VECTORS SCATTERING AND BOUNDARY CONDITIONS
IN KALUZA-KLEIN TOY MODEL

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Received (22 February 2012)

We study a simple higher-dimensional toy model of electroweak symmetry breaking,
in particular a pure gauge 5D theory on flat background with one extra finite space
dimension. The principle of least action and the requirement of gauge independence
of scattering amplitudes are used to determine the possible choices of boundary condi-
tions. We demonstrate that for any of these choices the scattering amplitudes of vector
bosons do not exhibit power-like growth in the high energy limit. Our analysis is an ex-
tension and generalization of the results obtained previously by other authors.

Keywords: Higher-dimensional theory; massive vector bosons; tree-level unitarity.
PACS Nos.: 11.10.Kk, 11.15.Bt, 14.70.-e, 14.80.Rt

1. Introduction

Electroweak symmetry breaking (EWSB), i.e. the mechanism of generating the $W$
and $Z$ boson masses, is one of the most important theoretical issues of the present-
day particle physics. Several viable scenarios are available in the current literature
(for a review, see e.g. Ref. [1] and it is clear that only experiments can resolve this
long-standing puzzle. In this respect, we are in a rather fortunate situation now,
since the first preliminary results from LHC experiments are already coming and
we can expect some important hints to the nature of EWSB in the horizon of one
year.

A simple way of implementing the EWSB is the “textbook” Higgs mechanism
that leads inevitably to one or several elementary scalar bosons in the physical spec-
trum. While the obvious paradigm for such a scheme is the current standard model
(SM), there are other highly popular theories built along these lines: most notably,
models involving supersymmetry have been intensely studied during the last two
decades or so, since they alleviate the famous hierarchy problem (i.e. that of stabi-
lizing the scale of the Higgs boson mass) considered by many to be a technical flaw
of the SM. Needless to say, models with elementary scalars are most convenient from the calculational point of view, since they are perturbatively renormalizable. It also means that the tree-level scattering amplitudes are unitarized automatically in the high energy limit if the Higgs particles are not too heavy.

Taking into account the hierarchy problem, a radical alternative would be a model with no Higgs scalars at all. The oldest example of such a higgsless version of the EWSB is the technicolor and its various ramifications (cf. Ref. 6 for a review), for which the original conceptual paradigm is the chiral symmetry breaking in QCD. While such a scheme is obviously quite attractive a priori, the application of the ideas of dynamical symmetry breaking in the area of electroweak interactions runs into specific difficulties and thus remains problematic so far.

With the advent of modern applications of the higher-dimensional theories of the Kaluza-Klein (KK) type, new attempts to attack the EWSB problem have been made during the last decade and models with compact extra dimensions have thus become increasingly popular (for a review, see e.g. Refs. 7, 8). A particularly attractive scenario is EWSB via a non-trivial choice of boundary conditions. Although the underlying higher-dimensional theory is non-renormalizable, the unitarity breakdown is postponed to the cutoff scale of the effective 4D theory, which is related to the size of the extra dimension. “Bad high energy behavior” of scattering amplitudes is prevented by the exchange of KK excitations rather than through elementary scalar particles. These models thus belong to the class of higgsless theories.

It is worth noting that the higher-dimensional theory can be viewed as a limiting case of deconstructed 4D “moose” theory. Such an approach received a considerable attention including the formulation of KK equivalence theorem. Although the deconstruction formalism can successfully restore the higher-dimensional theory, it is still quite instructive to study the formulation of the effective 4D theory from higher dimensions and the scattering of vector bosons without relying on the KK equivalence theorem. This approach has been already intensively studied (see in particular Refs. 8, 13–15), but in our opinion there are still some points that need clarification, because the results presented in the literature so far are not sufficiently general and complete, even at the level of simplified toy models.

We consider a pure gauge theory on flat background with one extra finite space dimension and shortly review its construction. We examine which choices of boundary conditions are allowed by the principle of least action and the requirement of gauge independence of scattering amplitudes in a simple $SU(2)$ toy model. We demonstrate that all of the allowed choices lead to the theory with scattering amplitudes that do not exhibit power-like growth in the high energy limit. In this way, previous results of other authors are extended and generalized.

2. Gauge theories on an interval

We start with the 5D Yang-Mills Lagrangian on flat background, where the extra space dimension is restricted to a finite interval, conventionally denoted as $(0, \pi R)$. 

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This may be written as
\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{MN} F^{aMN} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} F^a_{\mu5} F^{a\mu5}. \] (1)

The \( F^a_{\mu5} F^{a\mu5} \) part contains a quadratic term mixing fields \( A^a_\mu \) and \( A^a_5 \), but we can eliminate it by adding a suitable gauge fixing term to the Lagrangian. Since the compactification procedure generally breaks 5D Lorentz invariance, we do not need to limit ourselves to 5D invariant gauge fixing terms\(^\text{14} \) and are free to choose
\[ \mathcal{L}_{\text{g.f.}} = -\frac{1}{2\xi} \left( \partial_\mu A^a_\mu - \xi \partial_5 A^a_5 \right)^2. \] (2)

Such a term is still invariant under the usual 4D Lorentz transformations and exactly cancels the cross term. Furthermore, after the KK expansion, which we perform later in the Section \(^\text{4} \), the part independent of \( A^a_5 \) agrees with the usual Lorenz-type gauge fixing term for each KK mode of \( A^a_\mu \) and the propagators of vector modes have a form known from R\(^\xi \) gauge of the Standard Model. The unitary gauge is given by the limit \( \xi \to \infty \). All massive scalar modes are unphysical; they are eliminated in the unitary gauge, playing a similar role as the would-be Goldstone bosons in the Standard Model.

3. Principle of least action

When using the variational principle of least action, we have to keep the boundary terms coming from the integration by parts in the direction of the extra space dimension. Remember that it has a finite length, so there is no reason to assume a priori that the fields (or their variations) vanish at the endpoints of the interval. One thus gets
\[
\delta S_{\text{gauge}} = \int d^4x \int_0^{\pi R} dy \left[ (\partial_M F^{aM\nu} - g_5 f^{abc} F^{bM\nu} A^c_M) \delta A^a_\nu + \right. \\
+ \left. (\partial_\mu F^{a5\mu} - g_5 f^{abc} F^{b5\mu} A^c_5) \delta A^a_5 \right] - \int d^4x \left[ F^{a5\nu} \delta A^a_\nu \right]_0^{\pi R},
\] (3)
\[
\delta S_{\text{g.f.}} = \int d^4x \int_0^{\pi R} dy \left[ \left( \frac{1}{\xi} \partial^\nu \partial^\mu A^a_\mu - \partial^\nu \partial_5 A^a_5 \right) \delta A^a_\nu + \right. \\
+ \left. \left( \xi \partial_\mu \partial_5 A^a_\mu - \partial_\mu \partial^\mu A^a_0 \right) \delta A^a_5 \right] + \int d^4x \left[ (\partial^\mu A^a_\mu - \xi \partial_5 A^a_5) \delta A^a_5 \right]_0^{\pi R}. \] (4)

We thus have the equations of motion plus some consistency conditions on the fields at the endpoints 0 and \( \pi R \). Whatever boundary conditions we impose on the gauge fields, we obviously need to ensure that the two boundary terms vanish, i.e.
\[
\left[ F^{a5\nu} \delta A^a_\nu \right]_0^{\pi R} = 0, \] (5a)
\[
\left[ (\partial^\mu A^a_\mu - \xi \partial_5 A^a_5) \delta A^a_5 \right]_0^{\pi R} = 0. \] (5b)

\(^\text{a}\)We use capital Latin letters for 5D Lorentz indices running through 0, 1, 2, 3, 5. Small Greek letters as usual stand for 4D Lorentz indices.
There are many possibilities how to satisfy (5). The least complicated way is to ensure that the expressions vanish for every gauge field at each boundary separately, in other words require that the variation itself or its coefficient is zero. Assuming that we impose the same boundary conditions for all colors of gauge fields (but we can impose different conditions at each endpoint of the interval), we have three general choices of boundary conditions, namely

\[ A^a_\nu = \text{const} \quad \text{and} \quad \partial_5 A^a_\nu = 0 , \quad (6a) \]
\[ \partial_5 A^a_\nu = 0 \quad \text{and} \quad A^a_5 = 0 , \quad (6b) \]
\[ A^a_\nu = \text{const} \quad \text{and} \quad A^a_5 = \text{const} . \quad (6c) \]

The last option simply means requiring that the variations vanish (\( \delta A^a_\mu = 0 \)) at the boundary. Later on we show that this choice leads to the theory with \( \xi \)-dependent scattering amplitudes, thus we will omit this option from our discussion for the moment.

Before proceeding with our examination of possible boundary conditions, we make several assumptions to simplify the problem a little:

- The gauge group is \( SU(2) \).
- Since we want to construct two charged bosons \( W^\pm_M = (A^1_M \mp iA^2_M) / \sqrt{2} \) with the same masses, we will always impose the same boundary conditions on the fields \( A^1_M \) and \( A^2_M \).
- We consider only the Dirichlet (\( \psi = 0 \)) and Neumann (\( \partial_5 \psi = 0 \)) boundary conditions, thus we assume that all constants in (6) are zero.

When reading Ref. [13], one could easily get an impression that we can impose an arbitrary combination of those boundary conditions at each endpoint of the interval and for each color of gauge fields. This is not quite true, because the expression (5a) mixes fields of different colors, so when imposing different boundary conditions on different colors of gauge fields, we need to be sure that this term still vanishes.

The above simplification leaves us with 16 different combinations of boundary conditions – two boundaries \( \times \) two possible boundary conditions for each of two types of bosons, but seven (almost a half) of them do not satisfy the condition (5a). Those are the cases with the condition (6b) for \( A^1_M \) and (6a) for \( A^2_M \) at the same endpoint of the interval. As an explicit example let us write down the expression (5a) for the color \( a = 1 \):

\[ [\partial^\nu A^1_\nu - \partial_5 A^{1\nu} - g_5 \left( A^2_\mu A^{2\nu} - A^3_\mu A^{3\nu} \right)] \delta A^1_\nu = g_5 A^2_\nu A^{2\nu} \delta A^1_\nu \neq 0 \quad (7) \]

4. Effective 4D Lagrangian

To get the effective 4D fields one can use the KK expansion, i.e. decompose all fields into an infinite series of eigenfunctions \( \varphi_{a,n}(y) \) of the operator \( \partial_5 \partial_5 \). The decomposition of the vector field \( A^a_\mu(x,y) \) is then given by

\[ A^a_\mu(x,y) = \sum_n A^{a,n}_\mu(x) \varphi_{a,n}(y) \quad (8) \]
and the $\varphi_{a,n}(y)$ is then called the wave function (in the extra dimension) of the mode $A_{\mu}^{a,n}(x)$. Similar decomposition can be done for scalar fields $A_5^a$ with a different set of the wave functions due to the different boundary conditions.

As long as the imposed boundary conditions keep the operator $\partial_5 \partial_5$ hermitian\footnote{One can easily check that the Dirichlet or Neumann boundary conditions indeed keep the operator hermitian.} with respect to the scalar product $\langle f, g \rangle = \int_0^{\pi R} dy f^*(y)g(y)$, we are guaranteed that its eigenfunctions satisfying

$$\varphi''_{a,n}(y) = -m_{a,n}^2 \varphi_{a,n}(y)$$

form a complete orthonormal basis. In this basis each mode $A_{\mu}^{a,n}$ of the infinite KK tower obeys 4D equation of motion $(\square_4 + m_{a,n}^2)A_{\mu}^{a,n}(x) - \partial_\mu \partial^\mu A_{a,n}^{a,n}(x) = 0$, thus it is effectively 4D vector boson with mass $m_{a,n}$.

We get the effective 4D Lagrangian by means of a simple integration over the extra space dimension. The result\footnote{For the sake of simplicity we use a shorthand notation for multi-indices $a = (a, n)$.} can be split into several parts:

$$S = \int d^4x \left( \mathcal{L}_{\text{vector free}} + \mathcal{L}_{\text{scalar free}} + \mathcal{L}_{\text{vector int}} + \mathcal{L}_{\text{scalar int}} \right).$$

\begin{align*}
\mathcal{L}_{\text{vector free}} &= \frac{1}{2} \sum_a A_\mu^a \square_4 A^{\alpha \mu} + m_{a,n}^2 A_{\mu}^{a,n} A^{a \mu} + \left(1 - \frac{1}{\xi}\right) \left(\partial_\mu A^{a \mu}\right)^2 \quad (11a) \\
\mathcal{L}_{\text{scalar free}} &= \frac{1}{2} \sum_a (\partial_\mu A_5^a)(\partial_\mu A_5^a) - \xi m_{a,n}^2 A_5^a A_5^a \quad (11b) \\
\mathcal{L}_{\text{vector int}} &= \sum_{abc} g_{abc} f^{abc} A_5^a A_5^b \partial_\mu A_5^c - \frac{1}{4} \sum_{abcd} g^{abc} f^{cde} A_5^a A_5^b A_5^c A_5^d \quad (11c) \\
\mathcal{L}_{\text{scalar int}} &= \sum_{abc} \hat{e}_{abc} f^{abc} A_5^a A_5^b A_5^c + e_{abc} f^{abc} A_5^a A_5^b \partial_\mu A_5^c + \\
&\quad + \frac{1}{2} \sum_{abcd} e_{abcd} f^{abc} A_5^a A_5^b A_5^c A_5^d \quad (11d)
\end{align*}

The result contains the free field Lagrangian for the infinite tower of vector and scalar fields – each vector mode has the Lorenz-type gauge fixing term and clearly all scalars are unphysical and are eliminated in the unitary gauge (except if there is a massless mode). Further, there is an interaction of vector bosons only, which has a well known Yang-Mills structure and an interaction that involves at least one scalar particle in every vertex. All the effective 4D couplings $g_{abc}$, $\hat{e}_{abc}$, $e_{abc}$ and $g^{2}_{abcd}$, $e^{2}_{abcd}$ are defined by an integral of wave functions. In our analysis we will
need only three of them, explicitly

\[ g_{abcd} = g_5^2 \int_0^{\pi R} dy \varphi_a \varphi_b \varphi_c \varphi_d, \quad (12a) \]

\[ g_{abc} = g_5 \int_0^{\pi R} dy \varphi_a \varphi_b \varphi_c, \quad (12b) \]

\[ \hat{e}_{abc} = \frac{g_5}{2} \int_0^{\pi R} dy \tilde{\varphi}_a (\varphi_b \varphi_c - \varphi_b \varphi_c). \quad (12c) \]

5. Gauge independence of scattering amplitudes

We require that the scattering amplitude of process \(VV \rightarrow VV\) does not depend on the gauge parameter \(\xi\). The gauge parameter is present only in the part of vector propagator that is proportional to \(q_\mu q_\nu\) and the mass of a scalar field (thus also in its propagator), so the only relevant (lowest order) diagrams are the \(s\), \(t\) and \(u\) channel exchange of a scalar or vector particle. The gauge dependent terms must cancel out in each channel separately. Let us take a look at e.g. the \(s\) channel:

\[ M_s^{vector} = \sum_\epsilon - g_{eab} f^{reb} (m_b^2 - m_a^2) \frac{1}{q^2 - m_c^2} \frac{1 - \xi}{q^2 - \xi m_c^2} g_{ecd} f^{ced} \times \]

\[ \times (m_a^2 - m_c^2) [\varepsilon(k) \cdot \varepsilon(l)] [\varepsilon(p) \cdot \varepsilon(r)] + \ldots , \quad (13) \]

\[ M_s^{scalar} = \sum_\epsilon 2i \hat{e}_{eab} f^{reb} \frac{1}{q^2 - \xi m_c^2} 2i \hat{e}_{ecd} f^{ced} [\varepsilon(k) \cdot \varepsilon(l)] [\varepsilon(p) \cdot \varepsilon(r)] . \quad (14) \]

In order to have any chance of cancellation between the corresponding modes of exchanged vector and scalar particles, we obviously need \(m_\epsilon = \tilde{m}_\epsilon\). This means that we need to impose either the same boundary conditions on \(A_\mu^a\) as on \(A_\mu^s\), or the opposite boundary conditions (meaning every Dirichlet condition imposed on a vector field implies the Neumann condition on the scalar field of the same color at the same boundary and vice versa).

One can easily check that the gauge dependent parts cancel out, if the couplings and masses satisfy the relation

\[ g_{eab} (m_b^2 - m_a^2) g_{ecd} (m_d^2 - m_c^2) = 2 \hat{e}_{eab} 2 \hat{e}_{ecd} m_\epsilon^2 , \quad (15) \]

which can be recast in terms of the integrals of wave functions as follows

\[ g_5^2 \int_0^{\pi R} dy \varphi'_e (\varphi'_a \varphi'_b - \varphi'_a \varphi'_b) \int_0^{\pi R} dz \varphi'_e (\varphi'_c \varphi'_d - \varphi'_c \varphi'_d) = \]

\[ = g_5^2 m_\epsilon^2 \int_0^{\pi R} dy \tilde{\varphi}_e (\varphi'_a \varphi'_b - \varphi'_a \varphi'_b) \int_0^{\pi R} dz \tilde{\varphi}_e (\varphi'_c \varphi'_d - \varphi'_c \varphi'_d) . \quad (16) \]

Let us examine the relation between the wave functions \(\varphi'_e\) of the vector modes and \(\tilde{\varphi}_e\) of the scalar modes. If we use the boundary condition (1b), then the functions are the same and obviously we cannot get gauge independent scattering amplitudes.
The same conclusion was also reached in Ref. 15 using a different line of argumentation, based on the requirement of consistently defined restricted class of 5D gauge transformations. On the other hand, if we use an arbitrary combination of boundary conditions (6a) and (6b), then one of the functions is sine and the other cosine with the same arguments, and the relation for every massive mode of color $e$ reads

$$m^2_e \tilde{\varphi}_e(y) \tilde{\varphi}_e(z) = \varphi'_e(y) \varphi'_e(z).$$

There could still be a problem coming from massless modes, but since we have already established that the only consistent boundary conditions are (6a) and (6b), there can be only a massless scalar, or a massless vector particle, but not both of the same color. The massless scalar is not a problem, because the gauge parameter is present only in the term $\xi m^2_{e,0} = 0$. For a massless vector boson, the relevant term is proportional to the expression (16), which contains $\varphi'_{e,0} = 0$ under both integrals. Since the wave function of a massless mode is a simple constant, this term does not contribute to the scattering amplitude at all.

The conclusion of this section is that we can impose an arbitrary combination of boundary conditions (6a) and (6b) on the gauge fields as long as it satisfies (5). This fact allows us to pass to the unitary gauge, which simplifies significantly further calculations.

6. Tree-level unitarity of $V_L V_L \rightarrow V_L V_L$ process

Let us now calculate the energy dependence of the invariant matrix element for the (generally inelastic) scattering of gauge bosons without any assumptions regarding the color or the KK mode number of the gauge bosons in the initial or final state. We consider the high energy limit and expand all quantities in powers of energy (more precisely in the powers of Mandelstam invariant $s$) keeping only the divergent parts, and show that they indeed cancel out automatically without introducing an additional Higgs field. We do not employ a hard cutoff on the spectrum of the KK modes and keep the whole infinite towers of the KK excitations. This is justified due to the fact that the contributions from the highest KK modes are suppressed in the high energy limit (for a detailed discussion see e.g. Ref. 13). The lowest order diagrams for this process involve the direct four-boson interaction, the $s$, $t$ and $u$ channel exchange of KK vector excitations and possibly the exchange of massless scalar in all channels as well.

We carry out the calculation in the center of mass reference frame. Let us denote the scattering angle by $\theta$ and for the sake of simplicity introduce a shorthand notation $4\pi^2 = m^2_a + m^2_b + m^2_c + m^2_d$.

The energy expansion of the contribution of the contact four-boson interaction

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For completeness we should also show the gauge independence of scattering amplitudes for processes with one or more massless scalars in the initial or final state, but the reasoning remains the same, only the integrals in coupling definitions are a bit different.
is given by
\[ M(4V) = \left( \frac{s}{4} \right)^2 \frac{g_{abcd}}{m_a m_b m_c m_d} \left[ f^{eab} f^{ecd} (4 \cos \theta) + f^{eac} f^{ebd} (-3 + 2 \cos \theta + \cos^2 \theta) + f^{ead} f^{ebc} (-3 - 2 \cos \theta + \cos^2 \theta) \right] + \]
\[ + \left( \frac{s}{4} \right)^4 \frac{g_{abcd}}{m_a m_b m_c m_d} \left[ f^{eab} f^{ecd} (-\cos \theta) + f^{eac} f^{ebd} \frac{1 - \cos \theta}{2} + f^{ead} f^{ebc} \frac{1 + \cos \theta}{2} \right] + O(1). \] (17)

The contributions of s, t, u channel exchange of KK vector excitations are infinite sums over all KK modes, which may be massive as well as massless. Thus, with regard to the different form of vector propagator for massive and massless modes, it is convenient to split them in two parts, namely \( M^{(s,t,u)}_{(\text{long})} \) and \( M^{(s,t,u)}_{(\text{diag})} \) corresponding to the longitudinal \( (q^\mu q^\nu) \) and the diagonal \( (g_{\mu\nu}) \) parts of the propagator respectively. Since the t and u channels differ only in the simultaneous exchange of indices \( c \) and \( d \), and sign change of \( \cos \theta \), from now on we will explicitly display only the results for the s and t channel.

Terms corresponding to the longitudinal part of boson propagator can be expanded in the powers of energy as follows:

\[ M^{(s)}_{(\text{long})} = \sum_{k>0} \frac{s}{4} \frac{g_{abcd}}{m_a m_b m_c m_d} \left[ f^{eab} f^{ecd} (m_a^2 - m_b^2) (m_c^2 - m_d^2) \frac{1}{m_e^2} \right] + O(1), \] (18a)

\[ M^{(t)}_{(\text{long})} = \sum_{k>0} \frac{s}{4} \frac{g_{abcd}}{m_a m_b m_c m_d} \left[ f^{eab} f^{ecd} \frac{1 - \cos \theta}{2} + f^{ead} f^{ebc} \frac{1 + \cos \theta}{2} \right] + O(1). \] (18b)

Similarly, after quite a long calculation, one gets the terms corresponding to the diagonal part of boson propagator in the form

\[ M^{(s)}_{(\text{diag})} = \sum_{k>0} \frac{g_{abcd}}{m_a m_b m_c m_d} \left[ \left( \frac{s}{4} \right)^2 (-4 \cos \theta) + \frac{s}{4} (-m_e^2 \cos \theta) \right] + O(1), \] (19a)

\[ M^{(t)}_{(\text{diag})} = \sum_{k>0} \frac{g_{abcd}}{m_a m_b m_c m_d} \left[ \left( \frac{s}{4} \right)^2 (3 - 2 \cos \theta - \cos^2 \theta) + \frac{s}{4} \left( m_e^2 \frac{3 + \cos \theta}{2} + 8m_e^2 \cos \theta \right) \right] + O(1). \] (19b)

The only scalar mode that can be present in the theory is massless and the cor-

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*Note that due to the structure of SU(2) group the color e of exchanged gauge boson is fixed by the colors of the bosons in the initial and final state, thus we actually sum only over the KK index k.*
responding contributions to the invariant matrix elements read

\[ M^{(s)}(\text{scalar}) = \frac{s}{4} \frac{2 \hat{e}_{eab} 2 \hat{e}_{ecd} f_{eab} f_{ecd}}{m_a m_b m_c m_d} (-1) + \mathcal{O}(1), \quad (20a) \]

\[ M^{(t)}(\text{scalar}) = \frac{s}{4} \frac{2 \hat{e}_{eac} 2 \hat{e}_{ebd} f_{eac} f_{ebd}}{m_a m_b m_c m_d} \frac{1 - \cos \theta}{2} + \mathcal{O}(1). \quad (20b) \]

Owing to the relation (15) for all massive modes \((k > 0)\) of color \(e\) these terms give us in combination with \(M^{(s,t,u)}(\text{long})\) the sum over the complete orthonormal set of functions \(\tilde{\phi}_{e,k}\) in each channel.

It is a matter of simple exercise to derive the sum rule

\[ g_{abcd} = \sum_{k \geq 0} g_{eab} g_{ecd} \]

which implies that terms growing as the fourth power of energy in the \(s\), \(t\) and \(u\) channel contributions (19) cancel against the terms from (17) corresponding to the contact four-boson interaction.

Let us present some additional sum rules that are valid for all the remaining consistent boundary conditions. The combination of contributions (18) and (20) gives rise to the sum of terms containing two couplings of the type \(\hat{e}_{eab}\). The index of KK mode is present only through these couplings, thus we can write down the first sum rule

\[ \sum_{k \geq 0} 2 \hat{e}_{eab} 2 \hat{e}_{ecd} = \frac{s}{5} \int_0^{\pi R} dy \left( \tilde{\varphi}_e \tilde{\varphi}_d \tilde{\varphi}_e \tilde{\varphi}_d \right) \cdot (\tilde{\varphi}_e \tilde{\varphi}_d \tilde{\varphi}_e \tilde{\varphi}_d). \quad (22) \]

The second type of sum contains the KK index not only in the couplings, but also in the mass of the exchanged vector mode, explicitly

\[ \sum_{k \geq 0} m_e^2 g_{eab} g_{ecd} = \frac{g^2}{5} \int_0^{\pi R} dy \left( \tilde{\varphi}_e \tilde{\varphi}_d \right) \cdot (\tilde{\varphi}_e \tilde{\varphi}_d). \quad (23) \]

In order to get all terms of the invariant matrix element in a similar form, we need one more formula, which follows directly from the relation between wave functions and masses (9) and the coupling definition (12a):

\[ 4 m^2 g_{abcd} = 2 g^2 \int_0^{\pi R} dy \left[ (\tilde{\varphi}_e \tilde{\varphi}_d) (\tilde{\varphi}_e \tilde{\varphi}_d) + \tilde{\varphi}_e \tilde{\varphi}_b \tilde{\varphi}_c \tilde{\varphi}_d + \tilde{\varphi}_e \tilde{\varphi}_b \tilde{\varphi}_d \right]. \quad (24) \]

Now we gather all the remaining divergent terms from the contact four-boson interaction (17) and \(s, t, u\) channel exchange of vector and scalar modes (18), (19) and (20), employ the derived sum rules and the relation (24). Interestingly enough, the resulting invariant matrix element for the process in question then takes on
quite a simple form

\[
\mathcal{M} = \frac{8}{4m_\alpha m_\beta m_\gamma m_\delta} \left( f^{abc}f^{cde} - f^{ace}f^{bde} + f^{ade}f^{bce} \right) \times \\
\left\{ (1 - 3 \cos \theta) \int_0^{\pi R} dy \left( \varphi'_a \varphi'_b \varphi'_c \varphi'_d + \varphi_a \varphi'_b \varphi'_c \varphi'_d \right) \\
- (1 + 3 \cos \theta) \int_0^{\pi R} dy \left( \varphi'_a \varphi'_b \varphi'_c \varphi'_d + \varphi_a \varphi'_b \varphi'_c \varphi'_d \right) \\
- 2 \cos \theta \int_0^{\pi R} dy \left( \varphi'_a \varphi'_b \varphi'_c \varphi'_d + \varphi_a \varphi'_b \varphi'_c \varphi'_d \right) \right\} + \mathcal{O}(1). \tag{25}
\]

The whole divergent part of the matrix element is proportional to the expression

\[
f^{abc}f^{cde} - f^{ace}f^{bde} + f^{ade}f^{bce}. \]

However, this is zero due to the familiar Jacobi identity.

Thus we conclude that 2 → 2 scattering amplitude of longitudinal gauge bosons contains no terms growing indefinitely with the energy. We have shown this fact without any assumptions regarding the colors or the KK mode numbers of the gauge bosons in the initial and final state.

Note that the elastic scattering of two identical longitudinal vector modes studied in Ref. [13] is a special case contained in our general formulae. Since all the gauge fields satisfy the same boundary conditions, the masses and wave functions (thus, couplings as well) are color-insensitive and are uniquely identified by their KK indices. This implies that there is no contribution from (18) and (20) to the scattering amplitude. Furthermore, in this special case it is possible to combine (23) and (24) to one compact sum rule

\[
\sum_k m_k^2 \left( g_{nnn} \right)^2 = 4 m_n^2 \left( g_{nnn} \right)^2.
\]

7. Conclusions

We have studied the gauge sector of a 5D toy model with EWSB triggered by a non-trivial choice of boundary conditions in the fifth dimension. This class of models has already been intensively studied in the literature, but many authors prefer a more traditional approach to the extra dimensions known as orbifolding – one starts with an infinite extra dimension and compactifies it by a set of identifications (most commonly to $S^1/Z_2$ orbifold); such a procedure then implies certain boundary conditions for the fields. Another already studied possibility that we have also chosen in this work, is the interval approach, where one starts straight away with a finite space interval and then figures out, what the consistent boundary conditions are.

We have derived the set of consistent boundary conditions for a simple model with $SU(2)$ gauge symmetry solely from the principle of least action and the requirement of gauge independence of scattering amplitudes. Any choice belonging to this set leads to the theory with well-behaved scattering amplitudes of longitudinal vector bosons, i.e. all terms growing as positive power of energy cancel out.
This was explicitly demonstrated on a general $2 \rightarrow 2$ scattering process without any assumptions regarding the colors or KK mode numbers of the gauge bosons in the initial and final state (and without relying on the KK equivalence theorem). Previously published results of other authors (see Refs. 8, 10, 12–15) covered only certain special cases of this model (e.g., a special choice of boundary conditions, or the discussion of an elastic scattering process only). Our present work is therefore an improvement and generalization of these earlier results.

Acknowledgments

The work was supported by the grant of the Ministry of Education of the Czech Republic MSM 0021620859.

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\footnote{The technical details of all the calculations may be found in Ref. [16].}