We present an exploratory study consisting in the formulation of a relativistic quantum mechanics to describe the few-nucleon system at low energy, starting from the quantum field theoretical chiral Lagrangian involving pions and nucleons. To this aim we construct a Bakamjian-Thomas mass operator and perform a truncation of the Fock space which respects at each stage the relativistic covariance. Such truncation is justified, at sufficiently low energy, in the framework of a systematic chiral expansion. As an illustration we discuss the bound state observables and low-energy phaseshifts of the nucleon-nucleon and pion-nucleon scattering at the leading order of our scheme.

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I. INTRODUCTION

The quantitative understanding of the dynamics of few-nucleon systems, has reached such a high degree of accuracy that the need of including relativistic effects has been put forward [1, 2, 3]. Starting from a non-relativistic setting, for a given set of particles, it is possible to put constraints on the mutual interactions in order to fulfill the requirements of relativity. This problem has been posed in full generality by Dirac in 1949 [4], and developed among others by Krajcik and Foldy [5]. The relativistic corrections to the potential (“drift potentials”) can take the form of multinucleon forces and are currently being investigated quantitatively [2]; in some cases they are being used to help in the resolution of persisting puzzles in the few-nucleon physics, like the neutron-deuteron $^{A_2}$ problem [6].

Since the pioneering work of Weinberg and collaborators [8, 9], the nuclear interaction potential can nowadays be thought of as derived from a (relativistic) quantum field theory based on the chiral symmetry, which is the low-energy effective theory of QCD (see Ref. [10] for a recent review). A systematic perturbative expansion in powers of $p/\Lambda_H$ (Chiral Perturbation Theory, ChPT), the ratio of typical momenta divided by the typical hadronic scale $\Lambda_H \sim 1$ GeV, is formally possible, whose convergence properties have to be checked case by case in actual calculations. For power counting purposes a non-relativistic reduction is usually performed (heavy baryon expansion, HBChPT). This happens at two levels: in the calculation of Feynman amplitudes and in the definition of the effective potential, formulated by Weinberg in the framework of old-fashioned (time-ordered) perturbation theory. While the first step is not strictly necessary (a relativistic covariant scheme for the calculation of the two-pion exchange $NN$ potential exists [11] which relies on the covariant version of baryon ChPT formulated by Becher and Leutwyler [12]), the second step seems unavoidable. Thus relativistic and chiral corrections get intertwined, despite the different character of the corresponding symmetries: Lorentz invariance is an exact symmetry of Nature, whereas chiral symmetry, although useful as organizing principle, is anyhow approximate. One might therefore want to treat relativity exactly. The question then arises whether it is possible to set up a framework in which relativity is built in from the beginning, and corrections come only from dynamics (in our case from higher chiral orders). To the best of our knowledge this is not done in current ChPT approaches to few-nucleon systems. By separating the two expansions, one would also be able to compare relativistic and chiral corrections, and check quantitatively the effectiveness of the commonly adopted combined chiral and non-relativistic expansion. Indeed such a test is only possible, in our opinion, by performing a complete relativistic calculation, within a given framework, and comparing to its non-relativistic limit. Moreover a fully relativistic setting would allow to describe particle production, contrary to non-relativistic quantum-mechanical treatments.

The approach we take in the present paper is therefore to describe the few-nucleon systems with a relativistic coupled-channel wave equation, with simple transformation properties under the Lorentz symmetry, so that the...
behaviour of e.g. a $N - N$ subsystem in relative motion with respect to other nucleons, as demanded by relativity, is easily accounted for. In this perspective we consider the proposal of Ref. [13] (see also Ref. [14] for a similar point of view) consisting in a Bakamjian-Thomas construction formulated in the point-form of relativistic dynamics. This proposal has already been implemented for bound state problems in the framework of chiral quark models [15]. Compared to the above references we focus on nucleons instead of quarks and adopt a systematic ChPT-inspired point of view, instead of relying on specific models. We apply the above mentioned framework to nucleons interacting with pions according to the dictates of chiral symmetry, using the vertices of the chiral Lagrangian at the lowest order, and restrict ourselves to the 1 nucleon sector ($\pi - N$ scattering) and to the 2 nucleon sector (both the bound and scattering states of $N - N$). By adjusting the four low-energy constants which appear at the leading order, we obtain a reasonable unified description of these systems at low energy. However, the emphasis in the present paper is not on the accuracy of the description (limited to a leading order treatment), but rather on the presentation of a relativistic formalism which can naturally extend beyond the threshold of pion production.

The plan of the paper is as follows: a brief review of the adopted framework is given in the next section, in which the mass operator acting on the truncated Fock space is constructed from a Lagrangian density; in Section III we give a power-counting argument to justify the truncation of the Fock space in the low-energy expansion in the case in which the interactions are restricted by chiral symmetry. Section IV contains the application to the 1-nucleon sector, which yields the nucleon mass renormalization and a Lippman-Schwinger type equation for the $\pi - N$ scattering. We adjust the low-energy constant appearing at this order to reproduce the $S$ wave scattering lengths; the pion-nucleon axial coupling $g_A$ is fixed by the peripheral $N N$ phaseshifts, obtaining a reasonable description of the phaseshifts at small laboratory momentum (less than 50 MeV). In Section V we consider the 2-nucleon sector. Two more vertices (contact interactions) arise at the lowest order and we adjust the corresponding coupling constants to reproduce the $S$-wave scattering lengths. The agreement with experimental phaseshifts is again reasonable, up to center-of-mass kinetic energies of 100 MeV.

II. MASS OPERATOR FROM A LAGRANGIAN DENSITY

A. Bakamjian-Thomas construction in the point-form

The requirements of relativity are given by the Poincaré commutation relations among the generators of the group, written in terms of the particle coordinates. This translates into specific constraints for the possible interactions to include in the generators. Dirac classified three different possibilities each one associated with a particular spacelike hypersurface left invariant by a subgroup of the Poincaré group: for the instant-form the hypersurface is the hyperplane $t = \text{const.}$, for the point-form it is the hyperboloid $t^2 - x^2 = \tau^2$, for the light-front it is the hyperplane $t + z = 0$. The generators associated with these hypersurfaces are said to be “kinematical”, and do not contain interactions. In the case of the point-form the Lorentz transformations are kinematical and are the same as in the free case. Only the four momentum $P^\mu$ contains interactions, and the requirements of relativity are simply written covariantly as

$$[P^\mu, P^\nu] = 0, \quad U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu_\nu P^\nu,$$

where the generators of the Lorentz transformation $\Lambda$ are unaffected by interactions. The Bakamjian-Thomas construction [16, 17] in the point-form consists in the definition, starting from the non-interacting Poincaré generators, of a mass operator $M_0 = \sqrt{P_0^2 - P_B^2}$ and a four-velocity operator $V^\mu$ such that $P_0^\mu = M_0 V^\mu$; one then adds the interactions only to the mass operator $M = M_0 + M_I$, and reconstructs the interacting four-momentum as $P^\mu = M V^\mu$. Poincaré commutation relations are then satisfied provided the interacting mass operator is a Lorentz scalar which commutes with the four-velocity $V^\mu$. It is therefore particularly convenient to consider the “velocity states” [15, 18]: these are linear combinations of multiparticle momentum states which are eigenstates of the four-velocity operator. They have the nice property that all the particles transform with the same Wigner rotation under a Lorentz transformation.$v = \sum_i p_i / \sum_i \omega_k,$ where $\omega_k$ is the relativistic energy of a free particle with three-momentum $k$. A velocity state is obtained from a multiparticle momentum state defined in its rest frame after a boost to overall velocity $v$ by means of $B_v(v)$,$|v, k_1, \sigma_1, ..., k_n, \sigma_n\rangle \equiv U_{B_v(v)}|k_1, \sigma_1, ..., k_n, \sigma_n\rangle.$
With this definition, using the Lorentz invariant normalization for the momentum states, the velocity states are normalized as follows,

\[
\langle v, k_1, \sigma_1, \ldots, k_n, \sigma_n | v', k'_1, \sigma'_1, \ldots, k'_n, \sigma'_n \rangle = (2\pi)^3 \hbar \prod_{i=1}^{n} \frac{2\omega_{k_i} \delta_{\sigma_i, \sigma'_i}}{(\sum_{i=1}^{n} \omega_{k_i})^3} v_0 \delta^3(v - v') \prod_{i=1}^{n-1} \delta^3(k_i - k'_i).
\] (4)

A Bakamjian-Thomas construction in the point-form is thus accomplished in practice by defining the mass operator to be diagonal in the four velocity \[\| \text{ [13]} \], which is conveniently expressed in terms of velocity states as

\[
\langle v', k'_i, \sigma'_i | M_I | v, k_i, \sigma_i \rangle = \langle v', k'_i, \sigma'_i | H(0) | v, k_i, \sigma_i \rangle v_0 (2\pi)^3 \delta^3(v' - v) \frac{f(m, m')}{\sqrt{m^3 m'^3}},
\] (5)

\( m \) and \( m' \) being the initial and final relativistic energies, \( m = \sum_i \omega_{k_i}, \ m' = \sum_i \omega_{k'_i}. \) In this equation \( H(x) \) is the density of the interaction Hamiltonian, which is a Lorentz scalar. Therefore, the above definition is frame-independent, in view of Eq. \[\| \text{ [8]} \], and in principle any rotationally-invariant combination of the momenta \( k_i \) and \( k'_i \) would be allowed to appear. \( H(x) \) will be taken as a sum of vertices constructed as local products of field operators. The structure function \( f \) is introduced in order to compensate for the neglect of the off-diagonal terms in the four velocity, and to regulate the integrals as well. From its definition the structure function \( f \) is dimensionless. From now on we will take for \( f \) a real symmetric function of its arguments, further specified as a Gaussian function centered around zero with cutoff \( \Lambda \). Compared to Ref. \[\| \text{ [12]} \] we have introduced in Eq. \[\| \text{ [5]} \] a different normalization for the matrix elements of the interacting mass operator, in order to properly match, in the case when \( v = v' \) and \( m = m' \), with the quantum field theoretical result \[\| \text{ [19]} \]

\[
P_{\mu}^{\text{int}} = \int d^4x \frac{\partial F(x)}{\partial x^\mu} \delta(F(x) - \tau^2) \mathcal{H}^{\text{int}}(x),
\] (7)

where in the point-form \( F(x) = x^2 \) and the factor \( m^3 \) appears as a Jacobian in passing from the overall momentum conserving \( \delta \) function to the velocity conserving \( \delta \) function.

**B. A simple toy-model**

As an illustration of the general setting, we start by examining the simple example of a scalar nucleon field \( \Psi \) interacting with a pion field \( \phi \), where the interactions are provided by a Hamiltonian density of the form \( H(x) = g \Psi^\dagger(x) \Psi(x) \phi(x) \). (Here and in the following all products of field operators are understood as normally ordered.) Creation of nucleon-antinucleon pairs is neglected and a truncation of the Fock space to a given maximum number of pions is considered from the beginning. In the 1-nucleon sector, truncating the states containing two or more pions, the mass operator takes the form

\[
M = \begin{pmatrix} m_N + \delta_1^{\text{ren}} & gK \\ gK^\dagger & D_{1+1} \end{pmatrix},
\] (8)

where \( m_N \) is the physical nucleon mass, and \( D_{1+1} \) is the relativistic 1-nucleon + 1-pion free particle energy. The counterterm \( \delta_1^{\text{ren}} \) is needed for the mass renormalization. Due to the form of \( H(x) \), the interactions show up as off-diagonal entries in the mass operator. The nucleon mass renormalization and pion-nucleon scattering are described as eigenvalue-eigenvector problems for this mass operator. For instance, for the eigenvalue \( m_N \), the physical nucleon mass, one finds an equation for the counterterm

\[
\delta_1^{\text{ren}} = g^2 K^\dagger (D_{1+1} - m_N)^{-1} K,
\] (9)
with \( D_{1+1} = \omega_k + \omega_k^\pi \), having defined

\[
\omega_k \equiv \sqrt{m_N^2 + k^2}, \quad \omega_k^\pi = \sqrt{M_2^2 + k^2},
\]

and the operator \( K \), which connects 2-particle to 1-particle states,

\[
\langle v', k, -k | gK | v, 0 \rangle = g \frac{f^{(1)}(m, m')}{\sqrt{m^3 m'^3}} \langle v', k, -k | \psi_{\psi}(0) \psi(0) | v, 0 \rangle \nu_0 (2\pi)^3 \delta^3(v - v') = g \frac{f^{(1)}(m_N, \omega_k + \omega_k^\pi)}{\sqrt{m_N^3 (\omega_k + \omega_k^\pi)^3}} \nu_0 (2\pi)^3 \delta^3(v - v').
\]

The superscript \(^{(1)}\) refers to the sector of the Fock space with baryon number 1. The mass operator commutes with the baryon number, and there is the freedom to choose a different structure function \( f \) for each sector of the Fock space. Taking the expectation value of the above equation between 1-nucleon states and inserting a complete set of velocity states in the subspace of 1-nucleon + 1-pion states one arrives at the nucleon mass renormalization due to the \textquotedblleft pion cloud\textquotedblright,

\[
\delta_{1}^{\text{ren}} = \frac{g^2}{2m_N} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_k \omega_k^\pi} \frac{|f^{(1)}(m_N, \omega_k + \omega_k^\pi)|^2}{\omega_k + \omega_k^\pi - m_N}.
\]

Eq. (12) determines the counterterm \( \delta_{1}^{\text{ren}} \) for each choice of the cutoff \( \Lambda \) and coupling constant \( g \), and corresponds diagrammatically to the process shown in Fig. 1.

In the 2-nucleon sector, an analogous equation describes the deuteron,

\[
(D_2 + \delta_{2}^{\text{ren}}) \phi_2^D + g^2 K^1 (m_D - D_{2+1})^{-1} K \phi_2^D = m_D \phi_2^D,
\]

where \( \phi_2^D \) is a state vector in the subspace of 2-nucleon states, and the operators \( D_2 \) and \( D_{2+1} \) are respectively the relativistic 2-nucleon and 2-nucleon + 1-pion energy. As in the 1-nucleon sector, a counterterm \( \delta_{2}^{\text{ren}} \) is introduced in the corresponding diagonal element of the mass operator, in order to properly renormalize the 2-particle states. The kernel \( K \) connects 2-particle to 3-particle states and has matrix elements between velocity states

\[
\langle v', k_1, k_2, -k_1 - k_2 | gK | v, q, -q \rangle = g \frac{f^{(2)}(m, m')}{\sqrt{m^3 m'^3}} \nu_0 (2\pi)^3 \delta^3(v - v') \times (2\pi)^3 \left[ 2\omega_k \delta^3(k_1 - q) + 2\omega_k \delta^3(k_1 + q) + 2\omega_k \delta^3(k_2 - q) + 2\omega_k \delta^3(k_2 + q) \right],
\]

with \( m = 2\omega_q \) and \( m' = \omega_{k_1} + \omega_{k_2} + \omega_{k_1 + k_2}^\pi \). For what concerns the covariance properties, the structure function \( f^{(2)} \) for the 2-nucleon sector can be chosen different from \( f^{(1)} \). However, as we will see in subsection II C, in order to have a consistent renormalization we have to choose \( f^{(1)} = f^{(2)} \). By left-multiplying Eq. (13) with the bra \( \langle v, k, -k | \) representing a 2-nucleon state with four velocity \( v \) and relative momentum (in the center-of-mass system) \( 2k \), one arrives, after insertion of a complete set of states in the subspace of 2-nucleon + 1-pion states, to an eigenvalue wave equation for the center-of-mass wave function \( \phi_2^D(k) = \langle v = (1, 0), k, -k | \phi_2^D \rangle \). Using Bose symmetry \( \phi_2^D(k) = \phi_2^D(-k) \), the bound state equation becomes

\[
(2\omega_k + \delta_{2}^{\text{ren}}(k)) \phi_2^D(k) + 2\omega_k A(k) \phi_2^D(k) + \int \frac{d^3 q}{(2\pi)^3} B(k, q) \phi_2^D(q) = m_D \phi_2^D(k),
\]

with

\[
A(k) = \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{g^2}{16\omega_k \omega_q \omega_{k+q}^\pi} \frac{|f^{(2)}(2\omega_k, \omega_k + \omega_q + \omega_{k+q}^\pi)|^2}{m_D - \omega_k - \omega_q - \omega_{k+q}^\pi} + q \leftrightarrow -q \right\}
\]
The equation to fulfill is therefore

\[ B(k, q) = \frac{g^2}{8\omega_{k+q}^2} \frac{f^{(2)}(2\omega_q, \omega_k + \omega_{k+q}) f^{(2)}(\omega_k + \omega_{k+q}, 2\omega_k)}{m_D - \omega_k - \omega_q - \omega_{k+q}^2} + q \leftrightarrow -q. \]  

(17)

The term proportional to \( A(k) \) represents a wave function renormalization of the two-nucleon state: it describes diagrams in which the nucleon lines are disconnected and dressed with pion loops, cfr. Fig. 2.

It is possible to resum such diagrams and obtain a Lippmann-Schwinger equation for a reduced (renormalized) amplitude containing only a connected kernel. Alternatively, one can choose the counterterm \( \delta_2^{\text{ren}} \) so as to cancel the disconnected kernel, \( \delta_2^{\text{ren}}(k) = -2\omega_k A(k) \).

Correspondingly, the \( NN \) scattering is described by the Lippmann-Schwinger equation,

\[ \psi_2 = \phi_2 + g^2 \left[ \sqrt{s} - D_2 \right]^{-1} K^\dagger \left[ \sqrt{s} - D_{2+1} \right]^{-1} K \left| \psi_2 \right| \text{conn}, \]

(18)

where \( \sqrt{s} = E + i\epsilon \) is the scattering energy, and \( \phi_2 \) is an eigenstate of the free mass operator in the two-particle subspace. The above equation can be equivalently written in terms of the scattering amplitude, \( T(q, k) \) defined as

\[ T(q, k) = g^2 \langle v, q, -q | K^\dagger \left[ \sqrt{s} - D_{2+1} \right]^{-1} K \right| \text{conn} |\psi_2\rangle_k, \]

(19)

where \( k \) denotes the incident three-momentum of the interacting state \( \psi_2 \). Inserting a complete set of velocity states, and omitting the three-delta over the velocities (such factors will always be present because they appear in every interaction vertex) the LS equation takes the form

\[ T(q, k) = V(q, k) + \int \omega_p \frac{d^3p}{(2\pi)^3} \sqrt{s} - 2\omega_p + i\epsilon, \]

(20)

where the potential,

\[ V(q, k) = g^2 \langle v, q, -q | K^\dagger \left[ \sqrt{s} - D_{2+1} \right]^{-1} K \right| \text{conn} |v, k, -k\rangle = B(q, k), \]

(21)

consists only of the connected kernel \( B \) in Eq. (17), with the substitution \( m_D \rightarrow \sqrt{s} \).

C. Consistency of the renormalization procedure

The renormalization of the 2-nucleon lines describing \( NN \) scattering, realized by the choice of the counterterm \( \delta_2^{\text{ren}}(k) = -2\omega_k A(k) \), and of the 1-nucleon line, Eq. (12), correspond to the same physical processes, as can be seen by comparing Figs. 1 and 2. Physical considerations would require that, when the two nucleons are far apart and at rest, their energies should be renormalized as their respective masses. This implies the condition

\[ \delta_2^{\text{ren}}(0) = 2\delta_1^{\text{ren}}, \]

(22)

which can be regarded as the manifestation of the cluster decomposition principle in the simple case of two particles. The equation to fulfill is therefore

\[ -2m_N \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{g^2}{8m_N^2 \omega_q \omega_{k+q}} \left| f^{(2)}(2m_N, m_N + \omega_q + \omega_{k+q}) \right|^2 \right\} = 2g^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{4\omega_q \omega_{k+q}} \left| f^{(1)}(m_N, \omega_q + \omega_{k+q}) \right|^2, \]

(23)
where we have replaced in Eq. \[16\] $m_D$ by $\sqrt{s} = 2m_N$, since we are considering the case of two widely separated nucleons at rest. It is seen that our choice of the structure function, $f^{(1)} = f^{(2)} = f$ depending on $m - m'$ as in Eq. \[19\], independently of the baryon number sector, satisfies the requirement of a consistent renormalization procedure. Notice that this would not happen had we chosen the original formulation of Ref. \[13\]: the crucial point was the inclusion of a different normalization for the matrix elements of the interacting mass operator, Eq. \[5\], which in turn was dictated by a proper matching to the quantum field theory. The cluster decomposition principle, satisfied by local quantum field theories, could in general be violated by a truncation of the full quantum field theory to a relativistic quantum mechanics. In view of the above consideration, we will from now on drop the superscripts and use the same structure function $f$ for all sectors of the Fock space.

### III. EFFECTIVE THEORY IMPLEMENTATION

Having described the general features of the construction of the interacting mass operator from a vertex Lagrangian, we now proceed to make full use of the constraints given by chiral symmetry. Most importantly, the Goldstone theorem requires that the coupling between pion and nucleons be of derivative type (suppressed at low energy), and this will in turn provide a power-counting justification for the truncation of the Fock space, since the creation of pions brings more and more powers of momentum. A lowest order interaction Lagrangian which respects chiral symmetry is

$$
\mathcal{L}_{\pi N} = -\frac{g_{\pi A}}{2F_\pi} \bar{\psi} \gamma^\mu \gamma_5 \partial_\mu \pi \psi.
$$

In this expression $\psi$ is to be understood as an isospin spinor and the pion field $\pi = \tau^a \pi^a$ as the Goldstone boson SU(2) matrix, representing the coordinates of the coset space of chiral symmetry breaking. In addition to the pion-nucleon-nucleon vertex, at lowest order one has to consider other vertices such as the Weinberg-Tomozawa or SU(2) matrix, representing the coordinates of the coset space of chiral symmetry breaking. In addition to the nucleon-nucleon contact interaction, which comes into two independent operators, but let us for the time being concentrate on the above vertex. It will contribute to the mass operator as off-diagonal matrix elements. For instance, in the two-nucleon sector,

$$
\begin{pmatrix}
\hat{D}_2 
K 

K 
D_{2+1} 

0 
K 
D_{2+2} 

. 
. 
...
\end{pmatrix} \begin{pmatrix}
\phi_2 
\phi_{2+1} 

\phi_{2+1} 
\phi_{2+2} 

. 
. 
...
\end{pmatrix} = \sqrt{s} \begin{pmatrix}
\phi_2 
\phi_{2+1} 

\phi_{2+1} 
\phi_{2+2} 

. 
. 
...
\end{pmatrix},
$$

where we have defined for ease of notation $\hat{D}_2 = D_2 + \delta_2^{\text{ren}}$. Here $K$ connects states containing $n$ and $n + 1$ pions and is defined as in Eq. \[5\] using $\mathcal{H}(x) = -\mathcal{L}_{\pi N}$. The multi-particle states $\phi_{2+n}$ are expanded in velocity states which are normalized as in Eq. \[4\]. Counting three-momenta as small parameters of order $O(p)$, the chiral power of the phase space element $d_{2+n} \phi$, such that

$$
1 = \int d_{2+n} \phi |\phi_{2+n}\rangle \langle \phi_{2+n}|,
$$

in the two-nucleon sector is

$$
d_{2+n} \phi \sim O(p^{2n+3}).
$$

The formal solution of the above eigenvalue problem gives an equation for the two-particle component of the state vector,

$$
\left[\sqrt{s} - \hat{D}_2 - K \frac{1}{\sqrt{s} - D_{2+1} - K \frac{1}{\sqrt{s} - D_{2+2} - ...} K}\right] \phi_2 = 0.
$$

One can Taylor-expand the denominator in the continued fraction \[28\], and compare the different operators

$$
\left(\sqrt{s} - \hat{D}_2\right) - K \frac{1}{\sqrt{s} - D_{2+1}} K - K \left[\frac{1}{\sqrt{s} - D_{2+1}}\right]^2 K \frac{1}{\sqrt{s} - D_{2+2}} K K...,\n$$

contracted between two-particle states. Since we are interested in kinematical configurations in which $\sqrt{s} - \hat{D}_2 \sim O(p)$, the first term goes like

$$
\langle 2 | \sqrt{s} - \hat{D}_2 | 2' \rangle \sim O(p^{-2});
$$
FIG. 3: Connected diagrams contributing to the $\pi N$ scattering at the leading order, corresponding to the axial coupling (upper panel) and to the Weinberg-Tomozawa vertex (lower panel).

the second term, inserting a complete set of three-particle states,

$$\langle 2| K \frac{1}{\sqrt{s - D_{2+1}}} K |2' \rangle \sim O(p^0),$$

(31)

where we have used the fact that $\langle 2| K |2 + 1 \rangle \sim O(p^{-2})$, while the third term, since $\langle 2 + 1| K |2 + 2 \rangle \sim O(p^{-4})$,

$$\langle 2| K \left[ \frac{1}{\sqrt{s - D_{2+1}}} \right]^2 K \frac{1}{\sqrt{s - D_{2+2}}} K |2' \rangle \sim O(p^2).$$

(32)

The hierarchy is

$$O(p^{-2}) + O(p^0) + O(p^2) + \ldots$$

(33)

Therefore the inclusion of 4-particle states ($2$-nucleons + $2$-pions) $\phi_{2+2}$ yields a contribution suppressed by two orders in the chiral counting compared to the one of 3-particle states ($2$-nucleons + $1$-pion). This observation is the basis to justify the truncation of the Fock space. It is possible to convince oneself that the same mechanism applies for the most general vertex respecting chiral symmetry. In this paper we stick to the first order of the low-energy expansion.

IV. ONE NUCLEON SECTOR

The vertices to be considered at the lowest order in the 1-nucleon sector for the study of $\pi N$ scattering come from the Lagrangian

$$\mathcal{L}_{\pi N} = -\frac{g_A}{2F_\pi} \bar{\psi} \gamma^\mu \gamma_5 \partial_\mu \pi \psi + \frac{i}{8F_\pi^2} \bar{\psi} \gamma^\mu [\pi, \partial_\mu \pi] \psi.$$  

(34)

These vertices are part of the leading chiral Lagrangian, expanded up to terms quadratic in the pion field. From the argument presented above, the neglected terms start to contribute at the next to leading order. Only the $\pi NN$ vertex contributes to the nucleon mass renormalization, which reads,

$$\delta_1^{\text{ren}} = \frac{3g_A^2}{4F_\pi^2} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{4\omega_k \omega_k^\pi} \frac{[\omega_k^2 + \omega_k + m_N]^2}{\omega_k + m_N} \frac{|f_A(m_N, \omega_k + \omega_k^\pi)|^2}{\omega_k^2 - m_N - \delta_1^{\text{ren}}}. $$

(35)

The subscript in $f$ denotes that the form factor is associated with the axial $\pi NN$ vertex of the interaction Lagrangian. As in HBChPT, the mass renormalization counts formally as $O(p^3)$.

For the description of $\pi N$ scattering, one needs to take into account the truncation up to $2$ pions, because the $\pi N$ states are connected through the leading vertex to pure $N$ states and to $\pi \pi N$ states. This corresponds to the fact that the scattering contains the direct nucleon pole and the crossed one, as shown in Fig. 3 where the Weinberg-Tomozawa vertex is also included.

Consider then the pion-nucleon scattering in the center of mass frame, with $\{q, r, i\}$ $\{\{k, s, j\}\}$ the initial (final) nucleon momentum, spin and isospin indices, and $\{-q, a\}$ $\{-k, b\}$ the initial (final) pion momentum and isospin index. The kernel is written as

$$B_{sji}^{\tau\tau\eta}(q, k) = \frac{g_A^2}{4F_\pi^2} \frac{f_A(\omega_q + \omega_q^\pi, m_N) f_A(m_N, \omega_k + \omega_k^\pi)}{2m_N \sqrt{(\omega_q + \omega_q^\pi)^3(\omega_k + \omega_k^\pi)^3}} \frac{(\tau^\alpha r^\beta)_{ij}}{\sqrt{s - m_N - \delta_1^{\text{ren}}}}.$$
Analogously to what was done in the simple model of the previous section, the disconnected contribution to the
operator.

where we have denoted by \( q^\pi \) and \( k^\pi \) the pion 4-momenta; the first two terms come from the direct and the crossed nucleon pole, while the third one from the Weinberg-Tomozawa vertex, with an associated structure function \( f_{WT} \).

The Dirac 4-spinors are normalized so that \( \sum_s u(k,s)\bar{u}(k,s) = \delta + m_N \). The isospin structures are proportional to \( \delta_{ab} \), ("isoscalar") that we will denote with a \( +^+ \) superscript, and to \( i\epsilon^{abc}r^c \), ("isovector") that we will denote with a \( -^- \) superscript. Analogously, the spin structures are proportional to \( \delta_{rs} \) and to \( q \times k \cdot \sigma_{rs} \). The kernel can thus be written in the following operatorial form,

\[
B(q,k) = \delta^{ab} \left[ g^+(q,k,\cos \theta) + i q \times k \cdot \sigma h^+(q,k,\cos \theta) \right] + i\epsilon^{abc} \left[ g^-(q,k,\cos \theta) + i q \times k \cdot \sigma h^-(q,k,\cos \theta) \right].
\]

Notice that at the same order there is also a disconnected kernel, corresponding to the diagram shown in Fig. 4.

Analogously to what was done in the simple model of the previous section, the disconnected contribution to the scattering kernel can be absorbed by a local counterterm \( \delta^{ren}_{\ell+1} \) in the corresponding diagonal entry of the mass operator.

Thus the corresponding Lippmann-Schwinger equation reads,

\[
t_{\ell\pm}(q,k) = b_{\ell\pm}(q,k) + \int \frac{dp}{(2\pi)^2} \frac{b_{\ell\pm}(q,p)t_{\ell\pm}(p,k)}{\sqrt{s} - m_p - \frac{p^2}{2m_p} + i\epsilon}. \tag{38}
\]

where

\[
b_{\ell\pm}(q,k) = \int_{-1}^{1} dx \left\{ y^\ell(q,k,x)P_\ell(x) + qk h^\ell(q,k,x) [P_{\ell\pm}(x) - xP_\ell(x)] \right\}, \tag{39}
\]

the functions \( y^\ell \) (\( y = g, h \)) correspond to \( y^{1/2} = y^+ + 2y^- \) and \( y^{3/2} = y^+ - y^- \), and the subscript \( \pm \) corresponds to total angular momentum \( \ell = \pm 1/2 \). At the first order of the low-energy expansion, we can neglect the contributions of \( \delta^{ren}_{\ell} \). The equation is then solved for the off-shell amplitude \( t_{\ell\pm}(q,k) \) by discretizing the energy domain of integration, for different choices of the c.o.m. energy \( \sqrt{s} \). Phaseshifts are then calculated by putting the particles on energy shell,

\[
e^{2i\delta_{\ell\pm}(E)} = 1 - 2pi - \frac{E^2p}{4(2\pi)^2}t_{\ell\pm}(p,p) = 1 + 2ipf_{\ell\pm}(p), \tag{40}
\]

where \( p \) is the c.o.m. momentum, \( E = \sqrt{m_N^2 + p^2} + \sqrt{M^2 + p^2} \). In the non-relativistic limit, realized as \( m_N \to \infty \), our results coincide with the ones obtained in the heavy baryon formulation of ChPT, at the same chiral order. Although no complete fit procedure has been performed, we have chosen the Weinberg-Tomozawa coupling constant so as to reproduce the \( I = 1/2 \) S-wave, while \( g_A \) is fixed from the peripheral \( NN \) phaseshifts (to be discussed later). This is repeated for three values of the cutoff \( \Lambda = 300 - 400 - 500 \) MeV. In Fig. 5 the lowest partial waves are shown as function of laboratory three-momentum, compared with the experimental analyses (squares) [21]. The bands represent the uncertainties obtained by varying \( \Lambda \) between 300 and 500 MeV. However this is only a lower
bound on the theoretical uncertainty: in particular one should expect a larger uncertainty to come from the neglected higher orders, especially from the contribution of the NLO couplings $c_i$'s which are known to be anomalously large because of the proximity of the $\Delta$ [22]. The description of $P$-wave phaseshifs is rather poor in the channel of the $\Delta$, and reflects the lack of important physics not included in our leading order calculation. The dashed lines in Fig. 5 represent our result taken in the non-relativistic limit ($m_N \to \infty$) for $\Lambda = 400$ MeV, for which the fit procedure has been repeated, and show that relativistic corrections to this leading order calculation are sizeable in the case of $\pi N$ scattering. The dotted lines represent the tree-level HBChPT calculation, which should be of comparable accuracy as ours. One should emphasize that the latter calculation is perturbative while ours takes into account, through the LS equation, the loops due to rescattering. These loops in the (perturbative) HBChPT framework come always together with subleading vertices, which absorb the cut-off dependence. The renormalization of effective field theories at the non-perturbative level is an intensively studied problem, currently the subject of controversies among experts [23]. The importance of rescattering effects is especially apparent for the $I = 3/2$ $S$-wave.

FIG. 5: (Color online) $S$ and $P$ waves of $\pi N$ scattering as function of laboratory momentum. The (blue) bands are obtained by varying the cutoff $\Lambda$ between 300 and 500 MeV. Also shown are the experimental analysis[21] (squares), our results in the non-relativistic limit for $\Lambda = 400$ MeV (dashed) and the three-level HBChPT result (dotted).
V. TWO-NUCLEON SECTOR

In addition to $\mathcal{L}_N$, for the two-nucleon sector one has to consider, at the leading order of the low-energy expansion, also a contact interaction Lagrangian $\mathcal{L}_{NN}$ containing 4 nucleon fields and no derivatives. We therefore consider in this section $H(x) = -\mathcal{L}_N - \mathcal{L}_{NN}$ and discuss separately the one-pion exchange and the contact interaction.

A. One-pion exchange

The nucleon-nucleon wave equation can be written as Eq. (13). The two-particle component of the wave function in momentum space will carry in this case spin $(r,r')$ and isospin $(i,i')$ indices, and in the center-of-mass system is written as

$$(2\omega_k + \delta_{N}^{\text{ren}}(k))\phi_2(k)_{rr',ii'} + \int \omega_q \frac{d^3q}{(2\pi)^3} V(k,q)_{rr',ii'}^s s' \phi_2(q)_{ss',jj'} = \sqrt{s} \phi_2(k)_{rr',ii'},$$

where

$$\phi_2(k)_{rr',ii'} = (v = (1,0); k, r, i; -k, r', i') \phi_2.$$  

In analogy to Eq. (13), the Lippman-Schwinger equation will again contain disconnected contributions of the kind of the function $A$ and connected contributions of the kind of the function $B$,

$$V_{rr',ii'}^{ss',jj'}(k,q) = \left[ B_{rr',ii'}^{ss',jj'}(k,q) + (2\pi)^3 \delta(k-q) \delta_{r,r'} \delta_{s,s'} \delta_{j,j'} A(k) \right] - (q \leftrightarrow -q, s \leftrightarrow s', j \leftrightarrow j'),$$

with

$$B_{rr',ii'}^{ss',jj'}(k,q) = \frac{g_A^2}{4q^2} \left( \frac{1}{8\omega_q} \right)^2 \frac{1}{\omega_q} \int \frac{d^3q}{(2\pi)^3} \frac{2}{\omega_q} \left[ \frac{f_A(2\omega_k + \omega_q + \omega_k - \omega_q)}{\sqrt{8 - \omega_q - \omega_q - \omega_q}} \right] \times [\tilde{u}(k,r)\gamma^\mu \gamma_5 u(q,s)\tilde{u}(-k,r')\gamma^\nu \gamma_5 u(-q,s')] \gamma^\mu \gamma^\nu \tau_{i'i'j'j} A(k),$$

and

$$A(k) = \frac{g_A^2}{4q^2} \left( \frac{1}{8\omega_q} \right)^2 \frac{1}{\omega_q} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_q} \frac{2}{\omega_q} \left[ \frac{f_A(2\omega_k + \omega_q + \omega_k - \omega_q)}{\sqrt{8 - \omega_q - \omega_q - \omega_q}} \right] \times [\tilde{u}(k,r)\gamma^\mu \gamma_5 u(q,s)\tilde{u}(q,r')\gamma^\nu \gamma_5 u(k,r')\gamma^\mu \gamma^\nu \tau_{i'i'j'j}].$$

where $f_A$ is the form factor entering the definition of the interacting mass in Eq. (13) corresponding to the first operator in Eq. (33), and we have denoted by $p^T$ the 4-momentum of the intermediate pion, $p^T = (\omega_q, q-k)$. The disconnected contribution can be absorbed by the choice of the counterterm $\delta_{N}^{\text{ren}}(k) = -2\omega_k A(k)$. Counting the involved three momenta as small quantities of order $O(p)$, the pion mass $M_\pi \sim O(p)$ and the energy denominators $\omega_q \sim O(p^2)$. The remaining connected kernel $B$ (matrix in spin and isospin) reads explicitly,

$$B(q,k)^{ss',jj'}_{rr',ii'} = B_0(q,k) \left[ (\sigma \cdot p_1)^s_{i'}(\sigma \cdot p_1)^s_{i'} - (\sigma \cdot p_2)^s_{j'}(\sigma \cdot p_2)^s_{j'} \right] \left[ 2\delta^r_{j'} \delta^i_{i'} - \delta^r_{i'} \delta^i_{j'} \right] = -\{k \leftrightarrow -k; s \leftrightarrow s'; j \leftrightarrow j'\},$$

with the vectors $p_1$ and $p_2$ defined by

$$p_1 = q - k - \frac{k^2 q - q^2 k}{\omega_k + m_N},$$

$$p_2 = \frac{q}{\omega_q - k} - \frac{k}{\omega_k + m_N},$$

and the scalar kernel

$$B_0(q,k) = \frac{g_A^2}{4\pi^2} \left( \frac{1}{8\omega_q} \right)^2 \frac{1}{\omega_q} \int \frac{d^3q}{(2\pi)^3} \frac{f_A(2\omega_k + \omega_q + \omega_q - \omega_q)}{\sqrt{8 - \omega_q - \omega_q - \omega_q}} \frac{f_A(\omega_k + \omega_q - \omega_k + 2\omega_q)}{\omega_k + m_N}.$$  

B. Contact interactions

The most general chirally invariant leading order $NN$ Lagrangian can be written, after using Fierz reordering, as the sum of two terms

$$\mathcal{L}_{NN} = -\frac{1}{2} C_S \bar{\psi} \gamma_\mu \psi \bar{\psi} \gamma_\mu \gamma_5 \psi + \frac{1}{2} \bar{C}_T \gamma^\mu \psi \gamma^\nu \gamma_\mu \gamma_5 \psi,$$  

(48)
where the notation for the coupling constants has been chosen so as to conform with the usual ones in the non relativistic expansion. The two vertices are diagonal in the Fock space basis that we have chosen,

\[
\left( D_2 + \delta_2^{\text{en}} + C_2 \frac{g K \uparrow}{g K} \right) \left( \phi_2 \frac{\phi_2}{\phi_{2+1}} \right) = \sqrt{s} \left( \phi_2 \frac{\phi_2}{\phi_{2+1}} \right),
\]

where we have denoted by \( C_2 \) and \( C_{2+1} \) the contribution of the new interaction vertices to the interacting mass operator in the two-particle and three-particle subspace. According to the recipe, they contain in addition an associated structure function \( f \). The contribution of \( C_{2+1} \) to the wave equation is of higher chiral order, therefore in our leading order calculation, we can neglect it and only include the operator \( C_2 \) in the analysis. With this understanding the scattering equation projected in the two-particle subspace becomes

\[
D_2 \phi_2 + C_2 \phi_2 + g^2 K \uparrow [\sqrt{s} - D_{2+1}] g K_{\text{conn}} \phi_2 = \sqrt{s} \phi_2.
\]

The new vertices yield connected contributions to the wave equation. Let us denote this connected kernel, analogous to \( B(k, q)_{ss', jj'} \) in the one-pion exchange potential, with the letter \( C \). Its operatorial expression reads

\[
C(k, q) = C_S \left\{ \sqrt{(\omega_k + m_N)(\omega_q + m_N)} - \frac{q \cdot k}{\sqrt{(\omega_k + m_N)(\omega_q + m_N)}} \right\}^2 + i \left[ 1 - \frac{q \cdot k}{(\omega_k + m_N)(\omega_q + m_N)} \right] (k \times q) \cdot (\sigma_1 + \sigma_2) \\
+ \frac{1}{(\omega_k + m_N)(\omega_q + m_N)} (k + q) \cdot \sigma_1 (k + q) \cdot \sigma_2 \\
+ \frac{1}{(\omega_k + m_N)(\omega_q + m_N)} \left[ (\omega_q - \omega_k)(\omega_k + m_N) + k^2 + k \cdot q \right] q \cdot \sigma_1 q \cdot \sigma_2 \\
+ \frac{1}{(\omega_k + m_N)(\omega_q + m_N)} \left[ \omega_q - \omega_k \right] (\omega_q + m_N) + q^2 + k \cdot q \right] k \cdot \sigma_1 k \cdot \sigma_2 \\
+ i \left( q \cdot k \right) (k \times q) \cdot (\sigma_1 + \sigma_2) - \frac{q^2 k^2 - (q \cdot k)^2}{(\omega_k + m_N)(\omega_q + m_N)} \right\} \frac{f_S(2\omega_q, 2\omega_k)}{8 \sqrt{\omega_q^2 + \omega_k^2}} \\
- \{ q \leftrightarrow -q; 1 \leftrightarrow 2 \},
\]

where for the associated structure functions \( f_S \) and \( f_T \) we take the same expression as in Eq. (4), with the choice \( \xi = \exp[-(q^4 + k^4)/(2A^4)] \). Such additional cutoff for the contact interactions is needed to regulate the LS equation. It is chosen, analogously to what is done e.g. in Ref. [26], so that the induced modifications are of higher order in the chiral counting (actually more than needed for our leading order calculation). The wave equation now reads

\[
\int \omega_k \frac{d^3 q}{(2\pi)^3} \left[ B(k, q)_{ss', jj'} + C(k, q)_{ss', jj'} \right] \phi_2(q)_{ss', jj'} = \left[ \sqrt{s} - 2\omega_k \right] \phi_2(k)_{rr', ii'}. \tag{52}
\]

In the chiral counting \( B(k, q) \sim C(k, q) \sim O(1) \). The wave equation can be recast in the form of a Lippmann-Schwinger equation, from which phaseshifts are found by means of standard numerical methods [24].

As already mentioned in the previous section, we have fixed the coupling \( g_A \) from the \( 1^1S_0 \) wave, and the contact terms \( C_S \) and \( C_T \) from the \( 1^3S_1 \) and \( 1^3S_0 \) scattering lengths, requiring that

\[
a^3(S_1) = 5.4 \text{ fm}, \quad a^3(S_0) = -24 \text{ fm}. \tag{53}
\]

We have repeated this procedure for three values of the cutoff \( \Lambda = 300, 400, 500 \text{ MeV} \). The results are shown in Figs. [6][7]. The shaded bands represent the variations with \( \Lambda \) and can be considered as the intrinsic theoretical uncertainty of our calculation. Also plotted, as dashed lines, is the result corresponding to \( \Lambda = 400 \text{ MeV} \) in the non-relativistic limit, realized as \( m_N \to \infty \). The differences make it possible to quantify the size of the relativistic
FIG. 6: (Color online) Spin singlet $NN$ phaseshifts as function of laboratory kinetic energy. The (blue) bands show the effect of varying the cutoff $\Lambda$ between 300 and 500 MeV. The squares represent the Nijmegen phaseshifts [25]. Also shown in dashed are the results in the nonrelativistic limit ($m_N \to \infty$), computed with $\Lambda = 400$ MeV.

corrections included in our scheme: they are always smaller than the effect of varying $\Lambda$, therefore, at least for the leading order of the chiral expansion, relativistic corrections are smaller than the neglected chiral orders. The peripheral waves are well described by the one-pion exchange, but the agreement with data is poorer for the lowest waves. Clearly some important physics is missing in our leading order calculation (see e.g. the discussion of Ref. [26]). This is the case, for instance, of the $^3D_3$ wave: it is well known that two-pion exchange is very important for the description of this wave, and we do not include this process, not even implicitly, via subleading low-energy constants.

C. The deuteron

With all the parameters fixed by the scattering, for the chosen values of the cutoff $\Lambda$, we have computed the momentum space bound state mass and wave function $\phi^D_2(k)$ in the $^3S_1-^3D_1$ channel. The configuration-space wave functions $u_0$ and $u_2$ are found by Fourier transformation (cfr. Ref. [27]). Some bound-state observables are shown in
Table I. In Fig. 10 we show the configuration-space deuteron wave functions, in which as before the bands represent the variations with \( \Lambda \) between 300 and 500 MeV.

VI. CONCLUDING REMARKS

In this paper we have considered the proposal of Ref. [13] of a point-form formulation of relativistic quantum mechanics consisting in the construction of a Bakamjian-Thomas interacting mass operator for the few-nucleon system. Taking the most general chirally invariant Lagrangian describing pions and nucleons, we have implemented the above construction in the framework of an effective theory, where the needed truncation of the Fock space is justified by a systematic low-energy power counting. This in turn relies on the restrictions that the underlying chiral symmetry imposes on the interaction vertices. By introducing a different normalization for the structure functions \( f \), associated with each coupling of the Lagrangian, compared to the original proposal of Ref. [13], we obtain the correct matching with the point-form quantized field theory and show that the renormalization procedure is consistent with the cluster decomposition principle.
We have performed a complete leading order analysis of the $\pi N$ and $NN$ systems by solving numerically the eigenvalue equations originating by the diagonalization of the mass operator, and obtained a rather coherent picture of the low-energy phaseshifts. Of course, especially in the $\pi N$ system, important physics, such as the effect of the $\Delta$ resonance, is missed at leading order, and the examination of higher orders is mandatory for a better quantitative description. By comparison with the non-relativistic limit of our framework we have assessed the size of relativistic corrections included in our scheme, and shown that in the $NN$ sector they are always smaller than the accuracy of our leading order calculation: NLO chiral corrections are larger than our "all-order" relativistic corrections. Still, it could well be, provided the chiral expansion is convergent enough, that NNLO chiral corrections be smaller than our all-order relativistic corrections. This can only be checked by actual calculation. Moreover, the importance of relativistic corrections can depend on the observables: in Ref. [6] they are found to be surprisingly large for the neutron-deuteron $A_y$.

As a possible development then the next order can be analyzed, by including the 2-pion states and taking into account the next-to-leading order terms in $\mathcal{L}$. By examining the off-diagonal matrix elements of the mass operator, our framework allows to incorporate in a natural way the pion production channel whose understanding from the
field-theoretical point of view is only recently starting to emerge\cite{28}. Furthermore, once the Lagrangian is fixed in the two-nucleon sector, the 3N sector could be investigated. The point-form formulation, in which the boost operators have only a kinematical character, would allow to easily account for such relativistic effects as the boosting of the wave function of an interacting 2-nucleon subsystem, although the requirements of the cluster decomposition principle might be less simple to fulfill in this case.

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TABLE I: Bound state observables for the chosen values of the cutoff $\Lambda$.

| $\Lambda$ (MeV) | $B_d$ (MeV) | $\eta$ | $r_d$ (fm) | $Q_d$ (fm$^2$) | $P_D$ (%) |
|-----------------|-------------|-------|-------------|--------------|----------|
| 300             | 2.01        | 0.015 | 1.86        | 0.12         | 0.5      |
| 400             | 1.93        | 0.018 | 1.85        | 0.16         | 1.2      |
| 500             | 1.99        | 0.020 | 1.83        | 0.18         | 1.9      |

FIG. 10: (Color online) Configuration-space wave function of the S-wave (upper curves) and D-wave (lower curves) components of the deuteron. The (blue) bands show the effect of varying the cutoff between 300 and 500 MeV.

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