Indirect Inference for Locally Stationary Models

David T. Frazier* Bonsoo Koo†
Monash University

Abstract

We propose the use of indirect inference estimation for inference in locally stationary models. We develop a local indirect inference algorithm and establish the asymptotic properties of the proposed estimator. Due to the nonparametric nature of the model under study, the resulting estimators display nonparametric rates of convergence and behavior. We validate our methodology via simulation studies in the confines of a locally stationary moving average model and a locally stationary multiplicative stochastic volatility model. An application of the methodology gives evidence of non-linear, time-varying volatility for monthly returns on the Fama-French portfolios.

Key words: semiparametric, locally stationary, indirect inference, state-space models

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*Department of Econometrics and Business Statistics, Monash University, PO Box 11E, Clayton Campus, VIC 3800, Australia; e-mail: david.frazier@monash.edu

†Corresponding author. Department of Econometrics and Business Statistics, Monash University, PO Box 11E, Clayton Campus, VIC 3800, Australia; e-mail: bonsoo.koo@monash.edu
1 Introduction

Time-varying economic and financial variables, and relationships thereof, are stable features in applied econometrics. Notable examples include asset pricing models with time-varying features (Ghysels, 1998; Wang, 2003) and trending macroeconomic models (Stock and Watson, 1998; Phillips, 2001). While classical analyses of time series are built on the assumption of stationarity, data studied in finance and economics often exhibit nonstationary features.

Many different schools of modeling and estimation methods are proposed in order to accommodate the nonstationary behavior of observed time series data. In particular, statistical tools developed for locally stationary processes provide a convenient means of conducting analyses of trending economic and financial models. Heuristically, local stationarity implies that a process behaves in a stationary manner (at least) in the vicinity of a given time point but is nonstationary over the entire time horizon. For certain widely-studied time series models, slowly time-varying parameters ensure local stationarity under some regularity conditions; for instance, see Dahlhaus (1996) and Dahlhaus (1997) (AR(1)), Dahlhaus and Subba Rao (2006) (ARCH(∞)), Dahlhaus and Polonik (2009) (MA(∞)), Koo and Linton (2012) (Diffusion processes) and Koo and Linton (2015) (GARCH(1,1) with a time-varying unconditional variance) among many other classes of locally stationary processes.

While many classes of well-known time series models can be generalized to locally stationary processes, it is worth noting that estimation and inference procedures developed in one class of locally stationary processes often cannot be applied to a different class of locally stationary processes. In particular, many estimation methods for locally stationary processes are composed of estimation approaches that primarily focus on local regression with closed-form estimators, local maximum likelihood estimation (MLE) with a closed-form likelihood function (in the time domain) and spectral density approach (in the frequency domain), all of which could be intractable or simply difficult to implement for various locally stationary extensions of commonly used structural econometric models; we refer to Vogt (2012), Dahlhaus and Subba Rao (2006) and Dahlhaus and Polonik (2009), for examples. As such, model specifications compatible with the above statistical methods are rather limited and cannot be used for estimation and inference in more complicated locally stationary models, such as, for instance, models with latent variables or unobservable factors.

More importantly, structural models of economic and financial relationships commonly rely on the use of latent variables to represent information that is unavailable to the econometrician. This modeling approach implies, almost by definition, that simple (closed-form) representations for the conditional distributions of the endogenous variables are unavailable, with simple straightforward estimation methods often infeasible as a consequence. Taken together, the (seemingly) natural extension of many locally-stationary models to include latent variables commonly encountered in econometrics and finance often render the existing estimation methods used in that model infeasible. In a state-space model setting, such situations can arise when there is no closed form for the measurement, and or, state transition densities, in cases where the conditional distribution of the model has no closed-form, or in settings where the regularity conditions required for estimation of locally stationary processes lead to an intractable likelihood function. For example, estimation and asymptotic properties for univariate locally stationary diffusion models cannot be straightforwardly carried over to multivariate extensions or to stochastic volatility models. See Koo and Linton (2012) for more details. Nevertheless, these extensions are too important to be neglected.

To circumvent the above problem, and to help proliferate the use of these powerful locally stationary models and methods, we propose a novel nonparametric indirect inference (II) method to estimate locally stationary processes. Instead of estimating complex structural locally stationary models directly, we indirectly obtain our estimator by targeting consistent estimators of simpler auxiliary models, and use these
consistent estimates to conduct inference on the structural parameters. See, Smith (1993), Gourieroux et al. (1993) and Gourieroux and Monfort (1996) for discussion of indirect inference in parametric models.

To illustrate the main idea behind our nonparametric II approach for locally stationary processes, we consider the following motivating example. Suppose that the true data generating process evolves according to

\[ Y_{t,T} = \sqrt{\theta_0(t/T)} \exp(h_t/2) \varepsilon_t, \quad \text{where} \ h_t = \omega + \delta h_{t-1} + \sigma_v v_t, \ (\varepsilon_t, v_t)' \sim \mathcal{N}(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), \]

(1)

where \( \theta_0(t/T) > 0 \), for all \( t \leq T \). This locally stationary stochastic volatility (LS-SV) model decomposes volatility into a short-term, latent volatility process, \( h_t \), and a slowly time-varying component, \( \theta_0(\cdot) \), and can capture a wide range of volatility behaviors. The above model allows for non-stationary, but slowly changing, volatility dynamics, which may result from the transitory nature of the business cycle.

Suppose that we wish to estimate the unknown function \( \theta_0(\cdot) \) and conduct statistical inference on (1). While (G)ARCH-based versions of the locally stationary volatility model in (1) have been analyzed by several researchers (see, e.g., Dahlhaus and Subba Rao, 2006, Engle and Rangel, 2008, Fryzlewicz et al., 2008, and Koo and Linton, 2015), since the latent volatility process, \( h_t \), pollutes the observed data, \( Y_{t,T} \), it is not entirely clear how to estimate \( \theta_0(\cdot) \). Indeed, largely due to this fact, locally stationary volatility models have not been previously explored in the literature, even though their stationary counterparts form the backbone of many empirical studies in finance and financial econometrics.

In this paper, we generalize the II approach of Gourieroux et al. (1993) to present a convenient estimator for unknown functions in locally stationary models, such as the LS-SV model, where, due to the nature of the model, estimating the unknown functions of interest via other approaches would be infeasible or too cumbersome. This approach to II estimation relies on a locally stationary auxiliary model that can be easily estimated using the observed data and that captures the underlying features of interest in the structural model. For example, in the context of the LS-SV model, a reasonable auxiliary model would be the locally stationary GARCH model:

\[ Y_{t,T} = \sqrt{\rho(t/T)} \sigma_t z_t, \quad \text{where} \ \sigma_{t+1}^2 = \alpha_0 + \alpha_1 z_t^2 + \beta \sigma_t^2, \]

(2)

where \( \rho(t/T) > 0 \) for all \( t \leq T \), and \( z_t \sim \mathcal{WN}(0, 1) \), with \( \mathcal{WN}(0, 1) \) denoting a white noise process with zero mean and unit variance.

The remainder of this paper further develops the ideas behind this estimation method in the context of a general locally stationary model and establishes the asymptotic properties of the proposed estimation procedure under general (high-level) regularity conditions. To establish the asymptotic properties of these II estimators, we must first develop conditions that guarantee misspecified locally stationary models admit consistent estimators of their corresponding pseudo-true values. This is itself a novel result since the literature on locally stationary models has so far only treated estimators defined by relatively simple criterion functions, and all under the auspices of correct model specification. From these new results on locally stationary estimators of the auxiliary model, we can deduce the asymptotic properties of our proposed II estimator for the structural model parameters.

The estimation procedures proposed herein is demonstrated through two Monte Carlo examples, and an empirical application. The empirical application applies the LS-SV model to examine the volatility behavior exhibited by several Fama-French portfolios. We find that most of these portfolios display time-varying volatility patterns that broadly track the underlying (low-frequency) expansion and contractions of the United States economy.
The remainder of the paper is organized as follows. Section 2 introduces the general model and the related framework. In Section 2.3, we present our general approach and define the corresponding local II (L-II) estimators for a general locally stationary model. Section 3 develops asymptotic results that demonstrate the properties of this estimation procedure. Simulation results for a simple example of a locally stationary moving average model of order one are discussed in Section 4. In Section 5 we further analyze the locally stationary stochastic volatility model. We consider a small Monte Carlo to demonstrate our estimation method, then apply this method to analyze the volatility behavior of Fama-French portfolio returns, where we find ample evidence for smoothly time-varying nonlinear volatility dynamics over the sample period. All proofs are relegated to Appendix A. The tables and figures associated with the application in Section 5 are given in Appendix B.

Throughout this paper, the following notations are used. The tilde, \( \tilde{\cdot} \), over a variable denotes that the variable is simulated, whereas \( \hat{\cdot} \) over a parameter denotes the parameter value used for simulation. Let \( g \) be any function from \( \mathbb{R}^d \to \mathbb{R} \), \( \| g \|_r = (\int |g(x)|^r \, dx)^{1/r} \) and \( \| g \|_s = (\int |g(x)|^s \, dx)^{1/s} \). Particularly, \( \| g \| = \sup |g(\cdot)| \) and \( \| \cdot \|_\Omega \) is a norm weighted by \( \Omega \). A subscript, \( 0 \) denotes the true value or functional form of the corresponding parameters or functions. \( O_p(\cdot) \) and \( o_p(\cdot) \) denote the usual big \( O \) and little \( o \) in probability. \( \mathbb{N} \) denotes the set of all natural numbers. \( C \) denotes a generic constant which takes different values for different places.

2 The Model

2.1 Structural models

We assume the researcher is interested in conducting inference on some model within the class of locally stationary models. Let \( \{Y_{t,T}\}_{t=1,...,T;T=1,2,...} \) denote a triangular array of observations. The process \( \{Y_{t,T}\} \) is locally stationary if it satisfies the following definition.

**Definition 1.** The process \( \{Y_{t,T}\} \) is locally stationary if there exists a stationary process \( \{y_{t/T,t}\} \) for each re-scaled time point \( t/T \in [0,1] \), such that for all \( T \),

\[
P\left( \max_{1\leq t\leq T} |Y_{t,T} - y_{t/T,t}| \leq C_T T^{-1} \right) = 1,
\]

where \( \{C_T\} \) is a measurable process satisfying, for some \( \eta > 0 \), \( \sup_T \mathbb{E}(|C_T|^\eta) < \infty \).

The magnitude of \( \eta \) captures the degree of approximation of \( y_{t/T,t} \) to \( Y_{t,T} \), which reflects the characteristics of the underlying processes of interest. The larger \( \eta \), the better the approximation. We do not specify the magnitude of \( \eta \) to maintain generality, which allows us to represent various types of processes, and instead allow \( \eta \) to vary from model to model. See, for instance, Dahlhaus and Subba Rao (2006) for ARCH(\( \infty \)), Koo and Linton (2012) for diffusion processes, Vogt (2012) for AR processes and Dahlhaus and Polonik (2009) for MA processes among many other processes.

We consider that the process \( \{Y_{t,T}\} \) is generated from the following locally stationary structural model with time-varying parameters:

\[
Y_{t,T} = r(\epsilon_{t,T}; \theta_0(t/T)),
\]
\[
\epsilon_{t,T} = \varphi(\nu; \theta_0(t/T)),
\]

where both \( r(\cdot) \) and \( \varphi(\cdot) \) are real-valued functions that are known up to the unknown function \( \theta_0 \). The function of interest is \( \theta_0 \in \mathcal{H} \), where \( \mathcal{H} \) is some vector space of functions equipped with the norm \( \| \cdot \| \).

We assume that the structural model, and \( \theta_0 \) satisfy the following regularity conditions:
Assumption 1. (i) For all, \(\delta > 0\), and \(u = t/T \in [\delta, 1 - \delta]\), the function \(\theta_0(u)\) has uniformly bounded second-derivatives with respect to \(u\). (ii) The functions \(r(\cdot), \varphi(\cdot)\), known up to \(\theta_0(\cdot)\), are twice continuously differentiable with respect to \(\theta\), with uniformly bounded second derivatives. (iii) The error \(\nu_t\) is a strictly stationary process with \(\nu_t \sim D(0, \sigma)\), and with \(\sigma\) and \(D(\cdot)\) known to the econometrician.

The structural model in (3) is quite general and can accommodate many interesting processes, including models with complex time-varying features, such as time-varying autoregressive conditional heteroskedasticity (ARCH). In addition, the structural model in (3) can always be augmented by additional exogenous regressors at the cost of additional notation. Such regressors may be used, for instance, to capture some conditionally heteroskedastic features of the data. Critically for our purposes, under Assumption 1, if \(\theta_0(\cdot)\) were known, simulated realizations of \(\{Y_{t,T}\}\) could easily be generated from the model in equation (3).

If the process in (3) is locally stationary, inference on \(\theta_0(\cdot)\) can be carried out through an approximate structural model defining a stationary process indexed by \(u \in [0, 1]\):

\[
y_{u,t} = r(e_{u,t}; \theta_0(u)), \\
e_{u,t} = \varphi(\nu_t; \theta_0(u)).
\]  
(4)

In particular, if \(\{Y_{t,T}\}\) satisfies Definition 1, under Assumption 1, as \(T \to \infty\), we have that

\[
|Y_{t,T} - y_{u,t}| = O_p\left(\left|\frac{t}{T} - u\right| + T^{-1}\right),
\]  
(5)

To see this, from the triangle inequality we have

\[
|Y_{t,T} - y_{u,t}| \leq |Y_{t,T} - y_{t/T,t}| + |y_{t/T,t} - y_{u,t}| \leq O_p(T^{-1}) + |y_{t/T,t} - y_{u,t}|
\]

where the \(O_p(T^{-1})\) term follows from Definition 1. Now, consider \(|y_{t/T,t} - y_{u,t}|\) and expand \(y_{t/T,t}\), via (4), in a neighborhood of \(u_0\):

\[
y_{t/T,t} = y_{u_0,t} + \left[\frac{t}{T} - u_0\right] \frac{\partial y_{u_0,t}}{\partial u}
\]  
\[+ \frac{1}{2} \left[\frac{t}{T} - u\right]^2 \frac{\partial^2 y_{u_0,t}}{\partial u^2}
\]

\[
= y_{u_0,t} + \left[\frac{t}{T} - u_0\right] \left[\frac{\partial r}{\partial \theta} + \frac{\partial r}{\partial \varphi} \frac{\partial \varphi}{\partial \theta} \frac{\partial \theta_0}{\partial u}\right]
\]  
\[+ O_p\left(\left|\frac{t}{T} - u\right|\right).
\]

From Assumption 1, in particular the (uniform) bounded second-derivatives of \(r(u), \varphi(u), \theta_0(u)\) with respect to \(u\), it follows that \(|y_{t/T,t} - y_{u,t}| = O_p\left(\left|\frac{t}{T} - u\right|\right)\). We then have that

\[
|Y_{t,T} - y_{u,t}| \leq O_p(T^{-1}) + O_p\left(\left|\frac{t}{T} - u\right| + T^{-1}\right) = O_p\left(\left|\frac{t}{T} - u\right| + T^{-1}\right).
\]

Equation (5) implies that in the neighborhood of a re-scaled time point \(u \in [0, 1]\), statistical analysis can be conducted on the family \(\{y_{u,t}, u \in [0, 1]\}\) in a sense that the local moments from the distribution of \(\{Y_{t,T}\}\) can be approximated by those from the distribution of \(\{y_{u,t}\}\).

Under local stationarity, we will demonstrate that estimation of the unknown (vector) function \(\theta_0(\cdot)\) in (3) can proceed through a local version of indirect inference (L-II) that is conducted at the time points \(u = t/T\). This approach relies on the fact that, for any \(u \in [0, 1]\), \(\theta_0(u)\) in (4) satisfies \(\theta_0(u) \in \Theta \subseteq \mathbb{R}^d\);
i.e., in the locally stationary structural model we can view the function \( \theta_0 \) as \( \theta_0 : [0,1] \rightarrow \Theta \). In this way, the model is defined with the observed data \( \{Y_{i,T}\}_{t=1,\ldots,T;T=1,2,\ldots} \) while our statistical analysis is based on the collection of locally stationary processes with \( \{y_{u,t}\} \) where \( u \in [0,1] \). The assumption that \( \theta_0(\cdot) \) is our only parameter of interest is without loss of generality as we may always redefine \( \theta_0(\cdot) \) to include those time-varying elements of the distribution for \( \nu_t \). This paper is particularly concerned with estimation and inference for the model (3) when the model rules out direct estimation approaches developed in the existing literature, for instance due in large part to the computational difficulty or lack of a closed-form estimator or likelihood function.

The key to our L-II approach is the local stationarity of the process \( \{y_{u,t}\}_{1 \leq t \leq T} \). For fixed \( u, \theta(u) \in \Theta \subset \mathbb{R}^{d_{\theta}} \), the series \( \{y_{u,t}\}_{1 \leq t \leq T} \) can be simulated according to equation (4). Denote this simulated sample, based on \( \tilde{\theta} := \bar{\theta}(u) \in \Theta \), as \( \{\tilde{y}_{u,t}\} \)

\[
\begin{align*}
\tilde{y}_{u,t} &= r(\tilde{\epsilon}_{u,t}; \tilde{\theta}) \equiv r(\tilde{\epsilon}_{u,t}; \bar{\theta}(u)), \\
\tilde{\epsilon}_{u,t} &= \varphi(\tilde{\nu}_t; \tilde{\theta}) \equiv \varphi(\tilde{\nu}_t; \bar{\theta}(u)),
\end{align*}
\]

where \( \tilde{\nu}_t \) denotes a simulated realization of the random variable \( \nu_t \).

### 2.2 Auxiliary Models and Direct Estimation

To employ our L-II estimation method, we consider an auxiliary model defined by the unknown (vector) function \( \rho(\cdot) \), such that for any \( u \in [0,1] \), we have \( \rho(u) \in \Gamma \subset \mathbb{R}^{d_{\rho}} \), with \( d_{\rho} \geq d_{\theta} \). Reflecting the features of the true structural model, the auxiliary model is chosen such that it allows for direct estimation based on the observations \( \{Y_{i,T}\} \). That is, the auxiliary parameter is estimated by minimizing some sample objective function,

\[
\hat{\rho}(u; \theta_0(u)) = \arg \min_{\rho \in \Gamma} M_T[\{Y_{i,T}\}; \rho(u)],
\]

where \( M_T \) is defined by an auxiliary model. Throughout the remainder, when no confusion will result, we denote \( \hat{\rho}(u; \theta_0(u)) \) by \( \hat{\rho}(u) \); i.e., we drop \( \theta_0(u) \) from the representation of the estimator in (7). We further specialize the sample criterion, \( M_T \), by considering it is of a local form: for some kernel function, \( K(\cdot) \) and a bandwidth parameter, \( h \),

\[
M_T[\cdot; \rho(u)] := \frac{1}{Th} \sum_{t=1}^{T} g[\cdot; \rho(u)] K\left( \frac{u - t/T}{h} \right).
\]

It is natural to consider an auxiliary model that is well-behaved and allows for simple estimation of the auxiliary parameters. One such useful class of auxiliary models will be nonlinear regression models of the following form:

\[
Y_{i,T} = f(Z_{i,T}; \rho(t/T)) + \eta_t,
\]

where \( f(\cdot) \) is known and

\[
F := \{ f : |f(x, \rho) - f(x, \rho^*)| \leq b(x)|\rho - \rho^*|, \; \rho \in \Gamma \subset \mathbb{R}^{d_{\rho}} \},
\]

\(^1\)The use of slightly misspecified simulators in II is not uncommon, see, e.g., Dridi et al. (2007), Altonji et al. (2013), Bruins et al. (2015), and Frazier et al. (2018) for examples of misspecified simulators in the context of II estimation. In this sense, we follow the above papers in that the version of the structural model used to simulate data is a (locally) misspecified version of the true DGP.
and where $Z_{t,T}$ is a triangular array of variables that are measurable at time $t$ and exogenous with respect to the error term $\eta_t$. Under this specific nonlinear regression models with the least squares criterion, the sample criterion function, $M_T(\cdot)$ is given as

$$M_T[Y_{i,T}; f(Z_{i,T}, \rho(u))] := \frac{1}{Th} \sum_{t=1}^{T} g[Y_{i,T}; f(Z_{i,T}, \rho(u))] K\left(\frac{u - t/T}{h}\right)$$

$$\equiv \frac{1}{Th} \sum_{t=1}^{T} \left(Y_{i,T} - f(Z_{i,T}, \rho(u))\right)^2 K\left(\frac{u - t/T}{h}\right).$$

It is worth noting that the auxiliary model we consider is much more general than nonlinear regression models. Furthermore, instead of the least squares criterion discussed above, we could consider as our auxiliary criterion function a local quadratic distance criterion or a quadratic local log-likelihood function. However, to ensure our theory can capture situations where the auxiliary model has a nonlinear regression structure, throughout the remainder we further specialize the structure of the auxiliary criterion function $M_T$ as follows. For some kernel function, $K(\cdot)$ and a bandwidth parameter, $h$, some known function $f(\cdot)$ and observable exogenous variables $Z_{i,T}$, we assume that

$$M_T[Y_{i,T}; \rho(u)] := M_T[Y_{i,T}; f(Z_{i,T}, \rho(u))]$$

$$= \frac{1}{Th} \sum_{t=1}^{T} g[Y_{i,T}; f(Z_{i,T}, \rho(u))] K\left(\frac{u - t/T}{h}\right).$$

While direct estimation of $\rho(\cdot)$ will generally be relatively simple, the auxiliary model is a simplified version of the structural model, and in this way $\hat{\rho}(u)$ may be a poor estimate of the function of interest $\theta_0(\cdot)$. To this end, we propose to use simulation from the locally-stationary model in (4) and an II step to correctly estimate the unknown function of interest $\theta_0(\cdot)$.

2.3 Estimation of Structural Parameters

Suppose that we observe realizations $\{Y_{i,T}\}_{t=1,\ldots,T} \sim \cdots$ generated from the structural model (3), which are well-approximated, in the sense of Definition 1, by a family of stationary processes $\{y_{u, t}\}$ for $u \in [0, 1]$ as in (4). While $\hat{\rho}(\cdot)$ may be a poor estimator for $\theta_0(\cdot)$, an II approach can correct this issue. Such an II estimation approach is local in the sense that we are only able to estimate the features of $\theta_0(\cdot)$ in a neighborhood of the point $u$.

The local II (L-II) approach we propose proceeds in the same manner as standard II, namely, L-II matches observed and simulated estimates from the chosen auxiliary model to deliver an estimator of the quantity of interest. However, unlike standard II, our L-II approach repeats this process at a given set of time points $\{u_i\}_{i=1}^{m}$, where $\max_i \Delta u_i = O(T^{-1})$ and $\Delta u_i = u_i - u_{i-1}$, to uncover the shape of the unknown function $\theta_0(\cdot)$.

For some fixed $u_i$ and a corresponding candidate for $\theta_0(u_i)$, say, $\tilde{\theta} := \tilde{\theta}(u_i)$, L-II then proceeds by simulating data $\{\tilde{y}_{u_i, t}\}_{t=1,\ldots,T}$ from (4) by drawing simulated errors $\{\tilde{\nu}_t\}_{t=1,\ldots,T}$. It is important to note that, by the nature of the locally-stationary approximation from which $\{\tilde{y}_t\}_{t=1,\ldots,T}$ is simulated, the resulting series, $\{\tilde{y}_{u_i, t}\}_{t=1,\ldots,T}$ is stationary. Given $\{\tilde{y}_{u_i, t}\}_{t=1,\ldots,T}$, we can estimate a version of the auxiliary parameters using

$$\hat{\rho}(u_i; \tilde{\theta}) = \arg\min_{\rho \in \Gamma} \frac{1}{T} \sum_{t=1}^{T} g[\tilde{y}_{u_i, t}; f(\tilde{z}_{u_i, t}, \rho)],$$

(10)
which corresponds to a simulated version of the local criterion function $M_T$ based on the observed data, and in the vicinity of time point $u_i$.\(^2\)

Using \(\hat{\rho}(u_i)\) and \(\hat{\rho}(u_i; \tilde{\theta})\), the L-II estimator of \(\theta_0(u_i)\) can then be calculated, for positive definite weighting matrix \(\Omega\), as

\[
\hat{\theta}(u_i) := \arg\max_{\tilde{\theta} \in \Theta} -\|\hat{\rho}(u_i) - \hat{\rho}(u_i; \tilde{\theta})\|_\Omega.
\]

Using the simulated errors \(\{\tilde{v}_t\}_{t=1}^T\), we may repeat the above process for \(\{u_i\}_{i=1}^m\), with \(0 < u_1 < u_2 < \cdots < u_m < 1\), and \(\max_i \Delta u_i = O(T^{-1})\) to obtain an estimator of \(\theta_0(\cdot)\) at points \(t_i = u_i \cdot T\).

The key feature of the above L-II procedure is that, due to the locally-stationary nature of (4), the simulated series \(\tilde{y}_{u_i,t}\) is stationary for each $u_i$, $i = 1, \ldots, m$. In this way, at each time point $u_i$, L-II matches a nonparametric estimator against a parametric estimator. As we shall see more clearly in Section 3, a consequence of this estimation approach is that the estimator \(\hat{\theta}(u)\) will inherit the asymptotic properties of the nonparametric estimator \(\hat{\rho}(u)\).

## 3 Asymptotics

This section establishes the asymptotic properties of the L-II estimator \(\hat{\theta}(\cdot)\). We establish the convergence (in probability) of \(\hat{\theta}(u)\) to \(\theta_0(u)\) and provide the asymptotic distribution under a fairly general modeling setup.

Before presenting the details, we introduce limit quantities that will be needed for our results. Consider the limit objective function and its minimizer corresponding to sample quantities i.e. (7) and (8) such that, for \(U = [\delta, 1 - \delta]\) with small positive \(\delta = o(1)\),

\[
\rho_0(u; \theta_0(u)) = \arg\min_{\rho \in \Gamma} M_0[\rho(u)], \quad \text{where} \quad M_0[\rho(u)] = \lim_{T \to \infty} EM_T[\cdot; \rho(u)].
\]

Throughout the remainder, when no confusion will result, we denote \(\rho_0(u; \theta_0(u))\) by \(\rho_0(u)\).

The function \(\rho_0(u)\) is the minimizer of the limit map \(\rho(u) \mapsto M_0[\rho(u)]\), for all \(u \in U\), and can be interpreted as a pseudo-true (functional) value. This limit depends on features of the true distribution (in particular, local stationarity) and the true parameter of the structural model. The limit \(M_0\) will be well-defined and \(\rho_0\) unique under conditions on the data generating process given later.

Since the theory in relation to the auxiliary model is concerned with the local neighborhood of the pseudo-true parameter, we define a local neighborhood of the pseudo-true parameter \(\rho_0\) such that \(\mathcal{E} = \{\rho : \rho \in \Gamma, \|\rho - \rho_0\| \leq \varepsilon\}\) and \(\mathcal{E}^c = \{\rho : \rho \in \Gamma, \|\rho - \rho_0\| > \varepsilon\}\).

### 3.1 Consistency

Consistency of the L-II approach requires the uniform convergence (in probability) of the corresponding auxiliary estimators to some pseudo-true values uniformly in \(u \in U\)

Under regularity conditions given later, the estimator \(\hat{\rho}_0(u)\) will be a consistent estimator of \(\rho_0(u)\). Recalling that both \(\hat{\rho}(u)\) and \(\rho_0(u)\) implicitly rely on \(\theta_0(u)\), in what follows, we prove that \(\hat{\rho}(u)\) is uniformly convergent to \(\rho_0(u)\) in \(u \in U\). Likewise, we require that the simulated auxiliary estimator has a well-defined probability limit. Recalling the stationary nature of the simulated data, \(\tilde{y}_{u_i,t}\), such a requirement boils down to standard results for consistency of the estimator for the auxiliary model to the pseudo-true value;

\(^2\)Note that, similar to the notation we employ for \(\hat{\rho}(u_i)\), the notation \(\hat{\rho}(u_i; \tilde{\theta})\) is an abbreviation for \(\hat{\rho}(u_i; \tilde{\theta}(u_i))\).
where, e.g., White (1982) and White (1996). We define the pseudo-true value based on the simulated criterion as

$$\rho_0(u; \tilde{\theta}) = \arg \min_{\rho \in \Gamma} \tilde{M}_0(\rho) \equiv \arg \min_{\rho \in \Gamma} \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E g_t(\tilde{y}_{u,t}; f(\tilde{z}_{u,t}; \rho)).$$

From these definitions, consistency of $\hat{\rho}(u; \tilde{\theta})$ for $\rho_0(u; \tilde{\theta})$ can be based on a uniform law of large numbers (LLN) via stochastic equicontinuity and compactness of $\Gamma$. Regularity conditions for this are nested in conditions for the uniform convergence of $\hat{\rho}(u)$ to $\rho_0(u)$.

To demonstrate the asymptotic properties of our proposed L-II approach, we require the following regularity conditions.

**Assumption 2.** \{(Y_{i,T}, Z_{i,T}); t = 1, ..., T; T = 1, 2, ...\} are triangular arrays of locally stationary processes defined in Definition 1 and are $\phi$-mixing with its mixing coefficients $\phi(k)$ such that for all integers $0 < t < \infty$ and $k > 0$,

$$\phi(k) = \sup_{-T \leq t \leq T} \sup_{A \in \mathcal{F}_{t-k}^{T,t}, B \in \mathcal{F}_{t,T}^{\infty}, P(A) > 0} |P_T(B|A) - P_T(B)|$$

where $1 \geq \phi(0) \geq \phi(1) \geq ...$ and $\lim_{k \to \infty} \phi(k) = 0$ and $\mathcal{F}_{t,T}^{\infty}$ and $\mathcal{F}_{t,T}^{\infty+k}$ are $\sigma$-fields generated by $\{(Y_{i,T}, Z_{i,T}), i \leq t\}$ and $\{(Y_{i,T}, Z_{i,T}), i \geq t + k\}$ respectively and $\phi(k)$ converges to zero as $k \to \infty$ such that

$$\exists C < \infty : T\phi(m_T)/m_T \leq C, 1 \leq m_T \leq T, \forall T \in \mathbb{N}.$$

**Assumption 3.** (i) Let $g(Y_{i,T}; \rho(u)) = g(Y_{i,T}; f(Z_{i,T}, \rho))$ for some $f \in \mathcal{F}$. For all $u \in \mathcal{U}$, $g(\cdot)$ is uniformly bounded, twice continuously and boundedly differentiable with $\sup_{\rho \in \mathcal{E} \cup J} E|g(Y_{i,T}; f(Z_{i,T}, \rho))| < \infty$ where $\mathcal{J}$ is the set $\{\rho : \rho(u), u \in \mathcal{U}\}$.

(ii) For all $u \in \mathcal{U}$, $f(z; \rho)$ is a measurable function of $z$ for each $\rho \in \mathcal{E}$ and is twice differentiable at $\rho$ for each $z$, with $\sup_{\rho \in \mathcal{E} \cup J} E|f(z; \rho)| < \infty$. In addition, there exists a function $\tilde{f}(z)$ such that $\sup_{\rho \in \mathcal{E} \cup J} f(z; \rho) \leq \tilde{f}(z)$ with $E|\tilde{f}(z)| < \infty$.

(iii) Let $q(Y_{i,T}; f(Z_{i,T}, \rho)) = (\partial/\partial \rho)g(Y_{i,T}; f(Z_{i,T}, \rho))$. Then,

$$|q(Y_{i,T}; f(Z_{i,T}, \rho))| \leq c_q,$$

where $c_q$ is independent of $u$ and $\rho$. The function $q(\cdot)$ is differentiable for $\rho \in \mathcal{E}$ with strict monotonicity at the pseudo-true value, $\rho_0$.

(iv) For a given $u \in \mathcal{U}$, $\rho_0(u)$ is the unique zero with respect to $\rho$ such that

$$\rho \to \Psi_0(u, \rho) := E q[y_{u,t}; f(z_{u,t}, \rho)],$$

i.e. for an arbitrarily small $\varepsilon > 0$, there exists $\eta > 0$ such that $\inf_{\rho \in \mathcal{E}} M_0(\rho) - M_0(\rho_0) \geq \eta$.

(v) $\forall \varepsilon > 0, \exists a_1, a_2 > 0$

$$\sup_{u_1 \in \mathcal{U}} \sup_{u_2 \in \mathcal{U}} |E(g(y_{u_1,t}; f(z_{u_1,t}, \rho_1)) - E(g(y_{u_2,t}; f(z_{u_2,t}, \rho_2)))| \leq \varepsilon$$

is satisfied.

(vi) For any $u \in \mathcal{U}$, $\theta \mapsto \rho_0(u, \theta)$ is continuous and injective for all $\theta$ and $\Gamma \subset \mathbb{R}^{d_\theta}$ and $\Theta \subset \mathbb{R}^{d_\theta}$, with $d_\rho \geq d_\theta$, are compact.
Assumption 4. (i) The kernel $K(\cdot)$ is a positive bounded symmetric around zero function such that: (a) it is continuously differentiable up to order $r$ on $\mathbb{R}$ with $2 \leq r$; (b) it belongs to $L^2$, $\int |K(x)|dx < \infty$, $\int K(x)dx = 1$, and either $\sup_x K(x) < \infty$, $K(x) = 0$ for $|x| > L$ with $L < \infty$ or $|\partial K(x)/\partial x| \leq C$ and for some $v > 1$, $|\partial K(x)/\partial x| \leq C|x|^{-v}$ for $|x| > L$; (c) $\mu_i(K) = \int x^i K(x)dx = 0$, $i = 1, \ldots, r-1$, and: $\int x^r K(x)dx \neq 0$, $\int |x|^r |K(x)|dx < \infty$, $\lim_{||x|| \to \infty} ||x|| |K(x) = 0$; (d) $K(\cdot)$ is Lipschitz continuous, i.e. $|K(x) - K(x')| \leq C|x - x'|$ for all $x, x' \in \mathbb{R}$.

(ii) Let $h$ be the bandwidth such that as $T \to \infty$, $h \to 0$, $Th \to \infty$ and $Th/(m_T \log T) \to \infty$.

Assumption 2 states that we restrict our attention to locally stationary processes and allows us to utilize the asymptotic independence property for heterogeneous data. The decay rate of the $\phi$-mixing coefficient is quite weak. For instance, any exponential decay rate satisfies the condition. In addition, the $\phi$-mixing can be relaxed to strong-mixing if we restrict the form of $g(\cdot)$. For instance, for the regression objective function, (whether it is linear or nonlinear,) strong-mixing assumption suffices. Assumption 3 is concerned with the auxiliary model and its objective function. Assumptions 3.(i) and 3.(ii) ensure that uniform continuity of the objective function is ensured in the neighborhood of the pseudo-true value, $\rho_0$. They also ensure the existence of a well-behaved limit of the objective function due to the dominated convergence. Assumption 3.(iii) is concerned with the first order condition and the monotonicity warranted by the minimizer of the criterion. Assumption 3.(iv) is an asymptotic identification condition such that the unique minimizer of $M_0$ is well separated. Assumption 3.(v) states uniform equicontinuity for the uniform LLN. Assumptions 3.(vi) states the compact parameter spaces for the structural parameter $\theta(\cdot)$ and the corresponding auxiliary parameter $\rho(\cdot)$ respectively with injectivity between two given a fixed time point, $u$. Finally, Assumption 4 describes features of the kernel function and the bandwidth, which is standard in nonparametric kernel estimation.

Theorem 1. Under Assumptions 1-4, $\hat{\rho}(u)$ exists and is unique w.p.1. For all $u \in U$ and $\theta \in \Theta$, we have

$$\sup_{\theta \in \Theta} \sup_{u \in U} \|\hat{\rho}(u; \theta) - \rho_0(u; \theta)\| = o_p(1).$$

(12)

We note here that Theorem 1 is actually of independent interest. In particular, this is the first result, to our knowledge, on the uniform consistency of misspecified estimators in locally stationary models. Indeed, the only existing results on the consistency of estimators in a locally stationary context explicitly consider the case where the model is correctly specified. However, in this II estimation context we must explicitly treat the fact that these estimators are obtained from a misspecified auxiliary model.

As stated earlier, a potentially useful class of auxiliary models for use in L-II is the class of nonlinear regression models. Suppose that the auxiliary model can be represented by

$$Y_{l,T} = f(Z_{l,T}; \rho_0(t/T)) + \eta_l,$$

where $\eta_l$ is strictly stationary and $\phi$-mixing with $E|\eta_l| < \infty$ and independent of the explanatory variables $Z_{l,T}$.

The estimator of the auxiliary parameter, $\rho_0(u)$ is given as

$$\hat{\rho}(u; \theta_0(u)) = \arg\min_{\rho} M_T[Y_{l,T}, f(Z_{l,T}, \rho(u))],$$

(13)
where
\[
M_T[Y_t,T, f(Z_t,T, \rho(u))] := \frac{1}{Th} \sum_{t=1}^{T} (Y_t - f(Z_t,T, \rho(u)))^2 K \left( \frac{u - t/T}{h} \right) \tag{14}
\]

Under this specific choice of auxiliary model and criterion function, we have the following immediate corollary to Theorem 1.

**Corollary 1.** Under Assumptions 1-4, for \( \hat{\rho}(u; \theta) \) defined as in (13), we have
\[
\sup_{\theta \in \Theta} \sup_{u \in U} \| \hat{\rho}(u; \theta) - \rho_0(u; \theta) \| = o_p(1).
\]

From the (uniform) consistency of \( \hat{\rho}(u) \) and \( \hat{\rho}(u; \theta) \), in the general case we can deduce the following result for the L-II estimator of \( \theta_0(\cdot) \).

**Theorem 2.** Let Assumptions 1-4 be satisfied. For \( \Omega \) a positive definite weighting matrix, the estimator for \( \theta(u) \) in (3):
\[
\hat{\theta}(u) := \arg \max_{\theta \in \Theta} -\| \hat{\rho}(u) - \hat{\rho}(u; \hat{\theta}) \|_\Omega,
\]
satisfies
\[
\sup_{u \in U} \| \hat{\theta}(u) - \theta_0(u) \| = o_p(1).
\]

Theorem 2 requires, among other things, a condition guaranteeing identification of \( \theta_0(u) \) for any \( u \in U \). This requires that \( \rho_0(u; \hat{\theta}) \), for some \( \hat{\theta} \in \Theta \), be able to match \( \rho_0(u) \), for any \( u \in U \), and that this matching be unique. Recalling that \( \rho_0(u_i) = \rho_0(u_i; \theta_0(u_i)) \), identification requires \( \theta_0(u) \) be the unique solution, in \( \theta \), to
\[
\rho_0(u; \theta_0(u)) = \rho(u; \theta)
\]
for \( u \in U \). For \( \rho_0(\cdot; \theta(\cdot)) \) continuous and strictly monotonic in \( \theta \), for any \( u \), and in the case of \( d_\rho = d_\theta = 1 \), this defines \( \theta(\cdot) \) as
\[
\theta(\cdot) = \rho^{-1}(\cdot; \rho_0(\cdot; \theta_0(\cdot))).
\]

Therefore, under continuity and monotonicity of \( \rho(\cdot) \), for any \( u \in U \), \( \theta_0(\cdot) \) will be identified. Such a condition is equivalent to the injectivity conditions required by Theorem 2, which is a necessary condition required of parametric II (Gourieroux et al. 1993).

### 3.2 Asymptotic Normality

Standard II, based on parametric models, can be interpreted as attempting to match the biases from estimators based on observed and simulated data. The L-II estimator also attempts to match biases, however, due to the nonparametric nature of the problem, this matching is not as straightforward as with parametric II estimators. In particular, because the locally stationary processes of interest in this paper are semiparametric, i.e., the distributions under analysis are parametric but some of the parameters within those distributions are time-varying with unknown forms, two types of biases are in evidence: firstly, there is the bias that comes about from approximating the locally stationary process in the vicinity of a given time point using a stationary set of simulated data, and an additional bias that exists from using nonparametric estimators, based on kernel smoothing approaches, to capture the features of the true nonstationary data generating process.
This result has important implications for the interpretation of the L-II procedure: the first implication is that the L-II estimator of the structural parameters has bias of the same order as the nonparametric estimator \( \hat{\rho}(u) \); the second thing to note is that the asymptotic distribution of the simulated auxiliary estimators is degenerate when scaled by the factor \( \sqrt{T h} \), the nonparametric convergence rate of \( \hat{\rho}(u) \).

To deduce the asymptotic distribution of the L-II estimator, we employ the following high-level regularity conditions.\(^3\)

**Assumption 5.** (i) For any \( u \in U \), there exist some \( B(u) \) such that: for all \( u \in U \),
\[
\sqrt{T h}(\hat{\rho}(u) - \rho_0(u) - B(u)) \to_d N(0, V(u)),
\]
for some \( V(u) \) and \( B(u) \) such that \( 0 < \sup_{u \in U} ||V(u)|| < \infty \) and \( \sup_{u \in U} ||B(u)|| < \infty \).

(ii) For fixed \( u \), and some \( \delta > 0 \), recall \( \mathcal{E} := \{ \rho \in \Gamma : ||\rho - \rho_0|| < \delta \} \), \( \Psi_T(u, \rho) := \partial M_T \{ \{ y_{u,t} \}; \rho \} / \partial \rho \), and \( \Psi_0(u, \rho) := \lim_T E [\partial M_T \{ \{ y_{u,t} \}; \rho \} / \partial \rho] \).

The following are satisfied
1. \( \sup_{u \in U} \sup_{\rho \in \mathcal{E}} \left| \partial \Psi_T(u, \rho) / \partial \rho' - \partial \Psi_0(u, \rho) / \partial \rho' \right| = o_p(1) \)
2. \( \Psi_0(u, \rho) \) and \( \partial \Psi_0(u, \rho) / \partial \rho' \) are Lipschitz continuous in both \( u \) and \( \rho \).
3. \( \partial \Psi_0(u, \rho_0(u)) / \partial \rho' \) is invertible for all \( u \in U \).
4. \( \sup_{u \in U} ||\partial \rho(u) / \partial u|| < \infty \) and \( \sup_{u \in U} ||\partial^2 \rho(u) / \partial u^2|| < \infty \).

The following theorem gives the asymptotic distribution of the L-II estimator.

**Theorem 3.** Under Assumptions 1-5, for any \( u \in U \), the L-II estimator
\[
\hat{\theta}(u) := \arg \max_{\theta \in \Theta} -||\hat{\rho}(u) - \hat{\rho}(u; \hat{\theta})||_\Omega,
\]
satisfies
\[
\sqrt{T h} \left( \hat{\theta}(u) - \theta_0(u) - B(u) \right) \to_d \mathcal{N} \left( 0, Q^{-1}(u) W \Omega Q(u) \right),
\]
where
\[
Q = \left\{ \frac{\partial \rho_0(u, \theta)}{\partial \theta} \Omega \left| \frac{\partial \rho_0(u, \theta)}{\partial \theta} \right| \right|_{\theta_0(u)} \}
\]
\[
W = \lim_{T \to \infty} \text{Var} \left[ \sqrt{T h}(\hat{\rho}(u) - \rho_0(u) - B(u)) \right], \quad B(u) = \lim_{T \to \infty} \text{Bias} \left( \left\{ \sqrt{T h}(\hat{\rho}(u) - \rho_0(u)) \right\} \right)
\]

As is generally true of nonparametric estimation, the choice of the bandwidth \( h \) is critical for the performance of the proposed L-II estimator. However, unlike standard nonparametric estimation, the bandwidth, \( h \), affects the estimated structural parameters \( \hat{\theta}(\cdot) \) through the estimated auxiliary parameter \( \hat{\rho}(\cdot) \). Therefore, the bandwidth must be chosen with respect to the estimated auxiliary parameters. While there is as yet an incomplete theory on the choice of bandwidths in general locally stationary estimation, we note that when \( \rho_0(u) \) is estimated via nonparametric regression, the standard rule-of-thumb choice can be applied to yield reasonably accurate estimators (see, e.g., Koo and Linton, 2015). Given that the

\(^3\)We note that low-level conditions similar to those in Assumptions 1-3 could also be used in place of these high-level assumptions. However, such assumptions would require additional technicalities and would not greatly enhance the interpretation of the results.
majority of auxiliary models employed in II estimation have some regression-type basis, we believe that
the standard rule-of-thumb should yield reasonably accurate estimators of the structural parameters.

We note that our L-II approach and its asymptotic behavior differ from the “kernel-based” II approach
of Billio and Monfort (2003). In the confines of a fully parametric structural model, Billio and Monfort
(2003) generate a simulated conditional auxiliary criterion function, via kernel smoothing, which is used to
construct estimators of the auxiliary parameters. Matching auxiliary estimators based on the observed and
simulated data, then “knocks out” the bias of the nonparametric estimator in the asymptotic distribution
of the kernel II estimator. As such, in the authors parametric context, the bandwidth used in estimation
will have little impact on the behavior of the structural parameter estimates, and, hence, the researcher
has liberty to choose this tuning parameter as they see fit. However, unlike Billio and Monfort (2003), our
structural model is nonparametric and we must rely on a local simulation (and estimation) mechanism for
the structural model, in the neighborhood of the time point \( u \). This local approach is required since in
our context there is no reason to believe that a global approach will guarantee identification. As a result,
we must pay the price for nonparametric estimation, which results in a slower rate of convergence and the
introduction of nonparametric smoothing bias.

The bias term in Theorem 3 is proportional to the bias of the underlying nonparametric estimator \( \hat{\rho}(u) \). This is a direct consequence of matching a parametric estimate against a nonparametric estimate.
However, a couple of methods can be considered to alleviate this type of bias. A naive remedy could be
to employ an under-smoothed kernel estimate by adjusting the bandwidth \( h \). Since the bias from the
local approximation is not matched, under-smoothing could be used to alleviate this bias. However, this
approach is highly model specific, and may result in an increase in the estimators variance, which could
lead to an estimator with poor overall mean square-error properties. Alternatively, noting that \( \hat{\theta}(u) \)
is a consistent estimator of \( \theta_0(u) \) allows us to deduce a simple procue that can attenuate this bias and ensure
the resulting estimators are less sensitive to the choice of \( h \). Given \( \hat{\theta}(u) \), instead of simulating from the
locally stationary model (4), we could instead directly simulate from the non-stationary model (3); i.e.,
instead we simulate a locally stationary sequence of data according to the estimated function. This new
sequence can then be treated as ‘observed data, and a second round of L-II can be run using this ‘observed
data’. The result is that this second round of L-II will directly match the resulting nonparametric bias,
and yield an estimator that does not exhibit the asymptotic bias term \( B(u) \).

As is generally true of nonparametric estimators, the asymptotic distributions given in Theorem 3 will
only be an accurate reflection of the sampling properties of the estimator for relatively large samples. As
such, in cases with moderate sample sizes, we suggest the use of bootstrap techniques to conduct inference,
such as forming confidence intervals, in place of the asymptotic approximation in Theorem 3.

4 Simple Example and simulation study

In this section, we consider a simple generalization of the time-varying moving average model that allows
the roots of the moving average lag polynomial to be time-varying. After presenting the model, we
demonstrate how our L-II approach can be applied to estimate the model and present simulation results
on the effectiveness of this strategy.

4.1 MA(1) Time-Varying Parameters

We consider the semiparametric locally stationary MA(1)-process

\[
Y_{t,T} = \epsilon_t + \epsilon_{t-1}\theta_0(t/T), \text{ where } \epsilon_t \sim WN(0,1),
\]  

(15)
with \(E|\epsilon_t|^{4+\eta} < \infty\) for any arbitrarily small positive number \(\eta\), and where \(\sup_{u \in \mathcal{U}} |\theta_0(u)| < 1\).

Our goal is to estimate the unknown function \(\theta_0(\cdot)\) via our L-II approach. In doing so, we approximate (15) by a family of stationary MA(1) processes indexed by \(u \in \mathcal{U}\) where \(\mathcal{U} = [\delta, 1 - \delta]\) with some small trimming positive \(\delta = o(1)\),

\[
y_{u,t} = \epsilon_t + \epsilon_{t-1} \theta_0(u), \text{ where } \epsilon_t \sim \mathcal{WN}(0, 1), \quad \theta_0(u) \in [-1 + \delta, 1 - \delta] \ \forall u \in \mathcal{U}
\]

We specify as our auxiliary model the functional AR(1) model:

\[
y_{u,t} = \rho(u)y_{u,t-1} + \nu_t, \text{ where } \rho(u) \in [-1 + \delta, 1 - \delta].
\]

That is, for fixed \(u\) the auxiliary model is a simple AR(1) model.

Recall that, in parametric MA models, when the roots lie near the region of non-invertibility, the resulting estimators can display a loss in accuracy. Therefore, since for any fixed \(u\), the structural model is well-approximated by a parametric MA(1) model, it is likely that the same issue will be present if \(\sup_u |\theta_0(u)|\) is close to unity.

We use the above auxiliary model to present a L-II estimator of \(\theta_0(u)\).

**Algorithm 1** L-II algorithm for locally stationary MA(1) processes

1: Based on observed data and auxiliary model (17), the estimator \(\hat{\rho}(u)\) is defined as

\[
\hat{\rho}(u) = \frac{\sum_{t=1}^{T} Y_{t-1,T} Y_{t,T} K \left( \frac{u - t/T}{h} \right)}{\sum_{t=1}^{T} Y_{t-1,T}^2 K \left( \frac{u - t/T}{h} \right)},
\]

where \(K(\cdot)\) is a kernel function and \(h\) is a bandwidth parameter.

2: Based on the structural model (16), simulate \(H\) independent processes \(\{y_{u,t}^{[j]} \mid j = 1, \ldots, H\}\) corresponding to \(\{\epsilon_t^{[j]} \}_{t=1}^{T}\) for given fixed time point \(u_t\). That is, for fixed \(\tilde{\theta}\) and \(u\),

\[
y_{u,t}^{[j]} = \epsilon_t^{[j]} + \epsilon_{t-1}^{[j]} \tilde{\theta}.
\]

3: Based on the simulated data \(\{y_{u,t}^{[j]} \}_{j=1,...,H}\), obtain a set of estimators \(\{\hat{\rho}^{[j]}(u_t; \tilde{\theta}) \}_{j=1,...,H}\) defined as

\[
\hat{\rho}^{[j]}(u_t; \tilde{\theta}) = \frac{\sum_{t=1}^{T} y_{u,t-1}^{[j]}(\tilde{\theta}) y_{u,t}^{[j]}(\tilde{\theta})}{\sum_{t=1}^{T} \left( y_{u,t-1}^{[j]}(\tilde{\theta}) \right)^2}
\]

and define

\[
\hat{\rho}(u_t; \tilde{\theta}) = \frac{1}{H} \sum_{j=1}^{H} \hat{\rho}^{[j]}(u_t; \tilde{\theta}).
\]

4: Define the estimator \(\hat{\theta}(u_t)\) as the solution to the equation: \(\arg\min_{\tilde{\theta} \in \Theta} \|\hat{\rho}(u_t) - \hat{\rho}(u_t ; \tilde{\theta})\|_\Omega\) where \(\Theta = (-1, 1)\) is the parameter space for \(\theta_0(u)\).

5: Repeat the above procedure for different time points, say \(\{u_t\}_{t=1,...,m}\) to estimate the whole functional form of \(\theta_0(u)\).

In comparison with the general structure, the time-varying AR(1) auxiliary model in (17) corresponds to taking \(z_{u,t} = y_{u,t-1}\) and considering that \(g(y_{u,t}; \rho) = (y_{u,t} - \rho(u) y_{t-1})^2\). Note that it would also be
possible to consider additional lags of $y_{u,t}$ in $z_{u,t}$ to accommodate LS-MA models of higher order. It is also useful to note that under weak conditions on the error term, the process $y_{t,T}$ defined in the auxiliary model (17) is strong-mixing; see Orbe et al. (2005).

In this specific model, using the result of Corollary 1, we can deduce the consistency result in Theorem 2 to obtain the following uniform convergence of $\hat{\theta}(u)$ in the LS-MA(1) model to $\theta_0(u)$.

**Corollary 2.** Under Assumptions 1-4, if $\sup_{u \in U} |\rho_0(u)| < 1$ with uniformly bounded second-derivatives, the estimator $\hat{\theta}(u) := \arg\max_{\theta \in \Theta} -\|\hat{\rho}(u) - \hat{\rho}(u; \theta)\|$ satisfies $\sup_{u \in U} \|\hat{\theta}(u) - \theta_0(u)\| = o_p(1)$.

### 4.2 Monte Carlo Experiments

We demonstrate the usefulness of the L-II approach using a series of Monte Carlo experiments. We consider a sample size of $T = 1000$ generated according to the LS-MA(1) model

$$Y_{t,T} = \epsilon_t + \epsilon_{t-1} \theta_0(t/T), \quad \epsilon_t \sim \mathcal{N}(0, 1).$$

Data is generated according to one of three functional specifications for $\theta_0(u)$:

(a) $\theta_0(t/T) = 0.5 \cdot (t/T)^2$;

(b) $\theta_0(t/T) = 0.25 + (t/T) - (t/T)^2$;

(c) $\theta_0(t/T) = 0.5$.

For inference on $\theta_0(\cdot)$, we use Algorithm 1 with a Gaussian kernel and the rule of thumb bandwidth $h = 1.06 T^{-1/5}$. We estimate $\theta_0(\cdot)$ using the grid of points $\{u_i\}_{i=1}^{10}$, such that $u_i = i \ast 100/1000$.

We consider 5,000 replications of the above design across the three different specifications for $\theta_0(\cdot)$. The following three figures illustrate the sampling distribution, across the Monte Carlo replications for each of the three specifications.

Figure 1 demonstrates the ability of the L-II approach to obtain consistent estimators of the unknown function $\theta_0(\cdot)$ over $u \in [0.05, 0.95]$ across the three Monte Carlo designs. The bounds are truncated due to the well-known boundary bias associated with local constant nonparametric estimation. We note that, outside of these bounds, given the relatively short nature of the time series, these estimators are likely to be poorly behaved. This issue can be addressed through the use of local-linear smoothing approaches.

### 5 Multiplicative Stochastic Volatility

The use of stochastic volatility to capture the conditional heteroskedastic movements of asset returns is now commonplace in economics and finance. However, recently, several authors have suggested that volatility should be decomposed into short and long-term components (see, e.g, Engle and Rangel, 2008 and Engle et al., 2013). Such decompositions have given rise to the class of multiplicative time-varying GARCH models, e.g. Koo and Linton (2015). Such models decompose volatility into a short-run component, which is conveniently captured via a GARCH model, and a long-term component that slowly varies with larger macroeconomic factors that are captured nonparametrically.

Several approaches exist to estimate these multiplicative volatility models. Engle and Rangel (2008) and Engle et al. (2013) estimate the long-run volatility by a spline methodology, and Koo and Linton (2015)
propose a least absolute deviation type estimators based on the kernel approach. Given the heavy tailed nature of the error processes likely to be encountered in finance, the normalized least-squares and least absolute deviation type estimators are likely to yield robust estimators of these multiplicative volatility models.

While the class of multiplicative GARCH models can capture both short and long-run features, it is generally accepted that stochastic volatility models are superior to GARCH models in terms of modeling flexibility and their overall ability to capture fluctuations in short-run volatility. Given this feature, one would suspect that a multiplicative extension of the standard SV model should perform well in many cases. While such a model would be similar to multiplicative GARCH models, unlike GARCH models the introduction of the latent volatility would ensure that the aforementioned estimation are no longer feasible. In particular, implementation of such a SV model would be hindered by the fact that, unlike GARCH models, the short-term volatility process is a genuine latent process, which ensures that direct estimation approaches are infeasible in this context. However, this issue is immaterial for our L-II estimation approach since we can easily simulate the latent volatilities.

To this end, in this section we propose a new model where volatility evolves as the product of a short and long-run component: the long-run component is captured by a slowly time-varying function, and the short-run component is captured via an autoregressive SV model. In the context of simulation

Figure 1: Sampling distribution of $\hat{\theta}(u)$ in the LS-MA(1) model based on 5,000 Monte Carlo replications. The dashed dotted lines represent the 0.05 and 0.95 quantiles across the Monte Carlo replications, while the dotted line represents the 0.50 quantile. The solid line represents the respective true unknown function.
experiments, we demonstrate that our L-II approach can accurately estimate this new model. We then apply this model to analyzes the volatility of monthly returns on twenty-five Fama-French portfolios, with the results indicating that long-term volatility changes dramatically over the sample period under analysis.

Given the general nature of this paper, we leave a thorough discussion on the properties of this new SV model for future study.

5.1 Model
We now consider a non-stationary extension of the traditional stochastic volatility model. Given the observed demeaned data set \{Y_{t,T}\}_{t=0,1,...;T=1,2,...}, the model can be written in the following form:

\[
Y_{t,T} = \sqrt{\xi_{t,T}} \exp \left( h_t/2 \right) \nu_t, \\
h_{t+1} = \mu + \phi h_t + \eta_t,
\]

where \( \xi_{t,T} = \xi(t/T) \) and

\[
\begin{bmatrix} \nu_t \\ \eta_t \end{bmatrix} \sim iid \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \sigma_\eta \\ \rho \sigma_\eta & \sigma_\eta \end{bmatrix} \right),
\]

\( \rho \) is the correlation coefficient between \( \nu_t \) and \( \eta_t \). In this model, the long run trend is captured by the deterministic function \( \sqrt{\xi_{t,T}} \) whereas the short run dynamics, \( h_t \), are represented by the stochastic volatility model. Note that we implicitly assume that all the conditions for the local stationarity of \( \{Y_{t,T}\} \) are satisfied. In particular, \( \{Y_{t,T}\} \) changes smoothly over time and if it were not for the slowly time-varying long-run trend \( \xi_{t,T} \), \( \{Y_{t,T}\} \) would be stationary; i.e. \( \xi_{t,T} \) is uniformly positive and twice continuously differentiable and \( h_t \) is stationary, say \( |\phi| < 1 \).

However, directly estimating the structural model \( (18) \), and conducting statistical inference on the resulting estimates, is impossible with existing methods. Instead, we propose to conduct inference on the structural model through the following auxiliary model, namely a locally stationary multiplicative GJR-GARCH model given in the following form

\[
y_{u,t} = \sqrt{\tau(u)} \sigma_t z_t \\
\sigma_{t+1}^2 = \omega + \alpha \sigma_t^2 + \beta \left( \frac{y_{u,t}}{\sqrt{\tau(u)}} \right)^2 + \gamma \left( \frac{y_{u,t}}{\sqrt{\tau(u)}} \right)^2 I_{t-1},
\]

where \( z_t \sim iid \mathcal{N}(0,1) \) and \( I_t = 0 \) if \( y_{u,t}/\sqrt{\tau(u)} \geq 0 \), and \( I_t = 1 \) if \( y_{u,t}/\sqrt{\tau(u)} < 0 \). Again, the auxiliary model \( \{y_{u,t}\} \) is assumed to be locally stationary and related assumptions are assumed to be met. In particular, \( \tau(\cdot) \) is uniformly positive and twice continuously differentiable and \( \{\sigma_t\} \) is stationary, i.e., \( \alpha + \beta + \gamma/2 < 1 \) with positive parameters, \( \omega, \alpha, \beta \). In this setup, the parameters in the auxiliary model, \( \delta = (\tau(\cdot), \omega, \alpha, \beta, \gamma) \), are used as an attempt to estimate the parameters of interest in the structural model, \( \theta = (\xi(\cdot), \mu, \phi, \rho, \sigma_\eta) \).

As is shown in Koo and Linton (2015), the locally stationary multiplicative GARCH(1,1) can be estimated relatively easily in comparison to the estimation of the multiplicative stochastic volatility model. Note that the GARCH(1,1) model alone cannot make an appropriate auxiliary model for \( (18) \) because the GARCH(1,1) is symmetric by nature and therefore, there is no parameter that can be readily matched to the correlation coefficient, \( \rho \). Rather, GJR-GARCH(1,1) is employed so that the leverage effect modeled by \( \rho \) in the structural model is captured by asymmetry parameter \( \gamma \) in the auxiliary model.
5.1.1 Estimation procedure

Before we discuss our estimation strategy, note that $Y_{t,T}$ in (18) consists of unobservable long-run trend and short-run dynamics combined in a multiplicative way. This multiplicative relationship between two components requires an additional restriction for identification. The restriction can be imposed on either the long-run or the short-run part. This is in large part determined by the estimation strategy or the relative difficulty of estimation procedure arising from the structure of models. For instance, while Koo and Linton (2015) impose a restriction on the long-run component, Engle et al. (2013) impose a restriction on the short-run component. For our L-II, we impose a restriction on the short-run component because the L-II is applied over a finite number of fixed time points and therefore, a restriction on the long-run component is difficult to implement.

In particular, we impose the restriction that $\mu = 0$ for the structural model. Equivalently, for the auxiliary multiplicative GJR-GARCH model, we restrict $\omega = 1 - \alpha - \beta - \frac{\gamma}{2}$ such that the GJR-GARCH process has unit unconditional variance ($\frac{\omega}{1 - \alpha - \beta - \frac{\gamma}{2}} = 1$). Under this setup, we conduct our L-II as follows.

**Estimation of the auxiliary model:** Using the observations $\{Y_{t,T}\}$, we estimate the auxiliary multiplicative GJR-GARCH model à la Engle and Rangel (2008) and Koo and Linton (2015). Specifically, from (19),

$$
\log y^2_{u,t} = \log \tau(t/T) + \log \sigma_t^2 + \log z_t^2
$$

$$
\log y^2_{u,t} = \log \tau^*(t/T) + \log z_t^2
$$

where $\tau^*(t/T) = \tau(t/T)\sigma_t^2$. Then, we obtain a tentative estimate, $\log \hat{\tau}(u)$ as

$$
\log \hat{\tau}(u) = \arg\min_{\tau^* \in \mathbb{R}_+} \sum_{t=1}^{T} (\log y^2_{u,t} - \log \tau^*(u))^2 K_h(u - t/T)
$$

where $K_h(\cdot) = K(\cdot/h)/h$ with a bandwidth $h$. Once we obtain $\hat{\tau}(u)$,

$$
\hat{\tau}(u) = \frac{\hat{\tau}(u)}{\int_0^1 \hat{\tau}(u) du}
$$

with a restriction of $\int_0^1 \tau(u) du = 1$. This comes from

$$
E(\log y^2_{u,t}) = \log \tau(t/T) + E(\log \sigma_t^2)
$$

$$
= \log \tau(t/T) + C
$$

$$
= \log \tau^*(t/T)
$$

and therefore,

$$
\frac{\tau^*(u)}{\int_0^1 \tau^*(u) du} = \frac{\tau(u) \exp(C)}{\exp(C)} = \tau(u)
$$

For more details, see Koo and Linton (2015).

Note that the restriction, $\int_0^1 \tau(u) du = 1$ is not a model restriction but rather estimation restriction that can be re-normalized or reconstructed arbitrarily. Once $\hat{\tau}(u)$ is obtained, we estimate the GJR-GARCH parameters via the maximum likelihood estimation based on the following transformed data.
where $\Theta$ with $\tau$.

The estimation of the simulated structural model via the auxiliary model and L-II: Based on (18), for each given point in time, we simulate $H$ independent structural processes using an array of candidates under the restriction $\mu = 0$, $\{\hat{\theta}(u_i)\} \in \Theta_i$, where $\Theta_i$ is the parameter space $\theta(u_i)$ belongs to.

$$
\hat{y}_{u_t}^{[j]} = \sqrt{\hat{\xi}(u_i)} \exp(\hat{h}_t^{[j]} / 2) \hat{\nu}_t^{[j]},
\hat{h}_{t+1}^{[j]} = \phi \hat{h}_t^{[j]} + \hat{\eta}_t^{[j]},
$$

with $\begin{bmatrix} \hat{\nu}_t \\ \hat{\eta}_t \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \hat{\rho} \hat{\sigma}_\eta \\ \hat{\rho} \hat{\sigma}_\eta & \hat{\sigma}_\eta \end{bmatrix} \right)$. In the simulation step, we restrict $\mu = 0$ to impose unit unconditional variance for the multiplicative SV model, which is compatible with the restriction on the auxiliary model, $\omega = 1 - \alpha - \beta - \gamma / 2$.

Estimation of the simulated structural model via the auxiliary model and L-II: For a given time point $u_i$, based on the simulated data $\{\hat{y}_{u_t}^{[j]} ; j = 1, \ldots, H\}$, we obtain a set of estimators $\{\hat{\delta}^{[j]}(u_i; \theta)\}_{j=1}^H$ as follows. Note that when $\{\hat{\delta}^{[j]}(u_i; \theta)\}_{j=1}^H$ is estimated for each fixed time point, $u_i$, $\hat{\tau}(u_i)$ is an unknown constant not a function in the estimation of $\hat{\tau}(u)$. This implies that we just estimate the GJR-GARCH model based on the simulated data $\{\hat{y}_{u_t}^{[j]} ; j = 1, \ldots, H\}$, say $\{\hat{\omega}^{[j]}, \hat{\alpha}^{[j]}, \hat{\beta}^{[j]}, \hat{\gamma}^{[j]}\}_{j=1}^H$ and then, obtain, $\{\hat{\tau}^{[j]}(u_i)\}_{j=1}^H$ such that $\hat{\tau}^{[j]}(u_i) = \tau^{[j]}(u_i) = \frac{\hat{\omega}}{1 - \alpha - \beta - \gamma / 2}$ thanks to the restriction $\omega = 1 - \alpha - \beta - \gamma / 2$. Then, create a transformed or normalized data $\tilde{y}_{u_{t}} = \tilde{y}_{u_t}^{[j]} / \sqrt{\hat{\tau}^{[j]}(u_i)}$ and estimate $\{\tilde{\omega}^{[j]}, \tilde{\alpha}^{[j]}, \tilde{\beta}^{[j]}, \tilde{\gamma}^{[j]}\}_{j=1}^H$. Based on $\hat{\delta}(u_i)$ and $\{\hat{\delta}^{[j]}(u_i; \theta)\}_{j=1}^H$, we search for the best candidate for the given time point $u_i$ and define the estimator $\hat{\theta}(u_i)$ as the solution to the equation: $\arg\min_{\theta \in \Theta} \| \hat{\delta}(u_i) - \hat{\delta}(u_i; \theta) \|_2$ where $\Theta$ is the parameter space for $\theta_0(u_i)$. Repeat the above procedure for different time points, say $\{u_i\}_{i=1,\ldots,m}$ to estimate the whole functional form of $\theta_0(u)$.

Summing up, Algorithm 2 is employed for the L-II estimation of the locally stationary multiplicative stochastic volatility model.
Algorithm 2 L-II algorithm for multiplicative stochastic volatility processes

1: Using the auxiliary model (19), \( \hat{\delta}(u) \) is estimated based on the real observations where the identification restriction \( \omega = 1 - \alpha - \beta - \frac{\gamma}{2} \) is imposed. Note that \( \hat{\delta}(u) \) should depend on the true \( \theta_0(u) \). For details of estimation procedure, see the paragraph: Estimation of the auxiliary model.

2: For each fixed point in time, say \( u_i \), based on the structural model (18) with the restriction of \( \mu = 0 \), simulate \( H \) independent structural processes \( \{ \tilde{y}^{[j]}_{u_i,t}; j = 1, \ldots, H \} \) and \( \{ \tilde{h}^{[j]}_{u_i,t}; j = 1, \ldots, H \} \) corresponding to \( \tilde{\theta}(u_i) \in \Theta_i \), where \( \Theta_i \) is the parameter space \( \tilde{\theta}(u_i) \) belongs to. That is, for \( \tilde{\theta}^{[j]}(u_i) \in \Theta_i, j = 1, \ldots, H \) and \( u_i \),

\[
\begin{align*}
\tilde{y}^{[j]}_{u_i,t} &= \sqrt{\xi(u_i)} \exp(\tilde{h}^{[j]}_{u_i,t}/2) \tilde{\nu}_t^{[j]} \\
\tilde{h}^{[j]}_{t+1} &= \tilde{\phi}^{[j]} h_{t} + \tilde{\eta}_t^{[j]},
\end{align*}
\]

with

\[
\begin{bmatrix}
\tilde{\nu}_t \\
\tilde{\eta}_t
\end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \tilde{\rho} \tilde{\sigma}_\nu \\ \tilde{\rho} \tilde{\sigma}_\eta & \tilde{\sigma}_\eta \end{bmatrix} \right).
\]

3: Based on the simulated data \( \{ \tilde{y}^{[j]}_{u_i,t}; j = 1, \ldots, H \} \), using the auxiliary model (19), obtain a set of estimators \( \{ \hat{\delta}^{[j]}(u_i, \tilde{\theta}(u_i)) \} \) for given \( \tilde{\theta}(u_i) \), again, under the restriction of \( \omega = 1 - \alpha - \beta - \frac{\gamma}{2} \).

4: Based on \( \hat{\delta}(u_i) \) and \( \{ \hat{\delta}^{[j]}(u_i; \tilde{\theta}) \}^{H}_{j=1} \), we search for the best candidate for the given time point \( u_i \) and define the estimator \( \hat{\theta}(u_i) \) as the solution to the equation: \( \arg \min_{\tilde{\theta} \in \Theta_i} \| \hat{\delta}(u_i) - \delta(u_i; \tilde{\theta}) \|_\Omega \) where \( \Theta_i \) is the parameter space for \( \theta_0(u_i) \).

5: Repeat the above procedure for different time points, say \( \{ u_i \}^{i=1,\ldots,m} \) to estimate the whole set of \( \theta_0 = (\xi_0(u), \phi_0, \rho_0, \sigma_0) \).
5.1.2 Monte Carlo Experiments

Based on (18) and (19), we conduct a Monte Carlo experiment to illustrate the L-II approach in the locally stationary multiplicative stochastic volatility model. We consider a sample size of $T = 200$ observations generated from

$$Y_{t,T} = \sqrt{\xi_{t,T}} \exp\left(\frac{h_t}{2}\right)\nu_t$$

$$h_{t+1} = \mu + \phi h_t + \eta_t,$$

where $\xi_{t,T} = \xi(t/T)$ and

$$\begin{bmatrix} \nu_t \\ \eta_t \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \sigma_{\eta} \\ \rho \sigma_{\eta} & \sigma_{\eta} \end{bmatrix} \right),$$

where $\mu = 0$, $\phi = 0.2$, $\rho = -0.5$, $\sigma_{\eta} = 1$ and the long-run component $\xi_{t,T}$ is given by

$$\xi(t/T) = 0.2 \sin(0.5\pi t/T) + 0.8 \cos(0.5\pi t/T).$$

We take as our auxiliary model for this Monte Carlo experiment the LS-GJR-GARCH(1,1) auxiliary model:

$$y_{u,t} = \sqrt{\tau(u)} \sigma_t z_t$$

$$\sigma_{t+1}^2 = \omega + \alpha \sigma_t^2 + \beta \left( \frac{y_{u,t}}{\sqrt{\tau(u)}} \right)^2 + \gamma \left( \frac{y_{u,t}}{\sqrt{\tau(u)}} \right)^2 I_{t-1},$$

Like the Monte Carlo experiment for the LS-MA(1) model, we estimate the auxiliary parameter via local constant estimation with a Gaussian kernel and rule of thumb bandwidth. For the details of our estimation procedure, refer to Algorithm 2.

We consider 5000 replications of the above design across the Monte Carlo specification. Across each Monte Carlo trial we apply the LS-II approach, and record the estimated function $\hat{\xi}_{t,T}$. The estimation results for the unknown function are presented graphically in Figure 2. Similar to the results for the LS-MA(1) model, the LS-II procedure yields good estimates of the unknown function.\textsuperscript{67}

\textsuperscript{6}Similar to the previous Monte Carlo, we truncate the function estimate due to boundary bias problems associated with the local-constant smoothing approach considered in this implementation.

\textsuperscript{7}Results for the parametric components of the model are similar to those obtained for other II estimators, and are not presented for the sake of brevity.
Figure 2: Sampling distribution of the estimated function $\hat{\xi}(u)$ across the 5,000 Monte Carlo replications. The dashed dotted paths represent the 0.05 and 0.95 quantiles across the Monte Carlo replications, while the dashed path represents the 0.50 quantile. The solid line represents the true unknown function and is given by $\xi(t/T) = 0.2 \sin(0.5\pi t/T) + 0.8 \cos(0.5\pi t/T)$.

5.2 Empirical Application: LS-SV Model

Herein, we analyze the behavior of monthly returns from January 1952 until December 2018 on 25 Fama-French portfolios formed from the intersection of five portfolios on size and five portfolios on book-to-market, and where the breakpoints for the portfolios are taken from the NYSE quintiles. The monthly return series on the Fama-French portfolios covers an extremely long period of observation, and it is unlikely that these series display constant conditional covariance features over the entire sample period. In particular, while it is fairly widely accepted that these portfolios seem to display constant mean dynamics, the large fluctuations in the volatility of these series do not engender confidence that the conditional variance is constant throughout the sample period.

Firstly, we determine whether or not the series displays non-constant conditional variance. To this end, we consider ARCH testing of the demeaned returns for each of the 25 portfolios, where each test uses five lags. The resulting test statistics associated with the ARCH tests can be found in Table 1. The results overwhelmingly support the alternative hypothesis for all portfolios.

Given the long time-span over which the data is measured, we argue that it is not realistic to assume that the volatility dynamics that were present in the 1950s have persisted, unchanged, until 2018. In particular, it is likely that underlying macroeconomic factors would cause these portfolios to exhibit patterns of volatility that display both short-term and long-term fluctuations, which can not be adequately captured by a stationary volatility model. To capture the existence of any long-term volatility patterns in the data, we consider a LS-SV version of the Fama-French three factor model: for $r_{t,j}$, $j = 1, \ldots, 25$, denoting excess returns on the $j$-th portfolio, we assume that $r_{t,j}$ evolves according to

$$r_{t,j} = \alpha + \beta_1 r_{t,m} + \beta_2 \text{SMB}_t + \beta_3 \text{HML}_t + \epsilon_{t,j},$$

$$\epsilon_{t,j} = \sqrt{\xi_j(t/T)} \exp(h_{t,j}/2) v_{t,1j},$$

where $r_{t,m}$ denotes excess returns on the market factor, SMB$_t$ is the size factor, and HML$_t$ is the value factor. We model the short-term volatility component $h_{t,j}$ as a logarithmic AR(1) stochastic volatility

\footnote{The data is freely available from Kenneth French’s website.}
process with leverage effects:

$$h_{t,j} = \gamma_j h_{t-1,j} + \sigma_{v,j} v_{t,2j}, \quad \text{cov}(v_{t,1j}, v_{t,2j}) = \rho,$$

where we require that the mean of the short-term SV component be zero to ensure the scale of $\xi(\cdot)$ can be properly identified. The above LS-SV model considers that volatility is the composition of two components: a long-term volatility trend that moves slowly and is captured by $\xi_j(t/T)$, and a term, measured by $h_{t,j}$, that captures short-term fluctuations around $\xi_j(t/T)$.

Estimation and inference on the parameters in the above LS-SV model is carried out in two steps: first, we estimate the regression parameters to obtain $\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$; in the second step, the residuals

$$y_{t,j} = \left( r_{t,j} - \hat{\alpha} - \hat{\beta}_1 r_{t,m} - \hat{\beta}_2 \text{SMB}_t - \hat{\beta}_3 \text{HML}_t \right)$$

are used within the L-II algorithm for the LS-SV model, along with a LS-GJR-GARCH auxiliary model (we refer the reader to Algorithm 2 for specific implementation details). This estimation approach is carried out across all 25 portfolios, with each portfolio giving different estimates of the short and long-term volatility components. Given the nature of the above estimation approach, and the associated complications that result when calculating the standard errors, uncertainty quantification is carried out using the residual bootstrap and $B=999$ bootstrap replications.

Across the different portfolios, the results for the $\alpha$ and $\beta$ estimates are given in Table 2, while the results for the parametric SV components are given in Table 3. Focusing on the values of $\alpha, \beta$, we see that these estimated parameters are generally statistically significant and have the anticipated signs. Analyzing Table 3, we see that the short-term volatility parameters generally have statistically significant autocorrelation coefficients between 0.6 and 0.7, which indicates a moderate amount of short-term volatility persistence. The majority of the estimated values for $\sigma_v$ are between 0.6 and 1.0, indicating a relatively large level of noise in the short-term volatility process. Interestingly, none of the estimated leverage effects are statistically significant for the short-term volatility process. To ensure that this insignificance is not an artifact of the chosen auxiliary model, in Table 4 we report 99% confidence intervals for the corresponding LS-GJR-GARCH auxiliary parameter $\gamma$, which captures the impact of asymmetric news on volatility, and where the confidence intervals are calculated using QMLE sandwich form standard errors. For 24 out of the 25 portfolios, the resulting LS-GJR-GARCH asymmetry parameter is statistically insignificant at the one percent significance level. We believe that these findings provide meaningful evidence that the asymmetric volatility component often observed in returns is insignificant across the portfolios under analysis.

We present the long-term volatility component estimates for $\xi(\cdot)$ graphically in Figures 3-7 in the appendix. Similar to the parametric components of the model, the confidence bounds for these estimated functions are obtained using the residual bootstrap with $B=999$ replications. The confidence bounds are formed pointwise at each value of $u = t/T$ using the corresponding bootstrap output.

The goal of the long-term volatility component is to capture gradual changes in volatility that are potentially due to slowly varying macroeconomic factors that affect returns (see, e.g, Engle and Rangel, 2008 and Engle et al., 2013 for a detailed discussion). Given this aim, the results in Figures 3-7 are compelling as they closely align with the larger macroeconomic risk profile of returns over the sample period under analysis. In particular, during the 1950s to the early 1960s most series display relatively low volatility that is either flat or slightly increasing till the early-to-mid 1960s, with the overall trend of most series decreasing after about 1965. This overall trend is then maintained all the way through the great

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9While not reported here for brevity, this finding is further supported by the fact that the asymmetry parameter in a simple GJR-GARCH model is also insignificant for 24 out of the 25 portfolios under analysis.
moderation of the 1980s. However, after the end of the great moderation, virtually every series exhibits a significant upswing in long-term volatility. This pattern then continues and culminates around the time of the GFC in the late 2000s, after which there is another sustained decrease in long-term volatility. Perhaps worryingly, more than half of these return series now exhibit an additional steeping of long-term volatility, indicating that since around 2016 we have begun to enter a new period of long-term macroeconomic volatility.

6 Discussion

We propose a novel indirect inference estimator for locally stationary processes and thereby extend the use of indirect inference estimation to general classes of semiparametric models. As part of this study, we also propose a novel local stationary stochastic volatility (LS-SV) model. We leave two important topics for future research: the efficiency of the L-II estimator and the ensuing semiparametric efficiency bound for the class of locally stationary models considered in this paper; and the incorporation of shape restriction for nonparametric estimation within L-II procedure, which may improve efficiency, e.g. Horowitz and Lee (2017), at the cost of a more complicated estimation approach.
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A Proofs of Main Results

A.1 Proof of Theorem 1

Proof. Theorem 1 is concerned with uniform consistency of the estimator of the auxiliary model to the pseudo-true value.

\[ \sup_{u \in U} \sup_{\theta \in \Gamma} \| \hat{\theta}(u, \theta) - \theta_0(u, \theta) \| = o_p(1). \]

We break the proof down into two parts. Firstly, for a given \( \theta \in \Theta \), we show that

\[ \sup_{u \in U} \| \hat{\theta}(u; \theta) - \theta_0(u; \theta) \| = o_p(1). \] (23)

Then, for a given \( u \in \mathcal{U} \), we show that

\[ \sup_{\theta \in \Theta} \| \hat{\theta}(u; \theta) - \theta_0(u; \theta) \| = o_p(1). \] (24)

Using these results, we deduce the stated result of Theorem 1.

Part 1: Define \( \Psi_T(u, \rho(u)) = \sum_{t=1}^{T} w_t(u) q(Y_{t,T}; f(Z_{t,T}, \rho)) \) where \( w_t(u) = (Th)^{-1} K_{tu} \), where \( q(Y_{t,T}; f(Z_{t,T}, \rho)) = (\partial/\partial \rho) g(Y_{t,T}; f(Z_{t,T}, \rho)) \) and \( K_{tu} = K ((u - t/T)/h) \). By construction,

\[ \Psi_T(u, \rho(u)) = 0. \] (25)

For an arbitrarily small number \( \varepsilon > 0 \), let \( \| \hat{\rho}(u) - \rho_0(u) \| \leq \varepsilon \). Firstly, we focus on the existence of unique minimizer of \( M_T(\rho) \) or solution to (25). We consider w.l.o.g. \( D \) as a compact \( d_T \)-dimensional set in the vicinity of the origin. We divide \( D \) into \( N \) disjoint coverings of the form such that \( B_j = \{ \delta : \| \delta - \delta_j \| \leq \varepsilon_T \}; j = 1, ..., N \). Since \( D \) is compact, it can be covered by a finite number of \( B_j \)’s for \( j = 1, ..., N \) and \( N \leq c/\varepsilon_T \).

\[ \sup_{u \in U} \| \Psi_T(u, \rho_0(u) + \delta) - E\Psi_T(u, \rho_0(u) + \delta) \| \leq \sup_{u \in U} \max_{1 \leq j \leq N, D \cap B_j} \sup_{D \cap B_j} \| \Psi_T(u, \rho_0(u) + \delta_j) - \Psi_T(u, \rho_0(u) + \delta) \| + \sup_{u \in U} \max_{1 \leq j \leq N} \| E\Psi_T(u, \rho_0(u) + \delta_j) - E\Psi_T(u, \rho_0(u) + \delta) \| = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3. \]

Due to Assumption 3(iii) and Assumption 4(i),

\[ \mathcal{S}_1 \leq c_4 C \| \delta - \delta_j \| = O(r_T). \]

where \( r_T = ((m_T \log T)/Th)^{1/2} \). For \( \mathcal{S}_3 \), in a similar way,

\[ P \left( \max_{1 \leq j \leq N} \sup_{D \cap B_j} \| E\Psi_T(u, \rho_0(u) + \delta_j) - E\Psi_T(u, \rho_0(u) + \delta) \| \right) = O(r_T). \]

For \( \mathcal{S}_2 \),

\[ \mathcal{R}_T = \sup_{u \in U} \max_{1 \leq j \leq N} \| \Psi_T(u, \rho_0(u) + \delta_j) - E\Psi_T(u, \rho_0(u) + \delta) \| \]

\[ P(\mathcal{R}_T > \varepsilon) \leq \sum_{j=1}^{N} P(\sup_{u \in U} \| \Psi_T(u, \rho_0(u) + \delta_j) - E\Psi_T(u, \rho_0(u) + \delta_j) \| > \varepsilon) \] (26)

Due to Lemma 2, for some finite numbers, \( \varepsilon, N, C_1 \) and \( C_2 \),

\[ P(\mathcal{R}_T > \varepsilon) \leq NC_1 e^{-C_2 Th/m_T}. \]
Note that $e^{-CzT^h/m} < T^{-Cz}$ with $\tau \to \infty$ as $T \to \infty$. This implies that

$$\sum_{T=1}^{\infty} P(r_T R_T > \epsilon) < \infty.$$  

Combining all the above results with the Borel-Cantelli Lemma yields

$$\sup_{u \in U} \sup_{\rho \in \Gamma} P\left\{ |\Psi_T(u, \rho) - E\Psi_T(u, \rho)| \geq \varepsilon \right\} \to 0 \text{ w.p.1.} \quad (27)$$

Note that Assumption 3.(iii) and 3.(iv) and Assumption 4, for any $\delta \in \mathbb{R}^{d_p}$ which satisfies that $\Psi_T(u, \rho_0(u) + \delta) \neq 0$, $\Psi_0(u, \rho_0 + \delta) \neq 0$ so that (27) implies it with probability approaching to zero for all $u \in U$ as $T$ tends to infinity. For the uniform consistency, due to Assumption 3.(iii), the strict monotonicity of $q(\cdot)$ at the pseudo-true value, $\rho_0$ implies for $u \in U$, and for $\varepsilon$ a $d_p$ dimensional vector of ones,

$$[\Psi_0(u, \rho(u) + \varepsilon \cdot \iota)]_j < 0 < [\Psi_0(u, \rho(u) - \varepsilon \cdot \iota)]_j \text{, for } j = 1, \ldots, d_p.$$  

where $\Psi_0(u, \rho)$ is defined as in (11) and where, for $X \in \mathbb{R}^{d_p}$, $[X]_j$ denotes the $j$-th element of the vector. This implies that for all $u \in U$, as $T \to \infty$,

$$[\Psi_T(u, \rho(u) + \varepsilon \cdot \iota)]_j < 0 < [\Psi_T(u, \rho(u) - \varepsilon \cdot \iota)]_j \text{, for } j = 1, \ldots, d_p \quad (28)$$

By construction, (28) means that for all $u \in U$, w.p.1.,

$$\rho_0(u) - \varepsilon \cdot \iota < \hat{\rho}(u) < \rho_0(u) + \varepsilon \cdot \iota$$

due to (25) and $K(\cdot) > 0$ in Assumption 4. In combination with Assumption 3.(v),

$$\sup_{u \in U} \|\hat{\rho}(u) - \rho_0(u)\| \to 0 \text{ w.p.1.}$$

**Part 2:** Simplify notation by denoting $g(\rho) := g(Y_{t:T}; f(Z_{t:T}, \rho))$ and $g(\rho_0) := g(Y_{t:T}; f(Z_{t:T}, \rho_0))$ and define $p_t(\rho) = |g(\rho) - g(\rho_0)|$. Consider

$$M_T(\rho) - M_T(\rho_0)$$

$$= \frac{1}{Th} \sum_{t=1}^{T} g(Y_{t:T}; f(Z_{t:T}, \rho)) K_{ut} - \frac{1}{Th} \sum_{t=1}^{T} g(Y_{t:T}; f(Z_{t:T}, \rho_0)) K_{ut}$$

$$= E\left[ g(\rho) - g(\rho_0) \right] - E\left[ g(\rho) - g(\rho_0) \right]$$

$$+ \frac{1}{Th} \sum_{t=1}^{T} \left( [g(\rho) - g(\rho_0)] - E[g(\rho) - g(\rho_0)] \right) K_{ut}$$

$$+ \frac{1}{Th} \sum_{t=1}^{T} E\left[ g(\rho) - g(\rho_0) \right] K_{ut}$$

$$= E p_t(\rho) + E p_t \left[ \frac{1}{Th} \sum_{t=1}^{T} K_{ut} - 1 \right] + \frac{1}{Th} \sum_{t=1}^{T} [p_t(\rho) - E p_t(\rho)] K_{ut}$$

Firstly regarding $\mathcal{M}_1(\rho)$, due to Assumptions 3.(i), (ii) and (vi), with dominated convergence theorem, $\mathcal{M}_1(\rho)(= E[g(\rho) - g(\rho_0)]$ is continuous at $\rho_0(u)$, $\mathcal{M}_1(\rho)$ is nonstochastic and constant with respect to $\mathcal{E}$. For identifiability, due to Assumption 3.(iv), $|\mathcal{M}_1(\rho)| > 0$ for all $\rho \in \Gamma$ except for $\rho_0$, i.e. $|\mathcal{M}_1(\rho)| > 0$ whenever $\rho \neq \rho_0(u)$. This and
continuity of \( M_1(\rho) \) imply that \( M_1(\rho) \) is bounded away from 0 whenever \( \rho \in \mathcal{E}^c \), i.e. \( \rho \) is outside of a neighborhood of \( \rho_0(u) \). Furthermore, by compactness of \( \Gamma \) and continuity, \( \sup_{\rho \in \Gamma} |M_1(\rho)| < \infty \).

Meanwhile, with respect to \( M_2(\rho) \), we have two components. Firstly, for the first term of \( M_2 \),

\[
\sup_{\rho \in \Gamma} \left| Ep_\ell(\rho) \left[ \frac{1}{T} \sum_{t=1}^{T} K_{ut} - 1 \right] \right| \overset{P}{\to} 0
\]

since, as \( T \to \infty \), \( \frac{1}{Th} \sum_{t=1}^{T} K_{ut} \to 1 \) and \( \sup_{\rho \in \Gamma} |M_1(\rho)| < \infty \) as mentioned previously. For the second term of \( M_2 \), we need to show

\[
\sup_{\rho \in \Gamma} \left| \frac{1}{Th} \sum_{t=1}^{T} [p_t(\rho) - Ep_t(\rho)] \right| \overset{P}{\to} 0.
\]

We discuss two cases: 1) middle part 2) tail part. For some constant \( C < \infty \), let us define \( p^*_t(\rho) = p_t(\rho) \mathbb{1}(|p_t(\rho)| \leq C) \) where \( \mathbb{1}(\cdot) \) is the indicator function and \( p^{**}_t(\rho) = p_t(\rho) \mathbb{1}(|p_t(\rho)| > C) \) or \( p^{**}_t(\rho) = p_t(\rho) - p^*_t(\rho) \).

\[
\begin{align*}
E \left| \frac{1}{Th} \sum_{t=1}^{T} K_{ut} [p_t(\rho) - Ep_t(\rho)] \right| & \leq E \left| \frac{1}{Th} \sum_{t=1}^{T} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right| + E \left| \frac{1}{Th} \sum_{t=1}^{T} K_{ut} [p^{**}_t(\rho) - Ep^{**}_t(\rho)] \right|
\end{align*}
\]

For any fixed \( \rho \),

\[
E \left| \frac{1}{Th} \sum_{t=1}^{T} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right| \leq \frac{2}{Th} \sum_{t=1}^{T} |K_{ut}| E |p^{**}_t(\rho)|
\]

which can be arbitrarily small for \( C \) and \( T \) large enough irrespective of \( \rho \).

For some constant \( 0 < J < C \) such that data is selected via Kernel \((u - t/T)/h \leq J\),

\[
\begin{align*}
E \left| \frac{1}{Th} \sum_{t=1}^{T} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right| & = E \left| \frac{1}{Th} \sum_{|t-uT| \leq JTh} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right| + E \left| \frac{1}{Th} \sum_{|t-uT| > JTh} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right|
\end{align*}
\]

\[
\begin{align*}
\leq E \left| \frac{1}{Th} \sum_{|t-uT| \leq JTh} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right| + \frac{2C}{Th} \sum_{|t-uT| > JTh} |K_{ut}|
\end{align*}
\]

The second term tends to zero as \( T \to \infty \). For the first term,

\[
E \left[ \frac{1}{Th} \sum_{|t-u| \leq JTh} K_{ut} [p^*_t(\rho) - Ep^*_t(\rho)] \right]^2 \leq \frac{C^2}{T^2h^2} \sum_{|t-uT| \leq JTh} K_{ut} + \sum_{|t-uT| > JTh; s \neq t} \sum_{|s-uT| \leq JTh; s \neq t} |K_{utKst}| \phi(|t - s|)
\]

\[
= O \left( \frac{C^2}{Th} \sum_{j \leq Th} \phi(j) \right)
\]
where $\phi(\cdot)$ is the $\phi$-mixing coefficient defined as in Assumption 2. Due to Assumption 2, the term tends to zero in probability for each fixed $\rho \in \Gamma$. Consequently, $\sup_{\rho \in \Gamma} |\mathcal{M}_2[2]| \xrightarrow{P} 0$ due to compactness of $\Gamma$ and Assumption 3.(v). Finally, Assumption 1 and injectivity between $x$ and $\rho_0(u)$ and $\rho_0(u)$ leads to the following.

$$\sup_{\theta \in \Theta} \|\hat{\rho}(u; \theta) - \rho_0(u; \theta)\| = o_p(1)$$

Combining all the above results completes the proof. \hfill \Box

In what follows, we provide Lemma 2 and its proof. For the proof of Lemma 2, we need Lemma 1.

**Lemma 1.** Let $\{W_{t,T}\}$ be a triangular array such that

$$EW_{t,T} = 0$$

with $|W_{t,T}| \leq d$ and $E|W_{t,T}| \leq \delta$ and $EW_{t,T}^2 \leq D$. $\{W_{t,T}\}$ are also $\phi$-mixing and we denote $\phi(k)$ as the $\phi$-mixing coefficient such that $\phi(m) = \sum_{j=1}^{m} \phi(j)$. Let there exist an increasing sequence $m_T : T \in \mathbb{N}$ of positive integers such that

$$\exists C < \infty : T\phi(m_T)/m_T \leq C, 1 \leq m_T \leq T, \forall T \in \mathbb{N}. \quad (29)$$

Then, for any positive number $\epsilon$ and $c$, we have

$$P\left(\left\|\sum_{t=1}^{T} W_{t,T}\right\| \geq \epsilon\right) \leq c_1 \exp \left(-c\epsilon^2 c_2 T\right)$$

where $\phi(m_T) \rightarrow 0$ as $m_T \rightarrow 0$, $c_1 = 2e^{\frac{\delta^2}{2m_T}\phi(m)}$ and $c_2 = 6\delta^2[D + 4\delta \phi(m_T)]$.

**Proof of Lemma 1.** Define $S = \sum_{t=1}^{T} W_{t,T}$. Consider a number $n_0$ such that $2m(n_0 - 1) \leq T \leq 2mn_0$ with $m = m_T$. For all $j = 1, 2$ and $k = 1, \ldots, n_0$, we consider $A_{j,k} = \sum_{t=t_1}^{t_2} W_{t,T}$ where $t_1 = \inf[(2k + j - 3)m + 1, T]$ and $t_2 = \inf[t_1 + m - 1, T]$. Note that the size of block for $A_{j,k}$ is $m$. Then,

$$S = B_{1,n_0} + B_{2,n_0} \quad (30)$$

where $B_{j,0} = \sum_{t=1}^{T} A_{j,t}$ for $j = 1, 2$ with $B_{2,0} = 0$. By construction, for some constant $c$,

$$E \exp\{cS\} \leq (E \exp\{2cB_{1,n_0}\} + E \exp\{2cB_{2,n_0}\})/2. \quad (31)$$

From (30), applying (20.28) in Billingsley (1968, pp 171), we have

$$E \exp\{2cB_{j,k}\} = E \exp\{2cB_{j,k-1}\} \exp\{2cA_{j,k}\}. \leq E \exp\{2cB_{j,k-1}\} E \exp\{2cA_{j,k}\} + 2E \exp\{2cB_{j,k-1}\} \| \exp\{2cA_{j,k}\}\| \phi(m) \quad (32)$$

Setting $cmd = 1/4$ yields

$$|2cA_{j,k}| \leq 2cmd = \frac{1}{2} \quad (33)$$

This implies that since $e^x \leq 1 + x + x^2$ for $|x| \leq 1/2$,

$$\exp\{2cA_{j,k}\} \leq 1 + 2cA_{j,k} + 4c^2 A_{j,k}^2. \quad (34)$$

Moreover, from $1 + x \leq e^x$, $1 + 4c^2 EA_{j,k}^2 \leq e^{4c^2 EA_{j,k}^2}$. Combining the above two inequalities,

$$Ee^{2cA_{j,k}} \leq e^{4c^2 EA_{j,k}^2}. \quad (34)$$
From the definition of $A_{j,k}$,

$$EA_{j,k}^2 = \sum_{t=t_1}^{t_2} EW_{i,T}^2 + \sum_{t=t_1}^{t_2} \sum_{s=t_1, s \neq t}^{t_2} EW_{i,T} W_{s,T} \leq m[D + 4\delta d\hat{\phi}(m)].$$

where the inequality comes from $|EW_{i,T} W_{s,T}| \leq 2\delta d\hat{\phi}(|t - s|)$. With this and (34),

$$Ee^{2cA_{j,k}} \leq e^{4c^2 EA_{j,k}} \leq e^{4c^2 mC}$$

where $C = [D + 4\delta d\hat{\phi}(m)]$. In combination with (32) and (33), the inequality leads to

$$Ee^{2cB_{j,k}} \leq [e^{4c^2 mC} + 2e^{1/2}\phi(m)]Ee^{2cB_{j,k-1}} = e^{4c^2 mC}[1 + 2e^{1/2-4c^2 mC}\phi(m)]Ee^{2cB_{j,k-1}} \leq e^{4c^2 mC}[1 + 2e^{1/2}\phi(m)]Ee^{2cB_{j,k-1}}.$$

Iterating the same procedure yields

$$Ee^{2cB_{j,k-1}} \leq e^{4c^2 n_0mC}(1 + 2e^{1/2}\phi(m))^{n_0}$$

Recalling that $n_0$ is chosen such that $2m(n_0 - 1) \leq T \leq 2mn_0$, we set $n_0 \leq \frac{3T}{2m}$. From (31),

$$E \exp\{cS\} \leq c_1 \exp\{c_2 T\}$$

where $c_1 = [1 + 2e^{1/2}\phi(m)]^{3T} = \exp\{\frac{3T}{2m} \log(1 + 2e^{1/2}\phi(m))\} \leq \exp\{\frac{3T}{m} e^{1/2}\phi(m)\}$ and $c_2 = 6c^2[D + 4\delta d\hat{\phi}(m)]$. This is due to the fact that $\forall x \geq 0, \log(1 + x) \leq x$. Finally, due to Markov inequality,

$$P(|S| > \epsilon) \leq P(S > \epsilon) \leq e^{-c\epsilon} Ee^{c|S|} \leq 2e^{-c\epsilon} Ee^{S}.$$

This completes the proof.

\hfill $\Box$

**Lemma 2.** Under the Assumptions of Theorem 1, for some positive constants, $\epsilon, C_1$ and $C_2$,

$$P(\mathcal{R}_T > \epsilon) \leq NC_1 e^{-C_2 \epsilon Th/mT}.$$

where

$$\mathcal{R}_T = \max_{1 \leq j \leq N} \sup_{u \in \Omega} |\Psi_T(u, \rho_0(u) + \delta_j) - E\Psi_T(u, \rho_0(u) + \delta_j)|$$

with $\Psi_T(u, \rho(u)) = \sum_{t=1}^{T} w_t(u)g(Y_{t,T}: f(Z_{t,T}, \rho))$.

**Proof of Lemma 2.** Let $S(\delta_j) := \sum_{t=1}^{T} W_{i,T}(\delta_j) = \Psi_T(u, \rho(\mu) + \delta_j) - E\Psi_T(u, \rho(\mu) + \delta_j)$ where $|W_{i,T}(\delta_j)| \leq d_j$. Under the Assumptions of Theorem 1, $g(\cdot)$ is bounded and the Kernel function satisfies boundedness and Lipschitz continuity. Due to Assumption 2, there exists $m_T$ satisfying (29). Setting a constant $c$ proportional to $Th/mT$, applying Lemma 1 yields that, for some finite positive constants $C_1$ and $C_2$,

$$\sup_{\delta_j, j = 1, \ldots, N} P(|S(\delta_j)| > \epsilon) \leq \sup_{\delta_j, j = 1, \ldots, N} C_1 e^{-C_2(e^{-\kappa(d_j, \hat{\phi}(m_T)/m_T))T h/mT} \leq C_1 e^{-C_2 \epsilon Th/mT}$$

(35)

where $\kappa(d_j, \hat{\phi}(m_T)/m_T)$ is proportional to $c_2$ in Lemma 1.

Note that

$$P(\max_{1 \leq j \leq N} |S(\delta_j)| > \epsilon) \leq \sum_{1 \leq j \leq N} P(|S(\delta_j)| > \epsilon) \leq N \sup_{\delta_j, j = 1, \ldots, N} P(|S(\delta_j)| > \epsilon).$$

From (35),

$$P(\max_{1 \leq j \leq N} |S(\delta_j)| > \epsilon) \leq NC_1 e^{-C_2 \epsilon Th/mT},$$

which completes the proof.

\hfill $\Box$
A.2 Proof of Corollary 1

Proof. By construction, \( \sup_{u \in \mathcal{U} : |u - t/T| \leq T^{-1}} |f(Z_t; \rho_0(t/T)) - f(Z_t; \rho_0(u))| = O(T^{-1}) \) and therefore \( Y_{t,T} = f(Z_t, T; \rho(t/T)) + \eta_t = f(Z_t, T; \rho(u)) + \eta_t + O(T^{-1}) \). In what follows, \( O(T^{-1}) \) is suppressed.

Define \( p_t(\rho) = f(Z_t, T; \rho) - f(Z_t, T; \rho_0) \) for a given \( u \in \mathcal{U} \) where \( \rho := \rho(u) \in \Gamma \) and \( \rho_0 := \rho_0(u) \). Then, we have

\[
M_T(\rho) - M_T(\rho_0) = M_1(\rho) + M_2(\rho)
\]

where, for a given \( u \in \mathcal{U} \) such that \( |u - t/T| \leq T^{-1} \),

\[
M_1(\rho) = E \{ f(Z_t, T; \rho(u)) - f(Z_t, T; \rho_0(u)) \}^2 = E[p_t(\rho)]^2 \\
M_2(\rho) = M_1(\rho) \left( \frac{1}{Th} \sum_{t=1}^T K_{ut} - 1 \right) + \frac{1}{Th} \sum_{t=1}^T K_{ut} \left\{ p_t^2(\rho) - M_1(\rho) \right\} \\
- \frac{2}{Th} \sum_{t=1}^T K_{ut} \eta_t p_t(\rho).
\]

Once noting that the absolute summability implies the square summability, everything else is analogous to the proof of Theorem 1. This completes the proof.

A.3 Proof of Theorem 2

Proof. The proof is similar to others found in the literature on semiparametric estimation, see, e.g., Chen et al. (2003) (pg 1604), and in particular is similar to Lemma 1 in Frazier (2019) (pg 136-137).

From the definitions of \( \rho_0(u) \) and \( \rho_0(u, \cdot; \theta) \), and the injectivity and continuity of \( \rho_0(\cdot; \theta) \), for all \( \delta > 0 \), there exists some \( \epsilon > 0 \) such that, if \( \sup_u \| \theta - \theta_0(u) \| \geq \delta \), then

\[
\sup_u \{ Q_0[u, \theta_0(u)] - Q_0[u, \theta] \} \geq \epsilon.
\]

Applying this fact we see that

\[
P \left( \sup_u \| \hat{\theta}(u) - \theta_0(u) \| \geq \delta \right) \leq P \left( \sup_u \{ Q_0[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] \} \geq \epsilon \right)
\]

and the results follows if the right hand side of the above is \( o_p(1) \).

To this end, first note that, by the definitions of \( Q_T(u, \hat{\theta}) \) and \( Q_0(u, \hat{\theta}) \),

\[
\sup_{u \in \mathcal{U}} \sup_{\hat{\theta} \in \Theta} \left| Q_T(u, \hat{\theta}) - Q_0(u; \hat{\theta}) \right| = \sup_{u \in \mathcal{U}} \sup_{\hat{\theta} \in \Theta} \left| \| \hat{\theta}(u) - \hat{\theta}(u; \hat{\theta}) \| + \| \rho_0(u) - \rho_0(u; \hat{\theta}) \| \right| \\
\leq \sup_{u \in \mathcal{U}} \sup_{\hat{\theta} \in \Theta} \| \hat{\theta}(u) - \hat{\theta}(u; \hat{\theta}) \| + \| \rho_0(u) - \rho_0(u; \hat{\theta}) \| \\
\leq \sup_{u \in \mathcal{U}} \| \hat{\theta}(u) - \hat{\theta}(u) \| + \sup_{u \in \mathcal{U}} \sup_{\hat{\theta} \in \Theta} \| \rho_0(u; \hat{\theta}) - \hat{\theta}(u; \hat{\theta}) \|, 
\]

where the second inequality follows from the reverse triangle inequality and the third from the regular triangle inequality. The uniform convergence now follows from Theorem 1.

Now, we show that for any \( \tau > 0 \)

\[
\lim_{T \to \infty} P \left( \sup_{u \in \mathcal{U}} \left\{ Q_0[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] \right\} < \tau \right) = 1.
\]

From the definition of \( \hat{\theta}(u) \), for every \( u \in \mathcal{U} \),

\[
Q_T[u, \hat{\theta}(u)] \geq Q_T[u, \theta_0(u)],
\]

32
and
\[ \sup_{u \in U} \left\{ Q_T[u, \theta_0(u)] - Q_T[u, \hat{\theta}(u)] \right\} \leq 0. \] (37)

Moreover, by uniform convergence of \( Q_T[u, \theta] \) to \( Q_0[u, \theta] \) we have,
\[
\lim_{T \to \infty} P \left( \sup_{u \in U} \left\{ Q_T[u, \theta(u)] - Q_0[u, \hat{\theta}(u)] \right\} < \tau/2 \right) = 1 \quad \text{(38)}
\]
\[
\lim_{T \to \infty} P \left( \sup_{u \in U} \left\{ Q_0[u, \theta_0(u)] - Q_T[u, \theta_0(u)] \right\} < \tau/2 \right) = 1 \quad \text{(39)}
\]

Now, consider
\[
\sup_{u} \left\{ Q_0[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] \right\} = \sup_{u} \left\{ Q_0[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] + Q_T[u, \theta_0(u)] - Q_T[u, \theta_0(u)] \right\}
\leq \sup_{u} \left\{ Q_0[u, \theta_0(u)] - Q_T[u, \theta_0(u)] \right\} + \sup_{u} \left\{ Q_T[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] + Q_T[u, \theta_0(u)] - Q_T[u, \theta_0(u)] \right\}
\leq \sup_{u} \left\{ Q_0[u, \theta_0(u)] - Q_T[u, \theta_0(u)] \right\} + \sup_{u} \left\{ Q_T[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] \right\}
\]
where the last inequality comes from equation (37). Therefore, from the uniform convergence in (38) and (39),
\[
\lim_{T \to \infty} P \left( \sup_{u \in U} \left\{ Q_0[u, \theta_0(u)] - Q_0[u, \hat{\theta}(u)] \right\} < \tau \right) = 1. \quad \text{(40)}
\]
The result then follows by taking \( \tau = \epsilon \) in (36).

\[ \square \]

### A.4 Proof of Theorem 3

We break the proof down into two parts: first, we derive the asymptotic expansion of the estimating equations based on the observed estimator and derives the order of these expansions; we then use this result to deduce the stated result.

**Part 1:** By the definition of \( \hat{\rho}(u) \),
\[
0 = \frac{1}{Th} \sum_{t=1}^{T} q(Y_{t,T}, \hat{\rho}(u))K \left( \frac{u - t/T}{h} \right)
\]
\[
= \frac{1}{Th} \sum_{t=1}^{T} q(Y_{t,T}, \rho_0(u))K \left( \frac{u - t/T}{h} \right) + \frac{1}{Th} \sum_{t=1}^{T} \frac{\partial q(Y_{t,T}, \rho_0(u))}{\partial \rho}K \left( \frac{u - t/T}{h} \right) (\hat{\rho}(u) - \rho_0(u))
\]
\[
+ O_p(\|\hat{\rho}(u) - \rho_0(u)\|^2),
\]
where \( \partial q(x)/\partial x := \partial q(x)/\partial x \big|_{x=x_0} \). It can be rewritten as
\[
0 = \frac{1}{Th} \sum_{t=1}^{T} q(Y_{t,T}, \rho_0(u))K \left( \frac{u - t/T}{h} \right) + \frac{\partial \Psi_0(y_{u,t}, \rho_0(u))}{\partial \rho} (\hat{\rho}(u) - \rho_0(u)) + O_p(\|\hat{\rho}(u) - \rho_0(u)\|^2)
\]
\[
+ \left[ \frac{1}{Th} \sum_{t=1}^{T} \frac{\partial q(Y_{t,T}, \rho_0(u))}{\partial \rho}K \left( \frac{u - t/T}{h} \right) - \frac{\partial \Psi_0(y_{u,t}, \rho_0(u))}{\partial \rho} \right] (\hat{\rho}(u) - \rho_0(u))
\]
\[
\text{(41.1) 1}
\]
\[
\text{(41.2) 2}
\]
\[
\text{(41.3) 3}
\]

(41)
where \( \frac{\partial \Psi_0(y_{u,t}, \rho_0(u))}{\partial \rho} = \lim_{T \to \infty} (Th)^{-1} \sum_{t=1}^{T} E \left[ \frac{\partial q(Y_{t,T}, \rho_0(u))}{\partial \rho} \right] K((u-t/T)/h) \). Also, note that, from (3), \( Y_{t,T} \) implicitly depends on \( t/T \), i.e. \( Y_{t,T}(t/T) \).

Firstly, the term, (41.3) is of bounded variation. The convergence to zero in probability is ensured due to local stationarity of \( Y_{t,T} \) and Assumptions 2-5. To see this, for each \( u_0 = t_0/T \),

\[
\frac{1}{Th} \sum_{t=1}^{T} \frac{\partial q(Y_{t,T}, \rho_0(u))}{\partial \rho} K \left( \frac{u-t/T}{h} \right) = \frac{1}{Th} \sum_{k=-M}^{M} K \left( \frac{k}{Th} \right) Z_{k-t_0,T}
\]

where \( Z_{k-T} = \left[ \frac{\partial q(Y_{t,T}, \rho_0(u))}{\partial \rho} - E \left[ \frac{\partial q(Y_{t,T}, \rho_0(u))}{\partial \rho} \right] \right] \), and \( M = ThL \) with \( L \) being the bound of support of a Kernel function as in Assumption 4. Regarding (43.2),

\[
\left| \frac{1}{Th} \sum_{k=1}^{M} K \left( \frac{k}{Th} \right) Z_{k-t_0,T} \right| \leq \frac{C}{Th} \sup_{k \leq M} |S_k| \xrightarrow{P} 0
\]

Now, let us consider the first term in (41.1), for which we obtain, by the definition of local-stationarity of \( \{Y_{t,T}\} \),

\[
\frac{1}{Th} \sum_{t=1}^{T} q(Y_{t,T}, \rho_0(u)) K \left( \frac{u-t/T}{h} \right) = \frac{1}{Th} \sum_{t=1}^{T} q(y_{t,T}, \rho_0(u)) K \left( \frac{u-t/T}{h} \right) + O_p(T^{-1}).
\]

Moreover,

\[
\frac{1}{Th} \sum_{t=1}^{T} q(y_{t,T}, \rho_0(u)) K \left( \frac{u-t/T}{h} \right) = \frac{1}{Th} \sum_{t=1}^{T} q(y_{t,T}, \rho_0(t/T)) K \left( \frac{u-t/T}{h} \right)
\]

\[
+ \left[ \frac{1}{Th} \sum_{t=1}^{T} \left\{ q(y_{t,T}, \rho_0(u)) - q(y_{t,T}, \rho_0(t/T)) \right\} K \left( \frac{u-t/T}{h} \right) \right]
\]

\[
= \frac{1}{Th} \sum_{t=1}^{T} q(y_{t,T}, \rho_0(t/T)) K \left( \frac{u-t/T}{h} \right)
\]

\[
+ \frac{1}{Th} \sum_{t=1}^{T} \left[ \frac{\partial q(y_{t,T}, \rho_0(t/T))}{\partial \rho'} \right] (\rho_0(u) - \rho_0(t/T)) K \left( \frac{u-t/T}{h} \right) + O_p(|u-t/T|^2)
\]

\[
= A_{t/T} + B_{t/T} + \frac{1}{Th} \sum_{t=1}^{T} E \left[ \frac{\partial q(y_{t,T}, \rho_0(t/T))}{\partial \rho'} \right] (\rho_0(u) - \rho_0(t/T)) K \left( \frac{u-t/T}{h} \right)
\]

\[
= A_{t/T} + B_{t/T} + C_{t/T}
\]
where

\[ B_{i,T} = \left\{ \frac{1}{Th} \sum_{t=1}^{T} \left[ \frac{\partial q(y_{i,t},T, \rho_0(u_0(t/T)))}{\partial p'} \right] - \frac{1}{Th} \sum_{t=1}^{T} E \left[ \frac{\partial q(y_{i,T}, \rho_0(t/T))}{\partial p'} \right] \right\} (\rho_0(u) - \rho_0(t/T)) K \left[ \frac{u - t/T}{h} \right] , \]

and \( B_{i,T} = O_p(h^2T^{-1} + |u - t/T|^2) = o_p(1) \). Again, from Assumption 5, and the assumed continuity of \( \rho_0(\cdot) \), we have

\[ ||C_{i,T} - C_{i,T}|| = O_p(T^{-1}) , \]

where

\[ C_{i,T} = \frac{1}{h} \int_0^1 E \left[ \frac{\partial q(y_{i,T}(x), \rho_0(x))}{\partial p'} \right] (\rho_0(u) - \rho_0(x)) K_h[u - x] dx. \]

Using the change of variables \( v = (u - x)/h \) we again obtain

\[
C_{i/T} = E \left[ \int_{-\infty}^{\infty} \frac{\partial q(y_{lt,u}, \rho_0(u - vh))}{\partial p'} \rho_0(u) - \rho_0(u - vh) K[v] dv \right] \\
= E \left[ \frac{\partial q(y_{lt,u}, \rho_0(0))}{\partial p'} \rho_0(0) \int_{-\infty}^{\infty} K[v] dv \right] - E \left[ \frac{\partial q(y_{lt,u}, \rho_0(0))}{\partial p'} \rho_0(0) \int_{-\infty}^{\infty} vK[v] dv \right] \\
+ 2E \left[ \frac{\partial q(y_{lt,u}, \rho_0(0))}{\partial p'} \frac{\partial \rho_0(0)}{\partial u} \left( \frac{\partial \rho_0(0)}{\partial u} \right) \right] h^2 \int_{-\infty}^{\infty} v^2 K[v] dv \\
- E \left[ \frac{\partial q(y_{lt,u}, \rho_0(0))}{\partial p'} \frac{\partial^2 \rho_0(0)}{\partial u^2} h^2 \int_{-\infty}^{\infty} v^2 K[v] dv \right] \] (45)

Combining equation (45) with (44), and using the expansion in (41) we obtain:

\[ 0 = \frac{1}{Th} \sum_{t=1}^{T} q(y_{i,t}, \rho_0(u)) K \left[ \frac{u - t/T}{h} \right] = A_{i/T} + B_{i,T} + C_{i,T} \\
= A_{i/T} + O_p(h^2 + T^{-1} + |u - t/T|^2) + C_{i/T} \\
= A_{i/T} + O_p(h^2) + O_p(h^2 + T^{-1}) \]

Using this result within equation (41), and invoking Assumption 5 we obtain the result:

\[ (\hat{\rho}(u) - \rho_0(u)) = - \left\{ E \left[ \frac{\partial q(y_{lt,u}, \rho_0(0))}{\partial p'} \right] \right\}^{-1} A_{i/T} + O_p(h^2) + O_p(T^{-1}) \]

The stated result now follows by Assumption 5(i).

**Part 2:** We now use the above expansion to deduce the asymptotic distribution of the L-II estimator.

From the definition of \( \theta := \hat{\theta}(u) \),

\[ 0 = \frac{\partial \hat{\rho}(u; \hat{\theta})}{\partial \theta} \Omega(\hat{\rho}(u) - \hat{\rho}(u; \hat{\theta})) \]

Note that

\[ (\hat{\rho}(u) - \hat{\rho}(u; \hat{\theta})) = [\hat{\rho}(u) - \rho_0(u)] - [\hat{\rho}(u; \hat{\theta}) - \rho_0(u)] \]

and note that

\[ [\hat{\rho}(u; \hat{\theta}) - \rho_0(u)] = \frac{\partial \rho(u, \theta_0(u))}{\partial \theta}(\hat{\theta}(u) - \theta_0(u)) - [\rho_0(u) - \rho(u, \theta_0(u))] + O_p(T^{-1/2}) \] (46)
Using equation (46) within the FOCs, and the consistency of \( \hat{\theta}(u) \) obtained in Theorem 2, we obtain

\[
0 = \frac{\partial \rho(u, \theta_0(u))'}{\partial \theta} \Omega \left\{ (\hat{\rho}(u) - \rho_0(u)) - \frac{\partial \rho(u, \theta_0(u))}{\partial \theta} (\hat{\theta}(u) - \theta_0(u)) + [\rho_0(u) - \rho(u, \theta_0(u))] + O_p(T^{-1/2}) \right\}
\]

which implies

\[
\hat{\theta}(u) - \theta_0(u) = \left\{ \frac{\partial \rho(u, \theta_0(u))'}{\partial \theta} \Omega \frac{\partial \rho(u, \theta_0(u))}{\partial \theta} \right\}^{-1} \frac{\partial \rho(u, \theta_0(u))'}{\partial \theta} \Omega \left\{ (\hat{\rho}(u) - \rho_0(u)) + O_p(T^{-1/2}) \right\}, \tag{47}
\]

where we have used the injectivity of \( \rho(u, \theta) \) in \( \theta \).

The first part of the proposition, and Assumption 5(v) yield the stated result. It then follows directly that the optimal weighting matrix is

\[
\Omega = AV \left[ \sqrt{T} \hat{h}(\hat{\rho}(u) - \rho_0(u)) - B(u) \right]
\]
B  Figures and Tables

B.1  Figures

Figure 3: First five Fama-French portfolios: First quintile of Size, intersected with the five quintiles of book-to-market. The top most figure in the panel represents the results for the portfolio formed from the intersection of the first quintile of size, and the first quintile of book-to-market. Running from right to left, the results then correspond to the first quintile of size and the second through fifth quintiles of book-to-market. The confidence bands, CI-H and CI-L, were calculated using the residual bootstrap at each time interval \( u = t/T \) in the sample. For presentation of the results, the time span represented on the \( x \)-axis has been placed on the unit interval. The following correspondence can be used to aid interpretation of the results. The first time point in the sample corresponds to January 1952, the time point \( u = 0.5 \) corresponds to June 1985, and the time point \( u = .99 \) corresponds to December 2018.
Figure 4: Second five Fama-French portfolios: Second quintile of Size, intersected with the five quintiles of book-to-market. The figure has the same interpretation as Figure 3 but the results correspond to the second quintile of size.

Figure 5: Third five Fama-French portfolios: Third quintile of Size, intersected with the five quintiles of book-to-market. The figure has the same interpretation as Figure 3 but the results correspond to the third quintile of size.
Figure 6: Fourth five Fama-French portfolios: Fourth quintile of Size, intersected with the five quintiles of book-to-market. The figure has the same interpretation as Figure 3 but the results correspond to the fourth quintile of size.

Figure 7: Last five Fama-French portfolios: Fifth quintile of Size, intersected with the five quintiles of book-to-market. The figure has the same interpretation as Figure 3 but the results correspond to the fifth quintile of size.
### Table 1: ARCH test statistics for the 25 Fama-French portfolios, calculated using centered returns and five lags for the auxiliary regression. The corresponding $\chi_5^2(.01)$ critical value is 15.08. For each portfolio, we can reject the null at the 1% significance level. Furthermore, a similar conclusion remains at the .1% level for all but three of the 25 Fama-French portfolios. Size-$j$, and BM-$j$, $j = 1, ..., 5$, refer to the quintiles of size and book-to-market, respectively.

|        | Size-1 | Size-2 | Size-3 | Size-4 | Size-5 |
|--------|--------|--------|--------|--------|--------|
| BM-1   | 25.97  | 44.31  | 37.22  | 53.03  | 48.77  |
| BM-2   | 24.49  | 40.55  | 57.38  | 60.51  | 45.35  |
| BM-3   | 41.31  | 22.39  | 39.51  | 50.72  | 35.74  |
| BM-4   | 17.54  | 36.94  | 28.51  | 35.89  | 44.28  |
| BM-5   | 18.72  | 24.92  | 41.95  | 48.33  | 16.09  |
### Table 2: Estimated $\alpha$ and $\beta$ parameters from the modified three-factor Fama-French model.

For each column, Est denotes the point estimator and SE the standard error. Standard errors are calculated using the residual bootstrap. Standard errors are calculated using the residual bootstrap.

|       | Size-1     |       | Size-2     |       | Size-3     |       | Size-4     |       | Size-5     |       |
|-------|------------|-------|------------|-------|------------|-------|------------|-------|------------|-------|
|       | Est        | SE    | Est        | SE    | Est        | SE    | Est        | SE    | Est        | SE    |
| $\alpha$ | -0.5438    | 0.1201| -0.0562    | 0.1055| -0.0543    | 0.0768| 0.2285     | 0.0637| 0.2852     | 0.0551|
| $\beta_1$ | 1.1026     | 0.0182| 1.0027     | 0.0171| 0.9286     | 0.0129| 0.8881     | 0.0113| 0.9575     | 0.0101|
| BM-1   | $\beta_2$  | 1.3880| 0.0257     | 1.2981| 0.0254     | 1.0829| 0.0190     | 1.0541| 1.0856     | 0.0146|
|        | $\beta_3$  | -0.2198| 0.0281     | 0.0786| 0.0273     | 0.3102| 0.0196     | 0.4604| 0.0183     | 0.6905| 0.0153|
|        | $\beta_4$  | 1.1179| 0.3021     | 1.0438| 0.2648     | 1.0352| 0.2069     | 0.8360| 0.1777     | 0.5580| 0.1611|
| $\alpha$ | -0.5438    | 0.1201| -0.0562    | 0.1055| -0.0543    | 0.0768| 0.2285     | 0.0637| 0.2852     | 0.0551|
| $\beta_1$ | 1.1026     | 0.0182| 1.0027     | 0.0171| 0.9286     | 0.0129| 0.8881     | 0.0113| 0.9575     | 0.0101|
| BM-2   | $\beta_2$  | 1.0122| 0.0191     | 0.8861| 0.0187     | 0.7686| 0.0170     | 0.7221| 0.0145     | 0.8844| 0.0165|
|        | $\beta_3$  | -0.3607| 0.0205     | 0.1122| 0.0190     | 0.3684| 0.0179     | 0.5621| 0.0152     | 0.7729| 0.0176|
|        | $\beta_4$  | 1.1297| 0.2046     | 1.0988| 0.2157     | 0.8504| 0.1624     | 1.2760| 0.1589     | 1.0714| 0.1769|
| $\alpha$ | -0.1090    | 0.0716| 0.0640     | 0.0749| 0.0502     | 0.0613| 0.1633     | 0.0764| -0.0433    | 0.0549| -0.0322| 0.0629|
| $\beta_1$ | 1.0969     | 0.0120| 1.0164     | 0.0123| 0.9832     | 0.0104| 0.9802     | 0.0117| 1.0672     | 0.0141|
| BM-3   | $\beta_2$  | 0.7523| 0.0180     | 0.5531| 0.0194     | 0.4417| 0.0160     | 0.4332| 0.0181     | 0.5733| 0.0194|
|        | $\beta_3$  | -0.4256| 0.0196     | 0.1564| 0.0204     | 0.4065| 0.0175     | 0.5969| 0.0192     | 0.7952| 0.0226|
|        | $\beta_4$  | 1.1423| 0.2048     | 1.0354| 0.2059     | 0.8321| 0.1731     | 0.7456| 0.2058     | 1.2232| 0.2328|
| $\alpha$ | -0.0119    | 0.0659| 0.0595     | 0.0621| 0.0486     | 0.0779| 0.0862     | 0.0803| -0.2353    | 0.0785|
| $\beta_1$ | 1.0781     | 0.0123| 1.0415     | 0.0111| 1.0302     | 0.0122| 1.0053     | 0.0138| 1.1517     | 0.0138|
| BM-4   | $\beta_2$  | 0.4028| 0.0181     | 0.2123| 0.0180     | 0.1881| 0.0182     | 0.2234| 0.0207     | 0.2925| 0.0202|
|        | $\beta_3$  | -0.3953| 0.0191     | 0.1710| 0.0179     | 0.4054| 0.0206     | 0.5580| 0.0227     | 0.7945| 0.0220|
|        | $\beta_4$  | 1.2527| 0.1919     | 0.7168| 0.1823     | 0.8736| 0.2176     | 0.8652| 0.2188     | 1.2688| 0.2145|
| $\alpha$ | 0.1239      | 0.0494| -0.0227   | 0.0644| 0.1795     | 0.0804| -0.3252    | 0.0669| -0.1965    | 0.0990|
| $\beta_1$ | 0.9849     | 0.0093| 0.9833     | 0.0114| 0.9342     | 0.0137| 1.0327     | 0.0125| 1.1210     | 0.0152|
| BM-5   | $\beta_2$  | -0.2396| 0.0135     | -0.2036| 0.0171     | -0.2458| 0.0204     | -0.2110| 0.0185     | -0.0935| 0.0228|
|        | $\beta_3$  | -0.3649| 0.0142     | 0.0736| 0.0182     | 0.2976| 0.0210     | 0.6494| 0.0190     | 0.8387| 0.0255|
|        | $\beta_4$  | 1.0504| 0.1553     | 1.0980| 0.1962     | 0.6146| 0.2383     | 1.2650| 0.1908     | 0.9967| 0.2737|
Table 3: Short-term stochastic volatility parameter estimates across the 25 Fama-French portfolios. For each column, Est denotes the point estimator and SE the standard error. Standard errors are calculated using the residual bootstrap.

|       | Size-1 |        |        |        |        |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|       | Est.   | SE     | Est.   | SE     | Est.   | SE     | Est.   | SE     | Est.   | SE     |
| $\rho$ | 0.6111 | 0.1565 | 0.6379 | 0.1696 | 0.7392 | 0.2035 | 0.6195 | 0.2293 | 0.2012 | 0.2064 |
| BM-1  | $\sigma_v$ | 1.0222 | 0.2799 | 1.0993 | 0.2569 | 0.7398 | 0.2644 | 0.6318 | 0.2273 | 0.3776 | 0.1126 |
|       | $\tilde{\alpha}$ | 0.3850 | 0.5745 | -0.8244 | 0.7403 | 0.7727 | 0.7500 | 0.8011 | 0.7494 | -0.1141 | 0.5026 |
|       | $\rho$ | 0.5783 | 0.1585 | 0.7550 | 0.243  | 0.7641 | 0.2144 | 0.6441 | 0.2462 | 0.6450 | 0.2009 |
| BM-2  | $\sigma_v$ | 0.8726 | 0.1739 | 0.7032 | 0.2908 | 0.5016 | 0.1861 | 0.4784 | 0.1867 | 0.6740 | 0.1811 |
|       | $\tilde{\alpha}$ | 0.7949 | 0.6204 | -0.8072 | 0.773  | 0.9499 | 0.8427 | -0.9498 | 0.7699 | 0.9484 | 0.6576 |
|       | $\rho$ | 0.6240 | 0.2032 | 0.7147 | 0.232  | 0.6561 | 0.2669 | 0.7005 | 0.2493 | 0.4581 | 0.1696 |
| BM-3  | $\sigma_v$ | 0.6169 | 0.2615 | 0.7429 | 0.2817 | 0.4820 | 0.2324 | 0.6670 | 0.2683 | 0.8241 | 0.1787 |
|       | $\tilde{\alpha}$ | 0.0002 | 0.7717 | 0.8537 | 0.7931 | 0.6760 | 0.7421 | 0.8236 | 0.8209 | -0.3429 | 0.6489 |
|       | $\rho$ | 0.7021 | 0.1602 | 0.6199 | 0.2136 | 0.6860 | 0.208  | 0.5135 | 0.1594 | 0.3072 | 0.1752 |
| BM-4  | $\sigma_v$ | 0.7261 | 0.1818 | 0.6645 | 0.2319 | 0.7421 | 0.2559 | 1.0098 | 0.1985 | 0.9306 | 0.1335 |
|       | $\tilde{\alpha}$ | -0.9379 | 0.7853 | -0.8263 | 0.6644 | -0.8362 | 0.7045 | -0.8657 | 0.2639 | 0.5774 | 0.6281 |
|       | $\rho$ | 0.1009 | 0.2916 | 0.8377 | 0.3044 | 0.7673 | 0.2941 | 0.5379 | 0.196  | 0.6464 | 0.183  |
| BM-5  | $\sigma_v$ | 0.0100 | 0.1346 | 0.5029 | 0.2353 | 0.7215 | 0.403  | 0.7762 | 0.188  | 0.8946 | 0.3063 |
|       | $\tilde{\alpha}$ | 0.9499 | 0.4548 | 0.9497 | 0.7353 | -0.7100 | 0.8109 | 0.8475 | 0.7278 | 0.5857 | 0.5806 |
Table 4: 99% Confidence intervals for asymmetry parameter $\gamma$. For the entries in the table, $(x,y)$ refers to the lower and upper level of the confidence interval, respectively, as calculated using QMLE robust standard errors. Across all 25 portfolios, only a single asymmetry parameter is statistically significant, which we mark in bold text.

|     | Size-1          | Size-2          | Size-3          | Size-4          | Size-5          |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| BM-1| -0.0549, 0.1610 | -0.0622, 0.1516 | -0.0919, 0.1859 | -0.1203, 0.0712 | -0.0023, 0.1900 |
| BM-2| -0.1915, 0.0556 | -0.2716, 0.0399 | -0.0687, 0.0874 | -0.0324, 0.1940 | -0.1255, 0.0738 |
| BM-3| -0.2185, 0.0268 | -0.0597, 0.1614 | -0.0188, 0.2030 | -0.0383, 0.2178 | -0.1372, 0.0626 |
| BM-4| -0.1331, 0.0865 | -0.0403, 0.0981 | -0.0742, 0.1751 | -0.1258, 0.0755 | -0.0718, 0.1619 |
| BM-5| **0.0175, 0.1154** | -0.0762, 0.1446 | -0.0558, 0.1325 | -0.0072, 0.1698 | -0.0931, 0.1709 |
Supplementary Appendix

B.3 Proof of Corollary 2

Note that due to $\sup_{u \in U} |\theta_0(u)| < 1$ and local stationarity, there exists an invertible moving average process corresponding to the structural model (1) in the vicinity of any given time point $u \in U$. Therefore, there exists an Autoregressive process such that

$$y_{u,t} = \varepsilon_{u,t} + \sum_{s=1}^{\infty} (-\theta(u))^s y_{u,t-s},$$

in the neighborhood of a given time point $u \in U$. The auxiliary model AR(1) process is a misspecified version of the above model with a wrong order of lags. Define $M_T(\cdot)$ and $M_0(\cdot)$ as

$$M_T[\{Y_{t,T}\}_{t=0}^T; \rho(u)] := \frac{1}{T h} \sum_{t=1}^{T} (Y_{t,T} - \rho(t/T) Y_{t-1,T})^2 K \left( \frac{u - t/T}{h} \right)$$

and $M_0(\cdot) = \lim_{T \to \infty} EM_T(\cdot)$. $M_T(\cdot)$ and $M_0(\cdot)$ in this setting are well-defined and well behaved. Furthermore, for any given time point $u \in U$, the pseudo-true value, $\hat{\rho}_0(u)$ can be represented by $\theta_0(u) = \theta(u)/(1 + \theta(u)^2)$. The minimizer $\rho_0(u)$ for any $u \in \mathbb{U}$ is continuous and strictly monotonic in $\theta(u)$. All of these ensure that the map $\theta \mapsto \rho_0(u; \theta)$ is continuous and injective in $\theta$ and hence Assumptions 3.(vi) is met. Also, the assumption that $\sup_{u \in U} |\theta_0(u)| < 1$ implies the compactness of $\Theta$ and $\Gamma$ in conjunction with injectivity. With Lemma 3 that ensures Theorem 1 and Corollary 1, the proof of corollary 2 follows directly from verification of the requisite regularity conditions stated in Theorem 2.

Lemma 3. Under Assumptions 2 and 4, the following are satisfied,

$$\sup_{u \in \mathcal{U}} |\hat{\rho}(u) - \rho_0(u)| = o_p(1),$$

$$\sup_{\hat{\theta} \in [-1+\delta, 1-\delta]} |\hat{\rho}(u; \hat{\theta}(u)) - \rho_0(u; \hat{\theta}(u))| = o_p(1) \quad \text{a.s.}$$

Proof. For (48), under Assumptions 2, and 4, note that the following result follows straightforwardly from Theorem 2 in Kristensen (2009).

$$\sup_{u \in \mathcal{U}} |\hat{\rho}(u) - \rho_0(u)| = O_p(h^2) + O_p \left( \frac{\ln T}{T h} \right) = o_p(1)$$

For (49), recall that

$$\hat{\rho}(u; \hat{\theta}(u)) = \hat{\psi}_1(u; \hat{\theta}(u))/\hat{\psi}_2(u; \hat{\theta}(u)).$$

where $\hat{\psi}_1(u; \hat{\theta}(u)) = T^{-1} \sum_{t=1}^{T} \hat{y}_{t-1,u}(u; \hat{\theta}(u)) \hat{y}_{t,u}(u; \hat{\theta}(u))$ and $\hat{\psi}_2(u; \hat{\theta}(u)) = T^{-1} \sum_{t=1}^{T} \hat{y}_{t-1,u}(u; \hat{\theta}(u))$. For the sake of notation simplicity, we drop $(u)$ since it is clear that our argument is based on the fixed time point $u$.

Due to the mean value theorem,

$$\sup_{\delta \in \Theta} \left| \hat{\rho}(\hat{\theta}(u)) \right| = \sup_{\delta \in \Theta} \left| \frac{\hat{\psi}_1(\hat{\theta})}{\hat{\psi}_2(\hat{\theta})} - \frac{\psi_1(\hat{\theta})}{\psi_2(\hat{\theta})} \right|$$

$$\leq \sup_{\delta \in \Theta} \left| \frac{\hat{\psi}_1(\hat{\theta}) - \psi_1(\hat{\theta})}{\psi_2(\hat{\theta})} - \frac{\psi_1(\hat{\theta})}{\psi_2(\hat{\theta})} \left( \psi_2(\hat{\theta}) - \psi_2(\hat{\theta}) \right) \right| \leq \sup_{\delta \in \Theta} \left| \frac{\hat{\psi}_1(\hat{\theta}) - \psi_1(\hat{\theta})}{\psi_2(\hat{\theta})} \right| + \sup_{\delta \in \Theta} \left| \frac{\psi_1(\hat{\theta}) (\psi_2(\hat{\theta}) - \psi_2(\hat{\theta}))}{\psi_2(\hat{\theta})} \right|$$

44
where \( \hat{\Theta} = [-1 + \delta, 1 - \delta] \) and \( \tilde{\psi}_k(\hat{\theta}) \in [\psi_k(\hat{\theta}), \psi_k(\hat{\theta})] < \infty \) for \( k = 1, 2 \). This implies that the uniform convergence rate for the left hand side is determined by \( |\hat{\psi}_1(\hat{\theta}) - \psi_1(\hat{\theta})| \) and \( |\hat{\psi}_2(\hat{\theta}) - \psi_2(\hat{\theta})| \) only.

\[
\sup_{\hat{\theta} \in \hat{\Theta}} |\hat{\psi}_1(\hat{\theta}) - \psi_1(\hat{\theta})| \leq \sup_{\hat{\theta} \in \hat{\Theta}} |\hat{\psi}_1(\hat{\theta}) - E\hat{\psi}_1(\hat{\theta})| + \sup_{\hat{\theta} \in \hat{\Theta}} |E\hat{\psi}_1(\hat{\theta}) - \psi_1(\hat{\theta})| \tag{50}
\]

For A.2, \( o_p(1) \) by construction. For A.1, we have to show that

\[
\sup_{\hat{\theta} \in \hat{\Theta}} |\hat{\psi}_1(\hat{\theta}) - E\hat{\psi}_1(\hat{\theta})| \to 0 \quad \text{a.s.} \tag{51}
\]

The proof for (51) is organized as follows. Define \( Z_t(\hat{\theta}) = \tilde{y}_{t-1,\hat{\theta}}(\hat{\theta})\tilde{y}_{t,\hat{\theta}}(\hat{\theta}) \). We replace \( Z_t(\hat{\theta}) \) with the truncated process \( Z_t(\hat{\theta}) \mathbb{1}(|Z_t(\hat{\theta})| \leq \gamma_T) \) where \( \mathbb{1} \) is the indicator function and \( \gamma_T = \tau_T^{-1/(k-1)} \) such that \( \tau_T = \sqrt{\ln T/T} \) for some \( k > 2 \). Note that \( \tau_T = o(1) \). Then, we replace the supremum in (51) with a maximization over a finite \( N \) grids. Finally, we use the exponential inequality in Theorem 2.1. in Liebscher (1996) to bound the remainder.

First, consider truncation of \( Z_t(\hat{\theta}) \).

\[
R_T(\theta) = \hat{\psi}_1(\hat{\theta}) - \frac{1}{T} \sum_{t=1}^{T} Z_t(\hat{\theta}) \mathbb{1}(|Z_t(\hat{\theta})| \leq \gamma_T) = \frac{1}{T} \sum_{t=1}^{T} Z_t(\hat{\theta}) \mathbb{1}(|Z_t(\hat{\theta})| > \gamma_T)
\]

where \( Z_t(\hat{\theta}) = \tilde{y}_{t-1,\hat{\theta}}(\hat{\theta})\tilde{y}_{t,\hat{\theta}}(\hat{\theta}) \). Then,

\[
|E R_T(\theta)| \leq \mathbb{E} \left| \mathbb{E} \left[ Y_t(\hat{\theta}) \mathbb{I}(|Z_t(\hat{\theta})| > \gamma_T) \right] \right|
\]

\[
\leq \mathbb{E} \left[ \mathbb{E} \left[ Z_t(\hat{\theta}) \mathbb{I}(|Z_t(\hat{\theta})| > \gamma_T) \right] \right]
\]

\[
\leq \gamma_T^{-(k-1)} \mathbb{E} \left[ Z_t^{(k)}(\hat{\theta}) \right]
\]

Due to Markov’s inequality,

\[
|R_T(\hat{\theta}) - E R_T(\hat{\theta})| = O_p(\gamma_T^{-(k-1)}) = O_p(\tau_T).
\]

Therefore, we can focus on \( Z_t(\hat{\theta}) \mathbb{I}(|Z_t(\hat{\theta})| \leq \gamma_T) \) since replacing \( Z_t(\hat{\theta}) \) with \( Z_t(\hat{\theta}) \mathbb{I}(|Z_t(\hat{\theta})| \leq \gamma_T) \) incurs only an approximation error of order \( O_p(\tau_T) \), which can be made arbitrarily small. In what follows, \( |Y_t(\hat{\theta})| \leq \gamma_T \).

Next, consider a set of grids or coverings of the form such that \( B_j = \{ \hat{\theta} : ||\hat{\theta} - \hat{\theta}_j|| \leq \tau_T \} ; j = 1, ..., N \). Since \( \hat{\Theta} \) is compact, it can be covered by a finite number of \( B_j \)s for \( j = 1, ..., N \) and \( N \leq c/\tau_T \). Note that

\[
\sup_{\hat{\theta} \in \hat{\Theta}} |\hat{\psi}_1(\hat{\theta}) - E\hat{\psi}_1(\hat{\theta})| = \max_{1 \leq j \leq N} \sup_{\hat{\theta} \in \hat{\Theta} \cap B_j} |\hat{\psi}_1(\hat{\theta}) - E\hat{\psi}_1(\hat{\theta})|
\]

\[
\leq \max_{1 \leq j \leq N} \sup_{\hat{\theta} \in \hat{\Theta} \cap B_j} |\hat{\psi}_1(\hat{\theta}) - \hat{\psi}_1(\hat{\theta}_j)| + \max_{1 \leq j \leq N} |\hat{\psi}_1(\hat{\theta}_j) - E\hat{\psi}_1(\hat{\theta}_j)|
\]

\[
+ \max_{1 \leq j \leq N} \sup_{\hat{\theta} \in \hat{\Theta} \cap B_j} |E\hat{\psi}_1(\hat{\theta}_j) - \hat{\psi}_1(\hat{\theta}_j)| = S_1 + S_2 + S_3.
\]

For \( S_1 \), due to the assumption of Lipschitz condition and boundedness of the first derivative, \( \dot{Z}_t(\cdot) \),

\[
\max_{1 \leq j \leq N} \sup_{\hat{\theta} \in \hat{\Theta} \cap B_j} |\hat{\psi}_1(\hat{\theta}) - \hat{\psi}_1(\hat{\theta}_j)| \leq C \dot{Z}_t(\hat{\theta}) ||\hat{\theta} - \hat{\theta}_j|| = O_p(\tau_T) \tag{55}
\]
For $S_3$, the similar argument applies and hence

$$P \left( \max_{1 \leq j \leq N} \sup_{\hat{\theta} \in B_j} |E\hat{\psi}_1(\hat{\theta}) - E\hat{\psi}_1(\tilde{\theta})| \right) = O_p(\tau_T) \quad (56)$$

For $S_2$, let $T^{-1} \sum_{t=1}^T D_t(\hat{\theta}) = \hat{\psi}_1(\hat{\theta}) - E\hat{\psi}_1(\hat{\theta})$, i.e. $D_t(\hat{\theta}) = Z_t(\hat{\theta}) - E Z_t(\hat{\theta})$

$$P (S_2 > \tau_T) = P \left( \max_{1 \leq j \leq N} \left| \sum_{t=1}^T D_t(\hat{\theta}) \right| > T\tau_T \right)$$

$$\leq \sum_{j=1}^N P \left( \sum_{t=1}^T D_t(\hat{\theta}) \right) > T\tau_T \right)$$

$$\leq N \sup_{\delta \in \tilde{\Theta}} P \left( \sum_{t=1}^T D_t(\theta) \right) > T\tau_T \right)$$

$$\leq c\tau_T^{-1} \sup_{\delta \in \tilde{\Theta}} P \left( \sum_{t=1}^T D_t(\theta) \right) > T\tau_T \right)$$

Here, we apply the result of Theorem 2.1. in Liebscher (1996) (pg 71) on the strong convergence of sums of dependent strong mixing processes defined as follows with its mixing coefficients $\alpha(k)$ such that for $k > 0$,

$$\alpha(k) = \sup_{-T \leq t \leq T} \sup_{A \in \mathcal{F}_t^{-\infty}, B \in \mathcal{F}_t^{\infty}} |P_T(A \cap B) - P_T(A) P_T(B)|$$

where $\alpha(k)$ converges exponentially fast to zero as $k \to \infty$. For a stationary zero mean real valued process $M_t$ such that $|M_t| \leq b_T$ with strong mixing coefficients $\alpha_m$, we have

$$P \left( \left| \sum_{t=1}^T D_t \right| > \varepsilon \right) \leq 4 \exp \left[ -\frac{\varepsilon^2}{64\sigma_m^2 + \frac{\varepsilon}{3} \varepsilon b_T m} \right] + 4 \frac{T}{m} \alpha_m \quad (57)$$

where $\sigma_m^2 = E (\sum_{i=1}^m D_i)^2$. We will use this exponential inequality to prove (51). Set $m = \gamma_T^{-1} \tau_T^{-1}$ and note that $m < T$ and $m < \varepsilon b/4$ where $\varepsilon = T\tau_T$ and $b = \tau_T$ for any $\hat{\theta}$ and sufficiently large $T$. Also, note that

$$E \left( \sum_{i=1}^m D_i(\hat{\theta}) \right)^2 \leq C m.$$

From (57),

$$P \left( \left| \sum_{i=1}^T D_i(\hat{\theta}) \right| > T\tau_T \right) \leq 4 \exp \left[ -\frac{T^2 \tau_T^2}{64CT + \frac{\varepsilon}{3} \varepsilon b_T \tau_T} \right] + 4 \frac{T}{m} \alpha_m$$

$$\leq 4 \exp \left[ -\frac{\ln T}{C} \right] + 4 \frac{T}{m} \alpha_m$$

$$\leq 4 T^{-1/C} + o(1)$$

where $C$ is a constant and the second term tends to zero due to the assumption on the strong mixing coefficient, which is assumed in the statement of the result. Note that the last bound is independent of $\hat{\theta}$, it is the uniform bound. Then,

$$P (S_2 > \tau_T) \leq O_p(\tau_T^{-1} C)$$

$^{10}$Note that $\phi$-mixing in Assumption 2 implies strong mixing and hence it is consistent with Assumption 2.

46
Moreover, with sufficiently large strong mixing coefficient decay rate, $\beta$,

$$\sum_{t=1}^{\infty} P(S_2 > \tau_T) \leq \infty. \quad (58)$$

Then, the desired result follows from the Borel-Cantelli Lemma. Combining all the results, (55), (56) and (58) proves (51). The proof in relation to $\hat{\psi}_2(\theta)$ is similar and hence is omitted. This completes the proof. \qed