FANO VARIETIES AND LINEAR SECTIONS OF HYPERSURFACES

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Abstract. When $n$ satisfies an inequality which is almost best possible, we prove that the $k$-plane sections of every smooth, degree $d$, complex hypersurface in $\mathbb{P}^n$ dominate the moduli space of degree $d$ hypersurfaces in $\mathbb{P}^k$. As a corollary we prove that, for $n$ sufficiently large, every smooth, degree $d$ hypersurface in $\mathbb{P}^n$ satisfies a version of “rational simple connectedness”.

1. Statement of results

In their article [2], Harris, Mazur and Pandharipande prove that for fixed integers $d$ and $k$, there exists an integer $n_0 = n_0(d, k)$ such that for every $n \geq n_0$, every smooth degree $d$ hypersurface $X$ in $\mathbb{P}^n$ has a number of good properties:

(i) The hypersurface is unirational.
(ii) The Fano variety of $k$-planes in $X$ has the expected dimension.
(iii) The $k$-plane sections of the hypersurface dominate the moduli space of degree $d$ hypersurfaces in $\mathbb{P}^k$.

It is this last property which we consider. To be precise, the statement is that the following rational transformation

$$
\Phi : \mathbb{G}(k, n) \dashrightarrow \mathbb{P}^{N_d} // \text{PGL}_{k+1}
$$

is dominant. Here $\mathbb{G}(k, n)$ is the Grassmannian parametrizing linear $\mathbb{P}^k$’s in $\mathbb{P}^n$, $\mathbb{P}^{N_d}$ is the parameter space for degree $d$ hypersurface in $\mathbb{P}^k$, $\mathbb{P}^{N_d} // \text{PGL}_{k+1}$ is the moduli space of semistable degree $k$ hypersurface in $\mathbb{P}^k$, and $\Phi$ is the rational transformation sending a $k$-plane $\Lambda$ to the moduli point of the hypersurface $\Lambda \cap X \subset \Lambda$ (assuming $\Lambda \cap X$ is a semistable degree $k$ hypersurface in $\mathbb{P}^k$).

The bound $n_0(d, k)$ is very large, roughly a $d$-fold iterated exponential. Our result is the following.

Theorem 1.1. Let $X$ be a smooth degree $d$ hypersurface in $\mathbb{P}^n$. The map $\Phi$ is dominant if

$$n \geq \binom{d + k - 1}{k} + k - 1.$$

Question 1.2. For fixed $d$ and $k$, what is the smallest integer $n_0 = n_0(d, k)$ such that for every $n \geq n_0$ and every smooth, degree $d$ hypersurface in $\mathbb{P}^n$, the associated rational transformation $\Phi$ is dominant?

Theorem 1.1 is equivalent to the inequality

$$n_0(d, k) \leq \binom{d + k - 1}{k} + k - 1.$$
If $\Phi$ is dominant, then the dimension of the domain is at least the dimension of the target, i.e.,

$$(k + 1)(n - k) = \dim \mathbb{G}(k, n) \geq \dim(\mathbb{P}^{N_k}/\text{PGL}_{k+1}) = \binom{d + k}{k} - (k + 1)^2.$$ 

This is equivalent to the condition

$$n_0(d, k) = \frac{1}{k + 1} \left( \frac{d + k}{k} \right) - 1.$$ 

As far as we know, this is the correct bound. The bound from Theorem 1.1 differs from this optimal bound by roughly a factor of $k$.

The main step in the proof is a result of some independent interest.

**Proposition 1.3.** Let $X$ be a smooth degree $d$ hypersurface in $\mathbb{P}^n$. Let $F_k(X)$ be the Fano variety of $k$-planes in $X$. There exists an irreducible component $I$ of $F_k(X)$ of the expected dimension if

$$n \geq \left( \frac{d + k - 1}{k} \right) + k.$$ 

Moreover, if

$$n = \left( \frac{d + k - 1}{k} \right) + k - 1$$

then there is a nonempty open subset $U_{k-1} \subset F_{k-1}(X)$ such that for every $[\Lambda_{k-1}] \in U_{k-1}$, there exists no $k$-plane in $X$ containing $\Lambda_{k-1}$.

Theorem 1.1 implies a result about rational curves on every smooth hypersurface of sufficiently small degree. The Kontsevich moduli space $\overline{M}_{0,r}(X, e)$ parametrizes isomorphism classes of data $(C, q_1, \ldots, q_r, f)$ of a proper, connected, at-worst-nodal, arithmetic genus 0 curve $C$, an ordered collection $q_1, \ldots, q_r$ of distinct smooth points of $C$ and a morphism $f : C \to X$ satisfying a stability condition. The space $\overline{M}_{0,r}(X, e)$ is projective. There is an evaluation map

$$\text{ev} : \overline{M}_{0,r}(X, e) \to X^r$$

sending a datum $(C, q_1, \ldots, q_r, f)$ to the ordered collection $(f(q_1), \ldots, f(q_r))$.

**Corollary 1.4.** Let $X$ be a smooth degree $d$ hypersurface in $\mathbb{P}^n$. If

$$n \geq \left( \frac{d^2 + d - 1}{d - 1} \right) + d^2 - 1$$

then for every integer $e \geq 2$ there exists a canonically defined irreducible component $\mathcal{M} \subset \overline{M}_{0,2}(X, e)$ such that the evaluation morphism

$$\text{ev} : \mathcal{M} \to X \times X$$

is dominant with rationally connected generic fiber, i.e., $X$ satisfies a version of rational simple connectedness. Moreover $X$ has a very twisting family of pointed lines, cf. [1] Def. 3.7.

This is proved in [1] assuming $n$ satisfies a much weaker hypothesis

$$n \geq d^2$$

but only for general hypersurfaces, not for every smooth hypersurface. The goal here is to find a stronger hypothesis on $n$ that guarantees the theorem for every smooth hypersurface.
2. Flag Fano varieties

Naturally enough, the proof of Proposition 1.3 uses an induction on $k$. To set up the induction it is useful to consider not just $k$-planes in $X$, but flags of linear spaces

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^k \subset X.$$ 

The variety parametrizing such flags is the flag Fano variety of $X$. Also, although we are ultimately interested only in the case of a hypersurface in projective space, for the induction it is useful to allow a more general projective subvariety.

Let $S$ be a scheme such that $H^0(S, \mathcal{O}_S)$ contains $\mathbb{Q}$. Let $E$ be a locally free $\mathcal{O}_S$-module of rank $n+1$, and let $X \subset \mathbb{P}E$ be a closed subscheme such that the projection $\pi : X \to S$ is smooth and surjective of constant relative dimension $\dim(X/S)$. In other words, $X$ is a family of smooth, $\dim(X/S)$-dimensional subvarieties of $\mathbb{P}^n$ parametrized by $S$.

Let $0 \leq k \leq n$ be an integer. Denote by $\text{Fl}_k(E)$ the partial flag manifold representing the functor on $S$-schemes

$$T \mapsto \{(E_1 \subset E_2 \subset \cdots \subset E_{k+1} \subset E_T)|E_i \text{ locally free of rank } i, i = 1, \ldots, k+1\}.$$ 

For every $0 \leq j \leq k \leq n$, denote by $\rho^j_k : \text{Fl}_k(E) \to \text{Fl}_j(E)$ the obvious projection. The flag Fano variety is the locally closed subscheme $\text{Fl}_k(X) \subset \text{Fl}_k(E)$ parametrizing flags such that $\mathbb{P}(E_{k+1})$ is contained in $X$. In particular, $\text{Fl}_0(X) = X$. Denote by $\rho^j_k : \text{Fl}_k(X) \to \text{Fl}_j(X)$ the restriction of $\rho^j_k$.

2.1. Smoothness. There are two elementary observations about the schemes $\text{Fl}_k(X)$.

**Lemma 2.1.** \cite{1.1} There exists an open dense subset $U \subset X$ such that $U \times_X \text{Fl}_k(X)$ is smooth over $U$.

**Lemma 2.2.** Set $S^\text{new} = U$, the open subset from Lemma 2.1. Set $E^\text{new}$ to be the universal rank $n$ quotient bundle of $\pi^* E|_U$ so that $\mathbb{P}(E^\text{new}) = U \times_{\mathbb{P}(E)} \text{Fl}_k(E)$ and set $X^\text{new} = \text{Fl}_k(U)$. Then for every $0 \leq k \leq n$, $\text{Fl}_k(X^\text{new}) = U \times_X \text{Fl}_{k+1}(X)$.

**Proof.** This is obvious. □

**Proposition 2.3.** There exists a sequence of open subschemes $(U_k \subset \text{Fl}_k(X))_{0 \leq k \leq n}$ satisfying the following conditions.

(i) The open subset $U_0$ is dense in $\text{Fl}_0(X)$, and for every $1 \leq k \leq n$, $U_k$ is dense in $(\rho^{k-1}_k)^{-1}(U_{k-1})$.

(ii) For every $1 \leq k \leq n$, $\rho^{k-1}_k : (\rho^{k-1}_k)^{-1}(U_{k-1}) \to U_{k-1}$ is smooth.

**Proof.** Let $U_0$ be the open subscheme from Lemma 2.1. By way of induction, assume $k > 0$ and the open subscheme $U_{k-1}$ has been constructed. As in Lemma 2.2, replace $S$ by $U_{k-1}$, replace $E$ by the universal quotient bundle, and replace $X$ by $(\rho^{k-1}_k)^{-1}(U_{k-1})$. Now define $U_k \subset (\rho^{k-1}_k)^{-1}(U_{k-1})$ to be the open subscheme from Lemma 2.1. □

2.2. Dimension. Using the Grothendieck-Riemann-Roch formula, it is possible to express the Chern classes of $U \times_X \text{Fl}_1(X)$ in terms of the Chern classes of $U$.

Iterating this leads, in particular, to a formula for the dimension of $U_k$. Denote by $G_1$, resp. $G_2$, the restriction to $\text{Fl}_1(U)$ of $E_1$, resp. $E_2$. Denote by $L$ the invertible sheaf

$$L := (G_2/G_1)^\vee.$$
Denote by
\[ \pi : \mathbb{P}G_2 \to \text{Fl}_1(U), \]
\[ \sigma : \text{Fl}_1(U) = \mathbb{P}G_1 \to \mathbb{P}G_2, \]
and
\[ f : \mathbb{P}G_2 \to X \]
the obvious morphisms. In other words, \( \mathbb{P}G_2 \) is a family of \( \mathbb{P}^1 \)'s over \( \text{Fl}_1(U) \), \( \sigma \) is a marked point on each \( \mathbb{P}^1 \), and \( f \) is an embedding of each \( \mathbb{P}^1 \) as a line in \( X \). The formula for the Chern character of the vertical tangent bundle of \( \rho_1^0 \) is,
\[
\text{ch}(T_{\text{Fl}_1(U)/U}) = \pi^* f^* [(\text{ch}(T_{X/S}) - \text{dim}(X/S)) \text{Todd}(\mathcal{O}_{\mathbb{P}E(1)}|_X)] - \text{ch}(L) - 1.
\]
Given a flag \( \mathbb{P} = (\mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^k \subset \mathbb{P}^n) \) in \( U_k \), the formula for the fiber dimension of \( \rho_k^{k-1} \) at \( P \) is
\[
\text{dim}(U_k/U_{k-1}) = \sum_{m=1}^{k} b_{k,m} \langle \text{ch}_m(T_{X/S}), \mathbb{P}^m \rangle - k - 1
\]
where \( \text{ch}_m(E) \) is the \( m \)th graded piece of the Chern character of \( E \), and where the coefficients \( b_{k,m} \) are the unique rational numbers such that
\[
\left( \frac{x+k-1}{k} \right) = \sum_{m=1}^{k} \frac{b_{k,m}}{m!} x^m.
\]
Now define the numbers \( a_{k,m} \) to be
\[
a_{k,m} = \sum_{l=m}^{k} b_{l,m},
\]
in other words,
\[
\sum_{m=1}^{k} \frac{a_{k,m}}{m!} x^m = \sum_{l=1}^{k} \left( \frac{x+l-1}{l} \right).
\]
Then it follows from the previous formula that the dimension of \( U_k \) at \( \mathbb{P} \) equals
\[
\text{dim}(U_k) = \sum_{m=1}^{k} a_{k,m} \langle \text{ch}_m(T_{X/S}), \mathbb{P}^m \rangle + \text{dim}(X) - k^2.
\]
In a related direction, there is a class of complex projective varieties that is stable under the operation of replacing \( X \) by a general fiber of \( \text{Fl}_1(X) \to X \). Call a subvariety \( X \) of \( \mathbb{P}^n \) a quasi-complete-intersection of type
\[
\underline{d} = (d_1, \ldots, d_c)
\]
if there is a sequence
\[
X = X_c \subset X_{c-1} \subset \cdots \subset X_1 \subset X_0 = \mathbb{P}^n
\]
such that each \( X_k \) is a Cartier divisor in \( X_{k-1} \) in the linear equivalence class of \( \mathcal{O}_{\mathbb{P}^n}(d_k)|_{X_{k-1}} \). If \( X \) is a quasi-complete-intersection, then every fiber of \( U \times_X \text{Fl}_1(X) \to U \) is also a quasi-complete-intersection in \( \mathbb{P}^{n-1} \) of type
\[
(1, 2, \ldots, d_1, 1, 2, \ldots, d_2, \ldots, 1, 2, \ldots, d_c).
\]
Iterating this, every (non-empty) fiber of $(\rho_k^{k-1})^{-1}(U_{k-1}) \to U_{k-1}$ is a quasi-complete-intersection in $\mathbb{P}^{n-k}$ of dimension

$$N_k(n,d) = n - k - \sum_{i=1}^c \left( d_i + k - 1 \right).$$

Since the $m^{th}$ graded piece of the Chern character of $T_X$ equals

$$\text{ch}_m(T_X) = (n + 1 - \sum_{i=1}^c d_i^m) c_1(\mathcal{O}(1))^m / m!$$

this agrees with the previous formula for the fiber dimension.

**Corollary 2.4.** Let $X$ be a smooth quasi-complete-intersection of type $d$. If the integer $N_k(n,d)$ is nonnegative, there exists an irreducible component $I$ of $\mathcal{F}_k(X)$ having the expected dimension

$$\dim(I) = \sum_{m=0}^k N_m(n,d).$$

**Proof.** Of course we define $I$ to be the closure of any connected component of $U_k$. The issue is whether or not $U_k$ is empty. By construction $U_k$ is not empty if for every $m = 1, \ldots, k$ the morphism $\rho_m^{m-1}$ is surjective. By the argument above every fiber of $\rho_m^{m-1}$ is an iterated intersection in $\mathbb{P}^{n-m}$ of pseudo-divisors (in the sense of Def. 2.2.1]) in the linear equivalence class of an ample divisor. Thus the fiber is nonempty if the number of pseudo-divisors is $\leq n - m$. This follows from the hypothesis that $N_k(n,d) \geq 0$. \hfill \Box

3. Proofs

**Proof of Proposition 1.3.** The first part follows from Corollary 2.4. For the second part, observe that if $N_k(n,d) = -1$, then $N_{k-1}(n,d)$ is nonnegative. Therefore, by the first part, the open subset $U_{k-1}$ from Proposition 2.3 is nonempty. Since $(\rho_k^{k-1})^{-1}(U_{k-1}) \to U_{k-1}$ is smooth of the expected dimension, and since the expected dimension is negative, $(\rho_k^{k-1})^{-1}(U_{k-1})$ is empty. In other words, for every $(\Lambda_{k-1}) \in U_{k-1}$, there exists no $k$-plane in $X$ containing $\Lambda_{k-1}$. \hfill \Box

**Proof of Theorem 1.1.** Let $(H_{k,n}, e)$ be the universal pair of a scheme $H_{k,n}$ and a closed immersion of $H_{k,n}$-schemes

$$(pr_H, e) : H_{k,n} \times \mathbb{P}^k \to H_{k,n} \times \mathbb{P}^n$$

whose restriction to each fiber $\{h\} \times \mathbb{P}^k$ is a linear embedding. In other words, $H_{k,n}$ is the open subset of $\mathbb{P} \text{Hom}(\mathbb{C}^{k+1}, \mathbb{C}^{n+1})$ parametrizing injective matrices. Of course there is a natural action of $\text{PGL}_{k+1}$ on $H_{k,n}$, and the quotient is the Grassmannian $\mathbb{G}(k,n)$. Denote by $\tilde{F}_k(X)$ the inverse image of $F_k(X)$ in $H_{k,n}$, i.e., $\tilde{F}_k(X)$ parametrizes linear embeddings of $\mathbb{P}^k$ into $X$.

Let $F$ be a defining equation for the hypersurface $X$. Then $e^*F$ is a global section of $e^*\mathcal{O}_{\mathbb{P}^k}(d)$. By definition, this is canonically isomorphic to $pr_{\mathbb{P}^k}^*\mathcal{O}_{\mathbb{P}^k}(d)$. Therefore $e^*F$ determines a regular morphism

$$\tilde{\Phi} : H_{k,n} \to H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)).$$
Denote by $V$ the open subset of $H_{k,n}$ of points whose fiber dimension equals
\[ \dim H_{k,n} - \dim H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)). \]
The rational transformation $\Phi$ is dominant if and only if $\tilde{\Phi}$ is dominant. And the morphism $\Phi$ is dominant if and only if $V$ is nonempty.

The scheme $\tilde{F}_k(X)$ is the fiber $\tilde{\Phi}^{-1}(0)$. If
\[ n \geq \binom{d + k - 1}{k} + k \]
then Proposition 1.3 implies there exists an irreducible component $I$ of $F_k(X)$ of the expected dimension. Thus the inverse image $\tilde{I}$ in $H_{k,n}$ is an irreducible component of $\tilde{F}_k(X)$ of the expected dimension, or what is equivalent, the expected codimension. But the expected codimension is precisely
\[ h^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)) = \binom{d + k}{k}. \]
Thus, the generic point of $\tilde{I}$ is contained in $V$, i.e., $V$ is not empty.

This only leaves the case when
\[ n = \binom{d + k - 1}{k} + k - 1. \]
The argument is very similar. Let $y$ be a linear coordinate on $\mathbb{P}^k$, and let $\tilde{G}_k(X)$ be the closed subscheme of $H_{k,d}$ where $e^*F$ is a multiple of $y^d$. In other words, $\tilde{G}_k(X)$ parametrizes linear embeddings of $\mathbb{P}^k$ into $\mathbb{P}^n$ whose intersection with $X$ contains $dV(y)$. There is a projection morphism $\tilde{G}_k(X) \to F_{k-1}(X)$ associating to the linear embedding the $(k-1)$-plane
\[ \Lambda_{k-1} = \text{Image}(\mathcal{V}(y)). \]
Denote by $G_k(X)$ the image of $\tilde{G}_k(X)$ under the obvious morphism
\[ \tilde{G}_k(X) \to F_{k-1}(\mathbb{P}^n) \times F_k(\mathbb{P}^n). \]
Recall that for a quasi-complete-intersection $X$, the fiber of $F_1(X) \to X$ is an iterated intersection of ample pseudo-divisors in projective space. By a very similar argument, every fiber of $G_k(X) \to F_{k-1}(X)$ is an iterated intersection of ample pseudo-divisors in the projective space $\mathbb{P}^n/\Lambda_{k-1} \cong \mathbb{P}^{n-k}$. Moreover, the fiber of $\text{Fl}_k(X) \to \text{Fl}_{k-1}(X)$ (for any extension of $\Lambda_{k-1}$ to a flag in $\text{Fl}_{k-1}(X)$) is an ample pseudo-divisor in $G_k(X)$. By the second part of Proposition 1.3 there exists a nonempty open subset $U_{k-1} \subset \Lambda_{k-1}$ such that for every $\Lambda_{k-1} \in U_{k-1}$ this ample pseudo-divisor is empty. Therefore the fiber in $G_k(X)$ is finite or empty. But the equation
\[ n = \binom{d + k - 1}{k} + k - 1 \]
implies the expected dimension of the fiber is 0. Since an intersection of ample pseudo-divisors is nonempty if the expected dimension is nonnegative, the fiber of $G_k(X) \to F_{k-1}(X)$ is not empty and has the expected dimension 0. Since $U_{k-1}$ has the expected dimension, the open set $U_{k-1} \times F_{k-1}(X)$ is nonempty and has the expected dimension. Thus it has the expected codimension. Therefore a generic point of this nonempty open set is in $V$, i.e., $V$ is not empty.  
\[ \square \]
Proof of Corollary 1.4. Let $M_e$ be an irreducible component of $\overline{M}_{0,0}(X, e)$ not entirely contained in the boundary $\Delta$. Then for every integer $r \geq 0$ there exists a unique irreducible component $M_{e,r}$ of $\overline{M}_{0,r}(X, e)$ whose image in $\overline{M}_{0,0}(X, 2)$ equals $M_e$. Before defining the irreducible component $M$ of $\overline{M}_{0,2}(X, e)$, we will first inductively define an irreducible component $M_e$ of $\overline{M}_{0,0}(X, e)$ which is not entirely contained in the boundary $\Delta$ and such that the evaluation morphism 

$$ev : M_{e,1} \to X$$

is surjective. Then we define $M$ to be $M_{e,2}$.

Let $U$ denote the open subset of $\overline{M}_{0,1}(X, 1)$ where the evaluation morphism

$$ev : \overline{M}_{0,1}(X, 1) \to X$$

is smooth, i.e., $U$ parametrizes free pointed lines. By [3, 1.1], $U$ contains every general fiber of $ev$. By the argument in Subsection 2.2 (or any number of other references), a general fiber of $ev$ is connected if $d \leq n - 2$. Therefore $U \times X$ is irreducible. There is an obvious morphism $U \times X \to \overline{M}_{0,0}(X, 2)$. By elementary deformation theory, the morphism is unramified and $\overline{M}_{0,0}(X, 2)$ is smooth at every point of the image. Therefore there is a unique irreducible component $M_2$ of $\overline{M}_{0,0}(X, 2)$ containing the image of $U \times X$. Because $U \to X$ is dominant, $M_2 \to X$ is also dominant.

By way of induction assume $e \geq 3$ and $M_{e-1}$ is given. Form the fiber product $M_{e-1,1} \times_X U$. As above this is irreducible, and there is an unramified morphism

$$M_{e-1,1} \times_X U \to \overline{M}_{0,0}(X, e)$$

whose image is in the smooth locus. Therefore there exists a unique irreducible component $M_e$ of $\overline{M}_{0,0}(X, e)$ containing the image of $M_{e-1,1} \times_X U$. Because $M_{e-1,1} \to X$ is dominant, $M_{e,1} \to X$ is also dominant. This finishes the inductive construction of the irreducible components $M_e$, and thus also of $M_{e,2}$.

It remains to prove that

$$ev : M_{e,2} \to X \times X$$

is dominant with rationally connected generic fiber. The article [3] gives an inductive argument for proving this. To carry out the induction, one needs two results: the base of the induction and an important component of the induction argument. Set $k$ to be $d^2$. For a general degree $d$ hypersurface $Y$ in $\mathbb{P}^k$, [3, Prop. 4.6, Prop. 10.1] prove the two results for $Y$. By Theorem 1.1 since

$$n \geq \binom{d + k - 1}{k} + k - 1,$$

for a general $\mathbb{P}^k \subset \mathbb{P}^n$ the intersection $Y = \mathbb{P}^k \cap X$ is a general degree $d$ hypersurface in $\mathbb{P}^k$. Thus the two results hold for $Y$. As is clear from the proofs of [3, Prop. 4.6, Prop. 10.1], the results for $Y$ imply the corresponding results for $X$. \qed

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