About Generalized One-to-One Mappings between Sets of Order Homomorphisms

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Abstract
Structural properties of finite posets $R$ and $S$ are studied which enforce $\#\mathcal{H}(P,R) \leq \#\mathcal{H}(P,S)$ for every finite poset $P$, where $\mathcal{H}(P,Q)$ is the set of order homomorphisms from $P$ to $Q$. The concept of the strong Hom-scheme is introduced. In the case of existence, a strong Hom-scheme from $R$ to $S$ defines a one-to-one mapping $\rho_P : \mathcal{H}(P,R) \rightarrow \mathcal{H}(P,S)$ for every poset $P \in \Psi_r$, where $\Psi_r$ is a representation system of the non-isomorphic finite posets. By postulating regularity conditions for the way, how a strong Hom-scheme maps the elements of $\mathcal{H}(P,R)$ to the elements of $\mathcal{H}(P,S)$, the strong I-scheme from $R$ to $S$ is defined. The existence of a strong I-scheme from $R$ to $S$ turns out to be equivalent to the existence of a one-to-one homomorphism $\epsilon$ between the so-called EV-systems of $R$ and $S$, where $\epsilon$ has to fulfill an additional condition. Methods are developed which allow - in many cases - for given finite posets $R$ and $S$, the proof of the relation “$\#\mathcal{H}(P,R) \leq \#\mathcal{H}(P,S)$ for every finite poset $P$” or the refutation of this relation.

Mathematics Subject Classification:
Primary: 06A07. Secondary: 06A06.

Key words: poset, homomorphism, Hom-scheme, G-scheme, I-scheme, EV-system.

1 Introduction
In order to unify ordinal and cardinal arithmetic, Garret Birkhoff [2, 3] published two articles in 1937 and 1942 in which he introduced for posets $P$ and $Q$ their direct sum $P + Q$, their product $P \times Q$, and the homomorphism set $\mathcal{H}(P,Q)$, together with their (later on) usual partial order relations. By doing so, he opened the rich field of “order arithmetic”. Surveys about the respective state of the art are contained in Jónsson [15] from 1982, Duffus [8] from 1984, and McKenzie [22] from 2003.

In the beginning, Day [7] extended the operations introduced by Birkhoff to more general relations in 1945. In 1948 and 1951, Hashimoto [13, 14] proved that two product representations of a connected poset have always a common
refinement. Lovász [18] showed in 1967, that there is an in-depth connection between the structure of posets and the cardinalities of homomorphism sets related to them (see Theorem 1 below). He used the result in showing a cancellation rule for the product of finite posets: $P \times R \simeq P \times S \Rightarrow R \simeq S$, $P$ non-empty. In 1971, Lovász [19] extended his result to more general structures.

Birkhoff [3] conjectured already in 1942, that $\mathcal{H}(P,R) \simeq \mathcal{H}(P,S)$ implies $R \simeq S$ for finite posets $P,R$, and $S$, $P$ non-empty. This problem, called the “cancellation problem”, moved into focus at the end of the seventies. Bergman et al. [1] proved in 1977 that the cancellation rule holds for chains: for every finite, non-empty chain $C$, $\mathcal{H}(C,R) \simeq \mathcal{H}(C,S) \Rightarrow R \simeq S$. In 1978, Duffus et al. [9] worked with $\mathcal{H}(P,L)$, $L$ being a lattice, and in 1980, Wille [23] contributed to this topic as well. For finite, connected posets $P$ and $Q$, Duffus and Wille [11] showed in 1979 that $\mathcal{H}(P,P) \simeq \mathcal{H}(Q,Q)$ implies $P \simeq Q$.

Duffus and Rival [10] proved in 1978 a “logarithmic property”, which transfers certain cancellation problems for homomorphism sets to cancellation problems for products of posets. Duffus [8] developed this approach further in 1984.

In 1982, Jónsson [15] published an overview over the arithmetic of ordered sets, providing a rich spectrum of results and aspects. In the same year, Jónsson [16] presented results about the automorphism group of the sets $\mathcal{H}(P,Q)$, $P$ and $Q$ providing some special properties. As well in 1982, Jónsson and McKenzie [17] published four important cancellation and refinement rules.

Surprisingly, the rich flow of publications dried up now, until Farley [12] published a paper in 1996, in which he presented new results about the structure of the automorphism group of $\mathcal{H}(P,L)$ for certain posets $P$ and lattices $L$. According to [22], this article spurred McKenzie to take up Birkhoff’s cancellation problem again, and he succeeded in putting the keystone onto all the hard efforts: in 1999 and 2000, McKenzie [20, 21] proved that indeed $\mathcal{H}(P,R) \simeq \mathcal{H}(P,S)$ implies $R \simeq S$ for finite posets $P,R,S$, $P$ non-empty. In 2003, McKenzie [22] published an additional paper about this subject, using a different approach for the proof.

After this publication, the interest in homomorphism sets as such expired. The present paper takes the subject up again, but it is now the cardinality of homomorphism sets which is focussed. The starting points are two observations separated by 51 years. The first one has been published by Lovász [18] in 1967, showing an in-depth connection between the structure of posets and the cardinalities of homomorphism sets related to them:

**Theorem 1 (Lovász [18]).** Let $R, S$ be finite posets. Then

$$\#\mathcal{H}(P,R) = \#\mathcal{H}(P,S) \text{ for every finite poset } P \iff R \simeq S.$$ 

The second observation concerns the posets $R$ and $S$ shown in Figure 1. The author [4, Theorem 5] has proven that for these posets $\#\mathcal{H}(P,R) \leq \#\mathcal{H}(P,S)$ for every finite poset $P$. The method used in the proof was obviously an instance of a (so far unknown) general approach. Following the track, the author was able to prove the relation “$\mathcal{H}(P,R) \leq \mathcal{H}(P,S)$ for every finite poset $P$” for additional finite posets $R$ and $S$, too. Two simple non-trivial examples are shown in the Figures 1 and 1, more can be found in the Figures 5, 7 and 8 in the Appendix. Naturally, the question came up: **What is it in the structure of two finite posets $R$ and $S$ that results in $\#\mathcal{H}(P,R) \leq \#\mathcal{H}(P,S)$ for every finite poset $P$?**
In this paper, a first attempt is made to study the phenomenon in a systematic way. After the preparatory Section 2, the main mathematical work is done in the Sections 3 and 4:

- In Section 3.1, the concept of the strong Hom-scheme from $R$ to $S$ is introduced. In the case of existence, a strong Hom-scheme from $R$ to $S$ defines a one-to-one mapping $\rho_P : \mathcal{H}(P,R) \to \mathcal{H}(P,S)$ for every $P \in \mathcal{P}_r$, where $\mathcal{P}_r$ is a representation system of the non-isomorphic finite posets.

- In Section 3.3, the strong G-scheme and the strong I-scheme are defined by postulating regularity conditions for the way, how a strong Hom-scheme maps the elements of $\mathcal{H}(P,R)$ to the elements of $\mathcal{H}(P,S)$. (The existence of a strong G-scheme or a strong I-scheme from $R$ to $S$ implies thus $\#\mathcal{H}(P,R) \leq \#\mathcal{H}(P,S)$ for every finite poset $P$.) The definition and discussion of these two additional types of Hom-schemes requires the concept of the EV-system $\mathcal{E}(P)$ of a poset $P$ which is defined in Section 3.2.

- In Section 4, the structure theory of strong I-schemes is developed. It is shown in the Theorems 5 and 7, that the existence of a strong I-scheme from $R$ to $S$ is equivalent to the existence of a one-to-one homomorphism $\epsilon : \mathcal{E}(R) \to \mathcal{E}(S)$, where $\epsilon$ has to fulfill an additional condition. The additional condition on $\epsilon$ is unwieldy; therefore, it is shown in Theorem 8 how it can be replaced by more handy conditions.

The theoretical work is thoroughly illustrated by detailed discussions of the examples in the Figures 1b and 1c.

All three types of strong Hom-schemes define partial orders on the set of the non-isomorphic, finite partial orders (Theorems 2 and 3). Examples of the new partial order induced by strong I-schemes and strong G-schemes, respectively, are shown in the Figures 6, 7, and 8 in the Appendix. In Figure 9, advice is given how to construct strong G-schemes and strong I-schemes for the respective posets in the Figures 6, 7, and 8.

Finally, in Section 5.1, it is examined to which extent the three relations “from $R$ to $S$, there exists a strong Hom-scheme / a strong G-scheme / a strong I-scheme” are compatible with the operations of order arithmetic. Cancellation rules will be presented in a separate paper [5]. In Section 5.2, useful formulas about strong Hom-schemes are collected, and in Section 5.3, an inversion formula is presented which links the abstract structure theory with the hands-on construction of strong G-schemes and strong I-schemes between posets.

Structural aspects of strong G-schemes and sufficient criteria for their existence will be discussed in [6].
2 Preparation

2.1 Basics and Notation

A \textit{(binary) relation} \( R \) on a set \( X \) is a subset of \( X \times X \). For \( x, y \in X \), we write as usual \( xRy \) for \( (x, y) \in R \). Given a relation \( R \) on \( X \), the \textit{dual} relation \( R^d \) is defined by \( xR^dy \Leftrightarrow yRx \). A reflexive, antisymmetric, and transitive relation \( R \) on \( X \) is called a \textit{partial order relation}, the pair \( P = (X, R) \) is called a \textit{partially ordered set} or simply a \textit{poset}, and \( X \) is called the \textit{carrier} of \( P \). For a poset \( P \), the symbol \( \leq \) is generally used for its partial order relation. \textit{“} \( x < y \text{”} \text{ means } \textit{“} x \leq y \text{ and } x \neq y \text{”} \text{ and } \textit{“} \neq \text{y} \text{”} \text{ means } \textit{“} \neq \text{y} \text{ nor } y \leq x \text{”} \textit{.} \) Similarly, we use the notation \textit{“} \( A \subseteq B \text{”} \text{ for } \textit{“} A \subseteq B \text{ and } A \neq B \text{”} \text{ for sets } A \text{ and } B \text{.} \) Given a relation \( R \) on \( X \), the \textit{induced relation} on \( A \subseteq X \) is \( R \cap (A \times A) \).

For a set \( X \), the \textit{diagonal (relation)} is defined as \( \Delta_X \equiv \{(x, x) \mid x \in X\} \), and \((X, \Delta_X)\) is called an \textit{antichain}. A \textit{chain} is characterized by \( x \leq y \text{ or } y \leq x \) for all \( x, y \in X \); up to isomorphism, there is only one chain for every underlying set. For a finite set \( X \) of cardinality \( k \in \mathbb{N}_0 \), we write \( A_k \) for the antichain on \( X \), and \( C_k \) for the chain on \( X \) (defined up to isomorphism). The \textit{height} of a finite poset is the cardinality of a longest chain contained in it.

Let \( X, Y \) be disjoint sets, \( R \) a relation on \( X \), and \( S \) a relation on \( Y \). By means of \( R \) and \( S \), two relations are defined on \( X \cup Y \): the \textit{direct sum} \( R + S \equiv R \cup S \), and the \textit{ordinal sum} \( R \oplus S \equiv R \cup S \cup (X \times Y) \). If \( R \) and \( S \) are partial order relations, then \( R + S \) and \( R \oplus S \) are partial order relations, too. Both operations are associative.

Given posets \( P = (X, \leq_P) \) and \( Q = (Y, \leq_Q) \), their \textit{product} \( P \times Q \equiv (X \times Y, \leq_{P \times Q}) \) on \( X \times Y \) is defined by \((x_1, y_1) \leq_{P \times Q} (x_2, y_2) \Leftrightarrow x_1 \leq_P x_2 \text{ and } y_1 \leq_Q y_2 \). We write \( P^k \) for \( P \times \ldots \times P \) with \( k \) factors \( P \). A \textit{binary word} is an element of \((C_2)^k\) with \( 0 < 1 \) as carrier of \( C_2 \).

For relations \( R \) and \( S \) on \( X \) and \( Y \), respectively, a mapping \( \xi : X \to Y \) is called a \textit{homomorphism}, iff \( \xi(x) \leq_S (y) \) holds for all \( x, y \in X \) with \( xRy \). The concatenation of homomorphisms is a homomorphism. A homomorphism \( \xi \) is called \textit{strict} iff it fulfills additionally \((xRy \text{ and } x \neq y) \Rightarrow (\xi(x) \neq \xi(y)) \) for all \( x, y \in X \), and it is called an \textit{embedding} iff \( \xi(x) \neq \xi(y) \Rightarrow xRy \) for all \( x, y \in X \). Finally, an embedding is called an \textit{isomorphism} iff it is onto. For posets \( P \) and \( Q \), we let \( \mathcal{H}(P, Q) \) and \( \mathcal{S}(P, Q) \) denote the set of homomorphisms from \( P \) to \( Q \) and the set of strict homomorphisms from \( P \) to \( Q \), respectively. \( P \cong Q \) indicates isomorphism.

\( \mathfrak{P} \) is the class of all finite posets, and the set \( \mathfrak{P}_r \) is a representation system of the non-isomorphic posets in \( \mathfrak{P} \).

In order to avoid repetitions, we agree on that \( X \) is always the carrier of the poset \( P \), and that \( Y \) is always the carrier of the poset \( Q \). For a poset \( P \), we use the notation \( x \in P \) instead of \( x \in X \), and for posets \( P \) and \( Q \), we write \( \xi : P \to Q \) instead of \( \xi : X \to Y \) for a homomorphism \( \xi \in \mathcal{H}(P, Q) \). However, a homomorphism \( \xi \in \mathcal{H}(P, Q) \) is just a \textit{mapping}, i.e., a triplet \((X, F_\xi, Y)\) with \( F_\xi \subseteq X \times Y \) fulfilling certain conditions. Thus, the partial order relations on \( X \) and \( Y \) (making the mapping \( \xi \) a homomorphism) are not registered on \( \xi \). In the Sections 3 and 4, this fact will force us to maintain an accurate book-keeping about the posets to which a homomorphism is related.

From the rich concept of downsets and upsets, we need simple notation only.
Given a relation $R$ on $X$, we define for $A \subseteq X$
\[\downarrow A \equiv \{ y \in X \mid \exists a \in A : y \leq a \},\]
\[\uparrow A \equiv (\downarrow A) \setminus A,\]
\[\uparrow A \equiv \{ y \in X \mid \exists a \in A : a \leq y \},\]
\[\downarrow A \equiv (\uparrow A) \setminus A.\]

For $x \in X$, we write $\downarrow x$ and $\uparrow x$ instead of $\downarrow \{x\}$ and $\uparrow \{x\}$, respectively, and correspondingly $\uparrow x$ and $\uparrow x$. If required, we label the arrows with the relation they are referring to. For $x \in X$, we write $\downarrow x$ and $\uparrow x$ instead of $\downarrow \{x\}$ and $\uparrow \{x\}$, respectively, and correspondingly $\uparrow x$ and $\uparrow x$. If required, we label the arrows with the relation they are referring to. $A \subseteq X$ is called an upset iff $A = \uparrow A$.

For $x,y \in X$ we define the interval
\[\{y, x\} \equiv (\uparrow y) \cap (\downarrow x) = \{ z \in X \mid yRz \text{ and } zRx \}.\]

Additionally, we use the following notation from set theory:
\[\emptyset \equiv \emptyset,\]
\[\mathbb{N} \equiv \{1, \ldots, n\} \text{ for every } n \in \mathbb{N}.\]

For sets $X$ and $Y$, $A(X,Y)$ is the set of mappings from $X$ to $Y$. $id_X \in A(X,X)$ is the identity mapping. For a mapping $f \in A(X,Y)$, our symbols for the pre-image of $B \subseteq Y$ and of $y \in Y$ are
\[f^{-1}(B) \equiv \{ x \in X \mid f(x) \in B \},\]
\[f^{-1}(y) \equiv f^{-1}(\{y\}).\]

For $A \subseteq X$ and $B \subseteq Y$ with $f(X) \subseteq B$, let $f|_A : A \rightarrow Y$ and $f|_B : X \rightarrow B$ denote the pre-restriction and post-restriction of $f$, respectively.

Finally, we use the Cartesian product. Let $I$ be a non-empty set, and let $N_i$ be a non-empty set for every $i \in I$. Then the Cartesian product of the sets $N_i, i \in I$, is defined as
\[\prod_{i \in I} N_i \equiv \left\{ f \in A(\bigcup_{i \in I} N_i) \mid f(i) \in N_i \text{ for all } i \in I \right\}.\]

According to the axiom of choice, the Cartesian product is always non-empty.

### 2.2 Connectivity

**Definition 1.** Let $P \in \mathcal{P}$, $A \subseteq P$, and $x,y \in A$. We say that $x$ and $y$ are connected in $A$, iff there are $z_0, z_1, \ldots, z_L \in A$, $L \in \mathbb{N}_0$, with $x = z_0, y = z_L$ and $z_{\ell-1} < z_\ell$ or $z_{\ell-1} > z_\ell$ for all $\ell \in \mathbb{L}$. We call $z_0, \ldots, z_L$ a zigzag line connecting $x$ and $y$. We define for all $A \subseteq P$, $x \in A$, $B \subseteq A$
\[\gamma_A(x) \equiv \{ y \in A \mid x \text{ and } y \text{ are connected in } A \},\]
\[\gamma_A(B) \equiv \bigcup_{b \in B} \gamma_A(b).\]  

(1)
Of course, \( x \in \gamma_A(x) \) for every \( x \in A \subseteq P \). The following corollary has been proven in [4]:

**Corollary 1.** Let \( A \subseteq P \). The relation “connected in \( A \)” is an equivalence relation on \( A \) with partition \( \{ \gamma_A(a) \mid a \in A \} \). For \( B \subseteq A \subseteq A' \subseteq P \) we have

\[
\gamma_A(B) \subseteq \gamma_{A'}(B), \quad (2)
\]

\[
\gamma_A(B) = \gamma_{\gamma_A(B)}(B). \quad (3)
\]

The following definition is one of the central ones in this paper:

**Definition 2.** Let \( P = (X, \preceq) \) be a finite poset, let \( Y \) be a set, and let \( \xi \in \mathcal{A}(X, Y) \) be a mapping. We define for all \( x \in X \)

\[
G_\xi(x) \equiv \gamma_{\xi^{-1}(\xi(x))}(x).
\]

According to (3), \( G_\xi(x) = \gamma_{G_\xi(x)}(x) \) for every \( x \in P \): for \( y \in G_\xi(x) \), there is a zigzag line connecting \( x \) and \( y \) in \( \xi^{-1}(\xi(x)) \) that runs totally in \( G_\xi(x) \). Most mappings we are dealing with in what follows are order homomorphisms; however, we need the general case in the following corollary:

**Corollary 2.** Let \( P = (X, \preceq) \) be a finite poset, let \( Y, Z \) be sets, and let \( \xi, \zeta \in \mathcal{A}(X, Y), \sigma \in \mathcal{A}(Y, Z) \). Then \( G_\xi(x) \subseteq G_{\sigma \xi}(x) \) for all \( x \in P \). Equality holds for all \( x \in P \), if \( \sigma|_{\xi(P)} \) is one-to-one.

**Proof.** We have for every \( x \in X \): \((\sigma \circ \xi)^{-1}((\sigma \circ \xi)(x)) = \xi^{-1}(\sigma^{-1}(\sigma(\xi(x))) \supseteq \xi^{-1}(\xi(x))) \), with equality if \( \{\xi(x)\} = \sigma^{-1}(\sigma(\xi(x))) \). The first proposition follows with [2], the second one is clear.

**Corollary 3.** Let \( P, Q \in \mathcal{P} \). Then for every homomorphism \( \xi \in \mathcal{H}(P, Q) \)

\[
\xi \text{ strict } \iff \forall x \in P : G_\xi(x) = \{x\}.
\]

**Proof.** If \( \xi \) is strict, then \( \xi^{-1}(\xi(x)) \) is an antichain for every \( x \in P \), thus \( G_\xi(x) = \{x\} \). On the other hand, let \( x, y \in P \) with \( x < y \). \( G_\xi(x) = \{x\} \) yields \( \xi(y) \neq \xi(x) \), hence \( \xi(x) < \xi(y) \).

**Lemma 1.** Let \( P, R, S \in \mathcal{P} \), let \( \xi, \zeta \in \mathcal{H}(P, R) \) and \( \zeta \in \mathcal{H}(P, S) \) be homomorphisms, and let \( G_\xi(x) \subseteq G_\zeta(x) \) for an \( x \in P \). Then \( G_\xi(x) \subseteq G_\zeta(x) \) iff there are \( a, b \in G_\xi(x) \) with

\[
a < b \text{ and } \xi(a) < \xi(b) \quad \text{ and } \quad \zeta(a) = \zeta(b).
\]

**Proof.** Let \( y \in G_\zeta(x) \setminus G_\xi(x) \). The points \( y \) and \( x \) are connected by a zigzag line in \( G_\zeta(x) \), and on this line there are points \( a \) and \( b \) with \( a \in G_\xi(x), b \in G_\zeta(x) \setminus G_\xi(x) \), and \( a < b \) or \( b < a \). \( a \) and \( x \) are connected by a zigzag line in \( G_\xi(x) \). Therefore, \( b \notin G_\xi(x) \) means \( \xi(b) \neq \xi(x) = \xi(a) \), thus \( \xi(a) < \xi(b) \) or \( \xi(b) < \xi(a) \), depending on the relation between \( a \) and \( b \). \( a, b \in G_\xi(x) \) means \( \zeta(a) = \zeta(b) \).

On the other hand, let \( a, b \in G_\xi(x) \) with \( a < b \) and \( \xi(a) < \xi(b) \). \((\xi(a) = \zeta(b) \) is a direct consequence of \( a, b \in G_\xi(x) \)). Due to \( G_\xi(x) \subseteq \xi^{-1}(\xi(x)) \), we conclude that at least one of the points \( a, b \) is not an element of \( G_\xi(x) \).
For the proof of Proposition \[8\] we need:

**Lemma 2.** Let \( P \in \mathfrak{P} \), and let \( Y \) and \( Z \) be non-empty finite sets. For \( \xi \in A(P, Y \times Z) \) let \( \xi_1 \in A(P, Y) \) and \( \xi_2 \in A(P, Z) \) be defined by \( (\xi_1(x), \xi_2(x)) = \xi(x) \) for all \( x \in X \). Then, for all \( x \in X \),

\[
G_\xi(x) = \gamma_{G_{\xi_1}(x) \cap G_{\xi_2}}(x).
\]

**Proof.** For every \((a, b) \in Y \times Z\), we have \( \xi^{-1}(a, b) = \xi_1^{-1}(a) \cap \xi_2^{-1}(b) \), which yields \( \xi^{-1}(\xi(x)) \subseteq \xi_1^{-1}(\xi_1(x)) \) for every \( x \in P \) and \( j \in 2 \). With \( 2 \) we conclude \( G_\xi(x) \subseteq G_{\xi_1}(x) \cap G_{\xi_2}(x) \), and we get

\[
G_\xi(x) = \gamma_{G_{\xi_1}(x) \cap G_{\xi_2}}(x).
\]

Now let \( y \in \gamma_{G_{\xi_1}(x) \cap G_{\xi_2}}(x) \), and let \( z_0, \ldots, z_L \) be a zigzag line connecting \( x \) and \( y \) in \( G_{\xi_1}(x) \cap G_{\xi_2}(x) \). Then \( \xi_1(z_\ell) = \xi_1(x) \) and \( \xi_2(z_\ell) = \xi_2(x) \) for every \( \ell \in L \cup \{0\} \). The zigzag line \( z_0, \ldots, z_L \) connects thus \( x \) and \( y \) in \( \xi_1^{-1}(\xi_1(x)) \cap \xi_2^{-1}(\xi_2(x)) = \xi^{-1}(\xi(x)) \), and we conclude \( y \in G_\xi(x) \).

\[ \square \]

## 3 New concepts

With the exception of \( G_\xi(x) \), all mathematical objects in Section \[2\] are standard. In this section, new concepts are introduced: the (strong) Hom-scheme between posets in Section \[3.1\], the EV-system of a poset in Section \[3.2\], and the (strong) G-scheme and (strong) I-scheme between posets in Section \[3.3\].

### 3.1 Hom-schemes

Let \( R, S \in \mathfrak{P} \). Assume that there exists a one-to-one homomorphism \( \sigma : R \to S \). Then, for every \( P \in \mathfrak{P} \), we get a one-to-one mapping \( r_P : \mathcal{H}(P, R) \to \mathcal{H}(P, S) \) by setting for every \( \xi \in \mathcal{H}(P, R) \)

\[
r_P(\xi) \equiv \sigma \circ \xi, \tag{4}
\]

and according to Corollary \[2\] we have

\[
G_{r_P(\xi)}(x) = G_\xi(x)
\]

for every \( P \in \mathfrak{P}, \xi \in \mathcal{H}(P, R), x \in P \). Furthermore, \( r \) is controlled by the images of homomorphisms in the sense that for \( P, Q \in \mathfrak{P}, \xi \in \mathcal{H}(P, R), \zeta \in \mathcal{H}(Q, R) \), \( x \in P \), \( y \in Q \), we have \( \xi(x) \leq \zeta(y) \Rightarrow r_P(\xi)(x) \leq r_Q(\zeta)(y) \), where “\( \leq \)” on the left side implies “\( < \)” on the right side.

These three relations are generalized in what follows: the first one in the definition of the strong Hom-scheme below, the second and third one in the definitions of the G-scheme and the I-scheme later on in Section \[3.3\].

**Definition 3.** Let \( R, S \in \mathfrak{P} \). We call a mapping

\[
\rho \in \prod_{P \in \mathfrak{P}} \mathcal{A}(\mathcal{H}(P, R), \mathcal{H}(P, S)) \quad \text{(Cartesian product)}
\]

a Hom-scheme from \( R \) to \( S \). We call a Hom-scheme \( \rho \) from \( R \) to \( S \) strong iff the mapping \( \rho_P : \mathcal{H}(P, R) \to \mathcal{H}(P, S) \) is one-to-one for every \( P \in \mathfrak{P} \). If \( P \) is fixed, we write \( \rho(\xi) \) instead of \( \rho_P(\xi) \).
Figure 2: Two examples for the construction of strong Hom-schemes between posets. Explanations in text.

Due to the restriction to the set $\mathcal{P}_r$, we stay within ordinary set theory with this definition. The Hom-scheme (4) induced by a one-to-one homomorphism is always a strong Hom-scheme.

**Theorem 2.** Let $R, S \in \mathcal{P}$. We write

$$R \subseteq S$$

iff a strong Hom-scheme from $R$ to $S$ exists. $\subseteq$ is a partial order relation on $\mathcal{P}_r$.

**Proof.** For $R \in \mathcal{P}_r$, $id_R : R \to R$ is one-to-one, hence $R \subseteq R$. For $R, S \in \mathcal{P}_r$ with $R \subseteq S$ and $S \subseteq R$, we have $\#H(P, R) = \#H(P, S)$ for all $P \in \mathcal{P}_r$ which is equivalent to $R \simeq S$ according to Theorem 1. Finally, let $R, S, T \in \mathcal{P}_r$, let $\rho$ be a strong Hom-scheme from $R$ to $S$, and let $\tau$ be a strong Hom-scheme from $S$ to $T$. It is easily seen that $\tau \ast \rho$ defined by

$$(\tau \ast \rho)_P \equiv \tau_P \circ \rho_P$$

is a strong Hom-scheme from $R$ to $T$. 

For $R \in \mathcal{P}_r$, $k \equiv \#R$, there exist one-to-one homomorphisms from $A_k$ to $R$ and from $R$ to $C_k$, hence $A_k \subseteq R \subseteq C_k$. With respect to $\subseteq$, the posets $A_k$ and $C_k$ are thus the extrema of the posets with $k$ points.

In Figure 2, the posets from the Figures 1b and 1c are shown again, with their points represented by binary words. We want to prove $R \subseteq S$ for both pairs of posets. For the posets in Figure 2a, we define for $P \in \mathcal{P}_r$ and $\xi \in H(P, R)$ the following sets (right part of the figure):

$$a \equiv \xi^{-1}(100), \quad b \equiv \xi^{-1}(001), \quad A \equiv \xi^{-1}(110), \quad B \equiv \xi^{-1}(011), \quad \Gamma \equiv \gamma_A(A \cap \uparrow a).$$
Now, we define a mapping $\rho : P \to S$ by

$$
\begin{align*}
\rho(\xi)^{-1}(010) & \equiv A \setminus \Gamma, & \rho(\xi)^{-1}(100) & \equiv a, \\
\rho(\xi)^{-1}(001) & \equiv b, & \rho(\xi)^{-1}(101) & \equiv B \cup \Gamma.
\end{align*}
$$

Just as in the proof of [4, Theorem 5], we see that $\rho$ is strong Hom-scheme. Firstly, we show $\rho(\xi) \in H(P, S)$. Let $x, y \in P$ with $x < y$. $\rho(\xi)(x) \leq \rho(\xi)(y)$ is trivial for all but two cases: $x, y \in A$ and $x \in a, y \in A$. In the case of $x, y \in A$, nothing is to prove for $\Gamma = \emptyset$ or $\Gamma = A$. In the case of $\emptyset \neq \Gamma \subset A$, the set $\Gamma$ is a proper connectivity component of $A$. $x < y$ enforces therefore $x, y \in \Gamma$ or $x, y \in A \setminus \Gamma$, thus $\rho(\xi)(x) = \rho(\xi)(y)$ in both cases. And in the case of $x \in a, y \in A$, $x < y$ yields $y \in A \cup a \subseteq \Gamma$, hence $\rho(\xi)(x) = 100 < 101 = \rho(\xi)(y)$. Thus, $\rho(\xi) \in H(P, S)$.

It remains to show that $\rho$ is strong. Due to $100 \parallel \rho 011$, $a = \xi^{-1}(100)$, and $B = \xi^{-1}(011)$, we have $B \cap \uparrow a = \emptyset$. Furthermore, in the case of $B \neq \emptyset$, $\Gamma$ is empty or a proper connectivity component of $B \cup \Gamma$. Looking at the definition of $\Gamma$, we conclude

$$
\Gamma = \gamma_{B \cup \Gamma}((B \cup \Gamma) \cap \uparrow a) = \gamma_{\rho(\xi)(101)^{-1}}(\rho(\xi)^{-1}(100) \cap \uparrow \rho(\xi)^{-1}(001)).
$$

We can thus reconstruct $\Gamma$ by means of $\rho(\xi)$, and with the reconstructed set $\Gamma$ we can rebuild all the sets $a, b, A,$ and $B$. Because $\xi$ is totally determined by the pairs $(v, \xi^{-1}(v)), v \in R$, we can identify $\xi \in H(P, R)$ by means of $\rho(\xi) \in H(P, S)$, and $\rho$ is strong.

For the posets $R$ and $S$ in Figure 2(b), the relation $R \subseteq S$ is shown exactly in the same way. As indicated in the figure, we define for $P \in \mathcal{P}_R, \xi \in H(P, R)$

$$
a \equiv \xi^{-1}(100), \quad b \equiv \xi^{-1}(001),
$$

$$
A \equiv \xi^{-1}(101), \quad B \equiv \xi^{-1}(011), \quad \Gamma \equiv \gamma_{B \cap \uparrow b}
$$

and

$$
\rho(\xi)^{-1}(1000) \equiv b, \quad \rho(\xi)^{-1}(1100) \equiv a \cup \Gamma,
$$

$$
\rho(\xi)^{-1}(1101) \equiv A, \quad \rho(\xi)^{-1}(0001) \equiv B \setminus \Gamma.
$$

Again we get a well defined mapping $\rho(\xi) : P \to S$, and with the same reasoning as for the first example, we prove that $\rho(\xi)$ is a homomorphism and that we can reconstruct $\Gamma$ as $\gamma_{\mathcal{C} \cdot \Gamma}(a \cup \Gamma) \cap \uparrow b$.

The Figures 6, 7, and 8 in the Appendix show the diagrams of three posets defined by $\subset$. In Figure 6, it is the poset of the non-isomorphic posets with four points, in Figure 7 it is the poset of the non-isomorphic, flat posets with five points, and in Figure 8, it is the poset of the non-isomorphic posets with five points which are not isomorphic to $A_1 + Q$, $A_1 \circ Q$, or $Q \circ A_1$ for a poset $Q$ with four points. (The restriction is justified by the Propositions 6 and 7 in Section 5.1.)

In Figure 9 in the Appendix, it is indicated how (for the respective pairs of posets $R$ and $S$ in the Figures 6, 7, and 8) the pre-images of a homomorphism $\xi \in H(P, R)$ have to be redistributed on $S$ in order to get a strong Hom-scheme from $R$ to $S$. All proofs run exactly as for the example above. Other ways to prove the existence of a strong Hom-scheme will be demonstrated at the end of the Sections 4.1 and 4.3.
3.2 The EV-system of a poset

In mechanical engineering, the exploded-view drawing of an engine shows the relationship or order of assembly of its components by distributing them in the drawing area in a well-arranged and meaningful way. That is exactly what the EV-system of a poset does with respect to the relations between its points:

Definition 4. Let $P$ be a poset. The EV-system $E(P)$ of $P$ is defined as

$$E(P) \equiv \{(x, D, U) \mid x \in P, D \subseteq \downarrow x, U \subseteq \uparrow x\}.$$ 

For $a \in E(P)$, we refer to the three components of $a$ by $a_1$, $a_2$, and $a_3$. We equip $E(P)$ with a relation: For all $a, b \in E(P)$ we define

$$a <_+ b \equiv a_1 \in b_2 \text{ and } b_1 \in a_3,$$

and $\leq_+ \equiv <_+ \cup \Delta E(P)$. Additionally, we define for every $x \in P$

$$E(P; x) = \{a \in E(P) \mid a_1 = x\}.$$ 

The relation $\leq_+$ is reflexive and antisymmetric, but in general not transitive. Its transitive hull is a partial order on $E(P)$, but we will not use this fact. The mapping $E(P) \to P$ defined by $a \mapsto a_1$ is a strict homomorphism, and the mapping $P \to E(P)$ with $x \mapsto (x, \downarrow x, \uparrow x)$ is an embedding.

Figure 3 shows the EV-systems of the posets in Figure 2. In all EV-systems, the sets $E(P; x)$, $x \in P$, are encircled and labeled with $x$. For each point $a$ in the diagrams, $a_1$ is thus given by this label, and we get $a_2$ and $a_3$ by looking at the labels of the end points of lines starting in $a$ and going downwards and upwards, respectively.

Obviously, $E(R)$ can be embedded in $E(S)$ in both cases. (In checking this statement for Figure 3b, the reader should be aware that $\leq_+$ is not transitive.) Indeed, the existence of a suitable homomorphism between EV-systems will play an important role in what follows. We will see in Section 3.3 that a strict homomorphism between the EV-systems of two posets $R$ and $S$ induces a Hom-scheme from $R$ to $S$ fulfilling an additional regularity condition. And also in the
structure theory presented in Section 4, several types of strict homomorphisms between EV-systems are the key.

The following proposition is a direct consequence of the definition of the EV-system:

**Proposition 1.** Let $P,Q \in \mathfrak{P}$ be disjoint posets. Then

$$\mathcal{E}(P^d) \simeq \mathcal{E}(P)^d,$$

(5) $$\mathcal{E}(P + Q) = \mathcal{E}(P) + \mathcal{E}(Q).$$

(6)

Furthermore,

$$\mathcal{E}(P \oplus Q) = \{(x,D,U \cup V) \mid (x,D,U) \in \mathcal{E}(P), V \subseteq Q\} \cup \{(x,D,U) \mid (x,D,U) \in \mathcal{E}(Q), E \subseteq P\},$$

(7) and $\mathcal{E}(P \times Q)$ is for any posets $P,Q \in \mathfrak{P}$ the set of the $(x,y,D)$ with

$$\{(x,y) \in P \times Q, D \subseteq (\downarrow x \times \downarrow y) \setminus \{(x,y)\}, \up U \subseteq (\uparrow x \times \uparrow y) \setminus \{(x,y)\} \}.$$

(8)

In the beginning of Section 3.1, we have announced that we will define a generalization of the fact that the Hom-scheme 4 induced by a one-to-one homomorphism is controlled by the images of homomorphisms (in the sense explained above). For this generalization, we need an answer to the question: given $P \in \mathfrak{P}_r$, $\xi \in \mathcal{H}(P,R)$, $x,y \in P$ - under which condition shall $\xi(x) < \xi(y)$ imply $\rho(\xi)(x) < \rho(\xi)(y)$ for such an “image-controlled” Hom-scheme $\rho$ from $R$ to $S$?

Among the Hom-schemes, one extreme are the “pure” Hom-schemes defined in Section 3.1 for which the relation $\xi(x) < \xi(y)$ does not determine anything about the relation between $\rho(\xi)(x)$ and $\rho(\xi)(y)$. (For $x \| y$, we can even have $\rho(\xi)(y) < \rho(\xi)(x)$.) The other extreme are the Hom-schemes induced by one-to-one homomorphisms, for which $\xi(x) < \xi(y)$ stereotypically yields $\rho(\xi)(x) < \rho(\xi)(y)$. A first idea for an intermediate approach is to postulate “$\xi(x) < \xi(y) \Rightarrow \rho(\xi)(x) < \rho(\xi)(y)$” in the case of $\xi^{-1}(\xi(x)) \cap \downarrow \xi^{-1}(\xi(y)) \neq \emptyset$, i.e., in the case of $\xi(x) \in \xi(\downarrow \xi^{-1}(\xi(y)))$ (or, equivalently, $\xi(y) \in \xi(\uparrow \xi^{-1}(\xi(x)))$). However, this approach turned out to be too restrictive for providing something really new.

Therefore, we relax the condition once more and replace $\xi^{-1}(\xi(x))$ and $\xi^{-1}(\xi(y))$ partly by their respective connectivity components of $x$ and $y$, i.e., by $G_\xi(x)$ and $G_\xi(y)$. For an image-controlled Hom-scheme, we will enforce “$\xi(x) < \xi(y) \Rightarrow \rho(\xi)(x) < \rho(\xi)(y)$”, if $\xi^{-1}(\xi(x)) \cap G_\xi(y) \neq \emptyset$ and $\xi^{-1}(\xi(y)) \cap G_\xi(x) \neq \emptyset$ (two independent conditions!), which can be rewritten as $\xi(x) \in \xi(G_\xi(y))$ and $\xi(y) \in \xi(G_\xi(x))$. The book-keeping required by this approach is provided by the following objects:

**Definition 5.** Let $P,Q \in \mathfrak{P}$, and let $\xi \in \mathcal{H}(P,Q)$ be a homomorphism. We define for every $x \in P$

$$\alpha_{\rho_\xi}(x) \equiv \left(\xi(x),\xi(G_\xi(x)),\xi(G_\xi(x))\right).$$

If $P$ is fixed, we write $\alpha_\xi(x)$ instead of $\alpha_{\rho_\xi}(x)$. 

11
The reader should be aware that \( G_\xi(x) = G_\xi(y) \) implies \( \alpha_\xi(x) = \alpha_\xi(y) \).

**Lemma 3.** \( \alpha_\xi : P \to \mathcal{E}(Q) \) is a homomorphism for every \( P,Q \in \mathfrak{P} \), \( \xi \in \mathcal{H}(P,Q) \). In particular, we have

\[
\forall x, y \in P : x < y \text{ and } \xi(x) < \xi(y) \implies \alpha_\xi(x) <_+ \alpha_\xi(y).
\] (9)

**Proof.** Firstly, we have to show that indeed \( \alpha_\xi(x)_2 \subseteq \downarrow \xi(x) \) and \( \alpha_\xi(x)_3 \subseteq \uparrow \xi(x) \) for \( x \in P \). Let \( y \in \downarrow G_\xi(x) \). Then \( y < z \) for a \( z \in G_\xi(x) \), thus \( \xi(y) < \xi(z) = \xi(x) \).

In the case of “=", we get the contradiction \( y \in G_\xi(x) \). Therefore, \( \xi(y) \in \xi(x) \) which yields \( \xi(y) \in \downarrow \xi(x) \), and \( \alpha_\xi(x)_2 \subseteq \downarrow \xi(x) \) is proven. \( \alpha_\xi(x)_3 \subseteq \uparrow \xi(x) \) is dual. Thus, \( \alpha_\xi : P \to \mathcal{E}(Q) \) is a well-defined mapping.

We proceed with the proof of (9). Let \( x,y \in P \) with \( x < y \) and \( \xi(x) < \xi(y) \). Then \( x \in \downarrow G_\xi(y) \) and \( y \in \uparrow G_\xi(x) \), hence \( \xi(x) \in \xi(\downarrow G_\xi(y)) = \alpha_\xi(y)_2 \) and \( \xi(y) \in \xi(\uparrow G_\xi(x)) = \alpha_\xi(x)_3 \), and \( \alpha_\xi(x) <_+ \alpha_\xi(y) \) is proven.

The case \( x < y \) and \( \xi(x) = \xi(y) \) remains. In this case, \( y \in G_\xi(x) \), thus \( G_\xi(y) = G_\xi(x) \). But as observed above, \( G_\xi(y) = G_\xi(x) \) implies \( \alpha_\xi(y) = \alpha_\xi(x) \).

\( \Box \)

In Section 5.3, we present a formula for \( \alpha_\xi^{-1} \) as a useful link between the abstract world of EV-systems and the hands-on construction of strong Hom-schemes of a special type (cf. the discussion of Figure 4 below). We do not include it in the main text, because the formula and its proof are technical.

**Proposition 2.** Let \( R,S \in \mathfrak{P} \), and let \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) be a strict homomorphism. For given \( P \in \mathfrak{P} \) and \( \xi \in \mathcal{H}(P,R) \), we define for every \( x \in P \)

\[
\eta(x) \equiv \epsilon(\alpha_\xi(x))_1,
\]

Then \( \eta \in \mathcal{H}(P,S) \), and for every \( x \in P \) we have

\[
G_\eta(x) = G_\xi(x),
\]

\[
\alpha_\eta(x)_2 \subseteq \epsilon(\alpha_\xi(x))_2,
\]

\[
\alpha_\eta(x)_3 \subseteq \epsilon(\alpha_\xi(x))_3.
\]

**Proof.** Due to Lemma 3, the definition of \( \epsilon \), and the remark after Definition 9, \( \eta \) is a concatenation of homomorphisms, and thus a homomorphism. Let \( x \in P \) and \( y \in G_\xi(x) \). Then \( G_\xi(y) = G_\xi(x) \), hence \( \alpha_\xi(y) = \alpha_\xi(x) \), which yields \( \eta(y) = \epsilon(\alpha_\xi(y))_1 = \epsilon(\alpha_\xi(x))_1 = \eta(x) \). We have thus \( G_\xi(x) \subseteq \eta^{-1}(\eta(x)) \), and because \( x \) and \( y \) are connected by a zigzag line in \( G_\xi(x) \), we conclude \( y \in G_\eta(x) \), thus \( G_\xi(x) \subseteq G_\eta(x) \).

Let \( x \in P \) and assume that there is a \( y \in P \) with \( y \in G_\eta(x) \setminus G_\xi(x) \). Lemma 1 yields the existence of \( a,b \in G_\eta(x) \) with \( a < b \) and \( \xi(a) < \xi(b) \). With \( \Box \), we conclude \( \alpha_\xi(a) <_+ \alpha_\xi(b) \), hence \( \epsilon(\alpha_\xi(a)) <_+ \epsilon(\alpha_\xi(b)) \), which implies \( \eta(a) = \epsilon(\alpha_\xi(a))_1 <_+ \epsilon(\alpha_\xi(b))_1 = \eta(b) \) in contradiction to \( a,b \in G_\eta(x) \). Hence, \( G_\eta(x) \setminus G_\xi(x) = \emptyset \).

Let \( x \in P \) be fixed. For \( y \in \downarrow G_\xi(x) \), we have trivially \( \xi(y) \in \alpha_\xi(x)_2 \). Furthermore, there exists a \( z \in G_\xi(x) \) with \( y < z \) and \( \xi(y) < \xi(z) = \xi(x) \). Therefore, \( z \in \downarrow G_\xi(y) \), hence \( \xi(z) = \xi(y) \in \xi(\uparrow G_\xi(y)) = \alpha_\xi(y)_3 \). Thus, \( \alpha_\xi(y) <_+ \alpha_\xi(z) \) which yields \( \epsilon(\alpha_\xi(y)) <_+ \epsilon(\alpha_\xi(z)) \implies \eta(y) = \epsilon(\alpha_\xi(y))_1 = \epsilon(\alpha_\xi(z))_1 \). According to the definition of \( <_+ \), because \( G_\eta(x) = G_\xi(x) \), we conclude
\( \alpha_\xi(x)_2 = \eta(\downarrow \! G_\eta(x)) = \eta(\downarrow \! G_\xi(x)) \subseteq \epsilon(\alpha_\xi(x))_2 \). The proof of the last inclusion is dual.

Additionally, we need:

**Definition 6.** For \( P \in \mathfrak{P} \), we define for every \( a \in \mathcal{E}(P) \)

\[
M(a) \equiv a_2 \oplus \{a_1\} \oplus a_3,
\]

where the induced order is removed from \( a_2 \) and \( a_3 \), and instead, both are treated as antichains. \( \iota(a) : M(a) \to X \) is the canonical inclusion mapping of \( M(a) \) in \( X \).

\( M(a) \) is thus isomorphic to \( A_{\#a_2} \oplus A_1 \oplus A_{\#a_3} \), and \( \iota(a) \in \mathcal{H}(M(a), P) \) is a one-to-one homomorphism. It is easily seen that \( a = \alpha_{\mathcal{H}(a)}(a_1) \) for every \( a \in \mathcal{E}(P) \). In this way, we can jump back and forth between the elements \( a \in \mathcal{E}(P) \) and the posets \( M(a) \) related to them.

### 3.3 G-schemes and I-schemes

As announced, we will now define two additional types of Hom-schemes which generalize the second and third property provided by the Hom-scheme induced by a one-to-one homomorphism:

**Definition 7.** For \( R, S \in \mathfrak{P} \), let \( \rho \) be a Hom-scheme from \( R \) to \( S \). We call \( \rho \)

- a G-scheme iff for every \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), x \in P \)

\[
G_{\rho_P(\xi)}(x) = G_\xi(x);
\]

(10)

- image-controlled or an I-scheme, iff for every \( P, Q \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), \zeta \in \mathcal{H}(Q, R), x \in P, y \in Q \)

\[
\alpha_{P,\xi}(x) \leq \alpha_{Q,\zeta}(y) \Rightarrow \alpha_{P,\rho_P(\xi)}(x) \leq \alpha_{Q,\rho_Q(\zeta)}(y),
\]

(11)

where \(<_+\) on the left side implies \(<_+\) on the right side.

We call a G-scheme / I-scheme strong, if it is strong as a Hom-scheme.

(10) is a plausible condition if we regard a Hom-scheme as a technical apparatus which assigns to every \( \xi \in \mathcal{H}(P, R) \) a well-fitting \( \rho(\xi) \in \mathcal{H}(P, S) \). If we allow \( G_\xi(x) \subseteq G_{\rho(\xi)}(x) \) for \( x \in P \), then Lemma \[\] tells us that \( \rho(\xi) \) preserves the structure of \( P \) around \( x \) worse than \( \xi \), which is not satisfying. And in the case \( G_\xi(x) \not\subseteq G_{\rho(\xi)}(x) \), \( \rho(\xi) \) has to re-distribute the points of \( G_\xi(x) \setminus G_{\rho(\xi)}(x) \subseteq G_\xi(x) \setminus \{x\} \) in \( S \). Because the sets \( G_\xi(x) \setminus G_{\rho(\xi)}(x) \) can be arbitrarily complicated, this re-distribution process may require many single case decisions, which is out of the scope of a technical apparatus.

With Corollary \[\] we conclude that a G-scheme \( \rho \) preserves strictness: \( \xi \in \mathcal{S}(P, R) \Rightarrow \rho_{\mathcal{S}(\xi)}(x) \in \mathcal{S}(P, S) \).

Let \( \rho \) be an image-controlled Hom-scheme from \( R \) to \( S \), let \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R) \), and let \( x, y \in P \) with \( \xi(x) < \xi(y) \). In the case of \( \xi(x) \in \xi(G_\xi(y)) \) and \( \xi(y) \in \xi(G_\xi(x)) \), we have \( \alpha_\xi(x) <_+ \alpha_\xi(y) \), thus \( \alpha_{\rho_P(\xi)}(x) <_+ \alpha_{\rho_P(\xi)}(y) \).
Because the mapping \( b \mapsto b_1 \) from \( \mathcal{E}(S) \) to \( S \) is a strict homomorphism, we conclude \( \rho(\xi(x)) = \alpha_{\rho(\xi)}(x_1) < \alpha_{\rho(\xi)}(y_1) = \rho(\xi(y)) \). An image-controlled Hom-scheme acts thus with respect to \( \xi(x) < \xi(y) \) exactly in the way we considered in Section 3.2. Additionally, Condition (11) generalizes the third property of a Hom-scheme \( r \) defined by a one-to-one homomorphism (4), as it has been explained in the beginning of Section 3.1.

Indeed, for such a Hom-scheme \( r \), we have \( \alpha_{P,r(\xi)}(x) = \sigma(\alpha_{\xi}(x)) \), where \( \sigma \) is applied separately to all three components of \( \alpha_{\xi}(x) \). The Hom-scheme induced by a one-to-one homomorphism is thus always strong and image-controlled. In the Figures 6, 7, and 8, solid lines indicate thus (trivial) strong I-schemes.

For an image-controlled Hom-scheme \( \rho \), we have due to the antisymmetry of \( \leq_+ \)

\[
\alpha_{P,\xi}(x) = \alpha_{Q,\xi}(y) \Rightarrow \alpha_{P,\rho r(\xi)}(x) = \alpha_{Q,\rho(\xi)}(y) \\
\Rightarrow \rho_P(\xi)(x) = \rho_Q(\xi)(y).
\] (12)

Also strong G-schemes and strong I-schemes define partial orders:

**Theorem 3.** Let \( R, S \in \Psi \). We write

\[
R \subseteq_G S / R \subseteq_I S
\]

iff a strong G-scheme / a strong I-scheme from \( R to S \) exists. The relations \( \subseteq_G \) and \( \subseteq_I \) define partial orders on \( \Psi_r \).

**Proof.** Reflexivity and antisymmetry of \( \subseteq_G \) and \( \subseteq_I \) are seen as in the proof of Theorem 2. Let \( R, S, T \in \Psi_r \), let \( \rho \) be a strong G-scheme (strong I-scheme) from \( R to S \), and let \( \tau \) be a strong G-scheme (strong I-scheme) from \( S to T \). We have already seen in the proof of Theorem 2 that \( \tau * \rho \) defined by

\[
(\tau * \rho)_P \equiv \tau_P \circ \rho_P
\]

is a strong Hom-scheme from \( R to T \). A short calculation shows that \( \tau * \rho \) is a strong G-scheme (strong I-scheme), too.

\[ \square \]

There exist (strong) G-schemes which are not (strong) I-schemes. Let \( \rho \) be a (strong) G-scheme from \( R to S \neq \emptyset \). With two disjoint isomorphic copies \( S_1 \) and \( S_2 \) of \( S \) with isomorphisms \( \pi_1 : S \to S_1 \) and \( \pi_2 : S \to S_2 \), we define for every \( P \in \Psi_r, \xi \in \mathcal{H}(P,R) \)

\[
\tau_P(\xi) \equiv \begin{cases} 
\pi_1 \circ \rho_P(\xi), & \text{if } \#P \text{ is odd;} \\
\pi_2 \circ \rho_P(\xi), & \text{if } \#P \text{ is even.}
\end{cases}
\]

Then \( \tau \) is a (strong) G-scheme from \( R to S_1 + S_2 \), but \( \tau \) is not image-controlled: Take \( P \in \Psi_r \) with \( \#P \) odd, \( \xi \in \mathcal{H}(P,R) \), and \( \zeta \in \mathcal{H}(A_1 + P, R) \) with \( \zeta|_P = \xi \). Then, for every \( x \in P \), \( G_\xi(x) = G_\zeta(x) \) and \( \alpha_{P,\xi}(x) = \alpha_{P,\zeta}(x) \), but \( \alpha_{P,\rho r(\xi)}(x) \in \mathcal{E}(S_1) \) and \( \alpha_{A_1 + P,\rho(\zeta)}(x) \in \mathcal{E}(S_2) \), thus \( \alpha_{P,\rho r(\xi)}(x) \neq \alpha_{A_1 + P,\rho(\zeta)}(x) \).

Additional examples for strong G-schemes which are not strong I-schemes are mentioned at the end of Section 4.3.

However, an I-scheme is a G-scheme:
**Theorem 4.** Let \( R, S \in \mathfrak{P} \), and let \( \rho \) be an image-controlled Hom-scheme from \( R \) to \( S \). We define for every \( a \in \mathcal{E}(R) \)
\[
\epsilon(a) \equiv \alpha_{M(a),\rho(a)}(a_1).
\]
Then \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) is a strict homomorphism with
\[
\alpha_{P,\rho(\xi)}(x) = \epsilon(\alpha_{P,\xi}(x)).
\]
for all \( P \in \mathfrak{P}_r \), \( \xi \in \mathcal{H}(P,R) \), \( x \in P \). In particular, \( \rho \) is a G-scheme.

**Proof.** \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) is a well-defined mapping. Let \( a, b \in \mathcal{E}(R) \) with \( a \prec b \). Because \( \rho \) is image-controlled, we get due to \( a = \alpha_{M(a),\rho(a)}(a_1) \), \( b = \alpha_{M(b),\rho(b)}(b_1) \)
\[
\epsilon(a) = \alpha_{M(a),\rho(\epsilon(a))}(a_1) \prec \alpha_{M(b),\rho(\epsilon(b))}(b_1) = \epsilon(b),
\]
and \( \epsilon \) is a strict homomorphism.

Now let \( P \in \mathfrak{P}_r \), and \( \xi \in \mathcal{H}(P,R) \) be fixed. For every \( x \in P \) we have
\[
\alpha_{P,\xi}(x) = \alpha_{M(\alpha(\xi)),\rho(\alpha(\xi))}(\xi(x))
\]
and thus, according to the first implication in (12)
\[
\alpha_{P,\rho(\xi)}(x) = \alpha_{M(\alpha(\xi)),\rho(\alpha(\xi))}(\xi(x)) = \epsilon(\alpha_{P,\xi}(x)).
\]
In particular, \( \rho P(\xi)(x) = \epsilon(\alpha_{P,\xi}(x)) \) for all \( x \in P \). Proposition 2 delivers \( G_{\rho P(\xi)}(x) = G_{\xi}(x) \), and the proof is finished.

We conclude \( R \sqsubseteq_\xi S \Rightarrow R \sqsubseteq_\xi S \Rightarrow R \sqsubseteq S \) for all \( R, S \in \mathfrak{P}_r \).

It should be mentioned that even for a strongly I-scheme \( \rho \) from \( R \) to \( S \), \( \xi \in \mathcal{H}(P,R) \cap \mathcal{H}(Q,R) \) does not imply \( \rho P(\xi) = \rho Q(\xi) \) for posets \( P,Q \in \mathfrak{P}_r \) with \( P \neq Q \).

**Proposition 2** shows, that a strict homomorphism \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) on \( S \in \mathfrak{P} \), induces a G-scheme \( \eta \) from \( R \) to \( S \) by setting \( \eta P(\xi)(x) \equiv \epsilon(\alpha_{\xi}(x)) \) for every \( P \in \mathfrak{P} \), \( \xi \in \mathcal{H}(P,R) \), \( x \in P \). In this way, we can show that all relations indicated in the Figures 6, 7, and 8 are in fact induced by strong G-schemes. Due to Theorem 4 we have to show this only for relations indicated by dashed and dotted lines in the figures.

Figure 4 shows again the posets \( R \) and \( S \) from the Figures 2 and 3 on the left, their EV-systems in the middle, and additionally an embedding \( \epsilon \) from the respective \( \mathcal{E}(R) \) into the respective \( \mathcal{E}(S) \) on the right. Comparing the images of \( \epsilon \) with the sets \( \mathcal{E}(R;v) \) and \( \mathcal{E}(S;w) \), we see in the case of Figure 4a that for every \( P \in \mathfrak{P} \) and every \( \xi \in \mathcal{H}(P,R) \)
\[
\begin{align*}
\eta(\xi)^{-1}(100) &= \xi^{-1}(100), & \eta(\xi)^{-1}(001) &= \xi^{-1}(001), \\
\eta(\xi)^{-1}(010) &= \xi^{-1}(110) \setminus \alpha_{\xi}^{-1}(110,\{100\},\emptyset) \\
\eta(\xi)^{-1}(010) &= \xi^{-1}(110) \setminus \gamma_{\xi}^{-1}(110) \setminus \xi^{-1}(110) \cap \xi^{-1}(100), \\
\eta(\xi)^{-1}(101) &= \xi^{-1}(011) \cup \alpha_{\xi}^{-1}(110,\{100\},\emptyset) \\
\eta(\xi)^{-1}(011) &= \xi^{-1}(110) \cup \gamma_{\xi}^{-1}(110) \setminus \xi^{-1}(110) \cap \xi^{-1}(100).
\end{align*}
\]
Figure 4: The posets and EV-systems from the Figures 2 and 3 together with embeddings of their EV-systems.

It is thus the Hom-scheme in Figure 2a, which is induced by the embedding $\epsilon$ in Figure 3a. We conclude, that the Hom-scheme in Figure 2a is a G-scheme (and it is strong, as we already know).

In the same way we find for the embedding $\epsilon$ shown in Figure 4b for every $P \in \mathcal{P}$ and every $\xi \in \mathcal{H}(P, R)$

$$
\eta(\xi^{-1}(1000)) = \xi^{-1}(001),
\eta(\xi^{-1}(1100)) = \xi^{-1}(100) \cup \alpha_{\xi}^{-1}(011, \{001\}, \emptyset),
\eta(\xi^{-1}(0001)) = \xi^{-1}(011) \setminus \alpha_{\xi}^{-1}(011, \{001\}, \emptyset),
\eta(\xi^{-1}(1110)) = \xi^{-1}(101).
$$

This is the Hom-scheme indicated in Figure 2b. Again, we conclude that this Hom-scheme is in fact a strong G-scheme.

In a similar way, it can be shown that all dashed and dotted lines in the Figures 6, 7, and 8 in the Appendix are induced by strong G-schemes. The corresponding constructions are explained in Figure 9. (Another way of proving $R \sqsubseteq_G S$ will be provided by Proposition 3.) Additionally, for incomparable posets in the Figures 6, 7, and 8, $R \not\sqsubseteq_G S$ has in all cases been proven using the results of Section 5.2. Examples for these proofs are presented after Figure 6.
4 Structure theory of strong I-schemes

In the following two sections, we will prove that the existence of an image-controlled, strong Hom-scheme \( \rho \) from \( R \) to \( S \) is equivalent to the existence of a one-to-one homomorphism \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \), fulfilling an additional condition. In Section 4.1, Theorem 5, we construct a strong I-scheme for a given homomorphism \( \epsilon \) of this type, and in Section 4.2, Theorem 6, we show that a strong I-scheme \( \rho \) from \( R \) to \( S \) induces such a homomorphism from \( \mathcal{E}(R) \) to \( \mathcal{E}(S) \). Because the additional condition on \( \epsilon \) is complicated to check, Section 4.3 is devoted to its replacement by more handy conditions.

4.1 From homomorphisms of EV-systems to strong I-schemes

Definition 8. Let \( R, S \in \mathcal{P} \) be fixed. In what follows, \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) is a strict homomorphism. For every poset \( P \in \mathcal{P}_r \) and for every homomorphism \( \xi \in \mathcal{H}(P, R) \), we define for every \( x \in P \)

\[
\eta_P(\xi)(x) \equiv \epsilon(\alpha_{P, \xi}(x))_1,
\]

We write \( \eta(\xi)(x) \) as abbreviation for \( \eta_P(\xi)(x) \) in the case of a fixed poset \( P \).

Proposition 3. Assume that \( \epsilon(a) = \epsilon(b) \) implies \( a_1 = b_1 \) or every \( a, b \in \mathcal{E}(R) \). Assume additionally, that for every \( P \in \mathcal{P}_r \), \( \xi \in \mathcal{H}(P, R) \), \( x \in P \)

\[
\begin{align*}
\alpha_{P, \eta_P(\xi)}(x) & \in \epsilon(\mathcal{E}(R)) \quad \Rightarrow \quad \alpha_{P, \eta_P(\xi)}(x) \in \epsilon(\mathcal{E}(R; \xi(x))), \quad (13) \\
\alpha_{P, \eta_P(\xi)}(x) & \notin \epsilon(\mathcal{E}(R)) \quad \Rightarrow \quad \xi(x) \in Z \setminus Z^+, \quad (14)
\end{align*}
\]

where \( Z^+ \) is a subset of the carrier \( Z \) of \( R \) with \( \#(Z \setminus Z^+) \leq 1 \). Then \( \eta \) is a strong G-scheme from \( R \) to \( S \) with

\[
\xi^{-1}(v) = \bigcup_{a \in \mathcal{E}(R; v)} \alpha_{P, \eta(\xi)}^{-1}(\epsilon(a)) \quad (15)
\]

for every \( v \in Z^+ \), \( P \in \mathcal{P}_r \), \( \xi \in \mathcal{H}(P, R) \).

Proof. According to Proposition 2, \( \eta \) is a G-scheme. We proceed with the proof of equation (15). Let \( P \in \mathcal{P}_r \) and \( \xi \in \mathcal{H}(P, R) \) be fixed. Let \( v \in Z^+ \) and let \( V \) be the set on the right side of (15). For \( x \in \xi^{-1}(v) \), the assumptions (14) and (13) deliver an \( a \in \mathcal{E}(R; v) \) with \( \alpha_{P, \eta(\xi)}(x) = \epsilon(a) \), hence

\[
x \in \alpha_{P, \eta(\xi)}^{-1}(\alpha_{P, \eta(\xi)}(x)) = \alpha_{P, \eta(\xi)}^{-1}(\epsilon(a)).
\]

We conclude \( x \in V \) due to \( a_1 = v \).

Now let \( x \in V \), i.e., \( x \in \alpha_{P, \eta(\xi)}^{-1}(\epsilon(a)) \) for an \( a \in \mathcal{E}(R; v) \). Then \( \alpha_{P, \eta(\xi)}^{-1}(\epsilon(a)) \neq \emptyset \) which is equivalent to \( \epsilon(a) = \alpha_{P, \eta(\xi)}(y) \) for a \( y \in P \). Therefore, the point \( x \) is an element of \( \alpha_{P, \eta(\xi)}^{-1}(\alpha_{P, \eta(\xi)}(y)) \), which implies \( \alpha_{P, \eta(\xi)}(x) = \alpha_{P, \eta(\xi)}(y) \in \epsilon(\mathcal{E}(R)) \). According to (13), there exists a \( b \in \mathcal{E}(R; \xi(x)) \) with \( \alpha_{P, \eta(\xi)}(x) = \epsilon(b) \). Now we get

\[
\epsilon(a) = \alpha_{P, \eta(\xi)}(y) = \alpha_{P, \eta(\xi)}(x) = \epsilon(b),
\]

and our assumption yields \( v = a_1 = b_1 = \xi(x) \), hence \( x \in \xi^{-1}(v) \).
Let $P \in \mathcal{P}$, and $\xi, \zeta \in \mathcal{H}(P, R)$ with $\eta(\xi) = \eta(\zeta)$. Then $\alpha_{\eta(\xi)} = \alpha_{\eta(\zeta)}$, and \[ (15) \] yields $\xi^{-1}(v) = \zeta^{-1}(v)$ for every $v \in Z^+$. Let $A = Z \setminus Z^+$. In the case of $A = \emptyset$, we have $\xi^{-1}(v) = \zeta^{-1}(v)$ for every $v \in R$. In the case of $A \neq \emptyset$, there exists an $a \in R$ with $A = \{a\}$, hence $\xi^{-1}(a) = X \setminus \xi^{-1}(Z^+) = X \setminus \zeta^{-1}(Z^+) = \zeta^{-1}(a)$. Therefore, $\xi^{-1}(v) = \zeta^{-1}(v)$ for every $v \in R$ also in this case. We conclude $\xi(x) = \zeta(x)$ for every $x \in P$, hence $\xi = \zeta$.

In the proof of this proposition, condition \[ (13) \] has been used for both directions, “$\subseteq$” and “$\supseteq$”, of equation \[ (15) \]. It is thus crucial. We formulate a sharpened generalization of it as a separate item:

**Condition 1.** We say that $\epsilon$ fulfills Condition \[ (11) \] iff for every $P \in \mathcal{P}$, for every $\xi \in \mathcal{H}(P, R)$, and for every $x \in P$

\[
\alpha_{P,\eta(\xi)}(x) = \epsilon(\alpha_{P,\xi}(x)).
\] (16)

Condition \[ (11) \] is powerful, but it does not imply that $\epsilon$ is one-to-one. Let \((a, b)\) be the carrier of $A_2$, and let \((c)\) be the carrier of $A_1$. Then $\mathcal{E}(A_2) = \{(a, \emptyset, \emptyset), (b, \emptyset, \emptyset)\} \simeq A_2$ and $\mathcal{E}(A_1) = \{(c, \emptyset, \emptyset)\} \simeq A_1$. The constant mapping $\epsilon : \mathcal{E}(A_2) \rightarrow \mathcal{E}(A_1)$ is a strict homomorphism, and for every $P \in \mathcal{P}$, $\eta \in \mathcal{H}(P, A_2)$, and $x \in P$ we have trivially $\alpha_{P,\eta(\xi)}(x) = (c, \emptyset, \emptyset) = \epsilon(\alpha_{P,\xi}(x))$. $\epsilon$ thus fulfills Condition \[ (11) \] but is not one-to-one. We have to postulate this property separately in the following theorem:

**Theorem 5.** Let $\epsilon$ be one-to-one. If $\epsilon$ fulfills Condition \[ (11) \] then $\eta$ is a strong I-scheme from $R$ to $S$.

**Proof.** Condition \[ (11) \] implies $\alpha_{P,\eta(\xi)}(x) \in \epsilon(\mathcal{E}(R))$ for all $P \in \mathcal{P}$, $\xi \in \mathcal{H}(P, R)$, $x \in X$, and \[ (13) \] is trivially fulfilled with $Z^+ = Z$. Additionally, Condition \[ (11) \] implies \[ (13) \], and $\eta$ is a strong G-scheme according to Proposition \[ (3) \]. Furthermore, $\eta$ is image-controlled: for $\alpha_{P,\xi}(x) \leq_+ \alpha_{Q,\zeta}(y)$,

\[
\alpha_{P,\eta(\xi)}(x) \leq_+ \epsilon(\alpha_{P,\xi}(x)) \leq_+ \epsilon(\alpha_{Q,\zeta}(y)) \leq_+ \alpha_{Q,\eta(\zeta)}(y)
\] (15) (16)

with $\alpha_{P,\xi}(x) <_+ \alpha_{Q,\zeta}(y) \Rightarrow \epsilon(\alpha_{P,\xi}(x)) <_+ \epsilon(\alpha_{Q,\zeta}(y))$, because $\epsilon$ is one-to-one.

The following proposition provides a necessary criterion for Condition \[ (11) \] it may be useful in the construction of strong I-schemes because it allows sorting out improper candidates for a suitable one-to-one homomorphism between EV-systems:

**Proposition 4.** Let $\epsilon$ fulfill Condition \[ (11) \]. Then

\[
\bigcup_{\eta \in \mathcal{E}(R)} (\epsilon(a)_2 \cup \epsilon(a)_3) \subseteq \{\epsilon(a)_1 \mid a \in \mathcal{E}(R)\}.
\] (17)

**Proof.** Let $W$ be the set on the right side of \[ (17) \], and assume that there is an $(a, D, U) \equiv a \in \mathcal{E}(R)$ with $\epsilon(a)_2 \cup \epsilon(a)_3 \not\subseteq W$. $a = \alpha_{M(a),\eta(\xi(a))}(a)$ implies $G_{\eta(\xi(a))}(a) = D$, and we have $G_{\eta(\xi(a))}(a) = G_{\xi(a)}(a)$ according to Proposition \[ (2) \]. Now we get

\[
\alpha_{M(a),\eta(\xi(a))}(a)_2 = \eta(\epsilon(a))(D) = \{\epsilon(\alpha_{M(a),\eta(\xi(a))}(d))_1 \mid d \in D\} \subseteq W,
\]
and similarly \( \alpha_{M(a),\eta(i(a))}(a)_3 \subseteq W \). Therefore, \( \alpha_{M(a),\eta(i(a))}(a) \neq \epsilon(a) = \epsilon(\alpha_{M(a),\eta(i(a))}(a)) \).

Inclusion \([17]\) holds trivially, if \( \epsilon(\cdot)_1 : \mathcal{E}(R) \to S \) is onto, but the converse does not hold.

At the end of this section, we demonstrate how Proposition \([3]\) can be used in showing \( R \sqsubseteq_G S \). Let \( R, S \) be the posets in Figure \([4]\). The homomorphism \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) in the figure is an embedding, and according to Proposition \([2]\) \( \eta \) is a G-scheme. Furthermore, \( \mathcal{E}(S) \setminus \epsilon(\mathcal{E}(R)) = \{\epsilon \} \) where \( \epsilon = (101, \{100, 001\}, \emptyset) \). According to Proposition \([2]\) we have \( \alpha_{P,\eta(\xi)}(x)_2 \subseteq \epsilon(\alpha_{P,\xi}(x))_2 \) for every \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), x \in P \). Because there is no \( a \in \mathcal{E}(R) \) with \( \{100, 001\} \subseteq \epsilon(a)_2 \), we conclude \( \alpha_{P,\eta(\xi)}(x) \neq \epsilon \), hence \( \alpha_{P,\eta(\xi)}(x) \in \epsilon(\mathcal{E}(R)) \) for every \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), x \in P \). In Proposition \([3]\) we can therefore choose \( Z^+ = Z \) and neglect Condition \([14]\).

Let \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), x \in P \). For \( v \in \{100, 001, 101\} \), there exists a \( w \in S \) with \( \epsilon(\mathcal{E}(R); v) = \mathcal{E}(S; w) \), thus \( \xi(x) = v \Rightarrow \eta(\xi)(x) = \epsilon(\alpha_{P,\xi}(x))_1 = w \Rightarrow \alpha_{P,\eta(\xi)}(x) \in \mathcal{E}(S; w) \). Because of \( \mathcal{E}(S; 010) = \{(010, \emptyset, \emptyset)\} \), we have \( \eta(\xi)(x) = 010 \Rightarrow \alpha_{P,\eta(\xi)}(x) = (010, \emptyset, \emptyset) \in \epsilon(\mathcal{E}(R; v)) \).

Let \( \xi(x) = v = 110 \) and \( \eta(\xi)(x) = 101 \). Then \( \alpha_{\xi}(x) = (110, \{100\}, \emptyset) \), thus \( \emptyset \neq \downarrow G_{\xi}(x) \) with \( \xi(y) = 100 \) for every \( y \in \downarrow G_{\xi}(x) \), hence \( \alpha_{\xi}(y) \in \mathcal{E}(R; 100) \). Because \( \eta \) is a G-scheme, we have \( \downarrow G_{\eta(\xi)}(x) = \downarrow G_{\xi}(x) \neq \emptyset \), and due to \( \epsilon(\mathcal{E}(R; 100)) = \mathcal{E}(S; 100) \), this means \( \eta(\xi)(y) = \epsilon(\alpha_{\xi}(y))_1 = 100 \) for every \( y \in \downarrow G_{\eta(\xi)}(x) \). Now \( \alpha_{P,\eta(\xi)}(x) = (101, \{100\}, \emptyset) \in \epsilon(\mathcal{E}(R; v)) \) is shown. Also Condition \([15]\) in Proposition \([3]\) is thus fulfilled, and \( R \sqsubseteq_G S \) follows.

In this way, we can show \( R \sqsubseteq_G S \) for all pairs of posets in Figure \([4]\). (For some of them, e.g., for \( (C7) \), we need the “exception condition” \([13]\) in Proposition \([3]\).) Even if we regard this type of proof as simpler than the type of proof for \( R \sqsubseteq_G S \) presented at the end of Section \([3.1]\), it is still far away from being simple, and the case discriminations become tedious for more complicated posets \( R \) and \( S \). Easier ways to prove \( R \sqsubseteq_G S \) will be developed in a coming paper \([6]\).

### 4.2 From strong I-schemes to homomorphisms of EV-systems

In this section, \( R \) and \( S \) are fixed finite posets with \( R \sqsubseteq_I S \), and \( \rho \) is a strong I-scheme from \( R \) to \( S \). According to Theorem \([4]\) \( \rho \) is a G-scheme; we thus have

\[
G_{\rho P(\xi)}(x) = G_{\xi}(x)
\]

for all \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), \) and \( x \in P \). We define as in Theorem \([3]\) for all \( a \in \mathcal{E}(R) \)

\[
\epsilon(a) \equiv \alpha_{M(a),\rho(i(a))}(a)_1,
\]

and we know that \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) is a strict homomorphism with \( \rho P(\xi)(x) = \epsilon(\alpha_{P,\xi}(x))_1 \) for all \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R), \) \( x \in P \).
Corollary 4. For every \( a \in \mathcal{E}(R) \)
\[
\begin{align*}
\# \epsilon(a)_2 & \leq \# a_2, \\
\# \epsilon(a)_3 & \leq \# a_3.
\end{align*}
\]
Proof. We have \( \alpha_{M(a),n(a)}(a_1) = a \), thus \( \downarrow G_{i(a)}(a_1) = \iota(a)(\downarrow G_{i(a)}(a_1)) = a_2 \).
We conclude:
\[
\begin{align*}
\epsilon(a)_2 &= \alpha_{M(a),n(a)}(a_1)_2 \\
&= \rho_{M(a)}(\iota(a))(\downarrow G_{i(a)}(a_1)) \\
&= \rho_{M(a)}(\iota(a))(\downarrow G_{i(a)}(a_1)) = \rho_{M(a)}(\iota(a))(a_2),
\end{align*}
\]
and the first inequality is proven because \( \rho_{M(a)}(\iota(a)) \) is a mapping. The second one is dual.

Theorem 6. Let \( P \in \mathfrak{P} \), \( a \in \mathcal{E}(R) \), and \( \phi \in \mathcal{H}(P, R) \) with \( \phi(P) = M(a) \). If the post-restriction \( \phi_{i}^{M(a)} : P \to M(a) \) is an isomorphism, then \( \rho_{P}(\phi)(P) = M(\epsilon(a)) \), and \( \rho_{P}(\phi)|^{M(\epsilon(a))} : P \to M(\epsilon(a)) \) is an isomorphism, too. In particular,
\[
\rho_{M(a)}(\iota(a))|^{M(\epsilon(a))} : M(a) \to M(\epsilon(a))
\]
(20)
is an isomorphism.
Proof. We have \( P = D \oplus \{ \emptyset \} \oplus U \) with \( D \simeq A_{\# a_2} \) and \( U \simeq A_{\# a_3} \). Furthermore, \( \phi(D) = a_2, \phi(p) = a_1, \) and \( \phi(U) = a_3 \), thus
\[
\begin{align*}
\alpha_{P,\phi}(d) &= (\phi(d), \emptyset, \{ a_1 \} \cup a_3) = \alpha_{M(a),n(a)}(\phi(d)) \quad \text{for all } d \in D, \\
\alpha_{P,\phi}(p) &= a = \alpha_{M(a),n(a)}(a_1) = \alpha_{M(a),n(a)}(\phi(p)), \\
\alpha_{P,\phi}(u) &= (\phi(u), \{ a_1 \} \cup a_2, \emptyset) = \alpha_{M(a),n(a)}(\phi(u)) \quad \text{for all } u \in U.
\end{align*}
\]
Because \( \rho \) is image-controlled, this yields \( \alpha_{P,\rho(\phi)}(x) = \alpha_{M(a),n(\iota(a))}(\phi(x)) \) for all \( x \in P \) (first equation in (12)). We have thus
\[
\begin{align*}
\rho_{P}(\phi)(p) &= \alpha_{P,\rho(\phi)}(p)_1 = \alpha_{M(a),n(\iota(a))}(\phi(p))_1 \\
&= \alpha_{M(a),n(\iota(a))}(a_1)_1 = \epsilon(a)_1,
\end{align*}
\]
and for every \( d \in D \) we have
\[
\begin{align*}
\rho_{P}(\phi)(d) &= \alpha_{P,\rho(\phi)}(d)_1 = \alpha_{M(a),n(\iota(a))}(\phi(d))_1 \\
&= \rho_{M(a)}(\iota(a))(\phi(d)) \in \rho_{M(a)}(\iota(a))(\downarrow G_{i(a)}(a_1)) \\
&= \rho_{M(a)}(\iota(a))(a_1)_2 = \epsilon(a)_2.
\end{align*}
\]
Similarly, we get \( \rho_{P}(\phi)(u) \in \epsilon(a)_3 \) for every \( u \in U \). Therefore, \( \rho_{P}(\phi)(P) \subseteq M(\epsilon(a)) \) with \( \rho_{P}(\phi)(p) = \epsilon(a)_1, \rho_{P}(\phi)(D) \subseteq \epsilon(a)_2, \) and \( \rho_{P}(\phi)(U) \subseteq \epsilon(a)_3 \).

Now we prove that \( \rho_{P}(\phi) \) is one-to-one. It is sufficient to show that \( \rho_{P}(\phi)|_{D} \) and \( \rho_{P}(\phi)|_{U} \) are both one-to-one. Let \( c, d \in D \) with \( \rho_{P}(\phi)(c) = \rho_{P}(\phi)(d) \). We define \( \psi : P \to R \) by
\[
\psi(x) = \begin{cases} 
\phi(x), & \text{if } x \in P \setminus \{ c, d \}; \\
\phi(d), & \text{if } x = c; \\
\phi(c), & \text{if } x = d.
\end{cases}
\]
and as above we get \( \alpha_{P,\rho(\psi)}(x) = \alpha_{M(a),\rho_i(a)}(\psi(x)) \) for all \( x \in P \). Now we get for every \( x \in P \setminus \{c,d\} \)
\[
\rho_P(\psi)(x) = \alpha_{P,\rho(\psi)}(x)_1 = \alpha_{M(a),\rho_i(a)}(\psi(x))_1
= \alpha_{M(a),\rho_i(a)}(\phi(x))_1 = \alpha_{P,\rho(\phi)}(x)_1 = \rho_P(\phi)(x)
\]
and similarly
\[
\rho_P(\psi)(c) = \alpha_{M(a),\rho_i(a)}(\psi(c))_1 = \alpha_{M(a),\rho_i(a)}(\phi(c))_1
= \rho_P(\phi)(d) = \rho_P(\phi)(c),
\]
\[
\rho_P(\psi)(d) = \alpha_{M(a),\rho_i(a)}(\psi(d))_1 = \alpha_{M(a),\rho_i(a)}(\phi(d))_1
= \rho_P(\phi)(c) = \rho_P(\phi)(d).
\]
hence \( \rho_P(\psi) = \rho_P(\phi) \). Because \( \rho \) is strong, we have \( \psi = \phi \), thus \( c = d \), because \( \phi \) is one-to-one. In the same way we see that \( \rho_P(\phi)|_J \) is one-to-one. Therefore,
\[
\#a_2 \leq \#D \leq \#\epsilon(a)_2,
\]
\[
\#a_3 \leq \#U \leq \#\epsilon(a)_3.
\]
Corollary 4 yields equality, and \( \rho_P(\phi)|_{M(\epsilon(a))} : P \to M(\epsilon(a)) \) is an isomorphism. The last proposition follows with \( P = M(a) \) and \( \phi = i(a) \).

\[\square\]

**Theorem 7.** \( \epsilon \) is one-to-one and fulfills Condition 1.

**Proof.** Let \( a, b \in \mathcal{E}(R) \) with \( \epsilon(a) = \epsilon(b) \). Applying Theorem 3 we see that the posets \( M(a), P = M(\epsilon(a)) = M(\epsilon(b)) \), and \( M(b) \) are all three isomorphic. Let \( m \equiv \#a_2 \) and \( n \equiv \#a_3 \). There exist \( (m!) \cdot (n!) \) isomorphisms between \( P \) and \( M(a) \), between \( P \) and \( M(b) \), and between \( P \) and \( P \). Let \( \mathcal{I}(P, M(a)) \) and \( \mathcal{I}(P, M(b)) \) be the set of isomorphisms from \( P \) to \( M(a) \) and from \( P \) to \( M(b) \), respectively. With
\[
\mathcal{J}(P) \equiv \{ i(a) \circ \rho \mid \rho \in \mathcal{I}(P, M(a)) \} \cup \{ i(b) \circ \rho \mid \rho \in \mathcal{I}(P, M(b)) \}
\]
\( \mathcal{J}(P) \) is a subset of \( \mathcal{H}(P, R) \) with \( \#\mathcal{J}(P) \geq (m!) \cdot (n!) \); equality holds iff \( a = b \).

Theorem 3 delivers that for each \( \phi \in \mathcal{J}(P) \) the mapping \( \rho_P(\phi)^P_P : P \to P \) is an isomorphism. Because \( \rho \) is strong, we conclude \( \#\mathcal{J}(P) \leq (m!) \cdot (n!) \), thus \( a = b \), and \( \epsilon \) is one-to-one.

According to Theorem 4 \( \epsilon \) fulfills Condition 1.

\[\square\]

### 4.3 The replacement of Condition 1

We have seen in the last two sections that the existence of a strong I-scheme from \( R \) to \( S \) is equivalent to the existence of a one-to-one homomorphism \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) fulfilling Condition 1. Condition 1 is unwieldy to check because it refers to how \( \epsilon \) acts on the sets \( \alpha(\xi)(x) \) for all \( P \in \mathfrak{P}_r, \xi \in \mathcal{H}(P, R) \). It is desirable to replace it by more handy conditions referring to \( \epsilon : \mathcal{E}(R) \to \mathcal{E}(S) \) only. As in section 4.1 we define
\[
\eta(\xi)(x) \equiv \eta_P(\xi)(x) \equiv \epsilon(\alpha_{P,\xi}(x))_1,
\]

21
for every $P \in \mathcal{P}_r, \xi \in \mathcal{H}(P, R)$, and $x \in P$.

The first condition sharpens the demand on $\epsilon$ as a structure-preserving mapping: we will assume that $\epsilon$ is an embedding. The second and third condition concern the image of $\epsilon$:

**Condition 2.** For all $a \in \mathcal{E}(R)$
\begin{align*}
\forall c, d \in \mathcal{a}_2 : e \neq d & \implies \epsilon(\mathcal{E}(R; c)) \cap \epsilon(\mathcal{E}(R; d)) = \emptyset, \\
\forall c, d \in \mathcal{a}_3 : e \neq d & \implies \epsilon(\mathcal{E}(R; c)) \cap \epsilon(\mathcal{E}(R; d)) = \emptyset.
\end{align*}

**Condition 3.** For all $a \in \mathcal{E}(R)$
\begin{align*}
\forall w \in \epsilon(\mathcal{a})_2 : \exists b \in \mathcal{E}(R) : \epsilon(b) = (w, \emptyset, \{\epsilon(a)\}), \\
\forall w \in \epsilon(\mathcal{a})_3 : \exists b \in \mathcal{E}(R) : \epsilon(b) = (w, \{\epsilon(a)\}, \emptyset).
\end{align*}

**Lemma 4.** If $\epsilon$ is an embedding which fulfills the Conditions 2 and 3 then for all $a \in \mathcal{E}(R)$
\begin{align*}
\#a_2 &= \#\epsilon(a)_2, \\
\#a_3 &= \#\epsilon(a)_3.
\end{align*}

*Proof.* We provide a detailed proof for $a_2$ only; the proof for $a_3$ is dual.

We start with “$\leq$”. Let $c, d \in \mathcal{a}_2$ with $e \neq d$. With $c \equiv (c, \emptyset, \{a_1\}), d \equiv (d, \emptyset, \{a_1\})$, we have $e \in \mathcal{E}(R; c)$ and $\emptyset \in \mathcal{E}(R; d)$ with $c \prec e$ and $\emptyset \prec e$. Then $e(c) < e(a)$ and $\emptyset < a$, thus $e(c)_1 \in \epsilon(a)_2$ and $\emptyset \in \epsilon(a)_2$. According to Condition 2 $\epsilon(\emptyset)_1 \neq \epsilon(c)_1$.

The proof of “$\geq$” is not difficult, but technical. Let $a \in \mathcal{E}(R)$ be fixed. We start with the proof of
\[ \epsilon(a)_2 = \emptyset \iff a_2 = \emptyset \] (21)
and
\[ \#\epsilon(a)_2 = 1 \implies \#a_2 = 1. \] (22)

$a_2 \neq \emptyset \implies \epsilon(a)_2 \neq \emptyset$ has already been established. Assume $\epsilon(a)_2 \neq \emptyset$.

According to Condition 3 there is a $b \in \mathcal{E}(R)$ with $\epsilon(b) \prec e(a)$. We conclude $b \prec a$, hence $a_2 \neq \emptyset$, and (21) is shown. If $\#\epsilon(a)_2 = 1$, then (21) yields $\#a_2 \geq 1$. We have already seen $\#a_2 \leq \#\epsilon(a)_2$, and (22) is proven.

Now we prove $\#\epsilon(a)_2 \leq \#a_2$. For $\epsilon(a)_2 = \emptyset$, there is nothing to show.

Let $\epsilon(a)_2 = \{w_1, \ldots, w_n\}, n \in \mathbb{N}$. According to Condition 2 there exist $b(1), \ldots, b(n) \in \mathcal{E}(R)$ with
\[ \epsilon(b(i)) = (w_i, \emptyset, \{\epsilon(a)\}) \quad \text{for all } i \in \mathbb{N}. \]

Due to (21) and the dual of (22) (i.e., $\#\epsilon(a)_3 = 1 \implies \#a_3 = 1$), there exist $v_i, x_i \in R$ with
\[ b(i) = (v_i, \emptyset, \{x_i\}) \quad \text{for all } i \in \mathbb{N}. \]

We have $\epsilon(b(i)) \prec \epsilon(a)$ for all $i \in \mathbb{N}$, hence $b(i) \prec a$ for all $i \in \mathbb{N}$. We conclude $v_i \in a_2$ and $x_i = a_1$ for all $i \in \mathbb{N}$. Because the $b(i)$ are pairwise different, we have $v_i \neq v_j$ for $i, j \in \mathbb{N}$ with $i \neq j$. Therefore, $\#\epsilon(a)_2 \leq \#a_2$. 

\hfill \Box
Theorem 8. Let \( \epsilon \) be an embedding which fulfills the Conditions \( 3 \) and \( 5 \). Then \( \epsilon \) fulfills Condition \( 1 \) and \( \eta \) is a strong I-scheme.

Proof. Let \( P \in \mathcal{P}_r, \xi \in \mathcal{H}(P, R) \), and \( x \in P \). We have to show \( \alpha_\eta(\xi)(x) = \epsilon(\alpha_\xi(x)) \). Due to the definitions of \( \alpha_\eta(\xi)(x) \) and \( \eta \), we have \( \alpha_\eta(\xi)(x)_1 = \eta(\xi)(x) = \epsilon(\alpha_\xi(x))_1 \). Looking at Lemma \( 4 \) and Proposition \( 3 \) we realize that for the proof of \( \alpha_\eta(\xi)(x)_2 = \epsilon(\alpha_\xi(x))_2 \) and \( \alpha_\eta(\xi)(x)_3 = \epsilon(\alpha_\xi(x))_3 \), it is sufficient to prove \( \#\alpha_\xi(x)_j \leq \#\alpha_\eta(\xi)(x)_j, j = 2, 3 \).

Let \( c, d \in \alpha_\xi(x)_1 \) with \( c \neq d \). There exist \( y, z \in \mathcal{I}(R; d) \) with \( c = \xi(y) \) and \( d = \xi(z) \). According to Condition \( 7 \), we have \( \epsilon(\mathcal{I}(R; c))_1 \cap \epsilon(\mathcal{I}(R; d))_1 = \emptyset \), and due to \( \alpha_\xi(y) \in \mathcal{I}(R; c) \) and \( \alpha_\xi(z) \in \mathcal{I}(R; d) \), we conclude

\[
\eta(\xi)(y) = \epsilon(\alpha_\xi(y))_1 \neq \epsilon(\alpha_\xi(z))_1 = \eta(\xi)(z).
\]

Therefore, \( \eta(\xi)(y) \) and \( \eta(\xi)(z) \) are different elements of \( \alpha_\eta(\xi)(x)_1 \), and the proof of \( \#\alpha_\xi(x)_2 = \#\alpha_\eta(\xi)(x)_2 \) is finished. The proof for \( \alpha_\xi(x)_3 \) is dual. The proposition about \( \eta \) is a direct consequence of Theorem \( 5 \).

Using this result, we show \( R \sqsubseteq I S \) for the posets in Figure \( 4a \). The conditions in Theorem \( 8 \) deal with the homomorphism \( \epsilon : \mathcal{I}(R) \to \mathcal{I}(S) \) only, without taking into account the triplets \( \alpha_{P, \eta}(\xi)(x) \). Therefore, we may expect that the proof of \( R \sqsubseteq I S \) by means of Theorem \( 8 \) will be considerably simpler than the proof of \( R \sqsubseteq G S \) presented in Section \( 4.1 \). And indeed! The homomorphism \( \epsilon \) in Figure \( 4b \) is an embedding with

\[
\begin{align*}
\epsilon(\mathcal{I}(R; 100)) &= \{(1100, \emptyset, \emptyset), (1100, \emptyset, \{1110\})\}, \\
\epsilon(\mathcal{I}(R; 001)) &= \mathcal{I}(S; 1000), \\
\epsilon(\mathcal{I}(R; 101)) &= \mathcal{I}(S; 1110), \\
\epsilon(\mathcal{I}(R; 011)) &= \{(1100, \emptyset, \emptyset), (0001, \emptyset, \emptyset)\}.
\end{align*}
\]

Because there is no \( a \in \mathcal{I}(R) \) with \( \{100, 011\} \subseteq a \) or \( \{100, 011\} \subseteq a \), Condition \( 2 \) is fulfilled. Furthermore, \( \epsilon \) fulfills Condition \( 3 \) because \( \epsilon(\mathcal{I}(R)) \) contains all \( b \in \mathcal{I}(S) \) with \( \#b_2 = \emptyset, \#b_3 = 1 \) or \( \#b_2 = 1, b_3 = \emptyset \). Now \( R \sqsubseteq I S \) follows with Theorem \( 8 \).

Using Theorem \( 8 \) in this way, we see that the Constructions \( (C1)-(C6), (C8), \) and \( (C10) \) in Figure \( 9 \) in the Appendix define in fact strong I-schemes. However, the Constructions \( (C7), (C9), (C11), \) and \( (C12) \) violate Condition \( 2 \). Are they nevertheless I-schemes? No! Using the results of the previous sections we can show that there exists no strong I-scheme between the posets in these cases.
constructions. We show it for Construction (C7), the argumentation is similar for the Constructions (C9), (C11), and (C12).

Figure 5 shows the posets from Construction (C7). Assume, that there exists a strong I-scheme from $R$ to $S$. We define the homomorphism $\epsilon : E(R) \to E(S)$ as in (19), and we know by Theorem 7 that $\epsilon$ fulfills Condition 1. Let $P \equiv R, \xi \equiv id_P$. For $\alpha_\xi(101) = (101, \{100, 001\}, \emptyset)$, Theorem 6 delivers $\#\epsilon(\alpha_\xi(101))_2 = 2$. But we have $\alpha_\xi(100) = (100, \emptyset, \{110, 101\})$ and $\alpha_\xi(001) = (001, \emptyset, \{101, 011\})$, and again Theorem 6 enforces $\#\epsilon(\alpha_\xi(100))_3 = 2 = \#\epsilon(\alpha_\xi(001))_3$. Because $0100$ is the only point $w \in S$ for which a $b \in E(S; w)$ exists with $\#b_3 > 1$, we conclude $\epsilon(\alpha_\xi(100))_2 = \epsilon(\alpha_\xi(001))_2 \in E(S; 0100)$, hence $\eta(\xi)(100) = \epsilon(\alpha_\xi(100))_1 = 0100 = \epsilon(\alpha_\xi(001))_1 = \eta(\xi)(001)$.

Because $\eta$ is a G-scheme (Theorem 4), this means $\alpha_{\eta(\xi)}(101) = \eta(\xi)\{\uparrow G_{\eta(\xi)}(101)\} = \eta(\xi)\{\uparrow G_{\xi}(101)\} = \eta(\xi)\{\{100, 001\}\} = \{0100\}$, thus $\alpha_{\eta(\xi)}(101) \neq \epsilon(\alpha_\xi(101))_2$ in contradiction to Condition 1.

5 Miscellaneous

5.1 Compatibility with order arithmetic

As we have seen in the Theorems 2 and 3, the relations $\sqsubseteq, \sqsubseteq_G, \text{ and } \sqsubseteq_I$ define partial orders on $\mathfrak{P}_\rho$. In this section, we examine to which extent these partial orders are compatible with order arithmetic; cancellation rules will be presented in a separate paper [5]. In order to avoid repetitions, we agree that the carriers of posets involved in direct or ordinal sums are always disjoint. Additionally, we agree on that the index $j$ is always an element of $2$.

The simple equation (5) for $E(P^d)$ lets us expect that there will be a simple relation between the different types of strong Hom-schemes from $R$ to $S$ and from $R^d$ to $S^d$. Similarly, looking to (6), the direct sum of posets should also be easily manageable in the world of strong Hom-schemes. The formulas (7) and (8) for $E(P \oplus Q)$ and $E(P \times Q)$ refer to $E(P)$ and $E(Q)$ in a more involved way, and therefore, the situation will be more complicated for ordinal sums and products of posets.

**Proposition 5.** Let $R, S \in \mathfrak{P}$. Then

$$R \sqsubseteq S \iff R^d \sqsubseteq S^d,$$

$$R \sqsubseteq_G S \iff R^d \sqsubseteq_G S^d,$$

$$R \sqsubseteq_I S \iff R^d \sqsubseteq_I S^d,$$

$$R \sqsubseteq R^d \iff R \simeq R^d.$$

**Proof.** Let $\rho$ be a strong Hom-scheme from $R$ to $S$. Because of $H(P, Q) = H(P^d, Q^d)$ for every $P, Q \in \mathfrak{P}$, we get a strong Hom-scheme $\rho^d$ from $R^d$ to $S^d$ by setting $\rho_P^d \equiv \rho_{P^d}$ for all $P \in \mathfrak{P}$. Obviously, $\rho^d$ is a G-scheme (an I-scheme) iff $\rho$ is. The last proposition follows from the first one due to the antisymmetry of $\sqsubseteq$.

\square
Proposition 6. Let \( R^1, R^2, S^1, S^2 \in \mathcal{P} \). Then
\[
R^1 \subseteq S^1 \text{ and } R^2 \subseteq S^2 \Rightarrow R^1 + R^2 \subseteq S^1 + S^2,
\]
\[
R^1 \subseteq_G S^1 \text{ and } R^2 \subseteq_G S^2 \Rightarrow R^1 + R^2 \subseteq_G S^1 + S^2,
\]
\[
R^1 \subseteq_I S^1 \text{ and } R^2 \subseteq_I S^2 \Rightarrow R^1 + R^2 \subseteq_I S^1 + S^2.
\]

Proof. Let \( \rho^1 \) be a strong Hom-scheme from \( R^1 \) to \( S^1 \). For every \( P \in \mathcal{P} \), and for every \( \xi \in \mathcal{H}(P, R^1 + R^2) \), we define
\[
Q(\xi)^1 \equiv \xi^{-1}(R^1),
\]
\[
q(\xi)^1 \equiv \xi^{(R^1)}.
\]

\( Q(\xi)^1 \) and \( Q(\xi)^2 \) are disjoint subsets of \( P \) with \( Q(\xi)^1 \cup Q(\xi)^2 = P \). We equip \( Q(\xi)^1 \) with the partial order induced by \( P \) and define for every \( x \in P \)
\[
\rho_P(\xi)(x) = \begin{cases} 
\rho_{Q(\xi)^1}(q(\xi)^1)(x), & \text{if } x \in Q(\xi)^1; \\
\rho_{Q(\xi)^2}(q(\xi)^2)(x), & \text{if } x \in Q(\xi)^2.
\end{cases}
\]

Let \( \xi, \zeta \in \mathcal{H}(P, R^1 + R^2) \) with \( \xi \neq \zeta \). Then there exists an \( x \in P \) with \( \xi(x) \neq \zeta(x) \). Two cases are possible:

- **Case 1.** \( Q(\xi)^1 = Q(\zeta)^1 \) and \( Q(\xi)^2 = Q(\zeta)^2 \): Assume \( x \in Q(\xi)^1 \). Then we have \( q(\xi)^1, q(\zeta)^1 \in \mathcal{H}(Q(\xi)^1, R^1) \) with \( q(\xi)^1 \neq q(\zeta)^1 \). Because \( \rho^1 \) is strong, we get \( \rho_{Q(\xi)^1}(q(\xi)^1) \neq \rho_{Q(\zeta)^1}(q(\zeta)^1) \) which yields \( \rho_P(\xi)(x) \neq \rho_P(\zeta)(x) \). The same argument holds for \( x \in Q(\xi)^2 \).

- **Case 2.** \( Q(\xi)^1 \neq Q(\zeta)^1 \) and \( Q(\xi)^2 \neq Q(\zeta)^2 \): Assume that there exists a \( y \in Q(\xi)^2 \setminus Q(\zeta)^1 \). Then \( y \in Q(\xi)^2 \setminus Q(\xi)^1 \), and we get \( \rho_P(\xi)(y) \in S^1 \) but \( \rho_P(\zeta)(y) \in S^2 \), thus \( \rho_P(\xi) \neq \rho_P(\zeta) \) because \( S^1 \) and \( S^2 \) are disjoint. The same argumentation works for \( Q(\xi)^1 \setminus Q(\zeta)^1 \neq \emptyset \), too.

\( R^1 + R^2 \subseteq S^1 + S^2 \) is now proven.

Assume, that the \( \rho^1 \) are additionally G-schemes. For \( \xi(x) \in R^1 \), we have \( G_\xi(x) \subseteq \xi^{-1}(\xi(x)) \subseteq Q(\xi)^1 \) and \( G_{\rho_P(\xi)}(x) \subseteq \rho_P(\xi)^{-1}(\rho_P(\xi)(x)) \subseteq Q(\xi)^2 \), which implies
\[
G_\xi(x) = G_{q(\xi)^1}(x) = G_{\rho_{Q(\xi)^1}(q(\xi)^1)}(x) = G_{\rho_P(\xi)}(x),
\]
and \( \rho \) is a G-scheme from \( R^1 + R^2 \) to \( S^1 + S^2 \).

Now assume \( R^1 \subseteq_I S^1 \) and \( R^2 \subseteq_I S^2 \). According to Theorem \( \text{[\ref{thm:G-scheme}]} \) there exist one-to-one homomorphisms \( \epsilon^1 : \mathcal{E}(R^1) \to \mathcal{E}(S^1) \) which fulfill Condition \( \text{[\ref{cond:epsilon}]} \). Equation \( \text{[\ref{eq:epsilon}]} \) delivers \( \mathcal{E}(R^1 + R^2) = \mathcal{E}(R^1) + \mathcal{E}(R^2) \) and \( \mathcal{E}(S^1 + S^2) = \mathcal{E}(S^1) + \mathcal{E}(S^2) \). Now the mapping \( \epsilon^+ : \mathcal{E}(R^1 + R^2) \to \mathcal{E}(S^1 + S^2) \) defined by \( \epsilon^+(a) = \epsilon^1(a) \) for \( a \in \mathcal{E}(R^1) \) is a one-to-one homomorphism which fulfills Condition \( \text{[\ref{cond:epsilon}]} \). Theorem \( \text{[\ref{thm:G-scheme}]} \) delivers \( R^1 + R^2 \subseteq_I S^1 + S^2. \)
Proposition 7. Let $R^1, R^2, S^1, S^2 \in \mathfrak{P}$. Then

\[ R^1 \subseteq S^1 \text{ and } R^2 \subseteq S^2 \Rightarrow R^1 \oplus R^2 \subseteq S^1 \oplus S^2, \]
\[ R^1 \sqsubseteq_G S^1 \text{ and } R^2 \sqsubseteq_G S^2 \Rightarrow R^1 \oplus R^2 \sqsubseteq_G S^1 \oplus S^2. \]

Proof. For posets $A, B$, the relation $A \sqsubseteq B$ means $\#\mathcal{H}(P, A) \leq \#\mathcal{H}(P, B)$ for all $P \in \mathfrak{P}_A$. The first statement is now a direct consequence of

\[ \mathcal{H}(P, A \oplus B) \simeq \sum_{U \in \mathcal{U}(P)} \mathcal{H}(P \setminus U, A) \times \mathcal{H}(U, B) \]

for all finite posets $P, A, B$, where $\mathcal{U}(P)$ is the set of the upsets of $P$ and where $P \setminus U$ and $U$ are equipped with the induced partial order.

For the proof of the second statement, we start exactly as in the proof of Proposition 6. \( \rho^1 \) is a strong Hom-schemes from $R^1$ to $S^1$, and for every $P \in \mathfrak{P}$ and every $\xi \in \mathcal{H}(P, R^1 \oplus R^2)$ we define

\[ Q(\xi)^1 \equiv \xi^{-1}(R^1), \]
\[ q(\xi)^1 \equiv \xi^{R^1}(\xi), \]

and we equip $Q(\xi)^1 \subseteq P$ again with the partial order induced by $P$. Finally, for every $x \in P$

\[ \rho_P(\xi)(x) \equiv \begin{cases} \rho^1_{Q(\xi)^1}(q(\xi)^1)(x), & \text{if } x \in Q(\xi)^1; \\ \rho^2_{Q(\xi)^2}(q(\xi)^2)(x), & \text{if } x \in Q(\xi)^2. \end{cases} \]

Let $x, y \in P$ with $x \leq y$. In the cases $\xi(x), \xi(y) \in R^1$ and $\xi(x), \xi(y) \in R^2$, we have $\rho_P(\xi)(x) \leq \rho_P(\xi)(y)$ because $\rho^1$ and $\rho^2$ are both Hom-schemes. In the case $\xi(x) \in R^1, \xi(y) \in R^2$, we have $\rho_P(\xi)(x) < \rho_P(\xi)(y)$ due to $a < b$ for all $a \in S^1, b \in S^2$. The case $\xi(x) \in R^2, \xi(y) \in R^1$ is not possible due to $x \leq y$. $\rho_P(\xi)$ is thus a homomorphism.

\[ \rho_P(\xi) = \rho_P(\xi) \Rightarrow \xi = \zeta \text{ and } G_{\rho(\xi)}(x) = G_{\xi}(x) \text{ for all } P \in \mathfrak{P}_x, \xi, \zeta \in \mathcal{H}(P, R^1 \oplus R^2), x \in P, \] are shown exactly as in the proof of Proposition 6. \( \Box \)

Proposition 8. Let $R^1, R^2, S^1, S^2 \in \mathfrak{P}$. Then

\[ R^1 \subseteq S^1 \text{ and } R^2 \subseteq S^2 \Rightarrow R^1 \times R^2 \subseteq S^1 \times S^2, \]
\[ R^1 \sqsubseteq_G S^1 \text{ and } R^2 \sqsubseteq_G S^2 \Rightarrow R^1 \times R^2 \sqsubseteq_G S^1 \times S^2. \]

Proof. The first statement is a direct consequence of $\mathcal{H}(P, A \times B) \simeq \mathcal{H}(P, A) \times \mathcal{H}(P, B)$ for all finite posets $P, A, B$. For a constructive proof (which we need for the second statement), take strong Hom-schemes $\rho^1$ from $R^1$ to $S^1$. For every $P \in \mathfrak{P}$ and every $\xi = (\xi^1, \xi^2) \in \mathcal{H}(P, R^1 \times R^2)$ (with $\xi^2 \in \mathcal{H}(P, R^1)$), we define $\rho^1_P((\xi^1, \xi^2)) \equiv (\rho^1_P(\xi^1), \rho^2_P(\xi^2))$. Then $\rho^\times$ is a strong Hom-scheme from $R^1 \times R^2$ to $S^1 \times S^2$.

Now let $\rho^1$ and $\rho^2$ be strong G-schemes. For every poset $P \in \mathfrak{P}$, $\xi = (\xi^1, \xi^2) \in \mathcal{H}(P, R^1 \times R^2), x \in P$, we get by applying Lemma 2 twice

\[ G_{\rho^\times(\xi)}(x) = \gamma_{G_{\rho(\xi^1)}(x)} \circ G_{\rho(\xi^2)}(x) = \gamma_{\xi^1(x)} \circ G_{\xi^2}(x) = G_{\xi}(x). \]

\( \Box \)
Proposition 9. Let $R, S \in \mathcal{P}$. Then
\[ R \sqsubseteq S \implies \mathcal{H}(Q, R) \sqsubseteq_G \mathcal{H}(Q, S) \text{ for every } Q \in \mathcal{P}_r. \] (23)
Moreover,
\[ \forall Q \in \mathcal{P}_r : \mathcal{H}(Q, R) \sqsubseteq \mathcal{H}(Q, S) \implies R \sqsubseteq S, \] (24)
and thus
\[ \forall Q \in \mathcal{P}_r : \mathcal{H}(Q, R) \sqsubseteq \mathcal{H}(Q, S) \implies \forall Q \in \mathcal{P}_r : \mathcal{H}(Q, R) \sqsubseteq_G \mathcal{H}(Q, S). \] (25)
Finally,
\[ \forall Q \in \mathcal{P}_r : \mathcal{H}(Q, R) \sqsubseteq_I \mathcal{H}(Q, S) \implies R \sqsubseteq I. \] (26)

Proof. (23): Let $\rho$ be a strong Hom-scheme from $R$ to $S$, and let $Q \in \mathcal{P}_r$ be fixed. We define for every $P \in \mathcal{P}_r$ and every $\xi \in \mathcal{H}(P, \mathcal{H}(Q, R))$
\[ \tau_P(\xi) \equiv \rho_Q \circ \xi. \]
Then $\tau_P : \mathcal{H}(P, \mathcal{H}(Q, R)) \rightarrow \mathcal{H}(P, \mathcal{H}(Q, S))$ is a well-defined mapping and because $\rho_Q : \mathcal{H}(Q, R) \rightarrow \mathcal{H}(Q, S)$ is one-to-one, we have $G_{\tau_P(\xi)}(x) = G_\xi(x)$ for all $x \in P$ according to Corollary 2. $\tau$ is thus a G-scheme from $\mathcal{H}(Q, R)$ to $\mathcal{H}(Q, S)$. Let $\xi, \zeta \in \mathcal{H}(P, \mathcal{H}(Q, R))$ with $\tau_P(\xi) = \tau_P(\zeta)$. Then $\rho_Q(\xi(x)) = \tau_P(\xi(x)) = \tau_P(\zeta(x))$ for all $x \in P$. Because $\rho_Q$ is one-to-one, we get $\xi(x) = \zeta(x)$ for all $x \in P$, hence $\xi = \zeta$, and $\tau$ is strong.

(24): In the case of $\mathcal{H}(Q, R) \sqsubseteq \mathcal{H}(Q, S)$ for all $Q \in \mathcal{P}_r$, we have for all $P \in \mathcal{P}_r$:
\[ \#H(P, \mathcal{H}(Q, R)) = \#H(A_1, \mathcal{H}(P, R)) \leq \#H(A_1, \mathcal{H}(P, S)) = \#H(P, S), \]
thus $R \sqsubseteq S$.

(25): $\mathcal{H}(Q, R) \sqsubseteq \mathcal{H}(Q, S)$ for all $Q \in \mathcal{P}_r$ yields $R \sqsubseteq S$ according to (24), and
\[ \rho \circ \mathcal{H}(Q, R) \sqsubseteq \mathcal{H}(Q, S) \text{ for all } Q \in \mathcal{P}_r. \]
The direction “$\Leftarrow$” is trivial.

(26): $R \simeq \mathcal{H}(A_1, R) \sqsubseteq I \mathcal{H}(A_1, S) \simeq S$.

\[ \square \]

5.2 Necessary conditions for $R \sqsubseteq S$ and $R \sqsubseteq G S$

Given two finite posets $R$ and $S$, it is generally hard to show $R \sqsubseteq S$. But it may be as hard to prove $R \not\sqsubseteq S$. In this section, we collect criteria and formulas which are necessary conditions for $R \sqsubseteq S$ or $R \sqsubseteq_G S$ and which therefore are useful in demonstrating $R \not\sqsubseteq S$ or $R \not\sqsubseteq_G S$, respectively. Examples for their application are found in the Appendix.

Let $\rho$ be a strong Hom-scheme from $R$ to $S$. Because $\rho_{A_1} : \mathcal{H}(A_1, R) \rightarrow \mathcal{H}(A_1, S)$ and $\rho_{C_2} : \mathcal{H}(C_2, R) \rightarrow \mathcal{H}(C_2, S)$ are both one-to-one, we have $\#R \leq \#S$ and $(\# \leq_R) \leq (\# \leq_S)$. If $\rho$ is a strong G-scheme, we have $\#S(P, R) \leq \#S(P, S)$ for all $P \in \mathcal{P}_r$, because a G-scheme preserves strictness. In particular, in the case $R \sqsubseteq_G S$, the number of $k$-chains in $R$ is less than or equal to the number of $k$-chains in $S$, and the height of $R$ is less than or equal to the height of $S$. Moreover:

Lemma 5. Let $R, S \in \mathcal{P}$ with $R \sqsubseteq S$. Then for every $k, \ell \in \mathbb{N}_0$
\[ \sum_{v \in R} (\# \downarrow v)^k \cdot (\# \uparrow v)^\ell \leq \sum_{w \in S} (\# \downarrow w)^k \cdot (\# \uparrow w)^\ell, \] (27)
\[ \sum_{v \in R} \sum_{d \in \downarrow v} (\# [d, v])^k \leq \sum_{w \in S} \sum_{d \in \downarrow w} (\# [d, w])^k. \] (28)
Additionally, in the case $R \sqsubseteq G S$, we have for every $k, \ell \in \mathbb{N}_0$

$$\sum_{v \in R} (\#\downarrow v - 1)^k \cdot (\#\uparrow v - 1)^\ell \leq \sum_{w \in S} (\#\downarrow w - 1)^k \cdot (\#\uparrow w - 1)^\ell,$$  

(29)

$$\sum_{v \in R} \sum_{d \in \downarrow v} (\#[d, v] - 2)^k \leq \sum_{w \in S} \sum_{d \in \downarrow w} (\#[d, w] - 2)^k.$$  

(30)

Proof. In the first two inequalities, the sums on the left and on the right are $\#H(P, R)$ and $\#H(P, S)$, respectively, for $P \equiv A_k \oplus A_1 \oplus A_\ell$ and $P \equiv A_1 \oplus A_k \oplus A_1$, respectively. In the third and fourth inequality, the sums are $\#S(P, R)$ and $\#S(P, S)$ for these posets $P$.

From this lemma we get as immediate conclusions the two necessary conditions for $R \sqsubseteq S$ contained in the following corollary. The first one may look complicated at first glance. However, in the practical work it is easy to apply and powerful in showing $R \not\sqsubseteq S$ (which implies $R \not\sqsubseteq G S$):

Corollary 5. Let $R, S \in \mathfrak{P}$ with $R \sqsubseteq S$.

(a) We define for every poset $P \in \mathfrak{P}$ the sequence $d(P) = (d(P)_i)$ by setting for every $i \in \mathbb{N}$

$$d(P)_i \equiv \# \{ x \in P \mid \#\downarrow x = i \}.$$  

If $d(R) \neq d(S)$, then $d(R)_j < d(S)_j$, where $j \equiv \max \{ i \in \mathbb{N} \mid d(R)_i \neq d(S)_i \}$.

The same holds also for the sequences of the numbers

$$u(P)_i \equiv \# \{ x \in P \mid \#\uparrow x = i \},$$  

and

$$j(P)_i \equiv \# \{ x, y \in P \mid \#[x, y] = i \}.$$  

(b) In the case $\#R = \#S$, $S$ has a maximum/minimum/both, if $R$ has a maximum/minimum/both.

Proof. (a) Using inequality (27) with $\ell = 0$, yields for every $k \in \mathbb{N}_0$

$$\sum_{i=1}^{+\infty} d(R)_i \cdot i^k = \sum_{v \in R} (\#\downarrow v)^k \leq \sum_{w \in S} (\#\downarrow w)^k = \sum_{i=1}^{+\infty} d(S)_i \cdot i^k.$$  

Because $d(R)$ and $d(S)$ are finite sequences, the sums on the left and on the right side are exponential sums. Letting $k$ grow delivers the first proposition. The second one follows in the same way with $k = 0$ in (27) and the third one with (28).

(b) If $R$ has a maximum, $\sum_{i=1}^{+\infty} d(R)_i \cdot i^k$ contains the term $(\#R)^k$, and due to $\#R = \#S$, we have $i \leq \#R$ for all non-zero terms of $\sum_{i=1}^{+\infty} d(S)_i \cdot i^k$. Now apply (a). The results for the minimum and both extrema follow in the same way by using the two additional sequences mentioned in (a).

□
5.3 A formula for \( \alpha^{-1}_\xi \)

In Proposition 3, we have dealt with the pre-image of \( \alpha(x) \). The following formula has already been used in the discussion of the posets in Figure 4; it is a useful link between the abstract structure theory and the practical construction of strong G-schemes and strong I-schemes:

**Lemma 6.** Let \( P \in \mathcal{P}, \xi \in \mathcal{H}(P, R) \). Then for every \( a = (a, D, U) \in \mathcal{E}(R) \)

\[
\alpha^{-1}_\xi(a) = (\xi^{-1}(a) \cap B_\xi(a)) \setminus C_\xi(a)
\]

(31)

where

\[
B_\xi(a) \equiv \left( \bigcap_{d \in D} \gamma^{-1}_\xi(a) \left( \xi^{-1}(a) \cap \uparrow \xi^{-1}(d) \right) \right) \cap \left( \bigcap_{u \in U} \gamma^{-1}_\xi(a) \left( \xi^{-1}(a) \cap \downarrow \xi^{-1}(u) \right) \right),
\]

\[
C_\xi(a) \equiv \left( \bigcup_{d \in (\xi a) \setminus D} \gamma^{-1}_\xi(a) \left( \xi^{-1}(a) \cap \uparrow \xi^{-1}(d) \right) \right) \cup \left( \bigcup_{u \in (\xi a) \setminus U} \gamma^{-1}_\xi(a) \left( \xi^{-1}(a) \cap \downarrow \xi^{-1}(u) \right) \right).
\]

**Proof.** Let \( a = (a, D, U) \in \mathcal{E}(R) \) and \( d \in D \). Due to \( \xi^{-1}(a) = \bigcup_{x \in \xi^{-1}(a)} G_\xi(x) \), we have according to [1] for every \( d \in D \)

\[
\gamma^{-1}_\xi(a) \left( \xi^{-1}(a) \cap \uparrow \xi^{-1}(d) \right) = \bigcup_{x \in \xi^{-1}(a)} \gamma^{-1}_\xi(a) \left( G_\xi(x) \cap \uparrow \xi^{-1}(d) \right).
\]

For every \( x \in \xi^{-1}(a) \) and every \( d \in D \) we have

\[
\gamma^{-1}_\xi(a) \left( G_\xi(x) \cap \uparrow \xi^{-1}(d) \right) = \begin{cases} G_\xi(x), & \text{if } G_\xi(x) \cap \uparrow \xi^{-1}(d) \neq \emptyset; \\ \emptyset, & \text{if } G_\xi(x) \cap \uparrow \xi^{-1}(d) = \emptyset. \end{cases}
\]

and similarly for all \( u \in U \). Together with \( \cap_{d \in \emptyset} N_d = P \) for any system \( N_d \) of subsets of \( P \), we conclude that the set \( \xi^{-1}(a) \cap B_\xi(a) \) is the set of those \( x \in \xi^{-1}(a) \), for which \( G_\xi(x) \cap \uparrow \xi^{-1}(d) \neq \emptyset \) for all \( d \in D \) and \( G_\xi(x) \cap \downarrow \xi^{-1}(u) \neq \emptyset \) for all \( u \in U \). Due to \( D \subseteq \downarrow a \subseteq \uparrow \xi(x) \), we have \( G_\xi(x) \cap \uparrow \xi^{-1}(d) \neq \emptyset \Leftrightarrow \downarrow G_\xi(x) \cap \xi^{-1}(d) \neq \emptyset \) for all \( d \in D, x \in \xi^{-1}(a) \), and similarly for \( a \in U \). Therefore, \( \xi^{-1}(a) \cap B_\xi(a) \) is the set of those \( x \in P \) for which \( a_1 = a = \xi(x) = \alpha_\xi(x)_1 \), \( a_2 = D \subseteq \xi(\downarrow G_\xi(x)) = \alpha_\xi(x)_2 \), and \( a_3 = U \subseteq \xi(\uparrow G_\xi(x)) = \alpha_\xi(x)_3 \).

In order to get \( \xi^{-1}(a) \), we thus have to take away from \( \xi^{-1}(a) \cap B_\xi(a) \) exactly those \( x \in \xi^{-1}(a) \), for which \( D \subseteq \xi(\downarrow G_\xi(x)) \) or \( U \subseteq \xi(\uparrow G_\xi(x)) \). That is done by subtracting \( C_\xi(a) \) which can be seen by applying the same chain of transformations to \( C_\xi(a) \), which we have just applied to \( B_\xi(a) \) (use \( \cup_{d \in \emptyset} N_d = \emptyset \) for any system \( N_d \) of subsets of \( P \)).

\[\blacksquare\]
6 Appendix

The following Figures 6–8 and 9 show the diagrams of three posets ordered by $\sqsubseteq_J$ and $\sqsubseteq_G$, respectively. In Figure 6 it is the poset of the non-isomorphic posets with four points, in Figure 7 it is the poset of the non-isomorphic, flat posets with five points, and in Figure 8 it is the poset of the non-isomorphic posets with five points which are not isomorphic to $A_1 + Q$, $A_1 \oplus Q$, or $Q \oplus A_1$ for a poset $Q$ with four points. (The restriction is justified by the Propositions 6 and 7.)

In the figures, solid lines indicate (trivial) strong I-schemes induced by one-to-one homomorphisms, dashed lines indicate non-trivial strong I-schemes, and dotted lines indicate strong G-schemes between posets, for which no strong I-scheme exists. In Figure 6 all relations are induced by strong I-schemes. The construction of the strong G-schemes and strong I-schemes is explained in Figure 9.

It may be helpful to provide examples for how incomparableness with respect to $\sqsubseteq_G$ can be shown. I do so for the posets contained in Figure 6. Due to Proposition 5 we have $P \not\sqsubseteq_G P^d$ if $P$ is not self-dual. In the following table, the criteria established in Corollary 5 are applied. The implication $R \not\sqsubseteq_S S \Rightarrow R \not\sqsubseteq_G S$ is tacitly used. For $k \in \mathbb{N}$, we define the symbols $A_k = A_{k-1} \oplus A_1$ (the bug with $k-1$ legs), and $V_k = A_k^2$; $N$ is the poset with N-shaped diagram, and $N(2) = A_2 \oplus A_2$ is the double-N.

| $R$          | $S$          | $R \not\sqsubseteq_G S$ | $S \not\sqsubseteq_G R$ |
|--------------|--------------|--------------------------|--------------------------|
| $A_1 + V_3$  | $A_4$        | $u(R)_4 = 0 = u(S)_4$, $u(R)_3 = 1 > 0 = u(S)_3$ | $R$ has no maximum, but $S$ has |
| $N$          | $A_4$        | $u(R)_4 = 0 = u(S)_4$, $u(R)_3 = 1 > 0 = u(S)_3$ | $R$ has no maximum, but $S$ has |
| $A_1 + C_3$  | $N(2)$       | $\text{height } R = 3$ $\text{height } S = 2$ | $u(S)_4 = 0 = u(R)_4$, $u(S)_3 = 2 > 1 = u(R)_3$ |
| $A_4$        | $A_1 \oplus (A_1 + C_2)$ (the skew V) | $S$ has no maximum, but $R$ has | $R$ has no minimum, but $S$ has |
| $A_2 \oplus C_2$ (the Y turned upside-down) | $A_1 \oplus (A_1 + C_2)$ | $S$ has no maximum, but $R$ has | $R$ has no minimum, but $S$ has |

There are three pairs of posets in Figure 6 for which Proposition 5 and Corollary 5 are not sufficient to show incomparableness:

- $R = C_2 \times C_2$ (the diamond), $S = A_2 \oplus C_2$: $R \not\sqsubseteq_G S$ holds because $R$ has a minimum and $S$ not. $S \not\sqsubseteq_G R$ we see by 29 with $k = 2$ and $\ell = 0$:

$$\#S(A_3, S) = 13 > 11 = \#S(A_3, R).$$

- $R = C_2 \times C_2$, $S = C_2 \oplus A_2$ (the Y): dual to the previous pair.

- $R = C_2 \times C_2$, $S = N(2)$: $R \not\sqsubseteq_G S$ holds because $R$ has extrema and $S$ not, but for $S \not\sqsubseteq_G R$ we need

$$\#S(A_k \oplus A_k, S) = 4^k, \ #S(A_k \oplus A_k, R) = 2 \cdot 3^k - 1,$$

thus $\#S(A_k \oplus A_k, S) > \#S(A_k \oplus A_k, R)$ for $k \geq 3.
Figure 6: The non-isomorphic posets with four points ordered by $\sqsubseteq_I$. Solid lines indicate strong $I$-schemes induced by one-to-one homomorphisms, dashed lines indicate non-trivial strong $I$-schemes. For the construction of the non-trivial strong $I$-schemes, see Figure 9.
Figure 7: The non-isomorphic flat posets with five points ordered by $\sqsubseteq_G$. Solid lines indicate strong I-schemes induced by one-to-one homomorphisms, dashed lines indicate non-trivial strong I-schemes, and dotted lines indicate strong G-schemes between posets, for which no strong I-scheme exists. For the construction of the non-trivial strong G-schemes and I-schemes, see Figure 9.
Figure 8: The non-isomorphic posets with five points which are not isomorphic to $A_1 + Q$, $A_1 \oplus Q$, or $Q \oplus A_1$ for a poset $Q$ with four points. The posets are ordered by $\subseteq_G$. Solid lines indicate strong I-schemes induced by one-to-one homomorphisms, dashed lines indicate non-trivial strong I-schemes, and dotted lines indicate strong G-schemes between posets, for which no strong I-scheme exists. For the construction of the non-trivial strong G-schemes and I-schemes, see Figure 9.
In Figure 9, non-trivial strong G-schemes and I-schemes $\rho$ from $R$ to $S$ are shown for the posets in the Figures 6, 7, and 8 (dashed and dotted lines). Dual pairs of posets are omitted. For each construction, $R$ is on the left and $S$ is on the right. As in Figure 2, a label at a point of $R$ stands for the pre-image of the respective point under a homomorphism $\xi \in \mathcal{H}(P, R)$, and the label at a point of $S$ indicates the resulting pre-image of $\rho(\xi)$. Model proofs for showing $R \sqsubseteq S$, $R \sqsubseteq_G S$, and $R \sqsubseteq_I S$ are found in the sections 3.1, 4.1, and 4.3. No strong I-scheme exists for the Constructions (C7), (C9), (C11), and (C12); for a model proof, see Section 4.3.

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