Detecting unambiguously non-Abelian geometric phases with trapped ions

Xin-Ding Zhang¹, Z D Wang², Liang-Bin Hu¹, Zhi-Ming Zhang³ and Shi-Liang Zhu¹,²,⁴

¹ Institute for Condensed Matter Physics, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou, People’s Republic of China
² Department of Physics and Center of Theoretical and Computational Physics, University of Hong Kong, Pokfulam Road, Hong Kong, People’s Republic of China
³ Laboratory of Photonic Information Technology, South China Normal University, Guangzhou, People’s Republic of China
E-mail: slzhu@scnu.edu.cn

New Journal of Physics 10 (2008) 043031 (10pp)
Received 5 October 2007
Published 15 April 2008
Online at http://www.njp.org/
doi:10.1088/1367-2630/10/4/043031

Abstract. We propose an experimentally feasible scheme to disclose the noncommutative effects induced by a light-induced non-Abelian gauge structure with trapped ions. Under an appropriate configuration, a true non-Abelian gauge potential naturally arises in connection with the geometric phase associated with two degenerated dark states in a four-state atomic system interacting with three pulsed laser fields. We show that the population in the atomic state at the end of a composed path formed by two closed loops $C_1$ and $C_2$ in the parameter space can be significantly different from the composed counter-ordered path. This population difference is directly induced by the noncommutative feature of non-Abelian geometric phases and can be detected unambiguously with current technology.
Gauge structures have been used to describe almost all the fundamental interactions in nature and have been shown to play a key role in modern physics. Gauge fields are usually classified as Abelian and non-Abelian ones according to the commutative rule of the associated group elements. Remarkably, they also arise naturally in the adiabatic evolution of simple quantum systems, whose initial formulation has no apparent relationship to gauge fields [1, 2]. Berry demonstrated that the adiabatic cyclic evolution of a non-degenerated state leads to a scalar geometric phase factor corresponding to a $U(1)$ Abelian gauge field [1]. Wilczek and Zee further generalized this idea to the case of degenerate states and showed that a non-Abelian structure emerges if a set of quantum states remain degenerate as the Hamiltonian varies. When a system with $N$-fold degenerate levels undergoes cyclic process, the mapping corresponds to a geometric phase factor with $SU(N) \times U(1)$ matrix form [2, 3]. So far, gauge structures in such simple quantum systems have found many applications in diverse fields in physics [4]–[15].

Following the progress achieved in theory and application, direct experimental observations of gauge structures in quantum systems attract increasing interest. Although Abelian geometric phases have been tested experimentally by various means [4, 5], the noncommutative effect, a key property of non-Abelian structures has not been experimentally demonstrated. In an interesting experiment [16], Tycko demonstrated the effect of geometric phase on the magnetic resonance spectrum of two-fold degenerated states [16]. However, as pointed out in [3], because of the experimental limitation in nuclear magnetic resonance (NMR), only the (commutative) Abelian part of the non-Abelian gauge structure has been experimentally observed and the direct observation of noncommutativity of non-Abelian gauge structures in the NMR system is really difficult. Detection of non-Abelian geometric phases with two photonic qubits was also suggested in [17], but the corresponding phase factor has never been detected experimentally. Recent experiments in trapped-ion systems provide high-precise manipulations and measurements on internal states of atoms [18]–[24]. Comparing with the NMR systems, true non-Abelian structures can be easier to be realized by modulating controlling parameters in principle. Thus it is significant and important to explore whether a trapped-ion system is promising for detecting the fundamental non-Abelian characteristic.

In this paper, we analyze in detail why non-Abelian structures have not been observed in NMR experiments, and then propose an experimentally feasible scheme to directly detect the observable effect induced by the noncommutative property of the non-Abelian gauge potentials in a trapped-ion system$^5$. We show that a true non-Abelian gauge potential naturally arises in connection with the geometric phase associated with two degenerated dark states in a four-state atomic system interacting with three pulsed laser fields. The system under consideration is a four-level atom interacting with resonant external laser pulses. Under an appropriate configuration, there exist two-fold degenerated dark states and then a non-Abelian gauge potential naturally arises. We design two specific closed loops $C_1$ and $C_2$ in the parameter space, which generate two different geometric phase factors $U_1$ and $U_2$, respectively. Consider the composite path in which one traverses first $C_1$ and then $C_2$. After a cycle, a phase factor of $U = U_2U_1$ is generated. In contrast, the other phase factor of $U' = U_1U_2$ is accumulated if one traverses first $C_2$ and then $C_1$. We find that the population of the atoms in a specific state at the end of the two evolutions $U$ and $U'$ can be significantly different. For instance, the population difference can be as high as 43% for the typical experimental parameters. Note

$^5$ The scheme proposed here is equally suitable for cold atoms since the effective Hamiltonian for an atom interacting with three laser pulses may be exactly the same with the Hamiltonian (9).
that the observed precision of an atomic state can be higher than 99%, and thus the proposed noncommutative effect can be detected unambiguously. Since the variations of the parameters associated with the non-Abelian geometric phases can also be easily manipulated via the laser beams, the proposed scheme is very promising for detecting the fundamental non-Abelian gauge structures. In contrast to the arguments for observing the non-Abelian effects with photonic qubits [17] based on the interferometry [25], here we design explicitly a detailed experimental procedure to directly detect the noncommutative effects.

We start by reviewing a general framework of non-Abelian adiabatic geometric phase [2, 3] and its possible detection in nuclear quadrupole resonance [16]. Consider a family of Hamiltonians $H(\chi_\mu) (\mu = 1, 2, \ldots, n)$ depending continuously on parameters $\chi_\mu$, all of which have a set of $N$ degenerate levels. Let $\eta_a$ be an $N$-fold degenerate set of orthonormal instantaneous eigenstates of the Hamiltonian $H$ described by $H|\eta_a\rangle = E|\eta_a\rangle$ with $E$ being the eigenvalue. Under the adiabatic condition the instantaneous state $|\psi_a\rangle$ will not overflow the state vector space spanned by $\eta_a$. So $|\psi_a\rangle$ can always be expanded as a superposition of $|\eta_a\rangle$: $|\psi_a\rangle = \sum_b |\eta_a\rangle U_{ba}$. Substituting the wavefunction $|\psi_a\rangle$ into the time-dependent Schrödinger equation $i\frac{d}{dt}|\psi_a\rangle = H|\psi_a\rangle$, we have

$$\dot{U}_{ba} = -\sum_c \langle \eta_b | \dot{\eta}_c \rangle U_{ca},$$

where a trivial overall dynamical phase term has been dropped. A gauge structure appears when the Hamiltonian $H$ varies as the parameters $\chi_\mu(t)$ vary slowly with time $t$. We can define a gauge potential

$$A_{ab\mu} = \langle \eta_a | \frac{\partial}{\partial \chi_\mu} | \eta_b \rangle,$$

and have $\langle \eta_a | \dot{\eta}_b \rangle = \sum_{\mu} A_{ab\mu} (d\chi_\mu/dt)$. Integrating equation (2), it is straightforward to obtain

$$U_{ab} = \left[ P \exp \left( -\int A_{\mu} d\chi_\mu \right) \right]_{ab},$$

where $P$ denotes the path-ordered operator. The quantity $A = \sum_\mu A_{\mu} d\chi_\mu$ defined in equation (2) indeed plays the role of a gauge potential and the gauge group corresponds to the unitary freedom in choosing the basis states $\eta_a$ [2]. An intriguing feature of the gauge potential $A$ lies in that it depends only on the geometry of executed path in the space of degenerate levels.

In an earlier experiment, Tycko reported the effect of geometric phase on the nuclear-quadrupole-resonance spectra [16]. The effective Hamiltonian describing Tycko’s experiment is given by

$$H = (S \cdot B)^2 = (S_x \sin \theta \cos \phi + S_y \sin \theta \sin \phi + S_z \cos \theta)^2 B^2,$$

where $S$ denotes the nuclear spin operator for the sample used in the experiment (a spin-$\frac{3}{2}$ Cl atom in a NaClO$_3$ crystal is considered), and $B$ represents an external magnetic field. There always exists a pairwise degeneracy of states in the parameter space because the Hamiltonian is invariant under the operation $S \rightarrow -S$, thus non-Abelian gauge structure may appear in the system. To address clearly the gauge structure, one can parameterize the Hamiltonian (4) as follows:

$$H = B^2 e^{-i\phi S_z} e^{-i\theta S_y} e^{i\theta S_x} e^{i\phi S_z}.$$
As a result, the instantaneous eigenstates can be directly written as
\[ |\eta_a\rangle = e^{-i\omega S_z}e^{-i\theta S_x}|a\rangle, \]
where \(|a\rangle\) is one of the atomic states in the set \(|\pm \frac{3}{2}\rangle, |\pm \frac{1}{2}\rangle\). Here the state \(|m\rangle\) represents the eigenstate of the spin operator \(S_z\) defined by \(S_z|m\rangle = m|m\rangle\). The states \(|\pm m\rangle\) form a doubly degenerate sector, then the two sectors \(|\pm 3/2\rangle\) and \(|\pm 1/2\rangle\) can be studied separately. Substituting equation (6) into (2), the corresponding gauge potentials are given by
\[ A_{ab\theta} = -(i)\langle a| (\cos \theta S_z - \sin \theta S_x)|b\rangle, \quad A_{ab\theta} = (-i)\langle a| S_y|b\rangle. \]
As for the sector \(|\pm 3/2\rangle\), one can obtain an Abelian structure with the gauge potential \(A = -3i\cos \theta/2\) although the states there are doubly degenerate [3]. However, a true non-Abelian structure may appear for the sector \(|\pm 1/2\rangle\). Explicitly, the gauge potentials for this case are given by
\[ A = -i \left[ \frac{1}{2} \sigma_x \cos \theta - \frac{\alpha}{2} \sigma_y \sin \theta \right] d\varphi + \frac{\alpha}{2} \sigma_y d\theta, \]
with \(\alpha = S + 1/2\) an integer. In this sector, \(\{S_x, S_y, S_z\}\) in equations (7) are replaced by \(\{\alpha \sigma_x, \alpha \sigma_y, \sigma_z/2\}\) with \(\sigma_{x,y,z}\) the Pauli matrices in this sector.

To disclose unambiguously the non-Abelian characteristic of the geometric phase, we must detect the physically observable effects induced by the noncommutativity of the gauge structure. To this end, we consider two closed loops \(C_1\) and \(C_2\), both starting and ending at the same point \(\chi_0\) in the parameter space, where the two corresponding evolution operators are denoted as \(U_1\) and \(U_2\), respectively. For the composite path that the system first traverses along loop \(C_1\) and then \(C_2\), the total phase factor of \(U = U_2U_1\) is generated. If the counter-ordered evolution is executed, i.e. an evolution with first loop \(C_2\) and then \(C_1\), the phase factor generated is then given by that of \(U' = U_1U_2\). In a general experiment, if the observable effects are different for the phase factors of \(U\) and \(U'\), then non-Abelian geometric structures are confirmed there. Otherwise, only the Abelian part of the gauge structure is observed even if the quantum states of the system are degenerate.

From the above discussion, it is easy to understand that only the Abelian part of the gauge structures has been experimentally observed in Tycko’s experiment, as has been addressed in [3]. In Tycko’s experiment, the angle \(\theta\) is fixed as a constant \((\cos \theta = 1/\sqrt{3})\). It is due to that only the rotation around a fixed axis can be implemented fast enough to generate observable phase shift in the NMR systems. In this case gauge structure \(A\) is proportional to a fixed matrix \(\cos \theta \frac{\sigma_x}{2} - \alpha \sin \theta \frac{\sigma_y}{2}\) and the evolution operator can be written as \(U = \exp[-i\varphi(\sigma_z/\sqrt{3} - \sigma_x)]\). Thus the \(U\) and gauge potential \(A\) in different time sequences are always commutable. From this point of view, we say that the gauge potential loses its non-Abelian characteristic and is of Abelianization. In order to demonstrate the non-Abelian characteristic in nuclear quadrupole resonance, the system should traverse a non-Abelian path in which both \(\theta\) and \(\varphi\) vary with time. Unfortunately, it is difficult in an NMR set-up.

Recently, with the rapid experimental progress in trapped ions [18]–[24], it is quite possible to experimentally observe the non-Abelian characteristic of gauge potentials with current technology. We now start to present a scheme to detect the non-Abelian characteristic of the gauge potentials in such system. The system under consideration is an ion (or a set of ions) confined in a (linear) Paul trap. Each ion has three ground (or metastable) states \(|1\rangle, |2\rangle, |3\rangle\), and one excited state \(|0\rangle\). The ground states could be different hyperfine levels or in the same manifold but with different Zeeman sublevels, and they are coupled to the excited state \(|0\rangle\).
by resonant classical lasers with Rabi frequencies $\Omega_1$, $\Omega_2$ and $\Omega_3$, respectively, as shown in figure 1 [12]. The effective Hamiltonian for each ion in the rotating frame reads

$$H_I = -\hbar (\Omega_1 |0\rangle \langle 1| + \Omega_2 |0\rangle \langle 2| + \Omega_3 |0\rangle \langle 3|) + \text{h.c.}$$  \hspace{1cm} (9)

In a general case the Rabi frequencies $\Omega_\mu$, can be parameterized with angle and phase variables according to $\Omega_1 = \Omega \sin \theta \cos \phi e^{i\varphi_1}$, $\Omega_2 = \Omega \sin \theta \sin \phi e^{i\varphi_2}$ and $\Omega_3 = \Omega \cos \theta e^{i\varphi_3}$, where $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}$. Two adiabatic eigenstates of $H_I$ with zero eigenvalue (dark state) are then found to be

$$|D_1\rangle = \sin \phi e^{i\varphi} |1\rangle - \cos \phi e^{i\varphi} |2\rangle,$$

$$|D_2\rangle = \cos \theta \cos \phi e^{i\varphi} |1\rangle + \cos \theta \sin \phi e^{i\varphi} |2\rangle - \sin \theta |3\rangle,$$

where we have assumed that the laser phases satisfy the relations $\varphi = \varphi_3 - \varphi_1 = \varphi_3 - \varphi_2$. Substituting equation (10) into (2), we have

$$A_\theta = 0, \quad A_\phi = \begin{pmatrix} 0 & -\cos \theta \\ \cos \theta & 0 \end{pmatrix}, \quad A_\varphi = \begin{pmatrix} i & 0 \\ 0 & i \cos^2 \theta \end{pmatrix}.$$  \hspace{1cm} (11)

Then it is straightforward to find that the gauge potentials are given by

$$A = i \left( \frac{1 + \cos^2 \theta}{2} I + \frac{\sin^2 \theta}{2} \sigma_z \right) d\varphi - i \sigma_y \cos \theta d\phi,$$  \hspace{1cm} (12)

where $I$ is a $2 \times 2$ unit matrix. Similar to the above analysis in the NMR system, the non-Abelian characteristic occurs only when both the angle $\phi$ and phase $\varphi$ vary with time. For instance, if the phase $\varphi$ is fixed, we have

$$U = \exp(i \gamma \sigma_y) = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix},$$

where $\gamma = \int \cos \theta d\phi$. This evolution and its geometric phase have been addressed in [27]. However, it is clear that the non-Abelian characteristic (noncommutativity) of the potential $A$ is lost in this specific case. Therefore, rather than the non-Abelian geometric phase factor, the
scheme proposed in [27] can only be used to observe the Abelian part of the geometric phase, although the two-fold dark states remain degenerate as the Hamiltonian varies with time.

To observe the true non-Abelian structure, we propose two specific non-Abelian closed loops \( C_1 \) and \( C_2 \) in which both the phase \( \phi \) and angle \( \theta \) vary with time. In the closed loop \( C_1 \), we assume that the Rabi frequencies are given by

\[
\begin{align*}
\Omega_1 &= \Omega_0 f(t), \\
\Omega_2 &= \Omega_0 f^2(t), \\
\Omega_3 &= \Omega_0 e^{-r^2/\tau^2} e^{i\phi},
\end{align*}
\]

where \( \phi = \frac{\pi}{2} t \), and \( f(t) \) is set as

\[
f(t) = \begin{cases} 
\cos\left(\frac{\pi t}{2\tau}\right), & -\tau \leq t \leq \tau, \\
0, & \text{otherwise}.
\end{cases}
\]

In this case, the parameters \( \theta(t) \) and \( \phi(t) \) are determined by

\[
\tan \phi(t) = \frac{|\Omega_2|}{|\Omega_1|} = \cos \left( \frac{\pi t}{2\tau} \right),
\]

\[
\tan \theta(t) = \sqrt{\frac{|\Omega_1|^2 + |\Omega_2|^2}{|\Omega_3|^2}} = \sqrt{\frac{\cos^2(\pi t/2\tau) + \cos^4(\pi t/2\tau)}{\exp(-2r^2/\tau^2)}}.
\]

From time \(-\tau \to \tau\), the parameters \((\theta, \phi, \varphi)\) vary from \((0, 0, -\pi)\) to \((0, 0, \pi)\), and it just accomplishes a closed path in the parameter space. It is remarkable that all parameters \((\theta, \phi, \varphi)\) are completely independent of the laser parameters \( \Omega_0 \) and \( \tau \), provided that the adiabatic approximation is reasonable, thus it is easy to design the required laser pulses. By substituting equation (15) into (12), the gauge potential \( A_1 \) generated by the loop \( C_1 \) can be derived, and the associated evolution operator is given by

\[
U_1 = \mathcal{P} \exp \left( - \oint_{c_1} A_1^\mu d\chi^\mu \right).
\]

We set the Rabi frequencies for the loop \( C_2 \) as follows:

\[
\begin{align*}
\Omega_1 &= \Omega_0 f(t), \\
\Omega_2 &= \alpha \Omega_0 f^2(t), \\
\Omega_3 &= \Omega_0 e^{-(t-\beta \tau)^2/\tau^2} e^{i\phi},
\end{align*}
\]

where \( \alpha \) and \( \beta \) are two variables to differentiate the loops \( C_1 \) and \( C_2 \), and \( \beta \) can also be considered as a time delay among pulses \( \Omega_3 \) and \( \Omega_1, \Omega_2 \). In this loop both the start and end points in the parameter space \((\theta, \phi, \varphi)\) are the same as those in the loop \( C_1 \). Similarly, a gauge potential \( A_2 \) will be generated in this loop \( C_2 \), and the associated evolution operator is given by

\[
U_2 = \mathcal{P} \exp \left( - \oint_{c_2} A_2^\mu d\chi^\mu \right).
\]

The analytical result of the integral in equation (16) (or equation (18)) is difficult in a general case, because each segment in the time-ordered exponential loop integral does not commute with the next. Nevertheless, the integral can be evaluated easily by numerical methods.

We now start to show that the composite path formed by the loops \( C_1 \) and \( C_2 \) has noncommutative feature. It is straightforward to check that the initial eigenstates \(|D_{1,2}\rangle_i\) and the final eigenstates \(|D_{1,2}\rangle_f\) for both pathes \( C_1 \) and \( C_2 \) are given by \(|D_{1,2}\rangle_i = |D_{1,2}\rangle_f = -2\) and \(|D_{1,2}\rangle_i = |D_{1,2}\rangle_f = 1\). We may initially prepare the system as \(|\Psi\rangle_i = |D_{2}\rangle_i\). Then we let the
effective Hamiltonian undergo a composite path, first $C_1$ and then $C_2$. The total evolution operator is a $2 \times 2$ matrix denoted as $U = U_2 U_1$. After this evolution, the final state becomes

$$|\Psi\rangle_f = U_{12}|D_1\rangle_f + U_{22}|D_2\rangle_f = -U_{12}|2\rangle + U_{22}|1\rangle,$$

(19)

where $U_{12}$ and $U_{22}$ are the elements of the matrix $U$. On the other hand, if we let the Hamiltonian track the counter-ordered path, i.e. first $C_2$ and then $C_1$, the final state becomes

$$|\Psi'\rangle_f = -U_{12}'|2\rangle + U_{22}'|1\rangle,$$

(20)

where $U_{12}'$ and $U_{22}'$ are the elements of $U' = U_1 U_2$. Denote $P = |U_{22}|^2$ and $P' = |U_{22}'|^2$, which correspond to the possibilities to find the final state in the state $|1\rangle$ after composite paths $C_2 C_1$ and $C_1 C_2$, respectively, the population difference $P_d$ between the two composite paths is given by

$$P_d = P' - P = |U_{22}'|^2 - |U_{22}|^2.$$

(21)

The most essential feature of the non-Abelian gauge structure (the noncommutativity character) can be unambiguously observed if the population difference $P_d \neq 0$.

To show clearly the observable effect induced by the noncommutativity feature of the non-Abelian geometric phases, we here numerically calculate the population difference described by equation (21) for some typical parameters. The populations $P$, $P'$, and the difference $P_d$ between them as functions of the parameters $\alpha$ and $\beta$ are plotted in figures 2(a) and (b). In the calculations for the evolution $U (U')$, we let $U_1 (U_2)$ implement during time $-\tau \rightarrow \tau$, while $U_2 (U_1)$ implement from $\tau$ to $3\tau$. The effects induced by the noncommutative property are clear since the difference $P_d$ can be nonzero. To have a quantitative idea about the difference between the evolution $U$ and $U'$, the population difference $P_d$ as a function of $\beta$ for $\alpha = 7$ is shown in figure 2(b). In this case $P_d = 43.2\%$ is derived for $\beta = 0.5$. 

Figure 2. (a) The population difference $P' - P$ as a function of the parameters $\alpha$ and $\beta$. $P (P')$ is the population of the state $|1\rangle$ at the end of the evolution $U (U')$. (b) The populations $P$, $P'$ and $P' - P$ versus the parameter $\beta$ when $\alpha = 7$. 

New Journal of Physics 10 (2008) 043031 (http://www.njp.org/)
Figure 3. (a) The populations $P_{D_1}$ ($P_{D_2}$) for the dark state $|D_1\rangle$ ($|D_2\rangle$) during a complete composite evolution $U$. (b) The populations $P_{B_1}$ ($P_{B_2}$) for the bright state $|B_1\rangle$ ($|B_2\rangle$). (c) The maxima $P_{B_{\text{Max}}}$ of the sum $P_{B_1}$ and $P_{B_2}$ during the whole evolution time versus parameter $\beta$. The parameters $\Omega_0 = 2\pi \times 300$ KHz and $\tau = 80 \mu s$

Nowadays the measurement technique of population of atomic states has been well developed and the detection precision can be higher than 99%. This high-precision measurement really makes it experimentally feasible to directly ‘see’ the noncommutativity of non-Abelian gauge potentials by measuring the populations of atomic states.

We now turn to demonstrate that the adiabatic condition crucially required in the proposed scheme is well satisfied. Besides the two degenerate dark states described in equation (10), there exist two other bright states with eigenvalues $\pm \Omega$ in the system described by the Hamiltonian (9), and the corresponding eigenstates are given by

$$|B_1\rangle = \frac{1}{\sqrt{2}} [\sin \theta \cos \phi e^{i\psi}|1\rangle + \sin \theta \sin \phi e^{i\psi}|2\rangle + \cos \theta |3\rangle + e^{i\psi}|0\rangle],$$

$$|B_2\rangle = \frac{1}{\sqrt{2}} [\sin \theta \cos \phi e^{i\psi}|1\rangle + \sin \theta \sin \phi e^{i\psi}|2\rangle + \cos \theta |3\rangle - e^{i\psi}|0\rangle].$$

The adiabatic approximation is well satisfied if the overflow from the initial dark states to the two bright states (22) can be negligible during the whole evolution. In view of this point, we numerically calculate the populations of all dark and bright states. The results for the parameters $\alpha = 7$ and $\beta = 0.5$ are plotted in figures 3(a) and (b). It is shown that the populations for both bright states can be negligible. In particular, we calculate the maxima ($P_{B_{\text{Max}}}$) of the sum $P_{B_1}$ and $P_{B_2}$ during the whole evolution time versus the parameter $\beta$ with the other parameters being the same as those in figure 2(b). We note that the sum of the populations for the two bright states are always below 1.0% during the whole evolution. Therefore it clearly shown that the adiabatic approximation used in the calculations of data in figure 2(b) is reasonable. Usually the adiabatic approximation is well satisfied when $\Omega_0 \tau \gg 2\pi$ [26–28].

New Journal of Physics 10 (2008) 043031 (http://www.njp.org/)
We now address briefly the experimental feasibility in implementing our scheme. Firstly, the adiabatic condition should be fulfilled. A typical Rabi frequency $\Omega_0$ is about $2\pi \times 300$ KHz [29], then the adiabatic condition means that the operation time $\tau$ should be longer than $\frac{2\pi}{\Omega_0} = 3.3 \mu$s. As seen from the data in figure 3, the total population on the two bright states is reasonably small (less than 0.5% in almost whole operation time except a short interval near $t = \tau$, which is less than 2%) when $\tau = 80 \mu$s, and it will decrease further if the operation time is longer. Secondly, we need to avoid spontaneous emission with a rate $\gamma_0$ (typically being $10^{-1}$) of the excited state $|0\rangle$. Because of the adiabatic condition, the excited state is only weakly populated and then the effective spontaneous emission rate is reduced by a factor at least $1/p_0$ where $p_0(\leq P_{\text{Max}})$ is the population in the state $|0\rangle$. As a result, the leakage error $4\tau\gamma_0p_0$ due to the spontaneous emission is less than $3.2 \times 10^{-5}$ and thus could be neglected for the reasonable operation time $4\tau \approx 320 \mu$s. Finally, it is remarkable that the population difference $P_d$ is determined by the relative ratio between different coupling rates $\Omega_i (i = 1, 2, 3)$, being nearly independent of their absolute values, so the proposed scheme is actually a kind of stimulated Raman adiabatic passage and shares the same advantage that the results are highly insensitive to the fluctuations of coupling parameters $\Omega_i$ [28, 30].

In summary, we have proposed an experimentally feasible scheme to detect a true non-Abelian geometric phase effect in trapped ions coupled with laser beams. By designing two specific composite cyclic evolutions formed by $U_2U_1$ and $U_1U_2$, we show in detail how the effect induced by the noncommutativity of non-Abelian gauge structures can be observed through detecting the population of the internal states of trapped ions. The high-precision detection of the atomic states recently developed in the field of quantum information opens the great possibility to directly ‘see’ the noncommutative feature of the gauge potentials. It is quite promising for experimentalists to implement the interesting idea presented in this work.

Acknowledgments

This work was supported by the RGC of Hong Kong under grant nos HKU7045/05P and HKU7049/07P, the URC fund of HKU, the NSFC (nos 10429401, 10674049 and 60578055), NCET and the State Key Program for Basic Research of China (nos 2006CB921800 and 2007CB925204).

References

[1] Berry M V 1984 Proc. R. Soc. A 392 45
[2] Wilczek F and Zee A 1984 Phys. Rev. Lett. 52 2111
[3] Zee A 1988 Phys. Rev. A 38 1
[4] Shapere A and Wilczek F 1989 Geometric Phases in Physics (Singapore: World Scientific)
[5] Kwiat P G and Chiao R Y 1991 Phys. Rev. Lett. 66 588
[6] Satija I I, Dakin D C and Clark C W 2006 Phys. Rev. Lett. 97 216401
[7] Ruseckas J, Juzeliunas G, Ohberg P and Fleischhauer M 2005 Phys. Rev. Lett. 95 010404
[8] Osterloh K, Baig M, Santos L, Zoller P and Lewenstein M 2005 Phys. Rev. Lett. 95 010403
[9] Zhu S L, Fu H, Wu C-J, Zhang S-C and Duan L-M 2006 Phys. Rev. Lett. 97 240401

Li Y, Bruder C and Sun C P 2007 Phys. Rev. Lett. 99 130403

Zhu S L 2006 Phys. Rev. Lett. 96 077206
[10] Zanardi P and Rasetti M 1999 Phys. Lett. A 264 94

Pachos J, Zanardi P and Rasetti M 2000 Phys. Rev. A 61 010305

New Journal of Physics 10 (2008) 043031 (http://www.njp.org/)
[11] Sjoqvist E, Pati A K, Ekert A, Anandan J S, Ericsson M, Oi D K L and Vedral V 2000 Phys. Rev. Lett. 85 2845
Du J, Zou P, Shi M, Kwek L C, Pan J-W, Oh C H, Ekert A, Oi D K L and Ericsson M 2003 Phys. Rev. Lett. 91 100403
Yi X X, Wang L C and Zheng T Y 2004 Phys. Rev. Lett. 92 150406
[12] Duan L M, Cirac J I and Zoller P 2001 Science 292 1695
[13] Zhu S L and Wang Z D 2003 Phys. Rev. Lett. 91 187902
Zhu S L and Wang Z D 2002 Phys. Rev. Lett. 89 097902
Zhang X D, Zhang Q and Wang Z D 2006 Phys. Rev. A 74 034302
[14] Li Y, Zhang P, Zanardi P and Sun C P 2004 Phys. Rev. A 70 032330
[15] Recati A, Calarco T, Zanardi P, Cirac J I and Zoller P 2002 Phys. Rev. A 66 032309
[16] Tycko R 1987 Phys. Rev. Lett. 58 2281
[17] Marzlin K P, Bartlett S D and Sanders B C 2003 Phys. Rev. A 67 022316
[18] Monroe C, Meekhof D M, King B E, Itano W M and Wineland D J 1995 Phys. Rev. Lett. 75 4714
[19] Wineland D J, Monroe C, Itano W M, Leibfried D, King B E and Meekhof D M 1998 J. Res. Natl Inst. Stand. Technol. 103 259
[20] Barton P A, Donald C J S, Lucas D M, Stevens D A, Steane A M and Stacey D N 2000 Phys. Rev. A 62 032503
[21] Moehring D L, Blinov B B, Gidley D W, Kohn R N Jr, Madsen M J, Sanderson T D, Vallery R S and Monroe C 2006 Phys. Rev. A 73 023413
Acton M, Brickman K-A, Haljana P C, Leeb P J, Deslauriers L and Monroe C 2005 Preprint quant-ph/0511257v2
[22] Ozeri R et al 2005 Phys. Rev. Lett. 95 030403
Ben-Kish A et al 2003 Phys. Rev. Lett. 90 037902
[23] Kreuter A, Becher C, Lancaster G P T, Mundt A B, Russo C, Haffner H, Roos C, Eschner J, Schmidt-Kaler F and Blatt R 2004 Phys. Rev. Lett. 92 203002
[24] Rabl P, Steixner V and Zoller P 2005 Phys. Rev. A 72 043823
[25] Sanders B C, de Guise H, Bartlett S D and Zhang W 2001 Phys. Rev. Lett. 86 369
[26] Goto H and Ichimura K 2006 Phys. Rev. A 74 053410
Goto H and Ichimura K 2007 Phys. Rev. A 75 033404
[27] Unanyan R G, Shore B W and Bergmann K 1999 Phys. Rev. A 59 2910
Unanyan R G and Fleischhauer M 2004 Phys. Rev. A 69 050302
[28] Bergmann K, Theuer H and Shore B W 1998 Rev. Mod. Phys. 70 1003
[29] Leibfried D, Blatt R, Monroe C and Wineland D 2003 Rev. Mod. Phys. 75 281
[30] Duan L M, Kuzmich A and Kimble H J 2003 Phys. Rev. A 67 032305

New Journal of Physics 10 (2008) 043031 (http://www.njp.org/)