Incidence relations among the Schubert cells of
equivariant Hilbert schemes

Laurent Evain

Abstract: Let $H_{ab}(H)$ be the equivariant Hilbert scheme parametrizing the zero dimensional subschemes of the affine plane $k^2$, fixed under the one dimensional torus $T_{ab} = \{(t^{-b}, t^a), \ t \in k^*\}$ and whose Hilbert function is $H$. This Hilbert scheme admits a natural stratification in Schubert cells which extends the notion of Schubert cells on Grassmannians. However, the incidence relations between the cells become more complicated than in the case of Grassmannians. In this paper, we give a necessary condition for the closure of a cell to meet another cell. In the particular case of Grassmannians, it coincides with the well known necessary and sufficient incidence condition. There is no known example showing that the condition wouldn't be sufficient.

1 Introduction

The problem (general wording).
Fixing an algebraically closed field $k$, the Hilbert scheme $H$ parametrizing the zero dimensional subschemes of the affine plane $Spec \ k[x,y]$ admits a natural action of the two dimensional torus $k^* \times k^*$, induced by the linear action $(t_1, t_2).x^\alpha y^\beta = (t_1.x)^\alpha(t_2.y)^\beta$ of $k^* \times k^*$ on $k[x,y]$. If $T_{ab} = \{(t^{-b}, t^a), \ t \in k^*\}$ is a one dimensional sub-torus, the closed subscheme $H_{ab}$ which parametrizes by definition the zero dimensional subschemes invariant under the action of $T_{ab}$ is a disjoint union of irreducible subspaces $H_{ab}(H)$. Each $H_{ab}(H)$ can be embedded in a product of Grassmannians and inherits the stratification whose cells are the inverse images of the products of the Schubert cells. The problem is to understand the geometry of this stratification, and in particular to describe the incidences between the cells.

Motivations
Before explaining the results and the techniques, let’s explain the motivations. Roughly speaking, Grassmannians are easier to study than Hilbert schemes because they are stratified by Schubert cells. Those stratifications enable for instance to compute the intersection rings of Grassmannians, the Hilbert function of their natural embeddings in a projective space ([GH],[M]). On the other hand, Hilbert schemes admit a stratification furnished by the theory of Grobner bases (or standard basis) which is fussy to describe ([Gr]). The equivariant Hilbert scheme $H_{ab}(H)$ is an object which both looks like a Hilbert scheme and...
a Grassmannian, a sort of interface between them. It has two stratifications, the stratification in Schubert cells introduced above and the Gröbner stratification obtained by restricting the stratification of $H$ to $H_{ab}(H)$. Moreover these two stratifications coincide. So the philosophy is to try to describe the geometry of $H_{ab} = \cup H_{ab}(H)$, and then to lift the information on $H$. This is our motivation to study $H_{ab}(H)$. To see some precise examples where the geometry of $H_{ab}$ can be used to describe the geometry of $H$, see [I], where structures of bundles are highlighted, [ES] where the Betti numbers of the connected components of $H$ are computed via the study of the equivariant inclusion $H_{ab} \subset H$ for $(a, b)$ general enough, or [B] which describes links between the equivariant cohomology of $H_{ab}$ and the equivariant cohomology of $H$. Beyond this motivation, note that $H_{ab}$ is a classical object of study ([Gö], [I], [Y] for instance).

**Known results and precise formulation of the problem.**

For some special Hilbert functions $H$, the equivariant Hilbert scheme $H_{ab}(H)$ is a Grassmannian and the cells of $H_{ab}(H)$ are the Schubert cells (for this reason, we will say a “Schubert cell” to designate a cell on $H_{ab}(H)$). In this simple case, the closure of a cell is a union of cells and there is an explicit numerical condition to check incidence. This is no longer so simple when $H_{ab}(H)$ is not a Grassmannian: Yameogo has given in [Y] an example where the closure of a cell is not a union of cells. So the problem splits into two different pieces: the “weak incidence” problem which consists in deciding whether the closure of a cell meets another cell, and the “strong incidence” problem which consists in deciding whether the closure of a cell contains another cell. In [Y], Yameogo faced the weak incidence problem and gave a necessary but not sufficient condition to have a relation $C \cap C' \neq \emptyset$. In this paper, we address the weak incidence problem too.

**The results.**

Our main answer will be a necessary condition (theorem 8) for weak incidence. No counter-example to sufficiency is known to the author. The other propositions try to analyse the pertinence of the condition, by comparing it to the previous results (prop. 21, 22) and by testing it on small Hilbert functions (section 5).

We now formulate our results with more details. We associate with any Schubert cell $C$ a combinatorial data and we express the relation $C \cap C' \neq \emptyset$ by a combinatorial property linking $C$ and $C'$, as follows. Each Schubert cell is determined by a staircase $E$, i.e. a subset of $\mathbb{N}^2$ whose complementary is stable.
under the addition of \( \mathbb{N}^2 \): the ideal \( I^E \) generated by the monomials whose exponent is not in \( E \) is the unique monomial ideal of the cell. If \( n \) is the cardinal of \( E \), \( E \) is included in a box of size \( n \times n \) and we denote by \( E^\nu \) the dual staircase defined by the complementary of \( E \) in the box (see figure). A system of arrows \( S \) on \( E \) is the data for each element \( p \) of \( E \) of an arrow \((p, f_p)\) with origin \( p \), the set of arrows being compelled to compatibility conditions. Replacing each element \( p \) of \( E \) by \( f_p \) gives a new subset of \( \mathbb{N}^2 \) which turns out to be a staircase thanks to the compatibility conditions. Thinking of \( S \) as an operator on \( E \), we denote by \( S(E) \) the staircase obtained from \( E \) and \( S \) by the above procedure.

The system \( S \) on \( E \).

(Arrows of the form \((p, p)\).

\[ S(E) \]

The theorem says that two cells \( C(E) \) and \( C(F) \) corresponding respectively to staircases \( E \) and \( F \) verify

\[ C(E) \cap C(F) \neq \emptyset \]

only if

1. there exists a system of arrows \( S \) on \( E \) such that \( S(E) = F \)

2. there exists a system of arrows \( S^\nu \) on \( E^\nu \) such that \( S^\nu(E^\nu) = F^\nu \)

We then compare this condition to the known conditions and we test it on an example. In the particular case of Grassmannians, 1) and 2) are equivalent and reduce to the well known necessary and sufficient incidence condition (prop. 21). In general, they are not equivalent (ex. 10) but any of them imply the condition of [Y] (proposition 23). Finally, we study an example (section 5) which illustrates that when the Hilbert function \( H \) is small, we can solve the weak incidence problem because our condition can be shown to be necessary and sufficient.

The methods.
The first step consists in putting the problem in the context of Gröbner basis theory and to describe a cell \( C(E) \) as the locus formed by the ideals \( I \) having \( I^E \) as initial ideal. Then we interpret the incidence relation \( C(E) \cap C(F) \neq \emptyset \) as the existence of a rational morphism from \( \mathbb{A}^1 = \text{Spec } k[t] \) to \( C(E) \) which can be extended in \( \infty \) by putting \( f(\infty) = X_F \in C(F) \). The universal ideal on \( \mathbb{H}_{ab}(H) \) restricted on the affine curve defines an ideal \( I(t) \subset k[x, y, t] \). The key consists in using ideas coming from Gröbner basis theory to exhibit within \( I(t) \) a set of elements \( P_1, P_2, \ldots \) from which we read the system of arrows on \( E^\nu \). Then, using an argument of duality corresponding to the geometric notion of linkage of two zero dimensional schemes in a complete intersection, we get the system of arrows on \( E \).

I thank Michel Brion who told me about the problems that motivated this work and who answered several questions.
2 The stratification on \( \mathbb{H}_{ab}(H) \).

2.1 Notations

First, we keep the notations of the introduction: \( k \) is an algebraically closed field, \( a \) and \( b \) are two relatively prime integers, and \( T_{n,b} = \{(t^{-b}, t^n), t \in k^* \} \) is a one dimensional subtorus of \( k^* \times k^* =: T \). The torus \( T \) acts on the Hilbert scheme \( \mathbb{H} \) parametrizing the zero-dimensional subschemes of \( \text{Spec} \, k[x, y] \), and \( \mathbb{H}_{ab} \) is the subscheme of \( \mathbb{H} \) parametrizing the subschemes invariant under the action of \( T_{ab} \subset T \). Alternatively, the subschemes of \( \mathbb{H}_{ab} \) can be characterized by their ideals using degrees. The degrees \( d, d_x, d_y \) are defined on monomials by \( d(x^\alpha y^\beta) = -b\alpha + a\beta, d_x(x^\alpha y^\beta) = \alpha, d_y(x^\alpha y^\beta) = \beta \). If \( I \) is an ideal of \( k[x, y] \), we let \( I_n := I \cap k[x,y]_n \), where \( k[x,y]_n \) denotes the vector space generated by the monomials \( m \) of degree \( d(m) = n \). A subscheme \( Z \) is in \( \mathbb{H}_{ab} \), if and only if its ideal is quasi-homogeneous with respect to \( d \), ie. \( I(Z) = \oplus_{n\geq 0} I(Z)_n \).

We order the monomials of \( k[x,y] \) by the rule \( m_1 < m_2 \) if \( d(m_1) < d(m_2) \) or \( (d(m_1)_1 = d(m_2)_1 \) and \( d_y(m_1) < d_y(m_2) \)).

The scheme \( \mathbb{H}_{ab} \) is not connected but the connected components are determined by a Hilbert function. By semi-continuity, if a subscheme \( Z' \in \mathbb{H}_{ab} \) is a specialization of \( Z \in \mathbb{H}_{ab} \), then the codimensions of \( I_n(Z) \) and \( I_n(Z') \) in \( k[x,y]_n \) verify \( \text{codim} I_n(Z) \geq \text{codim} I_n(Z') \). But \( Z \) and \( Z' \) have the same length \( l = \sum_{n \geq 0} \text{codim} I_n(Z) = \sum_{n \geq 0} \text{codim} I_n(Z') \). It follows that the sequence \( H(Z) = (h_0, h_1, h_2, \ldots) \) where \( h_i = \text{codim} I_n(Z) \) is constant on the connected components of \( \mathbb{H}_{ab} \) and that \( h_n = 0 \) for \( n \) big enough. If \( H = (h_0, \ldots, h_r, 0, 0, \ldots) \) is any sequence, we denote by \( \mathbb{H}_{ab}(H) \) the closed subscheme of \( \mathbb{H} \) parametrizing the schemes \( Z \) verifying \( H(Z) = H \). One can verify that \( \mathbb{H}_{ab}(H) \) is an irreducible connected component of \( \mathbb{H}_{ab} \) (though we won’t use it in the sequel).

Recall that a staircase is a subset of \( \mathbb{N}^2 \) whose complementary is stable by addition of \( \mathbb{N}^2 \). Staircases will be used to parametrize the stratas on \( \mathbb{H}_{ab}(H) \). In this paper, we will identify freely the monomial \( x^p y^q \) with the couple \( (p, q) \) and therefore the expression “staircase of monomials” will make sense. More generally, we will transpose unscrupulously the definitions between couples of integers and monomials. If \( E \) is a staircase, then the vector space \( I^E \) generated by the monomials which are not in \( E \) is an ideal and reciprocally, every monomial ideal is an ideal \( I^E \) for a unique staircase \( E \). The subscheme \( Z(E) \) whose ideal is \( I^E \) is in \( \mathbb{H}_{ab}(H) \) if and only if \( E \) has \( h_i \) elements in degree \( i \).

2.2 The possible definitions

In this section, we explain that there are three ways to describe the stratification on \( \mathbb{H}_{ab}(H) \). The fact that the definitions coincide is shown in [Y]. We call a cell of this stratification a “Schubert cell” on \( \mathbb{H}_{ab}(H) \) as it is a Schubert cell when \( \mathbb{H}_{ab}(H) \) is a Grassmannian (section 1.1). Moreover each cell contains a unique subscheme \( Z(E) \) and we will denote this cell by \( C(E) \).

The Grobner basis point of view. The theory of Gröbner basis associates
with every ideal in \(k[x,y]\) a monomial ideal (with respect to the monomial order chosen in section 2.1) called initial ideal and \(C(E)\) is the locus in \(\mathbb{H}_{ab}(H)\) parametrizing the ideals whose initial ideal is \(I^E\). For the reader not familiar with Gröbner basis, we give a characterization which is sufficient for the sequel. Let \(m_1, m_2, \ldots\) be the monomials which don’t belong to \(E\). An ideal of \(\mathbb{H}_{ab}(H)\) is in \(C(E)\) if, regarding it as a \(k\)-vector space, it admits a base \(f_1, f_2, \ldots\) where \(f_i = m_i + R_i\), \(R_i\) being a linear combination of monomials strictly smaller than \(m_i\).

\textbf{Remark 1.} When the product \(a.b\) is zero, there is at most one staircase compatible with the Hilbert function \(H\) (i.e. such that \(E\) has \(h_i\) elements in degree \(i\)) and \(\mathbb{H}_{ab}(H)\) is either empty or is reduced to a unique cell (it is non empty when \(H\) is a decreasing sequence). These cases are not relevant for the incidence problem and we suppose from now on \(ab \neq 0\) and \(a > 0\).

\textbf{The Schubert cells point of view.} Let \(H = (h_0, \ldots, h_r, 0, 0, \ldots)\), and \(Z \in \mathbb{H}_{ab}(H)\). The ideal \(I_n(Z)\) being a vector space of codimension \(h_n\) in \(k[x,y]^n\), it corresponds to a point \(p_n\) in a Grassmannian \(G_n\). So \(I(Z) = \oplus I_n(Z)\) corresponds to a point \(p = (p_0, \ldots, p_r)\) in the product of Grassmannians \(G_0 \times G_1 \ldots \times G_r\). This set theoretical representation turns out to be a closed embedding \(g : \mathbb{H}_{ab}(H) \to G_0 \times G_1 \ldots \times G_r\). Let \(V_i\) be the subspace generated by the \(i\) smallest monomials of \(k[x,y]^n\). The flag \(V_0 \subset V_1 \ldots \subset V_{n+1}\) defines by a classical construction Schubert cells on \(G_n\) which stratify it. The stratification we consider on \(\mathbb{H}_{ab}(H)\) is the stratification whose strata are the locally closed subschemes \(g^{-1}(C_0 \times C_1 \times \ldots \times C_r)\), where \(C_i\) is a Schubert cell on \(G_i\).

\textbf{The Bialynicki-Birula point of view.} Let \(X\) be a smooth projective variety over \(k\) admitting an action of the torus \(k^*\). Suppose that the action has a finite number of fixed points \(x_1, \ldots, x_n\). Let \(T_{X,x_i}^+\) be the part of the tangent space to \(x_i\) in \(X\) where the weights of the \(k^*\)-action are positive, and let \(X_i := \{x \in X, \lim_{t \to 0} (t.x) = x_i\}\). Then a theorem of Bialynicki-Birula asserts that the \(X_i\) are a cellular decomposition of \(X\) in affine spaces and satisfy \(T_{X_i,x_i} = T_{X,x_i}^+\). In our case, fixing two integers \(p\) and \(q\) with \(ap + bq > 0\), the torus \(k^*\) acts on \(k[x,y]\) by \(t.x = t^p.x\) and \(t.y = t^q.y\). This action induces an action of \(k^*\) on \(\mathbb{H}_{ab}(H)\). The fixed points of \(\mathbb{H}_{ab}(H)\) under \(k^*\) are the monomial subschemes \(Z(E)\). Applying the Bialynicki-Birula theorem to the action of \(k^*\) on \(\mathbb{H}_{ab}(H)\), we get a stratification and \(C(E)\) is the cell associated with the fixed point \(Z(E)\).

\textbf{2.3 Incidence relations}

In this section, we recall the main theorem of [Y] about incidence relations. The monomials of \(k[x,y]\) can be ordered in an infinite sequence \(m_0 < m_1 < \ldots\) thanks to the monomial order. Let \(S_E\) be the function from \(\mathbb{N}\) to \(\mathbb{N}\) defined by \(S_E(k) =\) number of monomials in \(E\) smaller or equal to \(m_k\).

\textbf{Theorem 2.} If \(C(E) \cap C(F) \neq \emptyset\) then \(S_E \geq S_F\).
Remark 3. In [Y], the result says $\leq$ instead of $\geq$ as above. The reason is that we have slightly changed the definition of $S_E$ to get a shorter presentation. Moreover, the paper deals with the homogeneous case $a = 1, b = -1$ but the extension is immediate.

The condition is not sufficient: if $E$ and $F$ are the staircases whose monomial ideals are $I^E = (y^4, xy^2, x^2y, x^5)$ and $I^F = (y^5, xy^2, x^3)$ then $S_E \geq S_F$ but the incidence $C(E) \cap C(F) \neq \emptyset$ is not fulfilled([Y]).

3 A necessary condition for weak incidence

3.1 Statement of the result

In this section, we give a necessary condition on two staircases $E$ and $F$ to fulfill the relation $C(E) \cap C(F) \neq \emptyset$. The condition will rely on the notions of system of arrows and of dual staircase that we introduce now.

Definition 4. An arrow on $E$ is a couple of monomials $(P, Q)$, $(P$ being the origin of the arrow and $Q$ being the end of the arrow), such that $P$ is in $E$ and such that the vector $-\overrightarrow{PQ}$ of $\mathbb{N}^2$ is a negative multiple of $(a, b)$ (recall that we have adopted the convention $a > 0$). If $f = (P, Q)$, the multiplication by a monomial $m$ of $f$ is the arrow $m.f := (mP, mQ)$. An arrow $f = (P, Q)$ is shorter than $f' = (P', Q')$ if $-\overrightarrow{PQ} = \lambda (a, b)$, $-\overrightarrow{P'Q'} = \lambda' (a, b)$, and if the absolute values of $\lambda$ and $\lambda'$ verify $|\lambda| \leq |\lambda'|$.

A system of arrows on $E$ is the data for each element $p \in E$ of an arrow $(p, f_p)$ on $E$ such that set $S$ of arrows parametrized by $E$ satisfy the following conditions:

- $p \neq q \Rightarrow f_p \neq f_q$
- the arrows are compatible with division, which means that $\forall f = (P, Q) \in S$, $\forall Q'$ monomial of $k[x, y]$ dividing $Q$, $\exists g \in S$ such that $g$ has end $Q'$ and such that $g$ is shorter than $f$.

With a staircase $E$ and a system of arrows $S$, one can define a new staircase $S(E)$ as follows.

Proposition 5. Let $E$ be a staircase and $S$ a system of arrows on $E$. Let $S(E)$ be the set of monomials which are the end of an arrow. Then $S(E)$ is a staircase.

Proof: We have to verify that if $m$ and $Q'$ are two monomials with $mQ' \in S(E)$ then $Q' \in S(E)$. By definition of $S(E)$, $Q := mQ'$ is the end of an arrow. The compatibility of $S$ with division shows that $Q' \in S(E)$. £

Definition 6. If $E$ and $F$ are two staircases, and if $S$ is a system of arrows on $E$ such that $F = S(E)$, we will say that $S$ is a system of arrows from $E$ to $F$. 
Definition 7. The box of size $M \times N$ is $B_{M \times N} := \{(x, y) \in \mathbb{N}^2, x < M, y < N\}$. If a staircase $E$ is included in $B_{M \times N}$, the dual of $E$ in the box is by definition the set $E^\nu_{MN} := \{(x, y) \in \mathbb{N}^2 \text{such that } \forall (e_1, e_2) \in E, e_1 + x < M - 1 \text{ and } e_2 + y < N - 1\}$. In the sequel, we will often write $E^\nu$ instead of $E^\nu_{MN}$ for simplicity.

![Diagram of a staircase and its dual](image)

Theorem 8. Let $E$ and $F$ be two staircases included in a box $B_{M \times N}$. Let $E^\nu$ and $F^\nu$ be their respective dual in $B_{M \times N}$. If the incidence $C(E) \cap C(F) \neq \emptyset$ is fulfilled, then

1. there exists a system of arrows $S$ on $E$ such that $S(E) = F$
2. there exists a system of arrows $S^\nu$ on $E^\nu$ such that $S^\nu(E^\nu) = F^\nu$

Proof: see section 3.2.

Remark 9. One can prove that the existence of $S^\nu$ does not depend on the choice of the box $B_{M \times N}$ in which we construct the dual staircase $E^\nu$.

The following example shows that conditions 1) and 2) are not equivalent.

Example 10. Let $E$ and $F$ be the staircases whose monomial ideals are $I^E = (y^4, xy^2, x^2y, x^3)$ and $I^F = (y^5, xy^2, x^3)$. Let $(a, b) = (1, −1)$. There exist a system $S$ on $E$ such that $S(E) = F$ but there is no system of arrows $S$ on $E^\nu_5$ such that $S^\nu(E^\nu_5) = F^\nu_5$.

The set $S := \{(x^3, x^2y), (x^4, y^4), (p, p), p \in E, p \neq x^3, p \neq x^4\}$ is a system of arrows on $E$ such that $S(E) = F$. The dual staircases $E^\nu$ and $F^\nu$ in $B_{5 \times 5}$ are such that $I^{E^\nu} = (y^4, x^3y^3, x^4y, x^5)$ and $I^{F^\nu} = (y^5, x^2y^3, x^4)$. Suppose that there exists a system of arrows $S^\nu$ on $E^\nu$ such that $S^\nu(E^\nu) = F^\nu$. Because $x^2y^3$ is in $E^\nu$ but not in $F^\nu$, $S^\nu$ must contain an arrow $f$ from $x^2y^3$ to a point $p > x^2y^3$, $p \in F^\nu$ i.e. $p = xy^4$. The monomial $x^3y^2 \in E^\nu$ so there is an arrow $f^\prime$ from a monomial $p \in E^\nu$ and $\leq x^3y^2$ to $x^3y^2$, i.e. $p = x^3y^2$. The compatibility with division and the arrow $f^\prime$ show that all monomials dividing $x^2y^2$ admit an arrow to themselves. The compatibility with division by $y$ applied to the arrow $f$ ensures the existence of an arrow from $xy^3$ to itself. It follows that the compatibility with division by $x$ applied to $f$ cannot be satisfied.
3.2 Proof of theorem

3.2.1 Proof of point 2

To show that there exists a system $S^\nu$ on $E^\nu$ such that $S^\nu(E^\nu) = F^\nu$, we will exhibit a system $S^\nu$ on the complementary $E^\nu$ of $E$ in $\mathbb{N}^2$ such that $S^\nu(E^\nu) = F^\nu$. So the first step is to explain what is a system on $E^\nu$, and to reduce the proof to a construction of such a system $S^\nu$. The reduction is given in proposition 14.

**Definition-Proposition 11.** An arrow on $E^\nu$ is a couple of monomials $(P, Q)$ such that $P \notin E$ and such that the vector $\overrightarrow{PQ}$ of $\mathbb{N}^2$ is a positive multiple of $(a, b)$. A system of arrows on $E^\nu$ is the data for each element $p \notin E$ of an arrow $(p, f_p)$ on $E^\nu$ such that the set $S^\nu$ of arrows parametrized by $E^\nu$ satisfy the following conditions:

- $p \neq q \Rightarrow f_p \neq f_q$
- the arrows are compatible with multiplication, which means that $\forall f = (P, Q) \in S^\nu$, $\forall m$ monomial of $k[x, y]$, $\exists g \in S^\nu$ such that $g$ has end the product $mQ$ and such that $g$ is shorter than $f$.

If $S^\nu$ is a system of arrows on $E^\nu$, then the set

$$S^\nu(E^\nu) := \{(x, y) \text{ which are the end of an arrow of } S^\nu\}$$

is the complementary of a staircase $F$. The system $S^\nu$ will be called a system of arrows from $E^\nu$ to $F^\nu$.

The proof is similar to the proof of proposition 11.

The next lemma identifies systems on $E^\nu$ and systems on $E^\nu$. Let $S^\nu$ (resp. $S^\nu$) be the set of systems of arrows on $E^\nu$ (resp. on $E^\nu$). Let $\varphi : B_{MN} \to B_{MN}$, $(x, y) \mapsto (M - 1 - x, N - 1 - y)$ be the dualizing map. For an arrow $f = (p, q)$ on $E^\nu$, if $q \in B_{MN}$, we define $\varphi(f) := (\varphi(p), \varphi(q))$. Let

$$S^\nu_{MN} := \{S^\nu \text{ system of arrows on } E^\nu \text{ s.t. } s = (p, q) \in S^\nu \Rightarrow q \in B_{MN}\}.$$

If $S^\nu \in S^\nu_{MN}$, let $\psi(S^\nu) := \{\varphi(f), f \in S^\nu\} \cup \{(p, p), p \in E^\nu \setminus \varphi(E^\nu)\}$.

** Lemma 12.** The map $\psi : S^\nu_{MN} \to S^\nu$ is well defined, (i.e. $\psi(S^\nu)$ is a system of arrows on $E^\nu$) and injective. The image $\text{Im } \psi$ contains the systems $S^\nu$ such that: for all monomial $m \notin B_{MN}$, the arrow of $S^\nu$ whose origin is $m$ is the arrow $(m, m)$. Moreover if $S^\nu$ and $S^\nu$ are two systems such that $\psi(S^\nu) = S^\nu$, then $S^\nu(E^\nu) = (S^\nu(E^\nu))^\nu$.

**Proof:** it consists in a sequence of easy combinatorial verifications which are left to the reader.

Here is another criterion to check whether a system $S^\nu$ on $E^\nu$ is in $\text{Im } \psi$.

**Lemma 13.** Let $E$ be a staircase included in the box $B_{MN}$, $S^\nu$ be a system of arrows on $E^\nu$, and $F$ be the staircase such that $S^\nu(E^\nu) = F^\nu$. If $F \subset B_{MN}$, then $S^\nu \in \text{Im } \psi$. 

8
Proof: Let \( p_0, \ldots, p_s \) be the set of monomials of \( k[x, y]_s \). Up to reordering, one can suppose that \( p_0 > p_1 > \cdots > p_s \). There exist integers \( l \) and \( m \) such that \( k[x, y]_s \setminus B_{M \times N} = \{ p_0, \ldots, p_l \} \cup \{ p_{m}, \ldots, p_s \}, l < m, -1 \leq l \leq s, 0 \leq m \leq s + 1. \) By [12] to prove the proposition, we have to show that \( \forall i \in \{ 0, \ldots, l-1, l, m, m+1, \ldots, s \} \), the arrow \((p_i, p_i)\) is in \( S^c \). If \( m \leq s, p_s \notin B_{M \times N} \) so \( p_s \in E^c \). It follows that \( S^c \) contains an arrow \((p_s, *)\) by definition of a system on \( E^c \) and necessarily \( * = p_s \) since \( * \leq p_s \). Now, if \( m < s, p_{s-1} \in E^c \), so there is in \( S^c \) an arrow \((p_{s-1}, *)\). But distinct arrows have distinct ends so \( * = p_{s-1} \). By decreasing induction, one shows that \( S^c \) contains \((p_m, p_m), (p_{m+1}, p_{m+1}), \ldots, (p_s, p_s)\).

Similarly, if \( p_0 \notin B_{M \times N} \) then \( p_0 \in F^c \). By definition of \( S^c(E^c) \), this means that \( S^c \) contains an arrow \((*, p_0)\) and the only possibility is \((p_0, p_0)\). If the monomial \( p_1 \notin B_{M \times N} \), then \( p_1 \) is the end of an arrow \((*, p_1)\) and we must have \((* = p_1)\) because the arrow with origin \( p_0 \) is already determined. By induction, \( S^c \) contains the arrows \((p_0, p_0), \ldots, (p_1, p_1)\).

Proposition 14. To prove the existence of a system \( S^c \) such that \( S^c(E^c) = F^c \), it suffices to exhibit a system \( S^c \) on \( E^c \) such that \( S^c(E^c) = F^c \).

Proof: it is an immediate consequence of the last two lemmas.

The goal of the next proposition is to exhibit such a system \( S^c \). The cell \( C(E) \) is an affine space as a cell of a Bialynicki-Birula stratification. So, if \( C(E) \) meets \( C(F) \), there exists a morphism \( f \) from the affine line \( k^1 = \text{Spec } k[t] \) to \( C(E) \) such that the image of the generic point is in \( C(E) \) and such that the limit of \( f(t) \) when \( t \) tends to infinity is \( X_\infty \in C(F) \). Let \( p_1, \ldots, p_s \) be the points in \( A^1 \) whose image by \( f \) is in \( C(E) - C(E) \). By the universal property of \( C(E) \), the morphism \( f \) is defined by a closed subscheme \( U \) of \( A^1 - \{ p_1, \ldots, p_s \} \times \text{Spec } k[x, y] \). Let \( I(t) \subset k[x, y][t] \) be the ideal defining the closure of \( U \) in \( A^1 \times \text{Spec } k[x, y] \). The staircase over the generic point being \( E \), for each monomial \( m \notin E \), there exists a quasi-homogeneous element \( P \in I(t) \) with initial monomial \( m \). Among all possible \( P \), choose one as follows. Let \( d_t(P) \) be the degree of \( P \) in \( t \), and \( d_{m, t}(P) \) be the degree in \( t \) of the initial coefficient of \( P \). Let \( S_m \) be the set of \( P \) such that \( \Delta(P) := d_t(P) - d_{m, t}(P) \) is minimal. For a fixed \( P \), denote by \( \text{val}(P) \) the greatest monomial of \( P \) whose coefficient has degree \( d_t(P) \). Now choose a \( P \) in \( S_m \) such that \( \text{val}(P) \) is minimum. Call this element \( P(m) \) and put \( e(m) := \text{val}(P(m)) \).

Proposition 15. Let \( S^c := \{ (m, e(m)), m \in E^c \} \). Then \( S^c \) is a system of arrows on \( E^c \) and \( S^c(E^c) = F^c \).

To prove the proposition, we need two lemmas.

Lemma 16. Let \( P \) be a quasi-homogeneous element of \( I(t) \) with \( \text{in}(P) = m \) and \( \text{val}(P) = n \). Let \( Q \) be a quasi-homogeneous element of \( I(t) \) such that \( \text{in}(Q) < m \) and \( \text{val}(Q) = n \). Let \( b(t) \) be the coefficient of \( n \) in \( P \), \( d(t) \) be the coefficients of \( n \) in \( Q \). Then \( R = d(t) P - b(t)Q \) is an element of \( I(t) \) with initial term \( m \) satisfying \( \Delta(R) < \Delta(P) \) or \( (\Delta(R) = \Delta(P) \text{ and } \text{val}(R) < \text{val}(P)) \).
having degree $d$.

Definition 18. Let $Z$ be a zero-dimensional scheme included in the complete intersection $Y = (x^M, y^N)$. The scheme $Z^c$ defined by the ideal $(I(Y) : I(Z))$
is called the scheme obtained by linkage in the complete intersection \((x^M, y^N)\). Let \(\mathbb{H}(Y)\) be the reduced subscheme of \(\mathbb{H}\) parametrizing the subschemes included in \(Y\).

**Proposition 19.** The morphism \(\varphi : \mathbb{H}(Y) \to \mathbb{H}(Y), Z \mapsto Z'\) is well defined and sends \(C(E)\) to \(C(E_M^N)\). Moreover \(Z'' = Z\).

**Proof:** if \(Y\) is a 0-dimensional Gorenstein scheme, and if \(Z \subset Y\), then the ideal \((I(Y) : I(Y))\) defines a scheme of degree \(\text{deg}(Y) - \text{deg}(Z)\). A complete intersection being Gorenstein, the degree of \(Z'\) is constant on the connected components of \(\mathbb{H}(Y)\) and the morphism is well defined.

Let \(Z \in C(E)\), let \(\text{in}(I(Z'))\) be the set of initial monomials of the elements of \(I(Z')\), and let \(e = x_{e_1}y_{e_2} \in \text{in}(I(Z'))\). The inclusion \(\text{in}(I(Z'))\in \text{in}(I(Z))\) shows that: \(\nu(\alpha, \beta) \notin E, e_1 + \alpha \geq M\) or \(e_1 + \beta \geq N\). In other words, \(\text{in}(I(Z')) = (E^c_M)^c\). The complementary sets \(\text{in}(I(Z')) = (E^c_M)^c\) and \(E^c_M\) have the same cardinal \(M + \text{card}(E) = \text{card}(I(Z')) = (E^c_M)^c\). This shows that \(\varphi\) sends \(C(E)\) to \(C(E^c_M)\).

Finally, we have by definition of the dual, \(Z'' \subset Z\). But \(Z\) and \(Z''\) have the same degree so they are equal.

**Corollary 20.** Let \(E\) and \(F\) be two staircases of the same cardinal \(n\). Then \(C(E) \cap C(F) \neq \emptyset \Rightarrow C(E^c_M) \cap C(F^c_M) \neq \emptyset\).

**Proof:** apply the dualizing morphism \(\varphi\) of the last proposition with \(M = N = n\).

Now we come to the proof of point 1 of theorem \(\mathbb{G}\). If \(C(E) \cap C(F) \neq \emptyset\), then \(C(E^c_M) \cap C(F^c_M) \neq \emptyset\). Then, point 2 states that there exists a system of arrows \(S\) on \((E^c_M)^c\) such that \(S(E) = (E^c_M)^c = F\).

### 4 Comparison to the known results

#### 4.1 Grassmannians

In the case \(H = (1, 2, 3, \ldots, k-1, k, k+1-n, 0, 0, \ldots, 0)\), \(\mathbb{H}_{1-1}(H)\) is isomorphic to the Grassmannian \(G(n, k+1) = G(n, k[x, y]^k)\). The isomorphism consists in associating \(I = \oplus I_k\) with the vector space \(I_k\) and the inverse isomorphism associates \(I_k\) with the ideal \(I = I_k \oplus k[x, y] + k[x, y]_k+2 + \ldots\). A non increasing sequence of non negative integers \((p_1, \ldots, p_n)\) with \(p_i \leq k+1-n\) defines a staircase \(E(p_1, \ldots, p_n)\) which contains the monomials of degree at most \(k-1\) and the monomials of degree \(k\) which are not in \(\{x^{n-i+p_i}, y^{k-n-i-p_i} : 1 \leq i \leq n\}\). Let \(V_i\) be the vector space generated by the \(i\) smallest monomials of degree \(k\). The Schubert cell \(C(p_1, \ldots, p_n)\) on \(G(n, k[x, y]^k)\) is by definition \(C(p_1, \ldots, p_n) = \{W \in G(n, k[x, y]^k) : \dim(W \cap V_i) = i\ if\ k+1-n+i-p_i \leq j < k+1-n+i-p_{i+1}\}\). By construction, the Schubert cells \(C(E(p_1, \ldots, p_n))\) and \(C(p_1, \ldots, p_n)\) correspond under the above isomorphism. A classical result on Grassmannians asserts that the closure of a cell \(C(p_1, \ldots, p_n)\) is the union
of the cells \( C(q_1, \ldots, q_n) \) for which \( q_i \geq p_i \) for all \( i \). The following result says that in the particular case of Grassmannians, the two points of our necessary condition coincide, and they coincide with the known incidence condition on Grassmannians.

**Proposition 21.** The four following conditions are equivalent:

1. \( C(p_1, \ldots, p_n) \cap C(q_1, \ldots, q_n) \neq \emptyset \)
2. \( q_i \geq p_i \) for all \( i \).
3. There exists a system \( S \) on \( E(p_1, \ldots, p_n) \) such that \( S(E(p_1, \ldots, p_n)) = E(q_1, \ldots, q_n) \)
4. There exists a system \( S' \) on \( E(p_1, \ldots, p_n)' \) such that \( S'(E(p_1, \ldots, p_n)') = E(q_1, \ldots, q_n)' \)

**Proof:** 1 \( \Leftrightarrow \) 2 is a classical result on Grassmannians. 1 \( \Rightarrow \) 3 and 1 \( \Rightarrow \) 4 is exactly theorem 8. 4 \( \Rightarrow \) 1. If 4) is true, we have a system \( S' \) on \( E' \), which can be identified with a system \( S \) on \( E \) (thanks to lemma 12) such that \( S(E(p_1, \ldots, p_n)) = E(q_1, \ldots, q_n)' \). For each arrow \( f \) of \( S' \) in degree \( k \) with origin \( o(f) \) and end \( g(f) \), let \( e(f) := o(f) + tg(f) \in k[x,y,t] \) and denote by \( V_k \) the \( k[t] \)-module generated by \( \{e(f), f \in S \} \) of degree \( k \). Let \( I(t) \subset k[x,y][t] \) be the ideal defined by \( I(t) := V_k \oplus k[x,y][t+1] \oplus k[x,y][t+2] \oplus \ldots \). The scheme whose ideal is \( \lim_{t \to \infty} I(t) = I(E(p_1, \ldots, p_n)) \) is in \( C(E(p_1, \ldots, p_n)) \cap C(E(q_1, \ldots, q_n)) \).

3 \( \Rightarrow \) 4. We have 4 \( \Rightarrow \) 3. Using the duality of proposition 19 to exchange the roles of \( E(p_1, \ldots, p_n) \) and \( E(p_1, \ldots, p_n)' \), we have 3 \( \Rightarrow \) 4.

### 4.2 Comparison to Yameogo’s result

The next result asserts that any of the two conditions of theorem 8 implies strictly the condition of theorem 2.

**Proposition 22.** Let \( S \) be a system of arrows on \( E \). Then \( S_{S(E)} \leq S_E \). Moreover, there exist staircases \( E \) and \( F \) such that:

- \( S_E \geq S_F \)
- there is no system of arrows \( S \) on \( E \) verifying \( F = S(E) \)

Let \( S' \) be a system of arrows on \( E' \). Then \( S_{S'(E'_{MN})_{MN}} \leq S_E \). Moreover, there exist staircases \( E \) and \( F \) such that:

- \( S_E \geq S_F \)
- there is no system of arrows \( S' \) on \( E'_{MN} \) verifying \( F_{MN} = S'(E'_{MN}) \)
whose monomial ideals are $I_1$, verify $S_1$ for the incidence relation $\ast$.

For the second part of the proposition, we consider the staircase $s$ that the only possible relations are $E(x, y^i)$ and $F(x^i, y)$.

The same reasoning shows that the inequality $S_{S(E(M N))} \subseteq S_E$ always holds.

For the second part of the proposition, we consider the staircases $E$ and $F$ whose monomial ideals are $I_E = (y^i, x^j, x^k)$ and $I_F = (x^i, y^j, x^k)$. They verify $S_E \geq S_F$. Example 10 shows that there is no system of arrows $S'\subseteq E_5 \cup F_5$ verifying $F_5 \subseteq S'(E_5 \cup F_5)$. The dual example $I_E = (y^i, x^j, y^k)$ and $I_F = (x^i, y^j, x^k)$ verifies $S_E \geq S_F$, but there is no system $S$ on $E$ such that $F = S(E)$.

5 An example

For a small Hilbert function, it is easy to show that the necessary condition given by our theorem is in fact sufficient. It follows that we can solve the weak incidence problem on $\mathbb{H}_{ab}(H)$ for small $H$. In this section, we illustrate this fact in the case $(a, b) = (1, -1)$ and $H = (1, 2, 3, 2, 1, 0, 0, \ldots)$.

To solve the problem, we will use sufficient incidence conditions given by [Y2]. We could deal without these conditions, but we use them to shorten the proofs. There are nine possible staircases corresponding to the Hilbert function $H$, all of them drawn and named in the following figure. In the figure, a square whose position is $(i, j)$ relative to the square at the left bottom represents the monomial $x^i y^j$. We know that $C(E_1) \cap C(E_2) \neq \emptyset \Rightarrow S_{E_1} \geq S_{E_2}$. It follows that the only possible relations are

$$\{(E_{\text{gen}}, *) ; (a, c) ; (a, d) ; (a, e) ; (a, f) ; (a, g) ; (a, h) ; (b, d) ; (b, e) ; (b, f) ; (b, g) ; (b, h) ; (c, e) ; (c, f) ; (c, g) ; (c, h) ; (d, f) ; (d, g) ; (d, h) ; (e, g) ; (e, h) ; (f, h) ; (g, h)\}$$

where $*$ denotes any staircase and where we have adopted the notation $(E_1, E_2)$ for the incidence relation $C(E_1) \cap C(E_2) \neq \emptyset$.

In the figure, a thick arrow between two staircases $E_1$ and $E_2$ means that the relation $C(E_1) \supset C(E_2)$ occurs and these strong incidence relations come from [Y2]. It follows that if two staircases $E_1$ and $E_2$ are linked by a sequence of thick arrows, then the relation $(E_1, E_2)$ holds. The cases then remaining from the above list are:

$$\{(a, e) ; (b, d) ; (b, f) ; (c, e) ; (c, g) ; (d, f)\}$$
Example 10 shows that \((c, g)\) does not hold. The five remaining cases are compatible with our necessary condition. In particular, there are systems of arrows on the dual staircases \(a', b', c', d', e'\) coming from the condition 2) of theorem 8. We interpret these systems as systems on \(a', b', c', d', e'\) thanks to lemma 12. The following figure shows the systems obtained in this way. To exhibit ideals “compatible” with these systems, consider the sub-\(k[t]\)-modules of \(k[x, y, t]\) given by their generators: 

\[
T_3 = \langle y^3 + t^2xy^2 + tx^3, x^2y >, T_4 = \langle y^4 + t^2xy^3, xy^3 + tx^4, x^2y^2, x^3y >, P_{25} = \langle x^2y^4, \alpha + \beta \geq 5 >.
\]

Consider the ideal \(T := T_3 \oplus T_4 \oplus P_{25}\) of \(k[x, y, t]\). The quotient \(k[x, y, t]/T\) is flat over \(k[t]\). The associated universal morphism sends the generic point to a point in \(C(a)\) and can be extended by sending \(\infty\) to the point whose ideal is the monomial ideal with staircase \(e\). This shows that \((a, e)\) holds.

Similarly, if \(U_2 = \langle y^3, xy^2 + tx^2y + t^2x^3 >, U_4 = \langle y^4, xy^3, x^2y^2 + tx^3y, x^4 >, V_3 = \langle y^3 + tx^2y, xy^2 + tx^3 >, V_4 = \langle y^4, xy^3 + x^3y, x^2y^2, x^4 >, W_3 = \langle xy^2, x^2y >, W_4 = \langle y^3 + tx^4, xy^3, x^2y^2, x^3y >, Z_3 = \langle y^4 + txy^2 + t^2x^2y, x^3 >, Z_4 = \langle y^4, xy^3 + tx^2y, x^3y, x^4 >, U := U_3 \oplus U_4 \oplus P_{25}, V := V_3 \oplus V_4 \oplus P_{25}, W := W_3 \oplus W_4 \oplus P_{25}, Z := Z_3 \oplus Z_4 \oplus P_{25}\), the ideals \(U, V, W, Z\) show respectively that the relations \((b, d), (b, f), (c, e), (d, f)\) are true. The diagram sums up our results: the relation \((E, F)\) holds if and only if the staircases \(E\) and \(F\) are linked by a thin arrow or by a sequence of thick arrows.

**Bibliography:**

14
[BB]: Bialynicki-Birula, A: Some theorems on actions of algebraic groups . *Ann. of Math.* 98, 480-497, (1973)

[BB1]: Bialynicki-Birula, A: Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bulletin de l’académie Polonaise des sciences, Serie des Sciences math. astr. et phys.* 24, #9, 667-674, (1976)

[Br]: Brion, M: Equivariant Chow groups for torus actions, *Transformations Groups* 2, 225-267, (1997)

[ES]: Ellingsrud, G. and Stromme, S.: On the homology of the Hilbert scheme of points in the plane, *Invent. Math.* 87 (1987), no. 2, 343–352

[GH]: Griffiths, P. and Harris, J.: Principles of algebraic geometry, *J. Wiley, New York* (1977)

[Go]: Göttsche, L.: Betti numbers for the Hilbert function strata of the punctual Hilbert scheme in two variables. *Manuscripta Math.* 66, 253-259 (1990)

[Gr]: Granger, JM.: Géométrie des schémas de Hilbert ponctuels, *Mem. Soc. Math. Fr, Nouv. ser. 7-12*(1982-83)

[I]: Iarrobino, A: Punctual Hilbert Scheme. *Memoirs of AMS, vol. 10, #188,* (1977)

[M]: Manivel, L.: Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence, *Cours specialises de la SMF, 3* (1999)

[PS]: Peskine C., Szpiro L.: Liaison des variétés algébriques, *Invent. Math.* 26, (1974), 271-302

[Y]: Yameogo, J: Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $\mathbb{C}[[x,y]]$. Incidence des cellules I. *Compositio Math.* 90, #1, 81–98, (1994)

[Y2]: Yameogo, J: Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $\mathbb{C}[[x,y]]$. Incidence des cellules II. *J.reine angew. Math.* 450, 123-137, (1994)