ON THE GROWTH OF \(\mu\)-INARIANT IN IWASAWA THEORY OF SUPERSINGULAR ELLIPTIC CURVES

JISHNU RAY

Abstract. In this article, we provide a relation between the \(\mu\)-invariant of the dual signed Selmer groups for supersingular elliptic curves when we ascend from the cyclotomic \(\mathbb{Z}_p\)-extension to a \(\mathbb{Z}_p^2\)-extension of a number field. Further, we draw a remarkable analogy between our result and a similar result of Coates, Schneider and Sujatha for ordinary elliptic curves; the Iwasawa theoretic tower they consider is from the cyclotomic \(\mathbb{Z}_p\)-extension to a non-abelian \(GL_2(\mathbb{Z}_p)\)-extension.

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1. Introduction

This article is all about a striking similarity between a relation on \(\mu\)-invariants of the dual Selmer groups over two completely different Iwasawa theoretic towers when considered for two disjoint categories of elliptic curves. Let us first recall the setting of Coates-Schenider-Sujatha [CSS03a] which will help us to draw the analogy with the main result of this paper.

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1.1. Ordinary Elliptic curves and Coates-Schneider-Sujatha. Let $E$ be an elliptic curve over a number field $K$ without complex multiplication and $p$ be an odd prime. Let us suppose that $E$ has good ordinary reduction at all primes over $p$. For all $n \geq 0$, we define

$$E_{p^{n+1}} = \ker(E(\overline{\mathbb{Q}}) \xrightarrow{p^{n+1}} E(\overline{\mathbb{Q}})),$$

$$E_p = \bigcup_{n \geq 0} E_{p^{n+1}}.$$

We build up the $GL(2)$ Galois extensions of $K$ by defining $K_n = K(E_{p^{n+1}}), K_\infty = K(E_p)$. We write $G_n = \text{Gal}(K_\infty/K_n), G = \text{Gal}(K_\infty/K)$. By Serre [Ser72], $G$ is open in $GL_2(\mathbb{Z}_p)$ for all primes and $G = GL_2(\mathbb{Z}_p)$ for all but a finite number of primes. When $n$ is large, $G_n$ becomes the $n$-the congruence subgroup of $G$. We assume $G$ to be a pro-$p$ group and let $p \geq 5$ so that $G$ is torsion free. Note that $K_\infty$ contains $K_{\text{cyc}}$, the cyclotomic $\mathbb{Z}_p$-extension over $K$ with Galois group $\Gamma$, a $p$-adic Lie group of dimension 1. Let $\mathcal{H} = \text{Gal}(K_\infty/K_{\text{cyc}})$ which is a $p$-adic Lie group of dimension 3. The structure Theorem [Bou98, Chap. VII, Sec. 4] studies the structure of the Pontryagin dual of the Selmer group $\text{Sel}_p(E/K_{\text{cyc}})$ of $E$ over $K_{\text{cyc}}$ as a $\Lambda(\Gamma)$-module, where $\Lambda(\Gamma)$ is the Iwasawa algebra of $\Gamma = \text{Gal}(K_{\text{cyc}}/K)$. By this structure Theorem, we can attach Iwasawa $\mu$-invariant to the Pontryagin dual of the Selmer group $\text{Sel}_p(E/K_{\text{cyc}})$. Similarly, there is a structure theorem for the non-commutative extension $K_\infty$ [CSS03b] and we can attach $\mu$-invariant to the dual of $\text{Sel}_p(E/K_\infty)$. In [Maz72], Mazur conjectured that

**Conjecture 1.1 (Mazur).** $\text{Sel}_p(E/K_{\text{cyc}})$ is a finitely generated cotorsion $\Lambda(\Gamma)$-module.

(Cotorsion means that its Pontryagin dual is torsion). This conjecture is solved for elliptic curves over $\mathbb{Q}$ [Kat04].

In the non-commutative case, Coates, Schneider and Sujatha [CSS03a] developed Iwasawa theory over the $GL(2)$ extension $K_\infty$, generalizing results from $K_{\text{cyc}}$, thus going up the Iwasawa theoretic tower. Let $X_\infty$ and $X_{\text{cyc}}$ be the Pontryagin dual of $\text{Sel}_p(E/K_\infty)$ and $\text{Sel}_p(E/K_{\text{cyc}})$ respectively. Let $X_\infty(p)$ be the submodule of all the elements of $X_\infty$ which are annihilated by some power of $p$ and $X_{\text{cyc}}(p) = X_\infty/X_\infty(p)$.

Under this setting, Coates, Schneider and Sujatha proved the following result relating their $\mu$-invariants.

**Proposition 1.2.** [CSS03a, Prop. 2.12] Suppose $X_{\text{cyc}}$ is $\Lambda(\Gamma)$-torsion. Then we have

$$\mu_G(X_\infty) = \mu_\Gamma(X_{\text{cyc}}) + \delta + \epsilon,$$

where

$$\delta = \sum_{i=0}^{1} (-1)^{i+1} \mu_\Gamma(H_i(\mathcal{H}, X_\infty(p)))$$

and

$$\epsilon = \sum_{i=1}^{3} (-1)^{i} \mu_\Gamma(H_i(\mathcal{H}, X_\infty(p))).$$
Furthermore, if $X_{\infty,f}$ is finitely generated over $\Lambda(\mathcal{H})$ (this is commonly known as the Iwasawa conjecture in Iwasawa theory [Suj, Conjecture 5.2]), then $\mu_\mathcal{G}(X_\infty) = \mu_\Gamma(X_{\cyc})$.

1.2. **Supersingular Elliptic curves and main result.** Let $F$ be a number field. Our main theorem of this paper shows that a similar relation between $\mu$-invariant holds for elliptic curves with supersingular reduction at all primes above $p$ but when we ascend through a different Iwasawa theoretic tower; from $F_{\cyc}$ to a $\mathbb{Z}_p^2$-extension over $F$. Note that when $E$ has supersingular reduction at $p$ and is defined over $\mathbb{Q}$, the dual of the Selmer group $\text{Sel}_p(E/\mathbb{Q}_{\cyc})$ is not $\Lambda(\Gamma)$-torsion and Kobayashi defined the so-called plus and minus signed Selmer groups $\text{Sel}_p^\pm(E/\mathbb{Q}_{\cyc})$ which are cotorsion [Kob03, Thm. 2.2]. When $F$ is an imaginary quadratic field and $p$ splits completely in $F$, Kim generalized Kobayashi’s plus and minus Selmer groups over a $\mathbb{Z}_p^2$-extension of $F$ which contains $F_{\cyc}$ and proved similar cotorsion results [Kim14]. We prove a relation, very similar to that of Coates-Schneider-Sujatha, on the relation between $\mu$-invariants along this dimension two Iwasawa theoretic tower for plus and minus Selmer groups. This is the main theme of the paper. In the rest of the introduction, we provide the basic definitions of plus and minus Selmer groups which are useful for the reader to understand the main result of this paper. In the following sections we follow the exposition as in [LS19].

1.2.1. **Signed Selmer groups over cyclotomic extension.** Let $F'$ be a subfield of $F$, $E/F'$ be an elliptic curve with good supersingular reduction at all primes above $p$. Let $S$ be the set of primes of $F'$ above $p$ and the primes where $E$ has bad reduction. We write $S_p^{ss}$ to be the set of primes of $F'$ lying above $p$ where $E$ has supersingular reduction. Then, $S = S_p^{ss} \cup \{\text{bad primes}\}$. Let $F'_v$ be the completion of $F'$ at a prime $v \in S_p^{ss}$ with residue field $k'_v$. Let $\hat{E}(k'_v)$ be the $k'_v$-points of the reduction of $E$ at place $v$.

Assume the following.

(i) $S^{ss} \neq 0$,
(ii) For all $v \in S^{ss}$, the completion of $F'_v$ at $v$, denoted $F'_v$, is $\mathbb{Q}_p$,
(iii) $a_v = 1 + p - \# \hat{E}(k'_v) = 0$,
(iv) $v$ is unramified in $F$.

Recall that $F_{\cyc}$ is the cyclotomic $\mathbb{Z}_p$-extension of $F$. For each integer $n \geq 0$, let $F_n$ be the sub-extension of $F_{\cyc}$ such that $F_n$ is a cyclic extension of degree $p^n$ over $F$. Let $S_{p,F}^{ss}$ denote the set of primes of $F$ above $S^{ss}$. By abuse of notation, let $F_{n,v}$ be the completion of $F_n$ at the unique prime over $v \in S_{p,F}^{ss}$. For every $v \in S_{p,F}^{ss}$, following Kobayashi [Kob03], we define

$$E^+(F_{n,v}) = \{ P \in \hat{E}(F_{n,v}) \mid \text{Trace}_{n/m+1}P \in \hat{E}(F_{m,v}) \text{ for all even } m, 0 \leq m \leq n - 1 \},$$

and

$$E^-(F_{n,v}) = \{ P \in \hat{E}(F_{n,v}) \mid \text{Trace}_{n/m+1}P \in \hat{E}(F_{m,v}) \text{ for all odd } m, 0 \leq m \leq n - 1 \},$$
Here Trace_{n/m+1} is the trace map from \( \hat{E}(F_{n,v}) \) to \( \hat{E}(F_{m+1,v}) \). We define the \((p\text{-adic})\) Selmer group over \( F_n \) by the following sequence.

\[
0 \to \text{Sel}_p(E/F_n) \to H^1(F_S/F_n, E_{p^\infty}) \to \oplus_{w|S} H^1(F_{n,w}, E)(p).
\]

Here \( F_S \) is the maximal extension of \( F \) unramified outside primes of \( F \) over \( S \). The plus and minus Selmer group over \( F_n \) are defined by

\[
\text{Sel}_p^\pm(E/F_n) = \ker \left( \text{Sel}_p(E/F_n) \to \oplus_{v \in \mathbb{S}_{p,F}^n} \frac{H^1(F_{n,v}, E_{p^\infty})}{E^\pm(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right).
\]

We regard \( E^\pm(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \) as a subgroup of \( H^1(F_{n,v}, E_{p^\infty}) \) via the Kummer map. The plus and minus Selmer groups over \( F_{\text{cyc}} \) are defined by

\[
\text{Sel}_p^\pm(E/F_{\text{cyc}}) = \lim_{\rightarrow n} \text{Sel}_p^\pm(E/F_n).
\]

1.2.2. Signed Selmer groups over a \( \mathbb{Z}_p^2 \)-extension. Suppose \( F \) is an imaginary quadratic field where \( p \) splits completely. Let \( F_\infty \) denote the compositum of all \( \mathbb{Z}_p \)-extensions of \( F \).

(Here \( F_\infty \) is \textit{not} \( F(E_{p^\infty}) \) although we have used the same subscript \( \infty \) as in Section 1.1).

By Leopoldt’s conjecture we know that \( G = \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p^2 \), which implies that \( F_\infty \) over \( F_{\text{cyc}} \) is a \( \mathbb{Z}_p \)-extension. Let \( v \) be a place of \( F \) and \( w \) be a place of \( F_\infty \) above \( v \). If \( v \mid p \), then \( F_v \cong \mathbb{Q}_p \) and \( F_{\infty,w} \) is an abelian pro-\( p \) extension over \( F_v \). By local class field theory, \( \text{Gal}(F_{\infty,w}/F_v) \cong \mathbb{Z}_p^2 \). Under this setting, it is possible to define the plus and minus norm groups \( E^\pm(F_{\infty,w}) \subset \hat{E}(F_{\infty,w}) \) via Trace maps as in (1.1) and (1.2) (cf. see Section 5.2 of [LS19] which is a generalization of a construction by Kim [Kim14]). Identifying \( E^\pm(F_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \) with a subgroup of \( H^1(F_{\infty,w}, E_{p^\infty}) \) via the Kummer map, we may define the local terms

\[
J^\pm_v(E/F_\infty) = \oplus_{w|p} \frac{H^1(F_{\infty,w}, E_{p^\infty})}{E^\pm(F_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}.
\]

Then, the plus and minus Selmer groups over \( \mathbb{Z}_p^2 \)-extension \( F_\infty \) are defined by

\[
\text{Sel}_p^\pm(E/F_\infty) = \ker \left( \text{Sel}_p(E/F_\infty) \to \oplus_{v \in \mathbb{S}_{p,F}^\infty} J^\pm_v(E/F_\infty) \right).
\]

Here the (classical) Selmer group \( \text{Sel}_p(E/F_\infty) \) is defined by taking inductive limit of Selmer groups over all finite extensions contained in the \( p \)-adic Lie extension \( F_\infty \).

1.2.3. Main result. Let \( X_\infty^\pm \) denote the Pontryagin dual of \( \text{Sel}_p^\pm(E/F_\infty) \) and \( X_{\text{cyc}}^\pm \) denote the Pontryagin dual of \( \text{Sel}_p^\pm(E/F_{\text{cyc}}) \). Let \( G = \text{Gal}(F_\infty/F), H = \text{Gal}(F_\infty/F_{\text{cyc}}) \) and \( \Gamma = \text{Gal}(F_{\text{cyc}}/F) \). Then we show that
**Theorem 1.3** (See Theorem 3.7). Assume that $X_{\text{cyc}}^\pm$ is $\Lambda(\Gamma)$-torsion and
\[ H^2(F_S/F, E_{\nu^\infty}) = 0, \]
(the vanishing of $H^2$ is called weak Leopoldt’s conjecture). Then,
\[ \mu_G(X_{\infty}^\pm) = \mu_\Gamma(X_{\text{cyc}}^\pm) + \delta, \]
where $\delta = \mu_\Gamma(H_1(H, X_{\infty,f}^\pm)) - \mu_\Gamma(H_0(H, X_{\infty,f}^\pm)).$

Furthermore, if $X_{\infty,f}^\pm$ is finitely generated as a $\Lambda(H)$-module (equivalent version of $\mathfrak{M}_H(G)$ conjecture) then $\mu_G(X_{\infty}^\pm) = \mu_\Gamma(X_{\text{cyc}}^\pm)$.

(The $\epsilon$ part here is zero. In the ordinary case of Coates-Schneider-Sujatha, we know that weak Leopoldt’s conjecture is true [Coa99, Theorem 2.10]).

In Section 2, we recall basic facts on Hochschild-Serre spectral sequence which we will use throughout the article. The main result is included in Section 3.

**2. Recall on Hochschild-Serre spectral sequence**

Let $G$ be any profinite group. Let $M$ be an abelian group with discrete topology and a continuous action of $G$, $H$ be a closed normal subgroup of $G$, then there exists the following Hochschild-Serre spectral sequence.
\[ H^r(G/H, H^*(H, M)) \implies H^{r+s}(G, M). \]

This gives rise to the following inflation-restriction exact sequence
\[ 0 \to H^1(G/H, M^H) \to H^1(G, M) \to H^1(H, M)^{G/H} \to H^2(G/H, M^H) \to H^2(G, M). \]

Furthermore, if $H^i(H, M) = 0$ for $2 \leq i \leq m$, then we get the following exact sequence for higher cohomology groups.
\[ H^m(G/H, M^H) \to H^m(G, M) \to H^{m-1}(G/H, H^1(H, M)) \to H^{m+1}(G/H, M^H) \to H^{m+1}(G, M). \]

We are going to use the above two exact sequences throughout this article.

**3. Relation of $\mu$-invariants between cyclotomic and $\mathbb{Z}_p^2$-extension**

Recall that $H = \text{Gal}(F_\infty/F_{\text{cyc}})$, a $p$-adic Lie group of dimension 1, $\Gamma = \text{Gal}(F_{\text{cyc}}/F)$ also of dimension 1 and $\Lambda(\Gamma)$ is the Iwasawa algebra of $\Gamma$. This is a completed group algebra defined by
\[ \Lambda(\Gamma) = \varprojlim W \mathbb{Z}_p[\Gamma/W], \]
where $W$ runs over all open normal subgroups of $\Gamma$. The Iwasawa algebra $\Lambda(\Gamma)$ is a local Noetherian integral domain. The completed group algebra $\Lambda(\Gamma)$ can also be thought of as a distribution algebra of $\mathbb{Z}_p$-valued measures on the $p$-adic Lie group.
Γ. Let $G_S(F_\infty) = \text{Gal}(F_S/F_\infty)$ and $G_S(F_{\text{cyc}}) = \text{Gal}(F_S/F_{\text{cyc}})$. Consider the following fundamental diagram.

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Sel}_p^\pm(E/F_\infty)^H & \rightarrow & H^1(G_S(F_\infty), E_{p^\infty})^H \\
& & \alpha_\infty & \rightarrow & H^1(G_S(F_{\text{cyc}}), E_{p^\infty})^H \\
0 & \rightarrow & \text{Sel}_p^\pm(E/F_{\text{cyc}}) & \rightarrow & H^1(G_S(F_{\text{cyc}}), E_{p^\infty})^H \\
& & \beta_\infty & \rightarrow & \gamma_\infty^{\pm} = \oplus_{v \in S} J_v^{\pm}(E/F_\infty)^H \\
& & & & \oplus_{v \in S} J_v^{\pm}(E/F_{\text{cyc}})
\end{array}
\]  

The following Lemma is due to Lei and Sujatha.

**Lemma 3.1.** The maps $\alpha_\infty$, $\beta_\infty$ and $\gamma_\infty^{\pm}$ are isomorphisms.

**Proof.** See Corollary 5.7, Proposition 5.8 and Lemma 5.10 of [LS19] and use snake lemma for the fundamental diagram (3.1). Note that we have assumed that $E$ has supersingular reduction for all primes above $p$ and so the case of Lemma 5.9 of [LS19] does not appear. \hfill \square

Here are some Hypothesis that we will need.

(WLC)$_{F_\infty}$: Weak Leopoldt’s conjecture for $F_\infty$: that is $H^2(F_S/F_\infty, E_{p^\infty}) = 0$.

HYP$_1$: The Pontryagin dual of $\text{Sel}_p^\pm(E/F_{\text{cyc}})$ is $\Lambda(\Gamma) – \text{torsion}$.

**Lemma 3.2.** Under HYP$_1$, we have the following short exact sequence.

\[
0 \rightarrow \text{Sel}_p^\pm(E/F_\infty)^H \rightarrow H^1(G_S(F_\infty), E_{p^\infty})^H \xrightarrow{\lambda_{H,\infty}^{\pm}} \oplus_{v \in S} J_v^{\pm}(E/F_\infty)^H \rightarrow 0.
\]

That is, the map $\lambda_{H,\infty}^{H,\pm}$ is surjective.

**Proof.** This is Proposition 5.11 of Lei and Sujatha [LS19]. \hfill \square

**Lemma 3.3.** We show that under (WLC)$_{F_\infty}$ and HYP$_1$, we have

\[
H^1(H, H^1(G_S(F_\infty), E_{p^\infty})) = 0.
\]

**Proof.** Note that, as $p$ is odd, $G_S(F)$ has $p$-cohomological dimension 2. Therefore, as $G_S(F_\infty)$ is a closed subgroup of $G_S(F)$, the $p$-cohomological dimension of $G_S(F_\infty)$ is less than or equal to 2. This implies that $H^m(G_S(F_\infty), E_{p^\infty}) = 0$ for all $m \geq 2$. By Hochschild-Serre spectral sequence we then have

\[
H^2(G_S(F_{\text{cyc}}), E_{p^\infty}) \rightarrow H^1(H, H^1(G_S(F_\infty), E_{p^\infty})) \rightarrow H^3(H, E_{p^\infty}^{G_S(F_\infty)}).
\]

Now HYP$_1$ implies that $H^2(G_S(F_{\text{cyc}}), E_{p^\infty}) = 0$ (cf. [LS19, Prop. 4.4]). Therefore, the first term of the above sequence is zero; the third term is also zero as $H$ has $p$-cohomological dimension 1. Hence, the middle term has to be zero. \hfill \square
Proposition 3.4. Under $(WLC)_{F_\infty}$ and HYP, we have the following exact sequence.

\[(3.3) \quad 0 \to \text{Sel}_p^\pm(E/F_\infty) \to H^1(G_S(F_\infty), E_{p^\infty}) \xrightarrow{\lambda_\infty^\pm} \oplus_{v \in S} J_v^\pm(E/F_\infty) \to 0.\]

That is, the map $\lambda_\infty^\pm$ is surjective.

**Proof.** Let $A_\infty^\pm$ denote the image of the map $\lambda_\infty^\pm$. Therefore we have the short exact sequence

\[0 \to \text{Sel}_p^\pm(E/F_\infty) \to H^1(G_S(F_\infty), E_{p^\infty}) \to A_\infty^\pm \to 0.\]

Taking $H$-cohomology and writing the long exact sequence corresponding to the above short exact sequence and using Lemma 3.3, we get that

\[H^1(H, A_\infty^\pm) \cong H^2(H, \text{Sel}_p^\pm(E/F_\infty)).\]

But $H \cong \mathbb{Z}_p$ and so $H^2(H, \text{Sel}_p^\pm(E/F_\infty)) = 0$ which implies that $H^1(H, A_\infty^\pm) = 0$.

Hochschild-Serre spectral sequence gives

\[0 \to H^1(\Gamma, J_v^\pm(F_\infty)^H) \to H^1(G, J_v^\pm(F_\infty)) \to H^1(H, J_v^\pm(F_\infty)^\Gamma) \to H^2(\Gamma, J_v^\pm(F_\infty)^H) = 0.\]

By the proof of Proposition 5.14 of [LS19], we know that $H^1(G, J_v^\pm(F_\infty)) = 0$ for all $v \in S$.

(Note that the proof of Proposition 5.14 of [LS19] does not need $\text{Sel}_p(E/F)$ to be finite. It just needs the fact that $H^2(F_S/F_{\text{cycl}}, E_{p^\infty}) = 0$ which is true because of HYP and [LS19, Prop. 4.4].)

This implies $H^1(H, J_v^\pm(F_\infty))^\Gamma = 0$. Now $J_v^\pm(F_\infty)$ is $p$-primary and $H^1(H, J_v^\pm(F_\infty))$ is also $p$-primary and $\Gamma$ is pro-$p$. Therefore, by Nakayama lemma, $H^1(H, J_v^\pm(F_\infty))^\Gamma = 0$ implies

\[(3.4) \quad H^1(H, J_v^\pm(F_\infty)) = 0.\]

Let

\[0 \to A_\infty^\pm \to J_v^\pm(F_\infty) \to B_\infty^\pm \to 0\]

be exact where $B_\infty^\pm = \text{Coker}(\lambda_\infty^\pm)$. Therefore, by Lemma 3.2 and equation (3.4) we get that $(B_\infty^\pm)^H = H^1(H, A_\infty^\pm) = 0$. Again since $J_v^\pm(F_\infty)$ is $p$-primary, $B_\infty^\pm$ is $p$-primary and $H$ is pro-$p$. This again implies that $B_\infty^\pm = 0$ which completes the proof. \qed

**Lemma 3.5.** Under $(WLC)_{F_\infty}$ and HYP, we have $H^1(H, \text{Sel}_p^\pm(E/F_\infty)) = 0$.

**Proof.** It follows from Lemma 3.2, that the map $\lambda_\infty^H$ is surjective. By Proposition 3.4 we know that the map

\[\lambda_\infty^\pm : H^1(G_S(F_\infty), E_{p^\infty}) \to \oplus_{v \in S} J_v^\pm(E/F_\infty)\]

is surjective. Therefore the following sequence is short exact.

\[0 \to \text{Sel}_p^\pm(E/F_\infty) \to H^1(G_S(F_\infty), E_{p^\infty}) \xrightarrow{\lambda_\infty^\pm} \oplus_{v \in S} J_v^\pm(E/F_\infty) \to 0.\]
As $\lambda^{H,\pm}$ is also surjective, taking $H$-cohomology of the above exact sequence we get $H^1(H, \text{Sel}^\pm(E/F_\infty)) = 0$ since $H^1(H, H^1(G_S(F_\infty), E^{p,\infty})) = 0$ by Lemma 3.3.

Let $W$ be a finitely generated $\Lambda(G)$-module. We write $W(p)$ for the submodule of all elements of $W$ killed by some power of $p$, $W_f = W/W(p)$. The homology groups $H_i(G, W)$ are the Pontryagin duals of the cohomology groups $H^i(G, \hat{W})$. As shown in [How02], the modules $H^i(G, W)$ are finitely generated $\mathbb{Z}_p$-modules and so $H_i(G, W(p))$ are finite groups for all $i \geq 0$. Following Coates-Schneider-Sujatha [CSS03a, Equation 15, page 196] and [How02, Corollary 1.7], we define the $\mu$-invariant of $W$ using the Euler characteristic

$$p^{\mu_G(W)} = \prod_{i \geq 0} \# H_i(G, W(p))^{(-1)^i} = \chi(G, W(p)).$$

By Lemma 3.5, $H^1(H, \text{Sel}^\pm(E/F_\infty)) = 0$ which amounts to say that $H_1(H, X^\pm_\infty) = 0$. Now let us consider the following exact sequence $0 \to X^\pm_\infty(p) \to X^\pm_\infty \to X^\pm_{\infty,f} \to 0$ and take $H$-homology. We obtain

$$(3.5) \quad 0 \to H_1(H, X^\pm_{\infty,f}) \to H_0(H, X^\pm_\infty(p)) \to H_0(H, X^\pm_\infty) \to H_0(H, X^\pm_{\infty,f}) \to 0.$$  

Recall $(X^\pm_\infty)_H \cong X^\pm_{\text{cyc}}$ (cf. Lemma 3.1) which is finitely generated and torsion $\Lambda(\Gamma)$-module (under $\text{HYP}_1$). This implies that $H_0(H, X^\pm_{\infty,f})$ is also finitely generated and torsion (cf. equation (3.5)). Also $H_0(H, X^\pm_\infty(p))$ is finitely generated and killed by some power of $p$ and hence so is $H_1(H, X^\pm_{\infty,f})$ from (3.5). This proves the following Lemma.

**Lemma 3.6.** Under (WLC)$_{F_\infty}$ and $\text{HYP}_1$, the modules $H_0(H, X^\pm_{\infty,f})$ and $H_1(H, X^\pm_{\infty,f})$ are finitely generated torsion $\Lambda(\Gamma)$-modules and $H_1(H, X^\pm_{\infty,f})$ is killed by some power of $p$.

In the following, we are going to relate the $\mu$-invariant of $X^\pm_\infty$ and $X^\pm_{\text{cyc}}$ and their Euler characteristics.

**Theorem 3.7.** Under (WLC)$_{F_\infty}$ and $\text{HYP}_1$,

$$\mu_G(X^\pm_\infty) = \mu_\Gamma(X^\pm_{\text{cyc}}) + \delta,$$

where $\delta = \mu_\Gamma(H_1(H, X^\pm_{\infty,f})) - \mu_\Gamma(H_0(H, X^\pm_{\infty,f})).$

**Proof.** By Lemma 3.6, (3.5) is an exact sequence of finitely generated $\Lambda(\Gamma)$-torsion modules. Also $(X^\pm_\infty)_H \cong X^\pm_{\text{cyc}}$ which implies

$$\mu_\Gamma(H_0(H, X^\pm_\infty)) = \mu_\Gamma(X^\pm_{\text{cyc}}).$$

By Hochschild-Serre,

$$0 \to H_0(\Gamma, H_1(H, X^\pm_\infty(p))) \to H_1(G, X^\pm_\infty(p)) \to H_1(\Gamma, H_0(H, X^\pm_\infty(p))) \to 0$$

and
is exact and \( H_2(G, X^\pm_\infty(p)) \cong H_1(\Gamma, H_1(H, X^\pm_\infty(p))) \).

As \( G \) is a \( p \)-adic Lie group of dimension 2, \( H_1(G, X^\pm_\infty(p)) = 0 \) for \( i > 2 \). Therefore we have the equality of the following Euler characteristic formula
\[
\chi(G, X^\pm_\infty(p)) = \prod_{i=0}^{1} \chi(\Gamma, H_1(H, X^\pm_\infty(p))) (-1)^i,
\]
and this implies
\[
\mu(G, X^\pm_\infty) = \prod_{i=0}^{1} (-1)^i \mu(H_1(H, X^\pm_\infty(p))).
\]

But by (3.5), as \( \mu \) is additive along finitely generated torsion \( \Lambda(\Gamma) \)-modules,
\[
\mu(\Gamma(H_1(H, X^\pm_\infty,f))) + \mu(X^\pm_{\text{cyc}}) = \mu(\Gamma(H_0(H, X^\pm_\infty(p)))) + \mu(\Gamma(H_0(H, X^\pm_\infty,f))).
\]

Therefore,
\[
\mu(G, X^\pm_\infty) = \mu(\Gamma(X^\pm_{\text{cyc}}) + \delta + \epsilon,
\]
where \( \delta = \mu(\Gamma(H_1(H, X^\pm_\infty,f))) - \mu(\Gamma(H_0(H, X^\pm_\infty(f))) \) and \( \epsilon = -\mu(\Gamma(H_1(H, X^\pm_\infty(p))) \).

In the following, we are going to show that \( \epsilon = 0 \). By Lemma 3.5, \( H_1(H, X^\pm_\infty) = 0 \) for all \( i \geq 1 \). Taking \( H \)-homology of the following exact sequence
\[
0 \rightarrow X^\pm_\infty(p) \rightarrow X^\pm_\infty \rightarrow X^\pm_{\infty,f} \rightarrow 0,
\]
we obtain
\[
H_2(H, X^\pm_{\infty,f}) \cong H_1(H, X^\pm_\infty(p)).
\]
But \( H_2(H, X^\pm_{\infty,f}) = 0 \) which gives \( H_1(H, X^\pm_\infty(p)) = 0 \) and hence \( \epsilon = 0 \). This shows that \( \mu(G, X^\pm_\infty) = \mu(\Gamma(X^\pm_{\text{cyc}}) + \delta \) which completes the proof of our main Theorem. \( \Box \)

In the following, we give the equivalent version of \( \mathfrak{M}_H(G) \) conjecture for the supersingular case and prove the result analogous to Coates-Schneider-Sujatha about \( \mu \)-invariant of plus and minus Selmer groups.

**Conjecture 3.8 (\( \mathfrak{M}_H(G) \)-conjecture).** Under the notations of Section 3, when \( E \) has supersingular reduction at all primes above \( p \), then \( X^\pm_{\infty,f} \) is finitely generated \( \Lambda(H) \)-module.

Under the assumptions of Theorem 3.7 and \( \mathfrak{M}_H(G) \) conjecture, as \( X^\pm_{\infty,f} \) is finitely generated \( \Lambda(H) \)-module, then \( H_i(H, X^\pm_{\infty,f}) \) are finitely generated \( \mathbb{Z}_p \)-modules and hence their \( \mu \)-invariants are zero and so \( \delta = 0 \). In that case, we have \( \mu(G, X^\pm_\infty) = \mu(\Gamma(X^\pm_{\text{cyc}}) \).

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Department of Mathematics, The University of British Columbia

Current address: Room 121, 1984 Mathematics Road, Vancouver, BC, Canada V6T 1Z2

E-mail address: jishnuray1992@gmail.com