A SHORT PROOF ON THE TRANSITION MATRIX FROM THE SPECHT BASIS TO THE KAZHDAN–LUSZTIG BASIS

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Abstract. We provide a short proof on the change-of-basis coefficients from the Specht basis to the Kazhdan–Lusztig basis, using Kazhdan–Lusztig theory for parabolic Hecke algebra.

1. INTRODUCTION

It is well-known that the irreducible representations for the symmetric group $\Sigma_n$ are the Specht modules parametrized by the set of partitions of $n$, or equivalently, set of standard Young tableaux consisting of $n$ boxes. The Specht module has a purely combinatorial basis (called Specht basis) described in terms of certain alternating sums of so-called tabloids.

For the Young tableaux consisting of two rows of equal size, Russell–Tymoczko in [RT19] compare the Specht basis with another combinatorial basis (called the web basis) which arises from Temperley–Lieb algebra and knot theory. In their context, the web basis is a reincarnation of the Kazhdan–Lusztig basis, but this is not true in general. Their main result is a combinatorial model that gives a different proof to a special case of a classic theorem by Naruse [Nar89, Theorem 4.1] that the change-of-basis matrix is unitriangular, altogether with some vanishing conditions. The unitriangularity result is also found in [GM88], but without vanishing conditions.

In this paper, we give a short proof of [Nar89, Theorem 4.1] using Kazhdan–Lusztig theory for parabolic Hecke algebra. We also notice that the argument applies to all classical types, with a nonstandard notion of tableaux. For example, in type B we use certain centro-symmetric tableaux which are not the bi-tableaux that parametrize the irreducibles. That is to say, the analog of the Specht modules here are not the irreducibles. However, we hope that they correspond to the top cohomology of the Springer fiber corresponding to the parabolic subgroup. Furthermore, we omit the details for type D as future work. In this manuscript, we introduce a Specht basis for this module using certain alternating sum of centro-symmetric tabloids. We then prove a unitriangular theorem regarding the change-of-basis matrix between the Specht basis and the Kazhdan–Lusztig basis (cf. Theorem 3.1).

2. COMBINATORICS

2.1. Weyl groups. For type $\Phi = A$ or $B$, let $W^\Phi_d$ be the Weyl group of type $\Phi_d$ associated with the Dynkin diagrams in Figure 2.1.

Figure 1. Dynkin diagrams of type $A_d$ and $B_d$.

Write

$$[d] := [1, d] \cap \mathbb{Z}, \quad [\pm d] := [-d, d] \cap \mathbb{Z}. \quad (2.1)$$

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Set

\[ I_d^A = [d] \quad \text{and} \quad I_d^B = [0, d - 1] \cap \mathbb{Z}. \]  

(2.2)

We denote by \( S_d^\Phi = \{ s_i^\Phi : i \in I_d^\Phi \} \), the corresponding generators of \( W_d^\Phi \) as Coxeter groups. Denote by \( \ell^\Phi : W_d^\Phi \to \mathbb{Z}_{\geq 0} \) the length function.

For any set \( I \), let \( \text{Perm}(I) \) be the group of permutations on \( I \), and let \( (i, j) \in \text{Perm}(I) \) be the transposition for \( i, j \in I \). It is a standard fact that \( W_d^\Phi \) can be identified as a group of certain permutations (see [BB05]). Here we treat them as subgroups of \( \text{Perm}([\pm d]) \) as follows:

\[
W_d^A := \{ g \in \text{Perm}([\pm d]) : g(-i) = -g(i) \text{ for all } i \},
\]

(2.3)

\[
W_d^A_{d-1} := \{ g \in W_d^B : \text{neg}(g) = 0 \},
\]

where the function \( \text{neg} \) counts the total number of negative entries of \( \{1, \ldots, d\} \), i.e.,

\[
\text{neg} : W_d^B \to \mathbb{Z}_{\geq 0}, \quad g \mapsto \#(\{ k \in [d] : g(k) < 0 \}).
\]

(2.4)

The generators can be identified with the following permutations in \([\pm d] \):

\[
s_i = s_i^A = s_i^B = (i, i + 1)(-i, -i - 1) \quad \text{for} \quad 1 \leq i \leq d - 1,
\]

(2.5)

\[
s_0^B = (-1, 1).
\]

It is sometimes convenient to use the one-line notation:

\[ w \equiv |w(1), w(2), \ldots, w(d)| \quad \text{for } w \in W_d^\Phi. \]

(2.6)

We remark that the element \( w \) is uniquely determined by these \( d \) values due to the centro-symmetry condition: \( g(-i) = -g(i) \) for all \( i \).

2.2. Parabolic Hecke algebras. Let \( \mathcal{H}^\Phi = \mathcal{H}(W_d^\Phi) \) be the Hecke algebra of \( W_d^\Phi \) over \( \mathbb{C}[q, q^{-1}] \). As a free module, \( \mathcal{H}^\Phi \) has a basis \( \{ H_w^\Phi : w \in W_d^\Phi \} \). The multiplication in \( \mathcal{H}^\Phi \) is determined by

\[
H_w^\Phi H_x^\Phi = H_{wx}^\Phi \quad \text{if } \ell^\Phi(wx) = \ell^\Phi(w) + \ell^\Phi(x),
\]

\[
(H_s^\Phi)^2 = (q^{-1} - q) H_s^\Phi + H_e^\Phi \quad \text{if } s \in S^\Phi,
\]

(2.7)

where \( e \in W_d^\Phi \) is the identity, and hence \( H_e^\Phi \) is the identity element in \( \mathcal{H}^\Phi \). By a slight abuse of notation, we use the same symbol \( H_i \) (\( 1 \leq i \leq n - 1 \)) to denote the element \( H_{s_i^\Phi}^\Phi \in \mathcal{H}^\Phi \). We write \( H_0^\Phi = H_{s_0^\Phi}^\Phi \) for \( \Phi = B \). It is a standard fact that \( \mathcal{H}^\Phi \) is generated as a \( \mathbb{C}[q, q^{-1}] \)-algebra by \( \{ H_i^\Phi : i \in I_d^\Phi \} \).

For each subset \( J \subset I_d^\Phi \), denote the corresponding parabolic subgroup and parabolic Hecke algebra of \( W_d^\Phi \) by

\[ W_J^\Phi = \langle s_j^\Phi : j \in J \rangle \quad \text{and} \quad \mathcal{H}(W_J^\Phi) = \langle H_j^\Phi : j \in J \rangle,
\]

(2.8)

respectively. We also denote the set of shortest right coset representatives for \( W_J^\Phi \setminus W_d^\Phi \) by

\[ \mathcal{D}_J^\Phi = \{ w \in W_d^\Phi : \ell^\Phi(wg) = \ell^\Phi(w) + \ell^\Phi(g) \text{ for all } w \in W_J^\Phi \}. \]

(2.9)

Denote the induced trivial module corresponding to \( J \subset I_d^\Phi \) by

\[ M_J^\Phi = \mathbb{C}[q, q^{-1}] \otimes_{\mathcal{H}_J^\Phi} \mathcal{H}^\Phi, \]

(2.10)

where \( \mathbb{C}[q, q^{-1}] \) is regarded as a right \( \mathcal{H}^\Phi \)-module by setting

\[ f \cdot H_i^\Phi = q^{-1} f \quad \text{for } f \in \mathbb{C}[q, q^{-1}]. \]

(2.11)

The module \( M_J^\Phi \) admits a standard basis

\[ \{ M_w = 1 \otimes H_w^\Phi : w \in \mathcal{D}_J^\Phi \}. \]

(2.12)
We note that $M^J_\Phi$ can also be identified as the right $\mathcal{H}_\Phi$-module $y_J \mathcal{H}_\Phi$ under the assignment $M_w \mapsto y_J H^\Phi_w$, where $y_J = \sum_{w \in W_d^J} q^{\ell(w)} H^\Phi_w$, on which $\mathcal{H}_\Phi$ acts by a right multiplication. The action of $\mathcal{H}_\Phi$ on $M^J_\Phi$ below follows from (2.7) and a straight-forward calculation:

\[ M_w \cdot H^\Phi_i = \begin{cases} M_{w s_i} & \text{if } w s_i \in D^\Phi_J, \quad \ell^\Phi(w s_i) > \ell^\Phi(w), \\ M_{w s_i} + (q^{-1} - q)M_w & \text{if } w s_i \in D^\Phi_J, \quad \ell^\Phi(w s_i) < \ell^\Phi(w), \\ q^{-1}M_w & \text{if } w s_i \notin D^\Phi_J. \end{cases} \]

Following [Soc97] Theorem 3.1, $M^J_\Phi$ admits a Kazhdan–Lusztig basis $\{M_w : w \in D^\Phi_J\}$ such that

\[ M_w = \sum_{x \in D^\Phi_J} m^\Phi_{x,w}(q^{-1})M_x, \]

where $m^\Phi_{x,w} \in \mathbb{Z}[q^{-1}]$. We also define polynomials $p^\Phi_{w,x}(q^{-1}) \in \mathbb{Z}[q^{-1}]$ such that

\[ M_w = \sum_{x \in D^\Phi_J} p^\Phi_{w,x}(q^{-1})M_x. \]

Below we recall some properties of the (inverse) parabolic Kazhdan–Lusztig polynomials.

**Lemma 2.1.** We have the following:

(a) If $x \not\leq w$ with respect to the Bruhat order on $W^\Phi_d$, then $m^\Phi_{x,w} = 0 = p^\Phi_{x,w}$,

(b) $m^\Phi_{x,x} = 1 = p^\Phi_{x,x}$.

**Proof.** It follows from [Soc97] Theorem 3.1 that the matrix $(m_{x,w}(q^{-1}))_{x,w}$ is upper unitriangular, its inverse matrix, $(p_{x,w}(q^{-1}))_{x,w}$, is also upper unitriangular. \qed

2.3. Young tableaux. Now we want to associate each parabolic subgroup $W^J_\Phi$ of $W^\Phi_d$ a “Young tableau” beyond type A. The idea here is to use the corresponding composition (cf. [DJ86, BKLW18]). To be precise, we set

\[ \Lambda^A(d) = \bigsqcup_{n \geq 0} \Lambda^A(n, d) \quad \text{and} \quad \Lambda^B(d) = \bigsqcup_{n \geq 0} \Lambda^B(n, d), \]

where the legit $n$-part (signed) compositions are defined as

\[ \Lambda^A(n, d) = \left\{ \lambda = (\lambda_i)_{i \in [n]} \in \mathbb{Z}_{>0}^n : \sum_{i=1}^{n} \lambda_i = d \right\}, \]

\[ \Lambda^B(n, d) = \left\{ \lambda = (\lambda_i)_{i \in [-r]} \in \mathbb{Z}_{>0}^n : \begin{array}{c} \lambda_0 \in 1 + 2\mathbb{Z}, \quad \sum_{i=-r}^{r} \lambda_i = 2d + 1, \\ \lambda_{-i} = \lambda_i \quad \forall i \end{array} \right\} \quad \text{if } n = 2r + 1, \]

\[ \{ \lambda \in \Lambda^B(2r + 1, d) : \lambda_0 = 1 \} \quad \text{if } n = 2r. \]

In other words, the corresponding parabolic subgroups are generated by

\[ S^A(d) = \left\{ s_{\lambda_1}, s_{\lambda_1 + \lambda_2}, \ldots , s_{\lambda_1 + \ldots + \lambda_{n-1}} \right\} \quad \text{for} \quad \lambda \in \Lambda^A(n, d), \]

and generated by (abbreviate $\left\lfloor \frac{n}{2} \right\rfloor$ by $\lambda^k_0$ here)

\[ S^B(d) = \left\{ s_{\lambda_1}, s_{\lambda_1 + \lambda_2}, \ldots , s_{\lambda_1 + \ldots + \lambda_{r-1}} \right\} \quad \text{if } n = 2r, \]

\[ \left\{ s_{\lambda^k_0}, s_{\lambda^k_0 + \lambda_1}, \ldots , s_{\lambda^k_0 + \lambda_1 + \ldots + \lambda_{r-1}} \right\} \quad \text{if } n = 2r + 1, \]

for $\lambda \in \Lambda^B(n, d)$. \quad \text{We also write } W^\Phi_d \text{ and } \mathcal{H}_\lambda^\Phi \text{ to denote the corresponding parabolic subgroups and the parabolic Hecke algebras, respectively.}
For type A, any composition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^A(n, d)$ defines a Young subgroup

$$W^A_\lambda \simeq \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \times \Sigma_{\lambda_n}.$$ 

That is, any two compositions giving the same Young subgroup (and hence parabolic Hecke algebra) correspond to the same partition. We denote the set of type A partitions by $\Pi^A(d) = \bigsqcup_{n \geq 1} \Pi^A(n, d)$, where

$$\Pi^A(n, d) = \{ \lambda \in \Lambda^A(n, d) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}. \quad (2.20)$$

However, for type B, a composition $\lambda = (\lambda_0, \ldots, \lambda_n) \in \Lambda^B(n, d)$ defines a Young subgroup

$$W^B_\lambda \simeq W^B_{\lambda_0} \times \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \times \Sigma_{\lambda_n}.$$ 

In particular, $W^B_\lambda$ is always a product of symmetric groups when $n$ is even. Hence, compositions $\lambda, \mu$ describe the same Young subgroup (and hence parabolic Hecke algebra) if $\lambda_0 = \mu_0$ and also that $(\lambda_1, \ldots, \lambda_n)$ corresponds to the same partition as $(\mu_1, \ldots, \mu_n)$. We denote the set of type B "partitions" by $\Pi^B(d) = \bigsqcup_{n \geq 1} \Pi^B(n, d)$, where

$$\Pi^B(n, d) = \{ \lambda \in \Lambda^B(n, d) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}. \quad (2.21)$$

**Example 2.2.** Let $d = 3$. There are 8 subsets of $I^B_d$, and hence we have

| $\lambda \in \Lambda^B(3)$ | $W^B_\lambda$ | $J$ |
|--------------------------|-------------|-----|
| (7)                      | $W^B_3$     | $\{0, 1, 2\}$ |
| (1, 5, 1)                | $\langle s^B_0, s^B_1 \rangle \simeq W^B_2$ | $\{0, 1\}$ |
| (2, 3, 2)                | $\langle s^B_0, s^B_2 \rangle \simeq \Sigma_2 \times \Sigma_2$ | $\{0, 2\}$ |
| (3, 1, 3)                | $\langle s^B_1, s^B_2 \rangle \simeq \Sigma_3$ | $\{1, 2\}$ |
| (1, 1, 3, 1, 1)          | $\langle s^B_0 \rangle \simeq \Sigma_2$ | $\{0\}$ |
| (1, 2, 1, 2, 1)          | $\langle s^B_1 \rangle \simeq \Sigma_2$ | $\{1\}$ |
| (2, 1, 1, 1, 2)          | $\langle s^B_2 \rangle \simeq \Sigma_2$ | $\{2\}$ |
| (1, 1, 1, 1, 1, 1)       | $\{e\}$     | $\emptyset$ |

Let $\lambda \in \Lambda^B(n, d)$. For now we assume that $\Phi = A$. Let its corresponding Young diagram be $[\lambda] = \{(i, j) : i \geq 1, 1 \leq j \leq \lambda_i\}$. A Young tableau of shape $\lambda$ (or $\lambda$-tableau for short) is a bijection

$$T : [\lambda] \to [d].$$

For example, when $\lambda = (m, m)$ or $(m, m, m)$, a $\lambda$-tableau represents a $2 \times m$ or $3 \times m$ grid, respectively, whose boxes are filled in by every number, exactly once, from 1 to $d = 2m$ or $d = 3m$, respectively. We also write $\text{Sh}(T) = \lambda$. The set of all young tableaux admits a (right) action of the symmetric group $W^A_{d-1} = \Sigma_d$ by permuting the letters that are filled in the boxes.

A Young tableau is called row standard if the entries in each row are increasing; it is called standard if the entries in each row and each column are increasing. Denote by $\text{rStd}(\lambda)$ and $\text{Std}(\lambda)$ the sets of row standard and standard Young tableaux of shape $\lambda$, respectively. We remark that the size of $\text{Std}(\lambda)$ is counted by the hook formula, and hence

$$\#\text{Std}((m, m)) = \frac{1}{m+1} \binom{2m}{m} \quad \text{and} \quad \#\text{Std}((m, m, m)) = \frac{2 \cdot (3m)!}{m! \cdot (m+1)! \cdot (m+2)!}. \quad (2.22)$$

It is well-known that the following assignment is a bijection:

$$\text{rStd}(\lambda) \to D^A_{\lambda}, \quad T \mapsto w_T, \quad (2.23)$$

where $w_T$ is given by row-reading of tableau $T$.

We now generalize this bijection to type B by defining the Young tableaux of classical type. For $\Phi = B$, $\lambda \in \Lambda^B(n, d)$, denote the corresponding Young diagram by

$$[\lambda] = \begin{cases} 
\{(0, j) : -\lambda_1 \leq j \leq \lambda_1\} \cup \{\pm(i, j) : i \geq 1, 0 \leq j \leq \lambda_i - 1\} & \text{if } n = 2r+1, \\
\{\pm(i, j) : i \geq 1, 1 \leq j \leq \lambda_i\} & \text{if } n = 2r. \quad (2.24)
\end{cases}$$
Note that when $d$ is even, the corresponding Young diagram is just a type A Young diagram, with a copy obtained by rotating it 180 degrees.

**Example 2.3.** For $d = 2$, $\Pi^B(2) = \{(2, 1, 2), (1, 1, 1, 1, 1)\}$, and hence the tableaux are

\[
\begin{array}{c|c|c}
\lambda & 2 & 3 \\
\hline
4 & 5 & 1 \ 
\end{array}
\]

For $d = 3$, we then have

| $\lambda$ | (7) | (1, 5, 1) | (2, 3, 2) | (3, 1, 3) | (1, 1, 3, 1, 1) | (1, 2, 1, 2, 1) | (1, 1, 1, 1, 1, 1) |
|-----------|-----|-----------|-----------|-----------|----------------|----------------|----------------|

A $\lambda$-tableau of type B is again a bijection $T : [\lambda] \rightarrow [d]$. We also write $\text{Sh}(T) = \lambda$ by a slight abuse of notation. Such a $\lambda$-tableau is called row standard if the entries in each row are increasing, while it is called standard if the entries in each row and each column are increasing. Denote by $\text{rStd}^\Phi(\lambda)$ and $\text{Std}^\Phi(\lambda)$ the sets of type $\Phi$ row standard and standard $\lambda$-tableaux, respectively. The following assignment is an obvious bijection:

\[
v_T = \sum_{R \in \text{rStd}(d)} e^\lambda(R) \{ T \cdot w \},
\]

where $w_T$ is given by row-reading of $T$.

Let $\{T\}$ be the equivalence class (called a tabloid) of the $\lambda$-tableau $T$ under the (right) action of $W^A_\lambda \subset W^A_d$. It is clear that each class contains a unique row standard $\lambda$-tableau.

### 2.4. Specht module

We refer to [Spe35, Jam78, JK81] for an extensive background on Specht modules. For now we assume that $\Phi = A$. Define the Specht vector corresponding to $T \in \text{Sh}(\lambda)$ by

\[
v_T = \sum_{\lambda' \in \text{col}(T)} (-1)^{e^\lambda(w)} \{ T \cdot w \},
\]

where $\text{col}(T)$ is the subset of $\Sigma_d$ consisting of the permutations which reorder the columns of $T$. In other words, we have $\text{col}(T) = W^A_{\lambda'}$ where $\lambda'$ is the transposition of $\lambda$, i.e.,

\[
\lambda'_i = \# \{ j \in \mathbb{Z} : \lambda_j \geq i \}.
\]

For $T \in \text{Std}^\Phi(\lambda)$, $R \in \text{rStd}^\Phi(\lambda)$, and $\lambda \in \Lambda^\Phi(n, d)$, define $c^\Phi_{R, T} \in \mathbb{Z}$ by

\[
v_T = \sum_{R \in \text{rStd}(d)} c^\Phi_{R, T} \{ R \}.
\]

We now generalize this to type B by defining

\[
v_T = \sum_{R \in \text{rStd}^B(\lambda)} c^\Phi_{R, T} \{ R \} \text{ for } T \in \text{Sh}(\lambda) \text{ and } \lambda \in \Lambda^\Phi(n, d).
\]

**Lemma 2.4.** If $c^\Phi_{R, T} \neq 0$, then $w_R \leq w_T$ with respect to the Bruhat order.

We denote the (classical) Specht module over $\mathbb{C}$ corresponding to $\lambda \in \Lambda^\Phi(d)$ by

\[
S^\lambda_{\Phi} = \bigoplus_{T \in \text{Std}^\Phi(\lambda)} \mathbb{C}v_T,
\]

where the (right) action of $W^\Phi_d$ is given by $v^\Phi_T \cdot s_i = v^\Phi_{T \cdot s_i}$. We call $\{v^\Phi_T : T \in \text{Std}^\Phi(\lambda)\}$ the tableaux basis for $S^\lambda_{\Phi}$.
Note that for $T \in \text{Std}^\Phi(\lambda)$, it is not guaranteed that $T \cdot s_i \in \text{Std}^\Phi(\lambda)$, and hence $v_T^\Phi$ is, in general, not a single Specht vector corresponding to a standard $\lambda$-tableau, but a linear combination of Specht vectors.

3. The change-of-basis matrix

Now we embed the Specht module $S^\Phi_d$ into the induced trivial module $M^\Phi_d$ by identifying the tabloid \{R\} for $R \in r\text{Std}^\Phi(d)$, with the standard basis element $M^\Phi_w$ in $M^\Phi_d$, where $w \in D^\Phi_d$.

Under the embedding, we denote by $a^\Phi_{x,T}$ (for $T \in r\text{Std}^\Phi(J)$ and $x \in D^\Phi_J$) the change-of-basis coefficients from the Specht basis to the Kazhdan–Lusztig basis, i.e.,

$$v_T = \sum_{x \in D^\Phi_J} a^\Phi_{x,T} M_x. \quad (3.1)$$

Recall that $\leq$ is the Bruhat order on $D^\Phi_J \subset W^\Phi_d$.

**Theorem 3.1.** Fix total orders $\leq_D$ on $D^\Phi_J$ and $\leq_r$ on $r\text{Std}^\Phi(J)$ satisfying

$$x \leq y \in D^\Phi_J \Rightarrow x \leq_D y \quad \text{and} \quad w_R \leq w_T \in D^\Phi_J \Rightarrow R \leq_r T,$$

respectively. The matrix $(a^\Phi_{x,T})_{x,T}$ is upper unitriangular in the sense that

$$a^\Phi_{w_T,T} = 1 \quad \text{for} \quad T \in r\text{Std}^\Phi(J), \quad a^\Phi_{x,T} = 0 \quad \text{if} \quad x \not\leq_D w_T.$$

Moreover, $a^\Phi_{x,T} = 0$ if $x \not\leq w_T$.

**Proof.** Combining Lemmas 2.1 and 2.4 we obtain the following:

$$v_T = \sum_{R \in r\text{Std}^\Phi(J)} \sum_{x \leq w_R} c^\Phi_{R,T} m^\Phi_{x,w_R}(1) M_x = \sum_{T \in r\text{Std}^\Phi(J)} \sum_{w_R \leq T} c^\Phi_{R,T} m^\Phi_{w_T,w_R}(1) M_x. \quad (3.2)$$

It follows from the unitriangularity of $(c^\Phi_{R,T})$ and $(m^\Phi_{w_T,w_R}(1))$ that

$$a^\Phi_{w_T,T} = \sum_{R \in r\text{Std}^\Phi(J)} c^\Phi_{R,T} m^\Phi_{w_T,w_R}(1) = 1.$$

Also, if $x \not\leq_D w_T$ then $x \not\leq w_T$, and so $a^\Phi_{x,T} = 0$. \hfill \square

**Remark 3.2.** For type A, Theorem 3.1 recovers [Nar89, Theorem 4.1] for arbitrary $J$. If $J = [2n] \setminus \{n\}$, then it also recovers [RT19, Theorems 5.5 and 5.7].

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