ESCAPING SET AND JULIA SET OF TRANSCENDENTAL SEMIGROUPS

DINESH KUMAR AND SANJAY KUMAR

Abstract. We discuss the dynamics of an arbitrary semigroup of transcendental entire functions using Fatou-Julia theory and provide some condition for the complete invariance of escaping set and Julia set of transcendental semigroups. Some results on limit functions and postsingular set have been discussed. A class of hyperbolic transcendental semigroups and semigroups having no wandering domains have also been provided.

1. INTRODUCTION

Let \( f \) be a transcendental entire function and for \( n \in \mathbb{N} \) let \( f^n \) denote the \( n \)-th iterate of \( f \). The set \( F(f) = \{ z \in \mathbb{C} : \{ f^n \}_{n \in \mathbb{N}} \text{ is normal in some neighborhood of } z \} \) is called the Fatou set of \( f \) or the set of normality of \( f \) and its complement \( J(f) \) is called the Julia set of \( f \). An introduction to the basic properties of these sets can be found in [4]. The escaping set of \( f \) denoted by \( I(f) \) is the set of points in the complex plane that tend to infinity under iteration of \( f \). The set \( I(f) \) was introduced for the first time by Eremenko [7] who established that \( I(f) \) is non empty, and each component of \( I(f) \) is unbounded.

A complex number \( \xi \in \mathbb{C} \) is called a critical value of a transcendental entire function \( f \) if there exist some \( w \in \mathbb{C} \) such that \( f(w) = \xi \) and \( f'(w) = 0 \). Here \( w \) is called a critical point of \( f \) and its image under \( f \) is a critical value of \( f \). A complex number \( \zeta \in \mathbb{C} \) is an asymptotic value of a transcendental entire function \( f \) if there exist a curve \( \Gamma \) tending to infinity such that \( f(z) \to \zeta \) as \( z \to \infty \) along \( \Gamma \). Recall the Eremenko-Lyubich class

\[
\mathcal{B} = \{ f : \mathbb{C} \to \mathbb{C} \text{ transcendental entire : } \text{Sing}(f^{-1}) \text{ is bounded} \},
\]

where \( \text{Sing}(f^{-1}) \) is the set of critical values and asymptotic values of \( f \) and their finite limit points. Each \( f \in \mathcal{B} \) is said to be of bounded type. Moreover, if \( f \) and \( g \) are of bounded type, then so is \( f \circ g \) [6]. The definitions for critical point, critical value and asymptotic value of a transcendental semigroup \( G \) were provided in [12].

Two functions \( f \) and \( g \) are called permutable if \( f \circ g = g \circ f \). Fatou [3] proved that if \( f \) and \( g \) are two rational functions which are permutable, then \( F(f) = F(g) \). This was an important result that motivated the dynamics of composition of complex functions. Analogous results for transcendental entire functions is still not known, though it holds in some very special cases [2, Lemma 4.5]. The authors in [14] have constructed several examples where the dynamical behavior of \( f \) and \( g \) differ to a large extent, from the dynamical behavior...
of their compositions. Using approximation theory of entire functions, they have shown the existence of entire functions $f$ and $g$ having infinite number of domains satisfying various properties and relating it to their compositions. They explored and enlarged all the maximum possible ways of the solution in comparison to the past result worked out. It would be interesting to explore such relations in the context of transcendental semigroups and its constituent elements. In [13], the authors considered the relationship between Fatou sets and singular values of transcendental entire functions $f, g$ and its compositions. They provided several conditions under which Fatou sets of $f$ and $f \circ g$ coincide and also considered the relation between the singular values of $f, g$ and their compositions.

Recently, the dynamics of composite of two or more complex functions have been studied by many authors. The seminal work in this direction was done by Hinkkanen and Martin [9] related to semigroups of rational functions. In their papers, they extended the classical theory of the dynamics associated to the iteration of a rational function of one complex variable to the more general setting of an arbitrary semigroup of rational functions. Several results were extended to semigroups of transcendental entire functions in [11, 12, 15, 16, 18, 23]. It should be noted that Sumi has done an extensive work in the semigroup theory of rational functions and holomorphic maps. He has written a series of papers, for instance, [21, 22].

A transcendental semigroup $G$ is a semigroup generated by a family of transcendental entire functions $\{f_1, f_2, \ldots\}$ with the semigroup operation being functional composition. Denote the semigroup by $G = [f_1, f_2, \ldots]$. Thus, each $g \in G$ is a transcendental entire function and $G$ is closed under composition. The semigroup $G$ is called abelian if $f_i \circ f_j = f_j \circ f_i$, for all $i, j \in \mathbb{N}$. The Fatou set $F(G)$ of a transcendental semigroup $G$, is the largest open subset of $\mathbb{C}$ on which the family of functions in $G$ is normal and the Julia set $J(G)$ of $G$ is the complement of $F(G)$, that is, $J(G) = \bar{\mathbb{C}} \setminus F(G)$. The semigroup generated by a single function $g$ is denoted by $[g]$. The following definitions are well known in transcendental semigroup theory.

**Definition 1.1.** Let $G$ be a transcendental semigroup. A set $W$ is said to be forward invariant under $G$ if $g(W) \subseteq W$ for all $g \in G$ and $W$ is said to be backward invariant under $G$ if $g^{-1}(W) = \{w \in \mathbb{C} : g(w) \in W\} \subseteq W$ for all $g \in G$. Furthermore, $W$ is called completely invariant under $G$ if it is both forward and backward invariant under $G$.

For a transcendental semigroup $G$, $F(G)$ is forward invariant and $J(G)$ is backward invariant, [18, Theorem 2.1].

The contrast between the dynamics of a semigroup and those of a single function is that the semigroup dynamics is more complicated. For instance, $F(G)$ and $J(G)$ need not be completely invariant and $J(G)$ may have interior points without being entire complex plane $\mathbb{C}$, [9]. In [15], the authors extended the dynamics of a transcendental entire function on its escaping set to the dynamics of semigroups of transcendental entire functions on their escaping sets and initiated the study of escaping sets of semigroups of transcendental entire functions.
In this paper, we have considered the dynamics of an arbitrary semigroup of transcendental entire functions using Fatou-Julia theory. We have provided some condition for the complete invariance of escaping set and Julia set of a class of transcendental semigroups. Some results on limit functions and postsingular set have been discussed. A class of hyperbolic transcendental semigroups and semigroups having no wandering domains have also been provided.

2. THEOREMS AND THEIR PROOFS

Recall [15, Definition 2.1], for a transcendental semigroup $G$, the escaping set of $G$, denoted by $I(G)$ is defined as

$$I(G) = \{ z \in \mathbb{C} \mid \text{every sequence in } G \text{ has a subsequence which diverges to infinity at } z \}.$$ 

The following result provides backward invariance of $I(G)$.

**Theorem 2.1.** Let $G = [g_1, g_2, \ldots]$ be an abelian transcendental semigroup. Then the escaping set of $G$, $I(G)$ is backward invariant under $G$.

**Proof.** We have to show $g^{-1}(I(G)) \subset I(G)$ for all $g \in G$. Suppose $w \notin I(G)$. Then some sequence $\{g_n\}$ in $G$ has no subsequence which diverges to $\infty$ at $w$. That is, some sequence $\{g \circ g_n\}$ in $G$ has all its subsequences which are eventually bounded at $w$. Let $g \in G$ and consider the sequence $\{g_n \circ g\}$. As all subsequences of $\{g_n\}$ are eventually bounded at $w$, and $g$ is continuous therefore, all subsequences of $\{g \circ g_n\}$ are bounded at $w$. As $G$ is abelian therefore, $g_n \circ g = g \circ g_n$ for all $n \in \mathbb{N}$. So, all subsequences of the sequence $\{g_n \circ g\}$ are eventually bounded at $w$, that is, all subsequences of the sequence $\{g_n\}$ are eventually bounded at $g(w)$ and so $g(w) \notin I(G)$ for all $g \in G$, that is, $w \notin g^{-1}(I(G))$ for all $g \in G$. Therefore, $g^{-1}(I(G)) \subset I(G)$ for all $g \in G$ and this completes the proof of the theorem. \qed

**Remark 2.2.** In [15, Theorem 4.1], it was shown that for a transcendental semigroup $G$, $I(G)$ is forward invariant under $G$. Theorem 2.1, in particular, establishes that for an abelian transcendental semigroup $G$, $I(G)$ is completely invariant.

Even if a transcendental semigroup $G$ is non abelian, still $I(G)$ (and hence $\overline{I(G)}$) can be completely invariant. We first prove an elementary lemma.

**Lemma 2.3.** Let $f$ be a transcendental entire function. Then the closure of any forward invariant subset of $\mathbb{C}$ is forward invariant.

**Proof.** Suppose $A \subset \mathbb{C}$ is forward invariant and let $z \in \overline{A}$. Then there exist a sequence $\{z_n\}$ in $A$ such that $z_n \to z$ and so $f(z_n) \to f(z)$. As $A$ is forward invariant, so $f(z) \in \overline{A}$ and hence $\overline{A}$ is forward invariant. \qed

**Theorem 2.4.** Let $G = [g_1, \ldots, g_n]$ be a finitely generated transcendental semigroup in which each $g_i$, $1 \leq i \leq n$ is of bounded type. Then $\overline{I(G)}$ is completely invariant under $G$. 
Proof. As \( I(G) \) is forward invariant under \( G \), so from above lemma, its closure is forward invariant under \( G \). We now show \( \overline{I(G)} \) is backward invariant under \( G \), that is, \( g^{-1}(I(G)) \subset \overline{I(G)} \) for all \( g \in G \). Let \( w \notin \overline{I(G)} \). There exist a neighborhood \( U \) of \( w \) such that \( U \cap \overline{I(G)} = \emptyset \). As \( G \) is of bounded type so \( I(G) \subset J(G) \) and this will imply that \( U \subset J(G) \subset J(g) \) for all \( g \in G \). So \( U \cap I(g) = \emptyset \) for all \( g \in G \) which implies \( w \notin \overline{I(g)} \) for all \( g \in G \) and so \( g(w) \notin \overline{I(g)} \) for all \( g \in G \). Therefore, \( g(w) \notin \overline{I(G)} \) for all \( g \in G \), that is, \( w \notin g^{-1}\overline{I(G)} \) which proves the backward invariance of \( \overline{I(G)} \). \( \square \)

As a consequence, one obtains

**Corollary 2.5.** Let \( G = [g_1, \ldots, g_n] \) be a finitely generated transcendental semigroup in which each \( g_i, 1 \leq i \leq n \) is of bounded type. Then \( J(G) \) is completely invariant under \( G \).

**Proof.** From [15, Theorem 4.5(ii)] \( J(G) = \overline{I(G)} \) and so from above lemma, \( J(G) \) is completely invariant under \( G \). \( \square \)

In the next result, the relation between the escaping set of a semigroup and its generators has been provided.

**Theorem 2.6.** Let \( G = [g_1, \ldots, g_k] \) be a finitely generated abelian transcendental semigroup. Then \( I(G) = \bigcap_{i=1}^{k} g_i^{-1}(I(G)) \).

**Proof.** As \( I(G) \) is forward invariant under \( G \), so \( g_i(I(G)) \subset I(G) \), \( 1 \leq i \leq k \) which implies \( I(G) \subset \bigcap_{i=1}^{k} g_i^{-1}(I(G)) \). The argument for proving backward implication is similar to the one used in the proof of Theorem 2.1. We give it for completeness. Suppose \( w \notin I(G) \). Then some sequence \( \{f_n\} \) in \( G \) has no subsequence which diverges to \( \infty \) at \( w \). That is, some sequence \( \{f_n\} \) in \( G \) has all its subsequences which are eventually bounded at \( w \).

For \( g_i \in G, 1 \leq i \leq k \) consider the sequence \( \{f_n \circ g_i\} \). As all subsequences of \( \{f_n\} \) are eventually bounded at \( w \), and \( g_i \) is continuous therefore, all subsequences of \( \{g_i \circ f_n\} \) are bounded at \( w \). As \( G \) is abelian so \( f_n \circ g_i = g_i \circ f_n \) for all \( n \in \mathbb{N} \). So, all subsequences of the sequence \( \{f_n \circ g_i\} \) are eventually bounded at \( w \), that is, all subsequences of the sequence \( \{f_n\} \) are eventually bounded at \( g_i(w) \) and so \( g_i(w) \notin I(G) \) for all \( 1 \leq i \leq k \), that is, \( w \notin g_i^{-1}(I(G)) \) for all \( 1 \leq i \leq k \). Therefore, \( \bigcap_{i=1}^{k} g_i^{-1}(I(G)) \subset I(G) \) and this completes the proof of the theorem. \( \square \)

Recall, [10, p. 61] for a transcendental meromorphic function \( g \), a function \( \psi(z) \) is a limit function of \( \{g^n\} \) on a component \( V \subset F(g) \) if there is some subsequence of \( \{g^n\} \) which converges locally uniformly on \( V \) to \( \psi \). Denote by \( \mathfrak{L}(U) \) all such limit functions. The following result gives a criterion for connectivity of \( J(G) \).

**Theorem 2.7.** Let \( G = [g_1, \ldots, g_n] \) be a finitely generated abelian transcendental semigroup. If for each \( i, 1 \leq i \leq n \), every Fatou component of \( g_i \) is bounded and \( \infty \) is not a limit function of any sequence in \( G \) in a component of \( F(G) \), then \( J(G) \subset \mathbb{C} \) is connected.

**Proof.** Observe that for all \( g \in G \), every component of \( F(g) \) is bounded. As \( \infty \) is not a limit function of any sequence in \( G \) in a component of \( F(G) \), from [12, Theorem 4.1], every
component of $F(G)$ is simply connected and therefore, $J(G) \subset \mathbb{C}$ is connected using [16, Theorem 2.11].

\[ \square \]

**Remark 2.8.** For all $g \in G$, every Fatou component of $g$ is simply connected, and therefore, $J(g) \subset \mathbb{C}$ is connected.

**Theorem 2.9.** Let $f$ be a transcendental entire function of period of $c$ and let $g = f^m + c$ for some $m \in \mathbb{N}$. Let $U \subset F(f)$ be an invariant component and $\phi$ be a limit function of \{$(f \circ g)^n$\} on $U$. Then $\phi$ is a limit function of \{(f \circ g)^n\} on $U$ and $\phi + c$ is a limit function of \{(g \circ f)^n\} on $U$.

**Proof.** Observe that for all $n \in \mathbb{N}$, $(f \circ g)^n(z) = f^{n(m+1)}(z)$ and $(g \circ f)^n(z) = f^{n(m+1)}(z) + c$. As $\phi \in \mathcal{L}(U)$ for \{f^n\}, there exist a subsequence \{f^{n_i}\} of \{f^n\} with $\lim f^{n_i}(z) = \phi(z)$ on $U$. For $z \in U$, $\lim (f \circ g)^{n_i}(z) = \lim f^{n_i(m+1)}(z) = \phi(z)$ which implies $\phi$ is a limit function of \{(f \circ g)^n\} on $U$. On similar lines it can be seen that $\lim (g \circ f)^{n_i}(z) = \phi(z) + c$ on $U$ and hence the result. \[ \square \]

### 3. Postsingular Set

Recall that the postsingular set of an entire function $f$ is defined as

$$\mathcal{P}(f) = \left( \bigcup_{n \geq 0} f^n(\Sing(f^{-1})) \right).$$

For a transcendental semigroup $G$, let

$$\mathcal{P}(G) = \left( \bigcup_{f \in G} \Sing(f^{-1}) \right).$$

The next result gives the relation between postsingular set of compose of two entire functions with those of its factors.

**Lemma 3.1.** Let $f$ and $g$ be permutable transcendental entire functions of bounded type. Suppose $f(\Sing(g^{-1})) \subset \Sing(g^{-1})$ and $g(\Sing(f^{-1})) \subset \Sing(f^{-1})$. Then $\mathcal{P}(f \circ g) \subset \mathcal{P}(f) \cup \mathcal{P}(g)$. 

Proof. As \( f(\text{Sing}(g^{-1}))) \subset \text{Sing}(g^{-1}) \), so \( f^n(\text{Sing}(g^{-1})) \subset \text{Sing}(g^{-1}) \) for all \( n \in \mathbb{N} \). On similar lines, \( g^n(\text{Sing}(f^{-1})) \subset \text{Sing}(f^{-1}) \) for all \( n \in \mathbb{N} \). Using permutability of \( f \) and \( g \),

\[
\mathcal{P}(f \circ g) = \left( \bigcup_{n \geq 0} (f \circ g)^n(\text{Sing}(f \circ g)^{-1}) \right) \\
\subset \left( \bigcup_{n \geq 0} (f^n(\text{Sing}(f^{-1})) \bigcup f(\text{Sing}(g^{-1}))) \right) \\
= \left( \bigcup_{n \geq 0} ((f^n(\text{Sing}(f^{-1}))) \bigcup g^n(f^{n+1}(\text{Sing}(g^{-1})))) \right) \\
\subset \left( \bigcup_{n \geq 0} (f^n(\text{Sing}(f^{-1}))) \bigcup \left( \bigcup_{n \geq 0} g^n(\text{Sing}(g^{-1})) \right) \right) \\
= \mathcal{P}(f) \cup \mathcal{P}(g). \tag*{\Box}
\]

Remark 3.2. It can be seen by an induction argument that if \( g_1, \ldots, g_n \) are permutable entire functions of bounded type satisfying \( g_i(\text{Sing}(g_j)^{-1}) \subset \text{Sing}(g_j)^{-1} \), for all \( 1 \leq i, j \leq n, i \neq j \), then \( \mathcal{P}(g_1 \circ \cdots \circ g_n) \subset \bigcup_{i=1}^{n=1} \mathcal{P}(g_i) \).

Recall that an entire function \( f \) is called hyperbolic if the postsingular set \( \mathcal{P}(f) \) is a compact subset of \( F(f) \). For instance, \( e^{\lambda z}, \; 0 < \lambda < \frac{1}{e} \) are examples of hyperbolic entire functions. A transcendental semigroup \( G \) is called hyperbolic if each \( g \in G \) is hyperbolic \[15\]. Also recall that an entire function \( f \) is called postsingularly bounded if the postsingular set \( \mathcal{P}(f) \) is bounded. A transcendental semigroup \( G \) is called postsingularly bounded if each \( g \in G \) is postsingularly bounded \[15\]. The following result provides a class of hyperbolic transcendental semigroups.

Theorem 3.3. Let \( G = \{g_1, \ldots, g_n\} \) be a finitely generated abelian transcendental semigroup in which each generator is of bounded type. Suppose for each \( i, j, \ 1 \leq i, j \leq n, i \neq j \) \( g_i(\text{Sing}(g_j)^{-1}) \subset \text{Sing}(g_j)^{-1} \). Then, if the generators of \( G \) are hyperbolic, then so is \( G \).

Proof. From Remark \[3.2\], \( \mathcal{P}(g_1 \circ \cdots \circ g_n) \subset \bigcup_{i=1}^{n=1} \mathcal{P}(g_i) \). Using the permutability of each \( g_i \), any \( g \in G \) can be represented as \( g = g_1^{l_1} \circ \cdots \circ g_n^{l_n} \) which implies \( \mathcal{P}(g) \subset \bigcup_{i=1}^{n=1} \mathcal{P}(g_i^{l_i}) \). Also for any entire function \( f \), \( \mathcal{P}(f^k) = \mathcal{P}(f) \bigcup \), therefore, \( \mathcal{P}(g) \subset \bigcup_{i=1}^{n=1} \mathcal{P}(g_i) \). Using the hyperbolicity of the generators, \( \bigcup_{i=1}^{n=1} \mathcal{P}(g_i) \) is a compact subset of \( \mathbb{C} \) which further implies that \( \mathcal{P}(g) \subset \mathbb{C} \) is compact. As \( F(g_i) = F(g) \) \( 1 \leq i \leq n \), \[12\] Theorem 5.9, we get from \( \mathcal{P}(g) \subset \bigcup_{i=1}^{n=1} \mathcal{P}(g_i) \subset \bigcup_{i=1}^{n=1} F(g_i) \) that \( \mathcal{P}(g) \subset F(g) \). Thus, \( \mathcal{P}(g) \) is a compact subset of \( F(g) \) and so is hyperbolic. This completes the proof of the theorem. \( \Box \)

Remark 3.4. If the hypothesis of theorem are satisfied and the generators are postsingularly bounded, then so is \( G \).

Recall that a component \( U \) of \( F(G) \) is called a wandering domain of \( G \) if the set \( \{U_g : g \in G\} \) is infinite (where \( U_g \) is the component of \( F(G) \) containing \( g(U) \)). The next result rules out the existence of wandering domains for a transcendental semigroup. To prove the result, we require the following lemmas.
Lemma 3.5. [16, Lemma 4.6] For a transcendental semigroup $G = [g_1, g_2, \ldots]$, $P(G) = \left( \bigcup_{g \in G} P(g) \right)$.

Lemma 3.6. [18, Theorem 4.2] For a transcendental semigroup $G = [g_1, g_2, \ldots]$, $J(G) = \left( \bigcup_{g \in G} J(g) \right)$.

Theorem 3.7. Let $G = [g_1, \ldots, g_n]$ be a finitely generated abelian transcendental semigroup in which each generator is of bounded type. If for all $g \in G$, $P(g) \cap J(g) = \emptyset$, then $G$ has no wandering domains.

Proof. By hypothesis, $g$ has no wandering domains for all $g \in G$. It suffices to show that $P(G) \cap J(G) = \emptyset$. Using Lemmas 3.5 and 3.6,

$$P(G) \cap J(G) = \left( \bigcup_{g \in G} P(g) \right) \cap \left( \bigcup_{g \in G} J(g) \right) = \left( \bigcup_{g \in G} (P(g) \cap J(g)) \right).$$

Using hypothesis, we get that $P(G) \cap J(G) = \emptyset$. As a consequence of [12, Theorem 5.23], $G$ has no wandering domains. □

References

[1] I. N. Baker, Limit functions and sets of non-normality in iteration theory, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 467 (1970), 1-11.
[2] I. N. Baker, Wandering domains in the iteration of entire functions, Proc. London Math. Soc. 49 (1984), 563-576.
[3] A. F. Beardon, Iteration of rational functions, Springer Verlag, (1991).
[4] W. Bergweiler, Iteration of meromorphic functions, Bull. Amer. Math. Soc. 29 (1993), 151-188.
[5] W. Bergweiler and A. Hinkkanen, On the semiconjugation of entire functions, Math. Seminar Christian Albrechts-Univ. Kiel 21 (1997), 1-10.
[6] W. Bergweiler and Y. Wang, On the dynamics of composite entire functions. Ark. Math. 36 (1998), 31-39.
[7] A. E. Eremenko, On the iteration of entire functions, Ergodic Theory and Dynamical Systems, Banach Center Publications 23, Polish Scientific Publishers, Warsaw, (1989), 339-345.
[8] A. E. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier, Grenoble, 42 (1992), 989-1020.
[9] A. Hinkkanen and G. J. Martin, The dynamics of semigroups of rational functions I, Proc. London Math. Soc. (3) 73 (1996), 358-384.
[10] X. H. Hua, C. C. Yang, Dynamics of transcendental functions, Gordon and Breach Science Pub. (1998).
[11] Z. G. Huang and T. Cheng, Singularities and strictly wandering domains of transcendental semigroups, Bull. Korean Math. Soc. (1) 50 (2013), 343-351.
[12] D. Kumar and S. Kumar, The dynamics of semigroups of transcendental entire functions I, arXiv:math.DS/13027249, (2013) (accepted for publication in Indian J. Pure Appl. Math.)
[13] D. Kumar and S. Kumar, On dynamics of composite entire functions and singularities, arXiv:math.DS/13075785, (2013) (accepted for publication in Bull. Cal. Math. Soc.)
[14] D. Kumar, G. Datt and S. Kumar, Dynamics of composite entire functions, arXiv:math.DS/12075930, (2013) (accepted for publication in J. Ind. Math. Soc.)
[15] D. Kumar and S. Kumar, The dynamics of semigroups of transcendental entire functions II, arXiv:math.DS/14010425, (2014), submitted for publication.
[16] D. Kumar and S. Kumar, Semigroups of transcendental entire functions and their dynamics, arXiv:math.DS/14050224, (2014), submitted for publication.
[17] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, Holomorphic dynamics, Cambridge Univ. Press, (2000).
[18] K. K. Poon, Fatou-Julia theory on transcendental semigroups, Bull. Austral. Math. Soc. 58 (1998), 403-410.
[19] L. Rempe, On a question of Eremenko concerning escaping sets of entire functions, Bull. London Math. Soc. 39:4, (2007), 661-666.
[20] D. Schleicher and J. Zimmer, Escaping points of exponential maps, J. London Math. Soc. (2) 67 (2003), 380-400.
[21] H. Sumi, On dynamics of hyperbolic rational semigroups, J. Math. Kyoto Univ. 37 (1997), 717-733.
[22] H. Sumi, Random complex dynamics and semigroups of holomorphic maps, Proc. London Math. Soc. 102 (2011), 50-112.
[23] H. Zhigang, The dynamics of semigroups of transcendental meromorphic functions, Tsinghua Science and Technology, (4) 9 (2004), 472-474.

Department of Mathematics, University of Delhi, Delhi–110 007, India
E-mail address: dinukumar680@gmail.com

Department of Mathematics, Deen Dayal Upadhyaya College, University of Delhi, New Delhi–110 015, India
E-mail address: sanjpant@gmail.com