The Symmetric Space and Homogeneous sine-Gordon theories

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ABSTRACT

Two series of integrable theories are constructed which have soliton solutions and can be thought of as generalizations of the sine-Gordon theory. They exhibit internal symmetries and can be described as gauged WZW theories with a potential term. The spectrum of massive states is determined.

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1. Introduction

Integrability has proved to be a powerful tool in elucidating non-perturbative properties of field theory. For instance, in the sine-Gordon theory an exact quantum description of the solitons can be deduced [1,2], which can be shown to match precisely with semi-classical approaches in the appropriate limit. This work gives us confidence that similar semi-classical approaches are equally valid in higher dimensions, for instance for monopoles in four dimensional gauge theories. In that context, though, it is clear that additional complications can arise that are not captured by the sine-Gordon theory; namely the solitons, in this case t’ Hooft-Polyakov monopoles, can carry internal quantum numbers and have a complicated spectrum [3]. Therefore, it would be useful to have two-dimensional soliton theories which also have internal symmetries and are open to the same level of analysis as the sine-Gordon theory.

Many integrable generalizations of the sine-Gordon equations-of-motion have already been written down and are known as the non-abelian affine Toda equations [4,5,6]. However, not all these equations can be derived by extremizing an action with sensible properties like a positive-definite kinetic term and a real potential. In this paper, following on from [7], we show that non-abelian affine Toda equations give rise to two series of models, referred to as the Symmetric Space Sine-Gordon (SSSG) theories and the Homogeneous Sine-Gordon (HSG) theories, both of which are integrable, admit soliton solutions, and have a real positive-definite action.

One of the subtleties of these theories is that in general their potentials have flat directions [7]. In order to construct theories which will admit an S-matrix description, like the sine-Gordon theory, we will show how the flat directions in the potential can be removed by a gauging procedure. The resulting theories are of the form of a gauged Wess-Zumino-Witten (WZW) action plus a certain potential which deforms the theory away from the critical point along an integrable direction.

More specifically, the SSSG theories are related to a compact symmetric space $G/G_0$, with a $G_0$-valued field. As a particular example, the ordinary sine-Gordon theory corresponds to $G/G_0 = SU(2)/U(1)$. This class of theories was first considered in [8], and their Lagrangian formulation has been recently worked out in [9]. Nevertheless, neither the symmetries nor the mass spectrum of these theories have been investigated before. In fact the theories in [8,9] have flat directions which, as we remarked above, can be removed
by gauging. All the models constructed in [7] are of this type, with $G/G_0$ a symmetric space of type I.

The HSG theories are particular examples of the deformed coset models constructed in [10]. The fields of these theory take values in some compact semi-simple group $G_0$. It turns out that the equations-of-motion of the SSSG theories can be obtained by reduction from an appropriate HSG model. The simplest representative of this class is the complex sine-Gordon theory [10,11,12,13] for which $G_0 = SU(2)$. In fact, all the HSG theories can be understood as interacting sets of complex sine-Gordon fields.\(^1\)

The overall aim of the analysis is to completely determine the quantum properties of these theories by constructing a factorizable S-matrix which describes the solitons and particle states of the theory, in the same way that the sine-Gordon theory has been described [1,2]. In the present paper, our aims will be more modest, and we shall restrict ourselves to an explanation of how these theories are formulated at the Lagrangian level. The most important issue that we tackle, is to show how the flat directions of the potential can be removed by a gauging procedure. Actually, the condition that this procedure succeeds in removing all the flat directions requires that the theories are related to a coset $G_0/U(1)^{\times p}$. In other words, even though the field will be non-abelian, the gauge and global symmetries of these theories will always be abelian. In contrast to [9,10], we shall allow for more general types of gauging over, and above, the vector type. We will then derive the spectrum of particle states corresponding to small fluctuations around the vacuum. Finally, we will go on to explain briefly how soliton solutions can be found by the Leznov-Saveliev procedure and how soliton masses and classical scattering time-delays can be found.

2. Massive theories and non-abelian affine Toda equations

The quantum field theories that we will construct are characterized by the fact that their equations-of-motion are non-abelian affine Toda equations [4,5,6]. These equations are usually written down using affine (Kac-Moody) algebras, but for our purposes, it will be more convenient to describe them directly by means of finite Lie algebras.

In order to keep the construction as general as possible, let $g$ be a complex semi-simple finite-dimensional Lie algebra, and $\sigma$ a finite order automorphism of $g$.\(^2\) If $N$ is the order

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\(^{1}\) A complete discussion of the HSG theories will appear in [14].

\(^{2}\) On $g$, we denote the invariant and non-degenerate Killing form by $\langle \cdot \cdot \rangle$, normalized such that long roots have square length 2.
of \( \sigma, \sigma^N = I \), it induces a \( \mathbb{Z}/N\mathbb{Z} \) gradation of \( g \):

\[
g = \bigoplus_{j \in \mathbb{Z}} g_j, \quad [g_j, g_k] \subset g_{j+k},
\]

(2.1)

where \( \sigma(a) = \exp(2\pi i j/N)a \), for any \( a \in g_j \), and \( j \) stands for the residue of \( j \) mod \( N \). If \( g \) is simple, then let \( r = 1, 2 \) or \( 3 \) be the least positive integer such that \( \sigma^r \) is an inner automorphism of \( g \). In this case, \((g, \sigma)\) provide a realization of the (twisted) affine Kac-Moody algebra \( g^{(r)} \) in terms of the central extension of the (twisted) loop algebra

\[
\mathcal{L}(g, \sigma) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}^j \otimes g_j.
\]

(2.2)

The fields of the theory, \( h(x, t) \), take values in the group \( G_0 \) associated to the Lie algebra of the zero-graded component \( g_0 \), and the Toda equation involves the choice of two ad-diagonalizable (semi-simple) elements \( \Lambda_+ \in g_\mathcal{F} \) and \( \Lambda_- \in g_{N-k} \), for some non-negative integer number \( k \). The Toda equations are

\[
\partial_- (h^{-1} \partial_+ h) = -m^2 [\Lambda_+, h^{-1} \Lambda_- h]
\]

(2.3)

or equivalently

\[
\partial_+ (\partial_- h h^{-1}) = -m^2 [h \Lambda_+ h^{-1}, \Lambda_-],
\]

(2.4)

where \( x_{\pm} = t \pm x \) are the light-cone variables and \( m \) is a constant with dimensions of mass. The integrability of the Toda equations is manifested in the equivalent zero-curvature form

\[
[\partial_+ + h^{-1} \partial_+ h + im\Lambda_+, \partial_- + imh^{-1} \Lambda_- h] = 0.
\]

(2.5)

The equations (2.3) follow from extremizing the action

\[
S[h] = \frac{1}{\beta^2} \left\{ S_{\text{WZW}}[h] - \int d^2 x V(h) \right\},
\]

(2.6)

where the kinetic term \( S_{\text{WZW}}[h] \) is the Wess-Zumino-Witten action for the group \( G_0 \), and the potential is

\[
V(h) = -\frac{m^2}{2\pi} \langle \Lambda_+, h^{-1} \Lambda_- h \rangle.
\]

(2.7)

In (2.6), \( \beta \) is a coupling constant that plays no role in the classical theory; nevertheless, for non-abelian (compact) \( G_0 \), it becomes quantized if the quantum theory is to be well defined \((1/\beta^2 \in \mathbb{Z}^+)\) [15].
2.1 Theories with a real positive action.

The above construction of the non-abelian Toda equations makes perfect sense for any choice of \( \{g, \sigma, \Lambda_{\pm}\} \), without specifying a particular real form of the Lie group \( G_0 \). However, unless \( G_0 \) is abelian, the compact real form has to be chosen in order to ensure the theory has a positive-definite kinetic term [7]. This imposes a reality condition on the field \( h^\dagger = h^{-1} \) that is consistent with the equations-of-motion only if \( \Lambda_\pm^\dagger = \Lambda_\pm \), which ensures, moreover, that the potential \( V(h) \) is real. Since \( \Lambda_+, \Lambda_-^\dagger \in g_0^k \) and \( \Lambda_-, \Lambda_+^\dagger \in g_{N-k}^N \), the reality condition implies that \( \frac{2k}{2k} = 0 \).

The following cases can be distinguished:

(i) \( k = 0 \) and \( N \geq 1 \). Since \( h \in G_0 \) and \( \Lambda_\pm \in g_0^\dagger \), these theories can be described just in terms of \( g_0^\dagger \), which in general has the form \( u(1) \oplus \cdots \oplus u(1) \oplus g_{ss} \), where \( g_{ss} \) is semi-simple. However, we shall describe later how the \( u(1) \)-fields, associated to the centre of \( g_0^\dagger \), correspond to flat directions of the potential and have to be eliminated if we wish to consider theories with a mass-gap. Therefore, in this respect, only the semi-simple part of \( g_0^\dagger \) is important, and, hence, all the non-equivalent theories of this class can be associated to compact semi-simple Lie algebras. The complex sine-Gordon model lies in this class for \( g = su(2) \) [12,13].

(ii) \( N = 2 \) and \( k = 1 \). In this case, \( \sigma \) is an involutive automorphism of \( g \) that induces a gradation of the form

\[
g = g_\Pi \oplus g_\Gamma,
\]

where \( g_\Pi \) is a compact subalgebra of \( g \). This Lie algebra decomposition satisfies the commutation relations

\[
[g_\Pi, g_\Pi] \subset g_\Pi, \quad [g_\Pi, g_\Gamma] \subset g_\Gamma, \quad \text{and} \quad [g_\Gamma, g_\Gamma] \subset g_\Pi,
\]

which implies that these theories are associated to symmetric spaces [16].

These theories are known as the symmetric space sine-Gordon models [8,9] where the symmetric space is of the form \( G/G_0 \), where \( G \) is some suitable real form of the group associated to \( g \). However, since \( g_\Pi^\dagger \) is compact and \( (i\Lambda_\pm)^\dagger = -i\Lambda_\pm \) are semi-simple elements of \( g_\Gamma \), we will assume that the semi-simple Lie algebra \( g \) is also compact, implying that the relevant symmetric spaces will be of the compact type only. Nevertheless, we will keep on using the same notation for the compact Lie algebra \( g \) and its complexification.

It is worth mentioning that all the theories constructed in [7] in terms of the integral embeddings of \( sl(2) \) into \( g \) are included in this class. They are recovered by taking the
inner automorphism $\sigma = \exp(\pi i \text{ad} J_0)$ defined by the same Cartan element $J_0$ that specifies the embedding. The connection of this particular class of theories with symmetric spaces was originally realized in Ref. [9].

(iii) $k > 1$ and $N = 2k$. This case appears to be a generalization of the previous one where now the subalgebra $\hat{g} = g_0 \oplus g_k$ and the automorphism $\hat{\sigma} = \sigma^k$ play the role of $g$ and $\sigma$, respectively. In general, $\hat{g}$ has the form $\hat{g} = u(1) \oplus \cdots \oplus u(1) \oplus \hat{g}_{ss}$, where $\hat{g}_{ss}$ is semi-simple and $\text{cent}(\hat{g}) = u(1) \oplus \cdots \oplus u(1)$ is the centre of $\hat{g}$. However, since the zero graded generators of $\text{cent}(\hat{g})$ commute with $\Lambda_{\pm}$, later we shall see that the associated fields correspond to flat directions of the potential and so will be eliminated. In addition, if $\Lambda_{\pm}$ have non-vanishing components in $\text{cent}(\hat{g})$, they only induce a constant term in the potential (2.7) and it is obvious that they do not contribute to the equations of motion (2.3). All this shows that only the semi-simple part of $\hat{g}$ is important and, consequently, that the theories with $N = 2k$ and $k > 1$ are already included in the class (ii) by considering $\hat{g}_{ss}$ and the automorphism $\hat{\sigma} = \sigma^k$, which certainly satisfies $\hat{\sigma}^2 = I$.

In summary, for non-abelian $g_0$, the only inequivalent field theories with a positive-definite kinetic term and real potential are associated to automorphisms such that either $\sigma = I$ (the identity) or $\sigma^2 = I$ (an involution). The first class of theories, with $\sigma = I$, involve a compact semi-simple Lie algebra $g = g_0$, and they will be called homogeneous sine-Gordon models (HSG). The second class involves the decomposition $g = g_0 \oplus g_1$ of a compact semi-simple Lie algebra, and corresponds to the symmetric space sine-Gordon models (SSSG) [8] associated to symmetric spaces $G/G_0$ of compact type.

So far, the choice of $g$ has been kept as general as possible. In some cases the resulting theories can be decoupled into simpler ones. In particular, an HSG model associated to a semi-simple Lie algebra $g = g_1 \oplus g_2 \oplus \cdots$ can be decoupled into the set of HSG models associated to each simple factor $g_1, g_2, \ldots$. Therefore, it will be sufficient to study the HSG models associated to compact simple Lie algebras, and a thorough analysis of these theories will be presented in a subsequent publication [14]. On the other hand, it follows from the Cartan classification of compact symmetric spaces that the symmetric space sine-Gordon models corresponding to a compact Lie algebra $g$ can be decoupled into SSSG models associated to type I or type II symmetric spaces [16]. Recall that type I symmetric spaces are associated to a compact simple Lie algebra $g$ and an involutive automorphism $\sigma$, while

\[ h^\dagger = h. \]  

Nevertheless, in this case, one does not expect the theory to have soliton solutions since it is of the sinh-Gordon rather than sine-Gordon type [17], and consequently we will not consider this possibility here.
type II symmetric spaces involve a compact semi-simple Lie algebra \( g = g_1 \oplus g_2 \), where \( g_1 = g_2 \) are simple ideals, and an involutive automorphism \( \sigma \) of \( g \) that interchanges \( g_1 \) and \( g_2 \). Notice that it is always consistent to associate a different coupling constant \( \beta \) and mass scale \( m \) to each of the decoupled theories.

2.2 Theories with a mass-gap.

The previous discussion has identified the class of theories with a positive-definite kinetic term and real potential. Let us now address the problem of flat directions of the potential. The potential (2.7) exhibits the global symmetry \( V(\alpha_- h \alpha_+) = V(h) \) for any \( \alpha_\pm \) in the groups \( G_\pm \) associated to the subalgebras \( \text{Ker}(\text{ad}_{\Lambda_\pm}) \cap g_0^\perp \), in other words, \( V(h) \) has \( G_- \times G_+ \) (left-right) symmetry. This implies that the quantum theory corresponding to the action (2.6) will not have a mass-gap. However, according to the program of [7], we wish to study theories with a mass-gap because such theories are expected to admit an S-matrix description.

Recall that \( \Lambda_\pm \) are ad-diagonalizable (semi-simple) elements, which means that the Lie algebra \( g \) has the two following orthogonal decompositions with respect to their adjoint actions

\[
g = \text{Ker}(\text{ad}_{\Lambda_\pm}) \oplus \text{Im}(\text{ad}_{\Lambda_\pm}).
\]

Therefore, the equations of motion (2.3) imply that those field configurations such that

\[
h^\dagger \partial_+ h \in \text{Ker}(\text{ad}_{\Lambda_+}) \cap g_0^\perp \quad \text{or} \quad \partial_- h^\dagger h \in \text{Ker}(\text{ad}_{\Lambda_-}) \cap g_0^\perp,
\]

(2.11)

correspond to flat directions of the potential. In order to remove them, we have to somehow introduce the constraints \( P_+(h^\dagger \partial_+ h) = P_-(\partial_- h^\dagger h) = 0 \), where \( P_\pm \) are the projection operators onto the subalgebras \( \text{Ker}(\text{ad}_{\Lambda_\pm}) \cap g_0^\perp \).

The way to introduce these constraints was discussed in Ref. [10]. The idea is to gauge a subset of the symmetry transformations. It is well known that it is not possible to gauge an arbitrary subgroup of transformations since several conditions must be satisfied [18,19]. First of all, it has to correspond to the embedding of some common subgroup \( H \) of \( G_\pm \) into \( G_- \times G_+ \) of the form \( \alpha \mapsto (\alpha_L, \alpha_R) \). In other words, the local invariance has to be of the form \( h \mapsto \alpha_L(x,t) h \alpha_R^\dagger(x,t) \). In our case, this first condition implies that, at most, we will be able to gauge only the transformations generated by the compact group associated to the subalgebra

\[
(\text{Ker}(\text{ad}_{\Lambda_+}) \cap \text{Ker}(\text{ad}_{\Lambda_-})) \cap g_0^\perp,
\]

(2.12)

\(^4\) \text{Ker}(\text{ad}_{\Lambda_\pm}) \) is the centralizer of \( \Lambda_\pm \) in \( g \), i.e., the set of \( x \in g \) that commute with \( \Lambda_\pm \).
which, in general, is not sufficient to remove all the flat directions of the potential. Therefore, in order to ensure that this procedure leads to the required constraints, the two elements $\Lambda_{\pm}$ have to be chosen such that

$$\text{Ker}(\text{ad}_{\Lambda_{+}}) \cap g_{\sigma} = \text{Ker}(\text{ad}_{\Lambda_{-}}) \cap g_{\sigma}. \quad (2.13)$$

This additional requirement was not made explicit in [7] where it is enforced by the stronger condition $\Lambda_{+} = \Lambda_{-}$. Notice that (2.13) does not imply that $[\Lambda_{+}, \Lambda_{-}] = 0$ unless the theory is of the HSG type.

Assuming that $\Lambda_{\pm}$ satisfy (2.13), the flat directions would be associated to the subalgebra $g_{0}^{0} = \text{Ker}(\text{ad}_{\Lambda_{\pm}}) \cap g_{0}$. Then, $G_{+} = G_{-}$ is just the compact group $G_{0}^{0}$ associated to $g_{0}^{0}$, and one has to gauge some set of diagonal transformations of the form

$$h \mapsto \alpha_{L} h \alpha_{R}^{\dagger}, \quad \text{with} \quad \alpha_{L}, \alpha_{R} \in G_{0}^{0}. \quad (2.14)$$

These transformations correspond to an embedding of $G_{0}^{0}$ into $G_{0}^{0} \times G_{0}^{0}$, which is determined by an embedding of Lie algebras that can be written as

$$g_{0}^{0} \longrightarrow g_{0}^{0} \times g_{0}^{0} \quad \text{and} \quad u \longmapsto (u_{L}, u_{R}). \quad (2.15)$$

The transformations (2.14) can be gauged if $G_{0}^{0}$ is an anomaly free subgroup of $G_{0}^{0} \times G_{0}^{0}$, which simply means that [18,19]

$$\langle u_{L}, v_{L} \rangle = \langle u_{R}, v_{R} \rangle, \quad (2.16)$$

for all $u, v$ in $g_{0}^{0}$.

The most familiar solution of eq. (2.16) is $u_{R} = u_{L}$, which corresponds to the vector gauge transformations $h \mapsto \alpha h \alpha^{\dagger}$. For our purposes, we will need to consider more general gaugings. These can be described by considering a generic automorphism $\tau$ of $g_{0}^{0}$ and its lift $\hat{\tau}$ into the group $G_{0}^{0}$ such that $\hat{\tau}(\exp i \phi) = \exp i \tau(\phi)$, for all $i \phi \in g_{0}^{0}$. If the choice of $\tau$ is limited to the set of automorphisms that leave the restriction of the bilinear form $\langle , \rangle$ of $g$ to $g_{0}^{0}$ invariant, $u_{R} = \tau(u_{L})$ is the general solution of (2.16). It corresponds to the group of gauge transformations\(^5\)

$$h \mapsto \alpha h \hat{\tau}(\alpha^{\dagger}). \quad (2.17)$$

These transformations can be gauged by introducing a gauge field $A_{\pm}$ taking values in $g_{0}^{0}$, and substituting $S_{\text{WZW}}[h]$ with the gauged WZW action $S_{\text{WZW}}[h, A_{\pm}]$ associated to the coset $G_{0}/G_{0}^{0}$. Then, the gauge invariant action is

$$S[h, A_{\pm}] = \frac{1}{\beta^{2}} \left\{ S_{\text{WZW}}[h, A_{\pm}] - \int d^{2}x \, V(h) \right\}, \quad (2.18)$$

\(^5\) If $\tilde{\alpha} = \hat{\tau}(\alpha)$, notice that $\tilde{\alpha}^{\dagger} \partial_{\mu} \tilde{\alpha} = \tau(\alpha^{\dagger} \partial_{\mu} \alpha)$.
where the gauged WZW is explicitly given by

\[
S_{\text{WZW}}[h, A_{\pm}] = S_{\text{WZW}}[h] + \frac{1}{2\pi} \int d^2 x \left( -\langle A_+ , \partial_- h h^\dagger \rangle 
+ \langle \tau(A_-) , h^\dagger \partial_+ h \rangle + \langle h^\dagger A_+ h , \tau(A_-) \rangle - \langle A_+ , A_- \rangle \right),
\]

(2.19)

and the gauge field transforms as \( A_{\pm} \mapsto A_{\pm} + \alpha A_{\pm}^\dagger - \partial_{\pm} \alpha \alpha^\dagger \). The equations-of-motion, which follow from the variation of (2.18) with respect to \( h \), can be written in the zero-curvature form

\[
\left[ \partial_+ + h^\dagger \partial_+ h + h^\dagger A_+ h + im\Lambda_+ , \partial_- + \tau(A_-) + imh^\dagger \Lambda_- h \right] = 0.
\]

(2.20)

On the other hand, the variations with respect to \( A_{\pm} \) lead to the constraints

\[
P \left( h^\dagger \partial_+ h + h^\dagger A_+ h \right) - \tau(A_+) = 0,
\]

\[
P \left( -\partial_- h h^\dagger + h \tau(A_-) h^\dagger \right) - A_- = 0,
\]

(2.21)

where \( P \) is the projector onto the subalgebra \( g_0^0 \). By projecting (2.20) onto \( g_0^0 \) and using (2.21), one can see that the gauge field is flat: \( [\partial_+ + A_+ , \partial_- + A_-] = 0 \).

To show that the constraints (2.21) actually remove the flat directions of the potential, it suffices to choose the gauge \( A_{\pm} = 0 \), which is consistent due to the flatness of the gauge field considered on two-dimensional Minkowski space. In this gauge, the equations of motion reduce to the non-abelian Toda equation (2.3), along with the constraints

\[
P \left( h^\dagger \partial_+ h \right) = P \left( \partial_- h h^\dagger \right) = 0.
\]

(2.22)

At this stage, it is worth mentioning that, if we restrict the group of gauge transformations to be of the vector type, the only theories we end up with correspond exactly with either the deformed coset models of [10], if the theory is of the HSG type, or the symmetric space sine-Gordon models worked out in [9], if it is a SSSG theory. In contrast, the theories described by the action (2.18) involve a more general group of gauge transformations. They are associated with \( \{ g, \sigma, \Lambda_\pm, \tau \} \), where \( g, \sigma, \) and \( \Lambda_\pm \) have to satisfy the constraints discussed previously. In the following, we will single out the subclass of these theories that exhibit a mass-gap. There, the choice of the group of gauge transformations, \( i.e., \) of \( \tau \), will depend on the field configuration corresponding to the vacuum of the theory.

According to (2.3), the potential (2.7) has extrema when

\[
[\Lambda_+ , h_0^\dagger \Lambda_- h_0] = 0.
\]

(2.23)
Moreover, since $g_\pi$ is compact, the potential reaches an absolute minimum for some $x_\pm$-independent field configuration $h_0$ corresponding to the vacuum, and particles will be associated to the small fluctuations around $h_0$. The success of the previous procedure for constructing a theory with a mass-gap requires that all the flat directions of the potential around the vacuum $h_0$ correspond simply to gauge transformations. This means that for any $i\phi, i\psi \in g_0^0$, with $\phi^\dagger = \phi$ and $\psi^\dagger = \psi$, one can find $i\eta \in g_0^0$ such that

$$e^{i\phi} h_0 e^{i\psi} = e^{i\eta} h_0 e^{-i\tau(\eta)},$$

whose linearized form is

$$h_0^\dagger \eta h_0 - \tau(\eta) = h_0^\dagger \phi h_0 + \psi.$$ 

Since it has to be possible to solve this last equation for all the components of $\eta$, the automorphism $\tau$ has to be chosen such that

$$\tau(u) \neq h_0^\dagger u h_0$$

for any $u$ in $g_0^0$. The same conclusion is reached by demanding that gauge transformations do not leave the vacuum configuration invariant and, hence, that they do not correspond to flat directions of the potential around $h_0$. Notice that condition (2.26) implies that it will not be possible to construct massive theories with vector gauge transformations ($\tau = I$) if the vacuum configuration corresponds just to $h_0 = 1$.

On top of this, there is an additional constraint that the vacuum configuration $h_0$ has to satisfy. Let $(g_0^0)^\perp$ be the orthogonal complement of $g_0^0$ in $g_\pi$ with respect to $\langle , \rangle$, i.e. $g_\pi^0 = g_0^0 \oplus (g_0^0)^\perp$, and let $P$ and $P^\perp$ be the projectors onto $g_0^0$ and $(g_0^0)^\perp$, respectively. It is straightforward to see that eq. (2.25) implies

$$P\left(h_0^\dagger \eta h_0\right) - \tau(\eta) = P\left(h_0^\dagger \phi h_0\right) + \psi,$$

$$P^\perp\left(h_0^\dagger \eta h_0\right) = P^\perp\left(h_0^\dagger \phi h_0\right).$$

If (2.26) is satisfied, eq. (2.27) can be solved for all the components of $\eta$ as functions of $P\left(h_0^\dagger \phi h_0\right) + \psi$. Consequently, eq. (2.28) becomes an identity where the right-hand-side depends on $\phi$, while the left-hand-side depends on $P\left(h_0^\dagger \phi h_0\right) + \psi$. Therefore, since $i\phi$ and $i\psi$ are arbitrary elements of $g_0^0$, both sides of this equation have to vanish, which means that

$$h_0^\dagger g_0^0 h_0 = g_0^0.$$
In other words, \( h_0 \) has to induce an inner automorphism of \( g_\Pi \) that fixes the subalgebra \( g_0^0 \).

The constraint (2.26) is very restrictive and so the number of theories with positive and real action that exhibit a mass-gap is rather limited. Taking into account (2.29), \( \rho(u) = h_0 \tau(u) h_0^\dagger \) defines an automorphism of \( g_0^0 \), and eq. (2.26) requires that \( \rho \) does not leave fixed any element of \( g_0^0 \). The existence of fixed points under automorphisms of Lie algebras has been investigated, among other authors, by Borel and Mostow [20] and by Jacobson [21]. In particular, Jacobson proved that any automorphism of a non-solvable Lie algebra always have a fixed point (Theorem 9 of [21], see also the Theorem 4.5 of [20]).\(^6\) However, \( g_0^0 \) is a reductive Lie algebra, which means that it has the form \( u(1) \oplus \cdots \oplus u(1) \oplus (g_0^0)_{ss} \) where \((g_0^0)_{ss}\) is semi-simple. Therefore, the condition that \( \rho \) does not have any fixed point constrains \( g_0^0 \) to be an abelian subalgebra of \( g_\Pi \).

The condition that \( g_0^0 \) is abelian implies that the only HSG theories that have a mass-gap correspond to regular commuting elements \( \Lambda_\pm \) in \( g = g_\Pi \). This means that \( g_0^0 \) is a Cartan subalgebra of \( g \) and, hence, the massive HSG theories will be associated with the cosets \( G/U(1)^x r \), where \( G \) is a semi-simple compact Lie group of rank \( r \).

In contrast, for the SSSG theories, the condition that \( g_0^0 \) is abelian does not require that \( \Lambda_\pm \) are regular elements. In this case, it is convenient to introduce the notion of a Cartan subspace. Consider the decomposition \( g = g_\Pi \oplus g_\Pi \) associated with the (compact) symmetric space \( G/G_0 \). A Cartan subspace \( s \subseteq g_\Pi \) is a maximal subspace of ad-diagonalizable (semi-simple) elements which is also an abelian subalgebra of \( g \) [16,22]. For a given symmetric space, all such subspaces have the same dimension, which defines the rank of the symmetric space. Therefore, since \( g_0^0 \subset g_\Pi \) is an abelian set of semi-simple elements, it is easy to show that its dimension is bounded as

\[
0 \leq \text{rank}(G) - \text{rank}(G/G_0) \leq \dim(g_0^0) \leq \text{rank}(G) - 1.
\]

Moreover, in the particular case when \( \Lambda_\pm \) are regular elements and, hence, \( \text{Ker(ad}_{\Lambda_\pm} \) is a Cartan subalgebra of \( g \), the dimension of \( g_0^0 \) equals the lower bound \( \text{rank}(G) - \text{rank}(G/G_0) \). In any case, the massive SSSG theories will be associated with cosets of the form \( G_0/U(1)^x p \), where \( G/G_0 \) is a compact symmetric space and \( p = \dim(g_0^0) \). Notice that \( p \) may vanish if the rank of the symmetric space \( G/G_0 \) equals the rank of \( G \) and, then, the resulting massive SSSG theory is a perturbation of the WZW model corresponding to \( G_0 \).

\(^6\) Even though the proof of this important result is quite involved, it is possible to gain some intuition by considering the subset of inner automorphisms and of automorphisms that fix a Cartan subalgebra.
Assuming that the previous conditions are satisfied, all the non-gauge equivalent field configurations around the vacuum are expected to correspond to massive excitations in the quantum theory. Let us take the $A_\pm = 0$ gauge and $h = h_0 \exp(i\phi)$, with $i\phi \in g_\tau$ and $\phi^\dagger = \phi$. Then, the linearized constraints (2.22) and equations-of-motion (2.3) are

$$P(\phi) = 0, \quad \text{and} \quad \partial_\mu \partial^\mu \phi = -4 m^2 [\Lambda_+ , [h_0^\dagger \Lambda_- h_0^\dagger , \phi]] ,$$

(2.31)

which show that the fundamental particles are associated with the non-vanishing eigenvalues of $[\Lambda_+, [h_0^\dagger \Lambda_- h_0^\dagger , \bullet]]$ on $g_\tau$.

Let us introduce a Cartan-Weyl basis for the (complex) Lie algebra $g$, consisting of a Cartan subalgebra $h$ and step generators $E\alpha$, where $\alpha$ is a root of $g$. According to the previous discussion, the Cartan subalgebra can be chosen such that it contains both $\Lambda_+$ and $\tilde{\Lambda}_-$, $= h_0^\dagger \Lambda_- h_0$ (see eq. (2.23)), and such that its projection onto $g_\tau$ is just $g_0^\dagger$, $h \cap g_0^\dagger = g_0^\dagger$. Moreover, it is always possible to choose the basis of simple roots of $g$, $\{\alpha_1, \ldots, \alpha_r\}$, such that $\alpha_i \cdot \Lambda_+ \geq 0$ , i.e., such that $\Lambda_+$ lies in the fundamental Weyl chamber of $h$. With this choice, $\alpha \cdot \Lambda_+$ and $\alpha \cdot \tilde{\Lambda}_-$ cannot vanish for any root $\alpha$ unless the theory is of the SSSG type and $c E\alpha - c^* E_{-\alpha}$ is in $g_\tau$ for any complex number $c$.

Correspondingly, if $P_0$ is the projector onto $g_\tau$, the massive excitations are associated to the generators $P_0(c E\beta - c^* E_{-\beta})$ with $\beta \cdot \Lambda_+ \neq 0$ or, equivalently, $\beta \cdot \tilde{\Lambda}_- \neq 0$. Their masses are given by

$$m_{\beta}^2 = 4 m^2 (\beta \cdot \Lambda_+) (\beta \cdot \tilde{\Lambda}_-) ,$$

(2.32)

and they have to be positive because the potential has an absolute minimum at $h = h_0$. Therefore, whenever $P_0(c E\beta - c^* E_{-\beta}) \neq 0$ for a positive root $\beta$, we conclude that both $\beta \cdot \Lambda_+$ and $\beta \cdot \tilde{\Lambda}_-$ have to be strictly positive.

3. A classification

According to (2.32), the spectrum of particles states of these theories is characterised by $\Lambda_+$ and $\tilde{\Lambda}_- = h_0^\dagger \Lambda_- h_0$. Therefore, the theories constructed in the previous section are actually associated with the algebraic data $\{g, \sigma, \Lambda_+, \tilde{\Lambda}_-, h_0, \tau\}$, where $g$ is a semi-simple Lie algebra and $\sigma$ is either the identity ($k = 0$), for the HSG theories, or an involution of $g$ ($k = 1$), for the SSSG theories. In the second place, $\Lambda_+$ and $\tilde{\Lambda}_-$ are two elements in the

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7 The Cartan subalgebra is identified with the root space, which is viewed as a Euclidean vector space with dimension $r = \text{rank}(g)$, and we adopt the vector notation for the inner product induced there by the bilinear form of $g$. 

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projection of a Cartan subalgebra of \( g \) onto \( g_{\mathcal{F}} \). Their choice is constrained by the condition that the subalgebra \( g^0_0 = \ker(\text{ad}_{\Lambda^+}) \cap g_{\mathcal{F}} = \ker(\text{ad}_{\Lambda^+}) \cap g_{\mathcal{F}} \) is abelian. They specify the coset \( G_0/G^0_0 \) associated with the gauged WZW action (2.18). Finally, \( h_0 \) is a constant element of \( G_0 \) that conjugates the subalgebra \( g^0_0 \) into itself, and \( \tau \) is an automorphism of \( g^0_0 \) that leaves the bilinear form of \( g \) invariant, and such that \( \tau(\bullet) \neq h_0^\dagger(\bullet) h_0 \). The field configuration \( h_0 \) is the vacuum of the theory and \( \tau \) fixes the form of the group of gauge transformations.

However, not all the possible choices lead to non-equivalent theories. Let us consider the class of theories constructed from a fixed choice of \( g \) and \( \sigma \), \( i.e., \) the different HSG models corresponding to the same semi-simple Lie group \( G = G_0 \), if \( \{g, \sigma\} = \{g_{\mathcal{F}}, I\} \), or the SSSG theories associated with a given compact symmetric space \( G/G_0 \), if \( \{g, \sigma\} = \{g_{\mathcal{F}} \oplus g_{\mathcal{F}}, \sigma\} \) and \( \sigma^2 = I \). Then, for any constant element \( \phi \) in \( G_0 \), the theories specified by the data

\[
\{\Lambda^+, \Lambda^-, h_0, \tau\} \quad \text{and} \quad \{\phi^\dagger \Lambda^+ \phi, \phi^\dagger \Lambda^- \phi, \phi^\dagger h_0 \phi, \tau^*\}
\]  

(3.1)

are equivalent if the two groups of gauge transformations are conjugate, \( i.e., \) if \( \tau^*(\bullet) = \phi^\dagger \tau(\phi \bullet \phi^\dagger) \phi \). This can be easily proved by checking that the former theory is transformed into the latter through the invertible change of variables

\[
h \mapsto \phi h \phi^\dagger, \quad A_\pm \mapsto \phi A_\pm \phi^\dagger.
\]  

(3.2)

Therefore, the non-equivalent theories are obtained by restricting the choice of \( \Lambda^+ \) and \( \Lambda^- \) to a set of Cartan subalgebras of \( g \) that are non-conjugate under the adjoint action of \( G_0 \) on \( g \).

It is well known [16] that all the Cartan subalgebras of \( g \) are conjugate by the adjoint action of the Lie group \( G \). Therefore, all the different HSG theories associated to a given semi-simple Lie algebra \( g = g_{\mathcal{F}} \) can be recovered by considering all the possible choices of \( \Lambda^+ \) and \( \Lambda^- \) in a single Cartan subalgebra of \( g = g_{\mathcal{F}} \).

A parallel result for the non-equivalent SSSG theories corresponding to the same symmetric space requires the characterization of the orbits of the Cartan subalgebras of \( g \) under the adjoint action of \( G_0 \), which is only a subgroup of \( G \). For this reason, in the general case, the number of orbits is larger than one, and the non-equivalent SSSG theories involve all the possible choices of \( \Lambda^+ \) and \( \Lambda^- \) into more than one non-conjugate Cartan subalgebras. Some results about the \( G_0 \)-orbits of semi-simple elements of \( g_{\mathcal{F}} \) can be found in [16,22].

Finally, let us consider the different theories obtained from a given choice of \( \{g, \sigma, \Lambda^+, \Lambda^-, \}\), which are labelled by \( h_0 \) and \( \tau \). Although all these theories have the same
mass spectrum, they involve different groups of gauge transformations. Then, according to
the results of [19], the corresponding quantum theories are expected to exhibit some sort
of target-space duality and, hence, they should be considered as non-equivalent theories.
In any case, at the classical level, the theories associated with \{h_0, \tau\} and \{h_0^*, \tau^*\}, where
\[ \tau^*(\bullet) = \tau(h_0 (h_0^*)\dagger \bullet h_0^* h_0^\dagger), \] (3.3)
can be interchanged through the duality transformation
\[ h \mapsto h_0 (h_0^*)\dagger h, \quad A_\pm \mapsto h_0 (h_0^*)\dagger A_\pm h_0^* h_0^\dagger. \] (3.4)
Since the mass spectrum and, hence, \tilde{\Lambda}_- are kept fixed, notice that the change \( h_0 \mapsto h_0^* \)
also implies that
\[ \Lambda_- = h_0 \tilde{\Lambda}_- h_0^\dagger \mapsto h_0^* h_0^\dagger \Lambda_- h_0 (h_0^*)\dagger. \] (3.5)
Therefore, if \( h_0^* h_0^\dagger \Lambda_- h_0 (h_0^*)\dagger = -\Lambda_- \) for some particular choice of \( h_0 \) and \( h_0^* \), this change
of \( \Lambda_- \) is equivalent to \( m^2 \mapsto -m^2 \). This shows that the duality transformation (3.4)
is a generalization of the results of [13], where a similar transformation in the complex
sine-Gordon model was identified with the Krammers-Wannier duality in the context of
perturbed conformal field theory. Nevertheless, let us remark that the transformation (3.4)
is formulated without specifying any particular gauge fixing prescription, in contrast with
the results of [13].

For a given theory associated with \{\Lambda_+, \tilde{\Lambda}_-, h_0, \tau\}, eqs. (3.3) and (3.4) also provide a
condition for the possibility of formulating another theory with the same mass spectrum
but with vector gauge transformations. This requires that there exists some alternative
vacuum configuration \( h_0^* \) such that the change \( h_0 \mapsto h_0^* \) implies \( \tau \mapsto \tau^* = I \). According
to (3.3), this is only possible if \( \tau \) is the restriction onto \( g_0^0 \) of some inner automorphism of
\( G_0 \).

4. The fields associated to the centre of \( g_0^0 \)

The fields corresponding to the generators of \( \text{cent}(g_0^0) \) deserve special attention. First
of all, such fields are only present in the SSSG models. For simplicity, let us consider
the SSSG model corresponding to a (complex) simple Lie algebra \( g \) or, equivalently, to a
symmetric space of type I, even though it is straightforward to extend the analysis to the
general case of semi-simple algebras (type II).
Let $iu$ be an element in the centre of $g_\mathcal{P}$, and decompose the subalgebra $g_\mathcal{P}$ as
\[
g_\mathcal{P} = \mathbb{R} iu \oplus \widetilde{g}_0,
\]
where the elements of $\widetilde{g}_0$ are orthogonal to $u$ with respect to $\langle \cdot , \cdot \rangle$. Since $g_\mathcal{P}$ is compact, the adjoint action of $u$ can be diagonalized on the complex Lie algebra $g$, which defines an $u$-dependent integer gradation by adjoint action:
\[
g = \bigoplus_{j=-M}^{M} g_u^{(j)}, \quad [u, a] = j a \quad \text{for} \quad a \in g_u^{(j)}.
\]
Moreover, $u$ is in the centre of $g_\mathcal{P}$ and, hence, one has the following inclusions
\[
g_\mathcal{P} \subset g_u^{(0)} \quad \text{and} \quad g_u^{(j)} \subset g_\mathcal{P} \quad \text{for all} \quad j \neq 0.
\]
It is always possible to choose a Cartan-Weyl basis for the complex Lie algebra $g$ such that $u$ is in the fundamental Weyl chamber of the Cartan subalgebra. Then, if the system of simple roots is $\{\alpha_1, \ldots, \alpha_r\}$, the gradation (4.2) is specified by the non-negative integer numbers $s_i = \alpha_i \cdot u$, which give the grade of $E_{\alpha_i}$ in the gradation (4.2). The highest root of $g$ is $\theta = \sum_{i=1}^{r} k_i \alpha_i$, where $\{k_1, \ldots, k_r\}$ are the labels of the Dynkin diagram of $g$; hence, in (4.2), $E_{\pm \theta}$ is an element of the subspace $g_u^{(\pm M)}$ with $M = \sum_{i=1}^{r} k_i s_i$. The step operators $\{E_{\pm \alpha_1}, \ldots, E_{\pm \alpha_r}\}$ generate the Lie algebra $g$, and, taking into account eq. (4.3) and $[g_\mathcal{T}, g_\mathcal{T}] \subset g_\mathcal{P}$, one is led to the conclusion that $s_i$ can be non-vanishing for a single simple root $\alpha_j$ such that $k_j = 1$. In this gradation, $s_i = \delta_{i,j}$ and
\[
g = g_u^{(-1)} \oplus g_u^{(0)} \oplus g_u^{(1)}, \quad \text{with} \quad g_\mathcal{P} \subset g_u^{(0)} \quad \text{and} \quad g_u^{(\pm 1)} \subset g_\mathcal{T}.
\]
Using the last equation and taking into account $\Lambda_\pm^+ = \Lambda_\pm$, the elements $\Lambda_\pm$ can be decomposed as
\[
\Lambda_\pm = \Lambda_\pm^{(-1)} + \Lambda_\pm^{(0)} + \Lambda_\pm^{(1)},
\]
where $(\Lambda_\pm^{(0)})^\dagger = \Lambda_\pm^{(0)}$ and $(\Lambda_\pm^{(\pm 1)})^\dagger = \Lambda_\pm^{(\mp 1)}$.

According to (4.1), let us consider the field configuration
\[
h = \tilde{h} \exp(i \varphi u),
\]
where $\tilde{h}$ is a field taking values in the compact group associated to $\widetilde{g}_\mathcal{P}$, and $\varphi = \varphi(x, t)$ is the field associated to $u$. For simplicity, we will assume that $P(u) = 0$ and consider the
equations of motion in the $A_\pm = 0$ gauge. Then, the non-abelian affine Toda equation (2.3) yields two decoupled equations

$$\partial_- (\bar{h}^\dagger \partial_+ \bar{h}) = - m^2 [\Lambda_+, \bar{h}^\dagger \Lambda_- \bar{h}]$$

$$- m^2 \left( \cos \varphi - 1 \right) \left( [\Lambda_+^{(1)}, \bar{h}^\dagger \Lambda_-^{(-1)} \bar{h}] + [\Lambda_+^{(-1)}, \bar{h}^\dagger \Lambda_-^{(1)} \bar{h}] \right),$$

and

$$\partial_+ \partial_- \varphi = - \frac{2m^2}{\langle u, u \rangle} \sin \varphi \langle \Lambda_+^{(-1)}, \bar{h}^\dagger \Lambda_-^{(1)} \bar{h} \rangle,$$

which show that the equations-of-motion admit a reduction, preserving integrability, by taking $\varphi(x, t) = 0$. Since this result applies to a generic element $u$ in the centre of $g_0$, it implies that all the fields associated to the centre of $g_0$ can be decoupled whilst preserving integrability.

Notice that the analysis leading to (4.4) also shows that the dimension of the centre of $g_u^{(0)}$ is 1, which means that $\text{cent}(g_u^{(0)}) = \mathbb{R} i u$. Nevertheless, eq. (4.4) only implies that $\text{cent}(g_u^{(0)}) \subset \text{cent}(g_\tau)$, i.e., the centre of $g_\tau$ is not one-dimensional in the general case. However, if the identity $g_\tau = g_u^{(0)} = \text{Ker}(\text{ad}_u)$ is satisfied for some $u$ in cent($g_\tau$), one can ensure that, in this particular case, the dimension of the centre of $g_\tau$ is actually 1. For instance, this is the case of the theories constructed in [7] from the integral embeddings of $\text{sl}(2)$ into $g$. There, if $J_0$ is the Cartan element of the embedded $\text{sl}(2)$ subalgebra, $g_\tau = \text{Ker}(\text{ad}_{J_0}) = g_{J_0}^{(0)}$ and $J_0$ spans the one-dimensional centre of $g_\tau$. Moreover, in [7], $\Lambda_+ = \Lambda_- = J_+ + J_-$, which implies that

$$\Lambda_+^{(0)} = 0, \quad \Lambda_+^{(1)} = J_+, \quad \text{and} \quad \Lambda_-^{(-1)} = J_-.$$  

(4.9)

This shows that eq. (4.7) has the solution $\bar{h} = 1$ and, correspondingly, eq. (4.8) becomes the sine-Gordon equation for $\varphi$. Therefore, if $\varphi$ is not decoupled, the theories of [7] describe the interaction between some set of non-abelian Toda fields corresponding to $\bar{h}$, and the sine-Gordon field $\varphi$.

5. Parity invariant theories

As pointed out in Ref. [7], it is of particular interest to consider the class of theories that exhibit parity invariance. In particular, we require that the parity transformation fixes the vacuum.

By analysing the equations-of-motion (2.20), one can check that the theory has the symmetry $x \mapsto -x$, or $x_\pm \mapsto x_\mp$, along with $h \mapsto h_0 h^\dagger h_0$, $A_+ \mapsto h_0 \tau(A_-) h_0^\dagger$, and
\( A_- \mapsto \tau^{-1}(h^\dagger_0 A_+ h_0) \), only if
\[
\tilde{\Lambda}_- = h^\dagger_0 \Lambda_- h_0 = \mu \Lambda_+ \equiv \mu \Lambda, \tag{5.1}
\]
where \( \mu \) is some real number. Moreover, the constraints (2.21) have to be invariant under this transformation, which leads to the following relation
\[
\tau(h_0 \tau(A_\pm) h^\dagger_0) = h^\dagger_0 A_\pm h_0. \tag{5.2}
\]
It has to be satisfied independently of the particular values of the components of \( A_\pm \), which are functionals of \( h \). Therefore, taking into account (2.29), this means that the automorphism \( \rho(\bullet) = h_0 \tau(\bullet) h^\dagger_0 \) of \( g^0_0 \) has order two, \( \rho^2 = I \). Consequently, \( \rho \) can be diagonalized on \( g^0_0 \) with eigenvalues \( \pm 1 \), but the condition (2.26) implies that \( \rho = -I \).

Consequently, in the parity invariant theories, the automorphism \( \tau \) is given by
\[
\tau(u) = - h^\dagger_0 u h_0 \quad \text{for all} \quad u \in g^0_0, \tag{5.3}
\]
which indicates that the group of gauge transformations is completely specified by the vacuum configuration. Moreover, let us point out that the condition that \( \rho = -I \) is an automorphism of \( g^0_0 \) would constrain by itself the subalgebra \( g^0_0 \) to be abelian, as can be easily checked by considering the identities
\[
\rho([u, v]) = -[u, v] = [\rho(u), \rho(v)] = [-u, -v] = [u, v], \tag{5.4}
\]
for any \( u, v \in g^0_0 \).

Taking into account (2.32) and (5.1), the mass of the particle associated with a root \( \beta \) is given by \( m^2_\beta = 4 m^2 \mu (\beta \cdot \Lambda)^2 \). Since they have to be positive, \( \mu \) is a positive number that can be fixed to \( \mu = +1 \).

Therefore, parity invariant theories can be constructed in terms of a semi-simple Lie algebra \( g \) as follows. Let \( \sigma \) be an automorphism of \( g \) such that either \( \sigma = I \) and \( k = 0 \) (HSG), or \( \sigma^2 = I \) and \( k = 1 \) (SSSG), and let us choose a ad-diagonalizable element \( \Lambda = \Lambda^\dagger \in g_\mathbf{tr} \) such that \( g^0_0 = \text{Ker}(\text{ad}_\Lambda) \cap g_\mathbf{tr} \) is abelian. Then, for any element \( h_0 \in G_0 \) inducing an inner automorphism of \( g_\mathbf{tr} \) that fixes \( g^0_0 \), the theory is defined by the action (2.18) with the potential
\[
V(h) = -\frac{m^2}{2\pi} \langle \Lambda, h^\dagger (h_0 \Lambda h^\dagger_0) h \rangle, \tag{5.5}
\]
whose gauge symmetry is specified by the automorphism \( \tau \) given by (5.3). In this case, \( \hat{\tau}(\alpha) = h^\dagger_0 \alpha h^\dagger_0 \), and the resulting group of gauge transformations is\(^8\)
\[
\alpha h \mapsto \alpha h \hat{\tau}(\alpha^\dagger) = \alpha h (h^\dagger_0 \alpha h_0) \quad \text{for any} \quad \alpha \in G^0_0; \tag{5.6}
\]
\(^8\) If \( \alpha = \exp i\phi \), with \( i\phi \in g^0_0 \), then \( \hat{\tau}(\alpha) = \exp \sigma \tau(\phi) = \exp -i(h^\dagger_0 \phi h_0) = h^\dagger_0 \alpha h_0 \).
the corresponding transformation of the vacuum configuration is $h_0 \mapsto \alpha^2 h_0$. This theory is invariant under the parity transformation

$$x \mapsto -x, \quad h \mapsto h_0 h^\dagger h_0, \quad \text{and} \quad A_\pm \mapsto -A_\pm.$$  \hspace{1cm} (5.7)

Therefore, parity invariant theories can be labelled by the data $\{g, \sigma, \Lambda, h_0\}$, which, using the terminology of Section 3, correspond to $\Lambda_+ = \Lambda_- = \Lambda$ and $\tau(\bullet) = -h_0^\dagger(\bullet)h_0$.

The massive excitations are associated to those roots $\beta$ of $g$ such that $\beta \cdot \Lambda \neq 0$, and their masses are given by

$$m_\beta = 2 |\beta \cdot \Lambda|. \hspace{1cm} (5.8)$$

As explained in Section 2.2, it is always possible to choose a basis of simple roots $\{\alpha_1, \ldots, \alpha_r\}$ of $g$ such that $\Lambda$ is in the fundamental Weyl chamber of the Cartan subalgebra, i.e., $\alpha_i \cdot \Lambda \geq 0$ for all $i = 1, \ldots, r$. With this choice, the condition that $g_0^0$ is abelian means that $\alpha_i \cdot \Lambda = 0$ only if $P_0(c E_{\alpha_i} - c^* E_{-\alpha_i}) = 0$. With this choice, let $I$ be the set of integer numbers such that $\alpha_j \cdot \Lambda \neq 0$ for $j \in I$. Then, since any positive root is of the form $\beta = \sum_{i=1}^r n_i \alpha_i$ for some non-negative integers $n_i$, the mass of the fundamental particle associated to the roots $\pm \beta$ is

$$m_\beta = \sum_{j \in I} n_j m_{\alpha_j}. \hspace{1cm} (5.9)$$

This suggests that solitons corresponding to non-simple roots are bound-states at threshold of solitons associated to simple roots. An identical phenomenon occurs for monopoles in $N = 4$ supersymmetric gauge theories [3].

### 6. Some general properties

At this stage, let us make some comments about gauge fixing. The action (2.18) describes a theory that is invariant under the gauge transformations (2.17). Hence, it is pertinent to ask about the possible gauge fixing prescriptions. In this regard, there are two particular useful choices. The first one will be called the local gauge fixing prescription and consists in choosing some canonical form $h^\text{can}$ such that any $h$ can be taken to that form by means of a non-singular $h$-dependent gauge transformation. Therefore, for any $h \in G_0$ there exist two local functionals $i\phi^\text{can}[h] \in g^0_0$ and $h^\text{can}[h]$ such that

$$h = \exp(i\phi^\text{can}[h]) h^\text{can}[h] \exp(-i\tau(\phi^\text{can}[h])).$$ \hspace{1cm} (6.1)
Moreover, under a gauge transformation $h \mapsto h' = e^{iu} h e^{-i\tau(u)}$, with $iu$ taking values in $g^0_0$, $h^\text{can}[h]$ is gauge invariant, $h^\text{can}[h'] = h^\text{can}[h]$, while, since $g^0_0$ is abelian, $\phi^\text{can}[h]$ transforms as

$$\phi^\text{can}[h'] = \phi^\text{can}[h] + u .$$

The local gauge fixing prescription is simply $\phi^\text{can}[h] = 0$, and solving the constraints (2.21) for $A_\pm$ as local functionals of $h$ allows one to obtain a local gauge-invariant action: $S^\text{can}[h] \equiv S[h, A_\pm[h]]$.

The second gauge fixing prescription will be called the Leznov-Saveliev (LS) prescription. It consists in choosing $A_\pm = 0$, which can be done due to the on-shell flatness of the gauge field considered on two-dimensional Minkowski space; notice that this condition does not fix the global gauge transformations. In the LS gauge, the equations of motion (2.20) reduce to the non-abelian Toda equation (2.3), and the constraints (2.21) reduce to (2.22). These constraints, as pointed out in Ref. [10], cannot be solved locally. Nevertheless, using the method of Leznov and Saveliev, the explicit general solution of the non-abelian Toda equation along with the constraints (2.22) can be obtained using the representation theory of affine Kac-Moody algebras [4]. Actually, it is particularly easy, in this gauge, to obtain the multi-soliton solutions by means of the so-called solitonic specialization [23]. Moreover, many of the relevant calculations involving solitons, such as the calculation of their masses [6] and scattering time-delays, are greatly simplified.

In these theories, there exist conserved charges. In order to uncover them, recall that the potential has the $G^0_0 \times G^0_0$ symmetry property $V(e^{i\phi} h e^{i\psi}) = V(h)$ for any $i\phi, i\psi \in g^0_0$, with $\phi^\dagger = \phi$ and $\psi^\dagger = \psi$. Taking into account this, and requiring that the transformations fix the vacuum $h_0$, one can check that the theory exhibits an abelian global symmetry with respect to the tranformations

$$h \mapsto \alpha h (h_0^\dagger \alpha h_0) , \quad A_\pm \mapsto A_\pm ,$$

for each element $\alpha$ in the compact abelian group $G^0_0$.

Along with this abelian global symmetry there is a continuity equation

$$[\partial_+ + A_+ , \partial_- + A_-] = 0 ,$$

which is nothing else than the on-shell flatness condition for the gauge fields. Then, taking into account the gauge transformations of $A_\pm$ and of $\phi^\text{can} = \phi^\text{can}[h]$, defined in (6.1), the corresponding gauge invariant conserved Noether current is

$$J^\mu = \epsilon^{\mu \nu} (A_\nu + i\partial_\nu \phi^\text{can}) .$$
It is important to remark that, in the local gauge, \( J^\mu \) and the associated conserved Noether charge \( Q = \int_{-\infty}^{+\infty} dx \, J^0 \) are local functionals of \( h \).

Finally, let us briefly study the energy-momentum tensor of the theory described by the action (2.18). Its components are

\[
T_{++} = -\frac{1}{8\pi\beta^2} \langle \partial_+ h h^\dagger, \partial_+ h h^\dagger + 2 A_+ \rangle \\
T_{--} = -\frac{1}{8\pi\beta^2} \langle h^\dagger \partial_- h, h^\dagger \partial_- h - 2 \tau(A_-) \rangle \\
T_{+-} = -\frac{m^2}{4\pi\beta^2} \langle \Lambda_+, h^\dagger \Lambda_- h \rangle ,
\]

and it can be checked that it is explicitly gauge invariant. In the LS gauge, and using the solitonic specialization of the Leznov and Saveliev solution [23] and the results of [6], the calculation of the energy and momentum carried by solitons is greatly simplified, and their relation to the boundary conditions of the solitons can be clarified. Actually, in this gauge, eq. (6.5) shows that the values of the abelian conserved Noether charges are also explicitly related to the boundary conditions

\[
Q = \int_{-\infty}^{+\infty} dx \, J^0 = -i \phi^{\text{can}} \bigg|_{-\infty}^{+\infty} .
\]

7. Discussion

We have constructed two series of relativistic two-dimensional field theories whose equations-of-motion are related to the non-abelian Toda equations, namely the Symmetric Space (SSSG) and the Homogeneous (HSG) sine-Gordon models. They are singled out because they can be described by an action with a positive-definite kinetic term and a real potential, and because they also exhibit a mass-gap. The action consists of the gauged WZW action of a coset model plus a potential, which manifests their interpretation as perturbed conformal field theories. Moreover, the constructed theories are classically integrable, admit soliton solutions, and exhibit internal symmetries. Therefore, we expect that the semi-classical quantization of their soliton solutions will give rise to a spectrum consisting of massive charged particles.

The SSSG theories are associated to compact symmetric spaces, while the HSG theories involve compact semi-simple Lie groups. A compact symmetric space \( G/G_0 \) is related to a compact semi-simple Lie algebra \( g \) and an involutive automorphism \( \sigma \) that induces the decomposition \( g = g_0 \oplus \mathfrak{g}_\mathbf{T} \) into the eigenspaces where the eigenvalue of \( \sigma \) is +1 and −1;
$G$ and $G_0$ are the compact Lie groups whose Lie algebras are $g$ and $g_0$, respectively. If we also let $G_0$ and $g_\Pi$ denote a compact semi-simple Lie group and its Lie algebra, we can give a joint description of the data needed to define both series of theories, where, using this notation, the field always takes values in $G_0$. For a given compact symmetric space $G/G_0$ (compact semi-simple Lie group $G_0$), the different SSSG (HSG) theories are labelled by the data $\{\Lambda_+, \tilde{\Lambda}_-, h_0, \tau\}$. The first two, $\Lambda_+$ and $\tilde{\Lambda}_-$, are two semi-simple elements of $g_\Pi$ ($g_\Pi$) such that their centralizer in $g_0$ is an abelian subalgebra $g_0^0$. The choice of $\Lambda_+$ and $\tilde{\Lambda}_-$ determines the form of the potential, the mass-spectrum, and the coset $G_0/G_0^0$ corresponding to the gauged WZW term in the action, where $G_0^0$ is of the form $U(1)^{\times p}$. The precise form of the group of gauge transformations is specified by an automorphism $\tau$ of $g_0^0$ that preserves the bilinear form of $g$. Theories with the same spectrum but different groups of gauge transformations are expected to be related by target-space dualities [19].

The choice of the group of gauge transformations is constrained by the condition that it has to allow one to eliminate all the flat directions of the potential, which means that $\tau$ is related to the vacuum configuration $h_0$. Furthermore, the same condition requires that the adjoint action of $h_0$ on $g_0^0$ has to be an automorphism, which constrains the possible vacuum configurations. Therefore, different vacuum configurations imply different groups of gauge transformations and, because of this, we consider $h_0$ as an additional data. From this description, it is apparent that the SSSG theories associated to $G/G_0$ can also be viewed as the reduction of the HSG models related to $G$. Moreover, we have also described other reductions of the SSSG theories that maintain integrability. They consists in decoupling the fields associated to the centre of $G_0$.

The resulting theories are generalizations of the sine-Gordon [2] and complex sine-Gordon theories [12] and so we expect the spectrum of quantum states can be understood in terms of the semi-classical quantization of the soliton and other lump-like solutions. However, since in general the field takes values in a non-abelian group, an important difference between these theories and the sine-Gordon theory is that the coupling constant $\beta$ is quantized at the quantum level ($1/\beta^2 \in \mathbb{Z}^+$), something that has already been observed in the complex sine-Gordon theory [12].

The important question of the quantum integrability of these theories can be addressed by considering their description as perturbed conformal field theories. The existence of quantum conserved charges can then be investigated by using the method of Zamolodchikov [24]. The next stage of analysis involves trying to establish the form for the exact S-matrix for the scattering of the soliton and particle states. A powerful constraint on the soliton S-matrix arises from taking the semi-classical limit leading to a relation with the
time delays that occur in the classical scattering [25]. The time-delays themselves can be easily extracted from the solitonic specialization of Leznov and Saveliev [23].

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