Bi-invariant metric on symplectic diffeomorphisms group

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Abstract

We show the existence of a weak bi-invariant symmetric nondegenerate 2-form on the symplectic diffeomorphisms group $\mathcal{D}_\omega$ of a symplectic Riemannian manifold $(M, g, \omega)$ and study its properties. We describe the Euler’s equation on a Lie algebra of group $\mathcal{D}_\omega$ and calculate the sectional curvature of $\mathcal{D}_\omega(T^\omega)$.

1 Preface

Let $M$ be a compact manifold and let $G = Diff(M)$ be the group of all smooth (of class $C^\infty$) diffeomorphisms on $M$, with group operation being composition. The group $Diff(M)$ is an infinite-dimensional Frechét manifold and there are nontrivial problems with the notion of smooth maps between Frechét spaces. There is no canonical extension of the differential calculus from Banach spaces (which is the same as for $\mathbb{R}^n$) to Frechét spaces. It is possible to use the completion $Diff(M)$ in the Banach $C^k$-norm, $0 \leq k < \infty$, or in the Sobolev $H^s$-norm, $s > \text{dim}(M)/2$. Then $Diff^k(M)$ and $Diff^s(M)$ become Banach and Hilbert manifolds, respectively. Then we consider the inverse limits of these Banach and Hilbert manifolds, respectively: $Diff(M) = \lim_{\leftarrow} Diff^k(M)$ becomes a so-called ILB- (Inverse Limit of Banach) Lie group, or with the Sobolev topologies $Diff(M) = \lim_{\leftarrow} Diff^s(M)$ becomes a so-called ILH- (Inverse Limit of Hilbert) Lie group. The main theorems of differential calculus are true for (Banach) Hilbert manifolds. The results are proved for $Diff^s(M)$ and are then extended to the Lie-Frechét group $Diff(M)$. See the study by Omori in [18] for details. In [7], Ebin and Marsden showed that the group $\mathcal{D}_\omega = \{ \eta \in Diff(M); \eta^*(\omega) = \omega \}$ of smooth diffeomorphisms preserving a symplectic 2-form $\omega$ on $M$ is ILH-Lie group.

Let’s remind concept of ILH-Lie group. A topological vector space $E$ is called an ILH-space if $E$ is the inverse limit of the Hilbert spaces $\{E^s\}$ enumerated by integers $s \geq d \geq 0$, and, moreover, $E^{s+1}$ is linearly and densely embedded in $E^s$. Denote

$$E = \lim_{\leftarrow} E^s = \cap_{s \geq d} E^s.$$ 

A system of vector spaces $\{E, E^s, s \in \mathbb{N}(d)\}$, where $\mathbb{N}(d) = \{s \in \mathbb{N}; s \geq d \geq 0\}$, is also called a Sobolev chain.

Definition 1.1 ([18]). A topological group $G$ is called a strongly ILH-Lie group modeled on a chain $\{E, E^s, s \in \mathbb{N}(d)\}$ if there exists a system $\{G^s, s \in \mathbb{N}(d)\}$ of topological groups $G^s$ satisfying the following conditions:

(G1) every group $G^s$ is a smooth Hilbert manifold modeled on $E^s$;
(G2) $G^{s+1}$ is a dense subgroup in $G^s$, and the embedding $G^{s+1} \subset G^s$ is a mapping of class $C^\infty$;
(G3) $G = \cap G^s$ with the inverse limit topology;
(G4) the group multiplication $G \times G \rightarrow G$, $(\eta, \zeta) \rightarrow \eta \zeta$ extends to a mapping $G^{s+1} \times G^s \rightarrow G^s$ of class $C^1$;
(G5) the mapping $G \rightarrow G$, $\eta \rightarrow \eta^{-1}$ extends to a mapping $G^{s+1} \rightarrow G^s$ of class $C^1$;
(G6) for each $\eta \in G^s$, the right translation $R_\eta : G^s \rightarrow G^s$ is a mapping of class $C^\infty$. 

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(G7) let $T_eG^s$ be the tangent space of $G^s$ at the identity $e \in G^s$, and let $TG^s$ be the tangent bundle. The mapping $dR: T_eG^{s+1} \times G^s \to TG^s$ defined by $dR(u, \eta) = dR\eta u$ is a mapping of class $C^1$;

(G8) there exist an open neighborhood $U$ of zero in $T_eG^d$ and a $C^\infty$-diffeomorphism of $U$ onto an open neighborhood $\tilde{U}$ of the unity $\tilde{e} \in G^d$, $\varphi(0) = e$, such that the restriction of $\varphi$ to $U \cap T_eG^s$ is a $C^\infty$-diffeomorphism of the open subset $U \cap T_eG^s$ from $T_eG^s$ onto an open subset $\tilde{U} \cap G^s$ from $G^s$ for any $s \geq d$.

**Remark 1.** Roughly speaking, condition (G8) means that a coordinate neighborhood of the identity in each of the groups $G^s$ can be chosen independently of $s$. Then, setting $T_eG = \cap T_eG^s$ with the inverse limit topology, we see that $\varphi: U \cap T_eG \to \tilde{U} \cap G$ is a homeomorphism defining the Lie–Frechét group structure on $G$.

The pair $(U, \varphi)$ in condition (G8) is called the *ILH-coordinates* on $G$ in a neighborhood of the unity.

**Definition 1.2.** A topological group $G$ is called an ILH-Lie group if there exists a system of topological groups $\{G^s, s \in \mathbb{N}(d)\}$ satisfying conditions (G1)–(G7).

Omori showed in [16, 18] that the group $\mathcal{D}$ of smooth diffeomorphisms of a compact manifold $M$ is a strongly ILH-Lie group modeled on the space $\{\Gamma(TM), \Gamma^s(TM); s \in \mathbb{N}(\dim M + 5)\}$ of smooth vector fields on $M$, where $\Gamma^s(TM)$ is the space of vector fields of Sobolev class of smoothness $H^s$.

In [7], Ebin and Marsden showed that the following groups are ILH-Lie groups:

1. the group $\mathcal{D}_\mu$ of smooth diffeomorphisms preserving a volume element $\mu$ on the manifolds $M$;
2. the group $\mathcal{D}_\omega$ of smooth diffeomorphisms preserving a symplectic structure $\omega$ on $M$.

## 2 Symplectic diffeomorphisms group and bi-invariant metric

Let $M$ be a smooth (of class $C^\infty$) compact orientable manifold of dimension $2n$ without boundary. The manifold $M$ is said to be symplectic if a closed nondegenerate $2$-form $\omega$ is given on it. In this case, there exists an almost complex structure $J$ on $M$ having the following properties: $\omega(JX, JY) = \omega(X, Y)$, and $\omega(X, JX) > 0$ for any $X, Y \in \Gamma(TM)$. The formula

$$g(X, Y) = \omega(X, JY), \quad X, Y \in \Gamma(TM),$$

(1)

defines an almost Hermitian structure $(J, g)$ on $M$ whose fundamental form is $\omega$. Therefore, $(M, \omega, J, g)$ is an almost Kähler structure on $M$. Note that $\omega^n = n!\mu$, where $\mu$ is the Riemannian volume element on $M$. The form $\omega$ defines the bundle isomorphism $\iota: TM \to T^*M$, $\iota(V) = -i_V\omega = \omega(., V)$.

A transformation $\eta: M \to M$ is said to be symplectic if it preserves the symplectic form $\omega$, i.e., if $\eta^*\omega = \omega$. Let $\mathcal{D}_\omega$ be the group of all smooth symplectic diffeomorphisms of a manifold $M$. Ebin and Marsden showed in [7] that the group $\mathcal{D}_\omega$ is a closed ILH-Lie subgroup of the diffeomorphisms group $\mathcal{D}$. Omori proved in [18] that $\mathcal{D}_\omega$ is a closed strongly ILH-Lie subgroup of the group $\mathcal{D}$.

The Lie algebra of the group $\mathcal{D}_\omega$ consists of all vector fields that infinitesimally preserve the form $\omega$, i.e., those vector fields $X$ on $M$ for which $L_X\omega = 0$, where $L_X = i_X \circ d + d \circ i_X$ is
the Lie derivative. Such vector fields are said to be \textit{locally Hamiltonian}. Their characteristic property is that the form \( i_X \omega = \omega(X, \cdot) \) is closed. Indeed,

\[
L_X w = i_X (d \omega) + d(i_X \omega) = d(i_X \omega) = 0.
\]

A vector field \( X \) on \( M \) is said to be \textit{Hamiltonian} if the form \(-i_X \omega\) is exact, i.e., when it is the differential of a certain function \( F \) on \( M \): \( \omega(\cdot, X) = dF \). The function \( F \) is called the \textit{Hamiltonian function} of the field \( X \), and in this case, the vector field \( X \) is denoted by \( X_F \).

It well-known that \( X_H = J \text{grad} H \).

The set of Hamiltonian vector fields on \( M \) form a Lie algebra with respect to the Lie bracket of vector fields; moreover, \([X_F, X_H] = X_{\{F,H\}}\), where \( \{F,H\} = \omega(X_F, X_H) \) is the Poisson bracket of functions \( F \) and \( H \) on the symplectic manifold \( M \).

Denote by \( \Gamma_\omega(TM) \) the Lie algebra of smooth locally Hamiltonian vector fields on \( M \). Let

\[
\Gamma_{\omega\emptyset}(TM) = \{ X \in \Gamma_\omega(TM); \ i_X \omega \text{ is an exact form } \}.
\]

It is easy to see that \( \Gamma_{\omega\emptyset}(TM) \) is an ideal of \( \Gamma_\omega(TM) \), since \( i_{[X,Y]} \omega = d(i_X(i_Y \omega)) \) for \( X, Y \in \Gamma_\omega(TM) \). Since \( \Gamma_\omega(TM)/\Gamma_{\omega\emptyset}(TM) = H^1(M) \), it follows that \( \Gamma_{\omega\emptyset}(TM) \) is of finite codimension in \( \Gamma_\omega(TM) \). The ILH-Lie group corresponds to the ideal \( \Gamma_{\omega\emptyset}(TM) \).

**Theorem 2.1** (see \[18\], Theorem 8.5.1). \textit{There exists a strong ILH-subgroup} \( D_{\omega\emptyset} \) \textit{of the group} \( D_\omega \) \textit{such that} \( \Gamma_{\omega}(TM) \) \textit{is its Lie algebra.}

The connected component of the identity of the group \( D_{\omega\emptyset} \) is called the \textit{Hamiltonian transformation} group of the manifold \( M \) and is denoted by \( D_{\omega H} \) in what follows. The Lie algebra of the group \( D_{\omega H} \) is \( \Gamma_{\omega\emptyset}(TM) \).

**Remark 2.** Ebin and Marsden showed in \[7\] that the connected component \( D_0 \) of the diffeomorphism group of the manifold \( M \) is diffeomorphic to \( D_\mu \times \mathcal{V} \), the direct product of the group \( D_\mu \) of volume-preserving diffeomorphisms and the convex space \( \mathcal{V} \) of volume elements. In the symplectic case, this fact does not hold. McDuff presented examples \[14\] of cases where the group \( D_\omega \) is not homotopy equivalent to the group \( D_0 \).

**Bi-invariant metric on** \( D_\omega \)

Let \( X, Y \in T_e D_\omega = \Gamma_\omega(TM) \). The right-invariant weak Riemannian structure on the group \( D_\omega \) is defined by

\[
(X, Y)_e = \int_M g(X, Y) d\mu. \tag{2}
\]

Let \( X = X_H \) be a Hamiltonian vector field on \( M \). For the Hamiltonian \( H \) to be uniquely defined by its vector field \( X \), we assume that

\[
\int_M H(x) d\mu(x) = 0.
\]

Denote by \( C_0^\infty(M, \mathbb{R}) \) the space of such functions on \( M \). The Laplacian \( \Delta = -\text{div} \circ \text{grad} \) defines the isomorphism \( \Delta : C_0^\infty(M, \mathbb{R}) \to C_0^\infty(M, \mathbb{R}) \). Then the operator \( \Delta^{-1} \) inverse to the Laplacian is defined. Let \( H_0^s(M) \) be the completion of the space \( C_0^\infty(M, \mathbb{R}) \) with respect to the \( H^s \)-norm of the Sobolev space, \( s \geq 2n + 5 \). The operator \( \Delta \) extends to a Hilbert space isomorphism \( \Delta : H_0^s(M) \to H_0^{s-2}(M) \).

Consider the Lie algebra \( \Gamma_{\omega\emptyset}(TM) \) of smooth Hamiltonian vector fields on \( M \). In \[18\] and \[19\], it was shown that there exists a connected ILH-Lie group \( D_{\omega H} \) whose Lie algebra is \( \Gamma_{\omega\emptyset}(TM) \). If the class \([\omega] \in H^2(M, \mathbb{R})\) is integral, then the group \( D_{\omega H} \) is (see \[19\]) the
commutator $[D_{\omega}, D_{\omega}]$ for the connected component $D_{\omega_0}$ of the identity of the group $D_{\omega}$. If the first cohomology group is trivial, $H^1(M, \mathbb{R}) = 0$, then $D_{\omega H} = D_{\omega 0}$.

Define the inner product on the algebra $\Gamma_{\omega \theta}(TM)$ by

$$\langle X_F, X_H \rangle_e = \int_M F(x) H(x) d\mu(x).$$

(3)

It is easy to see that this inner product is bi-invariant, or, in other words, it is invariant with respect to the adjoint action of the group $D_{\omega}$ on the algebra $\Gamma_{\omega \theta}(TM)$. Indeed, for $\eta \in D_{\omega}$, we have $Ad_{\eta}X_F = dL_\eta dR_{\eta}^{-1}X_F = X_{F_{|\eta}}$, and $\eta^* \mu = \eta^* (c \omega^n) = c (\eta^* \omega)^n = c \omega^n = \mu$, where $c = 1/n!$. The infinitesimal variant of the invariance of the inner product (3) has the form

$$\langle [X_G, X_F], X_H \rangle_e + \langle X_F, [X_G, X_H] \rangle_e = 0,$$

(4)

where $[X_G, X_F]$ is the Lie bracket of Hamiltonian vector fields $X_G$ and $X_F$ on $M$. Recall that $[X_G, X_F] = X_{\{G,F\}}$, where $\{G,F\}$ is the Poisson bracket of functions $G$ and $F$ on a symplectic manifold. The bi-invariant metric (3) was introduced in [21], and it was also considered in [9], [8] and [11].

**Theorem 2.2** (see [21]). The inner product (2) is expressed through the bi-invariant inner product (3) on $\Gamma_{\omega \theta}(TM)$ as follows:

$$(X_F, X_H)_e = \langle X_{\Delta F}, X_H \rangle_e,$$

(5)

where $\Delta = -\text{div} \circ \text{grad}$ is the Laplace operator.

**Proof.** Indeed, we have

$$(X_F, X_H)_e = \int_M g(X_F, Y_H) d\mu = \int_M g(J \text{grad} F, J \text{grad} H) d\mu =$$

$$= \int_M g(\text{grad} F, \text{grad} H) d\mu = \int_M -\text{div}(\text{grad} F) H d\mu = \int_M (\Delta F) H d\mu.$$

\(\square\)

We can extend inner product (3) defined on the tangent space $T_e D_{\omega H} = \Gamma_{\omega \theta}(TM)$ at the identity to the whole group $D_{\omega H}$ using right translations. The invariance of form (3) implies the bi-invariance of the weak Riemannian structure on $D_{\omega H}$ being obtained.

For the group $D_{\omega H}$, there exists (see [13]) a system $\{D^s_{\omega H} ; s \geq 2n + 5\}$ of topological groups $D^s_{\omega H}$ satisfying properties (G1) – (G7) above. Each of the topological groups $D^s_{\omega H}$ is a smooth Hilbert manifold modeled on the space $\Gamma_{\omega \theta}(TM)^s$ of Hamiltonian vector fields of Sobolev class of smoothness $H^s$. Therefore the tangent space $T_e D^s_{\omega H}$ at the identity is identified with $\Gamma_{\omega \theta}(TM)^s$. According to property (G6) of an ILH-Lie group, the right translation $R_{\eta} : D_{\omega H} \to D^s_{\omega H}$ is a smooth mapping for any $\eta \in D^s_{\omega H}$. This allows us to obtain the right-invariant weak Riemannian structure on $D^s_{\omega H}$ from the inner product (3) on $\Gamma_{\omega \theta}(TM)^s = T_e D^s_{\omega H}$.

**Theorem 2.3** (see [21]). The weak Riemannian structure on the smooth Hilbert manifold $D^s_{\omega H}$ obtained from the inner product (3) on $T_e D^s_{\omega H}$ by using right translations on $D^s_{\omega H}$ is smooth. The corresponding weak Riemannian structure on the group $D_{\omega H} = \cap D^s_{\omega H}$ is bi-invariant and ILH-smooth.
3 Euler equation

On the algebra $\Gamma_{\omega\theta}(TM)$ of Hamiltonian vector fields, there exist the bi-invariant inner product $\langle \cdot, \cdot \rangle_e$ and the kinetic energy function $L$,

$$L(X_F) = \frac{1}{2}(X_F, X_F)_e = \frac{1}{2} \int_M g(X_F, X_F) d\mu, \quad X_F \in \Gamma_{\omega\theta}(TM).$$

(6)

The function $L$ is written as follows through the invariant inner product $(3)$:

$$L(X_F) = \frac{1}{2} \langle X_F, X_{\Delta F} \rangle_e.$$  

The Legendre transform \cite{1} corresponding to the function $L$ has the form $Y_F = X_{\Delta F}$. Then the Hamiltonian function has the expression $H(Y_F) = L(X_F) = \frac{1}{2} \langle Y_{\Delta^{-1} F}, Y_F \rangle_e$. It is easy to calculate the gradient of the function $H(Y_F)$ with respect to the invariant inner product $(3)$:

$$\text{grad} H(Y_F) = Y_{\Delta^{-1} F}.$$  

Recall that the operator $\Delta : \Gamma_{\omega\theta}(TM) \to \Gamma_{\omega\theta}(TM)$ is an isomorphism, and, therefore, it has the inverse operator $\Delta^{-1}$.

Following the general construction proposed in \cite{15}, we write the Euler equation

$$\frac{d}{dt} Y_F = [Y_F, \text{grad} H(Y_F)] = [Y_F, Y_{\Delta^{-1} F}].$$

(7)

Passing from the vector fields to their Hamiltonians, we arrive at the Euler equation of the form $\frac{d}{dt} F = \{F, \Delta^{-1} F\}$, where $\{\cdot, \cdot\}$ is the Poisson bracket on the symplectic manifold $M$. Make the change $F := \Delta^{-1} F$ then

$$\frac{\partial}{\partial t} \Delta F = \{\Delta F, F\}.$$  

(8)

We consider the Euler equation precisely in this form.

Remark 3. For $n = 1$, the Euler equation coincides with the Helmholtz equation of motion of the two-dimensional ideal incompressible fluid.

Theorem 3.1 (see \cite{21}). For any function $F_0(x) \in H^s_0(M)$, $s \geq 2n+7$, there exists a unique continuous solution $F(x, t)$ of Eq. (8) defined on $(-\varepsilon, \varepsilon) \times M$ for a certain $\varepsilon > 0$ and having the following properties:

1. $F(0, x) = F_0(x)$;
2. if $F_0(x) \in H^{s+l}_0(M)$, $l \geq 0$, then $F(t, x) \in H^{s+l}_0(M)$ for any $t \in (-\varepsilon, \varepsilon)$;
3. the flow $\eta_t$ on $M$ generated by the Hamiltonian vector field $X_{F(t, x)}$ is a geodesic on $\mathcal{D}_{\omega H}^s$ of the right-invariant metric $(2)$. Conversely, if $\eta_t$ is a geodesic on $\mathcal{D}_{\omega H}^s$, then the velocity field

$$X_F = dR^{-1}_{\omega H} \left( \frac{d}{dt} \eta_t \right)$$

has the Hamiltonian $F$ satisfying Eq. (8).

We write the Euler equation $\frac{\partial}{\partial t} \Delta F = \{\Delta F, F\}$ through vector fields in the form

$$\frac{\partial}{\partial t} X_{\Delta F} = [X_{\Delta F}, X_F] = -L_{X_F} X_{\Delta F}.$$
Therefore, the vector field $X_{\Delta F}$ on $M$ is transported by the flow $\eta_t$ of the vector field $X_F$. In other words, $X_{\Delta F}(t, x) = dL_{\eta_t} dR_{\eta_t}^{-1}(X_{\Delta F})$. Since $\text{Ad}_{\eta_t} X_H(t) = X_H(\eta_t^{-1}(x))$, the Hamiltonian $\Delta F(t, x)$ is transported by the flow $\eta_t$:

$$\Delta F(t, x) = (\Delta F)(\eta_t^{-1}(x)).$$

This immediately implies the following theorem.

**Theorem 3.2** (see [21]). Let $F = F(t, x)$ be a solution of the Euler equation (8), and let $\eta_t$ be the flow on $M$ generated by the vector field $X_F$. Then the following quantities are independent of time $t$:

\begin{align*}
L &= \frac{1}{2} (X_F, X_F) e = \frac{1}{2} \int_M F \Delta F \, d\mu, \\
I_k &= \int_M (\Delta F)^k \, d\mu.
\end{align*}

Remark 10.2. Let $F = F(t, x)$ be a solution of the Euler equation (8), and let $F_0(x) = F(0, x)$ be the initial value. Since $\Delta F(t, x) = (\Delta F_0)(\eta_t^{-1}(x))$, it follows that the vector field $dR_{\eta_t} X_{\Delta F}$ is the restriction of the left-invariant vector field $dL_{\eta_t} X_{\Delta F_0}$ on the group $D_\omega H$ to the geodesic $\eta_t$. Therefore, the velocity field $dR_{\eta_t} X_F$ along the geodesic $\eta_t$ on $D_\omega H$ assumes a unique value at each point of the geodesic $\eta_t$. This implies the following conclusion.

**Conclusion.** The geodesics on the group $D_\omega H$ of the right-invariant metric (2) cannot have self-intersections.

### 4 Curvature of the group $D_\omega$

The group $D_\omega H$ has the bi-invariant weak Riemannian structure (3). The curvature of the group $D_\omega H$ with respect to (3) is easily found. The covariant derivative $\nabla^0$ of the Riemannian connection of the bi-invariant metric (3) on the group $D_\omega H$ has the usual form:

$$\nabla^0_{X_F} X_H = \frac{1}{2} [X_F, X_H] = \frac{1}{2} X_{\{F, H\}}. \quad (11)$$

The curvature tensor of the connection $\nabla^0$ is:

$$R^0(X_F, X_H) X_G = -\frac{1}{4} [[X_F, X_H], X_G]. \quad (12)$$

The sectional curvature of the group $D_\omega H$ with respect to the bi-invariant metric in the direction of a 2-plane $\sigma$ given by an orthonormal pair of Hamiltonian vector fields $X_F, X_H \in \Gamma_\omega(TM)$ is expressed by the formula

$$K_\sigma = \frac{1}{4} \int_M \{F, H\}^2 \, d\mu. \quad (13)$$

Thus, the group $D_\omega H$ is of nonnegative sectional curvature and $K_\sigma = 0$ iff the Hamiltonians $F$ and $H$ commute: $\{F, H\} = 0$.

Now let us consider the problem on the curvature of the group $D_\omega$ with respect to the right-invariant weak Riemannian structure (2). The corresponding Riemannian connection $\tilde{\nabla}$ on $D_\omega$ is defined in the same way as in the case of the group $D_\mu$. If $X$ and $Y$ are two right-invariant vector fields on $D_\omega$, then

$$\left(\tilde{\nabla}_X Y\right)_e = P_e(\nabla_X Y), \quad (14)$$
where $P_e : \Gamma(TM) \to T_eD_\omega$ is the orthogonal projection of the space $\Gamma(TM)$ of vector fields on $M$ on the space $T_eD_\omega$ of locally Hamiltonian vector fields and $\nabla$ is the covariant derivative of the metric $g$ on $M$.

**Theorem 4.1** (see [22]). The sectional curvature of the group $\sigma$ in the direction of a 2-plane $\sigma$ given by an orthonormal pair of locally Hamiltonian vector fields $X, Y \in T_eD_\omega$ is expressed by the formula

$$K_\sigma = -\frac{1}{2}(X, [[X, Y], Y])_e - \frac{1}{2}([X, [X, Y]], Y)_e - \frac{3}{4}([X, Y], [X, Y])_e - (P_e(\nabla_X X), P_e(\nabla_Y Y))_e + \frac{1}{4}(P_e(\nabla_X Y + \nabla_Y X), P_e(\nabla_X Y + \nabla_Y X))_e,$$

where $P_e : \Gamma(TM) \to D_\omega$ is the orthogonal projection.

For the group $D_{\omega\theta}$, we can obtain a more convenient formula for the sectional curvatures expressed through the Hamiltonians of vector fields $X = X_F$ and $Y = X_H$ on $M$. First, we give the following characterization of the Hamiltonian component $P_e(\nabla_X Y) \in \Gamma_{\omega\theta}(TM)$ of the vector field of the covariant derivative $\nabla_X Y$ on $M$.

**Lemma 4.2.** For any Hamiltonian vector fields $X = X_F$ and $Y = X_H$ on $M$, the Hamiltonian $S$ of the vector field $X_S = P_e(\nabla_X Y) \in \Gamma_{\omega\theta}(TM)$ is connected with the Hamiltonians $F$ and $H$ by the relation

$$\Delta S = \frac{1}{2}(\Delta\{F, H\} + \{F, \Delta H\} + \{H, \Delta F\}),$$

where $\Delta = -\text{div} \circ \text{grad}$ is the Laplacian and $\{F, H\}$ is the Poisson bracket.

**Proof.** It was demonstrated in [7] that, given the weak right-invariant Riemannian structure $\omega$ on $D_{\omega\theta}$, there exists a Riemannian connection whose covariant derivative $\tilde{\nabla}$ at the identity $e \in D_{\omega\theta}$ is given by the formula

$$(\tilde{\nabla}_X Y)_e = P_e(\nabla_X Y)_e,$$

where $\nabla$ is the covariant derivative of the Riemannian connection on $M$ and $X(\eta) = X_e \circ \eta$, $Y(\eta) = Y_e \circ \eta$ are right-invariant vector fields on $D_{\omega\theta}$, $X_e, Y_e \in T_eD_{\omega\theta}$. For determining $P_e(\nabla_X Y)_e = (\tilde{\nabla}_X Y)_e$, we use the six-term formula

$$2(\tilde{\nabla}_X Y, Z)_e = X(Y, Z) + Y(Z, X) - Z(X, Y) + (Z, [X, Y])_e + (Y, [Z, X])_e - (X, [Y, Z])_e,$$

where $X = X_F$, $Y = X_H$ and $Z = X_G$ are regarded as right-invariant vector fields on $D_\theta$. Taking the right invariance of the weak Riemannian structure $\omega$ into account, we obtain $X(Y, Z) = Y(Z, X) = Z(X, Y) = 0$. Using the bi-invariant scalar product (3), we obtain

$$2(\tilde{\nabla}_X Y, Z)_e = (X_G, [X_F, X_H])_e + (X_H, [X_G, X_F])_e - (X_F, [X_H, X_G])_e =$$

$$= (X_G, [F, H])_e + (X_H, [G, F])_e - (X_F, [H, G])_e =$$

On the other hand, we have

$$2(\tilde{\nabla}_X Y, Z)_e = 2(P_e(\nabla_X Y), Z)_e = 2(X_S, X_G)_e = 2(X_{\Delta S}, X_G)_e.$$
Corollary 4.3. Let \( X = X_F \) and \( Y = X_H \). If \( P_e(\nabla_X X) = X_S \) and \( P_e(\nabla_X Y + \nabla_Y X) = X_T \), then
\[
\Delta S = \{ F, \Delta F \}, \quad (17)
\]
\[
\Delta T = \{ F, \Delta H \} + \{ H, \Delta F \}.
\]

Using the expressions for \( \Delta S \) and \( \Delta T \) we obtain from (15) the following expression for \( K_\sigma \).

Theorem 4.4. The sectional curvature of the group \( D_\omega H \) (with metric (2)) in the direction of a 2-plane \( \sigma \in T_e D_\omega H \) given by an orthonormal pair of Hamiltonian vector fields \( X_F, X_H \in \Gamma_\omega \partial \) is expressed by the formula
\[
K_\sigma = -\frac{3}{4} \int_M \Delta \{ F, H \} \{ F, H \} \, d\mu + \frac{1}{2} \int_M \{ F, H \} (\{ F, \Delta H \} + \{ \Delta F, H \}) \, d\mu - \int_M \{ F, \Delta F \} \Delta^{-1} (\{ H, \Delta H \}) \, d\mu + \frac{1}{4} \int_M (\{ F, \Delta H \} + \{ H, \Delta F \}) \Delta^{-1} (\{ F, \Delta H \} + \{ H, \Delta F \}) \, d\mu. \quad (19)
\]

In the case where, as the Hamiltonians \( F \) and \( H \), we take the eigenfunctions of the Laplace operator, \( \Delta F = \alpha F \) and \( \Delta H = \beta H \), the formula for the sectional curvatures of the group \( D_\omega H \) becomes
\[
K_\sigma = -\frac{3}{4} \int_M \Delta \{ F, H \} \{ F, H \} \, d\mu + \frac{\alpha + \beta}{2} \int_M \{ F, H \}^2 \, d\mu + \frac{(\alpha - \beta)^2}{4} \int_M \{ F, H \} \Delta^{-1} \{ F, H \} \, d\mu. \quad (20)
\]

Assume that the structural constants of the Lie algebra \( \Gamma_\omega \partial (TM) \),
\[
\{ F, H \} = C_{FH}^i F_i,
\]
where \( F_i \) is the orthonormal system of eigenfunctions of the Laplace operator corresponding to eigenvalues \( \lambda_i \), are known. In this case, the formula for the sectional curvature of the group \( D_\omega H \) assumes a simpler form
\[
K_\sigma = \frac{1}{\alpha \beta} \left( -\frac{3}{4} \sum_{i>0} \lambda_i (C_{FH}^i)^2 + \frac{\alpha + \beta}{2} \sum_{i>0} (C_{FH}^i)^2 + \frac{(\alpha - \beta)^2}{4} \sum_{i>0} \frac{(C_{FH}^i)^2}{\lambda_i} \right). \quad (21)
\]

In this formula, it is assumed that the functions \( F \) and \( G \) have unit \( L^2 \)-norms; then \( ||X_F||^2 = \alpha \) and \( ||X_H||^2 = \beta \). This explains the appearance of the coefficient \( \frac{1}{\alpha \beta} \) in formula (21).

Structural constants of the Lie algebra \( T_e D_\omega (S^2) \) in the basis of spherical functions

Introduce the coordinates \( (z, \varphi) \) on two-dimensional sphere \( S^2 \):
\[
x = \sqrt{1 - z^2} \sin \varphi, \quad y = \sqrt{1 - z^2} \cos \varphi, \quad 0 < \varphi < 2\pi, \ -1 < z < 1.
\]
Then the Riemannian volume element $\omega$ on $S^2$ has the form $\omega = dz \wedge d\varphi$. Therefore, $z$ and $\varphi$ are canonical variables on the sphere $S^2$. Therefore, the Poisson bracket of two functions $F$ and $H$ on $S^2$ is calculated according to the usual formula $\{F, H\} = \frac{\partial F}{\partial z} \frac{\partial H}{\partial \varphi} - \frac{\partial F}{\partial \varphi} \frac{\partial H}{\partial z}$.

The spherical functions

$$Y_l^m(z, \varphi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}} e^{i m \varphi} (1 - z^2)^{m/2} \frac{d^{l+m}}{dz^{l+m}}(1 - z^2)^l,$$

(22)

where $l \in \mathbb{N}$ and $-l \leq m \leq l$, compose a complete orthonormal function system on $S^2$ (see, e.g., [12]). The functions $Y_l^m(z, \varphi)$ are the eigenfunctions of the Laplace operator: $\Delta Y_l^m = l(l+1)Y_l^m$.

The structural constants of the Lie algebra $T_e \mathcal{D}_\omega(S^2)$ in the basis of spherical functions were calculated in the work [2] of Arakelyan and Savvidy; also, in this work, the curvature tensor was found. We set $Y_{lm}(z, \varphi) := Y_l^m(z, \varphi)$; then the structural constants $C_{nmkl}^{ij}$,

$$\{Y_{nm}, Y_{kl}\} = C_{nmkl}^{ij} Y_{ij},$$

have the following form [2]:

$$C_{nmkl}^{ij} = -i(-1)^j \sqrt{\frac{(2n+1)(2k+1)(2l+1)}{4\pi}} l \sum_p (2(n-2p-1)+1) \times$$

$$(n - |m|) \ldots (n - |m| - 2p) \sqrt{(n+|m|) \ldots (n+|m| - 2p)} \frac{2p - 1}{m} \begin{pmatrix} n - 2p - 1 & k & i \\ m & l & j \end{pmatrix} -$$

$$-m \sum_q (2(k-2q-1)+1) \sqrt{(k - |l|) \ldots (k - |l| - 2q)} \frac{2q}{k} \begin{pmatrix} n & k - 2q - 1 & i \\ m & l &-j \end{pmatrix} \begin{pmatrix} n & k - 2q - 1 & i \\ 0 & 0 & 0 \end{pmatrix},$$

where $\begin{pmatrix} n & k & i \\ m & l & j \end{pmatrix}$ are 3j-Wigner symbols (see, e.g., [12]).

4.1 Curvature of the symplectic diffeomorphism group of the torus

Let $T^{2q} = \mathbb{R}^{2q}/2\pi \mathbb{Z}^{2q}$ be a torus. We assume that the standard symplectic form $\omega = \sum_{i=1}^q dx_i \wedge dy_i$ is given on $T^{2q}$. The functions

$$\cos(n x + m y), \quad \sin(n x + m y),$$

where $n, m \in \mathbb{Z}^q$, $nx = \sum_{i=1}^q n_i x_i$ compose a complete orthogonal function system on the torus $T^{2q}$.

Consider a small two-dimensional area $\sigma \subset \Gamma_{\omega \theta}(TM)$ composed, e.g., by the vector fields $X_F$ and $X_H$ with the Hamiltonians

$$F = \cos(nx + my), \quad H = \cos(kx + ly),$$

under the condition $n^2 + m^2 \neq 0$ and $k^2 + l^2 \neq 0$. The Poisson bracket is

$$\{F, H\} = \frac{1}{2} (mk - nl) \left( \cos((n - k)x + (m - l)y) - \cos((n + k)x + (m + l)y) \right).$$

(23)
Under the integration over the torus $T = T^{2q}$ with respect to $d\mu = dx_1 \wedge ... \wedge dy_q$ we have
\[
\int_T \cos(nx + my) \cos(kx + ly) d\mu = \begin{cases} 
0, & (n, m) \neq (k, l) \\
\frac{1}{2}(2\pi)^{2q}, & n = k, m = l.
\end{cases}
\]
Therefore, $\|F\|^2 = \frac{1}{2}(2\pi)^{2q}$ and
\[
\int_T \{F, H\}^2 d\mu = \frac{1}{4} (mk - nl)^2 \left(\frac{1}{2}(2\pi)^{2q} + \frac{1}{2}(2\pi)^{2q}\right) = (2\pi)^{2q} \frac{(mk - nl)^2}{4}.
\]
For the sectional curvature of the bi-invariant metric
\[
K_\sigma = \frac{1}{4\|F\|^2\|H\|^2} \int_T \{F, H\}^2 d\mu,
\]
we obtain the expression
\[
K_\sigma = \frac{(mk - nl)^2}{4(2\pi)^{2q}}.
\] (24)
In the case $H = \sin(kx + ly)$ and two sines, we obtain exactly the same formula.

Therefore, the sectional curvatures of the bi-invariant metric are nonnegative and can assume arbitrarily large values. This yields a geometric explanation of the fact that the group exponential (i.e., the exponential of the bi-invariant metric) does not cover a neighborhood of the identity of the diffeomorphism group.

Let us calculate the sectional curvature of the right-invariant metric on the group $\mathcal{D}_{\omega H}(T^{2q})$. Consider a two-dimensional small area $\sigma \subset \Gamma_{\omega \phi}(TM)$ composed of the vector fields $X_F$ and $X_H$ with the Hamiltonians
\[
F = \cos(nx + my), \quad H = \cos(kx + ly),
\]
under the condition $n^2 + m^2 \neq 0$ and $k^2 + l^2 \neq 0$. These functions are eigenfunctions of the Laplace operator:
\[
\Delta F = (n^2 + m^2) F, \quad \lambda = n^2 + m^2, \quad \Delta H = (k^2 + l^2) H, \quad \mu = k^2 + l^2,
\]
Therefore:
\[
\|X_F\|^2 = \lambda \|F\|^2 = \frac{1}{2}(2\pi)^{2q}, \quad \|X_H\|^2 = \mu \|H\|^2 = \frac{1}{2}(2\pi)^{2q},
\]
\[
\int_T \{F, H\}^2 d\mu = (2\pi)^{2q} \frac{(mk - nl)^2}{4},
\]
\[
\int_T \{F, H\} \Delta \{F, H\} d\mu = (2\pi)^{2q} \left(\frac{(mk - nl)^2}{8}\right) \left((n - k)^2 + (m - l)^2 + (n + k)^2 + (m + l)^2\right),
\]
\[
\int_T \{F, H\} \Delta^{-1} \{F, H\} d\mu = (2\pi)^{2q} \left(\frac{(mk - nl)^2}{8}\right) \left(\frac{1}{(n - k)^2 + (m - l)^2} + \frac{1}{(n + k)^2 + (m + l)^2}\right).
\]
By formula (20), we obtain
\[
K_\sigma = -\frac{(mk - nl)^2 (n^2 + m^2 + k^2 + l^2)}{(2\pi)^{2q}(n^2 + m^2)(k^2 + l^2)((n - k)^2 + (m - l)^2)((n + k)^2 + (m + l)^2))}.
\] (25)

We see that the sectional curvatures of the right-invariant metric are nonpositive, and their modules grow in a lesser power, which completely corresponds to the fact that the Riemannian exponential of the right-invariant metric is a local diffeomorphism.
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