Conformal Field Theory Techniques in Random Matrix models

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In these notes we explain how the CFT description of random matrix models can be used to perform actual calculations. Our basic example is the hermitian matrix model, reformulated as a conformal invariant theory of free fermions. We give an explicit operator construction of the corresponding collective field theory in terms of a bosonic field on a hyperelliptic Riemann surface, with special operators associated with the branch points. The quasiclassical expressions for the spectral kernel and the joint eigenvalue probabilities are then easily obtained as correlation functions of current, fermionic and twist operators. The result for the spectral kernel is valid both in macroscopic and microscopic scales. At the end we briefly consider generalizations in different directions.

Based on the talk of the author at the Third Claude Itzykson-Meeting, "Integrable Models and Applications to Statistical Mechanics", Paris, July 27-29, 1998, and at the workshop "Random matrices and integrable systems", Univ. of Warwick, November 2-4 1998.

July 1999

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1. Introduction

The random matrix models have various applications in rather different domains, and sometimes language barriers prevents the flow of ideas and knowledge from one field to another. For example, such powerful techniques as the conformal field theory (CFT) description of the random matrix models and their relation with the integrable hierarchies, which were developed extensively by the string theorists in the early 90’s, are practically unknown to the mesoscopic physicists. This lecture is an attempt to explain the uses of the the CFT description in a language, which is acceptable by both communities. We therefore avoided any “physical” interpretation and concentrated on the method as such. The only thing the reader is supposed to know are the basics of the two-dimensional conformal field theory.

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The statistical ensembles of random matrices of large size (matrix models) have been introduced in 1951 by Wigner in order to analyze the spectral properties of complicated systems with chaotic behavior [1]. In this approach the Hamiltonian of a chaotic system is considered as a large matrix with random entries. Consequently, the analytical studies of random matrix ensembles carried out in the next 25 years (see the Mehta’s book [2]) were oriented to the calculation of the spectral correlation functions or joint eigenvalue probabilities.

If $M$ is an $N \times N$ hermitian random matrix, then the spectral correlation function $\rho(\lambda_1, ..., \lambda_n)$ is defined as the probability density that $\lambda_1, ..., \lambda_n$ are eigenvalues of $M$

$$\rho(\lambda_1, ..., \lambda_n) = \frac{(N-n)!}{N!} \left\langle \prod_{i=1}^{n} \delta(M - \lambda_i) \right\rangle.$$

All spectral correlation functions are expressed as determinants of a single kernel $K(\lambda, \mu)$ (the spectral kernel). The spectral kernel can be evaluated by the method of orthogonal polynomials [2]. Its large $N$ asymptotics is characterized by the interposition of a smooth behavior and fast oscillations with wavelength $\sim 1/N$ (in a scale where the total range of the spectrum is kept finite). The smooth large distance behavior depends on the concrete form of the matrix potential. On the contrary, the microscopic behavior characterized by oscillations depends only on the symmetry group (the unitary group in the case) and fall into several universality classes. It is the microscopic behavior of the spectral correlations, which is interesting from the point of view of applications to chaotic systems. A review of the latest developments in this direction, in particular these related to the recent Monte Carlo data in lattice QCD, can be found in [3].

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The discovery by ’t Hooft of the $1/N$ expansion [4] gave a meaning of the smooth part of the spectral correlators and opened the possibility of using random matrix models to solve various combinatorial problems, the simplest of which is the enumeration of planar graphs [5,6]. Here we can mention the exact solution of various statistical models defined on random surfaces [7], the matrix formulation of the 2d quantum gravity [10] and some more difficult combinatorial problems as the enumeration of the ”dually weighted” planar graphs [12] and the branched coverings of a compact Riemann surface [13].
In this kind of problems the solution is encoded in the $1/N$ expansion of the loop correlation functions, which are the correlation functions of the collective field variable

$$W(z) = \text{tr}\left(\frac{1}{z-M}\right).$$

(1.2)

In the large $N$ limit the correlation functions of the resolvent (1.2) are meromorphic functions with cuts along the intervals where the spectral density is nonzero. The discontinuity along the cuts gives the smooth part of the joint eigenvalue probabilities (1.1). Indeed, we have the evident relation

$$\langle W(z_1) \ldots W(z_n) \rangle_c = \int \frac{d\lambda_1 \ldots d\lambda_n}{(z_1 - \lambda_1) \ldots (z_n - \lambda_n)} \rho(\lambda_1, \ldots, \lambda_n)$$

(1.3)

where $\langle \quad \rangle_c$ means connected correlator.

The first exact results in the large $N$ limit were obtained by direct application of the saddle point method [5], but later it was recognized that a more powerful method is provided by the so called loop equations [14], whose iterative solution allows one to reconstruct order by order the $1/N$ expansion. The most efficient iterative procedure proved to be the ”moment’s description” [15], which allowed to calculate the free energy and loop correlators for an arbitrary potential up to $1/N^4$ terms.

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In the early 90’s, after the publication of the seminal papers [11], the two resolution techniques in random matrix models (orthogonal polynomials and loop equations) developed rapidly and were recognized as particular cases of well developed mathematical methods.

The method of orthogonal polynomials was reformulated in terms of the theory of $\tau$ functions of integrable hierarchies (see, for example, the review article [16]). On the other hand it was observed that the loop equations generate a representation of (half of) the Virasoro algebra and are therefore equivalent to the requirement that the theory is conformal invariant [17,18]. This allowed to describe a class of large $N$ matrix models near criticality in terms of the Hilbert space of twisted bosonic fields, and apply the well developed formalism of the conformal field theory. Finally, it was observed that the two approaches are closely related since the $\tau$ functions associated with the matrix models can be formulated as fermionic theories with conformal invariance [19]. (The approach based on integrable hierarchies is however less general since it works only for the unitary ensemble of random matrices.)

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It happened that the CFT techniques were developed exclusively from the point of view of their application to noncritical string theories, and therefore are known only to a relatively narrow circle of physicists. The aim of this lecture is to demonstrate the technical and conceptual advantages of the conformal field theory description in a more general setting. In particular, we will reproduce, using the CFT formalism, the microscopic oscillations of the spectral kernel and correlators. The CFT formalism allows to understand better the origin of the observed universality at mesoscopic and microscopic scales.
We will restrict ourselves to the simplest example of the hermitian one matrix model with arbitrary potential and will only briefly consider the generalizations at the end. First we will establish the equivalence of the unitary matrix ensemble with a system of two-dimensional free chiral fermions. Then we will construct the collective field theory, which is obtained from the fermionic system by the two-dimensional bosonization rules. The field possesses a nontrivial expectation value associated with a hyperelliptic Riemann surface. The latter is described, from the point of view of CFT, by a collection of twist operators associated with its branch points. with interaction concentrated at the branch points. We will demonstrate the power of the CFT description by performing some actual calculations. We will obtain the quasiclassical expressions for the correlator of the resolvent $W(z)$, the joint eigenvalue probabilities and the spectral kernel. Finally we will briefly mention how the formalism can be generalized to the case of a chain of coupled random matrices in presence of external matrix fields.

2. The hermitian one matrix model. Loop equations as Virasoro constraints

The partition function of the unitary ensemble is defined as the integral

$$Z_N[V] = \int dM \ e^{-\text{Tr} V(M)} \quad (2.1)$$

where $dM$ denotes the translational invariant measure in the space of $N \times N$ hermitian matrices $M = \{M_{ij}\}_{i,j=1}^N$. We normalize the measure as

$$dM = \frac{1}{\text{Vol}[U(N)]} \prod_{k=1}^N \frac{dM_{kk}}{2\pi} \prod_{k<j} 2 \ d\text{Re}(M_{kj}) \ d\text{Im}(M_{kj}) \quad (2.2)$$

where $\text{Vol}[U(N)] = \prod_{k=1}^N \frac{(2\pi)^k}{k!}$ is the volume of the unitary group. We assume that the potential is an entire function

$$V(M) = -\sum_{n=0}^\infty t_n M^n \quad (2.3)$$

which is the natural choice for the ensemble of hermitian matrices.

The integrand depends on the matrix variable $M$ only through its eigenvalues $\lambda_1, ..., \lambda_N$ and the integral $(2.1)$ is actually reduced to

$$Z_N[V] = \frac{1}{N!} \int \prod_{i=1}^N d\lambda_i \ e^{-V(\lambda_i)} \prod_{i<j}(\lambda_i - \lambda_j)^2. \quad (2.4)$$

The loop equations represent an infinite set of identities satisfied by the correlation functions of the collective field variable

$$W(z) = \sum_{i=1}^N \frac{1}{z - \lambda_i} = \text{tr} \left( \frac{1}{z - M} \right). \quad (2.5)$$
The derivation goes as follows. From the translational invariance of the integration measure \( d\lambda_i \) we find

\[
\left\langle \sum_{i=1}^{N} \left( \frac{\partial}{\partial \lambda_i} + 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} + \sum_{n \geq 0} nt_n \lambda_i^{n-1} \right) \frac{1}{z - \lambda_i} \right\rangle_{N,\{t\}} = 0
\]  \tag{2.6}

where \( \langle \cdots \rangle_{N,\{t\}} \) means the average with respect to this partition function. Using the identity

\[
\sum_i \frac{1}{(z - \lambda_i)^2} + 2 \sum_{i \neq j} \frac{1}{z - \lambda_i} \frac{1}{\lambda_i - \lambda_j} = \sum_{i,j} \frac{1}{z - \lambda_i} \frac{1}{z - \lambda_j}
\]  \tag{2.7}

we write (2.6) as

\[
\left\langle W^2(z) + \sum_{i=1}^{N} \frac{1}{z - \lambda_i} \sum_{n \geq 0} nt_n \lambda_i^{n-1} \right\rangle_{N,\{t\}} = 0.
\]  \tag{2.8}

Assume that the point \( z \) is very far from the origin. Then the sum in the second term can be viewed as the result of a contour integration along a big contour \( C \) circling all the eigenvalues \( \lambda_1, ..., \lambda_N \) but not the point \( z \):

\[
\oint_C \frac{dz'}{2\pi i} \frac{1}{z - z'} \left\langle T(z') \right\rangle_{N,\{t\}} = 0
\]  \tag{2.9}

where

\[
T(z) = \frac{1}{2} [\partial \Phi(z)]^2, \quad \Phi(z) = \frac{1}{\sqrt{2}} \sum_{n \geq 0} t_n z^n - \sqrt{2} \ln \left( \frac{1}{z - M} \right).
\]  \tag{2.10}

The insertion of the operator \( \text{tr} M^n = \sum_i \lambda_i^n \) can be realized by taking a partial derivative with respect to the coupling \( t_n \). This allows to represent the operator (2.10) as

\[
T(z) = \frac{1}{2} [\partial \Phi(z)]^2, \quad \Phi(z) = \frac{1}{\sqrt{2}} \sum_{n \geq 0} t_n z^n + \sqrt{2} N \ln z + \sqrt{2} \sum_{n \geq 0} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n}
\]  \tag{2.11}

and write the loop equations as linear differential constraints

\[
L_n \cdot Z_N[t] = 0 \quad (n \geq -1)
\]  \tag{2.12}

where

\[
L_n = \sum_{k=0}^{n} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k}} + \sum_{k=0}^{\infty} kt_k \frac{\partial}{\partial t_{n+k}}
\]  \tag{2.13}

(We used the notation \( Z_N[t] \) instead of \( Z_N[V] \) since a concrete representation of the potential \( V(z) \) by the set \( \{t_n\}_{n=0}^{\infty} \) of coordinates (coupling constants) is chosen.) The operators \( L_n \) satisfies the Virasoro algebra

\[
[L_m, L_n] = (m - n)L_{m+n}
\]  \tag{2.14}
which implies that the this matrix integral realizes a representation of the conformal group. In writing the Virasoro constraints we introduced the derivative with respect to the trivial variable $t_0$, which satisfies
\[ \frac{\partial}{\partial t_0} Z_N[t] = N Z_N[t]. \] (2.15)

It turns to be much simpler to use apparatus of the conformal field theory instead of solving directly the loop equations (2.12). The formal solution of these equations is given in terms of a chiral Dirac fermion or, through bosonization, of a chiral bosonic field with Liouville-like interaction, which is by its definition conformal invariant. Such a construction has been proposed in [18,19].

In the next sections we consider the CFT interpretation of the matrix model from more pragmatic point of view than in refs. [18,19]. Our aim is explore the large $N$ limit of the operator solution of the loop equations through the conformal field theory and reproduce the $1/N$ expansion for the free energy and the correlation functions. The bosonic field can be considered as the collective field describing the Dyson gas of eigenvalues. The $1/N$ expansion can be constructed in terms of a free bosonic field defined on a Riemann surface determined by the classical background (the spectral density in the thermodynamical limit). The spectral correlations can be determined directly from the correlation functions of this bosonic field.

3. Fermionic representation

3.1. Fock space representation of the hermitian matrix integral in terms of Dirac fermions

The model (2.1) was first solved by the so called method of orthogonal polynomials [20,21]. The method is based on the possibility to consider the matrix model as a system of free fermions. Later the fermionic partition function was identified as a $\tau$-function of the KP hierarchy [19]. (For a review of the theory of tau-functions see [22].)

There are different fermionic representations of the matrix integral depending on the choice of the fermionic wave functions. Below we will describe the one that is most natural from the point of view of the conformal symmetry. Let us introduce the chiral fermions
\[ \psi(z) = \left\{ \psi^{(1)}(z), \psi^{(2)}(z) \right\}, \quad \psi^\dagger(z) = \left\{ \psi^{*(1)}(z), \psi^{*(2)}(z) \right\}. \] (3.1)

Choosing a local coordinate $1/z$ at $z = \infty$ we expand
\[ \psi^{(a)}(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^{(a)}_r z^{-r - \frac{1}{2}}, \quad \psi^{*(a)}(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^{*(a)}_r z^{-r - \frac{1}{2}}. \] (3.2)

where the fermion modes satisfy the anticommutation relations
\[ [\psi^{(a)}_r, \psi^{*(b)}_{r'}]_+ = \delta_{a,b} \delta_{r+r',0}. \] (3.3)
The left and right vacuum states of given charge \( \vec{l} = (l^{(1)}, l^{(2)}) \) are defined by

\[
\langle \vec{l} | \psi^{(a)}_r \rangle = 0 \quad (r < l^{(a)})
\]
\[
\psi^{(a)}_r | \vec{l} \rangle = 0 \quad (r > l^{(a)})
\]
and the corresponding normal ordering is denoted by : :. The action of the chiral fermions is
\[
S = \int d^2 z \psi \dagger \partial \psi
\]
and the energy-momentum tensor is
\[
T = \frac{1}{2} [\partial \psi \dagger \psi - \psi \dagger \partial \psi].
\]

The fermion bilinears
\[
J^{(a)}(z) = : \psi^{* (a)}(z) \psi^{(a)}(z) : \quad (a = 1, 2)
\]
\[
J^+(z) = \psi^{* (1)}(z) \psi^{(2)}(z), \quad J^-(z) = \psi^{* (2)}(z) \psi^{(1)}(z)
\]
satisfy the \( u(2) \) current algebra.

The currents \( J^{(a)}(z) = \sum_n J^{(a)}_n z^{-n-1}, J_n^{(a)} = \sum_{r \in \mathbb{Z}+\frac{1}{2}} : \psi^{* (a)}_r \psi^{(a)}_{r+n} : \) commute with the fermionic fields as

\[
\begin{align*}
[\psi^{(a)}(z), J_n^{(a)}] &= z^n \psi^{(a)}(z), \quad [J_n^{(a)}, \psi^{* (a)}(z)] = z^n \psi^{* (a)}(z), \\
[\psi^{(a)}_r, J_n] &= \psi^{(a)}_{r+n}, \quad [J_n^{(a)}, \psi^{* (a)}_r] = \psi^{* (a)}_{r-n} \\
[J_n^{(a)}, J_m^{(a)}] &= n \delta_{n+m,0}.
\end{align*}
\]

We will also denote the \( su(2) \) current components by

\[
E_+(z) = J_+(z), \\
E_-(z) = J_-(z),
\]
\[
H(z) = \frac{1}{2} [J^{(1)}(z) - J^{(2)}(z)]
\]
and the \( u(1) \) current by

\[
\tilde{H}(z) = \frac{1}{2} [J^{(1)}(z) + J^{(2)}(z)].
\]

The \( su_2 \) currents are expressed in terms of the Pauli matrices as

\[
H = : \psi^* \sigma_3 \psi :, \\
E_\pm = : \psi^* \sigma_\pm \psi :.
\]

Now we are ready to write the Fock-space representation of the matrix partition function \( \mathcal{Z} \) in terms of the fermionic fields \( \psi^* \). For this purpose we define the ”Hamiltonian”

\[
H[V] = - \oint \frac{dz}{2\pi i} V(z) H(z)
\]
\[
= \frac{1}{2} \sum_{n \geq 0} \sum_r \left( : \psi^{* (1)}_{n-r} \psi^{(1)}_r - \psi^{* (2)}_{n-r} \psi^{(2)}_r : \right)
\]
and the "screening operator"

\[ Q_+ = \int_{-\infty}^{\infty} d\lambda \ E_+(\lambda). \] (3.11)

**Statement:**

The scalar product

\[ Z_N[V] = \langle N| e^{H[V]} e^{Q_+} |0 \rangle, \] (3.12)

where \(|0\rangle \equiv |0,0\rangle\) and \(|N\rangle \equiv |N, -N\rangle\), is equal to the integral (2.4).

**Proof:**

First we expand \(e^{Q_+}\) in fermion multilinear terms and move the operator \(e^{H[V]}\) to the right using the formulas

\[
\begin{align*}
  e^{H[V]} \psi^{(2)}(z) e^{-H[V]} &= e^{-\frac{1}{4}V(z)} \psi^{(2)}(z), \\
  e^{H[V]} \psi^{*(1)}(z) e^{-H[V]} &= e^{-\frac{1}{4}V(z)} \psi^{*(1)}(z),
\end{align*}
\] (3.13)

until it hits the right vacuum. The result is

\[ Z_N[V] = \langle N| e^{Q_Y} |0 \rangle, \quad Q_Y = \int_{-\infty}^{\infty} d\lambda \psi^{*(1)}(\lambda) \psi^{(2)}(\lambda) e^{-V(\lambda)}. \] (3.14)

Then, expanding the exponent in modes

\[
Q_Y = \sum_{r,r'>0} \int d\lambda \lambda^{r+r'-1} \psi_{-r}^{*(1)} \psi_{-r'}^{(2)} e^{-V(\lambda)}
\] (3.15)

and using (3.3) and (3.4), we reproduce the original integral.

The representation (3.12) of the partition function implies that the latter is a \(\tau\)-function of the KP hierarchy (this fact has been established in [23]) and therefore satisfies an infinite tower of differential equations, the first of which is the KP equation.

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1 Here we assume that the potential grows sufficiently fast at \(\lambda \rightarrow \pm \infty\) and therefore the integral is finite if the fermionic operator is regular at \(z = 0\). This is indeed the case since all \(\psi_r\) with \(r > 0\) are annihilated by the right vacuum.
3.2. Relation to the method of orthogonal polynomials

The method of orthogonal polynomials \[20\] consists, in this setting, in choosing a basis diagonalizing the quadratic form (3.15)

\[
\sum_{r \geq \frac{1}{2}} z^{r-\frac{1}{2}} \psi_{-r}^{(1)} = \sum_{n \geq 0} P_n(z) b_n^{(1)}, \quad \sum_{r \geq \frac{1}{2}} z^{r-\frac{1}{2}} \psi_{-r}^{(2)} |0\rangle = \sum_{n \geq 0} P_n(z) b_n^{(2)} \tag{3.16}
\]

where the polynomials \( P_n(z) = z^n + ... \) satisfy the orthogonal relations

\[
\int_{-\infty}^{\infty} d\lambda P_n(\lambda) P_m(\lambda) e^{-V(\lambda)} = h_{n+1} \delta_{m,n}. \tag{3.17}
\]

The operators \( b_n^{(1)} \) and \( b_n^{(2)} \) are linear combinations of \( \psi_{-r}^{(1)} \) and \( \psi_{-r}^{(2)} \) respectively:

\[
b_n^{(1)} = \sum_k B_{nk} \psi_{-k-\frac{1}{2}}, \quad b_n^{(2)} = \sum_k B_{nk} \psi_{-k-\frac{1}{2}} \tag{3.18}
\]

where the matrix \( B \) is triangular and with with unitary diagonal elements

\[
B_{nn} = 1, \quad B_{nk} = 0 \quad \text{for} \quad k > n \tag{3.19}
\]

and therefore its determinant is equal to one, \( \det B = 1 \).

Introducing the set of orthogonal functions

\[
f_n(\lambda) = P_{n-1}(\lambda) e^{-\frac{1}{2}V(z)}, \quad n = 1, 2, ...
\]

such that \( (f_k, f_j) \equiv \int d\lambda f_k(\lambda) f_j(\lambda) = \delta_{kj} h_k \), we represent the quadratic form (3.15) as

\[
Q^V_+ = \sum_{m,n=1}^\infty (f_m, f_n) b_m^{(1)} b_n^{(2)} = \sum_{n=1}^\infty h_n b_n^{(1)} b_n^{(2)} \tag{3.21}
\]

and the partition function equals

\[
Z_N[V] = \frac{1}{N!} (\det B)^2 \det_{N \times N} (f_k, f_j) = h_1 h_2 ... h_N. \tag{3.22}
\]

3.3. Spectral kernel and joint eigenvalue distributions

A complete set of observables in the matrix model is given by the joint distribution probabilities for \( n \) eigenvalues \((1 \leq n \leq N)\)

\[
\rho(\lambda_1, ..., \lambda_n) = \frac{(N-n)!}{N!} \langle \prod_{k=1}^n \delta(\lambda_k - M) \rangle. \tag{3.23}
\]
The density probabilities (3.23) are expressed as expectation values of fermionic bilinears. Indeed, keeping the only relevant term in the expansion $e^{Q_+} = \ldots + Q_+^N/N! + \ldots$ we write the partition function as

$$Z_N[V] = \int \frac{d\lambda_1 \ldots d\lambda_N}{N!} \left\langle N \left| e^{H[V]} \prod_{k=1}^{N} E_+(\lambda_k) \right| 0 \right\rangle = \int d\lambda_1 \ldots d\lambda_n \, \rho(\lambda_1, ..., \lambda_n)$$

(3.24)

from where the representation of the probability measure $\rho(\lambda_1, ..., \lambda_N)$ as the expectation value of $N$ operators $E_+$. Integrating with respect to $N - n$ of the $\lambda$'s gives

$$\rho(\lambda_1, ..., \lambda_n) = \frac{(N-n)!}{N!} \left\langle \prod_{k=1}^{n} E_+(\lambda_k) \right\rangle = \frac{(N-n)!}{N!} \left\langle \prod_{k=1}^{n} \psi^{(1)}(\lambda_k)\psi^{(2)}(\lambda_k) \right\rangle$$

(3.25)

where by definition $\langle O \rangle = \langle N | e^{H[V]} O e^{Q_+}|0 \rangle$. In particular, the spectral density is the expectation value of the fermionic current

$$\rho(\lambda) = \frac{1}{N} \langle \psi^{(1)}(\lambda)\psi^{(2)}(\lambda) \rangle.$$  

(3.26)

Since we have a system of free fermions, the average is equal to the determinant of the two-point correlators [20]

$$\rho(\lambda_1, ..., \lambda_n) = \frac{(N-n)!}{N!} \det_{n \times n} K(\lambda_i, \lambda_j)$$

(3.27)

where

$$K(\lambda, \lambda') = \langle \psi^{(1)}(\lambda)\psi^{(2)}(\lambda') \rangle.$$  

(3.28)

The two-point function (3.28) is called spectral kernel and is expressed in terms of the orthogonal functions (3.20) as

$$K(\lambda, \lambda') = \sum_{n=1}^{N} \frac{f_n(\lambda)f_n(\lambda')}{(f_n, f_n)}.$$  

(3.29)

Another useful representation of $\rho(\lambda_1, ..., \lambda_n)$ is as the correlators of the Cartan current $H$. One finds, using the operator product expansion for the currents $H$ and $E_+$,

$$H(z)e^{Q_+}|0\rangle = \left\langle \int_{-\infty}^{\infty} \frac{d\lambda}{z-\lambda} E_+(\lambda)e^{Q_+}|0\right\rangle.$$  

(3.30)

This relation generates a set of Ward identities the simplest of which is

$$\langle H(z) \rangle = \int_{-\infty}^{\infty} \frac{d\lambda}{z-\lambda} \langle E_+(\lambda) \rangle.$$  

(3.31)

This equation together with (3.26) gives the the representation of the resolvent $W(z) = \frac{1}{N} \langle H(z) \rangle$ as an integral of the spectral density $W(z) = \int_{-\infty}^{\infty} \frac{d\lambda \rho(\lambda)}{z-\lambda}$. Therefore all the information about the spectral correlators is also contained in the discontinuities of the correlators of $H(z)$.  

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3.4. Virasoro constraints and Hirota equations

We give here another derivation of the Virasoro constraints, using the Fock space representation of the partition function. The level-1 \( \hat{u}(2) \) current algebra satisfied by the operators \((3.3)\) decouples into \( \hat{u}(1) \) and \( \hat{su}(2)_1 \) parts. The corresponding Sugawara energy-momentum tensor also decouples to two noninteracting pieces

\[
\frac{1}{2} \sum_{a=1,2} (\partial \psi^*(a) \psi^{(a)} - \psi^*(a) \partial \psi^{(a)}) : = T(z) + \bar{T}(z),
\]

where

\[
T(z) = \frac{1}{6} : (2H^2 + E_+ E_- + E_- E_+) : = : H^2(z) :, \quad \bar{T}(z) = : \bar{H}^2(z) :
\]

and \( H \) and \( \bar{H} \) are given by \((3.7)\) and \((3.8)\) respectively. Both pieces commute with the operator \( e^{Q^+} \), which means that theory is conformal invariant. Only the operator \( T(z) = \sum_n L_n z^{-n-2} \) produces nontrivial Ward identities. For any \( n \geq -1 \),

\[
[L_n, Q_+] = \int_{-\infty}^{\infty} d\lambda [L_n, E_+(\lambda)] = \int_{-\infty}^{\infty} d\lambda \frac{d}{d\lambda} \left( \lambda^{n+1} E_+(\lambda) \right).
\]  

Since our potential diverges as \( \lambda \to \pm \infty \), the boundary terms of can be neglected, and we have

\[
\langle N | e^{H[V]} T(z) e^{Q^+} | 0 \rangle \} = \{\text{regular function at } z = 0 \}.
\]

On the other hand, by commuting \( L_n, n \geq -1 \) with \( e^{H[V]} \) we find the nontrivial linear differential equations \((2.12)\) known as Virasoro constraints.

The partition function is by construction a \( \tau \)-function of the KP hierarchy \[18\], \( Z_V[t] = \tau_N[t] \), where the “times” \( t_n \) are the coefficients in the expansion of the potential \( V(z) = -\sum_{n=1}^{\infty} t_n z^n \). It is a particular case of the “general solution” of the KP hierarchy presented in \[22\] and satisfies the bilinear Hirota equations

\[
\oint dz \prod_n e^{(t_n - t_n')} z^n \left\langle \det (z - M) \right\rangle_{N, \{t\}} \left\langle \det \frac{1}{z - M} \right\rangle_{N', \{t\}} = 0 \quad (N' \leq N).
\]

or

\[
\int_{\infty} dz \left( \mathbf{V}_+(z) \cdot Z_N[t] \right) \left( \mathbf{V}_-(z) \cdot Z_{N'}[t'] \right) = 0 \quad (N' \leq N)
\]

where the vertex operators \( \mathbf{V}_\pm(z) = \exp \left( \pm \sum_{n=0}^{\infty} t_n z^n \right) \exp \left( \mp \frac{1}{2} \frac{\partial}{\partial t_n} \mp \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{\partial}{\partial t_n} \right) \) are understood as formal series in \( t_n \) and \( \frac{\partial}{\partial t_n} \). They produce insertions of fermionic operators in the scalar product. Namely, \( \mathbf{V}_+(z) Z_N[t] = \langle N | \psi^{(1)} \mathbf{e}^{-H[V]} \psi^{(1)}(z) G | 0 \rangle \), \( \mathbf{V}_-(z) Z_{N}[t] = \langle N | \psi^{(1)} \mathbf{e}^{-H[V]} \psi^{(1)}(z) G | 0 \rangle \). The direct derivation of the Hirota equations from the matrix integral is given in \[24\].
The “susceptibility” \( u[t] = 2 \frac{\partial^2}{\partial t^2} \log Z_N \) satisfies an infinite hierarchy of differential equations with respect to the “times” \( t_n \) the first of which is the Kadomtsev-Petviashvili (KP) equation

\[
3 \frac{\partial^2 u}{\partial t_2^2} + \frac{\partial}{\partial t_1} \left[ -4 \frac{\partial u}{\partial t_3} + 6u \frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3} \right] = 0.
\] (3.37)

One can check directly that it is satisfied by the asymptotic expansion around the gaussian point

\[
\frac{Z_N[t_n]}{Z_N[-\delta n_2]} = (-t_2)^{-\frac{N^2}{2}} \exp \left( Nt_0 - \frac{N}{4} t_1^2 + \frac{N^2}{4} t_2^2 - \frac{N}{8} t_3^3 + \ldots \right),
\] (3.38)

The Hirota equations themselves are not sufficient to determine uniquely the partition functions \( Z_N[t] \). The missing information is supplied by the first two Virasoro constraints \( L_{0,-1} Z = 0 \) describing the reaction of the integral (2.1) to translations and rescaling of the variables,

\[
L_0 = N^2 + \sum_{n=1}^{\infty} nt_n \partial_n, \quad L_{-1} = t_1 N + \sum_{n=1}^{\infty} (n+1)t_{n+1} \partial_n.
\] (3.39)

Let us consider the example of the cubic potential \( V(z) = -t_1 z - t_2 z^2 - t_3 z^3 \). The partition function depends, due to the scaling laws (3.39) only on a single combination of the three couplings:

\[
Z_N[t_1, t_2, t_3] = e^{-\frac{t_1 t_2}{t_3} + \frac{2N^2}{27 t_3^2} \frac{N^2}{4} t_2^2 - \frac{N}{8} t_3^3 + \ldots} \frac{Z_N[x, 0, -1]}{Z_N[-\delta n_2]}, \quad x = \frac{t_2^2 - 3t_1 t_3}{3t_3^{4/3}}.
\]

and the susceptibility depends on the three couplings as

\[
u(t_1, t_2, t_3) = 2 \frac{\partial^2}{\partial t_1^2} \log Z_N = t_3^{-2/3} f(x).
\] (3.40)

The KP equation reduces to an ordinary differential equation

\[
\frac{\partial}{\partial t_1} \left( f + 2f' x - 9ff' - \frac{3}{2} f'''(x) \right) = 0.
\]

4. Bosonic representation

In most of the applications associated with the large \( N \) limit we are interested not in the exact expressions in terms of determinants of strongly oscillating fermionic eigenfunctions (typically they have \( N \) nodes), but in the smooth effective expressions. In this sense it is more advantageous to develop the theory of the collective excitations of fermions for which the small parameter \( 1/N^2 \) plays the role of a Planck constant and whose ground state is the equilibrium state of the Dyson gas. The free boson representation is very useful to study the large \( N \) limit of the theory.
By the 2D bosonization formulas we represent the chiral fermions as vertex operators

\[
\psi^{(a)}(z) = e^{\varphi^{(a)}(z)} : ; \psi^{\ast(a)}(x) = e^{-\varphi^{(a)}(z)} : ; \psi^{\ast(a)} \psi^{(a)} := \partial \varphi^{(a)} , \quad (a = 1, 2)
\]

where

\[
\varphi^{(a)}(z) = \hat{q}^{(a)} + \hat{p}^{(a)} \ln z - \sum_{n \neq 0} \frac{J_{n}^{(a)}}{n} z^{-n}
\]

are holomorphic two-component scalar fields with

\[
[J_{m}^{(a)}, J_{n}^{(b)}] = n \delta_{a,b} \delta_{n+m,0}, \quad [\hat{p}^{(a)}, \hat{q}^{(b)}] = \delta_{a,b}
\]
do that their operator product expansion is then \( \varphi^{(a)}(z) \varphi^{(b)}(z') \sim \delta_{a,b} \ln(z - z') \). The normal ordering \( : J_{n}^{(a)} J_{m}^{(a)} : \) is defined as \( J_{n}^{(a)} J_{m}^{(a)} \) for \( n \leq m \), \( J_{m}^{(a)} J_{n}^{(a)} \) for \( n > m \) and \( : \hat{q} \hat{p} : = : \hat{p} \hat{q} := \hat{q} \hat{p} \). The boson Fock space is generated by the oscillators with negative frequencies applied to the vacuum vector \( |0\rangle \) such that

\[
\hat{p}^{a} |0\rangle = 1, \quad J_{n}^{a} |0\rangle = 0 \quad (n > 0).
\]
The left vacuum \( \langle 0 | \) is similarly defined, with the normalization \( \langle 0 | 0 \rangle = 0 \), and the state \( \langle N | \) is constructed as

\[
\langle N | = \langle 0 | e^{N \hat{q}^{(1)}} e^{-N \hat{q}^{(2)}} | .
\]

It will be convenient to introduce the fields

\[
\phi = \frac{\varphi^{(1)} - \varphi^{(2)}}{\sqrt{2}}, \quad \hat{\phi} = \frac{\varphi^{(1)} + \varphi^{(2)}}{\sqrt{2}},
\]

associated with the \( \hat{su}(2) \) and the \( \hat{u}(1) \) algebras. The corresponding currents \( H = \frac{J_{1} - J_{2}}{2} \), \( \hat{H} = \frac{J_{1} + J_{2}}{2} \), \( E_{\pm} = J_{\pm} \) are given by

\[
H = \frac{1}{\sqrt{2}} \partial \phi(z), \quad E_{\pm} = : e^{\pm \sqrt{2} \phi} ;
\]

\[
\hat{H} = \frac{1}{\sqrt{2}} \partial \hat{\phi}.
\]
The Hamiltonian and the screening operator \( Q_{+} \) are represented as

\[
H[V] = \sum_{n \geq 0} t_{n} H_{n} = \frac{1}{\sqrt{2}} \oint_{\infty} \frac{dz}{2\pi i} V(z) \partial \phi(z) ,
\]

\[
Q_{+} = \int_{-\infty}^{\infty} d\lambda : e^{\sqrt{2} \phi(\lambda)} :.
\]
and the Virasoro generators are realized as follows

$$T(z) = \sum_n L_n z^{-n-2} = \frac{1}{2} : \partial \phi(z) \partial \phi(z) :$$  \hspace{1cm} (4.10)

The $U(N)$-invariant correlation functions of in the matrix model are obtained through the identification

$$\text{Tr} \frac{1}{z-M} = \sqrt{2} \partial \phi_+(z), \quad \det(z-M) = : e^{\sqrt{2} \phi_+(z)} :$$  \hspace{1cm} (4.11)

where $\phi_+(z)$ is the singular at $z = 0$ part of the operator $\phi(z)$.

The Virasoro constraints (2.12) follow from the representation of the bosonic field $\phi(z)$ by the operator

$$\Phi(z) = \frac{1}{\sqrt{2}} \sum_{n>0} t_n z^n + \frac{1}{\sqrt{2}} t_0 + \sqrt{2} \frac{\partial}{\partial t_0} \ln z + \sqrt{2} \sum_{n \geq 0} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n}$$  \hspace{1cm} (4.12)

acting on the partition function. The bosonic representation is easily generalized for the orthogonal and symplectic matrix integrals, see for example [26].

5. Quasi-classics

5.1. Ground state

In the large $N$ limit the bosonic field $\phi = \frac{\phi^{(1)} - \phi^{(2)}}{\sqrt{2}}$ develops a large (of order $N$) vacuum expectation value $\phi_c(z) = \langle \phi(z) \rangle$ which is a solution of classical Virasoro constraints

$$T_c \equiv \frac{1}{2} \partial \phi_c^2(z) = \{ \text{regular function at } z = 0 \}$$  \hspace{1cm} (5.1)

satisfying the boundary conditions at infinity

$$H_c(z) \equiv \frac{\partial \phi_c(z)}{\sqrt{2}} = -\frac{1}{2} V'(z) + \frac{N}{z} + O \left( \frac{1}{z^2} \right).$$  \hspace{1cm} (5.2)

Therefore the classical current $H_c$ is of the form

$$H_c(z) = -M(z) y(z), \quad y(z) = \sqrt{\prod_{k=1}^{2p} (z - a_k)}$$  \hspace{1cm} (5.3)

where $M(z)$ is an entire function of $z$.

The classical current has discontinuities along the intervals $[a_{2k-1}, a_{2k}]$. In the case of a single cut, the conditions (5.1) and (5.3) are sufficient for reconstructing the ground state. In the case of several cuts the situation is more complicated. The potential $V(z)$ then has $p$ minima along the real axis (we assume that it grows at infinity) so that the
spectral density $\rho(\lambda) = \frac{1}{2\pi i N} [H_c(\lambda + i0) - H_c(\lambda + i0)]$ splits into $p$ discontinuous pieces $\rho_k$. Each partition $N_1, ..., N_p$ ($N_1 + ... + N_p = N$) giving the distribution of the eigenvalues between the $p$ minima defines a metastable state. The tunneling events leading to the decay of the the metastable state are of order $e^{-N}$ which means that in the large $N$ limit the metastable states are actually stable. The true ground state is determined by requiring that the tunnelling amplitude between two neighboring vacua vanish, which yields $p - 1$ additional conditions fixing the filling numbers $N_1, ..., N_p$.

The classical field $\phi_c = \frac{1}{\sqrt{2}} \langle \varphi^{(1)} - \varphi^{(2)} \rangle$ changes its sign after circling a branch point. This means that the the expectation values of the two original bosonic fields $\varphi^{(1)}(z)$ and $\varphi^{(2)}(z)$ are given by the two branches of the meromorphic function (5.3). Therefore one can speak of a single field $\bar{\varphi}(z) = \{\varphi^{(1)}(z), \varphi^{(2)}(z)\}$ defined on the two-fold branched covering of the complex plane given by the hyper-elliptic Riemann surface $y^2 = \prod_{k=1}^{2p} (z - a_k)$, the two sheets of which are sewed along the $p$ cuts $[a_{2k-1}, a_{2k}]$.

### A. One-cut solution

The classical field depends on $N$ and on the potential $V(z)$ directly and through the positions $a_1, a_2$ of the two branch points. A simple but important observation (made first by V. Kazakov) is that the derivative with respect to $N$ of the resolvent of the random matrix (the expectation value $H_c(z)$ in our interpretation) depends only on the positions of the branch points. Indeed, according to the normalization condition

$$\oint H_c(z) \frac{dz}{2\pi i} = N,$$

the derivative $\partial H_c/\partial N$ satisfies

$$\oint_{A_1} \frac{\partial H_c}{\partial N} dz = 1$$

and therefore behaves as $1/z$ at $z \to \infty$. The only analytic function of the form (5.3) with this asymptotics is

$$\frac{\partial H_c(z)}{\partial N} = \frac{1}{y(z)} = \frac{1}{\sqrt{(z - a_1)(z - a_2)}}.$$  

If we make a conformal transformation $z \to \omega$ with

$$\omega = \int_{z}^{z} \frac{d\lambda}{y(\lambda)}$$

which maps the twice covered $z$-plane to the cylinder $-i\pi < \text{Im} \omega \leq i\pi$, the branch points are replaces by an orbifold structure. The rotation around a branch point acts as

$$\hat{\pi} : \bar{\varphi}(\omega) \to \bar{\varphi}(-\omega) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \bar{\varphi}(\omega),$$

or

$$\phi(-\omega) = -\phi(\omega), \quad \tilde{\phi}(-\omega) = \tilde{\phi}(\omega).$$
According to (5.6), the classical field is related to $\omega = \omega(z)$ by

$$\frac{\partial H_c}{\partial N} = \frac{d\omega}{dz}. \tag{5.7}$$

The explicit expression for the classical current is

$$H_c(z) = \frac{1}{2} \oint_{A_1} \frac{dz}{2\pi i} \frac{y(z) V'(z') - V'(z)}{z - z'}. \tag{5.7}$$

Expanding at $z = \infty$ and using the asymptotics $H_c(z) \sim N/z + ...$ one finds the conditions

$$\oint_{A_1} \frac{dz}{2\pi i} \frac{z^k V'(z)}{y(z)} = -2N\delta_{k,1} \quad (k = 0, 1) \tag{5.8}$$

which allow to determine the positions of the branch points corresponding to given potential.

**B. Multi-cut solutions**

The above analysis can be carried out in the general case of a classical solution with an arbitrary number of cuts. Assuming that the branch points are ordered as $a_1 < ... < a_{2p}$, we have

$$\int_{a_{2k-1}}^{a_{2k}} \rho_k(\lambda) d\lambda = \frac{N_k}{N}. \tag{5.9}$$

or, in terms of the zero modes of the classical current,

$$\oint_{A_k} H_c(z) \frac{dz}{2\pi i} = N_k \quad (k = 1, ..., p) \tag{5.9}$$

where $A_k$ is the contour circling the cut $[a_{2k-1}, a_{2k}]$.

In this case there the $p$ normalization conditions (5.9) which determine the filling numbers $N_1 + ... + N_p = N$ associated with the minima of the effective potential. The derivatives of the current with respect to the filling numbers $N_k$

$$\frac{\partial H_c(z)}{\partial N_j} = \frac{d\omega_j(z)}{dz} \tag{5.10}$$

form a basis of holomorphic abelian differentials $d\omega_k(z)$ associated with the canonical $A$-cycles encircling the cuts $[a_{2k-1}, a_{2k}]$ and satisfying

$$\frac{1}{2\pi i} \oint_{A_k} d\omega_j(z) = \delta_{kj}. \tag{5.11}$$
The period matrix of the hyperelliptic surface is given by the integrals\footnote{The period matrix is generally overdetermined. The condition that the the period matrix describes a Riemann surface (Shottkey problem) is that the corresponding theta function is a tau-function of KP (Krichever’s conjecture proved by Shiota and Mulase).}

$$\tau_{k,j} = \frac{1}{2\pi i} \oint_{B_k} d\omega_j \quad (k = 1, ..., p - 1, \ j = 1, ..., p) \quad (5.12)$$

where the contour $B_k$ encircles the points $a_{2k}, a_{2k+1}, ..., a_{2p-1}$ so that

$$A_k \circ B_j = \delta_{k,j}, \ (k, j = 1, ..., p - 1). \quad (5.13)$$

The explicit form of the abelian differentials is

$$\frac{\partial \omega_j(z)}{\partial z} = \sum_k [K^{-1}]_{kj} z^{k-1} \quad K_{kj} = \oint_{A_k} \frac{dz}{2\pi i} \frac{z^{j-1}}{y(z)}. \quad (5.14)$$

The functions $\partial \omega / \partial z$ behave as $1/z$ at infinity and are completely determined by the positions of the cuts. The explicit expression for the classical current is

$$H_c(z) = \frac{1}{2} \oint_{C} \frac{d\lambda}{2\pi i} \frac{y(z)}{y(\lambda)} \frac{V'(\lambda) - V'(z)}{z - \lambda} \quad (5.15)$$

where the contour $C$ encloses all cuts and the point $z$. Expanding in $1/z$ and using the asymptotics (5.2) we find

$$\oint_{C} \frac{dz}{2\pi i} \frac{z^k V'(z)}{y(z)} = -2N \delta_{k,p} \quad (k = 0, ..., p) \quad (5.16)$$

while the normalization conditions (5.4) give, after integrating along the contour $A_k$,

$$-\frac{1}{2} \oint_{a_{2k}}^{a_{2k+1}} \Pi_k(\lambda) \frac{V'(\lambda)}{y(\lambda)} = N_k \quad (k = 1, ..., p) \quad (5.17)$$

where

$$\Pi_k(\lambda) = \frac{1}{\pi} \int_{a_{2k-1}}^{a_{2k}} \frac{d\mu}{\mu - \lambda} y(\mu). \quad (5.18)$$

(It is sufficient to add only $p - 1$ such conditions in order to determine $a_1, ..., a_{2p}$.)

The quasiclassical problem with several cuts has been considered in [27,28,29]. The authors of [29] find a unique solution (the true vacuum) because they impose conditions equivalent to the requirement that the tunneling amplitudes between the different classical vacua are zero.
5.2. Quasi-classical expansion

The quasiclassical expansion of the free energy $\mathcal{F}_N = \ln \mathcal{Z}_N$ is actually the $1/N$ expansion

$$\mathcal{F}_N[V] = \sum_{g=0}^{\infty} N^{2-2g} \mathcal{F}^{(g)}[V/N] + \text{nonperturbative terms}. \quad (5.19)$$

It is also called genus expansion because the term $\mathcal{F}^{(g)}$ represents the sum of all connected Feynman diagrams of genus $g$.

The leading term ($O(N^2)$) is simply the action for the classical bosonic field

$$N^2 \mathcal{F}^{(0)} = N^2 \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln(\lambda - \lambda') - N \int d\lambda \rho(\lambda)V(\lambda)$$

$$= \frac{1}{4\pi} \int d^2 z \partial \phi_c \bar{\partial} \phi_c - \frac{1}{\sqrt{2}} \int \frac{dz}{2\pi i} V(z) \partial \phi_c(z). \quad (5.20)$$

The next term ($O(1)$) is produced by the gaussian fluctuations around the classical solution $\phi_c$. It is most easily calculated by considering the Riemann surface as the complex plane with two twist operators associated with its branch points. The notion of a twist operator has been introduced by Al. Zamolodchikov [30] (see also [31]). The twist operators, which are conformal operators with dimension $1/16$, can be represented as vertex operators

$$\sigma_+(z) = : e^{\frac{1}{2}[\varphi(1)(z) - \varphi(2)(z)]} : = : e^{\frac{1}{2}\varphi(z)} : \quad (z = a_{2k-1}),$$

$$\sigma_-(z) = : e^{\frac{1}{2}[\varphi(2)(z) - \varphi(1)(z)]} : = : e^{-\frac{1}{2}\varphi(z)} : \quad (z = a_{2k}). \quad (5.21)$$

The twist operators are not conformal invariant and have to be dressed by the modes of the bosonic field in order to obtain translational invariant invariant star operators [32]. One can construct perturbatively the star operators $S(a_k)$ out of the modes of the twisted bosonic field (the descendants of the twist operator $\sigma_{\pm}(a_k)$), by requiring that the singular terms with their OPE with the energy-momentum tensor vanish. Then the free energy can be evaluated as the logarithm of the correlation function of the $2p$ star operators.

Let us find the explicit expression for the $O(1)$ term $\mathcal{F}^{(1)}$. Up to $1/N^2$ corrections, the star operators read

$$S(a_{2k-1}) = [\mu(a_{2k-1})]^{-\frac{1}{2}} \sigma_+(a_{2k-1}), \quad S(a_{2k}) = [\mu(a_{2k})]^{-\frac{1}{2}} \sigma_-(a_{2k}) \quad (k = 1, \ldots, p)$$

where $\mu(a) \sim \int_a dz (z - a)^{-3/2} \partial \phi(z)$ is the coefficient in front of the power $(z - a)^{-1/2}$ in the mode expansion of $\partial \phi(z)$ at the point $z = a$. Using the explicit expression for the correlation function of the twist operators we find

$$\mathcal{F}^{(1)} = \log \langle S(a_1) \ldots S(a_{2p}) \rangle = -\frac{1}{24} \sum_{k=1}^{2p} \ln \mu(a_k) + \frac{1}{8} \log \left( \det_{kj} \left[ \frac{1}{a_{2k-1} - a_{2j}} \right] \right).$$

This expression generalizes the one-cut solution found in [15] by solving directly the loop equations.

The evaluation in this way of the next term $\mathcal{F}^{(2)}$ requires more work, but is still much more simple than the direct iteration of the loop equations. (The result of [15] was obtained with the help of Mathematica, and I reproduced all the coefficients, except the first one, using only my pen.)
6. Two-point correlators for one-cut background

6.1. The two-point correlator of the resolvent

The connected correlation function of the resolvent

\[ W(z, z') = \left\langle \operatorname{tr} \frac{1}{z - M} \operatorname{tr} \frac{1}{z' - M} \right\rangle_c \]

is related to that of the current \( H = \frac{1}{\sqrt{2}} \partial \phi \) by

\[ W(z, z') = \frac{1}{2} \left\langle \partial \phi_+ (z) \partial \phi_+ (z') \right\rangle = \frac{1}{2} \left( \left\langle \partial \phi (z) \partial \phi (z') \right\rangle - \frac{1}{(z - z')^2} \right). \tag{6.1} \]

(We remind that \( \phi_+ \) is the singular at \( z = 0 \) part of the field \( \phi \).) Therefore, in order to evaluate \( W(z, z') \), we need the expression for the two-point correlator of the gaussian field \( \varphi(z) \) defined on the Riemann surface with a cut along the interval \([a_1, a_2] \equiv [a, b] \). The latter can be evaluated as the 4-point function of two current and two twist operators

\[ \left\langle \varphi(z) \varphi(z') \right\rangle_c = \left\langle 0 | \varphi(z) \varphi(z') \sigma_+ (a) \sigma_- (b) | 0 \right\rangle = \ln \left( \frac{\sqrt{(z - a)(z' - b)}}{\sqrt{(z - b)(z' - a)}} \right). \tag{6.2} \]

This function has a nontrivial monodromy, and the determination of the square roots depends on whether the two fields belong to the same or to different sheets the Riemann surface. The correlator can be also determined from the condition that it behaves as \( \ln(z - z') \) at short distances and that it has the same analytic structure as the background field \( \varphi_c(z) \). The sum over the four terms gives of course the two-point correlator on the plane:

\[ \sum_{a, a'} \left\langle \varphi^{(a)}(z) \varphi^{(a')}(z') \right\rangle = \ln \left[ \zeta(z) - \zeta(z') \right], \quad \zeta(z) = (z - a)(z - b). \]

We have to calculate the correlator of the field \( \phi(z) = \frac{\varphi^{(1)} - \varphi^{(2)}}{\sqrt{2}} \) which is antisymmetric with respect to the \( \mathbb{Z}_2 \) monodromy. Taking the alternated sum over the four branches of (6.2), we get

\[ \left\langle \phi(z) \phi(z') \right\rangle_c = \ln \left( \frac{\sqrt{(z - a)(z' - b)}}{\sqrt{(z - a)(z' - b)}} - \frac{\sqrt{(z - b)(z' - a)}}{\sqrt{(z - b)(z' - a)}} \right). \tag{6.3} \]

Differentiating twice, we obtain for the current correlator

\[ \left\langle \partial \phi(z) \partial \phi(z') \right\rangle_c = \frac{\sqrt{(z - a)(z' - b)}}{\sqrt{(z - b)(z' - a)}} + \frac{\sqrt{(z' - a)(z - b)}}{\sqrt{(z' - b)(z - a)}} \left( \frac{1}{2(z - z')^2} \right). \tag{6.4} \]
6.2. The quasiclassical expression for the spectral kernel

The quasiclassical evaluation of the spectral kernel can be done as well, but in this case it is not so easy to control the approximation. Again we split the bosonic field $\varphi$ into a classical and quantum parts and consider the quantum part as a free field on the Riemann surface. The kernel (3.28) is expressed in the free field approximation as a correlation function of vertex operators

$$K(\lambda, \mu) = \langle : e^{\varphi(1)}(\lambda) : e^{-\varphi(2)}(\mu) : \rangle. \tag{6.5}$$

As in the case of the resolvent, we first write the quasiclassical expression for the four-point function of two vertex and two twist operators

$$G(z, z') = \langle 0 : e^{\varphi(z)} : e^{-\varphi(z')} : \sigma_+(a)\sigma_-(b) | 0 \rangle$$

$$= \frac{1}{2} e^{\frac{\phi_c(z) - \phi_c(z')}{\sqrt{2}}} \left( \frac{(z - a)(z' - b)}{(z - b)(z' - a)} \right)^{1/4} + \frac{(z - b)(z' - a)}{(z - a)(z' - b)}^{1/4} \tag{6.6}$$

where we have used that $\langle \varphi(z) \rangle = \pm \frac{1}{\sqrt{2}} \phi_c(z)$, the sign depending on whether the field is evaluated on the upper or the lower sheet of the Riemann surface. The overall coefficient is determined by the behavior at $z \to z'$ where we have to reproduce $\frac{1}{\sqrt{2}} \partial \phi_c(z) = \pm i\pi \rho(z)$. The branch of $\phi_c(z)$ has to be chosen so that $G(z)$ decays exponentially at infinity.

The kernel $K(\lambda, \mu)$ is defined on the real axis and is obtained as the average over the four values of this function on both sides of the cut $[a, b]$

$$K(\lambda, \mu) = \frac{1}{4\pi i} \sum_{\epsilon, \epsilon' = \pm} G(\lambda + i\epsilon 0, \mu + i\epsilon' 0). \tag{6.7}$$

(The average should be taken because of the ambiguity due to presence of the cut; this ambiguity of $G$ appears only in the large $N$ limit.) One finds explicitly, for $\lambda, \mu \in [a, b]$

$$K(\lambda, \mu) = \frac{1}{2\pi} \frac{\sin \left( \frac{\phi_c(\lambda) - \phi_c(\mu)}{\sqrt{2}} \right)}{\sqrt{(\lambda-a)(b-\mu)-(b-\lambda)(\mu-a)}} + \frac{\cos \left( \frac{\phi_c(\lambda) + \phi_c(\mu)}{\sqrt{2}} \right)}{\sqrt{(\lambda-a)(b-\mu)+(b-\lambda)(\mu-a)}} \tag{6.8}$$

This expression coincides with the one obtained in [33] and in [34,35] by solving the appropriate loop equations.

When $\lambda, \mu \in [a, b]$ and $\lambda - \mu \sim 1/N$, one obtains the well known short distance behavior

$$K(\lambda, \mu) = \frac{\sin \left[ \pi N(\lambda - \mu) \rho(\frac{\lambda+\mu}{2}) \right]}{\pi(\lambda - \mu)}. \tag{6.9}$$

In the case when one of the arguments is outside the eigenvalue interval, the exponent is real and negative and the spectral kernel decays rapidly.
7. Generalization to chains of random matrices

The most natural generalization of the CFT construction is given by the ADE matrix models [3], which were introduced as a nonperturbative microscopic realization of the rational string theories with C < 1. Each one of these models is associated with a rank r classical simply laced Lie algebra (that is, of type Ar, Dr, E6,7,8) or its affine extension, and represents a system of r coupled random matrices.

Here we will discuss only the models of the A-series, for which there exists a simple fermionic representation. The model associated with Ar = su(r + 1) represents a chain of r Hermitian matrices Ma of size Na × Na (a = 1, ..., r), interacting by means of r − 1 auxiliary gauge-field-like rectangular complex matrix variables Aa (a = 1, ..., r − 1) of size Na × Na+1. In this way the Ma and Aa are associated respectively with the nodes and the links of the Dynkin diagram of Ar. The partition function of the matrix chain is given by the following integral

\[ Z_N(V) = \int \prod_{a=1}^{r} dM_a \ e^{-\text{tr} V(M_a)} \int \prod_{a=1}^{r} dA_a dA_a^\dagger e^{-\text{tr} A_a A_a^\dagger M_a - \text{tr} A_a^\dagger A_a M_{a+1}}. \] (7.1)

After integrating with respect to the A-matrices and the angular variables of the M-matrices, the partition function reduces to an integral with respect of the eigenvalues λai, i = 1, ..., Na, of the hermitian matrices Ma

\[ Z_N(V) = \int \prod_{a=1}^{r} \prod_{i} d\lambda_{ai} \ e^{-\sum_{i} V^a(\lambda_{ai})} \prod_{i<j} (\lambda_{ai} - \lambda_{aj})^2 \prod_{a=1}^{r-1} \prod_{i,j} \frac{1}{(\lambda_{\bar{a}} + \lambda_{\bar{a}+1,j})}. \] (7.2)

Important remark: the integral with respect to the A-matrices exists only if all eigenvalues of the matrices Ma are positive. This can be achieved by an appropriate choice of the potential. We will thus assume that the integration is restricted to the positive real axis λa > 0.[3]

Let C+ be a contour representing the boundary of the half-plane Re z > 0. Using the fact that the eigenvalue integration is restricted to the positive real axis, we can write the following loop equations for each a

\[ \left\langle W_a(z) \right\rangle^2 + \oint_{C_+} \frac{dz'}{2\pi i} \frac{1}{z - z'} W(z') \left[ \sum_b G_{ab} W_b(-z') - \partial_z V^a(z) \right] \right\rangle_{N,t} = 0. \] (7.3)

The representation of this integral as a Fock space expectation value is a generalization of the su(2) construction of Sect.3. Now we consider r + 1 fermion fields \( \psi^{(a)}(z) \), a = 1, ..., r + 1, whose modes in the expansion (3.2) satisfy the anticommutation relations \[ [\psi^{(a)}_r, \psi^{(b)}_s]_+ = \delta_{rs} \delta_{ab} \], so that the bilinears \( J^{(a)}(z) =: \psi^{(a)*}(z) \psi^{(a)}(z) : \),

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3 Another way to achieve this is to take \( M_a = B_a B_a^\dagger \) where \( B_a \) are complex Na × Na matrices.
\( J^+_a(z) = \psi^{(a)}(z)\psi^{(a+1)}(-z) \) and \( J^-_a(z) = \psi^{(a+1)}(z)\psi^{(a)}(-z) \) generate an algebra related to the \( u(r+1) \) current algebra. As before, the \( u(1) \) current \( \tilde{J} = \frac{1}{r+1} \sum_a J^{(a)} \) completely decouples so that the Cartan currents of \( su(r) \) are given by the differences \( J^a(z) = J^{(a)}(z) - J^{(a+1)}(-z) \).

Let us express the potentials \( V^a(z) \) as differences \( V^a(z) = V^{(a)}(z) - V^{(a+1)}(-z) \), and define the Hamiltonians \( H^{(a)}[V^{(a)}] = \oint \frac{dz}{2\pi i} V^{(a)}(z)J^{(a)}(z) \). It is easy to see that the partition function (7.2) can be written similarly to (3.12) as

\[
\mathcal{Z}_N[\tilde{V}] = \langle \tilde{l} \prod_{a=1}^{r+1} e^{H^{(a)}[V^{(a)}]} \prod_a e^{|Q^+_a|} |0\rangle, \tag{7.4}
\]

where the vacuum state of charge \( \tilde{l} = (l^{(1)}, \ldots, l^{(r+1)}) \) is defined by (3.4) with \( l^{(a+1)} - l^{(a)} = 2N_a \), and the ”screening operators” are given by \( Q^+_a = \int_0^\infty d\lambda \left[ J^{(a)}(z) - J^{(a+1)}(-z) \right] \).

The bosonic representation is obtained according to (4.1) in terms of the bosonic fields \( \varphi^{(a)} \), \( a = 1, \ldots, r+1 \). They are split into \( r \) fields associated with the \( su(r) \) and the \( u(1) \) parts

\[
\phi^a(z) = \varphi^{(a)}(z) - \varphi^{(r+1)}(-z) \quad (a = 1, \ldots, r), \quad \tilde{\phi}(z) = \frac{1}{r+1} \sum_a \varphi^{(a)}((-)^{a-1}z). \tag{7.5}
\]

It is convenient to define another set of fields \( \phi_a \) by

\[
\phi^a(z) = 2\phi_a(z) - \sum_b G^{ab} \phi_b(-z) \tag{7.6}
\]

where \( G^{ab} = (\alpha^a \cdot \alpha^b) \) is the adjacency matrix of the \( A_r \) Dynkin diagram. The fields \( \phi^a \) and \( \phi_a \) are related as the contravariant and covariant components of a vector field in the base of the simple roots \( \tilde{\alpha}^a \) of the \( su(r) \). Note however the reflection \( z \rightarrow -z \) in the second term of (7.4).

If we define in a similar way the covariant components \( V_a \) of the potential by

\[
V^a(z) = 2V_a(z) - \sum_b G^{ab} V_b(-z) \tag{7.7}
\]

then

\[
\phi_a(z) = -V_a(z) + W_a(z), \quad W_a(z) = \sum_{i=1}^{N_a} \frac{1}{z - \lambda_{ai}}. \tag{7.8}
\]

The loop equations (7.3) read, in terms of the contravariant components \( \phi^a \)

\[
\left\langle \oint_{\mathcal{C}_r} \frac{dz'}{z - z'} [\partial \phi^a(z')]^2 \right\rangle = 0 \quad (a = 1, \ldots, r). \tag{7.9}
\]

\footnote{The difference originates in the minus sign in the definition of the currents \( J^\pm_a \) corresponding to the simple roots of \( u(r+1) \).}
The Riemann surface defined by the classical solution represents an \((r+1)\)-fold covering of the complex plane which has in the simplest case two cuts \([a,b]\) and \([-b,-a]\) on the real axis \((0 < a < b)\). The fields \(\varphi^{(a)}\) represent the values on the different sheets of a single meromorphic field \(\varphi(z)\). The quasiclassical expressions for the correlation functions and the free energy are obtained by introducing \(2r\) twist operators

\[
\sigma_k^+ \left[ \left( - \right)^k a \right] = e^{\frac{k}{2} \phi^k \left[ \left( - \right)^k a \right]} ; \quad \sigma_k^- \left[ \left( - \right)^k b \right] = e^{-\frac{k}{2} \phi^k \left[ \left( - \right)^k b \right]},
\]

associated with the points \(\pm a, \pm b\). In the limit \(a \to 0, b \to \infty\) the \(r\) twist operators located at the points \(b\) and \(-b\) merge into a \(\mathbb{Z}_r\) twist operator at the origin while the rest form a \(\mathbb{Z}_r\) twisted boundary condition at infinity. In this case the bosonic description is given by the \(\mathbb{Z}_n\) twisted field considered in refs. [17].

8. Another generalization: The hermitian matrix model as an amplitude between two \(GL(N)\) representations

Our CFT representation actually holds for a more general class of matrix models, containing external matrix sources. The source can be introduced by replacing the left vacuum state with an excited state. Consider the integral

\[
\mathcal{Z}_{R^{(1)}, R^{(2)}}[V] = \int dM e^{-\text{tr} V(M)} \chi_{R^{(1)}}(M) \chi_{R^{(2)}}(M)
\]

(8.1)

where \(\chi_{R^{(1)}}\) and \(\chi_{R^{(2)}}\) are the characters of two polynomial representations of the group \(U(N)\). The polynomial representation \(R\) is labeled by the Young tableau

\[
Y_R = \{l_1 \geq l_2 \geq ... \geq l_N\}
\]

and the corresponding character is given by a ratio of two determinants

\[
\chi_R(M) = \frac{\det \left( \lambda_i^{j-1 + l_{N-j+1}} \right)}{\det \left( \lambda_i^{j-1} \right)} \quad (i, j = 1, ..., N).
\]

(8.2)

Therefore the partition function can be written again as an integral over the eigenvalues

\[
\mathcal{Z}_{R^{(1)}, R^{(2)}}[V] = \int \prod_{i=1}^{N} d\lambda_i \; e^{-V(\lambda_i)} \det(\lambda_i^{j-1 + l_{N-j+1}^{(1)}}) \det(\lambda_i^{j-1 + l_{N-j+1}^{(2)}}).
\]

(8.3)

The \(U(N)\) characters are related to the expectation values of the fermions as

\[
\chi_{R^{(a)}}(M) = \left\langle 0 \right| \psi^{(a)}_1 \psi^{(a)}_N \ldots \psi^{(a)}_1 \psi^{(a)}_N (\lambda_N) \ldots (\lambda_1) \left| 0 \right\rangle \prod_{i<j} (\lambda_i - \lambda_j)
\]

(8.4)

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where \( r_k^{(a)} = k - \frac{1}{2} + l_{N-k+1}^{(a)} \) \((k = 1, \ldots, N)\). The Fock representation of the matrix integral (8.1) is an evident generalization of (3.12)

\[
Z_{R(1) R(2)}[V] = \langle R(1) R(2) | e^{H[V]} e^{Q^+} | 0 \rangle
\]

where

\[
\langle R(1) R(2) | = \langle 0 | \prod_{k=1}^{N} \psi^{(1)}_{r_k^{(1)}} \psi^{(2)}_{r_k^{(2)}}
\]

generalizes the vacuum state \( |N\rangle \). The analog of the diagonalization formula (3.22) is

\[
Z_{R(1), R(2)}[V] = \frac{\prod_{n=1}^{N} h_n}{\det P_{R(1)} \det P_{R(2)}}
\]

where the matrix \( P_R \) labeled by the Young diagram \( R = \{l_1 \geq l_2 \geq \ldots \geq l_N\} \), or equivalently, by the sequence of half-integers \( r_k = k - \frac{1}{2} + l_{N-k+1} \), is made of the coefficients of the orthogonal polynomials \( P_n(\lambda) = \sum_{k=1}^{N} P_n, k \lambda^{n-k} \): \( P^{(a)}_{ij} = P_{r_i - \frac{1}{2}, j} \). When \( l_{N-i-1}^{(a)} \equiv 0 \), the determinant of the triangular matrix \( P_{R(a)} \) equals 1 and (8.7) reduces to the standard formula (3.22).

The quasiclassical expressions for the spectral correlations depend on the nontrivial left vacuum only through the classical field \( \phi_c \) and are therefore given again by (6.4) and (6.8). The Virasoro constraints are however more complicated because the left vacuum state is not annihilated by the negative components of the current.

9. Concluding remarks

The CFT representation of the matrix models originates in the \( SU(N) \) invariant matrix integration measure and therefore can be actually constructed for all matrix models. In a system of several or even infinitely many interacting random matrices, one can associate with each matrix variable \( M_a \) a current \( J_a(z) = Tr z^{-1-M_a} \) and a vacuum \( |0_a\rangle \), which is invariant with respect to conformal transformations generated by the Sugawara stress-energy tensor \( T_a(x) \sim J_a^2(z) \). In the large \( N \) limit all other matrices interacting with \( M_a \) can be reduced to an effective matrix mean field, which will plays the role of the potential \( V \) in our analysis of the hermitian matrix model. The classical Virasoro constraints (5.1) hold again, but the effective potential can have cuts and poles. The Virasoro generators do not have in general a simple representation as linear differential operators as the Virasoro constraints (2.12). The classical background is described by a Riemann surface, which is in general not a hyperelliptic one. The Hilbert space associated with the matrix variable \( M_a \) corresponds to a contour separating the cuts of the resolvent \( Tr \frac{1}{z-M_a} \) from the cuts of the effective potential.

The usefulness of the CFT approach consists in the fact that once the classical solution \( \varphi_a(z) \) is found, the loop correlators and the spectral kernel are given by the universal expressions (6.4) and (6.8). In this sense observed universality of the short distance behavior of the loop correlators and the spectral kernel is a rather trivial consequence of the conformal invariance. It is also possible to construct a universal loop diagram technique for the \( 1/N \) corrections, one particular case of which is considered in ref. [36].
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