Scarf's theorems, simplices, and oriented matroids

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**Preface**

**Scarf proof of Brouwer’s fixed point theorem and Scarf combinatorial theorem.** In 1967 H. Scarf [Sc$_2$] suggested a new proof of Brouwer’s fixed point theorem. His paper opens with an outline of the well known proof of Brouwer’s theorem based on Sperner’s lemma and Knaster–Kuratowski–Mazurkiewicz argument. As is well known, this proof begins with a triangulation of the standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$ and eventually requires triangulations of $\Delta^n$ into arbitrarily small simplices.

Scarf’s proof follows a similar outline, but triangulations are replaced by sufficiently dense finite subsets $X \subset \Delta^n$. In particular, one can take set of vertices of a triangulation into small simplices as $X$. As in Sperner’s lemma, the points of $X$ are colored into $n + 1$ colors. Actually, *colorings* is a relatively modern terminology, neither Sperner, nor Scarf used it. Scarf’s key result is an analogue of Sperner’s lemma. It claims that under assumptions similar to the assumptions of Sperner’s lemma there exists $\sigma \subset X$ such that the colors of different elements of $\sigma$ are different and a technical condition holds. The latter ensures that $\sigma$ is small if $X$ is sufficiently dense. In contrast with Sperner’s lemma $\sigma$ may contain less than $n + 1$ points. Replacing sets of vertices of triangulations by sufficiently dense, but otherwise arbitrary, finite subsets of $\Delta^n$ requires a subtle modification of Knaster–Kuratowski–Mazurkiewicz coloring rule: the inequalities must be reversed.

While Scarf uses a very geometric language, his analogue of Sperner’s lemma is a truly combinatorial result and can be reformulated in terms of finite collections of linear orders on a finite set. See *Scarf’s combinatorial theorem* in Section 2. The origins of this theorem and its proof belong to the linear programming and the game theory. Actually, Scarf’s proof of Brouwer’s theorem is a byproduct of Scarf’s fundamental paper [Sc$_1$] in game theory.

**Vector colorings.** For the game theory Scarf needed to consider a much more general notion of colorings. Namely, he needed *vector colorings*, assigning to each point $x$ of $X \subset \Delta^n$ a vector $\varphi(x) \in \mathbb{R}^{n+1}$. The usual colorings are naturally interpreted as vector colorings taking as values the vertices of $\Delta^n$. Scarf’s analogue of Sperner’s lemma claims that under assumptions about a vector coloring $\varphi$ similar to the assumptions of Sperner’s lemma there exists a subset $\sigma \subset X$ such that $\varphi(\sigma)$ together with some vertices of $\Delta^n$ (which ones depends only on $\sigma$) forms a basis of $\mathbb{R}^{n+1}$ and, moreover, the coordinates of the center of the standard simplex $\Delta^n$ with respect to this basis are non-negative. The latter property replaces the classical property of having all available colors as colors of vertices of a simplex. See *Scarf theorem* in Section 7 for the details.

**Matroid colorings.** This theorem of Scarf appears to be a truly geometric one, in contrast with Scarf’s combinatorial theorem, and Scarf’s proof is based on geometric ideas from the linear programming. Nevertheless, as we will see, there is a purely combinatorial gener-
alization of this theorem of Scarf. Namely, one can replace vector colorings by colorings taking values in an oriented matroid, which may be thought as finite combinatorial models of vector spaces over ordered fields. We will call such colorings matroid colorings.

One of the main goals of the present paper is to prove a generalization of Scarf theorem to matroid colorings. See Sections 6 and 8. We attempted to made this generalization accessible to the readers not familiar even with the definition of oriented matroids. Section 5 is devoted to an introduction to oriented matroids, with more advanced results of Todd [T] and Las Vergnas [LV] relegated to two appendices. In Section 6 we prove our generalization in a non-degenerate case, where the main ideas are the most transparent. In Section 7 we prove the original Scarf’s theorem about vector colorings, including the degenerate case. Finally, in Section 8 we consider the general matroid colorings.

**Scarf’ methods and topology.** At the first sight Scarf’s methods appear to be radically different from the methods of algebraic (or combinatorial) topology. In any case, this was the first impression of the present author. For a topologist such as the present author, the most striking feature of Scarf’s methods is the absence of triangulations. Nevertheless, more general simplicial complexes are only hidden, and already H. Kuhn [Ku] related Scarf’s combinatorial theorem with Sperner’s lemma and classical topological concepts of pseudo-manifolds and simplicial subdivision.

In fact, the classical ideas of combinatorial topology provide the right framework for Scarf’s results. In the spirit of author’s papers [I1] – [I3] we approach the results of Scarf and their generalization to matroid colorings from the point of view of the classical combinatorial topology, with simplices, chains, and cochains being the main tools.

**Simplices, chains, and cochains.** The present paper begins with a generalization and refinement of the ideas of Kuhn [Ku]. The first basic notion is that of a pseudo-simplex, a simplicial complex equipped with a family of subcomplexes imitating the family of faces (of arbitrary codimension) of a simplex. For example, a closed pseudo-manifolds with a top-dimensional simplex removed, considered together with the collection of all proper faces of the removed simplex is a pseudo-simplex. Pseudo-manifolds with a top-dimensional simplex removed were used by Kuhn [Ku] in his approach to Scarf’s results, but in this situation the finer structure of faces is trivial, and was ignored by Kuhn.

The next basic notion is that of a chain-simplex. A chain-simplex is defined as a simplicial complex equipped with a family of subcomplexes imitating the family of faces of a simplex only at the level of chains with coefficients in the field of two elements \(\mathbb{F}_2\). Every pseudo-simplex is a chain-simplex in a natural way. The notion of a chain-simplex is an axiomatization of the main features behind the proofs of lemmas of Alexander and Sperner, as they are presented in [I2]. As we will see, the same structure is behind the proof of Scarf combinatorial theorem. Pseudo-simplices and chain-simplices are the topic of Section 1.
**Scarf's construction of pseudo-simplices.** From this point of view a key idea of Scarf can be interpreted as a construction of a pseudo-simplex from a family of linear orders on a finite set. This is purely combinatorial result. When combined with the analogue of Alexander's (or, what is the same, Sperner's) lemma from Section 1, this leads to Scarf's combinatorial theorem. This is the topic of Section 2.

In Section 3 we follow Scarf and apply Scarf's combinatorial theorem to finite subsets of $\Delta^n$ with orders defined by the barycentric coordinates, or, equivalently, by the usual coordinates in $\mathbb{R}^{n+1}$ (since $\Delta^n$ is the standard simplex in $\mathbb{R}^{n+1}$, they are the same). Together with the modified Knaster–Kuratowski–Mazurkiewicz argument this leads to Scarf's proof of Brouwer's fixed point theorem.

**Scarf's pseudo-simplices and triangulations.** One may suspect that the simplices of Scarf's pseudo-simplex constructed from a finite subset of $\Delta^n$ are, in fact, simplices of some triangulation of $\Delta^n$ with vertices belonging to $X$, and the whole proof is almost the standard proof in disguise. But, while there is a canonical surjective map from (the geometric realization of) this pseudo-simplex to $\Delta^n$, in general the images of the interiors of different simplices overlap and we don't get a triangulation of $\Delta^n$. Such examples were constructed by Scarf [Sc3]. We do not discuss them in the present paper. But for some natural choices of $X \subset \Delta^n$ one indeed gets triangulations of $\Delta^n$, and very beautiful ones. In Section 4 we discuss them in details following Scarf [Sc3]. These triangulations implicitly appeared in the works of Lebesgue. See [I3], Section 2 for the details. Later these triangularions were explicitly constructed by Freudenthal [Fr] as a solution of a problem of Brouwer. Simultaneously with Scarf these triangulations were rediscovered by Quillen in his work [Q] on the higher algebraic K-theory. See Segal [Se], Appendix 1.

**Scarf's proof of Kakutani fixed point theorem.** Kakutani fixed point theorem is a natural generalization of Brouwer's fixed point theorem to multivalued maps having closed convex sets as (sets of) values. Kakutani [Ka] deduced his theorem from Brouwer's. Scarf [Sc3] suggested a more natural proof and deduced Kakutani theorem directly from his theorem about vector colorings. In author's opinion, this proof clarifies both the Kakutani fixed point theorem and Scarf's theorem about vector colorings. In Section 9 we present a version of Scarf's proof making clear that for a vector coloring $\varphi$ the vectors $\varphi(x) - x$ should be thought as tangent vectors to $\Delta^n$.

**Combinatorial topology of $\mathbb{R}^n$.** In Section 10 we approach to this circle of questions from the viewpoint of the classical combinatorial topology. We consider rectilinear chains in $\mathbb{R}^n$ and prove the basic properties of the intersection numbers $c \cdot d$ of such chains in the case when the dimension of $c$ or $d$ is equal to 0 or 1. This leads to new proofs of Scarf and Kakutani theorems, and to a generalization of Kannai's [Ka] Generalized Sperner's lemma. But, naturally, these methods are not sufficient to deal with matroid colorings.
1. Pseudo-simplices

Simplex-families. Let $I$ be a finite set. A simplex-family based on $I$ is defined as a map $D : A \rightarrow D(A)$ assigning to every subset $A \subset I$ a simplicial complex $D(A)$ of dimension $|A| - 1$. Note that $D(B)$ is not assumed to be a subcomplex of $D(A)$ when $B \subset A$. It may be comfortable to think that all complexes $D(A)$ are subcomplexes of some universal simplicial complex.

For every finite set $A$ let $\Delta(A)$ be the abstract simplicial complex having $A$ as its set of vertices and all subsets of $A$ as its simplices. In other words, $\Delta(A)$ is the simplex having $A$ as the set of vertices and considered as a simplicial complex. A basic example of a simplex-family based on $I$ is the map $\Delta : A \rightarrow \Delta(A)$, where $A \subset I$.

Let $d(A)$ be the dimension of $\Delta(A)$, and let $e(A) = d(A) - 1$. So, $d(A) = |A| - 1$ and $e(A) = |A| - 2$. So, for a simplex-family $D$ based on $I$ and a subset $A \subset I$ the dimension of the simplicial complex $D(A)$ is equal to $d(A)$. We will be mostly interested in simplices of $D(A)$ of dimensions $d(A)$ and $e(A)$.

Pseudo-simplices. Suppose that $D$ is a simplex-family based on $I$. Let $A \subset I$ and let $\sigma$ be an $e(A)$-simplex of either $D(A)$ or $D(B)$ for some $e(A)$-subset $B \subset A$. Let us denote by $r_A(\sigma)$ the number of $d(A)$-simplices of $D(A)$ having $\sigma$ as a face, and by $s_A(\sigma)$ the number of $e(A)$-subsets $B \subset A$ such that $\sigma$ is a simplex of $D(B)$.

The simplex-family $D$ is said to be a pseudo-simplex if

$$(1) \quad r_A(\sigma) + s_A(\sigma) = 2$$

for every $A \subset I$ and $e(A)$-simplex $\sigma$ of either $D(A)$ or $D(B)$ for an $e(A)$-subset $B \subset A$.

Every triangulation $T$ of a geometric simplex $\delta$ leads to a pseudo-simplex. Let $I$ be the set of vertices of $\delta$. For $A \subset I$ let $\delta_A$ be the face of $\delta$ having $A$ as its set of vertices. In particular, $\delta = \delta_I$. The triangulation $T$ induces a triangulation $T_A$ of $\delta_A$. Let $D_T(A)$ be the abstract simplicial complex associated with $T_A$. The non-branching property of $T$ implies that the map $D_T : A \rightarrow D_T(A)$ is a pseudo-simplex. In fact, even more is true. Namely, every complex $D_T(A)$ is a pseudo-manifold of dimension $d(A)$. In particular, $D_T(A)$ is dimensionally homogeneous, i.e. every simplex is a face of a simplex of dimension $d(A)$. Therefore $r_A(\sigma) \geq 1$ for every simplex $\sigma$ of $D_T(A)$. In the next section we will deal with pseudo-simplices for which $r_A(\sigma) \geq 1$ does not necessarily hold.

Dimension $-1$. By a standard convention, for every simplicial complex the empty set $\emptyset$ is the only simplex of dimension $-1$. In an agreement with this, the boundary of every 0-simplex is interpreted as the unique $-1$-dimensional simplex $\emptyset$. 

The following lemma exploits these conventions. Perhaps, some readers will prefer to take its conclusion as a part of the definition of a pseudo-simplex.

1.1. Lemma. If $\mathcal{D}$ is a pseudo-simplex, then the complex $\mathcal{D}(A)$ has only one vertex for every 1-element subset $A \subset 1$.

**Proof.** If $|A| = 1$, then $e(A) = -1$ and hence $\emptyset$ there is the only $e(A)$-subset of $A$. The dimension of $\mathcal{D}(\emptyset)$ is $-1$ and hence $\emptyset$ is the only simplex of $\mathcal{D}(\emptyset)$. It follows that $s_A(\emptyset) = 1$. In view of (1) this implies that $r_A(\emptyset) = 1$. But every simplex has $\emptyset$ as a face and hence $r_A(\emptyset)$ is the number of 0-simplices of $\mathcal{D}(A)$. The lemma follows. ■

**Chain-simplices.** Let $\mathcal{D}$ be a simplex-family based on $I$. For every $A \subset I$ let $\mathcal{D}\llbracket A \rrbracket$ be the formal sum of all $d(A)$-simplices of $\mathcal{D}(A)$ considered as a chain with coefficients in $F_2$. In particular, $\mathcal{D}\llbracket \emptyset \rrbracket = \emptyset$, where $\emptyset$ in the right hand side is considered as $-1$-dimensional simplex. When there is no danger of confusion, the notation $\mathcal{D}\llbracket A \rrbracket$ will be abbreviated to $\llbracket A \rrbracket$. The simplex-family $\mathcal{D}$ is said to be a chain-simplex based on $I$ if for every $A \subset I$

\begin{equation}
\partial \mathcal{D}\llbracket A \rrbracket = \sum \mathcal{D}\llbracket B \rrbracket ,
\end{equation}

where the sum is taken over all simplices $B$ of $\Delta(A)$ of dimension $e(A)$, i.e. over all subsets $B$ of $A$ with $|B| = |A| - 1$. The notion of a chain-simplex is an axiomatization of the main homological relations underlying proofs of lemmas of Alexander and Sperner.

1.2. Lemma. Every pseudo-simplex is a chain-simplex.

**Proof.** Let us compute the both sides of (2). By the definition of the boundary operator $\partial$

\begin{equation}
\partial \mathcal{D}\llbracket A \rrbracket = \sum_{\sigma} r_A(\sigma) \sigma ,
\end{equation}

where the sum is taken over all $e(A)$-simplices $\sigma$ of $\mathcal{D}(A)$. On the other hand

\begin{equation}
\sum \mathcal{D}\llbracket B \rrbracket = \sum_{\sigma} s_A(\sigma) \sigma ,
\end{equation}

where the sum is taken over all $e(A)$-simplices $\sigma$ of complexes $\mathcal{D}(B)$ with $B \subset A$ and $|B| = |A| - 1$.

If an $e(A)$-simplex $\sigma$ is not a simplex of $\mathcal{D}(A)$, then $r_A(\sigma) = 0$. Similarly, if an $e(A)$-simplex $\sigma$ is not a simplex of $\mathcal{D}(B)$ for any $e(A)$-subset $B \subset A$, then $s_A(\sigma) = 0$. Therefore, the sums in the right hand sides of (3) and (4) can be both taken over all $e(A)$-simplices $\sigma$ of either $\mathcal{D}(A)$ or $\mathcal{D}(B)$ for some $e(A)$-subset $B \subset A$. Since $2 = 0$ in $F_2$, this observation together with (1) implies that these sums are equal. It follows that the left hand sides of (3) and (4) are equal and hence (2) holds. ■
Envelopes of simplex-families. Let $\mathcal{D}$ be a simplex-family based on $I$. The envelope of $\mathcal{D}$ is the simplex-family $\mathcal{E}$ based on $I$ and defined as follows.

Let $V_{\mathcal{D}}$ be the union of the sets of vertices of complexes $\mathcal{D}(A)$ with $A \subset I$. We may assume that $V_{\mathcal{D}}$ and $I$ are disjoint. In any case, nobody expects the sets $V_{\mathcal{D}}$, $I$ to intersect. If $A$ is a proper subset of $I$, then $\mathcal{E}(A) = \Delta(A)$. The simplicial complex $\mathcal{E}(I)$ has the union $V_{\mathcal{D}} \cup I$ as the set of vertices. For every $A \subset I$ the unions of the form $\sigma \cup K$, where $\sigma$ is a simplex of $\mathcal{D}(A)$ and $K$ is proper subset of $I$ disjoint from $A$, are simplices of $\mathcal{E}(I)$. Of course, if $A \neq \emptyset$, then $K$ is automatically a proper subset. There are no other simplices.

*-product of simplices. Suppose that $A \subset I$ and $A \neq \emptyset$. If $\sigma$ is a simplex of $\mathcal{D}(A)$ and $K \subset I \setminus A$, then $\sigma \cup K$ is a simplex of $\mathcal{E}(I)$. It will be convenient to denote the simplex $\sigma \cup K$ also by $\sigma \ast K$. The operation $(\sigma, K) \rightarrow \sigma \ast K$ extends to a bilinear product $\alpha \ast \Gamma$ between the chains $\alpha$ of $\mathcal{D}(A)$ and the chains $\Gamma$ of $\Delta(I \setminus A)$.

1.3. Lemma. $\partial(\sigma \ast K) = (\partial \tau) \ast K + \sigma \ast \partial K$.

Proof. The boundary $\partial(\sigma \ast K)$ is the sum of simplices obtained by removing from the simplex $\sigma \ast K$ either one vertex of $\sigma$ or one vertex of $K$. Clearly, $(\partial \tau) \ast K$ is the sum of the simplices of the first type and $\sigma \ast \partial L$ is the sum of the simplices of the second type. ■

1.4. Lemma. The dimension of a simplex of $\mathcal{E}(I)$ is equal to $d(I)$ if and only if it has the form $\sigma \ast (I \setminus A)$ for some non-empty subset $A \subset I$ and some $d(A)$-simplex $\sigma$ of $\mathcal{D}(A)$.

Proof. Let $A \subset I$. Let $\sigma$ be a simplex of $\mathcal{D}(A)$ and $K$ be a proper subset of $I$ disjoint from $A$. Clearly, $\sigma \ast K$ is a $d(I)$-simplex if and only if $|\sigma \cup K| = |I|$. Since the dimension of $\mathcal{D}(A)$ is equal to $d(A) = |A| - 1$ and hence $|\sigma| \leq |A|$, the latter condition holds if and only if $|\sigma| = |A|$ and $K = I \setminus A$. Since $K$ is a proper subset of $I$, the set $A$ has to be non-empty. This proves the “only if” part. The “if” part is obvious. ■

1.5. Theorem. If $\mathcal{D}$ is a pseudo-simplex, then its envelope $\mathcal{E}$ is also a pseudo-simplex. Also, $\mathcal{E}(I)$ is a non-branching complex and its boundary is equal to the boundary of $\Delta(I)$.

Proof. Let $A$ be a proper subset of $I$. Then $\mathcal{E}(A) = \Delta(A)$. Every $e(A)$-simplex of $\Delta(A)$ is a subset $C \subset A$ such that $|C| = |A| - 1$. Clearly, $A$ is the only $d(A)$-simplex of $\Delta(A)$ having $C$ as a face, and $C$ is a simplex of $\mathcal{E}(B) = \Delta(B)$ for a proper subset $B \subset A$ only if $B = C$. This proves (1) for $\mathcal{E}$ and proper subsets $A$ of $I$. It remains to deal with $\mathcal{E}(I)$.

Let $\tau \ast K$ be an $e(I)$-simplex of $\mathcal{E}(I)$, and let $D = I \setminus K$. Then $|\tau| = |D| - 1$ and $D \neq \emptyset$. By Lemma 1.4 every $d(I)$-simplex of $\mathcal{E}(I)$ has the form $\sigma \ast (I \setminus A)$ for some non-empty subset $A \subset I$ and some $d(A)$-simplex $\sigma$ of $\mathcal{D}(A)$. Obviously, $|\sigma| = |A|$. 

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If $\tau \ast K$ is a face of $\sigma \ast (I \smallsetminus A)$, then either $\tau$ is a face of $\sigma$ such that $|\sigma| = |\tau| + 1$ and $K = I \smallsetminus A$, or $\tau = \sigma$ and $K$ is a subset of $I \smallsetminus A$ such that $|K| = |I \smallsetminus A| - 1$. In the first case $A = I \smallsetminus K = D$, and in the second case $A$ is a subset of $I \smallsetminus K = D$ such that $|A| = |D| - 1$.

Conversely, if $\tau$ is a face of a simplex $\sigma$ of $D$ such that $|\sigma| = |\tau| + 1$, then the $\ast$-product $\sigma \ast (I \smallsetminus D) = \sigma \ast K$ is a simplex having $\tau \ast K$ as its face. The number of such simplices $\sigma \ast (I \smallsetminus D)$ is equal to $r_D(\tau)$. If $A$ is a subset of $D$ such that $|A| = |D| - 1$ and $A \neq \emptyset$, then the $\ast$-product $\tau \ast (I \smallsetminus A)$ is a simplex having $\tau \ast K$ as its face. Since $|A| = |D| - 1 = |\tau|$, the condition $A \neq \emptyset$ is equivalent to $\tau \neq \emptyset$. Hence if $\tau \neq \emptyset$, then the number of such simplices $\tau \ast (I \smallsetminus A)$ is equal to $s_D(\tau)$.

It follows that if $\tau \neq \emptyset$, then the number of $d(I)$-simplices of $E(I)$ having $\tau \ast K$ as a face is equal to $r_D(\tau) + s_D(\tau)$. By (1) this number is equal to 2. If $\tau = \emptyset$, then $|D| = 1$ and the only $d(I)$-simplices of $E(I)$ having $\tau \ast K = \emptyset \ast K = K$ as a face are simplices of the form $\sigma \ast K$, where $\sigma$ is a 0-simplex of $D$. By Lemma 1.1 the number of such simplices is equal to 1.

We see that every $e(I)$-simplex of $E(I)$ is either a face of exactly two $d(I)$-simplices, or is a subset $K \subset I$ such that $|K| = |I| - 1$ and is a face of exactly one $d(I)$-simplex. Therefore $E(I)$ is non-branching and (1) holds for $A = I$ and $E$ in the role of $D$. ■

1.6. Theorem. If $D$ is a chain-simplex, then $E$ is also a chain-simplex and $\partial E[I] = \partial I$.

Proof. With $E$ in the role of $D$, the equality (2) is trivial if $A \neq I$. Let us prove it for $A = I$. Lemma 1.4 implies that

$$E[I] = \sum_C \sum_\sigma \sigma \ast (I \smallsetminus C),$$

where $C$ runs over all non-empty subsets of $I$ and $\sigma$ over $d(C)$-simplices of $D(C)$ for each $C$. Together with Lemma 1.3 this implies that

$$\partial E[I] = \sum_C \sum_\sigma \partial \sigma \ast (I \smallsetminus C) + \sum_C \sum_\sigma \sigma \ast \partial (I \smallsetminus C).$$

Let us consider these two double sums separately. The first double sum is equal to

$$\sum_C \partial \left( \sum_\sigma \sigma \right) \ast (I \smallsetminus C) = \sum_C \left( \partial [\ ] \right) \ast (I \smallsetminus C),$$

where $[\ ]$ stands for $D[\ ]$. Now (2) implies that the first double sum is equal to

$$\sum_C \sum_D [D] \ast (I \smallsetminus C),$$
where $C$ runs over non-empty subsets of $I$ and $D$ runs over subsets of $I$ such that $D \subset C$ and $|D| = |C| - 1$. The second double sum is equal to

$$\sum_C \sum_\sigma \sigma \ast \left( \sum_E (I \setminus E) \right),$$

where $C$ runs over non-empty subsets of $I$ and $E$ runs over subsets of $I$ such that $C \subset E$ and $|E| = |C| + 1$. By interchanging the order of summation we see that it is equal to

$$\sum_C \sum_E \left( \sum_\sigma \sigma \ast (I \setminus E) \right) = \sum_C \sum_E \|C\| \ast (I \setminus E),$$

where $C$, $E$ are subject to the same conditions as above. By renaming $C$, $E$ into $D$, $C$ respectively, we see that the second double sum is equal to

$$(7) \quad \sum_D \sum_C \|D\| \ast (I \setminus C) = \sum_C \sum_D \|D\| \ast (I \setminus C),$$

where $D$ runs over non-empty subsets of $I$ and $C$ runs over subsets of $I$ such that $D \subset C$ and $|C| = |D| + 1$, or, what is the same, $|D| = |C| - 1$. The sums (6) and (7) differ only in that in (6) the set $D$ is allowed to be empty. Therefore in the sum of (6) and (7) all terms with non-empty $D$ cancel. Hence (5) implies that

$$\partial E \| I \| = \sum_C \| \emptyset \| \ast (I \setminus C)$$

$$= \sum_C \emptyset \ast (I \setminus C) = \sum_C I \setminus C = \partial I,$$

where $C$ runs over one-element subsets of $I$. It follows that

$$\partial \mathcal{E} \| I \| = \sum_D D = \sum_D \mathcal{E} \| D \|,$$

where $D$ runs over subsets of $I$ such that $|D| = |I| - 1$. This proves (2) for $A = I$ and $\mathcal{E}$ in the role of $\mathcal{D}$. The theorem follows. 

**Classical colorings.** Recall that $V_\mathcal{D}$ is the union of the sets of vertices of complexes $\mathcal{D}(A)$ with $A \subset I$. A classical coloring of $\mathcal{D}$ is a simply a map $V_\mathcal{D} \to I$. A classical coloring of $\mathcal{D}$ induces for every $A \subset I$ a map from the set of vertices of $\mathcal{D}(A)$, which we will consider as a simplicial map $\mathcal{D}(A) \to \Delta(I)$.

A classical coloring $c$ is called an Alexander–Sperner coloring if $c$ maps $\mathcal{D}(A)$ to $\Delta(A)$ for every $A \subset I$, i.e. if $c(v)$ belongs to $A$ for every subset $A \subset I$ and every vertex $v$ of $\mathcal{D}(A)$. A classical coloring $c$ is called a Scarf coloring if $c(v) \not\in A$ for every proper subset $A$ of $I$ and every vertex $v$ of $\mathcal{D}(A)$. Cf. [Sc4], Lemma 3.5.
It may happen that there are no Alexander–Sperner colorings. For example, if a vertex \( v \) is a vertex of complexes \( D(A) \) and \( D(A') \) for disjoint subsets \( A, A' \subset I \), then there are no Alexander–Sperner colorings. Similarly, it may happen that there are no Scarf colorings.

1.7. Theorem. Let \( c \) be an Alexander–Sperner coloring of a chain-simplex \( D \) based on \( I \). Then \( c(\sigma) = I \) for some \( d(1) \)-simplex \( \sigma \) of \( D(1) \) and the number of such \( \sigma \) is odd.

Proof. We will use the abbreviated notation \( \llbracket A \rrbracket = D(A) \). It is sufficient to prove that \( c_*[I] = I \). Using an induction by \( n = |I| - 1 \), we can assume that \( c_*[B] = B \) for every \( n \)-subset \( B \subset I \). Then (2) implies that

\[
\partial c_*[I] = c_*(\partial [I]) = c_*\left( \sum [B] \right) = \sum c_*[B] = \sum B = \partial I,
\]

where the sums are taken over all \( n \)-subsets \( B \) of \( I \). Since there is only one \( n \)-chain of \( \Delta(I) \) with the boundary equal to \( \partial I \), namely \( I \), it follows that \( c_*[I] = I \). ■

Extensions of classical colorings. Let \( c : V_D \to I \) be a map and let \( \varphi : V_D \cup I \to I \) be a map equal to \( c \) on \( V_D \) and inducing a bijection \( I \to I \). If \( \mathcal{E} \) is the envelope of \( D \) as above, then \( \varphi \) can be considered as a simplicial map \( \mathcal{E}(1) \to \Delta(I) \) equal to \( c \) on each complex \( D(A) \), where \( A \subset I \), and inducing an automorphism of the boundary \( \partial \Delta(I) \).

1.8. Lemma. If \( D \) is a chain-simplex, then \( \varphi_*([\mathcal{E} \llbracket I \rrbracket]) = I \).

Proof. By Theorem 1.6 the envelope \( \mathcal{E} \) is also a chain-simplex. By the definition of the envelope, \( \mathcal{E} \llbracket B \rrbracket = B \) for every proper subset \( B \subset I \). It follows that \( \partial \mathcal{E} \llbracket I \rrbracket = \partial I \) and

\[
\partial \varphi_*([\mathcal{E} \llbracket I \rrbracket]) = \varphi_*\left( \partial [\mathcal{E} \llbracket I \rrbracket] \right) = \varphi_*([\partial I]).
\]

But \( \varphi \) is equal to the identity on \( \partial \Delta(I) \) and hence \( \varphi_*([\partial I]) = \partial I \). It follows that

\[
\partial \varphi_*([\mathcal{E} \llbracket I \rrbracket]) = \partial I.
\]

Since the simplex \( I \) is the only \( d(1) \)-chain of \( \Delta(I) \) with the boundary \( \partial I \), this implies that \( \varphi_*([\mathcal{E} \llbracket I \rrbracket]) = I \). This completes the proof. ■

1.9. Theorem. If \( D \) is a chain-simplex, then for every classical coloring \( c : V_D \to I \) there exist a non-empty subset \( A \subset I \) and a simplex \( \sigma \) of \( D(A) \) such that \( c(\sigma) = A \). Moreover, the number of such pairs \( A, \sigma \) is odd.
Proof. Lemma 1.8 implies that \( \varphi(\rho) = I \) for some \( d(I) \)-simplex \( \rho \) of \( \mathcal{E}(I) \). By Lemma 1.4 the simplex \( \rho \) has the form \( \rho = \sigma \ast (I \setminus A) \) for a non-empty subset \( A \subset I \) and a \( d(A) \)-simplex \( \sigma \) of \( \mathcal{D}(A) \). Since \( |\rho| = |I| \) and \( \varphi(\rho) = I \), the map \( \rho \to I \) induced by \( \varphi \) is a bijection. Since \( \varphi \) is equal to the identity on \( I \), this implies that \( \varphi(\sigma) = A \). In turn, since \( \varphi \) is equal to \( c \) on \( \mathcal{D}(A) \), this implies that \( c(\sigma) = A \). In particular, there exist simplices \( \sigma \) with required properties.

Conversely, if \( A \) is a non-empty subset of \( I \) and \( \sigma \) is a simplex of the complex \( \mathcal{D}(A) \) such that \( c(\sigma) = A \), then \( \rho = \sigma \ast (I \setminus A) \) is a \( d(I) \)-simplex of \( \mathcal{E}(I) \) and \( \varphi(\rho) = I \). Therefore there is a one-to-one correspondence between pairs \( A, \sigma \) as in the theorem and \( d(I) \)-simplices \( \rho \) of \( \mathcal{E}(I) \) such that \( \varphi(\rho) = I \). But \( \varphi_*\left(\mathcal{E}[[I]]\right) = I \) implies that the number of such simplices \( \rho \) is odd. It follows that the number of pairs \( A, \sigma \) as in the theorem is also odd. This completes the proof. ■

1.10. Theorem. If \( \mathcal{D} \) is a chain-simplex and \( c : V_{\mathcal{D}} \to I \) is a Scarf coloring, then \( c(\sigma) = I \) for some simplex \( \sigma \) of \( \mathcal{D}(I) \) and the number of such simplices \( \sigma \) is odd.

Proof. By Lemma 1.4 every \( d(I) \)-simplex \( \rho \) of \( \mathcal{E}(I) \) has the form \( \rho = \sigma \ast (I \setminus A) \) for a non-empty subset \( A \subset I \) and a \( d(A) \)-simplex \( \sigma \) of \( \mathcal{D}(A) \). If \( \rho \) is not a simplex of \( \mathcal{D}(I) \), then \( A \) is a proper subset of \( I \). Since \( c \) is a Scarf coloring, this implies that \( c(\sigma) \) is disjoint from \( A \), i.e. \( c(\sigma) \subset I \setminus A \). Since \( \varphi \) is equal to the identity on \( I \), it follows that

\[
\varphi(\rho) = \varphi(\sigma \ast (I \setminus A)) \subset I \setminus A.
\]

But \( A \) is non-empty and hence \( \varphi(\rho) \) is an \( m \)-simplex for some \( m \leq d(I) - 1 \). Therefore \( \varphi_*(\rho) = 0 \) for every \( d(I) \)-simplex \( \rho \) which is not a simplex of \( \mathcal{D}(I) \). It follows that

\[
\varphi_*\left(\mathcal{D}[[I]]\right) = \varphi_*\left(\mathcal{E}[[I]]\right).
\]

Together with Lemma 1.8 this implies that \( c_*\left(\mathcal{D}[[I]]\right) = \varphi_*\left(\mathcal{D}[[I]]\right) = I \). Now the theorem follows by a standard argument. ■

Another proof of Theorem 1.7. In contrast with the first proof, this proof does not rely on induction by \( |I| \). Let \( f : I \to I \) be a bijection such that \( f(A) \neq A \) for every \( A \subset I \) different from \( \mathcal{D}, I \). Equivalently, let \( f \) be a cyclic permutation of \( I \) (i.e. a permutation consisting of one cycle). Let \( \varphi \) be the extension of \( c \) equal to \( f \) on \( I \). By Lemma 1.4 every \( d(I) \)-simplex \( \rho \) of \( \mathcal{E}(I) \) has the form \( \rho = \sigma \ast (I \setminus A) \) for a non-empty subset \( A \) of \( I \) and a simplex \( \sigma \) of \( \mathcal{D}(A) \). Suppose that \( \varphi(\rho) = I \). Since \( c \) is an Alexander–Sperner coloring, \( \varphi(\sigma) \subset A \) and hence \( \varphi(\rho) = I \) implies that \( \varphi(I \setminus A) = I \setminus A \). Since \( A \) is non-empty, by the choice of \( \varphi \) this is possible only if \( A = I \). In this case \( \rho = \sigma \) and \( \sigma \) is a simplex of \( \mathcal{D}(I) \). Therefore \( d(I) \)-simplex \( \rho \) of \( \mathcal{E}(I) \) is actually a simplex of \( \mathcal{D}(I) \). By a standard argument Lemma 1.8 implies that the number of such simplices \( \rho \) is odd. ■
2. Combinatorics of families of linear orders

**Linear orders and dominant sets.** Let $X$ be a non-empty finite set. Suppose that a family of linear orders $<_i$ on $X$, labeled by elements $i$ of a finite set $I$, is given. For a non-empty subset $\sigma \subset X$ let $\min_i \sigma$ be the minimal element of $\sigma$ with respect to the order $<_i$. A subset $\sigma \subset X$ is said to be **dominant with respect to** a non-empty subset $C$ of $I$ if there is no element $y \in X$ such that $\min_i \sigma <_i y$ for all $i \in C$.

By a convention, $\emptyset$ is dominant with respect to every $C \neq \emptyset$, but not with respect to $\emptyset$.

**2.1. Lemma.** If $\sigma \subset X$ is dominant with respect to $C \subset I$, then $\sigma = \{ \min_i \sigma \mid i \in C \}$.

**Proof.** Clearly, $\sigma$ contains all minima $\min_i \sigma$. Suppose that $x \in \sigma$ is different from all $\min_i \sigma$ with $i \in C$. Then $\min_i \sigma <_i x$ for all $i \in C$, contrary to the assumption. ■

**2.2. Corollary.** If $\sigma \subset X$ is dominant with respect to $C \subset I$, then $|\sigma| \leq |C|$. ■

**Deleting and adding elements.** For a set $A$ and an element $a \in A$ we will denote by $A - a$ the set $A \setminus \{a\}$. Similarly, for $b \notin A$ we will denote by $A + b$ the set $A \cup \{b\}$. The set $A - a$ is defined only if $a \in A$, and the set $A + b$ is defined only if $b \notin A$. We will interpret $A - a + b$ as $(A - a) + b$, and interpret similar expressions in the same way.

**Cells and faces.** Let us fix a subset $C \subset I$ and suppose that $\sigma$ is a subset of $X$. We will say that $\sigma$ is a $C$-cell if $\sigma$ is dominant with respect to $C$ and $|C| = |\sigma|$, and that $\sigma$ is a $C$-face if $\sigma$ is dominant with respect to $C$ and $|C| = |\sigma| + 1$.

Let $m_\sigma : C \to \sigma$ be the map defined by the rule

$$i \mapsto \min_i \sigma.$$ 

By Lemma 2.1, if $\sigma$ is dominant with respect to $C$, then $m_\sigma$ is a surjection. Therefore, if $\sigma$ is a $C$-cell, then $m_\sigma$ is a bijection and and for every $x \in \sigma$ there is a unique element $i \in C$ such that $x = \min_i \sigma$. We will denote this element $i$ by $i_\sigma(x)$. In other words, if $\sigma$ is a $C$-cell, then $i_\sigma$ is the inverse of $m_\sigma$.

**From cells to faces.** Clearly, if $\sigma$ is a $C$-cell and $i \notin C$, then $\sigma$ is also a $(C + i)$-face. Removing an element from a $C$-cell results in a $C$-face. Indeed, suppose that $\sigma$ is a $C$-cell and $x \in \sigma$. Since $\min_i \sigma \leq_i \min_i (\sigma - x)$ for every $i$, the set $\sigma - x$ is dominant with respect to $C$ together with $\sigma$. Also, $|C| = |\sigma - x| + 1$. Hence $\sigma - x$ is a $C$-face.
From faces to cells. Let $\sigma$ be a non-empty $C$-face. Then there are two elements $k, l \in C$ such that $m_\sigma(k) = m_\sigma(l)$ and $m_\sigma$ is injective on $C - k - l$. For every $j \in C$ let

$$M_j = \{ y \in X \mid \min_i \sigma <_i y \text{ for all } i \in C - j \}.$$ 

If $M_j \neq \emptyset$, then we will denote by $m_j$ the maximal element of $M_j$ with respect to $<_j$.

2.3. Lemma. The set $\sigma + a$ is a $C$-cell if and only if $a = m_j$ for some $j \in \{k, l\}$ such that $M_j \neq \emptyset$.

Proof. To begin with, let us observe that

\begin{align*}
\text{(8)} \quad \min_i (\sigma + a) &= a \quad \text{if} \quad a <_i \min_i \sigma, \quad \text{and} \\
\text{(9)} \quad \min_i (\sigma + a) &= \min_i \sigma \quad \text{if} \quad \min_i \sigma <_i a.
\end{align*}

In particular, $\min_i (\sigma + a) = \min_i \sigma$ or $a$ for every $i \in C$.

Since $\sigma$ is a $C$-face, $\sigma$ is dominant with respect to $C$ and Lemma 2.1 implies that

\begin{align*}
\text{(10)} \quad \{ \min_i \sigma \mid i \in C \} &= \sigma.
\end{align*}

If $\sigma + a$ is a $C$-cell, then $\sigma + a$ is also dominant with respect to $C$ and

\begin{align*}
\text{(11)} \quad \{ \min_i (\sigma + a) \mid i \in C \} &= \sigma + a
\end{align*}

by Lemma 2.1.

If (10) and (11) hold, then

$$\min_i (\sigma + a) = \min_i \sigma$$

for all $i \in C - \{k, l\}$ and for $i$ equal to one of the elements of the pair $\{k, l\}$, and

$$\min_i (\sigma + a) = a$$

for $i$ equal to the other element of $\{k, l\}$.

Therefore, if $\sigma + a$ is a $C$-cell, then we may assume that

\begin{align*}
\text{(12)} \quad \min_i (\sigma + a) &= \min_i \sigma \quad \text{for all} \quad i \in C - k \quad \text{and} \quad \min_k (\sigma + a) = a.
\end{align*}
By (8) and (9) in this case

\[ \min_i \sigma <_i a \quad \text{for all} \quad i \in C - k \quad \text{and} \quad a <_k \min_k \sigma. \]

It follows that \( a \in \mathcal{M}_k \). Since \( \sigma + a \) is dominant with respect to \( C \), the element \( a \) is the maximal in \( \mathcal{M}_k \) with respect to \( <_k \). In other terms, \( a = m_k \).

Conversely, if, say, \( \mathcal{M}_k \neq \emptyset \) and \( a \in \mathcal{M}_k \), then

\[ \min_i \sigma <_i a \quad \text{for all} \quad i \in C - k. \]

If also \( \min_k \sigma <_k a \), then \( \sigma \) is not dominant with respect to \( C \), contrary to the assumption. Therefore \( a <_k \min_k \sigma \). By applying (8) and (9) we see that (12) holds. It follows that if \( a = m_k \), then \( \sigma + a \) is dominant with respect to \( C \) and \( \sigma + a \) is a C-cell. 

2.4. Lemma. If \( \mathcal{M}_k \) and \( \mathcal{M}_l \) are both non-empty, then \( m_k \neq m_l \) and hence the C-cells \( \sigma + m_k \) and \( \sigma + m_l \) are not equal.

**Proof.** Since \( \sigma \) is dominant with respect to \( C \), the sets \( \mathcal{M}_k \) and \( \mathcal{M}_l \) are disjoint. The lemma follows. 

2.5. Lemma. The set \( \sigma \) is a \((C - j)\)-cell if and only if \( j \in \{k, l\} \) and \( \mathcal{M}_j = \emptyset \).

**Proof.** Since \( \sigma \) is a C-face, \( |\sigma| = |C| - 1 = |C - j| \). The condition \( \mathcal{M}_j = \emptyset \) is obviously equivalent to \( \sigma \) being dominant with respect to \( C - j \). Therefore it remains to show that \( \mathcal{M}_j \neq \emptyset \) if \( j \not\in \{k, l\} \). But if \( j \not\in \{k, l\} \), then

\[ \min_i \sigma \neq \min_j \sigma \]

for every \( i \neq j \). It follows that in this case

\[ \min_i \sigma <_i \min_j \sigma \]

for every \( i \neq j \) and hence \( \min_j \sigma \in \mathcal{M}_j \). 

2.6. Theorem (The main combinatorial lemma). If \( \sigma \) is a non-empty C-face, then \( \sigma \) is contained in no more than two C-cells. If \( r \) is the number of C-cells containing \( \sigma \) and \( s \) is the number of subsets \( D \subset C \) such that \( \sigma \) is a \( D \)-cell, then \( r + s = 2 \).

**Proof.** This follows from Lemmas 2.3 – 2.5. The number \( r \) is equal to the number of non-empty sets among \( \mathcal{M}_k \) and \( \mathcal{M}_l \).
The pseudo-simplex associated with a family of orders. Theorem 2.6 is an analogue of the non-branching property of triangulations of geometric simplices. Let us introduce an analogue of the pseudo-simplices defined by triangulations of geometric simplices (see Section 1). For every subset \( A \subset I \) let \( \mathcal{T}(A) \) be an abstract simplicial complex defined as follows. The set of vertices of \( \mathcal{T}(A) \) is the union of all \( A \)-cells, and a set of vertices is a simplex if and only if it is contained in a \( A \)-cell. Clearly, the map \( A \mapsto \mathcal{T}(A) \) is a simplex-family.

2.7. Theorem. The simplex-family \( \mathcal{T} \) is a pseudo-simplex.

Proof. Suppose that \( A \) be a subset of \( I \). Let us prove (1) for \( \mathcal{D} = \mathcal{T} \). Let us consider some \( e(A) \)-simplex \( \sigma \) of \( \mathcal{T}(A) \). By the definition the simplex \( \sigma \) is contained in some \( A \)-cell, i.e. \( \sigma + a \) is an \( A \)-cell for some \( a \). In particular, \( \sigma \) is an \( A \)-face. By Theorem 2.6 either \( \sigma \) is a face of exactly two \( d(A) \)-simplices, or \( \sigma \) is a face of exactly one \( d(A) \)-simplex and is a \( B \)-cell for exactly one subset \( B \subset A \). This implies that \( \mathcal{T} \) is a pseudo-simplex. ■

2.8. Corollary. Let \( \mathcal{I} \) be the envelope of \( \mathcal{T} \). Then \( \mathcal{I}(I) \) is a non-branching dimensionally homogeneous simplicial complex.

Proof. The envelope \( \mathcal{I}(I) \) is dimensionally homogenous by the construction. Theorem 1.5 implies that \( \mathcal{I}(I) \) is non-branching. ■

2.9. Corollary. The simplex-family \( \mathcal{T} \) is a chain-simplex.

Proof. It is sufficient to combine Theorem 2.7 and Lemma 1.2. ■

Classical colorings. In the present context the definitions of colorings from Section 1 take the following form. A classical coloring is a map \( c : X \rightarrow I \). The map \( c \) is called an Alexander–Sperner coloring if \( c \) maps \( \mathcal{T}(A) \) to \( \Delta(A) \) for every \( A \subset I \), i.e. if \( c(v) \in A \) for every element \( v \) of every \( A \)-cell. Similarly, \( c \) is called a Scarf coloring if \( c(v) \notin A \) for every element \( v \) of every \( A \)-cell such that \( A \neq I \).

Scarf’s combinatorial theorem. For every classical coloring \( c : X \rightarrow I \) there exist a non-empty subset \( A \subset I \) and an \( A \)-cell \( \sigma \) such that \( c(\sigma) = A \).

Proof. It is sufficient to combine Theorem 1.9 with Corollary 2.9. ■

Scarf’s dual form of Sperner’s lemma. If \( c : X \rightarrow I \) is an Scarf coloring, then \( c(\sigma) = I \) for some \( I \)-cell \( \sigma \) and the number of such \( I \)-cells \( \sigma \) is odd.

Proof. It is sufficient to combine Theorem 1.10 with Corollary 2.9. ■
3. Scarf’s proof of Brouwer’s fixed point theorem

The standard $n$-simplex. Let $n$ be a natural number and $I = \{0, 1, \ldots, n\}$. We will number the coordinates in $\mathbb{R}^{n+1}$ by $i \in I$. For a point $x \in \mathbb{R}^{n+1}$ we will denote by $x_i$ its $i$th coordinate, so that $x = (x_0, x_1, \ldots, x_n)$. Let $\Delta^n \subset \mathbb{R}^{n+1}$ be defined by the equation

$$x_0 + x_1 + \ldots + x_n = 1$$

and the inequalities $x_i \geq 0$ for each $i \in I$. Then $\Delta^n$ is the standard $n$-simplex.

Approximating $\Delta^n$ by finite subsets. Let $X \subset \Delta^n$ be a finite set. If $X$ is sufficiently dense in $\Delta^n$, then $X$ may serve as an approximation to $\Delta^n$. The set $X$ is not assumed to have any additional structure. In particular, $X$ is not assumed to be the set of vertices of a triangulation of $\Delta^n$. Instead, an additional structure is induced on $X$ from $\Delta^n$. Let us choose for each $i \in I$ a linear order $<_i$ on $X$ such that $x_i <_i y_i$ implies $x <_i y$ for every $x, y \in X$. Obviously, such orders exist. Moreover, under a mild non-degeneracy assumption such orders are unique. In fact, the order $<_i$ is uniquely determined if and only if no two different elements of $X$ have the same $i$th coordinate. This condition can be achieved by a small perturbation of $X$, but it precludes having many points of $X$ on faces of $\Delta^n$, which is an important feature of approximations by the vertices of a triangulation.

Geometric simplices associated with subsets of $X$. Suppose that $\sigma \subset X$ and $C \subset I$. Then the subsets $\sigma$ and $C$ lead to a geometric simplex $\Delta(\sigma, C)$ contained in $\Delta^n$. It is defined as follows. For each $i \in I$ let $m_i = \min \sigma_i$, and, in an agreement with the above notations, let $m_{ii}$ be the $i$th coordinate of $m_i$. Clearly, $m_{ii} \leq 1$ for every $i$. Let $\Delta(\sigma, C)$ be the subset of $\mathbb{R}^{n+1}$ defined by the equation

$$x_0 + x_1 + \ldots + x_n = 1$$

and the inequalities $x_i \geq m_{ii}$ with $i \in C$ and $x_i \geq 0$ with $i \in I \setminus C$. The definition of the numbers $m_{ii}$ implies that $\sigma \subset \Delta(\sigma, C)$. In particular, $\Delta(\sigma, C)$ is non-empty.

Let us introduce new coordinates $y_0, y_1, \ldots, y_n$ by setting

$$(13) \quad y_i = x_i - m_{ii} \quad \text{for} \quad i \in C$$

and $y_i = x_i$ for $i \in I \setminus C$. In these coordinates $\Delta(\sigma, C)$ is defined by the equation

$$y_0 + y_1 + \ldots + y_n = 1 - \sum_{i \in C} m_{ii}$$

and the inequalities $y_i \geq 0$ for all $i \in I$. Since $\Delta(\sigma, C) \neq \emptyset$, the right hand side of the last equation is $\geq 0$. If the right hand side is equal to 0, then $\Delta(\sigma, C)$ is a 1-point set.
i.e. a 0-simplex. Since $\sigma \subset \Delta(\sigma, C)$, in this case $\sigma$ is also a 1-point set. If the right hand side is $>0$, then $\Delta(\sigma, C)$ is a geometric $n$-simplex. Moreover, in this case $\Delta(\sigma, C)$ is similar to $\Delta^n$ in the sense of elementary geometry. In fact, up to a translation $\Delta(\sigma, C)$ is homothetic to $\Delta^n$ with the ratio equal to the right hand side of (13).

Clearly, the $(n-1)$-faces of $\Delta(\sigma, C)$ are defined by the equations $y_i = 0$. Equivalently, they are defined by the equations $x_i = m_{ii}$ with $i \in C$ and $x_i = 0$ with $i \in I \sim C$. In particular, the intersection of $\Delta(\sigma, C)$ with the face of $\Delta^n$ defined by the equations $x_i = 0$ with $i \in I \sim C$ is non-empty.

### 3.1. Lemma

Let $\varepsilon > 0$. If $\varepsilon' > 0$ is sufficiently small and $X$ is $\varepsilon'$-dense in $\Delta^n$, then the diameter of $\Delta(\sigma, C)$ is $< \varepsilon$ for every subset $\sigma \subset X$ dominant with respect to $C \subset I$.

**Proof.** Let $d$ be the diameter of the simplex $\Delta^n$ and let $r$ be the distance from its barycenter to its boundary. Suppose that $\varepsilon' < \varepsilon r/d$ and that $X$ is $\varepsilon'$-dense in $\Delta^n$.

Let $\sigma \subset X$ be a subset dominant with respect to $C \subset I$. If $\sigma$ consists of 1 point, then $\Delta(\sigma, C)$ is also consists of 1-point and hence its diameter is 0. Therefore we can assume that $|\sigma| > 1$ and hence $\Delta(\sigma, C)$ is an $n$-simplex. By the definition of dominant sets, the simplex $\Delta(\sigma, C)$ does not contain elements of $X$ in its interior.

Suppose that the diameter of $\Delta(\sigma, C)$ is $\geq \varepsilon$. Since $\Delta(\sigma, C)$ is similar to $\Delta^n$, in this case the distance from the barycenter of the simplex $\Delta(\sigma, C)$ to its boundary is $\geq \varepsilon r/d$. Since $X$ is $\varepsilon'$-dense in $\Delta^n$ and $\varepsilon' < \varepsilon r/d$, there is a point $x \in X$ with the distance $< \varepsilon r/d$ from the barycenter. Clearly, $x$ is contained in the interior of $\Delta(\sigma, C)$, contrary to the above observation. The contradiction shows that the diameter of $\Delta(\sigma, C)$ is $< \varepsilon$. ■

**Scarf’s modification of KKM colorings.** Let us consider a continuous map $f : \Delta^n \rightarrow \Delta^n$. If $x \in X$ and $y = f(x)$, then

$$x_0 + x_1 + \ldots + x_n = y_0 + y_1 + \ldots + y_n$$

and hence there exists $i \in I$ such that $x_i \leq y_i$. Let $c(x)$ be equal to some such $i$. Then $c$ is a map $X \rightarrow I$. This is the classical coloring associated by Scarf with $f$ and $X$.

Note that Scarf’s rule defining $c$ is nearly opposite to the Knaster–Kuratowski–Mazurkiewicz rule, which allows color $i$ if $x_i \geq y_i$. At the first sight it seems that the difference between these two rules is insignificant. But the KKM argument does not work with Scarf’s rule, and Scarf’s proof does not work with the KKM rule.

**Scarf’s proof of Brouwer’s theorem.** Let $X_1, X_2, X_3, \ldots$ be a sequence of finite subsets of $\Delta^n$. Every $X_k$ is equipped with the corresponding orders $<_i$, where $i \in I$. If we as-
assume that the above non-degeneracy assumption holds for every $X_k$, then these orders are uniquely determined. By Scarf’s combinatorial theorem there exist subsets $\sigma_k \subset X_k$ and $C_k \subset I$ such that $\sigma_k$ is dominant with respect to $C_k$ and $c(\sigma_k) = C_k$. Let
\[
\Delta_k = \Delta(\sigma_k, C_k).
\]

We claim that for each $i \in I$ the simplex $\Delta_k$ contains a point $x$ such that $x_i \leq y_i$, where $y = f(x)$. Indeed, if $i \in C_k$, then $c(x) = i$ for some $x \in \sigma_k$. Then $x_i \leq y_i$ by the choice of $c$. If $i \in I \setminus C_k$, then the intersection of $\Delta_k$ with the face of $\Delta^n$ defined by the equation $x_i = 0$ is non-empty and hence $x_i = 0 \leq y_i$ for any point $x$ in this intersection. This proves our claim.

Now suppose that the sets $X_k$ are chosen in such a way that $X_k$ is $\varepsilon_k$-dense in $X_k$ for some sequence $\varepsilon_k \rightarrow 0$. Lemma 3.1 implies that the diameters of the simplices $\Delta_k$ tend to 0. After passing to a subsequence we can assume that simplices $\Delta_k$ converge to a single point $x \in \Delta^n$, i.e. every sequence of points $x(k) \in \Delta_k$ converges to $x$. Since $f$ is continuous, the inequalities of the previous paragraph survive passing to the limit. It follows that $x_i \leq y_i$ for every $i \in I$, where $y = f(x)$. Together with (14) this implies that $x_i = y_i$ for every $i \in I$, i.e. $x = y = f(x)$. Hence $x$ is a fixed point of $f$. □

Orders and triangulations. One may ask if Scarf’s proof only avoids mentioning triangulations of $\Delta^n$, but nevertheless they are working in the background. Indeed, the complex $\mathcal{T}(I)$ from Section 2 played a role similar to triangulations. The set $X$ seems to be a natural candidate to be the set of the vertices of a hidden triangulation, but $X$ can be entirely contained in the interior of $\Delta^n$ and hence has not enough points to triangulate $\Delta^n$. A natural way to deal with this issue is to enlarge $\Delta^n$ to include vertices corresponding to elements of $I$. Let $E^n \subset \mathbb{R}^{n+1}$ be defined by the same equation as $\Delta^n$ and the inequalities $x_i \geq 1 - n$ with $i \in I$. Then $E^n$ is a simplex containing $\Delta^n$. Its vertices are
\[
v_i = (1, \ldots, 1, 1 - n, 1, \ldots, 1),
\]
where $1 - n$ stands at the $i$th place and 1’s at all other places. Let $\mathcal{I}(I)$ be the envelope of $\mathcal{T}(I)$ in the sense of Section 1. Let $V = \{v_i \mid i \in I\}$ and let us identify each vertex $i \in I$ of $\mathcal{I}(I)$ with the point $v_i \in E^n$. This identifies the set of vertices of $\mathcal{I}(I)$ with $X \cup V$. For every simplex $\sigma \subset X \cup V$ of $\mathcal{I}(I)$ let us denote by $\Delta(\sigma)$ the convex hull of $\sigma$ in $\mathbb{R}^{n+1}$. If we assume that no $n + 1$ elements of $X \cup V$ are linearly dependent (which can be achieved by a small perturbation of $X$), then every $\Delta(\sigma)$ is a simplex and the collection of these simplices seems to be a good candidate for a triangulation of $E^n$ behind the proof. This idea is supported by a theorem of Scarf to the effect that the union of simplices $\Delta(\sigma)$ is equal to $E^n$ (see [Sc3], Theorem 7.2.2). But, as an example of Scarf shows, these simplices do not always form a triangulation: different $n$-simplices $\Delta(\sigma)$ may have common interior points (see [Sc3], the end of Section 7.2). So, the envelope $\mathcal{I}(I)$ of $\mathcal{T}(I)$ seems to be the right analogue of a triangulation in Scarf’s method.
4. An example: integer points in a simplex

Cyclic permutations of coordinates. Let \( n \) be a natural number, \( I = \{0, 1, \ldots, n\} \), and let us number the coordinates in \( \mathbb{R}^{n+1} \) by \( i \in I \). In this section we will often treat elements of \( I \) as integers modulo \( n + 1 \). Actually, we will need only the resulting cyclic order on \( I \). Let \( t: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \) be the map induced by the cyclic shift of coordinates, i.e. let

\[
t(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_0)
\]

and let \( t_i \) be the \( i \)-fold composition \( t \circ t \circ \cdots \circ t \). Then

\[
t_i(x_0, x_1, \ldots, x_n) = (x_i, x_{i+1}, \ldots, x_{n+1}),
\]

where the addition in subscripts is understood modulo \( n + 1 \). In particular, \( t_0 = \text{id} \).

Lexicographic orders. Let \( x, y \in \mathbb{R}^{n+1} \) and

\[
x = (x_0, x_1, \ldots, x_n), \quad y = (y_0, y_1, \ldots, y_n).
\]

The inequality \( x < y \) in the lexicographic order \(<\) means that \( x \neq y \) and \( x_i < y_i \) for the smallest \( i \) such that \( x_i \neq y_i \). Together the maps \( t_i \) the order \(<\) generates a lexicographic order \(<_i\) for every \( i \in I \). Namely, the inequality \( x <_i y \) means that

\[
t_i(x) < t_i(y)
\]

Clearly, \(<_0\) is the same order as \(<\).

Integer points in a simplex. Let \( N \) be a natural number, and let \( D \) be the set of all points

\[
a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}
\]

such that \( a_i \) are integers \( \geq 0 \) and \( a_0 + a_1 + \ldots + a_n = N \). Equivalently, \( D \) is the set of integer points in the \( n \)-simplex \( \Delta \) defined by the equation \( x_0 + x_1 + \ldots + x_n = N \) and inequalities \( x_i \geq 0 \). We will consider \( D \) together with the orders \(<_i\) restricted to \( D \).

The operators \( S_i \). Let \( i \in I \) and let \( a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1} \). Then

\[
S_i(a) = (a_0, \ldots, a_{i-1} + 1, a_i - 1, \ldots, a_n),
\]

where the subtraction of \( 1 \) in the subscript is understood modulo \( n + 1 \). In other words, \( S_0(a) = (a_0 - 1, a_1, \ldots, a_{n-1}, a_n + 1) \).
4.1. Lemma. If \( i \geq 1 \), then \( a <_i S_0(a) \).

Proof. Since \( t_i(a) = (a_i, a_{i+1}, \ldots, a_n, \ldots) \) and
\[
t_i(S_0(a)) = (a_i, a_{i+1}, \ldots, a_n + 1, \ldots),
\]
this follows directly from the definitions.

The I-cells. Our first goal is to prove some necessary conditions for a subset of \( D \) to be an I-cell. Eventually these necessary conditions will turn out to be also sufficient. Suppose that \( \sigma \subset D \) is an I-cell. Then the set \( \sigma \) consists of \( n + 1 \) elements, which we will denote by \( a(0), a(1), \ldots, a(n) \). Let \( a_i(k) \) be the \( i \)th coordinate of \( a(k) \), i.e. let
\[
a(k) = (a_0(k), a_1(k), \ldots, a_n(k)).
\]
Without any loss of generality we can assume that
\[
(15) \quad a(0) <_0 a(1) <_0 \cdots <_0 a(n)
\]
and hence, in particular,
\[
(16) \quad a_0(0) \leq a_0(1) \leq \cdots \leq a_0(n).
\]

4.2. Lemma. For each \( i \in I \) not all coordinates \( a_i(k), 0 \leq k \leq n, \) are equal.

Proof. Indeed, if all \( a_i(k) \) are equal, the \( <_i \) and \( <_{i+1} \) induce the same orders on \( \sigma \) and hence \( \min_i \sigma = \min_{i+1} \sigma \). This contradicts to the fact that \( \sigma \) is a I-cell.

4.3. Lemma. For each \( i \) the coordinates \( a_i(k), 0 \leq k \leq n, \) differ by no more than 1.

Proof. Suppose that \( i = 0 \) and that some coordinates \( a_0(i) \) differ by at least 2. Then (16) implies that \( a_0(n) \geq a_0(0) + 2 \) and hence \( a_0(n) - 1 > a_0(0) \). Let
\[
a = S_0(a(n)).
\]
Then \( a_0(n) - 1 > a_0(0) \) implies that \( a \in D \) and \( a(0) <_0 a \). Lemma 4.1 implies that \( a(n) <_i a \) for \( i \geq 1 \). Therefore for every \( i \in I \) there exists \( b \in \sigma \) such that \( b <_i a \) and hence \( \min_i \sigma <_i a \). In view of Lemma 2.1 this contradicts to \( \sigma \) being an I-cell.

This proves the lemma for \( i = 0 \). The general case reduces to this one by a cyclic permutation of coordinates.
4.4. Lemma. If \( a_{i-1}(k) = a_{i-1}(k-1) + 1 \), then \( a(k) = \min_i \sigma \), the sequence

\[
a(k), a(k+1), \ldots, a(n), a(0), a(1), \ldots, a(k-1)
\]

is increasing with respect to the order \( <_i \), and \( a_i(k) = a_i(k-1) - 1 \).

Proof. Let us consider the case \( i = 1 \) first. Lemma 4.3 together with the inequalities \((16)\) implies that the coordinates \( a_0(l) \) with \( l \leq k - 1 \) are equal, as also the coordinates \( a_0(l) \) with \( l \geq k \). In turn, this implies that the orders \( <_0 \) and \( <_1 \) coincide on the sets

\[
\{ a(l) \mid l \leq k - 1 \} \quad \text{and} \quad \{ a(l) \mid l \geq k \}.
\]

Together with \((15)\) this implies that the sequence \( a(l) \) is increasing with respect to the order \( <_1 \) for \( l \leq k - 1 \) and for \( l \geq k \). This implies, in particular, that the minimum \( \min_1 \sigma \) could be equal only to \( a(k) \) or \( a(0) \). But \( a(0) = \min_0 \sigma \), and the minima with respect to different orders are different because \( \sigma \) is an I-cell. Therefore \( \min_1 \sigma = a(k) \).

Let us prove that \( a(n) <_1 a(0) \). Lemma 4.2 together with the inequalities \((16)\) implies that \( a_0(n) = a_0(0) + 1 \). Since the sums of coordinates of \( a(n) \) and \( a(0) \) are both equal to \( N \), this implies that \( a_j(n) \neq a_j(0) \) for some \( i \geq 1 \). Let \( j \) be the minimal such subscript. Suppose that \( a_j(0) < a_j(n) \) and let

\[
a = S_0(a(n)).
\]

We claim that \( a(0) <_0 a \). Indeed, \( a_0 = a_0(n) - 1 = a_0(0) \) and \( a_m = a_m(0) \) for \( 1 \leq m \leq j - 1 \) by the choice of \( j \). Since \( a_j(n) \leq a_j \), the assumption \( a_j(0) < a_j(n) \) implies that \( a_i(0) < a_i \). It follows that \( a(0) <_0 a \), proving our claim. At the same time Lemma 4.1 implies that \( a(n) <_j a \) for \( j \geq 1 \). Therefore for every \( j \in I \) there exists \( b \in \sigma \) such that \( b <_j a \). But this contradicts to \( \sigma \) being an I-cell. Therefore our assumption is wrong and actually \( a_j(n) < a_j(0) \). By the choice of \( j \) this implies that \( a(n) <_1 a(0) \). Together with the observations in the previous paragraph this implies that for \( i = 1 \) the sequence from the lemma is indeed increasing.

Since this sequence is increasing, Lemma 4.2 implies that \( a_1(k) < a_1(k-1) \). Now Lemma 4.3 implies that \( a_1(k) = a_1(k-1) - 1 \). This completes the proof in the case \( i = 1 \). The general case reduces to this one by a cyclic permutation of coordinates. 

The sets \( a + \sigma(\iota) \). Let \( a \in \mathbb{Z}^{n+1} \) and let \( \iota : I \to I \) be a permutation. Let us define a sequence \( \alpha(0), \alpha(1), \ldots, \alpha(n) \) by setting \( \alpha(0) = a \) and using the recursive rule

\[
(17) \quad \alpha(k) = S_i(\alpha(k-1)), \quad \text{where} \quad i = \iota(k),
\]

for \( k \geq 1 \). Let \( \sigma(\iota) \) be the set of terms of such sequence defined by \( a = 0 \) and the per-
mutation $\iota$. Clearly, the sequence defined by $a \in \mathbb{Z}^{n+1}$ and $\iota$ differs from the sequence defined by $0$ and $\iota$ by adding $a$ to each term. Therefore the set of terms of the sequence $\alpha(0), \alpha(1), \ldots, \alpha(n)$ is equal to $a + \sigma(\iota)$.

4.5. **Theorem.** If $\sigma \subset D$ is an I-cell, then $\sigma = a + \sigma(\iota)$ for some $a \in \sigma$ and some permutation $\iota : I \longrightarrow I$. Moreover, one can assume that $\iota(0) = 0$.

**Proof.** As above, we can assume that $\sigma = \{a(0), a(1), \ldots, a(n)\}$ and that the inequalities (15) hold. Since $\sigma$ is an I-cell, the map $i \longmapsto \min_i \sigma$ is a bijection (see Section 2). Therefore for every $k \in I$ there is a unique $i = \iota(k)$ such that $a(k) = \min_i \sigma$ and the map $\iota$ is a bijection $I \longrightarrow I$. The inequalities (15) imply that $a(0) = \min_0 \sigma$ and hence $\iota(0) = 0$. Let $a = a(0)$. As we will see, $\sigma = a + \sigma(\iota)$ for these $a$ and $\iota$.

Let $i \in I$ and let us consider the $i$th coordinates $a_i(k)$ as cyclicly ordered by $k \in I$. By Lemma 4.4 the $i$th coordinate decreases in this cyclic order only once and only by 1. Together with Lemma 4.3 this implies that $i$th coordinate increases also only once and only by 1. By Lemma 4.4 decreasing of $i$th coordinate happens simultaneously with increasing of $(i - 1)$th coordinate, namely, when one passes from $k - 1$ to $k$, where $k$ is uniquely determined by the condition $i = \iota(k)$.

Since the map $k \longmapsto \iota(k)$ is bijective, the above observations imply that $a(k)$ differs from $a(k - 1)$ only in two coordinates, namely, in coordinates numbered by $i$ and $i - 1$, where $i = \iota(k)$. The $i$th coordinate decreases by 1 and the $(i - 1)$th coordinate increases by 1. Therefore (17) holds for the sequence $a(k)$ in the role of the sequence $\alpha(k)$ and hence $\sigma = a + \sigma(\iota)$. This completes the proof. ■

**Remarks.** If in the assumption (15) the order $<_0$ is replaced by the order $<_i$, the above arguments show that $\sigma = b + \sigma(\kappa)$ for some $b \in \sigma$ and a permutation $\kappa : I \longrightarrow I$ such that $\kappa(i) = i$. One can also deduce this from Theorem 4.5 by using a cyclic permutation of coordinates. One can check that $\kappa$ is uniquely determined by the condition $\kappa(i) = i$ and that permutations corresponding to different $i \in I$ differ by cyclic permutations of $I$.

**Toward a converse to Theorem 4.5.** Let $I_0 = \{1, 2, \ldots, n\}$. We will identify permutations $\iota : I \longrightarrow I$ such that $\iota(0) = 0$ with the induced by them permutations $I_0 \longrightarrow I_0$. By Theorem 4.5 every I-cell $\sigma$ has the form $a + \sigma(\iota)$ for some $a \in \sigma$ and some permutation $\iota : I_0 \longrightarrow I_0$. It turns out that, conversely, every set of the form $a + \sigma(\iota)$ contained in $D$ is an I-cell. We will prove this in an indirect way. Namely, it turns out that a simple change of coordinates turns $D$ into the set of vertices of a canonical triangulation of a standard simplex. Moreover, this change of coordinates turns the sets of the form $a + \sigma(\iota)$ into sets of vertices of simplices of this triangulation. Once this is established, a homological argument (using only chains) implies that the sets of the form $a + \sigma(\iota)$ are I-cells. The same argument shows that $D$ is the set of the vertices of a triangulation.
Freudenthal triangulations. Let us summarize the basic facts about Freudenthal triangulations. See [I3], Section 9 and Appendix 1 for the details. The mentioned above standard simplex is the set \( \Gamma \) of all points \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) such that

\[
N \geq y_n \geq y_{n-1} \geq \cdots \geq y_1 \geq 0.
\]

The set \( \Gamma \) is an \( n \)-simplex with the vertices \( Nu_0, Nu_1, \ldots, Nu_n \in \mathbb{R}^n \), where

\[
u_i = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^n
\]

has 1 as the first \( i \) coordinates and 0 as the last \( n-i \). Let \( \Gamma_1 \) be the \( n \)-simplex with the vertices \( u_0, u_1, \ldots, u_n \). It is equal to \( \Gamma \) if \( N = 1 \). An \( (n-1) \)-face of \( \Gamma \) is defined by one of the following equations: \( N = y_n, \ y_i = y_{i-1} \) with \( n \geq i \geq 2 \), and \( y_1 = 0 \). Let

\[
e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n
\]

be the vector having 1 as the \( i \)th coordinate and 0 as all other coordinates. Then

\[
u_i = e_1 + e_2 + \cdots + e_i
\]

for every \( i \in I \), where the empty sum is interpreted as 0. For a permutation \( \omega \) of \( I_0 \) let

\[
L_\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

be the linear map taking \( e_i \) to \( e_{\omega(i)} \) for every \( i \in I_0 \), and let \( \Gamma(\omega) = L_\omega(\Gamma_1) \). Then \( \Gamma(\omega) \) is an \( n \)-simplex with the vertices \( u_0(\omega), u_1(\omega), \ldots, u_n(\omega) \), where

\[
u_i(\omega) = e_{\omega(1)} + e_{\omega(2)} + \cdots + e_{\omega(i)}.
\]

Clearly, \( u + \Gamma(\omega) \) is an \( n \)-simplex for every \( u \in \mathbb{Z}^n \) and every permutation \( \omega \) of \( I_0 \). It turns out that \( n \)-simplices having such form and contained in \( \Gamma \), together with their faces, form a triangulation of \( \Gamma \). The vertices of this triangulation are integer points of \( \Gamma \). This triangulation is the Freudenthal triangulation of \( \Gamma \). Freudenthal [Fr] considered only the case \( N = 2 \), but Kuhn and Scarf [Sc3] considered the general case also.

The sets of vertices of \( n \)-simplices. Let \( u \in \mathbb{Z}^n \) and let \( \omega \) be a permutation of \( I_0 \). Let \( v(0), v(1), \ldots, v(n) \) be the sequence defined by \( v(0) = u \) and the recursive rule

\[
v(k) = v(k-1) + e_i, \ \text{where} \ i = \omega(k)
\]

and \( k \geq 1 \). Let \( \tau(\omega) \) be the set of terms of such sequence defined by \( u = 0 \) and the permutation \( \omega \). Clearly, the sequence defined by \( u \in \mathbb{Z}^n \) and \( \omega \) differs from the sequence
defined by 0 and \( \omega \) by adding \( u \) to each term. Therefore the set of terms of the sequence \( v(0), v(1), \ldots, v(n) \) is equal to \( u + \tau(\omega) \). Clearly, \( u + \tau(\omega) \) is the set of vertices of the simplex \( u + \Gamma(\omega) \). It follows that a set \( \tau \) of integer points of \( \Gamma \) is the set of vertices of an \( n \)-simplex if and only if \( \tau \) has the form \( u + \tau(\omega) \) with \( u \) and \( \tau \) as above.

**The change of coordinates.** Recall that \( D \) is the set of integer points in the \( n \)-simplex \( \Delta \). Let \( s : \Delta \to \Gamma \) be the affine map defined by

\[
s(x_0, x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n),
\]

where \( y_i = x_0 + \ldots + x_{i-1} \) for each \( i \in I_0 \). The map \( t : \Gamma \to \Delta \) defined by

\[
t(y_1, y_2, \ldots, y_n) = (x_0, x_1, x_2, \ldots, x_n),
\]

where \( x_0 = y_1, x_i = y_{i+1} - y_i \) for \( 1 \leq i \leq n-1 \), and \( x_n = N - y_n \), is the inverse of \( s \). Clearly, \( s \) and \( t \) map integer points to integer points and hence establish a bijection between the set \( D \) of integer points in \( \Delta \) and the set of integer points in \( \Gamma \).

**4.6. Lemma.** A subset \( \sigma \subset D \) has the form \( a + \sigma(\iota) \) for a permutation \( \iota : I_0 \to I_0 \) if and only if \( s(\sigma) \) is the set of vertices of an \( n \)-simplex.

**Proof.** Suppose that \( a \in D \) and \( b = S_i(a) \) for some \( i \in I_0 \). Let

\[
s(a_0, a_1, \ldots, a_n) = (u_1, \ldots, u_n) \quad \text{and} \quad s(b_0, b_1, \ldots, b_n) = (v_1, \ldots, v_n).
\]

Then \( v_j = u_j \) for \( j \leq i - 1 \), \( v_i = u_i + 1 \), and \( v_j = u_j \) for \( j \geq i + 1 \). Therefore

\[
s(b) = s(a) + e_i.
\]

Therefore applying the operator \( S_i \) to \( a \in D \) corresponds to adding \( e_i \) to \( s(a) \). It follows that for every \( a, \iota \) the map \( s \) takes the set \( a + \sigma(\iota) \) to the set \( s(a) + \tau(\iota) \). Conversely, for every \( u, \omega \) the map \( t \) takes the set \( u + \tau(\omega) \) to the set \( t(u) + \sigma(\omega) \). Since the sets of integer points in \( \Gamma \) having the form \( u + \tau(\omega) \) are exactly the sets of vertices of \( n \)-simplices, this proves the lemma. ■

**4.7. Lemma.** Let \( C \subset I \). If \( \sigma \) is a C-cell, then \( a_k = 0 \) for every \( a \in \sigma \) and \( k \in I \sim C \).

**Proof.** Suppose that \( \sigma \) is a C-cell, \( a \in \sigma \), and \( a_k \neq 0 \) for some \( k \in I \sim C \). Let \( j \) be the last element of \( C \) appearing after \( k \) in the cyclic order of \( I \). Let \( b = (b_0, b_1, \ldots, b_n) \), where \( b_k = a_k - 1 \), \( b_j = a_j + 1 \), and \( b_i = a_i \) for \( i \neq k, j \). Then \( b \in D \) and \( a <_i b \) for every \( i \in C \). This contradicts to \( \sigma \) being a C-cell. ■
**Simplicial complexes and chains.** For each \( i \in I \) let \( \Delta_i \) be the face of \( \Delta \) defined by the equation \( x_i = 0 \). Let \( \Gamma_i \) be the face of \( \Gamma \) defined by the equation \( y_1 = 0 \) if \( i = 0 \), the equation \( y_{i+1} = y_i \) if \( 1 \leq i \leq n-1 \), and the equation \( y_n = N \) if \( i = n \). Then \( \Gamma_i = s(\Delta_i) \).

Let \( C \subset I \). Recall that the orders \( <_i \) define an abstract simplicial complex \( \mathcal{T}(C) \) having as its vertices and simplices the elements and subsets of \( C \)-cells respectively, and that the chain \( \mathcal{T}[C] \) is defined as the sum of all \( C \)-cells. We are interested mainly in cases \( C = I \) and \( C = I - i \) for some \( i \in I \). By Theorem 2.7

\[
\partial \mathcal{T}[I] = \sum_{i \in I} \mathcal{T}[I - i].
\]

Let \( \mathcal{F}(I) \) be the abstract simplicial complex associated with Freudenthal triangulation of \( \Gamma \) and \( \mathcal{F}(I - i) \) be the subcomplex of \( \mathcal{F}(I) \) having as its simplices all simplices of \( \mathcal{F}(I) \) contained in \( \Gamma_i \). Let us define the chain \( \mathcal{F}[I] \) as the sum of all \( n \)-simplices of \( \mathcal{F}(I) \) and the chain \( \mathcal{F}[I - i] \) as the sum of all \( (n-1) \)-simplices of \( \mathcal{F}(I - i) \). Since \( \mathcal{F}(I) \) is the abstract simplicial complex of a triangulation of \( \Gamma \),

\[
\partial \mathcal{F}[I] = \sum_{i \in I} \mathcal{F}[I - i].
\]

Theorem 4.5 together with Lemma 4.6 imply that the map \( s \) defines a simplicial map \( \varphi: \mathcal{T}(I) \to \mathcal{F}(I) \). Lemma 4.7 implies that for every \( i \in I \) the vertices of \( \mathcal{T}(I - i) \) belong to \( \Delta_i \) and hence \( \varphi \) maps \( \mathcal{T}(I - i) \) to \( \mathcal{F}(I - i) \).

**4.8. Theorem.** \( \varphi: \mathcal{T}(I) \to \mathcal{F}(I) \) is an isomorphism of simplicial complexes and

\[
\varphi_*(\mathcal{T}[I]) = \mathcal{F}[I].
\]

**Proof.** Since \( \varphi \) is induced by an injective affine map, it is an isomorphism if every \( n \)-simplex of \( \mathcal{F}(I) \) is the image of an \( n \)-simplex of \( \mathcal{T}(I) \). Clearly, this would follow once (20) is proved. Therefore it is sufficient to prove (20). We will prove (20) using an induction by \( n \), the case \( n = 0 \) (and even the case \( n = 1 \) being trivial.

Suppose that (20) holds with \( n - 1 \) in the role of \( n \). Let \( i \in I \). By omitting the coordinate \( x_i \) we can identify \( \Delta_i \) with the \( (n-1) \)-dimensional version of \( \Delta \). Lemma 4.7 implies that this identification turns \( I - i \)-cells into the \( (n-1) \)-dimensional version of \( I \)-cells. By omitting the coordinate \( y_i \) we can identify \( \Gamma_i \) with the \( (n-1) \)-dimensional version of \( \Gamma \). These identifications agree with \( s \) and hence our inductive assumption implies that

\[
\varphi_*(\mathcal{T}[I - i]) = \mathcal{F}[I - i].
\]
Since $\partial \circ \varphi_* = \varphi_* \circ \partial$, the equality (18) implies that
\[
\varphi_* (\mathcal{T}[I]) = \varphi_* (\partial \mathcal{T}[I]) = \varphi_* \left( \sum_{i \in I} \mathcal{T}[I - i] \right).
\]
In view of (21) and (19) the last expression is equal to
\[
\sum_{i \in I} \varphi_* (\mathcal{T}[I - i]) = \sum_{i \in I} \mathcal{T}[I - i] = \partial \mathcal{T}[I]
\]
and therefore
\[
\partial \varphi_* (\mathcal{T}[I]) = \partial \mathcal{T}[I].
\]
Since the complex $\mathcal{T}(I)$ is the abstract simplicial complex of a triangulation of a simplex, it is non-branching, strongly connected, and has non-empty boundary. It is well known and easy to see that these properties imply that $\mathcal{T}[I]$ is the only chain with the boundary $\partial \mathcal{T}[I]$. Therefore the last equality implies (20). This completes the proof. \(\blacksquare\)

4.9. Corollary. The $n$-simplices having $I$-cells as their sets of vertices are the $n$-simplices of a triangulation of $\Delta$, namely, of the image of Freudenthal triangulation of $\Gamma$ under $t$.

Proof. This follows from the theorem and the definitions of $\mathcal{T}(I)$ and $\mathcal{F}(I)$. \(\blacksquare\)

4.10. Corollary. A subset of $D$ is an $I$-cell if and only if it has the form $a + \sigma(1)$, where $a \in D$ and $\iota: I \rightarrow I$ is a permutation such that that $\iota(0) = 0$.

Proof. The “only if” part is Theorem 4.5. Let us prove the “if” part. Let $\sigma \subset D$ be a subset of the form $a + \sigma(1)$. Lemma 4.6 implies that $s(\sigma)$ is an $n$-simplex of $\mathcal{F}(I)$, and hence Theorem 4.8 implies that $\sigma$ is an $n$-simplex of $\mathcal{T}(I)$, i.e. is an $I$-cell. \(\blacksquare\)

Broken and restored symmetry. The set $D$ and the family of orders $<_I$ are invariant under the cyclic permutation of coordinates in $\mathbb{R}^{n+1}$, as is the notion of an $I$-cell. But the description of $I$-cells as subsets of $D$ having the form $a + \sigma(1)$ is not, because their definition in terms of sequences $\alpha(0), \alpha(1), \ldots, \alpha(n)$ does not require (17) to hold for $k = 0$. The cyclic symmetry of the order $n + 1$ is also broken in the very definition of $\Gamma$. This symmetry is restored by Corollary 4.10, which implies that if (17) holds for $k \geq 1$, then (17) automatically holds for $k = 0$. Since everything is invariant under translations by integer vectors, there is even no need to assume that the terms $\alpha(i)$ belong to $D$.

Remark. Scarf proved Corollaries 4.9 and 4.10 without explicitly invoking Freudenthal triangulations. See [Sc3], Theorems 7.1.8 and 6.2.1 respectively. But, for example, his formulas on p. 176 of [Sc3] amount to using an analogue of the map $s$. 

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5. Oriented matroids

**Signed subsets.** Let us fix a set $M$. A *signed subset* $\sigma$ of $M$ is a pair $(\sigma_+, \sigma_-)$ of two disjoint subsets $\sigma_+, \sigma_-$ of $M$. The *support* of a signed subset $\sigma$ is the union

$$\sigma = \sigma_+ \cup \sigma_-.$$ 

The opposite of a signed subset $\sigma = (\sigma_+, \sigma_-)$ is the signed subset $(\sigma_-, \sigma_+)$. Equivalently, the opposite of $\sigma$ is the unique signed subset $\tau$ such that

$$\tau_+ = \sigma_- \quad \text{and} \quad \tau_- = \sigma_+.$$ 

The opposite of $\sigma$ is usually denoted by $-\sigma$. A signed subset $\sigma$ can be identified with a map $\sigma \rightarrow \{+,-\}$, which is also denoted by $\sigma$ and defined by

$$\sigma(x) = + \text{ if } x \in \sigma_+ \quad \text{and} \quad \sigma(x) = - \text{ if } x \in \sigma_-.$$ 

This identification matches the notation for the opposite signed subset.

**Vector configurations.** Let $M \subset \mathbb{R}^n$ for some $n$. Suppose that $0 \not\in M$ and that different elements of $M$ are not proportional. If $X \subset M$ is a linearly dependent subset, then

$$\sum_{x \in X} \lambda_x x = 0$$

for some coefficients $\lambda_x \in \mathbb{R}$. Such a relation defines a signed subset $\sigma$ of $M$ as follows: the set $\sigma_+$ is equal to the set of elements $x \in X$ such that $\lambda_x > 0$, and $\sigma_-$ is equal to the set of elements $x \in X$ such that $\lambda_x < 0$. In general, $\sigma$ depends not only on $X$, but also on the relation. Let us suppose now that $X$ is a minimal linearly dependent subset, i.e. that no proper subset of $X$ is linearly dependent. In this case $\lambda_x \neq 0$ for all $x \in X$ and the coefficients $\lambda_x$ are uniquely determined by $X$ up to multiplication by a common non-zero factor. Therefore in this case $\sigma$ is determined by $X$ up to replacing $\sigma$ by $-\sigma$. The signed subsets $\sigma$ defined by minimal linearly dependent subsets $X \subset M$ in the above manner are called the *circuits* of the *vector configuration* $M$.

The notion of an *oriented matroid* is an axiomatization of the basic properties of these circuits. The first of these basic properties are obvious. Clearly, $\emptyset$ is not a circuit, and if $\sigma$ is a circuit of $M$, then $-\sigma$ is also a circuit. Since the circuits result from *minimal* linear dependencies, if $\sigma, \tau$ are circuits, then the inclusion $\sigma \subset \tau$ implies that $\sigma$ is equal to either $\tau$, or $-\tau$. The main property of circuits is concerned with an elimination procedure removing a element from two given relations.
An elimination procedure. Let $X, Y \subseteq M$ be two linearly dependent subsets. Then

$$\sum_{x \in X} \lambda_x x = 0 \quad \text{and} \quad \sum_{y \in Y} \mu_y y = 0$$

for some coefficients $\lambda_x, \mu_y \in \mathbb{R}$. Let $\sigma$ and $\tau$ be the signed subsets defined by these relations. Suppose that $u \in X \cap Y$ and the coefficients $\lambda_u$ and $\mu_u$ have opposite signs. Let us multiply the second relation by $\alpha = |\lambda_u/\mu_u|$ and add the resulting relation to the first one. This procedure eliminates $z$ from the given relations between the elements of $X$ and $Y$. Indeed, the result is a relation between the elements of the set $Z = (X \cup Y) \sim \{u\}$. Let $\omega$ be the signed set defined by this relation. The coefficient of $z \in Z$ is equal to

$$\begin{align*}
\lambda_z & \quad \text{for} \quad z \in X \sim Y, \\
\mu_z & \quad \text{for} \quad z \in Y \sim X, \quad \text{and} \\
\lambda_z + \alpha \mu_z & \quad \text{for} \quad z \in (X \cap Y) \sim \{u\}.
\end{align*}$$

Clearly, the coefficient $\lambda_z + \alpha \mu_z$ can be positive only if at least one of the coefficients $\lambda_z, \mu_z$ is positive, and negative if at least one of them is negative. It follows that

$$\omega_+ \subset (\sigma_+ \cup \tau_+) \sim \{u\} \quad \text{and} \quad \omega_- \subset (\sigma_- \cup \tau_-) \sim \{u\}. \quad \text{(22)}$$

Similarly, the coefficient of $z$ is positive if $\lambda_z > 0$ and either $z \not\in Y$, or $\mu_z \geq 0$, and is negative if $\lambda_z < 0$ and either $z \not\in Y$, or $\mu_z \leq 0$. It follows that

$$\omega_+ \supset \sigma_+ \sim \tau_- \quad \text{and} \quad \omega_- \supset \sigma_- \sim \tau_+. \quad \text{(23)}$$

5.1. Theorem. Let $\sigma, \tau$ be two circuits. Suppose that

$$u \in (\sigma_+ \cap \tau_-) \cup (\sigma_- \cap \tau_+) \quad \text{and}$$

$$v \in (\sigma_+ \sim \tau_-) \cup (\sigma_- \sim \tau_+).$$

Then there exists a circuit $\omega$ such that the inclusions (22) hold and $v \in \omega$.

Proof. Let $X, Y$ be minimal linearly dependent subsets defining circuits $\sigma, \tau$. The elimination procedure leads to a relation between the elements of $Z = (X \cup Y) \sim \{u\}$ such that the corresponding signed subset $\omega$ satisfies (22) and (23). In particular, $v \in \omega$. But it may happen that $Z$ is not minimal and $\omega$ is not a circuit. We will prove that some minimal dependent subset of $Z$ leads to a circuit with required properties. Let us consider relations

$$\sum_{z \in \omega} \alpha_z z = 0 \quad \text{(24)}$$
such that $\alpha_v \neq 0$ and the sign of $\alpha_z$ is equal to $\omega(z)$ if $\alpha_z \neq 0$. Obviously, the relation resulting from the elimination procedure has these properties. Suppose that (24) is a relation with the minimal possible number of non-zero coefficients among the relations with these properties. Let $V \subset \omega$ be the set of $z \in \omega$ such that $\alpha_z \neq 0$. We claim that $V$ is a minimal linearly dependent set. Suppose that $V$ is not minimal. Then there exists a proper subset $W \subset V$ and a relation of the form

$$\sum_{z \in W} \beta_z z = 0$$

such that all coefficients $\beta_z$ are non-zero. Moreover, the set $W$ and the relation (25) can be chosen in such a way that $v \not\in W$. Indeed, if $v \in W$, then, after multiplying (25) by $-1$ if necessary, we may assume that the signs of $\alpha_v$ and $\beta_v$ are opposite. By applying the elimination procedure to $W, V$ and $v$ in the roles of $X, Y$ and $u$ respectively, we will get a new relation of the form (25), this time such that $v \not\in W$. Let $\gamma = |\alpha_w / \beta_w|$ be the minimal number among $|\alpha_z / \beta_z|$ with $z \in W$. Then

$$|\gamma \beta_z| = |\alpha_w \beta_z / \beta_w| \leq |\alpha_z|$$

for every $z \in W$. It follows that the signs of $\alpha_z + \gamma \beta_z$ and $\alpha_z$ are the same for every $z \in W$ such that $\alpha_z + \gamma \beta_z \neq 0$. Multiplying (25) by $-1$ does not affects $\gamma$. Therefore we may assume that the signs of $\alpha_w$ and $\beta_w$ are opposite. Let us multiply (25) by $\alpha$ and add the resulting relation to (24). The coefficient of $z \in V$ in the resulting relation is

$\alpha_z$ if $z \in V \setminus W$ and

$\alpha_z + \gamma \beta_z$ if $z \in W$.

In particular, the coefficient of $z$ is either 0 or has the same sign $\omega(z)$ as $\alpha_z$ for every $z \in V$. Moreover, the coefficient of $w$ is equal to 0 and the coefficient of $v$ is non-zero. Hence the new relation has the same properties as (24), but the set of vectors with non-zero coefficient in the new relation is properly contained in $V$. This contradicts the minimality assumption about (24). The contradiction shows that $V$ is indeed a minimal linearly dependent set. The corresponding circuit satisfies the conditions of the theorem. ■

5.2. Corollary. Let $\sigma, \tau$ be two circuits. Suppose that $\sigma \neq -\tau$ and $\sigma(u) = -\tau(u)$ for some $u$. Then there exists a circuit $\omega$ such that the inclusions (22) hold.

Proof. If both differences $\sigma_+ \sim \tau_-$ and $\sigma_- \sim \tau_+$ are empty, then $\sigma \subset \tau$ and hence either $\sigma = \tau$ or $\sigma = -\tau$. The second equality is explicitly excluded, and the first one is impossible because $\sigma(u) = -\tau(u)$. Hence at least one of these differences is non-empty and hence Theorem 5.1 applies. ■
Oriented matroids. An oriented matroid is a set $M$ together with a collection of signed subsets of $M$, called its circuits, such that the above obvious properties of vector configurations together with the property of Corollary 5.2 hold. More formally, the axioms of oriented matroids are the following.

(i) The empty set is not a circuit.

(ii) If $\sigma$ is circuit, then $-\sigma$ is also a circuit.

(iii) If $\sigma, \tau$ are circuits and $\sigma \subset \tau$, then either $\sigma = \tau$ or $\sigma = -\tau$.

(iv) If $\sigma, \tau$ are circuits and $\sigma(u) = -\tau(u)$ for some $u$, then there exists a circuit $\omega$ such that $\omega_+ \subset (\sigma_+ \cup \tau_+) \setminus \{u\}$ and $\omega_- \subset (\sigma_- \cup \tau_-) \setminus \{u\}$.

The property (iv) is known as the weak elimination property. Remarkably, in every oriented matroid a stronger form of this property holds. Namely, every oriented matroid has the property proved in Theorem 5.1 for vector configurations. This property is known as the strong elimination property. See [B-Z], Theorem 3.2.5 for a proof, which is highly non-trivial. Cf. the situation with the symmetric exchange property of the usual matroids, which follows from the exchange property, but this is far from being obvious. For the purposes of the present paper one can take the strong elimination property as another axiom.

Let $M$ be an oriented matroid. Suppose that $X \subset M$ and $y \in M$. The element $y$ is said to belong to the convex hull of $X$ if either $y \in X$ or there exists a circuit $\sigma$ such that $\sigma_+ \subset X$ and $\sigma_- = \{y\}$. Naturally, the convex hull of $X$ is the set of all $y \in M$ belonging to the convex hull of $X$. It is denoted by $\langle X \rangle$.

A subset $X \subset M$ is said to be independent if there is no circuit $\sigma$ such that $\sigma \subset X$. A maximal independent subset is called a basis of $M$. Clearly, if $B$ is a basis and $y \in M \setminus B$, then there exists a circuit $\sigma$ such that $\sigma \subset B \cup \{y\}$ and $y \in \sigma$. One can check that the number of elements of a basis depends only on $M$.

Todd’s theorem. Let $\sigma, \tau$ be circuits of an oriented matroid and let $w \in \tau \setminus \sigma$. Suppose that there exists $e \in \sigma \cap \tau$ such that $\sigma(e) = -\tau(e)$. Then there is a circuit $\omega$ such that

$$\omega_+ \subset (\sigma_+ \cup \tau_+) \setminus \{e\},$$

$$\omega_- \subset (\sigma_- \cup \tau_-) \setminus \{e\},$$

$$w \in \omega, \quad \text{and} \quad \omega(w) = \tau(w).$$

Moreover, if $\tau \subset \sigma \cup \{w\}$, then such a circuit $\omega$ is unique and $\sigma \setminus \tau \subset \omega$.

Proof. See Appendix for a proof following M. Todd [T], Theorem 4.2. ■

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6. Oriented matroid colorings: the non-degenerate case

The oriented matroid framework. We need one more notion related to oriented matroids. An oriented matroid $M$ is said to be acyclic if there are no circuits $\sigma$ such that $\sigma_- = \emptyset$. Let $M$ be an acyclic oriented matroid and let $B = \{v_0, v_1, \ldots, v_n\}$ be a basis of $M$. Let $b \in M \setminus B$ and suppose that $b$ belongs to the convex hull of $B$. In this section we will also assume that the pair $M, b$ is non-degenerate in the sense that $b$ does not belong to the convex hull of $< n + 1$ elements of $M - b$. This is a sort of general position assumption, and if $M$ corresponds to a vector configuration, it holds, in particular, if $b$ is not a linear combination of $< n + 1$ elements of $M - b$.

6.1. Lemma. Let $X \subset M$. If $|X| = n + 1$ and $X$ is not a basis, then $b$ does not belong to the convex hull of $X$.

Proof. Suppose that $b$ belongs to the convex hull of $X$, i.e. there is a circuit $\tau$ such that $\tau_+ \subset X$ and $\tau_- = \{b\}$. Then $\tau_+$ has $> n + 1$ elements and hence $\tau_+ = X$.

If $X$ is independent, then $X$ is contained in some basis with $> n + 1$ elements. But $B$ is a basis consisting of $n + 1$ elements, and the number of elements of a basis depends only on $M$. It follows that $X$ is not independent, i.e. there is a circuit $\sigma$ such that $\sigma \subset X$. It follows that $\sigma \subset X \subset \tau$. By the axiom (iii) either $\sigma = \tau$ or $\sigma = -\tau$. But $b \in \tau$ and $b \notin \sigma$. The contradiction shows that $b$ does not belong to the convex hull of $X$. ■

6.2. Lemma. If $w \in M \setminus B$ and $w \neq b$, then there is a unique $i \in \{0, 1, \ldots, n\}$ such that $b$ belongs to the convex hull of $B - v_i + w$.

Proof. Since $b$ belongs to the convex hull of $B$ and $M$, $b$ is non-degenerate, there is a circuit $\sigma$ such that $\sigma_+ = B$ and $\sigma_- = \{b\}$. Since $B$ is a basis, there is a circuit $\tau$ such that $w \in \tau \subset B + w$. Replacing $\tau$ by $-\tau$ if necessary, we may assume that $\tau(w) = +$.

We would like to apply Todd’s theorem to $\sigma$, $\tau$, and $w$. Let us check its assumptions. Obviously, $w \in \tau \setminus \sigma$ and $\tau \subset \sigma \cup \{w\}$. Also,

$$\sigma \cap \tau = \tau - w = \tau \cap B.$$ 

Since $M$ is acyclic, $\tau_-$ is non-empty. Since $w \in \tau_+$, the set $\tau_-$ is contained in $B$ and hence the intersection $\tau_- \cap \sigma_+ = \tau_- \cap B$ is non-empty. If $e \in \tau_- \cap \sigma_+$, then $e \in \sigma \cap \tau$ and $\sigma(e) = -\tau(e)$. It follows that all assumptions of Todd’s theorem hold.
By Todd’s theorem there is a circuit \( \omega \) such that

\[
\begin{align*}
\omega_+ & \subset (\sigma_+ \cup \tau_+) \sim \sigma_- = B + w, \\
\omega_- & \subset (\sigma_- \cup \tau_-) \sim \sigma_+ = \{b\}, \\
w & \in \omega, \quad \text{and} \quad \omega(w) = \tau(w) = +.
\end{align*}
\]

If \( b \not\in \omega \), then \( \omega = \omega_+ \), contrary to the assumption that \( M \) is acyclic. If \( \omega \cap B \) consists of \( \leq n \) elements, then \( \omega \) is a circuit containing \( b \) and consisting of \( \leq n + 1 \) elements. This also contradicts to the assumptions of the oriented matroid framework. Therefore \( \omega \) has the form \( B - v_i + w - b \). This proves that \( i \) with the required properties exists.

In order to prove the uniqueness, suppose that \( \omega_+ = B - v_i + w \) and \( \omega_- = \{b\} \) for some circuit \( \omega \). Then \( \omega \) has all properties from Todd’s theorem. Since \( \tau \subset \sigma \cup \{w\} \), Todd’s theorem implies the uniqueness of such \( \omega \) and hence the uniqueness of \( i \). ■

6.3. Lemma. Let \( i \in \{0, 1, \ldots, n\} \). Suppose that \( D \subset M \) and \( b \) does not belong to the convex hull of \( D \). Then the number of elements \( d \in D \) such that \( D - d + v_i \) is a basis containing \( b \) in its convex hull is either 2 or 0.

Proof. Suppose that \( D - d + v_i \) is a basis containing \( b \) in its convex hull. If we replace \( B \) by the basis \( D - d + v_i \), all assumptions of the oriented matroid framework still hold. By applying Lemma 6.2 to the basis \( D - d + v_i \) in the role of \( B \) and to \( w = d \), we see that there is a unique element \( e \in D - d + v_i \) such that \( b \) belongs to the convex hull of

\[
D - d + v_i - e + d = D - e + v_i.
\]

If \( e = v_i \), then this set is equal to \( D \). But \( D \) does not contain \( b \) in its convex hull by the assumption of the lemma. Hence \( e \neq v_i \) and \( D - e + v_i \) is a basis containing \( b \) in its convex hull. It follows that if there is at least one element \( d \in D \) with the stated property, then there are exactly two such elements. This proves the lemma. ■

Colorings by elements of oriented matroids. Let \( \mathcal{D} \) be a chain-simplex based on \( I \) (see Section 1) and \( X = V_\mathcal{D} \) be the union of the sets of vertices of complexes \( \mathcal{D}(J) \) with \( J \subset I \). Suppose that \( |I| = |B| \). Then we can assume that \( I = \{0, 1, \ldots, n\} \) and that the map \( v_* : i \rightarrow v_i \) is a bijection \( I \rightarrow B \).

Let \( A = M - b \) and let \( \Delta(A) \) and \( \Delta(B) \) be simplicial complexes having, respectively, \( A \) and \( B \) as their sets of vertices and all subsets of \( A \) and \( B \) as simplices. Let \( \partial \Delta(B) \) be the simplicial complex having \( B \) as its set of vertices and all proper subsets of \( B \) as simplices. The pair \( \Delta(A), \partial \Delta(B) \) is an analogue of the pair \( \Delta(I), \partial \Delta(I) \) in the classical situation.
A matroid coloring of $\mathcal{D}$ is defined as an arbitrary map $X = V_{\mathcal{D}} \rightarrow A$. A matroid coloring $c : X \rightarrow A$ for every subset $J \subset I$ induces a simplicial map $\mathcal{D}(J) \rightarrow \Delta(A)$. The map $c$ canonically extends to a map $\varphi : X \cup I \rightarrow A$ equal to $v_*$ on $I$. Let $\mathcal{E}$ be the envelope of $\mathcal{D}$ in the sense of Section 1. Identification of $I$ with $B$ by the map $v_*$ turns $\varphi$ into an extension of $c$ in the sense of Section 1. In any case, $\varphi$ induces a simplicial map $\mathcal{E}(I) \rightarrow \Delta(A)$, which, in turn, induces an isomorphism $\partial \Delta(I) \rightarrow \partial \Delta(B)$.

The action of $\varphi$ on $n$-simplices can be described as follows. If $\sigma$ is an $n$-simplex of $\mathcal{E}(I)$ and $\sigma = \tau \ast (I \sim C)$, where $C \subset I$ and $\tau$ is a simplex of $\mathcal{D}(C)$, then

$$\varphi(\sigma) = c(\tau) \cup v_*(I \sim C).$$

By Lemma 1.4 every $n$-simplex $\sigma$ of $\mathcal{E}(I)$ has such form and hence (26) applies to $\sigma$.

**Cochains** $\delta_i$. These cochains are analogues of the cochains $\delta_i$ from the cochain-based proof of Sperner's lemma in Section 9 of [I2]. Let $i \in I$. The cochain $\delta_i$ is an $(n-1)$-cochain of $\Delta(A)$ defined as follows. Let $\epsilon$ be an $(n-1)$-simplex of $\Delta(A)$. Then

$$\delta_i(\epsilon) = 1 \quad \text{if} \quad b \in \langle \epsilon + v_i \rangle \quad \text{and}$$

$$\delta_i(\epsilon) = 0 \quad \text{otherwise}.$$

By Lemma 6.1 if $b \in \langle \epsilon + v_i \rangle$, then $\epsilon + v_i$ is a basis.

**6.4. Lemma.** Let $\sigma \subset A$ be an $(n-1)$-simplex of $\Delta(A)$. Then

$$\partial^* \delta_i(\sigma) = 1$$

if and only if $\sigma$ is a basis containing $b$ in its convex hull.

**Proof.** By the definition of the coboundary operator,

$$\partial^* \delta_i(\sigma) = \sum_{v \in \sigma} \delta_i(\sigma - v).$$

Let $\sigma$ be a basis containing $b$ in its convex hull. Suppose first that $v_i \not\in \sigma$. Then Lemma 6.2 with $\sigma$ in the role of $B$ and $v_i$ in the role of $w$ implies that there is a unique element $v \in s$ such that $\sigma - v + v_i$ is a basis containing $b$ in its convex hull. Therefore in this case $\partial^* \delta_i(\sigma) = 1$. Suppose now that $v_i \in \sigma$. Then $\sigma - v + v_i$ is defined only if $v = v_i$ and hence $\delta_i(\sigma - v) = 1$ if and only if $v = v_i$. Hence in this case $\partial^* \delta_i(\sigma) = 1$ also.

By Lemma 6.1, if $\sigma$ is not a basis, then $b$ does not belong to the convex hull of $\sigma$. Therefore it remains to consider the case when $\sigma$ does not contain $b$ in its convex hull. If all
summands of the sum representing $\partial^* \delta_i(\sigma)$ are 0, then $\partial^* \delta_i(\sigma) = 0$. Suppose that one of the summands is non-zero, i.e. $\delta_i(\sigma - v) = 1$ for some $v \in \sigma$. Then $\sigma - v + v_i$ is a basis containing $b$ in its convex hull. Since now $\sigma$ does not contain $b$ in its convex hull, Lemma 6.3 implies that in this case there are exactly 2 elements $v \in \delta$ with this property, and hence exactly 2 elements $v \in \sigma$ such that $\delta_i(\sigma - v) = 1$. It follows that $\partial^* \delta_i(\sigma) = 0$ if $\sigma$ is not a basis containing $b$ in its convex hull. ■

6.5. The main theorem for matroid colorings (the non-degenerate case). Let $c$ be a coloring. Then there exist a non-empty subset $C \subset I$ and a simplex $\tau$ of $D(\delta)$ such that $c(\tau) \cup v_*(I - C)$ is a basis containing $b$ in its convex hull. The number of such $\tau$ is odd.

Proof. We will use the action of the extended map $\phi$ on chains and cochains in a manner similar to the cochain-based proof of Sperner’s lemma in [I2]. Let us fix some $i \in I$. Let $e$ be the number of $n$-simplices $\sigma$ of $E(I)$ such that $\phi(\sigma)$ is a basis containing $b$ in its convex hull. In view of (26) the conclusion of the theorem means that $e$ is odd. Recall that $E[[I]]$ is the sum of all $n$-simplices of $E(I)$. By Lemma 6.4

$$\partial^* \delta_i(\phi_* E[[I]])$$

is equal to $e$ modulo 2. One the other hand, in view of the identification of $I$ with $B$ Lemma 1.8 implies that $\phi_* E[[I]] = B$, where $B$ is considered as an $n$-simplex of $\Delta(A)$. It follows that $e$ is equal modulo 2 to

$$\partial^* \delta_i(B) = \delta_i(\partial B).$$

It remains to prove that $\delta_i(\partial B) = 1$. Every $(n-1)$-face of $B$ has the form $B - v_k$, where $k \in B$. Since $B - v_k + v_i$ is not defined if $k \neq i$ and $B - v_i + v_i = B$, we see that $\delta_i(B - v_k) = 1$ if $k = i$ and $\delta_i(B - v_k) = 0$ otherwise. It follows that $\delta_i(\partial B) = 1$. This proves the theorem. ■

A cohomological interpretation of the proof. In the classical situation we are given a triangulation $T$ of an $n$-simplex $\Delta$ and a simplicial map $\varphi: T \rightarrow \Delta$ and we are interested in the number of simplices of $T$ mapped by $\varphi$ onto $\Delta$ considered modulo 2. A way to count them is to consider the $n$-cochain $\delta$ of $\Delta$ equal to 1 on the only $n$-simplex of $\Delta$ and its image $\varphi^*(\delta)$ with is equal to 1 exactly on $n$-simplices of $T$ mapped by $\varphi$ to $\Delta$. It turns out to be useful to represent $\delta$ as a coboundary. Namely, $\delta = \partial^* \delta_i$, where $\delta_i$ is the cochain equal to 1 on the $i$th $(n-1)$-face of $\Delta$. Moreover, the picture is clarified by passing from cochains to their cohomology classes. See [I2], Section 10 for the details. Now we have a simplicial map $\varphi: E(I) \rightarrow \Delta(A)$ and we are interested in the number of simplices of $E(I)$ mapped by $\varphi$ onto a basis containing $b$ in its convex hull. As in the classical situation, a way to count them is to consider the $n$-cochain $\delta$ of $\Delta(A)$ equal to 1 on each $n$-simplex which is a basis containing $b$ in its convex hull and equal to 0 on other
Then $\varphi^*(\delta)$ is equal to 1 on exactly the $n$-simplices we would like to count. As before, it is useful to represent $\delta$ as a coboundary. Lemma 6.4 tells us that $\delta = \partial^* \delta_i$, where now $\delta_i$ is the cochain defined before this lemma. This $\delta_i$ is an analogue of the classical $\delta_i$ in the sense that it works in the same way. But a naive analogue would be the cochain $\gamma_i$ equal to 1 on $B - v_i$ and to 0 on all other $(n-1)$-simplices. Remarkably, after passing to cohomology groups these two analogues of the classical cochain $\delta_i$ turn out to be equivalent. Let us consider $\varphi$ as a map of pairs

$$\left( \mathcal{E}(I), \partial \Delta(I) \right) \longrightarrow \left( \Delta(A), \partial \Delta(B) \right)$$

and let $\partial \varphi$ be the induced map $\partial \Delta(I) \longrightarrow \partial \Delta(B)$. Consider the following diagram.

$$
\begin{array}{ccc}
H^{n-1}(\partial \Delta(B)) & \longrightarrow & H^n(\Delta(A), \partial \Delta(B)) \\
(\partial \varphi)^* & \downarrow & \varphi^* \\
H^{n-1}(\partial \Delta(I)) & \longrightarrow & H^n(\mathcal{E}(I), \partial \Delta(I))
\end{array}
$$

Since $\partial \Delta(B)$ and $\partial \Delta(I)$ are the boundaries of the simplices $\Delta(B)$ and $\Delta(I)$ respectively, both cohomology groups in the left column are isomorphic to $\mathbb{F}_2$. Since $\varphi$ is equal to $v_\bullet$ on $\partial \Delta(I)$, the map $\partial \varphi : \partial \Delta(I) \longrightarrow \partial \Delta(B)$ is an isomorphism of simplicial complexes. It follows that the left vertical map $(\partial \varphi)^*$ is an isomorphism. While $\Delta(A)$ is a simplex, it is of uncontrollably big dimension and $\partial \Delta(B)$ is not its boundary. Still, the upper map $\partial^*$ is an isomorphism. By an easy standard homological argument this follows from the fact that all homology groups of a simplex are 0. In general, the cohomology group

$$(27) \quad H^n(\mathcal{E}(I), \partial \Delta(I))$$

is not isomorphic to $\mathbb{F}_2$. If $\mathcal{E}$ is not only a chain-simplex, but, moreover, is a pseudosimplex (as it is actually the case in the main examples), then Theorem 1.5 implies that the simplicial complex $\mathcal{E}(I)$ is a non-branching and its boundary is equal to the boundary of $\Delta(I)$. If $\mathcal{E}(I)$ is also strongly connected, then the cohomology group (27) is isomorphic to $\mathbb{F}_2$ and the lower map $\partial^*$ is an isomorphism (see [I_2], Section 10, Theorem 7).

In general, $\mathcal{E} \parallel I \parallel$ is a relative cycle of the pair $\left( \mathcal{E}(I), \partial \Delta(I) \right)$ and the evaluation of cohomology classes on this relative cycle defines a homomorphism

$$
\varepsilon : H^n(\mathcal{E}(I), \partial \Delta(I)) \longrightarrow \mathbb{F}_2.
$$

Theorem 1.6 implies that $\mathcal{E}$ is a chain-simplex and hence (2) implies that $\partial \mathcal{E} \parallel I \parallel = \partial I$. In turn, by a standard argument this implies that $\varepsilon \circ \partial^*$ is an isomorphism. Also, Lemma
1.8 implies that $\varphi^* \mathcal{E} [I] = B$, and by a standard argument this implies that $\varepsilon \circ \varphi^{**}$ is an isomorphism. Let us replace in the above diagram the lower right cohomology group by $F_2$, the lower horizontal map $\partial^{**}$ by $\varepsilon \circ \partial^{**}$, and the right vertical map $\varphi^{**}$ by $\varepsilon \circ \varphi^{**}$. One can identify all cohomology groups in the resulting diagram with $F_2$ and get the diagram

All maps in this diagram are isomorphisms. Let $1$ be the non-zero element of $F_2$. Then

$$\varepsilon \circ \partial^{**} \circ (\partial \varphi)^{**}(1) = 1$$

and the commutativity of the diagram implies that

$$(28) \quad \varepsilon \circ \varphi^{**} \circ \partial^{**}(1) = 1.$$

Now we have to identify the cohomology class $\partial^{**}(1)$. The cohomology class $[\gamma_i]$ of the cochain $\gamma_i$ is equal to $1$. By the definition of the connecting homomorphism $\partial^{**}$

$$\partial^{**}(1) = \partial^{**}([\gamma_i]) = [\partial \bar{\gamma}_i],$$

where $\bar{\gamma}_i$ is any extension of the cochain $\gamma_i$ to a cochain of $\Delta(A)$. As it is often the case, the tautological extension by 0 is not really useful (cf. [I2], Section 10). We claim that the cochain $\delta_i$ is an extension. Indeed, $B - v_i + v_i = B$ is a basis containing $b$ in its convex hull, and $B - v_i + v_k$ is not defined if $k \neq i$. Hence $\delta_i$ is equal to 1 on $B - v_i$ and to 0 on other $(n-1)$-simplices of $\partial \Delta(B)$ and hence extends $\gamma_i$. It follows that

$$\partial^{**}(1) = [\partial \delta_i] = [\delta],$$

where we used the fact that $\delta = \partial^* \delta_i$. Now (28) implies that $\varepsilon \circ \varphi^{**}([\delta]) = 1$. But

$$\varepsilon \circ \varphi^{**}([\delta]) = \sum \varphi^*(\delta)(\sigma),$$

where the sum is taken over all $n$-simplices of $\mathcal{E}(1)$. Since $\varphi^*(\delta)(\sigma) = 1$ if and only if $\varphi(\sigma)$ is a basis containing $b$ in its convex hull, it follows that the number of simplices $\sigma$ with this property is odd. In view of (26) this is equivalent to the main theorem. The key step of the both versions of the proof is an application of Lemma 6.4. From the cohomological point of view this lemma allows to identify the cohomology class $\partial^{**}[\gamma_i]$. 

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A graph-theoretical interpretation of the proof. Suppose now that \( \mathcal{D} \) is a pseudo-simplex. In this case the proof of the main theorem admits a graph-theoretical interpretation leading to a path-following algorithm in the spirit of Scarf. Cf. [I2], Section 9.

A basis of \( M \) is said to be a good basis if \( b \) is contained in its convex hull. Let \( i \in I \). The above proof can be understood as a computation of

\[
\partial^* \delta_i (\varphi_* (\mathcal{E}[I])) = \varphi^*(\partial^* \delta_i (\mathcal{E}[I]))
\]

Since \( \mathcal{E}[I] \) involves all \( n \)-simplices of \( \mathcal{E}(I) \), this is almost the same as computing

\[
\varphi^*(\partial^* \delta_i) = \partial^*(\varphi^* \delta_i)
\]

Let us treat cochains as sums of simplices. Then \( \varphi^* \delta_i \) is the sum of \( (n-1) \)-simplices \( \tau \) of \( \mathcal{E}(I) \) such that \( \varphi(\tau) + v_i \) is a good basis. The coboundary \( \partial^* \tau \) is the sum of all \( n \)-simplices having \( \tau \) as face, and \( \partial^*(\varphi^* \delta_i) \) is the sum of coboundaries \( \partial^* \tau \). Therefore the computation of \( \partial^*(\varphi^* \delta_i) \) involves only the \( (n-1) \)-simplices \( \tau \) as above, the \( n \)-simplices having them as faces, and the relation “\( \tau \) is a face of \( \sigma \)” between them.

This information can be represented in terms of a graph \( \mathbb{G}_i \). Its vertices are \( n \)-simplices \( \sigma \) of \( \mathcal{E}(I) \) such that \( \varphi(\tau) + v_i \) is a good basis for some \( (n-1) \)-face \( \tau \) of \( \sigma \). The edges of \( \mathbb{E}_i \) correspond to \( (n-1) \)-simplices \( \tau \) of \( \mathcal{E}(I) \) not contained in \( I \) and such that \( \varphi(\tau) + v_i \) is a good basis. Since \( \mathcal{D} \) is a pseudo-simplex, Theorem 1.5 implies that every such \( (n-1) \)-simplex \( \tau \) is a face of exactly two \( n \)-simplices of \( \mathcal{E}(I) \). By the definition, both these \( n \)-simplices are vertices of \( \mathbb{G}_i \). The edge corresponding to \( \tau \) connects these two vertices. The main properties of \( \mathbb{G}_i \) are proved in Lemma 6.7 below after a preliminary Lemma 6.6.

6.6. Lemma. Let \( \tau \) be an \((n-1)\)-simplex of \( \mathcal{E}(I) \) contained in \( I \). If \( \varphi(\tau) + v_i \) is defined, then \( \tau = I - i \). Conversely, \( \varphi(I - i) + v_i \) is a good basis. There is exactly one \( n \)-simplex of \( \mathcal{E}(I) \) having \( I - i \) as a face. This simplex \( \sigma_i \) is a vertex of \( \mathbb{G}_i \).

Proof. If \( \tau \subset I \), then \( \tau = I - k \) for some \( k \in I \). If \( k \neq i \), then \( i \in \tau \) and hence \( v_i = \varphi(i) \in \varphi(\tau) \). In this case \( \varphi(\tau) + v_i \) is undefined. Clearly, \( \varphi(I - i) + v_i = B \) is a good basis. This proves the first two claims of the lemma. If \( \rho \cup (I - i) \) is an \( n \)-simplex of \( \mathcal{E}(I) \), then \( \rho \) is a 0-simplex of \( \mathcal{D}([i]) \). By Lemma 1.1 the complex \( \mathcal{D}([i]) \) has only one vertex and hence only one 0-simplex. It follows that \( \rho \) is equal to this 0-simplex. In particular, \( \rho \) is unique. Therefore there is exactly one \( n \)-simplex \( \sigma_i \) of \( \mathcal{E}(I) \) having \( I - i \) as a face. Since \( \varphi(I - i) + v_i \) is a good basis, \( \sigma_i \) is a vertex of \( \mathbb{G}_i \). ■

6.7. Lemma. Let \( \sigma \) be a vertex of \( \mathbb{G}_i \). If \( \varphi(\sigma) \) is a good basis, then \( \sigma \) is an endpoint of exactly 1 edge if \( \sigma \neq \sigma_i \) and of 0 edges if \( \sigma = \sigma_i \). Otherwise \( \sigma \) is an endpoint of exactly 2 edges if \( \sigma \neq \sigma_i \) and of 1 edge if \( \sigma = \sigma_i \).
Proof. Suppose that $\varphi(\sigma)$ is a good basis. By applying Lemma 6.2 to $\varphi(\sigma)$ in the role of $B$ and $v_i$ in the role of $w$ we see that there is exactly one element $u \in \varphi(\sigma)$ such that $\varphi(\sigma) - u + v_i$ is a good basis. Since $\varphi(\sigma)$ is a basis, $\varphi$ is a bijection on $\sigma$ and elements $u \in \varphi(\sigma)$ correspond to $(n-1)$-faces $\tau$ of $\sigma$ such that $\varphi(\sigma) - u = \varphi(\tau)$. It follows that there is exactly one $(n-1)$-face $\tau$ of $\sigma$ such that $\varphi(\tau) + v_i$ is a good basis. If $\sigma = \sigma_i$, then $1 - i$ is such a face and hence $\tau = 1 - i$. In this case $\sigma$ is an endpoint of 0 edges. If $\sigma \neq \sigma_i$, then $\tau \not\in I$ by Lemma 6.6 and $\sigma$ is an endpoint of 1 edge. This completes the proof in the case when $\varphi(\sigma)$ is a good basis.

Suppose now that $\varphi(\sigma)$ is not a good basis. We claim that there exactly two $(n-1)$-faces $\tau$ of $\sigma$ such that $\varphi(\tau) + v_i$ is a good basis. Since $\sigma$ is a vertex of $G_i$, there is at least one such face. If $\tau$ is such a face, then the set $\varphi(\tau) + v_i$ is a basis and hence consists of $n + 1$ elements. It follows that $\varphi(\tau)$ consists of $n$ elements. Since $\tau$ also consists of $n$ elements, this implies that $\varphi$ is injective on $\tau$. Let $u$ be the vertex of $\sigma$ such that $\tau = \sigma - u$. There are two cases to consider, depending on if $\varphi(\sigma) = \varphi(\tau)$ or not.

Suppose first that $\varphi(\sigma) = \varphi(\tau)$. Since $\varphi$ is injective on $\tau$, in this case there is exactly one vertex $v \neq u$ of $\sigma$ such that $\varphi(v) = \varphi(u)$ and $\varphi(\sigma - u) = \varphi(\sigma - v)$. If $w$ is a vertex of $\sigma$ different from $u, v$, then $\varphi(\sigma - w)$ has $< n$ elements and hence the set $\varphi(\sigma - w) + v_i$ is not a basis. This implies our claim in the case when $\varphi(\sigma) = \varphi(\tau)$.

Suppose now that $\varphi(\sigma) \neq \varphi(\tau)$. Then $\varphi(\sigma)$ properly contains $\varphi(\tau)$ and hence consists of $n + 1$ elements. It follows that $\varphi$ is injective on $\sigma$, and $\varphi(\sigma) - \varphi(u) + v_i$ is a good basis. By applying Lemma 6.3 to $\varphi(\sigma)$ in the role of $D$ we see that there are 0 or 2 elements $d \in \varphi(\sigma)$ such that $\varphi(\sigma) - d + v_i$ is a good basis. Of course, $\varphi(u)$ is one of them. Since $\varphi$ is injective on $\sigma$, the other one is equal to $\varphi(u)$ for uniquely determined $v \in \sigma, \ v \neq u$. It follows that $\varphi(\sigma - w) + v_i$ is a good basis for $w = u$ or $v$ but not for any other vertex of $\sigma$. This implies our claim in the case when $\varphi(\sigma) \neq \varphi(\tau)$.

Therefore if $\varphi(\sigma)$ is not a good basis, then there exactly two $(n-1)$-faces $\tau$ of $\sigma$ such that $\varphi(\tau) + v_i$ is a good basis. Suppose that one of them is contained in $I$. In this case Lemma 6.6 implies that the other does not and $\sigma = \sigma_i$. Hence in this case $\sigma$ is an endpoint of 1 edge. If neither of these two faces is contained in $I$, then $\sigma$ is an endpoint of 2 edges. This completes the proof in the case when $\varphi(\sigma)$ is not a good basis.

A graph-based proof of Theorem 6.5. Lemma 6.7 implies that each component of the graph $G_i$ either is a cycle, or is a path connecting two $n$-simplices $\sigma, \sigma'$ such that $\varphi(\sigma)$ and $\varphi(\sigma')$ are good bases, or is a path connecting $\sigma_i$ with an $n$-simplex $\sigma$ such that $\varphi(\sigma)$ is a good basis. If $\varphi(\sigma_i)$ is a good basis, then the last path degenerates into a single vertex $\sigma_i$. If $\varphi(\sigma_i)$ is not a good basis, then $G_i$ contains a unique path connecting $\sigma_i$ with an $n$-simplex $\sigma$ such that $\varphi(\sigma)$ is a good basis. Other $n$-simplices $\sigma$ such that $\varphi(\sigma)$ is a good basis occur in pairs connected by a path of $G_i$. In particular, there is at least one $n$-simplex $\sigma$ such that $\varphi(\sigma)$ is a basis containing $b$ in its convex hull. Moreover, the number of such $n$-simplices $\sigma$ is odd. In view of (26) this implies Theorem 6.5.
7. Vector colorings

Introduction. The existence part of Theorem 6.5 remains valid without the non-degeneracy assumption (the part about the number of cells being odd does not). In this section we will prove this in the classical situation when the oriented matroid $M$ is defined by a vector configuration in $\mathbb{R}^{n+1}$. In this case the most natural approach is to perturb this vector configuration. It turns out that it is sufficient to perturb only the vector $b$, as suggested by Scarf [Sc1], [Sc3]. As an application we will prove the classical Scarf theorem [Sc1], [Sc3].

The vector framework. Let us number the standard coordinates of $\mathbb{R}^{n+1}$ by 0, 1, ..., $n$. Let $v_0, v_1, ..., v_n$ be the standard basis of $\mathbb{R}^{n+1}$. Let $B = \{v_0, v_1, ..., v_n\}$ and let $M \subset \mathbb{R}^{n+1}$ be a finite set containing $B$ and a vector $b \neq 0$ such that all its coordinates are non-negative. The last condition means that there are numbers $b_i \geq 0$ such that

\begin{equation}
(29) \quad b = \sum_{i \in I} b_i v_i,
\end{equation}

The circuits of the vector configuration $M$ define an oriented matroid which we also denote by $M$. Let $A = M - b$ and let us consider the equation

\begin{equation}
(30) \quad b = \sum_{v \in A} y_v v
\end{equation}

in real unknowns $y_v$. A solution $\left( y_v \right)_{v \in A}$ is a vector $y \in \mathbb{R}^A$. Following Scarf [Sc3], Theorem 4.2.3, we assume that the set of non-negative solutions of (30) is bounded.

7.1. Lemma. Under the above assumptions the oriented matroid $M$ is acyclic.

Proof. If $M$ is not acyclic, then there is a circuit $\sigma$ such that $\sigma_\sigma = \emptyset$. Equivalently, 0 is a non-trivial non-negative linear combination of vectors in $M$. Using (29) one can eliminate $b$ from this linear combination while keeping it non-negative. Therefore

\[ 0 = \sum_{v \in A} z_v v \]

for some non-negative numbers $z_v$, not all of which are 0. Clearly, if $y$ is a non-negative solution of (30), then $y + \lambda z$ is also a non-negative solution for every $\lambda > 0$ and hence the set of non-negative solutions is unbounded, contrary to the assumption. ■

The non-degeneracy property. The vector configuration $M$ together with $b \in M$ is said to be non-degenerate if the vector $b$ cannot be represented as a non-negative linear combination of $< n + 1$ elements of $M - b$. See [Sc3], the non-degeneracy assumption 4.3.1. In this case the corresponding matroid $M$ together with $b$ are, obviously, non-degenerate.
7.2. The main theorem for vector colorings (the general case). Suppose that $D$ is a chain-simplex based on $I = \{0, 1, \ldots, n\}$ and let $X = V_D$. For every coloring $c : X \rightarrow A$ there exist a non-empty subset $C \subset I$ and a $d(C)$-simplex $\tau$ of $D(C)$ such that

$$c(\tau) \cup \{v_i \mid i \in I \sim C\}$$

is a basis of $\mathbb{R}^{n+1}$ and $b$ is a non-negative linear combination of its elements.

**Proof.** Suppose first that $M, b$ is non-degenerate. Then Theorem 6.5 applies. By this theorem there exist a non-empty subset $C \subset I$ and a $d(C)$-simplex $\tau$ of $D(C)$ such that (31) is a basis of $\mathbb{R}^{n+1}$ and $b$ belongs to its convex hull in the sense of oriented matroids. By the definition, the latter means that $b$ is a non-negative linear combination of elements of the basis (31). This proves the theorem in the non-degenerate case.

The degenerate case reduces to the non-degenerate one by perturbing $b$. Obviously, there exists a sequence $b_1, b_2, b_3, \ldots$ of vectors $b_i \in \mathbb{R}^{n+1}$ such that $b_i$ tends to $b$ when $i \rightarrow \infty$ and $b_i$ is not a linear combination of $< n + 1$ elements of $M - b$ for every $i$. Let $M_i = M - b + b_i$ for every $i$. Then the configuration $M_i, b_i$ is non-degenerate for every $i$ and $c : X \rightarrow M - b = M_i - b_i$ is a matroid coloring.

By the already proved non-degenerate case of the theorem, for every $i$ there exist a non-empty subset $C_i \subset I$ and a simplex $\tau_i$ of $D(C_i)$ such that

$$c(\tau_i) \cup \{v_i \mid i \in I \sim C_i\}$$

is a basis and $b_i$ is a non-negative linear combination of its elements. Since the set $I$ and the complexes $D(C)$ are finite, after passing to a subsequence we may assume that $C_i = C$ for some subset $C \subset I$ and $\tau_i = \tau$ some simplex $\tau$ of $D(C)$ for all $i$. Then (31) is a basis and $b_i$ is a non-negative linear combination of its elements for every $i$. In other words, the coordinates of $b_i$ in this basis are non-negative for every $i$. When $i \rightarrow \infty$ these coordinates tend to the coordinates of $b$ in the same basis. By passing to the limit we see that the coordinates of $b$ are also non-negative. The theorem follows. ■

**Families of linear orders.** Let us move from the simplex-families of Section 1 to the families of linear orders of Section 2. So, let $X$ be a non-empty finite set and suppose that for every $i \in I$ a linear order $<_i$ on $X$ is given. As before, let us assume that $X \cap I = \emptyset$.

In order to be closer to Scarf’s form of his theorem, let us extend the orders $<_i$ to linear orders on the set $X \cup I$, which we will still denote by $<_i$. Following Scarf, we will require that every element $i \in I$ is minimal with respect to $<_i$ and that $x <_i k$ for every $x \in X$ and every $k \in I$, $k \neq i$. Clearly, such extensions exist, but are not unique in general: the restriction of the order $<_i$ to $I - i$ can be completely arbitrary. Let us fix some choice of such extensions. Now we can speak about dominant subsets of $X \cup I$. 40
7.3. Lemma. Let $\tau \subset X$ and $C \subset I$, $C \neq \emptyset$. Then $\tau$ is dominant with respect to $C$ if and only if $\tau \cup (I \smallsetminus C)$ is dominant with respect to $I$.

Proof. If $i \in C$ and $k \in I \smallsetminus C$, then $k \neq i$ and hence $x <_i k$ for every $x \in X$. It follows that $\min_i \tau \cup (I \smallsetminus C) = \min_i \tau$ for every $i \in C$. If $i \in I \smallsetminus C$, then $\tau \cup (I \smallsetminus C)$ contains the $<_i$-minimal element $i$ and hence $\min_i \tau \cup (I \smallsetminus C) = i$.

It follows that $\tau \cup (I \smallsetminus C)$ is dominant with respect to $I$ if there is no element $y \in X \cup I$ such that $\min_i \tau <_i y$ for every $i \in C$ and $i <_i y$ for every $i \in I \smallsetminus C$. Clearly, there is no such element $y \in X$ if and only if $\tau$ is dominant with respect to $C$ and the original orders on $X$. Therefore, it is sufficient to prove that there are no such elements $y \in I$. Indeed, if $y \in C$, then $y <_y \min_y \tau$ and hence the condition $\min_i \tau <_i y$ does not hold, and if $y \in I \smallsetminus C$, then the condition $y <_y y$ does not hold. The lemma follows. ■

Scarf theorem. Suppose that $\varphi : X \cup I \to \mathbb{R}^{n+1}$ is a map such that $\varphi(i) = v_i$ for every $i \in I$. Then there exists a subset $\sigma \subset X \cup I$ dominant with respect to $I$ and such that $\varphi(\sigma)$ is a basis of $\mathbb{R}^{n+1}$ and $b$ is a non-negative linear combination of elements of $\varphi(\sigma)$.

Proof. Let $\mathcal{T}$ be the simplex-family associated in Section 2 with $X$ and the orders $<_i$ on $X$. It is a chain-simplex by Corollary 2.9. If $C \subset I$, then every simplex $\tau$ of $\mathcal{T}(C)$ is a subset of a $C$-cell, i.e. of some set $\sigma \subset X$ dominant with respect to $C$ and such that $|\sigma| = |C|$. If $\tau$ is a $d(C)$-simplex, then $|\tau| = |C|$ and hence in this case $\tau = \sigma$. Therefore every $d(C)$-simplex of $\mathcal{T}(C)$ is a subset of $X$ dominant with respect to $C$.

Let $M = \varphi(X \cup I) + b$ and let $c$ be the restriction of $\varphi$ to $X$. By Theorem 7.2 applied to $M$ and $c$ there exist a non-empty set $C \subset I$ and a $d(C)$-simplex $\tau$ of $\mathcal{T}(C)$ such that (31) is a basis of $\mathbb{R}^{n+1}$ and $b$ is a non-negative linear combination of its elements. Let $\sigma = \tau \cup (I \smallsetminus C)$. Clearly, $|\sigma| = |I|$ and the image $\varphi(\sigma)$ is equal to the set (31). By the previous paragraph $\tau$ is dominant with respect to $C$ and hence $\sigma$ is dominant with respect to $I$ by Lemma 7.3. The theorem follows. ■

The classical Scarf theorem. In [Sc1] and [Sc3] Scarf considered only subsets $X$ of the standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$ and the orders $<_i$ which were almost completely determined by the standard order $<$ on the coordinates of points in $\mathbb{R}^{n+1}$, i.e. the orders from Section 3. At the same time he was aware that his methods work in more general situations. See [Sc2], Chapter 6 and especially the discussion at the bottom of p. 146. In fact, every family of $n + 1$ orders on a finite set $X$ can be realized by an embedding of $X$ into $\Delta^n$. So, the above theorem is more abstract, but is not more general than the original Scarf theorem. While the above proof incorporates a lot of ideas of Scarf, it depends on Theorem 6.5 about colorings into colors belonging to an oriented matroid. The latter notion was still in the future at the time of [Sc1] and [Sc3]. Also, our proof of Theorem 6.5 is based on topological ideas, in contrast with related, but different path-following arguments of Scarf.
8. Oriented matroid colorings: the general case

**Perturbations in oriented matroids.** Let $M$ be an acyclic oriented matroid and let

$$B = \{ v_0, v_1, \ldots, v_n \}$$

be a basis of $M$. Let $b \in M \sim B$ and suppose that $b$ belongs to the convex hull of $B$. In this section we do not assume that the pair $M, b$ is non-degenerate.

The theory of oriented matroids provides a combinatorial analogue of the perturbation argument used in the proof of Theorem 7.2. Namely, one can add to $M$ a new element $p$ playing the role of a particular perturbation of $b$ in the vector framework. Let us describe this particular perturbation. Clearly, in the vector framework $B - v_i + b$ is a basis for some $i \in I$ (usually for all such $i$). Without any loss of generality one can assume that $B - v_0 + b$ is a basis. Then one can take as the perturbation of $b$ the vector $p = b + \lambda b_1 + \lambda^2 b_2 + \ldots + \lambda^n b_n$ for any sufficiently small $\lambda > 0$.

Adding a new element is a special case of the notion of an extension of an oriented matroid. Namely, an oriented matroid $M'$ is said to be an extension of $M$ if $M \subset M'$ and a signed subset of $M$ is a circuit of $M'$ if and only if it is a circuit of $M$. We are interested in the one point extensions $M'$, i.e. extensions $M'$ such that $M' = M + p$ for some $p \not\in M$. Moreover, we are interested only in one-point extensions $M'$ such that a basis of $M$ is also a basis in $M'$ (if this is true for one basis of $M$, then this is true for every basis of $M$).

Among such extensions are the lexicographic extensions introduced by M. Las Vergnas [LV]. A lexicographic extension $L = M + p$ of $M$ is associated with every ordered basis

$$(32) \quad b = b_0, b_1, b_2, \ldots, b_n$$

of $M$. It was realized by M. Todd [T] that the new element $p$ of $L$ can play the role of the perturbation $b + \lambda b_1 + \lambda^2 b_2 + \ldots + \lambda^n b_n$ of $b$. In the following two lemmas we state, following M. Todd [T], the relevant properties of the lexicographic extension $L = M + p$ associated with (32). Appendix 2 contains some additional details (not used here).

**8.1. Lemma.** If $\sigma$ is a circuit of $L$ such that $p \in \sigma$, then $|\sigma| \geq n + 2$. The signed subset $\sigma = (\sigma_+, \sigma_-)$, where $\sigma_+ = \{b, b_1, b_2, \ldots, b_n\}$ and $\sigma_- = \{p\}$, is a circuit.

**8.2. Lemma.** Suppose that $\sigma$ is a circuit of $L$ such that $p \in \sigma_-$ and $b \not\in \sigma$. Then there exists exactly one circuit $\rho$ of $M$ such that $b \in \rho_-$, $\rho_- \sim \{b\} \subset \sigma_-$, and $\rho_+ \subset \sigma_+$.

**Proofs.** See [T], Propositions 5.3 and 5.4 and Corollary 5.6.
8.3. **Corollary.** If $M$ is acyclic, then $L - b$ is also acyclic.

**Proof.** Let $\tau$ be a circuit of $L$ such that $\tau \subseteq L - a$ and $\tau = \emptyset$. If $p \not\in \tau$, then $\tau$ is a circuit of $M$ and hence $M$ is not acyclic. Therefore we can assume that $p \in \tau = \tau_+$. Let $\sigma = -\tau$. Then $p \in \sigma_-$ and $a \not\in \sigma$. Let $\rho$ be the circuit of $M$ such as in Lemma 8.2. Then $\rho_+ \subseteq \sigma_+ = \tau_- = \emptyset$. Again, this is impossible for an acyclic $M$. ■

8.4. **Corollary.** If $p$ belongs to the convex hull of a subset $\varepsilon \subset M - b$ in $L$, then $b$ belongs to the convex hull of $\varepsilon$ in $M$.

**Proof.** The first statement immediately follows from Lemma 8.1. Suppose that $p$ belongs to the convex hull of $\varepsilon \subset M - b$. Then there is a circuit $\sigma$ of $L$ such that $\sigma_+ \subseteq \varepsilon$ and $\sigma_- = \{p\}$. Now Lemma 8.2 implies that there exists a circuit $\rho$ of $M$ such that $b \in \rho_-$, $\rho_- \setminus \{b\} \subseteq \sigma_- \cap M = \emptyset$, and $\rho_+ \subseteq \sigma_+ \subseteq \varepsilon$. Hence $\rho_- = \{b\}$ and $\rho_+ \subseteq \varepsilon$, and therefore $b$ belongs to the convex hull of $\varepsilon$. ■

8.5. **The main theorem for matroid colorings (the general case).** Let $\mathcal{D}$ is a chain-simplex based on $I = \{0, 1, \ldots, n\}$ and let $X = V_{\mathcal{D}}$. For every coloring $c : X \rightarrow M - b$ there exist a non-empty subset $C \subset I$ and a $d(C)$-simplex $\tau$ of $\mathcal{D}(C)$ such that

\[(33) \quad c(\tau) \cup \{v_i \mid i \in I \setminus C\}\]

is a basis of $M$ and $b$ is contained in its convex hull.

**Proof.** The axioms of oriented matroids immediately imply that the sets $\sigma$, where $\sigma$ is a circuit of $M$, are the circuits of a matroid, which we will denote by $\mathcal{M}$, and the oriented matroid $\mathcal{M}$ and the matroid $M$ have the same bases. By the symmetric exchange property of matroids the set $B - v_0 + b$ is a basis of $M$, and hence of $M$, for some $i \in I$. Without any loss of generality we can assume that $B - v_0 + b$ is a basis of $M$.

Let $L = M + p$ be the lexicographic extension of $M$ associated with the basis (32), where $b_i = v_i$ for $i \geq 1$. As an unordered set this is the basis $B - v_0 + b$. Let $M' = L - b$. Then $B$ is also a basis of $M'$ and the new element $p$ belongs to $M' \sim B$. By Corollary 8.3 the oriented matroid $M'$ is acyclic, and by Lemma 8.1 the pair $M', p$ is non-degenerate.

Let us check that $p$ belongs to the convex hull of $B$ in $M'$. Lemma 8.1 implies that the signed subset $\sigma = (\sigma_+, \sigma_-)$, where $\sigma_+ = B - v_0 + b$ and $\sigma_- = \{p\}$, is a circuit. Since $b$ belongs to the convex hull of $B$, there exists a circuit $\tau$ such that $\tau_- = \{b\}$ and $\tau_+ \subseteq B$. By the axiom (iv) of oriented matroids there exists a circuit $\omega$ such that

$\omega_+ \subseteq (\sigma_+ \cup \tau_+) \setminus \{b\}$ and $\omega_- \subseteq (\sigma_- \cup \tau_-) \setminus \{b\}$. 

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These inclusions imply that $\omega_+ \subset B$ and $\omega_- \subset \{p\}$. Since $B$ is a basis, $\omega_- \neq \emptyset$. Therefore $\omega_- = \{p\}$ and hence $p$ belongs to the convex hull of $B$.

Clearly, $M' - p = M - b$ and one can consider $c$ as a matroid coloring $X \rightarrow M' - p$. By applying Theorem 6.5 to $M'$, $p$ in the role of $M$, $b$, the same basis $B$, and the coloring $c$, we see that there exist a non-empty subset $C \subset I$ and a $d(C)$-simplex $\tau$ of $D(C)$ such that (33) is a basis of $M'$ containing $p$ in its convex hull. Let us denote this basis by $\epsilon$. Since $\epsilon$ is contained in $M' - p = M - b$, the basis $\epsilon$ is also a basis of $M$. Corollary 8.4 implies that $b$ belongs to the convex hull of $\epsilon$ in $M$. The theorem follows. ■

Families of linear orders. Let $X$ be a non-empty finite set and suppose that for every $i \in I$ a linear order $<_i$ on $X$ is given. As before, let us assume that $X \cap I = \emptyset$. Let us extend the orders $<_i$ to linear orders on $X \cup I$ subject to the same conditions as in Section 7.

Generalized Scarf theorem. Let $c : X \rightarrow M - b$ be a matroid coloring. Let us extend $c$ to a map $\varphi : X \cup I \rightarrow M - b$ by the rule $\varphi(i) = v_i$. Then there exists a subset $\sigma \subset X \cup I$ consisting of $|I|$ elements, dominant with respect to $I$, and such that $\varphi(\sigma)$ is a basis of $M$ containing $b$ in its convex hull.

Proof. The proof is completely similar to the proof of Scarf theorem in Section 7. The only difference is that one needs to apply Theorem 8.5 instead of Theorem 7.2. ■

Hedgehog colorings. One can apply Theorems 6.5 and 8.5 also to simplex-families arising from triangulations of geometric simplices. There is a version of these theorems which for such simplex-families is closer to the lemmas of Alexander and Sperner. See Theorem 8.7. It deals with matroid colorings called hedgehog colorings. Their definition is based on the notions of hyperplanes and cocircuits discussed in Appendix 2. See also [B-Z]. Let $H$ be a hyperplane of $M$. If $\eta$ is one of two cocircuits corresponding to $H$, then

$$\eta(\geq 0) = H \cup \{ v \in M \mid \eta(v) = + \}$$

is called the closed half-space defined by $\eta$. We will denote by $\eta(\leq 0)$ the closed half-space defined by the cocircuit $-\eta$. Closed half-spaces are convex in the sense that they are equal to their convex hulls. This follows, for example, from [B-Z], Exercise 3.9. For each $i \in I$ let $H_i$ be the hyperplane spanned by the set $B - v_i$ and let $\eta_i$ be one of two cocircuits associated with $H_i$, namely, the one for which $\eta_i(v_i) = +$.

A matroid coloring $c : X \rightarrow M - b$ is called a hedgehog coloring if $C \subset I - i$ implies that $c(v) \in \eta_i(\leq 0)$ for every vertex $v$ of $D(C)$ and every $i \in I$. If $D$ is the simplex-family associated with a triangulation of a geometric simplex, then $C \subset D$ implies that $D(C)$ is a subcomplex of $D(D)$ and $c$ is a hedgehog coloring if and only if $c(v) \in \eta_i(\leq 0)$ for every $i \in I$ and every vertex $v$ of $D(I - i)$. 44
8.6. Lemma. If \( b \) is not contained in the convex hull of any proper subset of the basis \( B \), then \( b \not\in \eta_i(\leq 0) \) for every \( i \in I \).

**Proof.** The point \( b \) is contained in the convex hull of \( B \) and hence belongs to \( \eta_i(\geq 0) \). Clearly, \( \eta_i(\geq 0) \cap \eta_i(\leq 0) = H_i \). Therefore, if \( b \in \eta_i(\leq 0) \), then \( b \in H_i \). Hence it is sufficient to show that \( b \) belongs to the convex hull of \( B - v_i \) if \( b \in H_i \). So, suppose that \( b \in H_i \). Then there exists a circuit \( \sigma \) such that \( b \in \sigma \) and \( \sigma \subseteq B - v_i + b \). Without any loss of generality we can assume that \( b \in \sigma_+ \). On the other hand, there exists a circuit \( \tau \) such that \( \tau_- = \{b\} \) and \( \tau_+ \subseteq B \). By the axiom (iv) there exists a circuit \( \omega \) such that \( \omega_+ \subseteq (\tau_+ \cup \sigma_+) \setminus \{b\} \subseteq B \) and \( \omega_- \subseteq (\tau_- \cup \sigma_-) \setminus \{b\} \subseteq B \). Hence \( \omega \subseteq B \). Since \( \omega \neq \emptyset \), by the axiom (i), this contradicts to \( B \) being a basis. The lemma follows. \( \blacksquare \)

8.7. The main theorem for hedgehog colorings. Suppose that \( b \) is not contained in the convex hull of any proper subset of the basis \( B \). Then for every hedgehog coloring \( c \) there exist a \( d(I) \)-simplex \( \tau \) of \( D(I) \) such that \( c(\tau) \) is a basis of \( M \) and \( b \) is contained in its convex hull. If the pair \( M, b \) is non-degenerate, then the number of such \( \tau \) is odd.

**Proof.** For each \( i \in I \) let \( w_i = v_{i+1} \), where subscripts are treated as integers modulo \( n + 1 \). We can apply Theorem 8.5 to the basis \( w_0, w_1, \ldots, w_n \) in the role of \( v_0, v_1, \ldots, v_n \) (it is the same as a set, but the order matters). Therefore there exist a non-empty subset \( C \subseteq I \) and a \( d(C) \)-simplex \( \tau \) of \( D(C) \) such that

\[
Y = c(\tau) \cup \{ v_{i+1} \mid i \in I \setminus C \} = c(\tau) \cup \{ w_i \mid i \in I \setminus C \}
\]

is a basis of \( M \) containing \( b \) in its convex hull. Suppose that \( C \neq I \). Then there exists \( k \in I \) such that \( k \not\in C \) and \( k - 1 \in C \). In particular, \( C \subset I - k \). Since \( c \) is a hedgehog coloring, this implies that \( c(v) \in \eta_k(\leq 0) \) for every \( v \in \tau \). Since \( k - 1 \in C \),

\[
v_k \not\in \{ v_{i+1} \mid i \in I \setminus C \}
\]

and hence \( \{ v_{i+1} \mid i \in I \setminus C \} \subseteq H_k \subseteq \eta_k(\leq 0) \). It follows that \( Y \subseteq \eta_k(\leq 0) \). Since \( b \) belongs to the convex hull of \( Y \) and \( \eta_k(\leq 0) \) is convex, this implies that \( b \in \eta_k(\leq 0) \). The contradiction with Lemma 8.6 shows that \( C = I \). This proves the existence claim. The claim about the non-degenerate pairs \( M, b \) similarly follows from Theorem 6.5. \( \blacksquare \)

Classical colorings and matroid colorings. Let us choose some element \( b \not\in I \) and let \( M = I + b \). Let \( \sigma \) be the signed subset of \( M \) defined by \( \sigma_+ = I \) and \( \sigma_- = \{b\} \). Taking \( \sigma \) and \( -\sigma \) as the only circuits turns \( M \) into an oriented matroid. Obviously, the sets \( I \) and \( I - i + b \) with \( i \in I \) are bases of \( M \) and there are no other bases. Clearly, the pair \( M, b \) is non-degenerate. This oriented matroid structure on \( M \) allows us to consider classical colorings \( c : X \to I \) as matroid colorings. For example, an Alexander–Sperner coloring is a hedgehog coloring and hence Theorem 1.7 is a special case of Theorem 8.7.
9. Scarf’s proof of Kakutani’s fixed point theorem

**Multi-valued maps.** Let $X, Y$ be two sets. A multi-valued map $F : X \rightarrow Y$ is simply a usual map $X \rightarrow \mathcal{P}(Y) \sim \{\emptyset\}$, where $\mathcal{P}(Y)$ is the set of all subsets of $Y$. The graph of a multi-valued map $F : X \rightarrow Y$ is the subset of $X \times Y$ consisting of all pairs $(x, y)$ such that $y \in F(x)$. A fixed point of $F$ is a point $x \in X$ such that $x \in F(x)$. Suppose that $X, Y$ are topological spaces. A multi-valued map $X \rightarrow Y$ is said to be closed if its graph is a closed subset of $X \times Y$. Usually closed multi-valued maps are called upper semicontinuous, but the term “closed”, borrowed from the operator theory, seems to invoke more relevant ideas. If $F : X \rightarrow Y$ is closed, then $F(x)$ is closed for every $x \in X$.

If the topological spaces $X, Y$ are sufficiently nice (for example, are subspaces of $\mathbb{R}^n$), then $F$ is closed if and only if the following condition holds. Suppose that $x_1, x_2, x_3, \ldots$ is a sequence of points of $X$ converging to a point $x \in X$. Suppose that $y_i \in F(x_i)$ for every $i$ and the sequence $y_1, y_2, y_3, \ldots$ converges to a point $y \in Y$. Then $y \in F(x)$.

We will consider only closed multi-valued maps $\Delta^n \rightarrow \Delta^n$, where $\Delta^n$ is the standard $n$-simplex in $\mathbb{R}^{n+1}$ (see Section 3). Let $F$ be such a map.

**Matroids and colorings related to $F$.** Let $B = \{v_0, v_1, \ldots, v_n\}$ be the standard basis of $\mathbb{R}^{n+1}$ as in the vector framework from Section 8. Let $X \subset \Delta^n$ be a finite set. It will be used in the same manner as in Scarf’s proof of Brouwer’s fixed point theorem. Let $b$ be the barycenter of $\Delta^n$, i.e. the point of $\mathbb{R}^{n+1}$ with all coordinates equal to $1/(n + 1)$.

Let $f : X \rightarrow \Delta^n$ be an arbitrary map, initially unrelated to $F$. The vector coloring associated with $f$ is the map $c : X \rightarrow \mathbb{R}^{n+1}$ defined by

$$c(x) = f(x) - x + b.$$

for every $x \in X$. The vector configuration associated with $f$ is the subset $M \subset \mathbb{R}^{n+1}$ consisting of $v_0, v_1, \ldots, v_n$, the point $b$, and the vectors $c(x)$ for all $x \in X$. The circuits of the vector configuration $M$ turn $M$ into an oriented matroid. Let $A = M - b$.

**9.1. Lemma.** Let $f : X \rightarrow \Delta^n$ be an arbitrary map and let $M$ be the vector configuration associated to $f$. If $(y_v)_{v \in A}$ is a non-negative solution of the equation (30), then $y_v \leq 1$ for every $v \in A$. In particular, the set of non-negative solutions is bounded and hence the vector configuration $M$ is acyclic as a matroid.

**Proof.** In the present situation the equation (30) takes the form

$$b = \sum_{i \in I} y_i v_i + \sum_{x \in X} y_x c(x).$$
Let us take the sum of all coordinates of each term of this equation. Obviously, the sum of coordinates of $b$ and of each $v_i$ is 1. For every $x \in X$ the sums of the coordinates of $x$ and $f(x)$ are both equal to 1 and hence the sum of coordinates of $c(x)$ is equal to that of $b$, i.e. to 1. Now the above equation implies that

$$1 = \sum_{i \in I} y_i + \sum_{x \in X} y_x.$$  

Since all numbers $y_i$ and $y_x$ are non-negative, this implies that all of them are $\leq 1$. ■

9.2. Lemma. Suppose that for every natural number $k$ a finite subset $X_k \subset \Delta^n$ and a map $f_k : X_k \rightarrow \Delta^n$ are given. Let $c_k$ be the vector coloring associated with $f_k$. Then for some subset $C \subset I$ the following property holds. For an infinite set of natural numbers $k$ there exist a $C$-cell $\sigma_k \subset X_k$ such that

$$c_k(\sigma_k) \cup \{ v_i \mid i \in I \sim C \}$$

is a basis of $\mathbb{R}^{n+1}$ containing $b$ in its convex hull.

Proof. Let $M_k$ be the vector configuration associated with $f_k$. Lemma 9.1 implies that $M_k$ together with $b$ satisfies the assumptions of the vector Scarf framework. By the classical Scarf theorem there are subsets $\sigma_k \subset X_k$ and $C_k \subset I$ such that $\sigma_k$ is a $C_k$-cell and the set (34) with $C = C_k$ is a basis of $\mathbb{R}^{n+1}$ containing $b$ in its convex hull. Since there is only a finite number of subsets of $I$, the same subset of $I$ occurs as $C_k$ an infinite number of times. Clearly, one can take such a subset as $C$. ■

Kakutani's theorem. Let $F : \Delta^n \rightarrow \Delta^n$ be a closed map. If the set $F(x)$ is convex for every $x \in \Delta^n$, then $F$ has a fixed point, i.e. there exists $z \in \Delta^n$ such that $z \in F(z)$.

Proof. The proof follows the outline of Scarf's proof of Brouwer's fixed point theorem from Section 3, and we will use the notions introduced in that section. Let $X_1, X_2, X_3, \ldots$ be a sequence of finite subsets of $\Delta^n$. Suppose that the sets $X_k$ are chosen in such a way that $X_k$ is $\varepsilon_k$-dense in $X_k$ for some sequence $\varepsilon_k \rightarrow 0$. Let us choose for every natural number $k$ a map $f_k : X_k \rightarrow \Delta^n$ such that

$$f_k(x) \in F(x)$$

for every $x \in X_k$ and let $c_k$ be the vector coloring associated with $f_k$.

Lemma 9.2 implies that after passing to a subsequence we can assume that there exist a subset $C \subset I$ independent of $k$ and $C$-cells $\sigma_k \subset X_k$ such that (34) is a basis of $\mathbb{R}^{n+1}$ containing $b$ in its convex hull. Then every $\sigma_k$ consists of $|C|$ elements. Moreover, Lemma 2.1 implies that maps $i \rightarrow \min_i \sigma_k$ are bijections $C \rightarrow \sigma_k$. 47
Let $c_k(i) = c_k(\min_i \sigma_k)$. Now the set (34) takes the form
\[
\{ c_k(i) \mid i \in C \} \cup \{ v_i \mid i \in I \sim C \}.
\]
Since this set is a basis containing $b$ in its convex hull,
\[
(35) \quad b = \sum_{i \in I \sim C} y_k(i) v_i + \sum_{i \in C} y_k(i) c_k(i),
\]
for some non-negative coefficients $y_k(i)$.

Let $\Delta_k = \Delta(\sigma_k, C)$. Since $X_k$ is $\varepsilon_k$-dense in $X_k$ and $\varepsilon_k \rightarrow 0$, Lemma 3.1 implies that the diameters of the simplices $\Delta_k$ tend to 0. After passing to a subsequence one can assume that the simplices $\Delta_k$ converge to a point $z \in \Delta^n$, i.e. that every sequence of points $x(k) \in \Delta_k$ converges to $z$. In particular, $\min_i \sigma_k$ converges to $z$ for every $i$. After passing to a further subsequence one can assume that for every $i \in I$ the sequence $f_k(\min_i \sigma_k)$ converges to a limit $f(i) \in \Delta^n$, and hence $c_k(i)$ converges to $f(i) - z + b$. For each $k$ the numbers $y_i = y_k(i)$ form a solution of (30) and hence Lemma 9.1 implies that $y_k(i) \leq 1$ for every $k, i$. Therefore, after passing to a subsequence once more one can assume that for every $i \in I$ the sequence $y_k(i)$ converges to a limit $y(i) \geq 0$. Now one can pass to the limit in the equation (35) and conclude that
\[
(36) \quad b = \sum_{i \in I \sim C} y(i) v_i + \sum_{i \in C} y(i) \left( f(i) - z + b \right).
\]

Claim. $y(i) = 0$ for every $i \in I \sim C$ and
\[
(37) \quad \sum_{i \in C} y(i) = 1.
\]

Proof of the claim. As we saw in Section 3, the intersection of the simplex $\Delta(\sigma_k, C)$ with the face of $\Delta^n$ defined by the equations $x_i = 0$ with $i \in I \sim C$ is non-empty. Since the diameters of the simplices $\Delta_k$ tend to 0, this implies that $z$ is contained in this face, i.e. that the coordinates $z_i$ of $z$ with $i \in I \sim C$ are equal to 0. Since the sum of coordinates of every point of $\Delta^n$ is 1, taking the sum of coordinates in (36) shows that
\[
(38) \quad 1 = \sum_{i \in I} y(i).
\]

Next, let us consider the sums of not all coordinates, but only of coordinates numbered by elements of $C$. For $b$ this sum is equal to $m/(n + 1)$, where $m = |C|$. For $v_i$ with $i \in I \sim C$ this sum is equal to 0. Since $z_i = 0$ if $i \in I \sim C$, for $z$ this sum is equal to 1.
Finally, since $f(i) \in \Delta^n$, for $f(i)$ this sum is $\leq 1$. Therefore (36) implies that

$$\frac{m}{n+1} \leq \sum_{i \in C} y(i) \frac{m}{n+1}.$$  

In turn, this implies that

$$1 \leq \sum_{i \in C} y(i). \tag{39}$$

Since $y(i)$ are non-negative, the inequalities (39) and (38) together imply that $y(i) = 0$ for every $i \in I \sim C$ and hence (37) holds. This completes the proof of the claim. \(\Box\)

In view of this claim the first sum in (36) is equal to 0 and hence (36) implies that

$$b = \sum_{i \in C} y(i) \left( f(i) - z + b \right).$$

Together with (37) this implies that

$$z = \sum_{i \in C} y(i) f(i)$$

and $z$ is a convex combination of the vectors $f(i)$ with $i \in C$. Since $F$ is a closed map, $f(i) \in F(z)$ for every $i \in I$. Since $F(z)$ is convex, this implies that $z \in F(z)$. \(\blacksquare\)

**A geometric interpretation of Scarf’s proof.** Scarf wrote that “there is an illuminating geometrical interpretation” of his theorem in the case when $X$ is a subset of the standard simplex $S = \Delta^n$ as in this section and Section 3, and the vector $b$ and the colors $c(x)$, where $x \in X$, also belong to the standard simplex. Scarf suggests to consider the latter standard simplex $S'$ as different from $S$ and to think about the coloring $c$ as a sort of a map $S \rightarrow S'$. He wrote that “the suggestion that the theorem is concerned with “inverting” an arbitrary mapping is both accurate and illuminating.” See [Sc$_3$], pp. 76–77.

But in Scarf’s proof of Kakutani theorem his theorem is used in a somewhat different manner. Namely, it is only natural to think about the vectors $f(x) - x$ as tangent vectors to $\Delta^n$. Then the colors $c(x) = f(x) - x + b$ are simply these tangent vectors with their origins moved from 0 to $b$, and both the coloring $c$ and the map $x \rightarrow f(x) - x$ are interpreted as a vector field on $\Delta^n$. Since $f$ maps $\Delta^n$ into $\Delta^n$, this vector field is directed inside of the simplex $\Delta^n$ on its boundary, and the proof amounts to showing that (under a continuity assumption) such a vector field always has a zero.

In other words, in Scarf’s proof of Kakutani theorem Scarf theorem is used not to invert a mapping, but to find a zero of a vector field. This agrees with Scarf’s intuition that the colors $c(x)$ should be thought as belonging not to $S$, but to another simplex $S'$. 49
10. Chains in $\mathbb{R}^n$ and their applications

Chains in $\mathbb{R}^n$. One can consider $m$-chains in $\mathbb{R}^n$ without introducing a simplicial complex in advance or even later. Let us define an *$m$-simplex in $\mathbb{R}^n$* as a subset of $\mathbb{R}^n$ consisting of $m+1$ points, and an *$m$-chain in $\mathbb{R}^n$* as a finite formal sum of $m$-simplices in $\mathbb{R}^n$. The boundary operator $\partial$ acting on $m$-chains in $\mathbb{R}^n$ is defined as before, and the identity $\partial \circ \partial = 0$ still holds, with the same proof. A simplex $\sigma$ is said to be a *simplex of the chain* $c$ if $\sigma$ occurs in the formal sum $c$ with the coefficient 1. If $\sigma$ is an $m$-simplex $\sigma$ in $\mathbb{R}^n$, then $\langle \sigma \rangle$ denotes the convex hull of the vertices (i.e. elements) of $\sigma$. An $m$-simplex $\sigma$ is said to be *generic* if its the vertices are affinely independent. In this case $\langle \sigma \rangle$ is a geometric $m$-simplex in $\mathbb{R}^n$. In general, $\langle \sigma \rangle$ is a polyhedron of dimension $\leq m$. An $m$-chain $c$ in $\mathbb{R}^n$ is said to be *generic* if every simplex of $c$ is generic.

One can introduce an abstract simplicial complex having $\mathbb{R}^n$ as its set of vertices and all finite subsets of $\mathbb{R}^n$ as its simplices. The $m$-simplices and $m$-chains in $\mathbb{R}^n$ are $m$-simplices and $m$-chains of this simplicial complex. This point of view includes these notions in the framework of the usual theory, but is not particularly useful otherwise.

Maps to $\mathbb{R}^n$. Suppose that $S$ is an abstract simplicial complex and $V$ is its set of vertices. Let $\varphi : V \rightarrow \mathbb{R}^n$ be an arbitrary map. For an $m$-simplex $\sigma$ of $S$ let

$$\varphi_*(\sigma) = \varphi(\sigma)$$

if $\varphi(\sigma)$ is an $m$-simplex in $\mathbb{R}^n$, i.e. if $\varphi$ is injective on $\sigma$, and let $\varphi_*(\sigma) = 0$ otherwise. Let us extend $\varphi_*$ to a map from $m$-chains of $S$ to $m$-chains in $\mathbb{R}^n$ by linearity. The resulting map, still denoted by $\varphi_*$, commutes with the boundary operators in the sense that $\partial \circ \varphi_* = \varphi_* \circ \partial$. The proof is the same as for the simplicial maps between simplicial complexes. See [I2], Section 1, Theorem 1, for example.

General position and the intersection numbers. We will need only the simplest instances of these notions. We will identify 0-simplices in $\mathbb{R}^n$ with the corresponding points in $\mathbb{R}^n$.

Let $\sigma$ be an $n$-simplex in $\mathbb{R}^n$ and let $z \in \mathbb{R}^n$. The point $z$ is said to be *in general position with respect to* $\sigma$ if $z$ does not belong to $\langle \tau \rangle$ for every $(n-1)$-face $\tau$ of $\sigma$. In this case the *intersection number* $\sigma \cdot z$ is defined as $1 \in \mathbb{F}_2$ if $z \in \langle \sigma \rangle$ and $0 \in \mathbb{F}_2$ otherwise. If $c$ is an $n$-chain in $\mathbb{R}^n$ and $d$ is a 0-chain in $\mathbb{R}^n$, then $c, d$ are said to be *in general position* if every simplex of $d$ is in general position with respect to every simplex of $c$. In this case the *intersection number* $c \cdot d$ is defined as the sum of the intersection numbers $\sigma \cdot z$ over all pairs $\sigma, z$ such that $\sigma$ is a simplex of $c$ and $z$ is a simplex of $d$.

Next, let $\tau$ be an $(n-1)$-simplex in $\mathbb{R}^n$, and let $\omega$ be a 1-simplex. Since $\omega$ is a two-points subset of $\mathbb{R}^n$, the convex hull $\langle \omega \rangle$ is a segment. The 1-simplex $\omega$ is said to be *in
general position with respect to $\tau$ if both vertices of $\omega$ do not belong to $\langle \tau \rangle$ and $\langle \omega \rangle$ is disjoint from $\langle \nu \rangle$ for every $(n-2)$-face $\nu$ of $\tau$. If this is the case, then the intersection number $\tau \cdot \omega$ is defined as $1 \in \mathbb{F}_2$ if $\langle \omega \rangle \cap \langle \tau \rangle \neq \emptyset$ and $0 \in \mathbb{F}_2$ otherwise. If $c$ is an $(n-1)$-chain in $\mathbb{R}^n$ and $d$ is a 1-chain in $\mathbb{R}^n$, then $c$, $d$ are said to be in general position if every simplex of $d$ is in general position with respect to every simplex of $c$. In this case the intersection number $c \cdot d$ is defined as the sum of the intersection numbers $\tau \cdot \omega$ over all pairs $\tau$, $\omega$ such that $\tau$ is a simplex of $c$ and $\omega$ is a simplex of $d$.

Finally, let $\sigma$ be an $n$-simplex in $\mathbb{R}^n$ and $\omega$ be a 1-simplex. Then $\omega$ is said to be in general position with respect to $\sigma$ if every vertex of $\omega$ is in general position with respect to $\sigma$ and $\omega$ is in general position with respect to every $(n-1)$-face of $\sigma$. If $c$ is an $n$-chain in $\mathbb{R}^n$ and $d$ is a 1-chain in $\mathbb{R}^n$, then $c$, $d$ are said to be in general position if every simplex of $d$ is in general position with respect to every simplex of $c$.

10.1. Lemma. Suppose that $\sigma$ and $\omega$ are a generic $n$-simplex and a 1-simplex in $\mathbb{R}^n$ respectively. If $\sigma$, $\omega$ are in general position, then $\partial \sigma \cdot \omega = \sigma \cdot \partial \omega$.

Proof. Being a 1-simplex, $\omega$ is always generic because two different affinely independent. Since $\sigma$ is generic, $\langle \sigma \rangle$ is a geometric $n$-simplex.

If both vertices of $\omega$ belong to $\langle \sigma \rangle$, then $\sigma \cdot \partial \omega = 2 = 0$. Also, in this case $\langle \omega \rangle$ is contained in $\langle \sigma \rangle$ and hence $\langle \omega \rangle$ is disjoint from the boundary of $\langle \sigma \rangle$. It follows that in this case $\partial \sigma \cdot \omega = 0$ and hence $\partial \sigma \cdot \omega = \sigma \cdot \partial \omega$.

If only one vertex of $\omega$ is contained in $\langle \sigma \rangle$, then $\sigma \cdot \partial \omega = 1$. Also, in this case the segment $\langle \omega \rangle$ intersects exactly one $(n-1)$-face of the geometric $n$-simplex $\langle \sigma \rangle$. It follows that in this case $\partial \sigma \cdot \omega = 1$ and hence $\partial \sigma \cdot \omega = \sigma \cdot \partial \omega$.

If neither of the vertices of $\omega$ is contained in $\langle \sigma \rangle$, then $\sigma \cdot \partial \omega = 0$. The intersection $\langle \omega \rangle \cap \langle \sigma \rangle$ is convex and contained in the segment $\langle \omega \rangle$. Therefore $\langle \omega \rangle \cap \langle \sigma \rangle$ is either empty, or is a segment, or consists of one point. In the first case $\partial \sigma \cdot \omega = 0 = \sigma \cdot \partial \omega$, in the second case $\langle \omega \rangle$ intersects two $(n-1)$-faces of $\langle \sigma \rangle$ and hence also $\partial \sigma \cdot \omega = 0$. In the last case $\langle \omega \rangle$ intersects an $(n-2)$-face of $\langle \sigma \rangle$, which is impossible because $\sigma$, $\omega$ are assumed to be in general position. The lemma follows.

10.2. Lemma. Suppose that $c$ is an $n$-chain in $\mathbb{R}^n$ such that $c \cdot \partial \kappa = 0$ for every 1-simplex $\kappa$ in $\mathbb{R}^n$ in general position with respect to $c$. Then $c \cdot z = 0$ for every point $z \in \mathbb{R}^n$ in general position with respect to $c$.

Proof. Let $C$ be the union of the geometric simplices $\langle \sigma \rangle$ over all $n$-simplices $\sigma$ of $c$. Since $c$ is a finite sum, the set $C$ is bounded. Clearly, there exists a point $u \in \mathbb{R}^n$ such that $u \notin C$ and the segment connecting $z$ and $u$ is disjoint from $\langle \nu \rangle$ for every $(n-2)$-face of every $n$-simplex of $c$. Then the 1-simplex $\kappa = \{z, u\}$ is in general position with
respect to \( c \). By the assumptions of the lemma, \( c \cdot \partial \kappa = 0 \). But \( \partial \kappa = z + u \) and hence \( c \cdot \partial \kappa = c \cdot z + c \cdot u \). Clearly, \( u \notin C \) implies that \( c \cdot u = 0 \). Together with \( c \cdot \partial \kappa = 0 \) this implies that \( c \cdot z = 0 \). ■

10.3. Lemma (Carathéodory). Let \( \sigma \) be an \( l \)-simplex in \( \mathbb{R}^n \) for some \( l \) and let \( m \) be the dimension of \( \langle \sigma \rangle \). Then every point of \( \langle \sigma \rangle \) belongs to \( \langle \tau \rangle \) for some \( m \)-face \( \tau \) of \( \sigma \).

Proof. Since \( m \) is the dimension of \( \langle \sigma \rangle \), the convex hull \( \langle \sigma \rangle \) is contained in an affine hyperplane \( A \) of dimension \( m \). Let \( x \in \langle \sigma \rangle \). By a classical theorem of Carathéodory applied to the subset \( \sigma \) of \( A \), there exists a subset \( \tau \subset \sigma \) such that \( |\tau| = m + 1 \) and \( x \in \langle \tau \rangle \). Clearly, \( \tau \) is an \( m \)-face of \( \sigma \). ■

10.4. Lemma. Suppose that \( \sigma \) and \( \omega \) are an \( n \)-simplex and a \( 1 \)-simplex in \( \mathbb{R}^n \) respectively. If \( \sigma \), \( \omega \) are in general position and \( \sigma \) is not generic, then \( \partial \sigma \cdot \omega = \sigma \cdot \partial \omega = 0 \).

Proof. Since \( \sigma \) is not generic, the vertices of \( \sigma \) are affinely dependent and \( \langle \sigma \rangle \) is a polyhedron of dimension \( \leq n - 1 \). Suppose first that the dimension of \( \langle \sigma \rangle \) is \( \leq n - 2 \). Let \( x \in \langle \sigma \rangle \). Lemma 10.3 implies that \( x \in \langle u \rangle \) for some \( m \leq n - 2 \) and some \( m \)-face \( v \) of \( \sigma \). In turn, this implies that \( v \) is an \( m \)-face of some \( (n - 1) \)-face \( \tau \) of \( \sigma \). Since \( \sigma \), \( \omega \) are in general position, this implies that \( \langle u \rangle \cap \langle \omega \rangle = \emptyset \) and hence \( x \notin \langle \omega \rangle \). Since \( x \) was an arbitrary point of \( \langle \sigma \rangle \), we see that \( \langle \sigma \rangle \cap \langle \omega \rangle = \emptyset \) and hence \( \partial \sigma \cdot \omega = \sigma \cdot \partial \omega = 0 \). This proves the theorem when the dimension of \( \langle \sigma \rangle \) is \( \leq n - 2 \).

Suppose that the dimension of \( \langle \sigma \rangle \) is \( = n - 1 \). Then \( \langle \sigma \rangle \) is contained in an affine plane \( H \subset \mathbb{R}^n \) of dimension \( n - 1 \). The segment \( \omega \) is either disjoint from \( H \), or intersects \( H \) in one of the two vertices of \( \omega \), or intersects \( H \) in an interior point, or is contained in \( H \).

If \( \langle \omega \rangle \) is disjoint from \( H \), then \( \langle \omega \rangle \cap \langle \sigma \rangle = \emptyset \) and hence \( \partial \sigma \cdot \omega = \sigma \cdot \partial \omega = 0 \).

Suppose that \( \langle \omega \rangle \) intersects \( H \) in a vertex \( z \) of \( \omega \). Lemma 10.3 implies that \( \langle \sigma \rangle \) is equal to the union of the sets \( \langle \tau \rangle \) over all \( (n - 1) \)-faces \( \tau \) of \( \sigma \). Since \( \sigma \), \( \omega \) are in general position, this implies that \( z \notin \langle \sigma \rangle \). In turn, this implies that \( \partial \sigma \cdot \omega = \sigma \cdot \partial \omega = 0 \).

Suppose now that \( \langle \omega \rangle \) intersects \( H \) only in a point \( u \) which is not a vertex of \( \omega \). In this case \( \sigma \cdot \partial \omega = 0 \). Also, in this case \( u \) is in general position with respect to \( \partial \sigma \) in \( H \) and \( \partial \sigma \cdot \omega = \partial \sigma \cdot u \), where the second intersection number is understood in \( H \). Let us identify \( H \) with \( \mathbb{R}^{n-1} \). Let \( c = \partial \sigma \) and let \( \kappa \) be a \( 1 \)-simplex in \( \mathbb{R}^{n-1} \). Arguing by induction, we can assume that the theorem is true for the intersections in \( \mathbb{R}^{n-1} \). Then

\[
    c \cdot \partial \kappa = \partial c \cdot \kappa = \partial \circ \partial (\sigma) \cdot \kappa = 0 \cdot \kappa = 0
\]

and hence Lemma 10.2 with \( n - 1 \) in the role of \( n \) implies that \( c \cdot u = 0 \). It follows that \( \partial \sigma \cdot \omega = \partial \sigma \cdot u = c \cdot u = 0 \). Therefore \( \partial \sigma \cdot \omega = \sigma \cdot \partial \omega = 0 \) in this case also.
It remains to consider the case when $\langle \omega \rangle \subset H$. Since $\sigma, \omega$ are in general position, $\omega$ is in general position with respect to every $(n - 1)$-face $\tau$ of $\sigma$. This means that the vertices of $\omega$ are not contained in $\langle \tau \rangle$ and $\langle \omega \rangle$ is disjoint from $(n - 2)$-faces of $\langle \tau \rangle$ and hence from the boundary of $\langle \tau \rangle$. It follows that $\langle \omega \rangle$ is disjoint from $\langle \sigma \rangle$ and hence $\partial \sigma \cdot \omega = \sigma \cdot \partial \omega = 0$. □

10.5. Theorem. Suppose that $c$ and $d$ are an $n$-chain and a 1-chain in $\mathbb{R}^n$ respectively. If the chains $c, d$ are in general position, then $\partial c \cdot d = c \cdot \partial d$.

Proof. In view of the definition of the intersection numbers of chains, it is sufficient to prove that $\partial \sigma \cdot \omega = \sigma \cdot \partial \omega$ if $\sigma$ is an $n$-simplex, $\omega$ is a 1-simplex, and $\sigma, \omega$ are in general position. If $\sigma$ is a generic $n$-simplex, then this equality follows from Lemma 10.1, if it is not, then from Lemma 10.4. □

10.6. Corollary. Suppose that $c$ is an $n$-chain in $\mathbb{R}^n$ and $z \in \mathbb{R}^n$ is a point in general position with respect to $c$. If $c$ is a cycle, i.e. if $\partial c = 0$, then $c \cdot z = 0$.

Proof. Let $\kappa$ be a 1-simplex in $\mathbb{R}^n$ in general position with respect to $c$. Then Theorem 10.5 implies that $c \cdot \partial \kappa = \partial c \cdot \kappa = 0$. It remains to apply Lemma 10.2. □

The affine framework. Let $\mathcal{D}$ be a chain-simplex based on $I = \{0, 1, \ldots, n\}$, $X = V_\mathcal{D}$, and let $\mathcal{E}$ be the envelope of $\mathcal{D}$. An affine coloring of $\mathcal{D}$ is simply a map $c : X \rightarrow \mathbb{R}^n$.

Let $w_0, w_1, \ldots, w_n$ be affinely independent points in $\mathbb{R}^n$ and $\Gamma$ be the geometric the $n$-simplex in $\mathbb{R}^n$ having these points as its vertices.

10.7. Lemma. Let $\varphi : X \cup I \rightarrow \mathbb{R}^n$ be a map such that $\varphi(i) = w_i$ for every $i \in I$. If $z \in \Gamma$ and $z$ is in general position with respect to the chain $\varphi_* \mathcal{E} \| I \|$ and the $n$-simplex $\varphi(I)$, then $z \in \langle \varphi(\sigma) \rangle$ for some $n$-simplex $\sigma$ of $\mathcal{E}(I)$.

Proof. Theorem 1.6 implies that $\partial \mathcal{E} \| I \| = \partial I$ and hence

$$\partial (\varphi_* \mathcal{E} \| I \|) = \varphi_* (\partial I) = \partial (\varphi_* (I)) .$$

In turn, this implies that $\varphi_* \mathcal{E} \| I \| \cdot z = \varphi_* (I) \cdot z$.

The simplex $\Gamma$ is the convex hull $\Gamma$ of $\varphi(I) = \varphi_* (I)$, and hence $z \in \Gamma$ implies that $\varphi_* (I) \cdot z = 1$. This, in turn, implies that $\varphi_* \mathcal{E} \| I \| \cdot z = 1$ and hence $z \in \langle \varphi(\sigma) \rangle$ for some $n$-simplex $\sigma$ of $\mathcal{E}(I)$. □
10.8. The main theorem for affine colorings. For every map \( c: X \rightarrow \mathbb{R}^n \) and every \( z \in \Gamma \) there exist a non-empty subset \( C \subset I \) and a \( d(C) \)-simplex \( \sigma \) of \( \mathcal{D}(C) \) such that the set

\[
(40) \quad c(\sigma) \cup \{ w_i \mid i \in I \sim C \}
\]

consists of \( n+1 \) elements, is an affinely independent, and contains \( z \) in its convex hull.

**Proof.** Let us extend \( c \) to a map \( \varphi: X \cup I \rightarrow \mathbb{R}^n \) by the rule \( \varphi(i) = w_i \). By Lemma 1.4 it is sufficient to prove that there exists an \( n \)-simplex \( \sigma \) of \( \mathcal{E}(I) \) such that the set \( \varphi(\sigma) \) is affinely independent and contains \( z \) in its convex hull. Cf. the proof of Theorem 6.5. Since \( z \in \Gamma \), there exists a sequence \( z_1, z_2, z_3, \ldots \) of points \( z_i \in \Gamma \) such that \( z_i \) tends to \( z \) when \( i \rightarrow \infty \) and for every \( i \) the point \( z_i \) is not an affine linear combination of \( <n+1 \) elements of the union of \( \varphi(X) \cup \{ w_0, w_1, \ldots, w_n \} \). This implies that \( z_i \) is in the general position with respect to \( \varphi, \mathcal{E} \) and \( \varphi(I) \) for every \( i \). Since

\[
\varphi(I) = \{ w_0, w_1, \ldots, w_n \},
\]

Lemma 10.7 implies that for every \( i \) there exists an \( n \)-simplex \( \sigma \) of \( \mathcal{E}(I) \) such that the set \( \varphi(\sigma) \) contains \( z_i \) in its convex hull. Since \( \mathcal{E}(I) \) has only a finite number of \( n \)-simplices, we can assume that this simplex \( \sigma \) is the same for all \( i \). By the choice of \( z_i \) the image \( \varphi(\sigma) \) consists of \( n+1 \) points and is affinely independent. Passing to the limit shows that \( z \) is contained in the convex hull of \( \varphi(\sigma) \). The theorem follows. \( \blacksquare \)

Families of linear orders. So, let \( X \) be a non-empty finite set and suppose that for every \( i \in I \) a linear order \( <_i \) on \( X \) is given. As in Section 7, we assume that \( X \cap I = \emptyset \) and extend the orders \( <_i \) to \( X \cup I \), subject to the same conditions.

The affine form of Scarf theorem. Let \( \varphi: X \cup I \rightarrow \mathbb{R}^n \) be a map such that \( \varphi(i) = w_i \) for every \( i \in I \) and let \( z \in \Gamma \). Then there exists a subset \( \sigma \subset X \cup I \) dominant with respect to \( I \) and such that \( z \in \langle \varphi(\sigma) \rangle \) and \( \varphi(\sigma) \) is an \( n \)-simplex in \( \mathbb{R}^n \).

**Proof.** This can be deduced from Theorem 10.8 in exactly the same way as Generalized Scarf Theorem was deduced from Theorem 8.5. \( \blacksquare \)

Another proof of Scarf theorem. See Section 7. Let \( M = \varphi(X \cup I) + b \) and let \( M \) be the convex hull of \( M \) in the usual geometric sense, i.e. the set of all linear combinations

\[
\sum_{v \in M} x_v v
\]

such that \( x_v \geq 0 \) for all \( v \in M \) and \( \sum_{v \in M} x_v = 1 \). Lemma 7.1 implies that 0 is not a non-negative linear combination of elements of \( M \). In particular, 0 is not contained
in the convex hull $M$. By a classical separation theorem there exists a linear functional $L : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $L(v) > 0$ for every $v \in M$. Let $r(v) = v / L(v)$ for every $v \in M$. Then each vector $r(v)$ belongs to the hyperplane $L^{-1}(1)$ and (29) implies that $r(b)$ is a convex combination of the vectors $r(v_i)$. Let $f : L^{-1}(1) \to \mathbb{R}^n$ be an affine isomorphism, and let $w_i = f \circ r(v_i)$, $z = f \circ r(b)$. Then the points $v_i$ are affinely independent and $z$ is a convex combination of these points, i.e. $z$ belongs to the $n$-simplex $\Gamma$ with the vertices $w_0, w_1, \ldots, w_n$. Therefore the affine form of Scarf theorem applies to the map $f \circ \varphi$ in the role of $\varphi$ and the point $z$. Clearly, every simplex $\sigma$ satisfying the conclusion of this affine form satisfies also the conclusion of Scarf theorem. ■

The simplex $\Gamma$ and related coordinates. The vertices $w_0, w_1, \ldots, w_n$ of $\Gamma$ are affinely independent and every point $x \in \mathbb{R}^n$ has a unique presentation in the form

$$x = a_0(x) w_0 + a_1(x) w_1 + \ldots + a_n(x) w_n$$

where coefficients $a_i(x)$, the barycentric coordinates of $x$, are such that

$$a_0(x) + a_1(x) + \ldots + a_n(x) = 1.$$

Clearly, $a_i(w_k) = 1$ if $i = k$ and $a_i(w_k) = 0$ if $i \neq k$. Also, the $(n-1)$-face $\Gamma_i$ of $\Gamma$ opposite to $w_i$ is contained in the hyperplane $H_i$ defined by the equation $a_i(x) = 0$. The half-space $\eta_i(>0)$ bounded by $H_i$ and containing $w_i$ is defined by the inequality $a_i(x) > 0$ and the other half-space $\eta_i(\leq 0)$ is defined by $a_i(x) \leq 0$. Let

$$l_i(x) = a_i(x) - a_i(0)$$

for every $i \in I$. Then $l_i : \mathbb{R}^n \to \mathbb{R}$ is a linear map and $l_i(y) \leq l_i(x)$ if and only if $a_i(y) \leq a_i(x)$.

For the rest of this section we will assume that $0$ is the barycenter of $\Gamma$, or, equivalently, that $w_0 + w_1 + \ldots + w_n = 0$. Then $a_i(0) = 1/(n+1)$ and hence

$$l_i(w_i) = \frac{n}{n+1} \quad \text{and} \quad l_i(w_k) = -\frac{1}{n+1}$$

for every $i$ and every $k \neq i$.

Triangulations and vector colorings. The simplex-families arising from triangulations of $\Gamma$ are pseudo-simplices and hence are chain-simplices, as we saw in Section 1. We leave the task of stating the corresponding special case of Theorem 10.8 to the reader. We will discuss in details two special cases dealing with vector hedgehog colorings and with inward tangent colorings, to be defined in a moment. The colors of the inward tangent colorings should be interpreted as tangent vectors to a simplex. Cf. remarks at the end of Section 9.
Suppose that $T$ is a triangulation of $\Gamma$ and let $X$ be the set of vertices of $T$. Let $\mathcal{D}_T$ be the simplex-family associated to $T$. Then $\mathcal{D}_T$ is a pseudo-simplex and hence is a chain-simplex. A map $c : X \rightarrow \mathbb{R}^n$ is called a vector hedgehog coloring if

$$c(x) \in \eta_i (\leq 0)$$

for every $i \in I$ and $x \in X \cap \Gamma_i$, and an inward tangent coloring if

$$x + c(x) \in \eta_i (\geq 0)$$

for every $i \in I$ and $x \in X \cap \Gamma_i$. Since $a_i(x) = 0$ for such $x$, the last condition holds if and only if $a_i(x + c(x)) \geq a_i(x)$, or, equivalently, $l_i(x + c(x)) \geq l_i(x)$ for every $i \in I$ and $x \in X \cap \delta_i$. Since $l$ is linear, this condition is equivalent to the condition

$$l_i(c(x)) \geq 0$$

for every $i \in I$ and $x \in X \cap \Gamma_i$.

10.9. The main theorem for vector hedgehog colorings. For every vector hedgehog coloring $c : X \rightarrow \mathbb{R}^n$ and every $z \in \Gamma$ there exists an $n$-simplex of $T$ such that if $\sigma$ is its set of vertices, then $z$ is contained in the convex hull of $c(\sigma)$.

Proof. Suppose first that $z$ is contained in the interior of $\Gamma$. Let us apply Theorem 10.8 to $\mathcal{D}_T$, the simplex $\Gamma$, the point $z \in \Gamma$, and the points $v_0, v_1, \ldots, v_n$ respectively. By this theorem there exists a non-empty subset $C \subset I$ and a $d(C)$-simplex $\sigma$ of $\mathcal{D}_T(C)$ such that the set

$$Y = c(\sigma) \cup \{v_{i+1} \mid i \in I \sim C\}$$

consists of $n + 1$ elements, is an affinely independent, and contains $z$ in its convex hull. If $C \neq I$, then there exists $k \in I$ such that $k \not\in C$ and $k - 1 \in C$. By arguing exactly as in the proof of Theorem 8.7 we conclude that $Y \subset \eta_k (\leq 0)$ and hence $z \in \eta_k (\leq 0)$. This contradicts to the assumption that $z$ is contained in the interior of $\Gamma$. It follows that $C = I$. In turn, this implies that $\sigma$ is the set of vertices of an $n$-simplex of $T$ and $c(\sigma)$ contains $z$ in its convex hull. This proves the theorem in the case when $z$ is contained in the interior of $\Gamma$. But the union of the convex hulls of the sets $c(\sigma)$ with $\sigma$ being the set of vertices of an $n$-simplex of $T$ is obviously closed. Therefore this special case implies the theorem for arbitrary $z \in \Gamma$. ■

10.10. The main theorem for inward tangent colorings. For every inward tangent coloring $c : X \rightarrow \mathbb{R}^n$ there exists an $n$-simplex of $T$ such that if $\sigma$ is its set of vertices, then $0$ is contained in the convex hull of $c(\sigma)$.
Proof. Let us apply Theorem 10.8 to $\mathcal{D}_T$, the simplex $\Gamma$, the points $w_0, w_1, \ldots, w_n$, and $z = 0$. By this theorem there exists a non-empty subset $C \subset I$ and a $d(C)$-simplex $\tau$ of $\mathcal{D}_T(C)$ such that the set (40) contains 0 in its convex hull. Then

$$\sum_{x \in \tau} a_x c(x) + \sum_{k \in I \setminus C} a_k w_k = 0$$

for some non-negative coefficients $a_x, a_i$ such that

$$\sum_{x \in \tau} a_x + \sum_{k \in I \setminus C} a_k = 1.$$ 

Suppose that $a_i = 0$ for every $i \in I \setminus C$ (this happens, in particular, when $C = I$). In this case 0 belongs to the convex hull of $c(\tau)$. The simplex $\tau$ is a simplex of the abstract simplicial complex associated with $T$ and hence is a face of an $n$-simplex $\sigma$ of that complex. It follows that $c(\tau) \subset c(\sigma)$ and hence 0 belongs to the convex hull of $c(\sigma)$. This proves the theorem in the case when $a_i = 0$ for every $i \in I \setminus C$.

Suppose now that $a_i \neq 0$ for some $i \in I \setminus C$. Let us choose some $i \in I \setminus C$ such that $a_i \geq a_k$ for all $k \in I \setminus C$. Then $a_i > 0$. By applying $l_i$ to (42) we conclude that

$$\sum_{x \in \tau} a_x l_i(c(x)) + \sum_{k \in I \setminus C} a_k l_i(w_k) = 0. \tag{43}$$

Since $c$ is an inward tangent coloring, the first sum at the left hand side is non-negative.

Let $K = (I \setminus C) \setminus i$. Then the second sum is equal to

$$a_i l_i(w_i) + \sum_{k \in K} a_k l_i(w_k).$$

In view of (41) the last expression is equal to

$$a_i \frac{n}{n + 1} - \sum_{k \in K} a_k \frac{1}{n + 1} \geq a_i \frac{n}{n + 1} - \sum_{k \in K} a_i \frac{1}{n + 1}$$

$$= a_i \frac{n}{n + 1} - a_i \frac{|K|}{n + 1} = a_i \frac{n - |K|}{n + 1}.$$ 

Since $C$ is non-empty, $|I \setminus C| \leq n$. Since $i \in I \setminus C$, this implies that $|K| \leq n - 1$ and hence $n - |K| > 0$. It follows that the second sum in (43) is $> 0$. Since the first sum is $\geq 0$, the left hand side of (43) is $> 0$, contrary to (43). The contradiction shows that $a_i = 0$ for every $i \in I \setminus C$. But we already saw that the theorem holds in this case. 

Another proof of Kakutani theorem. Let $\Gamma$ be as above, and let $F: \Gamma \rightarrow \Gamma$ be a closed multi-valued map. Let us choose a map $f: \Gamma \rightarrow \Gamma$ (not assumed to be continuous) such that $f(x) \in F(x)$ for every $x \in \Gamma$. Let $c(x) = f(x) - x$. 

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Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ be positive numbers such that $\varepsilon_k \to 0$ when $k \to \infty$. By considering triangulations of $\Gamma$ into simplices of diameter $< \varepsilon_k$ and applying Theorem 10.10, we see that there exist subsets $\sigma_k \subset \Gamma$ such that $|\sigma_k| = n + 1$, the diameter of $\sigma_k$ is $< \varepsilon_k$ and the image $c(\sigma_k)$ contains 0 in its convex hull. Let $w_0(k), w_1(k), \ldots, w_n(k)$ be the elements of $\sigma_k$. After passing to a subsequence we can assume that every sequence $w_i(1), w_i(2), w_i(3), \ldots$ converges to a limit when $k \to \infty$. Since the diameters of simplices $\sigma_k$ tend to 0, these limits are, in fact, equal. Let $u$ be the common values of these limits. After passing to a further subsequence we can assume that each sequence $f(w_i(1)), f(w_i(2)), f(w_i(3)), \ldots$ converges to a limit which we will denote by $w_i$. It follows that each sequence $c(w_i(1)), c(w_i(2)), c(w_i(3)), \ldots$ converges to the limit $c_i = w_i - u$. Since every set $c(\sigma_k)$ contains 0 in its convex hull, the set $\{c_0, c_1, \ldots, c_n\}$ also contains 0 in its convex hull. Clearly, every convex combination of the points $c_i$ has the form $w - u$, where $w$ is a convex combination of the points $w_i$. It follows that $u$ is a convex combination of the points $w_i$. Since $F$ is a closed multivalued map, the choice of the map $f$ implies that $w_i \in F(u)$ for every $i \in I$. Since $F(u)$ is convex, this implies that $u \in F(u)$, i.e. that $u$ is a fixed point of $F$. \[\blacksquare\]

Another version of Scarf theorem. The theorem called Scarf theorem above is Theorem 4.2.3 from [Sc3] written in a somewhat different language. In Theorem 7.1 of [Sc4] Scarf replaced the assumption that the set of non-negative solutions of (30) is bounded (which is a part of our definition of a vector framework) by the following. Suppose that

$$x = \sum_{v \in M - b} y_v v,$$

where $y_v \geq 0$ for all $v \in M - b$. Then, if all coordinates of the vector $x$ are non-positive, all coefficients $y_v$ should be equal to 0. This assumption immediately implies that 0 is not a non-negative linear combination of elements of $M$. Therefore the above proof works under this assumption also, and, moreover, there is no need to use Lemma 7.1.

Kannai’s Generalized Sperner lemma. Theorem 10.9 generalizes Generalized Sperner lemma of Kannai [K], who assumed that $c(x)$ belongs to the affine hyperplane spanned by the face $\Gamma_i$ for every $i \in I$ and $x \in X \cap \Gamma_i$, in contrast with our weaker assumption $c(x) \in \eta_i(\leq 0)$. The assumption used by Kannai is forced by his method of proof based on an induction by $n$ similar to Sperner’s arguments. In contrast with Kannai [K], our proof does not require any form of the simplicial approximation theorem. Using the latter is often considered as a topological, and hence non-elementary and non-combinatorial, argument. See Kannai [K], Section 4 and Ziegler [Z], Introduction, for example.
A.1. Todd’s theorem

Todd’s theorem. Let $\sigma, \tau$ be circuits of an oriented matroid and let $w \in \tau \setminus \sigma$. Suppose that there exists $e \in \sigma \cap \tau$ such that $\sigma(e) = -\tau(e)$. Then there is a circuit $\omega$ such that

\begin{align}
(44) & \quad \omega_+ \subset (\sigma_+ \cup \tau_+) \setminus \sigma_-, \\
(45) & \quad \omega_- \subset (\sigma_- \cup \tau_-) \setminus \sigma_+, \\
(46) & \quad w \in \omega, \quad \text{and} \quad \omega(w) = \tau(w).
\end{align}

Moreover, if $\tau \subset \sigma \cup \{w\}$, then such a circuit $\omega$ is unique and $\sigma \cup \tau \subset \omega$.

Proof. Let $d(\sigma, \tau) = |(\sigma_+ \cap \tau_-) \cup (\sigma_- \cap \tau_+)|$. Let us prove the first statement using an induction by $d(\sigma, \tau)$. By the assumptions of the theorem, $d(\sigma, \tau) \geq 1$. Without any loss of generality we can assume that $e \in \sigma_+ \cap \tau_-$. Suppose that $d(\sigma, \tau) = 1$. In this case $|\sigma_+ \cap \tau_-| = 1$ and $\sigma_- \cap \tau_+ = \emptyset$. It follows that $e \in \sigma_+ \cap \tau_-$. Let us apply the strong elimination property as stated in Theorem 5.1 to $u = e$ and $v = w$. We see that there exists a circuit $\omega$ such that $w \in \omega$,

\[ \omega_+ \subset (\sigma_+ \cup \tau_+) \setminus \{e\}, \quad \text{and} \quad \omega_- \subset (\sigma_- \cup \tau_-) \setminus \{e\}. \]

Since $\sigma_-$ is disjoint from $\tau_+$ and $\sigma_+$,

\[ \omega_+ \subset \sigma_+ \cup \tau_+ \subset (\sigma_+ \cup \tau_+) \setminus \sigma-. \]

On the other hand, $\sigma_+$ intersects $\tau_-$ only by $e$ and is disjoint from $\sigma_-$. Therefore

\[ \omega_- \subset (\sigma_- \cup \tau_-) \setminus \{e\} \subset (\sigma_- \cup \tau_-) \setminus \sigma_+. \]

It follows that the properties (44) and (45) hold. Since $w \in \omega$, these properties imply that $\omega(w) = \tau(w)$. This completes the proof of the first statement for $d(\sigma, \tau) = 1$.

Suppose now that $k > 1$ and that the first statement is proved for $d(\sigma, \tau) < k$. Suppose that $d(\sigma, \tau) = k$ and apply the strong elimination property as in the previous paragraph. If the properties (44) and (45) hold, then the property (46) also holds and we are done. Otherwise at least one of the sets $\sigma_+ \cap \omega_-$ and $\sigma_- \cap \omega_+$ is non-empty. Let $f$ be an element either of these sets. The inclusions (22) imply that

\[ \sigma_+ \cup \omega_+ \subset \sigma_+ \cup \tau_+ \quad \text{and} \quad \sigma_- \cup \omega_- \subset \sigma_- \cup \tau_. \]
Therefore, if the first statement holds for \( \sigma, \omega, \) and \( f \) in the role of \( \sigma, \tau, \) and \( e \) respectively, then it holds for \( \sigma, \tau, \) and \( e \) also. But

\[
\sigma_+ \cap \omega_- \subset (\sigma_+ \cap \tau_-) \sim \{e\}
\]

and hence \(|\sigma_+ \cap \omega_-| \leq |\sigma_+ \cap \tau_-| - 1\). On the other hand,

\[
\sigma_- \cap \omega_+ \subset \sigma_- \cap \tau_+
\]

and hence \(|\sigma_- \cap \omega_+| \leq |\sigma_- \cap \tau_+|\). It follows that \(d(\sigma, \omega) \leq d(\sigma, \tau) - 1\) and hence the first statement holds for \( \sigma, \omega, \) and \( f \) by the inductive assumption. This completes the step of the induction and hence the proof of the first statement.

Let us prove the second statement. If (44) – (46) hold and \( \tau \subset \sigma \cup \{w\} \), then

\[
\omega \subset \sigma \cup \tau \subset \sigma \cup (\sigma \cup \{w\}) = \sigma \cup \{w\}.
\]

By applying the elimination property to \( \tau, -\omega, \) and \( w \) we get a circuit \( \pi \) such that

\[
\pi \subset (\tau \cup \omega) \sim \{w\} \subset (\sigma \cup \{w\}) \sim \{w\} = \sigma.
\]

The axiom (iii) implies that \( \pi = \sigma \) or \( -\sigma \) and hence \( \pi = \sigma \). It follows that

\[
\sigma = (\tau \cup \omega) \sim \{w\}
\]

and hence \( \sigma \subset \tau \cup \omega \). In turn, this implies that \( \sigma \sim \tau \subset \omega \).

It remains to prove the uniqueness of \( \omega \) with the properties (44) – (46) under the assumption \( \tau \subset \sigma \cup \{w\} \). Suppose that \( \alpha \neq \omega \) also satisfies these properties. Since \( \alpha(w) = \omega(w) \) and hence \( \alpha \neq -\omega \), the axiom (iii) implies that

\[
\omega \sim \alpha \neq \emptyset \quad \text{and} \quad \alpha \sim \omega \neq \emptyset.
\]

By applying the elimination property to \( \omega, -\alpha, \) and \( w \) we get a circuit \( \pi \) such that

\[
\pi \subset (\omega \cup \alpha) \sim \{w\} \subset (\sigma \cup \{w\}) \sim \{w\} = \sigma.
\]

It follows that \( \pi = \sigma \) and \( \sigma = (\omega \cup \alpha) \sim \{w\} \). If \( x \in \omega \sim \alpha \), then

\[
\pi(x) = \omega(x) = \sigma(x)
\]

and hence \( \pi = \sigma \). A similar argument using some \( y \in \alpha \sim \omega \) leads to the conclusion that \( \pi = -\sigma \). The contradiction shows that \( \omega \) is unique. \( \blacksquare \)
### A.2. Cocircuits and lexicographic extensions

**Cocircuits.** Let $M$ be an oriented matroid. It is known that the number of elements in a base of $M$ depends only on $M$. It is called the *rank* of $M$. We will denote it by $n$. The *span* of a subset $X \subset M$ is defined as the set of all elements $a \in M$ such that there exists a circuit $\sigma$ of $M$ such that $a \in \sigma$ and $\sigma - a \subset X$. A subset of $M$ is called a *hyperplane* if it is equal to the span of some independent subset $X \subset M$ such that $|X| = n - 1$.

Let $X \subset M$ be an independent subset such that $|X| = n - 1$, and let $H$ be the hyperplane spanned by $X$. Clearly, there exists an element $e \in M$ such that $X + e$ is a basis. For every $u \in M \setminus (X + e)$ there is a unique circuit $\sigma$ such that $u \in \sigma$ and $\sigma - u \subset X + e$. By the definition of $H$, if $u \in M \setminus H$, then $e \in \sigma$. Let $\tau(u) = +$ if $e \in \sigma_+$ and $\tau(u) = -$ if $e \in \sigma_-$. Then $\tau$ is a signed subset of $M$ such that $\tau = M \setminus H$. One can prove that, up to replacing $\tau$ by $-\tau$, the signed subset $\tau$ depends only on $H$. It is called a *cocircuit* of $M$ corresponding to $H$. A signed subset of $M$ is called a *cocircuit* if it is a cocircuit corresponding to a hyperplane.

Two signed subsets $\sigma, \tau$ are said to be *orthogonal* if either $\sigma \cap \tau = \emptyset$, or there exist two elements $u, v \in \sigma \cap \tau$ such that $\sigma(u) = \tau(u)$ and $\sigma(v) = -\tau(v)$. If this is the case, then we write $\sigma \perp \tau$. It turns out that a signed subset $\tau$ of $M$ is a cocircuit if and only if $\sigma \perp \tau$ for every circuit $\sigma$ of $M$.

**Las Vergnas (lexicographic) extensions.** An oriented matroid $M'$ is said to be an *extension* of $M$ if $M \subset M'$ and a signed subset of $M$ is a circuit of $M'$ if and only if it is a circuit of $M$. Clearly, the rank of $M'$ is $\geq n$. We are interested in the *one point extensions* $M'$, i.e. extensions $M'$ such that $M' = M + p$ for some $p \notin M$. Up to isomorphism, there is only one such extension with the rank $n + 1$, and we are interested in the ones of rank $n$. Among such extensions are the *lexicographic extensions* introduced by M. Las Vergnas [LV]. Let us describe their basic properties following M. Todd [T], Theorem 5.1.

Recall that a signed set $\tau$ can be considered as a map $\tau \rightarrow \{+, -\}$. This allows to speak about the restriction of $\tau$ to a subset of $\tau$. Let $\{a_1, a_2, \ldots, a_k\} \subset M$ be an independent set and let $p \notin M$. Then there is a unique structure of an oriented matroid on $M' = M + p$ such that $M'$ is an extension of $M$ of the rank $n$ and the following two properties hold.

(a) If $\tau$ is a cocircuit of $M$ and $\tau \cap \{a_1, a_2, \ldots, a_k\} = \emptyset$, then $\tau$ is a cocircuit of $M'$.

(b) Let $\tau$ is a signed subset of $M'$ such that $p \in \tau$ and $\tau \cap \{a_1, a_2, \ldots, a_k\} \neq \emptyset$, and let $i$ be the minimal number such that $e_i \in \tau$. Then $\tau$ is a cocircuit of $M'$ if and only if the restriction of $\tau$ to $\tau \cap M$ is a cocircuit of $M$ and $\tau(p) = \tau(e_i)$.

Moreover, every cocircuit $\tau$ of $M'$ such that $p \in \tau$ has the form described in (b).
If $M$ is a subset of $\mathbb{R}^n$ considered as oriented matroid, then $M' = M + p$, where

$$p = a_1 + \lambda a_2 + \lambda^2 a_3 + \ldots + \lambda^{k-1} a_k$$

for a sufficiently small $\lambda > 0$, considers as an oriented matroid, has the above properties and so is a lexicographic extension. In this case the vector $p$ is a perturbation of the vector $a_1$. It turns out that the added element $p$ of a lexicographic extension can play the role of a perturbation of the vector $a_1$ also in general case. This was realized by M. Todd [T].

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