INTRINSIC $L_p$ METRICS FOR CONVEX BODIES
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Abstract. Intrinsic $L_p$ metrics are defined and shown to satisfy a dimension–free bound with respect to the Hausdorff metric.
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1 Introduction

$L_p$ metrics for convex bodies were originally introduced in the context of approximation questions ([23]; also [1], [4], [5], [22]) and later shown to be comparable to the Hausdorff metric:

$$\delta_p(K, L) \geq c_{p,d,K,L} \cdot \delta^{(p+d-1)/p}(K, L)$$  \hspace{1cm} (1)

([30, Corollary 1]; also, [19, Lemma 1]). Bounds like (1) have been useful for establishing stability results and others ([2], [3], [6]–[21], [25]–[27], [29]).

The dimension $d$ of the underlying space appears not only in the form of (1) but also in the definition of $\delta_p$ itself. A natural question to ask is whether these dependencies can be avoided. Analogous to the renormalization of quermassintegrals to intrinsic volumes ([24, 28]), this amounts to asking whether there are dimension–free, or intrinsic, versions of the $L_p$ metrics and of (1). A positive answer was given in [31]. In this note, we give an improved version of that result and some related comments.

Let $K, L$ be convex bodies in $\mathbb{R}^d$, and let $Z^{(d)} = (Z_1, Z_2, \ldots, Z_d)$ be a vector of independent, standard Gaussian variables. For $1 \leq p < \infty$, the intrinsic $L_p$ metric is given by

$$\delta_p^*(K, L) = \left[c_p E|h_K(Z^{(d)}) - h_L(Z^{(d)})|^p\right]^{1/p},$$  \hspace{1cm} (2)

where $c_p = 1/E|Z|^p = \pi^{1/2} \left[2^{2p/2} \Gamma((p+1)/2)\right]^{-1}$ is chosen so that $\delta_p^*(\{x\}, \{\tilde{x}\}) = \|x - \tilde{x}\|$ for any $x, \tilde{x} \in \mathbb{R}^d$ (cf. [31, Eqn. 22]). This coincides with the usual $L_p$ metric up to a multiplicative constant (which depends on both $p$ and $d$). We begin with an explicit proof of the following:

**Theorem 1** \( \delta_p^* \) is intrinsic, \( 1 \leq p < \infty \).
Proof Suppose that $K, L$ lie in a proper subspace of $\mathbb{R}^d$: without loss of generality, $K, L \subset \text{span}\{(x_1, x_2, \ldots, x_{d-\tilde{d}}, 0, 0, \ldots, 0)\}$. Let $\sigma: \mathbb{R}^d \to \mathbb{R}^{d-\tilde{d}}$ be the associated projection operator. For any $x \in \mathbb{R}^d$, one has $h_K(x) = h_{\sigma K}(x) = h_K(\sigma x)$, and the explicit form of $E|h_K(Z^{(d)}) - h_L(Z^{(d)})|^p$ gives

$$
\int_{z_1=-\infty}^{\infty} \cdots \int_{z_{d-\tilde{d}}=-\infty}^{\infty} |h_K(z) - h_L(z)|^p (2\pi)^{-d/2} \prod_{i=1}^{d-\tilde{d}} e^{-z_i^2/2} dz_1 \cdots dz_{d-\tilde{d}}
$$

$$
= \int_{z_1=-\infty}^{\infty} \cdots \int_{z_{d-\tilde{d}}=-\infty}^{\infty} |h_K(\sigma z) - h_L(\sigma z)|^p (2\pi)^{-\tilde{d}/2} \prod_{i=1}^{\tilde{d}} e^{-z_i^2/2} dz_1 \cdots dz_{\tilde{d}}
$$

$$
= E|h_{\sigma K}(Z^{(\tilde{d})}) - h_{\sigma L}(Z^{(\tilde{d})})|^p,
$$

so that $\delta_p^*(K, L) = \delta_p^*(\sigma K, \sigma L)$.

2 An Intrinsic Bound

We now give an intrinsic form of (1).

**Theorem 2** For $p \geq 1$ and finite dimensional convex bodies $K, L$:

$$
\delta_p^*(K, L) \geq (1/4)\delta(K, L)e^{-\frac{1}{2\pi} \left(\frac{V_1(\text{conv}(K \cup L))}{\delta(K, L)}\right)^2}
$$

(3)

**Proof** For notational convenience, let $\delta = \delta(K, L)$ and $V_1 = V_1(\text{conv}(K \cup L))$. Referring to [31, Theorem 1], let $M$ be the unique, positive solution to

$$
E(M - \delta Z)_+ = \frac{1}{\sqrt{2\pi}} V_1,
$$

(4)

which also satisfies $E|h_K(Z^{(d)}) - h_L(Z^{(d)})|^p \geq E[(\delta Z - M)_+]^p$. From (2), it follows that

$$
\delta_p^*(K, L) \geq (c_p E[(\delta Z - M)_+]^p)^{1/p}.
$$
Now
\[
E \left[ (\delta Z - M)_+ \right]^p = \delta^p E \left[ (Z - M/\delta)_+ \right]^p \\
= \delta^p \int_0^\infty \left( z - M/\delta \right)^p \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
= \delta^p \int_0^\infty \left( y - y^2 - (M/\delta)^2 \right) dy \\
\geq \delta^p e^{-2(M/\delta)^2} \int_0^\infty \left( y - y^2 \right) dy \\
\geq \delta^p e^{-2(M/\delta)^2} \frac{2}{2c_p}. 
\]

Therefore,
\[
\delta^*_p(K, L) \geq \left[ c_p \delta^p e^{-2(M/\delta)^2} \frac{2}{2c_p} \right]^{1/p} \geq (1/4)\delta e^{-2(M/\delta)^2}. 
\]

From (4), one has \( M = E(M - \delta Z) \leq E(M - \delta Z)_+ = \frac{1}{\sqrt{2\pi}} V_1 \), which can then be substituted into (5). \( \square \)

**Remarks**

1. The interested reader may want to compare (3) with [31, eqn. 24].
2. Theorem 3 has an equivalent formulation for Gaussian processes. Suppose that \( K, L \) are convex bodies in Hilbert space and that \( \{X_t\}_{t \in K}, \{X_t\}_{t \in L} \) are corresponding isonormally indexed, mean–zero, bounded Gaussian processes. Then
\[
E\left| \sup_{t \in K} X_t - \sup_{t \in L} X_t \right| \geq (1/4)\delta e^{-B^2/\delta^2},
\]
where \( \delta \) is the Hausdorff distance between \( K \) and \( L \), and \( B = E\max\{\sup_{t \in K} X_t, \sup_{t \in L} X_t\} \).
3. It is possible to extend Theorem 2 to all so-called GB convex bodies in Hilbert space. In this case, \( Z^{(d)} \) is replaced in (3) by \( Z^{(\infty)} = (Z_1, Z_2, \ldots) \), an infinite sequence of independent standard Gaussian variables. One can
ask then for the metric space completion of the class of convex bodies in Hilbert space under $\delta_p^*$. Unfortunately this turns out to have limited geometric significance. This can be seen using some facts from Gaussian processes: let $\{e_n\}_n$ be an orthonormal basis and $a_n = (\log(n+1))^{-1/2}$. Define $K_N = \text{conv}\{a_n e_n\}_1^N$. For any $p$, this is a Cauchy sequence, and the limit (in the completion) can be identified with $\text{conv}\{a_n e_n\}_1^\infty$. Now let $\tilde{K}_N = \text{conv}\{a_n e_n\}_1^\infty$. Each of these is also in the completion. Moreover, for any $p$, they form a Cauchy sequence whose limit is almost surely a (strictly) positive constant. But this cannot be a supremum $h_K(Z^{(\infty)})$ of Gaussian random variables for any $K$. Thus the completion goes beyond the natural geometric setting. An alternate approach is given in [32].

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