CONTINUUM LIMIT FOR LATTICE SCHRÖDINGER OPERATORS

HIROSHI ISOZAKI AND ARNE JENSEN

Abstract. We study the behavior of solutions of the Helmholtz equation \((-\Delta_{\text{disc},h} - E)u_h = f_h\) on a periodic lattice as the mesh size \(h\) tends to 0. Projecting to the eigenspace of a characteristic root \(\lambda_h(\xi)\) and using a gauge transformation associated with the Dirac point, we show that the gauge transformed solution \(u_h\) converges to that for the equation \((P(D_x) - E)v = g\) for a continuous model on \(\mathbb{R}^d\), where \(\lambda_h(\xi) \to P(\xi)\). For the case of the hexagonal and related lattices, in a suitable energy region, it converges to that for the Dirac equation. For the case of the square lattice, triangular lattice, hexagonal lattice (in another energy region) and subdivision of a square lattice, one can add a scalar potential, and the solution of the lattice Schrödinger equation \((-\Delta_{\text{disc},h} + V_{\text{disc},h} - E)u_h = f_h\) converges to that of the continuum Schrödinger equation \((P(D_x) + V(x) - E)u = f\).

1. Introduction

The lattice is a standard model to describe wave motions on periodic structures. The associated Laplacian \(\Delta_{\Gamma_h}\) is a difference operator. When the mesh size tends to 0, (a part of) \(H_{\text{disc},h} = \frac{1}{h^2}(\Delta_{\Gamma_h} - E_0)\) with a suitable scale factor \(h^\nu\) and a reference energy \(E_0\) has a formal limit \(H_{\text{cont}}\) as a (pseudo) differential operator \(P(D_x)\), and one expects the convergence of solutions of the equation \((H_{\text{disc},h} - E)u_h = f_h\) to those for the continuous model with Hamiltonian \(H_{\text{cont}}\). The aim of this paper is to study this continuum limit of discrete periodic systems. We are mainly interested in solutions representing the scattering wave, i.e. \(u_{\pm,h} = (H_{\text{disc},h} - E \mp i0)^{-1}f_h\), where \(E \in \sigma_{\text{cont}}(H_{\text{disc},h})\). We show that these scattering solutions of the lattice system converge to those for the continuous model as \(h \to 0\), namely, given a suitable relatively compact interval \(I \subset \mathbb{R}\),

\[ J_h(H_{\text{disc},h} - E \mp i0)^{-1}P_h \to (H_{\text{cont}} - E \mp i0)^{-1}P, \]

for all \(E \in I\) in the strong sense in \(L^2, -s, s > 1/2\), (see (2.13)), where \(J_h\) and \(P_h, P\) are suitable embedding and localization operators. This then yields

\[ J_h e^{-itH_{\text{disc},h}} E_h(I) P_h \to e^{-itH_{\text{cont}}} P, \]

where \(E_h(\cdot)\) is is the spectral decomposition of \(H_{\text{disc},h}\). Hence for any \(\varphi \in C_0^\infty(\mathbb{R})\),

one can show the convergence of the function of Hamiltonian:

\[ J_h \varphi(H_{\text{disc},h}) P_h \to \varphi(H_{\text{cont}}) P. \]

Our method is also able to deal with a complex energy parameter \(E\). In this case, one can derive (1.3), hence (1.2), even if one cannot show (1.1). On some lattices,
one can add a potential to $H_{\text{disc}, h}$, in which case, one can argue the convergence of observables in scattering phenomena, e.g. the S-matrix.

We pick up a characteristic root of $\frac{1}{\hbar^\nu} (\Delta - \Gamma - E_0)$ and pass to the gauge transformation to derive the convergence of characteristic root $\lambda_h(\xi)$ to $P(\xi)$. By projecting to the associated eigenspace, one derives the desired convergence. The gauge transformation is inspired by the expansion around Dirac points for the hexagonal lattice, hence our method covers carbonic lattices like graphene, graphite and the Kagome lattice. In this case, the associated continuous system is the two-dimensional massless Dirac operator. Discrete systems related to the square lattice, e.g. ladders or subdivisions, are also dealt with. In particular, for the case of square and triangular lattices, also for ladders, one can add compactly supported potentials.

Given an $h$-independent lattice Hamiltonian $\mathcal{L}(S)$, where $S = (S_1, \ldots, S_d)$ is a shift operator (see §2), one first chooses a reference energy $E_0$, and consider the scaled Hamiltonian

$$\mathcal{L}_h(S_h) = \frac{1}{\hbar^\nu} (\mathcal{L}(S_h) - E_0).$$

One should note that the scaling order $\nu$ depends on the energy region. For example, in the case of the hexagonal lattice, $\nu$'s are different near the middle of the spectrum and near the end points of the spectrum. This is due to the behavior of characteristic roots near the local extremal points.

The proof is based on the compactness argument in elementary topology: A pre-compact sequence $\{y_i\}$ in a complete metric space $Y$ having a unique accumulation point is convergent in $Y$. This basic argument has been used very often in the study of the continuous spectrum of Schrödinger operators in $L^2(\mathbb{R}^d)$. Let $H = -\Delta + V(x)$, and put $R(z) = (H - z)^{-1}$. To study the continuous spectrum of $H$, a first important step is the limiting absorption principle (LAP), i.e. the existence of the limit

$$R(E \pm i0) = \lim_{\epsilon \to 0} R(E \pm i\epsilon) : X \to Y, \quad E \in \sigma_c(H)$$

for suitable Banach spaces $X, Y$ rigging $L^2(\mathbb{R}^d)$, i.e. $X \subset L^2(\mathbb{R}^d) \subset Y$. The classical work of Eidus [8] proved the LAP by using the above compactness argument, and the uniqueness of solutions for Schrödinger equations satisfying the radiation condition played an important role. The LAP has been extended to more general differential operators by Agmon [1], Kato-Kuroda [19], [20], Jäger [17], Ikebe-Saito [16], Agmon-Hörmander [2] by using Fourier analysis, abstract operator theory, or integration by parts machinery. The commutator method of Mourre [21] is apparently different, however, it can be rewritten into the above mentioned form. The LAP is also valid for discrete Schrödinger operators. We can derive uniform estimates with respect to $0 < h < h_0$ of $u_h$ and the radiation condition for the discrete equation [3]. We shall use this argument also in the passage from discrete to continuous. We define $\tilde{u}_h(x)$ by (2.4). Using uniform estimates, we can show that $\{\tilde{u}_h(x)\}$ has the unique accumulation point as $h \to 0$, which guarantees that $\tilde{u}_h(x)$ itself converges to the unique solution $\tilde{u}$ to $(P(D_x) - E)\tilde{u} = f$.

In §2, we review basic assumptions for the lattices studied in [3]. In §3, we summarize the conditions on the characteristic roots of a discrete system needed for the passage to continuous system. In §4, we study the free system (the case without
potential) in a general form. The main results are Theorems 4.6 and 4.7. In §5, we study the case with potential also in a general form. In §6, we study the complex energy case. In §7 we apply these theorems to the square and triangular lattices, and also to the ladder and subdivision of square lattices. For these cases, one can add a potential, hence the solutions to lattice Schrödinger equations converge to those for the continuum Schrödinger operator $-\Delta + V(x)$. We then prove that the S-matrix for the continuum model is approximated by that for the discrete model. In §8, we consider the hexagonal lattice, the graphite lattice and the Kagome lattice. Here, we derive the Dirac equation as well as the Schrödinger equation. Some technical lemmas are proved in the Appendix.

Ignat-Zuazua [14] proved that for the time-dependent Schrödinger equation, the usual approximation scheme does not converge in some topologies and proposed a new approximation scheme. This scheme they then used to obtain fundamental estimates used in the study of the continuum limit of non-linear Schrödinger equations. The continuum limit of non-linear discrete Schrödinger operators was studied by Hong and Yang [10] employing mesh-uniform Strichartz estimates. See this paper for further references to the continuum limit of non-linear discrete Schrödinger operators. Nakamura-Tadano [22] proved the norm convergence of resolvents of Schrödinger operators for complex energies. Dirac operators are often used to study the spectral structure of graphene. See e.g. [9], [5], [6]. The resolvent of a discrete Schrödinger operator may have singularities at interior points of the continuous spectrum. For the discrete Laplacian on the square lattice the singularity structure was investigated in [15].

We use a standard notation. For $a \in \mathbb{R}$, $[a]$ denotes the greatest integer not exceeding $a$. For Banach spaces $X$, $Y$, $B(X,Y)$ is the space of all bounded operators from $X$ to $Y$. For $x \in \mathbb{R}^d$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, where $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. For an interval $I \subset \mathbb{R}$ and a Hilbert space $h$, $L^2(I, h, \rho(t)dt)$ denotes the set of $h$-valued $L^2$-functions on $I$ with respect to the measure $\rho(t)dt$.

The notation $P(x, D_x)$ denotes the pseudo-differential operator

$$P(x, D_x)u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} P(x, \xi)(\mathcal{F}_{\text{cont}}u)(\xi)d\xi,$$

where $\mathcal{F}_{\text{cont}}$ is defined in (2.5).

2. Preliminaries

2.1. Fourier transform. We consider the square lattice with mesh size $h$, i.e.

$$\mathbb{Z}_h^d = \{hn ; n \in \mathbb{Z}^d\}.$$

We equip $L^2(\mathbb{Z}_h^d)$ with norm

$$\|a\|_{L^2(\mathbb{Z}_h^d)} = h^{d/2} \left( \sum_{n \in \mathbb{Z}^d} |a_n|^2 \right)^{1/2}.$$  \hspace{1cm} (2.1)

Put

$$I_{hn} = hn + \left[ -\frac{h}{2}, \frac{h}{2} \right].$$
For $a = (a_n) \in L^2(Z_h^d)$, we define $f_a(x) \in L^2(R^d)$ by

$$f_a(x) = a_n, \quad \text{if } x \in I_{hn}.$$ 

Then by the definition (2.1), we have

$$\int_{R^d} |f_a(x)|^2 dx = \|a\|^2_{L^2(Z_h^d)}.$$

In the following $C$'s denote constants independent of $0 < h < h_0$, where $h_0 > 0$ is a suitable fixed small constant.

Let $S(Z_h^d)$ be the space of rapidly decreasing sequences on $Z_h^d$:

$$S(Z_h^d) \ni a = (a_n)_{n \in Z^d} \iff |a_n| \leq C_k(n)^{-k}, \quad \forall n \in Z^d, \quad \forall k \geq 0.$$

Its dual space is denoted by $S'(Z_h^d)$. It is embedded in $S'(R^d)$ by

$$S'(Z_h^d) \ni a \to \sum_{n \in Z^d} a_n \delta(x - hn) \in S'(R^d),$$

where $\delta(x)$ denotes the Dirac measure supported at $0 \in R^d$. Let $T_h^d$ be the $d$-dimensional torus of size $2\pi/h$:

$$T_h^d = (S_h^1)^d = \left[ -\frac{\pi}{h}, \frac{\pi}{h} \right]^d, \quad S_h^1 = \{ e^{ih\theta}; \quad \frac{\pi}{h} \leq \theta \leq \frac{\pi}{h} \}.$$

We use the following three symbols to denote functions on $Z_h^d$, $T_h^d$ and $R^d$:

(2.2) \quad \quad u_h = (u_h(n)) \quad \text{on} \quad Z_h^d,

(2.3) \quad \quad \tilde{u}_h(\xi) = \left( \frac{h}{2\pi} \right)^{d/2} \sum_{n \in Z^d} e^{-ihn \cdot \xi} u_h(n) \quad \text{on} \quad T_h^d,

(2.4) \quad \quad \tilde{\tilde{u}}_h(x) = \left( \frac{h}{2\pi} \right)^{d/2} \int_{T_h^d} e^{ix \cdot \xi} \tilde{u}_h(\xi) d\xi \quad \text{on} \quad R^d.

For $f \in S(R^d)$ consider the series of transformations:

$$f \Longrightarrow f_h \Longrightarrow \tilde{f}_h \Longrightarrow \tilde{\tilde{f}}_h,$$

where $f_h(n) = f(hn)$. Then $\tilde{f}_h$ is an interpolation of $f_h$, and $\tilde{\tilde{f}}_h \to f$ as $h \to 0$. In fact, since

$$\tilde{f}_h(x) = (2\pi)^{-d} \int_{T_h^d} \left( \sum_n e^{-ihn \cdot \xi} f(hn) h^d \right) d\xi,$$

it converges to

$$(2\pi)^{-d/2} \int_{R^d} e^{ix \cdot \xi} (\mathcal{F}_{\text{cont}} f)(\xi) d\xi$$

in the sense of distribution, where $\mathcal{F}_{\text{cont}}$ is the Fourier transformation on $R^d$:

(2.5) \quad (\mathcal{F}_{\text{cont}} f)(\xi) = (2\pi)^{-d/2} \int_{R^d} e^{-ix \cdot \xi} f(x) dx.$

Note that

$$\tilde{\tilde{f}}_h(x) = \sum_{n \in Z^d} \left( \prod_{j=1}^d \frac{\sin \frac{\pi}{h}(x_j - hn_j)}{\frac{\pi}{h}(x_j - hn_j)} \right) f(hn),$$

and the right-hand side is called the cardinal series (see [23]).
Define the discrete Fourier transform of \( a_h = (a_{h,n}) \in S(\mathbb{Z}^d_h) \) by
\[
(\mathcal{F}_{\text{disc}} a_h)(\xi) = \hat{a}_h(\xi) = \left( \frac{h}{2\pi} \right)^{d/2} \sum_{n \in \mathbb{Z}^d} a_{h,n} e^{-ihn \cdot \xi}.
\]

The inverse Fourier transform is
\[
a_h = (a_{h,n}) = (\mathcal{F}_{\text{disc}})^{-1} \hat{a}_h, \quad a_{h,n} = \left( \frac{h}{2\pi} \right)^{d/2} \int_{T_h^d} e^{ihn \cdot \xi} \hat{a}_h(\xi) d\xi.
\]

Since \( \left\{ \left( \frac{h}{2\pi} \right)^{d/2} e^{-ihn \cdot \xi}; \ n \in \mathbb{Z}^d \right\} \) is an orthonormal basis of \( L^2(T_h^d) \), we have the following lemma.

**Lemma 2.1.** The discrete Fourier transform
\[
(\mathcal{F}_{\text{disc}} : L^2(\mathbb{Z}^d_h) \ni a_h \rightarrow \hat{a}_h = \left( \frac{h}{2\pi} \right)^{d/2} \sum_{n \in \mathbb{Z}^d} a_{h,n} e^{-ihn \cdot \xi} \in L^2(T_h^d) \)
\]
is a bijection, in particular, it is isometric in the following sense:
\[
|h^{-d}\|a_h\|_{L^2(\mathbb{Z}^d_h)}^2 = \sum_{n \in \mathbb{Z}^d} |a_{h,n}|^2 = \|\hat{a}_h\|_{L^2(T_h^d)}^2 = h^{-d}\|\hat{a}_h\|_{L^2(\mathbb{R}^d)}^2.
\]

In particular, we have
\[
\|a_h\|_{L^2(\mathbb{Z}^d_h)} = \|\hat{a}_h\|_{L^2(\mathbb{R}^d)}.
\]

The shift operators \( S_{h,j} \) on \( S'(\mathbb{R}^d) \) and \( S'(\mathbb{Z}^d_h) \) are defined by
\[
(S_{h,j} f)(x) = f(x - he_j),
\]
\[
(S_{h,j} u_h)(n) = u_h(n - e_j),
\]
where
\[
e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)
\]
is the standard basis of \( \mathbb{R}^d \). We put
\[
S_h = (S_{h,1}, \ldots, S_{h,d}).
\]

Then we have
\[
\mathcal{F}_{\text{cont}} S_{h,j} = e^{-ih\xi_j} \mathcal{F}_{\text{cont}},
\]
\[
\mathcal{F}_{\text{disc}} S_{h,j} = e^{-ih\xi_j} \mathcal{F}_{\text{disc}}.
\]

**2.2. Function spaces.** First we recall the Agmon-Hörmander space \((\mathcal{B}, [12])\), which is a Besov space. Let
\[
\Omega_0 = \{ |x| < 1 \}, \quad \Omega_j = \{ 2^{j-1} < |x| < 2^j \}, \quad j \geq 1,
\]
\[
\mathcal{B}(\mathbb{R}^d) \ni u \iff \sum_{j=0}^{\infty} 2^{j/2} \| u \|_{L^2(\Omega_j)} < \infty,
\]
\[
\mathcal{B}^*(\mathbb{R}^d) \ni u \iff \sup_{R > 1} \frac{1}{R} \int_{|x| < R} |u(x)|^2 \, dx < \infty.
\]

For \( f, g \in \mathcal{B}(\mathbb{R}^d) \), we define
\[
f \simeq g \iff \lim_{R \to \infty} \frac{1}{R} \int_{|x| < R} |\mathcal{F}_{\text{cont}} f(\xi) - \mathcal{F}_{\text{cont}} g(\xi)|^2 \, d\xi = 0.
\]
Finally, we define for \( s \in \mathbb{R} \)
\[
L^{2,s}(\mathbb{R}^d) \ni u \iff \langle x \rangle^s u \in L^2(\mathbb{R}^d),
\]
\[
H_m(\mathbb{R}^d) \ni u \iff \langle \xi \rangle_m \mathcal{F}_{cont} u(\xi) \in L^2(\mathbb{R}^d),
\]
\[
H^{m,s}(\mathbb{R}^d) \ni u \iff \langle x \rangle^s u \in H^m(\mathbb{R}^d).
\]
We consider the discrete analogues of these spaces. For \( s \in \mathbb{R} \), we define
\[
L^{2,s}(Z^d_h) \ni u \iff \| u \|_{L^{2,s}(Z^d_h)} = \left( h^d \sum_{n \in \mathbb{Z}^d} \langle hn \rangle^s |u(n)|^2 \right)^{1/2}.
\]
Define
\[
\Omega_{h,0} = \{ n \in \mathbb{Z}^d; |hn| \leq 1 \},
\]
\[
\Omega_{h,\ell} = \{ n \in \mathbb{Z}^d; 2^{\ell-1} < |hn| \leq 2^\ell \}, \quad \ell \geq 1.
\]
Define \( \mathcal{B}(Z^d_h) \) by
\[
\mathcal{B}(Z^d_h) \ni u(n) \iff \sum_{\ell=0}^{\infty} 2^{\ell/2} \| u \|_{L^2(\Omega_{h,\ell})} < \infty,
\]
where \( \| u \|_{\Omega_{h,\ell}} = \left( h^d \sum_{n \in \Omega_{h,\ell}} |u(n)|^2 \right)^{1/2} \). The norm of the dual space of \( \mathcal{B}(Z^d_h) \) should be \( \sup_{\ell \geq 0} 2^{-\ell/2} \| u \|_{L^2(\Omega_{h,\ell})} \). However, by Lemmas 10.2 and 10.3 there exists a constant \( C > 0 \) independent of \( 0 < h < 1 \) such that
\[
C \left( \sup_{R>1} \frac{h^d}{R} \sum_{|hn| \leq R} |u(n)|^2 \right)^{1/2} \leq \sup_{\ell \geq 0} 2^{-\ell/2} \| u \|_{L^2(\Omega_{h,\ell})} \leq C^{-1} \left( \sup_{R>1} \frac{h^d}{R} \sum_{|hn| \leq R} |u(n)|^2 \right)^{1/2}.
\]
Therefore we employ
\[
\| u \|_{\mathcal{B}_2(Z^d)} = \left( \sup_{R>1} \frac{h^d}{R} \sum_{|hn| \leq R} |u(n)|^2 \right)^{1/2}
\]
as the norm of \( \mathcal{B}^*(Z^d_h) \). We define
\[
\mathcal{B}(T^d_h) \ni u \iff \left( \mathcal{F}_{disc,h} \right)^{-1} u \in \mathcal{B}(Z^d_h),
\]
\[
\mathcal{B}^*(T^d_h) \ni u \iff \left( \mathcal{F}_{disc,h} \right)^{-1} u \in \mathcal{B}^*(Z^d_h).
\]
An invariant way of defining the Besov spaces \( \mathcal{B} \) and \( \mathcal{B}^* \) on \( T^d_h \) is to use the Laplacian \( -\Delta_\xi = -\sum_{i=1}^{d} (\partial / \partial \xi_i)^2 \) on \( T^d_h \), which has eigenvalues \( (hn)^2 \) and eigenvectors \( (\frac{h}{2\pi})^{d/2} e^{-ihn \cdot \xi}, n \in \mathbb{Z}^d \). We then define
\[
\mathcal{B}(T^d_h) \ni u(\xi) \iff \sum_{\ell \geq 0} 2^{\ell/2} h^d/2 \| \chi_\ell(\sqrt{-\Delta_\xi}) u \| < \infty,
\]
where \( \chi_\ell \) is the characteristic function of the interval \( (c_{\ell-1}, c_\ell) \), \( c_{-1} = 0, \ c_\ell = 2^\ell, \ \ell \geq 0, \) and
\[
\mathcal{B}^*(T^d_h) \ni u(\xi) \iff \sup_{R>1} \frac{h^d}{R} \| \chi_R(\sqrt{-\Delta_\xi}) u \|^2 < \infty,
\]
where \( \chi_R \) is the characteristic functions of the interval \( (0, R) \).
2.3. **Lattice Schrödinger operators.** We introduce the mesh size parameter \( h > 0 \) in the definition of the lattice in [3]. Let \( \mathbf{L}_h \) be a lattice of rank \( d \) in \( \mathbb{R}^d \) \((d \geq 2)\) with basis \( v_j, j = 1, \ldots, d, \) and mesh size \( h, \) i.e.

\[
\mathbf{L}_h = \{ v(hn) ; n \in \mathbb{Z}^d \}, \quad v(hn) = \sum_{j=1}^{d} h n_j v_j, \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}^d.
\]

For \( h = 1, \) \( \mathbf{L}_h \) is denoted by \( \mathbf{L} \). Take points \( p_j \in \mathbb{R}^d, \) \( j = 1, \ldots, s, \) satisfying

\[
p_i - p_j \notin \mathbf{L}, \quad \text{if} \quad i \neq j,
\]

and define the vertex set \( \mathbf{V}_h \) by

\[
\mathbf{V}_h = \bigcup_{j=1}^{s} (hp_j + \mathbf{L}_h).
\]

For \( h = 1, \) \( \mathbf{V}_1 \) is denoted by \( \mathbf{V} \). There exists a bijection \( \mathbf{V} \ni a \rightarrow (j(a), n(a)) \in \{1, \ldots, s\} \times \mathbb{Z}^d \) such that

\[
a = p_{j(a)} + v(n(a)).
\]

The group \( \mathbb{Z}^d \) acts on \( \mathbf{V}_h \) as follows:

\[
\mathbb{Z}^d \times \mathbf{V}_h \ni (m, a) \rightarrow m \odot a := hp_{j(a)} + hv(m + n(a)) \in \mathbf{V}_h.
\]

The edge set \( \mathbf{E}_h \) is a subset of \( \mathbf{L}_h \times \mathbf{L}_h \) having the property

\[
\mathbf{E}_h \ni (a, b) \rightarrow (m \odot a, m \odot b) \in \mathbf{E}_h, \quad \forall m \in \mathbb{Z}^d.
\]

Then the triple \( \Gamma_h = \{ \mathbf{L}_h, \mathbf{V}_h, \mathbf{E}_h \} \) is a periodic graph in \( \mathbb{R}^d \) with mesh size \( h. \) As above, for \( h = 1, \) \( \Gamma_h \) is denoted by \( \Gamma = \{ \mathbf{L}, \mathbf{V}, \mathbf{E} \}. \) For \( a, b \in \mathbf{V}_h, \) \( a \sim b \) means that they are on the mutually opposite end points of an edge in \( \mathbf{E}_h, \)

and \( \deg(v) \) of \( v \in \mathbf{V}_h \) is the number of vertices adjacent to \( v. \) Then \( \deg(p_i + v(n)) \) depends only on \( i, \) which is denoted by \( \deg(i). \) In the following we assume that

\[
\deg(i) \text{ does not depend on } i, \text{ and is denoted by } d_g.
\]

Any function \( f \) on \( \mathbf{V}_h \) is written as \( f(n) = (f_1(n), \ldots, f_s(n)), \) \( n \in \mathbb{Z}^d, \) where \( f_j(n) \) is identified with a function on \( p_j + \mathbf{L}_0. \) Hence, \( L^2(\mathbf{V}_h) \) is the Hilbert space with inner product

\[
(f, g)_{L^2(\mathbf{V}_h)} = \sum_{j=1}^{s} (f_j, g_j)_{d_g},
\]

where

\[
(2.15) \quad (f_j, g_j)_{d_g} = d_g(f_j, g_j)_{L^2(\mathbf{L})}.
\]

The Laplacian \( \Delta_{\Gamma_h} \) on \( \Gamma_h \) is defined by the mean over the adjacent vertices

\[
(2.16) \quad (\Delta_{\Gamma_h} f)_j(n) = \frac{1}{d_g} \sum_{b \sim p_j + v(n)} f_{i(b)}(n(b)).
\]

We then define a unitary operator \( \mathcal{U}_{\Gamma_h} : L^2(\mathbf{V}_h) \rightarrow L^2(\mathbf{T}_h^d)^s \) by

\[
(\mathcal{U}_{\Gamma_h} f)_j = \sqrt{d_g} F_{\text{disc}, h} f_j,
\]

where \( F_{\text{disc}, h} f_j \) is the discrete Fourier transform of \( f_j \) with respect to the lattice \( \mathbf{L}_h. \)
where $L^2(T^d_\hbar)^s$ is equipped with the inner product
\[ (f,g)_{L^2(T^d_\hbar)} = \sum_{j=1}^s \int_{T^d_\hbar} f_j(\xi) \overline{g_j(\xi)} d\xi. \]

Note that the Laplacian is written as
\[ -\Delta_{T_\hbar} = \mathcal{L}(S_\hbar, S_\hbar^*) \]
where $\mathcal{L}(z,w)$ is a matrix whose entries are polynomials in $z,w \in \mathbb{C}^d$. Passing to the Fourier series, it is transferred to
\[ (2.17) \quad \mathcal{U}_{T_\hbar} \mathcal{L}(S_\hbar, S_\hbar^*) (\mathcal{U}_{T_\hbar})^{-1} = \mathcal{L}(e^{-i\hbar \xi}, e^{i\hbar \xi}), \]
where we use the notation
\[ e^{\pm i\hbar \xi} = (e^{\pm i\hbar \xi_1}, \ldots, e^{\pm i\hbar \xi_d}). \]

Then the spectrum of the operator $H_\hbar = -\Delta_{T_\hbar}$ is equal to that of the operator of multiplication by the function $\mathcal{L}(e^{-i\hbar \xi}, e^{i\hbar \xi})$ on $L^2(T^d_\hbar)^s$. Let us call $\mathcal{L}(e^{-i\hbar \xi}, e^{i\hbar \xi})$ the symbol of $-\Delta_{T_\hbar}$. The characteristic root of $\mathcal{L}(e^{-i\hbar \xi}, e^{i\hbar \xi})$ is a root $\lambda = \lambda_\hbar(\xi)$ of the equation $\det (\mathcal{L}(e^{-i\hbar \xi}, e^{i\hbar \xi}) - \lambda) = 0$, and the characteristic surface is the set $\{ \xi; \det (\mathcal{L}(e^{-i\hbar \xi}, e^{i\hbar \xi}) - \lambda) = 0 \}$.

We recall the assumptions on the lattice in [3]. We have only to state them for the case $\hbar = 1$, i.e. for $\Gamma$. Let $\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots \leq \lambda_\sigma(\xi)$ denote the eigenvalues of $\mathcal{L}(e^{-i\xi}, e^{i\xi})$. We put
\[ T^d = \mathbb{R}^d/(2\pi \mathbb{Z})^d, \quad T^d_C = \mathbb{C}^d/(2\pi \mathbb{Z})^d, \]
\[ p(\xi,\lambda) = \det (\mathcal{L}(e^{-i\xi}, e^{i\xi}) - \lambda), \]
\[ M_\lambda = \{ \xi \in T^d; p(\xi,\lambda) = 0 \}, \]
\[ M_{\lambda,j} = \{ \xi \in T^d; \lambda_j(\xi) = \lambda \}, \]
\[ M_\lambda^C = \{ z \in T^d_C; p(z,\lambda) = 0 \}, \]
\[ M_{\lambda,\text{reg}}^C = \{ z \in M_\lambda^C; \nabla_z p(z,\lambda) \neq 0 \}, \]
\[ M_{\lambda,\text{sng}}^C = \{ z \in M_\lambda^C; \nabla_z p(z,\lambda) = 0 \}, \]
\[ \bar{T} = \{ \lambda \in \sigma(H); M_{\lambda,\text{sng}}^C \cap T^d \neq \emptyset \}. \]

The assumptions are as follows. Let $H = -\Delta_{\Gamma}$.

**A-1** There exists a subset $T_1 \subset \sigma(H)$ such that for $\lambda \in \sigma(H) \setminus T_1$:

- **A-1-1** $M_{\lambda,\text{sng}}^C$ is discrete.
- **A-1-2** Each connected component of $M_{\lambda,\text{reg}}^C$ intersects with $T^d$ and the intersection is a $(d-1)$-dimensional analytic submanifold of $T^d$.

**A-2** There exists a finite set $T_0 \subset \sigma(H)$ such that
\[ M_{\lambda,i} \cap M_{\lambda,j} = \emptyset, \quad i \neq j, \quad \lambda \in \sigma(H_0) \setminus T_0. \]

**A-3** $\nabla_\xi p(\xi,\lambda) \neq 0$ on $M_\lambda$ for $\lambda \in \sigma(H) \setminus T_0$.

**A-4** The unique continuation property holds for $\Delta_{\Gamma}$ in $V$.

By the unique continuation property we mean the following assertion:
Assume \( u \) satisfies \((-\Delta - \lambda)u = 0\) on \( V \) for some constant \( \lambda \in \mathbb{C} \). If there exists \( R_0 > 0 \) such that \( u = 0 \) for \( |n| > R \), then \( u = 0 \) on \( V \).

The assumption on the potential is as follows:

(V-1) The potential \( V_{\text{disc},h} \) is a scalar multiplication operator defined by
\[
(V_{\text{disc},h}u)(n) = V(hn)u(n),
\]
where \( V(x) \) is a real-valued compactly supported continuous function on \( \mathbb{R}^d \).

Under these assumptions, for any \( 0 < h < h_0 \), we have the limiting absorption principle, spectral representation, completeness of wave operators, unitarity of the S-matrix for \(-\Delta_{\Gamma} + V_{\text{disc},h}\). Moreover, we can solve inverse scattering problems (see [4]).

3. Expansion at local extremal points

3.1. Local extremals.

In the sequel we use results from perturbation theory for matrices depending on several complex parameters. See [7] for these results.

For a subset \( K \subset \mathbb{R}^d \), we put
\[
K/h = \{ \xi/h ; \xi \in K \}.
\]

Let \( \Delta_{\Gamma} \) be the Laplacian in the previous section, and take \( E_0 \in \widetilde{T} \). Then at \( \lambda = E_0 \), \( p(\xi, \lambda) = \det(L(e^{-i\xi}, e^{i\xi}) - \lambda) \) may have multiple roots or, some simple characteristic root \( \lambda_j(\xi) \) satisfies \( \nabla_\xi \lambda_j(\xi) = 0 \) at some point in the characteristic surface \( \{ \xi ; \lambda_j(\xi) = E_0 \} \). We consider the behavior of solutions of the equation
\[
(-\frac{1}{h^\nu}\Delta_{\Gamma} + V_{\text{disc},h} - \lambda)u_h = f_h \text{ as } h \to 0, \text{ when } \lambda \text{ is close to } E_0/h^\nu.
\]
We call \( E_0 \) the reference energy. For the sake of simplicity of notation, we shift the Hamiltonian and consider
\[
\Delta_{\text{disc},h} = \frac{1}{h^\nu}(\Delta_{\Gamma} + E_0)
\]
instead of \( \Delta_{\Gamma} \). Then the symbol of \(-\Delta_{\text{disc},h}\) has the following form
\[
\mathcal{L}_h(z, w) = \frac{1}{h^\nu}(\mathcal{L}(z, w) - E_0I);
\]
\( I \) being the \( s \times s \) identity matrix.

We use the abbreviation:
\[
\mathcal{L}_h(e^{-ih\eta}) := \mathcal{L}_h(e^{-ih\eta}, e^{ih\eta}).
\]
For \( \eta \in \mathbb{R}^d \) let
\[
\lambda_{1,h}(\eta) \leq \cdots \leq \lambda_{s,h}(\eta)
\]
be the characteristic roots of \( \mathcal{L}_h(e^{-ih\eta}, e^{ih\eta}) \). They are rewritten as
\[
\lambda_{j,h}(\eta) = \frac{1}{h^\nu} \lambda_j(e^{-ih\eta}),
\]
where \( \lambda_j(z) \) is the characteristic root of \( \mathcal{L}(z, z) - E_0I \). Their behavior at local extremal points plays an important role. We consider only the minimum case, since the maximum case can be dealt with similarly. We restrict ourselves to two cases: (1) simple isolated roots and (2) double roots.
We assume for some \( j, 1 \leq j \leq s \), there exists an \( h \)-independent open set \( \mathcal{K}_0 \) in \( T^d \) with the following properties.

**(B-1)** \( \lambda_j(e^{-in}) \geq 0 \) on \( \mathcal{K}_0 \), and there exists a unique \( d_1 \in \mathcal{K}_0 \) such that \( \lambda_j(e^{-id_1}) = 0 \).

We assume one of the following conditions (B-2-1) or (B-2-2):

**(B-2-1)** There exist constants \( \epsilon_1, \epsilon_2 > 0 \) such that

\[
\lambda_{j-1}(e^{-in}) < -\epsilon_1 < \lambda_j(e^{-in}) < \epsilon_2 < \lambda_{j+1}(e^{-in}), \quad \forall \eta \in \overline{\mathcal{K}_0}.
\]

**(B-2-2)** \( \lambda_{j-1}(e^{-in}) = -\lambda_j(e^{-in}) \) on \( \mathcal{K}_0 \) and there exist constants \( \epsilon_1, \epsilon_2 > 0 \) such that

\[
\lambda_{j-2}(e^{-in}) < -\epsilon_1 < \lambda_{j-1}(e^{-in}) \leq \lambda_j(e^{-in}) < \epsilon_2 < \lambda_{j+1}(e^{-in}), \quad \forall \eta \in \overline{\mathcal{K}_0}.
\]

In both cases we put

\[
\mathcal{K} = \mathcal{K}_0 - d_1 = \{ \eta - d_1 ; \eta \in \mathcal{K}_0 \},
\]

**(3.4)**

\[
P_h(\xi) = \lambda_{j,h}(\xi + d_h), \quad d_h = d_1/h,
\]

and assume

**(B-3)** For \( 0 \neq \xi \in \mathcal{K}/h \) the limit

**(3.5)**

\[
P_h(\xi) \to P(\xi),
\]

together with that of all of its derivatives, exists as \( h \to 0 \), where \( P(\xi) \) is \( C^\infty \) for \( \xi \neq 0 \), homogeneous of degree \( \gamma > 0 \) and

\[
CP(\xi) \leq P_h(\xi) \leq C^{-1}P(\xi), \quad \text{on } \mathcal{K}/h
\]

for a constant \( C > 0 \).

Let \( \Pi_h(\xi) \) be the eigenprojection associated with the eigenvalue \( \lambda_{j,h}(\xi + d_h) \) for the case (B-2-1), and the sum of eigenprojections associated with \( \lambda_{j-1,h}(\xi + d_h) \) and \( \lambda_{j,h}(\xi + d_h) \) for the case (B-2-2). Since \( \lambda_{j,h} \) (or the pair \( \{ \lambda_{j-1,h}, \lambda_{j,h} \} \)) is isolated from the other eigenvalues, \( \Pi_h(\xi) \) is smooth for \( \xi \neq 0 \). We assume:

**(B-4)** For \( 0 \neq \xi \in \mathcal{K}/h \) there exists a projection \( \Pi_0(\xi) \) such that

\[
\Pi_h(\xi) \to \Pi_0(\xi),
\]

as \( h \to 0 \), together with all derivatives.

In our applications, the resolvent of the matrix \( L_h(e^{-i(\xi + d_h)}) \) converges together with its derivatives. Therefore, by using Risez’ formula

\[
\frac{1}{2\pi i} \int_C (z - L_h(e^{-i(\xi + d_h)}))^{-1} dz
\]

for the projection associated with (a group of) eigenvalues, where \( C \) is the contour enclosing (the group of) eigenvalues in question, the assumption (B-4) is verified. See also [IS], pp. 568-569.

By (B-2-1) and (B-2-2) there exist constants \( 0 < C_0 < C_1 \) and \( 0 < C_0' < C_1' \) such that

\[
\sup_{\xi \in \mathcal{K}} \lambda_{j,h}(\xi) \leq \frac{C_0}{h^\nu} < \frac{C_1}{h^\nu} < \inf_{\xi \in \mathcal{K}} \lambda_{k,h}(\xi), \quad k \geq j + 1,
\]
for both cases (B-2-1) and (B-2-2), and also
\[ \sup_{\xi \in \mathcal{K}} \lambda_{j,h}(\xi) \leq \frac{C'_0}{h^\nu} < \frac{C'_1}{h^\nu} < \inf_{\xi \in \mathcal{K}} |\lambda_{k,h}(\xi)|, \]
for \( k \leq j - 1 \) for the case (B-2-1), and \( k \leq j - 2 \) for the case (B-2-2).

Take an \( h \)-independent open interval \( I \subset (0, \infty) \) and \( h_0 \) small enough so that \( I \subset (0, C/h^\nu) \), for \( 0 < h < h_0 \), where \( C = \min\{C_0, C'_0\} \). Then there exists a constant \( C' > 0 \) such that
\[ |\lambda_{k,h}(\xi) - \lambda| \geq \frac{C'}{h^\nu}, \quad \forall \lambda \in I \]
for \( k \neq j \) in the case (B-2-1), and for \( k \neq j, j - 1 \) in the case (B-2-2). This implies that
\[ |(\mathcal{L}_h(e^{-ih(\xi+d_h)}) - z)^{-1}(1 - \Pi_h(\xi))| \leq C'h^\nu, \]
where \(| \cdot |\) denotes the matrix norm, for \( \text{Re } z \in I \) and \( \xi \in \mathcal{K}/h \).

**Lemma 3.1.** For any \( 0 \neq \xi^{(0)} \in \mathcal{K}/h \), there exists an \( h \)-independent neighborhood \( U \subset (\mathcal{K}/h) \setminus \{0\} \) of \( \xi^{(0)} \) and a unitary matrix \( A_h(\xi) \in C^\infty(U) \) such that
\[ A_h(\xi)^* \mathcal{L}_h(e^{-ih(\xi+d_h)}) A_h(\xi) = \begin{pmatrix} \lambda_j(e^{-ih(\xi+d_h)}) & 0 \\ 0 & L_h(\xi) \end{pmatrix}, \quad \forall \xi \in U \]
for the case (B-2-1), and
\[ A_h(\xi)^* \mathcal{L}_h(e^{-ih(\xi+d_h)}) A_h(\xi) = \begin{pmatrix} \lambda_j(e^{-ih(\xi+d_h)}) & 0 & 0 \\ 0 & -\lambda_j(e^{-ih(\xi+d_h)}) & 0 \\ 0 & 0 & L_h(\xi) \end{pmatrix}, \quad \forall \xi \in U \]
for the case (B-2-2), where \( \mathcal{L}_h(\xi) \) is smooth with respect to \( \xi \in U \) in both cases.

**Proof.** We consider the case (B-2-2). Take an orthonormal basis \( v_1, \ldots, v_s \) of \( \mathbb{R}^s \) such that \( v_1, v_2 \in \text{Ran } \Pi_0(\xi^{(0)}) \), \( v_3, \ldots, v_s \in \text{Ran } (1 - \Pi_0(\xi^{(0)})) \), and choose \( w_i \in \mathbb{R}^s \) such that \( v_i = \Pi_0(\xi^{(0)}) w_i, i = 1, 2, \) \( v_i = (1 - \Pi_0(\xi^{(0)})) v_i, i = 3, \ldots, s \). Let \( A_{h_i}(\xi) = \Pi_h(\xi) w_i, i = 1, 2, \) \( A_{h_i}(\xi) = (1 - \Pi_h(\xi)) w_i, i = 3, \ldots, s \). Letting \( A_h(\xi) \) be the unitary matrix with column vectors \( A_{h_i}(\xi) \) and taking a sufficiently small neighborhood of \( \xi^{(0)} \), we obtain the lemma. \( \square \)

### 3.2. Gauge transformation.

We define the gauge transformation \( \mathcal{G}_h \) by
\[ (\mathcal{G}_h a)(n) = e^{ihn \cdot d_h} a(n). \]
Then we have
\[ \mathcal{F}_{\text{disc},h} \mathcal{G}_h \mathcal{L}_h(S_h) \mathcal{G}_h^* \mathcal{F}_{\text{disc},h}^{-1} = \mathcal{L}_h(e^{-ih(\xi+d_h)}). \]
Multiplying by \( \Pi_h(\xi) \) the problem is reduced to the multiplication operator \( \lambda_{j,h}(\xi + d_h) = P_h(\xi) \). We show that, as \( h \to 0 \), it converges to the (pseudo-differential) operator \( P(-i\partial_x) \) in an appropriate sense.

---

\(^1\)By \( A \subset \subset O \) for an open set \( O \), we mean \( \overline{A} \subset O \) and compact.
4. The free equation

4.1. Characteristic surfaces. We use the notation introduced in the previous section. Recall that \( P_h(\xi) = P(\xi) + O(h) \). For \( 0 \neq E \in I \) the hypersurfaces

\[
M_{E,h} = \{ \xi \in \mathbb{R}^d ; P_h(\xi) = E \}, \quad M_E = \{ \xi \in \mathbb{R}^d ; P(\xi) = E \},
\]

are compact. By (B-3) there exist constants \( C_0, h_0 > 0 \) such that for all \( E \in I \)

\[
M_{E,h} \subset \{ \xi \in \mathbb{R}^d ; C_0 < |\xi| < C_0^{-1} \}, \quad 0 < \forall h < h_0.
\]

Let \( \epsilon_d > 0 \) be such that \( \{|\xi| < 2\epsilon_d\} \subset K \). Take \( \chi_d \in C_0^\infty(\mathbb{R}^d) \) such that \( \chi_d(\xi) = 1 \) on \( |\xi| < \epsilon_d \) and \( \chi_d(\xi) = 0 \) on \( |\xi| > 2\epsilon_d \). Note that

\[
\chi_d(h\xi) = \chi_d(h\xi)\chi_{T^d}(h\xi),
\]

where

\[
\chi_{T^d}(\xi) = \begin{cases} 1, & \text{if } \xi \in [-\pi, \pi]^d, \\ 0, & \text{if } \xi \not\in [-\pi, \pi]^d. \end{cases}
\]

Define \( f'_h \) by

\[
f'_h = \mathcal{F}_{\text{disc},h}^{-1}(\chi_d(h\xi)\hat{f}_h(\xi)),
\]

where \( f_h(n) = f(hn) \), and consider the equation

\[
(-\Delta_{\text{disc},h} - z)v'_h = G_h f'_h,
\]

where \( G_h \) is defined in Subsection 3.2. Then \( v_h = G_h^*v'_h \) satisfies the gauge transformed equation

\[
G_h^*(\mathcal{L}_h(S_h) - z)G_h v_h = f'_h.
\]

Assuming that \( f \in L^2(\mathbb{R}^d) \cap C(\mathbb{R}^d) \) and \( (f(hn))_{n \in \mathbb{Z}^d} \in L^1(\mathbb{Z}^d) \), we can solve it when \( z \not\in \mathbb{R} \), i.e. we have

\[
\hat{v}_h(\xi, z) = (\mathcal{L}_h(e^{-ih(\xi + dh)}) - z)^{-1}\chi_d(h\xi)\hat{f}_h(\xi) \in L^2(T_h^d),
\]

\[
\hat{f}_h(\xi) = \frac{h}{2\pi} d/2 \sum_{n \in \mathbb{Z}^d} f(hn)e^{-ihn\xi} \in L^2(T_h^d) \cap C(T_h^d).
\]

Note that \( (f(hn))_{n \in \mathbb{Z}^d} \in L^2(\mathbb{Z}^d) \).

By (3.6) and (10.1), we have for \( m > [d/2], s > d/2, \)

\[
\| (1 - \Pi_h(\xi))\hat{v}_h(\xi, z) \|_{L^2(T_h^d)} \leq C h^s \| \hat{f}_h \|_{L^2(T_h^d)} \leq C h^{s/2} \| f \|_{m,s}.
\]

Therefore the part \( (1 - \Pi_h(\xi))\hat{v}_h(\xi, z) \) disappears as \( h \to 0 \). We consider the part \( \Pi_h(\xi)\hat{v}_h(\xi, z) \). Let us first consider the case (B-2-1). We put

\[
u_h(n, z) = \left( \mathcal{F}_{\text{disc},h}^{-1} \right)^{-1}\chi_d(h\xi)\frac{\Pi_h(\xi)\hat{f}_h(\xi)}{P_h(\xi) - z}
\]

\[
= \left( \frac{h}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} e^{ihn\xi} \chi_{T^d}(h\xi)\chi_d(h\xi) \frac{\Pi_h(\xi)\hat{f}_h(\xi)}{P_h(\xi) - z} d\xi.
\]
A natural interpolation of \((u_h(n, z))_{n \in \mathbb{Z}^d}\) on \(\mathbb{R}^d\) is

\[
\tilde{u}_h(x, z) = h^{-d/2} \mathcal{F}^{-1}_{\text{cont}} \left( \chi_{T^d} (h\xi) \chi_d (h\xi) \frac{\Pi_h (\xi) \hat{f}_h (\xi)}{P_h (\xi) - z} \right)
\]

(4.8)

\[
= \left( \frac{h}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \chi_{T^d} (h\xi) \chi_d (h\xi) \frac{\Pi_h (\xi) \hat{f}_h (\xi)}{P_h (\xi) - z} \, d\xi.
\]

Take \(\chi_1(\xi) \in C^\infty (\mathbb{R}^d)\) such that

\[
\chi_1(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq C_0/3 \text{ or } 3/C_0 \leq |\xi|, \\
0, & \text{if } 2C_0/3 \leq |\xi| \leq 2/C_0,
\end{cases}
\]

(4.9)

where \(C_0\) is from (4.11), and put

\[
\chi_2(\xi) = 1 - \chi_1(\xi).
\]

Recalling that \(P(\xi)\) is homogeneous of degree \(\gamma\) by (B-3), on the support of \(\chi_1(\xi)\)

\[
|P_h (\xi) - z| \geq C(1 + |\xi|^\gamma), \quad \text{for \ Re z} \in I \quad \text{and} \quad \xi \in \mathcal{K}/h.
\]

(4.10)

In view of (4.8) we put

\[
\tilde{u}_h^{(i)}(x, z) = h^{-d/2} \mathcal{F}^{-1}_{\text{cont}} \left( \chi_{T^d} (h\xi) \chi_i (\xi) \chi_d (h\xi) \frac{\Pi_h (\xi) \hat{f}_h (\xi)}{P_h (\xi) - z} \right) \in L^2 (\mathbb{R}^d), \quad i = 1, 2
\]

(4.11)

\[
\tilde{u}_h^{(i)}(\xi, z) = \chi_{T^d} (h\xi) \chi_i (\xi) \chi_d (h\xi) \frac{\Pi_h (\xi) \hat{f}_h (\xi)}{P_h (\xi) - z} \in L^2 (T^d_h), \quad i = 1, 2
\]

(4.12)

4.2. Convergence outside \(M_{E, h}\). For \(E \in I\) (4.10) implies

\[
\lim_{\epsilon \to 0} \tilde{u}_h^{(i)}(\xi, E + i\epsilon) = \tilde{u}_h^{(i)}(\xi, E + i0) = \chi_{T^d} (h\xi) \chi_1 (\xi) \chi_d (h\xi) \frac{\Pi_h (\xi) \hat{f}_h (\xi)}{P_h (\xi) - E},
\]

since \(P_h (\xi) - E \neq 0\). Let

\[
\tilde{v}^{(i)}(\xi, E + i0) = \chi_1 (\xi) \Pi_0 (\xi) \frac{(\mathcal{F}_{\text{cont} f} (\xi))(\xi)}{P(\xi) - E} \in L^2 (\mathbb{R}^d),
\]

\[
\tilde{w}^{(i)}(x, E + i0) = \mathcal{F}_{\text{cont} \left( \tilde{v}^{(i)}(\xi, E + i0) \right)} \in L^2 (\mathbb{R}^d).
\]

Theorem 4.1. Assume that \(f \in H^{m,s} (\mathbb{R}^d)\) for some \(m > [d/2] + 1, s > d\). Then

\[
\tilde{u}_h^{(i)}(x, E + i0) \to \tilde{v}^{(i)}(x, E + i0) \quad \text{in} \quad L^{2, -\delta} (\mathbb{R}^d)
\]

for any \(\delta > 0\).

Proof. We first show that for \(h \to 0\)

\[
\tilde{u}_h^{(i)}(x, E + i0) \to \tilde{w}^{(i)}(x, E + i0) \quad \text{in} \quad \mathcal{S}' (\mathbb{R}^d).
\]

In fact, by Lemma (4.11)

\[
\hat{f}_h (\xi) h^{d/2} = (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} f(n) e^{-ihn \cdot \xi} h^d \to (\mathcal{F}_{\text{cont} f} (\xi)) \chi_1 (\xi).
\]

The assumption (B-4) then yields (4.14).

Next we prove the convergence in \(L^{2, -\delta} (\mathbb{R}^d)\) by showing the following two facts:
(1) \( \{\tilde{u}^{(1)}_h(x, E + i0)\}_{0 < h < h_0} \) is bounded in \( H^\gamma(\mathbb{R}^d) \).

(2) Compactness and uniqueness of accumulation point.

We show the assertion (1). Using (10.10) and (10.11), we have
\[
\|\tilde{u}^{(1)}_h(x)\|^2_{H^\gamma(\mathbb{R}^d)} \leq C h^d \|\tilde{f}_h\|^2_{L^2(T^*_h)} \leq C \|f\|^2_{H^{m,s}(\mathbb{R}^d)}
\]
from (10.1) in Lemma 10.1. This proves (1).

Omitting \( E + i0 \), take any sequence \( \tilde{u}^{(1)}_h(x) \), \( i = 1, 2, \cdots, h_i \to 0 \). For \( \delta > 0 \) the result (1) implies that there exists a constant \( C > 0 \) independent of \( h_i \) such that
\[
\int_{|x| > R} \langle x \rangle^{-\delta} |\tilde{u}^{(1)}_h(x)|^2 dx \leq CR^{-\delta}.
\]
In the bounded domain \( \{|x| < R\} \), applying Rellich’s selection theorem, one can choose a subsequence \( \{\tilde{u}^{(1)}_{h_i}'(x)\} \) of \( \{\tilde{u}^{(1)}_h(x)\} \) which converges to some \( w \) in \( L^{2-s}(\mathbb{R}^d) \).

Note by Plancherel’s formula, letting \( (\ , \ ) \) be the \( L^2(\mathbb{R}^d) \)-inner product,
\[
(\tilde{u}^{(1)}_h', \varphi) = (\tilde{u}^{(1)}_h, \mathcal{F}_{\text{cont}}\varphi), \quad \forall \mathcal{F}_{\text{cont}}\varphi \subseteq C^\infty_0(\mathbb{R}^d),
\]
Letting \( h_i' \to 0 \), we have \( (w, \varphi) = (\tilde{u}^{(1)}_h, \mathcal{F}_{\text{cont}}\varphi) \). This proves that the limit \( w \) of the subsequence \( \tilde{u}^{(1)}_{h_i}'(x) \) does not depend on this subsequence, which proves the uniqueness of the accumulation point of \( \{\tilde{u}_h(x)\} \). Then \( \tilde{u}^{(1)}_h(x) \) itself converges to \( \tilde{u}^{(1)}(x) \) in \( L^{2-s}(\mathbb{R}^d) \).

4.3. **Uniform estimates near** \( M_{E,h} \). Let \( \chi_1(\xi) \) be as in (4.9), \( \chi_2(\xi) = 1 - \chi_1(\xi) \), and
\[
\Xi_2 = \text{supp} \chi_2(\xi).
\]
There exists a constant \( C > 0 \) such that
\[
C|\xi|^{\gamma} \leq P_h(\xi) \leq C^{-1}|\xi|^{\gamma} \quad \text{on} \quad \Xi_2,
\]
moreover, as \( h \to 0 \) \( P_h(\xi) \to P(\xi) \). Since \( P(\xi) \) is homogeneous, \( \nabla_\xi P(\xi) \neq 0 \) on \( \Xi_2 \).

Therefore \( \nabla P_h(\xi) \neq 0 \) on \( \Xi_2 \). Take \( \xi^{(0)} = (\xi_1^{(0)}, \cdots, \xi_d^{(0)}) \in \Xi_2 \) arbitrarily. Around \( \xi^{(0)} \) we make a linear change of variables, \( \xi \to \eta \), hence \( \xi^{(0)} \to \eta^{(0)} \), so that in the \( \eta \)-coordinates \( (1, 0) \) is an outward transversal direction to \( M_{E,h} \) at \( \eta^{(0)} \), and the following factorization holds near \( \eta^{(0)} \):
\[
P_h(\xi(\eta)) = E - i\epsilon = (\eta_1 - p_h(\eta', E + i\epsilon))q_h(\eta, E + i\epsilon),
\]
\[
\text{Im} p_h(\eta', E + i\epsilon) \geq 0, \quad q_h(\eta, E + i\epsilon) \neq 0.
\]
For the simplicity of notation, we write \( \xi \) instead of \( \eta \). Take small \( \delta > 0 \) so that \( \xi_1 \neq 0 \) for \( |\xi - \xi^{(0)}| < \delta \). Let
\[
z = E + i\epsilon,
\]
and assume that \( \text{Re } z \in I \) and \( \text{Im } z > 0 \). Take \( \chi(\xi) \in C^\infty_0(\mathbb{R}^d) \) such that
\[
\chi(\xi) = \begin{cases} 0 & \text{for } |\xi - \xi^{(0)}| \geq 2\delta/3, \\ 1 & \text{on } |\xi - \xi^{(0)}| \leq \delta/3, \end{cases}
\]
and put
\[
(4.15) \quad \tilde{g}_h(\xi, z) = \frac{\chi(\xi)}{q_h(\xi, z)} \chi_{T^d}(h\xi) \chi_{\Omega}(h\xi) \chi_{\text{cont}}(h\xi) \chi_{\text{cont}}(h\xi) \Pi_h(\xi) \tilde{f}_h(\xi). \]
for a constant

\[ (4.16) \]

Lemma 4.2.
The function \( \chi \)

Note that

\[ (4.18) \]

Proof.

(1) For

we have by (4.17)

\[ (4.19) \]

Multiplying by

Lemma 10.4 then yields

\[ (4.21) \]

we see that the Fourier transform with respect to \( x \),

\[ (4.17) \]

\( \tilde{v}_h(x, z) = h^{d/2} F_{\text{cont}}^{-1}(\tilde{v}_h(x, z)) \).

Note that

\[ (4.22) \]

\[ (4.23) \]

we then have by (4.17)

\[ (4.20) \]

we see that the Fourier transform with respect to \( x_1 \) of \( \tilde{v}_h(x, \cdot, z) = \tilde{g}_h(x, z) \).

Multiplying by \( h^{d/2} \), and letting \( F_{x' \rightarrow \xi'} \) be the Fourier transform with respect to \( x' \), we then have by (4.17)

\[ (4.21) \]

\( \tilde{v}_h(x_1, \cdot, \xi', z) = i \int_{-\infty}^{\infty} e^{i(x_1 - y_1)p_h(\xi', z)} g_h_1(y_1, \xi', z) dy_1. \)

we see that the Fourier transform with respect to \( x_1 \) of \( v_h_1(x_1, \xi', z) \) solves (4.19).

Multiplying by \( h^{d/2} \), and letting \( F_{x' \rightarrow \xi'} \) be the Fourier transform with respect to \( x' \), we then have by (4.17)

\[ (4.21) \]

\( \tilde{v}_h(x_1, x', z) = i h^{d/2} \int_{-\infty}^{\#} (F_{x' \rightarrow \xi'})^{-1} \left( e^{i(x_1 - y_1)p_h(\xi', z)} g_h_1(y_1, \xi', z) \right) dy_1. \)

Letting \( \tilde{g}_h(y_1, x', z) = h^{d/2} (F_{x' \rightarrow \xi'})^{-1}(g_h_1(y_1, \xi', z)) \), we have

\[ (4.22) \]

\[ (4.23) \]

Note that

\[ \| \tilde{g}_h \|_{L^2(\mathbb{R}^d)} \leq C \| \tilde{g}_h \|_{L^2(\mathbb{R}^d)} \leq C \| \tilde{g}_h \|_{L^2(\mathbb{R}^d)} \leq C h^{d} \| \tilde{g}_h(\xi) \|_{H^1(\mathbb{R}^d)} \leq C h^{d} \| \tilde{h} \|_{H^1(\mathbb{R}^d)}^2. \)
By Sobolev’s inequality
\[ |f(x)| \leq C\|f\|_{H^{m,s}(\mathbb{R}^d)}^{-s}, \quad m > \left[ \frac{d}{2} \right], \quad s > 0. \]

Then we have
\[ h^d\|\hat{f}_h\|_{L^2(T^d)}^2 = h^d\sum_n |hn|^2 f(hn)^2 \leq C\|f\|_{H^{m,s}(\mathbb{R}^d)}^2 h^d\sum_n |hn|^2 (1 + |hn|)^{-2s} \leq C\|f\|_{H^{m,s}(\mathbb{R}^d)}^2 \int_{\mathbb{R}^d} |x|^2 (1 + |x|)^{-2s} dx. \]

Therefore the right-hand side of (4.23) is dominated from above by \( C\|f\|_{H^{m,s}(\mathbb{R}^d)} \) if \( s > d/2 + 1 \). We have thus proven (4.18).

The assertion (2) is a consequence of (4.21), and we have
\[ (4.24) \quad \tilde{v}_h(x, E + i0) = i \int_{-\infty}^{x_1} e^{i(x_1-y_1)p_\epsilon(x',z)} \tilde{g}_h(y_1, x', E) dy_1. \]

The inequality (4.22) implies (4.19). ∎

**Definition 4.3.** For \( u(\xi) \in S'(\mathbb{R}^d) \), the wave front set \( WF^*(u) \) is defined as follows. For \( (\omega, \xi_0) \in \mathbb{S}^{d-1} \times \mathbb{R}^d \), \( (\omega, \xi_0) \notin WF^*(u) \) if there exist \( 0 < \delta < 1 \) and \( \chi(\xi) \in C_0^\infty(\mathbb{R}^d) \) such that \( \chi(\xi_0) = 1 \) and
\[ (4.25) \quad \lim_{R \to \infty} \frac{1}{R} \int_{|x| < R} |C_{\omega,\delta}(x)\mathcal{F}_{cont}^{-1}(\chi u)(x)|^2 dx = 0, \]
where \( C_{\omega,\delta}(x) \) is the characteristic function of the cone \( \{ x \in \mathbb{R}^d ; \omega \cdot x > \delta|x| \} \).

**Definition 4.4.** Let \( P_h(\xi) \) and \( P(\xi) \) be as in (3.24) and (3.5). Let \( \mathcal{P}^{(h)} \) be the set of \( \Psi DO's \) whose symbol \( p_-(x, \xi) \) satisfies the following conditions:

1. \( |\partial_\xi^\alpha \partial_\xi^\beta p_-(x, \xi)| \leq C_{\alpha,\beta}(x)^{-|\alpha|}, \quad \forall \alpha, \beta, \)
2. There exist constants \( 0 < a < b < \infty \) and \(-1 < \delta < 1 \) such that
\[ p_-(x, \xi) = 0 \quad \text{for} \quad |\xi| \notin (a, b), \]
3. \( p_-(x, \xi) = 0 \quad \text{for} \quad \frac{x}{|x|} \cdot \nabla_\xi P_h(\xi) > \delta, \quad |x| > 1. \]

We define \( \mathcal{P}_- \) in the same way with \( P_h(\xi) \) replaced by \( P(\xi) \).

Let us now recall our notation. For \( \hat{f}_h(\xi) \) defined by (4.6), we define
\[ \hat{F}_h(\xi) = \chi_{T^d}(h\xi) \chi_d(h\xi) \Pi_h(\xi) \hat{f}_h(\xi), \]
\[ (4.27) \quad \hat{U}_h(\xi, z) = \hat{u}_h^{(1)}(\xi, z) + \hat{u}_h^{(2)}(\xi, z). \]
This is a solution to the equation
\[ (P_h(\xi) - z) \hat{U}_h(\xi, z) = \hat{F}_h(\xi). \]
Hence
\[ (4.28) \quad \hat{U}_h(x, z) = h^{d/2} \mathcal{F}_{cont}^{-1}(\hat{U}_h(\xi, z)) \]
converges as \( z \to E + i0 \in I \).
Lemma 4.5. Assume that $m > \lceil d/2 \rceil$, $s > \max\{d/2 + 1, 5/2\}$. Let $\widetilde{U}_h(\xi, E + i0)$ be as above. Then

\begin{equation}
WF^*(\widetilde{U}_h) \subset \{(\omega_h(\xi), \xi) ; \xi \in M_{E,h}\},
\end{equation}

where $\omega_h(\xi) = \nabla_{\xi} P_h(\xi)/|\nabla_{\xi} P_h(\xi)|$. Moreover, for any $p_-(x, D_x) \in \mathcal{P}_-$ and any $0 < \alpha < 1/2$, there exist constants $C > 0$, $h' > 0$ such that

\begin{equation}
\|p_-(x, D_x)\widetilde{U}_h\|_{L^{2,-s}(\mathbb{R}^d)} \leq C\|f\|_{m,s}, \quad 0 < \forall h < h'.
\end{equation}

Proof. By \((4.27), \ (4.28)\), $\widetilde{U}_h(x, z)$ is split into $\widetilde{U}_h(x, z) = \widetilde{u}_h^{(1)}(x, z) + \widetilde{u}_h^{(2)}(x, z)$. In the proof of Theorem 4.1, we showed that we have by integration by parts $
abla_{\xi} P_h(\xi)$ and letting $\alpha \to 0$, we obtain (4.30) for $\widetilde{u}_h^{(2)}(x, E + i0)$. Noting $\nabla_{\xi} P_h(\xi)$, we prove the lemma for $\widetilde{u}_h^{(2)}(x, E + i0)$. 

If $\xi \notin M_{E,h}$, we have $\widetilde{U}_h(\xi, E) = \widetilde{F}_h(\xi)/(P_h(\xi) - E)$, which implies that for any $\omega \in S^{d-1}$, $(\omega, \xi) \notin WF^*(\widetilde{u}_h^{(2)})$ if $\xi \notin M_{E,h}$. Take any $\xi(0) \in M_{E,h}$ and $\epsilon > 0$, and $p_-(x, \xi) \in \mathcal{P}_-$ satisfying $\text{supp}_x p_-(x, \xi) \subset \{ |\xi - \xi(0)| < \epsilon \}$. By virtue of \((4.26)\), for $t > 0$, we have on the support of $p_-(x, \xi)$

\begin{equation}
|\nabla_{\xi}(x \cdot \xi - tP_h(\xi))|^2 \geq |x|^2 - 2\delta |x| |\nabla_{\xi} P_h(\xi)| + t^2 |\nabla_{\xi} P_h(\xi)|^2
\end{equation}

\begin{equation}
\geq (1 - \delta) (|x|^2 + t^2 |\nabla_{\xi} P_h(\xi)|^2).
\end{equation}

Let $\tilde{G}_h(\xi) = \chi_2(\xi) \tilde{F}_h(\xi)$ and $\tilde{G}_h = h^{d/2} \mathcal{F}_{\text{cont}}^{-1} \tilde{G}_h$. Using the relation

\begin{equation}
e^{i(x \cdot \xi - tP_h(\xi))} = -i \frac{\nabla_{\xi}(x \cdot \xi - tP_h(\xi))}{|\nabla_{\xi}(x \cdot \xi - tP_h(\xi))|^2}, e^{i(x \cdot \xi - tP_h(\xi))},
\end{equation}

we have by integration by parts

\begin{equation}
\|(x)^{-\alpha} p_-(x, D_x) e^{-itP_h(D_x)} \tilde{G}_h \| \leq C \|q_h(t, x, D_x) e^{-itP_h(D_x)} \tilde{G}_h\|
\end{equation}

where $q_h(t, x, D_x)$ is a $\Psi$DO with symbol $q_h(t, x, \xi)$ satisfying

\begin{equation}
|q_h(t, x, \xi)| \leq C (1 + t + |x|)^{-2} \langle x \rangle^{-\alpha}
\end{equation}

together with its derivatives. This implies

\begin{equation}
\|p_-(x, D_x) e^{-itP_h(D_x)} \tilde{G}_h\|_{-\alpha} \leq C (1 + t)^{-1-\epsilon_0} \|f\|_{m,s}
\end{equation}

for some $\epsilon_0 > 0$. Noting

\begin{equation}
\int_0^\infty e^{-it(P_h(\xi) - E - i\epsilon)} = -i (P_h(\xi) - E - i\epsilon)^{-1}, \quad \epsilon > 0,
\end{equation}

and letting $\epsilon \to 0$, we obtain \((4.30)\) for $p_- \in \mathcal{P}_-$. This implies \((4.29)\). \hfill \Box

We say that the solution of the equation $(P(D_x) - E)u = f$ satisfies the outgoing radiation condition if $p_-(x, D_x)u \in B^s_0(\mathbb{R}^d)$ for any $p_- \in \mathcal{P}_-$. We call $u$ an outgoing solution.
4.4. Convergence near \( M_{E,h} \). We have now constructed a solution to the discrete Schrödinger equation

\[
(P_h - \xi \cdot \xi - E) \hat{U}_h(\xi, E + i0) = \hat{F}_h(\xi)
\]

having uniform estimates with respect to \( 0 < h < h_0 \). Assume that

\[
f \in H^{m,s}(\mathbb{R}^d), \quad s > d + 1, \quad m > \frac{d}{2} + 1.
\]

Then as \( h \to 0 \)

\[
\hat{f}_h(\xi) \to \hat{f}(\xi) \quad \text{pointwise.}
\]

In Theorem 4.1 the part outside of \( M_{E,h} \) was shown to converge to the free Schrödinger equation

\[
\left(-P(\xi) - \xi \cdot \xi - E\right) U(1) = \chi_1(\xi) \Pi_0(\xi) \hat{f}(\xi),
\]

\( \hat{f}(\xi) \) being the Fourier transform of \( f \). We consider the part near \( M_{E,h} \), i.e. \( \hat{v}_h(\xi, E + i0) \) constructed in the previous section.

In view of (4.20) and the fact that \( P_h(\xi) \to P(\xi) \), we then see that \( \tilde{v}_h(\xi) \) converges pointwise to some \( \tilde{v}(\xi) \), which satisfies the Schrödinger equation

\[
\left(P(D_x) - E\right) v = f_2,
\]

where \( \tilde{f}_2(\xi) = \chi_2(\xi) \Pi_0(\xi) \hat{f}(\xi) \). We have proven that \( \tilde{v}_h(x, E + i0) \) has the uniform estimates

\[
\|\tilde{v}_h\|_{B^s(\mathbb{R}^d)} < C,
\]

and satisfies the outgoing radiation condition uniformly in \( 0 < h < h_0 \), i.e. (4.25) is satisfied uniformly in \( 0 < h < h_0 \). Since

\[
C^{-1}|\xi|^7 \leq P_h(\xi) \leq C|\xi|^7
\]

holds, \( \tilde{v}_h \) is in \( H^7_{\text{loc}}(\mathbb{R}^d) \) uniformly in \( 0 < h < h_0 \). Then one can select a subsequence \( h_j \to 0 \) such that \( \tilde{v}_{h_j} \) converges to \( v \in L^2_{\text{loc}} \) and \( v \) satisfies

- the Schrödinger equation \( \left(P(D_x) - E\right) v = f_2\),
- \( v \in B^s(\mathbb{R}^d) \),
- the radiation condition.

Here, we introduce a new assumption.

(U-1). The solution of the equation

\[
(P(D_x) - E) u = f \in B
\]

satisfying \( u \in B^s \) and the radiation condition is unique.

Then \( \{v_h\}_{0 < h < h_0} \) converges as \( h \to 0 \). We have thus proven the following theorem.

**Theorem 4.6.** Assume that \( f \in H^{m,s}(\mathbb{R}^d) \) for some \( s > d + 1 \) and \( m > \left[ \frac{d}{2} \right] + 1 \). Assume (B-1), (B-2-1), (B-3), (B-4) and (U-1). Let \( u_h(n, E + i0) \) be an outgoing solution to the gauge transformed equation

\[
(-G_h^* \Delta_{\text{disc},h} G_h - E) u_h = f_h \quad \text{on} \quad \mathbb{Z}^d,
\]

where \( f_h(n) = f(hn) \). We put \( \tilde{u}_h(\xi, E + i0) = \mathcal{F}_{\text{disc},h} u_h \), and

\[
\tilde{v}_h(\xi, E + i0) = \chi_d(h\xi) \Pi_h(\xi) \tilde{u}_h(\xi, E + i0).
\]
\[
\tilde{v}_h(x, E + i0) = \left( \frac{h}{2\pi} \right)^{d/2} \int_{T^d_h} e^{i\xi \cdot \xi} \hat{u}_h(\xi, E + i0) d\xi.
\]

Then the strong limit
\[
\lim_{h \to 0} \tilde{v}_h(x, E + i\epsilon) = \tilde{v}(x, E + i0) \quad \text{exists in} \quad L^{2, -1/2 - \epsilon}(\mathbb{R}^d), \quad \epsilon > 0,
\]
and \(\tilde{v}(x, E + i0)\) is the unique outgoing solution to the Schrödinger equation
\[
(P(D_x) - E)\tilde{v} = g \quad \text{on} \quad \mathbb{R}^d,
\]
where \((\mathcal{F}_{\text{cont}} f)(\xi) = \Pi_0(\xi)(\mathcal{F}_{\text{cont}} f)(\xi)\), and \(\tilde{v}\) satisfies the radiation condition
\[
p_-(x, D_x)\tilde{v} \in L^{2, -1/2 + \epsilon}(\mathbb{R}^d), \quad p_+ \in \mathcal{P}_-.
\]

4.5. Dirac equation. We next consider the case (B-2-2). Take \(\xi^{(0)} \in (\mathcal{K}/h) \setminus \{0\}\) arbitrarily. By Lemma 3.1 there exists a neighborhood \(U\) of \(\xi^{(0)}\) on which \(\Pi_h(\xi)\) is into a sum
\[
\Pi_h(\xi) = \Pi_h^{(+)}(\xi) + \Pi_h^{(-)}(\xi),
\]
where \(\Pi_h^{(\pm)}(\xi)\) is the projection associated with the characteristic root \(\pm P_h(\xi)\). Thus
\[
\left( \mathcal{L}_h(e^{-ih(\xi + dz)} - z)^{-1} - z \right)^{-1} \Pi_h(\xi) = (P_h(\xi) - z)^{-1} \Pi_h^{(+)}(\xi) + (-P_h(\xi) - z)^{-1} \Pi_h^{(-)}(\xi).
\]
The 1st term of the right-hand side is treated in the same way as in the previous subsection, and the 2nd term is easier to deal with since \(-P_h(\xi) - E \neq 0\). We have thus obtained the following theorem.

**Theorem 4.7.** Assume that \(f \in H^{m,s}(\mathbb{R}^d)\) for some \(s > d + 1\) and \(m > [d/2] + 1\). Assume (B-1), (B-2-2), (B-3), (B-4) and (U-1). Let \(u_h(n, E + i0)\) be an outgoing solution to the gauge transformed equation
\[
(-\mathcal{G}_h^* \Delta_{\text{disc}, h} \mathcal{G}_h - E)u_h = f_h \quad \text{on} \quad \mathbb{Z}^d,
\]
where \(f_h(n) = f(hn)\). We put \(\hat{u}_h(\xi, E + i0) = \mathcal{F}_{\text{disc}, h} u_h\), and
\[
\hat{v}_h(\xi, E + i0) = \chi_d(h\xi)\Pi_h(\xi)\hat{u}_h(\xi, E + i0).
\]

\[
\tilde{v}_h(x, E + i0) = \left( \frac{h}{2\pi} \right)^{d/2} \int_{T^d_h} e^{i\xi \cdot \xi} \hat{u}_h(\xi, E + i0) d\xi,
\]
where \(\mathcal{F}_{\text{cont}} g^{(+)}(\xi) = \Pi_0^{(+)}(\xi)(\mathcal{F}_{\text{cont}} f)(\xi)\).

Then the strong limit
\[
\lim_{h \to 0} \tilde{v}_h(x, E + i\epsilon) = \tilde{v}(x, E + i0) \quad \text{exists in} \quad L^{2, -1/2 - \epsilon}(\mathbb{R}^d), \quad \epsilon > 0.
\]
Here \(\tilde{v}(x, E + i0)\) is split into two parts
\[
\tilde{v}(x, E + i0) = \tilde{v}^{(+)}(x, E + i0) + \tilde{v}^{(-)}(x, E + i0),
\]
\(\tilde{v}^{(+)}(x, E + i0)\) being the unique solution to the Schrödinger equation
\[
(P(D_x) - E)\tilde{v}^{(+)} = g^{(+)} \quad \text{on} \quad \mathbb{R}^d,
\]
satisfying the outgoing radiation condition
\[
p_-(x, D_x)\tilde{v}^{(+)} \in L^{2, -1/2 + \epsilon}(\mathbb{R}^d), \quad p_+ \in \mathcal{P}_-.
\]
and $\tilde{v}^-(x, E + i0)$ is the unique $L^2$-solution to the Schrödinger equation

$$(-P(D_x) - E)\tilde{v}^- = g^-(x, E + i0)$$ for $x \in \mathbb{R}^d$.

In the application, we encounter the case in which $d = 2$ and

$$P(\xi) = \left( \sum_{i,j=1}^{2} a_{ij} \xi_i \xi_j \right)^{1/2},$$

where $(a_{ij})$ is a positive definite symmetric matrix. We can make a linear transformation $\xi \to \eta$ so that

$$P(\xi) = (\eta_1^2 + \eta_2^2)^{1/2}.$$

Let us note the following equality for $w \in \mathbb{C}$

$$\begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} = P^* \begin{pmatrix} |w| & 0 \\ 0 & -|w| \end{pmatrix} P,$$

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & |w|/w \\ -1 & |w|/w \end{pmatrix}.$$

Letting $w = \eta_1 + i\eta_2$, we have

$$\begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eta_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \eta_2.$$

Replacing $\eta_1$ by $\frac{1}{i} \frac{\partial}{\partial y_1}$ and $\eta_2$ by $\frac{1}{i} \frac{\partial}{\partial y_2}$, where $x \to y$ is a linear transformation such that $x \cdot \xi = y \cdot \eta$, we have thus seen that the lattice Hamiltonian $-\Delta^{(0)}_{v,h}$ converges to the 2-dimensional massless Dirac operator.

4.6. On the assumption (U-1). The assumption (U-1) holds true for the above type of $P(D_x)$. Assume that

$$P(\xi) = \left( \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \right)^{1/2},$$

where $(a_{ij})$ is a positive definite symmetric matrix. It is well-known that if $V(x)$ is a real-valued function decaying to 0 at infinity, e.g. there exists $\epsilon > 0$ such that $\partial x^\alpha V(x) = O(|x|^{-\alpha} - \epsilon)$ for all $\alpha$, then (U-1) holds for the equation $(P(D_x)^2 + V(x) - E)u = f$, where $E > 0$.

**Lemma 4.8.** Let $u \in \mathcal{B}^*(\mathbb{R}^d)$ be a solution of the equation $(P(D_x) - E)u = 0$ for some $E > 0$. If $u$ satisfies the radiation condition, then $u = 0$.

**Proof.** Multiplying the equation by $P(D_x) + E$, we have $(P(D_x)^2 - E^2)u = 0$. Then the lemma follows. \qed
5. Perturbation by a potential

Let us consider the case where the characteristic root $\lambda_{j,h}(\eta)$, defined in (3.4), (3.3), has a unique global minimum. Assume that

\( C-1 \) \( \lambda_j(e^{-i\eta}) \geq 0, \) and there exists a unique \( d_1 \in \mathbb{T}^d \) such that \( \lambda_j(e^{-id_1}) = 0. \)

\( C-2 \) \( \max_{\eta \in \mathbb{T}^d} \lambda_j - 1(e^{-i\eta}) < 0 < \min_{\eta \in \mathbb{T}^d} \lambda_j + 1(e^{-i\eta}). \)

\( C-3 \) Letting \( P_h(\xi) = \lambda_{j,h}(\xi + d_1) \) be as in (3.4), we assume that \( P_h(\xi) \to P(\xi) \) on \( \mathbb{T}^d, \) where

\[
P(\xi) = \sum_{|\alpha| = 2m} a_\alpha \xi^\alpha,
\]

\( m \) being a positive integer.

We add a scalar potential \( V(x) \) to \( -\Delta_{\Gamma_h}. \) Assume that

\( V-2 \) \( V(x) \in H^s(\mathbb{R}^d) \) with \( s > d/2 \) and is compactly supported.

We finally assume the uniqueness of solutions to the Schrödinger equation.

\( U-2 \) The solution of the equation

\[
(P(D_x) + V(x) - E)u = f \in \mathcal{B}
\]

satisfying \( u \in \mathcal{B}^* \) and the radiation condition is unique.

We consider

\[
( -\Delta_{\text{disc},h} + V_{\text{disc},h} - z) u_h = f_h,
\]

where

\[
(V_{\text{disc},h} u_h)(n) = V(hn) u_h(n).
\]

Take an open interval

\[
I \subset \subset \left(0, \min_{\eta \in \mathbb{T}^d} \lambda_{j+1}(e^{-i\eta})\right),
\]

and assume that \( \text{Re} z \in I. \) As was discussed in the previous section, we prove

- fixed \( h \) limit (Lemma 5.1),
- uniform bound (Lemma 5.2),
- compactness (Lemma 5.3).

In [3] we have proven that there exists the unique solution to the equation (5.2) satisfying the outgoing radiation condition.

**Lemma 5.1.** Assume that \( f \in H^{m,s}(\mathbb{R}^d) \) for some \( s > d + 1 \) and \( m > \left\lfloor d/2 \right\rfloor + 1. \) Assume \( \text{Re} z \in I, \text{Im} z > 0, \) and let \( u_h(z) = \{u_h(n,z)\}_{n \in \mathbb{Z}^d} \) be the unique \( L^2 \)-solution to the equation (5.2). Then

\[
\lim_{\epsilon \to 0} u_h(E + i\epsilon) = u_h(E + i0) \quad \text{in} \quad L^{2-s}(\mathbb{Z}_h^d), \quad s > \frac{1}{2},
\]

and \( u_h(E + i0) \) is the unique outgoing solution to (5.2) with \( z = E. \)
Let \( \hat{u}_h(\xi) = F_{\text{disc}, h} u_h, \) \( \hat{u}_h(x) = \left( \frac{h}{2\pi} \right)^{d/2} \int_{T_h^d} e^{ix\xi} \hat{u}_h(\xi, z) d\xi. \)

Letting 
\[
\hat{w}_h = G_h^* u_h, \quad g_h = G_h f_h - V_{\text{disc}, h} w_h,
\]
we consider the gauge transformed equation
\[
( - G_h^* \Delta_{\text{disc}, h} G_h - \zeta) w_h = g_h.
\]

Then we have
\[
\hat{w}_h(E + i\eta) = \left( \mathcal{L}_h(e^{-i\xi\mathcal{D}^*_{\text{disc}, h}}) - E - i\eta \right)^{-1} \hat{g}_h.
\]

Let \( \Pi_0 \) be the eigenprojection associated with \( P_h(\xi) \). Arguing in the same way as in (3.6) and using the assumption that \( \Re z \in I \), we have the following inequality
\[
\| (1 - \Pi_0) \hat{w}_h(\xi, E + i\eta) \|_{L^2(T_h^d)} \leq C h^\nu \left( \| f \|_{m,s} + \| w_h \|_{-1-\epsilon} \right),
\]
for \( \epsilon > 0 \). Therefore we consider \( v_h \), where
\[
\hat{v}_h(\xi, z) = \Pi_0 \hat{w}_h(\xi, z).
\]

**Lemma 5.2.** Let \( u_h \) be as in Lemma 5.1.

1. For any \( s, 1/2 < s < 1 \), there exists a constant \( C_s > 0 \) such that
\[
\| \hat{u}_h \|_{L^{2-s}(\mathbb{R}^d)} \leq C \| \hat{f}_h \|_{L^{2-s}(\mathbb{R}^d)}, \quad \text{for all } \ h, \ 0 < h < h_0.
\]

2. Let \( p_-(x, D_x) \in P_- \). Then for any \( s, 1/2 < s < 1 \), there exists a constant \( C_s > 0 \) such that
\[
\| p_-(x, D_x) \hat{u}_h \|_{L^{2-s-1}(\mathbb{R}^d)} \leq C_s \| \hat{f}_h \|_{L^{2-s}(\mathbb{R}^d)}, \quad \text{for all } \ h, \ 0 < h < h_0.
\]

**Proof.** We prove (1). If (5.5) does not hold then for any \( n \geq 1 \) there exists \( u_{h_n}, f_{h_n} \) satisfying 
\[
( -\Delta_{\text{disc}, h_n} + V_{h_n} - E) u_{h_n} = f_{h_n} \text{ and } \| \hat{u}_{h_n} \|_{L^{2-s}(\mathbb{R}^d)} \geq n \| \hat{f}_{h_n} \|_{L^{2-s}(\mathbb{R}^d)}.
\]
Dividing by \( \| \hat{u}_{h_n} \|_{L^{2-s}(\mathbb{R}^d)} \), there exist an outgoing \( u_{h_n} \in B^*(Z_{h_n}^d) \) and \( f_{h_n} \in B(Z_{h_n}^d) \) such that
\[
( -\Delta_{\text{disc}, h_n} + V_{h_n} - E) u_{h_n} = f_{h_n}, \quad \| \hat{u}_{h_n} \|_{L^{2-s}(\mathbb{R}^d)} = 1, \quad \| \hat{f}_{h_n} \|_{L^{2-s}(\mathbb{R}^d)} \to 0, \quad n \to \infty.
\]

We put
\[
w_{h_n} = G_{h_n}^* u_{h_n}, \quad \hat{v}_{h_n}(\xi) = \Pi_0 \hat{w}_{h_n}(\xi),
\]
\[
\hat{v}_{h_n}'(\xi) = (1 - \Pi_0) \hat{w}_{h_n}(\xi).
\]

By (5.4), we have
\[
\hat{v}_{h_n}' \to 0, \quad \text{in } L^{2-s} \quad \text{for } \ s > 1/2.
\]

By the a priori estimate (10.14), one can select a subsequence of \( \{v_{h_n}\} \), which is denoted by \( \{v_{h_n}\} \), such that \( \{\hat{v}_{h_n}\} \) converges in \( L^2_{\text{loc}}(\mathbb{R}^d) \). Let \( \hat{u}_{h_n}(x) \to v(x) \). One can show that
\[
v_{h_n}(x) \to v(x) \quad \text{in } L^{2-s}(\mathbb{R}^d).
\]
In fact, take $1/2 < t < s$. The resolvent estimate for $-\Delta_{\text{disc},h}$ also holds between $L^{2,t}$ and $L^{2,-t}$. Therefore,
\[ \|\widetilde{v}_h\|_{L^{2,-t}(\mathbb{R}^d)} < C, \]
for a constant $C$ independent of $h_i$. This yields
\[ \int_{|x| > R} (1 + |x|^2)^{-s} |\widetilde{v}_h(x)|^2 dx \to 0, \quad (R \to \infty) \]
uniformly with respect to $h_i$. Thus we can choose $h_i$ so that (5.7) holds. In particular, we have
\[ \|v\|_{L^{2,-s}(\mathbb{R}^d)} = 1. \]
Recall that
\[ \widetilde{v}_h(hn) = v_h(n). \]
Then by the definition of the Riemann integral we have
\[ h_i^d \sum_{n \in \mathbb{Z}^d} V_h(n)v_h(n)\varphi(hn) \to \int_{\mathbb{R}^d} V(x)\widetilde{v}(x)\varphi(x) dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d). \]

We can assume that $h_i \to h_\infty \geq 0$. If $h_\infty > 0$, we easily arrive at the contradiction, since we have the resolvent estimate for $-\Delta_{\text{disc},h_\infty}$. We consider the case $h_\infty = 0$. In the equation
\[ (-\Delta_{\text{disc},h_i} + V_h - E)u_h(n) = f_h(n), \]
we split $u_{hn}$ into $u_{hn} = g_{hn}w_{hn} = g_{hn}(v_{hn} + v'_{hn})$. Recalling (5.6), we take the inner product with $\varphi(hn)$ and sum up in $n$. Letting $h_i \to 0$, we have
\[ (P(D_x) + V - E)v = 0. \]
By the well-known results on resolvent estimates for Schrödinger equations, we have
\[ v \in \mathcal{B}^s(\mathbb{R}^d), \quad p(x, D_x)v \in L^{2,s-1}(\mathbb{R}^d), \]
for $p(x, D_x) \in P_-$ and $1/2 < s < 1$. Therefore $v$ is a solution to the homogeneous Schrödinger equation satisfying the outgoing radiation condition, hence vanishes identically. This is a contradiction. We have thus proven (1). The assertion (2) follows from the resolvent estimate for $-\Delta_{\text{disc},h}$. \hfill \Box

**Lemma 5.3.** Let $\widetilde{u}_h(x, E + i0)$ be as in Lemma 5.1.
(1) The set $\{\widetilde{u}_h(x, E + i0) \mid 0 < h < h_0\}$ is compact in $L^2_{\text{loc}}(\mathbb{R}^d)$.
(2) The set $\{V_h\widetilde{u}_h(x, E + i0) \mid 0 < h < h_0\}$ is compact in $L^{2,s}(\mathbb{R}^d)$ for any $s > 0$, where $V_h$ is defined by
\[ (V_hw)(x) = V(hn)w(x), \quad \text{if} \quad x \in hn + [-h/2, h/2]^d. \]

**Proof.** The assertion (1) follows from Lemma 5.1. Since $V(x)$ is compactly supported, the assertion (2) follows from (1). \hfill \Box

These preparations and the assumption (U-2) are sufficient to show the following theorem.

**Theorem 5.4.** Let $s > 1/2$. As $h \to 0$, $\widetilde{u}_h(x) \to u(x)$, where $u(x) \in L^{2,-s}(\mathbb{R}^d)$ and satisfies
\[ u = \Pi_0 u, \quad (P(D_x) + V(x) - E)u = \Pi_0 f, \]
\[ p_-(x, D_x)u \in L^{2,s-1}(\mathbb{R}^d), \quad \forall p_-(x, D_x) \in P_- . \]
Corollary 5.5. With the same notation as in Theorem 5.4, \( \tilde{u}_h(x) \to u(x) \) locally uniformly on \( \mathbb{R}^d \).

Proof. We have for \( m > d/2 \) and \( s > 1/2 \),

\[
\sup_{0 < h < h_0} \| \tilde{u}_h \|_{H^{m-s}} < \infty.
\]

Using the a priori estimate and the Sobolev inequality, we get the assertion. \( \Box \)

6. Complex energy

In the above arguments the radiation condition was used at the step of the uniqueness of solutions to the equation \( P(D_x) + V - E \)\( u = f \) for \( E > 0 \). Usually this fact is proven by the Rellich type theorem and the unique continuation property for the Helmholtz equation. However, on some lattices, the latter result does not hold. Even for this case, if \( E \not\in \mathbb{R} \), one can employ a simpler condition for the uniqueness. For the sake of simplicity we consider here the case \( P(D_x) = -\Delta_{\text{cont}} \).

Lemma 6.1. Assume that \( u \in L^{2-s}(\mathbb{R}^d) \cap H^2_{\text{loc}}(\mathbb{R}^d) \) satisfies \( -\Delta_{\text{cont}} + V - z \)\( u = 0 \) on \( \mathbb{R}^d \). If \( z \not\in \mathbb{R} \) and \( 0 < s \leq 1/2 \), then \( u = 0 \).

Proof. Since \( u, \frac{\partial u}{\partial r} \in L^{2-s} \), we have \( \liminf_{r \to \infty} r^{n-2s} \int_{|x|=r} |\tilde{\nu}(\frac{\partial u}{\partial r})| dS = 0 \). By integration by parts, we have

\[
\int_{|x|<R} ((\Delta u)\tilde{u} - u\Delta \tilde{u}) \, dx = \int_{|x|=R} \left( \frac{\partial u}{\partial r} \tilde{u} - u \frac{\partial \tilde{u}}{\partial r} \right) \, dS.
\]

We then have, by taking the imaginary part and letting \( R \to \infty \) along a suitable sequence, \( \text{Im} \int_{\partial \Omega^*} |\tilde{u}|^2 dx = 0 \), which proves the lemma. \( \Box \)

Arguing as in the previous sections, noting that \( P_h(\xi) - z \not= 0 \) for \( z \not\in \mathbb{R} \), one can derive the estimates of \( \tilde{u}_h(x, z) \) in \( H^2(\mathbb{R}^d) \) uniformly with respect to \( 0 < h \leq h_0 \). Then by virtue of Lemma 6.1 for any \( 0 < s < 1/2 \), one can conclude the convergence of \( \tilde{u}_h(x, z) \) in \( L^{2-s}(\mathbb{R}^d) \) as \( h \to 0 \). In the following Theorems 6.2 and 6.3, we do not assume the unique continuation property for \( \Delta_{\text{disc},h} \).

Theorem 6.2. Assume that \( f \in H^{m,s}(\mathbb{R}^d) \) for some \( s > d + 1 \) and \( m > [d/2] + 1 \). Assume (B-1), (B-2-1) or (B-2-2), (B-3), (B-4) and (U-1). Let \( z \not\in \mathbb{R} \), and \( u_h(n, z) \) be an \( L^2 \)-solution to the gauge transformed equation

\[
(-G_{\lambda}^h \Delta_{\text{disc},h} \mathcal{G}_h - E)u_h = f_h \quad \text{on} \quad \mathbb{Z}^d,
\]

where \( f_h(n) = f(hn) \). Then the strong limit

\[
\lim_{h \to 0} \tilde{v}_h(x, z) = \tilde{v}(x, z) \quad \text{exists in} \quad L^{2-s}(\mathbb{R}^d), \quad 0 < s < 1/2.
\]

This convergence is locally uniform on \( \mathbb{R}^d \).

For the case (B-2-1), \( \tilde{v}(x, E + i\delta) \) is the unique \( L^2 \)-solution to the equation

\[
(P(D_x) - z)\tilde{v} = g \quad \text{on} \quad \mathbb{R}^d,
\]

where \( g \) is defined in \( 4.13 \).

For the case (B-2-2), \( \tilde{v}(x, z) \) split into two parts

\[
\tilde{v}(x, z) = \tilde{v}^{(+)}(x, z) + \tilde{v}^{(-)}(x, z),
\]

where
being the unique solution to the Schrödinger equation
\[(P(D_x) - z)v^{(\pm)} = g^{(\pm)} \text{ on } \mathbb{R}^d.\]

**Theorem 6.3.** Assume that \( f \in H^{m,s}(\mathbb{R}^d) \) for some \( s > d/2 + 1 \). In addition to (B-1), (B-2-1) or (B-2-2), (B-3), (B-4) and (U-1), assume (C-1), (C-2), (C-3), and (U-2). Let \( z \not\in \mathbb{R} \), and \( u_h(n, z) \) be an \( L^2 \)-solution of the equation
\[(-G_h \Delta_{\text{disc},h} G_h + V_{\text{disc},h} - z)u_h = f_h \text{ on } \mathbb{Z}^d,\]
where \( f_h(n) = f(hn) \). Then the strong limit
\[\lim_{h \to 0} \tilde{u}_h(x, z) = \tilde{u}(x, z) \text{ exists in } L^{2-s}(\mathbb{R}^d), 0 < s < 1/2.\]
The convergence is locally uniform on \( \mathbb{R}^d \). Moreover, \( u \) satisfies
\[u = \Pi_0 u, \quad (P(D_x) + V(x) - z)u = \Pi_0 f.\]

7. Derivation of Schrödinger equations

In the remaining sections, we apply the results in the previous sections to Hamiltonians on periodic lattices appearing in material science. We pick up basic examples whose spectral properties are studied in [3].

7.1. Square lattice. Define the Laplacian on the square lattice in \( \mathbb{R}^d \) by
\[\Delta_{\Gamma} = \sum_{j=1}^{d} (S_j + S_j^*).\]
In this case, we take the reference energy to be \( E_0 = -2d \) and consider
\[\frac{1}{\hbar^2} \sum_{j=1}^{d} (2 - S_{h,j} - S_{h,j}^*) + V_{\text{disc},h}.\]
The symbol of \( \frac{1}{\hbar^2} \sum_{j=1}^{d} (2 - S_{h,j} - S_{h,j}^*) \) is
\[\mathcal{L}_h(e^{-i\eta}) = P_h(\eta) = \frac{1}{\hbar^2} \sum_{j=1}^{d} (2 - e^{i\eta_j} - e^{-i\eta_j}) = \frac{4}{\hbar^2} \sum_{j=1}^{d} \sin^2 \frac{\hbar \eta_j}{2},\]
which satisfies the assumptions in §4 and §5 with \( G_h = 1 \). Therefore by Theorem 5.4 we obtain the following theorem.

**Theorem 7.1.** Assume that \( V(x) \in H^{s}(\mathbb{R}^d) \) with \( s > d/2 \) and compactly supported, and that \( f \in H^{m,s}(\mathbb{R}^d) \) with \( m > d/2 + 1 \). Let \( E > 0 \) and take \( h_0 > 0 \) such that \( E < 4/h_0^2 \). Then the solution of the discrete Schrödinger equation
\[(-\Delta + V_{\text{disc},h} - E)u_h = f_h, 0 < h < h_0,\]
satisfying the radiation condition converges to that of
\[(-\Delta + V(x) - E)u = f \text{ on } \mathbb{R}^d \]
as \( h \to 0 \).

Since this solution is written as \( u = (-\Delta + V(x) - E - i0)^{-1}f \), we can use it to represent the S-matrix.
7.2. Convergence of the S-matrix. We recall the definition
\[ M_{E,h} = \{ \xi \in T^d_h \mid P_h(\xi) = E \}. \]
This surface is not smooth for \( E = 4j/h^2, \ j = 1, 2, \ldots, d - 1 \). Therefore we impose the condition \( 0 < E < 4/h^2 \) in the sequel. In particular, we consider the S-matrix only in the energy interval \((0, 4/h^2)\).

Fix \( E \in (0, 4/h^2) \). We recall that for such \( E \) the surface \( M_{E,h} \) is diffeomorphic to \( S^{d-1} \). We use the parametrization
\[ \xi_j = \frac{2}{h} \arcsin \left( \frac{1}{2} \sqrt{E} \omega_j \right), \quad j = 1, 2, \ldots, d, \quad \omega \in S^{d-1}. \]
We also recall that \( \xi_j = \sqrt{E} \omega_j + O(h^2) \) as \( h \to 0 \).

Let \( dS_{E,h} \) denote the surface measure on \( M_{E,h} \) induced by \( d\xi \). Then
\[ d\xi = \frac{1}{|\nabla \xi P_h(\xi)|} dS_{E,h} dE. \]

Let \( L^2(M_{E,h}) \) be the Hilbert space equipped with the inner product
\[ (\phi, \psi)_{L^2(M_{E,h})} = \int_{M_{E,h}} \bar{\phi}(\xi) \psi(\xi) dS_E. \]

Let \( (7.3) \Omega_h = \bigcup_{0<E<4/h^2} M_{E,h} \).

The characteristic function of this set is denoted by \( \chi_{\Omega_h} \).

Let \( f \in S(\mathbb{R}^d) \). Put
\[ (\mathcal{F}_{0h}(E)f)(\xi) = \chi_{\Omega_h}(\xi)|\nabla \xi P_h(\xi)|^{-1/2} \left( \frac{h}{2\pi} \right)^{d/2} \sum_{n \in \mathbb{Z}^d} e^{-ihn \cdot \xi} f(n), \]
and then define
\[ (\mathcal{F}_{0h,f})(E, \xi) = (\mathcal{F}_{0h}(E)f)(\xi). \]

Let \( E_{0h} \) denote the spectral measure of the operator \( H_{0h} = -\Delta_{disc,h} \). Then
\[ \mathcal{F}_{0h}: E_{0h}((0, 4/h^2))L^2(\mathbb{Z}^d_h) \to L^2((0, 4/h^2); L^2(M_{E,h}); dE) \]
is unitary. Note that with the above definitions we have \( \mathcal{F}_{0h} = \mathcal{F}_{0h} E_{0h}((0, 4/h^2)) \).

Define \( H_{0h} = -\Delta_{disc,h} \) and \( H_h = -\Delta_{disc,h} + V_{disc,h} \). Then the wave operators
\[ W_h^{(\pm)} = \lim_{t \to \pm \infty} e^{itH_h} e^{-itH_{0h}} E_{0h}((0, 4/h^2)) \]
exist and are asymptotically complete in the localized sense. Define the localized scattering operator by
\[ S_h = (W_h^{(+)})^* W_h^{(-)}. \]

Then its localized Fourier transform
\[ \widehat{S}_h := \mathcal{F}_{0h} S_h (\mathcal{F}_{0h})^* \]
has the direct integral representation
\[ \widehat{S}_h = \int_0^{4/h^2} \widehat{S}_h(E) dE, \]
where $\hat{S}_h(E)$ is Heisenberg’s $S$-matrix for $E \in (0, 4/h^2)$, which is unitary on $L^2(M_{E,h})$. The scattering amplitude $A_h(E)$ is then defined by

$$\hat{S}_h(E) = I - 2\pi i A_h(E).$$

It has the representation

$$A_h(E) = \mathcal{F}_{0h}(E) V_h \mathcal{F}_{0h}(E)^* - \mathcal{F}_{0h}(E) V_h R_h(E + i0) V_h \mathcal{F}_{0h}(E)^*.$$ 

Therefore for $E \in (0, 4/h^2)$ its integral kernel is

$$A_h(E; \xi, \eta) = a_h(\xi, \eta) \left( \frac{h}{2\pi} \right)^d \sum_{n \in \mathbb{Z}^d} V(hn) e^{-ihn \cdot (\xi - \eta)} - a_h(\xi, \eta) \left( \frac{h}{2\pi} \right)^d \sum_{n \in \mathbb{Z}^d} V(hn) e^{ihn \cdot \eta} u_h(E, \xi, n),$$

where

$$u_h(E, \xi) = R_h(E + i0) \psi_h(\xi),$$

$\psi_h(\xi) \in L^2(\mathbb{Z}_h^d)$ is defined by

$$\psi_h(\xi, n) = V(hn) e^{ihn \cdot \xi},$$

and

$$a_h(\xi, \eta) = |\nabla P_h(\xi)|^{-1/2} |\nabla P_h(\eta)|^{-1/2}.$$

Using the parametrization $\xi = \xi_h(\omega)$ in (7.1), we can regard $A_h(E; \xi, \eta)$ as a function on $S^{d-1} \times S^{d-1}$, which we denote $A_h(E; \omega, \omega')$ for the sake of simplicity.

For the continuous case $A(E; \omega, \omega')$ is written as

$$A(E; \omega, \omega') = C(E) \int_{\mathbb{R}^d} e^{-i\sqrt{E}(\omega - \omega') \cdot x} V(x) dx - C(E) \int_{\mathbb{R}^d} e^{-i\sqrt{E}(\omega - \omega') \cdot x} V(x) u(E, x, \omega') dx,$$

$$u(E; x, \omega') = R(E + i0) (V \psi(E, \omega')),$$

$$\psi(E, \omega; x) = V(x) e^{i\sqrt{E}(\omega - \omega')} x.$$

**Theorem 7.2.** We have as $h \to 0$

$$A_h(E; \omega, \omega') \to A(E; \omega, \omega').$$

**Proof.** By Lemma 10.1

$$h^d \sum_{n \in \mathbb{Z}^d} V(hn) e^{ihn \cdot (\xi - \eta)} \to \int_{\mathbb{R}^d} V(x) e^{-ix \cdot (\xi - \eta)} dx,$$

$$h^d \sum_{n \in \mathbb{Z}^d} V(hn) e^{-ihn \cdot \eta} u_h(n) \to \int_{\mathbb{R}^d} V(x) e^{-ix \cdot \eta} \tilde{u}(x) dx,$$

where

$$u_h(n) = u_h(E, \xi, n), \quad \tilde{u}(x) = \tilde{u}(E, x, \omega').$$

Recall that $\tilde{u}_h(x)$ converges to $\tilde{u}(x)$ locally uniformly on $\mathbb{R}^d$, and $\tilde{u}_h(hn) = u_h(n)$.

$\square$
7.3. Triangular lattice. The Laplacian for the triangular lattice is

\[ \Delta_{\Gamma_h} = \frac{1}{6h^2} (S_{h,1} + S_{h,1}^* + S_{h,2} + S_{h,2}^* + S_{h,1}S_{h,2}^* + S_{h,1}^*S_{h,2}) . \]

The reference energy is \( E_0 = -1/h^2 \), and the symbol of \( -\Delta_{\Gamma_h} \) is given by

\[
P_h(\xi) = \frac{1}{3h^2} \left( 3 - \cos h\xi_1 - \cos h\xi_2 - \cos(h\xi_1 - h\xi_2) \right) \\
= \frac{2}{3h^2} \left( \sin^2 \frac{h\xi_1}{2} + \sin^2 \frac{h\xi_2}{2} + \sin^2 \frac{h(\xi_1 - \xi_2)}{2} \right).
\]

It has the unique global minimum at \( \xi = 0 \), and we have the asymptotic expansion

\[
P_h(\xi) = \frac{1}{3} (\xi_1^2 - \xi_1\xi_2 + \xi_2^2) + O(h^2) .
\]

Therefore by the same argument as above, the following theorem holds.

**Theorem 7.3.** Assume that \( V(x) \in H^s(\mathbb{R}^2) \) with \( s > 1 \) and compactly supported, and that \( f \in H^{m,s}(\mathbb{R}^2) \) with \( m > 2, s > 3 \). Then the solution of the Schrödinger equation \((-\Delta_{\text{disc},h} + V_{\text{disc},h} - E)u = f\) on the triangular lattice converges to the solution of the Schrödinger equation

\[
(-\frac{1}{3} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1\partial x_2} + \frac{\partial^2}{\partial x_2^2} \right) + V(x) - E)u = f \quad \text{in} \quad \mathbb{R}^2 .
\]

Theorem 7.2 also holds for this case. In this connection we note that the thresholds associated with \( P_h(\xi) \) are \( 0, 4/(3h^2), \) and \( 3/(2h^2) \). Thus in the proof of Theorem 7.2 we have a restriction to the energy interval \((0, 4/(3h^2))\). In the limit \( h \to 0 \) this restriction disappears, as for the square lattice. The same remark is valid for the following all examples. Therefore, in the theorems for the Schrödinger limit as in Theorems 7.1 and 7.2 to be given below, we do not mention this limitation for \( E \), i.e. \( 0 < E < C_1(h), \) \( C_1(h) \) being the first threshold in the spectrum of \( L_h(S_h) \).

7.4. Ladder of square lattice. In this case the Laplacian is written as

\[
L_h(S_h) = \frac{1}{h^2} L(S_h) ,
\]
Figure 2. 2-dim. ladder

\[ \mathcal{L}(S_h) = -\frac{1}{2d+1} \left( \sum_{j=1}^{d} \left( S_{h,j} + S_{h,j}^* \right) \frac{1}{2} \right) \sum_{j=1}^{d} \left( S_{h,j} + S_{h,j}^* \right) \right). \]

Letting

\[ T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \]

we have

\[ T^* \mathcal{L}(S_h) T = -\frac{1}{2d+1} \left( \sum_{j=1}^{d} \left( S_{h,j} + S_{h,j}^* \right) + 1 \right) \sum_{j=1}^{d} \left( S_{h,j} + S_{h,j}^* \right) - 1 \).

Then the reference energy is \( E_0 = -1/h^2 \). We then consider the Hamiltonian

\[ \mathcal{L}_h(S_h) = \frac{1}{(2d+1)h^2} \left( 2d - \sum_{j=1}^{d} \left( S_{h,j} + S_{h,j}^* \right) \right) \sum_{j=1}^{d} \left( S_{h,j} + S_{h,j}^* \right) \right). \]

The characteristic roots are

\[ \lambda_h^{(+)}(\eta) = \frac{2d + 2 - 2 \sum_{j=1}^{d} \cos h\eta_j}{(2d+1)h^2}, \]

\[ \lambda_h^{(-)}(\eta) = \frac{2d - 2 \sum_{j=1}^{d} \cos h\eta_j}{(2d+1)h^2}. \]

Letting \( P_h(\eta) = \lambda_h^{(-)}(\eta) \) we then have

\[ P_h(\eta) = \frac{4}{(2d+2)h^2} \sum_{j=1}^{d} \sin^2 \frac{\eta_j^2}{2} \to \frac{1}{2d+1} \sum_{j=1}^{d} \eta_j^2. \]

We then have the following theorem.

**Theorem 7.4.** Let \( V(x) \in H^s(\mathbb{R}^d) \) with \( s > d/2 \) and compactly supported, and \( f \in H^{m,s}(\mathbb{R}^d) \) with \( m > d/2 + 1, \ s > d + 1 \). Then the solution of the equation

\[ (-\Delta_{disc,h} + V_{disc,h} - E)u = f \]

converges to the solution of the Schrödinger equation

\[ \left( -\frac{1}{2d+1} \Delta + V(x) - E \right)u = f \quad \text{in} \quad \mathbb{R}^d. \]
Theorem 7.2 also holds for this case.

8. Derivation of Dirac equations

8.1. Hexagonal lattice. The Laplacian on the hexagonal lattice is

\[
\Delta_h = -\frac{1}{3} \begin{pmatrix} 0 & 1 + S_1^* + S_2^* \\ 1 + S_1 + S_2 & 0 \end{pmatrix}.
\]

See Figure 3. We put

\[
\mathcal{L}_h(e^{-i\eta}) = -\frac{1}{3h} \begin{pmatrix} 0 & 1 + e^{i\eta_1} + e^{i\eta_2} \\ 1 + e^{-i\eta_1} + e^{-i\eta_2} & 0 \end{pmatrix},
\]

and let \( \mathcal{L}(e^{-\eta}) = \mathcal{L}_1(e^{-i\eta}) \). Letting

\[
b(z) = \cos z_1 + \cos z_2 + \cos(z_1 - z_2), \quad z = (z_1, z_2) \in \mathbb{C}^2,
\]

we have

\[
|1 + e^{i\eta_1} + e^{i\eta_2}|^2 = 3 + 2b(\eta), \quad \eta = (\eta_1, \eta_2) \in \mathbb{R}^2.
\]

The characteristic roots of \( \mathcal{L}(e^{-\eta}) \) are given by \( \lambda^{(\pm)}(\eta) = \pm \sqrt{3 + 2b(\eta)} \). By elementary geometry, we have

\[
3 + 2b(\eta) = 0 \iff \eta = \left( \frac{2\pi}{3}, -\frac{2\pi}{3} \right), \quad \left( -\frac{2\pi}{3}, \frac{2\pi}{3} \right),
\]

\[
3 + 2b(\eta) = 9 \iff \eta = (0, 0).
\]

For the first two cases, the Hessian matrix of \( 3 + 2b(\eta) \) is

\[
\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix},
\]

and for the third case

\[
\begin{pmatrix} 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1 \end{pmatrix}.
\]

The characteristic roots of \( \mathcal{L}_h(S_h) \) are

\[
\lambda^{(\pm)}_h(\eta) = \pm \sqrt{3 + 2b(h\eta)}.
\]

The spectrum of \( -\frac{1}{h} \Delta_{\Gamma_h} \) is

\[
\sigma\left( -\frac{1}{h} \Delta_{\Gamma_h} \right) = [-1/h, 1/h].
\]
8.1.1. Expansion around the Dirac points. We take the reference energy $E_0 = 0$. We put

$$d_h^{(\pm)} = \pm \frac{1}{h} \left( \frac{2\pi}{3}, -\frac{2\pi}{3} \right),$$

$$\mathcal{K}_0^{(\pm)} = \{ \eta \in T^d; |\eta - d_h^{(\pm)}| < \frac{\pi}{3} \},$$

$$\mathcal{K}_0^{(0)} = T^d \setminus (\mathcal{K}_0^{(+) \cup \mathcal{K}_0^{(-)})}.$$  

Take $\chi^{(\pm)}, \chi^{(0)} \in C^\infty(T^d)$ such that

$$\chi^{(\pm)}(\eta) = \begin{cases} 
1, & \text{for } |\eta - d_h^{(\pm)}| < \frac{\pi}{6}, \\
0, & \text{for } |\eta - d_h^{(\pm)}| > \frac{\pi}{3}, 
\end{cases}$$

$$\chi^{(0)}(\eta) = 1 - \chi^{(+)}(\eta) - \chi^{(-)}(\eta).$$

We consider the Schrödinger equation

$$(8.6) \quad (-\Delta + z) u_h = f_h$$

on the hexagonal lattice. Let

$$u_h^{(\pm)} = \mathcal{F}^{-1}_{\text{disc,}h}(\chi^{(\pm)}(h\eta) \hat{u}_h(\eta)),$$

$$u_h^{(0)} = \mathcal{F}^{-1}_{\text{disc,}h}(\chi^{(0)}(h\eta) \hat{u}_h(\eta)),$$

$$f_h^{(\pm)} = \mathcal{F}^{-1}_{\text{disc,}h}(\chi^{(\pm)}(h\eta) \hat{f}_h(\eta)).$$

Take a compact interval $I \subset (0, \infty)$ and assume that $\Re z \in I$. We consider (8.6) on $T^d_h$. Since $\lambda^{(\pm)}(\eta) \neq 0$ on $\mathcal{K}_0^{(0)}$, there exists an $\epsilon_0 > 0$ such that

$$|\lambda^{(\pm)}(\eta)| \geq \frac{\epsilon_0}{h}, \quad \text{on } \mathcal{K}_0^{(0)}/h.$$ 

Therefore there exists $h_0 > 0$ such that for $0 < h < h_0$

$$|\det (L_h(e^{-h\eta}) - z)| \geq C/h^2 \quad \text{on } \text{supp} \chi^{(0)}(h\eta)$$

for a constant $C > 0$. We then have, letting $\| \cdot \|_s = \| \cdot \|_{L^2,s}$,

$$(8.7) \quad \| u_h^{(0)} \|_s \leq Ch (\| f_h \|_s + \| u_h \|_{-s}), \quad s > 1/2.$$

We put

$$P_h^{(\pm)}(\xi) = \frac{\sqrt{3 + 2b(h(\xi + d_h^{(\pm)}))}}{3h},$$

$$\mathcal{K}^{(\pm)} = \mathcal{K}_0^{(\pm)} - d_h^{(\pm)} = \{ \xi = \eta - d_h^{(\pm)}; \eta \in \mathcal{K}_0^{(\pm)} \}.$$  

In view of (8.3), (8.4) and Taylor expansion, we have the following lemma.

**Lemma 8.1.** On $\mathcal{K}^{(\pm)}/h$, $P_h^{(\pm)}(\xi)$ vanishes only at $\xi = 0$. Moreover, there exists a constant $C > 0$ independent of $h$ such that

$$P_h^{(\pm)}(\xi) \geq C|\xi|, \quad \xi \in \mathcal{K}^{(\pm)}/h.$$
In Subsection 4.5, we studied this equation by projecting onto each characteristic root. Here, we deal with \( L_h(S_h) \) without diagonalization. For the solution of the equation

\[
( - \Delta_{\text{disc}, h} - z ) u_h = f_h,
\]

we split \( u_h = u_h^+ + u_h^- \), where

\[
u_h^{(\pm)} = \mathcal{F}^{-1}_{\text{disc}, h}(\chi^{(\pm)}(h \eta) \check{u}_h(\eta)), \quad f_h^{(\pm)} = \mathcal{F}^{-1}_{\text{disc}, h}(\chi^{(\pm)}(h \eta) \check{f}_h(\eta)),
\]

which satisfy

\[
( L_h(S_h) - z ) u_h^{(\pm)} = f_h^{(\pm)}.
\]

Define the gauge transformation \( G^{(\pm)} \) by

\[
( G^{(\pm)} a )(n) = e^{i h n \cdot d^{(\pm)}_h} a(n) = e^{i n \cdot d^{(\pm)}_h} a(n), \quad a \in L^2(\mathbb{Z}^d).
\]

We put

\[
v_h^{(\pm)} = ( G^{(\pm)} )^* u_h^{(\pm)},
\]

and consider

\[
( ( G^{(\pm)} )^* L_h(S_h) G^{(\pm)} - z ) v_h^{(\pm)} = ( G^{(\pm)} )^* f_h^{(\pm)}.
\]

Note that

\[
( \mathcal{F}_{\text{disc}, h}( G^{(\pm)} )^* L_h(S_h) G^{(\pm)} - z ) v_h^{(\pm)} = \mathcal{F}_{\text{disc}, h}( G^{(\pm)} )^* L_h(e^{i h (\xi + d^{(\pm)}_h)})( \mathcal{F}_{\text{disc}, h} a)(\xi).
\]

We put

\[
q(\eta) = 1 + e^{i \eta_1} + e^{i \eta_2}.
\]

In view of (8.2), we have

\[
( L_h(e^{i h (\xi + d^{(\pm)}_h)} - z) )^{-1} = \frac{1}{P_h^{(\pm)}(\xi) - z^2} \begin{pmatrix} 0 & q(h(\xi + d^{(\pm)}_h)) \\ q(-h(\xi + d^{(\pm)}_h)) & 0 \end{pmatrix}.
\]

By virtue of Lemma 8.1, one can argue in the same way as in (i) to obtain the uniform estimates with respect to \( 0 < h < h_0 \). We take \( \psi^{(\pm)}(\eta) \in C_0^\infty(\mathbb{R}^d) \) and put

\[
f_h(x) = \left( \frac{h}{2 \pi} \right)^{d/2} \int_{\mathbb{R}^d} e^{i \eta \cdot x} \left( \psi^{(+)}(\eta - d^{(+)}_h) + \psi^{(-)}(\eta - d^{(-)}_h) \right) d\eta.
\]

Lemma 8.2. Let \( v_h^{(\pm)} = \tilde{v}_h^{(\pm)}(E + i 0) \) for \( E > 0 \). Then we have for \( \epsilon > 0 \)

\[
\| \tilde{v}_h^{(\pm)} \|_{-1/2-\epsilon} \leq C \| f \|_{m, s},
\]

\[
\| p_- (D_x) \tilde{v}_h^{(\pm)} \|_{-1/2+\epsilon} \leq C \| f \|_{m, s}, \quad p_- \in \mathcal{P}_-.
\]

Let \( u_h = u_h(E + i 0) = G^{(+)} v^{(+)} + G^{(-)} v^{(-)} \). Noting that

\[
G^{(\pm)} v^{(\pm)} = \hat{v}^{(\pm)}(\xi - d^{(\pm)}_h),
\]

we have

\[
\tilde{u}_h = e^{i x \cdot d^{(+)}_h} \tilde{v}_h^{(+)} + e^{i x \cdot d^{(-)}_h} \tilde{v}_h^{(-)}.
\]

Lemma 8.2 yields

\[
\| \tilde{u}_h \|_{-1/2-\epsilon} \leq C \| f \|_{m, s}.
\]

By the same argument as in (i), we see that \( v_h^{(\pm)} \to v^{(\pm)} \), hence \( \tilde{u}_h \) behaves like

\[
\tilde{u}_h \simeq e^{i x \cdot d^{(+)}_h} \tilde{v}_h^{(+)} + e^{i x \cdot d^{(-)}_h} \tilde{v}_h^{(-)}.
\]
We show that $\tilde{v}^{(\pm)}$ are solutions to massless Dirac equations.

We consider the case of $v_h^{(+)}$, and make the change of variables

$$\eta_1 = \xi_1 + \frac{2\pi}{3h}, \quad \eta_2 = \xi_2 - \frac{2\pi}{3h},$$

to obtain

$$1 + e^{ih\eta_1} + e^{ih\eta_2} = e^{2\pi i/3}(e^{ih\xi_1} - 1) + e^{-2\pi i/3}(e^{ih\xi_2} - 1)$$

$$\sim -hi\xi_1 + 2\sqrt{3}(\xi_1 - \xi_2)$$

as $h \to 0$. We put

$$\zeta_1 = \frac{\sqrt{3}}{6}(\xi_1 - \xi_2), \quad \zeta_2 = -\frac{1}{6}(\xi_1 + \xi_2).$$

Then we have

$$-\frac{1}{3h}(1 + e^{ih\eta_1} + e^{ih\eta_2}) \sim \zeta_1 - i\zeta_2.$$ 

We put

$$y_1 = \sqrt{3}(x_1 - x_2), \quad y_2 = -3(x_1 + x_2).$$

Then the map $(x, \eta) \to (y, \zeta)$ is a symplectic transformation. We then have

$$-\frac{1}{3h}(1 + e^{ih\eta_1} + e^{ih\eta_2}) \sim \zeta_1 - i\zeta_2,$$

as $h \to 0$. Similarly

$$-\frac{1}{3h}(1 + e^{-ih\eta_1} + e^{-ih\eta_2}) \sim \zeta_1 + i\zeta_2.$$ 

We have thus obtained

$$\mathcal{L}_h(e^{ih(\xi_1 + \xi_2)}) \sim \left( \begin{array}{cc} 0 & \zeta_1 - i\zeta_2 \\ \zeta_1 + i\zeta_2 & 0 \end{array} \right) = \zeta_1 \sigma_1 + \zeta_2 \sigma_2,$$

where $\sigma_1, \sigma_2$ are Pauli spin matrices. Therefore, $v^{(+)}$ satisfies

$$\left( \sigma_1 \frac{1}{i} \frac{\partial}{\partial y_1} + \sigma_2 \frac{1}{i} \frac{\partial}{\partial y_2} - E \right) v^{(+)} = g^{(+)}.$$ 

Similarly $v^{(-)}$ satisfies

$$\left( -\sigma_1 \frac{1}{i} \frac{\partial}{\partial y_1} + \sigma_2 \frac{1}{i} \frac{\partial}{\partial y_2} - E \right) v^{(-)} = g^{(-)},$$

where

$$g^{(\pm)} = \mathcal{F}^{-1}_{\text{cont}} v^{(\pm)}.$$ 

We have thus proven the following theorem.

**Theorem 8.3.** Assume that $f \in H^{m,s}(\mathbb{R}^2)$ with $m > 2, s > 3$. Then for the solution $u_h$ of (8.8), $\tilde{u}_h$ behaves like

$$\tilde{u}_h \simeq e^{ixd_h^{(+)}v^{(+)}} + e^{ixd_h^{(-)}v^{(-)}}.$$ 

Moreover, $\tilde{v}^{(\pm)}$ satisfy the massless Dirac equation (8.10) and (8.11) after the symplectic transformation.
8.1.2. Global minimum. To deal with the case near the lowest energy, instead of (8.2), we should consider
\[-\frac{1}{3\hbar^2} \begin{pmatrix} 0 & 1 + S_{h,1}^* + S_{h,2}^* \\ 1 + S_{h,1} + S_{h,2} & 0 \end{pmatrix} .\]

The reference energy is \( E_0 = -\hbar^2/2 \), and we consider the Hamiltonian
\[(8.12) \quad \mathcal{L}_h(e^{-i\hbar\eta}) = -\frac{1}{3\hbar^2} \begin{pmatrix} -3 & 1 + e^{-i\hbar\eta_1} + e^{-i\hbar\eta_2} \\ 1 + e^{-i\hbar\eta_1} + e^{-i\hbar\eta_2} & -3 \end{pmatrix} .\]

Then the characteristic roots are
\[\lambda_h(\eta) = \frac{3 \pm \sqrt{3 + 2b_2(h\eta)}}{3\hbar}.\]

The minimum is attained only at \( \eta = 0 \), and \( \lambda_h^-(\eta) \) has a Taylor expansion
\[\lambda_h^-(\eta) = \frac{1}{9} \left( \eta_2^2 - \eta_1 \eta_2 + \eta_2^2 \right) + O(h^2).\]

Therefore we obtain the following theorem.

**Theorem 8.4.** Assume that \( f \in H^{m,s}(\mathbb{R}^2) \) with \( m > 2 \), \( s > 3 \). Assume also that \( V \in H^2(\mathbb{R}^2) \) with \( s > 1 \), and \( z \notin \mathbb{R} \). Then the solution of the Schrödinger equation
\[(-\Delta_{\text{disc},h} + V_{\text{disc},h} - z)u_h = f_h \text{ on the hexagonal lattice, where } \Delta_{\text{disc},h} \text{ is the difference operator with symbol } \mathcal{L}_h, \]
converges to that for the continuum Schrödinger equation
\[\left( -\frac{1}{9} \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2^2} + V(x) - z \right) u = f, \quad \text{in } \mathbb{R}^2.\]

8.2. Graphite. The Laplacian for graphite is written as
\[H_{0h}(S_h, S_h^*) = \frac{1}{h} H_0(S_h, S_h^*),\]

\[H_0(S_h, S_h^*) = -\frac{1}{4} \begin{pmatrix} 0 & 1 + S_{h,1}^* + S_{h,2}^* \\ 1 + S_{h,1} + S_{h,2} & 0 \end{pmatrix} .\]

Put
\[T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .\]

Then it can be block diagonalized as follows
\[T^* H_0(S_h, S_h^*) T = -\frac{1}{4} \begin{pmatrix} C(S_h, S_h^*) + I_2 & 0 \\ 0 & C(S_h, S_h^*) - I_2 \end{pmatrix} , \quad C(S_h, S_h^*) = \begin{pmatrix} 0 & 1 + S_{h,1} + S_{h,2} \\ 1 + S_{h,1}^* + S_{h,2}^* & 0 \end{pmatrix} .\]

Therefore it can be dealt with in the same way as in the hexagonal lattice.
9. OTHER EXAMPLES

9.1. Kagome lattice. Passing to the Fourier series, the symbol of $-\Delta_G$ for the Kagome lattice is written as

$$L(e^{-i\eta}) = -\frac{1}{4} \begin{pmatrix} 0 & 1 + e^{i\eta_1}e^{-i\eta_2} & 1 + e^{i\eta_1} \\ 1 + e^{-i\eta_1}e^{i\eta_2} & 0 & 1 + e^{i\eta_2} \\ 1 + e^{-i\eta_1} & 1 + e^{-i\eta_2} & 0 \end{pmatrix}.$$ 

The characteristic determinant is

$$p(\eta, \lambda) = \det (L(e^{-i\eta}) - \lambda) = -\left(\lambda - \frac{1}{2}\right) \left(\lambda^2 + \frac{\lambda}{2} - \frac{\beta(\eta)}{8}\right),$$

where

$$\beta(\eta) = 1 + \cos \eta_1 + \cos \eta_2 + \cos(\eta_1 - \eta_2).$$

The characteristic roots are

$$\lambda = -\frac{1}{4} \pm \frac{\sqrt{2\beta(\eta)} + 1}{4}. \tag{9.1}$$

Then the spectrum of $-\Delta_G$ is $\sigma(-\Delta_G) = [-1, 1/2]$.

We first take the reference energy $E_0 = -1/h^2$, and consider

$$L_h(e^{-i\eta}) = -\frac{1}{4h^2} \begin{pmatrix} -4 & 1 + e^{ih\eta_1}e^{-i\eta_2} & 1 + e^{ih\eta_2} \\ 1 + e^{-ih\eta_1}e^{i\eta_2} & -4 & 1 + e^{ih\eta_2} \\ 1 + e^{-ih\eta_1} & 1 + e^{-i\eta_2} & -4 \end{pmatrix}.$$ 

The least characteristic root is then given by

$$P_h(\eta) = \frac{3 - \sqrt{2\beta(h\eta)} + 1}{4h^2}.$$ 

It is expanded as

$$P_h(\eta) = \frac{1}{6h^2} \left( \sin^2 \frac{h\eta_1}{2} + \sin^2 \frac{h\eta_2}{2} + \sin^2 \frac{h(\eta_1 - \eta_2)}{2} \right)$$

$$= \frac{1}{12} \left( \eta^2_1 - \eta_1 \eta_2 + \eta^2_2 \right) + O(h^2).$$
For the Kagome lattice the unique continuation theorem does not hold, and there may be embedded eigenvalues. Therefore we consider the continuum limit for the complex energy $z \not\in \mathbb{R}$.

**Theorem 9.1.** Assume that $f \in H^{m,s}(\mathbb{R}^d)$ with $s > 3$ and $m > 2$, and that $z \not\in \mathbb{R}$. Assume also that $V \in H^s(\mathbb{R}^2)$ with $s > 1$. Then the solution of \((-\Delta_{\text{disc},h} + V_{\text{disc},h} - z)u = f\) converges to that for the Schrödinger equation
\[
\left( -\Delta - \frac{1}{12}(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2^2}) + V(x) - z \right)v = g.
\]

For $\lambda = -1/4$, the characteristic roots (9.1) are double. In this case, we take the reference energy $E_0 = -1/4$ and consider
\[
\mathcal{L}_h(e^{-i\eta}) = -\frac{1}{4h} \begin{pmatrix}
-1 & 1 + e^{ih\eta} & 1 + e^{ih\eta} \\
1 + e^{-ih\eta} & 1 & 1 + e^{ih\eta} \\
1 + e^{-ih\eta} & 1 + e^{ih\eta} & 1
\end{pmatrix},
\]
then $2\beta(h\eta) + 1 = 0$ if and only if $\eta$ is the Dirac point, i.e. $\eta = d_h$, where $d_h$ is as in (8.5). One can then argue as in the case of the hexagonal lattice to show the following theorem.

**Theorem 9.2.** Assume that $f \in H^{m,s}(\mathbb{R}^d)$ with $s > 3$ and $m > 2$, and that $z \not\in \mathbb{R}$. Then the solution of \((-\mathcal{G}_h^* \Delta_{\text{disc},h} \mathcal{G}_h - z)u = f\) behaves like
\[
\tilde{u}_h \simeq e^{ix \cdot d_h^{(+)}} \tilde{v}^{(+)} + e^{ix \cdot d_h^{-}} \tilde{v}^{(-)},
\]
and $\tilde{v}^{(\pm)}$ satisfy the massless Dirac equation (8.10), (8.11).

9.2. **Subdivision of square lattice.** For this lattice the unique continuation property does not hold (see [3]). Therefore we apply Theorems 6.2 and 6.3.

Passing to the Fourier series in the case $h = 1$, $-\Delta_{\text{disc}}$ becomes the following matrix
\[
\mathcal{L}(e^{-i\eta}) = -\frac{1}{2\sqrt{d}} \begin{pmatrix}
0 & 1 + e^{-i\eta} & \cdots & 1 + e^{-i\eta} \\
1 + e^{i\eta} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 + e^{i\eta d} & 0 & \cdots & 0
\end{pmatrix},
\]
whose determinant is computed as

\[
\det(L(e^{-\eta}) - \lambda) = (-\lambda)^{d-1}(\lambda^2 - \frac{1}{2d}(d + \sum_{j=1}^{d} \cos \eta_j)).
\]

Therefore the characteristic roots are

\[
\lambda^{(\pm)}(\eta) = \pm \sqrt{\frac{1}{2d}(d + \sum_{j=1}^{d} \cos \eta_j)}.
\]

To observe the behavior near the bottom of energies, we take the reference energy \(E_0 = -1/h^2\), and consider

\[
L_h(e^{-ih\eta}) = -\frac{1}{2\sqrt{d}h^2} \begin{pmatrix}
-2\sqrt{d} & 1 + e^{-ih\eta_1} & \cdots & 1 + e^{-ih\eta_d} \\
1 + e^{ih\eta_1} & -2\sqrt{d} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 + e^{ih\eta_d} & 0 & \cdots & -2\sqrt{d}
\end{pmatrix}.
\]

We put

\[
\lambda_h^{(\pm)}(\eta) = \frac{1}{h^2}(1 \pm \sqrt{\frac{1}{2d}(d + \sum_{j=1}^{d} \cos h\eta_j)})
\]

As \(h \to 0\), it behaves like

\[
\lambda_h^{(-)}(\eta) = \frac{1}{8} \sum_{j=1}^{d} \eta_j^2 + O(h^2).
\]

**Theorem 9.3.** Let \(V(x) \in H^s(\mathbb{R}^d)\) with \(s > d/2\) and compactly supported. Assume that \(f \in H^{m,s}(\mathbb{R}^d)\) with \(s > d + 1\) and \(m > d/2 + 1\). If \(z \not\in \mathbb{R}\), the solution of the equation \((-\Delta_{\text{disc},h} + V_{\text{disc},h} - z)u = f\) converges to the solution of

\[
(-\frac{1}{8}\Delta + V(x) - z)v = g,
\]

where \(v\) and \(g\) are given in Theorem 6.2.
To study the behavior near the 0 energy of \(-\Delta_\Gamma\), we take the reference energy \(E_0 = 0\), and consider
\[
\mathcal{L}_h(e^{-ih\eta}) = -\frac{1}{2\sqrt{dh}} \begin{pmatrix}
0 & 1 + e^{-ih\eta} & \cdots & 1 + e^{-ih\eta_d} \\
1 + e^{ih\eta} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 + e^{ih\eta_d} & 0 & \cdots & 0
\end{pmatrix}.
\]
Then the characteristic roots are
\[
\lambda(\pm)(\eta) = \pm \frac{1}{h} \sqrt{\frac{1}{2d} (d + \sum_{j=1}^d \cos h\eta_j)}.
\]
They vanish if and only if \(\eta = d_h\), where
\[
d_h = \frac{1}{h}(\pi, \ldots, \pi).
\]
We then have
\[
\lambda_h(\pm)(\xi + d_h) = |\xi| + O(h^2).
\]

**Theorem 9.4.** Let \(z \not\in \mathbb{R}\), and assume that \(f \in H^{m,s}(\mathbb{R}^d)\) with \(s > d + 1\) and \(m > d/2 + 1\). Assume also that \(V \in H^s(\mathbb{R}^2)\) with \(s > 1\). Then the solution of the gauge transformed equation \((-G^*_k \Delta_{\text{disc},h} G_h + V_{\text{disc},h} - z)u = f\) converges to the solution of
\[
(-|D_x| + V(x) - z)v = g,
\]
where \(v\) and \(g\) are given in Theorem 6.2.

10. Appendix

10.1. The Riemann integral. We recall the notation
\[
I_{hn} = hn + [-h/2, h/2]^d, \quad n \in \mathbb{Z}^d.
\]
We recall (4.6). Assume \(f \in L^2(\mathbb{R}^d) \cap C(\mathbb{R}^d)\) and \((f(hn))_{n \in \mathbb{Z}^d} \in L^1(\mathbb{Z}^d)\). Then
\[
\hat{f}_h(\xi) = \left(\frac{h}{2\pi}\right)^{d/2} \sum_{n \in \mathbb{Z}^d} f(hn) e^{-ihn \cdot \xi}.
\]
Recall the weighted Sobolev space \((2,13)\), whose norm is denoted by \(\|\cdot\|_{m,s}\).

**Lemma 10.1.** (1) Assume \(m > [d/2]\) and \(s > d/2\). Let \(f \in H^{m,s}(\mathbb{R}^d)\). Then
\[
h^{d/2}\|\hat{f}_h\|_{L^2(\mathbb{T}^d_h)} \leq C\|f\|_{m,s}.
\]
(2) Assume \(m > [d/2]\) and \(s > d\). Let \(f \in H^{m,s}(\mathbb{R}^d)\). Then
\[
h^d \sum_{n \in \mathbb{Z}^d} |f(hn)| \leq C\|f\|_{m,s}.
\]
(3) Assume \(m > [d/2] + 1\) and \(s > d\). Let \(f \in H^{m,s}(\mathbb{R}^d)\). Then
\[
h^d \sum_{n \in \mathbb{Z}^d} f(hn) - \int_{\mathbb{R}^d} f(x)dx \leq C h \|f\|_{m,s}.
\]
Proof. Let $hx \in I_{hn}$. Then $hx-hn \in [-h/2, h/2]^d$, which implies $|hx| \leq |hn| + \frac{1}{2}\sqrt{dh}$. Thus for $h \in [0, 1]$ and $hx \in I_{hn}$ we have

$$1 + |hx| \leq C_d(1 + |hn|).$$

Assume $s > d$. Then we have

$$\sum_{n \in \mathbb{Z}^d} h^d \sum_{n \in \mathbb{Z}^d} (1 + |hn|)^{-s} = h^d \sum_{n \in \mathbb{Z}^d} (1 + |hn|)^{-s}dx$$

$$\leq C_d h^d \sum_{n \in \mathbb{Z}^d} (1 + |hx|)^{-s}dx$$

$$= C_d h^d \int_{\mathbb{R}^d} (1 + |hx|)^{-s}dx = C < \infty.$$ 

If $f \in H^{m,s}(\mathbb{R}^d)$, where $m > [d/2]$ and $s \geq 0$, then the Sobolev inequality implies

$$|f(x)| \leq C\|f\|_{m,s}(x)^{-s}.$$ 

Assume $m > [d/2]$ and $s > d/2$. By the Parseval equation, (10.6), and (10.5) we have

$$h^{d/2}\|\hat{f}(n)\|_{L^2(\mathbb{T}_d)} = h^d \sum_{n \in \mathbb{Z}^d} |\langle f, n \rangle|^2 \leq C\|f\|_{m,s} \sum_{n \in \mathbb{Z}^d} \langle h \rangle^{-2s}h^d \leq C\|f\|_{m,s}.$$ 

Thus part (1) follows.

Assume $m > [d/2]$ and $s > d$. Then

$$\sum_{n \in \mathbb{Z}^d} |f(hn)|h^d \leq \sum_{n \in \mathbb{Z}^d} (1 + |hn|)^{-s}h^d \leq C\|f\|_{m,s}$$

and part (2) follows.

Assume $m > [d/2] + 1$ and $s > d$. Let $f \in H^{m,s}(\mathbb{R}^d)$. Then note that

$$|\partial_\alpha f(x)| \leq C\|f\|_{m,s}(x)^{-s}, \quad |\alpha| \leq 1.$$ 

Define $g(t) = f(x + t(y - x)), x, y \in \mathbb{R}^d$. Then

$$f(y) - f(x) = \int_0^1 g'(t)dt = \int_0^1 (\nabla f)(x + t(y - x)) \cdot (y - x)dt.$$ 

The following estimate holds:

$$|(\nabla f)(x + t(y - x))| \leq C\|f\|_{m,s}(1 + |x + t(y - x)|)^{-s}.$$ 

Since $t \in [0, 1]$, we have for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$

$$|x + t(y - x)| \geq |x| - |t(y - x)| \geq |x| - 1,$$

which implies

$$1 + |x| \leq 2(1 + |x + t(y - x)|).$$

Thus for $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$ we have

$$|(\nabla f)(x + t(y - x))| \leq C\|f\|_{m,s}(1 + |x|)^{-s}$$

and then

$$|f(y) - f(x)| \leq C\|f\|_{m,s}(1 + |x|)^{-s}|x - y|.$$
Assume \( y \in I_{hn} \). Then
\[
|f(y) - f(hn)| \leq C\|f\|_{m,s}(1 + |hn|)^{-s}h.
\]
(Note that the above estimates hold with \(|x - y| \leq 1\) replaced by \(|x - y| \leq c_0\) for some fixed \(c_0\), depending on \(d\).) Integrating we get
\[
h^df(hn) = \int_{I_{hn}} f(y)dy + R(h, n),
\]
where
\[
|R(h, n)| \leq C\|f\|_{m,s}(1 + |hn|)^{-s}h^{d+1}.
\]
Then
\[
\sum_{n \in \mathbb{Z}^d} h^df(hn) = \sum_{n \in \mathbb{Z}^d} \int_{I_{hn}} f(y)dy + \sum_{n \in \mathbb{Z}^d} R(h, n)
\]
and
\[
\sum_{n \in \mathbb{Z}^d} |R(h, n)| \leq C\|f\|_{m,s} \sum_{n \in \mathbb{Z}^d} (1 + |hn|)^{-s}h^{d+1} \leq C\|f\|_{m,s}h
\]
by (10.5). This concludes the proof of part (3).

10.2. Lemmas for the Besov space. We use the following lemmas in 
§2 and §4, which are in [12], Chap. 14, §1 or follow from an adaptation of the proof there.

Lemma 10.2. Let \( b_n \geq 0, n = 0, 1, 2, \cdots \), and put
\[
\alpha = \sup_{n \geq 0} \frac{b_n}{2^n}, \quad \beta = \sup_{n \geq 0} \frac{1}{2^n}(b_0 + b_1 + \cdots + b_n).
\]
Then we have
\[
\alpha \leq \beta \leq 3\alpha.
\]

Lemma 10.3. Letting
\[
A = \sup_{j \geq 0} \frac{h^d}{2^j} \sum_{|hn| \leq 2^j} |u(n)|^2 h^d, \quad B = \sup_{R>1} \frac{h^d}{R} \sum_{|hn| \leq R} |u(n)|^2,
\]
we have
\[
A \leq B \leq 2A.
\]

Lemma 10.4. (1) We have
\[
\int_{-\infty}^{\infty} \|f(x_1, \cdot)\|_{L^2(\mathbb{R}^{d-1})} \leq \sqrt{2}\|f\|_{\mathcal{B}(\mathbb{R}^d)},
\]
\[
\|f\|_{\mathcal{B}^s(\mathbb{R}^d)} \leq \sqrt{2} \sup_{x_1 \in \mathbb{R}} \|f(x_1, \cdot)\|_{L^2(\mathbb{R}^{d-1})}.
\]
(2) If \( P \in S_{1,0}^0 \), we have
\[
P \in \mathcal{B}(\mathcal{B}; \mathcal{B}) \cap \mathcal{B}(\mathcal{B}^*; \mathcal{B}^*) \cap \mathcal{B}(\mathcal{B}^*_0; \mathcal{B}_0^*).
10.3. A priori estimates for discrete Schrödinger equations. We prove here a priori estimates needed in the proof in §5. For the sake of simplicity, we explain the proof for the case of square lattice. It works for the general case. Letting

\[ D_{h,j} = \frac{1}{h}(I - S_{h,j}), \]

we have

\[ - \Delta_{\text{disc},h} = \sum_{j=1}^{d} D_{h,j}^* D_{h,j} = \sum_{j=1}^{d} D_{h,j} D_{h,j}^*. \]

Take a function \( \chi(x) \) on \( \mathbb{R}^d \), and let \( \chi_h \) be the operator of multiplication by the function \( \chi(hn) \) on \( L^2(\mathbb{Z}^d_h) \). Then we have

\[ ([D_{h,j}, \chi_h]u)(n) = \frac{1}{h}(\chi(hn) - \chi(hn - he_j))(S_{h,j}u)(n), \]

\[ ([D_{h,j}^*, \chi_h]u)(n) = \frac{1}{h}(\chi(hn) - \chi(hn + he_j))(S_{h,j}^*u)(n). \]

We also have

\[ [\Delta_{\text{disc},h}, \chi_h] = \sum_{j=1}^{d} (D_{h,j}[D_{h,j}, \chi_h] + [D_{h,j}^*, \chi_h]D_{h,j}). \]

In the following, constants \( C \) are independent of \( 0 < h < h_0 \).

**Lemma 10.5.** Let \( \chi(x) = (1 + |x|^2)^{1/2} \). Then for any \( s > 0 \),

\[ \left| \frac{1}{h}(\chi^{-s}(hn) - \chi^{-s}(h(n-y))) \right| \leq C_s \chi^{-s+1}(hn), \text{ for } |y| \leq 1, \quad n \in \mathbb{Z}^d, \quad 0 < h < 1/2. \]

**Proof.** Letting \( g(t) = \chi^{-s}(thn+(1-t)h(n-y)) \), and noting that \( thn+(1-t)h(n-y) = hn - (1-t)hy \), we have

\[ \chi^{-s}(hn) - \chi^{-s}(h(n-y)) = \int_{0}^{1} g'(t)dt = h \int_{0}^{1} y \cdot \nabla \chi^{-s}(hn - (1-t)hy)dt. \]

Since \( 0 < h < 1/2 \), we have

\[ 1 + |hn - (1-t)hy| \geq 1 + |hn| - h \geq \frac{1}{2} + |hn| \geq \frac{1}{2}(1 + |hn|). \]

Then, we have

\[ |\chi^{-s}(hn) - \chi^{-s}(h(n-y))| \leq Ch \int_{0}^{1} (1 + |hn - (1-t)hy|)^{-s+1}dt \]

\[ \leq Ch(1 + |hn|)^{-s-1}, \]

which proves the lemma. \( \square \)

**Lemma 10.6.** For \( s \geq 0 \), we have

\[ \| [D_{h,j}, \chi_h^{-s}]u \| + \| [D_{h,j}^*, \chi_h^{-s}]u \| \leq C_s \| \chi_h^{-s+1}u \|. \]

**Proof.** Use (10.10), (10.11) and Lemma 10.5. \( \square \)

By virtue of Lemma 10.6, \([D_{h,j}, \chi_h^{-s}]\chi_h^{s+1}\) and \([D_{h,j}^*, \chi_h^{-s}]\chi_h^{s+1}\) are uniformly bounded with respect to \( 0 < h < h_0 \) in \( L^2(\mathbb{Z}^d) \).
Lemma 10.7. Let $u_h$ be a solution to the equation
\begin{equation}
(\Delta_{\text{disc},h} - z)u_h = f_h.
\end{equation}
Then for any $s \geq 0$,
\begin{equation}
\|\chi(x)^{-s}\tilde{u}_h(x)\|_{H^2(\mathbb{R}^d)} \leq C_s\left(\|\tilde{f}_h\|_{L^{2-s}(\mathbb{R}^d)} + \|\tilde{u}_h\|_{L^{2-s}(\mathbb{R}^d)}\right).
\end{equation}

Proof. Noting that
\[ C^{-1}|k| \leq \left|\frac{1}{h}(1 - e^{ih})\right| \leq C|k|, \]
\[ C^{-1}|k|^2 \leq \left|\frac{1}{h^2}(1 - \cos(hk))\right| \leq C|k|^2, \]
for $k \in \mathbb{R}$, we have
\[ C^{-1}\|\Delta\tilde{u}_h\|_{L^2(\mathbb{R}^d)} \leq \|\Delta_{\text{disc},h}u\|_{L^2(\mathbb{Z}_h^d)} \leq C\|\Delta\tilde{u}_h\|_{L^2(\mathbb{R}^d)}. \]
This and (10.13) imply (10.14) for $s = 0$.

Note that we also obtain
\begin{equation}
C^{-1}\|\xi\tilde{u}_h(\xi)\|_{L^2(T_h^d)} \leq \|D_{h;j}u_h\|_{L^2(\mathbb{Z}_h^d)} \leq C^{-1}\|\xi\tilde{u}_h(\xi)\|_{L^2(T_h^d)}.
\end{equation}

Letting $g_s(n) = \chi^{-s}(n)f_h(n)$ and $v_s = \chi^{-s}u_h$, we have
\[ (-\Delta_{\text{disc},h} - z)v_s = g_s - [\Delta_{\text{disc},h}, \chi^{-s}]u. \]
By (10.10) and (10.11), the $L^2(\mathbb{Z}_h^d)$ norm of the right-hand side is estimated from above by $\|g_s\|_{L^2(\mathbb{Z}_h^d)} + \|v_s\|_{L^2(\mathbb{Z}_h^d)}$. Then (10.14) for $s > 0$ follows from the case $s = 0$. \hfill \square

Acknowledgements. The authors were partially supported by the Danish Council for Independent Research | Natural Sciences, Grant DFF–8021-0084B, Grants-in-Aid for Scientific Research (S) 15H05740, and Grants-in-Aid for Scientific Research (C) 20K03667, Japan Society for the Promotion of Science.

References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, Ann. Sc. Norm. Super. Pisa. 2 (1975), 151-218.
[2] S. Agmon and L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics, J. d’Anal. Math. 30 (1976), 1-38.
[3] K. Ando, H. Isozaki and H. Morioka, Spectral properties of Schrödinger operators on perturbed lattices, Ann. Henri Poincaré 17 (2016), 2103-2171.
[4] K. Ando, H. Isozaki and H. Morioka, Inverse scattering for Schrödinger operators on perturbed lattices, Ann. Henri Poincaré 19 (2018), 3397-3455.
[5] J. M. Barbaroux, H. D. Cornean and E. Stockmeyer, Spectral gaps in graphene antidot lattices, Integ. Eq. and Op. Theor. 89, 631-646 (2017).
[6] J. M. Barbaroux, H. D. Cornean and S. Zalczer, Localization for gapped Dirac Hamiltonians with random perturbations: Applications to graphene antidot lattices, Doc. Math. 24 (2019), 6593.
[7] H. Baumgärtel, Analytic perturbation theory for matrices and operators. Operator Theory: Advances and Applications, 15. Birkhäuser Verlag, Basel, 1985. 427 pp. ISBN: 3-7643-1664-0
[8] D. M. Eidus, The principle of limit amplitude, Russian Math. Survey 24 (1969), 97-167.
[9] J. C. Guenin and H. Siedentop, Dipoles in graphene have infinitely many bound states, J. Math. Phys. 55 (12) : 122304 (2014).
[10] Y. Hong and C. Yang, Strong convergence for discrete nonlinear Schrödinger equations in the continuum limit SIAM J. Math. Anal. **51** no. 2, (2019), 1297-1320.

[11] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Distribution Theory and Fourier Analysis, Springer Verlag, Berlin-Heidelberg-New York Tokyo (1980).

[12] L. Hörmander, The Analysis of Linear Partial Differential Operators II, Differential Operators of Constant Coefficients, Springer Verlag, Berlin-Heidelberg-New York Tokyo (1983).

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Pseudodifferential operators, Springer Verlag, Berlin-Heidelberg-New York Tokyo (1985).

[14] L. I. Ignat and E. Zuazua, Numerical dispersive schemes for the nonlinear Schrödinger equation, SIAM J. Numer. Anal. **47** (2009), 1366-1390.

[15] K. Ito and A. Jensen, Branching form of the resolvent at threshold for ultra-hyperbolic operators and discrete Laplacians, J. Funct. Anal. **277**, No. 4, 15 (2019), 965-993.

[16] T. Ikebe and Y. Saito, Limiting absorption method and absolute continuity for Schrödinger operators, J. Math. Kyoto Univ. **12** (1972), 513-542.

[17] W. Jäger, Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem HilbertRaum, Math. Z. **113** (1970), 68-98.

[18] T. Kato, Perturbation Theory for Linear Operators, 2nd ed., Springer Berlin-Heidelberg-New York (1980).

[19] T. Kato and S. T. Kuroda, The abstract theory of scattering, Rock. Mt. J. Math. **1** (1971), 127-171.

[20] S. T. Kuroda, Scattering theory for differential operators, I, II, J. Math. Soc. Japan **25** (1973), 75-104, 222-234.

[21] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Commun. Math. Phys. **78** (1981), 391-408.

[22] S. Nakamura and Y. Tadano, On a continuum limit of discrete Schrödinger operators on square lattice, J. Spectr. Theory, to appear, arXiv:1903.10561v1.

[23] I. J. Schoenberg, Cardinal Spline Interpolation, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM (1973)

**Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba, 305-8571, Japan**

*E-mail address: isozakih@math.tsukuba.ac.jp*

**Department of Mathematical Sciences, Aalborg University, Skjernvej 4A, 9220 Aalborg Ø, Denmark**

*E-mail address: matarne@math.aau.dk*