CALOGERO-MOSER SPACES OVER ALGEBRAIC CURVES

YURI BEREST

Abstract. In these notes, we give a survey of the main results of [BC] and [BW]. Our aim is to generalize the geometric classification of (one-sided) ideals of the first Weyl algebra $A_1(C)$ (see [BW1, BW2]) to the ring $D(X)$ of differential operators on an arbitrary complex smooth affine curve $X$. We approach this problem in two steps: first, we classify the ideals of $D(X)$ up to stable isomorphism, in terms of the Picard group of $X$; then, we refine this classification by describing each stable isomorphism class as a disjoint union of (certain quotients of) generalized Calogero-Moser spaces $C_n(X, I)$. The latter are defined as representation varieties of deformed preprojective algebras over a one-point extension of the ring of regular functions on $X$ by the line bundle $I$. As in the classical case, $C_n(X, I)$ turn out to be smooth irreducible varieties of dimension $2n$.

1. Introduction

In recent years, there have been a number of interesting proposals in the area of smooth noncommutative algebraic geometry (see [CQ], [KR], [G], [LeH], [vdB], [CEG]). The algebras studied in this area are called quasi-free or formally smooth as they appear to be ‘smooth’ objects in the category of associative algebras (in the same sense as the rings of functions on nonsingular affine varieties in the category of commutative algebras). Over the complex numbers, quasi-free algebras can be characterized cohomologically as the ones having dimension $\leq 1$ with respect to Hochschild cohomology. This characterization shows that ‘quasi-freeness’ is a very restrictive property. Apart from semi-simple algebras, there are basically two sources of examples: the path algebras of quivers and the (commutative) rings of functions on smooth affine curves $^1$. Most developments in the area follow a familiar pattern of noncommutative geometry: translating geometric concepts and intuition into algebraic language and extending these to arbitrary quasi-free algebras. In this way, one gets noncommutative analogues of various results of smooth algebraic geometry.

On the other hand, quivers bring in a rich source of ideas and constructions originating from representation theory of finite-dimensional algebras. Reversing the logic (and alienating, perhaps, some classically educated geometers), one may try to apply these to commutative quasi-free algebras, i. e. to the ordinary curves viewed as objects of noncommutative geometry. It is this last approach that we adopt to define the Calogero-Moser spaces.

---

^1There are also a few natural constructions, which can be used to produce new quasi-free algebras from the old ones. For example, the class of quasi-free algebras is closed under products and coproducts in the category of associative algebras as well as (universal) localizations, see [CQ].
One construction, which plays a fundamental role in noncommutative geometry, is that of a representation variety of an algebra: it generalizes the variety of representations of a quiver. Another is that of a deformed preprojective algebra \([C]\): it generalizes the classical preprojective algebras associated to graphs (see [GP]). The third, perhaps less known, is a one-point extension of an algebra: this abstracts the idea of ‘framing’ a quiver (by adding to it a distinguished new vertex \(\infty\) and arrows from \(\infty\), see, e.g., [R]). These three constructions are key ingredients of our definition of Calogero-Moser spaces, and we will review them in some detail in Sections 3.

In Section 2, after some preliminaries, we explain our classification of ideals of \(\mathcal{D}(X)\) up to isomorphism in \(K_0(\mathcal{D})\), and relate this to an earlier work of Cannings and Holland [CH]. The main results of this section (Theorem 1 and Proposition 1) are proved in [BW].

In Section 3, we present our definition of the Calogero-Moser spaces \(C_n(X, \mathcal{I})\) for an arbitrary curve \(X\) and a line bundle \(\mathcal{I}\) over \(X\). We begin with the simplest example: \(X = \mathbb{A}^1\), in which case we observe that \(C_n(X, \mathcal{I})\) coincide with the classical Calogero-Moser spaces \(C_n\) (as defined in [W]). Apart from [BC1], this observation was the starting point for our work. The main result of this section (Theorem 4) is a generalization of a well-known theorem of Wilson [W] on irreducibility of the Calogero-Moser spaces. We note that the spaces \(C_n(X, \mathcal{I})\) behave functorially with respect to \(\mathcal{I}\), so the quotients \(\overline{C}_n(X, \mathcal{I}) := C_n(X, \mathcal{I})/\text{Aut}_X(\mathcal{I})\) depend only on the class of \(\mathcal{I}\) in Pic(\(X\)). We conclude Section 3 with an explicit description of \(\overline{C}_n(X, \mathcal{I})\), in terms of matrices satisfying a ‘rank one’ condition, and illustrate our theory with a broad class of examples, including a general plane curve.

Finally, in Section 4, we construct a natural action of the Picard group Pic(\(\mathcal{D}\)) on the reduced Calogero-Moser spaces \(\overline{C}_n(X, \mathcal{I})\) and state our main result (Theorem 5). This theorem provides a classification of left ideals of \(\mathcal{D}\) in terms of \(\overline{C}_n(X, \mathcal{I})\), which is, like in the Weyl algebra case, equivariant under the action of Pic(\(\mathcal{D}\)). The classifying map \(\omega\) from \(\overline{C}_n(X, \mathcal{I})\) to the space of ideals \(\mathcal{I}(\mathcal{D})\) is induced by a certain functor from the representation category of a deformed preprojective algebra to the category of \(\mathcal{D}\)-modules; in the special case when \(X = \mathbb{A}^1\), this map agrees with the Calogero-Moser map constructed in [BW1, BW2].

There still remain many questions. First of all, in the existing literature, there are (at least) two other definitions of Calogero-Moser spaces associated to curves. The first one, due to Etingof (see [E], Example 2.19), is given in terms of generalized Cherednik algebras (in the style of [EG]). The second, due to Ginzburg, employs the classical Hamiltonian reduction (see [BN], Definition 1.2). It is more or less clear that all three definitions should agree with each other, but it is not clear whether there exist canonical isomorphisms between them.

Next, there is an alternative description of torsion-free \(\mathcal{D}\)-modules on curves, using a noncommutative version of Beilinson’s resolution (see [BN]). Despite the fact that one of the starting points for [BN] was to extend [BC1] to general curves (which was also the starting point for the present work), the precise relation between the two approaches is not clear to us at the moment. It seems that the methods of [BN] are suitable for projective curves, while in the affine case, lead to a much more complicated classification of ideals than in [BC] (for example, no explicit map similar to our \(\omega\) appears in this classification). Comparing the two approaches is still an interesting problem, which we plan to discuss elsewhere.
We should also mention some generalizations. Many results of [BC] can be extended to an arbitrary (formally) smooth algebra, so it is natural to ask whether there is a general principle in noncommutative geometry behind our approach. On the other hand, it might be interesting to understand the results of [BW] and [BC] in purely geometric terms, with a view of extending them to complete and analytic curves.

In the end, I would like to thank W. Crawley-Boevey, P. Etingof, V. Ginzburg, I. Gordon, R. Rouquier, G. Segal, and especially my coauthors O. Chalykh and G. Wilson for interesting questions and comments. This paper evolved from notes of my talk at the conference on Cherednik algebras in June 2007. I would like to thank the organizers of this conference, in particular Iain Gordon, for inviting me to Edinburgh and giving an opportunity to speak. This work was partially supported by NSF grant DMS-0407502 and a LMS grant for visiting scholars.

**Notation.** Throughout this paper, $X$ will denote a smooth affine irreducible curve over $\mathbb{C}$, $\mathcal{O} = \mathcal{O}(X)$ the ring of regular functions on $X$, and $\mathcal{D} = \mathcal{D}(X)$ the ring of global (algebraic) differential operators on $X$. Unless otherwise specified, a module over a ring $R$ means a left module over $R$, and $\mathsf{Mod}(R)$ stands for the category of such modules.

## 2. RINGS OF DIFFERENTIAL OPERATORS ON CURVES

In this section, we state our first result (Theorem 1), which gives a $K$-theoretic classification of ideals of $\mathcal{D}$.

### 2.1. Basic properties.

Recall that $\mathcal{D}$ is a filtered algebra $\mathcal{D} = \bigcup_{k \geq 0} \mathcal{D}_k$, with filtration components $0 \subseteq \mathcal{D}_0 \subseteq \ldots \subseteq \mathcal{D}_{k-1} \subseteq \mathcal{D}_k \subseteq \ldots$ defined inductively by

$$\mathcal{D}_k := \{ D \in \text{End}_\mathbb{C} \mathcal{O} : [D, f] \in \mathcal{D}_{k-1} \text{ for all } f \in \mathcal{O} \}.$$  

The elements of $\mathcal{D}_k$ are called differential operators of order $\leq k$. In particular, the differential operators of order 0 are multiplication operators by regular functions on $X$, i.e. $\mathcal{D}_0 = \mathcal{O}$, while the differential operators of order $\leq 1$ are linear combinations of functions and (algebraic) vector fields on $X$, i.e. $\mathcal{D}_1$ is spanned by $\mathcal{O}$ and the space $\text{Der}(\mathcal{O})$ of derivations of $\mathcal{O}$. When $X$ is smooth and irreducible (as we always assume in this paper), $\mathcal{O}$ and $\text{Der}(\mathcal{O})$ generate $\mathcal{D}$ as an algebra, and $\mathcal{D}$ shares many properties with the first Weyl algebra $A_1(\mathbb{C})$, of which it is a generalization: $A_1(\mathbb{C}) \cong \mathcal{D}(\mathbb{C}^1)$. Thus, like $A_1(\mathbb{C})$, $\mathcal{D}$ is a simple Noetherian domain of homological dimension 1 (see [B], Ch. 2). However, unlike $A_1(\mathbb{C})$, $\mathcal{D}$ has a nontrivial $K$-group.

We write $\overline{\mathcal{D}} := \bigoplus_{k=0}^{\infty} \mathcal{D}_k/\mathcal{D}_{k-1}$ for the associated graded ring of $\mathcal{D}$, which is a commutative algebra isomorphic to the ring of regular functions on the cotangent bundle $T^*X$ of $X$. If $M$ is a $\mathcal{D}$-module equipped with a $\mathcal{D}$-module filtration $\{M_k\}$, we also write $\overline{M} := \bigoplus_{k=0}^{\infty} M_k/M_{k-1}$ for the associated graded $\overline{\mathcal{D}}$-module. Using the standard terminology, we say that $\{M_k\}$ is good if $\overline{M}$ is finitely generated.

### 2.2. Stable classification of ideals.

Let $K_0(X)$ and $\text{Pic}(X)$ denote the Grothendieck group and the Picard group of $X$ respectively. By definition, $K_0(X)$ is generated by the stable isomorphism classes of (algebraic) vector bundles on $X$, while the elements of $\text{Pic}(X)$ are the isomorphism classes of line bundles. As $X$ is affine, we may identify $K_0(X)$ with $K_0(\mathcal{O})$, the Grothendieck group of the ring $\mathcal{O}$, and $\text{Pic}(X)$ with $\text{Pic}(\mathcal{O})$, the ideal class group of $\mathcal{O}$. There are two natural maps $\text{rk} : K_0(X) \to \mathbb{Z}$ and $\text{det} : K_0(X) \to \text{Pic}(X)$ defined by taking the rank and the
Theorem 1. The main result of this section.

Assume that $\mathcal{O}$ is equipped with a good filtration. Assume that $\mathcal{O}$ is of smooth curves, $\text{rk} \oplus \text{det} : K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \text{Pic}(\mathcal{O})$ is a group isomorphism.

Now, let $\mathcal{I}(\mathcal{D})$ denote the set of isomorphism classes of (nonzero) left ideals of $\mathcal{D}$. Unlike Pic in the commutative case, $\mathcal{I}(\mathcal{D})$ carries no natural structure of a group. However, since $\mathcal{D}$ is a hereditary domain, $\mathcal{I}(\mathcal{D})$ can be identified with the space of isomorphism classes of rank 1 projective modules, and there is a natural map relating $\mathcal{I}(\mathcal{D})$ to $\text{Pic}(\mathcal{O})$:

\[ \gamma : \mathcal{I}(\mathcal{D}) \xrightarrow{\text{can}} K_0(\mathcal{D}) \to K_0(\mathcal{O}) \xrightarrow{\text{det}} \text{Pic}(\mathcal{O}). \]

Here, $K_0(\mathcal{D})$ denotes the Grothendieck group of the ring $\mathcal{D}$, 'can' is the canonical map assigning to the isomorphism class of an ideal of $\mathcal{D}$ its stable isomorphism class in $K_0(\mathcal{D})$, and $i_\ast : K_0(\mathcal{O}) \cong K_0(\mathcal{D})$ induced by the natural inclusion $i : \mathcal{O} \hookrightarrow \mathcal{D}$ (see [Q], Theorem 7).

The role of the map $\gamma$ becomes clear from the following theorem, which is the main result of this section.

**Theorem 1** ([BW], Proposition 2.1). Let $M$ be a projective $\mathcal{D}$-module of rank 1 equipped with a good filtration. Assume that $\overline{M}$ is torsion-free. Then

(a) there is a unique (up to isomorphism) ideal $\mathcal{I}_M \subseteq \mathcal{O}$, such that $\overline{\mathcal{M}}$ is isomorphic to a sub-$\mathcal{D}$-module of $\overline{\mathcal{D}\mathcal{I}_M}$ of finite codimension (over $\mathbb{C}$);

(b) the class of $\mathcal{I}_M$ in $\text{Pic}(\mathcal{O})$ and the codimension $n := \dim_{\mathbb{C}}(\overline{\mathcal{D}\mathcal{I}_M}/\overline{\mathcal{M}})$ are independent of the choice of filtration on $M$, and we have $\gamma[M] = [\mathcal{I}_M]$;

(c) if $M$ and $N$ are two projective $\mathcal{D}$-modules of rank 1, then

\[ [M] = [N] \quad \text{in} \quad K_0(\mathcal{D}) \iff [\mathcal{I}_M] = [\mathcal{I}_N] \quad \text{in} \quad \text{Pic}(\mathcal{O}). \]

It is easy to see that $\gamma[\mathcal{D}\mathcal{I}] = [\mathcal{I}]$ for any nonzero ideal $\mathcal{I} \subseteq \mathcal{O}$. Thus, by Theorem 1 the map $\gamma$ is a fibration over $\text{Pic}(\mathcal{O})$, with fibres being precisely the stable isomorphism classes of ideals of $\mathcal{D}$. The stably free ideals $M$ are characterized by the property that $\overline{\mathcal{M}}$ is isomorphic to an ideal in $\overline{\mathcal{D}}$ of finite codimension.

2.3. The Cannings-Holland correspondence. By a theory of Cannings and Holland (see [CH]), the ideals of $\mathcal{D}$ can be parametrized by primary decomposable subspaces of $\mathcal{O}$. We now describe the map (1) in terms of these subspaces.

First, we recall that a linear subspace $V \subseteq \mathcal{O}$ is called primary if it contains a power of the maximal ideal $m_x$ corresponding to a point $x \in X$ (we write $V = V_x$ in this case). Further, $V \subseteq \mathcal{O}(X)$ is called primary decomposable if it is an intersection of primary subspaces $V_x$, with $V_x = \mathcal{O}(\{x\})$ for almost all $x \in X$. By [CH], Theorem 2.4, the primary decomposition of $V$ is uniquely determined: in fact, we have $V_x = \cap_{i=1}^n (V + m_x^i)$ for all $x \in X$, and moreover, $\dim_{\mathbb{C}}\mathcal{O}/V = \sum_{x \in X} \dim_{\mathbb{C}}\mathcal{O}/V_x$.

Now, let $M$ be a nonzero left ideal of $\mathcal{D}$. Then, by [CH], Theorem 4.12, there is a unique (up to rational equivalence) primary decomposable subspace $V \subseteq \mathcal{O}$, such that $M \cong \mathcal{D}(V, \mathcal{O})$, where $\mathcal{D}(V, \mathcal{O})$ is the fractional ideal of $\mathcal{D}$ consisting of all differential operators with rational coefficients on $X$ mapping $V$ into $\mathcal{O}$. We write $V_x$ for the primary components of $V$, and $m_x \subset \mathcal{O}$ for the associated primes.

**Proposition 1** ([BW], Theorem 5.2). The map $\gamma$ sends the class of $M$ to the class of the ideal $\prod_{x \in X} m_x^{d_x}$ in $\text{Pic}(\mathcal{O})$, where $d_x := \dim_{\mathbb{C}}\mathcal{O}/V_x$. 

\[ \text{This assumption is very restrictive: if we identify } M \text{ with an ideal in } \mathcal{D}, \text{ the given filtration on } M \text{ coincides, up to a shift, with the induced filtration } \{M \cap D_k\} \text{ (see [BC], Lemma 5.12).} \]
Let $Gr^{ad}(X)$ be the deltic Grassmannian of $X$, i.e. the set of equivalence classes of primary decomposable subspaces of $O(X)$. There is a well-defined map from $Gr^{ad}(X)$ to the divisor class group of $X$: it takes the class of a subspace $V = \bigcap_{x \in X} V_x$ in $Gr^{ad}(X)$ to the class of the (Weil) divisor $d := \sum_{x \in X} d_x \cdot x$ in $\text{Div}(X)$. On the other hand, by the Cannings-Holland Theorem, we have the bijection: $Gr^{ad}(X) \sim \rightarrow I(D) , [V] \mapsto [D(V,O)]$, and, as $X$ is smooth, the natural isomorphism: $\text{Div}(X) \sim \rightarrow \text{Pic}(X) , [d] \mapsto [O_X(d)]$. In this way, we get the diagram

$$
\begin{array}{ccc}
Gr^{ad}(X) & \rightarrow & \text{Div}(X) \\
\cong & & \cong \\
\gamma & \sim & \text{Pic}(X)
\end{array}
$$

Now, Proposition \[\|\] immediately implies that (2) is commutative. This gives an alternative description of our map $\gamma$ in terms of primary decomposable subspaces:

$$
Gr^{ad}(X) \rightarrow \text{Div}(X) , \quad [V] \mapsto \sum d_x \cdot x ,
$$

where $d_x$ are codimensions of the primary components of $V$.

3. The Calogero-Moser Spaces

Theorem \[\|\] shows that the ideals of $D(X)$ are classified, up to stable isomorphism, by the elements of $\text{Pic}(X)$. Our goal now is to refine this classification by describing the fibres of the classifying map $\gamma : I(D) \rightarrow \text{Pic}(X)$. As we will see, each fibre of $\gamma$ breaks up into a countable union of the quotient spaces $\overline{C}_n(X,I) = C_n(X,I)/\text{Aut}_X(I)$ of smooth affine varieties $C_n(X,I)$. The varieties $C_n(X,I)$ will be introduced as representation varieties of deformed preprojective algebras over the one-point extension of the ring of regular functions on $X$ by the line bundle $I$. In the special case when $X$ is the affine line, $C_n(X,I)$ coincide with the ordinary Calogero-Moser spaces $\mathbb{W}$, and our classification of ideals of $D(X)$ agrees with the one given in $\mathbb{W}$, $\mathbb{W}$, and $\mathbb{W}$.

We begin by reviewing the basic ingredients of our construction.

3.1. Representation varieties. First, we recall the definition of representation varieties in the form they appear in representation theory of associative algebras (see, e.g., $\mathbb{K}$, Chap. II, Sect. 2.7).

Let $R$ be a finitely generated associative $C$-algebra, $S$, a finite-dimensional semisimple subalgebra of $R$, and $V$, a finite-dimensional $S$-module. By definition, the representation variety $\text{Rep}_S(R,V)$ of $R$ over $S$ parametrizes all $R$-module structures on the vector space $V$ extending the given $S$-module structure on it. The $S$-module structure on $V$ determines an algebra homomorphism $S \rightarrow \text{End}(V)$ making $\text{End}(V)$ an $S$-algebra. The points of $\text{Rep}_S(R,V)$ can thus be interpreted as $S$-algebra maps $R \rightarrow \text{End}(V)$.

If $S = C$, we simply write $\text{Rep}(R,V)$ for $\text{Rep}_C(R,V)$. Choosing a basis in $V$ and a presentation of $R$, say $R \cong \mathbb{C}\langle x_1, \ldots, x_m \rangle/I$, we can identify in this case

$$
\text{Rep}(R,V) \cong \{(X_1, \ldots, X_m) \in \text{Mat}(n, \mathbb{C})^m : r(X_1, \ldots, X_m) = 0 , \forall r \in I\} .
$$
Thus $\text{Rep}(R, V)$ is an affine variety\footnote{Here, by an affine variety we mean an affine scheme of finite type over $\mathbb{C}$.}. In general, for any semi-simple $S \subseteq R$, $\text{Rep}_S(R, V)$ can be identified with a fibre of the canonical morphism of affine varieties $\pi : \text{Rep}(R, V) \to \text{Rep}(S, V)$, and hence it is an affine variety as well.

The group $\text{Aut}_S(V)$ of $S$-linear automorphisms of $V$ acts on $\text{Rep}_S(R, V)$ in the natural way, with scalars $\mathbb{C}^\times \subseteq \text{Aut}_S(V)$ acting trivially. We set $\text{GL}_S(V) := \text{Aut}_S(V)/\mathbb{C}^\times$. Since $V$ is semi-simple, $V \cong \bigoplus_i V_i^{\otimes n_i}$, with $V_i$ non-isomorphic simple $S$-modules, and $\text{Aut}_S(V) \cong \prod_i \text{GL}(n_i, \mathbb{C})$. Thus $\text{GL}_S(V)$ is reductive.

The orbits of $\text{GL}_S(V)$ on $\text{Rep}_S(R, V)$ are in 1-1 correspondence with isomorphism classes of $S$-modules, which are isomorphic to $V$ as $S$-modules. The stabilizer of a point $\rho : R \to \text{End}(V)$ in $\text{Rep}_S(R, V)$ is canonically isomorphic to $\text{Aut}_R(V_\rho)/\mathbb{C}^\times \subseteq \text{GL}_S(V)$, where $V_\rho$ is the left $R$-module corresponding to $\rho$.

Now, one can show that the closure of any orbit $O_M$ contains a unique closed orbit, corresponding to a semi-simple $R$-module with the same composition factors and multiplicities as $M$. Thus the space $\text{Rep}_S(R, V)//\text{GL}_S(V)$ of closed orbits in $\text{Rep}_S(R, V)$ is an affine variety, whose (closed) points are in bijection with isomorphism classes of semi-simple $R$-modules $M$ isomorphic to $V$ as $S$-modules.

Typically, the representation varieties of $R$ are defined over subalgebras spanned by idempotents. For example, let $\{e_i\}_{i \in I}$ be a complete set of orthogonal idempotents in $R$. Set $S := \bigoplus_{i \in I} e_i \subseteq R$. A finite-dimensional $S$-module is then isomorphic to a direct sum $\mathbb{C}^n := \bigoplus_{i \in I} \mathbb{C}^n_i$, each $e_i$ acting as the projection onto the $i$-th component. The corresponding representation variety $\text{Rep}_S(R, \mathbb{C}^n)$, which we denote simply $\text{Rep}_S(R, n)$ in this case, consists of all algebra maps $R \to \text{End}(\mathbb{C}^n)$, sending $e_i$ to the projection onto $\mathbb{C}^n_i$. The group $\text{GL}_S(\mathbb{C}^n)$ (to be denoted $\text{GL}_S(n)$) is isomorphic to $\prod_{i \in I} \text{GL}(n_i, \mathbb{C})/\mathbb{C}^\times$, with $\mathbb{C}^\times$ embedded diagonally.

### 3.2. Deformed preprojective algebras

If $A$ is an associative algebra, its tensor square $A^{\otimes 2}$ over $\mathbb{C}$ has two commuting bimodule structures: one is defined by $a.(x \otimes y).b = ax \otimes yb$ and the other by $a.(x \otimes y).b = xb \otimes ay$. We will refer to the first structure as outer and to the second as inner.

The space $\text{Der}(A, A^{\otimes 2})$ of linear derivations $A \to A^{\otimes 2}$ taken with respect to the outer bimodule structure on $A^{\otimes 2}$ is naturally a bimodule with respect to the inner structure; thus, we can form the tensor algebra $T_A\text{Der}(A, A^{\otimes 2})$. If $A$ is unital, there is a canonical element in $\text{Der}(A, A^{\otimes 2})$: namely the derivation $\Delta = \Delta_A : A \to A^{\otimes 2}$, sending $x \in A$ to $(x \otimes 1 - 1 \otimes x) \in A^{\otimes 2}$. For any $\lambda \in A$, we can form then the two-sided ideal $\langle \Delta - \lambda \rangle$ in $T_A\text{Der}(A, A^{\otimes 2})$ and, following $\mathbb{C}$, define the quotient algebra

\[
\Pi^\lambda(A) := T_A\text{Der}(A, A^{\otimes 2})/\langle \Delta - \lambda \rangle .
\]

It turns out that, up to isomorphism, $\Pi^\lambda(A)$ depends only on the class of $\lambda$ in the Hochschild homology $\text{HH}_0(A) := A/[A, A]$ (see $\mathbb{C}$, Lemma 1.2). Moreover, instead of elements of $\text{HH}_0(A)$, it is convenient to parametrize the algebras $\mathbb{C}$ by the elements of $\mathbb{C} \otimes_\mathbb{Z} K_0(A)$, relating this last vector space to $\text{HH}_0(A)$ via a Chern character map. To be precise, let $\text{Tr}_A : K_0(A) \to \text{HH}_0(A)$ be the map, sending the class of a projective module $P$ to the class of the trace of any idempotent matrix $e \in \text{Mat}(n, A)$, satisfying $P \cong eA^{\otimes n}$. By additivity, this extends to a linear map $\mathbb{C} \otimes_\mathbb{Z} K_0(A) \to \text{HH}_0(A)$ to be denoted also $\text{Tr}_A$. Following $\mathbb{C}$, we call the elements of $\mathbb{C} \otimes_\mathbb{Z} K_0(A)$ weights and define the deformed preprojective algebra of weight
Example 1 (see [B], Lemma 4.1 and Theorem 0.3). If \( A = \mathcal{O}(X) \) is the coordinate ring of a smooth affine curve \( X \), then \( A \) is quasi-free. In this case, \( \Pi^0(A) \) is isomorphic to the coordinate ring \( \mathcal{O}(T^*X) \) of the cotangent bundle of \( X \), and \( \Pi^1(A) \) to the filtered algebra \( D(X) \) of differential operators.\(^4\)

In general, the \( \Pi^\lambda \)-construction is not functorial in \( A \); however, it does behave well with respect to a class of algebra maps called pseudo-flat\(^5\). To be precise, the pseudo-flat algebra homomorphisms \( \theta : B \to A \) are characterized by the condition: \( \text{Tor}_1^B(I, A) = 0 \), and the functoriality of \( \Pi^\lambda \) is stated as follows.

**Theorem 2** ([B], Theorem 0.7). If \( \theta : B \to A \) is a pseudo-flat ring epimorphism, then, for any \( \lambda \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(B) \), there is a canonical algebra map \( \theta : \Pi^\lambda(B) \to \Pi^{\theta^*(\lambda)}(A) \), where \( \theta^* : \mathbb{C} \otimes_{\mathbb{Z}} K_0(B) \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(A) \) is a linear map induced functorially by \( \theta \). If \( B \) is quasi-free and finitely generated, then \( \theta \) is also a pseudo-flat ring epimorphism.

Finally, the last theorem, which we want to state in this section, provides a simple homological principle for studying representations of \( \Pi^\lambda(A) \). It plays an important role in [BC], underlying several proofs and constructions.

**Theorem 3** ([BC], Theorem 2.2). Let \( A \) be a finitely generated quasi-free algebra, and let \( \varphi : A \to \text{End}(V) \) be a representation of \( A \) on a (not necessarily finite-dimensional) vector space \( V \). Then \( \varphi \) can be extended to a representation of \( \Pi^\lambda(A) \) if and only if the homology class of \( \varphi(\lambda) \) in \( \text{HH}_0(A, \text{End}(V)) \) is zero. If it exists, an extension of \( \varphi \) to \( \Pi^\lambda(A) \) is unique if and only if \( \text{HH}_1(A, \text{End}(V)) = 0 \).

**Remark.** To the best of our knowledge, Theorem 3 has not appeared in the literature in this form and generality. However, in the special case when \( A = \mathbb{C}Q \) is the path algebra of a quiver and \( V \) is a finite-dimensional representation of \( A \), this result is equivalent to [BC], Theorem 3.3 (see [BC], Proposition 2.1).

### 3.3. One-point extensions

If \( A \) is a unital associative algebra, and \( I \) a left module over \( A \), we define the one-point extension of \( A \) by \( I \) to be the ring of triangular matrices

\[
\begin{pmatrix}
A & I \\
0 & C
\end{pmatrix}
\]

\(\lambda \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)\) by

\[
(4) \quad \Pi^\lambda(A) := T_A \text{Der}(A, A^{\otimes 2})/(\Delta - \lambda),
\]

where \( \lambda \in A \) is any lifting of \( \text{Tr}_A(\lambda) \) to \( A \). Note, if \( A \) is commutative, then \( \text{HH}_0(A) = A \), and \( \lambda \) is uniquely determined by \( \text{Tr}_A(\lambda) \).

For basic properties and examples of the algebras \( \Pi^\lambda(A) \), we refer the reader to [C]. Here, we only review one important example and two theorems, which play a role in our construction.
with matrix addition and multiplication induced from the module structure of $I$. Clearly, $A[I]$ is a unital associative algebra, with identity element being the identity matrix. There are two distinguished idempotents in $A[I]$: namely

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

If $A$ is indecomposable (e.g., $A$ is a commutative integral domain), then [[[ ]] form a complete set of primitive orthogonal idempotents in $A[I]$.

A module over $A[I]$ can be identified with a triple $V = (V, V_\infty, \varphi)$, where $V$ is an $A$-module, $V_\infty$ is a $\mathbb{C}$-vector space and $\varphi : I \otimes V_\infty \rightarrow V$ is an $A$-module map. Using the standard matrix notation, we will write the elements of $V$ as column vectors $(v, w)^T$ with $v \in V$ and $w \in V_\infty$; the action of $A[I]$ is then given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a.v + \varphi(b \otimes w) \\ c.w \end{pmatrix}.$$  

If $V$ is finite-dimensional, with $\dim_\mathbb{C} V = n$ and $\dim_\mathbb{C} V_\infty = n_\infty$, we call $n = (n, n_\infty)$ the dimension vector of $V$.

The next lemma collects some basic properties of one-point extensions, which we will need for our construction.

**Lemma 1.** (1) $A[I]$ is canonically isomorphic to the tensor algebra $T_{A \times \mathbb{C}}(I)$.

(2) If $A$ is quasi-free and $I$ is f. g. projective, then $A[I]$ is quasi-free.

(3) $I \mapsto A[I]$ is a functor from $\text{Mod}(A)$ to the category of associative algebras.

(4) The natural projection $\theta : A[I] \rightarrow A$ is a flat ring epimorphism.

(5) There is an isomorphism of abelian groups $K_0(A[I]) \cong K_0(A) \oplus \mathbb{Z}$.

For the proof of Lemma 1 we refer the reader to [BC], Section 2.2.

### 3.4. The definition of Calogero-Moser spaces

We can now put pieces together and introduce our generalization of the Calogero-Moser varieties for an arbitrary smooth affine curve $X$. We begin with the simplest example: $X = \mathbb{A}^1$, which will provide motivation for our general construction.

**Example 2.** If $X = \mathbb{A}^1$, any line bundle $\mathcal{I}$ on $X$ is trivial. Choosing a coordinate on $X$ and a trivialization of $\mathcal{I}$, we identify $\mathcal{O} \cong \mathbb{C}[x]$ and $\mathcal{I} \cong \mathbb{C}[x]$ as an $\mathcal{O}$-module. The one-point extension of $\mathcal{O}$ by $\mathcal{I}$ is then isomorphic to the matrix algebra:

$$\mathcal{O}[\mathcal{I}] \cong \begin{pmatrix} \mathbb{C}[x] & \mathbb{C}[x] \\ 0 & \mathbb{C} \end{pmatrix},$$  

which is, in turn, isomorphic to the path algebra $\mathbb{C}Q$ of the quiver $Q$ consisting of two vertices $\{0, \infty\}$ and two arrows $X : 0 \rightarrow 0$ and $v : \infty \rightarrow 0$. In fact, the map sending the vertices 0 and $\infty$ to the idempotents $e$ and $e_\infty$ in $\mathcal{O}[\mathcal{I}]$, see [[[ ]], and

$$X \mapsto \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$  

extends to an algebra isomorphism $\mathbb{C}Q \cong \mathcal{O}[\mathcal{I}]$.

Now, let $Q$ be the double quiver of $Q$ obtained by adding the reverse arrows $Y := X^*$ and $w := v^*$ to the corresponding arrows of $Q$. Then, for any $\lambda = \lambda_0 + \lambda_\infty e_\infty$, with $(\lambda, \lambda_\infty) \in \mathbb{C}^2$, the algebra $\Pi^\lambda(Q)$ is isomorphic to the quotient of $\mathbb{C}Q$ modulo the relation $[X, Y] + [v, w] = \lambda$ (see [[C]], Theorem 3.1). The ideal generated by this last relation is the same as the ideal generated by the elements $[X, Y] + vw - \lambda e$.
and $w v + \lambda_\infty e_\infty$. Thus, the $\Pi^\lambda(Q)$-modules can be identified with representations $V = V \oplus V_\infty$ of $Q$, in which linear maps $\bar{X}, \bar{Y} \in \text{Hom}(V, V)$, $\bar{v} \in \text{Hom}(V_\infty, V)$, $\bar{w} \in \text{Hom}(V, V_\infty)$, given by the action of $X, Y, v, w$, satisfy

$$[\bar{X}, \bar{Y}] + \bar{v} \bar{w} = \lambda \text{Id}_V \quad \text{and} \quad \bar{w} \bar{v} = -\lambda_\infty \text{Id}_{V_\infty}. \quad (7)$$

Now, taking $\lambda = (1, -n)$, it is easy to see that all representations of $\Pi^\lambda(Q)$ of dimension vector $n = (n, 1)$ are simple, and the corresponding representation varieties coincide (in this special case) with the classical Calogero-Moser spaces $\mathcal{C}_n$.

This coincidence was first noticed by W. Crawley-Boevey (see \cite{C1}, remark on p. 45). For explanations and a detailed discussion of this example in relation to the Weyl algebra we refer the reader to \cite{BCP}.

Now, let $X$ be an arbitrary curve. As in the above example, we fix a line bundle $\mathcal{I}$ on $X$ and set $B := O[\mathcal{I}]$. Note that, by Lemma \cite[3]{BCP}, $B$ depends (up to isomorphism) only on the class of $\mathcal{I}$ in $\text{Pic}(X)$. More precisely, we have

**Lemma 2** (\cite{BCP}, Lemma 3.1). For two line bundles $\mathcal{I}$ and $\mathcal{J}$ on $X$, the algebras $O[\mathcal{I}]$ and $O[\mathcal{J}]$ are

(a) Morita equivalent;

(b) isomorphic if and only if $\mathcal{J} \cong \mathcal{I}^\tau$ for some $\tau \in \text{Aut}(X)$, where $\mathcal{I}^\tau := \tau^* \mathcal{I}$.

To define the deformed preprojective algebras over $B$ we need to compute the Chern character $\text{Tr}_B : K_0(B) \to \text{HH}_0(B)$. We recall that $\text{Tr}_*: K_0 \to \text{HH}_0$ is a natural transformation of functors on the category of associative algebras: corresponding to the algebra map $\theta : B \to O \times \mathbb{C}$, we have thus a commutative diagram

$$
\begin{array}{ccc}
K_0(B) & \xrightarrow{\text{Tr}_B} & \text{HH}_0(B) \\
\downarrow & & \downarrow \text{HH}_0(\text{Tr}_C) \\
K_0(O \times \mathbb{C}) & \xrightarrow{\text{Tr}_{O \times \mathbb{C}}} & \text{HH}_0(O \times \mathbb{C})
\end{array}
$$

The two vertical maps in (8) are isomorphisms: the first one is given by Lemma \cite[15]{BCP}, while the second has the obvious inverse (induced by the embedding $O \times \mathbb{C} \hookrightarrow B$ via diagonal matrices). We will use these isomorphisms to identify $\text{HH}_0(B) \cong \text{HH}_0(O \times \mathbb{C}) \cong O \oplus \mathbb{Z} \subset B$ and

$$K_0(B) \cong K_0(O \times \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \text{Pic}(X), \quad (9)$$

Now, for any commutative algebra (e.g., $O \times \mathbb{C}$), the Chern character map factors through the rank. Hence, with above identifications, $\text{Tr}_B$ is completely determined by its values on the first two summands in (9), while vanishing on the last. Since $\text{Tr}_B[(1, 0)] = e$ and $\text{Tr}_B[(0, 1)] = e_\infty$, the linear map $\text{Tr}_B : \mathbb{C} \otimes \mathbb{K}_0(B) \to \text{HH}_0(B)$ takes its values in the two-dimensional subspace $S$ of $B$ spanned by the idempotents $e$ and $e_\infty$. Identifying $S$ with $\mathbb{C}^2$, we may regard the vectors $\lambda := (\lambda, \lambda_\infty) = \lambda e + \lambda_\infty e_\infty \in S$ as weights for the family of deformed preprojective algebras associated to $B$:

$$\Pi^\lambda(B) = T_B \text{Der}(B, B \otimes \mathbb{C})/\langle \Delta_B - \lambda \rangle. \quad (10)$$

Since $O$ is an integral domain, $\{e, e_\infty\}$ is a complete set of primitive orthogonal idempotents in $\Pi^\lambda(B)$, and $S = \mathbb{C}e \oplus \mathbb{C}e_\infty$ is the associated semi-simple subalgebra of $\Pi^\lambda(B)$.
For each \(\mathbf{n} = (n_1, n_\infty) \in \mathbb{N}^2\), we now form the representation variety \(\text{Rep}_S(\Pi^\lambda(B), \mathbf{n})\) over \(S\) and define

\[
\mathcal{C}_{n,\lambda}(X, \mathcal{I}) := \text{Rep}_S(\Pi^\lambda(B), \mathbf{n})//\text{GL}_S(\mathbf{n}).
\]

As explained in Section 3.1, \(\mathcal{C}_{n,\lambda}(X, \mathcal{I})\) is an affine scheme, whose (closed) points are in bijection with isomorphism classes of semi-simple \(\Pi^\lambda(B)\)-modules of dimension vector \(\mathbf{n}\).

Now, by Lemma (13), every automorphism of \(\mathcal{I}\) induces an \(S\)-algebra automorphism of \(\Pi^\lambda(B)\) and hence an automorphism of the representation variety \(\mathcal{C}_{n,\lambda}(X, \mathcal{I})\). We let \(\mathcal{C}_{n,\lambda}(X, \mathcal{I})\) denote the corresponding quotient space:

\[
\mathcal{C}_{n,\lambda}(X, \mathcal{I}) := \mathcal{C}_{n,\lambda}(X, \mathcal{I})/\text{Aut}_X(\mathcal{I}).
\]

By definition, \(\mathcal{C}_{n,\lambda}(X, \mathcal{I})\) depends only on the class of \(\mathcal{I}\) in \(\text{Pic}(X)\).

More generally, from Lemma (21) and the fact that \(\Pi^\lambda\) behaves naturally under Morita equivalence (see [C], Corollary 5.5), it follows that

\[
\mathcal{C}_{n,\lambda}(X, \mathcal{I}) \cong \mathcal{C}_{n,\lambda}(X, \mathcal{J})
\]

for any line bundles \(\mathcal{I}\) and \(\mathcal{J}\); however, there is no natural choice for such an isomorphism.

Motivated by the above example, we will be interested in representations of \(\Pi^\lambda(B)\) of dimension \(\mathbf{n} = (n_1, 1)\). Using Theorem (3) it is not difficult to prove that such representations may exist only if \(\lambda = 0\) or \(\lambda = (\lambda, -n\lambda)\), with \(\lambda \neq 0\). In this last case, the algebras \(\Pi^\lambda(B)\) are all isomorphic to each other, so without loss of generality we may take \(\lambda = 1\).

**Proposition 2** ([BC], Proposition 3.2). Let \(\lambda = (1, -n)\) and \(\mathbf{n} = (n, 1)\) with \(n \in \mathbb{N}\). Then, for any \(\mathcal{I}\), \(\Pi^\lambda(B)\) has representations of dimension vector \(\mathbf{n}\), and every such representation is simple.

We are now in position to state the main definition and the main theorem of this section.

**Definition.** The variety \(\mathcal{C}_{n,\lambda}(X, \mathcal{I})\) with \(\lambda = (1, -n)\) and \(\mathbf{n} = (n, 1)\) will be denoted \(\mathcal{C}_{n}(X, \mathcal{I})\) and called the \(n\)-th Calogero-Moser space of type \((X, \mathcal{I})\). The corresponding quotient \((12)\) will be denoted \(\mathcal{C}_{n}(X, \mathcal{I})\) and called the \(n\)-th reduced Calogero-Moser space.

In view of Proposition (2) the varieties \(\mathcal{C}_{n}(X, \mathcal{I})\) parametrize the isomorphism classes of simple \(\Pi^\lambda(B)\)-modules of dimension \(\mathbf{n} = (n, 1)\); they are non-empty for any \([\mathcal{I}] \in \text{Pic}(X)\) and \(n \geq 0\). In the special case when \(X\) is the affine line, \(\mathcal{C}_{n}(X, \mathcal{I})\) coincide with the ordinary Calogero-Moser spaces \(\mathcal{C}_{n}\) as defined in [W] (see Example (2)). Now, one of the main results of [W] says that each \(\mathcal{C}_{n}\) is a smooth affine irreducible variety of dimension \(2n\). The following theorem shows that this is true in general.

**Theorem 4** ([BC], Theorem 3.2). For each \(n \geq 0\) and \([\mathcal{I}] \in \text{Pic}(X)\), \(\mathcal{C}_{n}(X, \mathcal{I})\) is a smooth irreducible affine variety of dimension \(2n\).

We close this section by describing generic points of the varieties \(\mathcal{C}_{n}(X, \mathcal{I})\) in geometric terms. First of all, using Theorem (3) it is not difficult to show that any \(\Pi^\lambda(B)\)-module of dimension vector \(\mathbf{n} = (n, 1)\) restricts to an indecomposable \(B\)-module, and conversely, every indecomposable \(B\)-module of dimension vector \(\mathbf{n} = (n, 1)\) extends to a \(\Pi^\lambda(B)\)-module. The generic points of \(\mathcal{C}_{n}(X, \mathcal{I})\) correspond...
under this restriction/extension to the $B$-modules $V$ with $\text{End}_B(V) \cong \mathbb{C}$. Now, as explained in Section 3.3, a $B$-module structure on $V = V \oplus V_\infty$ is determined by an $O$-module homomorphism $\varphi : I \otimes V_\infty \rightarrow V$, and if $\text{dim}_C V_\infty = 1$, it is easy to see that $\text{End}_B(V) \cong C$ is equivalent to $\varphi$ being surjective. Thus, for constructing generic points of $\mathcal{C}_n(X, I)$, it suffices to construct a torsion $O$-module $V$ on $X$ of length $n$ together with a surjective $O$-module map $\varphi : I \rightarrow V$. Geometrically, this can be done as follows.

Identify $I$ with an ideal in $O$ and fix $n$ distinct points $p_1, p_2, \ldots, p_n$ on $X$ outside the zero locus of $I$. Let $V := O/J$, where $J$ is the product of the maximal ideals $m_i \subset O$ corresponding to $p_i$'s. Clearly, $O/J \cong \bigoplus_{i=1}^n O/m_i$ and $\text{dim}_C V = n$. Now, since $O$ is a Dedekind domain and $I \not\subset m_i$ for any $i = 1, 2, \ldots, n$, we have $(O/J) \otimes_O (O/I) \cong \bigoplus_{i=1}^n (O/m_i) \otimes_O (O/I) = 0$ and $\text{Tor}_1(O/J, O/I) \cong (I \cap J)/IJ = 0$, so the canonical map $V \otimes_O I \rightarrow V$ is an isomorphism. On the other hand, as $V$ is a cyclic $O$-module, $I$ surjects naturally onto $V \otimes_O I$. Combining $I \rightarrow V \otimes_O I \rightarrow V$, we get the required homomorphism $\varphi$.

3.5. The structure of $\Pi^\lambda(B)$. One advantage of defining the Calogero-Moser spaces as representation varieties is that they can be described explicitly, like in the classical case, in terms of matrices satisfying the `rank-one condition' (see [15] below). For this, it suffices to find a suitable presentation of the algebras $\Pi^\lambda(B)$ in terms of generators and relations.

Recall that, following [11], we defined these algebras by

$$\Pi^\lambda(B) = T_B \text{Der}(B, B^{\otimes 2})/ (\Delta_B - \lambda) ,$$

where $\Delta_B$ is the distinguished derivation in $\text{Der}(B, B^{\otimes 2})$ mapping $x \mapsto x \otimes 1 - 1 \otimes x$. Now, $\text{Der}(B, B^{\otimes 2})$ contains a canonical sub-bimodule $\text{Der}_S(B, B^{\otimes 2})$, consisting of $S$-linear derivations. We write $\Delta_{B,S} : B \rightarrow B \otimes B$ for the inner derivation $x \mapsto \text{ad}_e(x)$, with $e := e \otimes e + e_\infty \otimes e_\infty \in B \otimes B$. It is easy to see that $\Delta_{B,S}(x) = 0$ for all $x \in S$, so $\Delta_{B,S} \in \text{Der}_S(B, B^{\otimes 2})$. This also follows immediately from the fact that $S$ is a separable algebra, and $e \in S \otimes S$ is the canonical separability element in $S$.

**Lemma 3.** For any $\lambda \in S$, there is a canonical algebra isomorphism

$$\Pi^\lambda(B) \cong T_B \text{Der}_S(B, B^{\otimes 2})/ (\Delta_{B,S} - \lambda) .$$

Thus, the structure of the algebras $\Pi^\lambda(B)$ is determined by the bimodule $\text{Der}_S(B, B^{\otimes 2})$. We now describe this bimodule explicitly, in terms of $O$, $I$ and the dual line bundle $I^* := \text{Hom}_O(I, O)$. To fix notation we begin with a few fairly obvious remarks on bimodules over one-point extensions.

A bimodule $\Lambda$ over $B = O[I]$ is characterized by the following data: an $O$-bimodule $T$, a left $O$-module $U$, a right $O$-module $V$ and a $\mathbb{C}$-vector space $W$ given together with three $O$-module homomorphisms $f_1 : I \otimes V \rightarrow T$, $f_2 : I \otimes W \rightarrow U$, $g_1 : T \otimes_O I \rightarrow U$ and a $\mathbb{C}$-linear map $g_2 : V \otimes_O I \rightarrow W$, which fit into the commutative diagram

$$\begin{array}{cccc}
I \otimes V \otimes_O I & \text{Id} \otimes g_2 & I \otimes W \\
\downarrow f_1 \otimes_O \text{Id} & & & \downarrow f_2 \\
T \otimes_O I & \rightarrow & U\end{array}$$

(13)
These data can be conveniently organized by using the matrix notation
\[ \Lambda = \begin{pmatrix} T & U \\ V & W \end{pmatrix}, \]
with understanding that \( B \) acts on \( \Lambda \) by the usual matrix multiplication, via the maps \( f_1, f_2, g_1 \) and \( g_2 \). Note that the commutativity of (13) ensures the associativity of the action of \( B \).

With this notation, the bimodule \( \text{Der}_S(B, B^{\otimes 2}) \) can be described as follows.

**Lemma 4.** There is an isomorphism of \( B \)-bimodules
\[ \text{Der}_S(B, B^{\otimes 2}) \cong \left( \begin{array}{c} \text{Der}(\mathcal{O}, \mathcal{O}^{\otimes 2}) & \text{Der}(\mathcal{O}, \mathcal{I} \otimes \mathcal{O}) \\ 0 & 0 \end{array} \right) \oplus \left( \begin{array}{cc} \mathcal{I} \otimes \mathcal{I}^* & \mathcal{I} \otimes \mathcal{O} \\ \mathcal{I}^* & \mathcal{O} \end{array} \right), \]
with \( \Delta_{B,S} \) corresponding to the element
\[ \left[ \begin{array}{cc} \Delta & 0 \\ 0 & 0 \end{array} \right], \left[ -\sum_i v_i \otimes w_i & 0 \\ 0 & 1 \end{array} \right], \]
where \( (v_i, w_i) \) is a pair of dual bases\(^6\) for the line bundles \( \mathcal{I} \) and \( \mathcal{I}^* \).

Now, as a consequence of Lemma 3 and Lemma 4, we get

**Proposition 3** ([BC], Proposition 5.1). The algebra \( \Pi^\Lambda(B) \) is generated by (the images of) the following elements
\[ \hat{a} := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \hat{v}_i := \begin{pmatrix} 0 & v_i \\ 0 & 0 \end{pmatrix}, \hat{d} := \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}, \hat{w}_i := \begin{pmatrix} 0 & 0 \\ 0 & w_i \end{pmatrix}, \]
where \( \hat{a}, \hat{v}_i \in B \) and \( \hat{d}, \hat{w}_i \in \text{Der}_S(B, B^{\otimes 2}) \) with \( d \in \text{Der}(\mathcal{O}, \mathcal{O}^{\otimes 2}) \). Apart from the obvious relations induced by matrix multiplication, these elements satisfy
\[ (14) \hat{\Delta} - \sum_{i=1}^N \hat{v}_i \cdot \hat{w}_i = \lambda e, \sum_{i=1}^N \hat{w}_i \cdot \hat{v}_i = \lambda_{\infty} e_{\infty}, \]
where ‘.’ denotes the action of \( B \) on the bimodule \( \text{Der}_S(B, B^{\otimes 2}) \).

With Proposition 3, we can describe the variety \( \mathcal{C}_n(X, \mathcal{I}) \) as the space of equivalence classes of linear maps (matrices)
\[ \{ (\bar{a}, \bar{d}, \bar{v}_i, \bar{w}_i) : \bar{a}, \bar{d} \in \text{End}(\mathbb{C}^n), \bar{v}_i \in \text{Hom}(\mathbb{C}, \mathbb{C}^n), \bar{w}_i \in \text{Hom}(\mathbb{C}^n, \mathbb{C}), \}, \]
satisfying the relations (cf. (13))
\[ (15) \hat{\Delta} - \sum_{i=1}^N \bar{v}_i \bar{w}_i = \text{Id}_n, \sum_{i=1}^N \bar{w}_i \bar{v}_i = -n. \]

Of course, in addition to (15), \( \bar{a} \) and \( \bar{d} \) should also obey the internal relations of the algebra \( \mathcal{O} \) and the bimodule \( \text{Der}(\mathcal{O}, \mathcal{O}^{\otimes 2}) \). Giving a matrix presentation of \( \mathcal{C}_n(X, \mathcal{I}) \) thus boils down to describing \( \mathcal{O} \) and \( \text{Der}(\mathcal{O}, \mathcal{O}^{\otimes 2}) \) in terms of generators and relations. This can be easily done in practice.

\(^6\)We recall that \( \{v_i\} \subset \mathcal{I} \) and \( \{w_i\} \subset \mathcal{I}^* \) form a ‘dual basis’ for f. g. projective modules \( \mathcal{I} \) and \( \mathcal{I}^* \) if \( s = \sum w_i(s)v_i \) for all \( s \in \mathcal{I} \). Of course, this is an abuse of terminology, since \( \{v_i\} \) and \( \{w_i\} \) are only generating sets of \( \mathcal{I} \) and \( \mathcal{I}^* \), not necessarily bases. The existence of such generating sets characterizes f. g. projective modules, see, e.g., [B], Ch. II, Prop. (4.5).
3.6. Example: plane curves. Let $X$ be a smooth curve on $\mathbb{C}^2$ defined by the equation $F(x, y) = 0$, with $F(x, y) := \sum_{r,s} a_{rs} x^r y^s \in \mathbb{C}[x, y]$. In this case, the algebra $\mathcal{O} \cong \mathbb{C}[x, y]/\langle F(x, y) \rangle$ is generated by $x$ and $y$, and the $\mathcal{O}$-module $\text{Der}(\mathcal{O})$ is (freely) generated by the derivation $\partial$ defined by

$$\partial(x) = F'_y(x, y), \quad \partial(y) = -F'_x(x, y).$$

Further, it is easy to show that the bimodule $\text{Der}(\mathcal{O}, \mathcal{O} \otimes \mathcal{O})$ is generated by the distinguished derivation $\Delta = \Delta_0$ and the element $z$ defined by

$$z(x) = \sum_{r,s} a_{rs} \sum_{k=0}^{s-1} x^r y^k \otimes y^{s-k-1},$$

$$z(y) = -\sum_{r,s} a_{rs} \sum_{l=0}^{r-1} x^l \otimes x^{r-l-1} y^s.$$ These generators satisfy the following commutation relations

$$(16) \quad [z, x] = \sum_{r,s} a_{rs} \sum_{k=0}^{s-1} y^{s-k-1} \Delta y^k x^r,$$

$$(17) \quad [z, y] = -\sum_{r,s} a_{rs} \sum_{l=0}^{r-1} y^s x^{r-l-1} \Delta x^l.$$ By Proposition 3, the algebra $\Pi^A(B)$ is then generated by the elements $\hat{x}, \hat{y}, \hat{z}, \hat{v}_i, \hat{w}_i$ and $\hat{\Delta}$, subject to the relations $(13)$ and $(10), (17)$.

Let us now explicitly describe the generic points of the varieties $C_n(X, I)$ (see remarks following Theorem 11). First, we consider the case when $I$ is trivial, i.e. $I \cong \mathcal{O}$. We choose $n$ distinct points $p_i = (x_i, y_i) \in X$, $i = 1, \ldots, n$ and define the matrices

$$(18) \quad (\hat{X}, \hat{Y}, \hat{Z}, \hat{v}, \hat{w}) \in \text{End}(\mathbb{C}^n) \times \text{End}(\mathbb{C}^n) \times \text{End}(\mathbb{C}^n) \times \text{Hom}(\mathbb{C}, \mathbb{C}^n) \times \text{Hom}(\mathbb{C}, \mathbb{C})$$

by $\hat{X} = \text{diag}(x_1, \ldots, x_n)$, $\hat{Y} = \text{diag}(y_1, \ldots, y_n)$, $\hat{v}^i = -\hat{w} = (1, \ldots, 1)$, and

$$(19) \quad \hat{Z}_{ii} = \alpha_i, \quad \hat{Z}_{ij} = \frac{F(x_i, y_j)}{(x_i - x_j)(y_i - y_j)} \quad (i \neq j),$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary scalars. A straightforward calculation, using the relations $(13)$ and $(17)$, shows that

$$\hat{x} \mapsto \hat{X}, \quad \hat{y} \mapsto \hat{Y}, \quad \hat{z} \mapsto \hat{Z}, \quad \hat{v} \mapsto \hat{v}, \quad \hat{w} \mapsto \hat{w}, \quad \hat{\Delta} \mapsto \text{Id}_n + \hat{v} \hat{w}$$

defines a representation of $\Pi^A(B)$ on the vector space $V = \mathbb{C}^n \oplus \mathbb{C}$. The equivalence classes of such representations correspond to generic points of $C_n(X, O)$.

Remark. The matrix $\hat{Z}$ defined by $(19)$ is a generalization of the classical Moser matrix in the theory of integrable systems (see [KKS]).

Now, let $I$ be an arbitrary line bundle on $X$. As before, we identify $I$ with an ideal in $\mathcal{O}$ and assume that the zero set $Z(I)$ of $I$ does not include the points $p_i$ (this can always be achieved by changing the embedding of $I$ in $\mathcal{O}$). Then, $I^*$ can be identified with a fractional ideal of $\mathcal{O}$ generated by rational functions with poles in $Z(I)$, and the pairing $I \times I^* \to \mathcal{O}$ is given by multiplication in $\mathbb{C}(X)$. The evaluation of $v \in I$ at $p_1, \ldots, p_n$ defines a vector $\hat{v} \in \mathbb{C}^n$; in a similar fashion, any $w \in I^*$ defines a row vector $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n) \in \text{Hom}(\mathbb{C}^n, \mathbb{C})$, with
\(\bar{w}_j = -w(p_j)\). If \(\{v_i\}, \{w_i\}\) are dual bases for \(I\) and \(I^*\), then \(\sum_i v_i \otimes w_i\) gives a rational function on \(X \times X\), which we denote by \(\phi\); the fact that the bases are dual implies \(\phi(p, p) = 1\) for all \(p \in X\). As a result, the \(n \times n\) matrix \(\sum_i \bar{v}_i \bar{w}_i\) equals \(\| - \phi(p_i, p_j)\|\), with all the diagonal entries being equal to \(-1\).

Now, let \(\bar{X}\) and \(\bar{Y}\) be the diagonal matrices as above, and let \(\hat{Z}\) be given by

\[
\hat{Z}_{ii} = \alpha_i , \quad \hat{Z}_{ij} = \frac{F(x_i, y_i) \phi(p_i, p_j)}{(x_i - x_j)(y_i - y_j)} \quad (i \neq j) .
\]

It is straightforward to check that the assignment

\[
\hat{x} \mapsto \hat{X} , \quad \hat{y} \mapsto \hat{Y} , \quad \hat{z} \mapsto \hat{Z} , \quad \hat{\varphi_i} \mapsto \hat{v_i} , \quad \hat{\bar{w}_i} \mapsto \hat{\bar{w}_i} , \quad \hat{\Delta} \mapsto \operatorname{Id}_n + \sum_i \hat{v}_i \hat{\bar{w}_i}
\]

extends to a representation of \(\Pi^!(B)\) on the vector space \(V = \mathbb{C}^n \oplus \mathbb{C}\); such representations correspond to generic points of the variety \(\mathcal{C}_n(X, I)\).

4. The Calogero-Moser Correspondence

We will keep the notation from the previous section: \(\mathcal{O} = \mathcal{O}(X)\) stands for the coordinate ring of a smooth affine irreducible curve \(X\), \(B = \mathcal{O}[I]\) for the one-point extension of \(\mathcal{O}\) by a line bundle \(I\), and \(\Pi = \Pi^!(B)\) for the deformed preprojective algebra over \(B\) of weight \(\lambda = (1, -n)\) with \(n \in \mathbb{N}\).

4.1. Recollement. We now explain the relation between representations of \(\Pi^!(B)\) and \(D\)-modules on \(X\). We begin with the following observation, which is a simple consequence of Theorem 2 (see \(\text{[BC]}\), Lemma 4.1).

**Lemma 5.** There is a canonical algebra map \(\theta : \Pi^!(B) \to \Pi^1(\mathcal{O})\), which is a surjective pseudo-flat ring epimorphism, with \(\operatorname{Ker}(\theta) = \langle e_\infty \rangle\).

With Proposition 3, the homomorphism \(\theta\) can be described explicitly, in terms of generators of \(\Pi^!(B)\); namely,

\[
\theta(a) = \bar{a} , \quad \theta(d) = \bar{d} , \quad \theta(\varphi_i) = \theta(\bar{w}_i) = 0 ,
\]

where \(\bar{a}\) and \(\bar{d}\) denote the classes of \(a \in \mathcal{O}\) and \(d \in \operatorname{Der}(\mathcal{O}, \mathcal{O}^{\otimes 2})\) in the tensor algebra of \(\operatorname{Der}(\mathcal{O}, \mathcal{O}^{\otimes 2})\) modulo the ideal \(\langle \Delta - 1\rangle\).

Now, by Example 1 (see Sect. 3.2), the algebra \(\Pi^1(\mathcal{O})\) is isomorphic to \(D\): we fix, once and for all, such an isomorphism to identify \(D = \Pi^1(\mathcal{O})\). In combination with Lemma 5, this yields an algebra map \(\theta : \Pi \to D\). We will use \(\theta\) to relate the derived module categories of \(\Pi\) and \(D\) in the following way (cf. \(\text{[BCE]}\)).

First, we let \(U\) denote the endomorphism ring of the projective module \(e_\infty \Pi\); this ring can be identified with the associative subalgebra \(e_\infty \Pi e_\infty\) of \(\Pi\) having \(e_\infty\) as an identity element. Next, we introduce six additive functors \((\theta^*, \theta_*, \theta)\) and \((j_i, j^*, j_*)\) between the module categories of \(\Pi\), \(D\) and \(U\). We define \(\theta_* : \mathcal{M}(D) \to \mathcal{M}(\Pi)\) to be the restriction functor associated to the algebra map \(\theta : \Pi \to D\). This functor is fully faithful and has both the right adjoint \(\theta^! := \operatorname{Hom}_\Pi(D, -)\) and the left adjoint \(\theta^* := D \otimes_\Pi -\), with adjunction maps \(\theta^* \theta_* \simeq \operatorname{Id} \simeq \theta^! \theta_*\) being isomorphisms. Now we define \(j^* : \mathcal{M}(\Pi) \to \mathcal{M}(U)\) by \(j^* V := e_\infty V\). Since \(e_\infty \in \Pi\) is an idempotent, \(j^*\) is exact and has also the right and the left adjoint functors: \(j_* := \operatorname{Hom}_U(e_\infty \Pi, -)\) and \(j_! := \Pi e_\infty \otimes_U -\) respectively, satisfying \(j_* j_* \simeq \operatorname{Id} \simeq j_* j^*\).
The functors \((θ^*, θ_*, θ^!\rangle)\) and \((j_1, j^*, j_*)\) defined above induce the six exact functors at the level of (bounded) derived categories:

\[
\begin{array}{ccc}
\mathcal{D}^-(\text{Mod } \mathcal{D}) & \xrightarrow{Lθ^*} & \mathcal{D}^-(\text{Mod } \Pi) \\
\xrightarrow{θ_*} & & \xleftarrow{j^*} \\
\xrightarrow{Rθ^!} & & \xleftarrow{Rj_*} \\
\mathcal{D}^-(\text{Mod } U) & & \\
\end{array}
\]

The properties of these functors can be summarized in the following way.

**Proposition 4** ([BC], Proposition 4.1). The diagram (21) is a recollement of triangulated categories.

**Remark.** The ‘recollement axioms’ were originally formulated in [BBD] to imitate a natural structure on the derived category \(\mathcal{D}(\text{Sh}_X)\) of abelian sheaves arising from the stratification of a topological space into a closed subspace and its open complement (see [BBD], Sect. 1.4). In an algebraic setting similar to ours, these axioms were studied in [CPS].

We will use the induction functor \(Lθ^*\) to define a natural map: \(C_n(X, I) \to \mathcal{I}(\mathcal{D})\). As a first step, we compute the values of \(Lθ^*\) on the finite-dimensional representations of \(Π^\lambda(B)\), regarding the latter as \(O\)-complexes in \(\mathcal{D}^-(\text{Mod } \Pi)\) (see [BC], Lemma 4.2). We recall that \(L_nθ^* \cong \text{Tor}_n^\Pi(D, -)\), where \(D\) is viewed as a \(Π\)-module via the algebra map \(θ\).

**Lemma 6.** If \(V\) is a \(Π\)-module of finite dimension over \(\mathbb{C}\), then \(L_nθ^*(V) = 0\) for \(n \neq 1\) and

\[
L_1θ^*(V) \cong \ker \left[ Πe_∞ ⊗_U e_∞ V \xrightarrow{μ} V \right],
\]

where \(μ\) is the natural multiplication-action map on \(V\).

Lemma 6 shows that \(Lθ^*(V)\) is isomorphic in \(\mathcal{D}^-(\text{Mod } \mathcal{D})\) to a single \(D\)-module \(M = L_1θ^*(V)\) located in cohomological degree \((-1)\). Abusing notation, we will simply write \(M = Lθ^*(V)\) in this case.

**4.2. The action of \(\text{Pic}(\mathcal{D})\) on Calogero-Moser spaces.** Next, we describe a natural action of the Picard group of the category of \(D\)-modules on representation varieties of \(Π^\lambda(B)\). It is known that \(\text{Pic}(\mathcal{D})\) has different descriptions depending on whether \(X\) is the affine line or not (see [CH1]). In this section, we will assume that \(X \neq A^1\). Our main result (Theorem 5) will still be true for all \(X\), since the case \(X = A^1\) is covered in [BW1] [BW2].

We recall (see, e.g., [B], Ch. II, § 5) that \(\text{Pic}(\mathcal{D})\) can be identified with the group of \(\mathbb{C}\)-linear auto-equivalences of the category \(\text{Mod}(\mathcal{D})\), and thus it acts naturally on \(\mathcal{I}(\mathcal{D})\) and \(K_0(\mathcal{D})\). To be precise, the elements of \(\text{Pic}(\mathcal{D})\) are the isomorphism classes \([P]\) of invertible bimodules over \(\mathcal{D}\), and the action of \(\text{Pic}(\mathcal{D})\) on \(\mathcal{I}(\mathcal{D})\) and \(K_0(\mathcal{D})\) is defined by \([M] \mapsto [P ⊗_D M]\). Observe that \(\text{Pic}(\mathcal{D})\) acts on \(K_0(\mathcal{D})\) preserving rank: hence, this action restricts to \(\text{Pic}(X)\) through the identification \(K_0(\mathcal{D}) \cong K_0(X) \cong \mathbb{Z} ⊕ \text{Pic}(X)\), see Section 2.2.

Now, let \(\text{Aut}(\mathcal{D})\) denote the group of \(\mathbb{C}\)-automorphisms of the algebra \(\mathcal{D}\). There is a natural homomorphism: \(\text{Aut}(\mathcal{D}) \to \text{Pic}(\mathcal{D})\), sending \(φ \in \text{Aut}(\mathcal{D})\) to (the class of) the bimodule \(D_φ\). The kernel of this homomorphism consists of the inner automorphisms of \(\mathcal{D}\), so the group \(Γ := \text{Out}(\mathcal{D})\) of outer automorphism classes can be identified with a subgroup of \(\text{Pic}(\mathcal{D})\). With this identification, we have
where $F \otimes (27)$ End

identify $D$ and, since $\text{End}(28)$ embedding $\tau$ deformed preprojective algebras: $\Pi \tau (26)$ Lemma 2 which is clearly a progenerator in the category of right $\Pi \lambda$

L where $\tau (25)$ $(24)$ $\text{Mod}$

$\Pi \lambda$ $\lambda$ $\Pi \lambda (B_r)$-modules. By Lemma 2(b), the algebra $B_r$ is isomorphic to $B$: the isomorphism is given by

$\tau : B \rightarrow B_r$, \[(a \ b) \rightarrow \begin{pmatrix} \tau(a) & \tau(b) \\ 0 & c \end{pmatrix}.\]

Since $\tau(\lambda) = \lambda$ for all $\lambda \in S$, (20) canonically extends to an isomorphism of deformed preprojective algebras: $\Pi \lambda (B) \sim \Pi \lambda (B_r)$, which we will also denote by $\tau$. Now, using this last isomorphism, we will regard $P$ as a $\Pi \lambda (B)$-module and identify

$\text{End}_{\Pi \lambda (B)} (P) = \hat{\mathcal{L}} \otimes A \Pi \lambda (B_r) \otimes A \hat{\mathcal{L}}^* \cong \hat{\mathcal{F}} \otimes A \Pi \lambda (B) \otimes A \hat{\mathcal{F}}^*$,

where $\mathcal{F} := \mathcal{L}^* = \tau^{-1} (\mathcal{L})$ and $\hat{\mathcal{F}} = \mathcal{F} \times \mathbb{C}$. With identification (27), we have the embedding

$\tau^{-1} : A[\mathcal{J}] \cong \hat{\mathcal{L}} \otimes A B_r \otimes A \hat{\mathcal{L}}^* \hookrightarrow \text{End}_{\Pi \lambda (B)} (P)$,

and, since $\text{End}_D (\mathcal{F}D) = \mathcal{F} \otimes A D \otimes A \mathcal{F}^* \cong \hat{\mathcal{F}} \otimes A D \otimes A \hat{\mathcal{F}}^*$, the natural map

$1 \otimes \theta \otimes 1 : \text{End}_{\Pi \lambda (B)} (P) \rightarrow \text{End}_D (\mathcal{F}D)$,
where $\theta : \Pi^\lambda(B) \to D$ is given by Lemma 5.

On the other hand, $\varphi(D) = \text{End}_D(D\mathcal{L}) = \mathcal{L}^* D \mathcal{L}$ implies $D = \mathcal{L} \varphi(D) \mathcal{L}^*$, so taking the inverse of $\varphi$ defines an algebra isomorphism

$$\psi = \varphi^{-1} : D \to \mathcal{F}D \mathcal{F}^* = \text{End}_D(D\mathcal{F}) .$$

Combining (29) and (30) together, we get the diagram of algebra maps

$$\begin{array}{ccc}
\Pi^\lambda(O[J]) & \longrightarrow & \text{End}_{\Pi^\lambda(B)}(P) \\
\begin{array}{c}
\theta \\
\psi
\end{array}
\downarrow & & \uparrow 1 \otimes \theta \otimes 1 \\
D & \longrightarrow & \text{End}_D(D\mathcal{F})
\end{array}$$

which obviously commutes when the dotted arrow is restricted to (28).

**Proposition 6** ([BC], Proposition 4.3). There is a unique algebra isomorphism $\psi : \Pi^\lambda(O[J]) \to \text{End}_{\Pi^\lambda(B)}(P)$, extending (28) and making (31) a commutative diagram.

The proof of this result in [BC] is rather indirect: it combines homological arguments of Theorem 5 with explicit calculations and description of automorphisms of $D$ given in [CH1].

Now, using the isomorphism $\psi$ of Proposition 6, we can make $P$ a left $\Pi^\lambda(A[I])$-module and thus a progenerator from $\Pi^\lambda(A[I])$ to $\Pi^\lambda(A[J])$. This assigns to $P = (D\mathcal{L})_\varphi$ the Morita equivalence:

$$\text{Mod} \Pi^\lambda(A[I]) \to \text{Mod} \Pi^\lambda(A[J]) , \quad V \mapsto P \otimes \Pi(V) ,$$

which, in turn, induces an isomorphism of representation varieties

$$f_P : \mathcal{C}_n(X, I) \simeq \mathcal{C}_n(X, J) .$$

We warn the reader that (32) depends on the choice of a specific representative in the class $[P] \in \text{Pic}(D)$, so, in general, we do not get an action of $\text{Pic}(D)$ on $\bigsqcup_{[I] \in \text{Pic}(X)} \mathcal{C}_n(X, I)$. However, it turns out that $f_P$ induce a well-defined action of $\text{Pic}(D)$ on the reduced Calogero-Moser spaces $\overline{\mathcal{C}}_n(X, I)$. Precisely, we have

**Lemma 7** ([BC], Lemma 4.3). The map (32) induces an isomorphism

$$\bar{f}_P : \overline{\mathcal{C}}_n(X, I) \simeq \overline{\mathcal{C}}_n(X, J) ,$$

which depends only on the class of $P$ in $\text{Pic}(D)$.

If we set $\overline{\mathcal{C}}_n(X) := \bigsqcup_{[I] \in \text{Pic}(X)} \overline{\mathcal{C}}_n(X, I)$, the assignment $[P] \mapsto \bar{f}_P$ defines an action of $\text{Pic}(D)$ on $\overline{\mathcal{C}}_n(X)$ for each $n \geq 0$.

4.3. The action of automorphisms. Assume now that $P$ comes from an automorphism of $D$, i.e. $[P] \in \Gamma \subseteq \text{Pic}(D)$, where $\Gamma := \text{Out}(D)$. Then, by Proposition 6, $[P]$ stabilizes $[I] \in \text{Pic}(X)$, so the isomorphisms $\bar{f}_P$ of Lemma 7 define an action of $\Gamma$ on each space $\overline{\mathcal{C}}_n(X, I)$ individually. We now describe this action in explicit terms.

First of all, when $X \neq \mathbb{A}^1$, we can identify (see [BC], Section 5.5)

$$\Gamma \cong \text{Aut}(X) \times (\Omega^1 X)/\Lambda ,$$
where $\Omega^1 X$ is the canonical bundle of $X$ and $\Lambda := \mathcal{O}^\times / \mathbb{C}^\times$ is the multiplicative group of (nontrivial) units of $\mathcal{O}(X)$ embedded in $\Omega^1 X$ via the logarithmic derivative map:

\[(33)\quad \mathrm{dlog} : \Lambda \hookrightarrow \Omega^1 X \quad u \mapsto u^{-1} du .\]

To simplify calculations, we will assume here that $\text{Aut}(X)$ is trivial, which is clearly the case for generic curves.

Let $\Omega^1 (B)$ denote the bimodule of noncommutative differential forms on $B$ (i.e. the kernel of the multiplication map $m : B \otimes B \to B$), and let $\text{DR}^1 (B) := \Omega^1 (B) / [B, \Omega^1 B]$ be the quotient of $\Omega^1 (B)$ by its commutator subspace (the Karoubi-de Rham differentials of $B$). Using the fact that $B$ is finitely generated and quasi-free (see Lemma 1), we identify

\[(34)\quad \text{DR}^1 (B) \cong B \otimes B^e \Omega^1 (B) \cong B \otimes B^e (\Omega^1 B)^{**} \cong \text{Hom}_{B^e}((\Omega^1 B)^*, B) ,\]

where $B^e := B \otimes B^{opp}$, and $(-)^*$ stands for the duality over $B^e$. Explicitly, under the identification \[(34), \quad \overline{\sigma} \in \text{DR}^1 (B) \] corresponds to the map $\hat{\sigma} : \delta \mapsto m[\delta(\omega)]$.

Now, we define an action of $\text{DR}^1 (B)$ on $T_B (\Omega^1 B)^*$ as follows: if $\overline{\sigma} \in \text{DR}^1 (B)$, we let $\tilde{\sigma} : \delta \mapsto \delta + \hat{\sigma}(\delta)$.

By the universal property of tensor algebras, this uniquely determines $\tilde{\sigma}$, and it is clear that this map is bijective. Moreover, if $\Delta_B \in (\Omega^1 B)^*$ is the canonical inclusion $\Omega^1 B \hookrightarrow B^e$, then $\hat{\sigma}(\Delta_B) = 0$, and hence $\tilde{\sigma}(\Delta_B) = \Delta_B$ for any $\overline{\sigma} \in \text{DR}^1 (B)$. The assignment $\overline{\sigma} \mapsto \tilde{\sigma}$ defines thus a homomorphism

\[(35)\] $\tilde{\sigma} : \text{DR}^1 (B) \to \text{Aut}_B [T_B (\Omega^1 B)^*]$

from the additive group of $\text{DR}^1 (B)$ to the subgroup of $B$-linear automorphisms of $T_B (\Omega^1 B)^*$ preserving the element $\Delta_B$.

Next, identifying the canonical bundle of $X$ with the module of Kähler differentials of $\mathcal{O}$, we construct an embedding of $\Omega^1 X$ into $\text{DR}^1 (B)$. For this, we consider the exact sequence

\[(36)\quad 0 \to \text{HH}_1 (B) \xrightarrow{\alpha} \text{DR}^1 (B) \to B \to \text{HH}_0 (B) \to 0 ,\]

obtained by tensoring the fundamental exact sequence

\[0 \to \Omega^1 (B) \to B^e \to B \to 0\]

with $B$, and compose the connecting map $\alpha$ in \[(36)\] with natural isomorphisms

\[(37)\quad \text{HH}_1 (B) \cong \text{HH}_1 (\mathcal{O}) \cong \Omega^1 X .\]

(The first isomorphism in \[(37)\] is induced by the projection $\theta : B \to \mathcal{O}$, while the second by the canonical map: $\mathcal{O} \otimes^2 \to \Omega^1 X$, $f \otimes g \mapsto f \, dg$.)

Finally, combining \[(36)\] with \[(37)\] and \[(35)\], we define

\[(38)\quad \sigma : \Omega^1 X \xrightarrow{\alpha} \text{DR}^1 (B) \xrightarrow{\tilde{\sigma}} \text{Aut}_B [T_B (\Omega^1 B)^*] \to \text{Aut}_B [\Pi^\lambda (B)] ,\]

where the last map is induced by the algebra projection: $T_B (\Omega^1 B)^* \to \Pi^\lambda (B)$. With Proposition 3, the action \[(38)\] can be easily described in terms of generators of $\Pi^\lambda (B)$. 

Lemma 8. The homomorphism $\sigma : \Omega^1 X \to \text{Aut}_S[\Pi^\lambda(B)]$ is determined by
\[
\sigma_\omega(\hat{a}) = \hat{a}, \quad \sigma_\omega(\hat{v}_i) = \hat{v}_i, \quad \sigma_\omega(\hat{w}_i) = \hat{w}_i, \quad \sigma_\omega(\hat{d}) = \hat{d} + \omega(\hat{d}),
\]
where $\omega \in \Omega^1 X$ acts on $\hat{d} \in \text{Der}(\mathcal{O}, \mathcal{O}^\otimes 2)$ via the natural identification, cf. (34):
\[
\Omega^1 X \cong DR^2(\mathcal{O}) \cong \text{Hom}_{O_{X \times X}}(\text{Der}(\mathcal{O}, \mathcal{O}^\otimes 2), \mathcal{O}).
\]

Now, the group $\text{Aut}_S[\Pi^\lambda(B)]$ acts on $\text{Rep}_S[\Pi^\lambda(B), n]$ in the obvious way: if $\varrho : \Pi^\lambda(B) \to \text{End}(\mathcal{V})$ represents a point in $\text{Rep}_S[\Pi^\lambda(B), n]$, then $\omega \mapsto \varrho \sigma_\omega^{-1}$ for $\omega \in \Omega^1 X$. Clearly, this commutes with the natural $\text{GL}_S(n)$-action on $\text{Rep}_S[\Pi^\lambda(B), n]$ and hence induces the action
\[
(39) \quad \sigma^* : \Omega^1 X \to \text{Aut}[\mathcal{C}_n(X, \mathcal{I})], \quad \omega \mapsto [\sigma^*_\omega : \varrho \mapsto \varrho \sigma_\omega^{-1}].
\]

By a straightforward calculation we get (see [BC], Lemma 4.3)

Lemma 9. The action (39) agrees with (32): that is, if $\mathcal{P} = \mathcal{D}_{\sigma_\omega}$, then
\[
f_\mathcal{P} = \sigma^*_\omega \quad \text{for all} \quad \omega \in \Omega^1 X.
\]

On the other hand, the restriction of (39) to the group $\Lambda$ via (38) agrees with the natural action of $\text{Aut}_S(\mathcal{I}) = \mathcal{O}^\times$ on $\mathcal{C}_n(X, \mathcal{I})$. Thus (39) induces an action of $\Gamma = (\Omega^1 X)/\Lambda$ on each of the spaces $\mathcal{C}_n(X, \mathcal{I})$. By Lemma 9 this coincides with the restriction to $\Gamma$ of the action of $\text{Pic}(\mathcal{D})$ constructed in Section 4.2.4.

4.4. The main theorem. We can now state the main result of [BC]. We recall the functor $L\theta^* = \text{Toilet}_1^m(D, -) : \text{Mod}(\Pi) \to \text{Mod}(\mathcal{D})$ associated to the algebra homomorphism $\theta : \Pi \to \mathcal{D}$, when restricted to finite-dimensional representations, this functor is given by (22).

Theorem 5 ([BC], Theorem 4.2). Let $X$ be a smooth affine irreducible curve.

(a) For each $n \geq 0$ and $[\mathcal{I}] \in \text{Pic}(X)$, the functor $L\theta^*$ induces an injective map
\[
\omega_n : \mathcal{C}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}],
\]
which is equivariant under the action of the group $\Gamma$.

(b) Amalgamating the maps $\omega_n$ for all $n \geq 0$ yields a bijective correspondence
\[
\omega : \bigsqcup_{n \geq 0} \mathcal{C}_n(X, \mathcal{I}) \cong \gamma^{-1}[\mathcal{I}].
\]

(c) For any $[\mathcal{I}]$ and $[\mathcal{J}]$ in $\text{Pic}(X)$ and for any $[\mathcal{P}] \in \text{Pic}(\mathcal{D})$, such that $[\mathcal{P}] \cdot [\mathcal{I}] = [\mathcal{J}]$, there is a commutative diagram:
\[
\begin{array}{ccc}
\mathcal{C}_n(X, \mathcal{I}) & \xrightarrow{f_\mathcal{P}} & \mathcal{C}_n(X, \mathcal{J}) \\
\omega_n \downarrow & & \downarrow \omega_n \\
\gamma^{-1}[\mathcal{I}] & \xrightarrow{[\mathcal{P}]} & \gamma^{-1}[\mathcal{J}]
\end{array}
\]

where $f_\mathcal{P}$ is an isomorphism induced by (22).

For technical reasons, we assumed above that $X \neq \Lambda^1$. However, Theorem 5 holds true in general: if $X = \Lambda^1$, by [BCF], Theorem 1, the map $\omega$ induced by $L\theta^*$ agrees with the Calogero-Moser map constructed in [BW1]. In this case, the ring $\mathcal{D}$ is isomorphic to the Weyl algebra $A_1(\mathbb{C})$ and $\text{Pic}(\mathcal{D}) \cong \text{Aut}(\mathcal{D}) = \text{Out}(\mathcal{D})$, see [S]. Since $\text{Pic}(\Lambda^1)$ is trivial, the last part of Theorem 5 implies the equivariance of $\omega$ under the action of $\text{Aut}(A_1)$, which was one of the main results in [BW1].
4.5. Explicit construction of ideals. To illustrate Theorem 5, we return to our basic example of plane curves (see Section 3.6). In this case, we will describe the map $\omega$ quite explicitly, in terms of generalized Calogero-Moser matrices (12). For simplicity, we consider only the case when $\mathcal{I}$ is trivial. A $\Pi$-module $V = \mathbb{C}^n \oplus \mathbb{C}$ can then be described by $(\bar{X}, \bar{Y}, \bar{Z}, \bar{v}, \bar{w})$, which, apart from (16) and (17), satisfy the following relations

$$F(\bar{X}, \bar{Y}) = 0, \quad [\bar{X}, \bar{Y}] = 0 \quad \text{and} \quad \bar{\Delta} = \text{Id}_n + \bar{v}\bar{w}.$$ 

The dual representation $\varrho^* : \Pi^{opp} \rightarrow \text{End}(V^*)$ is given by the transposed matrices $(\bar{X}^t, \bar{Y}^t, \bar{Z}^t, \bar{v}^t, \bar{w}^t)$.

Now, using these matrices, we define the following element in the field of fractions of the algebra $\mathcal{D}$:

$$\kappa := 1 + \bar{v}^t(Z^t - z\text{Id}_n)^{-1}(X^t - x\text{Id}_n)^{-1}(Y^t - y\text{Id}_n)^{-1}F(X^t, y\text{Id}_n)w^t.$$ 

and consider the (fractional) left ideal

$$M_V := \mathcal{D}\det(\bar{X} - x\text{Id}_n) + \mathcal{D}\det(\bar{Y} - y\text{Id}_n) + \mathcal{D}\det(\bar{Z} - z\text{Id}_n) \kappa.$$ 

Proposition 7. If $[V] \in \mathcal{C}_n(X, \mathcal{I})$ is determined by $(\bar{X}, \bar{Y}, \bar{Z}, \bar{v}, \bar{w})$, the corresponding ideal class $\omega[V]$ in $\mathfrak{I}(\mathcal{D})$ is represented by $M_V$.

For the proof of Proposition 7 and more examples, we refer the reader to [BC], Section 6. Here, we only mention that Theorem 6.1 of [BC] gives a similar explicit construction of $\omega$ for an arbitrary smooth curve, generalizing earlier calculations for the first Weyl algebra in [BC1].

References

[B] H. Bass, *Algebraic K-theory*, W. A. Benjamin Inc., New York-Amsterdam, 1968.
[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque 100, Soc. Math. France, 1982, pp. 5–171.
[BN] D. Ben-Zvi and T. Nevins, *Perverse bundles and Calogero-Moser spaces*, preprint, arXiv:math.AG/0610097.
[BC] Yu. Berest and O. Chalykh, *Ideals of rings of differential operators on algebraic curves* (with an Appendix by G. Wilson), preprint, 63 pp. [arXiv:0809.3289].
[BC1] Yu. Berest and O. Chalykh, $A_{\infty}$-modules and Calogero-Moser spaces, J. reine angew. Math. 607 (2007), 69–112.
[BCE] Yu. Berest, O. Chalykh and F. Eshmatov, *Recollement of deformed preprojective algebras and the Calogero-Moser correspondence*, Moscow Math. J. 8(1) (2008), 21–37.
[BW] Yu. Berest and G. Wilson, *Differential operators on affine curves: ideal classes and Picard groups*, preprint (2008), 15 pp. (to appear).
[BW1] Yu. Berest and G. Wilson, *Automorphisms and ideals of the Weyl algebra*, Math. Ann. 318(1) (2000), 127–147.
[BW2] Yu. Berest and G. Wilson, *Ideal classes of the Weyl algebra and noncommutative projective geometry* (with an Appendix by M. van den Bergh), Internat. Math. Res. Notices 26 (2002), 1347–1396.
[B] J.-E. Björk, *Rings of Differential Operators*, North-Holland Publishing, Amsterdam, 1979.
[CH] R. C. Cannings and M. P. Holland, *Right ideals of rings of differential operators*, J. Algebra 167 (1994), 116–141.
[CH1] R. C. Cannings and M. P. Holland, *Etale covers, bimodules and differential operators*, Math. Z. 216 (1994), 179–194.
[CPS] E. Cline, B. Parshall and L. Scott, *Algebraic stratification in representation categories*, J. Algebra 117 (1988), 504–521.
[C] W. Crawley-Boevey, *Preprojective algebras, differential operators and a Conze embedding for deformations of Kleinian singularities*, Comment. Math. Helv. 74(4) (1999), 548–574.
[C1] W. Crawley-Boevey, *Representations of quivers, preprojective algebras and deformations of quotient singularities*, lectures at the Workshop on “Quantizations of Kleinian singularities”, Oberwolfach, May 1999.

[C2] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. **126** (2001), 257–293.

[CEG] W. Crawley-Boevey, P. Etingof and V. Ginzburg, *Noncommutative geometry and quiver algebras*, Adv. Math. **209** (2007), 274–336.

[CBH] W. Crawley-Boevey and M. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), 605–636.

[CQ] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8**(2) (1995), 251–289.

[E] P. Etingof, *Cherednik and Hecke algebras of varieties with a finite group action*, preprint, arXiv:math.QA/0406499.

[EG] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243–348.

[GP] I. M. Gel’fand and V. A. Ponomarev, *Model algebras and representations of graphs*, Func. Anal. Appl. **13** (1979), 157–166.

[G] V. Ginzburg, *Lectures on Noncommutative Geometry*, preprint, arXiv:math.AG/0506603.

[KKS] D. Kazhdan, B. Kostant and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978), 481–507.

[K] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, D1. Friedr. Vieweg & Sohn, Braunschweig, 1984.

[KR] M. Kontsevich and A. Rosenberg, *Noncommutative smooth spaces*, The Gelfand Mathematical Seminars 1996-1999, 85–108, Birkhäuser, Boston, 2000.

[LeB] L. Le Bruyn, *Noncommutative Geometry 8n*, preprint, arXiv:math.AG/9904171.

[R] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math **1099**, 1984.

[Q] D. Quillen, *Higher algebraic K-theory*, Lecture Notes in Math. **341**, Springer, Berlin 1973, pp. 85–147.

[SS] S. P. Smith and J. T. Stafford, *Differential operators on an affine curve*, Proc. London Math. Soc. (3) **56** (1988), 229–259.

[S] J. T. Stafford, *Endomorphisms of right ideals of the Weyl algebra*, Trans. Amer. Math. Soc. **299** (1987), 623–639.

[vdB] M. van den Bergh, *Double Poisson algebras*, preprint, arXiv:math.QA/0410528 (to appear in Trans. Amer. Math. Soc.).

[W] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian* (with an Appendix by I. G. Macdonald), Invent. Math. **133** (1998), 1–41.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853-4201, USA

E-mail address: berest@math.cornell.edu