Research Article

Fixed Points and Stability for Integral-Type Multivalued Contractive Mappings

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The existence and iterative approximations of fixed points concerning two classes of integral-type multivalued contractive mappings in complete metric spaces are proved, and the stability of fixed point sets relative to these multivalued contractive mappings is established. The results obtained in this article generalize and improve some known results in the literature. An illustrative example is given.

1. Introduction

The famous Banach fixed point theorem has both various extensions and valuable applications in a mass of differential equations, difference equations, functional equations, matrix equations, and integral equations ([1–26]). In 2002, Branciari [3] obtained an interesting integral-type fixed point theorem for the contractive mapping of integral type, which is an integral version of the Banach contraction mapping.

Theorem 1 (see [3]). Let \( f \) be a mapping from a complete metric space \( (X, \rho) \) into itself satisfying

\[
\int_0^{p(x,y)} p(s) \, ds \leq c \int_0^{p(x,y)} p(s) \, ds, \forall x, y \in X,
\]

where \( c \in (0, 1) \) is a constant and \( p : [0, +\infty) \to [0, +\infty) \) is Lebesgue integrable, summable on each compact subset of \( [0, +\infty) \) and \( \int_0^c p(s) \, ds > 0 \) for each \( c > 0 \). Then, \( f \) has a unique fixed point \( u \in X \) such that \( \lim_{n \to \infty} f^n x = u \) for each \( x \in X \).

Later, the researchers [1, 2, 6, 7, 9, 10, 12, 14, 17–21, 26] generalized Theorem 1 from different directions and got a lot of fixed point results for various contractive mappings of integral type.

In 1969, Nadler [15] gave a multivalued analog of the Banach fixed point theorem by using the Hausdorff metric and introducing the multivalued contraction mapping, that is, he presented a nice fixed point theorem for the multivalued contraction mapping.

Theorem 2 (see [10]). Let \( (X, \rho) \) be a complete metric space and \( T : X \to CB(X) \) be a multivalued contraction mapping, that is, there exists a constant \( r \in (0, 1) \) satisfying

\[
H(Tx, Ty) \leq r \rho(x, y), \forall x, y \in X.
\]

Then, \( T \) has a fixed point in \( X \).

Czerwik [5] and Gordji et al. [8] extended Theorem 2 and proved fixed point theorems for some multivalued contractive mappings, which include (2) as special cases. The researchers [4, 11, 13, 22–24] gained fixed point theorems for several multivalued contractive mappings and studied also the stability of fixed point sets with respect to the multivalued contractive mappings. Lim [11] established the stability of fixed point sets associated with the multivalued
contraction mappings in Theorem 2. Choudhury et al. [24] proved that a uniformly convergent sequence of \(\alpha - \psi\) multivalued contractions has stable fixed point sets.

By combining the ideas of Nadler, Branciari, and Lim, in this article, we study the existence and iterative approximations of fixed points concerning two classes of integral-type multivalued contractive mappings in complete metric spaces and present stability of fixed point sets relative to a sequence of integral-type multivalued contractive mappings. Our results generalize and unify a few results in [5, 8, 11, 15]. An example is also presented to illustrate the efficiency of our results.

2. Preliminaries

Throughout this paper, \(\mathbb{N}\) denotes the set of all positive integers, \(\mathbb{R}^+ = [0, +\infty)\), \(\mathbb{N}_0 = \{0\} \cup \mathbb{N}\) and

\[
\Phi_1 = \left\{ p : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \mid \text{Lebesgue integrable, summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^\infty p(s)ds > 0 \text{ for each } \epsilon > 0 \right\},
\]

\[
\Phi_2 = \left\{ p \mid p \text{ is in } \Phi_1 \text{ and } \int_a^b p(s)ds \leq \int_a^b p(s)ds + \int_0^b p(s)ds \text{ for each } a, b \in \mathbb{R}^+ \right\}.
\]

(3)

Assume that \((X, \rho)\) is a metric space, \(CL(X)\) stands for the family of all nonempty closed subsets of \(X\), and \(CB(X)\) denotes the family of all nonempty closed bounded subsets of \(X\). For \(C, D \in CL(X)\) and \(T, \{T_i\}_{i \in \mathbb{N}_0} : X \rightarrow CL(X)\), define

\[
F(T) = \{ x \in X : x \in Tx \}, \rho(x, D) = \inf_{y \in D} \rho(x, y), \forall x \in X,
\]

\[
H(C, D) = \left\{ \begin{array}{ll}
\sum_{x \in C} \sup \{ \rho(x, D), \sup \{ \rho(y, C) \} \}, & \text{if the maximum exists,} \\
\infty, & \text{otherwise},
\end{array} \right.
\]

\[
N(x, y) = \max \left\{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)] \right\}, \forall x, y \in X,
\]

\[
M(x, y) = \max \left\{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)], \rho(x, Ty)\rho(y, Ty) + \rho(x, Ty)\rho(y, Tx) \right\}, \forall x, y \in X,
\]

\[
N_1(x, y) = \max \left\{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)] \right\}, \forall x, y \in X, i \in \mathbb{N}_0,
\]

\[
M_1(x, y) = \max \left\{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)], \rho(x, Ty)\rho(y, Ty) + \rho(x, Ty)\rho(y, Tx) \right\}, \forall x, y \in X, i \in \mathbb{N}_0.
\]

A sequence \(\{x_n\}_{n \in \mathbb{N}_0}\) in \(X\) is called an orbit of \(T\) at \(x_0\) if \(x_{n+1} \in Tx_n\) for each \(n \in \mathbb{N}_0\).

**Lemma 3** (see [12]). Let \(p \in \Phi_1\) and \(\{r_n\}_{n \in \mathbb{N}}\) be a nonnegative sequence and \(\lim_{n \rightarrow \infty} r_n = a\). Then

\[
\lim_{n \rightarrow \infty} \int_0^{r_n} p(s)ds = \int_0^a p(s)ds.
\]

(5)

**Lemma 4** (see [12]). Let \(p \in \Phi_1\) and \(\{r_n\}_{n \in \mathbb{N}}\) be a nonnegative sequence. Then

\[
\lim_{n \rightarrow \infty} \int_0^{r_n} p(s)ds = 0,
\]

if and only if \(\lim_{n \rightarrow \infty} r_n = 0\).

It follows from [13] the following.

**Lemma 5.** Assume that \((X, \rho)\) is a metric space and \(C, D \in CL(X)\). Then, for any \(r > 1\) and \(a \in C\), there exists \(b \in D\) such that

\[
\rho(a, b) \leq rH(C, D).
\]

(7)

**Lemma 6** (see [13]). Assume that \((X, \rho)\) is a metric space. Then

\[
|\rho(x, C) - \rho(y, C)| \leq \rho(x, y), \forall x, y \in X, C \in CL(X).
\]

(8)

**Lemma 7.** Assume that \((X, \rho)\) is a metric space, \(C \in CL(X)\), and \(\{x_n\}_{n \in \mathbb{N}} \subset X\) converges to \(a \in X\). Then

\[
\lim_{n \rightarrow \infty} \rho(x_n, C) = \rho(a, C).
\]

(9)
Proof. It follows from Lemma 6 that

\[
|\rho(x_n, C) - \rho(a, C)| \leq \rho(x_n, a) \quad \text{as} \quad n \to \infty, 
\]

that is

\[
\lim_{n \to \infty} \rho(x_n, C) = \rho(a, C).
\]

This completes the proof. \(\square\)

Lemma 8. Let \(C \subseteq \mathbb{R}^+\) and \(p \in \Phi_1\). Then

\[
\sup_{a \in C} \int_0^a p(s) \, ds = \int_0^C p(s) \, ds, 
\]

\[
\inf_{a \in C} \int_0^a p(s) \, ds = \int_0^C p(s) \, ds. 
\]

Proof. (see (12)). Let \(\sup C = c\). It follows that

\[
a \leq c, \forall a \in C, 
\]

and there exist a sequence \(\{a_n\}_{n \in \mathbb{N}}\) in \(C\) satisfying

\[
\lim_{n \to \infty} a_n = c. 
\]

Thus, (14) and \(p \in \Phi_1\) mean that

\[
\int_0^a p(s) \, ds \leq \int_0^c p(s) \, ds, \forall a \in C, 
\]

which yields that

\[
\sup_{a \in C} \int_0^a p(s) \, ds \leq \int_0^c p(s) \, ds. 
\]

On account of (15) and Lemma 3, we infer that

\[
\lim_{n \to \infty} \int_0^{a_n} p(s) \, ds = \int_0^c p(s) \, ds. 
\]

Clearly, (12) follows from (17) and (18). The proof of (13) is similar to that of (12) and is omitted. This completes the proof. \(\square\)

3. Fixed Point Theorems and an Example

Now, we investigate the existence and iterative approximations of fixed points for the integral-type multivalued contractive mappings (19) and (42), respectively.

Theorem 9. Assume that \((X, \rho)\) is a complete metric space and \(T : X \to CL(X)\) satisfies that

\[
\int_0^{\max\{\rho(x_n, x), 1\}} p(s) \, ds = q \int_0^{\max\{\rho(x_n, x), 1\}} p(s) \, ds, 
\]

where \(q\) is a constant in \((0, 1)\) and \(p \in \Phi_2\). Then, for each \(x_0 \in X\), there exists an orbit \(\{x_n\}_{n \in \mathbb{N}}\) of \(T\) at \(x_0\) such that it converges to some fixed point \(a \in X\) and

\[
\int_0^{\rho(x_n, a)} p(s) \, ds \leq \frac{q^n}{1 - q} \int_0^{\rho(x_0, a)} p(s) \, ds, \forall n \in \mathbb{N}. 
\]

Proof. For any \(x_0\) in \(X\) and \(x_1 \in Tx_0\), Lemma 5 guarantees that

\[
\rho(x_1, x_2) \leq \frac{1}{q} H(Tx_0, Tx_1) \quad \text{for some} \quad x_2 \in Tx_1. 
\]

Note that

\[
\rho(x_0, x_1) = \max \left\{ \rho(x_0, x_1), \rho(x_0, Tx_0), \rho(x_1, Tx_1), \rho(x_0, Tx_1), \frac{1}{q} \rho(x_0, x_1) + \rho(x_1, Tx_1) \right\}, 
\]

which together with (19), (21), and \(p \in \Phi_2\) yields.

\[
\int_0^{\rho(x_n, x)} p(s) \, ds \leq \int_0^{\max\{\rho(x_n, x), 1\}} p(s) \, ds, 
\]

and

\[
\rho(x_1, x_2) \leq \rho(x_0, x_1) = M(x_0, x_1) \quad \text{and} \quad \int_0^{\rho(x_n, x)} p(s) \, ds \leq q \int_0^{\rho(x_0, x)} p(s) \, ds, \forall n \in \mathbb{N}. 
\]

Lemma 5 reveals that

\[
\rho(x_2, x_3) \leq \frac{1}{q} H(Tx_1, Tx_2) \quad \text{for some} \quad x_3 \in Tx_2. 
\]
Notice that
\[
\rho(x_n, x_{n+1}) \leq M_n(x_n, x_{n+1})
\]
which together with (19), (25), and \( p \in \Phi_2 \) infers
\[
\int_0^{\rho(x_n, x_{n+1})} p(s) ds \leq \int_0^{\max \{ \rho(x_n, x_n), \rho(x_n, x_{n+1}) \}} p(s) ds \leq q \int_0^{\rho(x_n, x_{n+1})} p(s) ds = q \int_0^{\rho(x_n, x_{n+1})} p(s) ds,
\]
and
\[
\rho(x_{n+1}, x_{n+2}) \leq \rho(x_{n+1}, x_n) = M_n(x_n, x_{n+1}) \text{and} \int_0^{\rho(x_{n+1}, x_{n+2})} p(s) ds \leq q \int_0^{\rho(x_{n+1}, x_{n+2})} p(s) ds.
\]
Making use of (24) and (28), we deduce
\[
\int_0^{\rho(x_{n+1}, x_{n+2})} p(s) ds \leq q \int_0^{\rho(x_n, x_{n+1})} p(s) ds \leq q^2 \int_0^{\rho(x_n, x_{n+1})} p(s) ds.
\]
Continuing the process, we obtain an order \( \{ x_n \} \in \mathbb{N} \) of \( T \) at \( x_0 \) satisfying
\[
x_n \in T_{x_{n-1}}, \rho(x_n, x_{n+1}) \leq \rho(x_{n-1}, x_n), \forall n \in N \ni \int_0^{\rho(x_n, x_{n+1})} p(s) ds \leq q \int_0^{\rho(x_n, x_{n+1})} p(s) ds, \forall n \in N.
\]
Thus, (30), \( q \in (0, 1) \), and \( p \in \Phi_2 \) mean
\[
0 \leq \int_0^{\rho(x_n, x_{n+1})} p(s) ds \leq q^n \int_0^{\rho(x_0, x_1)} p(s) ds, \forall n \in N.
\]
Lemma 4 gives
\[
\lim_{n \to \infty} \rho(x_n, x_{n+1}) = 0.
\]
By (31) and \( p \in \Phi_2 \), we obtain
\[
\int_0^{\rho(x_n, x_{n+1})} p(s) ds \leq \int_0^{\rho(x_0, x_1)} p(s) ds + \int_0^{\rho(x_1, x_2)} p(s) ds + \cdots + \int_0^{\rho(x_{n-1}, x_n)} p(s) ds \leq q^n \int_0^{\rho(x_0, x_1)} p(s) ds
\]
that is, (20) holds.
Next, we prove that $a = Ta$. From (32), (37), and Lemmas 6 and 7, we get immediately

$$
\rho(a, Ta) \leq M(x_{n-1}, a)
$$

which generalizes Theorems 1 and 2 in [5], Theorem 2.1 in [8], and Theorem 5 in [15]. The example below shows that Theorem 10 extends properly Theorem 5 in [15].

**Example 1.** Let $X = [0, 1]$ be endowed with the Euclidean metric $\rho = | \cdot |$. Let $q = 9/10$, $T : X \rightarrow \text{CL}(X)$, and $p : R^+ \rightarrow R^+$ be defined by

$$
T_x = \left\{ \frac{1}{10} \right\}, \forall x \in [0, 1], \left\{ 0, \frac{1}{10} \right\}, x = 1,
$$

$$
p(t) = \frac{1}{2\sqrt{t} + 1/10}, \forall t \in R^+.
$$

It is clear that $p \in \Phi_2$. Let $x, y \in X$ with $y < x$. In order to verify (19), we consider below two possible cases:

**Case 1.** $0 \leq y < x < 1$. Clearly,

$$
\int_0^{(1/q)(H(T, Ty))} p(s)ds = 0 \leq q \int_0^{M(x_y)} p(s)ds.
$$

**Case 2.** $0 \leq y < 1$ and $x = 1$. It follows that

$$
\int_0^{(1/q)(H(T, Ty))} p(s)ds = \frac{1}{2\sqrt{t} + 1/10}dt = \sqrt{\frac{19}{10}} - \sqrt{\frac{1}{10}} < \sqrt{0.212} - 0.316 < 0.6147 < \frac{9}{10} < \frac{1 - \sqrt{10}}{10}
$$

$$
= q \int_0^{M(x_y)} p(s)ds.
$$

That is, (19) holds. Thus, Theorem 10 ensures that $T$ has a fixed point $1/10 \in X$. Observe that

$$
H \left( T_1, T, \frac{9}{10} \right) = H \left( \left\{ 0, \frac{1}{10} \right\}, \left\{ \frac{1}{10} \right\} \right) = \frac{1}{10} < a \frac{1}{10}
$$

$$
= \alpha \left( 1, \frac{9}{10} \right), \forall a \in (0, 1),
$$

which means that Theorem 5 in [15] cannot be used to show the existence of fixed points of $T$.

**4. On Stability of Fixed Point Sets**

Now, we discuss the stability of fixed point sets for the integral-type multivalued contractive mappings (19) and (42), respectively. Put $K = \sup_{x \in X} H(T_1x, T_2x)$.

**Theorem 12.** Assume that $(X, \rho)$ is a complete metric space and $T_1, T_2 : X \rightarrow \text{CL}(X)$ satisfy

$$
\int_0^{(1/q)(H(T, Ty))} p(s)ds \leq \int_0^{M(x_y)} p(s)ds, \forall x, y \in X, i \in \{1, 2\},
$$

where $q$ is a constant in $(0, 1)$ and $p \in \Phi_2$. Then, for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n \in N_0}$ of $T$ at $x_0$ such that it converges to some fixed point $a$ in $X$ of $T$ and (20) holds.

**Remark 11.** In case $p(t) = t$, $\forall t \in R^+$ and $q = \sqrt{a}$, where $a$ is a constant in $(0, 1)$, then Theorem 12 reduces to a result,
where \( q \) is a constant in \((0, 1)\) and \( p \in \Phi_2 \). Then,
\[
\int_0^{H(T_1), F(T_1)} p(s) ds \leq \frac{1}{1 - q} \int_0^\infty p(s) ds.
\]

**Proof.** Without loss of generality, we assume that \( K < \infty \). Note that Theorem 9 yields \( F(T_i) \neq \emptyset \) for \( i \in \{1, 2\} \). Put \( x_0 \) in \( F(T_i) \). Lemma 5 guarantees
\[
\rho(x_0, x_1) \leq \frac{1}{q} H(T_1 x_0, T_2 x_0) \leq \frac{1}{q} K \text{ for some } x_1 \in T_2 x_0, x_2 \in T_2 x_1,
\]
and
\[
\rho(x_1, x_2) \leq \frac{1}{q} H(T_2 x_0, T_2 x_1),
\]
which together with (47), (53), and \( p \in \Phi_2 \) gives
\[
\int_0^{\rho(x_0,x_2)} p(s) ds \leq \frac{1}{q} \int_0^{H(T_1 x_0, T_2 x_2)} p(s) ds \leq \frac{1}{q} \max \{\rho(x_1, x_2), \rho(x_1, T_2 x_2), \rho(x_0, x_1)\} \int_0^{\rho(x_0,x_2)} p(s) ds,
\]
and
\[
\rho(x_1, x_2) \leq \rho(x_0, x_1) = M_2(x_0, x_1) \text{ and } \rho(x_1, x_2) \leq \rho(x_0, x_1) = M_2(x_0, x_1) \text{ and } \int_0^{\rho(x_0,x_2)} p(s) ds \leq \frac{1}{q} \int_0^{\rho(x_0,x_2)} p(s) ds.
\]

Lemma 5 ensures with
\[
\rho(x_2, x_3) \leq \frac{1}{q} H(T_2 x_1, T_2 x_2) \text{ for some } x_3 \in T_2 x_2.
\]
In light of (58), \( q \in (0, 1) \), and \( p \in \Phi_2 \), we have

\[
0 \leq \int_0^{p(x_n, x_m)} p(s) ds \leq q^n \int_0^{p(x_n, x_1)} p(s) ds, \forall n \in N, \tag{59}
\]

\[
\lim_{n \to \infty} \int_0^{p(x_n, x_{n+1})} p(s) ds = 0. \tag{60}
\]

Combining (60) and Lemma 4, we get

\[
\lim_{n \to \infty} \rho(x_n, x_{n+1}) = 0. \tag{61}
\]

By (59) and \( \varphi \in \Phi_2 \), we infer

\[
\int_0^{p(x_n, x_m)} p(s) ds \leq q^n \int_0^{p(x_n, x_1)} p(s) ds + \int_0^{p(x_n, x_{n+1})} p(s) ds + \cdots + \int_0^{p(x_n, x_m)} p(s) ds
\]

\[
\leq q^n \int_0^{p(x_n, x_1)} p(s) ds + q^n \int_0^{p(x_n, x_{n+1})} p(s) ds + \cdots + q^n \int_0^{p(x_n, x_m)} p(s) ds
\]

\[
= q^n \int_0^{p(x_n, x_1)} p(s) ds + q^n \int_0^{p(x_n, x_{n+1})} p(s) ds + \cdots + q^n \int_0^{p(x_n, x_m)} p(s) ds
\]

\[
\leq q^n \int_0^{p(x_n, x_1)} p(s) ds, \forall n \in N \text{ with } m > n. \tag{62}
\]

It follows from (62), \( q \in (0, 1) \), \( \varphi \in \Phi_2 \), and Lemma 8 that

\[
0 \leq \int_0^{p(x_n, x_m): m > n} p(s) ds = \sup \left\{ \int_0^{p(x_n, x_m)} p(s) ds : m > n \right\}
\]

\[
\leq q^n \int_0^{p(x_n, x_1)} p(s) ds \to 0 \text{ as } n \to \infty,
\]

that is

\[
\lim_{n \to \infty} \sup \left\{ p(x_n, x_m): m > n \right\} = 0. \tag{64}
\]

which together with Lemma 4 means

\[
\lim_{n \to \infty} \sup \left\{ \rho(x_n, x_m): m > n \right\} = 0, \tag{65}
\]

that is, \( \{x_n\}_{n \in N} \) is a Cauchy sequence.

Completeness of \((X, \rho)\) guarantees

\[
\lim_{n \to \infty} x_n = a \text{ for some } a \in X. \tag{66}
\]

Letting \( m \to \infty \) in (62) and making use of (66) and Lemma 3, we deduce

\[
\int_0^{p(a, T_2 a)} p(s) ds \leq q^n \int_0^{p(x_n, x_1)} p(s) ds, \forall n \in N. \tag{67}
\]

Observe that (61), (66), and Lemma 7 ensure

\[
p(a, T_2 a) \leq M_2(x_{n-1}, a)
\]

\[
= \max \left\{ \rho(x_{n-1}, a), \rho(x_{n-1}, T_2 x_{n-1}), \rho(a, T_2 a), \rho(x_{n-1}, T_2 a) / C_1, \rho(x_{n-1}, T_2 a) / C_2, \rho(x_{n-1}, T_2 a) / C_3 \right\}
\]

\[
\leq \max \left\{ \rho(x_{n-1}, a), \rho(x_{n-1}, T_2 a) / C_1, \rho(x_{n-1}, T_2 a) / C_2, \rho(x_{n-1}, T_2 a) / C_3 \right\}
\]

\[
\to (a, T_2 a) \text{ as } n \to \infty,
\]

that is

\[
\lim_{n \to \infty} M_2(x_{n-1}, a) = \rho(a, T_2 a). \tag{69}
\]

By virtue of (47), (66), (69), \( p \in \Phi_2 \), and Lemmas 3 and 7, we have

\[
\int_0^{p(a, T_2 a)} p(s) ds = \limsup_{n \to \infty} \int_0^{p(x_n, x_m)} p(s) ds \leq \limsup_{n \to \infty} \int_0^{p(x_n, x_m) + \rho(x_n, x_m)} p(s) ds
\]

\[
\leq \limsup_{n \to \infty} \int_0^{p(x_n, x_m) + \rho(x_n, x_m)} p(s) ds \leq \limsup_{n \to \infty} \int_0^{p(x_n, x_m) + \rho(x_n, x_m)} p(s) ds
\]

\[
= q \int_0^{p(a, T_2 a)} p(s) ds, \tag{70}
\]

which means \( p(a, T_2 a) = 0 \) because \( q \in (0, 1) \), that is, \( a \in F(T_2) \).

Taking advantage of (48), (59), (67), \( \varphi \in \Phi_2 \), and Lemma 4, we get

\[
\int_0^{p(x_n, a)} p(s) ds \leq \int_0^{p(x_n, x_1) + \rho(x_n, x_1) + \cdots + \rho(x_n, x_m) + \rho(x_n, x_n)} p(s) ds
\]

\[
\leq \int_0^{p(x_n, x_1)} p(s) ds + \int_0^{p(x_n, x_{n+1})} p(s) ds + \cdots + \int_0^{p(x_n, x_m) + \rho(x_n, x_m)} p(s) ds
\]

\[
+ q \int_0^{p(x_n, x_m) + \rho(x_n, x_m) + \cdots + \rho(x_n, x_{n+1}) + \cdots + \rho(x_n, x_1)} p(s) ds
\]

\[
= 1 - q^n + q^{n+1} \int_0^{p(x_n, x_1)} p(s) ds + \int_0^{p(x_n, x_{n+1})} p(s) ds + \cdots + \int_0^{p(x_n, x_m)} p(s) ds
\]

\[
\leq 1 - q^n + q^{n+1} \int_0^{p(x_n, x_1)} p(s) ds, \forall n \in N. \tag{71}
\]
It follows from Lemma 8 that
\[
\int_0^{p(x_0,F(T_i))} p(s)ds = \inf_{y \in F(T_i)} \int_0^{p(x_0,y)} p(s)ds \leq \int_0^{p(x_0,a)} p(s)ds \leq \frac{1}{1-q} \int_0^{\\Phi} p(s)ds, \forall x_0 \in F(0T_i),
\]
and
\[
\sup_0 \sup_{x \in F(T_i)} \int_0^{p(x,F(T_i))} p(s)ds = \sup_{x \in F(T_i)} \int_0^{p(x,F(T_i))} p(s)ds \leq \frac{1}{1-q} \int_0^{\\Phi} p(s)ds.
\]
(72)

Reversing the roles of $T_1$ and $T_2$, we also conclude
\[
\sup_0 \sup_{x \in F(T_i)} \int_0^{p(x,F(T_i))} p(s)ds = \sup_{x \in F(T_i)} \int_0^{p(x,F(T_i))} p(s)ds \leq \frac{1}{1-q} \int_0^{\\Phi} p(s)ds.
\]
(73)

Using (73), (74), Lemma 8, and $\varphi \in \Phi_2$, we obtain
\[
\int_0^{H(T_i,F(T_i))} p(s)ds = \max \left\{ \int_0^{p(x,F(T_i))} p(s)ds, \int_0^{p(y,F(T_i))} p(s)ds \right\}
\leq \frac{1}{1-q} \int_0^{\\Phi} p(s)ds.
\]
(75)

This completes the proof. \(\square\)

**Theorem 13.** Assume that $(X, \rho)$ is a complete metric space and $\{T_i\}_{i \in N_0} : X \longrightarrow CL(X)$ satisfy
\[
\int_0^{H(T_i,F(T_i),y)} p(s)ds \leq q \int_0^{M_i(x,y)} p(s)ds, \forall x, y \in X, i \in N_0,
\]
(76)
where $q$ is a constant in $(0,1)$ and $p \in \Phi_2$. Assume that
\[
\lim_{i \longrightarrow \infty} H(T_i, x, T_i y) = 0 \text{ uniformly for all } x \in X.
\]
(77)
Then, $\lim_{i \longrightarrow \infty} H(F(T_i), F(T_0)) = 0$.

**Proof.** Let $\epsilon > 0$. It follows from Theorem 12.34 in [27] and $p \in \Phi_2$ that there exists $\delta > 0$ with
\[
\int_A p(s)ds < (1-q)\epsilon \text{ for every bounded subset } A \subset R^+ \text{ with } m(A) \leq \delta,
\]
(78)
where $m(A)$ is the Lebesgue measure of $A$. (77) guarantees that there exists $N \in N$ satisfying
\[
\sup_{x \in X} H(T_i, x, T_i y) < q\delta, \forall i \geq N.
\]
(79)
By means of (78), (79), $p \in \Phi_2$, and Theorem 12, we conclude
\[
0 \leq \int_0^{H(F(T_i),F(T_0))} p(s)ds \leq \frac{1}{1-q} \int_0^{\sup_{x \in X} H(T_i, x, T_i y)} p(s)ds \leq \frac{1}{1-q} \int_0^{\delta} p(s)ds < \epsilon, \forall i \geq N,
\]
which implies that
\[
\lim_{i \longrightarrow \infty} \int_0^{H(F(T_i),F(T_0))} p(s)ds = 0.
\]
(81)

Thus, $\lim_{i \longrightarrow \infty} H(F(T_i), F(T_0)) = 0$ follows from (81) and Lemma 4. This completes the proof. \(\square\)

Theorems 12 and 13 infer immediately the following.

**Theorem 14.** Assume that $(X, \rho)$ is a complete metric space and $T_1, T_2 : X \longrightarrow CL(X)$ satisfy that
\[
\int_0^{H(T_i,T_j)} p(s)ds \leq q \int_0^{N_i(x,y)} p(s)ds, \forall x, y \in X, i \in \{1,2\},
\]
(82)
where $q$ is a constant in $(0,1)$ and $p \in \Phi_2$. Then, $\int_0^{H(T_i,F(T_2))} p(s)ds \leq 1/1-q\int_0^{\delta} p(s)ds$.

**Theorem 15.** Assume that $(X, \rho)$ is a complete metric space and $\{T_i\}_{i \in N_0} : X \longrightarrow CL(X)$ satisfy (77) and
\[
\int_0^{H(T_i,T_j)} p(s)ds \leq q \int_0^{N_i(x,y)} p(s)ds, \forall x, y \in X, i \in N_0,
\]
(83)
where $q$ is a constant in $(0,1)$ and $p \in \Phi_2$. Then, $\lim_{i \longrightarrow \infty} H(F(T_i), F(T_0)) = 0$.

**Remark 16.** Theorems 14 and 15 extend, respectively, Lemma 1 and Theorem 1 in [11].

**5. Conclusion**

In this paper, we introduce two classes of integral-type multivalued contractive mappings, which include some known multivalued contractive mappings as special cases, and prove the existence, iterative approximations, and stability of fixed points for these integral-type multivalued contractive mappings under certain conditions. Our results extend several known results in the literature.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no competing interests.
Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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