ANALYSIS ON A HOMOGENEOUS SPACE OF A QUANTUM GROUP

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Abstract. A detailed account of the construction of a homogeneous space for the quantum
“az + b” group is presented. The homogeneous space is described by a commutative C∗-algebra
which means that it is a classical space. Then a covariant differential calculus on the homoge-
neous space is constructed and studied. A covariant measure and an analogue of the exponential
function are used to introduce elements of Fourier analysis.

1. Introduction

The concept of a homogeneous space for quantum groups has been studied for some time now. In
case of compact quantum groups the definition and many examples were worked out by Podle (16).
For non compact quantum groups even the definition is not agreed upon by the experts. In order
to have a better understanding of the problems we introduce an example of an object which should,
in our opinion, be called a homogeneous space. It turns out that despite a fairly abstract approach
to the definition of our homogeneous space for the quantum “az + b” group for real deformation
parameter (cf. [16]), the object turns out to be a classical space, i.e. is a quantum space described
by a commutative C∗-algebra. This relatively simple situation makes it easier to deal with such
aspects of quantum group covariant non commutative geometry as differential calculus, covariant
measures. Using some tools previously employed for the construction of examples of non compact
quantum groups we are able to introduce elements of Fourier analysis and prove that our “quantum
Fourier transform” has properties similar to those of its classical counterpart.

The paper is organizes as follows. In Section 2 we recall the necessary information about the
quantum “az + b” group needed for our construction. The homogeneous is defined and described in
detail in Section 3. Then we introduce a covariant differential calculus on the homogeneous space in
Section 4. At first we describe a general construction of, so called, embeddable covariant bimodules
for Hopf algebras. Then we introduce the class of function on the homogeneous space which play
the role of smooth functions in classical setting. In Subsection 4.3 we give an abstract description
of our covariant differential calculus and in the next subsection we describe its concrete realization
and covariance properties. Section 5 is devoted to developing integration on the homogeneous
space. A measure is introduced and properties of differential operators coming from the differential
calculus are studied from the point of view of functional analysis. The covariance and uniqueness
of the chosen measure are described in Subsection 5.1. Finally Section 6 contains the definition
and study of basic properties of the analogue of Fourier transform on the homogeneous space.

2. Quantum “az + b” group

Let us briefly recall the basic facts about the quantum “az + b” group (cf. [16, Appendix A]).
For a real parameter q such that 0 < q < 1 consider
\[ \Gamma = \{ z \in \mathbb{C} : |z| \in q^\mathbb{Z} \} . \]
Clearly \( \Gamma \) is a multiplicative subgroup of \( \mathbb{C} \setminus \{0\} \) and \( \Gamma \simeq \mathbb{Z} \times \mathbb{T} \). Moreover \( \Gamma \) is self dual, \( \Gamma \simeq \hat{\Gamma} \).

Any \( \gamma \in \Gamma \) is of the form \( \gamma = q^{k \varphi + k} \) for unique \( k \in \mathbb{Z} \) and \( \varphi \in \left[ 0, \frac{2\pi}{\log q} \right] \). Let
\[ \chi(\gamma, \gamma') = \chi(q^{i\varphi + k}, q^{i\varphi' + k'}) = q^{i(k - k') \log q} \]

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for all $\gamma, \gamma' \in \Gamma$. Then $\chi : \Gamma \times \Gamma \to \mathbb{T}$ is a non degenerate bicharacter on $\Gamma$. Using the non degeneracy of $\chi$ we shall identify $\overline{\Gamma}$ with $\Gamma$ via the formula

$$\langle \hat{\gamma}, \gamma \rangle = \chi(\hat{\gamma}, \gamma).$$

Finally let $\overline{\Gamma}$ be the closure of $\Gamma$, $\overline{\Gamma} = \Gamma \cup \{0\}$. The quantum “az + b” group is a pair $G = (\mathcal{A}, \Delta_G)$, where $\mathcal{A}$ is a C$^*$-algebra corresponding to the algebra of all continuous functions vanishing at infinity on the group and comultiplication $\Delta_G$ is a coassociative morphism: $\Delta_G \in \text{Mor}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$, i.e. a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{M}(\mathcal{A} \otimes \mathcal{A})$ encoding the group structure.

According to [16] there are normal elements $a$ and $b$ affiliated with the C$^*$-algebra $\mathcal{A}$ such that $\text{Sp} \ a, \text{Sp} \ b \subset \overline{\Gamma}$. Furthermore $a$ is invertible and $a^{-1} \eta \mathcal{A}$. Therefore for all $\gamma \in \Gamma$ we can form unitary elements $\chi(a, \gamma)$ of $\mathcal{M}(\mathcal{A})$ using functional calculus. Moreover $a$ and $b$ satisfy the relation

$$\chi(a, \gamma)b\chi(a, \gamma)^* = \gamma b$$

for any $\gamma \in \Gamma$. It turns out that $\mathcal{A}$ is generated by $a$, $a^{-1}$ and $b$ (cf. [15]). Let us also note that formula (2.1) implies that $ab$, $ba$, $ab^*$ and $b^*a$ are well defined normal operators satisfying

$$ab = q^2ba, \quad ab^* = b^*a$$

(cf. [16]). The reader should be warned, however, that (2.1) and (2.2) are not equivalent.

Let us remark (cf. [15]) that $\mathcal{A}$ is the universal C$^*$-algebra generated by $a$ and $b$ satisfying the described relations in the sense that for any C$^*$-algebra $C$ and $a_0$, $b_0 \in C$ satisfying

$$a_0, b_0 \text{ are normal},$$

$$\text{Sp} \ a_0, \text{Sp} \ b_0 \subset \Gamma,$$

$$a_0 \text{ is invertible and } a_0^{-1} \eta \mathcal{A}$$

$$\chi(a_0, \gamma)b_0\chi(a_0, \gamma)^* = \gamma b_0 \text{ for all } \gamma \in \Gamma$$

there exists a unique morphism $\Psi \in \text{Mor}(\mathcal{A}, C)$ such that

$$a_0 = \Psi(a), \quad b_0 = \Psi(b).$$

It turns out that the C$^*$-algebra $\mathcal{A}$ has a relatively simple structure. It is a C$^*$-crossed product

$$\mathcal{A} = C_{\infty}(\Gamma) \rtimes_\beta \Gamma,$$

where the action $\beta : \Gamma \ni \gamma \mapsto \beta_\gamma \in \text{Aut}(C_{\infty}(\Gamma))$ is given by

$$(\beta_\gamma f)(\gamma') = f(\gamma'^\gamma)$$

for all $f \in C_{\infty}(\Gamma)$.

To describe the generating elements $a$ and $b$ in this setting note that the natural inclusion $C_{\infty}(\Gamma) \hookrightarrow C_{\text{cont}}(\Gamma) \rtimes_\beta \Gamma$ is a morphism form $C_{\infty}(\Gamma)$ to $C_{\infty}(\Gamma) \rtimes_\beta \Gamma$. Let $z$ be the standard generator of $C_{\infty}(\Gamma)$, i.e. $z(\gamma) = \gamma$ for all $\gamma \in \Gamma$ then $b \eta C_{\infty}(\Gamma) \rtimes_\beta \Gamma$ is the image of $z$ under the canonical inclusion.

By definition of a crossed product $M(C_{\infty}(\Gamma) \rtimes_\beta \Gamma)$ contains a strictly continuous family of unitaries $(U_\gamma)_\gamma \in \Gamma$ such that

$$\beta_\gamma(f) = U_\gamma f U_\gamma^*$$

for any $f \in C_{\infty}(\Gamma)$. The results of Sections 4. and 5. of [16] show that there exists a normal element $a \eta C_{\infty}(\Gamma) \rtimes_\beta \Gamma$ such that $\text{Sp} \ a \subset \overline{\Gamma}$, $\text{ker} \ a = \{0\}$ and

$$U_\gamma = \chi(a, \gamma)$$

for all $\gamma \in \Gamma$. Moreover $a^{-1} \eta C_{\infty}(\Gamma) \rtimes_\beta \Gamma$. The elements $a$, $a^{-1}$ and $b$ generate $C_{\infty}(\Gamma) \rtimes_\beta \Gamma$ in the sense of [15].

The comultiplication is defined in the following way. Consider the C$^*$-tensor product $\mathcal{A} \otimes \mathcal{A}$. It turns out that $a \otimes b + b \otimes I$ is a closable operator and its closure $a \otimes b + b \otimes I$ is a normal element affiliated with $\mathcal{A} \otimes \mathcal{A}$. Moreover $\text{Sp} (a \otimes b + b \otimes a) \subset \overline{\Gamma}$. Clearly $(a \otimes a) \eta \mathcal{A} \otimes \mathcal{A}$ is normal, invertible, $(a^{-1} \otimes a^{-1}) \eta \mathcal{A} \otimes \mathcal{A}$ and $\text{Sp} (a \otimes a) \subset \overline{\Gamma}$. One can check that $a_0 = a \otimes a$ and $b_0 = a \otimes b + b \otimes I$
satisfy the commutation relations \(2.3\). Therefore there exists a unique \(\Delta_G \in \text{Mor}(A, A \otimes A)\) such that
\[
\begin{align*}
\Delta_G(a) &= a \otimes a, \\
\Delta_G(b) &= a \otimes b + b \otimes I.
\end{align*}
\]
Moreover \(\Delta_G\) is coassociative. This completes the description of the quantum “\(az + b\)” group on \(C^*\)-algebra level.

3. The homogeneous space

In this section we shall introduce a homogeneous space for \(G\) which is the main object of our considerations. First we shall observe that \(\hat{G}\) is a subgroup of \(G\). Indeed: for \(\hat{\gamma} \in \hat{G}\) let
\[
\begin{align*}
a_0(\hat{\gamma}) &= \hat{\gamma}, \\
b_0(\hat{\gamma}) &= 0. \tag{3.1}
\end{align*}
\]
Then \(a_0\) and \(b_0\) are continuous functions on \(\hat{G}\), i.e. elements affiliated with the \(C^*\)-algebra \(C_\infty(\hat{G})\). They satisfy the relations \(2.3\) and therefore there exists a unique morphism \(\pi \in \text{Mor}(A, C_\infty(\hat{G}))\) such that \(a_0 = \pi(a)\) and \(b_0 = \pi(b)\). Clearly \(\pi\) is surjective and the reader will easily check that
\[
(\pi \otimes \pi) \circ \Delta_G = \Delta_\pi \circ \pi, \tag{3.2}
\]
where \(\Delta_\pi\) is the standard comultiplication on \(C_\infty(\hat{G})\), i.e. \((\Delta_\pi f)(\hat{\gamma}_1, \hat{\gamma}_2) = f(\hat{\gamma}_1\hat{\gamma}_2)\). This means that \(\hat{G}\) is a subgroup of \(G\).

Our aim in this section is to describe a quantum analogue of the homogeneous space \(G/\hat{G}\). In the classical situation when \(G\) is a locally compact group and \(\hat{G}\) is a subgroup of \(G\), continuous functions on \(G/\hat{G}\) may be identified with continuous functions on \(G\) which are constant on left cosets, i.e. such \(x \in \hat{C}G\) that
\[
x(g\hat{\gamma}) = x(g) \tag{3.3}
\]
for all \(g \in G\) and \(\hat{\gamma} \in \hat{G}\). This space of functions carries a natural left action of \(G\) by left shifts. In contrast to the case of compact quantum groups (\([9\text{ Section 1]}\)) at the moment there seems to be no appropriate definition of a homogeneous space for non compact quantum groups in the \(C^*\)-algebra approach. The main problem is to describe the class of “functions on \(G\)” corresponding to continuous functions vanishing at infinity on \(G/\hat{G}\). Clearly they should be bounded and continuous.

In the case of the classical “\(az + b\)” group \(G\) is topologically the cartesian product \(\hat{G} \times \left(\hat{G}/\hat{G}\right)\), where \(\hat{G}\) is the subgroup of homoteties \((b = 0)\). Therefore in this case a bounded continuous function \(x\) on \(G\) corresponds to a function vanishing at infinity on \(G/\hat{G}\) if and only if \(x\) satisfies \(3.3\) and the function
\[
G \ni g = (a, b) \longmapsto x(a, b)f(a) \in \mathbb{C} \tag{3.4}
\]
vanishes at infinity on \(G\) for any \(f \in C_\infty(\hat{G})\). In other words \(x(a, b)f(a)\) belongs to \(C_\infty(G)\).

We shall follow the above ideas in the case of the quantum “\(az + b\)” group. Any “continuous function” on \(G/\hat{G}\) is realized by an element \(x \eta A\) such that (cf. \(3.3\))
\[
(id \otimes \pi)\Delta_\pi(x) = x \otimes I. \tag{3.5}
\]
One can check that \(x = b\) is a solution of Equation \(3.5\). Therefore for any \(f \in \hat{C}(\hat{G})\) the element \(f(b)\) affiliated with \(A\) satisfies \(3.5\). Since the \(C^*\)-algebra \(A\) is generated by \(a, a^{-1}\) and \(b\), and \(x = a\) does not fulfill the requirement \(3.5\), one expects that the algebra \(\hat{A}\) of all continuous functions vanishing at infinity on \(G/\hat{G}\) coincides with \(\{f(b) : f \in C_\infty(\hat{G})\}\) and the quotient map is given by the morphism
\[
C_\infty(\hat{G}) \ni f \longmapsto f(b) \in M(A). \tag{3.6}
\]

**Definition 3.1.** We define \(C_\infty(G/\hat{G})\) to be the set of those \(x \in M(A)\) which satisfy
\[
\begin{enumerate}
\item \((id \otimes \pi)\Delta_\pi(x) = x \otimes I,
\item xf(a) \in A \text{ for all } f \in C_\infty(\hat{G})
\end{enumerate}
\]
and
(3) the map $\Gamma \ni \gamma \mapsto U_{\gamma} x U_{\gamma}^* \in M(A)$ is norm continuous.

We shall refer to (3) of Definition 3.1 as the regularity condition. This condition was not
apparent in the classical setting and is of purely quantum nature. In introducing this condition
we followed the idea of Landstad (4).

**Proposition 3.2.** $C_{\infty}(G/\overline{\Gamma})$ is a $C^*$-algebra.

**Proof.** It is clear that the set of elements $x \in M(A)$ satisfying (1) and (3) of Definition 3.1 is a
$C^*$-algebra. Thus it is enough to show that if $x$ satisfies (2) and (3) of Definition 3.1 then so does
$x^*$.

It follows from Lemma 3.3 (see below) that for any $f \in C_{\infty}(\overline{\Gamma})$ the element
$$x^* f(a) = (\overline{f(a)} x)^*$$
belongs to $A$. Therefore $C_{\infty}(G/\overline{\Gamma})$ is a $C^*$-algebra. □

For any $g \in L^1(\Gamma)$ let $\hat{g}$ denote its Fourier transform:
$$\hat{g}(\bar{\gamma}) = \int_{\Gamma} g(\gamma) \overline{\chi(\bar{\gamma}, \gamma)} \, d\gamma = \int_{\Gamma} g(\gamma) \overline{\chi(\bar{\gamma}, \gamma)} \, d\gamma, \quad (3.7)$$
where $d\gamma$ is a fixed Haar measure on $\Gamma$. With this notation we have
$$\hat{g}(a) = \int_{\Gamma} g(\gamma) \overline{\chi(a, \gamma)} \, d\gamma = \int_{\Gamma} g(\gamma) U_{\gamma} \, d\gamma. \quad (3.8)$$

Clearly $\hat{g}(a) \in M(A)$.

**Lemma 3.3.** Let $x \in M(A)$ be such that the map $\Gamma \ni \gamma \mapsto U_{\gamma} x U_{\gamma}^*$ is norm continuous and assume
that $x f(a) \in A$ for all $f \in C_{\infty}(\overline{\Gamma})$. Then $f(a) x \in A$ for all $f \in C_{\infty}(\overline{\Gamma})$.

**Proof.** By the continuity of the map $\Gamma \ni \gamma \mapsto U_{\gamma} x U_{\gamma}^*$, for any $\varepsilon > 0$ there exists a compact
neighborhood $V_\varepsilon$ of $1 \in \Gamma$ such that
$$\|U_{\gamma} x - x U_{\gamma}\| < \varepsilon$$
for any $\gamma \in V_\varepsilon$. We shall choose $V_\varepsilon$ in such a way that $V_\varepsilon$ shrinks to $\{1\}$ when $\varepsilon \to 0$. Let
$$g_\varepsilon(\gamma) = \begin{cases} (\text{Haar measure of } V_\varepsilon)^{-1} \quad &\text{when } \gamma \in V_\varepsilon, \\ 0 \quad &\text{when } \gamma \notin V_\varepsilon. \end{cases}$$
and let $\hat{g}_\varepsilon$ be the Fourier transform of $g_\varepsilon$ (cf. (3.8)):
$$\hat{g}_\varepsilon(\bar{\gamma}) = \int_{\Gamma} g_\varepsilon(\gamma) \overline{\chi(\bar{\gamma}, \gamma)} \, d\gamma.$$

Using (3.8) with $g_\varepsilon$ instead of $g$ we obtain
$$\|\hat{g}_\varepsilon(a) x - x \hat{g}_\varepsilon(a)\| = \left\| \int_{\Gamma} g_\varepsilon(\gamma) (U_{\gamma} x - x U_{\gamma}) \, d\gamma \right\| \leq \int_{\Gamma} g_\varepsilon(\gamma) \|U_{\gamma} x - x U_{\gamma}\| \, d\gamma \leq \varepsilon.$$

Therefore for any $f \in C_{\infty}(\overline{\Gamma})$ we have
$$\|f(a) \hat{g}_\varepsilon(a) x - f(a) x \hat{g}_\varepsilon(a)\| \leq \|f(a)\| \varepsilon.$$
By construction $f(a) \hat{g}_\varepsilon(a) \to f(a)$ in norm when $\varepsilon \to 0$.

Since
$$\|f(a) x - f(a) x \hat{g}_\varepsilon(a)\| \leq \|f(a) - f(a) \hat{g}_\varepsilon(a)\| \|x\hat{g}_\varepsilon(a)\| \to 0 \quad \varepsilon \to 0$$
and $f(a) x \hat{g}_\varepsilon(a) = f(a) (x \hat{g}_\varepsilon(a)) \in M(A) A \subset A$ we see that $f(a) x \in A$. □

Now we have the following
Theorem 3.4. $C_\infty(G/\hat{\Gamma}) = \{ f(b) : f \in C_\infty(\hat{\Gamma}) \}$.

Proof. For $\hat{\gamma} \in \hat{\Gamma}$ let $\pi_\gamma$ be the morphism $\pi \in \text{Mor}(A,C_\infty(\hat{\Gamma}))$ (introduced in the beginning of this Section) composed with evaluation at $\hat{\gamma}$, $\pi_\gamma(x) = (\pi(x))(\hat{\gamma})$. Then $\pi_\gamma \in \text{Mor}(A,\mathbb{C})$ and due to (3.2)

$$\pi_{\gamma_1} * \pi_{\gamma_2} = (\pi_{\gamma_1} \otimes \pi_{\gamma_2}) \circ \Delta_c = \pi_{\gamma_1 \gamma_2}.$$  

(3.9)

For $x \in A$ and $\hat{\gamma} \in \hat{\Gamma}$ let

$$\hat{\alpha}_\gamma(x) = \pi_\gamma * x = (\text{id} \otimes \pi_\gamma) \Delta_c(x).$$  

(3.10)

Then by (3.9), $\hat{\alpha}_\gamma$ is an automorphism of $A$ and $\hat{\gamma} \mapsto \hat{\alpha}_\gamma \in \text{Aut} A$ is a continuous action of $\hat{\Gamma}$ on $A$. In other words $(A,\hat{\Gamma},\hat{\alpha})$ is a $C^*$-dynamical system. Clearly (cf. (3.10)) for any $x \in A$

$$\left( (\text{id} \otimes \pi) \Delta_c(x) = x \otimes I \right) \iff \left( \hat{\alpha}_\gamma(x) = x \text{ for all } \hat{\gamma} \in \hat{\Gamma} \right)$$  

(3.11)

and since by (2.7)

$$\Delta_c(U_\gamma) = \Delta_c(\chi(a,\gamma)) = \chi(a \otimes a, \gamma) = \chi(a,\gamma) \otimes \chi(a,\gamma) = U_\gamma \otimes U_\gamma$$

we have

$$\hat{\alpha}_\gamma(U_\gamma) = (\hat{\gamma},\gamma)U_\gamma$$

for all $\gamma \in \Gamma$ and $\hat{\gamma} \in \hat{\Gamma}$.

By Landstad’s theorem (see e.g. [8] Theorem 7.8.8) there exists a $C^*$-dynamical system $(B,\Gamma,\alpha)$ such that $A \simeq B \rtimes_\alpha \Gamma$. Moreover

$$B = \left\{ x \in M(A) : \begin{array}{l}
(i) \quad \hat{\alpha}_\gamma(x) = x \text{ for all } \hat{\gamma} \in \hat{\Gamma}, \\
(ii) \quad x\hat{g}(a), \hat{g}(a)x \in A \text{ for all } \hat{g} \in L^1(\Gamma), \\
(iii) \quad \text{the map } \Gamma \ni \gamma \mapsto U_\gamma x U_\gamma^* \in M(A) \text{ is norm continuous} \end{array} \right\}.$$  

(3.12)

The action $\alpha$ of $\Gamma$ on $B$ is given by

$$\alpha_\gamma(x) = U_\gamma x U_\gamma^*.$$  

Furthermore $(B,\Gamma,\alpha)$ is unique up to covariant isomorphism and thus by (2.6) we have $(B,\Gamma,\alpha) \simeq (C_\infty(\Gamma),\Gamma,\beta)$ . It turns out that in our case

$$B = C_\infty(G/\hat{\Gamma})$$

as subsets of $M(A)$. The condition (i) from (3.12) coincides with (1) of Definition 3.1 due to (3.3). Condition (3) of Definition 3.1 is the same as (iii) from (3.12). At first sight (ii) from (3.12) seems stronger than (2) of Definition 3.1 but in fact they are equivalent by Lemma 3.3.

Now let $x = f(b)$ with $f \in C_\infty(\hat{\Gamma})$. Clearly $x \in M(A)$ and $x$ satisfies (i). Since

$$U_\gamma f(b)U_\gamma^* = f \left( U_\gamma b U_\gamma^* \right) = f(\gamma b) = (\beta_\gamma(f))(b)$$  

(3.13)

condition (iii) of (3.12) is satisfied.

It is known (cf. [10] Formula (4.3))] that $f(b) g(a) \in A$ for all $f \in C_\infty(\Gamma)$ and $g \in C_\infty(\hat{\Gamma})$. In particular $xg(a) \in A$ for all $g \in C_\infty(\hat{\Gamma})$. By Lemma 3.3 also $g(a)x \in A$ for any $f \in C_\infty(\hat{\Gamma})$. It means that $x = f(b)$ satisfies condition (ii) of (3.12).

As a result $\{ f(b) : f \in C_\infty(\hat{\Gamma}) \} \subset B$.

Note that due to (3.13) the inclusion map

$$C_\infty(\Gamma) \ni f \mapsto f(b) \in B$$

intertwines the actions $\beta$ and $\alpha$ on $C_\infty(\hat{\Gamma})$ and $B$. By composing this map with the covariant isomorphism $B \simeq C_\infty(\hat{\Gamma})$ we obtain a covariant inclusion $j : C_\infty(\Gamma) \hookrightarrow C_\infty(\hat{\Gamma})$. Therefore the underlying map

$$\Psi : \Gamma \rightarrow \hat{\Gamma}.$$
is surjective and equivariant for the natural action of $\Gamma$ on $\overline{\Gamma}$.

Now for any $\gamma \in \Gamma$

$$\Psi(\gamma) = \Psi(1 \cdot \gamma) = \Psi(1)\gamma$$

and $\Psi(1) \neq 0$ because $\Psi$ is surjective. This shows that $\Psi$ is a homeomorphism and consequently $B = \{ f(b) : f \in C_\infty(\Gamma) \}$.

Let us remark that Theorem 3.4 proves that the quantum homogeneous space $G/\hat{\Gamma}$ is in fact a classical space: $G/\hat{\Gamma} = \Gamma$. More precisely $C_\infty(G/\hat{\Gamma}) = C_\infty(\Gamma)$ and the morphism $C_\infty(\Gamma) \ni f \mapsto f(b) \in M(A)$ (cf. (3.6)) is dual to the quotient map.

Using this morphism we shall identify the generator $z$ of $C_\infty(\Gamma)$ with $b \eta A$. Now we can define the action $\Phi_G$ of $G = (A, \Delta_G)$ on the homogeneous space $G/\hat{\Gamma} = \Gamma$ by restricting $\Delta_G$ to the image of $C_\infty(\Gamma)$ in $M(A)$. With such a definition we have (cf. (2.7)):

$$\Phi_G(z) = a \otimes z + b \otimes I.$$

Indeed using results of [15, Formula (2.6)] one can show that $(a \otimes z + b \otimes I) \in A \otimes C_\infty(\Gamma)$. It is normal and $\text{Sp}(a \otimes z + b \otimes I) \subset \Gamma$. Therefore $\Phi_G \in \text{Mor}(C_\infty(\Gamma), A \otimes C_\infty(\Gamma))$. Moreover the coassociativity of $\Delta_G$ leads to the following commutative diagram

which means that $G$ is acting on $\overline{\Gamma}$. This justifies our definition of the homogeneous space.

The homogeneous space $G/\hat{\Gamma}$ shall now play a role analogous to the role of the complex plane acted upon by the classical “$az + b$” group. The fact that $\Gamma$ is a closure in $\mathbb{C}$ of a multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ will not be important for what follows. We shall therefore from now on denote a generic element of $\overline{\Gamma}$ by the symbol $z$.

**4. Differential calculus**

**4.1. A general construction.** The following proposition describes a general method of constructing covariant differential calculi on coideals of Hopf algebras (so called embeddable homogeneous spaces cf. [9] Definition 1.8).

**Proposition 4.1.** Let $(A_0, \Delta_0)$ be a Hopf algebra with counit $e$ and let $\mathcal{B}_0$ be a left coideal of $A_0$.

1. Let $\varphi$ be a linear functional on $\mathcal{B}_0$. Define $d : \mathcal{B}_0 \to A_0$ by $d(b) = \varphi \ast b$ and let $\Omega$ be the image of $d$. Then
   - (a) $\Omega$ is a left coideal of $A_0$ and the diagram
     $$
     \begin{array}{ccc}
     \mathcal{B}_0 & \xrightarrow{d} & \Omega \\
     \downarrow{\Delta_0} & & \downarrow{\Delta_0} \\
     A_0 \otimes \mathcal{B}_0 & \xrightarrow{id \otimes d} & A_0 \otimes \Omega
     \end{array}
     $$
   - (b) $\varphi$ is a linear functional on $\Omega$.

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*The reader will notice that $j$ is not a morphism of $C^*$-algebras, so the existence of the map $\Psi$ is not obvious. However any character of $B$ can be treated as an irreducible representation of $j(B)$ seen as a subalgebra of $B$. This representation can be extended to $B$ as an irreducible representation ([11 Proposition 2.10.2]) which corresponds to a point of $\Gamma$. Thus the map $\Psi$ exists and is surjective. It is, however, a map from $\overline{\Gamma}$ to its one point compactification and it is not guaranteed to be continuous. Nevertheless it clearly has all of $\overline{\Gamma}$ in its range.*
is commutative;
(b) if \( \varphi \) satisfies
\[
\varphi(bc) = \varphi(b)e(c) + e(b)\varphi(c)
\]
for all \( b, c \in \mathcal{B}_0 \) then \( (\Omega, d) \) is a first order differential calculus over \( \mathcal{B}_0 \) which by statement (1) is left covariant under the action of \( (\mathcal{A}_0, \Delta_0) \).

(2) Let \( d : \mathcal{B}_0 \rightarrow \mathcal{A}_0 \) be a linear map. Let \( \Omega \) be the image of \( d \) and define \( \varphi = \circ d \). Then
\[
\begin{align*}
\text{(a) if } \Omega & \text{ is a left coideal of } \mathcal{A}_0 \text{ and the diagram is commutative then for any } b \in \mathcal{B}_0 \text{ we have } d(b) = \varphi \ast b; \\
\text{(b) if } (\Omega, d) & \text{ is a first order differential calculus over } \mathcal{B}_0 \text{ then } \varphi \text{ satisfies (4.2) for all } b, c \in \mathcal{B}_0.
\end{align*}
\]

We omit the proof which is purely computational. The proposition has an obvious analogue for \( * \)-calculi over \( * \)-algebras covariant under action of a Hopf \( * \)-algebra.

In our study to the quantum homogeneous space of the quantum \( \ast \)-group constructed in Section 3 we shall construct a differential calculus which corresponds to the algebraic construction given in Proposition 4.1.

4.2. Smooth functions on \( G/\Gamma \). Let \( \mathcal{B} \) be the algebra of all continuous functions on \( \Gamma \) which can be continued analytically from any circle \( \{ z \in \Gamma : |z| = q^n \} \) to a holomorphic function on \( \mathbb{C} \setminus \{0\} \). If \( f \in \mathcal{B} \) then for any \( z \in \Gamma \) the function
\[
\mathbb{R} \ni t \mapsto f(q^n z) \in \mathbb{C}
\]
has holomorphic continuation to an entire function. We shall denote the value of this continuation at \( t = -i \) by \( f(q^{-1} z) \). The algebra of bounded functions contained in \( \mathcal{B} \) will be denoted by \( \mathcal{B}_b \).

\( \mathcal{B} \) will play the role of smooth functions on the quotient space \( G/\Gamma = \overline{\Gamma} \). The symbol \( \mathcal{B} \) is chosen in analogy with the algebraic situation described in Subsection 4.1. We also need to define the algebra of smooth functions on the quantum group \( G \) which acts on \( \Gamma \).

Let \( \mathcal{A} \) be the set of those elements affiliated with \( \mathcal{A} \) which are entire analytic for the scaling group of \( G \) (for the natural topology, cf. [15]). Let \( \mathcal{A}_b \) be the intersection of \( \mathcal{A} \) with \( M(\mathcal{A}) \).

We shall now describe the analogy between the algebraic context of Subsection 4.1 and our situation. The algebra \( \mathcal{B} \) can be embedded in the set of elements affiliated with \( \mathcal{A} \) via the map \( j : f \mapsto f(b) \). The following algebraic statements:

(1) \( \mathcal{A}_0 \) is a Hopf \( * \)-algebra, in particular it has an antipode and for each \( x \in \mathcal{A}_0 \) we have \( \Delta_0(x) \in \mathcal{A}_0 \otimes \mathcal{A}_0 \);

(2) \( \mathcal{B}_0 \) is a left coideal in \( \mathcal{A}_0 \), i.e. \( \mathcal{B}_0 \) is a subset of \( \mathcal{A}_j \) and for each \( x \in \mathcal{B}_0 \) the element \( \Delta_0(x) \) belongs to \( \mathcal{A}_0 \otimes \mathcal{B}_0 \);

have the following counterparts in our situation:

(1) \( \mathcal{A}_b \) is a unital \( * \)-subalgebra of \( M(\mathcal{A}) \) contained in the domain of \( \overline{\kappa} \) and stable under \( \overline{\kappa} \), for each \( x \in \mathcal{A} \) the element \( \Delta(x) \) is entire analytic for \( (\tau_t \otimes \tau_t)_{t \in \mathbb{R}} \);

(2) \( \mathcal{B} \) is a subset of \( \mathcal{A} \) and for each \( x \in \mathcal{B} \) the element \( \Delta(x) \) is affiliated with \( \mathcal{A} \otimes C_\infty(\Gamma) \) and is entire analytic for \( (\tau_t \otimes \tau_t)_{t \in \mathbb{R}} \);

Let us comment on statements 11 and 12. The first one is a consequence of the properties of analytic generators of one parameter groups of automorphisms and the fact that for each \( t \in \mathbb{R} \) we have \( (\tau_t \otimes \tau_t) \circ \Delta = \Delta \circ \tau_t \). As for statement 12, the fact that \( \mathcal{B} \subset \mathcal{A} \) follows from the fact that \( \mathcal{B} \) is the set of elements which are entire analytic for the group \( (\beta_{q^t})_{t \in \mathbb{R}} \) and for any \( t \in \mathbb{R} \) we have \( \tau_t \circ j = j \circ \beta_{q^{-t}} \).

The differential calculus over \( \mathcal{B} \) we want to construct will be analogous to one described in Subsection 4.1 with \( \varphi \) corresponding to the map which sends \( b \) and \( b^* \) to 1 and all higher powers of \( b \) and \( b^* \) to 0.
4.3. **The calculus.** Let $\bar{\Omega}$ be the $B$ bimodule generated by two elements $\omega$ and $\bar{\omega}$ with the defining relations\(^b\)

\[
\omega f(z) = f(q \cdot qz)\omega, \\
\bar{\omega} f(z) = f(q \cdot q^{-1}z)\bar{\omega}
\]

for all $f \in B$. Clearly by putting $\omega^* = \bar{\omega}$ we obtain a $B$-* bimodule structure on $\bar{\Omega}$. Now we define the operator $d: B \rightarrow \bar{\Omega}$ by

\[
d f = [\omega - \bar{\omega}, f] = (\omega - \bar{\omega})f - f(\omega - \bar{\omega}).
\]

Clearly $d$ satisfies the Leibniz identity. The identity function $z : \Gamma \ni z \mapsto z \in \mathbb{C}$ and its adjoint $\bar{z}$ belong to $B$ and we can compute their differentials:

\[
d z = (q^2 - 1)z\omega, \\
d \bar{z} = (q^2 - 1)\bar{z}\bar{\omega}.
\] (4.3)

Now expressing $d f$ as a combination of $d z$ and $d \bar{z}$ we get the following expressions

\[
d f = \frac{\partial f}{\partial z} d z + \frac{\partial f}{\partial \bar{z}} d \bar{z}
\]

where

\[
\frac{\partial f}{\partial z} = \frac{f(q^{-1} \cdot q^{-1}z) - f(z)}{(q^2 - 1)z}, \\
\frac{\partial f}{\partial \bar{z}} = \frac{f(z) - f(q \cdot qz)}{(1 - q^2)z}, \\
\frac{\partial f}{\partial \bar{\omega}} = \frac{f(q \cdot q^{-1}z) - f(z)}{(q^2 - 1)\bar{\omega}}, \\
\frac{\partial f}{\partial \bar{\omega}} = \frac{f(z) - f(q^{-1} \cdot qz)}{(1 - q^2)\bar{\omega}}.
\] (4.4)

The above computations make sense for $z$ away from 0. The space $C_q^\infty(\Gamma)$ of smooth functions on $\Gamma$ is defined as

\[
C_q^\infty(\Gamma) = \left\{ f \in B : \exists \lim_{\Gamma \ni z \to 0} D f(z) \text{ for any product } D \text{ of operators (4.4)} \right\}.
\]

For any $f \in C_q^\infty(\Gamma)$ the right hand sides of (4.4) define elements of $C_q^\infty(\Gamma)$.

The properties of analytic generators of one parameter groups show that for any $f \in C_q^\infty(\Gamma)$ we have

\[
\frac{\partial h f}{\partial z}(z) = \frac{\partial h f}{\partial z}(q^{-1} \cdot q^{-1}z), \\
\frac{\partial h f}{\partial \bar{z}}(z) = \frac{\partial h f}{\partial \bar{z}}(q^{-1} \cdot qz)
\] (4.5)

as well as

\[
\frac{\partial h f}{\partial z} = \left( \frac{\partial h f}{\partial \bar{z}} \right)^* \text{ and } \frac{\partial h f}{\partial \bar{z}} = \left( \frac{\partial h f}{\partial z} \right)^*,
\] (4.6)

where $*$ denotes the standard involution on $B$ (complex conjugation).

\(^b\)To keep our notation less complicated we will sometimes be writing $f(z)$ instead of $f$. 

Using the Leibniz rule for $d$ we derive the value of our differential operators on products of functions. Indeed, if $f, g \in C^\infty_q(\overline{\Gamma})$ then we have
\[
d(fg) = df \cdot g + f \cdot dg = \left(\frac{\partial_h f}{\partial z} dz + d\overline{z} \frac{\partial_h f}{\partial \overline{z}} \right) g + f \left(\frac{\partial_h g}{\partial z} dz + d\overline{z} \frac{\partial_h g}{\partial \overline{z}} \right)
\]
\[
= \left(\frac{\partial_h f}{\partial z} g(q^{-1} \cdot q^{-1}) + f \frac{\partial_h g}{\partial z}\right) dz + d\overline{z} \left(\frac{\partial_h f}{\partial \overline{z}} g + f(q \cdot q^{-1}) \right) \frac{\partial_h g}{\partial \overline{z}}
\]
It follows that
\[
\frac{\partial_h f}{\partial z} g(q^{-1} \cdot q^{-1}) + f \frac{\partial_h g}{\partial z} = \left(\frac{\partial_h f}{\partial z} g + f \frac{\partial_h g}{\partial z}\right)
\]
\[
\frac{\partial_h f}{\partial \overline{z}} g + f(q \cdot q^{-1}) \frac{\partial_h g}{\partial \overline{z}} = \left(\frac{\partial_h f}{\partial \overline{z}} g + f \frac{\partial_h g}{\partial \overline{z}}\right)
\]
\[
\tag{4.7}
\]
Similarly
\[
\frac{\partial_h f}{\partial \overline{z}} g(q^{-1} \cdot q^{-1}) + f \frac{\partial_h g}{\partial \overline{z}} = \left(\frac{\partial_h f}{\partial \overline{z}} g + f \frac{\partial_h g}{\partial \overline{z}}\right)
\]
\[
\frac{\partial_h f}{\partial z} g + f(q \cdot q^{-1}) \frac{\partial_h g}{\partial z} = \left(\frac{\partial_h f}{\partial z} g + f \frac{\partial_h g}{\partial z}\right)
\]
\[
\tag{4.8}
\]
From \cite{14, 17, 18} we get

**Proposition 4.2.** $C^\infty_q(\overline{\Gamma})$ is a unital $\ast$-algebra.

Define
\[
\Omega^1 = \{ adz\alpha' + \beta d\overline{z}\beta' : \alpha, \alpha', \beta, \beta' \in C^\infty_q(\overline{\Gamma}) \}
\]

**Proposition 4.3.** $(\Omega^1, d)$ is a first order differential $\ast$-calculus over $C^\infty_q(\overline{\Gamma})$.

The differential operators defined by \cite{14} have some of the properties of the classical $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ operators:

**Proposition 4.4.** Let $f \in C^\infty_q(\overline{\Gamma})$ be a restriction to $\overline{\Gamma}$ of an entire function. Then
\[
\frac{\partial_h f}{\partial \overline{z}} = \frac{\partial_h f}{\partial z} = 0.
\]
Conversely, if $f \in C^\infty_q(\overline{\Gamma})$ and either $\frac{\partial_h f}{\partial \overline{z}} = 0$ or $\frac{\partial_h f}{\partial z} = 0$ then $f$ extends to an entire function.

We also have an analogous result for antiholomorphic functions.

Let us recall the special function $F_q$ introduced in \cite{13}. It is a continuous function on $\overline{\Gamma}$ defined for $z \notin \{ -q^{-2k} : k \in \mathbb{Z}_+ \}$ by
\[
F_q(z) = \prod_{k=0}^\infty \frac{1 + q^{2k} z}{1 + q^{2k} z}.
\]
For reasons explained in \cite{13} (cf. also \cite{10}) $F_q$ is called the quantum exponential function. This function is used in construction of the multiplicative unitary of the quantum “$a+b$” group (\cite{10}) as well as of the quantum $E(2)$ group (\cite{13}). $F_q$ does not belong to $C^\infty_q(\overline{\Gamma})$ because the analytic extension of the map $\mathbb{R} \ni t \mapsto F_q(q^t z)$ has singularities in the upper half plane (for generic $z$).

Nevertheless we can compute the action of the left differential operators on $F_q$:
\[
\frac{\partial_h F_q}{\partial z} = -\frac{1}{1 - q^2} F_q, \quad \frac{\partial_h F_q}{\partial \overline{z}} = \frac{1}{1 - q^2} F_q.
\]
A slightly more sophisticated version of these formulas will be needed later:
\[
\frac{\partial_h F_q}{\partial z} = -\frac{\zeta}{1 - q^2} F_q, \quad \frac{\partial_h F_q}{\partial \overline{z}} = \frac{\overline{\zeta}}{1 - q^2} F_q \tag{4.9}
\]
for any parameter $\zeta \in \overline{\Gamma}$. 

It is also possible to construct higher order differential calculus on $\overline{\Gamma}$. The requirement that $d^2 = 0$ and that for any form $\rho$ the exterior derivative $d\rho$ is given by the graded commutator with $\omega - \overline{\omega}$ leads to the definition

$$\Omega^2 = \left( \Omega^1 \otimes \text{C}^\infty_q(\overline{\Gamma}) \Omega^1 \right) / N,$$

where $N$ is the submodule generated by $dz \otimes dz$, $d\overline{\omega} \otimes d\overline{\omega}$ and $dz \otimes d\overline{\omega} + d\overline{\omega} \otimes dz$. There are no higher dimensional forms.

4.4. Covariance. Let us now relate the algebraic considerations of Subsection 4.1 and the abstract definition of differential calculus of Subsection 4.3 as before we shall identify the generator $z$ of the $\text{C}^\ast$-algebra $\text{C}^\infty(\overline{\Gamma})$ with the element $b$ affiliated with the $\text{C}^\ast$-algebra $A = \text{C}^\infty(\overline{\Gamma}) \rtimes \beta \Gamma$. Thus the algebra generated by $b$ becomes a left coideal in the $\text{C}^\ast$-algebra with comultiplication $(A, \Delta_G)$.

Now we need the functional $\varphi$ defined on the algebra generated by $b$. We cannot expect it to be continuous as it is designed to be the analogue of a tangent vector. Let $\varphi$ take the value 1 on $b$ and $b^\ast$ and zero on all higher powers of $b$ and $b^\ast$. Then $\varphi$ satisfies (4.2) in an appropriate sense: the counit of the quantum group $G$ is a morphism from $A$ to $\mathbb{C}$ taking value 1 on $a$ and 0 on $b$. Since $\varphi$ is equal to zero an any product of more than one element we get an analogue of (4.2).

With this definition of $\varphi$ we can formally apply $(\text{id} \otimes \varphi)$ to $\Delta_G(b)$ and call it $dz$:

$$dz = \varphi(b)a = a.$$

Then the first formula of (4.3) gives

$$\omega = \frac{1}{q^2 - 1} b^{-1} a$$

(notice that $b$ might not be invertible in some representations of $A$ and consequently $b^{-1}a$ is not affiliated with $A$).

Now the rules of differential calculus can be given a precise analytical meaning: for an element $f(b)$ of (the image in $A$ of) $\text{C}^\infty_q(\overline{\Gamma})$ the exterior derivative $df$ is $[\omega - \omega^\ast, f]$. For functions $f$ belonging to the algebra $\text{C}^\infty_q(\overline{\Gamma})$ the rules of differentiating obtained in Subsection 4.3 are valid. Thus obtained differential calculus is covariant by construction. However we are not claiming that a diagram analogous to (4.1) is commutative because the maps involved are not continuous.

The action of $G$ on differential forms on $G/\hat{\Gamma} = \overline{\Gamma}$ is obtained by applying $\Delta_G$ to the operator representing $dz$ affiliated with $A$. In other words $\Phi_G(dz) = a \otimes dz$.

5. Covariant measure

Let us introduce a measure on $G/\hat{\Gamma} = \overline{\Gamma}$. This measure will be covariant with respect to the action of the quantum “$az + b$” group. For a positive $f \in \text{C}^\infty(\overline{\Gamma})$ let

$$\int_\overline{\Gamma} f \, d\mu = \sum_{k \in \mathbb{Z}} q^{2k} \frac{2\pi}{2\pi} \int_0^{2\pi} f(q^k e^{i\varphi}) \, d\varphi.$$

Proposition 5.1. Let $f$ be a function in $\mathcal{B}$ integrable with respect to $\mu$. Then we have

$$\int_\overline{\Gamma} f(qz) \, d\mu(z) = q^{-2} \int_\overline{\Gamma} f(z) \, d\mu(z),$$

$$\int_\overline{\Gamma} f(q \cdot z) \, d\mu(z) = \int_\overline{\Gamma} f(qz) \, d\mu(z).$$

In particular the function $z \mapsto f(q \cdot z)$ is integrable.

Corollary 5.2. Let $f$ and $g$ be functions in $\text{C}^\infty_q(\overline{\Gamma})$. Then
(1) if \( f \frac{\partial g}{\partial z} \) and \( \frac{\partial f}{\partial z} g \) are integrable then
\[
\int_{\Gamma} f \frac{\partial g}{\partial z} \, d\mu = -q^2 \int_{\Gamma} \frac{\partial f}{\partial z} g \, d\mu.
\]

(2) if \( f \frac{\partial g}{\partial z} \) and \( \frac{\partial f}{\partial z} g \) are integrable then
\[
\int_{\Gamma} f \frac{\partial g}{\partial z} \, d\mu = -q^{-2} \int_{\Gamma} \frac{\partial f}{\partial z} g \, d\mu.
\]

Let \( L^2(\Gamma) \) be the space of \( \mu \)-square integrable functions on \( \Gamma \). The scalar product in \( L^2(\Gamma) \) will be denoted by \((\cdot,\cdot)\). With this notation.

(3) if \( f, g, \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \) are in \( L^2(\Gamma) \) then
\[
(f, \frac{\partial g}{\partial z}) = -q^2 \left( \frac{\partial f}{\partial z} \big| g \right);
\]

(4) if \( f, g, \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \) are in \( L^2(\Gamma) \) then
\[
(f, \frac{\partial g}{\partial z}) = -q^2 \left( \frac{\partial f}{\partial z} \big| g \right).
\]

Let us define the following family of functions on \( \Gamma \): for \( k, l \in \mathbb{Z} \) and \( z \in \Gamma \) let
\[
g_{k,l}(z) = \begin{cases} 
\chi(z,q^l) & \text{for } |z| = q^k, \\
0 & \text{for } |z| \neq q^k.
\end{cases} \tag{5.1}
\]

The family \( (g_{k,l})_{k,l \in \mathbb{Z}} \) is contained in \( C^\infty_q(\Gamma) \) and in \( L^2(\Gamma) \). Moreover these functions form a maximal orthogonal system in \( L^2(\Gamma) \). The action of the differential operators \( \frac{\partial \cdot}{\partial z} \) on functions \( g_{k,l} \) can be computed:
\[
\frac{\partial g_{k,l}}{\partial z} = \frac{1}{q^{-2} - 1} \left( q^{-(l-(k+1))} g_{k+1,l-1} - q^{-k} g_{k,l-1} \right),
\]
\[
\frac{\partial g_{k,l}}{\partial z} = \frac{1}{1 - q^2} \left( q^{-k} g_{k,l-1} - q^{l-(k-1)} g_{k-1,l} \right),
\]
\[
\frac{\partial g_{k,l}}{\partial z} = \frac{1}{1 - q^2} \left( q^{-k} g_{k+1,l} - q^{l-(k+1)} g_{k+1,l-1} \right),
\]
\[
\frac{\partial g_{k,l}}{\partial z} = \frac{1}{q^{-2} - 1} \left( q^{l-(k+1)} g_{k,l+1} - q^{-k} g_{k,l+1+1} \right).
\]

We see that the linear span of functions \( (g_{k,l})_{k,l \in \mathbb{Z}} \) is a dense subset of \( L^2(\Gamma) \) which is preserved by operators \( \frac{\partial \cdot}{\partial z} \) and left and right derivatives commute on this subset. Moreover, it follows from \( 3 \) and \( 4 \) of Corollary \( 5.2 \) that operators \( \frac{\partial \cdot}{\partial z} \) are not only densely defined, but also closable.

Without much trouble we obtain

**Proposition 5.3.** Denoting by the same symbols the closures of the operators \( \frac{\partial \cdot}{\partial z} \) in the space \( L^2(\Gamma) \) we have
\[
\left( \frac{\partial g}{\partial z} \right)^* = -q^2 \frac{\partial g}{\partial z} \quad \text{and} \quad \left( \frac{\partial g}{\partial z} \right)^* = -q^{-2} \frac{\partial g}{\partial z}.
\]

Moreover \( \frac{\partial \cdot}{\partial z} \frac{\partial \cdot}{\partial z} \frac{\partial \cdot}{\partial z} \) and \( \frac{\partial \cdot}{\partial z} \) are normal operators on \( L^2(\Gamma) \).

There is another interesting result which is very much analogous to the classical Stokes’ theorem:
Proposition 5.4. Let $f \in C_0^\infty(\overline{\Gamma})$ have a compact support. Then

$$\int_{\overline{\Gamma}} \frac{\partial_L f}{\partial z} d\mu = \int_{\overline{\Gamma}} \frac{\partial_R f}{\partial z} d\mu = \int_{\overline{\Gamma}} \frac{\partial_H f}{\partial z} d\mu = \int_{\overline{\Gamma}} \frac{\partial_R f}{\partial \overline{z}} d\mu = 0.$$ 

Let us remark that $\mu$ is the only positive (non-zero) measure on $\overline{\Gamma}$ for which the above theorem is true.

5.1. Covariance. The covariance of $\mu$ can be expressed imprecisely as the property that

$$(\text{id} \otimes \mu) \Phi_{\alpha}(f) = |a|^2 \mu(f)$$

(5.2)

for any $f \in C_\infty(G) = C_\infty(\overline{\Gamma})$. For a more precise definition let us pass to the von Neumann algebraic context. For this reason let us recall some basic facts about the Haar measure on $G$.

The right Haar measure $h$ of $\overline{\Gamma}$ was already introduced in 12 (cf. also 17). For an element $c = g(a)f(b)$ with $g \in C_\infty(\overline{\Gamma})$ and $f \in C_\infty(\overline{\Gamma})$ we have

$$h(c^*c) = \int_{\overline{\Gamma}} |g(\gamma)|^2 d\gamma \int_{\overline{\Gamma}} |f(z)|^2 d\mu(z),$$

where $d\gamma$ denotes the Haar measure on $\Gamma$:

$$\int_{\Gamma} g(\gamma) d\gamma = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \int_{0}^{2\pi} g(e^{it}) \, dt$$

(5.3)

(notice that $\mu$ is the push forward of the measure $|\gamma|^2 d\gamma$ via the inclusion of $\Gamma$ into $\overline{\Gamma}$).

Let us represent $\overline{\Gamma}$ in the GNS Hilbert space for the weight $h$ (one can show that $H = L^2(\Gamma) \otimes L^2(\overline{\Gamma})$ where $L^2(\overline{\Gamma})$ is the space of functions on $\overline{\Gamma}$ square integrable with respect to the Haar measure $\mathbb{E}$). We can form the von Neumann algebra $M = A''$. Then $h$ extends to a normal faithful semifinite weight on $M$ (cf. 5 or 7). One can show that $M$ is the von Neumann algebra crossed product of $N = L^\infty(\overline{\Gamma})$ (where on $\overline{\Gamma}$ we take the measure class of $\mu$) by the natural action of $\Gamma$, which clearly extends the action $\beta$ on $C_\infty(\overline{\Gamma})$. The weight $h$ is invariant under the dual action of $\overline{\Gamma}$ on $M$. Indeed, the Haar measure $h$ is right invariant, i.e. for any $\varphi \in A^*_+$ and any $x \in A_+$ we have $h(\varphi * x) = \varphi(1)h(x)$. In the proof of Theorem 3.3 we saw that the dual action of $\Gamma$ on $A$ is given by convolution with functionals $\pi_\gamma$. Therefore denoting the dual action by $\hat{\beta}$ we have $h = h \circ \hat{\beta}_\gamma$ for any $\gamma \in \overline{\Gamma}$ (in the proof of Theorem 3.3 we denoted $\hat{\beta}$ by $\hat{\alpha}$). It follows that the extensions $h$ and $h \circ \hat{\beta}_\gamma$ to $M$ are equal for all $\gamma \in \overline{\Gamma}$. As the dual action on $M$ is the natural extension of that on $A$ we have that the normal extension of $h$ is invariant under the (von Neumann algebraic) dual action.

Let $T$ be the operator valued weight $M \to N$ coming from the integration of the dual action (4). Then by 2 Theorem 3.7 and 3 Theorem 1.1 there exists a unique measure $\nu$ on $\overline{\Gamma}$ such that $h = \nu \otimes T$. One easily checks that for $c = g(a)f(b) \in A \subset M$ we have $(\nu \otimes T)(c^*c) = (\mu \otimes T)(c^*c)$.

By uniqueness of $\nu$ we have $\nu = \mu$ and $h = \mu \circ T$.

Let $\rho$ be the modular element of $G$ (the Radon-Nikodym derivative of the left Haar measure $h^L = h \circ R$ with respect to $h$). Then $\rho$ is a positive selfadjoint element affiliated with $A$, but we can also treat it as an unbounded operator on $H$. By 11 Proposition 2.5 we have

$$(\omega_{\xi,\xi} \otimes \text{id}) \Phi_{\alpha}(f) = \|\rho\xi\|^2 \mu(f)$$

(5.4)

for all $f \in C_\infty(G/\overline{\Gamma})$ and all $\xi \in D(\rho)$.

Since $\rho = |a|^2$ we shall take (5.4) as the precise formulation of (5.2).
6. Fourier transform

Our definition of the Fourier transformation on $\mathbb{T}$ will be motivated by the properties of the quantum exponential function $F_q$. Let $f$ be an element of $C^\infty_q(\mathbb{T})$ which is integrable for the measure $\mu$. Then we define

$$(Ff)(\zeta) = \int_{\mathbb{T}} F_q(\zeta z) f(z) \, d\mu(z).$$

In order to describe the properties of the operation $F$ let us introduce the following notation: for $t \in \mathbb{R}$ and any function $g$ on $\mathbb{T}$ let

$$(\sigma_t(g))(z) = g(q^{-it}z),$$
$$ (\theta(g))(z) = g(qz).$$

Then $(\sigma_t)_{t \in \mathbb{R}}$ becomes a one parameter group of automorphisms of the algebra of functions on $\mathbb{T}$ and similarly $\theta$ is an automorphism of this algebra commuting with the automorphisms $(\sigma_t)_{t \in \mathbb{R}}$. Restricting attention to the continuous functions the group $(\sigma_t)_{t \in \mathbb{R}}$ becomes continuous in the natural topology and we can form the analytic generator $\sigma_i$. The domain of $\sigma_i$ is precisely the algebra $B$ described in Subsection 4.2. From Proposition 5.1 it easily follows that

$$(\sigma_{-i} \circ \theta^{-1}) \circ F = q^2 F \circ (\sigma_i \circ \theta),$$
$$(\sigma_{-i} \circ \theta) \circ F = q^{-2} F \circ (\sigma_i \circ \theta^{-1}).$$

(6.1)

Indeed, the fact that $\theta \circ F = q^{-2} F \circ \theta$ is a direct consequence of the first formula of Proposition 5.1.

In order to see that $\sigma_{-i} \circ F = F \circ \sigma_i$ let us compute

$$(Ff)(q^{-it}\zeta) = \int_{\mathbb{T}} F_q(q^{-it}\zeta z) f(z) \, d\mu(z)$$
$$= \int_{\mathbb{T}} F_q(\zeta z) f(q^{it}z) \, d\mu(z).$$

Now if $f$ and $\sigma_i(f)$ are integrable then the last expression above has a holomorphic continuation to $t = -i$. Therefore $(Ff)(q^{-1} \cdot \zeta)$ exists and is equal to $(F \circ \sigma_i(f))(\zeta)$.

Let us denote the operator of multiplication by the identity function $z$ on $\mathbb{T}$ by the symbol $Z$ and let $Z^*$ be the multiplication by $\bar{z}$.

**Theorem 6.1.** The commutation relations between $F$, $Z$, $Z^*$ and the operators (13) are the following:

$$Z \circ F = (q^{-2} - 1) F \circ \frac{\partial_n}{\partial z},$$
$$Z^* \circ F = -(q^2 - q^4) F \circ \frac{\partial_n}{\partial \bar{z}},$$
$$\frac{\partial_n}{\partial z} \circ F = \frac{1}{1 - q^2} F \circ Z,$$
$$\frac{\partial_n}{\partial \bar{z}} \circ F = \frac{1}{1 - q^2} F \circ Z^*,$$
$$\frac{\partial_n}{\partial z} \circ F = \frac{q^4}{1 - q^2} F \circ Z^* \circ (\sigma_i \circ \theta^{-1}).$$
Proof. The formulas in the first line are easy consequences of (4.9) and statements (1) and (2) of Proposition 5.2. We shall only give the proof of the first one:

\[
(\mathcal{Z} \circ \mathcal{F}(f)) (\zeta) = \zeta \int_{\Gamma} F_\zeta(z) f(z) \, d\mu(z) = \int_{\Gamma} \zeta F_\zeta(z) f(z) \, d\mu(z)
\]

\[
= -(1 - q^2) \int_{\Gamma} \frac{-\zeta}{1 - q^2} F_\zeta(z) f(z) \, d\mu(z)
\]

\[
= -(1 - q^2) \int_{\Gamma} \frac{\partial_z}{\partial z} F_\zeta(z) f(z) \, d\mu(z)
\]

\[
= (1 - q^2) q^{-2} \int_{\Gamma} F_\zeta(z) \frac{\partial_z}{\partial z} f(z) \, d\mu(z) = (q^{-2} - 1) \left( F \frac{\partial_z f}{\partial z} \right)(\zeta).
\]

The two formulas in the last line are obtained from the two in the second line by using relations (4.5) and (6.1). Finally the formulas in the second line are obtained from (4.9). Again we only give proof of the first one:

\[
\left( \frac{\partial}{\partial \zeta} \mathcal{F} f \right) (\zeta) = \frac{\partial_z}{\partial \zeta} \int_{\Gamma} F_\zeta(z) f(z) \, d\mu(z)
\]

\[
= \int_{\Gamma} \frac{\partial_z}{\partial \zeta} F_\zeta(z) f(z) \, d\mu(z)
\]

\[
= \int_{\Gamma} \frac{-\zeta}{1 - q^2} F_\zeta(z) f(z) \, d\mu(z)
\]

\[
= \frac{-\zeta}{1 - q^2} \int_{\Gamma} F_\zeta(z) f(z) \, d\mu(z) = -\frac{1}{1 - q^2} (F \mathcal{Z} f)(\zeta).
\]

It is not difficult to show that the operator $\mathcal{F}$ can be extended to a closed operator on $L^2(\Gamma)$. The adjoint $\mathcal{F}^*$ is also an integral operator with kernel $\mathcal{F}_{q}$. With a detailed analysis of the special function $F_\zeta$ one can prove the following theorem:

**Theorem 6.2.**

1. The functions $(\mathcal{F} g_k)_k \in \mathbb{Z}$ introduced by (5.1) are in $C^\infty_q(\Gamma)$ and are integrable with respect to $\mu$.
2. We have $\mathcal{F}^* \mathcal{F} g_k, l = (q \sigma_{-1})^2 g_{k,l}$.
3. The operator $q^{-1} \mathcal{F} \sigma_{-1} \mathcal{F}$ extends to a unitary operator on $L^2(\Gamma)$.

Notice that if we formally put $q = 1$ then the group $(\sigma_t)_{t \in \mathbb{R}}$ is trivial and consequently $\mathcal{F}$ becomes a unitary operator.

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