Heterogeneity and aggregation in evolutionary dynamics: a general framework without aggregability

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Abstract

We consider general evolutionary dynamics under persistent payoff heterogeneity and study the dynamic relation between the strategy composition over different types and the aggregate strategy distribution of the entire population. It is rigorously proven that continuity of either the revision protocol or the type distribution guarantees the existence of a unique solution trajectory. In many major evolutionary dynamics, an agent’s switching rate between actions increases with the payoff gain from this switch, which causes nonaggregability: the current strategy composition must be identified to predict the transition of the aggregate strategy. Looking at the strategy composition, we retain equilibrium stationarity in general and stability in potential games under admissible dynamics. Local stability of an equilibrium composition under any admissible dynamic can be tested by local stability of the corresponding aggregate equilibrium under the best response dynamic with i.i.d. payoff perturbation. All the results are maintained under heterogeneity in revision protocols.

Keywords: evolutionary dynamics; heterogeneity; aggregation; continuous space; potential games; sorting

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1 Introduction

In a population game, a population of (infinitely or finitely many) agents plays a game. While it is commonly assumed that the agents in the population are homogeneous,

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there are a few studies that bring payoff heterogeneity into a game and discuss the relation between aggregation and dynamics: Blonski (1999) and Ely and Sandholm (2005). But, these studies rely on the aggregability of the dynamic—the change in the aggregate strategy distribution is wholly determined from the current state of the aggregate distribution alone, independently of the underlying correlation between strategy choices and payoff types. Such aggregability may be assumed as in Blonski (1999) or may be derived from some specific form of the agents’ strategy revision processes as in Ely and Sandholm (2005). In this paper, we consider a general class of evolutionary dynamics in a heterogeneous setting without requiring aggregability. We allow agents not only to have different payoff functions but also to follow different evolutionary dynamics (revision protocols). The ultimate goal of this paper is to provide a foundation for nonaggregable dynamics that would better capture the dynamic relations between heterogeneous microscopic behavior and the macroscopic aggregate state beyond the representative agent approach.

Aggregation in evolutionary dynamics. Generally, in an evolutionary dynamic, an agent occasionally switches her action. The switching decision of an agent is supposed to follow a revision protocol, which determines the switching rate from one action to another based on the payoff vector and (possibly, but not necessarily) the action distribution over the observed population of other agents. The population dynamic is obtained by collecting the switching processes of individual agents.

In a heterogeneous setting, there are two layers to describe the distribution of actions in the entire population. Strategy composition is a joint distribution of actions and types, while the aggregate strategy is its marginal distribution of actions, i.e., the distribution of actions collected over all the agents regardless of their types. We consider aggregate games with payoff heterogeneity: the payoff of an action for each agent changes with the aggregate strategy and differs depending on the type of agent. If the choice of a new action is solely based on the payoffs and not on the other agents’ action distribution as in the best response dynamic and payoff comparison dynamics, an individual agent’s switching rate is completely determined in an aggregate game only from the aggregate strategy and the agent’s own type without identifying the strategy composition.

However, different types of agents may have different switching rates, even if they follow the same revision protocol. In most evolutionary dynamics, the switching rate depends on the payoff gain from a switch. Under payoff heterogeneity, the payoff gains for agents vary not only with

1Of course, it is very common in evolutionary game theory to have finitely many populations, each of which represents a different player in the normal form of a game and consists of homogeneous agents. Lahkar (2017) considers logit dynamics in a potential game on a continuous strategy space with finitely many payoff types based on this approach. (Note that a logit dynamic is an aggregable dynamic, as we will see in Section 4.) Our theorems on the stationarity and stability of equilibrium compositions can be seen as an extension of the stationarity and stability of Nash equilibria in finitely many populations to (potentially) continuously many populations; however, our extension comes straight from those properties in a single homogeneous population setting. After all, our motivation and the central issue in this paper is the relationship between aggregation and dynamics.
their current actions but also with their payoff types. The population dynamic is constructed by collecting agents’ switches of actions over the population. Thus, to pin down the population dynamic, we need to identify which types of agents are more likely to switch actions, namely, which types have greater switching rates. The transition of the aggregate strategy is more dictated by the switches of those types of agents. For example, suppose that the population is divided into two types of agents with equal masses; switching rates for agents of either one type are much greater than those for the other type, which are relatively close to zero. Then, the transition of the aggregate strategy is determined mostly from the transition of the latter type’s action distribution, as illustrated in Figure 1. In sum, the transition of the aggregate strategy generally depends not only on the distribution of current actions alone, i.e., the current aggregate strategy, but also on the joint distribution of types and actions, i.e., the strategy composition.

**Aggregable dynamics.** Ely and Sandholm (2005) consider the standard best response dynamic (BRD) in a population of heterogeneous agents. In their heterogeneous standard BRD, every agent switches to the best response action for her own type at the constant and common rate. The heterogeneity in payoff types does not cause any difference in switching rates. Thus, the aggregate strategy moves straight toward the aggregate best response, in which every agent should be taking the best response of the own type given the current payoff vector; everyone switches to it at the same common rate. Since the aggregate strategy is enough to determine the best response of each agent in an aggregate game, the heterogeneous standard BRD is aggregable: that is, the transition of the aggregate strategy is completely determined by the current state of the aggregate strategy alone. The correlation between types and actions does not matter for the aggregate dynamic at all.

The aggregate dynamic—the homogenized smooth BRD—can be constructed by replacing persistent payoff heterogeneity among agents with transitory payoff perturbation of a representative agent. To homogenize the heterogeneity, an agent’s idiosyncratic payoff is supposed to change over time and follow a common i.i.d. process from the same probability distribution as that of permanent idiosyncratic payoffs in the heterogeneous setting. When agents switch to the best response after the realization of idiosyncratic payoffs, the ex-ante switching rate and the aggregate transition follow a smooth BRD; it is just a logit dynamic if the probability distribution of idiosyncratic payoffs is a double exponential distribution.

Aggregability eases the analysis significantly, as it reduces the dimension of the dynamic: while the strategy composition $X$ is a joint distribution over the product space $\mathcal{A} \times \Theta$ of actions and payoff types, the aggregate strategy $\bar{x}$ is just a distribution over action set $\mathcal{A}$. But this also suggests that we cannot employ such an aggregable dynamic to discuss, for example, how the correlation of choices and incentives at the microscopic agent level dynamically affects the macroscopic aggregate state beyond a representative agent approach. Besides, it is natural to expect that a huge disbalance in the incentives to switch actions among different types would cause volatil-

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2In their paper, the heterogeneous standard BRD is called Bayesian BRD, and the homogenized smooth BRD as the aggregate dynamic of the heterogeneous standard BRD is called the aggregate BRD.
A solid purple line indicates the trajectory of aggregate strategy $\bar{x}$, and sequences of red circles and blue crosses show the trajectories of Bayesian strategies $x(\theta^H)$ and $x(\theta^L)$. These trajectories are drawn from agent-based simulations of discrete-time dynamics with 10000 agents of each type for 5000 periods; an agent receives a revision opportunity with probability 0.005 in each period. In each dynamic except the standard BRD, the switching rate (conditional on the receipt of a revision opportunity) is basically set to $1/2$ of the payoff gain from the revision. (Note that the maximal payoff difference is $2 - 0$ and thus the conditional switching rate is at most 1.)

Figure 1: Dynamics of aggregate strategy in a symmetric 3-action coordination game. The population is divided to equal masses of two types, $\theta^H = 0.4$ and $\theta^L = 0.1$. The initial composition is set to $x_0(\theta^H) = (\epsilon, 1 - \epsilon, 0)$ (near $e^B = (0, 1, 0)$, the left bottom corner of a Kolm triangle) and $x_0(\theta^L) = (\epsilon, 0, 1 - \epsilon)$ (near $e^C = (0, 0, 1)$, the right bottom corner), with $\epsilon = 0.01$. In most dynamics except the standard BRD, the aggregate dynamic is more driven by the dynamic of a Bayesian strategy of type $\theta^H$ than that of type $\theta^L$. Type $\theta^H$ indeed has a greater payoff gain from revisions, especially around initial periods, and thus has a greater switching rate except in the standard BRD, in which the switching rate is constant. The asymmetry of trajectories (curved toward $e^C$) suggests the dependency of these dynamics on the initial strategy composition. See Supplementary Note for a more detailed analysis of this example and the trajectories when the initial composition is reversed between type $\theta^H$ and type $\theta^L$.

3Macroeconomists have been studying the relationship between heterogeneity and volatility of aggregate macroeconomy. For example, see Evans, Honkapohja, and Marimon (2001); Honkapohja and Mitra (2006) for theoretical study of heterogeneity in agents’ forecasting models, and Stiglitz (2012) for a broad (autobiographic) survey on macroeconomic volatility and inequality.
switching rate dependent on the amount of the payoff gain from a switch. In Theorem 2, we rigorously prove that, if different types of agents have different switching rates, the transition of the aggregate strategy varies with the current strategy composition, not only with the current aggregate strategy: that is, the dynamic is not aggregable. The nonaggregability condition in this theorem holds for all the above mentioned dynamics because of sensitivity of their switching rates to payoff gains.

**Stationarity and stability of equilibrium compositions.** In contrast to the generic nonaggregability, we verify that stationarity and stability of Nash equilibria extend to those of equilibrium compositions in the heterogeneous setting. These results have been established in the homogeneous setting for quite a wide class of evolutionary dynamics: see Sandholm (2010). For this, the least demanding properties are assumed based on the consistency between payoffs and switches in evolutionary dynamics. First, no agents should switch actions if and only if they are choosing the best response actions to their current payoff vectors (best response stationarity, Definition 3). Secondly, the net change in the mass of each action’s players should always be positively correlated with the relative payoff of the action (positive correlation, Definition 6). We call a dynamic that satisfies these two properties an admissible dynamic. We extend equilibrium stationarity in general and stability in potential games under admissible dynamics to the heterogeneous setting (Theorems 3 and 4).

We obtain these positive results by directly analyzing the dynamic of the strategy composition over heterogeneous types. The dynamic of the strategy composition over types is defined as a differential equation of a joint probability measure over the product space of actions and types. We rigorously define it from the individual agents’ revision protocol and verify the existence of a unique solution path under mild continuity conditions in Theorem 1. For this rigorous formulation of the dynamic over a (possibly) continuous-type space in our model, we basically borrow the formulation and techniques from the recent literature on continuous-strategy evolutionary dynamics, especially Oechssler and Riedel (2001, 2002) and Cheung (2014). However, we cannot simply apply it to our dynamics and need some twists peculiar to evolutionary dynamics on a (possibly) continuous type space; see Remarks 1 and 2 in Section 3.

Note that Milgrom and Weber (1985) consider a strategy composition in a general incomplete information game as a joint probability measure of types and actions, which they call a distributional strategy, and verify fundamental properties of an equilibrium distributional strategy such as existence and purification. The measure-theoretic formulation of heterogeneous dynamics in this paper provides a rigorous foundation for evolutionary dynamics of distributional strategy.

Specifically for potential games, we prove global asymptotic stability of the set of equilibrium compositions in Theorem 4. We further find in Corollary 2 that the equivalence between a local maximum of the potential function and local stability of an equilibrium holds for any admissible heterogeneous dynamic, which includes the standard BRD. Combined with the aggregability re-
sult on the heterogeneous standard BRD by Ely and Sandholm (2005), our local stability theorem suggests that local stability of an equilibrium composition under any heterogeneous admissible dynamic can be tested just by examining local stability of the corresponding aggregate equilibrium under the homogenized smooth BRD.

Yet, the homogenized smooth BRD may fail to predict transition and long-run outcomes from a given initial state, even in a potential game. The fallacy lies in the gap between the topology of strategy compositions and that of aggregate strategies. Local stability under a nonaggregable dynamic may require the initial strategy composition to be close to the equilibrium composition, while local stability under the homogenized smooth BRD requires only the initial aggregate strategy to be close to the aggregate equilibrium.

In Section 5.2, we clarify this gap by comparing Lyapunov functions for the heterogeneous dynamics and for the homogenized smooth BRD, while assuming additive separability of payoff heterogeneity. For this, we rigorously define a completely sorted strategy composition, where the difference in the action distributions over different types is completely explained by the difference in their idiosyncratic payoff vectors. It is an equilibrium composition if and only if the aggregate strategy is in equilibrium. From the comparison of the Lyapunov functions, we find that the homogenized smooth BRD can be interpreted as a heterogeneous dynamic where the strategy composition is kept to be completely sorted at every moment of time. Instantaneous complete sorting is not assured in evolutionary dynamics, where agents respond to payoffs through decentralized, individualistic strategy revisions. Thus, a decentralized gradual sorting process can account for the difference in aggregate behavior between heterogeneous evolutionary dynamics and the homogenized smooth BRD.

**Heterogeneity in revision protocols.** Difference in switching rates among agents may be caused not only by payoff heterogeneity but also directly by heterogeneity in revision protocols. All the results of this paper hold even if we allow different types of agents to follow different revision protocols; just like the base model with a homogeneous revision protocol, each theorem holds as long as the assumptions on the revision protocol are satisfied by each revision protocol adopted by any (positive mass of) agents.

With the other papers by the author of this paper, the extended results in this paper constitutes a series of studies on heterogeneous revision protocols. Zusai (2018a) provides unified proofs of equilibrium stability in contractive games and local stability of a (regular) ESS for non-observational dynamics, while allowing finitely many heterogeneous types to have different payoff functions and revision protocols. In contrast, here our measure-theoretical construction of heterogeneous dynamics allows us to have continuously many types, though we do not cover contractive games.

While these studies prove that asymptotic stability of equilibria retains under heterogeneity, there is a different approach to heterogeneity in revision protocols. Sawa and Zusai (2014) focus on imitative dynamics with finitely many heterogeneous types of agents who differ in aspiration levels (to trigger imitation of an observed action) and prove that the solution trajectory of such a
heterogeneous dynamic asymptotically coincides with the one under the homogeneous dynamic; thus, they share the same long-run outcomes, whether they converge to an equilibrium or behave chaotically. With the fundamental existence theorem (Theorem 1) in this paper, this series of research extends the scope of evolutionary dynamics to heterogeneous populations and clarifies what heterogeneity changes about the long-run behavior of the dynamics and what it does not.

**Implications on empirical and applied study.** In an empirical work on discrete choice, an economist may not have access to micro data and thus may need to use coarse aggregate data. The generic nonaggregability theorem suggests that, even if the economist could precisely identify the underlying payoff structure, the sensible difference between transitory payoff perturbation and persistent payoff heterogeneity results in qualitatively different predictions of an aggregate dynamic. This puts a limitation on the interpretation of heterogeneity accounted for by such empirical studies on aggregate data, as we argue in Section 4.

On the other hand, our positive results on asymptotic equivalence of local stability can relieve the concerns of applied economists, as long as their main focuses are on robustness of an equilibrium composition in a potential game. Corollary 2 suggests that local stability of an aggregate equilibrium in a potential game does not change by the specification of agents’ revision protocols or the distinction between transitory and persistent payoff heterogeneity.

**Outline of the paper.** In the next section, we define the game under payoff heterogeneity, paying attentions to the distinction between the aggregate strategy and the strategy composition; we build a heterogeneous evolutionary dynamic from an individual agent’s revision protocol. In Section 3, we verify the Lipschitz continuity of this dynamic in order to guarantee the existence of a unique solution path. In Section 4, we formally define aggregability of heterogeneous evolutionary dynamics and argue the generic nonaggregability of heterogeneous dynamics. Section 5 is devoted to presenting the positive results on stationarity and stability of equilibrium composition and comparing heterogeneous dynamics and the homogeneous smooth BRD in potential games. Until this section, we consider heterogeneity only in payoff functions and focus on non-observational evolutionary dynamics in which an agent’s switching rate depends only on the payoff vector for the agent but not on the other agents’ actions. In Section 6, we consider heterogeneity in revision protocols and observational dynamics such as imitative dynamics and excess payoff comparison dynamics; we confirm that the theorems in this paper are robust to these extensions. Appendices provide the proofs and a few technical details on the measure-theoretic construction of heterogeneous dynamics. Parts of proofs that essentially involve only heavy calculation are found in the Supplementary Note.
2 The base model

2.1 Aggregate games with payoff heterogeneity

Consider a large population of agents \( \Omega := [0, 1] \subset \mathbb{R} \) who share the same action set \( \mathcal{A} = \{1, \ldots, A\} \). We define probability measure \( \mathbb{P}_\Omega : \mathcal{B}_\Omega \to [0,1] \) as the Lebesgue measure so \( \mathbb{P}_\Omega(\Omega) = 1 \). Denote by \( \mathcal{B}_\Omega \) the Lebesgue \( \sigma \)-field over \( \Omega \). Let \( a(\omega) \) be the action taken by agent \( \omega \in \Omega \). We restrict action profile \( a : \Omega \to \mathcal{A} \) to a \( \mathcal{B}_\Omega \)-measurable function. Then, \( \bar{x}_a := \mathbb{P}_\Omega(\{\omega \in \Omega : a(\omega) = a\}) \in [0, 1] \) is the mass of agents who take action \( a \in \mathcal{A} \). We call \( \bar{x} := (\bar{x}_a)_{a \in \mathcal{A}} \in \Delta^A \) the aggregate strategy, where \( \Delta^A := \{ z \in \mathbb{R}^A : \sum_{a \in \mathcal{A}} z_a = 1 \} \) is the set of \( A \)-dimensional probability vectors.\(^5\)

While we will introduce heterogeneity in revision protocols in Section 6, here focus on payoff heterogeneity in an aggregate game, as follows. Each agent \( \omega \in \Omega \) is assigned to type \( \theta(\omega) \in \mathbb{R}^T \). Then, the agent’s payoff from action \( a \) is \( F_a(\bar{x})(\theta(\omega)) \) when the aggregate state is \( \bar{x} \). Thus, \( F(\bar{x})(\theta) = (F_a(\bar{x})(\theta(\omega)))_{a \in \mathcal{A}} \in \mathbb{R}^A \) is the payoff vector at aggregate state \( \bar{x} \) for type \( \theta \). Let \( | \cdot |^T \) be the sup norm on \( \mathbb{R}^T \) and \( \mathcal{B}_0 \) be the Borel \( \sigma \)-field on this metric space \( \mathbb{R}^T \).\(^6\) Agents’ type profile \( \theta : \Omega \to \mathbb{R}^T \) is assumed to be measurable with respect to \( \mathcal{B}_\Omega \). Then, it induces probability measure \( \mathbb{P}_\theta : \mathcal{B}_\theta \to [0,1] \) by \( \mathbb{P}_\theta(B_\theta) := \mathbb{P}_\Omega(\{\omega \in \Omega : \theta(\omega) \in B_\theta\}) \) for each \( B_\theta \in \mathcal{B}_\theta \). Denote by \( \Theta \subset \mathbb{R}^T \) the support of \( \mathbb{P}_\theta \); we call it the type space. Given \( \bar{x}, F(\bar{x}) : \Theta \to \mathbb{R}^A \) is assumed to belong to \( \mathcal{C}_\Theta \), the set of continuous functions from \( \Theta \) to \( \mathbb{R}^A \).

In contrast to the aggregate strategy, we define a Bayesian strategy \( x = (x_a)_{a \in \mathcal{A}} : \Theta \to \Delta^A \) to represent a strategy composition of actions and types, following the terminology of Ely and Sandholm (2005).\(^7\) More specifically, \( x_a(\theta) \) is the population share of action-\( a \) players in the subpopulation of type-\( \theta \) agents. For example, our formulation allows \( \mathbb{P}_\theta \) to have a finite support; then, the Bayesian strategy is obtained as

\[
x_a(\theta) = \frac{\mathbb{P}_\Omega(\{\omega \in \Omega : a(\omega) = a \text{ and } \theta(\omega) = \theta\})}{\mathbb{P}_\theta(\theta)}
\]

for each type in the support of \( \mathbb{P}_\theta \).

In general, we first define a strategy composition \( X = (X_a)_{a \in \mathcal{A}} : \mathcal{B}_\Theta \to \Delta^A \) as a joint distribution of actions and types such that the marginal distribution of types coincides with \( \mathbb{P}_\Theta \). For each action \( a \in \mathcal{A} \) and each Borel set \( B_\Theta \subset \mathcal{B}_\Theta \) of types, \( X_a(B_\Theta) \) is a mass of action-\( a \) players whose types belong to \( B_\Theta \). Since the mass of such types of agents is \( \mathbb{P}_\Theta(B_\Theta) \), \( X_a \) must satisfy \( X_a(B_\Theta) \leq \mathbb{P}_\Theta(B_\Theta) \) for each \( a \in \mathcal{A} \), which implies the absolute continuity of each \( X_a \) with respect to \( \mathbb{P}_\Theta \); see (A.1) in Appendix A.1. We denote this relationship of the absolute continuity by \( \mathbb{P}_\Theta \gg X \). By Radon-Nikodym theorem, the absolute continuity guarantees the existence of a unique density function

\(^5\)Let \( R_{+} = [0, +\infty) \) and \( R_{++} = (0, +\infty) \).

\(^6\)For vector \( \theta = (\theta_1, \ldots, \theta_T) \in \mathbb{R}^T \), the sup norm of the vector is \( |\theta|^T = \max\{|\theta_1|, \ldots, |\theta_T|\} \). (We omit a super/subscript when it is obvious.)

\(^7\)When calling \( x : \Theta \to \Delta^A \) a Bayesian strategy, we would imagine a Bayesian game where a player chooses a strategy (a contingent action plan) before she knows her own type. In a Bayesian game, we distinguish a ‘player’ and an ‘agent.’ A player comes to the game before knowing her type, and decides on a plan of the action contingent on the realized type: a Bayesian strategy is this contingent plan of one player. In a Bayesian population game, an agent comes to the game after knowing her type and decides on an action; the Bayesian strategy is essentially equivalent to an empirical joint distribution of type and actions.
An aggregate strategy is identified from a Bayesian strategy as

$$\bar{x} = \mathbb{E}_\Theta x_a$$

i.e., \( \bar{x} = \mathbb{E}_\Theta x \).

We consider Bayesian strategy \( x \) in the main body of this paper since it is the fundamental element to define the dynamic (see Remark 1 in Section 3) and also appealing to intuition; however, we prove most theorems by arguing strategy composition \( X \) to apply mathematical theorems (Theorems 8 and 9) on a dynamic of a probability measure. See Appendix A.1 for the measure-theoretic formulation of the model and dynamic.

Example 1 (Additively separable aggregate game (ASAG)). In the context of discrete choice models such as in Anderson, De Palma, and Thisse (1992), it is common to introduce payoff heterogeneity in an additively separable manner. That is, the payoff function is additively separated to the common part and the idiosyncratic part: with \( T = A \), type \( \theta = (\theta_a)_{a \in A} \in \mathbb{R}^A \) is defined as the idiosyncratic payoff vector for this type, which varies among agents but does not change over time regardless of the state of the population. Given aggregate state \( \bar{x} \), \( F^0(\bar{x}) = (F^0_a(\bar{x}))_{a \in A} \in \mathbb{R}^A \) is the common payoff vector, shared by all the agents in \( \Omega \). Thus, for each \( \tilde{x} \in \Delta^A \) and \( \theta \in \mathbb{R}^A \),

$$F[\tilde{x}](\theta) := F^0(\tilde{x}) + \theta, \quad \forall \theta \in \mathbb{R}^A \quad (1)$$

is the payoff vector for type \( \theta \) when the aggregate strategy is \( \bar{x} \). An agent of this type receives payoff \( F_a[\tilde{x}](\theta) = F^0_a(\tilde{x}) + \theta_a \) by taking action \( a \). We call an aggregate game with such additively separable idiosyncratic payoffs an additively separable aggregate game (ASAG).

Example 2 (Binary aggregate game). When arguing an aggregate game with two actions, we commonly denote the action set by \( A = \{I, O\} \). We can imagine an entry game in which \( I \) means participation (IN) in a certain platform and \( O \) means nonparticipation (OUT). We call such a game a binary aggregate game.\(^10\) If we further assume additive separability of payoff heterogeneity, it reduces without loss of generality to the payoff function defined by \( F^0_a(\tilde{x}) = 0 \) for each \( \tilde{x} \in \Delta^2 \) and \( \theta_1(\omega) \equiv 0 \) for each \( \omega \in \Omega \). Now an agent’s type \( \theta \) is identified by \( \theta_O \in \mathbb{R} \) alone. We can interpret \( \theta_O \) as the agent’s valuation of an outside option. Let \( P_\theta : \mathbb{R} \to [0, 1] \) be the cumulative distribution function (c.d.f.) of \( \theta_O \) and \( \Theta_O \subset \mathbb{R} \) be the support of \( \theta_O \); thus, \( \Theta = \{0\} \times \Theta_O \). Then, \( \theta \in \Theta \), \( \tilde{x} \in \Delta^2 \) and \( x : \Theta \to \Delta^2 \) are identified from \( \theta_O \in \Theta_O, x_I \in [0, 1] \) and \( x_I : \Theta_O \to [0, 1] \), respectively. We call such a binary aggregate game a binary ASAG.

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\(^8\)Two Bayesian strategies \( x, x' \in \mathcal{F}_X \) are considered identical, i.e., \( x = x' \) if \( x(\theta) = x'(\theta) \) for \( \mathbb{P}_\Theta \)-almost all \( \theta \in \Theta \). They indeed yield the same strategy composition.

\(^9\)Here \( \mathbb{E}_\Theta \) is the expectation operator on the probability space \( (\Theta, \mathcal{B}_\Theta, \mathbb{P}_\Theta) \), while \( \mathbb{E}_\Theta f \) is that on \( (\Omega, \mathcal{B}_\Theta, \mathbb{P}_\Theta) \); i.e., \( \mathbb{E}_\Theta f := \int f(\omega) \mathbb{P}_\Theta(\omega) \) for a \( \mathcal{B}_\Theta \)-measurable function \( f : \Omega \to \mathbb{R} \) and \( \mathbb{E}_\Theta f := \int f(\omega) \mathbb{P}_\Theta(\omega) \) for a \( \mathcal{B}_\Theta \)-measurable function \( f : \Theta \to \mathbb{R} \). If \( f = f \circ \theta \), then we have \( \mathbb{E}_\Theta f = \mathbb{E}_\Theta f \).

\(^10\)Blonski (1999) studies aggregate dynamics in a binary aggregate game in which \( \tilde{x} \) is assumed to be determined by the difference between the aggregate mass of agents for whom \( I \) is the best response to the current aggregate strategy and that of agents for whom \( O \) is the best response.
2.2 Bayesian equilibrium and aggregate equilibrium

In an aggregate game, each agent’s best response is determined from an aggregate strategy. Let \( b[\bar{x}](\theta) \subset \mathcal{A} \) be the set of type-\( \theta \)'s best response actions to aggregate strategy \( \bar{x} \) and \( F_\ast[\bar{x}](\theta) \in \mathbb{R} \) be the payoff from a best response action.

\[
 b[\bar{x}](\theta) := \arg \max_{a \in \mathcal{A}} F_a[\bar{x}](\theta), \quad F_\ast[\bar{x}](\theta) := \max_{a \in \mathcal{A}} F_a[\bar{x}](\theta).
\]

Given aggregate strategy \( \bar{x} \), \( b_\ast^{-1}[\bar{x}] := \{ \theta \in \Theta : a \in b[\bar{x}](\theta) \} \) is the set of types for which action \( a \) is a best response. Denote by \( \beta_a^{-1}[\bar{x}] := \{ \theta \in \Theta : \{ a \} = b[\bar{x}](\theta) \} \subset b_\ast^{-1}[\bar{x}] \) the set of types for which action \( a \) is the unique best response to \( \bar{x} \). Let \( B[\bar{x}](\theta) \) be the set of action distributions that assign positive probabilities only to the best response actions for type \( \theta \) given aggregate strategy \( \bar{x} \), i.e., the set of type-\( \theta \) agents’ best response mixed strategies to \( \bar{x} \):

\[
 B[\bar{x}](\theta) := \{ y \in \Delta^\mathcal{A} : y_a > 0 \Rightarrow a \in b[\bar{x}](\theta) \} \quad \text{for each } \bar{x} \in \Delta^\mathcal{A}, \theta \in \Theta.
\]

In a Nash equilibrium, (almost) every agent correctly predicts the strategy composition and takes the best response to it. Correspondingly, Bayesian strategy \( \bar{x} \in \mathcal{F}_\mathcal{X} \) is a Bayesian equilibrium, if

\[
 \bar{x}(\theta) \in B[\bar{x}](\theta) \quad \text{with} \quad \bar{x} = \mathbb{E}_\Theta \bar{x} \quad \text{for } \mathbb{P}_\Theta\text{-almost all } \theta \in \Theta, \quad (2)
\]

or equivalently,

\[
 x_a(\theta) = \begin{cases} \ 1 & \text{if } \theta \in \beta_a^{-1}[\bar{x}] \\ \ 0 & \text{if } \theta \notin \beta_a^{-1}[\bar{x}] \end{cases} \quad \text{with} \quad \bar{x} = \mathbb{E}_\Theta \bar{x} \quad \text{for all } a \in \mathcal{A} \text{ and } \mathbb{P}_\Theta\text{-almost all } \theta \in \Theta. \quad (2')
\]

That is, if \( a \) is the unique best response for type \( \theta \), (almost) all the agents of this type should take it; if \( a \) is not a best response, (almost) none of these agents should take it. We leave indeterminacy of \( x_a(\theta) \) in a Bayesian equilibrium when there are multiple best response actions for \( \theta \) and \( a \) is just one of them.

Aggregation of \( x_a(\theta) \) in a Bayesian equilibrium over all types \( \theta \in \Theta \) yields

\[
 \mathbb{P}_\Theta(\beta_a^{-1}[\bar{x}]) \leq \bar{x}_a \leq \mathbb{P}_\Theta(b_a^{-1}[\bar{x}]) \quad \text{for all } a \in \mathcal{A}. \quad (3)
\]

If the type distribution is continuous, then this condition reduces to \( \bar{x}_a = \mathbb{P}_\Theta(\beta_a^{-1}[\bar{x}]) = \mathbb{P}_\Theta(b_a^{-1}[\bar{x}]) \).

If aggregate strategy \( \bar{x} \) satisfies condition (3), it is called an aggregate equilibrium. Notice that an aggregate equilibrium does not assure that the underlying Bayesian strategy is a Bayesian equilibrium. A Bayesian equilibrium needs complete sorting of agents by types as seen in (2'). On the other hand, only the total mass of each action’s players matters to an aggregate equilibrium.

2.3 Construction of heterogeneous dynamics

In an evolutionary dynamic, an agent occasionally changes the action, following a Poisson process. The timing of a switch and the choice of which action to switch to are determined by revision protocol \( \rho = (\rho_{ij})_{i,j \in \mathcal{A}} : \mathcal{R}^\mathcal{A} \rightarrow \mathcal{R}_{+}^{\mathcal{A} \times \mathcal{A}} \), a collection of switching rate functions \( \rho_{ij} : \mathcal{R}^\mathcal{A} \rightarrow \mathcal{R}_+ \) over all the pairs \((i, j) \in \mathcal{A} \times \mathcal{A}\) of two actions. An economic agent should base the switching decision
on the payoff vector that the agent is facing. Let \( \pi \in \mathbb{R}^A \) be the payoff vector for the agent. The switching rate \( \rho_{ij}(\pi) \in \mathbb{R}_+ \) is a Poisson arrival rate at which this agent switches to action \( j \in A \) conditional on that the agent has been taking action \( i \in A \) so far and currently faces payoff vector \( \pi \). The analysis in this paper is applicable to observational dynamics, in which the switching rates also depend on the action distribution over others agents. In addition, all our theorems hold even when different types of agents can follow different revision protocols. We confirm applicability to these extensions in Section 6, while we focus on heterogeneity only in payoff functions and thus assume that all the types of agents share the same revision protocol \( \rho \) until that section.

From the common revision protocol \( \rho : \mathbb{R}^A \to \mathbb{R}_+^{A\times A} \) and payoff profile \( \pi : \Theta \to \mathbb{R}^A \) that specifies the payoff vector \( \pi(\theta) \) of each type \( \theta \), we construct the mean dynamic of Bayesian strategy \( x \) over \( \mathcal{F}_X \) with function \( v = (v_i)_{i \in A} : \mathbb{R}^A \times \Delta^A \to \mathbb{R}^A \) as

\[
\dot{x}_i(\theta) = v_i(\pi(\theta), x(\theta)) := \sum_{j \in A} x_j(\theta) \rho_{ji}(\pi(\theta)) - x_i(\theta) \sum_{j \in A} \rho_{ij}(\pi(\theta)),
\]

i.e.,

\[
\dot{x}(\theta) = v(\pi(\theta), x(\theta))
\]  

(4)

for each type \( \theta \in \Theta \) and each action \( i \in A \). In an infinitesimal length of time \( dt \in \mathbb{R} \), \( \sum_{j \in A} x_j(\theta) \rho_{ji}(\pi(\theta)) dt \) is approximately the mass of type-\( \theta \) agents who switch to action \( i \) from other actions \( j \in A \), namely, the gross inflow to \( x_i(\theta) \); similarly, \( x_i(\theta) \sum_{j \in A} \rho_{ij}(\pi(\theta)) dt \) is the gross outflow from \( x_i(\theta) \). Thus, \( v_i(\pi(\theta), x(\theta)) dt \) is the net flow to \( x_i(\theta) \) in this period of time \( dt \).

Coupled with a heterogeneous population game \( F \), the mean dynamic (4) of Bayesian strategy defines the heterogeneous Bayesian dynamic \( v^F \) over \( \mathcal{F}_X \) by

\[
\dot{x}(\theta) = v^F[x](\theta) := v[F[x](\theta), x(\theta)] \in \mathbb{R}^A
\]

with \( \bar{x} = \mathbb{E}_\Theta x \)

for each type \( \theta \in \Theta \).

### Examples of evolutionary dynamics

To make a concrete image of revision protocols, here we review major evolutionary dynamics. In particular, we separate the dynamics based on optimization from others because they need different regularity conditions to guarantee the existence of a unique solution trajectory.

#### Continuous revision protocols

Under a continuous revision protocol \( \rho \), the switching rate function \( \rho_{ij} \) continuously changes with the payoff vector.

**Definition 1** (Continuous revision protocols). In a continuous revision protocol \( \rho \), the switching rate function \( \rho_{ij} : \mathbb{R}^A \to \mathbb{R}_+ \) of each pair of actions \( i, j \in A \) is Lipschitz continuous.

**Example 3.** In a class of pairwise comparison dynamics, the switching rate \( \rho_{ij}(\pi) \) increases with the payoff difference \( \pi_j - \pi_i \). In particular, the revision protocol \( \rho_{ij}(\pi) = [\pi_j - \pi_i]_+ \) defines the

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11 Readers who are familiar with major evolutionary dynamics may just scan this subsection quickly and jump to Definitions 1 and 2. Yet, it is recommended to check the homogenized smooth BRD, since it will play a major role in this paper.
Smith dynamic (Smith, 1984).  

Example 4. Because of continuity, we see smooth best response dynamics (Fudenberg and Kreps, 1993) as constructed from continuous revision protocols. For example, the logit dynamic (Fudenberg and Levine, 1998) is constructed from \( \rho_{ij}(\pi) = \exp(\mu^{-1}\pi_i) / \sum_{a \in A} \exp(\mu^{-1}\pi_a) \) with noise level \( \mu > 0 \). This revision protocol can be obtained from perturbed optimization: upon the receipt of each revision opportunity, an agent draws each random perturbation in each action \( \varepsilon_a \) from a double exponential distribution\(^{13}\) and then switches to the action that maximizes \( \pi_a + \varepsilon_a \) among all actions \( a \in A \). In general, a smooth best response dynamic can be constructed from such perturbed optimization under some admissibility condition: see Hofbauer and Sandholm (2002, 2007). Note that, after the receipt of a revision opportunity and a draw of \( \varepsilon \in \mathbb{R}^A \), an agent always switches to the optimal action, however small the payoff gain by this switch is.

Note that payoff perturbation \( \varepsilon = (\varepsilon_a)_{a \in A} \) is transient: a different value of \( \varepsilon \) will be drawn at each revision opportunity from an i.i.d. distribution. So, there is no (ex ante) heterogeneity in \( \varepsilon \).

In contrast, the idiosyncratic payoff type \( \theta \) in our heterogeneous dynamics is persistent.

To make a comparison with heterogeneous dynamics, we can consider the homogenized smooth BRD with type distribution \( \mathbb{P}_\Theta \), following the idea of Ely and Sandholm (2005): upon the receipt of a revision opportunity, an agent draws a transient payoff type \( \varepsilon \in \mathbb{R}^T \) from the same distribution as \( \mathbb{P}_\Theta \).\(^{14}\) Then, the agent switches to the action that maximizes \( F_\varepsilon[\bar{x}](\varepsilon) \). Action \( a \) is the unique best response after drawing \( \varepsilon \) if \( \varepsilon \in \beta_a^{-1}[\bar{x}] \). This happens with probability \( \mathbb{P}_\Theta(\beta_a^{-1}[\bar{x}]) \). Therefore, the homogenized smooth BRD is obtained as\(^{15}\)

\[
\dot{x}_a = \mathbb{P}_\Theta(\beta_a^{-1}[\bar{x}]) - \bar{x}_a \quad \text{for each } a \in A.
\]

Note that, in a binary ASAG, this reduces to

\[
\dot{x}_i = P_\Theta(F_0^i(\bar{x}_i)) - \bar{x}_i,
\]

since \( \beta_i^{-1}[\bar{x}] = \{ \theta : \theta_O < F_0^i(\bar{x}_i) \} \). The sign of \( \dot{x}_i \) is thus identical with that of \( F_0^i(\bar{x}_i) - P_\Theta(\bar{x}_i) \).

Note that an idiosyncratic payoff type \( \varepsilon \) in the homogenized smooth BRD is newly drawn from \( \mathbb{P}_\Theta \) at each revision opportunity, while in idiosyncratic payoff vector \( \theta \) in a heterogeneous dynamic is drawn only once when an agent is born and the agent keeps it forever.

Exact optimization protocols. In an exact optimization protocol, an agent switches only to the best response given the current payoff vector: if action \( j \) does not yield the maximal payoff among \( \pi = (\pi_1, \ldots, \pi_A) \), then \( \rho_{ij}(\pi) = 0 \) regardless of the agent’s current action \( i \). We allow the switching rate to an optimal action to vary with \( \pi \) and \( i, j \in A \). Denote by \( Q_{ij}(\pi) \) the conditional switching rate from \( i \) to \( j \), provided that \( j \) is already designated as the new action. In the definition below, we extend the domain of \( Q_{ij} \) to \( \mathbb{R}^A \) while assuming its continuity over the whole domain. The

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\(^{12}\)\([\cdot]_+ \) is an operator to truncate the negative part of a number: i.e., \( [\pi]_+ = \pi \) if \( \pi \geq 0 \) and 0 otherwise.

\(^{13}\)Given the noise level \( \mu \), the cumulative distribution function of the double exponential distribution is \( \mathbb{P}(\varepsilon_a \leq c) = \exp(\gamma - \exp(-\mu^{-1}(c - \gamma)) \) where \( \gamma \approx 0.5772 \) is Euler’s constant.

\(^{14}\)The dimension of this \( \varepsilon \) does not have to be \( A \).

\(^{15}\)Continuity of the distribution \( \mathbb{P}_\Theta \) is assumed here to eliminate indeterminacy of the aggregate best response.
actual switching rate $\rho_{ij}$ is defined as the truncation of $Q_{ij}$ when $j$ is not a best response; the truncation causes discontinuity.

**Definition 2 (Exact optimization protocols).** In an exact optimization protocol, the switching rate function $\rho_{ij}: \mathbb{R}^A \to \mathbb{R}_+$ of each pair of actions $i, j \in A$ is expressed as

$$\rho_{ij}(\pi) = \begin{cases} 0 & \text{if } j \notin \text{argmax}_{a \in A} \pi_a, \\ Q_{ij}(\pi) & \text{if } \{j\} = \text{argmax}_{a \in A} \pi_a, \end{cases}$$

with a Lipschitz continuous function $Q_{ij}: \mathbb{R}^A \to \mathbb{R}_+$.

**Example 5.** In the standard BRD (Hofbauer, 1995b; Gilboa and Matsui, 1991), a revising agent switches to the optimal action that maximizes the current payoff with sure, however small the payoff gain by this optimization is. That is, the standard BRD is constructed from an exact optimization dynamic with $Q_{ij} \equiv 1$. The heterogeneous version is considered in Ely and Sandholm (2005); they prove that the aggregate strategy in the heterogeneous standard BRD follows the homogenized smooth BRD.

**Example 6.** Consider a version of BRD in which the switching rate to the unique best response $Q_{ij}$ depends on the payoff difference (the payoff deficit) between the current strategy $i$ and the best response $j$, i.e., $Q_{ij}(\pi) = Q(\pi_j - \pi_i)$ whenever $j \in \text{argmax}_{a \in A} \pi_a$. Function $Q: \mathbb{R}_+ \to [0, 1]$ is called a tempering function and assumed to be continuously differentiable and satisfy $Q(0) = 0$ and $Q(q) > 0$ whenever $q > 0$. Then this revision protocol yields the tempered BRD; Zusai (2018b) constructs this revision protocol from optimization with a stochastic switching cost whose cumulative distribution function is $Q$. Given payoff function $F$ in an aggregate game, we denote the payoff deficit of action $i$ for type $\theta$ at aggregate strategy $\bar{x}$ by $\hat{F}_i[\bar{x}](\theta) := F_{\bar{x}}[\bar{x}](\theta) - F_{\bar{x}}[\bar{x}](\theta)$.

Note that, if there are only two actions, a continuous revision protocol such as a pairwise comparison protocol reduces to an exact optimization protocol as long as an agent never switches to a worse action than the current action, i.e., $\rho_{ij}(\pi) = 0$ whenever $\pi_i > \pi_j$.

### 3 Existence of a unique solution trajectory

We verify Lipschitz continuity of a heterogeneous dynamic to guarantee the existence of a unique solution trajectory from an arbitrary initial strategy composition. For this goal, we impose regularity assumptions. First of all, to argue continuity on a Banach space, we extend the domain of $F(\theta)$ to $\mathbb{R}^A$: given $\theta$, $F(\theta)$ is a function that maps aggregate strategy $\bar{x} \in \mathbb{R}^A$ to payoff vector $F[\bar{x}](\theta) \in \mathbb{R}^A$. We assume that payoff function $F(\theta)$ is Lipschitz continuous. This assumption is satisfied in an ASAG, as long as the common payoff function $F^0: \mathbb{R}^A \to \mathbb{R}^A$ is Lipschitz continuous.

**Assumption 1 (Lipschitz continuity of the payoff function).** For $\mathbb{P}_\Theta$-almost every type $\theta \in \Theta$, $F(\theta): \mathbb{R}^A \to \mathbb{R}^A$ is Lipschitz continuous with Lipschitz constant $L_F(\theta)$. In addition, $L_F := \mathbb{E}_\Theta L_F < \infty$.  

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To ensure the existence of a unique solution trajectory, we assume that the switching rate is bounded. The assumption is satisfied in an ASAG, if the type distribution $P_\Theta$ has a bounded support and the common payoff function $F^0$ is continuous, even if the switching rate function itself is not bounded over the whole domain $\mathbb{R}^A$ like the Smith dynamic.

**Assumption 2** (Bounded switching rates). There exists $\bar{\rho} \in \mathbb{R}_+$ such that
\[
\rho_{ij}(F[m]|(\theta)) \leq \bar{\rho}
\]
for the density $m$ of any finite signed measure, any $i, j \in A$ and, $P_\Theta$-almost all $\theta \in \Theta$.

For an exact optimization protocol, the Lipschitz continuity of $Q_{ij}$ is not sufficient to guarantee Lipschitz continuity of revision protocol $\rho$ due to truncation when the best response action changes. The continuity of $Q_{ij}$ assures continuous change in switching rate $\rho_{ij}$ with the payoff vector, when action $j$ remains to be the unique best response. However, payoff changes may cause changes in the best responses, which triggers discontinuous changes in the switching rates: the switching rate $\rho_{ij}$ to the new best response changes from zero to some positive rate $Q_{ij}$ and the switching rate to the old one changes from positive to zero. The next assumption states that the mass of agents who experience switches of best responses due to payoff changes grows only continuously with the size of the payoff changes; thus, despite discontinuous changes in individual agents’ switching rates, the sum of these changes over all the agents becomes continuous.

**Assumption 3** (Continuous change in best response). If revision protocol $\rho : \mathbb{R}^A \rightarrow \mathbb{R}_+^{A \times A}$ is an exact optimization protocol, then there exists $L_\beta \in \mathbb{R}_+$ such that
\[
P_\Theta(\{\theta \in \Theta : c \leq \theta_b - \theta_a \leq d\}) \leq (d - c)\bar{\rho}_\Theta
\]
for any two distinct actions $b, c \in A$ such that $b \neq c$ and any two distinct aggregate states $\bar{x}, \bar{x}' \in \Delta^A$ such that $\bar{x}' \neq \bar{x}$.

Note that this assumption imposes the condition on the type distribution only if the revision protocol is an exact optimization protocol; continuous revision protocols do not need any such assumption on the type distribution for the existence of a unique solution trajectory.

In an ASAG, Assumption 3 is satisfied if the distribution of differences in idiosyncratic payoffs between every two actions satisfies a Lipschitz-like continuity in the sense that there exists $\rho_\Theta \in \mathbb{R}$ such that $P_\Theta(\{\theta \in \Theta : c \leq \theta_b - \theta_a \leq d\}) \leq (d - c)\rho_\Theta$ for any $a, b \in A$ and any $c, d \in \mathbb{R}$ such that $d > c$. This is verified at the end of the proof of Theorem 1 in Supplementary Note S2.2.

**Theorem 1** ( Existence of a unique solution trajectory under Bayesian dynamic). Consider a heterogeneous Bayesian dynamic $v^F$ in a population game $F$ under a continuous revision protocol or an exact optimization protocol. Under Assumptions 1 to 3, the following holds.

i) Function $v^F$ over $F_X$ is Lipschitz continuous in $L^1$-norm over $F_X$.

ii) There exists a unique solution trajectory $\{x_t\}_{t \in \mathbb{R}_+} \subset F_X$ of $\dot{x}_t = v^F[x]$ from any initial Bayesian strategy $x_0 \in F_X$. 

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Appendix A explains the whole outline of the proof as well as the measure-theoretic construction of heterogeneous dynamics, while heavy calculation in the proof is postponed to Section S2 of Supplementary Note. The basic idea of the proof follows a conventional proof for continuous-strategy dynamics such as in Oechssler and Riedel (2001) and Cheung (2014). Both ours and theirs deal with the dynamic of probability measure on a (possibly) continuous space. We use a standard mathematical theorem (Theorem 8) to obtain the existence of a unique solution trajectory from Lipschitz continuity of the dynamic. To apply this theorem, the domain of the dynamic must be a Banach (complete normed vector) space. Since the set of strategy compositions is not a vector space, we extend the domain to a space of finite signed measures $\mathcal{M}_{A \times \Theta}$ and then prove the Lipschitz continuity (A.5) of the dynamic on this extended domain. However, there are two major differences in the proof for our continuous-type dynamics, compared to that for their continuous-strategy dynamics, as discussed in the remarks below.

Remark 1. One of the differences comes from the essential defining nature of heterogeneous dynamics that each agent is born with a certain type $\theta$ and posses it persistently. Technically, it implies $P_\Theta \gg X$, i.e., the absolute continuity of $X$ with respect to $P_\Theta$. This enables us to obtain a Bayesian strategy $x$ as a density of $X$ w.r.t. $P_\Theta$ and to interpret $x_a(\theta)$ as a proportion of action-$a$ players among type-$\theta$ agents. Since an agent’s strategy revision crucially depends on the own type, it is natural to construct the dynamic $v$ of Bayesian strategy $x(\theta)$ at each $\theta$, as in (4); the dynamic $V$ of strategy composition $X$ is just derived from $v$, as in (A.4) in Appendix A.3.

When extending the domain of $V$ to $\mathcal{M}_{A \times \Theta}$, a finite signed measure may not be absolute continuous with respect to $P_\Theta$ and thus may not have a density. Then, how can we extend the density-based definition (4) of our dynamic? For this, we extract the absolute continuous part of a measure by Lebesgue decomposition theorem (Lemma 2) to keep the density-based definition of the dynamic.

On the other hand, in continuous-strategy evolutionary dynamics, an agent is assumed to be homogeneous and thus has no persistent characteristic. When they need some distribution that dominates the strategy distribution to obtain absolute continuity, they can create some ad hoc distribution artificially from the strategy distribution.\footnote{For example, see Oechssler and Riedel (2001, p.159) and Cheung (2014, p.2 in Online Appendix).} A continuous-strategy evolutionary dynamic typically defines the transition of the measure (the mass of players in a Borel set of strategies) directly; they obtain a density only to prove Lipschitz continuity. Thus, the issue of a dominating distribution for absolute continuity is only a technical issue for the study of continuous-strategy evolutionary dynamics, not an essential component of games or dynamics.

Remark 2. Another difference is that we cover exact optimization dynamics such as the standard and tempered BRDs, whose revision protocols $\rho$ are discontinuous. As far as the author is aware of, the studies on continuous-strategy evolutionary dynamics focus on continuous revision protocols: imitative dynamics (Oechssler and Riedel, 2001; Cheung, 2016), the BNN dynamic (Hofbauer, Oechssler, and Riedel, 2009), the gradient dynamic (Friedman and Ostrov, 2013), payoff comparison dynamics (Cheung, 2014) and the logit dynamic (Lahkar and Riedel, 2015).
In our heterogeneous exact optimization dynamics, we suppress discontinuity in switching rates by continuity in the mass of types of agents who experience discontinuous changes in switching rates, as assumed in Assumption 3. This continuity of the type distribution mitigates discontinuity in switching rates and helps to retain continuity of the dynamic, thanks to $\mathbb{P}_\Theta \gg X$.

For continuous-strategy dynamics, one might assume continuity of the ad hoc dominating distribution, argued in the above remark. But it restricts the strategy distribution to being continuous on the strategy space. However, since agents are homogeneous in these dynamics, it would be commonly expected that every agent eventually takes the same strategy, when the game has a unique pure-strategy Nash equilibrium. For this, the assumption of continuity in the strategy distribution is too demanding.

4 Nonaggregability of heterogeneous dynamics

4.1 Generic nonaggregability

Given population game $F$, Bayesian dynamic $\dot{x} = v^F[x]$ is defined over the Bayesian strategy space $F_X$: to predict the transition $\dot{x}$ of the Bayesian strategy, we need to identify the current Bayesian strategy or equivalently the current strategy composition over $A \times \Theta$. Then, the transition $\dot{x}$ of the aggregate strategy is obtained from aggregation of $\dot{x}(\theta)$ over all the types $\theta \in \Theta$: that is, $\dot{x}$ is obtained as

$$\dot{x} = E_\Theta x = E_\Theta v^F[x].$$

As noted in the introduction, the preceding literature on heterogeneous evolutionary dynamics focused on aggregable dynamics, in which the aggregate transition $\dot{x}$ can be identified by the current aggregate strategy $\dot{x}$ alone. More specifically, we say that Bayesian dynamic $v^F$ over $F_X$ is aggregable if there is an aggregate dynamic $\dot{v}^F : \Delta^A \to \mathbb{R}^A$ such that $\{E_\Theta x_t\}_{t \in \mathbb{R}_+} \subset \Delta^A$ is a solution trajectory of $\dot{v}^F$ whenever $\{x_t\}_{t \in \mathbb{R}_+} \subset F_X$ is a solution trajectory of $v^F$: that is,

$$[\dot{x}_t = E_\Theta x_t \quad \text{and} \quad x_t = v^F[x_t]] \quad \implies \quad \dot{x}_t = \dot{v}^F(x_t)$$

for all time $t \in \mathbb{R}_+$ and from any initial Bayesian strategy $x \in F_X$. As proven by Ely and Sandholm (2005, Theorem 5.4), this aggregability condition is equivalent to the interchangeability of aggregation and dynamic, or more specifically,

$$E_\Theta v^F[x] = v^F(E_\Theta x) \quad \text{for any } x \in F_X.$$

They further verify that the standard BRD is aggregable and the aggregate strategy under the heterogeneous standard BRD follows the homogenized smooth BRD. We can readily confirm that the smooth BRDs are also aggregable; a heterogeneous smooth BRD aggregates to a homogeneous smooth BRD in which both $\theta$ and $\epsilon$ are drawn independently at each revision opportunity.17

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17 Both these heterogeneous dynamics can be expressed as $x(\theta) \in T[x(\theta)] - x(\theta)$, where $T[x(\theta)]$ is the set of the possible distributions of new actions for type-$\theta$ revising agents: $T[x(\theta)] = B[x(\theta)]$ for the standard BRD and $T[x(\theta)] = \{y(\theta) : y_\theta(\theta) = P_\epsilon(b \in \arg\max_{a \in A} r_\theta[a(x) + \epsilon_a]) \text{ for each } b \in A\}$ for a smooth BRD with the distribution $P_\epsilon$ of the transitory idiosyncratic payoff vector. Then, the aggregation yields $\dot{x} = E_\Theta T[x] - \dot{x}$. (Note that the Bayesian strategy...
However, aggregability is indeed quite demanding for other evolutionary dynamics. The above condition requires the transition vector of an aggregate dynamic to vary only with the aggregate strategy, independently of the underlying strategy composition. An evolutionary dynamic is not aggregable when the difference in the payoff vector over different types causes different switching rates. Nonaggregability is held commonly in major evolutionary dynamics, since most of them—except the standard BRD and the smooth BRDs—have the switching rate continuously changing with the payoff gain.

Theorem 2 (Generic nonaggregability). Consider a heterogeneous evolutionary dynamic \( \mathbf{v} \) with revision protocol \( \rho \), in aggregate game \( \mathbf{F} \) with more than one types in \( \Theta \). The dynamic is not aggregable, unless \( \rho_j(\pi) + \sum_{k \neq i} \rho_{ik}(\mathbf{F}[\bar{x}](\pi)) \) is constant for any \( \pi \). In particular, \( \bar{x} \) is not wholly determined from \( \bar{x} \) alone, unless \( \bar{x} \) is an aggregate pure strategy (i.e., \( \bar{x}_a = 1 \) at either one action) or the variation in \( \rho_j(\mathbf{F}[\bar{x}](\theta)) + \sum_{k \neq i} \rho_{ik}(\mathbf{F}[\bar{x}](\theta)) \) over types is zero for every two distinct actions \( i, j \in A \) such that \( i \neq j \) and \( \bar{x}_j > 0 \).

In a binary aggregate game, the zero variation condition in the above theorem reduces to zero variation of \( \rho_{O}(\mathbf{F}[\bar{x}](\theta)) + \rho_{O}(\mathbf{F}[\bar{x}](\theta)) \), which we can call the unconditional total revision rate since it does not condition on the agent’s current action. In major evolutionary dynamics except smooth BRDs, an agent never switches to an action that is worse than the agent’s current action. Thus, the unconditional total switching rate in a binary aggregate game is simply the switching rate from a suboptimal action to the optimal action. In the tempered BRD and any payoff comparison dynamics such as the Smith dynamic, the revision rate from a suboptimal action to the optimal action in a binary game is an increasing function of the payoff deficit. Since the payoff deficits vary with payoff types, the above theorem suggests that these dynamics are not aggregable. By the same token, we can confirm nonaggregability of observational dynamics; see Section 6.1.

As Figure 1 suggests, nonaggregability may result in dependency of the long-run aggregate outcome on the initial strategy composition. Zusai (2017) focuses on binary ASAGs and finds that, when the switching rate is sufficiently sensitive to the amount of the payoff gain from a switch, a nonaggregable dynamic starting exactly from an aggregate equilibrium may leave it and converge to another equilibrium even if the starting aggregate equilibrium is stable under the homogenized smooth BRD. The following example illustrates such a case.

Example 7. Here we focus on a binary coordination ASAG, in which the common payoff function is specified as \( F^0_j(x_t) = (49x_t - 1)/20 \) and the c.d.f. is \( P_\Theta(\theta_\Theta) = \sqrt{\theta_\Theta + 1} - 1 \) with support \( \Theta_\Theta = [0,3] \). Under the homogenized BRD \( \dot{x}_t = P_\Theta(F^0_j(x_t)) - x_t \), aggregate equilibria \( \bar{x}_t = 0 \) and \( \bar{x}_t^* = 0.25 \) are stable, while another equilibrium \( \bar{x}_t^* = 0.2 \) is unstable, as shown in Figure 2a.

The initial strategy composition is set to \( x_1(\theta_\Theta) = 1 \) for all \( \theta_\Theta > P_\Theta^{-1}(1 - 0.25) = 33/16 \) and \( x_1(\theta_\Theta) = 0 \) for all the other types. The initial aggregate strategy coincides with \( \bar{x}_t^* = 0.25 \). This is the unique solution to \( \bar{x}_t^* = \bar{x}_t^* \) under the homogenized smooth BRD.

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18We can see nonaggregability hold generically in the following sense. Even if \( \rho_j + \sum_{k \neq i} \rho_{ik} \) is constant, it can be broken by introducing any small payoff sensitivity of switching rates. More specifically, given such a revision protocol \( \rho \), consider a mixture \( \rho' \) with the protocol of a tempered BRD \( \rho^{'BRD} \), such as \( \rho' = (1 - \varepsilon) \rho + \varepsilon \rho^{'BRD} \), with any \( \varepsilon > 0 \). Then, this new dynamic is not aggregable since \( \rho'_j + \sum_{k \neq i} \rho'_{ik} \) is not constant any more.
Figure 2: Numerical simulations of the BRD and the pairwise comparison dynamic/tBRD in a binary coordination ASGA. In Figure 2b, the thin solid line shows the set of compositions that keep the aggregate strategy to one of aggregate equilibria $\bar{x}_I = 0.25$; the dashed line corresponds to another aggregate equilibrium $\bar{x}_I = 0.20$ and the dotted line to $\bar{x}_I = 0.1$. In Figures 2c and 2d, the horizontal lines show these aggregate strategies as well.

not a Bayesian equilibrium and indeed sorted reversely against agents’ types (outside options) $\theta_O$, since action I is taken by those who have greater values of $\theta_O$. The best response at this aggregate strategy is $b[0.25](\theta_O) = O$ for any $\theta_O > \theta^*_O = F^I_O(0.25) = 9/16$. The agents with types lower than $\theta^*_O$ should take I and the others should take O in the Bayesian equilibrium with $\bar{x}_I^* = 0.25$, say $x^*$. In Figure 2, the former group of agents is called group I (to be IN) and the latter group is called group O (to be OUT).

Under the the heterogeneous standard BRD, the aggregate strategy remains at aggregate equilibrium $\bar{x}_I = 0.25$, as shown in Figure 2c; it is expected from its aggregability since the aggregate strategy follows the homogenized smooth BRD and $\bar{x}_I = 0.25$ is a (stable) stationary point under the homogenized smooth BRD. According to Figure 2b, the underlying strategy composition approaches the Bayesian equilibrium $x^*$.

As an example of nonaggregable dynamics, we consider a pairwise comparison dynamic with
revision protocol \( \rho_{ij}(\pi) = ([\pi_j - \pi_i]_+)^3 \). Figure 2d shows that this nonaggregable dynamic drives the aggregate strategy away from \( \bar{x}_I = 0.25 \) and leads it to \( \bar{x}_I = 0 \). From a close look at this figure, we can see that the switching agents in group I actually choose I in the first 100 periods, just as in the standard BRD; but their switches are slower than the switches of agents in group O. So the aggregate share of I-players decreases since the inflow to I does not compensate the outflow from I. Behind this, those who switch from I to O face greater payoff gains than those from O to I and the pairwise comparison dynamic allows the former to switch faster than the latter. This dominance of group O’s payoff gains over group I’s gains persists due to positive externality in the payoff function of this game, and it keeps pushing \( \bar{x}_{I,t} \) down from \( \bar{x}_{I,0} = \bar{x}^*_I = 0.25 \) to \( \bar{x}_I = 0 \).

### 4.2 Implication on empirical methodology

There is a growing literature on estimation of discrete choice models in the presence of switching costs and heterogeneity over individuals. In health economics and IO/marketing research, it is possible to obtain micro data and thus directly observe individual-level decision dynamics, i.e., the dynamic of strategic composition. But, in many cases, researchers may not have an access to desirable micro data. To study dynamics at individual level, one needs to track the behavior of each individual over time and thus needs to identify each individual in the sample and tag the identity with the data. Such well-designed micro data may not be collected if there is no particular intention for the data collecting agency to analyze the micro panel data. For example, a local transportation agency may be monitoring traffic volumes on major streets but does not track each individual car’s trip. Even if there is such data, privacy protection may be so strict or data collecting companies may charge so a high price for an access to data that the data is not available for researchers.

Hence, applied economists may hope to do some rough study from aggregate data, focusing on the aggregate dynamic. From theoretical analysis of Bertrand competition with consumers’ switching costs, Shy (2002) proposes a formula to calculate the switching costs from product prices and the aggregate strategy distribution by utilizing the equilibrium condition on aggregate variables; this method is widely used in applied empirical research. But, as Shy noted, his model assumes homogeneity of agents except their initial choices and also presumes equilibrium. In an empirical study on international migration, Artuç, Lederman, and Porto (2015) notice the limitation of such reduced-form approach: they run Monte Carlo simulation using micro data on labor mobility in the United States. They find that, if heterogeneity is persistent and also individual decisions incur switching costs, just as in the heterogeneous tempered BRD, then the logit estimation based on the aggregate data yields sample selection biases in the estimation of parameter values compared to the regression directly from micro data. The main issue of their study is international

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19 Due to binary actions, this can be also interpreted as a tempered BRD with \( Q(\pi) = \pi^3 \).

20 For example, see Goettler and Clay (2011) and Handel (2013).

21 Shy admits that an equilibrium does not exist in his Bertrand game, though he makes justification of the use of a necessary condition for an equilibrium (a variant of the condition for no profitable deviation). Besides, Shy’s paper belongs to applications of economic theory and is not intended to offer a rigorous econometric theory.
migrations from developing countries and they do not have micro data on this issue. Hence, they have to admit that only transient heterogeneity, as we assume in the homogenized smooth BRD, is captured in their econometric framework.

We share the same concern on over-simplification of aggregate dynamics in the presence of heterogeneity and indeed put it further. The theoretical study in this paper suggests that, even if an economist somehow precisely knows the payoff function, the distribution of idiosyncratic payoffs, and the revision protocols of individuals, an aggregate dynamic derived from the aggregate data may result in a wrong prediction of the true transition of the aggregate state, not only in the magnitudes of changes but also in the directions of changes and the long-run outcomes, as suggested from Example 7. Sensitivity of the timing of a switch to the incentive for the switch causes nonaggregability. This is typically a concern for a dynamic study of switching costs.

5 Bayesian equilibrium: stationarity and stability

5.1 Extension of stationarity and stability to heterogeneous dynamics

Our heterogeneous dynamics could be seen as an extension of evolutionary dynamics in a single homogeneous population to (possibly) continuously many heterogeneous subpopulations, though the existence of a unique solution trajectory requires careful formulation of the state space. Despite generic nonaggregability, it is natural to expect that stationarity and stability of Nash equilibria are extended to Bayesian equilibria in the heterogeneous setting.

We first define a homogeneous dynamic in a way comparable with a heterogeneous dynamic. For this, we fix revision protocol \( \rho = (\rho_{ij})_{i,j \in A} : \mathbb{R}^A \rightarrow \mathbb{R}^{A \times A} \). In the homogeneous setting, payoff vector \( \pi^0 \in \mathbb{R}^A \) is common to all the agents in the society and the action distribution is just \( A \)-dimensional, i.e., \( x^0 \in \Delta^A \). Thus, the homogeneous version of each evolutionary dynamic is straightforwardly obtained by plugging \( \pi^0 \) into the revision protocol \( \rho \) of the dynamic. That is, the homogeneous mean dynamic is obtained as

\[
\dot{x}^0_i = \sum_{j \in A} x^0_j \rho_{ji}(\pi^0) - x^0_i \sum_{j \in A} \rho_{ij}(\pi^0)
\]

for each \( i \in A \).

Compare this equation with (4) that defines the mean dynamic function \( v \), and we can find that (5) can be expressed as \( \dot{x}^0_i = v_i(\pi^0, x^0) \) for each \( i \) and thus \( \dot{x}^0 = v(\pi^0, x^0) \). In homogeneous population game \( F^0 : \Delta^A \rightarrow \mathbb{R}^A \), this induces a homogeneous evolutionary dynamic \( v^{F^0} \) over \( \Delta^A \) such as

\[
\dot{x}^0 = v^{F^0}(x^0) := v(F^0(x^0), x^0).
\]

**Stationarity of Bayesian equilibrium**

To link the homogeneous dynamic \( \dot{x}^0 = v^{F^0}(x^0) \) on \( \Delta^A \) and the heterogeneous Bayesian dynamic \( \dot{x} = v^F[x] \) on \( \mathcal{F}_X \), we first identify the properties of the mean dynamic \( v \) that induce stationarity
and stability of equilibria, separately from the population game.\(^{22}\) This separation is useful because both homogeneous and heterogeneous dynamics stem from the same mean dynamic \(v\) (constructed from the same revision protocol \(\rho\)). Their difference lies in the difference in the population game played by agents, namely the difference between \(F: \Delta^A \times \Theta \rightarrow \mathbb{R}^A\) and \(F^0: \Delta^A \rightarrow \mathbb{R}^A\).

In the homogeneous setting, the stationarity of a Nash equilibrium under \(v^F\) is an immediate consequence of the best response stationarity under \(v\); the mean dynamic stays at an action distribution if and only if agents are taking the best response to the current payoffs.

**Definition 3** (Best response stationarity of mean dynamic). Mean dynamic \(v: \Delta^A \times \mathbb{R}^A \rightarrow \mathbb{R}^A\) satisfies the **best response stationarity** if, for any \(\pi^0 \in \mathbb{R}^A, x^0 \in \Delta^A\),

\[
v(\pi^0, x^0) = 0 \iff \forall b \in A[x_b^0 > 0 \implies \pi_b^0 \geq \pi_a^0 \ \forall a \in A]. \tag{6}\]

All the evolutionary dynamics mentioned in Section 2.3, except smooth BRDs, satisfy the best response stationarity.\(^{23}\) In a homogeneous population game, the best response stationarity implies the stationarity of a Nash equilibrium and non-stationarity of non-equilibrium states:

\[
v^F(x^0) = 0 \iff x^0 \text{ is a Nash equilibrium in } F^0.\]

In the heterogeneous setting, the best response stationarity applies to each type: the action distribution of a particular type \(\theta\) remains unchanged if and only if almost all agents of this type choose the best response to the current payoff for this type. Thus, it is natural that the best response stationarity implies the stationarity of a Bayesian equilibrium and non-stationarity of non-equilibrium Bayesian strategies.

**Theorem 3** (Bayesian equilibrium stationarity). Suppose that mean dynamic \(v\) satisfies the best response stationarity (6). Then, in any heterogeneous population game \(F\), a Bayesian equilibrium is stationary under the heterogeneous evolutionary dynamic \(v^F = v(\cdot, F(\cdot)):\)

\[
v^F(x|\theta) = 0 \text{ for } \mathbb{P}_\Theta\text{-almost all } \theta \in \Theta \iff x \text{ is a Bayesian equilibrium in } F. \tag{7}\]

Example 7 suggests that stationarity of an aggregate equilibrium is not guaranteed any more, as generally studied in Zusai (2017). But Theorem 3 mitigates concerns about equilibrium stationarity. If the underlying Bayesian strategy is exactly a Bayesian equilibrium and thus is perfectly sorted as in (2'), it stays there; as a consequence, the aggregate strategy also remains at the corresponding aggregate equilibrium. This clarifies that the driving force to move the aggregate

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\(^{22}\)This separation accords with the view proposed by Sandholm (2010) and Fox and Shamma (2013). Sandholm (2010, especially, Sec. 1.2.2 and Ch.4) proposes to construct a mean dynamic \(v\) from agents’ revision protocol \(\rho\) and thus has guided our attention to individual decision rules behind the collective population dynamic. Pushing further in this direction, Fox and Shamma (2013) regard an evolutionary dynamic \(v^F\) as a hybrid of a mean dynamic \(v\) (a system that converts payoff vector \(\pi\) to the transition of state \(x\)) and a population game \(F\) (a feedback system that converts the current state \(x\) to payoff vector \(\pi\)); they propose to define properties of \(v\) (e.g. \(\delta\)-passivity in their paper) apart from those of \(v^F\) (e.g. equilibrium stability in a contractive game, derived by combining \(\delta\)-passivity of \(v\) and contractiveness of \(F\)).

\(^{23}\)In the homogeneous version of exact optimization dynamics, the best response stationarity needs to assume \(\rho_{ij}(\pi) = 0\) when the current action \(i\) is a best response to \(\pi\); this was not assumed in our definition in cases of multiple best responses. In the heterogeneous setting, this concern on multiple best responses is eliminated by the assumption of a continuous type distribution. Hence, Assumption 3 replaces the assumption of \(\rho_{ij}(\pi) = 0\) for best response \(i\) to \(\pi\).
strategy from an aggregate equilibrium is indeed the distributional sorting pressure on the underlying strategy composition toward a perfectly sorted composition; we will make this point more concrete in Section 5.2.

**Stability of Bayesian equilibrium in potential games**

While stability of a Nash equilibrium is not generally guaranteed even in the homogeneous setting, it is verified for potential games under a wide class of evolutionary dynamics. For a game played in large population, a potential game is defined as a game in which payoff vector can be derived as the derivative of some scalar-valued function, i.e., a potential function. It is equivalent to externality symmetry: the change in the payoff of an action by a change in the mass of another action’s players is symmetric between these two actions. The class of potential games includes random matching in common interest games, binary games and congestion games. Sandholm (2010, Chapter 3) provides further explanation and examples.

**Definition 4** (Potential game in the homogeneous setting). Homogeneous population game \( F^0 : \Delta^A \rightarrow \mathbb{R}^A \) is called a **potential game** if there is a scalar-valued continuously differentiable function \( f^0 : \mathbb{R}^A \rightarrow \mathbb{R} \) whose gradient vector always coincides with the payoff vector: for all \( \bar{x} \in \Delta^A \), \( f^0 \) satisfies

\[
\frac{\partial f^0}{\partial x_a}(\bar{x}) = F^0_a(\bar{x}) \quad \text{for all} \quad a \in A,
\]

i.e., \( \nabla f^0(\bar{x}) := \left( \frac{\partial f^0}{\partial x_1}(\bar{x}), \ldots, \frac{\partial f^0}{\partial x_A}(\bar{x}) \right) = F^0(\bar{x}) \).

**Definition 5** (Potential game in the heterogeneous setting). Aggregate heterogeneous population game \( F : \Delta^A \rightarrow C_\Theta \) is called an (aggregate) **potential game** if there is a scalar-valued Fréchet-differentiable function \( f : \mathcal{X} \rightarrow \mathbb{R} \) that is continuous in the weak topology on the strategy composition space \( \mathcal{X} \) and whose Fréchet-derivative at each composition \( \mathbf{X} \in \mathcal{X} \) coincides with \( F[\bar{x}] \) at the corresponding aggregate state \( \bar{x} \in \Delta^A \).

Both in the homogeneous and heterogeneous settings, all local maxima and interior local minima of a potential function, and indeed all the solutions of the Karash-Kuhn-Tucker first-order condition for maxima are equilibria in a potential game; see Sandholm (2001) for the proof for Nash equilibria in a homogeneous potential game and Sandholm (2005, Appendix A.3) for Bayesian equilibria in a heterogeneous potential game.

While the potential \( f \) is defined as a function of strategy composition \( \mathbf{X} \in \mathcal{X} \), we can say that the potential of Bayesian strategy \( \mathbf{x} \in \mathcal{F}_\mathcal{X} \) is \( f(\int \mathbf{x}d\mathbb{P}_\Theta) \) where \( \int \mathbf{x}d\mathbb{P}_\Theta \in \mathcal{X} \) is the corresponding

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24That is, at each strategy composition \( \mathbf{X} \in \mathcal{X} \) with the aggregate strategy \( \bar{x} = X(\Theta) \), the payoff vector function \( F[\bar{x}] \in C_\Theta \) satisfies \( f(X') = f(X) + \langle F[\bar{x}], X' - X \rangle + o(||X' - X||_{\mathcal{X}}) \) for any \( X' \in \mathcal{X} \). Here, operator \( \langle \cdot, \cdot \rangle \) is defined as \( \langle \pi, \Delta X \rangle = \int_\Theta \sum_{a \in A} \pi_a(\theta) \Delta X_a(d\theta) = \mathbb{E}_\Theta[\pi(\theta) \cdot \Delta x(\theta)] \) for each \( \pi \in C_\Theta \) and \( \Delta X = \int \Delta x d\mathbb{P}_\Theta \in M_{\mathcal{X}} \). The norm \( || \cdot ||_{\mathcal{X}} \) is the variational norm on \( \mathcal{X} \) to metrize the strong topology: we have \( \| \Delta X ||_{\mathcal{X}} = \sum_{a \in A} \mathbb{E}_\Theta[|\Delta x_a|] \) by Theorem 7 in Appendix A.2. Fréchet differentiability is defined for the strong topology and thus continuity in the weak topology is additionally required.
strategy composition. Below we abuse the notation of $f$ to denote $f(\int x \, dP_{\Theta})$ by $f(x)$, justified by the one-to-one correspondence between $X$ and $x$ as discussed in Appendix A.1.

The key property of evolutionary dynamics for equilibrium stability in potential games is the positive correlation: each action’s payoff and the net increase in the mass of the action’s players are positively correlated and the correlation is strictly positive unless the strategy distribution is unchanged. Major evolutionary dynamics, except smooth BRDs, satisfy the positive correlation.\(^{25}\)

**Definition 6** (Positive correlation of mean dynamic). Mean dynamic $\mathbf{v}: \Delta^A \times \mathbb{R}^A \rightarrow \mathbb{R}^A$ satisfies the positive correlation if

$$
\pi^0 \cdot \mathbf{v}(\pi^0, x^0) \begin{cases}
\geq 0 & \text{for any } \pi^0 \in \mathbb{R}^A, x^0 \in \Delta^A; \\
> 0 & \text{if } \mathbf{v}(\pi^0, x^0) \neq 0.
\end{cases} \tag{8}
$$

In a homogeneous potential game, the positive correlation immediately implies that the homogeneous potential function increases over time until it reaches a stationary point, which is indeed a Nash equilibrium by the equilibrium stationarity, since the definition of a potential function implies

$$
\frac{d}{dt} f^0(x^0) = Df^0(x^0) \dot{x}^0 = F^0(x^0) \cdot \dot{x}^0 \begin{cases}
\geq 0 & \text{for any } x^0 \in \Delta^A; \\
> 0 & \text{if } \mathbf{v}^F(x^0) \neq 0.
\end{cases}
$$

Thus, the homogeneous potential function $f^0$ works as a Lyapunov function commonly in these evolutionary dynamics. Therefore, the positive correlation guarantees that the set of local maxima of $f^0$ is globally attracting and a strict local maximum is asymptotically stable (Sandholm, 2001). As a local maximum of the potential function is a Nash equilibrium, this implies global convergence to the set of Nash equilibria.

In the heterogeneous setting, the positive correlation of the mean dynamic implies a positive correlation between the payoffs and the action distribution among each type of agents. Thus, by the same token as in a homogeneous potential game, this guarantees that the heterogeneous potential function $f$ works as a Lyapunov function for equilibrium stability in a heterogeneous potential game $F$. Again, since a local maximum of $f$ is a Bayesian equilibrium, the following theorem suggests global convergence to the set of Bayesian equilibria, which include all the local maxima of $f$.

**Theorem 4** (Stability of Bayesian equilibria in aggregate potential games). Suppose that mean dynamic $\mathbf{v}$ satisfies the best response stationarity (6) and the positive correlation (8). Then, in any aggregate potential game $F$, the following holds.

i) The set of Bayesian equilibria is globally attracting under $\mathbf{v}^F$; a local strict maximum of $f$ is locally asymptotically stable.

\(^{25}\)See Sandholm (2010, Chapter 5) for summary of the relationship between dynamics and the two properties in this section.
ii) Let $x^*$ be an isolated aggregate equilibrium in the sense that, in a neighborhood $O^*$ of the corresponding strategy composition $X^*$ in the composition space $X$, there is no other equilibrium composition than $X^*$. If $x^*$ is (locally) asymptotically stable, then it is a local strict maximum of $f$.

If a dynamic satisfies the best response stationarity and the positive correlation, we call it an admissible dynamic.

**Corollary 1.** Assume Assumptions 1 to 3. Pairwise comparison dynamics and exact optimization dynamics are admissible dynamics. Therefore, a Bayesian equilibrium and a stationarity Bayesian strategy are equivalent under these dynamics in any heterogeneous population games; besides, the stability of Bayesian equilibria holds for these dynamics in any aggregate potential games.

Combined with aggregability of the standard BRD verified by Ely and Sandholm (2005), Theorem 4 implies equivalence of local stability between (any) admissible dynamics—both aggregable and nonaggregable—and the homogenized smooth BRD. In a potential game, once an (isolated) Bayesian equilibrium is found to be locally stable in the heterogeneous standard BRD—or equivalently, if the corresponding aggregate equilibrium is locally stable in the homogenized smooth BRD, then its local stability is maintained in any admissible heterogeneous dynamics. Therefore, despite nonaggregability in general, we can test local stability of a Bayesian equilibrium under an arbitrary admissible heterogeneous dynamic just by examining stability under the homogenized smooth BRD, when we know that the aggregate game is a potential game.

**Corollary 2** (General aggregability of local stability). Consider an aggregate potential game $F$ with type distribution $P_\Theta$.

Let $\bar{x}^* \in \Delta^A$ be an isolated aggregate equilibrium in the sense that, in a neighborhood $O^*$ of $\bar{x}^*$ in the aggregate strategy space $\Delta^A$, there is no other aggregate equilibrium than $\bar{x}^*$. Correspondingly, let $x^* \in F_X$ be a Bayesian equilibrium such that $E_\Theta x^* = \bar{x}^*$. Then, the following statements are equivalent to each other:

i) $\bar{x}^*$ is locally asymptotically stable under the homogenized smooth BRD.

ii) $x^*$ is a local strict maximum of the heterogeneous potential $f$.

iii) $x^*$ is locally asymptotically stable under a heterogeneous admissible dynamic.\(^{27}\)

We see below applications of this positive result to ASAGs. In the next subsection, Theorem 5 verifies that a homogeneous potential game can be extended to an ASAG and presents how the potential function is modified through this extension.

**Example 8** (Convergence to a free-entry equilibrium.). Consider a binary homogeneous game $A = \{I, O\}$ with negative externality: $F^0_I(\bar{x}_I)$ decreases with $\bar{x}_I \in [0, 1]$. Then, the potential function

\(^{26}\)Observational dynamics such as the replicator dynamic (if $x(\theta) \gg 0$ for almost all $\theta$) and excess payoff dynamics can be also included in Corollary 1, as long as agents of each type $\theta \in \Theta$ observe $x(\theta)$ of the own type; see Section 6.1.

\(^{27}\)It is sufficient for the other conditions if it is asymptotically stable under some admissible heterogeneous dynamic, while each of the other conditions implies asymptotic stability under any admissible heterogeneous dynamics.
$f^0 : [0, 1] \to \mathbb{R}$ is given by $f^0(\bar{x}_I) = \int_{0}^{\bar{x}_I} F^1_0(\bar{y})d\bar{y}$ and strictly concave. With the boundedness of the domain $[0, 1]$, the strict concavity of $f^0$ implies that the global maximum exists uniquely and there is no other local maximum of $f^0$. The global maximum of $f^0$ is the only equilibrium of this game.

For an example in microeconomic theory to fall into this class of games, consider an entry-exit game played by suppliers in a particular industry. To make entry and exit symmetric, it is conventionally assumed that fixed costs exist but they are not sunk: fixed costs are paid only to maintain production capacities and they are revocable when the supplier becomes inactive. Further, the choice of entry or exit is conventionally regarded as a “long run” decision while the choice of the quantity supplied is a “short run” decision (as well as the underlying consumers’ decisions on the demand side); thus, it is commonly assumed that the market is settled to a market equilibrium (the state where the demand equals to the total supply) at each moment of time, given the mass (number) of active suppliers at the moment. A free-entry or so-called “long run” equilibrium is characterized in the homogeneous setting as a state in which the gross profit for an active producer is equal to the fixed cost.

One may want to introduce heterogeneity in the suppliers’ fixed costs; it not only sounds realistic but also eliminates indeterminacy of individual choices at a free-entry equilibrium. Under the heterogeneity in fixed costs, a free-entry equilibrium should be redefined as a state in which almost all the active producers have smaller fixed costs than the gross profit and almost all the inactive ones have greater fixed costs.

Under perfect competition in a standard setting as in Mas-Colell, Whinston, and Green (1995, Section 10.F), the instantaneous market-equilibrium profit of an active supplier decreases with the number of active suppliers. We can regard $F^1_j(\bar{x}_I)$ as the gross profit at this instantaneous competitive equilibrium given the current mass $\bar{x}_I$ of active suppliers and $\theta_O(\omega)$ as the fixed costs of supplier $\omega$; then, the choice between entry and exit in perfect competition falls into a binary ASAG with negative externality.

Thanks to our stability result, we can justify the free-entry equilibrium as the globally stable state in an evolutionary dynamic; indeed it is so strengthened to be stable in any admissible dynamics. As argued in Zusai (2018b), the tempered BRD is considered as a version of the BRD in which a revising agent pays a stochastic switching cost. Thus, the stability of the free-entry equilibrium under the tempered BRD suggests in this context that, even if entry and exit incur sunk costs to build or scrap the production capacity, the “long-run” equilibrium is indeed the long-run limit state under such an entry-exit dynamic.

By the same token, we can justify a free-entry equilibrium in the standard (static) monopolistic competition model such as Dixit and Stiglitz (1977) as a dynamically stable state under an arbitrary admissible dynamic.

Example 9 (Dynamic implementation of the social optimum.). Imagine a central planner whose goal is to maximize the total payoff of agents in an ASAG:

$$E_\Theta [F[\bar{x}](\theta) \cdot x(\theta)] = F^0(\bar{x}) \cdot \bar{x} + E_\Theta [\theta \cdot x(\theta)]$$

with $\bar{x} = E_\Theta x$. 25
To help the central planner achieve this goal, we introduce a monetary transfer to the agent’s payoff: now a type-θ agent’s payoff from action \( i \in \mathcal{A} \) is \( \tilde{F}^T_i(x)(\theta) := F_i(x)(\theta) - T_i(\bar{x}) \), where function \( T = (T_i)_{i \in \mathcal{A}} : \Delta^\mathcal{A} \to \mathbb{R}^\mathcal{A} \) is a pricing scheme to determine the amount of the monetary transfer (in terms of payoff) from the agent to the planner for taking each action at state \( x \in \mathcal{X} \).

In the homogeneous setting, a desirable aggregate state could be achieved by a very simple bang-bang control that gives a subsidy for actions that need more players and imposes a tax on actions that need less. By keeping the taxes and subsidies at extreme levels in their feasible ranges, convergence can be achieved in a finite time; and it is the fastest among all the pricing schemes. But, in the heterogeneous setting, such extreme pricing may result in excessive distortion of the underlying composition and practically unacceptable instability. To avoid these troubles, pricing should be less extreme and adjusted continuously over time.

Sandholm (2002, 2005) proposes the dynamic Pigouvian pricing scheme such as

\[
T_i(\bar{x}) = -\sum_{j \in \mathcal{A}} \bar{x}_j \frac{\partial F^0_j}{\partial \bar{x}_i}(\bar{x}) \quad \text{for each } \bar{x} \in \Delta^\mathcal{A}.
\]

Notice that this pricing scheme does not require the central planner to know agents’ revision protocols, the type distribution, or even the current strategy composition.

Strictly speaking, in a setting where there are finitely many payoff types, Sandholm (2002) verified that, with \( T \) being the above dynamic Pigouvian pricing scheme, \( \tilde{F}^T \) has a potential function \( \tilde{f}^T \) being the total payoff:

\[
\tilde{f}^T(x) = \mathbb{E}_\Theta [F[\bar{x}](\theta) \cdot x(\theta)] \quad \text{with } \bar{x} = \mathbb{E}_\Theta x.
\]

In particular, if the common payoff function \( F^0 \) exhibits negative externality, \( \tilde{f}^T \) is concave and thus the unique social optimum is achieved in the long run through this pricing scheme regardless of the initial state. Thanks to Theorem 4 and corollary 2, now we can extend this claim to the games with infinitely many payoff types.\(^{28}\)

### 5.2 Comparison with homogenized smooth BRD in potential ASAGs

Theorem 4 implies that the aggregate strategy must converge to the set of aggregate equilibria in a heterogeneous potential game even under nonaggregable dynamics. Recall that the binary coordination ASAG in Example 7 is indeed a potential game, and we witnessed that aggregate strategy moves from one aggregate equilibrium to another. Perhaps, one might feel uneasy to accord the positive result in the above theorem and the negative result in that example. The key to understand this gap is the topological difference between Bayesian strategies and aggregate strategies. Here we highlight this gap by focusing on an ASAG, which enables us to construct the homogenized version of a potential function.

\(^{28}\)Sandholm (2005, p.903) speculated it by referring to Ely and Sandholm (2005), which allows us to reduce stability in the heterogeneous standard BRD to the one in the homogenized smooth BRD. However, we have found that aggregation is not a valid approach for other heterogeneous dynamics including the Smith dynamic, which is first proposed by Smith (1984) and popularly used in transportation science to analyze congestion games.
Extension of a homogeneous potential game to an ASAG

We extend a homogeneous potential game $F^0$ to a heterogeneous aggregate game by keeping $F^0$ as a common payoff function and adding idiosyncratic payoffs $\theta$ just as in (1). Then, this heterogeneous game is an ASAG with a potential function; we call it a potential ASAG. Note that the additive separability of the idiosyncratic payoffs and continuity of $f^0$ imply Assumption 1.

**Theorem 5** (Extension of a homogenous potential game to an ASAG). Consider a common payoff function $F^0$ that admits a homogeneous potential function $f^0 : \Delta A \to \mathbb{R}$ such that $\nabla f^0 \equiv F^0$, and the ASAG $F$ derived from $F^0$ by (1). Then, $F$ is a potential ASAG with a heterogeneous potential function $f : \mathcal{F}_X \to \mathbb{R}$ given by

$$f(x) = f^0(E_\Theta x) + E_\Theta[\theta \cdot x(\theta)]$$

for each $x \in \mathcal{F}_X$. (9)

The heterogeneous potential function $f$ is defined on the space of Bayesian strategies $\mathcal{F}_X$ by adding a negative entropy term that accounts for the sortedness of the current strategy composition to the original potential function $f^0$.

**Homogenized potential and off-equilibrium complete sorting**

A Lyapunov function allows us to summarize the dynamic on a multi-dimensional space into a one-dimensional dynamic of the value of this scalar-valued function. In a potential game, the potential function serves as a Lyapunov function. As we saw in Example 7, a heterogeneous dynamic may behave differently from the homogenized smooth BRD and may even converge to a different equilibrium, despite the equivalence in local stability of each equilibrium in Corollary 2. The difference between these dynamics is clarified by looking at the difference in the Lyapunov functions that represent these dynamics.

Applying the result on a smooth BRD by Hofbauer and Sandholm (2002, 2007) to the homogenized smooth BRD, we can construct a Lyapunov function $\bar{f} : \Delta A \to \mathbb{R}$ in a potential ASAG as

$$\bar{f}(\bar{x}) := f^0(\bar{x}) + \min_{\pi \in \mathbb{R}^A} \left( E_\Theta[\max_{a \in A}(\pi a + \theta a)] - \pi \cdot \bar{x} \right)$$

from the original potential function $f^0 : \Delta A \to \mathbb{R}$. We call $\bar{f}$ the **homogenized potential function**, to compare it with the heterogeneous potential function $f$. The homogenized potential function $\bar{f}$ is defined on the space of aggregate strategies by adding to the original potential $f^0$ some extra term.

Hofbauer and Sandholm (2007) explain the additional term as a kind of the potential of payoff perturbation from the analysis based on Legendre transforms. But, we can interpret it in the

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29 This function $f$ appears in the study of evolutionary implementation by Sandholm (2005, Appendix A.3). But it was used there only to characterize a Bayesian equilibrium as a solution of the KKT condition for local maxima and minima of $f$.

30 If $P_\Theta$ is a double exponent distribution with noise level $\mu$ and thus the homogenized smooth BRD reduces to a logit dynamic, then the latter term becomes the entropy function $-\mu \sum_{a \in A} \bar{x}_a \ln \bar{x}_a$.

31 Sandholm (2010, §6.C) provides a succinct summary of Legendre transforms. Hofbauer and Sandholm (2002, 2007)
context of our heterogeneous setting: in short, this term represents the total expected idiosyncratic payoffs attained when agents are sorted completely according to their payoff types. While the notion of sorting may be obvious from (2') for a Bayesian equilibrium, it may be puzzling for non-equilibrium Bayesian strategies. We say that a Bayesian strategy \( x^\circ \in \mathcal{F}_x \) is completely sorted given aggregate strategy \( \bar{x} \) if

a) \( \bar{x} = E_{\Theta} x^\circ \mid \bar{x} \mid \);

b) for \( P_{\Theta} \)-almost every type \( \theta \in \Theta \), there exists a unique action assignment \( a^\circ \mid \bar{x} \mid \theta \) \( \in A \) such that \( x^a_{\theta}(\theta) = 1 \) only at \( a = a^\circ \mid \theta \); and

c) there exists a rationalizing common payoff vector \( \pi^\circ \mid \bar{x} \mid \in R^A \) such that \( a^\circ \mid \bar{x} \mid \theta \) = \( \arg\max_{i \in A} \pi_i^\circ \mid \bar{x} \mid + \theta_i \) for \( P_{\Theta} \)-almost every type \( \theta \in \Theta \).

This means that a) this Bayesian strategy \( x^\circ \mid \bar{x} \mid \) aggregates to the designated aggregate strategy \( \bar{x} \), b) for almost every type, \( x^\circ \mid \bar{x} \mid \) assigns almost all the agents of the type to either one action \( a^\circ \mid \bar{x} \mid \theta \) and c) the action assignment \( a^\circ \mid \bar{x} \mid \) for each type can be rationalized by common payoff vector \( \pi^\circ \mid \bar{x} \mid \) and thus the difference in actions among agents can be wholly explained by the difference in their idiosyncratic payoff vector \( \theta \).

In contrast, we can think of a completely unsorted composition, where the action distribution is the same across different types, i.e., \( x(\theta) = \bar{x} \) for all \( \theta \in \Theta \). Below, we consider the hypothetical aggregate payoff deficit of unsorting, i.e., the difference in the aggregate payoff given hypothetical common payoff vector \( \bar{\pi} \) between the one attained at a completely unsorted composition and the one at a composition in which each type of agents take the optimal action to maximize the hypothetical payoff \( \pi_i + \theta_i \). Given \( \bar{x} \) of the unsorted composition, the rationalizing common payoff vector \( \pi^\circ \mid \bar{x} \mid \) is the hypothetical payoff vector to minimize this hypothetical aggregate payoff deficit.

**Lemma 1** (Homogenized entropy and complete sorting). Assume a continuous type distribution \( P_{\Theta} \). Then, the following holds.

i) Given aggregate strategy \( \bar{x} \in \Delta^A \), \( \pi \in R^A \) is a rationalizing common payoff vector \( \pi^\circ \mid \bar{x} \mid \in R^A \) if and only if it minimizes the hypothetical aggregate payoff deficit:

\[
\min_{\pi \in \Delta^A} h(\pi, \bar{x}) := \left( E_{\Theta} \left[ \max_{a \in A} (\pi_a + \theta_a) \right] - (\pi + E_{\Theta} \theta) \cdot \bar{x} \right).
\]

construct a perturbed potential function from a smooth BRD with a deterministic payoff perturbation function \( v : \Delta^A \rightarrow R \), where the switching rate vector \( \rho_i(\pi) := (\rho_i(\pi))_{i \in A} \) from action \( i \) is defined as a maximizer of \( y \cdot \pi + v(y) \) over all mixed strategies \( \pi \in \Delta^A \). Then, they show the equivalence between the mean dynamics of this deterministically perturbed BRD and of the stochastically perturbed BRD with i.i.d. idiosyncratic payoffs \( \epsilon \) (our interpretation of a smooth BRD in Example 4), by finding \( v \) that derives the same switching rate function as derived from stochastic optimization under the given distribution of \( \epsilon \). Our homogenized potential function (10) is obtained by finding the deterministic payoff perturbation function \( v \) that matches with \( P_{\Theta} \) according to Hofbauer and Sandholm (2002, Theorem 2.1) and then plugging it into their perturbed potential function, which can be found in Hofbauer and Sandholm (2002, Proposition 4.1) and Hofbauer and Sandholm (2007, Theorem 3.2).

\[^{32}\text{Here we assume a continuous type distribution } P_{\Theta} \text{ so the action to maximize } \pi_a + \theta_a \text{ among all actions } a \in A \text{ is uniquely determined for } P_{\Theta} \text{-almost every type, given an arbitrary common payoff vector } \bar{\pi} \in R^A.\]
ii) Completely sorted Bayesian strategy $\tilde{x}^* \in \mathbb{R}^A$ satisfies
\[
E_\Theta [\theta \cdot \tilde{x}^*] = \min_{\tilde{\pi} \in \mathbb{R}^A} \left( E_\Theta [\max_{a \in A} (\tilde{\pi}_a + \theta_a)] - \tilde{\pi} \cdot \bar{x} \right).
\]

iii) Consider an ASAG with $F^0$. The following conditions on an aggregate strategy $\bar{x}^*$ are equivalent to each other: 1) $\bar{x}^*$ is an aggregate equilibrium in this ASAG; 2) Completely sorted Bayesian strategy $\tilde{x}^* \in \mathbb{R}^A$ is a Bayesian equilibrium in this ASAG; 3) The rationalizing common payoff vector $\tilde{\pi}^* \in \mathbb{R}^A$ coincides with the actual common payoff vector $F^0(\bar{x}^*)$.

Many applied models of heterogeneous dynamics assume that the Bayesian strategy is somehow completely sorted given the current aggregate strategy at every moment of time. In such a model, the homogenized potential $\tilde{f}$ can serve as a Lyapunov function to tell the direction of the dynamic. However, part iii) of the above lemma suggests that off-equilibrium complete sorting cannot be explained from rational choices of agents; there must be some centralized coordination to sort agents or some additional incentive to fill the gap between $F^0(\bar{x})$ and $\tilde{\pi}^* \in \mathbb{R}^A$.

From this lemma, we establish a close connection between $f$ and $\tilde{f}$ in the next theorem. In general, a Bayesian strategy $x$ may not be completely sorted: then, $E_\Theta [\theta \cdot x] \leq E_\Theta [\theta \cdot \tilde{x}] | x \in \mathcal{X}$. The heterogeneous potential of the Bayesian strategy is generally smaller than that of the completely aggregate strategy. The heterogeneous potential coincides with the homogenized potential if and only the current Bayesian strategy is completely sorted. Even if it is sorted completely, the rationalizing common payoff vector may be different from the actual common payoff vector $F^0(\bar{x})$; then it is not a Bayesian equilibrium. Until the heterogeneous potential $f$ reaches a local maximum at some Bayesian equilibrium, it can be increased by moving the Bayesian strategy to a completely sorted equilibrium or shifting the aggregate strategy to a local maximum of the homogenized potential $\tilde{f}$ or both. If it is already completely sorted, then any increase in $f$ must be accompanied with increase in $\tilde{f}$.

Theorem 6 (Homogenized and heterogeneous potentials). Assume a continuous type distribution $\Pi_\Theta$. Then, the following holds.

i) $\tilde{f}$ is an upper bound on $f$:
\[
\tilde{f}(E_\Theta x) \geq f(x) \quad \text{for any } x \in \mathcal{X}.
\]

ii) The equality $\tilde{f}(E_\Theta x) = f(x)$ holds if and only if $x$ is completely sorted.

iii) Let $\bar{x}^* = E_\Theta x^*$. Then, $x^*$ attains a local strict maximum of $f$ if and only if $\bar{x}^*$ attains a local strict maximum of $\tilde{f}$.

iv) If $x$ is completely sorted but the aggregate strategy $E_\Theta x^*$ does not attain a local maximum of $\tilde{f}$, then any increase in $f$ must be accompanied with increase in $\tilde{f}$.

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33 For example, see Dhebar and Oren (1985) for a seminal paper on platform pricing under network externality based on this assumption of instantaneous sorting. Baldwin and Okubo (2005) argue sorting in agglomeration dynamics in spatial economics. Duffy and Hopkins (2005) tested convergence to a sorted composition in lab experiments.
Difference in topology and basin of attraction

Since a local maximum of \( \bar{f} \) coincides with the aggregate strategy at a local maximum of \( f \), one might expect that \( \bar{f} \) would increase when \( f \) increases toward the local maximum under an admissible dynamic. However, \( \bar{f} \) may not increase, unless the current Bayesian strategy is completely sorted. We need to be careful about the difference between the topology—or the meaning of a neighborhood—in the space of Bayesian strategies/strategy compositions and the one in the space of aggregate strategies.

Local stability of a Bayesian equilibrium and local maxima of the heterogeneous potential \( f \) are defined on the space of Bayesian strategies. Unless it is global, this locality means proximity of a Bayesian strategy to the Bayesian equilibrium; that is, the joint distribution of types and actions should be close enough to the equilibrium joint distribution and thus is close to a completely sorted composition as in (2'). On the other hand, when arguing local stability of an aggregate equilibrium under the homogenized BRD or local maxima of the homogenized BRD, the locality means only proximity of the aggregate strategy: it only requires the total mass of each action’s players (regardless of types) to be close enough. Thus, the sortedness of the underlying strategy composition does not matter. In Example 7, the initial strategy composition is completely reversed and thus far from the equilibrium strategy composition on the space of strategy compositions, though they share the same aggregate strategy and thus the distance on the aggregate strategy space is zero.

The homogenized potential \( \bar{f} \) does not serve as a Lyapunov function to predict the transition of the aggregate strategy under nonaggregable dynamics. Let \( \bar{x}^* \) be an isolated aggregate equilibrium and \( x^* \) be the corresponding Bayesian equilibrium such that \( \mathbb{E}_\Theta x^* = \bar{x}^* \). Suppose that \( \bar{f} \) attains a local maximum at \( \bar{x}^* \). If the dynamic is aggregable, the aggregate strategy should stay there and \( \bar{f} \) should remain at the local maximum, whenever aggregate strategy \( \bar{x} \) reaches \( \bar{x}^* \). But, if the underlying Bayesian strategy \( x_0 \) at time 0 is not a Bayesian equilibrium but \( \mathbb{E}_\Theta x_0 = x^* \), Theorem 6 implies that \( f(x_0) < f(\bar{x}^*) = f(x^*) \). Hence, the heterogeneous potential \( f \) is not maximized at \( x_0 \). Since \( f \) is indeed a strictly increasing Lyapunov function on \( F_X \) under an admissible Bayesian dynamic, the fact that \( f \) is not maximized at \( x_0 \) implies that \( f \) still increases over time. This requires \( x \) to leave \( x_0 \). Under a nonaggregable dynamic, \( \bar{x} \) may leave the aggregate equilibrium \( \bar{x}^* \), which is the case in Example 7. On the other hand, because \( \bar{f} \) is locally maximized at \( \bar{x}^* = \mathbb{E}_\Theta x_0 \), any move of \( \bar{x} \) from \( \bar{x}^* \) decreases \( \bar{f} \) at least temporarily. Therefore, the homogenized potential \( \bar{f} \) does not tell whether \( \bar{x} \) is settled to equilibrium or not.

6 Extensions

6.1 Observational dynamics

In some of major evolutionary dynamics, an agent observes other agents’ actions and the observation influences the agents’ switching decision. For example, an agent may imitate other agents’
actions or the switching rate may depend on the relative payoffs compared to the average payoff of the observed population. We can generalize these dynamics as observational dynamics by having the action distribution among observed agents $\tilde{x} \in \Delta^A$, not only payoff vector $\pi \in \mathbb{R}^A$, in the argument of revision protocol $\rho$.

**Example 10.** With an **excess payoff protocol**, a revising agent calculates the average payoff $\tilde{x} \cdot \pi$ and switches to action $j$ with the rate that increases with the excess payoff of the new action $\pi_j - \tilde{x} \cdot \pi$. In particular, the revision protocol $\rho_{ij}(\pi, \tilde{x}) = [\pi_j - \tilde{x} \cdot \pi]_+$ defines the Brown-von Neumann-Nash (BNN) dynamic (Hofbauer, 2001).\(^{34}\)

**Example 11.** With an **imitative protocol**, a revising agent randomly picks another agent and switches to the observed agent’s action $j$ with the rate $I_{ij}(\pi) \in \mathbb{R}_+$: the overall switching rate is $\rho_{ij}(\pi, \tilde{x}) = \tilde{x}_i I_{ij}(\pi)$. There are several imitative protocols that yield the replicator dynamic (Taylor and Jonker, 1978): imitative pairwise comparison $I_{ij} = [\pi_j - \pi_i]_+$ (Schlag, 1998), imitation driven by dissatisfaction $I_{ij} = D - \pi_i$ with constant $D \in \mathbb{R}$ (Björnerstedt and Weibull, 1996), and imitation of success $I_{ij} = \pi_j - S$ with constant $S \in \mathbb{R}$ (Hofbauer, 1995a).

They fall into continuous revision protocols and satisfy Assumption 2.\(^{35}\) (Note that Assumption 3 is not needed for continuous revision protocols.) Nonaggregability is still confirmed by Theorem 2 for both excess payoff dynamics and imitative dynamics, as long as the switching rate function $\rho_{ij}$ is a strictly increasing function of the payoff gain from switch.\(^{36}\)

We can readily extend all the positive results, i.e., Theorems 1, 3 and 4 and corollary 2, to observational dynamics, if we assume that an agent observes the action distribution of the same type: a type-$\theta$ agent observes $x(\theta) \in \Delta^A$.\(^{37}\) This assumption of within-type observability matches with an assumption on imitative dynamics in the society of finitely many subpopulations where a member of each subpopulation imitates the behavior of those in the same subpopulation. The proofs of these theorems in the appendix are indeed written explicitly to include $x(\theta)$ as an argument of revision protocol $\rho$.

To maintain the existence of a unique solution trajectory (Theorem 1) and stationarity of Bayesian equilibrium (Theorem 3), this assumption of within-type observability can be replaced with an alternative assumption that an agent observes the aggregate strategy $\bar{x}$ instead of $x(\theta)$. But, then the positive correlation (Theorem 4) may not be extended from the homogeneous setting to the

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\(^{34}\)Excess payoff dynamics allow innovation of a new strategy, while imitative dynamics do not. Thus, stationarity and stability of equilibria are restricted to the interior of the strategy space for the latter dynamics while they are not for the former. Further, it is known that excess payoff dynamics guarantee global asymptotic stability of the Nash equilibrium set in a contractive game (as known as a negative semidefinite or stable game), while the latter guarantees only Lyapunov stability unless contractiveness is strict; see Sandholm (2010). Also, in a continuous strategy space, they result in different characterizations of local stability; see Hofbauer, Oechssler, and Riedel (2009).

\(^{35}\)Precisely for observational dynamics, $\bar{\rho}$ is an upper bound on $\rho_{ij}(F(m(\theta), m(\theta)))$.

\(^{36}\)As the least obvious case for nonaggregability, consider a binary aggregate game and let every agent observe $\xi = \tilde{s}$; thus every agent faces the same action distribution of samples. The unconditional total switching rate $\rho_{OIO} + \rho_{O1}$ increases with $\max_{i \in A} \pi_i(\theta) - \pi \cdot \tilde{s}$ under an excess payoff protocol. For an imitative protocol, the payoff monotonicity of $\rho_{ij}$ is implied by that of $I_{ij}$. It is indeed satisfied by any of the three protocols that induce the replicator dynamic.

\(^{37}\)Corollary 1 holds for excess payoff dynamics. Imitative dynamics such as the replicator dynamic satisfy the best response stationarity only if $x^0$ is in the interior of $\Delta^A$, and thus these theorems hold for imitative dynamics in the interior of $F_X$. 

31
heterogeneous setting. If the type of other agents are not distinguishable for an agent and observations are sampled from the entire population, stability analysis becomes essentially different from how we have investigated stability in this paper.38

### 6.2 Heterogeneity in revision protocols

All of our results are robust to heterogeneity in revision protocols. Now, let each type \( \theta \in \Theta \) of agents not only have its peculiar payoff function \( F(\theta) \) but also follow its own revision protocol \( \rho^\theta \); in the case of an exact optimization protocol, this should be constructed from the conditional switching rate function \((Q^\theta_{ij})_{(i,j) \in A^2}\). The mean dynamic \( v^\theta : \mathbb{R}^A \times \Delta^A \to \Delta^A \) is defined by tagging \( \theta \) to (4) as

\[
\dot{x}_i(\theta) = v^\theta_i(\pi(\theta), x(\theta)) := \sum_{j \in A} x_j(\theta) \rho^\theta_{ji}(\pi(\theta)) - x_i(\theta) \sum_{j \in A} \rho^\theta_{ij}(\pi(\theta)).
\]

Then, the Bayesian dynamic \( v^F \) is defined in the same fashion as \( \dot{x}(\theta) = v^F[x](\theta) := v^\theta(F[E_{\Theta}x](\theta), x(\theta)) \).

Again, these notations are explicitly shown in the proofs in the appendix. Theorems 3 and 4 and corollary 2 hold as long as the assumptions in each theorem are satisfied with \( v^\theta \) of (almost) every type \( \theta \in \Theta \).

The existence of a unique solution trajectory (Theorem 1) is also guaranteed, though we should clarify what the assumptions (including the Lipschitz continuities assumed in Definitions 1 and 2) are imposed on heterogeneous revision protocols. For this, let \( \Theta_C \) be the set of types that adopt any of continuous revision protocols and \( \Theta_E \) be the set of those who use exact optimization protocols with any conditional switching rate functions. Then, the assumptions for Theorem 1 should read as follows.

**Definition 1** There should be a common Lipschitz constant \( L_\rho \) of the switching rate functions over almost all the types in \( \Theta_C \): \( |\rho^\theta_{ij}(\pi) - \rho^\theta_{ij}(\pi')| \leq L_\rho |\pi - \pi'| \) for any \( i, j \in A, \pi, \pi' \in \mathbb{R}^A \) and \( \mathbb{P}_\Theta \)-almost all \( \theta \in \Theta_C \).

**Definition 2** There should be a common Lipschitz constants \( L_Q\) of the conditional switching rate functions \( Q^\theta_{ij} \) over almost all the types in \( \Theta_E \): \( |Q^\theta_{ij}(\pi) - Q^\theta_{ij}(\pi')| \leq L_Q |\pi - \pi'| \) for any \( i, j \in A, \pi, \pi' \in \mathbb{R}^A \) and \( \mathbb{P}_\Theta \)-almost all \( \theta \in \Theta_E \).

**Assumption 2** Since this is about payoff types, this needs no modification.

**Assumption 3** The assumption is needed as long as \( \mathbb{P}_\Theta(\Theta_E) > 0 \); otherwise, it is not needed.

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38About unobservable heterogeneity in aspiration levels in imitative dynamics, Sawa and Zusai (2014) verify that, although the dynamic becomes more complicated and basic properties such as positive correlation do not hold, long-run outcomes are robust to the introduction of unobservable heterogeneity.
Of course, this extension to heterogeneous revision protocols cover observational dynamics. While we assume within-type observability for Nash stationarity and stability, it is not needed for the existence of a unique solution trajectory. Thus, our existence theorem would provide the most fundamental starting point to study the effect of both observable and unobservable heterogeneity of revision protocols on population dynamics and equilibrium stability.

7 Concluding remarks

In this paper, we extend evolutionary dynamics to allow (possibly) continuously many payoff types under persistent heterogeneity in payoff functions and revision protocols. With a rigorous formulation of a heterogeneous evolutionary dynamic as a differential equation over the space of probability measures, we verify the existence of a unique solution path from an arbitrary initial state. Nonaggregability is confirmed for a general class of evolutionary dynamics, including pairwise comparison dynamics, tempered BRDs, excess payoff dynamics and imitative dynamics. When heterogeneity is persistent, such a dynamic may leave an aggregate equilibrium even if it would be stable when the heterogeneity was only transitional. Yet, we can retain equilibrium stationarity by shifting our attention to the strategy composition, i.e., the joint distribution of types and actions. Moreover, in a potential game, the set of locally stable equilibria under any of admissible heterogeneous dynamics coincides with that under the homogenized smooth BRD, whose aggregate dynamic is independent of the strategy composition over different types.

In an application to dynamic implementation of the social optimum, the dependency of the aggregate transition on the underlying strategy composition suggests that a bang-bang control results in excessive instability generally in the heterogeneous setting, though it achieves the fastest convergence in the homogeneous setting. Yet, the dynamic Pigouvian pricing, proposed by Sandholm (2002, 2005), still guarantees convergence to the social optimum, while not requiring any ex-ante information about the underlying dynamic or type distribution. Nevertheless, there might be a better pricing scheme that lies between the bang-bang control and the dynamic Pigovian pricing and achieves faster convergence than the Pigovian pricing without requiring too much information. Actually, nonaggregability also suggests that the direction of the transition in the aggregate strategy is related with the underlying strategy composition. If we can find a way to extract the information of the strategy composition from the transition of the aggregate state, it could be used to improve the pricing scheme.\textsuperscript{39}

In heterogeneous dynamics, it depends on the initial composition—not only on the aggregate strategy—\textit{which} aggregate equilibrium is eventually reached in the long run when the initial state is specifically given. On the positive side, we can use nonaggregability to select equilibria by requiring robustness of stability to any unsorted distortion in the strategy composition. Zusai (2017) explores this idea by presenting more detailed analysis of nonaggregability, while focusing

\textsuperscript{39}Fujishima (2012) considers a heterogeneous congestion game where payoff heterogeneity is not additively separable and the social planner does not exactly know its distribution, and proposes a modified Pigouvian pricing that is combined with estimation of the distribution.
We have found that payoff sensitivity of switching rates result in nonaggregability. The payoff sensitivity can be caused by stochastic switching costs or status quo biases to agents’ switching decision: an agent is more likely to switch an action when the payoff gain from the switch is greater, since the gain is more likely to exceed the switching cost/status quo bias. There are many experimental and empirical studies that suggest the existence and significance of switching costs or status quo bias in real economic choices. This paper suggests notable implications of switching costs on heterogeneous evolutionary dynamics.

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40 Zusai (2018b) constructs the tempered BRD as an optimization-based dynamic with a stochastic switching cost. Further, Zusai (2018a) presents a general framework to reconstruct nonobservational dynamics and excess payoff dynamics as an optimization-based dynamic with a stochastic switching cost and a stochastically restricted action set and finds that the payoff gain is a key variable to verify global stability of Nash equilibria in contractive games and local stability of a (regular) evolutionary stable state.

41 See Samuelson and Zeckhauser (1988); Hartman, Doane, and Woo (1991); Madrian and Shea (2001); Erev and Roth (1998) for example.
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A Appendix to Sections 2–3

A.1 Measure-theoretic definition of Bayesian strategy

This subsection provides mathematically rigorous definitions of a strategy composition and a Bayesian strategy based on measure theory. Proofs in the appendices deal with strategy compositions to properly utilize the Lyapunov stability theorem (Theorem 9 in Appendix C) and to borrow measure-theoretic construction of evolutionary dynamics on a continuous strategy space as in Oechssler and Riedel (2001, 2002) and Cheung (2014).

Combination of action profile \( a : \Omega \to A \) and type profile \( \theta : \Omega \to \Theta \) generates a finite measure \( X_a : B_\Theta \to \mathbb{R}_+ \) for each \( a \in A \) from \( \mathbb{P}_\Theta \):

\[
X_a(B_\Theta) := \mathbb{P}_\Theta(\{\omega \in \Omega : a(\omega) = a \text{ and } \theta(\omega) \in B_\Theta\}) \quad \text{for each } B_\Theta \in B_\Theta.
\]

\( X_a(B_\Theta) \) represents the mass of action-\( a \) players whose types belong to set \( B_\Theta \). \( X_a \) is dominated by \( \mathbb{P}_\Theta \) in the sense that

\[
\mathbb{P}_\Theta(B_\Theta) = 0 \implies X_a(B_\Theta) = 0 \quad \text{for each } B_\Theta \in B_\Theta. \tag{A.1}
\]

Denote by \( X_a \ll \mathbb{P}_\Theta \) this dominance relation, i.e., absolute continuity of \( X_a \) with respect to \( \mathbb{P}_\Theta \).

It follows by Radon-Nikodym theorem that there exists a \( B_\Theta \)-measurable nonnegative function \( x_a : \Theta \to \mathbb{R}_+ \) such that

\[
X_a(B_\Theta) = \int_{B_\Theta} x_a(\theta) \mathbb{P}_\Theta(d\theta) \quad \text{for any } B_\Theta \in B_\Theta.
\]

\( x_a \) is the density function of measure \( X_a \). The density is determined uniquely in the sense that, if another measurable function \( x'_a \) satisfies \( X_a(B_\Theta) = \int_{B_\Theta} x'_a(\theta) \mathbb{P}_\Theta(d\theta) \) for all \( B_\Theta \in B_\Theta \), then \( x'_a(\theta) = x_a(\theta) \) for \( \mathbb{P}_\Theta \)-almost all \( \theta \in \Theta \).

The distribution of strategies over different types is represented by \( X = (X_a)_{a \in A} : B_\Theta \to \Delta^A \), which we call strategy composition. We can see this vector measure as a joint probability measure over the product space \( A \times \Theta \). \(^{42}\) Let \( \mathcal{X} \) be the set of strategy compositions \( X = (X_a)_{a \in A} \), i.e., the set of probability measures over \( A \times \Theta \) that is dominated by \( \mathbb{P}_\Theta \) in the sense that \( X_a \ll \mathbb{P}_\Theta \) for all \( a \in A \); we abuse notation to denote this domination by \( X \ll \mathbb{P}_\Theta \). \( \mathbb{X}(\Theta) \) coincides with the aggregate strategy induced from \( X \). Note that the dominance of strategy composition \( X \) by the type distribution \( \mathbb{P}_\Theta \) is peculiar to heterogeneous dynamics, making a difference in the proof of Lipschitz continuity of the dynamic from the one for continuous strategy dynamics. See Remark 1 in Section 3.

The Radon-Nikodym density \( x = (x_a)_{a \in A} : \Theta \to \mathbb{R}_+^A \) is the Bayesian strategy corresponding to \( X \). We represent the relationship between \( X = (X_a)_{a \in A} \) and \( x = (x_a)_{a \in A} \) in the above integral equation by \( X = \int x d\mathbb{P}_\Theta \). From the fact that \( \sum_{a \in A} X_a(B_\Theta) = \mathbb{P}_\Theta(B_\Theta) \) and \( X_a(B_\Theta) \geq 0 \) for any

\[^{42}\text{Abusing notation, we could say that } X \text{ defines a measure of a Borel set } B_{A \times \Theta} \text{ on the product space } A \times \Theta \text{ by}
\]

\[
X(B_{A \times \Theta}) := \sum_{a \in A} X_a(\{\theta \in \Theta : (a, \theta) \in B_{A \times \Theta}\}) = \mathbb{P}_\Theta(\{\omega \in \Omega : (a(\omega), \theta(\omega)) \in B_{A \times \Theta}\}).
\]
B_Θ ∈ B_Θ and a ∈ A, we can confirm that x(θ) is a probability vector for almost all types:

\[ x(θ) ∈ Δ^A \quad \text{for } P_Θ\text{-almost all } θ ∈ Θ. \]

A Bayesian strategy is (P_Θ-almost) uniquely determined from a strategy composition by Radon-Nikodym theorem, and vice versa. So, X is equivalent to the set of Bayesian strategies F_X.

We call strategy composition X ∈ X an equilibrium composition, if

\[ P_Θ(β_a^{-1}(x) ∩ B_Θ) ≤ X_a(B_Θ) ≤ P_Θ(b_a^{-1}(x) ∩ B_Θ) \quad \text{with } x = X(Θ) \quad (A.2) \]

for all a ∈ A and B_Θ ∈ B_Θ. This condition is obtained by aggregation of the Bayesian equilibrium condition (2) on x_a(θ) over θ ∈ B_Θ. Among types in B_Θ, all those who have a as the unique best response must choose this action a in equilibrium; thus X_a(B_Θ) must be at least P_Θ(β_a^{-1}(x) ∩ B_Θ). On the other hand, those who have a as one of the best responses may or may not add to action-a players and thus X_a(B_Θ) is at most P_Θ(b_a^{-1}(x) ∩ B_Θ). X being an equilibrium composition (A.2) is equivalent to its density x being a Bayesian equilibrium (2).

### A.2 Topology of the space of strategy compositions

Choice of a topology is a sensitive issue when we argue dynamics of a probability measure over a continuous space. We follow the convention in the literature on evolutionary dynamics over a continuous strategy space, such as in Cheung (2014). That is, we use the strong topology to prove the existence of a unique solution path and the weak topology to obtain stability of equilibrium composition. See (Cheung, 2014, Section 4) for a detailed explanation on the strong and weak topology in evolutionary dynamics on a continuous space.

Below we define these two topologies on the space of finite signed measures M_{A×Θ}. Note that X ⊆ M_{A×Θ} and that M_{A×Θ} is the tangent space of X. This space M_{A×Θ} is a vector space and a transition vector stays in this extended space.

The strong topology is metrized by the variational norm \( \| M \|^{∞}_{A×Θ} \) defined as

\[ \| M \|^{∞}_{A×Θ} = \sup_{g} \left\{ \sum_{a ∈ A} \int_{Θ} g_a(θ) M_a(dθ) : \sup_{(a,θ) ∈ A×Θ} |g_a(θ)| ≤ 1 \right\}, \]

where the first sup is taken over the set of measurable functions g = (g_a)_{a ∈ A} on (A × Θ, B_{A×Θ}). Bayesian strategies belong to F_X, i.e., the space of B_Θ-measurable vector functions from Θ to Δ^A. Note that, if M ≪ P_Θ, then there uniquely exists a Radon-Nikodym density m ∈ F_X such that M = \( m dP_Θ \) in the sense we defined in Appendix A.1. The theorem below suggests that the variational norm on X is equivalent to the L^1-norm on F_X.43 The proof is provided in Section S2.1 of Supplementary Note.44

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43Ely and Sandholm (2005) define the standard BRD under payoff heterogeneity directly as a dynamic of x and adopt L^1 norm.

44This density-based formula of the variational norm comes essentially from Theorem 5 in Oechssler and Riedel (2001).
Theorem 7. For any finite signed measure \( M \in \mathcal{M}_{A \times \Theta} \) with density \( m = (m_a)_{a \in A} \), we have
\[
\|M\|_{\infty}^{A \times \Theta} = \sum_{a \in A} E_{\Theta} |m_a| = \sum_{a \in A} \int_{\Theta} |m_a(\theta)| \|P_\Theta(d\theta)\).
\] (A.3)

With the variational norm, the normed vector space \((\mathcal{M}_{A \times \Theta}, \|\cdot\|_{\infty}^{A \times \Theta})\) is a Banach space; but not with weak topology. By Zeidler (1986, Cor. 3.9), boundedness and Lipschitz continuity of the dynamic in the strong topology jointly guarantee the existence and uniqueness of a solution path of the dynamic. See Theorem 8 in Appendix A.3.

Under the weak topology on the set of measures over space \( S \), a mapping from \( \mathcal{M}(S) \to \mathbb{R} \) such as \( \mu \to \int_S f d\mu \) is continuous for any bounded and continuous function \( f : S \to \mathbb{R} \). In our model, the space \( S := A \times \Theta \) is separable with metric \( d_{A \times \Theta} : (A \times \Theta)^2 \to \mathbb{R}_+ \) given by
\[
d_{A \times \Theta}((a, \theta), (a', \theta')) := 1\{a \neq a'\} + |\theta - \theta'|_{\infty}.
\]
Then, the weak topology is metrized by Prokhorov metric \( d_M : \mathcal{M}_{A \times \Theta}^2 \to \mathbb{R}_+ \) such that
\[
d_M(M, M') := \inf\{\varepsilon > 0 : M(B_{A \times \Theta}) \leq M'(B'_{A \times \Theta}) + \varepsilon
\]
\[
\text{and } M'(B_{A \times \Theta}) \leq M(B'_{A \times \Theta}) + \varepsilon \text{ for all } B_{A \times \Theta} \in B_{A \times \Theta}^c\},
\]

where \( B_{A \times \Theta}^c \) is defined from \( B_{A \times \Theta} \) as \( B_{A \times \Theta}^c := \{(a, \theta) \in A \times \Theta : d_{A \times \Theta}((a, \theta), (a', \theta')) < \varepsilon \text{ with some } (a', \theta') \in B_{A \times \Theta}\}. \)

Under the weak topology, the space of probability measures, i.e., the space of strategy compositions becomes compact. Then, we can apply the Lyapunov stability theorem, as in Cheung (2014, Thm. 6). See Theorem 9 in Appendix C.

### A.3 Sketch of Proof of Theorem 1

We prove the existence of a unique solution trajectory under a Bayesian dynamic by verifying it for the corresponding dynamic of the strategy composition, appealing to the equivalence between Bayesian strategies and strategy compositions. Here we sketch the outline of the proof, while the complete presentation of the proof is provided in Section S2.2 of Supplementary Note.

First, we construct the mean dynamic of strategy composition \( \mathbf{V} = (V_i)_{i \in A} : \mathcal{X} \times C_\Theta \to \mathcal{M}_{A \times \Theta} \) by gathering the mean dynamic \( \mathbf{V}_\Theta \) of a Bayesian strategy, defined by (4), over all \( \theta \in \Theta \): for each action \( i \in \mathcal{A} \),
\[
X_i(B_\Theta) = V_i[X, \pi](B_\Theta) = \int_{B_\Theta} \nabla \theta_i^\theta[\pi(\theta), x(\theta)]P_\Theta(d\theta)
\]
\[
= \int_{B_\Theta} \sum_{j \in A} \rho_{ij}^\theta(\pi(\theta), x(\theta))X_j(d\theta) - \int_{B_\Theta} \left\{ \sum_{j \in A} \rho_{ij}^\theta(\pi(\theta), x(\theta)) \right\} X_i(d\theta)
\] (A.4)

---

45The metric \( d_{A \times \Theta} \) is a product metric constructed from the discrete norm on \( A \) and the sup norm on \( \Theta \subset \mathbb{R}^A \). Notice \( A < \infty \); so the product metric \( d_{A \times \Theta} \) makes \( A \times \Theta \) separable. Here \( 1\{a \neq a'\} \) is an indicator function and takes 1 if \( a \neq a' \) and 0 otherwise.

46If there is no payoff heterogeneity, i.e., \( \Theta = \{\theta_0\} \), then composition \( M \) can be simply represented by an \( A \)-dimensional vector \( (m_a)_{a \in A} \in \mathbb{R}^A \) such that \( m_a = \mathbf{M}(\{\theta_0\}) \). Then, \( d_M(M, M') = \varepsilon \) is equivalent to \( \sup_{a \in A} |m_a - m_a'| = \varepsilon \). So the metric \( d_M \) reduces to the sup norm on \( \mathbb{R}^A \).

47If \( \varepsilon < 1 \), the condition for \( (a, \theta) \in B_{A \times \Theta}^c \) is equivalent to the existence of \( \theta' \in \Theta \) such that \( |\theta - \theta'|_{\infty} < \varepsilon \) and \( (a, \theta') \in B_{A \times \Theta}^c \).
for each $B_\Theta \in B_\Theta$, given strategy composition $X = \int x d\Pi_\Theta \in \mathcal{X}$ and payoff profile $\pi : \Theta \to \mathbb{R}^A$. In short, we write $\dot{X} = V[X, \pi]$.

In a population game $F : \Delta^A \times \Theta \to \mathbb{R}^A$, the mean dynamic (A.4) of a strategy composition defines an autonomous dynamic $V^F$ over $\mathcal{X}$ by

$$X = V^F[X] := V[X, F(X(\Theta))] \in \mathcal{M}_{A \times \Theta}$$

for each strategy composition $X \in \mathcal{X}$. Then, this composition dynamic $V^F$ matches with the Bayesian dynamic $V^F$ defined in Section 2.3, in the sense that $V^F[X](B_\Theta) = \int_{B_\Theta} V^F[x](\theta) \Pi_\Theta(d\theta)$, where $x$ is the corresponding Bayesian strategy, i.e., the Radon-Nikodym density of $X$. Theorem 7 suggests that Lipschitz continuity of $V^F$ in the $L^1$-norm is equivalent to Lipschitz continuity of $V^F$ in the variational norm.

To argue the existence of a unique solution trajectory, we exploit the known result on a Lipschitz continuous dynamic over a Banach space as in the theorem below.\(^{48}\)

**Theorem 8** (Zeidler, 1986: Corollary 3.9). Consider a dynamic $\dot{z} = V(z)$ with $V : Z \to Z$. If the space $Z$ is a Banach space and the dynamic $V$ is Lipschitz continuous and bounded, then there exists a unique solution $\{z_t\}_{t \in \mathbb{R}_+}$ from any initial state in $z_0 \in Z$.

For this, we need a Banach space. But, the space of strategy compositions $\mathcal{X}$ is not a vector space. Thus, we extend the domain of the dynamic to the space of finite signed measures $\mathcal{M}_{A \times \Theta}$. Since the mean dynamic $V[X, \pi](B_\Theta)$ is defined by collecting the transition of the density $x(\theta)$ over types $\theta \in B_\Theta$, we still need a density of a measure on this extended space. However, a finite signed measure may not be absolutely continuous with respect to the type distribution $\Pi_\Theta$. We use the Lebesgue decomposition theorem to extract the absolutely continuous part.

**Lemma 2** (Rudin, 1987: §6.10). For any finite signed measure $M = (M_a)_{a \in A} \in \mathcal{M}_{A \times \Theta}$, there is a pair of finite signed measures $\tilde{M} = (\tilde{M}_a)_{a \in A}, \tilde{M} = (\tilde{M}_a)_{a \in A} \in \mathcal{M}_{A \times \Theta}$ such that, for each $a \in A$,

i) $M_a = \tilde{M}_a + \tilde{M}_a$;

ii) $\tilde{M}_a \ll \Pi_\Theta$, i.e., $\Pi_\Theta(B_\Theta) = 0 \implies \tilde{M}_a(B_\Theta) = 0$ for any $B_\Theta \in B_\Theta$.

iii) $\tilde{M}_a \perp \Pi_\Theta$, i.e., there exists $E_a \in B_\Theta$ such that $\tilde{M}_a(B_\Theta \cap E_a) = 0$ and $\Pi_\Theta(B_\Theta \setminus E_a) = 0$ for any $B_\Theta \in B_\Theta$.

The part (ii) implies that $\tilde{M}$ has density $\tilde{m} = (\tilde{m}_a)_{a \in A}$ with respect to $\Pi_\Theta$. Besides, $\|\tilde{M}\| \leq \|M\|$, since i) and ii) imply $\|M\| = \|\tilde{M}\| + \|\tilde{M}\|$. We extend $V$ to $\mathcal{M}_{A \times \Theta}$ by discarding the orthogonal part $\tilde{M}$ and applying (A.4) to the continuous part $\tilde{M}$. This need for the Lebesgue decomposition is the first difference from evolutionary dynamics on a continuous strategy space: see Remark 1 in Section 3. Let $\mathcal{M}_{A \times \Theta}$ be the space of $\Pi_\Theta$-absolutely continuous measures.

Yet, the density function $\tilde{m}$ of $\tilde{M}$ may not be bounded, while that of a strategy composition, i.e., a Bayesian strategy $x$ is bounded in the sense that $x(\theta)$ of almost every type $\theta$ belongs to $A$.

\(^{48}\)Ely and Sandholm (2005, Theorem A.3.) also guarantee the existence of a unique solution trajectory for a Bayesian dynamic on $\mathcal{F}_{\mathcal{X}}$ with $L^1$-norm from Lipschitz continuity of the dynamic.

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bounded set $\Delta^A$. We will utilize the assumptions that the payoff function $F$ and the switching rate function $\rho^\theta$ (or the conditional switching rate function $Q_{ij}$ for exact optimization protocols) are continuous and thus bounded if its domain is restricted to a compact set. To restrict the value of the density function into a compact domain, we truncate $\hat{\mathbf{m}}$ by a rounding function $\mu = (\mu_j)_{j \in A} : \mathbb{R}^A \rightarrow [-3, 3]^A$ such that $\mu(z) = z$ if $z \in \Delta^A$ and $\mu$ is Lipschitz continuous with constant $L_\mu$.\footnote{For example, define $\hat{\rho} : \mathbb{R} \rightarrow [-3, 3]$ such as $\hat{\rho}^\theta(z) := -3 + \exp(z + 2)$ for $z < -2$, $\hat{\rho}^\theta(z) := z$ for $z \in [-2, 2]$ and $\hat{\rho}^\theta(z) := 3 - \exp(2 - z)$ for $z > 2$. Then, define vector function $\mu = (\mu_a)_{a \in A}$ by $\mu_a(z) = \hat{\rho}^\theta(z_a)$ for each $a \in A$ and $z \in \Delta^A$.}

Then, we redefine function $v^F : \hat{M}_{A \times \Theta} \times \Theta \rightarrow \mathbb{R}$ on the extended domain by

$$v^F[\hat{\mathbf{M}}](\theta) = v^\theta(F[\mu(\mathbf{M}(\Theta))](\theta), \mu(\hat{\mathbf{m}}(\theta)))$$

for each $i \in A$ and any $\mathbb{P}_\Theta$-absolutely continuous finite signed vector measure $\hat{\mathbf{M}} \in \hat{M}_{A \times \Theta}$. Here $\hat{\mathbf{m}}$ is the Radon-Nikodym density of $\mathbf{M}$. This leads to the extension of $V^F := (V^F_i)_{i \in A}$ to $\hat{M}_{A \times \Theta}$, such as

$$V^F_i[\hat{\mathbf{M}}](B_\Theta) = \int_{B_\Theta} v^F_i[\hat{\mathbf{M}}](\theta) \mathbb{P}_\Theta(d\theta)$$

for each $i \in A$, any $B_\Theta \in B_\Theta$ and any finite signed vector measure $\mathbf{M} \in \hat{M}_{A \times \Theta}$; here $\hat{\mathbf{M}}$ is the $\mathbb{P}_\Theta$-absolutely continuous part of $\mathbf{M}$ in the Lebesgue decomposition of $\mathbf{M}$. As only this part matters to the value of $V^F$, we have $V^F[\hat{\mathbf{M}}] = V^F[\mathbf{M}]$.

To prove Lipschitz continuity of $V^F$, we look at $V^F$ on $\hat{M}_{A \times \Theta}$: in Supplementary Note S2.2, we find $L^F_i > 0$ such that\footnote{Here the norm $\| \cdot \|$ is the variational norm, defined in Appendix A.2.}

$$\| V^F[\hat{\mathbf{M}}] - V^F[\hat{\mathbf{M}}'] \| \leq L^F_i \| \hat{\mathbf{M}} - \hat{\mathbf{M}}' \| \quad \text{for any } \hat{\mathbf{M}}, \hat{\mathbf{M}}' \in \hat{M}_{A \times \Theta}. \quad (A.5)$$

Then, this implies Lipschitz continuity over the whole space $\hat{M}_{A \times \Theta}$, because $\| \hat{\mathbf{M}} - \hat{\mathbf{M}}' \| \leq \| \mathbf{M} - \mathbf{M}' \|$ for any $\mathbf{M} = \hat{\mathbf{M}} + \mathbf{M}'$, $\hat{\mathbf{M}}' = \mathbf{M}' + \hat{\mathbf{M}}' \in \hat{M}_{A \times \Theta}$.

For a continuous revision protocol, the Lipschitz continuity of $V^F$ is a natural consequence of the Lipschitz continuity of the switching rate function $\rho^\theta$ and of the payoff function $F$.

On the other hand, an exact optimization protocol is discontinuous. If the best response actions for some type of agents have changed by a change in the strategy composition from $\hat{\mathbf{M}}$ to $\hat{\mathbf{M}}'$, these agents should experience discontinuous changes in the switching rates. However, these discontinuous changes in their switching rates are bounded thanks to the boundedness of the switching rate function $\rho^\theta$. Further, thanks to Assumption 3, the mass of agents who belong to such types increases only (Lipschitz) continuously with the change in the strategy composition.\footnote{Note that this assumption also restricts the mass of types who have multiple best responses to a null set (zero measure) in $\mathbb{P}_\Theta$.} As a result, the aggregate change in their switching rates grows only continuously. This mitigation of discontinuity in exact optimization protocols by continuity of the type distribution makes the second difference from the preceding studies on continuous-strategy evolutionary dynamics on space: see Remark 2 in Section 3.
B Appendix to Section 4

B.1 Proof of Theorem 2

Henceforth, we let $\rho^F_{ij}(\theta) := \rho^F_{ij}(\theta, x(\theta))$ and $\hat{\rho}^F_{ij}(\theta) = \rho^F_{ij}(\theta) - \mathbb{E}_{\Theta} \rho^F_{ij}$ for each $\theta \in \Theta$. Likewise, let $\hat{X}_i(\Theta) = X_i(\Theta) - \bar{x}_i P_{\Theta}(\Theta)$ for each $\Theta \in \mathcal{B}_{\Theta}$.

First of all, recall the definition of the Bayesian mean dynamic (4):

$$\hat{x}_i(\theta) = \sum_{j \neq i} x_j(\theta) \rho^F_{ji}(\theta) - x_i(\theta) \sum_{j \neq i} \rho^F_{ij}(\theta).$$

Aggregating this over all the types $\theta \in \Theta$, we obtain the transition of the aggregate strategy:

$$\hat{x}_i = \sum_{j \neq i} \left\{ \int_{\Theta} \rho^F_{ij}(\theta) X_j(d\theta) - \int_{\Theta} \rho^F_{ij}(\theta) \hat{X}_j(d\theta) \right\}$$

$$= \sum_{j \neq i} \left\{ \hat{x}_j \mathbb{E}_{\Theta} \rho^F_{ji} - \hat{x}_i \mathbb{E}_{\Theta} \rho^F_{ij} + \int_{\Theta} \hat{\rho}^F_{ij}(\theta) \hat{X}_j(d\theta) - \int_{\Theta} \hat{\rho}^F_{ij}(\theta) \hat{X}_i(d\theta) \right\}$$

For the last equality, we use $\hat{X}_i(\Theta) = \hat{X}_j(\Theta) = 0$ and $\mathbb{E}_{\Theta} \hat{\rho}^F_{ij} = \mathbb{E}_{\Theta} \hat{\rho}^F_{ij} = 0$. Since $\hat{X}_i = -\sum_{k \neq i} \hat{X}_k$, the last term can be rearranged as

$$-\sum_{j \neq i} \int_{\Theta} \hat{\rho}^F_{ij}(\theta) \hat{X}_i(d\theta) = \sum_{j \neq k \neq i} \sum_{j \neq i} \int_{\Theta} \hat{\rho}^F_{ij}(\theta) \hat{X}_k(d\theta) = \sum_{j \neq k \neq i} \sum_{j \neq i} \int_{\Theta} \hat{\rho}^F_{ik}(\theta) \hat{X}_j(d\theta).$$

Therefore, we have

$$\hat{x}_i = \sum_{j \neq i} (\hat{x}_j \mathbb{E}_{\Theta} \rho^F_{ji} - \hat{x}_i \mathbb{E}_{\Theta} \rho^F_{ij}) + \sum_{j \neq i} \int_{\Theta} \{ \hat{\rho}^F_{ij}(\theta) + \sum_{k \neq i} \hat{\rho}^F_{ik}(\theta) \} \hat{X}_j(d\theta).$$

The first term is wholly determined only from the aggregate strategy. It indeed represents the transition of the aggregate strategy if the switching rate is the same over different types. The second term is the correlation between the unconditional total switching rate $\rho_{ji}^F + \sum_{k \neq i} \rho^F_{ik}$ and the strategy composition $X_j$. If the unconditional total switching rate varies among types, the transition of the aggregate strategy depends on the strategy composition through the difference in this correlation term, as we see below.

Proof. Suppose that there exists a pair of two distinctive actions $i, j$ such that $\hat{x}_j > 0$ and the variation of $\hat{\rho}^F_{ji}(\theta) + \sum_{k \neq i} \hat{\rho}^F_{ik}(\theta)$ is not zero. This implies the existence of $\Theta^+ \subset \Theta$ such that $P_{\Theta}(\Theta^+) > 0$ and

$$\hat{\rho}^F_{ji}(\theta) + \sum_{k \neq i} \hat{\rho}^F_{ik}(\theta) > 0 \quad \text{for all } \theta \in \Theta^+$$

and the existence of $\Theta^- \subset \Theta$ such that $P_{\Theta}(\Theta^-) > 0$ and

$$\hat{\rho}^F_{ji}(\theta) + \sum_{k \neq i} \hat{\rho}^F_{ik}(\theta) < 0 \quad \text{for all } \theta \in \Theta^-.$$

Note that $P_{\Theta}(\Theta^+) \leq 1 - P_{\Theta}(\Theta^-) < 1$ and similarly $P_{\Theta}(\Theta^-) < 1$. 

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We have \( \varepsilon = 0.5 \min \{ \bar{x}_j, 1 - \bar{x}_i \} > 0 \), since \( \bar{x}_j > 0 \) and \( \bar{x}_i \leq 1 - \bar{x}_j < 1 \). Define \( X^+ \) by

\[
X_i^+(\Theta) = (\bar{x}_i + \varepsilon)P_{\Theta}(\Theta) - \varepsilon P_{\Theta}(\Theta^+),
\]

\[
X_j^+(\Theta) = (\bar{x}_j - \varepsilon)P_{\Theta}(\Theta) + \varepsilon P_{\Theta}(\Theta^+),
\]

\[
X_k^+(\Theta) = \bar{x}_kP_{\Theta}(\Theta) \quad \text{for all } k \neq i, j
\]

for each \( P_{\Theta} \)-measurable set \( \Theta \subset \Theta \). Note that \( X^+(\Theta) = \bar{x} \). Then, the correlation term reduces to

\[
\int_{\Theta} \{ \hat{\rho}_{ji}(\theta) + \sum_{k \neq i} \hat{\rho}_{ik}(\theta) \} X_j^+(d\theta) = \varepsilon \left( \frac{1}{P_{\Theta}(\Theta^+)} - 1 \right) \int_{\Theta} \{ \hat{\rho}_{ji}^0(\theta) + \sum_{k \neq i} \hat{\rho}_{ik}^0(\theta) \} P_{\Theta}(d\theta) > 0.
\]

Therefore, from this strategy composition \( X^+ \), the transition of the aggregate strategy is

\[
\dot{x}_i > \sum_{j \neq i} (\bar{x}_j E_{\Theta} \rho_{ji}^F - \bar{x}_i E_{\Theta} \rho_{ji}^F).
\]

We define \( X^- \) by

\[
X_i^-(\Theta) = (\bar{x}_i + \varepsilon)P_{\Theta}(\Theta) - \varepsilon P_{\Theta}(\Theta^-),
\]

\[
X_j^-(\Theta) = (\bar{x}_j - \varepsilon)P_{\Theta}(\Theta) + \varepsilon P_{\Theta}(\Theta^-),
\]

\[
X_k^-(\Theta) = \bar{x}_kP_{\Theta}(\Theta) \quad \text{for all } k \neq i, j
\]

for each \( P_{\Theta} \)-measurable set \( \Theta \subset \Theta \). Again, \( X^-(\Theta) = \bar{x} \). Then, similarly to the above calculation, we obtain the negative correlation and

\[
\dot{x}_i < \sum_{j \neq i} (\bar{x}_j E_{\Theta} \rho_{ji}^F - \bar{x}_i E_{\Theta} \rho_{ji}^F).
\]

Thus, these two strategy compositions \( X^+ \) and \( X^- \) yield different transitions of \( \dot{x}_i \) and thus of \( \bar{x} \), though they share the same aggregate strategy \( \bar{x} \).

\[\square\]

### C Appendix to Section 5

For stability, we use the weak topology and apply the Lyapunov stability theorem, as in Cheung (2014).

**Theorem 9** (Cheung, 2014: Theorems 5–6, Corollary 2). Let \( Z \subset X \) be a closed set and let \( Y \subset X \) be a neighborhood of \( Z \) in the weak topology on \( X \). Let \( L : Y \to \mathbb{R} \) be a decreasing Lyapunov function for dynamic \( V \): that is, \( L \) is continuous with respect to the weak topology and Fréchet-differentiable with \( L(X) = \langle \nabla L(X), V|X \rangle \leq 0 \) for all \( X \in Y \). Then, the following holds.

i) Any solution path starting from \( Y \) converges to the set \( \{ X \in Y \mid \dot{L}(X) = 0 \} \) with respect to the weak topology; i.e., this set is attracting under \( V \).

ii) If \( L^{-1}(0) = Z \), \( Z \) is Lyapunov stable under \( V \) with respect to the weak topology. Furthermore, if \( \dot{L}(X) < 0 \) whenever \( X \in Y \setminus Z \), then \( Z \) is asymptotically stable under \( V \).
Part i) holds for an increasing Lyapunov function; part ii) is retained by defining \( Z \) as an isolated set of local maxima.

C.1 Proof of Theorem 3

Proof. \( x \) being a Bayesian equilibrium is equivalent to \( x(\theta) \in B[E_{\Theta}x](\theta) \) for \( \mathbb{P}_\Theta \)-almost all types \( \theta \). Then, for such \( \theta \), this is equivalent to \( v^F[x](\theta) = 0 \) by (6). It holds for \( \mathbb{P}_\Theta \)-almost all types \( \theta \), which means the stationarity of Bayesian strategy \( x \). Note that \( x \) being a Bayesian equilibrium is equivalent to the corresponding strategy composition \( X = \int xd\mathbb{P}_\Theta \) being an equilibrium composition and that the stationarity of Bayesian strategy \( x \) is equivalent to stationarity of strategy composition \( X \), i.e., \( V^F[X] = 0 \).

\[ \square \]

C.2 Proof of Theorem 4

Proof. i) Since \( f \) is a potential function for \( F \), we have

\[ \dot{f}(X) = \langle \nabla f(X), \dot{X} \rangle = \langle F[\dot{x}], V^F[X] \rangle = \mathbb{E}_{\Theta} [F[\dot{x}](\theta) \cdot v^F[x](\theta)], \]

where \( \dot{x} = X(\Theta) \) and \( X = \int x d\mathbb{P}_\Theta \).

Since \( v^F[x](\theta) = v^\theta(F[\dot{x}](\theta), x(\theta)) \), the first part of (8) implies \( F[\dot{x}](\theta) \cdot v^F[x](\theta) \geq 0 \) for all \( \theta \) and thus

\[ \dot{f}(X) = \mathbb{E}_{\Theta} [F[\dot{x}](\theta) \cdot v^F[x](\theta)] \geq 0. \]

Suppose \( x \) is not a Bayesian equilibrium. By Theorem 3, this is equivalent to non-stationarity of the Bayesian strategy \( x \), i.e., \( \mathbb{P}_\Theta \left( \{ \theta : v^F[x](\theta) \neq 0 \} \right) > 0 \). For a type with \( v^F[x](\theta) \neq 0 \), the second part of (8) implies \( F[\dot{x}](\theta) \cdot v^F[x](\theta) > 0 \). Since this holds for a positive mass of types, we have

\[ \dot{f}(X) = \mathbb{E}_{\Theta} [F[\dot{x}](\theta) \cdot v^F[x](\theta)] > 0. \]

Therefore, \( f \) is a strictly increasing Lyapunov function. By Theorem 9, this implies that the set of Bayesian equilibria is globally attracting; a local strict maximum of \( f \) is locally asymptotically stable.

ii) Suppose that the corresponding isolated equilibrium composition \( X^* \) is asymptotically stable, with a basin of attraction \( \mathcal{X}^* \subset \mathcal{X} \). Take an arbitrary strategy composition \( X_0 \) from \( \mathcal{X}^* \) and let \( \{ X_t \}_{t \in \mathbb{R}_+} \) be a solution trajectory under the heterogeneous dynamic from \( X_0 \). Since \( f \) is a strictly increasing Lyapunov function, it must be the case that \( \dot{f}(X_t) > 0 \) as long as \( X_t \) has not reached exactly \( X^* \). Thus, \( f(X^*) = f(X_0) + \int_0^\infty f(\dot{X}_t) dt > f(X_0) \). Since \( X_0 \) is taken arbitrarily from \( \mathcal{X}^* \), this verifies that \( X^* \) strictly maximizes \( f \) in this neighborhood \( \mathcal{X}^* \).

\[ \square \]

\[^{52}O = (O_a)_{a \in A} \in M_{A \times \Theta} \) denotes a zero vector measure such as \( O_a(B_\Theta) = 0 \) for any \( B_\Theta \in B_\Theta, a \in A. \]

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C.3 Proof of Theorem 5

In terms of strategy composition \( X = \int x d\Pi_{\Theta} \in \mathcal{X} \), we redefine the heterogeneous potential function \( f : \mathcal{X} \to \mathbb{R} \) as

\[
f(X) = f^0(X(\Theta)) + \int_{\Theta} \sum_{a \in A} \theta_a X_a(d\theta).
\]

**Proof.** As \( f(X) = f^0(X(\Theta)) + E_{\Theta}[\theta \cdot x(\Theta)] \), weak continuity of \( f \) is obtained from continuity of \( f^0 \) and the dominated convergence theorem. By applying the definition of \( f \) to \( \Delta X \), we have

\[
f(X + \Delta X) = f^0(\bar{x} + \Delta \bar{x}) + \int_{\Theta} \sum_{a \in A} \theta_a (X_a + \Delta X_a)(d\theta)
\]

\[
= \left\{ f^0(\bar{x}) + \nabla f^0(\bar{x}) \cdot \Delta \bar{x} + o(|\Delta \bar{x}|) \right\} + \left\{ \int_{\Theta} \sum_{a \in A} \theta_a X_a(d\theta) + \int_{\Theta} \sum_{a \in A} \theta_a \Delta X_a(d\theta) \right\}
\]

\[
= f(X) + F^0(\bar{x}) \cdot \Delta \bar{x} + \int_{\Theta} \sum_{a \in A} \theta_a \Delta X_a(d\theta) + o(|\Delta \bar{x}|)
\]

Here \( \bar{x} = X(\Theta) \) and \( \Delta \bar{x} = \Delta X(\Theta) \). The second equality comes from differentiability of \( f^0 \); the third is from the assumption that \( f^0 \) is a potential function of \( F^0 \) and the definition of \( f \) applied to \( X \). Then, we should recall \( F_\theta[X(\Theta)](\Theta) = F_\theta^0[X(\Theta)] + \theta_\theta \). So the second term is \( (F[X(\Theta)], \Delta X) \). About the third error term, note that \( |\Delta \bar{x}| = |\Delta X(\Theta)| \leq \|\Delta X\| \). Therefore, we obtain

\[
f(X + \Delta X) = f(X) + (F[X(\Theta)], \Delta X) + o(\|\Delta X\|).
\]

Thus, \( f \) is (Fréchet) differentiable with derivative \( \nabla f(X) \equiv F[X(\Theta)] \). So we have verified that \( f \) is a potential function of the game \( F \) defined on \( \mathcal{X} \).

C.4 Proofs of Lemma 1 and Theorem 6

**Proof of Lemma 1.** i) First of all, notice that the first order derivative of the hypothetical payoff deficit \( h(\cdot; \bar{x}) \) with respect to \( \bar{\pi}_i \) of each action \( i \in A \) is

\[
\frac{\partial h}{\partial \bar{\pi}_i}(\bar{\pi}; \bar{x}) = \Pi_{\Theta} (\{ \theta : i = \arg \max_{a \in A} (\bar{\pi}_a + \theta_a) \}) - \bar{x}_i.
\]

This objective function satisfies the first order condition for convexity and thus it is convex in \( \bar{\pi} \): for any \( \bar{\pi}, \Delta \bar{\pi} \in \mathbb{R}^A \), we have

\[
h(\bar{\pi} + \Delta \bar{\pi}; \bar{x}) - h(\bar{\pi}; \bar{x})
\]

\[
= \mathbb{E}_{\Theta} \left\{ \max_{i' \in A} \left( \bar{\pi}_{i'} + \Delta \bar{\pi}_{i'} + \theta_{i'} \right) - \max_{i \in A} (\bar{\pi}_i + \theta_i) \right\} - \Delta \bar{\pi} \cdot \bar{x}
\]

\[
= \mathbb{E}_{\Theta} \left\{ \max_{i' \in A} \left( \bar{\pi}_{i'} + \Delta \bar{\pi}_{i'} + \theta_{i'} \right) - (\bar{\pi}_{i_{s}(\theta)} + \theta_{i_{s}(\theta)}) \right\} - \Delta \bar{\pi} \cdot \bar{x}
\]

\[
\geq \mathbb{E}_{\Theta} \Delta \bar{\pi}_{i_{s}(\theta)} - \Delta \bar{\pi} \cdot \bar{x}
\]
= \sum_{i \in A} \Delta \pi_i(\{ \theta \in \Theta : i = \arg \max_{a \in A}(\pi_a + \theta_a) \}) - \bar{x}_i = \frac{\partial h}{\partial \bar{\pi}}(\bar{\pi}; \bar{x}) \Delta \bar{\pi},

where \(i^*(\theta) = \arg \max_{a \in A}(\pi_a + \theta_a)\). The first order condition for a minimum is

\[
\frac{\partial h}{\partial \bar{\pi}}(\bar{\pi}; \bar{x}) = \mathbb{P}_\Theta(\{ \theta \in \Theta : i = \arg \max_{a \in A}(\pi_a + \theta_a) \}) - \bar{x}_i = 0 \quad \text{with respect to each } i \in A,
\]

and it is both necessary and sufficient thanks to the convexity. (Note that \(\bar{\pi}\) can be any vector in \(\mathbb{R}^A\).) This first order condition is indeed satisfied by the rationalizing common payoff vector \(\bar{\pi}^\circ[\bar{x}]\).

ii) Conditions b) and c) of the definition of a completely sorted Bayesian strategy imply

\[
(\bar{\pi}^\circ[\bar{x}] + \theta) \cdot x^\circ[\bar{x}](\theta) = \max_{a \in A}(\pi_a^\circ[\bar{x}] + \theta_a) \quad \text{for } \mathbb{P}_\Theta\text{-almost every } \theta \in \Theta.
\]

Aggregating this over all the types \(\theta \in \Theta\), we obtain

\[
\mathbb{E}_\Theta[(\bar{\pi}^\circ[\bar{x}] + \theta) \cdot x^\circ[\bar{x}](\theta)] = \mathbb{E}_\Theta[\max_{a \in A}(\pi_a^\circ[\bar{x}] + \theta_a)] = h(\bar{\pi}^\circ[\bar{x}]; \bar{x}) + \mathbb{E}_\Theta \theta \cdot \bar{x}.
\]

Since \(\mathbb{E}_\Theta[\pi^\circ[\bar{x}] \cdot x^\circ[\bar{x}](\theta)] = \pi^\circ[\bar{x}] \cdot \bar{x}\) by condition a) of the definition of \(x^\circ[\bar{x}]\), we have

\[
\mathbb{E}_\Theta[\theta \cdot x^\circ[\bar{x}](\theta)] = \mathbb{E}_\Theta[\max_{a \in A}(\pi_a^\circ[\bar{x}] + \theta_a)] - \pi^\circ[\bar{x}] \cdot \bar{x} = h(\pi^\circ[\bar{x}]; \bar{x}) + \mathbb{E}_\Theta \theta \cdot \bar{x}.
\]

By part i) of the current lemma, we have

\[
h(\bar{\pi}^\circ[\bar{x}]; \bar{x}) = \min_{\pi \in \mathbb{R}^A} h(\bar{\pi}; \bar{x}) = \min_{\pi \in \mathbb{R}^A} \left( \mathbb{E}_\Theta[\max_{a \in A}(\pi_a + \theta_a)] - (\bar{\pi} + \mathbb{E}_\Theta \theta) \cdot \bar{x} \right).
\]

Combining these two equations, we obtain the equation in part ii).

iii) It is immediate from condition a) of a completely sorted Bayesian strategy that 2) implies 1). To see that 1) implies 3), notice that the aggregate equilibrium \(\bar{x}\) satisfies

\[
\bar{x}^*_i = \mathbb{P}_\Theta(b_i^{-1}(\bar{x}^*)) = \mathbb{P}_\Theta(\{ \theta \in \Theta : i = \arg \max_{a \in A}(\bar{F}_a^0(\bar{x}^*) + \theta_a) \})
\]

for each \(i \in A\) under the assumption of a continuous type distribution. Thus, \(\bar{\pi} = \bar{F}_0(\bar{x}^*)\) satisfies the first order condition to minimize \(h(\bar{\pi}; \bar{x}^*)\) and thus it is the rationalizing payoff vector. Now we verify that 3) implies 2) to close the loop of equivalence. With \(\pi^\circ[\bar{x}^*] = \bar{F}_0(\bar{x}^*)\), conditions b) and c) in the definition of \(x^\circ[\bar{x}^*]\) imply that almost every type attains the greatest of \(F_i^0(\bar{x}^*) + \theta_i\) among all actions \(i \in A\) at this Bayesian strategy \(x^\circ[\bar{x}^*]\): that is, \(x^\circ[\bar{x}^*]\) is a Bayesian equilibrium in the ASAG with \(\bar{F}_0\).

\[\square\]

Proof of Theorem 6. i) From (9), observe that, for any \(x \in \mathcal{F}_X\) and \(\pi \in \mathbb{R}^A\)

\[
f(x) = f^0(\mathbb{E}_\Theta x) + \mathbb{E}_\Theta[(\pi + \theta) \cdot x(\theta)] - \pi \cdot \mathbb{E}_\Theta x
\]

\[
\leq f^0(\mathbb{E}_\Theta x) + \mathbb{E}_\Theta[\max_{i \in A}(\pi_i + \theta_i)] - \pi \cdot \mathbb{E}_\Theta x.
\]

As this holds for any \(\pi \in \mathbb{R}^A\), the definition of \(\bar{f}\) implies

\[
f(x) \leq \bar{f}(\mathbb{E}_\Theta x) = \min_{\pi \in \mathbb{R}^A} \left( f^0(\mathbb{E}_\Theta x) + \mathbb{E}_\Theta[\max_{i \in A}(\pi_i + \theta_i)] - \pi \cdot \mathbb{E}_\Theta x \right).
\]

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First, suppose $\bar{f}(\bar{x}) = f(x)$ with $\bar{x} = \mathbb{E}_\Theta x(\theta)$. With the definition of the homogenized potential $\bar{f}$, part i) of Lemma 1 assures that $\bar{f}(\bar{x}) = f^0(\bar{x}) + \mathbb{E}_\Theta[\max_{i \in A}(\bar{\pi}_i + \theta_i)] - \bar{\pi} \cdot \bar{x}$ if and only if $\bar{\pi} = \pi^\circ[\bar{x}]$. Then, the equality holds in (C.6) if and only if $\mathbb{E}_\Theta[(\pi^\circ[\theta] + \theta) \cdot x(\theta)] = \mathbb{E}_\Theta[\max_{a \in A}(\pi^\circ_a[\theta] + \theta_a)]$, i.e.,

$$x(\theta) = \arg\max_{y(\theta) = (y_a(\theta))_{a \in A} \in \Delta^A} y(\theta) \cdot (\pi^\circ[\theta] + \theta) = \sum_{a \in A} (\pi^\circ_a[\theta] + \theta_a)y_a(\theta)$$

for $\mathbb{P}_\Theta$-almost all $\theta \in \Theta$; that is, in Bayesian strategy $x$, almost all agents take their optimal action given $\pi^\circ[\theta]$. This means that $x$ is a completely sorted Bayesian strategy with $\bar{x} = \mathbb{E}_\Theta x(\theta)$.

On the other hand, Suppose that $x$ is a completely sorted Bayesian strategy given aggregate strategy $\bar{x} = \mathbb{E}_\Theta x$. With the definition of the heterogeneous potential $\bar{f}$, part ii) of Lemma 1 implies

$$f(x) = f^0(\bar{x}) + \mathbb{E}_\Theta[\theta \cdot x] = \bar{f}(\mathbb{E}_\Theta x).$$

iii) It is immediate from parts i) and ii) that $x$ being a local maximum of $f$ implies $\bar{x}$ being a local maximum of $\bar{f}$.

Here we prove the opposite direction. Since we need to argue locality and thus topology, it is more accurate to discuss strategy compositions in $\mathcal{X}$, rather than Bayesian strategies in $\mathcal{F}_X$. For this, let $X^*$ be the equilibrium composition with $X^* = \int x^* d\mathbb{P}_\Theta$; then $X^*(\Theta) = \mathbb{E}_\Theta x^* = \bar{x}^*$.

$X^*$ being a strict local maximum of $\bar{f}$ means that there exists a neighborhood $\bar{O}^* \subset \Delta^A$ of $\bar{x}^*$ such that

$$\bar{x} \in \bar{O}^* \text{ and } \bar{x} \neq \bar{x}^* \implies \bar{f}(\bar{x}) < \bar{f}(\bar{x}^*).$$

Since aggregation operator $X(\Theta)$ is a continuous function of $X \in \mathcal{X}$, the preimage of $\bar{O}^*$ through $\mathbb{E}_\Theta$, i.e., $O^* := \{X \in \mathcal{X} \mid X(\Theta) \in \bar{O}^*\}$ is an open set in $\mathcal{X}$. Thus, (C.7) implies

$$X \in O^* \text{ and } X(\Theta) \neq x^* \implies \bar{f}(X(\Theta)) < \bar{f}(x^*) = \bar{f}(X^*(\Theta)).$$

Since $X^*$ is an equilibrium composition, part i) of Theorem 6 further implies

$$X \in O^* \text{ and } X(\Theta) \neq x^* \implies f(X) \leq \bar{f}(X(\Theta)) < \bar{f}(X^*(\Theta)) = f(X^*).$$

On the other hand, part ii) of Theorem 6 implies\(^{53}\)

$$X(\Theta) = x^* \text{ and } X \neq X^* \implies f(X) < \bar{f}(X(\Theta)) = \bar{f}(X^*(\Theta)) = f(X^*).$$

Combining these two cases, we obtain

$$X \in O^* \text{ and } X \neq X^* \implies f(X) < f(X^*).$$

That is, $X^*$ is a strict local maximum of $f$ in $\mathcal{X}$.

iv) This is immediate from parts i) and ii).

---

\(^{53}\)As $x^*$ is an isolated aggregate equilibrium, (3) holds with equality: $\mathbb{P}_\Theta(\beta_a^{-1}(x^*)) = \mathbb{P}_\Theta(\beta_a^{-1}(x^*))$ for all $a \in A$ and thus the best response is uniquely determined for $\mathbb{P}_\Theta$-almost all types. Thus the corresponding Bayesian equilibrium is uniquely determined.
Supplementary note

“Heterogeneity and aggregation in evolutionary dynamics ”

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April 5, 2018

S1 Supplementary note on Section 1

S1.1 Example in Figure 1

The transition vector at time 0

At the initial aggregate strategy $\bar{x}_0 = 0.5\bar{x}_0(\theta^H) + 0.5\bar{x}_0(\theta^L) = (\epsilon, 0.5(1 - \epsilon), 0.5(1 - \epsilon))$, the payoff vector for type $\theta$ is $F[\bar{x}_0](\theta) = (1 + \theta, 1, 1)$. Therefore, as long as $\theta > 0$, action A is the unique best response and the other two actions B and C are equally worse than A. Therefore, in all the major dynamics illustrated in the subfigures, all the agents switch to A and none switches between B and C. Besides, action B and action C initially yield the same payoffs and the aggregate masses of these two actions are initially equal. Hence, the switching rate from B to A and that from C to A are equal to each other at time 0, though these rates depend on the type of agents and they will differ later when the symmetry between B and C in the aggregate strategy breaks. Given the initial strategy composition $x_0$, none of type-$\theta^H$ agents takes action B at time 0 and none of type-$\theta^L$ agents take action C. But this symmetry in the switching rate is important below, when comparing the transition from this strategy composition and that from another flipped composition.

Denote type $\theta^H$’s initial switching rate from B or C to A by $\rho^H_0$ and that of type $\theta^L$ by $\rho^L_0$. These are obtained for each dynamic as in Table 1.

The transition vectors of each type’s Bayesian strategy at time 0 are obtained as follows:

$$\dot{x}_0(\theta^H) = \rho^H_0(1 - \epsilon)(e^A - e^B), \quad \dot{x}_0(\theta^L) = \rho^L_0(1 - \epsilon)(e^A - e^C).$$

By aggregating these transition vectors over the two types, we can get the transition vector of the

| Dynamic         | $\rho^H_0$ | $\rho^L_0$ |
|-----------------|------------|------------|
| Standard BRD    | 1          | 1          |
| Tempered BRD    | $Q(\theta^H)$ | $Q(\theta^L)$ |
| Smith           | $\theta^H$ | $\theta^L$ |
| BNN             | $(1 - \epsilon)\theta^H$ | $(1 - \epsilon)\theta^L$ |
| Replicator      | $\epsilon\theta^H$ | $\epsilon\theta^L$ |

*Table 1:* Switching rates from a suboptimal action B or C to the optimal action A, given the strategy composition $x_0$ as in Figure 1. For BNN, it is assumed that a revising agent compares her own payoff with the average payoff of agents of the same type. For replicator, since the proportion of action-A players is $\epsilon$ both in the aggregate strategy or in each type’s Bayesian strategy, it does not matter whether agents are sampling from the whole population or the same type of agents.
aggregate strategy at time 0:
\[ \dot{x}_0 = 0.5 \dot{x}_0(\theta^H) + 0.5 \dot{x}_0(\theta^L) = (1 - \epsilon) \left\{ \rho^H_0(e^A - e^B) + \rho^L_0(e^A - e^C) \right\}. \]

Thus, the initial transition vector is asymmetrically tilted toward \( e^A - e^B \), if \( \rho^H_0 > \rho^L_0 \). From Table 1, we can see that this is the case for all the dynamics, except the standard BRD.

**Comparison with another initial strategy composition**

To confirm the dependency of the aggregate strategy trajectory on the initial strategy composition, we consider another strategy composition \( x'_0 \):
\[ x'_0(\theta^H) = (\epsilon, 0, 1 - \epsilon), \quad x'_0(\theta^L) = (\epsilon, 1 - \epsilon, 0). \]

This yields the same aggregate strategy \( \dot{x}'_0 = 0.5 \dot{x}'_0(\theta^H) + 0.5 \dot{x}'_0(\theta^L) = (\epsilon, 0.5(1 - \epsilon), 0.5(1 - \epsilon)) \) as the first example. Figure 3 shows the trajectory of the aggregate strategy under each of the five major dynamics, starting from \( x'_0 \). Note that, by the same calculation as above, we can easily obtain the transition vector of the aggregate strategy at time 0 as
\[ \dot{x}'_0 = (1 - \epsilon) \left\{ \rho^H_0(e^A - e^C) + \rho^L_0(e^A - e^B) \right\}. \]

Note that the switching rate of each type from a suboptimal action to the optimal action A is the same as in the last example, i.e., the same as in Table 1. Therefore, if \( \rho^H_0 > \rho^L_0 \), i.e., in all the dynamics except the standard BRD the initial transition vector is now tilted toward \( e^A - e^C \).
S2 Supplementary note on Sections 2–3

S2.1 Norms on $X$ and on $F_{X}$

Proof of Theorem 7

Proof. An arbitrary measurable function $g : \Theta \rightarrow \mathbb{R}^{A}$ bounded by 1 satisfies

$$
\sum_{a \in A} \int_{\theta \in \Theta} g_a(\theta) M_a(d\theta) = \sum_{a \in A} \int_{\theta \in \Theta} g_a(\theta) m_a(\theta) P_\Theta(d\theta).
$$

It follows that

$$
\left| \sum_{a \in A} \int_{\theta \in \Theta} g_a(\theta) m_a(\theta) P_\Theta(d\theta) \right| \leq \sum_{a \in A} \int_{\theta \in \Theta} |g_a(\theta) m_a(\theta)| P_\Theta(d\theta) \leq \sum_{a \in A} \int_{\theta \in \Theta} |m_a(\theta)| P_\Theta(d\theta).
$$

The last inequality comes from $g_a$ being bounded by 1. As this holds for any such $g$, the supremum cannot exceed $\sum_{a} \int_{\Theta} |m_a| dP_\Theta.$

On the other hand, define function $\bar{g} : \Theta \rightarrow \mathbb{R}^{A}$ by $\bar{g}_a(\theta) = 1 \{ m_a(\theta) > 0 \} - 1 \{ m_a(\theta) \leq 0 \}$. Then,

$$
\| M \| \geq \sum_{a \in A} \int_{\theta \in \Theta} \bar{g}_a(\theta) M_a(d\theta)
$$

$$
= \sum_{a \in A} \int_{\theta \in m_a^{-1}(\mathbb{R}_{++})} 1 \cdot m_a(\theta) |P_\Theta(d\theta)| + \int_{\theta \in m_a^{-1}(\mathbb{R}_{-})} (-1) \cdot m_a(\theta) |P_\Theta(d\theta)|
$$

$$
= \sum_{a \in A} \int_{\theta \in m_a^{-1}(\mathbb{R}_{++})} |m_a(\theta)| |P_\Theta(d\theta)| + \int_{\theta \in m_a^{-1}(\mathbb{R}_{-})} |m_a(\theta)| |P_\Theta(d\theta)|
$$

$$
= \sum_{a \in A} \int_{\theta \in \Theta} |m_a(\theta)| P_\Theta(d\theta).
$$

Combining these two inequalities, we verify the claim. \qed

S2.2 Proof of Theorem 1

Proof of Lipschitz continuity of $V$ (part i of Theorem 1)

Henceforth, as we focus on the $P_\Theta$-absolutely continuous parts of finite signed measures, we omit the tilde from such measures. In the following proofs of part i, we consider two $P_\Theta$-absolutely continuous finite signed measures $M, M' \in \mathcal{M}_{A \times \Theta}$ with densities $m$ and $m'$. Let $\bar{m} = M(\Theta)$, $\bar{\mu} = \mu(\bar{m})$, $\mu(\cdot) := \mu(m(\cdot))$ and $\rho_{ij}^{F}(\theta) := \rho_{ij}^{F}(F|\tilde{\mu}|(\theta), \mu(\theta))$; similarly we define $\bar{m}', \bar{\mu}', \mu'(\cdot)$ and $\rho_{ij}^{F}(\theta)$. Denote $\Delta v_{i}^{F}(\theta) := v_{i}^{F}(\theta)\{M\} - v_{i}^{F}(\theta)\{M'\}$.

We divide $\Theta$ by the two classes of revision protocols: let $\Theta_C$ be the set of the types of agents who follow continuous revision protocols and $\Theta_E$ be the set of the types who follow exact optimization protocols; we have $\Theta_C \cup \Theta_E = \Theta$ and $\Theta_C \cap \Theta_E = \emptyset$. First we prove Lipschitz continuity of the transition of the strategy composition in each of these sets of types. Then, we merge them to get Lipschitz continuity of the transition of the entire strategy composition.

Proof. 1°: continuous revision protocols. Now we focus on $\Theta_C$. Let $\tilde{L}_{\rho} > 0$ be the upper bound
on the Lipschitz constants of functions \( \rho_{ji}^\theta \) over all pairs of actions \( i, j \in A \) and all types \( \theta \in \Theta_C \). The Lipschitz continuity of \( \rho_{ji}^\theta \) (Definition 1) and \( F \) (Assumption 1) implies

\[
|\rho_{ji}^F(\theta) - \rho_{ji}'^F(\theta)| \leq L_\rho |(F[\hat{\mu}](\theta), \mu(\theta)) - (F[\hat{\mu}'](\theta), \mu'(\theta))|
\leq L_\rho \left\{ |F[\hat{\mu}](\theta) - F(\hat{\mu}')(\theta)| + |\mu(\theta) - \mu'(\theta)| \right\}
\leq \bar{L}_\rho \left( L_F(\theta)|\hat{\mu} - \mu'| + |\mu(\theta) - \mu'(\theta)| \right)
\leq \bar{L}_\rho \left( L_F(\theta)L_\mu |\bar{m} - \bar{m}'| + L_\mu |\bar{m}(\theta) - \bar{m}'(\theta)| \right).
\] (S.1)

From the definition of \( v_i^{F^+} \), we have

\[
|v_i^{F^+}[M](\theta) - v_i^{F^+}[M'](\theta)| \leq \sum_{j \in A} |\rho_{ij}(\theta)\mu_j(\theta) - \rho_{ij}'(\theta)\mu_j'(\theta)|
\leq \sum_{j \in A} \left\{ |\rho_{ij}^F(\theta) - \rho_{ij}'^F(\theta)| |\mu_j(\theta)| + |\rho_{ij}'^F(\theta)| \cdot |\mu_j(\theta) - \mu_j'(\theta)| \right\}
\leq \sum_{j \in A} [3\bar{L}_\rho \left( L_F(\theta)L_\mu |\bar{m} - \bar{m}'| + L_\mu |\bar{m}(\theta) - \bar{m}'(\theta)| \right) + \rho L_\mu |\bar{m}(\theta) - \bar{m}'(\theta)|]
\leq A \left\{ 3\bar{L}_\rho L_F(\theta)L_\mu |\bar{m} - \bar{m}'| + (3\bar{L}_\rho + \rho)L_\mu |\bar{m}(\theta) - \bar{m}'(\theta)| \right\}
\] (S.2)

Here the third inequality comes from (S.1), Assumption 2 and \(|\mu_j(\cdot)| \leq 3 \). Similarly, we get

\[
|v_i^{F^-[M]}(\theta) - v_i^{F^-[M]'}(\theta)| \leq A \left\{ 3\bar{L}_\rho L_F(\theta)L_\mu |\bar{m} - \bar{m}'| + (3\bar{L}_\rho + \rho)L_\mu |\bar{m}(\theta) - \bar{m}'(\theta)| \right\}.
\]

Therefore, we have

\[
\int_{\Theta_C} \sum_{i \in A} |\Delta v_i^F(\theta)||P_\Theta(d\theta)
\leq \int_{\Theta_C} \sum_{i \in A} \left( |v_i^{F^+[M]}(\theta) - v_i^{F^+[M]'}(\theta)| + |v_i^{F^-[M]}(\theta) - v_i^{F^-[M]'}(\theta)| \right) P_\Theta(d\theta)
\leq \int_{\Theta_C} \left[ \sum_{i \in A} 2A \left\{ 3\bar{L}_\rho L_F(\theta)L_\mu |\bar{m} - \bar{m}'| + (3\bar{L}_\rho + \rho)L_\mu |\bar{m}(\theta) - \bar{m}'(\theta)| \right\} \right] P_\Theta(d\theta)
\leq 2A^2 \left( 3\bar{L}_\rho L_\mu |\bar{m} - \bar{m}'| \int_{\Theta_C} L_F(\theta)P_\Theta(d\theta) + 2A^2(3\bar{L}_\rho + \rho)L_\mu \int_{\Theta_C} |\bar{m}(\theta) - \bar{m}'(\theta)|P_\Theta(d\theta) \right)
\leq 2A^2(3\bar{L}_\rho L_F + 3\bar{L}_\rho + \rho)L_\mu \|M - M'\|.
\] (S.3)

The last inequality comes from \( \int_{\Theta_C} L_F(\theta)P_\Theta(d\theta) \leq \mathbb{E}_\Theta L_F = \bar{L}_F \) by \( L_F(\theta) \geq 0 \) and \( \|\bar{m} - \bar{m}'\| = |M(\Theta) - M'(\Theta)| \leq \|M - M'\| \).

2°: exact optimization protocols. Now we focus on \( \Theta_E \). In an exact optimization protocol, the dynamic reduces as the following: if action \( b \) is the unique maximizer of \( F_a(\bar{\mu}; \theta) \) among all actions \( a \in A \), i.e., the unique best response to \( \bar{\mu} \) for type \( \theta \), then

\[
v_b(\theta)[M] = \sum_{j \in A \{b\}} Q_{jb}(\bar{\mu}; \theta)\mu_j(\theta),
\]

\[
v_i(\theta)[M] = -Q_{ib}(\bar{\mu}; \theta)\mu_i(\theta) \quad \text{for any } i \in A \{b\}.
\]

Since Assumption 3 implies that the best response is unique for almost every type, this determines the composite dynamic without ambiguity.
Let $L_Q > 0$ be an upper bound on Lipschitz constants of functions $Q^\theta_{ij}$ over all pairs of actions $i, j \in A$ and almost all the types $\theta \in \Theta_F$. Let $\beta^{-1}_b(\mu)$ be the set of types for whom $b \in A$ is the unique optimal action given $F[\mu](\theta)$, and $N$ be the set of types for which there are multiple best responses at $F[\mu](\theta)$ or $F[\mu'](\theta)$. Assumption 3 implies $\mathbb{P}_\Theta(N) = 0$. Define partitions of $\Theta \setminus N$ by

$$\cap \beta := \beta^{-1}_b(\mu) \cap \beta^{-1}_c(\mu') \cap \Theta_E, \quad \Delta \beta := \beta^{-1}_b(\mu) \cap \beta^{-1}_c(\mu') \cap \Theta$$

for each $b \in A, c \in A \setminus \{b\}$.

Let $\eta_{\mu} := \beta^{-1}_b(\mu) \cap \beta^{-1}_c(\mu') \cap \Theta$.

Denote $Q^F_{\mu}(\theta) := Q^\theta_{\mu}(F[\mu](\theta))$ and $Q^{F}_{\mu}':(\theta) := Q^\theta_{\mu}'(F[\mu'](\theta))$. Similarly to (S.1), the Lipschitz continuity of $Q^\theta_{ji}$ (Definition 2) and (S.4) implies

$$|Q^F_{\mu}(\theta) - Q^{F}_{\mu}(')(\theta)| \leq L_{\mu} L_F (L_F(\theta)|\hat{m} - \hat{m}'| + |m(\theta) - m'(\theta)|)$$

for all $i, j \in A$, and $\mathbb{P}_\Theta$-almost all $\theta \in \Theta_E$. Note that, if $b \in b[\mu](\theta)$, then Assumption 2 assures the existence of an upper bound $\bar{\rho}$ on $Q^\theta_{ib}(\theta)$ such as $Q^F_{ib}(\theta) = \rho^\theta_{ib}(F[\mu](\theta)) \leq \bar{\rho}$ for any $j \in A$ and $\theta \in \Theta_F$.

Consider $i \in A$ for an arbitrary $b \in A$. Fix $\theta \in \cap \beta$; action $b$ is the optimal action for this type $\theta$ both in the state $M$ and the state $M'$. Then, similarly to (S.2), Assumption 2 and (S.4) imply

$$|\Delta \nu_i(\theta)| \leq \sum_{j \in A \setminus \{b\}} |Q^F_{\mu}(\theta) \mu_j(m(\theta)) - Q^{F}_{\mu} '(\theta) \mu_j(m'(\theta))|$$

$$\leq (A - 1) \left\{L_Q L_F (L_F(\theta)|\hat{m} - \hat{m}'| + (3L_Q + \bar{\rho})L_\mu |m(\theta) - m'(\theta)|) \right\}.$$

For action $i \not= b$,

$$\Delta \nu_i(\theta) = (-Q^F_{ib}(\theta) \mu_i(\theta)) - (-Q^{F}_{ib} (')(\theta) \mu_i(\theta)) = -\{Q^F_{ib}(\theta) - Q^{F}_{ib} (')(\theta)\} \mu_i(\theta) - Q^F_{ib}(\theta) \{\mu_i(\theta) - \mu_i'(\theta)\}.$$ (S.4) implies

$$|\Delta \nu_i(\theta)| \leq L_\mu (L_F(\theta)|\hat{m} - \hat{m}'| + |m(\theta) - m'(\theta)||) |\mu_i(\theta)| + Q^{F}_{ib} (')(\theta) L_\mu |m(\theta) - m'(\theta)|$$

$$\leq (3L_Q L_F (L_F(\theta)|\hat{m} - \hat{m}'| + (3L_Q + \bar{\rho})L_\mu |m(\theta) - m'(\theta)|.$$

The second inequality comes from boundeness of $Q^\theta$ and $\mu$.

Therefore, we have

$$\sum_{a \in A} |\Delta \nu_a(\theta)| \leq 2(A - 1)L_\mu \left\{3L_Q L_F (\theta)|\hat{m} - \hat{m}'| + (3L_Q + \bar{\rho})|m(\theta) - m'(\theta)| \right\}$$

and thus

$$\int_{\cap \beta} \sum_{a \in A} |\Delta \nu_a(\theta)| \mathbb{P}_\Theta(d\theta)$$

$$\leq 2(A - 1)L_\mu \left\{3L_Q \int_{\cap \beta} L_F (\theta)|\hat{m} - \hat{m}'| \mathbb{P}_\Theta(d\theta) + (3L_Q + \bar{\rho}) \int_{\cap \beta} |m(\theta) - m'(\theta)| \mathbb{P}_\Theta(d\theta) \right\}$$

$$\leq 2(A - 1)L_\mu (3L_Q L_F + 3L_Q + \bar{\rho})|M - M'|.$$(S.5)

The second inequality comes from $\mathbb{P}_\Theta(\cap \beta) \leq \mathbb{P}_\Theta(\Theta) = 1, |\hat{m} - \hat{m}'| \leq |M - M'|, \int_{\cap \beta} L_F (\theta) \mathbb{P}_\Theta(d\theta) \leq \mathbb{E}_\Theta L_F = \bar{L}_F$, and $\int_{\cap \beta} |m(\theta) - m'(\theta)| \mathbb{P}_\Theta(d\theta) \leq \|M - M'\|.$
ii) Consider $\Delta \beta_{bc}$ for two arbitrary distinct actions $b, c \in A$ with $b \neq c$. Fix $\theta \in \Delta \beta_{bc}$: action $b$ is the optimal action for this type $\theta$ in the state $M$ and $c$ is the optimal in the state $M'$.

Then,

$$0 \leq \sum_{j \in A \setminus \{b\}} Q^F_{ib}(\theta)\mu_j(\theta) - (-Q^F_{ib}(\theta)\mu'_b(\theta)) = \Delta v_b(\theta) \leq \sum_{j \in A \setminus \{b\}} \bar{\rho} \cdot 3 + \bar{\rho} \cdot 3 = 3A\bar{\rho}.$$ 

Similarly, we have $0 \geq \Delta v_c(\theta) \geq -3A\bar{\rho}$. For $i \neq b, c$,

$$\Delta v_i(\theta) = (-Q^F_{ib}(\theta)\mu_i(\theta)) - (-Q^F_{ib}(\theta)\mu'_i(\theta)).$$

Since $Q^\theta(\cdot) \in [0, \bar{\rho}]$ and $\mu(\cdot) \in [-3, 3]$, we have

$$|\Delta v_i(\theta)| \leq |Q^F_{ib}(\theta)\mu_i(\theta)| + |Q^F_{ib}(\theta)\mu'_i(\theta)| \leq 6\bar{\rho}.$$ 

Therefore,

$$\sum_{a \in A} |\Delta v_1(\theta)| \leq 2 \cdot 3A\bar{\rho} + (A - 2) \cdot 6\bar{\rho} = 12(A - 1)\bar{\rho}.$$ 

By Assumption 3, we have

$$\int_{\Delta \beta_{bc}} \sum_{a \in A} |\Delta v_a(\theta)|P_\Theta(d\theta) \leq 12(A - 1)\rho P_\Theta(\Delta \beta_{bc}) \leq 12(A - 1)\rho L_\beta |\bar{m} - \bar{m}'|$$

and thus

$$\int_{\Delta \beta} \sum_{a \in A} |\Delta v_a(\theta)|P_\Theta(d\theta) = \sum_{b \in A} \sum_{c \in A \setminus \{b\}} |\Delta v_a(\theta)|P_\Theta(d\theta) \leq 12A(A - 1)^2\rho L_\beta |\bar{m} - \bar{m}'| \leq 12A(A - 1)^2\rho L_\beta \|M - M'\|.$$ 

(S.6)

Again, the second inequality comes from $|\bar{m} - \bar{m}'| \leq \|M - M'\|$. 

3°: Merge them. As $\Theta = \Theta_C \cup \cap \beta \cup \Delta \beta \cup N$ and $P_\Theta(N) = 0$, we have

$$\|V[M] - V[M']\| = \int_{\Theta_C} \sum_{a \in A} |\Delta v_a(\theta)|P_\Theta(d\theta) + \int_{\cap \beta} \sum_{a \in A} |\Delta v_a(\theta)|P_\Theta(d\theta) + \int_{\Delta \beta} \sum_{a \in A} |\Delta v_a(\theta)|P_\Theta(d\theta).$$

(S.3), (S.5) and (S.6) imply (A.5), namely $\|V[M] - V[M']\| \leq L_V \|M - M'\|$ with

$$L_V := 2A^2(3\bar{L}_p\bar{L}_F + 3\bar{L}_p + \bar{\rho})L_\mu + 2(A - 1)L_\mu(3\bar{L}_Q\bar{L}_F + (3\bar{L}_Q + \bar{\rho})) + 12A(A - 1)^2\bar{\rho}L_\beta.$$ 

Consider an ASAG. Then, $\theta \in \Delta \beta_{bc}$ is equivalent to

$$\begin{cases} F^0_b(\bar{\mu}) + \theta_b > F^0_j(\bar{\mu}) + \theta_j & \text{for all } j \in A \setminus \{b\}, \\ F^0_c(\bar{\mu}') + \theta_c > F^0_j(\bar{\mu}') + \theta_j & \text{for all } j \in A \setminus \{c\}. \end{cases}$$

This implies

$$F^0_c(\bar{\mu}') - F^0_b(\bar{\mu}') > \theta_b - \theta_c > F^0_c(\bar{\mu}') - F^0_b(\bar{\mu}).$$

Hence, if there exists $\rho_\Theta \in \mathbb{R}$ such that $P_\Theta(\{\theta \in \Theta : c \leq \theta_b - \theta_a \leq d\}) \leq (d - c)\rho_\Theta$, we have

$$P_\Theta(\Delta \beta_{bc}) \leq \rho \left\{ (F^0_c(\bar{\mu}') - F^0_b(\bar{\mu}')) - (F^0_c(\bar{\mu}') - F^0_b(\bar{\mu})) \right\} \leq 2\rho \left\{ |F^0_c(\bar{\mu}') - F^0_b(\bar{\mu})| + |F^0_c(\bar{\mu}') - F^0_b(\bar{\mu})| \right\} \leq 2\rho L_\beta L_\mu |\bar{m} - \bar{m}'|. \ (S.7)$$
Thus, Assumption 3 is satisfied.

**Proof of part ii of Theorem 1**

For part ii, we use Theorem 8 in Appendix A.3, namely, Zeidler (1986, Corollary 3.9).

*Proof of parts ii.* We leave only the boundedness of the dynamic; it comes from Assumption 2. Using the formula (A.3) of the variational norm, we obtain $\|VF(M)\| \leq 3A\bar{\rho}$ for all $M \in \mathcal{M}$ since $\nu^f[M](\theta) \in [-3\bar{\rho}, 3\bar{\rho}]$ by Assumption 2 and $\mu(\cdot) \in [-3, 3]$.

Then, Theorem 8 implies the existence of a unique solution path of the dynamic on $\mathcal{M}$. Notice that $\mathcal{X}$ is forward invariant under $V$. Therefore, if the initial state $X_0$ lies in $\mathcal{X} \subset \mathcal{M}$, then the unique solution that passes $X_0$ at time 0 should remain in $\mathcal{X}$. 

\qed