BOUNDEDNESS OF SOLUTIONS TO A FULLY PARABOLIC KELLER-SEGEL SYSTEM WITH NONLINEAR SENSITIVITY

HAO YU, WEI WANG* AND SINING ZHENG
School of Mathematical Sciences, Dalian University of Technology
Dalian 116024, China

(Communicated by Michael Winkler)

ABSTRACT. This paper deals with the global boundedness of solutions to a fully parabolic Keller-Segel system $u_t = \Delta u - \nabla (u^\alpha \nabla v)$, $v_t = \Delta v - v + u$ under non-flux boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. The case of $\alpha \geq \max\{1, \frac{2}{n}\}$ with $n \geq 1$ was considered in a previous paper of the authors [Global boundedness of solutions to a Keller-Segel system with nonlinear sensitivity, Discrete Contin. Dyn. Syst. B, 21 (2016), 1317–1327]. In the present paper we prove for the other case $\alpha \in \left(\frac{2}{3}, 1\right)$ that if $\|u_0\|_{L^{n\alpha}(\Omega)}$ and $\|
abla v_0\|_{L^{n\alpha}(\Omega)}$ are small enough with $n \geq 3$, then the solutions are globally bounded with both $u$ and $v$ decaying to the same constant steady state $\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx$ exponentially in the $L^\infty$-norm as $t \to \infty$. Moreover, the above conclusions still hold for all $\alpha \geq 2$ and $n \geq 1$, provided $\|u_0\|_{L^{n\alpha-n}(\Omega)}$ and $\|
abla v_0\|_{L^\infty(\Omega)}$ sufficiently small.

1. Introduction. This paper considers the following fully parabolic Keller-Segel system with nonlinear sensitivity

\[
\begin{aligned}
&u_t = \Delta u - \nabla (u^\alpha \nabla v), &x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u, &x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, &x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), &x \in \Omega,
\end{aligned}
\]

(1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $\alpha \geq \frac{2}{n}$. Since 1970 the Keller-Segel system proposed by Keller and Segel [2] has been used as a classical chemotaxis model to describe the interaction between the random diffusion of cells and the aggregation of cells due to chemical signals produced by the cells themselves, where $u = u(x, t)$ and $v = v(x, t)$ denote the cell density and the signal concentration respectively.

The classical parabolic-parabolic Keller-Segel system with $\alpha = 1$ in (1) together with the corresponding parabolic-elliptic form with the second equation replaced by $0 = \Delta v - v + u$ have been well studied with various significant dynamics properties of solutions achieved. Refer to the currently published survey [1] and the references

2010 Mathematics Subject Classification. Primary: 35K55, 35B35, 35B40; Secondary: 92C17.
Key words and phrases. Keller-Segel system, boundedness, nonlinear sensitivity.

Supported by the National Natural Science Foundation of China (11171048) and the Fundamental Research Funds for the Central Universities (DUT16LK24).

* Corresponding author.
therein. Also see, e.g., [3][4][5][7][8][13][15][16] for results to quasilinear Keller-Segel systems.

Differently, there is a nonlinear sensitivity $u^\alpha$ (when $\alpha \neq 1$) contained in the fully parabolic Keller-Segel system \([1]\). The more general form is

\[
\begin{align*}
  u_t &= \Delta u - \nabla (f(u) \nabla v), \quad x \in \Omega, \; t > 0, \\
  v_t &= \Delta v - v + u, \quad x \in \Omega, \; t > 0,
\end{align*}
\]

where $f \in C^{1+\epsilon}([0, \infty))$ with $f(0) = 0$, for which it was shown that if $f(s) \leq cs^\alpha$ for $s \geq 1$, $0 < \alpha < \frac{2}{n}$ and $n \geq 1$, then the solution $(u, v)$ must be globally bounded for arbitrary value of $m := \int_\Omega u_0(x)dx = \int_\Omega u(x, t)dx$; if $f(s) \geq cs^\alpha$ for $s \geq 1$, $\alpha > \frac{2}{n}$ and $n \geq 2$, then the system admits unbounded solutions \([4]\). Moreover, for the case $f(u) = u^\alpha$ in \([1]\) with $\alpha \geq \max\{1, \frac{2}{n}\}$ and $n \geq 1$, the global boundedness of solutions was established in a previous paper of the authors \([17]\), similarly to the case of $\alpha = 1$ \([2][4]\), by using a new norm to describe the required smallness of $(u_0, v_0)$ instead. Precisely, it was shown that with notations

\[
q^* = \frac{n\alpha K}{n + K}, \quad p^* = \frac{n\alpha K}{n + K - \alpha n}, \quad K \in [n, 2n\alpha - n] \cap ((\alpha - 1)n, \infty),
\]

the smallness of $\|u_0\|_{L^{q^*}(\Omega)}$ and $\|v_0\|_{L^{p^*}(\Omega)}$ ensures the global boundedness of $(u, v)$ with both $u$ and $v$ converging to $\bar{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0(x)dx$ exponentially in the $L^\infty$-norm.

The case of $\alpha < 1$ in \([1]\) is more difficult, where $s^\alpha \not\in C^{1+\epsilon}([0, \infty))$, and even does not satisfy the Lipschitz continuity. Rather than the case $\alpha \geq 1$, the existence of classical solutions with $\alpha < 1$ is unclear yet, although they have been obtained for a parabolic-elliptic Keller-Segel system with $\alpha < 1$ and the second equation replaced by $-\Delta v = u$ in \([1]\) \([11][12]\). Recently, a mild solution was established to \([1]\) for $\alpha < 1$ \([10]\). The properties of these mild solutions can be found in Section \([4]\).

The goal of this paper is to extend our global boundedness results for \([1]\) obtained in \([17]\) with the following two points.

- The global boundedness of classical solutions under the critical $K = n\alpha - n$ and thus $p^* = \infty$ with $\alpha \geq 2$, corresponding to the minimal $q^* = n\alpha - n$.
- The global boundedness of mild solutions with $\frac{2}{n} \leq \alpha < 1$ and $n \geq 3$ (for which a new norm is necessary to describe the smallness of initial data).

Denote by $\lambda_1 > 0$ the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under the homogeneous Neumann boundary condition. The main results of the paper are the following two theorems.

**Theorem 1.1.** Suppose $n \geq 1$, $\alpha \geq 2$ with $0 < \lambda' < \lambda_1$. There exists $\delta_1$ such that if nonnegative initial data $(u_0, v_0) \in C^0(\Omega) \times W^{1,\infty}(\Omega)$ satisfies

\[
\|u_0\|_{L^{n-\alpha}(\Omega)}, \|v_0\|_{L^\infty(\Omega)} \leq \delta_1,
\]

then \([1]\) possesses a global classical solution $(u, v)$ with the decay estimates

\[
\|u(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq C_1 e^{-\lambda't}, \quad \|v(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq C_1 e^{-\min\{\lambda', 1\}t}
\]

for $t > 0$ and some $C_1 > 0$.

**Theorem 1.2.** Suppose $n \geq 3$, $\alpha \in (\frac{2}{n}, 1)$ with $0 < \lambda' < \lambda_1$. There exists $\delta_2 > 0$ such that if nonnegative initial data $(u_0, v_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ satisfies

\[
\|u_0\|_{L^{\frac{n}{n-\alpha}}(\Omega)}, \|v_0\|_{L^{n\alpha}(\Omega)} \leq \delta_2,
\]
then \( \text{Preliminaries.} \)

2. Proof of Theorem 1.1. The main techniques are motivated by those in \([2, 14]\).

3. Remark 1. It can be found that the involved norm for \( \alpha < 1 \) in Theorem 1.2 is unique, rather than those represented in \([3]\) for the case of \( \alpha > 1 \), and there is a gap left for \( \alpha \in \left[ \frac{2}{n}, \frac{2}{3} \right] \) when \( n \geq 3 \).

2. Preliminaries. We begin with the known \( L^p_L^q \) estimates for the Neumann heat semigroup on bounded domains as preliminaries, represented in the following two lemmas.

**Lemma 2.1.** \([(14, \text{Lemma 1.3}), (2, \text{Lemma 2.1})]\) Let \( (e^{t\Delta})_{t \geq 0} \) be the Neumann heat semigroup in \( \Omega \), with \( \lambda_1 > 0 \) denoting the first nonzero eigenvalue of \( -\Delta \) under the Neumann boundary condition. Then there exist \( K_1, \ldots, K_4 > 0 \) depending only on \( \Omega \) such that the following estimates hold.

(i) If \( 1 \leq q \leq p \leq \infty \), then

\[
\|e^{t\Delta}w\|_{L^p(\Omega)} \leq K_1(1 + t^{-\frac{2}{p}(\frac{1}{q} - \frac{1}{p})})e^{-\lambda_1 t\|w\|_{L^q(\Omega)}}, \quad t > 0
\]  

(ii) If \( 1 \leq q \leq p \leq \infty \), then

\[
\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \leq K_2(1 + t^{-\frac{2}{p}(\frac{1}{q} - \frac{1}{p})})e^{-\lambda_1 t\|w\|_{L^q(\Omega)}}, \quad t > 0
\]  

(iii) If \( 2 \leq q \leq p \leq \infty \), then

\[
\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \leq K_3(1 + t^{-\frac{2}{p}(\frac{1}{q} - \frac{1}{p})})e^{-\lambda_1 t\|\nabla w\|_{L^q(\Omega)}}, \quad t > 0
\]  

(iv) If \( 1 < q \leq p < \infty \) or \( 1 < q < p = \infty \), then

\[
\|e^{t\Delta}\nabla \cdot w\|_{L^p(\Omega)} \leq K_4(1 + t^{-\frac{2}{p}(\frac{1}{q} - \frac{1}{p})})e^{-\lambda_1 t\|w\|_{L^q(\Omega)}}, \quad t > 0
\]  

for \( w \in W^{1,q}(\Omega) \).

**Lemma 2.2.** \([(14, \text{Lemma 1.2})]\) For \( \mu, \beta < 1, \gamma, \delta > 0 \) with \( \gamma \neq \delta \), we have

\[
\int_0^t (1 + (t-s)^{-\mu})e^{-\gamma(t-s)}(1 + s^{-\beta})e^{-\delta s}ds 
\]

\[
\leq C(1 + t^{\min(0,1-\mu-\beta)})e^{-\min(\gamma,\delta)t}, \quad t > 0
\]  

with \( C = O(\frac{1}{n-\gamma} + \frac{1}{1-\mu} + \frac{1}{1-\beta}) \).

Next, we will prove Theorems 1.1 and 1.2 in the next two sections respectively. The main techniques are motivated by those in \([2, 14]\).

3. Proof of Theorem 1.1. Suppose \( K = na - n \) and thus \( q^* = na - n, p^* = \infty \) in \([3]\) with \( \alpha \geq 2 \). The local existence of classical solutions of \([1]\) with \( \alpha \geq 1 \) has been proved in \([6, \text{Theorem 3.1}]\). To establish the global boundedness of solutions, we introduce an important elementary inequality

\[
|u|^\alpha \leq (|u - \bar{u}| + |\bar{u}|)^\alpha \leq 2^{\alpha-1}(|u - \bar{u}| + |\bar{u}|^\alpha)
\]  

(12) to treat the nonlinearity \( u^\alpha \) with \( \alpha \geq 1 \). The proof of Theorem 1.1 relies on the following proposition.
Proposition 1. Under the conditions of Theorem 1.1, there exists $\varepsilon_1 > 0$ such that if initial data $(u_0, v_0) \in C^0(\Omega) \times W^{1,\infty}(\Omega)$ nonnegative satisfies

$$\|u_0\|_{L^{n-\alpha}(\Omega)}, \|\nabla v_0\|_{L^\infty(\Omega)} \leq \varepsilon_1,$$

then \( \bullet \) possesses a global classical solution satisfying

$$\|u(\cdot, t) - \tilde{u}_0\|_{L^p(\Omega)} \leq C_1\varepsilon_1(1 + \frac{1}{\alpha - 1} + \frac{\theta^2}{n}) e^{-\lambda t}, \quad \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1\varepsilon_1 e^{-\lambda t}$$

for $t > 0$ and some $C_1 > 0$, where $\theta \in [q_0, \infty)$ with $q_0 = \frac{\alpha - \frac{3}{2}n}{\theta}$.

Proof. The framework of the proof is similar to that for [17, Proposition 2] of ours.

Suppose (13) holds with $\theta = 0$ to be determined (and denoted by $\varepsilon > 0$ below for simplicity). Let $T_{\text{max}} > 0$ be the maximal existence time of the local solution under initial data $(u_0, v_0)$, and

$$T := \sup \left\{ \tilde{T} \in (0, T_{\text{max}}) \mid \|u(\cdot, t) - e^{t\Delta}u_0\|_{L^p(\Omega)} \leq \varepsilon(1 + t^{-\frac{\theta}{\alpha - 1} + \frac{\theta^2}{n}}) e^{-\lambda t}, \quad \text{for all } t \in (0, \tilde{T}) \right\}.$$

Next, we will show that $T = \infty$ if $\varepsilon$ is sufficiently small. Denote

$$k = \max\{1, |\Omega|, |\Omega|^{-1}\}$$

in the sequel. Similar to (22) of [17], we have for each $\theta \in [q_0, \infty)$ that

$$\|u(\cdot, t) - \tilde{u}_0\|_{L^p(\Omega)} \leq \|u(\cdot, t) - e^{t\Delta}u_0\|_{L^p(\Omega)} + \|e^{t\Delta}(u_0 - \tilde{u}_0)\|_{L^\infty(\Omega)}$$

$$\leq C_1\varepsilon(1 + t^{-\frac{\theta}{\alpha - 1} + \frac{\theta^2}{n}}) e^{-\lambda t}, \quad t \in (0, T).$$

Taking $\theta = q_0$ in (15), since $\frac{1}{2} + \frac{\alpha - 1}{2p} < \frac{1}{2} + \frac{2}{2p} < 1$ and $\frac{1}{2(\alpha - 1)} - \frac{\alpha - 1}{2p} \leq \frac{1}{2}$ with $\alpha \geq 2$, we have by (5) and (11), for $p \in [q_0, \infty)$ that

$$\|\nabla v(t) - e^{t(\Delta - 1)}v_0\|_{L^p(\Omega)}$$

$$\leq \int_0^t \|\nabla e^{(t-s)(\Delta - 1)}(u(\cdot, s) - \tilde{u}_0)\|_{L^p(\Omega)} ds$$

$$\leq C_1C_2\varepsilon \int_0^t (1 + (t - s))^{-\frac{1}{2}} \|e^{-\frac{1}{2}}(\frac{\alpha - 1}{2p} - \frac{\theta^2}{n}) e^{-\lambda_1(t-1)}(t-s)^{\frac{\alpha - 1}{2p} - \frac{\theta^2}{n}} e^{-\lambda t} ds$$

$$\leq C_3\varepsilon e^{-\lambda t}, \quad t \in (0, T).$$

By (9) and $\|\nabla v_0\|_{L^\infty(\Omega)} \leq \varepsilon$, we have for $p \in [2, \infty)$

$$\|\nabla e^{t(\Delta - 1)}v_0\|_{L^p(\Omega)} \leq C_4e^{-\lambda_1 t}\|\nabla v_0\|_{L^p(\Omega)}$$

$$\leq C_4e^{-\lambda_1 t}\|\nabla v_0\|_{L^\infty(\Omega)}$$

$$\leq C_4ke^{-\lambda_1 t}, \quad t \in (0, T),$$

and for $p \in [q_0, 2)$

$$\|\nabla e^{t(\Delta - 1)}v_0\|_{L^p(\Omega)} \leq \|\nabla e^{t(\Delta - 1)}v_0\|_{L^2(\Omega)}$$

$$\leq C_5\|\nabla e^{t(\Delta - 1)}v_0\|_{L^2(\Omega)}$$

$$\leq C_5ke^{-\lambda_1 t}\|\nabla v_0\|_{L^\infty(\Omega)}$$

$$\leq C_5ke^{-\lambda_1 t}, \quad t \in (0, T).$$
Therefore, together with (16), we conclude
\[\| \nabla v(t, \cdot) \|_{L^p(\Omega)} \leq c_6 \varepsilon e^{-\lambda t}, \ t \in (0, T)\]
for all \( p \in [g_0, \infty) \) with \( c_6 > 0 \) independent of \( p \), and hence
\[\| \nabla v(t, \cdot) \|_{L^\infty(\Omega)} \leq c_6 \varepsilon e^{-\lambda t}, \ t \in (0, T). \tag{17}\]
Due to (10), (11) and (12), we have for each \( \theta \in [g_0, \infty] \) that
\[\| u(\cdot, t) - e^{t \Delta} u_0 \|_{L^p(\Omega)} \]
\[\leq c_7 \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} \| u^\alpha(\cdot, s) \nabla v(\cdot, s) \|_{L^p(\Omega)} ds\]
\[\leq c_7 2^{\alpha - 1} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} \| u(\cdot, s) - \bar{u}_0 \|_{L^p(\Omega)} \| \nabla v(\cdot, s) \|_{L^p(\Omega)} ds\]
\[+ c_7 2^{\alpha - 1} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} \bar{u}_0 \| \nabla v(\cdot, s) \|_{L^p(\Omega)} ds\]
\[\leq c_7 2^{\alpha - 1} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} \| u(\cdot, s) - \bar{u}_0 \|_{L^p(\Omega)} \| \nabla v(\cdot, s) \|_{L^\infty(\Omega)} ds\]
\[+ c_7 2^{\alpha - 1} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} \bar{u}_0 \| \nabla v(\cdot, s) \|_{L^\infty(\Omega)} ds\]
\[=: I_1 + I_2.\]
By (17) with \( \bar{u}_0 \leq k \varepsilon \), we know
\[I_2 \leq c_6 2^{\alpha - 1} k \varepsilon c_7 \varepsilon^{1 + \alpha} |\Omega|^\frac{\alpha}{2} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} e^{-\lambda t} ds\]
\[\leq c_6 \varepsilon e^{1 + \alpha} e^{-\lambda t}, \ t \in (0, T).\]
And by (15),
\[\| u(\cdot, s) - \bar{u}_0 \|_{L^p(\Omega)} \leq 2^{\alpha - 1} c_4 \varepsilon^{\alpha} (1 + s^{-\frac{n}{2(\alpha - 1)} + \frac{m}{2}}) e^{-\lambda t}, \ t \in (0, T).\]
Noticing \( \frac{\alpha}{2(\alpha - 1)} - \frac{n}{2p} < 1 \) with \( \alpha \geq 2 \), we combine this with (17) to get
\[I_1 \leq c_7 4^{\alpha - 1} c_4 c_6 \varepsilon^{1 + \alpha} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{n}{2} (\frac{1}{m} - \frac{1}{2})}) e^{-\lambda_1 (t-s)} (1 + s^{-\frac{n}{2(\alpha - 1)}} + \frac{m}{2}) e^{-\lambda t} ds\]
\[\leq c_6 \varepsilon^{1 + \alpha} (1 + t^{\min(0, \frac{1}{2} - \frac{n}{2(\alpha - 1)} + \frac{m}{2})}) e^{-\lambda t}\]
\[= c_6 \varepsilon^{1 + \alpha} (1 + t^{-\frac{1}{2(\alpha - 1)}} + \frac{m}{2}) e^{-\lambda t}, \ t \in (0, T).\]
The estimates for \( I_1 \) and \( I_2 \) yield
\[\| u(\cdot, t) - e^{t \Delta} u_0 \|_{L^p(\Omega)} \leq c_6 \varepsilon^{1 + \alpha} (1 + t^{-\frac{n}{2(\alpha - 1)}} + \frac{m}{2}) e^{-\lambda t}, \ t \in (0, T). \tag{18}\]
We conclude from (15) and (18) that \( T = \infty \) provided \( \varepsilon < (\frac{1}{2c_{10})} \frac{1}{2}. \) This completes the proof by (15) and (17) with \( T = \infty. \)

**Proof of Theorem 1.4** By Proposition 1 with \( \theta = g_0 = n \alpha - \frac{n}{2} \), we have
\[\| u(\cdot, t) \|_{L^{\alpha - \frac{n}{2}}(\Omega)} \leq \overline{C}_1 \delta_1 (1 + t^{-\frac{1}{2(\alpha - 1)}} + \frac{m}{2}) e^{-\lambda t} + \delta_1 |\Omega|^{-\frac{1}{2(\alpha - 1)}} - \frac{m}{2}, \ t > 0,\]
\[\| \nabla v(\cdot, t) \|_{L^{\alpha - \frac{3n}{2}}(\Omega)} \leq \overline{C}_1 k \delta_1 e^{-\lambda t}, \ t > 0\]
Lemma 4.1. (Theorem 2.1) Let \( \lambda > 0 \) and \( \alpha > 0 \), then \( \|u\|_{L^{\infty}(\Omega)} \leq \delta_1 \) with \( \delta_1 \in (0, \varepsilon_1) \), and hence
\[
\|u(\cdot,1)\|_{L^{\infty}(\Omega)} \leq 2C_1 \delta_1 e^{-\lambda'},
\]
with \( 0 < \lambda' \leq \alpha \), and therefore give a comparison between the superlinear and sublinear sensitivity to those with \( \alpha < \lambda \).

Proof of Theorem 1.2. In this section, we treat the case of \( \alpha < 1 \) in \([17, \text{Theorem 1.1}]\) for establishing the global boundedness and asymptotic behavior to the mild solutions, corresponding to those with \( \alpha > 1 \) (refer to \([17, \text{Theorem 1.1}]\)) for the classical solutions, and thereby give a comparison between the superlinear and sublinear sensitivity \( u' \).

Denote \( u_+ = \max\{u, 0\} \), and define the mild solution of \([1]\) on \([0, T_0]\) with nonnegative initial data \((u_0, v_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)\) by a couple of \((u, v)\) satisfying
\[
\begin{align*}
u &\in L^\infty([0, T_0] \cup \{0\}; L^\infty(\Omega)), \quad v \in L^\infty([0, T_0] \cup \{0\}; W^{1,\infty}(\Omega)), \quad \text{(21)}
\end{align*}
\]
and
\[
\begin{align*}
u'(t) &= e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot [u_+ \nabla v](s)ds, & t \in (0, T_0), \\
u'(t) &= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} [u - v](s)ds, & t \in (0, T_0).
\end{align*}
\]

The local existence this mild solution can be obtained by the following lemma.

Lemma 4.1. (Theorem 2.1) Let \( 0 < \alpha < 1 \) with nonnegative \((u_0, v_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)\). Then there exist \( T_{\max} \in (0, \infty] \) and functions \( u, v \) on \( \Omega \times [0, T_{\max}] \) with the following properties:

\( u, v \) is a mild solution of \((21) - (22)\) on \([0, T_{\max}]\):

\( \lim_{t \to T_{\max}} \|u(t)\|_{L^\infty} = \infty \).

Remark 2. Let \((u, v)\) be a mild solution of \((1)\) with \( \alpha < 1 \) under initial data \((u_0, v_0)\). Then due to \([10, \text{Proposition 2.1}]\), \((u, v)\) is a classical solution to \((1)\) in \( \{(x, t) \in \Omega \times (0, T_{\max}); u(x, t) > 0\}\). Theorem 1.2 implies that \( u > 0 \) after a time \( t_0 \), and thus the mild solution \((u, v)\) becomes the classical one whenever \( t > t_0 \).

Next we will prove Theorem 1.2 via two propositions with the following elementary fact with \( \alpha \in (0, 1) \) that
\[
|u|^\alpha \leq |u - \bar{u}|^\alpha + |\bar{u}|^\alpha \leq |u - \bar{u}|^\alpha + |\bar{u}|^\alpha.
\]

Proposition 2. Suppose \( n \geq 3 \), \( 0 < \lambda' < \lambda_1 \), \( 0 < \alpha < 1 \), and let \((u, v)\) be a mild solution of \((1)\). There exists \( \varepsilon_2 > 0 \) such that if initial data \((u_0, v_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)\) nonnegative satisfies
\[
\|u_0\|_{L^1(\Omega)}, \|\nabla v_0\|_{L^p(\Omega)} \leq \varepsilon_2
\]
and \( 0 > \frac{2}{q} \) and \( p > n \), then \((u, v)\) is global and satisfies the decay estimates in Theorem 1.2 with some \( C_2 > 0 \).
Proof. It is known from [10] Proposition 2.1 that the mild solution \((u, v)\) is non-negative and satisfies
\[
\|u(\cdot, t) - \bar{u}_0\|_{L^p(\Omega)} \leq c_0 \varepsilon_2 (1 + t^{-\frac{nq - n\gamma}{p_0}}) e^{-\lambda_1 t} \|u(\cdot, s) - \bar{u}_0\|_{L^{p_0}(\Omega)} ds,
\]
By using the framework of [14] Theorem 2.1 with the constants \(q_0, p_0\) and \(\theta \in [p_0, \infty]\) there, we have for \(\lambda' \in (0, \lambda_1)\) with \(c_0 > 0\) that
\[
\|u(\cdot, t) - \bar{u}_0\|_{L^p(\Omega)} \leq c_0 \varepsilon_2 (1 + t^{-\frac{nq - n\gamma}{p_0}}) e^{-\lambda_1 t} \|\nabla v(\cdot, t)\|_{L^{p_0}(\Omega)} \leq c_0 \varepsilon_2 e^{-\lambda_1 t}
\]
for \(t \in (0, T)\). Due to Lemma 2.1 (iv), it follows for each \(\theta \in [p_0, \infty]\) that
\[
\|u(\cdot, t) - e^{t \Delta} u_0\|_{L^q(\Omega)} \leq K_4 \int_0^t (1 + (t - s)^{-\frac{nq - n\gamma}{p_0}}) e^{-\lambda_1(t-s)} (1 + s^{-\frac{nq - n\gamma}{p_0}}) e^{-\lambda_1(s-r)} ds,
\]
where the elementary inequality [23] with \(\alpha < 1\) is employed. Consequently, the decay estimates of the form [5] can be obtained by repeating the arguments in [13] Theorem 2.1.

\[\square\]

Proposition 3. Under the conditions of Theorem 1.2 with \(q_0 \in (\frac{n}{2}, \frac{nm}{n - m})\), \(\theta_0 \in (n, \frac{nm}{n - m})\), there exists \(\tilde{\varepsilon}_2 > 0\) such that if nonnegative initial data \((u_0, v_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)\) satisfies
\[
\|u_0\|_{L^p(\Omega)} \|\nabla v_0\|_{L^{q_0}(\Omega)} \leq \tilde{\varepsilon}_2,
\]
then [1] possesses a global mild solution \((u, v)\) decaying with the way
\[
\|u(\cdot, t) - e^{t \Delta} u_0\|_{L^p(\Omega)} \leq \varepsilon_2 (1 + t^{-\alpha + \frac{n\gamma}{n - m}}) e^{-\lambda t}, \quad t > 0
\]
for \(\theta \in [\theta_0, \theta_0]\).

Proof. Suppose [25] holds with \(\tilde{\varepsilon}_2\) to be determined, and denoted by \(\varepsilon\) below for simplicity. Define
\[
T := \sup \left\{ \tilde{T} \in (0, T_{\text{max}}) \mid \|u(\cdot, t) - e^{t \Delta} u_0\|_{L^p(\Omega)} \leq \varepsilon (1 + t^{-\alpha + \frac{n\gamma}{n - m}}) e^{-\lambda t}, \quad \text{for all } t \in (0, \tilde{T}) \text{ and } \theta \in [\theta_0, \theta_0]\right\}
\]
with \( \theta_0 \in (\frac{n}{n - q_0}, \frac{nq_0}{n - q_0}) \). Notice here both \( u(\cdot, t) \) and \( e^t \Delta u_0 \) are bounded near \( t = 0 \).

Similar to \([15]\), we obtain for \( \theta \in [q_0, \theta_0] \) that

\[
\|u(\cdot, t) - \bar{u}_0\|_{L^p(\Omega)} \leq c_1 \varepsilon (1 + t^{-\frac{1}{2} + \frac{\alpha}{2p}}) e^{-\lambda't}, \quad t \in (0, T), \tag{28}
\]

Since \( \frac{1}{p} - \frac{n}{2p} < \frac{1}{2}, \frac{1}{2} + \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p}) < 1 \) with \( q_0 < \frac{n\alpha}{2 - \alpha}, p < \frac{nq_0}{n - q_0} \), we know from \([8],[11]\) and \([28]\) that for \( p \in (q_0, \frac{nq_0}{n - q_0}) \),

\[
\|\nabla v(\cdot, t) - e^{t(\Delta - 1)}v_0\|_{L^p(\Omega)}
\leq c_1 c_2 \varepsilon \int_0^t \left( 1 + (t - s)^{-\frac{1}{2} - \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})} \right) e^{-(\lambda_1 + 1)(t - s)}(1 + s^{-\frac{1}{2} + \frac{\alpha}{2p}}) e^{-\lambda's} ds
\leq c_3 \varepsilon (1 + t^{\frac{1}{2} - \frac{1}{2} + \frac{\alpha}{2p}}) e^{-\lambda't}, \quad t \in (0, T),
\]

with \( c_3 = c_3(p) > 0 \). Due to \([9]\) and \([25]\) with \( \alpha_n > 2 \), we obtain for \( p \in (n\alpha, \frac{nq_0}{n - q_0}) \) that

\[
\|\nabla e^{t(\Delta - 1)}v_0\|_{L^p(\Omega)} \leq c_4 e^{-\lambda_1 t}(1 + t^{-\frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})})\|\nabla v_0\|_{L^{n\alpha}(\Omega)}
\leq c_4 e^{-\lambda_1 t}(1 + t^{-\frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})})
\leq 2c_4 \varepsilon e^{-\lambda_1 t}(1 + t^{\frac{1}{2} - \frac{1}{2} + \frac{\alpha}{2p}}), \quad t \in (0, T),
\]

because \( \frac{1}{\alpha} - \frac{1}{2} - \frac{2}{q_0} > \frac{1}{\alpha} - \frac{1}{2} > 0 \) with \( p > n\alpha \) and \( \alpha < 1 \). While for \( p \in [q_0, n\alpha] \),

\[
\|\nabla e^{t(\Delta - 1)}v_0\|_{L^p(\Omega)} \leq c_5 k e^{-\lambda_1 t}(1 + t^{\frac{1}{2} - \frac{1}{2} + \frac{\alpha}{2p}}), \quad t \in (0, T).
\]

Therefore, for \( p \in (q_0, \frac{nq_0}{n - q_0}) \), we have the estimate

\[
\|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leq c_6 \varepsilon (1 + t^{\frac{1}{2} - \frac{1}{2} + \frac{\alpha}{2p}}) e^{-\lambda't}, \quad t \in (0, T)
\]

with \( c_6 = c_6(p) > 0 \).

By \([10]\) and \([23]\) with \( \theta \in [q_0, \theta_0] \), we have

\[
\|u(\cdot, t) - e^t \Delta u_0\|_{L^p(\Omega)}
\leq c_7 \int_0^t \left( 1 + (t - s)^{-\frac{1}{2} - \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})} \right) e^{-\lambda_1 (t - s)} \|u^\alpha(\cdot, s) \nabla v(\cdot, s)\|_{L^{n\alpha}(\Omega)} ds
\leq c_7 \int_0^t \left( 1 + (t - s)^{-\frac{1}{2} - \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})} \right) e^{-\lambda_1 (t - s)} \|u - \bar{u}_0\|^\alpha \|\nabla v(\cdot, s)\|_{L^{n\alpha}(\Omega)} ds
\]

\[ + c_7 \int_0^t \left( 1 + (t - s)^{-\frac{1}{2} - \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})} \right) e^{-\lambda_1 (t - s)} \bar{u}_0^\alpha \|\nabla v(\cdot, s)\|_{L^{n\alpha}(\Omega)} ds
\]

\[ =: I_1 + I_2.
\]

Combining \([11]\) and \([29]\), we obtain for \( \theta \in [q_0, \theta_0] \) that

\[
I_2 \leq c_7 k^\alpha c_6 e^{1 + \alpha} \int_0^t \left( 1 + (t - s)^{-\frac{1}{2} - \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p})} \right) e^{-\lambda_1 (t - s)}(1 + s^{\frac{1}{2} - \frac{1}{2} + \frac{\alpha}{2p}}) e^{-\lambda's} ds
\]

\[ \leq c_8 e^{1 + \alpha}(1 + t^{\min(0,1 - \frac{1}{2} + \frac{\alpha}{2p})}) e^{-\lambda't}
\]

\[ \leq 2c_8 e^{1 + \alpha}(1 + t^{-\frac{1}{2} + \frac{\alpha}{2p}}) e^{-\lambda't}, \quad t \in (0, T),
\]

since \( \frac{1}{\alpha} - \frac{1}{2} + \frac{\alpha}{2p} > 0 \) with \( q_0 < \frac{n\alpha}{2 - \alpha} \) and \( \frac{1}{2} + \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p}) < \frac{1}{2} + \frac{2}{q_0} (\frac{1}{q_0} - \frac{1}{p}) < 1 \) with \( \theta < \theta_0 < \frac{nq_0}{n - q_0} \).

Due to \( q_0 \in (\frac{2n}{2 - \alpha}, \frac{nq_0}{n - q_0}) \) with \( \alpha \in (\frac{2}{3}, 1) \), \( \theta_0 > n \), we can take \( q_1 \in (\frac{2n}{2 - \alpha}, \theta_0) \), \( q_2 \in (\frac{nq_0}{n - q_0}, \frac{nq_0}{n - q_0}) \) such that \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_2} \). Noticing that \(-1 + \frac{n}{2q_1} < 0, \frac{1}{2} - \frac{1}{2} + \frac{n}{2q_2} < 0, \)
This concludes Consequence. Consequently, we have for each $\theta \in [q_0, 0]$ that

$$
I_1 \leq c_7 \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{\alpha}{2}} (\frac{1}{2} - \frac{\alpha}{2})) e^{-\lambda_1(t-s)} \|u - \bar{u}\|_{L^{q_1,\alpha}(\Omega)}^2 \|\nabla u\|_{L^{q_2}(\Omega)} ds
$$

$$
\leq c_9 \varepsilon^{1+\alpha} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{\alpha}{2}} (\frac{1}{2} - \frac{\alpha}{2})) e^{-\lambda_1(t-s)} (1 + s^{1+\frac{\alpha}{2}}) (1 + s^{-\frac{1}{2} - \frac{\alpha}{2}} (\frac{1}{2} + \frac{\alpha}{2})) e^{-\lambda s} ds
$$

$$
\leq c_{10} \varepsilon^{1+\alpha} \int_0^t (1 + (t - s)^{-\frac{1}{2} - \frac{\alpha}{2}} (\frac{1}{2} - \frac{\alpha}{2})) e^{-\lambda_1(t-s)} (1 + s^{-\frac{1}{2} - \frac{\alpha}{2}} (\frac{1}{2} + \frac{\alpha}{2})) e^{-\lambda s} ds
$$

$$
\leq c_{11} \varepsilon^{1+\alpha} (1 + t^{-\frac{1}{2} + \frac{\alpha}{2}}) e^{-\lambda s}, \quad t \in (0, T).
$$

Consequently,

$$
\|u(\cdot, t) - e^{t\Delta} u_0\|_{L^p(\Omega)} \leq c_{12} \varepsilon^{1+\alpha} (1 + t^{-\frac{1}{2} + \frac{\alpha}{2}}) e^{-\lambda s}, \quad t \in (0, T).
$$

This concludes $T = \infty$ provided $\varepsilon < (\frac{1}{2c_{12}})^{\frac{1}{2}}$ by Lemma [4.1] with (24). The proof is complete.

Proof of Theorem 1.2. By using the estimates of (28)-(29) and Proposition 2 similarly to the proof of Theorem 1.1, we can choose $\theta > \frac{\alpha}{2}$ and $p > n$ to arrive at the conclusion.

REFERENCES

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.

[2] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, Discrete Contin. Dyn. Syst., 35 (2015), 1891–1904.

[3] T. Cieślak and C. Stinner, Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions, J. Differential Equations, 252 (2012), 5832–5851.

[4] T. Cieślak and C. Stinner, Finite-time blowup in a supercritical quasilinear parabolic-Keller-Segel system in dimension $2$, Acta Appl. Math., 129 (2014), 135–146.

[5] T. Cieślak and C. Stinner, New critical exponents in a fully parabolic quasilinear Keller-Segel system and applications to volume filling models, J. Differential Equations, 258 (2015), 2080–2113.

[6] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system J. Differential Equations, 215 (2005), 52–107.

[7] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller-Segel system of parabolic-parabolic type on non-convex bounded domains, J. Differential Equations, 256 (2014), 2993–3010.

[8] S. Ishida and T. Yokota, Blow-up in finite or infinite time for quasilinear degenerate Keller-Segel system of parabolic-parabolic type, Discrete Contin. Dyn. Syst. Ser. B, 18 (2013), 2569–2596.

[9] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol., 26 (1970), 399–415.

[10] N. Mizoguchi and P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), 851–875.

[11] A. Montaru, A semilinear parabolic-elliptic chemotaxis system with critical mass in any space dimension Nonlinearity, 26 (2013), 2669–2701.

[12] A. Montaru, Wellposedness and regularity for a degenerate parabolic equation arising in a model of chemotaxis with nonlinear sensitivity, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), 231–256.
[13] Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Differential Equations*, **252** (2012), 692–715.

[14] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differential Equations*, **248** (2010), 2889–2905.

[15] M. Winkler, Does a ‘volume-filling effect’ always prevent chemotactic collapse? *Math. Methods Appl. Sci.*, **33** (2010), 12–24.

[16] M. Winkler and K. C. Djie, Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, *Nonlinear Anal.*, **72** (2010), 1044–1064.

[17] H. Yu, W. Wang and S. Zheng, Global boundedness of solutions to a Keller-Segel system with nonlinear sensitivity, *Discrete Contin. Dyn. Syst. B.*, **21** (2016), 1317–1327.

Received May 2016; revised September 2016.

E-mail address: yuhaomuu@126.com
E-mail address: weiwang@dlut.edu.cn
E-mail address: snzheng@dlut.edu.cn