Existence results for nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space

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Abstract
The main aim of this manuscript is to analyze the existence and uniqueness of $PC$-mild solution of nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space. Based on the Banach contraction principle, we develop the main results. An example is ultimately given for the theoretical results to be justified.

Keywords
Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification
34K30, 35R12, 26A33.

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1. Introduction

Differential systems with fractional order have acquired a great deal of consideration from mathematicians and researchers because of its broad application in different fields of science and designing. Truth be told, one may discover various applications in visco-variability, seismology, electromagnetism, etc. More subtleties and applications can be found in the monograph by Kilbas et al. [5] and the papers of [2, 3, 6, 7].

Motivated by [1, 4], in this paper, we consider a class of nonlocal fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

$$^cD^\alpha x(t) = f(t,x(t),K_1x(t),K_2x(t)),\quad \text{for } t \in (s_i,s_{i+1}], i = 0,1,2,\ldots,m$$

$$x(t) = g_i(t,x(t)), \quad \text{for } t \in (t_i,s_i], i = 1,2,\ldots,m$$

(1.1)

where $^cD^\alpha$ is the Caputo fractional derivative of order $0<\alpha \leq 1, t \in [0,T]; x_0 \in X, 0=t_0= s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C((t_i,s_i] \times \mathbb{R};\mathbb{R}), f: [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a nonlinear function, $h: PC(J,\mathbb{R}) \to \mathbb{R}$ and the functions $K_1$ and $K_2$ are defined by

$$K_1x(t) = \int_0^t u(t,s,x(s))ds \quad \text{and} \quad K_2x(t) = \int_0^T \bar{u}(t,s,x(s))ds,$$

where $u, \bar{u}: \Delta \times \mathbb{R} \to \mathbb{R}$, where $\Delta = \{(x,s): 0 \leq s \leq x \leq \tau\}$ are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence results of problem (1.1). An example is given in Section 4 to illustrate the results.
2. Preliminaries

Let us set \( J = [0,T], J_0 = [0,t_1], J_1 = (t_1,t_2], \ldots, J_{m-1} = (t_{m-1},t_m], J_m = (t_m,\infty) \) and introduce the space \( PC(J,X) := \{ u : J \to X \mid u \in C(J_k,X), k=0,1,2,\ldots,m \) \] and there exist \( u(t_0^+), u(t_0^-) \), \( k=1,2,\ldots,m \), with \( u(t_0^-) = u(t_0^+) \). It is clear that \( PC(J,X) \) is a Banach space with the norm \( \| u \|_{PC} = \sup\{ \| u(t) \| : t \in J \} \).

Let us recall the following well-known definitions [5].

**Definition 2.1.** A real function \( f(t) \) is said to be in the space \( C_a \), \( \alpha \in \mathbb{R} \), if there exists a real number \( p > \alpha \), such that \( f(t) = t^p g(t) \), where \( g \in C[0,\infty) \) and it is said to be in the space \( C^a \), \( \alpha \in \mathbb{N} \) if and only if \( f(n) \in C_a, n \in \mathbb{N} \).

**Definition 2.2.** The Riemann-Liouville derivative of order \( \alpha \geq 0 \) for a function \( f \in C^a \), \( n \in \mathbb{N} \), is defined as

\[
D^\alpha f(t) = D^p D^{\alpha-p} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds, t > 0, n - 1 < \alpha < n.
\]

**Definition 2.3.** The Caputo fractional derivative of order \( \alpha > 0 \) for a function \( f \in C_a, n \in \mathbb{N} \), is defined as

\[
CD^\alpha f(t) = D^{\alpha-n} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)ds, t > 0, n - 1 < \alpha < n.
\]

**Lemma 2.4.** Let \( f : J \to \mathbb{R} \) be a continuous function. A function \( x \in C(J,\mathbb{R}) \) is a solution of the fractional integral equation

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,
\]

if and only if \( x(t) \) is a solution of the following fractional Cauchy problem:

\[
CD^\alpha x(t) = f(t), \quad t \in J
\]

\[
x(b) = x_0, \quad b > 0.
\]

In view of the Lemma 2.1, we define the PC-mild solution for the given system (1.1).

**Definition 2.5.** A function \( x \in PC(J,\mathbb{R}) \) is a mild solution of the problem (1.1) if \( x(0) + h(x) = x_0, x(t) = g_i(t,x(t)), t \in (t,i,s_i], i = 1,2,\ldots,m \) and

\[
x(t) = x_0 - h(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \quad t \in [0,t_1]
\]

and

\[
x(t) = g_i(s_i,x(s_i)) + \frac{1}{\Gamma(\alpha)} \int_{s_i}^t (s_i-t)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds,
\]

where, \( t \in (s_i,t_i+1] \).

3. Existence and Uniqueness Results

In this section, we present and prove the existence and uniquenes of the system (1.1) under Banach contraction principle fixed point theorem.

To establish our results on the existence of solutions, we consider the following hypotheses:

(A1) Let \( \Omega \subset \text{Dom}(f) \) be an open subset of \( J \times \mathbb{R}^3 \). For each \( (t,x_1,y_1,z_1) \in \Omega \) there is a neighbourhood \( V_1 \subset \Omega \) of \( (t,x_1,y_1,z_1) \) such that the nonlinear map

\[
f : J \times \mathbb{R}^3 \to \mathbb{R}
\]

satisfies the following condition

\[
|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leq L_f (|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|) \forall (t,x_1,y_1,z_1), (t,x_2,y_2,z_2) \in V_1 \text{ and } L_f = L_f (t,x_1,y_1,z_1) > 0,
\]

a constant.

(A2) The functions \( u, \tilde{u} : \Delta \times \mathbb{R} \to \mathbb{R} \) are continuous and there exist constants \( L_u, L_{\tilde{u}} > 0 \) such that

\[
\left| \int_0^t [u(t,s,x(s)) - u(t,s,y(s))]ds \right| \leq L_u |x - y|,
\]

for all \( x, y \in \mathbb{R} \); and

\[
\left| \int_0^t [\tilde{u}(t,s,x(s)) - \tilde{u}(t,s,y(s))]ds \right| \leq L_{\tilde{u}} |x - y|,
\]

for all \( x, y \in \mathbb{R} \);

(A3) For \( i = 1,2,\ldots,m \), the functions \( g_i \in C((t_i,s_i] \times \mathbb{R}; \mathbb{R}) \) and there exists \( L_{g_i} \in C(J,\mathbb{R}^+ \) such that

\[
|g_i(t,x) - g_i(t,y)| \leq L_{g_i} |x - y|
\]

for all \( x, y \in \mathbb{R} \) and \( t \in (t_i,s_i] \).

(A4) \( h : PC(J,\mathbb{R}) \to \mathbb{R} \) is continuous and there exists a positive constant \( L_h > 0 \) such that

\[
|h(x) - h(y)| \leq L_h \| x - y \|_{PC}, \quad \text{for all } x, y \in PC(J,\mathbb{R}).
\]

**Theorem 3.1.** Assume that the hypotheses (A1) – (A4) hold and

\[
\Lambda = \max \left\{ \max_{1 \leq i \leq m} \left\{ L_{g_i} + \left( s_i^a + t_{i+1}^a \right) L_f \left( 1 + L_a + L_{\tilde{u}} \right) \right\}, L_h + \frac{t_{i+1}^a}{\Gamma(\alpha + 1)} L_f \left( 1 + L_a + L_{\tilde{u}} \right) \right\} < 1
\]

Then there exists a unique mild solution \( x \in PC(J,\mathbb{R}) \) for the problem (1.1).
Proof. Define the operator $Y : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$Y(t) = x_0 - h(x) + \frac{1}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s), K_1(x(s)), K_2(x(s))) ds, \quad t \in [0, t_1]$$

$$Y(t) = g_i(t, x(t)), \quad t \in (t_i, t_{i+1}], i = 1, 2, \ldots, m.$$ 

From the assumptions, this definition is well-defined.

We prove that $Y$ is a contraction mapping on $PC(J, \mathbb{R})$. Let $x, y \in PC(J, \mathbb{R})$. For $t \in [0, t_1]$, we have

$$|Y(t) - Y(t)| = |g_i(t, x(t)) - g_i(t, y(t))| \leq L_{g_i} |x(t) - y(t)|$$

and hence,

$$\|Y_t - Y_t\|_{C([0, t_1], \mathbb{R})} \leq L_{g_i} \|x(t) - y(t)\|_{PC}.$$ 

For $t \in (t_i, t_{i+1}]$, we have

$$|Y(t) - Y(t)| = |g_i(t, x(t)) - g_i(t, y(t))| \leq L_{g_i} |x(t) - y(t)|$$

and hence,

$$\|Y_t - Y_t\|_{C((t_i, t_{i+1}], \mathbb{R})} \leq L_{g_i} \|x(t) - y(t)\|_{PC}.$$ 

4. Application

In this section we present an application of the result in Section 3.

Consider the following nonlocal fractional mixed type integro-differential equation with impulsive condition of the form

$$CD^\alpha x(t) = \frac{|x(t)|}{(9 + e^6)(1 + |x(t)|)} + \frac{1}{9} \int_0^t e^{-\frac{1}{2} \lambda(s)} ds + \frac{1}{9} \int_0^t e^{-\frac{1}{2} \lambda(s)} ds, \quad t \in (0, 1) \cup (2, 3)$$

$$x(t) = \frac{|x(t)|}{(9 + e^6)(1 + |x(t)|)}, \quad t \in (1, 2)$$

$$x(0) = x_0 + \sum_{i=1}^{10} c_i x(p_i),$$

where, $c_i$ are constants with $\max_{1 \leq i \leq 10} c_i < 0.01$ and $0 < p_1 < p_2 \ldots < p_{10} < 3$, $\alpha = \frac{1}{2}$, $J = [0, 3]$, $0 = t_0 < t_1 = 1 < t_2 = 2 < t_3 = 3$.

Take

$$f(t, x(t), K_1(x(t)), K_2(x(t))) = \frac{|x(t)|}{(9 + e^6)(1 + |x(t)|)} + \frac{1}{9} \int_0^t e^{-\frac{1}{2} \lambda(s)} ds + \frac{1}{9} \int_0^t e^{-\frac{1}{2} \lambda(s)} ds, \quad t \in (0, 1) \cup (2, 3)$$

$$K_1(x(t)) = \frac{1}{9} \int_0^t e^{-\frac{1}{2} \lambda(s)} ds, \quad t \in (0, 1) \cup (2, 3)$$

$$K_2(x(t)) = \frac{1}{9} \int_0^t e^{-\frac{1}{2} \lambda(s)} ds, \quad t \in (0, 1) \cup (2, 3)$$

$$g_i(t, x(t)) = \frac{|x(t)|}{(9 + e^6)(1 + |x(t)|)}, \quad t \in (1, 2)$$

$$h(x) = \sum_{i=1}^{10} c_i x(p_i).$$

For $x, y \in PC(J, \mathbb{R})$ and $t \in (0, 1) \cup (2, 3)$

$$|g_i(t, x(t)) - g_i(t, y(t))| \leq \frac{1}{10} |x(t) - y(t)| \leq \frac{1}{10} \|x(t) - y(t)\|_{PC}.$$ 

For $x, y \in PC(J, \mathbb{R})$, and $t \in (0, 1) \cup (2, 3)$

$$|f(t, x(t), K_1(x(t)), K_2(x(t))) - f(t, y(t), K_1(y(t)), K_2(y(t)))| \leq \frac{1}{10} \|x(t) - y(t)\|_{PC} + \frac{1}{9} \|K_1(x(t)) - K_1(y(t))\|_{PC} + \frac{1}{9} \|K_2(x(t)) - K_2(y(t))\|_{PC}$$

For $x, y \in PC(J, \mathbb{R})$ and $t \in (0, 1) \cup (2, 3)$

$$\|K_1(x(t)) - K_1(y(t))\| \leq \frac{1}{4} \|x(t) - y(t)\|_{PC}$$

$$\|K_2(x(t)) - K_2(y(t))\| \leq \frac{1}{8} \|x(t) - y(t)\|_{PC}$$

which implies that $Y$ is a contraction and hence there exists a mild solution to the problem (1.1).
and
\[ |h(x) - h(y)| < \frac{1}{10} ||x - y||_{PC}. \]
We can verify that the assumptions (A1)-(A4) hold by putting
\[ L_{g_1} = L_h = L_{f} = \frac{1}{10}, L_u = \frac{1}{4}, L_{\tilde{u}} = \frac{1}{8}, \]
we have
\[
\Lambda = \max \left\{ \max_{1 \leq i \leq m} \left\{ L_{g_i} + (s^a_i + r^a_{i+1}) \frac{L_{f}(1+L_u + L_{\tilde{u}})}{\Gamma(\alpha + 1)} \right\},
\right.
\]
\[
L_h + \frac{r^a_1}{\Gamma(\alpha + 1)} L_{f}(1+L_u + L_{\tilde{u}}) \right\}
\]
\[
\leq \max \left\{ \frac{1}{10} + (\sqrt{2} + \sqrt{3}) \frac{0.1(1+0.25+0.125)}{\Gamma(\frac{3}{2})}, \frac{1}{10}
\right.
\]
\[
+ \frac{1}{\Gamma(\frac{3}{2})}(0.1)(1.375) \right\} = 0.588 < 1.
\]
Thus, since all the assumptions in Theorem 3.1 are fulfilled, our results can be applied to the problem (4.1)-(4.3).

References

[1] J. Borah and S. N. Bora, Existence of mild solution of a class of a class of nonlocal fractional order differential equation with not instantaneous impulses, *Fractional Calculus & Applied Analysis*, 22(2)(2019), 495–508.

[2] X. Fu, X. Liu and B. Lu, On a new class of impulsive fractional evolution equations, *Advances in Difference Equations*, (2015) 2015:227.

[3] Ganga Ram Gautham and Jeydev Dabas, Mild solution for nonlocal fractional functional differential equation with not instantaneous impulse, *International Journal of Nonlinear Science*, 21(3)(2016), 151–160.

[4] E. Hernández and D. O’Regan, On a new class of abstract impulsive differential equations, *Proc.Amer. Math. Soc.*, 141 (2013), 1641–1649.

[5] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, 2006.

[6] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal., Real World Appl.* 11(2010), 4465–4475.

[7] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, 59(2010), 1063–1077.

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ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
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