DUALITY INDUCED REFLECTIONS AND CPT

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Abstract

The linear particle-antiparticle conjugation $C$ and position space reflection $P$ as well as the antilinear time reflection $T$ are shown to be inducable by the selfduality of representations for the operation groups $SU(2)$, $SL(\mathbb{C}^2)$ and $\mathbb{R}$ for spin, Lorentz transformations and time translations resp. The definition of a colour compatible linear $\mathbb{CP}$-reflection for quarks as selfduality induced is impossible since triplet and antitriplet $SU(3)$-representations are not linearly equivalent.
1 Reflections

1.1 Reflections

A reflection will be defined to be an involution of a finite dimensional vector space $V$

$$ V \leftarrow^R V, \quad R \circ R = \text{id}_V \iff R = R^{-1} $$

i.e. a realization of the parity group $\mathbb{Z}_2 \equiv \{\pm 1\}$ in the $V$-bijections which is linear for a real space and may be linear or antilinear for a complex space

$$ R(v + w) = R(v) + R(w), \quad R(\alpha v) = \begin{cases} \alpha R(v) & \text{for } \alpha \in \mathbb{R} \text{ or } \mathbb{C} \quad (\text{linear}) \\ \bar{\alpha}R(v) & \text{for } \alpha \in \mathbb{C} \quad (\text{antilinear}) \end{cases} $$

An antilinear reflection for a complex space $V \cong \mathbb{C}^n$ is a real linear one for its real forms $V \cong \mathbb{R}^{2n}$.

The inversion of the real numbers $\alpha \leftrightarrow -\alpha$ is the simplest nontrivial linear reflection, the canonical conjugation $\alpha \leftrightarrow \bar{\alpha}$ is the simplest nontrivial

\footnote{Since the parity group is used as multiplicative group, I do not use the additive notation $\mathbb{Z}_2 = \{0, 1\}$.}
antilinear one being a linear one of \( C \) considered as real 2-dimensional space \( C = \mathbb{R} \oplus i\mathbb{R} \).

Any (anti)linear isomorphism \( \iota : V \longrightarrow W \) of two vector spaces defines an (anti)linear reflection of the direct sum \( V \oplus W \xrightarrow{\iota \oplus \iota^{-1}} V \oplus W \) which will be denoted in short also by \( V \xrightarrow{\iota} W \).

### 1.2 Mirrors

The fixpoints of a linear reflection \( V_R^+ = \{ v \mid R(v) = v \} \), i.e. the elements with even parity, in an \( n \)-dimensional space constitute a vector subspace, the mirror for the reflection \( R \), with dimension \( 0 \leq m \leq n \) with the complement \( V_R^- = \{ v \mid R(v) = -v \} \), i.e. the elements with odd parity, for the direct decomposition \( V = V_R^+ \oplus V_R^- \). The central reflection \( R = -\text{id}_V \) has the origin as a 0-dimensional mirror. Linear reflections are diagonalizable \( R \sim \left( \begin{array}{cc} 1_m & 0 \\ 0 & -1_{n-m} \end{array} \right) \) with \( (m, n - m) \) the signature characterizing the degeneracy of \( \pm 1 \) in the spectrum of \( R \). And vice versa: Any direct decomposition \( V = V^+ \oplus V^- \) defines two reflections with the mirror either \( V^+ \) or \( V^- \).

With \( (\det R)^2 = 1 \) any linear reflection has either a positive or a negative orientation. Looking in the 2-dimensional bathroom mirror is formalized by the negatively oriented 3-space reflection \((x, y, z) \leftrightarrow (-x, y, z)\). The position space \( \mathbb{R}^3 \) reflection \( \vec{x} \xleftarrow{13} -\vec{x} \) with negative orientation or the Minkowski spacetime translation \( \mathbb{R}^4 \) reflection \( x \xleftarrow{14} -x \) with positive orientation are central reflections with the origins ‘here’ and ‘here-now’ as point mirrors. A space reflection \((x_0, \vec{x}) \xleftarrow{p} (x_0, -\vec{x})\) in Minkowski space or a time reflection \((x_0, \vec{x}) \xleftarrow{T} (-x_0, \vec{x})\) have both negative orientation with a 1-dimensional time and 3-dimensional position space mirror resp.

### 1.3 Reflections in Orthogonal Groups

A real linear reflection \( R \cong \left( \begin{array}{cc} 1_m & 0 \\ 0 & -1_{n-m} \end{array} \right) \) can be considered to be an element of an orthogonal group \( \mathbf{O}(p, q) \) for any \( 3 \mid (p, q) \) with \( p + q = n \). A positively oriented reflection, \( \det R = 1 \), is is an element even of the special orthogonal groups, \( R \in \mathbf{SO}(p, q), p + q \geq 1 \).

Orthogonal groups have discrete (semi)direct factor parity subgroups \( \mathbb{I}(2) \) as seen in the simplest compact and noncompact examples

\[
\mathbf{O}(2) \ni \epsilon \left( \begin{array}{cc} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{array} \right), \quad \epsilon \in \mathbb{I}(2) = \{ \pm 1 \}, \quad \alpha \in [0, 2\pi[ \\
\mathbf{O}(1, 1) \ni \epsilon' \left( \begin{array}{cc} \cosh \beta & \sinh \beta \\ \sinh \beta & \cosh \beta \end{array} \right), \quad \epsilon, \epsilon' \in \mathbb{I}(2), \quad \beta \in \mathbb{R}
\]

In general, the classes of a real orthogonal groups with respect to its special normal subgroup constitute a reflection group

\[ \mathbf{O}(p, q)/\mathbf{SO}(p, q) \cong \mathbb{I}(2) \]

For real odd dimensional spaces \( V \), e.g. for position space \( \mathbb{R}^3 \), one has direct products of the special groups with the central reflection group, whereas for

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\[3\]The orthogonal signature \((p, q)\) has nothing to do with the reflection signature \((n, m)\).
even dimensional spaces, e.g. a Minkowski space $\mathbb{R}^4$, there arise semidirect products (denoted by $\rtimes$) of the special group with a reflection group which can be generated by any negatively oriented reflection

\[
O(p, q) \cong \begin{cases} 
\mathbb{I}(2) \times SO(p, q), & p + q = 1, 3, \ldots \\
\mathbb{I}(2) \rtimes SO(p, q), & p + q = 2, 4, \ldots 
\end{cases} \quad \mathbb{I}(2) \cong \{ \pm \text{id}_V \}
\]

In the semidirect case the product is given as follows

\[(I, \Lambda) \in \mathbb{I}(2) \rtimes SO(p, q) \Rightarrow (I_1, \Lambda_1)(I_2, \Lambda_2) = (I_1 \circ I_2, \Lambda_1 \circ I_1 \circ \Lambda_2 \circ I_1)\]

Obviously, in the semidirect case the reflection group $\mathbb{I}(2)$ is not compatible with the action of the (special) orthogonal group.

\[p + q = 2, 4, \ldots, \det R = -1 \Rightarrow [R, SO(p, q)] \neq \{0\}\]

E.g. the group $O(2)$ is nonabelian, or, a space reflection and a time reflection of Minkowski space is not Lorentz group $SO(1, 3)$ compatible.

For noncompact orthogonal groups there is another discrete reflection group: The connected subgroup $G_0$ (unit connection component and Lie algebra exponent) of a Lie group $G$ is normal with a discrete quotient group $G/G_0$. The connected components of the full orthogonal groups are those of the special groups $O_0(p, q) = SO_0(p, q)$. For the compact case they are the special groups, for the noncompact ones one has two components

\[SO_0(n) = SO(n) \quad \text{if } pq \geq 1 \Rightarrow SO(p, q)/SO_0(p, q) \cong \mathbb{I}(2)\]

Summarizing: A compact orthogonal group gives rise to a reflection group $\mathbb{I}(2)$

\[O(n) \cong \begin{cases} 
\{ \pm 1_n \} \times SO(n), & n = 1, 3, \ldots \\
\mathbb{I}(2) \rtimes SO(n), & n = 2, 4, \ldots 
\end{cases} \quad \mathbb{I}(2) \cong \{ R, 1_n \}, \quad \det R = -1
\]

a noncompact one to a reflection Klein group $\mathbb{I}(2) \times \mathbb{I}(2)$

\[O(p, q) \cong \begin{cases} 
\{ \pm 1_{p+q} \} \times \mathbb{I}(2) \rtimes SO_0(p, q), & p + q = 3, 5, \ldots \\
\mathbb{I}(2) \rtimes \{ [\pm 1_{p+q}] \times SO_0(p, q) \}, & p + q = 2, 4, \ldots
\end{cases} \quad \mathbb{I}(2) \cong \{ R, 1_n \}, \quad \det R = -1
\]

For a noncompact $O(p, q)$ with $p = 1$ the connected subgroup is the orthochronous group, compatible with the order on the vector space $V \cong \mathbb{R}^{1+q}$, e.g. for Minkowski spacetime

\[O(1, 3) \cong \mathbb{I}(2) \rtimes \mathbb{I}(2) \rtimes SO_0(1, 3)\]

where the reflection Klein group can be generated by the central reflection $-1_4$ and a position space reflection $P$

\[\mathbb{I}(2) \times \mathbb{I}(2) \cong \{ P, 1_4 \} \times \{ \pm 1_4 \} = \{ \pm 1_4, P, T = -P \}, \quad [SO_0(1, 3), P] \neq \{0\}
\]

\[P = \begin{pmatrix} 1 & 0 \\ 0 & -1_3 \end{pmatrix}, \quad T = -1_4 \circ P = \begin{pmatrix} -1 & 0 \\ 0 & 1_3 \end{pmatrix}\]
Also the connected subgroup $\text{SO}_0(p, q)$ may contain positively oriented reflections which are called continuous since they can be written as exponentials $R = e^l$ with an element of the orthogonal Lie algebra $l \in \log \text{SO}_0(p, q)$. E.g. the central reflections $-1_{2n} \in \text{SO}(2n)$ in even dimensional Euclidean spaces, e.g. in the Euclidean 2-plane. A negatively oriented reflection $R$ of a space $V$ can be embedded as a reflection $R \oplus S$ with any orientation of a strictly higher dimensional space $V \oplus W$

$$V \xleftarrow{R} V, \quad \det R = -1$$
$$V \oplus W \xleftarrow{R \oplus S} V \oplus W, \quad \det (R \oplus S) = -\det S$$

where, for compact orthogonal groups on $V$ and $V \oplus W$, a reflection $R \oplus S$ with $\det S = -1$ is a continuous reflection, i.e. a rotation. There are the familiar examples$^2$ for $\text{O}(n) \hookrightarrow \text{SO}(n + 1)$: Two letter noodles in $L$-form, lying with opposite helicity on the kitchen table, can be 3-space rotated into each other, or, a left and a right handed glove are identical up to Euclidean 4-space rotations. The embedding of the central position space reflection into Minkowski spacetime can go into a positively or negatively oriented reflection which are both not continuous, i.e. they are in the discrete Klein reflection group

$$-1_3 \hookrightarrow \left( \begin{array}{cc} 1 & 0 \\ 0 & -1_3 \end{array} \right), \quad \{0, -1_4\} \subset \text{O}(1, 3)/\text{SO}_0(1, 3)$$

## 2 Reflections for Spinors

The doubly connected groups $\text{SO}(3)$ and $\text{SO}_0(1, 3)$ can be complex represented via their simply connected covering groups $\text{SU}(2)$ and $^5\text{SL}(\mathbb{C}^2)$ resp.

$$\text{SO}(3) \cong \text{SU}(2)/\{\pm 1_2\}, \quad \text{SO}_0(1, 3) \cong \text{SL}(\mathbb{C}^2)/\{\pm 1_2\},$$

The reflection group $\{\pm 1_2\}$ for the $\text{SO}(3)$-classes in $\text{SU}(2)$ and the $\text{SO}_0(1, 3)$-classes in $\text{SL}(\mathbb{C}^2)$ contains the continuous central $\mathbb{C}^2$-reflection $-1_2 = e^{i\pi\sigma_3} \in \text{SU}(2)$.

### 2.1 The Pauli Spinor Reflection

The fundamental defining $\text{SU}(2)$-representation for the rotations acts on Pauli spinors $W \cong \mathbb{C}^2$

$$u = e^{i\vec{a} \vec{\sigma}} \in \text{SU}(2) \quad (\text{Pauli matrices} \ \vec{\sigma})$$

They have an invariant antisymmetric bilinear form (spinor ‘metric’)

$$\epsilon : W \times W \rightarrow \mathbb{C}, \quad \epsilon(\psi^A, \psi^B) = \epsilon^{AB} = -\epsilon^{BA}, \quad A, B = 1, 2$$

$^4\log G$ denotes the Lie algebra of the Lie group $G$.

$^5$Throughout this paper the group $\text{SL}(\mathbb{C}^2)$ is used as real 6-dimensional Lie group.
which defines an isomorphism with the dual space $W^T \cong \mathbb{C}^2$ is compatible with the $SU(2)$-action - on the dual space as dual representation $\tilde{u}$ (inverse transposed)

$$
\begin{array}{c}
W \xrightarrow{u} W \\
W^T \xrightarrow{\tilde{u}} W^T
\end{array}
$$

$\epsilon$ connects the two Pauli representations with reflected transformations of the spin Lie algebra $\log SU(2)$, i.e. it defines a central reflection for the three compact rotation parameters $\vec{\alpha}$

$$
e^{i\vec{\alpha}\vec{\sigma}} \leftrightarrow \epsilon (e^{-i\vec{\alpha}\vec{\sigma}})^T$$

and will be called the **Pauli spinor reflection**

$$W \xleftrightarrow{\epsilon} W^T, \ \psi^A \leftrightarrow \epsilon^{AB}\psi^*_B, \ \ [\epsilon, SU(2)] = \{0\}$$

The mathematical structure of selfduality as a reflection generating mechanism is given in the appendix.

### 2.2 Reflections $C$ and $P$ for Weyl Spinors

The two fundamental $SL(\mathbb{C}^2)$-representations for the Lorentz group are the the left and right handed Weyl representation on $W_L, W_R \cong \mathbb{C}^2$ with the dual representations on the linear forms $W^T_L,R$

left: $\lambda = e^{(i\vec{\alpha}+\vec{\beta}\vec{\sigma})}$, right: $\tilde{\lambda} = \lambda^{-1*} = e^{(i\vec{\alpha}-\vec{\beta}\vec{\sigma})}$

left dual: $\tilde{\lambda} = \lambda^{-1T} = [e^{(-i\vec{\alpha}+\vec{\beta}\vec{\sigma})}]^T$, right dual: $\lambda^{T*} = \overline{\lambda} = [e^{(-i\vec{\alpha}+\vec{\beta}\vec{\sigma})}]^T$

The Weyl representations with dual bases in the conventional notations with dotted and undotted indices

$$
\begin{array}{c}
\text{left: } l^A \in W_L \cong \mathbb{C}^2, \ \text{right: } r^A \in W_R \cong \mathbb{C}^2 \\
\text{left dual: } r^*_A \in W^*_L \cong \mathbb{C}^2, \ \text{right dual: } l^*_A \in W^*_R \cong \mathbb{C}^2
\end{array}
$$

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6The linear forms $V^T$ of a vector space $V$ define the dual product $V^T \times V \rightarrow \mathbb{C}$ by $\langle \omega, v \rangle = \omega(v)$ and dual bases by $\langle e_j, e^k \rangle = \delta^k_j$. Transposed mappings $f : V \rightarrow W$ are denoted by $f^T : W^T \rightarrow V^T$ with $(f^T(\omega), v) = \langle \omega, f(v) \rangle$.

7The usual strange looking crossover association of the letters $l^*$ and $r^*$ for right and left handed dual spinors resp. will be discussed later.
are selfdual with the $\text{SL}(\mathbb{C}^2)$-invariant volume form on $\mathbb{C}^2$, i.e. the dual isomorphisms are Lorentz compatible

\[
\begin{array}{ccc}
W_L & \xrightarrow{\lambda} & W_L \\
\epsilon_L & & \epsilon_L \\
W'_L & \xrightarrow{\lambda} & W'_L \\
\end{array}
\quad
\begin{array}{ccc}
W_R & \xrightarrow{\lambda} & W_R \\
\epsilon_R & & \epsilon_R \\
W'_R & \xrightarrow{\lambda} & W'_R \\
\end{array}
\]

For the Lorentz group the spinor ‘metric’ will prove to be related to the particle-antiparticle conjugation, and will be called Weyl spinor reflection, denoted by $C \in \{\epsilon_L, \epsilon_R\}$

\[
\begin{align*}
W_L & \xleftarrow{\epsilon} W'_L, & l^A & \leftrightarrow \epsilon^{AB} r^*_B \\
W_R & \xleftarrow{\epsilon} W'_R, & r^A & \leftrightarrow \epsilon^{AB} l^*_B \\
\end{align*}
\]

There exist isomorphisms $\delta$ between left and right handed Weyl spinors, compatible with the spin group action, however not with the Lorentz group $\text{SL}(\mathbb{C}^2)$

\[
\begin{array}{ccc}
W_L & \xrightarrow{u_L} & W_L \\
\delta & & \delta \\
W_R & \xrightarrow{u_R} & W_R \\
\end{array}
\quad
\begin{array}{ccc}
W_L & \xrightarrow{u_L} & W_L \\
\delta & & \delta \\
W_R & \xrightarrow{u_R} & W_R \\
\end{array}
\]

They connect representations with a reflected boost transformation, i.e. they define a central reflection for the three noncompact boost parameters $\tilde{\beta}$

\[
e^{(i\tilde{\alpha}+\tilde{\beta})\bar{\sigma}} \leftarrow \delta \rightarrow e^{(i\tilde{\alpha}-\tilde{\beta})\bar{\sigma}} \\
\tilde{\sigma}\tilde{\beta} \in \log \text{SL}(\mathbb{C}^2)/\log \text{SU}(2) \cong \mathbb{R}^3, \quad \tilde{\beta} \xrightarrow{\delta} -\tilde{\beta}
\]

These isomorphisms induce nontrivial reflections of the Dirac spinors $\Psi \in W_L \oplus W_R \cong \mathbb{C}^4$

\[
\Psi = \begin{pmatrix} l^A \\ r^A \end{pmatrix} \xleftarrow{\delta} \begin{pmatrix} 0 \\ \delta_B^A \end{pmatrix} \begin{pmatrix} l^B \\ r^B \end{pmatrix} = \gamma^0 \Psi
\]

with the chiral representation of the Dirac matrices

\[
\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \tilde{\sigma}^j & 0 \end{pmatrix}, \quad \sigma^j = (1_2, \tilde{\sigma}), \quad \tilde{\sigma}^j = (1_2, -\tilde{\sigma})
\]

and will be called Weyl spinor boost reflections $P = \delta$, later used for the central position space reflection representation

\[
\begin{align*}
W_L & \xleftarrow{P} W_R, & l^A & \leftrightarrow \delta^A_{\tilde{\alpha}} r^A \\
W'_L & \xleftarrow{P} W'_R, & r^*_A & \leftrightarrow \delta^A_{\tilde{\alpha}} l^*_A \\
\end{align*}
\]
Therewith all four Weyl spinor spaces are connected to each other by linear reflections

\[
\begin{align*}
W_L \xleftarrow{p} & W_R, \\
W_L^T \xleftarrow{c} & W_R^T,
\end{align*}
\]

\[[P, SL(\mathbb{C}^2)] \neq \{0\}, \quad [P, SU(2)] = \{0\}\]

\[[C, SL(\mathbb{C}^2)] = \{0\}\]

\[
3 \text{ Time Reflection}
\]

The time representations define the antilinear reflection \(T\) for time translation. The different duality with respect to \(SL(\mathbb{C}^2)\) and Lorentz group representations, on the one side, and time representations, on the other side, leads to the nontrivial \(C, P, T\) cooperation.

### 3.1 Reflection \(T\) of Time Translations

The irreducible time representations, familiar from the quantum mechanical harmonic oscillator with time action eigenvalue (frequency) \(\omega\), with their duals (inverse transposed) are complex 1-dimensional

\[
t \mapsto e^{i\omega t} \in GL(U), \quad t \mapsto e^{-i\omega t} \in GL(U^T), \quad U \cong \mathbb{C} \cong U^T
\]

They are selfdual (equivalent) with an antilinear dual isomorphism which is the \(U(1)\)-conjugation for a dual basis \(u \in U, \ u^* \in U^T\)

\[
\begin{array}{ccc}
U & \xrightarrow{e^{i\omega t}} & U \\
U^T & \xrightarrow{e^{-i\omega t}} & U^T \\
\end{array}
\]

\(*, \ u \leftrightarrow u^*\)

The antilinear isomorphism \(*\) defines a scalar product which gives rise to the quantum mechanical probability amplitudes (Fock state for the harmonic oscillator)

\[
U \times U \rightarrow \mathbb{C}, \quad \langle u | u \rangle = \langle u^*, u \rangle = 1
\]

and defines the time reflection \(T = *\) for the time translations

\[
e^{i\omega t} \xleftarrow{T} e^{-i\omega t}, \quad t \xleftarrow{T} -t
\]
3.2 Lorentz Duality versus Time Duality

As anticipated in the conventional, on first sight strange looking dual Weyl spinor notation, e.g. \( l \in W_L \) and \( l^* \in W^T_R \), the Weyl spinor spaces \( W_L, W^T_R \) with the dual left and right handed \( \text{SL}(\mathbb{C}^2) \)-representations are not the spaces with the dual time representations as exemplified in the harmonic analysis of the left and right handed components in a Dirac field

\[
\begin{align*}
1^A(x) &= \int \frac{d^3q}{(2\pi)^3} \ s\left( \frac{q}{m} \right)^A_C \ C e^{+ixq}u^C(q) + C e^{-ixq}a^C(q) \sqrt{2} \\
1^*_A(x) &= \int \frac{d^3q}{(2\pi)^3} \ s^*\left( \frac{q}{m} \right)^C_A \ \sqrt{2} \\
r^A(x) &= \int \frac{d^3q}{(2\pi)^3} \ s^{-1}\left( \frac{q}{m} \right)^A_C \ C e^{-ixq}u^C(q) + C e^{ixq}a^C(q) \sqrt{2} \\
r^*_A(x) &= \int \frac{d^3q}{(2\pi)^3} \ s^{-1}\left( \frac{q}{m} \right)^C_A \ \sqrt{2} \\
s\left( \frac{q}{m} \right) &= \sqrt{q_0^2 + m^2} (1 + \frac{\vec{q} \cdot \vec{q}}{q_0 + m}) \quad q = (q_0, \vec{q}), \quad q_0 = \sqrt{m^2 + \vec{q}^2}
\end{align*}
\]

Here, \( s\left( \frac{q}{m} \right) \in \text{SL}(\mathbb{C}^2) \) is the Weyl representation of the boost from the rest system of the particle to a frame moving with velocity \( \frac{\vec{q}}{q_0} \) (solution of the Dirac equation), \( u^C \) and \( a^C \) are the creation operators for particle and antiparticles with spin \( \frac{1}{2} \) and opposite charge number \( \pm 1 \) and 3rd spin direction, e.g. for electron and positron, \( u^*_C \) and \( a^*_C \) are the corresponding annihilation operators.

\( \star \) denotes the time representation dual \( U \leftrightarrow U^* \), and \( T \) the Lorentz representation dual \( W \leftrightarrow W^T \) (with spinor indices up and down), i.e. for the four types of Weyl spinors

\[
\begin{align*}
1^A &\in W_L \quad \text{time dual} \\
1^*_A &\in W^T_R = W_L^* \quad \text{Lorentz dual} \\
r^*_A &\in W^T_L = W^*_R \quad \text{Lorentz dual} \\
r^A &\in W_R \quad \text{time dual}
\end{align*}
\]

Time representation duality does not coincide with Lorentz group representation duality.

The antilinear time reflection \( (U(1)-\text{conjugation}) \ T = \star \) is compatible with the action of the little group \( \text{SU}(2) \), not with the full Lorentz group

\[
\begin{align*}
W_L &\leftrightarrow W^T_R, \quad 1^A \leftrightarrow \delta^{A\dot{A}}_A 1^*_A \\
W_R &\leftrightarrow W^T_L, \quad r^A \leftrightarrow \delta^{A\dot{A}}_A r^*_A \\
[\text{SU}(2), \text{SU}(2)] &= 0 \quad \text{not compatible}
\end{align*}
\]

3.3 The Cooperation of \( C, P, T \) in the Lorentz Group

It is useful to summarize the action of the linear Weyl spinor reflections \( C \) (particle-antiparticle conjugation) and \( P \) (position space central reflection) and
the antilinear time reflection $T$ in the two types of commuting diagrams

\[
\begin{align*}
\text{with } \mathcal{C}, \mathbf{SL}(&\mathbb{C}^2) = \{0\}, \quad [\mathbf{P} \text{ and } T, \mathbf{SL}(&\mathbb{C}^2)] \neq \{0\}, \quad [\mathbf{P} \text{ and } T, \mathbf{SU}(2)] = \{0\} \\
&[\mathcal{C}, \mathbf{P}] = 0, \quad [\mathcal{C}, T] = 0, \quad [\mathbf{P}, T] = 0
\end{align*}
\]

The product $\mathbf{CPT}$ is an antilinear reflection of each Weyl spinor space, e.g. for the left handed spinors

\[
\begin{align*}
\mathbf{CPT} \sim &\delta^A_{\dot{A}} \delta_{\dot{B}B}^B \mapsto \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} \in \mathbf{SO}(3), \quad (x, y, z) \leftrightarrow (-x, y, -z)
\end{align*}
\]

The fact that the antilinear $\mathbf{CPT}$-reflection is - up to a number conjugation (indicated by overlining) - an element of $\mathbf{SL}(&\mathbb{C}^2)$, covering the connected Lorentz group $\mathbf{SO}_0(1, 3)$, is decisive for the proof of the well known $\mathbf{CPT}$-theorem\[4,3\]

\[
\mathbf{CPT} \in \mathbf{SL}(&\mathbb{C}^2)
\]

**4 Spinor Induced Reflections**

The linear spinor reflections $\epsilon$ for Pauli spinors and $\mathcal{C}, \mathbf{P}$ for Weyl spinors are inducable on all irreducible finite dimensional representations of $\mathbf{SU}(2)$ and
\( \text{SL}(\mathbb{C}^2) \) with their adjoint groups \( \text{SO}(3) \) and \( \text{SO}_0(1,3) \) resp. via the general procedure: Given the group \( G \) action on two vector spaces its tensor product representation reads
\[
G \times (V_1 \otimes V_2) \rightarrow V_1 \otimes V_2, \quad g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2)
\]
A realization of the simple reflection group \( \mathbb{I}(2) = \{ \pm 1 \} \) is either faithful or trivial.

### 4.1 Spinor Induced Reflection of Position Space

The reflection \( W \xleftarrow{\epsilon} W^T \) for a Pauli spinor space \( W \cong \mathbb{C}^2 \) induces the central reflection of position space whose elements come - in the Pauli representation of position space - as traceless hermitian \( (2 \times 2) \)-matrices
\[
\vec{x} : W \rightarrow W, \quad \text{tr} \, \vec{x} = 0, \quad \vec{x} = \vec{x}^* = \left( \begin{array}{cc} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_3 \end{array} \right)
\]
i.e. as elements\(^8\) of the tensor product \( W \otimes W^T \) with the induced \( \epsilon \)-reflection
\[
-\vec{\sigma} = \epsilon^{-1} \circ \sigma^T \circ \epsilon \Rightarrow \vec{x} \xleftarrow{\epsilon} \epsilon^{-1} \circ \vec{x}^T \circ \epsilon = -\vec{x}
\]
In the Cartan representation the Minkowski spacetime translations are hermitian mappings from right handed to left handed spinors
\[
x : W_R \rightarrow W_L, \quad x = x^* = \left( \begin{array}{cc} x_0 + x_1 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{array} \right)
\]
i.e. tensors in the product \( W_L \otimes W_R^T \). The linear \( \mathbb{C}P \)-reflection for Weyl spinors
\[
W_L \xleftarrow{\mathbb{C}P} W_R^T, \quad W_R \xleftarrow{\mathbb{C}P} W_L^T
\]
induces the position space reflection of Minkowski spacetime
\[
\sigma^j = (1_2, \vec{\sigma}), \quad \epsilon^{-1} \circ (\sigma^j)^T \circ \epsilon = \sigma^j = (1_2 - \vec{\sigma})
\]
\[
x \cong (x_0, \vec{x}) \xleftarrow{\mathbb{C}P} \epsilon^{-1} \circ x^T \circ \epsilon = \left( \begin{array}{cc} x_0 - x_3 & -x_1 + ix_2 \\ -x_1 - ix_2 & x_0 + x_3 \end{array} \right) \cong (x_0, -\vec{x})
\]

### 4.2 Induced Reflections of Spin Representation Spaces

All irreducible complex representations of the spin group \( \text{SU}(2) \) with \( 2J = 0, 1, 2, \ldots \) have an invariant bilinear form arising as a symmetric tensor product of the antisymmetric spinor ‘metric’ \( \epsilon \). The bilinear form is given for the irreducible representation \( [2J] \cong \bigvee^u W \) on the vector space \( \bigvee^u W \cong \mathbb{C}^{2J+1} \) by the corresponding totally symmetric\(^9\) power and is antisymmetric for halfinteger spin and symmetric for integer spin
\[
\epsilon^{2J} = \bigvee^2 \epsilon, \quad \epsilon^{2J}(v,w) = \begin{cases} +\epsilon^{2J}(w,v), & 2J = 0, 2, 4 \ldots \\ -\epsilon^{2J}(w,v), & 2J = 1, 3, \ldots \end{cases}
\]
\(^8\)The linear mappings \( \{ V \rightarrow W \} \) for finite dimensional vector spaces are naturally isomorphic to the tensor product \( W \otimes V^T \) with the linear \( V \)-forms \( V^T \).
\(^9\)\( \bigvee \) and \( \bigwedge \) denotes symmetrized and antisymmetrized tensor products.
The complex representation spaces for integer spin \( J = 0, 1, \ldots \), acted upon faithfully only with the special rotations \( \text{SO}(3) \cong \text{SU}(2)/\{\pm 1\} \), are direct sums of two irreducible real \( \text{SO}(3) \)-representation spaces \( \mathbb{R}^{2J+1} \) where the invariant bilinear form is symmetric and definite, e.g. the negative definite Killing form \(-\mathbf{1}_3\) for the adjoint representation \([2] \cong u \lor u\) on \( \mathbb{R}^3 \).

The Pauli spinor reflection induces the reflections for the irreducible spin representation spaces

\[
V \cong \sqrt{2W} \cong \Phi^{2J+1} : \quad V \xleftarrow{\epsilon^{2J}} V^T
\]

For integer spin (odd dimensional representation spaces) the two real subspaces with irreducible real \( \text{SO}(3) \)-representation come with a trivial \(-\mathbf{1}_3\mapsto \mathbf{1}_3\) and a faithful \(-\mathbf{1}_3 \mapsto -\mathbf{1}_3 \in \text{O}(2J+1)/\text{SO}(2J+1)\) representation of the central position space reflection, as seen in the diagonalization of the induced reflection

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\cong
\begin{pmatrix}
1 & 0 \\
0 & -1 \\
\end{pmatrix}, \quad J = 0
\]

\[
\begin{pmatrix}
0 & \epsilon \\
\epsilon & 0 \\
\end{pmatrix}, \quad J = \frac{1}{2}
\]

\[
\begin{pmatrix}
0 & -1_3 \\
-1_3 & 0 \\
\end{pmatrix}
\cong
\begin{pmatrix}
1_3 & 0 \\
0 & -1_3 \\
\end{pmatrix}, \quad J = 1
\]

etc.

The decomposition for the integer spin representation spaces uses symmetric and antisymmetric tensor products as illustrated for the scalar and vector spin representation with a Pauli spinor basis

\[
W \xleftarrow{\epsilon} W^T,
\]

\[
W^T \otimes W \xleftarrow{\epsilon} W \otimes W^T,
\]

\[
\begin{aligned}
\psi^A &\leftrightarrow \epsilon^{AB}\psi^*_B, & J = \frac{1}{2} \\
\psi^*_A \otimes \psi^A &\leftrightarrow \psi^A \otimes \psi^*_A, & J = 0 \\
\bar{\sigma}^A_B\psi^*_A \otimes \psi^B &\leftrightarrow -\bar{\sigma}^A_B\psi^B \otimes \psi^*_A, & J = 1
\end{aligned}
\]

Writing for the tensor (anti)commutator \( [a, b]_\epsilon = a \otimes b + \epsilon b \otimes a \) with \( \epsilon = \pm 1 \) one has in both cases one trivial and one faithful reflection representation

\[
\begin{aligned}
[\psi^*_A, \psi^A]_\epsilon \leftrightarrow \epsilon[\psi^*_A, \psi^A]_\epsilon, & \quad J = 0 \\
[\psi^*_A \bar{\sigma}^A_B, \psi^B]_\epsilon \leftrightarrow -\epsilon[\psi^*_A \bar{\sigma}^A_B, \psi^B]_\epsilon, & \quad J = 1
\end{aligned}
\]

\[4.3\] Induced Reflections of Lorentz Group Representation Spaces

The generating structure of the two Weyl representations induces \( C, P \)-reflections of \( \text{SL}(\mathbb{C}^2) \)-representation spaces.

The complex finite dimensional irreducible representations of the group \( \text{SL}(\mathbb{C}^2) \) are characterized by two spins \([2L|2R]\) with integer and halfinteger
$L, R = 0, \frac{1}{2}, 1, \ldots$. They are equivalent to the totally symmetric products of the left and right handed Weyl representations

Weyl left: $[1|0] = \lambda = e^{(i\vec{a} + \vec{b})\vec{\sigma}}$, Weyl right: $[0|1] = \hat{\lambda} = e^{(i\vec{a} - \vec{b})\vec{\sigma}}$

$[2L|2R] \cong \sqrt{\lambda} \otimes \sqrt{\hat{\lambda}}$ acting on $V \cong \sqrt{W_L} \otimes \sqrt{W_R} \cong \mathbb{C}^{(2L+1)(2R+1)}$

$[2L|2R]$ and $[2R|2L]$ are equivalent with respect to the subgroup SU(2)-representations. The induced reflections are given by the corresponding products of the Weyl spinor reflections.

The real representation spaces for the Lorentz group $SO_0(1, 3)$ are characterized by integer spin $L + R = 0, 1, 2, \ldots$ They are all generated by the Minkowski representation $[1|1] \cong \lambda \otimes \bar{\lambda}$ where the complex 4-dimensional representation space is decomposable into two real 4-dimensional ones, a hermitian and an antihermitian tensor

$\mathbb{C}^4 \cong W_L \otimes W^T_R \ni l \otimes l^* = z = x + i\alpha \in \mathbb{R}^4 \oplus i\mathbb{R}^4$

With Weyl spinor bases the induced linear reflections for the Minkowski representation look as follows (with $\sigma^j = (1_2, \vec{\sigma}) = \vec{\sigma}$ and $\sigma_j = (1_2, -\vec{\sigma}) = \vec{\sigma}^j$)

$\sigma^j \leftarrow \vec{P} \rightarrow \vec{\sigma}^T_j$, $\text{I}^* \sigma^j \leftarrow \vec{P} \rightarrow \text{I}^* \vec{\sigma}^T_j$

$\sigma_j \leftarrow \vec{C} \rightarrow \vec{\sigma}^T_j$, $\text{I}^* \sigma_j \leftarrow \vec{C} \rightarrow \text{I}^* \vec{\sigma}^T_j$

$\sigma^j \leftarrow \text{CP} \rightarrow \sigma^T_j$, $\text{I}^* \sigma^j \leftarrow \text{CP} \rightarrow \text{I}^* \sigma^T_j$, $\text{I}^* \vec{\sigma}^j \leftarrow \text{CP} \rightarrow \text{I}^* \vec{\sigma}^T_j$

and can be arranged in combinations of definite parity, e.g. for $\mathbb{P}$ with Dirac spinors in a vector $\bar{\Psi} \gamma^j \Psi$ and an axial vector $\bar{\Psi} \gamma^j \gamma^5 \Psi$. The antilinear time reflection has to change in addition the order in the product

$\sigma^j \leftarrow \vec{T} \rightarrow \sigma_j$, $\text{I}^* \sigma^j \leftarrow \vec{C} \rightarrow \text{I}^* \sigma_j$, $\text{I}^* \vec{\sigma}^j \leftarrow \vec{P} \rightarrow \text{I}^* \vec{\sigma}_j$

4.4 Reflections of Spacetime Fields

A field $\Phi$ is a mapping from position space $\mathbb{R}^3$ or, as relativistic field, from Minkowski spacetime $\mathbb{R}^4$ with values in a complex vector space $V$ with the action of a group $G$ both on space(time) and on $V$. This defines the action of the group on the field $\Phi \mapsto g \cdot \Phi = g\Phi$ by the commutativity of the diagram

$$
\Phi \quad \begin{array}{c}
\mathbb{R}^3 \rightarrow \mathbb{R}^4 \\
V \rightarrow V
\end{array} \quad \begin{array}{c}
o(g) \\
d(g)
\end{array} \quad \begin{array}{c}
\mathbb{R}^3 \rightarrow \mathbb{R}^4 \\
V \rightarrow V
\end{array} \quad \begin{array}{c}
s_\Phi \\
g\Phi = D(g)\Phi(O(g^{-1}).x)
\end{array}
$$

for $g \in G$. 
For position space the external action group is the Euclidean group $O(3) \times \mathbb{R}^3$, for Minkowski spacetime the Poincaré group $O(1,3) \times \mathbb{R}^4$. The value space may have additional internal action groups, e.g. $U(1)$, $SU(2)$ and $SU(3)$ hypercharge, isospin and colour resp. in the standard model for quark and lepton fields.

For Pauli spinor fields on position space the $O(3)$-action has a direct $SU(2)$-factor and a reflection factor $\mathbb{I}(2)$

$$\psi : \mathbb{R}^3 \rightarrow W \cong \mathbb{C}^2, \quad \left\{ \begin{array}{ll}
u \psi(\vec{x}) = D(u)\psi(O(u^{-1}).\vec{x}), & u \in SU(2), \; O(u) \in SO(3) \\ \psi^A(\vec{x}) & \mapsto \epsilon^{AB}\psi^B(\vec{x}), \end{array} \right.$$

Position reflection $\mathbb{I}(2)$

Spacetime fields have the Lorentz group behaviour

$$\lambda \Phi(x) = D(\lambda).\Phi(O(\lambda^{-1}).x), \; \lambda \in SL(\mathbb{C}^2), \; O(\lambda) \in SO_0(1,3)$$

The antilinear time reflection uses the conjugation to the time dual field

$$\Phi(x_0, \vec{x}) \xrightarrow{T} \Phi^*(-x_0, \vec{x})$$

The reflections for Weyl spinor fields on Minkowski spacetime are

$$\begin{array}{cccc}
1^A & (x_0, \vec{x}) & \xrightarrow{P} & \delta^A_{\bar{A}} (x_0, -\vec{x}) \\
(l^A, r^\bar{A}) & (x_0, \vec{x}) & \xrightarrow{C} & (\epsilon^{AB}_{\bar{A}}\epsilon^{\bar{B}}_{\bar{B}}, \epsilon^{\bar{B}}_{\bar{B}}) (x_0, \vec{x}) \\
(l^A, r^\bar{A}) & (x_0, \vec{x}) & \xrightarrow{CP} & (\delta^A_{\bar{A}}\epsilon_{\bar{B}}^B, \epsilon_{\bar{A}}^B\epsilon_{\bar{B}}^A) (x_0, -\vec{x}) \\
(l^A, r^\bar{A}) & (x_0, \vec{x}) & \xrightarrow{T} & (\delta^A_{\bar{A}}\epsilon_{\bar{B}}^B, \delta^A_{\bar{A}}\epsilon_{\bar{B}}^A) (-x_0, \vec{x})
\end{array}$$

which is inducable on product representations.

5 The Standard Model Breakdown of $P$ and $CP$

A relativistic dynamics, characterized by a Lagrangian for the fields involved, may be invariant with respect to an operation group $G$, e.g. the $C$, $P$ and $T$ reflections, or not. A breakdown of the symmetry can occur in two different ways: Either the symmetry is represented on the field value space $V$, but the Lagrangian is not $G$-invariant, or there does not even exist a $G$-representation on $V$. Both cases occur in the standard model for quark and lepton fields.

5.1 Standard Model Breakdown of $P$

The charge $U(1)$ vertex in electrodynamics for a Dirac electron-positron field $\Psi$ interacting with an electromagnetic gauge field $\Gamma_j$

$$-\Gamma_j \bar{\Psi} \gamma^i \gamma^j \Psi = -\Gamma_j (l^* \sigma^j l + r^* \bar{\sigma}^j \bar{r})$$

is invariant under $P$ and $T$ if the fields have the Weyl spinor induced behaviour given above.
In the standard model of leptons with a left handed isospin doublet field $L$ and a right handed isospin singlet field $r$ the hypercharge $U(1)$ and isospin $SU(2)$ vertex with gauge fields $A_j$ and $\vec{B}_j$ resp. and internal Pauli matrices $\vec{\tau}$ reads

$$-A_j(L^\dagger \sigma^j \frac{1}{2} L + r^\dagger \vec{\sigma}^j r) + \vec{B}_j L^\dagger \sigma^j \frac{\vec{\tau}}{2} L$$

All gauge fields are assumed with the spinor induced reflection behaviour. The $P$-invariance is broken in two different ways: One component of the lepton isodoublet, e.g. $l = \frac{1-\tau_3}{2} L \in W^- \cong \mathbb{C}^2$, can be used together with the right handed isosinglet $r$ as a basis of a Dirac space $\Psi \in W^- \oplus W^R \cong \mathbb{C}^4$ with a representation of $P$. This is impossible for the remaining unpaired left handed field $\frac{1+\tau_3}{2} L \in W^+ \cong \mathbb{C}^2$ - here $P$ cannot even be defined. However, also for the left-right pair $(l, r)$ the resulting gauge vertex breaks position space reflection $P$ invariance via the familiar neutral weak interactions, induced by a vector field $Z_j$ arising in addition to the $U(1)$-electromagnetic gauge field $\Gamma_j$

$$-\frac{A_j + B_j^3}{2} l^\dagger \sigma^j 1 = -A_j \vec{\sigma}^j r = -\Gamma_j \overline{\psi} \gamma^j \psi - Z_j \overline{\psi} \gamma^j \gamma_5 \psi$$

with \( \begin{pmatrix} \Gamma_j \\ Z_j \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix} \)

There is no parameter involved whose vanishing would lead to a $P$-invariant dynamics.

### 5.2 GP-Invariance in the Standard Model of Leptons

The CP-reflection induced by the spinor ‘metric’

$$W_L \leftrightarrow_{CP} W^T_R, \quad 1^A \leftrightarrow \delta^A_B \epsilon \hat{B}^B$$

$$W_R \leftrightarrow_{CP} W^T_L, \quad r^A \leftrightarrow \delta^A_B \epsilon \hat{B}^B$$

has to include also a linear reflection of internal operation representations spaces in the case of Weyl spinors with nonabelian internal degrees of freedom.

For isospin $SU(2)$-doublets this reflection is given by the Pauli isospinor reflection discussed above and is denoted as internal reflection by $I = \epsilon$

\[
\begin{array}{cccc}
U & \rightarrow & U & , \quad u \in SU(2) \ (\text{isospin}) \\
\epsilon & \downarrow & \epsilon & , \\
U^T & \rightarrow & U^T & , \\
\hat{u} & \downarrow & \hat{u} & , \\
\end{array}
\]

\[
\psi^a \rightarrow \epsilon \psi^b \epsilon, \quad a, b = 1, 2 \\
\overline{\psi} = \epsilon^{-1} \circ \overline{\psi} \circ \epsilon
\]

Therewith the linear GP-reflection as particle-antiparticle conjugation including nontrivial isospin eigenvalues

$$G = IC, \quad GP = ICP$$

reads for left handed Weyl spinors isospinors

$$W_L \otimes U \leftrightarrow_{GP} W^T_R \otimes U^T, \quad L^A a \leftrightarrow \delta^A_B \epsilon \hat{B}^B e^{ab} \epsilon \hat{B}^B$$
The antilinear $T$-reflection uses the $U(2)$-scalar product

$$U \leftrightarrow U^T, \quad U \times U \rightarrow \mathbb{C}, \quad \langle \psi^a | \psi^b \rangle = \delta^{ab}$$

$$W_L \otimes U \leftrightarrow W_R^T \otimes U^T, \quad L^A \leftrightarrow \delta^{AB} \delta_{aB} L^a$$

The isospin dual coincides with the time dual $U^T = U^\star$.

In the product CP there arises - in the basis chosen - an isospin transformation $\epsilon^{ac} \delta_{cb} \sim e^{i \pi/2} \in SU(2)$

$$W_L \otimes U \leftrightarrow W_L \otimes U, \quad L^a \leftrightarrow \delta^{AB} \delta_{B B} \epsilon^{ac} \delta_{cb} L^b$$

decisive to prove the GPT-theorem with

$$\mathcal{CPT} \in SU(2) \times SL(\mathfrak{q}^2)$$

With the spinor induced reflection behaviour for the gauge fields the standard model for leptons, i.e. with internal hupercharge-isospin action, allows the representation of $\text{GP}$ and $T$ with the gauge vertex above being $\text{GP}$ and $T$ invariant.

### 5.3 CP-Problems for Quarks

If quark triplets and antitriplets which come with the dual defining $SU(3)$-representations, are included in the standard model, an extended CP-reflection has to employ a linear reflection $\gamma$ between dual representation spaces of colour $SU(3)$, i.e. an $SU(3)$-invariant bilinear form of the representation space

$$\gamma^T \gamma \rightarrow \gamma^T \gamma, \quad \gamma^{-1} \circ D(u)^T \circ \gamma = D(u^{-1}) \text{ for all } u \in SU(3)$$

The situation for isospin $SU(2)$ and colour $SU(3)$ is completely different with respect to the existence of such a linear dual isomorphism $\gamma$: All irreducible $SU(2)$-representations $[2T]$ with isospin $T = 0, \frac{1}{2}, 1, \ldots$ have an - up to a scalar factor - unique invariant bilinear form $\sqrt{\epsilon}$ as product of the spinor ‘metric’, discussed above.

That is not the case for the colour representations. Some representations are linearly selfdual, some are not.

The complex irreducible representations of $SU(3)$ are characterized by $[N_1, N_2]$ with two integers $N_{1,2} = 0, 1, 2, \ldots$. They arise from the two fundamental triplet representations, dual to each other and parametrizable with eight Gell-Mann matrices $\lambda$

triplet: $[1, 0] = u = e^{i \lambda^X}$, antitriplet: $[0, 1] = \bar{u} = u^{-1T} = (e^{-i \lambda^X})^T$

$[N_1, N_2]$ acting on vector space $U$ with $\dim_\mathbb{C} U = (N_1 + 1)(N_2 + 1)(N_1 + N_2 + 2)/2$
Dual representations have reflected integer values \([N_1, N_2] \leftrightarrow [N_2, N_1]\). Only those SU(3)-representations whose weight diagram is central reflection symmetric in the real 2-dimensional weight vector space (appendix) have one, and only one, SU(3)-invariant bilinear form \([1]\), i.e. they are linearly selfdual. Dual representations have weights which are reflected to each other(weights) \([N_1, N_2]\) \[\leftarrow \{1\} \rightarrow\] (weights) \([N_2, N_1]\). Therefore, one obtains as selfdual irreducible SU(3)-representations\([1]\), i.e. they are linearly selfdual. Dual representations have weights which are reflected to each other.

\[\text{weights} [N_1, N_2] \leftrightarrow [N_2, N_1]\]

E.g. for the octet \([1, 1]\) as adjoint SU(3)-representation, the Killing form defines its selfduality.

A central reflections of Lie Algebras

A representation of a group \(G\) on a vector space \(V\) is selfdual if it is equivalent to its dual representation, defined by the inversed transposed action on the linear forms \(V^T\)

\[
\begin{align*}
D : G &\rightarrow \text{GL}(V) \\
\check{D} : G &\rightarrow \text{GL}(V^T)
\end{align*}
\]

\(\check{D}(g) = D(g^{-1})^T\)
i.e. if the following diagram with a linear or antilinear isomorphism $\zeta : V \rightarrow V^T$ commutes with the action of all group elements

$$
\begin{array}{c}
V & \xrightarrow{D(g)} & V \\
\downarrow \zeta & & \downarrow \zeta \\
V^T & \xrightarrow{D(g)} & V^T \\
\end{array}
$$

$\zeta^{-1} \circ D(g)^T \circ \zeta = D(g^{-1})$ for all $g \in G$

Selfduality is equivalent to the existence of a nondegenerate bilinear (for linear $\zeta$) or sesquilinear form (for antilinear $\zeta$) of the vector space $V$

$$
V \times V \rightarrow \mathbb{C}, \quad \zeta(w, v) = \langle \zeta(w), v \rangle \\
\text{selfdual} \quad \zeta(g \bullet w, g \bullet v) = \zeta(w, v), \quad g \bullet v = D(g).v
$$

For the Lie algebra $L = \log G$ of a Lie group $G$ with dual representations in the endomorphism algebras $\mathbf{AL}(V)$ and $\mathbf{AL}(V^T)$ which are negative transposed to each other

$$
\begin{array}{c}
\varnothing : L \rightarrow \mathbf{AL}(V) \\
\check{\varnothing} : L \rightarrow \mathbf{AL}(V^T) \\
\end{array}
\quad \check{\varnothing}(l) = -\varnothing(l)^T
$$

a selfduality isomorphism, i.e. the reflection $V \xleftarrow{\zeta} V^T$ fulfills

$$
\zeta(l \bullet w, v) = -\zeta(w, l \bullet v), \quad l \bullet v = \varnothing(l).v
$$

and defines the central reflection of the Lie algebra in the representation

$$
\begin{array}{c}
V & \xrightarrow{\varnothing(l)} & V \\
\downarrow \zeta & & \downarrow \zeta \\
V^T & \xrightarrow{\check{\varnothing}(l)} & V^T \\
\end{array}
$$

$\zeta^{-1} \circ \varnothing(l)^T \circ \zeta = \varnothing(l)$ for all $l \in \log G$

With Schur’s lemma, an irreducible complex finite dimensional representation of a group or Lie algebra can have at most - up to a constant - one invariant bilinear and one invariant sesquilinear form. E.g. Pauli spinors for $\text{SU}(2)$ have both, $\epsilon^{AB}$ (bilinear) and $\delta^{AB}$ (sesquilinear, scalar product), $A,B = 1, 2$, quark triplets have only a scalar product $\delta^{ab}$, $a,b = 1, 2, 3$, Weyl spinors for $\text{SL}(\mathbb{C}^2)$ have only the bilinear ‘metric’ $\epsilon^{AB}$.

For a simple Lie algebra $L$ of rank $r$, the weights (eigenvalue vectors for a Cartan subalgebra) of dual representations $\varnothing$ and $\check{\varnothing}$ are related to each other by the central reflection of the weight vector space $\mathbb{R}^r$

$$
\text{weights } \varnothing[L] \xleftarrow{1_r} \text{weights } \check{\varnothing}[L]
$$
which may be induced by a linear isomorphism $\zeta$ of the dual representation spaces. Therewith: Such a linear isomorphism for an $L$-representation exists if, and only if, the weights of the representation $\mathcal{D}: L \to \text{AL}(V)$ are invariant under central reflection

$$V \overset{\zeta}{\leftarrow} V^T \iff \text{weights } \mathcal{D}[L] = -\text{weights } \mathcal{D}[L]$$

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