osp(1,2)–covariant Lagrangian quantization of irreducible massive gauge theories with generic background configurations

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Abstract
In the framework of the osp(1,2)-symmetric quantization of irreducible massive gauge theories the background field method is studied for the simplest case of a linear splitting of the gauge field into a background configuration $A^i$ and the quantum fluctuations $Q^i$. The entire set of symmetries of that approach, including three types of background–dependent gauge transformations, is expressed by the corresponding set of Ward identities. Making use of these identities, together with the equation of motion of the auxiliary field, the background dependence of the generating functionals of the 1PI vertex functions and the Green’s functions is completely determined by the dependence of these functionals upon the gauge fields and the associated antifields. It is proven that the introduction of a background field does not change the ultraviolet asymptotics of the theory.

1 Introduction and general results

Lagrangian quantization of general gauge theories is usually considered in the Batalin–Vilkovisky (BV) approach [1, 2] as well as in the $Sp(2)$–covariant formalism of Batalin, Lavrov and Tyutin [3, 4] which is a generalization of the former observing the extended BRST–symmetry, the symmetry under BRST– and antiBRST–transformations. Recently, the $Sp(2)$–covariant quantization has been extended to a formalism which is based on the orthosymplectic superalgebra osp(1,2) [5] and which can be applied to massive theories.

The very general quantization procedures [1] – [4] may be applied also to gauge theories with external fields and/or composite fields [6]. However, there is a wide class of physically interesting situations where these procedures seem not to be optimally adapted. This is the case if quantum effects of gauge fields in the presence of classical gauge configurations, like instantons or merons or monopoles, have to be evaluated or, more simply,
if the background field method (BFM) \cite{7} should be applied in order to facilitate explicit computations.

The BFM is very convenient also for the investigation of general properties of gauge theories and, therefore, deserves a comprehensive study. Among the many attractive features of the BFM one should mention that by choosing a background covariant gauge it is possible to compute quantum effects without losing manifest gauge invariance. This is due to the fact that in its original formulation the BFM introduces a linear splitting of the gauge field into quantum fluctuations $Q$ around the background field $A$ such that the gauge transformation of the latter may be chosen inhomogeneous whereas the gauge transformations of the quantum field occur homogeneous like those of the (anti)ghost and the matter fields.

The BFM may also be used in the proof of renormalizability of gauge theories within the BRST approach. Concerning the role of the background field an essential step has been made by Kluberg-Stern and Zuber \cite{8} distinguishing between BRST and type–I transformations. In principle, the procedure of quantizing gauge fields in the background of a generic classical field configuration, as has been used in Ref. \cite{8}, could be taken over to the BV–formalism by choosing background covariant gauges. However, since the BV–formalism essentially works in the minimal sector of the theory – the nonminimal sector being trivial – one will not be able to make use of the full symmetry content of the classical theory.

Therefore, the aim of the present paper is to offer some general aspects of the BFM in the $osp(1,2)$–covariant quantization scheme where the results of Ref. \cite{8} are extended nontrivially to the non–minimal sector of the theory (A short exposition of that approach already has been given in Ref. \cite{9}; in a more restricted context these problems have been considered in Ref. \cite{10}). In that approach all symmetries of the classical action can be imposed as Ward identities also for the full quantum action. This leads, on the one hand, to the symmetries of the $osp(1,2)$–superalgebra and, on the other hand, to the so-called type I – type III symmetries related to the background field.

Here, we prefer the $osp(1,2)$–covariant quantization scheme, rather than the $Sp(2)$–covariant formalism, since the quantization of gauge theories in the presence of classical background configurations, like instanton or meron solutions, may be plagued by severe infrared problems which requires the consideration of massive gauge fields. On the other hand, the incorporation of mass terms into the quantum action is necessary, at least intermediate, in the renormalization scheme of Bogoliubov, Parasiuk, Hepp, Zimmermann and Lowenstein(BPHZL) \cite{11}. In order to deal with massless theories in that scheme an essential ingredient consists in the introduction of a regularizing mass $m_s = (s - 1)m$ for any massless field and performing ultraviolet as well as infrared subtractions thereby avoiding spurious infrared singularities in the limit $s \to 1$. By using that infrared regularization – without violating the extended BRST symmetries – the $osp(1,2)$–superalgebra occurs necessarily. Moreover, the BPHZL renormalization scheme is surely the mathematical best founded one in order to formulate the quantum master equations on the level of algebraic renormalization theory and to properly compute higher–loop anomalies \cite{12}.

Now, to make our assertions more explicit let us relate our method to the realm of general gauge theories, thereby introducing also the necessary prerequisites of gauge theories with a generic background field. In order to simplify the complexity of the $osp(1,2)$–formalism, we restrict ourselves to the consideration of irreducible gauge theories of first rank with semi–simple Lie group in 4–dimensional space–time. Thorough this paper we
use the condensed notation introduced by DeWitt \[13\] and conventions concerning the derivatives with respect to fields and antifields adopted in Ref. \[3\]. However, for later notational simplicity all the antifields are introduced with opposite sign in comparison with Ref. \[1 – 3\]!}

We consider a set of gauge fields \( G^i \) which, for the sake of simplicity, are supposed to be bosonic and whose classical action \( S_{cl}(G) \) is invariant under the gauge transformations

\[
\delta G^i = R^i_\alpha(G) \xi^\alpha, \quad R^i_\alpha(G) S_{cl,i}(G) = 0 \quad \text{with} \quad S_{cl,i}(G) \equiv \frac{\delta}{\delta G^i} S_{cl}(G),
\]

where \( \xi^\alpha \) and \( R^i_\alpha(G) \) are the parameters of the gauge transformations and the gauge generators, respectively. We assume the set of generators \( R^i_\alpha(G) \) to be linearly independent and complete (irreducible or zero-stage theories). Furthermore, to be able to introduce the background field \( A^i \) by a linear quantum–background splitting of the gauge fields, \( G^i = A^i + Q^i \), we make the assumption that \( R^i_\alpha(G) \) depends on \( G^i \) only linearly,

\[
R^i_\alpha(G) = R^i_\alpha(0) + R^i_{\alpha,j} G^j, \quad R^i_\alpha(0) \equiv r^{i\mu}_\alpha \partial_\mu,
\]

and, for the sake of renormalizability, that \( R^i_\alpha(0) \) are linear differential operators. (Notice, that supersymmetric theories in general do not obey this restriction but that the majority of physically interesting theories, including, e.g., \( N = 2 \) super Yang–Mills theories in harmonic superspace, comply it.) The (closed) gauge algebra is expressed by

\[
R^i_{\alpha,j} R^j_{\beta,j}(G) - R^i_{\beta,j} R^j_{\alpha,i}(G) = -R^i_\gamma(G) f^\gamma_{\alpha\beta}.
\]

Here, \( f^\gamma_{\alpha\beta} \) are the structure constants of a semi-simple Lie group which obey the usual Jacobi identity and, by the help of the (nonsingular) Killing metric \( g_{\alpha\beta} \equiv f^{\delta}_{\alpha\gamma} f^{\gamma}_{\beta\delta} \), may be chosen totally antisymmetric after lowering all the indices. Furthermore, introducing the metric tensor with respect to the field components, \( g_{ij} \), with \( g_{ij} g^{jk} = \delta^k_i \), the following similar relations hold: \( g_{ki} R^i_{\alpha,j} = -g_{ji} R^i_{\alpha,k} \), \( g^{ik} R^i_{\alpha,j} = -g^{jk} R^k_{\alpha,i} \); of course, they are compatible with the gauge algebra \([13]\) (observe that according to DeWitt’s notation \( i = (\mu, \alpha, x, \ldots) \) summarizes Lorentz and group indices, space-time points, etc.).

Now we point to the fact that, because of the linear quantum–background splitting the symmetry transformations \([1,1]\) of \( S_{cl}(A + Q) \) can be realized in three different ways: as background or type–I gauge transformations \([1]\), and two forms of gauge transformations of the quantum field, one with \( A^i \) held fixed (called type–II) and another one with \( A^i \) transforming homogenously (called type–III):

- type–I: \( \delta A^i = R^i_\alpha(A) \xi^\alpha, \quad \delta Q^i = R^i_{\alpha,j} Q^j \xi^\alpha, \)
- type–II: \( \delta A^i = 0, \quad \delta Q^i = R^i_\alpha(A + Q) \xi^\alpha, \)
- type–III: \( \delta A^i = R^i_{\alpha,j} A^j \xi^\alpha, \quad \delta Q^i = R^i_\alpha(Q) \xi^\alpha. \)

Notice that all these symmetries are written down with the parameters \( \xi^\alpha \) of the gauge transformations. To our knowledge, so far neither type–II nor type–III symmetry have been considered in the literature as quantum symmetries. However, this is possible if the theory, as in our case, is appropriately extended to the non–minimal sector! Thereby, in order to remove the gauge degeneracy of the classical action \( S_{cl}(A + Q) \), one should choose the gauge fixing term in such a way that the invariance under type–II and type–III.
transformations is fixed whereas its invariance under type–I transformations, i.e., under a gauge change of the classical background, is not to be affected.

In the $Sp(2)$–covariant Lagrangian quantization [3] the effective (gauge fixed) action $S_{\text{eff}}(A|\phi)$ with $\phi^I = (Q^i, B^a, C^{ab})$ denoting the (quantum part of the) gauge field, the auxiliary field and the (anti)ghost fields, respectively, is invariant under properly generalized type–I and (anti)BRST–transformations. Introducing for each field $\phi^I$ two additional sets of antifields $\phi^*_a = (Q^*_a, B^*_a, C^{*_ab})$, $\bar{\phi}_I = (\bar{Q}_i, \bar{B}_a, \bar{C}_{ab})$ the extended action $S_{\text{ext}}(A|\phi, \phi^*_a, \bar{\phi})$ is required to obey the quantum master equations of the (anti)BRST–symmetry together with the type–I Ward identity, both being properly extended to the additional antifields.

If this approach is extended to the $osp(1,2)$–symmetric Lagrangian quantization [3] the corresponding gauge fixed action $S_{m,\text{eff}}(A|\phi)$ additionally depends on a mass parameter $m$ and is required to be $osp(1,2)$– as well as type–I invariant; the extended action $S_{m,\text{ext}}(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ is required to satisfy the quantum master equations of both the (anti)BRST– and $Sp(2)$–symmetry. Thereby, in comparison with the $Sp(2)$–approach, an additional source $\eta_i = (D_i, E_a, F_{ab})$ has to be included in order to fulfil the $osp(1,2)$–superalgebra for $m \neq 0$ (for basic definitions see Section 2 below).

Whereas the requirement of gauge covariance with respect to $A^i$, i.e., type–I symmetry, is completely independent of the master equations, i.e., on the postulated (anti)BRST–symmetries, the type–II and type–III Ward identities are related to the master equations, i.e., they are consistency conditions of the theory. Indeed, both identities can be derived from the quantum master equations by imposing the equations of motion for the auxiliary fields $B^a$ and the (anti)ghosts $C^{ab}$. Even more, if $S_{m,\text{ext}}(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ is constructed in such a way that for $A^i \neq 0$ the type–I and for $A^i = 0$ the type–II (or type–III) Ward identity is satisfied, then its $A^i$–dependence is completely determined by these symmetry requirements! Using this the vertex functional of massive irreducible gauge theories with background field $A^I$, $\Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta)$, can be related to the vertex functional without background field,

$$\Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta) = O_m(A|\delta/\delta \phi, \delta/\delta \bar{\phi}) \cdot \Gamma_m(0|\phi, \phi^*_a, \bar{\phi}, \eta),$$

by a well defined operation $O_m(A|\delta/\delta \phi, \delta/\delta \bar{\phi})$. In this way the determination of the background dependence is reduced to quantities being completely independent of $A^i$! An analogous relation holds for the generating functional of Green’s functions $Z_m(A|j, \phi^*_a, \bar{\phi}, \eta)$ where $j_I = (J_i, K_a, L_{ab})$. Within the $osp(1,2)$–symmetric formalism it was also possible to give a general solution of a problem which has been posed by Rouet [14] but solved by him only for the physical Green’s functions: There exists a relationship between the Green’s functions with a generic background field and without them. Furthermore, it has been proven that the introduction of a background gauge does not change the ultraviolet asymptotics of the theory. This is by no means a trivial result since $S_{m,\text{ext}}(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ in general depends nonlinear on the antifields and, therefore, the antifields never more may be interpreted as sources of the (anti)BRST transforms of the corresponding fields.

The paper is organized as follows. In Section 2 the $osp(1,2)$–symmetric Lagrangian quantization is shortly reviewed. In Section 3 it is shown how, starting from a proper solution of the quantum master equations, by generalized canonical transformations any admissible solution of the master equations may be obtained. Section 4 is devoted to the construction of a proper solution of the classical master equations in the presence of generic background fields. This solution $\tilde{S}_{m,\text{ext}}^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta)$, which is obtained by assuming that it depends only linearly on the antifields, after choosing a gauge obeys all
the symmetry requirements of the classical theory $S_{cl}(A + Q)$ we started from. In Section 5, again making use of generalized canonical transformations, the general solution of the classical master equations, $S_{m,cl}^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta)$, will be obtained which, however, now depends \textit{nonlinear} on the antifields. It is also shown that this solution is stable against small perturbations and, imposing the equations of motion of the auxiliary field $B^\alpha$, that it depends on three independent parameters ($z$–factors) only. In Section 6 the Ward identities for the generating functional $\Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ of the one–particle–irreducible (1PI) vertex functions are derived. Besides the generalizations of the Slavnov–Taylor and the Delduc–Sorella identities we derive the Kluberg-Stern–Zuber and the Lee identity which are related to type–I and type–II symmetry, respectively. In Section 7, combining both the last two identities, the $A^i$–dependence of $\Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ is determined. In addition, the Ward identity related to type–III symmetry is derived and $\Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ is shown to be type–III invariant. As an application Rouet’s result [14] concerning physical Green’s functions is proven independently. Finally, in Section 8 the renormalization group equation for $\Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta)$ is derived and a rigorous proof is given that the $\beta$–function and the anomalous dimensions are independent of the background also in this very general setting.

2 \textit{osp}(1,2)–covariant quantization

Now we shortly review the $\textit{osp}(1,2)$–symmetric quantization procedure [3]. Let us start with the widely known $Sp(2)$–covariant formalism [3]. It is characterized by introducing to each field $\phi^I = (Q^i, B^i, C^{ab})$ with Grassmann parity $\epsilon(\phi^I) \equiv \epsilon_I$, three kinds of antifields, $\phi^*_{Ia} = (Q^*_a, B^*_a, C^{*a}_{ab})$, $\epsilon(\phi^*_{Ia}) = \epsilon_I + 1$ and $\bar{\phi}^I = (\bar{Q}^i, \bar{B}_a, \bar{C}^{ab})$, $\epsilon(\bar{\phi}^I) = \epsilon_I$. Here $\phi^*_{Ia}$ correspond to (the negative of) the sources of the (anti)BRST–transforms, $s^a \phi^I$, while $\bar{\phi}^I$ corresponds to (the negative of the) source of their combined transforms, $\epsilon_{ab}s^a \phi^I$; the index $a = 1, 2$ labels the $Sp(2)$–doublets of the (anti)BRST–operators $s^a$ as well as of the (anti)ghosts fields. Raising and lowering of $Sp(2)$–indices is obtained by the invariant antisymmetric tensor

$$
\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{ac}\epsilon_{cb} = \delta_b^a.
$$

In the $Sp(2)$–approach a doublet of odd graded symplectic structures, the extended antibrackets $(F, G)^a$, is introduced by

$$
(F, G)^a = \frac{\delta F}{\delta \phi^I} \frac{\delta G}{\delta \phi_{Ia}^*} - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(F \leftrightarrow G),
$$

and a doublet of odd graded, nilpotent generating operators is defined through

$$
\bar{\Delta}^a = \Delta^a - (i/\hbar)V^a \quad \text{with} \quad \Delta^a = (-1)^{\epsilon_I} \frac{\delta L^I}{\delta \phi^I} \frac{\delta}{\delta \phi_{Ia}^*} \quad \text{and} \quad V^a = \epsilon^{ab} \phi_{Ib}^* \frac{\delta}{\delta \phi^I}.
$$

(Here, we remind the reader being familiar with the $Sp(2)$–approach that according to our changement in the definition of the antifields a minus sign appears in the definition of $\Delta^a$ above; the same occurs at various other places). Let us remark that the odd graded antibrackets may be defined alternatively by the Leibniz rule for $\Delta^a$ according to

$$
\Delta^a(FG) = (\Delta^a F)G + (-1)^{(F)}((F, G)^a + F(\Delta^a G)).
$$
The relative nilpotency of the (anti)BRST–operators, \{s^a, s^b\} = 0, repeats itself in the important relations \{\bar{\Delta}^a, \bar{\Delta}^b\} = 0.

The quantum action \(S(\phi, \phi^*_a, \bar{\phi})\) is required to satisfy the quantum master equations:

\[
\frac{1}{\hbar} (S, S)^a - V^a S = i\hbar \Delta^a S \quad \iff \quad \bar{\Delta}^a \exp{(i/\hbar)S} = 0
\]

with the boundary condition \(S|_{\phi^*_a=\bar{\phi}=\hbar=0} = S_{cl}(Q)\). Of course, the solution of the master equations is not unique. Analogous to the case of the BV–approach the extended action \(S_{ext}(\phi, \phi^*_a, \bar{\phi})\) which by construction also satisfies Eq. (2.3):

\[
\exp{(i/\hbar)S_{ext}} = \exp{-i\hbar \hat{T}(F)} \exp{(i/\hbar)S} \quad \text{with} \quad \hat{T}(F) = \frac{1}{2} \epsilon_{ab} \{\bar{\Delta}^b, [\bar{\Delta}^a, F]\}.
\]

This formalism despite appearing manifest \(Sp(2)\)–symmetric leads to solutions of the quantum master equations which also may be \(Sp(2)\)–nonsymmetric. The reason for this can be traced back to the fact that the general transformation properties of the solutions of (2.3) do not restrict \(F\) to be a \(Sp(2)\)–scalar. Therefore, in order to ensure \(Sp(2)\)–symmetry the extended quantum action has to be subjected to further requirements. In addition, the formalism may be generalized to contain also massive gauge and (anti)ghost fields, which are necessary at least intermediately in the BPHZL renormalization procedure to avoid unwanted infrared singularities. Of course, then the (anti)BRST–transformations, and the operators \(\bar{\Delta}^a\), must be extended to include also mass terms.

Let us now state the essential modifications of the \(Sp(2)\)–formalism leading to the \(osp(1, 2)\)–symmetric quantization of irreducible gauge theories with massive fields. First, the linear part of the operators (2.2) has to be extended to observe the \(m\)–dependence,

\[
\bar{\Delta}^a_m = \Delta^a - (i/\hbar)V^a_m
\]

with

\[
\Delta^a = (-1)^{\epsilon_f} \frac{d L}{d \phi^*_I} \frac{\delta}{\delta \phi^*_I a}
\]

and, in addition, three even graded, second order differential operators \(\bar{\Delta}_A, A = (0, \pm)\), generating the symplectic group, have to be introduced,

\[
\bar{\Delta}_A = \Delta_A - (i/\hbar)V_A
\]

with

\[
\Delta_A = (-1)^{\epsilon_f} (\sigma_A)_I \frac{J^f L}{\delta \phi^*_I} \frac{\delta}{\delta \eta_I}
\]

and

\[
V_A = \tilde{\phi}_f (\sigma_A)_I \frac{\delta}{\delta \phi^*_I} + (\phi^*_I (\sigma_L)_a + \phi_J (\sigma_L)_I) \frac{\delta}{\delta \phi^*_I a} + \eta_J (\sigma_A)_I \frac{\delta}{\delta \eta_I}.
\]

Here, the following abbreviations have been used:

\[
(P_+)^{Ja}_{lb} \equiv (P_+)_{lb}^{Ja} + (\delta^f_I - (P_+)^Jf_c \delta^b_c) \
(P_+)^Ja_{lb} \equiv \sigma_A \delta^{Ja}_{lb}.
\]
where, for irreducible gauge theories, the matrix \((P_+)^{Ja}_{1b}\) is defined by

\[
(P_+)^{Ja}_{1b} = \begin{cases} 
\frac{\delta^I_J}{\delta^a_b} & \text{for } I = i, J = j, \\
\delta^I_J & \text{for } I = \alpha, J = \beta, \\
\delta^I_J (\delta^d_e \delta^a_b + \delta^d_e \delta^a_b) & \text{for } I = \alpha c, J = \beta d, \\
0 & \text{otherwise}.
\end{cases}
\]

Obviously, the operator \(P_+\) projects – up to a factor of 2 for the (anti)ghost components – onto the nontrivial representations of the symplectic group. Notice that the linear parts \(V_m^a\) and \(V_A\) depend and act only on the antifields \(\phi^*_I, \phi^*_I\) and \(\eta_I\).

The operators \(\Delta^a_m\) and \(\Delta_A\) generate a Lie superalgebra isomorphic to \(osp(1, 2)\) \([15]\):

\[
\begin{align*}
[\Delta_A, \Delta_B] &= -i(\hbar)\epsilon_{AB}^\ C \Delta_C, \\
[\Delta_A, \Delta^a_m] &= -i(\hbar)\Delta^b_m (\sigma_A)_a^b, \\
\{\Delta^a_m, \Delta^b_m\} &= (i/\hbar)m^2 (\sigma^A)^{ab}_m \Delta^A;
\end{align*}
\]

here, the matrices \(\sigma_A\) generate the algebra \(sl(2, R)\),

\[
(\sigma_A)_a^c (\sigma_B)_c^b = g_{AB}\delta^b_a + \frac{i}{2}\epsilon_{ABC}(\sigma^C)_a^b,
\]

with Cartan metric

\[
g^{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad g^{AC}g_{CB} = \delta^A_B,
\]

and \(\epsilon_{ABC}\) is the antisymmetric tensor, \(\epsilon_{0+++} = 1\). The spinorial indices \(a, b\) of \(\sigma_A\) are raised or lowered according to

\[
(\sigma_A)^{ab} = \epsilon^{ac}(\sigma_A)_c^b = (\sigma_A)_a^c \epsilon^{cb} = \epsilon^{ac}(\sigma_A)_{cd} \epsilon^{db}, \quad (\sigma_A)_a^b = -(\sigma_A)^{ba}.
\]

Notice, that \(sl(2, R)\), the even part of \(osp(1, 2)\), is isomorphic to \(sp(2, R)\). For \(\sigma_A\) we choose the representation \((\sigma_0)_a^b = \tau_3\) and \((\sigma_\pm)_a^b = \mp \frac{1}{2}(\tau_1 \pm i\tau_2)\), where \(\tau_A\) (\(A = 1, 2, 3\)) are the Pauli matrices. Obviously, as long as \(m \neq 0\) the operators \(\Delta^a_m\) are neither nilpotent nor do they anticommute among themselves.

In addition to the \(m\)-extended quantum master equations

\[
\frac{1}{2}(S_m, S_m)_a^b - V_m S = \hbar \Delta^a_m S_m \quad \iff \quad \Delta^a_m \exp\{(i/\hbar)S_m\} = 0,
\]

which ensure (anti)BRST- invariance, the mass dependent quantum action \(S_m(\phi, \phi^*_I, \phi, \eta)\) is required to satisfy the symplectic master equations which ensure \(Sp(2)\)-invariance:

\[
\frac{1}{2}(S_m, S_m)_A - V_A S_m = i\hbar \Delta_A S_m \quad \iff \quad \Delta_A \exp\{(i/\hbar)S_m\} = 0,
\]

where the new, even graded brackets are defined by

\[
\{F, G\}_A = (\sigma_A)_J^I \frac{\delta F}{\delta \phi^I} \frac{\delta G}{\delta \eta_J} + (-1)^{(F)(G)}(F \leftrightarrow G); \quad (2.16)
\]

alternatively, these brackets again may be defined by the Leibniz rule for \(\Delta_A\):

\[
\Delta_A(FG) = (\Delta_A F)G + \{F, G\}_A + F(\Delta_A G).
\]
In the limit $\hbar \to 0$ from Eqs. (2.14) and (2.15) the classical master equations are obtained.

From Eqs. (2.14) and (2.15) it follows that, if and only if the action $S_m$ is $Sp(2)$ invariant, it can be (anti)BRST invariant as well. Moreover, in order to express the algebra (2.11) – (2.13) by operator identities, and to decompose the set of antifields into linear spaces being irreducible under the $osp(1,2)$ superalgebra, one is forced to enlarge the set of antifields by additional sources $\eta_i = (D_i, E_{\alpha}, F_{ab})$ with $\epsilon(\eta_i) = \epsilon_i$.

In order remove the gauge arbitrariness, analogous to Eq. (2.4), the $m$-extended quantum action $S_{m,ext}(\phi, \phi^*_a, \phi, \eta)$ will be introduced according to

$$\exp\{(i/h)S_{m,ext}\} = \hat{U}_m(F) \exp\{(i/h)S_m\},$$  \hspace{1cm} (2.17)

$$\hat{U}_m(F) = \exp\{-ih\hat{T}_m(F)\} \quad \text{with} \quad \hat{T}_m(F) = \frac{1}{2}\epsilon_{ab}\{\Delta^a_m, [\Delta^b_m, F]\} + (i/h)^2m^2F.$$  \hspace{1cm} (2.18)

Written explicitly the operator $\hat{T}_m(F)$ becomes

$$\hat{T}_m(F) = \frac{\delta F}{\delta \phi^J} \left( \frac{\delta}{\delta \phi_I} - \frac{1}{2}m^2(P_-)_{Jc} \frac{\delta}{\delta \eta_J} \right) - \frac{1}{2}(h/i)\epsilon_{ab}\frac{\delta}{\delta \phi^*_I} \frac{\delta^2 F}{\delta \phi^*_I \delta \phi^*_J} \frac{\delta}{\delta \phi^*_J} + (i/h)m^2F.$$  \hspace{1cm} (2.19)

If the gauge fixing functional $F(\phi)$ is chosen as a $Sp(2)$–scalar, and if $S_m$ is restricted to depend on $\eta_i$ only linearly, namely $\delta S_m/\delta \eta_i = -\phi^I$, then the following relations may be shown to hold: $[\Delta^a_m, \hat{U}_m(F)]\exp\{(i/h)S_m\} = 0$ and $[\Delta_A, \hat{U}_m(F)]\exp\{(i/h)S_m\} = 0$. Hence, because $S_m$ being a proper solution of the master equations, the extended action $S_{m,ext}$ satisfies the quantum master equations (2.14) and (2.15) as well. A more detailed discussion of these aspects is given in the next Section.

Before going on we present the explicit expressions for the operators $\Delta^a, V^a_m$ and $\Delta_A$. $V_A$ as well as for the odd and even bracket structures $(F,G)^a$ and $\{F,G\}_A$. Thereby we restrict ourselves to the case of irreducible gauge theories of first rank with semi–simple gauge group and bosonic gauge fields, $\epsilon_1 = (\epsilon_1, \epsilon_2, \epsilon_3 + 1)$ with $\epsilon_1 = \epsilon_2 = 0$; in that case the components $D_i$ and $E_{\alpha}$ of $\eta_i$ may be choosed equal to zero:

$$\Delta^a = \frac{\delta L}{\delta Q^i} \frac{\delta}{\delta Q^{*i}} + \frac{\delta L}{\delta B_{ab}} \frac{\delta}{\delta B_{ab}^*},$$  \hspace{1cm} (2.20)

$$V^a_m = \epsilon^{ab}Q^*_{ib}\frac{\delta}{\delta Q^i} + m^2\bar{Q}^i - \frac{\delta}{\delta \bar{Q}^i} \frac{\delta}{\delta \bar{Q}^i} + \epsilon^{ab}B_{ab}^* \frac{\delta}{\delta B_{ab}} + m^2\bar{B}_{ab} \frac{\delta}{\delta \bar{B}_{ab}} + \epsilon^{ab}C_{ab}^* \frac{\delta}{\delta C_{ac}},$$  \hspace{1cm} (2.21)

$$\Delta_A = - \frac{1}{2} \frac{\delta}{\delta C^{*ab}} \frac{\delta}{\delta F_{aa}},$$  \hspace{1cm} (2.22)

$$V_A = Q_{ia}^*(\sigma_A)^a_b \frac{\delta}{\delta Q^{*i}} + B_{ab}(\sigma_A)^a\frac{\delta}{\delta B_{ab}} + \bar{C}_{aa}(\sigma_A)^a_b \frac{\delta}{\delta C_{ac}},$$  \hspace{1cm} (2.23)

and

$$(F,G)^a = \frac{\delta F}{\delta Q^i} \frac{\delta}{\delta Q^{*i}} + \frac{\delta F}{\delta B_{ab}} \frac{\delta}{\delta B_{ab}^*} + \frac{\delta G}{\delta C_{ab}^*} \frac{\delta}{\delta C_{ab}},$$  \hspace{1cm} (2.24)

$$\{F, G\}_A = \frac{\delta F}{\delta C^{*ab}} \frac{\delta}{\delta F_{aa}} + (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(F \leftrightarrow G),$$  \hspace{1cm} (2.25)
3 Generalized canonical transformations

As became obvious from the considerations of the last Section there exists a whole set of proper solutions of the master equations. In the BV–approach a solution $S$ of the master equation is called proper if (a) it has a stationary point where the variations with respect to the fields and the antifields vanish and (b) its Hessian with respect to the fields and antifields at that stationary point is nonsingular. It has been shown that two proper solutions of the master equations are connected by a canonical transformation [2]. These notions have a natural generalization to the $Sp(2)$–covariant approach–where a corresponding connection between two proper solutions already has been proven [4]–as well as to the $osp(1, 2)$–symmetric quantization procedure.

In this Section we show how, starting from a proper solution of the quantum master equations (2.14) and (2.15), by generalized canonical transformations (see Eq. (3.12) below) any admissible solution of the same equations may be obtained. The organization of the proof – as well as the resulting solution – is quite similar to that presented in Ref. [4].

Thereby, it is very remarkable that there exists a formal extension of these results (up to mass terms) to the case of $osp(1, 2)$–symmetry. Actually, we are only interested in generalized canonical transformation of the classical master equations. However, in order to get such a transformation we solve this problem quite generally, i.e., first we study transformations that allow one to consider the characteristic arbitrariness of a solution of the quantum master equations. Then, by taking the limit $\hbar \to 0$ we recover the transformation we are looking for. Finally, we discuss how in this way the general solution of the classical master equations may be obtained (the explicit construction of this solution will be given in Sections 4 and 5).

To begin with let us assume (a) that two arbitrary proper solutions, $S_m(0)$ and $S_m(1)$, of the quantum master equations are given and (b) that a continuous path in the manifold of solutions $S_m(\zeta), 0 \leq \zeta \leq 1$, i.e., an interpolating functional, exists which connects them. Such a functional, for every value of $\zeta$, is obliged to satisfy the generating equations

$$\frac{1}{2}(S_m(\zeta), S_m(\zeta)) - i\hbar \bar{\Delta}_m S_m(\zeta) = 0, \quad \frac{1}{2}\{S_m(\zeta), S_m(\zeta)\}_A - i\hbar \bar{\Delta}_A S_m(\zeta) = 0.$$  \tag{3.1}

Differentiating these equations with respect to $\zeta$, then for the derivation $\partial S_m(\zeta)/\partial \zeta$ one gets the following consistency conditions:

$$Q^a_m(\zeta) \frac{\partial S_m(\zeta)}{\partial \zeta} = 0, \quad Q_A(\zeta) \frac{\partial S_m(\zeta)}{\partial \zeta} = 0; \tag{3.2}$$

here the operators $Q^a_m(\zeta)$ and $Q_A(\zeta)$ depend explicitly on $S_m(\zeta)$ and their action on an arbitrary functional $X$ is defined according to

$$Q^a_m(\zeta) X \equiv (S_m(\zeta), X)^a - i\hbar \bar{\Delta}_m X, \quad Q_A(\zeta) X \equiv \{S_m(\zeta), X\}_A - i\hbar \bar{\Delta}_A X.$$  \tag{3.3}

These operators, like $\bar{\Delta}_m$ and $\bar{\Delta}_A$, are a realization of the $osp(1, 2)$–superalgebra:

$$[Q_A(\zeta), Q_B(\zeta)] = -\epsilon_{AB}^C Q_C(\zeta),$$

$$[Q_A(\zeta), Q^a_m(\zeta)] = -Q^b_m(\zeta)(\sigma_A)_b^a,$$

$$\{Q^a_m(\zeta), Q^b_m(\zeta)\} = m^2(\sigma^A)^{ab} Q_A(\zeta).$$  \tag{3.4}
In order to find an explicit expression for the interpolating functional $S_m(\zeta)$ subjected to the consistency conditions (3.2) let us make for $\partial S_m(\zeta)/\partial \zeta$ the following ansatz:

$$\frac{\partial S_m(\zeta)}{\partial \zeta} = \hat{W}_m(\zeta)Y \quad \text{with} \quad \hat{W}_m(\zeta) \equiv \frac{1}{2} \epsilon_{ab} Q^a_m(\zeta)Q^b_m(\zeta) + m^2,$$  \hspace{1cm} (3.5)

$Y = Y(\phi, \phi^*_a, \bar{\phi}, \eta)$ being an arbitrary local $Sp(2)$–symmetric functional, i.e.,

$$(\sigma_A)_J \frac{\delta Y}{\delta \phi^J} \phi^J + V_A Y = 0.$$ \hspace{1cm} (3.6)

The justification of that ansatz will be given in Appendix A.

Now, taking into account the $osp(1,2)$–symmetry, Eq. (3.4), by a straightforward, but tedious direct computation one obtains

$$Q^a_m(\zeta)\hat{W}_m(\zeta) = -\frac{1}{2} m^2 (\sigma^A)^a_b Q^b_m(\zeta)Q_A(\zeta), \quad Q_A(\zeta)\hat{W}_m(\zeta) = \hat{W}_m(\zeta)Q_A(\zeta), \quad (3.7)$$

and the consistency conditions (3.2) are fulfilled provided it holds

$$Q_A(\zeta)Y \equiv (\sigma_A)_J \frac{\delta Y}{\delta \eta_J} + \frac{\delta Y}{\delta \phi^J} \left( \frac{\delta S_m(\zeta)}{\delta \eta_J} + \phi^I \right) - \hbar (-1)^{\epsilon_I} \frac{\delta Y}{\delta \phi^I} \frac{\delta S_m(\zeta)}{\delta \eta_J} = 0.$$ \hspace{1cm} (3.8)

Obviously, this equation has a solution which is given by

$$\frac{\delta S_m(\zeta)}{\delta \eta_J} + \phi^I = 0, \quad \frac{\delta Y}{\delta \eta_J} = 0,$$ \hspace{1cm} (3.9)

where the first of these equations is a nontrivial condition concerning the dependence of $S_m(\zeta)$ on $\eta_J$. However, it is a natural condition. This may be seen if Eqs. (3.1) are differentiated with respect to $\eta_J$ leading to the following requirements

$$Q^a_m(\zeta)\left( \frac{\delta S_m(\zeta)}{\delta \eta_J} + \phi^I \right) = 0, \quad Q_A(\zeta)\left( \frac{\delta S_m(\zeta)}{\delta \eta_J} + \phi^I \right) = 0.$$ \hspace{1cm} (3.10)

Therefore, we are forced to require that $S_m(\zeta)$ is linear in $\eta_J$ and that $Y = Y(\phi, \phi^*_a, \bar{\phi})$ is independent of $\eta_J$. Then, as a consequence of the restrictions (3.3), the second of the equations (3.1) simplifies essentially,

$$(\sigma_A)_J \frac{\delta S_m(\zeta)}{\delta \phi^J} \phi^J + V_A S_m(\zeta) = 0,$$ \hspace{1cm} (3.11)

showing that, analogous to Eq. (3.6), the interpolating functional is $Sp(2)$–symmetric.

Now, we are left with the problem to integrate Eq. (3.5). The solution of that differential equation is given by

$$\exp\{(i/\hbar)S_m(\zeta)\} = \hat{U}_m(\zeta Y)\exp\{(i/\hbar)S_m(0)\},$$ \hspace{1cm} (3.12)

$$\hat{U}_m(\zeta Y) = \exp\{(h/\i)\hat{T}_m(\zeta Y)\} \quad \text{with} \quad \hat{T}_m(\zeta Y) = \frac{1}{2} \epsilon_{ab} \{\hat{\Delta}_m^b, [\hat{\Delta}_m^a, Y]\} + (i/\hbar)^2 m^2 Y.$$  

The proof is as follows. First, we rewrite the action of $Q^a_m$ on an arbitrary functional $X$:

$$Q^a_m(\zeta)X = \exp\{- (i/\hbar)S_m(\zeta)\}(h/\i)[\hat{\Delta}_m^a, X] \exp\{(i/\hbar)S_m(\zeta)\}.$$
With the help of this expression the differential equation \((3.15)\) obtains the following form:

\[
\frac{\partial S_m(\zeta)}{\partial \zeta} = \exp\{-i(\hbar)S_m(\zeta)\}\left\{ \frac{1}{2}\epsilon_{ab}(\hbar/2)^2\{\bar{\Delta}_m^a, [\bar{\Delta}_m^a, Y]\} + m^2Y \right\}\exp\{i(\hbar)S_m(\zeta)\}.
\]

Using this result, and the definition of \(\hat{T}_m(Y)\), we obtain

\[
\frac{\partial \exp\{(i/\hbar)S_m(\zeta)\}}{\partial \zeta} = \exp\{(i/\hbar)S_m(\zeta)\}(i/\hbar)\frac{\partial S_m(\zeta)}{\partial \zeta} = (\hbar/i)\hat{T}_m(Y)\exp\{(i/\hbar)S_m(\zeta)\}.
\]

Because of \(\zeta\hat{T}_m(Y) = \hat{T}_m(\zeta Y)\) this finishes the proof of Eq. (3.12).

Now, by virtue of

\[
[\bar{\Delta}_m^a, -i\hbar\frac{\delta}{\delta \eta_I} + \phi^I] = 0, \quad [\bar{\Delta}_A, -i\hbar\frac{\delta}{\delta \eta_I} + \phi^I] = 0, \quad [Y, -i\hbar\frac{\delta}{\delta \eta_I} + \phi^I] = 0, \quad (3.13)
\]

from (3.12) it follows that imposing condition (3.9) together with (3.6) for \(\zeta = 0\) is sufficient to ensure their validity also for the whole range \(0 < \zeta \leq 1\). This proves that \(S_m(\zeta)\), as given by Eq. (3.12), is a solution of the quantum master equations, i.e., it fulfills the first of the equations (3.1) as well as (3.11) for any \(\zeta\). Obviously, performing a generalized canonical transformation (3.12) of a proper solution \(S_m(0)\) by an appropriately chosen functional \(Y\) appears as the general procedure of introducing a gauge (see Eq. (2.17)). According to this result the assumption (b) above has been justified a posteriori. Therefore, the manifold of proper solutions is connected, i.e., any two arbitrary proper solutions may be related through Eq. (3.12) in an analogous manner as in the case of the BV and the \(Sp(2)\)-covariant formalism.

If Eq. (3.12) is solved iteratively one gets a series expansion of \(S_m(\zeta)\) in powers of \(\zeta\):

\[
S_m(\zeta) = \sum_{n=0}^{\infty} \zeta^n S_m^{(n)}, \quad S_m^{(0)} = S_m(0), \quad (3.14)
\]

\[
(n + 1) S_m^{(n+1)} = \frac{1}{2}\epsilon_{ab} \left\{ \sum_{k=0}^{n} \binom{n}{k} (S_m, (S_m, Y)^a)^b + (\hbar/i)(S_m, [\bar{\Delta}_m^a, Y]^b
\right.

\left. + (\hbar/i)\bar{\Delta}_m^b (S_m, Y)^a + (\hbar/i)^2\delta_{n,0}\bar{\Delta}_m^b \bar{\Delta}_m^a Y + \delta_{n,0}m^2Y, \quad n \geq 0;
\right.

this shows that if both \(S_m(0)\) and \(Y\) are local then \(S_m(\zeta)\) is a local functional as well.

Next, if we are interested in a general solution of the classical master equations, which will be the case in the next two Sections, we only have to take the limit \(\hbar \to 0\) in the expansion (3.14). As a result we get a quite nontrivial transformation which converts a proper solution \(S_m(0)\) of the classical master equations into another solution \(S_m(1)\) of the same equations (for notational simplicity we do not distinguish them by an additional index from solutions of the quantum master equations)

\[
S_m(\zeta) = \sum_{n=0}^{\infty} \zeta^n S_m^{(n)}, \quad S_m^{(0)} = S_m(0), \quad (3.15)
\]

\[
(n + 1) S_m^{(n+1)} = \frac{1}{2}\epsilon_{ab} \left\{ \sum_{k=0}^{n} \binom{n}{k} (S_m, (S_m, Y)^a)^b - (S_m, V_m^a Y)^b
\right.

\left. - V_m^b (S_m, Y)^a + \delta_{n,0}V_m^b V_m^a Y + \delta_{n,0}m^2Y, \quad n \geq 0.
\right.
\]
Finally, we outline the steps how through the use of two special transformations of this kind the general solution of the classical master equations may be obtained. Let us assume that the general proper solution $S_m(0) \equiv W_m^{(0)}$, being linear with respect to the antifields, has been constructed. In the general case of arbitrary $L$–stage reducible theories with closed or open algebra for $S_m(0)$ one has to choose the general proper solution of the classical master equations with vanishing new ghost number, i.e., $\text{ngh}(S_m(0)) = 0$ \cite{2}. In the present case of irreducible theories of first–rank with closed algebra this requirement is equivalent to the determination of the general proper solution being linear with respect to the antifields. Then, a more general solution, being nonlinear with respect to the antifields, and denoted by $S_m(1) \equiv S_m^{(1)}$, can be constructed by choosing

$$S_m(0) \equiv W_m^{(0)} \text{ (linear)}, \quad Y \equiv G \quad \Rightarrow \quad S_m(1) = S_m^{(0)} \text{ (nonlinear)}, \quad (3.16)$$

where the most general ansatz for the functional $G(\bar{\phi})$, which has mass dimension two in four space–time dimension, is uniquely given by $G(\bar{\phi}) \sim g^{I,J} \bar{\phi}_I \phi_J$. Furthermore, using the transformation (3.15) once more, the general solution, being denoted by $S_m(1) = S_m^{(0)}_{m,\text{ext}}$, is obtained from the previous solution $S_m^{(0)}$ by choosing

$$S_m(0) \equiv S_m^{(0)} \text{ (nonlinear)}, \quad Y \equiv F \quad \Rightarrow \quad S_m(1) = S_m^{(0)}_{m,\text{ext}} \text{ (nonlinear)}, \quad (3.17)$$

with a gauge–fixing functional $F(\phi) \sim g_{I,J} \phi^I \phi^J$ being the most general ansatz with respect to the fields $\phi^I$. Of course, the solution $S_m^{(0)}_{m,\text{ext}}$ could have been constructed also by interchanging both steps above, i.e., there holds the following commutative diagram:

$$\begin{array}{c}
W_m^{(0)} \xrightarrow{Y=G} S_m^{(0)} \\
Y=F \downarrow \quad \downarrow Y=F \\
W_m^{(0)}_{m,\text{ext}} \xrightarrow{Y=G} S_m^{(0)}_{m,\text{ext}}
\end{array} \quad (3.18)$$

thereby, the solution $S_m(1) = W_m^{(0)}_{m,\text{ext}}$ is constructed from $S_m(0) = W_m^{(0)}$ by choosing

$$S_m(0) \equiv W_m^{(0)} \text{ (linear)}, \quad Y \equiv F \quad \Rightarrow \quad S_m(1) = W_m^{(0)}_{m,\text{ext}} \text{ (linear)}. \quad (3.19)$$

4 Proper Solution of classical master equations

The general problem of how to construct a solution of the quantum master equations in the $\text{osp}(1,2)$–approach has been solved in Ref. \cite{3}. Here we show how to proceed in the case of a linear quantum–background splitting of the gauge field. The consecutive steps are the following:

1. Starting with the classical action $S_{\text{cl}}(A + Q)$, we first construct a proper solution $W_m^{(0)}(A|\phi, \phi^a, \bar{\phi}, \eta)$ of the classical master equations,

$$\frac{1}{2}(W_m^{(0)}, W_m^{(0)})^a - V_m^a W_m^{(0)} = 0, \quad \frac{1}{2}\{W_m^{(0)}, W_m^{(0)}\}_A - V_A W_m^{(0)} = 0, \quad (4.1)$$

being linear in the antifields.
2. In order to remove the gauge degeneracy of that solution \( W_m^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta) \) a gauge is introduced by an appropriate generating functional \( F(\phi) \) according to Eq. (3.12) in the tree approximation. This defines the extended action \( W_{m,\text{ext}}^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta) \) being a solution of the classical master equations which, in addition, is invariant under type–I as well as (up to mass terms) under type–II and type–III transformations.

Despite these nice properties, that solution can not be used for the construction of the generating functional of vertex functions, \( \Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta) \), since it is not stable under small perturbations, i.e., not all of the counter terms which may occur in the process of renormalization could be absorbed by redefining the independent parameters of the theory. This circumstance is due the fact that \( \bar{\phi}^I \) and \( \bar{\phi}_I \) mix under renormalization. Therefore, in the next Section, we continue our programme by the following steps.

3. The proper solution \( W_m^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta) \) by the help of the extension (3.16) will be generalized to a solution of the classical master equations, \( S_m^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta) \), which depends \textit{nonlinear} on the antifields \( \phi^*_I \), and \( \bar{\phi}_I \) (however, because of the requirement (3.9), it must be linear in \( \eta_I \)).

4. Now, fixing the gauge as before by the generating functional \( F(\phi) \) we arrive at the most general solution of the classical master equations, \( S_{m,\text{ext}}^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta) \), which will be the starting point for the solution of the quantum master equations, i.e., the generalization to any order of perturbation theory. It may be shown that \( S_{m,\text{ext}}^{(0)}(A|\phi, \phi^*_a, \bar{\phi}, \eta) \) depends on seven independent parameters and that it is invariant (up to mass terms) under type–I, type–II and type–III transformations.

(A) \textit{Construction of a proper solution}

The symmetry operators of the action \( W_m^{(0)} \), the \( m \)--extended (anti)BRST operator and the generators of symplectic transformations, will be denoted by \( s^a_m \) \((a = 1, 2)\) and \( d_A \) \((A = 0, +, -, \) respectively. They fulfil the \( osp(1,2) \)--superalgebra in the following form:

\[
[d_A, d_B] = \epsilon_{ABC} d^C, \quad [d_A, s^a_m] = s^b_m (\sigma_A)^a_b, \quad \{s^a_m, s^b_m\} = -m^2 (\sigma^A)^{ab} d_A. \quad (4.2)
\]

In order to ensure the \( osp(1,2) \)--invariance of \( W_m^{(0)} \) we make the following ansatz:

\[
W_m^{(0)} = S_{\text{cl}}(A + Q) - (\frac{1}{2} \epsilon_{abc} s^b_m s^a_m + m^2) X \quad \text{with} \quad X = \bar{Q}_i Q^i + \bar{B}_a B^a + \bar{C}_{aa} C^{aa}. \quad (4.3)
\]

Obviously, the term to be added to \( S_{\text{cl}}(A + Q) \) has a structure which is analogous to the right hand side of Eq. (3.3). Furthermore, \( X \) is chosen to be a \( Sp(2) \)--scalar, \( d_A X = 0 \); in fact, it is the only one we are able to build up \textit{linear} in the antifields. Now, because \( S_{\text{cl}}(A + Q) \) is gauge invariant and since, by virtue of (1.2), one verifies the equalities \( s^c_m (\frac{1}{2} \epsilon_{abc} s^b_m s^a_m + m^2) X = \frac{1}{2} m^2 (\sigma^A)^c_a s^a_m d_A X = 0 \) and \( [d_A, \frac{1}{2} \epsilon_{abc} s^b_m s^a_m + m^2] X = 0 \), it easily follows that \( W_m^{(0)} \) is both (anti)BRST-- and \( Sp(2) \)--invariant,

\[
s^a_m W_m^{(0)} = 0, \quad d_A W_m^{(0)} = 0.
\]
For fixed $A^i$ and for the case of irreducible closed gauge algebra, the explicit expressions of the (anti)BRST- and $Sp(2)$-transformations of the fields are uniquely defined by

\[
\begin{align*}
  s^a_m A^i &= 0, & d_A A^i &= 0, \\
  s^a_m Q^i &= R^i_{\alpha}(A + Q)C^{\alpha a}, & d_A Q^i &= 0, \\
  s^a_m C^{ab} &= \varepsilon^{ab} B^a - \frac{1}{2} f^{\alpha}_{\beta\gamma} C^{\alpha b} C^{\gamma a}, & d_A C^{ab} &= C^{\alpha d}(\sigma_A)^d_b, \\
  s^a_m B^a &= \frac{1}{2} f^{\alpha}_{\beta\gamma} B^\beta C^{\gamma a} + \frac{1}{12} \varepsilon_{cd} f^{\alpha}_{\eta\beta} f^{\eta}_{\omega\gamma} C^{\gamma a} C^{\delta c} C^{\beta d} - m^2 C^{\alpha a}, & d_A B^a &= 0.
\end{align*}
\]

Here, the (anti)BRST- transformations of $A^i$ and $Q^i$ are obtained from the type–II transformations (1.3) with $C^a$ replaced by the (anti)ghosts $C^{\alpha a}$. (Let us point to the fact that for irreducible gauge theories the auxiliary field $B^a$ and the (anti)ghosts $C^{ab}$ are related to each other according to $B^a = \frac{1}{2} \epsilon_{abcd} s^a_m C^{ab}$. In principle, one could replace also the type–III transformations (1.3) by defining $s^a_m A^i = R^i_{\alpha} A^i C^{\alpha a}$ and $s^a_m Q^i = R^i_{\alpha}(Q) C^{\alpha a}$; but, such a definition comes into conflict with the background gauge covariance of the theory.

The corresponding transformations of the antifields, taking into account their Grassmann parity and dimension as well as their transformation properties under $Sp(2)$, are uniquely given as

\[
\begin{align*}
  s^a_m \tilde{Q}_i &= \varepsilon^{ab} Q^a_{ib}, & d_A \tilde{Q}_i &= 0, \\
  s^a_m \tilde{Q}^i_{ab} &= m^2 \delta^a_{\beta} \tilde{Q}^i_{\beta b}, & d_A \tilde{Q}^i_{ab} &= Q^i_{ib}(\sigma_A)^d_b, \\
  s^a_m \tilde{B}_{\alpha} &= \varepsilon^{ab} B^b_{\alpha}, & d_A \tilde{B}_{\alpha} &= 0, \\
  s^a_m \tilde{B}^a_{\alpha} &= m^2 \delta_c^a \tilde{B}^a_{\alpha}, & d_A \tilde{B}^a_{\alpha} &= B^a_{\alpha d}(\sigma_A)^d_b, \\
  s^a_m \tilde{C}^{ab} &= \varepsilon^{ab} C^{\alpha b}_{\alpha c}, & d_A \tilde{C}^{ab} &= C^{\alpha d}(\sigma_A)^d_c, \\
  s^a_m \tilde{C}^a_{abc} &= m^2 (\delta^a_c \tilde{C}^{bc}_{\alpha c} + \delta^a_c \tilde{C}^{ab}_{\alpha c}) - \delta_c^a F^{abc}_{\alpha c}, & d_A \tilde{C}^a_{abc} &= C^{\alpha d}(\sigma_A)^d_b + C^{\alpha d}(\sigma_A)^d_c.
\end{align*}
\]

Using the explicit expressions (1.4) and (1.5) for the proper solution (1.3) one obtains

\[
\begin{align*}
  W^{(0)}_m &= S_c(A + Q) - (\varepsilon^{ab} C^{\alpha ba}_{\alpha a} - m^2 \tilde{B}^a_{\alpha}) B^a - (F^a_{\alpha a} - m^2 B^a_{\alpha a}) C^{\alpha a} \\
  &- Q^i_{\alpha a} R^i_{\alpha}(A + Q) C^{\alpha a} - \tilde{Q}_i (R^i_{\alpha}(A + Q) B^a + \frac{1}{2} \epsilon_{abcd} R^i_{\alpha d}, R^j_{\beta d}(A + Q) C^{\beta b} C^{\alpha a}) \\
  &+ C^{\alpha a}_{\alpha a b} (\frac{1}{2} f^{\alpha}_{\beta\gamma} C^{\beta a} C^{\alpha b}) + (\tilde{C}^{ab}_{\alpha a} - \frac{1}{2} B^a_{\alpha a}) (f^{\alpha}_{\beta\gamma} B^\beta C^{\gamma a} + \frac{1}{6} \varepsilon_{cd} f^{\alpha}_{\eta\beta} f^{\eta}_{\omega\gamma} C^{\gamma a} C^{\delta c} C^{\beta d}).
\end{align*}
\]

From this explicit expression for $W^{(0)}_m$ one reads off the relations

\[
\frac{1}{2} \epsilon_{abcd} \frac{\delta}{\delta C^{\alpha a}_{\alpha a b}} W^{(0)}_m = B^a \quad \text{and} \quad \frac{\delta}{\delta F^a_{\alpha a}} W^{(0)}_m = -C^{\alpha a},
\]

which imply, by using the master equations (1.1), two additional conditions for $W^{(0)}_m$,

\[
\left( \frac{\delta}{\delta B^a_{\alpha a}} + \frac{1}{2} \frac{\delta}{\delta C^{\alpha a}_{\alpha a}} \right) W^{(0)}_m = -m^2 \frac{\delta}{\delta F^a_{\alpha a}} W^{(0)}_m \quad \text{and} \quad \frac{\delta}{\delta B^a_{\alpha a}} W^{(0)}_m = \frac{1}{2} m^2 \epsilon_{abcd} \frac{\delta}{\delta C^{\alpha a}_{\alpha a b}} W^{(0)}_m.
\]
Thus, \( W_m^{(0)} \) depends on \( B_{a\alpha}^* \) and \( \bar{B}_\alpha \) only through the combinations

\[
E_{\alpha a} \equiv \bar{C}_{\alpha a} - \frac{1}{2} B_{a\alpha}^*, \quad E_{a\alpha}^* \equiv C_{a\alpha}^* - \frac{1}{2} m^2 \epsilon_{ab} \bar{B}_\alpha, \quad H_{a\alpha} \equiv F_{a\alpha} - m^2 B_{a\alpha}^*, \quad (4.9)
\]

respectively, which may be read off also from Eq. (4.9). Of course, these combinations transform under \( s_m^a \) and \( d_A \) exactly as their first components \( \bar{C}_{\alpha a}, C_{a\alpha}^* \) and \( F_{a\alpha} \),

\[
s_m^a E_{a\alpha} = \epsilon^{ab} E_{abc}^*, \quad d_A E_{a\alpha} = E_{a\alpha d}(\sigma_A)^d_c, \quad s_m^a H_{a\alpha} = m^2 \epsilon^{ab}(E_{abc} - E_{a\alpha c}), \quad d_A H_{a\alpha} = H_{a\alpha d}(\sigma_A)^d_c, \quad (4.10)
\]

Thus, the dependence of \( W_m^{(0)} \) on the antifields \( B_{a\alpha}^* \) and \( \bar{B}_\alpha \) is completely determined by the classical master equations (4.7) together with the relations (4.9) and (4.8). (The \( B_{a\alpha}^* \) dependence is much more involved; it will be considered later on.) Of course, Eqs. (4.11) replace the last five sets of Eqs. in (4.5). This allows to rewrite the action (4.3) as

\[
W_m^{(0)} = S_{cl}(A + Q) - \left( \frac{1}{2} \epsilon_{ab} s_m^a s_m^b + m^2 \right) Y \quad \text{with} \quad Y = \bar{Q}_i Q^i + \bar{E}_{\alpha a} C^{\alpha a},
\]

where the combination \( \bar{B}_\alpha B_{a\alpha}^* + \bar{C}_{a\alpha} C^{a\alpha} \) in the earlier definition of \( X \) are now replaced by \( \bar{E}_{\alpha a} C^{\alpha a} \). Then, for \( W_m^{(0)} \) one obtains

\[
W_m^{(0)} = S_{cl}(A + Q) - H_{a\alpha a} C^{a\alpha} - Q_{a_i} R_{a_i}^{\alpha} (A + Q) B_{a\alpha}^* - \frac{1}{2} \epsilon_{ab} R_{a_i\alpha}^i j_{a_i\alpha} R_{b}^j (A + Q) C^{b\alpha} C^{a\alpha} - E_{a\alpha b} (\epsilon^{ab} B_{a\alpha}^* - \frac{1}{2} j_{a\alpha b} C^{a\alpha} C^{b\alpha}) + E_{a\alpha b} (\epsilon^{ab} B_{a\alpha}^* - \frac{1}{2} j_{a\alpha b} C^{a\alpha} C^{b\alpha}) + \bar{E}_{\alpha a} (\epsilon^{ab} B_{a\alpha}^* - \frac{1}{2} j_{a\alpha b} C^{a\alpha} C^{b\alpha}) + \frac{1}{4} \epsilon_{cd} j_{a\alpha b}^i j_{a\alpha c}^j C^{\alpha a} C^{b\alpha} C^{c\alpha} C^{d\alpha} \quad (4.11)
\]

which is a proper solution of the classical master equations being regular in the neighborhood of the stationary point. The proof that expression (4.11) is the most general solution of the classical master equations depending only linearly on the antifields is given in Appendix A.

(B) Gauging of the proper solution

In order to lift the degeneracy of \( W_m^{(0)} \) we still have to introduce a gauge. The corresponding extended action \( W_{m,ext}^{(0)} \) is obtained from the canonical transformation Eq. (3.13) according to the choice Eq. (3.19). In the case of linear dependence on the antifields this transformation reduces considerably. Namely, choosing a minimal gauge, i.e., a gauge which depends only on the independent dynamical fields \( Q^i \) and \( C^{a\alpha} \), the extended action \( W_{m,ext}^{(0)} \) is obtained through the following construction

\[
W_{m,ext}^{(0)} = W_m^{(0)} + \left( \frac{1}{2} \epsilon_{ab} s_m^a s_m^b + m^2 \right) F \quad \text{with} \quad F = \frac{1}{2} (g_{ij} Q^i Q^j + \xi g_{a\beta} \epsilon_{ab} C^{a\alpha} C^{b\alpha}), \quad (4.12)
\]

\( \xi \) being an arbitrary gauge parameter. Again, the gauge-fixing functional \( F \) is assumed to be a \( Sp(2) \)–scalar, \( d_A F = 0 \). The gauge fixing terms in (4.12) extend the action \( W_m^{(0)} \) according to

\[
W_{m,ext}^{(0)} = W_m^{(0)} + g_{ij} (Q^i R_{a_i}^\alpha (A) B_{a\alpha}^* + \frac{1}{2} \epsilon_{ab} R_{a_i}^i (A) C^{a\alpha} R_{b}^j (A + Q) C^{b\alpha} + \frac{1}{2} m^2 Q^i Q^j) + \xi g_{a\beta} (B_{a\alpha}^* B_{b\beta}^* - \frac{1}{2} \epsilon_{cd} \epsilon_{e\gamma} (f_{a\alpha b}^e C^{e\alpha} C^{c\alpha})(f_{b\beta d}^b C^{b\beta} C^{d\beta}) + m^2 \epsilon_{ab} C^{a\alpha} C^{b\alpha}), \quad (4.13)
\]
where use has been made of the symmetry properties of the structure constants and the relations $g_{ki}R^i_{\alpha,j} = -g_{ji}R^i_{\alpha,k}$ and $g^{ik}R^j_{\alpha,i} = -g^{ij}R^k_{\alpha,i}$. Note, that this background gauge is nonlinear due the occurrence of quartic (anti)ghosts terms. By construction, $W^{(0)}_{\text{m,ext}}$ is both (anti)BRST– and $Sp(2)$–invariant,

$$s^a_m W^{(0)}_{\text{m,ext}} = 0, \quad d_A W^{(0)}_{\text{m,ext}} = 0.$$  

Hence, it provides also a solution of the Eqs. (4.1) and, in addition, it satisfies the constraints implied by the Eqs. (1.7) and (1.8).

Furthermore, it can be verified that $W^{(0)}_{\text{m,ext}}$, in accordance with (1.4) – (1.6), is invariant under both background gauge (type–I) as well as quantum gauge (type–II and type–III) transformations (except for the mass term of $Q^i$):

\begin{align}
\text{type–I} : & \quad \delta W^{(0)}_{\text{m,ext}} = 0, \quad (4.14) \\
& \delta Q^i = R^i_{\alpha,j} Q^j \xi^\alpha, \quad \delta A^i = R^i_{\alpha}(A) \xi^\alpha, \quad \delta \bar{Q}^i = -R^i_{\alpha,i} \bar{Q}^i \xi^\alpha, \quad \ldots, \\
\text{type–II} : & \quad \delta W^{(0)}_{\text{m,ext}} = m^2 g_{ij} R^i_{\alpha}(A) Q^j \xi^\alpha, \quad (4.15) \\
& \delta Q^i = R^i_{\alpha}(A + Q) \xi^\alpha, \quad \delta A^i = 0, \quad \delta \bar{Q}^i = g_{ij} R^i_{\alpha} (A + \bar{Q}) \xi^\alpha, \quad \ldots, \\
\text{type–III} : & \quad \delta W^{(0)}_{\text{m,ext}} = m^2 g_{ij} R^i_{\alpha}(0) Q^j \xi^\alpha, \quad (4.16) \\
& \delta Q^i = R^i_{\alpha}(Q) \xi^\alpha, \quad \delta A^i = R^i_{\alpha,j} A^j \xi^\alpha, \quad \delta \bar{Q}^i = g_{ij} R^i_{\alpha} (Q) \xi^\alpha, \quad \ldots;
\end{align}

here the ellipses \ldots indicate the transformations of all the other (anti)fields which transform according to the adjoint representation:

$$\delta B^\alpha = f^\alpha_{\beta\gamma} B^\beta \xi^\gamma, \quad \delta C^{\alpha b} = f^\alpha_{\beta\gamma} C^{\beta b} \xi^\gamma,$$

$$\delta F_{ab} = -f^\beta_{\alpha\gamma} F_{\beta b} \xi^\gamma, \quad \delta \bar{B}_\alpha = -f^\beta_{\alpha\gamma} \bar{B}_{\beta \gamma} \xi^\gamma, \quad \delta \bar{C}_{ab} = -f^\beta_{\alpha\gamma} \bar{C}_{\beta b} \xi^\gamma, \quad (4.17)$$

$$\delta Q^*_{ia} = -R^j_{\alpha,i} Q^*_j \xi^\alpha, \quad \delta B^*_{aa} = -f^\beta_{\alpha\gamma} B^*_{\beta a} \xi^\gamma, \quad \delta C^{* a}_{aab} = -f^\beta_{\alpha\gamma} C^{* a}_{\beta ab} \xi^\gamma.$$  

Let us mention, that the symmetries (4.13) and (4.16) do not hold for the effective action $S^{(0)}_{m,\text{eff}}$ but only for the extended action $W^{(0)}_{\text{m,ext}}$.

5 General solution of the master equations

So far, we have ignored the important question whether the action (4.13), which was supposed to be linear with respect to the antifields, is the general solution of the classical master equations (1.7) subjected to the conditions (1.7) and (1.8). Unfortunately, this is not the case because $W^{(0)}_m$ is not stable against small perturbations. This may be traced back to the fact that the fields $Q^i$, $C^{\alpha a}$ and the related antifields $\bar{Q}^j$, $\bar{E}_{\alpha a}$ have the same quantum numbers, respectively, and therefore mix under renormalization. More precisely, without violating the $osp(1,2)$–superalgebra, the (anti)BRST– and $Sp(2)$–transformations (4.3) may be altered with $Q^i$, $C^{\alpha a}$ and $B^\alpha = \frac{1}{2} \epsilon_{ba} s^a_m C^{\alpha b}$ being replaced by

$$\tilde{Q}^i \equiv Q^i + \bar{Q}^j Q^j, \quad (5.1)$$

$$\tilde{C}^{\alpha a} \equiv C^{\alpha a} + \bar{\rho} \xi^{-1} g^{\alpha \beta} \epsilon^{ab} E_{\beta b} \quad \text{and} \quad \tilde{B}^\alpha \equiv B^\alpha + \frac{1}{2} \bar{\rho} \xi^{-1} g^{\alpha \beta} \epsilon^{ab} E_{\beta ab},$$
respectively, where the following abbreviations are introduced:

\[ \tilde{\sigma} \equiv 1 - \sigma, \quad \tilde{\rho} \equiv 1 - \rho, \]

\( \sigma \) and \( \rho \) being independent dimensionless parameters; \( \xi^{-1} \) has been introduced for later convenience where it will be identified with the (inverse) gauge parameter (before introducing a gauge we could use \( \xi = 1 \)). Now, in order to ensure stability of the action \( W_m^{(0)} \) we introduce the antifields in a \textit{nonlinear} way by terms which depend on \( \sigma \) and \( \rho \). Of course, if the action depends nonlinearly on the antifields these quantities lose their interpretation as sources of the (anti)BRST transforms of the fields.

(A) General solution of the classical master equations prior to gauge fixing

Let us ignore the procedure of generalized canonical transformations as introduced in Section 3 and simply change the solution \( W_m^{(0)} \) in an analogous way as we have done for fixing the gauge, namely by adding the following \textit{nonlinear} terms

\[ X_m = W_m^{(0)} + \left( \frac{1}{2} \epsilon_{ab} S_m^b s_m^a + m^2 \right) G \quad \text{with} \quad G = \frac{1}{2} \left( \tilde{\sigma} g^{ij} Q_i Q_j + \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \tilde{E}_{\alpha a} \tilde{E}_{\beta b} \right). \]

Carrying out the replacements (5.1) in the proper solution (4.11) and making use of (4.5) and (4.10) then for the resulting functional, \( \tilde{X}_m \), one gets

\[ \tilde{X}_m = S_{\text{cl}}(A + \tilde{Q}) - H_{\alpha a} (\tilde{C}^\alpha \tilde{a} - \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \tilde{E}_{\beta b}) \]

\[ + \frac{1}{2} \tilde{\sigma} g^{ij} (\epsilon^{ab} Q_i^* Q_j^* - m^2 \tilde{Q}_i \tilde{Q}_j) - \frac{1}{2} \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \left( \frac{1}{2} \epsilon^{cd} E_{\alpha ac} E_{\beta bd} + m^2 \tilde{E}_{\alpha a} \tilde{E}_{\beta b} \right) \]

\[ - Q_i^* R^i_\alpha (A + \tilde{Q}) \tilde{C}_i^{\alpha a} - \tilde{Q}_i (R^i_\alpha (A + \tilde{Q}) \tilde{B}^{a} + \frac{1}{2} \epsilon_{ab} R^i_{\alpha j} R^j_{\beta} (A + \tilde{Q}) \tilde{C}^{\beta b} \tilde{C}_i^{\alpha a}) \]

\[ + E_{\alpha ab} (\epsilon^{ab} \tilde{B}^{a} - \frac{1}{2} \epsilon^{\alpha cd} F^{cd}_{\beta} \tilde{C}^\beta \tilde{C}^{\gamma a}) + E_{\alpha a} \left( f_{\beta}^{\alpha} \tilde{B}^{\beta} \tilde{C}^{\gamma a} + \frac{1}{2} \epsilon_{cd} f_{\eta}^{\alpha} \tilde{C}^{\gamma a} \tilde{C}^{\delta c} \tilde{C}^{\beta d} \right). \]

By construction \( \tilde{X}_m \) is both (anti)BRST- and \( Sp(2) \)-invariant,

\[ \tilde{s}_m^a \tilde{X}_m = 0, \quad d_A \tilde{X}_m = 0, \]

where the action of \( \tilde{s}_m^a \) on the fields is defined by carrying out in (4.4) the replacements (3.1). But, unfortunately, owing to its nonlinear dependence on the antifields it does not solve the master equations (1.1); instead it holds

\[ \frac{1}{2} (\tilde{X}_m, \tilde{X}_m)^a - V_m^a \tilde{X}_m = \tilde{\rho} \xi^{-1} g_{\alpha \beta} \delta \tilde{X}_m \delta \tilde{X}_m, \quad \frac{1}{2} \{ \tilde{X}_m, \tilde{X}_m \}_A - V_A \tilde{X}_m = 0. \]

The reason for this failure is that we did not really apply the correct transformation law, Eq. (3.15). However, the symmetry breaking term on the right hand side may be compensated by the following construction

\[ S_m^{(0)} = \tilde{X}_m + \frac{1}{3} \tilde{\rho} \xi^{-1} g_{\alpha \beta} \delta \tilde{X}_m \delta \tilde{X}_m. \]

Of course, by using the more sophisticated transformation law, Eq. (3.15), with the choice Eq. (3.16) for the generating functional \( G \), we would be led directly to expression (5.2) (for a detailed proof see Appendix B).

(B) Determination of the extended action \( S_{m,\text{ext}}^{(0)} \)

Now, we are left with the problem to introduce a gauge. Again, we choose the \textit{minimal}
solution (5.3) may be multiplicatively redefined by corresponding:

\[ s^{(0)}_{m, \text{ext}} = \tilde{Y}_m + \frac{1}{4} (1 - \tilde{\rho})^{-1} \tilde{\rho} \xi^{-1} g^{\alpha \beta} \delta \tilde{Y}_m \delta \tilde{Y}_m, \]

where the difficulty consists in determining the factor \((1 - \tilde{\rho})^{-1} \tilde{\rho}\) in front of the second term. Here, \(\tilde{Y}_m\) is obtained from the functional

\[ Y_m = X_m + \left( \frac{1}{2} \epsilon_{a b} s^{b}_m s^{a}_m + m^2 \right) F \quad \text{with} \quad F = \frac{1}{2} (g_{i j} Q^i Q^j + \xi g_{a \beta} \epsilon_{a b} C^{a a} C^{b b}) \]

by performing the above replacements (5.1). As is quite obvious the gauge fixing terms are already known from \(W^{(0)}_{m, \text{ext}}\), Eq. (1.12), so that, putting all terms together, \(\tilde{Y}_m\) becomes

\[ \tilde{Y}_m = S_\alpha (A + \bar{Q}) - H_{a a} (\bar{C}^{a a} - \tilde{\rho} \xi^{-1} g^{\alpha \beta} \epsilon_{a b} E_{b b}) + \frac{1}{2} g^{i j} (\epsilon^{a b} Q^i a Q^j b - m^2 \bar{Q}^i \bar{Q}^j) - \tilde{\rho} \xi^{-1} g^{\alpha \beta} \epsilon_{a b} (\frac{1}{2} \epsilon^{c d} E_{a a} \bar{E}_{b b} + m^2 \bar{E}_{a a} \bar{E}_{b b}) + g_{i j} (\tilde{Q}^i R^i_{\alpha} (A) \bar{B}^{\alpha} + \frac{1}{2} \epsilon_{a b} R^i_{\alpha} (A) \tilde{\bar{C}}^{a b} \tilde{R}_j^b (A + \bar{Q}) \bar{C}^{b \beta} + \frac{1}{2} m^2 \tilde{Q}^i \tilde{Q}^j) + \frac{\xi g_{a \beta}}{2} (\bar{B}^a \bar{B}^\beta - \frac{1}{2} \epsilon_{a b} F_{\gamma \rho} \bar{C}^{a \gamma} \bar{C}^{b \rho} (f_{a b}^{-1} \bar{C}^{a b} \bar{C}^{c d} + m^2 \epsilon_{a b} \bar{C}^{a a} \bar{C}^{b b})) - Q^i a R^i_{\alpha} (A + \bar{Q}) \bar{C}^{a a} - \tilde{Q}^i (R^i_{\alpha} (A + \bar{Q}) \bar{B}^a + \frac{1}{2} \epsilon_{a b} R^i_{\alpha} \tilde{R}_j^b (A + \bar{Q}) \bar{C}^{b \beta} \bar{C}^{a a}) + E^{a b} (\epsilon^{a b} \bar{B}^a \bar{B}^\beta - \frac{1}{2} F_{\gamma \rho} \bar{C}^{a \gamma} \bar{C}^{b \rho} + \bar{E}_{a a} (f_{a b}^{-1} \bar{B}^a \bar{B}^\beta \bar{C}^{a a} + \frac{1}{2} \epsilon_{a b} F_{\gamma \rho} f_{a b}^{-1} \bar{C}^{a \gamma} \bar{C}^{b \rho} \bar{C}^{c d} \bar{C}^{d b})). \]

In order to facilitate a check of Eq. (5.3) and for later convenience we also write down the first and second derivative of \(\tilde{Y}_m\) with respect to the auxiliary field \(B^\alpha\):

\[ \frac{\delta^2 \tilde{Y}_m}{\delta B^\alpha \delta B^\beta} = 2 g_{a \beta} \xi, \]

\[ \frac{\delta \tilde{Y}_m}{\delta B^\alpha} = 2 \xi g_{a \beta} \bar{B}^\beta + R^i_{\alpha}(A) (g_{i j} \tilde{Q}^i \tilde{Q}^j - Q^i) - R^i_{\alpha} \tilde{Q}^i \tilde{Q}^j + f_{\gamma \rho}^{a b} \bar{C}^{a \gamma} \bar{C}^{b \rho}, \]

As has been shown in Appendices A and B, \(S^{(0)}_{m, \text{ext}}\) is the general solution of the classical master equations. By a straightforward, but lengthy calculation it can be verified also that \(S^{(0)}_{m, \text{ext}}\) obeys the master equations (4.14) subjected to the conditions (4.7) and (4.8). As usual, it serves as the tree approximation \(\Gamma^{(0)}_m\) of the generating functional of the one–particle–irreducible vertex functions of the theory. In addition, it may be shown that \(S^{(0)}_{m, \text{ext}}\) obeys the properly generalized symmetries (1.14) – (1.16).

After all, the fields and antifields together with the independent parameters of the solution (5.3) may be multiplicatively redefined by corresponding \(z\)-factors,

\[ z_g, \ z_\sigma, \ z_\rho, \ z_\xi, \ z_m = z_Q^{-1} z_{\bar{Q}}^{-1}, \]

\[ z_Q, \ z_{\bar{Q}}, \ z_{Q^*} = z_m^{-1} z_{\bar{Q}}^{-1}, \ z_F = z_m z_B^{-1}, \]

\[ z_B, \ z_B = z_m^{-2} z_B^{-1}, \ z_B^* = z_m^{-1} z_B^{-1}, \ z_C = z_m^{-1} z_B^{-1}, \ z_C = z_m^{-2} z_B^{-1}, \]

Here, \(z_g\) is a factor by which the classical action can be multiplied, corresponding to a redefinition of the gauge coupling constant \(g\), and \(z_\sigma, z_\rho, z_\xi\) and \(z_m\) are normalization
factors related to the parameters $\sigma$, $\rho$, $\xi$ and the mass $m$, respectively. Therefore, the general solution of the Eqs. (4.1) contains, in total, seven independent $z$-factors. Let us emphasize that this is not in contradiction to Ref. [16] where, reanalysing the renormalization of massive gauge theories due to Curci and Ferrari in Delbourgo–Jarvis gauge [17], only five independent $z$-factors have been found. The reason is, that in Ref. [16] neither antiBRST$^-$ nor $Sp(2)$-invariance has been required thus avoiding the $z$-factors $z_\sigma$ and $z_\rho$.

Let us now consider the $B^\alpha$-dependence of the general solution $S_{m, ext}^{(0)}$. From Eqs. (5.3) and (5.5), (5.6) it follows

$$\rho \frac{\delta L}{\delta B^\alpha} S_{m, ext}^{(0)} = \frac{\delta L}{\delta B^\alpha} \tilde{S}_m,$$

which, by making use of (5.1), may be expressed by

$$\rho \left( \frac{\delta L}{\delta B^\alpha} S_{m, ext}^{(0)} + \epsilon^{cd} E^*_e d e d \right) = 2 \xi g_{\alpha \beta} B^\beta + R^i_\alpha (A)(g_{ij} Q^i - \sigma \tilde{Q}_j) - R_{\alpha, j} \tilde{Q}_i Q^j + f^\beta_\alpha \tilde{E}_{\beta b} C^{\alpha b}.$$  

(5.8)

This is just the equation of motion of the auxiliary field $B^\alpha$. As we will show in the next Section the equations of motion for the (anti)ghost fields $C^{\alpha b}$ are direct consequences of Eq. (5.8) and the master equations.

Now, if the equation of motion (5.8) is required the $z$-factors $z_\tilde{Q}$ as well as $z_\xi$, $z_\sigma$ and $z_\rho$ are fixed according to

$$z_\tilde{Q} = 1, \quad z_\xi = z_Q z_B^{-1}, \quad z_\sigma = z_Q, \quad z_\rho = z_Q z_B,$$  

(5.9)

and the general solution depends only on the three independent $z$-factors $z_Q$, $z_Q$ and $z_B$, which are to be fixed by suitable normalization conditions. Obviously, $\tilde{Q}_i$ is not subjected to any renormalization; this property turns out to be essential for the study of the dependence of the Green’s functions on the background field $A^i$ (see Section 7 below).

(C) Generalization to higher orders

Now the solution (5.3), including the $z$-factors, has to be generalized to any order of perturbation theory such that

$$S_{m, ext} = \sum_{n=0}^{\infty} \hbar^n S_{m, ext}^{(n)},$$

fulfills the quantum master equations (2.14) and (2.13) as well as obeys the type–I and the (broken) type–II and type–III symmetry. Let us emphasize that $S_{m, ext}$ is a local functional in the fields and the antifields. At lowest order $S_{m, ext}$ coincides with the solution $S_{m, ext}^{(0)}$, Eq. (5.3). Furthermore, in order to ensure that $S_{m, ext}$ depends on $B_\alpha$ and $B^{*}_{ab}$ only through the combinations (4.9) the constraints (4.7) and (4.8) will be required to be valid at any order of $\hbar$; due their linearity these requirements can be realized for any renormalized action $S_{m, ext}$, e.g., by the help of the renormalized quantum action principles [11]. In the same manner, also the validity of the equation of motion for the auxiliary field $B^\alpha$ could be required.

In the following, however, we are mainly interested in the generating functionals of the Green’s functions, $Z_m$, and the 1PI vertex functions, $\Gamma_m$. However, the vertex functional

$$\Gamma_m = \sum_{n=0}^{\infty} \hbar^n \Gamma_m^{(n)},$$

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is a nonlocal solution of the classical master equations (compare Ward identities (6.3) and (6.4) below) which at leading order coincides with $S_{m,\text{ext}}^{(0)}$.

### 6 Ward identities and equations of motion

Now we shall derive the Ward identities, being a consequence of the symmetry properties of the (renormalized) extended action $S_{m,\text{ext}}(A|\phi, \phi^*, \bar{\phi}, \eta)$ which are due to the $osp(1,2)$–superalgebra observed by the generators $\Delta_a^m$ and $\Delta_A$. Moreover, in the case of background gauges, besides of type–I invariance, additional (broken) Ward identities arise which are due to the type–II and type–III symmetry.

Let us introduce sources $j_I = (J_I, K_\alpha, L_{\alpha b})$ for the fields $\phi^I = (Q^i, B^\alpha, C_{\alpha b})$. Then the extended generating functional of Green’s functions $Z_m(A|j, \phi^*, \bar{\phi}, \eta)$ is defined as

$$Z_m(A|j, \phi^*, \bar{\phi}, \eta) = \int d\phi \exp\{(i/\hbar)(S_{m,\text{ext}}(A|\phi, \phi^*, \bar{\phi}, \eta) + j_I \phi^I)\}. \quad (6.1)$$

Multiplying Eqs. (2.14) and (2.15) by $\exp\{(i/\hbar)j_I \phi^I\}$, integrating them over $\phi^I$,

$$\int d\phi \exp\{(i/\hbar)j_I \phi^I\} \Delta_a^m \exp\{(i/\hbar)S_{m,\text{ext}}\} = 0,$$

$$\int d\phi \exp\{(i/\hbar)j_I \phi^I\} \Delta_A \exp\{(i/\hbar)S_{m,\text{ext}}\} = 0,$$

and assuming, after integrating by parts, that the integrated expressions vanish, one can rewrite the resulting equalities by the help of the definition (6.1) as

$$\left\{ J_I \frac{\delta}{\delta Q^i} + K_\alpha \frac{\delta}{\delta B^\alpha} + L_{\alpha b} \frac{\delta}{\delta C_{\alpha b}} + V^a_m \right\} Z_m = 0, \quad \left\{ (\sigma_A)_b^a L_{\alpha a} \frac{\delta}{\delta F_{ab}} + V_A \right\} Z_m = 0,$$

which are just the Ward identities for the generating functional of Green’s functions due to the $osp(1,2)$–symmetry of the theory.

Introducing, as usual, the 1PI vertex functional, $\Gamma_m(A|\phi, \phi^*, \bar{\phi}, \eta)$, according to

$$\Gamma_m = -i\hbar \ln Z_m - j_I \phi^I \quad \text{with} \quad \phi^I = -i\hbar \frac{\delta}{\delta j_I} \ln Z_m, \quad (6.2)$$

we obtain

$$S^a_m(\Gamma_m) \equiv \frac{1}{2}(\Gamma_m, \Gamma_m)^a - V^a_m \Gamma_m = 0, \quad (6.3)$$

$$D_A(\Gamma_m) \equiv \frac{1}{2}(\Gamma_m, \Gamma_m)_A - V_A \Gamma_m = 0. \quad (6.4)$$

For Yang-Mills theories the Eqs. (6.3) are the Slavnov-Taylor identities of the extended BRST–symmetry. Furthermore, the Eqs. (6.4) for $A = 0$ express the ghost number conservation and, in Yang-Mills theories, for $A = \pm$ they are the Delduc-Sorella identities [18] of the $Sp(2)$–symmetry.

In order to ensure type–I invariance $\Gamma_m$ will be required to fulfil the Kluberg-Stern–Zuber identity [8] which is governed by

$$K_\alpha \Gamma_m = 0 \quad (6.5)$$

with $$K_\alpha \equiv R^a_i(A) \frac{\delta}{\delta A^i} + R^i_{\alpha j} Q^j \frac{\delta L}{\delta Q^i} - R^i_{\alpha a} \bar{Q}_j \frac{\delta}{\delta \bar{Q}_i} + \ldots.$$
Here, the ellipses \ldots indicate the contributions of all the other fields and antifields which transform homogeneously (see Eqs. (1.17)) and which for the following considerations are irrelevant (since they are identical for analogous expressions occurring below). This identity expresses the fact that \( A' \) may be gauged arbitrarily if \( \Gamma_m \) depends gauge-covariant on it. Let us notice, that this identity is defined only for \( A^i \neq 0 \). Hence, to fix the \( A^i \)-dependence of \( \Gamma_m \) by means of an operator equation, we still need another identity which is valid also for \( A^i = 0 \). Such an identity appears as a \textit{consistency condition}:

If the \( B^a \)-dependence of \( \Gamma_m \) is restricted by imposing the same equation of motion as it holds for \( S_m^{(0)} \), then, by virtue of the Ward identities (6.3), that requirement leads, first of all, to the (anti)ghost equations of motion. Next, taking into account the Ward identities (6.3) from the (anti)ghost equations of motion we obtain the so-called \textit{Lee identity}, which is nothing else but the Ward identity of type-II symmetry.

\textit{Derivation of the Lee identity}

To begin with, let us require that the \( B^a \)-dependence of \( \Gamma_m \) is governed by Eq. (6.8) with \( S_m^{(0)} \) replaced by \( \Gamma_m \),

\[
\rho \left( \frac{\delta L}{\delta B^a} \Gamma_m + \epsilon^{cd} E_{adc}^* \right) = 2 \xi g_{a\beta} B^\beta + R^i_o(A)(g_{ij} Q^j - \sigma \tilde{Q}_j) - R^i_{a\beta} \tilde{Q}_i Q^j + f^{ij}_{\beta\gamma} \bar{E}_{\beta\gamma} C^\gamma. \quad (6.6)
\]

Since this equation contains only terms being linear with respect to \( Q^j \), \( B^a \) and \( C^{ab} \), its validity can be simply established by the help of the renormalized quantum action principles [11]. Of course, this requirement is equivalent to the related one that the \( z \)-factors \( \tilde{Q}_j, z_\xi, z_\sigma \) and \( z_\rho \), being formal power series in \( \hbar \), are determined through Eqs. (5.9) at any order of perturbation theory.

Let us now apply \( \delta L/\delta B^a \) on the identities (6.3) and (6.4) we find as necessary conditions for \( \Gamma_m \):

\[
0 = \frac{\delta L}{\delta B^a} S^a_m(\Gamma_m) = \tilde{Q}_m^a \frac{\delta L}{\delta B^a} \Gamma_m, \quad 0 = \frac{\delta L}{\delta B^a} D^a_A(\Gamma_m) = \tilde{Q}_A \frac{\delta L}{\delta B^a} \Gamma_m, \quad (6.7)
\]

where the differential operators

\[
\tilde{Q}_m^a X \equiv (\Gamma_m, X)^a - V^a_m X, \quad \tilde{Q}_A X \equiv \{\Gamma_m, X\}_A - V_A X,
\]

satisfy the \textit{osp}(1,2)-superalgebra:

\[
[\tilde{Q}_A, \tilde{Q}_B] = -\epsilon_{ABC} \tilde{Q}_C^*, \quad [\tilde{Q}_A, \tilde{Q}_m] = -\tilde{Q}_m^a (\sigma_A)_b^a, \quad \{\tilde{Q}_m^a, \tilde{Q}_m^b\} = m^2 (\sigma^a)^{ab} \tilde{Q}_A.
\]

Let note, that these operators are related to the untilded ones introduced by Eqs. (3.2); however, they are first-order differential operators. The first of the conditions (6.7) can be rewritten as

\[
0 = \tilde{Q}_m^a \frac{\delta L}{\delta B^a} \Gamma_m \equiv \tilde{Q}_m^a \left( \frac{\delta L}{\delta B^a} \Gamma_m + \epsilon^{cd} E_{adc}^* \right) - \epsilon^{ab} \left( \frac{\delta L}{\delta C_{abc}} \Gamma_m - H_{abc} \right); \quad (6.9)
\]

the second condition (6.7) gives no further constraint. Thus, if the \( B^a \)-dependence of \( \Gamma_m \) is fixed by imposing the equation of motion (6.6) its \( C^{ab} \)-dependence is restricted by

\[
\frac{\delta L}{\delta C_{abc}} \Gamma_m - H_{abc} = -\epsilon_{ab} \tilde{Q}_m^a \left( \frac{\delta L}{\delta B^a} \Gamma_m + \epsilon^{cd} E_{adc}^* \right). \quad (6.10)
\]
In fact, this is nothing else then the \textit{(anti)ghost equation of motion}. This may be shown by using the explicit expression (6.6) for \( \delta L \Gamma_m / \delta B^a \):

\[
\rho \left( \frac{\delta L}{\delta C_{ab}} \Gamma_m - H_{ab} \right) = -2\xi_{ab}g_{a\beta} \frac{\delta}{\delta B_{\beta a}} \Gamma_m
\]

\[
- R^i_A(A) \left( \epsilon_{ab}g_{ij} \frac{\delta}{\delta Q^*_{ja}} \Gamma_m - \sigma Q^*_{jb} \right) + \frac{1}{2} f_{\alpha \beta \gamma} \epsilon_{ab}g_{c\delta} \frac{\delta L}{\delta B^\gamma} \Gamma_m
\]

\[
+ R_{\alpha,j}^i \left( \epsilon_{ab} \bar{Q}^*_i \frac{\delta}{\delta Q^*_{ja}} \Gamma_m + Q^*_{ib} Q^j \right) - f_{\beta \alpha} \left( \epsilon_{ab} \tilde{E}_{\beta c} \frac{\delta}{\delta C^*_{\gamma c}} \Gamma_m + E^*_{\beta bc} C^*_{\gamma c} \right).
\]

Proceeding in the same manner as before we apply \( \delta L / \delta C^{\alpha a} \) on the identities (6.3) and (6.4). This yields another set of consistency conditions:

\[
0 = \frac{\delta L}{\delta C^{\alpha a}} S^a_m(\Gamma_m) = -\tilde{Q}^a_m \frac{\delta L}{\delta C^{\alpha a}} \Gamma_m, \quad 0 = \frac{\delta L}{\delta C^{\alpha a}} D_A(\Gamma_m) = \tilde{Q}^a_A \frac{\delta L}{\delta C^{\alpha a}} \Gamma_m.
\]

The first of these conditions, analogous to Eq. (6.9), will be rewritten as

\[
0 = -\frac{1}{2} \tilde{Q}^a_m \frac{\delta L}{\delta C^{\alpha a}} \Gamma_m \equiv -\frac{1}{2} \tilde{Q}^a_m \left( \frac{\delta L}{\delta C^{\alpha a}} \Gamma_m \right) - m^2 \left( \frac{\delta L}{\delta B^a} \Gamma_m + \epsilon^{cd} E^*_a \right),
\]

while the other condition again gives no further constraint. From this, together with (6.10), we get another identity,

\[
0 = -(\frac{1}{2} \epsilon_{ab} \tilde{Q}^b_m \tilde{Q}^a_m + m^2) \rho \left( \frac{\delta L}{\delta B^a} \Gamma_m + \epsilon^{cd} E^*_a \right) \equiv L_\alpha \Gamma_m - m^2 g_{ij} R^i_A(A) Q^j.
\]

Inserting for \( \delta L \Gamma_m / \delta B^a \) the expression (6.6) we obtain the explicit form of the Lee identity:

\[
L_\alpha \Gamma_m = m^2 g_{ij} R^i_A(A) Q^j
\]

with \( L_\alpha \equiv \sigma R^i_A(A) + \sigma^{-1} Q^i Q^j \frac{\delta L}{\delta Q^j} + g_{ij} R^i_A(A + \bar{Q}) \frac{\delta}{\delta Q^j} + \ldots \),

where \( R_\alpha^i(Q) \equiv R_\alpha^i(0) + R_{\alpha,j}^i Q^j \) and \( R_\alpha^i(\bar{Q}) \equiv R_\alpha^i(0) - g_{ij} R_{\alpha,j}^k \bar{Q}^k \). It can be checked that the Lee operator \( L_\alpha \) is independent of \( \rho \) and \( \xi \). The term \( m^2 g_{ij} R^i_A(A) Q^j \) stems from the fact that the type–II invariance is broken by that mass term (compare Eq. (4.13)).

In writing down the identity (6.13) one still has to require that \( \Gamma_m \) obeys constraints implied by the equations:

\[
\frac{1}{2} \epsilon^{cd} \frac{\delta}{\delta C^{\alpha d}} \Gamma_m = B^\alpha, \quad \frac{\delta}{\delta B^\alpha} \Gamma_m = m^2 B^\alpha,
\]

\[
\frac{\delta}{\delta F^{aa}} \Gamma_m = -C^{aa}, \quad \frac{\delta}{\delta B^{aa}} \Gamma_m + \frac{1}{2} \frac{\delta}{\delta C^{aa}} \Gamma_m = m^2 C^{aa},
\]

which generalize the corresponding requirements for \( \Gamma_m^{(0)} \) to any orders. Again, their validity can be simply established by using the quantum action principles [11].
7 Background dependence of Green’s functions

Now, having completely characterized the symmetry properties of \( \Gamma_m \) let us enquire into its \( A^i \)-dependence. This is achieved by comparing the Kluberg-Stern–Zuber identity (6.5) with the Lee identity (6.13) which leads to the differential equation

\[
\frac{\delta}{\delta A^i} \Gamma_m = \left( \sigma \frac{\delta L}{\delta Q^i} + g_{ij} \frac{\delta}{\delta Q_j} \right) \Gamma_m - m^2 g_{ij} Q^j \tag{7.1}
\]
describing the \( A^i \)-dependence of \( \Gamma_m(A|\phi, \phi_a^*, \bar{\phi}, \eta) \). Its integration yields

\[
\Gamma_m(A|\phi, \phi_a^*, \bar{\phi}, \eta) = \exp \left\{ A^i \left( \sigma \frac{\delta L}{\delta Q^i} + g_{ij} \frac{\delta}{\delta Q_j} \right) \right\} \Gamma_m(0|\phi, \phi_a^*, \bar{\phi}, \eta) \tag{7.2}
\]

\[- m^2 g_{ij} \left( A^i Q^j + \frac{1}{2} \sigma A^i A^j \right),
\]
i.e., the \( A^i \)-dependence is completely determined by the \( Q^i \)- and \( \bar{Q}^i \)-dependence of the corresponding functional \( \Gamma_m(0|\phi, \phi_a^*, \bar{\phi}, \eta) \) for \( A^i = 0 \).

By making use of Eq. (7.1) the Lee identity (6.13) can be cast into the form

\[
W_\alpha \Gamma_m = m^2 g_{ij} R^i_\alpha(0) Q^j \tag{7.3}
\]

with

\[
W_\alpha \equiv \sigma R^i_\alpha (\sigma^{-1} Q) \frac{\delta L}{\delta Q^i} + g_{ij} R^i_\alpha (\bar{Q}) \frac{\delta}{\delta Q^j} + \ldots,
\]

which is just the Ward identity of type–III symmetry. Integrating (7.3) over space-time we recover the Ward identity of the rigid symmetry,

\[
\int d^4x \; R_\alpha \Gamma_m = 0 \tag{7.4}
\]

with

\[
R_\alpha \equiv R^i_{\alpha,j} A^j \frac{\delta}{\delta A^i} + R^i_{\alpha,j} Q^j \frac{\delta L}{\delta Q^i} - R^i_{\alpha,i} \bar{Q}_j \frac{\delta}{\delta \bar{Q}^i} + \ldots.
\]

For the relation between \( Z_m(A|J, \phi_a^*, \bar{\phi}, \eta) \) and \( Z_m(0|J, \phi_a^*, \bar{\phi}, \eta) \) corresponding to Eq. (7.1), by virtue of (6.2), we obtain the differential equation

\[
\frac{\delta}{\delta A^i} Z_m = g_{ij} \left( \frac{\delta}{\delta Q^j} - m^2 \frac{\delta}{\delta J^j} \right) Z_m - (i/h) \sigma J_i Z_m.
\]

Its solution is simply given by

\[
Z_m(A|J, \phi_a^*, \bar{\phi}, \eta) = \exp \left\{ - (i/h) \sigma (J_i A^i - \frac{1}{2} m^2 g_{ij} A^i A^j) \right\} \times \exp \left\{ g_{ij} A^i \left( \frac{\delta}{\delta Q^j} - m^2 \frac{\delta}{\delta J^j} \right) \right\} Z_m(0|J, \phi_a^*, \bar{\phi}, \eta). \tag{7.5}
\]

Together with Eq. (7.2) this is the main result of this Section.

**Background dependence of physical Green’s functions**

In order to apply this exact relation we consider physical Green’s functions, i.e., on–shall Green’s functions \( Z_{\text{phys}}(A|J) := Z(A|J, \{ K, L_b; \phi_a^*, \bar{\phi}, \eta \} = 0) \) of transverse gauge fields, \( g_{ij} R^i_\alpha(A)(\delta/\delta J^j)Z_{\text{phys}}(A|J) = 0 \), for \( m = 0 \) and choosing the Landau gauge \( \xi = 0 \). In
Here, we can put $A^i = 0$ and put $-$ with the exception of $J_i$ and $\bar{Q}_i$ -- any of the sources in $Z_m(0|\bar{J}, \phi_\alpha, \bar{\phi}, \eta)$ equal to zero. Then, the problem consists in determining the explicit $\bar{Q}_i$–dependence of $Z_m(0|J, \bar{Q})$.

To begin with, let us write down the rigid Ward identity (7.4) and the Lee identity (7.3) for the particular case under consideration,

$$\int d^4 x \, R_\alpha Z_m(0|J, \bar{Q}) = 0, \quad R_\alpha = - R_{\alpha, j}^i \left( J_i \frac{\delta}{\delta J_j} + \bar{Q}_i \frac{\delta}{\delta \bar{Q}_j} \right), \quad (7.6)$$

$$\left( L_\alpha - m^2 g_{ij} R_\alpha^i (0) \frac{\delta}{\delta J_j} \right) Z_m(0|J, \bar{Q}) = 0, \quad L_\alpha = R_\alpha^i (0) \left( - \frac{i}{\hbar} \sigma \frac{\delta}{\delta \bar{Q}_i} + g_{ij} \frac{\delta}{\delta \bar{Q}_j} \right) + R_\alpha,$$

where the operator $L_\alpha$ differs from $R_\alpha$ only by divergence terms, i.e., terms proportional to $R_{\alpha, j}^i (0)$ which vanish after integration over space–time.

In the limit $\bar{Q}_i = 0$ the following general ansatz for $L_\alpha$ has to be taken when applied to $Z_m(0|J, 0)$:

$$\int d^4 x \, R_\alpha Z_m(0|J, 0) = 0, \quad R_\alpha = - R_{\alpha, j}^i \frac{\delta}{\delta J_j}, \quad (7.7)$$

$$\left( L_\alpha - m^2 g_{ij} R_\alpha^i (0) \frac{\delta}{\delta J_j} \right) Z_m(0|J, 0) = 0, \quad L_\alpha = R_\alpha^i (0) \left( - \frac{i}{\hbar} \sigma \frac{\delta}{\delta \bar{Q}_i} + g_{ij} \frac{\delta}{\delta \bar{Q}_j} \right) + R_\alpha,$$

with $z^{-1}_Q$ being the $z$–factor of $J_i$. The necessity for making that ansatz results from the fact that in the case $\bar{Q}_i = 0$ the identities (6.13) and (7.3) are undefined. Comparing (7.6) and (7.7) the $Q_i$–dependence of $Z_m(0|J, \bar{Q})$ is obtained:

$$Z_m(0|J, \bar{Q}) = \exp \left\{ (i/\hbar) \left( \sigma - z^{-1}_Q \right) g^{ij} \left( J_i \bar{Q}_j + \frac{1}{2} m^2 \bar{Q}_i \bar{Q}_j \right) \right\} \exp \left\{ m^2 \bar{Q}_i \frac{\delta}{\delta \bar{Q}_i} \right\} Z_m(0|J). \quad (7.8)$$

This relation immediately leads to the conclusion that, in the absence of sources $J_i$ and $\bar{Q}_i$, the following equality holds $\delta Z(0)/\delta \bar{Q}_i \equiv (\delta Z_m(0|J, \bar{Q})/\delta \bar{Q}_i)|_{J=\bar{Q}=0} = 0$. This property, which for the analogous case of the background field $A^i$ instead of $\bar{Q}_i$ already has been noted in the first of Refs. [13], turns out to be essential for the study of the $A^i$–dependence.

Now, we are to consider the case $A^i \neq 0$. Inserting the expression (7.8) for $Z_m(0|J, \bar{Q})$ into the relation (see Eq. (7.3))

$$Z_m(A|J, \bar{Q}) = \exp \left\{ - \frac{i}{\hbar} \sigma \left( J_i A^i - \frac{1}{2} m^2 g_{ij} A^i A^j \right) \right\} \exp \left\{ g_{ij} A^i \left( \frac{\delta}{\delta \bar{Q}_j} - m^2 \frac{\delta}{\delta J_j} \right) \right\} Z_m(0|J, \bar{Q}),$$

we get

$$Z_m(A|J, \bar{Q}) = \exp \left\{ (i/\hbar) \left( \sigma - z^{-1}_Q \right) g^{ij} \left( J_i \bar{Q}_j + \frac{1}{2} m^2 \bar{Q}_i \bar{Q}_j \right) \right\} \times$$

$$\exp \left\{ - (i/\hbar) z^{-1}_Q \left( J_i A^i - \frac{1}{2} m^2 g_{ij} A^i A^j \right) \right\} \exp \left\{ - m^2 g_{ij} A^i \frac{\delta}{\delta J_j} \right\} Z_m(0|J, 0).$$

Here, we can put $\bar{Q}_i = 0$ and obtain the relation between $Z_m(A|J)$ and $Z_m(0|J)$ we are looking for, namely

$$Z_m(A|J) = \exp \left\{ - (i/\hbar) z^{-1}_Q \left( J_i A^i - \frac{1}{2} m^2 g_{ij} A^i A^j \right) \right\} \exp \left\{ - m^2 g_{ij} A^i \frac{\delta}{\delta J_j} \right\} Z_m(0|J). \quad (7.9)$$
Notice, that for $m = 0$ and choosing the Landau gauge $\xi = 0$ this relation coincides with a corresponding result obtained by Rouet using a quite different method \cite{14}:

$$Z_{phys}(A|J) = \exp\left\{ -\frac{i}{\hbar}z_Q^{-1}J_i A^i \right\} Z_{phys}(0|J).$$

This relation states that physical Green's functions in a background gauge are obtained from the Green's functions without background field by a mere translation.

Finally, let us point at the similarity between the relations (7.8) and (7.9), which indicates that $\bar{Q}_i$ for $A^i = 0$ plays the role of a background field (because of the above mentioned property $\delta Z(0)/\delta \bar{Q}_i = 0$ in the absence of $J_i$ and $\bar{Q}_i$). That is the reason why generally one should require not only background gauge (type–I) invariance but also quantum gauge (type–II and type–III) invariance: both symmetry requirements together ensure that neither $A^i$ nor $\bar{Q}_i$ are subjected to any renormalization.

8 Ultraviolet asymptotics

Now we are going to show that the ultraviolet asymptotics of $\Gamma_m$ is independent of the background field $A^i$. In order to obtain this result we derive the renormalization group equation and prove, making use of the basic relation (7.3), that the $\beta$–function and the anomalous dimensions in the presence of a background field $A^i$ agree with the corresponding ones for $A^i = 0$.

For a consistent treatment of the ultraviolet divergences emerging from the Feynman graphs the BPHZL subtraction scheme \cite{19} will be employed which allows the application of the renormalized quantum action principles \cite{11}. In this scheme the mass $m$ is replaced by $m_s = (1 - s)m$, where the parameter $s$ ($0 \leq s \leq 1$) is incorporated in order to avoid spurious infrared singularities which would occur by ultraviolet subtractions at $s = 1$; it interpolates between the massive ($s = 0$) and the massless ($s = 1$) case.

In order to fix $\Gamma_m$ completely we have to choose suitable normalization conditions for the independent $z$-factors $z_g$, $z_Q$ and $z_B$, being formal power series in $\hbar$ (not to be specified further). Denoting the (Euclidean) normalization point by $\mu^2$, the physical content of the theory must be independent on how $\mu^2$ is chosen. This requirement is expressed by the renormalization group equation which relates the action of $\nabla \equiv \mu^2 \partial / \partial \mu^2$ on $\Gamma_m$ to a corresponding change of the independent parameters of the theory.

As is well known $\nabla \Gamma_m$, by the quantum action principle, defines an insertion $\Delta$:

$$\nabla \Gamma_m = \Delta \cdot \Gamma_m = \Delta + O(\hbar \Delta). \quad (8.1)$$

Now, it is our task to expand this insertion $\Delta$ into a suitable basis of independent symmetric insertions. In general, an insertion $\Delta$ is called symmetric if the corresponding differential operator $\nabla$ satisfies a set of constraints which are related to the symmetry properties of $\Gamma_m$:

First, as $\Gamma_m$ is assumed to fulfil the Ward identities of both the (anti)BRST– and $Sp(2)$–symmetry,

$$S^a_m(\Gamma_m) = 0, \quad D_A(\Gamma_m) = 0,$$

the operator $\nabla$ is restricted to

$$[\bar{Q}_m^a, \nabla] \Gamma_m = 0, \quad [\bar{Q}_A, \nabla] \Gamma_m = 0, \quad (8.2)$$
where $\tilde{Q}_m^a$ and $\bar{Q}_A$ are defined by Eqs. (5.8). Second, because $\Gamma_m$ obeys both the equations of motion for the auxiliary fields (see Eq. (6.6)),

$$B_\alpha \Gamma_m = -\rho \epsilon^{cd} E_{ad}^* + 2\xi g_{a\beta} B^\beta + R^i_\alpha (A)(g_{ij} Q^i - \sigma \bar{Q}_j) - R^i_{\alpha j} \bar{Q}_i Q^j + f^\beta_{\alpha \gamma} E_{\beta \gamma} C^\gamma ;$$

$$B_\alpha = \rho \frac{\delta}{\delta B_\alpha} ,$$

and the (anti)ghost fields (see Eq. (6.11)):

$$C_{ab} \Gamma_m = \rho H_{ab} - \sigma R^i_{\alpha j} (A) Q^*_{jb} + R^i_{\alpha j} Q^*_{ab} Q^j - f^\beta_{\alpha \gamma} E_{\beta \gamma} C^\gamma ;$$

$$C_{ab} = \rho \frac{\delta}{\delta C_{ab}} + 2\xi \epsilon_{ab g_{\alpha \beta}} \frac{\delta}{\delta B^\beta} + \epsilon_{ab} (g_{ij} R^i_{\alpha j} (A) - R^i_{\alpha j} \bar{Q}_i) \frac{\delta}{\delta Q^*_{ia}} - \frac{1}{2} f^\gamma_{\alpha \beta} \epsilon_{ab} C^\gamma_{\alpha \beta} \frac{\delta}{\delta B^\gamma} + f^\beta_{\alpha \gamma} \epsilon_{ab} E_{\beta \gamma} \frac{\delta}{\delta C^\gamma} ,$$

$$\nabla$$

will be required further to satisfy the following relations:

$$[B_\alpha, \nabla] \Gamma_m = 0, \quad [C_{ab}, \nabla] \Gamma_m = 0; \quad (8.3)$$

finally, since $\Gamma_m$ obeys the Ward identities of type–I, type–II and type–III invariance (see Eqs. (5.7), (6.13) and (7.3)), respectively,

$$K_\alpha \Gamma_m = 0, \quad L_\alpha \Gamma_m = m^2 g_{ij} R^i_\alpha (A) Q^j, \quad W_\alpha \Gamma_m = m^2 g_{ij} R^i_\alpha (0) Q^j,$$

$$\nabla$$

should also satisfy the following relations:

$$[K_\alpha, \nabla] \Gamma_m = 0, \quad [L_\alpha, \nabla] \Gamma_m = 0, \quad [W_\alpha, \nabla] \Gamma_m = 0. \quad (8.4)$$

Since neither $\tilde{Q}_m^a$ nor $\bar{Q}_A$ nor $B_\alpha$, $C_{ab}$, $K_\alpha$, $L_\alpha$ and $W_\alpha$ depend on the normalization point $\mu$, the operator $\nabla$ obviously is symmetric. Hence, our task is to construct a basis for any symmetric differential operator in a form which permits a generalization to all orders of $\hbar$. This is most easily done by using the results of Section 5; actually one needs only the Eqs. (5.7) and (5.4) for the $z$-factors.

A basis of symmetric operators which fulfill the requirements (5.2), as can be deduced from Eqs. (5.7), consists of seven differential operators which are given by

$$g \frac{\partial}{\partial g}, \quad \xi \frac{\partial}{\partial \xi}, \quad \sigma \frac{\partial}{\partial \sigma}, \quad \rho \frac{\partial}{\partial \rho}$$

and the following counting operators:

$$N_Q = \bar{B}_\alpha \frac{\delta}{\delta B_\alpha} + Q^i \frac{\delta}{\delta Q^i} + \frac{1}{2} \left( B^\alpha_{\alpha a} \frac{\delta}{\delta B^*_{\alpha a}} - Q^*_{\alpha a} \frac{\delta}{\delta Q^*_{\alpha a}} \right)$$

$$+ \frac{1}{2} \left( C_{ab} \frac{\delta}{\delta C_{ab}} + \bar{C}_{ab} \frac{\delta}{\delta \bar{C}_{ab}} - F_{ab} \frac{\delta}{\delta F_{ab}} \right) - m^2 \frac{\partial}{\partial m^2} ,$$

$$N_Q = \bar{B}_\alpha \frac{\delta}{\delta B_\alpha} + \tilde{Q}_i \frac{\delta}{\delta \tilde{Q}_i} + \frac{1}{2} \left( B^\alpha_{\alpha a} \frac{\delta}{\delta B^*_{\alpha a}} + Q^*_{\alpha a} \frac{\delta}{\delta Q^*_{\alpha a}} \right)$$

$$+ \frac{1}{2} \left( C_{ab} \frac{\delta}{\delta C_{ab}} + \bar{C}_{ab} \frac{\delta}{\delta \bar{C}_{ab}} - F_{ab} \frac{\delta}{\delta F_{ab}} \right) - m^2 \frac{\partial}{\partial m^2} ,$$

$$N_B = B^\alpha \frac{\delta}{\delta B^\alpha} - \bar{B}_\alpha \frac{\delta}{\delta B_\alpha} - B_{\alpha a} \frac{\delta}{\delta B_{\alpha a}}$$

$$+ C_{ab} \frac{\delta}{\delta C_{ab}} - \bar{C}_{ab} \frac{\delta}{\delta \bar{C}_{ab}} - C^*_{ab} \frac{\delta}{\delta C^*_{ab}} - F_{ab} \frac{\delta}{\delta F_{ab}} .$$
Hence, any $osp(1, 2)$–symmetric differential operator $\nabla$ can be expanded with respect to this basis:

$$\nabla = \beta_g \frac{\partial}{\partial g} + \beta_\xi \frac{\partial}{\partial \xi} + \beta_\sigma \frac{\partial}{\partial \sigma} + \beta_\rho \frac{\partial}{\partial \rho} + \gamma_Q N_Q + \gamma_B N_B.$$

(8.5)

Obviously, these symmetric differential operators are independent of the background field.

Now, requiring that $\nabla$ satisfies also the constraints (8.3) and (8.4) leads, in accordance with (5.9), to the restrictions

$$\gamma_\bar{Q} = 0, \quad \beta_\xi = \gamma_Q - \gamma_B, \quad \beta_\sigma = \gamma_Q, \quad \beta_\rho = \gamma_Q + \gamma_B,$$

so that the expansion (8.5) reduces to

$$\nabla = \beta_g \frac{\partial}{\partial g} + \gamma_Q \tilde{N}_Q + \gamma_B \tilde{N}_B,$$

(8.6)

where $\tilde{N}_Q$ and $\tilde{N}_B$ are given by

$$\tilde{N}_Q \equiv N_Q + \xi \frac{\partial}{\partial \xi} + \sigma \frac{\partial}{\partial \sigma} + \rho \frac{\partial}{\partial \rho}, \quad \tilde{N}_B \equiv N_B - \xi \frac{\partial}{\partial \xi} + \rho \frac{\partial}{\partial \rho}.$$

Now, having defined to any order of $\hbar$ a basis of symmetric differential operators let us return to our starting point: Expanding $\Delta \cdot \Gamma_m$ on the right–hand side of Eq. (8.1) on the basis of symmetric insertions generated by the operator (8.6) the renormalization group equation in the background gauge reads

$$\mu^2 \frac{\partial}{\partial \mu^2} \Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta) = \left(\beta_g \frac{\partial}{\partial g} + \gamma_Q \tilde{N}_Q + \gamma_B \tilde{N}_B\right) \Gamma_m(A|\phi, \phi^*_a, \bar{\phi}, \eta),$$

(8.7)

where the $\beta$–function $\beta_g$ and the anomalous dimensions $\gamma_Q, \gamma_B$ start at first order in $\hbar$.

Next, by virtue of Eq. (7.2), from the previous relation (8.7) one obtains

$$\mu^2 \frac{\partial}{\partial \mu^2} \Gamma_m(0|\phi, \phi^*_a, \bar{\phi}, \eta) = \left(\beta_g \frac{\partial}{\partial g} + \gamma_Q \tilde{N}_Q + \gamma_B \tilde{N}_B\right) \Gamma_m(0|\phi, \phi^*_a, \bar{\phi}, \eta),$$

(8.8)

which is just the renormalization group equation for $A^i = 0$. Obviously, the coefficients $\beta_g$, $\gamma_Q$ and $\gamma_B$ in both renormalization group equations are the same. In addition, it should be emphasized that by choosing suitable normalization conditions for $A^i = 0$ the solution of the renormalization group equation (8.7) is uniquely determined by the corresponding solution of the renormalization group equation (8.8) because by virtue of Eq. (7.2) the normalization conditions for $A^i \neq 0$ are fixed as well. Thus, it is proven that the ultraviolet asymptotics of the vertex functions is in fact independent of the background field $A^i$ to all orders of perturbation theory.

9 Concluding remarks

Let us first state the essential results which have been obtained. Under the assumption of a linear quantum–background splitting it has been shown that for irreducible massive gauge theories of first rank with a generic background field the renormalized generating
functional $\Gamma_m(A)$ is invariant under background (type–I) as well as quantum (type–II and type–III) gauge transformations, thereby generalizing the results of Ref. [3] by exploiting the full non–minimal sector of the theory. As a consequence of this we were able to determine the $A^i$–dependence of $\Gamma_m(A)$ completely by these symmetry requirements. In order to determine the $A^i$–dependence of $\Gamma_m(A)$ explicitly one only has to determine the dependence of $\Gamma_m(0)$ on the gauge fields $Q^i$ and the associated antifields $\bar{Q}_i$. As an application we independently recovered an earlier result of Rouet [14]. Furthermore, it was proven that introducing a background gauge does not change the ultraviolet asymptotics of the theory.

In this paper only generic background configurations, i.e., the gauge zero modes, are taken into account. However, within the quite general frame we introduced here our results may be generalized. If the classical action is not only gauge invariant but has additional symmetries and if $A^i$ is a solution of the equation of motion for $Q^i$ depending on collective coordinates which break the additional symmetries, then similar conclusions can be drawn. In that case, according to the method of Gervais and Sakita [20], one first has to factor out the dependence of the generating functional $Z_m(A)$ on the collective coordinates of $A^i$. After that, our method can be applied in order to reduce the determination of the background dependence of $\Gamma_m(A)$ to quantities being independent of $A^i$, i.e., to the determination of the antifield dependence of $\Gamma_m(0)$. If the antifield dependence of $\Gamma_m$ has been determined once and for all the result is valid for any background configuration having the same dependence on the collective coordinates.

In the case of a nonlinear quantum–background splitting – which takes place, e.g., for $N = 1$ supersymmetric theories – the situation is much more involved, since then both the type–II and type–III symmetries are nonlinear ones. In this case the determination of the background dependence turns out to be very complicated.

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A Construction of the general solution of the classical master equations (4.11) with vanishing new ghost number

In this Appendix, we show that the expression (4.11) is the general solution of the classical master equations (4.1), with vanishing new ghost number, $ngh(W_m^{(0)}) = 0$, i.e. being linear in the antifields,

$$\frac{1}{2}(W_m^{(0)}, W_m^{(0)})^a - V_m^a W_m^{(0)} = 0, \quad \frac{1}{2}\{W_m^{(0)}, W_m^{(0)}\}_A - V_A W_m^{(0)} = 0,$$

(A.1)

and being subjected to the constraints (4.7), (4.8). For that reason we ascribe according to [2] to all quantities, including the background field $A^i$ and formally also the mass
parameter $m$, the following new ghost numbers:

$$\text{ngth}(Q^i, C^{ab}, B^\alpha) = (0, 1, 2), \quad \text{ngth}(A^i) = 0,$$

$$\text{ngth}(Q^i_{aa}, C_{aab}^{*}, B_{aa}^{*}) = (-1, -2, -3), \quad \text{ngth}(m) = 1,$$

$$\text{ngth}(\bar{Q}^i, \bar{C}_{ab}, \bar{B}_\alpha) = (-2, -3, -4), \quad \text{ngth}(F_{ab}) = -1.$$

Now, adopting the method of Ref. [2] the solution $W_m^{(0)}$ will be sought in the form of an terminating expansion

$$W_m^{(0)} = S_{cl}(A + Q) + \sum_{n=1} W_{m(n)}, \quad \text{ngth}(W_{m(n)}) = 0, \quad (A.2)$$

where $W_{m(n)}$ are polynomials of $n$th order in powers of the fields $B^\alpha$ and $C^{\alpha a}$. Furthermore, $W_{m(n)}$ is subjected to the above mentioned constraints (4.7) and (4.8) which now read

$$\frac{1}{2} \epsilon_{ab} \frac{\delta}{\delta C_{aba}^{*}} W_{m(n)} = B_{aa}^{*}, \quad \frac{\delta}{\delta F_{aa}} W_{m(n)} = -C_{aa}^{*},$$

$$\left( \frac{\delta}{\delta B_{aa}^{*}} + \frac{1}{2} \frac{\delta}{\delta C_{aa}^{*}} \right) W_{m(n)} = -m^2 \frac{\delta}{\delta F_{aa}} W_{m(n)}, \quad \frac{\delta}{\delta B_{\alpha}} W_{m(n)} = \frac{1}{2} m^2 \epsilon_{ab} \frac{\delta}{\delta C_{aba}^{*}} W_{m(n)};$$

thus, the polynomials $W_{m(n)}$ depend on $B_{aa}^{*}$ and $\bar{B}_\alpha$ only through the combinations (4.9),

$$\bar{E}_{aa} = \bar{C}_{aa} - \frac{1}{2} B_{aa}^{*}, \quad E_{aab}^{*} = C_{aab}^{*} - \frac{1}{2} m^2 \epsilon_{ab} \bar{B}_\alpha, \quad H_{aa} = F_{aa} - m^2 B_{aa}^{*}.$$

Let us consider the first approximation $W_{m(1)}$. The most general form of $W_{m(1)}$ which fulfills the above mentioned requirements is

$$W_{m(1)} = - (\bar{Q}^i \Lambda_{aa}^i - \epsilon^{ab} E_{aab}^{*}) B_{aa}^{*} - (Q_{aa}^{*} \Lambda_{aa}^{iab} + H_{aa}) C_{aa}^{*} + \frac{1}{2} Q_{ia}^{*} Q_{jb}^{*} \Lambda_{aa}^{iab} B_{aa}^{*},$$

where $\Lambda_{aa}^i$, $\Lambda_{aa}^{iab}$ and $\Lambda_{aa}^{iakb}$ are some unknown matrices depending on the fields $A^i$ and $Q^i$. Next, we require that $S_{cl}(A + Q) + W_{m(1)}$ fulfills the classical master equations to first order. This leads to the result that $\Lambda_{aa}^i = R_{aa}^i (A + Q)$ and $\Lambda_{aa}^{iab} = R_{aa}^i (A + Q) \delta_{ab}^{i}$ has to be identified with the gauge generators and that $\Lambda_{aa}^{iakb} = 0$ has to be put equal to zero. Thus, the first approximation is given by

$$W_{m(1)} = - (\bar{Q}^i R_{aa}^i (A + Q) - \epsilon^{ab} E_{aab}^{*}) B_{aa}^{*} - (Q_{ia}^{*} R_{aa}^i (A + Q) + H_{aa}) C_{aa}^{*}. \quad (A.3)$$

In order to explain how by iteration the $(n + 1)$th order is obtained from the $n$th order approximation let us introduce

$$W_{m[n]} \equiv S_{cl}(A + Q) + \sum_{k=1}^{n} W_{m(k)} \quad (A.4)$$

satisfying the $n$th order classical master equations

$$\frac{1}{2} \langle W_{m[n]}, W_{m[n]} \rangle_{(k)} - V_{m[n] a} W_{m(k)} = 0, \quad \frac{1}{2} \{ W_{m[n]}, W_{m[n]} \}_{A_{(k)}} - V_A W_{m(k)} = 0,$$
for \( k = 1, 2, \ldots, n \). Here, \( \{ \, , \}^{(k)}_A \) and \( \{ \, , \}_A \) denote the brackets of \( k \)th order in powers of the fields \( B^\alpha \) and \( C^{\alpha a} \). Furthermore, functionals \( Y^a_{m(n+1)} \) and \( Y_{A(n+1)} \) are constructed from \( W_{m(k)} \), \( k \leq n \), by the formulas

\[
Y^a_{m(n+1)} = -\frac{1}{2}(W_m[n], W_m[n])^a_{(n+1)}, \quad Y_{A(n+1)} = -\frac{1}{2}(W_m[n], W_m[n])_A^{(n+1)}. \tag{A.5}
\]

Then, the \( (n + 1) \)th approximation \( W_{m(n+1)} \) of the solution \( (A.2) \) is obtained by the help of the equations

\[
O^a_m W_{m(n+1)} = Y^a_m, \quad O_A W_{m(n+1)} = Y_{A(n+1)}, \quad n \geq 1. \tag{A.6}
\]

where the operators \( O^a_m \) and \( O_A \) are given by

\[
O^a_m = S_{\text{cl}}(A + Q), i \frac{\delta}{\delta Q_{ia}} - (Q_{ia} R^{i}_{\alpha}(A + Q) + H_{\alpha a}) \delta C^*_{\alpha ab} - \epsilon^{ab} B^\alpha \frac{\delta L}{\delta C_{ab}} - \frac{\delta (Q_{ia} R^{i}_{\alpha}(A + Q) - \epsilon^{bc} E_{\alpha bc})}{\delta B^\alpha} + m^2 C^{\alpha a} \frac{\delta L}{\delta B^\alpha} - V^a_m \tag{A.7}
\]

\[
O_A = -(\sigma_A)_{a}^b H_{ab} \frac{\delta}{\delta F_{\alpha a}} - (\sigma_A)^{b}_{a} C^{\alpha a} \frac{\delta L}{\delta C_{ab}} - V_A. \tag{A.8}
\]

By construction they obey the \( osp(1,2) \) superalgebra

\[
\{O^a_m, O^b_m\} = m^2 (\sigma^4)^{ab} O_A, \quad [O_A, O^a_m] = -O^b_m (\sigma_A)^{a}_{b}, \quad [O_A, O_B] = -\epsilon^{AB} C O_C. \]

The functionals \( Y^a_{m(n+1)} \) and \( Y_{A(n+1)} \) obey the equations

\[
O_A Y_B^{(n+1)} - O_B Y_A^{(n+1)} = -\epsilon^{AB} C Y_{C(n+1)}, \\
O_A Y^a_{m(n+1)} - O^a_m Y_{A(n+1)} = -Y^b_{m(n+1)} (\sigma_A)^{a}_{b}, \\
O^b_m Y^a_{m(n+1)} + O^a_m Y^b_{m(n+1)} = m^2 (\sigma^4)^{ab} Y_{A(n+1)},
\]

being the compatibility conditions for the Eqs. \( (A.6) \). Note, that in solving Eqs. \( (A.6) \) the \( (n + 1) \)th approximation \( W_{m(n+1)} \) is uniquely determined up to terms of the form \( \frac{1}{2} \epsilon_{abc} O^b_m O^a_m + m^2 \), with \( O_A X_{m(n+1)} = 0 \), which still could be added to \( W_{m(n+1)} \); this is a consequence of the relations \( O^c_m (\frac{1}{2} \epsilon_{abc} O^b_m O^a_m + m^2) = \frac{1}{2} m^2 (\sigma^4)^{cd} O^d_m O_A \) and \( [O_A, \frac{1}{2} \epsilon_{abc} O^b_m O^a_m + m^2] = 0 \). But, in the present case it turns out that \( X_{m(n+1)} = 0 \) for \( n \geq 1 \).

In order to obtain the higher order approximations, \( W_{m(n+1)}, n \geq 1 \), one first has to construct the functionals \( Y^a_{m(n+1)} \) and \( Y_{A(n+1)} \) according to the rules \( (A.4) \) which requires, by virtue of \( (A.4) \), the explicit knowledge of all lower order approximations \( W_{m(k)}, k = 1, 2, \ldots, n \). After that, one has to determine the solutions of the equations \( (A.6) \). Thereby, one has to employ all gauge structure relations, i.e., the algebra of the generators, \( R^i_{\alpha} R^j_{\beta} (A + Q) - R^j_{\beta} R^i_{\alpha} (A + Q) = -R^{i}_{\gamma} (A + Q) f^j_{\alpha \beta}, \) the Jacobi identity \( f^{\delta}_{\eta \alpha} f^{\eta}_{\beta \gamma} + \text{cyclic perm.} (\alpha, \beta, \gamma) = 0, \) and the antisymmetry of the structure constants \( f^\gamma_{\alpha \beta} = -f^\gamma_{\beta \alpha}. \)
Omitting any details of the cumbersome algebraic manipulations, one gets at second order
\[ \frac{1}{2} Q_{ab} R^i_{\alpha} (A + Q) f^a_{\beta \gamma} C^{\beta a} C^{\gamma b} - \frac{1}{2} \epsilon^{ab} Q_{ab} R^i_{\alpha} R^j_{\beta} (A + Q) \epsilon_{cd} C^{\alpha c} C^{\beta d} \]
\[ + \frac{1}{2} Q_i R^i_{\alpha} (A + Q) f^a_{\beta \gamma} B^{\beta} C^{\gamma a} - \frac{1}{2} Q_i (R_{\alpha j} R^j_{\beta} (A + Q) + R_{\beta j} R^j_{\alpha} (A + Q)) B^\alpha C^{\beta a}, \]
\[ Y_{A(2)} = 0, \]

which leads to
\[ W_{m(2)} = -\frac{1}{2} E^{\gamma b}_{\alpha \beta} f^i_{\alpha \beta} C^{\gamma b} + E_{\gamma a} f^i_{\alpha \beta} B^\alpha C^{\beta a} + \frac{1}{2} \epsilon^{ab} Q_i R^i_{\alpha} R^j_{\beta} (A + Q) C^{\beta b} C^{\alpha a}. \]

Iterating the preceding step at the next order one gets
\[ \frac{1}{4} \tilde{Q}_i (R_{\alpha j} R^j_{\beta} (A + Q) + R^j_{\beta j} R^j_{\alpha} (A + Q)) f^a_{\beta \gamma} C^{\alpha c} C^{\beta d} C^{\gamma a} \]
\[ + \frac{1}{2} R_{\alpha j} R^j_{\beta k} R^k_{\gamma} (A + Q) \epsilon_{cd} C^{\alpha c} C^{\beta d} C^{\gamma a} - \frac{1}{12} \epsilon^{ab} (C_{\alpha e} + C_{\beta e}) f^a_{\eta \gamma} f^\eta_{\gamma \beta} \epsilon_{cd} C^{\beta b} C^{\alpha b} C^{\gamma c} \]
\[ + \frac{1}{4} \tilde{E}_{ab} (f^a_{\eta \gamma} f^\eta_{\gamma \beta} + f^a_{\eta \beta} f^\eta_{\gamma \gamma}) C^{\alpha a} C^{\beta b} B^\gamma \]
\[ Y_{A(3)} = 0, \]

which yields
\[ W_{m(3)} = \frac{1}{6} \epsilon_{cd} \tilde{E}_{ab} f^a_{\eta \gamma} f^\eta_{\gamma \beta} C^{\alpha a} C^{\beta b} C^{\gamma c} C^{\beta d}, \]

for the third order approximation. The contributions to all the higher approximations \( W_{m(n)}, n \geq 4 \), vanish identically due to the gauge structure relations. Inserting into (A.2) for \( W_{m(n)} \) the explicit expressions (A.3), (A.9) and (A.10) we recover the solution (4.11). Moreover, as a consequence of the boundary conditions, which means that the expansion (A.2) must start with the classical action \( S_{cl}(A + Q) \), the solution (A.2) is also type–I, type–II and type–III invariant.

Finally, let us justify the ansatz (3.3). It has been established in Ref. [3] for \( Sp(2) \)–symmetric irreducible theories that the characteristic arbitrariness of a solution of the first master equation (A.1) for \( m = 0 \) of a fixed order in the fields \( C^{\alpha a} \) and \( B^a \) has the form:
\[ \delta W^{(0)} = \frac{1}{2} \epsilon_{ab} O^b O^a X, \]

where the nilpotent operators \( O^a \) are given by the \( m \)–independent part of (A.7), while \( X \) is an arbitrary \( Sp(2) \)–scalar. This proof can be extended without any problems to the \( osp(1,2) \)–symmetric case. The characteristic arbitrariness of the solutions of both the master equations (A.1) for \( m \neq 0 \) of a fixed order in the fields \( C^{\alpha a} \) and \( B^a \) has the following form:
\[ \delta W^{(0)} = (\frac{1}{2} \epsilon_{ab} O^b O^a + m^2) X \quad \text{with} \quad O_A X = 0. \]

If we consider now the variation \( \partial S_m(\zeta) / \partial \zeta \) of the functional exp\{\( i / \hbar \)\( S_m(\zeta) \)\} with \( S_m(\zeta) \) satisfying (3.1) it is obvious that the ansatz (3.3) is just the counterpart of Eq. (A.11).
B  Construction of the solutions (5.2) and (5.3) of the classical master equations

In this Appendix we show how the solutions $S_m^0$ and $S_{m, \text{ext}}^0$, Eqs. (5.2) and (5.3), are constructed from the solutions $W_m^{(0)}$ and $W_{m, \text{ext}}^{(0)}$, Eqs. (4.11) and (4.13), by means of the generalized canonical transformation (3.15). Thereby it will be proven that (5.3) is the most general (gauge fixed) solution of the classical master equations with arbitrary dependence on the antifields.

According to (3.15) and the commutative diagram (3.18) the solution $S_m(1) \equiv S_m^0$ is obtained from $S_m(0) \equiv W_m^{(0)}$ by the expansion

$$S_m^0 = \sum_{n=0}^{\infty} \langle n \rangle S_m \quad \text{s.t.} \quad S_m \equiv W_m^{(0)}, \quad (B.1)$$

$$(n + 1) \langle n+1 \rangle S_m = \frac{1}{2} \epsilon_{ab} \left\{ \sum_{k=0}^{n} \langle n \rangle S_m (S_m, G)^b - (S_m, V_m^a G)^b \right\}$$

$$+ V_m^b (S_m, G)^a + \delta_{n,0} V_m^b V_m^a G \} + \delta_{n,0} m^2 G, \quad n \geq 0,$$

with

$$G = \frac{1}{2} (\tilde{\sigma} g^{ij} \tilde{Q}_i \tilde{Q}_j + \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \tilde{E}_{\alpha a} \tilde{E}_{\beta b}),$$

where the antibrackets $( , )^a$ and the operators $V_m^a$ are defined in (2.23) and (2.20).

The evaluation of the second term in the expansion (B.1) yields

$$S_m^{(1)} = \left\{ \tilde{\sigma} g^{ij} \tilde{Q}_j \delta \delta Q_i + \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \left( \tilde{E}_{\beta b} \delta \delta C_{aa} + \frac{1}{2} \tilde{E}_{\beta b}^* \delta \delta B_{aa} \right) \right\} W_m^{(0)} \quad (B.2)$$

$$+ \tilde{\sigma} X - \tilde{\rho} \xi^{-1} (Y + g^{\alpha\beta} \epsilon^{ab} H_{\alpha a} E_{\beta b}) + \frac{1}{4} \tilde{\rho} \xi^{-1} g^{\alpha\beta} \frac{\delta W_m^{(0)}}{\delta B^a} \frac{\delta W_m^{(0)}}{\delta B^b},$$

where the quantities $X$ and $Y$ are given by

$$X \equiv \frac{1}{2} g^{ij} (\epsilon^{ab} Q_{ia} Q_{jb} - m^2 \tilde{Q}_i \tilde{Q}_j), \quad Y \equiv \epsilon^{\alpha\beta} g^{ab} \left( \frac{1}{2} \epsilon^{cd} E_{\alpha ac}^* E_{\beta bd}^* + m^2 \tilde{E}_{\alpha a} \tilde{E}_{\beta b} \right).$$

For the other terms, after tedious but straightforward calculations, one obtains

$$S_m^{(2)} = \frac{1}{2!} \left\{ \tilde{\sigma} g^{ij} \tilde{Q}_j \delta \delta Q_i + \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \left( \tilde{E}_{\beta b} \delta \delta C_{aa} + \frac{1}{2} \tilde{E}_{\beta b}^* \delta \delta B_{aa} \right) \right\}^2 W_m^{(0)} \quad (B.3)$$

$$+ \frac{1}{2} \tilde{\rho} \xi^{-1} g^{\alpha\beta} \frac{\delta W_m^{(0)}}{\delta B^a} (\tilde{\sigma} X_{\beta} + \tilde{\rho} \xi^{-1} Y_{\beta}),$$

$$S_m^{(3)} = \frac{1}{3!} \left\{ \tilde{\sigma} g^{ij} \tilde{Q}_j \delta \delta Q_i + \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \left( \tilde{E}_{\beta b} \delta \delta C_{aa} + \frac{1}{2} \tilde{E}_{\beta b}^* \delta \delta B_{aa} \right) \right\}^3 W_m^{(0)} \quad (B.4)$$

$$+ \frac{1}{4} \tilde{\rho} \xi^{-1} g^{\alpha\beta} (\tilde{\sigma} X_{\alpha} + \tilde{\rho} \xi^{-1} Y_{\alpha}) (\tilde{\sigma} X_{\beta} + \tilde{\rho} \xi^{-1} Y_{\beta})$$

and

$$S_m^{(n)} = \frac{1}{n!} \left\{ \tilde{\sigma} g^{ij} \tilde{Q}_j \delta \delta Q_i + \tilde{\rho} \xi^{-1} g^{\alpha\beta} \epsilon^{ab} \left( \tilde{E}_{\beta b} \delta \delta C_{aa} + \frac{1}{2} \tilde{E}_{\beta b}^* \delta \delta B_{aa} \right) \right\}^n W_m^{(0)}, \quad (B.5)$$
for $n \geq 4$, with the abbreviations

$$X_\alpha \equiv -R^i_{x\alpha j} \bar{Q}_i g^{jk} \bar{Q}_k, \quad Y_\alpha \equiv f^\beta_{\alpha \gamma} \bar{E}_{\beta \alpha} g^{\gamma \delta} \bar{E}_{\delta \beta}.$$  \hfill (B.6)

Inserting into (B.1) for $S_m, n \geq 1$, the quantities (B.2) – (B.5) it is easily seen that the resulting expression for $S_m^{(0)}$ can be cast into the form (B.2). Notice, that adding to $W_m^{(0)}(A^i|i Q^i, C^{aa}, B^a)$ only the first terms of the quantities (B.2) – (B.5) gives the same functional $W_m^{(0)}(A^i | \bar{Q}^i, \bar{C}^{aa}, \bar{B}^a)$ but with $Q^i$, $C^{aa}$ and $B^a$ being replaced according to the relations (5.1).

The simplest way to construct the general solution $S_m^{(0, \text{ext})}$ consists in using the same generalized canonical transformation as in (B.1) and merely changing the boundary conditions. Indeed, according to (3.13) and (3.18) the solution $S_m(1) \equiv S_m^{(0)}$ is obtained from $S_m(0) \equiv W_m^{(0)}$ by the expansion

$$S_m^{(0, \text{ext})} = \sum_{n=0}^{\infty} (n+1) S_m^{(n)}, \quad S_m^{(n)} \equiv W_m^{(0)}.$$  \hfill (B.7)

For the second term in that expansion one gets

$$S_m^{(1)} = \left\{ \bar{\sigma} g^{ij} \bar{Q}_j \delta \bar{Q}^i + \bar{\rho} \xi^{-1} g^{\alpha \beta} \epsilon_{\alpha \beta} \left( \bar{E}_{\beta \alpha} \delta \bar{C}^{aa} + \frac{1}{2} F^*_{\beta \alpha} \delta \bar{B}^a \right) \right\} W_m^{(0)}$$

$$+ \bar{\sigma} X - \bar{\rho} \xi^{-1} (Y + g^{\alpha \beta} \epsilon_{\alpha \beta} H_{\alpha \alpha} \bar{E}_{\beta \beta}) + \frac{1}{4} \bar{\rho} \xi^{-1} g^{\alpha \beta} \delta W_m^{(0)} \frac{\delta W_m^{(0)}}{\delta B^a} \frac{\delta W_m^{(0)}}{\delta B^a},$$

with the same expression for $X$ and $Y$ as in (B.2). For the other terms one obtains

$$S_m^{(2)} = \frac{1}{2!} \left\{ \bar{\sigma} g^{ij} \bar{Q}_j \delta \bar{Q}^i + \bar{\rho} \xi^{-1} g^{\alpha \beta} \epsilon_{\alpha \beta} \left( \bar{E}_{\beta \alpha} \delta \bar{C}^{aa} + \frac{1}{2} F^*_{\beta \alpha} \delta \bar{B}^a \right) \right\}^2 W_m^{(0)}$$

$$+ \frac{1}{4} \bar{\rho} \xi^{-1} g^{\alpha \beta} \left\{ \rho \frac{\delta W_m^{(0)}}{\delta B^a} \frac{\delta W_m^{(0)}}{\delta B^a} + 2 \frac{\delta W_m^{(0)}}{\delta B^a} \frac{\delta W_m^{(0)}}{\delta B^a} (\bar{\sigma} \bar{X}_\beta + \bar{\rho} \xi^{-1} \bar{Y}_\beta) \right\},$$

and

$$S_m^{(n)} = \frac{1}{n!} \left\{ \bar{\sigma} g^{ij} \bar{Q}_j \delta \bar{Q}^i + \bar{\rho} \xi^{-1} g^{\alpha \beta} \epsilon_{\alpha \beta} \left( \bar{E}_{\beta \alpha} \delta \bar{C}^{aa} + \frac{1}{2} F^*_{\beta \alpha} \delta \bar{B}^a \right) \right\}^n W_m^{(0)}$$

$$+ \frac{1}{4} \bar{\rho}^{n-1} \xi^{-1} g^{\alpha \beta} \left\{ \rho^2 \frac{\delta W_m^{(0)}}{\delta B^a} \frac{\delta W_m^{(0)}}{\delta B^a} + 2 \rho \frac{\delta W_m^{(0)}}{\delta B^a} \frac{\delta W_m^{(0)}}{\delta B^a} (\bar{\sigma} \bar{X}_\beta + \bar{\rho} \xi^{-1} \bar{Y}_\beta) \right.$$}

$$+ (\bar{\sigma} \bar{X}_\alpha + \bar{\rho} \xi^{-1} \bar{Y}_\alpha)) (\bar{\sigma} \bar{X}_\beta + \bar{\rho} \xi^{-1} \bar{Y}_\beta) \right\},$$

for $n \geq 3$, where $\bar{X}_\alpha$ and $\bar{Y}_\alpha$ are obtained from the expressions $X_\alpha$ and $Y_\alpha$, Eqs. (B.6), according to

$$\bar{X}_\alpha \equiv X_\alpha + R^i_{\alpha j} (A) \bar{Q}_i, \quad \bar{Y}_\alpha \equiv Y_\alpha + \xi \epsilon_{\alpha \beta} F^*_{\alpha \beta \beta}.$$
Inserting into (B.7) for $S_m$, $n \geq 1$, the quantities (B.8) – (B.10) and using the equality $(1 - \tilde{\rho})^{-1} = \sum_{n=0}^{\infty} \tilde{\rho}^n$ one simply establishes that the resulting expression for $S_{m,\text{ext}}^{(0)}$ can be expressed in the form (5.3).

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