Magnetic compressibility and ion-temperature-gradient-driven microinstabilities in magnetically confined plasmas

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Received 20 February 2015, revised 22 May 2015
Accepted for publication 4 June 2015
Published 2 July 2015

Abstract

The electromagnetic theory of the strongly driven ion-temperature-gradient (ITG) instability in magnetically confined toroidal plasmas is developed. Stabilizing and destabilizing effects are identified, and a critical $\beta_e$ (the ratio of the electron to magnetic pressure) for stabilization of the toroidal branch of the mode is calculated for magnetic equilibria independent of the coordinate along the magnetic field. Its scaling is $\beta_e \sim L_{Te} / R$, where $L_{Te}$ is the characteristic electron temperature gradient length, and $R$ the major radius of the torus. We conjecture that a fast particle population can cause a similar stabilization due to its contribution to the equilibrium pressure gradient. For sheared equilibria, the boundary of marginal stability of the electromagnetic correction to the electrostatic mode is also given. For a general magnetic equilibrium, we find a critical length (for electromagnetic stabilization) of the extent of the unfavourable curvature along the magnetic field. This is a decreasing function of the local magnetic shear.

Keywords: electromagnetic ITG, fusion, turbulence

(Some figures may appear in colour only in the online journal)
but as we shall see they become important at surprisingly low values of $\beta$ because of other small parameters present in the problem. A general finite-$\beta$ theory must necessarily describe several families of instabilities, such as ITG Alfvénic modes [7], $\beta$–induced Alfvénic eigenmodes [19], $\beta$–induced temperature gradient eigenmodes [20], and kinetic [4, 5] and ideal ballooning modes [21]. In this work, we limit ourselves to the analysis of curvature-driven ITG modes by adopting an ordering scheme which excludes other instabilities but, at the same time, allows a small value of $\beta (\beta \ll 1)$ to affect the ITG mode through both $A_1$ and $\delta B_i$. The result is a simple formulation shedding light on why and when electromagnetic effects are important for toroidal ITG instabilities.

From a numerical point of view, early gyrokinetic simulations [12, 13] had already found magnetic compressibility to be important, in particular to cancel the stabilizing effect of the ‘self-dug’ magnetic well for drift instabilities [22]. Waltz and Miller reported on such a cancellation, resulting in a substitution rule for the magnetic drift: magnetic compressibility could be dropped if the magnetic drift were replaced by the curvature drift [12]. While this fact now seems to be common knowledge in part of the gyrokinetic community [17, 23], the picture that emerges from systematic electromagnetic gyrokinetic simulations of the ITG mode is more complicated [16] and difficult to disentangle. A simple analytical explanation is therefore helpful.

In the present work, we build on the recent electrostatic linear theory of Plunk et al [24], exploit asymptotic techniques to solve the kinetic problem of the ITG instability, and identify the conditions that allow this theory to accommodate electromagnetic perturbations. Somewhat to our surprise, we find that, for strongly driven modes, magnetic compressibility can be as important as perpendicular magnetic perturbations for values of $\beta$ accessible to both tokamaks and stellarators. The ions contribute to magnetic perturbations to maintain pressure balance, whereas the electrons can have both a stabilizing and destabilizing effect, depending on the value of $\beta$. In the case of a uniform equilibrium magnetic field, a new critical $\beta$ for the electromagnetic stabilization of the toroidal ITG is calculated. This differs from the one given by Kim et al [6] in a fundamental way. A similar stabilization is predicted when an additional fast particle population is considered. For sheared magnetic equilibria, the boundary of marginal stability for the electromagnetic component of the ITG is given, for the first time, using a local approximation of the magnetic drifts.

2. Physical picture

To understand the role of magnetic perturbations for ITG modes, it is useful to start with a physical picture of the instability. We follow Rosenbluth and Longmire, who first described the physical mechanism responsible for interchange modes [25]. The same description works for the curvature-driven branch of the ITG modes and will be used here.

Consider a plasma with gradients of the temperature and the magnetic field strength in the direction of $-\nabla T$. The magnetic field points in the $z$-direction, and for simplicity we take the density gradient to vanish. The ion guiding centers drift in the direction $\mathbf{B} \times \mathbf{V B}$, i.e. in the negative $y$-direction, and do so with a speed that decreases with increasing $x$, since the drift velocity is proportional to the energy.

If the plasma is displaced by an $\mathbf{E} \times \mathbf{B}$ drift in the $x$-direction by the distance

$$\xi = \xi_0 \sin (k_{iY}),$$

the ion pressure is perturbed according to

$$\delta p_i = -\xi \cdot \nabla p_i,$$

where $p_i$ is the equilibrium ion plasma pressure. The ion guiding centers will then start accumulating at $k_{iY} = 2n\pi$ and a corresponding deficit of ion guiding centers arises at $k_{iY} = (2n + 1)\pi$, see figure 1. An electrostatic potential, $\phi = \phi_0 \cos k_{iY}$, thus builds up (with $\phi_0$ having the same sign as $\xi_0$) and gives rise to an $\mathbf{E} \times \mathbf{B}$ drift,

$$\frac{d\xi}{dt} = \frac{B \times V \phi}{B} = \frac{\phi_0}{B} 2 \times V \cos k_{iY} = \frac{\phi_0}{B} \sin k_{iY} = \frac{\phi_0}{B} \xi,$$

that amplifies the initial perturbation (1). In this picture of the instability, the motion of the ions parallel to the magnetic field is neglected, so it is tacitly assumed that $k_{iY} \ll \omega_i$, where $\nu_{thi} = (2T/m_i)^{1/2}$ denotes the ion thermal speed and $\omega_i k_i$ the parallel phase velocity of the instability. The electrons, on the
other hand, can be expected to move quickly compared with the instability, $k_y v_{\|} \gg \omega$, and will therefore only experience a small $E \times B$ displacement.

How is this mechanism affected by electromagnetic terms within the gyrokinetic description of the instability? As already mentioned, there are two such terms, proportional to $A_k$ and $\partial B_y/\partial t$ (the perturbation of the magnetic field strength), respectively. The first one describes the effect of the inductive electric field and is important to the electrons, which unlike the ions have time to move significant distances along the magnetic field during the evolution of the instability. They are therefore sensitive to the parallel electric field,

$$E_\parallel = -\nabla_\parallel \phi - \frac{\partial A_k}{\partial t}.$$  

Instead of $A_k$, we introduce the quantity $\psi$, defined by

$$\nabla_\parallel \psi = -\frac{\partial A_k}{\partial t},$$

so that $E_\parallel = -\nabla_\parallel (\phi - \psi)$. Ampère’s law, $k_y^2 A_k = \mu_0 J_\parallel$, then implies

$$\nabla_\parallel^2 \psi = -\frac{\mu_0}{k_y^2} \frac{\partial}{\partial t} \nabla_\parallel J_\parallel$$

(2)

where $\nabla_\parallel J_\parallel$ describes the local accumulation of electrons due to their parallel motion. If the inductive field is weak, $\psi \ll \phi$, the electrons are approximately Boltzmann-distributed,

$$\frac{\delta n}{n} = \frac{e \phi_0}{T_e} \cos k_y \psi,$$

so that

$$\nabla_\parallel J_\parallel \frac{\delta n}{\partial t} = \frac{en^2}{T_e} \frac{\partial \phi_0}{\partial t} \cos k_y \psi.$$  

Hence and from equation (2) we obtain the estimate

$$\psi \sim \frac{\mu_0 n e^2}{k_y^2 T_e} \frac{\partial^2 \phi_0}{\partial t^2} \cos k_y \psi,$$

and we conclude that the critical $\beta = 2 \mu_0 n T_e B^2$ above which electromagnetic effects are important, $\psi \sim \phi$, scales as

$$\beta_c \sim \left( \frac{k_y^2 T_e}{\omega_c} \right)^2.$$  

(3)

where $\omega_c$ is the sound speed, and $\rho_i$ the ion Larmor radius. The ITG mode has a frequency of order $\omega_c \sim (k_y \rho_i)\omega_c/L_A$, where $L_A$ is the length scale of the cross-field gradients, it thus follows that the critical beta is $\beta_c \sim (k_y L_A)^2 \sim \epsilon^2$; and, in a standard tokamak, can be ordered as the square of the inverse aspect ratio. This is the basic reason why electromagnetic effects are already important in standard tokamak situations when $\beta \sim 10^{-2}$ rather than when $\beta = O(1)$.

The other electromagnetic term in the gyrokinetic equation involves $\partial B_y/\partial t$ and is sometimes neglected in analytical treatments and numerical simulations of the gyrokinetic equation. Physically, it accounts for the perturbation in the $VB$ drift due to the variation in the total magnetic field strength, $B_y$, with the perpendicular component $\nabla B_T = \nabla |B + \delta B| \approx \nabla (B + \delta B_T)$.

The latter is determined by perpendicular pressure balance,

$$\delta \left( p_e + \frac{B_y^2}{2\mu_0} \right) = 0,$$

with $p_e = p_{e,i} + p_{e,c}$, which implies

$$\delta B_y = -\frac{\mu_0 p_e}{B} \frac{\partial B_y}{\partial t} = -\frac{\mu_0 \nabla_\parallel J_\parallel}{B} \sin k_y \psi,$$

and thus gives rise to a perturbed $VB$-drift of the ions

$$\delta v_{id} = \frac{v_{id}^2}{2\Omega_i B} \frac{\partial B_y}{\partial t} = \frac{\mu_0 \nabla_\parallel J_\parallel}{2\Omega_i B^2} \nabla B_T \cos k_y \psi,$$

(4)

where $\mathbf{b} = \mathbf{B}/B$ and $\Omega_i = eB/m_i$. As is clear from figure 1, this extra drift reinforces the density accumulation around $k_y \psi = 2\pi x$ and thus amplifies the instability. It does so even in the absence of a density gradient, since the $VB$-drift is proportional to the perpendicular kinetic energy and we assume that a temperature gradient is present.

There is, however, also a third effect of finite plasma pressure, since this affects the equilibrium magnetic field by making the curvature vector deviate from the gradient of the field strength,

$$\kappa = \frac{\nabla B}{B} + \frac{\mu_0 V_P}{B^2},$$

where $p = p_{e,i} + p_{e,c}$. The equilibrium ion drift velocity can thus be written

$$v_i = \left( \frac{v_i^2}{2} + \psi^2 \right) \frac{\frac{1}{\Omega_i} \mathbf{b} \times \nabla \psi}{\frac{1}{\Omega_i} \mathbf{b} \times \nabla (\frac{p_{e,i}}{\Omega_i} + \frac{p_{e,c}}{\Omega_i})} = v_i + \Delta v_i,$$

(5)

where the second term on the right (III+IV) opposes the basic curvature drift (term II) causing the instability. Thus, if the plasma pressure is increased whilst the magnetic curvature is kept fixed, then the drift velocity is reduced and the instability is weakened. As has been discussed in the literature [5, 16, 17, 22], this effect from the ions partly cancels that from $\delta B_k$, but it is important to keep in mind that this cancellation only holds if $\kappa$, rather than $\nabla \psi$, is held constant. A simple mathematical argument for the cancellation is given in an appendix.

Even though the electrons contribute relatively little to the ion instability, their pressure gradient exerts a stabilizing effect. While term III in equation (5) tends to cancel the perturbed grad-$B$ drift I of equation (4), the diamagnetic electron contribution (term IV in equation (5)), tends to oppose the drive of the mode II. When these terms balance, we have

$$\beta_c \sim \left( \frac{L_p}{R} \right)^2,$$

(6)

where we used $VB \sim B/R$, and $Vp_e \sim p_e/L_p$. This stabilizing influence of finite $\beta$ was studied by Hastie and Taylor for MHD instabilities in a combined mirror-cusp magnetic configuration [26], and by Rosenbluth and Sloan for electrostatic
and weakly electromagnetic instabilities [27]. It will be confirmed quantitatively in the context of the electromagnetic ITG instability below.

It is worth noticing that a similar stabilization can be expected when a population of fast ions is present [28]. Just like the electrons, fast ions move quickly along the magnetic field, and in addition they have large gyroradii. If the typical velocity of the fast ions exceeds the phase velocity of the instability along the field, \( \omega/k_v \rho_{fast} < 1 \), or their gyroradius exceeds the perpendicular wavelength, \( k_v \rho_{fast} > 1 \), such ions will experience relatively small \( \mathbf{E} \times \mathbf{B} \) displacement and therefore contribute little to the magnetic-drift perturbation in equation (4). The fast ions will then contribute relatively little to the instability. On the other hand, their equilibrium pressure can be significant and acts to reduce the equilibrium drift in equation (5) by a new additive term giving \( p_t + p_e > p_t + p_e + p_{fast} \). We thus expect a net stabilisation from fast ions. Gyrokinetic simulations of plasmas with such particles indeed indicate the presence of a critical \( \beta \) for electromagnetic ITG stabilization that decreases as \( L_p/R \) [29]. Moreover, the effect of fast ions is even more significant in nonlinear simulations.

3. Reduction of the gyrokinetic equations

Bearing in mind the qualitative picture from the preceding section, we now give quantitative substance to our findings. We proceed by first deriving from gyrokinetics a set of second order differential equations for the electrostatic and the magnetic potentials. These equations support the electromagnetic ITG mode in the limit of vanishing \( \beta \) Alfvenic perturbations, magnetic compressibility and finite-\( \eta \)-Larmor radius effects. They are derived in a large-\( \eta \) expansion, where \( \eta = d \log T_i / d \log n_i = L_n / L_T \), with \( T_i \) and \( n_i \) the equilibrium temperature and density, respectively. Kinetic ballooning modes are therefore diamagnetically stabilized within our ordering.

Our starting point is the linearized gyrokinetic equation in ballooning space [4, 5, 30]

\[
iv_v \mathbf{B} \cdot \nabla \mathbf{B} + (\omega - \nabla \phi) h = \left( \omega - \omega_a^T \right) \frac{eF_{0||}}{T_{0||}}
\]

\[
\times \left\{ J_\parallel (\omega_a^T - v_B \phi) + \frac{T_{1\parallel}}{v_{B1\parallel}} J_\parallel (\omega_a^T - \Omega_i^T) \frac{\delta B_\parallel}{B} \right\},
\]

(7)

where \( \phi \) is the electrostatic potential, \( A_\parallel \) the perturbed magnetic potential parallel to the equilibrium magnetic field in the Coulomb gauge, \( \mathbf{V} \cdot \mathbf{A} = 0 \), \( \delta B_\parallel \) the parallel magnetic field perturbation, and \( B \) the modulus of the equilibrium magnetic field. For ease of comparison with previous analytical results, we prefer to work in guiding-centre co-ordinates as opposed to gyro-centre ones [31–34]. The form of the perturbations used is \( \exp[-i \omega t + i \mathbf{k}_v \cdot \mathbf{x}] \). The function \( h_\parallel \), defined by \( h_\parallel = \exp(i L_z) = \delta f_\parallel + e\phi F_{0||}/T_i \), denotes the nonadiabatic part of the perturbed distribution function, \( \delta f_\parallel \), where \( f_\parallel = F_{0||} + \delta f_\parallel \), with \( \delta f_\parallel \ll F_{0||}, F_{0||} \) is a Maxwellian equilibrium with temperature \( T_i = m_v v_{B1\parallel}^2 / 2 \) and density \( n_{0||} \), \( L_z = \mathbf{k} \times \mathbf{v}_1 \cdot \mathbf{B}/\Omega_i, \mathbf{B} = \mathbf{B}/B \), with \( \mathbf{v}_1 \) the parallel particle velocity.

Here \( \Omega_i = eB/m_i \) is the cyclotron frequency, \( J_0 \) and \( J_1 \) are Bessel function of the first kind of argument \( \alpha_i = v_B k_v \rho_i \equiv \nu v_B \rho_i \), where \( \nu = v_B \rho_{th} \), \( \rho_i = v_{th}/\Omega_i \) is the Larmor radius, \( k_v^2 = k_1^2 (1 + \frac{\delta^2}{2}) \), with \( k_1 \) the mode wave number, \( \delta \) the local magnetic shear and \( z \) the distance along the equilibrium field lines. Furthermore, \( \delta f_\parallel = 2(\omega_p^2 \mathbf{v}_1^2/2 + \omega_a^T \mathbf{v}_1^2) \), \( 2\omega_a^T = \mathbf{k} \cdot \mathbf{v}_1 \cdot \mathbf{B} \times \mathbf{V}/B \), \( 2\omega_a^T = \mathbf{k} \cdot \mathbf{v}_1 \cdot \mathbf{B} \times (\mathbf{v} \times \mathbf{B}/B) \), with \( v_B \) the parallel particle velocity. Finally, \( \omega_a^T = \omega_{a||} + \eta_1 \omega_{a\perp}(\delta^2 - 3/2) \), and \( \omega_{a\perp} = (1/2)k_1 \rho_i \rho_{th}/L_{ns} \).

The gyrokinetic equation (7) is most easily solved for the electrons, which we take to be sufficiently light that the terms multiplied by \( v_B \) dominate. Neglecting magnetic trapping, we thus obtain the electron response, being described by the solution

\[
h_e \approx - \left( 1 - \frac{\omega_{a\perp}}{\omega} \right) \exp \left[ - \frac{v_B}{\omega} \right],
\]

(8)

where we have written \( \mathbf{v} = i\omega \mathbf{A} \).

For the ions, the equation (7) is solved iteratively using the ordering [3, 24]

\[
\frac{k_{v1\parallel}^2}{\omega^2} \approx \frac{\omega}{\eta \omega_{a\perp}} \approx \frac{\omega_{a\perp} + \omega_B}{\omega} \approx \frac{\omega_B}{\omega} - \epsilon \ll 1,
\]

(9)

which retains the strongly driven \( \omega_{a\perp} \) toroidal and slab ITG instability and finite Larmor radius (FLR) effects. To include electromagnetic perturbations in the electrostatic picture, we use a maximal ordering for the fields, \( v_{th}/\Omega_i \sim v_B \phi, \) and find in lowest order

\[
h_i^{(0)} = \frac{\omega - \omega_a^T}{\omega - \omega_{a\perp}} \left[ J_\parallel (\omega_a^T + v_B \phi) + \frac{v_B^2 - 1}{v_{B1\parallel}} J_\parallel (\omega_a^T - \Omega_i^T) \frac{\delta B_\parallel}{B} \right] F_{0||},
\]

(10)

No ordering in \( \beta \) has been used yet, therefore, at this stage, the perturbed parallel magnetic field must be kept. The electrostatic potential is obtained from the quasineutrality condition,

\[
n_{0||}(T_{1\perp}^{-1} + T_{1\parallel}^{-1}) \phi = \int d^3v \mathbf{v} h_i^{(0)} - \int d^3v \mathbf{v} \epsilon_i^{(0)},
\]

(11)

to which the contribution from \( h_i^{(0)} \) becomes

\[
\int d^3v \mathbf{v} h_i^{(0)} = \frac{e\phi}{T_i} \left[ 1 - \frac{\omega_{a\perp}}{\omega} (1 - \eta \beta) - \frac{\eta_1 \omega_{a\perp} (\omega_B + \omega_{a\perp})}{\omega^2} \right]
\]

\[
 \times \frac{\omega_{a\perp} \omega_{a\perp} \delta B_\parallel}{\omega B}.
\]

(12)

in lowest order. This density perturbation is a factor \( \eta_1^{-1} \ll 1 \) smaller than expected from the size of \( h_i^{(0)} \) \( \sim (\eta_1 \omega_{a\perp} / a)(e\phi/T_i)F_{0||} \), compelling us to find the solution to higher order. We thus iterate the solution,

\[
h_i \approx h_i^{(0)} - \frac{v_{th}}{\omega - \omega_{a\perp}} \left[ (\omega - \omega_{a\perp}) J_\parallel (\omega_a^T + \frac{eA_\parallel}{T_i} F_{0||} + i\mathbf{v}_1) \right]
\]

\[
\times \left( \left( \frac{v_B}{\omega} \right) + \frac{v_B}{\omega} \left[ (\omega - \omega_{a\perp}) J_\parallel (\omega_a^T + eA_\parallel / T_i F_{0||} + i\mathbf{v} h_i^{(0)}) \right] \right),
\]

(13)
and find that a sufficiently accurate expression for the ion density perturbation is

\[
\int d^3v J_i h_i = \frac{e \phi}{T_i} \left[ 1 - \frac{\omega_{ai}}{\omega} (1 - \eta \beta) - \frac{\eta_\omega \omega_{ai} (\omega_B + \omega_c)}{\omega^2} \right] - \frac{\eta \omega_{ai} \delta B}{\omega B} + \frac{\eta_\omega \omega_{ai} B}{m_i \omega^3 V_i} \left[ \frac{e \nabla_i (\phi - \psi)}{B} \right]
\]

(14)

The magnetic field strength fluctuations are determined by the perpendicular Ampère’s law,

\[
\frac{\delta B}{B} = -\frac{\mu_0}{B^2} \sum_s \int d^3v m_s \nabla_i a_s^{-1} J_i(a_i) h_i,
\]

(15)

implying that \( \delta B/B \) is proportional to \( \beta e \phi / T_i \). At this point, a traditional and popular approach would be to neglect the magnetic compressibility altogether \([6, 8–10, 18, 35]\), since \( \beta \) in fusion relevant plasmas is of the order of 1%–5%. However, even such a small \( \beta \) is not necessarily negligible, since the perturbed fields in \( k_i^{(0)} \) of equation (11) get multiplied by a large factor of order \( \epsilon^{-1} \) in equation (14), and, at the same time, the explicit \( \nabla_i \) dependence of the integrand in equation (15) generates a nonzero result, as opposed to what happens for the quasineutrality condition in equation (11). As a result, we reach the conclusion that \( \beta \) must be ordered with respect to \( \eta \).

Thus, using equations (8) and (13) to calculate the integrals in Ampère’s law, we find that \( \beta \sim \omega^2 (\eta_i^2 \omega_{ai}^2) \sim \epsilon^2 \ll 1 \) is the correct ordering that allows us to calculate

\[
\frac{T_i}{e B} \frac{\delta B}{B} = \frac{1}{2} \frac{\beta_i}{\omega} \frac{\eta_\omega \omega_{ai}}{\omega} \left[ \frac{e}{T_i} \left( \phi - \psi \right) + \frac{1}{\tau} \frac{\eta_i}{\eta_i} \right],
\]

(16)

where \( \beta_i = 2 \mu_0 n_i T_i / B^2 \) and \( \tau = T_i / T_e \). This result is a special case of a general formula derived in the work of Tang et al \([5]\) on kinetic ballooning modes. It is derived by substituting equation (10) in equation (15), performing the \( v \)-space integrals, and solving for \( \delta B/B \) in a low-\( \beta \) expansion. The procedure is similar to the one proposed by Tang and co-authors, details are in appendix B. Finally, using equations (8), (14) and (16) in the quasineutrality condition (11), we obtain

\[
\left[ \frac{\omega_{ai} + \beta_i}{\omega} - \frac{\beta_i}{2 \tau} \frac{\eta_i}{\eta_i} \frac{\eta_\omega \omega_{ai}^2}{\omega^2} - \frac{\eta_\omega \omega_{ai}^2 \nabla_i^2}{2 \omega^2} \frac{\partial^2}{\partial x^2} \right] \left( \phi - \psi \right) = -\left( 2 \frac{\eta_i \omega_{ai}}{\omega} - \frac{\eta_\omega \omega_{ai} b}{\omega} \right) \phi
\]

(17)

where we have defined the normalization length \( l_i \) and the coordinate \( z \) along the field so that \( l_i \eta_i \equiv \partial_z \). We have also used the result

\[
\omega_{ai} - \omega_B = 1 + \eta_i (\eta_i T_i) \eta_i \omega_{ai} \beta_i / 2,
\]

(18)

which follows directly from the force balance equation \([5]\)

\[
\mathbf{j} \times \mathbf{B} = \nabla \mathbf{p}.
\]

(19)

Equation (17) is similar to previous results in the literature, but is different in a couple of ways. In particular, the third term on the LHS is absent from previous electromagnetic theories of ITG instabilities \([6]\). Another novelty of this equation is that the inclusion of the ion contribution to magnetic compressibility (the term proportional to \( \phi \) in equation (16)) resulted in the ‘rule’ that the drive of the toroidal branch of the ITG (the first term on the RHS of equation (17)) is the curvature drift only. This result has been confirmed by various numerical works \([12, 17, 23]\).

We close the system of equations calculating the divergence of the current \([4, 5]\) to obtain

\[
\frac{1}{\beta B} \frac{\partial}{\partial z} \left( \frac{\nabla_i}{\omega} \frac{\partial B}{\partial z} \right) = B \eta_i \omega_{ai} \phi - \frac{2 \eta_\omega \omega_{ai} \nabla_i}{\omega^2} \left( \phi + \frac{1}{\tau} \frac{\eta_i}{\eta_i} \right)
\]

(20)

This is obtained by using Ampère’s law after taking the \( \Sigma \phi \int d^3v_0 \) moment of the gyrokinetic equation (7)

\[
B \int d^3v \frac{\nabla_i}{\omega} \frac{\partial B}{\partial z} = \frac{\eta_i \omega_{ai} b \phi}{T_i} + \sum_s \int d^3v_0 \frac{J_i(a_i)}{\omega} \nabla_i \left[ \frac{1}{2} \frac{\partial \delta B}{\partial x} \right] + \sum_s \int d^3v_0 \delta d_i \frac{\partial d_i}{\omega}.
\]

(21)

where the ordering in equation (9) as been used. Velocity-space integrals are performed using solutions (8) and (10). Thus, from equation (21), we have

\[
\frac{B}{\mu_0 \omega^2} \int d^3v \frac{k_i^2}{B} \nabla_i \psi = \frac{\eta_i \omega_{ai} b \phi}{T_i} - \frac{1 + \eta_i}{\tau} \frac{\eta_i}{\omega} \frac{\omega_{ai} T_i}{\omega^2} \frac{\partial B}{\partial z} - \frac{\eta_\omega \omega_{ai} (\omega_c + \omega_B)}{\omega^2} \frac{\partial \psi}{\partial z} - \frac{\eta_\omega \omega_{ai} (\omega_c + \omega_B)}{\omega^2} \frac{\partial \psi}{\partial z}.
\]

(22)

Substitution of equations (16) and (18) now yields equation (20).

4. Critical \( \beta \) for stability

Let us first consider the simple case in which the equilibrium magnetic field is independent of the coordinate along \( B \). Then \( \delta \equiv 0, k_i^2 = k_y^2 \), and we can Fourier transform equations (17)–(20), to obtain

\[
\left( \frac{\omega_{ai}}{\omega} - \frac{\beta_i}{\tau} \frac{\eta_i}{\omega} \frac{\omega_{ai}^2}{2 \tau} + \frac{\omega}{\omega} \right) \Lambda = -\frac{2 \omega r \omega_{ai}}{\omega^2} + \frac{b \omega r}{\omega},
\]

(23)

where \( \omega r = \eta_i \omega_{ai} \),

\[
\Lambda = \frac{\frac{1}{\eta_i} \beta_{MHD} - \beta_i}{\eta_i \omega_{ai}} + \frac{\omega}{\omega_{ai} \omega_{ai}} b \beta_i,
\]

(24)

and

\[
\beta_{MHD} = \frac{b}{2 \omega r \eta_i \omega_{ai} (1 + \eta_i)}
\]

(25)
is the value of $\beta$ above which ideal MHD modes would be destabilized if they were not suppressed by diamagnetic effects.

For $\beta_{\text{MHD}} \ll \beta_i$, $\Lambda \approx \eta \tau [1 + \frac{\eta}{(\eta \tau)} - \text{boil}(2\omega_\lambda)]/\eta_\nu$, while for $\beta_{\text{MHD}} \gg \beta_i$, $\Lambda \approx 1$. Similarly, for $\beta_i \rightarrow 0$, $\Lambda \rightarrow 1$, and equation (23) reduces to the dispersion relation for the electrostatic ITG mode [1, 3, 24],

$$\tau + \frac{\omega_{\text{oni}}}{\omega} + \frac{\omega_T}{2\omega} \frac{k^2}{2} \frac{\omega_{\text{oni}}^2}{\omega^2} = -2 \frac{\omega_T \omega_{\text{oni}}}{\omega^2} + b \frac{\omega_T}{\omega}. \quad (26)$$

Equation (23) agrees with the large-$\eta$ limit of equation (25) in [6] only if the electron contribution to the magnetic compressibility (the third term on the LHS) is neglected. In general, the coupling of all the roots of equation (23) is essential to understand the electromagnetic stabilization of the toroidal ITG mode. To illustrate some what typical case, we solve equation (23) numerically for $\tau = 1$, $b = 1/(2^{3/4}) \approx 0.12$, $R/L_{T1} = 25$, $k_i q = k_q R = 0.8$, and $q = \sqrt{2}$, where $R$ is the major radius of the toroidal device, and $q$ measures the pitch of the magnetic field. We consider the flat density limit for simplicity, $\omega_\nu = 0$, but $\omega_T \neq 0$. For these values $\beta_{\text{MHD}} = 0.64\%$. We note the normalised frequencies $\omega_{\text{oni}}(v_{\text{th}}qR) = \sqrt{b/2} q R/L_{T1}$ and $\omega_T(v_{\text{th}}qR) = q \sqrt{b/2} = \sqrt{b}$, for this particular value of $q$. Thus, we have $\omega_{\text{oni}}(v_{\text{th}}qR) \approx 0.6$, and $\omega_T(v_{\text{th}}qR) \approx 0.34$. Results are in figure 2. Here, several electromagnetic branches can be observed, depending on the value of $\beta_i$. For small $\beta_i$ we find two complex conjugated ion roots ($\omega_{\text{oni}}$ and $\omega_T$). We identify the toroidal ITG branch, $0 < \Re[\omega(v_{\text{th}}qR)] < 1$, and $\Im[\omega(v_{\text{th}}qR)] \sim 1$. Its $\beta$–stabilization occurs at a critical $\beta_i^{\text{crit}}$ for which the imaginary parts of the two complex conjugated roots coalesce. At low $\beta_i$, a further stable electron mode $\Re[\omega(v_{\text{th}}qR)] < 0$ is present ($\omega_{\text{onab}}$). Its real part changes sign when the $\beta$ – stabilization of the ITG becomes effective for $\beta_i \approx 1\%$, see figure 2. All these branches will be shortly derived analytically. We remark that for these values $\omega_{\text{oni}}/\omega_T \approx 0.3$, $k^2 v_{\text{th}}/\omega^2 \approx 0.09$, and $\omega_{\text{oni}}/\omega_T \approx 0.13$, thus satisfying the ordering in equation (9). It is evident that the large $R/L_{T1}$ is, the better the ordering is satisfied.

To establish the scaling of the observed $\beta$ for stabilization with $\omega_T$, we solve equation (23) for several values of $\omega_{\text{oni}}(v_{\text{th}}qR) = \sqrt{b/2} q R/L_{T1}$ but fixing $\omega_{\text{oni}}(v_{\text{th}}qR) = q \sqrt{b/2} \approx \sqrt{12}$, for the above values of $b$ and $q$. This means $\omega_{\text{oni}}(v_{\text{th}}qR) \approx \sqrt{0.12} R/L_{T1}$. We then record the value of $\beta_i$ at which the mode is completely stable. To determine the scaling of the observed $\beta$ for stabilization with $k_i$, we repeat the same evaluation of $\beta_i^{\text{crit}}$ for constant $\omega_{\text{oni}}(v_{\text{th}}qR) = \sqrt{b/2} q R/L_{T1} \approx 0.12$ 25, but varying $k_q R$. As figures 3 and 4 show, the critical $\beta$ for stabilization scales as $\beta_i^{\text{crit}} \sim \beta_{\text{MHD}}$, which implies [36]

$$\beta_i^{\text{crit}} \sim \frac{1}{2\tau q} \frac{L_{E}}{R}. \quad (27)$$

However, as is evident from the figures, $\beta_i^{\text{crit}}$ lies somewhat above $\beta_{\text{MHD}}$, which means that, for these parameters, the stabilization occurs only for values of $\beta_i$ above the ideal MHD threshold.

It is interesting to analyze the stability below this threshold, for $\beta_i \ll \beta_{\text{MHD}}$. This situation corresponds to $\Lambda \approx 1$. For a strongly toroidal mode $4\omega T \omega_{\text{oni}}(v_{\text{th}}qR) \gg b \omega_T/v_{\text{oni}}(v_{\text{th}}qR)$, the new term on the RHS of equation (23) cannot be neglected, and indeed it is responsible for a new critical electron $\beta_i$, for stabilization. After neglecting the stabilizing FLR term on the RHS of equation (23), we obtain $\tau \omega_T = -2\omega T \omega_{\text{oni}}(v_{\text{th}}qR) + \beta_i \eta T \omega_{\text{oni}}^2(2\eta \tau)$. Hence, the electron contribution to magnetic compressibility suppresses the instability when

$$\beta_i > \beta_i^{\text{crit}} = \frac{\eta}{\eta_\nu} \frac{4\omega T}{\Lambda \omega_T}, \quad \text{for } \Lambda > 0. \quad (29)$$

In the limit $\beta_i \ll \beta_{\text{MHD}}$, $\Lambda \approx 1$. The same critical $\beta_i^{\text{crit}}$ for stabilization is obtained in the $\beta_{\text{MHD}} \ll \beta_i$ limit, but now $\Lambda \neq 1$. In both cases, we find

$$\beta_i^{\text{crit}} \sim \frac{L_{E}}{R}. \quad (30)$$

It is perhaps interesting to notice that $\beta_i^{\text{crit}}$ and $\beta_i^{\text{crit}}$ show different explicit scalings with $\omega_{\text{oni}}$, however they follow the same scaling with $R/L_{T1}$.

To verify the estimate in equation (29), we now solve equation (23) numerically in the asymptotic regime $\omega_{\text{oni}} = 0$, $k_q R = 0.001$, $\omega_{\text{oni}}(v_{\text{th}}qR) = 10$, $\omega_T(v_{\text{th}}qR) = 0.25$, $\tau = 1$, $b = 0.05$, and $q = 1.58$. For these values $\Lambda = 2 - \omega_T(2v_{\text{oni}}qR) \approx 2$, when $\omega_{\text{oni}}(v_{\text{th}}qR) < 1$. Again, we solve equation (23) for several values of $\omega_{\text{oni}}(v_{\text{th}}qR)$ at fixed $\omega_T(v_{\text{th}}qR) = 0.25$, and $\omega_{\text{oni}}(v_{\text{th}}qR) = q \sqrt{b/2}$ at fixed

Figure 2. Real and imaginary part of the roots of equation (23) for $\tau = 1$, $k_i q = k_q R = 0.8$, $b = 0.12$, $R/L_{T1} = 25$, $\omega T(v_{\text{th}}qR) = \sqrt{b/2} q R/L_{T1}$. The unstable root at low-$\beta$ is the toroidal branch of the ITG mode. The mode is stabilized for $\beta_i \approx 1\%$.
\[ \omega_{\text{stab}} = -\frac{k_{\perp}^2 v_{thi}^2}{4 \omega_c^2} \left( 1 + \frac{\eta_c}{\tau_c} \right) \beta_{\text{MHD}} - \frac{\beta_i}{1 + b k_{\perp}^2 v_{thi}^2/\omega_c^2}, \]  
and a cubic

\[ \omega^3 + a_{\omega} \omega + a_0 = 0, \]  

with

\[ a_1 = -\frac{1}{\tau} \left( \frac{1}{1 + \tau \eta_c/\tau_c} \right) \frac{k_{\perp}^2 v_{thi}^2}{4 \omega_c^2} b + 2 \right) \omega_c \omega_T, \]  

and

\[ a_0 = \frac{1}{\tau} \left( \frac{\omega_T k_{\perp}^2 v_{thi}^2}{2} + 4 \frac{\omega_c^2 \omega_T}{b} \right). \]  

4.1. Electromagnetic roots at \( \beta_i \approx \beta_{\text{MHD}} \)

Equation (23), when \( \omega_{\text{si}} \equiv 0 \), is in general a quartic for \( \omega \). However, near \( \beta_i \approx \beta_{\text{MHD}} \), it can be factored into a stable solution.
For $\beta_r \rightarrow \beta_{\text{MHD}}$, $\omega_{\text{stab}} \rightarrow 0$, whereas the roots of equation (32) are

$$\omega_1 = A + B,$$

$$\omega_2 = -\frac{1}{2}(A + B) + i\sqrt{\frac{3}{2}}(A - B),$$

and

$$\omega_3 = -\frac{1}{2}(A + B) - i\sqrt{\frac{3}{2}}(A - B),$$

with $A^3 = C + \sqrt{C^2 + D^3}$, $B^3 = C - \sqrt{C^2 + D^3}$,

$$C = -\frac{1}{2}a_0 < 0,$$

and

$$D = \frac{1}{3}a_0 < 0.$$

For

$$\left(\frac{a_1}{3}\right)^3 + \left(\frac{a_0}{2}\right)^2 < 0,$$

all three roots are real.

In the limit $\left(\frac{a_1}{3}\right)^3 \ll \left(\frac{a_0}{2}\right)^2$, $A^3 \sim a_0\left(\frac{a_1}{a_0}\right)^3 \ll 1$, and $B^3 \approx -a_0 = O(1)$. Thus, we find the unstable mode

$$\omega_1 \approx \frac{\omega_{\text{stab}}}{\omega_0} \approx \left\{ \frac{1}{\tau} + \frac{\omega_{\text{stab}}}{2} + \frac{4\alpha_2^2\omega_T}{b} \right\}^{1/3},$$

while $\omega_2 = \omega_1^*$ is damped, and $\omega_3 = -|\omega_1|$ is marginally stable. In the case of negligible slab drive, we have

$$\omega_1 \approx \frac{\omega_{\text{stab}}}{\omega_0} \approx \frac{2}{\eta_0/\tau} \frac{\sqrt{2}b_2}{L_T} \left[ \frac{1}{R} \right]^{1/2},$$

for $\frac{R}{L_T} \ll 27/2$. (42)

These roots are shown in figure 2. It is interesting to note that, close to ideal MHD marginality, the growth rate of the electromagnetic ITG shows a scaling different from the usual electrostatic one $3|\omega| \sim b^{1/2}v_{\text{thi}}/\sqrt{R L_T}$.

In the opposite limit $(\omega_0/2)^2 \gg (a_1/3)^3$, we have $A^3 \approx -i(a_1/3)^3$, and $B^3 \approx -A^3$. Therefore, we obtain one stable ion root

$$\omega_1 \approx \frac{1}{\tau} \left[ \frac{1}{\tau} + \frac{k_0^2v_{\text{thi}}}{4a_0^2b_2 + 2} \right]^{1/2},$$

which, for negligible slab drive is

$$\omega_1 \approx \frac{1}{\tau} \left[ \frac{1}{\tau} + \frac{k_0^2v_{\text{thi}}}{4a_0^2b_2 + 2} \right]^{1/2} \frac{\omega_{\text{stab}}}{\omega_0},$$

for $\frac{R}{L_T} \gg 27/2$. (44)

The second root is

$$\omega_2 \approx 0.$$ (45)

Finally, we find the stable electron mode

$$\omega_3 = -\omega_1.$$ (46)

5. Electromagnetic boundary of marginal stability

For the more realistic case of finite shear, we have $k_2^2 = k_1^2(1 + 2\delta^2 z^2)$, and the previous analysis does not apply. Nevertheless, we can still construct a perturbative electromagnetic theory of the ITG instability similar to that introduced in [9], if we use a local approximation of the curvature drift, $\omega_k (z) = \omega_k (1 - \alpha z^2)$ [24]. We calculate the electromagnetic correction to the electrostatic eigenvalue using a low $\beta, \omega_r^2/\omega_0^2 \ll 1$ subsidiary expansion. The zeroth order electrostatic response is given by equation (17) with $\psi^{(0)} = 0$, and $\phi^{(0)} = \exp[-\lambda z^2]$, with [24]

$$4\lambda^2 = -2\omega_0\omega_{\text{stab}}/\omega_0^2 = (b_2\omega_0^2/\omega_0^3)^2,$$

$$\tau + \omega_{\eta_0/\omega_0} + 2\omega_{T\omega_{\text{stab}}} = -b_2\omega_T/\omega_0 + 2\omega_{\eta_0/\omega_0} = 0,$$

and $\omega_{\text{stab}}^2 = \nu_T^2/(2L_T^2)$. Equations (47) and (48) constitute the electrostatic eigenvalue equation; they determine $\lambda$ and $\omega_0$ which have complex values. After writing equation (17) to first order, we can calculate $\delta \omega$ such that $\omega = \omega_0 + \delta \omega$, with $\delta \omega/\omega_0 = O(\beta)$. Since the zeroth order operator acting on $\phi^{(0)}$, $L^{(0)} = -\left(\tau + \omega_{\eta_0/\omega_0} + 2\omega_{T\omega_{\text{stab}}}/\omega_0^2 - b_2\omega_T/\omega_0\right) + \omega_{\eta_0/\omega_0} + \omega_{T\omega_{\text{stab}}}^2/(2L_T^2\omega_0^3)$, is self-adjoint, we obtain

$$\frac{\delta \omega}{\omega_0} = \frac{1}{\omega_0} \left\{ \int_{-\infty}^{\infty} d\zeta \frac{\partial \psi^{(0)}}{\partial \zeta} \right\}^{(1)} - \frac{\delta \psi^{(0)}}{\omega_0} = \left( \tau + \omega_{\eta_0/\omega_0} + 2\omega_{T\omega_{\text{stab}}} = -b_2\omega_T/\omega_0 + 2\omega_{\eta_0/\omega_0} + \omega_{T\omega_{\text{stab}}}^2/(2L_T^2\omega_0^3) \right)^2,$$

where

$$L^{(1)} = \frac{3}{\omega_0} \frac{\partial \psi^{(0)}}{\partial \zeta} \frac{\partial \psi^{(0)}}{\partial \zeta} - \frac{\partial \omega_{\eta_0/\omega_0}}{\omega_0} - 4\omega_{\eta_0/\omega_0}^2 - \frac{\omega_{T\omega_{\text{stab}}}}{\omega_0} + \frac{\omega_{T\omega_{\text{stab}}}^2}{\omega_0}.$$ (50)

Note that the expression for $\delta \omega$ only requires knowledge of the eigenfunction $\phi^{(0)}$ to zeroth order. To perform the integrations in equation (49), we need the first order electromagnetic component, $\psi^{(1)}$, given by equation (20). We find

$$\psi^{(1)} = \frac{\beta_1}{b_0} \frac{\omega_0^2}{\omega_0} \int_{-\infty}^{\infty} d\zeta \frac{\partial \psi^{(0)}}{\partial \zeta} + \nu_T \text{Erf} (\sqrt{2\alpha} \zeta),$$

with $2\mu = -b_2\omega_T/\omega_0 (\omega_{\eta_0/\omega_0} - 2\omega_{\eta_0/\omega_0}^2 (\omega_{\eta_0/\omega_0} - 2\lambda))$, and

$$\nu = \frac{\sqrt{\pi}}{4\lambda} \left[ b_0 \frac{\omega_{\eta_0/\omega_0}^2 (\delta^2 + 2\lambda) + 2\omega_{\eta_0/\omega_0} (\omega_{\eta_0/\omega_0} - 2\lambda)}{\omega_0} \right].$$ (52)
Thus, the electromagnetic correction to the electrostatic ITG
for finite shear is
\[
\frac{\delta \omega}{\omega_0} = \sqrt{2} \left\{ \frac{\omega_T \beta_i}{\omega_0} b_0 [\mu J_1 + \nu J_2] + \left( 1 - \frac{\omega_T e}{\omega_0} \right) \right\} \times \frac{\beta_i}{b_0} \frac{\omega_0^2}{\eta_0} \sqrt{2} \left\{ \frac{\eta_0}{\omega_0} \sqrt{2} \lambda \right\} \times \frac{3}{2} \left[ \omega_T \omega_a \lambda \right]^{1/2} + \left( \frac{\omega_T}{\omega_0} + 4 \frac{\omega_T \omega_a}{\omega_0} - \frac{\omega_T b_0}{\omega_0} \right) \frac{1}{\lambda^{3/2}}
\]
\[\times \left\{ - \frac{1}{4} \left( 4 \frac{\omega_T \omega_a}{\omega_0} - \frac{\omega_T b_0}{\omega_0} \right) \right\} \left( \frac{\lambda}{\lambda^{3/2}} \right)^{-1}, \tag{53}\]
with \( J_1 = \int_0^\infty d\zeta \text{Erf}(\sqrt{2} \lambda (1 + \hat{s}^2 \zeta)^{-1}) \), \( J_2 = \int_0^\infty d\zeta \text{Erf}(\sqrt{2} \lambda) \), and \( J_3 = \int_0^\infty d\zeta \text{Erf}(\sqrt{2} \lambda (1 + \hat{s}^2 \zeta)^{-1}) \). We find an
analytic closed form of equation (53) if we introduce the Padé
approximants for the two asymptotic limits \( \hat{s}^2 \gg \lambda \) and \( \hat{s}^2 \ll \lambda \).
For the integral \( J_1 \), we find
\[
\lambda^{1/2} J_1(\hat{s}^2, \lambda) \approx \frac{1}{\lambda^{1/2}} \left\{ 1 - \frac{5 \hat{s}^2}{4 \lambda} \left[ 1 - \log(1 + \sqrt{2} \left( \frac{\hat{s}^2}{\lambda} \right)^{1/3}) \right] \right\}, \tag{54}\]
see figure 7.

The Padé approximant of \( J_2 \) for the two asymptotic limits,
\( \hat{s}^2 \gg \lambda \) and \( \hat{s}^2 \ll \lambda \), is
\[
\lambda^{3/2} J_2(\hat{s}^2, \lambda) \approx \frac{1}{\lambda^{3/2}} \left\{ 1 - \frac{3 \hat{s}^2}{16 \lambda} \left[ 1 - \left( \frac{\hat{s}^2}{\lambda} \right)^{1/3} \right] \right\}, \tag{55}\]
see figure 8.

For the integral \( J_3 \), we have
\[
\lambda^{3/2} J_3(\hat{s}^2, \lambda) \approx \frac{1}{\lambda^{3/2}} \left\{ 1 - \frac{3 \hat{s}^2}{16 \lambda} \left[ 1 - \left( \frac{\hat{s}^2}{\lambda} \right)^{1/3} \right] \right\}, \tag{56}\]
see figure 9.
In the present work, we have revisited the problem of how curvature-driven ITG instabilities are affected by finite plasma pressure. As is well known, the latter affects both the equilibrium and the perturbed magnetic drifts of the ions, and these effects partly cancel each other. If the magnetic-field curvature is held constant while the electron + ion pressure is increased, the equilibrium $\mathbf{V} \times \mathbf{B}$-drift is reduced in bad-curvature regions, see equation (6), which is stabilizing. On the other hand, the finite ion pressure gradient also introduces a new $\mathbf{B} \times \nabla B_i$ ion drift, which is destabilizing by a mechanism identified in figure 1 and tends to cancel the stabilizing effect of the ion pressure gradient (if the curvature $\kappa$ is held constant). There remains, however, the stabilizing action of the equilibrium $\mathbf{v}_d$ drift velocity. Mathematically, this cancellation can be seen directly from the kinetic equation for the distribution function $f = f_0 + \delta f$, where the following combination of terms appear in first order,

$$\mathbf{v}_d \cdot \nabla \delta f + \delta \mathbf{v}_d \cdot \nabla f_0.$$  

Substituting the expressions (4) and (5) for $\mathbf{v}_d$ and $\delta \mathbf{v}_d$ from the Introduction gives

$$\mathbf{v}_d \cdot \nabla \delta f + \delta \mathbf{v}_d \cdot \nabla f_0 = \mathbf{v}_e \cdot \nabla \delta f - \frac{\mu_0 v_i^2}{2 \Omega B} \mathbf{B} \cdot \left( \mathbf{V}_p \times \nabla \delta f - \nabla f_0 \times \nabla \delta \mathbf{B}_i \right).$$

A third critical $\beta_i^{\text{crit}}$ for stabilization might be caused by the presence of a fast particle species. We argue that the scaling for $\beta_i^{\text{crit}}$ should be in qualitative agreement with $\beta_i^{\text{crit}} \sim L_T/(2q^2 R)$, due to some similarities in the response of a fast population and electrons. Also in this case, a key role is played by the stabilizing action of the equilibrium $\mathbf{B} \times \nabla \mathbf{B}_i$ ion drift will be missed and the code will tend to underestimate curvature-driven ITG instability.

The results obtained from the local dispersion relation equation (23) are valid when the magnetic shear and the finite extent (along the field) of the bad-curvature region are negligible, unlike in a toroidal device. When these are retained, we have shown that the effect of a small plasma pressure gradient can be determined by perturbation theory. Since the unperturbed (zero-$\beta$) operator is self-adjoint, the amount of stabilization or destabilization can be determined without calculating the perturbed eigenfunctions. The resulting expression (53) is nevertheless complicated but predicts that the extent of the unfavourable curvature along the magnetic field needed for electromagnetic stabilization is a decreasing function of the magnetic shear.

### Appendix A. Cancellation

As mentioned in the Introduction and at several places in the literature [5, 16, 17, 22], the destabilizing effect of the $\mathbf{B} \times \nabla \mathbf{B}_i$ drift is approximately cancelled by the stabilizing influence of the finite-$\beta$ modification of the equilibrium drift velocity. Mathematically, this cancellation can be seen directly from the kinetic equation for the distribution function $f = f_0 + \delta f$, where the following combination of terms appear in first order,

$$\mathbf{v}_d \cdot \nabla \delta f + \delta \mathbf{v}_d \cdot \nabla f_0.$$
The terms within the brackets obviously have the tendency to cancel, and indeed do so exactly when the divergence of the current is calculated, which is effectively what is done in deriving equation (20). If we multiply by the charge, integrate over velocity space and sum over all species \( s \), these terms disappear:

\[
\sum_s e_s \int (\mathbf{v}_s \cdot \nabla \delta f_s + \delta \mathbf{v}_s \cdot \nabla f_s) \, d^3v = \sum_s e_s \int \mathbf{v}_s \cdot \nabla \delta f_s \, d^3v.
\]

**Appendix B. Velocity space integrals**

In equation (12) we have the integral

\[
\int d^3\mathbf{v} J_0(k_x \mathbf{v}_i) h_0, (B.1)
\]

After using equation (11) of the text, we find that the following integrals need to be evaluated:

\[
\alpha = \frac{2}{\sqrt{\pi}} \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \frac{\omega}{\omega_i} f_v^T \frac{J_v}{k_x \mathbf{v}_i} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)} \tag{B.1}
\]

and

\[
\alpha_1 = \frac{2}{\sqrt{\pi}} \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \frac{\omega}{\omega_i} f_v^T J_0(k_x \mathbf{v}_i) \frac{J_v}{k_x \mathbf{v}_i} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

\[
\sim O(e^{-1}) \left( \frac{1}{\sqrt{\pi}} \right) \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \delta \mathbf{v}_1 (\mathbf{v}_1^2 + \mathbf{v}_2^2)^{3/2} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

However, the velocity space integral is exactly zero. Then we are left with order \( O(1) \) quantities, and we find

\[
\alpha \approx \frac{2}{\sqrt{\pi}} \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \delta \mathbf{v}_1 e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

\[
\times \left\{ \frac{\omega}{\omega_i} - \eta \frac{\omega}{\omega_i} (\mathbf{v}_1^2 + \mathbf{v}_2^2) - 3/2 \left( \frac{\omega}{\omega_i} + 2 \frac{\omega}{\omega_i} \right) \right\}
\]

\[
+ \eta \frac{\omega}{\omega_i} (\mathbf{v}_1^2 + \mathbf{v}_2^2) - 3/2 \frac{1}{k_x^2} (\mathbf{v}_1^2 + \mathbf{v}_2^2)
\]

\[
= 1 - \frac{\omega}{\omega_i} + \beta \frac{\omega}{\omega_i} - \eta \omega \frac{\omega}{\omega_i} + \frac{\omega}{\omega_i}, \tag{B.3}
\]

which is the term proportional to the electrostatic potential in equation (14) of the text. The last term of equation (14) is generated by the integral

\[
\beta = \frac{2}{\sqrt{\pi}} \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \delta \mathbf{v}_1 e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

which comes from the second order correction to \( h_0^{(0)} \) in equation (13). The integral \( \beta \) is performed by using the same expansion that led to equation (B.3). Velocity space integrals on the RHS of equation (21) are performed similarly.

The integral \( \alpha_1 \) in equation (B.2) deserves more attentions. This is generated by the \( \delta \mathbf{B}/B \) component of \( h_0^{(0)} \). We have

\[
\alpha_1 \approx - \frac{2}{\sqrt{\pi}} \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \omega \frac{\omega}{\omega_i} f_v^T \frac{J_v}{k_x \mathbf{v}_i} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

\[
\sim O(e^{-1}) \left( \frac{1}{\sqrt{\pi}} \right) \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \delta \mathbf{v}_1 (\mathbf{v}_1^2 + \mathbf{v}_2^2)^{3/2} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

since \( 2J_v(k_x \mathbf{v}_i)k_x \mathbf{v}_i \rightarrow 1 \) for \( k_x \mathbf{v}_i \rightarrow 0 \). The velocity space integral no longer integrates to zero, thus we are required to choose an ordering for \( \delta \mathbf{B}/B \) when compared to \( e \phi / T_i \). Since from equation (12) we have

\[
\int d^3\mathbf{v} J_0(k_x \mathbf{v}_i) h_0^{(0)} = \left[ 1 - \frac{\omega}{\omega_i} + \frac{\omega}{\omega_i} - \frac{\omega}{\omega_i} \right] \frac{e \phi}{T_i} - \frac{\eta \omega}{\omega_i} \frac{\delta B_i}{B},
\]

we must find that the first nonzero contribution to \( \delta \mathbf{B}/B \) is of order one, when solving for \( \delta \mathbf{B}/B \) using the perpendicular Ampère’s law equation (15). This can be achieved only if

\[
\beta_1 \sim e^2.
\]

To illustrate this point, we calculate the ion contribution to the perpendicular Ampère’s law by using the zeroth order solution \( h_0^{(0)} \). We have

\[
(1 + \beta \mu) \frac{\delta B_i}{B} = -\frac{\mu}{2} \frac{\beta_1}{\beta_1} \frac{e \phi}{T_i}, \tag{B.4}
\]

where

\[
\mu = \frac{2}{\sqrt{\pi}} \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \delta \mathbf{v}_1 (\mathbf{v}_1^2 + \mathbf{v}_2^2)^{3/2} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

\[
\sim O(e^{-1}) \left( \frac{1}{\sqrt{\pi}} \right) \int_0^\infty d^3v \mathbf{u}_v \int_0^\infty d^3v \delta \mathbf{v}_1 (\mathbf{v}_1^2 + \mathbf{v}_2^2)^{3/2} e^{-(\mathbf{v}_1^2 + \mathbf{v}_2^2)}
\]

Therefore, for \( \beta_1 \sim e^2 \), we can solve iteratively in \( \beta_1 \) to obtain

\[
\left( \frac{\delta B_i}{B} \right)^{(0)}_{\text{ions}} = 0,
\]

and

\[
\left( \frac{\delta B_i}{B} \right)^{(1)}_{\text{ions}} = \frac{\mu}{2} \frac{\beta_1 \eta \omega}{\omega_i} \frac{e \phi}{T_i}.
\]

which is the first contribution on the RHS of equation (16) of the text. Equation (B.4) coincides with equation (3.35) of [5] when the ordering in equation (10) is applied. The electron contribution is calculated similarly.
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