TRANSCENDENTAL GROUPS

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Abstract. In this note we introduce the notion of a transcendental group, that is, a subgroup $G$ of the topological group $\mathbb{C}$ of all complex numbers such that every element of $G$ except 0 is a transcendental number. All such topological groups are separable metrizable zero-dimensional torsion-free abelian groups. Further, each transcendental group is homeomorphic to a subspace of $\mathbb{N}^{\aleph_0}$, where $\mathbb{N}$ denotes the discrete space of natural numbers. It is shown that (i) each countably infinite transcendental group is a member of one of three classes, where each class has $\mathfrak{c}$ (the cardinality of the continuum) members – the first class consists of those isomorphic as a topological group to the discrete group $\mathbb{Z}$ of integers, the second class consists of those isomorphic as a topological group to $\mathbb{Z} \times \mathbb{Z}$, and the third class consists of those homeomorphic to the topological space $\mathbb{Q}$ of all rational numbers; (ii) for each cardinal number $\aleph$ with $\aleph_0 < \aleph \leq \mathfrak{c}$, there exist $2^\aleph$ transcendental groups of cardinality $\aleph$ such that no two of the transcendental groups are isomorphic as topological groups or even homeomorphic; (iii) there exist $\mathfrak{c}$ countably infinite transcendental groups each of which is homeomorphic to $\mathbb{Q}$ and algebraically isomorphic to a vector space over the field $\mathbb{A}$ of all algebraic numbers (and hence also over $\mathbb{Q}$) of countably infinite dimension; (iv) $\mathbb{R}$ has $2^\mathfrak{c}$ transcendental subgroups, each being a zero-dimensional metrizable torsion-free abelian group, such that no two of the transcendental groups are isomorphic as topological groups or even homeomorphic.

1. Introduction

This paper initiates the study of transcendental groups which is an interesting combination of number theory, algebra, and topology. We shall see that transcendental groups are a wonderfully rich source of examples of zero-dimensional separable metrizable topological groups. Transcendental groups may also lead to a new way of looking at problems in transcendental number theory.

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Dedicated to Ralph Kopperman.
Remarks 1.1. We shall discuss four fields: \( \mathbb{C} \), the field of all complex numbers; \( \mathbb{R} \), the field of all real numbers; \( \mathbb{A} \), the field of all algebraic numbers; and \( \mathbb{Q} \), the field of all rational numbers. Observe the following easily verified facts:

(i) the fields \( \mathbb{C} \) and \( \mathbb{R} \) have cardinality \( \aleph_1 \), the cardinalty of the continuum;
(ii) the fields \( \mathbb{A} \) and \( \mathbb{Q} \) have cardinality \( \aleph_0 \);
(iii) \( \mathbb{C} \) with its euclidean topology is isomorphic as a topological group to \( \mathbb{R} \times \mathbb{R} \), where \( \mathbb{R} \) has its euclidean topology;
(iv) each of these four fields has a natural topology; \( \mathbb{C} \) and \( \mathbb{R} \) have euclidean topologies, while \( \mathbb{A} \) and \( \mathbb{Q} \) inherit a natural topology as a subspace of \( \mathbb{C} \);
(v) the topological group \( \mathbb{Q} \) is a dense subgroup of the topological group \( \mathbb{R} \) (that is, the closure, in the topological sense, of \( \mathbb{Q} \) is \( \mathbb{R} \));
(vi) the topological group \( \mathbb{A} \) is a dense subgroup of the topological group \( \mathbb{C} \);
(vii) \( \mathbb{C} \supset \mathbb{A} \supset \mathbb{A} \cap \mathbb{R} \supset \mathbb{Q} \), but \( \mathbb{A} \) is not a subset of \( \mathbb{R} \);
(viii) the field \( \mathbb{C} \) is a vector space of dimension \( \aleph_1 \) over \( \mathbb{A} \) and it is also a vector space of dimension \( \aleph_1 \) over \( \mathbb{Q} \);
(ix) using the Axiom of Choice, we see that for any set of linearly independent vectors in a vector space \( V \), there is another linearly independent set of vectors in \( V \) such that the union of the two linearly independent sets is a basis for the vector space \( V \); this implies from (viii) that there exists a vector space \( \mathbb{B} \) over the field \( \mathbb{Q} \) such that the vector space \( \mathbb{C} \) is isomorphic as a vector space to the direct sum of the vector spaces \( \mathbb{A} \) and \( \mathbb{B} \) over \( \mathbb{Q} \); that is, \( \mathbb{C} \cong \mathbb{A} \oplus \mathbb{B} \); (This is an algebraic isomorphism and definitely not a topological group isomorphism since \( \mathbb{C} \) is a connected topological space while \( \mathbb{A} \), being countable, is not a connected topological space.)
(x) from (ix), the vector space \( \mathbb{B} \) has dimension \( \aleph_1 \) over \( \mathbb{Q} \);
(xi) \( \mathbb{B} \), \( \mathbb{R} \), and \( \mathbb{C} \) are each a vector space of dimension \( \aleph_1 \) over \( \mathbb{Q} \); so \( \mathbb{B} \), \( \mathbb{R} \), and \( \mathbb{C} \) are algebraically isomorphic as groups to each other and to a restricted direct sum of \( \aleph_1 \) copies of \( \mathbb{Q} \);
(xii) \( \mathbb{A} \) is a vector space of countably infinite dimension over \( \mathbb{Q} \);
(xiii) \( \mathbb{B} \) and \( \mathbb{C} \) are each a vector space of dimension \( \aleph_1 \) over \( \mathbb{A} \);
(xiv) \( \mathbb{B} \) is a topological group which is algebraically isomorphic to both \( \mathbb{R} \) and \( \mathbb{C} \).

We shall focus on \( \mathcal{T} \), the topological space of all transcendental numbers, where \( \mathcal{T} = \mathbb{C} \setminus \mathbb{A} \) and has a natural topology as a subspace of \( \mathbb{C} \). The
topology of $\mathcal{T}$ is separable, metrizable, and zero-dimensional. Also the cardinality of $\mathcal{T}$ is $\mathfrak{c}$. □

**Definition 1.2.** A topological group $G$ is said to be a **transcendental group** if it is a subgroup of the topological group $\mathbb{C}$ of all complex numbers such that every element of $G$ except $0$ is a transcendental number. □

As transcendental groups are topological subgroups of the separable metrizable group $\mathbb{C}$, each is separable and metrizable and has cardinality not greater than $\mathfrak{c}$. As $\mathbb{C}$ is a torsion-free abelian group, every transcendental group is an infinite torsion-free abelian group. Since transcendental groups are subspaces of the space $\mathcal{T}$, they are zero-dimensional. This is summarized in Proposition 1.3.

**Proposition 1.3.** Every transcendental group is an infinite separable metrizable zero-dimensional torsion-free abelian group of cardinality not greater than $\mathfrak{c}$. □

**Proposition 1.4.** The topological group $\mathbb{B}$ introduced in Remarks 1.1 is a transcendental group. Further, every transcendental group is algebraically isomorphic to a subgroup of $\mathbb{B}$.

**Proof.** Clearly from the definition of $\mathbb{B}$ in Remarks 1.1, $\mathbb{B}$ is a subset of $\mathcal{T}$ and so it is a transcendental group. Let $p$ be the projection homomorphism of $\mathbb{C}$ onto $\mathbb{B}$. Let the subgroup $T$ of $\mathbb{C}$ be a transcendental group. If $t \in T$, $t \neq 0$, then $p(t) \neq 0$ since otherwise $t \in \mathbb{A}$, which is false as $t$ is transcendental. So $p$ is a one-to-one homomorphism of $T$ onto the subgroup $p(T)$ of $\mathbb{B}$. Thus $T$ is algebraically isomorphic to a subgroup of $\mathbb{B}$, and the proposition is proved. □

**2. Discrete Transcendental Groups**

If $S$ is a subset of a group $G$, then $\text{gp}(S)$ denotes the subgroup of $G$ generated algebraically by the set $S$. If $S$ is the singleton set $\{g\}$, then this group is denoted by $\text{gp}\{g\}$ and equals $\{ng : n \in \mathbb{Z}\}$.

Our first proposition is obvious.

**Proposition 2.1.** Let $t$ be any transcendental number in $\mathbb{C}$. Then $\text{gp}\{t\}$ is a discrete countably infinite transcendental group and is isomorphic as a topological group to the discrete group $\mathbb{Z}$ of all integers. □

**Corollary 2.2.** There exist $\mathfrak{c}$ distinct transcendental groups each of which is topologically isomorphic to $\mathbb{Z}$.

**Proof.** If $\text{gp}\{t_1\} = \text{gp}\{t_2\}$, for positive transcendental numbers $t_1$ and $t_2$, then $t_1 = n_1t_2$ and $t_2 = n_2t_1$, for $n_1, n_2 \in \mathbb{N}$. But then $t_1 = n_1n_2t_1$, which implies $n_1 = n_2 = 1$ and so $t_1 = t_2$. Thus distinct positive transcendental
numbers generate distinct transcendental groups each of which is topologically isomorphic to $\mathbb{Z}$. The result now immediately follows from the fact that there are $\epsilon$ distinct positive transcendental numbers.

Let us now turn to groups generated by two (unequal) transcendental numbers $t_1$ and $t_2$. If $t_2 = t_1 + 1$, then $gp\{t_1, t_2\}$ contains $t_2 - t_1 = 1$. So $gp\{t_1, t_2\}$ is not a transcendental group. But we are very close to open questions. For example, while the Euler number $e$ and the number $\pi$ are transcendental numbers, it is not known if either $\pi + e$ or $\pi - e$ is transcendental (although one of them must obviously be transcendental).

Much modern transcendental number theory centres on linear independence. (See [7].) In this context we have our next proposition.

**Proposition 2.3.** Let $a$ and $b$ be transcendental numbers with $a, b \neq 0$. Then $gp\{a, b\}$ is a cyclic transcendental group if and only if $a$ and $b$ are linearly dependent over $\mathbb{Q}$.

**Proof.** If $gp\{a, b\}$ is a cyclic group, then there exists $c \in gp\{a, b\}$ such that $a = mc$ and $b = nc$, for some $m, n \in \mathbb{Z} \setminus \{0\}$. So $na - mb = 0$. So $a$ and $b$ are linearly dependent over $\mathbb{Q}$.

Now consider the case that $a$ and $b$ are linearly dependent over $\mathbb{Q}$; that is, there exist $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, with $m_1, m_2, n_1, n_2 \neq 0$, such that

$$\frac{m_1}{n_1} a + \frac{m_2}{n_2} b = 0$$

Consider the element $d \in \mathbb{C}$, defined by $d = \frac{a}{m_2 n_1}$. The number $d$ is transcendental and $a \in gp\{d\}$. As $b = -\frac{m_1 n_2}{m_2 n_1} a = -m_1 n_2 d$ we also have $b \in gp\{d\}$. Thus $gp\{a, b\}$ is a subgroup of the cyclic transcendental group $gp\{d\}$. Hence $gp\{a, b\}$ is a cyclic transcendental group. □

Proposition 2.3 tells us, for example, that the numbers $e$ and $\pi$ are linearly independent over $\mathbb{Q}$ if and only if $gp\{e, \pi\}$ is not a cyclic transcendental group. If $e + \pi$ is an algebraic number, then $gp\{e, \pi\}$ is not a transcendental group. So $e$ and $\pi$ must be linearly independent over $\mathbb{Q}$. Note that if Schanuel’s Conjecture [4, 7] is true, then $e$ and $\pi$ are indeed linearly independent over $\mathbb{Q}$.

**Proposition 2.4.** If $G$ is a discrete transcendental group and $G \neq \{0\}$, then $G$ is topologically isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

**Proof.** By Theorem 6 of [6] or Theorem A.12 of [3], every nontrivial closed subgroup of $\mathbb{R}^n$, for $n \in \mathbb{N}$, is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b$, where $a, b \in \mathbb{N} \cup \{0\}$ and $a + b \leq n$. As $\mathbb{C}$ is topologically isomorphic to $\mathbb{R}^2$, every discrete subgroup of $\mathbb{C}$ is topologically isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. □
We shall see in Theorem 4.5 that, if $i$ as usual denotes an imaginary number who square is $-1$, the transcendental group $\text{gp}\{e, ei\}$ is topologically isomorphic to the discrete $\mathbb{Z} \times \mathbb{Z}$.

3. Counting Transcendental Groups

Firstly, in this section, we identify concrete non-cyclic countably infinite transcendental groups. We begin by recalling the following theorem:

Generalized Lindemann Theorem. [5, Theorem 9.1] For any distinct algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, $m \in \mathbb{N}$, the values $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_m}$ are linearly independent over the field $\mathbb{A}$ of algebraic numbers. □

**Theorem 3.1.** Let $\Omega$ be any finite or infinite set of distinct algebraic numbers such that $0 \notin \Omega$. Then the set $\Gamma = \{e^a : a \in \Omega\}$ generates the transcendental group $\text{gp}(\Gamma)$.

**Proof.** Let $\alpha_2, \alpha_3, \ldots, \alpha_m \in \Omega$, $m \in \mathbb{N}$, and put $\alpha_1 = 0$. The Generalized Lindemann Theorem implies that $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_m}$ are linearly independent over the field $\mathbb{A}$ of algebraic numbers; that is, for $a_1, a_2, \ldots, a_m \in \mathbb{A}$ with $a_1, a_2, \ldots, a_m \neq 0$,

$$a_1 e^{\alpha_1} + a_2 e^{\alpha_2} + \cdots + a_m e^{\alpha_m} \neq 0.$$ 

As $a_1 e^{\alpha_1} = a_1 \in \mathbb{A}$, this says

$$a_2 e^{\alpha_2} + \cdots + a_m e^{\alpha_m} \notin \mathbb{A}.$$ 

Thus $\Gamma$ is a transcendental group. □

**Remark 3.2.** Noting that there are $c$ distinct sets $\Omega$ satisfying the conditions of Theorem 3.1, we obtain $c$ distinct countably infinite transcendental groups $\text{gp}(\Gamma)$ each of which is algebraically isomorphic to a vector space over $\mathbb{A}$ (and hence also over $\mathbb{Q}$) of countably infinite dimension. □

**Theorem 3.3.** Let $S = \{\alpha_i : i \in I\}$ for some index set $I$ such that each $\alpha_i$ is an algebraic number with $\alpha_i \neq 0$. Put $T = \{\log(\alpha_i) : \alpha_i \in S\}$. Then $\text{gp}(T)$ is a transcendental group.

**Proof.** Let $g \in \text{gp}(T)$, $g \neq 0$. Then for each $i = 1, 2, \ldots, n$ with $n \in \mathbb{N}$,

$$g = m_1 \log(\alpha_1) + m_2 \log(\alpha_2) + \cdots + m_n \log(\alpha_n), \quad m_i \in \mathbb{Z} \setminus \{0\}$$

$$= \log(\alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_n^{m_n}).$$

So $e^g = \alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_n^{m_n}$.

Suppose $g$ is an algebraic number. Then by Theorem 9.11 of [5], $e^g$ is a transcendental number, while $\alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_n^{m_n}$ is an algebraic number, which is a contradiction. So $g$ is a transcendental number, and $\text{gp}(T)$ is a transcendental group. □
Theorem 3.4. There exist $2^\mathfrak{c}$ transcendental groups.

Proof. Consider $\mathbb{C}$ as a vector space over $\mathbb{A}$ of dimension $\mathfrak{c}$. Let $R$ be a basis for this vector space such that $1 \in R$. Put $T = \{x \in R : x \notin \mathbb{A}\}$. For each subset $S'$ of $T$, let $S = S' \cup \{1\}$. Then $S$ is linearly independent over $\mathbb{A}$.

Let $g \in \text{gp}(S'), g \neq 0$ and suppose $g$ is not a transcendental number. Then $g = m_1s_1 + m_2s_2 + \cdots + m_n s_n$, where $s_1, s_2, \ldots, s_n \in S'$ and $m_1, m_2, \ldots, m_n \in \mathbb{Z} \setminus \{0\}$. Thus

$$g = m_1s_1 + m_2s_2 + \cdots + m_n s_n = a \in \mathbb{A}.$$ 

So $m_1s_1 + m_2s_2 + \cdots + m_n s_n + (-a)1 = 0$. But this contradicts the fact that $s_1, s_2, \ldots, s_n, 1 \in S$ and so are linearly independent over $\mathbb{A}$. Hence $g$ is indeed a transcendental number. Therefore $\text{gp}(S')$ is a transcendental group. As there are $2^\mathfrak{c}$ different such $S'$ each generating a different group, the theorem is proved. $\square$

As a consequence of the proof of Theorem 3.4 we have the following theorem.

Theorem 3.5. For each cardinal number $\aleph$ with $\aleph_0 \leq \aleph \leq \mathfrak{c}$, there exist $2^\aleph$ transcendental groups of cardinality $\aleph$. $\square$

If in the proof of Theorem 3.4 we replace $\mathbb{C}$ by $\mathbb{R}$ and $\mathbb{A}$ by $\mathbb{A} \cap \mathbb{R}$, then we obtain the following theorem:

Theorem 3.6. For each cardinal number $\aleph$ with $\aleph_0 \leq \aleph \leq \mathfrak{c}$, there exist $2^\aleph$ transcendental groups, each of which is a topological subgroup of $\mathbb{R}$ and has cardinality $\aleph$. $\square$

Remark 3.7. Corollary 1.2 of [2] states that if $G$ is an abelian group of cardinality $\aleph > \aleph_0$, then $G$ has $2^\aleph$ subgroups. If one knew that $G$ has as a subgroup a transcendental group of cardinality $\aleph$, then one could deduce Theorem 3.5 except for the case $\aleph = \aleph_0$. However, the proof in [2] is not shorter than the one presented here. $\square$

4. The Topology of Transcendental Groups

Remark 4.1. Jan van Mill drew my attention to Theorem 1.9.6, Theorem 1.9.8, and Corollary 1.9.9 of [9] where it is proved that

(i) the space $\mathbb{Q}$ of all rational numbers up to homeomorphism is the unique nonempty countable separable metrizable space without isolated points.

(ii) the space $\mathbb{P}$ of all irrational numbers up to homeomorphism is the unique nonempty separable metrizable topologically complete nowhere locally compact zero-dimensional space.
(iii) \( \mathbb{P} \) is homeomorphic to \( \mathbb{N}^{\aleph_0} \).

We have already noted that all transcendental groups are separable, metrizable and zero-dimensional.

As immediate corollaries of the observations in Remark 4.1 we have:

**Corollary 4.2.** The topological space \( A \) of all algebraic numbers is homeomorphic to \( \mathbb{Q} \).

**Corollary 4.3.** Every countably infinite separable metrizable topological group is either discrete or homeomorphic to \( \mathbb{Q} \).

**Lemma 4.4.** Let \( a \) be any transcendental number which is also a nonzero real number. Then \( \text{gp}\{a, ai\} \) is a transcendental group isomorphic as a topological group to \( \mathbb{Z} \times \mathbb{Z} \).

**Proof.** By Remarks 4.1(i), \( \text{gp}\{a, ai\} \) is either a discrete group or has 0 as a nonisolated point. But for \( z \in \text{gp}\{a, ai\}, z \neq 0 \), for \( m, n \in \mathbb{Z} \), not both zero,

\[
|z| = |ma + nai| = \sqrt{(m^2 + n^2)|a^2|} \geq |a|.
\]

So 0 is indeed an isolated point. Thus \( \text{gp}\{a, ai\} \) is discrete.

**Theorem 4.5.** There exist \( \mathfrak{c} \) countably infinite transcendental groups of the form \( \text{gp}\{a, ai\} \) where each is isomorphic as a topological group to \( \mathbb{Z} \times \mathbb{Z} \).

**Proof.** Using a similar argument used to prove those statements in Remark 4.1 one readily obtains:

**Corollary 4.6.** The topological space of transcendental numbers is homeomorphic to both \( \mathbb{P} \) and \( \mathbb{N}^{\aleph_0} \).

**Remark 4.7.** It follows from Corollary 4.6 that every transcendental group is homeomorphic to a subspace of \( \mathbb{N}^{\aleph_0} \).

**Remark 4.8.** From Corollary 4.3 and Theorem 3.3 we see that if \( S = \{\alpha_n = 1 + \frac{1}{n} : n \in \mathbb{N}\} \), then 0 is a limit point of \( \text{gp}(T) \). So the topological group \( \text{gp}(T) \) has no isolated points and therefore is homeomorphic to \( \mathbb{Q} \).

**Theorem 4.9.** There exist \( \mathfrak{c} \) transcendental groups homeomorphic to \( \mathbb{Q} \).

**Proof.** Let \( S' \) be any subset of \( S \) such that \( \frac{1}{n} \notin S' \), for all \( n \in \mathbb{N} \). Put \( S = S' \cup \{\frac{1}{n} : n \in \mathbb{N}\} \). Define \( T_S = \{e^{\alpha} : \alpha \in S\} \). As observed in Theorem 5.1 \( \text{gp}(T_S) \) is a transcendental group. Clearly \( e^x \to 1 \) as \( n \to \infty \). So the countably infinite group \( \text{gp}(T_S) \) is not discrete. By Corollary 4.3 \( \text{gp}(T_S) \) is therefore homeomorphic to \( \mathbb{Q} \). As there are \( \mathfrak{c} \) different possible choices of \( S' \), the theorem is proved.
Theorem 4.10. Each countably infinite transcendental group is a member of one of three classes, where each class consists of those isomorphic as a topological group to the discrete group $\mathbb{Z}$ of integers, the second class consists of those isomorphic as a topological group to $\mathbb{Z} \times \mathbb{Z}$, and the third class consists of those homeomorphic to the topological space $\mathbb{Q}$ of all rational numbers.

Proof. The theorem is an immediate consequence of Corollary 2.2, Proposition 2.4, Corollary 4.3, Theorem 4.5, and Theorem 4.9.

Remark 4.11. Jan van Mill has pointed out to me a beautiful consequence of Theorem 3.4. The Laverentieff Theorem, Theorem A8.5 of [9], implies that there are at most $\mathfrak{c}$ subspaces of $\mathbb{C}$ which are homeomorphic. So from Theorem 3.4 there are $2^\omega$ transcendental groups no two of which are homeomorphic.

Of course this trivially has the consequence there are $2^\omega$ transcendental groups no two of which are isomorphic as topological groups.

So the topological group $\mathbb{B}$, introduced in Remarks 1.1 which is algebraically isomorphic to $\mathbb{R}$, has $2^\omega$ subgroups no two of which are isomorphic as topological groups.

Noting Remark 4.11 and Theorem 3.6 we obtain:

Theorem 4.12. For each cardinal $\aleph$ with $\aleph_0 < \aleph \leq \mathfrak{c}$, the topological groups $\mathbb{C}$ and $\mathbb{R}$ each have $2^\aleph$ transcendental subgroups no two of which are isomorphic as topological groups or even homeomorphic.

Our next corollary is clear since each Banach space has a subgroup isomorphic as a topological group to $\mathbb{R}$. The importance of closed totally disconnected subgroups of Banach is well known, see [1, 8].

Corollary 4.13. Let the separable Banach space $E$ be finite or infinite dimensional. For each cardinal $\aleph$ with $\aleph_0 < \aleph \leq \mathfrak{c}$, $E$ has $2^\aleph$ transcendental subgroups no two of which are isomorphic as topological groups or even homeomorphic.

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Dedication. This paper is dedicated to Ralph Kopperman who was a friend and coauthor. He visited me in Australia a number of times and I visited him in New York. I enjoyed doing research with him. We discussed coauthoring a book but we were both too strong-minded for that to be a success. He worked tirelessly for the Summer Topology Conference. My fondest memory is of his visiting me in Armidale, New
South Wales a regional Australian city of 25,000 people hundreds of miles from any large city and his thoroughly enjoying the wonderful Southern Hemisphere night sky. He was particularly impressed with the Clouds of Magellan (two dwarf galaxies visible in the Southern Hemisphere sky).

REFERENCES

[1] F. D. Ancel and T. Dobrowolski and J. Grabowski, Closed Subgroups in Banach Spaces, Studia Mathematica. 109, (1994), 278–289.
[2] S. Berhanu and W. W. Comfort and J. D. Reid, Counting Subgroups and Topological Group Topologies, Pacific J. Math. 116 (1985), 217–241.
[3] K. H. Hofmann and S. A. Morris, The Structure of Compact Groups. Walter de Gruyter, 2020.
[4] S. Lang, Introduction to Transcendental Groups. Addison-Wesley Publishing Company 1966.
[5] I. Niven, Irrational Numbers, Carus Mathematical Monographs, No. 11. Mathematical Association of America, 1967.
[6] S. A. Morris, Pontryagin Duality and the Structure of Locally Compact Abelian Groups. Cambridge University Press, 1977.
[7] M. R. Murty and P. Rath, Transcendental Numbers. Springer, 2014.
[8] S. J. Sidney, Weakly Dense Subgroups of Banach Spaces, Indiana University Math. J. 26 (1977), 981–986.
[9] J. van Mill, The Infinite-Dimensional Topology of Function Spaces. Elsevier, 2001.

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