The Asymptotic Quasinormal Modes of Dilatonic Black Holes

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Abstract. We investigate highly damped quasinormal mode of dilatonic black holes motivated by its relation to loop quantum gravity. Using the WKB approximation, we show that the real part of the frequency approaches the value \(T_H \ln 3\) for dilatonic black holes, as conjectured by Medved et al. and Padmanabhan. This is surprising since the area spectrum of the black hole determined by the Bohr’s correspondence principle completely agrees with that of a Schwarzschild black hole for any values of the electromagnetic charge or the dilaton coupling. We discuss the generality of this result for single-horizon black holes and its meaning in loop quantum gravity.

1. Introduction

Progress in loop quantum gravity (LQG) has been remarkable particularly after the introduction of the spin network formalism [1]. Due to this formalism, general expressions for the spectrum of the area and the volume operators can be derived [2, 3]. For example, the area spectrum \(A\) is

\[ A = 8\pi\gamma \sum \sqrt{j(j+1)} , \]

where \(\gamma\) is the Immirzi parameter related to an ambiguity in the choice of canonically conjugate variables [4]. The sum is over all intersections between a surface and a spin network carrying a label \(j = 0, 1/2, 1, 3/2, \ldots\) reflecting the SU(2) nature of the gauge group. The statistical origin of the black hole entropy \(S\) is also derived using this formalism (and the introduction of the isolated horizon [5] and the U(1) Chern-Simons theory). The result is summarized as [6]

\[ S = \frac{A \ln(2j_{\text{min}} + 1)}{8\pi\gamma \sqrt{j_{\text{min}}}(j_{\text{min}} + 1)} , \]

where \(A\) and \(j_{\text{min}}\) are the horizon area, and the lowest non-trivial representation usually taken to be 1/2 because of SU(2). In this case, the Immirzi parameter is determined as \(\gamma = \ln 2/(\pi\sqrt{3})\) to produce the Bekenstein-Hawking entropy formula \(S = A/4\). This is one of the important achievements of LQG.
Recently, quite a new relation between LQG and the quasinormal modes was considered in Ref. [7]. We explain the idea briefly. If we apply the first law of black hole thermodynamics,

$$dA = \frac{4}{T_H} dM,$$  \hfill (3)

where we have only considered an “infinitesimal” change in gravitational mass for simplicity. Then we seek for a possibility that there is a lower bound in the area change. The discrete area spectrum is also favorable from the observation that the horizon area of non-extremal black holes behaves as a classical adiabatic invariant [8], since the Ehrenfest principle says that any classical invariant corresponds to a quantum entity with discrete spectrum. We identify the minimum change $dM$ as the real part of the highly damped quasinormal mode $\text{Re}(\omega)$, based on the Bohr’s correspondence principle “transition frequency at large numbers should equal classical oscillation frequencies” followed by [9]. For a Schwarzschild black hole, we have [10, 11]

$$\text{Re}(\omega) = T_H \ln 3 \text{ for } \text{Im}(\omega) \to \infty.$$ \hfill (4)

In this case, we obtain

$$dA = 4 \ln 3.$$ \hfill (5)

At this point, there is no direct relation to LQG. An interesting and debatable issue is that we identify (5) with the minimum area change in the area spectrum (1), i.e.,

$$dA = 4 \ln 3 = 8\pi\gamma \sqrt{j_{\min}(j_{\min}+1)}.$$ \hfill (6)

By substituting this formula into (2), we obtain $j_{\min} = 1$ to produce $S = A/4$. In this case, the Immirzi parameter is modified as $\gamma = \ln 3/(2\pi\sqrt{2})$. This consideration calls various arguments such as modification of the gauge group SU(2) to SO(3) or the modification of the area spectrum in LQG and so on which we will discuss later [12, 13, 14, 15, 16].

We must also suspect that only a Schwarzschild black hole has the relation (5) and the identification (6) has no universality. We should notice that the formulae (1) and (2) in LQG do not depend on matter fields since their symplectic structures do not have a contribution for the horizon surface term [6]. Thus, it is important to investigate these properties in other black holes in determining whether or not the discussion above is related to LQG.

The works we should mention are Ref. [17, 18] which show that the imaginary part of the highly damped quasinormal modes have a period proportional to the Hawking temperature for the single-horizon black holes. This result suggests a generalization of the Schwarzschild black hole case, i.e.,

$$\omega = T_H \ln 3 - 2\pi T_H i \left( n + \frac{1}{2} \right).$$ \hfill (7)

For a Schwarzschild black hole, this formula applies to scalar and gravitational perturbations. For electromagnetic perturbations, the real part disappears in this limit. What this means in the context of Hod’s proposal is not clear at present. Their work and Ref [19] also suggest that if we are between two horizons, we will see a mixed contribution from the two horizons. Thus, we cannot see a periodic behavior in the imaginary part in general which was also confirmed numerically in Ref. [20] for a Schwarzschild-de Sitter black hole. The analysis for a Reissner-Nordström black hole in Ref. [10, 11] also shows that existence of the inner horizon prevents the imaginary part to be periodic. This result agrees with numerical results in Ref. [21]. This would also be true for a Kerr black hole where the contribution of the angular momentum also makes things more complicated [22].

Therefore, the strategy we take here is whether or not the formula (7) holds for single-horizon black holes. From this viewpoint, we examine the WKB analysis following Ref. [11] by
exemplifying the case for dilatonic black hole [23] (for quasinormal modes of dilatonic black hole, see Refs. [24].) Surprisingly, the answer is in the affirmative. If one see its derivation, one would confirm the generality for the single-horizon black holes. Notice that a dilatonic black hole is a charged black hole with a single-horizon. Thus, considering this model provides the evidence that the essential thing that determines whether or not (7) holds is not the electromagnetic charge but the complex space-time structure.

2. The QNMs for single-horizon black holes.
As a background, we consider the static and spherically symmetric metric as
\[
ds^2 = -f(r)e^{-2\delta(r)}dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2,
\]
where \( f(r) := 1 - 2m(r)/r \). We define
\[
g(r) = e^{-\delta}f(r).
\]
Notice that \([17, 18] g'(r_H) = 4\pi T_H \),
where \( \delta := d/dr \) and \( r_H \) is the event horizon. Our basic equation for black hole perturbations is
\[
d^2\psi/dr^2 + [\omega^2 - V(r)]\psi = 0,
\]
where the time dependence of the perturbations is assumed to be \( e^{-i\omega t} \). The tortoise coordinate \( r_\ast \) is defined as \( dr_\ast/dr = 1/g(r) \). The potential \( V(r) \) for the general case (8) is written following by \([17, 25]\) as
\[
V(r) = g\left[\frac{l(l+1)}{r^2}e^{-\delta} + (1 - k^2)\frac{2m}{r^3}e^{-\delta} + (1 - k)(\frac{g'}{r} - \frac{2m}{r^3}e^{-\delta})\right].
\]
For \( k = 0, 1 \) and 2, \( V(r) \) corresponds to the case for the scalar, electromagnetic and the odd parity gravitational perturbations, respectively. At present, we cannot obtain an expression like (11) for the even parity mode. First, we concentrate on the odd parity gravitational perturbations, i.e., \( k = 2 \). We also define \( \Psi = g^{1/2}\psi \). Using (9), our basic equation can be rewritten as
\[
\Psi'' + R(r)\Psi = 0,
\]
where
\[
R(r) = g^{-2}[\omega^2 - V + (g')^2/4 - gg''/2] .
\]
Then, we consider the WKB analysis combined with the complex-integration technique which is a good approximation in the limit \( \text{Im}(\omega) \to -\infty \).

First, we summarize the analysis for a Schwarzschild black hole and consider the complex \( r \)-plane below. Two WKB solutions of (13) can be written as
\[
\Psi^{(s)}_{1,2}(r) = Q^{-1/2}\exp\left[\pm i\int^r_s Q(x)dx\right] ,
\]
where \( Q^2 = R + \text{extra term} \). Here, the extra term is chosen for \( \Psi \) to behave near the origin appropriately. From (13), \( \Psi(r) \sim r^{1/2}\pm 2 \) at \( r \to 0 \). Since \( R \sim -15/4r^2 \) at \( r \to 0 \) in Schwarzschild black hole, we should choose \( Q^2 := R - 1/(4r^2) \) for the WKB solution (15) to behave correctly.
Figure 1. Zeros and poles of $Q^2(r)$ and for Schwarzschild black hole in the complex $r$-plane in the limit $\text{Im}(\omega) \to -\infty$. The related Stokes and anti-Stokes lines are shown by dashed lines and solid lines, respectively.

We should consider the problem concerning the “Stokes phenomenon” related to the zeros and poles of $Q^2$ [26], which are shown in Fig. 1 in the limit $\text{Im}(\omega) \to -\infty$. One of the important points are that the zeros of $Q^2$ approach the origin in the limit $\text{Im}(\omega) \to -\infty$. Near the origin, we can write

$$Q^2 = g^{-2} \left[ \omega^2 - \frac{4g^2}{r^2} \right].$$

(16)

Since $g \to -2M/r$ for $r \to 0$, where $M$ is the mass of Schwarzschild black hole, $Q^2$ has four zeros. When we start the outgoing solution at the point $a$ as

$$\Psi_a = \psi_1^{(r_1)},$$

(17)

and proceed along anti-Stokes lines, encircle the pole at the horizon clockwise, and turn back to $a$, we investigate what conditions must be imposed to reproduce the original solution (17). Using Stokes phenomenon (see [11] for the details), we can get the final condition

$$e^{2i\Gamma} = -1 - 2 \cos 2I,$$

(18)

where $\Gamma = \oint Qdr$.

We should also perform the same analysis for the ingoing solution near the event horizon. The result is the same as (18).

Let us evaluate $\Gamma$ and $I$. $\Gamma$ is written as

$$\Gamma = -2\pi i \lim_{r \to r_H} \frac{r - r_H}{g} \sqrt{\omega^2 + (g r)^2/4}$$

$$= -2\pi i \lim_{r \to r_H} \frac{r - r_H}{g} \omega = -i \frac{\omega}{2T_H}.$$ 

(19)

Notice that this result does not depend on the species of black holes, which becomes important later. Setting $y = \omega r^2/4M$, we can perform the integral $I$ as

$$I = - \int_{-1}^{1} \sqrt{1 - \frac{1}{y^2}} dr = \pi.$$ 

(20)
By substituting (19) and (20) into (18), we have (7) as derived in previous papers.

Next, we consider generalization of the above argument by exemplifying the case of a dilatonic black hole. The crux of the point we now show is that \( Q^2(r) \) for dilatonic black holes has two second order poles and four zeros in the limit \( \text{Im}(\omega) \to -\infty \), which is qualitatively the same as for a Schwarzschild black hole. Dilatonic black holes can be expressed using the coordinate [23]

\[
ds^2 = -\lambda^2(\rho)dt^2 + \frac{1}{\lambda^2}d\rho^2 + r^2(\rho)d\Omega^2,
\]

where

\[
\lambda^2 = \left(1 - \frac{\rho_+}{\rho}\right)\left(1 - \frac{\rho_-}{\rho}\right)^{(1-\alpha^2)/(1+\alpha^2)}, \quad r = \rho \left(1 - \frac{\rho_-}{\rho}\right)^{\alpha^2/(1+\alpha^2)}.
\]

\( \rho_+, \rho_- \) and \( \alpha \) are the event horizon, the “inner horizon”, and the dilaton coupling, respectively. We can see from (22) that the “inner horizon” corresponds to the origin in the area radius.

By comparing (21) and (8), we obtain

\[
g(r) = \left(1 - \frac{\rho_+}{\rho}\right)\left(1 - \frac{\rho_-}{\rho}\right)^{1/(1+\alpha^2)} \left(1 + \frac{\alpha^2}{1 + \alpha^2} \frac{\rho_-}{\rho - \rho_-}\right),
\]

\[
e^{-\delta} = \left(1 - \frac{\rho_-}{\rho}\right)^{-\alpha^2/(1+\alpha^2)} \left(1 + \frac{\alpha^2}{1 + \alpha^2} \frac{\rho_-}{\rho - \rho_-}\right)^{-1}.
\]

At first glance, it is not evident whether or not the zeros of \( Q^2 \) approach the origin in the limit \( \text{Im}(\omega) \to -\infty \). However, we can find from (23) and (24) that \( e^{-\delta} \) and \( g(r) \) do not show singular behavior for \( r \neq 0, r_H (\rho \neq \rho_-, \rho_+) \) as it is expected from the fact that dilatonic black hole is a single-horizon black hole. Thus, the zeros approach the origin as in the Schwarzschild case. We evaluate \( g(r) \) in the limit \( r \to 0 \), which is

\[
g(r) \approx \frac{\alpha^2}{1 + \alpha^2} \frac{\rho_- - \rho_+}{(\rho - \rho_-)^{1+\alpha^2}}.
\]

If we substitute (22) in this relation, we obtain

\[
g(r) \approx \frac{\alpha^2}{1 + \alpha^2} \frac{\rho_- - \rho_+}{r}.
\]

Using this asymptotic relation in (14), we have \( Q^2(r) = R - 1/(4r^2) \) again for \( \Psi \) to behave near the origin appropriately. Then, we have the form (16) near the origin and using the fact that a dilatonic black hole has one horizon, we find that \( Q^2(r) \) has four zeros and two second order poles as for Schwarzschild black hole.

Therefore, the WKB condition to obtain the global solution is quite analogous to the case of Schwarzschild black hole and is written as (18). As we noted above, the expression (19) is also not changed for a dilatonic black hole. The non-trivial factor is \( I \). However, since the only difference of \( g(r) \) in (26) from the Schwarzschild case is its coefficient, if we define

\[
y = \frac{\omega r^2}{1 + \alpha^2 (\rho_+ - \rho_-)},
\]

we can also perform the integral \( I \) as (20). Thus, we obtain (7) again which is the realization of the conjecture in [17, 18].
As for scalar and electromagnetic perturbations, we can show that (7) also holds for scalar perturbations and the real part of electromagnetic perturbations disappears as for the case of a Schwarzschild black hole.

For even parity gravitational perturbations of a dilatonic black hole, isospectrality between odd and even parity mode does not hold and the corresponding basic equation becomes complicated as shown in Ref. [24]. However, there remains a possibility that isospectrality is restored in the highly damped mode. This is under investigation.

From the observation for the case of a dilatonic black hole, the important things are: (i) the number of poles in $Q^2$ which is restricted to two in the single-horizon black holes. (ii) the number of zeros in $Q^2$ near the origin. (iii) asymptotically flatness that guarantees our boundary conditions.

3. Conclusion

We investigated the highly damped quasinormal mode of single-horizon black holes and obtained the relation (7) for a dilatonic black hole and considered the possibility of its generality. Our results are important since we supply the first example which shows (7) for black holes with matter fields. This suggest the generality of (7) in single-horizon black holes. Then, what about the confrontation in determining the Immirzi parameter $\gamma$ and the case in multi-horizon black holes? It would be worth examining the present proposals [12, 13, 15] since the results $j_{\text{min}}$ and $\gamma$ in both cases (would) turn out to be general for single-horizon black holes, and are too close to ignore and suggest some relations.

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