Vortex interactions in a thin platelet superconductor

Cedric Yen-Yu Lin and Ian Affleck
Department of Physics and Astronomy, University of British Columbia,
Vancouver, BC, V6T 1Z1, Canada
E-mail: cedricl@interchange.ubc.ca and iaffleck@phas.ubc.ca

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Abstract. The thermal fluctuations of vortices in a superconductor can be usefully mapped onto the quantum fluctuations of a collection of bosons at $T = 0$ moving in two dimensions. When the superconductor is a thin platelet with the magnetic field parallel to its surface, the interacting quantum bosons are effectively moving in one dimension, allowing for powerful Tomonaga–Luttinger liquid methods to be applied. Here we consider how this one-dimensional limit is approached, studying the interaction of vortices with the platelet surfaces and each other. Using realistic parameters and vortex interactions for an underdoped YBCO (yttrium–barium–copper oxide) platelet we determine the scattering length, $a$, characterizing the low energy interaction of a vortex pair as a function of the platelet thickness. $a$ determines the Luttinger parameter, $g$, for the quantum system at low densities, $n_0$: $g \rightarrow 1 - 2an_0$.

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1. Introduction

Thermal fluctuations of vortices, taking into account pinning by impurities and vortex–vortex interactions, is a challenging and technologically important problem in statistical physics. An elegant approach to this subject is to map each fluctuating vortex line into the world line of a quantum particle in a Feynman path integral, with the magnetic field direction becoming the imaginary time direction and the bosons moving in the other two spatial directions [1, 2]. Such a mapping is especially powerful for studying columnar defects which become static point defects in the quantum model. When the field direction is tilted relative to the (parallel) pins a novel non-Hermitian two-dimensional many-body quantum problem arises [3]. If the superconductor is a thin platelet, of thickness of order the penetration depth, with the field lying in the plane, then the quantum bosons are essentially restricted to one dimension. This allows theoretical techniques including Tomonaga–Luttinger liquid (TLL) theory and density matrix renormalization group (DMRG) ones to be brought to bear [4, 5], rendering tractable a formidable problem. It may be feasible to realize this classical analogue of a Luttinger liquid experimentally using high-$T_c$ superconductors. A promising candidate would be a very clean highly underdoped YBCO single crystal in which the penetration depth $\lambda_c$ can be as large as 50 $\mu$m. A platelet should be cleaved with thickness in the $a$ direction of order 0.1–1 mm. and then a magnetic field should be applied in the $b$ direction.

It was shown in [4, 5] that critical phenomena connected with rotating the field direction away from the pin direction are controlled by the dimensionless Luttinger parameter, $g$. In the case $g > 1$ columnar defects are irrelevant and have little effect on the long-distance properties of the vortices. On the other hand, for $g < 1$ they are relevant and an arbitrarily weak pinning potential drastically alters the system. Thus it is of considerable interest to determine $g$ and how it depends on the parameters of the system, including the density, $n_0$. In the dilute limit, $g$ approaches unity and the interacting boson system becomes equivalent to non-interacting fermions. The leading density dependent correction is [5]

$$g = 1 - 2an_0.$$ (1.1)

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Here $a$ is the one-dimensional scattering length. This is defined in terms of the one-dimensional even channel phase shift, $\delta(k)$. The even wavefunctions have the asymptotic long-distance behaviour

$$\psi_e(x) \rightarrow \sin[k|x| - \delta(k)], \quad (1.2)$$

where $x$ is the separation of the two bosons. At $k \to 0$ the phase shift is linear in $k$:

$$\delta(k) \to ak, \quad (1.3)$$

implying

$$\psi_e(x) \to \sin k(|x| - a). \quad (1.4)$$

Thus the relevance or irrelevance of pinning in the dilute limit is determined by the sign of $a$. It is important to realize that this crucial sign is not fixed by the requirement that the boson–boson interaction be repulsive. For example, an infinite hard core repulsion of range $a_0$ leads to a scattering length $a = a_0 > 0$. On the other hand, a repulsive $\delta$ function interaction, $\psi_\delta(x)$, leads to a negative scattering length, $a = -1/(\mu v)$ (where $\mu$ is the reduced mass). In a confined geometry the one-dimensional scattering length depends not only on the direct inter-particle interaction but also on the effects of the boundaries. This problem was solved by Olshanii [6] for the case of ultra-cold atoms in a harmonic cylindrical trap, where it was shown that the sign of $a$ can be positive or negative depending on the ratio of the (positive) three-dimensional scattering length to the trap radius.

In this paper we study the properties of two interacting vortices in a thin platelet, or equivalently of two interacting bosons restricted to a narrow strip. We begin with the usual modified Bessel function interaction between vortices given by anisotropic London theory. Standard boundary conditions at the edges of the platelet imply the existence of an infinite set of image vortices for each physical vortex. The interaction of an isolated vortex with its images and with the external magnetic field determines its wavefunction, $f(y)$, in the quantum mechanical analogue, determining the probability of the vortex being at a distance $y$ from the centre of the platelet. It is peaked near the centre, $y = 0$. We then consider the scattering of two physical vortices, taking into account the interactions with all image vortices. In general the two vortices could move off centre (away from $y_1 = y_2 = 0$) as they scatter. Thus the calculation of the effective 1D scattering length requires, in principle, solving for a two-dimensional two-body wavefunction.

Fortunately, there is a very large dimensionless number that appears quite generally in the thermodynamics of vortices, and which simplifies our calculations considerably. Consider, for simplicity, a macroscopic isotropic London superconductor of penetration depth $\lambda$ and coherence length $\xi$. We approximate the Gibbs free energy for $N$ vortices as

$$G \approx \int d\tau \left[ \tilde{\epsilon}_1 \sum_{i=1}^N \left( \frac{d\tilde{r}_i}{d\tau} \right)^2 + \frac{\phi_0^2}{8\pi^2\lambda^2} \sum_{i<j} K_0(|\tilde{r}_i(\tau) - \tilde{r}_j(\tau)|/\lambda) \right]. \quad (1.5)$$

Here $\phi_0 = hc/(2e) \approx 2 \times 10^{-7}$ G cm$^2$ is the flux quantum and

$$\tilde{\epsilon}_1 \approx \frac{\phi_0^2}{16\pi^2\lambda^2} \ln(\lambda/\xi), \quad (1.6)$$

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the tilt modulus, is simply the energy per unit length of the vortex \[ \gamma \]. \( \tau \) is the spatial coordinate along the field direction and \( \vec{r}_i(\tau) \) describes the shape of the \( i \)th vortex. We have approximated the vortex–vortex interaction as only depending on the difference of the \( \vec{r}_i \)'s at the same value of \( \tau \) and used the standard London model result for the interaction energy per unit length between straight parallel vortices \[ \gamma \], given by the modified Bessel function, \( K_0 \). (This needs to be cut off at short distances of order \( r_{ij} \approx \xi \).) We approximate the partition function by an integral over vortex paths, \( \vec{r}_i(\tau) \), weighted by the Boltzmann factor, \( \exp[-G/(k_B T)] \). On identifying \( G/(k_B T) \) with \( S/\hbar \) where \( S \) is the classical action for \( N \) interacting bosons, the classical partition function describing thermal fluctuations of vortices becomes equivalent to the Feynman path integral for interacting bosons. In this way various thermal properties of the vortex system can be conveniently obtained from the quantum system \[ \gamma, \gamma \]. (We set \( \hbar \) and \( k_B = 1 \).) The corresponding Hamiltonian is

\[
H = -\frac{1}{2m} \sum_{i=1}^{N} \nabla_i^2 + \frac{\phi_0^2}{8\pi^2 \lambda^2 T} \sum_{i<j} K_0(|\vec{r}_i(\tau) - \vec{r}_j(\tau)|/\lambda),
\]

with

\[
m = \tilde{\epsilon}_1/T. \tag{1.8}
\]

(Strictly speaking, even after inserting appropriate factors of \( \hbar \) and \( k_B \), both terms in this Hamiltonian have dimensions of inverse length rather than energy. This can be traced back to the fact that in equation (1.5) \( \tau \) is a spatial coordinate in the classical model but is treated as an imaginary time in the analogue quantum one. This creates no problems for our analysis since physical quantities calculated using this quantum approach involve appropriate ratios of parameters with the correct dimensions, as we shall see.) It is convenient to change to dimensionless length variables, letting

\[
\bar{u}_i = \vec{r}_i/\lambda. \tag{1.9}
\]

We then may write the Hamiltonian in dimensionless form:

\[
2m\lambda^2 H = -\sum_{i=1}^{N} \nabla_i^2 + V_0 \sum_{i<j} K_0(|\bar{u}_i - \bar{u}_j|),
\]

where the dimensionless parameter which measures the interaction strength is

\[
V_0 = \frac{\tilde{\epsilon}_1 \phi_0^2}{4\pi^2 T^2} \approx \left( \frac{\phi_0^2}{8\pi^2 T \lambda} \right)^2 \ln(\lambda/\xi). \tag{1.11}
\]

Noting that

\[
\phi_0^2/(8\pi^2 k_B) = 3.9223 \times 10^4 \mu m K,
\]

we see that \( V_0 \gg 1 \) for essentially any superconductor at any \( T < T_c \). This means that the analogue quantum mechanical bosons have very strong short-range interactions when measured in dimensionless units. Variants of this large number will appear when we consider the potential energy function that holds the vortices in the middle of the slab and the interaction between vortices inside the slab. This implies that the vortices stay near the centre of the slab up to rather large slab widths justifying a one-dimensional
approximation. It also allows an unusual but powerful semi-classical approximation to be applied to the one-dimensional problem yielding an explicit formula for the scattering length as a function the platelet thickness $d$ and other parameters ($\lambda_a$, $\lambda_c$, $\xi_a$, $\xi_c$ and $T$).

Our conclusion is that $a$ is positive and large for narrow platelets, increasing with $d$ and having a value $a \approx 19\lambda_a$ for $d = 10\lambda_c \approx 0.5$ mm. This implies that columnar pins are relevant and also that the system rapidly leaves the dilute regime at low densities of order $1/(20\lambda_a)$, corresponding to fields $H$ only slightly above $H_{c1}$.

In section 2 we briefly review London theory and discuss properties of a single vortex in a thin platelet. In section 3 we consider two interacting vortices, determining the scattering length. Section 4 contains conclusions.

2. A single vortex in a thin platelet superconductor

We make the London approximation [7], valid when the penetration depth is much longer than the coherence length, $\lambda \gg \xi$. We label the direction perpendicular to the platelet the $y$ direction, and label the direction of the magnetic field the $z$ direction. In a YBCO crystal the most promising geometry may be choosing $y$ and $z$ to be the $a$ and $b$ directions (or vice versa). See figure 1. Thus the thin direction of the platelet is the $a$ direction, not the usual growth direction, which is the $c$ direction. Such a sample could presumably be obtained by cleaving a macroscopic sample. The magnetic field of a single vortex centred at $\vec{r} = (x, y) = 0$ thus obeys

$$ h - \left[ \lambda_a^2 \frac{\partial^2 h}{\partial x^2} + \lambda_c^2 \frac{\partial^2 h}{\partial y^2} \right] = \phi_0 \delta^2(\vec{r}). \quad (2.1) $$

The Dirac $\delta$ function at the vortex core should actually be smeared over a distance of order $\xi$, the coherence length. Note that the decay of the magnetic field in the $x = c$ direction is governed by supercurrents running in the $y = a$ direction and hence involves $\lambda_a$, whereas the decay in the $y = a$ direction is governed by supercurrents running in the $x = c$ direction and hence involves $\lambda_c$. For extremely underdoped YBCO crystals typical parameter values are

$$ \begin{align*}
\lambda_c &= 50 \, \mu\text{m}, \\
\lambda_a &= 0.5 \, \mu\text{m}, \\
\xi_a &= 5 \, \text{nm}, \\
\xi_c &= 0.05 \, \text{nm}, \\
T_c &= 17 \, \text{K}. 
\end{align*} \quad (2.2) $$

(The value of $\xi_c$ may be a bit small compared to existing measurements but it is convenient to assume the result which follows from anisotropic Ginzburg–Landau theory: $\lambda_c/\lambda_a = \xi_a/\xi_c$. In any event our results only depend logarithmically on the $\xi$'s.) Thus the vortex is extremely elliptical: much more extended in the $y = a$ direction. To achieve the two-dimensional limit, we need the sample thickness to be of order the vortex size. (Actually, as we shall see that a thickness of up to ten times the vortex size or more is acceptable.) Thus we can take advantage of the larger $\lambda_c$ by cleaving our crystal in the $a$ direction. We will refer to the parameters in equation (2.2) at temperature $T \approx T_c$, with $a = y$ the thin direction, as the standard parameters. However, our results should

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also apply at lower temperatures. Note that the vortex wandering that we are concerned with here occurs primarily in the $x = c$ direction. The vortices presumably feel a periodic potential with wavelength given by the lattice constant in the $c$ direction. We will ignore this here. In the dilute limit that we are considering it is not expected to have an important effect.

The solution of equation (2.1) at distances $r \gg \xi$, ignoring for now the boundaries, can be found by using a Fourier transform and is

$$h = \frac{\phi_0}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(xk_x + yk_y)} \frac{\lambda^2 k_x^2 + \lambda^2 k_y^2 + 1}{\lambda^2 k_x^2 + \lambda^2 k_y^2 + 1} \, dk_x \, dk_y$$

(2.3)

$$= \frac{\phi_0}{2\pi \lambda a \lambda c} K_0 \left( \sqrt{\left( \frac{x}{\lambda a} \right)^2 + \left( \frac{y}{\lambda c} \right)^2} \right),$$

(2.4)

where $K_0$ is a modified Bessel function.

The energy per unit length of a vortex is

$$\epsilon = -\frac{1}{8\pi} \int h \left[ \lambda^2_a \frac{\partial h}{\partial x} \hat{x} + \lambda^2_c \frac{\partial h}{\partial y} \hat{y} \right] \cdot d\sigma,$$

(2.5)

where the integral is taken over an ellipse of axes $\xi_a, \xi_c$ around the vortex core. Thus

$$\epsilon \approx \frac{\phi_0^2}{16\pi^2 \lambda a \lambda c} \ln \left( \frac{\lambda c}{\xi_a} \right),$$

(2.6)

to logarithmic accuracy. For the parameters in equation (2.2) we have $\epsilon = 10^{-8}$ erg cm$^{-1}$.

We remark that the interaction energy per unit length between two straight parallel vortices separated by a vector $\vec{r}$ is simply

$$U_{12} = \frac{\phi_0 h_{12}}{4\pi},$$

(2.7)

where $h_{12}(\vec{r})$ is the magnetic field at the location of one vortex produced by the other, equation (2.4).

We now consider a single straight vortex in an infinite slab, of thickness $d$, extending from $-d/2$ to $d/2$. The presence of boundaries of the superconductor at $y = \pm d/2$ imposes

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boundary conditions,

\[ 0 = j_y = \frac{\partial h}{\partial x}. \tag{2.8} \]

A vortex at position \( y \) in a slab of thickness \( d \) creates image vortices with magnetic field in the opposite direction at positions \((2n+1)d - y\) for integral \( n \), and creates image vortices with field in the same direction at positions \(2nd + y\) for all non-zero integral \( n \). Thus the magnetic field at location \((x, y)\) for a vortex at \((0, y_1)\) is

\[
h(x; y_2, y_1) = \frac{\phi_0}{2\pi \lambda_a \lambda_c} \left[ \sum_{n=-\infty}^{\infty} K_0 \left( \sqrt{\left( \frac{x}{\lambda_a} \right)^2 + \left( \frac{2nd + y_1 - y_2}{\lambda_c} \right)^2} \right) - \sum_{n=-\infty}^{\infty} K_0 \left( \sqrt{\left( \frac{x}{\lambda_a} \right)^2 + \left( \frac{(2n+1)d - y_1 - y_2}{\lambda_c} \right)^2} \right) \right]. \tag{2.9} \]

In addition to the field of the vortex and its images an additional field occurs inside the superconductor when a field \( H \) is applied outside of it:

\[
h_1(y) = H \frac{\cosh(y/\lambda_c)}{\cosh(d/2\lambda_c)}. \tag{2.10} \]

The magnetic field at the position of the vortex, \((0, y)\) due to all its image vortices is

\[
h_2(y) = \frac{\phi_0}{2\pi \lambda_a \lambda_c} \left[ - \sum_{n=\pm1,\pm3,...} K_0 \left( \frac{nd - 2y}{\lambda_c} \right) + \sum_{n=\pm2,\pm4,...} K_0 \left( \frac{nd}{\lambda_c} \right) \right]. \tag{2.11} \]

The Gibbs free energy depends on the position \( y \) of the vortex as

\[
V_1(y) = \frac{\phi_0}{4\pi} \left[ h_1(y) + \frac{1}{2} h_2(y) \right] + \text{constant}. \tag{2.12} \]

This is plotted in figures 2(a) and (b) at \( H = H_{c1} \). It has a minimum at \( y = 0 \), large barriers at intermediate \( y \) and then appears to diverge to \(-\infty\) at \( y \rightarrow \pm d/2 \):

\[
V_1(y) \rightarrow -\frac{\phi_0^2}{16\pi^2 \lambda_a \lambda_c} \ln[\lambda_c/(d \pm 2y)]. \tag{2.13} \]

This divergence is due to the interaction of the vortex with its image at \( y \mp d \). This divergence should actually be cut off at \( y \mp d/2 \) of order \( \xi \) due to corrections to London theory. We take this into account by replacing \( K_0[(nd - 2y)/\lambda_c] \) by \( K_0[(nd - 2y)/\lambda_c] - K_0[(nd - 2y)/\xi] \) in equation (2.11). For \( H > H_{c1} \), we might expect the true minimum energy to be at \( y = 0 \). This gives the usual formula for \( H_{c1} \) for an anisotropic superconductor:

\[
H_{c1} \approx \frac{4\pi}{\phi_0} \epsilon, \tag{2.14} \]

where \( \epsilon \) is the energy per unit length of a bulk vortex, equation (2.6).

We now turn to a study of thermal fluctuations of a single vortex inside the slab for such an anisotropic superconductor. With the magnetic field along the \( b \) axis, the tilt
modulus is very different for vortex tilting in the $a = y$ or $c = x$ direction. The elastic energy is written as

$$G_0 = \int d\tau \left[ \frac{\tilde{\epsilon}_a}{2} \left( \frac{dy}{d\tau} \right)^2 + \frac{\tilde{\epsilon}_c}{2} \left( \frac{dx}{d\tau} \right)^2 \right].$$  \hspace{1cm} (2.15)$$

Due to the assumed symmetry under rotations in the $a-b$ plane, the tilt modulus for tilting in the $a$ direction is just given by the energy per unit length:

$$\tilde{\epsilon}_a = \epsilon,$$  \hspace{1cm} (2.16)

where $\epsilon$ is given in equation (2.6). On the other hand, the energy per unit length of a vortex aligned parallel to the $c = x$ axis is much larger, resulting in the tilt modulus [8]

$$\tilde{\epsilon}_c = \frac{\phi_0^2 \lambda_c}{8\pi^2 \lambda_a^2} \ln(\lambda_c/\xi) \approx \frac{2\lambda^2_c}{\lambda_a^2} \tilde{\epsilon}_a.$$  \hspace{1cm} (2.17)

To this must be added the $y$ dependent free energy:

$$G_1 = \int d\tau V_1[y(\tau)].$$  \hspace{1cm} (2.18)$$

Here we have again considered only ‘instantaneous’ interactions between the vortex and its images, at a fixed value of $\tau$. We do not expect this approximation to qualitatively change the long-distance physics in the dilute limit. If we consider a very long vortex, in a sample of macroscopic length in the $z = \tau = b$ direction, then the probability of the displacement of the vortex from the centre of the slab having some value $y$ is
simply given by $|f(y)|^2$ where $f$ is the ground state wavefunction of the one-dimensional Hamiltonian:

$$H_1 = -\frac{1}{2m_a} \left( \frac{d}{dy} \right)^2 + \frac{V_1(y)}{T},$$

with

$$m_a \equiv \frac{\tilde{\epsilon}_a}{T}. \quad (2.20)$$

It is now convenient to define the dimensionless length variable:

$$\tilde{y} \equiv y/\lambda_c,$$

in terms of which

$$2m_a\lambda_c^2 H = -\left( \frac{d}{dy} \right)^2 + V_0[y - \sum_n K_0[(2n + 1)d/\lambda_c - 2\tilde{y}] + \ln(\lambda_c/\xi_a) \frac{\cosh(\tilde{y}/2)}{\cosh(d/2\lambda_c)}]. \quad (2.22)$$

where

$$V_0[y] = \frac{2\tilde{\epsilon}_a \phi_0^2 \lambda_c}{16\pi^2 \lambda_a T^2} = 2 \left( \frac{\phi_0^2}{16\pi^2 \lambda_a T} \right)^2 \ln \left( \frac{\lambda_c}{\xi_a} \right). \quad (2.23)$$

(We have set $H = H_{c1}$ given in equation (2.14).) Since we will choose $d$ of order $\lambda_c$, we see that the dimensionless number $V_0[y]$ characterizes the height of the barriers holding the vortex at the centre of the platelet. Using the numbers in equation (2.2) and choosing $T = T_c$ we find

$$V_0[y] \approx 10^8. \quad (2.24)$$

We solve this Schrödinger equation numerically for the ground state, for $d = 10\lambda_c$, figure 2(c), finding that the particle makes only very small quantum fluctuations away from $y = 0$ due to the huge barriers. For the above parameters we find

$$\sqrt{\langle y^2 \rangle} = 0.01419 \lambda_c = 0.001419 d. \quad (2.25)$$

We should estimate the true critical field, $H_{c1}(d)$, using the ground state energy of the quantum Hamiltonian. If the field is too low the particle can tunnel through the barrier corresponding to a vortex terminating at some value of $\tau$ with the magnetic flux leaving the superconductor. However, due to the exceedingly high barrier, the tunnelling probability is minuscule and the system will not be very sensitive to the precise value of $H$. We should also remark that to solve the Schrödinger equation precisely we need to specify some boundary conditions on the wavefunction at $y = \pm d/2$. We imposed vanishing boundary conditions. Fortunately, the extremely large dimensionless barrier also renders our results very insensitive to this choice.
3. Two vortices

Consider two straight parallel vortices inside the platelet, at locations \((x_i, y_i)\). The Gibbs free energy per unit length is

\[
V(x_1 - x_2; y_1, y_2) = \frac{\phi_0}{4\pi} h(x_1 - x_2; y_1, y_2) + V_1(y_1) + V_1(y_2). \tag{3.1}
\]

By translational invariance in the \(x\) direction, \(V\) depends only on the difference of the \(x\) coordinates of the two vortices:

\[
x \equiv x_1 - x_2. \tag{3.2}
\]

Here \(h(x, y_1, y_2)\) is given in equation (2.9) and \(V_1(y)\) by equation (2.12). The first term in equation (3.1) represents the interaction of one vortex with the other one and with the images of the other one. The second and third terms represent the interaction of each vortex with its own images and with the screened external field. \(V(x, y_1, y_2)\) has a deep minimum at \(y_1 = y_2 = 0\) for large \(|x| \gg \lambda_a\). \(V(x, 0, 0)\) has a large peak centred at \(x = 0\) with a weak, logarithmic divergence right at \(x = 0\). This logarithmic divergence should be cut off at scales of order \(\xi\); however this cut off has essentially no effect on the scattering length, as we shall see.

Again we may study the thermodynamics of two wiggling vortices in the platelet by mapping onto a quantum mechanics model. We make the fundamental assumption that the vortices only bend on long length scales (compared to \(\lambda_a\)) and that we may approximate the vortex–vortex (and vortex–image vortex) interaction by an ‘instantaneous’ one at a fixed value of \(\tau = z\). The Boltzmann sum is now over the configuration of two vortices and we must include the vortex–vortex interaction in the free energy. Again identifying the free energy with the imaginary time action, we see that the corresponding quantum Hamiltonian is

\[
H = -\frac{1}{2} \sum_{i=1}^{2} \left[ \frac{1}{m_a} \left( \frac{d}{dy_i} \right)^2 + \frac{1}{m_c} \left( \frac{d}{dx_i} \right)^2 \right] + \frac{V(x, y_1, y_2)}{T}. \tag{3.3}
\]

Here \(m_a\) is given in equation (2.20) and

\[
m_c \equiv \frac{\tilde{\epsilon}_c}{T}, \tag{3.4}
\]

where \(\tilde{\epsilon}_c\) is given in equation (2.17). In the quantum analogue, the vortices obey Bose statistics [1] and consequently the two-body wavefunction must be symmetric: even under \(x \rightarrow -x\). The asymptotic behaviour of the low energy wavefunctions at \(|x| \gg \lambda_a\) is given by

\[
\psi(x, y_1, y_2) \rightarrow f(y_1) f(y_2) \sin[k|x| - \delta(k)], \tag{3.5}
\]

where \(f(y)\) is the ground state wavefunction for a single vortex, discussed in section 2. The energy of this scattering state is

\[
E = 2E_1 + \frac{k^2}{2\mu}, \tag{3.6}
\]

where \(E_1\) is the ground state energy for a single vortex, discussed in the previous section, and the reduced mass which governs the relative motion is

\[
\mu = m_c/2. \tag{3.7}
\]

The scattering length is defined by equation (1.3).
It turns out that, due to the large barrier near $x = 0$, for small $y_i$, the scattering length is determined almost completely by the large $x$ asymptotic form of the potential, until $d$ gets very large compared to $\lambda_c$. The $x$ dependent part of the exact potential can be written as a sum of exponentials:

$$V(x; y_1, y_2) = \frac{\phi_0^2}{4\pi\lambda_c^2d} \sum_{n=1}^{\infty} f(n, \frac{y_1}{d}, \frac{y_2}{d}) \frac{e^{-|x|/\sqrt{1 + (n\pi\lambda_c/d)^2/\lambda_c}}}{\sqrt{1 + (n\pi\lambda_c/d)^2/\lambda_c}} + V_1(y_1) + V_1(y_2), \quad (3.8)$$

where we define

$$f(n, y_1/d, y_2/d) \equiv \begin{cases} \cos(n\pi y_1/d) \cos(n\pi y_2/d) & \text{if } n \text{ is odd} \\ \sin(n\pi y_1/d) \sin(n\pi y_2/d) & \text{if } n \text{ is even}. \end{cases} \quad (3.9)$$

At large $|x|$ we may approximate the sum by the first term only:

$$V \approx \frac{\phi_0^2}{4\pi\lambda_c^2d} e^{-|x|/\lambda} \cos\left(\frac{n\pi y_1}{d}\right) \cos\left(\frac{n\pi y_2}{d}\right) + V_1(y_1) + V_1(y_2), \quad (3.10)$$

where we have defined, for convenience, a reduced value of $\lambda_a$:

$$\lambda \equiv \lambda_a \left[1 + \left(\frac{\pi\lambda_c}{d}\right)^2\right]^{-1/2}. \quad (3.11)$$

For this approximation to hold, we need the first term to dominate over all other terms. In the one-dimensional limit (i.e. $y_1 = y_2 = 0$) the condition for large $x$ is

$$\exp\left(\frac{|x|}{\lambda} \frac{4\pi^2\lambda_c^2}{\lambda_a^2 d^2}\right) \gg 1. \quad (3.12)$$

Note that, unlike the case for the direct vortex–vortex interaction, this potential has a simple exponential dependence on $x$ at large $|x|$, albeit with a reduced penetration depth.

Due to the large barriers in $V_1(y)$ we expect the low energy scattering states to be confined to $y_i \approx 0$. We first calculate $a$ assuming that the vortices stay exactly in the middle of the slab, $y_i = 0$, throughout the scattering process. We return to a further discussion of why this is reasonable at the end of this section. This reduces the problem to a one-dimensional quantum mechanics model with Hamiltonian

$$H = \frac{1}{2\mu} \frac{d^2}{dx^2} + \frac{V(x)}{T}, \quad (3.13)$$

where

$$V(x) = V(x; 0, 0) \rightarrow \frac{\phi_0^2\lambda}{4\pi\lambda_c^2d} e^{-|x|/\lambda}. \quad (3.14)$$

We look for parity even solutions of this Schrödinger equation with asymptotic behaviour $\psi(x) \rightarrow \sin[k|x| - \delta(k)]$ with $\delta(k) \rightarrow ak$ as $k \rightarrow 0$. Note that in the small $k$ limit, $\psi(x) \rightarrow \sin[k(|x| - a)]$ for $x \gg \lambda$. Then, if we consider an intermediate range of $x$,

$$\lambda \ll |x| \ll 1/k, \quad (3.15)$$

we may approximate the wavefunction by a linear form:

$$\psi(x) \propto |x| - a. \quad (3.16)$$

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Thus, to find the scattering length we need to simply solve the zero-energy Schrödinger equation:
\[
\left[- \frac{1}{2\mu} \frac{d^2}{dx^2} + \frac{V(x)}{T}\right] \psi = 0. \tag{3.17}
\]
Note that this reduces the eigenvalue problem to a simple initial value problem. We simply impose the initial conditions:
\[
\begin{align*}
\frac{d\psi}{dx}(0) &= 0, \\
\psi(0) &= 1,
\end{align*} \tag{3.18}
\]
and solve the zero-energy equation. The asymptotic behaviour of the solution at \(|x| \gg \lambda\) is given by equation (3.16) which determines the scattering length, \(a\).

It is convenient to introduce a dimensionless length variable:
\[
\tilde{x} \equiv x/\lambda, \tag{3.19}
\]
in terms of which the Schrödinger equation becomes, at large \(|\tilde{x}|\),
\[
\left[- \frac{d^2}{d\tilde{x}^2} + V_0 e^{-\tilde{x}}\right] \psi(\tilde{x}) = 2\mu \lambda^2 E \psi(\tilde{x}). \tag{3.20}
\]
Here the dimensionless number which measures the strength of the repulsive potential is
\[
V_0 = \frac{\tilde{\epsilon}_c \phi_0^2 \lambda^3}{4\pi \lambda_a^2 d T^2} = \frac{1}{2\pi} \left( \frac{\phi_0^2}{4\pi \lambda_a T} \right)^2 \ln(\lambda_c/\xi_a) \frac{\lambda_c/d}{[1 + (\pi \lambda_c/d)^2]^{3/2}}. \tag{3.21}
\]
Using our estimates of the parameters in equation (2.2) with \(T = T_c\) and \(d = 10\lambda_c\) we find
\[
V_0 = 1.07 \times 10^8. \tag{3.22}
\]
Note that \(V_0 \propto 1/d\) at \(d \ll \lambda_c\). Importantly \(V_0 \gg 1\) when \(d\) is of order \(\lambda_c\) and remains large out to extremely large values of \(d/\lambda_c\). The largeness of \(V_0\) leads to a large scattering length, allows for an unusual semi-classical solution approximation and also helps to justify setting \(y_i = 0\) as we shall see below. It is of course, possible to solve the Schrödinger equation numerically for specified values of the parameters. However, the largeness of \(V_0\) and \(V_0 y_i\) creates numerical difficulties for standard algorithms, when one attempts to solve the full three-dimensional problem, including the \(y_i\). In this case it is much easier, and more transparent, to use the semi-classical approximation.

Our semi-classical approximation for the scattering length at \(V_0 \gg 1\) begins with the observation that the classical turning point for two particles approaching each other with a small relative momentum, \(k\), occurs at \(\tilde{x} \gg 1\). Therefore \(a\) is determined almost completely by the large \(\tilde{x}\) form of the potential in equation (3.10). Using the large \(\tilde{x}\) form of the potential, we may solve the one-dimensional Schrödinger equation exactly. To do this we change variables to
\[
u = 2 \sqrt{V_0 e^{-|\tilde{x}|/2}}. \tag{3.23}
\]
The zero-energy Schrödinger equation, (3.17), simplifies to
\[
u^2 \psi'' + w \psi' - u^2 \psi = 0, \tag{3.24}
\]
where the primes denote differentiation with respect to $u$. This is the zeroth-order modified Bessel’s differential equation, and the general solution is given in terms of the modified Bessel functions:

$$
\psi = c_1 I_0(u) + c_2 K_0(u)
= c_1 I_0 \left( 2\sqrt{V_0}e^{-|x|/2\lambda} \right) + c_2 K_0 \left( 2\sqrt{V_0}e^{-|x|/2\lambda} \right).
$$  \hspace{1cm} (3.25)

For large $u$ (i.e. $|\tilde{x}| \ll \ln V_0$) we have

$$
\psi \approx \frac{1}{\sqrt{2\pi u}} (c_1e^u + c_2\pi e^{-u}),
$$  \hspace{1cm} (3.26)

or, writing in terms of the original variables and putting back factors of $\lambda$, we have

$$
\psi \approx \frac{1}{\sqrt{4\pi V_0^{1/4}e^{-|x|/4\lambda}}} \left( c_1e^{2\sqrt{V_0}e^{-|x|/2\lambda}} + c_2\pi e^{-2\sqrt{V_0}e^{-|x|/2\lambda}} \right).
$$  \hspace{1cm} (3.27)

On the other hand, for small $u$ (i.e. $|\tilde{x}| \gg \ln V_0$) we have

$$
\psi \approx c_1 - c_2(ln(u/2) + \gamma) = c_1 + c_2 \left( \frac{|\tilde{x}|}{2} - \frac{1}{2} \ln V_0 - \gamma \right),
$$  \hspace{1cm} (3.28)

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. Note that the last formula can be written as

$$
\psi \approx \frac{c_2}{2\lambda} (|x| - a),
$$  \hspace{1cm} (3.29)

where the scattering length $a$ is (putting back factors of $\lambda$)

$$
da = \lambda(\ln V_0 + 2\gamma - 2c_1/c_2).
$$  \hspace{1cm} (3.30)

To determine $c_1/c_2$ we match our solution in the region $1 \ll |\tilde{x}| \ll \ln V_0$ to the WKB solution which works for $|x|$ not too large, in the region where $V(x)$ is large. The even WKB wavefunction for $E = 0$ is given by

$$
\psi(x) = A \left\{ \exp \left[ \int_0^x \sqrt{V'(x')} \, dx' \right] + \exp \left[ - \int_0^x \sqrt{V'(x')} \, dx' \right] \right\},
$$  \hspace{1cm} (3.31)

where

$$
V'(x) = 2\mu V(x;0,0),
$$  \hspace{1cm} (3.32)

and $V$ is the exact potential of equation (3.1). (Note that we do not make any large $x$ approximation to $V$ now.) We can rewrite this as

$$
\psi(x) = A \left\{ e^{-\alpha} \exp \left[ \int_x^\infty \sqrt{V'(x')} \, dx' \right] + e^\alpha \exp \left[ - \int_x^\infty \sqrt{V'(x')} \, dx' \right] \right\},
$$  \hspace{1cm} (3.33)

where

$$
\alpha = \int_0^\infty \sqrt{V'(x)} \, dx.
$$  \hspace{1cm} (3.34)
Now, if \( x \) is large enough for our asymptotic expression \( V'(x) \approx V_{0x} e^{-x/\lambda}/\lambda^2 \) to hold, then the integration can be done quite readily:

\[
\psi(x) = A \left\{ e^{-\alpha} \exp \left[ 2\sqrt{V_{0x} e^{-x/2\lambda}} \right] + e^{\alpha} \exp \left[ -2\sqrt{V_{0x} e^{-x/2\lambda}} \right] \right\}.
\]  

(3.35)

Comparison with equation (3.27) thus gives

\[
c_1/c_2 = \pi e^{-2\alpha} = \pi \exp \left[ -\int_{-\infty}^{\infty} \sqrt{V'(x)} \, dx \right].
\]  

(3.36)

This quantity is exponentially small in the large quantity \( V_{0x} \) so it is completely negligible. Note also that the logarithmic divergence of \( V(x; 0, 0) \) at \( x \to 0 \) has no important effects, leaving the integral finite in equation (3.36). The last term in equation (3.30) essentially vanishes, and we simply have that

\[
a = \lambda (\ln V_{0x} + 2\gamma).
\]  

(3.37)

Asymptotically, the scattering length is linearly dependent on the logarithm of the size of the potential.

Interestingly, we get almost the same result for the odd wavefunctions, except that the sign of \( c_1/c_2 \) is reversed. The even channel and odd channel scattering lengths are therefore almost exactly the same. However, it is the difference between the even and odd channel scattering lengths that determines the transmission coefficient, and it is only then that \( c_1/c_2 \) plays an important role.

We have based this approximation on the assumption that there exists a region of separation \( x \) such that the approximation equation (3.27) holds and the WKB approximation to the wavefunction also holds. Typically the matching is done around the region \( x \approx a \); therefore we need that (using equations (3.37) and (3.12), and noticing \( \lambda \approx \lambda_a \) for large \( d \))

\[
\exp \left( \frac{2\pi \lambda_a}{d} \right)^2 \ln(e^{2\gamma} V_{0x}) \gg 1.
\]  

(3.38)

This is the condition that must be satisfied for the formula equation (3.37) to hold.

The wavefunction calculated numerically to high precision (in the one-dimensional approximation) and the semi-classical wavefunction are compared in figure 3 for our standard parameters and \( d = 10\lambda_a \). As can be seen, the two wavefunctions give good agreement in the large \( x \) regime (with an error <1% for \( x > 18\lambda_a \)). The semi-classical wavefunction is grossly inaccurate in the small \( x \) region (\( x < 15\lambda_a \)) but the wavefunction is negligible there anyway. The predicted semi-classical scattering length is \( a = \lambda (\ln V_0 + 2\gamma) = 18.7398\lambda_a \). The actual scattering length of the numerically determined wavefunction (obtained by fitting the wavefunction in the large \( x \) regime to a linear function) is 18.7409\( \lambda_a \). The semi-classical wavefunction gives an error of less than 0.01%. In figure 4 we show the scattering length versus \( d \), comparing our numerical results to the semi-classical approximation (in both cases making the one-dimensional approximation). Most of the \( d \) dependence in our semi-classical formula, equation (3.37), arises from the \( d \) dependence of the reduced penetration depth, \( \lambda \), given in equation (3.11). As the sample thickness decreases, the effective range of the interaction potential, \( \lambda \), also decreases, and the scattering length simply scales with it, up to logarithmic corrections coming from \( V_{0x} \).
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**Figure 3.** The wavefunctions in the $k \to 0$ limit comparing a precise numerical calculation (in the one-dimensional approximation) to the semi-classical approximation for our standard parameters and $d = 10\lambda_c$. Lengths are in units of $\lambda_a$.

**Figure 4.** The scattering lengths based on a precise numerical calculation (in the one-dimensional approximation) compared to our semi-classical approximation for our standard parameters. $d$ is in units of $\lambda_c$, the scattering length in units of $\lambda_a$.

Defined in equation (3.21). The semi-classical and numerical values agree within 1% up to about $d = 22\lambda_c$. For $d = 22\lambda_c$, we have that (refer to condition (3.38))

$$
\left( \frac{2\pi\lambda_c}{d} \right)^2 \ln(e^{2\gamma V_0x}) = 1.56,
$$

(3.39)

which is already not far from unity. There will therefore be significant deviations of the true value from equation (3.37). We could also see that the scattering length tends towards a finite value for large $d$. This is because as the thickness of the sample grows, the image vortices move further away from the original vortices until their effects become negligible.

Next, we discuss the validity of our one-dimensional approximation, setting $y_i = 0$, which is justified by the fact that the single-vortex wavefunction, $f(y)$, is so sharply peaked near $y = 0$. If we look at the shape of the potentials, $V_1(y)$, we see that for small $d$...
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Figure 5. The spread of the single-vortex wavefunction, $\sqrt{\langle y^2 \rangle}$, plotted against $d$ using standard parameters. Both lengths are in units of $\lambda_c$.

(figure 2(a)) the potential has an obvious minimum in the centre, and is approximately simple harmonic near the centre. For larger $d$ (figure 2(b)), however, the potential is almost negligible except for a large potential barrier close to (but not at) the edges; the potential will be qualitatively more similar to a square well. Therefore as $d$ increases, we would expect the shape of the wavefunction to morph from a confined Gaussian to a spread out sinusoidal form (figures 2(c) and (d)). The spread of the single-vortex wavefunction ($\langle y^2 \rangle^{1/2}$) is plotted in figure 5. The thickness reaches 1% of the platelet thickness at around $d = 35\lambda_c$, an indication that our one-dimensional approximation fails above this value. The fact that the thickness becomes linearly dependent on $d$ at large $d$ also suggests that the wavefunction tends to a fixed shape (a sinusoidal form).

A more systematic approximation for solving the full two-body two-dimensional Schrödinger equation of equation (3.3) would involve writing

$$\psi(x; y_1, y_2) \approx \psi_1(x) f(y_1) f(y_2),$$

(3.40)

where $\psi_1(x)$ is the one-dimensional wavefunction found above and $f(y_1)$ is the single-vortex wavefunction. We could then improve our estimate of the one-dimensional effective potential by using

$$V(x) \approx \int dy_1 dy_2 |f(y_1)|^2 |f(y_2)|^2 V(x; y_1, y_2)$$

(3.41)

rather than simply $V(x) \approx V(x; 0, 0)$. However, because $f(y)$ is so sharply peaked at $y \approx 0$ this makes a negligible difference.

Note that our calculation of $a$ depended essentially only on $V(x)$ in the large $x$ region, $x \gg \lambda$. Our consideration of the small $x$ region only served to determine $c_1/c_2$ which was exponentially small anyway and can simply be ignored. In this large $x$ region, equation (3.10) is a good approximation, the wavefunction approximately factorizes and the large barriers in $V_1(y)$ ensure that the wavefunction is strongly peaked near $y_i = 0$. For smaller values of $x$ the wavefunction presumably spreads out more in the $y$ direction. However, at smaller $x$ the wavefunction is exponentially small anyway.

Finally, we consider the case where the thin direction of the YBCO platelet is the $c$ direction: $y = c$ (and $x = a$, $z = b$). In this case the roles of $\lambda_a$ and $\lambda_c$ are switched, as
are the roles of $\tilde{\epsilon}_a$ and $\tilde{\epsilon}_c$, and we get

\begin{align*}
V_{0y} = 2\frac{\tilde{\epsilon}_c \phi_0^2 \lambda_a}{16\pi^2 T^2 \lambda_c} &= 4 \left(\frac{\phi_0^2}{16\pi^2 \lambda_a T}\right)^2 \ln(\lambda_c/\xi_a), \\
V_{0x} = \frac{\tilde{\epsilon}_a \phi_0^2 \lambda_c^3}{4\pi \lambda_c^2 d T^2} &= \frac{1}{4\pi} \frac{\phi_0^2}{4\pi \lambda_a T} \ln(\lambda_c/\xi_a) \frac{\lambda_a/d}{[1 + (\pi \lambda_a/d)^2]^{3/2}},
\end{align*}

where we now define

\begin{equation}
\lambda \equiv \lambda_c \left[1 + \left(\frac{\pi \lambda_a}{d}\right)^2\right]^{-1/2}.
\end{equation}

Apart from some unimportant factors of 2, the formulae for $V_{0y}$ and $V_{0x}$ are the same as for the other geometry except that $d$ now appears in the dimensionless ratio $\lambda_a/d$ rather than $\lambda_c/d$. So we now conclude that the one-dimensional approximation holds for $d \leq 20\lambda_a \approx 10\mu m$ and the semi-classical approximation holds out to roughly the same value of $d$.

4. Conclusions

Our main result is the formula equation (3.37) for the scattering length. This is plotted versus the platelet thickness, $d$, for our standard parameters, in figure 4. Note that $a$ is everywhere positive and $a/\lambda_a$ is everywhere quite large, having the value $a/\lambda_a \approx 19$ at $d \approx 10\lambda_c$. At somewhat larger values of $d$ we expect our semi-classical approximation to the one-dimensional problem to break down and, more problematically, the one-dimensional approximation itself to start to fail.

The main use of our formula for $a$ is not, of course, for studying the system with only two vortices, but rather for studying the thermodynamic limit of many vortices. In the dilute limit, $n_0a \ll 1$ (where $n_0$ is the vortex density per unit length), $a$ determines the Luttinger parameter via equation (1.1). Of course the Luttinger liquid treatment of the problem assumes that it is fundamentally one-dimensional. Our calculations here indicate that the one-dimensional approximation should be good, at least in the dilute limit $n_0a \ll 1$, up to platelet thicknesses of order $d = 10\lambda_c$ or more, since the vortices stay very close to the centre of the platelet. Furthermore, we have determined the Luttinger parameter for this range of thicknesses and vortex densities. When $a$ is large the Luttinger parameter decreases rapidly for increasing vortex density. It was argued in [5] that, at high densities, $g \ll 1$. Taken together, these results suggest a rapid monotonic drop of $g$ from 1 with increasing density. In this case, columnar pins would be highly relevant for essentially all fields above $H_{c1}$. Thus a promising region to look at experimentally might be one very close to $H_{c1}$ with low vortex densities, $n_0 \ll 1/\lambda_a$, and samples of thickness around $10\lambda_c$.

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