Approximate Analytical Solutions of the Baby Skyrme Model

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In this paper we show that many properties of the baby skyrmions, which have been determined numerically, can be understood in terms of an analytic approximation. In particular, we show that the approximation captures properties of the multiskyrmion solutions (derived numerically) such as their stability towards decay into various channels, and that it is more accurate for the “\textit{new baby Skyrme model}” which describes anisotropic physical systems in terms of multiskyrmion fields with axial symmetry. Some universal characteristics of configurations of this kind are demonstrated, which do not depend on their topological number.

1 Introduction

It is known that the two-dimensional $O(3)$ $\sigma$-model\textsuperscript{[1]} possesses metastable states which when perturbed can shrink or spread out due to the conformal (scale) invariance of the model\textsuperscript{[2, 3, 4]}. This implies that the metastable states can be of any size and so a fourth order in derivatives term, the so-called Skyrme term, needs to be added to break the scale invariance of the model. However the resulting energy functional has no minima and a further extra term is needed to stabilize the size of the corresponding solitons, ie a term which contains no derivatives of the field, often called the potential (or mass) term. In this case the field can be thought of as the magnetization vector of a two-dimensional ferromagnetic substance\textsuperscript{[1]}, and the potential term describes the coupling of the magnetization vector to a constant external magnetic field. As the extra terms contribute to the masses of the solitons their dependence deviates from a simple law in which the skyrmion mass is proportional to the skyrmion (topological) number and the two-skyrmion configuration becomes stable showing that the model possesses bound states\textsuperscript{[5]}.

In this paper we demonstrate that the simple analytical method used for the description of the three-dimensional Skyrme model presented in\textsuperscript{[6]} can be used also to study various properties of the low-energy states of the corresponding two-dimensional $\sigma$-model when the parameters which determine the contributions of the Skyrme and the potential term are not large. More precisely, it was possible to describe analytically the basic properties of the three-dimensional skyrmions for large baryon numbers\textsuperscript{[6]}, and so it is worthwhile to derive such a description for the two-dimensional $O(3)$ $\sigma$-model as well. In general, such analytical discussions of soliton models are useful as they lead to a better understanding of the soliton properties. The two-dimensional $O(3)$ $\sigma$-model is widely used to describe ferromagnetic systems, high-temperature superconductivity, etc and so the results obtained here can be useful for the understanding of these phenomena.

Our method is based on the ansatz introduced in\textsuperscript{[6]} and is accurate for the so-called “\textit{new baby Skyrme model}”\textsuperscript{[7]} which describes anisotropic physical systems. In fact, its accuracy increases as the skyrmion number $n$ increases, and this method allows to predict some universal properties of the ring-like configurations for large $n$, independently on its particular value. Although such models are not integrable, in the case where $n$ is large the “\textit{new baby Skyrme model}” appears to have the properties of an integrable system.
2 Near the Nonlinear $O(3)$ $\sigma$-Model

The Lagrangian density of the $O(3)$ $\sigma$-model with the additional terms introduced and discussed in [5, 7, 8] is:

$$L = \frac{g^2}{2} (\partial_{a} \vec{n})^2 - \frac{1}{4e^2} [\partial_{a} \vec{n}, \partial_{b} \vec{n}]^2 - g^2 V.$$  (1)

Here $\partial_{a} = \partial/\partial x^a$; $x^a$, $a = 0, 1, 2$, refer to both time and spatial components of $(t, x, y)$; and the field $\vec{n}$ is a scalar field with three components $n_a$, $a = 1, 2, 3$, satisfying the condition $\vec{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1$. The constants $g$, $e$ are free parameters, i.e., $g^2$ has the dimension of energy. It is useful to think of $g^2$ and $1/ge$ as natural units of energy and length respectively. The first term in (1) is familiar from $\sigma$-models; the second term, fourth order in derivatives, is the analogue of the Skyrme term; while the last term is the potential term. In fact, the potentials for the “old baby Skyrme model” (OBM) and the “new baby Skyrme model” (NBM) describing anisotropic systems are given by

$$V_{\text{OBM}} = \mu^2 (1 - n_3), \quad V_{\text{NBM}} = \frac{1}{2} \mu^2 (1 - n_3^2)$$  (2)

respectively, and $\mu$ has the dimension of energy, so $1/\mu$ defines a second length scale in our model. Evidently, $V_{\text{NBM}} \leq V_{\text{OBM}}$ at fixed value of $\mu$.

In three spatial dimensions the Skyrme term is necessary for the existence of soliton solutions, but the inclusion of a potential is optional from the mathematical point of view. Physically, however, a potential of a certain form is required to give the pions a mass [9]. By contrast, in two dimensions a potential term has to be included in the above Lagrangian in order soliton solutions to exist. As it has been shown in [10], the different potential terms give quite different properties to the multiskyrmion configurations when the skyrmion number is large. Our analytical treatment here supports this conclusion, as shown in sections 3-5.

We are only interested in configurations with finite energy, so we define the configuration space to be the space of all maps $\vec{n} : R^2 \to S^2$ which tend to the constant field $(0, 0, 1)$ (so-called vacuum) at spatial infinity

$$\lim_{|x| \to \infty} \vec{n}(\vec{x}) = (0, 0, 1).$$  (3)

Thus every configuration $\vec{n}$ may be regarded as a representative of a homotopy class in $\pi_2(S^2) = \mathbb{Z}$ and has a corresponding integer degree of the form

$$\deg[\vec{n}] = \frac{1}{8\pi} \int d^2 x \epsilon^{ab} \vec{n} (\partial_a \vec{n} \times \partial_b \vec{n}).$$  (4)

The vacuum field is invariant under the symmetry group $G = E_2 \times SO(2)_{\text{iso}} \times P$, where $E_2$ is the Euclidean group of translations and rotations in two dimensions which acts on fields via pull-back. $SO(2)_{\text{iso}}$ is the subgroup of the three-dimensional rotation group acting on $S^2$ which leaves the vacuum fixed. [We call its elements iso-rotations to distinguish them from rotations in physical space]. Finally $P$ is a combined reflection in both space and the target space $S^2$.

We are interested in stationary points of $\deg[\vec{n}] \neq 0$, and the maximal subgroups of $G$ under which such fields can be invariant are labelled by a nonzero integer $n$ and consist

\footnote{The first few paragraphs of this section follow quite closely to \cite{3, 4} and are included to make the paper more selfconsistent.}
of spatial rotations by some angle $\alpha \in [0, 2\pi]$ and simultaneous iso-rotation by $-n\alpha$. Fields invariant under such a group are of the form

$$n_1 = \sin f(\tilde{r}) \cos(n\phi), \quad n_2 = \sin f(\tilde{r}) \sin(n\phi), \quad n_3 = \cos f(\tilde{r})$$

(5)

where $(\tilde{r}, \phi)$ are polar coordinates and $f(\tilde{r})$ is the profile function. Such fields are the analogues and generalizations of the hedgehog fields in the Skyrme model. This parametrisation which involves azimuthal symmetry of the fields assumes that all the skyrmions sit on top of each other while forming the multiskyrmion configuration.

It is easy to show that the degree of the field (6) is

$$\deg [\tilde{n}] = n$$

ie equal to the azimuthal winding number $n$.

The corresponding static energy functional connected with the Lagrangian (1) for the OBM and NBM is equal to

$$E_{cl}(n)_{\text{OBM}} = \frac{g^2}{2} \int r dr \left( f'^2 + \frac{n^2 \sin^2 f}{r^2} + a \left[ \frac{n^2 f'^2 \sin^2 f}{r^2} + 2 (1 - \cos f) \right] \right), \quad (7)$$

$$E_{cl}(n)_{\text{NBM}} = \frac{g^2}{2} \int r dr \left( f'^2 + \frac{n^2 \sin^2 f}{r^2} + a \left[ \frac{n^2 f'^2 \sin^2 f}{r^2} + (1 - \cos^2 f) \right] \right)$$

(8)

respectively. In (7) and (8) the length $\sqrt{\mu g}$ has been absorbed so that the scale size of the localized structures is a function of the dimensionless spatial coordinate $r = \sqrt{\mu g \tilde{r}}$ while the dimensionless parameter $a = \mu / ge$ becomes the only nontrivial parameter of the model. Finiteness of the energy functional requires that the profile function has to satisfy the following boundary conditions: $f(0) = \pi$ and $f(\infty) = 0$.

By setting $\phi = \cos f$ into (7) the energy functional becomes

$$E_{cl}(n)_{\text{OBM}} = \frac{g^2}{2} \int r dr \left( \frac{\phi'^2}{1 - \phi^2} + \frac{n^2 (1 - \phi^2)}{r^2} + a \left[ \frac{n^2 \phi'^2}{r^2} + 2 (1 - \phi) \right] \right)$$

(9)

and a similar expression for $E_{cl}(n)_{\text{NBM}}$. Parametrizing the field $\phi$, using the ansatz introduced in (4) for the description of the three-dimensional skyrmions, as

$$\phi = \cos f = \frac{(r/r_n)^p - 1}{(r/r_n)^p + 1}, \quad \phi' = \frac{p}{2r} (1 - \phi^2)$$

(10)

leads after integration with respect to $r$ to the following analytic energy expressions

$$E_{cl}(n)_{\text{OBM}} = \pi g^2 \left( \frac{4n^2}{p} + p + \frac{4\pi a}{p \sin(2\pi/p)} \left[ \frac{n^2 (p^2 - 4)}{3r_n^2 p} + r_n^2 \right] \right), \quad (11)$$

$$E_{cl}(n)_{\text{NBM}} = \pi g^2 \left( \frac{4n^2}{p} + p + \frac{4\pi a}{p \sin(2\pi/p)} \left[ \frac{n^2 (p^2 - 4)}{3r_n^2 p} + \frac{2}{p} r_n^2 \right] \right)$$

(12)

Here $p$ and $r_n$ are parameters which still need to be determined by minimizing the energy. In fact, $r_n$ corresponds to the radius of the $n$th-soliton configuration. Remark: The following Euler-type integrals have been used for the derivation of (11) and (12), see also [6]

$$\int_0^\infty \frac{2r \, dr}{1 + (r/r_n)^p} = \frac{2\pi r_n^2}{p \sin(2\pi/p)} \quad p > 2$$

$$\int_0^\infty \frac{dr (r/r_n)^p}{r [1 + (r/r_n)^p]^2} = \frac{1}{p} \quad p > 0$$
\[
\int_0^\infty \frac{dr}{r^3 (1 + (r/r_n)^p)^3} = \frac{\pi}{3} \frac{(p^2 - 4)}{2p^3 \sin(2\pi/p)}, \quad p > 1
\]
\[
\int_0^\infty \frac{2r dr}{[1 + (r/r_n)^p]^2} = \left(1 - \frac{2}{p}\right) \frac{2\pi r_n^2}{p \sin(2\pi/p)}, \quad p > 1.
\] (13)

It can be easily proved that the minimization of the energies (11) and (12) implies that

\[
(r_n^{\text{min}})^2_{\text{OBM}} = \frac{n}{\sqrt{3}} \frac{\sqrt{p^2 - 4}}{p}, \quad (r_n^{\text{min}})^2_{\text{NBM}} = n \frac{\sqrt{p^2 - 4}}{6}.
\] (14)

ie \( (r_n^{\text{min}})^2_{\text{OBM}} = \sqrt{2} (r_n^{\text{min}})^2_{\text{NBM}} \) and so the minimum value of the energies is equal to

\[
E_{cl}(n)_{\text{OBM}} = 4\pi g^2 \left[ \frac{n^2}{p} + \frac{p}{4} + \frac{2an\pi}{\sqrt{3}p \sin(2\pi/p)} \frac{\sqrt{p^2 - 4}}{\sqrt{p}} \right],
\]
\[
E_{cl}(n)_{\text{NBM}} = 4\pi g^2 \left[ \frac{n^2}{p} + \frac{p}{4} + \frac{2\sqrt{3}an\pi}{\sqrt{3} \sin(2\pi/p)} \frac{\sqrt{p^2 - 4}}{p^2} \right].
\] (15) (16)

It is obvious that the energy contributions of the Skyrme and the potential term are equal due to (14), which is in agreement with the result obtained from Derrick’s theorem. Equations (15) (16) provide an upper bound for the energies of baby-skyrmions for any value of \( p \). To get lower bound, we should minimize the right-hand sides of (15) (16) with respect to the parameter \( p \). In what follows we investigate various cases which correspond to different values of the only nontrivial parameter of the model, \( a \).

First consider the case where \( a \ll 1 \), ie for very small values of the model parameter. Observe that for \( a = 0 \) the ansatz (10) is a solution of the model for \( p = 2n \), which implies that \( p \to 2n \) as \( a \to 0 \). In this case, due to (14), the radius of the multiskyrmion configuration increases with \( n \): \( (r_n^{\text{min}})^2_{\text{OBM}} \sim n^{3/2} \) and \( (r_n^{\text{min}})^2_{\text{NBM}} \sim n^2 \). Moreover, the configuration consists of a ring of thickness: \( \delta \sim 4r_n/p \) thus \( \delta_{\text{OBM}} \sim 2n^{-1/4} \) and \( \delta_{\text{NBM}} \sim \text{const} \). Remark: The ring thickness is determined as the difference of the values of \( \phi \) inside (which is equal to -1) and outside (which is equal to +1) of the ring (ie \( d\phi = 2 \)) divided by its derivative at \( r = r_n \) where, due to (10), \( \phi(r_n) = 0 \) and so \( \phi'(r_n) = p/2r_n \).

This kind of magnetic solitons have been observed in (11), (12) as solutions of the Landau-Lifshitz equations defining the dynamics of ferromagnets. [Note that the static solutions of the baby Skyrme model and the Landau-Lifshitz equations are related.] In general, \( \phi \) given by (10) for \( p = 2n \) is a low-energy approximation of multiskyrmion configurations (for \( n > 1 \)), since for \( n = 1 \) the corresponding energies given by (15) and (16) are infinite. Indeed, it is a matter of simple algebra to show that

\[
E_{cl}(n = 2)_{\text{OBM}} = 4\pi g^2 (2 + a\pi), \quad E_{cl}(n = 2)_{\text{NBM}} = 4\pi g^2 \left(2 + \frac{a\pi}{\sqrt{2}}\right),
\]
\[
E_{cl}(n = 3)_{\text{OBM}} = 4\pi g^2 \left(3 + a\pi \frac{8}{3\sqrt{3}}\right), \quad E_{cl}(n = 3)_{\text{NBM}} = 4\pi g^2 \left(3 + a\pi \frac{8}{9}\right),
\]
\[
E_{cl}(n = 4)_{\text{OBM}} = 4\pi g^2 \left(4 + a\pi \sqrt{3}\right), \quad E_{cl}(n = 4)_{\text{NBM}} = 4\pi g^2 \left(4 + a\pi \frac{\sqrt{3}}{2}\right). \] (17)

For large \( n \), the energies take the asymptotic values

\[
E_{cl}(n)_{\text{OBM}} = 4\pi ng^2 \left(1 + \sqrt{\frac{2n}{3} a}\right),
\]
\[
E_{cl}(n)_{\text{NBM}} = 4\pi ng^2 \left(1 + \sqrt{\frac{2}{3} a}\right). \] (18)
Note that the energy of the OBM per unit skyrmion number increases as $n$ increases, while the energy of the NBM per skyrmion decreases with increasing $n$ to become constant for $n$ large. In fact, the energies given by (17) are the upper bounds of the multiskyrmion energies since the exact profile function corresponding to the minimum of the energy differs from that given by (10).

3 Perturbation Theory for the Model Parameter

In this section energy corrections up to second or higher order with respect to the model parameter $a$ have been obtained. The corresponding energies for OMB and NBM can be written as

$$E_{cl}(n) = 4\pi g^2 [f(p) + a h(p)]$$  \hspace{1cm} (19)

where $f(p)$ and $h(p)$ can be evaluated from (13) and (16), respectively. By letting $p = 2n + \epsilon$ and expanding the energies (13) and (16) up to second order in $\epsilon$ we get $f(p) = n^2 / (8n)$ and $h(p) = h_0 + \epsilon h_1$ where $h_1 = (2n)^{-1} \beta h_0$. In fact, the corresponding functions for the OBM and the NBM are given by

$$\frac{h_{0\text{OBM}}}{n} = \sqrt{\frac{2n}{3}} \frac{\pi}{n \sin(\pi/n)} \sqrt{1 - 1/n^2}, \quad \beta_{\text{OBM}} = \frac{\pi}{n} \cot(\pi/n) - \frac{1}{2} + \frac{1}{n^2 - 1},$$  \hspace{1cm} (20)

$$\frac{h_{0\text{NBM}}}{n} = \sqrt{\frac{2}{3}} \frac{\pi}{n \sin(\pi/n)} \sqrt{1 - 1/n^2}, \quad \beta_{\text{NBM}} = \frac{\pi}{n} \cot(\pi/n) - 1 + \frac{1}{n^2 - 1}.$$  \hspace{1cm} (21)

Minimization of (19) with respect to $\epsilon$ implies that $\epsilon_{\text{min}}^{\text{OBM}} = -4ah_1 = -2a\beta h_0$. At large values of $n$, the parameteres $\epsilon$ and $p = 2n + \epsilon$ take the values

$$\epsilon(n)_{\text{OBM}} \approx -an \sqrt{\frac{2n}{3}}, \quad \epsilon(n)_{\text{NBM}} \approx 2a \sqrt{\frac{2 \pi^2 / 3 - 1}{n}},$$

$$p(n)_{\text{OBM}} \approx 2n - an \sqrt{\frac{2n}{3}}, \quad p(n)_{\text{NBM}} \approx 2n + 2a \sqrt{\frac{2 \pi^2 / 3 - 1}{n}}.$$  \hspace{1cm} (22)

For any $a$, as $n$ increases, the effective power $p(n)_{\text{OBM}}$ becomes negative and the approach based on the assumption that $\epsilon_{\text{OBM}}$ is small is not self-consistent (also, see next section). On the contrary, for NBM, $p(n)_{\text{NBM}} \approx 2n$ as $n$ increases which implies that our consideration is self-consistent in this case. In terms of (19)-(21), the energy per skyrmion of the $n$-th skyrmion configuration takes the value

$$\frac{E_{cl}(n)}{4\pi g^2 n} = 1 + \frac{a h_0}{n} - a^2 \frac{h_0^2 \beta^2}{2n^2},$$  \hspace{1cm} (23)

which gives us

$$\frac{E_{cl}(2)_{\text{OBM}}}{4\pi g^2} = 1 + 1.5708 a - 0.034 a^2, \quad \frac{E_{cl}(2)_{\text{NBM}}}{4\pi g^2} = 1 + 1.1107 a - 0.2741 a^2,$$

$$\frac{E_{cl}(3)_{\text{OBM}}}{4\pi g^2} = 1 + 1.6120 a - 0.068 a^2, \quad \frac{E_{cl}(3)_{\text{NBM}}}{4\pi g^2} = 1 + 0.9308 a - 0.0317 a^2,$$

$$\frac{E_{cl}(4)_{\text{OBM}}}{4\pi g^2} = 1 + 1.7562 a - 0.191 a^2, \quad \frac{E_{cl}(4)_{\text{NBM}}}{4\pi g^2} = 1 + 0.8781 a - 0.0084 a^2,$$

$$\frac{E_{cl}(5)_{\text{OBM}}}{4\pi g^2} = 1 + 1.9122 a - 0.302 a^2, \quad \frac{E_{cl}(5)_{\text{NBM}}}{4\pi g^2} = 1 + 0.8552 a - 0.0032 a^2,$$

$$\frac{E_{cl}(6)_{\text{OBM}}}{4\pi g^2} = 1 + 2.0649 a - 0.404 a^2, \quad \frac{E_{cl}(6)_{\text{NBM}}}{4\pi g^2} = 1 + 0.8430 a - 0.0015 a^2.$$  \hspace{1cm} (24)
For large \( n \), the energies (24) take the asymptotic values

\[
\frac{E_{cl}(n)}{4\pi g^2n} \quad \text{OBM:} \quad \left( 1 + a\sqrt{\frac{2n}{3}} - a^2 \frac{n}{12} \right),
\]

\[
\frac{E_{cl}(n)}{4\pi g^2n} \quad \text{NBM:} \quad \left( 1 + a\sqrt{\frac{2}{3}} - a^2 \frac{(\pi^2/3 - 1)^2}{3n^2} \right).
\]

(26)

Note that the energies of the two models behave differently when we consider terms of the second order in the model parameter, i.e. terms \( \sim a^2 \). Indeed, for the OBM the contribution to the energy is linearly proportional to the skyrmion number \( n \), while for the NBM the contribution decreases rapidly as the skyrmion number increases. This implies that the linear approximation in \( a \) is accurate for the NBM since the quadratic term becomes negligible for large \( n \). Numerical results obtained for different values of \( a \) for the OBM and NBM are presented in Table 1 and Table 2, respectively.

As we have pointed out earlier, our method cannot describe the one-skyrmion configuration since the corresponding energies become infinite. However, by setting \( p = 2 + \varepsilon \) to (15) and (16) and expanding all terms up to third order in \( \varepsilon \ll 1 \) we obtain

\[
E_{cl}(n = 1) = 4\pi g^2 \left( 1 + \frac{\varepsilon^2}{8} - \frac{\varepsilon^3}{16} + 2a\sqrt{\frac{2}{3\varepsilon}}(1 - \gamma \varepsilon) \right).
\]

(27)

where \( \gamma \) has different value for each model, i.e

\[
\gamma_{\text{OBM}} = \frac{1}{8}, \quad \gamma_{\text{NBM}} = \frac{3}{8}.
\]

(28)

Note, that when only terms up to second order in \( \varepsilon \) have been considered, the corresponding energy (27) simplified to: \( E_{cl} = 4\pi g^2 \left( 1 + \frac{\varepsilon^2}{8} + 2a\sqrt{\frac{2}{3\varepsilon}} \right) \), and the minimum occurs at: \( \varepsilon_1 = 2(a/\sqrt{3})^{2/5} \). Finally, the minimum of (27) occurs at

\[
\varepsilon_{\text{min}} = 2 \left( \frac{a}{\sqrt{3}} \right)^{2/5} \left[ 1 + \frac{4}{5} \left( \frac{a}{\sqrt{3}} \right)^{2/5} \left( \gamma + \frac{3}{4} \right) \right]
\]

(29)

and corresponds to a shift of \( \varepsilon_1 \) since higher order corrections of \( \varepsilon \) have been considered in (27). The energy of the one-skyrmion configuration is

\[
\frac{E_{cl}(n = 1)}{4\pi g^2} = \left\{ 1 + \frac{5}{2} \left( \frac{a}{\sqrt{3}} \right)^{4/5} \left[ 1 - \frac{1}{5} \left( \frac{a}{\sqrt{3}} \right)^{2/5} (8\gamma + 1) \right] \right\}
\]

\[
\simeq \left[ 1 + 1.611a^{4/5} \left( 1 - 0.1605a^{2/5} (8\gamma + 1) \right) \right].
\]

(30)

Equation (30) implies that, for a single skyrmion, the energy expansion in \( a \) is proportional to a power of \( a \) instead of being linearly proportional to \( a \) (which is the case for the multiskyrmion configurations with \( n \geq 2 \)), while its convergence is worse than for multiskyrmions, especially for the NBM. In fact, when \( a = 0.4213 \) the first two terms in (30) are equal to 1.807, while the next order term lowers this value down to 1.44, which compared with the exact value 1.564 obtained from numerical simulations gives an error of 7%. Note, that our one-skyrmion parametrization gives the same energy for both models, when expansions only up to the lowest order in \( a \) have been considered: the difference appears only in the term \( \sim a\gamma\sqrt{\varepsilon} \) in (27).
Table 1: Energy per unit skyrmion number (in \(4\pi g^2\)) for different values of the parameter \(a\) for the OBM case where corrections of second order in \(a\) have been taken into account. The last two lines contain the exact results obtained from the numerical simulations of multiskyrmions \((n \geq 2)\) with ring-like shapes and \((n \geq 3)\) with shapes other than ring-like [10], respectively. For the first case, we have solved numerically the equations using the hedgehog ansatz [3].

| \(a/n\) | \(n = 1\) | \(n = 2\) | \(n = 3\) | \(n = 4\) | \(n = 5\) | \(n = 6\) | \(n = 8\) |
|--------|---------|---------|---------|---------|---------|---------|---------|
| \(a = 0.001\) | 1.0063 | 1.00157 | 1.0016 | 1.0017 | 1.0019 | 1.0021 | 1.0023 |
| \(a = 0.01\) | 1.0384 | 1.0157 | 1.0161 | 1.0176 | 1.0191 | 1.0206 | 1.0234 |
| \(a = 0.0316\) | 1.0933 | 1.0496 | 1.0508 | 1.0553 | 1.0601 | 1.0649 | 1.0737 |
| \(a = 0.1\) | 1.2227 | 1.1567 | 1.1605 | 1.1737 | 1.1882 | 1.2025 | 1.2291 |
| \(a = 0.316\) | 1.5113 | 1.4930 | 1.5026 | 1.5358 | 1.5638 | 1.6126 | 1.6835 |
| \(\nu_{\text{hed}} = 0.316\) (num) | 1.5647 | 1.4681 | 1.4901 | 1.5284 | 1.5692 | 1.6092 | 1.6832 |
| \(a = 0.316\) (num) | 1.564 | 1.468 | 1.460 | 1.450 | 1.456 | 1.449 | — |

Table 2: Energy per unit skyrmion number for different values of the parameter \(a\) for the NBM. The last line contains the exact results determined by the numerical simulations [10] of multiskyrmions with ring-like shapes when \(a = 0.4213\), for \(n \leq 6\) coinciding with ours.

| \(a/n\) | \(n = 1\) | \(n = 2\) | \(n = 3\) | \(n = 4\) | \(n = 5\) | \(n = 6\) | \(n = 8\) | \(n = 12\) | \(n = 16\) |
|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \(a = 0.01\) | 1.0363 | 1.0111 | 1.0093 | 1.0088 | 1.0085 | 1.0084 | 1.0083 | 1.0082 |
| \(a = 0.0316\) | 1.0851 | 1.0348 | 1.0294 | 1.0277 | 1.0270 | 1.0266 | 1.0262 | 1.0260 | 1.0259 |
| \(a = 0.1\) | 1.1887 | 1.1083 | 1.0928 | 1.0877 | 1.0855 | 1.0843 | 1.0831 | 1.0823 | 1.0820 |
| \(a = 0.316\) | 1.3814 | 1.3238 | 1.2912 | 1.2768 | 1.2699 | 1.2662 | 1.2626 | 1.2602 | 1.2593 |
| \(a = 0.4213\) | 1.44 | 1.4193 | 1.3865 | 1.3684 | 1.3597 | 1.3549 | 1.3501 | 1.3467 | 1.3455 |
| \(a = 0.4213\) (num) | 1.564 | 1.405 | 1.371 | 1.358 | 1.352 | 1.349 | 1.3447 | 1.3407 | 1.3385 |

By looking at the results presented in Table 1 and Table 2 it is clear that our approximate method gives the energy values which are quite close to the exact values obtained by numerical simulations, especially for the NBM. In particular, the difference between the exact and approximate energy for \(a = 0.4213\) is less than 0.5% for \(n \geq 6\). For smaller values of \(a\) the agreement between analytical and numerical results is even better. In evident agreement with [2], the energies of the NBM skyrmions given in Table 2 are smaller than those of the OBM skyrmions (see Table 1) at the same values of the model parameters.

Note that, for the OBM (when \(a\) is small) the energy per skyrmion of a multiskyrmion configuration with \(n \geq 2\) is smaller compared to the single skyrmion energy and therefore, these configurations are bound states, stable with respect to the decay into \(n\) individual skyrmions. On the contrary, the ring-like OBM multiskyrmions with even \(n\) (where \(n \geq 4\)) are unstable with respect to the decay into two-skyrmion configurations, while configurations with odd \(n\) (where \(n \geq 5\)) are unstable with respect to the breakup into a two- and a three-skyrmion configurations. In addition, Table 1 and (30) show that for any \(n\) (where \(n \neq 1\)) there is an upper limit for the model parameter: \(a \leq a_{\nu}(n)\) above which the \(n\)-th ring-like skyrmion configuration can decay into \(n\) individual skyrmions.

Let us consider the case of \(n = 3\) in more detail. As it can be observed from the energies (17) and (30) when \(a \leq 0.77\) the ring-like three-skyrmion configuration is stable with respect to the decay into a single and a two-skyrmion configuration since

\[
E_1 + E_2 - E_3 \simeq 1.611a^{4/5} - a\pi \left(\frac{8}{3\sqrt{3}} - 1\right),
\]
and this difference becomes positive when and only when

\[ a \leq \left( \frac{3\sqrt{3} \cdot 1.611}{\pi (8 - 3\sqrt{3})} \right)^5 \approx 0.77. \]  

(32)

Corrections to the energy for the skyrmion configurations with \( n = 1, 2, 3 \) of the higher order in \( a \), lead to smaller critical values \( a_{cr}(n) \).

Since our fields with axial symmetry \((5)\) and \((10)\) correspond to ring-like solutions of the Euler-Lagrange equations \([1]\) for \( a = 0 \), they have to be solutions of the corresponding equations also as \( a \to 0 \), ie when \( a \) takes values in a small region close to zero. [In fact, this region becomes more narrow as \( n \) increases as in this limit the expansion in \( a \) becomes less convergent]. On the other hand, the lattice-like configurations (tripole for \( n = 3 \), quadrupole for \( n = 4 \), etc.) are solutions of the equations when \( a \geq a_{cr}(n) \) for given \( n \) \([3, 10, 13]\). However, the transition from the ring-like configuration to any other minimal energy configuration is a phenomenon which has not been studied in much detail yet and deserves further investigation.

Finally, it should be stressed that, in contrast to the linear approximation, the quadratic approximation given by \((25)\) does not provide an upper bound for the energy.

4 Away from the Nonlinear \( O(3) \) \( \sigma \)-Model

In the general case, for arbitrary values of the parameter \( a \) and the skyrmion number \( n \), soliton solutions can be obtained by minimizing numerically the energy \((13)\) and \((16)\) with respect to the variable \( p \). This way an upper bound is obtained for the corresponding energies since the profile function is given by \((17)\).

For large \( a \) at fixed \( n \) (or for large \( n \) at fixed \( a \)), the expansion \((20)\) is not self-consistent for the OBM. However, some analytical results can also be obtained in this case since \((13)\) for large \( a \) can be approximated by

\[ E_{cl}(n)_{OBM} \simeq 4\pi g^2 \frac{2an\pi}{\sqrt{3}p \sin(2\pi/p)} \frac{\sqrt{p^2 - 4}}{\sqrt{p}}. \]  

(33)

Expansion, up to second order terms of \((33)\) with respect to \( p \), gives

\[ E_{cl}(n)_{OBM} \simeq 4\pi g^2 \frac{an}{\sqrt{3}} \frac{\sqrt{p}}{\sqrt{3}} \left( 1 + \frac{c_2}{p^2} \right) \]  

(34)

where \( c_2 = 2(\pi^2/3 - 1) \); while its minimization implies that \( p_{min} \simeq \sqrt{3c2} = 3.71 \) and so the corresponding energy is

\[ \frac{E_{cl}(n)_{OBM}}{4\pi g^2} \simeq \frac{4}{3} an \left( \frac{c_2}{3} \right)^{1/4} = 1.48 an \]  

(35)

Note that, by contrast with the results obtained near the nonlinear \( O(3) \) \( \sigma \)-model, for large \( a \) the parameter \( p \) is independent of the skyrmion number \( n \). Also, for \( a \gg n \) the skyrmion radius is proportional to the square root of the skyrmion number: \( r_n \sim n^{1/2} \); while the skyrmion thickness is given by: \( \delta \sim r_n/p \sim n^{1/2} \), and so the ring-like structure of the configuration is not very pronounced.
Direct numerical minimization of (33) with respect to $p$ gives $p_{\text{min}} = 4.5$ and the corresponding value of the energy is

$$\frac{E_{cl}(n)_{\text{OBM}}}{4\pi g^2} = 1.55 \text{an.}$$

(36)

The energy, which has been obtained by solving numerically the Euler-Lagrange equation [8], is:

$$\frac{E_{cl}(n)}{4\pi g^2} = 1.333 \text{an.}$$

The profile function corresponding to this solution is given by:

$$\cos f = \frac{r^2}{2\pi}(r_n^2 - r^2) + \frac{2}{r_n^2} - 1$$

for $r \leq r_n$ and $f = 0$ for $r > r_n$. This solution is quite different from our parametrization (14) and thus, the 16% difference between the exact and the approximate solution is understandable.

To conclude, we recall that for NBM the parametrization (10) works well for arbitrary large $n$, and its accuracy increases with increasing $n$, as illustrated in Table 2.

5 Properties of the Skyrmions: Mean Square Radii, Energy Density, Moment of Inertia

Many properties of multiskyrmions can be determined using the ansatz (10). For example, the mean square radius of the $n$-th multiskyrmion configuration takes the simple form

$$< r^2 >_n = \frac{1}{2} \int dr r^2 \phi'$$

$$= \frac{2\pi r_n^2}{p \sin(2\pi/p)}$$

(37)

where $r_n$ is given by (14) for the OBM and NBM. For small $a$, it was shown in section 2 that $p = 2n$, which implies that the mean square radius becomes

$$< r^2 >_{\text{OBM}} \approx \frac{\pi \sqrt{2(n^2 - 1)}}{\sin(\pi/n)\sqrt{3n}},$$

$$< r^2 >_{\text{NBM}} \approx \frac{\pi \sqrt{2(n^2 - 1)}}{n \sin(\pi/n)\sqrt{3}}$$

(38)

which is equal to $\pi, 8\pi/3\sqrt{3}, \pi \sqrt{5}, \ldots$ and $\sqrt{2}\pi, 8\pi/3, 2\pi \sqrt{3}, \ldots$ for $n = 2, 3, 4, \ldots$, respectively.

For the NBM, even for a large enough value of the parameter $a$, the analytical formula (14) with the power $p$ taken from (23) gives the values of $< r^2 >_{\text{NBM}}$ in a remarkably good agreement with those obtained in numerical calculations. E.g., for $n = 3$ the analytical result is $\sqrt{< r^2 >_3} = 2.987$, in natural units of the model $1/(g \epsilon \mu)$, to be compared with 2.872 obtained numerically. This agreement improves with increasing $n$, and for $n = 12$ we have $\sqrt{< r^2 >_{12}} \sim 10.92$ to be compared with 10.85 determined numerically. A similar agreement between analytical and numerical results takes place for the mean square radius of the energy distribution of multiskyrmions (the 3D-case has been considered in detail in [8]).

Note that the one-skyrmion configuration is (still) a singular case since (37) is not defined for $n = 1$. However, as we have shown earlier, by expressing $p = 2 + \epsilon$ and expanding (14) in $\epsilon$ we get $r_{n=1}^2 = 2\sqrt{2}\epsilon/3$ which leads to

$$< r^2 >_1 = 2\sqrt{\frac{2}{3\epsilon_{\text{min}}}}$$

(39)

for $\epsilon_{\text{min}}$ given by (29). So, our approximate method shows that as the model parameter tends to zero the mean square radius of the one skyrmion field tends to infinity since $< r^2 >_1 \sim a^{-1/5}$, while since $< r^2 >_{\text{NBM}}(n) = \sqrt{n} < r^2 >_{\text{OBM}}(n)$ in this case the mean square radius is given by (39) for both models.
The average energy density per unit surface element is defined as

$$\rho_E = \frac{E_{cl}(n)}{2\pi r_n \delta}$$

(40)

with $\delta \simeq 2r_n/n$, see discussion after (16). For the NBM, when $n$ is large, (41) takes the constant value

$$\rho_{ENBM} \simeq e \mu g^3 \left( \sqrt{\frac{3}{2}} + a \right)$$

(41)

ie is independent of $n$. So, (41) represents the fundamental property of this kind of multiskyrmions. On the contrary, for the OBM when ring-like configurations (which do not correspond to the minimum of the energy [5, 10]) are considered, the energy density increases with $n$ like $\sim \sqrt{n}$ at small values of $a$.

Another quantity of physical significance determining the quantum corrections to the energy of skyrmions is the moment of inertia which has been considered for two-dimensional models in [13]. In order to obtain the energy quantum correction of the soliton, due to its rotation around the axis perpendicular to the plane in which the soliton is located, we have to take the $r$-dependent ansatz of the form

$$n_1 = \sin f(\tilde{r}) \cos[n(\phi - \omega t)], \quad n_2 = \sin f(\tilde{r}) \sin[n(\phi - \omega t)], \quad n_3 = \cos f(\tilde{r}).$$

(42)

Then the $\omega$-dependence of the energy is given by the simple formula:

$$E_{\text{rot}} = \frac{\Theta_J}{2} \omega^2$$

(43)

where $\Theta_J$, the so-called moment of inertia, is given by [13]

$$\Theta_J(n) = g^2 n^2 \int d^2r \sin^2 f \left( 1 + af'^2 \right).$$

(44)

Using (10) and the relations

$$\frac{1}{4} \int (1 - \phi^2) r \, dr = \int_0^\infty \frac{(r/r_n)^p \, dr}{[1 + (r/r_n)^p]^2}$$

$$= \frac{2\pi r_n^2}{p^2 \sin(2\pi/p)}, \quad p > 2$$

$$\frac{1}{16} \int (1 - \phi^2)^2 \frac{dr}{r} = \int_0^\infty \frac{(r/r_n)^{2p} \, dr}{[1 + (r/r_n)^p]^4 r}$$

$$= \frac{1}{6p}, \quad p > 0$$

(45)

we find that at large values of $n$ the moment of inertia simplifies to

$$\Theta_J(n) \simeq 4\pi g^2 n r_n^2 \left( \frac{2n}{p} + \frac{anp}{3r_n^2} \right)$$

(46)

which holds for any multiskyrmion configuration described by ansatz (10), for both models. For small values of $a$, letting $p = 2n$ and taking $r_n^2$ given by (14) we find that

$$\Theta_J(n)_{\text{OBM}} \simeq 4\pi g^2 n r_n^2 \left( 1 + a \sqrt{\frac{2n}{3}} \right)$$

$$\Theta_J(n)_{\text{NBM}} \simeq 4\pi g^2 n r_n^2 \left( 1 + a \sqrt{\frac{2}{3}} \right).$$

(47)
which implies that the moment of inertia, for large \( n \), is
\[
\Theta_J(n) \simeq E_{cl}(n) r_n^2.
\] (48)

in agreement with simple quasi-classical arguments for the thin massive ring. Similar quasi-classical formulae have been obtained for the three-dimensional skyrmions (see, for example, Ref. \[3, 14\]) where the moment of inertia was shown to be given by \( \Theta_J = 2M_B r_B^2/3 \) for large baryon numbers; this expression is valid for a classical spherical bubble with the mass concentrated in its shell.

6 Conclusions

In this paper we have presented an analytical approach for deriving approximate expressions to skyrmion solutions of the two-dimensional \( O(3) \) \( \sigma \)-model. These approximations are very accurate for small values of the parameter \( a \) which determines the weight of the Skyrme and the potential term in the Lagrangian. For other values of the model parameter we have performed some numerical calculations and then combined them with further analytical work to investigate the binding and other properties of multiskyrmion states.

Two models have been studied: the “old baby Skyrme model” and the “new baby Skyrme model” which differ from each other in the form of the potentials (2). For both models the \( a \) dependence of the energy of a single skyrmion differs from the cases where topological number \( n \geq 2 \). For the OBM, when \( a \) is small, the \( n = 3 \) skyrmion configuration is stable with respect to the decay into a single skyrmion and a two-skyrmion configuration while the ring-like multiskyrmion configurations with \( n \geq 4 \) are unstable with respect to the breakup into two- and three-skyrmion configurations.

For the NBM, on the other hand, the hedgehog multiskyrmion configurations considered in \[10\] and here describe bound states since the energy per skyrmion decreases as the skyrmion number increases. We note that the results obtained for the NBM are similar to the ones obtained for the three-dimensional model studied in \[6\]. In both cases the energy per skyrmion decreases as the skyrmion number increases. The three-dimensional skyrmions obtained using the rational map ansatz \[15\], for large \( n \), have the form of a bubble with energy and baryon number concentrated in the shell. The thickness and the energy density of the shell (which is analogous to the thickness of the ring in the two-dimensional case) do not depend on the skyrmion number \[6\]. Similarly, in this paper we have shown that the two-dimensional baby skyrmions of the NBM, for large \( n \), correspond to ring-like configurations with constant thickness and constant energy density per unit surface of the ring. The building material for these objects is a band of matter with constant thickness and average energy density per unit surface. Thus the baby skyrmions can be obtained as dimensional reductions of the three-dimensional skyrmions at large \( n \); while the three-dimensional skyrmions can be derived from the two-dimensional baby skyrmions as dimensional extensions.

In \[8\] it has been concluded that the Casimir energy, or quantum loop corrections, can destroy the binding properties of the two-skyrmion bound states. It would be worth to investigate the validity of this argument for the two- and three-skyrmion bound states of the NBM. Another interesting question is to determine to what extent the region of small enough \( a \) is of importance from the point of view of physics. For large \( a \) the method overestimates the skyrmion masses for the OBM but is accurate for the NBM, especially for large \( n \).
The existence of bound states of the three-dimensional skyrmions has rich phenomenological consequences in elementary particles and nuclear physics. It suggests possible existence of multibaryons with nontrivial flavor, strangeness, charm or beauty; more details are given in \[14\] and references therein. Similarly, the existence of bound states of two-dimensional baby skyrmions with universal properties in the NBM, which describes anisotropic systems, can also have some consequences for the condensed state physics, which would be worth to investigate in detail.

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