COHOMOLOGICAL AND CYCLE-THEORETIC CONNECTIVITY

KAPIL H. PARANJAPE

Tata Institute of Fundamental Research, Bombay.

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ABSTRACT. One of the themes in algebraic geometry is the study of the relation between the “topology” of a smooth projective variety and a (“general”) hyperplane section. Recent results of Nori produce cohomological evidence for a conjecture that a general hypersurface of sufficiently large degree should have no “interesting” cycles. We compute precise bounds for these results and show by example that there are indeed interesting cycles for degrees that are not high enough. In a different direction Esnault, Nori and Srinivas have shown connectivity for intersections of small multidegree. We show analogous cycle-theoretic connectivity results.

INTRODUCTION

We study the relation between the “topology” of a smooth projective variety and a “general” subvariety. One of the measures of topology is a suitable cohomology theory, another measure is the group of cycles modulo rational equivalence—the Chow group. These two are conjecturally related by series of conjectures of A. A. Beilinson and S. Bloch (see [11]). We formulate and examine some concrete cases.

The classical “weak” Lefschetz theorem states that if $X$ is a smooth projective variety and $Y$ an ample divisor then $H^l(X) \to H^l(Y)$ is an isomorphism for $l < \dim Y$. The conjectural cycle-theoretic analogue is that $\text{CH}^p(X)_\mathbb{Q} \to \text{CH}^p(Y)_\mathbb{Q}$ is an isomorphism for $p < \dim Y/2$ (see (1.5)). The case $p = 1$ is a classical theorem of S. Lefschetz and A. Grothendieck (see [9]). We prove the conjecture in some cases (a more precise statement is (5.4)).

Theorem 0.1. Given integers $1 \leq d_1 \leq \ldots \leq d_r$ and any non-negative integer $l$, let $X \subset \mathbb{P}^n$ be a smooth subvariety of multidegree $(d_1, \ldots, d_r)$. If $n$ is sufficiently large then $\text{CH}^l(X)_\mathbb{Q} \cong \mathbb{Q}$.

The classical theorem of M. Noether and S. Lefschetz states that if $X = \mathbb{P}^3$ and $Y \subset X$ is a “general” surface with $\deg Y \geq 4$, then $\text{CH}^1(Y) = \mathbb{Z}$. There are also some recent generalisations by M. Green [6]. In a recent paper [14] M. V. Nori shows that we have a cohomological analogue of this statement as well. In particular, from the conjectural

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framework mentioned above one should expect that if $Y$ is a “general” complete intersection in $X$ of sufficiently high multidegree then $\text{CH}^p(X) \to \text{CH}^p(Y)$ is an isomorphism for $p < \dim Y$ (see (2.9)). We compute precise bounds for the degrees required. As a result we find that if $X = \mathbb{P}^n$ then we need $\deg Y \geq (2n - 2)$ in Nori’s result (see Section 2). To contrast this we show (see also [18]):

**Proposition 0.2.** Let $Y$ be a “general” hypersurface in $\mathbb{P}^n$ for $n \geq 3$ of degree between $n$ and $2n - 3$. Then $Y$ contains a pair of lines $L_1$ and $L_2$ such that the difference is not rationally equivalent to zero in the Chow group of 1-cycles on $Y$.

It was shown by A. A. Roitman [16] that the Chow group of 0-cycles on intersections of small multidegree is $\mathbb{Z}$. Recent results of H. Esnault, M. V. Nori and V. Srinivas [4] give a cohomological analogue for schemes $Y \subset \mathbb{P}^n$ defined by a small number of equations of low degree. These results led Srinivas to suggest that there are other “rational-like” connectivity properties of such varieties; i.e. we should have $\text{CH}_p(Y)_\mathbb{Q} = \mathbb{Q}$ for small enough $p$ (see (1.9)). The precise prediction for cubic hypersurfaces $Y$ is that if $\dim Y > 4$ then $\text{CH}_1(Y)_\mathbb{Q} = \mathbb{Q}$. We prove this for the general cubic hypersurface in (4.1.4). More generally, we prove the following theorem (a more precise formulation of the result is (4.2.5)).

**Theorem 0.3.** Given integers $1 \leq d_1 \leq \ldots \leq d_r$ and any non-negative integer $l$, choose any $r$ sections $f_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$ and let $X = V(f_1, \ldots, f_r)$ be the scheme defined by these. If $n$ is sufficiently large then $\text{CH}_l(X) \cong \mathbb{Z}$ and is generated by the class of a linear subspace $\mathbb{P}^l \subset X$.

In Section 1 we define the notion of cohomological and cycle-theoretic connectivity. We also state the general conjectures that form a basis for the kind of results one is looking for. We follow up in Section 2 with a sketch of the proof of the Theorem of Nori where we also obtain the bounds for the numbers $N(e)$ appearing in this Theorem. We produce examples in Section 3 to show the sharpness of these bounds. In Section 4 we study cycles of small dimension on intersections of small multidegree in projective space. Finally, in Section 5 we obtain results on cycles of small codimension on such varieties.

I am very grateful to M. V. Nori for explaining his ideas and results related to the papers [14] and [4]. C. Schoen gave me his preprint [18] on connectivity and also clarified many ideas on cycles. The question raised therein on cubic hypersurfaces led to Section (4.1) of this paper. Discussions with K. Joshi, B. Kahn, N. Mohan Kumar and V. Srinivas were very helpful. In particular, B. Kahn pointed out the paper of Leep and Schmidt [12]. V. Srinivas helped me solve the optimisation problem that appears in Section 2 and provided much guidance and encouragement.

**1. Preliminaries on Connectivity**

We work over a fixed base field of characteristic zero (say $\mathbb{C}$) which we denote by $k$ and a fixed universal domain $K \supset k$; i.e. $K$ is an algebraically closed field of infinite transcendence degree over $k$. Let $X$ be a scheme of finite type over $K$. Let $H^*(X)$ denote the de Rham cohomology of $X$ relative to $k$ (see [10]). This is the direct limit of the de Rham cohomology groups $H^*(X)$ as $X$ runs over all models of $X$ which are of finite type over $k$. 


Definition 1.1. Let $X$ be a variety over $K$ and $Y \subset X$ a closed subvariety. We say that the inclusion of $Y$ in $X$ is a cohomological $r$-equivalence if the restriction morphism $H^l(X) \rightarrow H^l(Y)$ is an isomorphism for $l \leq r$.

Definition 1.2. Let $X$ be a variety over $K$ and $Y \subset X$ a closed subvariety. We say that the inclusion of $Y$ in $X$ is a cycle-theoretic $c$-equivalence if the restriction morphism $\text{CH}^p(X)_\mathbb{Q} \rightarrow \text{CH}^p(Y)_\mathbb{Q}$ is an isomorphism for $p \leq c$.

A typical situation is the following. We have a smooth projective variety $X$ over $k$ and $Y \subset X \times S$ is a family of smooth subvarieties. We choose a $k$-embedding of $k(S)$ into $K$, where $k(S)$ is the function field of $S$, and put $Y = Y \times_S \text{Spec} K$. This corresponds to the study of restriction of cycles (or cohomology classes) to the “general” subvariety of $X$ in the family parametrised by $S$.

Remark 1.3. The classical theorem of S. Lefschetz (the “weak” Lefschetz theorem) says that if $Y$ is an ample divisor in a smooth projective variety $X$, then the inclusion $Y \hookrightarrow X$ is a cohomological $r$-equivalence, for $r = \dim Y - 1$. By induction we have the same result for a complete intersection subvariety.

Remark 1.4. One expects the Chow group $\text{CH}^p(X)_\mathbb{Q}$ to carry a natural filtration $F$ such that we have a relation between $\text{gr}_F \text{CH}^p(X)_\mathbb{Q}$ and $H^{2p-l}(X)$ (see [11]). More precisely, we may expect cohomological $r$-equivalence to imply (tensor $\mathbb{Q}$) cycle-theoretic $c$-equivalence where $2c \leq r$. In view of (1.3) there is some interest in determining cycle-theoretic connectivity for ample divisors (and more generally complete intersection subvarieties).

With notation as in (1.3), the theorems of S. Lefschetz and A. Grothendieck (see e.g. [9]) say that the inclusion $Y \hookrightarrow X$ is a cycle-theoretic 1-equivalence if $\dim Y \geq 3$. This is a particular case of the following:

Conjecture 1.5. Let $Y$ be a smooth ample divisor in $X$, then the inclusion of $Y$ in $X$ is a cycle-theoretic $c$-equivalence for $2c \leq \dim Y - 1$.

We obtain some results in this direction in Section 5.

Remark 1.6. The theorem of M. Noether and S. Lefschetz says that if we take $Y$ to be the “general” complete intersection in $X = \mathbb{P}^n$ then the inclusion of $Y$ in $X$ is a cycle-theoretic 1-equivalence if $\dim Y \geq 2$. The following theorem of Nori is a cohomological generalisation:

Theorem. (M. V. Nori) Let $X$ be a smooth projective variety over $k$ and let $\mathcal{O}_X(1)$ be an ample line bundle on $X$. Let $V \rightarrow \Gamma(X, \mathcal{O}_X(1))$ be a space of sections which generates $\mathcal{O}_X(1)$ at stalks. Let $d_1 \leq \cdots \leq d_r$ be a multidegree and $Y$ be the “general” complete intersection in $X_K$ of this multidegree. Then the inclusion $Y \hookrightarrow X_K$ is a cohomological $(\dim Y + e - 1)$-equivalence, if $d_1 \geq N(e)$, where $N(e)$ is an integer depending only on $(X, \mathcal{O}_X(1), V)$ and $e \leq \dim Y$.

By (1.4) this leads to the following conjecture also due to Nori:

Conjecture. (M. V. Nori) In the situation of the above theorem, the inclusion of $Y$ in $X_K$ is a cycle-theoretic $c$-equivalence for $2c \leq \dim Y + e - 1$. 

Note that we are free to choose $e$; however, if we want an isomorphism $\text{CH}^i(X_K) \cong \text{CH}^i(Y)$ for all $l \leq (\dim Y + e - 1)/2$ as predicted, then the best one can hope for is $e = \dim Y - 1$ by the results of D. Mumford and A. A. Roitman [17] (see also Example (3.1)).

We will make this conjecture more precise in Section 2 by computing a value for $N(e)$. In Section 3 we will show that for $X = \mathbb{P}^n$ this gives a sharp bound for cycle-theoretic connectivity.

Remark 1.7. Reverting back to varieties over $k$ we know that the de Rham cohomology theory satisfies the Hard Lefschetz theorem for smooth projective varieties $X$ over $k$. As shown in [11] a consequence of the conjectured relation mentioned in (1.3) and the Hard Lefschetz theorem is that $\text{gr}^k_{N^i} \text{CH}^p(X) \mathbb{Q}$ should depend only on $\text{H}^{2p-l}(X)/N^{p-l+1} \text{H}^{2p-l}(X)$ for $l \leq p$, where $N^i$ denotes the filtration by coniveau (see [8]).

There is a natural filtration on de Rham cohomology—the Hodge filtration—which conjecturally yields the coniveau filtration (via the comparison with singular cohomology; see [8]). With this in mind we examine a result of Esnault-Nori-Srinivas [4]:

**Theorem.** (H. Esnault, M. V. Nori, V. Srinivas) Let $X = V(f_1, \ldots, f_r) \subset \mathbb{P}^n$ be the zero locus of a collection of homogeneous equations $f_i$ of degrees $d_i$, where we assume $d_1 \leq \cdots \leq d_r$. Then for any $i \geq 0$,

$$F^k \text{H}^i_c(\mathbb{P}^n - X) = \text{H}^i_c(\mathbb{P}^n - X),$$

where $F^k$ denotes the Hodge filtration and

$$k = \left\lfloor \frac{n - \sum_{j=2}^r d_r}{d_1} \right\rfloor.$$

This leads us to the following:

**Conjecture 1.9.** If $X$ is a smooth subvariety of $\mathbb{P}^n$ of multidegree $d_1 \leq \cdots \leq d_r$, then $\text{CH}_p(X) \mathbb{Q} = \mathbb{Q}$ for $p < k$ with $k$ as above.

Remark 1.10. For $n \geq \sum_{j=1}^r d_r$, this conjecture says that $\text{CH}_0(X) \mathbb{Q} = \mathbb{Q}$. This particular case has been proved by A. A. Roitman [16] even with $\mathbb{Z}$ instead of $\mathbb{Q}$. We will obtain generalisations in Section 4.

2. Cohomological connectivity

In this section we study cohomological connectivity for the “general” member of the family of all complete intersection subvarieties of a fixed multidegree in a smooth projective variety. We sketch a proof of the theorem of Nori stated in Section 1. In addition, we will obtain bounds (see (2.8)) for the number $N(e)$ appearing in this theorem in terms of some natural invariants for the triple $(X, O_X(1), V)$.

A more general situation than the one in the Theorem is the following. Let $X$ be a smooth projective variety and $F$ a vector bundle on $X$ which is generated at stalks by a space of sections $W \to \Gamma(X, F)$. We define the vector bundle $M$ by the exact sequence

$$0 \to M \to W \otimes O_X \to F \to 0$$
Let $S$ be the projective space $\mathbf{P}(W^*)$. We have $\mathcal{Y} = \mathbf{P}_X(M^*)$ which is a closed subvariety of $X \times S$.

The morphism $\mathcal{Y} \to S$ makes it the family of subvarieties of $X$ defined by zeroes of sections of $E$. Choose a $k$-embedding into $K$ of the function field of $S$ and let $Y = \mathcal{Y} \times_S \text{Spec} K$. Let the sheaves $\Omega^i_{(X \times S, \mathcal{Y})}$ be defined by the exact sequence

$$0 \to \Omega^i_{(X \times S, \mathcal{Y})} \to \Omega^i_{X \times S} \to \Omega^i_Y \to 0$$

Madhav Nori [14] has shown that his Theorem follows from the vanishing of the direct image sheaves $R^ap_{S*}(\Omega^b_{(X \times S, \mathcal{Y})})$ for $a \leq \dim Y$ and $a + b \leq \dim Y + e$.

**Remark 2.4.** V. Srinivas [19] has defined Hodge cohomology for any scheme over $k$ as follows:

$$H^n(X) = \oplus_{p=0}^{n} H^{n-p}(X, \Omega^p_{X/k}).$$

He has constructed Gysin and cycle class maps for this theory when $X$ and $Y$ are smooth varieties over any field extension $K$ of $k$. If this is a Poincaré duality theory, then one may view the result of M. V. Nori stated above as a result on Hodge-theoretic connectivity.

**Lemma 2.5.** The higher direct image sheaf $R^ap_{S*}(\Omega^b_{(X \times S, \mathcal{Y})})$ vanishes if the cohomology groups $H^k(X, \Lambda^m \otimes \Lambda^l F \otimes \Omega^c_X)$ vanish for all non-negative integers $k$, $l$, $m$, and $c$ such that $k \leq \dim X - a$, $m + (\dim X - c) \leq b$ and $k + l + m - c = b + 1 - a$.

**Sketch of Proof.** As in [14] an $E_1$ spectral sequence argument shows that $R^ap_{S*}(\Omega^b_{(X \times S, \mathcal{Y})})$ vanishes if the sheaves $R^ap_{S*}(\Omega^b_{(X \times S, \mathcal{Y})}/X \otimes \Omega^c_X)$ vanish for $\beta + \gamma = b$; where the sheaf $\Omega^b_{(X \times S, \mathcal{Y})}/X$ is defined by an exact sequence analogous to (2.3) with relative differentials. These sheaves can be computed by using a natural resolution for the sheaf $\Omega^b_{(X \times S, \mathcal{Y})}/X$ which we now explain.

Let $G_r$ be the filtration of $\Lambda W^*$ obtained by taking exterior powers of the exact sequence (2.2); the numbering is so chosen that $\text{gr}^G_r(\Lambda W^*) = \Lambda^r F^* \otimes \Lambda^r M^*$. Let $O_S(1)$ be the tautological line bundle. We use the notation $G_b(\Lambda W^*)(-k)$ to denote the sheaf $p_X^*G_b(\Lambda W^*) \otimes p_S^*O_S(-k)$ on $X \times S$. We have natural homomorphisms of sheaves on $S$, for each $k$ and $b$,

$$G_b^k(\Lambda W^*)(-k) \to G_b^k(\Lambda W^*)(-k)$$

which fit together for $k \geq b + 1$ and give a resolution for $\Omega^b_{(X \times S, \mathcal{Y})}/X$, i.e. we have an exact sequence.

$$0 \leftarrow \Omega^b_{Y/X} \leftarrow \Omega^b_{X \times S/S} \leftarrow G_b^b(\Lambda W^*)(-b - 1) \leftarrow G_b^{b+1}(\Lambda W^*)(-b - 2) \cdots$$

Thus the sheaf $R^ap_{S*}(\Omega^b_{(X \times S, \mathcal{Y})}/X \otimes \Omega^c_X)$ is the zero sheaf if for all $\alpha \geq a$ and $\beta = b + 1 + (\alpha - a)$, the sheaves $R^ap_{S*}(G_b(\Lambda W^*)(-\beta) \otimes \Omega^c_X)$ are zero. By the projection formula
we see that this sheaf is a twist of the trivial sheaf on \( S \) with fibre \( H^a(X, G_b(\Lambda W^*) \otimes \Omega_X) \) by \( \mathcal{O}_S(-\beta) \). Now, by the \( E_1 \) spectral sequence associated with the filtration \( G_* \), we see that the group \( H^a(X, G_l(\Lambda W^*) \otimes \Omega_X) \) vanishes if the groups \( H^a(X, \Lambda^* \otimes \Lambda^* F^* \otimes \Omega_X) \) vanish for all \( d \leq l \). Applying Serre’s duality theorem to the latter groups we obtain 
\[
H^{n-a}(X, \Lambda^* \otimes \Lambda^* F \otimes \Omega_X^{-c}),
\]
where \( n = \dim X \). Combining the equations and inequalities above we have the result. \( \square \)

Let us now revert to the earlier situation where we are given \((X, \mathcal{O}_X(1), V)\) as in the Theorem of Nori. Assume we are given \( d_1 \leq \cdots \leq d_r \), a multidegree and let \( F = \oplus_i \mathcal{O}_X(d_i) \).
Let us choose \( W \) to be the direct sum of \( \text{Sym}^d_i(V) \) over all \( i \). If \( M_k \) is defined by the exact sequence
\[
0 \to M_k \to \text{Sym}^k(V) \otimes \mathcal{O}_X \to \mathcal{O}_X(k) \to 0,
\]
then \( M = \oplus_i M_{d_i} \) and so we can apply the following:

**Lemma 2.6.** Let \( m_b \) be such that \( H^a(X, \Omega_X^b \otimes \mathcal{O}_X(m_b - a)) = 0 \) for all \( a \geq 1 \). Then
\[
H^a(X, \Omega_X^b \otimes \Lambda^* \otimes \mathcal{O}_X(k)) = 0
\]
for all \( a \geq 1 \) such that \( a + k \geq m_b + c \).

**Proof.** It is well known that the given condition on \( \Omega_X^b \) is the condition for \( m_b \)-regularity in the sense of Castelnuovo and Mumford (see e.g. [13]), which in turn implies that
\[
H^a(X, \Omega_X^b \otimes \mathcal{O}_X(k)) = 0
\]
for all \( a \geq 1 \) such that \( a + k \geq m_b \). Let \( f : X \to \mathbf{P}(V) \) be the morphism given by \((V, \mathcal{O}_X(1))\). We have a vector bundle \( E_k \) on \( \mathbf{P}(V) \) defined by the exact sequence
\[
0 \to E_k \to \text{Sym}^k(V) \otimes \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(k) \to 0.
\]
Clearly, we have \( M_k = f^*E_k \). Moreover, \( f \) is finite and so we may compute cohomologies of sheaves on \( X \) via their direct images under \( f \). In particular, we see that \( f_*(\Omega_X^b) \) is \( m_b \)-regular on \( \mathbf{P}(V) \). By the projection formula we have
\[
f_*(\Omega_X^b \otimes \Lambda^* \otimes \mathcal{O}_X(k)) = f_*\Omega_X^b \otimes \Lambda^*(\oplus_i E_{d_i}) \otimes \mathcal{O}_{\mathbf{P}(V)}(k).
\]
It is well known (see e.g. [5]) that \( E_k \) is 1-regular. A direct sum of 1-regular sheaves is also 1-regular. Moreover, the regularity of the tensor product of a coherent sheaf with a vector bundle on \( \mathbf{P}(V) \) is the sum of their regularities (see [5]). Thus we have the result. \( \square \)

By this lemma we only need \( k + d^{(l)} \geq m + m_c \) and \( k \geq 1 \) for all integers \( k, l, m \) and \( c \) occurring in Lemma (2.5); here \( d^{(l)} = \sum_{i=1}^l d_i \). An easy computation shows that \( k \geq 1 \) and \( l \geq 1 \) for all tuples \((k, l, m, c)\) satisfying the conditions in Lemma (2.5). Thus we need only
take maximum of \((m + m_c - k)/l\) over all such tuples as \((a, b)\) vary over all pairs such that \(a \leq \dim Y\) and \(a + b \leq \dim Y + e\). This maximum will serve as \(N(e)\).

For any triple \((X, \mathcal{O}_X(1), V)\) we define

\[
(2.7) \quad m_X = \max\{m_c - c - 1 : 0 \leq c \leq \dim X\}.
\]

One can solve the above optimisation problem after substituting \(m_c\) by \(m_X + c + 1\); the maximum obtained this way is

\[
(2.8) \quad N(e) = \dim Y + e + 1 + m_X.
\]

Thus we may restate the conjecture of M. V. Nori as follows:

**Conjecture 2.9.** Let \(Y\) be the general complete intersection of multidegree \(d_1 \leq \cdots \leq d_r\) in \(X\) and assume that \(d_1 \geq \dim Y + e + 1 + m_X\) for some \(e \leq \dim Y - 1\). Then the inclusion of \(Y\) in \(X_{\mathbb{K}}\) is a cycle-theoretic \(c\)-equivalence for \(2c < \dim Y + e\).

**Remark 2.10.** The number \(m_c\) does not seem to have been computed for general triples \((X, \mathcal{O}_X(1), V)\), even when one assumes reasonable properties. Work of L. Ein and R. Lazarsfeld [3] shows that \(m_0 \leq (d_1 + \cdots + d_{c} - c + 1)\) where \(c = \text{codim}_{\mathbb{P}(V)} X\).

At the other extreme, by the Kodaira Vanishing Theorem we have \(m_n \leq n + 1\) for \(n = \dim X\).

**Remark 2.11.** For \(X = \mathbb{P}^n\), we see easily that \(m_c = c + 1\) for all \(c\) in the range \(0 \leq c \leq n\); thus \(m_{\mathbb{P}^n} = 0\). Thus we have a special case of (2.9):

**Conjecture 2.12.** Let \(Y\) be the general hypersurface in \(\mathbb{P}^n\) of degree \(\geq 2n - 2\), then \(\text{CH}^p(Y)_{\mathbb{Q}} = \mathbb{Q}\) for \(p < \dim Y\).

In Section 3 we will show that this bound is sharp by showing that the general hypersurface of degree \(2n - 3\) does in fact contain interesting cycles.

### 3. Examples of large degree

In this section we work over a universal domain (say \(\mathbb{C}\)). We will construct examples of hypersurfaces \(X\) of degree \(\leq 2n - 3\) in \(\mathbb{P}^n\) such that \(\text{CH}_k(X)\) is not \(\mathbb{Z}\) for \(k = 0, 1\); i.e. cycle-theoretic connectivity in codimension \((\dim X - k)\) does not hold.

**Example 3.1.** Let \(X\) be a smooth projective variety and \(\{p_1, \ldots, p_n\} = S\) be a set of distinct points on \(X\). There is a divisor \(Y \subset X\) such that \(S \subset Y\) and

\[
\bigoplus_{i=1}^n \mathbb{Z} \cdot p_i \hookrightarrow \text{CH}_0(Y),
\]

i.e. the points are linearly independent in the group of 0-cycles modulo rational equivalence on \(Y\). In fact, if \(\mathcal{O}_X(1)\) is ample and \(d\) sufficiently large, then a “general” element \(Y\) of the complete linear system \(|\mathcal{O}_X(d)|\) will have this property.

**Proof.** Let \(A\) be any line bundle on \(X\) with the property that there is a base point free linear system \(V \subset \Gamma(X, \mathcal{O}_X)\) such that the evaluation map \(V \otimes \mathcal{O}_X \rightarrow \bigoplus_{i=1}^n A_{p_i}\) surjects. Then we have a morphism \(f : X \rightarrow \mathbb{P}(V)\) such that \(f(S)\) consists of linearly independent points in \(\mathbb{P}(V)\).
Sublemma 3.2. If \( H \subset \mathbf{P}(V) \) is a general hypersurface of degree \( \geq \dim V \) which contains \( f(S) \) then
\[
\bigoplus_{i=1}^{n} \mathbb{Z} \cdot f(p_i) \hookrightarrow \text{CH}_0(H).
\]

Assuming this, we see that is \( s \in \text{Sym}^k(V) \) is a general element and \( k \geq \dim V \), then the zero locus \( Y \) of \( s \) in \( X \) is a smooth divisor. Now from the commutative diagram
\[
\begin{array}{ccc}
\bigoplus_{i=1}^{n} \mathbb{Z} \cdot p_i & \to & \text{CH}_0(Y) \\
\downarrow f_* & & \downarrow f_* \\
\bigoplus_{i=1}^{n} \mathbb{Z} \cdot f(p_i) & \hookrightarrow & \text{CH}_0(H)
\end{array}
\]
we see that we have the result. \( \square \)

Proof. (of (3.2)) Let \( H' \) be a general hypersurface in \( \mathbf{P}(V) \) of degree \( \geq \dim V \). Then by the theorem of Mumford and Roitman (see e.g. [17]), any \( n \) general points \( q_1, \ldots, q_n \) on \( H' \) are linearly independent in the Chow group of 0-cycles on \( H' \).

Now the set \( f(S) \) consists of \( n \) linearly independent points in \( \mathbf{P}(V) \). Hence there is an automorphism of \( \mathbf{P}(V) \) that takes this set into the set \( q_1, \ldots, q_n \) (which by generality may be assumed to be linearly independent also). Under this automorphism we pull back the hypersurface \( H' \) to give us the required hypersurface \( H \). \( \square \)

Example 3.3. Let \( L_1, L_2 \) be a pair of skew lines in \( \mathbf{P}^n \) and \( X \) a general hypersurface of degree \( \geq n \) containing \( L_1 \) and \( L_2 \). Then we have
\[
\mathbb{Z} \cdot L_1 \oplus \mathbb{Z} \cdot L_2 \hookrightarrow \text{CH}_1(X).
\]

Proof. Let \( \mathbf{P}^{n-1} \) be a general linear space and \( p_i = L_i \cap \mathbf{P}^{n-1} \). We have seen during the proof of the earlier example that if \( Y \) is a general hypersurface of degree \( \geq n \) in \( \mathbf{P}^{n-1} \) containing \( p_1 \) and \( p_2 \), then
\[
\mathbb{Z} \cdot p_1 \oplus \mathbb{Z} \cdot p_2 \hookrightarrow \text{CH}_0(Y).
\]

We have a short exact sequence
\[
0 \to \mathcal{I}_{L_1 \cup L_2/\mathbf{P}^n}(d-1) \to \mathcal{I}_{L_1 \cup L_2/\mathbf{P}^n}(d) \to \mathcal{I}_{p_1 \cup p_2/\mathbf{P}^{n-1}}(d) \to 0.
\]
Thus, the general hypersurface of degree \( d \) in \( \mathbf{P}^n \) containing \( L_i \) restricts to the general hypersurface of the same degree in \( \mathbf{P}^{n-1} \) containing \( p_i \), providing \( H^1(\mathbf{P}^n, \mathcal{I}_{L_1 \cup L_2/\mathbf{P}^n}(d-1)) \) vanishes. Now we use the exact sequence
\[
0 \to \mathcal{I}_{L_1 \cup L_2/\mathbf{P}^n}(d) \to \mathcal{O}_{\mathbf{P}^n}(d) \to \mathcal{O}_{L_1}(d) \oplus \mathcal{O}_{L_2}(d) \to 0
\]
to see that \( H^1(\mathbf{P}^n, \mathcal{I}_{L_1 \cup L_2/\mathbf{P}^n}(d)) = 0 \) for all \( d \geq 1 \). We have a diagram
\[
\begin{array}{ccc}
\mathbb{Z} \cdot L_1 \oplus \mathbb{Z} \cdot L_2 & \to & \text{CH}_0(X) \\
\downarrow i^* & & \downarrow i^* \\
\mathbb{Z} \cdot p_1 \oplus \mathbb{Z} \cdot p_2 & \hookrightarrow & \text{CH}_0(Y)
\end{array}
\]
where \( i : \mathbf{P}^{n-1} \to \mathbf{P}^n \) is the inclusion. Hence the result. \( \square \)

Remark 3.4. By general considerations (see e.g. [15]) we can show that a hypersurface in \( \mathbf{P}^n \) of degree \( \leq 2n - 3 \) always contains a pair of lines. Hence we have Proposition (0.2).
4. Examples of small degree

Fix a multidegree \( d_1 \leq \cdots \leq d_r \). If we take a large \( n \) and look at the varieties \( X \) given by a collections of homogeneous equations of the given multidegree in \( \mathbb{P}^n \), then we get connectivity statements for \( \text{CH}_k(X) \) for \( k \) small. In the first part we study the Chow group of 1-cycles on cubics as an illustrative example. In the second part we prove the more general assertion (0.2).

4.1 Cubic hypersurfaces.

We illustrate the general case by deducing that the Chow group of 1-cycles on a general cubic hypersurface of dimension at least 2 (i.e. \( n \geq 3 \)) is \( \mathbb{Z} \), using as starting point the following well known facts.

**Fact 4.1.1.** The Chow group of 1-cycles on a quadric hypersurface \( X \subset \mathbb{P}^n_k \) of dimension at least 2 (i.e. \( n \geq 3 \)) is generated by lines, providing \( k \) is algebraically closed.

**Fact 4.1.2.** A quadric hypersurface \( X \subset \mathbb{P}^n_k \) of dimension at least 1 (i.e. \( n \geq 2 \)) contains a \( k \)-rational point \( p \) provided that \( k \) is a \( C_1 \)-field (in the sense of Lang; see e.g. [7]), and hence \( \text{CH}_0(X) \cong \mathbb{Z} \cdot p \).

By general principles (see e.g. [15]) we see that if \( k \) is an algebraically closed field, a cubic hypersurface \( X \subset \mathbb{P}^n_k \) contains a plane \( P \). We project from this plane to get a morphism \( \tilde{X} \to \mathbb{P}^3_k \), where \( \tilde{X} \to X \) is the blow up of \( X \) along the plane. Let \( E \subset \tilde{X} \) be the exceptional divisor of this blow up. Then we see that \( E \to \mathbb{P}^3_k \) is a family of conics and \( \tilde{X} \to \mathbb{P}^3_k \) is a family of quadric surfaces.

Let \( C \) be any curve in \( \tilde{X} \). If its image \( D \) in \( \mathbb{P}^3_k \) is a curve, then the field \( k(D) \) is \( C_1 \). The conic \( E_{k(D)} = E \times_{\mathbb{P}^3_k} \text{Spec} \ k(D) \) then has a \( k(D) \)-rational point and so we have a morphism \( \tilde{D} \to E \) over \( D \), where \( \tilde{D} \) is the normalisation of \( D \). The 1-cycle \( \xi = C - \deg(C/D) \cdot \tilde{D} \) restricts to a 0-cycle of degree 0 on the fibre of \( \tilde{X} \) over the point \( \text{Spec} \ k(D) \to \mathbb{P}^3_k \). By Fact 1.2, we see that this class is then rationally trivial in this fibre. We can then apply the following well known lemma to the cycle \( \xi \) on \( \tilde{X} \times \mathbb{P}^3_k D \).

**Lemma 4.1.3.** Let \( X \to Y \) be a proper morphism where \( Y \) is integral. Let \( K \) be the function field of \( Y \) and \( X_K = X \times_Y \text{Spec} \ K \). If \( \xi \in \text{CH}_k(X) \) is such that it restricts to zero in \( \text{CH}_k'(X_K) \), where \( k' = k - \dim Y \), then there is a proper subscheme \( Z \subsetneq Y \) such that \( \xi \) is supported on \( X \times_Y Z \).

Thus \( \xi \) is rationally equivalent on \( \tilde{X} \) to a cycle supported on the fibres of \( \tilde{X} \to \mathbb{P}^3_k \). By Fact 1.1 such 1-cycles are generated by lines. Hence we have a surjection

\[
\text{CH}_1(E) \oplus \{ \text{subgroup generated by lines} \} \twoheadrightarrow \text{CH}_1(\tilde{X}).
\]

But the morphism \( \text{CH}_1(E) \to \text{CH}_1(X) \) factors through \( \text{CH}_1(P) \) which is also generated by lines. Thus we see that the Chow group of 1-cycles on \( X \) is generated by lines. An easy computation shows that the variety of lines on such a cube is a Fano variety. If this variety is smooth then it is known (see [2]) that all lines on \( X \) are rationally equivalent. A similar argument will also give the following result for \( X \) of dimension bigger than 5. An analogous statement appears in the paper of Schoen (see Theorem 5.1 in [18]) for cubics of dimension at least 6.
Proposition 4.1.4. Let $X \subset P^m_k$ be a smooth cubic hypersurface of dimension at least 5, such that the variety of lines on it is smooth; then $\text{CH}_1(X) \cong \mathbb{Z}$.

4.2 Intersections of small multidegree.

Let $k$ be any field. Given positive integers $d_1 \leq \cdots \leq d_r$, and a non-negative integer $m$, we define $n = n(d_1, \ldots, d_r; m; k)$ to be the smallest integer (possibly infinite!) such that for all homogeneous polynomials (in $n + 1$ variables) $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ respectively, the subscheme $X = V(f_1, \ldots, f_r) \subset P^m_k$ defined by these equations contains a linear space $P^n$ of dimension $m$.

We also define $n = l(d_1, \ldots, d_r; m; k)$ to be the smallest integer (possibly infinite!) such that every subscheme $X \subset P^m_k$ as above, contains a linear space $P \cong P^n_k$ and satisfies $\text{CH}_m(X) = \mathbb{Z} \cdot P$. Note that in this situation, the $m$-fold intersection with the hyperplane class will induce an isomorphism $\text{CH}_m(X) \cong \mathbb{Z}$. Hence, if $P' \cong P^m_k$ is any other linear subspace contained in $X$ then we also have $\text{CH}_m(X) = \mathbb{Z} \cdot P'$.

We now prove various inequalities about these numbers. The first and trivial inequality is

\begin{equation}
(4.2.1) \quad n(d_1, \ldots, d_r; m; k) \leq n(d_1, \ldots, d_s; n(d_{s+1}, \ldots, d_r; m; k); k).
\end{equation}

for any $s$ between 1 and $r$ such that $n(d_{s+1}, \ldots, d_r; m; k)$ is finite.

Lemma 4.2.2. We have the inequality

$$n(d; m; k) \leq \max\{n(d; 0; k), n(1, \ldots, d; m - 1; k)\}.$$ 

Proof. Let us choose $n$ greater than or equal to the number on the right hand side of the inequality. Then we have an $k$-rational point $p$ on $X$. We blow up the point $p$ and project to obtain $f : \tilde{X} \to P^{n-1}$ and $g : P^n \to P^{n-1}$. The condition that the fibre of $f$ at a point of $P^{n-1}$ is the same as the fibre of $g$ at this point gives us one equation of each degree between 1 and $d$ on $P^{n-1}$ (these are the polars of the equation defining $X$, with respect to the point $p$). Let $Y$ be the zero locus of these equations. Since $n \geq n(1, \ldots, d; m - 1; k)$ we have a linear space of dimension $m - 1$ in $Y$. The inverse image in $\tilde{X}$ pushes down to a linear subspace of dimension $m$ in $X$. □

We now observe that the inequalities (4.2.1) and (4.2.2) give us a weaker version of a result of Leep and Schmidt [12]:

Proposition 4.2.3. (Leep-Schmidt) Let $k$ be a field such that $n(d_r; 0; k)$ is finite. Then $n(d_1, \ldots, d_r; m; k)$ is finite for all integers $m$.

We now obtain an inequality for $l(d_1, \ldots, d_r; m; k)$.

Lemma 4.2.4. Let us define $l$ to be the supremum of the numbers $l(d_1 - 1, \ldots, d_r - 1; m'; k')$, where $m'$ runs over all numbers between 0 and $m$, and $k'$ runs over all finitely generated field extensions of $k$ which have transcendence degree $m - m'$. If $l$ is finite, then we have the inequality

$$l(d_1, \ldots, d_r; m; k) \leq n(d_1, \ldots, d_r; l; k).$$
Proof. Let $n$ be a number greater than or equal to the right hand side of the inequality. Then we have a linear subspace $\mathbb{P}^l_k$ in $X = V(f_1, \ldots, f_r)$. We blow up this subspace and project to obtain a morphism $\tilde{X} \to \mathbb{P}^{n-l-1}$ and also $\mathbb{P}^n \to \mathbb{P}^{n-l-1}$. Let $E$ and $\mathbb{P}^l_k \times \mathbb{P}^{n-l-1}$ be the exceptional divisors for the blow ups. The morphisms $\tilde{X} \to \mathbb{P}^{n-l-1}$ and $E \to \mathbb{P}^{n-l-1}$ are families of intersections of multidegree $(d_1 - 1, \ldots, d_r - 1)$ in the $\mathbb{P}^{l+1}$-bundle (resp. $\mathbb{P}^l$-bundle) $\tilde{\mathbb{P}}^n \to \mathbb{P}^{n-l-1}$ (resp. $\mathbb{P}^l \times \mathbb{P}^{n-l-1} \to \mathbb{P}^{n-l-1}$).

Let $A$ be any subvariety of $\tilde{X}$ of dimension $m$. Let $B$ be its image in $\mathbb{P}^{n-l-1}$. By induction on the dimension of $B$, we will show that $A$ is rationally equivalent to a $m$-cycle supported on $E$. If $\dim B = -\infty$, then $A$ is empty and hence we are done.

Let $\dim(A/B) = m'$ and $k' = k(B)$, the function field of $B$; let

$$E_{k'} = E \times_{\mathbb{P}^{n-l-1}} \text{Spec } k(B) \subset X_{k'} = \tilde{X} \times_{\mathbb{P}^{n-l-1}} \text{Spec } k(B)$$

Since $l$ has so been chosen that $l \geq l(d_1 - 1, \ldots, d_r - 1; m'; k')$, we have a $P_{k'} = \mathbb{P}^{m'}_{k'}$ contained in $E_{k'}$ such that

(A) \[ \mathbf{Z} \cdot P_{k'} \mapsto \text{CH}_{m'}(E_{k'}) \mapsto \text{CH}(X_{k'}). \]

Let $C \subset E$ be the closure of $P_{k'}$. Then by (A) we have an integer $a$ such that the $m$-cycle $A - a \cdot C$ on $X_B = \tilde{X} \times_{\mathbb{P}^{n-l-1}} B$ restricts to a rationally trivial cycle in $X_{k'}$. If $B$ has dimension zero then we note that $X_B = X_{k'}$ so that we are done. In any case, by (4.1.3) we see that $A - a \cdot C$ is rationally equivalent in $X_B$ to a cycle supported over a proper subscheme of $B$. By induction on the dimension of $B$ we are done.

Hence we have shown that the natural morphism $\text{CH}_m(E) \mapsto \text{CH}_m(\tilde{X})$ is a surjection. Now the image of $\text{CH}_m(E) \to \text{CH}_m(X)$ factors through $\text{CH}_m(\mathbb{P}^l_k)$ and so we have the result. □

Combining Proposition (4.2.3) and Lemma (4.2.4) we have

**Theorem 4.2.5.** Let $k$ be a field such that for all finitely generated field extensions $k' \supset k$ of transcendence degree at most $m$ there is a uniform bound for $n(d_1, 0; k')$. Then, $l(d_1, \ldots, d_r; m; k)$ is finite.

**Remark 4.2.6.** Note that if $F$ is an algebraically closed field, and $k$ is finitely generated extension of $F$ then it always satisfies the hypothesis of (4.2.3) and (4.2.5) by results of Lang and Nagata (see e.g. [7]).

**Remark 4.2.7.** If we only wish to obtain (4.2.5) upto torsion where $k$ is replaced by a universal domain $K$, the bound obtained in (4.2.4) can be improved. The same proof will show that we need only take $l$ to be the supremum of the integers $l(d_1 - 1, \ldots, d_r - 1; m'; K)$, where $m'$ runs over all numbers between 0 and $m$.

5. The Method of Bloch and Srinivas

We first generalise the key Proposition of [1].
Proposition 5.1. Let $X$ be a smooth projective variety over $k$ and $K$ be a universal domain over $k$. Let $V_l \subset X$ be subschemes for each $l \in [0, m]$ such that $\text{CH}_l((X - V_l)_K) = 0$ for each $l$. Then for each $l$ we have cycles $\Gamma_l \in \text{CH}^{\dim X}(X \times X)_Q$ such that support of $\Gamma_l$ is contained in $V_l \times X$ and a cycle $\Gamma^{m+1} \in \text{CH}^{\dim X}(X \times X)_Q$ such that support of $\Gamma^{m+1}$ is contained in $X \times W$, where $W \subset X$ is pure of codimension $m + 1$, so that if $\Delta_X \in \text{CH}^{\dim X}(X \times X)_Q$ is the class of the diagonal we have an equation

$$(5.2) \quad \Delta_X = \Gamma_0 + \cdots + \Gamma_m + \Gamma^{m+1}.$$ 

Proof. The result is obvious for $m = -1$ with $\Gamma^0 = \Delta_X$. We prove the general case by induction on $m$. Assume that the result is known for $m - 1$. We then have $\Gamma^m$ which has support contained in $X \times W'$ where $W'$ is pure of codimension $m$ in $X$. Let $P$ be any geometric generic point of $W'$ with values in $K$, i.e. choose a $k$-embedding of the function field of some component of $W'$ into $K$. The restriction of $\Gamma^m$ to $X \times P$ gives an element of $\text{CH}_m(X_K)$. By assumption, this cycle is then rationally equivalent to a cycle $\gamma^m_P \in \text{CH}_m((V_m)_K)$. Choose one such $P$ for each component of $W'$ and let $\Gamma_m$ be the sum over these $P$'s of the closures in $X \times W'$ of $\gamma^m_P$. The difference $\xi = \Gamma^m - \Gamma_m$ restricts to zero on $X \times P$ for each choice of $P$. By (4.1.3) we see that there is a subscheme $W$ of $W'$ of pure codimension 1 and a cycle $\Gamma^{m+1} \in \text{CH}^{\dim X}(X \times X)$ supported on $X \times W$ such that $\Gamma^m - \Gamma_m$ is rationally equivalent to $\Gamma^{m+1}$. Hence we have the result. □

Lemma 5.3. Let $X$ be a smooth projective variety over $k$, $V \subset X$ be a closed subscheme and $W \subset X$ be a closed subscheme of pure codimension $m + 1$. Moreover assume that for each $l \in [0, k]$ we have cycles $\Gamma_l \in \text{CH}^{\dim X}(X \times X)_Q$ with support contained in $V \times X$, and a cycle $\Gamma^{m+1} \in \text{CH}^{\dim X}(X \times X)_Q$ with support in $X \times W$, and that these cycles satisfy (5.2). Then for each $l$ we have a morphism $\text{CH}^l(V)_Q \to \text{CH}^l(X)_Q$ given by $\sum [\Gamma_l]_\ast$.

1. The morphism $\text{CH}^l(V)_Q \to \text{CH}^l(X)_Q$ is a surjection for $l \leq m$.
2. The cokernel of $\text{CH}^{m+1}(V)_Q \to \text{CH}^{m+1}(X)_Q$ is finite dimensional.
3. The cokernel of $\text{CH}^{m+2}(V)_Q \to \text{CH}^{m+2}(X)_Q$ is weakly representable.

Proof. By (5.2) we only need to look at the image of $[\Gamma^{m+1}]_\ast$, for each $l$. We have a factoring of this via the Gysin morphism $\text{CH}^{l-m-1}(W)_Q \to \text{CH}^l(X)_Q$. The results then follow from the facts: (1) For $a < 0$ we have $\text{CH}^a(W) = 0$, (2) $\text{CH}^0(W)_Q$ is a finite dimensional $Q$-vector space, and (3) $\text{CH}^1(W)$ is representable. □

Now in the situation where $X$ is a smooth subvariety of $\mathbf{P}^n$ defined by $r$ equations of degrees $d_1 \leq \cdots \leq d_r$, and $n \geq l(d_1, \ldots, d_r; m; k)$ for an algebraically closed field $k$ (for notation see (4.2.1)). We have a linear subspace $\mathbf{P}^m \subset X$ such that we can apply (5.1) with $V_l = \mathbf{P}^l \subset \mathbf{P}^m$ for all $l \in [0, m]$. Then we can apply (5.3) with $V = \mathbf{P}^m$ to conclude

Theorem 5.4. Let $X$ be a smooth subvariety of $\mathbf{P}^n_K$ defined by $r$ equations of degrees $d_1 \leq \cdots \leq d_r$, and let $m$ be a positive integer. If $n \geq l(d_1, \ldots, d_r; m; k)$, then

1. For $l \in [0, m]$ we have $\text{CH}^l(X)_Q = Q$.
2. $\text{CH}^{m+1}(X)_Q$ is finite dimensional.
3. $\text{CH}^{m+2}(X)_Q$ is representable.
Further applications of (5.1) may be found by applying the methods of [1]. We mention one consequence which is to conclude the validity of the generalised Hodge conjecture of Grothendieck (see [8]) in some cases.

**Proposition 5.5.** *In the situation of (5.4) we have \( H^l(X) = N^{m+1} H^l(X) \) for all \( l \) such that \( l \geq (2m+2) \).*

**Proof.** We note that \([\Gamma^{m+1}]_*\) is an endomorphism of \( H^l(X) \) that factors through the Gysin homomorphism \( H^{l-2(m+1)}(\tilde{W}) \to H^l(X) \), where \( \tilde{W} \) is the desingularization of \( W \). By definition, the image of this Gysin homomorphism is in \( N^{m+1} H^l(X) \). Further, the endomorphism \([\Gamma_l]^*\) of \( H^l(X) \) factors through \( H^l(V) \). In our case \( V = \mathbb{P}^m \), so that \( H^l(V) = N^{l/2} H^l(V) \). □

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

E-mail address: KAPIL@TIFRVAX.bitnet

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