Alternative approaches to the Casalbuoni-Brink-Schwarz Superparticle

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Wigner’s method of induced representations is applied to the $N = 1$ super-Poincaré group, and by using a state corresponding to the basic vector of the little group as a Clifford vacuum we show that the spin operator of a supersymmetric point particle obeys Wigner’s constraints. As dynamical variables for the particle we use canonical coordinates on the symmetry group manifold. The physical phase space is then constructed using a vielbein formalism. We find that the Casalbuoni-Brink-Schwarz superparticle appears as a special case of our general construction. Finally, the theory is reformulated as a gauge theory where the gauge freedom corresponds to the choice of spin constraints or, equivalently, the free choice of relativistic center of mass. In a special case the gauge symmetry reduces to the well known $\kappa$-symmetry.

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1. INTRODUCTION

The Casalbuoni-Brink-Schwarz (CBS) [1, 2, 3] superparticle has played an important conceptual role for superstring theory as a source of new inspiration for the solution of the problems encountered in the attempt to quantize supersymmetric string theories in a manifestly covariant manner. An extensive literature on the quantization of the superparticle exists (see [4] and references given there). In these papers the quantization includ-

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ing the construction of the physical Hilbert space is carried out by means of the covariant Becchi-Rouet-Stora-Tyutin or Batalin-Fradkin-Vilkovisky methods.

As we shall see in the present paper an alternative method for construction of the superparticle state space is provided by Wigner’s method of induced representations [5] applied to the $N = 1$ super-Poincaré group. Our other main concern is the connection between the conventional formulation of the superparticle and the basic understanding of spinning point particles provided by the classical papers of Pryce [6] who showed that there is a freedom in the choice of spin operator corresponding to the arbitrariness of the relativistic center of mass.

For a spinning particle the total angular momentum is the sum of the orbital angular momentum $L^{\mu\nu}$ and the spin $S^{\mu\nu}$ which must obey some constraints. In particular, for a massive particle the Pryce constraints

$$S^{\mu\nu} P_\nu = 0,$$  \hspace{1cm} (1)

with the nice feature of covariance, are equivalent to the Wigner constraints

$$(P^0 + m) S^{ij} - P^i S^{ij} = 0,$$  \hspace{1cm} (2)

which arise naturally by the method of induced representations of the Poincaré group [5].

Here we shall mainly deal with the connection between (1) and (2). The spin operators and the corresponding position operators are related by linear transformations the details of which can be found in Appendix A.3. Remarkably, in the customary formulation of the superparticle the constraints (1) are realized automatically, and as a consequence the components of the position operator do not commute, cf. Eq. (A.36). However, it was observed by Brink and Schwarz [2] that a redefinition is possible in such a way that the components of the new position operator commute mutually. Their transformation formula is reminiscent of Eq. (A.28) relating the position operators in the two cases where (1) and (2) apply.

In an earlier publication [7] it was shown that the freedom of choice of spin constraints actually can be viewed as a gauge symmetry. This gauge symmetry is in the present paper extended to the supersymmetric point particle in a generalized version of the CBS particle that, however, is physically equivalent to the original one. This gauge symmetry reduces in a special case to the $\kappa$-symmetry of Siegel [3].

In [7] it was also found that the dynamical degrees of freedom of a non-supersymmetric point particle can be described by canonical coordinates on the Poincaré group manifold. A similar analysis of the supersymmetric point particle is made below. The particle will in this case move on the super-Poincaré group manifold projected onto the physical superspace by
means of supervielbeins constructed in accordance with the supermanifold formalism of De Witt [8]. In addition to the usual mass shell constraint and spin constraints one must now impose a set of fermionic constraints on the supercoordinate. We find that this system is equivalent to the superparticle considered in [1, 2]. Using a modified method of gauge unfixing [9] it is seen that the gauge freedom in the choice of the spin constraints found for the non-supersymmetric particle is still present. Furthermore, it is demonstrated that the gauge unfixed theory can be gauge fixed again in such a way that the Dirac brackets lead to the same commutation relations as the method of induced representations. Finally it is indeed found that the transition from Eq. (1) to (2) as well as the redefinition of position variables of Brink and Schwarz (see [2] Eq. (21)) constitute gauge transformations.

The analysis will be carried out in several steps.

First, in Sec. 2 a factorization of a general super-Poincaré transformation into an ordinary Poincaré transformation and a supertranslation is found. Next, in Sec. 3, the Wigner construction of the representations of the Poincaré group [5] is extended to the super-Poincaré group by means of this factorization. The Clifford vacuum method of Salam and Strathdee [10] is used in the basic frame of the little group and we restrict ourselves for simplicity to the case of a spinless Clifford vacuum. In this way one finds that the constraints (2) still occurs as a natural candidate for a set of spin constraints. An explicit expression for the spatial part of the spin operator is derived from the structure relations in Sec. 4, and by use of the constraints (2) the rest of the components of the spin operator is determined.

The superparticle is then in Sec. 5 identified with a particle moving on the super-Poincaré manifold. The constraints on the spin operator are obtained by imposing a set of constraints on the fermionic degrees of freedom. These fermionic constraints give rise to a second class constraint algebra (see [11] and references given there) and thus do not define a gauge theory. In Sec. 6 we use gauge unfixing [9], where half of these constraints are singled out to form a first class algebra, and next gauge fix the resulting gauge theory suitably, thus obtaining a spin operator which obeys either one of the sets of constraints (1)-(2) or some other constraints depending on the choice of gauge. We present two varieties of gauge unfixing the theory, using projection operators constructed either by means of $\gamma^5$ (in four dimensions) or the free massless Dirac operator, and in each case construct Dirac brackets. The appendices contain details on vielbeins and on commutators involving the spin operator subject to the constraints (1) and (2), respectively.
We use a metric $\eta_{\mu \nu} = \text{diag}(-1, 1, \cdots, 1)$ and the Dirac matrices are in a Majorana representation, with
\[
\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu \nu}.
\]
(3)
The charge conjugation matrix is $-\gamma^0$ and the number of space-time dimensions is denoted $D$.

2. FACTORIZATIONS

A general Poincaré transformation can be factorized into a translation and a Lorentz transformation. Such a factorization is necessary in order to use Wigner’s method of induced representations [5] on a semidirect product group. This factorization can be obtained just by applying two successive Poincaré transformations on a general vector. It is, however, instructive to see how the factorization can be obtained from a construction involving left and right vielbeins, since the superspace part of a super-Poincaré transformation in a similar way can be factorized from the ordinary Poincaré transformation. We use the general formulation of supermanifolds developed in [8]. Definitions of and explicit expressions for the vielbeins can be found in the appendices A.1 and A.2.

2.1. Factorization of the Poincaré transformation

The Poincaré group consists of translations and Lorentz transformations. The generators of infinitesimal transformations are $P_\mu$ and $M_{\mu \nu}$, respectively. The generators fulfil the commutation relations
\[
[M_{\mu \nu}, M_{\lambda \kappa}] = \frac{i}{2} C^{\xi \eta}_{\mu \nu, \lambda \kappa} M_{\xi \eta}, \quad [M_{\mu \nu}, P_\lambda] = i C^{\xi \eta}_{\mu \nu, \lambda} P_\xi, \quad [P_\mu, P_\nu] = 0 \quad (4)
\]
where the structure constants are
\[
C^{\xi \eta}_{\mu \nu, \lambda \kappa} = \delta^\xi_\eta \eta_{\mu \lambda} - \delta^\xi_\mu \eta_{\nu \kappa}, \quad C^{\xi \eta}_{\mu \nu, \sigma \rho} = \delta^\xi_\sigma \eta_{\mu \rho} - \delta^\xi_\rho \eta_{\nu \sigma} - \delta^\xi_\sigma C^{\eta \rho}_{\mu \nu, \sigma} + \delta^\eta_\rho C^{\xi \rho}_{\mu \nu, \sigma} \quad (5)
\]
\[
C^{\xi \eta}_{\mu \nu, \sigma \rho} = \delta^\xi_\sigma \eta_{\mu \rho} - \delta^\xi_\rho \eta_{\nu \sigma} - \delta^\xi_\sigma C^{\eta \rho}_{\mu \nu, \sigma} + \delta^\eta_\rho C^{\xi \rho}_{\mu \nu, \sigma} \quad (6)
\]

Poincaré group elements are specified by $(a, \lambda)$, their canonical coordinates in the sense of [8]. Here $a$ and $\lambda$ correspond to translations and Lorentz transformations, respectively. A Lorentz transformation $\Lambda$ is given by $\lambda^{\mu \nu}$ through
\[
\Lambda^\mu_\nu = (e^{-C \cdot \lambda})^\mu_\nu \quad (7)
\]
with

\[(C \cdot \lambda)_{\nu}^{\mu} = \frac{1}{2}C_{\rho \sigma, \nu}^{\mu} \lambda^{\rho \sigma}.\]  

Let composition of Poincaré group elements be given by the function \(F\) according to

\[(a_1, \lambda_1) \cdot (a_2, \lambda_2) = F[(a_1, \lambda_1), (a_2, \lambda_2)].\]  

If \((da, 0)\) is an infinitesimal translation, we get by a Taylor expansion, using (A.1):

\[F[(da, 0), (0, \lambda)] = (0, \lambda) + da \cdot u^{-1}[\lambda]\]  

that by means of (A.13) explicitly is

\[F[(da, 0), (0, \lambda)] = (da^\mu u^{-1}[\lambda]_{\mu}^{\nu}, \lambda^{\rho \sigma}).\]  

This formula is only valid when an infinitesimal translation is considered. However, when the product of two infinitesimal transformations is applied group associativity allows one to use (11) twice in succession:

\[F[(2da, 0), (0, \lambda)] = F[(da, 0), F[(da, 0), (0, \lambda)]] = (2da \cdot u^{-1}[\lambda], \lambda).\]  

Repeating this procedure \(n\) times gives us

\[F[(nda, 0), (0, \lambda)] = (nda \cdot u^{-1}[\lambda], \lambda)\]  

Taking \(n \to \infty\) with \(nda\) fixed, \(nda = a \cdot u[\lambda]\), this becomes

\[(a, \lambda) = F[(X, 0), (0, \lambda)]\]  

where we define \(X^\lambda = a^\nu u^\lambda_{\nu}\) as the physical translation vector.

We do the same thing for \(F[(0, \lambda), (da, 0)]\):

\[F[(0, \lambda), (da, 0)] = (0, \lambda) + da \cdot v^{-1}[\lambda]\]  

with \(v_{\mu}^{\nu} = u_{\mu}^{\alpha} \Lambda^{\nu}_{\alpha}\) the right vielbein. By the same procedure as was used above we get:

\[(a, \lambda) = F[(0, \lambda), (\Lambda X, 0)].\]  

By equating (14) and (16) one finds the well known factorization.
2.2. Factorization of the super-Poincaré transformation

The $N = 1$ super-Poincaré group is an extension of the Poincaré group. The algebra is enlarged by the generator of infinitesimal supertranslations $Q$ which is a Majorana spinor. The canonical coordinate corresponding to a supertranslation is a Grassman variable denoted $\xi^\alpha$. New structure relations are

$$[M_{\mu\nu}, Q_\alpha] = -\frac{i}{4} ([\gamma_\mu, \gamma_\nu])^\beta_\alpha Q_\beta, \quad \{Q_\alpha, Q_\beta\} = -2(\gamma \cdot P_0)_{\alpha\beta}, \quad [Q_\alpha, P_\mu] = 0$$

(17)

Following the procedure of sec. 2.1 one can factorize a supersymmetry transformation into a Poincaré transformation and a supertranslation. If $(d\xi, 0, 0)$ is an infinitesimal supertranslation a Taylor expansion leads to the following identity

$$F[(d\xi, 0, 0), (\xi, a, \lambda)] = (\xi, a, \lambda) + d\xi \cdot u^{-1}[(\xi, a, \lambda)].$$

(18)

Comparing this to the explicit expressions for the vielbeins found in appendix (A.2) it is seen that the multiplication of a general group element by an infinitesimal supertranslation affects the translation and supertranslation parts, while the Lorentz part is unaffected. Using the group associativity in a way similar to (12) one finds

$$F[(2d\xi, 0, 0), (0, a, \lambda)] = F[(d\xi, 0, 0), F[(d\xi, 0, 0), (0, a, \lambda)]]$$

$$= \left(2d\xi^\beta u^{-1}[\lambda]_\beta^\alpha, a^\mu + d\xi^\beta u^{-1}[0, \lambda]_\beta^\mu + d\xi^\beta u^{-1}[(d\xi \cdot u^{-1}, \lambda)]_\beta^\mu, \lambda\right)$$

(19)

Repeating this process $n$ times gives

$$F[(nd\xi, 0, 0), (0, a, \lambda)] = \left(nd\xi^\beta u^{-1}[\lambda]_\beta^\alpha, a^\mu + d\xi^\beta \sum_{k=0}^{n-1} u^{-1}[(kd\xi \cdot u^{-1}, \lambda)]_\beta^\mu, \lambda\right)$$

(20)

Now let $n \to \infty$ and $d\xi \to 0$ with $nd\xi$ fixed: $nd\xi^\alpha = \xi^\beta u[\lambda]_\beta^\alpha$. Then Eq. (20) reads

$$(\xi, a, \lambda) = F[(\xi^\alpha u[\lambda]_\alpha^\beta, 0, 0), (0, a^\mu + \frac{1}{2}\xi^\alpha u[(\xi^\gamma, \lambda)]_\alpha^\nu u^{-1}[\lambda]_\nu^\mu, \lambda)]$$

(21)

Introducing the spacetime translation vector $X$ and the supertranslation spinor $\theta$ by

$$X^\lambda = a^\nu u_\nu^\lambda + \frac{1}{2}\xi^\beta u_\beta^\lambda, \quad \theta^\alpha = \xi^\beta u_\beta^\alpha$$

(22)
we find

\[(\xi, a, \lambda) = F[\theta, 0, 0, 0, X^\nu (u^{-1})_\nu, \lambda]\]

(23)

Thus, the super-Poincaré transformation has been factorized into a Poincaré transformation followed by a supertranslation.

The same procedure can be applied to the right vielbeins:

\[F[(\xi, a, \lambda), (d\xi, 0, 0)] = \left(\xi^\alpha + d\xi^\beta v^{-1}[\lambda]_\beta^\alpha, a^\mu + d\xi^\beta v^{-1}[(\xi, \lambda)]_\beta^\mu, \lambda\right)\]

(24)

with \(v\) denoting right vielbeins, and by the same procedure as before we get

\[(\xi, a, \lambda) = F \left[0, a^\mu + \frac{1}{2}\xi^\alpha v[(\xi, \lambda)]_\alpha^\rho v^{-1}[\lambda]_\rho^\mu, \lambda, (\xi^\alpha v[\lambda]_\alpha^\beta, 0, 0)\right].\]

(25)

Under a Lorentz transformation given by the canonical coordinate \(\lambda^\mu\nu\) the spinor transformation matrix is

\[S^\alpha_\beta = \left(e^{-\frac{i}{2}\lambda^\mu\nu[\gamma_\mu, \gamma_\nu]}\right)^\alpha_\beta\]

(26)

It is seen from Eq. (A.13) in connection with the definition of the structure constants of Eq. (A.12) that the following relation holds

\[v_\beta^\alpha = u_\beta^\gamma S^\alpha_\gamma\]

(27)

By means of the identity (A.17) and in terms of the translation vector and the supertranslation spinor given in Eq. (22) we can rewrite Eq. (25)

\[(\xi, a, \lambda) = F \left[0, X^\rho u^{-1}[\lambda]_\rho^\mu, \lambda, (S^\alpha_\beta \theta^\beta, 0, 0)\right].\]

(28)

The factorizations of the super-Poincaré transformation (23) and (28) are central for the construction of induced representations of the super-Poincaré group and are used for this purpose in section 3.2.

3. SPIN CONSTRAINTS

The factorization of general Poincaré and super-Poincaré group elements of (14)-(16) as well as (23) and (28) is used for the construction of induced representations in the present section. First we review how the Wigner constraints (2) appears naturally for the Poincaré group when a group theoretical analysis is made, and then it is shown how this analysis carries over to the super-Poincaré group
3.1. The Poincaré group

Let a state vector be denoted by $|p, s\rangle$, where $p$ refers to momentum quantum numbers and $s$ to spin quantum numbers and other internal degrees of freedom. Let a translation $X$ be represented by an operator $T[X]$, a Lorentz transformation $\Lambda$ by an operator $T[\Lambda]$ and a Poincare transformation $(a, \lambda)$ by an operator $T[(a, \lambda)]$ in this space, where $a$ and $\lambda$ are canonical coordinates, and $X^\nu = a^\mu u^\nu_\mu$ and $\Lambda = e^{-C \cdot \lambda}$ the associated translation and Lorentz transformation, respectively. The operators $T[X]$, $T[\Lambda]$ and $T[(a, \lambda)]$ exist as unitary operators on the Hilbert space as demonstrated by Wigner [5, 12]. The canonical coordinates $(a, \lambda)$ refer to passive transformations, i.e. the system is unchanged but the observer is transformed. In contrast, the Lorentz transformation $\Lambda$ is an active transformation, where the system is transformed.

The little group [5, 13] corresponding to a fixed vector $q$, $G_q$, is the subgroup of the Lorentz group which leaves $q$ invariant. For each momentum vector $p$ one singles out one Lorentz transformation $\Lambda_{qp}$ with corresponding canonical coordinate $\lambda_{qp}$, which transforms $q$ into $p$, and uses this transformation to define a general state:

$$\Lambda_{qp} q = p, \quad |p, s\rangle \equiv T[\Lambda_{qp}][q, s].$$

(29)

To an arbitrary Lorentz transformation $\Lambda$ corresponds a Wigner transformation

$$\Lambda_q[\Lambda, p] \equiv \Lambda_{q(\Lambda p)}^{-1} \Lambda \Lambda_{qp} \in G_q$$

(30)

which belongs to the little group. The general formula for the representations of the Poincaré group is:

$$T[(a, \lambda)][p, s] = e^{-iX \cdot (\Lambda^{-1} p)} \sum_{s'} T_{s's}[\lambda_q[\Lambda^{-1} p]] |\Lambda^{-1} p, s\rangle$$

(31)

Thus all that is needed to perform a Poincaré transformation on a state vector is the representations of the little group.

3.1.1. Massive case

For a particle with non-zero rest mass $m$ the rest system vector, $q^\mu = (m, \vec{0})$, is used as the basis of the Wigner analysis. In this frame the little group $G_q$ consists of all spatial rotations, $G_q = SO(D - 1)$. Boosts are

$$\Lambda_{qp} = \hat{\Lambda}(p) P$$

(32)
with the definitions

\[ \hat{\Lambda}(p)^{\mu\nu} = \frac{2\tilde{p}^\mu \tilde{p}^\nu}{p^2} - \eta^{\mu\nu}, \quad \tilde{p}^\mu = (p^0 + m, \vec{p}) \]  

(33)

and with \( P = \text{diag}(1, -1, -1, \cdots) \) the parity operator. In the case of an infinitesimal Lorentz transformation with antisymmetric parameter \( \delta \lambda^{\mu\nu} \) the transformation (30) is an infinitesimal rotation with rotation parameter

\[ \hat{\delta} \lambda^{ij} = \delta \lambda^{ij} - \frac{p^i \delta \lambda^0_j - \delta \lambda^0_i p^j}{p^0 + m} \]  

(34)

The general expression for the transformation matrix of an infinitesimal Lorentz transformation applied to a rest state is

\[ T_{s's}(\lambda_q[\Lambda, p]) = \delta_{s's} - \frac{i}{2} \delta \lambda^{\mu\nu} (S_{\mu\nu})_{s's} \]  

(35)

while application of the general formula (31) gives:

\[ T_{s's}(\lambda_q[\Lambda, p]) = \delta_{s's} - \frac{i}{2} \hat{\delta} \lambda^{ij} (S_{ij})_{s's} \]  

(36)

where only spatial components of the spin operator are present. Equating the two expressions using (34) we obtain

\[ S_{j0} = \frac{p^j}{p^0 + m} S_{ij} \]  

(37)

which in operator form are the Wigner constraints (2).

3.1.2. Massless case

For a massless particle a similar analysis can be carried out using \( q^\mu = (E, E, 0, \cdots, 0) \) as the basis for the Wigner analysis. In this case the little group consists of \( D - 2 \) dimensional rotations as well as some combinations of rotations and boosts. The generators of these boost/rotation components of the little group is denoted \( K_i \). Introducing light cone coordinates as \( p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1) \) and correspondingly for the spin operator this operator must fulfil

\[ S_{-i} = -\frac{1}{\sqrt{2}} K_i, \quad S_{+i} = \frac{1}{p^+} (p^j S_{ij} - p^- S_{-i} - \sqrt{2} E S_{-i}), \quad S_{+-} = \frac{p^i S_{-i}}{p^+} \]  

(38)
Because of the boosts contained in the $K_i$ generators the little group of the massless particle is not compact, so all unitary representations are infinite-dimensional or trivial. Therefore, in order to have a finite-dimensional unitary representation we must demand that the generators $K_i$ vanishes. In this case the spin constraints become

$$S_{-i} = S_{+-} = 0, \quad S_{+i} = \frac{1}{p^i} p^j S_{ij}. \quad (39)$$

### 3.2. The super-Poincaré group

An analysis similar to the one of the previous section can be carried out for the super-Poincaré group (cp. also [14]).

If (26) is examined in the case of a pure boost giving (29) we find that the corresponding spinor transformation is given by

massive case:

$$(S_p)_{\alpha \beta} = \frac{1}{\sqrt{2m(p^0 + m)}} \left( \delta_{\alpha \beta} (p^0 + m) + (\gamma^0 \gamma \cdot \vec{p})_{\alpha \beta} \right) \quad (40)$$

massless case:

$$(S_p)_{\alpha \beta} = \frac{1}{\sqrt{2 E(p^0 + p^1)}} \left( \delta_{\alpha \beta} (p^0 + E) + E(\gamma^0 \gamma^1)_{\alpha \beta} + (\gamma^0 \gamma \cdot \vec{p})_{\alpha \beta} \right) \quad (41)$$

In the rest frame of a massive particle or the light-cone frame of a massless particle the general structure relations reduce considerably. We then use the $Q_q$ operators to carry out a supertranslation on a particle state according to [10, 15]:

$$T[\theta]|q, s\rangle = \sum_{s'} (e^{-iQ_q \theta})_{s' s} |q, s'\rangle = \sum_{s'} R_{s' s}(\theta)|q, s'\rangle \quad (42)$$

In this way one can construct supermultiplets from a Clifford vacuum characterized by its innate spin. Using the results (23) and (28) one gets for a general state

$$T[\theta]|p, s\rangle = T[\theta]T[\Lambda_p]|q, s\rangle = T[\Lambda_p]T[S^{-1}_p \theta]|q, s\rangle \quad (43)$$

This is a supertranslation on a particle in the rest frame, followed by a Lorentz transformation, so the effect of a supertranslation on a general
state is given by

\[ T[\theta]|p, s\rangle = \sum_{s'} R_{s's}(S^{-1}_p\theta)|p, s'\rangle. \quad (44) \]

Here it is important to note three things:

1. The effect of the supertranslation is to bring the particle into a linear combination of the elements of the supermultiplet to which the particle belongs.

2. The coefficients of this linear combination \( R_{s's} \) are defined in the rest frame of a massive particle or the light-cone frame of a massless particle. So even when these coefficients appear in the supertranslation of a general state we have to go to one of these frames to evaluate the coefficients.

3. Eq. (42) implies

\[ R_{s's}(S^{-1}_p\theta) = \left( e^{-i\bar{Q}_p S^{-1}_p\theta} \right)_{s's} = \left( e^{-iQ_p\theta} \right)_{s's} \quad (45) \]

so when supertranslating a general state instead of a basic state one can change \( \theta \) as in (44) or one can use boosted \( Q_s \) instead of the basic frame \( Q_q \).

According to (23) the operator representing a general super-Poincaré transformation factorizes into

\[ T[(\xi, a, \lambda)] = T[\theta]T[(0, X^\nu(u^{-1})_\nu, \lambda)] \quad (46) \]

and acts consequently on a general quantum mechanical state \(|p, s\rangle\) as follows:

\[ T[(\xi, a, \lambda)]|p, s\rangle = e^{-iX^\nu\Lambda^{-1}_\nu} \sum_{s', s''} T_{s's}(\lambda[\Lambda^{-1}_s, p]) R_{s's''}(S^{-1}_p\theta)|\Lambda^{-1}_s, p, s''\rangle \quad (47) \]

where (31) has been used.

We can compare this result, accomplished by use of the factorization (23), with the general Poincaré transformation in Eq. (31). The new thing is the appearance of the \( R_{s's} \) coefficients. However, we still have the \( T_{s's} \) coefficients appearing in exactly the same way leading again to the Wigner constraints (2) for the spin operator, but this time the spin operator operates within a supermultiplet. Especially, the supermultiplet
may involve a Clifford vacuum with spin 0, in which case the construction leads to the Casalbuoni-Brink-Schwarz superparticle.

4. THE SPIN OPERATOR

Having verified in the previous section that the spin operator when constructed by the method of induced representations obeys the constraints (2) also in the supersymmetric case, we are now ready for an explicit construction of the spin operator, using the structure relations of the little group basic frame.

4.1. The massive particle

To determine $S_{\mu\nu}$ one uses the nontrivial structure relations (17) in the rest frame:

\[
[S_{ij}, Q_r] = -\frac{i}{4} [\gamma_i, \gamma_j] Q_r,
\]

\[
\{Q_{r\alpha}, Q_{r\beta}\} = 2m \delta_{\alpha\beta},
\]

where $Q_r$ is the supersymmetry generator in the rest frame. From Eqs. (48)-(49) one determines the following spin operator:

\[
S_{ij} = -\frac{i}{8m} \bar{Q}_r \gamma^{0ij} Q_r
\]

with $\gamma^{\mu\nu\lambda} = \gamma^{[\mu} \gamma^{\nu} \gamma^{\lambda]}$ the completely antisymmetric product. The spin operator can be expressed in terms of the boosted $Q_s$ by inverting (40). The outcome is:

\[
S_{ij} = -\frac{i}{8m^2} \bar{Q}_p \gamma^{ij\mu} Q_p p^\mu + \frac{i}{8m^2 (p^0 + m)} \bar{Q}_p (p^\gamma \gamma^{0j\mu} - p^\gamma \gamma^{0i\mu}) p^\mu Q_p
\]

Since this spin operator obeys the Wigner constraints (2) the last components of the spin operator can also be determined. Defining the new spin operator

\[
\hat{S}_{\mu\nu} = \frac{i}{8m^2} Q_p \gamma^{\mu\nu\lambda} Q_p p^\lambda
\]

which obviously fulfills the Pryce constraints (1) we can incorporate also the time components of this spin operator such that

\[
S_{\mu\nu} = \hat{S}_{\mu\nu} - \frac{1}{p^0 + m} p^\mu \hat{S}_{0\nu} + \frac{1}{p^0 + m} p^\nu \hat{S}_{0\mu}.
\]
This is the spin operator transformation formula describing a transition between a system where the Wigner constraints are valid and the same system where the Pryce constraints are valid, cf. eqs. (A.33)-(A.35).

To establish the connection of \( S^{\mu\nu} \) to the spin operator of [2] one introduces the coordinate \( \theta \) according to

\[
Q_p = 2i \gamma \cdot p \theta. \tag{54}
\]

The anticommutator \( \{Q_p, Q_{\rho\beta}\} = 2(\gamma \cdot p \gamma^0)_{\alpha\beta} \) then leads to

\[
\{\theta_\alpha, \theta_\beta\} = \frac{1}{2m^2} (\gamma \cdot p \gamma^0)_{\alpha\beta} \tag{55}
\]

and the spin operator (52) becomes

\[
\hat{S}^{\mu\nu} = -\frac{i}{2} \theta^{\mu\nu} \theta P_{\lambda}. \tag{56}
\]

These formulas can be compared to ref. [2] and are recognized as the anticommutation relations of the superspace coordinates and the expression giving the spin operator in the case of a massive superparticle. Thus the massive version of the CBS superparticle has been constructed by means of the method of induced representations since the state space was determined in Sec. 4.

4.2. The massless particle

In the light-cone system of a massless particle where \( p^\mu = (E, E, 0, \ldots) \) the two nontrivial structure relations (17) reduce to

\[
\{Q_{ic\alpha}, Q_{ic\beta}\} = 2\sqrt{2} E (\gamma^0 \gamma^0)_{\alpha\beta}, \tag{57}
\]

\[
[S_{ij}, Q_{lc}] = -\frac{i}{4} [\gamma_i, \gamma_j] Q_{lc}, \tag{58}
\]

\[
[S_{-j}, Q_{lc}] = -\frac{i}{2} \gamma^- \gamma_j Q_{lc}. \tag{59}
\]

In these expressions \( Q_{ic} \) is the supersymmetry generator in the light-cone system and the indices \( i, j \) are in the range \( 2, \ldots, D-1 \). In order to express the spin operator in terms of \( Q_{ic} \) one must require

\[
Q_{lc} = \frac{1}{\sqrt{2}} \gamma^- \gamma^0 Q_{ic} \tag{60}
\]

thus halving the number of independent components of \( Q_{ic} \). A consequence is

\[
S_{-i} = 0. \tag{61}
\]
From Eqs. (57)-(58) one finds:

\[
S^{ij} = \frac{i}{32E} Q_{tc}[\gamma^i, \gamma^j]Q_{tc} = \frac{i}{16\sqrt{2}E} Q_{tc} \gamma^{ij} Q_{tc}.
\]  

(62)

The remaining components of the spin operator are fixed by the constraints (39).

To find the boosted Q’s one uses Eq. (41) and Eq. (60):

\[
Q_p = -\frac{1}{2\sqrt{E}p^\gamma} \gamma \cdot p \gamma^0 Q_{tc}.
\]  

(63)

By comparison with [2] one realizes that \( \gamma^0 Q_{tc} \) is proportional to the \( S \)-variable of that paper. The method thus also has allowed construction of the massless superparticle.

5. PARTICLE ON A GROUP MANIFOLD

Having accomplished the construction of the CBS superparticle by means of the method of induced representations, we next use a different starting point to elucidate the relationships between the two sets of spin constraints (1) and (2). In the course of the construction, the spin operator had to be transformed according to Eq. (53) (cf. Eq. (A.35)). For a non-supersymmetric particle with spin, it was demonstrated in [7] that this transformation can be considered a gauge transformation. However, this identification is not possible within the framework developed so far.

Following [11, 7] we identify the superparticle with a particle moving on the super-Poincaré group manifold. First the naive action for a free particle moving on the supergroup manifold is considered. Next the connection between our construction and the Casalbuoni-Brink-Schwarz superparticle is made.

5.1. Naive action

In analogy to the case of the Poincaré group [11, 7] we now imagine a supersymmetric particle with spin moving on the super-Poincaré group manifold, where we use the general formulation of supermanifolds developed in [8]. The group manifold is parametrized by the canonical coordinates \( a^\mu, \lambda^{\mu\nu} \) and \( \xi^\alpha \). We project the corresponding canonical momenta onto the physical generators according to (A.7) with the vielbeins listed in Appendix A.2:

\[
\Pi_\mu = P_\nu u_\mu^{\nu}, \quad \Pi_\alpha = (\bar{Q}_\beta u_\beta^{\beta} + P_\nu u_\nu^{\nu}) (\gamma^0)^{\alpha}_{\beta}.
\]
\[ \Pi_{\mu\nu} = \frac{1}{2} M_{\rho\sigma} u_{\rho\mu} u_{\sigma\nu} + P_\lambda u_{\mu\lambda} + \bar{Q}_\alpha u_{\mu\alpha} \]  \hspace{1cm} (64)

to ensure the correct structure relations, while coordinates are
\[ X^\lambda = u_\nu^\lambda a^\nu + \frac{1}{2} u_\beta^\lambda \xi^\beta, \quad \theta^\alpha = u_\beta^\alpha \xi^\beta. \]  \hspace{1cm} (65)

The naive action on the group manifold is
\[ S_{\text{naive}} = \int d\tau (\Pi_\mu \dot{a}^\mu + \frac{1}{2} \Pi_{\mu\nu} \dot{\lambda}^{\mu\nu} + \bar{\Pi} \dot{\xi}). \]  \hspace{1cm} (66)

Now one uses the Cartan-Maurer equations (A.9) as well as (64), (65) and
\[ P_\theta \equiv Q - i \gamma \cdot P\theta. \]  \hspace{1cm} (67)

Here \( P_\theta \) is the conjugate momentum to \( \theta \), with \( \{ \theta^\alpha, P_\beta^\theta \}_P = (\gamma^\mu)^{\alpha\beta} \) with the subscript \( PB \) denoting Poisson brackets (Poisson brackets involving Grassmann variables are defined in Eq. (A.6)). After lengthy calculations one obtains the naive action
\[ S_{\text{naive}} = \int d\tau \left( P_\mu \dot{X}^\mu + P_\theta \dot{\theta} + \frac{1}{2} \Sigma_{\mu\nu} \sigma^{\mu\nu} \right) \]  \hspace{1cm} (68)

where
\[ \sigma^{\mu\nu} = (\Lambda^{-1})^{\mu}_\lambda \dot{\Lambda}^{\lambda\nu} = \frac{1}{2} \dot{\lambda}^{\rho\sigma} u_{\rho\sigma}^{\mu\nu} \]  \hspace{1cm} (69)

and where the spin of the Clifford vacuum appears
\[ \Sigma_{\mu\nu} = M_{\mu\nu} - X_\mu P_\nu + X_\nu P_\mu - S_{\mu\nu} \]  \hspace{1cm} (70)

as we would expect. Here
\[ S_{\mu\nu} = -\frac{1}{4} \theta [\gamma^\mu, \gamma^\nu] P_\theta \]  \hspace{1cm} (71)

is the part of the angular momentum which comes from the Grassmann part of superspace. We still have to impose constraints on \( S_{\mu\nu} \) in order to reduce the number of independent components.

### 5.2. Connection to the CBS superparticle

In order to regain the Casalbuoni-Brink-Schwarz superparticle we change the coordinates such that the spin operator fulfils the Pryce constraints and
assume a spinless Clifford vacuum ($\Sigma_{\mu\nu} = 0$). Comparing (67) to [1, 2] we require

$$\psi = P_\theta - i\gamma \cdot P\theta = 0$$

so the generator becomes

$$Q = P_\theta + i\gamma \cdot P\theta = 2i\gamma \cdot P\theta.$$  

(72)

The anticommutator $\{Q_\alpha, Q_\beta\} = 2(\gamma \cdot P\gamma^0)_{\alpha\beta}$ then leads to

$$\{\theta_\alpha, \theta_\beta\} = -\frac{1}{2P^2}(\gamma \cdot P\gamma^0)_{\alpha\beta}$$

(73)

and the spin operator (52) becomes

$$\hat{S}^{\mu\nu} = -\frac{i}{2}\bar{\theta}\gamma^{\mu\nu}\lambda P_\lambda.$$  

(74)

The action is obtained from (68) with $\Sigma = 0$ and with constraint terms added:

$$S = \int d\tau \left( P_\mu \dot{X}^\mu + \bar{P}_\theta \dot{\theta} - \bar{\lambda}(P_\theta - i\gamma \cdot P\theta) - e(P^2 + m^2) \right)$$

(75)

with $m$ a mass parameter and $e$ and $\lambda$ Lagrange multipliers. This is recognized as the action of the CBS superparticle in first order form, while eqs. (74) and (75) are identical to eqs. (55) and (56).

6. GAUGE THEORY OF THE SUPERPARTICLE

The results obtained so far can be summarized in the following way: Using the method of induced representations, we obtained in Eq. (56) the spin operator of the massive CBS superparticle, with the anticommutation relations (55) in the same form as in [1, 2]. In the massless case the CBS superparticle also emerged from the method of induced representations. Next it was shown how the superparticle also could be interpreted as moving on the super-Poincaré group manifold, when the set of constraints (72) is imposed upon the naive action of a free particle.

In this connection it is puzzling that the superparticle spin operator obeys Eq. (1). When determining the spin operator by the method of induced representations we had to carry out a redefinition according to Eq. (53) to obtain this relation, while it was ensured by the set of constraints
(72) when the superparticle was considered moving on the super Poincaré manifold.

In the case of a nonsupersymmetric point particle it is known that the arbitrariness in the constraints of the spin operator as reflected in the mutually exclusive conditions given in Eqs. (1)-(2) reflects a deep symmetry of a particle with spin (or an extended object) related to the arbitrariness of the relativistic center of mass [6]. Furthermore, it has been demonstrated that this symmetry can be formulated as a gauge symmetry such that Eq. (53) expresses a gauge transformation [7]. This raises the interesting possibility that an equivalent formulation of the CBS superparticle exists, such that it has the same physical contents but allows a gauge symmetry corresponding to the transformation (53). In this relation it should be mentioned that a gauge symmetry of the massless CBS superparticle was found by Siegel [3] and has given rise to an extensive literature on the covariant quantization problem for this theory (see [4] and references quoted there). What we shall determine below is a more general scheme containing Siegel’s gauge symmetry as a special case.

The constraints (72) are second class while the constraints of a gauge theory are first class. The task at hand consists of halving the number of these constraints in such a way that those remaining are first class and thus define a gauge theory. This gauge theory should reduce to the CBS superparticle in a particular gauge. The Dirac brackets of the super-Poincaré generators should be unaffected by the choice of gauge. The procedure of obtaining a gauge theory from a theory defined by second class constraints is known as gauge unfixing [9].

6.1. General framework

Gauge unfixing on the set of constraints of Eq. (72) is carried out by means of two projection operators $Y_{\pm}$, in such a way that one obtains the new constraints:

$$\psi_{\pm} = Y_{\pm} (P_{0} - i\gamma \cdot P_{\pm} \theta) = 0$$

(77)

where $P_{+} \neq P_{-}$, and only the constraints $\psi_{+}$ are kept, while $\psi_{-}$ are taken as gauge fixing conditions. The constraints $\psi_{+}$ should

1. be first class:

$$\{\psi_{+}, \psi_{+}\}_{PB} = 0,$$

(78)

2. have weakly vanishing Poisson brackets with the generators of the super Poincaré group in order to ensure the correct algebra of the generators
after Dirac quantization:

$$\{ M_{\mu\nu}, \psi^+ \}_P B \simeq 0, \quad \{ P_{\mu}, \psi^+ \}_P B \simeq 0, \quad \{ Q, \psi^+ \}_P B \simeq 0. \quad (79)$$

In order to obtain Eq. (79) one has to choose $P_+ = P$. On the other hand $P_-$ is arbitrary and a particular choice means fixing the gauge freedom. If one chooses the gauge $P_- = P$ the resulting model is thus identical to the CBS superparticle.

Having obtained a first class constraint algebra one can determine gauge transformations of a general variable $A$. The generator of an infinitesimal gauge transformation with gauge parameter $\lambda$ is

$$Q = \bar{\lambda} \psi^+, \quad (80)$$

and $A$ transforms according to

$$\delta A = \{ Q, A \}_P B. \quad (81)$$

Eq. (79) then implies that the generators $M_{\mu\nu}, P_{\mu}$ and $Q$ are gauge invariant.

Two sets of projection operators $Y_{\pm}$ are considered. Projection using chiral constraints is the simplest one in terms of the algebra involved, but the success relies on an antisymmetric $\gamma^{D+1}$. Hence this procedure is only useful in some dimensionalities e.g. four dimensions, but not ten dimensions. The other set of projection operators involves the free massless Dirac operator and works in any number of dimensions. However, these projection operators may be ill defined at $P^2 \rightarrow 0$ and hence cannot immediately be applied to massless particles.

On the mass shell the spin constraints should be given by [7]

$$S^{\mu\nu}(P + P_-)_\nu = 0 \quad (82)$$

which e.g. gives each of the sets of constraints (1)-(2) with the proper choice of $P_-$. The aim of this section is to show that there exists a choice of gauge where (82) reduce to the Wigner constraints (2) and where we obtain Dirac brackets (see [11] and references given there) involving position and spin operators consistent with the commutation relations obtained directly when the spin operator is assumed to obey the Wigner constraints (2) as per the induced representation theory. These commutators are given in Eqs. (A.24)-(A.25).
This brings us to the main point of our construction. The corresponding commutators obtained in [1, 2] and given in Eqs. (A.36)-(A.37) on the mass shell are those that apply when the spin operator obeys the Pryce constraints (1). Since gauges giving respectively the Pryce and the Wigner constraints have been determined it is concluded in analogy with the non-supersymmetric case that the two versions of the theory differ only by a gauge transformation.

6.2. Chiral projections in four dimensions

Consider first the case where the projection operators \( Y_\pm \) are the chiral projections such that

\[
Y_\pm = \frac{1 \pm \gamma^5}{2}.
\]  

(83)

In four dimensions each chirality separately has vanishing Poisson brackets:

\[
\{ \psi_+, \psi_+ \} = \{ \psi_-, \psi_- \} = 0.
\]  

(84)

For \( P_- \) we choose

\[
P_- = \bar{P} = (m, \vec{0}).
\]  

(85)

6.2.1. Dirac quantization

Dirac brackets of arbitrary variables \( A \) and \( B \) are

\[
\{ A, B \} = \{ A, B \}_{PB} - \{ A, (\psi_+)_\alpha \}_{PB} (C^\alpha)_{\beta}^\gamma \{ (\psi_-)_\beta, B \}_{PB}
\]

\[
- \{ A, (\psi_-)_\alpha \}_{PB} (C^-\alpha)_{\beta}^\gamma \{ (\psi_+)_\beta, B \}_{PB}
\]  

(86)

with

\[
(C^{-1})_{++} = ((C^{-1})_{++})^T = -\frac{i}{(P + \bar{P})^2} Y_+ \gamma^0 \gamma \cdot (P + \bar{P}).
\]  

(87)

The spin operator \( S^{\mu\nu} \) as given in (71) is according to the constraints (77) with the projection operators \( Y_\pm \) specified in (83):

\[
S^{\mu\nu} \simeq S_V^{\mu\nu} - A^\mu (P - \bar{P})\nu + A^\nu (P - \bar{P})\mu,
\]  

(88)

where

\[
S_V^{\mu\nu} \simeq -\frac{i}{4} \bar{\theta} \gamma^\mu \gamma^\nu \gamma \cdot \frac{P + \bar{P}}{2} \theta, \quad A^\mu = \frac{i}{4} \bar{\theta} \gamma^\mu \gamma^5 \theta
\]  

(89)
In the present case we get, using that $\theta$ is a Majorana spinor:

$$S^{\mu\nu}(P + \bar{P})_{\nu} = -A^\mu(P^2 - \bar{P}^2) + A \cdot (P + \bar{P})(P - \bar{P})^\mu$$  \hspace{1cm} (90)

that does not vanish even for $P^2 = \bar{P}^2$. Thus, to obtain constraints of the form (82) one has to modify $S^{\mu\nu}$. This modification amounts to

$$\hat{S}^{\mu\nu} = S^{\mu\nu} + 2(\bar{P}^\mu P^\nu - \bar{P}^\nu P^\mu) \frac{A \cdot (P + \bar{P})}{(P + \bar{P})^2}$$  \hspace{1cm} (91)

that only affects $S^{\mu\nu}$ for $\mu = 0$ or $\nu = 0$, and for which

$$\hat{S}^{\mu\nu}(P + \bar{P})_{\nu} = -\left( A^\mu - (P + \bar{P})^\mu \frac{A \cdot (P + \bar{P})}{(P + \bar{P})^2} \right) (P^2 - \bar{P}^2).$$  \hspace{1cm} (92)

Dirac brackets for the spin operator (91) are

$$\{ \hat{S}^{\mu\nu}, \hat{S}^{\lambda\rho} \}' = \Delta^{\mu\lambda} \hat{S}^{\nu\rho} - \Delta^{\nu\lambda} \hat{S}^{\mu\rho} + \Delta^{\nu\rho} \hat{S}^{\mu\lambda} - \Delta^{\mu\rho} \hat{S}^{\nu\lambda}$$  \hspace{1cm} (93)

with

$$\Delta^{\mu\nu} = \eta^{\mu\nu} - 2 \frac{P^\mu \bar{P}^\nu + P^\nu \bar{P}^\mu}{(P + \bar{P})^2}.$$  \hspace{1cm} (94)

Here was used

$$\{ A^\mu, A \cdot (P + \bar{P}) \}' = \{ S^{\mu\nu}, A \cdot (P + \bar{P}) \} = 0.$$  \hspace{1cm} (95)

Having redefined the spin operator $S^{\mu\nu}$ according to (91) we have to redefine the position operator $X^\mu$ also since the total Lorentz transformation generator $M_{\mu\nu}$ should be unmodified. In this way $\hat{X}^\mu$ is fixed (apart from a term proportional to $P^\mu$) to

$$\hat{X}^\mu = X^\mu - 2 \bar{P}^\mu \frac{A \cdot (P + \bar{P})}{(P + \bar{P})^2}.$$  \hspace{1cm} (96)

The redefinition (96) only affects the time component of the position operator. For the new position operator the following Dirac brackets are obtained:

$$\{ \hat{X}^\mu, \hat{X}^{\nu} \}' = 4(P^\mu P^{\nu} - \bar{P}^{\nu} P^\mu) \frac{A \cdot (P + \bar{P})}{((P + \bar{P})^2)^2}.$$  \hspace{1cm} (97)
\[
\{\hat{X}^\mu, \hat{S}^{\nu\lambda}\}^* = 2 \frac{P^\mu \hat{S}^{\lambda\mu} - \bar{P}^\lambda \hat{S}^{\nu\mu}}{(P + \bar{P})^2} - \frac{4P^\mu}{(P + \bar{P})^2} (\bar{P}^\lambda D^{\nu\rho} A_\rho - \bar{P}^\nu D^{\lambda\rho} A_\rho),
\]

For the spinorial coordinate we get
\[
\{\theta^\alpha, \bar{\theta}^\beta\}^* = -\frac{i}{(P + \bar{P})^2} (\gamma \cdot (P + \bar{P}) \gamma^0)^{\alpha\beta}.
\]

### 6.2.2. The mass shell constraint

The final Dirac brackets are obtained by the mass shell constraint
\[
\phi_1 = P^2 + m^2
\]
and the corresponding gauge condition
\[
\phi_2 = \bar{X}^0 - \tau
\]
in terms of which the final Dirac brackets of the variables \(A\) and \(B\) are
\[
\{A, B\}^* = \{A, B\}' - \{A, \phi_1\}' \frac{1}{2P^0} \{\phi_2, B\}' + \{A, \phi_2\}' \frac{1}{2P^0} \{\phi_1, B\}'.
\]

In this way one obtains the Dirac brackets equivalent to the commutators (A.24) and (A.25) obtained by the method of induced representations:
\[
\{\hat{X}^\mu, P^\nu\}^* = \eta^{\mu\nu} - \frac{P^\mu}{P^0} \eta^{0\nu},
\]
\[
\{\hat{X}^\mu, \hat{X}^\nu\}^* = 0,
\]
\[
\{\hat{X}^\mu, \hat{S}^{\nu\lambda}\}^* = \{\hat{X}^\mu, \hat{S}^{\nu\lambda}\}' - \frac{P^\mu}{P^0} \{\hat{X}^0, \hat{S}^{\nu\lambda}\}',
\]
whence
\[
\{\hat{X}^i, \hat{S}^{ijk}\}^* = 0,
\]
\[
\{\hat{X}^i, \hat{S}^{0ij}\}^* = \frac{1}{P^0 + m} \hat{S}^{ij} - \frac{P^i}{P^0 (P^0 + m)} \hat{S}^{0ij},
\]
and finally
\[
\{\hat{S}^{\mu\nu}, \hat{S}^{\lambda\rho}\}^* = \{\hat{S}^{\mu\nu}, \hat{S}^{\lambda\rho}\}'.
\]
6.3. Projection by the Dirac operator

Instead of the chiral projections in (83) consider the projection operators

\[ Y_{\pm} = \frac{1}{2} \left( 1 \pm \frac{i \gamma \cdot P}{\sqrt{P^2}} \right), \quad \bar{Y}_{\pm} = \frac{1}{2} \left( 1 \pm \frac{i \gamma \cdot \bar{P}}{\sqrt{P^2}} \right) \]

where \( Y_{\pm} \) is used to define the new constraints and \( \bar{Y}_{\pm} \) are used to simplify the following calculations. We specify \( P_- \) according to:

\[ P_- = \bar{P} = (-i \sqrt{P^2}, 0) \]  

Eq. (110) in contrast to Eq. (85) shows explicit dependence on the momentum operator \( P \). This is necessary in order to obtain the Dirac brackets algebra that after quantization leads to the commutation relations (A.24) and (A.25).

From Eqs. (71) and (77) one finds in this case

\[ S^{\mu\nu} \simeq \hat{S}^{\mu\nu} + \frac{1}{4 \sqrt{P^2}} (P^\mu \bar{P}^\nu - P^\nu \bar{P}^\mu) \bar{\theta} \theta \]

with

\[ \hat{S}^{\mu\nu} = -i \frac{1}{4 \sqrt{P^2}} \bar{\theta} \gamma^{\mu\nu\lambda\tau} \theta \bar{P}_\lambda \bar{P}_\tau. \]

6.3.1. Dirac quantization

After finding and inverting the constraint algebra one finds preliminary Dirac brackets for arbitrary variables \( A \) and \( B \):

\[ \{ A, B \}' = \{ A, B \} P_B + \frac{2i P^2}{(P \cdot (P + P))^2} \{ A, \psi_+ \} P_B \gamma^0 Y_+ \bar{\gamma} \cdot \bar{P} Y_+ \{ \psi_+, B \} P_B \]

\[ -\frac{\sqrt{P^2}}{P \cdot (P + P)} \left( \{ A, \psi_+ \} P_B \gamma^0 Y_- \{ \psi_-, B \} P_B - \{ A, \psi_- \} P_B \gamma^0 Y_+ \{ \psi_+, B \} P_B \right), \]

Dirac brackets for the spin operator are:

\[ \{ S^{\mu\nu}, S^{\rho\sigma} \}' - \{ S^{\mu\nu}, S^{\rho\sigma} \} = -\frac{P^\mu \bar{P}^\rho + \bar{P}^\mu P^\rho}{P \cdot (P + P)} \hat{S}^{\nu\sigma} \]

\[ -\eta^{\mu\rho} \frac{1}{4 \sqrt{P^2}} (P^\nu \bar{P}^\rho - P^\rho \bar{P}^\nu) \bar{\theta} \theta + \text{permutations}. \]
This result indicates that one should consider instead the modified spin operator \( \hat{S}^\mu_\nu \) for which:

\[
\{ \hat{S}^\mu_\nu, \hat{S}^\rho_\sigma \} = \{ S^\mu_\nu, S^\rho_\sigma \} = \Delta^{\mu\rho} \hat{S}^\nu_\sigma - \Delta^{\nu\rho} \hat{S}^\mu_\sigma + \Delta^{\nu\sigma} \hat{S}^\mu_\rho - \Delta^{\mu\sigma} \hat{S}^\nu_\rho
\]

(115)

where

\[
\Delta^{\mu\rho} = \eta^{\mu\rho} - \frac{P^\mu \bar{P}^\rho + \bar{P}^\mu P^\rho}{P \cdot (P + \bar{P})}
\]

(116)

which is identical to (94) on the mass shell. The redefinition of the spin operator must be accompanied by a redefinition of the position operator:

\[
\hat{X}^\mu = X^\mu - \frac{1}{4\sqrt{P^2}} \bar{\theta} \theta \bar{P}^\mu.
\]

(117)

The Dirac brackets involving the new position variable are

\[
\{ \hat{X}_\lambda, \hat{S}^{\rho\sigma} \} = \frac{1}{P \cdot (P + \bar{P})} \left( \eta^{\lambda\mu} + \frac{P^\lambda \bar{P}^\mu}{P^2} \right) \left( \hat{S}^\mu_\rho \bar{P}^\sigma - \hat{S}^\mu_\sigma \bar{P}^\rho \right).
\]

(118)

Finally, we find

\[
\{ \theta^\alpha, \theta^\beta \} = -\frac{P^2}{(P \cdot (P + \bar{P}))^2} \left( \left( i \gamma \cdot (P + \bar{P}) + \frac{[\gamma \cdot P, \gamma \cdot \bar{P}]}{2\sqrt{P^2}} \right) \gamma^0 \right)^{\alpha\beta}
\]

(120)

that is different from Eq. (99).

6.3.2. The mass shell constraint

The final Dirac brackets are found by the mass shell constraint along with a gauge fixing condition according to Eqs. (100), (101) and (102). For the position operator we find

\[
\{ \hat{X}^\mu, \hat{X}^\nu \}^* = 0
\]

(121)

and for the spin operator the brackets are unchanged and given by Eq. (115). For the spatial components of the spin operator

\[
\{ \hat{X}^\lambda, \hat{S}^{ij} \}^* = 0
\]

(122)
while for the remaining components of the spin operator we find
\[
\{ \hat{X}^\lambda, \hat{S}^{0j} \}^* = \frac{1}{P^0 + m} \left( \hat{S}^{0j} - \frac{P^\lambda}{P^0} \hat{S}^{0j} \right).
\]  
(123)

It is again seen that the Dirac brackets (115) and (123) are in accordance with the commutators (A.24)-(A.25).

6.3.3. Gauge Transformations

With Eq. (81) defining gauge transformations, where \( Y_\pm \) are fixed according to Eq. (109), the resulting gauge symmetry is a generalization of the local fermionic symmetry of the massless CBS superparticle discovered by Siegel [3].

The following gauge transformations are found by insertion into (80) and (81):
\[
\delta \theta = \bar{\lambda} Y_+ \theta, 
\]  
(124)
\[
\delta \hat{X}^\mu = -\bar{\lambda} Y_+ i\gamma^\mu \theta + \bar{\lambda} Y_+ Y_\mu - \frac{1}{2} (\gamma^\mu - P^\mu \frac{P}{P^2}) \theta - \frac{1}{2} P^\mu P \cdot \bar{P} \sqrt{P^2} \bar{\lambda} Y_+ \theta.
\]  
(125)

Introducing here \( \kappa = \frac{2}{\sqrt{P^2}} \lambda \), choosing the Pryce gauge condition \( \bar{P}^\mu = P^\mu \) and finally using the mass shell condition \( P^2 = 0 \) one regains the gauge symmetry of [3]. Here it should be noted that the position operator can always be redefined by addition of a term proportional to the momentum operator. This amounts to a gauge transformation generated by the mass shell constraint.

Another consequence of Eq. (125) follows in the special case
\[
\lambda = \theta \delta t
\]  
(126)
with \( \delta t \) an infinitesimal real parameter. In this case Eq. (125) reduces to
\[
\delta \hat{X}^\mu = \frac{i}{8 P^2} \delta t \bar{\theta} \gamma^{\mu\nu\lambda} \theta P_\nu \bar{P}_\lambda
\]  
(127)
where terms proportional to \( P^\mu \) are disregarded. Eq. (127) holds for any choice of \( \bar{P} \) and is the infinitesimal version of Eq. (A.28) (with \( \bar{P} \) specified in (110)) and of the Brink and Schwarz coordinate transformation formula.
(see [2] Eq. (21)) for $\bar{P} = P^-$. The finite version of Eq. (127) can be obtained through integration of infinitesimal gauge transformations.

Needless to say, similar considerations on gauge transformations can be made in the case where chiral constraints are used for gauge unfixing.

7. CONCLUSION AND OUTLOOK

Using the factorization of a general super-Poincaré transformation, which was carried out by means of the vielbein formalism, we have applied the method of induced representations to the $N = 1$ super-Poincaré group. By combining this with the Clifford vacuum method of Salam and Strathdee we have then shown that the Wigner constraints for the spin operator occur in a natural way. This allows one to find an explicit expression for the spin operator using only the structure relations of the super-Poincaré group, and the relation to the Casalbuoni-Brink-Schwarz superparticle is demonstrated.

Next a different analysis was performed. The superparticle was considered moving on the $N = 1$ super-Poincaré group manifold. By imposing the proper constraints the CBS particle is the result of this. By the use of projection operators half of the constraints could be selected to serve as the generators of gauge transformations, while the other half was considered fixing the gauge. It is immediately obvious how one should fix the gauge to recover the CBS superparticle where the spin operator obeys the Pryce constraints. Using Dirac quantization we then showed that for another gauge choice the resulting commutation relations corresponds to those expected if the Wigner constraints are valid. By analogy to similar calculations for the nonsupersymmetric case it is concluded that this is in fact a gauge theory where the gauge freedom corresponds to the choice of spin constraints or, equivalently, the free choice of relativistic center of mass. We also showed how in a special case the gauge symmetry reduces to the well known $\kappa$-symmetry.

One can imagine several interesting ways to generalize this work: A Clifford vacuum with nonzero spin, $N = 2$ supersymmetry, and strings and branes.

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APPENDIX

A.1. SUPER LIE GROUPS

Let $F^a[x, y]$ be the multiplication functional of a general supergroup [8]. Left- and right derivatives of $F$ are

$$
L_b^a[z] \equiv \left. \frac{\delta}{\delta x^b} F^a[x, z] \right|_{\bar{x}=1}, \quad R^a_b[z] \equiv \left. F^a[\bar{z}, y] \frac{\delta}{\delta y^b} \right|_{y=1}
$$

with the explicit representations

$$
(L^{-1})^b_a = \left( \frac{e^{C \bar{x}} - 1}{C \cdot \bar{x}} \right)^b_a = \int_0^1 dt \left( e^{tC \bar{x}} \right)^b_a, \quad (R^{-1})^b_a = \left( \frac{1 - e^{-C \bar{x}}}{C \cdot \bar{x}} \right)^b_a = \int_0^1 dt \left( e^{-tC \bar{x}} \right)^b_a
$$

where

$$(C \cdot x)^a_b = (-1)^{bc} x^c C^a_{cb} \quad \text{(A.4)}$$

Here $C^a_{cb}$ are the supergroup structure constants and $x^c$ are the canonical coordinates on the group manifold. The supertranspose is defined by

$$(M^a_b)^T = (-1)^{b(a+b)} M^a_b \quad \text{(A.5)}$$

We introduce the notation $u^a_b = (L^{-1})^a_b$ and $v^b_a = (R^{-1})^b_a$. The right vielbeins $v$ are obtained from the left vielbeins $u$ by the replacement $x^a \rightarrow -x^a$.

Poisson brackets are in the presence of Grassmann variables defined according to

$$\{A, B\}_P = \sum_{\alpha, \beta} (\frac{\partial}{\partial q^\alpha}) \Gamma^\alpha_{\beta \gamma} \left( \frac{\partial}{\partial p^\gamma} A \right) - (-1)^{\beta A} (\frac{\partial}{\partial q^\alpha}) \Gamma^\alpha_{\beta \gamma} \left( \frac{\partial}{\partial p^\gamma} B \right).$$

(A.6)

A derivative with respect to a coordinate is a right-derivative (denoted $\frac{\partial}{\partial q^\alpha}$) while a derivative with respect to a momentum is a left-derivative.

1In the sign factor $(-1)^a$ corresponding to the quantity $A$ one ascribes to $a$ the value 0 for $A$ an ordinary number and 1 for $A$ a Grassmann number.
(denoted $\frac{\partial}{\partial p_i}$). For Grassmann variables this distinction is important. The quantity $\Gamma^{\alpha \beta}$ is equal to $\delta^{\alpha \beta}$ if $\alpha, \beta$ refer to ordinary numbers while one must require

$$ (\Gamma^{\alpha \beta})^* = -\Gamma^{\alpha \beta} $$

if $\alpha, \beta$ refer to Grassmann variables. $\Gamma^{\alpha \beta}$ should also be nonsingular but is otherwise unrestricted.

With the definition of Poisson brackets given above the following projection formula leads to group generators $I_\alpha$:

$$ I_\alpha = \Pi_\beta (\Gamma^{-1})^\beta_\delta (u^{-1})^\delta_\gamma K^{\gamma \delta}_\alpha $$

with $K$ a nonsingular matrix that ensures that $I_\alpha$ is real if $\Pi_\alpha$ is real. The Cartan-Maurer equation is

$$ ((u^{-1})^\gamma_\delta, (u^{-1})^\delta_\alpha - (u^{-1})^\alpha_\beta (u^{-1})^\gamma_\delta, (u^{-1})^\delta_\alpha - (u^{-1})^\gamma_\delta, (u^{-1})^\delta_\alpha ) = C^{\delta \gamma}_{\alpha \beta} $$

(A.8)

where the derivatives are right derivatives, or equivalently

$$ (u^\gamma_\beta, -u^\delta_\alpha - (u^{-1})^\alpha_\beta C^{\gamma \delta}_{\alpha \beta} ) $$

(A.9)

The Cartan-Maurer equation leads to the structure relation of generators

$$ \{I_\alpha, I_\beta\} = I_\gamma (K^{-1})^\gamma_{\xi} C^{\xi}_{\gamma \delta} K^{\delta}_{\alpha} $$

(A.10)

where the structure constants $C^{\gamma}_{\alpha \beta}$ have the property

$$ C^{\gamma *}_{\alpha \beta} = (-1)^{\alpha \beta} C^{\gamma}_{\alpha \beta}. $$

(A.11)

### A.2. THE SUPER-POINCARÉ GROUP

For the $N = 1$ super-Poincaré group we take $\Gamma = K = \gamma^0$, with $\gamma^0$ a Dirac matrix from the Majorana representation constructed according to Eq. (3). By comparison of Eq. (A.10) with the structure relations (17) one obtains the following structure constants complementing those of Eqs. (5)-(6):

$$ C^\mu_{\alpha \beta} = -2i(\gamma^0 \gamma^\mu)_{\alpha \beta}, \quad C^{\beta \gamma}_{\mu \nu, \alpha} = -\frac{1}{4}\gamma^0 [\gamma_{\mu}, \gamma_{\nu}] \gamma^0 \gamma^\beta. $$

(A.12)
The nontrivial left vielbeins are:

\[ u^\lambda_\mu = \int_0^1 dt \ (e^{t C \cdot \lambda})^{\lambda}_\mu, \quad u_{\mu \nu} \lambda^\kappa = \int_0^1 dt \ (e^{t C \cdot \lambda})^{\lambda^\kappa}_{\mu \nu}, \quad u^\beta_\alpha = \int_0^1 dt \ (e^{t C \cdot \lambda})^{\beta}_\alpha \]  

(A.13)

as well as

\[ u_\beta^\lambda = \xi^\lambda \delta_{\alpha} \int_0^1 dt \int_0^t du \ (e^{u C \cdot \lambda})^{^\alpha}_{\beta} \]  

(A.14)

\[ u_{\mu \nu} = \frac{1}{2} \xi^\delta \ C_{\alpha, \kappa \tau} \int_0^1 dt \int_0^t du \ (e^{u C \cdot \lambda})^{^\alpha}_{\delta} (e^{t C \cdot \lambda})^{^\kappa \tau}_{\mu \nu} \]  

(A.15)

and

\[ u_{\mu \nu}^\lambda = \frac{1}{2} \alpha^\xi C^{\lambda}_{\alpha, \kappa \tau} \int_0^1 dt \int_0^t du \ (e^{u C \cdot \lambda})^{^\alpha}_{\xi} (e^{t C \cdot \lambda})^{^\kappa \tau}_{\mu \nu} \]  

\[ + \frac{1}{2} \int_0^1 dt \frac{1}{2} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \]  

\[ (e^{t \alpha_1 C \cdot \lambda})^{^\lambda}_{\sigma} (C \cdot \xi)^{^\sigma}_{\alpha} (e^{t \alpha_2 C \cdot \lambda})^{^\alpha}_{\gamma} (C \cdot \xi)^{^\gamma}_{\kappa \tau} (e^{t \alpha_3 C \cdot \lambda})^{^\kappa \tau}_{\mu \nu}. \]  

(A.16)

In the last expression one should remember the sign in the term \((C \cdot \xi)^{^\sigma}_{\alpha}\) specified according to (A.4).

From these expressions the corresponding right vielbeins \(v\) are obtained by a change of sign of the canonical coordinates. The following identity applies:

\[ \frac{1}{2} \xi^\alpha u^\rho_\alpha u^{-1}_\rho = \frac{1}{2} \xi^\alpha u^\rho_\alpha u^{-1}_\rho. \]  

(A.17)

The Poincaré group vielbeins are the first two of Eq. (A.13) as well as that of Eq. (A.16) where one should take \(\xi = 0\).

A.3. SPIN CONSTRAINTS

The space components of the spin operator \(S_{ij}\) obey the usual \(SO(D-1)\) algebra:

\[ [S_{ij}, S_{kl}] = i(\delta_{ik} S_{jl} + \delta_{il} S_{jk} - \delta_{jl} S_{ik} - \delta_{jk} S_{il}). \]  

(A.18)
while the position operator commutes with the spin components
\[ [X^i, S^{jk}] = 0. \tag{A.19} \]

The remaining components are determined by the Wigner constraints:
\[ S_0^i = \frac{1}{P^0 + m} P^j S^{ji} \tag{A.20} \]

and we keep the mass-shell condition:
\[ P^2 + m^2 = 0 \tag{A.21} \]

Then we compute:
\[ [S_0^i, S^{jk}] = i \frac{1}{P^0 + m} (P^j S^{ik} - P^k S^{ij}) + i(\delta^{ik} S^{0j} - \delta^{ij} S^{0k}) \tag{A.22} \]
\[ [S_0^i, S^{0j}] = i \frac{i}{(P^0 + m)^2} P^2 S^{ij} + i \frac{i}{P^0 + m} (P^j S^{0i} - P^i S^{0j}) \tag{A.23} \]

and
\[ [S_0^i, X^j] = i \frac{1}{P^0 + m} S^{ij} + i \frac{1}{P^0(P^0 + m)} P^j S^{0i}. \tag{A.24} \]

The commutation relations of the spin operator are summarized:
\[ [S^{\mu\nu}, S^{\lambda\rho}] = i(\Delta^{\mu\lambda} S^{\nu\rho} - \Delta^{\mu\rho} S^{\nu\lambda} + \Delta^{\nu\rho} S^{\mu\lambda} - \Delta^{\nu\lambda} S^{\mu\rho}) \tag{A.25} \]

with:
\[ \Delta^{\mu\nu} = \eta^{\mu\nu} - a^{\mu} P^\nu + a^{\nu} P^\mu, \tag{A.26} \]

that agrees with (94) on the mass shell, and where
\[ a^{\mu} = \eta^{\mu0}. \tag{A.27} \]

We can enforce the Pryce constraints by using new coordinate and spin variables \( \hat{X}^i \) and \( \hat{S}^{0j} \), defined through the relations:
\[ X^i - \hat{X}^i = \frac{1}{P^0} (S_0^i - \hat{S}^{0i}), \tag{A.28} \]
\[ S^{ij} - \hat{S}^{ij} = -(X^i - \hat{X}^i)P^j + (X^j - \hat{X}^j)P^i. \] (A.29)

These relations ensure that the Lorentz generators \( M^{\mu\nu} \) and therefore also their commutation relations are unchanged. The Pryce constraints are:

\[ \hat{S}^{0i} = \frac{1}{P^0} P^j \hat{S}^{ji}. \] (A.30)

Obviously:

\[ P^i (X^i - \hat{X}^i) = 0, \] (A.31)

so:

\[ P^i (S^{ij} - \hat{S}^{ij}) = \frac{P^2}{P^0} (S^{0j} - \hat{S}^{0j}) = (P^0 + m)S^{0j} - \hat{P}^0 \hat{S}^{0j}, \] (A.32)

i.e.:

\[ (P^0 + m)S^{0j} = m \hat{S}^{0j} \] (A.33)

and thus:

\[ X^i - \hat{X}^i = -\frac{1}{P^0 + m} \hat{S}^{0i}, \] (A.34)

\[ S^{ij} - \hat{S}^{ij} = \frac{1}{P^0 + m} (P^j \hat{S}^{0i} - P^i \hat{S}^{0j}). \] (A.35)

Commutation relations are from (A.33)-(A.35) combined with (A.24) and (A.25) (with \( \hat{X}^0 \) commuting with everything):

\[ [\hat{X}^\mu, \hat{X}^\nu] = \frac{i}{m^2} \hat{S}^{\mu\nu} - \frac{i}{m^2 P^0} (P^\mu \hat{S}^{0\nu} - P^\nu \hat{S}^{0\mu}), \] (A.36)

\[ [\hat{X}^\mu, \hat{S}^{\nu\lambda}] = -\frac{i P^\mu}{m^2 P^0} (P^\nu \hat{S}^{0\lambda} - P^\lambda \hat{S}^{0\nu}) + \frac{i}{m^2} (P^\nu \hat{S}^{\mu\lambda} - P^\lambda \hat{S}^{\mu\nu}). \] (A.37)

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