Domain Walls in MQCD
and
Monge-Ampère Equation

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Abstract

We study Witten’s proposal that a domain wall exists in M-theory fivebrane version of QCD (MQCD) and that it can be represented as a supersymmetric three-cycle in $G_2$ holonomy manifold. It is shown that equations defining the $U(1)$ invariant domain wall for $SU(2)$ group can be reduced to the Monge-Ampère equation. A proof of an algebraic formula of Kaplunovsky, Sonnenschein and Yankielowicz is presented. The formal solution of equations for domain wall is constructed.

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1 Introduction

Starting from the pioneering work by Hanany and Witten [1], the study of the low-energy dynamics of a certain class of intersecting brane configurations has shed much light on non-perturbative properties of gauge theories [1]-[18]. Recently, Witten explored the minimal $N=1$ model with an $SU(n)$ vector multiplet in four dimensions [10]. He showed how for this model some of the outstanding properties of the ordinary QCD such as confinement, a mass gap and spontaneous breaking of a discrete chiral symmetry can be approached from M-theory point of view. The consideration of $N=1$ gauge theory in the geometric engineering approach was performed in [1].

To describe the M5-brane version of QCD (MQCD) [10], one starts from the brane configuration in type IIA superstring theory with space-time coordinates $(x^0, x^1, ..., x^9)$ and studies a configuration arising from $n$ D4-branes suspended between two NS5-branes located at $x^6 = 0$ and $x^6 = S_0$ [3], [11]. D4-branes world-volumes occupy $(x^0, x^1, x^2, x^3, x^6)$, with $0 \leq x_6 \leq S_0$, NS5-brane’s world-volume is spanned by $(x^0, x^1, x^2, x^3, x^4, x^5)$ and another NS5-brane’s by $(x^0, x^1, x^2, x^3, x^7, x^8)$, where $S_0$ is an arbitrary length. Then the world-volume theory on D4s is four dimensional Super Yang Mills with gauge group $SU(n)$ and $N = 1$ supersymmetry. Elevating to the M-theory picture by adding the coordinate $x^{10}$ makes possible a solution of the theory as follows [10]. Reinterpreted as a brane configuration embedded in eleven dimensional spacetime, the entire brane configuration corresponds to a single smooth M5-brane with world-volume $R^{1,3} \times \Sigma$, where $\Sigma$ is a Riemann surface, embedded in three-dimensional space $Y$ with coordinates $v, w, t = e^{-s}$ as $vw = \zeta$, $v^n = t$, here $v = x^4 + ix^5$, $w = x^7 + ix^8$, $s = x^6 + ix^{10}, 0 \leq x^{10} \leq 2\pi$ and $\zeta$ is a complex constant. Analyzing the symmetries, one can notice [10] that $Z_n$ symmetry: $t \to t, v \to v, w \to e^{2\pi i/n}w$ is only symmetry at infinity, which doesn’t leave the first equation defining $\Sigma$ invariant. Thus [10], this symmetry is spontaneously broken and the theory has $n$ distinct vacua, specified by the curves $w = exp(2\pi i/n)\zeta v^{-1}, t = v^n$.

A consequence of the spontaneously broken chiral symmetry is that there can be a domain wall separating different vacua. BPS–saturated domain walls in four dimensional supersymmetric gauge theories have been considered in [21], [22]. Witten has suggested that a BPS–saturated domain wall exists in MQCD and that it can be represented as a supersymmetric three-cycle in the sense of Becker et al [19], [20] with a prescribed asymptotic behavior. The domain wall is described [10] as an M-theory fivebrane of the form $R^3 \times S$, where $R^3$ is parameterized by $x^0, x^1, x^2$ and $S$ is a three-surface in the seven manifold $Y = R \times Y$, here $R$ is the copy of $x^3$ direction. Near $x^3 = -\infty$, $S$ should look like $R \times \Sigma$, where $\Sigma$ is the Riemann surface defined by $w = \zeta v^{-1}, t = v^n$. Near $x^3 = +\infty$, $S$ should look like $R \times \Sigma'$, where $\Sigma'$ is the Riemann surface of an "adjacent" vacuum, defined by $w = exp(2\pi i/n)\zeta v^{-1}, t = v^n$. MQCD is by no means identical to QCD, it depends on one extra parameter – type IIA string coupling constant. For the domain wall to be in the universality class of SQCD, $S$ must be invariant under $U(1)$ symmetry $t \to e^{i\delta}t, v \to e^{i\delta}v, w \to e^{-i\delta}w$. Different approaches to the problem of domain walls in MQCD have been explored in [23], [24], [25], [26]. Equations defining the domain wall have been derived and studied in
The aim of this note is the consideration of such $U(1)$ symmetric $S$ which is a supersymmetric three-cycle in $\tilde{Y}$ with the described asymptotic behavior. We use Witten’s $U(1)$ invariant ansatz for $SU(2)$ group and an algebraic formula of Kaplunovsky, Sonnenschein and Yankielowicz (KSY). We consider two gauges when one of equations has the form of conservation law. This permits to reduce the system of equations to one equation. We show that the equations defining the domain wall can be reduced to the Monge-Ampère equation. A proof of the KSY formula is presented. This formula is very useful for the consideration of domain walls in MQCD. The formal solution of equations for the domain wall is constructed using a special separation of variables for group $SU(2)$ in the spirit of [23].

2 Supersymmetric Cycles in Various Dimensions

A supersymmetric cycle is defined by the property that the world-volume theory of a brane wrapping around it is supersymmetric. To study supersymmetric cycles one uses the concept of calibration [20], i.e. a closed $p$-form $\Phi$ on a Riemannian manifold of dimension $n$ such that $\Phi$ has comass 1. Submanifolds for which there is equality are said to be calibrated by $\Phi$. The calibrated submanifold has the least volume in its homology class. This provides a natural geometrical interpretation of the BPS bound for D-branes wrapped around such submanifolds, with the calibrated submanifolds corresponding to BPS-states, which saturate the bound.

The conditions for the supersymmetric cycles in Calabi-Yau 3-folds have been analyzed in [19]. It was shown that a supersymmetric three-cycle is one for which the pullback of Kähler form $J$ vanishes and the pullback of the holomorphic 3-form $\Omega$ is a constant multiple of volume element, namely $\ast X(J) = 0$, $\ast X(\Omega) \sim 1$, where $X(\cdot)$ denotes the pullback and $\ast$ is a Hodge dual on membrane world-volume.

In the case of domain walls in MQCD [10] one deals with a seven dimensional flat manifold $\tilde{Y}$ of $G_2$ holonomy and with the associative calibration $\Phi$. The group $G_2$ is most naturally defined as the automorphism group of the octonions or Cayley numbers $O = H(+)$, the euclidean algebras obtained from the quaternions by Cayley-Dickson process [27]. If we choose the local veilbein so that the metric on $\tilde{Y}$ is $\sum_{i=1}^{n} e_i \otimes e_i$, locally the $G_2$ invariant 3-form $\Phi$ can be written as [28]

$$\Phi = e_1 \wedge e_2 \wedge e_7 + e_1 \wedge e_3 \wedge e_6 + e_1 \wedge e_4 \wedge e_5 + e_2 \wedge e_3 \wedge e_5 - e_2 \wedge e_4 \wedge e_6 + e_3 \wedge e_4 \wedge e_7 + e_5 \wedge e_6 \wedge e_7.$$ (1)

A supersymmetric three-cycle $S$ in $\tilde{Y}$ is one for which the pullback of this three-form is a constant multiple of the volume element [20]. The invariant forms are related by the dimensional reduction. If we set

$$e_1 = dx^{10}, \quad e_2 = dx^5, \quad e_3 = dx^3, \quad e_4 = dx^7, \quad e_5 = dx^4, \quad e_6 = dx^6, \quad e_7 = dx^8.$$ (2)
then the form $\bar{\Phi}$ can be written as

$$
\bar{\Phi} = \text{Im}(\Omega) + \frac{i}{2} dx^3 \wedge J,
$$

(3)

where

$$
\Omega = dv \wedge dw \wedge dt/t
$$

(4)

and

$$
J = dv \wedge d\bar{v} + dw \wedge d\bar{w} + dt \wedge d\bar{t}/|t|^2
$$

(5)

are Kähler and holomorphic forms in Calabi-Yau 3-fold $Y$. If one equates the pullback of the $J$ to zero, then from the condition for supersymmetric cycle in 7 dimensional manifold one gets the condition in 6 dimensional manifold. This is probably a relation between the equations considered here and in the recent paper [27], where another approach to the problem of the domain wall in MQCD was suggested.

Baulieu, Kanno, and Singer have developed an almost topological theory, so called BRSTQFT in 8 dimensions [30, 31]. It seems that supersymmetric cycles in various dimensions can be obtained by the dimensional reduction from the BRSTQFT.

3 Equivalent Form of Supersymmetric Cycles

We will be looking for a supersymmetric three-cycle $S$ with worldvolume coordinates $(y_1, y_2, y_3)$ in a 7-manifold $\tilde{Y}$ with coordinates $(x^3, v, w, t)$ which near $x^3 = -\infty$ looks like $\mathbb{R} \times \Sigma$, where $\Sigma$ is the Riemann surface defined by $w = \zeta_v^{-1}, t = v^n$ and near $x^3 = +\infty, S$ like $\mathbb{R} \times \Sigma'$, where $\Sigma'$ is the Riemann surface of an "adjacent" vacuum, defined by $w = \zeta e^{2\pi i/n}v^{-1}, t = v^n$. In this note we will consider group $SU(2)$.

Let us make an embedding of $S$ into $\tilde{Y}$

$$
v = z_1(y_1, y_2, y_3),
$$

(6)

$$
w = z_2(y_1, y_2, y_3),
$$

(7)

$$
s = z_3(y_1, y_2, y_3)
$$

(8)

$$
x^3 = y_3
$$

(9)

and introduce the complex 3-vectors $a_k^i$ with components

$$
a_k^i = \frac{\partial z_i}{\partial y_k},
$$

(10)

where $i, k = 1, 2, 3$.

The condition for $S$ to be a supersymmetric 3-cycle in these notations is [19, 20, 23]

$$
\sqrt{\text{det}|h_{mn}|} dy_1 \wedge dy_2 \wedge dy_3 = \bar{\Phi} \quad \text{or} \quad \text{det}|h_{mn}| = \Phi^2,
$$

(11)
where
\[ h_{mn} = \text{Re}(a_m^* \cdot a_n) + \delta_m^3 \delta_n^3 \] (12)
is an induced metric and
\[ \Phi = \text{Im}[(a_1^* \cdot a_2) - (a_3 \cdot a_1 \times a_2)] \] (13)
is a pullback of $G_2$ invariant form and $(a \cdot b) = \sum_{i=1}^{3} a^i b^i$.

Due to Theorem proved in Appendix we have
\[ \det|h_{mn}| = \Phi^2 + |R|^2 + \text{Re}^2(a_3 \cdot R), \] (14)
where
\[ R = a_1 \times a_2 + \frac{i}{2} \epsilon_m^{3k} a_m^* \text{Im}(a_3^* \cdot a_k) \] (15)
and the requirement for the surface $S$ to be a supersymmetric three cycle ([11]), is reduced to the equation derived by Kaplunovsky, Sonnenschein and Yankielowicz [28, 29]
\[ R = 0, \] (16)
or
\[ a_1 \times a_2 + \frac{i}{2} \epsilon_m^{3k} a_m^* \text{Im}(a_3^* \cdot a_k) = 0. \] (17)

Let us note the following

**Proposition.** The relation
\[ ia_3^* \text{Im}(a_1^* \cdot a_2) + a_1 \times a_2 = 0 \] (18)
implies (17).

### 4 U(1) Ansatz for Domain Wall

Let us consider the group $SU(2)$ and make an embedding:
\[ v = z_1 = f(y_1, y_3)e^{iy_2}, \] (19)
\[ w = z_2 = g(y_1, y_3)e^{-iy_2}, \] (20)
\[ s = z_3 = -h(y_1, y_3) - 2iy_2. \] (21)

Under this U(1) invariant ansatz the equation (17)
\[ -ia_1 \times a_2 + a_1^* \text{Im}(a_2^3 \cdot a_3) + a_2^* \text{Im}(a_3^3 \cdot a_1) + a_3^* \text{Im}(a_1^3 \cdot a_2) = 0, \] (22)
where
\[ a_1 = (\partial_1 f \cdot e^{iy_2}, \partial_1 g \cdot e^{-iy_2}, -\partial_1 h), \]

\[ a_2 = (if \cdot e^{iy_2}, -ig \cdot e^{-iy_2}, -2i), \]

\[ a_3 = (\partial_3 f \cdot e^{iy_2}, \partial_3 g \cdot e^{-iy_2}, -\partial_3 h) \]  

is reduced to the following equations for complex functions \( f, g, h \):

\[
\{K, f^*\}_{(3,1)} - iP f^* - 2(2\partial_1 g + g\partial_1 h) = 0; \tag{24}
\]

\[
\{K, g^*\}_{(3,1)} + iP g^* + 2(2\partial_1 f - f\partial_1 h) = 0; \tag{25}
\]

\[
- \{K, h^*\}_{(3,1)} + 2iP - 2\partial_1(fg) = 0, \tag{26}
\]

where

\[ P = -Im\{\{f, f^*\}_{3,1} + \{g, g^*\}_{3,1} + \{h, h^*\}_{3,1}\}, \quad K = |g|^2 - |f|^2 - 2h - 2h^* \]

and the Poisson brackets are defined as

\[ \{g, f\}_{(i,j)} = \partial_i g \partial_j f - \partial_j g \partial_i f, \quad i, j = 1, 3. \]

The boundary conditions for group \( SU(2) \) read:

\[ fg|_{y_3 = \mp \infty} = \pm \zeta, \quad f^2 e^{-h}|_{y_3 = \pm \infty} = 1, \quad Im \zeta = 0. \tag{27} \]

Note that we have 3 complex equations for 3 complex functions, but not all of them are independent. From equation (24) and its complex conjugated one gets the following equation

\[ \{K, h + h^*\}_{(3,1)} + 4\partial_1 Re(fg) = 0, \tag{28} \]

which in fact follows from (24) and (25).

### 4.1 Real Functions

Let us now assume that functions \( f, g, h \) are real. Then \( P = 0 \) and equations (24)-(26) are reduced to the following equations

\[
\{K, f\}_{(3,1)} - 2(2\partial_1 g + g\partial_1 h) = 0, \tag{29}
\]

\[
\{K, g\}_{(3,1)} + 2(2\partial_1 f - f\partial_1 h) = 0, \tag{30}
\]

\[
\{K, h\}_{(3,1)} + 2\partial_1(fg) = 0, \tag{31}
\]

with

\[ K = g^2 - f^2 - 4h. \tag{32} \]
Equation (30) is the combination of equations (29) and (31), thus it can be dropped out and we are left with the following system of equations:

\[
\begin{align*}
\{4h - g^2, f\}_{(1,3)} - 4\partial_1 g - 2g\partial_1 h &= 0, \\
\{g^2 - f^2, h\}_{(1,3)} - 2\partial_1 (fg) &= 0.
\end{align*}
\] (33) (34)

4.2 Formal Solution

To check a self consistency of the above equations let us derive a formal solution of these equations for group SU(2). The ansatz for functions \( f, g, h \) considered here is of the form:

\[
\begin{align*}
f(y_1, y_3) &= f_0(y_1) + \sum_{k=1}^{\infty} \gamma^{2k}(y_3)f_{2k}(y_1), \\
g(y_1, y_3) &= -\zeta \beta(y_3) \cdot (g_0(y_1) + \sum_{k=1}^{\infty} \gamma^{2k}(y_3)g_{2k}(y_1)), \\
h(y_1, y_3) &= h_0(y_1) + \sum_{k=1}^{\infty} \gamma^{2k}(y_3)h_{2k}(y_1),
\end{align*}
\] (35) (36) (37)

where

\[
\begin{align*}
\gamma(y_3) &= \frac{1}{e^{y_3} + e^{-y_3}} = \frac{1}{2 \cosh y_3}, \\
\beta(y_3) &= \frac{e^{y_3} - e^{-y_3}}{e^{y_3} + e^{-y_3}} = \tanh y_3.
\end{align*}
\]

We notice that for the above ansatz the boundary conditions are trivially satisfied if \( f_0g_0 = 1 \), and \( f_0^2e^{-h_0} = 1 \).

Due to the simple differential algebra \( \partial_3\beta = 4\gamma^2, \partial_3\gamma^k = -k\beta\gamma^k, \beta^2 = 1 - 4\gamma^2 \) the equations (33-34) will take the form of the equations on \( f_{2k}, g_{2k}, h_{2k} \):

\[
\begin{align*}
-\zeta \partial_1 f_{2k} + (2k f_0 \partial_1 h_0 - \zeta \partial_1 g_0) f_{2k} - \zeta f_0 \partial_1 g_{2k} - (2k \zeta^2 g_0 \partial_1 h_0 + \zeta \partial_1 f_0) g_{2k} + \frac{k(2\zeta^2 g_0 \partial_1 g_0 - 2f_0 \partial_1 f_0)h_{2k}}{L_{2k}},
\end{align*}
\] (38)

\[
\begin{align*}
k(2\zeta^2 g_0 \partial_1 g_0 - 4\partial_1 h_0) f_{2k} + 2\zeta \partial_1 g_{2k} - (2k \zeta^2 g_0 \partial_1 f_0 - \zeta \partial_1 h_0) g_{2k} + \frac{k(2\zeta^2 g_0 \partial_1 g_0 - 4\partial_1 h_0) f_{2k}}{M_{2k}},
\end{align*}
\] (39)

where

\[
\begin{align*}
L_{2k} &= \zeta \partial_1 (f_{2k-2m}g_{2m}) - mh_{2m}K^{(1)}_{2k-2m} - \partial_1 h_{2k-2m}K^{(3)}_{2m} - \partial_1 h_0 K^{(3)}_{2k}, \\
M_{2k} &= -\zeta g_{2k-2m} \partial_1 h_{2m} - \partial_1 f_{2k-2m}K^{(3)}_{2m} + mf_{2m}K^{(1)}_{2k-2m} - \partial_1 f_0 K^{(3)}_{2k}, \\
K^{(1)}_{2k} &= \tilde{K}^{(1)}_{2k} + K^{(1)}_{2k}, \quad K^{(3)}_{2k} = \tilde{K}^{(3)}_{2k} + K^{(3)}_{2k}, \\
K^{(1)}_{2k} &= 2\zeta^2 \partial_1 (g_0 g_{2k}) - 2\partial_1 (f_0 f_{2k}) - 4\partial_1 h_{2k}, \quad K^{(3)}_{2k} = -2k\zeta^2 g_0 g_{2k} + 2k f_0 f_{2k} + 4k h_{2k}.
\end{align*}
\] (40) (41) (42) (43)
\[ K'_{2k}^{(1)} = -8\zeta^2 (\delta_{k1} g_0 \partial_1 g_0 + \partial_1 (g_0 g_{2k-2})) + \]
\[ \sum_{m=1}^{k-1} 2\zeta^2 g_{2k-2m} \partial_1 g_{2m} - 8\zeta^2 g_{2k-2m-2} \partial_1 g_{2m} - 2f_{2k-2m} \partial_1 f_{2m}, \]
\[ K'_{2k}^{(3)} = g_0 \zeta^2 (4g_0 \delta_{k1} + 4(2k-1)g_{2k-2}) - \]
\[ \sum_{m=1}^{k-1} \zeta^2 g_{2k-2m} (2mg_{2m} - 4g_0 \delta_{m1} - 4(2m-1)g_{2m-2}) + 2mf_{2k-2m} f_{2m}. \]

We notice that we have two equations for three functions, so it seems useful to assume that for example \( h_{2k} = 0 \). Let us also take a parametrization of the form
\[ h_0 = y_1, \quad f_0 = e^{y_1/2}, \quad g_0 = e^{-y_1/2}. \]

Then the equations will take the form:
\[ -\zeta e^{-y_1/2} \partial_1 f_2 + (2ke^{y_1/2} + \frac{\zeta}{2} e^{-y_1/2}) f_2 - \zeta e^{y_1/2} \partial_1 g_2 - (2k\zeta^2 e^{-y_1/2} + \frac{\zeta}{2} e^{y_1/2}) g_2 = L_{2k} \]
\[ k(-\zeta^2 e^{-y_1} - 4) f_2 + 2\zeta \partial_1 g_2 - (k\zeta^2 - \zeta) g_2 = M_{2k} \]

For example, in the first order one can easily get
\[ -\zeta e^{-y_1/2} \partial_1 f_2 + (2e^{y_1/2} + \frac{\zeta}{2} e^{-y_1/2}) f_2 - \zeta e^{y_1/2} \partial_1 g_2 - (2\zeta^2 e^{-y_1/2} + \frac{\zeta}{2} e^{y_1/2}) g_2 = -4\zeta^2 e^{-y_1/2}, \]
\[ - (\zeta^2 e^{-y_1} + 4) f_2 + 2\zeta \partial_1 g_2 - (\zeta^2 - \zeta) g_2 = -2\zeta^2 e^{-y_1/2}. \]

From equation (50) one can express \( f_2 \), substitute this expression to (49) and get one equation on \( g_2 \), which can be solved in quadratures. Similarly, for the \( k \)th order. So we get the following

**Proposition.** There exists a solution of equations (33) and (34) of the following form
\[ f(y_1, y_3) = e^{y_1/2} + \sum_{k=1}^{\infty} \gamma^{2k}(y_3) f_{2k}(y_1), \]
\[ g(y_1, y_3) = -\zeta \beta(y_3) \cdot (e^{-y_1/2} + \sum_{k=1}^{\infty} \gamma^{2k}(y_3) g_{2k}(y_1)), \]
\[ h(y_1, y_3) = y_1, \]

where \( f_{2k} \) and \( g_{2k} \) satisfy the recursive relations (37) and (38). The solution (51)-(53) satisfies the boundary conditions (27).

**Remark. Small \( \zeta \)**

Let us take \( \zeta \to 0 \). As it was pointed in [10] \( \zeta \to 0 \) and \( R \to \infty \) corresponds to the small QCD scale and one gets the ordinary super Yang-Mills.

In this case the equations will take the form
\[ 2ke^{y_1/2} f_{2k} = -2mf_{2k-2m} f_{2m}, \]
so the solution is \( f_{2k} = 0 \) and \( h_{2k} \) is arbitrary. So, in principle, one can take \( h_{2k} = 0 \) then the surface

\[
vw = -\zeta \tanh(x_3), \quad t = v^2
\]

will correspond to the domain wall for small \( \zeta \).

From (51) one notices that if \( K = y_1 \) or \( h = y_1 \) then this equation becomes the conservation law. We consider these two cases separately.

### 4.3 Monge-Ampère equation

Let us first take a parameterization of the surface defined by (29)-(31) as

\[ h = y_1. \]  

(57)

This “gauge” was considered also in [28, 29]. Then we get

\[
4\partial_3 f + \{ g^2, f \}_{(3,1)} - 2(2\partial_1 g + g) = 0
\]

(58)

\[
4\partial_3 g - \{ f^2, g \}_{(3,1)} + 2(2\partial_1 f - f) = 0
\]

(59)

\[
\partial_3 (g^2 - f^2) + \partial_1 (2fg) = 0.
\]

(60)

Equation (60) has the form of a conservation law. From this equation it follows that there exists a function \( \chi \) such that

\[
f g = -\frac{1}{2} \partial_3 \chi,
\]

(61)

\[ g^2 - f^2 = \partial_1 \chi. \]

(62)

One can express \( g^2 \) and \( f^2 \) in term of \( \chi \) as

\[
g^2 = \frac{1}{2} (\partial_1 \chi + \sqrt{(\partial_1 \chi)^2 + (\partial_3 \chi)^2}),
\]

(63)

\[
f^2 = \frac{1}{2} (-\partial_1 \chi + \sqrt{(\partial_1 \chi)^2 + (\partial_3 \chi)^2}),
\]

(64)

\[
f^2 + g^2 = \sqrt{(\partial_1 \chi)^2 + (\partial_3 \chi)^2}.
\]

(65)

Multiplying (58) on \( g \) and (59) on \( f \) and sum up we get

\[
4\partial_3 (fg) + \{ g^2 - f^2, fg \}_{(3,1)} - 2\partial_1 (g^2 - f^2) - 2(f^2 + g^2) = 0.
\]

(66)

Substituting (51), (52) and (55) in (66) one gets

\[
- \partial_1^2 \chi - \partial_3^2 \chi + \frac{1}{4} \partial_1^2 \chi \cdot \partial_3^2 \chi - \frac{1}{4} (\partial_1^2 \chi)^2 - \sqrt{(\partial_3 \chi)^2 + (\partial_1 \chi)^2} = 0.
\]

(67)
One can easily derive the boundary conditions for function $\chi$ from (27)

$$\partial_3 \chi|_{y_3 \to \pm \infty} = \pm 2\zeta, \quad \partial_1 \chi|_{y_3 \to \pm \infty} = \zeta^2 e^{-y_1} - e^{y_1} = -2e^\xi \sinh(y_1 - \xi).$$

(68)

where $\zeta = e^\xi$. Equation (67) is in fact a Monge-Ampère equation [32, 33, 34]. One can write it in the canonical form if one sets $\phi(x, y) = \frac{1}{4} \chi(y_1, y_3)$. Then (67) reads

$$\phi_{xx} \phi_{yy} - \phi_{xy}^2 = \phi_{xx} + \phi_{yy} + \sqrt{\phi_x^2 + \phi_y^2}.$$ (69)

We have to find a solution of (69) in the plane with the following boundary conditions:

$$\lim_{y \to \pm \infty} \phi_x = \frac{1}{4}(\zeta^2 e^{-x} - e^x), \quad \lim_{y \to \pm \infty} \phi_y = \pm \frac{1}{2}\zeta.$$ (70)

Here $\zeta$ is a real parameter.

Notice vacuum solutions of (69)

$$\phi = -\frac{e^\xi}{2}(\mp y + \cosh(x - \xi)).$$ (71)

Let us recall that the general Monge-Ampère equation has the form [32]

$$\phi_{xx} \phi_{yy} - \phi_{xy}^2 = a\phi_{xx} + 2b\phi_{xy} + c\phi_{yy} + g.$$ (72)

where $a, b, c$ and $g$ are functions of $x, y, \phi, \phi_x$ and $\phi_y$. In our case $a = c = 1, b = 0$ and $g = \sqrt{\phi_x^2 + \phi_y^2}$.

The Monge-Ampère equation (72) is called strongly elliptic if $g > 0$ and the quadratic form $a\rho^2 + 2b\rho\eta + c\eta^2$ is non-negatively defined [32]. For such equations Pogorelov [32] has proved the existence of a generalized solution in any convex domain on the plane. Also the Dirichlet problem has been solved and properties of regularity of the solution have been investigated. We cannot directly apply to our case these results because in our case $g = \sqrt{\phi_x^2 + \phi_y^2} > 0$, i.e. it is positive but not strictly positive and moreover we have a boundary problem which is not of a Dirichlet type, but rather of the Neumann type (70). We will consider equation (69) in another work.

### 4.4 Quasi-linear Equation

Let us now take a parameterization ("gauge") of the surface defined by (33) and (34) as

$$K(y_1, y_3) = y_1$$ (73)

This parameterization is consistent with the boundary conditions. One can see that this parameterization is related with condition (18) after a change of variables $y_1 \to y_1'$, such that $(\partial_1 K)^{-1} \partial / \partial y_1 \to \partial / \partial y_1'$.

In parameterization (73) we have

$$\partial_3 f = -2(2\partial_1 g + g \partial_1 h);$$ (74)
\[ \partial_3 g = 2(2\partial_1 f - f\partial_1 h); \quad (75) \]

\[ \partial_3 h = \partial_1 (2fg). \quad (76) \]

Equation (76) can be rewritten in the form of conservation law for functions \( f \) and \( g \)

\[ \partial_3 (g^2 - f^2) - \partial_1 (8fg) = 0. \quad (77) \]

From this equation follows that there exists a function \( \Psi \) such that

\[ fg = \frac{1}{8}\partial_3 \Psi, \quad (78) \]

\[ g^2 - f^2 = \partial_1 \Psi. \quad (79) \]

One can express \( g^2 \) and \( f^2 \) in term of \( \Psi \) as

\[ g^2 = \frac{1}{2}(\partial_1 \Psi + \sqrt{(\partial_1 \Psi)^2 + (1/4\partial_3 \Psi)^2}) \quad (80) \]

\[ f^2 = \frac{1}{2}(-\partial_1 \Psi + \sqrt{(\partial_1 \Psi)^2 + (1/4\partial_3 \Psi)^2}) \quad (81) \]

\[ f^2 + g^2 = \sqrt{(\partial_1 \Psi)^2 + (1/4\partial_3 \Psi)^2} \quad (82) \]

Substituting (78), (79) and (82) to the equation

\[ \partial_3 (fg) = 2\partial_1 (f^2 - g^2) - 2\partial_1 h(g^2 + f^2) \quad (83) \]

which follows from (74), (74) and (73) one gets the following

**Proposition.** If \( \Psi \) is a solution of

\[ \frac{1}{4}\partial_3^2 \Psi + 4\partial_1^2 \Psi + (\partial_2^2 \Psi - 1) \cdot \sqrt{(1/4\partial_3 \Psi)^2 + (\partial_1 \Psi)^2} = 0 \quad (84) \]

then \( f \) and \( g \) defined by (84) and (80), and \( h = \frac{1}{4}(g^2 - f^2) \) satisfy (74)-(76).

Boundary conditions for equation (84) are the following

\[ \partial_3 \Psi|_{y_3 \to \pm \infty} = \pm 8\zeta; \quad \partial_1 \Psi|_{y_3 \to \pm \infty} = a(y_1); \quad (85) \]

where \( a(y_1) \) is a solution of the following equation

\[ ae^{\frac{1}{x}(a-y_1)} + e^{\frac{1}{x}(a-y_1)} - \zeta^2 = 0. \quad (86) \]

Introducing \( \phi(x, y) = \Psi(y_1, 2y_3) \) one rewrites equation (84) as

\[ \phi_{yy} + \phi_{xx} \cdot (4 + \sqrt{\phi_x^2 + \phi_y^2}) = \sqrt{\phi_x^2 + \phi_y^2}, \quad (87) \]
5 Conclusion

In this note we have shown that equations for the domain wall for $SU(2)$ group can be reduced to the Monge-Ampère equation (69) or to equation (87). The boundary conditions for these equations are nonstandard and they require a further investigation. We also constructed a formal solution for $U(1)$ symmetric domain wall.

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Appendix

A Algebraic relations

We will prove here the algebraic Kaplunovsky-Sonnenschein-Yankielowicz (KSY) formula [28, 29] which is used in the investigation of the supersymmetric cycles.

We use the following notations:

$A_m$, $m = 1, 2, 3$ are complex 3-vectors, $A_m \in C^3$ with components $A_{m\alpha} \in C$, $\alpha = 1, 2, 3$. The scalar product

$(A_m, A_n) = \sum_{\alpha=1}^{3} \bar{A}_{m\alpha} A_{n\alpha},$  \hspace{1cm} (88)

here $\bar{A}_{m\alpha}$ means the complex conjugation. Note that

$(A_n, A_n) \geq 0, \hspace{1cm} \text{(no summation)}$  \hspace{1cm} (89)

$Re(A_m, A_n) = Re(A_n, A_m), \hspace{1cm} Im(A_m, A_n) = -Im(A_n, A_m).$  \hspace{1cm} (90)

One has the known formula for the Gram determinant

$det\||(A_m, A_n)|| = det\|\bar{A}_{m\alpha}\| \cdot det\|A_{m\alpha}\|,$  \hspace{1cm} (91)

which follows from the matrix relation

$\||(A_m, A_n)|| = \|\bar{A}_{m\alpha}\| \cdot \|A_{m\alpha}^T\|.$  \hspace{1cm} (92)

One also has

$det\|A_{m\alpha}\| = \epsilon_{\alpha\beta\gamma} A_{1\alpha} A_{2\beta} A_{3\gamma} = (\bar{A}_3, A_1 \times A_2).$  \hspace{1cm} (93)

Therefore the Gram determinant is

$det\||(A_m, A_n)|| = |(\bar{A}_1, A_2 \times A_3)|^2.$  \hspace{1cm} (94)
One has the following

**Lemma 1.**

\[
\det ||(A_m, A_n)|| = \det ||\text{Re}(A_m, A_n)|| - \left( (A_1, A_1)\text{Im}^2(A_2, A_3) - (A_2, A_2)\text{Im}^2(A_3, A_1) - (A_3, A_3)\text{Im}^2(A_1, A_2). \right)
\]  \hspace{1cm} (95)

**Proof.** Let us denote

\[ x_{mn} = \text{Re}(A_m, A_n), \quad y_{mn} = \text{Im}(A_m, A_n). \]

Then we have

\[
\det ||(A_m, A_n)|| = \sum_p (-)^p (x_{1p(1)} + iy_{1p(1)}) \cdot (x_{2p(2)} + iy_{2p(2)}) \cdot (x_{3p(3)} + iy_{3p(3)}) = 
\]

\[
\sum_p (-)^p (x_{1p(1)}x_{2p(2)}x_{3p(3)} - x_{1p(1)}y_{2p(2)}y_{3p(3)} - x_{2p(2)}y_{1p(1)}y_{3p(3)} - x_{3p(3)}y_{1p(1)}y_{2p(2)}) = \]  \hspace{1cm} (96)

\[
\det ||\text{Re}(A_m, A_n)|| - \sum_p (-)^p (x_{1p(1)}y_{2p(2)}y_{3p(3)} + x_{2p(2)}y_{1p(1)}y_{3p(3)} + x_{3p(3)}y_{1p(1)}y_{2p(2)}) = \]  \hspace{1cm} (97)

\[
= \det ||\text{Re}(A_m, A_n)|| - \\
\left( (A_1, A_1)\text{Im}^2(A_2, A_3) - (A_2, A_2)\text{Im}^2(A_3, A_1) - (A_3, A_3)\text{Im}^2(A_1, A_2). \right) \hspace{1cm} (98)
\]

In the line (96) we have used that the Gram determinant is real. To get line (98) from (97) note that \( y_{11} = y_{22} = y_{33} = 0 \) and that terms of the form \( x_{12}y_{23}y_{32} \) are canceled out due to the factor \((-)^p\).

**Lemma 2.**

\[
\det ||\text{Re}(A_m, A_n)|| = |(\bar{A}_3, A_1 \times A_2)|^2 + \frac{1}{2} \epsilon_{mnk} A_m \text{Im}(A_n, A_k)|^2. \]  \hspace{1cm} (99)

**Proof.** The relation (99) follows from (93) due to (94) and

\[
\frac{1}{2} \epsilon_{mnk} A_m \text{Im}(A_n, A_k)|^2 = 
\]

\[
(\bar{A}_1, A_1)\text{Im}^2(A_2, A_3) + (\bar{A}_2, A_2)\text{Im}^2(A_3, A_1) + (\bar{A}_3, A_3)\text{Im}^2(A_1, A_2). \]  \hspace{1cm} (100)
Let us set
\[ h_{mn} = \text{Re}(A_m, A_n) + \delta_m \delta_n, \quad m, n = 1, 2, 3. \] (101)

**Lemma 3.** One has
\[ \text{det}||h_{mn}|| = \text{det}||\text{Re}(A_m, A_n)|| + |A_1 \times A_2|^2 - \text{Im}^2(A_1, A_2). \] (102)

**Proof.** The matrix $||h_{mn}||$ has the form:
\[ ||h_{mn}|| = \begin{pmatrix} (A_1, A_1) & \text{Re}(A_1, A_2) & \text{Re}(A_1, A_3) \\ \text{Re}(A_2, A_1) & (A_2, A_2) & \text{Re}(A_2, A_3) \\ \text{Re}(A_3, A_1) & \text{Re}(A_3, A_2) & (A_3, A_3) + 1 \end{pmatrix} \] (103)

The determinant of $||h_{mn}||$ can be represented as
\[ \text{det} ||h_{mn}|| = \text{det} ||\text{Re}(A_m, A_n)|| + (A_1, A_1)(A_2, A_2) - \text{Re}^2(A_1, A_2). \] (104)

Using the identities
\[ |A_1 \times A_2|^2 = (A_1 \times A_2, A_1 \times A_2) = (A_1, A_1)(A_2, A_2) - |(A_1, A_2)|^2, \] (105)
\[ |(A_1, A_2)|^2 = \text{Re}^2(A_1, A_2) + \text{Im}^2(A_1, A_2) \] (106)
we get the proof of Lemma 3.

One has the following basic algebraic formula \[28, 29\]

**Proposition** (V. Kaplunovsky, J. Sonnenschein and S. Yankielowicz). One has the following representation for the determinant of the matrix $||h_{mn}||$:
\[ \text{det} ||h_{mn}|| = \text{Im}^2[(A_1, A_2) - \bar{A}_3, A_1 \times A_2] + \text{Re}^2(\bar{A}_3, A_1 \times A_2) + \]
\[ + |A_1 \times A_2 + \frac{i}{2} \epsilon_{mnk} \bar{A}_m \text{Im}(A_n, A_k)|^2. \] (107)

**Proof** of the KSY formula follows from Lemmas 1, 2 and 3.

Let us set
\[ \Phi = \text{Im}[(A_1, A_2) - (\bar{A}_3, A_1 \times A_2)] \] (108)
Corollary (V.Kaplunovsky, J.Sonnenschein and S.Yankielowicz). The relation
\[ \det ||h_{mn}|| = \Phi^2 \] (109)
is equivalent to
\[ A_1 \times A_2 + \frac{i}{2} \epsilon_{mnk} \bar{A}_m \text{Im}(A_n, A_k) = 0 \] (110)

Proof. From the KSY formula (107) it follows that to prove the Corollary one has to show that the relation
\[ A_1 \times A_2 + i \bar{A}_1 \text{Im}(A_2, A_3) + i \bar{A}_2 \text{Im}(A_3, A_1) + i \bar{A}_3 \text{Im}(A_1, A_2) = 0 \] (111)
implies the equality
\[ \text{Re}(\bar{A}_3, A_1 \times A_2) = 0. \] (112)
Let us take the scalar product of (111) with \( \bar{A}_3 \). Then we get
\[ (\bar{A}_3, A_1 \times A_2) + i(\bar{A}_3, \bar{A}_1) \text{Im}(A_2, A_3) + i(\bar{A}_3, \bar{A}_2) \text{Im}(A_3, A_1) + i(\bar{A}_3, \bar{A}_3) \text{Im}(A_1, A_2) = 0, \] (113)
or
\[ (\bar{A}_3, A_1 \times A_2) + i(A_1, A_3) \text{Im}(A_2, A_3) + i(A_2, A_3) \text{Im}(A_3, A_1) + i(A_3, A_3) \text{Im}(A_1, A_2) = 0. \] (114)
Now let us take the real part of (114)
\[ \text{Re}(\bar{A}_3, A_1 \times A_2) = \text{Im}(A_1, A_3) \text{Im}(A_2, A_3) + \text{Im}(A_2, A_3) \text{Im}(A_3, A_1). \] (115)
The right hand side of (113) vanishes since \( \text{Im}(A_2, A_3) = -\text{Im}(A_3, A_2) \). The Corollary is proved.

Let us notice that there is also an equivalent formulation of the KSY proposition and Corollary, that can be formulated as

Theorem. One has the following relation
\[ \det ||h_{mn}|| = \Phi^2 + |R|^2 + \text{Re}^2(\bar{A}_3, R), \] (116)
where
\[ R = A_1 \times A_2 + \frac{i}{2} \epsilon_{mnk} \bar{A}_m \text{Im}(A_n, A_k). \] (117)

Proof. The theorem follows from Lemmas 1,2,3 and the relation
\[ \text{Re}(\bar{A}_3, A_1 \times A_2) = \text{Re}(\bar{A}_3, R). \] (118)
This formulation is convenient because the corollary now is more clear since
\[ |R|^2 + \text{Re}^2(\bar{A}_3, R) = 0 \] (119)
is obviously equivalent to \( R = 0 \).
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