SINGULARITIES IN MIXED CHARACTERISTIC VIA BIG
COHEN-MACAULAY ALGEBRAS

LINQUAN MA AND KARL SCHWEDE

Abstract. We utilize a recent result of André and Gabber on the existence of weakly
functorial (integral perfectoid) big Cohen-Macaulay (BCM) algebras to study singularities
of local rings in mixed characteristic. In particular, we introduce a mixed characteristic
BCM-variant of rational/$F$-rational singularities, of log terminal/$F$-regular singularities and
of multiplier/test ideals of divisor pairs. We prove a number of results about these objects
including a restriction theorem for BCM multiplier/test ideals and deformation statements
for BCM-regular and BCM-rational singularities. As an application, we obtain results on
the behavior of $F$-regular and $F$-rational singularities in arithmetic families.

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1. Introduction

For the past nearly 40 years, researchers have known about a connection between classes of
singularities defined by resolution of singularities and connected to Kodaira-type vanishing
theorems, with singularities defined by Frobenius in positive characteristic [Fed83, MR85,
HH90, FW89, Smi97, Har98, MS97, Smi00, Har01, Tak04, HY03]. For instance, a variety
$X$ has rational singularities in characteristic zero if and only if its modulo $p$ reductions have
$F$-rational singularities for $p \gg 0$. Analogous statements hold for KLT singularities and
$F$-regular singularities, and also with the multiplier ideals and test ideals.

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In this paper, we begin to extend this story to mixed characteristic. Using André’s recent breakthrough result on the existence of weakly functorial (integral perfectoid) big Cohen-Macaulay $R^+$-algebras [And18c] (also see [And18b, And18a, Bha18, HM17]), we produce mixed characteristic analogs of

- log terminal / strongly $F$-regular singularities, which we call $BCM_B$-regular.
- rational / $F$-rational singularities, which we call $BCM_B$-rational.
- multiplier / test ideals, which we denote by $\tau_B(R, \Delta)$.
- Grauert-Riemenschneider / parameter test submodules, which we denote by $\tau_B(\omega_R)$.

Here $B$ is any fixed big Cohen-Macaulay $R^+$-algebra, in fact our first main theorem is that the choice of this $B$ is not so important. From our perspective the role of $B$ is analogous to a resolution of singularities in characteristic zero or the perfection (up to a small perturbation) in characteristic $p > 0$.

**Theorem A** (Proposition 5.7, Proposition 6.10). The objects $\tau_B(\omega_R)$ and $\tau_B(R, \Delta)$ (and hence the notions of $BCM_B$-regular and $BCM_B$-rational singularities) are independent of the big Cohen-Macaulay $R^+$-algebra $B$ as long as $B$ is chosen to be sufficiently large.

Note that similar definitions were produced by [PR18] who also emphasized a more ideal/module closure theoretic approach. On the other hand, the two authors of this paper previously defined in [MS17] a closely related multiplier/test ideal of pairs $(R, a^t)$ in the case that $R$ is complete and regular.

We prove a number of results about these objects, which we now describe. We first point out that $BCM_B$-regular singularities are always $BCM_B$-rational (and in particular they are Cohen-Macaulay), see Theorem 6.12, and that in characteristic $p > 0$, $BCM_B$-rational, $BCM_B$-regular singularities and $BCM$ test ideals are exactly the same as $F$-rational, $F$-regular and test ideals in characteristic $p > 0$, see Proposition 3.5 and Corollary 6.23.

**Theorem B** (Comparison with characteristic zero definitions). Suppose that $(R, m)$ is a complete local domain of mixed characteristic $(0, p)$.

(a) If $R$ is $BCM_B$-rational for all (or even some sufficiently large) big Cohen-Macaulay algebras $B$, then $R$ is pseudo-rational. See Proposition 3.7 and Proposition 5.7.

(b) For all sufficiently large big Cohen-Macaulay $R^+$-algebras $B$, we have $\tau_B(\omega_R) \subseteq \pi_*\omega_Y$ for all proper birational maps $\pi : Y \to X = \text{Spec } R$. See Proposition 5.11.

(c) If $R$ is normal and $\Delta \geq 0$ is an effective $Q$-divisor such that $K_R + \Delta$ is $Q$-Cartier, and if $(R, \Delta)$ is $BCM_B$-regular for all (or even some sufficiently large) big Cohen-Macaulay $R^+$-algebras $B$, then $(R, \Delta)$ is KLT. See Corollary 6.22 and Proposition 6.10.

(d) Suppose $R$ is normal and $\Delta \geq 0$ is an effective $Q$-divisor such that $K_R + \Delta$ is $Q$-Cartier. Then for all sufficiently large big Cohen-Macaulay $R^+$-algebras $B$, we have $\tau_B(R, \Delta) \subseteq \pi_* \omega_Y([K_Y - \pi^!(K_X + \Delta)])$ for all proper birational maps $\pi : Y \to X = \text{Spec } R$. In other words:

$$\tau_B(R, \Delta) \subseteq \mathfrak{J}(R, \Delta),$$

the $BCM$ test ideal is contained in the usual multiplier ideal (if resolutions of singularities exist or if one runs over all proper birational maps). See Theorem 6.21.

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1The existence of weakly functorial big Cohen-Macaulay algebras was also obtained independently by Gabber.
Here (a) and (b) should be viewed as the mixed characteristic analog of the main result of [Smi97]. Likewise (c) is analogous to the main result of [HW02], whereas the characteristic $p > 0$ analog of (d) appeared essentially in [Tak04, Theorem 2.13]. We should note that our strategy when proving these results is quite distinct from the characteristic $p > 0$ case, we are inspired by the ideas from [Ma18]. A related mixed characteristic result was also proved in [MS17, Lemma 5.6] but again using a different strategy.

We show that our BCM test ideals are stable under small perturbations as long as $B$ is large enough. This is analogous to multiplier ideals and test ideals.

**Theorem C** (Proposition 6.10). Suppose $R$ is a complete normal local domain of mixed characteristic $(0,p)$ and $\Delta \geq 0$ is an effective $\mathbb{Q}$-divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. Then for all sufficiently large big Cohen-Macaulay $R^+$-algebras $B$, we have

$$\tau_B(R, \Delta) = \tau_B(R, \Delta + \varepsilon \text{div}_R(g))$$

for any $0 \neq g \in R$ and any rational number $\varepsilon \ll 1$.

We also obtain transformation rules for $\tau_B(R, \Delta)$ under finite maps analogous to the main result of [STI14].

**Theorem D** (Transformation under finite maps, Theorem 6.17). Suppose that $R \subseteq S$ is a finite extension of complete normal local domains of mixed characteristic $(0,p)$ with induced $\phi: \text{Spec } S \to \text{Spec } R$. Further assume that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on $\text{Spec } R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier and such that $\phi^* \Delta \geq \text{Ram}$, the ramification divisor. Then:

$$(1.0.1) \quad \tau_B(R, \Delta) = Tr(\tau_B(S, \phi^* \Delta - \text{Ram})).$$

One concern the reader may have at this point is that our $\tau_B(\omega_R)$ and $\tau_B(R, \Delta)$ are too small to be useful. We prove the following result which should be viewed as an analog of [HH90, Theorem 6.13].

**Theorem E** (Theorem 5.13). Let $(A, m_A) \to (R, m)$ be a module-finite extension such that $A$ is a complete regular local ring of mixed characteristic $(0,p)$ and $R$ is a complete local domain. Suppose $h \in A$ is such that $A_h \to R_h$ is finite étale. Then there exists an integer $N$ such that $h^N \omega_R \subseteq \tau_B(\omega_R)$ for every big Cohen-Macaulay $R$-algebra $B$.

We also show that ideal generated by all those $h$ such that $h^N \omega_R \subseteq \tau_B(\omega_R)$ contains the defining ideal of the singular locus of $R/p$, see Theorem 5.17. This gives us a weak analog of [HH89, Theorem 3.4].

The previous result can be viewed as showing that certain elements are contained in $\tau_B(R, \Delta)$. Another way we can obtain similar results is to prove a restriction theorem, one of the most useful results of this paper in our opinion. In particular, we are proving a form of inversion of adjunction in mixed characteristic. Compare with for instance [Tak04, Proposition 2.12(1)], [HY03, Theorem 4.1], [EV92], [KM98] and [Laz04, Theorem 9.5.1].

**Theorem F** (Restriction theorem, Theorem 6.27 Corollary 6.29). Let $(R, m)$ be a complete normal local domain of mixed characteristic $(0,p)$ and fix $\Delta \geq 0$ a $\mathbb{Q}$-divisor on $R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p$. Choose $0 \neq h \in R$ so that $V(h)$ and $\Delta$ have no common components. Then for any big Cohen-Macaulay $R^+$-algebra $B$ and any $1 > \varepsilon > 0$, there exists a big Cohen-Macaulay $(R/hR)^+$-algebra $C$ (with a compatible map $B \to C$) so that:

$$\tau_C(R/hR, \Delta|_{R/hR}) \subseteq \tau_B(R, \Delta + (1 - \varepsilon) \text{div}_R(h)) \cdot (R/hR).$$
As a consequence, if $R/hR$ is of characteristic $p > 0$ and $(R/hR, \Delta|_{R/hR})$ is strongly $F$-regular, then $(R, \Delta + (1 - \varepsilon) \text{div}_R(h))$ is KLT and thus $(R, \Delta + \text{div}_R(h))$ is log canonical.

We also obtain a parameter test submodule version of the same result in Theorem 5.9 hence we obtain the analog of one of the main results of [Elk78] and of [FW89].

Our key technical result which lurks behind many of the aforementioned theorems is that we prove any two big Cohen-Macaulay $R^+$-algebras can be mapped to another one. This is the mixed characteristic analog of the main result of [Die07] and answers a question of Shimomoto [Shi17 Problem 1]. Our method of proving this result is largely inspired by the ideas from [And18c], and this is the only place in this paper that we use the machinery of perfectoid spaces.

**Theorem G** (**Theorem 4.10**). Let $(R, m)$ be a complete local domain of mixed characteristic $(0, p)$ and $B_1, B_2$ be two big Cohen-Macaulay $R$-algebras. Then $B_1 \otimes_R B_2$ maps to another (integral perfectoid) big Cohen-Macaulay $R$-algebra $B$. Moreover, if $B_1$ and $B_2$ are in addition $R^+$-algebras, then $B_1 \otimes_R B_2$ maps to another (integral perfectoid) big Cohen-Macaulay $R^+$-algebra $B$ (via an $R^+$-linear map).

As an application of our theory and the main result of [Har98, MS97, Tak04] we obtain the following result, again compare with [Elk78] in characteristic zero and also [Has01, PSZ13] in characteristic $p > 0$.

**Theorem H** (**Theorem 7.2** [Theorem 7.9]). Let $X \to U \subseteq \text{Spec } \mathbb{Z}$ be a proper flat family (resp. let $(X, \Delta \geq 0) \to U \subseteq \text{Spec } \mathbb{Z}$ be a proper flat family of pairs such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier of index $N$). Suppose the fiber $X_p$ is $F$-rational for some point $p \in U$ (resp. $(X_p, \Delta_p)$ is strongly $F$-regular for some $p$ whose residual characteristic does not divide $N$). Then the general fiber $X_\mathbb{Q}$ has rational singularities (resp. the general fiber $(X_\mathbb{Q}, \Delta_\mathbb{Q})$ is KLT). Furthermore $X_q$ is $F$-rational (resp. $(X_q, \Delta_q)$ is strongly $F$-regular) for a Zariski dense and open set of closed points $q \in U$.

It is not hard to see that the statement is false for arbitrary affine schemes, [Example 7.4]. However, a properly formulated local version of it is true for both rational singularities [Theorem 7.5] and for KLT singularities [Theorem 7.10]. The point is we must consider a family of local rings over $\text{Spec } \mathbb{Z}$ (or more general bases). Finally, we also point out that we can replace $\mathbb{Z}$ with more general mixed characteristic Dedekind domains.

In Section 8, as an application of these ideas we obtain a new effective way of proving that a singularity in characteristic zero is rational or KLT. In particular if one starts in characteristic 0 with a singularity $(R, m)$ and if one spreads it out to a mixed characteristic domain $R_A$ which restricts to a characteristic $p > 0$ local ring $R_p$ with $F$-rational singularities, then $R$ has rational singularities. The point is that there is no requirement that $p \gg 0$. Note that in [Smi97], Smith observed that one needs to check only a single $R_p$, but $p > 0$ had to be large enough so that $\omega_A/\pi^* \omega_B$ is $p$-torsion free (which essentially means one must already compute whether the singularity is rational or not). We can avoid this restriction. This idea also generalizes to KLT and strongly $F$-regular pairs (as long as $p$ does not divide the index of $K_R + \Delta$). This is important because $F$-rationality and $F$-regularity checking are implemented in Macaulay2 [GS] in [BBB7] for more general cases than rational or log terminal checking is done in characteristic zero [LT]. Finally, also compare with [Zhu17] where related results were proved for the log canonical threshold via jet schemes.
We describe the organization of the paper. In Section 2 we briefly discuss preliminaries and notations. In Section 3 we prove many of the main results of this paper for BCM$_B$-rational singularities, the proofs of which motivate and are simple cases of many of the other main results in the paper. In Section 4 we review André’s recent result on weakly functorial big Cohen-Macaulay algebras and we prove Theorem G above. In Section 5 we introduce $\tau_B(\omega_R)$ and prove many of the results above in that setting and we prove Theorem E. In Section 6 we introduce $\tau_B(R, \Delta)$ and when a pair $(R, \Delta)$ is BCM$_B$-regular, and we prove Theorem C, Theorem D, and Theorem F. In Section 7 we use these ideas to study $F$-singularities in families where the characteristic varies and we prove Theorem H. Finally in Section 8 we discuss algorithmic consequences mentioned above.

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2. Preliminaries

Throughout this paper, all rings will be commutative with unity. Local rings $(R, \mathfrak{m})$ are always assumed to be Noetherian, although we frequently will consider non-Noetherian rings (including those with unique maximal ideals).

2.1. Perfectoid algebras and big Cohen-Macaulay algebras. Throughout this paper we will use the language of integral perfectoid algebras and almost mathematics as in And18c, Bha18, Sch12, GR03. We will work over a fixed perfectoid field $K = \hat{\mathbb{Q}}_{p}(p^{1/p^\infty})$ and its ring of integers $K^\circ = \hat{\mathbb{Z}}_p[p^{1/p^\infty}]$. We collect some definitions.

A perfectoid $K$-algebra is a Banach $K$-algebra $R$ such that the set of powerbounded elements $R^\circ \subseteq R$ is bounded and the Frobenius is surjective on $R^\circ/p$. A $K^\circ$-algebra $S$ is called integral perfectoid if it is $p$-adically complete, $p$-torsion free, and the Frobenius induces an isomorphism $S/p^{1/p} \to S/p$. If $R$ is a perfectoid $K$-algebra, then the ring of powerbounded elements $R^\circ$ is integral perfectoid, and if $S$ is integral perfectoid, then $S[1/p]$ perfectoid, see Sch12, Theorem 5.2.

Remark 2.1. In Bha18, there is an extra condition in the definition of integral perfectoid algebra: one requires that $S = S_* = \{x \in S[1/p] \mid p^{1/p^n}x \in S \text{ for all } n\}$. However, in practice one can safely ignore the difference between $S$ and $S_*$ (or simply pass from $S$ to $S_*$) because they are $p^{1/p^n}$-almost isomorphic to each other. Our definitions of perfectoid and integral perfectoid algebras are the same as in And18c, Section 2.2.

Let $(R, \mathfrak{m})$ be a local ring and let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of $R$. Recall that an $R$-algebra $B$ is big Cohen-Macaulay with respect to $\underline{x}$ if $\underline{\mathfrak{x}}$ is a regular sequence on $B$ (this means $(x_1, \ldots, x_i) :_B x_{i+1} = (x_1, \ldots, x_i)B$ and $B/(x_1, \ldots, x_d)B \neq 0$), and $B$ is called a (balanced) big Cohen-Macaulay algebra if it is big Cohen-Macaulay with respect to $\underline{x}$ for every system of parameters $\underline{x}$. It is well known that if $B$ is big Cohen-Macaulay with respect to $\underline{x}$, then $\hat{B}^\mathfrak{m}$ is (balanced) big Cohen-Macaulay BH93, Corollary 8.5.3. Big Cohen-Macaulay
algebras always exist: in equal characteristic, this follows from [HH92, HH95], and in mixed characteristic, this is settled by André in [And18a] (see also [HM17]).

Let \((R, m)\) be a local ring of mixed characteristic \((0, p)\) or equal characteristic \(p > 0\), and let \(x = x_1, \ldots, x_d\) be a system of parameters of \(R\). Let \(B\) be an \(R\)-algebra and \(g\) be an element of \(R\) such that \(g\) has a compatible system of \(p\)-power roots \(\{g^{1/p^i}\}_{i=1}^{\infty}\) in \(B\). For example, this holds if \(R\) is a domain and \(B\) is an \(R^+\)-algebra, where \(R^+\) denote the absolute integral closure of \(R\): this is the integral closure of \(R\) inside an algebraic closure of its fraction field. We say \(B\) is \(g^{1/p^\infty}\)-almost big Cohen-Macaulay with respect to \(x\) if \(x\) is a \(g^{1/p^\infty}\)-almost regular sequence on \(B\), that is, \(\{(x_1, \ldots, x_i)^{1/p^{i+1}}\}_{i=1}^{\infty}\) is \(g^{1/p^\infty}\)-almost zero and \(B/(x_1, \ldots, x_d)B\) is not \(g^{1/p^\infty}\)-almost zero. This terminology is slightly misleading since “big Cohen-Macaulay” does not formally imply “almost big Cohen-Macaulay” because the last condition that \(B/(x_1, \ldots, x_d)B\) is not \(g^{1/p^\infty}\)-almost zero is stronger than \(B/(x_1, \ldots, x_d)B \neq 0\). However in most cases this is not an issue: see for example [And18c, Proposition 2.5.1] and [Claim 4.3].

The importance of introducing this almost big Cohen-Macaulay notion is because in characteristic \(p > 0\), it is not hard to show that under mild assumptions, \(R^{1/p^\infty}\) is \(g^{1/p^\infty}\)-almost big Cohen-Macaulay for suitable choice of \(g\) (e.g., \(R_g\) is Cohen-Macaulay) with respect to any \(x\). Hochster essentially proved that every almost big Cohen-Macaulay algebra can be mapped to a big Cohen-Macaulay algebra (see [Hoc75, Hoc94, Hoc02, Die07]).

Throughout this paper, we will often work with (integral perfectoid) big Cohen-Macaulay \(R^+\)-algebras for a complete local domain \((R, m)\) of mixed characteristic \((0, p)\). The existence of such algebras follows from recent work of André [And18c] and Shimomoto [Shi17]. For the convenience of the reader we re-define integral perfectoid in this context (note that, since any \(p\)-adically complete \(R^+\)-algebra \(B\) is automatically an algebra over \(K^\circ\), the definition of integral perfectoid algebra below is compatible with our general definition above).

**Definition 2.2.** Let \((R, m)\) be a complete local domain of mixed characteristic \((0, p)\), fix an algebraic closure of its fraction field and fix an \(R^+\). Furthermore let \(g\) be a nonzero element in \(R^+\). Then an integral perfectoid \(R^+\)-algebra (resp., a \((pg)^{1/p^\infty}\)-almost integral perfectoid \(R^+\)-algebra) is an \(R^+\)-algebra \(S\) such that

(a) \(S\) is \(p\)-adically complete and \(p\)-torsion free;

(b) The Frobenius map \(S/p^{1/p} \rightarrow S/p\) is an isomorphism (resp., the Frobenius map \(S/p^{1/p} \rightarrow S/p\) is an injection and an \((pg)^{1/p^\infty}\)-almost surjection).

Given a \((pg)^{1/p^\infty}\)-almost integral perfectoid algebra \(B\), we can tilt and then untilt to obtain an honest integral perfectoid algebra \(B^\circ = (B^\circ)^\circ\). Moreover, there always exists a natural map \(B^\circ \rightarrow B\) that is injective and is \((pg)^{1/p^\infty}\)-almost surjective, see [And18c 2.3.1] or [And18b, Section 2].

2.2. **Singularities coming from characteristic 0 and singularities in characteristic** \(p\). In characteristic zero, for higher dimensional varieties, we frequently measure singularities by considering a resolution of singularities \(\pi : Y \rightarrow X\) and comparing top differential forms on \(Y\) with those on \(X\). We begin with rational and log terminal singularities, noting some alternate definitions which hopefully are suggestive for some of our definitions later. We will not need all of these definitions but we include them for motivation. We refer the reader to [Har66] for standard notations and facts on local and Grothendieck duality. For a local ring \((R, m)\), we use \(E\) to denote the injective hull of \(R/m\).
Definition-Proposition 2.3 (Singularities in characteristic zero). Suppose that $(R, \mathfrak{m})$ is a $d$-dimensional local ring essentially of finite type over a field of characteristic zero. Suppose that $\pi : Y \to X = \text{Spec} R$ is a log resolution of singularities (all the definitions below are independent of the choice of resolution).

**Rational singularities:** We say that $R$ has rational singularities if $R \simeq_{\text{qis}} R\Gamma(Y, \mathcal{O}_Y)$.

This is equivalent to

(a) $R$ is Cohen-Macaulay and one of the following equivalent conditions hold.
(b) $\Gamma(Y, \omega_Y) \to \omega_R$ surjects.
(b') $H^d_m(R) \to \mathbb{H}^d_m(R\Gamma(Y, \mathcal{O}_Y))$ injects.

**Grauert-Riemenschneider (multiplier) submodules:** We next define $\mathcal{J}(\omega_R)$, called the Grauert Riemenschneider (multiplier) submodule of $R$, to be $\Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(\omega_X + \Delta) \rceil)) \subseteq \omega_R$. If further $\Lambda \geq 0$ is a $\mathbb{Q}$-Cartier divisor on $X$ and $\pi$ is a log resolution, then we define $\mathcal{J}(\omega_R, \Lambda)$ to be $\Gamma(Y, \omega_Y(\lceil \pi^*(\omega_X + \Delta) \rceil)) \subseteq \omega_R$.

**Kawamata-log terminal singularities:** Suppose further that $R$ is normal and there exists a $\mathbb{Q}$-divisor $\Delta \geq 0$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier. We now require $Y$ to be a log resolution for $(X, \Delta)$. Then we say that the pair $(R, \Delta)$ is Kawamata-log terminal (or KLT) if the following equivalent conditions hold.

(c) $\lceil K_Y - \pi^*(K_X + \Delta) \rceil \geq 0$.
(c') $H^d_m(\omega_R) \to \mathbb{H}^d_m(R\Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil)))$ injects.

**Multiplier ideals:** With the same assumptions as we had for KLT singularities, we define the multiplier ideal $\mathcal{J}(X, \Delta)$ to be:

(d) $\Gamma(Y, \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil)) \subseteq R$.
(d') $\text{Ann}_R \ker \left( H^d_m(\omega_R) \to \mathbb{H}^d_m(R\Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil))) \right)$.

Note that $(X, \Delta)$ is KLT if and only if $\mathcal{J}(X, \Delta) = R$.

**Proof.** Using local and Grothendieck duality, we prove the equivalence of (c) with (c') below. The equivalence of (d) and (d') is essentially the same and the equivalence of (b) and (b') also follows from local and Grothendieck duality, but it is less involved. Therefore we omit those proofs. For (c) and (c'), notice that the trace map

\[
\text{R} \Gamma(Y, \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil)) \to R
\]
surjects on 0th-cohomology if and only if the Matlis dual

\[
\text{Hom}_R \left( \text{R} \Gamma(Y, \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil)), E \right) \leftarrow E = H^d_m(\omega_R).
\]

injects. By local and Grothendieck duality and using the fact that $Y$ is regular so that $\omega_Y = \omega_Y^{-d}$, we see that

\[
\text{Hom}_R \left( \text{R} \Gamma(Y, \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil)), E \right) \simeq_{\text{qis}} \text{Hom}_R \left( \text{R} \Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil)), E \right)
\]

\[
\simeq_{\text{qis}} \text{Hom}_R \left( \text{R} \text{Hom}_R(\text{R} \Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil)), \omega_Y^{-d}), E \right)
\]

\[
\simeq_{\text{qis}} \text{R} \Gamma_m(\text{R} \Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil)))[d].
\]

Hence $\Gamma(Y, \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil))$ is Matlis dual to $\mathbb{H}_m^d(\text{R} \Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil)))$. The statement follows. Notice that the Grothendieck dual to (2.3.1) is

\[
\omega_R[d] \to \omega_R^* \to \text{R} \Gamma(Y, \mathcal{O}_Y(\lceil \pi^*(K_X + \Delta) \rceil))[d]
\]
which induces the inclusion in (c').

\[\Box\]
**Definition 2.4** (Pseudo-rational and KLT singularities in all characteristics). If \((R, m)\) is a local ring with a dualizing complex then we say that \(R\) has pseudo-rational singularities if it satisfies the conditions (a) and (b) (or (b')) in **Definition-Proposition 2.3** for all proper birational maps \(\pi : Y \to X = \text{Spec } R\) with \(Y\) normal.

Additionally, if \(R\) is normal and \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier for an effective \(\mathbb{Q}\)-divisor \(\Delta \geq 0\), then we say that \((R, \Delta)\) is KLT if it satisfies the equivalent conditions (c) or (c') above in **Definition-Proposition 2.3** for all proper birational maps \(\pi : Y \to X = \text{Spec } R\) with \(Y\) normal.

One can likewise define \(\mathcal{J}(R, \Delta)\) to be the intersection of \(\Gamma(Y, \mathcal{O}_Y([K_Y - \pi^*(K_X + \Delta)])) \subseteq R\) where \(Y\) runs over all proper birational maps and define \(\mathcal{J}(\omega_R)\) to be the intersection of \(\Gamma(Y, \omega_Y) \subseteq \omega_R\) where \(Y\) again runs over all proper birational maps. Note that by Chow’s Lemma, in any of these definitions, one can restrict attention to projective birational maps.

We next move to positive characteristic and we record the definitions of \(F\)-rational and \(F\)-regular singularities. Recall that in characteristic \(p > 0\), the Frobenius map \(R \to F^e_* R\) induces a natural Frobenius action \(F^e\) on the local cohomology module \(H^d_m(R)\) (which can be identified with \(H^d_m(F^e_* R)\)).

**Definition-Proposition 2.5** (Singularities in characteristic \(p > 0\)). Suppose \((R, m)\) is an excellent \(d\)-dimensional local domain of characteristic \(p > 0\) with normalized dualizing complex \(\omega_R\) and canonical module \(\omega_R = h^{-d}(\omega_R)\).

**\(F\)-rational singularities**: We say that \(R\) has \(F\)-rational singularities if

(a) \(R\) is Cohen-Macaulay and one of the following equivalent conditions hold:

(b) If \(N \subseteq H^d_m(R)\) is such that \(F(N) \subseteq N\), then \(N = 0\) or \(N = H^d_m(R)\).

(b') For any \(0 \neq c \in R\), there exists an integer \(e > 0\) so that \(H^d_m(R) \xrightarrow{1 \to F^e_*} H^d_m(F^e_* R)\) is injective.

(b'') If \(R^+\) is the absolute integral closure of \(R\) and \(B\) is any big Cohen-Macaulay \(R^+\)-algebra, then \(H^d_m(R) \to H^d_m(B)\) is injective.

**Parameter test submodules**: The parameter test submodule \(\tau(\omega_R)\) is the \(\omega_R\)-annihilator of the kernel of \(H^d_m(R) \to H^d_m(R^+)\). Further, if \(\Lambda \geq 0\) is a \(\mathbb{Q}\)-Cartier divisor with \(n\Lambda = \text{div}(f)\), then we define \(\tau(\omega_R, \Lambda)\) to be the \(\omega_R\)-annihilator of the kernel of \(H^d_m(R) \xrightarrow{1 \to f^{1/n}} H^d_m(R^+)\).

**Strongly \(F\)-regular singularities**: Further assume that \(R\) is normal and \(\Delta \geq 0\) is an effective \(\mathbb{Q}\)-divisor on \(\text{Spec } R\). Fix a choice of \(K_R \geq 0\) and suppose \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier with \(n(K_R + \Delta) = \text{div}_R(f)\) for some \(f \in R\) and \(n > 0\). Let \(B\) be a big Cohen-Macaulay \(R^+\)-algebra (and hence contains a copy of \(f^{1/n}\) coming from \(R^+\)).

Notice that the map \(H^d_m(R) \xrightarrow{f^{1/n}} H^d_m(B)\) factors through \(H^d_m(R(K_R)) \cong E\).

\[
\begin{array}{ccc}
H^d_m(R) & \xrightarrow{1} & H^d_m(B) \\
 & \downarrow H^d_m(R(K_R)) & \\
 & \psi & \\
\end{array}
\]

In this case, we say that \((R, \Delta)\) has strongly \(F\)-regular singularities if the induced map

\[
\psi : H^d_m(R(K_R)) \to H^d_m(B)
\]
The canonical map $R$ (Proposition 3.5) essentially contained in [Smi94, Section 5]. However, we will work it out in detail later in Section 5 (see also Proposition 5.3). Thus so is the kernel of the composition. This proves the independence of the choice of $B$.

**Proof.** The equivalence of (b) and (b') is [Smi97, Theorem 2.6]. The equivalence of (b'') is essentially contained in [Smi94, Section 5]. However, we will work it out in detail later in Proposition 3.5.

We first establish the factorization in the digram before the definition of strongly $F$-regularity. Notice that for any finite extension $R \subseteq S \subseteq R^+$ with $S$ normal and containing $f^{1/n}$ and with induced map $\rho : \text{Spec} \, S \rightarrow \text{Spec} \, R$, we have that $S \xrightarrow{f^{1/n}} R$ factors as $S \xhookleftarrow{S(\rho^*K_R)} \xrightarrow{\nu} S$. This is because $(1/n) \text{div}(f) \geq \rho^*K_R$ by construction. But $R \xhookleftarrow{S(\rho^*K_R)}$ factors through $R \xhookleftarrow{R(K_R)}$. Applying $H^d_m(-)$ then gives us our desired factorization.

It remains to prove that strongly $F$-regularity of pairs and the test ideal are independent of the choice of $B$. With $S$ as above, we have the factorization

$$H^d_m(R(K_R)) \xrightarrow{\nu} H^d_m(S(\rho^*K_R)) \xrightarrow{\nu} H^d_m(S) \xrightarrow{\nu} H^d_m(B).$$

Note the map $H^d_m(S) \rightarrow H^d_m(B)$ is induced by $S \rightarrow B$. Now, $B$ is a big Cohen-Macaulay $R^+$-algebra, hence the kernel of $H^d_m(S) \rightarrow H^d_m(B)$ is independent of $B$ by [Smi94, Section 5] (see also Proposition 5.3). Thus so is the kernel of the composition. This proves the independence of the choice of $B$ for both strong $F$-regularity and for the test ideal. \qed

Our definition of strong $F$-regularity and test ideal for pairs $(R, \Delta)$ is not the usual definition (we made our choice for motivational purposes as an analogous definition will be made in the mixed characteristic setting). However, it is equivalent to the usual definitions. Note that $F$-singularities of pairs in the literature have typically (perhaps nearly always) been defined in the $F$-finite setting, see for instance [HW02, H104, Tak04]. In that case, $(R,\Delta \geq 0)$ is called strongly $F$-regular if for every $0 \neq c \in R$, there exists an $e > 0$ so that the canonical map $R \rightarrow F^eR([p^e-1]\Delta + \text{div}_R(c))$ splits (this definition holds without the condition that $K_R + \Delta$ is $\mathbb{Q}$-Cartier).

**Proposition 2.6.** Our definition of strongly $F$-regular pairs and the test ideal from Definition-Proposition 2 agrees with the classical definition of [HW02] if $R$ is $F$-finite and $K_R + \Delta$ is $\mathbb{Q}$-Cartier.

**Proof sketch.** We prove the statement for the test ideal since strong $F$-regularity is just the condition that $\tau(R, \Delta) = R$. Again write that $n(K_R + \Delta) = \text{div}_R(f)$. By [BST15], we know that $\tau(R, \Delta) = \text{Image}(\text{Tr} : S([K_S - \rho^*(K_R + \Delta)]) \rightarrow R)$ for any sufficiently larger finite extension $R \subseteq S$ with induced $\rho : \text{Spec} \, S \rightarrow \text{Spec} \, R$ (here $\text{Tr}$ is the Grothendieck trace). We may assume that $f^{1/n} \in S$ so that $\text{div}_S(f^{1/n}) = \rho^*(K_R + \Delta)$. Thus $\tau(R, \Delta) = \text{Image}(\text{Tr} : f^{1/n} \cdot S(K_S) \rightarrow R)$. We observe that $\text{Hom}_R(S(K_S), E) = \text{Hom}_R(\text{Hom}_R(S, \omega_R), E) = H^d_m(S)$ Applying Matlis duality, we then have

$$\tau(R, \Delta) = \text{Ann}_R(\ker(E \rightarrow H^d_m(S))).$$

and it is easy to see that the map $E \rightarrow H^d_m(S)$ factors $H^d_m(R) \xrightarrow{f^{1/n}} H^d_m(S)$ as in Definition-Proposition 2.5. Because we chose $S$ large enough, the kernel is independent of the choice of $S$ and hence
taking a limit we see that
\[ \tau(R, \Delta) = \text{Ann}_R(\ker(E \to H^d_m(R^+))) \]
where the map \( E \to H^d_m(R^+) \) is the map in the diagram of Definition-Proposition 2.5 (one can take \( B = R^+ \) since \( R^+ \) itself is big Cohen-Macaulay by [HH92]). □

Another advantage of our definition of test ideals is that it makes transformation rules under finite maps quite transparent, see Section 6.2.

Remark 2.7. It is natural to expect that, even without the \( F \)-finite hypothesis, we have that our definition of \( \tau(R, \Delta) \) is equal to the annihilator of \( 0^* \Delta E \), which can be defined to be
\[ \{ \eta \in E \mid \text{there exists nonzero } c \in R \text{ such that } \forall e > 0, 0 = \eta \otimes c^{1/p^e} \in E \otimes_R (R((p^e \Delta)))^{1/p^e} \}. \]
We believe this is straightforward but we do not work it out here since the literature on \( F \)-singularities of pairs includes the \( F \)-finite hypothesis.

3. Big Cohen-Macaulay rational singularities

In this section we prove our result on rational and \( F \)-rational singularities. Most results of this section, at least when \( R \) is a complete normal local domain, will also follow from our more general results in Section 5. However, our treatment here is less technical and it only requires the version of weakly functorial big Cohen-Macaulay algebras established by Heitmann and the first author in [HM17]. On the other hand, those later arguments in many cases are simply jazzed up versions of what is done in this section.

Definition 3.1. Let \((R, m)\) be an excellent local ring of dimension \( d \) and let \( B \) be a big Cohen-Macaulay \( R \)-algebra. We say \( R \) is big Cohen-Macaulay-rational with respect to \( B \) (or simply BCM\(_B\)-rational) if
- \( R \) is Cohen-Macaulay and
- \( H^d_m(R) \to H^d_m(B) \) is injective.

We say \( R \) is BCM-rational if \( R \) is BCM\(_B\)-rational for all big Cohen-Macaulay \( R \)-algebras \( B \).

Remark 3.2. \( R \) is BCM\(_B\)-rational if and only if \( \hat{R} \) is BCM\(_\hat{B}\)-rational, this is because \( \hat{B} \) is a big Cohen-Macaulay \( \hat{R} \)-algebra [BH93 Corollary 8.5.3] and \( H^d_m(M) \cong H^d_m(\hat{M}) \) for any \( R \)-module \( M \). As a consequence, \( R \) is BCM-rational if and only if \( \hat{R} \) is BCM-rational since every big Cohen-Macaulay \( \hat{R} \)-algebra is certainly a big Cohen-Macaulay \( R \)-algebra.

Lemma 3.3. If \( R \) is BCM-rational, then \( \hat{R} \) is a normal domain.

Proof. By Remark 3.2, we may assume \( R \) is complete. Let \( P \) be a minimal prime of \( R \) such that \( \dim R/P = \dim R \) and let \( B \) be a big Cohen-Macaulay \( R/P \)-algebra (and hence a big Cohen-Macaulay \( R \)-algebra). Since \( R \) is BCM-rational, the composition \( H^d_m(R) \to H^d_m(R/P) \to H^d_m(B) \) is injective, thus \( H^d_m(R) \to H^d_m(R/P) \) is injective. For every system of

\(^2\)The version of weakly functorial big Cohen-Macaulay algebras established in [HM17] is weaker, but the method is substantially shorter than [And18c], which requires [And18b].
parameters \( x_1, \ldots, x_d \) of \( R \), consider the following diagram:

\[
\begin{array}{ccc}
H^d_m(R) & \xrightarrow{\cdot x} & H^d_m(R/P) \\
\uparrow & & \uparrow \\
R/(x_1, \ldots, x_d) & \rightarrow & (R/P)/(x_1, \ldots, x_d)(R/P)
\end{array}
\]

where the injectivity of the left vertical map is because \( R \) is Cohen-Macaulay (by the definition of BCM-rational) and thus \( x_1, \ldots, x_d \) is a regular sequence on \( R \). Chasing the diagram, we know that \( R/(x_1, \ldots, x_d) \rightarrow (R/P)/(x_1, \ldots, x_d)(R/P) \) is injective. Thus \( P \subseteq (x_1, \ldots, x_d) \) for every system of parameters \( x_1, \ldots, x_d \) of \( R \). Hence \( P \subseteq \cap_t (x_1^t, \ldots, x_d^t) = 0 \). This proves that \( R \) is a domain.

Now let \( R^N \) denote the normalization of \( R \), which is also a complete normal domain. A big Cohen-Macaulay \( R^N \)-algebra \( C \) is still a big Cohen-Macaulay \( R \)-algebra, and hence it follows that \( H^d_m(R) \rightarrow H^d_m(R^N) \) is injective. But this already implies that \( R \) is normal since the Grothendieck trace map \( \omega_{R^N} \rightarrow \omega_R \) is not surjective at any height one prime where \( R \) is non-normal (if it was surjective, then since it is always injective it would follow that \( R \) is already normal at that point).

\[ \square \]

Our first observation is that, with the above definition, BCM-rational singularities deform in the following sense.

**Proposition 3.4.** Let \( (R, m) \) be an excellent local ring of dimension \( d \) and \( x \in R \) be a nonzerodivisor. Let \( B \) be a big Cohen-Macaulay \( R \)-algebra. If \( R/xR \) is BCM\(_{B/xB}\)-rational, then \( R \) is BCM\(_B\)-rational. In particular, if \( R/xR \) is BCM-rational, then so is \( R \).

Compare our proof with the main argument of [Elk78].

**Proof.** Since \( R/xR \) is Cohen-Macaulay and \( x \) is a nonzerodivisor, \( R \) is also Cohen-Macaulay. It suffices to prove \( H^d_m(R) \rightarrow H^d_m(B) \) is injective. The commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & R \\
\downarrow & & \downarrow x \\
0 & \rightarrow & R/xR \\
\downarrow & & \downarrow \\
0 & \rightarrow & R^N \\
\downarrow & & \downarrow \\
0 & \rightarrow & B \\
\downarrow & & \downarrow x \\
0 & \rightarrow & B/xB \\
\end{array}
\]

induces

\[
\begin{array}{ccc}
0 & \rightarrow & H^d_m(R/xR) \\
\downarrow & & \downarrow \alpha \\
0 & \rightarrow & H^d_m(R) \\
\downarrow & & \downarrow \alpha \\
0 & \rightarrow & H^d_m(B/xB) \\
\downarrow & & \downarrow \alpha \\
0 & \rightarrow & H^d_m(B) \\
\end{array}
\]

Chasing this diagram, it is easy to see that \( \alpha \) is injective provided \( H^d_m(R/xR) \rightarrow H^d_m(B/xB) \) is injective. But this follows because \( R/xR \) is BCM\(_{B/xB}\)-rational.

Next we prove that in characteristic \( p > 0 \), BCM-rational singularities are the same as \( F \)-rational singularities, this should be well-known to experts in tight closure theory.

**Proposition 3.5.** Let \( (R, m) \) be an excellent local ring of characteristic \( p > 0 \). Then \( R \) is BCM-rational if and only if \( R \) is \( F \)-rational.
Proof. Suppose $R$ is BCM-rational, it is enough to prove $\hat{R}$ is $F$-rational. Thus by Remark 3.2 and Lemma 3.3 we may assume $R$ is a complete local domain. If $R$ is not $F$-rational, then there exists a system of parameters $(x_1, \ldots, x_d)$ of $R$ and an element $u \in (x_1, \ldots, x_d)^* - (x_1, \ldots, x_d)$. Since $R$ is a complete local domain, by [Hoc94, Theorem 11.1], there exists a big Cohen-Macaulay $R$-algebra $B$ such that $u \in (x_1, \ldots, x_d)B$. But then the map $H^d_m(R) \rightarrow H^d_m(B)$ is not injective, because the nonzero class $[\frac{u}{x_1 \cdots x_d}]$ in $H^d_m(R)$ is mapped to zero in $H^d_m(B)$, which contradicts our assumption that $R$ is BCM-rational.

Conversely, suppose $R$ is $F$-rational. Then $H^d_m(R)$ is simple in the category of $R$-modules with Frobenius action [Smi97]. Therefore the kernel of $H^d_m(R) \rightarrow H^d_m(B)$, being $F$-stable, is either 0 or $H^d_m(R)$. But the kernel cannot be $H^d_m(R)$ because $B$ is big Cohen-Macaulay: pick any system of parameters $x_1, \ldots, x_d$ of $R$, the class $[\frac{1}{x_1x_2\cdots x_d}]$ is not zero in $H^d_m(B)$, since $x_1, \ldots, x_d$ is a regular sequence on $B$. \qed

Remark 3.6. Let $(R, m)$ be a complete local domain. In characteristic $p > 0$, Proposition 3.5 implies that $R$ is BCM-rational if and only if $R$ is $BCM_B$-rational for one single large enough big Cohen-Macaulay $B$ (in fact, one can take $B = R^+$ or any big Cohen-Macaulay $R^+$-algebra $B$ [Smi94]). We will eventually prove that in mixed characteristic, $R$ is BCM-rational if and only if $R$ is $BCM_B$-rational for one (large enough) big Cohen-Macaulay $R^+$-algebra $B$, see Proposition 5.7.

Next we show that BCM-rational singularities are pseudo-rational.

Proposition 3.7. Let $(R, m)$ be an excellent local ring that is BCM-rational. Then $R$ is pseudo-rational. In particular, if $R$ is essentially of finite type over a field of characteristic 0, then $R$ has rational singularities.

Proof. By Remark 3.2 we may assume $R$ is complete. We know $R$ is normal by Lemma 3.3 and Cohen-Macaulay (by the definition of BCM-rational). Therefore to show $\hat{R}$ is pseudo-rational, it suffices to prove that $H^d_m(R) \rightarrow H^d_E(X, O_X)$ is injective for every projective birational map $\pi$: $X \rightarrow Spec R$ with $E$ the pre-image of $\{m\}$. Let $X = \text{Proj } R[Jt]$ for some ideal $J \subseteq R$. By the Sancho-de-Salas exact sequence [SdS87], we have

$$[H^d_{m+Jt}(R[Jt])]_0 \rightarrow H^d_m(R) \rightarrow H^d_E(X, O_X).$$

Thus in order to prove $H^d_m(R) \rightarrow H^d_E(X, O_X)$ is injective, it is enough to show the natural map $H^d_{m+Jt}(R[Jt]) \rightarrow H^d_m(R)$ vanishes.

We let $S = R[Jt]$ and let $n \subseteq S$ denote the maximal ideal $m + Jt$. We consider the surjective ring homomorphism $\bar{S}_n \rightarrow R$, since $R$ is a domain this map factors through $\bar{S}_n/P$ for some minimal prime $P$ of $\bar{S}_n$. We note that $\dim(\bar{S}_n/P) = \dim R + 1$ and hence $R$ is obtained from $\bar{S}_n/P$ by killing a height one prime $Q$. Since $R$ and $\bar{S}_n$ clearly have the same characteristic, by [HH95, Theorem 3.9] in equal characteristic and [HM17, Theorem 3.1] in mixed characteristic, we have a commutative diagram:

$$\begin{array}{ccc}
\bar{S}_n/P & \rightarrow & R \\
\downarrow & & \downarrow \\
B & \rightarrow & C
\end{array}$$
where $B, C$ are balanced big Cohen-Macaulay algebras for $\hat{S}_n/P$ and $R$ respectively. This induces a commutative diagram:

$$
\begin{array}{cccc}
H^d_{m+Jt}(R[Jt]) & \longrightarrow & H^d_n(\hat{S}_n) & \longrightarrow & H^d_n(\hat{S}_n/P) & \longrightarrow & H^d_m(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^d_n(B) & \longrightarrow & H^d_m(C)
\end{array}
$$

where the bottom left 0 is because $B$ is balanced big Cohen-Macaulay $\hat{S}_n/P$-algebra and $\dim(\hat{S}_n/P) > d$. Chasing this diagram, to show the map $H^d_{m+Jt}(R[Jt]) \to H^d_m(R)$ vanishes it suffices to prove $\alpha$ is injective, which follows because $R$ is BCM-rational.

Combining the above three Propositions, we have the following theorem:

**Theorem 3.8.** Let $(R, \mathfrak{m})$ be an excellent local ring of mixed characteristic $(0, p)$ and $x \in R$ be a nonzerodivisor such that $R/xR$ has equal characteristic $p > 0$ (e.g. $R$ is an excellent local domain and $x = p$). If $R/xR$ is F-rational, then $R$ is pseudo-rational.

**Proof.** Since $R/xR$ is F-rational, by Proposition 3.5, $R/xR$ is BCM-rational. By Proposition 3.4, we know $R$ is BCM-rational. This implies $R$ is pseudo-rational by Proposition 3.7.

Motivated by Proposition 3.5 and Proposition 3.7, we conjecture the following:

**Conjecture 3.9.** Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over a field of characteristic 0. Then $R$ is BCM-rational if and only if $R$ has rational singularities.

4. **Dominating big Cohen-Macaulay $R^+$-algebras**

Our main goal in this section is to prove that, in mixed characteristic, any big Cohen-Macaulay algebra can be mapped to an integral perfectoid big Cohen-Macaulay algebra, see Lemma 4.6 and that any two big Cohen-Macaulay algebras can be dominated by another, see Theorem 4.10. These results can be viewed as mixed characteristic analog of the main results in [Die07]. A consequence is that given any set of big Cohen-Macaulay algebras, we can find an (integral perfectoid) big Cohen-Macaulay $R^+$-algebra that dominates all of them.

**Theorem 4.1.** (André [And18c], Shimomoto [Shi17]) Let $(R, \mathfrak{m})$ be a complete local domain of mixed characteristic $(0, p)$ with $\underline{a} = p, x_1, \ldots, x_{d-1}$ a system of parameters. Let $B$ be a $(pg)^{1/p^n}$-almost integral perfectoid $R$-algebra that is $(pg)^{1/p^n}$-almost big Cohen-Macaulay with respect to $\underline{a}$. Then

(a) There exists an integral perfectoid (balanced) big Cohen-Macaulay $R$-algebra $C$ and a morphism $B' = (B')^d \to C$.

(b) We may further map $C$ to an integral perfectoid (balanced) big Cohen-Macaulay $R^+$-algebra $C''$.

**Remark 4.2.** We caution the reader that, in the above theorem, if $B$ is already an $R^+$-algebra, then the construction of Theorem 4.1 (a) yields a map $R^+ \to B' \to C$ (i.e., $B^e \to C$ is $R^+$-linear). However, if we follow the construction of Theorem 4.1 (b), we may result in a different $R^+$-structure on $C''$ (i.e., the map $B^e \to C''$ may not be $R^+$-linear).

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Using Theorem 4.1, André proved the following theorem on the weak functoriality of big Cohen-Macaulay algebras [And18c, Theorem 1.2.1, 4.1.1]. Similar results were also obtained independently by Gabber.

**Theorem 4.3.** Any local homomorphism $R \rightarrow R'$ of complete local domains, with $R$ of mixed characteristic $(0, p)$, fits into a commutative diagram:

\[
\begin{array}{ccc}
R & \rightarrow & R' \\
\downarrow & & \downarrow \\
R^+ & \rightarrow & R'^+ \\
\downarrow & & \downarrow \\
B & \rightarrow & B'
\end{array}
\]

where $B$ and $B'$ are integral perfectoid (balanced) big Cohen-Macaulay $R^+$-algebra and $R'^+$-algebra respectively (when $R'$ is of characteristic $p > 0$, this means $B'$ is perfect and $p^\infty$-adically complete). Moreover, if $R \rightarrow R'$ is surjective, then $B$ can be given in advance.

We will eventually generalize this theorem slightly. We begin with the following, which is our key lemma.

**Lemma 4.4.** Let $(R, m)$ be a complete local domain of mixed characteristic $(0, p)$ and let $B_1, B_2$ be two integral perfectoid big Cohen-Macaulay $R^+$-algebras. Then $B_1 \otimes_{R^+} B_2$ maps $R^+$-linearly to another integral perfectoid big Cohen-Macaulay $R^+$-algebra $B$, i.e., we have a commutative diagram of $R^+$-algebras and $R^+$-linear maps

\[
\begin{array}{ccc}
B_1 & \rightarrow & B \\
\downarrow & & \downarrow \\
R^+ & \rightarrow & B_2
\end{array}
\]

**Proof.** Without loss of generality, we can assume the residue field $k = R/m$ is algebraically closed. By Cohen’s structure theorem we have $A \rightarrow R$ that is a module-finite extension such that $A \cong W(k)[[x_1, \ldots, x_{d-1}]]$ is a complete and unramified regular local ring. Let $A_{\infty,0}$ be the $p$-adic completion of $A[[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}]] \subseteq A^+ = R^+$, which is an integral perfectoid algebra faithfully flat over $A$. Because $B_1$ and $B_2$ are integral perfectoid $R^+$-algebras, they are automatically $A_{\infty,0}$-algebras.

We consider $B_1 \hat{\otimes}_{A_{\infty,0}} B_2$. Since $B_1$ is a (balanced) big Cohen-Macaulay $A$-algebra and $A$ is regular, $B_1$ is faithfully flat over $A$. The same reasoning shows $B_i$ is faithfully flat over $A[[p^{1/p^k}, x_1^{1/p^k}, \ldots, x_{d-1}^{1/p^k}]]$ for every $k$ and thus $B_i$ is faithfully flat over $A[[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}]]$. This implies $B_i/p$ is faithfully flat over $A_{\infty,0}/p$ and thus $(B_1 \hat{\otimes}_{A_{\infty,0}} B_2)/p \cong B_1/p \otimes_{A_{\infty,0}/p} B_2/p$ is faithfully flat over $A_{\infty,0}/p$. Therefore $B_1 \hat{\otimes}_{A_{\infty,0}} B_2$ is big Cohen-Macaulay with respect to $\mathfrak{a} = p, x_1, \ldots, x_{d-1}$. We next consider

\[
B_1 \hat{\otimes}_{A_{\infty,0}} B_2 \rightarrow \tilde{B} = (B_1 \hat{\otimes}_{A_{\infty,0}} B_2)[1/p].
\]

Here $\tilde{B}$ is an integral perfectoid $A_{\infty,0}$-algebra that is $p^{1/p^\infty}$-almost isomorphic to $B_1 \hat{\otimes}_{A_{\infty,0}} B_2$ ([Sch12, Proposition 6.18]). In particular, $\tilde{B}$ is reduced, has no $p$-torsion, and is $p^{1/p^\infty}$-almost big Cohen-Macaulay with respect to $\mathfrak{a}$. 

\[\text{14}\]
We next consider the ring $R \otimes_A R$, this is a commutative ring and is a finite extension of $R$, and if we tensor with the fraction field $K$ of $A$, then this ring becomes

$$(R \otimes_A R) \otimes_A K \cong L \otimes_K L,$$

where $L$ is the fraction field of $R$. Since $K \to L$ is finite separable (as they have characteristic 0), we know that $L \otimes_K L$ is reduced. Therefore the kernel of $R \otimes_A R \to (R \otimes_A R) \otimes_A K$ is a radical ideal of $R \otimes_A R$, which we call it $J$. We now let $S = (R \otimes_A R)/J$, then $S$ is a reduced ring. Because $\tilde{B}$ is torsion-free over $A$, $J$ clearly maps to 0 in $\tilde{B}$, thus we have an induced map $S \to \tilde{B}$. The multiplication map $S = (R \otimes_A R)/J \to R$ is surjective (since $J$ maps to 0 under the multiplication map because $R$ is torsion-free over $A$), and since $R \otimes_A R$ is module-finite over $R$, $S$ has the same dimension with $R$, therefore $R = S/P$ for a minimal prime $P$ of $S$ which defines the diagonal. In summary:

(i) $S = (R \otimes_A R)/J$ is reduced.
(ii) We have an induced map $S \to \tilde{B}$.
(iii) The multiplication map induces $S \to R$, the kernel of which is $P$.

Since $S$ is reduced and $P$ is a minimal prime of $S$, there exists an element $g \in S$ such that

(iv) $g \notin P$ and
(v) $gP = 0$.

We now set

$$T = \tilde{B}(g^{1/p^\infty})[\frac{1}{p}],$$

i.e., this is André’s construction of adjoining $p^{1/p^\infty}$-root of $g$ to $\tilde{B}$. By [And18a] or [Bha18, Theorem 2.3], $T$ is an integral perfectoid $\tilde{B}$-algebra (so in particular it is reduced) and is $p^{1/p^\infty}$-almost faithfully flat over $\tilde{B}$ mod $p$. In particular,

- $T$ is $p^{1/p^\infty}$-almost big Cohen-Macaulay with respect to $x$.

We next consider $g^{-1/p^\infty}T = \cap_i (g^{-1/p^e_i}T)$. By [And18c, 2.3.2], $g^{-1/p^\infty}T$ is a $(pg)^{1/p^\infty}$-almost integral perfectoid $R$-algebra. The key point here is that, since $T$ is reduced, any $g$-torsion in $T$ is automatically $g^{1/p^\infty}$-torsion and thus its image in $g^{-1/p^\infty}T$ vanishes, and thus also in $(g^{-1/p^\infty}T)^\natural$. Therefore the map $S \to (g^{-1/p^\infty}T)^\natural$ factors through $R = S/P$ because $gP = 0$ in $S$. In sum, we have a commutative diagram:

$$\begin{array}{ccc}
A & \to & S = (R \otimes_A R)/J \\
\downarrow & & \downarrow \\
B_1 \otimes_{A_{x_0}} B_2 & \to & \tilde{B} \\
\downarrow & & \downarrow \\
& & T \\
& & (g^{-1/p^\infty}T)^\natural \\
& & g^{-1/p^\infty}T
\end{array}$$

We now come to the heart of the argument.

**Claim 4.5.** $g^{-1/p^\infty}T$ is a $(pg)^{1/p^\infty}$-almost big Cohen-Macaulay $R$-algebra with respect to $x$.

**Proof of Claim.** Since $g^{-1/p^\infty}T$ is $g^{1/p^\infty}$-almost isomorphic to $T$ and $T$ is $p^{1/p^\infty}$-almost big Cohen-Macaulay with respect to $x$, the colon ideal

$$\frac{(p, x_1, \ldots, x_{i-1}) : g^{-1/p^\infty}T x_i}{(p, x_1, \ldots, x_{i-1})(g^{-1/p^\infty}T)}$$
is annihilated by \((pg)^{1/p^\infty}\) for every \(i\). Therefore it is enough to prove that
\[
g^{-1/p^\infty}T/(p, x_1, \ldots, x_{d-1})(g^{-1/p^\infty}T)
\]
is not \((pg)^{1/p^\infty}\)-almost zero. Suppose on the contrary, we have
\[
(pg)^{1/p^e} \in (p, x_1, \ldots, x_{d-1})(g^{-1/p^\infty}T)
\]
for every \(e\), it follows that
\[
pg \in (p, x_1, \ldots, x_{d-1})^{p^e}(g^{-1/p^\infty}T)
\]
for every \(e\). Since \(g^{-1/p^\infty}T\) is \(g^{1/p^\infty}\)-almost isomorphic to \(T\), \(T\) is \(p^{1/p^\infty}\)-almost faithfully flat over \(\tilde{B}\) mod \(p^m\) for every \(m\), and \(\tilde{B}\) is \(p^{1/p^\infty}\)-almost isomorphic to \(B_1 \hat{\otimes}_{A_{\infty,0}} B_2\). This implies that
\[
(pg)^2 \in (p, x_1, \ldots, x_{d-1})^{p^{\hat{e}}}B_1 \hat{\otimes}_{A_{\infty,0}} B_2
\]
for every \(e\), where we are abusing notation and interpreting \(g \in S\) as a lift to an element of \(R \otimes_A R\). Next we note that modulo \(p^m\), we have
\[
(B_1 \hat{\otimes}_{A_{\infty,0}} B_2)/p^m \cong (B_1 \otimes_{A[p^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}]} B_2)/p^m.
\]
Therefore \((pg)^2 \in (p, x_1, \ldots, x_{d-1})^{p^{\hat{e}}}(B_1 \otimes_{A[p^{1/p^n}, \ldots, x_{d-1}^{1/p^n}]} B_2)\) for every \(e\). But then we know that there exists \(A_n = A[p^{1/p^n}, \ldots, x_{d-1}^{1/p^n}]\) for some \(n \gg 0\) depending on \(e\) such that
\[
(pg)^2 \in (p, x_1, \ldots, x_{d-1})^{p^{\hat{e}}}(B_1 \otimes_{A_n} B_2).
\]
At this point, we mod out by the prime ideal \(P\) (again, abuse notation a bit and identify \(P\) with a minimal prime ideal of \(R \otimes_A R\) such that \((R \otimes_A R)/P = R\)). Since we have
\[
\frac{B_1 \otimes_{A_n} B_2}{P(B_1 \otimes_{A_n} B_2)} \to B_1 \otimes_{A_n[R]} B_2,
\]
it follows that, after we mod out by \(P\) we obtain that
\[
(pg)^2 \in (p, x_1, \ldots, x_{d-1})^{p^{\hat{e}}}(B_1 \otimes_{A_n[R]} B_2) \cap R.
\]
Now \(A_n[R]\) is a module-finite domain extension of \(R\), so it is a complete local domain. Since \(B_1, B_2\) are big Cohen-Macaulay algebras over \(A_n[R]\), they are solid \(A_n[R]\)-algebras by [Hoc94, Corollary 2.4], thus \(B_1 \otimes_{A_n[R]} B_2\) is a solid \(A_n[R]\)-algebra by [Hoc94, Proposition 2.1 (a)], and hence \(B_1 \otimes_{A_n[R]} B_2\) is a solid \(R\)-algebra by [Hoc94, Corollary 2.3]. Therefore in \(R\), we have
\[
(pg)^2 \in ((p, x_1, \ldots, x_{d-1})^{p^{\hat{e}}})^\star \subseteq (x_1, \ldots, x_d)^{p^{\hat{e}}}
\]
for every \(e\), where \(I^\star\) denotes the solid closure of \(I\), which is always contained inside the integral closure by [Hoc94, Theorem 5.10]. However, since \(g \notin P\), the element \((pg)^2\) is nonzero in \(R\) and hence
\[
0 \neq (pg)^2 \in \cap_\omega (x_1, \ldots, x_d)^{p^\omega} = 0
\]
which is a contradiction. This completes the proof of Claim 4.5. □
By Claim 4.5, we can apply Theorem 4.1 to the $(pg)^{1/p^\infty}$-almost integral perfectoid, $(pg)^{1/p^\infty}$-almost big Cohen-Macaulay $R$-algebra $g^{-1/p^\infty}T$, we know that there exists an integral perfectoid (balanced) big Cohen-Macaulay $(g^{-1/p^\infty}T)^2$-algebra $B'$ such that $B_1 \otimes_R B_2$ maps to $B'$: because the kernel of $B_1 \otimes_A B_2 \to B_1 \otimes_R B_2$ is precisely the extended ideal $P(B_1 \otimes A B_2)$. Next we need to refine $B'$ to an $R^+$-algebra $B$ such that we have an $R^+$-linear map $B_1 \otimes_{R^+} B_2 \to B$, this does not follow immediately from Theorem 4.1 (for example see Remark 4.2) so we proceed carefully below.

Making $B'$ a big Cohen-Macaulay $R^+$ algebra. The rough idea is to run the above construction for all module-finite domain extensions of $R$ and take the direct limit. Below we give the details. We write $R^+ = \varinjlim_{\beta} R_\beta$ for a directed system of normal module-finite domain extensions $\{R_\beta\}_{\beta}$ of $R$.

We let $S_\beta = (R_\beta \otimes_A R_\beta)/J_\beta$ and let $R_\beta = S_\beta/P_\beta$ for the minimal prime $P_\beta$ of $S_\beta$ defining the diagonal. By the same reasoning as above, there exists $g_\beta \in S_\beta$ such that $g_\beta \notin P_\beta$ and $g_\beta P_\beta = 0$. Moreover, if $\alpha < \beta$, then it is straightforward to check that the image of $g_\alpha$ in $S_\beta/P_\beta$ is nonzero. Let $\tilde{g}_\beta$ be a finite sequence of elements of $S_\beta$ such that each element in this sequence is the image of $g_\alpha$ for some $\alpha \leq \beta$ (so each element is not contained in $P_\beta$) and that $(\prod \tilde{g}_\beta) P_\beta = 0$ in $S_\beta$. We recall that we have an almost isomorphism

$$B_1 \otimes_{A_{\infty,0}} B_2 \to \tilde{B} = (B_1 \otimes_{A_{\infty,0}} B_2)[\frac{1}{p}]^\circ$$

where $\tilde{B}$ is an integral perfectoid $A_{\infty,0}$-algebra.

We next consider the directed system formed by tuples $\beta = (\beta, g_\beta)$, with order $(\beta, g_\beta) \leq (\beta', g_{\beta'})$ if $\beta \leq \beta'$ and the image of $g_\alpha$ forms part of the sequence $g_{\beta'}$. Also note that $\{S_\beta\}_{\beta}$ also forms a direct limit system, and all the $S_\beta$ map to $\tilde{B}$ compatibly because $B_1$ and $B_2$ are $R^+$-algebras. For each $\beta$, we set

$$T_\beta = \tilde{B}((g_\beta)^{1/p^\infty})[\frac{1}{p}]^\circ.$$

By Claim 4.5, $(\prod \tilde{g}_\beta)^{-1/p^\infty}T_\beta$ is a $(p \prod \tilde{g}_\beta)^{1/p^\infty}$-almost integral perfectoid, $(p \prod \tilde{g}_\beta)^{1/p^\infty}$-almost big Cohen-Macaulay (with respect to $x$) $R$-algebra, and we have a commutative diagram for every $\beta$:

$$\begin{array}{ccc}
A & \longrightarrow & S_\beta \\
\downarrow & & \downarrow \\
B_1 \otimes_{A_{\infty,0}} B_2 & \longrightarrow & \tilde{B} \\
\downarrow & & \downarrow \\
B_1 \otimes_{A_{\infty,0}} B_2 & \longrightarrow & \tilde{B} & \longrightarrow & T_\beta & \longrightarrow & ((\prod \tilde{g}_\beta)^{-1/p^\infty}T_\beta)^2 & \longrightarrow & (\prod \tilde{g}_\beta)^{-1/p^\infty}T_\beta
\end{array}$$

where the existence of the map $R_\beta \to ((\prod \tilde{g}_\beta)^{-1/p^\infty}T_\beta)^2$ follows from $(\prod \tilde{g}_\beta) P_\beta = 0$. We set $B_\beta = ((\prod \tilde{g}_\beta)^{-1/p^\infty}T_\beta)^2$, by the same reasoning as above, we know that $B_\beta$ is an integral perfectoid, $(p \prod \tilde{g}_\beta)^{1/p^\infty}$-almost big Cohen-Macaulay (with respect to $x$), $B_1 \otimes_{R_\beta} B_2$-algebra. Moreover, the constructions are compatible in the sense that whenever $\beta \leq \beta'$ and $\beta \leq \beta'$,
we have the following commutative diagram:

\[
\begin{array}{ccc}
R_\beta & \rightarrow & R'_{\beta'} \\
\downarrow & & \downarrow \\
B_\beta & \rightarrow & B'_{\beta'}
\end{array}
\]

Taking the \( p \)-adic completion of the direct limit, we obtain a map of integral perfectoid \( R^{+} \)-algebras

\[
\hat{R}^+ \rightarrow \lim_{\beta} \hat{B}_\beta.
\]

Moreover, we know that this map factors through \( B_1 \) and \( B_2 \) since \( B_1 \otimes_{R_\beta} B_2 \) maps to \( B_\beta \) for every \( \beta \). Thus we have \( B_1 \otimes_{R^{+}} B_2 \rightarrow \lim_{\beta} \hat{B}_\beta \).

Therefore it remains to prove that \( \lim_{\beta} \hat{B}_\beta \) can be mapped to an integral perfectoid big Cohen-Macaulay \( R^{+} \)-algebra \( B \). But since \( \hat{B}_\beta \) is \((p \prod g_\beta)^{1/p^\infty}\)-almost big Cohen-Macaulay with respect to \( \hat{x} = px_1, \ldots, x_{d-1} \), \( \hat{B}_\beta \) maps to a balanced big Cohen-Macaulay algebra, and hence so is the direct limit \( \lim_{\beta} \hat{B}_\beta \) by [Die07, Lemma 3.2].

Since we are in characteristic \( p > 0 \), \( \lim_{\beta} \hat{B}_\beta \) maps to a perfect (balanced) big Cohen-Macaulay algebra \( C \) that is \((p^{\hat{\beta}}, x_1^{\hat{\beta}}, \ldots, x_{d-1}^{\hat{\beta}})\)-adically complete by [Die07, Proposition 3.7]. Thus we have an induced map of perfect, \( p^{\hat{\beta}} \)-adically complete rings of characteristic \( p > 0 \):

\[
\lim_{\beta} \hat{B}_\beta^{p^{\hat{\beta}}} \rightarrow C.
\]

Untilting, we get:

\[
\lim_{\beta} \hat{B}_\beta \rightarrow C^\sharp \rightarrow B = \hat{C}^\sharp_m
\]

Since \( C \) is perfect, \( p^{\hat{\beta}} \)-adically complete, and big Cohen-Macaulay with respect to \( \hat{x} \), \( C^\sharp \) is integral perfectoid and big Cohen-Macaulay with respect to \( x \). Therefore \( B \) is balanced big Cohen-Macaulay by [BH93, Corollary 8.5.3]. This implies that \( B \) is \( p \)-torsion free, and thus \( B \) is also integral perfectoid by [And18c, Proposition 2.2.1].

In both [Theorem 4.3] and [Lemma 4.4] we need to assume that our big Cohen-Macaulay algebras are integral perfectoid. To get rid of this assumption, we next prove the following.

**Lemma 4.6.** Let \((R, \mathfrak{m})\) be a complete local domain of mixed characteristic \((0, p)\) and \( B \) be a big Cohen-Macaulay \( R \)-algebra. Then there exists an integral perfectoid big Cohen-Macaulay \( R^{+} \)-algebra \( C \) and an \( R \)-linear map \( B \rightarrow C \). Moreover, if \( B \) is in addition an \( R^{+} \)-algebra, we can construct \( C \) such that the map \( B \rightarrow C \) is \( R^{+} \)-linear.

**Proof.** Without loss of generality, we can assume the residue field \( k = R/\mathfrak{m} \) is algebraically closed. By Cohen’s structure theorem we have \( A \rightarrow R \) that is a module-finite extension such that \( A \cong W(k)[[x_1, \ldots, x_{d-1}]] \) is a complete and unramified regular local ring. Let \( A_{\infty, 0} \) be the \( p \)-adic completion of \( A[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}] \), which is integral perfectoid and faithfully
flat over $A$. Now we write $B = A[\hat{z}_i]/(\hat{g}_j)$ where $\hat{z}_i$ and $\hat{g}_j$ denote some (possibly infinite) set of variables and relations. We next consider the maps

$$B = A[\hat{z}_i] / (\hat{g}_j) \rightarrow A_{\infty,0}(\zeta^{1/p_\infty}) / (\hat{g}_j) \rightarrow A_{\infty,0}(\zeta^{1/p_\infty}) / (\hat{g}_j) \rightarrow A_{\infty,0}(\zeta^{1/p_\infty}) / (\hat{g}_j)\mod(\zeta^{1/p_\infty})^-,$$

where $(\hat{g}_j^{1/p_\infty})^-$ denote the closure of this ideal in the $p$-adic topology. For the rest of the proof we write $S = A_{\infty,0}(\zeta^{1/p_\infty}) / (\hat{g}_j^{1/p_\infty}) [1/p].$ We next prove:

**Claim 4.7.** $S/(\hat{g}_j^{1/p_\infty})^-$ is integral perfectoid and is $p^{1/p_\infty}$-almost big Cohen-Macaulay with respect to $p, x_1, \ldots, x_{d-1}$.

**Proof of Claim.** $S$ is integral perfectoid by construction, therefore $S/(\hat{g}_j^{1/p_\infty}) j \in \Lambda)^-$ is integral perfectoid for any finite set $\Lambda$ by [Bha17 Example 6.2.11]. But then we have

$$S/(\hat{g}_j^{1/p_\infty})^- = \colim_{\Lambda} S/(\hat{g}_j^{1/p_\infty}) j \in \Lambda)^-$$

is integral perfectoid.

Next we show $S/(\hat{g}_j^{1/p_\infty})^-$ is $p^{1/p_\infty}$-almost big Cohen-Macaulay with respect to $p, x_1, \ldots, x_{d-1}$. We first note that, by [And18a] or [Bha18 Theorem 2.3], $S$ is $p^{1/p_\infty}$-almost faithfully flat over $A_{\infty,0}(\zeta^{1/p_\infty}) \mod p.$ Since $A_{\infty,0}(\zeta^{1/p_\infty}) / (\hat{g}_j)$ is faithfully flat over $B \mod p$ by construction, it follows that $p, x_1, \ldots, x_{d-1}$ is a $p^{1/p_\infty}$-almost regular sequence on $S/(\hat{g}_j)$. Since $S$ is integral perfectoid, the $e$-th Frobenius map $S/p^{1/p^e} \rightarrow S/p$ is an isomorphism. It follows that we have an induced isomorphism:

$$S/(p^{1/p^e}, \hat{g}_j^{1/p^e}) \rightarrow S/(p, \hat{g}_j).$$

The isomorphism above implies that $x_1^{1/p^e}, \ldots, x_{d-1}^{1/p^e}$ is a $p^{1/p_\infty}$-almost regular sequence on $S/(p^{1/p^e}, \hat{g}_j^{1/p^e}).$ Thus $p^{1/p_\infty}, x_1^{1/p^e}, \ldots, x_{d-1}^{1/p^e}$ is a $p^{1/p_\infty}$-almost regular sequence on $S/(\hat{g}_j^{1/p^e}).$ But then by taking filtration, it is easy to see that $p, x_1, \ldots, x_{d-1}$ is also a $p^{1/p_\infty}$-almost regular sequence on $S/(\hat{g}_j^{1/p^e}).$ This means for every $e$, we have

- $\frac{(p, x_1, \ldots, x_{d-1})(S/(\hat{g}_j^{1/p^e}))}{(p, x_1, \ldots, x_{d-1})(S/(\hat{g}_j^{1/p^e}))}$ is $p^{1/p_\infty}$-almost zero.
- $\frac{S/(\hat{g}_j^{1/p^e})}{S/(\hat{g}_j^{1/p^e})}$ is not $p^{1/p_\infty}$-almost zero.

It follows that for every $e$, $H^d_m(S/(\hat{g}_j^{1/p^e})) \neq 0$: in fact the class $\frac{1}{px_1 \cdots x_{d-1}}$ is nonzero in $H^d_m(S/(\hat{g}_j^{1/p^e}))$ since otherwise there exists $t$ such that

$$(px_1 \cdots x_{d-1})^{t-1} \in (p^t, x_1^t, \ldots, x_{d-1}^t) S/(\hat{g}_j^{1/p^e})$$

and thus $p^{1/p_\infty} \in (p, x_1, \ldots, x_{d-1}) S/(\hat{g}_j^{1/p^e}),$ which is a contradiction. This implies that $S/(\hat{g}_j^{1/p^e})$ is a solid $R$-algebra by [Hoc94 Corollary 2.4].
At this point we take the direct limit over \( e \), we still have \( S/((g_j^{1/p^\infty} \cap (S/(g_j^{1/p^\infty}))) \) is \( p^{1/p^\infty} \)-almost zero. Now we show that \( p^{1/p^e} \in (p, x_1, \ldots, x_{d-1})(S/(g_j^{1/p^\infty})) \) for every \( e \), which implies \( p \in (p, x_1, \ldots, x_{d-1})^{p^e}(S/(g_j^{1/p^\infty})) \) for every \( e \). From this we know there exists \( e' \gg 0 \) depending on \( e \) such that

\[
p \in (p, x_1, \ldots, x_{d-1})^{p^{e'}}(S/(g_j^{1/p^{e'}})) \cap R.
\]

Since \( S/(g_j^{1/p^{e'}}) \) is a solid \( R \)-algebra,

\[
p \in ((p, x_1, \ldots, x_{d-1})^{p^{e'}})^\ast \subseteq (p, x_1, \ldots, x_{d-1})^{p^e}
\]

for every \( e \), where \( I^\ast \) denotes the solid closure of \( I \), which is contained inside the integral closure by [Hoc94, Theorem 5.10]. But then we have

\[
0 \neq p \in \cap_e (p, x_1, \ldots, x_{d-1})^{p^e} = 0
\]

which is a contradiction. Therefore \( p, x_1, \ldots, x_{d-1} \) is a \( p^{1/p^\infty} \)-almost regular sequence on \( S/(g_j^{1/p^\infty}) \), and thus on \( S/(g_j^{1/p^\infty})^- \). This finishes the proof of the claim. \( \square \)

Finally, we apply [Theorem 4.1] (b) to \( S/(g_j^{1/p^\infty})^- \) (the conditions are satisfied by [Claim 4.7]) and we find that there exists an integral perfectoid big Cohen-Macaulay \( R^+ \)-algebra \( C \) and an \( R \)-linear map \( S/(g_j^{1/p^\infty})^- \cong (S/(g_j^{1/p^\infty})^-)^2 \to C \). In particular, there is an \( R \)-linear map \( B \to C \). When \( B \) is an \( R^+ \)-algebra, we similarly apply [Theorem 4.1] (a) to get an \( R^+ \)-linear map \( B \to S/(g_j^{1/p^\infty})^- \cong (S/(g_j^{1/p^\infty})^-)^2 \to C \) with \( C \) an integral perfectoid big Cohen-Macaulay \( R^+ \)-algebra (see [Remark 4.2]). \( \square \)

[Lemma 4.6] combined with [Theorem 4.3] gives an immediate extension of [Theorem 4.3] We will need this next result in [Section 5] and [Section 6].

**Corollary 4.8.** Let \( R \to S \) be a surjective local homomorphism of complete local domains, with \( R \) of mixed characteristic \((0, p)\). Given any big Cohen-Macaulay \( R \)-algebra \( B \), there exists a big Cohen-Macaulay \( S \)-algebra \( C \) and a commutative diagram:

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
B & \longrightarrow & C.
\end{array}
\]
Furthermore, if \( B \) is in addition an \( R^+ \)-algebra, then there exists a big Cohen-Macaulay \( S^+ \)-algebra \( C \) and a commutative diagram:

\[
\begin{array}{c}
R \\
\downarrow \\
R^+ \\
\downarrow \\
B \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
S \\
\downarrow \\
S^+ \\
\downarrow \\
C
\end{array}
\]

Moreover, in both cases, we can assume \( C \) is an integral perfectoid big Cohen-Macaulay \( S^+ \)-algebra (when \( S \) is of characteristic \( p > 0 \), this means \( C \) is perfect and \( p^\flat \)-adically complete).

**Proof.** We first map \( B \) to an integral perfectoid big Cohen-Macaulay \( R^+ \)-algebra \( B' \) using Lemma 4.6. Replacing \( B \) by \( B' \), we then apply Theorem 4.3 (where \( B' \) is given in advance now) to find the desired \( C \). \( \square \)

**Remark 4.9.** With assumptions as otherwise in Corollary 4.8, suppose that \( S = R/I \) is not necessarily a domain but instead is reduced and equidimensional (we still assume \( R \) is complete). Let \( Q_i \subseteq R \) be the minimal primes of \( I \) so that \( I = \bigcap_{i=1}^n Q_i \) and set \( S_i = R/Q_i \). It follows that \( S^+ = (R/I)^+ = \prod_{i=1}^n (R/Q_i)^+ = \prod_{i=1}^n S_i^+ \). We can form \( C_i \) for each \( S_i^+ \) as in Corollary 4.8 and set \( C = \prod_{i=1}^n C_i \). It follows immediately that still have a commutative diagram

\[
\begin{array}{c}
R \\
\downarrow \\
R^+ \\
\downarrow \\
B \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
S \\
\downarrow \\
S^+ \\
\downarrow \\
C
\end{array}
\]

Notice that \( C \) is still a big Cohen-Macaulay \( S^+ \)-algebra and in particular \( H_j^m(C) = 0 \) for every \( j < \dim S \).

The following is the main result of this section, which is the mixed characteristic analog of [Die07, Theorem 8.10].

**Theorem 4.10.** Let \((R, \mathfrak{m})\) be a complete local domain of mixed characteristic \((0, p)\) and \( B_1, B_2 \) be two big Cohen-Macaulay \( R \)-algebras. Then \( B_1 \otimes_R B_2 \) maps to an integral perfectoid big Cohen-Macaulay \( R \)-algebra \( B \). Moreover, if \( B_1 \) and \( B_2 \) are in addition \( R^+ \)-algebras, then \( B_1 \otimes_{R^+} B_2 \) maps to an integral perfectoid big Cohen-Macaulay \( R^+ \)-algebra \( B \) (via an \( R^+ \)-algebra map).

**Proof.** By Lemma 4.6 we can map \( B_1, B_2 \) to two integral perfectoid \( R^+ \)-algebras \( B_1', B_2' \), and when \( B_1, B_2 \) are already \( R^+ \)-algebras, we can construct \( B_1', B_2' \) such that the corresponding maps are \( R^+ \)-linear. Therefore it is enough to prove the second statement when \( B_1, B_2 \) are integral perfectoid. This follows from Lemma 4.4 \( \square \)

We will use the following corollary of Theorem 4.10 in Section 5 and Section 6.
Corollary 4.11. Let \( (R, \mathfrak{m}) \) be a complete local domain of mixed characteristic \((0, p)\). Let \( \{B_\gamma\}_{\gamma \in \Gamma} \) be any set of big Cohen-Macaulay \( R \)-algebras (resp. big Cohen-Macaulay \( R^+ \)-algebras), then there is an integral perfectoid big Cohen-Macaulay \( R^+ \)-algebra \( B \) such that the map \( R \to B \) (resp. \( R^+ \to B \)) factors through all the maps \( R \to B_\gamma \) (resp. \( R^+ \to B_\gamma \)).

Proof. For any finite subset \( \Lambda = \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma \), we set \( B_\Lambda = B_{\gamma_1} \otimes_R B_{\gamma_2} \otimes_R \cdots \otimes_R B_{\gamma_n} \) (resp. \( B_\Lambda^+ = B_{\gamma_1} \otimes_R B_{\gamma_2} \otimes_R \cdots \otimes_R B_{\gamma_n} \)). Now we consider the following directed set of \( R \)-algebras (resp. \( R^+ \)-algebras):

\[
\{B_\Lambda\}_{\Lambda \subseteq \Gamma, |\Lambda| < \infty} \quad (\text{resp. } \{B_\Lambda^+\}_{\Lambda \subseteq \Gamma, |\Lambda| < \infty})
\]

where the transition maps are the obvious ones. By Theorem 4.10, we know that each \( B_\Lambda \) (resp. \( B_\Lambda^+ \)) maps to a big Cohen-Macaulay algebra, and hence so does the direct limit \( \varprojlim B_\Lambda \) (resp. \( \varprojlim B_\Lambda^+ \)) by [Die07, Lemma 3.2]. But then by Lemma 4.6 we know that \( \varprojlim B_\Lambda \) (resp. \( \varprojlim B_\Lambda^+ \)) maps to an integral perfectoid big Cohen-Macaulay \( R^+ \)-algebra \( B \) (via an \( R^+ \)-linear map in the latter). Therefore the map to this \( B \) factors through all \( B_\gamma \) as desired. \( \square \)

5. Big Cohen-Macaulay Parameter Test Submodules

In this section we define big Cohen-Macaulay parameter test submodule and prove many properties of it in analogy with the parameter test submodule in characteristic \( p > 0 \) as well as the Grauert-Riemenschneider multiplier submodule in characteristic 0.

Definition 5.1. Let \( (R, \mathfrak{m}) \) be an excellent local ring of dimension \( d \) with a normalized dualizing complex \( \omega_R^* \) and canonical module \( \omega_R \). Let \( B \) denote a big Cohen-Macaulay \( R \)-algebra. We define

\[
0^B_{H^d_m(R)} = \ker \left( H^d_m(R) \to H^d_m(B) \right),
\]

\[
0^{\omega}_{H^d_m(R)} = \{ \eta \in H^d_m(R) \mid \exists B \text{ such that } \eta = 0 \text{ in } H^d_m(B) \} \tag{3}
\]

We then define

\[
\tau_B(\omega_R) = \text{Ann}_{\omega_R} 0^B_{H^d_m(R)} \subseteq \omega_R, \quad \text{and } \tau_{\omega}(\omega_R) = \text{Ann}_{\omega_R} 0^{\omega}_{H^d_m(R)} \subseteq \omega_R.
\]

We call \( \tau_B(\omega_R) \) (resp. \( \tau_{\omega}(\omega_R) \)) the BCM test submodule of \( \omega_R \) with respect to \( B \) (resp. the BCM test submodule of \( \omega_R \)). We note that if \( R \) is complete, then we have

\[
\tau_B(\omega_R) = \left( H^d_m(R)/0^B_{H^d_m(R)} \right)^\vee \quad \text{and } \tau_{\omega}(\omega_R) = \left( H^d_m(R)/0^{\omega}_{H^d_m(R)} \right)^\vee.
\]

Remark 5.2. It is clear from Definition 5.1 that \( 0^B_{H^d_m(R)} \subseteq 0^{\omega}_{H^d_m(R)} \) and \( \tau_{\omega}(\omega_R) \subseteq \tau_B(\omega_R) \) for all big Cohen-Macaulay algebras \( B \). We have \( 0^B_{H^d_m(R)} = 0^\wedge_{H^d_m(R)} \) as in Remark 3.2. However, it is not clear that \( \tau_B(\omega_R) \otimes_R \widehat{R} = \tau_B(\omega_R) \). Because of this, we will mostly work with complete local rings.

We start by proving that in characteristic \( p > 0 \), \( \tau_B(\omega_R) \) is the same as the parameter test submodule for large enough \( B \). This should be well-known to experts.

\( 0^\wedge_{H^d_m(R)} \) is a submodule of \( H^d_m(R) \): if \( \eta_1 = 0 \) in \( H^d_m(B_1) \) and \( \eta_2 = 0 \) in \( H^d_m(B_1) \), take \( B \) such that \( B_1 \otimes_R B_2 \to B \) by Theorem 4.10 then the image of both \( \eta_1 \) and \( \eta_2 \) are 0 in \( H^d_m(B) \) so \( \eta_1 + \eta_2 \in 0^\wedge_{H^d_m(R)} \).
**Proposition 5.3.** Let $R$ be a complete local domain of dimension $d$ and characteristic $p > 0$. Then for every big Cohen-Macaulay $R$-algebra $B$, $\tau_B(\omega_R) \supseteq \tau_{\mathcal{B}}(\omega_R) \supseteq \tau(\omega_R)$. Moreover, if $B$ is an $R^+$-algebra, then $\tau_B(\omega_R) = \tau_{\mathcal{B}}(\omega_R) = \tau(\omega_R)$.

**Proof.** By local duality, it is enough to show that for every big Cohen-Macaulay $R$-algebra $B$, $0^B_{\mathcal{H}_m^d}(R) \subseteq 0^\mathcal{B}_{\mathcal{H}_m^d}(R)$. However, $0^B_{\mathcal{H}_m^d}(R) = \ker(H_m^d(R) \to H_m^d(B))$ is a proper $F$-stable submodule of $H_m^d(R)$ (to see it is proper, pick any system of parameters $x_1, \ldots, x_d$ of $R$, the class $[1_{x_1x_2\cdots x_d}]$ is not zero in $H_m^d(B)$ since $x_1, \ldots, x_d$ is a regular sequence on $B$). But since $R$ is a complete local domain, $0^\mathcal{B}_{\mathcal{H}_m^d}(R)$ is the unique largest proper $F$-stable submodule of $H_m^d(R)$, see Proposition 2.5 or [EH08, Proposition 2.11]. Thus $0^B_{\mathcal{H}_m^d}(R) \subseteq 0^\mathcal{B}_{\mathcal{H}_m^d}(R)$.

Finally, when $B$ is an $R^+$-algebra, $0^B_{\mathcal{H}_m^d}(R) \supseteq 0^+_{\mathcal{H}_m^d}(R) = 0^\mathcal{B}_{\mathcal{H}_m^d}(R)$ by [Smi94, Theorem 5.1] and so $\tau_B(\omega_R) \subseteq \tau(\omega_R)$. \qed

**Remark 5.4.** The conclusion of Proposition 5.3 is not true in general if we do not assume $R$ is a domain: for example take $R = k[x,y]/(xy)$ and $B = k[x,y]/(x)$, then $B$ is a big Cohen-Macaulay $R$-algebra but $0^B_{\mathcal{H}_m^d}(R) \not\subseteq 0^\mathcal{B}_{\mathcal{H}_m^d}(R)$.

Next we will show that, even in mixed characteristic, $\tau_{\mathcal{B}}(\omega_R) = \tau_B(\omega_R)$ for all sufficiently large choice of $B$. In fact, if we assume the axiom of global choice, then we can pick a big Cohen-Macaulay $R$-algebra $B_\eta$ for each element $\eta \in 0^\mathcal{B}_{\mathcal{H}_m^d}(R)$, since $0^\mathcal{B}_{\mathcal{H}_m^d}(R)$ is a set, by Corollary 4.11 there exists a big Cohen-Macaulay $R$-algebra $B_\eta$ that dominates every $B_\eta$. It follows that $0^B_{\mathcal{H}_m^d}(R) = 0^B_{\mathcal{H}_m^d}(R)$ and so $\tau_{\mathcal{B}}(\omega_R) = \tau_B(\omega_R)$. Our goal is to construct $B_\eta$ in a more canonical way to get rid of the use of axiom of global choice.

**Definition 5.5.** Let $(R, m)$ be a complete local domain of dimension $d$ and let $B$ denote a big Cohen-Macaulay $R^+$-algebra. We define

$$0^B_{\mathcal{H}_m^d}(R) = \{ \eta \in H_m^d(R) \mid \exists 0 \neq g \in R^+ \text{ such that } g^{1/p^\infty} \eta = 0 \text{ in } H_m^d(B) \}.$$  

$$0^\mathcal{B}_{\mathcal{H}_m^d}(R) = \{ \eta \in H_m^d(R) \mid \exists 0 \neq g \in R^+ \text{ and } B \text{ such that } g^{1/p^\infty} \eta = 0 \text{ in } H_m^d(B) \}$$

Moreover, given $\eta = \frac{z}{x_1 \cdots x_d} \in H_m^d(R)$, we define

$$B_\eta = \lim_{\to} \left( R^+ \to R_0 = \frac{R^+[Y_1, \ldots, Y_d]}{z - Y_1 x_1 - \cdots - Y_d x_d} \to R_1 \to R_2 \to \cdots \right)$$

where for each $i \geq 0$, $R_{i+1}$ is the total algebra modification of $R_i$.

The next lemma is the key to prove $\tau_{\mathcal{B}}(\omega_R) = \tau_B(\omega_R)$ for all sufficiently large choice of $B$ (compare with [Hoc97, Theorem on page 250] or [Hoc94, Section 11]).

**Lemma 5.6.** With notations as in Definition 5.5, if $\eta \in 0^B_{\mathcal{H}_m^d}(R)$, then $B_\eta$ is a (balanced) big Cohen-Macaulay $R^+$-algebra and $\eta \in 0^\mathcal{B}_{\mathcal{H}_m^d}(R)$.

\[ \text{Again, } 0^\mathcal{B}_{\mathcal{H}_m^d}(R) \text{ is a submodule: if } g^{1/p^\infty} \eta_1 = 0 \in H_m^d(B_1) \text{ and } h^{1/p^\infty} \eta_2 = 0 \in H_m^d(B_2), \text{ then take } B \text{ such that } B_1 \otimes_R B_2 \to B \text{ by Theorem 4.10 we have } (gh)^{1/p^\infty} (\eta_1 + \eta_2) = 0 \text{ in } H_m^d(B) \text{ and hence } \eta_1 + \eta_2 \in 0^\mathcal{B}_{\mathcal{H}_m^d}(R). \]

\[ \text{We refer to [Hoc97], Page 252-253} \text{ for details on the construction. Roughly speaking, if we have part of a system of parameters } x_1, \ldots, x_{k+1} \text{ of } R \text{ and a relation } ax_{k+1} \in (x_1, \ldots, x_k)R_e \text{ for some } u \in R_e, \text{ we adjoin variables to force } a \text{ into the ideal } (x_1, \ldots, x_k), \text{ and we do this for all system of parameters and all relations.} \]
Proof. The condition \( \eta \in \overline{0}_{H^d_m(R)} \) implies there exists \( 0 \neq g \in R^+ \) such that \( g^{1/p^e} z \in (x_1, \ldots, x_d)B \) for all \( e > 0 \). We consider the following sequence:

\[
(5.6.1) \quad R^+ \rightarrow T_0 = \left( \frac{R^+[Y_1, \ldots, Y_d]}{z - Y_1 x_1 - \cdots - Y_d x_d} \right)_{\leq N} \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_r
\]

where \( T_{i+1} \) is a partial algebra modification of \( T_i \) for \( i \geq 0 \) in the sense of [Hoc02, Secton 4]. Following [Hoc02, Theorem 4.2] or [Hoc94, Section 11], in order to show that \( B_\eta \) is a (balanced) big Cohen-Macaulay algebra, it suffices to show that there is no such sequence where the image of 1 in \( T_r \) is in \( mT_r \).

Now suppose \( 1 \in mT_r \) in \((5.6.1)\) and suppose \( g^{1/p^e} z = a_1 x_1 + \cdots a_d x_d \). We look at the following commutative diagram:

\[
\begin{array}{cccccccc}
R^+ & \rightarrow & T_0 & \rightarrow & T_1 & \rightarrow & \cdots & \rightarrow & T_r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B & \rightarrow & B[1/(g^{1/p^e})^N] & \rightarrow & B[1/(g^{1/p^e})^{N_1}] & \rightarrow & \cdots & \rightarrow & B[1/(g^{1/p^e})^{N_r}] \\
\end{array}
\]

We briefly explain why such diagram exists. The first vertical map is the natural map \( R^+ \rightarrow B \), the second vertical map sends \( Y_i \) to \( a_i/g^{1/p^e} \), one can check that this is well defined because \( g^{1/p^e} z = a_1 x_1 + \cdots a_d x_d \). The other vertical maps exist because \( B \) is a big Cohen-Macaulay algebra (so all bad relations are trivialized in \( B \)) by our hypothesis, so we can define these maps as in [Hoc02]. The key point is that, the numbers \( N_1, \ldots, N_r \) depend only on \( N \) and the sequence \[(5.6.1)\] but not on \( e \)!

Tracing the image of 1 in \( B \) in the above commutative diagram in two different ways, we find that 1 in \( B[1/(g^{1/p^e})^{N_r}] \) is inside \( mB[1/(g^{1/p^e})^{N_r}] \). This means \( (g^{1/p^e})^{N_r} \in mB \) for all \( e \). Since \( N_r \) does not depend on \( e \), this implies \( g^{1/p^e} \in mB \) for all \( e \). Since \( B \) is a big Cohen-Macaulay \( R^+ \)-algebra, it is a solid \( S \)-algebra for every complete local domain \( S \) that is module-finite over \( R \) by [Hoc94, Corollary 2.4]. Pick such an \( S \) such that \( g \in S \), we have

\[
g \in m^e B \cap S \subseteq (m^e)^\star
\]

for every \( e \), where \( I^\star \) denotes the solid closure of \( I \) in \( S \), which is always contained inside the integral closure by [Hoc94, Theorem 5.10]. But then we have

\[
0 \neq g \in \cap_e m^e = 0
\]

which is a contradiction.

The last conclusion is obvious because \( B_\eta \) factors through \( R_0 = \frac{R^+[Y_1, \ldots, Y_d]}{z - Y_1 x_1 - \cdots - Y_d x_d} \) thus \( \eta = \frac{z}{x_1 \cdots x_d} = 0 \) in \( H^d_{m}(B_\eta) \). \( \square \)

**Proposition 5.7.** Let \((R, m)\) be a complete local domain of mixed characteristic \((0, p)\). Then there exists a big Cohen-Macaulay \( R^+ \)-algebra \( B \) such that

\[
0^B_{H^d_m(R)} = 0^B_{H^d_m(R)} = 0^{\sigma^B}_{H^d_m(R)} = 0^{\sigma^B}_{H^d_m(R)}.
\]

As a consequence, \( \tau^{B}(\omega_R) = \tau_B(\omega_R) \) (it follows that the same is true for all \( B' \) such that \( B \rightarrow B' \)). In particular, \( R \) is BCM-rational if and only if \( R \) is BCM_{B'}-rational for one (large enough) big Cohen-Macaulay \( R^+ \)-algebra \( B \).
Proof. Since every big Cohen-Macaulay algebra maps to a big Cohen-Macaulay $R^+$-algebra by Lemma 4.6, we have $0^B_{H^n_m(R)} \subseteq 0^\omega_{H^n_m(R)}$. Therefore it is enough to find $B$ such that $0^\omega_{H^n_m(R)} \subseteq 0^B_{H^n_m(R)}$. Now by Lemma 5.6, we have that $\{B_\eta\}_{\eta \in 0^\omega_{H^n_m(R)}}$ is a set of big Cohen-Macaulay $R^+$-algebras. By Corollary 4.11 there exists a big Cohen-Macaulay $R^+$-algebra $B$ such that $B_\eta \rightarrow B$ for all $\eta$. This implies $\eta \in 0^B_{H^n_m(R)}$ for all $\eta \in 0^\omega_{H^n_m(R)}$ and so $0^\omega_{H^n_m(R)} \subseteq 0^B_{H^n_m(R)}$ as desired. The last statement follows because by definition $R$ is BCM$_R$-rational (resp. BCM-rational) if and only if $0^B_{H^n_m(R)} = 0$ (resp. $0^\omega_{H^n_m(R)} = 0$). \hfill \Box

We have the following transformation rule of the BCM parameter test submodules under finite maps.

**Lemma 5.8.** Suppose that $(R, m) \subseteq (S, n)$ is a finite extension of complete local rings and that $B$ is a big Cohen-Macaulay $S$-algebra (and hence also a big Cohen-Macaulay $R$-algebra). Then $\text{Tr}(\tau_B(\omega_S)) = \tau_B(\omega_R)$ (here $\text{Tr}$ is the Grothendieck trace which coincides with the field trace of $R \subseteq S$ is generically separable).

**Proof.** Observe that we have a factorization:

$$H^d_n(m) \xrightarrow{\phi} H^d_n(S) \xrightarrow{} H^d_n(B)$$

and that $\phi$ is the Matlis dual of $\text{Tr} : \omega_S \rightarrow \omega_R$. We have that

$$H^d_n(m)/(0^B_{H^n_m(R)}) \xrightarrow{} H^d_n(S)/(0^B_{H^n_n(S)})$$

injects. The result follows by Matlis duality. \hfill \Box

Now we prove our restriction type theorem for the BCM test submodule.

**Theorem 5.9.** Let $(R, m)$ be a complete local ring of dimension $d$ and let $B$ be a big Cohen-Macaulay $R$-algebra. Then for every nonzero divisor $x \in R$, the image of $\tau_B(\omega_R)$ under the natural map $\omega_R \rightarrow \omega_R/x\omega_R \rightarrow \omega_{R/xR}$ equals $\tau_B(\omega_R/xR)$.

In particular, if $C$ is any big Cohen-Macaulay $B/xB$-algebra, then we have that the image of $\tau_B(\omega_R)$ under the natural map $\omega_R \rightarrow \omega_R/x\omega_R \rightarrow \omega_{R/xR}$ contains $\tau_C(\omega_{R/xR})$.

**Proof.** We consider the commutative diagram

$$
\begin{array}{ccccc}
0 & \rightarrow & R & \xrightarrow{x} & R & \rightarrow & R/xR & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B & \xrightarrow{x} & B & \rightarrow & B/xB & \rightarrow & 0
\end{array}
$$

which induces

$$
\begin{array}{cccc}
H^{d-1}_m(R/xR) & \rightarrow & H^d_n(m) & \xrightarrow{x} & H^d_n(m) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^{d-1}_m(B/xB) & \rightarrow & H^d_n(m) & \xrightarrow{x} & H^d_n(m) & \rightarrow & 0.
\end{array}
$$

Chasing this diagram we know that $\text{im}(\beta) \rightarrow \text{im}(\alpha)$. But the Matlis dual of $\text{im}(\beta)$ is $\tau_{B/xB}(\omega_{R/xR})$ and the Matlis dual of $\text{im}(\alpha)$ is $\tau_B(\omega_R)$. The result follows. \hfill \Box
Corollary 5.10. Let $R$ be a complete local domain of mixed characteristic $(0, p)$. If $0 \neq x \in R$ is such that $R/xR$ is reduced and of characteristic $p$, then for every big Cohen-Macaulay $R$-algebra $B$, we have $\tau_{B/xB}(\omega_{R/xR}) \supseteq \tau(\omega_{R/xR})$.

Proof. By Corollary 4.8, for every minimal prime $Q$ of $(x)$, the surjective ring map $R \to R/xR \to R/Q$ fits into a commutative diagram:

$$
\begin{array}{ccc}
R & \longrightarrow & R/xR \\
\downarrow & & \downarrow \\
B & \longrightarrow & B/xB \\
\downarrow & & \downarrow \\
& & B(Q)
\end{array}
$$

where $B(Q)$ is a big Cohen-Macaulay $(R/Q)^+$-algebra. This induces a commutative diagram:

$$
\begin{array}{ccc}
H^d_m(R/xR) & \longrightarrow & H^d_m(R/Q) \\
\downarrow & & \downarrow \\
H^d_m(B/xB) & \longrightarrow & H^d_m(B(Q))
\end{array}
$$

Chasing this diagram it is easy to see that the image of $0^{B/xB}_{H^d_m(R/xR)}$ in $H^d_m(R/Q)$ is contained in $0^{B(Q)}_{H^d_m(R/Q)} = 0^*_{H^d_m(R/xR)}$ by Proposition 5.3. Since this is true for every minimal prime $Q$ of $R/xR$ and $R/xR$ is reduced, it follows that $0^{B/xB}_{H^d_m(R/xR)} \subseteq 0^*_{H^d_m(R/xR)}$ and hence $\tau_{B/xB}(\omega_{R/xR}) \supseteq \tau(\omega_{R/xR})$ by applying Matlis duality.

We now compare our parameter test module with the Grauert-Riemenschneider multiplier submodule producing a generalization of Proposition 3.7. Let $(R, \mathfrak{m})$ be an excellent local domain and let $\pi: X \to \text{Spec } R$ be a projective birational map with $E$ the pre-image of $\{\mathfrak{m}\}$. Let $K$ be the kernel of the natural map $H^d_m(R) \to H^d_E(X, \mathcal{O}_X) = h^d(\Gamma_\mathfrak{m}(R_\pi, \mathcal{O}_X))$. Then by local and Grothendieck duality we know that $\text{Ann}_{\omega_R} K$ can be identified with $\pi_*\omega_X$, see Definition-Proposition 2.3.

Proposition 5.11. Let $(R, \mathfrak{m})$ be a complete local domain of dimension $d$, and let $\pi: X \to \text{Spec } R$ be a proper birational map. Then there exists a big Cohen-Macaulay $R^+$-algebra $C$ such that $\pi_*\omega_X \supseteq \tau_C(\omega_R)$. In particular, $\tau_{\mathfrak{m}}(\omega_R) \subseteq \pi_*\omega_X$ for all such $\pi$.

Proof. Using Chow’s lemma, it is harmless to assume that $\pi$ is projective. Since $X \to \text{Spec } R$ is projective and birational, $X = \text{Proj } R[\mathfrak{I}]$ for some ideal $\mathfrak{I} \subseteq R$. We let $S = R[\mathfrak{I}]$, which is an excellent local domain. Let $\mathfrak{n} \subseteq S$ denote the maximal ideal $\mathfrak{m} + \mathfrak{I}$. Let

$$
K = \ker \left( H^d_m(R) \to H^d_E(X, \mathcal{O}_X) \right)
$$

with $E$ the pre-image of $\{\mathfrak{m}\}$. By the Sancho-de-Salas exact sequence [SdS87], we have

$$
[H^d_n(S)]_0 \to H^d_m(R) \to H^d_E(X, \mathcal{O}_X).
$$

This sequence implies that

$$
(5.11.1) \quad K \subseteq \text{Image } \left( H^d_n(S) \to H^d_m(R) \right).
$$

6Here $R/xR$ is complete but not necessarily a domain, we can still define $\tau(\omega_{R/xR})$ as in [Sm95], which is the Matlis dual of $0^*_{H^d_m(R/xR)}$. 

26
Next we consider the surjective map $\tilde{S}_n \to R$. Since $R$ is a domain this map factors through $\tilde{S}_n/P$ for some minimal prime $P$ of $\tilde{S}_n$. We note that $\dim(\tilde{S}_n/P) = \dim R + 1$ and hence $R$ is obtained from $\tilde{S}_n/P$ by killing a height one prime $Q$. We now apply [HH95, Theorem 3.9] in equal characteristic, and apply [Theorem 4.3 or Corollary 4.8] in mixed characteristic to obtain a commutative diagram:

$$
\begin{array}{ccc}
\tilde{S}_n/P & \xrightarrow{} & R \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & C
\end{array}
$$

where $B$, $C$ are big Cohen-Macaulay algebras over $(\tilde{S}_n/P)^+$ and $R^+$ respectively. This induces a commutative diagram of local cohomology:

$$
\begin{array}{ccc}
H^d_n(S) & \xrightarrow{} & H^d_n(\tilde{S}_n) \\
\downarrow & & \downarrow \\
H^d_n(\tilde{S}_n/P) & \xrightarrow{} & H^d_m(R) \\
\downarrow & & \downarrow \\
0 & = & H^d_n(B) \xrightarrow{} H^d_m(C)
\end{array}
$$

By [5.11.1], $K$ is in the image of $H^d_n(S) \to H^d_m(R)$. Thus chasing the diagram we find that $K \subseteq 0_{H^d_m(R)}$. By local duality we find that

$$
\pi_+ \omega_X = \text{Ann}_{\omega_R} K \supseteq \tau_C(\omega_R).
$$

This finishes the proof. \(\square\)

Combining the earlier results in the section, we obtain the following.

**Theorem 5.12.** Let $(R, m)$ be an excellent analytically irreducible local domain of mixed characteristic $(0, p)$. Suppose $x \in R$ is a nonzerodivisor such that $R/xR$ is reduced and of characteristic $p$. Let $\pi: X \to \text{Spec } R$ be a projective birational map. Then the image of $\pi_+ \omega_X$ in $\omega_{R/xR}$ under the natural map $\pi_+ \omega_X \to \omega_R \to \omega_R/x\omega_R \to \omega_{R/xR}$ contains $\tau(\omega_{R/xR})$.

In particular, if $R/xR$ is $F$-rational, then $R$ is pseudo-rational.

**Proof.** First of all we will replace $R$ by its completion $\hat{R}$ and replace $X$ by $X \times_{\text{Spec } R} \text{Spec } \hat{R}$. Since $R$ is excellent and $R/xR$ is reduced, $\hat{R}/x\hat{R}$ is still reduced. It is clear that under this flat base change $\pi_+ \omega_X$ and $\tau(\omega_{R/xR})$ will be changed to $\pi_+ \omega_X \otimes \hat{R}$ and $\tau(\omega_{R/xR}) \otimes \hat{R}$. Therefore without loss of generality, we can assume $R$ is a complete local domain.

Applying Proposition 5.11 we know that there exists a big Cohen-Macaulay $R^+$-algebra $B$ such that $\pi_+ \omega_X \supseteq \tau_B(\omega_R)$. By [Theorem 5.9], we know that the image of $\tau_B(\omega_R)$ under the natural map $\tau_B(\omega_R) \to \omega_R \to \omega_R/x\omega_R \to \omega_{R/xR}$ contains $\tau_{B/xB}(\omega_{R/xR})$. But then by [Corollary 5.10] we know that $\tau_{B/xB}(\omega_{R/xR}) \supseteq \tau(\omega_{R/xR})$.

The last assertion is already proved in [Theorem 3.8], but we pointed out that it follows directly from the more general statement. So suppose $R/xR$ is $F$-rational, then $R/xR$ is normal and Cohen-Macaulay. We know that $R$ is normal and Cohen-Macaulay (because these two properties deform). Therefore to show $R$ is pseudo-rational, it suffices to show that $\pi_+ \omega_X = \omega_R$ for every projective birational map $\pi: X \to \text{Spec } R$. But we already know that the image of $\pi_+ \omega_X$ in $\omega_{R/xR}$ contains $\tau(\omega_{R/xR}) = \omega_{R/xR}$ by $F$-rationality. Thus $\pi_+ \omega_X = \omega_R$ as desired. \(\square\)
5.1. Big Cohen-Macaulay parameter test submodules and the singular locus. In characteristic \( p > 0 \), the parameter test submodule \( \tau(\omega_R) \) basically measures how far \( R \) is from being \( F \)-rational. In particular, for every \( h \) such that \( R_h \) is regular, \( \tau(\omega_R)_h = \tau(\omega_{R_h}) = R_h \) and hence a fixed power of \( h \), \( h^N \), multiplies \( \omega_R \) into the parameter test submodule. It is natural to ask whether analogous results holds in mixed characteristic. In this subsection we will partially answer this question.

**Theorem 5.13** (Uniform annihilation). Let \( (A, m_A) \to (R, m) \) be a module-finite extension such that \( A \) is a complete regular local ring of mixed characteristic \( (0, p) \) and \( R \) is a complete local domain. Suppose \( h \in A \) is such that \( A_h \to R_h \) is finite étale. Then there exists an integer \( N \) such that \( h^N \omega_R \subset \tau_{\mathcal{B}}(\omega_R) \).

To prove this theorem we need a simple lemma on Galois theory of rings.

**Lemma 5.14.** Let \( A \to R \) be a finite extension of Noetherian normal domains. Suppose the induced map on the fraction field \( L_A \to L_R \) is Galois with Galois group \( \mathcal{G} \). If \( I \subseteq R \) is an \( \mathcal{G} \)-invariant ideal of \( R \) such that \( \text{Tr}_{L_R/L_A}(I) = A \), then \( I = R \).

**Proof.** Since \( L_A \to L_R \) is Galois, we know that

\[
\text{Tr}_{L_R/L_A}(x) = \sum_{\sigma \in \mathcal{G}} \sigma(x)
\]

for every \( x \in L_R \). Since \( A \) is normal, we know that \( \text{Tr}_{L_R/L_A}(R) \subseteq A \). Now suppose \( z \in I \), since \( I \) is \( \mathcal{G} \)-invariant it follows directly from the above formula that \( \text{Tr}_{L_R/L_A}(z) \in I \). Therefore

\[
\text{Tr}_{L_R/L_A}(I) \subseteq A \cap I,
\]

thus \( \text{Tr}_{L_R/L_A}(I) = A \) implies \( I = R \). \( \Box \)

**Proof of Theorem 5.13.** We pick a (sufficiently large) big Cohen-Macaulay \( R^+ \)-algebra \( B \) that satisfies the conclusion of Proposition 5.7 it is enough to prove that there exists \( N \) such that \( h^N \omega_{R^+} = 0 \).

Let \( L_A \) and \( L_R \) be the fraction field of \( A \) and \( R \) respectively. We let \( L' \) be the Galois closure of \( L_R \) over \( L_A \) inside an algebraic closure \( \overline{L}_A \) of \( L_A \). We write \( \mathcal{G} = \text{Gal}(\overline{L}_A/L_A) \), and write \( \mathcal{G}' = \text{Gal}(L'/L_A) \). Note that \( \mathcal{G}' = \mathcal{G}/\mathcal{H} \) for some normal subgroup \( \mathcal{H} \) of \( \mathcal{G} \). We further let \( R' \) be the integral closure of \( R \) in \( L' \). Then \( A \) is the ring of invariants of \( R' \) under \( \mathcal{G}' \) (and is the ring of invariants of \( A^+ = R^+ \) under \( \mathcal{G} \)), and \( R' \) is the ring of invariants of \( A^+ = R^+ \) under \( \mathcal{H} \). More importantly, since \( A_h \to R_h \) is finite étale, \( A_h \to R'_h \) is also finite étale. Since the \( \mathcal{G}' \)-action on \( R' \) induces a \( \mathcal{G} \)-action on \( H_m^d(R') \), we set

\[
W = \sum_{\sigma' \in \mathcal{G}'} \sigma'(\omega_{R'}^B). \tag{5.15.1}
\]

**Claim 5.15.** We have \( H_m^d(A) \cap W = 0 \) in \( H_m^d(R') \). Consequently, the composite map

\[
(5.15.1) \quad \text{Ann}_{\omega_{R'}} W \hookrightarrow \omega_{R'} \xrightarrow{\text{Tr}} \omega_A
\]

is a surjection.

**Proof of Claim.** For every \( \sigma' \in \mathcal{G}' \), we pick a lift \( \sigma \in \mathcal{G} \) of \( \sigma' \). Let \( B_\sigma \) denote the big Cohen-Macaulay \( R^+ \)-algebra \( B \) with its \( R^+ \)-algebra structure coming from the composition
\( R^+ \overset{\sigma}{\rightarrow} R^+ \rightarrow B \). Since every \( \eta \in H^d_m(R') \) is fixed by \( \mathcal{H} \), we have \( \sigma'(\eta) = \sigma(\eta) \) as elements in \( H^d_m(R^+) \) and hence their images in \( H^d_m(B) \) are also the same. It follows that

\[
\sigma'(0^B_{H^d_m(R)}) = 0^B_{H^d_m(R')}.
\]

(5.15.2)

Now applying Corollary 4.11, we can find a big Cohen-Macaulay \( R^+ \)-algebra \( C \) such that there exists an \( R^+ \)-linear map \( B_\sigma \rightarrow C \) for every \( \sigma \). Since \( A \) is regular and \( C \) is a big Cohen-Macaulay \( R^+ \)-algebra, \( C \) is faithfully flat over \( A \). In particular, we know that

\[
0^C_{H^d_m(A)} = \{ \eta \in H^d_m(A) \mid \eta = 0 \text{ in } H^d_m(C) \} = 0.
\]

(5.15.3)

But since \( B_\sigma \) maps to \( C \) for every \( \sigma \) (which is a fixed lift of \( \sigma' \)), it follows from (5.15.2) that

\[
\sigma'(0^B_{H^d_m(R')}) = 0^B_{H^d_m(R')} \subseteq 0^C_{H^d_m(R')}
\]

for every \( \sigma' \) and thus \( W = \sum_{\sigma' \in \mathcal{G}'} \sigma'(0^B_{H^d_m(R')}) \subseteq 0^C_{H^d_m(R')} \). Therefore it must intersect with \( H^d_m(A) \) trivially by (5.15.3). This completes the proof of Claim 5.15 by Matlis duality. \( \square \)

Since \( \omega_{R'} \) can be canonically viewed as \( \text{Hom}_A(H^d_m(R'), H^d_m(A)) = \text{Hom}_A(R', A) \), the \( \mathcal{G}' \)-action on \( R' \) induces a natural \( \mathcal{G}' \)-action on \( \omega_{R'} \). Since \( W \) is \( \mathcal{G}' \)-invariant by construction, \( \text{Ann}\omega_{R'} W \subseteq \omega_{R'} \) is a \( \mathcal{G}' \)-submodule. Now localizing (5.15.1) at \( h \), since \( A_h \rightarrow R'_h \) is finite étale, we can identify \( (\omega_A)_h \) and \( (\omega_{R'})_h \) with \( A_h \) and \( R'_h \) respectively. Thus we have:

\[
(\text{Ann}\omega_{R'} W)_h \sqrt{\omega_{R'}}_h \text{Tr} (\omega_A)_h \\
I \cong \quad \cong \quad \cong \\
I \rightarrow R'_h \rightarrow A_h \\
\| \| \| \\
0 \rightarrow 0^B_{H^d_m(R')} \rightarrow H^d_m(R') \rightarrow H^d_m(R) \rightarrow 0 \\
= \quad \quad = \quad \quad =
\]

where we identify \( (\text{Ann}\omega_{R'} W)_h \) as a \( \mathcal{G}' \)-invariant ideal \( I \subseteq R'_h \). Since the composite map \( I \rightarrow R'_h \rightarrow A_h \) is surjective by (5.15.1) Applying Lemma 5.14 (since the map is finite étale after inverting \( h \), the trace map is the same as the field trace map up to multiplication by a unit), we know that \( I = R'_h \), i.e., \( (\text{Ann}\omega_{R'} W)_h = (\omega_{R'})_h \). This implies \( (W^\vee)_h \cong (\omega_{R'}/\text{Ann}\omega_{R'} W)_h = 0 \). From this we know that the finitely generated module \( W^\vee \) is annihilated by a power of \( h \), thus so is \( W \). Therefore \( 0^B_{H^d_m(R')} \subseteq W \) is also annihilated by a power of \( h \), and thus so is \( (0^B_{H^d_m(R')})^\vee \). This implies \( (0^B_{H^d_m(R')})^\vee h = 0 \). Finally, we consider the commutative diagram:

\[
0 \rightarrow 0^B_{H^d_m(R')} \rightarrow H^d_m(R') \rightarrow H^d_m(R) \rightarrow 0
\]

Taking the Matlis dual and localizing \( h \), the above diagram induces:

\[
0 = (0^B_{H^d_m(R')})^\vee h \leftarrow (\omega_R)_h \cong \omega_R^e \leftarrow H^d_m(B)^\vee_h \\
\]

\[
0 = (0^B_{H^d_m(R')})^\vee h \leftarrow (\omega_R)_h \cong \omega_R^e \leftarrow H^d_m(B)^\vee_h \\
\]

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where the middle surjectivity is because $R_h$ is regular (since it is finite étale over $A_h$) hence $R_h \to R^*_h$ splits. Chasing this diagram it is easy to see that $(0^B_{H^d_m(R)})^\vee = 0$, thus the finitely generated module $(0^B_{H^d_m(R)})^\vee$ is annihilated by $h^N$ for some $N \gg 0$. Therefore $0^B_{H^d_m(R)}$ is also annihilated by $h^N$.

Recall that a regular local ring $(A, m_A)$ is called unramified, if either $A$ has equal characteristic, or $A$ has mixed characteristic $(0, p)$ and $p \notin m^2_A$. We need the following lemma which is a consequence of the Cohen-Gabber theorem.

**Lemma 5.16.** Let $(R, m, k)$ be a complete local domain of mixed characteristic $(0, p)$ and dimension $d$. Suppose $Q \in \text{Spec} R$ is such that $R_Q$ is unramified regular. Then there exists a complete and unramified regular local ring $A \cong W(k)[z_1, \ldots, z_{d-1}]$ and an element $h \in A$, $h \notin Q$ such that $A \to R$ is a module-finite extension and that $A_h \to R_h$ is finite étale.

**Proof.** In the case $p \notin Q$, we pick $x_1, \ldots, x_s \subseteq Q$ such that they form a regular system of parameters of $R_Q$ and $x_1, \ldots, x_s$ is part of a system of parameters on $R$. We extend $x_1, \ldots, x_s$ to a full system of parameters $x_1, \ldots, x_s, p, y_1, \ldots, y_t$ of $R$. In the case $p \in Q$, by the Cohen-Gabber theorem, there exists $\overline{\eta}_1, \ldots, \overline{\eta}_t \in R/Q$ such that $k[\overline{\eta}_1, \ldots, \overline{\eta}_t] \to R/Q$ is module-finite and generically étale. We pick lifts $y_1, \ldots, y_t$ to $R$ such that $y_1, \ldots, y_t$ is part of a system of parameters on $R$. Then we can find $p, x_1, \ldots, x_s \in Q$ such that they form a regular system of parameters of $R_Q$ and that $p, x_1, \ldots, x_s, y_1, \ldots, y_t$ is a system of parameters of $R$. This is because $R_Q$ is unramified and thus $p$ is a part of a minimal generator of $QR_Q$. In both cases, by Cohen’s structure theorem we have $A = W(k)[x_1, \ldots, x_s, y_1, \ldots, y_t] \to R$ is a module-finite extension.

In the case $p \notin Q$, $Q \cap A = (x_1, \ldots, x_s)$ and the map $A_{(x_1, \ldots, x_s)} \to R_Q$ is essentially étale: this is because the map is flat, $(x_1, \ldots, x_s) R_Q = QR_Q$, and the residue field extension is finite separable (the residue field has characteristic 0). In the case $p \in Q$, $Q \cap A = (p, x_1, \ldots, x_s)$ and we consider the map $A_{(p, x_1, \ldots, x_s)} \to R_Q$. This map is flat, satisfies $(p, x_1, \ldots, x_s) R_Q = QR_Q$, and by our construction $R_Q/QR_Q$ is a finite separable field extension of $A_{(p, x_1, \ldots, x_s)}/(p, x_1, \ldots, x_s) A_{(p, x_1, \ldots, x_s)}$ (because this is the fraction field of $k[\overline{\eta}_1, \ldots, \overline{\eta}_t]$). Therefore the map $A_{(p, x_1, \ldots, x_s)} \to R_Q$ is again essentially étale. In particular, in both cases, the discriminant $h \in A$ of the map $A \to R$ is not contained in $Q$.

**Theorem 5.17.** Let $(R, m)$ be a complete local domain of mixed characteristic $(0, p)$ with $R/p$ reduced. Then there exists an ideal $J \subseteq R$ such that $J(R/p)$ defines the singular locus of $R/p$, and such that $J$ multiplies $\omega_R$ into $\tau_{\partial}(\omega_R)$.

**Proof.** By **Lemma 5.16** for every $Q \in \text{Spec} R$ such that $p \in Q$ and $(R/p)_Q$ is regular, we can find $A \to R$ and $h \notin Q$ that satisfies the hypothesis of **Theorem 5.13**. It follows that there exists a sequence of elements $h_1, \ldots, h_s$ of $R$ such that $(h_1, \ldots, h_s)$ generates the singular locus of $\text{Spec}(R/p)$ up to radical and such that there exists $N_1, \ldots, N_s$ such that $h_i^N \omega_R \subseteq \tau_{\partial}(\omega_R)$ by **Theorem 5.13**. Now take $J = (h_1^{N_1}, \ldots, h_s^{N_s})$, the result follows.

### 6. Big Cohen-Macaulay test ideals of pairs

In this section we develop a theory of BCM test ideals for a pair $(R, \Delta \geq 0)$ and also BCM parameter test submodules for a pair $(\omega_R, \Gamma)$. 


**Setting 6.1.** Let \((R, \mathfrak{m})\) be a complete normal local domain and \(\Gamma \geq 0\) an effective \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \text{Spec} \(R\). Usually we consider the mixed characteristic case, but this setting makes sense in all characteristics.

The divisor \(\Gamma \geq 0\) will always denote an effective \(\mathbb{Q}\)-Cartier divisor. When discussing a divisor \(\Delta \geq 0\), it is an effective \(\mathbb{Q}\)-divisor on \text{Spec} \(R\) such that \(K_\mathbb{R} + \Delta\) is \(\mathbb{Q}\)-Cartier. In this setting we also choose an effective canonical divisor \(K_\mathbb{R}\) (or equivalently, up to units, fix an embedding \(R \subseteq \omega_R \subseteq K(R)\)) and we set \(\Gamma = K_\mathbb{R} + \Delta\) whenever \(\Delta\) is defined.

We will use \(B\) to denote a big Cohen-Macaulay \(R^+\)-algebra. By hypothesis \(n\Gamma\) is Cartier for some integer \(n > 0\) and so we can write \(n\Gamma = \text{div}_R(f)\) where \(f \in R\) (we have that \(\mathfrak{f} \subseteq R\) and not just in \(K(R)\)) by our effectivity hypotheses). Since \(B\) is an \(R^+\)-algebra, \(f^{1/n}\) makes sense in \(B\).

**Definition 6.2.** With notation as in **Setting 6.1** we define

\[
0^{B, \Gamma}_{H^d_\mathbb{m}(R)} = \ker \left( H^d_\mathbb{m}(R) \xrightarrow{f^{1/n}} H^d_\mathbb{m}(B) \right)
\]

\[
0^{\mathscr{D}, \Gamma}_{H^d_\mathbb{m}(R)} = \{ \eta \in H^d_\mathbb{m}(R) \mid \exists B \text{ such that } f^{1/n}\eta = 0 \text{ in } H^d_\mathbb{m}(B) \}.
\]

We then define

\[
\tau_B(\omega_R, \Gamma) = \text{Ann}_{\omega_R} 0^{B, \Gamma}_{H^d_\mathbb{m}(R)} \quad \text{and} \quad \tau\mathscr{D}(\omega_R, \Gamma) = \text{Ann}_{\omega_R} 0^{\mathscr{D}, \Gamma}_{H^d_\mathbb{m}(R)}.
\]

We call \(\tau_B(\omega_R, \Gamma)\) the BCM parameter test submodule of \((\omega_R, \Gamma)\) with respect to \(B\) and \(\tau\mathscr{D}(\omega_R, \Gamma)\) the BCM parameter test submodule of \((\omega_R, \Gamma)\). Clearly we have

\[
0^{B, \Gamma}_{H^d_\mathbb{m}(R)} \subseteq 0^{\mathscr{D}, \Gamma}_{H^d_\mathbb{m}(R)} \quad \text{and} \quad \tau_B(\omega_R, \Gamma) \supseteq \tau\mathscr{D}(\omega_R, \Gamma).
\]

**Remark 6.3.**

(a) It is easy to see that these definitions are independent of the choice of \(f\) or \(f^{1/n}\) since any two choices only differ by a unit and so the choice does not impact \(0^{B, \Gamma}_{H^d_\mathbb{m}(R)}\) and \(0^{\mathscr{D}, \Gamma}_{H^d_\mathbb{m}(R)}\).

(b) It is clear that when \(\Gamma = 0\), \(\tau_B(\omega_R, 0) = \tau_B(\omega_R)\) and \(\tau\mathscr{D}(\omega_R, 0) = \tau\mathscr{D}(\omega_R)\) are the same as the ones given in **Definition 5.1**.

We now prove the pair variant of **Proposition 5.7**. This implies that our BCM parameter test submodule is stable under small perturbation, and thus everything we will prove later in this section about \(\tau_B\) also holds for \(\tau\mathscr{D}\).

**Proposition 6.4.** With notation as in **Setting 6.1** and **Definition 6.2**, and suppose \((R, \mathfrak{m})\) has mixed characteristic \((0, p)\). Then there exists a big Cohen-Macaulay \(R^+\)-algebra \(B\) such that for any \(0 \neq g \in R\) and any rational number \(\varepsilon \ll 1\),

\[
0^{B, \Gamma}_{H^d_\mathbb{m}(R)} = 0^{B, \Gamma + \varepsilon \text{ div}_R(g)}_{H^d_\mathbb{m}(R)} = 0^{\mathscr{D}, \Gamma}_{H^d_\mathbb{m}(R)} = 0^{\mathscr{D}, \Gamma + \varepsilon \text{ div}_R(g)}_{H^d_\mathbb{m}(R)}.
\]

It follows that

\[
\tau_B(\omega_R, \Gamma) = \tau_B(\omega_R, \Gamma + \varepsilon \text{ div}_R(g)) = \tau\mathscr{D}(\omega_R, \Gamma) = \tau\mathscr{D}(\omega_R, \Gamma + \varepsilon \text{ div}_R(g)).
\]

**Proof.** Notice first that \(0^{B, \Gamma + \varepsilon \text{ div}_R(g)}_{H^d_\mathbb{m}(R)}\) stabilizes for \(1 \gg \varepsilon > 0\) by the Artinian property of \(H^d_\mathbb{m}(R)\) (and likewise for the version with \(\mathscr{D}\)), and so working with an arbitrary sufficiently small \(\varepsilon\) does make sense. It is enough to find a big Cohen-Macaulay \(R^+\)-algebra \(B\) such that \(0^{\mathscr{D}, \Gamma + \varepsilon \text{ div}_R(g)}_{H^d_\mathbb{m}(R)} \subseteq 0^{B, \Gamma}_{H^d_\mathbb{m}(R)}\) for all \(0 \neq g \in R\) and \(\varepsilon \ll 1\). For every \(\eta \in 0^{\mathscr{D}, \Gamma + \varepsilon \text{ div}_R(g)}_{H^d_\mathbb{m}(R)}, \) we
have $f^{1/n} g^e \eta = 0$ in $H^d_m(C)$ for some big Cohen-Macaulay $R^+$-algebra $C$ and for all $\varepsilon \ll 1$. Therefore $f^{1/n} \eta \in 0^C_{H^d_m(R)}$ and thus by Lemma 5.6 $\{B_{f^{1/n} \eta}\}_{g \in \mathbb{Z}}$ is a set of big Cohen-Macaulay $R^+$-algebras. By Corollary 4.11 there exists a big Cohen-Macaulay $R^+$-algebra $B$ such that $B_{f^{1/n} \eta} \rightarrow B$ for all $\eta$. It follows that $f^{1/n} \eta = 0$ in $H^d_m(B)$ for all $\eta \in 0^C_{H^d_m(R)}$ and hence $0^C_{H^d_m(R)} \subseteq 0^{B,\Gamma}_{H^d_m(R)}$ as desired. □

Remark 6.5 (More general big Cohen-Macaulay algebras). Note that in order for our definition to make sense we did not really need $B$ to be an $R^+$ algebra. We only required $B$ to be a big Cohen-Macaulay $R$-algebra containing an element $f^{1/n}$ whose $n$th power is the image of $f$ inside of $B$. The advantage of working with $R^+$ algebras is two-fold. First, no matter which $\Delta$ you pick, if $K_R + \Delta$ is $\mathbb{Q}$-Cartier, you can find a corresponding $f$. Second, if one works with $R^+$ algebras, then we may pick our $f^{1/n} \in R^+$ and any such choice differs by a unit, an $n$th root of unity and hence it does not matter which $f^{1/n}$ you pick in the definition of the test ideal. For a more general big Cohen-Macaulay algebra however, it may be hard to guarantee whether any two $n$th roots of $f$ differ by a unit, and so the choice of $f^{1/n}$ actually matters in the definition. One can address it for instance by requiring that $B$ is a $R[f^{1/n}]^N$ algebra, where $R[f^{1/n}]^N \subseteq R^+$ denotes the normalization of $R[f^{1/n}]$.

Next we prove the following very basic version of Skoda’s theorem.

Lemma 6.6. With notation as in Setting 6.1, assume additionally that $\Gamma' = \Gamma + \text{div}_R(h)$ for some $0 \neq h \in R$. Then

$$\tau_B(R, \Gamma') = h \cdot \tau_B(R, \Gamma)$$

Proof. We notice that $0^{B,\Gamma'}_{H^d_m(R)}$ is the kernel of

$$H^d_m(R) \xrightarrow{h \cdot f^{1/n}} H^d_m(B)$$

which can be factored as

$$H^d_m(R) \xrightarrow{h} H^d_m(R) \xrightarrow{f^{1/n}} H^d_m(B).$$

Thus

$$0^{B,\Gamma'}_{H^d_m(R)} = 0^{B,\Gamma}_{H^d_m(R)} :_R \text{div}_R(h).$$

The result follows. □

Next we switch to $\Gamma = K_R + \Delta$ as in Setting 6.1 We prove that the definition is independence of the choice of $K_R$.

Lemma 6.7. Suppose $K_R$ and $K'_R$ are two canonical divisors corresponding to embeddings $i : \omega_R \rightarrow K(R)$ and $i' : \omega_R \rightarrow K(R)$ respectively. Then,

$$i(\tau_B(\omega_R, K_R + \Delta)) = i'(\tau_B(\omega_R, K'_R + \Delta)).$$

as fractional ideals.

Proof. Without loss of generality we may assume that $K_R = K'_R + \text{div}_R(h)$ where $h \in R$, then $i(\omega_R) = h \cdot i'(\omega_R)$. On the other hand, it follows from Lemma 6.6 that $\tau_B(\omega_R, K_R + \Delta) = h \cdot \tau_B(\omega_R, K'_R + \Delta)$. Combining these proves the lemma. □
Lemma 6.8. With notation as in Setting 6.1, \( \tau_B(\omega_R, K_R + \Delta) \subseteq \omega_R \) is in fact a subset and hence an ideal of \( R \). The same is true for \( \tau_B(\omega_R, K_R + \Delta) \).

Proof. Let \( T \subseteq R^+ \) denote the normalization of \( R[f^{1/n}] \subseteq R^+ \). By Matlis duality and that fact that \( R \to B \) factors through \( T \), it suffices to show that \( f^{1/n} \cdot \omega_T \xrightarrow{\text{Tr}} \omega_R \) has image contained in \( R \). However, this follows because \( f^{1/n} \cdot \omega_T = \omega_T(-\pi^*(K_R + \Delta)) \subseteq \omega_{T/R} \) and \( \text{Tr}(\omega_{T/R}) \subseteq R \). The second conclusion follows from the first and Proposition 6.4. \( \square \)

Based on the above lemma, we make the following definition.

Definition 6.9. With notation as in Setting 6.1, we define

\[ \tau_B(R, \Delta) = \tau_B(\omega_R, K_R + \Delta) \subseteq R \text{ and } \tau_B(\Delta) = \tau_B(\omega_R, K_R + \Delta) \subseteq R \]

We call \( \tau_B(R, \Delta) \) the BCM test ideal of \((R, \Delta)\) with respect to \( B \) and \( \tau_B(\Delta) \) the BCM test ideal of \((R, \Delta)\). We say that \((R, \Delta)\) is big Cohen-Macaulay-regular with respect to \( B \) (or simply BCM\(_B\)-regular) if \( \tau_B(R, \Delta) = R \), and we say \((R, \Delta)\) is BCM-regular if it \( \tau_B(\Delta) = R \).

The following result summarizes basic properties of BCM test ideals and BCM-regular singularities.

Proposition 6.10. With notation as in Setting 6.1 and Definition 6.9, and suppose \((R, m)\) has mixed characteristic \((0, p)\). Then there exists a big Cohen-Macaulay \( R^+ \)-algebra \( B \) such that for any \( 0 \neq g \in R \) and any rational number \( \varepsilon \ll 1 \),

\[ \tau_B(R, \Delta) = \tau_B(R, \Delta + \varepsilon \text{div}_R(g)) = \tau_B(\Delta) = \tau_B(\Delta + \varepsilon \text{div}_R(g)). \]

In particular, \( R \) is BCM-regular if and only if \( R \) is BCM\(_B\)-regular for one (large enough) big Cohen-Macaulay \( R^+ \)-algebra \( B \) (and equivalently \( R \) is BCM\(_B\)-regular for every big Cohen-Macaulay \( R^+ \)-algebra \( B \)).

Proof. This follows immediately from Definition 6.9 and Proposition 6.4. \( \square \)

Lemma 6.11. With notation as in Setting 6.1 and Definition 6.9, assume we have \( \Gamma \leq \Gamma' \), both \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisors. Then we have

\[ \tau_B(\omega_R, \Gamma) \supseteq \tau_B(\omega_R, \Gamma') \text{ and so } \tau_B(\omega_R, \Gamma) \supseteq f \tau_B(\omega_R). \]

In particular if \( \Delta \leq \Delta' \) with \( \Gamma = \Delta + K_R \) and \( \Gamma' = \Delta' + K_R \), as above, then

\[ \tau_B(R, \Delta) \supseteq \tau_B(R, \Delta') \]

In particular, both \( \tau_B(\omega_R, \Gamma) \) and \( \tau_B(R, \Delta) \) are nonzero in mixed characteristic. The same conclusions hold for \( \tau_B \).

Proof. The first part of the first line follows simply by writing \( \text{div}_R(f) = n\Gamma, \text{div}_R(f') = m\Gamma' \) and noting that \( f \) divides \( f' \) in \( R^+ \). For the second part of the first line simply use Lemma 6.6. The second line follows from the first by definition. To show that they are nonzero in mixed characteristic, choose a regular local subring \( A \subseteq R \) by the Cohen-Structure theorem, and then apply Theorem 5.13 we know that \( \tau_B(\omega_R) \) is nonzero, therefore so is \( \tau_B(\omega, \Gamma) \supseteq f \tau_B(\omega_R) \). The result for \( \tau_B \) follows from the result for \( \tau_B \) and Proposition 6.10. \( \square \)
6.1. **Comparison between BCM\(_B\)-regular and BCM\(_B\)-rational singularities.** In this subsection we relate BCM\(_B\)-regular and BCM\(_B\)-rational singularities, and in particular we will prove that BCM\(_B\)-regular rings are Cohen-Macaulay.

**Theorem 6.12.** Suppose that \((R,\mathfrak{m})\) is a complete normal local domain and that \(\Delta \geq 0\) is an effective \(\mathbb{Q}\)-divisor such that \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier. If \((R,\Delta)\) is BCM\(_B\)-regular for some big Cohen-Macaulay \(R^+\)-algebra \(B\), then \(R \to B\) is pure and so \(R\) is BCM\(_B\)-rational and in particular, \(R\) is Cohen-Macaulay.

Thus when \((R,\mathfrak{m})\) has equal characteristic \(p > 0\) or mixed characteristic \((0,p)\), if \(R\) is BCM-regular then \(R\) is BCM-rational.

**Proof.** Assume the notations of [Setting 6.1](#) and in particular fix \(R \subseteq \omega_R\). We suppose \(n > 0\) is the index of \(K_R + \Delta\), and write \(\text{div}_R(f) = n(K_R + \Delta)\) for some \(f \in R\), we fix \(f^{1/n} \in R^+\).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
R & \rightarrow & B \\
\downarrow & & \downarrow \\
\omega_R & \rightarrow & B \otimes \omega_R \quad \text{by } f^{1/n} \\
\end{array}
\]

Taking local cohomology we obtain

\[
\begin{array}{ccc}
H^d_m(R) & \rightarrow & H^d_m(B) \\
\downarrow & & \downarrow \\
H^d_m(\omega_R) & \rightarrow & H^d_m(B \otimes \omega_R) \quad \text{by } f^{1/n} \\
\end{array}
\]

Since \((R,\Delta)\) is BCM-regular and \(\tau_B(R,\Delta) = \tau_B(\omega_R, K_R + \Delta) \subseteq R\) by [Lemma 6.8](#) we know that \(\mu \circ \psi\) is an injection and hence \(\psi\) is injective.

Now we observe that, since \(H^d_m(B \otimes \omega_R) \cong B \otimes H^d_m(\omega_R)\) and \(H^d_m(\omega_R) \cong E\) (which is the injective hull of the residue field of \(R\)), \(\psi\) can be identified with the canonical map \(E \to E \otimes_R B\). Thus the Matlis dual \(\psi^!\) of \(\psi\) is

\[
R \xleftarrow{\psi^!} \text{Hom}_R(E \otimes_R B, E) \cong \text{Hom}_R(B, \text{Hom}_R(E, E)) \cong \text{Hom}_R(B, R).
\]

It is not difficult to see that this map \(\text{Hom}_R(B, R) \to R\) is evaluation at 1. Hence the map \(R \to B\) splits. This completes the proof of the first statement.

Finally, if \(R\) is BCM-regular, then the first statement shows that \(R \to B\) is pure for every big Cohen-Macaulay \(R^+\)-algebra \(B\). But since every big Cohen-Macaulay \(R\)-algebra maps to a big Cohen-Macaulay \(R^+\)-algebra by [Die07, Theorem 8.10](#) in characteristic \(p > 0\) and by [Lemma 4.6](#) in mixed characteristic. We know that \(R \to B\) is pure for every big Cohen-Macaulay algebra \(B\). Hence \(R\) is BCM-rational. \(\square\)

It is quite natural to ask whether the converse of [Theorem 6.12](#) holds.
Question 6.13. Suppose \( R \) is a complete local normal domain. Further suppose that \( R \to B \) is pure for some (or sufficiently large) big Cohen-Macaulay \( R^+ \)-algebra \( B \). Does there exist an effective \( \mathbb{Q} \)-divisor \( \Delta \) such that \( K_R + \Delta \) is \( \mathbb{Q} \)-Cartier and \( (R, \Delta) \) is BCM\(_B\)-regular?

In fact, an affirmative answer to the above question in characteristic \( p > 0 \) will imply that weakly \( F \)-regular and strongly \( F \)-regular are equivalent. Thus we expect this question is very difficult in general. We note that in characteristic \( p > 0 \), some related results were shown in [SS10] (also see [DH09] in characteristic zero).

The next proposition shows that this question has an affirmative answer when \( R \) is \( \mathbb{Q} \)-Gorenstein.

**Proposition 6.14.** Suppose that \( (R, \mathfrak{m}) \) is a normal complete domain and is \( \mathbb{Q} \)-Gorenstein with index \( n \). Suppose that \( B \) is a big Cohen-Macaulay \( R^+ \)-algebra and that \( R \to B \) is pure. Then \( (R, 0) \) is BCM\(_B\)-regular.

**Proof.** Write \( nK_R = \text{div}_R(f) \) and choose \( f^{1/n} \in R^+ \). We form the same diagram as in the proof of Theorem 6.12.

\[
\begin{array}{ccc}
H^d_m(R) & \to & H^d_m(B) \\
\downarrow & & \downarrow \\
H^d_m(\omega_R) & \xrightarrow{\psi} & H^d_m(B \otimes \omega_R) \\
\downarrow & \searrow \mu & \\
H^d_m(B) & & \\
\end{array}
\]

Since \( R \to B \) is pure, \( \psi \) is injective. To prove that \( H^d_m(R) \xrightarrow{f^{1/n}} H^d_m(B) \) is injective (which is equivalent to \( (R, 0) \) being BCM\(_B\)-regular by definition), it is enough to show \( H^d_m(\omega_R) \to H^d_m(B) \) is injective (see Lemma 6.8). Therefore it suffices to show that the map \( \mu \) is injective.

Let \( S \) be the normalization of \( R[f^{1/n}] \). Since \( nK_R = \text{div}(f) \), \( S(\text{div}_S(f^{1/n})) \) is the reflexification of \( S \otimes \omega_R \). Thus \( H^d_m(S \otimes \omega_R) \to H^d_m(S(\text{div}_S(f^{1/n}))) \) is an isomorphism. This implies the second map in

\[
.f^{1/n} : H^d_m(S) \to H^d_m(S \otimes \omega_R) \to H^d_m(S)
\]

is an isomorphism. Therefore since \( B \) is an \( S \)-algebra, after base change to \( B \), we know that \( \mu : H^d_m(B \otimes \omega_R) \to H^d_m(B) \) is an isomorphism.

**Corollary 6.15.** Suppose \( (R, \mathfrak{m}) \) is a complete normal Gorenstein local domain. If \( R \) is BCM\(_B\)-rational for some big Cohen-Macaulay \( R^+ \)-algebra \( B \), then \( R \) is BCM\(_B\)-regular. In particular, if \( R \) is Gorenstein and BCM-rational, then \( R \) is BCM-regular.

**Proof.** Since \( R \) is Gorenstein, \( H^d_m(R) \) is the injective hull of \( R/\mathfrak{m} \). Therefore the injectivity of \( H^d_m(R) \to H^d_m(B) \) implies that \( R \to B \) is pure. But then \( R \) is BCM\(_B\)-regular by Proposition 6.14. The last statement follows from Proposition 6.10.

**6.2. Transformation rule under finite maps.** We can now state a transformation rule for BCM test ideals under finite maps. This is completely analogous to the main results of [ST14]. First we recall some setup.
Setup 6.16 (cf. [ST14]). For a separable finite extension $R \subseteq S$ of normal domains we have that the field trace $\text{Tr}$ sends $S$ into $R$. Furthermore, if we fix the ramification divisor $\text{Ram}$ and set $K_S := \phi^*K_R + \text{Ram}$, then the field trace also restricts to

$$\text{Tr} : \omega_S := S(K_S) \to R(K_R) := \omega_R$$

which is also identified with the Grothendieck trace and hence is Matlis dual to $H^d_m(R) \to H^d_m(S)$.

More generally for an arbitrary finite extension $R \subseteq S$ (if $R \subseteq S$ is inseparable), we consider Grothendieck trace $\text{Tr} \in \text{Hom}_R(\omega_S, \omega_R) \cong S$ generating the set as an $S$-module, and then choose embeddings $R \subseteq \omega_R$ and $S \subseteq \omega_S$ so that $\text{Tr}(S) \subseteq R$. In this case, $\text{Tr} \in \text{Hom}_R(S, R) = \omega_{S/R}$ corresponds to a divisor which we also call Ram. Note that if $S = R$ is quasi-Gorenstein and the inclusion is an iterate of Frobenius, then we may identify $R$ and $\omega_R$ in which case the corresponding Ram $= 0$.

We generalize Lemma 5.8 to the setting of pairs.

Theorem 6.17. Suppose that $R \subseteq S$ is a finite extension of complete normal local domains with induced $\phi : \text{Spec } S \to \text{Spec } R$. Then for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\Gamma \geq 0$ as in Setting 6.1 (where we implicitly choose a map $S \to R^+ \to B$ and so also view $B$ as an $S$-algebra) and $\text{Tr} \in \text{Hom}_R(\omega_S, \omega_R)$ the Grothendieck trace, we have

$$(6.17.1) \quad \text{Tr}(\tau_B(\omega_S, \phi^*\Gamma)) = \tau_B(\omega_R, \Gamma).$$

Further assume that $\Delta$ is a $\mathbb{Q}$-divisor on $\text{Spec } R$ satisfying the conditions of Setting 6.1 and such that $\phi^*\Delta \geq \text{Ram}$, the ramification divisor corresponding to the fixed $\text{Tr} \in \text{Hom}_R(S, R)$ as in Setup 6.16. Then:

$$(6.17.2) \quad \tau_B(R, \Delta) = \text{Tr}(\tau_B(S, \phi^*\Delta - \text{Ram})).$$

Proof. Note (6.17.2) is simply a special case of (6.17.1) based on our choice of $\tau(R, \Delta)$ and so we only need to prove the first statement. We first notice that $n\phi^*\Gamma = \text{div}_S(f)$. Consider the factorization

$$H^d_m(R) \to H^d_m(S) \overset{f^{1/n}}{\to} H^d_m(B).$$

The kernel of the composition is those elements of $H^d_m(R)$ that map to the kernel of $f^{1/n}$. The result follows by Matlis duality. \hfill \square

Remark 6.18 (The non-complete case). Suppose $R \subseteq S$ is a finite extension of excellent domains with $(R, m)$ local and $S$ semi-local with maximal ideals $n_i$. Completing this extension we obtain a map $\hat{R} \to \hat{S} = \prod_i \hat{S}_i := \prod_i \hat{S}_n$. If one fixes $B$, a big Cohen-Macaulay $\hat{R}^+$-algebra, then we can choose an embedding $\hat{S}_i \subseteq \hat{R}^+$ (which is determined only up to an element of the Galois group of $\hat{S}_i/\hat{R}$). Based on such an embedding, we obtain that $B$ is an $\hat{S}_i$-algebra for each $i$.

In this way, if $\Gamma$ is a $\mathbb{Q}$-divisor on $S$ such that $K_S + \Gamma$ is $\mathbb{Q}$-Cartier, we can define

$$\tau_B(S, \Gamma) := S \cap \left( \bigcap_i \tau_B(\hat{S}_i, \Gamma_i) \right)$$

where $\Gamma_i$ is the pullback of $\Gamma$ to $\text{Spec } S_i$. In view of this definition and the fact that the trace map behaves well under completion, we expect that that

$$\tau_B(R, \Delta) \overset{?}{=} \text{Tr}(\tau_B(S, \phi^*\Delta - \text{Ram}))$$
holds even without the complete hypothesis. However this is not clear to us since we do not know if the formation of \( \tau_B(R, \Delta) \) commutes with completion.

We now obtain two corollaries analogous to those obtained in [ST14].

**Corollary 6.19.** Suppose \( R \subseteq S \) is a finite extension of excellent normal domains with \( R \) local and with induced \( \phi : \text{Spec } S \rightarrow \text{Spec } R \). Suppose \( (R, \Delta) \) is a pair so that \( (\widehat{R}, \widehat{\Delta}) \) is BCM\(_B\)-regular for some \( B \) as in Setting 6.7, and \( \phi^* \widehat{\Delta} \geq \text{Ram}_S = \text{Ram}_S^\ast \) (for instance if \( R \subseteq S \) is étale-in-codimension 1). Then \( \text{Tr} : S \rightarrow R \) is surjective and hence if \( R \subseteq S \) is generically Galois, \( R \subseteq S \) is tamely ramified everywhere in any of the senses of [KS10].

**Proof.** We work using the notation of Remark 6.18 note each \( \widehat{S}_i \) is a finite extension of \( \widehat{R} \), and so we may assume that \( R \) is complete and that \( S \) is local. The result follows then since \( R = \text{Tr}(\tau_B(S, \phi^*\Delta - \text{Ram})) \subseteq \text{Tr}(S) \). \( \square \)

**Corollary 6.20.** Suppose \( (R, \mathfrak{m}) \subseteq (S, \mathfrak{n}) \) is a finite and étale-in-codimension 1 extension of complete local normal domains with induced \( \phi : \text{Spec } S \rightarrow \text{Spec } R \). Suppose further that \( (R, \Delta) \) is BCM\(_B\)-regular for some \( B \) as in Setting 6.7. Then \( (S, \phi^*\Delta) \) is also BCM\(_B\)-regular.

**Proof.** We notice that \( \text{Tr} : S \rightarrow R \) is surjective by Corollary 6.19. Furthermore, if we have \( \tau_B(S, \phi^*\Delta) \neq S \), then \( \tau_B(S, \phi^*\Delta) \subseteq \mathfrak{n} \). It follows from Theorem 6.17 that
\[
\tau_B(R, \Delta) = \text{Tr}(\tau_B(S, \phi^*\Delta - \text{Ram})) = \text{Tr}(\tau_B(S, \phi^*\Delta)) \subseteq \text{Tr}(\mathfrak{n}) \subseteq \mathfrak{m}
\]
which contradicts that \( (R, \Delta) \) is BCM\(_B\)-regular. \( \square \)

6.3. **Comparison with multiplier ideals and test ideals.** In this subsection we prove that our BCM test ideal is always contained in the multiplier ideal and agrees with the test ideal in characteristic \( p > 0 \).

**Theorem 6.21.** Suppose \( R \) is a complete normal local domain and \( \Delta \geq 0 \) is a \( \mathbb{Q} \)-divisor such that \( K_R + \Delta \) is \( \mathbb{Q} \)-Cartier. If \( \pi : Y \rightarrow X = \text{Spec } R \) is a proper birational map with \( Y \) normal, then there exists a big Cohen-Macaulay \( R^+ \)-algebra \( B \) such that
\[
\tau_B(R, \Delta) \subseteq \pi_* \mathcal{O}_Y([K_Y - \pi^*(K_X + \Delta)]).
\]

Note in the case that \( \pi \) is a resolution of singularities, the right side of the displayed equation is the multiplier ideal. It follows that
\[
\tau_B(R, \Delta) \subseteq \pi_* \mathcal{O}_Y([K_Y - \pi^*(K_X + \Delta)])
\]
for all such \( \pi \).

Compare the following proof also to [BST13], Theorem 8.1.

**Proof.** Write \( \Gamma = K_R + \Delta \) as in Setting 6.1. It suffices to consider the case where \( \pi \) is projective, the blowup of \( J \subseteq R \), by Chow’s lemma. Let \( T \) be the normalization of \( \text{Spec } R[f^{1/n}] \) and let \( Z := Y \times_X T \) denote the normalization of the blowup of \( J \cdot \mathcal{O}_T \). Let \( \pi_{Z/X} : Z \rightarrow X \) and \( \pi_{Z/T} : Z \rightarrow T \) denote the induced maps and notice that \( \pi_{Z/X} \) factors through \( \pi \).

By Proposition 5.11 there exists a big Cohen-Macaulay \( R^+ \)-algebra \( B \) such that
\[
\tau_B(\omega_T) \subseteq \Gamma(Z, \omega_Z).
\]
Multiplying both sides by \( f^{1/n} \) and applying \( \text{Tr} \), by Lemma 6.6 and Theorem 6.17 we have
\[
\tau_B(R, \Delta) = \tau_B(\omega_T, \Gamma) \subseteq \text{Tr} \left( \Gamma(Z, \mathcal{O}_Z(K_Z - \pi_{Z/X}^*(K_X + \Delta))) \right).
\]
Finally, using the argument of [BST15, Theorem 8.1], we see that
\[\text{Tr}(\Gamma(Z, \mathcal{O}_Z(K_Z - \pi^*_Z(K_X + \Delta))) ) \subseteq \Gamma(Y, \mathcal{O}_Y([K_Y - \pi^*(K_X + \Delta)])).\]
This completes the proof.

**Corollary 6.22.** If \((X = \text{Spec } R, \Delta)\) is a pair with \(R\) an excellent normal local domain such that \((\hat{X} = \text{Spec } \hat{R}, \hat{\Delta})\) is BCM-regular, then \((X, \Delta)\) is KLT.

**Proof.** This follows immediately from [Theorem 6.21] by noting that being KLT can be checked after completion.

It follows directly from our [Definition-Proposition 2.3] and [Definition 6.9] that our BCM test ideal is the same as the test ideal in equal characteristic \(p > 0\) (and hence is the same as the usual test ideal when \(R\) is \(F\)-finite by [Proposition 2.6]).

**Corollary 6.23.** With notation as in [Setting 6.7] suppose \((R, \mathfrak{m})\) is a complete local normal domain of characteristic \(p > 0\). Then \(\tau_B(R, \Delta) = \tau_{\mathfrak{m}}(R, \Delta) = \tau(R, \Delta)\) for every big Cohen-Macaulay \(R^+\)-algebra \(B\). In particular, \(R\) is BCM-regular if and only if \(R\) is strongly \(F\)-regular in characteristic \(p > 0\).

6.4. **More general restriction theorems.** Our goal in this subsection is to prove a general restriction theorem for BCM test ideals. We first include lemmas on the construction of canonical divisors. These are obvious to experts but we do not know a reference in this generality. The first is a slight divisorsial variation of the usual prime avoidance lemma.

**Lemma 6.24.** Suppose \(R\) is a normal domain and \(\omega\) is a rank one reflexive module. Further suppose that \(D_1, \ldots, D_n\) are a list of distinct prime divisors. Then there exists an effective divisor \(D\) with no common components with the \(D_i\) and such that \(R(D) \cong \omega\).

**Proof.** We proceed by induction on \(n\). In the case that \(n = 1\), simply choose an element \(z \in \omega \setminus \omega(-D_1)\). Then the effective divisor corresponding to the section \(z \in \omega\) does not have \(D_1\) as a component. For the inductive case, suppose we have a section \(z \in \omega\) such that
\[z \notin \omega(-D_i)\]
for \(i = 1, \ldots, n - 1\). If \(z \notin \omega(-D_n)\), we are done, so suppose \(z \in \omega(-D_n)\). Next notice that \(\omega(-D_1 - \cdots - D_{n-1}) \not\subseteq \omega(-D_n)\) since the \(D_i\) are distinct, we choose
\[y \in \omega(-D_1 - \cdots - D_{n-1}) \setminus \omega(-D_n)\]
Finally we consider \(y + z\), this cannot be in \(\omega(-D_i)\) for \(i = 1, \ldots, n\) by construction. The divisor corresponding to the section \(y + z \in \omega\) is effective and has no common components with the \(D_i\). \(\square\)

**Lemma 6.25.** Suppose that \((R, \mathfrak{m})\) is a normal local ring of mixed characteristic \((0, p)\) with a canonical module \(\omega_R\) and an element \(0 \neq v \in R\). Additionally fix \(\Delta \geq 0\) a \(\mathbb{Q}\)-divisor on \(\text{Spec } R\) such that \(\Delta\) and \(\text{div}(v)\) have no common components, and suppose that \(K_R + \Delta\) is \(\mathbb{Q}\)-Cartier. Then there exists a choice of canonical divisor \(K_R \geq 0\) (corresponding to an embedding \(R \subseteq \omega_R\)) satisfying the following conditions.

(a) The cokernel \(\omega_R/R\) is unmixed (all associated primes are minimal primes of height 1 in \(R\)).
(b) \(v\) is a regular element on \(\omega_R/R\).
(c) If additionally, the index of $K_R + \Delta$ is not divisible by $p > 0$, then we may form the normalization of a local cyclic index-1 cover $R \subseteq R'$ that is étale over the generic points of $V(v)$ and such that if $\pi : \text{Spec } R' \to \text{Spec } R$ is the induced map, then $\pi^*(K_R + \Delta)$ is Cartier.

Proof. First use Lemma 6.24 choose an effective canonical divisor $K_R$ with no common components with any component of $\Delta$ or $\text{div}(v)$. The choice of $K_R$ fixes an embedding $R \subseteq \omega_R = R(K_R)$ and so we have

$$0 \to R \to \omega_R \to \omega_R/R \to 0.$$  

We claim that $\omega_R/R$ has no associated primes $Q$ of height $\geq 2$ in $R$. Indeed, suppose $Q$ is a height 2 prime,

$$0 = H^0_Q(\omega_R) \to H^0_Q(\omega_R/R) \to H^1_Q(R) = 0$$

where the first vanishing is since $\omega_R$ is torsion free and $H^1_Q(R) = 0$ since $R$ is S2. It follows that $H^0_Q(\omega_R/R) = 0$ and so (a) holds. Then (b) follows since $\text{Supp } \omega_R/R = \text{Supp } K_R$, thus any associated prime of $\omega_R/R$ which contains $v$ also has height at least 2 in $R$ (and there are no such primes by (a)).

Finally, we prove (c). We fix $R''$ to be a ramified cyclic cover of $R$ along the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $K_R + \Delta$, see for instance [Kol13, Section 2.3]. Further let $R'$ denote the normalization of $R''$. Since $K_R + \Delta$ has no common components with $V(v)$, we see that $R \subseteq R''$ is étale where claimed, and so $R''$ is normal over the generic points of $V(v)$, and thus $R \subseteq R'$ is étale over the generic points of $V(v)$ as well. Since the pullback of $K_R + \Delta$ to $\text{Spec } R''$ is Cartier, it remains Cartier after pullback to $R'$. This proves (c).

Next we generalize Theorem 5.9 to the context of pairs.

Theorem 6.26. Let $(R, m)$ be a complete normal local domain of dimension $d$ and let $B$ be a big Cohen-Macaulay $R^+$-algebra. Then for every nonzero element $x \in R$ and every rational $1 > \varepsilon > 0$, the image of $\tau_B(\omega_R, (1 - \varepsilon) \text{div}_R(x))$ under the natural map $\omega_R \to \omega_R/\omega_R \to \omega_R/\omega_R$ equals $\tau_B(\omega_R, (1 - \varepsilon) \text{div}_R(x))$. In particular, if $C$ is a big Cohen-Macaulay $R/xR$-algebra which is also canonically a $B/x^\varepsilon B$-algebra, then we have that the image of $\tau_B(\omega_R, (1 - \varepsilon) \text{div}_R(x))$ under the natural map $\omega_R \to \omega_R/\omega_R \to \omega_R/\omega_R$ contains $\tau_C(\omega_R/\omega_R)$.

Proof. We first note that $B/x^\varepsilon B$ is canonically an algebra over $R/xR$ via the map $R/xR \to B/xB \to B/x^\varepsilon B$. The remainder of the proof is essentially the same as that of Theorem 5.9 but we start instead with the diagram

$$
\begin{array}{cccccc}
0 & \to & R & \xrightarrow{x} & R/xR & \to 0 \\
& & \downarrow{x-1} & & \downarrow & \\
0 & \to & B & \xrightarrow{x^\varepsilon} & B/x^\varepsilon B & \to 0
\end{array}
$$

which induces

$$
\begin{array}{cccccc}
0 & \to & H^d_m(R/xR) & \xrightarrow{\beta} & H^d_m(R) & \xrightarrow{\alpha=(x^{-1-\varepsilon})_\gamma} & H^d_m(R) & \to 0 \\
& & \downarrow{\gamma} & & \downarrow & & \downarrow & \\
0 & \to & H^d_m(B/x^\varepsilon B) & \xrightarrow{x^\varepsilon} & H^d_m(B) & \to 0.
\end{array}
$$
We know that \( \text{im}(\beta) \hookrightarrow \text{im}(\alpha) \). But the Matlis dual of \( \text{im}(\beta) \) is \( \tau_{B/x'B}(\omega_{R/x'R}) \) and the Matlis dual of \( \text{im}(\alpha) \) is \( \tau_B(\omega_R, (1-\varepsilon) \text{div}_R(x)) \). The result follows. \( \square \)

Using the above and a cyclic cover, we obtain the following rather general restriction theorem.

**Theorem 6.27.** Suppose that \( R \) is a complete normal local domain of mixed characteristic \((0,p)\), and that \( R/hR \) is such that \( R/hR \) is also a normal domain. Additionally fix \( \Delta \) a \( \mathbb{Q} \)-divisor on \( R \) such that \( K_R + \Delta \) is \( \mathbb{Q} \)-Cartier with index not divisible by \( p \) and that \( V(h) \) and \( \Delta \) have no common components. Then for every big Cohen-Macaulay \( R^+\)-algebra \( B \) and any \( 1 > \varepsilon > 0 \), there exists a big Cohen-Macaulay \( (R/hR)^+\)-algebra \( C \) (with a compatible map \( B \to C \)) so that:

\[
\tau_C(R/hR, \Delta|_{R/hR}) \subseteq \tau_B(R, \Delta + (1-\varepsilon) \text{div}_R(h)) \cdot (R/hR).
\]

In particular, we have

\[
\tau_B(R/hR, \Delta|_{R/hR}) \subseteq \tau_B(R, \Delta + (1-\varepsilon) \text{div}_R(h)) \cdot (R/hR),
\]

and if \( (R/hR, \Delta|_{R/hR}) \) is BCM-regular, then \( (R, \Delta + (1-\varepsilon) \text{div}_R(h)) \) is BCM-regular for every \( 1 > \varepsilon > 0 \).

**Proof.** The statement on \( \tau_B \) and BCM-regularity follows immediately from the statement on \( \tau_C \) by considering [Proposition 6.10](#). Therefore we only need to prove the containment involving \( \tau_C \) and \( \tau_B \).

We fix some rational \( \varepsilon > 0 \) for the rest of the proof. We first describe \( \Delta|_{R/hR} \). Since \( R/hR \) is normal, we know that each point of \( V(h) \subseteq \text{Spec} \, R \), which has height \( \leq 2 \) as an ideal of \( R \), is regular. Hence each point in \( \text{Supp}(\Delta) \cap V(h) \) that has codimension 1 in \( V(h) \) is regular in \( \text{Spec} \, R \). It follows that one may define \( \Delta|_{R/hR} \) in codimension 1 and so we can define it everywhere. Finally, set \( K_R + \Delta = \frac{1}{n} \text{div}_R(f) \) where \( f \) has no common components with \( V(h) \) (we can do this by [Lemma 6.23](#) and [Lemma 6.25](#)).

**Claim 6.28.** If \( \overline{f} \in R/hR \) denotes the image of \( f \), then we may choose \( K_{R/hR} \) so that

\[
(K_R + \Delta)|_{V(h)} = \frac{1}{n} \text{div}_R(f)|_{V(h)} = \frac{1}{n} \text{div}_{R/hR}(\overline{f}) = K_{R/hR} + \Delta|_{V(h)}.
\]

**Proof of Claim.** This is obvious for regular schemes, set \( K_{R/hR} = K_R|_{V(h)} \) and use the fact that we are working locally. However, the computation can be done at codimension 1 points of \( V(h) \), which correspond to codimension-2 regular points of \( \text{Spec} \, R \). In particular, we can reduce to the regular case. This proves the claim. \( \square \)

Fix a normalization of a ramified cyclic cover \( R \subseteq R' \) with respect to \( K_R + \Delta \) and \( h =: v \) as in [Lemma 6.25](#). Because \( R' \) is normal and \( R' \supseteq R \) is étale over \( V(h) \), we see that \( R'/hR' \) is generically reduced and all the irreducible components of \( \text{Spec}(R'/hR') \) are of dimension \( \dim R - 1 \). Because \( R' \) is \( S2 \), we see that \( R'/hR' \) is \( S1 \), and so \( R'/hR' \) is reduced. Finally, observe that \( B \) is also an \( R' \)-algebra since we may embed \( R' \subseteq R^+ \). Now using the weakly functorial big Cohen-Macaulay algebras in the form of [Remark 4.9](#), we fix \( C \) a big Cohen-Macaulay \( (R'/hR')^+\)-algebra which admits a map from \( B \) and so satisfies the role of \( C \) in the diagram in [Corollary 4.8](#). Notice that since \( C \) is an \( (R'/hR')^+\)-algebra, when we view \( C \) as an \( R^+ = R^+\)-algebra, \( h^\varepsilon C = 0 \) because \( h^\varepsilon \cdot (R'/hR')^+ = 0 \). Thus \( C \) is a \( B/h^\varepsilon B \)-algebra as well.
Next notice that we have an induced finite generically étale map $R/hR \hookrightarrow R'/hR'$ and so $C$ is also big Cohen-Macaulay $R/hR$-algebra. If $S$ is the normalization of $R'/hR'$, then it is a product of complete normal local domains $S = \prod S_i$, each of which is a finite extension of $R/hR$. By our construction from Remark 4.9 $C$ is equal to a product of $S_i$ -algebras. By Lemma 5.8 and Theorem 6.17 we obtain the following diagram:

\[
\begin{array}{ccc}
\tau_B(\omega_R, (1 - \varepsilon) \text{ div}_R(h)) & \xleftarrow{\text{Tr}} & \tau_B(\omega_{R'}, (1 - \varepsilon) \text{ div}_{R'}(h)) \\
\phi & & \psi \\
\omega_R & \xleftrightarrow{\text{Tr}} & \omega_{R'} \\
\omega_{R/hR} & \xleftrightarrow{\mu} & \omega_{R'/hR'} \\
\tau_C(\omega_{R/hR}) & \xleftrightarrow{\text{Tr}} & \tau_C(\omega_{R'/hR'}) \\
\end{array}
\]

where by Theorem 6.26 the maps $\phi$ and $\psi$ contain the images of $\mu$ and $\nu$ respectively.

We will now consider the map $\text{Tr} : f^{1/n} \cdot \tau_C(\omega_S) \to \tau_C(\omega_{R/hR})$. By construction, note that $\tau_C(\omega_{R/hR}) = \sum_{i=1}^n \tau_C(\omega_{R/hR})$ and likewise $\tau_C(R/hR, \Delta|_{V(h)}) = \sum_{i=1}^n \tau_C(R/hR, \Delta|_{V(h)})$. Next observe that by Lemma 6.6 $f^{1/n} \cdot \tau_C(\omega_S) = \tau_C(\omega_S, \psi_i(K_R + \Delta)|_{V(h)})$ where $\psi_i : \text{Spec } S_i \to \text{Spec } R/hR$ is the induced map. But now by Theorem 6.17

\[
\text{Tr} (f^{1/n} \cdot \tau_C(\omega_S)) = \tau_C(\omega_{R/hR}, (K_R + \Delta)|_{V(h)}) = \tau_C(\text{Spec } R/hR, \Delta|_{V(h)}).
\]

Summing over all $S_i$ we see that $\text{Tr} : f^{1/n} \cdot \tau_C(\omega_S) \to \tau_C(R/hR, \Delta|_{V(h)})$ surjects. But this is factored by the surjective map $f^{1/n} \cdot \tau_C(\omega_S) \to f^{1/n} \cdot \tau_C(\omega_{R'/hR'})$ and hence we have that

\[
\text{Tr}(f^{1/n} \cdot \tau_C(\omega_{R'/hR'})) = \tau_C(R/hR, \Delta|_{V(h)}).
\]

Multiplying the middle column of the large diagram above by $f^{1/n}$ and using again Lemma 6.6 and Theorem 6.17 we may factor the rows of the above diagram as follows:

\[
\begin{array}{ccc}
\tau_B(\omega_R, (1 - \varepsilon) \text{ div}_R(h)) & \xleftarrow{\text{Tr}} & \tau_B(R, \Delta + (1 - \varepsilon) \text{ div}_R(h)) & \xleftarrow{\text{Tr}} & f^{1/n} \cdot \tau_B(\omega_{R'}, (1 - \varepsilon) \text{ div}_{R'}(h)) \\
\phi & & \psi & & \phi \\
\omega_R & \xleftrightarrow{\text{Tr}} & \omega_R & \xleftrightarrow{\text{Tr}} & \omega_{R'/hR'} \\
\omega_{R/hR} & \xleftrightarrow{\mu} & \omega_{R/hR} & \xleftrightarrow{\mu} & \omega_{R'/hR'} \\
\tau_C(\omega_{R/hR}) & \xleftrightarrow{\text{Tr}} & \tau_C(\omega_{R/hR}, \Delta|_{V(h)}) & \xleftrightarrow{\text{Tr}} & f^{1/n} \cdot \tau_C(\omega_{R'/hR'}) \\
\end{array}
\]

Since the image of $\psi$ contains the image of $\nu$, it follows from the diagram that the image of $\tau_B(R, \Delta + (1 - \varepsilon) \text{ div}_R(h))$ in $R/hR$ contains $\tau_C(R/hR, \Delta|_{V(h)})$. This proves the theorem. □

Combining Theorem 6.27 Corollary 6.23 and Corollary 6.22 we immediately obtain the following result.
Corollary 6.29. Use the notation of Theorem 6.27 but instead of assuming that $R$ is complete assume that $R$ is only excellent. Assume additionally that $R/hR$ has equal characteristic $p > 0$ (e.g., $h = p$). If the pair $(R/hR, \Delta|_{R/hR})$ is strongly $F$-regular, then the pair $(R, \Delta + (1 - \varepsilon)\text{div}_R(h))$ is KLT for every $\varepsilon > 0$ and hence $(R, \Delta + \text{div}_R(h))$ is log canonical.

Proof. Since strongly $F$-regularity is preserved by passing to completion, we know that $(\hat{R}/h\hat{R}, \hat{\Delta}|_{\hat{R}/h\hat{R}})$ is strongly $F$-regular. By Corollary 6.23, $(\hat{R}/h\hat{R}, \hat{\Delta}|_{\hat{R}/h\hat{R}})$ is BCM-regular. But then Theorem 6.27 implies $(\hat{R}, \hat{\Delta} + (1 - \varepsilon)\text{div}_R(h))$ is BCM-regular and hence $(R, \Delta + (1 - \varepsilon)\text{div}_R(h))$ is KLT by Corollary 6.22. $\square$

Remark 6.30. One should also expect more general inversion of adjunction / restriction theorem statements where the subscheme we are restricting to is a Weil divisor and not necessarily Cartier. For example, if $D \subseteq \text{Spec } R$ is a normal prime $\mathbb{Q}$-Cartier divisor and $K_R + \Delta$ is also $\mathbb{Q}$-Cartier, then if $(D, \text{Diff}_D(K_R + \Delta))$ is BCM-regular (recall this means it is BCM$_{B_D}$-regular for all big Cohen-Macaulay $\mathcal{O}_D$-algebras $B_D$), we should expect that $(R, \Delta + (1 - \varepsilon)D)$ is BCM-regular. In fact, one would even expect that $(R, \Delta + D)$ is an analog of PLT. Likewise one would expect sharper restriction theorems analogous to [Tak08 Theorem 4.4] or [Laz04, Theorem 9.5.16].

In view of this remark, we prove one more result in this direction in the situation when there is no Shokurov’s different [Sho92].

Proposition 6.31. Suppose $(R, m)$ is a complete $S2$ local domain of dimension $d$ and $I \cong \omega_R$ is a proper canonical ideal (note if $R$ is normal and $D = \text{div}_R(I)$ is an anti-canonical divisor, then $K_R + D = \text{div}_R(f)$, thus this fits into the framework of pairs we discussed above). Suppose that $R/I$ is BCM$_C$-rational for some big Cohen-Macaulay $R/I$-algebra $C$ admitting a compatible map from a big Cohen-Macaulay $R$-algebra $B$. Then $R \to B$ splits. In particular, if $R/I$ is BCM-rational and $R$ is $\mathbb{Q}$-Gorenstein, the $R$ is KLT.

Proof. We begin with a claim.

Claim 6.32. $H^d_m(I \otimes_R B) \cong H^d_m(I \cdot B)$ via the canonical map.

Proof of claim. It suffices to proves that for every finitely generated $R$-module $N \subseteq B$ we have that $H^d_m(I \otimes_R N) \to H^d_m(I \cdot N)$ is an isomorphism. We notice that if we write $0 \to K \to I \otimes N \to I \cdot M \to 0$, then $K$ is supported on a set of codimension $\leq 1$ (since $I$ is free in codimension 0). Hence $H^d_m(K) = 0$. This proves the claim. $\square$

Consider the diagram

\[
\begin{array}{c}
0 \to I \to R \to R/I \to 0 \\
| \quad | \quad | \\
0 \to I \cdot B \to B \to B/(I \cdot B) \to 0.
\end{array}
\]
Taking local cohomology and using the claim we obtain

\[
\begin{array}{cccc}
H^{d-1}_m(R/I) & \rightarrow & H^d_m(I) & \rightarrow & H^d_m(R) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^{d-1}_m(B/(I \cdot B)) & \rightarrow & H^d_m(I \otimes B) & \rightarrow & H^d_m(B).
\end{array}
\]

We next claim that \( H^d_m(I) \rightarrow H^d_m(I \otimes B) \) is injective. To see this, note that \( H^d_m(I) \cong E \), the injective hull of the residue field, and hence it has a 1-dimension socle. By local duality, the Matlis dual of \( H^d_m(I) \rightarrow H^d_m(R) \) is \( R \hookrightarrow \omega_R \), and this is the embedding of \( I \cong \omega_R \) as a proper ideal. Thus \( H^d_m(I) \rightarrow H^d_m(R) \) is not injective and so a socle generator \( z \) is mapped to zero in \( H^d_m(R) \) and so has a pre-image \( y \in H^{d-1}_m(R/I) \). Since \( H^{d-1}_m(R/I) \rightarrow H^{d-1}_m(C) \) is injective, \( y \) has nonzero image also in \( H^{d-1}_m(B/(I \cdot B)) \). In conclusion, \( z \) has nonzero image in \( H^d_m(I \otimes B) \) and so

\[
H^d_m(I) \hookrightarrow H^d_m(I \otimes B)
\]

injects as claimed. Since \( H^d_m(I) \cong E \) is the injective hull of the residue field of \( R \) and \( H^d_m(I \otimes B) \cong E \otimes B \), the above injection says that \( E \rightarrow E \otimes B \) is injective. Taking the Matlis dual we observe that \( R \rightarrow B \) is split, as desired. The final statement follows from Proposition 6.14. \( \square \)

We conclude the section with a corollary which follows immediately from combining Corollary 6.19 and Theorem 6.27.

**Corollary 6.33** (Purity of the tamely ramified locus). Suppose \( R \subseteq S \) is a finite Galois extension of complete normal domains such that \( R \) is \( Q \)-Gorenstein. Suppose \( 0 \neq f \in R \) is an element so that \( R/fR \) is BCM-regular. Suppose also that \( R[f^{-1}] \subseteq S[f^{-1}] \) is étale and \( R \subseteq S \) is tamely ramified in codimension 1. Then \( \text{Tr} : S \rightarrow R \) surjects and so \( R \subseteq S \) is tame everywhere.

**Proof.** By Theorem 6.27, \((R, (1-\varepsilon) \text{div}_R(f))\) is BCM-regular for all \( 1 > \varepsilon > 0 \). Since \( R \subseteq S \) is tamely ramified, if \( \pi : \text{Spec } S \rightarrow \text{Spec } R \) is the induced map, then \( \pi^*(1-\varepsilon) \text{div}_R(f) \geq \text{Ram}_\pi \). The result follows by Corollary 6.19. \( \square \)

7. **Application: F-rational and strongly F-regular singularities in families**

Our goal in this section is to study the following question:

*How do F-rationality and F-regularity vary as one varies the characteristic?*

We begin with a simple case which will illustrate the larger point.

**Setting 7.1.** Suppose a Dedekind domain \( D \) is a quasi-finite extension of \( Z \). In particular, \( D \) is a localization of a finite extension of \( Z \). We fix \( K \) to be the fraction field, a field of characteristic zero. Let \( \phi : X \rightarrow U := \text{Spec } D \) be a flat family essentially of finite type. When talking about a \( Q \)-divisor \( \Delta \) on \( X \), we always assume that \( \Delta \) is effective and has no vertical components (in other words, every component of \( \Delta \) dominates \( D \)). For any point \( p \in D \), we use \( X_p \) (resp. \( \Delta_p \)) to denote the fiber.
Little is lost if you simply assume that $D = \mathbb{Z}$ (or a localization of $\mathbb{Z}$).

**Theorem 7.2.** With notation as in Setting 7.1, let $\phi : X \to U := \text{Spec} D$ be a proper flat family. Suppose $X_p$ is $F$-rational for some closed point $p \in U$. Then $X_K$ has rational singularities. Furthermore $X_q$ is $F$-rational for a Zariski dense and open set $V$ of closed points $q \in U$.

*Proof.* Let $Y$ be the non-pseudo-rational locus of $X$. Since we do not have resolution of singularities, it is not clear to us that this set is closed. Hence, we let $\overline{Y}$ denote the closure of $Y$ and we will eventually show that $\overline{Y} \cap X_p = \emptyset$. So suppose not, then we pick $z \in \overline{Y} \cap X_p$ viewed as a point of $X$, and let $W$ be an irreducible component of $\overline{Y}$ that contains $z$.

**Claim 7.3.** We may assume that $\phi(W) = U$.

*Proof of Claim.* Note that $\phi(W)$ is closed since $W$ is closed and $\phi$ is proper. If $\phi(W) \subseteq U$ then $\phi(W)$ is discrete, which means $W \subseteq X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_n}$ for some points $p_1 = p, p_2, \ldots, p_n \in U$. Since $W$ is irreducible and $X_{p_i}$ are disjoint, we must have $W \subseteq X_p$ (since $z \in W$ and $z \in X_p$). But for every $y \in X_p$, $O_{X,y}/p$ is a local ring of $X_p$ and thus is $F$-rational by hypothesis. Hence $O_{X,y}$ is pseudo-rational for every $y \in X_p$ by Theorem 3.8. However, then we have $X_p \cap Y \supseteq W \cap Y \neq \emptyset$, a contradiction. Thus we may assume $\phi(W) = U$. \qed

Now we fix $\eta$ to be the generic point of $W$. We see that the local ring $O_{X,\eta}$ is pseudo-rational (because this is a localization of $O_{X,z}$) and hence rational since $O_{X,\eta}$ has characteristic 0. The key point for us is that being rational is an open condition in characteristic zero since resolution of singularities exist. We next choose $\text{Spec} R$, an open neighborhood of $\eta \in X$, such that $\text{Spec}(R \otimes_D K)$ has rational singularities. By the main result of [Har98, MS97], for an open dense set $V$ of points $q$ of $U$, $R/qR$ has $F$-rational singularities. It follows from Theorem 3.8 that all points in the open set $\phi^{-1}V \cap \text{Spec} R$ have pseudo-rational singularities. In particular, $\overline{Y} \cap (\phi^{-1}V \cap \text{Spec} R) = \emptyset$ and so $\overline{Y}$ is a proper closed subset of $X$. However, we notice that $\eta \in \overline{Y}$, $\eta \in \text{Spec} R$ and $\eta \in \phi^{-1}(V)$ (since $\eta$ lies over $(0) \subseteq D$), but then $\eta \in \overline{Y} \cap (\phi^{-1}V \cap \text{Spec} R)$ a contradiction. Therefore we conclude that $X_p \cap \overline{Y} = \emptyset$.

Finally, since $\overline{Y} \cap X_p = \emptyset$, $\phi(\overline{Y})$ is a closed subset of $U$ that does not contain $p$. Thus $\phi(\overline{Y})$ does not contain the generic point of $U$ and hence $X_K$ is pseudo-rational, therefore has rational singularities. \qed

Without the properness hypothesis (or a suitable locality hypothesis, see Theorem 7.5 below), the above result is false. Consider the following example:

**Example 7.4.** Consider the diagram of rings

$$\{ \mathbb{Z}[x] \xrightarrow{x \mapsto y} \mathbb{Z}[x]/(px - 1) \xleftarrow{y \mapsto x} \mathbb{Z}[y] \}$$

and let $R$ denote the pullback. In other words

$$R = \{ (f(x), g(y)) \in \mathbb{Z}[x] \times \mathbb{Z}[y] \mid f(1/p) = g(1/p) \in \mathbb{Z}[1/p] \} \cong \text{ker} \left( \mathbb{Z}[x] \times \mathbb{Z}[y] \xrightarrow{(f(x), g(y)) \mapsto f(1/p) - g(1/p)} \mathbb{Z}[1/p] \right)$$

First observe that $R/p \cong \mathbb{F}_p[x] \oplus \mathbb{F}_p[y]$ and in particular, $R/p$ has $F$-rational singularities. However, for any prime $t \neq p$,

$$R/t \cong \{ (f(x), g(y)) \in \mathbb{F}_t[x] \times \mathbb{F}_t[y] \mid f(1/p) = g(1/p) \in \mathbb{F}_t \} \cong \text{ker} \left( \mathbb{F}_t[x] \times \mathbb{F}_t[y] \xrightarrow{(f(x), g(y)) \mapsto f(1/p) - g(1/p)} \mathbb{F}_t \right).$$
But this is a nodal singularity, and in particular not normal or $F$-rational. In particular $R/p$ is $F$-rational but $R/t$ is not $F$-rational for any other prime $t \in \mathbb{Z}$.

Of course, the same sort of example as the one above occurs in characteristic zero. Consider an affine family over a smooth curve, $X \to C$. One could imagine that the badly singular locus (for example, the locus with non-rational singularities) of $X$ as a hyperbola which has a vertical asymptote over the closed fiber $X_c$ (for some $c \in C$). Then it can happen that the closed fiber is smooth but no other fiber is smooth. Of course, this cannot happen if the family is projective or proper, since then the hyperbola describing the singular locus would intersect the closed fiber $X_c$ at infinity.

There is another way to obtain the openness of the $F$-rational locus, and that is by working even more locally.

**Theorem 7.5.** Suppose that $R$ is of finite type and flat over $D$ as in Setting 7.1 and choose a prime ideal $Q \subseteq R$ with $(0) = Q \cap D$. For each $t \in \text{Spec} D$, write $\sqrt{tR + Q} = \bigcap_{i=1}^{n_t} q_{t,i}$ a decomposition into minimal primes. Then the following set is open in $\mathfrak{m}$-Spec$D$:

$$W = \{ t \in \mathfrak{m}$-$\text{Spec} D \mid \{ q_{t,i} \}_{i=1}^{n_t} \text{ is non-empty and } (R/t)_{q_{t,i}} \text{ is } F\text{-rational for all } q_{t,i} \}.$$

Note a picture outlining the idea of the proof follows the proof.

**Proof.** Let $\rho : \text{Spec} R \to \text{Spec} D$ be the induced map. Note that $D \to R/Q$ is flat since $Q \cap D = (0)$. Thus $\rho(V(Q))$ is open and nonempty in $D$. Hence the set

$$\rho(V(Q)) \cap \mathfrak{m}$-$\text{Spec}(D) = \{ t \in \mathfrak{m}$-$\text{Spec} D \mid \{ q_{t,i} \}_{i=1}^{n_t} \text{ is non-empty } \}$$

is nonempty and open in $\mathfrak{m}$-$\text{Spec} D$. Since $Q \cap D = (0)$ we have $K \subseteq R_Q$ is of characteristic zero. Further, we may assume that for some $t \in \mathfrak{m}$-$\text{Spec} D$, $(R/t)_{q_{t,i}}$ is $F$-rational for all $q_{t,i}$ since if not, then the set $W$ is empty.

Since $(R/t)_{q_{t,i}}$ is $F$-rational we see by Theorem 3.8 that $R_{q_{t,i}}$ is pseudo-rational, and so the further localization $R_Q$ has rational singularities (since it has characteristic 0). By inverting an element of $R$ not contained in $Q$, we can choose $R' \supseteq R$ a finitely generated $D$-algebra such that $\text{Spec}(R' \otimes_D K)$ is an open neighborhood of $Q \in \text{Spec}(R \otimes_D K)$ that has rational singularities (we abuse notation here and refer to more than one ideal as $Q$). It follows from [Har98, MS97] that for all but finitely many $t \in \text{Spec} D$, $R'/t$ has $F$-rational singularities.
Claim 7.6. For all but finitely many \( t, \) \( \{q_{t,i}\}_{i=1}^{n_t} \subseteq \text{Spec } R' \subseteq \text{Spec } R. \)

**Proof of Claim.** Note that \( R' = R[h^{-1}] \) for some \( h \in R \setminus Q. \) Observe that any set of infinitely many \( q_{t,i} \)'s, is dense in \( V(Q) \subseteq \text{Spec } R \) (since it must vary over an infinite set of \( t \) s). Thus if \( V(h) \) contains infinitely many \( q_{t,i} \)'s, then \( V(h) \) contains \( Q, \) a contradiction. This proves the claim.

Now that we have the claim, we simply observe that \( (R'/t)_{q_{t,i}} = (R/t)_{q_{t,i}} \) for all \( i = 1, \ldots, n_t \) and all but finitely many \( t. \)

**Remark 7.7.** The picture to keep in mind for the above theorem is the following:

Here a point \( q_{t_2,2} \in X_{t_2} = \text{Spec}(R/t_2) \) is not \( F \)-rational but those \( t_2 \in \text{Spec } D \) are a closed subset that do not contain most points (like \( t_1 \)).

A slight rephrasing yields the following corollary.

**Corollary 7.8.** With notation as in Setting 7.1, suppose that \( (R_K, m_K) \) is a local ring essentially of finite type over \( K. \) Suppose that \( (R_D, Q_D \in \text{Spec } R_D) \) is a finite type and flat-over-\( D \) model for \( (R_K, m_K), \) so that \( (R_D)_{Q_D} = (R_D \otimes_D K)_{Q_D} = R_K. \) Then the set of primes \( t \in m - \text{Spec } D \) that satisfy the following condition

\[
\circ tR + Q_D \neq R_D \quad \text{and if } q_{t,i} \text{ is a minimal prime of } tR + Q_D, \text{ then } R_{q_{t,i}}/t \text{ is } F \text{-rational.}
\]

is an open set of \( m - \text{Spec } D. \)

7.1. **\( F \)-regularity and log terminal singularities.** With minimal change, the results we have already obtained for \( F \)-rational/rational singularities also hold for \( F \)-regular/log terminal singularities.

**Theorem 7.9.** Let \( \phi : (X, \Delta \geq 0) \to U := \text{Spec } D \) be a proper and flat family of pairs with notation as in Setting 7.1. Suppose \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier of index \( N \) and suppose \( (X_p, \Delta_p) \) is strongly \( F \)-regular for some closed point \( p \in U \) whose residual characteristic does not divide \( N. \) Then \( (X_K, \Delta_K) \) is KLT. Furthermore \( (X_q, \Delta_q) \) is KLT for a Zariski dense and open set \( V \) of closed points \( q \in U. \)

**Proof.** Let \( Y \) be the non-KLT locus of \( (X, \Delta). \) The proof then follows essentially as in Theorem 7.2 with the following modifications.
We replace pseudo-rational/rational with KLT.

We replace Theorem 3.8 with Corollary 6.29 and notice that we must use the hypothesis that $\Delta$ has no vertical components to guarantee that it and anything we mod out by have no common components.

We replace [Har98, MS97] by [Tak04] (or see [Sch10, Proposition 6.10] for the non-local case).

Likewise we also obtain the following.

**Theorem 7.10.** Suppose that $R$ is of finite type, flat, and generically geometrically normal over $D$ and $\Delta$ an effective $\mathbb{Q}$-divisor as in [Setting 7.1]. Suppose that $K_{R} + \Delta$ is $\mathbb{Q}$-Cartier with index $N$. Finally choose a prime ideal $Q \subseteq R$ such that $Q \cap D = (0)$. For each $t \in \text{Spec} D$, write $\sqrt{tR + Q} = \bigcap_{i=1}^{\nu_t} q_{t,i}$ a decomposition into minimal primes. Then the set $W$ below is open in $m$-$\text{Spec} D$:

$$\{ t \in m$-$\text{Spec} D \mid \text{char}(D/t) \nmid N, \{q_{t,i}\}_{i=1}^{\nu_t} \neq \emptyset \text{ and each } ((R/tR)_{q_{t,i}}, \Delta) \text{ is strongly } F\text{-regular} \}.$$ 

**Proof.** As in Theorem 7.5, the set

$$\rho(V(Q)) \cap m$-$\text{Spec}(D) = \{ t \in m$-$\text{Spec} D \mid \{q_{t,i}\}_{i=1}^{\nu_t} \text{ is non-empty } \}$$

is nonempty and open in $m$-$\text{Spec} D$. We may further assume that there is some $t \in W$ as otherwise the statement is vacuous.

By [Corollary 6.29] we know that $(R_Q, \Delta|_{R_Q})$ is KLT. By inverting an element of $R$ not contained in $Q$, we can choose $R' \supseteq R$ a finitely generated $D$-algebra such that $\text{Spec}(R' \otimes_D K)$ is an open neighborhood of $Q \in \text{Spec}(R \otimes_D K)$ so that $(R', \Delta|_{R'})$ has KLT singularities (we abuse notation here and refer to more than one ideal as $Q$). It follows from [Har98, Theorem 5.2] or [Tak04, Corollary 3.4] that for all but finitely many $t \in \text{Spec} D$, $(R'/t, \Delta_t)$ has strongly $F$-regular singularities.

The following claim from [Theorem 7.5] is unchanged.

**Claim 7.11.** For all but finitely many $t$, $\{q_{t,i}\} \subseteq \text{Spec} R' \subseteq \text{Spec} R$.

Now that we have the claim, we simply see that $(R'/t)_{q_{t,i}} = (R/t)_{q_{t,i}}$ for all but finitely many $t$ and the result follows as in [Theorem 7.5].

We make the following conjecture over higher dimensional bases. We do not pursue this here however.

**Conjecture 7.12.** If one is working over a higher dimension base $D$ (instead of simply the spectrum of a Dedekind domain), then for a general finite type family $X \to \text{Spec} D$, the locus of $t \in m$-$\text{Spec} D$ with $F$-rational $X_t$ is constructible. Likewise if the family is flat and proper (or sufficiently local), then the same locus is open.

8. Discussion of algorithmic consequences

Suppose that $R$ is a ring of finite type over $\mathbb{Q}$. We would like to decide if $R$ has rational or log terminal singularities using a computer algebra system such as Macaulay2 [GS]. This appears to be quite difficult (not least because the most obvious strategy requires that one must first implement resolution of singularities in these environments, and that can be quite
slow) and, at least to the authors knowledge, has not been done outside of the case where the ring is a complete intersection (done via D-module techniques, see [LT]). However, we do have methods for verifying that a ring is $F$-rational in a fixed characteristic $p > 0$. And indeed, by [Smi97], it was known that if $(R_Z)/p$ has $F$-rational singularities for some $p \gg 0$, then $R_Q$ has rational singularities as well. The problem is that to determine if $p > 0$ is big enough, one had to already compute $\pi_*\omega_{\tilde{X}_Z} \subseteq \omega_{X_Z}$ where $\pi : \tilde{X}_Z \to X_Z = \text{Spec} R_Z$ is a resolution of singularities.

The main results of this paper imply that one can use the following method to verify that a ring has rational singularities (although it cannot be used to show that a ring does not have rational singularities). Note that related results for the log canonical threshold vs the $F$-pure threshold in a regular ambient ring were obtained in [Zhu17].

**Algorithm 8.1.** With $R$ as above, choose a prime $Q \subseteq R$ such that you want to verify $R_Q$ has rational singularities:

**Step 1:** Spread out $R$ to a domain $R_Z \subseteq R$ over $\mathbb{Z}$ and also spread out $Q$ to a prime $Q_Z$ of $R_Z$.

**Step 2:** Choose a prime $p$ such that $Q_Z + (t) \neq R_Z$.

**Step 3:** Use the TestIdeals package of Macaulay2 to check if $(R_Z)/p$ has $F$-rational singularities.

**Step 4a:** If the answer to Step 3 is affirmative, then $R_Q$ has rational singularities.

**Step 4b:** If the answer to Step 3 is not affirmative, then return to Step 2 and choose a different prime.

**Caveat 8.2.** Note that spreading out $R$ to a domain already requires some checking, as one needs to verify that the given presentation $R_Z$ is in fact a domain.

As mentioned, this algorithm can verify that a ring has rational singularities but cannot show that a ring is not $F$-rational. However, there are already a number of ways to do that.

- Show that $R$ is not Cohen-Macaulay.
- Find a blowup $Y \to \text{Spec} R$ such that $\pi_*\omega_Y \subset \omega_R$ (this can be done without a resolution of singularities).

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Department of Mathematics, University of Utah, Salt Lake City, UT 84112

E-mail address: lquanma@math.utah.edu

E-mail address: schwede@math.utah.edu