THE ABSOLUTE GALOIS GROUP ACTS FAITHFULLY ON THE CONNECTED COMPONENTS OF THE MODULI SPACES OF SURFACES OF GENERAL TYPE

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ABSTRACT. We show that the Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ operates faithfully on the set of connected components of the moduli spaces of surfaces of general type, and also that for each element $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ different from the identity and from complex conjugation, there is a surface of general type such that $X$ and the Galois conjugate variety $X^\sigma$ have nonisomorphic fundamental groups. The result was announced by the second author at the Alghero Conference 'Topology of algebraic varieties' in september 2006. Before the present paper was actually written, we received a very interesting preprint by Robert Easton and Ravi Vakil ([E-V07]), where it is proven, with a completely different type of examples, that the Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ operates faithfully on the set of irreducible components of the moduli spaces of surfaces of general type. We also give other simpler examples of surfaces with nonisomorphic fundamental groups which are Galois conjugate, hence have isomorphic algebraic fundamental groups.

Introduction

In the 60’s J. P. Serre showed in [Ser64] that there exists a field automorphism $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety $X$ defined over $\bar{\mathbb{Q}}$ such that $X$ and the Galois conjugate variety $X^\sigma$ have non isomorphic fundamental groups, in particular they are not homeomorphic.

In this note, using completely different methods, we show the following two results, which give a strong sharpening of the phenomenon discovered by Serre.

Theorem 0.1. The absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli spaces of minimal surfaces of general type.

Theorem 0.2. Assume that $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is different from the identity and from complex conjugation. Then there is a minimal surface of general type $X$ such that $X$ and $X^\sigma$ have non isomorphic fundamental groups.

In order not to get confused by the above two statements, note that while the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of connected components of the (coarse) moduli spaces of minimal surfaces of general type, it does not act on the set of isomorphism classes of fundamental groups of surfaces of general type. Observe that obviously complex conjugation does not change the isomorphism class of the fundamental group ($X$ and $X$ are diffeomorphic). Now, if we had an action on the set of isomorphism classes of fundamental groups, then all the normal closure of the $\mathbb{Z}/2$ generated by complex conjugation would
act trivially. This would contradict theorem 2 since complex conjugation does not lie in the centre of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The surfaces we consider in this note are the so-called 'surfaces isogenous to a product' whose weak rigidity was proven in [Cat00], and which by definition are the quotient of a product of curves $(C_1 \times C_2)$ by the free action of a finite group $G$.

Therefore our method is strictly related to the so-called theory of 'dessins d’enfants' (see [Gro97]). Dessins d’enfants are, in view of Riemann’s existence theorem (generalized by Grauert and Remmert in [GR58]), a combinatorial way to look at the monodromies of algebraic functions with only three branch points. It goes without saying that we make here an essential use of Belyi functions ([Belyi79]) and of their functoriality.

It would be interesting to obtain similar types of results with examples involving rigid varieties belonging to the class of varieties isogenous to a product. For instance we expect that similar results also hold if we restrict ourselves to consider only the so-called Beauville surfaces (see [Cat00] for the definition of Beauville surfaces and [Cat03], [BCG05], [BCG06] for further properties of these).

In the last section we use Beauville surfaces and polynomials with two critical values in order to produce simple examples of pairs of surfaces with nonisomorphic fundamental groups which are conjugate under the absolute Galois group (hence the two groups have isomorphic profinite completions).

1. Very special hyperelliptic curves

Fix a positive integer $g \in \mathbb{N}$, $g \geq 3$, and define, for any complex number $a \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g - 1\}$, $C_a$ as the hyperelliptic curve of genus $g$

$$w^2 = (z - a)(z + 2g)\prod_{i=0}^{2g-1}(z - i)$$

branched over $\{-2g, 0, 1, \ldots, 2g - 1, a\} \subset \mathbb{P}^1_{\mathbb{C}}$.

Lemma 1.1. Consider two complex numbers $a, b$ such that $a \in \mathbb{C} \setminus \mathbb{Q}$: then $C_a \cong C_b$ if and only if $a = b$.

Proof.

One direction being obvious, assume that $C_a \cong C_b$. Then the two sets with $2g + 2$ elements $B_a := \{-2g, 0, 1, \ldots, 2g - 1, a\}$ and $B_b := \{-2g, 0, 1, \ldots, 2g - 1, b\}$ are projectively equivalent over $\mathbb{C}$ (the latter set $B_b$ has also cardinality $2g + 2$ since $C_a \cong C_b$ and $C_a$ smooth implies that also $C_b$ is smooth).

In fact, this projectivity $\varphi$ is defined over $\mathbb{Q}$, since there are three rational numbers which are carried into three rational numbers (since $g \geq 2$).

Since $a \notin \mathbb{Q}$ it follows that $b \notin \mathbb{Q}$ and $\varphi$ maps $B := \{-2g, 0, 1, \ldots, 2g - 1\}$ to $B$, and in particular $\varphi$ has finite order. Since $\varphi$ either leaves the cyclical order of $(-2g, 0, 1, \ldots, 2g - 1)$ invariant or reverses it, and $g \geq 3$ we see that there are 3 consecutive integers such that $\varphi$ maps them to 3 consecutive integers. Therefore $\varphi$ is either an integer translation, or an affine symmetry of the form $x \mapsto -x + 2n$. In the former case $\varphi = \text{id}$, since it has finite order, and in
particular, \( a = b \). In the latter case it must be \( 2g + 2n = \varphi(-2g) = 2g - 1 \) and \( 2n = \varphi(0) = 2g - 2 \), a contradiction.

\[ \square \]

**Remark 1.2.** The previous lemma holds more generally under the assumption that \( a, b \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g - 1\} \), provided \( g \geq 6 \).

**Proof.**

The case where \( a, b \in \mathbb{Q} \) is similar to the previous one: \( \varphi \) preserves the cyclical order of the two sets, and we are done if \( \varphi(a) = b \) or there are 3 consecutive integers which are mapped by \( \varphi \) to 3 consecutive integers.

set \( B_a \) each consecutive triple of points is a triple of consecutive integers, unless one element in the triple is \(-2g\) or \(b\). This excludes six triples. Keep in mind that \( a \in B_a \) and consider all the consecutive triples of integers in the set \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \); at most two such triples are not a consecutive triple of points of \( B_a \). We conclude that there is a triple of consecutive integers in the set \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \) mapping to a triple of consecutive integers under \( \varphi \). Then either \( \varphi \) is an integer translation \( x \mapsto x + n \), or it is a symmetry \( x \mapsto -x + 2n \). In both case the intervals equal to the convex spans of the sets \( B_a, B_b \) are sent to each other by \( \varphi \), in particular the length is preserved and the extremal point are permuted. If \( a \in [-2g, 2g - 1] \) also \( b \in [-2g, 2g - 1] \) and in the translation case \( n = 0 \), so that \( \varphi(x) = x \) and \( a = b \). We see right away that \( \varphi \) cannot be a symmetry, because only two points belong to the left half of the interval. If \( a < -2g \) the interval has length \( 2g - 1 - a \), if \( a > 2g - 1 \) the interval has length \( 2g + a \). We only need to exclude \( a < -2g, b > 2g - 1, 2g - 1 - a = 2g + b \), i.e., \( a = -b - 1 \). In this case \( b = \varphi(a) \) leads to a contradiction since we should have \( b = \varphi(a) = -a + 2n = b + 1 + 2n \).

We shall assume from now on that \( a, b \in \overline{\mathbb{Q}} \setminus \mathbb{Q} \) and that there is a field automorphism \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) such that \( \sigma(a) = b \). (Obviously, for any \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) different from the identity, there are \( a, b \in \overline{\mathbb{Q}} \setminus \mathbb{Q} \) with \( \sigma(a) = b \) and \( a \neq b \).

**Proposition 1.3.** Let \( P \) be the minimal polynomial of \( a \) and consider the field \( L := \mathbb{Q}[x]/(P) \). Let \( C_x \) be the hyperelliptic curve over \( L \)

\[
w^2 = (z - a)(z + 2g)\prod_{i=0}^{2g-1}(z - i).
\]

Then there is a rational function \( F_x : C_x \to \mathbb{P}^1_L \) such that for each \( a \in \mathbb{C} \) with \( P(a) = 0 \) it holds that the rational function \( F_a \) (obtained under the specialization \( x \mapsto a \)) is a Belyi function for \( C_a \).

**Proof.** Let \( f_x : C_x \to \mathbb{P}^1_L \) be the hyperelliptic involution, branched in \( \{-2g, 0, 1, \ldots, 2g - 1, x\} \). Then \( P \circ f_x \) has as critical values:

- the images of the critical values of \( f_x \) under \( P \), which are \( \in \mathbb{Q} \),
- the critical values \( y \) of \( P \), i.e. the zeroes of the discriminant \( h_1(y) \) of \( P(z) - y \) with respect to the variable \( z \).
The degree $\deg(P) - 1$, whence, inductively, we obtain $\tilde{f}_x := h \circ P \circ f$ whose critical values are all contained in $\mathbb{Q} \cup \{\infty\}$. If we take any root $a$ of $P$, then obviously $\tilde{f}_a$ has the same critical values.

Let now $r_1, \ldots, r_n \in \mathbb{Q}$ be the (pairwise distinct) finite critical values of $\tilde{f}_x$. We set:

$$y_i := \frac{1}{\prod_{j \neq i} (r_i - r_j)}.$$  

Let $N \in \mathbb{N}$ be a positive integer such that $m_i := Ny_i \in \mathbb{Z}$. Then we have that the rational function

$$g(t) := \Pi_i (t - r_i)^{m_i} \in \mathbb{Q}(t)$$

is ramified at most in $\infty$ and $r_1, \ldots, r_n$. In fact, $g(t)$ vanishes exactly when the logarithmic derivative $G(t) := \frac{g'(t)}{g(t)} = \sum_i m_i \frac{1}{t - r_i}$ has a zero or a pole, but the choice made yields a zero of order $n$ at $\infty$.

Therefore the critical values of $g \circ \tilde{f}_x$ are at most 0, $\infty$, $g(\infty)$ (for details see [Wo01]).

We set $F_x := \Phi \circ g \circ \tilde{f}_x$ where $\Phi$ is the affine map $z \mapsto g(\infty)^{-1}z$, so that the critical values of $F_x$ are equal to $\{0, 1, \infty\}$. It is obvious by our construction that for any root $a$ of $P$, $F_a$ has the same critical values as $F_x$, in particular, $F_a$ is a Belyi function for $C_a$.

Since in the sequel we shall consider the normal closure (we prefer here, to avoid confusion, not to use the term 'Galois closure' for the geometric setting) $\psi_a : D_a \to \mathbb{P}_C^1$ of each of the functions $F_a : C_a \to \mathbb{P}_C^1$, we recall in the next section the 'scheme theoretic' construction of the normal closure.

2. Effective construction of normal closures

In this section we consider algebraic varieties over the complex numbers, endowed with their Hausdorff topology, and, more generally, 'good' covering spaces (i.e., between topological spaces which are arcwise connected and semilocally simply connected).

**Lemma 2.1.** Let $\pi : X \to Y$ be a finite 'good' unramified covering space of degree $d$ between connected spaces $X$ and $Y$. Then the normal closure $Z$ of $\pi : X \to Y$ is isomorphic to any connected component of

$$W := W_\pi := (X \times_Y \ldots \times_Y X) \setminus \Delta \subset X^d \setminus \Delta,$$

where $\Delta := \{(x_1, \ldots, x_d) \in X \times_Y \ldots \times_Y X | \exists i \neq j, x_i = x_j\}$ is the big diagonal.

**Proof.** Choose base points $x_0 \in X$, $y_0 \in Y$ such that $\pi(x_0) = y_0$ and denote by $F_0$ the fibre over $y_0$, $F_0 := \pi^{-1}(\{y_0\})$.

We consider the monodromy $\mu : \pi_1(Y, y_0) \to \mathfrak{S}_d = \mathfrak{S}(F_0)$ of the unramified covering $\pi$. The monodromy of $\phi : W \to Y$ is induced by the diagonal product monodromy $\mu^d : \pi_1(Y, y_0) \to \mathfrak{S}(F_0^d)$, such that, for $(x_1, \ldots, x_d) \in F_0^d$, we have $\mu^d(\gamma)(x_1, \ldots, x_d) = (\mu(\gamma)(x_1), \ldots, \mu(\gamma)(x_d))$.

It follows that the monodromy of $\phi : W \to Y$, $\mu_W : \pi_1(Y, y_0) \to \mathfrak{S}(\mathfrak{S}_d)$ is given by left translation $\mu_W(\gamma)(\tau) = \mu(\gamma) \circ (\tau)$. 
If we denote by $G := \mu(\pi_1(Y, y_0)) \subset \mathfrak{S}_d$ the monodromy group, it follows right away that the components of $W$ correspond to the cosets $G\tau$ of $G$. Thus all the components yield isomorphic covering spaces.

\[ \square \]

The theorem of Grauert and Remmert ([GR58]) allows to extend the above construction to yield normal closures of morphisms between normal algebraic varieties.

**Corollary 2.2.** Let $\pi : X \to Y$ be a finite map between normal projective varieties, let $B \subset Y$ be the branch locus of $\pi$ and set $X^0 := X \setminus \pi^{-1}(B)$, $Y^0 := Y \setminus B$.

If $X$ is connected, then the normal closure $Z$ of $\pi$ is isomorphic to any connected component of the closure of $W^0 := (X^0 \times_{Y^0} \ldots \times_{Y^0} X^0) \setminus \Delta$ in the normalization $W^\prime$ of $W := (X \times_Y \ldots \times_Y X) \setminus \Delta$.

**Proof.**

The irreducible components of $W$ correspond to the connected components of $W^0$, as well as to the connected components $Z$ of $W^\prime$. So, our component $Z$ is the closure of a connected component $Z^0$ of $W^0$. We know that the monodromy group $G$ acts on $Z^0$ as a group of covering transformations and simply transitively on the fibre of $Z^0$ over $y_0$: by normality the action extends biholomorphically to $Z$, and clearly $Z/G \cong Y$.

\[ \square \]

3. Connected components of moduli spaces associated to very special hyperelliptic curves

Let $a$ be an algebraic number, $g \geq 2$, and consider as in section 1 the hyperelliptic curve $C_a$ of genus $g$ defined by the equation

$w^2 = (z - a)(z + 2g)\prod_{i=0}^{2g-1}(z - i)$.

Let $F_a : C_a \to \mathbb{P}^1$ be the Belyi function constructed in proposition 1.3 and denote by $\psi_a : D_a \to \mathbb{P}^1$ the normal closure of $C_a$ as in corollary 2.2.

**Remark 3.1.** We denote by $G_a$ the monodromy group of $D_a$ and observe that there is a subgroup $H_a \subset G_a$ acting on $D_a$ such that $D_a/H_a \cong C_a$.

We choose a monodromy representation $\mu : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to G_a$ corresponding to the normal ramified covering $\psi_a : D_a \to \mathbb{P}^1$ and we denote the images of geometric loops around $0$, $1$, $\infty$ by $\tau_0$, $\tau_1$, $\tau_\infty$. Then we have that $G_a$ is generated by $\tau_0$, $\tau_1$, $\tau_\infty$ and $\tau_0 \cdot \tau_1 \cdot \tau_\infty = 1$.

Fix now another integer $g' \geq 2$ and consider all the possible smooth complex curves $C'$ of genus $g'$. Observe that the fundamental group of $C'$ is isomorphic to the standard group

$\Pi_{g'} := \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{g'}[\alpha_i, \beta_i] = 1 \rangle$.

Since $g' \geq 2$ there are epimorphisms (surjective homomorphisms) $\rho : \Pi_{g'} \to G_a$. For instance it suffices to consider the epimorphism $\theta : \Pi_{g'} \to \mathbb{F}_{g'}$ from $\Pi_{g'}$ to the free group $\mathbb{F}_{g'} := \langle \lambda_1, \ldots, \lambda_{g'} \rangle$ in $g'$ letters given by $\theta(\alpha_i) = \theta(\beta_i) = \lambda_i$. 


$\forall 1 \leq i \leq g'$, and to compose $\theta$ with the surjection $\phi : \mathbb{P}g' \to G_a$, given by $\phi(\lambda_1) = \tau_0$, $\phi(\lambda_2) = \tau_1$, and $\phi(\lambda_i) = 1$ for $3 \leq i \leq g'$.

Consider all the possible epimorphisms $\rho : \Pi_g \to G_a$. Each such $\rho$ gives a normal unramified covering $D' \to C'$ with monodromy group $G_a$.

**Definition 3.2.** Let $\mathcal{M}_a$ be the subset of the moduli space of surfaces of general type given by surfaces $S \cong (D_a \times D')/G_a$, where $D,a,D'$ are as above and the group $G_a$ acts by a diagonal action.

From [Cat00] and especially Theorem 3.3 of [Cat03] it follows:

**Proposition 3.3.** For each $a \in \hat{\mathbb{Q}}$, $\mathcal{M}_a$ is a union of connected components of the moduli spaces of surfaces of general type.

Moreover, for $\sigma \in \text{Gal}(\hat{\mathbb{Q}}/\mathbb{Q})$, $\sigma(\mathcal{M}_a) = \mathcal{M}_{\sigma(a)}$.

**Proof.** Since $D_a$ is a triangle curve, the pair $(D_a,G_a)$ is rigid, whereas, varying $C'$ and $\rho$, we obtain the full union of the moduli spaces for the pairs $(D',G_a)$, corresponding to the possible free topological actions of the group $G_a$ on a curve $D'$ of genus $|G_a|(g'-1)+1$.

Thus, the surfaces $S \cong (D_a \times D')/G_a$ give, according to the cited theorem 3.3 of [Cat03], a union of connected components of the moduli spaces of surfaces of general type.

Choose now a surface $S$ as above (thus, $[S] \in \mathcal{M}_a$) and apply the field automorphism $\sigma \in \text{Gal}(\hat{\mathbb{Q}}/\mathbb{Q})$ to a point of the Hilbert scheme corresponding to the 5-canonical image of $S$ (which is isomorphic to $S$, since the canonical divisor of $S$ is ample). We obtain a surface which we denote by $S^\sigma$.

By taking the fibre product of $\sigma$ with $D_a \times D' \to S$ it follows that $S^\sigma$ has an étale covering with group $G_a$ which is the product $(D_a)^{\sigma} \times (D')^{\sigma}$.

Since $(C_a)^{\sigma} = C_{\sigma(a)}$ and since $\sigma(a)$ corresponds to another embedding of the field $L$ into $\mathbb{C}$, it follows that $(F_a)^{\sigma} = F_{\sigma(a)}$, whence $(D_a)^{\sigma} = D_{\sigma(a)}$.

On the other hand, the quotient of $(D')^{\sigma}$ by the action of the group $G_a$ has genus equal to the dimension of the space of invariants $\dim(H^0(\Omega_{(D')^{\sigma}}^{1}))^{G_a}$, but this dimension is the same $g' = \dim(H^0(\Omega_{D'}^{1}))^{G_a}$. Hence the action of $G_a$ on $(D')^{\sigma}$ is also free (by Hurwitz’ formula), and we have shown that $S^\sigma$ is a surface whose moduli point is in $\mathcal{M}_{\sigma(a)}$.

□

4. **Proof of the main theorems**

**Proof of theorem 1.**

Given $a \in \hat{\mathbb{Q}}$, consider a connected component $\mathcal{N}_a$ of $\mathcal{M}_a$. Our main theorem 1 follows easily from the following

**Main Claim:** if $\mathcal{N}_a = \sigma(\mathcal{N}_a)$, then necessarily $a = \sigma(a)$.

**Proof.** Denote $\sigma(a)$ by $b$. We know that $C_a \cong D_a/H_a$, that $C_b \cong D_b/H_b$, and we have already shown that $\sigma$ yields an isomorphism $G_a \cong G_b$. The assumption that $\mathcal{N}_a = \sigma(\mathcal{N}_a)$ implies, by theorem (3.3) of [Cat03], the condition that the pairs $(D_a,G_a)$ and $(D_b,G_b)$ are isomorphic as complex triangle curves.
It suffices therefore to show that under this isomorphism the subgroup \( H_a \) corresponds to the subgroup \( H_b \), because then we conclude that the curves \( C_a \), \( C_b \) are not only Galois conjugate, but also isomorphic.

And then by lemma 1 we conclude that \( a = b \).

To show that \( H_a \) corresponds to the subgroup \( H_b \), let \( K \) be the Galois closure of the field \( L \) (= splitting field of the field extension \( \mathbb{Q} \subset L \)), and view \( L \) as embedded in \( \mathbb{C} \) under the isomorphism sending \( x \mapsto a \).

Consider the curve \( \hat{C}_x \) obtained from \( C_x \) by scalar extension \( \hat{C}_x := C_x \otimes_L K \). Let also \( \hat{F}_x := F_x \otimes_L K \) the corresponding Belyi function with values in \( \mathbb{P}^1_K \).

Apply now the effective construction of the normal closure of section \( \text{2} \) and, taking a connected component of \( (\hat{C}_x \times_{\mathbb{P}^1_K} \ldots \times_{\mathbb{P}^1_K} \hat{C}_x) \setminus \Delta \) we obtain a curve \( D_x \) defined over \( K \).

Note that \( D_x \) is not geometrically irreducible, but once we tensor with \( \mathbb{C} \) it splits into several components which are Galois conjugate and which are isomorphic to the conjugates of \( D_a \).

Apply now the Galois automorphism \( \sigma \) to the triple \( D_a \to C_a \to \mathbb{P}^1 \). Since the triple is induced by the triple \( D_x \to C_x \to \mathbb{P}^1_K \) by taking a tensor product \( \otimes_K \mathbb{C} \) via the embedding sending \( x \mapsto a \), the morphisms are induced by the composition of the inclusion \( D_x \subset (C_x)^d \) with the coordinate projections, respectively buy the fibre product equation, it follows from proposition \( \text{1.3} \) that \( \sigma \) carries the triple \( D_a \to C_a \to \mathbb{P}^1 \) to the triple \( D_b \to C_b \to \mathbb{P}^1 \).

If we want to interpret our argument in terms of Grothendieck’s étale fundamental group, we define \( C_0^x := F^{-1}_x(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \), and accordingly \( \hat{C}_x \) and \( \hat{D}_x \).

There are the following exact sequences for the Grothendieck étale fundamental group (compare Theorem 6.1 of \( \text{[SGA1]} \)):

\[
1 \to \pi_1^{\text{alg}}(D_a) \to \pi_1^{\text{alg}}(D_x) \to \text{Gal}(\overline{\mathbb{Q}}/K) \to 1
\]

\[
1 \to \pi_1^{\text{alg}}(C_a) \to \pi_1^{\text{alg}}(C_x) \to \text{Gal}(\overline{\mathbb{Q}}/K) \to 1
\]

\[
1 \to \pi_1^{\text{alg}}(\mathbb{P}_C^1 \setminus \{0, 1, \infty\}) \to \pi_1^{\text{alg}}(\mathbb{P}_K^1 \setminus \{0, 1, \infty\}) \to \text{Gal}(\overline{\mathbb{Q}}/K) \to 1
\]

where \( H_a \) and \( G_a \) are the respective factor groups for the inclusions of the left hand sides, corresponding to the first and second sequence, and corresponding to the first and third sequence.

On the other hand, we also have the exact sequence

\[
1 \to \pi_1^{\text{alg}}(\mathbb{P}_C^1 \setminus \{0, 1, \infty\}) \to \pi_1^{\text{alg}}(\mathbb{P}_Q^1 \setminus \{0, 1, \infty\}) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.
\]

The finite quotient \( G_a \) of \( \pi_1^{\text{alg}}(\mathbb{P}_C^1 \setminus \{0, 1, \infty\}) \) (defined over \( K \)) is sent by \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) to another quotient, corresponding to \( D_{\sigma(a)} \), and the subgroup \( H_a \) yields the quotient \( C_a \), is sent to the subgroup \( H_{\sigma(a)} \).

\textit{Proof of theorem 2.}
Assume that $\sigma(a) = b$, and that $\sigma$ is neither complex conjugation nor the identity. Consider a surface $X_a$ with $[X_a] \in \mathcal{M}_a$ and its Galois conjugate $(X_a)^\sigma = X_{\sigma(a)} = X_b$.

Denote as before by $\mathfrak{N}_a$ the connected component of the moduli space of surfaces of general type containing $[X_a]$.

Assume now that the $X_a$ and $X_b$ have isomorphic fundamental groups. Since obviously the two surfaces have the same Euler number $e(X_a) = e((X_a)^\sigma)$ (since they have the same Hodge numbers $h^i(\Omega^j)$) we can apply again theorem 3.3. of [Cat03] and there are only two possibilities: either $[X_a]$ and $[X_b]$ belong to the same connected component $\mathfrak{N}_a$ of the moduli space, or $[X_b]$ belongs to the complex conjugate component $\mathfrak{N}_{\bar{a}}$ of the connected component $\mathfrak{N}_a$.

In the first case the main claim says that $\sigma(a) = a$, in the second case it says that $\sigma(a) = \bar{a}$.

Since however $\sigma$ is neither complex conjugation nor the identity, we find an $a \in \bar{Q}$ such that $b := \sigma(a) \neq a$ and $b := \sigma(a) \neq \bar{a}$.

Therefore the corresponding surfaces $X_a$ and $(X_a)^\sigma$ have nonisomorphic fundamental groups.

\[ \Box \]

Remark 4.1. We observe that $X_a$ and $(X_a)^\sigma$ have isomorphic Grothendieck étale fundamental groups. In particular, the profinite completion of $\pi_1(X_a)$ and $\pi_1((X_a)^\sigma)$ are isomorphic.

It is not so easy to calculate explicitly the fundamental groups of the surfaces constructed above, since one has to explicitly calculate the monodromy of the Belyi function of the very special hyperelliptic curves.

Therefore we give in the next section an explicit example of two rigid surfaces with non isomorphic fundamental groups which are Galois conjugate.

5. AN EXPLICIT EXAMPLE

In this section we provide, as we already mentioned, an explicit example of two surfaces with non isomorphic fundamental groups which are conjugate under the absolute Galois group, hence with isomorphic profinite completions of their respective fundamental groups. These surfaces are rigid.

We consider (see [BCG06] for an elementary treatment of what follows) polynomials with only two critical values: $\{0, 1\}$.

Let $P \in \mathbb{C}[z]$ be a polynomial with critical values $\{0, 1\}$.

In order not to have infinitely many polynomials with the same branching behaviour, one considers normalized polynomials $P(z) := z^n + a_{n-2}z^{n-2} + \ldots + a_0$. The condition that $P$ has only $\{0, 1\}$ as critical values, implies, as we shall briefly recall, that $P$ has coefficients in $\bar{Q}$. Denote by $K$ the number field generated by the coefficients of $P$.

Fix the types the types $(m_1, \ldots, m_r)$ and $(n_1, \ldots, n_s)$ of the cycle decompositions of the respective local monodromies around 0 and 1: we can write our
polynomial $P$ in two ways, namely as:

$$P(z) = \prod_{i=1}^{r}(z - \beta_i)^{m_i},$$

and

$$P(z) = 1 + \prod_{k=1}^{s}(z - \gamma_k)^{n_k}.$$ 

We have the equations $F_1 = \sum m_i\beta_i = 0$ and $F_2 = \sum n_k\gamma_k = 0$ (since $P$ is normalized). Moreover, $m_1 + \ldots + m_r = n_1 + \ldots + n_s = n = \text{deg}P$ and therefore, since $\sum_{j}(m_j - 1) + \sum_{i}(n_i - 1) = n - 1$, we get $r + s = n + 1$.

Since we have $\prod_{i=1}^{r}(z - \beta_i)^{m_i} = 1 + \prod_{k=1}^{s}(z - \gamma_k)^{n_k}$, comparing coefficients we obtain further $n - 1$ polynomial equations with integer coefficients in the variables $\beta_i$, $\gamma_k$, which we denote by $F_3 = 0, \ldots, F_{n+1} = 0$. Let $\mathbb{V}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$ be the algebraic set in affine $(n + 1)$-space defined by the equations $F_1 = 0, \ldots, F_{n+1} = 0$. Mapping a point of this algebraic set to the vector $(a_0, \ldots, a_{n-2})$ of coefficients of the corresponding polynomial $P$ we obtain a set

$$\mathbb{W}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$$

(by elimination of variables) in affine $(n - 1)$ space. Both these are finite algebraic sets defined over $\mathbb{Q}$ since by Riemann’s existence theorem they are either empty or have dimension 0.

Observe also that the equivalence classes of monodromies $\mu: \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to S_n$ correspond to the orbits of the group of $n$-th roots of 1 (we refer to [BCG06] for more details).

We recall also the following

**Definition 5.1.** A smooth algebraic curve $C$ is called a triangle curve iff there is a finite group $G$ acting effectively on $C$ with the property that $C/G \cong \mathbb{P}^1$ in such a way that $f: C \to \mathbb{P}^1$ is ramified only in $\{0,1,\infty\}$.

**Example 5.2.** We calculate (here and in the following, either by a MAGMA routine, or, sometimes, more painfully by direct calculation) that $\mathbb{W} := \mathbb{W}(7; (2,2,1,1,1); (3,2,2))$ is irreducible over $\mathbb{Q}$. This implies that $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ acts transitively on $\mathbb{W}$. Looking at the possible monodromies, one sees that there are exactly two real non equivalent polynomials. Observe also that the equivalence classes of monodromies $\mu: \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to (S)_n$ correspond to the orbits of the group $\mu_n$ of $n$-th roots of 1 (we refer to [BCG06] for more details). In both cases, which will be explicitly described later on, the two permutations, of types $(2,2)$ and $(3,2,2)$, are seen to generate $\mathfrak{A}_7$ and the respective normal closures of the two polynomial maps are easily seen to give (we use here the fact that the automorphism group of $\mathfrak{A}_7$ is $\mathfrak{S}_7$) nonequivalent triangle curves $C_1$, $C_2$.

By Hurwitz’s formula, we see that $g(C_i) = \frac{|\mathfrak{A}_7|}{2}(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7}) + 1 = 241$.

We remark that $\mathfrak{A}_7$ admits generators $a_1, a_2$ of order 5 such that their product has order 5, hence we get a triangle curve $C$ (of genus 505).

Consider the triangle curve $C$ given by a **spherical system of generators of type $(5,5,5)$ of $\mathfrak{A}_7$**, i.e., generators $a_1, a_2, a_3$ of $\mathfrak{A}_7$ such that $a_1 \cdot a_2 \cdot a_3 = 1$. 

Definition 5.3. Let \( (a_1,a_2,a_3) \) and \( (b_1,b_2,b_3) \) be two spherical systems of generators of a finite group \( G \) of the same unordered type, i.e., 
\[
\{\text{ord}(a_1), \text{ord}(a_2), \text{ord}(a_3)\} = \{\text{ord}(b_1), \text{ord}(b_2), \text{ord}(b_3)\}. 
\]
Then \( (a_1,a_2,a_3) \) and \( (b_1,b_2,b_3) \) are called \textit{Hurwitz equivalent} iff they are equivalent under the equivalence relation generated by
\[
(a_1,a_2,a_3) \equiv (a_2,a_2^{-1}a_1a_2,a_3), \\
(a_1,a_2,a_3) \equiv (a_1,a_3,a_3^{-1}a_2a_3). 
\]

It is well known that two such triangle curves are isomorphic, compatibly with the action of the group \( G \), if and only if the two spherical systems of generators are \textit{Hurwitz equivalent}.

Remark 5.4. An easy MAGMA routine shows that there is exactly one Hurwitz equivalence class of triangle curves given by a spherical system of generators of type \((5,5,5)\) of \( \mathfrak{A}_7 \). In other words, if \( D_1 \) and \( D_2 \) are two triangle curves given by spherical systems of generators of type \((5,5,5)\) of \( \mathfrak{A}_7 \), then \( D_1 \) and \( D_2 \) are not only isomorphic as algebraic curves, but they have the same action of \( G \).

Let \( C \) be as above the triangle curve given by a(ny) spherical systems of generators of type \((5,5,5)\) of \( \mathfrak{A}_7 \), and consider the two triangle curves \( C_1 \) and \( C_2 \) as in example 5.2. Clearly \( \mathfrak{A}_7 \) acts freely on \( C_1 \times C \) as well as on \( C_2 \times C \) and we obtain two non isomorphic so-called Beauville surfaces \( S_1 := (C_1 \times C)/G, S_2 := (C_2 \times C)/G. \)

Obviously, these two surfaces have the same topological Euler characteristic. If they had isomorphic fundamental groups, by theorem 3.3 of \cite{Cat03}, \( S_2 \) would be the complex conjugate surface of \( S_1 \). In particular, \( C_1 \) would be the complex conjugate triangle curve of \( C_2 \): but this is absurd since we shall show that both \( C_1 \) and \( C_2 \) are real triangle curves.

Proposition 5.5. There is a field automorphism \( \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) such that \( S_2 = (S_1)^\sigma. \)

\textit{Proof.} We know that \((S_1)^\sigma = ((C_1)^\sigma \times (C)^\sigma)/G. \) Since there is only one Hurwitz class of triangle curves given by a spherical system of generators of type \((5,5,5)\) of \( \mathfrak{A}_7 \), we have \((C)^\sigma \cong C \) (with the same action of \( G \)). \hfill \Box

We determine now explicitly the respective fundamental groups of \( S_1 \) and \( S_2 \).

In general, let \( (a_1,\ldots,a_n) \) and \( (b_1,\ldots,b_m) \) be two sets of spherical generators of a finite group \( G \) of respective order types \( r := (r_1,\ldots,r_n), s := (s_1,\ldots,s_m) \). We denote the corresponding ‘polygonal’ curves by \( D_1, \) resp. \( D_2. \)

Assume now that the diagonal action of \( G \) on \( D_1 \times D_2 \) is free. We get then the smooth surface \( S := (D_1 \times D_2)/G, \) isogenous to a product.

Denote by \( T_r := T(r_1,\ldots,r_n) \) the \textit{polygonal group}
\[
\langle x_1, \ldots, x_{n-1}|x_1^{r_1} = \ldots = x_{n-1}^{r_{n-1}} = (x_1x_2\ldots x_{n-1})^{r_n} = 1 \rangle.
\]
We have the exact sequence (cf. [Cat00] cor. 4.7)
\[ 1 \to \pi_1 \times \pi_2 \to T_r \times T_s \to G \times G \to 1, \]
where \(\pi_i := \pi_1(D_i)\).

Let \(\Delta_G\) be the diagonal in \(G \times G\) and let \(H\) be the inverse image of \(\Delta_G\) under \(\Phi: T_r \times T_s \to G \times G\). We get the exact sequence
\[ 1 \to \pi_1 \times \pi_2 \to H \to G \cong \Delta_G \to 1. \]

**Remark 5.6.** \(\pi_1(S) \cong H\) (cf. [Cat00] cor. 4.7).

We choose now an arbitrary spherical system of generators of type \((5,5,5)\) of \(\mathfrak{A}_7\), for instance \(((1,7,6,5,4), (1,3,2,6,7), (2,3,4,5,6))\). Note that we use here MAGMA’s notation, where permutations act on the right (i.e., \(ab\) sends \(x\) to \((xa)b\)).

A MAGMA routine shows that
\[ (1) \quad ((1,2)(3,4), (1,5,7)(2,3)(4,6), (1,7,5,2,4,6,3)) \]
and
\[ (2) \quad ((1,2)(3,4), (1,7,4)(2,5)(3,6), (1,3,6,4,7,2,5)) \]
are two representatives of spherical generators of type \((2,6,7)\) yielding two non isomorphic triangle curves \(C_1\) and \(C_2\), each of which is isomorphic to its complex conjugate. In fact, an alternative direct argument is as follows: first of all \(C_i\) is isomorphic to its complex conjugate triangle curve since, for an appropriate choice of the real base point, complex conjugation sends \(a \mapsto a^{-1}, b \mapsto b^{-1}\) and then one sees that the two corresponding monodromies are permutation equivalent (see Figure 1 and Figure 2).

Moreover, since \(Aut(\mathfrak{A}_7) = \mathfrak{S}_7\), if the two triangle curves were isomorphic, then the two monodromies were conjugate in \(\mathfrak{S}_7\). That this is not the case is seen again by the following pictures.

![Figure 1. Monodromy corresponding to (1)](image1.png)

![Figure 2. Monodromy corresponding to (2)](image2.png)

The two corresponding homomorphisms \(\Phi_1: T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7\) and \(\Phi_2: T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7\) give two exact sequences
1 \rightarrow \pi_1(C_1) \times \pi_1(C) \rightarrow T_{(2,6,7)} \times T_{(5,5,5)} \rightarrow A_7 \times A_7 \rightarrow 1,

and

1 \rightarrow \pi_1(C_2) \times \pi_1(C) \rightarrow T_{(2,6,7)} \times T_{(5,5,5)} \rightarrow A_7 \times A_7 \rightarrow 1,

yielding two non isomorphic fundamental groups \( \pi_1(S_1) = \Phi_1^{-1}(\Delta_{37}) \) and \( \pi_1(S_2) = \Phi_2^{-1}(\Delta_{37}) \) fitting both in an exact sequence of type

\[ 1 \rightarrow \Pi_{241} \times \Pi_{505} \rightarrow \pi_1(S_j) \rightarrow \Delta_{37} \cong A_7 \rightarrow 1, \]

where \( \Pi_{241} \cong \pi_1(C_1) \cong \pi_1(C_2), \Pi_{505} = \pi_1(C) \).

Remark 5.7. 1) Using the same trick that we used for our main theorems, namely, using a surjection of a group \( \Pi_g \rightarrow A_7 \) we get infinitely many examples of pairs of fundamental groups which are nonisomorphic, but which have isomorphic profinite completions. Each pair fits into an exact sequence

\[ 1 \rightarrow \Pi_{241} \times \Pi_{g} \rightarrow \pi_1(S_j) \rightarrow A_7 \rightarrow 1. \]

2) Many more explicit examples as the one above (but with cokernel group different from \( A_7 \)) can be obtained using polynomials with two critical values.

A construction of polynomials with two critical values having a very large Galois orbit was proposed to us by D. van Straten.

Remark 5.8. In the sequel to this work we plan to show that the absolute Galois group in fact acts faithfully also on the set of Beauville surfaces.

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