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Hartle–Hawking no-boundary proposal in dRGT massive gravity: making inflation exponentially more probable

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Abstract

It is known that the no-boundary proposal in the traditional Einstein gravity does not prefer inflation, that is, the probability of realizing a large number of $e$-folds is exponentially suppressed. This situation may be changed drastically in a class of nonlinear massive gravity theories recently proposed by de Rham, Gabadadze and Tolley, called dRGT massive gravity. We show that the contribution from the massive gravity sector can enhance the probability of a large number of $e$-folds substantially for a sufficiently large mass parameter $m_g$ comparable to the Hubble parameter during inflation, say $m_g \gtrsim 10^{12}$ GeV. We illustrate possible models to trigger such a large mass parameter in the early universe while it is negligibly small in the present universe. This opens a new window to explore the inflationary scenario in the context of quantum cosmology.

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(Some figures may appear in colour only in the online journal)

1. Introduction

It is known that quantum gravity is hoped to resolve the initial singularity of the universe and assign a proper initial condition with a reasonable probability, where the corresponding wavefunction of the universe is described by the Wheeler–DeWitt equation [1]. In order to obtain a unique solution to this equation, one natural idea to fix the boundary condition is to choose the initial state such that it most closely resembles the ground state, or the Euclidean vacuum state. Moreover, it is known that the Euclidean vacuum state wavefunction in standard quantum theory can be obtained by the Euclidean path integral. Hence, Hartle and Hawking [2] generalized it to the case of gravity, and introduced the Euclidean path integral to choose the wavefunction of the universe,

$$\Psi[h_{ij}, \chi] = \int_{h = h_0, \phi = \chi} \mathcal{D}g \mathcal{D}\phi e^{-S_E[g,\phi]},$$

(1)
where $h_{ij}$ is the metric of a compact 3-geometry $\Sigma$ and $\chi$ is the matter (inflaton) field $\phi$ on $\Sigma$, which form the boundary of all possible, regular 4-geometries and matter configurations over which the path integral is to be performed. This is called the no-boundary proposal. However it is known that the Hartle–Hawking no-boundary proposal is in severe conflict with the realization of successful inflation [3].

It has been proved that an $O(4)$-symmetric solution gives the lowest action for a wide class of scalar-field theories, hence dominating the path integral (1). So it is reasonable to assume that the metric owns the $O(4)$ symmetry even in the presence of gravity [4],

$$d\Sigma_E^2 = dr^2 + a^2(\tau)d\Omega_3^2.$$ (2)

On the other hand, under the steepest-descent approximation (or equivalently, WKB approximation) where the wavefunction is dominated by sum-over on-shell histories, the corresponding actions are complex-valued,

$$\Psi[\tilde{a}, \tilde{\phi}] \simeq A[\tilde{a}, \tilde{\phi}] e^{iS[\tilde{a}, \tilde{\phi}]} ,$$ (3)

where $\tilde{a}$ and $\tilde{\phi}$ are the boundary values of $a$ and $\phi$, respectively, and $A$ and $S$ are real-valued functions. When the phase $S$ varies much faster than $A$,

$$|\nabla_I A[\tilde{a}, \tilde{\phi}]| \ll |\nabla_I S[\tilde{a}, \tilde{\phi}]| , \quad I = \tilde{a}, \tilde{\phi} ,$$ (4)

these histories are classical since it satisfies the semi-classical Hamilton–Jacobi equation [5].

Thus, provided that the potential is exactly flat, if the values $\tilde{a}$ and $\tilde{\phi}$ are such that they form a maximal slice of an $O(4)$ symmetric regular Euclidean solution, that is, $\tilde{a}$ and $\tilde{\phi}$ are the values at which $\dot{a} = \dot{\phi} = 0$, then the probability of the existence of such a classical universe is evaluated as

$$P \simeq e^{-2S_E} ,$$ (5)

where $S_E$ is half the Euclidean action of the solution with the maximal slice at which $a = \tilde{a}$ and $\phi = \tilde{\phi}$. At the maximal slice the Euclidean solution can be analytically continued to a Lorentzian classical solution, and a classical universe is born with its initial data given by $a = \tilde{a}$ and $\phi = \tilde{\phi}$.

This analysis can be applied to evaluate the probability of creation of a universe that has witnessed an early inflationary era, at which the potential satisfies the slow-roll conditions. For simplicity, let us consider the chaotic inflationary scenario where the potential $V(\phi) \propto \phi^n$ with $n > 0$ has reflection symmetric around $\phi = 0$ and monotonically increases as $|\phi|$ increases. In Einstein gravity, there is a continuous distribution of complex-valued instantons for $|\phi| > \phi_{cut} \sim 1$ (hereafter, we adopt the natural units $8\pi G = M_{Pl}^2 = 1$), where $\phi_{cut}$ is a cutoff scale below which the classicality condition is no longer satisfied [5–7]. When there are approximately classical Euclidean solutions, the slow-roll condition is generally satisfied in the Lorentzian regime, and the Euclidean action is approximately given by

$$S_E \simeq -\frac{12\pi^2}{V(\phi)} .$$ (6)

Inserting this into equation (5), one immediately obtains the probability of creation of such a universe:

$$P \propto \exp \left( \frac{24\pi^2}{V(\phi)} \right) \implies \ln P \propto N^{-\frac{5}{2}} ,$$ (7)

where $N$ is the corresponding $e$-folding number defined by $dN = H dt$. It is obvious from equation (7) that an early universe with a large vacuum energy or large number of $e$-folds is exponentially suppressed. Hence, a successful inflationary scenario with sufficient $e$-folding number (e.g. 50–60 $e$-folding numbers) is disfavored, which is thought to be a defect of the
no-boundary proposal (for alternative explanations and reviews, see [8]). However, this severe problem can be naturally solved in a modified gravity theory with a non-vanishing graviton mass, that is, the so-called massive gravity theory.

2. dRGT massive gravity

Recently, with various motivations, massive gravity models have been extensively studied. In this paper, we consider nonlinear massive gravity proposed by de Rham, Gabadadze and Tolley [9], the so-called dRGT massive gravity, where \( \phi \) is the inflaton and \( \mathcal{L}_{mg} \) is the massive gravity term given by

\[
\mathcal{L}_{mg} = m_g^2 \left[ \frac{1}{2} \left( |\mathcal{K}|^2 - |\mathcal{K}_2| \right) + \alpha_3 \frac{1}{2} (|\mathcal{K}|^3 - 3|\mathcal{K}|^2 + |\mathcal{K}_2|^2) + \alpha_4 \right. \\
\left. + \frac{1}{2 \Omega_m^2} (|\mathcal{K}|^4 - 6|\mathcal{K}_2|^2|\mathcal{K}_2|^2 + 8|\mathcal{K}|^2|\mathcal{K}_2|^3 - 6|\mathcal{K}_2|^4) \right],
\]

(9)

with \( m_g^2 \) being a graviton mass parameter, \( \alpha_3 \) and \( \alpha_4 \) being non-dimensional parameters, and

\[
K_{\mu \nu} = \delta_{\mu \nu} - \sqrt{g^{\alpha \beta} G_{ab}(\phi)} \partial_\alpha \phi \partial_\beta \phi.
\]

(10)

The fields \( \phi^a (a = 0, 1, 2, 3) \) are called the St"uckelberg fields whose role is to recover the general covariance.

We set the fiducial, field space metric \( G_{ab}(\phi) \) to be a de Sitter metric \([10, 11],
\[
G_{ab}(\phi) d\phi^a d\phi^b \equiv -(d\phi^0)^2 + b^2 (\phi^0) d\Omega_3^2,
\]

(11)

where \( b(\phi^0) = F^{-1} \cosh(\phi^0) \) with \( F^{-1} \) being the curvature radius of the fiducial metric. For simplicity, we assume \( G_{ab}(\phi) \) to be non-dynamical as in the original dRGT gravity [9]. However, the theory may be more consistently formulated if \( G_{ab} \) is made dynamical [12]. We will not discuss further this issue here since it is beyond the scope of this short communication.

Under the assumption of \( O(4) \) symmetry, we set \( \phi^0 = f(\tau) \). Then after a straightforward calculation, we obtain \( b(\phi^0) = X_\pm a(\tau) \), where [13]

\[
X_\pm = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_4} - \alpha_4}{\alpha_3 + \alpha_4}.
\]

(12)

We note that \( X_\pm \) must be positive. This constrains the parameter space of the theory. The equations of motion are given by

\[
a^2 - 1 - \frac{\dot{\phi}^2}{3} = 0,
\]

(13)

\[
\ddot{\phi} + 3\frac{a}{a} \dot{\phi} - V' = 0,
\]

(14)

where a dot (\( \cdot \)) denotes \( d/d\tau \), a prime (\( \prime \)) denotes \( d/d\phi \), and the effective potential is given by

\[
V_{eff}(\phi) = V(\phi) + \Lambda_\pm
\]

(15)

with \( \Lambda_\pm \) being the cosmological constant due to the massive gravity terms,

\[
\Lambda_\pm = -m_g^2 (1 - X_\pm) [3(2 - X_\pm) + \alpha_3 (1 - X_\pm)(4 - X_\pm) + \alpha_4 (1 - X_\pm)^2].
\]

(16)

Using these equations of motion, we obtain the on-shell action as [10]

\[
S_{on} = 2\pi^2 \int d\tau \left[ 2a^3 V_{eff} - 6a - m_g^2 a^3 Y_\pm \sqrt{-f^2} \right],
\]

(17)
where
\[ Y_\pm \equiv 3(1 - X_\pm) + 3\alpha_3(1 - X_\pm)^2 + \alpha_4(1 - X_\pm)^3. \]  

(18)

Now we assume that the potential is sufficiently flat so that at leading order approximation we can set \( V'_{\text{eff}} = 0 \), hence \( \dot{\phi} = 0 \). Thus we obtain
\[ \phi = \phi_0, \]  

(19)

\[ a = \frac{1}{H} \cos H \tau, \]  

(20)

where \( H^2 = V'_{\text{eff}}(\phi_0)/3 \). Then the action (over the half hemisphere) is given by [10]
\[ S_E \simeq -\frac{4\pi^2}{X_\pm F^2} \left( 1 - \frac{m^2 Y_\pm}{X_\pm} \right). \]  

(21)

where \( \alpha = X_\pm F/H \) and
\[ C(\alpha^2) \equiv \frac{2 - \sqrt{1 - \alpha^2}(2 + \alpha^2)}{6\alpha^4}. \]  

(22)

Comparing equation (21) to (6), using the Friedman equation \( 3H^2 = V(\phi) \), one finds that a counter term proportional to \( m^2 \) appears in dRGT massive gravity theory, which drastically changes the behavior of wavefunction so that the distribution of probability may not peak at \( H^2 \simeq 0 \). It should be noted that the above approximation is valid as long as the slow-roll condition is satisfied and the field value is well outside the cutoff, i.e., \( |\phi_0| > \phi_{\text{cut}} \).

3. Sufficient e-folding number for inflation

As mentioned above, we must have \( X_\pm > 0 \). On the other hand, \( \alpha \) must be in the range \( 0 < \alpha < 1 \) [10]. Moreover, for definiteness, we also assume \( Y_\pm > 0 \). From equation (22), \( C(\alpha) > 0 \), which implies that the absolute value of \( S_E \) (or \( -S_E \)) is smaller in the massive gravity case than in the Einstein case, provided with the same value of \( H \).

Alternatively, if we fix the model parameters and vary \( H, \alpha^2 C(\alpha^2) \to 0 \) as \( H \to \infty \) while it approaches 1/3 as \( H \to H_{\text{min}} \), where \( \alpha(H_{\text{min}}) = 1 \) or \( H_{\text{min}} = X_\pm F \). Then there arises a hope that the probability of a universe with small \( H \) may be substantially suppressed, or conversely the probability of a universe with larger \( H \) is exponentially enhanced.

To see if this is the case or not, let us introduce variable \( u \equiv \alpha^2 = X_\pm^2 F^2/H^2 \) and rewrite equation (21) in the following form:
\[ S_E = -\frac{4\pi^2}{X_\pm^2 F^2} \left[ u - \frac{m^2 Y_\pm}{X_\pm} u^2 C(u) \right] \]  

\[ = -\frac{4\pi^2}{X_\pm^2 F^2} Q(u), \]  

(23)

where \( Q(u) \equiv -X_\pm^2 F^2 S_E/(4\pi^2) \). Then one finds
\[ \frac{\partial Q}{\partial H^2} = -\frac{\alpha^2}{H^2} \frac{\partial Q}{\partial \alpha^2} \]  

\[ = -\frac{\alpha^2}{H^2} \left( 1 - \frac{m^2 Y_\pm}{X_\pm} \frac{\alpha^2}{4\sqrt{1 - \alpha^2}} \right). \]  

(24)

The function \( Q(u) \) is plotted in figure 1 as a function of the normalized Hubble parameter \( h^2 = 1/u \), for \( m^2 Y_\pm/(F^2 X_\pm) = 0, 3, 6, \) and 10, respectively. It is readily seen that unlike the
Figure 1. The function $Q(u)$ as a function of the normalized Hubble parameter $h^2 = 1/u = H^2/(X^2 F^2)$ for $m_s^2 Y_{±}/(F^2 X_{±}) = 0$ (dashed), 3 (red), 6 (green), and 10 (blue) from top to bottom. As has been expected, in traditional Einstein gravity where $m_s = 0$, the probability will exponentially decrease when $H^2$ increases, which implies disfavor of inflation with a large number of $e$-foldings. However, in dRGT massive gravity, counter terms proportional to $m_s^2$ appear so that the probability peaks at much larger $H^2$. This implies the preference of a much larger $e$-folding number for a successful inflationary scenario, hence offers a way to realize inflation in the context of quantum gravity.

case for traditional Einstein gravity where $m_s = 0$ (dashed line), in dRGT massive gravity theory, the function $Q$, hence $−2\Sigma$ will be maximized at $\alpha^2 = \alpha_m^2$ where

$$\frac{\alpha_m^2}{\sqrt{1 - \alpha_m^2}} = \frac{4F^2 X_{±}}{m_s^2 Y_{±}}. \tag{25}$$

Since the left-hand side varies monotonically from zero to infinity as $\alpha^2$ varies in the range $0 < \alpha^2 < 1$, there always exists a unique maximum provided that both $X_{±}$ and $Y_{±}$ are positive. Thus the probability is maximized at $\alpha^2 = \alpha_m^2$ with the exponent given by

$$\ln P \approx −2\Sigma = \frac{8\pi^2}{X^2 F^2} G(\alpha_m^2);$$

$$G(u) \equiv \frac{u^2 - 2u + 4 - 4\sqrt{1 - u}}{3u}, \tag{26}$$

which may be well approximated by $G(u) \approx u/2$ when $u \ll 1$.

We see that the problem of the no-boundary proposal may be solved for a sufficiently wide range of the parameter space. For example, assuming both $X_{±}$ and $Y_{±}$ are of order unity, we may consider the case $m_s^2 \gg F^2$ which implies $\alpha_m^2 \approx 4X_{±}F^2/(Y_{±}m_s^2) \ll 1$. Inserting this into equation (26), one finds

$$\ln P \approx \frac{16\pi^2}{m_s^2 X_{±} Y_{±}} \approx \frac{4\pi^2}{H_m^2}, \tag{27}$$

where $H_m^2 \approx X_{±} Y_{±} m_s^2/4$ is the Hubble parameter at which the probability is maximized. Thus a classical universe emerges most probably with the Hubble parameter $H^2 = O(m_s^2)$. For $m_s^2$ close to the Planck scale this gives sufficient inflation.
4. Triggering mass parameter

We have shown that for a sufficiently large mass parameter $m_g$, the problem associated with the no-boundary proposal that the expected number of e-folds is too small to realize successful inflation may be solved. However, since we know that the graviton mass should be extremely small today, we need a mechanism to make it large only in the very early universe.

Here we present a couple of speculations for such a mechanism.

(1) Field-dependent mass: let us consider the case when $m_g^2$ is a function of the inflaton $\phi$ [14].

$$m_g^2 = m_0^2(\phi). \tag{28}$$

If $m_g^2$ is finite for $|\phi| > \phi_{\text{cut}}$ but exponentially small for $|\phi| < \phi_{\text{cut}}$, then our analysis in the case of massive gravity is still valid and the fact that there is no classical histories at $|\phi| < \phi_{\text{cut}}$ remains the same as in the Einstein case [5]. Therefore, there will be a long enough inflationary stage, and Einstein gravity will be recovered when the inflation ends at $|\phi| < \phi_{\text{cut}}$. For example, a simple function like $m_g^2 = m_0^2 \exp[-(\phi_{\text{cut}}/\phi)^2]$ seems to satisfy the requirement. Of course, however, we need a more thorough analysis before we may conclude that such a model can actually lead to the scenario described above (some discussions on its dynamics have been done in some references, e.g. [15]).

(2) Running mass parameter: quantum effects may change $m_g^2$ through energy scales. If gravitational interactions are not asymptotically free, then one may have a large graviton mass $m_g \sim M_{\text{Pl}}$ at the Planck energy scale, while it becomes small $m_g \sim H_0$ at the current energy scale, where $H_0$ is the current expansion rate of the universe. If this scenario works, it may be also possible to explain the accelerated expansion of the current universe simultaneously.

5. Conclusion

Studies of Hartle–Hawking no-boundary proposal for the wavefunction of the universe in the context of dRGT massive gravity opens a window to discuss inflationary scenario in quantum gravity theories. Traditionally, the no-boundary wavefunction exponentially prefers small number of e-foldings near the minimum of the inflaton potential, and hence it does not seem to predict the universe we observe today. However, we found that the contribution from the massive gravity sector can drastically change this situation. We showed that, for a fairly wide range of the parameters of the theory, the no-boundary wavefunction can have a peak at a sufficiently large value of the Hubble parameter so that one obtains a sufficient number of e-folds of inflation.

To make this model to work, however, we need to find a way to trigger the mass parameter in the very early universe while it should remain to be extremely small in the current universe. We speculated a couple of mechanisms for this purpose. It is a future issue to see if these mechanisms can be actually implemented in massive gravity. In addition, we remain a future issue to consider the implication of potential problems of the dRGT model [16] and applications for possible generalized massive gravity models that may not suffer from such problems, e.g., [17].

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