Positivity-Preserving Well-Balanced Central Discontinuous Galerkin Schemes
for the Euler Equations under Gravitational Fields

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Abstract

This paper designs and analyzes positivity-preserving well-balanced (WB) central discontinuous Galerkin (CDG) schemes for the Euler equations with gravity. A distinctive feature of these schemes is that they not only are WB for a general known stationary hydrostatic solution, but also can preserve the positivity of the fluid density and pressure. The standard CDG method does not possess this feature, while directly applying some existing WB techniques to the CDG framework may not accommodate the positivity and keep other important properties at the same time. In order to obtain the WB and positivity-preserving properties simultaneously while also maintaining the conservativeness and stability of the schemes, a novel spatial discretization is devised in the CDG framework based on suitable modifications to the numerical dissipation term and the source term approximation. The modifications are based on a crucial projection operator for the stationary hydrostatic solution, which is proposed for the first time in this work. This novel projection has the same order of accuracy as the standard $L^2$-projection, can be explicitly calculated, and is easy to implement without solving any optimization problems. More importantly, it ensures that the projected stationary solution has the same cell averages on both the primal and dual meshes, which is a key to achieve the desired properties of our schemes. Based on some convex decomposition techniques, rigorous positivity-preserving analyses for the resulting WB CDG schemes are carried out. Several one- and two-dimensional numerical examples are performed to illustrate the desired properties of these schemes, including the high-order accuracy, the WB property, the robustness for simulations involving the low pressure or density, high resolution for the discontinuous solutions and the small perturbations around the equilibrium state.

Keywords: Euler equations; central discontinuous Galerkin method; well-balanced schemes; positivity-preserving property; gravitational field

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1 Introduction

The Euler equations under gravitational fields are widely adopted to model physical phenomena in the atmospheric science and astrophysics, such as numerical weather forecasting [4], climate modeling, and supernova explosions [17]. In the one-dimensional case, this nonlinear system can be written into the form of the hyperbolic balance laws as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}, \phi_x),$$  \hspace{1cm} (1)

with

$$\mathbf{U} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{pmatrix}, \quad \mathbf{S}(\mathbf{U}, \phi_x) = \begin{pmatrix} 0 \\ -\rho \phi_x \\ -m \phi_x \end{pmatrix}.$$  

Here $\rho$ denotes the fluid density, $u$ is the velocity, $m = \rho u$ is the momentum, $p$ represents the pressure, $E = \frac{1}{2} \rho u^2 + \rho e$ denotes the total non-gravitational energy, and $e$ is the specific internal energy. The function $\phi(x)$ in the source terms is the static gravitational potential. In order to close the system (1), an equation of state (EOS) is needed and can be written as $p = p(\rho, e)$; for the ideal gas it is given by

$$p = (\gamma - 1) \rho e = (\gamma - 1) \left( E - \frac{m^2}{2 \rho} \right),$$  \hspace{1cm} (2)

where the constant $\gamma > 1$ denotes the ratio of specific heats. This paper will mainly focus on the ideal EOS (2), and the proposed schemes are readily extensible to a general EOS.

The Euler system (1) under the gravitational potential $\phi(x)$ admits non-trivial stationary hydrostatic solutions, where the velocity is zero and the gravity is exactly balanced by the pressure gradient:

$$\rho = \rho(x), \quad u = 0, \quad p_x = -\rho \phi_x.$$  \hspace{1cm} (3)

Two types of equilibria appear frequently in practical applications. They are the isothermal [41] and polytropic [17] hydrostatic equilibrium states. The temperature $T(x) = T_0$ is a constant under the isothermal assumption. For an isothermal ideal gas with $p = \rho RT$, integrating (3) yields

$$\rho = \hat{\rho}_0 \exp \left( -\frac{\phi}{RT_0} \right), \quad u = 0, \quad p = \hat{\rho}_0 \exp \left( -\frac{\phi}{RT_0} \right),$$

where $R$ is the gas constant, $\hat{\rho}_0$ is the pressure at a reference position $x_0$, and $\hat{\rho}_0 = \hat{\rho}_0 RT_0$. A polytropic hydrostatic equilibrium, which arises from the astrophysical applications, is characterized by $p = K_0 \rho^\nu$, and for this equilibrium, integrating (3) gives

$$\rho = \left( \frac{\nu - 1}{K_0 \nu} (C - \phi) \right)^{\frac{1}{\nu - 1}}, \quad u = 0, \quad p = \frac{1}{K_0^{\frac{1}{\nu - 1}}} \left( \frac{\nu - 1}{\nu} (C - \phi) \right)^{\frac{\nu}{\nu - 1}},$$

with $K_0$ and $C$ being constants. A special case is $\nu = \gamma$, which corresponds to a constant entropy.

In order to correctly and accurately capture small perturbations around the equilibrium state (3), it is desirable to develop well-balanced (WB) numerical methods that preserve the discrete version
of those steady state solutions exactly up to machine accuracy. In fact, a straightforward numerical
discretization may not be WB and can lead to a numerical solution which is inaccurate or oscillates
around the hydrostatic equilibrium after a long time simulation. This problem may be improved if the
mesh size is extremely refined, which, however, may cause the simulation time-consuming especially
in the multidimensional cases. To reduce the computational cost, the exploration of the WB schemes
has attracted much attention in the past few decades. Most of those schemes were devised for the
nonlinear shallow water equations over varied bottom topology, another typical model of hyperbolic
balance laws; see, e.g., [29, 1, 3, 14, 21, 58, 60, 61, 62] for more details. In recent years, various
WB schemes for the Euler equations under gravitational fields have been developed within several
different frameworks, including but not limited to the non-central finite volume methods [4, 34, 63,
17, 18, 5, 25], the central finite volume methods [43, 16], the finite difference methods [41, 11, 28],
and the discontinuous Galerkin (DG) methods [26, 6, 27], etc. A numerical comparison between the
high-order DG method and the WB DG method was carried out in [45]. Most of those works assumed
that the target equilibrium is explicitly known, which is also adopted in our present work. It is worth
mentioning that, recently, there exist some efforts [2, 7, 10, 18, 44] on developing the WB schemes
for the Euler system under gravitational field, without requiring a prior knowledge of the stationary
hydrostatic solution.

In physics, the fluid density and thermal pressure are positive, implying that the conservative
variables $U$ must stay in the set of admissible states

$$G := \left\{ U = (\rho, m, E)^\top : \rho > 0, \ p(U) = (\gamma - 1) \left( E - \frac{m^2}{2\rho} \right) > 0 \right\}.$$ 

Given that the initial data in the set $G$, a scheme is defined to be positivity-preserving if its solutions
are always belong to $G$. Over the past decade, studying the positivity-preserving and more generally
bound-preserving high-order numerical methods has attracted much attention and achieved signifi-
cant progresses for hyperbolic systems. Most of those high-order accurate schemes are designed with
two types of limiters: the simple scaling limiter proposed in [68, 69] and the flux-correction limit-
ers proposed in [15, 64]. Based on the simple scaling limiter, the high-order positivity-preserving
DG schemes were designed for the Euler equations without source term in [69, 71] and with various
source terms including the gravitational source term in [70]. The bound-preserving methods
were also extended to, for example, the shallow water equations [61], the special relativistic Euler
equations [54, 36, 56, 49], the compressible Navier–Stokes equations [67], and the compressible
magnetohydrodynamic systems [48, 50, 51, 53], and the general relativistic Euler equations under
strong gravitational fields [47]. Recently, a universal framework, called geometric quasilinearization
(GQL), was proposed in [52] for studying general bound-preserving problems involving nonlinear
constraints, with applications to a wide variety of physical systems including the Euler equations. For
more developments and applications, the readers are referred to the review articles [40, 65] and the
references therein.

The present paper is concerned with the central DG (CDG) methods for solving the Euler equa-
tions under gravitational fields. The CDG methods are a family of high-order numerical schemes
based on the DG methods [8] and the central scheme framework [35, 20, 31], which were originally
introduced for solving the hyperbolic conservation laws [32], and have been applied to the Hamilton-
Jacobi equations [24], the ideal magnetohydrodynamic equations [23, 22], and the special relativistic
hydrodynamic [72] and magnetohydrodynamic equations [73]. Based on the simple scaling limiter,
the high-order bound-preserving CDG schemes were constructed for the scalar conservation laws and
the Euler equations [30], the relativistic Euler equations [56], and the shallow water equations [29].
The CDG methods evolve two copies of numerical solutions defined on two sets of meshes (e.g. the
primal mesh and its dual mesh), avoiding using any exact or approximate Riemann solvers at the cell
interfaces which can be extremely complicated and time-consuming in some cases. Although needing
more memory space than standard DG methods, the CDG methods were proven to allow relatively
larger time step-sizes [38] and be more accurate in some numerical tests [33].

The aim of this work is to design and rigorously analyze the high-order positivity-preserving
WB CDG schemes for the Euler equations under gravitational fields. A second-order positivity-
preserving WB finite volume scheme based on a relaxation Riemann solver was developed in [42]
for the Euler equations with gravity for arbitrary hydrostatic equilibria. Based on the (non-central)
DG framework, the arbitrarily high-order positivity-preserving WB methods were proposed in [57]
for the Euler equations with gravitation. It is also worth mentioning that, in the context of the shallow
water equations, several positivity-preserving WB schemes have been developed in the literature [19,
59, 61, 29, 66]. However, within the CDG framework, the study of the positivity-preserving WB
schemes for the Euler equations with gravitation is still blank.

For the regular (non-central) DG methods [57], a key to achieve the WB and positivity-preserving
properties simultaneously is based on a suitable modification of the HLLC numerical flux, which
satisfies both the contact property and the positivity. By contrast, the CDG methods have no numerical
flux but possess an extra numerical dissipation term (not existing in the regular DG methods). As such,
some existing WB and positivity-preserving techniques in the regular DG case [57] do not apply to
the CDG case. Therefore, the design and analysis of positivity-preserving WB schemes in the CDG
framework have quite different difficulties and require the development of new techniques. Most
notably, we need to carefully deal with the numerical dissipation term by proposing a novel critical
projection operator, so as to obtain the WB and positivity-preserving properties simultaneously while
also maintaining the conservativeness and stability of the CDG schemes. The main efforts in this
paper are summarized as follows.

- In order to obtain the WB and positivity-preserving properties simultaneously while also keep-
ing the conservativeness and stability of the schemes, a novel spatial discretization is devised in
the CDG framework based on suitable modifications to the numerical dissipation term and the
source term approximation.

- The modifications are based on a crucial projection operator for the stationary hydrostatic solu-
tion, which is proposed for the first time in this work. This novel projection has the same order
of accuracy as the standard $L^2$-projection, can be explicitly calculated, and is easy to implement
without solving any optimization problems. More importantly, it ensures that the projected sta-
tionary solution has the same cell averages on both the primal and dual meshes, which is key to
achieve the desired properties of our schemes.

- Based on some convex decomposition techniques, a weak positivity property of the resulting WB CDG schemes is rigorously proved, which implies that a simple limiter [69, 46] can ensure the positivity-preserving property without losing the high-order accuracy. The WB modifications of the numerical dissipation term and the approximate source term lead to additional difficulties in our positivity-preserving analyses, which are more complicated than the analysis for the standard CDG schemes.

The rest of the paper is organized as follows. Section 2 proposes the novel projection of the stationary hydrostatic solutions. Section 3 constructs the high-order positivity-preserving WB CDG method for the one-dimensional Euler equations under gravitational fields. The proposed CDG schemes are extended to the two-dimensional case in Section 4. Section 5 gives several numerical examples to verify the high-order accuracy, robustness, and effectiveness of our schemes. Concluding remarks are finally presented in Section 6.

2 Novel projection of the stationary hydrostatic solutions

This section presents a novel projection of the stationary hydrostatic solutions, which will play a crucial role in designing our positivity-preserving WB CDG method.

2.1 Notations

Let us introduce some standard notations. The spatial domain $\Omega$ is uniformly divided into $\{I_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}$ with constant stepsize $\Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. If denoting $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$, then $\{I_{j+\frac{1}{2}} := (x_j, x_{j+1})\}$ forms a dual partition. To approximate the exact solution $U(x,t)$ in the CDG framework, two discrete function spaces are defined associated with the primal mesh $\{I_j\}$ and the dual mesh $\{I_{j+\frac{1}{2}}\}$, respectively, as

$$V_h^{C,k} = \left\{v \in L^2(\Omega) : v|_{I_j} \in P^k(I_j) \forall j \right\}, \quad V_h^{D,k} = \left\{w \in L^2(\Omega) : w|_{I_{j+\frac{1}{2}}} \in P^k(I_{j+\frac{1}{2}}) \forall j \right\},$$

where $P^k(I_j)$ and $P^k(I_{j+\frac{1}{2}})$ denote the space of the polynomials with degree at most $k$ on the cells $I_j$ and $I_{j+\frac{1}{2}}$, respectively.

2.2 Motivation of the novel projection

Assume that the target equilibrium state is known and denoted by $\{\rho^s(x), u^s(x), p^s(x)\}$. This yields

$$u^s(x) = 0, \quad (p^s(x))_x = -\rho^s(x)\phi_x.$$ 

Let $U^s(x) = (\rho^s(x), 0, p^s(x)/(\gamma - 1))^\top$. The standard CDG method is generally not WB for the stationary hydrostatic solutions of the Euler system (1), and some modifications are required. As the WB
DG and finite volume schemes in [57, 26, 25], our WB CDG methods proposed in the Section 3 are also achieved by suitable modifications based on the projection of the target stationary hydrostatic solution. However, the standard \(L^2\)-projection is not a good choice in the present CDG framework, as it may lose the conservative property and affect the positivity-preserving property of the CDG schemes, which will be clarified in Remarks 3.1 and 3.3. In order to maintain the WB, positivity-preserving and conservative properties at the same time, we need to seek a new projection, which ensures that the projected stationary solutions \(U^s_C \in [\mathcal{V}_h^{C,k}]^3\) and \(U^s_D \in [\mathcal{V}_h^{D,k}]^3\), have the same cell averages, namely,

\[
\int_{I_j} U^s_C \, dx = \int_{I_j} U^s_D \, dx, \quad \int_{I_{j+1}^+} U^s_D \, dx = \int_{I_{j+1}^+} U^s_C \, dx, \quad \forall j.
\]

(4)

The projected stationary solutions \(U^s_C\) and \(U^s_D\) will be used to modify the numerical dissipation term and discretized source term for the WB property, and the desired condition (4) will be important in guaranteeing the provably positivity-preserving and conservative properties; see more details in Section 3.3.1 and Section 3.4.2.

**Remark 2.1.** Note that, for the shallow-water equations with (non-flat) bottom topography function \(b(x)\), a similar projection of \(b(x)\) is also required in designing the positivity-preserving WB CDG methods in [29], where the projection of \(b(x)\) is defined by by solving a constrained minimization problem

\[
\min_{b^C_h \in \mathcal{V}_h^{C,k}, \ b^D_h \in \mathcal{V}_h^{D,k}} \|b^C_h - b\|_{L^2(\Omega)} + \|b^D_h - b\|_{L^2(\Omega)},
\]

subject to

\[
\int_{I_j} b^C_h \, dx = \int_{I_j} b^D_h \, dx, \quad \int_{I_{j+1}^+} b^D_h \, dx = \int_{I_{j+1}^+} b^C_h \, dx.
\]

(5)

Although there is no rigorous proof, numerical tests in [29] indicate that the above projected approximations \(b^C_h\) and \(b^D_h\) have the same high-order accuracy as the standard \(L^2\)-projection, provided that \(b(x)\) is a smooth function. The notion of (5) can be extended to our case to construct a projection satisfies (4). However, this projection is not easy to implement due to the involved optimization problem, and its accuracy has not yet been theoretically justified.

We find a new projection, which satisfies (4) and is more efficient than (5). Our new projection can be explicitly calculated without solving any (constrained) optimization problems, and thus can be easily implemented. Moreover, we can rigorously prove that our new projection also has the same order of accuracy as the standard \(L^2\)-projection.

### 2.3 Definition of the novel projection

We first define our new projection operator on the primal mesh. Let \(\mathcal{P}^C_h : L^2(\Omega) \longrightarrow \mathcal{V}_h^{C,k}\) denote an operator, which maps any function \(f(x) \in L^2(\Omega)\) onto the piecewise polynomial space \(\mathcal{V}_h^{C,k}\), and satisfies

\[
\int_{I_j} \mathcal{P}^C_h(f) \, dx = \int_{I_j} f \, dx, \quad \int_{I_{j+1}^+} \mathcal{P}^C_h(f) \, dx = \int_{I_{j+1}^+} f \, dx, \quad \forall j,
\]

(6)
with \( I_j^- = (x_{j-\frac{1}{2}}, x_j) \), \( I_j^+ = (x_j, x_{j+\frac{1}{2}}) \). Note that with only the condition (6), the mapping \( P_h^C \) is not uniquely determined. We uniquely define the mapping \( P_h^C \) by

\[
\int_{I_j^+} P_h^C(f)dx = \int_{I_j^-} f dx, \quad \int_{I_j^+} P_h^C(f)vdx = \int_{I_j^-} f vdx, \quad \forall \ v \in \mathbb{P}^k(I_j) \setminus \text{span}\{\Phi_1\}, \tag{7}
\]

where \( \mathbb{P}^k(I_j) \setminus \text{span}\{\Phi_1\} := \text{span}\{\Phi_0, \Phi_2, \Phi_3, \cdots, \Phi_k\} \), and \( \{\Phi_i\}_{i=0}^k \) denotes an orthogonal basis \( \mathbb{P}^k(I_j) \) and is taken as the scaled Legendre polynomials

\[
\Phi_0(\xi) = 1, \quad \Phi_1(\xi) = \xi, \quad \Phi_2(\xi) = \xi^2 - \frac{1}{3}, \quad \Phi_3(\xi) = \xi^3 - \frac{3}{5}\xi, \cdots, \tag{8}
\]

with \( \xi = 2(x-x_j)/\Delta x \). Take \( v = \Phi_0 = 1 \) in (7), one has the equality \( \int_{I_j} P_h^C(f)dx = \int_{I_j} f dx \), which implies that the projection \( P_h^C \) satisfies the desired condition (6). We will show in Lemma 2.1 that the mapping \( P_h^C \) defined by (7) is a projection operator from \( L^2(\Omega) \) onto the space \( \mathbb{V}_h^{C,k} \).

Similarly, we can define the projection \( P_h^D : L^2(\Omega) \rightarrow \mathbb{V}_h^{D,k} \) on the dual mesh by

\[
\int_{I_j^+} P_h^D(f)dx = \int_{I_j^-} f dx, \quad \int_{I_{j+\frac{1}{2}}} P_h^D(f)vdx = \int_{I_{j+\frac{1}{2}}} f vdx, \quad \forall \ v \in \mathbb{P}^k(I_{j+\frac{1}{2}}) \setminus \text{span}\{\Phi_1\}, \tag{9}
\]

where \( \Phi_i(\xi), \ i = 0, 1, \cdots, k \), defined in (8) are with \( \xi = 2(x-x_{j+\frac{1}{2}})/\Delta x \). It can be shown that \( P_h^D \) satisfies

\[
\int_{I_j} P_h^D(f)dx = \int_{I_j} f dx, \quad \int_{I_{j+1}} P_h^D(f)dx = \int_{I_{j+1}} f dx, \quad \forall \ j. \tag{10}
\]

Combining (6) with (9) gives

\[
\int_{I_j} P_h^C(f)dx = \int_{I_j} P_h^D(f)dx, \quad \int_{I_{j+\frac{1}{2}}} P_h^D(f)dx = \int_{I_{j+\frac{1}{2}}} P_h^C(f)dx, \quad \forall \ j. \tag{10}
\]

If assuming that \( U_h^{s,C} \) and \( U_h^{s,D} \) denote the new projections of each component of the stationary solutions \( U^s(x) \) onto the space \( \mathbb{V}_h^{C,k} \) and \( \mathbb{V}_h^{D,k} \), respectively, then it follows from (10) that

\[
\int_{I_j} U_h^{s,C}dx = \int_{I_j} U^sdx = \int_{I_j} U_h^{s,D}dx, \quad \int_{I_{j+\frac{1}{2}}} U_h^{s,D}dx = \int_{I_{j+\frac{1}{2}}} U^sdx = \int_{I_{j+\frac{1}{2}}} U_h^{s,C}dx, \quad \forall \ j. \tag{11}
\]

which yields (4). The identity (11) plays an important role in the positivity-preserving analyses in Section 3.4.2.

### 2.4 Properties of the novel projection

For convenience, we will mainly discuss the properties of the operator \( P_h^C \) on the primal mesh in detail, as those of \( P_h^D \) on the dual mesh are very similar.

Note that, on each cell \( I_j \), the projected solution \( P_h^C(f) \) is a piecewise polynomial and can be explicitly written as

\[
f_h^C(x) = \sum_{i=0}^{k} N_i(f)\Phi_i(x), \quad x \in I_j,
\]
with the coefficients \( \{N_i(f)\}_{i=0}^k \) given by
\[
N_i(f) = \frac{\int_{I_j} f \Phi_i \text{d}x}{\int_{I_j} \Phi_i^2 \text{d}x}, \quad i \neq 1, \quad N_1(f) = \frac{\int_{I_j} \left( f - \sum_{i \neq 1} N_i(f) \Phi_i \right) \text{d}x}{\int_{I_j} \Phi_1 \text{d}x},
\]
where \( \int_{I_j} \Phi_1 \text{d}x = -\frac{\Delta x}{4} \neq 0 \). It is easy to show that the operator \( \mathcal{P}_h^C \) satisfies the following properties.

**Lemma 2.1.** For any function \( g(x) \in \mathbb{V}_h^k \), one has \( \mathcal{P}_h^C(g)(x) = g(x) \), and consequently \( \mathcal{P}_h^C \) is a projection operator.

**Proof.** For each \( j \), on the cell \( I_j \) the function \( g(x) \in \mathbb{V}_h^k \) can be expressed as a linear combination of the orthogonal basis \( \{\Phi_i\}_{i=0}^k \), i.e.
\[
g(x) = \sum_{i=0}^k \alpha_i \Phi_i(x), \quad x \in I_j.
\]
Noting that
\[
N_i(g) = \alpha_i, \quad 0 \leq i \leq k,
\]
we obtain
\[
\mathcal{P}_h^C(g)(x) = \sum_{i=0}^k N_i(g) \Phi_i(x) = \sum_{i=0}^k \alpha_i \Phi_i(x) = g(x), \quad x \in I_j.
\]
For any functions \( f_1, f_2 \in L^2(\Omega) \) and any real numbers \( \alpha_1, \alpha_2 \), one has
\[
N_i(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 N_i(f_1) + \alpha_2 N_i(f_2), \quad 0 \leq i \leq k,
\]
It follows that
\[
\mathcal{P}_h^C(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{P}_h^C(f_1) + \alpha_2 \mathcal{P}_h^C(f_2),
\]
which implies \( \mathcal{P}_h^C \) is a linear operator. Furthermore, for any \( f \in L^2(\Omega) \), we have \( \mathcal{P}_h^C(f) \in \mathbb{V}_h^k \) and thus \( \mathcal{P}_h^C(\mathcal{P}_h^C(f)) = \mathcal{P}_h^C(f) \). This implies \( \mathcal{P}_h^C \) is a projection operator. The proof is completed. \( \blacksquare \)

**Lemma 2.2.** The projection operator \( \mathcal{P}_h^C \) is bounded.

**Proof.** Let us consider an arbitrary function \( f \in L^2(I_j) \). Applying the triangular inequality gives
\[
\| \mathcal{P}_h^C(f) \|_{L^2(I_j)} \leq \sum_{i=0}^k |N_i(f)| \| \Phi_i \|_{L^2(I_j)}.
\]
If \( i \neq 1 \), one can derive that
\[
|N_i(f)| = \frac{|\int_{I_j} f \Phi_i \text{d}x|}{\| \Phi_i \|^2_{L^2(I_j)}} \leq \frac{\| f \|_{L^2(I_j)} \| \Phi_i \|_{L^2(I_j)}}{\| \Phi_i \|^2_{L^2(I_j)}} = \frac{\| f \|_{L^2(I_j)}}{\| \Phi_i \|_{L^2(I_j)}},
\]

\[
8
\]
which leads to
\[
|N_i(f)| \leq \|\Phi_i\|_{L^2(I_j)} \leq M_i \|f\|_{L^2(I_j)}, \quad \text{with} \quad M_i = 1.
\] (12)

If \( i = 1 \), we have
\[
|N_1(f)| \leq \left| \int_{I_j} f \, dx \right| + \sum_{i \neq 1} |N_i(f)| \int_{I_j} \Phi_i \, dx | \leq \left| \int_{I_j} f \, dx \right| + \sum_{i \neq 1} |N_i(f)| \int_{I_j} \Phi_i \, dx |
\]
\[
= \left| \int_{I_j} f \, dx \right| + \sum_{i \neq 1} |N_i(f)| \int_{I_j} \Phi_i \, dx |N_i(f)|.
\]

Note that \( I_j \subseteq I_i \) and \( \Phi_0 = 1 \), one has
\[
\left| \int_{I_j} f \, dx \right| = \left| \int_{I_j} f \Phi_0 \, dx \right| \leq \|f\|_{L^2(I_j)} \|\Phi_0\|_{L^2(I_j)}, \quad \left| \int_{I_j} \Phi_i \, dx \right| \leq \|\Phi_i\|_{L^2(I_j)} \|\Phi_0\|_{L^2(I_j)}.
\]
It follows that
\[
|N_1(f)| \leq \frac{\|f\|_{L^2(I_j)} \|\Phi_0\|_{L^2(I_j)}}{\int_{I_j} \Phi_1 \, dx } \sum_{i \neq 1} \frac{\|\Phi_i\|_{L^2(I_j)}}{\|\Phi_1\|_{L^2(I_j)}} \|N_i(f)\|
\]
\[
\leq \frac{\|f\|_{L^2(I_j)} \|\Phi_0\|_{L^2(I_j)}}{\int_{I_j} \Phi_1 \, dx } \sum_{i \neq 1} \frac{\|\Phi_i\|_{L^2(I_j)}}{\|\Phi_1\|_{L^2(I_j)}} \|f\|_{L^2(I_j)}
\]
\[
= (k+1) \frac{\|\Phi_0\|_{L^2(I_j)}}{\int_{I_j} \Phi_1 \, dx } \|f\|_{L^2(I_j)},
\]
where (12) has been used in the second inequality. Therefore, we obtain
\[
|N_1(f)| \leq \frac{\|\Phi_1\|_{L^2(I_j)} \|\Phi_0\|_{L^2(I_j)}}{\int_{I_j} \Phi_1 \, dx } \|f\|_{L^2(I_j)}
\] (13)

with
\[
M_1 := (k+1) \times \frac{\|\Phi_1\|_{L^2(I_j)} \|\Phi_0\|_{L^2(I_j)}}{\int_{I_j} \Phi_1 \, dx } = \frac{2\sqrt{6}}{3} (k+1).
\]
Combining (12) with (13) yields
\[
\|P^C_h(f)\|_{L^2(I_j)} \leq M \|f\|_{L^2(I_j)}, \quad M = \sum_{i=0}^k M_i = k + \frac{2\sqrt{6}}{3} (k+1).
\]
This finishes the proof.

**Theorem 2.1.** For any function \( f \in W^{k+1}_2(I_j) \), one has
\[
\|f - P^C_h(f)\|_{L^2(I_j)} \leq C_{k+1} (\Delta x)^{k+1} |f|_{W^{k+1}_2(I_j)}.
\]

where \( W^{k+1}_2(I_j) \) is the standard Sobolev space, \( |\cdot|_{W^{k+1}_2(I_j)} \) is the Sobolev seminorm of order \( k + 1 \), and \( C_{k+1} \) is a constant only depending on \( k \).
Consequently,

\[ \| f - \mathcal{P}_h^C(f) \|_{L^2(I_j)} \leq \| f - g \|_{L^2(I_j)} + \| g - \mathcal{P}_h^C(f) \|_{L^2(I_j)} \]

= \| f - g \|_{L^2(I_j)} + \| \mathcal{P}_h^C(f - g) \|_{L^2(I_j)} \quad \text{(Lemma 2.1)}

\leq \| f - g \|_{L^2(I_j)} + M \| f - g \|_{L^2(I_j)} \quad \text{(Lemma 2.2)}

= (1 + M) \| f - g \|_{L^2(I_j)}. \quad (14)

Consequently,

\[ \| f - \mathcal{P}_h^D(f) \|_{L^2(I_j)} \leq (1 + M) \inf_{g \in \mathbb{P}_k(I_j)} \| f - g \|_{L^2(I_j)} \]

\leq (1 + M) \tilde{C}_{k+1}(\Delta x)^{k+1} |f|_{W^{k+1}_2(I_j)} \quad \text{(Bramble – Hilbert)}

\[ = C_{k+1}(\Delta x)^{k+1} |f|_{W^{k+1}_2(I_j)}. \]

where have used the Bramble-Hilbert Lemma [9], and \( \tilde{C}_{k+1} \) is a constant only depending on \( k \). \( \blacksquare \)

**Remark 2.2.** Similarly, the operator \( \mathcal{P}_h^D \) on the dual mesh is also linear, bounded, and a projection. Furthermore, for any function \( f \in W^{k+1}_2(I_{j-\frac{1}{2}}) \)

\[ \| f - \mathcal{P}_h^D(f) \|_{L^2(I_{j-\frac{1}{2}})} \leq C_{k+1}(\Delta x)^{k+1} |f|_{W^{k+1}_2(I_{j-\frac{1}{2}})}, \]

where \( C_{k+1} \) is a constant only depending on \( k \).

**Remark 2.3.** By similar arguments, one can prove that the errors \( \| f - \mathcal{P}_h^C(f) \|_{L^q(\Omega)} \) and \( \| f - \mathcal{P}_h^D(f) \|_{L^q(\Omega)} \) are of order \( O((\Delta x)^{k+1}) \) for a general \( q \) \((1 \leq q \leq +\infty)\) and \( f \in W^{k+1}_q(\Omega) \). The details are omitted here.

### 3  Positivity-preserving WB CDG schemes in one dimension

To solve the system (1), the CDG schemes evolve two copies of numerical solutions, denoted by \( U_h^C(x,t) \) and \( U_h^D(x,t) \), on the primal and dual meshes, respectively.

#### 3.1  Review of the standard CDG method

The semi-discrete formulations of the standard CDG method are given as follows: for any test function \( v \in \mathbb{V}_h^{C,k} \) and \( w \in \mathbb{V}_h^{D,k} \), look for the numerical solutions \( U_h^C \in [\mathbb{V}_h^{C,k}]^3 \) and \( U_h^D \in [\mathbb{V}_h^{D,k}]^3 \) satisfying

\[ \int_{I_j} \frac{\partial U_h^C}{\partial t} v dx = \frac{1}{\tau_{\max}} \int_{I_j} (U_h^D - U_h^C) v dx + \int_{I_j} F(U_h^D) v dx \]

\[ - \left( F(U_h^D(x_{j+\frac{1}{2}})) v(x_{j+\frac{1}{2}}^-) - F(U_h^D(x_{j-\frac{1}{2}})) v(x_{j-\frac{1}{2}}^+) \right) + \int_{I_j} S(U_h^D, (\phi_h^D) x) v dx, \quad (15) \]
To define the positivity-preserving CDG schemes, we introduce the following two sets:

\[ \mathcal{G}_h^{C,k} := \left\{ v \in [\mathcal{V}_h^{C,k}]^3 : \frac{1}{\Delta x} \int_{I_j} v(x) \, dx \in G, \forall j \right\}, \]

\[ \mathcal{G}_h^{D,k} := \left\{ v \in [\mathcal{V}_h^{D,k}]^3 : \frac{1}{\Delta x} \int_{I_j} v(x) \, dx \in \mathcal{S}_j, \forall j \right\}, \]

where \( \mathcal{S}_j \) denotes the set of some critical points in \( I_j \) which will be specified in (25). Similarly, we can define the sets \( \mathcal{D}_h^{D,k} \) and \( \mathcal{G}_h^{D,k} \) on the dual mesh as

\[ \mathcal{D}_h^{D,k} := \left\{ w \in [\mathcal{V}_h^{D,k}]^3 : \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} w(x) \, dx \in G, \forall j \right\}, \]

\[ \mathcal{G}_h^{D,k} := \left\{ w \in [\mathcal{V}_h^{D,k}]^3 : w|_{I_{j+\frac{1}{2}}} \in G, \forall x \in \mathcal{S}_{j+\frac{1}{2}}, \forall j \right\}, \]

where \( \mathcal{S}_{j+\frac{1}{2}} \) denotes the set of some critical points in \( I_{j+\frac{1}{2}} \) which will be specified in (26).

We will describe in Section 3.3 a suitable CDG spatial discretization of the Euler system (1), and the resulting semi-discrete CDG schemes can be written in the ODE form as follows

\[ \frac{dU_h^C}{dt} = L_h^C(U_h^C, U_h^D), \quad \frac{dU_h^D}{dt} = L_h^D(U_h^D, U_h^C), \]

where \( L_h^C \) and \( L_h^D \) are respectively spatial discretization operators on the primal and dual meshes obtained from suitable modifications to the standard CDG discretization.

**Definition 3.1.** Suppose the initial data satisfy \( U_h^C(x,0) = U_h^{s,C} \), \( U_h^D(x,0) = U_h^{s,D} \), a CDG scheme is defined to be WB if the flux and source term approximations balance each other, namely \( L_h^C(U_h^{s,C}, U_h^{s,D}) = 0 \), \( L_h^D(U_h^{s,D}, U_h^{s,C}) = 0 \).

**Definition 3.2.** A CDG scheme is defined to be positivity-preserving if its numerical solutions \( U_h^C \), \( U_h^D \) stay in sets \( \mathcal{G}_h^{C,k} \) and \( \mathcal{G}_h^{D,k} \), respectively. For clarity, if a CDG scheme preserves the numerical solutions in set \( \mathcal{G}_h^{C,k} \) and \( \mathcal{G}_h^{D,k} \), then we say that it satisfies a weak positivity-preserving property.

We aim at designing the high-order accurate CDG schemes that satisfy the WB and positivity-preserving properties simultaneously. This goal will be achieved by following three steps:
First, we seek spatial discretization operators $L_h^C(U_h^C, U_h^D)$ and $L_h^D(U_h^D, U_h^C)$ satisfying both the WB property:

\[ L_h^C(U_h^C, U_h^D) = 0, \quad L_h^D(U_h^D, U_h^C) = 0, \]

and the weak positivity-preserving property: if $U_h^C \in \mathbb{G}_{h}^{C,k}$, $U_h^D \in \mathbb{G}_{h}^{D,k}$, then

\[ U_h^C + \Delta t L_h^C(U_h^C, U_h^D) \in \overline{\mathbb{G}_{h}^{C,k}}, \quad U_h^D + \Delta t L_h^D(U_h^D, U_h^C) \in \overline{\mathbb{G}_{h}^{D,k}}, \]

under some CFL-type condition on time stepsize $\Delta t$.

Second, we further discretize the ODE system (17) in time using a strong-stability-preserving (SSP) explicit Runge-Kutta method [13].

Finally, a local scaling positivity-preserving limiting procedure, which will be introduced in Section 3.5, is applied to the intermediate solutions of the Runge-Kutta discretization. This procedure corresponds to two operators $\Pi_h^C : \overline{\mathbb{G}_{h}^{C,k}} \rightarrow \mathbb{G}_{h}^{C,k}$ and $\Pi_h^D : \overline{\mathbb{G}_{h}^{D,k}} \rightarrow \mathbb{G}_{h}^{D,k}$, which satisfy the conservative property

\[ \frac{1}{\Delta x} \int_{I_j} \Pi_h^C(v) dx = \frac{1}{\Delta x} \int_{I_j} v dx \quad \forall j, \forall v \in \overline{\mathbb{G}_{h}^{C,k}}, \]

\[ \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} \Pi_h^D(w) dx = \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} w dx \quad \forall j, \forall w \in \overline{\mathbb{G}_{h}^{D,k}}. \]

For the first order CDG scheme ($k = 0$), both $\Pi_h^C$ and $\Pi_h^D$ become the identity operators, so that the positivity-preserving limiting procedure is only operated for the high-order CDG schemes with $k \geq 1$.

Let $U_{h}^{C,n}$ and $U_{h}^{D,n}$ denote the numerical solution at time $t = t_n$. The resulting fully discrete positivity-preserving WB CDG method, with the third-order accurate SSP Runge-Kutta time discretization as example, is then given as follows.

- Set $U_{h}^{C,0} = \Pi_h^C[U_h^{C}(x,0)]$ and $U_{h}^{D,0} = \Pi_h^D[U_h^{D}(x,0)]$, where $U_h^{C}(x,0)$ and $U_h^{D}(x,0)$ denote the novel projections of the initial data $U(x,0)$ onto the space $[V_h^{C,3}]^3$ and $[V_h^{D,3}]^3$, respectively.

- For $n = 0, \cdots, N - 1$, compute $U_{h}^{C,n+1}$ and $U_{h}^{D,n+1}$ as follows:
  
  \begin{enumerate}
  \item Compute the intermediate solutions $U_{h}^{C,(1)}$ and $U_{h}^{D,(1)}$ via
  \[ U_{h}^{C,(1)} = \Pi_h^C[U_h^{C,n} + \Delta t L_h^C(U_h^{C,n}, U_h^{D,n})], \]
  \[ U_{h}^{D,(1)} = \Pi_h^D[U_h^{D,n} + \Delta t L_h^D(U_h^{D,n}, U_h^{C,n})]. \]
  
  \item Compute the intermediate solutions $U_{h}^{C,(2)}$ and $U_{h}^{D,(2)}$ via
  \[ U_{h}^{C,(2)} = \Pi_h^C\left[\frac{3}{4} U_h^{C,n} + \frac{1}{4} \left( U_{h}^{C,(1)} + \Delta t L_h^C(U_{h}^{C,(1)}, U_{h}^{D,(1)}) \right) \right], \]
  \[ U_{h}^{D,(2)} = \Pi_h^D\left[\frac{3}{4} U_h^{D,n} + \frac{1}{4} \left( U_{h}^{D,(1)} + \Delta t L_h^D(U_{h}^{D,(1)}, U_{h}^{C,(1)}) \right) \right]. \]
  \end{enumerate}
Assume Proposition 3.1.

the CDG schemes as indicated by the following proposition. we can also prove that our modification of dissipation term will not affect the high-order accuracy of does not lead to this desirable feature.

Thanks to the key property of our schemes. It is worth noting that using the standard $L^2$ which implies that the modified numerical dissipation term (20)

As well-known, the numerical dissipation term in the standard CDG method (15) is essential for the numerical stability [33]. However, it would destroy the WB property at the steady state. To address this issue, we propose to modify it into

\[
\tilde{\mathcal{d}}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) = \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^D - \mathbf{U}_h^C) v dx + \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^{s,C} - \mathbf{U}_h^{s,D}) v dx, \tag{20}
\]

so that

\[
\tilde{\mathcal{d}}_j^C(\mathbf{U}_h^{s,C}, \mathbf{U}_h^{s,D}, v) = \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^{s,D} - \mathbf{U}_h^{s,C}) v dx + \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^{s,C} - \mathbf{U}_h^{s,D}) v dx = 0.
\]

Remark 3.1. Thanks to the key property (11) of our novel projection, it holds

\[
\tilde{\mathcal{d}}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, 1) = \mathcal{d}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, 1),
\]

which implies that the modified numerical dissipation term (20) would not destroy the conservative property of our schemes. It is worth noting that using the standard $L^2$-projection for $\mathbf{U}_h^{s,C}$ and $\mathbf{U}_h^{s,D}$ does not lead to this desirable feature.

In addition to the above-mentioned advantages of the modified numerical dissipation term (20), we can also prove that our modification of dissipation term will not affect the high-order accuracy of the CDG schemes as indicated by the following proposition.

Proposition 3.1. Assume $\mathbf{U}^s \in W^{k+1}_2(\Omega)$, $\tau_{\max} = O(\Delta x)$, then on each cell $I_j$, it holds that

\[
\left| \mathcal{d}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) - \tilde{\mathcal{d}}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) \right| = \frac{1}{\tau_{\max}} \left| \int_{I_j} (\mathbf{U}_h^{s,C} - \mathbf{U}_h^{s,D}) v dx \right| \leq O((\Delta x)^{k+1}).
\]

3.3 Spatial discretization operators $L_C^h$ and $L_D^h$

For convenience, we will mainly present the CDG spatial discretization on the primal mesh in detail, as that on the dual mesh is very similar.

3.3.1 WB dissipation term

As well-known, the numerical dissipation term

\[
d_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) = \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^D - \mathbf{U}_h^C) v dx, \quad v \in \mathcal{V}_h^{C,k},
\]

in the standard CDG method (15) is essential for the numerical stability [33]. However, it would destroy the WB property at the steady state. To address this issue, we propose to modify it into

\[
\tilde{d}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) = \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^D - \mathbf{U}_h^C) v dx + \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^{s,C} - \mathbf{U}_h^{s,D}) v dx, \tag{20}
\]

so that

\[
\tilde{d}_j^C(\mathbf{U}_h^{s,C}, \mathbf{U}_h^{s,D}, v) = \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^{s,D} - \mathbf{U}_h^{s,C}) v dx + \frac{1}{\tau_{\max}} \int_{I_j} (\mathbf{U}_h^{s,C} - \mathbf{U}_h^{s,D}) v dx = 0.
\]

Remark 3.1. Thanks to the key property (11) of our novel projection, it holds

\[
\tilde{d}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, 1) = d_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, 1),
\]

which implies that the modified numerical dissipation term (20) would not destroy the conservative property of our schemes. It is worth noting that using the standard $L^2$-projection for $\mathbf{U}_h^{s,C}$ and $\mathbf{U}_h^{s,D}$ does not lead to this desirable feature.

In addition to the above-mentioned advantages of the modified numerical dissipation term (20), we can also prove that our modification of dissipation term will not affect the high-order accuracy of the CDG schemes as indicated by the following proposition.

Proposition 3.1. Assume $\mathbf{U}^s \in W^{k+1}_2(\Omega)$, $\tau_{\max} = O(\Delta x)$, then on each cell $I_j$, it holds that

\[
\left| \mathcal{d}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) - \tilde{d}_j^C(\mathbf{U}_h^C, \mathbf{U}_h^D, v) \right| = \frac{1}{\tau_{\max}} \left| \int_{I_j} (\mathbf{U}_h^{s,C} - \mathbf{U}_h^{s,D}) v dx \right| \leq O((\Delta x)^{k+1}).
\]
Proof. Using the triangular inequality gives
\[
\frac{1}{\tau_{\text{max}}} \left| \int_{I_j} \left( U_h^{s,C} - U_h^{s,D} \right) v \, dx \right| \leq \frac{1}{\tau_{\text{max}}} \int_{I_j} \left| (U_h^{s,C} - U_h^s) v \right| dx + \frac{1}{\tau_{\text{max}}} \int_{I_j} \left| (U_h^s - U_h^{s,D}) v \right| dx.
\]
Under the condition $\| U_h^s \|_{W^{k+1}_2(I_j)} \| v \|_{L^2(I_j)} = \mathcal{O}(\Delta x)$, Theorem 2.1 implies
\[
\frac{1}{\tau_{\text{max}}} \int_{I_j} \left( U_h^{s,C} - U_h^{s,D} \right) v \, dx = \frac{1}{\tau_{\text{max}}} \left( \int_{I_j} \left( U_h^s - U_h^{s,D} \right) v \, dx + \int_{I_j} \left( U_h^s - U_h^{s,D} \right) v \, dx \right)
\leq \frac{1}{\tau_{\text{max}}} \left( \| U_h^s - U_h^{s,D} \|_{L^2(I_j)} \| v \|_{L^2(I_j)} + \| U_h^s - U_h^{s,D} \|_{L^2(I_j)} \| v \|_{L^2(I_j)} \right) = \mathcal{O}(\Delta x)^{k+1}.
\]
Similarly, one has
\[
\frac{1}{\tau_{\text{max}}} \int_{I_j} \left( U_h^s - U_h^{s,D} \right) v \, dx = \frac{1}{\tau_{\text{max}}} \left( \int_{I_j} \left( U_h^s - U_h^{s,D} \right) v \, dx + \int_{I_j} \left( U_h^s - U_h^{s,D} \right) v \, dx \right)
\leq \frac{1}{\tau_{\text{max}}} \left( \| U_h^s - U_h^{s,D} \|_{L^2(I_j)} \| v \|_{L^2(I_j)} + \| U_h^s - U_h^{s,D} \|_{L^2(I_j)} \| v \|_{L^2(I_j)} \right) = \mathcal{O}(\Delta x)^{k+1}.
\]
Combining these results, we conclude that $\frac{1}{\tau_{\text{max}}} \left| \int_{I_j} \left( U_h^{s,C} - U_h^{s,D} \right) v \, dx \right| \leq \mathcal{O}(\Delta x)^{k+1}$.

3.3.2 Numerical flux and source term

Let $\xi_{j,x} = \{ x_j^1, \alpha \}_{\alpha=1}^N$ and $\xi_{j,x}^2 = \{ x_j^2, \alpha \}_{\alpha=1}^N$ denote the $N$-point Gauss quadrature nodes transformed into the interval $[x_{j-\frac{1}{2}}, x_j]$ and $[x_j, x_{j+\frac{1}{2}}]$, respectively, and $\{ \omega_{\alpha} \}_{\alpha=1}^N$ are the associated weights satisfying $\sum_{\alpha=1}^N \omega_{\alpha} = 1$, with $N \geq k+1$ for the CDG accuracy requirement.

Because $U_h^D(x,t)$ can be discontinuous at $x = x_j$, the element integral $\int_{I_j} F(U_h^D) v_s \, dx$ in (15) is usually divided into two parts
\[
\int_{I_j} F(U_h^D) v_s \, dx = \int_{I_j} F(U_h^D) v_s \, dx + \int_{I_j} F(U_h^D) v_s \, dx,
\]
which is then approximately by the numerical quadrature rule
\[
\int_{I_j} F(U_h^D) v_s \, dx \approx \frac{\Delta x}{2} \sum_{\kappa=1}^2 \sum_{\alpha=1}^N \omega_{\alpha} F(U_h^D(x_j^{\kappa,\alpha})) v_s(x_j^{\kappa,\alpha}).
\]

Next, we introduce a non-standard approximation to the source term integral in (15) to achieve the WB property. This idea is similar to [26] but has some key differences owing to carefully accommodate the positivity-preserving property; see Remark 3.2. Reformulate and decompose the integral of the source term in the momentum equation as
\[
\int_{I_j} S_2(U, \phi_s) \, dx = - \int_{I_j} \rho \phi_s \, dx = \int_{I_j} \frac{\rho}{\rho^s} p_s^x \, dx = \int_{I_j} \left( \frac{\rho}{\rho^s} - \frac{\rho^D}{\rho_j^D} + \frac{p_j}{\rho_j^D} \right) p_s^x \, dx,
\]
where $\rho_j = \frac{1}{\Delta x} \int_{I_j} \rho \, dx$ denotes the cell average. Our numerical approximation to the source term takes the form of
\[
\int_{I_j} S_2(U_h^D, (\phi_h^D)_s) \, dx \approx \int_{I_j} \left( \frac{\rho_h^D}{\rho_h^D} - \frac{(\rho_h^D)^D}{(\rho_h^D)^D} \right) (p_h^s, \alpha) \, dx + \frac{(\rho_h^D)}{(\rho_h^D)^D} \int_{I_j} (p_h^s, \alpha) \, dx.
\]
Similarly, we approximate the source term

\[ \int I_j (p_h^{s,D})_x v dx = - \int I_j p_h^{s,D} v_x dx - \left( p_h^{s,D}(x^+_j) - p_h^{s,D}(x^-_j) \right) v(x_j) + \left( p_h^{s,D}(x_{j+\frac{1}{2}}^-) - p_h^{s,D}(x_{j-\frac{1}{2}}^-) \right) v(x_{j-\frac{1}{2}}) \approx - \int I_j p_h^{s,D} v_x dx + \left( p_h^{s,D}(x_{j+\frac{1}{2}}) v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}(x_{j-\frac{1}{2}}^-) v(x_{j-\frac{1}{2}}) \right), \]

where the following term has been omitted

\[ p_h^{s,D}(x^+_j) - p_h^{s,D}(x^-_j) = (p_h^{s,D}(x^+_j) - p^*(x_j)) + (p^*(x_j) - p_h^{s,D}(x^-_j)) = O(\Delta x^{k+1}). \]

This leads to

\[ \int I_j S_2(U_h^D, (\phi_h^D)_x) v dx \approx \int I_j \left( \frac{p_h^D}{\rho_h^{s,D}} - \frac{\rho_h^{D}}{(\rho_h^{s,D})_j} \right) (p_h^{s,D})_x v dx - \frac{(\rho_h^{D})_j}{(\rho_h^{s,D})_j} \int I_j p_h^{s,D} v_x dx + \frac{(\rho_h^{D})_j}{(\rho_h^{s,D})_j} \left( p_h^{s,D}(x_{j+\frac{1}{2}}) v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}(x_{j-\frac{1}{2}}) v(x_{j-\frac{1}{2}}^-) \right). \]

Therefore, the source term \( \int I_j S_2(U_h^D, (\phi_h^D)_x) v dx \) in the momentum equation can further be approximated by

\[ \langle S_{h,2}^{D}, v \rangle_j = \frac{(\rho_h^{D})_j}{(\rho_h^{s,D})_j} \left( p_h^{s,D}(x_{j+\frac{1}{2}}) v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}(x_{j-\frac{1}{2}}) v(x_{j-\frac{1}{2}}^-) \right) \]

\[ + \frac{\Delta x}{2} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \left[ \frac{(\rho_h^D)_{x_{j+\frac{1}{2}}}}{(\rho_h^{s,D})_j} - \frac{(\rho_h^D)_{x_{j-\frac{1}{2}}}}{(\rho_h^{s,D})_j} \right] \left( p_h^{s,D}_{x_{j+\frac{1}{2}}} v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}_{x_{j-\frac{1}{2}}} v(x_{j-\frac{1}{2}}^-) \right) \]

\[ + \frac{\Delta x}{2} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \left[ \frac{m_h^D_{x_{j+\frac{1}{2}}}}{(\rho_h^{s,D})_j} - \frac{m_h^D_{x_{j-\frac{1}{2}}}}{(\rho_h^{s,D})_j} \right] \left( p_h^{s,D}_{x_{j+\frac{1}{2}}} v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}_{x_{j-\frac{1}{2}}} v(x_{j-\frac{1}{2}}^-) \right). \]

Similarly, we approximate the source term \( \int I_j S_3(U_h^D, (\phi_h^D)_x) v dx \) in the energy equation by

\[ \langle S_{h,3}^{D}, v \rangle_j = \frac{(m_h^D)}{(\rho_h^{s,D})_j} \left( p_h^{s,D}(x_{j+\frac{1}{2}}) v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}(x_{j-\frac{1}{2}}) v(x_{j-\frac{1}{2}}^-) \right) \]

\[ + \frac{\Delta x}{2} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \left[ \frac{m_h^D_{x_{j+\frac{1}{2}}}}{(\rho_h^{s,D})_j} - \frac{m_h^D_{x_{j-\frac{1}{2}}}}{(\rho_h^{s,D})_j} \right] \left( p_h^{s,D}_{x_{j+\frac{1}{2}}} v(x_{j+\frac{1}{2}}^-) - p_h^{s,D}_{x_{j-\frac{1}{2}}} v(x_{j-\frac{1}{2}}^-) \right). \]

(21)
3.3.3 Semi-discrete WB CDG schemes

Combining the modified dissipation term in Section 3.3.1 with the discrete source term in Section 3.3.2, we obtain the final semi-discrete WB CDG method on the primal mesh

$$
\int_{I_j} \frac{\partial U^C_h}{\partial t} \, dx = \frac{1}{\tau} \int_{I_j} (U^D_h - U^C_h) \, dx + \frac{1}{\tau} \int_{I_j} (U^s_h - U^C_h) \, dx

- \left( F(U^D_h(x) + v(x^-)) - F(U^D_h(x) - v(x^+)) \right)

+ \frac{\Delta x}{2} \sum_{\kappa=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} F(U^C_h(x, x)) w(x, x) + \langle S^D_h, v \rangle \quad \forall v \in \mathbb{V}_h^C
$$

(22)

where $\langle S^D_h, v \rangle = (0, \langle S^D_h, v \rangle, \langle S^D_h, v \rangle)^\top$.

The WB CDG spatial discretization on the dual mesh is very similar. Denote $Q = (0, \rho, m)^\top$, one has $S(U, \phi, x) = -\phi, Q$, and the modified source term approximation $\langle S^C_h, w \rangle_{j+\frac{1}{2}}$ is given by

$$
\langle S^C_h, w \rangle_{j+\frac{1}{2}} = \frac{(Q^C_h)_{j+\frac{1}{2}}}{(\rho^C_h)_{j+\frac{1}{2}}} \left( p^C_h(x, x) w(x, x) - p^C_h(x, x) w(x, x) \right)

+ \frac{\Delta x}{2} \sum_{\kappa=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \left( \frac{(Q^C_h(x, x))_{j+\frac{1}{2}}}{(\rho^C_h(x, x))_{j+\frac{1}{2}}} \right) \left( \frac{(p^C_h(x, x))_{j+\frac{1}{2}}}{(\rho^C_h(x, x))_{j+\frac{1}{2}}} \right) \left( \frac{(p^C_h(x, x))_{j+\frac{1}{2}}}{(\rho^C_h(x, x))_{j+\frac{1}{2}}} \right)

$$

where $w \in \mathbb{V}_h^D$, $\{x^1, x^2\}^N_{\kappa=1} = Q^2_{j+\frac{1}{2}}$ and $\{x^1, x^2\}^N_{\alpha=1} = Q^1_{j+\frac{1}{2}}$ denote the Gauss quadrature nodes transformed into the interval $[x, x]_{j+\frac{1}{2}}$ and $[x, x]_{j+\frac{1}{2}}$, respectively. Then the WB CDG method on the dual mesh reads

$$
\int_{I_{j+\frac{1}{2}}} \frac{\partial U^D_h}{\partial t} \, dx = \frac{1}{\tau} \int_{I_{j+\frac{1}{2}}} (U^C_h - U^D_h) \, dx + \frac{1}{\tau} \int_{I_{j+\frac{1}{2}}} (U^s_h - U^C_h) \, dx

- \left( F(U^D_h(x)) w(x) + F(U^C_h(x)) w(x) \right)

+ \frac{\Delta x}{2} \sum_{\kappa=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} F(U^C_h(x, x)) w(x, x) + \langle S^C_h, w \rangle_{j+\frac{1}{2}} \quad \forall w \in \mathbb{V}_h^D
$$

(23)

As the standard CDG schemes (15) and (16), the semi-discrete WB CDG schemes (22) and (23) can be rewritten in the ODE form as

$$
\frac{dU^C_h}{dt} = L^C_h(U^C_h, U^D_h), \quad \frac{dU^D_h}{dt} = L^D_h(U^D_h, U^C_h),
$$

after choosing suitable bases of $\mathbb{V}_h^C$, $\mathbb{V}_h^D$ and representing $U^C_h, U^D_h$ as linear combinations of the basis functions.

**Remark 3.2.** It is worth noting that the above WB discretization has carefully accommodated the positivity-preserving property. For example, if we are only concerned with the WB property (see, for
The modified dissipation term becomes
\[
\text{and } \int_{I_j} (\rho u)^D_j (\phi^D_j)_x vdx
\]
by using any standard quadrature rule and does not affect the WB property. However, our analyses indicate that it is crucial to employ a “unified” discretization for the source terms in the momentum and energy equations to simultaneously accommodate the positivity-preserving property.

### 3.4 Proofs of WB and positivity-preserving properties

#### 3.4.1 WB property

**Theorem 3.1.** For the one-dimensional Euler equations \((1)\) under the gravitational field, the modified semi-discrete CDG schemes, given by \((22)\) and \((23)\), are WB for a general stationary hydrostatic solution \((3)\).

**Proof.** Suppose that the initial solution is \(U^s\). By the construction of \(U_h^{s,C}, U_h^{s,D}\), one has
\[
U_h^C(x,0) = U_h^{s,C}, \quad U_h^D(x,0) = U_h^{s,D},
\]
and
\[
\rho_h^D(x,0) = \rho_h^{s,D}, \quad u_h^D(x,0) = 0, \quad p_h^D(x,0) = p_h^{s,D}.
\]
The modified dissipation term becomes
\[
\frac{1}{\tau_{\text{max}}} \int_{I_j} (U_h^D - U_h^C) vdx + \frac{1}{\tau_{\text{max}}} \int_{I_j} (U_h^{s,C} - U_h^{s,D}) vdx = 0.
\]
It is observed that the WB property holds for the density and energy equations, as both the flux and source term approximations in those equations become zero. For the momentum equation, because \(\frac{\langle (\rho^D_j) \rangle}{\langle (\rho^D j)_j \rangle} = 1\), the modified source term becomes
\[
\langle S_{h,2}, v \rangle_j = \left( p_h^{s,D}(x_{j+\frac{1}{2}}) v(x_{j+\frac{1}{2}}) - p_h^{s,D}(x_{j-\frac{1}{2}}) v(x_{j-\frac{1}{2}}) \right) - \frac{\Delta x}{2} \sum_{\alpha=1}^{N} \omega_{\alpha} p_h^{s,D}(x_{j+\frac{1}{2}}) v_{\alpha} x_{j}^{\alpha}.
\]
Since \(u = 0\), the flux term \(F_2 = \rho u^2 + p\) reduces to \(p\), and its numerical approximation is given by
\[
\frac{\Delta x}{2} \sum_{\alpha=1}^{N} \omega_{\alpha} (F_2(U_h^{s,D}(x_{j+\frac{1}{2}})) v_{\alpha} x_{j}^{\alpha} - (F_2(U_h^{s,D}(x_{j+\frac{1}{2}})) v_{\alpha} x_{j}^{\alpha}) - (F_2(U_h^{s,D}(x_{j-\frac{1}{2}})) v_{\alpha} x_{j}^{\alpha}) - (F_2(U_h^{s,D}(x_{j-\frac{1}{2}})) v_{\alpha} x_{j}^{\alpha}) = -\langle S_{h,2}, v \rangle_j.
\]
Therefore, the flux and source term approximations balance each other, implying
\[
\int_{I_j} \frac{\partial U_h^C}{\partial t} v \, dx = 0, \quad \forall v \in \mathcal{V}_h^{C,k}.
\]
Similarly, on the dual mesh, one can establish
\[
\int_{I_{j+\frac{1}{2}}} \frac{\partial U_h^D}{\partial t} w \, dx = 0, \quad \forall w \in \mathcal{V}_h^{D,k}.
\]
Hence our CDG schemes, given by (22) and (23), are WB for a general stationary hydrostatic solution (3).

\[\blacksquare\]

### 3.4.2 Positivity-preserving property

This subsection will discuss the positivity-preserving property of the WB CDG schemes (22) and (23). The WB modifications of the numerical dissipation and source terms lead to additional difficulties in the positivity-preserving analyses, which are more complicated than that for the standard CDG method. We introduce several basic properties of the admissible state set \(G\), which will be useful in our positivity-preserving analyses.

**Lemma 3.1** (Convexity). *The set \(G\) is a convex set.*

**Proof.** This property can be verified by definition and Jensen’s inequality; see [69, Page 8919].

**Lemma 3.2.** *For any \(U \in G\) and \(b \in \mathbb{R}\), the state \(U + \lambda S(U, b) \in G\) under the condition
\[|\lambda| < \frac{1}{|b|} \sqrt{\frac{2p}{(\gamma - 1)p}}.\]

**Proof.** The proof can be found in [57, Page A476] for the details.

**Lemma 3.3.** *For any \(U \in G\) and \(\lambda \in \mathbb{R}\), the state \(U - \lambda F(U) \in G\) under the condition
\[|\lambda| a_s(U) \leq 1, \quad \text{with} \quad a_s(U) := |u| + \sqrt{\frac{\gamma p}{\rho}}.\]

**Proof.** A proof can be found in [69, Page 8921]. See also [52] for another simple proof based on the GQL approach.

Next, we consider the semi-discrete scheme satisfied by the cell averages of the WB CDG solution. Denote
\[
\bar{U}_j^C(t) = \frac{1}{\Delta x} \int_{I_j} U_h^C(x, t) \, dx, \quad \bar{U}_{j+\frac{1}{2}}^D(t) = \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} U_h^D(x, t) \, dx.
\]
Taking the test function \(v = 1\) in (22) and \(w = 1\) in (23) and using the identities in (11) gives
\[
\begin{align*}
\frac{d\bar{U}_j^C}{dt} &= L_j^C(U_h^C, U_h^D) := \left(\frac{U_j^D - U_j^C}{\tau_{max}}\right) - \left(\frac{F(U_{j+\frac{1}{2}}) - F(U_{j-\frac{1}{2}})}{\Delta x}\right) + \frac{\langle S_h^D, 1 \rangle_j}{\Delta x}, \\
\frac{d\bar{U}_{j+\frac{1}{2}}^D}{dt} &= L_{j+\frac{1}{2}}^D(U_h^D, U_h^C) := \left(\frac{U_{j+\frac{1}{2}}^C - U_{j+\frac{1}{2}}^D}{\tau_{max}}\right) - \left(\frac{F(U_{j+1}) - F(U_{j+\frac{1}{2}})}{\Delta x}\right) + \frac{\langle S_h^C, 1 \rangle_{j+\frac{1}{2}}}{\Delta x},
\end{align*}
\]
where \(U_{j+\frac{1}{2}}^D = U_h^D(x_{j+\frac{1}{2}}), \, U_{j+1}^C = U_h^C(x_{j+1}).\)
**Remark 3.3.** Recall that the two key identities in (11) are derived from the novel projection operators \( \mathcal{P}_h^C \) and \( \mathcal{P}_h^D \) and do not hold for the standard \( L^2 \)-projection. Benefited from this remarkable feature, our modification of the numerical dissipation term does not destroy the positivity-preserving property.

Let \( \mathbb{L}^1_j = \{ x_j^{1, \beta} \}_\beta = 1 \) and \( \mathbb{L}^2_j = \{ x_j^{2, \beta} \}_\beta = 1 \) denote the Gauss-Lobatto quadrature nodes transformed into the interval \([x_{j-1}, x_j]\) and \([x_j, x_{j+1}]\), respectively, and \( \{ \phi_\beta \}_\beta = 1 \) are the associated weights satisfying \( \sum_{\beta = 1}^L \phi_\beta = 1 \). We take \( L = N/2 \), which gives \( 2L - 3 \geq k \), so that the \( L \)-point Gauss-Lobatto quadrature rule is exact for polynomials of degree up to \( k \). For each primal cell \( I_j \), we define the point set

\[
S_j = \mathbb{Q}^1_{j,x} \cup \mathbb{Q}^2_{j,x} \cup \mathcal{L}^1_j \cup \mathcal{L}^2_j, \quad \hat{Q}_j = \mathbb{Q}^1_{j,x} \cup \mathbb{Q}^2_{j,x},
\]

and the parameter \( \tilde{\alpha}_j^D \) as

\[
\tilde{\alpha}_j^D = \alpha_{1, j}^D + \alpha_{2, j}^D, \quad \alpha_{1, j} = \max_{x \in \{x_{j-1/2}, x_{j+1/2}\}} a_x(U_h^D),
\]

\[
\tilde{\alpha}_{2,j}^D = \frac{\omega_1 \Delta x}{2} \max_{x \in \hat{Q}_j} \left\{|(\phi_\beta^D)_x| \sqrt{(\gamma - 1) \rho^D} / 2 \rho^D \right\}, \quad \hat{\phi}_j^D = \frac{p^s_j(x^-_j) - p^s_j(x^+_j)}{\left(\rho^s_j \right)_{j+1/2} \Delta x} - \frac{(p^s_j)_x}{\rho^s_j}.
\]

Similarly, in each dual cell \( I_{j+1/2} \), we define the point set

\[
S_{j+1/2} = \mathbb{Q}^2_{j,x} \cup \mathbb{Q}^1_{j+1} \cup \mathcal{L}^1_j \cup \mathcal{L}^2_j, \quad \hat{Q}_{j+1/2} = \mathbb{Q}^2_{j,x} \cup \mathbb{Q}^1_{j+1}
\]

and the parameter \( \tilde{\alpha}_{j+1/2}^C \) as

\[
\tilde{\alpha}_{j+1/2}^C = \alpha_{1, j+1/2}^C + \alpha_{2, j+1/2}^C, \quad \alpha_{1, j+1/2} = \max_{x \in \{x_{j+1/2}, x_{j+3/2}\}} a_x(U_h^C),
\]

\[
\alpha_{2,j+1/2}^C = \frac{\omega_1 \Delta x}{2} \max_{x \in \hat{Q}_{j+1/2}} \left\{|(\phi_\beta^C)_x| \sqrt{(\gamma - 1) \rho^C} / 2 \rho^C \right\}, \quad \hat{\phi}_j^C = \frac{p^s_j(x^-_{j+1/2}) - p^s_j(x^+_{j+1/2})}{\left(\rho^s_j \right)_{j+1/2} \Delta x} - \frac{(p^s_j)_x}{\rho^s_j}.
\]

Then we have the following CFL-type condition for the high-order CDG schemes (22) and (23) to be positivity-preserving.

**Theorem 3.2.** Assume that the numerical solutions \( U_h^C(x,t), U_h^D(x,t) \) and the projected stationary hydrostatic solutions \( U_h^s(x), U_h^s(x) \) satisfy

\[
\begin{cases}
  U_h^C(x,t) \in G, & U_h^s(x) \in G \quad \forall x \in S_j, \forall j, \\
  U_h^D(x,t) \in G, & U_h^s(x) \in G \quad \forall x \in S_{j+1/2}, \forall j.
\end{cases}
\]

If \( U_j^C, U_{j+1/2}^D \in G \), then the weak positivity-preserving property

\[
U_j^C + \Delta t \mathcal{L}_j^C(U_h^C, U_h^D) \in G, \quad U_{j+1/2}^D + \Delta t \mathcal{L}_{j+1/2}^D(U_h^D, U_h^C) \in G, \quad \forall j,
\]

holds under the CFL-type condition

\[
\frac{\Delta t}{\Delta x} \tilde{\alpha}_x < \frac{\theta \delta_1}{2}, \quad \tilde{\alpha}_x = \max_{j} \{ \tilde{\alpha}_j^D, \tilde{\alpha}_{j+1/2}^C \}, \quad \theta = \frac{\Delta t}{\tau_{\max}} \in (0,1].
\]
Proof. Using (24) gives
\[ \bar{U}_j^C + \Delta t L_j^C (U_h^C, U_h^D) = (1 - \theta) \bar{U}_j^C + \left[ \eta \theta \bar{U}_j^D + \lambda_x \langle S_h^D, 1 \rangle_j \right] \\
+ \left[ (1 - \eta) \theta \bar{U}_j^D - \alpha_x \left( F(U_{j+1}^D) - F(U_{j-1}^D) \right) \right] \\
= (1 - \theta) \bar{U}_j^C + \eta \theta L_{h,F} + (1 - \eta) \theta L_{h,S}, \]
where \( \lambda_x = \frac{\Delta t}{\Delta x}, \theta = \frac{\Delta t}{\Delta t_{\text{max}}}, \eta \in (0, 1) \) is a constant, and \( L_{h,F}, L_{h,S} \) are given by
\[ L_{h,F} = U_j^D - \frac{\lambda_x}{(1 - \eta) \theta} \left( F(U_{j+1}^D) - F(U_{j-1}^D) \right), \quad L_{h,S} = U_j^D + \frac{\lambda_x}{\eta \theta} \langle S_h^D, 1 \rangle_j. \]

Due to the exactness of the Gauss-Lobatto quadrature rule, one has
\[ U_j^D = \sum_{\kappa=1}^{2} \sum_{\beta=1}^{L} \frac{\hat{\omega}_\beta}{2} U_h^D(x_j^\kappa \beta). \]

Let us first consider \( L_{h,F} \) and reformulate it as follows
\[ L_{h,F} = \sum_{\kappa=1}^{2} \sum_{\beta=1}^{L} \frac{\hat{\omega}_\beta}{2} U_h^D(x_j^\kappa \beta) - \frac{\lambda_x}{(1 - \eta) \theta} \left( F(U_{j+1}^D) - F(U_{j-1}^D) \right) \\
= \sum_{\beta=2}^{L} \frac{\hat{\omega}_\beta}{2} U_h^D(x_j^1 \beta) + \frac{\hat{\omega}_1}{2} \left( U_{j-1}^D + \frac{2\lambda_x}{(1 - \eta) \theta} F(U_{j-1}^D) \right) \\
+ \sum_{\beta=1}^{L-1} \frac{\hat{\omega}_\beta}{2} U_h^D(x_j^2 \beta) + \frac{\hat{\omega}_L}{2} \left( U_{j+1}^D - \frac{2\lambda_x}{(1 - \eta) \theta} F(U_{j+1}^D) \right) \\
= \sum_{\beta=2}^{L} \frac{\hat{\omega}_\beta}{2} U_h^D(x_j^1 \beta) + \frac{\hat{\omega}_1}{2} \psi_j^{+} + \sum_{\beta=1}^{L-1} \frac{\hat{\omega}_\beta}{2} U_h^D(x_j^2 \beta) + \frac{\hat{\omega}_L}{2} \psi_j^{-}, \]
where
\[ \psi_j^{+} = U_{j-1}^D + \frac{2\lambda_x}{(1 - \eta) \theta} F(U_{j-1}^D), \quad \psi_j^{-} = U_{j+1}^D - \frac{2\lambda_x}{(1 - \eta) \theta} F(U_{j+1}^D). \]

Thanks to the Lax-Friedrichs splitting property, we have \( \psi_j^{+} \in G \) and \( \psi_j^{-} \in G \), as long as
\[ \lambda_x \max_{x \in \{x_{j-\frac{1}{2}} \rightarrow x_{j+\frac{1}{2}}\}} a_s(U_h^D) = \lambda_x \alpha_{s,\frac{1}{2}} < \frac{(1 - \eta) \theta \hat{\omega}_1}{2}. \]

Using the convexity of set \( G \), we obtain \( L_{h,F} \in G \). Next, we discuss the term \( L_{h,S} \), and reformulate the source term \( \langle S_h^D, 1 \rangle_j \) as follows
\[ \langle S_h^D, 1 \rangle_j = \frac{(\rho_h^D)_j}{(\rho_h^D)^{\gamma}} \left( \left( \rho_h^D s_{x_{j-\frac{1}{2}}} - \rho_h^D s_{x_{j+\frac{1}{2}}} \right) - \frac{\Delta t}{2} \sum_{\kappa=1}^{N} \sum_{\alpha=1}^{N} \omega_\alpha (\rho_h^D s_{x_j^\alpha}) \left( \rho_h^D s_{x_j^\alpha} \right) \right) \\
+ \frac{\Delta t}{2} \sum_{\kappa=1}^{N} \sum_{\alpha=1}^{N} \omega_\alpha \frac{\rho_h^D s_{x_j^\alpha}}{\rho_h^D s_{x_j^\alpha}} \left( \rho_h^D s_{x_j^\alpha} \right) \left( \rho_h^D s_{x_j^\alpha} \right). \]
Notice that
\[
\frac{\Delta x}{2} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} (p_{h}^{s,D})_{x}(x_{j}^{\kappa,\alpha}) = \int_{I_{j}} (p_{h}^{s,D})_{x} \, dx,
\]
which leads to
\[
\langle S_{h,2}, 1 \rangle_{j} = \frac{(\rho_{h}^{D})_{j}}{D_{j}} \left( p_{h}^{s,D}(x_{j}^{+}) - p_{h}^{s,D}(x_{j}^{-}) \right) + \frac{\Delta x}{2} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \frac{\rho_{h}^{D}(x_{j}^{\kappa,\alpha})}{D_{j}} (p_{h}^{s,D})_{x}(x_{j}^{\kappa,\alpha})
\]
with
\[
-(\hat{\phi}_{h}^{D})_{x}(x_{j}^{\kappa,\alpha}) := \frac{p_{h}^{s,D}(x_{j}^{+}) - p_{h}^{s,D}(x_{j}^{-})}{(\rho_{h}^{D})_{j} \Delta x} + \frac{(p_{h}^{s,D})_{x}(x_{j}^{\kappa,\alpha})}{D_{j}}
\]
Similarly, one can derive
\[
\langle S_{h,3}, 1 \rangle_{j} = \frac{(m_{h}^{D})_{j}}{D_{j}} \left( p_{h}^{s,D}(x_{j}^{+}) - p_{h}^{s,D}(x_{j}^{-}) \right) + \frac{\Delta x}{2} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \frac{m_{h}^{D}(x_{j}^{\kappa,\alpha})}{D_{j}} (p_{h}^{s,D})_{x}(x_{j}^{\kappa,\alpha})
\]
Thus \( L_{h,s} \) is reformulated as
\[
L_{h,s} = U_{j}^{D} + \frac{\lambda_{t}}{\eta \theta} \langle S_{h}, 1 \rangle_{j} = U_{j}^{D} + \frac{\Delta t}{\eta \theta} \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{\alpha} \dot{S}_{h}^{D}(x_{j}^{\kappa,\alpha})
\]
where
\[
\dot{S}_{h}^{D} := (0, -\rho_{h}^{D}(\hat{\phi}_{h}^{D})_{x}, -m_{h}^{D}(\hat{\phi}_{h}^{D})_{x})^{\top}
\]
Thanks to Lemma 3.2, we have \( L_{h,s} \in G \) under the condition
\[
\Delta t < \eta \theta \min_{x \in Q_{j}} \left\{ \frac{1}{|((\dot{\phi}_{h}^{D})_{x})|} \sqrt{\frac{2p_{h}^{D}}{\gamma - 1} p_{h}^{D}} \right\}
\]
or equivalently
\[
\lambda_{t} \hat{\alpha}_{2,j}^{D} < \frac{\eta \theta \hat{\omega}_{1}}{2}.
\]
Combining those results, we conclude that if
\[
\lambda_{t} \in \left\{ \lambda \in \mathbb{R}^{+} : \lambda \hat{\alpha}_{1,j}^{D} < \frac{(1 - \eta) \theta \hat{\omega}_{1}}{2}, \lambda \hat{\alpha}_{2,j}^{D} < \frac{\eta \theta \hat{\omega}_{1}}{2} \right\},
\] (28)
then \( \mathbf{U}_j^C + \Delta t \mathbf{L}^C_j (\mathbf{U}^C_{h_1}, \mathbf{U}^D_{h_2}) \in G \). Since the parameters \( \eta \) can be chosen arbitrarily in this proof, we specify \( \eta = \alpha^D_{\tau,j} / \bar{\alpha}^D_{\tau,j} = \lambda_{\tau,j} / (\bar{\lambda}_{\tau,j} + \alpha^D_{\tau,j}) \) such that the condition (28) becomes

\[
\lambda_{\tau}(\bar{\lambda}_{\tau,j} + \lambda_{\tau,j}^D) = \lambda_{\tau,j}^D < \frac{\theta \bar{\lambda}_1}{2}.
\]

Similar arguments show that \( \mathbf{U}^D_{j+\frac{1}{2}} + \Delta t \mathbf{L}^D_{j+\frac{1}{2}} (\mathbf{U}^D_{h_1}, \mathbf{U}^D_{h_2}) \in G \). The proof is completed. \( \blacksquare \)

### 3.5 Positivity-preserving limiting operators \( \Pi^C_h \) and \( \Pi^D_h \)

A simple positivity-preserving limiter can be applied to enforce the condition (27). Because the limiting procedures for \( \mathbf{U}^C_{h_1}(x) \) and \( \mathbf{U}^D_{h_1}(x) \) are similar and implemented separately, we only present that for \( \mathbf{U}^C_{h_1}(x) \). For any \( \mathbf{U}^C_{h_1} \in \mathbb{G}^{C,k}_h \) with \( \mathbf{U}^C_{h_1}|_{t_j} := \mathbf{U}^C_{j}(x) \), we follow [46, 69] and define the positivity-preserving limiting operator \( \Pi^C_h : \mathbb{G}^{C,k}_h \rightarrow \mathbb{G}^{C,k}_h \) as follows

\[
(\Pi^C_h \mathbf{U}^C_{h_1})|_{t_j} = \theta^{(2)}_{j} (\hat{\mathbf{U}}^C_{j}(x) - \mathbf{U}^C_{j}) + \mathbf{U}^C_{j}, \quad \theta^{(2)}_{j} = \min \left\{ 1, \frac{p(\mathbf{U}^C_{j}) - \epsilon_2}{p(\mathbf{U}^C_{j}) - \min_{x \in \mathcal{S}_j} p(\mathbf{U}^C_{j}(x))} \right\},
\]

where \( \hat{\mathbf{U}}^C_{j}(x) = (\hat{\rho}^C_{j}(x), m^C_{j}(x), E^C_{j}(x))^T \), and \( \hat{\rho}^C_{j}(x) \) is a modification of the density \( \rho^C_{j}(x) \) given by

\[
\hat{\rho}^C_{j}(x) = \theta^{(1)}_{j} (\rho^C_{j}(x) - \bar{\rho}^C_{j}) + \bar{\rho}^C_{j}, \quad \theta^{(1)}_{j} = \min \left\{ 1, \frac{\bar{\rho}^C_{j} - \epsilon_1}{\bar{\rho}^C_{j} - \min_{x \in \mathcal{S}_j} \rho^C_{j}(x)} \right\}.
\]

Here \( \epsilon_1 \) and \( \epsilon_2 \) are two small positive numbers for avoiding the effect of round-off error, and in the computation, one can take \( \epsilon_1 = \min \{10^{-13}, \bar{\rho}^C_{j} \}, \epsilon_2 = \min \{10^{-13}, p(\hat{\mathbf{U}}^C_{j})\} \). Note that such a local scaling limiter keeps the local conservation and does not destroy the high-order accuracy; see [68, 67] for more details. The positivity-preserving limiting operator \( \Pi^D_h : \mathbb{G}^{D,k}_h \rightarrow \mathbb{G}^{D,k}_h \) defined on the dual mesh is similar.

Suppose the initial numerical solutions are defined as \( \mathbf{U}^{C,0}_{h_1} = \Pi^C_h [\mathbf{U}^C_{h_1}(x, 0)] \), \( \mathbf{U}^{D,0}_{h_1} = \Pi^D_h [\mathbf{U}^D_{h_1}(x, 0)] \).

For the WB CDG schemes (22) and (23) coupled with an third order SSP Runge-Kutta method, if the positivity-preserving limiter is used at each Runge-Kutta stage, then our fully discrete CDG schemes are positivity-preserving, namely, \( \mathbf{U}^{C,n}_{h_1} \in \mathbb{G}^{C,k}_h \) and \( \mathbf{U}^{D,n}_{h_1} \in \mathbb{G}^{D,k}_h \).

**Remark 3.4 (WB Implementation of Non-oscillatory Limiters).** When the exact solution contains strong discontinuities, the above positivity-preserving limiter may not control the nonphysical numerical oscillations in the CDG solutions, and a standard non-oscillatory limiter, such as the TVD/TVB or WENO limiter, is still needed in the “troubled” cells. We will adopt the WENO limiter [37] in the numerical examples involving discontinuities (Examples 4, 9, and 11 in Section 5). However, the traditional use of non-oscillatory limiters may destroy the WB property of our schemes. This issue can be easily addressed by slightly modifying the procedure of identifying the “troubled” cells, based on the perturbations of the solutions and cell averages

\[
\hat{\mathbf{U}}^C_{h_1}(x) = \mathbf{U}^C_{h_1}(x) - \mathbf{U}^{\pi,C}_{h_1}(x), \quad \bar{\mathbf{U}}^C_{j} = \mathbf{U}^C_{j} - \mathbf{U}^{\pi,C}_{j}.
\]
More specifically, for each \( j \) we first use the TVB corrected minmod function (see, e.g., [37])

\[
\bar{m}(a_1, a_2, \ldots, a_n) = \begin{cases} 
    a_1, & \text{if } |a_1| \leq M(\Delta x)^2, \\
    m(a_1, a_2, \ldots, a_n), & \text{otherwise},
\end{cases}
\]

(29)
to check if the cell \( I_j \) is “troubled” based on the cell-averaged values \( \overline{U}_j^C, \overline{U}_{j+1}^C \), and the endpoint values \( U_h^C(x^{j-\frac{1}{2}}), U_h^C(x^{j+\frac{1}{2}}) \) on the cell \( I_j \). Only if cell \( I_j \) is identified as “troubled” cell, we then apply the WENO limiter on \( U_h^C(x) \) as usual before the positivity-preserving limiter. The same implementation is also used separately on the dual mesh. Note that if the steady state is reached, then \( \overline{U}_h^C(x) \) becomes zero so that no cell will be flagged as “troubled”, and thus the WB property is preserved. Numerical results in Example 11 will further confirm that our implementation of the WENO limiter does not affect the WB property.

4 Extension to the two-dimensional case

This section will extend the positivity-preserving WB CDG schemes to the two-dimensional Euler equations under the gravitational field \( \phi(x,y) \)

\[
U_t + \nabla \cdot F(U) = S(U, \nabla \phi),
\]

(30)

where \( U = (\rho, \rho u_1, \rho u_2, E)^\top \) denotes the conservative variables, \( F(U) = (F_1(U), F_2(U)) \) with

\[
F_1(U) = (\rho u_1, \rho u_1^2 + p, \rho u_1 u_2, (E + p) u_1)^\top,
\]

\[
F_2(U) = (\rho u_2, \rho u_2 u_1, \rho u_2^2 + p, (E + p) u_2)^\top,
\]
denote the fluxes, and \( S(U, \nabla \phi) = (0, -\rho \phi_x, -\rho \phi_y, -m \cdot \nabla \phi)^\top \) is the source term with \( m = (\rho u_1, \rho u_2) \) being the momentum vector.

Let \( \mathcal{T}_h^C = \{I_{i,j}, \forall i,j\} \) and \( \mathcal{T}_h^D = \{I_{i+\frac{1}{2},j+\frac{1}{2}}, \forall i,j\} \) respectively denote two overlapping uniform meshes for the rectangular computational domain \( \Omega = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}] \) with \( I_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \) and \( I_{i+\frac{1}{2},j+\frac{1}{2}} = (x_i, x_{i+1}) \times (y_j, y_{j+1}) \). The spatial stepsizes are \( \Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \) in the \( x \)-direction and \( \Delta y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \) in the \( y \)-direction. We define two discrete function spaces associated with the overlapping meshes \( \{I_{i,j}\} \) and \( \{I_{i+\frac{1}{2},j+\frac{1}{2}}\} \)

\[
\mathbb{V}_h^{C,k} = \left\{ v : v|_{I_{i,j}} \in \mathbb{P}_k(I_{i,j}), \forall i,j \right\}, \quad \mathbb{V}_h^{D,k} = \left\{ v : v|_{I_{i+\frac{1}{2},j+\frac{1}{2}}} \in \mathbb{P}_k(I_{i+\frac{1}{2},j+\frac{1}{2}}), \forall i,j \right\},
\]

where \( \mathbb{P}_k(I_{i,j}) \) and \( \mathbb{P}_k(I_{i+\frac{1}{2},j+\frac{1}{2}}) \) denote the space of two-dimensional polynomials in the cells \( I_{i,j} \) and \( I_{i+\frac{1}{2},j+\frac{1}{2}} \) with degree of at most \( k \), respectively. To solve the system (30), the standard CDG method
in the semi-discrete form looks for two numerical solutions \( \mathbf{U}^C_h \in [\mathbb{V}^{C,k}]^4 \) and \( \mathbf{U}^D_h \in [\mathbb{V}^{D,k}]^4 \) such that

\[
\int_{l_{i,j}} \frac{\partial \mathbf{U}^C_h}{\partial t} \, v \, dx \, dy = \frac{1}{\tau_{\text{max}}} \int_{l_{i,j}} (\mathbf{U}^D_h - \mathbf{U}^C_h) \, v \, dx \, dy + \int_{l_{i,j}} \mathbf{F}(\mathbf{U}^C_h) \cdot \nabla v \, dx \, dy
\]

\[
- \int_{y_{j+\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left( F_1(U^D_h(x_{i+\frac{1}{2}},y))v(x_{i+\frac{1}{2}},y) - F_1(U^C_h(x_{i},y))v(x_{i},y) \right) dy
\]

\[
- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( F_2(U^D_h(x,y_{j+\frac{1}{2}}))v(x,y_{j+\frac{1}{2}}) - F_2(U^C_h(x,y_{j}))v(x,y_{j}) \right) dx
\]

\[
+ \int_{l_{i,j}} \mathbf{S}(\mathbf{U}^D_h, \nabla \phi^D_h) \, v \, dx \, dy \quad \forall v \in \mathbb{V}^{D,k}_h, \tag{31}
\]

\[
\int_{l_{i+\frac{1}{2},j+\frac{1}{2}}} \frac{\partial \mathbf{U}^D_h}{\partial t} \, w \, dx \, dy = \frac{1}{\tau_{\text{max}}} \int_{l_{i+\frac{1}{2},j+\frac{1}{2}}} (\mathbf{U}^C_h - \mathbf{U}^D_h) \, w \, dx \, dy + \int_{l_{i+\frac{1}{2},j+\frac{1}{2}}} \mathbf{F}(\mathbf{U}^C_h) \cdot \nabla w \, dx \, dy
\]

\[
- \int_{y_{j+\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left( F_1(U^C_h(x_{i+1},y))w(x_{i+1},y) - F_1(U^C_h(x_i,y))w(x_i,y) \right) dy
\]

\[
- \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left( F_2(U^C_h(x,y_{j+1}))w(x,y_{j+1}) - F_2(U^C_h(x,y_j))w(x,y_j) \right) dx
\]

\[
+ \int_{l_{i+\frac{1}{2},j+\frac{1}{2}}} \mathbf{S}(\mathbf{U}^C_h, \nabla \phi^C_h) \, w \, dx \, dy \quad \forall w \in \mathbb{V}^{C,k}_h, \tag{32}
\]

where \( \tau_{\text{max}} = \tau_{\text{max}}(t) \) is the maximal time step allowed by the CFL restriction at time \( t \). As the one-dimensional case, the two-dimensional standard CDG method (31)–(32) is generally not WB for the stationary hydrostatic solutions.

### 4.1 Novel projection of the stationary hydrostatic solutions

Assume that the target equilibrium state of the system (30) is known and denoted by

\[
\{ \rho^s(x,y), u_1^s(x,y), u_2^s(x,y), p^s(x,y) \},
\]

which satisfies

\[
u_1^s(x,y) = 0, \quad u_2^s(x,y) = 0, \quad \nabla p^s = -\rho^s \nabla \phi. \tag{33}
\]

Let \( \mathbf{U}^s(x,y) = (\rho^s(x,y), 0, 0, p^s(x,y)/(\gamma - 1))^T \), and define

\[
I_{0,i,j} = (x_{i-\frac{1}{2}},x_i) \times (y_{j-\frac{1}{2}},y_j), \quad I_{1,i,j} = (x_i,x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}},y_j),
\]

\[
I_{2,i,j} = (x_{i-\frac{1}{2}},x_i) \times (y_j,y_{j+\frac{1}{2}}), \quad I_{3,i,j} = (x_i,x_{i+\frac{1}{2}}) \times (y_j,y_{j+\frac{1}{2}}).
\]

Following the ideas in the one-dimensional case, we first introduce the novel projection of the stationary solution \( \mathbf{U}^s(x,y) \) on the primal mesh. Define the operator \( \mathbb{P}_h^C : L^2(\Omega) \rightarrow \mathbb{V}^{C,k}_h \), such that for any function \( f \in L^2(\Omega) \),

\[
\int_{I_{m,i,j}} \mathbb{P}_h^C(f) \, dx \, dy = \int_{I_{m,i,j}} f \, dx \, dy, \quad m \in \{0, 1, 2\},
\]

\[
\int_{I_{i,j}} \mathbb{P}_h^D(f) \, dx \, dy = \int_{I_{i,j}} f \, dx \, dy, \quad \forall f \in \text{span}\{ \Phi_0, \Phi_4, \ldots, \Phi_K \}, \tag{34}
\]

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where $K = k(k+3)/2$, and $\{\Phi_l\}_{l=0}^K$ is an orthogonal basis of $\mathbb{P}^k(I_{i,j})$ and taken as the scaled Legendre polynomials

$$
\Phi_0(\xi, \eta) = 1, \quad \Phi_1(\xi, \eta) = \xi, \quad \Phi_2(\xi, \eta) = \eta, \quad \Phi_3(\xi, \eta) = \xi \eta,
$$

$$
\Phi_4(\xi, \eta) = \xi^2 - \frac{1}{3}, \quad \Phi_5(\xi, \eta) = \eta^2 - \frac{1}{3}, \quad \ldots,
$$

with $\xi = 2(x-x_i)/\Delta x$ and $\eta = 2(y-y_j)/\Delta y$. It follows from (34) that the operator $\mathcal{P}_h^C$ satisfies

$$
\int_{I_{i,j}} \mathcal{P}_h^C(f) \ dx \ dy = \int_{I_{i,j}} f \ dx \ dy, \quad \forall m \in \{0, 1, 2, 3\}, \quad \forall i, j, \quad (35)
$$

for any function $f \in L^2(\Omega)$. As the one-dimensional case, the operator $\mathcal{P}_h^C$ defined by (34) can be explicitly expressed. In fact, the piecewise polynomial $\mathcal{P}_h^C(f)$ on each cell $I_{i,j}$ takes the form of

$$
\mathcal{P}_h^C(f) = \sum_{l=0}^{K} N_l(f) \Phi_l,
$$

with the polynomial coefficients $\{N_l(f)\}_{l=0}^K$ given by

$$
N_l(f) = \frac{\int_{I_{i,j}} f \Phi_l \ dx \ dy}{\int_{I_{i,j}} \Phi_l^2 \ dx \ dy}, \quad l \neq 1, 2, 3,
$$

$$
N_1(f) = \frac{-4(b_0 + b_2)}{\Delta x \Delta y}, \quad N_2(f) = \frac{-4(b_0 + b_1)}{\Delta x \Delta y}, \quad N_3(f) = \frac{-8(b_1 + b_2)}{\Delta x \Delta y},
$$

where the parameter $b_m = \int_{I_{i,j}} (f - \sum_{l \in \{1,2,3\}} N_l(f) \Phi_l) \ dx \ dy, m \in \{0, 1, 2\}$.

Similarly, we can define the projection $\mathcal{P}_h^D : L^2(\Omega) \rightarrow \mathbb{V}_h^{D,k}$ on the dual mesh such that

$$
\int_{I_{i,j}} \mathcal{P}_h^D(f) \ dx \ dy = \int_{I_{i,j}} f \ dx \ dy, \quad \forall m \in \{0, 1, 2, 3\}, \quad \forall i, j, \quad (36)
$$

where $I_{i,j}^{m_{\frac{1}{2}, \frac{1}{2}}}$ is a shift of $I_{i,j}^m$ with $\frac{\Delta x}{2}$ in the $x$-direction and $\frac{\Delta y}{2}$ in the $y$-direction. Combining (35) with (36) leads to the following crucial identities

$$
\int_{I_{i,j}} \mathcal{P}_h^C(f) \ dx \ dy = \int_{I_{i,j}} \mathcal{P}_h^D(f) \ dx \ dy, \quad \int_{I_{i,j}^{\frac{1}{2}, \frac{1}{2}}} \mathcal{P}_h^D(f) \ dx \ dy = \int_{I_{i,j}^{\frac{1}{2}, \frac{1}{2}}} \mathcal{P}_h^C(f) \ dx \ dy. \quad (37)
$$

If let $\mathbf{U}_h^C$ and $\mathbf{U}_h^D$ denote the above novel projections of the steady state solutions $\mathbf{U}^s(x,y)$ onto the space $[\mathbb{V}_h^{C,k}]^4$ and $[\mathbb{V}_h^{D,k}]^4$, respectively, then the identities in (37) imply

$$
\int_{I_{i,j}} \mathbf{U}_h^C \ dx \ dy = \int_{I_{i,j}} \mathbf{U}_h^D \ dx \ dy, \quad \int_{I_{i,j}^{\frac{1}{2}, \frac{1}{2}}} \mathbf{U}_h^D \ dx \ dy = \int_{I_{i,j}^{\frac{1}{2}, \frac{1}{2}}} \mathbf{U}_h^C \ dx \ dy, \quad \forall i, j. \quad (38)
$$
4.2 WB CDG schemes

The design of our two-dimensional WB CDG method on the rectangular mesh is similar to the procedure described in the one-dimensional case.

The numerical dissipation term in the semi-discrete CDG method (31) is modified as

\[
\tilde{d}_{ij}^C(U_h^C, U_h^D, v) = \frac{1}{\tau_{\max}} \int_{I_{i,j}} (U_h^D - U_h^C) v dx dy + \frac{1}{\tau_{\max}} \int_{I_{i,j}} (U_h^{s,C} - U_h^{s,D}) v dx dy,
\]

where \( \tilde{d}_{ij}^C \) satisfies the WB property \( \tilde{d}_{ij}^C(U_h^C, U_h^D, v) = 0 \). Such modification does not affect the spatial accuracy.

In order to discretize the flux and source term integrals, we need to introduce the two-dimensional notations for the quadrature points in the \( x \)-direction. For the \( y \)-direction, let \( \mathcal{Q}_{1,j}^y = \{ y_j^{1,\mu} \}_{\mu=1}^N \) and \( \mathcal{Q}_{2,j}^y = \{ y_j^{2,\mu} \}_{\mu=1}^N \) denote the \( N \)-point Gauss quadrature nodes transformed into the interval \([ y_j - \frac{1}{2}, y_j ]\) and \([ y_j, y_j + \frac{1}{2} ]\), respectively, and \( \{ \omega_\mu \}_{\mu=1}^N \) are the associated weights satisfying \( \sum_{\mu=1}^N \omega_\mu = 1 \), with \( N \geq k + 1 \) for the CDG accuracy requirement. Then the flux integrals can be approximated by the numerical quadrature

\[
\int_{I_{i,j}} F(U_h^D) v N dx dy \approx \frac{\Delta x \Delta y}{4} \sum_{\kappa,\alpha} \sum_{\sigma,\mu} \omega_\alpha \omega_\mu F(U_h^D(x_i^{\kappa,\alpha}, y_j^{\sigma,\mu})) \nabla v(x_i^{\kappa,\alpha}, y_j^{\sigma,\mu}),
\]

\[
\int_{\partial I_{i,j}} (F(U_h^D) \cdot n) v ds \approx \Delta y \delta_{ij}^x(F_1(U_h^D)v) + \Delta x \delta_{ij}^y(F_2(U_h^D)v),
\]

where \( \sigma, \kappa \in \{ 1, 2 \} \) and \( \mu, \alpha \in \{ 1, \ldots, N \} \), \( n \) is the outward unit normal vector of the cell \( I_{i,j} \), and the operators \( \delta_{ij}^x, \delta_{ij}^y \) are defined by

\[
\delta_{ij}^x(f) := \sum_{\kappa,\alpha} \frac{\omega_\alpha}{2} \left( f(x_{i+\frac{1}{2}}^{\kappa,\alpha}, y_j^{\sigma,\mu}) - f(x_{i-\frac{1}{2}}^{\kappa,\alpha}, y_j^{\sigma,\mu}) \right),
\]

\[
\delta_{ij}^y(f) := \sum_{\kappa,\alpha} \frac{\omega_\alpha}{2} \left( f(x_i^{\kappa,\alpha}, y_{j+\frac{1}{2}}) - f(x_i^{\kappa,\alpha}, y_{j-\frac{1}{2}}) \right).
\]

The last step for designing our WB CDG spatial discretization is to suitably discretize the source term integral. The source term integrals in the momentum equations are reformulated into

\[
\int_{I_{i,j}} (S_2, S_3)^T v dx dy = - \int_{I_{i,j}} \rho \nabla \phi v dx dy = \int_{I_{i,j}} \frac{\rho}{\rho^s} \nabla p^s v dx dy
\]

\[
= \int_{I_{i,j}} \left( \frac{\rho}{\rho^s} - \frac{\bar{\rho}_{ij}}{\bar{\rho}_{ij}} + \frac{\bar{\rho}_{ij}}{\bar{\rho}_{ij}} \right) \nabla p^s v dx dy,
\]

where \( \bar{\rho}_{ij} = \frac{1}{\Delta x \Delta y} \int_{I_{ij}} \rho dx dy \) is the cell average. Following the one-dimensional design, we observe
that

\[
\int_{I_{h,j}} (S_2, S_3)^T v dx dy \approx \int_{I_{h,j}} \left( \frac{\rho_h^D p_h^{s,D}}{\rho_h^{s,D}} - \frac{(\rho_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \right) \nabla p_h^{s,D} v dx dy + \int_{I_{h,j}} \nabla p_h^{s,D} v dx dy
\]

\[
\approx \int_{I_{h,j}} \left( \frac{\rho_h^D p_h^{s,D}}{\rho_h^{s,D}} - \frac{(\rho_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \right) \nabla p_h^{s,D} v dx dy - \frac{(\rho_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \int_{I_{h,j}} p_h^{s,D} v dx dy
\]

\[
+ \frac{(\rho_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \int_{\partial I_{h,j}} p_h^{s,D} v n ds.
\]

Therefore, the source term integrals \( \int_{I_{h,j}} (S_2, S_3)^T v dx dy \) can be approximated by

\[
\langle (S_2 h, S_3 h)^T, v \rangle_{ij} := \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} \left( \frac{m_h^{D}(x_i^{\alpha}, y_j^{\mu})}{\rho_h^{s,D}(x_i^{\alpha}, y_j^{\mu})} - \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \right) \nabla p_h^{s,D}(x_i^{\alpha}, y_j^{\mu}) + \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \nabla p_h^{s,D}(x_i^{\alpha}, y_j^{\mu})
\]

\[
- \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} (p_h^{s,D}(x_i^{\alpha}, y_j^{\sigma, \mu}) v) + \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \frac{\Delta y \delta_{ij}^x (p_h^{s,D}(y))}{\Delta x \delta_{ij}^y (p_h^{s,D}(y))}
\]

Similarly, the source term integral \( \int_{I_{h,j}} S_4 v dx dy \) in the energy equation can be approximated by

\[
\langle S_4 h, v \rangle_{ij} := \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} \left( \frac{m_h^{D}(x_i^{\alpha}, y_j^{\sigma, \mu})}{\rho_h^{s,D}(x_i^{\alpha}, y_j^{\sigma, \mu})} - \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \right) \nabla p_h^{s,D}(x_i^{\alpha}, y_j^{\sigma, \mu}) + \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \nabla p_h^{s,D}(x_i^{\alpha}, y_j^{\sigma, \mu})
\]

\[
- \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} (p_h^{s,D}(x_i^{\alpha}, y_j^{\sigma, \mu}) v) + \frac{(m_h^{D})_{ij}}{(\rho_h^{s,D})_{ij}} \frac{\Delta y \delta_{ij}^x (p_h^{s,D}(y))}{\Delta x \delta_{ij}^y (p_h^{s,D}(y))}
\]

Combining those leads to the following WB CDG discretization for the two-dimensional Euler equations with gravity on the primal mesh

\[
\int_{I_{h,j}} \frac{\partial U_h^C}{\partial t} v dx dy = \frac{1}{\tau_{\text{max}}} \int_{I_{h,j}} (U_h^D - U_h^C) v dx dy + \frac{1}{\tau_{\text{max}}} \int_{I_{h,j}} (U_h^C - U_h^{s,D}) v dx dy + \langle S_h^D, v \rangle_{ij}
\]

\[
+ \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} F(U_h^D(x_i^{\alpha}, y_j^{\sigma, \mu})) \nabla v(x_i^{\alpha}, y_j^{\sigma, \mu}) - \Delta y \delta_{ij}^x (F_1(U_h^D) v) - \Delta x \delta_{ij}^y (F_2(U_h^D) v),
\]

where \( \langle S_h^D, v \rangle_{ij} = \langle (S_2 h, S_3 h)^T, v \rangle_{ij} \). The WB CDG spatial discretization on the dual mesh is very similar. Denote \( Q = (Q_1, Q_2) \), with \( Q_1 = (0, \rho, 0, \rho u_1)^T \), \( Q_2 = (0, 0, 0, \rho u_2)^T \), one has \( S(U) = -Q \cdot \nabla \phi \), and the source term integrals on the dual mesh are approximated by

\[
\langle S_h^C, w \rangle_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} \left( \frac{Q_h^C(x_i^{\alpha}, y_j^{\sigma, \mu})}{\rho_h^{s,C}(x_i^{\alpha}, y_j^{\sigma, \mu})} - \frac{(Q_h^C)_{ij+\frac{1}{2},j+\frac{1}{2}}}{(\rho_h^{s,C})_{ij+\frac{1}{2},j+\frac{1}{2}}} \right) \nabla p_h^{s,C}(x_i^{\alpha}, y_j^{\sigma, \mu}) + \frac{(Q_h^C)_{ij+\frac{1}{2},j+\frac{1}{2}}}{(\rho_h^{s,C})_{ij+\frac{1}{2},j+\frac{1}{2}}}
\]

\[
- \frac{(Q_h^{C})_{ij+\frac{1}{2},j+\frac{1}{2}}}{(\rho_h^{s,C})_{ij+\frac{1}{2},j+\frac{1}{2}}} \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} (p_h^{s,C}(x_i^{\alpha}, y_j^{\sigma, \mu}) v) + \frac{(Q_h^{C})_{ij+\frac{1}{2},j+\frac{1}{2}}}{(\rho_h^{s,C})_{ij+\frac{1}{2},j+\frac{1}{2}}} \frac{\Delta y \delta_{ij}^x (p_h^{s,C}(y))}{\Delta x \delta_{ij}^y (p_h^{s,C}(y))}
\]

\[
- \frac{(Q_h^{C})_{ij+\frac{1}{2},j+\frac{1}{2}}}{(\rho_h^{s,C})_{ij+\frac{1}{2},j+\frac{1}{2}}} \frac{\Delta x \Delta y}{4} \sum_{k, \alpha, \sigma, \mu} \omega_{\alpha} \omega_{\mu} (p_h^{s,C}(x_i^{\alpha}, y_j^{\sigma, \mu}) v) + \frac{(Q_h^{C})_{ij+\frac{1}{2},j+\frac{1}{2}}}{(\rho_h^{s,C})_{ij+\frac{1}{2},j+\frac{1}{2}}} \frac{\Delta y \delta_{ij}^x (p_h^{s,C}(y))}{\Delta x \delta_{ij}^y (p_h^{s,C}(y))}
\]
Proof. For the two-dimensional Euler equations (30) with gravity, our semi-discrete CDG schemes (39)–(40) are WB for the stationary hydrostatic solution. Suppose the spatial domain conditions is also essential for preserving the WB property. Take the solid wall boundary as example, Remark 4.1 (WB Implementation of Boundary Conditions).

\[
\begin{align*}
\delta^x_{i+\frac{1}{2}, j+\frac{1}{2}}(f) & := \sum_{\kappa, \alpha} \frac{\omega_\alpha}{2} \left( f(x_{i+1}, y^\kappa_{j+\frac{1}{2}}) - f(x_i, y^\kappa_{j+\frac{1}{2}}) \right), \\
\delta^y_{i+\frac{1}{2}, j+\frac{1}{2}}(f) & := \sum_{\kappa, \alpha} \frac{\omega_\alpha}{2} \left( f(x^\kappa_{i+\frac{1}{2}}, y_j+1) - f(x^\kappa_{i+\frac{1}{2}}, y_j) \right).
\end{align*}
\]

Then the WB CDG discretization on the dual mesh is given by

\[
\int_{I_{i+\frac{1}{2}, j+\frac{1}{2}}} \frac{\partial U^D_h}{\partial t} w dx dy = \frac{1}{\tau_{\text{max}}} \int_{I_{i+\frac{1}{2}, j+\frac{1}{2}}} (U^C_h - U^D_h) w dx dy + \frac{1}{\tau_{\text{max}}} \int_{I_{i+\frac{1}{2}, j+\frac{1}{2}}} (U^{s,D}_h - U^{s,C}_h) w dx dy \\
+ \frac{\Delta x \Delta y}{4} \sum_{\kappa, \alpha, \sigma, \mu} \omega_\alpha \omega_\mu F(U^C_h(x^\kappa_{i+\frac{1}{2}}, y^\sigma_{j+\frac{1}{2}})) \nabla w(x^\kappa_{i+\frac{1}{2}}, y^\sigma_{j+\frac{1}{2}}) \\
- \Delta y \delta^x_{i+\frac{1}{2}, j+\frac{1}{2}}(F_1(U^C_h)w) - \Delta x \delta^y_{i+\frac{1}{2}, j+\frac{1}{2}}(F_2(U^C_h)w) + \langle S^C_h, w \rangle_{i+\frac{1}{2}, j+\frac{1}{2}}. \tag{40}
\]

**Theorem 4.1.** For the two-dimensional Euler equations (30) with gravity, our semi-discrete CDG schemes (39)–(40) are WB for the stationary hydrostatic solution (33).

**Proof.** The proof is similar to that of Theorem 3.1 and thus is omitted here. ■

**Remark 4.1 (WB Implementation of Boundary Conditions).** A suitable implementation of boundary conditions is also essential for preserving the WB property. Take the solid wall boundary as example, which may appear in the bottom of atmosphere as in weather modeling (see, e.g., the rising thermal bubble problem in Section 5.2.6). Suppose the spatial domain \(\Omega = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]\) is divided into \(N_x \times N_y\) uniform cells, with

\[
x_{\min} = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = x_{\max}, \quad y_{\min} = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y+\frac{1}{2}} = y_{\max}.
\]

We implement the reflective boundary conditions on the solid wall as follows:

- **For the right boundary,** we set
  \[
  \rho^C_{N_x+1,j}(x,y) = \rho^C_{N_x,j}(2x_{\max} - x,y), \quad \rho^D_{N_x+\frac{1}{2},j+\frac{1}{2}}(x,y) = \rho^D_{N_x+\frac{1}{2},j+\frac{1}{2}}(2x_{\max} - x,y),
  \]
  \[
  m^C_{1,N_x+1,j}(x,y) = -m^C_{1,N_x,j}(2x_{\max} - x,y), \quad m^D_{1,N_x+\frac{1}{2},j+\frac{1}{2}}(x,y) = -m^D_{1,N_x+\frac{1}{2},j+\frac{1}{2}}(2x_{\max} - x,y).
  \]

The boundary conditions for \(m^C_{1,h}(x,y), E^C_h(x,y)\) and \(m^D_{1,h}(x,y), E^D_h(x,y)\) are same as the density. The left boundary condition is similar to the right.

- **Analogously, for the top boundary,** we set
  \[
  \rho^C_{i,N_y+1}(x,y) = \rho^C_{i,N_y}(2y_{\max} - y), \quad \rho^D_{i+\frac{1}{2},N_y+\frac{1}{2}}(x,y) = \rho^D_{i+\frac{1}{2},N_y+\frac{1}{2}}(2y_{\max} - y),
  \]
  \[
  m^C_{2,i,N_y+1}(x,y) = -m^C_{2,i,N_y}(2y_{\max} - y), \quad m^D_{2,i+\frac{1}{2},N_y+\frac{1}{2}}(x,y) = -m^D_{2,i+\frac{1}{2},N_y+\frac{1}{2}}(2y_{\max} - y).
  \]

The boundary conditions for \(m^C_{2,h}(x,y), E^C_h(x,y)\) and \(m^D_{2,h}(x,y), E^D_h(x,y)\) are same as the density. The bottom boundary condition is similar to the top.
It is worth noting that, in order to preserve the WB property, we should also apply the same reflective boundary conditions to the projected hydrostatic solutions $\rho^s_{h, C}, \rho^s_{h, C}$ and $\rho^s_{h, D}, \rho^s_{h, D}$ for consistency. The implementations for other boundary conditions are similar and omitted here.

### 4.3 Positivity-preserving WB CDG schemes

#### 4.3.1 Properties of admissible states

The set of admissible states of the two-dimensional Euler equations (30) is defined by

$$ G = \left\{ \mathbf{U} = (\rho, \mathbf{m}, E)^T : \rho > 0, \ p(\mathbf{U}) = (\gamma - 1) \left( E - \frac{|\mathbf{m}|^2}{2\rho} \right) > 0 \right\}, $$

which is a convex set [69].

**Lemma 4.1.** For any $\mathbf{U} \in G$ and $\mathbf{b} \in \mathbb{R}^2$, one has $\mathbf{U} + \lambda \mathbf{S} \mathbf{U} + \mathbf{b} \in G$ under the condition

$$ |\lambda| < \frac{1}{|\mathbf{b}|} \sqrt{\frac{2p}{(\gamma - 1)\rho}}. $$

**Proof.** The proof can be found in, for example, [57, Page A476].

**Lemma 4.2.** For any $\mathbf{U} \in G$ and $\lambda \in \mathbb{R}$, the states $\mathbf{U} - \lambda \mathbf{F}_1(\mathbf{U}) \in G$ and $\mathbf{U} - \lambda \mathbf{F}_2(\mathbf{U}) \in G$ under the conditions $|\lambda|a_x(\mathbf{U}) \leq 1$ and $|\lambda|a_y(\mathbf{U}) \leq 1$, respectively. Here $a_x(\mathbf{U}) := |u_1| + \sqrt{\gamma p/\rho}$ and $a_y(\mathbf{U}) := |u_2| + \sqrt{\gamma p/\rho}$.

**Proof.** The proof is similar to that of Lemma 3.3 and can be found in, for example, [69, 52].

#### 4.3.2 Positivity-preserving analysis

Let us derive the semi-discrete scheme satisfied by the cell averages of the WB CDG method (39)–(40). Denote

$$ \mathbf{U}^c_{ij}(t) = \frac{1}{\Delta x \Delta y} \int_{I_{i,j}} \mathbf{U}^c_h(x, y, t)dx\,dy, \quad \mathbf{U}^D_{i+\frac{1}{2},j+\frac{1}{2}}(t) = \frac{1}{\Delta x \Delta y} \int_{I_{i+\frac{1}{2},j+\frac{1}{2}}} \mathbf{U}^D_h(x, y, t)dx\,dy. $$

Taking the test function $v = 1$ in (39) and $w = 1$ in (40) and using the crucial identities in (38) gives

$$ \frac{d\mathbf{U}^c_{ij}}{dt} = \frac{\delta^c_{ij}(\mathbf{F}_1(\mathbf{U}^c_h))}{\tau_{max}} - \frac{\delta^c_{ij}(\mathbf{F}_2(\mathbf{U}^c_h))}{\Delta x} + \frac{\mathbf{S}^c_{ij}(\mathbf{F}_1(\mathbf{U}^c_h))}{\Delta x \Delta y}, \quad (41) $$

$$ \frac{d\mathbf{U}^{D}_{i+\frac{1}{2},j+\frac{1}{2}}}{dt} = \frac{\delta^D_{i+\frac{1}{2},j+\frac{1}{2}}(\mathbf{F}_1(\mathbf{U}^c_h))}{\tau_{max}} - \frac{\mathbf{S}^D_{ij}(\mathbf{F}_1(\mathbf{U}^c_h))}{\Delta x} \Delta x \Delta y + \frac{\mathbf{S}^D_{ij}(\mathbf{F}_1(\mathbf{U}^c_h))}{\Delta x \Delta y}, \quad (42) $$

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Denote the right-hand sides of (41) and (42) by \( \mathbf{L}^C_{ij}(\mathbf{U}^C_h, \mathbf{U}^D_h) \) and \( \mathbf{L}^D_{i+rac{1}{2},j+rac{1}{2}}(\mathbf{U}^D_h, \mathbf{U}^C_h) \), respectively. In other words, we have
\[
\frac{d\mathbf{U}^C_{ij}}{dt} = \mathbf{L}^C_{ij}(\mathbf{U}^C_h, \mathbf{U}^D_h) , \quad \frac{d\mathbf{U}^D_{i+rac{1}{2},j+rac{1}{2}}}{dt} = \mathbf{L}^D_{i+rac{1}{2},j+rac{1}{2}}(\mathbf{U}^D_h, \mathbf{U}^C_h) .
\]
(43)

On each primal cell \( I_{i,j} \), we define the point set
\[
S_{ij} = S_{ij}^{1,1} \cup S_{ij}^{1,2} \cup S_{ij}^{2,1} \cup S_{ij}^{2,2} , \quad Q_{ij} = Q_{ij}^{1,1} \cup Q_{ij}^{1,2} \cup Q_{ij}^{2,1} \cup Q_{ij}^{2,2} ,
\]
and the parameters \( \tilde{\alpha}^D_{x,ij} \) and \( \tilde{\alpha}^D_{y,ij} \) by
\[
\tilde{\alpha}^D_{x,ij} = \tilde{\alpha}^D_{x,1} + \tilde{\alpha}^D_{x,2} , \quad \tilde{\alpha}^D_{x,1} = \max_{(x,y) \in S_{ij}} a_x(U^D_h) , \quad \tilde{\alpha}^D_{x,2} = \frac{\hat{\omega}_1 \Delta x}{4} \tilde{\alpha}^D_x ,
\]
\[
\tilde{\alpha}^D_{y,ij} = \tilde{\alpha}^D_{y,1} + \tilde{\alpha}^D_{y,2} , \quad \tilde{\alpha}^D_{y,1} = \max_{(x,y) \in S_{ij}} a_y(U^D_h) , \quad \tilde{\alpha}^D_{y,2} = \frac{\hat{\omega}_1 \Delta y}{4} \tilde{\alpha}^D_y ,
\]
where
\[
\tilde{\alpha}^D_x = \max_{(x,y) \in Q_{ij}} \left\{ \left| \nabla \hat{\phi}^D \right| \sqrt{\frac{(\gamma - 1) \rho^D_h}{2 p^D_h}} \right\} , \quad \nabla \hat{\phi}^D = -\frac{\nabla p^x_h}{p^x_h} - \left( \frac{[p^x_h]_{i+rac{1}{2},j+rac{1}{2}}, [p^y_h]_{i+rac{1}{2},j+rac{1}{2}}^y}{(p^x_h)_{i+rac{1}{2},j+rac{1}{2}} \Delta x \Delta y} \right)^T ,
\]
\[
[p^x_h]_{i+rac{1}{2},j+rac{1}{2}} := \int_{y_{i+rac{1}{2}}}^{y_{i+rac{1}{2}}} \left( p^x_h(x_i^+,y) - p^x_h(x_i^-,y) \right) dy , \quad [p^y_h]_{i+rac{1}{2},j+rac{1}{2}}^y := \int_{x_{i+rac{1}{2}}}^{x_{i+rac{1}{2}}} \left( p^y_h(x,y_i^+) - p^y_h(x,y_i^-) \right) dx .
\]

Similarly, on each dual cell \( I_{i+rac{1}{2},j+rac{1}{2}} \), we define the point set \( S_{i+rac{1}{2},j+rac{1}{2}} \) and \( Q_{i+rac{1}{2},j+rac{1}{2}} \), which are respectively the shifts of the point sets \( S_{ij} \) and \( Q_{ij} \) with \( \frac{\Delta x}{2} \) in the \( x \)-direction and \( \frac{\Delta y}{2} \) in the \( y \)-direction, and the parameters \( \tilde{\alpha}^C_{x,i+rac{1}{2},j+rac{1}{2}} \) and \( \tilde{\alpha}^C_{y,i+rac{1}{2},j+rac{1}{2}} \) by
\[
\tilde{\alpha}^C_{x,i+rac{1}{2},j+rac{1}{2}} = \tilde{\alpha}^C_{x,1} + \tilde{\alpha}^C_{x,2} , \quad \tilde{\alpha}^C_{x,1} = \max_{(x,y) \in S_{i+rac{1}{2},j+rac{1}{2}}} a_x(U^C_h) , \quad \tilde{\alpha}^C_{x,2} = \frac{\hat{\omega}_1 \Delta x}{4} \tilde{\alpha}^C_x ,
\]
\[
\tilde{\alpha}^C_{y,i+rac{1}{2},j+rac{1}{2}} = \tilde{\alpha}^C_{y,1} + \tilde{\alpha}^C_{y,2} , \quad \tilde{\alpha}^C_{y,1} = \max_{(x,y) \in S_{i+rac{1}{2},j+rac{1}{2}}} a_y(U^C_h) , \quad \tilde{\alpha}^C_{y,2} = \frac{\hat{\omega}_1 \Delta y}{4} \tilde{\alpha}^C_y ,
\]
where
\[
\tilde{\alpha}^C = \max_{(x,y) \in Q_{i+rac{1}{2},j+rac{1}{2}}} \left\{ \left| \nabla \hat{\phi}^C \right| \sqrt{\frac{(\gamma - 1) \rho^C_h}{2 p^C_h}} \right\} , \quad \nabla \hat{\phi}^C = -\frac{\nabla p^x_h}{p^x_h} - \left( \frac{[p^x_h]_{i+rac{1}{2},j+rac{1}{2}}, [p^y_h]_{i+rac{1}{2},j+rac{1}{2}}^y}{(p^x_h)_{i+rac{1}{2},j+rac{1}{2}} \Delta x \Delta y} \right)^T ,
\]
with
\[
[p^x_h]_{i+rac{1}{2},j+rac{1}{2}} := \int_{y_{i+rac{1}{2}}}^{y_{i+rac{1}{2}}} \left( p^x_h(x_i^+,y) - p^x_h(x_i^-,y) \right) dy , \quad [p^y_h]_{i+rac{1}{2},j+rac{1}{2}}^y := \int_{x_{i+rac{1}{2}}}^{x_{i+rac{1}{2}}} \left( p^y_h(x,y_i^+) - p^y_h(x,y_i^-) \right) dx .
\]
Theorem 4.2. Assume that the numerical solutions $U_h^C(x, y), U_h^D(x, y)$ satisfy

$$U_h^C(x, y) \in G \ \forall (x, y) \in \Omega, \quad U_h^D(x, y) \in G \ \forall (x, y) \in \Omega$$

and the projected stationary hydrostatic solutions $U_h^sC(x, y), U_h^sD(x, y)$ satisfy

$$U_h^sC(x, y) \in G \ \forall (x, y) \in \Omega, \quad U_h^sD(x, y) \in G \ \forall (x, y) \in \Omega$$

If $U_i^C, U_i^D \in G$, then the positivity-preserving property

$$U_i^C + \Delta t I_i^C(U_h^C, U_h^D) \in G, \quad \Delta t I_i^D(U_h^D, U_h^C) \in G, \ \forall i, j,$$

holds under the CFL-type condition

$$\frac{\Delta t}{\Delta x} \alpha_x + \frac{\Delta t}{\Delta y} \alpha_y < \frac{\theta_0}{2}, \quad \theta = \frac{\Delta t}{\tau_{max}} \in (0, 1),$$

with

$$\alpha_x = \max_{i,j} \{ \alpha_{ix}, \alpha_{x \in \Omega} \}, \quad \alpha_y = \max_{i,j} \{ \alpha_{iy}, \alpha_{y \in \Omega} \}.$$ 

Proof. Using (43) gives

$$U_i^C + \Delta t I_i^C(U_h^C, U_h^D) = (1 - \theta) U_i^C + \theta U_i^D + \frac{\Delta t}{\Delta x} \Delta y \langle S_h^D, 1 \rangle_{ij}$$

$$= \lambda_x \delta_{ij}^x (F_1(U_h^D)) - \lambda_y \delta_{ij}^y (F_2(U_h^D))$$

$$= (1 - \theta) U_i^C + \left( \eta \theta U_i^D + \frac{\Delta t}{\Delta x} \Delta y \langle S_h^D, 1 \rangle_{ij} \right)$$

$$+ \left( (1 - \eta) \theta U_i^D - \lambda_x \delta_{ij}^x (F_1(U_h^D)) - \lambda_y \delta_{ij}^y (F_2(U_h^D)) \right)$$

$$= (1 - \theta) U_i^C + \eta \theta L_{h,S} + (1 - \eta) \theta L_{h,F},$$

where $\lambda_x = \frac{\Delta t}{\Delta x}, \lambda_y = \frac{\Delta t}{\Delta y}, \eta \in (0, 1)$ is a constant, and $L_{h,F}$ and $L_{h,S}$ are defined by

$$L_{h,F} = \frac{\lambda_x}{(1 - \eta) \theta} \delta_{ij}^x (F_1(U_h^D)) - \frac{\lambda_y}{(1 - \eta) \theta} \delta_{ij}^y (F_2(U_h^D)),$$

$$L_{h,S} = \frac{\Delta t}{\eta \theta \Delta x \Delta y} \langle S_h^D, 1 \rangle_{ij}.$$

Using the convexity of set $G$ and the exactness of the quadrature rule can derive that

$$U_i^D = \frac{1}{\Delta x \Delta y} \int_{I_i} U_h^D(x, y) dy dx = \frac{1}{\Delta x} \int_{x_i - \frac{1}{2}}^{x_i + \frac{1}{2}} \left( \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{x} U_h^D(x, y) J_{i, \alpha} \right) dx$$

$$= \frac{1}{2} \left( \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{x} \int_{x_i - \frac{1}{2}}^{x_i + \frac{1}{2}} U_h^D(x, y) J_{i, \alpha} \right) = \frac{1}{2} \left( \sum_{k=1}^{N} \sum_{\alpha=1}^{N} \omega_{x} \int_{x_i - \frac{1}{2}}^{x_i + \frac{1}{2}} \left( \frac{\partial}{2} U_h^D(x, y) J_{i, \alpha} \right) + \left( 1 - \omega_i \right) J_{i, \alpha} \right),$$

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where \( \hat{\omega}_1 = \hat{\omega}_L \) has been used, and

\[
\mathbb{E}^\alpha_{1,ij} = \frac{1}{(1 - \hat{\omega}_1)} \left( \frac{1}{2} \sum_{\beta=2}^{L} \hat{\omega}_\beta U_h^D(x_i^{1,\beta}, y_j^{\kappa,\alpha}) + \frac{1}{2} \sum_{\beta=2}^{L} \hat{\omega}_\beta U_h^D(x_i^{2,\beta}, y_j^{\kappa,\alpha}) \right) \in G.
\]

Similarly, one has

\[
U_{ij}^D = \frac{1}{2} \sum_{\kappa=1}^{2} \sum_{\alpha=1}^{N} \omega_\alpha \left( \hat{\omega}_1 U_h^D(x_i^{\kappa,\alpha}, y_{j-1}^{1}) + \hat{\omega}_1 U_h^D(x_i^{\kappa,\alpha}, y_{j+1}^{1}) + (1 - \hat{\omega}_1) \mathbb{E}^\alpha_{2,ij} \right),
\]

with

\[
\mathbb{E}^\alpha_{2,ij} = \frac{1}{(1 - \hat{\omega}_1)} \left( \frac{1}{2} \sum_{\beta=2}^{L} \hat{\omega}_\beta U_h^D(x_i^{\kappa,\alpha}, y_j^{1,\beta}) + \frac{1}{2} \sum_{\beta=2}^{L} \hat{\omega}_\beta U_h^D(x_i^{\kappa,\alpha}, y_j^{1,\beta}) \right) \in G.
\]

Note that \( L_{h,F} \) can be reformulated as

\[
L_{h,F} = \frac{a_x \lambda_x + a_y \lambda_y}{\lambda} \left( U_{ij}^D - \frac{1}{1 - \eta} \hat{\omega} \sigma_{ij}(F_1(U_h^D)) \right) + \frac{a_y \lambda_y}{\lambda} \left( U_{ij}^D - \frac{1}{1 - \eta} \hat{\omega} \sigma_{ij}(F_2(U_h^D)) \right),
\]

where \( \lambda = a_x \lambda_x + a_y \lambda_y, a_x = \hat{\epsilon}_{x,1}, a_y = \hat{\epsilon}_{y,1} \), and

\[
U_{ij}^D - \frac{1}{1 - \eta} \hat{\omega} \sigma_{ij}(F_1(U_h^D)) = \frac{1}{2} \sum_{\kappa=1}^{2} \sum_{\alpha=1}^{N} \omega_\alpha \left( \frac{\hat{\omega}_1}{2} \mathbb{E}_{1,i,j}^{+} - \frac{\hat{\omega}_1}{2} \mathbb{E}_{1,i,j}^{-} + (1 - \hat{\omega}_1) \mathbb{E}^\alpha_{1,ij} \right),
\]

\[
U_{ij}^D - \frac{1}{1 - \eta} \hat{\omega} \sigma_{ij}(F_2(U_h^D)) = \frac{1}{2} \sum_{\kappa=1}^{2} \sum_{\alpha=1}^{N} \omega_\alpha \left( \frac{\hat{\omega}_1}{2} \mathbb{E}_{2,i,j}^{+} - \frac{\hat{\omega}_1}{2} \mathbb{E}_{2,i,j}^{-} + (1 - \hat{\omega}_1) \mathbb{E}^\alpha_{2,ij} \right),
\]

with

\[
\mathbb{E}_{1,i,j}^{\pm} = U_h^D(x_{i+\frac{1}{2}}, y_j^{\kappa,\alpha}) \pm \frac{2\hat{\omega}_1}{\hat{\omega}_1 (1 - \eta) \omega} F_1(U_h^D(x_{i+\frac{1}{2}}, y_j^{\kappa,\alpha})),
\]

\[
\mathbb{E}_{2,i,j}^{\pm} = U_h^D(x_i^{\kappa,\alpha}, y_{j+\frac{1}{2}}) \pm \frac{2\hat{\omega}_1}{\hat{\omega}_1 (1 - \eta) \omega} F_2(U_h^D(x_i^{\kappa,\alpha}, y_{j+\frac{1}{2}})).
\]

Thanks to Lemma 4.2, we have \( \mathbb{E}_{1,i,j,\frac{1}{2}}^{\pm} \in G \) and \( \mathbb{E}_{2,i,j,\frac{1}{2}}^{\pm} \in G \), as long as

\[
\lambda_x \alpha_{x,1} + \lambda_y \alpha_{y,1} < \frac{(1 - \eta) \hat{\omega}_1}{2}.
\]

Using the convexity of \( G \), we further obtain \( L_{h,F} \in G \) under (45). Next, we discuss the term \( L_{h,S} \). The source term approximations \( \langle (S_{2,h}^D, S_{3,h}^D)^\top, 1 \rangle_{ij} \) in the momentum equations are

\[
\langle (S_{2,h}^D, S_{3,h}^D)^\top, 1 \rangle_{ij} = \frac{\Delta x \Delta y}{4} \sum_{\kappa,\alpha, \tau, \beta} \omega_\alpha \omega_\beta \left( \frac{\rho_h^D(x_i^{\kappa,\alpha}, y_j^{\tau,\beta})}{\rho_h^{s,D}(x_i^{\kappa,\alpha}, y_j^{\tau,\beta})} - \frac{\rho_h^{s,D}(x_i^{\kappa,\alpha}, y_j^{\tau,\beta})}{\rho_h^{s,D}(x_i^{\kappa,\alpha}, y_j^{\tau,\beta})} \right) (\nabla p_h^{s,D})(x_i^{\kappa,\alpha}, y_j^{\tau,\beta})
\]

\[
+ \frac{(\rho_h^D)^{ij}}{(\rho_h^{s,D})^{ij}} (\Delta y \delta_i^\tau(p_h^{s,D}), \Delta x \delta_j^\beta(p_h^{s,D}))^\top.
\]
Based on

\[
\int_{ij} \nabla p_h^{s,D} \, dx \, dy = \frac{\Delta t \Delta y}{4} \sum_{\kappa, \alpha, \tau, \beta} \omega_\kappa \omega_\beta \nabla \rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}),
\]

\[
\int_{\partial ij} p_h^{s,D} \, ds = \left( \Delta y \delta_{ij} \right) \left( \rho_h^{s,D} \right) \left( x_i^{\kappa, \alpha}, y_j^{\tau, \beta} \right),
\]

and the identity

\[
\int_{\partial ij} p_h^{s,D} \, ds - \int_{ij} \nabla p_h^{s,D} \, dx \, dy = \left( \left[ p_h^{s,D} \right]_{ij}, \left[ p_h^{s,D} \right]_{ij} \right) \tau,
\]

we reformulate \( \langle (S_{2h}^D, S_{3h}^D), 1 \rangle_{ij} \) as

\[
\langle (S_{2h}^D, S_{3h}^D), 1 \rangle_{ij} = \frac{\Delta t \Delta y}{4} \sum_{\kappa, \alpha, \tau, \beta} \omega_\kappa \omega_\beta \rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) \nabla \rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta})
\]

\[
+ \frac{\langle \rho_h^{s,D} \rangle_{ij}}{\langle p_h^{s,D} \rangle_{ij}} \left( \left[ p_h^{s,D} \right]_{ij}, \left[ p_h^{s,D} \right]_{ij} \right) \tau = -\frac{\Delta t \Delta y}{4} \sum_{\kappa, \alpha, \tau, \beta} \omega_\kappa \omega_\beta \rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) \nabla \rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}),
\]

where

\[
-\nabla \hat{\phi}_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) = \frac{\nabla \rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta})}{\rho_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta})} \left( \left[ p_h^{s,D} \right]_{ij}, \left[ p_h^{s,D} \right]_{ij} \right) \tau.
\]

Similarly, we can rewrite \( \langle S_{4h}^{D}, 1 \rangle_{ij} \) in the energy equation as

\[
\langle S_{4h}^{D}, 1 \rangle_{ij} = -\frac{\Delta t \Delta y}{4} \sum_{\kappa, \alpha, \tau, \beta} \omega_\kappa \omega_\beta \mathbf{m}_h^{D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) \cdot \nabla \hat{\phi}_h^{s,D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}).
\]

Thus, \( \mathbf{L}_{h,S} = \mathbf{U}_{ij}^{D} + \frac{\Delta t}{\eta \theta \Delta x \Delta y} \langle S_{h}^{D}, 1 \rangle_{ij} \) can be reformulated as

\[
\mathbf{L}_{h,S} = \sum_{\kappa, \alpha, \tau, \beta} \frac{\omega_\kappa \omega_\beta}{4} \left( \mathbf{U}_h^{D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) + \frac{\Delta t}{\eta \theta} \hat{S}_h^{D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) \right),
\]

with \( \hat{S}_h^{D}(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) := \left( 0, -(\lambda_p^{D} \nabla \hat{\phi}_h^{D})(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}), -(\mathbf{m}_h^{D} \cdot \nabla \hat{\phi}_h^{D})(x_i^{\kappa, \alpha}, y_j^{\tau, \beta}) \right)^{\tau} \). Thanks to Lemma 4.1, we have \( \mathbf{L}_{h,S} \in \mathcal{G} \) under the condition

\[
\Delta t < \eta \theta \min_{(x,y) \in \Omega_{ij}} \left\{ \frac{1}{|\nabla \hat{\phi}_h^{D}|} \sqrt{\frac{2 \rho_h^{D}}{(y-1) \rho_h^{D}}} \right\},
\]

or equivalently

\[
\lambda_x \hat{\alpha}_x^{D} + \lambda_y \hat{\alpha}_y^{D} < \frac{\eta \theta \delta_1}{2}.
\]

Combining those results, we conclude that if

\[
(\lambda_x, \lambda_y) \in \left\{ (\lambda_x, \lambda_y) \in \mathbb{R}^+ : \lambda_x \hat{\alpha}_x^{D} + \lambda_y \hat{\alpha}_y^{D} < \frac{(1 - \eta \theta \delta_1)}{2}, \lambda_x \hat{\alpha}_x^{D} + \lambda_y \hat{\alpha}_y^{D} < \frac{\eta \theta \delta_1}{2} \right\}, \quad (46)
\]
then $\mathbf{U}_{ij}^C + \Delta t \mathbf{L}_{ij}^C (\mathbf{U}_{h_i}^C, \mathbf{U}_{h_j}^D) \in G$. Since the parameters $\eta$ can be chosen arbitrarily in this proof, we specify

$$\eta = \frac{\lambda_x \alpha_{x,2} + \lambda_y \alpha_{y,2}}{\lambda_x \alpha_{x,ij} + \lambda_y \alpha_{y,ij}} = \frac{\lambda_x (\alpha_{x,1} + \alpha_{x,2}) + \lambda_y (\alpha_{y,1} + \alpha_{y,2})}{\lambda_x (\alpha_{x,1} + \alpha_{x,2}) + \lambda_y (\alpha_{y,1} + \alpha_{y,2})},$$

so that the condition (46) becomes

$$\lambda_x (\alpha_{x,1} + \alpha_{x,2}) + \lambda_y (\alpha_{y,1} + \alpha_{y,2}) = \lambda_x \alpha_{x,ij} + \lambda_y \alpha_{y,ij} < \frac{\theta \Delta t}{2}.$$  

Similar arguments can show that $\mathbf{U}_{i+\frac{1}{2},j+\frac{1}{2}}^D + \Delta t \mathbf{L}_{i+\frac{1}{2},j+\frac{1}{2}}^D (\mathbf{U}_{h_i}^D, \mathbf{U}_{h_j}^C) \in G$. The proof is completed. ■

Theorem 4.2 provides a sufficient condition for the proposed high-order WB CDG schemes (39) and (40) to be positivity-preserving, when the SSP time discretization is used. The condition (44) can again be enforced by a simple positivity-preserving limiter similar to the one-dimensional case; see Section 4.4. With the positivity-preserving limiter applied at each stage of the SSP Runge-Kutta method, the resulting fully discrete CDG schemes are positivity-preserving.

### 4.4 Positivity-preserving limiting operators

Introduce the following two sets

$$\mathbb{C}_h^{C,k} := \left\{ \mathbf{v} \in \mathbb{V}_h^{C,k} : \frac{1}{\Delta x \Delta y} \int_{L_{ij}} \mathbf{v}(x,y) \, dx \, dy \in G, \forall i, j \right\},$$

$$\mathbb{G}_h^{C,k} := \left\{ \mathbf{v} \in \mathbb{C}_h^{C,k} : \mathbf{v} \mid_{L_{ij}} (x,y) \in G, \forall (x,y) \in \mathbb{S}_{ij}, \forall i, j \right\}.$$

For any $\mathbf{U}_{h_i}^C, \mathbf{U}_{h_j}^C$ with $\mathbf{U}_{h_i}^C \mid_{L_{ij}} := \mathbf{U}_{h_i}^C (x,y)$, following [46, 69] we define the positivity-preserving limiting operator $\Pi_h^{C,k} : \mathbb{C}_h^{C,k} \rightarrow \mathbb{G}_h^{C,k}$ as follows

$$\Pi_h^{C,k} \mathbf{U}_{h_i}^C \mid_{L_{ij}} = \theta_{ij}^{(2)} (\mathbf{U}_{h_i}^C (x,y) - \mathbf{U}_{ij}^C) + \mathbf{U}_{ij}^C, \quad \theta_{ij}^{(2)} = \min \left\{ 1, \frac{p(C) - \varepsilon_2}{p(C) - \min_{(x,y) \in S_{ij}} p(C)} \right\},$$

where $\mathbf{U}_{ij}^C (x,y) = (\mathbf{p}_{ij}^C (x,y), \mathbf{m}_{ij}^C (x,y), \mathbf{E}_{ij}^C (x,y))^\top$, and $\hat{\mathbf{p}}_{ij}^C (x,y)$ is a modification of the density $\rho_{ij}^C (x,y)$ given by

$$\hat{\mathbf{p}}_{ij}^C (x,y) = \theta_{ij}^{(1)} (\mathbf{p}_{ij}^C (x,y) - \mathbf{p}_{ij}^C) + \mathbf{p}_{ij}^C, \quad \theta_{ij}^{(1)} = \min \left\{ 1, \frac{\hat{\mathbf{p}}_{ij}^C - \varepsilon_1}{\hat{\mathbf{p}}_{ij}^C - \min_{(x,y) \in S_{ij}} \hat{\mathbf{p}}_{ij}^C} \right\},$$

We take $\varepsilon_1 = \min \{ 10^{-13}, \bar{\mathbf{p}}_{ij}^C \}$, $\varepsilon_2 = \min \{ 10^{-13}, p(\mathbf{U}_{ij}^C) \}$. The sets $\mathbb{C}_h^{D,k}, \mathbb{G}_h^{D,k}$ and the positivity-preserving limiting operator $\Pi_h^{D,k} : \mathbb{C}_h^{D,k} \rightarrow \mathbb{G}_h^{D,k}$ defined on the dual mesh $I_{i+\frac{1}{2},j+\frac{1}{2}}$ are very similar, and the details are omitted here.
5 Numerical examples

This section presents several one- and two-dimensional tests to demonstrate the WB and positivity-preserving properties of the proposed CDG methods on uniform Cartesian meshes. The explicit third-order SSP Runge-Kutta method is employed for the time discretization. For comparison, we will also show the numerical results of the standard non-WB CDG schemes with the straightforward source term approximation and the original numerical dissipation term. Unless explained specifically, we use the ideal EOS (2) with $\gamma = 1.4$, the CFL numbers for the third-order and fourth-order CDG methods are taken as 0.25 and 0.15, respectively, and the parameter $\theta = \frac{\Delta t}{\tau_{\text{max}}} = 1$. In all the numerical examples, the schemes are implemented by using C/C++ language with double precision.

5.1 One-dimensional tests

5.1.1 Example 1: Accuracy test

We start with a one-dimensional example [41] to demonstrate the accuracy of the proposed WB CDG schemes for the Euler equations under a linear gravitational field $\phi_x = g = 1$. The time-dependent exact solution of this example is given by

$$\rho(x,t) = 1 + 0.2\sin(\pi(x-u_0t)), \quad u(x,t) = u_0, \quad p(x,t) = p_0 + u_0 t - x + 0.2\cos(\pi(x-u_0t))/\pi,$$

where the constants $u_0 = 1.0$, $p_0 = 4.5$. The computational domain $\Omega = [0, 2]$ is divided into $N$ uniform cells and the boundary condition is set as the exact solution on $\partial \Omega$. To match the temporal and spatial accuracy, we use $\Delta t = 0.15(\Delta x)^{\frac{3}{4}}/\alpha_x$ for the fourth-order WB CDG scheme. The $L^1$ errors and corresponding convergence rates at $t = 0.1$ are shown in Tables 1 and 2. We clearly observe that the expected third-order and fourth-order convergence rates are achieved by the WB CDG schemes. This indicates that our novel projection, modification of the dissipation term and WB source term approximation do not destroy the accuracy of our proposed WB CDG schemes.

Table 1: Example 1: $L^1$ errors at $t = 0.1$ and corresponding convergence rates for the third-order WB CDG scheme at different grid resolutions.

|   | $\rho$  | $\rho u$ | $E$    |
|---|---------|----------|--------|
|   | $L^1$ error | Order | $L^1$ error | Order | $L^1$ error | Order |
| 8 | 1.99e-04 | - | 2.01e-04 | - | 1.94e-04 | - |
| 16 | 2.54e-05 | 2.97 | 2.54e-05 | 2.98 | 2.39e-05 | 3.02 |
| 32 | 3.16e-06 | 3.01 | 3.17e-06 | 3.00 | 2.98e-06 | 3.00 |
| 64 | 3.96e-07 | 3.00 | 3.96e-07 | 3.00 | 3.72e-07 | 3.00 |
| 128 | 4.94e-08 | 3.00 | 4.95e-08 | 3.00 | 4.65e-08 | 3.00 |
Table 2: Same as Table 1, except for the fourth-order WB CDG scheme.

| Mesh | $L^1$ error $\rho$ | Order | $L^1$ error $\rho u$ | Order | $L^1$ error $E$ | Order |
|------|-----------------|-------|-----------------|-------|-----------------|-------|
| 8    | 3.12e-06        | -     | 2.93e-06        | -     | 2.70e-06        | -     |
| 16   | 1.69e-07        | 4.21  | 1.67e-07        | 4.13  | 1.61e-07        | 4.07  |
| 32   | 9.62e-09        | 4.13  | 9.68e-09        | 4.11  | 9.74e-09        | 4.05  |
| 64   | 6.67e-10        | 3.85  | 6.75e-10        | 3.84  | 7.65e-10        | 3.67  |
| 128  | 4.05e-11        | 4.04  | 4.12e-11        | 4.03  | 4.79e-11        | 4.00  |

5.1.2 Example 2: Isothermal equilibrium test

We consider the isothermal equilibrium problem [6] under a linear gravitational field $\phi_x = g = 1$. The initial data is taken as an isothermal steady state solution

$$
\rho(x) = \exp(-x), \quad u(x) = 0, \quad p(x) = \exp(-x).
$$

(47)

The computational domain is taken as $\Omega = [0, 1]$, and the adiabatic index is $\gamma = 5/3$. We first use this example to verify the WB property of the proposed CDG method. The numerical solution is computed until $t = 2$ by using our third-order WB CDG scheme with respectively 50 and 100 uniform cells. Table 3 lists the $L^1$ errors between the numerical solution and the projected stationary hydrostatic solution (47). It is clearly observed that all the numerical errors are at the level of rounding error, demonstrating that the proposed CDG method satisfies the WB property.

Table 3: $L^1$ errors for the isothermal equilibrium test in Example 2.

| Mesh | errors in $\rho$ | errors in $\rho u$ | errors in $E$ |
|------|-----------------|-----------------|-----------------|
| 50   | 7.71e-15        | 1.97e-15        | 4.00e-15        |
| 100  | 1.63e-14        | 4.50e-15        | 7.27e-15        |

Next, in order to check the effectiveness of our WB CDG method in capturing a small perturbation near the isothermal equilibrium solution (47), we modify the initial pressure state as

$$
p(x, 0) = \exp(-x) + \eta \exp(-100(x - 0.5)^2),
$$

where $\eta$ is a non-zero perturbation parameter. We simulate two cases: $\eta = 10^{-2}$ and $\eta = 10^{-3}$. Outflow boundary conditions are used at $x = 0$ and $x = 1$. The pressure perturbations at $t = 0.25$ computed by our third-order WB CDG scheme on the mesh of 50 uniform cells, compared with a reference solution with 1000 cells, are displayed in Figure 1. For comparison, we also present the results simulated by the third-order non-WB CDG scheme in the same figure. The initial pressure perturbations are also plotted in the dashed curves. As we can see that the results of the WB CDG scheme agree well with the reference solutions for both cases, while the results obtained by the non-WB CDG scheme fail to capture the small perturbations. This demonstrates that our WB method is more accurate and advantageous in resolving small perturbations near the equilibrium states.
5.1.3 Example 3: Rarefaction test with low density and pressure

The purpose of this example is to investigate the positivity-preserving property of our WB CDG method. We consider an extreme rarefaction problem [57] under a quadratic gravitational potential function $\phi(x) = x^2/2$, and the initial data are given by

$$\rho(x,0) = 7, \quad p(x,0) = 0.2, \quad u(x,0) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

The computational domain is set as $\Omega = [-1,1]$ with outflow boundary conditions at $x = -1$ and $x = 1$. This test involves extremely low density and pressure, so that the positivity-preserving limiting operators are used in our simulation. Figure 2 shows the numerical solutions at $t = 0.6$ computed by our third-order positivity-preserving WB CDG scheme with 400 uniform cells, along with a reference solution with 1000 cells. As we can see that the low density and pressure wave structures are well captured by the proposed method. Our numerical scheme exhibits good robustness, and no negative density or pressure is encountered during the entire simulation.

5.1.4 Example 4: Leblanc shock tube problem under gravitational field

This example considers a Leblanc shock tube problem [57] under a linear gravitational field $\phi(x) = gx$ with $g = 1$. The initial condition is given by

$$(\rho, u, p)(x,0) = \begin{cases} (2, 0, 10^9), & x < 0, \\ (10^{-3}, 0, 1), & x > 0. \end{cases}$$
Figure 2: Example 3: Numerical results for the rarefaction test at $t = 0.6$ obtained by the third-order positivity-preserving WB CDG scheme with 400 cells (circles) and 1000 cells (solid lines).

The computational domain is taken as $\Omega = [-10, 10]$ with outflow boundary condition at $x = -10$ and $x = 10$. As the initial data contain a strong discontinuity in the density and the pressure, the WB implementation (see Remark 3.4) of the WENO limiter [37] is applied to the local characteristic fields in some troubled cells, before the positivity-preserving limiting procedure in our simulation. The troubled cells are adaptively identified with parameters $(M_1, M_2, M_3) = (10^{10}, 2 \times 10^6, 5 \times 10^{10})$, where $M_i$ denotes the parameter $M$ in (29) for the $i$-th component of $U$. Figure 3 displays the density, the velocity, and the pressure at $t = 0.0001$ computed by our third-order positivity-preserving WB CDG scheme on a mesh with 800 cells, compared with a reference solution with a refined mesh of 1600 cells. It is seen that our positivity-preserving scheme is highly robust, and the strong discontinuities are captured with high resolution.

Figure 3: Example 4: Numerical results for the Leblanc shock tube problem at $t = 0.0001$ obtained by the third-order positivity-preserving WB CDG scheme with 800 cells (circles) and 1600 cells (solid lines).
5.2 Two-dimensional tests

5.2.1 Example 5: Accuracy test

This test checks the convergence rates of the WB CDG schemes for the two-dimensional Euler equations under a linear gravitational field $\phi_x = \phi_y = 1$. A time-dependent exact solution [41] takes the form of

$$
\rho(x,y,t) = 1 + 0.2 \sin(\pi(x+y-(u_0+v_0)t)),
$$

$$
u_1(x,y,t) = u_0, \quad u_2(x,y,t) = v_0,$$

$$p(x,y,t) = p_0 + (u_0+v_0)t - x - y + 0.2 \cos(\pi(x+y-(u_0+v_0)t))/\pi,$$

where the parameters $u_0 = v_0 = 1$ and $p_0 = 4.5$. The computational domain $\Omega = [0,2] \times [0,2]$ is divided into $N \times N$ uniform cells, and the boundary condition is set as the exact solution on $\partial \Omega$. The $L^1$ errors and corresponding convergence rates at $t = 0.1$ are displayed in Tables 4 and 5. It is seen that the theoretical convergence rates are achieved by our WB CDG schemes, as expected. Our novel projection, modification of the dissipation term, and WB source term approximation do not affect the accuracy of the CDG method.

Table 4: Example 5: $L^1$ errors at $t = 0.1$ and corresponding convergence rates for the third-order WB CDG scheme at different grid resolutions.

| $N \times N$ | $\rho$ | $\rho u_1$ | $\rho u_2$ | $E$ |
|--------------|-------|-----------|-----------|-----|
| $L^1$ error  | Order | $L^1$ error | Order | $L^1$ error | Order | $L^1$ error | Order |
| 8 x 8        | 7.18e-04 | - | 7.09e-04 | - | 7.08e-04 | - | 8.99e-04 | - |
| 16 x 16      | 8.53e-05 | 3.07 | 8.48e-05 | 3.06 | 8.48e-05 | 3.06 | 1.08e-04 | 3.06 |
| 32 x 32      | 1.05e-05 | 3.02 | 1.05e-05 | 3.01 | 1.05e-05 | 3.01 | 1.34e-05 | 3.01 |
| 64 x 64      | 1.31e-06 | 3.00 | 1.30e-06 | 3.01 | 1.30e-06 | 3.01 | 1.67e-06 | 3.00 |
| 128 x 128    | 1.63e-07 | 3.01 | 1.63e-07 | 3.00 | 1.63e-07 | 3.00 | 2.09e-07 | 3.00 |

Table 5: Same as Table 4, except for the fourth-order WB CDG scheme.

| $N \times N$ | $\rho$ | $\rho u_1$ | $\rho u_2$ | $E$ |
|--------------|-------|-----------|-----------|-----|
| $L^1$ error  | Order | $L^1$ error | Order | $L^1$ error | Order | $L^1$ error | Order |
| 8 x 8        | 9.65e-05 | - | 9.35e-05 | - | 9.35e-05 | - | 1.16e-04 | - |
| 16 x 16      | 5.51e-06 | 4.13 | 5.43e-06 | 4.11 | 5.43e-06 | 4.11 | 6.87e-06 | 4.08 |
| 32 x 32      | 3.33e-07 | 4.05 | 3.30e-07 | 4.04 | 3.30e-07 | 4.04 | 4.22e-07 | 4.02 |
| 64 x 64      | 2.06e-08 | 4.01 | 2.05e-08 | 4.01 | 2.05e-08 | 4.01 | 2.62e-08 | 4.01 |
| 128 x 128    | 1.29e-09 | 4.00 | 1.29e-09 | 3.99 | 1.29e-09 | 3.99 | 1.65e-09 | 3.99 |
5.2.2 Example 6: Isothermal equilibrium solution

This example considers a two-dimensional isothermal equilibrium problem [41] under the linear gravitational field $\phi_x = \phi_y = g$. The initial condition is specified as

$$
\rho(x,y) = \rho_0 \exp\left(-\frac{\rho_0 g}{p_0}(x+y)\right),
$$

$$
u_1(x,y) = u_2(x,y) = 0,
$$

$$p(x,y) = p_0 \exp\left(-\frac{\rho_0 g}{p_0}(x+y)\right),
$$

where the parameters $\rho_0 = 1.21$, $p_0 = 1$ and $g = 1$. The computational domain is a unit square $\Omega = [0,1] \times [0,1]$. We first use this test to demonstrate the WB property of the proposed CDG method. The numerical results at $t = 1$ are obtained by using our third-order WB CDG scheme on two different uniform meshes. Table 6 lists the $L^1$ errors between the numerical solution and the isothermal equilibrium solution (48). It is clearly observed that all the numerical errors are at the level of machine precision, demonstrating that the proposed CDG method satisfies the WB property in two-dimensional case.

Table 6: $L^1$ errors for the isothermal equilibrium solution in Example 6.

| Mesh   | errors in $\rho$ | errors in $\rho u_1$ | errors in $\rho u_2$ | errors in $E$ |
|--------|-----------------|----------------------|----------------------|---------------|
| 50×50  | 2.26e-15        | 8.43e-16             | 8.44e-16             | 4.07e-15      |
| 80×80  | 3.96e-15        | 1.40e-15             | 1.38e-15             | 6.49e-15      |

Next, we investigate the effectiveness of our WB CDG method in capturing the propagation of a small wave perturbation around the isothermal equilibrium solution. Initially, a small Gaussian perturbation is imposed in the pressure state as follows

$$p(x,y) = p_0 \exp\left(-\frac{\rho_0 g}{p_0}(x+y)\right) + \eta \exp\left(-100\frac{\rho_0 g}{p_0}(x-0.3)^2 + (y-0.3)^2\right),$$

where parameter $\eta = 10^{-3}$, and the density and the velocities are given by the equilibrium state (48). We evolve the numerical solution until $t = 0.15$ on the mesh of $50 \times 50$ cells, with the transmissive boundary conditions specified on $\partial\Omega$. The contour plots of the density and pressure perturbations obtained by the third-order WB and non-WB CDG schemes are shown in Figure 4. One can see that the non-WB CDG scheme cannot capture those small perturbations well on the relatively coarse mesh, while the WB method resolves them accurately.

5.2.3 Example 7: Polytropic equilibrium solution

This example considers a two-dimensional polytropic problem arising from astrophysics [17]. This model can be constructed from the hydrostatic equilibrium in spherical symmetry case

$$\frac{dp}{dr} = -\rho \frac{d\phi}{dr},$$
Figure 4: Example 6: Contour plots of the pressure and density perturbations for the two-dimensional isothermal equilibrium problem at $t = 0.15$ obtained by the WB and non-WB CDG schemes with $50 \times 50$ uniform cells.
where \( r := \sqrt{x^2 + y^2} \) denotes the radial variable, and the adiabatic index is \( \gamma = 2 \). One equilibrium solution of this model is given by

\[
\rho(r) = \rho_c \frac{\sin(\alpha r)}{\alpha r}, \quad u_1(r) = 0, \quad u_2(r) = 0, \quad p(r) = K \rho(r)^2, \tag{49}
\]
coupled with a gravitational potential function

\[
\phi(r) = -2K \rho_c \frac{\sin(\alpha r)}{\alpha r}, \tag{50}
\]
where \( \alpha = \sqrt{\frac{2\pi g}{K}} \) and \( K = g = \rho_c = 1 \). The computational domain is taken as \( \Omega = [-0.5, 0.5] \times [-0.5, 0.5] \).

We first use this example to verify the WB property of the proposed CDG schemes. The initial data are set as the equilibrium solution (49), and we perform the numerical simulations up to \( t = 14.8 \) on two different uniform meshes. Table 7 lists the \( L^1 \) errors between the numerical solution and the projected equilibrium solution (49). It is observed that all the numerical errors are at the level of rounding error, which confirms that the proposed CDG method is WB.

**Table 7: \( L^1 \) errors for the polytropic equilibrium solution in Example 7.**

| Mesh    | errors in \( \rho \) | errors in \( \rho u_1 \) | errors in \( \rho u_2 \) | errors in \( E \) |
|---------|----------------------|--------------------------|--------------------------|-------------------|
| 50\times50 | 1.31e-13  | 1.48e-14  | 1.55e-14  | 3.68e-14  |
| 80\times80 | 2.25e-13  | 2.02e-14  | 2.03e-14  | 6.39e-14  |

In order to investigate the capability of our WB CDG method in capturing small perturbations near the polytropic equilibrium solution, we impose a small Gaussian hump perturbation in pressure as follows

\[
p(r) = K \rho(r)^2 + \eta \exp(-100r^2),
\]
where the parameter \( \eta = 10^{-3} \). We perform the numerical simulation up to \( t = 0.2 \) on the mesh of 50 \( \times \) 50 cells with outflow boundary conditions specified on \( \partial \Omega \). The contour plots of the pressure perturbation and the velocity magnitude \( \sqrt{u^2 + v^2} \) are displayed in Figure 5. The results show that the non-WB scheme are not capable of capturing the small perturbations on the relatively coarse mesh, while our WB scheme can resolve them accurately. In addition, the WB scheme is able to preserve the axial symmetry, but the non-WB scheme cannot well maintain the symmetry.

**5.2.4 Example 8: Rarefaction test with low density and pressure**

This example is used to demonstrate the positivity-preserving property of the proposed CDG schemes. The initial condition is given by

\[
\rho(x,y,0) = \exp(-\phi(x,y)/0.4), \quad p(x,y,0) = 0.4 \exp(-\phi(x,y)/0.4),
\]

\[
u_1(x,y,0) = \begin{cases} 
-2, & x < 0.5, \\
2, & x > 0.5,
\end{cases} \quad u_2(x,y,0) = 0,
\]
Figure 5: Example 7: Contour plots of the velocity magnitude $\sqrt{u_1^2 + u_2^2}$ and pressure perturbation for the two-dimensional polytropic equilibrium problem at $t = 0.2$ obtained by the third-order WB and non-WB CDG schemes with $100 \times 100$ uniform cells. 10 uniformly spaced contour lines are displayed.
with a quadratic gravitational potential $\phi(x, y) = \frac{1}{2}[(x - 0.5)^2 + (y - 0.5)^2]$. The computational domain $\Omega = [0, 1] \times [0, 1]$ is divided into $100 \times 100$ uniform cells with outflow boundary conditions on $\partial \Omega$. Figure 6 displays the numerical solutions obtained by our third-order positivity-preserving WB CDG method. We observe that the density and the pressure get close to zero but remain positive throughout the simulation. It is noticed that the CDG code would blow-up, if the positivity-preserving limiter is not employed.

Figure 6: Example 8: Contour plots for the two-dimensional rarefaction test at $t = 0.1$ obtained by the positivity-preserving WB CDG scheme with $100 \times 100$ uniform cells.

5.2.5 Example 9: Blast problem

In order to demonstrate the positivity-preserving property and the capability of the proposed WB CDG method in resolving strong discontinuities, we consider a two-dimensional blast problem [57] under the gravitational field (50). The initial condition is obtained by adding a huge jump to the pressure of the polytropic equilibrium solution (49). Specially, the initial pressure is given by

$$p(r) = K \rho(r)^2 + \begin{cases} 100, & r < 0.1, \\ 0, & r \geq 0.1. \end{cases}$$

We set the parameters $K = g = 1$ and $\rho_c = 0.01$, so that the low pressure and the low density appear in the solution and make this test challenging. The computational domain is set as $\Omega = [-0.5, 0.5] \times [-0.5, 0.5]$, and the adiabatic index is $\gamma = 2$.

In this test, both the adaptive WENO limiter [37] (see Remark 3.4 for its WB implementation with the TVB parameter $M = 200$) and the positivity-preserving limiter are implemented. Figure 7 displays the contour plots of the density and the pressure at $t = 0.005$ computed by the third-order positivity-preserving WB CDG method with $200 \times 200$ cells. Figure 7 also gives the plots along the line $y = 0$, from which we can clearly observe a strong shock at $|x| \approx 0.4$. It is seen that the discontinuities are captured with high resolution, and the proposed CDG method preserves the positivity of the density and the pressure as well as the axisymmetric structure of the solution.
Figure 7: Example 9: Contour plots of the density (top-left) and the pressure (top-right) and corresponding plots (bottom) along the line $y = 0$ for the two-dimensional blast problem at $t = 0.005$, obtained by the positivity-preserving WB CDG scheme on a mesh with $200 \times 200$ uniform cells.
5.2.6 Example 10: Rising thermal bubble

This is a benchmark test problem arising from the atmospheric flows [11, 12, 57]. It shows the evolution of a warm bubble in a constant potential temperature environment. Because the bubble is warmer than the ambient air, it rises while deforming as a consequence of the shearing motion caused by the velocity field gradients until it forms a mushroom cloud. The computational domain is set as $\Omega = [0, 1000] \times [0, 1000] \text{ m}^2$. The boundary conditions on all sides are set as the solid walls and the reflective boundary conditions are specified. The initial solution is a stratified atmosphere in hydrostatic balance; see, e.g., the second example in the Appendix of [11]. The constant potential temperature (and thus the reference temperature at $y = 0$) is $300 \text{ K}$, and the reference pressure is $10^5 \text{ N/m}^2$. The ambient flow is at rest (i.e. $u = 0 \text{ m/s}$) and experiences a constant gravitational force per unit mass of $g = 9.8 \text{ m/s}^2$, which implies a linear gravitational field with $\phi_x = 0 \text{ m/s}^2$ and $\phi_y = g$. The potential temperature and the Exner pressure of the ambient air are $\Theta = T_0 = 300 \text{ K}$ and $\Pi = 1 - \frac{(r-1)gy_0}{R\Theta}$, respectively, where $R = 287.058 \text{ J/(kg \cdot K)}$ is the gas constant for dry air. Initially, the warm bubble is added as a potential temperature perturbation to the hydrostatic balance:

$$\Delta \Theta(x,y,t=0) = \begin{cases} 0, & r > r_c, \\ \frac{\theta_c}{2}(1 + \cos(\pi r/r_c)), & r \leq r_c, \end{cases} \quad r = \sqrt{(x-x_c)^2 + (y-y_c)^2},$$

where $\theta_c = 0.5 \text{ K}$, $(x_c,y_c) = (500 \text{ m},350 \text{ m})$, and $r_c = 250 \text{ m}$. The pressure and density are computed by $\Theta$ and $\Pi$ via the following formulas:

$$p = p_0 \Pi^{\frac{r}{\Theta}}, \quad \rho = \frac{p_0}{R\Theta} \Pi^{\frac{1}{T-1}},$$

with the reference pressure $p_0 = 10^5 \text{ N/m}^2$. Figure 8 shows the evolution of the potential temperature perturbation $\Delta \Theta$ obtained by the proposed fourth-order accurate WB CDG method on the meshes of $100 \times 100$ cells ($10 \text{ m}$ resolution). We observe clearly that the initial circular bubble is deformed to a mushroom-like cloud and the flow structures are well resolved.

5.2.7 Example 11: Rayleigh–Taylor (RT) instability tests

This example simulates three tests, which involve discontinuous stationary hydrostatic solutions.

For the first two tests, we use the same setups as in [6], with the gravitational potential function $\phi(x,y) = y$ and the computational domain $\Omega = [-0.25,0.25] \times [-1,1]$. The initial solution is stationary hydrostatic with the pressure and density given by

$$p(x,y,0) = \begin{cases} p_0 \exp(-y/T_1), & y < 0, \\ p_0 \exp(-y/T_a), & y > 0, \end{cases} \quad \rho(x,y,0) = \begin{cases} p/T_1, & y < 0, \\ p/T_a, & y > 0, \end{cases}$$

where $p_0 = 1$, $\{T_1,T_a\}$ are two different constant temperatures. Note that the density $\rho(x,y,0)$ is discontinuous because of the jump in temperature at $y = 0$, while the pressure is continuous at $y = 0$. We consider two configurations [6]:
Figure 8: Example 10: Contour plots of the potential temperature perturbation $\Delta \Theta$ at $t = 400$ s, 500 s, 600 s, and 700 s, respectively, obtained by our fourth-order WB CDG method. 10 uniformly spaced contour lines are displayed.
• **RT test 1**: \( T_l = 1 \) and \( T_u = 2 \). This is a stable case, because the light fluid is above the heavy fluid.

• **RT test 2**: \( T_l = 2 \) and \( T_u = 1 \). This is a physically unstable case, because heavy fluid is above the light fluid.

The numerical solutions for both tests are computed until \( t = 0.1 \) by using the third-order WB CDG scheme with respectively \( 25 \times 100 \) and \( 50 \times 200 \) uniform cells. In order to demonstrate our WB implementation of the WENO limiter (see Remark 3.4), we perform the tests with and without the WENO limiter, respectively. Tables 8 and 9 list the \( L^1 \) errors between the numerical solutions and the projection of the initial solution (52). We clearly see that all the numerical errors are at the level of rounding error, confirming that the proposed CDG method and our implementation of the WENO limiter exactly preserve the WB property. As the solution remains at the stationary hydrostatic state, we observe that no cell is flagged as “troubled cells” up to \( t = 0.1 \). Following [6], we also continue the simulation for a very long time. As shown in Figure 9, the steady state solution in the stable case (RT test 1) is still exactly preserved in a long-time simulation until \( t = 10 \), thanks to the WB property. However, because the configuration of RT test 2 is the physically unstable, in the long-time simulation the small rounding errors will accumulate, as time evolves, and eventually cause the RT instability near the interface \( y = 0 \) (the WB property is locally preserved away from the interface); see Figure 9.

| Limiter    | mesh     | \( \text{errors in } \rho \) | \( \text{errors in } \rho u_1 \) | \( \text{errors in } \rho u_2 \) | \( \text{errors in } E \) |
|------------|----------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| No limiter | \( 25 \times 100 \) | 1.43e-15                    | 3.41e-16                    | 5.22e-16                    | 5.54e-15                    |
|            | \( 50 \times 200 \) | 2.93e-15                    | 6.39e-16                    | 9.07e-16                    | 1.17e-14                    |
| WENO limiter | \( 25 \times 100 \) | 1.43e-15                    | 3.41e-16                    | 5.22e-16                    | 5.54e-15                    |
|            | \( 50 \times 200 \) | 2.93e-15                    | 6.39e-16                    | 9.07e-16                    | 1.17e-14                    |

| Limiter    | mesh     | \( \text{errors in } \rho \) | \( \text{errors in } \rho u_1 \) | \( \text{errors in } \rho u_2 \) | \( \text{errors in } E \) |
|------------|----------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| No limiter | \( 25 \times 100 \) | 9.48e-16                    | 2.92e-16                    | 3.20e-16                    | 5.01e-15                    |
|            | \( 50 \times 200 \) | 2.02e-15                    | 5.34e-16                    | 6.05e-16                    | 1.01e-14                    |
| WENO limiter | \( 25 \times 100 \) | 9.48e-16                    | 2.92e-16                    | 3.20e-16                    | 5.01e-15                    |
|            | \( 50 \times 200 \) | 2.02e-15                    | 5.34e-16                    | 6.05e-16                    | 1.01e-14                    |

To check the effectiveness of our WB CDG method in capturing small perturbations near discontinuous equilibrium solutions, we simulate another classical benchmark RT test [39], which is abbreviated as “**RT test 3**” for convenience. This test is usually used to validate the ability of a high-order numerical scheme in capturing complicated small wave structures. For comparison purpose, we
Figure 9: Long-time simulations of RT test 1 (top row; the stable configuration) and RT test 2 (bottom row; the unstable configuration) by using the third-order WB CDG scheme with $50 \times 200$ uniform cells. The WB implementation of the WENO limiter is used with the parameter $M = 200$ in the TVB corrected minmod function (29).
use the same setup as in [39]. The gravitational potential function is taken as \( \phi(x, y) = -y \), so that the acceleration is in the positive \( y \)-direction. The initial condition is a small perturbation to an unstable stationary hydrostatic solution involving a discontinuity in density:

\[
(\rho, u_1, u_2, p)(x, y, 0) = \begin{cases} 
(2, 0, \tilde{u}(x), 2y + 1)^\top, & 0 \leq y < 0.5, \\
(1, 0, \tilde{u}(x), y + 1.5)^\top, & 0.5 \leq y \leq 1, 
\end{cases}
\]

where \( \tilde{u}(x) = -0.025\sqrt{\gamma p/\rho \cos(8\pi x)} \). The variables \((\rho, u_1, u_2, p)\) are set as \((1, 0, 0, 2.5)\) on the top boundary and as \((2, 0, 0, 1)\) on the bottom. Reflective boundary conditions are imposed on both the left and right boundaries. The WB implementation of the WENO limiter is used with the parameter \( M = 200 \) in the TVB corrected minmod function (29). As in [39], the mesh refinement study is carried out here by using three different uniform square meshes: \( h = \frac{1}{240}, \frac{1}{480}, \frac{1}{960} \), where \( h \) is the spatial step-size in both the \( x \)- and \( y \)-directions. Figure 10 displays the density contours at time \( t = 1.95 \). We see that our third-order WB CDG method can clearly resolve the complicated wave structures and that the numerical resolutions are comparable to those computed with WENO9 (a ninth-order WENO scheme) in [39] with the same mesh sizes.

Figure 10: RT test 3 (the perturbation configuration): Contour plots of the density at \( t = 1.95 \) obtained by the third-order WB CDG scheme with 15 equally spaced contour lines from 0.952269 to 2.14589.
6 Conclusions

This paper designed a high-order positivity-preserving well-balanced (WB) central discontinuous Galerkin (CDG) method for the Euler equations under gravitational fields. A novel WB spatial discretization in the CDG framework was devised with suitable modifications to the numerical dissipation term and the source term approximation, while the desired conservative and positivity-preserving properties were also simultaneously preserved in the discretization. The modifications were based on a novel projection for the stationary hydrostatic solution, which had the same order of accuracy as the standard $L^2$-projection, could be explicitly calculated, and was easy to implement without solving any optimization problems. Moreover, the novel projection guaranteed the projected stationary solution having the same cell averages on both the primal and dual meshes. This feature was a key to obtain the desired properties of our schemes. Based on some convex decomposition techniques and several key properties of the admissible states, we rigorously proved that the resulting WB CDG method satisfied a weak positivity-preserving property, which implied that a simple limiter could ensure the positivity-preserving property without losing the high-order accuracy and conservativeness. Extensive one- and two-dimensional numerical examples were provided to demonstrate the robustness, high-order accuracy, WB and positivity-preserving properties of the proposed schemes.

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