1 Introduction

Rascal ([Klint et al., 2009, 2011]) is a high-level transformation language that aims to simplify software language engineering tasks like defining program syntax, analyzing and transforming programs, and performing code generation. The language provides several features including built-in collections (lists, sets, maps), algebraic data-types, powerful pattern matching operations with backtracking, and high-level traversals supporting multiple strategies.

Interaction between different language features can be difficult to comprehend, since most features are semantically rich. My goal is to provide a well-defined formal semantics for a large subset of Rascal called Rascal Light—in the spirit of (core) Caml Light ([Leroy, 1997], Clight ([Blazy and Leroy, 2009]), and Middleweight Java ([Bierman et al., 2003])—suitable for developing formal techniques, e.g., type systems and static analyses.

Scope

Rascal Light aims to model a realistic set of high-level transformation language features, by capturing a large subset of the Rascal operational language. Rascal Light targets being practically usable, i.e., it should be possible to translate many realistic pure Rascal programs to this subset by an expert programmer without losing the high level of abstraction. The following Rascal features are captured in Rascal Light:

- Fundamental definitions including algebraic data-types, functions with type parameters, and global variables.

- Basic expressions, including variable assignment, definition and access, exceptions, collections (including sets, lists and maps), if-expressions, switch-statements, for-loops and while-loops including control flow operators.

- Powerful pattern matching operations including type patterns, non-linear pattern matching, (sub)collection patterns, and descendant pattern matching.

- Backtracking using the fail operator, including roll-back of state.

- Traversals using generic visit-expressions, supporting the different kinds of available strategies: bottom-up(-break), top-down(-break), innermost, and outermost.
• Fixed-point iteration using the solve-loop.

The following Rascal features are considered out of scope:

• Concrete syntax declaration and literals, string interpolation, regular expression matching, date literals and path literals

• The standard library including Input/Output and the Java foreign function interface

• The module system, include modular extension of language elements such as datatypes and functions

• Advanced type system features, like parametric polymorphism, the numerical hierarchy and type inference

In Rascal, Boolean expressions can contain pattern matching subexpressions and backtracking is affected by the various Boolean operators (conjunction, disjunction, implication). In Rascal Light, backtracking is more restricted, which I believe heavily simplifies the semantics while losing little expressiveness in practice; most programs could be rewritten to support the required backtracking at the cost of increased verbosity.

Method

I have used the language documentation\(^1\) and the open source implementation of Rascal\(^2\) to derive the formalism. The syntax is primarily based on the \(\mu\text{Rascal} \) syntax description (CWI Amsterdam, 2017), but altered to focus on the high-level features. The semantics is based on the description of individual language features in the documentation. In case of under-specification or ambiguity, I used small test programs to check what the expected behavior is. I thank the Rascal developers for our personal correspondence which further clarified ambiguity and limitations of the semantics compared to Rascal.

To discover possible issues, the semantics has been implemented as a prototype and tested against a series of Rascal programs. To strengthen the correctness claims, I have proven a series of theorems of interest in Section 4, since as Milner et al. (1990) states:

\emph{The robustness of the semantics depends upon theorems}

— The Definition of Standard ML

Notation

A sequence of \(e\)s is represented by either \(e\) or more explicitly \(e_1, \ldots, e_n\), the empty sequence is denoted by \(\varepsilon\) and the concatenation of sequences \(e_1\) and \(e_2\) is denoted \(e_1 \cdot e_2\). We overload notation in an intuitive manner for operations on sequences, so given a sequence \(v\), \(v_i\) denotes the \(i\)th element in the sequence, and \(v : t\) denotes the sequence \(v_1 : t_1, \ldots, v_n : t_n\).

\(^1\)http://tutor.rascal- mpl.org/Rascal/Rascal.html
\(^2\)https://github.com/usethesource/rascal
Overview

The abstract syntax of Rascal Light is presented in Section 2 and the dynamic (operational) semantics is presented in Section 3. Finally, Section 4 presents relevant theorems regarding the soundness of the semantics.

2 Abstract Syntax

Rascal Light programs are organized into modules that consist of definitions of global variables, functions and algebraic data types. I will in the rest of this report assume that modules are well-formed: top-level definitions and function parameters have unique names, all function calls, constructors and datatypes used have corresponding definitions, and all variables are well-scoped with no shadowing. To maintain a clean presentation, I will not write the definition environment explicitly, but mention the necessary definitions as premises when required.

Rascal Light has three kinds of definitions: global variables, functions and algebraic data types. Global variables are typed and are initialized with a target expression at module loading time. Functions have a return type, a unique name, a list of typed uniquely named parameters, and have an expression as a body. Algebraic data types have unique names and declare a set of possible alternative constructors, each taking a list of typed fields as arguments.

\[
\begin{align*}
  d & := \text{global } t y = e \\
  & \mid \text{fun } t f(t x) = e \\
  & \mid \text{data } at = k_1(t x) \ldots k_n(t x) 
\end{align*}
\]  

(Global variables) (Function Definition) (Algebraic Data Type Definition)

The operational part of Rascal consists of syntactically distinct categories of statements and expressions; I have chosen to collapse the two categories in Rascal Light since statements also have result values that can be used inside expressions. Most constructs are standard imperative ones, such as blocks, assignment, branching and loops (including control operations); I have chosen to use local-in for representing blocks in Rascal Light instead of curly braces like in Rascal, to distinguish them from set literal expressions; blocks contain locally declared variables and a sequence of expressions to evaluate.

Notable Rascal-specific constructs include a generalized version of the switch-expression that supports rich pattern matching over both basic values and algebraic data types, the visit-expression which allows traversing data types using various strategies (Example 1), and the solve-expression which continuously evaluates its body expression until target variables reach a fixed-point in assigned values (Example 2). The fail control operator allows backtracking inside switch and visit statements to try other possible matches (Example 3).

Example 1 (Expression simplifier). An expression simplifier can use visit to completely simplify all sub-expressions no matter where they are in the input expression:

\[
data \text{ Expr = intlit(int v) } | \text{ plus(Expr lop, Expr rop) } | \ldots;
\]
Expr simplify(Expr e) =
  bottom-up visit(e) {
    case plus(intlit(0), y) => y
    case plus(x, intlit(0)) => x
  };

Example 2 (Lattice fixed-point). The Kleene fixed-point of a continuous function in a lattice for a domain Val with functions for the bottom element and least-upper bound can be computed using solve:

Val fix(Fun f) = {
  Val v = bottom();
  solve(v) {
    v = lub(v, apply(f, v))
  }
};

Example 3 (Knapsack problem). The knapsack problem concerns finding a subset of items with greatest value under a specified maximum weight. The function below uses backtracking to (slowly) find the optimal solution.

set[Item] slowknap(sack(set[Item] items, int maxWeight) = {
  set[Item] res = {};
  solve(res) {
    switch(items) {
      case (**xs, *ys): |
        if (sumweights(xs) > maxWeight |
          sumvalues(xs) < sumvalues(res)) fail;
        else res = xs;
      }
  res
  }
};

$e ::= vb$ (Basic Values, $vb \in \{1, \text{"foo"}, 3.14, \ldots\}$)

| $x$  | (Variables, $x, y \in \text{Var}$)
| $\ominus e$ | (Unary Operations, $\ominus \in \{-, \ldots\}$)
| $e_1 \oplus e_2$ | (Binary Operations, $\ominus \in \{+, -, \times, \ldots\}$)
| $k(e)$ | (Constructor Applications, $k \in \text{Constructor}$)
| $[e]$ | (List Expressions)
| $\{e\}$ | (Set Expressions)
| $(e : e')$ | (Map Expressions)
| $e_1[e_2]$ | (Map Lookup)
| $e_1[e_2 = e_3]$ | (Map Update)
| $f(e)$ | (Function Calls, $f \in \text{Function}$)
| return $e$ | (Return Expressions)
| $x = e$ | (Variable Update Assignments)
| if $e$ then $e_1$ else $e_2$ | (Conditionals)
| switch $e$ do $cs$ | (Case Matching)
| st visit $e$ do $cs$ | (Deep Traversal)
| break | continue | fail | (Control Operations)
| local $t$ $x$ in $e$ end            (Blocks) |
| for $g$ $e$                           (Iteration) |
| while $e$ $e$                          (While Expressions) |
| solve $x$ $e$                         (Fixedpoint Expressions) |
| throw $e$                              (Exception Throwing) |
| try $e_1$ catch $x$ $\Rightarrow$ $e_2$ (Exception Catching) |
| try $e_1$ finally $e_2$                (Exception Finalization) |

```plaintext
cs ::= case $p$ $\Rightarrow$ $e$
```

There are two kinds of generator expressions: enumerating assignment where the range variable is assigned to each element in the result of a collection-producing expression, and matching assignment that produces all possible assignments that match target patterns (defined later).

```plaintext
g ::= $x$ $\leftarrow$ $e$  (Enumerating Assignment) |
| $p$ $::=$ $e$       (Matching Assignment)
```

The visit-expression supports various strategies that determine the order a particular value is traversed w.r.t. its contained values. The top-down strategy traverses the value itself before contained values, and conversely bottom-up traverses contained values before itself. The break versions stop the traversal at first successful match, and the outermost and innermost respectively apply the top-down and bottom-up until a fixed-point is reached.

```plaintext
st ::= top-down                 (Preorder Traversal) |
| bottom-up                       (Postorder Traversal) |
| top-down-break                  (First-match Preorder Traversal) |
| bottom-up-break                 (First-match Postorder Traversal) |
| outermost                       (Fixedpoint Preorder Traversal) |
| innermost                       (Fixedpoint Postorder Traversal)
```

Like Rascal, Rascal Light has a rich pattern language that not only includes matching on basic values and constructors, but also powerful matching inside collections and descendant patterns that allow matching arbitrarily deeply contained values.

```plaintext
p ::= vb                       (Basic Value Patterns) |
| $x$                            (Variable Patterns) |
| $k(p)$                         (Deconstructor Patterns) |
| $t$ $x$ : $p$                  (Typed Labelled Patterns) |
| $*p$                           (List Patterns) |
| $\{*p\}$                      (Set Patterns) |
| $!p$                           (Negation pattern)
```
Patterns inside collections can be either ordinary patterns or star patterns that match a subcollection with arbitrary number of elements.

\[ *p ::= p \quad \text{(Ordinary Pattern)} \]
\[ \quad | *x \quad \text{(Star Pattern)} \]

Rascal Light programs expressions evaluate to values, which either are basic values, constructor values, collections or the undefined value (■).

\[ v ::= vb \quad \text{(Basic Values)} \]
\[ \quad | k(v) \quad \text{(Constructor Values, } k \in \text{Constructor)} \]
\[ \quad | [v] \quad \text{(List Values)} \]
\[ \quad | \{v\} \quad \text{(Set Values)} \]
\[ \quad | (v : v') \quad \text{(Map Values)} \]
\[ \quad | ■ \quad \text{(Undefined Value)} \]

3 Semantics

I present a formal development of the dynamic aspects of Rascal Light, using a natural semantics specification. Natural (big-step) semantics (Kahn, 1987) is particularly suitable for Rascal, because it closely mimics semantics of an interpreter and the high-level features—exceptions, backtracking, traversals—introduce a rich control-flow that depends not only on the structure of the program but also on the provided input. There is no concurrency or interleaving in Rascal, and so there is no need for a more fine-grained operational semantics like (small-step) structural operational semantics (SOS) (Plotkin, 2004).

Value Typing

Rascal (Light) is strongly typed and so all values are typed. The types are fairly straightforward in the sense that most values have a canonical type and there is a sub-typing hierarchy (explained shortly) with a bottom type void and a top type value.

\[ t ::= tb \quad \text{(Base Types, } tb \in \{\text{Int, Rational, String, \ldots\})} \]
\[ \quad | at \quad \text{(Algebraic Data Types, } at \in \text{DataType)} \]
\[ \quad | \text{set}(t) \quad \text{(Sets)} \]
\[ \quad | \text{list}(t) \quad \text{(Lists)} \]
\[ \quad | \text{map}(t_1, t_2) \quad \text{(Maps)} \]
\[ \quad | \text{void} \quad \text{(Bottom Type)} \]
\[ \quad | \text{value} \quad \text{(Top Type)} \]
I provide a typing judgment for values of form $v : t$, which states that value $v$ has type $t$. Basic values are typed by their defining basic type, the undefined value ($\top$) has the bottom type $\text{void}$, and constructor values are typed by their corresponding data type definition assuming the contained values are well-typed. The type of collections is determined by the contained values, and so a least upper bound operator is defined in types—following the sub-type ordering—which is used to infer a precise type for the value parameters.

\[
\begin{align*}
\text{T-Basic} & \quad \frac{vb \in \llbracket tb \rrbracket}{vb : tb} & \text{T-Bot} & \quad \frac{}{\top : \text{void}} \\
\text{T-Constructor} & \quad \frac{\text{data } at = \ldots | k(t) | \ldots \quad v : t' \quad t' :<: t}{k(v_1, \ldots, v_n) : at} \\
\text{T-Set} & \quad \frac{v : t}{\{v\} : \text{set}(\llbracket t\rrbracket)} & \text{T-List} & \quad \frac{v : t}{[v] : \text{list}(\llbracket t\rrbracket)} & \text{T-Map} & \quad \frac{v : t \quad v' : t'}{(v : v') : \text{map}(\llbracket t\rrbracket, \llbracket t'\rrbracket)}
\end{align*}
\]

The subtyping relation has form $t <: t'$, stating $t$ is a subtype of $t'$. We let $t :<: t'$ denote the negated form where none of the given cases below matches. Sub-typing is reflexive, so every type is a sub-type of itself; $\text{void}$ and $\text{value}$ act as bottom type and top type respectively.

\[
\begin{align*}
\text{ST-Refl} & \quad t <: t & \text{ST-Void} & \quad \text{void} <: t & \text{ST-Value} & \quad t <: \text{value}
\end{align*}
\]

Collections are covariant in their type parameters, which is safe since all values are immutable.

\[
\begin{align*}
\text{ST-List} & \quad \frac{t <: t'}{\text{list}(t) <: \text{list}(t')} & \text{ST-Set} & \quad \frac{t <: t'}{\text{set}(t) <: \text{set}(t')} \\
\text{ST-Map} & \quad \frac{t_{\text{key}} <: t'_{\text{key}} \quad t_{\text{val}} <: t'_{\text{val}}}{\text{map}(t_{\text{key}}, t_{\text{val}}) <: \text{map}(t'_{\text{key}}, t'_{\text{val}})}
\end{align*}
\]

The least upper bound on types is defined as follows:

\[
\begin{align*}
\bigcup \varepsilon & = \text{void} & \bigcup t, t' & = t \sqcup t' \\
\begin{cases}
  t_1 & \text{if } t_2 = \text{void} \lor t_1 = t_2 \\
  t_2 & \text{if } t_1 = \text{void} \\
  \text{list}(t'_1 \sqcup t'_2) & \text{if } t_1 = \text{list}(t'_1) \land t_2 = \text{list}(t'_2) \\
  \text{set}(t'_1 \sqcup t'_2) & \text{if } t_1 = \text{set}(t'_1) \land t_2 = \text{set}(t'_2) \\
  \text{map}(t'_1 \sqcup t'_2, t''_1 \sqcup t''_2) & \text{if } t_1 = \text{map}(t'_1, t''_1) \land t_2 = \text{map}(t'_2, t''_2) \\
  \text{value} & \text{otherwise}
\end{cases}
\end{align*}
\]
Expression Evaluation

The main judgment for Rascal Light expressions has the form $e;\sigma \Rightarrow vres;\sigma'$, where the expression $e$ is evaluated in an initial store $\sigma \in \mathbf{Var} \rightarrow \mathbf{Val}$—mapping variables to values—returning a result $vres$ and updated store $\sigma'$ as a side-effect. The result $vres$ is either an ordinary success $v$ signifying successful execution or an exceptional result $exres$.

$$vres ::= \text{success } v \mid \text{exres}$$

An exceptional result is either a control operation ($\text{break}$, $\text{continue}$, $\text{fail}$), an error that happened during execution, a thrown exception $\text{throw } v$ or an early return value $\text{return } v$. The difference between success $v$ and return $v$ is that the latter should propagate directly from sub-expressions to the surrounding function call boundary, while the value in the former can be used further in intermediate computations (Example 4).

$$exres ::= \text{return } v \mid \text{throw } v \mid \text{break} \mid \text{continue} \mid \text{fail} \mid \text{error}$$

**Example 4 (Early Return).** The function that calculates products, uses the early return functionality to short-circuit the rest of the calculation when a factor in the provided list is zero. If this branch is hit during execution, evaluating the expression produces return 0 as result, which is then propagated directly to function call boundary, skipping the rest of the loop and sequenced expressions.

```
1 int prod(list[int] xs) {
2     int res = 1;
3     for (x <- xs) {
4         if (x == 0) return 0;
5         else res *= x;
6     }
7     res
8 }
```

Basic values evaluate to their semantic equivalent result values without any side effect (E-VAL).

$$\text{E-VAL} \quad vb;\sigma \Rightarrow \text{success } vb;\sigma$$

A variable evaluates to the value it is assigned to in the store if available (E-VAR-SUCS), and otherwise result in an error (E-VAR-ERR).

$$\text{E-VAR-SUCS} \quad x \in \text{dom } \sigma \quad x;\sigma \Rightarrow \text{success } \sigma(x);\sigma$$

$$\text{E-VAR-ERR} \quad x \notin \text{dom } \sigma \quad x;\sigma \Rightarrow \text{error};\sigma$$

Unary expressions evaluate their operands, applying possible side-effects; if successful the corresponding semantic unary operator $[\ominus]$ is applied on the result value (E-UN-SUCS), and otherwise it propagates the exceptional result (E-UN-EXC).

$$\text{E-UN-SUCS} \quad e;\sigma \Rightarrow \text{success } v;\sigma' \quad \ominus e;\sigma \Rightarrow \ominus (v);\sigma'$$

$$\text{E-UN-EXC} \quad e;\sigma \Rightarrow \text{exres};\sigma' \quad \ominus e;\sigma \Rightarrow \ominus \text{exres};\sigma'$$
Example 5 (Unary operator semantics). \([-\](3) will evaluate to success \(-3\), while \([-\](\{\}) will evaluate to error.

Evaluating binary expressions is similar to unary expressions, requiring both operands to evaluate successfully (E-Bin-Sucs) to apply the corresponding semantic binary operator \([\oplus]\); otherwise the exceptional results of the operands are propagated in order from left (E-Bin-Exc1) to right (E-Bin-Exc2).

\[
\begin{align*}
e_1; \sigma \rightarrow \text{success } v_1; \sigma'' & \quad e_2; \sigma'' \rightarrow \text{success } v_2; \sigma' \\
E-\text{Bin-Sucs} & \quad e_1 \oplus e_2; \sigma \rightarrow \oplus (v_1, v_2); \sigma'
\end{align*}
\]

\[
\begin{align*}
e_1; \sigma \rightarrow \text{exres}_1; \sigma'' & \\
E-\text{Bin-Exc1} & \quad e_1 \oplus e_2; \sigma \rightarrow \text{exres}_1; \sigma'
\end{align*}
\]

\[
\begin{align*}
e_1; \sigma \rightarrow \text{success } v_1; \sigma'' & \quad e_2; \sigma'' \rightarrow \text{exres}_2; \sigma' \\
E-\text{Bin-Exc2} & \quad e_1 \oplus e_2; \sigma \rightarrow \text{exres}_2; \sigma'
\end{align*}
\]

Constructor expressions evaluate their arguments first, and if they all successfully evaluate to values, then check whether the types of values match those expected in the declaration. If the result values have the right types and are not \(\text{■}\), a constructor value is constructed (E-Cons-Sucs), and otherwise a (type) error is produced (E-Cons-Err). In case any of the arguments has an exceptional result the evaluation of the rest of the arguments halts and the exceptional result is propagated (E-Cons-Exc).

\[
\begin{align*}
data \ at = \ldots | k(t) | \ldots & \quad e; \sigma \rightarrow \text{success } v; \sigma' \\
E-\text{Cons-Sucs} & \quad k(e); \sigma \rightarrow \text{success } k(v); \sigma' \\
& \quad v : t' \quad v \neq \text{■} \quad t' < ; t
\end{align*}
\]

\[
\begin{align*}
data \ at = \ldots | k(t) | \ldots & \quad e; \sigma \rightarrow \text{success } v; \sigma' \\
E-\text{Cons-Err} & \quad k(e); \sigma \rightarrow \text{error}; \sigma' \\
& \quad v : t' \quad \exists i. v_i = \text{■} \lor t'_i \notin ; t_i
\end{align*}
\]

\[
\begin{align*}
& \quad e; \sigma \rightarrow \text{exres}; \sigma' \\
E-\text{Cons-Exc} & \quad k(e); \sigma \rightarrow \text{exres}; \sigma'
\end{align*}
\]

Evaluating list expressions also requires evaluating all subexpressions to a series of values; because of sequencing and necessity of early propagation of exceptional results, evaluation of series of subexpressions is done using a mutually recursive sequence evaluation judgment (see page 17). If the evaluation is successful then a list value is constructed (E-List-Sucs), unless any value is undefined (■) in which case we produce an error (E-List-Err), and otherwise the exceptional result is propagated (E-List-Exc).

\[
\begin{align*}
e; \sigma \rightarrow \text{success } v; \sigma' & \quad v \neq \text{■} \\
E-\text{List-Sucs} & \quad [e]; \sigma \rightarrow \text{success } [v]; \sigma'
\end{align*}
\]
Set expression evaluation mirror the one for lists, except that values are constructed using a set constructor (E-Set-Sucs), which may reorder values and ensures that there are no duplicates. If any contained value was undefined (■) then an error is produced instead (E-Set-Err), and exceptional results are propagated (E-Set-Exc).

E-Set-Sucs
e, e′; σ ⇒⇒ success v, v′; σ′  v ≠ ■  v′ ≠ ■

E-Set-Err
e; σ ⇒⇒ success v; σ′  v = ■

E-Set-Exc
e; σ ⇒⇒ exres; σ′

Map expressions evaluate their keys and values in the declaration sequence, and if successful construct a map (E-Map-Sucs). Similarly to other collection expressions, errors are produced if any value is undefined (E-Map-Err) and exceptional results are propagated (E-Map-Exc).

E-Map-Sucs
e, e′; σ ⇒⇒ success v, v′; σ′  v ≠ ■  v′ ≠ ■

E-Map-Err
e, e′; σ ⇒⇒ error; σ′

E-Map-Exc
e, e′; σ ⇒⇒ exres; σ′

Lookup expressions require evaluating the outer expression to a map—otherwise producing an error (E-Lookup-Err)—and the index expression to a value. If the index is an existing key then the corresponding value is produced as result (E-Lookup-Sucs) and otherwise the nokey exception is thrown (E-Lookup-NoKey); here, I assume that

data NoKey = nokey(value key)

is a built-in data-type definition. Exceptional results are propagated from sub-terms (E-Lookup-Exc1, E-Lookup-Exc2).

E-Lookup-Sucs
e1; σ ⇒⇒ success (. . . , v : v′, . . . ); σ″
e2; σ″ ⇒⇒ success v; σ′

E-Lookup-NoKey
e1; σ ⇒⇒ success v; σ′
e2; σ″ ⇒⇒ success v″; σ′  ∀i.v″ ≠ v_i
Map update expressions also require the outer expression to evaluate to a map—otherwise producing an error (E-UPDATE-ERR1)—and the index and target expressions to evaluate to values. On successful evaluation of both index and target value the map is updated, overriding the old value of the corresponding index if necessary (E-UPDATE-SUCS) unless the index or target value is equal to in which case an error is produced (E-UPDATE-ERR2). Finally, exceptional results are propagated left-to-right if necessary (E-UPDATE-Exc1, E-UPDATE-Exc2, E-UPDATE-Exc3).

Evaluation of function calls is more elaborate. Function definitions are statically scoped, and the semantics is eager, so arguments are evaluated using call-by-value. The initial step is thus to evaluate all the arguments to values if possible—propagating the exceptional result
otherwise (E-CALL-ARG-Exc)—and then check whether the values have the right type
(otherwise producing an error, E-CALL-ARG-Err). The evaluation proceeds by evaluating
the body of the function with a fresh store that contains the values of global variables and
the parameters bound to their respective argument values. There are then four cases:

1. If the body successfully evaluates to a correctly typed value, then that value is provided
as the result (E-CALL-Sucs).

2. If the body evaluates to a value that does not have the expected type, then it produces
an error (E-CALL-Res-Err1).

3. If the result is a thrown exception or error, then it is propagated (E-CALL-Res-Exc).

4. Otherwise if the result is a control operator, then an error is produced (E-CALL-Res-
Err2).

In all cases the resulting store of executing the body is discarded—since local assignments
fall out of scope—except the global variable values which are added to the store that was
there before the function call.
The return-expression evaluates its argument expression first, and if it successfully produces a value then the result would be an early return with that value (E-Ret-Sucs); recall that early returns are treated as exceptional values and so propagated through most evaluation rules, except at function call boundaries (rules E-CALL-SUCS and E-CALL-RES-ERR1). Otherwise, the exceptional result is propagated (E-Ret-Exc).

In Rascal Light, variables must be declared before being assigned, and declarations are unique since shadowing is disallowed. Evaluating an assignment proceeds by evaluating the right-hand side expression—propagating exceptional results (E-ASGN-EXC)—and then checking whether the produced value is compatible with the declared type. If it is compatible then the store is updated (E-ASGN-SUCS), and otherwise an error is produced (E-ASGN-Err).

The if-expression works like other languages: the then-branch is evaluated if the condition is true (E-If-True), otherwise the else-branch is evaluated (E-If-False). If the conditional produces a non-Boolean value then an error is raised (E-If-Err) and otherwise exceptional results are propagated (E-If-Exc).
The `switch`-expression initially evaluates the scrutinee expression `(e)`, and then proceeds to execute the cases (discussed on page 18) on the result value `(E-Switch-Sucs)`. The evaluation of cases is allowed to fail, in which case the evaluation is successful and has the special value ■ (E-Switch-Fail); other exceptional results are propagated as usual (E-Switch-Exc1, E-Switch-Exc2).

The `visit`-expression has similar evaluation cases to `switch` (E-Visit-Sucs, E-Visit-Fail, E-Visit-Exc1, E-Visit-Exc2), except that the cases are evaluated using the visit relation that traverses the produced value of target expression given the provided strategy.
The control operations `break`, `continue` and `fail` evaluate to themselves without any side-effects (E-Break, E-Continue, E-Fail).

Blocks allow evaluating inner expressions using a local declaration of variables, which are then afterwards removed from the resulting store (E-BLOCK-SUCS, E-BLOCK-EXC). Recall, that we consider an implicit definition environment based on scoping, and so the local declarations in the block will be implicitly available in the evaluation of the body subexpression sequence.

The auxiliary function `last` is here used to extract the last element in the sequence (or return ■ if empty).

\[
\text{last}(v_1, \ldots, v_n, v') = v' \\
\text{last}(\varepsilon) = \boxed{\text{■}}
\]

The `for`-loop evaluates target generator expression to a set of possible environments that represent possible assignments of variables to values—propagating exceptions if necessary (E-FOR-EXC)—and then it iterates over each possible assignment using the each-relation (E-FOR-SUCS).

The evaluation of `while`-loops is analogous to other imperative languages with control operations, in that the body of the while loop is continuously executed until the target condition does not hold (E-WHILE-FALSE). If the body successfully finishes with an value
or continue then iteration continues (E-WHILE-TRUE-SUCS), if the body finishes with break the iteration stops with value ■ (E-WHILE-TRUE-BREAK), if the conditional evaluates to a non-Boolean value it errors out (E-WHILE-ERR) and otherwise if another kind of exceptional result is produced then it is propagated (E-WHILE-EXC1, E-WHILE-EXC2).

\[
\text{E-WHILE-False}
\]

\[
\text{E-WHILE-True-SUCS}
\]

\[
\text{E-WHILE-True-Break}
\]

\[
\text{E-WHILE-Exc1}
\]

\[
\text{E-WHILE-Exc2}
\]

\[
\text{E-WHILE-Err}
\]

The solve-loop keeps evaluating the body expression until the provided variables reach a fixed-point. Initially, the body expression is evaluated and then the values of target variables is compared from before and after iteration; if the values are equal after an iteration, then evaluation stops (E-SOLVE-EQ) and otherwise the iteration continues (E-SOLVE-NEQ). If any of the variables do not have a value assigned, an error is produced (E-SOLVE-ERR), and otherwise if an exceptional result is produced, it is propagated (E-SOLVE-EXC).

\[
\text{E-SOLVE-EQ}
\]

\[
\text{E-SOLVE-NEQ}
\]

\[
\text{E-SOLVE-Exc}
\]
The throw-expression, evaluates its inner expression first—propagating exceptional results if necessary (E-Thr-Exc)—and then produces a throw result with result value (E-Thr-Sucs).

The try-finally expression executes the try-body first and then the finally-body. If the finally-body produces an exceptional result during execution then that result is propagated (E-Fin-Exc) and otherwise the try-body result value is used (E-Fin-Sucs).

The try-catch expression evaluates the try-body and if it produces a thrown value, then it binds the value in the body of catch and continues evaluation (E-Try-Catch). For all other results, it simply propagates them without evaluating the catch-body (E-Try-Ord).

Expression Sequences Evaluating a sequence of expressions proceeds by evaluating each expression, combining the results if successful (ES-Emp, ES-More) and otherwise propagating the first exceptional result encountered (ES-Exc1, ES-Exc2).
**Cases**  The evaluation relation for evaluating a series of cases has the form \( cs; v; \sigma \Longrightarrow vres; \sigma' \), and intuitively proceeds by sequentially evaluating each case (in \( cs \)) against value \( v \) until one of them produces a non-
\( \text{fail} \) result. For each case, the first step is to match the given value against target pattern and then evaluate the target expression under the set of possible matches; if the evaluation of the target expression produces a \( \text{fail} \) as result, the rest of the cases are evaluated in a restored initial state (ECS-MORE-FAIL) and otherwise the result is propagated (ECS-MORE-ORD). If all possible cases are exhausted, the result is \( \text{fail} \) (ECS-EMP).

\[
\begin{align*}
ECS-EMP & : \varepsilon; v; \sigma \Longrightarrow \text{fail}; \sigma \\
ECS-More-Fail & : \sigma \vdash p := v \Longrightarrow \rho; \sigma \Longrightarrow \text{fail}; \sigma'' \\
ECS-More-Ord & : \sigma \vdash p := v \Longrightarrow \rho; \sigma \Longrightarrow \text{vres}; \sigma' \\
& \quad \text{vres} \neq \text{fail}
\end{align*}
\]

Evaluating a single case—with relation \( \rho; e; \sigma \Longrightarrow \text{vres}; \sigma' \)—requires trying each possible binding (in \( \rho \)) sequentially, producing \( \text{fail} \) if no binding is available (EC-EMP). If evaluating target expression produces a non-
\( \text{fail} \) value then it is propagated (EC-MORE-ORD), otherwise the rest of the possible bindings are tried in a restored initial state (EC-MORE-FAIL).

\[
\begin{align*}
EC-More-Fail & : e; \sigma \Longrightarrow \text{fail}; \sigma'' \\
EC-More-Ord & : \rho, \rho'; e; \sigma \Longrightarrow \text{vres}; \sigma' \\
& \quad \text{vres} \neq \text{fail}
\end{align*}
\]

**Traversals** One of the key features of Rascal is visit-expressions which provide generic traversals over data values and collections, allowing for multiple strategies to determine the traversal ordering and halting conditions. In a traditional object-oriented language or functional language, transforming large structures is cumbersome and requires a great amount of boilerplate, requiring a function for each type of datatype, where each function must deconstruct the input data, applying target changes, recursively calling the right traversal
functions for traversal of further contained data and reconstructing the data with new values. Precisely, the first-class handling of these aspects combined with the other features such as the powerful pattern matching makes Rascal particularly suitable as a high-level transformation language.

The main traversal relation $\text{cs}; v; \sigma \xrightarrow{\text{visit}} \text{vres}; \sigma'$ delegates execution to the correct strategy-dependent traversal, and performs fixed-point calculation if necessary. For the top-down-break and top-down strategies it uses the top-down traversal relation $\text{cs}; v; \sigma \xrightarrow{\text{br}} \text{vres}; \sigma'$ specifying break and no no-break as breaking strategies respectively (EV-TD and EV-TDB); this works analogously with the bottom-up-break (EV-BU and EV-BUB) and bottom-up strategies using the $\text{cs}; v; \sigma \xrightarrow{\text{br}} \text{vres}; \sigma'$ relation.

The innermost strategy evaluates the bottom-up traversal as long as it produces a resulting value not equal to the one from the previous iteration (EV-IM-Neq), returning the result value when a fixed-point is reached (EV-IM-Eq); if any exceptional result happens during evaluation it will be propagated (EV-IM-Exc). Analogous evaluation steps happens with outermost strategy and top-down traversal (EV-OM-Neq, EV-OM-Eq, EV-OM-Exc).
The top-down traversal strategy starts by executing all cases on the target value, scrutinee, applying possible replacements and effects to produce an intermediate result value; the traversal then continues on the sequence of contained values of the this intermediate result, finally reconstructing a new output containing the possible replacement values obtained (ETV-Ord-Sucs1, ETV-Ord-Sucs2). If using the break strategy, the traversal will stop at the first value that produces a successful result (ETV-Break-Sucs); otherwise, if any sub-result produces a non-fail exceptional result it is propagated (ETV-Exc1, ETV-Exc2).

ETV-Break-Sucs

\[
\text{cs}; \ v; \ \sigma \longrightarrow \text{success} \ v'; \ \sigma' \quad \text{br} = \text{break}
\]

ETV-Ord-Sucs1

\[
\text{cs}; \ v; \ \sigma \longrightarrow \text{vres}; \ \sigma'' \quad \text{br} = \text{break} \Rightarrow \text{vres} = \text{fail}
\]

ETV-Ord-Sucs2

\[
\text{cs}; \ v''; \ \sigma'' \longrightarrow \text{success} \ v'''; \ \sigma' \quad \text{recons} \ v'' \text{ using } v''' \text{ to } \text{rcres}
\]

ETV-Exc1

\[
\text{cs}; \ v; \ \sigma \longrightarrow \text{exres}; \ \sigma' \quad \text{exres} \neq \text{fail}
\]

ETV-Exc2

\[
\text{cs}; \ v; \ \sigma \longrightarrow \text{exres}; \ \sigma'
\]

Evaluating a sequence of top-down traversals, requires executing a top-down traversal for each element, failing if the input sequence is empty (ETVS-Emp) and otherwise combining the results (ETVS-More). If the break strategy is used, then the iteration will instead stop at first successful result (ETVS-Break), and any non-fail exceptional result is propagated (ETVS-Exc1, ETV-Exc2).
Bottom-up traversals work analogously to top-down traversals, except that traversal of children and reconstruction happens before traversing the final reconstructed value (EBU-Break-Sucs, EBU-No-Break-Sucs, EBU-Fail-Sucs). The analogy also holds with propagation of exceptional results and errors (EBU-Exc, EBU-No-Break-Err, EBU-No-Break-Exc).
Evaluating a sequence of bottom-up traversals is analogous to evaluating a sequence of top-down traversals. Each element in the sequence is evaluated and their results is combined (EBUS-Emp, EBUS-More), stopping at the first successful result when using the break strategy (EBUS-Break). Otherwise, non-fail exceptional results are propagated (EBUS-Exc1, EBUS-Exc2).
**Auxiliary**  The children function extracts the directly contained values of the given input value.

\[
\begin{align*}
\text{children}(v_b) &= \varepsilon \quad \text{children}(k(v)) = v \\
\text{children}(v) &= v \quad \text{children}([v]) = v \\
\text{children}(\{v\}) &= v \quad \text{children}(\{\}) = \varepsilon
\end{align*}
\]

\[vfres ::= \text{success } v \mid \text{fail}\]

The if-fail function will return a provided default value if the first argument is fail, and otherwise it will use the provided value in the first argument.

\[
\text{if-fail}(\text{fail}, v) = v \quad \text{if-fail}(\text{success } v', v) = v'
\]

The vcombine function will combine success and fail results from visitor, producing fail if both result arguments are fail otherwise producing success result, possibly using default values.

\[
vcombine(vfres, vfres', v, v') =
\begin{cases}
\text{fail} & \text{if } vfres = \text{fail} \land vfres' = \text{fail} \\
\text{success} \left( \text{if-fail}(vfres, v), \text{if-fail}(vfres', v') \right) & \text{otherwise}
\end{cases}
\]

The reconstruction relation \(\text{recons } v \text{ using } v' \text{ to } \text{rcres}\) tries to update the elements of a value, checking whether provided values are type correct and defined (not \(\Box\)), otherwise producing an error.

\[\text{rcres} ::= \text{success } v \mid \text{error}\]

\[
\begin{align*}
\text{RC-VAL-SUCS} & \quad \text{recons } v_b \text{ using } \varepsilon \text{ to } \text{success } v_b \\
\text{RC-VAL-ERR} & \quad \text{recons } v_b \text{ using } v', v'' \text{ to } \text{error} \\
\text{RC-CONS-SUCS} & \quad \text{data } at = \ldots | k(t) | \ldots | v' : t' | v' \neq \Box | t' <: t \\
& \quad \text{recons } k(v) \text{ using } v' \text{ to } \text{success } k(v') \\
\text{RC-CONS-ERR} & \quad \text{data } at = \ldots | k(t) | \ldots | v' : t' | v_i = \Box \lor t_i \not<: t_i \\
& \quad \text{recons } k(v) \text{ using } v' \text{ to } \text{error} \\
\text{RC-LIST-SUCS} & \quad v' \neq \Box \\
& \quad \text{recons } [v] \text{ using } v' \text{ to } \text{success } [v']
\end{align*}
\]
### Enumeration

The enumeration relation $\{e; \rho; \sigma \Rightarrow v; \sigma'\}$ iterates over all provided bindings (EE-MORE-SUCS) until there are none left (EE-EMP) or the result is neither an ordinary value or continue from one of the iterations (EE-MORE-Exc); in case the result is break the evaluation will terminate early with a successful result (EE-MORE-Break).

#### EE-EMP

$\{e; \rho; \sigma \Rightarrow success \; \square; \sigma\}$

#### EE-MORE-SUCS

$\{e; \rho; \sigma \Rightarrow v; \sigma'\}$

$\{e; \rho; \sigma \Rightarrow v \lor v; \sigma'\}$

#### EE-MORE-Break

$\{e; \rho; \sigma \Rightarrow break; \sigma'\}$

#### EE-MORE-Exc

$\{e; \rho; \sigma \Rightarrow exres; \sigma'\}$

where $exres \in \{\text{throw } v, \text{return } v, \text{fail, error}\}$
Generator expressions

The evaluation relation for generator expressions has form $g; \sigma \xrightarrow{\text{geexpr}} \text{enres}; \sigma'$. For matching assignments the target right-hand side expression is evaluated first—propagating possible exceptional results (G-Pat-Exc) and then the value is matched against target pattern (G-Pat-Sucs).

\[
\text{enres ::= success } \bar{\rho} \mid \text{exres}
\]

\[
\begin{align*}
\text{G-PAT-Sucs} & : e; \sigma \xrightarrow{\text{expr}} \text{success } v; \sigma' \quad \sigma' \vdash p := v \xrightarrow{\text{match}} \bar{\rho} \\
& \quad p := e; \sigma \xrightarrow{\text{geexpr}} \text{success } \bar{\rho}; \sigma' \\
& \quad e; \sigma \xrightarrow{\text{expr}} \text{exres}; \sigma'
\end{align*}
\]

For enumerating assignments, each possible value in a collection is provided as a possible binding to the range variable in the output (G-Enum-List, G-Enum-Set); for maps in particular, only the keys are bounds (G-Enum-Map). An error is raised if the result value is not a collection (G-Enum-Err), and exceptions are propagated as always (G-Enum-Exc).

\[
\begin{align*}
\text{G-ENUM-LIST} & : e; \sigma \xrightarrow{\text{expr}} \text{success } [v_1, \ldots, v_n]; \sigma' \\
& \quad x \leftarrow e; \sigma \xrightarrow{\text{geexpr}} \text{success } [x \mapsto v_1], \ldots, [x \mapsto v_n]; \sigma'
\end{align*}
\]

\[
\begin{align*}
\text{G-ENUM-SET} & : e; \sigma \xrightarrow{\text{expr}} \text{success } \{v_1, \ldots, v_n\}; \sigma' \\
& \quad x \leftarrow e; \sigma \xrightarrow{\text{geexpr}} \text{success } [x \mapsto v_1], \ldots, [x \mapsto v_n]; \sigma'
\end{align*}
\]

\[
\begin{align*}
\text{G-ENUM-MAP} & : e; \sigma \xrightarrow{\text{expr}} \text{success } (v_1 : v_1', \ldots, v_n : v_n'); \sigma' \\
& \quad x \leftarrow e; \sigma \xrightarrow{\text{geexpr}} \text{success } [x \mapsto v_1], \ldots, [x \mapsto v_n]; \sigma'
\end{align*}
\]

\[
\begin{align*}
\text{G-ENUM-ERR} & : e; \sigma \xrightarrow{\text{expr}} \text{success } v; \sigma' \\
& \quad x \leftarrow e; \sigma \xrightarrow{\text{geexpr}} \text{error}; \sigma'
\end{align*}
\]

\[
\begin{align*}
\text{G-ENUM-EXC} & : e; \sigma \xrightarrow{\text{exres}} \text{exres}; \sigma'
\end{align*}
\]

Pattern Matching

The pattern matching relation $v \vdash \sigma := p \xrightarrow{\text{match}} \bar{\rho}$ takes as input the current store $\sigma$, a pattern $p$ and a target value $v$ and produces a sequence of compatible environments that represent possible bindings of variables to values. Pattern matching a basic value against target value produces a single environment that does not bind any variable ([]) if the target value is the same basic value (P-Val-Sucs) and otherwise does not produce any binding environment ($\varepsilon$), (P-Val-Fail).
Pattern matching against a variable depends on whether the variable already exists in the current store. If it is assigned in the current store, then the target value must the assigned value to return a possible binding (P-VAR-UNI) and otherwise failing with no bindings (P-VAR-FAIL); if it is not in the current store, it will simply bind the variable to the target value (P-VAR-BIND).

\[
\begin{align*}
P-VAR-UNI & : x \in \text{dom } \sigma \quad v = \sigma(x) \\
& \quad \sigma \vdash x :? \quad v \xrightarrow{\text{match}} \epsilon \\
\end{align*}
\]

\[
\begin{align*}
P-VAR-FAIL & : x \in \text{dom } \sigma \quad v \neq \sigma(x) \\
& \quad \sigma \vdash x :? \quad v \xrightarrow{\text{match}} \epsilon \\
\end{align*}
\]

\[
\begin{align*}
P-VAR-BIND & : x \notin \text{dom } \sigma \\
& \quad \sigma \vdash x :? \quad v \xrightarrow{\text{match}} [x \mapsto v] \\
\end{align*}
\]

When pattern matching against a constructor pattern, it is first checked whether the target value has the same constructor. If it does, then the sub-patterns are matched against the contained values of the target value, merging their resulting environments (P-CONS-SUCS), and otherwise failing with no bindings (P-CONS-FAIL). The merging procedure is described formally later in this section, but it intuitively it takes the union of all bindings that have consistent assignments to the same variables.

\[
\begin{align*}
P-CONS-SUCS & : \sigma \vdash p_1 :? \quad v_1 \xrightarrow{\text{match}} \rho_1 \quad \ldots \\
& \quad \sigma \vdash p_n :? \quad v_n \xrightarrow{\text{match}} \rho_n \\
& \quad \sigma \vdash k(p) :? \quad v \xrightarrow{\text{match}} \text{merge}(\rho_1, \ldots, \rho_n) \\
\end{align*}
\]

\[
\begin{align*}
P-CONS-FAIL & : v \neq k(v') \\
& \quad \sigma \vdash k(p) :? \quad v \xrightarrow{\text{match}} \epsilon \\
\end{align*}
\]

When pattern matching against a typed labeled pattern, the target value is checked to have a compatible type—failing with no bindings otherwise (P-TYPE-FAIL)— and then the inner pattern is matched against the same value. The result of the sub-pattern match is merged with the environment where the target value is bound to the label variable (P-TYPE-SUCS).

\[
\begin{align*}
P-TYPE-SUCS & : v : t' \quad t' <: t \\
& \quad \sigma \vdash p :? \quad v \xrightarrow{\text{match}} \rho \\
& \quad \sigma \vdash t \quad x : p :? \quad v \xrightarrow{\text{match}} \text{merge}([x \mapsto v], \rho) \\
\end{align*}
\]

\[
\begin{align*}
P-TYPE-FAIL & : v : t' \quad t' \not<: t \\
& \quad \sigma \vdash t \quad x : p :? \quad v \xrightarrow{\text{match}} \epsilon \\
\end{align*}
\]
Pattern matching against a list pattern first checks whether the target value is a list—otherwise failing (P-List-Fail)—and then pattern matches against the sub-patterns returning their result (P-List-Sucs).

\[
\frac{\sigma \vdash *p \triangleright v | \emptyset \triangleright \rho}{\text{match}} \quad \frac{v \neq [v']}{\text{P-List-Fail}}
\]

Pattern matching against a set pattern is analogous to pattern matching against list patterns (P-Set-Sucs, P-Set-Fail).

\[
\frac{\sigma \vdash *p \triangleright v | \emptyset \{ v \} \triangleright \rho}{\text{match}} \quad \frac{v \neq \{ v' \}}{\text{P-Set-Fail}}
\]

Negation pattern !p matching succeeds with no variables bound if the sub-pattern \( p \) produces no binding environment (P-Neg-Sucs), and otherwise fails (P-Neg-Fail).

\[
\frac{\sigma \vdash p \triangleright v \triangleright \varepsilon}{\text{match}} \quad \frac{\sigma \vdash p \triangleright v \triangleright \rho}{\text{match}}
\]

\[
\sigma \vdash !p \triangleright v \triangleright \varepsilon
\]

Descendant pattern matching applies the sub-pattern against target value, and keeps applying the deep matching pattern against the children values, concatenating their results (P-Deep).

\[
\frac{\sigma \vdash p \triangleright v \triangleright \rho}{\text{match}} \quad v', \ldots, v' = \text{children}(v)
\]

\[
\frac{\sigma \vdash /p \triangleright v' \triangleright \rho'_{1}}{\text{match}} \quad \ldots \quad \frac{\sigma \vdash /p \triangleright v' \triangleright \rho'_{n}}{\text{match}}
\]

\[
\sigma \vdash /p \triangleright v \triangleright \rho, \rho'_{1}, \ldots, \rho'_{n}
\]

The star pattern matching relation has form

\[
\sigma \vdash *p \triangleright v \mid \emptyset \triangleright \rho\]

which tries to match the sequence of patterns *\( p \) on the left-hand side with sequence of provided values \( v \) on the right-hand side; the relations is parameterized over the construction function (\( \langle \rangle \)) and the partition relation (\( \otimes \)), and because matching arbitrary elements patterns *\( x \) is non-deterministic we keep track of a set of values \( \mathbb{V} \) that have already been tried for the latest available variable.

If both the pattern sequence and the value sequence is empty, then we have successfully finished matching (PL-EMP-BOTH). Otherwise if any of the sequences finish while the other is non-empty we produce no possible bindings (PL-EMP-PAT, PL-EMP-VAL); an exemption is made for arbitrary match patterns because they can match empty sequences of values.

\[
\frac{\sigma \vdash \varepsilon \triangleright \varepsilon}{\text{match}} \quad \frac{\sigma \vdash \varepsilon \triangleright v, v'}{\text{match}}
\]
When the initial element of the starred pattern sequence is an ordinary pattern then it is matched against the initial element of the value sequence, and the rest of the pattern sequence is matched against the rest of the value sequence. The results of both submatches are then merged together (PL-MORE-PAT).

\[
\begin{align*}
\sigma \vdash p \star p &::= v \\
\mathsf{match} &\quad \sigma \vdash \star p ::= v' \\
\mathsf{match*} &\quad \mathsf{merge}(\rho, \rho')
\end{align*}
\]

Like ordinary variables, arbitrary matching patterns depend on whether the binding variable already exists in the current store. If the variable is assigned in the current store then either there must exist a partition of values that has a matching subcollection (PL-MORE-STAR-UNI), or the matching fails producing any consistent binding environment (PL-MORE-STAR-PAT-FAIL, PL-MORE-STAR-VAL-FAIL).

\[
\begin{align*}
x \in \mathsf{dom} \sigma \quad \sigma(x) &= \langle v' \rangle \\
\sigma \vdash \star p ::= v' \otimes v'' &\quad \sigma \vdash \star p ::= v' \otimes v''
\end{align*}
\]

If the variable is not in the current store then there are two options: either i) there still exist a partition that is possible to try, or ii) we have exhausted all possible partitions of the value sequence. In the first case, an arbitrary partition is bound to the target variable and the rest of the patterns are matched against the rest of the values, merging their results; additionally, the other partitions are also tried concatenating their results with the merged one (PL-MORE-STAR-RE). In the exhausted case, the pattern match produces no bindings (PL-MORE-STAR-EXH).

\[
\begin{align*}
x \notin \mathsf{dom} \sigma \quad v &= v' \otimes v'' \\
\sigma \vdash \star p ::= v' &\quad \sigma \vdash \star p ::= v' \\
\mathsf{match*} &\quad \mathsf{merge}(\rho, \rho')
\end{align*}
\]

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Merging a sequence of possible variable bindings produces a sequence containing consistent variable bindings from the sequence: that is, all possible environments that assign consistent values to the same variables are merged.

\[
\text{merge}(\epsilon) = [] \\
\text{merge}(\rho, \rho'_1, \ldots, \rho'_n) = \text{merge-pairs}(\rho \times \text{merge}(\rho'_1, \ldots, \rho'_n))
\]

\[
\text{merge-pairs}(\langle \rho_1, \rho'_1 \rangle, \ldots, \langle \rho_n, \rho'_n \rangle) = \text{merge-pair}(\rho_1, \rho'_1), \ldots, \text{merge-pair}(\rho_n, \rho'_n)
\]

\[
\text{merge-pair}(\rho, \rho') = \begin{cases} 
\rho\rho' & \text{if } \forall x \in \text{dom } \rho \cap \text{dom } \rho' \rho'(x) = \rho'(x) \\
\epsilon & \text{otherwise}
\end{cases}
\]

4 Semantics Properties

Backtracking is pure in Rascal (Light) programs, and so if evaluating a set cases produces fail as result, the initial state is restored.

**Theorem 1** (Backtracking purity). If \( CS; v; \sigma \xrightarrow{\text{fail; } \sigma'} \) then \( \sigma' = \sigma \)

Strong typing is an important safety property of Rascal, which I capture by specifying a theorem with two result properties: one about the well-typedness of state, and the other about well-typedness of resulting values.

**Theorem 2** (Strong typing). Assume that semantic unary \([\ominus]\) and binary operators \([\oplus]\) are strongly typed. If \( e; \sigma \xrightarrow{\text{expr; } \sigma'} \) and there exists a type \( t \) such that \( \forall v : t \) for each value in the input store \( v \in \text{img } \sigma \), then

1. There exists a type \( t' \) such that \( v' : t' \) for each value in the result store \( v' \in \text{img } \sigma' \).

2. If the result value \( v' \) is either success \( v'' \), return \( v'' \), or throw \( v'' \), then there exists a type \( t'' \) such that \( v'': t'' \).

Consider an augmented version of the operational semantics where each execution relation is annotated with partiality fuel ([Amin and Rompf, 2017])—represented by a superscript natural number \( n \)—which specifies the maximal number of recursive premises allowed in the derivation. The fuel is subtracted on each recursion, resulting in a timeout value when it
reaches zero, and the rule set is amended by congruence rules which propagate timeout results from the recursive premises to the conclusion.

\[ \text{vtres ::= vres | timeout} \]

For this version of the semantics, we can specify the property that execution will either produce a result or it will timeout; that is, the semantics does not get stuck.

**Theorem 3** (Partial progress). *It is possible to construct a derivation* \( e; \sigma \xrightarrow{n} \text{vres}; \sigma' \) *for any input expression* \( e \), *well-typed store* \( \sigma \) *and fuel* \( n \).

Finally, consider a subset of the expression language described by syntactic category \( e_{\text{fin}} \), where while-loops, solve-loops and function calls are disallowed, and similarly with all traversal strategies except bottom-up and bottom-up-break. This subset is known to be terminating:

**Theorem 4** (Terminating expressions). *There exists* \( n \) *such that derivation* \( e_{\text{fin}}; \sigma \xrightarrow{n} \text{vres}; \sigma' \) *has a result* \( \text{vres} \) *which is not timeout for expression* \( e_{\text{fin}} \) *in the terminating subset.*

**Remark 1.** Why is the top-down traversal strategy potentially non-terminating while the bottom-up traversal strategy is terminating? The answer lies in that the top-down traverses the children of the target expression after evaluating the required case, which makes it possible to keep add children ad infinitum as illustrated by the following example:

```
1  data Nat = zero() | succ(Nat pred);
2  Nat infincrement(Nat n) =
3    top-down visit(n) {
4      case succ(n) => succ(succ(n))
5    }
```

## 5 Formal Semantics, Types and Intented Semantics

This formal specification of Rascal Light was largely performed when the official type checker of Rascal (developed by CWI Amsterdam) was only at the experimental stage. Useful extension of this formalization include a deductive type system that includes polymorphic aspects of Rascal to prove the consistency of the type system, and type safety with regards to the dynamic semantics provided in this formalization.

The presented semantics was checked against the Rascal implementation in various ways: the existing implementation was used as a reference point, rules for individual were held up against the documentation, and correspondence with the Rascal developers was used to clarify remaining questions. A more formal way to check whether the captured semantics is the intended one, is to construct an independent formal semantics which is related to the natural semantics presented in this report. This could be an axiomatic semantics [Hoare (1969)], which modularly specifies for each construct the intented effects using logical formulae.
as pre and post-conditions, without necessarily specifying how each construct is evaluated. The natural semantics would then be checked to satisfy the axiomatic semantics for each construct, which will further increase confidence that the captured semantics is the intended one.

6 Recap

I presented the formalization of a large subset of the operational part of Rascal\cite{Klint2009,Klint2011}, called Rascal Light. The formalization was primarily based on the available open source implementation\footnote{https://github.com/usethesource/rascal} and the accompanying documentation\footnote{http://tutor.rascal-mpl.org/}, and personal correspondence with the developers further clarified previous ambiguities and mismatches.

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\footnote{http://tutor.rascal-mpl.org/}
A Semantics Properties Proofs

Theorem 1 (Backtracking purity). If $\text{CS} \overset{\text{cases}}{\Rightarrow} \text{fail}; \sigma' \Rightarrow \text{fail}; \sigma''$ then $\sigma' = \sigma$

Proof. By induction on the derivation $\text{CS}$:

- Case $\text{CS} = \text{ECS-EMP}$ \hspace{1cm} $\varepsilon; v; \sigma \overset{\text{cases}}{\Rightarrow} \text{fail}; \sigma'$, so $\text{cs} = \varepsilon$ and $\sigma' = \sigma$. Holds by definition.

$$
\begin{align*}
\sigma \vdash p \overset{\text{match}}{\Rightarrow} \rho ; c; \sigma \overset{\text{case}}{\Rightarrow} \text{fail}; \sigma'' \\
\text{CS}' ; v; \sigma \overset{\text{cases}}{\Rightarrow} \text{fail}; \sigma'
\end{align*}
$$

- Case $\text{CS} = \text{ECS-MORE-FAIL}$ \hspace{1cm} $\text{(case p \Rightarrow e, cs')}$ and $\text{vres} = \text{fail}$.
By inductive hypothesis of $C S'$ we get $\sigma' = \sigma$.

• Note: the rule ECS-MORE-ORD is inapplicable since its premise states that the result value $v_{res} \neq \text{fail}$.

In order to prove Theorem \(2\), we need to state some helper lemmas about sub-derivations. We have a lemma for the auxiliary merge function:

**Lemma 1.** If we have $\rho' = \text{merge}(\rho_1, \ldots, \rho_n)$ and for each value $v \in \text{img} \rho_{i,j}$ in an environment in the input sequence of environment sequences $\rho_1, \ldots, \rho_n$ there exists a type $t$ so that $v : t$, then we have that for each value $v' \in \text{img} \rho'_{k}$ in an environment in the resulting environment sequence $\rho'$ we have a type $t'$ so that $v' : t'$

**Proof.** Follows directly from the premises by induction of the input sequence of environment sequences $\rho_1, \ldots, \rho_n$.

We have a lemma for the auxiliary children function:

**Lemma 2.** For a value $v$ such that we have $\overline{v'} = \text{children}(v)$ and $\overline{T} = \overline{v : t}$, then there exists a type sequence $\overline{t'}$ such that $\overline{v' : t'}$

**Proof.** By induction on the syntax of $v$:

• Cases $v = vb$ and $v = \mathbb{L}$, so $\overline{v'} = \varepsilon$. Holds trivially.

• Case $v = k(v')$
  
  - By inversion we get
  
  \[
  \begin{array}{c}
  \text{data at} = \ldots | k(t'') | \ldots \\
  T-\text{Constructor}
  \end{array}
  \]

  $k(v') : at$

  so $t = at$

  – Here, $\overline{T'}$ exactly represents our target goal

• Case $v = [v']$
  
  - By inversion we get $\overline{T} = T-\text{List} \overline{[v'] : \text{list}(\bigcup t')}$, so $t = \text{list}(\bigcup t')$

  – Here, $\overline{T'}$ exactly represents our target goal

• Case $v = \{v'\}$
  
  - By inversion we get $\overline{T} = T-\text{Set} \overline{\{v'\} : \text{set}(\bigcup t')}$, so $t = \text{set}(\bigcup t')$
- Here, $\mathcal{T}'$ exactly represents our target goal

- Case $v = (v'' : v'''')$, so $v' = v'' , v'''$

  - By inversion we get

  $\mathcal{T} = T\text{-MAP} \frac{v'' : t'' \quad v''' : t'''}{(v'' : v''') : \text{map}(t'' \cup t''')}$

- Here we take the concatenation of the premises $\mathcal{T}'' , \mathcal{T}'''$ to fulfill our goal.

We have two mutually recursive lemmas on pattern matching: Lemma 3 and Lemma 4.

**Lemma 3.** If $\sigma \vdash p \overset{?}{=} v \overset{\text{match}}{\Rightarrow} \rho$ and there exists a type $t$ such that $\mathcal{T} \overset{v : t}{\Rightarrow}$ and a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then we have a type $t''$ for each value $v'' \in \text{img} \rho_i$ in an environment in the output sequence $\rho$.

**Proof.** By induction on the derivation $M$:

- Cases $M = P\text{-VAL\text{-SUCS}}, M = P\text{-VAL\text{-FAIL}}, M = P\text{-VAR\text{-UNI}}, M = P\text{-VAR\text{-FAIL}}, M = P\text{-CONS\text{-FAIL}}, M = P\text{-TYPE\text{-FAIL}}, M = P\text{-LIST\text{-FAIL}}, M = P\text{-SET\text{-FAIL}}, M = P\text{-NEG\text{-SUCS}}, M = P\text{-NEG\text{-FAIL}}$ hold trivially since we have that the output environment sequence $\rho = [\] or $\rho = \varepsilon$, in both cases containing no values.

- Case $M = P\text{-VAR\text{-BIND}}$ holds by the premise derivation $\mathcal{T}$.

  - Case $M = P\text{-CONS\text{-SUCS}}$ holds by the premise derivation $\mathcal{T}$.

    $\sigma \vdash p' \overset{?}{=} v' \overset{\text{match}}{\Rightarrow} \rho'_1 \quad \ldots \quad \sigma \vdash p'_n \overset{?}{=} v'_n \overset{\text{match}}{\Rightarrow} \rho'_n$, so $p = k(p')$.

    $v = k(v')$ and $\rho = \text{merge}(\rho'_1 , \ldots , \rho'_n)$

    - By inversion we get

      $\mathcal{T} = T\text{-CONSTRUCTOR} \frac{\text{data at} = \ldots | k(v') | \ldots \quad \mathcal{T}' \quad v : t' \quad t' : t''}{k(v'_1 , \ldots , v'_n) : \text{at}}$

      so $t = \text{at}$

    - Now by induction hypotheses of $\mathcal{T}'$ using $\mathcal{T}'$, and then using Lemma 4 we get that $\rho$ has well-typed values.

- Case $M = P\text{-TYPE\text{-SUCS}}$ holds by the premise derivation $\mathcal{T}$ and Lemma 4.
Case $M = \text{P-List-Sucs}$

\[
\frac{\sigma \vdash \ast p' \coloneqq v' \mid \emptyset \xrightarrow{\text{match}} \rho'}{\sigma \vdash [\ast p'] \coloneqq [v'] \xrightarrow{\text{match}} \rho'}
\]

so $p = [\ast p']$ and $v = [v']$

- By inversion we get $T = \text{T-List}$

\[
\frac{\text{v'} : t'}{\text{list}(\bigcup t')} \]

so $t = \text{list}(\bigcup t')$

- Partitioning lists by splitting on concatenation preserves typing since we can for each value pick the corresponding type derivation in the sequence.

- By induction hypothesis (using Lemma 4) of $MS'$ using $T'$ and above fact we get that $\rho$ has well-typed values.

Case $M = \text{P-Set-Sucs}$

\[
\frac{\sigma \vdash \{ \ast p' \} \coloneqq \{ v' \} \xrightarrow{\text{match}} \rho}{\sigma \vdash \ast p' \coloneqq \{ v' \} \xrightarrow{\text{match}} \rho}
\]

- By inversion we get $T = \text{T-Set}$

\[
\frac{\text{v'} : t'}{\text{set}(\bigcup t')} \]

so $t = \text{set}(\bigcup t')$

- Partitioning sets using disjoint union $\uplus$ preserves typing of value sub-sequences, since we can for each value $v_i$ in a subsequence pick the corresponding typing derivation $T'_i$.

- By induction hypothesis (using Lemma 4) of $MS'$ using $T'$ and above fact we get that $\rho$ has well-typed values.

Case $M = \text{P-Deep}$

\[
\frac{\sigma \vdash p' \coloneqq \text{children}(v)}{\sigma \vdash [p'] \coloneqq \text{children}(v)}
\]

$v_1, \ldots, v_n = \text{children}(v)$

\[
\frac{\sigma \vdash /p' \coloneqq v_1 \xrightarrow{\text{match}} \rho_1'}{\sigma \vdash /p' \coloneqq v_1 \xrightarrow{\text{match}} \rho_1'}
\]

\[
\frac{\sigma \vdash /p' \coloneqq v_n \xrightarrow{\text{match}} \rho_n'}{\sigma \vdash /p' \coloneqq v_n \xrightarrow{\text{match}} \rho_n'}
\]

- By induction hypothesis on $M'$, we get that $\rho'$ has well-typed values.

- By using Lemma 2 on $v'$, we get $T'$

\[
\frac{\text{v'} : t'}{\text{set}(\bigcup t')} \]

- By induction hypotheses on $M''$ using $T'$ we get that $\rho_1', \ldots, \rho_n'$ is well-typed.

- Now, we can show that $\rho', \rho_1', \ldots, \rho_n'$ is well-typed using above facts.

\[\blacksquare\]

**Lemma 4.** If $\sigma \vdash \ast p \coloneqq v \mid \emptyset \xrightarrow{\text{match}} \rho$ and the following properties hold:
1. There exists a type sequence \( t \) such that \( v : t \)

2. There exists a type \( t' \) such that \( v' : t' \) for each value in the input store \( v' \in \text{img} \sigma \)

3. There exists a type \( t' \) such that \( v' : t' \) for each value in the visited value set \( v' \in V \)

4. If for all value sequences \( v', v'', v''' \) where we have \( v' = v'' \otimes v''' \) and there exists a type sequence \( t' \) so \( v' : t' \), then we have that there exists two type sequences \( t'' \) and \( t''' \) so that \( v'' : t'' \) and \( v''' : t''' \)

Then we have a type \( t' \) so that \( v' : t' \) for each value \( v' \in \text{img} \rho_i \) in an environment in the output sequence \( \rho \).

**Proof.** By induction on the derivation \( M_S \):

- Cases \( M_S = \text{PL-Emp-Both}, \ M_S = \text{PL-Emp-Pat}, \ M_S = \text{PL-Emp-Val}, \ M_S = \text{PL-More-Star-Pat-Fail}, \) and \( M_S = \text{PL-More-Star-Val-Fail} \) hold trivially since we have \( \rho = [\] \) or \( \rho = \varepsilon \), and there are no values in either [ ] or \( \varepsilon \).

- Case \( M_S = \text{PL-More-Pat} \) holds by induction hypotheses (including Lemma 3), and Lemma [1]

  \[
  x \in \text{dom} \sigma \quad \sigma(x) = (v') \quad v = v' \otimes v''
  \]

  \[
  \begin{array}{c}
  \sigma \vdash \star p' \overset{?}{=} v'' | \emptyset \\
  \sigma \vdash \star x, \star p' \overset{?}{=} V | \emptyset \\
  \end{array}
  \]

  By property [1] we can derive that there exists a type sequence \( t'' \) such that \( v'' : t'' \). Then we can apply induction hypothesis on derivation \( M_S' \), and get our target result.

We have a lemma on the auxiliary function if-fail:

**Lemma 5.** If we have \( v' = \text{if-fail}(vfres, v'') \), and:

1. If \( vfres = \text{success} \ v \), then we have \( v : t \) for some type \( t \)

2. We have \( v'' : t'' \) for some type \( t'' \)

Then there exists a type \( t' \) such that \( v' : t' \)

**Proof.** Straightforwardly by case analysis on \( vfres \)

Similarly, we have a lemma on the auxiliary function vcombine

**Lemma 6.** If we have \( vfres *'' = \text{vcombine}(vfres, vfres', v, v') \), and:

1. If \( vfres = \text{success} \ v'' \), then we have \( v'' : t'' \) for some type \( t \)

2. If \( vfres' = \text{success} \ v''' \), then we have \( v''' : t''' \) for some typing sequence \( t''' \)
3. We have $v : t$ for some type $t$

4. We have $v' : t'$ for some type $t'$

Then if $v_fres^* = \text{success } v''$ there exists a type sequence $t''$ such that $v'' : t''$

**Proof.** Straightforwardly by case analysis on $v_fres$ and $v_fres^*$ and using Lemma 5.

We have a lemma on the reconstruct derivation:

**Lemma 7.** If we have $\text{recons } v \text{ using } v' \text{ to } \text{rcres } v : t$ for some type $t$, and $v' : t'$ for some type sequence $t'$, then when $\text{rcres } v = \text{success } v''$ there exists a type $t''$ such that if $v' : t''$

**Proof.** By induction on the derivation $\text{rcres } v$:

- Cases $\text{RC } = \text{RC-VAL-ERR}$, $\text{RC } = \text{RC-CONS-ERR}$, $\text{RC } = \text{RC-LIST-ERR}$, $\text{RC } = \text{RC-SET-ERR}$, $\text{RC } = \text{RC-MAP-ERR}$, $\text{RC } = \text{RC-BOT-ERR}$ hold trivially.
- Case $\text{RC } = \text{RC-VAL-SUCS}$ holds using the premises.
- Case $\text{RC } = \text{RC-CONS-SUCS}$ holds using the premises and rule $\text{T-CONSTRUCTOR}$.
- Case $\text{RC } = \text{RC-LIST-SUCS}$ holds using the premises and rule $\text{T-LIST}$.
- Case $\text{RC } = \text{RC-SET-SUCS}$ holds using the premises and rule $\text{T-SET}$.
- Case $\text{RC } = \text{RC-MAP-SUCS}$ holds using the premises and rule $\text{T-MAP}$.
- Case $\text{RC } = \text{RC-BOT-SUCS}$ holds using the premises and rule $\text{T-BOT}$.

We now have a series of mutually inductive lemmas with our Theorem 2, since the operational semantics rules are mutually inductive themselves. The lemmas are Lemma 8, Lemma 9, Lemma 10, Lemma 11, Lemma 12, Lemma 13, Lemma 14, Lemma 15, Lemma 16, and Lemma 17.

**Lemma 8.** If $e ; \sigma \overset{\text{expr } v_fres^* ; \sigma'}{\longrightarrow} v$ and there exists a type $t$ such that $v : t$ for each value in the input store $v \in \text{img } \sigma$, then

1. There exists a type $t'$ such that $v' : t'$ for each value in the result store $v' \in \text{img } \sigma'$.

2. If the result value $v_fres^*$ is $\text{success } v''$, then there exists a type sequence $t''$ such that $v'' : t''$.

3. If the result value $v_fres^*$ is either $\text{return } v''$, or $\text{throw } v''$, then there exists a type $t''$ such that $v'' : t''$

**Proof.** By induction on the derivation $\text{expr } v_fres^*$:

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• Case $\mathcal{ES} = \text{ES-EMP}$ holds directly using premises.

• Cases $\mathcal{ES} = \text{ES-MORE}$, $\mathcal{ES} = \text{ES-Exc1}$ and $\mathcal{ES} = \text{ES-Exc2}$ hold directly from the induction hypotheses (including the one given by Theorem 2).

Lemma 9. If $e; \rho; \sigma \Rightarrow v_{\text{res}}; \sigma'$, there exists a type $t$ such that $v : t$ for each value in the input store $v \in \text{img} \sigma$, and there exists a type $t$ such that $v : t$ for each value in an environment $v \in \text{img} \rho_i$ in the environment sequence $\rho$ then

1. There exists a type $t'$ such that $v' : t'$ for each value in the result store $v' \in \text{img} \sigma'$.

2. If the result value $v_{\text{res}}$ is either $\text{success} v''$, $\text{return} v''$, or $\text{throw} v''$, then there exists a type $t''$ such that $v'' : t''$

Proof. By induction on the derivation $\mathcal{ES}$:

• Case $\mathcal{ES} = \text{EE-EMP}$ holds directly using the premises.

• Cases $\mathcal{ES} = \text{EE-MORE-SUCS}$, $\mathcal{ES} = \text{EE-MORE-BREAK}$ and $\mathcal{ES} = \text{EE-MORE-Exc}$ hold directly from the induction hypotheses (including Theorem 2), and the (trivial) facts that if any store $\sigma$ is well-typed then $\sigma \setminus X$ is well-typed for any set of variables $X$ and $\sigma\rho$ is well-typed for any well-typed environment $\rho$.

Lemma 10. If $g; \sigma \Rightarrow^{\text{expr}} v_{\text{res}}; \sigma'$, and there exists a type $t$ such that $v : t$ for each value in the input store $v \in \text{img} \sigma$ then

1. There exists a type $t'$ such that $v' : t'$ for each value in the result store $v' \in \text{img} \sigma'$.

2. If the result value $v_{\text{res}}$ is $\text{success} \rho$, then there exists a type $t''$ for each value in an environment $v'' \in \rho_i$ in the environment sequence $\rho$ such that $v'' : t''$

3. If the result value $v_{\text{res}}$ is either $\text{return} v'''$, or $\text{throw} v'''$, then there exists a type $t'''$ such that $v''' : t'''$

Proof. By induction on the derivation $G$:

• Cases $G = \text{G-Pat-Sucs}$ and $G = \text{G-Pat-Exc}$ hold directly by induction hypothesis given by Theorem 2 and then using Lemma 3 if necessary.

• Cases $G = \text{G-Enum-List}$, $G = \text{G-Enum-Set}$ and $G = \text{G-Enum-Map}$ hold by the induction hypothesis given by Theorem 2 and then using inversion on the type derivation of the result collection value to extract the type derivations of the contained values in the result environments (from rules T-List, T-Set, and T-Map respectively).
• Cases $G = \text{G-Enum-Err}$ and $G = \text{G-Enum-Exc}$ hold directly by the induction hypothesis given by Theorem 2.

Lemma 11. If $C \vdash_\text{case} \rho; e; \sigma \Rightarrow v\text{res}; \sigma'$, there exists a type $t$ such that $v : t$ for each value in an environment $v \in \text{img} \rho$ in the environment sequence $\rho$, and there exists a type $t$ such that $v : t$ for each value in the input store $v \in \text{img} \sigma$, then

1. There exists a type $t'$ such that $v' : t'$ for each value in the result store $v' \in \text{img} \sigma'$.

2. If the result value $v\text{res}$ is either success $v''$, return $v''$, or throw $v''$, then there exists a type $t''$ such that $v'' : t''$

Proof. By induction on derivation $C$:

• Case $C = \text{EC-EMP}$ holds directly using the premises.

• Case $C = \text{EC-MORE-FAIL}$ and $C = \text{EC-MORE-ORD}$ hold directly using induction hypotheses (including Theorem 2) and the facts of well-typedness of store extension by well-typed environments and well-typedness of variable removals from stores.

Lemma 12. If $C S \vdash_\text{cases} \epsilon S ; v; \sigma \Rightarrow v\text{res}; \sigma'$, there exists a type $t$ such that $v : t$, and there exists a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then

1. There exists a type $t''$ such that $v'' : t''$ for each value in the result store $v'' \in \text{img} \sigma'$.

2. If the result value $v\text{res}$ is either success $v''$, return $v''$, or throw $v''$, then there exists a type $t''$ such that $v'' : t''$

Proof. By induction on the derivation $C S$:

• Case $C S = \text{ECS-EMP}$ holds directly using the premises.

• Cases $C S = \text{ECS-MORE-FAIL}$ and $C S = \text{ECS-MORE-ORD}$ hold using Lemma 3 and the induction hypotheses (including Lemma 11).

Lemma 13. If $C S \vdash_\text{visit} \epsilon S ; v; \sigma \Rightarrow v\text{res}; \sigma'$, there exists a type $t$ such that $v : t$, and there exists a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then

1. There exists a type $t''$ such that $v'' : t''$ for each value in the result store $v'' \in \text{img} \sigma'$.

2. If the result value $v\text{res}$ is either success $v''$, return $v''$, or throw $v''$, then there exists a type $t''$ such that $v'' : t''$
Proof. By induction on the derivation $\mathcal{V}$:

- Cases $\mathcal{V} = \text{EV-TD}, \mathcal{V} = \text{EV-TDB}, \mathcal{V} = \text{EV-QM-EQ}, \mathcal{V} = \text{EV-QM-NEQ}$ and $\mathcal{V} = \text{EV-QM-Exc}$ hold by induction hypotheses (including Lemma 14).

- Cases $\mathcal{V} = \text{EV-BU}, \mathcal{V} = \text{EV-BUB}, \mathcal{V} = \text{EV-IM-EQ}, \mathcal{V} = \text{EV-IM-NEQ}$ and $\mathcal{V} = \text{EV-IM-Exc}$ hold by induction hypotheses (including Lemma 16).

Lemma 14. If $\mathcal{V}$

$\frac{\text{CS}; v; \sigma \xrightarrow{br} vres; \sigma'}{\text{td-visit}}$

there exists a type $t$ such that $v : t$, and there exists a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then

1. There exists a type $t''$ such that $v'' : t''$ for each value in the result store $v'' \in \text{img} \sigma'$.

2. If the result value $vres$ is either success $v''$, return $v''$, or throw $v''$, then there exists a type $t'''$ such that $v''' : t'''$

Proof. By induction on the derivation $\mathcal{V}$:

- All cases ($\mathcal{V} = \text{ETV-Break-Sucs}, \mathcal{V} = \text{ETV-Ord-Sucs1}, \mathcal{V} = \text{ETV-Ord-Sucs2}, \mathcal{V} = \text{ETV-Exc1}$ and $\mathcal{V} = \text{ETV-Exc2}$) hold by induction hypotheses (including Lemma 11 and Lemma 15), Lemma 5, Lemma 2 and Lemma 7.

Lemma 15. If $\mathcal{V}$

$\frac{\text{CS}; v; \sigma \xrightarrow{br} vres \star; \sigma'}{\text{td-visit}}$

there exists a type sequence $t$ such that $v : t$, and there exists a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then

1. There exists a type $t''$ such that $v'' : t''$ for each value in the result store $v'' \in \text{img} \sigma'$.

2. If the result value $vres \star$ is success $v''$, then there exists a type sequence $t'''$ such that $v''' : t'''$

3. If the result value $vres \star$ is either return $v'''$, or throw $v'''$, then there exists a type $t'''$ such that $v''' : t'''$

Proof. By induction on the derivation $\mathcal{V}$

- Case $\mathcal{V} = \text{ETV-EMP}$ holds using the premises.

- Cases $\mathcal{V} = \text{ETV-Break}, \mathcal{V} = \text{ETV-More}, \mathcal{V} = \text{ETV-Exc1}$ and $\mathcal{V} = \text{ETV-Exc2}$ holds using the induction hypotheses (including Lemma 14 and Lemma 6).
Lemma 16. If $V_B \xrightarrow{cs; v; \sigma} vres; \sigma'$, there exists a type $t$ such that $v : t$, and there exists a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then

1. There exists a type $t''$ such that $v'' : t''$ for each value in the result store $v'' \in \text{img} \sigma'$.

2. If the result value $vres$ is either success $v'''$, return $v'''$, or throw $v'''$, then there exists a type $t'''$ such that $v''' : t'''$.

Proof. By induction on the derivation $V_B$:

- All cases ($V_B = \text{EBU-No-Break-SUCS}$, $V_B = \text{EBU-Break-SUCS}$, $V_B = \text{EBU-Fail-SUCS}$, $V_B = \text{EBU-No-Break-EXC}$, $V_B = \text{EBU-EXC}$, and $V_B = \text{EBU-No-BreakErr}$) hold by induction hypotheses (including Lemma 11 and Lemma 17).

Lemma 17. If $V_BS \xrightarrow{cs; v; \sigma} vres \star; \sigma'$, there exists a type sequence $t$ such that $v : t$, and there exists a type $t'$ such that $v' : t'$ for each value in the input store $v' \in \text{img} \sigma$, then

1. There exists a type $t''$ such that $v'' : t''$ for each value in the result store $v'' \in \text{img} \sigma'$.

2. If the result value $vres \star$ is success $v'''$, then there exists a type sequence $t'''$ such that $v''' : t'''$.

3. If the result value $vres \star$ is either return $v'''$, or throw $v'''$, then there exists a type $t'''$ such that $v''' : t'''$.

Proof. By induction on the derivation $V_BS$:

- Case $V_BS = \text{EBUS-EMP}$ holds using the premises.

- Cases $V_BS = \text{EBUS-BREAK}$, $V_BS = \text{EBUS-MORE}$, $V_BS = \text{EBUS-EXC1}$ and $V_BS = \text{EBUS-EXC2}$ holds using the induction hypotheses (including Lemma 16) and Lemma 6.

Theorem 2 (Strong typing). Assume that semantic unary $[\odot]$ and binary operators $[\oplus]$ are strongly typed. If $e; \sigma \xrightarrow{\text{expr}} vres; \sigma'$ and there exists a type $t$ such that $v : t$ for each value in the input store $v \in \text{img} \sigma$, then

1. There exists a type $t'$ such that $v' : t'$ for each value in the result store $v' \in \text{img} \sigma'$.

2. If the result value $vres$ is either success $v''$, return $v''$, or throw $v''$, then there exists a type $t''$ such that $v'' : t''$. 

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Proof. By induction on the derivation $\delta$:

- Cases $\delta = E-Va1$, $\delta = E-Va1-Suc$, $\delta = E-Va1-Err$, $\delta = E-Break$, $\delta = E-Continue$ and $\delta = E-Fail$ hold directly using the premises.

- Cases $\delta = E-Un-Exc1$, $\delta = E-Bin-Exc1$, $\delta = E-Bin-Exc2$, $\delta = E-Cons-Err$, $\delta = E-Cons-Exc$, $\delta = E-List-Err$, $\delta = E-List-Exc$, $\delta = E-Set-Err$, $\delta = E-Set-Exc$, $\delta = E-Map-Err$, $\delta = E-Map-Exc$, $\delta = E-Lookup-Err$, $\delta = E-Lookup-Exc1$, $\delta = E-Lookup-Exc2$, $\delta = E-Update-Err1$, $\delta = E-Update-Err2$, $\delta = E-Update-Exc1$, $\delta = E-Update-Exc2$, $\delta = E-Update-Exc3$, $\delta = E-Call-Arg-Err$, $\delta = E-Call-Arg-Exc$, $\delta = E-Ret-Suc$, $\delta = E-Ret-Exc$, $\delta = E-Asgn-Err$, $\delta = E-Asgn-Exc$, $\delta = E-If-True$, $\delta = E-If-False$, $\delta = E-If-Err$, $\delta = E-If-Exc$, $\delta = E-Switch-Suc$, $\delta = E-Switch-Exc1$, $\delta = E-Switch-Exc2$, $\delta = E-Visit-Suc$, $\delta = E-Visit-Fail$, $\delta = E-Visit-Exc1$, and $\delta = E-Visit-Exc2$, $\delta = E-For-Suc$, $\delta = E-For-Exc$, $\delta = E-While-True-Suc$, $\delta = E-While-Exc1$, $\delta = E-While-Exc2$, $\delta = E-While-Err$, $\delta = E-Solve-Neq$, $\delta = E-Solve-Exc$, $\delta = E-Solve-Err$, $\delta = E-Thr-Suc$, $\delta = E-Thr-Exc$, $\delta = E-Fin-Suc$, $\delta = E-Fin-Exc$, and $\delta = E-Try-Ord$ hold by induction hypotheses (including Lemma 8, Lemma 10, Lemma 12 and Lemma 13).

- Case $\delta = E-Un-Suc$ holds by its induction hypothesis and strong typing of $[\oplus]$.

- Case $\delta = E-Bin-Suc$ holds by its induction hypothesis and strong typing of $[\ominus]$.

- Case $\delta = E-Cons-Suc$ holds by its induction hypothesis given by Lemma 8 and then by using T-CONSTRUCTOR with the provided typing premises.

- Case $\delta = E-List-Suc$ holds by its induction hypothesis given by Lemma 8 and then by using T-List.

- Case $\delta = E-Set-Suc$ holds by its induction hypothesis given by Lemma 8 and then by using T-Set.

- Case $\delta = E-Map-Suc$ holds by its induction hypothesis given by Lemma 8 and then by using T-Map.

- Case $\delta = E-Lookup-Suc$ holds by its induction hypotheses and inversion of $T$ to T-Map.

- Case $\delta = E-Lookup-NoKey$ holds by its induction hypotheses and using T-Cons on the definition of NoKey.

- Case $\delta = E-Update-Suc$ holds by its induction hypotheses and using T-Map to reconstruct the type derivation for the result.

- Cases $\delta = E-Call-Suc$, $\delta = E-Call-Res-Exc$, $\delta = E-Call-Res-Err1$, and $\delta = E-Call-Res-Err2$ hold by induction hypotheses (including Lemma 8) and by using the fact that extracting variables from and extending well-typed stores produces well-typed stores.
• Cases $\mathcal{E} = \text{E-ASGN-SUCS}$ and $\mathcal{E} = \text{E-TRY-CATCH}$ hold by induction hypotheses and the fact that extending well-typed stores with well-typed environments preserves well-typedness.

• Cases $\mathcal{E} = \text{E-SWITCH-FAIL}$, $\mathcal{E} = \text{E-WHILE-FALSE}$, and $\mathcal{E} = \text{E-WHILE-TRUE-BREAK}$ hold by induction hypotheses and using $\text{T-VOID}$.

• Cases $\mathcal{E} = \text{E-BLOCK-SUCS}$, $\mathcal{E} = \text{E-BLOCK-EXC}$ hold by the induction hypothesis given by Lemma 8 and the fact that removing variables from a well-typed store produces a well-typed store.

We will for Theorem 3 also need to state some lemmas. We will for the proof focus mainly on cases where the induction hypothesis does not timeout, since if it does it is trivially possible to construct a timeout derivation for the result syntax.

First, we need a lemma that specifies that the sequence of values produced by the children is strictly smaller than the input value. Let $v \prec v'$ denote the relation that $v$ is syntactically contained in $v'$, then our target property is specified in Lemma 18.

**Lemma 18.** If $v' = \text{children}(v)$ then $v'_i \prec v$ for all $i$.

*Proof.* Directly by induction on $v$. □

We have a lemma for progress on reconstruction:

**Lemma 19.** It is possible to construct a derivation $\text{recons } v$ using $v'$ to $\text{rcres } v'$ for any well-typed value $v$ and well-typed value sequence $v'$.

*Proof.* Straightforwardly by case analysis on $v$. □

We have two mutually recursive lemmas for pattern matching: Lemma 20 and Lemma 21.

**Lemma 20.** It is possible to construct a derivation $\sigma \vdash \text{match } p := v \implies \rho$ for any pattern $p$, well-typed value $v$, well-typed store $\sigma$.

*Proof.* By induction on syntax of $p$:

• Case $p = \text{vb}$: We proceed by testing whether $v$ is equal to $vb$:
  
  – Case $v = \text{vb}$ then use P-VAL-SUCS.
  
  – Case $v \neq \text{vb}$ then use P-VAL-FAIL.

• Case $p = x$: We proceed by testing whether $x$ is in dom $\sigma$
  
  – Case $x \in \text{dom } \sigma$: we proceed to test whether $v$ is equal to $\sigma(x)$
    
    * Case $v = \sigma(x)$ then use P-VAR-UNI.
    
    * Case $v \neq \sigma(x)$ then use P-VAR-FAIL.
Case $x \notin \text{dom } \sigma$ then use P-VAR-BIND.

- Case $p = k(p')$: We proceed to test whether $v$ is equal to $k(v')$ for some $v'$
  - Case $v = k(v')$ then use P-CONS-SUCS using the induction hypotheses of $p'$ with $v'$.
  - Case $v \neq k(v')$ then use P-CONS-FAIL.

- Case $p = t : p'$: From our premise we know that $v$ is well-typed, i.e. that there exists a $t'$ such that $v : t'$. We proceed to test whether $t' <: t$.
  - Case $t' <: t$ then use P-TYPE-SUCS using the induction hypothesis on $p'$ with $v$.
  - Case $t' \nsubseteq t$ then use P-TYPE-FAIL.

- Case $p = \star p'$:
  We proceed to test whether $v = \star v'$ for some value sequence $v'$.
  - Case $v = \star v'$ then use P-LIST-SUCS induction hypothesis given by Lemma 21 on $\star p'$ with $v'$.
  - Case $v \neq \star v'$ then use P-LIST-FAIL.

- Case $p = \{ \star p' \}$:
  We proceed to test whether $v = \{ v' \}$ for some value sequence $v'$.
  - Case $v = \{ v' \}$ then use P-SET-SUCS induction hypothesis given by Lemma 21 on $\star p'$ with $v'$.
  - Case $v \neq \{ v' \}$ then use P-SET-FAIL.

- Case $p = / p'$:
  - Using the induction hypothesis on $p'$ with $v$, we get $\sigma \vdash p' :? v \Rightarrow \rho$.
  - Now, let $v' = \text{children}(v)$. In order to handle the self-recursive calls using $/ p'$, we proceed by inner well-founded induction on the relation $<$ using value sequence $v'$:
    * Using the inner induction hypothesis we get derivations $\sigma \vdash / p' :? v'_i \Rightarrow \rho_i$ for all $i$.
  - Finally, we use P-DEEP on the derivations we got from the outer and inner induction hypotheses.

\[\square\]

\textbf{Lemma 21.} It is possible to construct a derivation $\sigma \vdash \star p :/? v \mid \forall \frac{0}{\text{match}} \otimes \rho$ for any pattern $p$, well-typed value $v$, well-typed store $\sigma$, well-typed visited value set $\forall$, type-preserving partition operator $\otimes$.  

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Proof. By induction on the syntax of $\star p$:

- Case $\star p = \varepsilon$
  
  By case analysis on $v$:
  
  - Case $v = \varepsilon$ then use PL-EMP-BOTH
  - Case $v = v', v''$ then use PL-EMP-PAT

- Case $\star p = p', \star p''$:
  
  By case analysis on $v$:
  
  - Case $v = \varepsilon$ then use PL-EMP-VAL
  - Case $v = v', v''$ then use PL-MORE-PAT using induction hypotheses (including Lemma 20).

- Case $\star p = \star x, \star p''$: We proceed to test whether $x$ is in dom $\sigma$
  
  - Case $x \in \text{dom} \ \sigma$:
    
    - Case $x \notin \text{dom} \ \sigma$:

      Because of backtracking we need to do an inner induction to handle the cases which recurse to the same star pattern sequence. Let $V_{\text{all}} = \{ v'| \exists v'', v = v' \otimes v'' \}$, then we proceed by inner induction on $|V_{\text{all}} - V|

      * Case $|V_{\text{all}} - V| = 0$:
        Then we have $V = V_{\text{all}}$ and so the only applicable rule is PL-MORE-STAR-EXH since $V_{\text{all}}$ covers all partitions and thus $\exists v', v''. v = v' \otimes v'' \wedge v' \notin V$ holds.

      * Case $|V_{\text{all}} - V| > 0$:
        Then there exists $V' \neq \emptyset$ such that $V_{\text{all}} = V' \uplus V$ and we can pick $v' \in V'$ such that $v = v' \uplus v''$ for some $v''$.

        We can now use PL-MORE-STAR-RE by using the outer hypothesis on $\star p$ with $v''$ and $\emptyset$, and the inner hypothesis on $V \uplus v'$ (since the size decreases by 1).

We now have a series of lemmas that are mutually recursive with Theorem 3, Lemma 22, Lemma 23, Lemma 24, Lemma 25, Lemma 26, Lemma 27, Lemma 28, Lemma 29, Lemma 30, and Lemma 31.

**Lemma 22.** It is possible to construct a derivation $e; \sigma \xrightarrow{\text{expr}}^n \text{vres} \star; \sigma'$, for any expression sequence $e$, well-typed store $\sigma$ and fuel $n$.

*Proof.* By induction on $n$:

- Case $n = 0$ then use corresponding timeout-rule.
• Case $n > 0$, then by case analysis on the expression sequence $e$:
  
  - Case $e = \varepsilon$ then use ES-EMP.
  - Case $e = e', e''$:
    * Using induction hypothesis given by Theorem $\mathbb{3}$ on $n - 1$ with $e'$ and $\sigma$ we get a derivation $e'; \sigma \xrightarrow{\text{expr}} vres'; \sigma''$
    
    By case analysis on $vres'$:
    
    - Case $vres' = \text{success } v'$:
      
      By Theorem $\mathbb{2}$ we get that $\sigma''$ is well-typed.
      
      By induction hypothesis on $n - 1$ with $e''$ and $\sigma''$ we get a derivation $e''; \sigma'' \xrightarrow{\text{expr}*} vres *'; \sigma'$
      
      By case analysis on $vres *'$:
      
      - Case $vres *' = \text{success } v''$ then use ES-MORE
      - Case $vres *' = \text{exres}$ then use ES-Exc2
      - Case $vres' = \text{exres}$ then use ES-Exc1

\[\square\]

Lemma 23. It is possible to construct a derivation $e; \rho; \sigma \xrightarrow{\text{each}} vres; \sigma'$ for any expression $e$, well-typed environment sequence $\rho$, well-typed store $\sigma$ and fuel $n$.

Proof. By induction on $n$:

• Case $n = 0$ then use the corresponding timeout-derivation.

• Case $n > 0$:
  
  By case analysis on the environment sequence $\rho$:
  
  - Case $\rho = \varepsilon$ then use EE-EMP
  - Case $\rho = \rho', \rho''$:
    
    By induction hypothesis given by Theorem $\mathbb{3}$ on $n - 1$ with $e$ and $\sigma\rho$ we get a derivation $e; \sigma\rho \xrightarrow{\text{expr}} vres'; \sigma''$.
    
    By case analysis on $vres'$:
    
    * Cases $vres' = \text{success } v'$ and $vres' = \text{continue}$ then use EE-MORE-SUCS with above derivation and the induction hypothesis on $n - 1$ with $\rho''$, $e$, and $\sigma'' = \sigma'' \setminus \text{dom } \rho'$.
    * Cases $vres' = \text{break}$ then use EE-MORE-BREAK
    * Cases $vres' = \text{throw } v'$, $vres' = \text{return } v'$, $vres' = \text{fail}$, and $vres' = \text{error}$ then use EE-MORE-Exc

\[\square\]

Lemma 24. It is possible to construct a derivation $g; \sigma \xrightarrow{\text{gexpr}} vres; \sigma'$ for any generator expression $g$, well-typed store $\sigma$ and fuel $n$. 

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Proof. By induction on $n$:

- Case $n = 0$ then use the corresponding timeout-derivation.
- Case $n > 0$:
  
  By case analysis on $g$:

  - Case $g = p := e$:
    
    By induction hypothesis given by Theorem [8] on $n - 1$ with $e$ and $\sigma$ we get a derivation $e;\sigma \xrightarrow{\text{expr}}^{n-1} vres';\sigma'$

    By case analysis on $vres'$:

    * Case $vres' = \text{success } v'$ then use G-PAT-SUCS with above derivation and the derivation from applying Lemma [20] on $p$ with $\sigma'$ and $v'$.

    * Case $vres' = exres$ then use G-PAT-Exc:

  - Case $g = x \leftarrow e$:
    
    By induction hypothesis given by Theorem [8] on $n - 1$ with $e$ and $\sigma$ we get a derivation $e;\sigma \xrightarrow{\text{expr}}^{n-1} vres';\sigma'$

    By case analysis on $vres'$

    * Case $vres' = \text{success } v'$

      By case analysis on $v'$:

      - Case $v' = [v'']$ then use $G - \text{Enum} - \text{List}$

      - Case $v' = \{v''\}$ then use $G - \text{Enum} - \text{Set}$

      - Case $v' = (v'' : {v''})$ then use $G - \text{Enum} - \text{Map}$

      - Cases $v' = vb$, $v' = k(v'')$ and $v' = \Box$ then use G-ENUM-ERR

    * Case $vres' = exres$ then use G-ENUM-Exc

\[ \square \]

Lemma 25. It is possible to construct a derivation $\rho; e;\sigma \xrightarrow{\text{case}}^{n} vres;\sigma'$ for any well-typed environment sequence $\rho$, expression $e$, well-typed store $\sigma$ and fuel $n$.

Proof. By induction on $n$:

- Case $n = 0$ then use the corresponding timeout-derivation.
- Case $n > 0$:

  By case analysis on the environment sequence $\rho$

  - Case $\rho = 0$ then use EC-EMP.

  - Case $\rho = \rho', \rho''$:

    Using the induction hypothesis given by Theorem [8] on $n - 1$ with $e$ and $\sigma$ we get a derivation $e;\sigma \xrightarrow{\text{expr}}^{n-1} vres';\sigma'$.

    By case analysis on $vres'$

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Lemma 26. It is possible to construct a derivation $cs; v; \sigma \xrightarrow{\text{cases}}^n vres; \sigma'$ for any case sequence $cs$, well-typed value $v$, well-typed store $\sigma$ and fuel $n$.

Proof. By induction on $n$:

- Case $n = 0$ then use the corresponding timeout-derivation.
- Case $n > 0$:
  
  By case analysis on the case sequence $cs$
  
  - Case $cs = \varepsilon$ then use ECS-EMP
  
  - Case $cs = \text{case } p \Rightarrow e, cs'$:
    
    Using Lemma 20 on $p$ with $v$ and $\sigma$ gives us a derivation $\sigma \vdash p :? \xrightarrow{\text{match}} \rho$. By Lemma 3 we know that $\rho$ is well-typed.
    
    Using induction hypothesis given by Lemma 25 on $n - 1$ with $\rho, e$ and $\sigma$ we get a derivation $\rho; e; \sigma \xrightarrow{\text{case}}^{n-1} vres'; \sigma'$.
    
    By case analysis on $vres'$:
    
    * Case $vres' = \text{fail}$ then use ECS-MORE-FAIL using above derivation and the derivation given by the induction hypothesis on $n - 1$ with $cs', v$ and $\sigma$.
    
    * Case $vres' \neq \text{fail}$ then use ECS-MORE-ORD

Lemma 27. It is possible to construct a derivation $cs; v; \sigma \xrightarrow{\text{visit}}^n vres; \sigma'$ for any case sequence $cs$, well-typed value $v$, well-typed store $\sigma$, traversal strategy $st$ and fuel $n$.

Proof. By induction on $n$:

- Case $n = 0$ then use the corresponding timeout-derivation.
- Case $n > 0$:

  By case analysis on syntax of $st$

  - Case $st = \text{top-down}$ then use EV-TD with the derivation from the induction hypothesis given by Lemma 28 on $n - 1$ with $cs, v, \sigma$ and no-break.
  
  - Case $st = \text{top-down-break}$ then use EV-TDB with the derivation from the induction hypothesis given by Lemma 28 on $n - 1$ with $cs, v, \sigma$ and break.
  
  - Case $st = \text{bottom-up}$ then use EV-BU with the derivation from the induction hypothesis given by Lemma 30 on $n - 1$ with $cs, v, \sigma$ and no-break.
Case $st = \text{bottom-up-break}$ then use EV-BUB with the derivation from the induction hypothesis given by Lemma 30 on $n - 1$ with $cs$, $v$, $\sigma$ and $\text{break}$.

Case $st = \text{outermost}$:
Using the induction hypothesis by Lemma 28 on $n - 1$ with $cs$, $v$, $\sigma$ and $\text{no-break}$ we get a derivation $cs; v; \sigma \xrightarrow{\text{no-break}}_{\text{td-visit}}^{n-1} vres'; \sigma''$. We know the well-typedness of the output components from Lemma 14.

By case analysis on $vres'$:

* Case $vres' = \text{success } v'$:
  We proceed by checking whether $v = v'$:
  
  * Case $v = v'$ then use EV-OM-Eq using above derivation.
  
  * Case $v \neq v'$ then use EV-OM-Neq using above derivation and the derivation from the induction hypothesis given by Lemma 28 on $n - 1$ with $cs$, $v'$, $\sigma''$ and $\text{no-break}$.

* Case $vres' = \text{exres}$ then use EV-OM-Exc using above derivation.

Case $st = \text{innermost}$: Using the induction hypothesis by Lemma 30 on $n - 1$ with $cs$, $v$, $\sigma$ and $\text{no-break}$ we get a derivation $cs; v; \sigma \xrightarrow{\text{no-break}}_{\text{bu-visit}}^{n-1} vres'; \sigma''$. We know the well-typedness of the output components from Lemma 16.

By case analysis on $vres'$:

* Case $vres' = \text{success } v'$:
  We proceed by checking whether $v = v'$:
  
  * Case $v = v'$ then use EV-IM-Eq using above derivation.
  
  * Case $v \neq v'$ then use EV-IM-Neq using above derivation and the derivation from the induction hypothesis given by Lemma 30 on $n - 1$ with $cs$, $v'$, $\sigma''$ and $\text{no-break}$.

* Case $vres' = \text{exres}$ then use EV-IM-Exc using above derivation.

\[ \square \]

Lemma 28. It is possible to construct a derivation $cs; v; \sigma \xrightarrow{br} vres'; \sigma'$ for any $cs$, well-typed value $v$, well-typed store $\sigma$, breaking strategy $br$ and fuel $n$.

Proof. By induction on $n$:

* Case $n = 0$ then use the corresponding timeout-derivation.

* Case $n > 0$:
  Applying the induction hypothesis given by Lemma 26 on $n - 1$ with $cs$, $v$ and $\text{sigma}$ we get a derivation $cs; v; \sigma \xrightarrow{\text{cases}}_{\text{cases}}^{n-1} vres'; \sigma''$.

  We proceed to check whether $vres'$ has syntax $vfres'$ for some $vfres'$.
We proceed to check whether \( \text{vfres}' = \text{success} \ v' \) and \( \text{br} = \text{break} \):

* Case \( \text{vfres}' = \text{success} \ v' \) and \( \text{br} = \text{break} \) then use ETV-Break-Sucs with \( \mathcal{CS} \).

* Case \( \text{vfres}' \neq \text{success} \ v' \) or \( \text{br} \neq \text{break} \):

By Boolean logic, we have that \( \text{br} \neq \text{break} \Rightarrow \text{vfres}' = \text{fail} \) is satisfied.

Let \( v'' = \text{if-fail}(\text{vfres}, v) \) and \( v''' = \text{children}(v'') \) (all well-typed from Lemma \ref{lem:children} and Lemma \ref{lem:if-fail}).

By induction hypothesis given by Lemma \ref{lem:induction} on \( n - 1 \) with \( \mathcal{CS}, v'' \) and \( \sigma'' \), we get a derivation
\[
\frac{\mathcal{CS}; v''; \sigma'' \xrightarrow{\text{br} \ \text{td-visit}^*} \ vres''; \sigma'''}{
\mathcal{CS}'', v'''; \sigma'''' \xrightarrow{\text{br} \ \text{td-visit}^*} \ vres''''.}
\]

We know that the results are well typed from Lemma \ref{lem:well-typed}.

By case analysis on \( vres'' \):

- Case \( vres'' = \text{success} \ v''' \) then use ETV-Ord-Sucs2 with \( \mathcal{CS}, \mathcal{CS}' \) and the result derivation from applying Lemma \ref{lem:success}.

- Case \( vres'' = \text{fail} \) then use ETV-Ord-Sucs1 with \( \mathcal{CS}, \mathcal{CS}' \).

- Case \( vres'' = \text{exres} \) where \( \text{exres} \neq \text{fail} \) then use ETV-Exc2 with \( \mathcal{CS}, \mathcal{CS}' \).

Case \( vres' \neq vres' \): Necessarily, we then have that \( vres' = \text{exres} \) where \( \text{exres} \neq \text{fail} \) and so we use ETV-Exc1 with \( \mathcal{CS} \).

\[\square\]

\textbf{Lemma 29.} It is possible to construct a derivation \( \mathcal{CS}; v; \sigma \xrightarrow{\text{br} \ \text{td-visit}^*} vres; \sigma' \) for any \( \mathcal{CS}, \text{well-typed value sequence } v, \text{well-typed store } \sigma, \text{breaking strategy } \text{br} \) and fuel \( n \).

\textit{Proof.} By induction on \( n \):

- Case \( n = 0 \) then use the corresponding timeout-derivation.

- Case \( n > 0 \): By case analysis on \( v \):

  - Case \( v = \varepsilon \) then use ETVS-EMP.

  - Case \( v = v', v'' \):

    By induction hypothesis given by Lemma \ref{lem:induction} on \( n - 1 \) with \( \mathcal{CS}, v \) and \( \sigma \) we get a derivation
    \[
    \frac{\mathcal{CS}; v; \sigma \xrightarrow{\text{br} \ \text{td-visit}^*} vres''; \sigma''}{vres''''.}
    \]

    We proceed by checking whether \( vres'' \) is syntactically a \( \text{vfres}'' \) for some \( \text{vfres}'' \):

    * Case \( vres'' = \text{vfres}'' \):

      We proceed by checking whether \( \text{vfres}'' = \text{success} v''' \) and \( \text{br} = \text{break} \):

      - Case \( \text{vfres}'' = \text{success} v''' \) and \( \text{br} = \text{break} \) then use ETVS-Break with \( \forall \mathcal{T} \).

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Case $vfres'' \neq success v'''$ or $br \neq break$:
By Boolean logic, we have that $br \neq break \Rightarrow vfres'' = fail$ is satisfied.
By induction hypothesis on $n - 1$ with $cs, v''$ and $\sigma''$ we get a derivation $vBS$
\[
\frac{br}{bu-visit*}^{n-1} cs; v''; \sigma'' \xrightarrow{br} vres'''; \sigma'.
\]
By case analysis on $vres'''$:
- Case $vres''' = vfres'''$ then use ETVS-MORE with $\forall T$ and $\forall TS$.
- Case $vres''' = exres$ where $exres \neq fail$ then use ETVS-Exc2 with $\forall T$ and $\forall TS$.
  * Case $vres'' \neq vfres''$:
    Here we then have that $vres'' = exres$, where $exres \neq fail$ and so we use ETVS-Exc1.

\[\square\]

Lemma 30. It is possible to construct a derivation $cs; v; \sigma \xrightarrow{bu-visit*} vres; \sigma'$ for any $cs$, well-typed value $v$, well-typed store $\sigma$, breaking strategy $br$ and fuel $n$.

Proof. By case analysis on $n$:
- Case $n = 0$ then use the corresponding timeout-derivation.
- Case $n > 0$:
  Let $v'' = children(v)$. We know that $v''$ is well-typed from Lemma 2.
  By induction hypothesis given by Lemma 31 on $n - 1$ with $cs, v''$ and $\sigma$ we get a derivation $vBS$
  \[
  \frac{br}{bu-visit*}^{n-1} cs; v''; \sigma \xrightarrow{br} vres'''; \sigma'.
  \]
  Recall that the output is well-typed from Lemma 17.
  By case analysis on $vres'''$:
  - Case $vres''' = vfres'''$:
    We proceed by case analysis on $vfres''$
    * Case $vfres'' = success v'''$: We proceed by case analysis on $br$:
      - Case $br = break$ then use EBU-BREAK-SUCS with $\forall BS$ and the derivation from applying Lemma 19 on $v$ and $v'''$.
      - Case $br = no-break$:
        By applying Lemma 19 on $v$ and $v'''$ we get a derivation $\forall RC$
        \[
        \text{recons } v \text{ using } v''' \text{ to } rcres'.
        \]
        By case analysis on $rcres$:
• Case \( rcres = success \):  
By induction hypothesis given by Lemma 26 on \( n - 1 \) with \( cs, v' \) and \( \sigma'' \) we get a derivation \( \underbrace{cs; v'; \sigma''}_{CS} \xrightarrow{n-1} \text{cases} vres'; \sigma' \).  
By case analysis on \( vres' \):  
– Case \( vres' = vfres' \) then use EBU-No-Break-Sucs with \( VBS, RC \) and \( CS \).  
– Case \( exres \) then use EBU-No-Break-Exc with \( VBS, RC \) and \( CS \).  
• Case \( rcres = error \) then use EBU-No-Break-Err.

* Case \( vfres'' = fail \) then use EBU-Fail-Sucs with the induction hypothesis given by Lemma 26 on \( n - 1 \) with \( cs, v \) and \( \sigma'' \).  
– Case \( vres^*'' = exres \) where \( exres \neq fail \) then use EBU-Exc with \( VBS \).

\[ \square \]

**Lemma 31.** It is possible to construct a derivation \( \underbrace{cs; v; \sigma}_{bu-visit} \xrightarrow{n} vres*; \sigma' \) for any \( cs \), well-typed value sequence \( v \), well-typed store \( \sigma \), breaking strategy \( br \) and fuel \( n \).

**Proof.** By induction on \( n \):

- Case \( n = 0 \) then use the corresponding timeout-derivation.

- Case \( n > 0 \):
  
  By case analysis on \( v \):
  
  – Case \( v = \varepsilon \) then use EBUS-Emp.
  
  – Case \( v = v', v'' \): By induction hypothesis given by Lemma 26 on \( n - 1 \) with \( cs, v' \) and \( \sigma \) we get a derivation \( \underbrace{cs; v'; \sigma}_{VBS} \xrightarrow{n-1} vres''; \sigma'' \).  
  
  By case analysis on \( vres'' \):
  
  * Case \( vres'' = vfres'' \):
    
    By case analysis on \( vfres'' \) and \( br \):
    
    • Case \( vfres'' = success \) then use EBUS-Break with \( VBS \).
    
    • Case \( vfres'' \neq success \) or \( br \neq break \):  
      
      By Boolean logic we have \( br = break \Rightarrow vfres = fail \).  
      
      By induction hypothesis on \( n - 1 \) with \( cs, v'' \) and \( \sigma \) we get a derivation \( \underbrace{cs; v''; \sigma}_{VBS} \xrightarrow{n-1} vres*' \).  
      
      By case analysis on \( vres*' \):
      
      • Case \( vres*' = vfres*' \) then use EBUS-More with \( VBS \) and \( VBS \).  
      
      • Case \( vres*' = exres \) then use EBUS-Exc2 with \( VBS \) and \( VBS \).
* Case $vres'' = exres$ then use EBUS-Exc1 with $\forall B$. □

**Theorem 3** (Partial progress). *It is possible to construct a derivation $e; \sigma \xrightarrow{\text{expr}}^n vres; \sigma'$ for any input expression $e$, well-typed store $\sigma$ and fuel $n$.*

**Proof.** By induction on $n$:

- Case $n = 0$ then use the corresponding timeout-derivation.

- Case $n > 0$:
  By case analysis on syntax $e$:
  - Case $e = vb$ then use T-Basic.
  - Case $e = x$
    We proceed by checking $x \in \text{dom} \sigma$:
      * Case $x \in \text{dom} \sigma$ then use E-VAR-SUCS.
      * Case $x \notin \text{dom} \sigma$ then use E-VAR-ERR.
  - Case $e = \ominus e'$:
    By induction hypothesis on $n - 1$ with $e'$ and $\sigma$ we get a derivation $e'; \sigma \xrightarrow{\text{expr}}^{n-1} \vres'; \sigma'$.
    By case analysis on $\vres'$:
      * Case $\vres' = \text{success}$ then use E-Un-SUCS with above derivation.
      * Case $\vres' = exres$ then use E-Un-Exc with above derivation.
  - Case $e = e_1 \oplus e_2$:
    By induction hypothesis on $n - 1$ with $e_1$ and $\sigma$ we get a derivation $e_1; \sigma \xrightarrow{\text{expr}}^{n-1} \vres_1; \sigma''$.
    By case analysis on $\vres_1$:
      * Case $\vres_1 = \text{success}$ $v_1$:
        By induction hypothesis on $n - 1$ with $e_2$ and $\sigma''$ we get a derivation $e_2; \sigma'' \xrightarrow{\text{expr}}^{n-1} \vres_2; \sigma'$.
        By case analysis on $\vres_2$:
          * Case $\vres_2 = \text{success}$ $v_2$ then use E-Bin-SUCS with $\vres_1$ and $\vres_2$.
          * Case $\vres_2 = exres$ then use E-Bin-Exc2 with $\vres_1$ and $\vres_2$.
      * Case $\vres_1 = exres$ then use E-Bin-Exc1 with $\vres_1$.
  - Case $e = k(e')$:
    Recall that all derivations in our paper are assumed to be well-scoped so there must exist a corresponding data-type $at$ that has $k(t)$ as a constructor.
    By using induction hypothesis given by Lemma 22 on $n - 1$ with $e'$ and $\sigma$ we get a derivation $e'; \sigma \xrightarrow{\text{expr}}^{n-1} \vres'; \sigma'$.
    By case analysis on $\vres'$:
* Case \( vres^* = \text{success } v' \):
  By Lemma 8 we know that \( v' \) is well-typed, i.e. that we have \( v' : t' \) for some type sequence \( t' \). We proceed to check whether all values in the sequence are non-\( \Box \) and each have a type \( t'_i \) that is a subtype of the target type \( t_i \).

  · Case \( v \neq \Box \) and \( t' < : t \) then use \( \text{E-Cons-Sucs} \) with \&\$ and the required typing derivations.
  · Case \( v_i = \Box \) or \( t'_i < : t_i \) for some \( i \) then use \( \text{E-Cons-Err} \) with \&\$.

* Case \( vres^* = \text{exres} \) then use \( \text{E-Cons-Exc} \) with \&\$.

  – Case \( e = [e'] \):
    By induction hypothesis given by Lemma 22 on \( n - 1 \) with \( e' \) and \( \sigma \) we get a derivation \( e'; \sigma \xrightarrow{\text{expr}*} vres^*; \sigma' \).
    By case analysis on \( vres^* \):

    * Case \( vres^* = \text{success } v \):
      We proceed by checking whether all values \( v \) are non-\( \Box \):
      · Case \( v \neq \Box \) then use \( \text{E-List-Sucs} \) with above derivation.
      · Case \( v_i \neq \Box \) for some \( i \) then use \( \text{E-List-Err} \) with above derivation.
    * Case \( vres^* = \text{exres} \) then use \( \text{E-List-Exc} \) with above derivation.

  – Case \( e = \{e'\} \):
    By induction hypothesis given by Lemma 22 on \( n - 1 \) with \( e' \) and \( \sigma \) we get a derivation \( e'; \sigma \xrightarrow{\text{expr}*} vres^*; \sigma' \).
    By case analysis on \( vres^* \):

    * Case \( vres^* = \text{success } v \):
      We proceed by checking whether all values \( v \) are non-\( \Box \):
      · Case \( v \neq \Box \) then use \( \text{E-Set-Sucs} \) with above derivation.
      · Case \( v_i \neq \Box \) for some \( i \) then use \( \text{E-Set-Err} \) with above derivation.
    * Case \( vres^* = \text{exres} \) then use \( \text{E-Set-Exc} \) with above derivation.

  – Case \( e = (e' : e'') \):
    By induction hypothesis given by Lemma 22 on \( n - 1 \) with \( e', e'' \) and \( \sigma \) we get a derivation \( e', e''; \sigma \xrightarrow{\text{expr}*} vres^*; \sigma' \).
    By case analysis on \( vres^* \):

    * Case \( vres^* = \text{success } v, v' \):
      We proceed by checking whether all values \( v \) and \( v' \) are non-\( \Box \):
      · Case \( v \neq \Box \) and \( v' \neq \Box \) then use \( \text{E-Map-Sucs} \) with above derivation.
      · Case \( v_i \neq \Box \) or \( v'_i \neq \Box \) for some \( i \) then use \( \text{E-Map-Err} \) with above derivation.
    * Case \( vres^* = \text{exres} \) then use \( \text{E-Map-Exc} \) with above derivation.

  – Case \( e = e_1[e_2] \)
By induction hypothesis on \( n - 1 \) with \( e_1 \) and \( \sigma \) we get a derivation
\[
e_1; \sigma \xrightarrow{\text{expr}}_{n-1} vres_1; \sigma''.
\]

By case analysis on \( vres_1 \):

* Case \( vres_1 = \text{success} \ v_1 \):

By case analysis on \( v_1 \):

· Case \( v_1 = (v': v'') \):

By induction hypothesis on \( n - 1 \) with \( e_2 \) and \( \sigma'' \) we get a derivation
\[
e_2; \sigma'' \xrightarrow{\text{expr}}_{n-1} vres_2; \sigma'.
\]

By case analysis on \( vres_2 \):

· Case \( vres_2 = \text{success} \ v_2 \):

We proceed to check whether \( \exists i. v'_i = v_2 \):

· Case \( v'_i = v_2 \) then use \( \text{E-LOOKUP-SUCS} \) with \( \mathcal{E} \) and \( \mathcal{E}' \).

· Case \( \nexists i. v'_i = v_2 \) then use \( \text{E-LOOKUP-NoKey} \) with \( \mathcal{E} \) and \( \mathcal{E}' \).

· Case \( vres_2 = \text{exres} \) then use \( \text{E-LOOKUP-Exc2} \) with \( \mathcal{E} \) and \( \mathcal{E}' \).

· Case \( v_1 \neq (v': v'') \) then use \( \text{E-LOOKUP-Err} \) with \( \mathcal{E} \).

· Case \( vres_1 = \text{exres} \) then use \( \text{E-LOOKUP-Exc1} \) with \( \mathcal{E} \).

– Case \( e = e_1[e_2 = e_3] \)

By induction hypothesis on \( n - 1 \) with \( e_1 \) and \( \sigma \) we get a derivation
\[
e_1; \sigma \xrightarrow{\text{expr}}_{n-1} vres_1; \sigma''.
\]

By case analysis on \( vres_1 \):

* Case \( vres_1 = \text{success} \ v_1 \)

By case analysis on \( v_1 \):

· Case \( v_1 = (v': v'') \):

By induction hypothesis on \( n - 1 \) with \( e_2 \) and \( \sigma'' \) we get a derivation
\[
e_2; \sigma'' \xrightarrow{\text{expr}}_{n-1} vres_2; \sigma''.
\]

By case analysis on \( vres_2 \):

· Case \( vres_2 = \text{success} \ v_2 \):

By induction hypothesis on \( n - 1 \) with \( e_3 \) and \( \sigma'' \) we get a derivation
\[
e_3; \sigma'' \xrightarrow{\text{expr}}_{n-1} vres_3; \sigma'.
\]

By case analysis on \( vres_3 \):

– Case \( vres_3 = \text{success} \ v_3 \):

We proceed to check whether \( v_2 \) and \( v_3 \) are non-\( \blacksquare \):

* Case \( v_2 \neq \blacksquare \) and \( v_3 \neq \blacksquare \) then use \( \text{E-UPDATE-SUCS} \) with \( \mathcal{E} \), \( \mathcal{E}' \) and \( \mathcal{E}'' \).

* Case \( v_2 = \blacksquare \) or \( v_3 = \blacksquare \) then use \( \text{E-UPDATE-Err2} \) with \( \mathcal{E} \), \( \mathcal{E}' \) and \( \mathcal{E}'' \).

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– Case $vres_3 = exres$ then use E-UPDATE-EXC3 with $\xi$ and $\xi'$.

• Case $vres_2 = exres$ then use E-UPDATE-EXC2 on $\xi$ and $\xi'$.
  • Case $v_1 \neq (v': v'')$ then use E-UPDATE-ERR1 with $\xi$.
  * Case $vres_1 = exres$ then use E-UPDATE-EXC1 with $\xi$.

– Case $e = f(e')$

By induction hypothesis given by Lemma 22 on $n - 1$ with $e'$ and $\sigma$ we get a derivation 

$$\xi \frac{e' \sigma}{\text{expr}} \xrightarrow{n-1} \text{vres}'''; \sigma''$$

By case analysis on $\text{vres}'''$:

* Case $\text{vres}''' = \text{success} v''$
  Recall that we assume that our function calls are well-scope d and so there must exist a corresponding function definition $\text{fun} t' f(t x) = e''$. By Lemma 8 we know that we have $v'' : t''$ for some type sequence $\overrightarrow{t''}$.
  We proceed to check whether $t'' <: t$:
    • Case $t'' <: t$
      Let $\text{global} t_y y$ represent all global variable definitions.
      By induction hypothesis on $n - 1$ with $e''$ and $[y \mapsto \sigma''(y), x \mapsto v'']$ we get a derivation 

$$\xi \frac{e' [y \mapsto \sigma''(y), x \mapsto v'']}{\text{expr}} \xrightarrow{n-1} \text{vres}'; \sigma''$$

By case analysis on $\text{vres}'$:

• Case $\text{vres}' = \text{success} v'$ or $\text{vres}' = \text{return} v'$.
  By Theorem 2 we know that $v' : t''$ for some type $t''$. We proceed to check whether $t'' <: t'$:
    – Case $t'' <: t'$ then use E-CALL-RES-SUCS with $\xi$ and $\xi$.
    – Case $t'' \nleq t'$ then use E-CALL-RES-ERR1 with $\xi$ and $\xi$.

• Case $\text{vres}' = \text{throw} v'$ then use E-CALL-RES-EXC with $\xi$ and $\xi$.
• Case $\text{vres}' \in \{\text{break}, \text{continue}, \text{fail}, \text{error}\}$ then use E-CALL-RES-ERR2 with $\xi$ and $\xi$.
  • Case $t'' \nleq t$ for some $i$ then use E-CALL-ARG-ERR with $\xi$.
  * Case $\text{vres}'''' = exres$ then use E-CALL-ARG-EXC with $\xi$.

– Case $e = \text{return} e'$

By induction hypothesis on $n - 1$ with $e'$ and $\sigma$ we get a derivation 

$$\xi \frac{e' \sigma}{\text{expr}} \xrightarrow{n-1} \text{vres}''; \sigma'$$

* Case $\text{vres}' = \text{success} v$ then use E-RET-SUCS with above derivation.
  * Case $\text{vres}' = exres$ then use E-RET-EXC with above derivation.

– Case $e = (x = e')$

Recall that our definitions are assumed to be well-scoped and so there must exists either a local $t x$ or global $t x$ declaration for the variable (with no overshadowing).
By induction hypothesis on $n - 1$ with $e'$ and $\sigma$ we get a derivation $e' ; \sigma \xrightarrow{\text{expr}}^{n-1} vres'; \sigma''$.

By case analysis on $vres'$:

* Case $vres' = \text{success} \; v$: By Theorem \[2\] we know that there exists a $t'$ such that $v : t'$.
We proceed by checking whether $t' <: t$:
  - Case $t' <: t$ then use E-ASGN-SUCS with above evaluation and typing derivations.
  - Case $t' \not <: t$ then use E-ASGN-ERR with above evaluation and typing derivations.

* Case $vres' = \text{exres}$ then use E-ASGN-Exc with above derivation.

Case $e = \text{if} \; e_1 \; \text{then} \; e_2 \; \text{else} \; e_3$
By induction hypothesis on $n - 1$ with $e_1$ and $\sigma$ we get a derivation $e_1 ; \sigma \xrightarrow{\text{expr}}^{n-1} vres'' ; \sigma''$.

By case analysis on $vres''$:

* Case $vres'' = \text{success} \; v''$
  By induction hypothesis given by Lemma \[26\] on $n - 1$ with $cs$, $v''$ and $\sigma''$ we get a derivation $cs ; v'' ; \sigma'' \xrightarrow{\text{cases}}_{\text{case}_2}^{n-1} vres' ; \sigma'$.

* Case $vres'' = \text{exres}$ then use E-Switch-Exc1 with $\delta$.
– Case $e = \text{st visit } e'$ do $cs$:
  By induction hypothesis on $n - 1$ with $e'$ and $\sigma$ we get a derivation
  $\begin{array}{c}
e'; \sigma \\ \xrightarrow{\text{expr } n - 1} \text{vres''; } \sigma''
  \end{array}$.

  By case analysis on $\text{vres''}$:
  * Case $\text{vres''} = \text{success } v''$:
    By induction hypothesis given by Lemma \text{[27]} on $n - 1$ with $cs$, $v''$ and $\sigma''$ we get a derivation
    $\begin{array}{c}
    cs; v''; \sigma'' \\ \xrightarrow{\text{st visit } n - 1} vres'; \sigma'.
    \end{array}$
  
    By case analysis on $\text{vres'}$:
    · Case $\text{vres'} = \text{success } v'$ then use $\text{E-Visit-Sucs}$ with $\delta$ and $\mathcal{V}$.
    · Case $\text{vres'} = \text{fail}$ then use $\text{E-Visit-Fail}$ with $\delta$ and $\mathcal{V}$.
    · Case $\text{vres'} = \text{exres}$ where $\text{exres} \neq \text{fail}$ then use $\text{E-Visit-Exc2}$ with $\delta$ and $\mathcal{V}$.
    
    * Case $\text{vres''} = \text{exres}$ then use $\text{E-Visit-Exc1}$ with $\delta$.

– Case $e = \text{break}$ then use $\text{E-Break}$.

– Case $e = \text{continue}$ then use $\text{E-Continue}$.

– Case $e = \text{fail}$ then use $\text{E-Fail}$.

– Case $e = \text{local } t x \text{ in } e' \text{ end}$ then use either $\text{E-Block-Sucs}$ (when it produces a successful result) or $\text{E-Block-Exc}$ (otherwise) with the derivation of the induction hypothesis given by Lemma \text{[22]} on $n - 1$ with $e'$ and $\sigma$.

– Case $e = \text{for } ge'$:
  By induction hypothesis given by Lemma \text{[24]} on $n - 1$ with $g$ and $\sigma$ we get a derivation
  $\begin{array}{c}
g; \sigma \\ \xrightarrow{\text{gexpr } n - 1} \text{envres; } \sigma''
  \end{array}$.

  By case analysis on $\text{envres}$:
  * Case $\text{envres} = \text{success } \rho$ then use $\text{E-For-Sucs}$ with above derivation and the derivation from the induction hypothesis given by Lemma \text{[23]} on $n - 1$ with $e'$, $\rho$ and $\sigma''$.
  * Case $\text{envres} = \text{exres}$ then use $\text{E-For-Exc}$ with above derivation.

– Case $e = \text{while } e_1 \ e_2$
  By induction hypothesis on $n - 1$ with $e_1$ and $\sigma$ we get a derivation:
  $\begin{array}{c}
e_1; \sigma \\ \xrightarrow{\text{expr } n - 1} \text{vres''; } \sigma''
  \end{array}$.

  By case analysis on $\text{vres''}$:
  * Case $\text{vres''} = \text{success } v''$
    By case analysis on $v''$:
      · Case $v'' = \text{false}()$ then use $\text{E-While-False}$ with $\delta$.
· Case $v'' = \text{true}()$:

By induction hypothesis on $n - 1$ with $e_2$ and $\sigma''$ we get a derivation $e_2; \sigma'' \xrightarrow{\text{expr}} vres'''; \sigma'''$.

By case analysis on $vres'''$:

- Case $vres''' = \text{success}$ or $vres''' = \text{continue}$ then use $E\text{-While-True-Sucs}$ with $e$, $\sigma$ and the derivation from the induction hypothesis on $n - 1$ with while $e_1 e_2$ and $\sigma'''$.
- Case $vres''' = \text{break}$ then use $E\text{-While-True-Break}$ with $e$ and $\sigma$.
- Case $vres''' = \text{exres} \in \{\text{throw } v'', \text{return } v'', \text{fail}, \text{error}\}$ then use $E\text{-While-Exc2}$ with $e$ and $\sigma$.

· Case $v'' \neq \text{true}()$ and $v'' \neq \text{false}()$ then use $E\text{-While-Err}$

  * Case $vres'' = \text{exres}$ then use $E\text{-While-Exc1}$ with $e$.

- Case $e = \text{solve } x e'$:

By induction hypothesis on $n - 1$ with $e'$ and $\sigma$ we get a derivation $e'; \sigma \xrightarrow{\text{expr}} vres''; \sigma''$.

By case analysis on $vres''$:

  * Case $vres'' = \text{success}$ or $vres'' = \text{continue}$ then use $E\text{-Solve-Eq}$ with $e$.
  * Case $vres'' = \text{break}$ then use $E\text{-Solve-Neq}$ with $\sigma$ and the derivation from the induction hypothesis on $n - 1$ with solve $x e'$ and $\sigma''$.

  · Case $x \notin \text{dom } \sigma \cap \text{dom } \sigma''$ for some $i$ then use $E\text{-Solve-Err}$ with $e$.

- Case $e = \text{throw } e'$:

By induction hypothesis on $n - 1$ with $e'$ and $\sigma$ we get a derivation $e'; \sigma \xrightarrow{\text{expr}} vres'; \sigma'$.

  * Case $vres' = \text{success}$ then use $E\text{-THR-Sucs}$ with above derivation.
  * Case $vres' = \text{exres}$ then use $E\text{-THR-Exc}$ with above derivation.

- Case $e = \text{try } e_1 \text{ catch } x \Rightarrow e_2$:

By induction hypothesis on $n - 1$ with $e_1$ and $\sigma$ we get a derivation $e_1; \sigma \xrightarrow{\text{expr}} vres_1; \sigma''$.

By case analysis on $vres_1$:
* Case $vres_1 = \text{throw } v_1$ then use E-TRY-CATCH with $\mathcal{E}$ and the derivation
  from the induction hypothesis on $n-1$ with $e_2$ and $\sigma''[x \mapsto v_1]$.

* Case $vres_1 \neq \text{throw } v_1$ then use E-TRY-ORD with $\mathcal{E}$.

- Case $e = \text{try } e_1 \text{ finally } e_2$

  By induction hypothesis on $n-1$ with $e_1$ and $\sigma$ we get a derivation
  
  $e_1; \sigma \mathrel{\xrightarrow{\text{expr}}^{n-1}} vres_1; \sigma''$.

  By induction hypothesis on $n-1$ with $e_2$ and $\sigma''$ we get a derivation
  
  $e_2; \sigma'' \mathrel{\xrightarrow{\text{expr}}^{n-1}} vres_2; \sigma'$.

  By case analysis on $vres_2$:

  * Case $vres_2 = \text{success } v_2$ then use E-FIN-SUCS with $\mathcal{E}$ and $\mathcal{E}'$

  * Case $vres_2 = \text{exres}$ then use E-FIN-EXC with $\mathcal{E}$ and $\mathcal{E}'$.

\[\square\]

**Theorem 4** (Terminating expressions). There exists $n$ such that derivation

\[\mathcal{E} \quad e_\text{fin}; \sigma \mathrel{\xrightarrow{\text{expr}}^n} vres; \sigma'\]

has a result $vres$ which is not timeout for expression $e_\text{fin}$ in the
terminating subset.

**Proof.** The proof proceeds similarly to Theorem 3 except that instead of doing the induction
on $n$—which we know need to provide—we do the induction on the relevant syntactic element
starting with the $e_\text{fin}$ for this theorem. The only major complication is that for the bottom-up
visit rules, we need to do an inner well-founded induction on the $\prec$ relation on values when
traversing the children in order to terminate.

The result $n$ is simply taking to be $n' = 1 + n$ where $n$ is the maximal fuel used in a
sub-term. \[\square\]