Heavy-Traffic Behavior of the MaxWeight Algorithm in a Switch with Uniform Traffic

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Abstract

We consider a switch with uniform traffic operating under the MaxWeight scheduling algorithm. This traffic pattern is interesting to study in the heavy-traffic regime since the queue lengths exhibit a multi-dimensional state-space collapse. We use a Lyapunov-type drift technique to characterize the heavy-traffic behavior of the expectation of the sum queue lengths in steady-state. Specifically, in the case of Bernoulli arrivals, we show that the heavy-traffic scaled queue length is $n^{-3/2} + 1/2n$. Our result implies that the MaxWeight algorithm has optimal queue-length scaling behavior in the heavy-traffic regime with respect to the size of a switch with a uniform traffic pattern. This settles the heavy-traffic version of an open conjecture.

1 Introduction

Consider a collection of queues arranged in the form of an $n \times n$ matrix. The queues are assumed to operate in discrete-time and jobs arriving to the queues will be called packets. The following constraints are imposed on the service process of the queueing system: (a) at most one queue can be served in each time slot in each row of the matrix, (b) at most one queue can be served in each time slot in each column of the matrix, and (c) when a queue is served, at most one packet can be removed from the queue. Such a queueing system is called a switch.

A scheduling algorithm for the switch is a rule which selects the queues to be served in each time slot. A well-known algorithm called the MaxWeight algorithm is known to optimize the throughput in a switch. The algorithm was derived in a more general context in [1] and for the special context of the switch considered in here in [2], where it was also shown that other seemingly good policies are not throughput-optimal. An important open question that is not fully understood is whether the MaxWeight algorithm is also queue length or delay optimal in any sense. In [3], it was shown that the MaxWeight algorithm minimizes the sum of the squares of the queue lengths in heavy-traffic under a condition called Complete Resource Pooling (CRP). For the switch, the CRP condition means that the arriving traffic saturates at most one column or one row of the switch. The result relies on the fact that, under CRP and in the heavy-traffic regime, there is a one-dimensional state-space collapse, i.e., the state of the system collapses to a line. When the CRP condition is not met, the state-space collapses to a lower-dimension, but is not one-dimensional. State-space collapse without the CRP condition has been studied in [4], and a diffusion limit has been established in [5]. However, a characterization of the steady-state behavior of the diffusion limit was still open.
In this paper, we use the Lyapunov-type drift technique introduced in [6]. The basic idea is to set the drift of an appropriately chosen function equal to zero in steady-state to obtain both upper and lower bounds on quantities of interest, such as the moments of the queue lengths. To obtain upper bounds one has to establish state-space collapse in a sense that is somewhat different than the one in [3]: the main difference being that the state-space collapse is expressed in terms of the moments of the queue length is steady-state. This form of state-space collapse can then be readily used in the drift condition to obtain the upper bound. However, in [9], the usefulness of the drift technique was only established under the CRP condition. In this paper, we consider the switch with uniform traffic, i.e., where the arrival rates to all queues are equal. Thus, in the heavy-traffic regime, when the traffic in one column (or row) approaches its capacity, the traffic in all rows and columns approach capacity, and the CRP condition is violated. The main contribution of the paper is to characterize the expected steady-state queue lengths in heavy-traffic even though the CRP condition is violated. As mentioned earlier, when the CRP condition is violated, the state does not typically collapse to a single dimension. The main challenge in our proof is due to the difficulty in characterizing the behavior of the queue length process under such a multi-dimensional state-space collapse. Characterizing the behavior of the queue lengths under multi-dimensional state-space collapse has been difficult, in general, except in rare cases; see [7, 8] for two such examples in other contexts.

The difficulty in understanding the steady-state queue length behavior of the MaxWeight algorithm has meant that it is unknown whether the the MaxWeight algorithm minimizes the expected total queue length in steady-state. One way to pose the optimality question is to increase the number of queues in the system, or increase the arrival to a point close to the boundary of the capacity region (the heavy-traffic regime), or do both, and study whether the MaxWeight algorithm is queue-length-optimal in a scaling sense. A conjecture regarding the scaling behavior for any algorithm, both in heavy-traffic and under all traffic conditions, has been stated in [9]. The authors first heard about the non-heavy-traffic version of this conjecture from A. L. Stolyar in 2005. The conjecture seemed to be difficult to verify for the MaxWeight algorithm, and so a number of other algorithms have been developed to achieve either optimal or near-optimal scaling behavior; see [10, 11, 12]. The results in this paper establish the validity of one version of the conjecture (pertaining to uniform traffic in the heavy-traffic regime) for the MaxWeight algorithm.

Note on Notation: The set of real numbers, and the set of non-negative real numbers are denoted by \( \mathbb{R} \) and \( \mathbb{R}_+ \), respectively. We work in the \( n \times n \) dimensional Euclidean space \( \mathbb{R}^{n^2} \). We represent vectors in this space in bold font, by \( \mathbf{x} \). We use two indices \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \) for different components of \( \mathbf{x} \). We represent the \((i, j)\)th component by \( x_{ij} \) and thus, \( \mathbf{x} = (x_{ij})_{ij} \). For two vectors \( \mathbf{x} \) and \( \mathbf{y} \) in \( \mathbb{R}^{n^2} \), their inner product \( \langle \mathbf{x}, \mathbf{y} \rangle \) and Euclidean norm \( \| \mathbf{x} \| \) are defined by

\[
\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}, \quad \| \mathbf{x} \| \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2}.
\]

For two vectors \( \mathbf{x} \) and \( \mathbf{y} \) in \( \mathbb{R}^{n^2} \), \( \mathbf{x} \leq \mathbf{y} \) means \( x_{ij} \leq y_{ij} \) for every \((i, j)\). We use \( \mathbf{1} \) to denote the all ones vector. Let \( \mathbf{e}^{(i)} \) denote the vector defined by \( e_{ij}^{(i)} = 1 \) for all \( j \) and \( e_{ij}^{(i)} = 0 \) for all \( j' \neq i \) and for all \( j \). Thus, \( \mathbf{e}^{(i)} \) is a matrix with \( i \)th row being all ones and zeros everywhere else. Similarly, let \( \mathbf{e}^{(j)} \) denote the vector defined by \( e_{ij}^{(j)} = 1 \) for all \( i \) and \( e_{ij}^{(j)} = 0 \) for all \( j' \neq j \) and for all \( i \), i.e., it is a matrix with \( j \)th column being all ones and zeros everywhere else. For a random process \( \mathbf{q}(t) \) and a Lyapunov function \( V(\cdot) \), we will sometimes use \( V(t) \) to denote \( V(\mathbf{q}(t)) \).
2 Preliminaries

In this section, we will present the model of an input queued switch, MaxWeight scheduling algorithm, some observations on the geometry of the capacity region and other preliminaries.

2.1 System Model and MaxWeight Algorithm

An input queued switch is a model for cross-bar switches that are widely used. An \( n \times n \) switch has \( n \) input ports and \( n \) output ports. We consider a discrete time system. In each time slot \( t \), packets arrive at any of the input ports to be delivered to any of the output ports. When scheduled, each packet needs one time slot to be transmitted across.

Each input port maintains \( n \) separate queues, one each for packets to be delivered to each of the \( n \) output ports. We denote the queue length of packets at input port \( i \) to be delivered at output port \( j \) at time \( t \) by \( q_{ij}(t) \). Let \( \mathbf{q} \in \mathbb{R}^{n^2} \) denote the vector of all queue lengths. Let \( a_{ij}(t) \) denote the number of packet arrivals at input port \( i \) at time \( t \) to be delivered to output port \( j \), and we let \( \mathbf{a} \in \mathbb{R}^{n^2} \) denote the vector \((a_{ij})_{ij}\). For every input-output pair \((i,j)\), we assume that \( a_{ij}(t) \) is a Bernoulli process with rate \( \lambda_{ij} \) and is independent of the arrival processes at other input-output pairs. The arrival rate vector is denoted by \( \mathbf{\lambda} = (\lambda_{ij})_{ij} \).

In each time slot, each input port can be matched to only one output port and similarly, each output port can be matched to only one input port. These constraints can be captured in a graph. Let \( G \) denote a complete \( n \times n \) bipartite graph with \( n^2 \) edges between the set of input ports and the set of output ports. The schedule in each time slot is a matching on this graph \( G \). Let \( s_{ij} = 1 \) if the link between input port \( i \) and output port \( j \) is matched or scheduled and \( s_{ij} = 0 \) otherwise.

Then, the set of feasible schedules is \( \mathcal{S} \) where \( \mathbf{s} \) is the \( \mathbb{R}^{n^2} \) vector \((s_{ij})_{ij}\) and

\[
\mathcal{S} = \left\{ \mathbf{s} \in \{0,1\}^{n^2} : \sum_{i=1}^{n} s_{ij} \leq 1, \sum_{j=1}^{n} s_{ij} \leq 1 \forall i,j \in \{1,2,\ldots,n\} \right\}.
\]

Let \( \mathcal{S}^* \) denote the set of maximal feasible schedules. Then, it is easy to see that

\[
\mathcal{S}^* = \left\{ \mathbf{s} \in \{0,1\}^{n^2} : \sum_{i=1}^{n} s_{ij} = 1, \sum_{j=1}^{n} s_{ij} = 1 \forall i,j \in \{1,2,\ldots,n\} \right\}.
\]

Each element in this set corresponds to a perfect matching on the graph \( G \). Each of these maximal feasible schedules is also a permutation \( \pi \) on the set \( 1,2,\ldots,n \) with \( \pi(i) = j \) if \( s_{ij} = 1 \).

A scheduling policy or algorithm picks a schedule \( \mathbf{s}(t) \) in every time slot based on the current queue length vector, \( \mathbf{q}(t) \). In each time slot, the order of events is as follows. Queue lengths at the beginning of time slot \( t \) are \( \mathbf{q}(t) \). A schedule \( \mathbf{s}(t) \) is then picked for that time slot based on the queue lengths. Then, arrivals for that time \( \mathbf{a}(t) \) happen. Finally the packets are served and there is unused service if there are no packets in a scheduled queue. The queue lengths are then updated to give the queue lengths for the next time slot. The queue lengths therefore evolve as follows.

\[
q_{ij}(t+1) = [q_{ij}(t) + a_{ij}(t) - s_{ij}(t)]^+
= q_{ij}(t) + a_{ij}(t) - s_{ij}(t) + u_{ij}(t)
\]

\[
\mathbf{q}(t+1) = \mathbf{q}(t) + \mathbf{a}(t) - \mathbf{s}(t) + \mathbf{u}(t)
\]

where \( [x]^+ = \max(0,x) \) is the projection onto positive real axis, \( u_{ij}(t) \) is the unused service on link \((i,j)\). Unused service is 1 only when link \((i,j)\) is scheduled, but has zero queue length; and it is 0.
in all other cases. Thus, we have that when \( u_{ij}(t) = 1 \), we have \( q_{ij}(t) = 0 \), \( a_{ij}(t) = 0 \), \( s_{ij}(t) = 1 \) and \( q_{ij}(t + 1) = 0 \). Therefore, we have \( u_{ij}(t)q_{ij}(t) = 0 \), \( u_{ij}(t)a_{ij}(t) = 0 \) and \( u_{ij}(t)q_{ij}(t + 1) = 0 \). Also note that since \( u_{ij}(t) = s_{ij}(t) \), we have that \( \sum_{i=1}^{n} u_{ij} \in \{0, 1\} \) and \( \sum_{j=1}^{n} u_{ij} \in \{0, 1\} \) for all \( i, j \).

The queue lengths process \( q(t) \) is a Markov chain. The switch is said to be stable under a scheduling policy if the sum of all the queue lengths is finite, i.e.,

\[
\lim sup \lim sup_{t \to \infty} \mathbb{P} \left( \sum_{ij} q_{ij}(t) \geq C \right) = 0.
\]

If the queue lengths process \( q(t) \) is positive recurrent under a scheduling policy, then we have stability. The capacity region of the switch is the set of arrival rates \( \lambda \) for which the switch is stable under some scheduling policy. A policy that stabilizes the switch under any arrival rate in the capacity region is said to be throughput optimal. The MaxWeight Algorithm is a popular scheduling policy if the sum of all the queue lengths is finite, i.e.,

\[
\lim sup \lim sup_{t \to \infty} \mathbb{P} \left( \sum_{ij} q_{ij}(t) \geq C \right) = 0.
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If the queue lengths process \( q(t) \) is positive recurrent under a scheduling policy, then we have stability. The capacity region of the switch is the set of arrival rates \( \lambda \) for which the switch is stable under some scheduling policy. A policy that stabilizes the switch under any arrival rate in the capacity region is said to be throughput optimal. The MaxWeight Algorithm is a popular scheduling algorithm for the switches. In every time slot \( t \), each link \((i, j)\) is given a weight equal to its queue length \( q_{ij}(t) \) and the schedule with the maximum weight among the feasible schedules \( S \) is chosen at that time slot. This algorithm is presented in Algorithm 1. It is easy to see that the Markov chain \( q(t) \) is irreducible and aperiodic under the MaxWeight algorithm. It is well known [1][2] that the capacity region \( C \) of the switch is convex hull of all feasible schedules,

\[
C = \text{Conv}(S) = \left\{ \lambda \in \mathbb{R}^{n^2}_+ : \sum_{i=1}^{n} \lambda_{ij} \leq 1, \sum_{j=1}^{n} \lambda_{ij} \leq 1 \forall i, j \in \{1, 2, \ldots, n\} \right\}
\]

\[
= \left\{ \lambda \in \mathbb{R}^{n^2}_+ : \langle \lambda, e^{(i)} \rangle \leq 1, \langle \lambda, e^{(j)} \rangle \leq 1 \forall i, j \in \{1, 2, \ldots, n\} \right\}.
\]

It is also known that the queue lengths process is positive recurrent under the MaxWeight algorithm whenever the arrival rate is in the capacity region \( C \) and therefore is throughput optimal.

**Algorithm 1** MaxWeight Scheduling Algorithm for an input-queued switch

Consider the complete bipartite graph between the input ports and output ports. Let the queue length \( q_{ij}(t) \) be the weight of the edge between input port \( i \) and output port \( j \). A maximum weight matching in this graph is chosen as the schedule in every time slot, i.e.,

\[
s(t) = \arg \max_{s \in S} \sum_{ij} q_{ij}(t)s_{ij} = \arg \max_{s \in S} \langle q(t), s \rangle
\]

Ties are broken uniformly at random.

Note that there is always a maximum weight schedule that is maximal. If the MaxWeight schedule chosen at time \( t \), \( s \) is not maximal, there exists a maximal schedule \( s^* \in S^* \) such that \( s \leq s^* \). For any link \((i, j)\) such that \( s_{ij} = 0 \) and \( s^*_{ij} = 1 \), \( q_{ij}(t) = 0 \). If not, \( s \) would not have been a maximum weight schedule. Therefore, we can pretend that the actual schedule chosen is \( s^* \) and the links \((i, j)\) that are in \( s \) and \( s^* \) have an unused service of 1. Note that this does not change the scheduling algorithm, but it is just a convenience in the proof. Therefore, without loss of generality, we assume that the schedule chosen in each time slot is a maximal schedule, i.e.,

\[
s(t) \in S^* \text{ for all time } t.
\]

Hence the MaxWeight schedule picks one of the \( n! \) possible permutations from the set \( S^* \) in each time slot.
For any arrival rate in the capacity region $C$, due to positive recurrence of $q(t)$, we have that a steady state distribution exists under MaxWeight policy. Let $\overline{q}$ denote the steady state random vector. In this paper, we focus on the average queue length under the steady state distribution, i.e., $E[\sum_{i,j} \overline{q}_{ij}]$. We consider a sequence of systems indexed by $\epsilon$ with arrival rate $\lambda^\epsilon = \frac{1}{n^\epsilon} \mathbf{1}$. We will study the switch when $\epsilon \downarrow 0$. This is called the heavy traffic limit. We first show an $O(n)$ universal lower bound on the average queue length in heavy traffic limit, i.e., on $\lim_{\epsilon \to 0} E[\sum_{i,j} \overline{q}_{ij}]$. We then show that under MaxWeight policy, the limiting average queue length is $O(n)$ and thus MaxWeight has optimal average queue length scaling. We will show these bounds using Lyapunov drift conditions. We will use several different quadratic Lyapunov functions throughout the paper.

2.2 Geometry of the Capacity Region

The capacity region $C$ is a coordinate convex polytope in $\mathbb{R}^n$. Here, we review some basic definitions. For any set $P \subseteq \mathbb{R}^m$, its dimension is defined by

$$\dim(P) \triangleq \min\{\dim(A) | P \subseteq A, A \text{ is an affine space}\}$$

So the capacity region $C$ has dimension $n^2$. A hyperplane $H$ is said to be a supporting hyperplane of $P$ if $P \cap H \neq \emptyset$, $P \cap H_+ \neq \emptyset$ and $P \cap H_- = \emptyset$ where $H_+$ and $H_-$ are the open half-spaces determined by the hyperplane $H$. For any supporting hyperplane $H$ of $P$, $P \cap H$ is called a face [13]. A face of a polytope is a polytope with lower dimension. A face $F$ with dimension $\dim(F) = \dim(P) - 1$ is called a facet. Heavy traffic optimality of MaxWeight algorithm for generalized switches is shown in [3, 6] when approaching an arrival rate vector on a facet of the capacity region. However, the rate vector $\frac{1}{n} \mathbf{1}$ does not lie on a facet and so, that result is not applicable here.

Let $F$ denote the face of the capacity region $C$ containing the rate vector $\frac{1}{n} \mathbf{1}$. Then,

$$F = \left\{ \lambda \in \mathbb{R}^{n^2}_+ : \sum_{i=1}^n \lambda_{ij} = 1, \sum_{j=1}^n \lambda_{ij} = 1 \forall i, j \in \{1, 2, \ldots, n\} \right\}$$

$$= \left\{ \lambda \in \mathbb{R}^{n^2}_+ : \langle \lambda, e^{(i)} \rangle = 1, \langle \lambda, e^{(j)} \rangle = 1 \forall i, j \in \{1, 2, \ldots, n\} \right\}.$$

This is easy to see by observing that the hyperplane $\langle \lambda, 1 \rangle = n$ containing the rate vector $\frac{1}{n} \mathbf{1}$ is a supporting hyperplane of the capacity region $C$. The face $F$ has dimension $(n-1)^2 = n^2 - (2n-1)$, and lies in the affine space formed by the intersection of the $2n - 1$ constraints $\sum_{i=1}^n \lambda_{ij} = 1$ for all $i$, and $\lambda_{ij} = 1$ for all $j$. Of these $2n$ constraints, one is linearly dependent of the others and so we have $2n - 1$ linearly independent constraints. The face $F$ is actually the convex combination of the maximal feasible schedules $S^*$, i.e., $F = \text{Conv}(S^*)$. These results follow from the fact that the face $F$ is the Birkhoff polytope $B_n$ that contains all the $n \times n$ doubly stochastic matrices. It is known [14, page 20] that $B_n$ lies in the $(n-1)^2$ dimensional affine space of the constraints and is a convex hull of the permutation matrices.

A facet of a polytope has a unique supporting hyperplane defining the facet. It was shown in [6] that when the arrival rate vector approaches a rate vector in the relative interior of a facet, in the limit, the queue length vector concentrates along the direction of the normal vector of the unique supporting hyperplane. However, a lower dimensional face can be defined by one of several hyperplanes, and so there is no unique normal vector. A lower dimensional face is always an intersection of two or more facets. We are interested in the case when the arrival rate vector approaches the vector $\frac{1}{n} \mathbf{1}$ that lies on the face $F$. The face $F$ is the intersection of the $2n$ facets, $\{\langle e^{(i)}, \lambda \rangle = 1\} \cap C$ for all $i$, and $\{\langle e^{(j)}, \lambda \rangle = 1\} \cap C$ for all $j$. We will show in section 4 that in
the heavy traffic limit, the queue length vector concentrates within the cone spanned by the 2\(n\) normal vectors, \(\{e^{(i)}\text{ for all } i\} \cup \{\tilde{e}^{(j)}\text{ for all } j\}\). We will call this cone \(K\). Here, we will present some definitions and other results related to this cone. More formally, the cone \(K\) can be defined as follows.

\[
K = \left\{ x \in \mathbb{R}^{n^2} : x = \sum_i w_i e^{(i)} + \sum_j \tilde{w}_j \tilde{e}^{(j)} \text{ where } w_i \in \mathbb{R}_+ \text{ and } \tilde{w}_j \in \mathbb{R}_+ \text{ for all } i, j \right\}
\]

Note that this means that for any \(x \in K\) there are \(w_i \in \mathbb{R}_+\) and \(\tilde{w}_j \in \mathbb{R}_+\) for all \(i, j \in \{1, 2, \ldots, n\}\) such that \(x_{ij} = w_i + \tilde{w}_j\). However, such a representation need not be unique. For example, suppose that \(w_i \geq 1\) for all \(i\), then setting \(w'_i = w_i - 1\) for each \(i\) and \(\tilde{w}'_j = \tilde{w}_j + 1\) for each \(j\), we again have that \(w'_i \in \mathbb{R}_+, \tilde{w}'_j \in \mathbb{R}_+\) and \(x_{ij} = w'_i + \tilde{w}'_j\) for all \(i, j\).

The cone \(K\) lies in the \(2n - 1\) dimensional subspace spanned by the \(2n - 1\) independent vectors among the \(2n\) vectors, \(\{e^{(i)}\text{ for all } i\} \cup \{\tilde{e}^{(j)}\text{ for all } j\}\). Call this space \(V_K\). For any two vectors \(x, y \in F\), \(x - y\) is orthogonal to the subspace \(V_K\), i.e.,

\[
x - y \perp V_K.
\]

This is easy to see since \(\langle x, e^{(i)} \rangle = \langle y, e^{(i)} \rangle = 1\) for all \(i\) and \(\langle x, \tilde{e}^{(j)} \rangle = \langle y, \tilde{e}^{(j)} \rangle = 1\) for all \(j\). If \(\mathcal{V}_F\) denotes the subspace obtained by translating the affine space spanned by \(F\), it follows that the spaces \(V_K\) and \(\mathcal{V}_F\) are orthogonal because translation means subtraction by a vector. Moreover, they span the entire space \(\mathbb{R}^{n^2}\) since their dimensions sum to \(n^2\).

### 2.2.1 Projection onto the cone \(K\)

The cone \(K\) is closed and convex. For any \(x \in \mathbb{R}^{n^2}\), the closest point in the cone \(K\) to \(x\) is called the projection of \(x\) on to the cone \(K\) and we will denote it by \(x_\parallel\). More formally,

\[
x_\parallel = \arg \min_{y \in K} \|x - y\|
\]

For a closed convex cone \(K\), the projection \(x_\parallel\) is well defined and is unique. [13] Appendix E.9.2. We will use \(x_\perp\) to denote \(x - x_\parallel\). We will use \(x_{\parallel ij}\) to denote the \((i, j)\)th component of \(x_\parallel\). Similarly, \(x_{\perp ij}\).

Note that unlike projection on to a subspace, projection on to a cone is not linear, i.e., \((x + y)_\parallel \neq x_\parallel + y_\parallel\). A simple counter example is the following. In \(\mathbb{R}^2\), let \(x = (2, 2)\) and \(y = (-1, -1)\). Consider the positive quadrant as the cone of interest. Then, \(x_\parallel = (2, 2), y_\parallel = (0,0)\) and \((x + y)_\parallel = (1,1)\).

Since for any \(x \in \mathbb{R}^{n^2}, x_\parallel \in K\), from the definition of the cone \(K\), we have that every component of \(x_\parallel\) is non negative, i.e., \(x_{\parallel ij} \geq 0\). However, \(x_\perp\) could have negative components.

The polar cone \(K^\circ\) of cone \(K\) is defined as

\[
K^\circ = \left\{ x \in \mathbb{R}^{n^2} : \langle x, y \rangle \leq 0 \text{ for all } y \in K \right\}.
\]

The polar cone \(K^\circ\) is negative of the dual cone \(K^{\ast}\) of the cone \(K\). For any \(x \in \mathbb{R}^{n^2}, x_\perp \in K^\circ\) and \(\langle x_\parallel, x_\perp \rangle = 0\) [15] Appendix E.9.2. Therefore, pythagoras theorem is applicable, i.e.,

\[
\|x\|^2 = \|x_\parallel\|^2 + \|x_\perp\|^2
\]

and so, \(\|x_\parallel\| \leq \|x\|\) and \(\|x_\perp\| \leq \|x\|\).
Projection onto any closed convex set in \( \mathbb{R}^{n^2} \) (and so onto a closed convex cone) is nonexpansive \([15, \text{Appendix E.9.3}]\). Therefore, we have \( \|x\| - \|y\| \leq \|x - y\| \). Since \( x_\perp \) is a projection onto \( K^\circ \), we also have
\[
\|x_\perp - y_\perp\| \leq \|x - y\|.
\] (5)

2.3 Moment bounds from Lyapunov drift conditions

In this paper, we will use the Lyapunov drift based approach presented in \([6]\) to obtain bounds of average queue length under MaxWeight. A key ingredient in this approach is to obtain moment bounds from drift conditions. The following lemma \([16]\) is a key tool in this approach. We state it here as it was stated in \([6]\).

**Lemma 1.** For an irreducible and aperiodic Markov chain \( \{X(t)\}_{t \geq 0} \) over a countable state space \( \mathcal{X} \), suppose \( Z : \mathcal{X} \rightarrow \mathbb{R}_+ \) is a nonnegative-valued Lyapunov function. We define the drift of \( Z \) at \( X \) as
\[
\Delta Z(X) \triangleq [Z(X(t+1)) - Z(X(t))] I(X(t) = X),
\]
where \( I(\cdot) \) is the indicator function. Thus, \( \Delta Z(X) \) is a random variable that measures the amount of change in the value of \( Z \) in one step, starting from state \( X \). This drift is assumed to satisfy the following conditions:

\begin{align*}
\textbf{C1} & \text{ There exists an } \eta > 0, \text{ and a } \kappa < \infty \text{ such that for all } X \in \mathcal{X} \text{ with } Z(X) \geq \kappa, \quad E[\Delta Z(X)|X(t) = X] \leq -\eta. \\
\textbf{C2} & \text{ There exists a } D < \infty \text{ such that for all } X \in \mathcal{X}, \quad P(|\Delta Z(X)| \leq D) = 1.
\end{align*}

Then, there exists a \( \theta^* > 0 \) and a \( C^* < \infty \) such that
\[
\limsup_{t \to \infty} E\left[e^{\theta^*Z(X(t))}\right] \leq C^*.
\]

If we further assume that the Markov chain \( \{X(t)\}_t \) is positive recurrent, then \( Z(X(t)) \) converges in distribution to a random variable \( Z \) for which
\[
E\left[e^{\theta Z}\right] \leq C^*,
\]
which directly implies that all moments of \( Z \) exist and are finite.

3 Universal Lower Bound

In this section, we will prove the following lower bound on the average queue lengths, which is valid under any scheduling policy.

**Proposition 1.** Consider a set of switch systems with the Bernoulli arrival processes \( a^{(\epsilon)}(t) \) parameterized by \( 0 < \epsilon < 1 \), such that the mean arrival rate vector is \( \lambda^\epsilon = \frac{1-\epsilon}{n} \mathbf{1} \). Fix a scheduling policy under which the switch system is stable for any \( \epsilon > 0 \). Let \( q^{(\epsilon)}(t) \) denote the queue lengths process under this policy for each system. Suppose that this process converges in distribution to a
steady state random vector $\mathbf{q}^{(e)}$. Then, for each of these systems, the average queue length is lower bounded by

$$
\mathbb{E} \left[ \sum_{ij} q_{ij}^{(e)} \right] \geq \frac{(1 - \epsilon)^2}{2\epsilon} (n - 1)
$$

Therefore, in the heavy-traffic limit as $\epsilon \downarrow 0$, we have

$$
\liminf_{\epsilon \downarrow 0} \epsilon \mathbb{E} \left[ \sum_{ij} q_{ij}^{(e)} \right] \geq \frac{n - 1}{2}
$$

Proof. We will obtain a lower bound on sum of all the queue lengths by lower bounding the queue lengths at each input port, i.e., we will first bound $\mathbb{E} \left[ \sum_{ij} q_{ij}^{(e)} \right]$ for a fixed input port $i$. We do this by considering a single queue that is coupled to the process $\sum_{ij} q_{ij}^{(e)}(t)$. Consider a single server queue $\phi_i^{(e)}(t)$ in discrete time. Packets arrive into this queue to be served. Each packet needs exactly one time slot of service. The arrival process to this queue is $\alpha_i^{(e)}(t) = \sum_j a_{ij}^{(e)}(t)$. As long as the queue is non empty, one packet is served in every time slot. Thus, this queue evolves as

$$
\phi_i^{(e)}(t + 1) = \left[ \phi_i^{(e)}(t) + \alpha_i^{(e)}(t) - 1 \right]^+
$$

where $v^{(e)}(t)$ is the unused service and so $v^{(e)}(t)\phi_i^{(e)}(t + 1) = 0$. Clearly, $\phi_i^{(e)}(t)$ is positive recurrent and let $\phi_i^{(e)}$ denote the steady state random variable to which it converges in distribution.

Claim 1. In steady state

$$
\mathbb{E} \left[ \sum_j q_{ij}^{(e)} \right] \geq \mathbb{E} \left[ \phi_i^{(e)} \right]
$$

Proof. Suppose that at time zero, the queue $\phi_i^{(e)}$ starts with $\phi_i^{(e)}(0) = \sum_j q_{ij}^{(e)}(0)$. Then, for any time $t$, the queue $\phi_i^{(e)}(t)$ is stochastically no greater than $\sum_j q_{ij}^{(e)}(t)$. This can easily be seen using induction. For $t = 0$, we have $\sum_j q_{ij}(0) \geq \phi_i(0)$. Suppose that $\sum_j q_{ij}^{(e)}(t) \geq \phi_i^{(e)}(t)$. Then, at time $(t + 1)$,

$$
\sum_j q_{ij}^{(e)}(t + 1) = \sum_j \left[ q_{ij}^{(e)}(t) + a_{ij}^{(e)}(t) - s_{ij}^{(e)}(t) \right]^+
$$

where the last inequality follows from the inductive hypothesis, definition of $\alpha^{(e)}(t)$, the constraint $s_{ij}^{(e)}(t) \leq 1$ and the fact that if $x \geq y$, we have that $[x]^+ \geq [y]^+$. Thus, we have that in steady state,
Since steady state distribution of $\sum_j q_{ij}^{(e)}$ and $\varphi_i^{(e)}$ does not depend on the initial state at time zero, we have the lower bound $\mathbb{E}[\sum_j q_{ij}^{(e)}] \geq \mathbb{E}[\varphi_i^{(e)}]$ independent of the initial states $\varphi_i^{(e)}(0)$ and $\sum_j q_{ij}^{(e)}(0)$. □

Consider the drift of $\mathbb{E}[(\varphi_i^{(e)}(t))^2]$.

$$
\mathbb{E}[(\varphi_i^{(e)}(t + 1))^2 - (\varphi_i^{(e)}(t))^2] = \mathbb{E}[(\varphi_i^{(e)}(t) + \alpha_i^{(e)}(t) - 1 + v^{(e)}(t))^2 - (\varphi_i(t)^{(e)})^2] \\
= \mathbb{E}[(\varphi_i^{(e)}(t) + \alpha_i^{(e)}(t) - 1)^2 - (v^{(e)}(t))^2 - (\varphi_i(t)^{(e)})^2] \\
= \mathbb{E}[(\alpha_i^{(e)}(t) - 1)^2 + 2(\varphi_i^{(e)}(t))(\alpha_i^{(e)}(t) - 1) - (v^{(e)}(t))^2] \\
= (1 - \epsilon)^2 \frac{n - 1}{n} + \epsilon - 2\epsilon \mathbb{E}[\varphi_i^{(e)}(t)] - \mathbb{E}[v^{(e)}(t)]
$$

where $(a)$ follows from noting that $(v^{(e)}(t))(\varphi_i^{(e)}(t) + \alpha_i^{(e)}(t) - 1 + v^{(e)}(t))) = 0$. The last equality follows from the definition of $\alpha_i^{(e)}(t)$ and since $v^{(e)}(t) \in \{0, 1\}$. It can easily be shown that $\mathbb{E}[(\varphi_i^{(e)}(t))^2]$ is finite. Therefore, the steady state drift of $\mathbb{E}[(\varphi_i^{(e)}(t))^2]$ is zero, i.e., in steady-state, we get

$$
2\epsilon \mathbb{E}[\varphi_i^{(e)}] = (1 - \epsilon)^2 \frac{n - 1}{n} + \epsilon - \mathbb{E}[v^{(e)}] 	ag{6}
$$

Consider the drift of $\mathbb{E}[\varphi_i^{(e)}(t)]$.

$$
\mathbb{E}[\varphi_i^{(e)}(t + 1) - \varphi_i^{(e)}(t)] = \mathbb{E}[\alpha_i^{(e)}(t) - 1 + v^{(e)}(t)] \\
= -\epsilon + \mathbb{E}[v^{(e)}(t)]
$$

Since $\varphi_i^{(e)}(t) \in \mathbb{Z}_+$, we have $\varphi_i^{(e)}(t) \leq (\varphi_i^{(e)}(t))^2$, and so we get finiteness of $\mathbb{E}[\varphi_i^{(e)}]$ from that of $\mathbb{E}[(\varphi_i^{(e)}(t))^2]$. Therefore, the drift of $\mathbb{E}[\varphi_i^{(e)}(t)]$ is zero in steady state. Thus, we get that in steady state, $\mathbb{E}[v^{(e)}] = \epsilon$. Substituting this in (6), and using the claim, we get

$$
\mathbb{E}[\sum_j q_{ij}^{(e)}] \geq \mathbb{E}[\varphi_i^{(e)}] = \frac{(1 - \epsilon)^2 n - 1}{2\epsilon}
$$

Since this lower bound is true for any input port $i$, summing over all the input ports, we have the proposition. Note that we could have obtained the same bound by similarly lower bounding the sum of lengths of all the queues destined to port $j$, i.e., $\sum_i q_{ij}^{(e)}(t)$. □

We do not know if this lower bound is tight, i.e., if there is a scheduling policy that attains this lower bound. However, in section 5 we show that under MaxWeight scheduling algorithm, the average queue lengths are $O(n)$, thus showing that MaxWeight has optimal scaling. Queue lengths under MaxWeight are less than factor 2 away from this universal lower bound. Closing this gap is an open question.

4 State Space Collapse under MaxWeight policy

In this section, we will show that under the MaxWeight scheduling algorithm, in the heavy traffic limit, the steady state queue length vector concentrates within the cone $K$ in the following sense.
As the parameter $\epsilon$ approaches zero, the mean arrival rate approaches the boundary of the capacity region and we know from the lower bound that the average queue lengths go to infinity $O(1/\epsilon)$. We will show that under the MaxWeight algorithm, the $\mathbf{q}_1^{(\epsilon)}$ component of the queue length vector is upper bounded independent of $\epsilon$. Thus the $\mathbf{q}_1^{(\epsilon)}$ component is negligible compared to the $\mathbf{q}_\parallel^{(\epsilon)}$ component of $\mathbf{q}^{(\epsilon)}$. This is called state space collapse. We say that the state space collapses to the cone $\mathcal{K}$. It was shown in [18] that the state space collapses to the subspace containing the cone $\mathcal{K}$. A similar result was also shown in [19]. Here, we show the stronger result that the state space collapses to the cone, which is essential to obtain upper bounds with optimal scaling.

We define the following Lyapunov functions and their corresponding drifts.

\[
V(q) \triangleq \|q\|^2 = \sum_{ij} q_{ij}^2 \quad W_{\perp}(q) \triangleq \|q_{\perp}\| \quad V_{\perp}(q) \triangleq \|q_{\perp}\|^2 = \sum_{ij} q_{ij}^2 \quad V_{\parallel}(q) \triangleq \|q_{\parallel}\|^2 = \sum_{ij} q_{ij}^2
\]

\[
\Delta V(q) \triangleq [V(q(t+1)) - V(q(t))] I(q(t) = q)
\]

\[
\Delta W_{\perp}(q) \triangleq [W_{\perp}(q(t+1)) - W_{\perp}(q(t))] I(q(t) = q)
\]

\[
\Delta V_{\perp}(q) \triangleq [V_{\perp}(q(t+1)) - V_{\perp}(q(t))] I(q(t) = q)
\]

\[
\Delta V_{\parallel}(q) \triangleq [V_{\parallel}(q(t+1)) - V_{\parallel}(q(t))] I(q(t) = q)
\]

We will use Lemma 1 using the Lyapunov function $W_{\perp}(q)(\cdot)$ to bound the $\mathbf{q}_1^{(\epsilon)}$ component in steady state. We need the following lemma, which follows from concavity of square root function and the pythagorean theorem [1]. The proof of this lemma is similar to the proof of Lemma 7 in [6] and so we skip it here.

**Lemma 2.** Drift of $W_{\perp}(\cdot)$ can be bounded in terms of drift of $V(\cdot)$ and $V_{\parallel}(\cdot)$ as follows.

\[
\Delta W_{\perp}(q) \leq \frac{1}{2\|q_{\perp}\|} (\Delta V(q) - \Delta V_{\parallel}(q)) \quad \forall q \in \mathbb{R}^{n^2}
\]

We will now formally state the state space collapse result.

**Proposition 2.** Consider a set of switch systems under MaxWeight scheduling algorithm, with the Bernoulli arrival processes $a^{(\epsilon)}(t)$ parameterized by $0 < \epsilon < 1$, such that the mean arrival rate vector is $\lambda^{\epsilon} = \frac{1-\epsilon}{n} \mathbf{1}$. Let $q^{(\epsilon)}(t)$ denote the queue lengths process of each system, which is positive recurrent. Therefore, the process $q^{(\epsilon)}(t)$ converges to a steady state random vector in distribution, which we denote by $\mathbf{q}^{(\epsilon)}$. Then, there exist $\{M_r\}_{r=1,2,...}$ not depending on $\epsilon$ such that, for each system with $0 < \epsilon \leq 1/2n$, the steady state average queue length satisfies

\[
\mathbb{E} \left[\|q^{(\epsilon)}_r\|^r\right] \leq M_r \forall r \in \{1,2,\ldots\}
\]

**Proof.** We will verify both the conditions in Lemma 1 for the Markov chain $q(t)$ and Lyapunov function $W_{\perp}(q(\cdot))$. First we consider condition [C.2]

\[
|\Delta W_{\perp}(q)| = \|q_{\perp}(t+1)\| - \|q_{\perp}(t)\| I(q(t) = q)
\]

\[
\leq \|q_{\perp}(t+1) - q_{\perp}(t)\| I(q(t) = q)
\]

\[
\leq \|q(t+1) - q(t)\| I(q(t) = q)
\]

\[
\leq n^2 \max_{ij} |q_{ij}(t+1) - q_{ij}(t)| I(q(t) = q)
\]

\[
\leq n^2
\]
where (a) follows from triangle inequality, i.e., $||x|| - ||y|| \leq ||x - y||$; (b) follows from nonexpansivity of projection operator \([5]\); and the last inequality is true because under Bernoulli arrivals, in a switch the maximum possible increase or decrease in any queue length is 1. Thus condition (C.2) of Lemma \([1]\) is true with $D = n^2$.

We will now verify \([C.1]\) using Lemma \([2]\) by bounding the drifts $\Delta V(q)$ and $\Delta V_\parallel(q)$.

\[
\begin{align*}
E[\Delta V(q)|q(t) &= q] \\
&= E[||q(t + 1)||^2 - ||q(t)||^2 | q(t) = q] \\
&= E[||q(t) + a(t) - s(t) + u(t)||^2 - ||q(t)||^2 | q(t) = q] \\
&= E[||q(t) + a(t) - s(t)||^2 + ||u(t)||^2 + 2(q(t + 1) - u(t), u(t)) - ||q(t)||^2 | q(t) = q] \\
&\leq E[||a(t) - s(t)||^2 + 2(q(t), a(t) - s(t)) | q(t) = q] \\
&\leq E \left[ \sum_{ij} (a_{ij}(t) + s_{ij}(t) - 2a_{ij}(t)s_{ij}(t)) \right] q(t) = q \\
&\leq 2n(1 - \epsilon) + 2 \epsilon \left< q, \frac{1}{n} \right> + 2 \epsilon \left< q, \frac{1}{n} - 1 - E[s(t)|q(t) = q] \right> \\
&\leq 2n - 2 \epsilon \left< q, \frac{1}{n} \right> + 2 \min_{r \in C} \left< q, \frac{1}{n} - r \right> \\
\end{align*}
\]

where (a) follows from the fact that $\langle q(t + 1), u(t) \rangle = 0$ and dropping the $-||u(t)||^2$ term; (b) is true because $a_{ij} \in \{0, 1\}$ and $s_{ij} \in \{0, 1\}$; (c) follows because the arrivals are independent of the queue lengths and the service process and from \([2]\); (d) again follows from \([2]\). Since we use MaxWeight scheduling algorithm, from \([1]\), and since $\epsilon < 1$, we have \([5]\). In order to bound the last term in \([8]\), we present the following claim.

**Claim 2.** For any $q \in \mathbb{R}^{n^2}$ and $0 \leq \delta \leq 1/n$,

\[
\frac{1}{n} 1 + \frac{\delta}{||q_\perp||} q_\perp \in \mathcal{C}
\]

**Proof.** For $0 \leq \delta \leq 1/n$, we have $\left| \frac{\delta}{||q_\perp||} q_\perp \right| \leq 1/n$. Therefore independent of direction of $q_\perp$, $\frac{1}{n} 1 + \frac{\delta}{||q_\perp||} q_\perp \in \mathbb{R}^{n^2}$. We know that $q_\perp \in \mathcal{K}^c$ and $e^{(i)} \in \mathcal{K}$, and so $\langle q_\perp, e^{(i)} \rangle \leq 0$. Thus, for any $i$, we have

\[
\left< \frac{1}{n} 1 + \frac{\delta}{||q_\perp||} q_\perp, e^{(i)} \right> = \frac{1}{n} \left< 1, e^{(i)} \right> + \frac{\delta}{||q_\perp||} \left< q_\perp, e^{(i)} \right> \\
\leq \frac{1}{n} \left< 1, e^{(i)} \right> = 1
\]

Similarly, we can show that $\left< \frac{1}{n} 1 + \frac{\delta}{||q_\perp||} q_\perp, e^{(j)} \right> \leq 1$ for any $j$, proving the claim. \(\square\)
Using the claim with $\delta = 1/n$ in (8), we get

$$\mathbb{E} [\Delta V(q)|q(t) = q] \leq 2n - 2\epsilon \left( q, \frac{1}{n} \right) + 2 \left( q, \frac{1}{n} - \left( \frac{1}{n} + \frac{1}{n \|q\|} \right) \right)$$

$$= 2n - 2\epsilon \left( q, \frac{1}{n} \right) - \frac{2}{n \|q\|} \left( q\| + q\perp, q\perp \right)$$

$$= 2n - 2\epsilon \left( q, \frac{1}{n} \right) - \frac{2}{n \|q\|} \tag{9}$$

where the last equality follows from the fact that $\langle q\|, q\perp \rangle = 0$. We will now bound the drift $\Delta V\| (q)$.

$$\mathbb{E} [\Delta V\| (q)|q(t) = q]$$

$$= \mathbb{E} \left[ \|q\| (t + 1) \right] - \|q\| (t) \right| q(t) = q]$$

$$= \mathbb{E} \left[ \left( q\| (t + 1) + \|q\| (t), q\| (t + 1) - \|q\| (t) \right) \right| q(t) = q]$$

$$= \mathbb{E} \left[ \langle q\| (t), q\| (t + 1) - q\| (t) \rangle \right| q(t) = q]$$

$$\geq 2\mathbb{E} \left[ \langle q\| (t), q\| (t + 1) - q\| (t) \rangle \right| q(t) = q]$$

$$\geq 2\mathbb{E} \left[ \langle q\| (t), a(t) - s(t) + u(t) \rangle \right| q(t) = q]$$

$$\geq 2\mathbb{E} \left[ \langle q\|, \lambda \rangle \right] - 2\mathbb{E} \left[ \langle q\|, s(t) \rangle \right| q(t) = q]$$

$$= - 2\epsilon \left( q, \frac{1}{n} \right) - 2\mathbb{E} \left[ \left( q\|, s(t) - \frac{1}{n} \right) \right| q(t) = q]$$

$$= - 2\epsilon \left( q, \frac{1}{n} \right) \tag{10}$$

Equation (a) is true because $\langle q\| (t), q\perp (t) \rangle = 0$ and $\langle q\| (t), q\perp (t + 1) \rangle \leq 0$ since $q\| (t) \in \mathcal{K}$ and $q\perp (t + 1) \in \mathcal{K}°$. All the components of $q\|$ and $u(t)$ are nonnegative. Using this fact with independence of the arrivals and the queue lengths gives Equation (b). The last equality follows from (3) since $q\| \in \mathcal{K} \subseteq \mathcal{V}_K$ and $s(t), \frac{1}{n} \in \mathcal{F}$ from (2). Now substituting (3) and (10) in Lemma 2 we get

$$\mathbb{E} [\Delta W\perp (q)|q(t) = q] \leq \frac{1}{2 \|q\perp\|} \left( 2n - 2\epsilon \left( q, \frac{1}{n} \right) - \frac{2}{n \|q\|} \right) + 2\epsilon \left( q\|, \frac{1}{n} \right)$$

$$= \frac{n}{\|q\perp\|} - \frac{1}{n} - \frac{\epsilon}{\|q\|}$$

$$\leq \frac{n}{\|q\perp\|} - \frac{1}{n} + \epsilon$$

$$\leq \frac{n}{\|q\perp\|} - \frac{1}{2n} \text{ whenever } \epsilon \leq 1/2n$$

where (a) is due to the Cauchy Schwartz inequality $\left( \frac{q\|}{\|q\perp\|}, \frac{1}{n} \right) \leq \frac{1}{\|q\perp\|} \frac{1}{n} \leq 1$. Thus condition $C.1$ of Lemma 1 is valid and the proposition follows from the lemma. \hfill $\square$

Note that the parameters $\{M_r\}_{r=1,2,\ldots}$ depend on $n$, the number of input ports of the switch, and are not universal constant.
Recall that there are $n!$ maximal schedules (permutations or perfect matchings). For each of them, MaxWeight assigns a weight which is the sum of corresponding queue lengths and then picks the one with the maximum weight. In this process, it is equalizing the weights of all the schedules by serving the matching with maximum weight and thereby decreasing it. The cone $K$ has the property that if the queue lengths vector $\mathbf{q}$ is in the cone $K$, we have $w_i + \tilde{w}_j$ such that $q_{ij} = w_i + \tilde{w}_j$. This means that all the maximal schedules have the same weight $\sum_i w_i + \sum_j \tilde{w}_j$ and the MaxWeight algorithm is agnostic between them. Thus, the state space collapse result states that in steady state, MaxWeight is (almost) successful in being able to equalize the weights of all maximal schedules in the heavy traffic limit. This behavior is very similar to Join-the-Shortest queue (JSQ) routing policy in a supermarket checkout system. In such a system, there are a few servers, each with a queue. When a customer arrives to be served, under JSQ policy, (s)he picks the server with the shortest queue. It was shown in [6] that in the heavy traffic limit, the state of this system collapses to a state where all the queues are equal, and thus, JSQ is agnostic between all the queues when such a state space collapse occurs. Here JSQ policy is trying to equalize all the queues by increasing the shortest one, and it is (almost) successful in doing that in steady state in heavy traffic limit.

5 Asymptotically tight Upper and Lower bounds under MaxWeight policy

In the previous section, we have shown that the queue length vector collapses within the cone $K$ in the steady state. We will use this result to obtain lower and upper bounds on the average queue lengths under MaxWeight algorithm. The lower and upper bounds differ only in $o(1/\varepsilon)$ and so match in the heavy traffic limit.

We will obtain these bounds using Lyapunov drifts. We first define a few Lyapunov functions and their drifts, in addition to the already defined $V(\mathbf{q}) = \|\mathbf{q}\|^2$. The following lemma states that all these Lyapunov functions have finite expectations in steady state.

$$V_1(\mathbf{q}) \triangleq \sum_i \left( \sum_j q_{ij} \right)^2, \quad V_2(\mathbf{q}) \triangleq \sum_j \left( \sum_i q_{ij} \right)^2, \quad V_3(\mathbf{q}) \triangleq \left( \sum_{ij} q_{ij} \right)^2$$

$$\Delta V_1(\mathbf{q}) \triangleq [V_1(\mathbf{q}(t+1)) - V_1(\mathbf{q}(t))]|\mathcal{I}(\mathbf{q}(t) = \mathbf{q})$$

$$\Delta V_2(\mathbf{q}) \triangleq [V_2(\mathbf{q}(t+1)) - V_2(\mathbf{q}(t))]|\mathcal{I}(\mathbf{q}(t) = \mathbf{q})$$

$$\Delta V_3(\mathbf{q}) \triangleq [V_3(\mathbf{q}(t+1)) - V_3(\mathbf{q}(t))]|\mathcal{I}(\mathbf{q}(t) = \mathbf{q})$$

**Lemma 3.** Consider the switch under MaxWeight scheduling algorithm. For any arrival rate vector $\lambda$ in the interior of the capacity region $\mathcal{C}$, the steady state means $\mathbb{E}[V(\mathbf{q})], \mathbb{E}[V_1(\mathbf{q})], \mathbb{E}[V_2(\mathbf{q})], \mathbb{E}[V_3(\mathbf{q})]$ are finite.

The lemma is proved in the appendix. We will now state and prove the main result of this paper.

**Theorem 1.** Consider a set of switch systems under MaxWeight scheduling algorithm, with Bernoulli arrival processes $A^{(i)}(t)$ parameterized by $0 < \varepsilon < 1$, such that the mean arrival rate vector is $\lambda^\varepsilon = \frac{1}{n} \mathbf{1}$. The queue length process $\mathbf{q}^{(i)}(t)$ for each system converges in distribution to the steady state random vector $\overline{\mathbf{q}}^{(i)}$. For each system with $0 < \varepsilon \leq 1/2n$, the steady state average queue length
satisfies
\[
\frac{1}{\epsilon} \left( n - \frac{3}{2} + \frac{1}{2n} \right) - B^{(\epsilon)} - B^{(\epsilon)}_{2} \leq \mathbb{E} \left[ \sum_{ij} q_{ij}^{(\epsilon)} \right] \leq \frac{1}{\epsilon} \left( n - \frac{3}{2} + \frac{1}{2n} \right) + B^{(\epsilon)}
\]

where \(B^{(\epsilon)}_{1}\) and \(B^{(\epsilon)}_{2}\) are \(o(\frac{1}{\epsilon})\), i.e., \(\lim_{\epsilon \to 0} \epsilon B^{(\epsilon)}_{1} = 0\) and \(\lim_{\epsilon \to 0} \epsilon B^{(\epsilon)}_{2} = 0\). Therefore, in the heavy traffic limit as \(\epsilon \downarrow 0\) which means the mean arrival rate \(\lambda^t \rightarrow \frac{1}{n} \mathbf{1}\), we have

\[
\lim_{\epsilon \downarrow 0} \epsilon \mathbb{E} \left[ \sum_{ij} q_{ij}^{(\epsilon)} \right] = \left( n - \frac{3}{2} + \frac{1}{2n} \right)
\]

**Proof.** Fix an \(0 < \epsilon \leq 1/2n\) and consider the \(\epsilon\)th system. For simplicity of notation, we will skip the superscript \((\epsilon)\) in the proof and use \(\overline{q}\) to denote the steady state queue length vector. We will use \(\overline{a}\) to denote the arrival vector in steady state, which is just a Bernoulli random vector with mean \(\lambda^t = \frac{1}{n} \mathbf{1}\). We will use \(s(\overline{q})\) and \(u(\overline{q})\) to denote the schedule and unused service to show their dependence on the queue lengths. We will use \(\overline{q}^+\) to denote \(\overline{q} + \overline{a} - s(\overline{q}) + u(\overline{q})\), which is the queue lengths vector at time \(t + 1\) if it was \(\overline{q}\) at time \(t\). Clearly, \(\overline{q}^+\) and \(\overline{q}\) have the same distribution.

Define a new function \(V_4(\overline{q})\) and its drift as follows.

\[
V_4(\overline{q}) = V_1(\overline{q}) + V_2(\overline{q}) - \frac{1}{n} V_3(\overline{q})
\]

\[
= \sum_{i} \left( \sum_{j} q_{ij} \right)^2 + \sum_{j} \left( \sum_{i} q_{ij} \right)^2 - \frac{1}{n} \left( \sum_{ij} q_{ij} \right)^2
\]

\[
\Delta V_4(\overline{q}) \equiv [V_4(\overline{q}(t+1)) - V_4(\overline{q}(t))] \mathcal{I}(\overline{q}(t) = \overline{q})
\]

\[
= \Delta V_1(\overline{q}) + \Delta V_2(\overline{q}) - \frac{1}{n} \Delta V_3(\overline{q})
\]

Since \(-\frac{1}{n} V_3(\overline{q}) \leq V_4(\overline{q}) \leq V_1(\overline{q}) + V_2(\overline{q})\), the steady state mean \(\mathbb{E}[V_4(\overline{q})]\) is finite. Therefore, the mean drift of \(V_4(\cdot)\) in steady state is zero, i.e.,

\[
\mathbb{E}[\Delta V_4(\overline{q})] = \mathbb{E}[[V_4(\overline{q}(t+1)) - V_4(\overline{q}(t))] \mathcal{I}(\overline{q}(t) = \overline{q})] = \mathbb{E}[V_4(\overline{q}^+)] - \mathbb{E}[V_4(\overline{q})] = \mathbb{E}[V_4(\overline{q})] - \mathbb{E}[V_4(\overline{q})] = 0
\]

\[
0 = \mathbb{E} [\Delta V_1(\overline{q})] + \mathbb{E} [\Delta V_2(\overline{q})] - \frac{1}{n} \mathbb{E} [\Delta V_3(\overline{q})] = \mathbb{E} [\Delta V_1(\overline{q})] + \mathbb{E} [\Delta V_2(\overline{q})] - \frac{1}{n} \mathbb{E} [\Delta V_3(\overline{q})]
\]

(11)

Expanding the drift of \(V_1(\cdot)\), we get

\[
\mathbb{E} [\Delta V_1(\overline{q})]
\]

\[
= \mathbb{E} [V_1(\overline{q} + \overline{a} - s(\overline{q}) + u(\overline{q})) - V_1(\overline{q})]
\]

\[
= \mathbb{E} \left[ \sum_{i} \left( \sum_{j} (q_{ij} + \overline{a}_{ij} - s_{ij}(\overline{q}) + u_{ij}(\overline{q})) \right)^2 - \sum_{i} \left( \sum_{j} q_{ij} \right)^2 \right]
\]

\[
= \mathbb{E} \left[ \sum_{i} \left( \sum_{j} (\overline{q}_{ij} - s_{ij}(\overline{q})) \right)^2 + 2 \sum_{i} \left( \sum_{j} (\overline{q}_{ij} + \overline{a}_{ij} - s_{ij}(\overline{q})) \right) \left( \sum_{ij} u_{ij}(\overline{q}) \right) \right]
\]
Similarly expanding drifts of $V_2(.)$ and $V_3(.)$ and substituting in (11), we get the following expression. Since this is a lengthy equation, we split into various terms which we denote by $T_1$, $T_2$, $T_3$ and $T_4$. For simplicity of notation, we suppress all the dependencies in terms of $\bar{q}$, $\bar{r}$, $s(q)$ and $u(q)$.

$$T_1 = T_2 + T_3 + T_4$$

where

$$T_1 = 2\mathbb{E} \left[ \sum_i \left( \sum_j q_{ij} \right) \left( \sum_{j'} (s_{ij'} - \bar{u}_{ij'}) \right) \right] + 2\mathbb{E} \left[ \sum_j \left( \sum_i q_{ij} \right) \left( \sum_{i'} (s_{i'j} - \bar{u}_{ij}) \right) \right] - \frac{2}{n} \mathbb{E} \left[ \left( \sum_{ij} q_{ij} \right) \left( \sum_{i'j'} (s_{i'j'} - \bar{u}_{i'j'}) \right) \right]$$

$$T_2 = \mathbb{E} \left[ \sum_i \left( \sum_j (u_{ij} - s_{ij}(q)) \right)^2 \right] + \mathbb{E} \left[ \sum_j \left( \sum_i (u_{ij} - s_{ij}(q)) \right)^2 \right] - \frac{1}{n} \mathbb{E} \left[ \left( \sum_{ij} u_{ij} - s_{ij}(q) \right)^2 \right]$$

$$T_3 = - \mathbb{E} \left[ \sum_i \left( \sum_j u_{ij}(q) \right)^2 \right] - \mathbb{E} \left[ \sum_j \left( \sum_i u_{ij}(q) \right)^2 \right] + \frac{1}{n} \mathbb{E} \left[ \left( \sum_{ij} u_{ij}(q) \right)^2 \right]$$

$$T_4 = 2\mathbb{E} \left[ \sum_i \left( \sum_j q^+_{ij} \right) \left( \sum_{j'} u_{ij'}(q) \right) \right] + 2\mathbb{E} \left[ \sum_j \left( \sum_i q^+_{ij} \right) \left( \sum_{i'} u_{i'j}(q) \right) \right] - \frac{2}{n} \mathbb{E} \left[ \left( \sum_{ij} q^+_{ij} \right) \left( \sum_{i'j'} u_{i'j'}(q) \right) \right]$$

We will now bound each of the four terms. The schedule in each time slot is maximal and so $\sum_i s_{ij} = 1$, $\sum_j s_{ij} = 1$ and $\sum_{ij} s_{ij} = n$. Noting that the arrivals are independent of queue lengths, we can simplify the term $T_1$ as follows.

$$T_1 = 2\mathbb{E} \left[ \sum_i \left( \sum_j \bar{q}_{ij} \right) \left( 1 - \sum_{j'} \lambda_{ij'} \right) \right] + 2\mathbb{E} \left[ \sum_j \left( \sum_i \bar{q}_{ij} \right) \left( 1 - \sum_{i'} \lambda_{i'j} \right) \right] - \frac{2}{n} \mathbb{E} \left[ \left( \sum_{ij} \bar{q}_{ij} \right) \left( n - \sum_{i'j'} \lambda_{i'j'} \right) \right]$$
Thus, from (12), we have
\[ 2\epsilon E \left[ \sum_{ij} q_{ij} \right] = T_2 + T_3 + T_4. \] (13)

Now the rest of the proof involves bounding the term \( T_2, T_3 \) and \( T_4 \). We start with the term \( T_2 \). Consider the first term of \( T_2 \). Again noting that the schedules are maximal, we get
\[
2\epsilon E \left[ \sum_{ij} q_{ij} \right] = \sum_i E \left[ \left( \sum_j \pi_{ij} - s_{ij}(\pi) \right)^2 \right] = \sum_i E \left[ \left( \sum_j \pi_{ij} - 1 \right)^2 \right]
\]
\[
= \sum_i E \left[ \left( \sum_j \pi_{ij} - (1 - \epsilon) - \epsilon \right)^2 \right]
\]
\[
= \sum_i E \left[ \left( \sum_j \pi_{ij} - (1 - \epsilon) \right)^2 \right] + \sum_i \epsilon^2 - \sum_i 2\epsilon E \left[ \left( \sum_j \pi_{ij} - (1 - \epsilon) \right) \right]
\]
\[
= n\epsilon^2 + \sum_i \text{Var} \left( \sum_j \pi_{ij} \right)
\]
\[
= n\epsilon^2 + n^2 \left( \frac{1 - \epsilon}{n} \right) \left( 1 - \frac{1 - \epsilon}{n} \right)
\]
\[
= n\epsilon + (n - 1)(1 - \epsilon)^2
\]
where \( \text{Var}(\cdot) \) denotes the variance. Similarly, we can show that the second term in \( T_2 \) evaluates to
\[
E \left[ \sum_i \left( \sum_j (\pi_{ij} - s_{ij}(\pi)) \right)^2 \right] = n\epsilon + (n - 1)(1 - \epsilon)^2.
\]
The last term can likewise be evaluated as follows.
\[
\frac{1}{n} E \left[ \left( \sum_{ij} (\pi_{ij} - s_{ij}(\pi)) \right)^2 \right] = \frac{1}{n} E \left[ \left( \sum_{ij} \pi_{ij} - n \right)^2 \right]
\]
\[
= \frac{1}{n} E \left[ \left( \sum_{ij} \pi_{ij} - n(1 - \epsilon) - n\epsilon \right)^2 \right]
\]
\[
= \frac{1}{n} E \left[ \left( \sum_{ij} \pi_{ij} - n(1 - \epsilon) \right)^2 \right] + n\epsilon^2 - 2\epsilon E \left[ \left( \sum_{ij} \pi_{ij} - n(1 - \epsilon) \right) \right]
\]
\[
= n\epsilon^2 + \frac{1}{n} \text{Var} \left( \sum_{ij} \pi_{ij} \right)
\]
\[ n \epsilon^2 + \frac{1}{n} \epsilon^2 \left( \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right) + \frac{1}{n} \epsilon \]

Putting all the terms of \( T_2 \) together, we get

\[ T_2 = 2n \epsilon + 2(n - 1)(1 - \epsilon)^2 - n \epsilon^2 - (1 - \epsilon)^2 + \frac{(1 - \epsilon)^2}{n} \]

\[ = n - 1 + (1 - \epsilon)^2 \left( \frac{1}{n} + n - 2 \right) + \epsilon. \]

Thus,

\[ \left( 2n - 3 + \frac{1}{n} \right) - 2 \epsilon \left( \frac{1}{n} + n - 2 \right) \leq T_2 \leq \left( 2n - 3 + \frac{1}{n} + \epsilon \right). \quad (14) \]

Since \( \sum_{ij} q_{ij} \in \mathbb{Z}_+ \), we have \( \sum_{ij} q_{ij} \leq (\sum_{ij} q_{ij})^2 \). Using the fact that \( \mathbb{E} \left[ (\sum_{ij} q_{ij})^2 \right] \) is finite from Lemma 3, we have that \( \mathbb{E} \left[ \sum_{ij} q_{ij} \right] \) is finite and so its drift is zero in steady state. Thus, we get

\[ 0 = \mathbb{E} \left[ \sum_{ij} q_{ij}(t + 1) - \sum_{ij} q_{ij}(t) \right] I(q(t) = \overline{q}) \]

\[ = \mathbb{E} \left[ \sum_{ij} \overline{q}_{ij} - \sum_{ij} s_{ij}(\overline{q}) + \sum_{ij} u_{ij}(\overline{q}) \right] \]

\[ \mathbb{E} \left[ \sum_{ij} u_{ij}(\overline{q}) \right] = n - n(1 - \epsilon) \]

\[ = n \epsilon \]

\[ (15) \]

We will now bound the term \( T_3 \). Since \( u_{ij}(t) \leq s_{ij}(t) \), we have \( \sum_i u_{ij} \leq 1, \sum_j u_{ij} \leq 1 \) and \( \sum_{ij} u_{ij} \leq n \). Therefore,

\[ -\mathbb{E} \left[ \sum_i \left( \sum_j u_{ij}(\overline{q}) \right)^2 \right] - \mathbb{E} \left[ \sum_j \left( \sum_i u_{ij}(\overline{q}) \right) \right] \leq T_3 \leq \frac{1}{n} \mathbb{E} \left[ \left( \sum_{ij} u_{ij}(\overline{q}) \right)^2 \right] \]

\[ -\mathbb{E} \left[ \sum_i \left( \sum_j u_{ij}(\overline{q}) \right) \right] - \mathbb{E} \left[ \sum_j \left( \sum_i u_{ij}(\overline{q}) \right) \right] \leq T_3 \leq \frac{1}{n} \mathbb{E} \left[ n \left( \sum_{ij} u_{ij}(\overline{q}) \right) \right] \]

\[ -2n \epsilon \leq T_3 \leq n \epsilon \] \quad (16)

We now consider the term \( T_4 \). It can be rewritten as follows, and can be split into two parts,
one each corresponding to $\mathbf{q}_i^+$ and $\mathbf{q}_i^-$. 

$$T_4 = 2E \left[ \sum_{ij} u_{ij}(\mathbf{q}) \left( \sum_{j'} q_{ij'}^+ + \sum_{i'} q_{ij'j}^+ - \frac{1}{n} \sum_{i'j'} q_{ij'}^+ \right) \right]$$

$$= 2E \left[ \sum_{ij} u_{ij}(\mathbf{q}) \left( \sum_{j'} q_{||ij'}^+ + \sum_{i'} q_{||ij'j}^+ - \frac{1}{n} \sum_{i'j'} q_{||ij'}^+ \right) \right]$$

$$+ 2E \left[ \sum_{ij} u_{ij}(\mathbf{q}) \left( \sum_{j'} q_{\perp ij'}^+ + \sum_{i'} q_{\perp ij'j}^+ - \frac{1}{n} \sum_{i'j'} q_{\perp ij'}^+ \right) \right]$$

Since the vector $\mathbf{q}_i^+$ is in cone $\mathcal{K}$ by definition, there exist $w_1, w_2, \ldots, w_n \in \mathbb{R}_+$ such that $\mathbf{q}_i^+ = w_i + \bar{w}_j$ for all $i, j \in \{1, 2, \ldots, n\}$. Recall that when $u_{ij}(t) = 1$, $q_{ij}(t + 1) = 0$. Similarly, when $u_{ij}(\mathbf{q}) = 1$, we have

$$q_{ij}^+ = 0$$

$$q_{||ij}^+ + q_{\perp ij}^+ = 0$$

$$w_i + \bar{w}_j = - q_{\perp ij}$$

$$\sum_{j'=1}^{n} q_{||ij'}^+ + \sum_{i'=1}^{n} q_{||i'j}^+ - \frac{1}{n} \sum_{i'=1}^{n} q_{||i'j}^+ = \sum_{j'=1}^{n} (w_i + \bar{w}_j) + \sum_{i'=1}^{n} (w_i + \bar{w}_j) - \frac{1}{n} \sum_{i'=1}^{n} (w_i + \bar{w}_j)$$

$$= nw_i + n\bar{w}_j$$

$$= - nq_{\perp ij}$$

Therefore, we get

$$u_{ij}(\mathbf{q}) \left( \sum_{j'} q_{||ij'}^+ + \sum_{i'} q_{||i'j}^+ - \frac{1}{n} \sum_{i'j'} q_{\perp ij'}^+ \right) = -nu_{ij}(\mathbf{q})q_{\perp ij}$$

and the term $T_4$ reduces to

$$T_4 = 2E \left[ \sum_{ij} u_{ij}(\mathbf{q}) \left( -nq_{\perp ij} + \sum_{j'} q_{\perp ij'}^+ + \sum_{i'} q_{\perp ij'j}^+ - \frac{1}{n} \sum_{i'j'} q_{\perp ij'}^+ \right) \right]$$

$$= 2E \left[ \mathbf{u}(\mathbf{q}) - n\mathbf{q}_i^- + \sum_{i} \left( \mathbf{q}_i^+, \mathbf{e}^{(i)} \right) \mathbf{e}^{(i)} + \sum_{i} \left( \mathbf{q}_i^+, \mathbf{e}^{(i)} \right) \mathbf{e}^{(j)} - \frac{1}{n} \left( \mathbf{q}_i^+, 1 \right) 1 \right] .$$

Suppose the queue lengths lie in the cone $\mathcal{K}$ and so $\mathbf{q}_i^+$ is zero, then the term $T_4$ would be zero. This is the main reason for our choice of the function $V_4(.)$. For a $G/G/1$ queueing system, one can use a quadratic Lyapunov function to obtain Kingman style upper bound that is optimal in the heavy traffic limit [17]. There, we again have four terms that have similar form to the four terms $T_1, T_2, T_3$ and $T_4$. However the fourth term there would be of the form $u(\mathbf{q})\mathbf{q}_i^+$, which is zero by definition of unused service. In a Join the shortest queue (JSQ ) routing system, the fourth term is of the form $(\sum_i u_i(\mathbf{q}))(\sum_i \mathbf{q}_i^+)$, which is not zero in general. However, it was shown in [3] that in the heavy traffic limit, the queue lengths collapse so that they are (almost) equal. Then, this term
is effectively like \( u(q)q^+ \), which is zero. This is called resource pooling. In all these problems, when using a quadratic Lyapunov function, the fourth term \( T_4 \) is the most important and challenging one to bound correctly. The key idea in our upper bound proof is the choice of the function \( V_4(.) \). We picked the function \( V_4(.) \) so that it matches with the geometry of the cone \( K \) in the sense that if the queue length vector is in the cone \( K \), the fourth term \( T_4 \) is zero.

From state space collapse, we know that \( \bar{q}^+ \) is not zero, but is bounded. We will now use this result to show that \( T_4 \) is \( o(\epsilon) \). Since \( \bar{q}^+ \in \mathcal{K}^o \) and \( e^{(i)}, \bar{e}^{(j)}, 1 \in \mathcal{K} \) for all \( i, j \), we have that \( \langle \bar{q}^+, e^{(i)} \rangle \leq 0, \langle \bar{q}^+, \bar{e}^{(j)} \rangle \leq 0 \) and \( \langle \bar{q}^+, 1 \rangle \leq 0 \). Moreover all components of \( u, e^{(i)}, \bar{e}^{(j)} \) and \( 1 \) take values 0 and 1. Therefore,

\[
T_4 \leq 2E \left[ \left( u(q), -n\bar{q}^+ - \frac{1}{n} \langle \bar{q}^+, 1 \rangle \right) \right]
\]

\[
\leq 2 \sqrt{E[\|u(q)\|^2] E\left[ \left\| -n\bar{q}^+ - \frac{1}{n} \langle \bar{q}^+, 1 \rangle \right\|^2 \right]}
\]

\[
\leq 2 \sqrt{n\epsilon E\left[ \left( n\|\bar{q}^+\| + \frac{1}{n} \langle \bar{q}^+, 1 \rangle \right) \right]^2}
\]

\[
= 4n \sqrt{n\epsilon E[\|\bar{q}^+\|^2]}
\]

\[
\leq 4n \sqrt{n\epsilon M_2}
\]

where (a) follows from the Cauchy-Schwartz inequality. Since \( u_{ij} \in \{0,1\} \), from \( \text{(15)} \), we have \( E[\|u(q)\|^2] = E\left[ \sum_{ij} u_{ij}(q) \right] = n\epsilon \). This fact along with using triangle inequality on the second term gives (b). Noting that \( \|1\| = n \) and again using Cauchy-Schwartz inequality gives (c). Equality (d) follows from the fact that \( \bar{q}^+ \) has same distribution as \( q^+ \). The last inequality follows from state space collapse in Proposition 2. Similarly, we can lower bound \( T_4 \) as follows.

\[
T_4 \geq 2E \left[ \left( u(q), -n\bar{q}^+ + \sum_i \langle \bar{q}^+, e^{(i)} \rangle e^{(i)} + \sum_j \langle \bar{q}^+, \bar{e}^{(j)} \rangle \bar{e}^{(j)} \right) \right]
\]

\[
\geq -2 \sqrt{E[\|u(q)\|^2] E\left[ \left\| -n\bar{q}^+ + \sum_i \langle \bar{q}^+, e^{(i)} \rangle e^{(i)} + \sum_j \langle \bar{q}^+, \bar{e}^{(j)} \rangle \bar{e}^{(j)} \right\|^2 \right]}
\]

\[
\geq -2 \sqrt{n\epsilon E\left[ \left( n\|\bar{q}^+\| + \sum_i \|\bar{q}^+\| \|e^{(i)}\| + \sum_j \|\bar{q}^+\| \|\bar{e}^{(j)}\| \right) \right]^2}
\]

\[
\geq -2 \sqrt{n\epsilon E\left[ \left( n\|\bar{q}^+\| + \sum_i \|\bar{q}^+\| \|e^{(i)}\|^2 + \sum_j \|\bar{q}^+\| \|\bar{e}^{(j)}\|^2 \right) \right]}
\]

\[
\geq -2(2n^2 + n) \sqrt{n\epsilon E[\|\bar{q}^+\|^2]}
\]
\[ \geq -2(2n^2 + n)\sqrt{n\epsilon M_2} \]

where we use the fact that \( \|e^{(i)}\|^2 = n \) and \( \|e^{(i)}\|^2 = n \) to get (a). Therefore, we have

\[ -2(2n^2 + n)\sqrt{n\epsilon M_2} \leq T_4 \leq 4n\sqrt{n\epsilon M_2}. \tag{17} \]

Using (14), (16) and (17) to bound (13), we get the theorem with

\begin{align*}
B_1^{(\epsilon)} &= -\left(\frac{1}{n} + n - 2\right) - n - \frac{1}{\sqrt{\epsilon}}(2n^2 + n)\sqrt{nM_2} \\
B_2^{(\epsilon)} &= \frac{1}{2} + \frac{1}{\sqrt{\epsilon}} 2n\sqrt{nM_2}
\end{align*}

which are clearly \( o(\frac{1}{\epsilon}) \).

\section{Conclusion}

We have obtained a characterization of the heavy-traffic behavior of the sum queue length in steady-state in an \( n \times n \) switch operating under the MaxWeight scheduling policy with an uniform arrival pattern. The result settles one version of a conjecture regarding the performance of the MaxWeight policy. A number of extensions can be considered:

- The assumption about Bernoulli arrivals can be replaced by an i.i.d. arrival process with an upper bound on the number of arrivals per time slot. The results can also be extended to the non-uniform traffic case where all rows and columns are saturated and the arrival rate to each queue is non-zero. Both these extensions should be straightforward. Extensions to more general traffic patterns is an open problem.

- We believe that one may be also be able to allow correlations across time slots by making an assumption similar to the assumption in Section II.C of [20], and considering the drift of the Lyapunov function over multiple time slots. This extension may require a bit of additional work.

- A Brownian limit has been established in the heavy-traffic regime in [5], but a characterization of the behavior of this limit in steady-state is not known. We expect the mean of the sum queue lengths (multiplied by \( \epsilon \) and in the limit \( \epsilon \to 0 \)) in steady-state that we have derived to be equal to the sum of the steady-state expectations of the components of the Brownian motion in [5]. This would be interesting to verify.

- Verifying whether the MaxWeight algorithm achieves the optimal queue-length scaling in the size of the switch in non-heavy-traffic regimes is still an open problem.

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A Proof of Lemma 3

Proof. We will use Lemma 1 to first show that $E[V(q)]$ is finite. Define the Lyapunov function $W(q) \triangleq \|q\| = \sqrt{V(q)}$, and its drift

$$\Delta W(q) \triangleq [W(q(t+1)) - W(q(t))] I(q(t) = q)$$

We will first verify condition C.2 of Lemma 1. Using the same arguments as in, we get

$$|\Delta W(q)| = \|q(t+1)\| - \|q(t)\| I(q(t) = q)$$
$$\leq \|q(t+1) - q(t)\| I(q(t) = q)$$
$$\leq n^2 \max_{ij} |q_{ij}(t+1) - q_{ij}(t)| I(q(t) = q)$$
$$\leq n^2,$$

thus verifying condition C.2. We will now verify condition C.1

$$E[\Delta W(q)|q(t) = q] = E[\|q(t+1)\| - \|q(t)\| |q(t) = q]$$
$$= E\left[\frac{1}{2\|q(t)\|} \|q(t+1)\| - \|q(t)\| \bigg| q(t) = q\right]$$
$$\overset{(a)}{\leq} \frac{1}{2\|q\|} E[\Delta V(q)|q(t) = q]$$
$$\overset{(b)}{\leq} \frac{1}{2\|q\|} \left(2n - 2\epsilon \left< q, \frac{1}{n}\right> - \frac{2}{n}\|q\|_1\right)$$
$$\leq \frac{1}{2\|q\|} \left(2n - 2\epsilon \|q\|_1\right)$$
$$\leq \frac{1}{2\|q\|} \left(2n - 2\epsilon \|q\|\right)$$
$$\leq \frac{n}{\|q(t)\|} - \frac{\epsilon}{n}$$

where $\|q\|_1 = \sum_{ij} q_{ij}$ denotes the $\ell_1$ norm of $q$. Inequality (a) follows from the concavity of square root function, due to which we have that $\sqrt{y} - \sqrt{x} \leq \frac{1}{2\sqrt{x}}(y - x)$. Inequality (b) follows from the bound on drift of $V(.)$ obtained in (9) in the proof of the proof of Proposition 2. For any vector $x$,
its $\ell_1$ norm is at least its $\ell_2$ norm, i.e., $\|x\|_1 \geq \|x\|$. This gives inequality (c). Thus, condition[C.1] is verified and we have that all moments of $W(q)$ exist in steady state. In particular, we have that $\mathbb{E}[V(q)]$ is finite.

Now, note that

$$V_3(q) = \left( \sum_{ij} q_{ij} \right)^2 \leq \left( \sum_{ij} \max_{ij} q_{ij} \right)^2 = n^4 \max_{ij} q_{ij}^2 \leq n^4 \sum_{ij} q_{ij}^2 = n^4 V(q).$$

Thus, $\mathbb{E}[V_3(q)]$ is also finite. The lemma follows by noting that $V_1(q) \leq V_3(q)$ and $V_2(q) \leq V_3(q)$.