Nonextremal Black Hole Microstates 
and U-Duality

GARY T. HOROWITZ†, JUAN M. MALDACENA‡ and ANDREW STROMINGER†

†Department of Physics, University of California, Santa Barbara, CA 93106, USA
gary@cosmic.physics.ucsb.edu  andy@denali.physics.ucsb.edu

‡Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA
malda@puhep1.Princeton.edu

Abstract

A six-parameter family of five-dimensional black hole solutions is constructed which are labeled by their mass, two asymptotic scalar fields and three charges. It is shown that the Bekenstein-Hawking entropy is exactly matched, arbitrarily far from extremality, by a simple but mysterious duality-invariant extension of previously derived formulae for the number of D-brane states in string theory.
1. Introduction

Recently a precise accounting of the microstates responsible for the Bekenstein-Hawking entropy of certain extremal BPS black holes has been given in string theory [1-4]. Since black holes are nonperturbative objects, the calculations required considerable understanding of non-perturbative string theory and a certain class of solitons known as D-branes [5]. Nevertheless, due to the special character of the BPS states involved, there is a sense in which these calculations “had to work”. The (weighted) number of BPS states is a topological invariant and so a string calculation at weak coupling can be compared to a semiclassical calculation at strong coupling. It would have been very strange indeed had the two calculations yielded different results. A failure of the string counting of states to match the Bekenstein-Hawking entropy would have been a serious blow to the notion that string theory is a complete quantum theory of gravity.

Following [1] several attempts were made [6-8] to extend these results to leading order away from extremality. These endeavors were on much shakier ground, because the number of non-BPS states is not topologically protected, and strong coupling effects could ruin the ability to extrapolate from the weakly-coupled stringy regime in which calculations are possible to the semiclassical regime in which the black hole picture is valid. Nevertheless it was argued that strong coupling effects could be avoided in certain corners of the parameter space sufficiently near to extremality. The striking agreement discovered in [6-8] between the string and black hole calculations indicate that under favorable circumstances strong coupling effects can indeed be avoided.\footnote{However, as a note of caution, a similar analysis of near-extremal threebrane states in ten dimensions fails to match by a very puzzling $4/3$.}

In this paper we construct a six-parameter family of five-dimensional black hole solutions with arbitrary mass, three charges and arbitrary asymptotic values of two scalar fields. The familiar Reissner-Nordstrom solution is included as a special case. These black holes may be uniquely decomposed into a collection of D-branes, anti-D-branes and strings, whose numbers we denote $(N_1, N_\bar{1}, N_5, N_\bar{5}, n_R, n_L)$. (An anti-D-brane is simply a D-brane which is oriented in the opposite direction, and hence carries the opposite sign of the RR charge.) These numbers are defined in section 2.4 by matching thermodynamic properties of the black hole (under variation of the asymptotic parameters) to the thermodynamic properties of a collection of $(N_1, N_\bar{1}, N_5, N_\bar{5}, n_R, n_L)$ non-interacting branes,
anti-branes and strings. In terms of these numbers the entropy takes the surprisingly simple, but mysterious, form

\[ S = 2\pi(\sqrt{N_1} + \sqrt{\bar{N}_1})(\sqrt{N_5} + \sqrt{\bar{N}_5})(\sqrt{n_L} + \sqrt{n_R}). \tag{1.1} \]

Expressed in these variables the entropy is independent of both the string coupling and the internal five-volume. This expression reduces, in several different limits, to expressions derived in several different weak coupling expansions in [6], [7]. It is also the simplest duality-invariant extension of those expressions.

The most surprising feature of (1.1) is that it is an exact expression which is valid arbitrarily far from extremality, even for large neutral Schwarzschild black holes. We will derive (1.1) from the stringy D-brane picture in several different weakly coupled limits, and motivate the full expression from duality. We have not been able to obtain a stringy derivation of the full expression. This would require more than a weak-coupling analysis and seems to be quite challenging. The fact that the entropy is a product, rather than a sum, of terms suggests that the structure of the black hole Hilbert space may be quite different than anything previously imagined. At the very least it indicates that black holes and string theory have more lessons in store for us.

In section 2 we describe the black hole solutions and discuss their properties. The N’s are introduced and the above formula for the entropy is derived. The D-brane analysis is given in section 3 and some possible extensions of our results are discussed in section 4.

2. The Black Hole Picture

2.1. The Solutions

The low-energy action for ten-dimensional type IIB string theory contains the terms

\[ \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{12} e^\phi H^2 \right] \tag{2.1} \]

in the ten-dimensional Einstein frame. \( H \) denotes the RR three form field strength, and \( \phi \) is the dilaton. The NS three form, self-dual five form, and second scalar are set to zero. We will let \( g \) denote ten-dimensional string coupling and define the zero mode of \( \phi \) so that \( \phi \) vanishes asymptotically. The ten-dimensional Newton’s constant is then \( G_{10} = 8\pi^6 g^2 \) with \( \alpha' = 1 \). The metric in (2.1) differs from the string metric by a factor of \( e^{\phi/2} \). We wish to consider toroidal compactification to five dimensions with an \( S^1 \) of length \( 2\pi R \), a
$T^4$ of four-volume $(2\pi)^4V$, and momentum along the $S^1$. This implies that the metric takes the form

$$ds_{10}^2 = e^{2\chi} dx_i dx^i + e^{2\psi} (dx_5 + A_\mu dx^\mu)^2 + e^{-2(4\chi + \psi)/3} ds_5^2$$

(2.2)

where $\mu = 0, 1, ..., 4$, $i = 6, ..., 9$, and all fields depend only on $x^\mu$. The factor in front of $ds_5^2$ insures that this is the Einstein metric in five dimensions. We will assume that $x_5$ is periodically identified with period $2\pi R$, $x_i$ are each identified with period $2\pi V^{1/4}$, and $\chi, \psi$ go to zero asymptotically.

A five parameter solution of the equations of motion following from (2.1) was considered in [7] which was labeled by the energy, three charges, and $R$. The volume $V$ was fixed in terms of the charges by the requirement that the field $\chi$ in (2.2) remains constant. We now remove that restriction and present a six parameter family of solutions. These can be obtained by first applying a $U$-duality transformation (discussed in section 2.3) to the solution in [7] which permutes the three charges [6], and then applying a boost. The resulting ten dimensional solution is given by

$$e^{-2\phi} = \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right) \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-1}$$

(2.3)

$$ds^2 = \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-3/4} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{-1/4} \left[-dt^2 + dx_5^2ight] + \frac{r_0^2}{r^2} (\cosh \sigma dt + \sinh \sigma dx_5)^2 + \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right) dx_i dx^i$$

$$+ \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{1/4} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{3/4} \left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right]$$

(2.4)

This solution is parameterized by the six independent quantities $\alpha$, $\gamma$, $\sigma$, $r_0$, $R$ and $V$. These may be traded for three charges, the mass, $R$ and $V$. The three charges are

$$Q_1 = \frac{V}{4\pi^2 g} \int e^\phi * H = \frac{V r_0^2}{2g} \sinh 2\alpha,$$

$$Q_5 = \frac{1}{4\pi^2 g} \int H = \frac{r_0^2}{2g} \sinh 2\gamma,$$

$$n = \frac{R^2 V r_0^2}{2g^2} \sinh 2\sigma,$$

(2.5)

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2 With these conventions, T-duality sends $R$ to $1/R$ or $V$ to $1/V$, and S-duality sends $g$ to $1/g$.

3 After completion of this work, reference [11] appeared in which these solutions are also constructed.
where $\ast$ is the Hodge dual in the six dimensions $x^0, \ldots, x^5$. The last charge $n$ is related to the momentum around the $S^1$ by $P = n/R$. All charges are normalized to be integers. The energy is
\[
E = \frac{RVr_0^2}{2g^2}(\cosh 2\alpha + \cosh 2\gamma + \cosh 2\sigma) \quad (2.6)
\]

If one reduces to five dimensions using (2.2), the solution takes the remarkably simple and symmetric form:
\[
ds_5^2 = -f^{-2/3} \left(1 - \frac{r_0^2}{r^2}\right) dt^2 + f^{1/3} \left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right] \quad (2.7)
\]
where
\[
f = \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right) \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right) \left(1 + \frac{r_0^2 \sinh^2 \sigma}{r^2}\right) \quad (2.8)
\]
This is just the five-dimensional Schwarzschild metric with the time and space components rescaled by different powers of $f$. The factored form of $f$ was known to hold for extremal solutions [12,13]. It is surprising that it continues to hold even in the nonextremal case.

The solution is manifestly invariant under permutations of the three boost parameters as required by U-duality (see section 2.3). The event horizon is clearly at $r = r_0$. The coordinates we have used present the solution in a simple and symmetric form, but they do not always cover the entire spacetime. When all three charges are nonzero, the surface $r = 0$ is a smooth inner horizon. This is analogous to the situation in four dimensions with four charges [14]. When at least one of the charges is zero, the surface $r = 0$ becomes singular.

Several thermodynamic quantities can be associated to this solution. They can be computed in either the ten dimensional or five dimensional metrics and yield the same answer. For example, the ADM energy (2.6) is the total energy of either solution. The Bekenstein-Hawking entropy is
\[
S = \frac{A_{10}}{4G_{10}} = \frac{A_5}{4G_5} = \frac{2\pi RVr_0^3}{g^2} \cosh \alpha \cosh \gamma \cosh \sigma. \quad (2.9)
\]
where $A$ is the area of the horizon, and we have used the fact that $8\pi^6 g^2 = G_{10} = G_5(2\pi)^5 RV$. The Hawking temperature is
\[
T = \frac{1}{2\pi r_0 \cosh \alpha \cosh \gamma \cosh \sigma}. \quad (2.10)
\]
In ten dimensions, the black hole is characterized by pressures which describe how the energy changes for isentropic variations in $R$ and $V$. In five dimensions, these are ‘charges’ associated with the two scalar fields. In either case, they are directly related to the asymptotic fall-off of $\psi$ and $\chi$ in (2.2) and are given by

\[
P_1 = \frac{R V r_0^2}{2 g^2} \left[ \cosh 2\sigma - \frac{1}{2} (\cosh 2\alpha + \cosh 2\gamma) \right]
\]
\[
P_2 = \frac{R V r_0^2}{2 g^2} (\cosh 2\alpha - \cosh 2\gamma)
\] (2.11)

The extremal limit corresponds to the limit $r_0 \to 0$ with at least one of the boost parameters $\alpha, \gamma, \sigma \to \pm \infty$ keeping $R, V$ and the associated charges (2.5) fixed. If we keep all three charges nonzero in this limit, one obtains

\[
E_{\text{ext}} = \frac{R |Q_1|}{g} + \frac{RV |Q_5|}{g} + \frac{|n|}{R},
\]
\[
S_{\text{ext}} = 2\pi \sqrt{|Q_1 Q_5 n|},
\]
\[
T_{\text{ext}} = 0,
\]
\[
P_{1\text{ext}} = \frac{|n|}{R} - \frac{R |Q_1|}{2g} - \frac{RV |Q_5|}{2g},
\]
\[
P_{2\text{ext}} = \frac{R |Q_1|}{g} - \frac{RV |Q_5|}{g}.
\] (2.12)

The first equation is the Bogomolnyi bound for this theory.

2.2. Special cases

The solution (2.3), (2.4) contains many well known solutions as special cases. For example, suppose all three boost parameters are equal: $\alpha = \gamma = \sigma$. Then the dilaton is constant, and the internal five torus is constant. Letting $\hat{r}^2 = r^2 + r_0^2 \sinh^2 \alpha$, the metric (2.7) is immediately recognized as the five dimensional Reissner-Nordström solution. The five dimensional Schwarzschild metric corresponds to $\alpha = \gamma = \sigma = 0$.

Next, suppose $\gamma = \sigma = 0$, so the only nonzero charge is $Q_1$. Then the ten-dimensional metric becomes

\[
ds^2 = \left( 1 + \frac{r_0^2 \sinh^2 \alpha}{r^2} \right)^{-3/4} \left[ - \left( 1 - \frac{r_0^2}{r^2} \right) dt^2 + dx_5^2 \right]
\]
\[
+ \left( 1 + \frac{r_0^2 \sinh^2 \alpha}{r^2} \right)^{1/4} \left[ \left( 1 - \frac{r_0^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 + dx^i dx_i \right]
\] (2.13)
This is the Einstein metric for the six dimensional black string solution (cross $T^4$) \[13\].

Now suppose only $\gamma$ is nonzero, so the solution has only $Q_5$ charge. Then we obtain

\[
ds^2 = \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{-1/4} \left[-\left(1 - \frac{r_0^2}{r^2}\right) dt^2 + dx_5^2 + dx_i dx^i\right] \\
+ \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{3/4} \left[-\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right]
\] \hspace{1cm} (2.14)

This is the black five-brane solution of \[16\].

If $\sigma$ is the only boost parameter which is nonzero, then the metric becomes

\[
ds^2 = \left[-dt^2 + dx_5^2 + \frac{r_0^2}{r^2} (\cosh \sigma dt + \sinh \sigma dx_5)^2 + dx_i dx^i\right] \\
+ \left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2
\] \hspace{1cm} (2.15)

This is just the five dimensional Schwarzschild metric cross $T^5$, with a boost in the $x_5$ direction. Similarly, adding nonzero $\sigma$ to either (2.13) or (2.14) corresponds to adding a boost in the $x_5$ direction. Finally, if $\alpha = \gamma$, then the dilaton and volume of $T^4$ remain constant. This is just the dyonic black string solution discussed in \[7\] with $r_+^2 = r_0^2 \cosh^2 \gamma, \ r_-^2 = r_0^2 \sinh^2 \gamma$.

2.3. Duality Transformations

The full type IIB string theory compactified on $T^5$ yields $N = 8$ supergravity in five dimensions. This theory has a global $E_{6(6)}$ symmetry and contains 27 gauge fields which transform in the 27 of $E_{6(6)}$, and 42 scalar fields which parameterize the coset $E_{6(6)}/Sp(8)$ \[17\]. In string theory this symmetry is believed to be broken down to an $E_{6(6)}(Z)$ duality group. Since we have only kept three of the gauge fields, we will be interested in an $S_3$ subgroup which permutes the three charges $(Q_1, Q_5, n)$\[6\]. This is generated by $A$ and $B$.

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\footnote{The metric in \[13\] is $e^{-\phi/2}$ times the above Einstein metric. This is the string metric of the associated S-dual solution in which $H$ represents the NS three form.}

\footnote{Once again, the metric of \[16\] is $e^{-\phi/2}$ times the above metric, and expressed in terms of the radial coordinate $\hat{r}^2 = r^2 + r_0^2 \sinh^2 \gamma$.}
which act as

\[ A = T_6 T_7 T_8 T_9 : \quad Q'_1 = Q_5, \quad g' = \frac{g}{R_6 R_7 R_8 R_9}, \]
\[ Q'_5 = Q_1, \quad R'_5 = R_5, \]
\[ n' = n, \quad R'_i = \frac{1}{R_i}, \quad i = 6, 7, 8, 9, \]

\[ B = T_9 T_8 T_7 T_6 S T_5 T_6 : \quad Q'_1 = Q_5, \quad g' = \frac{g}{R_5 R_7 R_8 R_9}, \]
\[ Q'_5 = n, \quad R'_5 = \sqrt{\frac{R_6}{g R_5}}, \]
\[ n' = Q_1, \quad R'_6 = \sqrt{\frac{g R_6}{R_5}}, \]
\[ R'_i = \sqrt{\frac{g}{R_5 R_6}} \frac{1}{R_i}, \quad i = 7, 8, 9 \]

where \( R_i \) is the radius of the internal direction \( i \), similarly \( T_i \) is T duality along the internal direction \( i \). Note that after the last transformation the string is pointing along a different direction, we could use a simple rotation in the internal space to take it to the original configuration.

2.4. Relation to Fundamental Strings and D-branes

We now show that there is a formal sense in which the entire family of solutions discussed in section 2.2 can be viewed as “built up” of branes, anti-branes, and strings. The extremal limits of (2.13), (2.14), and (2.15), are obtained by letting \( r_0 \) go to zero, the boost parameter go to infinity such that the charge is fixed. These extremal metrics represent a D-onebrane wrapping the \( S^1 \), a D-fivebrane wrapping the \( T^5 \), or the momentum mode of a fundamental string around the \( S^1 \). From (2.6) and (2.11) we see that a single onebrane or anti-onebrane has mass and pressures

\[ M = \frac{R}{g}, \quad P_1 = -\frac{R}{2g}, \quad P_2 = \frac{R}{g} \]

Of course a onebrane has \( Q_1 = 1 \), while an anti-onebrane has \( Q_1 = -1 \). A single fivebrane or anti-fivebrane has

\[ M = \frac{R V}{g}, \quad P_1 = -\frac{R V}{2g}, \quad P_2 = -\frac{R V}{g} \]
For left- or right-moving strings

\[
M = \frac{1}{R}, \quad P_1 = \frac{1}{R}, \quad P_2 = 0 \quad (2.19)
\]

Given (2.17) - (2.19), and the relations (2.5), (2.6), and (2.11), it is possible to trade the six parameters of the general solution for the six quantities \((N_1, N_{\bar{1}}, N_5, N_{\bar{5}}, n_R, n_L)\) which are the “numbers” of onebranes, anti-onebranes, fivebranes, anti-fivebranes, right-moving strings and left-moving strings respectively. This is accomplished by equating the total mass, pressures and charges of the black hole with those of a collection of \((N_1, N_{\bar{1}}, N_5, N_{\bar{5}}, n_R, n_L)\) non-interacting “constituent” branes, antibranes and strings. By non-interacting we mean that the masses and pressures are simply the sums of the masses and pressures of the constituents. The resulting expression for the \(N\)’s are

\[
\begin{align*}
N_1 &= \frac{Vr_0^2}{4g} e^{2\alpha}, \\
N_{\bar{1}} &= \frac{Vr_0^2}{4g} e^{-2\alpha}, \\
N_5 &= \frac{r_0^2}{4g} e^{2\gamma}, \\
N_{\bar{5}} &= \frac{r_0^2}{4g} e^{-2\gamma}, \\
n_R &= \frac{R^2 V^2}{4g^2} e^{2\sigma}, \\
n_L &= \frac{R^2 V^2}{4g^2} e^{-2\sigma}.
\end{align*}
\]

(2.20) is the definition of the \(N\)’s, but we will refer to them as the numbers of branes, antibranes and strings because (as will be seen) they reduce to those numbers in certain limits where these concepts are well defined.

In terms of the numbers (2.20), the charges are simply \(Q_1 = N_1 - N_{\bar{1}}, Q_5 = N_5 - N_{\bar{5}}, n = n_R - n_L\), the total energy is

\[
E = \frac{R}{g} (N_1 + N_{\bar{1}}) + \frac{RV}{g} (N_5 + N_{\bar{5}}) + \frac{1}{R} (n_R + n_L) \quad (2.21)
\]

and the volume and radius are

\[
\begin{align*}
V &= \left( \frac{N_1 N_{\bar{1}}}{N_5 N_{\bar{5}}} \right)^{1/2} \\
R &= \left( \frac{g^2 n_R n_L}{N_1 N_{\bar{1}}} \right)^{1/4}
\end{align*}
\]

8
From (2.12) we see that the extremal solutions correspond to including either branes or anti-branes, but not both. Notice that for the general Reissner-Nordstrom solutions ($\alpha = \gamma = \sigma$) the contribution to the total energy from onebranes, fivebranes, and strings are all equal:

$$\frac{R}{g}(N_1 + N_\bar{1}) = \frac{RV}{g}(N_5 + N_\bar{5}) = \frac{1}{R}(n_R + n_L). \quad (2.24)$$

The actual number of branes of each type depends on $R$ and $V$ and can be very different.

Of course there seems to be no reason for neglecting interactions between collections of branes and strings composing a highly non-extremal black hole at strong or intermediate coupling. Hence the definitions (2.20) would seem to be inappropriate for describing a generic black hole. However, the utility of these definitions can be seen when we reexpress the black hole entropy (2.9) in terms of the $N$'s. It takes the remarkably simple form

$$S = 2\pi(\sqrt{N_1} + \sqrt{N_\bar{1}})(\sqrt{N_5} + \sqrt{N_\bar{5}})(\sqrt{n_L} + \sqrt{n_R}). \quad (2.25)$$

In the next section we shall see that this entropy formula arises naturally in the D-brane picture.

3. The D-Brane Picture

In this section we will describe and compute the entropy of a collection of D-branes and strings and reproduce the formula (2.25) in various limits. Throughout most of this section we will presume that the ten-dimensional string coupling is extremely small, so that D-brane perturbation theory is accurate. This implies that the string scale is large in Planck units and the “black hole” is surrounded by a stringy halo which is large compared to its Schwarzschild radius. Hence the best physical description is as a bound collection of D-branes and strings rather than as a semiclassical black hole solution.

To begin, consider a BPS-saturated state consisting of $N_1$ onebranes, $N_5$ fivebranes, $n_R$ right moving strings, and no anti-branes or left-moving momenta. We have seen that this corresponds to an extremal black hole. The fivebranes are wrapped around the $T^5$ with size $VR$ and the onebranes are wrapped around the $S^1$ of radius $R$. If $N_5 = 1$, the $N_1$ onebranes are marginally bound to the fivebrane \[18,19\] but are free to move within the transverse $T^4$. This motion is generated by the (1,1) Dirichlet strings both of whose ends are stuck to the onebranes, but can carry momentum along the $S^1$. (The (1,5) strings carry charge and so are confined.) The extremal black hole with nonzero $n_R$-charge
corresponds to a BPS state with nonzero momentum along the $S^1$, i.e. with right-moving but no left-moving Dirichlet strings. The number of such states for fixed $n_R$ follows from the standard thermodynamic formula for the entropy of $N_B$ ($N_F$) species of right-moving bosons (fermions) with total energy $E_R$ in a box of length $L$

$$S = \sqrt{\frac{\pi(2N_B + \alpha N_F)E_R L}{6}}. \quad (3.1)$$

Using $N_F = N_B = 4N_1N_5$, $L = 2\pi R$, $E_R = n_R/R$, (3.1) becomes

$$S = 2\pi \sqrt{N_1N_5n_R} \quad (3.2)$$

which agrees with the formula (2.9) for $N_1 = N_5 = n_L = 0$.

For $N_5 > 1$ the picture is somewhat different. In that case the number of $(1,5)$ strings $(N_1N_5)$ is greater than the rank of the gauge group. Their potential accordingly has $D$-flat directions and they can condense (they are five-dimensional hypermultiplets) and break all the gauge symmetries. However in this phase there are still $N_1N_5$ Dirichlet strings so the formula (3.2) remains valid, as promised by duality. The best way to describe this situation is to represent the onebranes as instantons in the fivebrane [20,21], but in the following we concentrate on the simplest case $N_5 = 1$.

Since the above counting does not depend on the sign of the charges, it is clear that if we had started with $N_1$ anti-onebranes and no onebranes, the counting would have been identical with the result

$$S = 2\pi \sqrt{N_1N_5n_R}, \quad (3.3)$$

for $N_1 = N_5 = n_L = 0$. Similarly, we could have started with anti-fivebranes or left moving strings and obtained analogous formula. This shows that the entropy (2.9) correctly reproduces the number of states when one integer in each of the three factors is zero, which are the extremal black holes.

We now consider small deviations from extremality (we assume $N_1 = N_5 = n_L = 0$ initially) in three regimes where one of the excitations is much lighter than the other two. In this case, the nonextremal entropy will come from adding only left moving momentum or only anti-onebranes or only anti-fivebranes.

Let us review first the case considered in [7] which corresponds to the case in which the momentum modes are light. It can be seen from (2.21) that this case corresponds to taking $R$ very large, we always consider low energies $\delta E$ above extremality (but still big
enough so that we have a large number of right movers \( n_R \gg 1 \). In this case the D-branes get very heavy and the light modes of the system are left and right moving excitations of the \((1,1)\) Dirichlet strings which carry energy \( E_{R,L} = n_{R,L}/R \). For nonzero \( \delta E \) both left- and right-moving modes will be present. The rate of interactions between them is proportional to the string coupling \( g \) as well as the density of strings, which is inversely proportional to \( R \). Hence for fixed coupling interactions can be ignored to leading order in a \( 1/R \) expansion. The entropy is then just the sum of left- and right-moving contributions

\[
S = 2\pi \sqrt{N_1 N_5} (\sqrt{n_R} + \sqrt{n_L})
\]

which again agrees with (2.25) for \( N_1 = N_5 = 0 \).

A second way to get light modes is to take \( R \) to be very small with \( RV \) fixed. In that case momentum modes of strings and wrapping modes of fivebranes are heavy but winding modes of onebranes are light. The best way to analyze this is to T-dualize along the \( S^1 \) to a IIA theory with large \( \tilde{R} = 1/R \).\(^6\) The onebranes become zero-branes, the fivebranes become four-branes and momentum becomes fundamental string winding. Let us first reproduce the extremal entropy (3.2) in this picture. For \( N_1 = 0, N_5 = 1 \) and arbitrary \( n_R \) one has \( n_R \) fundamental strings which wind once around the \( S^1 \) and have both ends stuck on the fourbrane. To reproduce (3.2) we must count the number of ways of adding zero-branes with total charge \( N_1 \). The zero-branes are like beads (in a zero-momentum wave function) threaded on any one of the \( n_R \) fundamental strings. Since \( V \) is large the strings are far apart and the beads can be threaded by only one string at a time. There is one species of bead for every integer value of the charge. (This was first postulated in \(^{22}\) as required for compatibility of string theory and eleven-dimensional supergravity, and has subsequently been confirmed in a variety of contexts.) Furthermore each such bead is an \( N = 2 \) hypermultiplet with 4 bosonic and 4 fermionic states. Supersymmetry of the extremal configuration requires that we use only positively charged beads. Counting the number of such states with total charge \( N_1 \) is isomorphic to counting the number of states of \( 4n_R \) bosonic and \( 4n_R \) fermionic oscillators with total energy \( N_1 \). This reproduces exactly the extremal entropy (3.2). In the case \( N_5 > 1 \) we can think that the four branes are separated and there are \( N_5 \) spaces between them where \( n_R \) strings can start and end on different points on the four branes, so there are \( N_5 n_R \) distinct strings where we can put the zero branes. Now let us consider nonzero \( \delta E \). This requires both charges of beads.

\(^6\) We wish to keep the IIA string coupling fixed as \( R \) gets small.
However for large $\tilde{R}$ (small $R$) the beads are dilute. Over most of phase space the forces between the beads are small and so interactions can be ignored. The entropy is additive and given by
\[
S = 2\pi \sqrt{N_5n_R}(\sqrt{N_1} + \sqrt{\bar{N}_1}) ,
\] (3.5)
where $N_1$ ($\bar{N}_1$) is the number of zero-branes (anti-zero-branes). This is again in agreement with (2.25) for $N_5 = n_L = 0$.

A third way to get light modes is to keep $R$ fixed and take $V$ to be very small. In this case the fivebranes become light and dominate the near-extremal entropy. This is related by T-duality to the preceding case. The near-extremal entropy is then
\[
S = 2\pi \sqrt{N_1n_R}(\sqrt{N_5} + \sqrt{\bar{N}_5}) ,
\] (3.6)
where $N_5$ ($\bar{N}_5$) is the number of zero-branes (anti-zero-branes).

Using the U-duality transformations (2.16) the role of fivebranes, onebranes and strings in the preceding can be permuted. A check on this is given by the fact that that (3.4), (3.5) and (3.6) are permuted into one another under interchanges of fivebranes, onebranes and strings.

Away from extremality and the special limits of moduli space discussed above one expects all six quantities $N_1, \bar{N}_1, N_5, \bar{N}_5, n_R$ and $n_L$ to be nonzero. There is a natural and simple expression for the entropy which reduces to (3.4) (3.5) (3.6) and also is invariant under the permutations as required by duality. That expression is
\[
S = 2\pi(\sqrt{N_1} + \sqrt{\bar{N}_1})(\sqrt{N_5} + \sqrt{\bar{N}_5})(\sqrt{n_L} + \sqrt{n_R}) .
\] (3.7)
We do not know of a systematic derivation of this formula using D-brane technology. However miraculously it agrees with the Bekenstein-Hawking entropy calculated in the previous section from the area of the event horizon.

From the D-brane point of view, if (3.7) is the correct formula and the $N$’s can be interpreted as the number of branes and anti-branes, then there seems to be a discrepancy in the number of free parameters: in the D-brane picture, one can specify the six numbers in (3.7) plus $V$ and $R$. However, it turns out that if one maximizes the entropy (3.7) keeping the charges (2.5) mass (2.21) and $R, V$ fixed, then the proportion of branes and anti-branes that results is precisely the amount present in the black hole solution (2.20). This supports the picture of the black hole as an ensemble of branes in thermodynamic equilibrium.\footnote{Even in the black hole picture, $R$ and $V$ are not completely arbitrary, since (away from extremality) if they are too large, the solution becomes classically unstable. [23].}

\[12\]
4. Discussion

In principle one might hope to go beyond the special weakly-coupled limits in (3.4) - (3.6) and derive (3.7) by counting D-brane configurations. The problem is that one immediately runs into strong coupling. There are interactions between D-branes which are independent of the string coupling, e.g., since $G \sim g^2$ and $E \sim 1/g$, gravitational interactions are independent of $g$, and so cannot be suppressed by making $g$ small. For example consider the regime in which $g$ is small, $R$ is large and $V$ is order one. Then the string modes are weakly coupled and (3.4) can be derived. However, the onebrane-anti-onebrane pairs are on top of each other with coupling of order one and would seem to immediately annihilate, even if $g$ is small. Hence it does not appear to be sensible to discuss onebrane-anti-onebrane pairs in this regime. Nevertheless the simplicity of (3.7) begs for an explanation.

If one simply expands (3.7) one obtains a representation of the entropy of a nonextreme black hole in terms of the sum of eight extremal black holes. This suggests that the entropy of a generic black hole might be explained as the sums of the entropies of extremal constituents. However, the eight extremal black holes have a total mass which is four times the mass of the initial black hole (since each integer $N$ appears four times in the sum). So this approach cannot explain the entropy formula.

While it is probably not possible to derive all of (3.7) from known techniques it may be both possible and instructive to go beyond what we have done. For small $R$ and fixed $V$ the couplings between both onebrane-anti-onebrane pairs and fivebrane-anti-fivebrane pairs are weak in the sense that (in the T-dual picture) they are dilute and take a long time to annihilate. If we add a small amount of energy to the extremal configuration in this small $R$ regime, it should go into configurations which look approximately like collections of strings, onebranes, anti-onebranes, fivebranes and anti-fivebranes (note from (2.20) that $n_L \to 0$ for small $R$). By counting such configurations one should be able to determine the ratio of anti-onebranes to antifivebranes as a function of the energy, $R$, $V$ and the three charges to leading order in $R$. After transforming to the $N$’s, this would give a check on (3.7) with only $n_L = 0$.

A puzzling feature of (3.7) is that it only involves onebranes, fivebranes, and strings. This is understandable for extremal solutions with these charges, but when one moves away from extremality, one might expect pairs of threebranes and anti-threebranes or

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8 It is conceivable that this annihilation is suppressed by some new strong coupling effects.
fundamental string winding modes to contribute to the entropy. To see the contributions of these other objects, one should start with the full Type II string theory compactified on $T^5$. The low energy limit of this theory is $N = 8$ supergravity in five dimensions. This theory has 27 gauge fields, 42 scalars and a global $E_6$ symmetry. Since only the scalar fields which couple to the gauge fields are nontrivial in a black hole background, we expect the general solution to be characterized by 27 scalars in addition to the 27 charges. (Since an overall shift of all the scalars can be compensated by rescaling the metric, one can interpret the 27 scalar parameters as 26 scalars plus the ADM energy.) Each charge corresponds to a type of soliton or string. Thus we expect the solution to again be characterized by the number of solitons and anti-solitons. For an extremal black hole, the entropy can be written in the $E_6$ invariant form \[24,25\]

$$S = 2\pi |T_{ABC} V^A V^B V^C|^{1/2}$$

(4.1)

where $V^A$ is the 27 dimensional charge vector and $T_{ABC}$ is a symmetric cubic invariant in $E_6$. For the nonextremal black holes, the above argument suggests that one can introduce two vectors $V_i^A$ $i = 1, 2$ which represent the number of solitons and anti-solitons. Although we have not done the calculation, the general black hole entropy might take the $E_6$ invariant form

$$S = 2\pi \sum_{i,j,k} |T_{ABC} V_i^A V_j^B V_k^C|^{1/2}$$

(4.2)

where the sum is over the two possible values of each of $i$, $j$ and $k$. If this is the case, the entropy of nonextremal black holes could be represented in terms of solitons and anti-solitons in many different (equivalent) ways which are related by $E_6$ transformations.

The entropy of four dimensional black holes can also be expressed in a form similar to (3.7) \[26\], using the representation given in \[3,4\].

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