Abstract
We suggest a method of reduction of mixed absolute and relative boundary conditions to pure ones. The case of rank two tensor is studied in detail. For four-dimensional disk the corresponding heat kernel is expressed in terms of scalar heat kernels. The result for scaling behavior $\zeta(0)$ agrees with previous calculations.

1. Introduction
The modern interest to heat kernel expansion for manifolds with boundaries is connected with applications to quantum cosmology and quantum state of the universe (see monograph [1] and references therein). This topic is also studied by mathematicians, especially when related to $p$-forms and de Rham complex. It is worth noting that the absolute and relative boundary conditions, which are natural for the de Rham complex [2], are also singled out by the BRST-invariance of quantum theory [3,4]. The absolute boundary conditions are defined [2] as the Neumann boundary conditions for tangential components of a $p$-form and the Dirichlet conditions for other components

$$\partial_0 B_{i...k}|_{\partial M} = 0, \quad B_{0i...k}|_{\partial M} = 0$$ (1)

The relative boundary conditions are dual to the set (1). These conditions are mixed. There were contradictions in computations of the heat kernel expansion for mixed boundary conditions even for QED in covariant gauge [3,5]. These contradictions were removed recently [6,7] using the corrections [7] to the analytic expressions [8]. However, further checks are highly desirable.

The aim of this work is two-fold. First, we suggest a procedure of reduction of the mixed boundary conditions on a disk to pure ones and derive

1Permanent address, e-mail: dvassil @ sph.spb.su
expression for all the Seeley-Gilkey coefficients in terms of the heat kernel expansion for scalar field. This expression is especially efficient in analyzing higher terms in the heat kernel expansion. Second, in a particular case we compare our results with calculations [6] based on analytic formulae [8,7] and find that these two approaches agree. To achieve these goals we do the following. We analyze the Laplace operator for $p$-forms on $d + 1$-dimensional disk and find that the spectrum can be expressed in terms of $p, p - 1, \ldots, 0$-forms with tangential indices only satisfying pure boundary conditions. We find that the degeneracies $D_n$ and the eigenvalues $-\lambda_n^2$ have the same form as for scalar fields and express the heat kernel

$$K(t) = \sum_n D_n \exp(-\lambda_n^2)$$

in terms of the scalar heat kernels. The case of two-forms is analyzed in detail.

2. Harmonic expansion

Consider $d + 1$ dimensional unit disk with the metric

$$ds^2 = dr^2 + r^2 d\Omega^2, \quad 0 \leq r \leq 1 \quad (2)$$

where $d\Omega^2$ is the metric on unit sphere $S^d$. Throughout this paper we use the notations $\{x_\mu\} = \{x_0, x_i\}$, $x^0 = r$, $\mu = 0, 1, \ldots, d$. The $d + 1$ dimensional Laplace operator $\Delta = \nabla^\mu \nabla_\mu$ acting on $p$-form $B$ can be written as

$$(\Delta B)_{i_1 \ldots i_p} = \left( \partial_0^2 + \frac{d - 2p + 2}{r} \partial_0 + \frac{p^2 - dp - 1}{r^2} + (d)\Delta \right) B_{i_1 \ldots i_p} - \frac{2}{r} (d)\nabla^k B_{i_1 \ldots i_p} \quad (3)$$

$$(\Delta B)_{i_1 \ldots i_p} = \left( \partial_0^2 + \frac{d - 2p}{r} \partial_0 + \frac{p^2 - dp}{r^2} + (d)\Delta \right) B_{i_1 \ldots i_p} + \frac{2}{r} \sum_{a=1}^p (d)\nabla_{i_a} B_{i_1 \ldots i_{a-1} i_{a+1} \ldots i_p}, \quad (4)$$

where $(d)\nabla$ and $(d)\Delta$ are the covariant derivative and Laplace operator corresponding to $d$-dimensional metric $g_{ik}$.
We assume that the \( p \)-forms from \( B \) satisfy absolute or relative boundary conditions. Let us outline the strategy of finding the spectrum of the Laplace operator (3), (4). First we make the Hodge-de Rham decomposition, which is orthogonal for the boundary conditions in question. Since the metric (2) is flat, the spectrum of the Laplace operator on longitudinal \( p \)-forms is identical to that on the transversal \( p - 1 \)-forms. This is a well known fact from the cohomology theory. Hence at the first step our problem is reduced to two eigenvalue problems for transversal \( p \) and \( p - 1 \)-forms. We shall assume that all the cohomology groups are trivial (this is indeed restriction on \( p \)). The modifications for the case of non-trivial cohomology are straightforward.

Consider the \( d + 1 \) dimensional transversality condition

\[
\nabla^\mu B_{\mu\nu...\rho} = 0.
\]

On a disk it can be written as

\[
(\nabla B)_{i_1...i_{p-2}0} = (d) \nabla^i B_{ii_1...i_{p-2}0} = 0
\]

\[
(\nabla B)_{i_1...i_{p-1}} = (\partial_0 + \frac{d - 2p + 2}{r}) B_{0i_1...i_{p-1}} + (d) \nabla^i B_{ii_1...i_{p-1}} = 0
\]

Of course, for \( p = 1 \) the condition (6) is absent. For \( p = d + 1 \) the theory is easily reduced to the scalar one with pure boundary conditions. For \( p \leq d \) the equations (6), (7) have two types of solutions depending on whether \( B_{i_1...i_p} \) is transversal or longitudinal as a \( p \)-form on \( S^d \). For the solutions of the first type \( B_{0i_1...} \) is identically zero. For the solutions of the second type both \( B_{0i_1...} \) and \( B_{i_1...i_p} \) can be expressed via transversal \( p - 1 \)-form on \( S^d \). The eigenvalue problem for the second type solutions is the same as for the first type but with \( p - 1 \)-form instead of \( p \)-form. Thus as a result of the second step the problem is reduced to the eigenvalue equation for operator (4) acting on anisymmetric transversal tensors on \( S^d \). Note, that now these tensors satisfy pure boundary conditions if either absolute or relative boundary conditions were chosen initially. The corresponding eigenfunctions can be expressed in terms of Bessel functions. The eigenvalues are defined by zeros of these functions or their derivatives.

Furthermore, the heat kernels can be related to that for scalar fields. Eigenfunctions are typically proportional to \( J_n(\lambda r) Y^n(x_j) \) for odd \( d \) and to \( J_{n+\frac{1}{2}}(\lambda r) Y^n(x_j) \) for even \( d \), where \( J_n \) are the Bessel functions and \( Y^n \) are
tensor spherical harmonics. The eigenvalues $\lambda$ are defined by boundary conditions. The degeneracies $D_n$ for odd $d$ are polynomials of $n^2$. For even $d$ the $D_n$ contain only odd powers of $n+\frac{1}{2}$. This structure is common for all $p$-forms. This fact allows us to express heat kernel for $p$-forms in terms of scalar heat kernels in $d$, $d-2$, etc. dimensions.

To illustrate this procedure let us consider the case of $p = 2$. The absolute and relative boundary conditions are respectively

$$\partial_0 B_{ik}|_{\partial M} = (\nabla_0 + \frac{2}{r}) B_{ik}|_{\partial M} = 0, \quad B_{\alpha i}|_{\partial M} = 0 \quad (8A)$$

$$(\partial_0 + \frac{d-2}{r}) B_{i\alpha}|_{\partial M} = (\nabla_0 + \frac{d-1}{r}) B_{i\alpha}|_{\partial M} = 0, \quad B_{ik}|_{\partial M} = 0 \quad (8R)$$

One can see that for longitudinal forms $B^L_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ the corresponding 1-forms $A$ also satisfy absolute or relative boundary conditions

$$\partial_0 A_i|_{\partial M} = (\nabla_0 + \frac{1}{r}) A_i|_{\partial M} = 0, \quad A_0|_{\partial M} = 0 \quad (9A)$$

$$(\partial_0 + \frac{d}{r}) A_0|_{\partial M} = (\nabla_0 + \frac{d}{r}) A_0|_{\partial M} = 0, \quad A_i|_{\partial M} = 0 \quad (9R)$$

This follows from the general theory [2] and can be verified by direct calculations. It is obvious that

$$\Delta (\nabla_\mu A_\nu - \nabla_\nu A_\mu) = \nabla_\mu \Delta A_\nu - \nabla_\nu \Delta A_\mu \quad (10)$$

Hence the spectrum of the Laplace operator $\Delta$ on $B^L_{\mu\nu}$ is defined by the spectrum of the Laplace operator on transversal vector fields $A^T_\mu$.

Consider now transversal 2-forms. As it was mentioned earlier, the equations (6), (7) have solutions for which $B_{0i} = 0$ and $B_{ik}$ is arbitrary transversal 2-form on $S^d$, $(^d)\nabla^i B_{ik} = 0$. Such modes decouple from other modes and satisfy pure boundary conditions (see (8A) and (8R)). The corresponding eigenvalue problem will be considered in the next section.

After some algebra one can find the rest of the solutions of (6) and (7).

$$B_{ik}(\phi) = (\partial_0 + \frac{d-4}{r}) r (\partial_i \phi_k - \partial_k \phi_i),$$

$$B_{i\alpha}(\phi) = (^{(d)}\Delta - \frac{d-1}{r^2} r) \phi_i, \quad (11)$$
where $\phi_i$ is arbitrary transversal vector on $S^d$, $^{(d)}\nabla^i \phi_i = 0$. After lengthy but straightforward calculations one obtains

$$\Delta B_{\mu\nu}(\phi) = B_{\mu\nu}(\Delta \phi)$$  \hspace{1cm} (12)

The boundary conditions for $\phi$ leading to (8A) and (8R) are respectively

$$\phi_i|_{\partial M} = 0$$  \hspace{1cm} (13A)

$$(\partial_0 + \frac{d-3}{r})\phi_i|_{\partial M} = (\nabla_0 + \frac{d-2}{r})\phi_i|_{\partial M} = 0$$  \hspace{1cm} (13R)

To prove that the boundary conditions (8A) and (8R) are ensured by (13A) and (13R) it is useful to expand $\phi_i$ in a sum of harmonics which are eigenvalues of both $\Delta$ and the Laplace operator on unit $S^d$, $\tilde{\Delta} = r^2 \Delta$. Equations (13R) and the second equation of (13A) become evident. Note a helpful operator identity

$$\partial_0 \Delta = \Delta (\partial_0 - \frac{2}{r}).$$  \hspace{1cm} (14)

For the first of the conditions (8A) we have

$$\partial_0 B_{ik}(\phi)|_{\partial M} = \left( ^{(d)}\nabla_i r (\partial_0 + \frac{1}{r})(\partial_0 + \frac{d-3}{r})\phi_k - (i \leftrightarrow k) \right) |_{\partial M} =$$

$$= \left( ^{(d)}\nabla_i r (\Delta - \frac{\tilde{\Delta}}{r^2} + \frac{d-1}{r^2})\phi_k - (i \leftrightarrow k) \right) |_{\partial M}$$  \hspace{1cm} (15)

where we used transversality of $\phi_k$ and the equation (4) for $p = 1$. If $\phi_k$ satisfy Dirichlet boundary condition (13A), the last line vanishes term by term for every eigenmode of $\Delta$ and $\tilde{\Delta}$.

It is important to note that the orthogonal harmonics of $\phi_i$ generate orthogonal harmonics of $B_{\mu\nu}(\phi)$. In both cases we use standard inner product without surface terms.

Now we are to reduce the harmonic expansion for $A^T_\mu$ to pure boundary condition problem. This was done in Ref. [7], thus here we give the results only. By solving the transversality condition (7) we arrive again at two types of solutions. The one is described by $A_0 = 0$, $^{(d)}\nabla^i A_i = 0$ with pure boundary conditions (9A) and (9R) for $A_i$, and the other is expressed in terms of scalar field $\psi$

$$A_0(\psi) = ^{(d)}\Delta r \psi, \quad A_i(\psi) = - ^{(d)}\nabla_i (\partial_0 + \frac{d-2}{r})r \psi.$$  \hspace{1cm} (16)
The boundary conditions for $\psi$ are given by the following equations

$$\psi|_{\partial M} = 0$$  \hspace{1cm} (17A)

$$(\partial_0 + \frac{d - 1}{r})\psi|_{\partial M} = 0$$  \hspace{1cm} (17R)

for absolute and relative boundary conditions on $A_\mu$ respectively. The decomposition commutes with the Laplace operator, $\Delta A_\mu(\psi) = A_\mu(\Delta \psi)$, and satisfy all necessary orthogonality properties.

In this section we reduced the harmonic expansion for the 2-form $B_{\mu\nu}$, satisfying absolute or relative mixed boundary conditions, to the expansions of the fields $B^T_{ik}$, $\phi^T_i$, $A^T_i$ and $\psi$. All of them are subject to pure boundary conditions.

3. Heat kernel expansion

Let us express the heat kernel for 2-form in $d + 1 = 4$ dimensions via scalar heat kernels. First recall the eigenvalues $a_l$ and degeneracies $D_l$ of the Laplace operator $\tilde{\Delta}$ on unit $S^3$ acting on scalars, transversal vectors and transversal two-forms.

$$a_l^0 = -l(l + 2), \quad D_l^0 = (l + 1)^2, \quad l = 0, 1, \ldots$$
$$a_l^1 = -l(l + 2) + 1, \quad D_l^1 = 2l(l + 2), \quad l = 1, 2, \ldots$$
$$a_l^2 = -l(l + 2) + 2, \quad D_l^2 = (l + 1)^2, \quad l = 1, 2, \ldots$$  \hspace{1cm} (18)

As it was mentioned before, the transversal 3-dimensional harmonics decouple from other components of corresponding tensors. By substituting (18) in (4) we obtain the eigenfunctions of the Laplace operator for scalar, vector and tensor fields respectively

$$r^{-1}J_{l+1}(\lambda r)Y^l(x_j), \quad J_{l+1}(\lambda r)Y^l_i(x_j), \quad rJ_{l+1}(\lambda r)Y^l_{ik}(x_j),$$  \hspace{1cm} (19)

where $Y^l$, $Y^l_i$ and $Y^l_{ik}$ are spherical harmonics. We used $^{(3)}\Delta = \frac{1}{r^2}\tilde{\Delta}$. The eigenvalues $-\lambda^2$ are defined by boundary conditions.

Denote by $S$ the number appearing in the Neumann boundary conditions for some field $\Phi$

$$(\nabla_0 - S)\Phi|_{\partial M} = 0$$  \hspace{1cm} (20)
Consider absolute boundary conditions. The integrated heat kernel for the field $B^T_{ik}$ has the form

$$K_B(t) = \sum_{l=1}^{\infty} \sum_{\lambda_l} (l+1)^2 \exp(-\lambda_l^2 t) =$$

$$= \sum_{l=0}^{\infty} \sum_{\lambda_l} (l+1)^2 \exp(-\lambda_l^2 t) - \sum_{\lambda_0} \exp(-\lambda_0^2 t)$$

(21)

The eigenvalues $\lambda$ are defined by the condition $\partial_0 r J_{l+1}(r \lambda)|_{r=1} = 0$. This condition is equivalent to $(\partial_0 + 2) r^{-1} J_{l+1}(r \lambda)|_{r=1} = 0$. Hence the first term in last line of (21) is just a scalar heat kernel,

$$K_B(t) = K_N(S = -2, t) - \sum_{\lambda_0} \exp(-\lambda_0^2 t)$$

(22)

where $N$ stands for the Neumann boundary conditions. The same procedure can be done for vectors $\phi_i$ [7] satisfying Dirichlet boundary conditions.

$$K_{\phi}(t) = \sum_{l=1}^{\infty} 2((l+1)^2 - 1) \exp(-\lambda_l^2 t) =$$

$$= \sum_{l=0}^{\infty} 2((l+1)^2 - 1) \exp(-\lambda_l^2 t) =$$

$$= 2 \sum_{l=0}^{\infty} (l+1)^2 \exp(-\lambda_l^2 t) - [2 \sum_{n=1}^{\infty} \sum_{\lambda_n} \exp(-\tilde{\lambda}_n^2 t) + \sum_{\lambda_0} \exp(-\tilde{\lambda}_0^2 t)] +$$

$$+ \sum_{\lambda_0} \exp(-\tilde{\lambda}_0^2 t)$$

(23)

where we changed the summation index $n = l+1$ and introduced tilde over $\lambda_n$. The first sum is just the heat kernel $K_D(t)$ for scalar fields satisfying Dirichlet boundary conditions. The terms in square brackets can be identified with the scalar heat kernel on two-dimensional unit disk. Making use of the identity $(\partial_0 + \frac{1}{4}) J_1 = J_0$, we obtain $\tilde{\lambda}_0 = \lambda_0$, with $\lambda_0$ from the equations (21) and (22). Hence,

$$K_B(t) + K_{\phi}(t) = K_N(S = -2, t) + 2K_D(t) - K_D(d = 1, t)$$

(24)

For the heat kernels in $d = 3$ the dimensionality is not shown manifestly.
The contribution of the field $A^T_i$ can be evaluated in a similar way

$$K_A(t) = 2K_N(S = -1, t) - K_N(S = 0, d = 1, t) + \sum_\kappa \exp(-\kappa^2 t),$$

(25)

where $\kappa$ is defined by the condition $\partial_0 J_0(\kappa) = 0$.

The contribution of the scalar field $\psi$ is the standard scalar heat kernel up to the subtraction of the $l = 0$ modes which do not generate any 3-dimensional vector fields. For $l = 0$ the condition $J_1(\kappa) = 0$ selects the same eigenvalues as in eq (25), $\partial_0 J_0 = -J_1$. However, now we should not exclude the constant mode $\kappa = 0$, because it is already excluded by the Dirichlet boundary condition.

$$K_\psi(t) = K_D(t) - \sum_\kappa \exp(-\kappa^2 t) + 1$$

(26)

Collecting together eqs. (24)-(26) one obtains the heat kernel for two-form satisfying absolute boundary condition

$$K(A; t) = 3K_D(t) + K_N(S = -2, t) + 2K_N(S = -1, t) - K_D(d = 1, t) - K_N(S = 0, d = 1, t) + 1$$

(27)

By repeating this procedure for relative boundary conditions we obtain

$$K(R; t) = K(A; t)$$

(28)

The relation (28) was obvious before any calculations because relative boundary conditions are dual to the absolute ones. For the two-forms in $d + 1 = 4$ the duality transformation maps the functional space on itself. One can consider (28) as a consistency check.

One can evaluate the $\zeta(0)$ for two-form by using the expressions for the heat kernel expansion in the case of pure boundary conditions. We obtain $\zeta(0) = \frac{7}{15}$. This value agrees with calculations [6] based on analytic formulae [8,7] for mixed boundary conditions.

In a conclusion, let us formulate main results of this work. We suggested a method of the reduction of the eigenvalue problem for the Laplace operator acting on $p$-forms obeying mixed boundary conditions to the eigenvalue problem for pure boundary conditions. The case of rank two forms was studied in some detail. We expressed the tensor harmonics on a disk via transverse
tensor, transverse vector and scalar harmonics on $S^d$. A complete analysis
was performed for $d = 3$. We express the heat kernel as a whole in terms of
heat kernels for scalar fields satisfying pure boundary conditions. In a partic-
ular case of $\zeta(0)$ we find complete agreement with previous calculations [6].
Our results are especially efficient for higher coefficients of the Seeley-Gilkey
expansion and can be applied to other fields [1,3-6,9].

Acknowledgements
I am indebted to Peter Gilkey for sending me preliminary results of his
computations of the coefficient $a_5$ on manifolds with boundaries. This work
was inspired by these computations and was originally intended for cross-
checking. I am also grateful to Giampiero Esposito for fruitful discussions. I
would like to thank Professor Abdus Salam, IAEA and UNESCO for hospi-
tality at the International Centre for Theoretical Physics, Trieste. This work
was partially supported by the Russian Foundation for Fundamental Studies,
grant 93-02-14378.

REFERENCES
[1] G. Esposito, Quantum gravity, quantum cosmology and Lorentzian geomet-
ries (Lecture Notes in Physics m12, Berlin, Springer, 1992).
[2] P.B.Gilkey, Invariance theory, the heat equation and the Atiyah-Singer
theorem (Publish or Perish, Delaware, 1984).
[3] I.G.Moss and S.J.Poletti, Phys. Lett. B245 (1990) 355; Nucl. Phys. B341
(1990) 155; S.J.Poletti, Phys. Lett. B249 (1990) 355.
[4] H.C.Luckock and I.G.Moss, Class. Quantum Grav. 6 (1989) 1993; H.C.Luckock, J. Math. Phys. 32 (1991) 1755.
[5] G.Esposito, Nuovo Cimento 109B (1994) 203; G.Esposito, A.Yu.Kamenschchik, I.V.Mishakov and G.Pollifrone, DSF preprint
94/4 (1994), to appear in Class. Quantum Grav.
[6] I.G.Moss and S.Poletti, Phys. Lett. B333 (1994) 326.
[7] D.V.Vassilevich, Vector fields on a disk with mixed boundary conditions,
St.Petersburg preprint SPbU-94-6, [gr-qc/9404052], to be published.
[8] T.Branson and P.B.Gilkey, Commun. Part. Diff. Eqs. 15 (1990) 245.
[9] P.D.D’Eath and G.Esposito, Phys. Rev. D43 (1991) 3234; D44 (1991)
1713; A.Yu.Kamenschchik and I.V.Mishakov, Phys. Rev. D47 (1993) 1380;
A.O.Barvinsky, A.Yu.Kamenshchik and I.P.Karmazin, Ann. Phys. N.Y. 219 (1992) 201;
G.Esposito, A.Yu.Kamenshchik, I.V.Mishakov and G.Pollifrone, DSF preprint 92/14, to appear in Phys. Rev. D.