Stability and bifurcations of heteroclinic cycles of type Z

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Abstract
Dynamical systems that are invariant under the action of a non-trivial symmetry
group can possess structurally stable heteroclinic cycles. In this paper, we
study stability properties of a class of structurally stable heteroclinic cycles
in \( \mathbb{R}^n \) which we call heteroclinic cycles of type Z. It is well known that a
heteroclinic cycle that is not asymptotically stable can nevertheless attract a
positive measure set from its neighbourhood. We say that an invariant set
\( X \) is fragmentarily asymptotically stable, if for any \( \delta > 0 \) the measure of
its local basin of attraction \( B_\delta(X) \) is positive. A local basin of attraction
\( B_\delta(X) \) is the set of such points that trajectories starting there remain in the
\( \delta \)-neighbourhood of \( X \) for all \( t > 0 \), and are attracted by \( X \) as \( t \to \infty \).
 Necessary and sufficient conditions for fragmentary asymptotic stability are
expressed in terms of eigenvalues and eigenvectors of transition matrices. If
all transverse eigenvalues of linearizations near steady states involved in the
cycle are negative, then fragmentary asymptotic stability implies asymptotic
stability. In the latter case the condition for asymptotic stability is that the
transition matrices have an eigenvalue larger than one in absolute value. Finally,
we discuss bifurcations occurring when the conditions for asymptotic stability
or for fragmentary asymptotic stability are broken.

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1. Introduction

A smooth dynamical system
\[
\dot{x} = f(x), \quad f: \mathbb{R}^n \to \mathbb{R}^n
\]  
(1)
can possess various kinds of invariant sets—steady states, periodic orbits, tori, heteroclinic
cycles and strange attractors. Conditions for asymptotic stability and (local) bifurcations of
steady states and periodic orbits are well known (see e.g. [13]). For a steady state the conditions
for stability are formulated in terms of eigenvalues of the linearization near the steady state. For a periodic orbit they are expressed in terms of eigenvalues of linearization of the Poincaré return map near the periodic orbit. When the conditions for stability cease to be satisfied, a bifurcation of the steady state or of the periodic orbit takes place. No complete theory for stability and bifurcations of heteroclinic cycles is yet available.

Let \( \xi_1, \ldots, \xi_m \in \mathbb{R}^n \) be hyperbolic equilibria of (1) and \( \kappa_j : \xi_j \to \xi_{j+1}, \) \( j = 1, \ldots, m, \) \( \xi_{m+1} = \xi_1, \) be a set of trajectories from \( \xi_j \) to \( \xi_{j+1}. \) The union of the equilibria and the connecting trajectories is called a heteroclinic cycle. Generically heteroclinic cycles are structurally unstable, because an arbitrary small perturbation of \( f \) breaks a connection between two saddle steady states. However, the connections can be structurally stable (or robust) if the dynamical system has a non-trivial symmetry group and only symmetric perturbations are considered [2, 15, 25], or if the system is constrained to preserve certain invariant subspaces [15].

Heteroclinic cycles that are not asymptotically stable can attract a positive measure set from its small neighbourhood [4, 5, 8, 14, 18, 22]. We call such heteroclinic cycles fragmentarily asymptotically stable. In earlier papers several types of stability were employed to describe locally attracting, but not asymptotically stable invariant sets: essential asymptotic stability [4, 5, 17, 19], relative asymptotic stability [5, 26], predominant asymptotic stability [21]. If a set is stable in any of these senses, then it is fragmentarily asymptotically stable. If a heteroclinic cycle is not fragmentarily asymptotically stable, we call it completely unstable.

Asymptotic stability or fragmentary asymptotic stability of structurally stable heteroclinic cycles was considered in a number of papers. A sufficient condition for asymptotic stability of heteroclinic cycles is given in [16]. A heteroclinic cycle is called simple, if all eigenvalues of \( d f(\xi_j) \) are different and the connecting orbits \( \kappa_j \) are one-dimensional. Necessary and sufficient conditions for asymptotic stability of simple homoclinic and heteroclinic cycles in \( \mathbb{R}^4 \) are given in [6, 9, 18]; necessary and sufficient conditions for fragmentary asymptotic stability of simple heteroclinic cycles in \( \mathbb{R}^4 \) are given in [21] (the term fragmentary asymptotic stability is not used there). Conditions for asymptotic stability, essential asymptotic stability or relative asymptotic stability for heteroclinic cycles in particular systems are presented in [4–6, 8, 9, 11, 14, 17, 22, 23]. In some of these papers [6, 8, 9, 22, 23] bifurcations of homoclinic and heteroclinic cycles are also studied.

In this paper, we introduce a class of (structurally stable simple) heteroclinic cycles in \( \mathbb{R}^n. \) All simple heteroclinic cycles studied in the papers cited above, except for the so-called type A cycles, belong to this class. We call this class type Z heteroclinic cycles.

For type Z heteroclinic cycles we derive necessary and sufficient conditions for asymptotic stability and fragmentary asymptotic stability. A cycle is fragmentarily asymptotically stable, whenever certain inequalities on eigenvalues and eigenvectors of transition matrices associated with the cycle are satisfied. If for all \( j \) all transverse eigenvalues of \( d f(\xi_j) \) are negative and the cycle is fragmentarily asymptotically stable, then it is asymptotically stable. For each inequality determining asymptotic stability or fragmentary asymptotic stability we discuss, a bifurcation of which kind happens when the inequality ceases to be satisfied as a control parameter is varied.

2. Definitions

2.1. Stability

Denote by \( \Phi_t(x) \) a trajectory of system (1) starting at point \( x. \) For a set \( X \) and a number \( \epsilon > 0, \) an \( \epsilon \)-neighbourhood of \( X \) is the set of points satisfying

\[
B_\epsilon(X) = \{ x \in \mathbb{R}^n : d(x, X) < \epsilon \}.
\]
Let $X$ be a compact invariant set of (1). Denote by $B_\delta(X)$ its $\delta$-local basin of attraction defined as

$$B_\delta(X) = \{ x \in \mathbb{R}^n : d(\Phi_t(x), X) < \delta \text{ for any } t \geq 0 \text{ and } \lim_{t \to \infty} d(\Phi_t(x), X) = 0 \}.$$  \hspace{1cm} (3)

**Definition 1.** A compact invariant set $X$ is called asymptotically stable, if for any $\delta > 0$ there exists an $\epsilon > 0$ such that

$$B_\epsilon(X) \subset B_\delta(X).$$

**Definition 2.** We call a compact invariant set $X$ fragmentarily asymptotically stable, if for any $\delta > 0$

$$\mu(B_\delta(X)) > 0.$$

(Here $\mu$ is the Lebesgue measure of a set in $\mathbb{R}^n$.)

Evidently, if a set is asymptotically stable, then it is fragmentarily asymptotically stable.

**Definition 3.** A set $X$ is called completely unstable, if there exists $\delta > 0$ such that

$$\mu(B_\delta(X)) = 0.$$

Recall definitions of invariant sets which are not asymptotically stable, but are attractors in a weaker sense.

**Definition 4** ([3, 20]). A compact invariant set $X$ is called a weak attractor, if

$$\mu(B(X)) > 0$$

(here $B(X)$ denotes the basin of attraction of $X$). A compact invariant set $X$ is called a Milnor attractor, if it is a weak attractor and any proper compact invariant subset $Y \subset X$ satisfies

$$\mu(B(X) \setminus B(Y)) > 0.$$

As proved in [3], any weak attractor contains a Milnor attractor. Due to the inclusion $B_\delta(X) \subset B(X)$ which takes place for any $\delta > 0$, if a set is fragmentarily asymptotically stable, then it is a weak attractor. The converse implication is, in general, wrong; it does not hold when $B_\delta(X) > 0$ for some $\delta > \delta_0$ and $B_\delta(X) = 0$ for all $\delta < \delta_0$.

### 2.2. Heteroclinic cycles

In this paper, we consider dynamical systems that have a group of symmetries, which we denote by $\Gamma$. A dynamical system (1) is called $\Gamma$-equivariant, where $\Gamma \subset O(n)$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\Gamma$-equivariant vector field, i.e.

$$f(\gamma x) = \gamma f(x), \quad \text{for all } \gamma \in \Gamma.$$

We assume that the group $\Gamma$ is finite.

Let $\xi_1, \ldots, \xi_m$ be hyperbolic equilibria of (1) with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$, respectively, and $\kappa_j = W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset$, $j = 1, \ldots, m$, $\xi_{m+1} = \xi_1$, be a set of trajectories from $\xi_j$ to $\xi_{j+1}$.

**Definition 5.** A heteroclinic cycle is an invariant set $X \subset \mathbb{R}^n$ comprised of a set of equilibria $\{\xi_1, \ldots, \xi_m\}$ and a set of connecting orbits $\{\kappa_1, \ldots, \kappa_m\}$.

Recall that for a group $\Gamma$ acting on $\mathbb{R}^n$ the *isotropy group* of the point $x \in \mathbb{R}^n$ is the subgroup

$$\Sigma_x = \{ \gamma \in \Gamma : \gamma x = x \},$$

and a *fixed-point subspace* of a subgroup $\Sigma \subset \Gamma$ is the linear subspace

$$\text{Fix}(\Sigma) = \{ x \in \mathbb{R}^n : \sigma x = x \text{ for all } \sigma \in \Sigma \}.$$
Definition 6. A heteroclinic cycle is called structurally stable (or robust), if for any $j$, $1 \leq j \leq m$, there exists a fixed-point subspace $P_j = \text{Fix} (\Sigma_j)$, where $\Sigma_j \subset \Gamma$, such that

- $\xi_{j+1}$ is a sink in $P_j$;
- $\kappa_j \subset P_j$.

We denote $L_j = P_{j-1} \cap P_j$ and the isotropy subgroup of $L_j$ by $T_j$; evidently, $\xi_j \in L_j$.

2.3. Eigenspaces, simple cycles and type Z cycles

For a structurally stable heteroclinic cycle, eigenvalues of $d f(\xi_j)$ can be divided into four classes [16–18]:

- Eigenvalues with associated eigenvectors in $L_j$ are called radial.
- Eigenvalues with associated eigenvectors in $P_{j-1} \ominus L_j$ are called contracting.
- Eigenvalues with associated eigenvectors in $P_j \ominus L_j$ are called expanding.
- Eigenvalues not belonging to any of the three above classes are called transverse.

Definition 7 (adapted from [18]). We call a robust heteroclinic cycle $X \in \mathbb{R}^n \setminus \{0\}$ simple, if for any $j$

- all eigenvalues of $d f(\xi_j)$ are distinct;
- $\dim(P_{j-1} \ominus L_j) = 1$.

Note that definitions of simple heteroclinic cycles different from the one in [18] can be found in the literature. According to Hofbauer and Sigmund [12], a heteroclinic cycle is simple, if for each $j$ the linearization $d f(\xi_j)$ has only one expanding eigenvector and all transverse eigenvalues are negative. Field [10] calls a heteroclinic cycle simple, if all transverse eigenvalues are negative and the cycle is a compact set.

Denote by $P_j^\perp$ the orthogonal complement to $P_j$ in $\mathbb{R}^n$.

Definition 8. We call a simple robust heteroclinic cycle $X$ to be of type Z, if for any $j$

- $\dim P_j = \dim P_{j+1}$;
- the isotropy subgroup of $P_j$, $\Sigma_j$, decomposes $P_j^\perp$ into one-dimensional isotypic components.

The letter Z in the name of the cycle is chosen as the ‘opposite’ one to A: consider four eigenspaces that are associated with the dominant (i.e. having the largest real part) contracting eigenvalue of $d f(\xi_j)$, the weakest (having the smallest real part) transverse eigenvalue of $d f(\xi_j)$, the dominant expanding eigenvalue of $d f(\xi_{j+1})$ and the weakest transverse eigenvalue of $d f(\xi_{j+1})$. A heteroclinic cycle is called to be of type A, if all the four eigenspaces belong to the same $\Sigma_j$-isotypic component [16, 21]. This isotypic component of $P_j^\perp$ is therefore at least two-dimensional. For type Z cycles, by contrast, all isotypic components of $P_j^\perp$ are required to be one-dimensional.

The condition $\dim(P_{j-1} \ominus L_j) = 1$ implies that for any $j$ the contracting eigenspace at $\xi_j$ is one-dimensional. Together with the condition $\dim P_j = \dim P_{j+1}$, this implies that the dimension of the expanding eigenspace is also one. Denote by $n_r$ the number of radial eigenvalues and by $n_t$ the number of transverse eigenvalues; for a type Z heteroclinic cycle $n_r$ and $n_t$ are the same for all equilibria, and $n = n_r + n_t + 2$. The radial eigenvalues and the associated eigenvectors near $\xi_j$ are denoted by $-c_j$ and $v_j$, the expanding ones by $e_j$ and $v_j^e$, and the transverse ones by $t_j = \{t_{j,l}\}$ and $u_j = \{u_{j,l}\}$, $1 \leq l \leq n_r$, respectively.
The basis in $P_j^\perp$ can be chosen to be comprised of contracting and transverse eigenvectors at $\xi_j$, or of expanding and transverse eigenvectors at $\xi_{j+1}$. For type $Z$ cycles all isotypic components of $P_j^\perp$ are one-dimensional. Since any eigenvector belongs to an isotypic component, the basis $\{v_{j+1}^\sigma, v_j^\sigma\}$ is a permutation of the basis $\{v_j^\sigma, v_{j+1}^\sigma\}$ to components of the vector in the basis $\{v_{j+1}^\sigma, v_j^\sigma\}$ is a product $A_j^\pm A_j$, where $A_j$ is a permutation matrix and $A_j^\pm$ is a diagonal matrix with elements $+1$ and $-1$ on the diagonal.

In fact, if a dynamical system has a heteroclinic cycle of type $Z$, then the system possesses a variety of symmetry-invariant subspaces in addition to the subspaces $P_j$ required in definition 7. Existence of these subspaces follows from the following lemma.

**Lemma 1.** Let a group $\Sigma$ act on a linear space $V$. Consider the isotypic decomposition of the linear space under the action of $\Sigma$:

$$V = U_0 \oplus U_1 \oplus \cdots \oplus U_K.$$

Suppose

- the action of $\Sigma$ on $U_0$ is trivial;
- any $\sigma \in \Sigma$ acts on a $U_k$, $1 \leq k \leq K$, either as $I$ or as $-I$.

Then for any collection of indices $1 \leq i_1, \ldots, i_l \leq K$ there exists a subgroup $G_{i_1, \ldots, i_l} \subset \Sigma$ such that the subspace

$$V_{i_1, \ldots, i_l} = U_0 \oplus U_{i_1} \oplus \cdots \oplus U_{i_l}$$

is a fixed-point subspace of the group $G_{i_1, \ldots, i_l}$.

**Proof.** By definition of the isotypic decomposition, the conditions of the lemma imply that for any $1 \leq k \leq K$ we can find an element $\gamma_k \in \Sigma$ such that $\gamma_k U_k = -U_k$. For any $0 \leq k \leq K$ and $1 \leq s \leq K$ we can also find $\gamma_s \in \Sigma$ such that

$$\gamma_k U_k = U_k \quad \text{and} \quad \gamma_s U_s = -U_s.$$

Hence for any $1 \leq k \leq K$ there exists $\sigma_k \in \Sigma$ such that

$$\sigma_k U_k = -U_k \quad \text{and} \quad \sigma_k U_s = U_s \quad \text{for any} \ s \neq k.$$

(This can be proved by induction in $K$. The proof is omitted.) Evidently, the subspace $V_{i_1, \ldots, i_l}$ is a fixed-point subspace of the subgroup of $\Sigma$ generated by all $\sigma_k$ with $k \neq i_1, \ldots, i_l$. \textbf{QED}

Definition 7 requires a $\Sigma_j$-invariant map to have no multiple eigenvalues; consequently, all $\Sigma_j$ satisfy the conditions of the lemma. Alternatively, by definition of a type $Z$ cycle all isotypic components of $P_j^\perp$ are one-dimensional, hence elements of $\Sigma_j$ act on these components either as $I$ or $-I$.

Following [14, 18, 21], in order to examine stability we construct a Poincaré map in the vicinity of the cycle.
2.4. Collection of maps associated with a heteroclinic cycle

In section 2.3 we have given definitions for radial, contracting, expanding and transverse eigenvalues of the linearization $df(\xi_j)$. Let $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ be local coordinates near $\xi_j$ in the basis, where radial eigenvectors come the first (the respective coordinates are $\tilde{u}$), followed by the contracting and the expanding eigenvectors, the transverse eigenvectors being the last. Suppose $\delta$ is small. In a $2\delta$-neighbourhood of $\xi_j$, $B_{2\delta}(\xi_j)$, defined as
\[ B_{2\delta}(\xi_j) = \{(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z}) : \max(|\tilde{u}|, |\tilde{v}|, |\tilde{w}|, |\tilde{z}|) < 2\delta \}, \]

system (1) can be approximated by the linear system$^1$
\begin{align*}
\dot{u}_l &= -r_{j,l} u_l, \quad 1 \leq l \leq n_r, \\
\dot{v} &= -c_j v, \\
\dot{w} &= e_j w, \\
\dot{z}_s &= t_j,s z_s, \quad 1 \leq s \leq n_t.
\end{align*}

We denote by $(u, v, w, z)$ the scaled coordinates $(u, v, w, z) = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})/\delta$.

Consider a neighbourhood of a steady state $\xi_j$. Let $(u_0, v_0)$ be the point in $P_{j-1}$ where trajectory $\kappa_{j-1}$ intersects with the sphere $|u|^2 + v^2 = 1$, and $q$ be local coordinates in the hyperplane tangent to the sphere at the point $(u_0, v_0)$. Coordinates $(u, v)$ of a point in the hyperplane are related to coordinates $q$ as follows:
\begin{align*}
\begin{pmatrix} u \\ v \end{pmatrix} &= D_j^q q = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + D_j^q q,
\end{align*}

where $D_j^q$ is a $n_r \times (n_r + 1)$ matrix. Some components of $u_0$ can vanish, if e.g. $\kappa_{j-1}$ belongs to an invariant subspace in $P_{j-1}$. (Note that $P_j$ is not required to be the smallest possible subspace.) $v_0$ does not vanish, because it is the component in the contracting direction.

Near $\xi_j$ we define two cross-sections of the heteroclinic cycle. One, denoted by $H_j^{(in)}$, is an $(n - 1)$-dimensional hyperplane intersecting connection $\kappa_{j-1}$ at the point $(u_0, v_0, 0, 0)$; coordinates in the hyperplane are $(q, w, z)$. Another one, $H_j^{(out)}$, is parallel to the hyperplane $w = 0$ and intersects connection $\kappa_j$ at the point $w = 1$; coordinates in the hyperplane are $(u, v, z)$. Near $\xi_j$ trajectories of system (1) can be approximated by a local map (called the first return map)$^2$ $\phi_j : H_j^{(in)} \rightarrow H_j^{(out)}$ relating a point, where a trajectory enters the neighbourhood, to the point, where it exits. In the leading order (see (4) and (5)), the local map is
\begin{align*}
\phi_j((q_0, w, \{z_s\})) &= (\left\{ (u_l + \sum_{s=1}^{n_t} D_j^{q_0} q_s w^s / f_j^s, v_0 w^s / f_j^s, \{z_s w^{-t_j,s / f_j^s}\} \right\}).
\end{align*}

The map can be expressed as a superposition $\phi_j = C_j^{tot} D_j^{tot}$, where $C_j^{tot} : R^n \rightarrow R^{n-1}$ and $D_j^{tot} : R^{n-1} \rightarrow R^n$. The action of the map $D_j^{tot}$ on the $q$ coordinates is presented by (5), and

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$^1$ Below we assume that all eigenvalues are real. Definition 8 implies that for type Z cycles transverse, contracting and expanding eigenvalues are real. Radial eigenvalues can be complex, but this does not change our proof significantly.

$^2$ If some components $t_j,s$ of $t_j$ are positive, then the local map is defined for $z_s$ satisfying the inequality $|z_s| < K(1 - \delta)|u|^{1 / f_j}$, where $K$ is a constant and $\delta$ is small (see [14, 21]). However, this restriction is not important, because in order to study stability of a cycle we study stability of a fixed point of a collection of maps $R^N \rightarrow R^N$, and the maps are defined for all $x \in R^N$. Moreover, the local map is defined only for particular signs of $v$ and $w$. To overcome this complication, we consider group orbits of heteroclinic cycles, see section 2.5.
its action on the $w$ and $z$ coordinates is trivial. The map $C_j^{\text{tot}}$ is

\[
C_j^{\text{tot}} \begin{pmatrix} u_t \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} u_t, u_t^j, u_t^j, u_t^j \\ v, v, v, v \\ w^j, w^j, w^j, w^j \\ z, z, z, z \end{pmatrix}.
\]

(7)

Near connection $\kappa_j$ system (1) can be approximated by a global map (also called a connecting diffeomorphism) $\psi_j : H_j^{\text{out}} \rightarrow H_j^{\text{in}}$,

\[
\begin{pmatrix} q^{j+1} \\ w^{j+1} \\ z^{j+1} \end{pmatrix} := \psi_j \begin{pmatrix} u^j \\ v^j \\ w^j \\ z^j \end{pmatrix} = A_j^{\text{tot}} B_j^{\text{tot}} \begin{pmatrix} u^j \\ v^j \\ w^j \\ z^j \end{pmatrix},
\]

(8)

where superscripts in the notation of components indicate, whether the respective vector is decomposed in the local basis near $\xi_j$ or near $\xi_{j+1}$. The $(n-1) \times (n-1)$ matrix $B_j^{\text{tot}}$ presents the map $\psi_j$ in the local coordinates near $\xi_j$ (i.e. the basis near $\xi_{j+1}$ is the same, as near $\xi_j$, and the origin is shifted to $\xi_{j+1}$), and matrix $A_j^{\text{tot}}$ relates the coordinates in the two local bases. Each matrix is comprised of two diagonal blocks (the respective non-diagonal blocks vanish). The first $n_t \times n_t$ blocks, $B_j^1$ and $A_j^1$, approximate the maps acting in $P_j$, and the second blocks, $B_j$ and $A_j$, the maps acting in $P_j$. Lemma 1 implies that the matrix $B_j$ is diagonal. As discussed in section 2.3, $A_j = A_j^\pm A_j$.

Denote the superpositions of the local, $\phi_j$, and global, $\psi_j$, maps by $\tilde{g}_j := \psi_j \circ \phi_j : H_j^{\text{in}} \rightarrow \tilde{H}_j^{\text{in}}$. The Poincaré map $H_j^{\text{in}} \rightarrow H_j^{\text{out}}$ for the cycle is the superposition $\tilde{\pi}_1 = \tilde{g}_m \circ \cdots \circ \tilde{g}_1$: for $j > 1$ the Poincaré maps $H_j^{\text{in}} \rightarrow H_j^{\text{in}}$ are constructed similarly:

\[
\tilde{\pi}_j = \tilde{g}_j \circ \cdots \circ \tilde{g}_1 \circ \tilde{g}_n \circ \cdots \circ \tilde{g}_1.
\]

The coordinates $(w, z)$ used to define the maps $\tilde{g}_j$ are independent of $q$. Hence, we can define maps $g_j$ which are restrictions of the maps $\tilde{g}_j$ into the $(w, z)$-subspace:

\[
g_j(w, z) = A_j^\pm A_j B_j \begin{pmatrix} v, v, v, v \\ w^j, w^j, w^j, w^j \\ z, z, z, z \end{pmatrix}.
\]

(9)

We call the set of maps $\{g_j^m\} = \{g_1, \ldots, g_m\}$, where $g_j : \mathbb{R}^{n_t+1} \rightarrow \mathbb{R}^{n_t+1}$ have been constructed above, a collection of maps associated with the heteroclinic cycle $\{\xi_1, \ldots, \xi_m\}$. The collection of maps $\{g^{-1}_j\} = \{g_1^{-1}, \ldots, g_m^{-1}, g_{m+1}, \ldots, g_{l-1}\}$ is associated with the heteroclinic cycle $\{\xi_1, \ldots, \xi_m, \xi_1, \ldots, \xi_{l-1}\}$ which geometrically coincides with the former cycle.

2.5. Comments on definitions of a heteroclinic cycle

Note that a heteroclinic cycle satisfying the commonly used definition 5 is never asymptotically stable in the sense of definition 1 for the following reasons. Denote by $\gamma^{\text{out}}$ a symmetry which belongs to $T_j$ but does not belong to $\Sigma_j$. The symmetry fixes $\xi_j$, reverses the sign of $w$, maps $\tilde{H}_j^{\text{in}}$ into itself, and maps the connection $\kappa_j$ into a different connection $\gamma^{\text{out}} \kappa_j$ exiting a neighbourhood of $\xi_j$ via $\gamma^{\text{out}} \tilde{H}_j^{\text{out}}$ (which differs from $\tilde{H}_j^{\text{out}}$ by the sign of $w$). Hence, for any small $\epsilon$ we can find points in $B(X) \cap \tilde{H}_j^{\text{in}}$ such that trajectories starting there do not follow the connection $\kappa_j$. Similarly, the sign of $v$ is the same throughout $\tilde{H}_j^{\text{in}}$, and the symmetry $\gamma^{\text{in}} \in T_j$ such that $\gamma^{\text{in}} \not\in \Sigma_{j-1}$ maps $\tilde{H}_j^{\text{in}}$ into a different cross-section with the opposite sign of $v$ and the connection $\kappa_{j-1}$ into $\gamma^{\text{in}} \kappa_{j-1}$. Hence, the map $\phi_j : \tilde{H}_j^{\text{in}} \rightarrow \tilde{H}_j^{\text{out}}$
is defined for points with certain fixed sign of \( w \), and its image contains points with a particular fixed sign of \( v \).

For this reason, a heteroclinic cycle is often defined as (a connected component of) an orbit under the action of the group of symmetries \( \Gamma \) of a heteroclinic cycle satisfying definition 5. For a group orbit, the maps \( \Phi_j \) are defined for all signs of \( w \) and \( v \), because in addition to \( k \) and \( \kappa_j \) the orbit involves \( \gamma^{(in)} \kappa_{j-1} \) and \( \gamma^{(out)} \kappa_j \) as well. A group orbit can be asymptotically stable according to definition 1. Below, when speaking about a heteroclinic cycle we always assume a group orbit.

Consider a sample trajectory close to a heteroclinic cycle. After the trajectory passes near \( \xi_j \), it can head for \( \xi_{j+1} \) following connection \( \kappa_j \), or for \( \gamma^{(out)} \xi_{j+1} \) following connection \( \gamma^{(out)} \). The choice depends on the sign of \( w \). Hence, different trajectories can visit different sets of equilibria. Suppose that all subspaces \( P_j \) are maximal, i.e. there does not exist an invariant proper subspace of \( P_j \), \( \hat{P}_j \) such that \( \kappa_j \subset \hat{P}_j \). Then generically none of the \( (u_0, v_0) \) components vanish. The signs of components of \( z \) are preserved by the local maps. Any global map preserves the sign of each component of \( z \) for all trajectories, or reverses it for all trajectories. Hence, a particular path along the cycle followed by the trajectory (i.e. the sequence of equilibria visited by the trajectory) \( \Phi_t(x) \) for \( x \in \tilde{H}^{(in)}_j \) is uniquely determined by the signs of coordinates of \( x \) (i.e. all points in each orthant of \( \mathbb{R}^{n-1} \) of \( \tilde{H}^{(in)}_j \) follow the same path).

When definition 5 of a heteroclinic cycle is used, often a different object is tacitly assumed. Suppose there exists a symmetry \( \gamma \in \Gamma \) such that the heteroclinic cycle \( \{ \xi_1, \ldots, \xi_m \} \) can be generated from its smaller subcycle \( \{ \xi_1, \ldots, \xi_l \} \), \( m = lK \), by applying the symmetry \( \gamma \):

\[
\gamma \xi_{s+(k-1)l} = \xi_{s+kl} \quad \text{for all } 1 \leq s \leq l, 1 \leq k \leq K.
\]

Then the subcycle \( \{ \xi_1, \ldots, \xi_l \} \) is called sometimes a heteroclinic cycle referring to its group orbit. If \( l = 1 \), then the heteroclinic cycle is called a homoclinic cycle and the symmetry \( \gamma \) is called a twist [2, 25].

### 2.6. Collection of maps: definitions of stability

For a collection of maps \( \{ g^m_i \} = \{ g_1, \ldots, g_m \} \) we define superpositions

\[
\pi_j = g_{j-1} \circ \cdots \circ g_1 \circ g_m \circ \cdots \circ g_{j+1} \circ g_j
\]

(for \( j = 1 \) and \( j = 2 \) this reduces to \( \pi_1 = g_m \circ \cdots \circ g_2 \circ g_1 \) and \( \pi_2 = g_1 \circ g_m \circ \cdots \circ g_2 \), respectively) and

\[
g_{l,j} = \begin{cases} g_{l} \circ \cdots \circ g_{j}, & l > j, \\ g_{l} \circ \cdots \circ g_{m} \circ g_{1} \circ \cdots \circ g_{j}, & l < j. \end{cases}
\]

Given a collection of maps \( \{ g^m_i \} = \{ g_1, \ldots, g_m \} \), \( g_j : \mathbb{R}^N \to \mathbb{R}^N \), we define a discrete dynamical system by the relation \( y_{n+1} = \pi_1 y_n \). We call \( y^1 \in \mathbb{R}^N \) a fixed point of the collection of maps \( \{ g^m_i \} \), if \( \pi_1 y^1 = y^1 \). Evidently, \( y^1 = g_{0,1} y^1 \) is then a fixed point of the collection of maps \( \{ g^m_i \} \).

**Definition 9.** We say that a fixed point \( y^1 \in \mathbb{R}^N \) of a collection of maps

\[
\{ g^m_i \} = \{ g_1, \ldots, g_m \}, \quad g_j : \mathbb{R}^N \to \mathbb{R}^N,
\]

is asymptotically stable, if for any \( \delta > 0 \) there exists an \( \epsilon > 0 \) such that for any \( 1 \leq l \leq m \)

\[
d(x, y^1) < \epsilon, \quad \text{where } y^1 = g_{l-1,1} y^1,
\]

implies

\[
d(\pi_j^k g_{l,j} x, g_{l,j} y^1) < \delta \quad \text{for all } 1 \leq j \leq m, \quad k \geq 0.
\]
and
\[ \lim_{k \to \infty} d(\pi_j^k g(j-1,i)x, g(j-1,i)y^l) = 0 \quad \text{for all } 1 \leq j \leq m. \]

**Definition 10.** We say that a fixed point \( y^l \in \mathbb{R}^N \) of a collection of maps \( \{g^n_l\} \) is fragmentarily asymptotically stable, if for any \( \delta > 0 \)
\[ \mu(B((g^n_l), y^l)) > 0, \]
where
\[ B((g^n_l), y^l) := \{x : x \in \mathbb{R}^N, d(\pi_j^k g(j-1,i)x, g(j-1,i)y^l) < \delta \text{ for all } 1 \leq j \leq m, k \geq 0 \}\]
and \[ \lim_{k \to \infty} d(\pi_j^k g(j-1,i)x, g(j-1,i)y^l) = 0 \text{ for all } 1 \leq j \leq m. \]

**Definition 11.** We say that a fixed point \( y^l \in \mathbb{R}^N \) of a collection of maps \( \{g^n_l\} \) is completely unstable, if there exists \( \delta > 0 \) such that
\[ \mu(B((g^n_l), y^l)) = 0. \]

### 3. Stability of a cycle and a collection of maps

In this section we prove two theorems relating the asymptotic stability of a type Z heteroclinic cycle with the asymptotic stability of the fixed point \((w, z) = 0\) of the collection of maps associated with the cycle. (The point \((w, z) = 0\) is evidently a fixed point of the collection of maps constructed in section 2.4.)

**Theorem 1.** Let \( \{g^n_l\}, g_j : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \), be the collection of maps associated with a heteroclinic cycle of type Z. The cycle is asymptotically stable, if and only if the fixed point \((w, z) = 0\) of the collection of maps is asymptotically stable.

**Proof.** A necessary condition for asymptotic stability of both the cycle and the collection of maps is that all transverse eigenvalues are negative. We assume henceforth in this proof that this is the case.

For a trajectory \( \Phi_t(x) \) belonging to \( B_t(X) \) for all \( t > 0 \), denote by \( \Phi_{j,k}^{(in)}(x) \) the \( k \)th intersection of \( \Phi_t(x) \) with \( \tilde{H}_j^{(in)} \), by \( \Phi_{j,k}^{(out)}(x) \) the \( k \)th intersection of \( \Phi_t(x) \) with \( \tilde{H}_j^{(out)} \), and by \( t_{j,k}^{(in)} \) and \( t_{j,k}^{(out)} \) the times of occurrence of the intersections.

For a sufficiently small \( \delta > 0 \) the collection of maps \( \{\tilde{g}^n_l\} \), where \( \tilde{g}_j : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \), accurately approximates trajectories in the vicinity of the cycle, i.e. if \( x \in \tilde{H}_t^{(in)} \) and \( \Phi_t(x) \in B_t(X) \) for all \( t > 0 \), then
\[ \Phi_{j,k}^{(in)}(x) \simeq \tilde{\pi}_j^{k-1} \tilde{g}_{j-1,i} x. \quad (13) \]

Stability of the cycle implies stability of the fixed point \( 0 \) of the collection of maps, because

- if \( \Phi_t(x) \in B_t(X) \) for any \( x \) satisfying \( d(x, X) < \epsilon \), then
  \[ d(\Phi_{j,k}^{(in)}(x), X) < \delta \quad \text{for any } j, k \text{ and } x \in \tilde{H}_t^{(in)}, |x| < \epsilon, \]
  and hence, by \( (13) \), \[ |\pi_j^k \tilde{g}_{j-1,i} x| < |\tilde{\pi}_j^{k-1} \tilde{g}_{j-1,i} x| < \delta; \]
- if \( \lim_{k \to \infty} d(\Phi_t(x), X) = 0 \) for any \( x \) satisfying \( d(x, X) < \epsilon \), then
  \[ \lim_{k \to \infty} d(\Phi_{j,k}^{(in)}(x), X) = 0 \quad \text{for any } j, k \text{ and } x \in \tilde{H}_t^{(in)}, |x| < \epsilon, \]
  and hence, by \( (13) \),
  \[ \lim_{k \to \infty} |\pi_j^k \tilde{g}_{j-1,i} x| < \lim_{k \to \infty} |\tilde{\pi}_j^{k-1} \tilde{g}_{j-1,i} x| = 0. \]

Therefore, by definition 9, \( 0 \) is an asymptotically stable fixed point of the collection of maps.
To prove that asymptotic stability of the cycle follows from asymptotic stability of the fixed point \(0\) of the collection of maps, we first show that asymptotic stability of the fixed point \((w, z) = 0\) of the collection \(\{g^n\}\) implies asymptotic stability of the fixed point \((q, w, z) = 0\) of the collection \(\{\tilde{g}^n\}\). Denote by \(\tilde{g}_j\) the map \(\tilde{g}_j\), restricted to the subspace \((w = 0, z = 0)\) which is equipped with the \(q\) coordinates. By virtue of (5)–(8) and (14)–(16), the map takes the form

\[
g^n_j (q, w, z) = A^n_j B^n_j C^n_j D^n_j \left( \begin{array}{c} q \\ w \\ z \end{array} \right),
\]

where

\[
C^n_j \left( \begin{array}{ccc} u \\ v \\ w \end{array} \right) = \left( \begin{array}{ccc} u \\ v \\ w \end{array} \right) = \left( |u_1 u_2 | \right).
\]

Let \(K^n_j = n_r \max_{i,s} |A^n_j|, K^n_B = n_r \max_{i,s} |B^n_j|, K^n_D = n_r \max(|u_0|, 1, |D^n_j|)\) and \(r^n_j = \min_{i,j} (r_{i,j}/e_j)\). For a given \(\delta > 0\), choose \(\delta_j\) satisfying

\[
K^n_j K^n_B K^n_D r^n_j < \delta/2.
\]

Since \(0\) is a stable fixed point of the collection \(\{g^n\}\), we can find \(\epsilon > 0\) such that

\[
|\pi^n_{i,j} \tilde{g}_{j-1,i} x| < \min(\delta/2, \min \delta_j) \quad \text{for all} \quad 1 \leq i, j \leq m, \quad k \geq 0, \quad |x| < \epsilon.
\]

If (17) holds true, then

\[
|\pi^n_{i,j} \tilde{g}_{j-1,i} x| < \delta \quad \text{for all} \quad 1 \leq i, \quad l \leq m, \quad k \geq 0, \quad |x| < \epsilon
\]

by virtue of (5)–(8) and (14)–(16). The proof that \(\lim_{k \to \infty} \pi^n_{i,j} \tilde{g}_{j-1,i} x = 0\) implies \(\lim_{k \to \infty} \tilde{g}_{j}^{i} \tilde{g}_{j-1,i} x = 0\) is similar and we omit it.

Second, we prove that if the fixed point \(0\) of the collection \(\{\tilde{g}^m\}\) is asymptotically stable, then for any \(\delta > 0\) we can find \(\tilde{\epsilon} > 0\) such that for any \(x \in \tilde{H}^{(m)}\) and \(|x| < \tilde{\epsilon}\) the trajectories \(\Phi_t(x)\) satisfy

\[
d(\Phi_t(x), X) < \delta \quad \text{for any} \quad t \geq 0.
\]

In a sufficiently small neighbourhood of the origin at the hyperplane \(\tilde{H}^{(m)}\), the map \(\psi_j : \tilde{H}^{(m)}_{j-1} \to \tilde{H}^{(m)}_j\) is predominantly linear. For any intermediate cross-section \(\tilde{H}^{(m)}_j\) between \(\tilde{H}^{(m)}_{j-1}\) and \(\tilde{H}^{(m)}_j\), the induced map \(\tilde{H}^{(m)}_j \to \tilde{H}^{(m)}_j\) is also predominantly linear. Hence we can find a positive \(K^{(m)}_j\) such that

\[
d(\Phi_t(x), X) < K^{(m)}_j d(\Phi_t(x), X) \quad \text{for any} \quad t \geq 0.
\]

Now consider the trajectory \(\Phi_t(x)\) at the time interval \(t_{j,k}^{(m)} < t < t_{j+1,k}^{(m)}\). Near \(\xi_j\), we project the cycle \(X\) and the trajectory \(\Phi_t(x)\) onto the plane \((v, w)\) and onto the orthogonal hyperplane \((u, z)\). By \(d^{(v,w)}(\cdot, \cdot)\) and \(d^{(u,z)}(\cdot, \cdot)\) we denote the distances between the projections onto the \((v, w)\) plane and onto the \((u, z)\) hyperplane, respectively. Simple algebra (not presented here) attests that

\[
d^{(v,w)}(\Phi_t(x), X) < (d(\Phi_t(x), X) \beta_j/\theta_j)^{1/\theta_j},
\]

where \(\beta_j = e_j/\theta_j\). The estimate

\[
d^{(u,z)}(\Phi_t(x), X) < d(\Phi_t(x), X) + d(\Phi_t(x), X)
\]
follows from (4). We denote
\[ K^{(\text{glob})} = \max_j (K_j^{(\text{glob})}), \quad \beta = \min_j (\beta_j). \]
For a given \( \delta > 0 \), let \( \delta_1 > 0 \) satisfy
\[ K^{(\text{glob})} \delta_1^{\beta/(1+\beta)} < \delta/2, \quad (K^{(\text{glob})} + 1) \delta_1 < \delta/2. \]
Since \( \emptyset \) is asymptotically stable, we can find \( \tilde{\epsilon} > 0 \) such that \( d(\Phi_i^{(\text{in})}(x), X) < \delta_1 \) for all \( j \) and \( k \), provided \( x \in \tilde{H}_j^{(\text{in})} \) for some \( l \) and \( d(x, X) < \tilde{\epsilon} \). The estimates presented above imply that \( d(\Phi_i(x), X) < \delta \) for all \( t > 0 \).

Finally, we consider \( x \notin H_l^{(\text{in})} \) for all \( l \). Denote by \( \tilde{x} \) the first intersection of \( \Phi_i(x) \) with some \( H_l^{(\text{in})} \). By the arguments similar to those presented above, at least one of the estimates \( d(\tilde{x}, X) < K d(x, X) \) holds true. Each of the two inequalities imply that, for \( \tilde{\epsilon} \) defined in the previous paragraph, we can find \( \epsilon > 0 \) such that \( d(x, X) < \epsilon \) implies \( d(\tilde{x}, X) < \tilde{\epsilon} \). Hence for \( d(x, X) < \epsilon \) the estimate \( d(\Phi_i(x), X) < \delta \) holds true for all \( t > 0 \).

The proof that \( \lim_{t \to \infty} d(\Phi_i(x), X) = 0 \) follows from \( \lim_{k \to \infty} \tilde{x}^{k} (j_{j-1}, j_{j}) x = 0 \) is similar, and we omit it.

**QED**

**Lemma 2.** Let \( X \) be a type Z heteroclinic cycle. If \( X \) is fragmentarily asymptotically stable, then for any \( \delta > 0 \) and any \( j \)
\[ \mu^{n-1} (\tilde{H}_j^{(\text{in})} \cap B_0(X)) > 0, \]
where \( \mu^{n-1} \) is the Lebesgue measure in \( \mathbb{R}^{n-1} \).

**Proof.** Denote \( Q_j = \tilde{H}_j^{(\text{in})} \cap B_0(X) \). Suppose (19) is not satisfied for some \( j \), i.e.
\[ \mu^{n-1}(Q_j) = 0. \]
(20)

The set \( B_0(X) \) can be regarded as the union of segments of trajectories going from \( Q_j \) in the direction of negative \( t \) till an intersection either with \( \tilde{H}_j^{(\text{in})} \) or with the boundary of \( B_0(X) \) occurs:
\[ B_0 = \{ \Phi_i(x) : x \in Q_j, t_{\text{final}} < t \leq 0 \text{ where } \Phi_i^{(\text{out})}(x) \in \tilde{H}_j^{(\text{in})} \text{ or } d(\Phi_i^{(\text{out})}(x), X) = \delta \}. \]

Denote by \( B_{j, \delta}^{(\text{glob})} \) the part of \( B_0(X) \) bounded by \( \tilde{H}_j^{(\text{out})} \) and \( \tilde{H}_{j+1}^{(\text{in})} \), and by \( B_{j, \delta}^{(\text{loc})} \) the part bounded by \( \tilde{H}_j^{(\text{in})} \) and \( \tilde{H}_{j+1}^{(\text{out})} \). By linearity of the global map, (20) implies \( \mu(B_{j, \delta}^{(\text{glob})}(X)) = 0 \) and
\[ \mu^{n-1}(\tilde{H}_j^{(\text{out})} \cap B_0(X)) = 0. \]
(21)

Since the local map satisfies (4) and due to (21),
\[ \mu(B_{j-1, \delta}^{(\text{loc})}(X)) = 0 \quad \text{and} \quad \mu^{n-1}(\tilde{H}_j^{(\text{in})} \cap B_0(X)) = 0. \]

The same arguments applied \( m - 1 \) times to the sets \( Q_{j-1} = \tilde{H}_{j-1}^{(\text{in})} \cap B_0(X), Q_{j-2}, \ldots, \) imply that \( \mu(B_0(X)) = 0 \), in contradiction with the statement of the lemma. Therefore assumption (20) is false, i.e. \( \mu^{n-1}(Q_j) > 0 \) for all \( j \).

**QED**
Theorem 2. Let \([g^n]\), \(g_j : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}\), be the collection of maps associated with a type \(Z\) heteroclinic cycle. The cycle is fragmentarily asymptotically stable, if and only if the fixed point \((w, z) = 0\) of the collection of maps is fragmentarily asymptotically stable.

Proof. By lemma 2, for any \(j\) (19) holds true. Therefore the measure \(\mu^{n+1}\) (in \(\mathbb{R}^{n+1}\)) of the orthogonal projection of the set \(\tilde{H}^{(m)}_j \cap B_\delta(X)\) into the plane \(q = 0\) is positive. For a small \(\delta\) the collection of maps gives accurate predictions for trajectories \(\Phi_t(x), \Phi_t(x) \subset B_\delta(X)\) for \(t > 0\), and the coordinates \(w\) and \(z\) are independent of \(q\). Hence \(0\) is a fragmentarily asymptotically stable fixed point of the collection of maps.

The proof, that fragmentary asymptotic stability of \(0\) of the collection of maps \([g^n]\) implies fragmentary asymptotic stability of \(0\) of the collection of maps \([\tilde{g}^m]\), is similar to the one in the proof of theorem 1 and is omitted.

Let the constants \(K^{(glob)}, \tilde{K}\) and \(\tilde{\beta}\) be defined as in the proof of theorem 1. Same arguments as employed in this proof imply that the inequalities

\[
d(\Phi_j(x), X) < K^{(glob)}d(\Phi_j^{(in)}(x), X) \text{ for any } j, k \text{ and } t, t_{j,k}^\text{in} < t < t_{j,k}^\text{out}
\]

if \(d(\Phi_j^{(in)}(x), X) < \delta\);

\[
d^{(v,u)}(\Phi_j(x), X) < (d(\Phi_j^{(in)}(x), X))^{\beta/(1+\beta)} \text{ for any } j, k \text{ and } t, t_{j,k}^\text{in} < t < t_{j,k}^\text{out}
\]

if \(d(\Phi_j^{(in)}(x), X) < \delta\);

\[
d^{(u,z)}(\Phi_j(x), X) < d(\Phi_j^{(in)}(x), X) + d(\Phi_j^{(out)}(x), X) \text{ for any } j, k \text{ and } t, t_{j,k}^\text{in} < t < t_{j,k}^\text{out},
\]

if \(d(\Phi_j^{(in)}(x), X) < \delta\) and \(d(\Phi_j^{(out)}(x), X) < \delta\)

hold true for a sufficiently small \(\delta > 0\).

Defining \(\tilde{\delta}_1\) as in the proof of theorem 1, we find that \(d(\Phi_j(x), X) < \delta\) for all \(t > 0\) provided \(x \in \tilde{H}^{(in)}_j\) satisfies \(x \in B_{\tilde{\delta}_1}([\tilde{g}^m], 0)\). The proof, that \(x \in B_{\tilde{\delta}_1}([\tilde{g}^m], 0)\) implies \(\lim_{t \to \infty} d(\Phi_j(x), X) = 0\), is similar.

Let \(\tilde{H}^{(in)}_j\) and \(\tilde{H}^{(out)}_j\) be two hyperplanes parallel to \(\tilde{H}^{(in)}_1\) and located at distance \(s\) from this hyperplane. Consider the set \(\tilde{Q}_1\) comprised of pieces of trajectories contained between the hyperplanes \(\tilde{H}^{(in)}_1\) and \(\tilde{H}^{(out)}_1\), whose points of intersection with the hyperplane \(\tilde{H}^{(in)}_1\) constitute the set \(\tilde{Q}_1 := B_{\delta_1}([\tilde{g}^m], 0)\). For small \(s\), \(\tilde{Q}_1 \subset B_\delta(X)\) and \(\mu(\tilde{Q}_1) = 2s\mu^{n-1}(Q_j) > 0\). Thus the set \(X\) is fragmentarily asymptotically stable.

QED

4. Stability of fixed points of a collection of maps

4.1. Transition matrix

Denote by \(M_j\) the maps \(g_j\) in the new coordinates\(^3\) \(\eta\), where

\[
\eta = (\ln |w|, \ln |z_1|, \ldots, \ln |z_n|).
\]

As discussed in section 2.4, the maps are linear, their structure being

\[
M_j \eta = M_j \eta + F_j,
\]

\(^3\) Here we ignore the matrix \(A_k\) which is irrelevant in the study of stability. It becomes important in the study of bifurcations—\(A_k\) determines the length of periodic orbit(s) bifurcating from a heteroclinic cycle (see section 5.2).
Stability and bifurcations of heteroclinic cycles of type Z

where

\[ M_j := A_j B_j = A_j \begin{pmatrix} b_{j,1} & 0 & 0 & \ldots & 0 \\ b_{j,2} & 1 & 0 & \ldots & 0 \\ b_{j,3} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{j,N} & 0 & 0 & \ldots & 0 \end{pmatrix} \] (24)

are the basic transition matrices of the maps. Here \( A_j \) and \( B_j \) are \( N \times N \) matrices, \( N = n_t + 1 \), \( A_j \) is a permutation matrix, and the entries \( b_{j,l} \) of the matrix \( B_j \) depend on the eigenvalues of the linearization \( d f(\xi_j) \) of (1) near \( \xi_j \) via the relation

\[ b_{j,1} = c_j/e_j \quad \text{and} \quad b_{j,l+1} = -t_{j,l}/e_j, \quad 1 \leq l \leq n_t, \quad 1 \leq j \leq m. \] (25)

We call \( \{ M_m \} \), as \( \{ g_m \} \), a collection of maps associated with the heteroclinic cycle. A fixed point \((w, z) = 0\) of the collection \( \{ g_m \} \) becomes a fixed point \( \eta = -\infty \) of the collection \( \{ M_m \} \). In the study of stability of the point \((w, z) = 0\) we consider asymptotically small \( z \) and \( w \), i.e. asymptotically large negative \( \eta \), and hence finite \( F_j \) can be ignored.

Transition matrices of the superposition of maps \( \pi_j \) and \( g_{j,l} \) are the products \( M(j) = M_{j-1} \ldots M_1 M_m M_{j+1} \ldots M_l \) and \( M_{j,l} = M_1 \ldots M_m M_{j+1} M_l \) (or \( M_{j,l} \) if \( j > l \)), respectively. For a collection of permutation matrices \( A_j \) we define \( A(j) \) and \( A_{j,l} \) in the same way. Denote by \( \lambda_s \) the eigenvalues of matrices \( M(j) \) (they are independent of \( j \), since all matrices \( M(j) \) are similar) enumerated in the descending order of their real parts (generically all the real parts are distinct except for pairs of complex conjugate eigenvalues). Let also \( w_{j,s} \) denote the eigenvector of the matrix \( M(j) \) associated with the eigenvalue \( \lambda_s \).

4.2. Two types of eigenvalues of a transition matrix

Consider a matrix \( M := M^{(1)} = M_n \ldots M_1 : \mathbb{R}^N \to \mathbb{R}^N \); it is a product of the basic transition matrices of the form (24). We separate the coordinate vectors \( e_l, 1 \leq l \leq N \), into two groups. The first group is comprised of the vectors \( e_l \) for which there exist such \( k \) and \( j \) that \( (A(j))_{k,l} A_{j-1,l} = e_1 \) (recall that \( A_j \) are permutation matrices), the second one incorporates the remaining vectors. Denote by \( V^{\text{sig}} \) and \( V^{\text{ins}} \) the subspaces spanned by vectors from the first and second group, respectively (the superscripts ‘ins’ and ‘sig’ stand for significant and insignificant).

**Theorem 3.** Let \( V^{\text{sig}} \) and \( V^{\text{ins}} \) be the subspaces defined above.

(a) The subspace \( V^{\text{ins}} \) is \( M \)-invariant and the absolute value of all eigenvalues associated with the eigenvectors from this subspace is one.

(b) Generically all components of eigenvectors that do not belong to \( V^{\text{ins}} \) are non-zero.

**Proof.**

(a) Denote by \( V^{\text{ins}}_j, 2 \leq j \leq m, \) the subspaces constructed for matrices \( M^{(j)} \) similarly to \( V^{\text{ins}} \). Since \( e_1 \notin V^{\text{ins}}_j \), the action of \( B_j \) on \( V^{\text{ins}}_j \) is trivial. Evidently, \( A_j V^{\text{ins}}_j = V^{\text{ins}}_{j+1} \). Hence the actions of \( M \) and \( A \) coincide on the subspace \( V^{\text{ins}} \). The \( A \)-invariance of \( V^{\text{ins}} \) follows from the definition of \( V^{\text{ins}} \), and hence \( V^{\text{ins}} \) is \( M \)-invariant. The matrix \( A \) is a permutation matrix, since it is a product of permutation matrices, and thus all its eigenvalues have the unit absolute value. Consequently, the same holds true for the restriction of \( M \) on \( V^{\text{ins}} \).
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(b) Let \( w^{1,s} \) be an eigenvector of \( M \) that does not belong to \( V_{\text{ins}} \). At least one significant component of \( w^{1,s} \) does not vanish. Denote it by \( w^{1,s}_1 \). By definition of \( V_{\text{sig}} \), there exist such \( i \) and \( j \) that \( (A^{(i)})^j \epsilon_j = e_1 \). Eigenvectors \( w^{1,s} \) and \( w^{1,s}_1 \) are related as follows:

\[
w^{1,s} = (M^{(j)})^k M_{1,1} w^{1,s}_1
\]

(up to an arbitrary factor). Hence generically the first component of \( w^{1,s} \) does not vanish. Since eigenvectors satisfy the relation

\[
w^{1,s} = M_{m,j} w^{1,s}
\]

and (24) holds true for all matrices \( M_j \) in the product \( M_{m,j} \), generically all components of \( w^{1,s} \) are non-zero.

QED

We call \textit{insignificant} the eigenvalues associated with eigenvectors from \( V_{\text{ins}} \), and \textit{significant} the rest ones. Generically the absolute values of all significant eigenvalues differ from one.

4.3. Two lemmas

Here we prove two lemmas which will be used in the next subsection to establish some results on asymptotic stability of a fixed point of a collection of maps associated with a heteroclinic cycle; as we have proved in section 3, stability of the fixed point implies asymptotic stability of the cycle. Arbitrary matrices \( M \) are considered in this subsection (except for they are supposed to have non-negative entries in lemma 4); in particular, they are not assumed to be products of matrices of the form (24).

We consider a criterion for positivity of the measure of the set

\[
U^{-\infty}(M) = \{ y : y \in \mathbb{R}^N, \lim_{k \to \infty} M^k y = -\infty \}
\]

in terms of the dominant eigenvalue and the associated eigenvector of matrix \( M : \mathbb{R}^N \to \mathbb{R}^N \).

We have denoted

\[
\mathbb{R}_+^N = \{ y = (y_1, \ldots, y_N) : y_j < 0 \text{ for all } j \}, \quad \mathbb{R}_-^N = \{ y = (y_1, \ldots, y_N) : y_j \leq 0 \text{ for all } j \},
\]

\[
U_S = \{ y : \max_{1 \leq i \leq N} y_i < S \}, \quad \bar{U}_S = \{ y : \max_{1 \leq i \leq N} y_i \leq S \}.
\]

Upon the change of variables (22), the Lebesgue measure of a set initially of a finite measure can become infinite. To avoid this, the measure of a set \( V \) in the original variables is regarded as its measure. Note that \( \mu(V) \) is strictly positive, if and only if this is true for the image of \( V \) under the mapping \( y \to e^y \) inverse to (22).

Denote by \( \lambda_{\text{max}} \) and \( w^{\text{max}} \) the maximum in absolute value eigenvalue of the matrix \( M : \mathbb{R}^N \to \mathbb{R}^N \) and the associated eigenvector, respectively, and by \( w^{\text{max}}_l \) the components of the eigenvector. If \( |\lambda_{\text{max}}| > 1 \), all other eigenvalues (except for \( \lambda_{\text{max}} \) if \( \lambda_{\text{max}} \) is complex) are supposed to be strictly smaller in absolute value than \( \lambda_{\text{max}} \).

**Lemma 3.** \( \mu(U^{-\infty}(M)) \) depends on \( \lambda_{\text{max}} \) and \( w^{\text{max}} \) as follows:

(i) If \( |\lambda_{\text{max}}| < 1 \), then \( U^{-\infty}(M) = \emptyset \).
(ii) If \( \lambda_{\text{max}} \) is real and \( \lambda_{\text{max}} < -1 \), then \( \mu(U^{-\infty}(M)) = 0 \).
(iii) If \( \lambda_{\text{max}} \) is complex and \( |\lambda_{\text{max}}| > 1 \), then \( \mu(U^{-\infty}(M)) = 0 \).
(iv) If \( \lambda_{\text{max}} \) is real, \( \lambda_{\text{max}} > 1 \) and \( w^{\text{max}}_{\max l} w^{\text{max}}_q \leq 0 \) for some \( l \) and \( q \), then \( \mu(U^{-\infty}(M)) = 0 \).
(v) If \( \lambda_{\text{max}} \) is real, \( \lambda_{\text{max}} > 1 \) and \( w^{\text{max}}_{l} w^{\text{max}}_{q} > 0 \) for all \( 1 \leq l, q \leq N \), then \( \mu(U^{-\infty}(M)) > 0 \).
Proof. Consider the expansion of $y \in \mathbb{R}^N$

$$y = \sum_{i=1}^{N} a_i w_i$$

(26)

in the basis comprised of eigenvectors of $M$, $w_i$, $1 \leq i \leq N$, and denote by $a_{\text{max}}$ the coefficient in front of $w_{\text{max}}$ in the sum (26). Then the $k$th iterate is

$$M^k y = \sum_{i=1}^{N} \lambda_i^k a_i w_i.$$  

(27)

Consequently

(1) If $|\lambda_{\text{max}}| \leq 1$, then

$$\lim_{k \to \infty} M^k y \neq -\infty$$

for any $y \in \mathbb{R}^N$, because if $\lim_{k \to \infty} \lambda_i^k a_i$ exists, it is zero or finite.

(2) If $\lambda_{\text{max}}$ is real and $\lambda_{\text{max}} < -1$, then

for any $y \in \mathbb{R}^N$ such that $a_{\text{max}} \neq 0$ the limit $\lim_{k \to \infty} M^k y$ does not exist,

because for $k \to \infty$ the iterates $M^k y$ become aligned with $\lambda_{\text{max}}^k w_{\text{max}}$ and the signs of individual components of $\lambda_{\text{max}}^k w_{\text{max}}$ alternate for odd and even $k$.

(3) If $\lambda_{\text{max}}$ is complex and $|\lambda_{\text{max}}| > 1$, then

for any $y \in \mathbb{R}^N$ such that $a_{1,\text{max}} \neq 0$ or $a_{2,\text{max}} \neq 0$, the limit $\lim_{k \to \infty} M^k y$ does not exist,

because for $k \to \infty$ the iterates $M^k y$ are attracted by the plane spanned by $w_{1,\text{max}}$ and $w_{2,\text{max}}$. The map $M$ expressed in the polar coordinates in the plane amounts to multiplication by $r e^{i\psi}$ (i.e. it involves rotation by an angle $\psi \neq 2\pi$ and multiplication by $r > 1$). However, only one quadrant of the plane belongs to $\mathbb{R}^N$. Hence, the limit does not exist.

(4) If $\lambda_{\text{max}}$ is real, $\lambda_{\text{max}} > 1$ and $a_{l,\text{max}} w_{l,\text{max}} a_{q,\text{max}} > 0$ for some $l$ and $q$, then

for any $y$ such that $a_{\text{max}} \neq 0$ $\lim_{k \to \infty} M^k y \neq \mathbb{R}^N$.

To show this note that for $k \to \infty$ the iterates $M^k y$ become asymptotically close to $a_{\text{max}} \lambda_{\text{max}}^k w_{\text{max}}$, the signs of two individual components, $a_{l,\text{max}} \lambda_{l,\text{max}}^k w_{l,\text{max}}$ and $a_{q,\text{max}} \lambda_{q,\text{max}}^k w_{q,\text{max}}$, are opposite and hence one of them is positive (unless one or both of the two components vanish).

(5) If $\lambda_{\text{max}}$ is real, $\lambda_{\text{max}} > 1$ and $a_{l,\text{max}} w_{l,\text{max}} a_{q,\text{max}} > 0$ for all $l$ and $q$ (to be specific, we can assume all $a_{l,\text{max}} > 0$), then

$$\lim_{n \to \infty} M^n y = -\infty$$

for any $y \in \mathbb{R}^N$ such that $a_{\text{max}} < 0$,

because for $k \to \infty$ the iterates $M^k y$ become asymptotically close to $a_{\text{max}} \lambda_{\text{max}}^k w_{\text{max}}$.

Clearly, (1)–(4) imply (i)–(iv), respectively.

To prove (v), note that the point $-w_{\text{max}}$ belongs to $\mathbb{R}^N$. Therefore there exists a neighbourhood $V \subset \mathbb{R}^N$ of $-w_{\text{max}}$ such that $a_{\text{max}} < 0$ for any $y \in V$. The measure of this neighbourhood is positive. Thus (5) implies (v). QED
Lemma 4. Let $M$ be a matrix with non-negative entries, $|\lambda_{\text{max}}| > 1$ and $w^\text{max}_l \neq 0$ for all $1 \leq l \leq N$. Then

(i) $\lambda_{\text{max}}$ is real and positive.
(ii) $w^\text{max}_l w^\text{max}_q > 0$ for all $1 \leq l, q \leq N$.
(iii) $U^{\rightarrow \infty}(M) = \mathbb{R}^N_-$.

Proof. Consider $y \in \mathbb{R}^N$. Since all entries of $M$ are non-negative, $M^k(y) \in \bar{\mathbb{R}}^N_-$ for all $k$. (28)

Consider expansion (26) for $y$. If $\lambda_{\text{max}}$ is complex or real negative, then some iterates $M^k y$ are not in $\bar{\mathbb{R}}^N_-$, as noted in the proof of lemma 3. Similarly, if $w^\text{max}_l w^\text{max}_q < 0$ for some $l$ and $q$, then (28) does not hold true for sufficiently large $k$. No components of the eigenvector $w^\text{max}$ vanish. Consequently, $w^\text{max}_l w^\text{max}_q \neq 0$ for all $l$ and $q$, and therefore (ii) holds true.

To prove (iii), note that if $a^\text{max} > 0$ in the expansion (26) for $y$, then $\lim_{k \to \infty} M^k y = \infty$ which is prohibited by (28). Now we show that $a^\text{max}$ in expansion (26) is non-zero. Suppose $a^\text{max} = 0$ for some $y \in \mathbb{R}^N$. There exists a neighbourhood $U \subset \mathbb{R}^N$ of $y$, and hence there exists $\tilde{y} \in U$ such that in expansion (26) for $\tilde{y}$ the factor in front of $w^\text{max}$ is positive, which contradicts (28). Hence, $a^\text{max} < 0$ for all $y \in \mathbb{R}^N$; this implies $\lim_{k \to \infty} M^k y = -\infty$. QED

4.4. Properties of maps

In this subsection we prove two theorems concerning the Poincaré maps for heteroclinic cycles constructed in section 2.4. These theorems will be used in investigation of stability of fixed points of a collection of maps. The Poincaré maps $M^{(j)}$ are superpositions of maps (23):

$M^{(j)} = M_{j-1} \ldots M_1 M_m \ldots M_{j+1} M_j$. (29)

In the coordinates $\eta$ (22), they reduce to

$M^{(j)} \eta = M^{(j)} \eta + C^{(j)}$. (30)

We denote $M_{j,l} = M_j \ldots M_{l+1} M_l \ldots M_{j+1} M_j$ (or $M_{j,l} = M_j \ldots M_l$ if $j > l$). We will consider matrices $M$ (for instance, $M = M^{(j)}$) that are products of basic transition matrices of the form (24).

For a linear map $M$, where

$M \eta = M \eta + C$, (31)

we define

$U^{\rightarrow \infty}(M) = \{ y : y \in \mathbb{R}^N, \lim_{k \to \infty} M^k y = -\infty \}$.

Lemma 5. Let $\lambda_{\text{max}}$ be the largest in absolute value significant eigenvalue of the matrix $M$ in (31) and $w^\text{max}$ be the associated eigenvector. Suppose $\lambda_{\text{max}} \neq 1$ (as noted in section 4.2, generically this is true). The measure $\mu(U^{\rightarrow \infty}(M))$ is positive, if and only if the three following conditions are satisfied:

(i) $\lambda_{\text{max}}$ is real;
(ii) $\lambda_{\text{max}} > 1$;
(iii) $w^\text{max}_l w^\text{max}_q > 0$ for all $l$ and $q$, $1 \leq l, q \leq N$. 
Proof. For a vector \( y = y_1, \ldots, y_N \in \mathbb{R}^N \) expressed as a linear combination (26) of eigenvectors \( w_i \), we have
\[
\mathcal{M}y = \sum_{i=1}^{N} (\lambda_i a_i + d_i)w_i,
\]
where \( d_i \) is the component of \( C \) in the direction of \( w_i \). By simple algebra,
\[
\mathcal{M}^k y = \sum_{i=1; \lambda_i \neq 1}^{N} ((a_i - d_i(1 - \lambda_i))^{-1})^k \lambda_i^k w_i + \sum_{i=1; \lambda_i \neq 1}^{N} (a_i + kd_i)w_i, \quad (32)
\]
Suppose that (i)--(iii) hold true and \( a_{\text{max}} - d_{\text{max}}(\lambda_{\text{max}} - 1)^{-1} < 0 \). Thus for \( k \to \infty \) the iterates \( \mathcal{M}^k y \) become asymptotically close to
\[
\mathcal{M}^k y = (a_{\text{max}} - d_{\text{max}}(\lambda_{\text{max}} - 1)^{-1})^{\infty} \lambda_{\text{max}}^k w_{\text{max}}^k + r_k, \quad (33)
\]
and for \( k \to \infty \) the residual term \( r_k \) is infinitely small compared with the first one. Thus \( \mu(U^{-\infty}(\mathcal{M})) > 0 \) by the same arguments as in the proof of part (v) of lemma 3.

Suppose \( |\lambda_{\text{max}}| > 1 \). For \( k \to \infty \), the first term in (33) is again asymptotically the largest one and hence predominantly determines \( \mathcal{M}^k y \). Therefore, if (i) or (iii) are not satisfied or \( \lambda_{\text{max}} < 0 \), then \( \mu(U^{-\infty}(\mathcal{M})) = 0 \) by the arguments employed in the proof of lemma 3.

Consider now the case \( |\lambda_{\text{max}}| < 1 \). Suppose that (upon Cartesian coordinates are reordered, if necessary) the insignificant subspace of \( M \) consists of vectors \( y = 0, \ldots, 0, y_{N-n+1}, \ldots, y_N \in \mathbb{R}^N \) (the insignificant subspace is defined in section 4.2). We can express \( \mathcal{M}^k y = y_{1,k}, \ldots, y_{N,k} \) (32) as
\[
\mathcal{M}^k y = \sum_{i=1; \lambda_i \neq 1}^{N} ((a_i - d_i(1 - \lambda_i))^{-1})^k \lambda_i^k w_i + \sum_{i=1; \lambda_i \neq 1}^{N} (a_i + kd_i)w_i.
\]
The first sum has a finite limit as \( k \to \infty \). In the second and third sums, \( y_{1,k}, \ldots, y_{n,k} \), vanish as proved in theorem 3. Therefore \( \lim_{k \to \infty} \mathcal{M}^k y \neq -\infty \).

Below \( \lambda_{\text{max}} \neq 1 \) denotes the largest significant eigenvalue of a transition matrix \( M^{(j)} \).

Theorem 4. Let \( M_j \) be basic transition matrices of a collection of maps \( \{g^{(m)}_j\} \) associated with a heteroclinic cycle of type Z. Suppose that for all \( j, 1 \leq j \leq m \), all transverse eigenvalues of \( d f(\xi_j) \) are negative. Then
(a) If the inequality \( |\lambda_{\text{max}}| > 1 \) holds true for the transition matrix \( M := M^{(1)} = M_m \ldots M_1 \),
then \( 0 \) is an asymptotically stable fixed point of the collection of maps \( \{g^{(m)}_j\} \).
(b) If \( |\lambda_{\text{max}}| < 1 \), then \( 0 \) is completely unstable.

Proof.
(a) The matrices \( M^{(j)} \) are similar, hence if the maximum absolute value of eigenvalues of matrix \( M \) is larger than unity, this is also the case for \( M^{(j)} \) for any \( j \). All transverse eigenvalues of \( d f(\xi_j) \) being negative, (24) and (25) imply that all entries of matrices \( M_j \) are non-negative. Since \( |\lambda_{\text{max}}| > 1 \), by theorem 3 \( w_{\text{max}} \neq 0 \) for all \( 1 \leq q \leq N \). Hence by lemma 4
\[
\lim_{k \to \infty} (M^{(j)}_j)^k y = -\infty \quad (34)
\]
for all \( 1 \leq j, l \leq m \) and any \( y \in \mathbb{R}^N \).
As noted in the proof of lemma 4, \( y \in \mathbb{R}^N \) implies the inequality \( a_{\max} < 0 \) for the coefficient in front of \( w_{\max} \) in expansion (26) for \( y \). Therefore there exists \( \tilde{S} < 0 \) such that \( a_{\max} - d_{\max}(1 - \lambda_{\max})^{-1} < 0 \) for any \( y \in U_{\tilde{S}} \) in (26). Thus, by the same arguments as employed in the proof of lemma 5,

\[
U_{\tilde{S}} \subset U^{-\infty}(\mathcal{M}).
\]  

(35)

Denote by \( V^\perp(\tilde{S}) \) the intersection of the set \( \tilde{U}_{\tilde{S}} \) with the \( N - 1 \)-dimensional hyperplane orthogonal to the line \((1, \ldots, 1)\) and crossing this line at the point \((2\tilde{S}, \ldots, 2\tilde{S})\). The set \( V^\perp(\tilde{S}) \) is compact; therefore, by virtue of the inclusion (35), for any \( l \) and \( j \) there exists a constant \( Q^{l,j} \) such that

\[
\max_{1 \leq s \leq N} e_s \cdot (\mathcal{M}^{(j)})^k \mathcal{M}_{j-1,l} y < Q^{l,j}
\]

for all \( k > 0 \) and \( y \in V^\perp(\tilde{S}) \); here \( \{e_s\} \) are Cartesian coordinate system vectors. Denote

\[
Q_{\max} = \max_{1 \leq l, j \leq N} Q^{l,j}.
\]

All entries of the matrices \( M_j \) are non-negative and the set \( V^\perp(\tilde{S}) \) is compact; hence for any \( l \) and \( j \) (34) implies existence of a constant \( R^{l,j} < 0 \) such that

\[
\max_{1 \leq s \leq N} e_s \cdot (\mathcal{M}^{(j)})^k M_{j-1,l} y < R^{l,j}
\]

(37)

for all \( k > 0 \) and \( y \in V^\perp(\tilde{S}) \). Denote

\[
R_{\max} = \max_{1 \leq l, j \leq N} R^{l,j}.
\]

By virtue of (32),

\[
\mathcal{M}^k y = \mathcal{M}^k (y - \tilde{y}) + \tilde{y} + \sum_{i=1, \lambda_i = 1}^N k d_i w_i,
\]

(38)

where

\[
\tilde{y} = \sum_{i=1, \lambda_i \neq 1}^N d_i (1 - \lambda_i)^{-1} w_i.
\]

For a given \( R < 0 \), set \( \bar{S} = 2\tilde{S}(R + R_{\max} - Q_{\max})/R_{\max} \). The linearity of maps \( M_j \) and (38) imply

\[
\max_{1 \leq s \leq N} e_s \cdot (\mathcal{M}^{(j)})^k M_{j-1,l} y < R
\]

for all \( k > 0 \), \( 1 \leq j, l \leq m \) and \( y \in U_{\tilde{S}} \).

Changing coordinates from \( \eta \) back to \((w, z)\), we conclude that by definition 9 the point \( 0 \) is an asymptotically stable fixed point of the collection \( \{g^m_i\} \).

(b) The statement of the theorem follows from lemma 5. QED

The theorem 4 is proved in [12] for the particular case of a heteroclinic cycle defined by the replicator equation. This proof relies on the Perron–Frobenius theorem and can be applied to any heteroclinic cycle of type Z with an empty insignificant subspace.

**Theorem 5.** Let \( M_j \) be basic transition matrices of a collection of maps \( \{g^m_i\} \) associated with a heteroclinic cycle of type Z. (For type Z heteroclinic cycles the matrices are of the form (24).)
Denote by \( j = j_1, \ldots, j_L \) the indices, for which \( M_j \) involves negative entries; all entries are non-negative for all remaining \( j \). Assume \( L > 0 \) (the case \( L = 0 \) is treated by theorem 4).

(a) If for at least one \( j = j_1 + 1 \) the matrix \( M^{(j)} \) does not satisfy conditions (i)–(iii) of lemma 5, then \( 0 \) is a completely unstable fixed point of the collection \( \{g^n\} \).

(b) If the matrices \( M^{(j)} \) satisfy conditions (i)–(iii) of lemma 5 for all \( j \) such that \( j = j_1 + 1 \), then \( 0 \) is a fragmentarily asymptotically stable fixed point of the collection \( \{g^n\} \).

**Proof.**

(a) Matrices \( M^{(j)} \) satisfy, or not satisfy, conditions (i)–(ii) of lemma 5 simultaneously for all \( j \), because all \( M^{(j)} \) are similar. Hence, by lemma 5, \( \mu(U^{-\infty}(M^{(j)})) = 0 \) for all \( j \).

Suppose condition (iii) is not satisfied for some \( j = J \). The iterates \((M^{(j)})^kM_{J-1,1}y\) tend to align with \( w^{max,J} \) in the limit \( k \to \infty \) (see (32)). Since \( w^{max,J} \) is negative and sufficiently large in magnitude, then \( (M^{(j)})^kM_{J-1,1}y^{stab} \in \mathbb{R}^N_+ \) for all \( 1 \leq j \leq m \) and \( k \geq 0 \)

and

\[
\lim_{k \to \infty} (M^{(j)})^kM_{J-1,1}y^{stab} = -\infty \quad \text{for all} \quad 1 \leq j \leq m.
\]

Let \( V^{compl} \) be the \( N - 1 \)-dimensional \( M \)-invariant complement to \( w^{max} \) in \( \mathbb{R}^N \); and \( B_1^{N-1}(0) \) the unit ball in \( V^{compl} \) centred at \( 0 \). If \( y \in B_1^{N-1}(0) \), then

\[
\lim_{k \to \infty} \left( \frac{1}{\lambda_{\max}} M \right)^k y = 0.
\]

Therefore there exists a constant \( K_1 \) such that

\[
\left| \left( \frac{1}{\lambda_{\max}} M \right)^k y \right| < K_1
\]

for all \( y \in B_1^{N-1}(0) \) and \( k \geq 0 \). Similarly there exist constants \( K_j, j = 2, \ldots, N \), such that

\[
\left| \left( \frac{1}{\lambda_{\max}} M^{(j)} \right)^k M_{J-1,1}y \right| < K_j
\]

for all \( y \in B_1^{N-1}(0) \) and \( k > 0 \). Denote \( K_{\max} = \max_j K_j \) and assume that upon normalization \( w^{max,j} \) are unit eigenvectors. Let \( d_{min} \) be the minimum distance from \((M^{(j)})^kM_{J-1,1}y^{stab} \in \mathbb{R}^N_+ \) to the \( N - 1 \)-dimensional coordinate hyperplanes \( y_p = 0 \) (the minimum is over all hyperplanes, all \( k \) and all \( j \)).

Denote by \( B_{d_{min}/K_{\max}}^{N-1}(y^{stab}) \) the ball of radius \( d_{min}/K_{\max} \) centred at \( y^{stab} \) in the \( N - 1 \)-dimensional hyperplane parallel to \( V^{compl} \), and by \( C_{d_{min}/K_{\max}}^{N-1}(y^{stab}) \) the semi-infinite cylinder comprised of rays originating at \( B_{d_{min}/K_{\max}}^{N-1}(y^{stab}) \) (which is the base of the cylinder) that are
parallel to $w^{\text{max}}$ and extend towards $-\infty$. Suppose $y \in C^{N-1}_{d_{\text{max}}/K_{\text{max}}}(y^{\text{stab}})$; then, by linearity of maps $M_j$ (and hence of their superpositions) and by construction of the set $C^{N-1}_{d_{\text{max}}/K_{\text{max}}}(y^{\text{stab}})$,

$$(M^{(j)})^kM_{j-1,1}y \in \mathbb{R}^N$$

for all $1 \leq j \leq m$ and $k > 0$

and

$$\lim_{k \to \infty} (M^{(j)})^kM_{j-1,1}y = -\infty$$

for all $1 \leq j \leq m$.

Using the same arguments as employed in the proof of theorem 4, we can show that for any $R < 0$ there exists $S < 0$ such that $y \in C^{N-1}_{d_{\text{max}}/K_{\text{max}}}(y^{\text{stab}}) \cap U_S$ implies

$$(M^{(j)})^kM_{j-1,1}y \in U_R$$

for all $1 \leq j \leq m$ and $k > 0$.

Since the measure of the set $C^{N-1}_{d_{\text{max}}/K_{\text{max}}}(y^{\text{stab}}) \cap U_S$ is positive, part (b) is proved. QED

5. Bifurcations of heteroclinic cycles

In this section we consider codimension-one bifurcations of type Z heteroclinic cycles. We assume that the system (1) depends on a scalar parameter, $\alpha$, i.e. it is

$$\dot{x} = f(x, \alpha), \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n. \quad (39)$$

A bifurcation takes place at $\alpha = 0$ and the cycle exists at the interval $[\alpha_-, \alpha_+]$, where $\alpha_- < 0$ and $\alpha_+ > 0$. We do not study bifurcations occurring when the cycle ceases to exist, which happens, e.g., if for some $j$ a contracting, expanding or radial eigenvalue of $d f(\xi_j)$ vanishes at $\alpha = 0$.

5.1. Transverse bifurcations

Suppose a transverse eigenvalue of a steady state $\xi_j$ becomes positive at $\alpha = 0$. We have proved in section 2.4 that in a small neighbourhood of the cycle the Poincaré map can be approximated by a superposition of local and global maps. Near steady states only the linear part of $f$ has been taken into account. The approximation is accurate in a neighbourhood of a steady state provided in this neighbourhood the linear part is significantly larger than the omitted nonlinear terms in Taylor’s expansion of $f$. As an eigenvalue of $d f(\xi_j)$ tends to zero, the neighbourhood of $\xi_j$, where the approximation is valid, shrinks. At $\alpha = 0$, for which the eigenvalue of $d f(\xi_j)$ is zero, the collection of maps cannot be used to study the behaviour of trajectories in the vicinity of the cycle, but nevertheless we can still comment on bifurcations of the cycle. Stability of the cycle can change in a bifurcation of a steady state $\xi_j$, where a transverse eigenvalue of $d f(\xi_j)$ becomes positive for $\alpha > 0$, in the following ways:

(a) The cycle is asymptotically stable for $\alpha < 0$ and fragmentarily asymptotically stable for $\alpha > 0$. This happens, if the conditions (i)–(iii) of lemma 5 remain satisfied for $\alpha > 0$. An example of such a bifurcation is studied in the paper [8].

(b) The cycle is asymptotically stable for $\alpha < 0$ and completely unstable for $\alpha > 0$. This occurs, if at least one of the conditions (i)–(iii) is violated for $\alpha > 0$. Examples of such bifurcations of homoclinic cycles are studied in [6].

(c) The cycle is fragmentarily asymptotically stable for $\alpha < 0$ and remains such for $\alpha > 0$ (the conditions (i)–(iii) hold true for $\alpha > 0$).

(d) The cycle is fragmentarily asymptotically stable for $\alpha < 0$ and completely unstable for $\alpha > 0$ (at least one of the conditions (i)–(iii) is violated for $\alpha > 0$).
When a transverse eigenvalue of $d f(\xi_j)$ changes its sign, a pair of mutually symmetric steady states, $\xi'_j$ and $\xi''_j$, emerges in a pitchfork bifurcation. Denote by $v^{\text{crit}}$ the eigenvector spanning the eigenspace of $d f(\xi_j)$, where the bifurcation takes place. By lemma 1, the subspace $L_j \oplus v^{\text{crit}}$ is invariant for the dynamical system (39) implying that the heteroclinic connection $\xi_j \rightarrow \xi'_j$ (and, due to the symmetry, $\xi_j \rightarrow \xi''_j$) exists that is structurally stable. The type of the global object appearing in the bifurcation depends on whether there exists another structurally stable heteroclinic connection, namely, $\xi'_j \rightarrow \xi_{j+1}$ (or possibly $\xi''_j \rightarrow \xi_{j+l}$ with $l > 1$). If such a connection exists, a new heteroclinic cycle $(\xi_1, \ldots, \xi_j, \xi'_j, \xi_{j+l}, \ldots, \xi_m)$ is created in the bifurcation, as, e.g., in the system considered in [8]. If the connection does not exist, more complex objects can possibly emerge, for instance, a depth-two heteroclinic cycle involving a connection from $\xi'_j$ to the original heteroclinic cycle $X$.

We mention certain particular examples of bifurcating cycles:

- If the cycle $X$ is homoclinic (i.e. the equilibria are related by a symmetry $\gamma$, $\xi_{j+1} = \gamma \xi_j$ and $v^{\text{crit}} = \gamma v^{\text{crit}}$), then the cycle $(\xi'_1, \xi'_2, \ldots, \xi'_m)$ emerges that does not involve any $\xi_j$.
- Suppose that in the cycle $X$ any $\xi_j$ and $\xi_{j+2}$ are related by a symmetry $\gamma$, i.e. $\xi_{j+2} = \gamma \xi_j$ (this does happen for homoclinic cycles and can occur in other cases) and $v^{\text{crit}} = \gamma v^{\text{crit}}$.

Then, assuming that the bifurcation takes place at an even $j$, the bifurcating cycle is $(\xi'_2, \xi'_4, \ldots, \xi'_m)$ (note $m$ is even).

Such bifurcations were studied in [6]. Note that when the dimension of the transverse eigenspace is higher than one (this is impossible for cycles of type Z), more complex transverse bifurcations are possible, for instance, the ones discussed in [7].

5.2. Resonance bifurcations

In this section we assume that for any $j$ none of the eigenvalues of $d f(\xi_j)$ crosses the imaginary axis at the interval $\alpha_- < \alpha < \alpha_+$. Hence there exists a neighbourhood of the heteroclinic cycle, where trajectories of system (39) are accurately approximated by the superposition of maps $g_j$, $1 \leq j \leq m$. Thus bifurcations of the cycles can be investigated by studying bifurcations of fixed points of the collection of maps associated with the cycle. Here we explore what happens when the conditions (i)–(iii) of lemma 5 for fragmentary asymptotic stability of a fixed point $(w, z) = 0$ of the collection of maps are broken.

In section 4.2 we have divided eigenvectors and eigenvalues of a transition matrix $M^{(j)}$ into two groups, which we have called significant and insignificant. We arrange vectors in the basis in $H^{(j)}$ (recall that the basis consists of eigenvectors of $d f(\xi_j)$) in such a way that the first $n_1$ vectors are significant and the last $n_2$ ones are insignificant ($N = n_1 + n_2$).

The first basis vector is the expanding eigenvector of $d f(\xi_j)$. Significant eigenvalues are the eigenvalues of the left upper $n_1 \times n_2$ submatrix of $M^{(j)}$; their absolute values generically differ from unity. All components of the associated eigenvectors generically do not vanish. We order the significant eigenvalues so that $\Re \lambda_j \geq \Re \lambda_{j+1}$ for $1 \leq j < n_1$. Insignificant eigenvalues are one in absolute value, and the first $n_2$ components of the associated eigenvectors are zero. The transition matrix $M^{(j)}$ restricted to $V^{\text{ins}}$ is a matrix of permutation of the basis vectors. A permutation is a combination of cyclic permutations. We arrange the insignificant eigenvectors in such a way that each consequent $n_1$ vectors are involved in the same cyclic permutation of length $n_1$ and the permutation is $v^{N_1,1} \rightarrow v^{N_1,2} \rightarrow \ldots \rightarrow v^{N_1} \rightarrow v^{N_1,1}$ where $N_j = n_1 + n_2 + \ldots + n_1$ and $n_1 = n_1 + \ldots + n_L$.

Conditions (i) and (ii) of lemma 5 for $\lambda_{\text{max}}$ are now replaced by the following ones:

(i') $\lambda_1$ is real and $\lambda_1 > 1$;
(ii') $|\lambda_{1+2}| > |\lambda_j|$ for any $2 \leq j \leq N$. 

Denote by $\zeta$ a vector in the basis comprised of eigenvectors of matrix $M$. As above, we set $M := M^{(1)}$ (our arguments are applicable for any $M = M^{(j)}$). The map $M(j)$, which is the superposition (29) of maps (23), defines the iterates

\[ \eta_{n+1} = M \eta_n := M \eta_n + c. \]

In these coordinates the iterates reduce to

\[ \zeta_{j,n+1} = \lambda_j \zeta_{j,n} + d_j \]

for a real $\lambda_j$, and

\[ \zeta_{j,n+1} = \alpha_j \zeta_{j,n} + \beta_j \zeta_{j+1,n} + d_j, \quad \zeta_{j+1,n+1} = \alpha_j \zeta_{j+1,n} - \beta_j \zeta_{j,n} + d_{j+1} \]

Lemma 6. Consider the following dynamical systems in $\mathbb{R}$ ((a)–(c)) and $\mathbb{R}^2$ (d):

(a) $x_{n+1} = \lambda x_n + d$, where $\lambda \neq \pm 1$ is real;
(b) $x_{n+1} = \lambda x_n + d$, where $\lambda = -1$;
(c) $x_{n+1} = \lambda x_n + d$, where $\lambda = 1$;
(d) $x_{n+1} = \alpha x_n^2 + \beta x_n + d^2$, $x_{n+1}^2 = \alpha x_n - \beta x_n^2 + d^2$.

Fixed points and periodic orbits of these systems are:

(a) $x = d/(1-\lambda)$ and $x = \pm \infty$ (if $\lambda < 0$, $x = \pm \infty$ is a period-two orbit). The fixed point $x = d/(1-\lambda)$ is stable for $|\lambda| < 1$, and the fixed points $x = \pm \infty$ are stable for $|\lambda| > 1$;
(b) $x = d/2$ is a fixed point, any real number is a period-two orbit;
(c) $x = \pm \infty$ are the only fixed points. The fixed point $x = \infty$ is stable for $d > 0$ and the fixed point $x = -\infty$ is stable for $d < 0$;
(d) $x^1 = (d^2(1 - \alpha) - \beta d^2)/((\alpha - 1)^2 + \beta^2)$, $x^2 = (d^2(1 - \alpha) - \beta d^1)/((\alpha - 1)^2 + \beta^2)$ is a unique fixed point, stable for $|\lambda| < 1$.

The statements of the lemma are directly verified by a simple algebra.

Theorem 6. Let $M_j, 1 \leq j \leq N$, be basic transition matrices of the collection of maps $\{g^n_m\}$ associated with a type Z heteroclinic cycle (this implies that the transition matrices have the form (24)). Suppose

(i) the entries $b_{j,l}$ of the basic transition matrices depend continuously on $\alpha$;
(ii) for $\alpha_- \leq \alpha < \alpha_+$ condition (iii) of lemma 5 is satisfied for all $M^{(j)}$;
(iii) for $\alpha_- \leq \alpha < 0$ significant eigenvalues of matrix $M$ satisfy the conditions $\lambda_1 > 1$ and $|\lambda_j| < 1$ for $2 \leq j \leq n$;
(iv) for $0 < \alpha \leq \alpha_+$ all significant eigenvalues of matrix $M$ satisfy $|\lambda_j| < 1$ for all $1 \leq j \leq n$.

Then

(a) For $\alpha_- < \alpha < 0$ the fixed point $0$ of the collection of maps $\{g^n_m\}$ is fragmentarily asymptotically stable, and for $\alpha > 0$ it is completely unstable;
(b) Suppose $d_j < 0$ in (41) and the point $\eta_j$ is defined by the relations

\[ \eta_j = \sum_{l=1}^{n_s} \frac{d_l}{1 - \lambda_j} w_{j,l} \]

for $1 \leq j \leq n_s$;

\[ \eta_j = -\infty \text{ for } n_s + 1 \leq j \leq N. \]

(In the second sum the sign $\pm$ coincides with the sign of the imaginary part of $\lambda_j$. A component is $-\infty$ whenever in the original coordinates $(w, z)$ the respective component
vanishes.) Then \( \eta^1 \) is an asymptotically stable fixed point of the collection \( \{ g^n \} \) for \( 0 < \alpha \leq \alpha_c \). Components \( \eta^1_j (43) \) for \( 1 \leq j \leq n_s \) satisfy

\[
\lim_{\alpha \to +0} \eta^1_j = -\infty.
\]

(c) If \( d_1 > 0 \), then for \( \alpha_- \leq \alpha < 0 \) the point (43) is an unstable fixed point of the collection \( \{ g^n \} \). Components \( \eta^1_j (43) \) for \( 1 \leq j \leq n_s \) satisfy

\[
\lim_{\alpha \to -0} \eta^1_j = -\infty.
\]

**Proof.** (a) follows from lemma 5.

(b) Recall that insignificant eigenvectors are arranged in such a way, that each consequent \( n_l \) vectors are involved in the same cyclic permutation. The subspace spanned by these \( n_l \) vectors, \( \{ v_{1, N_l+1}, v_{1, N_l+2}, \ldots, v_{1, N_l+n_s} \} \), is an invariant subspace of map \( M \). The action of the map on this subspace is determined by the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\] (44)

The \( n_l \) eigenvalues of the restriction of \( M \) into this invariant subspace are roots of unity. One of the eigenvalues is unity, all \( n_l \) components of the associated eigenvector are equal.

Consider the map \( M \) in the basis comprised of eigenvectors of \( M \). We apply lemma 6 in each eigenspace separately to find a fixed point of the map. For eigenvalues different from unity, we choose finite fixed points in the associated eigenspaces. For the eigenvalues 1, we choose the fixed points \(-\infty\) in the associated eigenspaces. The change of coordinates from \( \zeta \) to \( \eta \) yields the desirable expression (43) for the fixed point.

In the sum (43), the factor \( d_1/(1 - \lambda_1) \) is asymptotically larger than any other factor in all other terms. For \( \alpha \to +0 \) this factor tends to infinity, while others have finite limits. Hence the components \( \eta^1_j \) for \( 1 \leq j \leq n_s \) tend to \(-\infty\) for \( \alpha \to +0 \), as required.

Next we prove that the fixed point (43) is asymptotically stable. Suppose we know the first \( n_s \) components, \( (w_{1,1}, \ldots, w_{1,n_s}) \), of the eigenvector \( u_{1,1} \) (they can be found by calculating the eigenvectors of the \( n_s \times n_s \) upper left submatrix of \( M \)). The last \( n_l \) components of \( u_{1,1} \) can be obtained from the equations

\[
\begin{aligned}
\sum_{q=1}^{n_s} a_{N_l+1,q} w_{q}^{1,1} + w_{N_l+1}^{1,1} &= \lambda_1 w_{N_l+1}^{1,1}, \\
\sum_{q=1}^{n_s} a_{N_l+2,q} w_{q}^{1,1} + w_{N_l+2}^{1,1} &= \lambda_1 w_{N_l+2}^{1,1}, \\
& \vdots \\
\sum_{q=1}^{n_s} a_{N_l+n_l,q} w_{q}^{1,1} + w_{N_l+n_l}^{1,1} &= \lambda_1 w_{N_l+n_l}^{1,1},
\end{aligned}
\]

(45)

where \( a_{i,j} \) are the entries of the matrix \( M \) (a separate system of dimension \( n_l \) is obtained for each invariant insignificant subspace of \( M \)).
We solve system (45) as follows. Multiplying the first equation by \( \lambda_1^{n_1-1} \), the second one by \( \lambda_1^{n_1-2} \), etc, and adding up the resultant equations we obtain

\[
w_{N_1+1}^{1,1} = \frac{1}{\lambda_1^{n_1-1}} \sum_{j=1}^{n_1} \sum_{q=1}^{n_1} a_{N_1+1,j,q} w_q^{1,1}. \tag{46}
\]

We proceed similarly to find \( w_{N_1+1}^{1,1} \) if \( N_{j-1}+1 < j \leq N_j \).

Since \( \lambda_1 \) tends to unity for \( \alpha \to -0 \), the condition \( w_{N_1+1}^{1,1} > 0 \) for \( N_{j-1}+1 \leq j \leq N_j \) in this limit is equivalent to

\[
\sum_{j=1}^{n_1} \sum_{q=1}^{n_1} a_{N_1+1,j,q} w_q^{1,1} > 0. \tag{47}
\]

Stability of the point (43) to a perturbation from the significant subspace of \( M \) follows from lemma 6 and the fact that all eigenvalues in the significant subspace are less than one in absolute value. For \( \alpha \) close to +0, the fixed point \( \eta^1 \) has the asymptotics \( C(\alpha)w^{1,1} \), where \( C(\alpha) < 0 \) is a large constant.

Consider now a perturbation from an invariant subspace of dimension \( n_s \) of the insignificant subspace. Choose a positive \( D \) such that

\[
D > -\min \sum_{j=1}^{n_1} \sum_{q=1}^{n_1} a_{N_1+1,j,q} w_q^{1,1}. \tag{48}
\]

For a given \( S < 0 \), set \( R = S - n_s C(\alpha)D \). An initial perturbation \( v_1 \) from this subspace evolves according to the following equations:

\[
v_{N_1+1}^{1} = C(\alpha) \sum_{q=1}^{n_1} a_{N_1+1,q} w_q^{1,1} + v_{n_1}^{1},
\]

\[
v_{N_1+1}^{2} = C(\alpha) \sum_{q=1}^{n_1} a_{N_1+2,q} w_q^{1,1} + v_{n_1}^{2},
\]

\[
\vdots
\]

\[
v_{N_1} = C(\alpha) \sum_{q=1}^{n_1} a_{N_1,q} w_q^{1,1} + v_{n_1}^{1}. \tag{49}
\]

Due to our choice of \( R \) and (47), if initially all components of \( v_1 \) satisfy \( v_j^1 < R \) for \( N_{j-1}+1 \leq j \leq N_j \), then for any \( k \geq 1 \) the estimate \( v_j^k < S \) for \( N_{j-1}+1 \leq j \leq N_j \) holds true. Hence, the point (43) is stable in the insignificant subspace.

To prove (c), note that expression (43) still defines a fixed point of the collection of maps. It bifurcates from \( -\infty \) at \( \alpha = 0 \) and remains close to \( -\infty \) for negative \( \alpha \) (recall that \( d_1 > 0 \)). The point is unstable in the direction of \( w^{1,1} \), because the associated eigenvalue is larger than one.

QED

Note that if a fixed point of a collection of maps associated with a heteroclinic cycle exists near \( \theta \), then there exists a periodic orbit close to the cycle. Stability (or instability) of the periodic orbit follows from stability (or instability, respectively) of the fixed point of the collection of maps. The distance of the bifurcating fixed point from \( \theta \) is \( O(\exp(D/(\lambda_1 - 1))) \). If the eigenvalue depends linearly on \( \alpha \) (the generic case), the distance from the periodic orbit to the heteroclinic cycle in \( H^{(in)}(\alpha) \) is proportional to \( e^{-d/|\alpha|} \), and hence for any \( j \) the time required for the orbit to get from \( H^{(in)}(\alpha) \) to \( H^{(out)}(\alpha) \) is proportional to \( 1/|\alpha| \). Therefore near the point of bifurcation the temporal period of the bifurcating periodic orbit behaves as \( 1/|\alpha| \).
In the study of stability we have ignored matrices $A_j^\pm$, because only the distance to $0$ matters for stability. Denote $A^\pm = A_m^\pm \ldots A_1^\pm$. Since the matrices $A_j^\pm$ are diagonal, the product is independent of the order of multiplication of the matrices. Suppose a bifurcating periodic orbit intersects $H_j^{(0)}$ at a point $x$. The next intersection is at the point $A^\pm x$ coinciding with $x$, if the matrix $A^\pm$ is the identity, or is a different point otherwise. Therefore the bifurcating orbit can have the same length as the heteroclinic cycle under consideration or a twice larger length, depending on the signs of the diagonal entries of matrix $A^\pm$.

In the next theorem we consider bifurcations occurring when $|\lambda_1| > 1$, but one or more conditions of lemma 5 are broken at the point of bifurcation. In this case the fixed point $0$ of the collection of maps $\{g^n_m\}$ ceases to be fragmentarily asymptotically stable at the point of bifurcation and no new objects (fixed points or invariant sets of other types) bifurcate from it. Thus, the respective heteroclinic cycle looses fragmentary asymptotic stability without emergence of bifurcating objects.

If the entries of a matrix depend on a single parameter, generically only two real eigenvalues of the matrix can become equal as the parameter is varied, and at the critical parameter value the restriction of the matrix onto the eigenspace associated with this eigenvalue is a Jordan cell. On further variation of the parameter the two eigenvalues in this two-dimensional subspace change into a pair of complex conjugate ones.

**Theorem 7.** Let $M_j$, $1 \leq j \leq m$, be basic transition matrices of the collection of maps $\{g^n_m\}$ associated with a type Z heteroclinic cycle. (In particular, $M_j$ have the form (24).) Suppose

(i) the entries $b_{j,1}$ of the transition matrices depend continuously on $\alpha$;
(ii) for $\alpha_- < \alpha < \alpha_+$ the inequalities $|\lambda_1| > 1$ and $\lambda_j \neq 1$ for $1 < j \leq n_\alpha$ are satisfied;
(iii) for $\alpha_- < \alpha < 0$ all conditions of lemma 5 on eigenvectors and eigenvalues of matrices $M^{(1)}$ are satisfied;
(iv) for $0 < \alpha < \alpha_+$ some conditions of lemma 5 are not satisfied, i.e. at least one of the following possibilities is realized:

(iv.1) $\lambda_1$ is complex, $\lambda_1 = \bar{\lambda}_2$ and $\lim_{\alpha \to 0} \Im(\lambda_1) = \lim_{\alpha \to 0} \Im(\lambda_2) = 0$;
(iv.2) there exists $\lambda_j < 0$ such that $|\lambda_j| > |\lambda_1|$ and $\lim_{\alpha \to 0} |\lambda_j| = |\lambda_1|$;
(iv.3) two eigenvalues $\lambda_j$ and $\lambda_{j+1} = \bar{\lambda}_j$ are complex, $|\lambda_j| > |\lambda_1|$ and $\lim_{\alpha \to 0} |\lambda_j| = |\lambda_1|$;
(iv.4) there exists $q$ such that $w_q^{1,1} < 0, w_q^{1,1} > 0$ for $l \neq q$ and $\lim_{\alpha \to 0} w_q^{1,1} = 0$.

Then

(a) For $\alpha_- < \alpha < 0$ the fixed point $0$ of the collection of maps $\{g^n_m\}$ is fragmentarily asymptotically stable, and for $0 < \alpha < \alpha_+$ it is completely unstable.
(b) There exists a negative $S$ such that the map $M := M^{(1)}$ does not have fixed points in $U_S$ (other than $-\infty$) for any $\alpha_- < \alpha < \alpha_+$.
(c) There exists a negative $S$ such that for any $y \in U_S$ one of the following statements is true:

- there exist $K > 0$ and $h > 1$ such that $|M^{k+1}y| > h|M^ky|$ for any $k > K$ or
- for any $K > 0$ there exists $k > K$ such that $M^ky \not\in U_S$.

(In case (c) the map $M$ has no periodic orbits or other invariant sets in $U_S$ other than $-\infty$.)

---

4 We use the term length in a loose sense here. The length of a heteroclinic cycle and of the bifurcating periodic orbit can be measured as a number of connecting trajectories comprising the cycle. Alternatively, it can be just the Lebesgue measure $\mu^1$ of a one-dimensional set.
Proof. (a) follows from lemma 5.

To prove (b), note that by lemma 6 the first $n_1$ coordinates of any fixed point $\eta^1$ of the map $M$ are $-\infty$, or otherwise they are given by (43). For $S < 0$ satisfying

$$S < \min_{1 \leq j \leq n_1, \alpha_- < \alpha < \alpha_+} \eta^1_j(\alpha),$$

the map $M$ has no fixed points in $U_S$ other than $-\infty$. (The values of $\eta^1_j$, $1 \leq j \leq n_1$, are bounded for $\alpha_- < \alpha < \alpha_+$, because no significant $\lambda_j$ is equal to 1 for such $\alpha$.)

To prove (c), set $S$ as above. Define $\tilde{\eta}^1_j$ by relations

$$\tilde{\eta}^1_j = \sum_{l=1, \lambda_j \text{ is real}}^N \frac{d_l}{1 - \lambda_j} w^{1,l}_j + \sum_{l=1, \lambda_j \text{ is complex}}^N \frac{d_l(1 - \text{Re}(\lambda_j)) \pm d_l \text{Im}(\lambda_j)}{|\lambda_j - 1|^2} w^{1,l}_j,$$

for $1 \leq j \leq n_s$, $\tilde{\eta}^1_j = 0$ for $n_s + 1 \leq j \leq N$.

Let $y \in U_S$ be decomposed in the basis comprised of eigenvectors of $M$, with the origin shifted to $\tilde{\eta}^1$. First, we note that at least one of the first $n_s$ coefficients $\tilde{\zeta}_j$ in the sum

$$y = \sum_{q=1}^N \tilde{\zeta}_j w^{1,j}$$

is non-zero (otherwise by (49) the point $y$ would be outside $U_S$). Let $\tilde{\lambda}_{\max}$ be the largest in absolute value significant eigenvalue associated with a non-vanishing coefficient. For $k \to \infty$, the iterates $M^k y$ tend to be aligned with the associated eigenvector $\tilde{w}_{\max}$ (or with the respective two-dimensional eigenspace, if $\tilde{\lambda}_{\max}$ is complex). If $\tilde{\lambda}_{\max}$ is real and $\tilde{\lambda}_{\max} > 1$, then for large $k$ the iterates $M^k y$ behave as $(\tilde{\lambda}_{\max})^k \tilde{w}_{\max}$; hence depending on the signs of components of $\tilde{w}_{\max}$ they either tend to $-\infty$ or escape from $U_S$. If $|\tilde{\lambda}_{\max}| < 1$, then the iterates $M^k y$ are attracted by (43), which is outside $U_S$ by our choice of $S$. If $\tilde{\lambda}_{\max}$ is complex, or if it is real and $\tilde{\lambda}_{\max} < -1$, then for any $K > 0$ there exists $M^K y$ outside $\mathbb{R}^N$ for some $k > K$. QED

The bifurcation considered in theorem 7 was studied by Postlethwaite [22] for a particular dynamical system in $\mathbb{R}^4$. She proved that ‘there are no dynamical structures which merge with the cycle at the point of stability loss’, in agreement with our theorem.

6. Homoclinic cycles

6.1. Transition matrices

A homoclinic cycle is a heteroclinic cycle, where all equilibria are related by a symmetry $\gamma$, $\gamma \xi_j = \xi_{j+1}$. The transition matrix of a homoclinic cycle is a basic transition matrix (24) which is a product of a permutation matrix $A$ and a local matrix $B$. Any permutation is a composition of cyclic permutations. Suppose the permutation defined by matrix $A$ is a combination of $L + 1$ cyclic permutations. The first permutation involves $n_s$ significant basis vectors, and the last $n_1$ basis vectors are insignificant. We order basis vectors in agreement with the relations

$$A e_j = \begin{cases} e_{j+1}, & j \neq N_l \text{ for any } 1 \leq l \leq L, \\ e_{j-n_1+1}, & j = N_l. \end{cases}$$

where, as above, we have denoted $N_l = n_s + n_1 + n_2 + \cdots + n_l$. Without any loss of genericity we assume that $e_1$ is the expanding eigenvector of $df(\xi)$. 

For the basis vectors ordered according to (51), the permutation matrix $A$ is comprised of $L$ non-vanishing diagonal blocks $n_l \times n_l$, each being of the form (44). Thus, the transition matrix is

$$M = \begin{pmatrix}
    b_2 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & \ldots \\
    b_3 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \ldots \\
    b_{n_l+2} & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & \ldots \\
    b_{n_l+3} & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n_l+1} & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots 
\end{pmatrix}.$$  

(52)

6.2. Two examples of type Z homoclinic cycles

In this section we present two general examples of type Z homoclinic cycles. In both cases, our constructions employ a known homoclinic cycle in $\mathbb{R}^n$ with an empty insignificant subspace (i.e. all eigenvalues are significant and the permutation matrix $A$ is just a single cyclic permutation involving $n_l$ vectors). We add $n_1$ insignificant transverse directions, where the acceptable $n_1$ depends on $n$. This step (enlargement of the insignificant subspace by adding new dimensions) can be repeated any number of times.

The first example. Postlethwaite and Dawes [23] presented an example of system (1) with a $\mathbb{Z}_n \ltimes \mathbb{Z}_2^n$ symmetry group possessing a type Z homoclinic cycle. The subgroup $\mathbb{Z}_2^n$ acts on $\mathbb{R}^n$ as $n$ reflections

$$s_j(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, -x_j, \ldots, x_n);$$

the subgroup $\mathbb{Z}_n$ is generated by the cyclic permutation

$$\rho(x_1, x_2, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1}).$$

Suppose

– system (1) has $n$ equilibria $\xi_j$ related by the symmetry $\rho$. $\rho \xi_j = \xi_{j+1}$; each equilibrium has an isotropy subgroup $\mathbb{Z}_n^{n_1+2}$;

– unstable manifold of $\xi_j$ is one-dimensional, has the isotropy subgroup $\mathbb{Z}_2^n$ and is attracted by $\xi_{j+1}$.

Under these assumptions the system has a heteroclinic cycle. The transition matrix of the cycle is a block of size $(n_1 + 1) \times (n_1 + 1)$ (recall that $n = n_1 + n_r + 2$). Note that instead of the condition that the unstable manifold of $\xi_j$ is one-dimensional we can impose the weaker condition that the connection $\kappa_j = W^u(\xi_j) \cap W^s(\xi_{j+1})$ is one-dimensional and it belongs to $\text{Fix}(\mathbb{Z}_n^{n_1+1})$.

We can enlarge the dimension of (1) by any $K$ dividing $n$ an arbitrary number of times as follows. Consider the action of the group $\mathbb{Z}_n \ltimes \mathbb{Z}_2^{n+K}$, where the subgroup $\mathbb{Z}_2^{n+K}$ acts on $\mathbb{R}^{n+K}$ as $n + K$ reflections

$$s_j(x_1, \ldots, x_j, \ldots, x_{n+K}) = (x_1, \ldots, -x_j, \ldots, x_{n+K})$$

and the subgroup $\mathbb{Z}_n$ is generated by a combination of cyclic permutations

$$\rho(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+K}) = (x_n, x_1, \ldots, x_{n-1}, x_{n+K}, x_{n+1}, \ldots, x_{n+K-1}).$$  

(53)
As a result of this modification the size of the transition matrix increases to $(n_1 + 1 + K) \times (n_1 + 1 + K)$ and a diagonal block (44) of size $K \times K$ is added to the matrix $A$.

A general third-order system in $\mathbb{R}^{n+K}$ with the above symmetry group is as follows:

$$\dot{x}_1 = x_1 \left( v_1 + \sum_{l=1}^{n+K} c_l x_l^2 \right),$$

$$\dot{x}_{n+1} = x_{n+1} \left( v_2 + \sum_{l=1}^{n/K-1} d_l x_l^2 + \sum_{l=n/K}^{K} d_n x_{n+l}^2 \right).$$

Equations for other $\dot{x}_j$ are obtained by application of the symmetry $\rho$. If $v_1 c_1 < 0$, the system has $n$ equilibria on coordinate axes, related by the symmetry $\rho$. Sufficient conditions on $v_1$ and $c_1$, $1 \leq l \leq n$, for existence in $\mathbb{R}^n$ of a homoclinic cycle connecting the equilibria are presented in [23]. Our construction thus implies existence of a homoclinic cycle in the extended $(n + K)$-dimensional space for all values of $v_2, c_l, n + 1 \leq l \leq n + K$ and $d_l$.

The second example follows [8]. The authors consider a heteroclinic cycle in $\mathbb{R}^3$ [1, 24] in a system (1) with the symmetry group $\mathbb{Z}^2 \ltimes \mathbb{Z}_2$ generated by the symmetries

$$s_1(x_1, x_2, x_3) = (x_1, x_2, -x_3),$$

$$s_2(x_1, x_2, x_3) = (x_1, -x_2, x_3),$$

$$\rho(x_1, x_2, x_3) = (-x_1, x_3, x_2).$$

In [8], the dimension of the system is increased to 5 and the symmetry group is enlarged to $\mathbb{Z}^4 \ltimes \mathbb{Z}_2$ by adding the symmetry

$$s_3(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4, -x_5).$$

The symmetries $s_1$ and $s_2$ act on the added dimensions $(x_4, x_5)$ trivially, and $\rho$ is modified to become

$$\rho(x_1, x_2, x_3, x_4, x_5) = (-x_1, x_3, x_2, -x_5, x_4).$$

The system in $\mathbb{R}^5$ with the above symmetry group truncated at the third order is as follows:

$$\dot{x}_1 = v_1 x_1 + c_1(x_2^2 - x_3^2) + c_2(x_4^2 - x_5^2) + x_1(c_3 x_1^2 + c_4(x_2^2 + x_3^2) + c_5(x_4^2 + x_5^2)),$$

$$\dot{x}_2 = x_2(v_2 + c_6 x_1 + c_7 x_1^2 + c_8 x_2^2 + c_9 x_4^2 + c_{10} x_2^2 + c_{11} x_3^2),$$

$$\dot{x}_3 = x_3(v_3 - c_6 x_1 + c_7 x_1^2 + c_8 x_2^2 + c_9 x_4^2 + c_{10} x_2^2 + c_{11} x_3^2),$$

$$\dot{x}_4 = x_4(v_1 + d_1 x_1 + d_2 x_1^2 + d_3 x_2^2 + d_4 x_2^2 + d_5 x_2^2 + d_6 x_2^2),$$

$$\dot{x}_5 = x_5(v_1 - d_1 x_1 + d_2 x_1^2 + d_3 x_2^2 + d_4 x_2^2 + d_5 x_2^2 + d_6 x_2^2).$$

If $v_1 c_3 < 0$, the system possesses two equilibria on the $x$ axis related by the symmetry $\rho$. Sufficient conditions for existence of a homoclinic cycle in $\mathbb{R}^3$ involving the equilibria are presented in [1, 24]. In [8] this homoclinic cycle in $\mathbb{R}^5$ is studied for particular values of coefficients $v, c$ and $d$ in (55).

By the same procedure dimension of (1) can be incremented by 1 or 2 any number of times. The dimension of the system can be also expanded by choosing the dimension of $L_j$ larger than unity, as in the first example considered in this subsection (the resultant transition matrix is not modified).
6.3. **Stability of a cycle when all transverse eigenvalues are negative**

If all transverse eigenvalues of \( d f(\xi) \) are negative, the cycle can be completely unstable or asymptotically stable (see theorem 4). Suppose vectors comprising the basis are ordered in accordance with (51). The eigenvalues associated with eigenvectors from the insignificant subspace are 1 in absolute value (see section 4.2). As proved in [23], the dominant eigenvalue of the left upper \( n_s \times n_s \) submatrix is real and larger than one if and only if

\[
b_1 + b_2 + \cdots + b_{n_s} > 1.
\]

By theorem 1, condition (56) is necessary and sufficient for a homoclinic cycle of type Z, whose all transverse eigenvalues are negative, to be asymptotically stable.

6.4. **Two simple cases**

If the dimension of the significant subspace is one or two, then the dominant eigenvalue and the associated eigenvector can be explicitly calculated in terms of eigenvalues of \( d f(\xi) \). The left upper submatrix is then

\[
\begin{pmatrix}
 b_1 \\
 1 \\
 b_0 
\end{pmatrix}
\]

for the dimension one or two, respectively.

If the dimension is one, the eigenvalue is

\[
\lambda = b_1
\]

and the necessary condition for stability is \( b_1 > 1 \). If the dimension is two, the dominant eigenvalue is

\[
\lambda = \frac{b_2 + \sqrt{b_2^2 - 4b_1}}{2},
\]

the necessary conditions for stability are \( b_2 > 0 \) and \( b_1 + b_2 > 1 \) (see [21]). These conditions imply that the first two components of the associated eigenvector have the same sign. We assume that they are positive.

We calculate other components of the eigenvector \( w \) of the transition matrix, associated with the dominant eigenvalue \( \lambda \). (For simplicity, the upper indices are omitted.) Consider the eigenvector components \( (w_{Nl+1}, w_{Nl+2}, \ldots, w_{Nl}) \) related to the same permutation cycle of length \( n_l \). The components satisfy the equations

\[
\begin{align*}
b_{Nl+1}w_1 + w_{Nl+2} &= \lambda w_{Nl+1} , \\
b_{Nl+2}w_1 + w_{Nl+3} &= \lambda w_{Nl+2} , \\
&\vdots \\
b_{Nl}w_1 + w_{Nl+1} &= \lambda w_{Nl} .
\end{align*}
\]

Denote \( h(k, n_l) = \text{mod}_{n_l}(k) \) and suppose \( 0 \leq k \leq n_l - 1 \). Multiplying the equations by \( \lambda^{h(k+1,n_l)} \), \( \lambda^{h(k+2,n_l)} \), respectively, and adding them up we obtain

\[
(b_{Nl+1}\lambda^{h(k+1,n_l)} + \cdots + b_{Nl}\lambda^{h(k+n_l,n_l)})w_1 = (\lambda^{n_l} - 1)w_{Nl+1}.
\]

Thus the components \( w_{Nl+1}, \ldots, w_{Nl} \) all have the same sign as \( w_1 \) as long as

\[
b_{Nl+1}\lambda^{h(k+1,n_l)} + \cdots + b_{Nl}\lambda^{h(k+n_l,n_l)} > 0 \quad \text{for any} \ 0 \leq k \leq n_l - 1.
\]

If the length of the first permutation cycle is larger than two, the same conditions guarantee stability of the cycle, namely, conditions (i)–(iii) of lemma 5 for eigenvalues and eigenvectors of the significant \( n_s \times n_s \) submatrix, and condition (57) for each insignificant permutation.
cycle. However we cannot express the eigenvalue $\lambda$ in terms of eigenvalues of $df(\xi)$ except for $n_s = 3$ and 4, but in these cases the expressions are too complex to be of any practical use.

7. Conclusion

We have defined type Z heteroclinic cycles as simple cycles with a certain action of subgroups of the symmetry group of the dynamical system. We have introduced the notion of fragmentary asymptotic stability of an invariant set and have derived necessary and sufficient conditions for asymptotic stability or fragmentary asymptotic stability of type Z heteroclinic cycles. For a type Z cycle we have calculated the basic transition matrices; the matrices depend on eigenvalues of the linearizations near steady states comprising the cycle and the action of the system symmetry group.

If all transverse eigenvalues of the linearizations near steady states are negative, such a heteroclinic cycle can be either asymptotically stable or completely unstable. The stability depends on whether the largest in absolute value eigenvalue of transition matrices (which are products of the basic transition matrices; all transition matrices are similar) is larger than one in absolute value. For such type Z cycles, eigenvalues of transition matrices play the same role, as eigenvalues of the linearizations in the study of stability of steady states and eigenvalues of the linearizations of the Poincaré maps in the study of stability of periodic orbits.

A type Z cycle, some of whose transverse eigenvalues are positive, can be fragmentarily asymptotically stable. We have derived a criterion for fragmentary asymptotic stability in terms of eigenvalues and eigenvectors of its transition matrices.

We have studied bifurcations occurring when conditions for fragmentary asymptotic stability are broken. Two types of bifurcations have been identified, transverse and resonance ones. A detailed study of transverse bifurcations is problematic, because the collection of maps that we use to study stability does not approximate correctly trajectories of the system for the critical parameter value. Nevertheless we have commented on how the stability of a cycle can change in a bifurcation of a steady state, where a transverse eigenvalue of the linearization near a steady state vanishes. For a resonance bifurcation, we prove that either a periodic orbit is born in it (if the dominant eigenvalue of a transition matrix becomes smaller than one), or no new invariant objects emerge in it (if other conditions for fragmentary asymptotic stability are broken).

We anticipate the following continuations of our study.

We have not considered all issues related to asymptotic stability and bifurcations of type Z heteroclinic cycles. As we have noted in section 5, bifurcations resulting in destruction of the cycle are yet to be investigated. Our study of transverse bifurcations is by far incomplete. We have assumed that in a neighbourhood of the cycle the behaviour of trajectories is accurately approximated by a collection of maps which only linearizations are taken into account. This is true in a neighbourhood, where the linear part is significantly larger than the nonlinearity, but when a (transverse) eigenvalue of the linearization vanishes, the neighbourhood of a steady state where this holds true shrinks to zero. For a more detailed analysis of the bifurcation, some nonlinear terms in the local expansion of the r.h.s. $f$ should also be included into the local map.

In section 6.2 we have presented two general examples of type Z homoclinic cycles. The question arises whether type Z homoclinic cycles of other kinds exist; if they do exist, their classification is wanted. (Heteroclinic cycles are apparently too diverse for any such classification to be useful.)
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