Locally Encodable and Decodable Codes for Distributed Storage Systems

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Abstract—We consider the locality of encoding and decoding operations in distributed storage systems (DSS), and propose a new class of codes, called locally encodable and decodable codes (LEDC), that provides a higher degree of operational locality compared to currently known codes. For a given locality structure, we derive an upper bound on the global distance and demonstrate the existence of an optimal LEDC for sufficiently large field size. In addition, we also construct two families of optimal LEDC for fields with size linear in code length.

I. INTRODUCTION

Motivated by practical applications in data storage and communication, locality has become an increasingly important notion in the study of erasure codes. A code with some locality property allows certain operations to be performed by accessing only a portion of a codeword or a data vector. As a result, significant amount of resources (bandwidth, computational power, memory) can be saved. We list below some representative families of codes with certain locality properties.

- **Locally decodable codes** were proposed by Katz and Trevisan [1], which allow a single symbol of the original data to be decoded with high probability by only querying a small number of coded symbols of a possibly corrupted codeword.

- **Locally testable code** was first systematically studied by Goldreich and Sudan [2]. For such a code, there exists a test that checks whether a given string is a codeword, or rather far from the code, by reading only a constant number of symbols of the string.

- **Locally repairable codes** (LRC) (see e.g. [3, 4]) were tailored-made for distributed storage systems (DSS), which guarantee that any erased coded symbol (corresponding to a failed storage node) can be reconstructed from a small subset of other coded symbols (corresponding to other surviving nodes). LRC with a given locality network topology were studied in [5, 6].

- **Update efficient codes**, also known as **locally updatable codes** (see e.g. [7, 8, 9]), require updating as few nodes as possible whenever one data symbol is changed.

- **Coding with constraints** [10, 11, 12, 13, 14, 15] requires that each coded symbol must be a function of a given subset of data symbols. In a classical code, each coded symbol can be a function of all data symbols.

This work was completed when Han Mao Kiah visited Singapore University of Technology and Design.

![Fig. 1: An example of an LEDC with two local subcodes. Here \( \mathcal{K}_1 = \{x_1, x_2, x_3\}, \mathcal{K}_2 = \{x_2, \ldots, x_5\}, \mathcal{N}_1 = \{c_1, \ldots, c_4\}, \) and \( \mathcal{N}_2 = \{c_5, \ldots, c_{10}\} \). The underlying field is \( \mathbb{F}_7 \) - the set of integers modulo 7. Coded symbols in each group \( \mathcal{N}_i \) are encoded from the corresponding data symbols in \( \mathcal{K}_i \) and conversely can be used to decode these data symbols. Each local subcode is an MDS code and moreover, the LEDC, which encodes \( k = |\mathcal{K}_1 \cup \mathcal{K}_2| = 5 \) data symbols into \( n = |\mathcal{N}_1| + |\mathcal{N}_2| = 10 \) coded symbols, can reach the optimal minimum distance \( d = 4 \). In this example, the subcodes are encoded by simply using two Vandermonde matrices. In general, this naive construction yields LEDCs with optimal distances only in certain cases (see Theorem 3 and Example 2).

Continuing along this line of research, we propose the class of **locally encodable and decodable code** (LEDC) that provides a higher level of operational locality in distributed storage systems (DSS) compared to currently known codes. In an LEDC, the set of \( n \) coded symbols (corresponding to \( n \) storage nodes) is partitioned into \( m \) disjoint subsets \( \mathcal{N}_i \)'s, each of which is responsible for encoding and decoding of a given subset \( \mathcal{K}_i \) of some data symbols. Each pair \((\mathcal{N}_i, \mathcal{K}_i)\) forms a **maximum distance separable** (MDS) code, referred to as a local subcode. The parameter of interest is the minimum (Hamming) distance of the global LEDC. An LEDC that has the largest minimum distance is called **optimal**.
An LEDC provides extra protection for node failures: any form an optimal LEDC with optimal distance $d$. The underlying field is $\mathbb{F}_7$ - the set of integers modulo 7.

An LEDC distinguishes itself from the family of locally repairable codes in the following aspects.

- First, both encoding and decoding can be done locally for each local subcode of an LEDC - to create coded symbols in $\mathcal{N}_i$, only data symbols from $\mathcal{K}_i$ are involved, and conversely, to decode data symbols in $\mathcal{K}_i$, only coded symbols in $\mathcal{N}_i$ are involved. In contrast, the encoding operation in each local group in an LRC may involve all data symbols and only local repair is required, not local decidability. In other words, an LRC does not provide local encoding and decoding and only guarantees local repair, which can also be done by an LEDC.

- Secondly, in the context of LEDC, the structure of the local groups (i.e. $\mathcal{K}_i$’s and $\mathcal{N}_i$’s) are given, whereas in the context of LRC, the size and the repair capability of each group are given as input.

LEDC falls under the regime of coding with constraints as each coded symbol must be a function of a given set of data symbols. However, while LEDC provides local decidability, coding with constraints does not. Notice also that the notion of local decidability in the setting of LEDC is different from that in the setting of locally decodable codes (LDC) \[\text{I}\]. Indeed, while an LEDC guarantees that each given subset of data symbols can be decoded (with probability one) from a specific subset of coded symbol (possibly under some errors/erasures), an LDC requires that each data symbol can be decoded (with high probability) from a small subset of coded symbols (possibly under some errors).

A crucial feature of an optimal LEDC is that the repair capability of the global system can be greater than its individual local systems. To illustrate this, we consider the example in Fig. 1 and regard the local subcodes as codes operating over independent storage systems. The first storage system utilizes an MDS code of minimum distance $d_1 = 2$ and hence tolerates one node failure. On the other hand, the second storage system utilizes an MDS code of minimum distance $d_2 = 3$ and hence tolerates two node failures. If the two codes are co-designed to form an optimal LEDC with optimal distance $d = 4$, then the LEDC provides extra protection for node failures: any three node failures across the two systems can be tolerated.

The improvement in fault tolerance results from the fact that the two systems share some common data symbols ($x_2$ and $x_3$ in this toy example). Each system also has some private data symbols ($x_1$ in the first system and $x_4$ and $x_5$ in the second system). In normal condition where the node failures are within the local fault tolerance, each storage system can work independently to repair the failed nodes. No sharing of private data is required. However, in a catastrophic scenario where the number of node failures exceeds the local fault tolerance, the two systems can cooperate to repair the failed nodes by sharing the private data.

Furthermore, there exist LEDCs whose fault tolerances exceed the sum of the fault tolerances of their local subcodes.

**Example 1.** For example, we can construct an LEDC with the following locality structure:

\[
\mathcal{N}_1 = \{c_1, c_2, \ldots, c_5\}, \quad \mathcal{K}_1 = \{x_1, x_2, x_3, x_4\}, \\
\mathcal{N}_2 = \{c_6, c_7, \ldots, c_{12}\}, \quad \mathcal{K}_2 = \{x_2, x_3, \ldots, x_7\}.
\]

Here, the local subcodes are $[5, 4]$ and $[7, 6]$ MDS codes that each tolerate up to one erasure. We demonstrate later in Example 5 that the LEDC can achieve a minimum distance of five and hence is able to tolerate up to four erasures.

Our key results are summarized below.

- We prove that for any given locality structure (i.e. $\mathcal{K}_i$’s and $\mathcal{N}_i$’s), there always exists an optimal LEDC over any sufficiently large finite fields. The optimal minimum distance, however, can be determined in polynomial time.

- When $m = 2$ (i.e. there are only two local subcodes), we provide constructions for two families of optimal LEDC.

  (I) A straightforward construction using nested MDS codes for the case where $|\mathcal{K}_1 \cap \mathcal{K}_2|$ is small.

  (II) An algebraic construction of LEDC as a (punctured) subcode of a cyclic code of length $q - 1$, where $q$ is the size of the finite field, in the case where $|\mathcal{N}_1 - \mathcal{K}_1| = |\mathcal{N}_2 - \mathcal{K}_2|$.

Our paper is organized as follows. Necessary definitions and notation are provided in Section II. We prove the existence of optimal LEDC over sufficiently large finite fields in Section III. Section IV is devoted for the construction of optimal LEDC over small fields when there are two local subcodes.

II. Preliminaries

Let $\mathbb{F}_q$ denote the finite field with $q$ elements. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For a $k \times n$ matrix $M$, for $i \in [k]$ and $j \in [n]$, let $M_i$ and $M[j]$ denote the row $i$ and the column $j$ of $M$, respectively. We define below standard notions from coding theory (for instance, see \[\text{I}\]).

The support of a vector $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ is the set $\text{supp}(u) = \{i \in [n]: u_i \neq 0\}$. The (Hamming) weight of a vector $u \in \mathbb{F}_q^n$ is $|\text{supp}(u)|$. The (Hamming) distance between two vectors $u$ and $v$ of $\mathbb{F}_q^n$ is defined to be $d(u, v) = |\{i \in [n]: u_i \neq v_i\}|$. A $k$-dimensional subspace $\mathcal{C}$ of $\mathbb{F}_q^n$ is called a linear $[n, k, d]$ (erasure) code over $\mathbb{F}_q$ if the minimum distance $d(\mathcal{C})$ between any pair of distinct vectors in $\mathcal{C}$ is equal to $d$. Sometimes we may use the notation $[n, k, d]$ or just $[n, k]$ for the sake of simplicity. The vectors in $\mathcal{C}$ are called codewords. It is known that the minimum weight of a nonzero codeword in a linear code $\mathcal{C}$ is equal to its minimum distance $d(\mathcal{C})$. The well-known Singleton bound (\[\text{I}\], Ch. 1) states that for any $[n, k, d]$ code, it holds that $d \leq n - k + 1$. If the equality is attained, the code is called maximum distance separable (MDS).
A generator matrix $G$ of an $[n, k]$ code $C$ is a $k \times n$ matrix whose rows are linearly independent codewords of $C$. Then $C = \{xG : x \in \mathbb{F}_q^n\}$. It is also well known that if $d(C) = d$ then $C$ can correct any $d - 1$ erasures. In other words, $x$ can be recovered from any $n - d + 1$ coordinates of the codeword $c = xG$. An $[n, k, d]$ code can be used in a DSS as follows. A vector $x$ of $k$ data symbols can be encoded into $n$ coded symbols $c = xG$, each of which is stored at one node in the system. Then $x$ can be recovered from any set of $n - d + 1$ nodes. Hence the DSS can tolerate $d - 1$ node failures.

**Definition 1.** Let $n \geq k \geq 1$ and $m \geq 1$ be some integers. Let $K_1, \ldots, K_m$ be $m$ nonempty (possibly overlapping) subsets of $[k]$ such that $[k] = \bigcup_{i=1}^m K_i$. Let $N_1, \ldots, N_m$ be $m$ nonempty non-overlapping subsets of $[n]$ that partition $[n]$, i.e. $N_i \cap N_j = \emptyset$ if $i \neq j$ and $[n] = \bigcup_{i=1}^m N_i$. Suppose that $n_i = |N_i| \geq |K_i| = k_i$ for every $i \in [m]$. An $(N_1, K_1)^m, n_1, \ldots, n_m$ such an LEDC over an alphabet $\Sigma$ is a mapping $\mathcal{E} : \Sigma^k \rightarrow \Sigma^n$ that maps a vector of $k$ data symbols $x = (x_1, \ldots, x_k) \in \Sigma^k$ into a vector of $n = \sum_{i=1}^m n_i$ coded symbols $c = (c_1, \ldots, c_n) \in \Sigma^n$ and satisfies the following properties.

1. The coded symbols in $N_i \triangleq \{c_j : j \in N_i\}$ only depend on the data symbols in $K_i \triangleq \{x_j : j \in K_i\}$ for every $i \in [m]$.
2. The set of data symbols $K_i$ can be determined from any subset of $k_i$ coded symbols of the set $N_i \forall i \in [m]$.

When $\Sigma$ is a finite field $\mathbb{F}_q$ for some prime power $q$ and the mapping $\mathcal{E}$ is linear, the corresponding LEDC is called linear. For linear LEDC, the mapping $\mathcal{E}$ can be represented by a $k \times n$ matrix $G$ over $\mathbb{F}_q$ such that $\mathcal{E}(x) = xG$. Such a matrix $G$ is referred to as a generator matrix of the LEDC. In this work we only are interested in linear LEDC. The second property (P2) in Definition 1, which $\mathcal{E}$ must satisfy, states that each pair of $n_i$ coded symbols in $N_i$ and $k_i$ data symbols in $K_i$ forms a linear $[n_i, k_i]$ MDS code. We refer to these $m$ MDS codes as the local subcodes of the LEDC. An example of a linear LEDC with two local subcodes over $\mathbb{F}_2$ is given in Fig. 1.

### III. Existence of Optimal Locally Encodable and Decodable Codes Over Large Fields

We first discuss the closely related concept of coding with constraints. The upper bound on the minimum distance for a code with coding constraints [11], [15] still applies in the setting of LEDC. The existence proof for optimal codes over large finite fields [15] Lemma 12], however, needs to be appropriately modified to take into account the new locality feature of LEDC.

#### A. Coding With Constraints

In the setting of linear coding with constraints [10], [11], [12], [13], [14], [15], the data vector $x \in \mathbb{F}_q^n$ is encoded into the coded vector $c = xG \in \mathbb{F}_q^n$ for some $k \times n$ matrix $G$ in $\mathbb{F}_q$, subjected to the following constraints: each coded symbol $c_j$ is a function of a given subset of the data symbols indexed by $C_j \subseteq [k]$. In a classical code, $C_j \equiv [k]$ for all $j \in [n]$. For $i \in [k]$ let $R_i \triangleq \{j : i \in C_j\}$. Then it is obvious that the support of the $i$th row of any valid generator matrix $G$ of a code with coding constraints must be included in $R_i$. Similarly, the support of the $j$th column of $G$ must be included in $C_j$. The following theorem presents an upper bound on the minimum distance of a code with coding constraints and states that an optimal code attaining this upper bound does exist over a sufficiently large finite field.

**Theorem 1.** ([11] Corollary 1), ([15] Lemma 12]) Suppose that $C$ is a linear code that encodes the vector of data symbols $x \in \mathbb{F}_q^n$ into the vector of coded symbols $c \in \mathbb{F}_q^n$ under the following constraints: each $c_j$ is a function of a given subset of the data symbols indexed by $C_j \subseteq [k]$. Let $R_i \triangleq \{j : i \in C_j\}$. Then

$$d(C) \leq d_{\max} \triangleq 1 + \min_{\emptyset \neq I \subseteq [k]} \{| \cup_{i \in I} R_i | - | I | \}.$$  

Moreover, when $q$ is sufficiently large, there exists a code with minimum distance attaining this bound.

**Proposition 1.** The upper bound $d_{\max}$ in Theorem 1 can be found in polynomial time.

**Proof:** Note that $d_{\max}$ is the largest $d$ satisfying

$$| \cup_{i \in I} R_i | \geq d - 1 + | I |, \quad \forall \emptyset \neq I \subseteq [k].$$  

From the Singleton bound, $d \leq n - k + 1$. Hence, we can find $d_{\max}$ by verifying $[\ref{summary}]$ for each $d$ ranging from $n - k + 1$ down to 1. As long as $[\ref{summary}]$ can be verified in polynomial time for every $d \in [n - k + 1]$, we can find $d_{\max}$ in polynomial time.

Note that any $d \in [n - k + 1]$, can be written as $d = (n - k + 1) - \delta$, for some $0 \leq \delta \leq n - k$. Hence, $[\ref{summary}]$ can be rewritten as

$$| \cup_{i \in I} R_i | \geq n - k - \delta + | I |, \quad \forall \emptyset \neq I \subseteq [k].$$  

In the proof of [14] Lemma 10], we provide a polynomial time algorithm to verify the so-called MDS Condition

$$| \cup_{i \in I} R_i | \geq n - k + | I |, \quad \forall \emptyset \neq I \subseteq [k],$$  

where $R_1, \ldots, R_k$ are arbitrary nonempty subsets of $[n]$. We do so by creating a network with one source and $k$ sinks and prove that $[\ref{summary}]$ holds if and only if the capacity of a minimum cut between the source and any sink is at least $n - k$.

Using exactly the same proof, we can show that $[\ref{summary}]$ holds if and only if the capacity of a minimum cut between the source and any sink of that network is at least $n - 2$. As the capacity of such a minimum cut can be computed in polynomial time, $[\ref{summary}]$ can be verified in polynomial time and so can $[\ref{summary}]$. 

#### B. Optimal Locally Encodable and Decodable Codes Over Large Fields

In this subsection we establish that an optimal LEDC always exists over a sufficiently large field.

**Theorem 2.** Suppose that $C$ is a linear $(N_1, K_1)^m, n_1, \ldots, n_m$ LEDC. Let $n_i = |N_i|$, $n = \sum_{i=1}^m n_i$, and $k = |\bigcup_{i=1}^m K_i|$. For $i \in [m]$ and $j \in N_i$ let $C_j = K_i$. For each $i \in [k]$ let $R_i \triangleq \{j : i \in C_j\}$. Then

$$d(C) \leq d_{\max} \triangleq 1 + \min_{\emptyset \neq I \subseteq [k]} \{| \cup_{i \in I} R_i | - | I | \}.$$  

(5)
Moreover, when \( q \) is sufficiently large, there exists a linear \( \{ N_i, K_i \}_{i=1}^m \)-LED\(C \) over \( \mathbb{F}_q \) with minimum distance attaining this bound.

The upper bound \( \delta \) simply follows from Theorem \( \ref{thm:upper-bound} \). Note that by Proposition \( \ref{prop:matrix} \), this upper bound can be determined in polynomial time. We present below a proof of the existence of optimal LEDCs over large fields. We aim to show that when \( q \) is sufficiently large, there always exists a \( k \times n \) matrix \( G \) over \( \mathbb{F}_q \) that generates an \( \{ N_i, K_i \}_{i=1}^m \)-LED\(C \) with minimum distance attaining \( \delta \).

Firstly, observe that if \( G \) is a generator matrix of an \( \{ N_i, K_i \}_{i=1}^m \)-LED\(C \) then by (P1), \( \text{supp}(G_i) \subseteq R_i \) and \( \text{supp}(G[j]) \subseteq C_j \) for all \( i \in [k] \) and \( j \in [n] \). Let \( G^\xi = (g^\xi_{i,j})_{k \times n} \) where

\[
g^\xi_{i,j} = \begin{cases} 
\xi_{i,j}, & \text{if } j \in R_i, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \xi_{i,j} \)'s are indeterminates.

For any subset \( J \subseteq [n] \) of size \( n - d_{\text{max}} + 1 \) let \( \mathcal{G}^J = (\mathcal{V}, \mathcal{E}) \) be the bipartite graph defined as follows. The vertex set \( \mathcal{V} \) can be partitioned into two parts, namely, the left part \( L = \{ \ell_1, \ldots, \ell_k \} \), and the right part \( R = \{ r_j : j \in J \} \). The edge set is

\[
\mathcal{E} = \{ (\ell_i, r_j) : i \in [k], j \in J \cap R_i \}.
\]

Then for every \( \emptyset \neq I \subseteq [k] \), the neighbor set of \( \{ \ell_i : i \in I \} \) has size at least

\[
|N(\{ \ell_i : i \in I \})| = | \cup_{i \in I} (J \cap R_i) | \\
= | J \cap ( \cup_{i \in I} R_i) | \\
= | \{ \cup_{i \in I} R_i \} \setminus [n] \setminus J | \\
\geq | \{ \cup_{i \in I} R_i \} | - (n - |J|) \\
\geq (d_{\text{max}} + |I| - 1) - (n - (n - d_{\text{max}} + 1)) \\
= |I|.
\]

Hence, according to the famous Hall’s marriage theorem, there is a matching of size \( k \) in \( \mathcal{G}^J \). In other words, there exist \( k \) distinct indices \( j_1, \ldots, j_k \) in \( J \) such that \( j_i \in R_i \) for all \( i \in [k] \).

For each subset \( J \subseteq [n] \) of size \( n - d_{\text{max}} + 1 \), we consider the submatrix \( P^J \) of \( G^\xi \) that consists of the columns indexed by \( j_1, \ldots, j_k \) as discussed above. Then the determinant \( \det(P^J) \), which is a multivariable polynomial in \( \mathbb{F}_q[\ldots, \xi_{i,j}, \ldots] \), is not identically zero. The reason is that in the expression of \( \det(P^J) \) as a sum of monomials, there is one monomial that cannot be canceled out, namely \( \prod_{J \subseteq [n]} \xi_{i,j} \). Let

\[
\delta_{\text{dist}} = \prod_{|J| = n - d_{\text{max}} + 1} \det(P^J) \in \mathbb{F}_q[\ldots, \xi_{i,j}, \ldots].
\]

Then \( \delta_{\text{dist}} \) is not identically zero. Roughly speaking, the polynomial \( \delta_{\text{dist}} \) captures the locality structure \( \{ N_i, K_i \}_{i=1}^m \) of the LED\(C \) and the desired minimum distance \( d_{\text{max}} \).

Regarding the locality for decoding, i.e. every \( k_i \) coded symbols in \( N_i \) can be used to recover all data symbols in \( K_i \), let

\[
\delta_{\text{MDS}} = \prod_{i=1}^m \det(Q_i) \in \mathbb{F}_q[\ldots, \xi_{i,j}, \ldots],
\]

where the second product is taken over all \( k_i \times k_i \) submatrices \( Q_i \) that consist of \( k_i \) rows of \( G^\xi \) indexed by \( K_i \) and some \( k_i \) columns of \( G^\xi \) among those indexed by \( N_i \). By definition of \( G^\xi \), each of such matrices \( Q_i \) has a nontrivial determinant. Therefore, \( \delta_{\text{MDS}} \) is not identically zero.

Thus,

\[
\delta = \delta_{\text{dist}} \times \delta_{\text{MDS}} \neq 0.
\]

Therefore, by Lemma 4, for sufficiently large \( q \), there exist \( g_{i,j} \in \mathbb{F}_q \) such that \( \delta(\ldots, g_{i,j}, \ldots) \neq 0 \). Let \( G = (g_{i,j}) \) for \( i,j \) where \( j \notin R_i \), simply let \( g_{i,j} = 0 \). Since \( \delta_{\text{dist}}(g_{i,j}) \neq 0 \), the linear code generated by \( G \) has minimum distance at least \( d_{\text{max}} \). Moreover, since \( \delta_{\text{MDS}}(g_{i,j}) \neq 0 \), the coded symbols in each \( N_i \) form an \( [n_i, k_i] \) MDS code.

Hence, \( G \) generates an optimal LED\(C \). We complete the proof of Theorem \( \ref{thm:main} \).

IV. OPTIMAL LOCALLY ENCODEABLE AND DECODABLE CODES WITH TWO LOCAL SUBCODES

In this section we restrict ourselves to the case \( m = 2 \), i.e. when the LED\(C \) has exactly two local subcodes. We provide two constructions of optimal LEDCs over fields of sizes linear in \( n \).

Without loss of generality, we consider the following locality structure:

\[
K_1 = \{ 1, 2, \ldots, k_1 \}, \quad K_2 = \{ k_1 - t + 1, k_1 - t + 2, \ldots, k \}, \\
N_1 = \{ 1, 2, \ldots, n_1 \}, \quad N_2 = \{ n_1 + 1, n_1 + 2, \ldots, n \}.
\]

Here \( t = |K_1 \cap K_2| \) is the number of common data symbols shared by the two local subcodes. For brevity purpose, we denote an LED\(C \) with this locality structure as an \( [n_1, k_1; n_2, k_2; t] \)-LED\(C \).

When \( t = k \), i.e. \( K_1 = K_2 = [k] \), there is no locality in encoding, as both subcodes use all of the data symbols in their encoding process. An optimal LED\(C \) is simply an \( [n, k] \) MDS code with minimum distance \( d = n - k + 1 \). In the remainder of this section, we always assume that \( t < k \). The following is a straightforward corollary of Theorem \( \ref{thm:main} \).

**Corollary 1.** Suppose that \( t = |K_1 \cap K_2| < k \). Additionally, assume that either \( t < \min\{k_1, k_2\} \) or \( n_1 - k_1 = n_2 - k_2 \).
If there exists a linear \( [n_1, k_1; n_2, k_2; t] \)-LED\(C \) with minimum distance \( d \) then

\[
d \leq 1 + t + \min\{n_1 - k_1, n_2 - k_2\}.
\]

(6)

The upper bound \( \delta_{\text{dist}} \) reflects the fact that the more common data the local subcodes have, the larger the global minimum distance of the LED\(C \). In one extreme case when \( t = 0 \), i.e. the two subcodes share no common data symbols, the global minimum distance is equal to one local minimum distance, whichever smaller. Hence, the global LED\(C \) offers no further protection against erasures to each local subcode. When \( t \)
is sufficiently large, however, the global LEDC can offer considerable amount of additional protection against erasures.

We henceforth assume, without loss of generality, that $n_1 - k_1 \leq n_2 - k_2$. Then (6) can be rewritten as

$$d \leq 1 + t + n_1 - k_1. \tag{7}$$

We aim to construct optimal LEDCs whose minimum distances meet the upper bound (7).

In what follows, we consider a linear $[n_1, k_1; n_2, k_2; t]$-LEDC $\mathcal{C}$ and describe $\mathcal{C}$ via its generator matrix $G$. From the local encoding property, we may write $G$ as

$$
\begin{pmatrix}
U & 0 \\
A & B \\
0 & V
\end{pmatrix}
\begin{pmatrix}
k_1 - t \\
t \\
k_2 - t
\end{pmatrix}
\tag{8}
$$

where $U$, $A$, $B$, and $V$ are $(k_1 - t) \times n_1$, $t \times n_1$, $t \times n_2$, and $(k_2 - t) \times n_2$ matrices, respectively. From the properties of an LEDC, the linear code $\mathcal{A}$ with generator matrix $G_{\mathcal{A}} \triangleq \begin{pmatrix} U \\ A \end{pmatrix}$ is an $[n_1, k_1]$ MDS code and the linear code $\mathcal{B}$ with generator matrix $G_{\mathcal{B}} \triangleq \begin{pmatrix} B \\ V \end{pmatrix}$ is an $[n_2, k_2]$ MDS code.

A. Optimal LEDCs from Nested MDS codes

Our first construction uses pairs of nested MDS codes. More formally, for $1 \leq k' < k \leq n$, the pair $(\mathcal{C}', \mathcal{C})$ is a pair of nested $(k', k; n)$ MDS codes if $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{C}' \subseteq \mathcal{C}$ are $[n, k']$ and $[n, k]$ MDS codes, respectively.

**Theorem 3** (Construction 1). Suppose $n_2 - k_2 + 1 \geq t$ and there exist pairs of nested $(k_1 - t, k_1; n_1)$ and $(k_2 - t, k_2; n_2)$ MDS codes. Then there exists an optimal $[n_1, k_1; n_2, k_2; t]$-LEDC $\mathcal{C}$ whose minimum distance is $n_1 - k_1 + t + 1$.

**Proof:** Let $(\mathcal{A}', \mathcal{A})$ be the pair of nested $(k_1 - t, k_1; n_1)$ MDS codes with generator matrices $U$ and $\begin{pmatrix} U \\ A \end{pmatrix}$. Similarly, let $(\mathcal{B}', \mathcal{B})$ be the pair of nested $(k_2 - t, k_2; n_2)$ MDS codes with generator matrices $V$ and $\begin{pmatrix} B \\ V \end{pmatrix}$.

Then we consider a typical nonzero codeword $c = (x_1, x_2, x_3)G$, where $x_1 = (x_i : i \in K_1 \setminus K_2)$, $x_2 = (x_i : i \in K_1 \cap K_2)$ and $x_3 = (x_i : i \in K_2 \setminus K_1)$. Hence,

$$c = (x_1U + x_2A, x_2B + x_3V).$$

We have the following cases.

(i) When $x_3 = 0$ and $x_1 \neq 0$, then the first $n_2$ coordinates $x_1U$ of $c$ is a nonzero codeword from $\mathcal{A}'$ and has weight at least $n_1 - k_1 + t + 1$.

(ii) When $x_2 = 0$ and $x_3 \neq 0$, then the last $n_2$ coordinates $x_2B$ of $c$ is a nonzero codeword from $\mathcal{B}'$ and has weight at least $n_2 - k_2 + t + 1 \geq n_1 - k_1 + t + 1$.

(iii) When $x_2 \neq 0$, then $c$ consists of nonzero codewords from $\mathcal{A}$ and $\mathcal{B}$. Hence, $c$ has weight at least $(n_1 - k_1 + 1) + (n_2 - k_2 + 1) \geq n_1 - k_1 + t + 1$, because we assume $n_2 - k_2 + 1 \geq t$.

Therefore, the minimum distance of $\mathcal{C}$ is $n_1 - k_1 + t + 1$, achieving the upper bound in (7).

Pairs of nested MDS codes were studied by Ezerman et al. [18] in the context of quantum codes. Specifically, pairs of nested MDS codes of all possible parameters (assuming the MDS conjecture) were constructed and in general, a pair of $(k', k; n)$ MDS codes exists if $n \leq q$. Therefore, $\max\{n_1, n_2\} \leq q$ suffices for Construction I. The LEDC in Fig 1 is, in fact, constructed using two pairs of nested codes over $F_7$, generated by Vandermonde matrices. Note that a Cauchy matrix also generates a pair of nested MDS codes, while requires a larger field size than a Vandermonde matrix.

When the assumption in Theorem 3 does not hold, i.e. $t > 1 + \max\{n_1 - k_1, n_2 - k_2\}$, Construction I may fail to produce an optimal LEDC. We demonstrate this fact below.

**Example 2.** Let $K_1 = \{1, 2, 3, 4\}$, $K_2 = \{2, 3, 4, 5\}$, and $n_1 = n_2 = 5$. In this case, $t = |K_1 \cap K_2| = 3 > 1 + \max\{n_1 - k_1, n_2 - k_2\}$. Hence, the assumption of Theorem 3 is violated. In fact, using two Vandermonde matrices over $F_7$ for the local subcodes results in the following generator matrix:

$$G = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 1 & 1 & 1 & 1 \\
1 & 4 & 2 & 2 & 3 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 6 & 1 & 6 & 1 & 4 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 6 & 1 & 6
\end{pmatrix}.$$
In this subsection, we demonstrate the existence of field elements $u_j$, $v_j$, $a_j$, and $b_j$ such that the above construction yields a generator matrix $G$ of an optimal LEDC. In particular, we prove the following theorem.

**Theorem 4 (Construction II).** Suppose $n_1 - k_1 = n_2 - k_2 = r$ and $n_1 + n_2 \leq q - 1$. Then there exists an optimal $[n_1, k_1; n_2, k_2; t]$-LEDC $C$ whose minimum distance is $r + t + 1$.

To derive the distance properties, our strategy is to show that the linear codes $\mathcal{A}$, $\mathcal{B}$, and $C$ are subcodes of certain cyclic codes $[16]$ Ch. 7. To this end, suppose that $q - 1 \geq n_1 + n_2$ and identify a vector $(a_0, c_1, \ldots, c_{q-2}) \in \mathbb{F}_q^{q-1}$ with the polynomial $\sum_{j=0}^{q-2} c_j x^j$ in the quotient ring $\mathbb{F}_q[x]/(x^{q-1} - 1)$. Note, however, that while the cyclic codes that we consider are of length $q - 1$ over $\mathbb{F}_q$, our codes $\mathcal{A}$, $\mathcal{B}$, and $C$ may have shorter lengths, namely $n_1$, $n_2$, and $n$, respectively. However, we can regard $\mathcal{A}$, $\mathcal{B}$, and $C$ as codes of length $q - 1$ by simply adding a sufficient number of zero coordinates to the right of each codeword. Doing so obviously does not affect the polynomial representation of each codeword. Thus, from now on we can treat $\mathcal{A}$, $\mathcal{B}$, and $C$ as subspaces of $\mathbb{F}_q^{q-1}$.

Under the mapping of vectors to polynomials, let $(u_0, u_1, \ldots, u_{r+t})$ and $(v_0, v_1, \ldots, v_{r+t})$ correspond to $u(x)$ and $v(x)$, which are polynomials of degree at most $r + t$, respectively. For $1 \leq \ell \leq t$, let $\left(\bar{a}_0^{(\ell)}, a_0^{(\ell)}, \ldots, a^{(\ell)}_{r+t-\ell}\right)$ and $\left(\bar{b}_0^{(\ell)}, b_0^{(\ell)}, \ldots, b^{(\ell)}_{r+t-\ell}\right)$ correspond to $a^{(\ell)}(x)$ and $b^{(\ell)}(x)$, which are polynomials of degree at most $r + t - \ell$ and $r + \ell - 1$, respectively.

Furthermore, the codewords in the linear codes $C$, $\mathcal{A}$, and $\mathcal{B}$, in their polynomial representations, can be obtained via linear combinations of certain polynomials described by $u(x)$, $v(x)$, $a^{(\ell)}(x)$, and $b^{(\ell)}(x)$. Specifically,

1) $C$ is the vector subspace of $\mathbb{F}_q^{q-1}$ spanned by the polynomials $x^i u(x)$ for $0 \leq i \leq k_1 - t - 1$, $x^j v(x)$ for $n_1 \leq j \leq n_1 + k_2 - t - 1$, and $c^{(\ell)}(x) \triangleq x^{k_1 + t - \ell} a^{(\ell)}(x) + x^{n_1 + k_2 - t - \ell} b^{(\ell)}(x)$ for $1 \leq \ell \leq t$, respectively.

2) $\mathcal{A}$ is the vector subspace of $\mathbb{F}_q^{q-1}$ spanned by the polynomials $x^i u(x)$ for $0 \leq i \leq k_1 - t - 1$ and $x^{k_1 + t - \ell} a^{(\ell)}(x)$ for $1 \leq \ell \leq t$.

3) $\mathcal{B}$ is the vector subspace of $\mathbb{F}_q^{q-1}$ spanned by the polynomials $x^j v(x)$ for $n_1 \leq j \leq n_1 + k_2 - t - 1$ and $x^{n_1 + k_2 - t - \ell} b^{(\ell)}(x)$ for $1 \leq \ell \leq t$.

Next, we provide sufficient conditions for the polynomials $u(x)$, $v(x)$, $a^{(\ell)}(x)$, and $b^{(\ell)}(x)$ so that the codes $C$, $\mathcal{A}$, and $\mathcal{B}$ have the desired dimension and distance properties.

**Proposition 2.** Let $\omega$ be a primitive element in $\mathbb{F}_q$. Suppose that the polynomials $u(x)$, $v(x)$, $a^{(\ell)}(x)$, and $b^{(\ell)}(x)$, $1 \leq \ell \leq t$, satisfy the following conditions.

- **(D1)** $u_0, v_0, a_0$, and $b_0$ are nonzero for $1 \leq \ell \leq t$.
- **(D2)** $u(\omega^j) = v(\omega^j) = 0$ for $0 \leq j \leq r + t - 1$.
- **(D3)** For $1 \leq \ell \leq t$, $a^{(\ell)}(\omega^j) = b^{(\ell)}(\omega^j) = 0$ for $0 \leq j \leq r + t - 1$.

Then the LEDC $C$ defined as above is an optimal $[n_1, k_1; n_2, k_2; t]$-LEDC of minimum distance $r + t + 1$.

To prove this proposition, we consider the well-known BCH bound on minimum distance of a cyclic code. We first recall some necessary notations from $[16]$ Ch. 7. Let $C$ be a cyclic code of length $q - 1$ over $\mathbb{F}_q$. An element $z \in \mathbb{F}_q$ is called a zero of $C$ if $c(z) = 0$ for every codeword $c(x) \in C$. Let $B$ be the set of all zeros of $C$. The polynomial $g(x) \triangleq \prod_{z \in B} (x - \alpha)$ is called the generator polynomial of $C$. Then $c(x) \in C$ if and only if $g(x)|c(x)$.

**Theorem 5 (BCH bound).** Let $\omega$ be the primitive element of $\mathbb{F}_q$ and $r$ be an integer. The cyclic code with the generator polynomial $(x - \omega)(x - \omega^r)(x - \omega^{r^2})$ has minimum distance at least $r + 1$.

**Proof of Proposition 2.** First observe that from condition (D1), the matrices $G$ given in $[G_{\mathcal{A}} = \begin{pmatrix} U \\ A \end{pmatrix}$, and $G_{\mathcal{B}} = \begin{pmatrix} B \\ V \end{pmatrix}$ all have full rank. Therefore, the corresponding linear codes $C$, $\mathcal{A}$, and $\mathcal{B}$ have the desired dimensions. For the distance properties, we make the following two claims. Recall that we may regard $C$, $\mathcal{A}$, and $\mathcal{B}$ as codes of length $q - 1$ by adding a sufficient number of zero coordinates to the right of each codeword.

- **(C1)** $C$ is a subcode of the cyclic code with generator polynomial $g_1(x) = (x - 1)(x - \omega) \ldots (x - \omega^{r+1})$.
- **(C2)** $\mathcal{A}$ and $\mathcal{B}$ are subcodes of the cyclic code with generator polynomial $g_2(x) = (x - 1)(x - \omega) \ldots (x - \omega^{r-1})$.

Then by the BCH bound, the codes $C$, $\mathcal{A}$, and $\mathcal{B}$ have minimum distance $r + t + 1$, $r + 1$ and $r + 1$, respectively and the proposition is immediate.

Hence, it suffices to prove the claim. From conditions (D2) and (D4), we deduce that

$u(\omega^j) = v(\omega^j) = c^{(i)}(\omega^j) = \cdots = c^{(\ell)}(\omega^j), 0 \leq j \leq r + t - 1$.

Therefore, $g_1(x) = (x - 1)(x - \omega) \ldots (x - \omega^{r+1})$ divides all polynomials in the basis of $C$. Hence, (C1) follows.

It can be also verified that $g_2(x) = (x - 1)(x - \omega) \ldots (x - \omega^{r-1})$ divides all polynomials in the bases of $\mathcal{A}$ and $\mathcal{B}$, respectively. Hence, (C2) follows.

The following proposition shows that the polynomials satisfying conditions (D1)-(D4) in Proposition 2 do exist. Hence, Theorem 4 follows.

**Proposition 3.** There exist polynomials $u(x)$, $v(x)$, $a^{(\ell)}(x)$, and $b^{(\ell)}(x)$ for $1 \leq \ell \leq t$ that satisfy conditions (D1)-(D4).

To prove this proposition, we consider the following lemma.

**Lemma 1.** Fix $1 \leq \ell \leq t$, $r \geq 1$ and $t - \ell < T < q - \ell$. Then there exists polynomials $a(x)$ with degree at most $t - \ell$ and $b(x)$ with degree at most $\ell - 1$, having nonzero constants,
such that
\[ a(\omega^j) + x^T b(\omega^j) = 0 \quad \text{for} \quad r \leq j \leq r + t - 1. \tag{9} \]

Proof: Let \( a(x) = \sum_{j=0}^{t-\ell} a_j x^j \) and \( b(x) = \sum_{j=0}^{t-1} b_j x^j \). Then (9) is equivalent to the linear system of equations \( M(a_0, a_1, \ldots, a_{t-\ell}, b_0, b_1, \ldots, b_{t-1})^T = 0 \), where \( M \) is the \( t \times (t + 1) \) matrix:

\[
\begin{pmatrix}
1 & \cdots & (\omega^r)^{t-\ell} & (\omega^{r+1})^T & \cdots & (\omega^{r+\ell})^T & \cdots & (\omega^{r+t-1})^T \\
1 & \cdots & (\omega^{r+1})^{t-\ell} & (\omega^{r+2})^T & \cdots & (\omega^{r+\ell+1})^T & \cdots & (\omega^{r+t})^T \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & (\omega^{r+t-1})^{t-\ell} & (\omega^{r+t})^T & \cdots & (\omega^{r+2t-2})^T & \cdots & (\omega^{r+\ell})^T \\
\end{pmatrix}.
\]

Then \( M \) can be rewritten as

\[
\begin{pmatrix}
1 & \cdots & (\omega^{r-\ell})^r & (\omega^{r})^r & \cdots & (\omega^{r+t-1})^r \\
1 & \cdots & (\omega^{r-\ell+1})^r & (\omega^{r+1})^r & \cdots & (\omega^{r+t})^r \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & (\omega^{r+t-1})^r & (\omega^{r+t})^r & \cdots & (\omega^{r+2t-2})^r \\
\end{pmatrix}.
\]

Since \( t - \ell < T \) and \( T + \ell - 1 < q - 1 \), the values \( 1, \ldots, \omega^{t-\ell}, \omega^T, \ldots, \omega^{T+t-1} \) are distinct nonzero elements in \( \mathbb{F}_q \). Therefore \( M \) is the generator matrix of a \([t + 1, t, 2] \) MDS code. Hence, \((a_0, a_1, \ldots, a_{t-\ell}, b_0, b_1, \ldots, b_{t-1})\) belongs to its dual code. Since the dual code is a \([t + 1, 1, t + 1] \) MDS code, which has minimum distance \( t + 1 \), we can choose \((a_0, a_1, \ldots, a_{t-\ell}, b_0, b_1, \ldots, b_{t-1})\) such that \( a_0, a_1, \ldots, a_{t-\ell}, b_0, b_1, \ldots, b_{t-1} \) are nonzero. In particular, \( a_0 \) and \( b_0 \) are nonzero. Thus, \( a(x) \) and \( b(x) \) are the desired polynomials.

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