New Upper Bounds on $A(n,d)$

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Abstract—Upper bounds on the maximum number of codewords in a binary code of a given length and minimum Hamming distance are considered. New bounds are derived by a combination of linear programming and counting arguments. Some of these bounds improve on the best known analytic bounds. Several new record bounds are obtained for codes with small lengths.

I. INTRODUCTION

Let $A(n,d)$ denote the maximum number of codewords in a binary code of length $n$ and minimum Hamming distance $d$. $A(n,d)$ is a basic quantity in coding theory. Lower bounds on $A(n,d)$ are obtained by constructions. For survey on the known lower bounds the reader is referred to [9].

In this work we consider upper bounds on $A(n,d)$. The most basic upper bound on $A(n,d)$, $d = 2e + 1$, is the sphere packing bound, also known as the Hamming bound:

$$A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^{2e} \binom{n}{i}}.$$  \hspace{1cm} (1)

Johnson [8] has improved the sphere packing bound. In his theorem, Johnson used the quantity $A(n, d, w)$, which is the maximum number of codewords in a binary code of length $n$, constant weight $w$, and minimum distance $d$:

$$A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^{2e} \binom{n}{i} + \frac{(n+1) - (2e+1)}{(2e+2)} A(n, 2e+2, e+1)}.$$  \hspace{1cm} (2)

In [11] a new bound was obtained:

$$A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^{2e} \binom{n}{i} + \frac{(n+2)}{(2e+2)} A(n+1, 2e+2, e+2).}$$  \hspace{1cm} (3)

This bound is at least as good as the Johnson bound for all values of $n$ and $d$, and for each $d$ there are infinitely many values of $n$ for which the new bound is better than the Johnson bound.

When someone is given specific, relatively small values, of $n$ and $d$, usually the best method to find upper bound on $A(n,d)$ is the linear programming (LP) bound. A summary about this method and some new upper bounds appeared in [11]. However, the computation of this bound is not tractable for large values of $n$. In this work we will present new upper bounds on $A(n, 2e + 1)$, $e \geq 1$.

Let $F_2 = \{0, 1\}$ and let $F_2^n$ denote the set of all binary words of length $n$. For $x, y \in F_2^n$, $d(x, y)$ denote the Hamming distance between $x$ and $y$. Given $x, y \in F_2^n$ such that $d(x, y) = k$, we denote by $p_{i,j}^k$ the number of words $z \in F_2^n$ such that $d(x, z) = i$ and $d(z, y) = j$. This number is independent of choice of $x$ and $y$ and equal to

$$p_{i,j}^k = \begin{cases} \frac{(\frac{1}{2})^{k-i}}{\frac{(n-k)}{2}!} & \text{if } i + j - k \text{ is even,} \\ 0 & \text{if } i + j - k \text{ is odd.} \end{cases}$$

If $x = y$, then $p_{i,j}^0 = \delta_{i,j}v_i$, where $v_i = \binom{n}{i}$ is the number of words of distance $i$ from $x \in F_2^n$, and $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. We also denote $v = 2^n$. The $p_{i,j}^k$’s are the intersection numbers of the Hamming scheme and $v_i$ is the valency of the relation $R_i$. For the connection between association schemes and coding theory the reader is referred to [6], [10, Chapter 21].

An $(n, M, 2e + 1)$ code $C$ is a nonempty subset of $F_2^n$ of cardinality $M$ and minimum Hamming distance $2e + 1$. For a word $x \in F_2^n$, $d(x, C)$ is the Hamming distance between $x$ and $C$, i.e., $d(x, C) = \min_{c \in C} d(x, c)$. A word $h \in F_2^n$ is called a hole if $d(h, C) > e$ and $H = \{ h \in F_2^n : d(h, C) > e \}$ is the set of all holes. Clearly, we have

$$|H| = v - |C|V(n,e),$$  \hspace{1cm} (4)

where $V(n,e) = \sum_{v=0}^{n} v_j$ is the volume of sphere of radius $e$. The distance distribution of $C$ is defined as the sequence $A_i = |\{(c_1, c_2) \in C \times C : d(c_1, c_2) = i\}|/|C|$ for $0 \leq i \leq n$ and $A_i(e)$ denote the number of codewords at distance $i$ from $c \in C$. We also define the (non-normalized) holes distance distribution $\{D_i\}_{i=0}^{n}$ by $D_i = |\{(h_1, h_2) \in H \times H : d(h_1, h_2) = i\}|$, and $D_i(h)$ denote the number of holes at distance $i$ from $h \in H$. Finally, we define $NC(h, C, \Delta)$ to be the number of codewords of $C$ at distance $\Delta$ from a hole $h$.

II. HOLES DISTANCE DISTRIBUTION

In the first theorem we state that for a given $(n, M, 2e + 1)$ code $C$, the distance distribution of the holes is uniquely determined by the distance distribution $\{A_i\}_{i=0}^{n}$ of the code $C$.

**Theorem 1:** If $C$ is an $(n, M, 2e + 1)$ code with distance distribution $\{A_i\}_{i=0}^{n}$, then

$$D_i = v\delta_{i} + |C| (R(C, i) - 2V(n,e)v_i),$$

where $R(C, i)$ is the number of $C$-holes at distance $i$. The statement is equivalent to the identity $\{A_i\}_{i=0}^{n}$.

For the proof of this theorem and for more information about the bounds and linear programming we refer the reader to [8].
for each $i$, $0 \leq i \leq n$, where

$$R(C, i) = \sum_{k=0}^{n} \left( \delta_{i,k} + 2 \sum_{j=1}^{e} p_{i,j}^k + \sum_{l=1}^{n} \sum_{m=1}^{n} p_{i,m}^k \sum_{j=1}^{e} p_{i,j}^l \right) A_k.$$  \hspace{1cm} (5)

**Corollary 1:** Let $C$ be an $(n, M, 2e+1)$ code with distance distribution $\{A_i\}_{i=0}^{n}$. If $\{q_i\}_{i=0}^{n}$ is a sequence of real numbers, then

$$\sum_{i=0}^{n} q_i D_i = v \sum_{i=0}^{n} q_i v_i + |C| \sum_{i=0}^{n} q_i (R(C, i) - 2V(n, e)v_i).$$  \hspace{1cm} (6)

By using Corollary 1 for any given sequence $\{q_i\}_{i=0}^{n}$ we obtain a linear combination of the $D_i$’s. By finding a lower bound on this combination we can obtain an upper bound on the size of $C$.

**Example 1:** Let $q_0 = 1$ and $q_i = 0$, $i > 0$. Clearly $v_0 = 1$ and by (5) we have $R(C, 0) = V(n, e)$. After substituting the trivial bound $D_0 \geq 0$ to (6) we obtain the sphere packing bound.

The sequence $\{q_i\}_{i=0}^{n}$ of Corollary 1 will be called the holes distance indices (HDI) sequence. For convenience, in the rest of the paper we will write $\{q_i\}$ instead of $\{q_i\}_{i=0}^{n}$. In the next two sections we will find some good HDI sequences $\{q_i\}$ and develop methods to find lower bounds on $\sum_{i=0}^{n} q_i D_i$.

### III. HDI Sequences with Small Indices

In this section we consider HDI sequences, where nonzero $q_i$’s correspond to small indices. The following lemma gives an alternative expression for $D_i$.

**Lemma 1:** For each $i$, $0 \leq i \leq n$,

$$D_i = \sum_{h \in H} \left( v_i - \sum_{k=e+1}^{e+i} NC(h, C, k) \sum_{j=0}^{e} p_{i,j}^k \right).$$

Given a sequence $\{q_i\}$, by using Lemma 1 and (4) we estimate $\sum_{i=0}^{n} q_i D_i$ in the following way.

$$\sum_{i=0}^{n} q_i D_i = \sum_{i=0}^{n} q_i \sum_{h \in H} \left( v_i - \sum_{k=e+1}^{e+i} NC(h, C, k) \sum_{j=0}^{e} p_{i,j}^k \right)$$

$$= \sum_{h \in H} \left( \sum_{i=0}^{n} q_i v_i - \sum_{i=0}^{n} q_i \sum_{k=e+1}^{e+i} NC(h, C, k) \sum_{j=0}^{e} p_{i,j}^k \right)$$

$$\geq (v - |C|V(n, e)) \left( \sum_{i=0}^{n} q_i v_i - \xi(C, \{q_i\}) \right),$$  \hspace{1cm} (7)

where

$$\xi(C, \{q_i\}) = \max_{h \in H} \left\{ \sum_{i=0}^{n} q_i \sum_{k=e+1}^{e+i} NC(h, C, k) \sum_{j=0}^{e} p_{i,j}^k \right\}.$$  \hspace{1cm} (8)

By combining (6) and (7) we obtain

**Theorem 2:** If $C$ is an $(n, M, 2e+1)$ code with distance distribution $\{A_i\}_{i=0}^{n}$, then

$$|C| \leq \frac{v}{V(n, e) + \sum_{i=0}^{n} q_i (V(n, e)v_i - R(C, i))},$$

provided $\xi(C, \{q_i\})$ is not zero, where $\xi(C, \{q_i\})$ is given by (8) and $R(C, i)$ is given by (5).

**Example 2:** Let $q_1 = 1$ and $q_i = 0$ for $i \neq 1$. From (5) and (6) we have

$$R(C, 1) = V(n, e)v_1 - p_{1,e}^{e+1}v_{e+1}^{e+1} + p_{1,e}^{e+1}p_{e+1,e}^{2e+1} A_{2e+1}$$

and

$$\xi(C, \{q_i\}) = p_{1,e}^{e+1} \max_{h \in H} \{NC(h, C, e + 1)\}.$$

Thus, using Theorem 2 we obtain

**Theorem 3:** If $C$ is an $(n, M, 2e+1)$ code with distance distribution $\{A_i\}_{i=0}^{n}$, then

$$|C| \leq \frac{v}{V(n, e) + \sum_{i=0}^{n} q_i (V(n, e)v_i - R(C, i))},$$  \hspace{1cm} (9)

By substituting

$$A_{2e+1} \leq A(n, 2e + 2, 2e + 1)$$

and

$$\max_{h \in H} \{NC(h, C, e + 1)\} \leq A(n, 2e + 2, e + 1)$$

in (9) we obtain the Johnson upper bound.

**Example 3:** Let $q_1 = p_{1,e}^{e+1} - p_{1,e}^{e} - p_{1,e}^{e} A_{2e+1}^{e+1}$, $q_2 = 1$, and $q_i = 0$ for $i \notin \{1, 2\}$. From (5) and (8) we have

$$q_1 R(C, 1) + q_2 R(C, 2) = V(n, e)(q_1 v_1 + q_2 v_2) - p_{1,e}^{e+1}v_{e+1} + v_{e+2} - (p_{e+1,e}^{2e+1} + p_{e+1,e}^{2} A_{2e+1} - p_{e+2,e}^{2e+2} A_{2e+2})$$

and

$$\xi(C, \{q_i\}) = p_{1,e}^{e+1} \max_{h \in H} \{NC(h, C, e + 1) + NC(h, C, e + 2)\}$$

Thus, using Theorem 3 we obtain
Theorem 4: If $C$ is an $(n, M, 2e + 1)$ code with distance distribution $\{A_i\}_{i=0}^n$, then
\[
|C| \leq \frac{2^n}{\sum_{i=0}^n \binom{i}{e} + \frac{(n+1) - (2e+2)}{A_{n+1,2e+2}} + \frac{2^n}{A_{n+1,2e+2}}},
\] (10)
where
\[
\gamma = (p_{e+1} + p_{e+2} + 1)A_{e+1} + p_{e+2}A_{e+2}.
\]

By substituting
\[
A_{e+1} + A_{e+2} \leq A(n + 1, 2e + 2, 2e + 2)
\]
and
\[
\max_{h \in \mathcal{H}} \{NC(h, C, e + 1) + NC(h, C, e + 2)\}
\]
in (10) we obtain the bound of 4.

Next, we want to improve the trivial bound on $A_i$ given by $A_i \leq A(n, 2e + 2, i)$. We will find upper bounds on distance distribution coefficients $A_i$'s using linear programming. For an $(n, M, 2e + 1)$ code $C$ with distance distribution $\{A_i\}_{i=0}^n$ let us denote by $LP[n, 2e+1]$ the following system of Delsarte's linear constraints:
\[
\begin{align*}
\sum_{i=0}^k A_i P_k(i) &\geq 0 & \text{for } 0 \leq k \leq n, \\
0 &\leq A_i \leq A(n, 2e + 2, i) & \text{for } i = 2e + 1, 2e + 2, \ldots, n, \\
A_0 = 1, A_1 = 0 & \quad & \text{for } 1 \leq i < 2e + 1,
\end{align*}
\]
where $P_k(i) = \sum_{j=0}^k (-1)^i(i)^j \binom{n-j}{k-j}$ denote Krawtchouk polynomial of degree $k$. We also denote $\tilde{n} = n + 1$ and let $\{\tilde{A}_i\}_{i=0}^n$ be the distance distribution of the $(n + 1, M, 2e + 2)$ extended code $C_e$ which is obtained from the $(n, M, 2e + 1)$ code $C$ with distance distribution $\{A_i\}_{i=0}^n$ by adding an even parity bit to each codeword of $C$. It’s easy to verify that for each $i, e + 1 \leq i \leq \lfloor \tilde{n}/2 \rfloor$,
\[
\tilde{A}_{2i} = A_{2i-1} + A_{2i}.
\]

For the even weight code $C_e$ of length $\tilde{n}$ and distance $d = 2e + 2$ we denote by $LP[n, 2e+2]$ the following system of Delsarte’s linear constraints:
\[
\begin{align*}
\sum_{i=0}^k \tilde{A}_i P_k(i) &\geq 0 & \text{for } 0 \leq k \leq \lfloor \tilde{n}/2 \rfloor, \\
0 &\leq \tilde{A}_i \leq \tilde{A}(\tilde{n}, d, i) & \text{for } i = 2e + 2, 2e + 4, \ldots, 2\lfloor \tilde{n}/2 \rfloor, \\
\tilde{A}_0 = 1, \tilde{A}_1 = 0 & \quad & \text{for } 1 \leq i < 2e + 2.
\end{align*}
\]
In some cases we will add more constraints to obtain some specific bounds as in [5], [7], [11], [12].

By Theorem 4 we have that for an $(n, M, 2e + 1)$ code $C$ with distance distribution $\{A_i\}_{i=0}^n$ the following holds:
\[
|C| \leq \frac{2^n}{\sum_{i=0}^n \binom{i}{e} + \frac{(n+1) - (2e+2)}{A_{n+1,2e+2}} + \frac{2^n}{A_{n+1,2e+2}}},
\]
Using (11) we obtain
\[
\tilde{A}_{2i} = \tilde{A}_{2i-1} + \tilde{A}_{2i}.
\]

Theorem 5:
\[
A(n, 2e + 1) \leq \frac{2^n}{\sum_{i=0}^n \binom{i}{e} + \frac{(n+1) - (2e+2) \text{max} \{A_{2e+2} \}}{A_{n+1,2e+2} + \frac{2^n}{A_{n+1,2e+2}}}}
\]
where $max \{A_{2e+2} \}$ is taken subject to $LP_e[\tilde{n}, 2e + 2]$.

For the next result we need the following theorem which is a generalization of a theorem given by [3].

Theorem 6: Let $C$ be a code of length $n$, minimum Hamming distance $d$, and distance distribution $\{A_i\}_{i=0}^n$. Let $\{p_i\}_{i=0}^n$ be a sequence of real numbers. Then there exists a code $C'$ of length $n - 1$, distance $d$, with distance distribution $\{A'_i\}_{i=0}^{n-1}$ satisfying
\[
\sum_{i=0}^n (n - i)p_i A_i \leq \sum_{i=0}^{n-1} p_i A'_i.
\]

It was proved in [13] by using LP that for an even weight code $C$ of length $\tilde{n} \equiv 1(\text{mod } 4)$, distance $d = 4$, and distance distribution $\{A_i\}_{i=0}^{\tilde{n}}$, then
\[
\tilde{A}_4 \leq \frac{(\tilde{n} - 1)(\tilde{n} - 2)(\tilde{n} - 3)}{24}.
\]
We substitute $p_i = \delta_{i,4}$ in (12) and (13) for the upper bound on $A'_4$ to obtain

Lemma 2: If $C$ is an even weight code of length $\tilde{n} \equiv 2(\text{mod } 4)$, distance $d = 4$, and distance distribution $\{A_i\}_{i=0}^{\tilde{n}}$, then
\[
\tilde{A}_4 \leq \frac{\tilde{n}(\tilde{n} - 2)(\tilde{n} - 3)}{24}.
\]

The previous best known bound $A(n, 3) \leq 2^n/(n + 3)$ for $n \equiv 1(\text{mod } 4)$ was obtained in [4] by LP. In particular, we have $A(21, 3) \leq 87348$ which improves on the previous best known bound $A(21, 3) \leq 87376$ [11].

IV. HDI SEQUENCES WITH LARGE INDICES

We demonstrate another approach to estimating $\sum_{i=0}^n q_i D_i$, where nonzero elements of $\{q_i\}$ correspond to large indices. For each $t$, $0 \leq t \leq e$, we denote
\[
E_{n-t} = \{h \in \mathcal{H} : NC(h, C, n - t) = 1\}.
\]
Note, that for any hole $h \in \mathcal{H}$ we have $NC(h, C, n - t) \in \{0, 1\}$, where $0 \leq t \leq e$. 

Lemma 3: For each $t$, $0 \leq t \leq e$, 
$$|E_{n-t}| = |C| \left( v_{n-t} - \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t} \right).$$  \hspace{1cm} (14)

Let $q_{n-1} = q_n = 1$ and $q_i = 0$ for $i \notin \{n-1, n\}$. If $h \in E_{n-t}$ for $t \in \{0, 1, \ldots, e-1\}$, then
$$D_{n-1}(h) + D_n(h) = 0. \hspace{1cm} (15)$$

If $h \in E_{n-e}$, then 
$$D_{n-1}(h) + D_n(h) \geq n - e - (e + 1)A(n - e, 2e + 1) = n - e - (e + 1)\left\lfloor \frac{n - e}{e + 1} \right\rfloor. \hspace{1cm} (16)$$

If for a given hole $h$ there exists no codeword at distance $k \in \{n-e, n-(e-1), \ldots, n-1, n\}$, then
$$D_{n-1}(h) + D_n(h) \geq n + 1 - (e + 1)A(n, 2e + 1) = n + 1 - (e + 1)\left\lfloor \frac{n}{e + 1} \right\rfloor. \hspace{1cm} (17)$$

By combining (14)-(17) with Corollary 1, we obtain

Theorem 8: 
$$A(n, 2e + 1) \leq 2 \sum_{i=0}^{e} (\binom{n}{i}) + \frac{2^n}{(e+1)(e+2)(e+3)} \sum_{t=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t} \sum_{i=0}^{e} R(C, n-1) + R(C, n)$$
$$- \left( n + 1 - (e + 1)\left\lfloor \frac{n}{e + 1} \right\rfloor \right) \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t}$$
$$- \left( e + 1 \right) \left( 1 + \left\lfloor \frac{n-e}{e+1} \right\rfloor - \left\lfloor \frac{n}{e+1} \right\rfloor \right) \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t},$$

subject to $LP[n, 2e+1]$ and $R(C, n-1)$, $R(C, n)$ are given by (15).

By Theorem 8, we obtain $A(22, 3) \leq 172361$ and $A(24, 5) \leq 47538$ which improve the previous best known bounds $A(22, 3) \leq 173015$ [11] and $A(24, 5) \leq 48008$ [14].

Let $n$ be even integer and let $e = 1$. By Theorem 8 and 11, we obtain

Theorem 9: If $n$ is an even integer, then
$$A(n, 3) \leq \frac{2^n}{2n + 3 - \frac{\Delta(n)}{n}},$$

where 
$$\Delta(n) = \max \{ 6\tilde{A}_n - 3n\tilde{A}_{n-1} \}$$

subject to $LP_c[n, 4]$.

Using LP, we can prove the following lemma.

Lemma 4: If $C$ is an even weight code of length $\tilde{n} \equiv 11 \mod 12$, distance $d = 4$, and distance distribution \{$A_i$ \}_{i=0}^{5}$, then
$$6\tilde{A}_{\tilde{n}-3} + 3(\tilde{n} - 1)\tilde{A}_{\tilde{n}-1} \leq \left( \tilde{n} - 1 \right) \tilde{A}_{\tilde{n}-1} \leq \frac{(\tilde{n} - 1)(\tilde{n} - 2)(\tilde{n} + 4)}{\tilde{n} + 2}.$$ 

Therefore, by Theorem 9 and Lemma 3, we have

Theorem 10: For $n \equiv 10 \mod 12$
$$A(n, 3) \leq \frac{2^n}{n + 2 + \frac{8}{n+3}},$$

was obtained by (3).

By similar arguments, if \{ $q_i$ \} is a sequence with $q_i = 0$, except for $q_{n-2} = q_{n-1} = q_n = 1$, we obtain the following bound.

Theorem 11: If $C$ is an $(n, M, 2e+1)$ code, then
$$|C| \leq \frac{2^n}{2 \sum_{i=0}^{e} (\binom{n}{i}) + \frac{\Delta(n)}{n} \sum_{t=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t} \sum_{i=0}^{e} R(C, n-1) + R(C, n) - \left( n + 1 - (e + 1)\left\lfloor \frac{n}{e + 1} \right\rfloor \right) \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t}$$
$$- \left( e + 1 \right) \left( 1 + \left\lfloor \frac{n-e}{e+1} \right\rfloor - \left\lfloor \frac{n}{e+1} \right\rfloor \right) \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t} \right.$$ 

and
$$U(n) = \max \{ \sum_{i=n-2}^{n} R(C, i) - (1 + \left\lfloor \frac{n+1}{2} \right\rfloor \right\}$$
$$- \left( e + 2 \right) A(n + 1, 2e + 2, e + 2) \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t}$$ 
$$- (1 + e(n-e) + \left( e + 1 \right) - \left( e + 2 \right) \left( e + 2 \right) (A(n + 1, 2e + 2, e + 2)$$
$$- A(n + 1 - e, 2e + 2, e + 2)) \sum_{i=0}^{e-t} A_{n-i} \sum_{j=0}^{e} p_{n-1,ij}^{t} \},$$

subject to $LP[n, 2e+1]$ and $R(C, n-2)$, $R(C, n-1)$, $R(C, n)$ are given by (5).

Applying Theorem 11, we obtain $A(21, 3) \leq 87333$ which is better than the best previously known bound (see Section 11).
V. Generalization for Arbitrary Metric Association Schemes

We can generalize our approach to arbitrary metric association scheme \((\mathcal{X}, R)\) with distance function \(d\), which consists of a finite set \(\mathcal{X}\) together with a set \(R\) of \(n+1\) relations defined on \(\mathcal{X}\) with certain properties. For the complete definition and brief introduction to the association schemes, the reader is referred to [10, Chapter 21]. We extend the definitions from the first section as follows. \(|\mathcal{X}| = v\) is the number of points of a finite set \(\mathcal{X}\), \(v_i\) is the valency of the relation \(R_i\), and \(p_{i,j}^k\) are the intersection numbers of the scheme. A code \(C\) is a nonempty subset of \(\mathcal{X}\) with minimum distance \(2e + 1\).

The definitions related to holes and distance distribution are easily generalized. The results of (4), Lemma 1, Theorems 10 through 13 and Corollary 11 are valid for arbitrary metric association schemes.

As an example we consider the Johnson scheme. In this scheme \(\mathcal{X}\) is the set of all binary vectors of length \(n\) and weight \(w\). Note, that in this scheme the number of relations is \(w + 1\) and \(n\) has different meaning. The distance between two vectors is defined to be the half of the Hamming distance between them. One can verify, that \(v = \binom{n}{w}\), \(v_i = \binom{w}{i}\binom{n-w}{i}\), and \(p_{i,j}^k\) is given by

\[
\sum_{l=0}^{w-k}\binom{w-k}{l}\binom{k}{w-i-l}\binom{k}{w-j-l}\binom{n-w-k}{i+j+l-w}.
\]

Denote by \(T(w_1, w_2, w, d)\) the maximum number of binary vectors of length \(w_1 + w_2\), having mutual Hamming distance of at least \(d\), where each vector has exactly \(w_1\) ones in the first \(n_1\) coordinates and exactly \(w_2\) ones in the last \(n_2\) coordinates. By substituting

\[
\max_{h \in R} \{|NC(h, C, e + 1)| \leq T(e + 1, w, e + 1, n - w, 4e + 2)\}
\]

in [9], we obtain the following bound.

**Theorem 12:**

\[
A(n, 4e+2, w) \leq \frac{\binom{n}{w}}{\sum_{i=0}^{e} \binom{w}{i} \binom{n-w}{i} + \frac{e}{T(e+1, w, e+1, n-w, 4e+2)} U_w(n)},
\]

where

\[
U_w(n) = \max\{A_{2e+1}\},
\]

subject to Delsarte’s linear constraints for Johnson scheme (see [10, Theorem 12, p. 666]).

Applying Theorem 12 for \(e = 1\) we obtain the following improvements (the values in the parentheses are the best bounds previously known [11, 14]):

- \(A(19, 6, 7) \leq 519 (520)\), \(A(22, 6, 11) \leq 5033 (5064)\), \(A(26, 6, 11) \leq 42017 (42080)\).

We would like to remark, that LP can be applied for upper bounds that obtained by centering a spheres around a codewords. We give an example of such bound.

**Theorem 13:**

\[
A(n, 10, w) \leq \frac{\binom{n}{w}}{\sum_{i=0}^{w} \binom{w}{i} \binom{n-w}{i} + \frac{\binom{n-w}{i} - \binom{w}{i} \binom{n-w}{i}}{T(3, w, 3, n-w, 10)} + \frac{\binom{n-w}{i} \binom{w}{w-i-1}}{T(4, w, 4, n-w, 10)} - U_w(n)},
\]

where

\[
U_w(n) = \max\{A_{2e+2}\}
\]

subject to Delsarte’s linear constraints for Johnson scheme.

By Theorem 13 we have: \(A(23, 10, 9) \leq 78 (81)\), \(A(24, 10, 9) \leq 116 (119)\), \(A(25, 10, 9) \leq 157 (158)\), \(A(27, 10, 9) \leq 293 (299)\), \(A(28, 10, 10) \leq 785 (821)\).

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