New bounds and constructions for constant weighted X-codes

Xiangliang Kong, Xin Wang and Gennian Ge

Abstract

As a crucial technique for integrated circuits (IC) test response compaction, X-compact employs a special kind of codes called X-codes for reliable compressions of the test response in the presence of unknown logic values (Xs). From a combinatorial viewpoint, Fujiwara and Colbourn [13] introduced an equivalent definition of X-codes and studied X-codes of small weights that have good detectability and X-tolerance.

In this paper, bounds and constructions for constant weighted X-codes are investigated. First, we prove a general lower bound on the maximum number of codewords for an \((m,n,d,x)\) X-code of weight \(w\), and we further improve this lower bound for the case with \(x = 2\) and \(w = 3\) through a probabilistic hypergraph independent set approach. Then, using tools from additive combinatorics and finite fields, we present some explicit constructions for constant weighted X-codes with \(d = 3, 5\) and \(x = 2\), which are nearly optimal for the case when \(d = 3\) and \(w = 3\). We also consider a special class of X-codes introduced in [13] and improve the best known lower bound on the maximum number of codewords for this kind of X-codes.

Keywords and phrases: circuit testing, constant weighted X-codes, additive combinatorics, hypergraph independent set, \(r\)-even-free triple packing.

I. Introduction

Typical digital circuit testing applies test patterns to the circuit and observes the circuit’s responses to the applied patterns. The observed responses are compared to a test pattern with expected responses, and a chip in the circuit is determined to be defective if the comparison mismatches. With the development of the large scale integrated circuits (IC), although the comparison for each testing output is simple, the ever increasing amount of testing data costs much more time and space for processing. This leads to the requirement of more advanced test compression techniques [27]. Since then, various related techniques have been studied such as automatic test pattern generation (ATPG) (see [6], [15], [21], [30] and the reference therein) and compression-based approaches (e.g., [26], [28]). The technique of X-compact is one of the compression-based approaches that have high error detection ability in actual digital systems [26].

Usually, test engineers obtain the expected responses through fault-free simulations of the circuit for the applied test patterns. But fault-free simulations cannot always determine the expected responses. In some cases, due to uninitialized memory elements, bus contention, inaccurate simulation models, etc (see Table 2 in [26]), the responses may contain unknown logic values. These unknown bits are denoted by Xs, and the idea of X-compact provides a technique for reliable test response compaction in the presence of Xs [25].

X-compact uses X-codes as linear maps to compress test responses. An \((m,n,d,x)\) X-code is an \(m \times n\) binary matrix with column vectors as its codewords. The parameters \(d, x\) correspond to the test quality of the code. The weight of a codeword \(c\) is the number of 1s in \(c\). The value of \(\frac{d}{m}\) is called the compaction ratio and X-codes with large compaction ratios are desirable for actual IC testing.

For X-codes of arbitrary weight, let \(M(m,d,x)\) be the maximum number \(n\) of codewords for which there exists an \((m,n,d,x)\) X-code. In [13], based on a combinatorial approach, Fujiwara and Colbourn obtained a general lower bound \(2^{\frac{x-1}{d}} \cdot \binom{m+1}{d}\) on \(M(m,d,x)\) using probabilistic method (see Theorem 4.6, [13]). And this lower bound was further improved to \(e^\frac{c(x+1)(d+2)}{m+c}\) by Tsuboda et al. in [36].

For constant weighted X-codes, let \(M_w(m,d,x)\) be the maximum number \(n\) of codewords for which there exists an \((m,n,d,x)\) X-code of constant weight \(w\). Since factors like power requirements, compactor delay and wireability require the weight of each codeword to be small to meet the practical limitations (see [26], [37]) and codewords with weight at most \(x\) are not essential when considering the compaction ratio (see [13], [23]). Therefore, aiming to achieve a large compaction ratio while minimizing the weight of each codeword, many works have been done about \((m,n,d,x)\) X-codes of constant weight \(x + 1\).

In [37], by viewing the matrix of an \((m,n,d,1)\) X-code as an incidence matrix of a graph, Wohl and Huisman build a connection between \((m,n,d,1)\) X-codes of constant weight 2 and graphs with girth at least \(d + 2\). For the cases with
multiple $X$’s, using results from combinatorial design theory and superimposed codes, Fujiwara and Colbourn [13] proved that $M_3(m, d, 2) = O(m^2)$ and $M_3(m, 1, 2) = \Theta(m^2)$. And they studied a special class of $(m, n, 1, 2)$ $X$-codes of constant weight 3 with a property that boosts test quality when there are fewer unknowable bits than anticipated. Recently, Tsunoda and Fujiwara [35] proved that $M_3(m, d, 2) = o(m^2)$ for $d \geq 4$ and they also improved the lower bound on the maximum number of codewords for the above special class of $(m, n, 1, 2)$ $X$-codes of constant weight 3 introduced in [13].

In this paper, we focus on the constant weighted $X$-codes. Based on the results from additive combinatorics and extremal graph (hypergraph) theory, we obtain the following results:

- A general lower bound for constant weighted $X$-codes:
  $$M_w(m, d, x) \geq (1 - o(1)) \frac{m}{\lfloor w/(x+d-1) \rfloor}.$$

- An improved lower bound for $X$-codes of constant weight 3 with $x = 2$ for any $d$:
  $$M_3(m, d, 2) \geq c \cdot m^\frac{5}{2} (\log m)^{\frac{1}{2}}.$$

for some absolute constant $c > 0$.

- Explicit constructions for constant weighted $X$-codes with $d = 3, 5$ and $x = 2$. These constructions further improve the general lower bound and obtain nearly optimal lower bounds $m^{2-\varepsilon}$ for the cases $M_3(m, 3, 2)$ and $M_3(m, 3, 2)$, when $m$ is large enough.

- An improvement of $(\log m)^{\frac{1}{2}}$ of the best known lower bound on the maximum number of codewords for the special class of $(m, n, 1, 2)$ $X$-codes of constant weight 3 introduced in [13]. This improvement is also extended to the general case where higher error tolerances are required.

This paper is organised as follows: In Section II, we introduce the formal definitions for $X$-codes and superimposed codes, we also include a well-known lower bound for hypergraph independent sets. In Section III, we investigate the bounds and constructions for constant weighted $X$-codes. We prove a general lower bound on $M_w(m, d, x)$ and a non-trivial lower bound on $M_3(m, d, 2)$. We also present some explicit constructions for constant weighted $X$-codes with $d = 3, 5$ and $x = 2$ based on the results from additive combinatorics and finite fields. In Section IV, we improve the lower bound on the maximum number of codewords for a special class of $(m, n, 1, 2)$ $X$-codes of constant weight 3 and extend this result to a general case. In Section V, we conclude our work with some remarks.

II. Preliminaries

A. Notation

We use the following notations throughout this paper.

- Let $q$ be the power of a prime $p$, $\mathbb{F}_q$ be the finite field with $q$ elements, $\mathbb{F}_q^n$ be the vector space of dimension $n$ over $\mathbb{F}_q$.
- For any vector $v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$, let $\text{supp}(v) = \{i \in [n] : v_i \neq 0\}$ and $W(v) = |\text{supp}(v)|$. For a set $S \subseteq [n]$, define $v|_S = (v_{i_1}, \ldots, v_{i_{|S|}})$, where $i_j \in S$ for $1 \leq j \leq |S|$ and $1 \leq i_1 < \cdots < i_{|S|} \leq n$.
- For any integer $n > 0$, denote $[n]$ as the set of the first $n$ consecutive positive integers $\{1, 2, \ldots, n\}$.
- For simplicity, denote $AP$ as the shortened form of arithmetic progression and $k$-$AP$ as the shortened form of arithmetic progression of length $k$.

B. $X$-Codes

Consider two $m$-dimensional vectors $s_1 = (s_1^{(1)}, s_2^{(1)}, \ldots, s_m^{(1)})$ and $s_2 = (s_1^{(2)}, s_2^{(2)}, \ldots, s_m^{(2)})$ where $s_i^{(j)} \in \mathbb{F}_2$. The addition of $s_1$ and $s_2$ is bit-by-bit addition, denoted by $s_1 \oplus s_2$; that is

$$s_1 \oplus s_2 = (s_1^{(1)} + s_2^{(1)}, s_1^{(2)} + s_2^{(2)}, \ldots, s_m^{(1)} + s_m^{(2)}).$$

The superimposed sum of $s_1$ and $s_2$, denoted by $s_1 \vee s_2$, is

$$s_1 \vee s_2 = (s_1^{(1)} \vee s_2^{(1)}, s_1^{(2)} \vee s_2^{(2)}, \ldots, s_m^{(1)} \vee s_m^{(2)}),$$

where $s_i^{(j)} \vee s_i^{(k)} = 0$ if $s_i^{(j)} = s_i^{(k)} = 0$, otherwise 1. And we say an $m$-dimensional vector $s_1$ covers an $m$-dimensional vector $s_2$ if $s_1 \vee s_2 = s_1$.

For a finite set $S = \{s_1, \ldots, s_s\}$ of $m$-dimensional vectors, define

$$\bigoplus S = s_1 \oplus \cdots \oplus s_s,$$

and

$$\bigvee S = s_1 \vee \cdots \vee s_s.$$  

When $s = 1$, $\bigoplus S = \bigvee S = \{s_1\}$, and when $S = \emptyset$, define $\bigoplus S = \bigvee S = \emptyset$ (i.e. the zero vector).
Definition II.1. \[13\] Let \( d \) be a positive integer and \( x \) a nonnegative integer. An \((m, n, d, x)\) X-code \( \mathcal{X} = \{s_1, \ldots, s_n\} \) is a set of \( m \)-dimensional vectors over \( \mathbb{F}_2 \) such that \( |\mathcal{X}| = n \) and

\[
\bigvee S_i \lor \bigoplus S_2 \neq \bigwedge S_i \tag{1}
\]

for any pair of mutually disjoint subsets \( S_1 \) and \( S_2 \) of \( \mathcal{X} \) with \( |S_1| = x \) and \( 1 \leq |S_2| \leq d \). A vector \( s_i \in \mathcal{X} \) is called a codeword. The weight of the code word \( s_i \) is \( |\text{supp}(s_i)| \). The ratio \( \frac{n}{m} \) is called the compaction ratio of \( \mathcal{X} \).

Let \( M(m, d, x) \) be the maximum number \( n \) of codewords for which there exists an \((m, n, d, x)\) X-code. From the definition above, when \( x = 0 \), the codewords of an \((m, n, d, 0)\) X-code actually form an \( m \times n \) parity check matrix of a binary linear code of length \( n \) with minimum distance \( d \). Therefore, \((m, n, d, 0)\) X-codes can be viewed as a special kind of traditional error-correcting codes.

For the case when \( x \geq 1 \) and \( d = 1 \), as pointed out in \[23\], an \((m, n, 1, x)\) X-code is equivalent to a \((1, x)\)-superimposed code of size \( m \times n \).

Definition II.2. \[19\] A \((1, x)\)-superimposed code of size \( m \times n \) is an \( m \times n \) matrix \( S \) with entries in \( \mathbb{F}_2 \) such that no superimposed sum of any \( x \) columns of \( S \) covers any other column of \( S \).

Superimposed codes are also called cover-free families and disjunct matrices. These kinds of structures have been extensively studied in information theory, combinatorics and group testing. Thus, the bounds and constructions of \((1, x)\)-superimposed codes can also be regarded as those for \((m, n, 1, x)\) X-codes (see, for example, \[5\], \[8\]–\[10\], \[14\], \[16\], \[33\]).

When \( x \geq 1 \) and \( d \geq 2 \), according to the definition, an \((m, n, d, x)\) X-code is also an \((m, n, d-1, x)\) X-code and an \((m, n, d-1, x)\) X-code is an \((m, n, d, x-1)\) X-code as well.

Given an \((m, n, d, x)\) X-code, Fujiwara and Colbourn \[13\] showed that a codeword of weight less than or equal to \( x \) does not essentially contribute to the compaction ratio (see also \[23\]). Therefore, when considering X-codes of constant weight \( w \), we always assume that \( w \geq x + 1 \).

C. Independent sets in hypergraphs

A hypergraph is a pair \((V, E)\), where \( V \) is a finite set and \( E \subseteq 2^V \) is a family of subsets of \( V \). The elements of \( V \) are called vertices and the subsets in \( E \) are called hyperedges. We call \( H \) a \( k \)-uniform hypergraph, if all the hyperedges have the same size \( k \), i.e., \( E \subseteq \binom{V}{k} \). For any vertex \( v \in V \), we define the degree of \( v \) to be the number of hyperedges containing \( v \), denoted by \( d(v) \). The maximum of the degrees of all the vertices is called the maximum degree of \( H \) and denoted by \( \Delta(H) \).

An independent set of a hypergraph is a set of vertices containing no hyperedges and the independence number of a hypergraph is the size of its largest independent set. There are many results on the independence number of hypergraphs obtained through different methods (see \[2\], \[3\], \[7\], \[18\]). Recall that a hypergraph \( H \) is linear if every pair of distinct hyperedges from \( E \) intersects in at most one vertex. In this paper, we shall use the following version of the famous result of Ajtai et al. \[2\] due to Duke et al. \[7\] to derive some lower bounds on \( M(m, d, x) \).

Lemma II.3. \[7\] Let \( k \geq 3 \) and let \( H \) be a \( k \)-uniform hypergraph with \( \Delta(H) \leq D \). If \( H \) is linear, then

\[
\alpha(H) \geq c \cdot v(H) \cdot \left( \frac{\log D}{D} \right)^{1/4},
\]

for some constant \( c \) that depends only on \( k \).

III. BOUNDS AND CONSTRUCTIONS OF CONSTANT WEIGHTED X-CODES

In this section, we consider X-codes of constant weight. This section is divided into four subsections. Section III-A includes some known results and a general upper bound on the number of codewords of constant weighted X-codes. In Section III-B, based on a result of packing, we obtain a general lower bound on the maximum number of codewords of constant weighted X-codes. Then in Section III-C, we give some explicit constructions for constant weighted X-codes with \( d = 3, 5 \) and \( x = 2 \). And in Section III-D, we improve the general lower bound for X-codes of constant weight 3 with \( x = 2 \).

A. A general upper bound and known results

Denote \( M_w(m, d, x) \) as the maximum number of codewords of an \((m, n, d, x)\) X-code of constant weight \( w \). Since the restrictions for X-codes get more rigid with the growing of \( d \), thus we have

\[
M_w(m, d, x) \leq M_w(m, 1, x). \tag{3}
\]

In 1985, Erdős et al. \[10\] proved the following bounds on the maximum number of codewords of a \((1, x)\)-superimposed code of constant weight \( w \).
Theorem III.1. \[10\] Denote \(f_x(m, w)\) as the maximum number of columns of a \((1, x)\)-superimposed code of constant weight \(w\). Let \(t = \lceil \frac{w}{x} \rceil\). Then, we have
\[
\left( \frac{m}{t} \right) \left( \frac{t}{w} \right)^2 \leq f_x(m, w) \leq \left( \frac{m}{t} \right) \left( \frac{t}{w-1} \right).
\]
Moreover, if we take \(w = x(t - 1) + 1 + \delta\) where \(0 \leq \delta < x\), then for \(m > m_0(w)\),
\[
f_x(m, w) \geq (1 - o(1)) \left( \frac{m - \delta}{t} \right) \left( \frac{t}{w} \right).
\]
and \(f_x(m, w) \leq \left( \frac{m-d}{t} \right) \left( \frac{t}{w-d} \right)\) holds in the following cases: 1) \(\delta = 0, 1, 2\) \(\delta < \frac{x}{2t^2}\); 3) \(t = 2\) and \(\delta < \left\lceil \frac{2x}{3} \right\rceil\). Moreover, equality of the latter upper bound holds if and only if there exists a Steiner \(t\)-design \(S(t, w - \delta, n - \delta)\).

By the equivalency between an \(X\)-code and a \((1, x)\)-superimposed code, we have the following immediate consequence:

Theorem III.2. Let \(w = x(t - 1) + 1 + d\) where \(0 \leq d < x\). Then, for all \(m \geq 1\),
\[
M_w(m, d, x) \leq M_w(m, 1, x) \leq \left( \frac{m}{t} \right) \left( \frac{t}{w-1} \right).
\]
And for \(m > m_0(k)\),
\[
M_w(m, d, x) \leq M_w(m, 1, x) \leq \left( \frac{m-d}{t} \right) \left( \frac{t}{w-d} \right)
\]
holds in the following cases: 1) \(d = 0, 1, 2\) \(d < \frac{x}{2t^2}\); 3) \(t = 2\) and \(d < \left\lceil \frac{2x}{3} \right\rceil\).

In particular, for the case \(x = 2\), Theorem III.2 actually gives the following upper bound
\[
M_w(m, d, 2) \leq \begin{cases} \left( \frac{m}{w/2} \right), & \text{when } w \text{ is even}; \\ \left( \frac{m}{w+1/2} \right), & \text{when } w \text{ is odd}. \end{cases}
\]
According to the results from design theory, Fujiwara and Colbourn [13] proved the above upper bound is tight for the case \(w = 3\) and \(d = 1\), when there exists a corresponding Steiner triple system. Using the well-known graph removal lemma, Tsunoda and Fujiwara [35] improved this upper bound on \(M_0(m, d, 2)\) to \(o(m^2)\) for \(d \geq 4\). So far as we know, for \(d \geq 2\) and \(x = 2\), no upper or lower bounds better than these can be found in the literature.

B. A general lower bound from maximum \((t, w, m)\)-packings

Let \(V\) be an \(m\)-element set. A \(\mathcal{P} \subseteq \binom{V}{w}\) is called a \((t, w, m)\)-packing if \(|P_1 \cap P_2| < t\) holds for every pair \(P_1, P_2 \in \mathcal{P}\). In [29], V. Rödl proved the following lower bound on the size of the maximum \((t, w, m)\)-packing.

Theorem III.3. [29] For all positive integers \(w, t\), there exists a constant \(M_0 = M_0(t, w)\), such that when \(m \geq M_0\), we have
\[
\max\{|\mathcal{P} : \mathcal{P} \text{ is a } (t, w, m)\text{-packing}\| = (1 - o(1)) \left( \frac{m}{t} \right) \left( \frac{t}{w} \right).
\]

Based on the above result for maximum \((t, w, m)\)-packings, we can obtain the following asymptotic general lower bound for constant weighted \(X\)-codes.

Theorem III.4. For all positive integers \(d, x\) and \(w \geq x + 1\), there exists a constant \(M_1 = M_1(w, d, x)\), such that when \(m \geq M_1\), we have
\[
M_w(m, d, x) \geq (1 - o(1)) \left( \frac{m}{w/(x+d-1)} \right).
\]

Proof of Theorem III.4. For given \(d, x\) and \(w \geq x + 1\), we only need to show that the indicator vectors of a \((\lceil w/(x+d-1) \rceil, w, m)\)-packing of size \(n\) form an \((m, n, d, x)\) \(X\)-code of constant weight \(w\).

Consider a \((\lceil w/(x+d-1) \rceil, w, m)\)-packing \(\mathcal{P}\) of size \(n\), fix any \(x\) distinct \(w\)-subsets \(\{P_1, P_2, \ldots, P_x\}\) from \(\mathcal{P}\). For each \(P \in \mathcal{P}\), denote \(v_P\) as its indicator vector. Assume that there exist \(l\) distinct \(w\)-subsets \(\{Q_1, Q_2, \ldots, Q_l\}\) in \(\mathcal{P}\) for some \(1 \leq l \leq d\), such that
\[
\bigvee_{i=1}^{x} v_{P_i} \cup \bigoplus_{j=1}^{l} v_{Q_j} = \bigvee_{i=1}^{x} v_{P_i}.
\]
Then, denote $Q_0$ as the subset of $V$ with indicator vector $\bigoplus_{j=1}^l v_{Q_j}$, we have
\[(Q_0 \cap Q_j) \subseteq \bigcup_{i=1}^x (P_i \cap Q_j),\]
for every $1 \leq j \leq l$. W.l.o.g. take $j = 1$. Since $Q_1 \setminus Q_0 = \{ v \in Q_1 : \text{there exist even number of } Q_j \text{s such that } v \in Q_j \}$, we have
\[|Q_1 \setminus Q_0| \leq \bigcup_{j=2}^l |Q_1 \cap Q_j|\]
Therefore,
\[|Q_1| = |Q_1 \cap Q_0| + |Q_1 \setminus Q_0| \leq \bigcup_{i=1}^x |Q_1 \cap P_i| + \bigcup_{j=2}^l |Q_1 \cap Q_j|\]
\[\leq \sum_{i=1}^x |Q_1 \cap P_i| + \sum_{j=2}^l |Q_1 \cap Q_j|\]
\[\leq (x + l - 1) \cdot \left( \left\lceil \frac{w}{x + d - 1} \right\rceil - 1 \right) < w,\]
this leads to a contradiction, which indicates that $\{v_P\}_{P \in P}$ is an $(m, n, d, x)$ X-code of constant weight $w$. This completes the proof.

C. Explicit constructions of constant weighted X-codes

1) Construction of constant weighted X-codes with $d = 3$ and $x = 2$: In this part, based on some results from additive combinatorics, we shall prove the following asymptotic lower bound for constant weighted X-codes with $d = 3$ and $x = 2$.

**Theorem III.5.** For any $\varepsilon > 0$ and $w \geq 3$, there exists a constant $M = M(w, \varepsilon) > 0$, such that for $m \geq M$, there exists an $(m, m^2 - \varepsilon, 3, 2)$ X-code of constant weight $w$.

For the proof of Theorem III.5 we need the following lemma from [11].

**Lemma III.6.** ([11]) For positive integers $w$ and $m$, there exists a set of positive integers $A \subseteq \{1, 2, \ldots, m\}$ of size
\[|A| \geq \frac{m}{e^c \log w \sqrt{\log m}}\]
for some absolute constant $c$, such that $A$ contains no three terms of any arithmetic progressions of length $w$.

The specific construction of the set $A$ from Lemma III.6 can be regarded as an extension of the 3-AP-free subset of $[n]$ given by Behrend [4]: Let $\beta \geq 2$ and $w \geq 1$, for any $1 \leq a \leq m$, $a$ can be written as
\[a = a_0 + a_1(2\beta w) + a_2(2\beta w)^2 + \cdots + a_k(2\beta w)^k.\]
Set $N(\alpha) = (\sum_{i=0}^k a_i^2)^{1/2}$, where $\alpha = (a_0, a_1, \ldots, a_k)$. For $s \geq 1$, set
\[A = A_{m, \beta, s} = \{a : 1 \leq a \leq n, 0 \leq a_i \leq d \text{ for all } i, \ (N(\alpha))^2 = s\}.

Based on the construction from Lemma III.6 we proceed to prove Theorem III.5

**Proof of Theorem III.5.** Let $m = \lceil \frac{m'}{w} \rceil$ and $A \subseteq m'$ be the subset constructed from Lemma III.6 such that $A$ contains no three terms of any progressions of length $w$.

Take $w$ distinct sets $X_0, X_1, \ldots, X_{w-1}$, where $X_i = [(i + 1) \cdot m']$. Define
\[\mathcal{P} = \{ (x, x + a, \ldots, x + (w - 1)a) : 1 \leq x \leq m', a \in A \},\]
as a family of ordered $w$-subsets in $[m']$, then $\mathcal{P} \subseteq X_0 \times X_1 \times \cdots \times X_{w-1}$. Since $|A| \geq \frac{m'}{e^c \log w \sqrt{\log m}}$ for some $c > 0$, we have $|\mathcal{P}| \geq m'|A| \geq m^{2-\varepsilon}$ for every $\varepsilon > 0$ and $m \geq M$.

Now, considering the indicator vectors $\mathcal{C}$ corresponding to the $w$-subsets in $\mathcal{P}$, we have the following claim.

**Claim 1.** $\mathcal{C}$ is an $(m, |\mathcal{P}|, 3, 2)$ X-code of constant weight $w$.

For each $P \in \mathcal{P}$, denote $v_P$ as its indicator vector. First, noted that $|P_1 \cap P_2| \leq 1$ for any two distinct $P_1, P_2 \in \mathcal{P}$ and $w \geq x + 1 = 3$, thus the superimposed sum of any two vectors in $\mathcal{C}$ cannot cover any other vector in $\mathcal{C}$.

Meanwhile, we can obtain $W(v_{P_1} \oplus v_{P_2}) \geq 2w - 2$. Therefore, when $w \geq 4$, we have $2w - 2 > 4$. If there exist four distinct $P_1, P_2, P_3, P_4 \in \mathcal{P}$ such that $v_{P_1} \oplus v_{P_2}$ is covered by $v_{P_3} \vee v_{P_4}$, one of the four intersections $|P_1 \cap P_2|, |P_1 \cap P_4|, |P_2 \cap P_3|, |P_2 \cap P_4|$ must be strictly larger than one, which leads to a contradiction. When $w = 3$, if there exist distinct $P_1, P_2, P_3, P_4 \in \mathcal{P}$...
such that $v_{P_1} \oplus v_{P_2}$ is covered by $v_{P_3} \vee v_{P_4}$, then we have $P_1 \cap P_2 = \{\theta_0\}$, $P_1 \cap P_3 = \{\theta_1\}$, $P_2 \cap P_3 = \{\theta_2\}$, $P_2 \cap P_4 = \{\theta_3\}$, $P_3 \cap P_4 = \{\theta_4\}$, where for $0 \leq j \leq 4$, elements $\theta_j \in X_{i_j}$ are pairwise distinct for some different $0 \leq i_j \leq w-1$. From the definition of $P_t$, we have
\[
\begin{align*}
\theta_0 &= x_1 + i_0 a_1 = x_2 + i_0 a_2; \\
\theta_1 &= x_1 + i_1 a_1 = x_3 + i_1 a_3; \\
\theta_2 &= x_1 + i_2 a_1 = x_4 + i_2 a_4; \\
\theta_3 &= x_2 + i_3 a_2 = x_3 + i_3 a_3; \\
\theta_4 &= x_2 + i_4 a_2 = x_4 + i_4 a_4,
\end{align*}
\]
where element $x_i \in X_0$ is the leading term in $P_i$, and element $a_i \in A$ is the common difference corresponding to $P_i$. Combining the first two identities with the $4_{th}$ one above, we have
\[
(i_1 - i_0)a_1 + (i_0 - i_3)a_2 = (i_1 - i_3)a_3.
\]
This means that $a_1, a_2, a_3$ are three distinct terms in a $w$-AP, which contradicts the choice of set $A$.

For any three distinct $P_1, P_2, P_3 \in \mathcal{P}$, we have $W(v_{P_1} \oplus v_{P_2}) \geq 3w - 6$. Therefore, when $w \geq 7$, the addition of any three distinct vectors in $C$ cannot be covered by the superimposed sum of any two other vectors.

Now, assume that there exist $\{P_i\}_{i=1}^5 \subseteq \mathcal{P}$ such that $v_{P_1} \oplus v_{P_2} \oplus v_{P_3}$ is covered by $v_{P_4} \vee v_{P_5}$.

When $w = 3$, we have either $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 3 \lor W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 5$. For the case $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 3$, we have $P_1 \cap P_2 = \{\theta_0\}, P_1 \cap P_3 = \{\theta_1\}, P_2 \cap P_3 = \{\theta_2\}$, for three pairwise distinct $\theta_j \in X_{i_j}$. For the case $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 5$, we have $P_1 \cap P_2 = \{\theta_0\}, P_1 \cap P_3 = \{\theta_1\}, P_2 \cap P_3 = \{\theta_2\}$, $P_2 \cap P_4 = \{\theta_3\}$, $P_3 \cap P_4 = \{\theta_4\}$, for five pairwise distinct $\theta_j \in X_{i_j}$. For both cases, we have three distinct $P_i$s pairwise intersecting at three distinct elements $\theta_j$. From the analysis above, we know that this will lead to $A$ contains three distinct terms in a $w$-AP, a contradiction.

When $w = 4$, we have either $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 6$ or $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 8$. For the case $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) = 6$, we have $P_1 \cap P_2 = \{\theta_1\}, P_1 \cap P_3 = \{\theta_3\}, P_2 \cap P_3 = \{\theta_2\}$ and $P_i \cap P_j = \{\theta_{ij}\}$ for each $i \in \{1, 2, 3\}, j \in \{4, 5\}$, where elements $\theta_{ij} \in X_{s_{ij}}$ are pairwise distinct. From the definition of $P_t$, we have
\[
\begin{align*}
\theta_{12} &= x_1 + s_{12} a_1 = x_2 + s_{12} a_2, \text{ for some } 0 \leq s_{12} \leq w-1; \\
\theta_{14} &= x_1 + s_{14} a_1 = x_4 + s_{14} a_4, \text{ for some } 0 \leq s_{14} \leq w-1; \\
\theta_{24} &= x_2 + s_{24} a_2 = x_4 + s_{24} a_4, \text{ for some } 0 \leq s_{24} \leq w-1.
\end{align*}
\]
And this also leads to $A$ contains three distinct terms in a $w$-AP, a contradiction.

For the cases when $w = 5$ and $w = 6$, we have $W(v_{P_1} \oplus v_{P_2} \oplus v_{P_3}) \geq 9$. Therefore, for both cases, there exist $P_i, P_j$ such that $|P_i \cap P_j| > 1$ for some $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$, a contradiction.

In conclusion, any three or fewer vectors in $C$ cannot be covered by the superimposed sum of any other two vectors. Therefore, **Claim 1** is verified and this completes the proof.

**Remark III.7.** According to the upper bound given by (6), we have
\[
\begin{align*}
M_3(m, 3, 2) &\leq \frac{m(m-1)}{6}, \\
M_4(m, 3, 2) &\leq \frac{(m-1)(m-2)}{6}.
\end{align*}
\]
This implies that for cases $w = 3$ and $w = 4$, the lower bounds $m^{2+\varepsilon}$ from Theorem III.5 are nearly optimal. For cases when $w \geq 5$, Theorem III.4 provide better lower bounds $(1 - o(1))\frac{m}{(\log m)}\frac{1}{(\log w)}$, but the gaps between the upper bounds and the lower bounds are still quite large.

It is also worth noting that the construction from Theorem III.5 was originally proposed by Erdős et al. [2] to construct $w$-uniform hypergraphs on $m$ vertices such that no $3w - 3$ vertices span $3$ or more hyperedges. This kind of hypergraphs is a special kind of sparse hypergraphs which we will discuss later in Section III.D.

2) **Construction of constant weighted $X$-codes with $d = 5$ and $x = 2$:** Before we present the construction, we shall prove a proposition which establishes a connection between constant weighted $X$-codes with $d = 5, x = 2$ and uniform hypergraphs of girth five.

Given a $k$-uniform hypergraph $\mathcal{H} = (V, E)$ and a positive integer $l \geq 2$, a cycle of length $l$ in $\mathcal{H}$, denoted by $C_l$, is an alternating sequence of distinct vertices and hyperedges of the form: $v_1, E_1, v_2, E_2, \ldots, v_l, E_l, v_1$, such that $\{v_i, v_{i+1}\} \subseteq E_i$ for each $i \in \{1, 2, \ldots, l\}$ and $\{v_1, v_l\} \subseteq E_l$. A linear path of length $l$, denoted by $P_l$, is an alternating sequence of distinct vertices and hyperedges of the form: $E_1, v_2, E_2, v_3, \ldots, v_l, E_l$, such that $E_1 \cap E_{i+1} = \{v_{i+1}\}$ for each $i$ and $E_i \cap E_j = \emptyset$ whenever $|j-i| > 1$. And the *girth* of hypergraph $\mathcal{H}$ is the minimum length of a cycle in $\mathcal{H}$. **Proposition III.8.** Let $w \geq 3$ be a positive integer. For any $w$-uniform hypergraph $\mathcal{H} = (V, E)$ of girth at least 5, the set of all the indicator vectors of hyperedges in $E$ forms a $(\lfloor |V|/|E| \rfloor, 5, 2)$ $X$-code of constant weight $w$. 
Proof of Proposition III.8 \ First, noted that the girth of $H$ is at least 5, we know that $H$ is a linear hypergraph, i.e., $|E_1 \cap E_2| \leq 1$ for any $E_1, E_2 \in \mathcal{E}$. Hence, if we denote $v_{E_i}$ as the the indicator vector of hyperedge $E_i$, then for any $\{E_1, E_2, E_3, E_4, E_5\} \subseteq \mathcal{E}$, we have

\[
\begin{cases}
W(\bigoplus_{i \in I_2} v_{E_i}) \geq 2(w - 1), \text{ for any 2-subset } I_2 \subseteq [5]; \\
W(\bigoplus_{i \in I_3} v_{E_i}) \geq 3(w - 2), \text{ for any 3-subset } I_3 \subseteq [5]; \\
W(\bigoplus_{i \in I_4} v_{E_i}) \geq 4(w - 3), \text{ for any 4-subset } I_4 \subseteq [5]; \\
W(\bigoplus_{i \in I_5} v_{E_i}) \geq 5(w - 4), \text{ for any 5-subset } I_5 \subseteq [5].
\end{cases}
\]

**Case 1.** Assume that there exist $\{E_i\}_{i=1}^5 \subseteq \mathcal{E}$ such that $v_{E_1} \oplus v_{E_2} \oplus v_{E_3} \oplus v_{E_4} \oplus v_{E_5}$ is covered by $v_{E_6} \lor v_{E_7}$. From the linearity, $v_{E_6} \lor v_{E_7}$ can cover at most 2 distinct vertices in each $v_{E_i}$ $(1 \leq i \leq 5)$, thus we have $5(w - 4) \leq 10$, which indicates that $w \leq 6$.

When the length of the longest linear path in the configuration formed by $\{E_i\}_{i=1}^5$ is at most 3, the assumption that $v_{E_1} \oplus v_{E_2} \oplus v_{E_3} \oplus v_{E_4} \oplus v_{E_5}$ is covered by $v_{E_6} \lor v_{E_7}$ forces $E_6$ (or $E_7$) together with three distinct hyperedges from $\{E_i\}_{i=1}^5$ to form a cycle of length 4, this contradicts the requirement that $H$ is a hypergraph of girth at least 5.

When the length of the longest linear path in the configuration formed by $\{E_i\}_{i=1}^5$ equals to 4, one can easily determine all the possible types of this configuration. As those shown in Fig. 1, there are four types. Based on this characterization, in order to cover all the vertices with odd degree, $E_6$ (or $E_7$) has to form a cycle of length at most 4 with two or three distinct hyperedges from $\{E_i\}_{i=1}^5$, a contradiction.

Similarly, when the length of the longest linear path in the configuration formed by $\{E_i\}_{i=1}^5$ is 5, then this configuration itself is a linear path of length 5, as shown in Fig. 2. Again, the assumption forces the existence of a cycle of length at most 4, a contradiction.

**Case 2.** Assume that there exist $\{E_i\}_{i=1}^6 \subseteq \mathcal{E}$ such that $v_{E_1} \oplus v_{E_2} \oplus v_{E_3} \oplus v_{E_4}$ is covered by $v_{E_6} \lor v_{E_7}$. From the linearity, we have $4(w - 3) \leq 8$, which indicates that $w = 3, 4$ or 5. Similar to the analysis in Case 1, for all these three cases, either $E_1, E_2, E_3, E_4$ form a cycle of length four in $H$, or $E_5, E_6$ together with two hyperedges from $\{E_1, E_2, E_3, E_4\}$ form a cycle of length four in $H$, a contradiction.

**Case 3.** Assume that there exist $\{E_i\}_{i=1}^5 \subseteq \mathcal{E}$ such that $v_{E_1} \oplus v_{E_2} \oplus v_{E_3}$ is covered by $v_{E_4} \lor v_{E_5}$. From the linearity, we have $3(w - 2) \leq 6$, which indicates that $w = 3$ or $w = 4$. For both cases, either $E_1, E_2, E_3$ form a cycle of length three in $H$, or $E_4, E_5$ together with two hyperedges from $\{E_1, E_2, E_3\}$ form a cycle of length four in $H$, a contradiction.

**Case 4.** Assume that there exist $\{E_i\}_{i=1}^4 \subseteq \mathcal{E}$ such that $v_{E_1} \oplus v_{E_2}$ is covered by $v_{E_3} \lor v_{E_4}$. From the linearity, we have $2(w - 1) \leq 4$. Therefore $w = 3$ and in this case, $E_1, E_2, E_3, E_4$ form a cycle of length four in $H$, a contradiction.

In conclusion, any five or fewer distinct indicator vectors of hyperedges in $\mathcal{E}$ can not be covered by the superimposed sum of any other two indicator vectors. Therefore, these indicator vectors form a $([|V|, |\mathcal{E}|, 5, 2])$ X-code of constant weight $w$. \qed

Based on a construction of 3-uniform hypergraphs of girth at least five in [20], by Proposition III.8 we have the following result.

**Theorem III.9.** For any odd prime power $q$, there exists a $(q(q - 1), (q^2))$, $5, 2)$ X-code of constant weight 3.

**Proof of Theorem III.9** For any odd prime power $q$, consider the finite field $\mathbb{F}_q$, let $C_q$ denote the set of points on the curve $2x_2 = x_1^2$, where $(x_1, x_2) \in \mathbb{F}_q^2$. 

Define a hypergraph $G_q$ with vertex set $V(G_q) = \mathbb{F}_q^2 \setminus C_q$. Three distinct vertices $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ form a hyperedge $\{a, b, c\}$ in $G_q$ if and only if the following three equations hold:

$$\begin{align*}
a_2 + b_2 &= a_1 b_1; \\
b_2 + c_2 &= b_1 c_1; \\
c_2 + a_2 &= c_1 a_1.
\end{align*}$$

From case by case, one can check that $G_q$ has girth at least five. Clearly, there are $q^3$ choices for distinct numbers $a_1$, $b_1$ and $c_1$, and each choice uniquely specifies $a_2$, $b_2$ and $c_2$. Since any two of the $a_1, b_1, c_1$ being the same will lead to identical corresponding vertices, thus the number of hyperedges in $G_q$ is precisely $q^3$. Therefore, by Proposition III.8, we obtain a $(q(q-1), (q^3)/3, 5, 2)$ $X$-code of constant weight 3.

**Remark III.10.** The construction from Theorem III.9 actually gives a lower bound on $M_3(m, 5, 2)$ of the form

$$M_3(m, 5, 2) = \Omega(m^{2}),$$

for sufficiently large $m$, which is better than that given by Theorem III.4 in this case. But, compared to the upper bound $o(m^2)$ given by Tsunoda and Fujiwara [35], there is still a gap.

Unfortunately, this construction can not be extended to obtain general constant weighted $X$-codes. But at least, together with Proposition III.8 it provides a way for constructing large constant weighted $X$-codes with $d = 5$ and $x = 2$.

**D. An improved lower bound for $X$-codes of constant weight 3 with $x = 2$**

Noticed that when taking $w = x + 1$ in Theorem III.4, the general lower bound given by [3] is only a linear function of $m$ for $d \geq 2$. Inspired by a new probabilistic hypergraph independent set approach introduced by Ferber et al. [12], we prove the following theorem, which improves this linear lower bound to $\Omega(m^{\frac{2}{3}} (\log m)^{\frac{1}{3}})$ for the case when $x = 2$.

**Theorem III.11.** For any positive integer $d \geq 6$ and sufficiently large $m$, there exists an $(m, c \cdot m^{\frac{2}{3}} (\log m)^{\frac{1}{3}}, d, 2)$ $X$-code of constant weight 3, where $c > 0$ is an absolute constant.

In graph theory, a $k$-uniform hypergraph $H$ is called $G_k(v, e)$-free if the union of any $e$ distinct hyperedges contains at least $v + 1$ vertices. These kinds of hypergraphs are called sparse hypergraphs. They are important structures in extremal graph theory and have been well-studied since 1970s (see [1], [17], [34] and the reference therein). Before we present the proof of Theorem III.11 we need the following generalized version of Proposition III.8 related to sparse hypergraphs.

**Proposition III.12.** For any 3-uniform hypergraph $H = (V, E)$ of girth at least 5, if for each $6 \leq s \leq d$, $H$ is also $G_3(\lceil \frac{3s}{2} \rceil + 3, s)$-free, then the set of all the indicator vectors of hyperedges in $E$ forms a $(|V|, |E|, d, 2)$ $X$-code of constant weight 3.

**Proof of Proposition III.12.** By Proposition III.8 we know that the set of all the indicator vectors of hyperedges in $E$ is already a $(|V|, |E|, 5, 2)$ $X$-code of constant weight 3.

For each $e \in E$, denote $v_e$ as the indicator vector of $e$. Now, for a fixed $6 \leq s_0 \leq d$, consider $s_0$ distinct hyperedges $\{e_1, \ldots, e_{s_0}\} \subset E$. Assume that there exist two other hyperedges $f_1$ and $f_2$ such that $v_{e_1} \oplus \cdots \oplus v_{e_{s_0}}$ can be covered by $v_{f_1} \lor v_{f_2}$. Denote $V_0$ as the set of vertices in $\bigcup_{i=1}^{s_0} e_i$ that are contained in even number of hyperedges in $\{e_1, \ldots, e_{s_0}\}$ and $V_1$ as the set of vertices in $\bigcup_{i=1}^{s_0} e_i$ that are contained in odd number of hyperedges in $\{e_1, \ldots, e_{s_0}\}$. Then the assumption indicates that $V_1 \subseteq f_1 \lor f_2$.

Since $H$ is a 3-uniform hypergraph, we have

$$|V_1| \leq 6$$

This implies that $|\bigcup_{i=1}^{s_0} e_i| = |V_0| + |V_1| \leq \frac{3}{2} s_0 + 3$, i.e., these $s_0$ distinct hyperedges $\{e_1, \ldots, e_{s_0}\}$ are spanned by at most $\frac{3}{2} s_0 + 3$ distinct vertices in $H$, a contradiction. Thus, for each $6 \leq s \leq d$, the addition of the indicator vectors corresponding to any distinct $s$ hyperedges in $E$ can not be covered by the superimposed sum of the indicator vectors corresponding to any other 2 hyperedges in $E$. Therefore, combined with the direct conclusion from Proposition III.8 the set of all the indicator vectors of hyperedges in $E$ forms a $(|V|, |E|, d, 2)$ $X$-code of constant weight 3.

**Now, we present the proof of Theorem III.11.**

**Proof of Theorem III.11.** By Proposition III.12, we only need to construct a 3-uniform hypergraph $H_0 = (V, E)$ of girth at least 5 that is also $G_3(\lceil \frac{3s}{2} \rceil + 3, s)$-free for each $6 \leq s \leq d$.

Noticed that a 3-uniform hypergraph $H_0$ that is $G_3(2t, t)$-free for each $2 \leq t \leq 5$ has girth at least 5, thus our goal is to construct a 3-uniform hypergraph $H_0$ that is simultaneously $G_3(2t, t)$-free for each $2 \leq t \leq 5$ and $G_3(\lceil \frac{3s}{2} \rceil + 3, s)$-free for each $6 \leq s \leq d$. 

Let $V$ be a finite set of points and $|V| = m$, take a subset $B$ of triples by picking elements of $\binom{V}{3}$ uniformly and independently at random with probability $p$. Then we have
\[
\mathbb{E}[|B|] = p \cdot \binom{|V|}{3}.
\]

For each $2 \leq t \leq 5$, denote $D_t$ as the set of $t$-subsets in $B$ that are spanned by at most $2t$ points in $V$, i.e., for each $\{B_1, \ldots, B_t\} \in D_t \subseteq \binom{\binom{V}{3}}{t}$, $|\bigcup_{i=1}^{t} B_i| \leq 2t$. Then we have
\[
p^t \cdot \binom{|V|}{2t} \leq \mathbb{E}[|D_t|] \leq 2t \cdot \binom{2t}{3} \cdot p^t \cdot \binom{|V|}{2t},
\]
for each $2 \leq t \leq 5$.

For each $6 \leq s \leq d$, denote $D_s$ as the set of $s$-subsets in $B$ that are spanned by at most $\left\lceil \frac{3s}{2} \right\rceil$ points in $V$, i.e., for each $\{B_1, \ldots, B_s\} \in D_s \subseteq \binom{\binom{V}{3}}{s}$, $|\bigcup_{i=1}^{s} B_i| \leq \left\lceil \frac{3s}{2} \right\rceil + 3$. Then we have
\[
p^s \cdot \binom{|V|}{\left\lceil \frac{3s}{2} \right\rceil + 3} \leq \mathbb{E}[|D_s|] \leq \left( \left\lceil \frac{3s}{2} \right\rceil + 3 \right)^s \cdot p^s \cdot \binom{|V|}{\left\lceil \frac{3s}{2} \right\rceil + 3},
\]
for each $6 \leq s \leq d$.

Let $Y = \{\{C_1, C_2\} : C_1, C_2 \in D_7 \land |C_1 \cap C_2| \geq 2\}$. For each $2 \leq t \leq 6$, take
\[Y_t = \{\{C_1, C_2\} \in Y : \exists e_1, \ldots, e_t \in C_1 \cap C_2 \text{ such that } \bigcup_{i=1}^{t} e_i \leq 2t\}
\]
and $Y_0 = Y \setminus \left( \bigcup_{2 \leq t \leq 6} Y_t \right)$. Then, for each $\{C_1, C_2\} \in Y_0$, $C_1 \cap C_2$ can be viewed as a 3-uniform hypergraph with vertex set $V$ that is simultaneously $G_3(2t, t)$-free for each $2 \leq t \leq 5$ and $G_3(\left\lceil \frac{3s}{2} \right\rceil + 3, s)$-free for $s = 6$. Thus, we have
\[
\mathbb{E}[|Y_0|] \leq \sum_{2 \leq t \leq 6} \sum_{j=2+1}^{13} \left( \binom{j}{3}^t \cdot p^t \cdot \binom{|V|}{j} \cdot \left( \frac{14}{3} \right)^{7-t} \cdot p^{7-t} \cdot \binom{|V|}{14-j} \right)^2.
\]

Now, take $H$ as a random 7-uniform hypergraph with vertex set $B$ and hyperedge set
\[\mathcal{E}(H) = \{\{B_1, \ldots, B_7\} : \{B_1, \ldots, B_7\} \in D_7\},
\]
and set $p = m^{-\left(\frac{3s}{2} + \varepsilon\right)}$ for some small enough such that $0 < \varepsilon < \frac{1}{60}$.

Then, when $m$ is large enough, we have for each $2 \leq t \leq 5$ and each $6 \leq s \leq d$, $s \neq 7$,
\[
\mathbb{E}[|D_t|], \mathbb{E}[|D_s|], \mathbb{E}[|Y_0|] \ll \mathbb{E}[|B|].
\]

Thus, with probability at least $\frac{3}{4}$, deleting at most one triple from each $t$-subset in $D_t$ for all $2 \leq t \leq 5$, each $s$-subset in $D_s$ for all $6 \leq s \leq d$, $s \neq 7$ and each $C_1 \cup C_2$ such that $\{C_1, C_2\} \in Y_0$, we can obtain an induced 7-uniform subhypergraph $H'$ in $H$ with at least $\frac{3}{4} \cdot |V(H)|$ vertices such that the vertex set $V(H')$ of $H'$ is simultaneously $G_3(2t, t)$-free for all $2 \leq t \leq 5$ and $G_3(\left\lceil \frac{3s}{2} \right\rceil + 3, s)$-free for all $6 \leq s \leq d$, $s \neq 7$. Moreover, since the pairs in $Y_0$ are destructured during the process of deleting triples from $t$-subsets in $D_t$ for $2 \leq t \leq 6$, therefore, the deletion of triples in $C_1 \cup C_2$ for $\{C_1, C_2\} \in Y_0$ guarantees that $H'$ is also a linear hypergraph.

Meanwhile, fix any $A \in \binom{V}{3}$, by symmetry, we have
\[
\mathbb{E}[\deg_H A] = \mathbb{P}[A \in H] \cdot \mathbb{E}[\text{average degree of } H].
\]

Moreover,
\[
\mathbb{E}[|V(H)|] = \mathbb{E}[|B|] = \left( \frac{1}{6} - o(1) \right) \cdot m^{\frac{14}{11}} - \varepsilon
\]
and
\[
c_1 \cdot m^{\frac{14}{11} - 7\varepsilon} \leq \mathbb{E}[|\mathcal{E}(H)|] = \mathbb{E}[|D_t|] \leq c_2 \cdot m^{\frac{14}{11} - 7\varepsilon}
\]
for some absolute constant $0 < c_1 < c_2$. Since both $|V(H)|$ and $|\mathcal{E}(H)|$ can be regarded as the sum of several independent Bernoulli random variables, thus by Chernoff bound, for $m$ large enough, we have
\[
\frac{m^{\frac{14}{11} - \varepsilon}}{12} \leq |V(H)| \leq \frac{m^{\frac{14}{11} - \varepsilon}}{3}
\]
and
\[
\frac{c_1}{2} \cdot m^{\frac{14}{11} - 7\varepsilon} \leq |\mathcal{E}(H)| \leq 2c_2 \cdot m^{\frac{14}{11} - 7\varepsilon}.
\]
with probability at least $\frac{7}{8}$. Meanwhile, we also have
$$E[\deg_H A] = p \cdot E\left[\frac{T_{\gamma}(E(H))}{|V(H)|}\right] \leq 200c_2 \cdot p \cdot m^{\frac{1}{11} - 6\varepsilon}.$$ 

Thus, by Markov’s inequality, with probability at least $\frac{7}{8}$,
$$Pr[\deg_H A > c_3 \cdot m^\frac{1}{11} - 6\varepsilon] < \frac{p}{32},$$
for some absolute constant $c_3 \geq 10^4 \cdot c_2$. Thus, denote $N_0$ as the number of vertices in $V(H)$ of degree exceeding $c_3 \cdot m^\frac{1}{11} - 6\varepsilon$, we have $E[N_0] < \frac{p^2}{32} |V(H)|$, with probability at least $\frac{7}{8}$. Again, by Markov’s inequality, with probability at least $\frac{7}{8}$, the hypergraph $H$ contains at most $\frac{1}{8} \cdot |V(H)|$ vertices of degree exceeding $c_3 \cdot m^\frac{1}{11} - 6\varepsilon$. In particular, with probability at least $\frac{7}{8}$, we can delete these vertices from $H'$ to obtain a linear induced subhypergraph $H''$ of $H'$ with at least $\frac{m^{\frac{1}{11}} - \varepsilon}{24}$ vertices and maximum degree at most $c_3 \cdot m^\frac{1}{11} - 6\varepsilon$.

Therefore, by Lemma [I.3] we have
$$\alpha(H'') \geq c \cdot m^{\frac{\varepsilon}{4}} (\log m)^{\frac{1}{8}},$$
for some absolute constant $c > 0$. Since an independent set $I$ in $H''$ is a 3-uniform hypergraph with vertex set $V$ that is simultaneously $G_2(2t, t)$-free for each $2 \leq t \leq 5$ and $G_3(\lceil \frac{m^2}{4} \rceil + 3, s)$-free for each $6 \leq s \leq d$, thus the above inequality guarantees the existence of the corresponding $(m, c \cdot m^{\frac{\varepsilon}{4}} (\log m)^{\frac{1}{8}}, d, 2)$ $X$-code of constant weight 3. This completes the proof.

**IV. $r$-even-free triple packings and $X$-codes with higher error tolerance**

To construct $X$-codes with $x = 2$ and weight 3, Fujiwara and Colbourn [13] introduced the notion of $r$-even-free triple packing, which was further studied in [35]. In this section, by obtaining an existence result of the corresponding 6-even-free triple packing, we prove a lower bound on the maximum number of codewords of an $(m, n, 1, 2)$ $X$-code of constant weight 3 which can detect up to three erroneous bits if there is only one $X$ in the raw response data and up to six erroneous bits if there is no $X$, this improves the lower bound given in [35]. And we also extend this lower bound to a general case.

A triple packing of order $v$ is a set system $(V, B)$ such that $B$ is a family of triples of a finite set $V$ and any pair of elements of $V$ appears in $B$ at most once. Given a triple packing $(V, B)$, we call subset $C$ in $B$ an i-configuration if $|C| = i$. A configuration $C$ is even if for every vertex $v \in V$ appearing in $C$, the number $|\{B : v \in B \in C\}|$ of triples containing $v$ is even. And a triple packing $(V, B)$ is r-even-free if for every integer $i$ satisfying $1 \leq i \leq r$, $B$ contains no even i-configurations.

By carefully analysing the structure of r-even-free triple packing, Fujiwara and Colbourn [13] obtained the following theorem which relates the r-even-free triple packing to a special kind of $X$-codes.

**Theorem IV.1.** [13] For $r \geq 4$, if there exists an r-even-free triple packing $(V, B)$, there exists a $(|V|, |B|, 1, 2)$ $X$-code of constant weight 3 that is also a $(|V|, |B|, 3, 1)$ $X$-code and a $(|V|, |B|, r, 0)$ $X$-code.

Using the existence results of anti-Pasch Steiner triple systems, Fujiwara and Colbourn [13] proved that for every $m \equiv 1, 3 \pmod{6}$ and $m \notin \{7, 13\}$, there exists an $(m, m(m - 1)/6, 1, 2)$ $X$-code of constant weight 3 that is an $(m, m(m - 1)/6, 3, 1)$ $X$-code and an $(m, m(m - 1)/6, 5, 0)$ $X$-code. And they also proved the existence of a 6-even-free triple packing $B$ of order $m$ with $|B| = 6.31 \times 10^{-3} \times m^{1.8}$ using the probabilistic method, which gives a lower bound on the size of the corresponding $X$-code given by Theorem [IV.1].

Recently, according to a complete characterization of all the forbidden even configurations in the 6-even-free triple packing, Tsuruoka and Fujiwara [35] obtained the following result, which improves the lower bound $6.31 \times 10^{-3} \times m^{1.8}$ given in [13].

**Theorem IV.2.** [35] For sufficiently large $m$, there exists an $(m, c' \cdot m^{1.8}, 1, 2)$ $X$-code of constant weight 3 that is also an $(m, c' \cdot m^{1.8}, 3, 1)$ $X$-code and an $(m, c' \cdot m^{1.8}, 6, 0)$ $X$-code, where $c' = \frac{1}{36} (\frac{1}{2^5})^{\frac{1}{8}}$.

Similar to Section III.D, using the probabilistic hypergraph independent set approach, we prove the following theorem, which improves the order of magnitude of the lower bound in Theorem [IV.2] by a factor of $(\log m)^{\frac{1}{8}}$.

**Theorem IV.3.** For sufficiently large $m$, there exists an $(m, c_0 \cdot m^{\frac{\varepsilon}{4}} (\log m)^{\frac{1}{8}}, 1, 2)$ $X$-code of constant weight 3 that is also an $(m, c_0 \cdot m^{\frac{\varepsilon}{4}} (\log m)^{\frac{1}{8}}, 3, 1)$ $X$-code and an $(m, c_0 \cdot m^{\frac{\varepsilon}{4}} (\log m)^{\frac{1}{8}}, 6, 0)$ $X$-code, where $c_0 > 0$ is an absolute constant.

An even 4-configuration is called a Pasch, if it has the form $\{a, b, c\}, \{a, c, f\}, \{b, d, f\}, \{c, d, e\}$. An even 6-configuration is called a grid if it has the form $\{a, b, c\}, \{d, e, f\}, \{g, h, i\}, \{a, d, g\}, \{b, e, h\}, \{c, f, i\}$, and a double triangle if it has the form $\{a, b, c\}, \{c, d, e\}, \{e, f, g\}, \{a, g, h\}, \{b, h, i\}, \{d, f, i\}$. Before we present the proof of Theorem [IV.3], we need the following proposition.

**Proposition IV.4.** [35] A triple packing contains no Pasches, grids or double triangles is 6-even-free.

**Proof of Theorem [IV.3]** The idea of the proof here is the same as that of Theorem [II.1]
By Theorem [V.1] and Proposition [V.4], we only need to construct a triple packing without Pasches, grids and double triangles. Let $V$ be a finite set of points and $|V| = m$, take a subset $B$ of triples by picking elements of $\binom{V}{3}$ uniformly and independently at random with probability $p$.

Denote $D_2$ as the set of non-linear triple pairs in $B$, i.e., for each $\{B_1, B_2\} \in D_2 \subseteq \binom{B}{2}$, $|B_1 \cap B_2| \geq 2$. Then we have

$$\mathbb{E}[|D_2|] \leq \binom{4}{3} \cdot p^2 \cdot \binom{|V|}{4}.$$

Denote $D_4$ as the set of Pasches, $D_{61}$ as the set of grids and $D_{62}$ as the set of double triangles in $B$, we have

$$\mathbb{E}[|D_4|] \leq 6! \cdot p^4 \cdot \binom{|V|}{6},$$

and

$$\left(\frac{9}{3}\right) \cdot \left(\frac{6}{3}\right) \cdot p^6 \cdot \frac{|V|^3}{9} \leq \mathbb{E}[|D_{61}|], \mathbb{E}[|D_{62}|] \leq 9! \cdot p^6 \cdot \frac{|V|^3}{9}.$$

Let $Y = \{(C_1, C_2) : C_1, C_2 \in D_{61} \cup D_{62} \text{ and } |C_1 \cap C_2| \geq 2\}$, then $Y = Y_1 \cup Y_2 \cup Y_3$, where $Y_1 = Y \cap (D_{61} \times D_{61})$, $Y_2 = Y \cap (D_{62} \times D_{62})$ and $Y_3 = Y \cap (D_{61} \times D_{62} \cup D_{62} \times D_{61})$. Since

$$\begin{cases}
\mathbb{E}[Y_1] \leq c_1 \cdot (p^{10}m^{13} + p^9m^{12} + p^8m^{11} + p^9m^9 + p^8m^{10} + p^8m^9); \\
\mathbb{E}[Y_2] \leq c_2 \cdot (p^{10}m^{13} + p^9m^{12} + p^9m^{11} + p^9m^{10} + p^8m^{10} + p^8m^9); \\
\mathbb{E}[Y_3] \leq c_3 \cdot (p^{10}m^{13} + p^{10}m^{12} + p^9m^{11} + p^8m^{10}),
\end{cases}$$

for some large absolute constants $c_1, c_2, c_3$, we have

$$\mathbb{E}[|Y|] \leq C_0 \cdot (p^{10}m^{13} + p^9m^{12} + p^9m^{11} + p^9m^{10} + p^8m^{10} + p^8m^9),$$

for some absolute constant $C_0 \geq (c_1 + c_2 + c_3)$.

Now, take $\mathcal{H}$ as a random 6-uniform hypergraph with vertex set $B$ and hyperedge set

$$\mathcal{E}(\mathcal{H}) = \{(B_1, \ldots, B_6) : \{B_1, \ldots, B_6\} \text{ forms a grid or a double triangle in } B\},$$

and set $p = m^{-\left(\frac{3}{4} + \varepsilon\right)}$ for some $\varepsilon$ small enough such that $0 < \varepsilon < \frac{3}{40}$.

Then, for $m$ large enough, we have

$$\mathbb{E}[|D_2|], \mathbb{E}[|D_4|], \mathbb{E}[|Y|] \ll \mathbb{E}[|B|].$$

Thus, with probability at least $\frac{3}{4}$, we can delete at most one triple from each non-linear pair, Pasche and $C_1 \cup C_2$ such that $(C_1, C_2) \in Y$, obtaining a linear induced 6-uniform subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ with at least $\frac{3}{4} \cdot |V(\mathcal{H})|$ vertices such that the vertex set of $\mathcal{H}'$ contains no non-linear triple pairs and Pasches.

Meanwhile, fix any $A \in \binom{V}{3}$, since

$$\mathbb{E}[|V(\mathcal{H})|] = \mathbb{E}[|B|] = \left(\frac{1}{6} - o(1)\right) \cdot m^{\frac{15}{2} - \varepsilon}$$

and

$$m^{\frac{5}{2} - 6\varepsilon} \leq \frac{532}{m^{2\varepsilon}} \leq \mathbb{E}[|\mathcal{E}(\mathcal{H})|] = \mathbb{E}[|D_{61} \cup D_{62}|] \leq (2 - o(1)) \cdot m^{\frac{5}{2} - 6\varepsilon}.$$

By Chernoff bound, for $m$ large enough, we have

$$\begin{cases}
m^{\frac{15}{2} - \varepsilon} \leq |V(\mathcal{H})| \leq m^{\frac{15}{2} - \varepsilon}; \\
m^{\frac{9}{2} - 6\varepsilon} \leq |\mathcal{E}(\mathcal{H})| \leq 3m^{\frac{9}{2} - 6\varepsilon},
\end{cases}$$

and

$$\mathbb{E}[deg_\mathcal{H}A] \leq p \cdot 216m^{\frac{3}{2} - 5\varepsilon},$$

with probability at least $\frac{7}{8}$. Thus, by Markov’s inequality,

$$Pr[deg_\mathcal{H}A \geq 10^4 \cdot m^{\frac{3}{2} - 5\varepsilon}] \leq \frac{p}{32},$$

with probability at least $\frac{7}{8}$. Again, by Markov’s inequality, with probability at least $\frac{3}{4}$, the hypergraph $\mathcal{H}$ contains at most $\frac{1}{4} \cdot |V(\mathcal{H})|$ vertices of degree exceeding $10^4 \cdot m^{\frac{3}{2} - 5\varepsilon}$. Therefore, with probability at least $\frac{1}{2}$, we can delete these vertices and obtain a linear subhypergraph $\mathcal{H}''$ of $\mathcal{H}'$ with at least $\left(\frac{1}{17}\right) \cdot m^{\frac{15}{2} - \varepsilon}$ vertices and maximum degree at most $10^4 \cdot m^{\frac{3}{2} - 5\varepsilon}$.

Finally, by Lemma [IV.3] we have

$$\alpha(\mathcal{H}'') \geq c_0 \cdot m^{\frac{5}{2} \log m},$$
for some absolute constant $c_0 > 0$. Since an independent set $I$ in $\mathcal{H}'$ is a triple packing that contains no Pasche, grid or double triangle, thus the above inequality guarantees the existence of a 6-even-free triple packing of order $c_0 \cdot m^{2} (\log m)^{2}$. This completes the proof.

The above new approach can also be applied to obtain general $r$-even-free triple packings. Noted that for any even $i$-configuration $\mathcal{C}$, $1 \leq i \leq r$, we have

$$\deg_{\mathcal{C}}(v) \equiv 0 \mod 2,$$

for every $v \in V$. Since $(V, \mathcal{C})$ is a triple system, we also have

$$\sum_{v \in V} \deg_{\mathcal{C}}(v) = 3 \cdot |\mathcal{C}| = 3i.$$

Thus, for odd $i$, an $i$-configuration $\mathcal{C}$ cannot be even, and for even $i$, $\mathcal{C}$ involves at most $\frac{3i}{2}$ points in $V$.

Now, take a triple packing $(V, \mathcal{B})$ as a 3-uniform linear hypergraph with vertex set $V$, from the perspective of sparse hypergraphs, for even $i$, a $\mathcal{G}_3(\frac{3i}{2}, i)$-free 3-uniform linear hypergraph is a triple packing that contains no even $i$-configurations. Ranging $i$ from 1 to $r$, we have the following proposition.

**Proposition IV.5.** If a 3-uniform linear hypergraph $\mathcal{H}$ is simultaneously $\mathcal{G}_3(\frac{3i}{2}, i)$-free for every even $1 \leq i \leq r$, then $\mathcal{H}$ is an $r$-even-free triple packing.

Let $r' = \lfloor \frac{r}{2} \rfloor$ and $V$ be a finite set of points, consider a random triple system $(V, \mathcal{B})$ by picking elements of $\binom{n}{3}$ uniformly and independently with a proper probability $p$. First, estimate the expectations of the number of non-linear triple pairs and the number of forbidden $\mathcal{G}_3(\frac{3i}{2}, i)$s for every even $1 \leq i \leq r$. Then, construct a $2r'$-uniform random hypergraph with the set of triples $\mathcal{B}$ as its vertex set such that any $2r'$ triples form a hyperedge if and only if they involve at most $3r'$ points in $V$. Using a similar probabilistic hypergraph independent set approach as that for Theorem IV.3, one can obtain the following theorem.

**Theorem IV.6.** For sufficiently large $m$, there exists an $r$-even-free triple packing $\mathcal{B}$ of order $m$ such that

$$|\mathcal{B}| = \Omega(m^{\frac{2r'}{3r' - 1}} (\log m)^{\frac{1}{2r' - 1}}),$$

where $r' = \lfloor \frac{r}{2} \rfloor$.

Combining the above result with Theorem IV.1 we immediately have

**Corollary IV.7.** For sufficiently large $m$, there exists an $(m, \Omega(m^{\frac{2r'}{3r' - 1}} (\log m)^{\frac{1}{2r' - 1}}), 1, 2)$ X-code of constant weight 3 that is also an $(m, \Omega(m^{\frac{2r'}{3r' - 1}} (\log m)^{\frac{1}{2r' - 1}}), 3, 1)$ X-code and an $(m, \Omega(m^{\frac{2r'}{3r' - 1}} (\log m)^{\frac{1}{2r' - 1}}), r, 0)$ X-code, where $r' = \lfloor \frac{r}{2} \rfloor$.

**Remark IV.8.** A little different from the case $r = 6$, for general $r$, we can not fully characterize the specific even configurations that shall be forbidden to obtain an $r$-even-free triple packing. Thus, a stronger restriction has been required in Proposition IV.5.

Recently, Shangguan and Tamo [32] used a similar method and improved the lower bound on the maximum number of hyperedges in a $\mathcal{G}_k(v, e)$-free $k$-uniform hypergraph on $m$ vertices. It is worth noting that the hypergraph they constructed is universally $\mathcal{G}_3(ik - \lceil \frac{k-1}{2} \rceil k, i)$-free for every $2 \leq i \leq e$, which is a sparser structure than that in Proposition IV.5 when $k = 3$. Therefore, Theorem IV.6 can also be viewed as an application of Theorem 3 in [32].

V. CONCLUDING REMARKS AND FURTHER RESEARCH

In this paper, we investigate the maximum number $M_w(m, d, x)$ of an X-code of constant weight $w$ with testing quality parameters $d$ and $x$. We obtain a general lower bound for $M_w(m, d, x)$ and further improve this bound for the case with $w = 3$ and $x = 2$. Using tools from additive combinatorics and finite fields, we also obtain some explicit constructions for cases $d = 3, 5$ and $x = 2$, which improve the corresponding general lower bounds. Moreover, we study a special class of $(m, n, 1, 2)$ X-codes of constant weight 3 which can also detect many erroneous bits if there is at most one $X$.

It is worth noting that the results of Proposition III.8 and Theorem III.9 also hold for some larger $d$. One can check that for $d = 6$, the indicator vectors of the hyperedge set of a uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ with girth at least 5 also form a constant weighted $(|V|, |\mathcal{E}|, 6, 2)$ X-code. But, due to the complexity of verification, we still don’t know when these results no longer hold. And it is also an interesting direction using other kinds of sparse hypergraphs to construct X-codes.

Although many works have been done about bounding $M_w(m, n, d, x)$, in most cases, the gaps between the upper bounds and the lower bounds are still quite large. For cases $d = 3, x = 2$ and $w = 3, 4$, constructions given by Theorem III.5 narrow the gaps between the upper bounds and the lower bounds to an $\varepsilon$ over the exponent. We expect methods from other aspects can provide some better constructions.
REFERENCES

[1] N. Alon and A. Shapira, “On an extremal hypergraph problem of Brown, Erdős and Sós,” *Combinatorica*, vol. 26, no. 6, pp. 627–645, 2006.

[2] M. Ajtai, J. Komlós, J. Pintz, J. Spencer, and E. Szemerédi, “Extremal uncrowded hypergraphs,” *J. Combin. Theory Ser. A*, vol. 32, no. 3, pp. 321–335, 1982.

[3] M. Ajtai, J. Komlós, and E. Szemerédi, “A note on Ramsey numbers,” *J. Combin. Theory Ser. A*, vol. 29, no. 3, pp. 354–360, 1980.

[4] F. A. Berend, “On Sets of Integers Which Contain No Three Terms in Arithmetical Progression,” *Proc. Nat. Acad. Sci. U. S. A.*, vol. 32, pp. 561–563, 1946.

[5] K. A. Bush, W. T. Federer, H. Pesotan, and D. Raghavarao, “New combinatorial designs and their application to group testing,” *J. Statist. Plann. and Inference*, vol. 10, no. 3, pp. 335–343, 1984.

[6] G. D. Cohen and G. Zemor, “Intersecting codes and independent families,” *IEEE Trans. Inform. Theory*, vol. 40, no. 6, pp. 1872–1881, 1994.

[7] R. Duke, H. Lefmann, and V. Rödl, “On uncrowded hypergraphs,” *Random Structures & Algorithms*, vol. 6, no. 2-3, pp. 209–212, 1995.

[8] A. G. Dyachkov and V. V. Rykov, “Bounds on the length of disjunctive codes,” *Problemy Peredachi Informatsii*, vol. 18, no. 3, pp. 7–13, 1982. [In Russian]

[9] P. Erdős, P. Frankl, and Z. Füredi, “Families of finite sets in which no set is covered by the union of two others,” *J. Combin. Theory Ser. A*, vol. 33, no. 2, pp. 158–166, 1982.

[10] P. Erdős, P. Frankl, and Z. Füredi, “Families of finite sets in which no set is covered by the union of r others,” *Israel J. Math.*, vol. 51, no. 1-2, pp. 75–89, 1985.

[11] P. Erdős, P. Frankl, and V. Rödl, “The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent,” *Graphs Combin.*, vol. 2, no. 2, pp. 113–121, 1986.

[12] A. Ferber, G. McKinley, and W. Samotij, “Supersaturated sparse graphs and hypergraphs,” *International Mathematics Research Notices*, pp. 1–25, 2018.

[13] Y. Fujiwara and C. J. Colbourn, “A Combinatorial Approach to X-Tolerant Compaction Circuits,” *IEEE Trans. Inform. Theory*, vol. 56, no. 7, pp. 3196–3206, 2010.

[14] Z. Füredi, “On r-cover-free families,” *J. Combin. Theory Ser. A*, vol. 73, no. 1, pp. 172–173, 1996.

[15] H. Hollmann, “Design of test sequences for VLSI self-testing using LFSR,” *IEEE Trans. Inform. Theory*, vol. 36, no. 32, pp. 386–392, 1990.

[16] F. K. Hwang and V. T. Sós, “Non-adaptive hypergeometric group testing,” *Studia Sci. Math. Hungar.*, vol. 22, no. 1, pp. 257–263, 1987.

[17] P. Keevash, “Hypergraph Turán problems,” in *Surveys in combinatorics 2011*, volume 392 of London Math. Soc. Lecture Note Ser., pages 83–139, Cambridge Univ. Press, Cambridge, 2011.

[18] A. Kostochka, D. Mubayi, and J. Verstraëte, “On independent sets in hypergraphs,” *Random Structures & Algorithms*, vol. 44, no. 2, pp. 224–239, 2014.

[19] W. H. Kautz and R. Singleton, “Nonrandom binary superimposed codes,” *IEEE Trans. Inform. Theory*, vol. 10, no. 4, pp. 363–377, 1964.

[20] F. Lazebnik and J. Verstraete, “On hypergraphs of girth five,” *Electron. J. Combin.*, vol. 10, no. R25, pp. 1–15, 2003.

[21] A. Lempel and M. Cohn, “Design of universal test sequences for VLSI,” *IEEE Trans. Inform. Theory*, vol. 31, no. 1, pp. 10–17, 1985.

[22] R. Lidl and N. Niederreiter, *Finite Fields*. Cambridge Univ. Press, Cambridge, 1983.

[23] S. S. Lumetta and S. Mitra, “X-codes: Theory and Applications of Unknowable Inputs Center for Reliable and High-Performance Computing,” Univ. Illinois at Urbana Champaign, 2003, Tech. Rep. CRHC-03-08 (also UILU-ENG-03-2217).

[24] S. S. Lumetta and S. Mitra, “X-codes: Error control with unknowable inputs,” in *Proc. IEEE Int. Symp. Inf. Theory*, p. 102, 2003.

[25] S. Mitra and K. S. Kim, “X-compact: An efficient response compaction technique,” *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 23, no. 3, pp. 421–432, 2004.

[26] S. Mitra, S. S. Lumetta, M. Mitzenmacher, and N. Patil, “X-tolerant test response compaction,” *IEEE Design & Test of Computers*, vol. 22, no. 6, pp. 566–574, 2005.

[27] E. J. McCluskey, D. Burek, B. Koenemann, S. Mitra, J. H. Patel, J. Rajski, and J. A. Waicukauski, “Test Data Compression,” *IEEE Design & Test of Computers*, vol. 20, no. 2, pp. 76–87, 2003.

[28] J. H. Patel, S. S. Lumetta, and S. M. Reddy, “Application of Saluja-Karpovsky compactors to test responses with many unknowns,” in *Proc.21st IEEE VLSI Test Symp.*, pp. 107–112, 2003.

[29] V. Rödl, “On a packing and covering problem,” *European J. Combin.*, vol. 5, no. 1, pp. 69–78, 1985.

[30] G. Seroussi and N. H. Bshouty, “Vector sets for exhaustive testing of logic circuits,” *IEEE Trans. Inform. Theory*, vol. 34, no. 3, pp. 513–522, 1988.

[31] K. K. Saluja and M. Karpovsky, “Testing computer hardware through data compression in space and time,” in *Proc. Int. Test Conf.*, pp.83–88, 1983.

[32] C. Shangguan and I. Tamo, “Universally Sparse Hypergraphs with Applications to Coding Theory,” *arXiv:1902.03503*, 2019.

[33] D. R. Stinson and R. Wei, “Some new upper bounds for cover-free families,” *J. Combin. Theory Ser. A*, vol. 90, no. 1, pp. 224–234, 2000.

[34] B. Sudakov, “Recent developments in extremal combinatorics: Ramsey and Turán type problems,” in *Proceedings of the International Congress of Mathematicians*, Volume IV, pages 2579–2606, Hindustan Book Agency, New Delhi, 2010.

[35] Y. Tsuboda and Y. Fujiwara, “Bounds and polynomial-time construction algorithm for X-codes of constant weight three,” in *Proc. IEEE Int. Symp. Inf. Theory*, pp. 2515–2519, 2018.

[36] Y. Tsuboda, Y. Fujiwara, Hana Ando, and Peter Vandendriessche, “Bounds on Separating Redundancy of Linear Codes and Rates of X-Codes,” *IEEE Trans. Inform. Theory*, vol. 64, no. 12, pp. 7577–7593, 2018.

[37] P. Wohl and L. Huismann, “Analysis and design of optimal combinational compactors,” in *Proc. 21st IEEE VLSI Test Symp.*, pp. 101–106, 2003.