On the mass difference between proton and neutron

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The Cottingham formula expresses the electromagnetic part of the mass of a particle in terms of the virtual Compton scattering amplitude. We show that the formula can be rewritten such that the subtraction function in the dispersive representation of this amplitude dominates the contributions from the inelastic region. The numerical evaluation on the basis of a sum rule that follows from
Reggeon dominance indicates that the inelastic contributions to the mass difference between proton and neutron are very small: $m_{QED}^{p} = 0.58 \pm 0.16 \text{ MeV}$, $m_{QED}^{n} = -1.87 \pm 0.16 \text{ MeV}$.

The present paper concerns the energy $m_\gamma$ of the cloud of virtual photons surrounding a particle. As shown by Cottingham [1], the first term in the expansion of $m_\gamma$ in powers of the electromagnetic coupling constant can be represented as an integral over the spin averaged forward Compton scattering amplitude

$$T^{\mu\nu}(p,q) = \frac{i}{2} \int d^4x e^{i\vec{x} \cdot \vec{q}} \langle p | T j^\mu(x) j^\nu(0) | p \rangle. \quad (1)$$

Current conservation and Lorentz invariance imply that $T^{\mu\nu}$ can be expressed in terms of two functions $T_1(\nu, q^2), T_2(\nu, q^2)$ that only depend on $\nu = p \cdot q/m$ and $q^2$ ($m$ is the mass of the particle). As such, the following analysis applies to any stable particle, but the application we have in mind concerns the nucleon and we adopt the notation specified in Ref. [2] for this case. The amplitudes $t_1, t_2$ used in Ref. [1] differ from those we are working with only in normalization and sign: $t_1 = -n T_1, t_2 = n T_2$, $n = a_{em}/\pi m$.

In the rest frame of the particle, the analytic properties of the time-ordered product allow one to perform a Wick rotation that turns the path of integration in the variable $q_0$ from the real axis into the imaginary axis, $q_0 = iQ_4$. The variable $\nu$ coincides with $q^0$ and thus becomes purely imaginary. Identifying $Q_1, Q_2, Q_3$ with the space components of the physical momentum, we have $q^2 = -Q^2$, where $Q$ is the length of the euclidean four-vector $Q_\mu$. Eq. (1.9) of Ref. [1] can be written in the form of an integral over euclidean space:

$$m_\gamma = \frac{e^2}{2m(2\pi)^4} \int \frac{d^4Q}{Q^2} \phi(Q_4, Q^2), \quad (2)$$

$$\phi = 3Q^2T_1(iQ_4, -Q^2) + (2Q_4^2 + Q^2)T_2(iQ_4, -Q^2).$$

The asymptotic behaviour of the integrand is controlled by the operator product expansion [3, 4]. The leading contributions are determined by the Wilson coefficients of the operators of lowest dimension, which carry either spin 0 or spin 2. The explicit expressions [7, 8] show that the contributions from $T_1$ and $T_2$ both diverge – the integral (2) must be regularized, e.g. by cutting it off with $Q^2 \leq \Lambda^2$.

The leading term in the asymptotic behaviour of $T_2$ arises from operators of spin 2 and only depends on $Q^2$. Since the euclidean integral averages over the directions of the vector $Q_\mu$, the factor $Q_2^3$ can be replaced by $\frac{1}{2}Q^2$: asymptotically, the Cottingham formula is dominated by

$$\tilde{T}(\nu, q^2) = T_1(\nu, q^2) + \frac{1}{2} T_2(\nu, q^2). \quad (3)$$

The logarithmic divergence stems from the first term in the decomposition $\phi = 3Q^2\tilde{T} - 2(\frac{1}{2}Q^2 - Q_2^3)T_2$. The angular integration suppresses the second term when $Q^2 \to \infty$.

The imaginary part of $\tilde{T}$ is given by a combination of the structure functions $F_L \equiv F_2 - 2x F_1$ and $F_2$:

$$\text{Im} \tilde{T} = \pi \frac{F}{2sxQ^2}, \quad \bar{F} = F_L + 2m^2 x^2 F_2/Q^2, \quad (4)$$

with $x = Q^2/2mv$. Regge asymptotics implies that $T_2$ obeys an unsubtracted dispersion relation, while $\tilde{T}$ requires a subtraction. Replacing the variable of integration by Bjorken $x$, the dispersion relations take the form:

$$T(\nu, -Q^2) = \bar{T}^{el}(\nu, -Q^2) + \mathcal{S}(-Q^2) + \left( Q^2 + 4\nu^2 \right) \int_0^{x_{th}} \frac{dx}{(Q^2 + m^2 x^2)(Q^4 - 4m^2 x^2 v^2 - i\epsilon)} = 0 ,$$

$$T_2(\nu, -Q^2) = T_2^{el}(\nu, -Q^2) + \frac{4\nu^2}{4m^2 F_2(x, Q^2)} \int_0^{x_{th}} \frac{dx}{Q^4 - 4m^2 x^2 v^2 - i\epsilon} ,$$

where $x_{th} = Q^2/(Q^2 + 2mM_\pi + M_\pi^2)$ represents the boundary of the inelastic region. The explicit expressions for the elastic contributions to $T_1, T_2$ are listed in Eq. (15) of Ref. [2] and $T^{el} = T_1^{el} + \frac{1}{2} T_2^{el}$.

Note that we are not subtracting the dispersion integral for $\tilde{T}$ at $\nu = 0$, but at $\nu = \frac{1}{2} Q$. This ensures that,
at \( \nu = iQ^4 \), the subtracted integral picks up the factor \( (Q^2 - 4Q^2) \), so that the angular average suppresses it when \( Q^2 \to \infty \).

In the framework of QCD+QED, the mass of a particle is determined by the bare parameters that occur in the Lagrangian and the cutoff used to regularize the theory. If the electromagnetic interaction is turned off, only the QCD coupling constant, the quark masses and the cutoff are relevant. To order \( e^2 \), the e.m. interaction changes the mass of a particle – the explicit expression relevant for this in the notation: we write \( \bar{m} = m_{\gamma} + \Delta m^\Lambda \).

The crude estimate \( m_{\text{QCD}} \approx -2 \text{MeV} \) and the values of the quark mass ratios \( m_s/m_{ud} = 27.23(10) \). \( Q = 22.1(7) \) imply that \( C \) is tiny: \( C \approx 6 \cdot 10^{-3} \text{GeV}^2 \). The approximate chiral symmetry of the Standard Model very strongly suppresses the asymptotic behaviour of \( \bar{S} \).

In contrast to the mass itself, \( m_{\text{QCD}} \) depends on the renormalization scale \( \mu \): the splitting into a contribution from QCD and one from QED is a matter of convention. The decomposition \[ m_{\text{QED}} = m_{\text{el}} + m_F + m_{F_2} + m_{\bar{S}}. \] The first term involves an integral over the e.m. form factors of the particle – the explicit expression relevant in the case of the nucleon is given in Eqs. (10) and (60) of Ref. [2]. The second and third terms represent convergent integrals over the structure functions:

\[
m_F = 4N \int_0^\infty dQ^2 \int_0^{\frac{x_{\min}}{x}} dx \frac{y f(y)}{x^4 + 4y} F(x, Q^2),
\]

\[
m_{F_2} = -2N \int_0^\infty dQ^2 \int_0^{\frac{x_{\min}}{x}} dx \frac{y f(y)}{x^4 + 4y} F_2(x, Q^2),
\]

\[
f(y) = \frac{1 + 4y}{2} \sqrt{1 + \frac{1}{y} - \frac{3 + 4y}{2}} = y = \frac{Q^2}{4m^2x^2}, \]

with \( N = 3\alpha_{\text{em}}/8\pi m \). The fourth term contains the contributions from subtraction function and counter term:

\[
m_{\bar{S}} = \lim_{\Lambda \to \infty} \left\{ N \int_0^{\Lambda^2} dQ^2 Q^2 \bar{S}(-Q^2) + \Delta m^\Lambda \right\}. \tag{9}
\]

In the decomposition we are using, the angular integration suppresses the integrals relevant for \( m_F \) and \( m_{F_2} \): the function \( f(y) \) falls off in proportion to \( 1/y^2 \) when \( y \) becomes large – in Eqs. [8], the limit \( \Lambda \to \infty \) has been taken. Only the integral over the subtraction function diverges – the divergence is absorbed by the counter term (non-leading contributions are briefly discussed below):

\[
\bar{S}(-Q^2) \to \frac{C}{Q^4}, \quad \Delta m^\Lambda \to -N C \ln \frac{\Lambda^2}{\mu^2}. \tag{10}
\]

The constant \( C \) is related to the matrix elements of the lowest dimensionless operators of spin zero.

Throughout the following, we consider the difference between proton and neutron, without explicitly indicating this in the notation: we write \( T \) for \( T^{-} \) and likewise for \( S, C, T_2, F, F_2, m_{\text{QED}}, m_F, m_{F_2} \). In the isospin limit, only the matrix element of the non-singlet operator \( \bar{u}u - \bar{d}d \) is determined by the bare parameters that occur in the leading Reggeon as:

\[
C = \frac{4m_u - m_d}{9} \langle p|\bar{u}u - \bar{d}d|p \rangle. \tag{11}
\]

We neglect the contributions from operators with \( I = 0 \), which are suppressed by isospin symmetry. In the chiral limit, \( C \) vanishes. The matrix element of the operator \( \bar{u}u - \bar{d}d \) also determines the leading contribution to the QCD part of the proton-neutron mass difference \( m_{\text{QCD}} \):

\[
m_{\text{QCD}} = \frac{m_u - m_d}{2m} \langle p|\bar{u}u - \bar{d}d|p \rangle \left\{ 1 + O(m_u - m_d) \right\}. \tag{12}
\]

The crude estimate \( m_{\text{QCD}} \approx -2 \text{MeV} \) and the values of the quark mass ratios \( m_s/m_{ud} = 27.23(10) \). \( Q = 22.1(7) \) imply that \( C \) is tiny: \( C \approx 6 \cdot 10^{-3} \text{GeV}^2 \). The approximate chiral symmetry of the Standard Model very strongly suppresses the asymptotic behaviour of \( \bar{S} \).

In view of the fact that the latter are beyond the accuracy of our calculation, we ignore these corrections.

Traditionally, the subtraction function is identified with a multiple of \( S_1(q^2) \equiv T_1(0, q^2) \). The operator product expansion shows that, while the asymptotic behaviour of \( \bar{S} \) is dominated by the contributions from the scalar operators, \( S_1 \) picks up additional contributions from operators of spin 2 [2]. These also dominate the asymptotic behaviour of \( T_2 \). In the traditional approach, the contribution to the Cottingham formula from \( S_1 \) contains an additional divergence and the one from \( F_2 \) diverges as well. The problem is of purely technical nature: in the sum of the contributions from \( S_1 \) and \( F_2 \), the spin 2 terms cancel [3]. It is difficult, however, to numerically evaluate the contributions from \( S_1 \) and \( F_2 \), in particular also because the matrix elements of the spin 2 operators are not known. Our framework avoids these problems.

We assume that, at high energies, the amplitude exhibits Regge-behaviour and represent the contribution from the leading Reggeon as:

\[
\bar{T}^R(\nu, q^2) = -\frac{\pi \bar{S}(q^2)}{\sin \pi \alpha} \left\{ (-s)^\alpha + (-u)^\alpha \right\}, \tag{13}
\]
where $s = m^2 + 2mv + q^2$ and $u = m^2 - 2mv + q^2$ represent the square of the centre of mass energy in the $s$- and $u$-channels, respectively. In general, the power $\alpha$ depends on $t$: $\alpha(t)$ moves on a Regge trajectory, but for the forward scattering amplitude, only the intercept $\alpha = \alpha(0)$ is relevant. In $\bar{T}$, the Reggeons generate singularities at $x = 0$. The leading singularity is of the form:

$$\bar{F}^R = b(Q^2)x^{-\alpha}, \quad b(Q^2) = 2Q^{2(\alpha+1)}\beta(Q^2).$$

(14)

We assume that the Reggeons dominate the asymptotics,

$$\lim_{\nu \to \infty} (\bar{T} - \bar{T}^R) = 0,$$

(15)

and that the remainder disappears sufficiently fast for the difference $\bar{T} - \bar{T}^R$ to obey an unsubtracted dispersion relation. This leads to a sum rule that fully determines the subtraction function $\bar{F}$. To derive the sum rule, we return to the dispersion relation $[5]$. In the limit $\nu \to \infty$, $\bar{T}^\text{el}$ disappears. The dispersion integral can be split into two parts with $\bar{F} = (\bar{F} - \bar{F}^R) + \bar{F}^R$. In the first part, the integration can be interchanged with the limit. The second part contains a contribution that approaches $\bar{T}^R$ when $\nu \to \infty$. Collecting terms, Eq. (15) leads to

$$Q^2 \mathcal{S}(Q^2) = \int_0^{x_\text{th}} \frac{dx}{x^2} \bar{F}(x, Q^2) - \frac{b(Q^2)}{x^{\alpha + 1}} \bar{F}^R(x, Q^2) - \int_0^{x_\text{th}} \frac{dx}{Q^2 + m^2 x^2}.$$

(16)

In Ref. [16], the violations of Bjorken scaling were ignored: it was assumed that for $Q^2 \to \infty$, the structure function $\bar{F}$ tends to $(2xH_1 + F_2)x^2m^2/Q^2$, where $H_1$ and $F_2$ only depend on $x$. One readily checks that the sum rule (16) then indeed reduces to the relation between the operator matrix element $C$ and the structure functions given in (5.2), (5.3), (13.14) of Ref. [16]. Scaling would imply that the last term on the r.h.s. of Eq. (16) tends to zero $\propto 1/Q^4$. The scaling violations merely make it disappear less rapidly, in proportion to $1/Q^2/(\ln Q^2)^{1+\gamma_2}$ with $\gamma_2 > 0$.

It is instructive to compare the sum rule (16) with the fixed pole occurring in the high energy behaviour of $T_2$ [17, 18]. This amplitude is dominated by Reggeon exchange as well: $T_2 - T_2^R \to 0$ for $\nu \to \infty$. While $T_2^R$ falls off in proportion to $\nu^{-a-2}$, the dispersion relation shows that the low energy singularities necessarily generate contributions that fall off with $\nu^{-2}$. In the amplitude $\bar{T}$, Reggeons contribute with $\nu^a$. The dispersion relation implies that this amplitude contains contributions proportional to $\nu^{-2}$ as well. In Regge language, such contributions correspond to a fixed pole with $\alpha = 0$ in $T_2$ and a fixed pole with $\alpha = -2$ in $\bar{T}$. Reggeon dominance is perfectly consistent with that, but excludes the presence of a fixed pole with $\alpha = 0$ in $\bar{T}$. We do not know of a source that could produce such a contribution – neither causality nor the short-distance singularities nor the Reggeons generate terms of this sort. We rely on Reggeon dominance, which in particular also ensures that the continuation of the structure functions from the space-like region, where they are measurable, into the time-like region is unique (see appendix C of Ref. [2] for a discussion of this aspect).

Note that the amplitudes used in the literature often have kinematic zeros – this can make it difficult not only to sort out the asymptotic behaviour, but also to identify the elastic part of the dispersive representation (“Born term”) with the contribution generated by the one-particle intermediate state [2, 19]. In Refs. [20–22], for instance, it is assumed that the amplitude $\bar{T} = q^2T_1 + \nu^2 T_2$ satisfies the asymptotic condition (15). That assumption, however, requires $q^2T_1$ to contain a fixed pole which compensates the one in $\nu^2T_2$ and hence violates Reggeon dominance.

We now turn to the numerical evaluation of the mass difference and start with the elastic contribution. In Ref. [16], the dipole approximation for the Sachs form factors was used, which yields 0.63 MeV for the proton and $-0.13$ MeV for the neutron, so that the elastic contribution to the self-energy difference amounts to $m_{\text{el}} = 0.76$ MeV. In the meantime, the precision to which the form factors are known has increased significantly. For a thorough review of the experimental information, we refer to Ref. [23]. With the parametrization given there, we obtain $m_{\text{el}} = 0.752$ MeV, indicating that, in the difference between the e.m. self-energies of proton and neutron, the departures from the dipole formulae only generate a small shift. These departures are now firmly established – our estimate for the remaining uncertainty is of the order of a keV, negligible compared to the uncertainties in the inelastic contributions.

For the numerical evaluation of the sum rule (16), we need a representation for the difference between the structure functions of MAID and DMT – we refer to these as MD. Both of these are accessible on the MAID home page [24].

(i) For the range $W < 1.3$, we rely on the parametrization of the structure functions of MAID and DMT – we refer to these as MD. Both of these are accessible on the MAID home page [24].

(ii) In the interval $1.3 < W < 3$, we make use of the representation due to Bosted and Christy (BC) [25, 26].

(iii) For $W > 3$, $Q^2 < 1$, we rely on the Reggeon parametrization of the proton structure functions due to Alwall and Ingelman (AI) [27], invoking SU(3) to arrive at a corresponding representation for the neutron.

(iv) In the region $W > 3$, $Q^2 > 1$, we rely on the solution of the DGLAP equations constructed by Alekhin, Blümlein and Moch (ABM) [28, 30], who provide numerical values for the structure functions over a wide range: $1 \leq Q^2 \leq 2 \cdot 10^5$ and $1 \cdot 10^{-7} \leq x \leq 0.99$. The values of $F_2(x, Q^2)$ and $F_L(x, Q^2)$ are listed for the proton as well.
as for the neutron on a grid of 60 \times 98 points.

Next, we discuss the solution of the sum rule (10) for $Q^2 < 1$, where the parametrizations listed in (i) –(iii) suffice. For details concerning these, in particular also for estimates of the uncertainties attached to them, we refer to Ref. [2]. It is straightforward to solve the sum rule for $\bar{S}$ with this representation of the structure functions and to calculate the corresponding contribution to $m_{\bar{S}}$ with Eq. (9). Isospin conservation prevents the most prominent feature in this region, the $\Delta(1232)$, to make a significant contribution. Moreover, the regions below and above $W = 3$ contribute with opposite sign – within errors, they cancel: $m_{\bar{S}}^{Q^2 < 1} = -0.034(68)\text{MeV}$. Note that the error is twice as large as the central value. It is dominated by the uncertainties in the resonance region and is of systematic nature, as it stems from the simplification used in the data analysis of Refs. [25, 26]: the ratio $R = \sigma_L/\sigma_T$ is assumed to be the same for proton and neutron. In the region where the Pomeron dominates, this holds to good accuracy, but we need the difference between the two, where Pomeron exchange drops out.

To account for this shortcoming, we follow the procedure used in Ref. [2], treat the transverse and longitudinal cross sections as independent quantities and attach an uncertainty to $\sigma^n_L, \sigma^n_T$ of 0.08 $\sigma^n_T, 0.08 \sigma^n_T$, respectively. In part of phase space, this may well overestimate the uncertainties considerably – a reanalysis of the data in the resonance region would be most welcome.

Photoproduction provides a check at $Q^2 = 0$, where the value of $\bar{S}$ is related to the polarizabilities. Reggeon dominance leads to a prediction for $\alpha^n_B$ [2]. Moreover, the Baldin sum rule determines $\alpha^n_B + \beta_M$ in terms of the cross sections for photoproduction. The numbers quoted in Ref. [2] lead to $\bar{S}(0) = -1.5(0.8)\text{GeV}^{-2}$. This is consistent with the value $\bar{S}(0) = -0.4(2.7)\text{GeV}^{-2}$, obtained if the prediction for $\alpha^n_B$ is replaced by the experimental value [31], but more precise. The threshold singularities generate a pronounced structure at small $Q^2$. In this region, chiral perturbation theory represents a useful method of analysis [31, 32].

Next, we take up the contributions to $m_{\bar{S}}$ arising from $Q^2 > 1$, where we rely on ABM. The leading term in Eq. (14) stems from the Reggeon with the quantum numbers of the $f_2$ and $\alpha \simeq 0.55$. In order to determine the coefficient $b$, we focus on small values of $x$ and approximate the numbers for $\bar{F}$ obtained from the ABM table at a given value of $Q^2$ with a parametrization of the form $\bar{F} = x^{1-\alpha}(b + b_1 x + b_2 x^2)$. At very small values of $x$, the numerical noise hides the signal while if $x$ is too large, the approximation used breaks down – we find that $10^{-4} < x < x_1$ with $x_1 = 3 \cdot 10^{-2}$ represents a suitable range, fix $b_2$ with continuity at $x_1$ and determine $b, b_1$ by fitting the parametrization to the table. The result not only determines the residue $\beta(Q^2)$ of the Regge pole, but Eq. (10) then also allows us to determine the subtraction function that belongs to the ABM-representation of the structure functions.

At $Q^2 = 1$, where the representations $AI$ and ABM meet, the results for the contributions to $\bar{S}$ from $W > 3$ agree within errors: the two entirely different sources match, both in sign and in size. It is questionable, however, whether the ABM table can be trusted down to $Q^2 = 1$, because it relies on perturbation theory. In the deep inelastic region, the structure functions of proton and neutron are nearly the same – it is difficult to reliably determine the difference from the data on elastic scattering, even if the DGLAP equations provide a strong theoretical constraint. In the ABM table, the numerical noise becomes visible at $Q^2 \sim 3.5$ and, for $Q^2 > 6$, it hides the signal completely: there, $\bar{S}$ vanishes within errors.

In order to interpolate between the values of $Q^2$ where the ABM table provides significant information and the region where asymptotics sets in, we make use of the Generalized Vector Dominance Model of Sakurai and Schiffknecht [33], parametrizing the subtraction function in terms of the contributions from $\rho, \omega$ and $\phi$. In the difference between proton and neutron, only the off-diagonal terms survive:

$$
\bar{S}_{\text{VMD}}(-Q^2) = \frac{1}{m_{\rho}^2 + Q^2} \left( \frac{c_\omega}{m_{\omega}^2 + Q^2} + \frac{c_\phi}{m_{\phi}^2 + Q^2} \right).
$$

(17)

The asymptotic condition (10) requires the two terms in the bracket to nearly cancel: $c_\omega + c_\phi = C$. This leaves a single parameter free, say $c_\omega$. Fitting the model to the sum of the contributions from MD, BC and ABM in the range $2 < Q^2 < 6$, we obtain $c_\omega = -0.74(49)\text{GeV}^2$. We have checked that the outcome for $m_{\bar{S}}$ is not sensitive to the range used in the fit. For the corresponding contributions to $m_{\bar{S}}$ from the intervals $1 < Q^2 < 2$ and $2 < Q^2 < \infty$, this yields $-0.040(27)\text{MeV}$ and $-0.092(61)\text{MeV}$ respectively. To account for the correlations between the contributions from the various regions, we determine the net error in $m_{\bar{S}}$ by evaluating the integral in Eq. (9) for the upper and lower edges of the error band. This amounts to adding the errors linearly and yields $m_{\bar{S}} = -0.17(16)\text{MeV}$.

Finally, we evaluate the convergent integrals $m_{\bar{P}}, m_{F_2}$ in Eqs. (8). In these integrals, the small $x$ region does not require special care. As mentioned above, the angular integration suppresses the contributions from the deep inelastic region. In fact, a very strong suppression also occurs at low values of $Q^2$. Numerically, these integrals are tiny: $m_{\bar{P}} + m_{F_2} = -0.004(1)\text{MeV}$.

Collecting the various contributions and using the experimental value of the proton-neutron mass difference, the parts due to the e.m. interaction and to the difference between $m_u$ and $m_d$ become

$$
m_{\text{QED}} = 0.58(16)\text{MeV}, \quad m_{\text{QCD}} = -1.87(16)\text{MeV}.
$$

(18)

The result for $m_{\text{QCD}}$ yields a more precise estimate for the leading Wilson coefficient: $C = 5.7(1.1) \times 10^{-4}\text{GeV}^2$, but the corresponding shift in our results is negligibly small.

The conclusions reached in Ref. [10] are thus confirmed: $m_{\text{QED}}$ is dominated by the elastic contribution.
The uncertainty in the old result, \( m_{\text{QED}} = 0.7(3) \) MeV, is reduced by about a factor of two.

It is by no means puzzling that the inelastic contributions are so small: (a) the angular integration suppresses the contributions from the dispersion integrals, (b) in the deep inelastic region, the subtraction function is nearly the same for proton and neutron – in the chiral limit, there is no difference, (c) in the region where Reggeon exchange dominates, the leading term, the Pomeron, is the same, (d) isospin symmetry ensures that the most important resonance, the \( \Delta(1232) \), contributes equally to proton and neutron and (e) the leading terms of the chiral perturbation series are also the same.

The determination of \( m_{\text{QED}} \) on a lattice is a very demanding goal. While the numbers in [34] cluster around \( m_{\text{QED}} \approx 0.7 \) MeV, in agreement with our result, the values 1.00(7)(14) MeV [35], 1.03(17) [36] and 1.53(25)(50) MeV [37] are higher than ours. Adding statistical and systematic errors in quadrature, all of the differences amount to less than two standard deviations. In the framework we are relying on, values like \( m_{\text{QED}} = 1 \) MeV or even higher require sizeable positive contributions from the subtraction function – this is not compatible with Reggeon dominance.

The main difference between our analysis and the work reported in Refs. [38, 40] is that, there, the subtraction function is not calculated, but parametrized with an ansatz in terms of its value at \( Q^2 = 0 \) (taken from experiment) and a scale \( m_0 \) that specifies the momentum dependence. If we replace our representation for \( S \) with the ansatz \( S(−Q^2) = 1 + Q^2/m_0^2 \), take \( S(0) \) from experiment, set \( \Lambda^2 = 2 \) GeV², pick the scale at \( m_0^2 = 0.71 \) GeV² and drop the counter term [38, 40], we obtain \( m_\bar{S} = -0.11(76) \) MeV, where the error exclusively accounts for the uncertainty in \( S(0) \).

Since chiral symmetry suppresses the coefficient \( C \), the leading asymptotic term dominates \( \bar{S} \) only if \( Q^2 \) becomes large. The contributions from the pre-asymptotic region can be parametrized with the model proposed in Ref. [39]: \( S(−Q^2) = (S(0) + C Q^2/m_0^6)(1+Q^2/m_0^2)^{-3} \). It does have the proper asymptotic behaviour, so that the limit \( \Lambda \rightarrow \infty \) can be taken in Eq. (9). Simply retaining the value of \( m_0 \), setting \( \mu = 2 \) GeV and again only accounting for the uncertainty in \( S(0) \), this parametrization yields \( m_\bar{S} = -0.09(63) \) MeV. Note that applying the same ansatz to \( S_1 \) instead of \( S \) fails, because the counter term required by mass renormalization does not remove the divergence associated with the spin 2 operators.

Replacing Reggeon dominance by an ansatz for the subtraction function \( S \) of the type proposed in Refs. [38] or [39] thus leads to results that are consistent with ours. They come with comparatively large uncertainties, in particular because the experimental information about \( S(0) \) is significantly less precise than our prediction. A more accurate measurement of the polarizabilities would be most welcome as it would subject Reggeon dominance to a more stringent test.

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