THE STABILIZATION THEOREM FOR PROPER GROUPOIDS

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ABSTRACT. The stabilization theorem for \( A \)-Hilbert modules was established by G. G. Kasparov. The equivariant version, in which a locally compact group \( H \) acts properly on a locally compact space \( Y \), was proved by N. C. Phillips. This equivariant theorem involves the Hilbert \((H, C_0(Y))\)-module \( C_0(Y, L^2(H)\infty) \). It can naturally be interpreted in terms of a stabilization theorem for proper groupoids, and the paper establishes this theorem within the general proper groupoid context. The theorem has applications in equivariant KK-theory and groupoid index theory.

1. Introduction

The Kasparov stabilization theorem ([11]) asserts that for a C*-algebra \( A \), the standard Hilbert module \( A^\infty \) "absorbs" every other (countably generated) Hilbert \( A \)-module \( P \) in the sense that
\[
P \oplus A^\infty \cong A^\infty.
\]
The theorem is of central importance for the development of KK-theory, and can be regarded as an extension of Swan’s theorem for vector bundles. Accounts of the theorem are given in the books by Blackadar and Wegge-Olsen ([3, 27]). In [13, Part 1, §2, Theorem 1], Kasparov obtained a stabilization theorem involving a group action: if \( H \) is a locally compact group acting on \( A \) and \( P \) is a Hilbert \((H - A)\)-module that is countably generated as a Hilbert \( A \)-module, then
\[
P \oplus L^2(H, A)^\infty \cong L^2(H, A)^\infty
\]
in the sense that there exists an \( H \)-continuous isomorphism from \( P \oplus L^2(H, A)^\infty \) onto \( L^2(H, A)^\infty \). The isomorphism, however, need not be equivariant. An elegant, self-contained account of all of this is contained in the paper [19] of J. A. Mingo and W. J. Phillips.

For an equivariant stabilization theorem, one needs a properness condition, and N. C. Phillips has obtained such a theorem in the case of group actions ([24, Theorem 2.9]). Here, a locally compact group \( H \) is assumed to act properly on a locally compact Hausdorff space \( Y \). This action gives in the
obvious way an action of $H$ on the $C^*$-algebra $C_0(Y)$. A Hilbert $(H, C_0(Y))$-module is defined to be a Hilbert $C_0(Y)$-module with a compatible action of $H$ which is strong operator continuous - for the precise definition, see, for example, [11, Definition 1] or [19, Definition 2.1]. The theorem then asserts that for any Hilbert $(H, C_0(Y))$-module $P$, there is an equivariant isomorphism of Hilbert $(H, C_0(Y))$-modules:

$$P \oplus (C_0(Y) \otimes L^2(H)) \cong C_0(Y) \otimes L^2(H).$$

Phillips uses this stabilization theorem in his proof of the generalized Green-Rosenberg theorem (that equivariant K-theory (in terms of $H$-Hilbert bundles over $Y$) is the same as the K-theory of the transformation groupoid $C^*$-algebra). The starting point for the present paper is the observation (below) that Phillips’s stabilization theorem (and the generalized Green-Rosenberg theorem) can be expressed very naturally in terms of locally compact proper groupoids. (Accounts of the theory of locally compact groupoids are given in [25, 20].) Groupoid versions of these theorems are, of course, required for the development of groupoid equivariant KK-theory, as well as for index theory in noncommutative geometry ([6]), in particular, to orbifold theory. (In connection with the latter, the properness condition is automatically satisfied since the structure of an orbifold with underlying space $X$ is completely described by the Morita equivalence class of a proper, effective, étale Lie groupoid with orbit space homeomorphic to $X$ ([11, pp.19-23]).) The groupoid stabilization theorem is also necessary for extending Higson’s K-theory proof of the index theorem ([10]) to the equivariant case.

In this paper, we will prove the stabilization theorem for proper groupoids; the generalized Green-Rosenberg theorem will be discussed elsewhere. The proof of this stabilization theorem follows similar lines to that of Phillips’s stabilization theorem, but also requires groupoid versions of results of [19]. The main additional technical issues to be dealt with arise from the fact that, unlike the Hilbert bundles of [24], the Hilbert bundles involved in this paper are not usually locally trivial. Indeed, the $G$-Hilbert module $P_G$ for a proper groupoid $G$, whose Hilbert module $P_G^\infty$ of infinite sequences stabilizes (as we will see) all the other $G$-Hilbert modules, is associated with a $G$-Hilbert bundle that is not usually locally trivial.

We now translate the Phillips stabilization theorem into groupoid terms. We are given a locally compact group $H$ acting properly on the left on $Y$. One forms the transformation groupoid $G = H \times Y$: so multiplication is given by composition - $(h',hy)(h,y) = (h'fh, y)$ - and inversion by $(h,y)^{-1} = (h^{-1},hy)$. The unit space of $H \times Y$ can be identified with $Y$, and the properness condition translates into the requirement that the groupoid be proper: the map $g \rightarrow (r(g), s(g))$ (i.e. $(h,y) \rightarrow (hy, y)$) is proper (inverse image of compact is compact). The next objective is to interpret in groupoid terms the $C_0(Y) \otimes L^2(H)$ occurring in the Phillips stabilization theorem. A dense pre-Hilbert $(G, C_0(Y))$-module of $C_0(Y) \otimes L^2(H) = C_0(Y, L^2(H))$ is $C_c(H \times Y) = C_c(G)$ - so for a general proper groupoid $G$, we should replace
$C_0(Y) \otimes L^2(H)$ by the completion $P_G$ of the pre-Hilbert module $C_c(G)$. The stabilization theorem for proper groupoids is then:

$$P \oplus P_G^\infty \cong P_G^\infty$$

where $P$ is (in the appropriate sense) a $G$-Hilbert module.

All groupoids in the paper are assumed to be locally compact, Hausdorff, proper and second countable, and all Hilbert spaces and Hilbert modules second countable.

For lack of a convenient reference, we state the following elementary partition of unity result which is proved as in, for example, [9, Theorem 1.3]. Let $X$ be a second countable locally compact Hausdorff space, $C$ a compact subset of $X$ and $\{V_1, \ldots, V_n\}$ a cover of $C$ by relatively compact, open subsets of $X$. Then there exist $f_i \in C_c(V_i) \subset C_c(X)$ with $0 \leq f_i \leq 1$, $\sum_{i=1}^n f_i(y) \leq 1$ for all $y \in Y$, $\sum_{i=1}^n f_i(y) = 1$ for all $y \in C$.

2. Groupoid Hilbert bundles

We start by discussing the class of Hilbert bundles that we will need for $G$-actions. The correspondence between Hilbert bundles over $Y$ and Hilbert $C_0(Y)$-modules seems to be well known, but for lack of a reference we sketch the details that we will need. (Note that a Hilbert $C_0(Y)$-module $P$ can be regarded as a left $C_0(Y)$-module - $fp$ is the same as $pf$ for $p \in P, f \in C_0(Y)$.) In the transformation groupoid case developed by Phillips, one uses locally trivial bundles with fiber $L$ and structure group $U(L)$ with the strong operator topology. However, as noted above, the bundle associated with $C_c(G)$, required for the groupoid stabilization theorem, is not always locally trivial (though in the transformation groupoid case, it is trivial ($= Y \times L^2(H)$)), and we extend the class of bundles to be considered as follows. Our approach, based on the work of Fell and Hoffman, is modelled on the account of the Dauns-Hoffman theorem in [8] with bundles of Banach spaces and $C^*$-algebras replaced by Hilbert bundles over $Y$ and Hilbert $C_0(Y)$-modules. For the results of [8, Chapter 2], the Banach modules are modules over $C_0(X)$ where $X$ is completely regular. In our case, we wish to obtain similar results for Hilbert modules over $C_0(Y)$. (The corresponding modifications needed for $C_0(Y)$-algebras are given in [23]. See also [28, C.2].) Since the Hilbert bundles that we will need are usually not locally trivial, it is natural to define such a bundle in terms of a space of sections deemed to be continuous and vanishing at infinity (cf. [7, Ch. 10]). This can be done. However, for our purposes, it is more convenient to use a topological approach which is in some respects akin to the classical definition of vector bundles. In the following definition of Hilbert bundle, we are given a topology on the total space and the set of continuous sections that vanish at infinity has to satisfy certain properties.

**Definition 2.1.** Let $\{H_y\}_{y \in Y}$ be a family of Hilbert spaces, $E$ a second countable, topological space which is the disjoint union of the $H_y$'s, and
\( \pi : E \to Y \) be the projection map. Let \( C_0(Y, E) \) be the set of continuous sections \( F \) of \( E \) such \( \lim_{y \to \infty} \| F(y) \| = 0 \). Then \( E \) is called a Hilbert bundle over \( Y \) if the following properties hold:

(i) the addition map \( E \oplus_Y E \to E \) and the scalar multiplication map \( (Y \times C) \oplus_Y E \to E \) are continuous;

(ii) For each \( F \in C_0(Y, E) \), the map \( y \to \| F(y) \| \) is continuous;

(iii) for each \( y \), \( \{ F(y) : F \in C_0(Y, E) \} = H_y \).

(iv) The topology on \( E \) is determined by \( C_0(Y, E) \) in the sense that a base for it is given by the sets of the form \( U_{F, \epsilon} \), where \( U \) is an open subset of \( Y \) and

\[
U_{F, \epsilon} = \{ h_y : y \in U, h_y \in H_y, \| h_y - F(y) \| < \epsilon \}.
\]

Here are some comments on the preceding definition. From (i) and (iii), \( C_0(Y, E) \) is a vector space. It follows from (iv) and (iii) that \( \pi \) is open and continuous, and each \( H_y \) has its Hilbert space norm topology in the relative topology of \( E \). Using (ii), (iii) and (iv), the norm function \( \| \cdot \| : E \to \mathbb{R} \) is continuous. By a simple triangular inequality argument - use the continuity of \( y \to \| F(y) - F'(y) \| \) for \( F, F' \in C_0(Y, E) \) - if \( \xi \in H_{y_0} \) and \( F \in C_0(Y, E) \) is fixed such that \( F(y_0) = \xi \), then the family of sets \( U(F, \epsilon) \) with \( y_0 \in U, \epsilon > 0 \), is a base of neighborhoods for \( \xi \) in \( E \). By [15, p.57], there is a countable base for the topology of \( E \) consisting of sets of the form \( U(F, \epsilon) \). We note that \( E \) is Hausdorff though we will not use this fact. We also note that in (iv), we get the same topology if the functions \( F \) are restricted to lie in a subspace of \( C_0(Y, E) \) which is dense in the uniform norm topology (below).

**Proposition 1.** Let \( E \) be a Hilbert bundle over \( Y \). Then \( C_0(Y, E) \) is a separable \( C_0(Y) \)-Hilbert module in the uniform norm topology: \( \| F \| = \sup_{y \in Y} \| F(y) \| \).

**Proof.** To show that \( C_0(Y, E) \) is a Banach space, one modifies the proof for the corresponding elementary result on uniform convergence of functions. Let \( \{ F_n \} \) be a Cauchy sequence in \( C_0(Y, E) \). Then \( F_n \to F \) pointwise for some section \( F \) of \( E \). We now show that \( F \in C_0(Y, E) \). It is obvious that \( \| F(y) \| \to 0 \) as \( y \to \infty \). It remains to show that \( F \) is continuous. Let \( y_k \to y_0 \) in \( Y \). We have to show that \( F(y_k) \to F(y_0) \). Let \( F' \in C_0(Y, E) \) be such that \( F'(y_0) = F(y_0) \). Let \( U \) be an open neighborhood of \( y_0 \) and \( \epsilon > 0 \). One shows that eventually, \( F(y_k) \in U(F', \epsilon) \) and the continuity of \( F \) follows by the preceding comments on the definition. For \( F_1, F_2 \in C_0(Y, E) \), define \( \langle F_1, F_2 \rangle : Y \to \mathbb{C} \) in the obvious way: \( \langle F_1, F_2 \rangle(y) = \langle F_1(y), F_2(y) \rangle \). By the polarization identity and (ii) of the definition, \( \langle F_1, F_2 \rangle \in C_0(Y) \). It is easy to check that \( C_0(Y, E) \) is a Hilbert \( C_0(Y) \)-module with inner product \( \langle \cdot, \cdot \rangle \) and module action given by: \( F f(y) = f(y) F(y) \).

We now prove that \( C_0(Y, E) \) is separable. Let \( A \) be a countable base for \( E \) whose elements are of the form \( U(F, \eta) \). It suffices to show that for a compact subset \( C \) of \( Y \), the space of sections \( A \subset C_0(Y, E) \) with support in \( C \) is separable. Let \( F' \in A \) and \( \epsilon > 0 \). For each \( y \in C \), let \( U_y \) be a
relatively compact, open neighborhood of \( y \) in \( Y \). Then \( F'(y) \in U_y(F', \epsilon) \), and there exists a \( V_y(F_y, \epsilon_y) \in \mathcal{A} \) such that \( F'(y) \in V_y(F_y, \epsilon_y) \subset U_y(F', \epsilon) \).

In particular, \( y \in V_y \subset U_y \) and \( \|F'(y') - F_y(y')\| < \epsilon \) for all \( y' \in V_y \). Since \( C \) is compact, there exists a finite cover \( \{V_{y_1}, \ldots V_{y_n}\} \) of \( C \). Let \( \{f_i\} \) \((1 \leq i \leq n)\) be a partition of unity for \( C \) subordinate to the \( \{V_{y_i}\} \), and let \( F'' = \sum_{i=1}^n f_i F_{y_i} \). Then \( \|F'(y) - F''(y)\| < \epsilon \) for all \( y \in Y \). The span of such functions \( F'' \) in \( C_0(Y, E) \) is separable, and the separability of \( C_0(Y, E) \) then follows.

As a simple example of a Hilbert bundle, let \( Y = (0, 2) \), \( F \) be the trivial Hilbert bundle \( Y \times C^2 \) and \( \{e_1, e_2\} \) the standard orthonormal basis for \( C^2 \). Then \( C_0(Y, F) = C_0((0, 2)) \times C_0((0, 2)) \) in the obvious way. Let \( E \) be the subbundle \([0, 1] \times C e_1 \cup (1, 2) \times C^2 \) of \( F \) with the relative topology. Then \( E \) is a Hilbert subbundle of \( F \) though it is neither locally constant nor locally compact. (Note that \( C_0(Y, E) \) can be identified with \( C_0((0, 2)) \times \{f \in C_0((0, 2)) : f(y) = 0 \text{ for } 0 < y \leq 1\} \).

A morphism between two Hilbert bundles \( E, F \) over \( Y \) is (cf. \[24\] Definition 1.5]) a continuous bundle map \( \Phi : E \to F \) whose restriction \( \Phi_y : E_y \to F_y \) for each \( y \in Y \) is a bounded linear map and \( \sup_{y \in Y} \|\Phi_y\| = \|\Phi\| < \infty \), and such that the adjoint map \( \Phi^* : F \to E \), where \( \Phi^*(\xi_y) = (\Phi_y)^{-1}(\xi_y) \) for \( \xi_y \in F_y \) is also continuous. It is obvious that any such morphism \( \Phi \) determines an adjointable Hilbert module map \( \hat{\Phi} : C_0(Y, E) \to C_0(Y, F) \) by setting \( \hat{\Phi}(F)(y) = \Phi_y(F(y)) \). It is also obvious that with these morphisms, the class of Hilbert bundles over \( Y \) is a category.

We have seen that every \( C_0(Y, E) \) is a second countable \( C_0(Y) \)-Hilbert module. We will show that every second countable \( C_0(Y) \)-Hilbert module \( P \) is of this form. We recall first that a morphism between two Hilbert \( C_0(Y) \)-modules \( P, Q \) is an adjointable map \( T : P \to Q \). This gives the category of Hilbert \( C_0(Y) \)-modules. Two Hilbert \( C_0(Y) \)-modules \( P, Q \) are said to be equivalent - written \( P \cong Q \) - if there exists a unitary morphism \( U : P \to Q \). Next, a result of Kasparov (\[11\] Theorem 1, \[27\] Lemma 15.2.9]) gives that in any Hilbert \( A \)-module \( P \) and for any \( p \in P \),

\[
(2) \quad p = \lim_{\epsilon \to 0^+} p(p, p)[\langle p, p \rangle + \epsilon]^{-1}.
\]

It follows by Cohen’s factorization theorem and \(\langle 2 \rangle\) that \( P = \{fp : f \in C_0(Y), p \in P\} \). In the stabilization theorem of Kasparov, the Hilbert \( A \)-modules are assumed to be countably generated. It is obvious that in our situation (\( P \) second countable) \( P \) is automatically countably generated.

Let \( P \) be a \( C_0(Y) \)-Hilbert module. We construct an associated Hilbert bundle \( E \) in the familiar way (e.g. \[8\]). For \( y \in Y \), let \( I_y = \{f \in C_0(Y) : f(y) = 0\} \), a closed ideal in \( C_0(Y) \). By Cohen’s factorization theorem, \( I_y P \) is closed in \( P \). Let \( P/(I_y P) = P_y \). We claim that the norm on \( P_y \) is a Hilbert space norm, with inner product given by \( \langle p + I_y P, q + I_y P \rangle = \langle p, q \rangle(y) \). This inner product is well-defined. To see that it is non-degenerate, suppose that \( \langle p, p \rangle(y) = 0 \). Then \( \langle p, p \rangle \in I_y \) and by \(\langle 2 \rangle\), \( p \in (I_y P) = I_y P \), and
non-degeneracy follows. Let $E = \cup_{y \in Y} P_y$. If we wish to emphasize the
connection of $E$ with $P$, we write $E_P$ in place of $E$. (If $Q$ is just a pre-
Hilbert $C_0(Y)$-submodule, we define $E_Q$ to be $E_\overline{Q}$.) For each $p \in P$, let
$\hat{p}(y) = p + I_y P \in H_y$. We sometimes write $p_y$ in place of $\hat{p}(y)$. For each
open subset $U$ of $Y$ and each $\epsilon > 0$, define $U_{p,\epsilon} = U_{\hat{p},\epsilon}$, the latter being
defined as in [1].

We now show that the functor $E \to C_0(Y, E)$ is an equivalence for the
categories of Hilbert bundles over $Y$ and of Hilbert $C_0(Y)$-modules.

**Proposition 2.** Let $P$ be a Hilbert $C_0(Y)$-module. Then the family of $U_{p,\epsilon}$'s
$(p \in P)$ is a base for a second countable topology $T_P$ on $E$ which makes $E$
into a Hilbert bundle over $Y$. Further, the map $p \to \hat{p}$ is a Hilbert $C_0(Y)$-
module unitary from $P$ onto $C_0(Y, E)$, and the map $P \to E$ is an equivalence
between the category of Hilbert $C_0(Y)$-modules $P$ and the category of Hilbert
bundles $E$ over $Y$.

**Proof.** Give each $\hat{p}$ the uniform norm as a section of $E$. The proposition
is an easier version of corresponding results for Banach $A$-modules in [8].
It is easier because, as earlier, by the polarization identity, the maps $y \to
\|\hat{p}(y)\| = \sqrt{\langle p, p \rangle}(y)$ are continuous (instead of just upper semicontinuous)
and vanish at infinity. Then $\|\hat{p}\|^2 = \|\langle p, p \rangle\| = \|p\|^2$, giving $p \to \hat{p}$ an
isometry. We now check the conditions of Definition 2.1 to show that $E$ is a
Hilbert bundle over $Y$. One easily checks that the family of $U_{p,\epsilon}$'s $(p \in P)$
is a base for a topology $T_P$ on $E$, each $\hat{p}$ is continuous and the addition and
scalar multiplication maps for $E$ are continuous. The topology $T_P$ on $E$ is
second countable since $P$ is. This gives (i) of Definition 2.1 while (iii) of
that definition is trivial. The remaining requirements, (ii) and (iv) will follow
once we have shown that $\hat{P} = C_0(Y, E)$. As in the proof of Proposition 1
(cf. [8] Proposition 2.3]) $\hat{P}$ is dense in $C_0(Y, E)$. Further, $\langle \hat{p}, \hat{q} \rangle = \langle p, q \rangle$
giving the map $p \to \hat{p}$ unitary. Then $\hat{P} = C_0(Y, E)$ since the map $p \to \hat{p}$
is isometric and $P$ is complete. A morphism $T : P \to Q$ of Hilbert $C_0(Y)$-
modules determines a Hilbert bundle morphism $\Phi = \Phi_T : E_P \to E_Q$ in
the natural way: set $\Phi = \{T_y\}$ where $T_y$ is defined: $T_y p_y = (T_p)_y$. Then
$\Phi : E_P \to E_Q$ is a continuous bundle map, and $\|\Phi\| = \|T\|$. □

For a Hilbert bundle $E$ over $Y$, let $G \ast E = \{(g, \xi) : s(g) = \pi(\xi)\}$ with
the relative topology inherited from $G \times E$. Then $E$ is called a $G$-Hilbert bundle if
there is a continuous map $(g, \xi) \to g\xi$ from $G \ast E \to E$ which is algebraically
a left groupoid action (by unitaries). (The unitary condition means that
for each fixed $g \in G$, the map $\xi \to g\xi$ is unitary from $H_s(g)$ onto $H_t(g)$.)
One can also define this notion in terms of pull-back bundles as in [17], [18],
but the approach adopted here is more elementary, and closer in spirit to
the usual definition of a group Hilbert bundle. A Hilbert $C_0(Y)$-module $P$
is called a $G$-Hilbert module if $E_P$ is a $G$-Hilbert bundle. The corollary to
the following proposition shows that when $G$ is a transformation groupoid
$H \times Y$, a $G$-Hilbert module is the same as a Hilbert $(H, C_0(Y))$-module in
the notation of [24]. (In [24, Proposition 1.3], it is shown that if \( E \) is an \( H \)-Hilbert bundle over \( Y \), then \( C_0(Y,E) \) is a Hilbert \((H,C_0(Y))\)-module. The corollary shows that the opposite direction holds as well as long as we use the wider category of Hilbert bundles of the present paper.)

**Proposition 3.** A left groupoid action of \( G \) on \( E \) is continuous if and only if, for each \( F \in C_0(Y,E) \), the map \( g \to gF_{s(g)} \) is continuous from \( G \to E \).

*Proof.* If the action is continuous, then trivially, the maps \( g \to gF_{s(g)} \) are continuous. The converse is very similar to [23, Corollary 1], and so we give only a brief sketch of the proof. Suppose then that for each \( F \in C_0(Y,E) \), the map \( g \to gF_{s(g)} \) is continuous from \( G \to E \). Let \( \{g_n\} \) be a sequence in \( G \) and \( \{\xi_n\} \) a sequence in \( E \) with \( \xi_n \in E_{s(g_n)} \) such that \( g_n \to g \) in \( G \) and \( \xi_n \to \xi \) in \( E \). We have to show that \( g_n\xi_n \to g\xi \) in \( E \). By Definition 2.1(iii), there exist \( F \in C_0(Y,E) \) such that \( g\xi = F_{r(g)} \) and \( F' \in C_0(Y,E) \) such that \( \xi = F'_{s(g)} \). Then \( \|g_n\xi_n - g\xi\| \to 0 \), so that \( \|g_n\xi_n - g_nF'_{s(g_n)}\| \to 0 \) as well. Next, by assumption, \( g_nF'_{s(g_n)} \to gF'_{s(g)} = g\xi = F_{r(g)} \) and so by the continuity of \( F \), \( \|g_nF'_{s(g_n)} - F_{r(g_n)}\| \to 0 \). So \( g_n\xi_n \to g\xi \). \( \square 

**Corollary 2.2.** Let \( G \) be a transformation groupoid \( H \times Y \). Then the map \( E \to C_0(Y,E) \) is an equivalence between the category of \( H \)-Hilbert bundles over \( Y \) and the category of Hilbert \((H,C_0(Y))\)-modules.

*Proof.* We recall [11, 19] that a Hilbert \( C_0(Y) \)-module \( S \) is an \((H,C_0(Y))\)-module if it is a left \( H \)-module such that \( h(Ff) = (hf)(hf) \), the map \( h \to hF \) is continuous, and \( \langle hF,hF' \rangle = h\langle F,F' \rangle \) for all \( h \in H, F,F' \in S \) and \( f \in C_0(Y) \). (Of course, \( (hf)(y) = f(h^{-1}y) \).) An \( H \)-Hilbert bundle over \( Y \) (cf. [24, Definition 1.2]) is a Hilbert bundle over \( Y \) (in the sense of this paper) with a continuous action \( (h,\xi) \to h\xi \) from \( H \times E \) into \( E \) such that for each \( y \), the action of \( h \) on \( E_y \) is a unitary onto \( E_{hy} \). (Recalling that \((H \times Y)_0 = H \) for all \( y \), it is obvious that \( H \)-Hilbert bundles over \( Y \) are just the same as the groupoid \((H \times Y)\)-Hilbert bundles.) Suppose, first that \( E \) is an \( H \)-Hilbert bundle. Then (as in [24, Proposition 1.3]) the Hilbert \( C_0(Y) \)-module \( C_0(Y,E) \) is a Hilbert \((H,C_0(Y))\)-module, where \((FF)(y) = f(y)F(y) \) and \((hF)(y) = h[F(h^{-1}y)] \) \( (F \in C_0(Y,E)) \). For the converse, let \( P \) be a Hilbert \((H,C_0(Y))\)-module, \( E = E_P \). By Proposition 2 we can canonically identify \( P \) with \( C_0(Y,E) \). It is obvious that \( hI_y = I_{hy} \). We define a groupoid action of \( H \times Y \) on \( E \) by setting \((h,y)(p+I_yP) = hp+I_{hy}P \), i.e. \((h,y)p_y = (hp)_{hy} \). We now check that this is indeed a groupoid action (in the sense of this paper). The algebraic properties are obvious using the formulas for multiplication and inversion in \( H \times Y \) given in the introduction. To prove that \( H \times Y \) acts on \( E \) by unitaries,

\[
(h,y)p_y, (h,y)q_y = (hp,hq)(hy) = (p,q)(h^{-1}hy) = (p_y,q_y).
\]

Last, to prove the continuity of the groupoid action on \( E \), we have, by Proposition 3 to show, identifying \( \hat{P} \) with \( C_0(Y,E) \), that for each \( p \in P \), the map \((h,y) \to (h,y)p_y \) is continuous from \( H \times Y \) into \( E \), i.e. that the map
If \( P, Q \) are \( G \)-Hilbert modules, then a Hilbert \( C_0(Y) \)-module morphism \( T : P \to Q \) is called \( G \)-equivariant if for all \( g \in G \), \( T_r(g)g = gT_s(g) \) on \((E_P)_s\). Using the fact that the groupoid action is unitary, \( T^* \) is also \( G \)-equivariant. Of course, \( P \) and \( Q \) are said to be equivalent \((P \cong Q)\) if there exists \( G \)-equivariant unitary between them.

A pre-Hilbert \( C_0(Y) \)-module \( Q \) is called a \( pre-G \) Hilbert module if \( \overline{Q} \) is a \( G \)-Hilbert module, and the action of \( G \) on \( E = \overline{Q} \) leaves invariant the \( Q_y \)'s, where \( Q_y \) is the image of \( Q \) in \( E_y \). As we will see below, an important example of a pre-\( G \)-Hilbert module is the case \( Q = C_c(G) \). The \( C_0(Y) \)-module action on \( C_c(G) \) is given by: \((F, f) \to F(f \circ r)\) and the \( C_0(Y) \)-valued inner product on \( C_c(G) \) by: \((F_1, F_2)_y = \langle (F_1)_y, (F_2)_y \rangle \) \((F_y = F|_{G^y})\). One uses the axioms for a locally compact groupoid to check the required properties. For example, the continuity of \( y \to \langle (F_1)_y, (F_2)_y \rangle \) follows from the axiom that for \( \phi \in C_c(G) \), the function \( y \to \int_{G^y} \phi(g) d\lambda^y(g) \) is continuous. Let \( P_G \) be the Hilbert \( C_0(Y) \)-module completion of \( C_c(G) \), and \( L^2(G) = E_{P_G} \), the Hilbert bundle determined by \( P_G \) as in Proposition 2. It is easy to check that for each \( y \), the image of \( C_c(G) \) in \( H_y \) is naturally identified as a pre-Hilbert space with \( C_c(G^y) \) with the \( L^2(G^y) \) inner product. So the Hilbert space \((E_{P_G})_y = L^2(G^y)\), which justifies writing \( E_{P_G} \) as \( L^2(G) \). The isomorphism \( F \to \hat{F} \) from \( C_c(G) \) into \( C_c(Y, L^2(G)) \) takes \( F \) to the section \( y \to \hat{F}_y = \hat{F}(y) \), and the family of sets \( U(F, \epsilon) \) forms a base for the topology of \( L^2(G) \). The \( G \)-action on \( L^2(G) \) is the natural one: \( g\xi_{s(g)}(h) = \xi_{s(g)}(g^{-1}h) \) \((h \in \mathbb{G}^r(g))\) for \( \xi_{s(g)} \in L^2(G^s(g)) \). We now show that this action is continuous for the topology of \( L^2(G) \).

**Proposition 4.** The \( G \)-action is continuous on \( L^2(G) \) (so that \( L^2(G) \) is a \( G \)-Hilbert bundle and \( P_G \) a \( G \)-Hilbert module).

**Proof.** From Proposition 3 it suffices to show that if \( \psi \in C_0(Y, L^2(G)) \) and \( g_n \to g \) in \( G \), then \( g_n\psi_{s(g_n)} \to g\psi_{s(g)} \). Since \( \hat{C}(G) \) is uniformly dense in \( C_0(Y, L^2(G)) = \hat{P}_G \) (Proposition 2), we can suppose that \( \psi = \hat{F} \) where \( F \in C_c(G) \). By Tietze’s extension theorem, there exists \( F' \in C_c(G) \) such that \( F'_r(g) = gF_s(g) \). It is sufficient, then, to show that \( \|F'_{r(g_n)} - g_nF_{s(g_n)}\|_2 \to 0 \) since the \( U(F', \epsilon)'s \) \((r(g) \in U)\) form a base of neighborhoods for \( F_{r(g)} \) in \( L^2(G) \). Arguing by contradiction, suppose that the sequence \( \{\|F'_{r(g_n)} - g_nF_{s(g_n)}\|_2\} \) does not converge to 0. We can then suppose that for some \( k > 0 \), \( \|F'_{r(g_n)} - g_nF_{s(g_n)}\|_2 \geq k \) for all \( n \). Let \( D \) be a compact subset of \( G \) containing the sequence \( \{g_n\} \) and let \( C = Dsupp(F) \cup supp(F') \subset G \). Since \( C \) is compact, \( M = \sup_{u \in Y} \lambda^u(C^u) < \infty \). Then

\[ [\sup\{ | F'(h) - F(g_n^{-1}h) | ; h \in C^r(g_n) \cap C \} ]^2 M \geq \| F'_{r(g_n)} - g_nF_{s(g_n)} \|_2 \geq k^2. \]
So we can find $h_n \in G^r(g_n) \cap C$ such that $|F'(h_n) - F(g_n^{-1}h_n)| > \sqrt{k^2/(2M)}$. By the compactness of $C$, we can suppose that $h_n \to h \in G^r(g)$, and thus obtain $|F'(h) - gF_s(g)(h)| > 0$, contradicting $F'_r = gF_s(g)$. $\square$

$C_0(Y)$ itself is naturally a $G$-Hilbert module. To see this, $C_0(Y)$ is, like every $C^*$-algebra, a Hilbert module over itself. The Hilbert bundle determined by $C_0(Y)$ is, of course, just $Y \times C$. It is left to the reader to check that the topology determined on $E = Y \times C$ is just the product topology. The $G$-action on $Y$ is given by $(g, s(g), a) \to (r(g), a)$ (trivially continuous).

Let $E(i) = \{E(i)_y\}$ ($1 \leq i \leq n$) be Hilbert bundles over $Y$ and $P(i)$ the Hilbert $C_0(Y)$-module $C_0(Y, E(i))$. Let $E = \oplus_{i=1}^n E(i)$. It is easy to check that $E = \oplus_{i=1}^n E(i)$ with the relative topology inherited from $E(1) \times \ldots \times E(n)$. (Note also that the elements of $C_0(Y, E)$ are of the form $F = (F_1, \ldots, F_n)$ where $F_i \in C_0(Y, E(i))$.)

Similarly if $1 \leq i < \infty$, then $E = \oplus_{i=1}^\infty E(i)$ is defined to be $E(\oplus_{i=1}^\infty P(i))$. (Here (e.g. [17, 2.2.1]) $\oplus_{i=1}^\infty P_i$ consists of all sequences $\{p_i\}$, $p_i \in P_i$, such that $\sum_{i=1}^\infty (p_i, p_i)$ is convergent in $C_0(Y)$. The argument of, for example, [27], pp.237-238, shows that $\oplus_{i=1}^\infty P_i$ is a Hilbert $C_0(Y)$-module with $C_0(Y)$-valued inner product given by $\langle \{p_i\}, \{q_i\}\rangle = \sum_{i=1}^\infty (p_i, q_i)$. Then for each $y$, $E_y$ is the Hilbert space direct sum $\oplus_{i=1}^\infty E(i)_y$. Using Proposition 2, the topology on $E$ can be conveniently described in terms of convergent sequences: $\xi_n \to \xi$ ($\xi_n \in \{\xi_i\}$, $\xi \in \{\xi_i\}$) if and only if $\xi_n \to \xi_i$ in $E(i)$ for all $i$ and $\sum_{i=N}^\infty \|\xi_i^n\|^2 \to 0$ as $N, n \to \infty$. When $E(i) = E(1)$ for all $i$, then we write $E = E(1)^\infty$, corresponding to the module $P = P(1)^\infty$. Using the preceding criterion for convergent sequences, it is straightforward to show that the Hilbert bundles $\oplus_{i=1}^n E(i), \oplus_{i=1}^\infty E(i)$ are $G$-Hilbert bundles in the natural way if the $E(i)$’s are $G$-Hilbert bundles. Of course, $\oplus_{i=1}^n P(i), \oplus_{i=1}^\infty P(i)$ are then $G$-Hilbert modules.

We also require that for any $G$-Hilbert module $P$,

$$P^{\infty}\otimes G \cong P^{\infty}.$$

To prove this, using the Cantor diagonal process, one “rearranges” a sequence $\{\xi_i\} \in (P^{\infty})^\infty$, $\xi_i = \{\xi_{ij}\}$, $\xi_{ij} \in P$, as a sequence in $P^{\infty}$, and checks that the $C_0(Y)$-Hilbert module structure and the $G$-action are preserved.

A number of natural $G$-Hilbert $C_0(Y)$-modules arise from other such modules as tensor products over $C_0(Y)$ (cf. [4, 5], see [17, 3.2.2] for a pull-back approach to the construction of tensor product $G$-Hilbert modules. Let $P, Q$ be pre-Hilbert $C_0(Y)$-modules and form the algebraic balanced tensor product $P \otimes_{alg,C_0(Y)} Q$. This is a pre-Hilbert $C_0(Y)$-module in the natural way, i.e. with $(p \otimes q)f = p \otimes qf = p \otimes fQ = pf \otimes q$ and inner product given by $\langle p_1 \otimes q_1, p_2 \otimes q_2\rangle = \langle p_1, p_2\rangle \langle q_1, q_2\rangle$. The completion of $P \otimes_{alg,C_0(Y)} Q$, quotiented out by the null space of the norm induced by the inner product, is a Hilbert $C_0(Y)$-module $P \otimes_{C_0(Y)} Q$. (When $P, Q$ are Hilbert modules, the construction is a special case of the inner tensor product $P \otimes_{\phi} Q$ ([3, 13.5]) with $\phi : C_0(Y) \to B(Q)$ where $\phi(f)q =fq$ - see [16] and [28, I.1] for details of the construction of the inner tensor product.) Note that $P \otimes_{alg,C_0(Y)} Q$ is
a dense Hilbert submodule of $\overline{P \otimes_{C_0(Y)} Q}$, so that $\overline{P \otimes_{C_0(Y)} Q} = P \otimes_{C_0(Y)} Q$. Canonicly, $(P \otimes_{C_0(Y)} Q)_y$ is the Hilbert space tensor product $P_y \otimes Q_y$ and for $p \in P, q \in Q$, $p \otimes q(y) = \hat{p}(y) \otimes \hat{q}(y)$. We write $E_{P \otimes_{C_0(Y)} Q} = E_P \otimes E_Q$. (We note that this construction of the tensor product of two Hilbert bundles over $Y$ cannot be defined, as for vector bundles, using charts in the usual way (as, for example, in [23 1.2]).)

**Proposition 5.** If $P, Q$ are $G$-pre-Hilbert $C_0(Y)$-modules, then $P \otimes_{C_0(Y)} Q$ is a $G$-Hilbert module, the $G$-action being the diagonal one.

**Proof.** By definition of $P \otimes_{C_0(Y)} Q$, we can assume that $P, Q$ are $G$-Hilbert modules. It is obvious that $G$ acts isometrically on $E_{P \otimes_{C_0(Y)} Q} = \bigcup_{y \in Y} P_y \otimes Q_y$. For $G$-continuity, we only need to check Proposition 3 when $F = \tilde{v}$ where $v = \sum_{i=1}^n p_i \otimes q_i \in P \otimes_{alg,C_0(Y)} Q$. Suppose then that $y_r \to y$ in $Y$, $g_r \to g$ in $G$ with $s(g_r) = y_r$. Since $F$ is continuous, $F(y_r) \to \sum_{i=1}^n \hat{p}_i(y) \otimes \hat{q}_i(y)$. Since $P, Q$ are $G$-Hilbert modules, for each $i$, $g_r \hat{p}_i(y) \to g \hat{p}_i(y), g \hat{q}_i(y) \to g \hat{q}_i(y)$ in $E_P, E_Q$ respectively. Let $p'_i \in P, q'_i \in Q$ be such that $p'_i((r(g)) = \hat{g}\hat{p}_i(y), q'_i((r(g)) = \hat{g}\hat{q}_i(y)$, and set $w = \sum_{i=1}^n p'_i \otimes q'_i \in P \otimes_{alg,C_0(Y)} Q$. Let $z_r = r(g_r)$. Then $\|\bar{w}(z_r) - \sum_{i=1}^n g_r \hat{p}_i(y) \otimes g_r \hat{q}_i(y)\| \leq \sum_{i=1}^n \|p'_i(z_r) - g_r \hat{p}_i(y)\| \|g_r \hat{q}_i(y)\| + \|q'_i(z_r)\| \|g_r \hat{q}_i(y)\| \to 0$. Since $w(z_r) \to \overline{w(r(g))} = gF(y), g, F(y) = \sum_{i=1}^n g_r \hat{p}_i(y) \otimes g_r \hat{q}_i(y) \to gF(y)$. So $P \otimes_{C_0(Y)} Q$ is a $G$-Hilbert module.

Next, we require the result that for any $G$-Hilbert modules $P, Q$, we have that as $G$-Hilbert modules,

$$(P^\infty \otimes_{C_0(Y)} Q) \cong (P \otimes_{C_0(Y)} Q^\infty) \cong (P \otimes_{C_0(Y)} Q)\infty.$$

Let us prove that $(P^\infty \otimes_{C_0(Y)} Q) \cong (P \otimes_{C_0(Y)} Q^\infty)$, the other equality being proved similarly. Let $R$ be the dense subspace of $P^\infty$ whose elements are the finite sequences $r = (p_1, \ldots, p_n, 0, 0, \ldots)$ with $p_i \in P$. Define a $C_0(Y)$-module map $\alpha : R \otimes_{alg,C_0(Y)} Q \to (P \otimes_{C_0(Y)} Q)^\infty$ by setting $\alpha(r \otimes q) = (p_1 \otimes q, \ldots, p_n \otimes q, 0, 0, \ldots)$. It is easily checked that $\alpha$ is well-defined, and preserves the $C_0(Y)$-inner product: $\langle \alpha(r \otimes q), \alpha(r' \otimes q') \rangle = \langle r, r' \rangle \langle q, q' \rangle$. The range of $\alpha$ is onto a dense subspace of $(P \otimes_{C_0(Y)} Q)^\infty$ and preserves the $G$-action, so the result follows.

Of particular importance is the case of the $G$-Hilbert module $P \otimes_{C_0(Y)} PG$. We write $E_{P \otimes_{C_0(Y)} PG} = L^2(G) \otimes E$ (or $E \otimes L^2(G)$) where $E = E_P$. Here $L^2(G) \otimes E$ is the Hilbert bundle over $Y$ with $(L^2(G) \otimes E)_y = L^2(G^y, E_y)$ and a dense subspace of $C_0(Y, L^2(G) \otimes E)$, determining its topology as earlier, is given by the span of sections of the form $\hat{h} \otimes \hat{p} (h \in C_c(G))$ where $(\hat{h} \otimes \hat{p})(y) = h|_{G^y} \otimes \hat{p}(y)$. A section $k$ of $L^2(G) \otimes E$ is invariant if for all $g \in G$, $g k_{s(g)}(g^{-1} h) = k_r(h) (h \in G^r(g))$ as maps in $L^2(G^r(g), E_{r(g)})$. We now identify a certain dense linear subspace $C_c(G, r^* E)$ of $C_0(Y, L^2(G) \otimes E)$ (cf. [26]). Here, $C_c(G, r^* E)$ is the set of continuous, compactly supported functions $\phi$ from $G$ into $E$ such that for all $g \in G$, $\phi(g) \in E_{r(g)}$. For
each \( y \in Y \) and \( \phi \in C_c(G, r^*E) \), let \( \hat{\phi}(y) = \phi_y \), the restriction of \( \phi \) to \( G^y \). Then \( \hat{\phi}(y) \in C_c(G^y, E_y) \subset L^2(G^y, E_y) = (L^2(G) \otimes E)_y \) so that \( \hat{\phi} \) is a section of \( L^2(G) \otimes E \). The section norm on \( C_c(G, r^*E) \) is then given by: 
\[
\|\phi\| = \sup_{y \in Y} \|\phi_y\|.
\]

**Proposition 6.** Let \( P \) be a \( G \)-Hilbert module and \( E = E_P \). Then \( C_c(G, r^*E) \) is a dense subspace of \( C_0(Y, L^2(G) \otimes E) \), and contains all functions of the form \( \hat{h} \otimes \hat{p} \) above.

**Proof.** Clearly, \( \hat{h} \otimes \hat{p} \in C_c(G, r^*E) \) since the map \( g \to h(g)\hat{p}(r(g)) \) is continuous. For the rest of the proposition, the span of such functions \( \hat{h} \otimes \hat{p} \) is uniformly dense in \( C_0(Y, L^2(G) \otimes E) \), so it is enough to show that every \( \hat{\phi} \) \( (\phi \in C_c(G, r^*E)) \) is in the uniform closure of this span.

To this end, let \( H = \text{supp}(\phi) \). Let \( y_0 \in Y \). Let \( W \) be a compact subset of \( G \) such that \( H \subset W^0 \). Let \( \epsilon > 0 \). For each \( g \in H \), let \( p_g \in P \) be such that \( \hat{p}_g(r(g)) = \phi(g) \). Let \( h_g \in C_c(G) \) be such that \( h_g(g) = 1 \). By continuity, there exists an open neighborhood \( U_g \) of \( g \) in \( G \) such that \( U_g \subset W \) and such that for all \( g' \in U_g \),
\[
\|\phi(g') - h_g(g')\hat{p}_g(r(g'))\| < \eta = \epsilon/\sup_{y \in Y} \lambda^u(W)^{1/2} + 1.
\]

Since \( H \) is compact, it is covered by a finite number of the \( U_g \)'s, say \( U_{g_1}, \ldots, U_{g_n} \). Taking a partition of unity, there exist functions \( f_i \in C_c(U_{g_i}) \), \( f_i \geq 0 \), \( \sum_{i=1}^n f_i = 1 \) on \( H \) and \( \sum_{i=1}^n f_i \leq 1 \) on \( G \). Then for \( g' \in W \),
\[
\|\phi(g') - \sum_{i=1}^n f_i(g')h_{g_i}(g')\hat{p}_{g_i}(r(g'))\| < \eta. \text{ It follows that for } y \in Y,
\]
\[
\|\phi_y - \sum_{i=1}^n (f_i h_{g_i} \otimes p_{g_i})y\|_2 < \epsilon.
\]

So \( \phi \in C_0(Y, L^2(G) \otimes E) \). \( \square \)

We now note two simple results on the tensor products of two \( G \)-Hilbert modules. First, if \( P \) is a \( G \)-Hilbert module then
\[
C_0(Y) \otimes C_0(Y) \overset{\cong}{\to} P.
\]

The natural isomorphism is given by the equivariant Hilbert module map determined by: \( f \otimes P \to fp \) \((f \in C_0(Y), p \in P)\). Next, it is left to the reader to check that if \( P, Q, R \) are \( G \)-Hilbert modules, then the Hilbert module direct sum \( P \oplus Q \) is a \( G \)-Hilbert module in the obvious way, and
\[
(P \oplus Q) \otimes C_0(Y) \overset{\cong}{\to} (P \otimes C_0(Y) \oplus Q \otimes C_0(Y)) \overset{\cong}{\to} R).
\]

The final proposition of this section is a groupoid version of [19 Lemma 2.3] (which applies to the group case).

**Proposition 7.** Let \( P, Q \) be \( G \)-Hilbert modules with \( P \cong Q \) as Hilbert \( C_0(Y) \)-modules. Then \( P \otimes C_0(Y) \overset{\cong}{\to} Q \otimes C_0(Y) \overset{\cong}{\to} P \) as \( G \)-Hilbert modules.

**Proof.** Let \( E = E_P, F = E_Q \). By assumption, there exists a Hilbert module unitary \( U : P \to Q \). For \( \phi \in C_c(G, r^*E) \), define \( V\phi : G \to r^*F \) by:
\[
V\phi(g) = gU_{s(g)}(g^{-1}\phi(g)).
\]
Using the continuity of $\Phi_U = \{U_y\}$ and of the $G$-actions of $E, F$, we see that $V$ belongs to $C_c(G, r^*F)$. Regard, as earlier, $C_c(G, r^*E), C_c(G, r^*F)$ fibered over $Y$ (with $\phi \to \{\phi_y\}$). Then $V$ is a fiber preserving isomorphism onto $C_c(G, r^*F)$ with $V^{-1}(\chi(g)) = gU_{s(g)}(g^{-1}\chi(g))$. Further, 

$$\langle (V\phi)_y, (V\psi)_y \rangle = \int \langle gU_{s(g)}(g^{-1}\phi_y(g)), gU_{s(g)}(g^{-1}\psi_y(g)) \rangle \, d\lambda^y(g) = \langle \phi_y, \psi_y \rangle$$

so that $V$ preserves inner products. So $V$ extends to a Hilbert module unitary from $C_0(Y, L^2(G) \otimes E) \to C_0(Y, L^2(G) \otimes F)$, using Proposition 5 and Proposition 2. It remains to show that $V$ is $G$-equivariant. We note first that by (7), $V_y$ is given by: $V_y\xi(g) = gU_{s(g)}(g^{-1}\xi(g))$ for $\xi \in L^2(G^g, E_y), g \in G^g$. Then for $g, h \in G^y$, $[g[V_y\xi_s(g)]]}(h) = g[V_y\xi_s(g)](g^{-1}h) = g(g^{-1}h)[U_s(h)(g^{-1}\xi_s(g))g^{-1}h])] = h[U_s(h)(h^{-1}[g\xi_s(g)](h))] = V_{r(g)}(g\xi_s(g))(h)$, so that $gV_y = V_{r(g)}g$ and $V$ is equivariant. 

3. Stabilization

In this section we establish the proper groupoid stabilization theorem. Throughout, $G$ is a proper groupoid and $P$ a $G$-Hilbert module. We require two preliminary propositions. The first of these is the general groupoid version of [24 Lemma 2.8].

Proposition 8. There exists a continuous, invariant section $\phi$ of the Hilbert bundle $L^2(G)\infty$ such that $\|\phi(y)\|_2 = 1$ for all $y$. Locally, $\phi(y)$ is of the form 

$$((\psi_1)|_{G^y}, \ldots, (\psi_n)|_{G^y}, 0, \ldots)$$

where $\psi_i \in C_c(G)$. 

Proof. For $y_0 \in Y$, let $a_{y_0} \in C_c(G)$ be such that $a_{y_0} \geq 0, a_{y_0}(y_0) > 0$. Let $\eta_{y_0} : G \to R^+$ be given by:

$$\eta_{y_0}(g) = \int_{G^{r(g)}} a_{y_0}(h^{-1}g) \, d\lambda^r(g)(h).$$

We want to regard $k = \eta_{y_0}$ as a continuous, invariant section $y \to k_y$ of $L^2(G)$. To prove this, the invariance of $k$ (i.e. that $g_0k_{s(g_0)} = k_{r(g_0)}$, or equivalently, that $k(g_0^{-1}g) = k(g)$ for all $g_0, g \in G, r(g_0) = r(g)$) follows from an axiom for left Haar systems. For the continuity of the section $y \to k_y$ of $L^2(G)$, we will show that for any compact subset $A$ of $Y$, $k_{|r^{-1}A} \in C_c(r^{-1}A)$. The continuity of $k$ as a section of $L^2(G)$ then follows, since for every relatively compact open subset $U$ of $Y$, there will then exist an $F \in C_c(G)$ such that $F = k$ on $r^{-1}U$ (so that $F_y = k_y$ for all $y \in U$). Since $F$ is continuous as a section $y \to F_y$ of $L^2(G)$, so also is $k$. (Of course, $y \to k_y$ need not vanish at infinity.) 

To show that $k_{|r^{-1}A} \in C_c(r^{-1}A)$, let $C$ be the (compact) support of $a_{y_0}$ and let $g \in r^{-1}A$. If $a_{y_0}(h^{-1}g) > 0$, then $r(h) = r(g) \in A$, and $s(h) \in r(C)$. By the properness of $G$, $h$ belongs to the compact set $D = \{g' \in G : (r(g'), s(g')) \in A \times r(C)\}$. Let $F \in C_c(G)$ be such that $F = 1$ on $D$. Then
on \( r^{-1}(A) \), \( k \) coincides with the convolution \( F \ast a_{y_0} \) of two \( C_c(G) \)-functions, and so is the restriction of a \( C_c(G) \)-function as required.

By the continuity and positivity assumptions on \( a_{y_0} \), the function \( \eta_{y_0}(y_0) > 0 \). So \( (\eta_{y_0})_{y_0} \neq 0 \). By the continuity of \( y \mapsto \|(\eta_{y_0})_y\|_2 \), the set \( U_{y_0} = \{ y \in Y : (\eta_{y_0})_y \neq 0 \} \) is an open neighborhood of \( y_0 \) in \( Y \). Since \( \eta_{y_0} \) is invariant, it follows that \( U_{y_0} \) is an invariant subset of \( Y \), i.e. is such that for \( g \in G \), \( s(g) \in U_{y_0} \) if and only if \( r(g) \in U_{y_0} \). Further, the \( U_{y_0} \)'s cover \( Y \). Since the action of \( G \) on \( Y \) is proper, there is a \( G \)-partition of unity \( \{ f_\gamma : \gamma \in S \} \), where \( S \) can be taken to be infinitely countable (and so identified with \( \{1, 2, 3, \ldots \} \)), subordinate to the \( U_\gamma \)'s ([21, 22 Proposition 4]). This means that for each \( \gamma, f_\gamma \in C_c(Y), \ 0 \leq f_\gamma \), there exists a \( y(\gamma) \in Y \) such that \( \text{supp}(f_\gamma) \subset U_{y(\gamma)} \), and with \( m_\gamma : Y \to \mathbb{R} \) given by \( m_\gamma(y) = \int_{G^y} f_\gamma(s(g)) \, d\lambda^y(g) \), we have

\[
\sum_\gamma m_\gamma(y) = 1,
\]

the sum being locally finite.

Using the properness of \( G \) and the continuity of the maps \( y \mapsto \int_{G^y} F(g) \, d\lambda^y(g) \) for \( F \in C_c(G) \), \( m_\gamma \) is invariant (i.e. \( m_\gamma(s(g)) = m_\gamma(r(g)) \) for all \( g \in G \) and continuous. Define a section \( \phi = \{ \phi_\gamma \} \) of \( L^2(G)^\infty \) by setting

\[
\phi_\gamma(y) = m_\gamma(y)^{1/2}(\|(\eta_{y(\gamma)})_{G^y}\|_2)^{-1}(\eta_{y(\gamma)})_{G^y}.
\]

We take \( \phi_\gamma(y) \) to be 0 whenever \( (\eta_{y(\gamma)})_{G^y} = 0 \). For continuity reasons, we need to know that if \( (\eta_{y(\gamma)})_{G^y} = 0 \) then \( m_\gamma(y) = 0 \). To prove this, suppose then that \( (\eta_{y(\gamma)})_{G^y} = 0 \). Then \( y \in Y \setminus U_{y(\gamma)} \), which is invariant since \( U_{y(\gamma)} \) is. So if \( g \in G^y \), then \( s(g) \in Y \setminus U_{y(\gamma)} \), and in that case, \( f_\gamma(s(g)) = 0 \) (since the support of \( f_\gamma \) lies inside \( U_{y(\gamma)} \)), so that \( m_\gamma(y) = 0 \) from the definition of \( m_\gamma \).

We now claim that \( \phi = \{ \phi_\gamma \} \) is continuous and \( G \)-invariant. For the continuity of \( \phi \), we note that \( \|\phi_\gamma(y)\|_2^2 = m_\gamma(y) \), and use the preceding paragraph, the local finiteness of the sum in (S), and the continuity of the maps \( y \mapsto (\eta_{y(\gamma)})_{G^y}, y \mapsto \|(\eta_{y(\gamma)})_{G^y}\|_2 \) to obtain that locally \( \phi \) takes values in some \( L^2(G)^n \) with \( n \) finite and components the restrictions of \( C_c(G) \)-functions. Since the \( \eta_{y(\gamma)} \), \( m_\gamma \) are \( G \)-invariant so also is \( \phi \).

Last, from (S), \( \|\phi(y)\|_2 = \left[ \sum_{\gamma \in S} \|\phi_\gamma(y)\|_2^2 \right]^{1/2} = 1. \)

Proposition 9.

\[
P \oplus (P \otimes_{C_0(Y)} P_{G}^\infty) \cong P \otimes_{C_0(Y)} P_{G}^\infty.
\]

Proof. Let \( \phi \) be the continuous, invariant section of \( L^2(G)^\infty \) given by Proposition(S). For each \( p \in P \), define a section \( \hat{W}p \) of \( E \otimes L^2(G)^\infty \) by: \( \hat{W}p = \hat{p} \otimes \phi \). We claim that \( \hat{W}p \in C_0(Y, E \otimes L^2(G)^\infty) \). To prove that \( \|\hat{W}p(y)\| \to 0 \) as \( y \to \infty \), by Proposition(S) \( \|\hat{p} \otimes \phi(y)\|_2 = \|\hat{p}(y)\|_2 \|\phi(y)\|_2 = \|\hat{p}(y)\|_2 \to 0 \) as \( y \to \infty \). The continuity of \( \hat{W}p \) follows from the fact that locally, it is the restriction of an element of \( (P \otimes_{\text{alg}, C_0(Y)} C_c(G)^n) \) (which is a subspace of the space of continuous sections of \( E \otimes L^2(G)^\infty \)). It is easy to check that
$W : C_0(Y, E) \to C_0(Y, E \otimes L^2(G)^\infty)$ is a linear, $C_0(Y)$-module map, and that $(W\hat{p}, W\hat{p}') = \langle \hat{p}, \hat{p}' \rangle$. Further, using the invariance of $\phi$,

$W_{r(g)}(g\hat{p}_s(g)) = g\hat{p}_s(g) \otimes \phi_{r(g)} = g\hat{p}_s(g) \otimes gW(\hat{p}_s(g)) = gW_s(g)(\hat{p}_s(g))$

so that $W$ is $G$-invariant. By Proposition [2], there exists a map $V : P \to P \otimes C_0(Y) P_G^\infty$ such that $Vp = W\hat{p}$. Note that $Vyr_y = (W\hat{p})y$.

From the corresponding properties for $W$, $(V(p), V(p')) = (p, p')$ and $V$ is a $G$-equivariant $C_0(Y)$-module map. We claim that $V$ is adjoinable with adjoint $V^*$ determined by: $V^*(p \otimes \psi) = p(\phi, \hat{\psi})$ for $\psi \in \cup_{n=1}^\infty C_c(G)^n$, a dense subspace of $P_G^\infty$. Note that by definition, $(\phi, \hat{\psi})(y) = (\phi_y, \psi_y)$ (the inner product evaluated in $L^2(G)^\infty$), and using Proposition [5] $\langle \phi, \hat{\psi} \rangle \in C_0(Y)$ and $|\langle \phi, \hat{\psi} \rangle(y)| \leq ||\psi_y||$. Now

$$\langle p \otimes \psi, Vp' \rangle = \langle \hat{p} \otimes \hat{\psi}, \hat{V}p' \rangle = \langle \hat{p} \otimes \hat{\psi}, (p, \hat{p}') \rangle = \langle p(\phi, \hat{\psi}), p' \rangle.$$ 

It is easy to check that the bilinear map $p \otimes \psi \to p(\phi, \hat{\psi})$ extends to a linear map $V^*$ from $P \otimes_{alg, C_0(Y)} P_G^\infty \to P$, and so $(t, Vp') = \langle V^*t, p' \rangle$ for all $t \in P \otimes_{alg, C_0(Y)} P_G^\infty$, $p' \in P$. Since $||\langle V^*(\sum_{i=1}^n p_i \otimes \psi_i), p' \rangle|| = ||\langle \sum_{i=1}^n p_i \otimes \psi_i, Vp' \rangle|| \leq ||\sum_{i=1}^n p_i \otimes \psi_i|| ||p'||$, $V^*$ is continuous on $P \otimes_{alg, C_0(Y)} P_G^\infty$ and so extends by continuity to $P \otimes C_0(Y) P_G^\infty$. This extension is the adjoint of $V$ as claimed.

Using the approach of Mingo and Phillips [19], define $U : P \oplus (P \otimes C_0(Y) P_G^\infty) \to (P \otimes C_0(Y) P_G^\infty)$ by:

$$U(p_0, \xi_1, \xi_2, \ldots) = (Vp_0 + (1 - VV^*)\xi_1, VV^*\xi_1 + (1 - VV^*)\xi_2, \ldots).$$

One checks that for each $w = (p_0, \xi)$, $U(w) = (b_1, b_2, \ldots)$ belongs to $(P \otimes C_0(Y) P_G^\infty)^\infty$, i.e. that $\sum_{i=1}^\infty (b_i, b_i)$ converges in $C_0(Y)$. By [3] and [4], $(P \otimes_{alg} C_0(Y) P_G^\infty)^\infty = P \otimes_{alg} C_0(Y) P_G^\infty$. Further, $U$ preserves the $C_0(Y)$-valued inner product. Direct calculation shows that $U$ has an adjoint given by:

$$U^*(\eta_1, \eta_2, \ldots) = (V^*\eta_1, VV^*\eta_2 + (1 - VV^*)\eta_1, VV^*\eta_3 + (1 - VV^*)\eta_2, \ldots),$$

that $U$ is unitary and, using the invariance of $V$, that $U$ preserves the groupoid action. □

**Theorem 3.1.** (Groupoid stabilization theorem) If $P$ is a $G$-Hilbert module, then

$$P \oplus P_G^\infty \cong P_G^\infty.$$

**Proof.** We claim first that

$$P_G^\infty \cong (P \otimes C_0(Y) P_G^\infty) \oplus P_G^\infty.$$ 

For using [5], [8], [9], the non-equivariant stabilization theorem, Proposition [7] and [10],

$$P_G^\infty \cong (C_0(Y) \otimes C_0(Y) P_G^\infty) \cong C_0(Y)^\infty \otimes C_0(Y) P_G^\infty \cong (P \oplus C_0(Y)^\infty \otimes C_0(Y)) P_G^\infty$$

$$\cong (P \otimes C_0(Y) P_G^\infty) \oplus (C_0(Y)^\infty \otimes C_0(Y) P_G^\infty) \cong (P \otimes C_0(Y) P_G^\infty) \oplus P_G^\infty.$$
Using (10) and (9),

\[ P \oplus P_G^\infty \cong P \oplus (P \otimes C_0(Y) P_G^\infty) \oplus P_G^\infty = [P \oplus (P \otimes C_0(Y) P_G^\infty)] \oplus P_G^\infty \cong (P \otimes C_0(Y) P_G^\infty) \oplus P_G^\infty \cong P_G^\infty. \]

\[ \square \]

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