Fundamental physics and solar flares

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Solar flares exemplify a universal aspect of magnetic fields in our universe. Differential rotation, shear and thermal convection, among other things, produce complex patterns of turbulent flows in magnetized astrophysical systems such as galaxies, stars and accretions disks. Such a complex flow will in general entangle the field, by means of stochastic flux freezing, in an extremely complicated way. Once very entangled, the built-up magnetic tension will make the field slip through the fluid to reduce its spatial complexity level, otherwise the observed large scale fields in astrophysical objects could never be generated and evolved over cosmological time scales. Serving as a more direct evidence are highly entangled solar magnetic fields which, after escalating the solar activity, are succeeded by smoother magnetic configurations. Such a spontaneous slippage of magnetic fields launching jets of fluid—magnetic reconnection—is usually interpreted and described as a change in the topology of the stochastic magnetic fields in the literature of plasma physics and astrophysics. However, neither magnetic topology nor magnetic stochasticity level is given a precise mathematical definition, and such technical terms are usually used rather loosely. Most often, magnetic field lines are presumed to be uniquely defined and magnetohydrodynamic (MHD) equations to be well-defined, yet without rigorous mathematical justifications.

In the present paper, after illustrating the inaccuracy of such assumptions based on previous work, we show that in fact magnetic topology is well-defined only in the phase space corresponding to a dynamical system governed by the induction equation. Hence the field’s topology and stochasticity should be studied in terms of the corresponding phase space trajectories rather than the field lines in real Euclidean space. In fact, our results suggest that magnetic field lines should not be taken too literally because their existence and uniqueness and more importantly continuity in time require strong mathematical conditions, hardly satisfied in astrophysical systems. As for magnetic topology change, it is shown that the phase space topology is preserved in time for a magnetic field which, besides satisfying few continuity conditions, solves a time reversal invariant induction equation. What breaks the time symmetry in the induction equation is the presence of non-ideal plasma effects at small scales such as resistivity, which results from random collisions between diffusing electrons and other particles. The small scale, stochastic disturbances produced thereby are super-linearly amplified by Richardson diffusion in the turbulent cascade, which are eventually manifested as large scale reconnection events, somehow similar to stretching quantum fluctuations during inflation to seed large scale cosmological structures. This suggests that reconnection is rooted in the second law of thermodynamics that dictates entropy increase which in turn breaks the time symmetry.

I. INTRODUCTION

The sun, studied more than any other star for obvious reasons, has long played an important role in our attempts to understand nature, from helium which was first discovered on the sun to the evolution of magnetic fields in electrically conducting fluids. The internally generated solar magnetic fields, by the magnetic dynamo action working probably near the tachocline, affect the dynamics on the solar surface, e.g., the evolution of sunspots, and also more distant structures including our communication systems and, as a matter of fact, the biosphere. The sun’s magnetic signature is also carried to space by means of a stream of high energy charged particles, known as the solar wind, whose interaction with the magnetosphere leads to phenomena such as aurora (northern lights). Different magnetic processes observed on the sun, the corona and in the solar wind are also found abundantly in a spectrum of other systems from controlled fusion devices to the accretion disks around massive black holes. One example of such ubiquitous processes is the spontaneous acceleration of fluid particles in regions with strong magnetic shear, a process dubbed magnetic reconnection. A variety of non-ideal plasma effects such as resistivity (e.g., see [1]; [2]; [3]; [4]) as well as different non-linear turbulent effects (e.g., see [5]; [6]; [7]; [8]) have been proposed as mechanisms driving reconnection. These mechanisms are believed to enforce magnetic field lines to disconnect and reconnect again giving rise to a different field configuration which has a lower energy—a process of energy relaxation. The rapidly reconnecting field lines, as widely believed, can accelerate fluid elements on their way, whose collective effect at larger scales may be observed as spontaneous, eruptive fluid motions. There exists, of course, a large number of competing reconnection models, some more plausible than the others. For instance, the rapid motions caused by reconnection can make the flow turbulent ([5]; [9]), even if it was initially laminar, which demands taking into account the effects of turbulence, ubiquitous in astrophysical systems (see also [10]; [11]). Nevertheless, even with such more general considerations, neither of current reconnection models seems to satisfactorily explain observations. It is not also clear at present what may be the underlying mechanism governing reconnection in terms of the most fundamental laws of physics, not to mention the lack of complete consensus on the definition of reconnection itself.

Majority, if not all, of magnetic reconnection models are

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described in some way or another employing the concept of magnetic field lines, put forward long time ago by Faraday\(^1\).

In fact, reconnection is usually described, or even interchangeably referred to, as the magnetic topology change visualized in terms of magnetic field lines. Nonetheless, in the literature of magnetic reconnection, less attention is paid to the mathematical conditions required for magnetic field lines to be well-defined. The same goes for the definition of topology and topology change. Interestingly, in fact, it turns out that the concept of field lines is well-defined only for magnetic fields satisfying a strong mathematical condition (Lipschitz continuity; see §II), which is violated e.g., in turbulence. In addition, it also turns out that for time dependent fields in dissipative or turbulent media, the magnetic field lines do not continuously evolve in time, i.e., they do not deform continuously in space as time advances and consequently lack a preserved identity. Moreover, as we show in this paper, magnetic topology is not in fact preserved in time even for well-behaved and smooth fields. The term topology change too is usually employed in an inaccurate way, paying less attention, if at all, to certain mathematical technicalities (see §IV). Finally, in addition to the notion of topology, specially in turbulent reconnection and dynamo models, the concept of weak or strong magnetic stochasticity is also widely used in an almost casual way without providing a precise quantitative measure. These mathematical issues are physically remarkable since they point to fundamental misunderstandings and misrepresentations, which might in fact be responsible for our failure in achieving a fully satisfactory theory for magnetic field reconnection and magnetic field generation in cosmological bodies. Taking for granted the validity of apparently simple mathematical concepts, without actually validating them, can lead us astray.

Reconnection in laminar and turbulent flows is in fact a problem in plasma physics and magnetohydrodynamics which constitutes an intense research field developed primarily by plasma physicists and astrophysicists. In this paper, we approach the more fundamental aspects of this phenomenon as a general problem in theoretical physics to show how mathematical subtleties involved are connected to fundamental laws of physics. We revisit the notions of magnetic field lines, magnetic topology and topology change in particular for stochastic fields in the presence of turbulence. More detailed mathematical results applicable to any physical vector field can be found in [12]. In order to quantify the notions of weak and strong magnetic stochasticity, we employ a statistical formalism recently developed in [9], which was successfully tested in a subsequent work [13] using an MHD numerical simulation. The magnetic stochasticity level developed in these works was quantitatively related to magnetic diffusion in [14] which was also accompanied with a numerical test. We start off, in §II, by a brief, quantitative consideration to show that a renormalized version of magnetic and velocity fields should be employed in order to study the magnetic field evolution and reconnection in turbulent media. We also show that magnetic field lines are not well-defined in majority of physical situations e.g., in astrophysical systems. In §III, we reformulate the problem of magnetic reconnection based on the recent mathematical developments presented in [9]. In §IV, we show that magnetic topology should be considered in the phase space \((x, B)\) governed by the induction equation instead of the field lines in real Euclidean space. Moreover, the mathematical conditions required to keep the phase space topology are discussed in terms of the induction equation. A short discussion of magnetic diffusion and its relationship with spatial complexity of the magnetic field is presented in §V. Finally, it is shown in §VI that magnetic topology change in the corresponding phase space is related to the presence of non-ideal plasma effects and the non-linear turbulent effects, which break the time reversal invariance in the induction equation. This latter statement, in fact, illustrates that reconnection is rooted in the second law of thermodynamics, for the entropy increase in isolated, diffusive systems breaks the time symmetry. We summarize and discuss our results in §VI.

II. TIME EVOLUTION AND FIELD LINES

From the Maxwell’s equations, the governing equation for magnetic field \(B(x, t)\) is given by

\[
\frac{\partial B}{\partial t} = -\nabla \times E, \tag{1}
\]

where \(E(x, t)\) is the electric field. In a magnetized, and electrically conducting fluid, e.g., a plasma, the momentum equation for electrons dictates the generalized Ohm’s law \(E + u \times B = P\), hence eq.(1) becomes the induction equation:

\[
\frac{\partial B}{\partial t} = \nabla \times (u \times B - P), \tag{2}
\]

where \(u(x, t)\) is the velocity field (which is usually a solution of the Navier-Stokes equation although this assumption is not necessary here), and \(P\) represents non-ideal plasma effects, such as the Hall effect or the resistive electric field (which arises due to electron-ion collisions; \(P = \eta J\) with \(\eta\) being resistivity and \(J = \nabla \times B\) the electric current). In the limit \(P \to 0\), e.g., a vanishing resistive electric field in the limit \(\eta \to 0\), one can use the induction equation as \(D_t B = B.\nabla u - B.\nabla u + \eta \nabla^2 B\) with Lagrangian derivative \(D_t \equiv (\partial_t + u.\nabla)\) and combine it with the continuity equation \(D_t \rho + \rho \nabla . u = 0\) to write \(D_t (B/\rho) = (B/\rho).\nabla u\) which represents the conventional flux freezing theorem. This result is based on the presumption that MHD equations remain well-behaved in the limit \(\eta \to 0\), hence the integral curves (field lines) of \(B/\rho\) are advected with the fluid. This approximation, widely used in plasma physics and astrophysics, should be applied with care however—for flux freezing will not hold

\(^1\)It seems that for Faraday, magnetic field lines, which he called the lines of magnetic force, had more physical character than what an abstract mathematical object would have. In 1864, Maxwell adopted a slightly different terminology of magnetic force tubes but neither of them, of course, scrutinized the validity or usefulness of such notions for time-dependent fields in magnetized fluids.
even as an approximation if the flow becomes turbulent. It turns out, as a matter of fact, that in a fully turbulent fluid, in the limit of vanishing viscosity and resistivity, the velocity and magnetic fields will be in general Hölder singular\footnote{The real field $\mathbf{B}(x)$ is Hölder continuous in $x \in \mathbb{R}^n$ if $||\mathbf{B}(x) - \mathbf{B}(y)|| \leq C||x-y||^h$ for some $C > 0$ and $h > 0$. If $h = 1$, then $\mathbf{B}$ is uniformly Lipschitz continuous. Also $\mathbf{B}$ is called Hölder singular if $0 < h < 1$. Roughly speaking, a uniformly Lipschitz function $f(x)$ has a bounded derivative, i.e., $|f'(x)| < M$ for some $M > 0$. Hence, the derivative of a Hölder singular function can blow up; $|f'(x)| \to \infty$. However, experiments and simulations show that viscous energy dissipation rate $\nu |\nabla u|^2$ in a turbulent fluid with viscosity $\nu$ and velocity field $u$ does not vanish in the limit $\nu \to 0$. Similarly, magnetic energy dissipation rate $\eta |\nabla \mathbf{B}|^2$ in a turbulent fluid with magnetic diffusivity $\eta$ does not approach zero when $\eta \to 0$. These dissipative anomalies indicate that in the limit $\nu, \eta \to 0$, the velocity and magnetic divergences blow up, i.e., $|\nabla u| \to \infty$ and $|\nabla \mathbf{B}| \to \infty$. These ultra-violet (UV) divergences, historically first encountered in quantum field theories, imply that turbulent velocity and magnetic fields are not Lipschitz continuous. For details see \cite{15}; \cite{9} and references therein.} which implies ill-defined spatial derivatives (gradients) and hence ill-defined MHD equations\footnote{Without loss of generality, we also assume $G(r) \geq 0$, $\lim_{r \to \infty} G(r) \to 0$, $\int_{\mathbb{R}^n} d^3r G(r) = 1$, $\int_{\mathbb{R}^n} d^3r r G(r) = 0$, $\int_{\mathbb{R}^n} d^3r |r|^2 G(r) = 1$ and $G(r) = G(|r|)$ with $|r| = r$. Mathematically, $G \in C^0_\infty(\mathbb{R}^n)$; the space of infinitely-differentiable functions with compact support. A function $g$ is said to have a compact support (set for its arguments for which $g \neq 0$ if $g = 0$ outside of a compact set (equivalent to closed and bounded sets in $\mathbb{R}^n$).}.

In order to remove these Hölder singularities from a given field $\mathbf{B}(x, t)$, defined in a spatial volume $V$, we can renormalize it at a spatial scale $l > 0$ using distributions, e.g., by writing

$$\mathbf{B}_l(x, t) = \int_V G_l(r)\mathbf{B}(x + r, t) d^3r,$$  \hspace{1cm} (3)

where $G_l(r) = l^{-3}G(r/l)$ with $G(r)$ being a smooth and rapidly decaying kernel, e.g., the Gaussian kernel $G(r) \sim e^{-r^2}$. We will call $\mathbf{B}_l$ the bare field whereas $\mathbf{B}_l$ is the renormalized, or coarse-grained, field at scale $l$\footnote{This is the Picard-Lindelöf theorem for a system of differential equations.}. Integration by parts in $\nabla_x \mathbf{B}_l$ makes the derivative act on $G$, implying that $\mathbf{B}_l$ is Hölder continuous even if $\mathbf{B}_l$ is not. Differential equations containing the field $\mathbf{B}$ can also be multiplied by the kernel $G$ and integrated to get the corresponding renormalized equations. The renormalized induction equation thus reads

$$\frac{\partial \mathbf{B}_l}{\partial t} = \nabla \times \left( \mathbf{u}_l \times \mathbf{B}_l - \mathbf{R}_l - \mathbf{P}_l \right),$$  \hspace{1cm} (4)

using the renormalized Ohm’s law $\mathbf{E}_l + (\mathbf{u}_l \times \mathbf{B}_l) = \mathbf{P}_l,$ which can also be written as

$$\mathbf{E}_l = \mathbf{P}_l + \mathbf{R}_l - \mathbf{u}_l \times \mathbf{B}_l.$$  \hspace{1cm} (5)

Even with a negligible non-ideal term $\mathbf{P}_l$, the non-linear term $\mathbf{R}_l = - (\mathbf{u}_l \times \mathbf{B}_l) + \mathbf{u}_l \times \mathbf{B}_l$ can be large. This term, known as the turbulent electromagnetic force (EMF) $\mathbf{E}_l = - \mathbf{R}_l$, is the motional electric field induced by turbulent eddies of scales smaller than $l$ and plays a crucial role in magnetic dynamo

Theories. One can use the renormalized induction equation, eq.(4), to study the time evolution of the unit tangent vector, $\mathbf{B}_l = \mathbf{B}_l/|\mathbf{B}_l|$ (related to magnetic topology) and magnitude $|\mathbf{B}_l| = |\mathbf{B}_l|$ (related to magnetic energy) separately \cite{9}. The induction equation implies

$$\begin{align*}
\partial_t \mathbf{B}_l &= - (\nabla \times \mathbf{u}_l) \times \mathbf{B}_l - (\mathbf{R}_l + \mathbf{P}_l), \\
\partial_t |\mathbf{B}_l| &= - (\mathbf{R}_l + \mathbf{P}_l) \cdot |\mathbf{B}_l|,
\end{align*}$$  \hspace{1cm} (6)

where $(\cdot)_\parallel$ and $(\cdot)_\perp$ respectively refer to the parallel and perpendicular direction with respect to $\mathbf{B}_l$ and

$$\begin{align*}
\mathbf{R}_l &= (\nabla \times \mathbf{u}_l), \\
\mathbf{P}_l &= (\nabla \times \mathbf{R}_l), \\
(\sigma)_\parallel &= (\mathbf{R}_l - \mathbf{P}_l)/|\mathbf{B}_l|.
\end{align*}$$  \hspace{1cm} (7)

These terms, as we will see in the next section, are associated with field-fluid slippage and reconnection. It is also important to note that $\mathbf{R}_l$, $\mathbf{P}_l$ at any scale $l$ are associated with non-ideal plasma and non-linear turbulent phenomena affecting the field’s topology whereas $\mathbf{P}_l$, $\mathbf{R}_l$ dissipate magnetic energy without any direct effect on the evolution of magnetic topology.

Let us also shortly comment on the concept of field lines or integral curves of a given magnetic field $\mathbf{B}$. At time $t_0$, the solution of the following initial value problem is a curve, a magnetic filed line, passing through an arbitrary point $x$:

$$\begin{align*}
\frac{d\xi_\nu(s, t_0)}{ds} &= \mathbf{B}(\xi_\nu(s, t_0), t_0), \\
\xi_\nu(0, t_0) &= x.
\end{align*}$$  \hspace{1cm} (8)

This initial value problem has a unique solution\footnote{This uniformity implies that magnetic field lines do not generally deform in a continuous manner in dissipative and turbulent media. It is indeed a commonplace oversimplification to formulate or describe phenomena such as reconnection in such environments appealing to the notion of field lines, which are abstract mathematical, and not physical, objects useful only in certain situations. Nevertheless, to be very clear, it should be emphasized that the Hölder singularities of magnetic and velocity fields in MHD \cite{10}; \cite{15}; \cite{9}] are usually asymptotic and the fields are in fact differentiable at very small scales. But, as a general remark, one should keep in mind the simple fact that even the} if $\mathbf{B}(x, t)$ is uniformly Lipschitz in $x$ \cite{12}, which may not hold in turbulence (see below). Moreover, even with a unique solution at time $t_0$, the field lines will not necessarily continuously evolve in time unless $\mathbf{B}$ is uniformly Lipschitz in spacetime position vector $\vec{x} = (x, t)$ (see Appendix A and \cite{12} for details). Finally, we note that the Lipschitz continuity of $\mathbf{B}(\mathbf{B})$ follows from that of $\mathbf{B}(\mathbf{B})$ if magnetic energy $B^2/2$ is bounded from above (below), i.e., $B \leq B_{\text{max}} > 0$ ($B \geq B_{\text{min}} > 0$); see Appendix B and \cite{12}. The physical implication is that magnetic field lines do not generally deform in a continuous manner in dissipative and turbulent media. It is indeed a commonplace oversimplification to formulate or describe phenomena such as reconnection in such environments appealing to the notion of field lines, which are abstract mathematical, and not physical, objects useful only in certain situations. Nevertheless, to be very clear, it should be emphasized that the Hölder singularities of magnetic and velocity fields in MHD \cite{10}; \cite{15}; \cite{9}] are usually asymptotic and the fields are in fact differentiable at very small scales. But, as a general remark, one should keep in mind the simple fact that even the
concept of a fluid may lose its meaning if one naively goes down to extremely small scales. On the other hand, and more importantly, even at scales which are large enough to let the fluid approximation be valid but otherwise are very small, e.g., compared to the system size or down in the turbulence inertial range, an extreme sensitivity to initial conditions is observed. Nevertheless, these integral curves, in general, will show extreme sensitivity to the initial conditions (usually characterized by Lyapunov exponents) may plague differential equations such as the initial value problem that defines magnetic field lines; eqs. (8). Take for example two field lines, $\xi_x(s)$ and $\xi_x(s)$ starting from two nearby points $x$ and $x'$. With a small but still non-zero magnetic diffusivity, at very small scales, the field can be Lipschitz and its corresponding field lines unique. Nevertheless, these integral curves, in general, will show extreme sensitivity to the initial conditions such that the distance between them, i.e., $|\xi_x(s) - \xi_x(s)|$, may become independent of $|x' - x|$ at distances $s \gg |x' - x|$. The corresponding large Lyapunov exponents indicate explosive separation of magnetic field lines which will not be present if the field lines exhibit standard deterministic chaos in which, unlike turbulence, the system preserves a memory of the initial conditions (see also [16]; [6]; [15]).

III. SLIPPAGE AND RECONNECTION

The non-ideal effects in the reorganized Ohm’s law, eq.(5), denoted collectively by $V_l$ at scale $l$, arise from micro-scale plasma effects such as the resistive electric field, Hall effect etc., which drive reconnection at small diffusive scales and are mathematically represented by $\sigma_l$ in the induction equation; eqs.(6). On the other hand, the non-linear term $R_l$ in the Ohm’s law arises from the non-linear interactions below the arbitrary scale $l > 0$ which correspond to the non-linear term $\Sigma_l$ in the induction equation; eqs.(6). At larger scales in the turbulent inertial range, $\Sigma_l$ dominates $\sigma_l$. However, $\Sigma_l$ decreases with decreasing scale and eventually becomes comparable to $\sigma_l$ at the dissipative scale down the inertial range. Below the dissipative scale, $\sigma_l$ dominates $\Sigma_l$. One may argue that the explosive nature of super-linear Richardson diffusion in the inertial range brings distant field lines to small distances set by resistivity where they may reconnect while it also causes explosive separations between initially close field lines. Nevertheless, this argument, in particular its first part, takes the notion of field lines in a very literal sense which we try to avoid here. The notion of close or distant field lines in real space can be avoided altogether employing instead the phase space trajectories in the context of dynamical systems. What happens in real space, during reconnection, can be understood in terms of the super-linear amplification of small scale magnetic disturbances, generated by the non-ideal plasma effects at the dissipative scale, by the turbulence. This is somehow similar to cosmological inflation which might have stretched sub-atomic quantum fluctuations to astrophysical density perturbations which in turn seed large scale cosmological structures.

Physically, $B_l(x, t)$ is the weighted-average magnetic field of a fluid parcel of size $l$ at point $x$. Since $G(\mathbf{r}/l)$ is a rapidly decaying function so the integral $B_l(x, t) = \int_V G_l(\mathbf{r}) B(x + \mathbf{r}, t) d^3 \mathbf{r}$ gets smaller and smaller contributions from points at distances $\gg l$ from $x$. If we renormalize the field at a larger scale $L > l$, on the other hand, we will get the average magnetic field of a fluid parcel of scale $L$ at point $x$, i.e., $\tilde{B}_L(x, t)$ which is in general different from $B_l(x, t)$ because the weight function $G(\mathbf{r}/L)$ gets major contributions only from points with distance $\sim L > l$ from $x$. In a laminar flow threaded by a smooth magnetic field with a large curvature radius $\gg L$, we expect $\tilde{B}_L \tilde{B}_L \approx 1$. For a stochastic magnetic field in a fully turbulent medium, on the other hand, $L > 1 \leq B \tilde{B}_L \lesssim 1$ becomes a rapidly varying stochastic variable which measures the spatial complexity (or stochasticity level) of $B$ at point $x$. Its root-mean-square (rms) value tells us how spatially complex (or stochastic) the field is on average in a given volume $V$. For a stochastic field, it is also a measure of the field’s stochasticity. To obtain a non-negative quantity, we can volume average $\frac{1}{2} [\tilde{B}_L(x, t) \tilde{B}_L(x, t) - 1]$. In fact, applied to the velocity field $u$ in a turbulent flow, the quantity $\frac{1}{2}[u(x, t) \tilde{u}_L(x, t) - 1]$ measures the spatial complexity (or the level of randomness) of the fluid motions at point $(x, t)$. Here $\tilde{u}_L$ is the velocity field renormalized at scale $l (L)$. The spatial complexity, or stochasticity level, of the magnetic (or velocity) field in an arbitrary spatial volume $V$ is defined as the root-mean-square of this quantity \footnote{A more general definition employs the $L_p$ norm: $S_p(t) = \frac{1}{2} [B_l \tilde{B}_L - 1] \|p\| = \frac{1}{2} \left[ \int_V |B_l \tilde{B}_L - 1|^p \, d^3 x \right]^{1/p}$ for any arbitrary $p \in \mathbb{N}$. We take $p = 2$ which corresponds to rms value; $S_2(t) = \frac{1}{2} (B_l \tilde{B}_L - 1)_{\text{rms}}$. See [9] and [13] for more details.}

\begin{equation}
S(t) = \frac{1}{2} (\tilde{B}_L \tilde{B}_L - 1)_{\text{rms}}.
\end{equation}

In the language of topological dynamics in mathematics, a topological entropy can be defined in the magnetic phase space $(x, B)$, corresponding to magnetic dynamical system defined by $\dot{x} = B$ and $\partial B = -\nabla \times E$ as a measure of the complexity of the system [12]. Similarly, in real Euclidean space $\mathbb{R}^3$ (or in general on a manifold), the function $S(t)$ is a measure of spatial complexity of the magnetic field (for mathematical details see [12]).

Conventional magnetic flux freezing (Alfvén theorem), which asserts that the magnetic field is perfectly frozen into a fluid with a vanishingly small resistivity, fails in turbulent flows. However, a more general form known as stochastic flux freezing [16] holds which can be interpreted statistically; the field follows the turbulent fluid motions only in an average and stochastic manner ([9]; [13]). Therefore, turbulence will tend to tangle the field and increase its spatial complexity (stochasticity); $T(t) = \partial_t S(t) > 0$. On the other hand, the field resists the increase in its spatial complexity (by means of magnetic tension force) and at some point slips through the fluid to relax, during which $T(t) < 0$. Thus, when magnetic complexity...
(stochasticity) level reaches a local maximum, i.e., \( T(t) = 0 \) and \( \partial_t T(t) < 0 \), and then starts to decrease, i.e., \( T(t) < 0 \), the rapidly relaxing field will accelerate fluid particles (by means of Lorentz force). This in turn increases kinetic complexity (stochasticity) level \( s(t) \):

\[
s(t) = \frac{1}{2} (\mathbf{u}_L \cdot \mathbf{u}_L - 1)_{\text{rms}},
\]

This picture associates magnetic reconnection with \( T(t) = 0 \) and \( \partial_t T(t) < 0 \) which is almost simultaneous with \( \tau(t) = \partial_t s(t) > 0 \)—magnetic reconnection enhances turbulence thus kinetic stochasticity.

Quite apart from the mathematical subtleties regarding the field lines and field topology, as discussed above, reconnection or field-fluid slippage in a small volume \( \sim l^3 \) of the fluid is expected to correspond to spontaneous, large changes in the magnetic direction vector *\( \mathbf{B}_l \)*. Here, the weighted average of *\( \mathbf{B}_l \)*, measured with respect to the large scale field \( \mathbf{B}_L \) with \( L \gg l \). Likewise, a global topology change in a larger volume \( \sim L^3 \) corresponds to large changes in the magnetic direction vector \( \mathbf{B}_L \) measured with respect to the small scale field \( \mathbf{B}_l \). In other words, we may use the average of \( \mathbf{B}_l \cdot \partial_t \mathbf{B}_L + \mathbf{B}_L \partial_t \mathbf{B}_l \) as a measure of the field reconnection at point \((x,t)\). The weighted average of this quantity, which is incidentally equal to \( \partial_t \mathbf{B}_L \), is in fact proportional to \( T(t) = \partial_t S(t) \):

\[
T(t) = \frac{1}{4S(t)} \int_V (\mathbf{B}_l \cdot \mathbf{B}_L - 1) \partial_t (\mathbf{B}_l \cdot \mathbf{B}_L) \frac{d^3x}{V},
\]

which represents the rate of change of the magnetic field configuration (but not necessarily rate of change in the velocity field, see eq.(16) below). A similar quantity can be written for the velocity field, \( \tau(t) = \partial_t s(t) \):

\[
\tau(t) = \frac{1}{(\mathbf{u}_L \cdot \mathbf{u}_L - 1)_{\text{rms}}} \times \int_V \frac{1}{2} (\mathbf{u}_l \cdot \mathbf{u}_L - 1) \partial_t (\mathbf{u}_l \cdot \mathbf{u}_L) \frac{d^3x}{V}.
\]

Combining the first equation in (6) with (11) shows that \( T(t) \) depends on weighted averages of \( \sigma_{l\perp}, \Sigma_{l\perp} \) and their counterparts at scale \( L \). It has already been shown [6] that these quantities act as source terms in differential equations governing the relative velocity between the field and fluid. Here, we will avoid such literal interpretations in terms of field lines and their relative motion with respect to the fluid, however, one may still think of \( \sigma_{l\perp}, \Sigma_{l\perp} \) as a measure of the field-fluid slippage ([9]; [13]).

Fig.(1) plots \( T(t) \) and \( \tau(t) \) in an incompressible, homogeneous MHD numerical simulation stored online ([17];[18];[19]). Turbulence gradually increases an initially small magnetic stochasticity, hence \( T = \partial_t S > 0 \). At some point, the field’s resistance against tangling leads to a spontaneous field-fluid slippage (reconnection) which decreases the magnetic stochasticity from its maximum \( T = \partial_t S = 0 \) and \( \partial_t^2 S = \partial_t T < 0 \) and accelerates fluid elements (points A, B and C in Fig.(1)). This in turn increases kinetic stochasticity, i.e., \( \tau = \partial_t s > 0 \), which corresponds to the positive values of \( \tau \) in Fig.(1). A more detailed theoretical and numerical approach to reconnection using this formalism can be found in [13].

The statistical formalism presented so far can also be used to define a rate for field-fluid slippage and reconnection. The incompressible Navier-Stokes equation can be written as

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) = \mathbf{f} + \mathbf{j} \times \mathbf{B},
\]

where \( \mathbf{j} = \nabla \times \mathbf{B} \) is the electric current and \( \mathbf{f} \) represents all other non-magnetic force densities including pressure gradient, viscous and any external force. The renormalized form of this equation reads

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) = \mathbf{f} + \mathbf{j} \times \mathbf{B} + \mathbf{N}_t,
\]

where \( \mathbf{M}_t = (\mathbf{u} \mathbf{u})_t - \mathbf{u}_t \mathbf{u}_t \) is the turbulent stress tensor. The non-linear term

\[
\mathbf{N}_t = (\mathbf{j} \times \mathbf{B})_t - \mathbf{j}_t \times \mathbf{B}_t,
\]
may be dubbed the reconnection field, for it is the magnetic force responsible for spontaneously driving fluid jets and, con-
sequently, increasing the kinetic stochasticity \( s(t) \). In fact, \( N_L \) is the turbulent magnetic force generated by eddies at scales 
\(< l \). The reconnection rate, at which this force changes the kinetic stochasticity \( s(t) \), is determined by the contribution of \( N_L \) to \( \tau = \partial_t s \), that is

\[
\tau(t) \bigg|_{\text{rec}} = \frac{1}{(\tilde{u}_L - 1)_{\text{rms}}} \times \int_V \frac{1}{2} (\tilde{u}_L - 1) \left( \frac{N_L}{u_L} + \tilde{u}_L \right) \frac{d^3x}{V},
\]

where \((.)_{\text{L}}\) and \((.)_{\text{L+}}\) denote, respectively, the perpendicular components with respect to \( u_L \) and \( u_L \). For a reconnection region of scale \( l \), embedded in a system of size \( L \gg l \), the global field can be initially assumed undisturbed while the local field \( B_1 \) undergoes reconnection/slippage, in which case the last term in eq.(16), i.e., \( \tilde{u}_l(N_L/u_L) \), can be neglected. Hence, the reconnection rate will be

\[
\tau(t) \bigg|_{\text{rec}} = \int_V W(x,t) \frac{\tilde{u}_L N_{L+}}{u_L} d^3x,
\]

with the weight function

\[
W(x,t) = \frac{1}{2} \frac{\tilde{u}_L - 1}{(\tilde{u}_L - 1)_{\text{rms}}}. \tag{17}
\]

In passing, we also note that the new formalism presented above is based on the concept of scale split energy density \( \psi(x,t) = \frac{1}{2} B_1 B_L \) which can be written in terms of two other scalar fields in the form \( \psi = \chi \phi \). Here \( \phi = B_1 B_L \) is associated with magnetic topology used to define spatial complexity (or stochasticity level) while \( \chi = \frac{1}{2} B_1 B_L \) is related to magnetic energy and is in fact the geometric mean of the energy densities at scales \( l \) and \( L \); \( \chi = \sqrt{\frac{B_1^2 + B_L^2}{2}} \). These scalar fields have different evolution equations, obtained using the renormalized induction equation, which can be used to study magnetic field evolution in a statistical context; see [9].

IV. TOPOLOGY AND DYNAMICAL SYSTEMS

The Euclidean vector norm, \(|| \cdot |||\), defines a notion of distance (i.e., metric) for the magnetic field \( B \), e.g., \(||B(x) - B(y)|||\) can be taken as the distance between the vectors \( B(x) \) and \( B(y) \). We are interested, however, in magnetic vectors which are not close in the above sense but also are located at nearby points in real space. That is to say, we are interested in magnetic vectors \( B(x) \) and \( B(y) \) for which both the quantities \(||B(x) - B(y)|||\) and \(||x - y|||\) are smaller than an arbitrary positive number. For example, in reconnection, we are concerned with the magnetic vectors located in a spatial volume, i.e., the reconnection region. This condition can be represented mathematically as

\[
\sqrt{||x - y||^2 + ||B(x) - B(y)||^2} < r,
\]

for a positive number \( r > 0 \). Note that this expression defines the distance between two points in the phase space \( (x,B) \) [12]. Any trajectory \( x(t) \) in this phase space is a solution to an initial value problem in the form

\[
\frac{dx(t)}{dt} = B(x(t)); x(0) = x_0,
\]

where the field solves the induction equation, eq.(2). Magnetic field topology, therefore, is naturally defined in the phase space. In order to ensure that time evolution preserves the topological properties of the phase space, the field \( B(x,t) \) is required to be (i) Lipschitz continuous in \( x \), (ii) uniformly continuous in \( t \), (iii) odd under time reversal \( B(x,t) = -B(x,t) \) and solve a (iv) time reversal invariant induction equation; see Appendix C or [12] for details.

One can always renormalize the field and MHD equations to remove any Hölder singularities, as discussed before. Therefore, the time evolution of the magnetic field, resolved at an arbitrary scale \( l > 0 \) (whose vanishing limit, \( l \to 0 \), corresponds to the real, bare field \( B \) translates into the dynamics of a particle moving in the phase space with velocity \( dy(t)/dt = B_l(y(t),t) \) whose trajectory solves the following equation:

\[
\frac{d^2y}{dt^2} = B_l \nabla B_l + \nabla \times (u_l \times B_l) - B_l(\sigma_l + \Sigma_l), \tag{19}
\]

with appropriate initial conditions. The first term on the right hand side of eq.(19) is the acceleration due to the magnetic tension force while the second term represents the effects of the flow, and the last two terms are non-ideal plasma and non-linear turbulent effects, respectively. These two last terms break the time symmetry in this equation of motion.

V. THE SECOND LAW AND TIME SYMMETRY

In vacuum, electric field is even under time reversal, \( E(-t) = -E(t) \) while magnetic field is not, i.e., \( B(-t) = -B(t) \). Consequently, the Faraday equation, \( \partial_t B = -\nabla \times E \), respects time reversal invariance as expected. In resistive and turbulent fluids, frequently encountered in astrophysics, however, the time reversal symmetry can be broken by various non-idealities. On the one hand, both magnetic and velocity fields are odd under time reversal, i.e., \( B(-t) = -B(t) \) and

\[8\]

A method of nondimensionalization should obviously be applied to ensure that expressions which mix \( B \) and \( x \) are dimensionally correct.
can break the time reversal invariance symmetry in eq.(2), that is \( \partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B} - \mathbf{P}) \). For resistive electric field, for instance, \( \mathbf{P}(-t) = \eta \nabla \times (\nabla \times \mathbf{B}(-t)) = -\mathbf{P}(t) \). As a consequence of this broken symmetry, the magnetic field topology is not preserved in a resistive fluid. Since electrons and ions interact with the magnetic field, a sudden change in magnetic topology can in general accelerate these charged particles and therefore convert magnetic energy into other forms. Hence, even with smooth magnetic and velocity fields, the presence of a non-ideality such as resistivity will prevent magnetic field from keeping its topological properties intact. This corresponds to magnetic reconnection. It is important to note also that turbulent flows are not invariant under time reversal. Therefore, the non-linear electromotive force, \( \mathbf{R}_t \), will also break the time reversal invariance in the induction equation, eq.(4).

Time symmetry implies entropy conservation while the second law of thermodynamics indicates that entropy never decreases in isolated systems as time flows forward. Dissipative phenomena, which break time symmetry, are originated in the second law of thermodynamics. Therefore the time symmetry breaking effect of the non-ideal dissipative terms, denoted by \( \mathbf{P} \) in the Ohm’s law, is directly related to the second law. The most fundamental physical law behind magnetic topology change, or reconnection, is the second law of thermodynamics.

VI. SUMMARY AND DISCUSSION

In this paper, we have advanced physical arguments to support the idea that the dynamics of magnetic fields in dissipative or turbulent systems should be studied in terms of the phase space trajectories instead of magnetic field lines in real space. In fact, as it turns out, for time dependent magnetic fields in dissipative and turbulent fluids, the corresponding field lines do not in general evolve continuously in real space, i.e., the field lines at a given time \( t_0 \) do not continuously deform to the field lines at a later time \( t_1 \). This strongly restricts their usefulness in describing, let alone formulating, phenomena such as magnetic reconnection. Magnetic field lines are uniquely defined only if the tangent vector \( \mathbf{\hat{B}} \) is uniformly Lipschitz continuous, a mathematical condition which is seldom satisfied in astrophysical plasmas. Moreover, even if defined uniquely, field lines are not generally continuous in time unless \( \mathbf{\hat{B}} \) is uniformly Lipschitz continuous in space-time vector \( \mathbf{x} = (x, t) \). Hence appealing to the magnetic field lines as continuously deforming curves in space is mathematically problematic. Phase space trajectories defined by \( \mathbf{x}(t) = \mathbf{B}(\mathbf{x}(t), t) \) in the 6-dimensional phase space \((\mathbf{x}, \mathbf{B})\) with \( \mathbf{B} \) solving the induction equation, on the other hand, are more fundamental objects which allow one to evoke a variety of standard methodologies in the mathematical theory of dynamical systems, such as stochasticity and entropy, to study magnetic phenomena.

We have also argued that the term magnetic topology, in the context of reconnection and dynamo theories, is often employed without paying attention to its real mathematical meaning. In principle, topology for a given vector field \( \mathbf{B} \) should be defined e.g., in terms of open balls in real space as the Euclidean norm \( ||\mathbf{B}(x) - \mathbf{B}(y)|| \) defines a notion of distance (metric) \([12]\). However, even if defined rigorously, e.g., using the fact that a vector field defines a metric space and thus also a topological space, such a topology is not associated with magnetic vectors that are nearby in both vector space and real Euclidean space. The phase space topology, however, satisfies these conditions constituting a more desirable picture in dealing with problems such as magnetic field generation and reconnection which occur in a finite volume of real space. Moreover, the phase space topology is preserved if the field satisfies certain mathematical conditions, unlike the metric topology in Euclidean space mentioned above which is not generally preserved even for well-behaved smooth fields. Not only are magnetic topology and topology change well-defined, in terms of physical plausibility, only in the phase space \((\mathbf{x}, \mathbf{B})\) but also this formalism leads to a deeper physical understanding of magnetic phenomena in turbulent fluids. These considerations, all in all, suggest the study of magnetic fields in the context of dynamical systems. This approach avoids the difficulties associated with the definition of continuously evolving field lines and leads to a mathematically accurate picture in terms of magnetic topology. In addition, this formalism makes direct contact with fundamental concepts such as entropy and time symmetry in theoretical physics. For instance, the magnetic topology change in the phase space \((\mathbf{x}, \mathbf{B})\), which is intimately related to field-fluid slippage and reconnection, turns out to be a consequence of the second law of thermodynamics.

Stochastic flux freezing \([16]\) in turbulence—a statistical generalization of Alfvén flux freezing which holds only in laminar flows—allows the turbulent flow to entangle the magnetic field in a complicated way which can only be quantified statistically. The result is twofold: the field becomes stochastic and also Hölder singular with ill-defined spatial derivatives. Renormalization (coarse-graining) of the field \( \mathbf{B} \), using the mathematical concept of distributions, at an arbitrary scale \( l > 0 \) can be employed to remove the singularities and obtain a smooth field \( \mathbf{B}_l \)—the very quantity that we can actually measure in the lab unlike the real, mathematical, bare field \( \mathbf{B} \). This methodology is incidentally similar to renormalization methods used in quantum field theories to deal with ultra-violet infinities which resemble the infinities we encounter for derivatives of Hölder singular fields; \( |\nabla \mathbf{B}| > \infty \). In such an approach to remove singularities and infinities, the coarse-grained field \( \mathbf{B}_l(x, t) \) corresponds to the average field in a parcel of fluid of size \( l \) at spacetime point \((x, t)\), which is smooth with finite derivatives \( |\nabla \mathbf{B}_l| < \infty \). Renormalizing MHD equations in a general flow, it follows that the evolution equation of the unit direction vector \( \mathbf{B}_l = \mathbf{B}_l/B_l \) reads

\[
\partial_t \mathbf{B}_l = \frac{\nabla \times (\mathbf{u}_l \times \mathbf{B}_l)_\perp}{B_l} - \frac{\sigma_l}{B_l} \mathbf{B}_l + \frac{\Sigma_l}{B_l} \mathbf{B}_l \nonumber
\]

For the case of laminar flows, for which \( \Sigma_l \) is unimportant, a large \( \sigma_l = (\nabla \times \mathbf{P}_l)/B_l \), e.g., encountered in regions with
intense electric current corresponding to $\mathbf{P} = \eta \nabla \times (\nabla \times \mathbf{B})$, can lead to appreciable changes in $\mathbf{B}_t$, associated with field-fluid slippage (negligible energy dissipation) or reconnection events (appreciable energy dissipation). These arise, at a more fundamental level, as a result of magnetic topology change in the phase space $(x, \mathbf{B})$, which is allowed whenever the time symmetry is broken by the presence of non-ideal effects such as random collisions between electrons and other particles (resistivity). In turbulence, on the one hand, $\sigma_r$ dominates $\Sigma_L = \nabla \times \left( \mathbf{u}_L \times \mathbf{B}_L - (\mathbf{u} \times \mathbf{B}_t) \right) / B_t$ at small dissipative scales, therefore, similar to the laminar case, $\sigma_r$ will still affect the evolution of the unit direction vector $\hat{\mathbf{B}}_t$ at small scales. Super-linear Richardson diffusion in the turbulent cascade, on the other hand, will amplify these small scale random variations in $\mathbf{B}_t$ leading to considerable variations at much larger scales $L \gg l$ in the turbulence inertial range where $\Sigma_L$ dominates $\sigma_L$:

$$\partial_t \hat{\mathbf{B}}_L = \frac{\nabla \times \left( \mathbf{u}_L \times \mathbf{B}_L \right) \perp}{B_L} - \underbrace{\sigma_{L \perp} - \Sigma_{L \perp}}_{\text{sub-dominant}} \cdot$$

Such large scale, spontaneous changes in $\hat{\mathbf{B}}_L$ can accelerate fluid particles producing jets of fluid—large scale reconnection. This process can be formulated statistically by quantifying the notion of magnetic spatial complexity level using the quantity $S = \frac{1}{2} (\mathbf{B}_t, \mathbf{B}_L - 1)_{rms}$. In fact, $S$ turns out to be related to magnetic diffusion [14]. An increasing (decreasing) $S$ is also accompanied with a decreasing (increasing) magnetic energy density $B^2 / 2$ in MHD turbulence [9]. Convective flux freezing in laminar flows and stochastic flux freezing in turbulence both indicate that a complex flow will in general entangle the magnetic field increasing the magnetic complexity level; $\partial_t S > 0$. At some point, the field’s resistance against tangling will lead to a spontaneous field-fluid slippage which consequently decreases $S(t)$ from its maximum, where $\partial_t S = 0$ and $\partial^2_t S < 0$. This relaxation process in turn can accelerate fluid elements, by means of Lorentz force, which translates into an increase in the spatial complexity of the velocity field: $s = \frac{1}{2} (\mathbf{u}_L, \mathbf{B}_L - 1)_{rms}$, i.e., $\partial_t s > 0$. As a result, reconnection may be defined using the conditions $\partial_t S = 0$, $\partial^2_t S < 0$ and $\partial_t s > 0$. Reconnection seems to involve stochastic, eruptive fluid motions which proceed at time-dependent rates on different scales, a picture best studied by statistical methods and hardly captured in conventional models.

**Appendix A: Continuity of Field Lines**

In this appendix, we obtain the mathematical conditions for the uniform continuity of magnetic field lines in time (for details see [12]). The equation defining the integral curves, i.e., magnetic field lines, at time $t_0 + \epsilon$ for a real $\epsilon$ reads

$$\frac{d \xi_x(s, t_0 + \epsilon)}{d s} = \hat{\mathbf{B}}(\xi_x(s, t_0 + \epsilon), t_0 + \epsilon),$$

$$\xi_x(0, t_0 + \epsilon) = x.$$

The condition for $\xi_x$ to be continuous in $t$ is $\lim_{\epsilon \to 0} ||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| \to 0$ for any $s$. We write

$$||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)||$$

$$= ||\int_0^s ds' \left[ \hat{\mathbf{B}}(\xi_x(s', t_0 + \epsilon), t_0 + \epsilon) - \hat{\mathbf{B}}(\xi_x(s', t_0), t_0) \right]||$$

$$\leq \int_0^s ds' ||\hat{\mathbf{B}}(\xi_x(s', t_0 + \epsilon), t_0 + \epsilon) - \hat{\mathbf{B}}(\xi_x(s', t_0), t_0)||.$$

Assuming that $\hat{\mathbf{B}}$ is Lipschitz in spacetime position vector $\vec{x} = (x, t)$, i.e.,

$$||\hat{\mathbf{B}}(\vec{x}_2) - \hat{\mathbf{B}}(\vec{x}_1)|| \leq K_0 ||\vec{x}_2 - \vec{x}_1||$$

$$= K_0 \sqrt{||x_2 - x_1||^2 + |t_2 - t_1|^2},$$

for some $K_0 > 0$, we can write

$$||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)||$$

$$\leq K_0 \int_0^s ds' \sqrt{||\xi_x(s', t_0 + \epsilon) - \xi_x(s', t_0)||^2 + \epsilon^2}.$$

Therefore, we find

$$\frac{d}{d s} \left[ ||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| \right]$$

$$\leq K_0 \sqrt{||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)||^2 + \epsilon^2}$$

$$\leq K_0 \left( ||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| + |\epsilon| \right),$$

which implies

$$||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| \leq |\epsilon| \left( \frac{K_0 s}{K_0} - 1 \right).$$

(A2)

For any finite but arbitrarily large $s > 0$, we can take $|\epsilon|$ small enough to make the RHS of (A2) arbitrarily small, which indicates that $\xi_x$ is uniformly continuous in time. Consequently, $\xi_x(s, t)$ is uniformly continuous in $t$, provided that $\hat{\mathbf{B}}(x, t)$ is Lipschitz in $\vec{x} = (x, t)$. Lipschitz continuity of $\hat{\mathbf{B}}$ in $\vec{x} = (x, t)$ indicates that $\hat{\mathbf{B}}$ is Lipschitz in both $x$ and $t$ which can be seen from the last line of (A1).
Appendix B: Lipschitz Continuity of $B$ and $\hat{B}$

Here, we briefly discuss the mathematical conditions for Lipschitz continuity of $B$ and $\hat{B} = B/B$ and the relationship between the two [12]. First, we note that if $B$ is uniformly Lipschitz, i.e.,

$$||\hat{B}(x) - \hat{B}(y)|| \leq K'||x - y||$$

for some $K' > 0$, and $B$ has an upper bound, i.e., $\exists \Lambda > 0$ s.t.

$$||B|| \leq \Lambda,$$

then $B$ will be uniformly Lipschitz continuous;

$$||B(x) - B(y)|| \leq \Lambda'||B(x) - \hat{B}(y)|| \leq \Lambda'K'||x - y||.$$  \hspace{1cm} (B1)

On the other hand, if $B$ is uniformly Lipschitz, i.e.,

$$||B(x) - B(y)|| \leq K||x - y||,$$

for some $K > 0$ and has a lower bound, i.e., $\exists \Lambda > 0$ s.t.

$$||B|| \geq \Lambda,$$

then $\hat{B}$ will be uniformly Lipschitz;

$$||\hat{B}(x) - \hat{B}(y)|| \leq \frac{1}{\Lambda}||B(x) - B(y)|| \leq \frac{K}{\Lambda}||x - y||.$$  \hspace{1cm} (B2)

Appendix C: Magnetic Topology

Here, we obtain conditions under which the magnetic field $B$ will keep its phase space topology for all times [12]. The trajectories are solutions of the following non-autonomous differential equation:

$$\begin{cases}
\frac{dx(t)}{dt} = B(x(t), t), \\
x(t_0) = x_0,
\end{cases}$$

which has a unique solution if $B$ is uniformly Lipschitz continuous in $x$ and continuous in $t$. The time translation operator, acting at any point $(x, B(x, t))$ in the phase space, can be represented as

$$\hat{T}_\epsilon(x, B(x, t)) = (x, B(x, t + \epsilon)).$$

It is easy to see that $\hat{T}_\epsilon$ is an onto, one-to-one, and continuous map with continuous inverse. For its continuity, for example, we note that $\hat{T}_\epsilon(\epsilon)$, for any $\epsilon \in \mathbb{R}$, is continuous (so is its inverse for $\hat{T}_\epsilon^{-1}(\epsilon) = \hat{T}_\epsilon(-\epsilon)$) if it is continuous at $\epsilon = 0$. In order to show this for any $t$, the following $L_1$-norm should vanish in the limit $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \int_{t \in \epsilon} dt \left||\hat{T}_\epsilon(\epsilon)(x, B(x, t)) - \hat{T}_\epsilon(0)(x, B(x, t))\right||.$$  \hspace{1cm} (C1)

Thus the condition for the continuity of $\hat{T}_\epsilon^{-1}(\epsilon)$ is

$$\lim_{\epsilon \to 0} \int_{t \in \epsilon} dt \left||B(x, t + \epsilon) - B(x, t)\right|| \to 0,$$  \hspace{1cm} (C2)

which follows if $B$ is uniformly continuous in $t$. In order to keep the phase space topology preserved in time, we need to ensure that the phase space at any given time $t_0$, as a topological space, is homeomorphic to the phase space at another time $t_1$. The condition of continuity for $\hat{T}_\epsilon^{-1}(\epsilon) = \hat{T}_\epsilon(-\epsilon)$, on the other hand, requires equations

$$\begin{cases}
\frac{dx}{dt} = B \\
\frac{d\hat{B}}{dt} = -\nabla \times E,
\end{cases}$$

to be time reversal invariant, which requires $B$ to be odd, i.e., $B(x, -t) = -B(x, t)$ and $E$ to be even, i.e., $E(x, -t) = E(x, t)$. In resistive MHD, the Ohm’s law contains a non-ideal term $\mathbf{P}$, $\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \mathbf{P}$, which breaks the time symmetry.

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