \mathbb{N}$.

Applying Lemma 5.2 to $N^{i,k} \equiv \{\theta_i - \bar{\theta}_1 \leq \varepsilon_k \}$, we obtain two sequences $(N^k)$ and $(\hat{N}^k)$ of stationary integer rectifiable varifolds and integral currents with no boundary respectively. Its Radon measures are denoted by $\mu_k$ and $\hat{\mu}_k$ respectively. Federer and Fleming compactness Theorem implies that $(\hat{N}^k)$ has a subsequence that converges in $B_{2R}(0)$ to an integral Lagrangian current $L_1$ with no boundary. Moreover, $L_1$ is an integral Special Lagrangian because it is calibrated by $\vartheta \equiv \text{Re} \left( e^{-i\bar{\theta}_1} \Omega \right)$ and it is nonempty because, using (8), we obtain that for every nonnegative smooth $\phi$ compactly supported in $B_{2R}(0)$
\[\int_{L_1} \phi \, d\mathcal{H}^n \geq \lim_{k \to \infty} \hat{\mu}_k(\vartheta \phi) = \lim_{k \to \infty} \lim_{i \to \infty} \int_{N^{i,k}} \vartheta \phi \geq \lim_{k \to \infty} \lim_{i \to \infty} \int_{N^{i,k}} \cos \varepsilon_k \phi \, d\mathcal{H}^n = \lim_{k \to \infty} \cos \varepsilon_k \mu_k(\phi) \geq K\rho^n.\]

Furthermore, the support of each integral current $\hat{N}^k$ is a stationary rectifiable varifold which, combined with the fact that
\[\hat{\mu}_{k+1}(\phi) \leq \hat{\mu}_k(\phi)\]
for every nonnegative $\phi$ compactly supported in $B_{2R}(0)$ and every $k \in \mathbb{N}$, implies that, for all $k$ sufficiently large, $\hat{N}^k$ must coincide with $L_1$ in $B_R(0)$.

\[\square\]

Before proving Theorem A, we recall the monotonicity formula, found by Huisken in [5], valid for any smooth family of $k$-dimensional submanifolds $(N_t)_{t \geq 0}$ moving by mean curvature flow in $\mathbb{R}^m$. Consider the backward heat kernel
\[\Phi_{x_0,T}(x,t) = \frac{1}{(4\pi(T-t))^{k/2}} e^{-\frac{|x-x_0|^2}{4(T-t)}},\]
When \((x_0, T) = (0, 0)\), we denote it simply by \(\Phi\). The following formula holds
\[
\frac{d}{dt} \int_{N_t} f_t \Phi_{x_0, T} \, dH^n = \int_{N_t} \left( \frac{d}{dt} f_t - \Delta f_t - \left| H + \frac{(x - x_0)^\perp}{2(T - t)} \right|^2 \right) \Phi_{x_0, T} \, dH^n,
\]
where \(f_t\) is a smooth function with polynomial growth at infinity and \((x - x_0)^\perp\) denotes the orthogonal projection on \((T_x N)^\perp\) of the vector determined by the point \((x - x_0)\) in \(\mathbb{R}^m\).

Let \(L_t\) be a solution to Lagrangian mean curvature flow with a singularity at time \(T\).

**Theorem A.** If \(L_0\) is zero-Maslov class with bounded Lagrangian angle, then for any sequence of rescaled flows \((L_i^s)_{s < 0}\) at a singularity, there exist a finite set \(\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}\) and integral Special Lagrangian cones \(L_1, \ldots, L_N\) such that, after passing to a subsequence, we have for every smooth function \(\phi\) compactly supported, every \(f\) in \(C^2(\mathbb{R})\), and every \(s < 0\)
\[
\lim_{i \to \infty} \int_{L_i^s} f(\theta_{i,s}) \phi \, dH^n = \sum_{j=1}^N m_j f(\bar{\theta}_j) \mu_j(\phi),
\]
where \(\mu_j\) and \(m_j\) denote the Radon measure of the support of \(L_j\) and its multiplicity respectively.

Furthermore, the set \(\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}\) does not depend on the sequence of rescalings chosen.

**Proof.** We start with the following lemma

**Lemma 5.4.** For any \(a < b < 0\) and any \(R > 0\), we have
\[
\lim_{i \to \infty} \int_a^b \int_{L_i^s \cap B_R(0)} \left( |x^\perp|^2 + |H|^2 \right) \, dH^n \, ds = 0.
\]

**Proof.** From Huisken’s monotonicity formula we have that, for all \(i \in \mathbb{N}\),
\[
\frac{d}{ds} \int_{L_i^s} \theta_{i,s}^2 \Phi \, dH^n = \int_{L_i^s} \left( -2 |H|^2 - \left| H - \frac{x^\perp}{2s} \right|^2 \theta_{i,s}^2 \right) \Phi \, dH^n
\]
and
\[
\frac{d}{ds} \int_{L_i^s} \Phi \, dH^n = \int_{L_i^s} \left| H - \frac{x^\perp}{2s} \right|^2 \Phi \, dH^n.
\]
Using the scale invariance properties of the backward heat kernel, we obtain that
\[
\lim_{i \to \infty} 2 \int_a^b |H|^2 \Phi \, dH^n \, ds \leq \lim_{i \to \infty} \left( \int_{L_i^a} \theta_{i,a}^2 \Phi \, dH^n - \int_{L_i^b} \theta_{i,b}^2 \Phi \, dH^n \right) = 0
\]
and
\[ \lim_{i \to \infty} \int_a^b \int_{L_i^a} \left| H - \frac{x^\perp}{2s} \right|^2 \Phi \, d\mathcal{H}^n \, ds = \lim_{i \to \infty} \left( \int_{L_i^a} \Phi \, d\mathcal{H}^n - \int_{L_i^b} \Phi \, d\mathcal{H}^n \right) = 0. \]

Therefore
\[ \lim_{i \to \infty} \int_a^b \int_{L_i^a} \left| \frac{x^\perp}{2s} \right|^2 \Phi \, d\mathcal{H}^n \, ds \]
\[ \leq \lim_{i \to \infty} \int_a^b \int_{L_i^a} \left( \left| H - \frac{x^\perp}{2s} \right|^2 + |H|^2 \right) \Phi \, d\mathcal{H}^n \, ds = 0 \]
and so the result follows. \(\square\)

Pick \(a < 0\) for which
\[ \lim_{i \to \infty} \int_{L_i^a \cap B_R(0)} \left| \frac{x^\perp}{2s} \right|^2 + |H|^2 = 0 \]
for all positive \(R\).

The maximum principle implies that the Lagrangian angle \(\theta_t\) is uniformly bounded and hence, by scale invariance, the same is true for the Lagrangian angle of \(L_i^a\). Lemma B.1 implies the existence of a constant \(D_0\) for which
\[ \mathcal{H}^n \left( L_i^a \cap B_R(0) \right) \leq D_0 R^n \]
for all positive \(R\). We can, therefore, apply Proposition 5.1 to the sequence \((L_i^a)\) and, after a simple diagonalization argument, obtain a subsequence for which there are integral Special Lagrangian currents
\[ L_1, \ldots, L_N \]
and a finite set \(\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}\) such that, for every smooth function \(\phi\) compactly supported and every \(f \in C^2(\mathbb{R})\),
\[ \lim_{i \to \infty} \int_{L_i^a} f(\theta_{i,a}) \phi \, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\bar{\theta}_j) \mu_j(\phi), \]
where \(\mu_j\) and \(m_j\) denote the Radon measure and the multiplicity of \(L_j\) respectively. The fact that
\[ \lim_{i \to \infty} \int_{L_i^a \cap B_R(0)} \left| \frac{x^\perp}{2s} \right|^2 \, d\mathcal{H}^n = 0 \]
for all positive \(R\) implies that the Special Lagrangians \(L_j\) are all cones. Next, we want to show that, for all \(b < 0\),
\[ \lim_{i \to \infty} \int_{L_i^b} f(\theta_{i,b}) \phi \, d\mathcal{H}^n = \lim_{i \to \infty} \int_{L_i^a} f(\theta_{i,a}) \phi \, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\bar{\theta}_j) \mu_j(\phi). \]
This comes from
\[
\frac{d}{ds} \int_{L_s}^b f(\theta_{i,s}) \phi \, dH^n \, ds = \int_{L_s}^b f'(\theta_{i,s}) \Delta \theta_{i,s} \phi \, dH^n \, ds + \int_{L_s}^b f(\theta_{i,s}) \langle H, D\phi \rangle \, dH^n \, ds - \int_{L_s}^b f(\theta_{i,s}) |H|^2 \phi \, dH^n \, ds
\]
because, after integration with respect to the \(s\) variable, all terms on the right hand side vanish when \(i\) goes to infinity. We check this for the first term. Integrating by parts (and assuming \(a < b\) for simplicity), we obtain
\[
\int_a^b \int_{L_s}^b f'(\theta_{i,s}) \Delta \theta_{i,s} \phi \, dH^n \, ds = -\int_a^b \int_{L_s}^b f''(\theta_{i,s}) |\nabla \theta_{i,s}|^2 \phi \, dH^n \, ds - \int_a^b \int_{L_s}^b f'(\theta_{i,s}) \langle \nabla \theta_{i,s}, D\phi \rangle \, dH^n \, ds
\]
and hence, by Hölder’s inequality, there is a constant \(C = C(\phi, f, D\theta_0, a, b)\) such that, for all \(i \in \mathbb{N}\),
\[
\int_a^b \int_{L_s}^b |f''(\theta_{i,s})| |\nabla \theta_{i,s}|^2 \phi \, dH^n \, ds \leq C \int_a^b \int_{L_s}^b |H|^2 \Phi \, dH^n \, ds
\]
and
\[
\int_a^b \int_{L_s}^b |f'(\theta_{i,s})| |\nabla \theta_{i,s}, D\phi \rangle \, dH^n \, ds \leq C \left( \int_a^b \int_{L_s}^b |H|^2 \Phi \, dH^n \, ds \right)^{1/2}
\]
Finally, we show that \(\{\tilde{\theta}_1, \ldots, \tilde{\theta}_N\}\) does not depend on the sequence of rescalings chosen. Let \(\{(\tilde{L}_s)^k\}_{s < 0}\) be another sequence of rescaled flows for which there are Special Lagrangian cones
\[\hat{L}_1, \ldots, \hat{L}_P\]
and a finite set \(\{\tilde{\theta}_1, \ldots, \tilde{\theta}_P\}\) such that, for every smooth function \(\phi\) compactly supported, every \(f\) in \(C^2(\mathbb{R})\), and every \(s < 0\)
\[
\lim_{k \to \infty} \int_{\hat{L}_s}^b f(\theta_{k,s}) \phi \, dH^n = \sum_{j=1}^P \hat{m}_j f(\tilde{\theta}_j) \hat{\mu}_j(\phi),
\]
where \(\hat{\mu}_j\) and \(\hat{m}_j\) denote the Radon measure of the support of \(L_j\) and its multiplicity respectively.

For any real number \(y\) and any integer \(q\), we have the following evolution equation
\[
\frac{d}{dt} (\theta_t - y)^{2q} = \Delta (\theta_t - y)^{2q} - 2q(2q - 1)(\theta_t - y)^{2p-2} |H|^2.
\]
Applying the monotonicity formula to \( (\theta_t - y)^{2q} \), we get that
\[
\frac{d}{dt} \int_{L_t} (\theta_t - y)^{2q} \Phi_{x_0, T} \, d\mathcal{H}^n \leq 0
\]
and thus, by scale invariance, we obtain for any \( s, \bar{s} < 0 \)
\[
\lim_{i \to \infty} \int_{L_i^{s}} (\theta_{i,s} - y)^{2q} \Phi \, d\mathcal{H}^n = \lim_{k \to \infty} \int_{L_i^{\bar{s}}} (\theta_{i,\bar{s}} - y)^{2q} \Phi \, d\mathcal{H}^n
\]
Therefore
\[
\sum_{j=1}^{N} m_j (\bar{\theta}_j - y)^{2q} \mu_j(\Phi) = \sum_{j=1}^{P} \hat{m}_j (\hat{\theta}_j - y)^{2q} \hat{\mu}_j(\Phi)
\]
for all positive integer \( q \) and all \( y \) in \( \mathbb{R} \) and this implies that
\[
\{ \theta_1, \ldots, \theta_N \} = \{ \hat{\theta}_1, \ldots, \hat{\theta}_P \}.
\]

6. EVOLUTION EQUATIONS

Let \( L_0 \) be a rational and zero-Maslov Lagrangian submanifold of \( \mathbb{C}^n \). We will argue now that the rational condition is preserved by the flow. Denoting by \( F_t \) the normal deformation by mean curvature, we have
\[
\frac{d}{dt} \int_{F_t(\gamma)} \lambda = \frac{d}{dt} \int_{\gamma} F_t^* \lambda = \int_{\gamma} L_H F_t^* \lambda = \int_{\gamma} dF_t^*(H \lambda) + F_t^*(H \cdot \omega) = \int_{\gamma} dF_t^*(H \lambda - 2\theta_t) = 0
\]
for every \( [\gamma] \) in \( H^1(L_0) \). Hence
\[
[\lambda] = [F_t^*(\lambda)] \quad \text{in} \quad H^1(L_0)
\]
for all times where the solution exists smoothly and therefore it follows that
\[
\lambda(H_1(L_t, \mathbb{Z})) = \lambda(H_1(L_0, \mathbb{Z})) = \{a2k\pi \mid k \in \mathbb{Z}\}.
\]
Thus, there is a smooth family of multivalued functions
\[
\beta_t : L_t \to \mathbb{R}/2\pi a\mathbb{Z}
\]
such that
\[
\nabla \beta_t(x) = (Jx)^T \quad \text{for all} \ x \in L_t.
\]

**Proposition 6.1.** The functions \( \beta_t \) can be chosen so that
\[
\frac{d\beta_t}{dt} = \Delta \beta_t - 2\theta_t.
\]

**Proof.** Assume, without loss of generality, that the family of functions \( \beta_t \) is smooth with respect to the time parameter. We have
Lemma 6.2. \[ \Delta \beta_t = H \lambda. \]

Proof. We use a normal coordinate system around the point \( x \) and denote the coordinate vectors by \( \{ \partial_1, \cdots, \partial_n \} \). The result follows from

\[
\langle \nabla_{\partial_i} (Jx)^\top, \partial_j \rangle = \partial_i \langle Jx, \partial_j \rangle - \langle (Jx)^\top, D\partial_i \partial_j \rangle = \langle Jx, A_{ij} \rangle.
\]

Thus,

\[
d\left( \frac{d\beta_t}{dt} \right) = d\lambda = \Delta H \lambda = H \lambda + 2H \omega = d(\Delta \beta_t - 2\theta_t)
\]

and so we can add a time dependent constant to each \( \beta_t \) so that the desired result follows. \[ \square \]

Given any \( t_0 \) in \( \mathbb{R} \) and any \( k \) in \( \mathbb{Z} \), the function

\[ u_t \equiv \cos \left( \frac{k(\beta_t + 2(t - t_0)\theta_t)}{a} \right) \]

is well defined on \( L_t \). If \( L_0 \) is exact, take \( a = 1 \). A straightforward computation using Proposition 6.1 and

\[ Jx^\top = (Jx)^\top \]

gives

Corollary 6.3.

\[
\frac{du_t}{dt} = \Delta u_t + u_t \left| \frac{k(x^\top + 2(t_0 - t)H)}{a} \right|^2.
\]

7. Proof of Compactness Theorem B

Theorem B. If \( L_0 \) is almost-calibrated and rational, then, after passing to a subsequence of \( (L^i_s)_{s<0} \), the following property holds for all \( R > 0 \) and almost all \( s < 0 \).

For any convergent subsequence \( \Sigma_t \) of connected components of \( B_{2R}(0) \cap L^i_s \) intersecting \( B_R(0) \), there exists a Special Lagrangian cone \( L \) in \( B_{2R}(0) \) with Lagrangian angle \( \bar{\theta} \) such that

\[
\lim_{i \to \infty} \int_{\Sigma_t} f(\theta_i, \phi) d\mathcal{H}^n = mf(\bar{\theta})\mu(\phi),
\]

for every \( f \) in \( C(\mathbb{R}) \) and every smooth \( \phi \) compactly supported in \( B_{2R}(0) \), where \( \mu \) and \( m \) denote the Radon measure of the support of \( L \) and its multiplicity respectively.

Proof. The almost-calibrated condition is preserved by the flow and implies the following lemma.
**Lemma 7.1.** There is a constant $C_1$ such that, for all $s < 0$,

$$(\mathcal{H}^n(A))^{(n-1)/n} \leq C_1 \mathcal{H}^{n-1}(\partial A),$$

where $A$ is any open subset of $L^i_s$ with rectifiable boundary.

**Proof.** The Isoperimetric Theorem [10, Theorem 30.1] guarantees the existence of an integral current $B$ with compact support such that $\partial B = \partial A$ and for which

$$(\mathcal{H}(B))^{(n-1)/n} \leq C \mathcal{H}^{n-1}(\partial A),$$

where $C = C(n)$. If $T$ denotes the cone over the current $A - B$ (see [10 page 141]), then $\partial T = A - B$ and thus, because

$$\text{Re} \Omega|_{L^i_s} = \cos \theta_{i,s} \geq \varepsilon_0$$

for some positive $\varepsilon_0$, we obtain

$$\mathcal{H}^n(A) \leq \varepsilon_0^{-1} \int_A \text{Re} \Omega = \varepsilon_0^{-1} \int_B \text{Re} \Omega + \partial T(\text{Re} \Omega)$$

$$\leq \varepsilon_0^{-1} \mathcal{H}^n(B) + T(d\text{Re} \Omega) \leq \varepsilon_0^{-1} (C \mathcal{H}^{n-1}(\partial A))^{n/(n-1)}.$$

□

The discussion in Section 6 implies the existence of $a \in \mathbb{R}$ and of a family of multivalued functions

$$\beta_{i,s} : L^i_s \longrightarrow \mathbb{R}/\sigma^2_i a 2\pi \mathbb{Z}$$

such that

$$\nabla \beta_{i,s}(x) = (Jx)^\top$$

for all $x \in L^i_s$ and all $s < 0$. Furthermore, we can choose a bounded sequence $(b_i)$ so that, for any real number $s_0$,

$$u_{i,s} \equiv \cos \left( \frac{\beta_{i,s} + 2(s - s_0)\theta_{i,s}}{b_i} \right)$$

is a well defined function. After passing to a subsequence, the sequence $(b_i)$ converges to $b \neq 0$ and, for simplicity, we assume that $b = 1$. Furthermore, from Lemma 5.4 we can also assume that

$$\lim_{i \to \infty} \int_{L^i_{-1} \cap B_R(0)} \left( |H|^2 + |x^+|^2 \right) \ d\mathcal{H}^n = 0$$

for all $R > 0$.

**Lemma 7.2.** There is a set

$$\{(\cos \bar{\beta}_1, \sin \bar{\beta}_1), \ldots, (\cos \bar{\beta}_Q, \sin \bar{\beta}_Q)\}$$

and integral Special Lagrangian cones $P_1, \ldots, P_Q$
such that, after passing to a subsequence, we have for all smooth \( \phi \) with compact support and all \( f \) in \( C(\mathbb{R}) \),

\[
\lim_{i \to \infty} \int_{L^i} f(\cos(\beta_{i,-1}/b_i)) \phi \, d\mathcal{H}^n = \sum_{k=1}^{Q} p_k f(\cos \bar{\beta}_k) \nu_k(\phi),
\]

\[
\lim_{i \to \infty} \int_{L^i} f(\sin(\beta_{i,-1}/b_i)) \phi \, d\mathcal{H}^n = \sum_{k=1}^{Q} p_k f(\sin \bar{\beta}_k) \nu_k(\phi),
\]

where \( \nu_k \) and the positive integer \( p_k \) denote the Radon measure of the support of \( P_k \) and its multiplicity respectively.

**Proof.** Let \((R_k)\) denote a sequence of positive numbers going to infinity. We start by arguing the existence of a uniform bound on the number of connected components of \( L^i \cap B_{2R_k}(0) \) that intersect \( B_{R_k}(0) \). For any \( x \) in \( L^i \cap B_{2R_k}(0) \), denote the intrinsic ball of radius \( r \) around \( x \) by \( \widehat{B}_i(x,r) \).

Set

\[
\psi_i(r) = \mathcal{H}^n \left( \widehat{B}_i(x,r) \right)
\]

which has, for almost all \( r \), derivative given by

\[
\psi'_i(r) = \mathcal{H}^{n-1} \left( \partial \widehat{B}_i(x,r) \right).
\]

We know from Lemma 7.1 that, for all \( r < R_k \),

\[
(\psi_i(r))^{(n-1)/n} \leq C_1 \psi'_i(r)
\]

and so

\[
\mathcal{H}^n \left( \widehat{B}_i(x,r) \right) \geq Kr^n
\]

for all \( x \) in \( L^i \cap B_{2R_k}(0) \), where \( K = K(n, C_1) \). Hence, each connected component has area bigger than \( KR^n \) and so the claim follows from the uniform area bounds for \( L^i \) (Lemma B.1).

From Proposition 5.1 we know that, after passing to a subsequence, all the connected components of \( L^i \cap B_{4R_k}(0) \) intersecting \( B_{R_k}(0) \) converge to a union of Special Lagrangian cones in \( B_{2R_k}(0) \). Moreover,

\[
|\nabla \beta_{i,-1}(x)| = |(Jx)^\top| = |x^\perp|
\]

and thus the functions \( \cos(\beta_{i,-1}/b_i) \) and \( \sin(\beta_{i,-1}/b_i) \) satisfy the conditions of Proposition A.1. We can, therefore, apply this result to all the connected components of \( L^i \cap B_{2R_k}(0) \) intersecting \( B_{R_k}(0) \). A standard diagonalization method finds a subsequence that works for all \( R_k \) and so the lemma is proved.

Combining this lemma with Theorem A we obtain that, after a rearrangement of the supports of the Special Lagrangian cones and its multiplicities (which we still denote by \( L_1, \ldots, L_N \)
and \(m_1, \ldots, m_N\) respectively), we can assume that for all \(\phi\) with compact support, all \(f\) in \(C(\mathbb{R})\), and all \(y \in \mathbb{R}\),

\[
\lim_{i \to \infty} \int_{L_{i-1}^b} f \left( \cos \left( \frac{\beta_{i-1} + 2y\theta_{i-1}}{b_i} \right) \right) \phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\cos(\beta_j + 2y\theta_j)) \mu_j(\phi)
\]

where \(\mu_j\) denotes the Radon measure of the support of \(L_j\) and the elements of the set

\[
\{ (\cos \beta_1, \sin \beta_1, \theta_1), \ldots, (\cos \beta_N, \sin \beta_Q, \theta_N) \}
\]

are all distinct.

Using the evolution equation for \(u_{i,s}\) we show

**Lemma 7.3.** For all \(\phi\) with compact support, all \(f\) in \(C^2(\mathbb{R})\), and all \(s < 0\),

\[
\lim_{i \to \infty} \int_{L_i^b} f(\cos(\beta_i/s/b_i)) \phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\cos(\beta_j - 2(s+1)\theta_j)) \mu_j(\phi).
\]

**Proof.** Corollary 6.1 implies that, for all \(\phi\) with compact support, all \(f\) in \(C^2(\mathbb{R})\), and all \(s_0 < 0\),

\[
(9) \quad \frac{d}{ds} \int_{L_i^b} f(u_{i,s}) \phi \, d\mathcal{H}^n = \int_{L_i^b} f'(u_{i,s}) \Delta u_{i,s} \phi \, d\mathcal{H}^n + \int_{L_i^b} f(u_{i,s}) \langle H, D\phi \rangle \, d\mathcal{H}^n
\]

From Lemma 5.4 we obtain that (assuming \(-1 < s_0 < 0\) for simplicity)

\[
\lim_{i \to \infty} \int_{-1}^{s_0} \int_{L_i^b \cap B_R(0)} \left| \frac{x^\perp + 2(s_0 - s)H}{b_i} \right|^2 \, d\mathcal{H}^n \leq \lim_{i \to \infty} 8 \int_{-1}^{s_0} \int_{L_i^b \cap B_R(0)} \left( \frac{(s - s_0)^2 |H|^2 + |x^\perp|^2}{b_i^2} \right) \, d\mathcal{H}^n = 0
\]

for all positive \(R\).

This inequality allows us to argue in the same way as it was done in the proof of Theorem A and show that, after integration with respect to the \(s\) variable, all terms on the right hand side of (9) converge to zero when \(i\) goes to infinity. Thus, because

\[
u_{i,s_0} = \cos(\beta_i/s_0/b_i) \quad \text{and} \quad \nu_{i,-1} = \cos \left( \frac{\beta_{i-1} - 2(1+s_0)\theta_{i-1}}{b_i} \right),
\]
we obtain from Lemma 7.3:

\[
\lim_{i \to \infty} \int_{L_{s_0}^i} f(\cos(\beta_{i,s_0}/b_i)) \phi \, d\mathcal{H}^n = \lim_{i \to \infty} \int_{L_{i-1}^i} f\left(\cos\left(\frac{\beta_{i,-1} - 2(1 + s_0)\theta_{i,-1}}{b_i}\right)\right) \phi \, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\cos(\tilde{\beta}_j - 2(1 + s_0)\theta_j)) \mu_j(\phi).
\]

The result follows from the arbitrariness of \(s_0\).

The proof of the theorem can now be completed. Because the elements of the set

\[
\{(\cos \tilde{\beta}_1, \sin \tilde{\beta}_1, \bar{\theta}_1), \ldots, (\cos \tilde{\beta}_N, \sin \tilde{\beta}_N, \bar{\theta}_N)\}
\]

are all distinct, we get that, for all but countably many \(s\), the real numbers

\[
\cos(\tilde{\beta}_1 - 2(s + 1)\bar{\theta}_1), \ldots, \cos(\tilde{\beta}_N - 2(s + 1)\bar{\theta}_N)
\]

are all distinct. Moreover, Lemma 5.4 implies that, for almost all \(s < 0\),

\[
\lim_{i \to \infty} \int_{L_i \cap B_R(0)} \left(|H|^2 + |\mathbf{x}^+|^2\right) \, d\mathcal{H}^n = 0
\]

for all \(R > 0\).

Pick \(s\) so that both conditions described above hold and consider a subsequence of connected components \(\Sigma^i\) of \(B_{4R}(0) \cap L_i^i\) intersecting \(B_R(0)\) that converges weakly to \(\Sigma\). The arguments presented in the proof of Lemma 5.4 imply that \(\Sigma\) has positive measure. We first show that \(\Sigma\) is a Special Lagrangian cone.

Proposition A.1 can be applied to the sequence \(\Sigma^i\) and thus, after passing to a subsequence, \((\cos(\beta_{i,s}/b_i))\) converges to a constant \(\gamma\). Define \(f \in C^2(\mathbb{R})\) to be a nonnegative cutoff function that is one in small neighborhood of \(\gamma\) and zero everywhere else.

Denoting by \(\mu_\Sigma\) the Radon measure of \(\Sigma\), we obtain from Lemma 7.3 that for every nonnegative test function \(\phi\) with support in \(B_{2R}(0)\)

\[
\mu_\Sigma(\phi) = \lim_{i \to \infty} \int_{\Sigma_i} \phi \, d\mathcal{H}^n = \lim_{i \to \infty} \int_{\Sigma_i} f(\cos(\beta_{i,s}/b_i)) \phi \, d\mathcal{H}^n \\
\leq \lim_{i \to \infty} \int_{L_i} f(\cos(\beta_{i,s}/b_i)) \phi \, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\cos(\tilde{\beta}_j - 2(s + 1)\bar{\theta}_j)) \mu_j(\phi).
\]

Because the support of \(f\) can be chosen arbitrarily small and the real numbers

\[
\cos(\tilde{\beta}_1 - 2(s + 1)\bar{\theta}_1), \ldots, \cos(\tilde{\beta}_N - 2(s + 1)\bar{\theta}_N)
\]

are all distinct, the above inequality implies that

\[
\gamma = \cos(\tilde{\beta}_{j_0} - 2(s + 1)\bar{\theta}_{j_0})
\]
for a unique \( j_0 \). Thus

\[ \mu_{\Sigma}(\phi) \leq m_{j_0} \mu_{j_0}(\phi) \]

for every \( \phi \geq 0 \) and, as a result, the support of \( \Sigma \) must be contained in \( L_{j_0} \).

Finally, suppose there are \( f \) continuous and \( \phi \) compactly supported in \( B_{2R}(0) \) such that

\[ \int_{\Sigma} f(\theta_{i,s}) \phi d\mathcal{H}^n \]

has two distinct convergent subsequences. We can use Proposition 5.1 to get a contradiction because \( L_0 \) being almost-calibrated implies that any two Special Lagrangian cones with support contained in the support of \( \Sigma \) have the same Lagrangian angle. \( \square \)

Appendix A.

Suppose we have a sequence of functions \( (\alpha_i) \) defined on a sequence of manifolds \( (N^i) \) converging weakly to \( N \) and such that the \( L^2 \)-norm of \( |\nabla \alpha_i| \) converges to zero. The next proposition gives conditions under which, after passing to a subsequence, \( (\alpha_i) \) converges to a constant. Before giving its proof, we comment on the necessity of all the hypothesis.

Proposition A.1. Let \( (N^i) \) and \( (\alpha_i) \) be a sequence of smooth \( k \)-submanifolds in \( \mathbb{R}^n \) and smooth functions on \( N^i \) respectively, such that \( (N^i) \) converges weakly to an integer rectifiable stationary \( k \)-varifold \( N \). We assume that, for some \( R > 0 \), the following properties hold:

a) There exists a constant \( D_0 \) such that

\[ \mathcal{H}^k(N^i \cap B_{3R}) \leq D_0 R^k \]

for all \( i \in \mathbb{N} \), and

\[ \left( \mathcal{H}^k(A) \right)^{(k-1)/k} \leq D_0 \mathcal{H}^{k-1}(\partial A) \]

for all open subsets \( A \) of \( N^i \cap B_{3R} \) with rectifiable boundary.

b) \[ \lim_{i \to \infty} \int_{N^i \cap B_{3R}(0)} (|H|^2 + |\nabla \alpha_i|^2) \, d\mathcal{H}^n = 0. \]

c) There exists a constant \( D_1 \) for which

\[ \sup_{N^i \cap B_{3R}(0)} |\nabla \alpha_i| + R^{-1} \sup_{N^i \cap B_{3R}(0)} |\alpha_i| \leq D_1 \]

for all \( i \in \mathbb{N} \).

d) For all \( i \in \mathbb{N} \),

\[ N^i \cap B_{2R}(0) \]

is connected

and

\[ \partial(N^i \cap B_{3R}(0)) \subset \partial B_{3R}(0). \]
Then, there is a real number $\alpha$ such that, after passing to a subsequence, we have for all $\phi$ with compact support in $B_R(0)$ and all $f$ in $C(\mathbb{R})$

$$\lim_{i \to \infty} \int_{N^i} f(\alpha_i) \phi = f(\alpha) \mu_N(\phi),$$

where $\mu_N$ denotes the Radon measure associated to $N$.

The first hypothesis is needed in order to ensure lower density bounds on $N^i$. The third hypothesis is essential because, without the pointwise bounds on $|\nabla \alpha_i|$ and $\alpha_i$, the result would be false. Finally, the last hypothesis is needed because otherwise the proposition would fail for trivial reasons.

**Proof.** It suffices to find $\alpha \in \mathbb{R}$ and a sequence $(\varepsilon_j)$ converging to zero such that, for some appropriate subsequence, we have for all $j \in \mathbb{N}$

$$\lim_{i \to \infty} \mathcal{H}^{k-1}(\{|\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_R(0)) = \mathcal{H}^{k-1}(N \cap B_R(0)).$$

For the rest of this proof, $K = K(D_0, D_1, k)$ will denote a generic constant depending only on the mentioned quantities. Choose any sequence $(x_i)$ in $N_i \cap B_R(0)$. After passing to a subsequence, we have that

$$\lim_{i \to \infty} x_i = x_0 \quad \text{and} \quad \lim_{i \to \infty} \alpha_i(x_i) = \alpha$$

for some $x_0 \in B_R(0)$ and $\alpha \in \mathbb{R}$. Furthermore, consider also a sequence $(\varepsilon_j)$ converging to zero such that, for all $j \in \mathbb{N}$,

$$\lim_{i \to \infty} \mathcal{H}^{k-1}(\{\alpha_i = \alpha \pm \varepsilon_j\} \cap B_{3R}) = 0.$$

Such a subsequence exists because, by the coarea formula, we have

$$\lim_{i \to \infty} \int_{-\infty}^{\infty} \mathcal{H}^{k-1}(\{\alpha_i = s\} \cap B_{3R}) ds = \lim_{i \to \infty} \int_{N^i \cap B_{3R}} |\nabla \alpha_i| d\mathcal{H}^n$$

$$\leq \lim_{i \to \infty} KR^{k/2} \left( \int_{N^i \cap B_{3R}} |\nabla \alpha_i|^2 d\mathcal{H}^n \right)^{1/2} = 0.$$

Define

$$N^{i,\alpha,j} \equiv \{|\alpha_i - \alpha| \leq \varepsilon_j\}.$$  

The first variation formula yields, for any vector field $Y$ supported in $B_{3R}$,

$$\delta N^{i,\alpha,j}(Y) = -\int_{N^{i,\alpha,j} \cap B_{2R}} \langle H, Y \rangle d\mathcal{H}^n + \int_{\partial \{||\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_{2R}} \langle Y, \nu \rangle d\mathcal{H}^{n-1}$$

where $\nu$ denotes the exterior unit normal. Hence, whenever the sup norm of $Y$ satisfies $|Y|_{\infty} \leq 1$, we get

$$|\delta N^{i,\alpha,j}(Y)| \leq KR^{k/2} \left( \int_{N^{i,\alpha,j} \cap B_{2R}} |H|^2 d\mathcal{H}^n \right)^{1/2}$$

$$+ \mathcal{H}^{k-1}(\{\alpha_i = \alpha \pm \varepsilon_j\} \cap B_{2R}).$$
Lemma A.2. For all \( j \in \mathbb{N} \),
\[
\mathcal{H}^k(N^{\alpha,j} \cap B_R(x_0)) \geq KR^k.
\]

Proof. Set
\[
\psi_i(s) \equiv \mathcal{H}^k(\{|\alpha_i - \alpha_i(x_i)| \leq s\} \cap B_s(x_i))
\]
which, by the coarea formula, has derivative equal to
\[
\psi'_i(s) = \int_{\partial B_s(x_i) \cap \{|\alpha_i - \alpha_i(x_i)| \leq s\}} \frac{|x - x_i|}{\|x - x_i\|^{n-1}} d\mathcal{H}^{n-1} + \int_{B_s(x_i) \cap \partial \{|\alpha_i - \alpha_i(x_i)| \leq s\}} \frac{1}{|\nabla \alpha_i|} d\mathcal{H}^{n-1}
\]
for almost all \( s \). We can estimate
\[
\psi'_i(s) \geq \mathcal{H}^{k-1}(\partial B_s(x_i) \cap \{|\alpha_i - \alpha_i(x_i)| \leq s\})
\[
+ K \mathcal{H}^{k-1}(B_s(x_i) \cap \partial \{|\alpha_i - \alpha_i(x_i)| \leq s\})
\]
and so, using the isoperimetric condition a), we obtain
\[
(\psi_i(s))^{(k-1)/k} \leq D_0 \mathcal{H}^{k-1}(\partial(B_s(x_i) \cap \{|\alpha_i - \alpha_i(x_i)| \leq s\})) \leq K \psi'_i(s)
\]
for almost all \( s \leq R \). This implies that
\[
s^{-k} \mathcal{H}^k(\{|\alpha_i - \alpha_i(x_i)| \leq s\} \cap B_s(x_i)) \geq K
\]
for all \( s \leq R \). This inequality and the inclusion
\[
\{|\alpha_i - \alpha_i(x_i)| \leq \varepsilon_j/2\} \cap B_{\varepsilon_j/2}(x_i) \subset \{|\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_{\varepsilon_j}(x_0),
\]
valid for all \( i \) sufficiently large, imply that
\[
\varepsilon_j^{-k} \mathcal{H}^k(N^{\alpha,j} \cap B_{\varepsilon_j}(x_0)) \geq \varepsilon_j^{-k} \mathcal{H}^k(\{|\alpha_i - \alpha(x_i)| \leq \varepsilon_j/2\} \cap B_{\varepsilon_j/2}(x_i)) \geq K
\]
for all \( i \) sufficiently large. Taking the limit when \( i \) goes to infinity and recalling that \( N^{\alpha,j} \) is a stationary varifold we get, by the monotonicity formula, that
\[
R^{-k} \mathcal{H}^k(N^{\alpha,j} \cap B_R(x_0)) \geq \varepsilon_j^{-k} \mathcal{H}^k(N^{\alpha,j} \cap B_{\varepsilon_j}(x_0)) \geq K
\]
for all \( j \in \mathbb{N} \). 

Suppose that for some positive integer \( j \) we have
\[
\mathcal{H}^k(N^{\alpha,j} \cap B_R(0)) < \mathcal{H}^k(N \cap B_R(0)).
\]
We can now apply Allard compactness theorem to conclude that, after passing to a subsequence, we have convergence to an integer rectifiable stationary varifold \( N^{\alpha,j} \). By a standard diagonalization argument, we can find a subsequence that works for every positive integer \( j \).
Repeating the same type of arguments, we can find $y_0$ in $B_R(0)$ and a closed interval $I$ disjoint from $[\alpha - \varepsilon_j, \alpha + \varepsilon_j]$ so that, after passing to a subsequence,

$$\lim_{i \to \infty} \mathcal{H}^k(\alpha_i^{-1}(I) \cap B_R(y_0)) \geq KR^k.$$ 

Given any positive integer $p$, pick disjoint closed intervals $I_1, \ldots, I_p$ lying between $I$ and $[\alpha - \varepsilon_j, \alpha + \varepsilon_j]$. The connectedness of $N_i \cap B_2R(0)$ implies that all $\alpha_i^{-1}(I_l) \cap B_2R(0)$ are nonempty for $i$ sufficiently large. Hence, arguing as before, we find $y_1, \ldots, y_p$ in $B_2R(0)$ such that, after passing to a subsequence,

$$\lim_{i \to \infty} \mathcal{H}^k(\alpha_i^{-1}(I_l) \cap B_R(y_l)) \geq KR^k,$$

for all $l$ in $\{1, \ldots, p\}$. This implies that

$$\lim_{i \to \infty} \mathcal{H}^k(N_i \cap B_2R(0)) \geq \lim_{i \to \infty} \sum_{l=1}^{p} \mathcal{H}^k(\alpha_i^{-1}(I_l) \cap B_R(y_l)) \geq pKR^k.$$

Choosing $p$ sufficiently large we get a contradiction. \hfill \Box

**Appendix B.**

The next lemma is a simple modification of a result that can be found in Ecker’s book [5] and Ilmanen’s preprint [9]. The proof is the same but we write it here for the sake of completeness.

**Lemma B.1.** Let $(M_t)_{t \geq 0}$ be family of $k$-dimensional submanifolds $(M_t)_{t \geq 0}$ moving by mean curvature flow in $\mathbb{R}^m$. Assume there are constants $A_0$ and $R_0$ such that

$$\mathcal{H}^k(M_0 \cap B_r(0)) \leq A_0 r^k,$$

for all $r \geq R_0$. Then, for all $t \geq t_0$ and $x_0 \in \mathbb{R}^m$, there is a constant $C = C(A_0, R_0/\sqrt{t_0}, |x_0|)$ such that

$$\mathcal{H}^k(M_t \cap B_r(x_0)) \leq Cr^k$$

for all $r > 0$.

**Proof.** In what follows, $C = C(A_0, t_0^{-1}, R_0, |x_0|)$ will denote a constant depending only on the mentioned quantities. Using the monotonicity formula
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we obtain

\[ H^k(M_t \cap B_r(x_0)) \leq C \int_{M_t} \frac{1}{(4\pi t)^{k/2}} e^{-\frac{|x-x_0|^2}{4t}} \, dH^n \]
\[ \leq C \int_{M_0} \frac{1}{(4\pi (t+r^2))^{k/2}} e^{-\frac{|x-x_0|^2}{8(t+r^2)}} \, dH^n \]
\[ \leq C \int_{M_0} \frac{1}{(4\pi (t+r^2))^{k/2}} e^{-\frac{|x|^2}{8(t+r^2)}} \, dH^n \]
\[ \leq C \int_{\lambda M_0} e^{-|x|^2} \, dH^n, \]

where \( \lambda \equiv (8(t+r^2))^{-1/2}. \) For all \( s \geq \lambda R_0 \) we have

\[ H^k(\lambda M_0 \cap B_s(0)) \leq A_0 s^k \]

and thus, setting \( R_1 \equiv \max\{2, (8t_0)^{-1/2} R_0\}, \) the result follows from

\[ \int_{\lambda M_0} e^{-|x|^2} \, dH^n \leq A_0 R_1^k + \int_{\lambda M_0 \setminus B_{R_1}} e^{-|x|^2} \, dH^n \]
\[ = A_0 R_1^k + \sum_{j \geq 0} \int_{\lambda M_0 \cap (B_{R_1^{j+1}} \setminus B_{R_1^j})} e^{-|x|^2} \, dH^n \]
\[ \leq A_0 R_1^k + \sum_{j \geq 0} A_0 R_1^{j+1} e^{-R_1^{2j}}. \]

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