A note on the automorphism group of the Hamming graph

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Abstract. Let $\Omega$ be a $m$-set, where $m > 1$, is an integer. The Hamming graph $H(n, m)$, has $\Omega^n$ as its vertex-set, with two vertices are adjacent if and only if they differ in exactly one coordinate. In this paper, we provide a proof on the automorphism group of the Hamming graph $H(n, m)$, by using elementary facts of group theory and graph theory.

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1. Introduction

Let $\Omega$ be a $m$-set, where $m > 1$, is an integer. The Hamming graph $H(n, m)$, has $\Omega^n$ as its vertex-set, with two vertices are adjacent if and only if they differ in exactly one coordinate. This graph is very famous and much is known about it, for instance this graph is actually the Cartesian product of $n$ complete graphs $K_m$, that is, $K_m \square \cdots \square K_m$. In general, the connection between Hamming graphs and coding theory is of major importance. If $m = 2$, then $H(n, m) = Q_n$, where $Q_n$ is the hypercube of dimension $n$. Since, the automorphism group of the hypercube $Q_n$ has been already determined [10], in the sequel, we assume that $m \geq 3$. Figure 1. displays $H(2, 3)$ in the plane. Note that in this figure, we denote the vertex $(x, y)$ by $xy$.

It follows from the definition of the Hamming graph $H(n, m)$ that if $\theta \in \text{Sym}([n])$, where $\Omega = [n] = \{1, \ldots, n\}$, then

$$f_\theta : V(H(n, m)) \rightarrow V(H(n, m)), f_\theta(x_1, \ldots, x_n) = (x_{\theta(1)}, \ldots, x_{\theta(n)}),$$

is an automorphism of the Hamming graph $H(n, m)$, and the mapping $\psi : \text{Sym}([n]) \rightarrow \text{Aut}(H(n, m))$, defined by this rule, $\psi(\theta) = f_\theta$, is an injection. Therefore, the set $H = \{f_\theta \mid \theta \in \text{Sym}([n])\}$, is a subgroup of $\text{Aut}(H(n, m))$, which is isomorphic with $\text{Sym}([n])$. Hence, we have $\text{Sym}([n]) \leq \text{Aut}(H(n, m))$. 

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Let $A, B$, be non-empty sets. Let $\text{Fun}(A, B)$, be the set of functions from $A$ to $B$, in other words, $\text{Fun}(A, B) = \{ f \mid f : A \to B \}$. If $B$ is a group, then we can turn $\text{Fun}(A, B)$ into a group by defining a product, $(fg)(a) = f(a)g(a), \quad f, g \in \text{Fun}(A, B), \quad a \in A,$ where the product on the right of the equation is in $B$. If $f \in \text{Fun}([n], \text{Sym}([m]))$, then we define the mapping, 

$$A_f : V(\Gamma) \to V(\Gamma),$$

by this rule, $A_f(x_1, \cdots, x_n) = (f(1)(x_1), \cdots, f(n)(x_n))$. It is easy to show that the mapping $A_f$ is an automorphism of the Hamming graph $\Gamma = H(n, m)$, and hence the group, $F = \{ A_f \mid f \in \text{Fun}([n], \text{Sym}([m])) \}$, is also a subgroup of the Hamming graph $\Gamma = H(n, m)$. Therefore, the subgroup which is generated by $H$ and $F$ in the group $\text{Aut}(\Gamma)$, namely, $W = < H, F >$ is a subgroup of $\text{Aut}(\Gamma)$. In this paper, we want to show that:

$$\text{Aut}(H(n, m)) = W = < H, F > = \text{Sym}(\Omega) \text{wr} \text{Sym}([n])$$

There are various important families of graphs $\Gamma$, in which we know that for a particular group $G$, we have $G \leq \text{Aut}(\Gamma)$, but showing that in fact we have $G = \text{Aut}(\Gamma)$, is a difficult task. For example note to the following cases.

(1) The Boolean lattice $BL_n$, $n \geq 1$, is the graph whose vertex set is the set of all subsets of $[n] = \{1, 2, \ldots, n\}$, where two subsets $x$ and $y$ are adjacent if and only if their symmetric difference has precisely one element. The hypercube $Q_n$ is the graph whose vertex set is $\{0, 1\}^n$, where two $n$-tuples are adjacent if they differ in precisely one coordinates. It is an easy task to show that $Q_n \cong BL_n$, and $Q_n \cong \text{Cay}(\mathbb{Z}_2^n, S)$, where $\mathbb{Z}_2$ is the cyclic group of order 2, and $S = \{ e_i \mid 1 \leq i \leq n \}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 at the $i$th position. It is an easy task to show that the set $H = \{ f_{\theta} \mid \theta \in \text{Sym}([n]) \}$, $f_{\theta}(\{x_1, \ldots, x_n\}) = \{\theta(x_1), \ldots, \theta(x_n)\}$ is a subgroup of $\text{Aut}(BL_n)$, and hence $H$ is a subgroup of the group $\text{Aut}(Q_n)$. We know that in every Cayley
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graph $\Gamma = \text{Cay}(G,S)$, the group $\text{Aut}(\Gamma)$ contains a subgroup isomorphic with the group $G$. Therefore, $\mathbb{Z}_2^n$ is a subgroup of $\text{Aut}(Q_n)$. Now, showing that $\text{Aut}(Q_n) = \langle \mathbb{Z}_2^n, \text{Sym}([n]) \rangle$ (or $\mathbb{Z}_2 \times \text{Sym}([n])$), is not an easy task [10].

(2) Let $n, k \in \mathbb{N}$ with $k < \frac{n}{2}$ and Let $[n] = \{1, ..., n\}$. The Kneser graph $K(n,k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices $v$ and $w$ are adjacent if and only if $|v \cap w| = 0$. The Kneser graph $K(n,k)$ is a vertex-transitive graph [6]. It is an easy task to show that the set $H = \{f_\theta \mid \theta \in \text{Sym}([n])\}$, $f_\theta(\{x_1, ..., x_k\}) = \{\theta(x_1), ..., \theta(x_k)\}$, is a subgroup of $\text{Aut}(K(n,k))$ [6]. But, showing that

$$H = \{f_\theta \mid \theta \in \text{Sym}([n])\} = \text{Aut}(K(n,k))$$

is not very easy [6 chapter 7, 13].

(3) Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, ..., n\}$. The Johnson graph $J(n,k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices $v$ and $w$ are adjacent if and only if $|v \cap w| = k - 1$. The Johnson graph $J(n,k)$ is a vertex-transitive graph [6]. It is an easy task to show that the set $H = \{f_\theta \mid \theta \in \text{Sym}([n])\}$, $f_\theta(\{x_1, ..., x_k\}) = \{\theta(x_1), ..., \theta(x_k)\}$, is a subgroup of $\text{Aut}(J(n,k))$ [6]. It has been shown that $\text{Aut}(J(n,k)) \cong \text{Sym}([n])$, if $n \neq 2k$, and $\text{Aut}(J(n,k)) \cong \text{Sym}([n]) \times \mathbb{Z}_2$, if $n = 2k$, where $\mathbb{Z}_2$ is the cyclic group of order 2 [3,7,12].

2. Preliminaries

In this paper, a graph $\Gamma = (V, E)$ is considered as a simple undirected graph with vertex-set $V(\Gamma) = V$, and edge-set $E(\Gamma) = E$. For all the terminology and notation not defined here, we follow [1,2,5,6].

The group of all permutations of a set $V$ is denoted by $\text{Sym}(V)$ or just $\text{Sym}(n)$ when $|V| = n$. A permutation group $G$ on $V$ is a subgroup of $\text{Sym}(V)$. In this case we say that $G$ act on $V$. If $\Gamma$ is a graph with vertex set $V$, then we can view each automorphism as a permutation of $V$, and so $\text{Aut}(\Gamma)$ is a permutation group. Let $G$ act on $V$, we say that $G$ is transitive (or $G$ acts transitively on $V$), if there is just one orbit. This means that given any two elements $u$ and $v$ of $V$, there is an element $\beta$ of $G$ such that $\beta(u) = v$.

Let $\Gamma, \Lambda$ be arbitrary graphs with vertex-set $V_1, V_2$, respectively. An isomorphism from $\Gamma$ to $\Lambda$ is a bijection $\psi : V_1 \rightarrow V_2$ such that $\{x, y\}$ is an edge in $\Gamma$ if and only if $\{\psi(x), \psi(y)\}$ is an edge in $\Lambda$. An isomorphism from a graph $\Gamma$ to itself is called an automorphism of the graph $\Gamma$. The set of automorphisms of graph $\Gamma$ with the operation of composition of functions is a group, called the automorphism group of $\Gamma$ and denoted by $\text{Aut}(\Gamma)$. In most situations, it is difficult to determine the automorphism group of a graph, but there are various in the literature and some of the recent works appear in the references [7,8,9,11,13,14,15,16,17].
The graph $\Gamma$ is called vertex-transitive, if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. In other words, given any vertices $u, v$ of $\Gamma$, there is an $f \in \text{Aut}(\Gamma)$ such that $f(u) = v$.

For $v \in V(\Gamma)$ and $G = \text{Aut}(\Gamma)$, the stabilizer subgroup $G_v$ is the subgroup of $G$ containing of all automorphisms which fix $v$. In the vertex-transitive case all stabilizer subgroups $G_v$ are conjugate in $G$, and consequently isomorphic, in this case, the index of $G_v$ in $G$ is given by the equation, $|G : G_v| = |G|/|G_v| = |V(\Gamma)|$. If each stabilizer $G_v$ is the identity group, then every element of $G$, except the identity, does not fix any vertex and we say that $G$ act semiregularly on $V$. We say that $G$ act regularly on $V$ if and only if $G$ acts transitively and semiregularly on $V$, and in this case we have $|V| = |G|$.

Let $N$ and $H$ be groups, and let $\phi : H \rightarrow \text{Aut}(N)$ be a group homomorphism. In other words, the group $H$ acts on the group $N$, by this rule $n^h = \phi(h)(n)$, $n \in N, h \in H$. Note that in this case we have $(n_1 n_2)^h = n_1^h n_2^h$, $n_1, n_2 \in N$. The semidirect product $N$ by $H$ which is denoted by $N \rtimes H$ is a group on the set $N \times H = \{(n, h) \mid n \in N, h \in H\}$, with the multiplication $(n, h)(n_1, h_1) = (n(n_1)^{-h}, hh_1)$. Note that the identity element of the group $N \rtimes H$ is $(1_N, 1_H)$, and the inverse of the element $(n, h)$ is the element $((n^{-1})^h, h^{-1})$.

3. Main Results

Let $\Gamma$ be a connected graph with diameter $d$. Then we can partition the vertex-set $V(\Gamma)$ with respect to the distances of vertices from a fixed vertex. Let $v$ be a fixed vertex of the graph $\Gamma$. We denote the set of vertices at distance $i$ from $v$, by $\Gamma_i(v)$. Thus it is obvious that $\{v\} = \Gamma_0(v)$ and $\Gamma_1(v) = N(v)$, the set of adjacent vertices to vertex $v$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(x), ..., \Gamma_D(x)$. If $\Gamma = H(n, m)$, then it is clear that two vertices are at distance $k$ if and only if they differ in exactly $k$ coordinates. Then the maximum distance occurs when the two vertices (regarded as ordered $n$-tuples) differ in all $n$ coordinates. Thus the diameter of $H(n, m)$ is equal to $n$.

Lemma 3.1. Let $m \geq 3$ and $\Gamma = H(n, m)$. Let $x \in V(\Gamma)$, $\Gamma_i = \Gamma_i(x)$ and $v \in \Gamma_i$. Then we have;

$$\bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i) = \{v\}.$$

Proof. It is obvious that

$$v \in \bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i).$$

Let $x = (x_1, \cdots, x_n)$. Since the Hamming graph $H(n, m)$ is a distance-transitive graph [3], then we can assume that, $v = (x_1, \cdots, x_{n-i}, y_{n-i+1}, \cdots, y_n)$, where $y_j \in \mathbb{Z}_m - \{x_j\}$ for all $j = n - i + 1, \cdots, n$. 


Let $w \in \Gamma_{i-1} \cap N(v)$. Then $w, x$ differ in exactly $i - 1$ coordinates and $w, v$ differ in exactly one coordinate. Note that, if in $v$ we change one of $x_j s$, where $j = 1, \cdots, n - i$, then we obtain a vertex $u$ such that $d(u, x) \geq i + 1$. Thus, $w$ has a form such as;

$$w = w_r = (x_1, \cdots, x_{n-i}, y_{n-i+1}, \cdots, y_{r-1}, y, y_{r+1}, \cdots, y_n)$$

We show that if $u \in \Gamma_i$ and $u \neq v$ and $u$ is adjacent to some $w_r$, then there is some $w_p$ such that $u$ is not adjacent to $w_p$.

If $v \neq u \in \Gamma_i$ is adjacent to $w_r$ then $u$ has one of the following forms:

(i) $u_1 = (x_1, \cdots, x_{n-i}, y_{n-i+1}, \cdots, y_{r-1}, y, y_{r+1}, \cdots, y_n)$, where $y \in \mathbb{Z}_m$, and $y \neq y_r, x_r$ (note that since $m \geq 3$, hence there is such a $y$).

(ii) $u_2 = (x_1, \cdots, x_{j-1}, y, x_{j+1}, \cdots, x_{n-i}, y_{n-i+1}, \cdots, y_{r-1}, y, y_{r+1}, \cdots, y_n)$, where $y \in \mathbb{Z}_m$, and $y \neq x_j$.

In the case (i), $u_1$ is not adjacent to $w_t$, for all possible $t, t \neq r$. In the case (ii), it is obvious that $u_2$ is also not adjacent to $w_t$ for all possible $t, t \neq r$.

Our argument shows that if $u \in \Gamma_i$, and $u \neq v$, then there is some $w_r$ such that $u$ is not adjacent to $w_r$, in other words $u \notin N(w_r)$. Thus we have;

$$\bigcap_{w \in N(v) \cap \Gamma_{i-1}} (N(w) \cap \Gamma_i) = \{v\}. \qed$$

Let $I = \{\gamma_1, \ldots, \gamma_n\}$ be a set and $K$ be a group. Let $\text{Fun}(I, K)$ be the set of all functions from $I$ into $K$. We can turn $\text{Fun}(I, K)$ into a group by defining a product:

$$(fg)(\gamma) = f(\gamma)g(\gamma), \ f, g \in \text{Fun}(I, K), \ \gamma \in I,$$

where the product on the right of the equation is in $K$. Since $I$ is finite, the group $\text{Fun}(I, K)$ is isomorphic to $K^n$ (the direct product of $n$ copies of $K$), by the isomorphism $f \mapsto (f(\gamma_1), \ldots, f(\gamma_n))$. Let $H$ be a group and assume that $H$ acts on the nonempty set $I$. Then, the wreath product of $K$ by $H$ with respect to this action is the semidirect product $\text{Fun}(I, K) \rtimes H$ where $H$ acts on the group $\text{Fun}(I, K)$, by the following rule,

$$f^x(\gamma) = f(\gamma^x), \ f \in \text{Fun}(I, K), \ \gamma \in I, \ x \in H.$$ 

We denote this group by $K \text{wr}_I H$. Consider the wreath product $G = K \text{wr}_I H$.

If $K$ acts on a set $\Delta$ then we can define an action of $G$ on $\Delta \times I$ by the following rule,

$$(\delta, \gamma)^{(f, h)} = (\delta f(\gamma), \gamma^h), \ (\delta, \gamma) \in \Delta \times I,$$

where $(f, h) \in \text{Fun}(I, K) \rtimes H = K \text{wr}_I H$. It is clear that if $I, K$ and $H$, are finite sets, then $G = K \text{wr}_I H$, is a finite group, and we have $|G| = |K|^{|I|}|H|$.

We have the following theorem [4].
Theorem 3.2. Let $\Gamma$ be a graph with $n$ connected components $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$, where $\Gamma_i$ is isomorphic to $\Gamma_1$ for all $i \in [n] = \{1, \cdots, n\} = I$. Then we have, $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \cup_r \text{Sym}(|n|)$.

Lemma 3.3. Let $n \geq 2$, $m \geq 3$. Let $v$ be a vertex of the Hamming graph $H(n,m)$. Then, $\Gamma_1 = <N(v)>$, the induced subgraph of $N(v)$ in $H(n,m)$, is isomorphic with $nK_{m-1}$, where $nK_{m-1}$ is the disjoint union of $n$ copies of the complete graph $K_{m-1}$.

Proof. Let $v = (v_1, \cdots, v_n)$. Then, for all $i$, $1 = i = 1, \cdots, n$, there are $m - 1$ elements $w_j, w_j \in \mathbb{Z}_m - \{v_i\}$. Let $x_{ij} = (v_1, \cdots, v_{i-1}, w_j, v_{i+1}, \cdots, v_n)$, $1 \leq i \leq n, 1 \leq j \leq n - 1$. Then, $N(v) = \{x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m - 1\}$. Let $x_{ij}, x_{rs}$, be two vertices in $\Gamma_1 = <N(v)>$, then $x_{ij}, x_{rs}$ are adjacent in $\Gamma_1$ if and only if $i = r$. Note that two vertices, $(v_1, \cdots, v_{i-1}, w_j, v_{i+1}, \cdots, v_n)$ and $(v_1, \cdots, v_{i-1}, w_s, v_{i+1}, \cdots, v_n)$ differ in only one coordinate. Therefore, for each $i = 1, \cdots, n$, there are $m - 2$ vertices $w_j$ in $\Gamma_1$ which are adjacent to the vertex $w_{ij}$, where $r \neq j$. Now, it is obvious that the subgraph induced by the set $\{x_{ij} : 1 \leq j \leq m - 1\}$, is isomorphic with $K_{m-1}$, the complete graph of order $m - 1$. Now, it is easy to see that, the subgraph induced by the set $\{x_{ij} : i = 1, \cdots, n, j = 1, \cdots, m - 1\}$, is isomorphic with $nK_{m-1}$, the disjoint union of $n$ copies of the complete graph $K_{m-1}$.

We now are ready to prove the main result of this paper.

Theorem 3.4. Let $n \geq 2$, $m \geq 3$, and $\Gamma = H(n,m)$ be a Hamming graph. Then $\text{Aut}(\Gamma) \cong \text{Sym}(|n|) \cup_r \text{Sym}(|m|)$, where $I = [n] = \{1, 2, \cdots n\}$.

Proof. Let $G = \text{Aut}(\Gamma)$. Let $x \in V = V(\Gamma)$, and $G_x = \{f \in G \mid f(x) = x\}$ be the stabilizer subgroup of the vertex $x$ in $\text{Aut}(\Gamma)$. Let $<N(x)> = \Gamma_1$ be the induced subgroup of $N(x)$ in $\Gamma$. If $f \in G_x$ then $f|_{N(x)}$, the restriction of $f$ to $N(x)$ is an automorphism of the graph $\Gamma_1$. We define the mapping $\psi : G_x \rightarrow \text{Aut}(\Gamma_1)$ by this rule, $\psi(f) = f|_{N(x)}$. It is an easy task to show that $\psi$ is a group homomorphism. We show that $\text{Ker}(\psi)$ is the identity group. If $f \in \text{Ker}(\psi)$, then $f(x) = x$ and $f(w) = w$ for every $w \in N(x)$. Let $\Gamma_i$ be the set of vertices of $\Gamma$ which are at distance $i$ from the vertex $x$. Since, the diameter of the graph $\Gamma = H(n,m)$, is $n$, then $V = V(\Gamma) = \bigcup_{i=0}^{n} \Gamma_i$. We prove by induction on $i$, that $f(u) = u$ for every $u \in \Gamma_i$. Let $d(u, x)$ be the distance of the vertex $u$ from $x$. If $d(u, x) = 1$, then $u \in \Gamma_1$ and we have $f(u) = u$. Assume that $f(u) = u$, when $d(u, x) = i - 1$. If $d(u, x) = i$, then by Lemma 1, $\{u\} = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(w) \cap \Gamma_i)$. Note that if $w \in \Gamma_{i-1}$, then $d(w, x) = i - 1$, and hence $f(w) = w$. Therefore,

$$\{f(u)\} = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(f(w)) \cap \Gamma_i) = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} (N(w) \cap \Gamma_i) = u.$$
Thus, $f(u) = u$ for all $u \in V(\Gamma)$, hence we have $\text{Ker}(\psi) = \{1\}$. On the other hand,

$$\frac{G_v}{\text{Ker}(\psi)} \cong \psi(G_v) \leq \text{Aut}(\Gamma_1), \text{ hence } G_v \cong \psi(G_v) \leq \text{Aut}(\Gamma_1).$$

Thus, $|G_v| \leq |\text{Aut}(\Gamma_1)|$.

We know by Lemma 3. that $\Gamma_1 \cong nK_{m-1}$. We know that, $\text{Aut}(K_{m-1}) \cong \text{Sym}([m-1])$. Therefore, the set $H$ is an automorphism of the Hamming graph $\Gamma$.

Then, by the above equation, we have;

$$|G_v| \leq |\text{Aut}(\Gamma_1)| = |\text{Sym}([m-1]) wr \text{Sym}([n])| = ((m-1)!)^n n!,$$

where $I = [n] = \{1, \cdots, n\}$.

Since $\Gamma = H(n, m)$ is a vertex-transitive graph, then we have $|V(\Gamma)| = |G||G_v|$, and therefore;

$$|G| = |G_v||V(\Gamma)| \leq |\text{Aut}(nK_{m-1})| m^n = m^n((m-1)!)^n n! = (m!)^n n! \quad (*)$$

We have seen (in the introduction section of this paper) that if $\theta \in \text{Sym}([n])$, where $\Omega = [n] = \{1, \cdots, n\}$, then

$$f_\theta : V(H(n, m)) \rightarrow V(H(n, m)), f_\theta(x_1, \cdots, x_n) = (x_{\theta(1)}, \cdots, x_{\theta(n)}),$$

is an automorphism of the Hamming graph $H(n, m)$, and the mapping $\psi : \text{Sym}([n]) \rightarrow \text{Aut}(H(n, m))$, defined by this rule, $\psi(\theta) = f_\theta$, is an injection. Therefore, the set $H = \{f_\theta \mid \theta \in \text{Sym}([n])\}$, is a subgroup of $\text{Aut}((H(n, m)))$, which is isomorphic with $\text{Sym}([n])$. Hence, we have $\text{Sym}([n]) \leq \text{Aut}(H(n, m))$.

On the other hand, if $f \in \text{Fun}([n], \text{Sym}([m]))$, then we define the mapping; $A_f : V(\Gamma) \rightarrow V(\Gamma)$, by this rule,

$$A_f(x_1, \cdots, x_n) = (f(1)(x_1), \cdots, f(n)(x_n)).$$

It is an easy task to show that the mapping $A_f$ is an automorphism of the Hamming graph $\Gamma$, and hence the group, $F = \{A_f \mid f \in \text{Fun}([n], ([m]))\}$, is a subgroup of the Hamming graph $\Gamma = H(n, m)$. Therefore, the subgroup which is generated by $H$ and $F$ is in the group $\text{Aut}(\Gamma)$, namely, $W = < H, F >$ is a subgroup of $\text{Aut}(\Gamma)$. Note that $W = \text{Sym}([m]) wr I \text{Sym}([n])$, where $I = [n] = \{1, 2, \cdots, n\}$. Since, the subgroup $W$ has $(m!)^n n!$ elements, then by $(*)$, we conclude that;

$$\text{Aut}(\Gamma) = W = \text{Sym}([m]) wr I \text{Sym}([n])$$

□

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