THE KRASNOSEL’SKII FORMULA FOR PARABOLIC
DIFFERENTIAL INCLUSIONS WITH STATE CONSTRAINTS

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Abstract. We consider a constrained semilinear evolution inclusion of paraboly type involving an m-dissipative linear operator and a source term of multivalued type in a Banach space and topological properties of the solution map. We establish the \( R_{δ} \)-description of the set of solutions surviving in the constraining area and show a relation between the fixed point index of the Krasnosel’ski–Poincaré operator of translation along trajectories associated with the problem and the appropriately defined constrained degree of the right-hand side in the equation. This provides topological tools appropriate to obtain results on the existence of periodic solutions to studied differential problems.

1. Introduction. In the paper we present a topological approach to study dynamics and periodic behaviour of a reaction-diffusion system of the form

\[
\partial_t u_i = \Delta u_i + f_i(t,x,u), \quad i = 1, \ldots, N,
\]

or, shortly,

\[
u_i = Lu + f(t,x,u), \quad \text{where} \quad Lu = (\Delta u_1, \ldots, \Delta u_N),
\]

along with initial and Dirichlet or Neumann boundary conditions on the boundary \( \partial \Omega \) of a bounded open smooth domain \( \Omega \subset \mathbb{R}^M \); the unknown state \( u = (u_1, \ldots, u_N) \) depends on spatial variables \( x = (x_1, \ldots, x_M) \in \Omega \) and time \( t \in [0,T], T > 0 \).

Our interest is concerned with a chemical system of \( N \) reacting components, \( u_i(t,x) \) is the concentration at time \( t \) and location \( x \) in the unstirred bounded reactor \( \Omega \) of the \( i \)-th reactant \( i = 1, \ldots, N \), subject to diffusion and the source term \( f_i \) depending on \( (t,x,u) \). A natural requirement is that the initial \( u_i(\cdot,0) \) as well as the intermediate concentrations \( u_i(\cdot,t), i = 1, \ldots, N \), are nonnegative. There are natural upper bounds, too, e.g. some threshold values \( R_i > 0 \) beyond which the component \( u_i \) is saturated. Sometimes implicit bounds follow e.g. from the mass constraint: the total mass cannot exceed the threshold value \( \bar{R} \). It, therefore, makes sense to

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† If \( v : \Omega \to \mathbb{R} \) is \( C^2 \)-smooth, then the Laplacian is given by \( \Delta v = \sum_{j=1}^M \partial_j^2 v \).
look for solutions $u = u(x,t)$ taking values in a rectangle $P := \{ u \in \mathbb{R}^N \mid 0 \leq u_i \leq R_i, \; i = 1, \ldots, N \}$ or in a simplex $S := \{ u \in \mathbb{R}^N \mid \sum_{i=1}^N u_i \leq R, \; u_i \geq 0, \; i = 1, \ldots, N \}$. Of course the imposed state constraints must be compatible with initial boundary conditions.

The presence of state constraints puts also certain limitations to the source term $f_i$ responsible for the increase (production) or decrease (vanishing) of $u_i$. Firstly it may make sense to define $f$ for $u$ in the constraining set only; secondly if, for instance, we postulate that solutions are to take values in the rectangle $P$, then it is clear that we need to require $f_i(t,x,u) \geq 0$ if $u_i = 0$ since the concentration $u_i$ may only increase in this case, and $f_i(t,x,u) \leq 0$ if $u_i = R_i$ since the concentration may only decrease then. The same argument implies that if we impose state constraints in the simplex $S$, then $\sum_{i=1}^N f_i(t,x,u) \leq 0$ if $\sum_{i=1}^N u_i = R$ since the total mass can only decrease in this situation and $f_i(t,x,u) \geq 0$ if $u_i = 0$. In both cases the natural requirements on $f$ may be subsumed by saying, for instance, that $f(t,x,u) \in TP(u)$ for all $t \in [0,T]$, $x \in \Omega$ and $u \in P$, where $TP(u)$ stands for the tangent cone to $P$ at $u$ (see (4)). Similar assumptions should be made if the state of the system is constrained to the simplex $S$.

In general we will look for solutions to (1) in a prescribed closed convex set $C \subset \mathbb{R}^N$ of state constraints requiring additionally that $f$ is tangent to $C$, i.e., $f(t,x,u) \in TC(u)$ for all $t \in [0,T]$, $x \in \Omega$ and $u \in C$.

We will also admit discontinuous or even multivalued nonlinear term $f$. This is motivated by numerous applications when $f$ is e.g. accretive, or when system data are determined by measurements, or when systems are subject to the presence of Coulomb friction, or phase transitions, or the studying phenomena display the hysteresis effect. In this case it is usual to replace $f$ by an appropriate set-valued regularization $\varphi : [0,T] \times \Omega \times C \rightarrow \mathbb{R}^N$ (see e.g. [29]) and to deal with problems of the form

$$u_t \in Lu + \varphi(t,x,u), \quad u(0,\cdot) = u_0,$$

where $u_0(x) \in C$ for $x \in \Omega$, subject to Dirichlet of Neumann boundary conditions.

We are going to study the existence and the structure of (strong) solutions to (2) and the existence of periodic solutions. To this end, we shall put the problem into an appropriate abstract setting of a constrained semilinear differential inclusion in a Banach space (see (14)) and consider an approach based on the $t$-Poincaré map (or the translation along trajectories operator) assigning to the initial state $x_0$ the set $\Sigma_t(x_0)$ of reachable states of the system after time $t > 0$. A standard observation is that fixed points of $\Sigma_T$ correspond to ‘periodic’ solutions $u$, i.e. such that $u(0) = u(T)$. This approach to periodic orbits is well-known an used by numerous authors (see the book [39] and references therein). The important obstacle here is that we deal with generic nonuniqueness of solutions and the presence of constraints. In order to apply appropriate fixed point theoretical results relying on the fixed point index approach and get conditions guaranteeing the existence of fixed points of the Poincaré operator associated to (2) we shall characterize the set of its solutions and study its dependence on initial conditions. Finally we shall establish a counterpart of the famous Krasnosel’skii formula for such problems. In this way we will obtain a convenient topological tool to study the dynamics of (2) as well as its abstract versions and the existence of periodic solutions as well as the branching (or bifurcation) of periodic solutions in the case of the parameterized problems.
Recall the classical Krasnosel’skii formula that concerns an ODE
\[ u = f(t, u), \quad u \in \mathbb{R}^N, \ t \in [0,T], \]
with locally Lipschitz \( f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N \), admitting global solutions. If \( U \subset \mathbb{R}^N \) is open bounded and \( f(0, u) \neq 0 \) for \( u \) in the boundary \( \partial U \) of \( U \), then \( P_t(x) \neq x \) for \( x \in \partial U \) and sufficiently small \( t > 0 \), where \( P_t(x) = u(t) \) with \( u(\cdot) \) being the unique solution to \((*)\) starting at \( x \in U \) (\( P_t \) is the \( t \)-Poincaré operator associated to \((*)\)), and the Brouwer degrees are equal
\[ \deg_B(-f(0, \cdot), U) = \deg_B(I - P_t, U), \]
with \( I \) denoting the identity operator on \( \mathbb{R}^N \) (cf. [41, Lem. 13.1, 13.2]). The Krasnosel’skii formula shows that the right-hand side data allows to compute the fixed point index of the Poincaré operator associated to \((*)\). An infinite dimensional variant of the Krasnosel’skii formula was obtained in [21, Thm. 4.5] in the case of \((14)\) with \( K = E \), single-valued, time-independent and locally Lipschitz nonlinearity and in [22, Thm. 4.1] for \( K \) being a closed convex cone (see also [23]).

After this introduction the paper is organized as follows: below we recall some relevant concepts and auxiliary results, present standing assumptions concerning \((2)\) and introduce the convenient functional setting. In the second section we present the abstract version of \((2)\), namely problem \((14)\), and present a result on a topological characterization of the set of its solutions. In the third section we introduce topological tools necessary to establish the announced version of the Krasnosel’skii formula. The setting we propose is convenient and general. In the fourth section the Krasnosel’skii formula is proven, while the fifth section is devoted to the study of periodic orbits. We present several results for a bounded, as well unbounded set of constraints. In the unbounded case we introduce a guiding function approach. This attitude is well-known in the finite-dimensional situation (see the extensive and up to date references in [45]). To the best of our knowledge however, this approach was used here in the infinite-dimensional case for the first time.

Remark 1. In the paper we deal mostly with the set-valued setting. It is to be noted however that the results we propose are new in the single-valued case, too.

1.1. Preliminaries. The notation used in this paper is rather standard and so is the use of functional spaces \( (L^p, \text{Sobolev etc.}) \), the theory of operators, etc. In what follows \( (E, \| \cdot \|) \) denotes a real Banach space, while \( E^* \) is the normed topological dual of \( E \); we write \( \langle x, p \rangle \) instead of \( p(x) \) for \( x \in E, p \in E^* \); \( L(E) \) denotes the space of bounded linear operators on \( E \). By \( L^p(0,T;E), \ p \geq 1 \) (resp. \( C([0,T],E) \)) we denote the space of Bochner \( p \)-integrable (resp. continuous) functions \( u : [0,T] \to E \). Recall that \( A \subset L^1(0,T;E) \) is integrably bounded if there is \( \lambda \in L^1(0,T) \) such that \( \| f \| \leq \lambda \) a.e. for every \( f \in A \). If \( X \) is a metric space, \( \varepsilon > 0 \) then \( B_X(A, \varepsilon) := \{ x \in X \mid d_A(x) := \inf_{a \in A} d(x, a) < \varepsilon \} \) is the \( \varepsilon \)-neighborhood of \( A \); in particular \( B_X(x,r) \) (resp. \( D_X(x,r) \)) stands for the open (resp. closed) ball at \( x \in X \) of radius \( r > 0 \). If \( X \subset E \), \( Y \) is a topological space, then a continuous \( f : X \to Y \) is compact or completely continuous if \( f(B) \) is relatively compact for each bounded \( B \subset X \). Set-valued analysis terminology and detailed discussion of many ‘set-valued’ objects is taken from [7]; in particular a set-valued map \( \varphi : X \to Y \) assigns to each \( x \in X \) a nonempty subset \( \varphi(x) \subset Y \). If \( X,Y \) are topological spaces, then \( \varphi \) is upper semicontinuous or usc (resp. lower semicontinuous or lsc) if \( \varphi^{-1}(A) := \{ x \in X \mid \varphi(x) \cap A \neq \emptyset \} \) is closed (resp. open) for every closed (resp. open) \( A \subset Y \). If \( X \subset E \), then \( \varphi : X \to Y \) is compact if it is upper semicontinuous
and \( \varphi(B) := \bigcup_{x \in B} \varphi(x) \) is relatively compact for any bounded \( B \subset X \). If \( X, Y \) are metric spaces, then \( \varphi : X \to Y \) is \( H \)-upper semicontinuous (resp. \( H \)-lower semicontinuous) if for any \( x_0 \in X \) and \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \varphi(x) \subset B_Y(\varphi(x_0), \varepsilon) \) (resp. \( \varphi(x_0) \subset B_Y(\varphi(x), \varepsilon) \)), for \( x \in B_X(x_0, \delta) \) (see e.g. [31] for details and examples concerning set-valued maps).

If \( K \) is a closed convex subset of a Banach space \( E \) and \( x \in K \), then

\[
T_K(x) := \text{cl} \bigcup_{h > 0} \frac{K - x}{h}
\]

stands for tangent cone to \( K \) at \( x \). If \( x \) belongs to the interior of \( K \), then \( T_K(x) = E \).

It is easy to see that (cf. [7, Prop 4.2.1])

\[
T_K(x) = \{ v \in E \mid \lim_{h \to 0^+} \inf d_K(x + hv)/h = 0 \} = \{ v \in E \mid \lim_{h \to 0^+} d_K(y + hv)/h = 0 \}.
\]

Note that \( T_K(x) \) is the polar to the normal cone \( N_K(x) = \{ p \in E^* \mid \forall y \in K \ (y - x, p) \leq 0 \} \), i.e.

\[
v \in T_K(x) \iff \forall p \in N_K(x) \ (v, p) \leq 0.
\]

We present now two auxiliary results in a form adopted for our needs.

**Lemma 1.1.** [10, Lem. 17.], [42, Lem. 3.2] Let \( K \subset E \) be closed convex, \( F : [0, T] \times K \to E \) be tangent to \( K \), i.e. \( F(t,x) \cap T_K(x) \neq \emptyset \) for any \( t \in [0,T] \) and \( x \in K \), and \( H \)-upper semicontinuous with convex values. For any continuous \( \alpha : [0, T] \times K \to (0, \infty) \) there is a locally Lipschitz \( f : [0, T] \times K \to E \) such that for any \( t \in [0, T] \), \( x \in K \), \( f(t,x) \in T_K(x) \) and

\[
f(t,x) \in F(t', x') + B_E(0, \alpha(t, x))
\]

for some \( t' \in [0,T], x' \in K \) with \( |t' - t|, \| x' - x \| \leq \alpha(t, x) \).

**Theorem 1.2.** [38, Th. 3.2] Let \( E \) be a separable Banach space, \( K \subset E \) be closed convex and let \( F, G : [0, T] \times K \to E \) be product measurable (\( ^2 \)) with closed convex values and such that \( F(t,x) \cap G(t,x) \neq \emptyset \) for all \( (t,x) \). If for \( t \in [0,T], F(t, \cdot) \) is \( H \)-upper semicontinuous and \( G(t, \cdot) \) is lower semicontinuous, then for every \( \varepsilon > 0 \) there is a Carathéodory map \( f : [0, T] \times K \to E \) (i.e. \( f(\cdot, x) \) is continuous for any \( t \in [0,T] \) and \( f(\cdot, x) \) is measurable for any \( x \in K \)) such that for all \( t \in [0,T] \), \( x \in K \),

\[
f(t,x) \in G(t,x) \quad \text{and} \quad f(t,x) \in F(t, x') + B_E(0, \varepsilon)
\]

for some \( x' \in K \) with \( \| x' - x \| < \varepsilon \).

**1.2. The functional setting.** While studying (2) we shall assume that:

(\( \varphi_1 \)) the set-valued map \( \varphi : [0, T] \times \Omega \times C \to \mathbb{R}^N \) is upper semicontinuous, has compact convex values; \( C \subset \mathbb{R}^N \) is closed convex;

(\( \varphi_2 \)) \( \varphi \) has sublinear growth: there are \( \alpha \in L^2([0, T] \times \Omega) \) and \( \beta \geq 0 \) such that

\[
\sup_{y \in \varphi(t,x,u)} |y| \leq \alpha(t,x) + \beta |u| \quad \text{for all} \quad t \in [0,T], \ x \in \Omega, \ u \in C;
\]

(\( \varphi_3 \)) \( \varphi \) satisfies the weak tangency condition:

\[
\text{for all} \quad t \in [0,T], \ x \in \Omega \quad \text{and} \quad u \in C \quad \varphi(t,x,u) \cap T_C(u) \neq \emptyset,
\]

\(^2\)It means that given an open \( V \subset E \), \( F^{-1}(V) := \{(t,u) \in [0, T] \times K \mid F(t,u) \cap V \neq \emptyset \} \) belongs to the product of the Lebesgue \( \sigma \)-algebra in \([0, T]\) and the Borel \( \sigma \)-algebra in \( K \).
and study (2) along with the initial condition \( u(0,\cdot) = u_0 \in L^2(\Omega, \mathbb{R}^N) \) and the homogeneous Dirichlet (or Neumann) boundary conditions, i.e. \( u|_{\partial \Omega} = 0 \) (or \( \frac{\partial u}{\partial n} |_{\partial \Omega} = 0 \)).

We will look for strong solutions to (2), i.e., functions \( u : [0, T] \times \Omega \to C \) such that \( u(0,\cdot) = u_0 \) a.e. on \( \Omega \), \( u(t, \cdot) \in H^2(\Omega, \mathbb{R}^N) \) with \( u(t, \cdot) = 0 \) (or \( \frac{\partial u(t, \cdot)}{\partial n} = 0 \)) on \( \partial \Omega \) in the sense of trace for all \( t \in (0, T] \), \( u \in W^{1,1}_{loc}((0, T], L^2(\Omega, \mathbb{R}^N)) \) and for a.a. \( x \in \Omega \) and all \( t \in (0, T] \), \( u_t(t,x) = \Delta_x u(t,x) + w(t,x) \), where \( w \in L^1(0, T; L^2(\Omega, \mathbb{R}^N)) \) and \( w(t,x) \in \varphi(t,x,u(t,x)) \) on \( [0, T] \times \Omega \).

It is convenient and natural to put (2) into the functional setting of a semilinear differential inclusion in \( E := L^2(\Omega, \mathbb{R}^N) \)

\[
\begin{cases}
\dot{u} \in Au + F(t,u), & t \in [0,T], \; u \in K, \\
u(0) = x_0 \in K,
\end{cases}
\tag{7}
\]

where

(i) \( K := \{ u \in E \mid u(x) \in C \; \text{for a.a.} \; x \in \Omega \}; \)
(ii) \( A : D(A) \to E, \; Au = (\Delta u_1, ..., \Delta u_N), \) for \( u = (u_1, ..., u_N) \in D(A) \), where \( D(A) := H^2 \cap H^1_0(\Omega, \mathbb{R}^N) \) (or \( D(A) := \{ u \in H^2(\Omega, \mathbb{R}^N) \mid \frac{\partial u}{\partial n} = 0 \; \text{on} \; \partial \Omega \} \)).
(iii) \( F : [0, T] \times K \to E \) is given by
\[
F(t,u) := \{ v \in E \mid v(x) \in \varphi(t,x,u(x)) \; \text{for a.a.} \; x \in \Omega, \; t \in [0,T], \; u \in E \}.
\]

Let us collect some properties of \( K, \; A \) and \( F \) introduced above. This is important since it justifies assumptions undertaken in the next abstract sections.

**Remark 2.** (a) In view of [48, Theorem 7.2.7], \( A \) generates a holomorphic and resolvent compact, hence compact \( C_0 \) semigroup \( \{ S(t) \}_{t \geq 0} \) of contractions. It is clear that \( K \) is closed and convex. Arguing as in [42, Proposition 4.2] one proves that \( J_h(K) \subset K \) for any \( h > 0 \), where
\[
J_h := (I - hA)^{-1} : E \to E
\tag{8}
\]
is the (modified) resolvent of \( A \) (in the Dirichlet case this holds provided \( 0 \in C \)). For the sake of completeness we give an argument in the Dirichlet case. Assume that \( 0 \in C \). In view of [3, Cor. 7.49], \( C = \bigcap_{n \geq 1} C_n \), where \( C_n := \{ x \in \mathbb{R}^N \mid p_n \cdot x \leq a_n \} \) for some \( p_n \in \mathbb{R}^N \) and \( a_n \geq 0 \). We thus have \( K = \bigcap_{n=1}^\infty K_n \), where \( K_n = \{ u \in E \mid u(x) \in C_n \; \text{for a.a.} \; x \in \Omega \} \) and the countable collection plays a role. We shall show that \( J_h(K_n) \subset K_n \) for each \( n \geq 1 \). Then
\[
J_h(K) = J_h \left( \bigcap_{n \geq 1} K_n \right) \subset \bigcap_{n \geq 1} J_h(K_n) \subset \bigcap_{n \geq 1} K_n = K.
\]

To this end we assume that \( C = \{ x \in \mathbb{R}^N \mid p \cdot x \leq a \} \), where \( p \in \mathbb{R}^N \), \( a \geq 0 \), and \( K = \{ u \in E \mid p \cdot u(x) \leq a \; \text{for a.a.} \; x \in \Omega \} \). Let \( f \in K \) and put \( u = J_h(f) \). By definition \( u \in D(A) \) and \( u - hAu = f \). Let \( \bar{f}(x) := p \cdot f(x), \; \bar{u}(x) := p \cdot u(x) \) for \( x \in \Omega \). Then \( \bar{f} \leq a \) a.e., \( \bar{u} \in H^2 \cap H^1_0(\Omega) \) and for every \( \xi \in H^1_0(\Omega) \)
\[
\int_\Omega (\bar{u}(x) - a)\xi(x) \, dx = \int_\Omega (\bar{f}(x) - a)\xi(x) \, dx - h \int_\Omega \nabla(\bar{u} - a)(x)\nabla\xi(x) \, dx.
\]
Taking $\xi = (\bar{u} - a)_+ := \max \{0, \bar{u} - a\}$, we have $\xi \in H^1_0(\Omega)$ and $\nabla \xi = \chi \nabla (\bar{u} - a)$ by [18, Cor. 1.3.6], where $\chi$ is the indicator of the set $\{\bar{u} > a\}$, and

$$0 \leq \int_{\Omega} (\bar{u} - a)^2 dx = \int_{\Omega} (\bar{f}(x) - a)(\bar{u} - a)_+ dx - h \int_{\{\bar{u} > a\}} |\nabla (\bar{u} - a)_+|^2 dx \leq 0.$$ 

Hence $\bar{u} \leq a$ a.e. and $J_h(f) \in K$.

By the Post-Widder formula (see [28, Corollary 5.5, 5.6]) $K$ is also semigroup invariant, i.e. $S(t)(K) \subset K$ for all $t \geq 0$. The physical meaning is that if the reaction term $F$ vanishes, then the pure diffusion process $u(t) := S(t)x, x \in K$, survives in $K$.

(b) In view of $(\varphi_1)$, for each $t \in [0, T]$ and $u \in K$, the set $F(t, u)$ is nonempty since the map $\Omega \ni x \mapsto \varphi(t, x, u(x)) \subset \mathbb{R}^N$ is measurable and, thus, has a selector $v$ which belongs to $E$ in view of $(\varphi_2)$. Moreover, the set $F(t, u)$ is closed and convex; $F(t, u)$ is weakly compact since, by $(\varphi_2)$, it is bounded and $E$ is reflexive. Observe, however, that $F(t, u)$ is never compact. Arguing as in [42, Lemma 4.1], one proves that $F$ is $H$-upper semicontinuous. Therefore (see e.g. [14, Prop. 2.3]), $F$ is upper semicontinuous when the target space is endowed with the weak topology: given sequences $(t_n) \subset [0, T), (u_n) \subset E$ and $v_n \in F(t_n, u_n)$, if $t_n \to t \in [0, T], u_n \to u \in E$, there is a subsequence $(v_{n_k})$ such that $v_{n_k} \to y_0 \in F(t, u)$ ($\to$ denotes the weak convergence in $E$). This implies that given a weakly closed $C \subset E$, the preimage $F^{-1}(C)$ is closed. As a consequence, we see that if $U \subset E$ is open then $F^{-1}(U)$ is an $F$-set since $U$ may be represented as the union of countable many closed balls. Hence $F$ is product-measurable.

By $(\varphi_2)$ for all $t \in [0, T]$ and $u \in K$,

$$\sup_{v \in F(t, u)} \|v\| \leq a(t) + b\|u\|,$$

where $a(t) := \|a(t, \cdot)\|_{L^2}$ and $b := \beta$.

(c) It is well-known that $C \ni y \mapsto T_C(y)$ is lower semicontinuous. Let $u \in K$ and $t \in [0, T]$. Then maps $T_C(u(\cdot))$ and $\varphi(t, \cdot, u(\cdot))$ are measurable and so is the intersection $T_C(u(\cdot)) \cap \varphi(t, \cdot, u(\cdot))$. In view of $(\varphi_3)$ and the Kuratowski–Ryll-Nardzewski theorem, there is $v \in E$ such that $v(x) \in \varphi(t, x, u(x)) \cap T_C(u(x))$ for almost all $x \in \Omega$. Hence $v \in F(t, u) \cap T_K(u)$ since, by [7, Cor. 8.5.2], $T_K(u) = \{v \in E \mid v(x) \in T_C(u(x)) \text{ a.e. in } \Omega\}$. In other words, for all $t \in [0, T]$ and $u \in K$,

$$F(t, u) \cap T_K(u) \neq \emptyset.$$ 

(d) Recall that a continuous $u : [0, T] \to K$ is a *mild solution* to (7) if

$$u(t) = S(t)x_0 + \int_0^t S(t-s)w(s) \, ds, \quad t \in [0, T],$$

where $w \in L^1(0, T; E)$ and $w(s) \in F(s, u(s))$ a.e. on $[0, T]$ (it is easy to see that such $w$ exists; observe that $w$ belongs actually to $L^2(0, T; E)$ in view of (9)). In particular $u$ is a unique mild solution to

$$\dot{u} = Au + w(t), \quad u(0) = x_0 \in E.$$ 

Observe that (in both cases of the Dirichlet or Neumann boundary conditions) $-A$ is the subdiifferential of the lower semicontinuous quadratic functional $\gamma : E \to \mathbb{R}$. 


where $\gamma(u)$ is the Frobenius square of the derivative $Du = [\partial_t u]$ in (3). In view of the classical result of Brezis in [16, Théorème 3.6] (see also [51, Theorem 1.9.3] or [50, Proposition III.2.5, Corollary III.2.4], $u$ is a unique strong solution to (12), i.e. $u(0) = x_0$, $u(t) \in D(A)$ for $t \in (0,T]$, $u \in W^{1,1}_{loc}((0,T],E)$ and $u'(t) = Au(t) + w(t)$ for a.a. $t \in (0,T)$ ($u'(t)$ denotes the strong derivative of $u$ which exists a.a. since $u \in W^{1,1}_{loc}$). This implies that actually any mild solution $u$ of (7) (treated as a function of variables $t \in [0,T]$ and $x \in \Omega$) is actually a required strong solution to problem (2). \hfill \Box

2. Existence and structure of solutions. In this Section we are going to consider an abstract version of (7), i.e. a semilinear differential inclusion

$$u'(t) \in Au + F(t,u), \quad t \in [0,T], \quad u \in K \tag{14}$$

subject to the initial value problem

$$u(0) = x_0 \in K \subset E, \tag{15}$$

where:

(A) $(E, \|\cdot\|)$ is a separable Banach space and $A : D(A) \to E$ generates a compact $C_0$ semigroup $\{S(t)\}_{t \geq 0}$ of linear operators on $E$;

(K) a closed convex $K \subset E$ is semigroup invariant, i.e. $S(t)(K) \subset K$ for every $t \geq 0$;

(F1) $F : [0,T] \times K \to E$ has convex weakly compact values;

(F2) $F$ is product measurable and for every $t \in [0,T]$, the map $K \ni u \mapsto F(t,u) \in E$ is $H$-upper semicontinuous;

(F3) there are $a \in L^1([0,T],E)$, $b \geq 0$ such that $\sup_{y \in F(t,u)} \|y\| \leq a(t) + b\|u\|$ for $t \in [0,T]$ and $u \in K$;

(F4) $F$ is weakly tangent to $K$, i.e. $F(t,u) \cap T_K(u) \neq \emptyset$ for all $t \in [0,T]$ and $u \in K$ (see (4)).

We will study the existence and the structure of mild solutions to (14) surviving $K$, i.e. such that $u(t) \in K$ for $t \in [0,T]$.

The study of (14) is motivated by (1), (2) and its functional form (7). Assumptions (A), (K), (F1) – (F4) are justified by $(\varphi_1) - (\varphi_3)$ and the setting in front of Remark 2.

Let us briefly comment on these assumptions.

Remark 3. (a) (A) implies that there are $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$. For $h > 0$ and $h\omega < 1$, the resolvent $J_h := (I - hA)^{-1} : E \to D(A) \subset E$ is well-defined and bounded $\|J_h\| \leq M(1 - h\omega)^{-1}$. By [48, Th. 2.3.3], $\{S(t)\}_{t \geq 0}$ is compact (i.e. for any $t > 0$, $S(t)$ is compact) if and only if it is resolvent compact, i.e. for $h > 0$, $h\omega < 1$, $J_h$ is compact and $(0, +\infty) \ni t \mapsto S(t) \in L(E)$ is continuous. The resolvent compactness implies that a bounded sequence $(x_n)$ in $D(A)$ such that the sequence $(Ax_n)$ is bounded possesses a convergent subsequence.

---

3Given two $N \times M$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $A \cdot B := \sum_{i=1}^{N} \sum_{j=1}^{M} a_{ij} b_{ij}$. **KRASNOSEL’SKI FORMULA 301**
if for any $y$ is nonempty. The following characterization is well-known:

$$J_0(y)(t) := \int_0^t S(t-s)y(s)\,ds \quad \text{for } y \in L^1(0,T;E), \; t \in [0,T],$$

sends integrably bounded subsets of $L^1(0,T;E)$ into compact subsets of $C([0,T],E)$.

(c) It is standard to see (comp. e.g. [42, Rem. 3.5.]) that $(K)$ holds if and only if $K$ is resolvent invariant, i.e. $J_h(K) \subset K$ for $h > 0$ with $h\omega < 1$.

Suppose that $E$ is reflexive. Then for any $x \in E$, the metric projection

$$\pi_K(x) = \{ y \in K \mid \| y - x \| = d_K(x) \}$$

is nonempty. The following characterization is well-known: $y \in \pi_K(x)$ if and only if for any $p \in J(x - y)$ and $v \in K$, $\langle v - y, p \rangle \leq 0$, where $J$ stands for the duality map in $E$ (see e.g. [51]). Arguing similarly as in [4, Th. (6.2)] we show that: if for any $x \in D(A)$ there is $y \in \pi_K(x)$ and $p \in J(x - y)$ such that

$$\langle Ax, p \rangle \leq \omega \| x - y \|^2,$$

then $K$ is resolvent invariant. If $M = 1$ (see (a) above) and $K$ is semigroup invariant, then for each $x \in D(A)$, $y \in \pi_K(x)$ and $p \in J(x - y)$ condition (16) is satisfied.

(d) The relevance of the tangency assumption $(F_4)$ and the resolvent invariance has already been discussed. These assumptions are not only justified by the model application, as it was explained in the Introduction, but they are responsible for the so-called viability properties. It is easy to see that, as a consequence of (5), $(K)$ implies that for all $u \in K$

$$T_K(u) \subset T_K^A(u) := \left\{ v \in E \left| \liminf_{t \to 0^+} \frac{1}{t} d_K(S(t)u + tv) = 0 \right. \right\}.$$

Hence $(K)$ together with $(F_4)$ imply that

$$F(t, u) \cap T_K^A(u) \neq \emptyset, \quad u \in K, \; t \in [0,T].$$

The sets $T_K^A(x)$, $u \in K$, were introduced by Pavel [46] and condition (17) was shown to be necessary and sufficient for the viability, i.e. the existence of (mild) solutions to (14) with single-valued continuous $F$ surviving in $K$. This condition is also sufficient for the existence in case of a $H$-upper semicontinuous set-valued perturbation $F$ (see [14, §4.5] and [47]); see also [17, Chap. 9] and [5] for a detailed discussion of different tangency issues and relations to viability. \hfill \Box

We are going to show that the set of all (mild) solutions to (14) surviving in $K$ is an $R_3$-subset of $C([0,T],K)$, the space of continuous functions on $[0,T]$ taking values in $K$, i.e. can be represented as the intersection of a decreasing sequence of compact absolute retracts (see also e.g. [31, p. 14] and [37] for a detailed discussion of the class of $R_3$-sets). In particular $R_3$-sets are nonempty.

**Remark 4.** It is easy to see that a compact $X \subset C([0,T],K)$ is $R_3$ as a subset of $C([0,T],K)$ if and only if $X$ is $R_3$ as a subset of $C([0,T],E)$. \hfill \Box

The question of topological characterization of solution to (14) with $K = E$ was discussed by numerous authors (see e.g. monographies [35, 36, 39, 27] and Górniewicz [32] and references therein) and a diversity of results have been obtained. Most of these results relied on a restrictive assumption about the compactness of values of $F$. As noted in Remark 2 (b) the abstract substitution operator $F$ related to compact convex valued perturbation $\varphi$ has only convex weakly compact values.
In Bothe [14] (see also some of his earlier papers) and Chen et al. [19] the case of weakly upper semicontinuous and weakly compact convex case was studied.

Here we deal additionally with the presence of constraints. Suppose for a while that $F$ is defined on $[0, T] \times E$. As mentioned above assumptions $(K)$ and $(E_4)$ imply the existence of solutions in $K$, but certainly do not prevent that some solutions escape from $K$.

**Example 1.** Let $E = \mathbb{R}^2$, $K = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$, let $F(x,y) = (1, 2\sqrt{|y|})$ on $\mathbb{R}^2$ and $A \equiv \{1\}$. Then $F(u) \in T_K(u)$ for all $u \in K$. The only solution to $\dot{u} = F(u)$, $u(0) = (0,0)$ surviving in $K$ is $u_0(t) = (t, t^2)$, $t \geq 0$, while e.g. the solution $u(t) = (t, 0)$ escapes immediately from $K$.

Hence results concerning the structure of the set of ‘unconstrained’ solutions gives no information about the structure of solutions surviving in $K$. Such problems have been addressed by Bader [9], Bader, Kryszewski [10] and Bothe [14, Thm. 5.2]. The important drawback is that in [9] there is an assumption $\text{int} K \neq \emptyset$, in both [9, 10] $F$ has compact values, while in [14, Thm. 5.2] $F$ is $H$-upper semicontinuous and there is an extra assumption that $E^*$ is uniformly convex.

**Theorem 2.1.** For any $x_0 \in K$, the set $X_0$ of all mild solutions in $K$ of (14) starting at $x_0$ is an $R_3$ subset of $C([0, T], K)$.

**Proof.** In the proof the following characterization will be used (see [14, Lem. 5.1.]):

If a decreasing family $\{X_n\}_{n=1}^\infty$ of closed contractible subset of a metric space is such that the Hausdorff measure of noncompactness $\beta(X_n) \to 0$, then $X_0 := \bigcap_{n=1}^\infty X_n$ is an $R_3$-set.

**Step 1.** Take a sequence $(\varepsilon_n)_{n \geq 1}$ in $(0, 1)$ such that $\varepsilon_n \searrow 0$. Since $K \ni x \mapsto T_{K}(x)$ is Carathéodory lower semicontinuous we can apply Theorem 1.2: for every $n \geq 1$, there is a Carathéodory $f_n : [0, T] \times K \to E$ such that $f_n(t,x) \in T_{K}(x)$ and $f_n(t,x) \in F(\{t\} \times B_K(x, \varepsilon_n)) + B_E(0, \varepsilon_n)$ for all $t \in [0, T]$, $x \in K$.

For each $k \geq 1$, by a version of the Scorza-Dragoni theorem (cf. [8, 43]), there is a closed subset $\bar{I}_k \subset [0, T]$ such that the Lebesgue measure $\mu([0, T] \setminus \bar{I}_k) \leq \min\{\varepsilon_n/2^{k-m+1} \mid m = 1, \ldots, k\}$ and the restriction $f_k|_{\bar{I}_k \times K}$ is continuous (with respect to both variables). Let $I_n := \bigcap_{k \geq n} \bar{I}_k$, $n \geq 1$. The family $\{I_n\}$ increases, consists of compact sets and $f_n|_{I_n \times K}$ is continuous. Moreover $\mu([0, T] \setminus I_n) \leq \varepsilon_n$ and $\mu \left( \bigcup_{n \geq 1} I_n \right) = T$.

Fix $n \geq 1$. The complement of $I_n$ in $(0, T)$ is a countable union of open intervals $(a_k, b_k)$, $k \geq 1$, i.e. $(0, T) \setminus I_n = \bigcup_{k \geq 1} (a_k, b_k)$. Define $\hat{f}_n : [0, T] \times K \to E$ by

$$\hat{f}_n(t,x) := \begin{cases} f_n(t,x) & \text{for } t \in I_n, \ x \in K; \\ \frac{b_k-a_k}{b_k-a_k} f_n(a_k,x) + \frac{t-a_k}{b_k-a_k} f_n(b_k,x) & \text{for } t \in [a_k, b_k], \ x \in K. \end{cases}$$

Obviously, $\hat{f}_n$ is continuous and for $t \in [0, T]$ and $x \in K$ (below conv stands for the convex hull)

$$\hat{f}_n(t,x) \in \text{conv} F(t', x') + B_E(0, \varepsilon_n) \quad (18)$$

for some $t' \in [0, T]$ and $x' \in K$ with $|t' - t|, ||x' - x|| < \varepsilon_n$ since $b_k - a_k < \varepsilon_n$ for all $k \geq 1$. If $t \in I_n$, $x \in K$ then

$$\hat{f}_n(t,x) = f_n(t,x) \in F(\{t\} \times B_K(x, \varepsilon_n)) + B_E(0, \varepsilon_n). \quad (19)$$
For any \( n \geq 1 \) we easily find a continuous \( \alpha_n : [0, T] \times K \to (0, \infty) \) such that if a function \( g : [0, T] \times K \to E \) for all \( t \in [0, T], x \in K \) satisfies
\[
g(t, x) \in \hat{f}_n(t', x') + B_E(0, \alpha_n(t, x),)
\]
where \( t' \in [0, T], x' \in K \) with \( |t' - t|, ||x' - x|| < \alpha_n(t, x) \), then \( g(t, x) \in \hat{f}_n(t, x) + B_E(0, \varepsilon_n) \) on \([0, T] \times K\). Applying Lemma 1.1 to \( F = \hat{f}_n \) and \( \alpha = \alpha_n \), we get a locally Lipschitz \( g = g_n : [0, 1] \times K \to E \) such that
\[
g_n(t, x) \in T_K(x)
\]
and (20) holds, i.e.,
\[
g_n(t, x) \in \hat{f}_n(t, x) + B_E(0, \varepsilon_n).
\]
In view of (21), (A) and [12, Thm 1, Lemma 2 (b)], the problem
\[
\dot{u} = Au + g_n(t, u), \quad u(0) = x_0
\]
admits a unique (mild) solution \( \bar{u}_n \in C([0, T], K) \).

For any \( n \geq 1 \), let \( X_n \) be the set of mild solutions (in \( K \)) of the problem
\[
\begin{cases}
\dot{u} \in Au + F_n(t, u), \\
u(0) = x_0,
\end{cases}
\]
where \( F_n : [0, T] \times K \to E \) is given by
\[
F_n(t, x) := F(\{t\} \times B_K(x, \varepsilon_n)) + B_E(0, 2\varepsilon_n) \quad \text{for} \ t \in I_n, \ x \in K,
\]
and
\[
F_n(t, x) := \text{conv} \left\{ F(t', x') \mid t' \in [0, T], x' \in K, \text{where } |t' - t|, ||x' - x|| < \varepsilon_n \right\} + B_E(0, 2\varepsilon_n), \quad \text{for} \ t \in [0, T] \setminus I_n, \ x \in K.
\]

By (22) and (18), \( g_n(t, x) \in F_n(t, x) \) on \([0, T] \times K\). Hence \( \bar{u}_n \in X_n \) and thus \( X_n \neq \emptyset \). Clearly,
\[
X_0 \subset \bigcap_{n=1}^{\infty} X_n.
\]

**Step 2.** We shall see that given a sequence \( (u_n) \), where \( u_n \in X_n \) for \( n \geq 1 \), then (up to a subsequence) \( u_n \to u_0 \in X_0 \). To this end, observe that for each \( n \geq 1 \) there is \( u_n \in L^1(0, T; E) \) such that \( u_n(t) \in F_n(t, u_n(t)) \) for a.e. \( t \in [0, T] \) and \( u_n(t) = S(t)x_0 + J_0(w_n)(t) \) for \( t \in [0, T] \) (see Remark 3 (b)). The Gronwall inequality and \( (F_3) \) imply that \( \sup_{n \geq 1} \|u_n\|_{\infty} \leq C \) (here \( \|\cdot\|_{\infty} \) is the norm in \( C([0, T], E) \)) for some \( C > 0 \). Thus, again by \( (F_3) \), the set \( \{w_n\}_{n \geq 1} \) is integrably bounded and, by Remark 3 (b), the set \( \{u_n\}_{n \geq 1} \) is relatively compact in \( C([0, T], E) \), i.e., (up to a subsequence) \( u_n \to u_0 \) uniformly; clearly \( u_0 \in C([0, T], K) \).

Observe now that the set \( \{\chi_n w_n\}_{n \geq 1} \), where \( \chi_n \) is the indicator of \( I_n \), is integrably bounded. If \( t \in \bigcup_{n \geq 1} I_n \), i.e., \( t \in I_n \) for large \( n \), say \( n \geq N \), then
\[
\chi_n(t) w_n(t) = w_n(t) \in F(\{t\} \times B_K(u_n(t), \varepsilon_n)) + B_E(0, 2\varepsilon_n),
\]
\[
\|u_0(t) - u_n(t)\| \leq \|u_0(t) - u_n(t)\| + \|u_n(t) - v_n\| \to 0 \quad \text{as} \ n \to \infty.
\]
Note that \( \{\chi(t) w_n\}_{n \geq 1} \subset F(\{t\} \times \{v_n\}_{n \geq 1}) + \{b_n\}_{n \geq 1} \cup \{0\} \) and that, by the \( H \)-upper semicontinuity of \( F(t, \cdot) \) and since \( F \) has weakly compact values, the set \( F(\{t\} \times \{v_n\}_{n \geq N}) \) is relatively weakly compact in \( E \). By the weak compactness
Some topological tools.

On the other hand \(||J_0((1 - \chi_n)w_n)|| \to 0\), since the measure \(\mu([0, T] \setminus I_n) \to 0\) as \(n \to \infty\). Therefore,

\[
u_n - S(\cdot)x_0 = J_0(\chi_n w_n) + J_0((1 - \chi_n)w_n) \to J_0(w_0).
\]

This shows \(u_0(t) = S(t)x_0 + \int_0^t S(t - s)w_0(s)\,ds\) for \(t \in [0, T]\). In view of (24) and the ‘convergence theorem’ [6, Th. 3.2.6], \(w_0(t) \in F(t, u_0(t))\) for a.e. \(t \in [0, T]\), i.e. \(u_0 \in X_0\).

The assertion we have just proved and (23) implies that \(\sup_{v \in X_n} d_{X_n}(v) \to 0\).

**Step 3.** Now we shall show that the closure \(\text{cl} X_n\) is contractible. To see this, fix \(n \geq 1\) and recall the above constructed locally Lipschitz \(g_n : [0, T] \times K \to E\) being tangent to \(K\) and having the sublinear growth. Take \(z \in [0, T]\) and \(y \in K\). The problem

\[
\begin{cases}
\dot{u} = A u + g_n(t, u), \\
u(z) = y,
\end{cases}
\]

admits a unique solution \(v(\cdot, z, y) : [z, T] \to K\). The strong continuity of \(\{S(t)\}_{t \geq 0}\) along with local lipschitzeanity of \(g_n\) imply that \(v(\cdot, z, y)\) depends continuously on \(z \in [0, T]\) and \(y \in K\). Precisely, given \(\varepsilon > 0\), \(z_0 \in [0, T]\) and \(y_0 \in K\) there is \(\delta > 0\) such that \(|v(t; z_0, y_0) - v(t; z, y)| < \varepsilon\) for all \(t \in [\max\{z_0, z\}, 1]\), if \(|z - z_0| < \delta, |y - y_0| < \delta|\).

Let us consider the homotopy \(h : \text{cl} X_n \times [0, 1] \to C([0, T], K)\) given by

\[
h(u, \lambda)(s) := \begin{cases} u(s) & \text{for } s \in [0, \lambda T]; \\
v(s; \lambda T, u(\lambda T)) & \text{for } s \in [\lambda T, T]
\end{cases}
\]

where \(u \in \text{cl} X_n, \lambda \in [0, 1]\) and \(s \in [0, T]\). It is easy to see that \(h\) is well-defined, continuous (comp. [14, Th. 5.1]) and \(h(X_n \times [0, 1]) \subset X_n\) since \(g_n\) is the selection of \(F_n\); thus \(h(\text{cl} X_n \times [0, 1]) \subset \text{cl} X_n\). Furthermore, \(h(\cdot, 0) = v(\cdot, 0, u(0)) = v(\cdot, 0, x_0)\) and \(h(\cdot, 1) = \text{id}_{\text{cl} X_n}\) proving the contractibility of \(\text{cl} X_n\).

**Example 2.** If we are in the setting of Subsection 1.2 and problem (2) then it follows that the set of all strong solution to (2) surviving in \(C\) is an \(R_3\)-set in \(C([0, T], L^2(\Omega, \mathbb{R}^N))\). \(\Box\)

3. Some topological tools. Let the solution map \(\Sigma : K \to C([0, T], K)\) assign to \(x \in K\) the set of all solutions in \(K\) to (14) starting at \(x\). For a fixed \(t \in [0, T]\), the evaluation \(e_t : C([0, T], K) \to K, e_t(u) := u(t)\) for \(u \in C([0, T], K)\), is well-defined and continuous. With (14) we associate the Poincaré \(t\)-operator \(\Sigma_t : K \to K\),

\[
\Sigma_t := e_t \circ \Sigma, \quad \text{i.e., } \Sigma_t(x) = \{u(t) \mid u \in \Sigma(x)\}, \quad x \in K. \tag{25}
\]

By Theorem 2.1, \(\Sigma_t\) is the superposition of a set-valued map \(\Sigma\) with \(R_3\)-values with a continuous map \(e_t\). It makes sense to discuss the class of such maps carefully and introduce appropriate tools to study their fixed points. Here, after [30], we recall some relevant concepts and results (comp. also [31]).
3.1. c - admissible maps. A compact metric space $S$ is cell-like if it can be represented as the intersection of a decreasing sequence of compact contractible spaces. The following conditions are equivalent (see e.g. [37, 44]): $S$ is cell-like; $S$ has the shape of a point; $S$ is an $R_3$-set; $S$ has the $UV^\infty$-property, i.e. if $S$ is embedded into an ANR, then it is contractible in any of its neighbourhoods. It is clear that cell-like sets are acyclic with respect to any continuous homology or cohomology theory, e.g. the Čech homology, the Čech or the Alexander-Spanier cohomology. Theorem 2.1 states that the set of all (mild) solutions to (14) starting at $x_0 \in K$ and surviving in $K$ is cell-like

Let $X, Y$ be metric spaces; an upper semicontinuous map $\varphi : X \to Y$ is cell-like if $\varphi(x), x \in X$, is cell-like. A map $\varphi : X \to Y$ is c-admissible if there is a metric space $Z$, a cell-like map $\psi : X \to Z$ and a continuous $f : Z \to Y$ such that $\varphi = f \circ \psi$. Equivalently (see [30, Section 3]), $\varphi : X \to Y$ is c-admissible if it is represented by a c-admissible pair $(p, q)$, i.e., $\varphi(x) = q(p^{-1}(x))$ for $x \in X$, where $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$, $\Gamma$ is a metric space, $p, q$ are continuous and $p$ is a proper (i.e. the preimages of compact sets are compact) surjection with cell-like fibres $p^{-1}(x), x \in X$. It is easy to see that c-admissible maps are admissible in the sense of [31, Definition (40.1)].

Properties of a c-admissible $\varphi$ strongly depend on a decomposition $\varphi = f \circ \psi$ or a pair $(p, q)$ representing it. When studying c-admissible maps one has to take into account representing pairs (for a detailed discussion of c-admissible maps, related topics and some references – see [30]). In particular: if $\varphi : X \to Y$ is cell-like, then the canonical pair $(p_\varphi, q_\varphi)$, where $p_\varphi : Gr(\varphi) \to X, q_\varphi : Gr(\varphi) \to Y$ are projections and $Gr(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ is the graph of $\varphi$, is c-admissible and represents $\varphi$. A c-admissible pair $(p, q)$ is compact if $\text{cl } q(p^{-1}(B))$ is compact for any bounded $B \subset X$. It is clear that a c-admissible $\varphi : X \to Y$ is compact if and only if it is represented by a compact c-admissible pair.

After [30, Definition 3.5] we say that c-admissible pairs $X \xleftarrow{p_k} \Gamma_k \xrightarrow{q_k} Y, k = 0, 1,$ (and set-valued maps represented by them) are c-homotopic (written $(p_0, q_0) \simeq (p_1, q_1)$) if there is a c-admissible pair $X \times \{0, 1\} \xleftarrow{\varphi} \Gamma \xrightarrow{q} Y$ and continuous maps $j_k : \Gamma_k \to \Gamma, k = 0, 1,$ such that the following diagram

$$
\begin{array}{ccc}
X & \xleftarrow{p_0} & \Gamma_0 \\
\downarrow{i_0} & & \downarrow{j_0} \\
X \times \{0, 1\} & \xleftarrow{p} & \Gamma & \xrightarrow{q} & Y, \\
\downarrow{i} & & \downarrow{j_1} & & \downarrow{q_1} \\
X & \xleftarrow{p_1} & \Gamma_1 \\
\end{array}
$$

where $i_k(x) := (x, k)$ for $x \in X$ and $k = 0, 1,$ is commutative. The pair $(p, q)$ is called a c-homotopy joining $(p_0, q_0)$ to $(p_1, q_1)$.

**Lemma 3.1.** The solution map $\Sigma$ is cell-like and maps bounded sets onto bounded ones.

**Proof.** The second assertion follows from the Gronwall inequality and $(F_3)$. In view of Theorem 2.1, we only need to show that $\Sigma$ is upper semicontinuous. Let $x_n \to x \in K$ and $u_n \in \Sigma(x_n)$ for $n \geq 1$. Then $u_n = S(\cdot)x_n + J_0(u_n)$ for some $w_n \in L^1(0, T; E)$ such that $w_n(t) \in F(t, u_n(t))$ for a.e. $t \in [0, T]$. $(F_3)$ and the Gronwall inequality imply that $\{u_n\}_{n \geq 1}$ is bounded, so $\{w_n\}_{n \geq 1}$ is integrably bounded. As in Step 2 of the proof of Theorem 2.1, (up to a subsequence) $u_n - S(\cdot)x_n \to u - S(\cdot)x$ in
C([0, T], E). Thus, again up to a subsequence \( w_n \to w \in L^1(0, T; E) \) and \( w(t) \in F(t, u(t)) \) for a.a. \( t \in [0, T] \). As a result, \( u = S(\cdot)x + J_0(w) \in \Sigma(x) \).

In what follows \( \Sigma \) will be identified with its canonical pair

\[
K \xrightarrow{p_\Sigma} \Gamma \xrightarrow{q_\Sigma} C([0, T], K), \quad \Sigma(x) = q_\Sigma(p_\Sigma^{-1}(x)), \quad x \in K;
\]

this means that \( \Gamma := \{(x, u) \in K \times C([0, T], K) \mid u \in \Sigma(x)\} \) is the graph of \( \Sigma \), \( p_\Sigma \) and \( q_\Sigma \) are the projections onto \( K \) and into \( C([0, T], K) \), respectively.

Clearly, the Poincaré \( t \)-map \( \Sigma_t \) (see (25)) is \( c \)-admissible (cf. [30, Rem. 3.4. (2)])

\[
\text{it is represented by the } c \text{-admissible pair}
\]

\[
K \xrightarrow{p_t} \Gamma \xrightarrow{q_t} K, \text{ where } p_t := p_\Sigma, \ q_t := e_t \circ q_\Sigma.
\]

**Remark 5.** (1) The mapping \( K \times [0, T] \ni (t, x) \mapsto \Sigma_t(x) \subset K \) is \( c \)-admissible. It is represented by the pair

\[
K \times [0, T] \xrightarrow{p} \Gamma \xrightarrow{q} K,
\]

where \( p := p_\Sigma \times \text{id}_{[0, T]}, \ q(\gamma, t) := e_t \circ q_\Sigma(\gamma) \) for \( t \in [0, T], \ \gamma \in \Gamma \).

(2) For any numbers \( 0 < a \leq b \leq T \), the restriction \( [a, b] \times K \ni (t, x) \mapsto \Sigma_t(x) \subset K \) is compact, what is a consequence of the compactness of the semigroup \( \{S(t)\}_{t \geq 0} \).

Observe, in particular, that \( \Sigma_t \) and the pair \( (p_t, q_t) \) are compact.

**Remark 6.** (a) A parameterized version of the above results will also be useful.

Let \( \Lambda \) be a compact metric space and let \( F : \Lambda \times [0, T] \times K \to E \) be (product) measurable, \( F(\cdot, t, \cdot), t \in [0, T], \) be \( H \)-upper semicontinuous and \( F(\lambda, \cdot, \cdot), \lambda \in \Lambda \), satisfy assumptions \((F_1) - (F_3)\). Consider a parameterized problem

\[
u(t) \in Au(t) + F(\lambda, t, u(t)), \quad u \in K, \ t \in [0, T].
\]

Then all above results remain true: for any \( x \in K \), the set \( \Sigma(\lambda, x) \) of all mild solutions to this problem starting at \( x \) is an \( R_3 \)-set; arguing as in Lemma 3.1 one shows that the solution map \( \Sigma : \Lambda \times K \to C([0, T], K) \) is cell-like. Hence, for all \( t \in [0, T] \), the Poincaré map \( \Sigma_t : \Lambda \times K \to K \) given by \( \Sigma_t(\lambda, x) = \{u(t) \mid u \in \Sigma(\lambda, x)\} \) is compact and \( c \)-admissible: it is represented by the pair

\[
\Lambda \times K \xrightarrow{p} \Gamma \xrightarrow{q} K,
\]

where here \( \Gamma \) is the graph of \( \Sigma, \ p \) is the projection onto \( \Lambda \times K \) and \( q_t(\lambda, x, u) = u(t) \) for any \( (\lambda, x, u) \in \Gamma \) (i.e. \( u \in \Sigma(\lambda, x), \ \lambda \in \Lambda, \ x \in K \)).

(b) In particular, if \( \Lambda \) is the unit interval \([0, 1]\), then for any \( t \in [0, T] \), the \( c \)-admissible pair representing \( \Sigma_t : [0, 1] \times K \to K \) provides a \( c \)-homotopy joining the \( c \)-admissible pairs representing \( \Sigma^0_t \) and \( \Sigma^1_t \), where \( \Sigma^i_t \) is the \( t \)-Poincaré operator associated to \( F_i := F(i, \cdot, \cdot), \ i = 0, 1 \).

**3.2. Fixed point index for \( c \)-admissible maps.** Given a compact \( c \)-admissible pair \( \text{cl}V \xleftarrow{p} \Gamma \xrightarrow{q} E \), where \( V \subset E \) is open bounded, such that \( x \notin q(p^{-1}(x)) \) for \( x \in \partial V \), the fixed point index \( \text{Ind}(p, q, V) \) is well-defined (cf. [30, Th. 4.5]). This index has the usual properties such as: the existence (of fixed points), the localization, the additivity and the homotopy invariance (see [30]).

It is easy to get a generalization to a constrained case in a standard way. Let \( K \subset E \) be convex closed and let \( U \subset K \) be (relatively) open and bounded. Let \( r : E \to K \) be an arbitrary retraction and \( j : K \hookrightarrow E \) be the inclusion. Given a \( c \)-admissible compact pair \( K \xleftarrow{p} \Gamma \xrightarrow{q} K \) such that \( x \notin q(p^{-1}(x)) \) for \( x \in \partial_K U \) (\( \text{cl}_K U \) and \( \partial_K U \) denote the closure and the boundary of \( U \) in \( K \), respectively),
we let $U_r := r^{-1}(U) \cap B$, where $B$ is open bounded and $B \supset U$, $\Gamma_r := \{(x, \gamma) \in \partial U \times \Gamma \mid r(x) = p(\gamma)\}$, $p_r : \Gamma_r \to \partial U_r$ and $q_r : \Gamma_r \to E$ by $p_r(x, \gamma) := x$ and $q_r(x, \gamma) := j_q(q(\gamma))$ for $(x, \gamma) \in \Gamma_r$. Note that $q_r \circ p_r^{-1} = j_q \circ p^{-1} \circ \partial |_{\partial U_r}$, the pair $(p_r, q_r)$ is $c$-admissible and $x \not\in q_r(p_r^{-1}(x))$ for $x \in \partial U_r$. Thus, we are in a position to define the constrained fixed point index by

$$\text{Ind}_K((p, q, U)) := \text{Ind}(p_r, q_r, U_r).$$

It is easy to see that this definition is correct, i.e. it does not depend on the choice of $r$ and $B$; furthermore $\text{Ind}_K$ has the same properties as $\text{Ind}$ does.

**Remark 7.** (i) In particular, if two $c$-admissible pairs $K \xleftarrow{\quad} \Gamma_j \xrightarrow{\quad} K$, $j = 0, 1$, are $c$-homotopic and the $c$-homotopy $K \times [0, 1] \xrightarrow{\quad} \Gamma \xrightarrow{\quad} K$ is compact and such that $x \not\in q(p^{-1}(x, t))$, for $x \in \partial K U, t \in [0, 1]$, then $\text{Ind}_K((p_j, q_j), U)$, $j = 0, 1$, are defined and equal.

(ii) If a $c$-admissible pair $(p, q)$ represents a compact single-valued $f : K \to K$ (that is $f(x) = q(p^{-1}(x))$ on $K$) and $x \not\in f(x)$ for $x \in \partial U$, then it can be proved (see [30]) that $\text{Ind}_K((p, q), U) = \text{Ind}_K(f, U)$, where $\text{Ind}_K(f, U)$ stands for the fixed point index as defined in [33, §12]. Observe that, in particular, $f$ is represented by the pair $K \xleftarrow{\text{id}} K \xrightarrow{f} K$. The same holds true if $K = E$; then $\text{Ind}_K(f, U)$ is the usual Leray-Schauder fixed point index of $f$ on $U$.

(iii) If $\text{Ind}_K((p, q), U) \neq 0$ then the map $\varphi$ represented by $(p, q)$ has a fixed point, i.e. there is $x \in U$ such that $x \in q(p^{-1}(x))$. If $U = K$ is bounded then any $c$-admissible pair $(p, q)$ is $c$-homotopic to the constant map $f(\cdot) \equiv u_0 \in K$. Hence $\text{Ind}_K((p, q), U) = 1$ the existence of fixed points follows.

3.3. The degree of the right hand side. We will construct a homotopy invariant (the so-called constrained topological degree) responsible for the existence of zeros of maps of the form $A + G$, where:

(G1) $G : K \to E$ is $H$-upper semicontinuous, has convex weakly compact values, maps bounded sets onto bounded ones and $G(x) \cap T_K(x) \neq \emptyset$ for every $x \in K$, i.e. $G$ is tangent to $K$;

(G2) $K \subset E$ is convex closed; $A : D(A) \to E$ satisfies (A) and (K).

Let $U \subset K$ be bounded and relatively open in $K$. We assume that

$$0 \not\in A x + G(x) \quad \text{for} \quad x \in D(A) \cap \partial U; \quad (28)$$

here $\partial U = \partial_K U$ stands for the boundary of $U$ relative to $K$.

**Lemma 3.2.** There is $\alpha_0 > 0$ such that if $0 < \alpha \leq \alpha_0$, then

$$0 \not\in A x + G(B_K(x, \alpha)) + B_{E}(0, \alpha) \quad \text{for} \quad x \in D(A) \cap \partial U.$$ 

**Proof.** Fix $h > 0$ with $h\omega < 1$ and suppose to the contrary that for $n \geq 1$ there is $x_n \in D(A) \cap \partial U$, $y_n \in G(x_n)$, where $\|x_n - \bar{x}_n\| < 1/n$ and $\xi_n \in E$ with $\|\xi_n\| < 1/n$ such that

$$0 = Ax_n + y_n + \xi_n \quad \iff \quad x_n = J_h(x_n + h(y_n + \xi_n)).$$

Clearly, the sequence $(y_n)_n$ is bounded and so is $(Ax_n)$. In view of Remark 3 (a), up to a subsequence, $x_n \to x_0 \in \partial U$ and $\bar{x}_n \to \bar{x}_0$. $H$-upper semicontinuity of $G$ (see Remark 2 (b)) implies that, again up to a subsequence, $y_n \to y_0 \in G(x_0)$. Moreover, $x_n = J_h(x_n + h(y_n + \xi_n)) \to J_h(x_0 + h\bar{y}_0)$, since $J_h$ is compact. Thus, $x_0 \in D(A) \cap \partial U$ and $0 = Ax_0 + y_0$: a contradiction with (28).
Lemma 3.3. If a continuous mapping \( g : K \to E \) is tangent to \( K \) then for every \( x \in K \) we have
\[
\lim_{h \to 0^+} \frac{d_K(J_h(y + hg(y)))}{h} = 0.
\]
Proof. Take \( x \in K \). The continuity and the tangency of \( g \) together with [7, Prop. 4.2.1] imply
\[
\lim_{h \to 0^+, y \to x, y \in K} \frac{d_K(y + hg(y))}{h} = 0 \quad \text{for} \quad x \in K.
\]
Let \( \varepsilon > 0 \); there is \( \delta > 0 \) such that if \( \|y - x\| < \delta \), \( 0 < h < \delta \) and \( h\omega < 1 \), then
\[
d_K(y + hg(y)) < \frac{\varepsilon}{2M} h \quad \text{and} \quad \frac{M}{1 - h\omega} < 2M.
\]
(see Remark 3 (a)). Take \( k \in K \) with \( \|y + hg(y) - k\| < \varepsilon h(2M)^{-1} \). Putting \( e := h^{-1}(k - y - hg(y)) \), we have \( \|e\| < \varepsilon / 2M \) and \( y + h(g(y) + e) = k \in K \). Then \( J_h(y + h(g(y) + e)) \in K \), in view of (K). Thus,
\[
d_K(J_h(y + hg(y))) \leq \|J_h(y + hg(y)) - J_h(y + h(g(y) + e))\| \leq h\|J_h\|\|e\| < h\varepsilon
\]
if \( \|y - x\| < \delta \), \( 0 < h < \delta \). \( \Box \)

Let \( r : E \to K \) be a retraction such that \( \|x - r(x)\| \leq 2d_K(x) \), for \( x \in E \); such retractions exist (see [11, Corollary II.3.4]).

Lemma 3.4. Assume that \( 0 < \alpha \leq \alpha_0 \) and \( g : K \to E \) is a continuous and tangent \( \alpha \)-approximation of \( G \), i.e. \( g(x) \in G(B_K(x, \alpha)) + B_E(0, \alpha) \) for \( x \in K \). Then there is \( h_0 > 0 \) with \( h_\omega < 1 \) such that for \( h \in (0, h_0) \)
\[
x \neq r \circ J_h(x + hg(x)) \quad \text{for} \quad x \in \partial U.
\]
Proof. If not then for each \( n \geq 1 \) there is \( x_n \in \partial U \) such that \( x_n = r(u_n) \), where \( u_n := J_{h_n}(x_n + h_ng(x_n)) \in D(A) \) and \( 0 < h_n < 1/n \) with \( h_n\omega < 1 \). For each \( n \geq 1 \) we have
\[
u_n - r(u_n) = u_n - x_n = h_n(Au_n + g(x_n))
\]
and, recalling that \( J_h(x_n) \in K \),
\[
\|u_n - r(u_n)\| \leq 2d_K(u_n) \leq 2\|u_n - J_h(x_n)\| = 2\|J_h(x_n + h_ng(x_n)) - J_h(x_n)\| \leq 2h_n\|J_h\|\|g(x_n)\|.
\]
Hence the sequence \((Au_n)\) is bounded since so is \((g(x_n))\). Clearly, \((u_n)\) is bounded, too. Therefore, by Remark 3 (a), (up to a subsequence) \( u_n \to u_0 \). Thus, \( x_n = r(u_n) \to r(u_0) \) and, by (29), \( u_0 = r(u_0) \in K \). In view of Lemma 3.3, we see that
\[
h_n^{-1}d_K(u_n) \to 0.
\]
Hence
\[
\|Au_n + g(x_n)\| = \frac{1}{h_n}\|u_n - r(u_n)\| \leq \frac{2}{h_n}d_K(u_n) \to 0.
\]
This implies that \( Au_n \to -g(u_0) \) and since \( A \) is closed we have \( u_0 \in D(A) \cap \partial U \) and \( Au_0 = -g(u_0) \). This contradicts Lemma 3.2. \( \Box \)

Let \( \alpha \in (0, \alpha_0] \) and \( h \in (0, h_0) \) as given in above Lemmas. By Lemma 1.1, there is a locally Lipschitz \( g : K \to E \) being an \( \alpha \)-approximation of \( G \) tangent to \( K \). Consider \( f : cU \to K \) defined by
\[
f(x) := r \circ J_h(x + hg(x)) \quad \text{for} \quad x \in cU.
\]
Then $f$ is compact and, by Lemma 3.4, $x \neq f(x)$ for $x \in \partial U$. Thus, the fixed point index $\text{Ind}_K(f,U)$ is well-defined (see again [33, §12]).

**Lemma 3.5.** The number $\text{Ind}_K(f,U)$ does not depend on the choice of a sufficiently close approximation $g$, a retraction $r$ and sufficiently small $h > 0$.

**Proof.** Take two retraction $r_0,r_1 : E \to K$ such that $\|x - r_i(x)\| \leq 2d_K(x), x \in E$, $i = 0,1$, and two locally Lipschitz $\alpha$-approximations $g_0,g_1 : K \to E$ of $G$ tangent to $K$, where $0 < \alpha \leq \alpha_0$. Repeating arguments from Lemmas 3.2 and 3.4 we find a sufficiently small $\alpha > 0$ and $h \leq h_0$ such that for any $t \in [0,1]$

$$x \neq f_t(x) := r_t \circ J_h(x + hg_t(x)) \text{ on } \partial U,$$

where $r_t := (1-t)r_0 + tr_1$ and $g_t = (1-t)g_0 + tg_1$. Thus, $\overline{cl}\, U \times [0,1] \ni (x,t) \mapsto f_t(x)$ provides a (compact) homotopy joining $f_0$ to $f_1$ showing that $\text{Ind}_K(f_0,U) = \text{Ind}_K(f_1,U)$. The independence of (small) $h$ of $\text{Ind}_K(f,U)$ follows easily from the resolvent identity

$$J_b = J_a \left( \frac{a}{b} I + \frac{b-a}{b} J_b \right), \quad (30)$$

valid for any $a,b > 0$ with $a\omega, b\omega < 1$ and the homotopy invariance of the fixed point index.

Thus, we are in a position to define the degree $\text{deg}_K$ by

$$\text{deg}_K(A + G,U) := \lim_{h \to 0^+} \text{Ind}_K(r \circ J_h(I + hg),U), \quad (31)$$

where $g : K \to E$ is a tangent and sufficiently close locally Lipschitz approximation of $G$.

**Proposition 1.** The degree $\text{deg}_K$ has the following basic properties:

1. (Existence) If $\text{deg}_K(A + G,U) \neq 0$, then there is $x \in D(A) \cap U$ such that $0 \in Ax + G(x)$;
2. (Additivity) If $U_1, U_2 \subset U$ are disjoint open in $K$ and $0 \notin Ax + G(x)$ for $x \in D(A) \cap (\overline{cl}\, U \setminus (U_1 \cup U_2))$, then

$$\text{deg}_K(A + G,U) = \text{deg}_K(A + G,U_1) + \text{deg}_K(A + G,U_2).$$

3. (Homotopy invariance) If $H : [0,1] \times K \to E$ is $H$-upper semicontinuous with convex weakly compact values, maps bounded sets onto bounded ones and is tangent to $K$, i.e. $H(t,x) \cap T_K(x) \neq \emptyset$, $t \in [0,1], x \in K$, and such that $0 \notin Ax + H(t,x)$ for $t \in [0,1], x \in \partial U$, then

$$\text{deg}_K(A + H(0,\cdot),U) = \text{deg}_K(A + H(1,\cdot),U).$$

**Proof.** (1) Suppose to the contrary that $0 \notin Ax + G(x)$ for $x \in \overline{cl}\, U \cap D(A)$. Arguing as in Lemmas 3.2 and 3.4, we find $0 < \alpha_1 \leq \alpha_0$ and $0 < h_1 \leq h_0$ such that for any $0 < \alpha \leq \alpha_1$ and any locally Lipschitz and tangent $\alpha$-approximation $g : K \to E$, $x \neq r \circ J_k(x + hg(x))$ for $x \in \overline{cl}\, U$, where $0 < h \leq h_1$. This shows that $\text{deg}_K(A + G,U) = 0$.

The remaining assertions are standard and left to the reader. \qed
4. The Krasnosel’skii type formula. In this section we will prove the following counterpart of the Krasnosel’skii formula (3) by showing a relation between the constrained degree of the operator \( A + F(0, \cdot) \) in the right-hand side of (14) and the fixed point index of the Poincaré operator \( \Sigma_t \) (with sufficiently small \( t > 0 \)) associated to (14); see (25), (27). The result should be considered as a generalization of [22, Thm. 4.1].

Theorem 4.1. Assume that operator \( A : D(A) \to E \), where \( E \) is a separable Hilbert space, and \( K \) satisfy hypotheses (A), (K) and, additionally let

\[
\|S(t)\| \leq e^{\omega t} \quad \text{for some } \omega \in \mathbb{R} \text{ and all } t \geq 0.
\]

Let \( F : [0, T] \times K \to E \) satisfy conditions (F_1), (F_4) and, instead of (F_2) and (F_3), we assume that\( F : [0, T] \times K \to E \) is \( H \)-upper semicontinuous,

\[
\sup_{y \in F(t,u)} \|y\| \leq a + b\|u\| \quad \text{for some } a, b \geq 0.
\]

If \( U \subset K \) is bounded and open (in \( K \)) and \( 0 \notin Ax + F(0, x) \), for \( x \in \partial U \cap D(A) \), then there is \( t_0 \in (0, T] \) such that for \( t \in (0, t_0) \) the fixed point index \( \text{Ind}_K((p_1, q_1), U) \) is well-defined and equal to \( \text{deg}_K(A + F(0, \cdot), U) \) \(^4\).

Observe that \( F(0, \cdot) \) satisfies (G_1) and (G_2); hence \( \text{deg}_K(A + F(0, \cdot), U) \) is well-defined.

Proof. The long proof of Theorem 4.1 will be presented in a series of steps and auxiliary lemmas.

Step 1. Define \( \hat{F} : [0, T] \times K \to E \) by the formula

\[
\hat{F}(t, x) := \text{conv} F([0, t], x), \quad t \in [0, T], \quad x \in K \quad \hat{F} \quad \text{(5)}.
\]

Lemma 4.2. \( \hat{F} \) has convex weakly compact values, is \( H \)-upper semicontinuous, has sublinear growth and is tangent to \( K \).

Proof. We shall show that \( [0, T] \times K \ni (t, x) \mapsto F([0, t], x) \subset E \) is \( H \)-upper semicontinuous; then the assertion will follow easily. Take \( t_0 \in [0, T] \), \( x_0 \in K \) and \( \varepsilon > 0 \). There is \( \delta_0 > 0 \) such that if \( t \in [0, T] \) and \( |t - t_0| < \delta_0 \) then

\[
F(t, x_0) \subset F(t_0, x_0) + B_E(0, \varepsilon/2)
\]

For every \( 0 \leq t \leq t_0 + \delta_0/2 \) there is \( \delta(t) > 0 \) such that

\[
F(s, x) \subset F(t, x_0) + B_E(0, \varepsilon/2),
\]

provided \( s \in [0, T], \ |s - t| < \delta(t) \) and \( x \in B_K(x_0, \delta(t)) \). There are \( t_i, \ i = 1, \ldots, k \) such that \( [0, t_0 + \delta_0/2] \subset \bigcup_{i=1}^{k} (t_i - \delta_i, t_i + \delta_i) \), where \( \delta_i := \delta(t_i) \).

Put \( \delta := \min_{i=1,\ldots,k}\{\delta_0/2, \delta_i\} \), let \( t \in [0, T], \ |t - t_0| < \delta, \ x \in B_K(x_0, \delta) \) and \( y \in F([0, t], x) \). Then \( y \in F(s, x) \) for some \( s \in [0, t] \). There is \( i = 1, \ldots, k \) with \( |s - t_i| < \delta_i \). By (35), \( y \in F(t_i, x_0) + B_E(0, \varepsilon/2) \). If \( t_i \leq t_0 \), then \( y \in F(t_i, x_0) + B_E(0, \varepsilon/2) \subset F([0, t_0], x_0) + B_E(0, \varepsilon) \);

\(^4\)Since \( F \) is \( H \)-upper semicontinuous, then, arguing as in Remark 2 (b), \( F \) is product measurable, i.e. (33) implies (F_2) and (F_1). Note also that if \( A \) is defined by the Dirichlet (or Neumann) Laplacian then assumption (32) is fulfilled.

\(^5\)Here \( F([0, t], x) := F([0, t] \times \{x\}) \).
while if \( t_i > t_0 \) then \( t_i - t_0 \leq \delta_0/2 \) and, by (34),

\[
y \in F(t, x_0) + B_E(0, \varepsilon/2) \subset F(t, x_0) + B_E(0, \varepsilon) \subset F([0, t_0], x_0) + B_E(0, \varepsilon).
\]

Hence \( F([0, t], x) \subset F([0, t_0], x_0) + B_E(0, \varepsilon) \).

\[ \square \]

Using the same methods as in Lemma 3.2 we get:

**Lemma 4.3.** There are \( \alpha > 0 \) and \( \tau \in (0, T] \) small enough that \( 0 \notin Ax + \hat{F}(\tau, B_K(x, \alpha)) + B_E(0, \alpha) \) for \( x \in \partial U \).

\[ \square \]

**Step 2.** By Lemma 1.1, there is a locally Lipschitz map \( f : K \to E \) being a tangent to \( K \alpha \)-approximation of \( F(0, \cdot) \). Obviously, there is a constant \( c > 0 \) such that for all \( u \in K \)

\[
\|f(u)\| \leq c(1 + \|u\|).
\]

By Lemma 4.3, since \( f(u) \in \hat{F}(\tau, B(x, \alpha)) + B(0, \alpha) \), \( 0 \neq Au + f(u) \) for \( u \in \partial U \).

Hence arguing as in Lemma 3.4, we find \( h_0 > 0 \) with \( h_0 \omega < 1 \) such that

\[
x \neq r \circ J_h(x + hf(x)) \text{ for } x \in \partial U, \ h \in (0, h_0],
\]

where \( r : E \to K \) is a retraction. Since \( E \) is the Hilbert space, without loss of generality we may assume that \( r \) is a metric projection, i.e., \( \|x - r(x)\| = d_K(x) \) for any \( x \in E \).

Observe that, by definition (see (31)),

\[
\deg_K(A + F(0, \cdot), U) = \text{Ind}_K(r \circ J_h(I + hf), U).
\]

Define the auxiliary set-valued map \( G : [0, 1] \times K \to E \) by the formula

\[
G(z, x) := (1 - z)f(x) + z\hat{F}(\tau, x) \quad z \in [0, 1], \ x \in K.
\]

Obviously, \( G \) is \( H \)-upper semicontinuous, tangent to \( K \), has sublinear growth and convex weakly compact values. By Theorem 2.1 (see also Remark 6), for any \( z \in [0, 1] \) the set of all (mild) solutions to

\[
\dot{u} = Au + G(z, u), \ u \in K
\]

starting at \( x \in K \) is an \( R_{\delta} \)-set.

**Lemma 4.4.** There is \( t_0 \in (0, \tau] \) such that, for every \( t \in (0, t_0], \ x \in \partial U \) and \( z \in [0, 1] \), if \( u \) is a solution to (39) then \( u(t) \neq x \).

**Proof.** Suppose to the contrary that for each integer \( n \geq n_0 \), where \( n_0^{-1} < \tau \) there are \( x_n \in \partial U, \ t_n \in (0, n^{-1}], \ z_n \in [0, 1] \) and a solution \( u_n : [0, t_n] \to K \) of (39) such that \( u_n(0) = x_n = u_n(t_n) \). Then there is \( w_n \in L^1(0, t_n; E) \) such that \( w_n(s) \in G(z_n, u_n(s)) \) for a.e. \( s \in [0, t_n] \) and

\[
u_n(t) = S(t)x_n + \int_0^t S(t - s)w_n(s) \, ds, \quad t \in [0, t_n].
\]

Extending periodically, we may assume that \( u_n \) and \( w_n \) are defined on \( [0, \tau] \), i.e. \( u_n \in C([0, \tau], K), \ w_n \in L^1([0, \tau; E) \). The semigroup property ensures that formula (40) is valid for every \( t \in [0, \tau] \) and \( w_n \in G(z_n, u_n(s)) \) for a.e. \( s \in [0, \tau] \). Thus, \( u_n \) is a solution on \( [0, \tau] \) of (39).

The growth condition and Gronwall’s inequality imply that the set \( \{w_n\}_{n \geq 1} \) is bounded. Therefore, there is some constant \( c \) such that set \( \{w_n(s)\}_{n \geq 1} \subset D(0, c) \)
being weakly compact in \( E \), i.e. \( \{w_n\} \) is weakly relatively compact in \( L^1(0, \tau; E) \) (cf. [26, Cor. 2.6]). Up to a subsequence we may assume that \( w_n \rightharpoonup w_0 \in L^1(0, \tau; E) \) and \( z_n \to z_0 \in [0, 1] \).

To prove that the set \( \{u_n\}_{n \geq 1} \) is relatively compact it is enough to show that so is \( \{x_n\}_{n \geq 1} \) (cf. Remark 3 (b)). Take \( \tau_0 \in (0, \tau) \) and put \( k_n := \lfloor \tau_0/t_n \rfloor + 1 \). Then

\[
x_n = u_n(k_n t_n) = S(\tau_0 + r_n) x_n + \int_{\tau_0}^{\tau_0 + r_n} S(\tau_0 + r_n - s) w_n(s) \, ds.
\]

In other words, \( x_n \in \Sigma_{\tau_0 + r_n}(x_n) \) for all \( n \geq 1 \) and, by Remark 5 (2) \( \{x_n\} \) is indeed relatively compact and so, up to a subsequence, \( x_n \to x_0 \in \partial U \). Thus, again for a subsequence, \( u_n \rightharpoonup u_0 \in C([0, \tau], K) \) and, by the uniform equicontinuity of \( \{u_n\} \)

\[
\| u_0(t) - x_0 \| \leq \| u_0 - u_n \| \to\infty + \| u_n(t) - u_n([t/t_n]t_n) \| + \| x_n - x_0 \| \to 0,
\]

hence \( u_0(t) = x_0 \) for all \( t \in [0, \tau] \). Therefore, for all \( t \in [0, \tau] \)

\[
x_0 = S(t) x_0 + \int_0^t S(t-s) w_0(s) \, ds \quad (41)
\]

and \( w_0(s) \in G(z_0, x_0) \) for a.e. \( s \in [0, \tau] \). We will show that this implies that \( x_0 \in D(A) \cap \partial U \) and \( 0 \in A x_0 + G(z_0, x_0) \).

The function \( [0, \tau] \ni t \mapsto \int_0^t w_0(s) \, ds \) is a.e. differentiable; take \( \xi \in (0, \tau) \) such that \( w_0(\xi) \in G(z_0, x_0) \) and \( \frac{d}{dt} t=\xi \int_0^t w_0(s) \, ds = w_0(\xi) \). By (41) for small \( \eta > 0 \), we have

\[
x_0 = S(\eta) x_0 + \int_\xi^{\xi+\eta} S(\xi+\eta-s) w_0(s) \, ds.
\]

We shall show that

\[
\frac{1}{\eta} \int_\xi^{\xi+\eta} (w_0(s) - S(\xi+\eta-s) w_0(s)) \, ds \to 0 \quad \text{as } \eta \to 0^+.
\]

To this end take \( p \in E^* \) and \( \varepsilon > 0 \). Then

\[
\left\langle \frac{1}{\eta} \int_\xi^{\xi+\eta} (w_0(s) - S(\xi+\eta-s) w_0(s)) \, ds, \ p \right\rangle \\
= \frac{1}{\eta} \int_\xi^{\xi+\eta} \langle w_0(s), p - S^*(\xi+\eta-s)p \rangle \, ds
eq
\]

The dual semigroup \( \{S^*(t)\}_{t \geq 0} \) is strongly continuous since \( E \) is reflexive. Thus, there is \( \delta > 0 \) such that \( \|S^*(t)p - p\| < \varepsilon C^{-1} \), if \( 0 \leq t < \delta \), where \( C := \sup_{y \in G(z_0, x_0)} \|y\| \). If \( 0 < \eta < \delta \), then for a.a. \( \xi \leq s \leq \xi + \eta \),

\[
|\langle w_0(s), p - S^*(\xi+\eta-s)p \rangle| < \varepsilon \quad \text{for a.e. } s \in [\xi, \xi+\eta],
\]

what proves (42). As a result,

\[
\frac{S(\eta) x_0 - x_0}{\eta} \\
= \frac{1}{\eta} \int_\xi^{\xi+\eta} (S(\xi+\eta-s) w_0(s)) \, ds - \frac{1}{\eta} \int_\xi^{\xi+\eta} w_0(s) \, ds \rightharpoonup -w_0(\xi).
\]
In view of [48, Th. 2.1.3], \(x_0 \in D(A) \cap \partial U\) and \(Ax_0 = -w_0(\xi)\). Hence \(0 = Ax_0 + w_0(\xi) \in Ax_0 + G(z_0, x_0)\). This is a contradiction with Lemma 4.3, since \(G(z_0, x_0) \subset \tilde{F}(\tau, B_K(x_0, \alpha)) + B_E(0, \alpha)\). \(\square\)

**Step 3.** Recall the solution operator \(\Sigma : K \rightarrow C([0, T], K)\) (see (26)) and the \(t\)-Poincaré operator \(\Sigma_t : K \rightarrow K\) associated with (14) (see (26) and (27)) and consider their restrictions to \(\text{cl} U\). By a slight abuse of notation, we will still denote these restrictions by the same symbols, i.e. \(\Sigma : \text{cl} U \rightarrow C([0, T], K)\) is represented by \(\text{cl} U \ni (x, t) \mapsto (\Sigma(x), t)\) having the same sense as in (26), while \(\Sigma_t : \text{cl} U \rightarrow K\) is represented by \(\text{cl} U \ni (x, t) \mapsto (\Sigma(x), t)\) with \(p_1 := p_S, q_t := e_t \circ q_S\).

Taking into account (38) and Remark 7, we are to show that for sufficiently small \(t > 0, h > 0\), the \(c\)-admissible pairs \((p_t, q_t)\) and \((\text{id}, r \circ J_h(I + hf))\), where \(\text{id} = \text{id}_{\text{cl} U}\) stands for the identity on \(\text{cl} U\), are \(c\)-homotopic via a compact \(c\)-homotopy without fixed points on the boundary \(\partial U\). This will be done in two stages.

**Stage 1.** For any \(x \in \text{cl} U\), the problem

\[
\begin{cases}
\dot{u} = Au + f(u), & t \in [0, T], \ u \in K \\
u(0) = x \in K
\end{cases}
\]  

(43)

possesses the unique solution \(P(x) \in C([0, T], K)\); the map \(P : \text{cl} U \rightarrow C([0, T], K)\) is continuous. For \(t \in [0, T]\), the Poincaré \(t\)-operator \(P_t : \text{cl} U \rightarrow K\) associated to (43) (given by \(P_t(x) := P(x)(t)\) for \(x \in \text{cl} U\)) is compact (see Remark 5).

Let us consider \(\Phi : [0, 1] \times \text{cl} U \rightarrow C([0, T], K)\), where given \(z \in [0, 1]\), the map \(\Phi(z, \cdot) : K \rightarrow C([0, T], K)\) is the Poincaré operator associated with the parameterized problem

\[
\dot{u} = Au + (1 - z)f(u) + zF(t, u), \quad t \in [0, T], \ u \in K.
\]  

(44)

It is clear that \(\Phi\) is a cell-like map (cf. Remark 6). Fix \(t \in (0, T]\) and consider the Poincaré \(t\)-operator \(\Phi_t : [0, 1] \times \text{cl} U \rightarrow K\) defined by

\[
\Phi_t(z, x) := \{u(t) \in K \mid u \in \Phi(z, x)\}.
\]

As before \(\Phi_t\) is compact and \(c\)-admissible. If \(x \in \Phi(z, x)\), for some \(z \in [0, 1]\) and \(x \in K\), then \(u|_{[0, t_0]}\) is also the solution of the problem (39) on the segment \([0, t_0]\). Hence, by Lemma 4.4,

\[
x \notin \Phi_t(z, x) \quad \text{for} \quad t \in [0, t_0], \ x \in \partial U, \ z \in [0, 1].
\]

Clearly, \(\Phi(1, \cdot) = \Sigma\) and \(\Phi(0, \cdot) = P\); hence \(\Phi_t(1, \cdot) = \Sigma_t\) and \(\Phi_t(0, \cdot) = P_t\) on \(\text{cl} U\). Therefore, the canonical pair \((p_\Phi, q_\Phi)\) representing \(\Phi\) is the \(c\)-homotopy joining \((p_S, q_S)\) to the canonical pair representing \(P\). At the same time, the pair \((p_\Phi, e_t \circ q_\Phi)\) representing \(\Phi_t\) is a \(c\)-homotopy joining \((p_t, q_t)\) to \((\text{id}_{\text{cl} U}, P_t)\). We have thus shown that:

**Lemma 4.5.** If \(t \in (0, t_0]\), where \(t_0\) is given by Lemma 4.4, then the pairs \((p_t, q_t)\) and \((\text{id}_{\text{cl} U}, P_t)\) are \(c\)-homotopic via the compact \(c\)-homotopy without fixed points on \(\partial U\). \(\square\)

**Stage 2.** We are going now to establish the following

**Proposition 2.** There are \(0 < t_1 \leq t_0\) and \(h_1 \in (0, h_0]\), where \(h_0\) was chosen at the beginning of Step 2 (see (37)), such that for \(t \in (0, t_1]\) and \(h \in (0, h_1]\), the maps \(P_t\) and \(r \circ J_h(I + hf)\), see (38), are homotopic via a compact homotopy without fixed points on \(\partial U\).
Proof. Claim 1. First we shall show that for sufficiently small \( h \in (0, h_0] \) and \( t > 0 \) the Poincaré \( t \)-operator \( P_t \) associated to (43) and the Poincaré operator associated to the parameterized problem

\[
\dot{u} = r \circ J_h(u + hf(u)) - u,
\]

are homotopic via a \( \beta \)-contracting homotopy without fixed points on \( \partial U \) \(^6\).

To this end, fix \( h' \in (0, h_0] \) such that

\[
(\omega + 1)h' < 1
\]

(46)
take \( h \in (0, h'] \) and consider a parameterized semilinear problem

\[
\dot{u} = zA + g_z(u), \quad z \in [0, 1], \ u \in K,
\]

(47)
where \( g_z : K \to E \) is defined by

\[
g_z(x) := zf(x) + (1 - z)(r \circ J_h(x + hf(x)) - x) \quad \text{for} \ z \in [0, 1], \ x \in K.
\]

For each \( z \in [0, 1] \), \( g_z \) is locally Lipschitz, since so are \( f \) and \( r \) (recall that \( r \), as the metric projection onto the convex \( K \), is nonexpansive). Moreover, for any \( x \in K \), \( f(x) \in T_K(x) \) and \( r \circ J_h(x + hf(x)) - x \in K - x \subset T_K(x) \), for \( x \in K \); and, hence, \( g_z(x) \in T_K(x) \), for \( x \in K \). It is easy to see that, for each \( z \in [0, 1] \), \( g_z \) has sublinear growth and the semigroup \( \{S(zt)\}_{t \geq 0} \) generated by \( zA \) leaves the set \( K \) invariant. Thus, in view of Remark 3 (d) and Theorem 2.1, for any \( h \) depends on \( \Theta(1, x) \) is the solution to (45) (resp. (43)) starting at \( x \). Note that \( \Theta \) implicitly depends on \( h \), too.

To see that the map

\[
[0, 1] \times \text{cl} U \ni (z, x) \mapsto \Theta_t(z, x) := \Theta(z, x)(t) \in K,
\]

with sufficiently small \( t > 0 \) and \( h \), is the required homotopy joining the Poincaré \( t \)-operators associated to problems of (45) and (43) we need to consider a different form of (47).

Namely, following [21], we shall consider the following families \( \{A_z : D(A) \to E\}_{z \in [0, 1]} \) and \( \{f_z : K \to E\}_{z \in [0, 1]} \) of linear operators and maps defined for \( z \in [0, 1] \) by

\[
A_z := (z - 1 - zh^{-1})I + zA,
\]

(48)

\[
f_z(x) := zf_1(x) + (1 - z)f_2(x),
\]

(49)
where \( f_1, f_2 : K \to E \) and \( f_1(x) := h^{-1}(x + hf(x)), \ f_2(x) := r \circ J_h(x + hf(x)) \) for \( x \in K \).

It is easy to see that

\[
A_z x + f_z(x) = zAx + g_z(x), \quad x \in K \cap D(A).
\]

For any \( z \in [0, 1] \), \( A_z \) generates the strongly continuous semigroup \( \{S_z(t)\}_{t \geq 0} \), where

\[
S_z(t) := e^{(z - 1 - zh^{-1})t} S(zt), \quad t \geq 0
\]

\(^6\)Recall that \( \beta \) stands for the Hausdorff measure of noncompactness; a map \( f : X \to Y \) between metric spaces is \( \beta \)-contracting if there is \( \kappa \in (0, 1) \) such that \( \beta(f(B)) \leq \kappa \beta(B) \) for any bounded \( B \subset X \).
\( f_z \) is locally Lipschitz and has sublinear growth. By [48, Chapter 3.1. (1.2)] and the Fubini theorem, we gather that \( \Theta(z,x) \) is the unique (mild) solution to the problem
\[
\begin{cases}
\dot{u} = A_z u + f_z(u), \\
u(0) = x.
\end{cases}
\tag{50}
\]

In order to establish the necessary properties of \( \Theta_t \) with small \( t > 0 \) we need to collect some facts about the family \( \{A_z\}_{z \in [0,1]} \).

(a) By (46), for each \( z \in [0,1] \) and \( t \geq 0 \), \( S_z(t) \) is a contraction (hence \( (I - \lambda A_z)^{-1} \) exists for all \( \lambda > 0 \)) and
\[
\|S_z(t)\| \leq e^{t(z(1 + \omega - h^{-1}) - 1)} \leq e^{-t}.
\]

(b) One shows easily that the family \( \{A_z\}_{z \in [0,1]} \) is resolvent continuous, i.e. for any \( \lambda > 0 \) and \( x \in E \) the map
\[
[0,1] \ni z \mapsto (I - \lambda A_z)^{-1}x \in E
\]
is continuous. The Trotter-Kato theorem (see [28, Thm. III.4.8]) implies that, for each \( x \in E \), the map
\[
[0,1] \ni z \mapsto S_z(\cdot)x \in C([0,T],E)
\]
is continuous.

c) Moreover (see [24, Ex. 2.6]), for any \( 0 < \gamma < 1 \) the family \( \{A_z\}_{z \in [\gamma,1]} \) is resolvent compact, i.e. for any \( \lambda > 0 \) the map (51) restricted to \( [\gamma,1] \) is compact.

d) Conditions (a) and (52) imply immediately that for any \( t > 0 \) and a bounded \( B \subset E \)
\[
\beta(B') \leq e^{-t}\beta(B), \quad \text{where } B' := \{S_z(t)x \mid x \in B, z \in [0,1]\}.
\tag{53}
\]
To this end, assume that for some \( r > 0 \) a set \( \{x_1, \ldots, x_n\} \) is an \( r \)-net for \( B \). Take \( \varepsilon > 0 \); using (52) we find \( z_1, \ldots, z_m \in [0,1] \) such that for any \( z \in [0,1] \),
\[
\sup_{t \in [0,T]} \|S_z(t)x_i - S_z(t)x_j\| < \varepsilon \text{ for some } j = 1, \ldots, m \text{ and all } i = 1, \ldots, n.
\]
Hence \( \{S_z(t)x_i \}_{i=1, \ldots, j=1, \ldots, m} \) is an \( (e^{-t}r + \varepsilon) \)-net for \( B' \).

We are now ready to collect properties of \( \Theta_t \).

1. (Continuity) Condition (52) together with [36, Thm. III.3.20] imply that the map
\[
\Theta : [0,1] \times K \ni (z,x) \mapsto \Theta(z,x) \in C([0,T],K)
\]
is continuous; hence for any \( t \in [0,T] \), the map \( \Theta_t \) is continuous, as well. \(^7\)

2. (\( \beta \)-Contractivity) Take \( t \in [0,T] \), a bounded \( B \subset K \) and let \( B' \) be as in (53). The set \( \Theta_t([0,1] \times B) \) is bounded and
\[
\Theta_t([0,1] \times B) \subset B' + \bigcup_{z \in [0,1]} (zB_1(z) + (1 - z)B_2(z)),
\]
where
\[
B_i(z) := \left\{ \int_0^t S_z(t-s)f_i(\Theta(z,x)(s)) \, ds \mid x \in B \right\}, \quad z \in [0,1], \ i = 1, 2.
\]
The resolvent continuity together with compactness of \( f_2 \) implies that the set
\[
\bigcup_{z \in [0,1]} (1 - z)B_2(z)
\]
is relatively compact. The sets \( B_1(z) \) are uniformly bounded, i.e. there is \( R > 0 \) such that for any \( z \in [0,1] \), \( \|y\| \leq R \) for \( y \in B_1(z) \).

Take \( \gamma \in (0,1] \). Using the resolvent compactness of the family \( \{A_z\}_{z \in [\gamma,1]} \) we see

\(^7\) A direct proof of the continuity of \( \Theta_t \) may also follow as in [21, page 460].
in the standard way that the set $\bigcup_{z \in [\gamma, 1]} zB_1(z)$ is relatively compact. Hence and in view of (53)
\[
\beta(\Theta_t([0, 1] \times B)) \leq e^{-t}\beta(B) + \beta\left( \bigcup_{z \in [0, \gamma]} zB_1(z) \right) \leq e^{-t}\beta(B) + \gamma R.
\]

Taking an arbitrary $\varepsilon > 0$ and $\gamma < \varepsilon R^{-1}$ we see that $\beta(\Theta_t([0, 1] \times B)) \leq e^{-t}\beta(B) + \varepsilon$. We have shown that for any $t \in [0, T]$, $\Theta_t$ is a $\beta$-contraction with constant $e^{-t}$, i.e., for a bounded $B \subset K$,
\[
\beta(\Theta_t([0, 1] \times B)) \leq e^{-t}\beta(B). \tag{54}
\]

Observe that the contraction constant depends on $t$ but it does not depend on $h$.

3. (No fixed points on the boundary) We claim that there is $h_1 \in (0, h'_1]$ and $t'_1 > 0$ such that for $h \in (0, h_1]$ and $t \in (0, t'_1)$
\[
\Theta_t(z, x) \neq x \quad \text{for } z \in [0, 1], \quad x \in \partial U. \tag{55}
\]

Suppose to the contrary and fix a small $h$. There are sequences $(x_n)$ in $\partial U$, $(z_n)$ in $[0, 1]$ and $r_n \to 0^+$ such that $x_n = \Theta_{t_n}(z_n, x_n)$. Take $t \in [0, T)$, then $x_n = \Theta_t + r_n(z_n, x_n)$, where $r_n := \left(\lfloor t/t_n \rfloor + 1\right)t_n - t$ (note that $t + r_n < T$ for large $n$). In view of (54) and the continuity of $\Theta$, $(x_n)$ is relatively compact. Taking subsequences if necessary, $x_n \to x_h \in \partial U$, $z_n \to z_h \in [0, 1]$, we see that $u_n := \Theta(z_n, x_n) \to u_h := \Theta(z_h, x_h)$ (remember that the sequences $(x_n)$, $(z_n)$ as well as their existing limiting points depend on $h$). Having this and proceeding as in the proof of Lemma 4.4, we show that $u_h$ is constantly equal to $x_h$, $x_h \in D(Az_h) = D(A)$ and
\[
0 = Az_h x_h + f(z_h). \tag{56}
\]

Let
\[
h' := z_h(1 + z_h h^{-1} - z_h)^{-1}
\]
and observe that $h' \omega < 1$, $h' > 0$ if $z_h \neq 0$ and
\[
\frac{1 - z_h}{1 + z_h h^{-1} - z_h} = 1 - \frac{h'}{h}.
\]
Hence, by (48), (49), the equality (56) reads
\[
x_h - h' Ax_h = \frac{h'}{h} y_h + \left(1 - \frac{h'}{h}\right) r(J_h(y_h)), \tag{57}
\]
where $y_h := x_h + hf(x_h)$.

If $z_h = 0$ then $h' = 0$ and we have $x_h = r(J_h(y_h))$. Therefore, putting $u_h := J_h(y_h)$, $u_h \in D(A)$ and
\[
\frac{r(u_h) - u_h}{h} = h^{-1}(x_h - u_h) = Au_h + f(x_h). \tag{58}
\]

If $z_h > 0$, then $h' > 0$. Inverting $I - h'A$ in (57), we get by (30) that
\[
x_h = J_{h'} \left( h' h^{-1} y_h + \left(1 - \frac{h'}{h}\right) J_h(y_h) \right) + \left(1 - \frac{h'}{h}\right) J_{h'}(r(u_h) - u_h)
\]
\[
= u_h + \left(1 - \frac{h'}{h}\right) J_{h'}(r(u_h) - u_h).
\]

Therefore,
\[
\left(1 - \frac{h'}{h}\right) \frac{J_{h'}(r(u_h) - u_h)}{h} = h^{-1}(x_h - u_h) = Au_h + f(x_h). \tag{59}
\]
Now let $h \to 0^+$. Arguing as in the proof of Lemma 3.4, we see that quantities $h^{-1}|r(u_h) - u_h|$ in (58) and $h^{-1}(1 - h^{-1})|r(u_h) - u_h|$ in (59) are bounded; hence also $\|Au_h\|$ is bounded and so is $\|u_h\|$. By Remark 3 (a), we gather that (up to a subsequence) $u_h \to x_0$ and $x_h \to x_0 \in D(A) \cap \partial U$. Hence, by Lemma 3.3, we see that $h^{-1}|r(u_h) - u_h|$ is bounded. Moreover, the expression $(1 - h^{-1}h')$ is bounded. Thus, in both cases (58) and (59), $Au_h + f(x_h) \to 0$, i.e. $0 = Ax_0 + f(x_0)$ in contradiction to (36).

Claim 1 is established: if $0 < t < t_1'$ and $h \in (0, h_1']$, then the Poincaré $t$-operator $P_t$, associated to (43), is homotopic to the Poincaré $t$-operator $\Theta_t(0, \cdot)$, associated to (45), via a $\beta$-contracting homotopy without fixed points on $\partial U$.

Claim 2. For $h \in (0, h_1]$ and sufficiently small $t > 0$ the Poincaré $t$-operator $\Theta_t(0, \cdot)$ associated with (45) is homotopic to $r \circ J_h(I + hf)$ via a $\beta$-contracting homotopy without fixed points on $\partial U$.

Indeed, for a fixed $t \in [0, T]$ and for $x \in \text{cl}U$, $z \in [0, 1]$ let

$$
\hat{\Psi}_t(z, x) := \begin{cases} 
(1 - \frac{1}{z(t + z - zt)} ) x + \frac{1}{z(t + z - zt)} \Theta_xt(0, x) & \text{for } z \in (0, 1], \\
r \circ J_h(x + hf(x)) & \text{for } z = 0,
\end{cases}
$$

As in [21, Prop. 4.3], one shows that $\hat{\Psi}_t$ is continuous and there is $t_1 \in (0, T)$, $t_1 < t_1'$, such that $\hat{\Psi}_t$ is $\beta$-contracting and $\hat{\Psi}_t(z, x) \neq x$ for $z \in [0, 1], x \in \partial U$ and $t \in (0, t_1]$.

Take $0 < t < t_1$ and let

$$
\Psi_t := r \circ \hat{\Psi}_t : [0, 1] \times \text{cl}U \to K.
$$

Then $\Psi_t$ is continuous and $\beta$-contracting as the superposition of $\hat{\Psi}_t$ with the non-expansive retraction $r$. To see that for all $z \in [0, 1]$, $\Psi_t(z, \cdot)$ has no fixed points on $\partial U$, we shall make use of the following general observation.

Lemma 4.6. If $x \in K$ then $y \in x + \bigcup_{h > 0} h(K - x)$ and $r(y) = x$ if and only if $y = x$.

Proof. There are $h_0 > 0$ and $k_0 \in K$ such that $y = x + h_0(k_0 - x)$. The so-called variational characterization of $r$ (see e.g. [15, Th. 5.2]) yields that for all $k \in K$, $\langle k - r(y), y - r(y) \rangle = \langle k - x, y - x \rangle \leq 0$. Hence $\langle k - x, (x + h_0(k_0 - x)) - x \rangle = h_0(k - x, k_0 - x) \leq 0$ for every $k \in K$. Thus, $k_0 = x$ and $y = x$. \hfill $\square$

Observe that $\Psi_t(0, x) = r \circ J_h(x + hf(x))$ and, in view of (37), $\Psi_t(0, x) \neq x$ for $x \in \partial U$. If $z \in (0, 1]$ then for $x \in \text{cl}U$

$$
\hat{\Psi}_t(z, x) = x + \frac{1}{z(t + z - zt )} (\Theta_xt(0, x) - x) \in x + \bigcup_{h \geq 0} h(K - x).
$$

For such $z$ and $x$, by Lemma 4.6, $x = \Psi_t(z, x) = r \circ \hat{\Psi}_t(z, x)$ if and only if $x = \hat{\Psi}_t(z, x)$. Therefore, $\Psi_t(z, x) \neq x$ for $z \in (0, 1], x \in \partial U$.

Connecting $\beta$-contracting homotopies provided by Claim 1 and Claim 2 we get the $\beta$-contracting homotopy $H$ joining $r \circ J_h(I + hf)$ to $P_t$ when $t > 0$ and $h > 0$ are sufficiently small.

Claim 3. There is a compact homotopy joining $r \circ J_h(I + hf)$ to $P_t$ without fixed points on $\partial U$.

To this end we will rely on the following general result.

Lemma 4.7. [2, Th. 3.1.4.], [2, Def. 3.1.7.] Let $X \subset K$ be bounded closed and $f_0, f_1 : X \to K$ be compact maps. If $h : [0, 1] \times X \to K$ is a condensing (in
particular, a \( \beta \)-contracting) homotopy joining \( f_0 \) to \( f_1 \) then there is the compact homotopy \( H : [0,1] \times X \to K \) joining \( f_0 \) to \( f_1 \) having the same fixed points as \( h \) does.

This establishes Proposition 2, since Lemma 4.7 produces a compact homotopy out of \( H \) (recall that \( r \circ J_h(I+hf) \) and \( P_t \) are compact).

To sum up, during the proof we have shown that for sufficiently small \( t > 0 \) and \( h > 0 \):

1. the \( c \)-admissible pair \((p_t, q_t)\) is \( c \)-homotopic to the pair \((\text{id}_C, P_t)\) via the compact \( c \)-homotopy without fixed point on \( \partial U \) (cf. Lemma 4.5);
2. the Poincaré \( t \)-operator \( P_t : \text{cl} U \to K \) is homotopic to \( r \circ J_h(I+hf) : \text{cl} U \to K \) via the compact homotopy without fixed points on \( \partial U \) (cf. Proposition 2).

Thus, in view of Remark 7 (ii) and (38), we have

\[
\text{Ind}_K((p_t, q_t), U) = \text{Ind}_K((\text{id}_C, P_t), U) = \text{Ind}_K(P_t, U) = \deg_K(A + f(0, \cdot), U).
\]

This concludes the proof of Theorem 4.1.

Let us finally formulate a direct single-valued counterpart of this result being a direct generalization of [21, Thm 4.5].

**Corollary 1.** Assume that \( A \) and \( U \) are the same as in Theorem 4.1. Additionally, let \( f : [0,1] \times K \to E \) be tangent to \( K \) locally Lipschitz function with sublinear growth. If \( 0 \neq Ax + f(0,x), \ x \in \partial U, \) then there is \( t_0 \in (0,1] \) such that for every \( t \in (0, t_0] \)

\[
\text{Ind}_K(P_t, U) = \deg_K(A + f(0, \cdot), U),
\]

where \( P_t : \text{cl} U \to K \) is the Poincaré \( t \)-operator associated with the problem \( \dot{u} = Au + f(t,u) \).

5. **Periodic solutions.** It is common to establish the existence of periodic solutions to different type of ODE’s by applying the fixed point theory to the associated Poincaré time map. Such approach, in the case of a semilinear constrained evolution equation has been presented by e.g. Prüss [49] and, for the evolution inclusion of the form (14) by Bothe [13] (see also [14]), Bader [9, 10]. Fully nonlinear evolution equations/inclusions (also including \( m \)-dissipative ‘diffusion’ \( A \)), in general ‘unconstrained’, have been studied by numerous authors and many interesting results were obtained, see for instance Vrabie [52], Hirano et al. [34], Bothe [14], Ćwiszewski [21] and monographs [36, 39]. The quite recent paper of Aizicovici et al. [1] gives a nice historical overview of the topic.

We would like to show how the theory developed in the present paper may be used in the study of periodic orbits. We shall show some results that follow immediately from was done above. They correspond directly to results of [9, 10] and [52]; in these papers, however, either assumptions concerning the constraining set or the nonlinearity are stronger then ours.

**Proposition 3.** Assume \((A), \ (K)\) and that \( F : [0,T] \times K \to E \) satisfies \((F_1)\) – \((F_4)\). If \( K \) is bounded then (14) has a ‘periodic’ solution, i.e. there is a mild solution \( u : [0,T] \to K \) such that \( u(0) = u(T) \).

If, additionally, \( F : [0, +\infty) \times K \to E \) and is \( T \)-periodic, i.e. \( F(t+T,u) = F(t,u) \) for all \( t \geq 0 \), then, there is a \( T \)-periodic \( u : [0, +\infty) \to K \) solving (1).
Proof. It is clear that if \( x \in K \) and \( x \in \Sigma_T(x) \) then there is a solution \( u : [0, T] \rightarrow K \) of (14) such that \( x = u(0) = u(T) \). Extending \( u \) periodically onto \( \mathbb{R} \) one gets a periodic solution. Since \( K \) is closed, convex and bounded, \( \Sigma_T : K \rightarrow K \) is a compact \( c \)-admissible map (i.e., \( \text{cl} \Sigma_T(K) \) is compact). In view of the Schauder fixed point theorem for admissible maps (see [31, Theorem (41.13)]), \( \Sigma_T \) has a fixed point. Equivalently, one may use Remark 7 (iii).

In what follows we let \( K \) be not necessarily bounded. We begin with the following generalization of [9, Cor. 11].

**Theorem 5.1.** Assume that \( K = E \) and, in addition to hypotheses of Proposition 3, \( A \) satisfies condition (32). If

\[
\limsup_{\|x\| \to \infty, x \in K} \frac{1}{\|x\|^2} \sup_{p \in J(x)} \inf_{y \in F(t,x)} \langle y, p \rangle < -\omega
\]

uniformly for \( t \in [0, T] \), where \( J : E \to E^* \) is the duality map (see e.g. [51]), then (14) has a periodic solution.

Proof. In view of (60), there is \( R > 0 \) such that if \( x \in E \) with \( \|x\| \geq R \) then for any \( t \in [0, T] \) there is \( y = y(t, x) \in F(t, x) \) such that for all \( p \in J(x) \), \( \langle y, p \rangle \leq -\omega \|x\|^2 \).

Consider an auxiliary problem

\[
\dot{u} \in Bu + G(t, u),
\]

where \( B := A - \omega I \) and \( G(t, x) = F(t, x) + \omega x \) for \( t \in [0, T] \), \( x \in E \). It is clear that \( G \) has the same properties as \( F \) does. Moreover, for any \( x \in E \), the set of all solutions to (61) coincides with \( \Sigma(x) \).

Let \( t \in [0, T] \), \( x \in E \) and \( \|x\| = R \). If \( z = y(t, x) + \omega x \) then

\[
\forall p \in J(x) \quad \langle z, p \rangle = \langle y, p \rangle + \omega \|x\|^2 \leq 0.
\]

Let \( \xi(x) := \frac{1}{2}\|x\|^2 \), \( x \in E \). It is well-known that \( J(x) = \partial \xi(x) \), the subdifferential of \( \xi \) at \( x \). In view of [6, Prop. 7.3.16],

\[
\{ v \in E \mid \forall p \in J(x) \quad \langle v, p \rangle \leq 0 \} \subset T_D(x)
\]

and, thus, (62) implies that \( z \in T_D(x) \), where \( D := D(0, R) \). Hence \( G(t, x) \cap T_D(x) \neq \emptyset \). The semigroup generated by \( B \) is of the form \( \{ e^{-\omega t} S(t) \}_{t \geq 0} \). In view of (32), this semigroup consists of contractions; hence \( D \) is invariant with respect to this semigroup. The assertion follows from Proposition 3 after replacing \( K \) by \( D \).

**Remark 8.** (i) In [52], the author studies (14) with a general accretive \( A \) and single-valued \( F \). Specifying for a closed densely defined linear \( A \), the main results asserts the existence of a periodic solution provided there is \( \omega > 0 \) such that \( A - \omega I \) is accretive and

\[
\lim_{r \to \infty} \frac{1}{r} \sup_{t \in [0, T], x \in E, \|x\| \leq r} \| F(t, x) \| = m < \omega.
\]

It is easy to see that this condition implies (60). Hence Theorem 5.1 may be considered as a (partial) generalization of [52, Thm. 1].

(ii) The result stated in Theorem 5.1 stays true if \( K \subseteq E \), \( \omega \leq 0 \) in (32) and for sufficiently large \( R > 0 \)

\[
\forall x \in K, \|x\| = R \sup_{y \in F(t, x), p \in J(x)} \langle y, p \rangle \leq 0.
\]
In this case, one does not introduce $B$ and $G$ but shows that $F(t, x) \cap T_{D \cap K}(x)$ for any $x \in D \cap K$, where $D := D(0, R)$ with sufficiently large $R > 0$. For large $R$, $K \cap B(0, R) \neq \emptyset$ and $T_K(u) \cap T_D(u) = T_{K \cap D}(u)$, in view of [6, Cor. 4.1.19]. Therefore, $F(t, x) \cap T_{K \cap D}(x) \neq \emptyset$ if $x \in K \cap D$. Moreover, $D$ is semigroup invariant since it consists of contractions. The assertion follows from Corollary 3 after replacing $K$ by $K \cap D$.

(iii) Finally, observe that if $E^*$ is uniformly convex, then $J$ is single-valued and the result stated in (ii) is a straightforward generalization of the (semilinear) version of Theorem 5 in [1].

In order to further deal with unbounded $K$ we need to collect some additional facts concerning compact $C_0$ semigroups.

**Remark 9.** (1) Assume that $B : D(B) \to E$ generates a $C_0$-semigroup $\{U(t)\}_{t \geq 0}$. Then, for any $t \geq 0$, the spectral radius of $r(U(t))$ of $U(t)$ is given by

$$r(U(t)) = e^{\omega_0(B)},$$

(64)

where $\omega_0(B) := \lim_{t \to \infty} \frac{1}{t} \log \|U(t)\|$ is the growth bound of $B$ (see [28, Def. IV.2.1, Prop. VI.2.2.]).

(2) If the semigroup $\{U(t)\}_{t \geq 0}$ is compact then, in view of [28, Cor IV.3.12], the so-called spectral determined growth condition is satisfied, i.e. the spectral bound $s(B)$ coincides with the growth bound $\omega_0(B)$:

$$s(B) := \sup \{ \Re \lambda \mid \lambda \in \sigma(B) \} = \omega_0(B).$$

Suppose that $C$ is a closed convex cone in $E$ such that the closure $\text{cl}(C - C) = E$. If $C$ is invariant with respect to $\{U(t)\}$ then, in view of [40, Corollary 2.9 and Example 2.4], $s(B)$ is an eigenvalue of $B$ with a corresponding (nontrivial) eigenvector in $C$.

Assume $(A), (K)$ and suppose now that $0 \in K$ and $K_0 := T_K(0)$ is the tangent cone to $K$ at $0$. Then $E_0 := \text{cl}(K_0 - K_0)$ is the minimal closed subspace containing $K$ and the so-called blade of $K_0$, i.e. $b(K_0) := K_0 \cap (-K_0)$ is the maximal closed subspace contained in $K_0$. The semigroup invariance of $K$ implies that that $E_0$ and $b(K_0)$ are invariant, too. Therefore, the restrictions $S_0(t) := S(t) \upharpoonright E_0$, $t \geq 0$, form the compact $C_0$ semigroup generated by the restriction $A_0 : D(A_0) \to E_0$, where $D(A_0) := \{ x \in D(A) \cap E_0 \mid Ax \in E_0 \}$; see [28, Sections I.5.12, II.2.3]. Observe that since $0 \in K$, we have $K \subseteq K_0$ and $S(t)x = S_0(t)x$ for $t \geq 0$ and $x \in K$. In view of [28, Sections I.5.13, II.2.3], the quotient operators $\tilde{S}_0(t) : \tilde{E}_0 \to \tilde{E}_0$, where $\tilde{E}_0 := E_0 / b(K_0)$, determined by $S_0(t)$ (8), $t \geq 0$, form the compact $C_0$ semigroup generated by $\tilde{A}_0 : D(\tilde{A}_0) \to \tilde{E}_0$, $\tilde{A}_0 q(x) = q(A_0 x)$ for $x \in D(\tilde{A}_0) := q(D(A_0))$.

Since, for every $t \geq 0$, $\|\tilde{S}_0(t)\| \leq \|S_0(t)\| \leq \|S(t)\|$, the growth bounds of $A_0$, $A_0$ and $A$ are related as follows

$$\omega_0(\tilde{A}_0) \leq \omega_0(A_0) \leq \omega_0(A).$$

(65)

**Theorem 5.2.** Assume $(A), (K)$, $0 \in K$, the growth bound $\omega_0(\tilde{A}_0) < 0$ and that no number of the form $2\pi k T^{-1}$, where $k \in \mathbb{N}$, belongs to the spectrum of $A_0$. If $F$ satisfies $(F_1), (F_2), (F_3)$ and, instead of $(F_3)$, $F$ is bounded, i.e. there is $c > 0$ such that $\sup_{y \in F(t,x)} \|y\| \leq c$, for all $t \in [0, T]$ and $x \in K$, then (14) has a periodic solution.

---

*I.e. $\tilde{S}_0(t)q(x) := q(S_0(t)x)$, where $q : E_0 \to \tilde{E}_0$ is the quotient map.*
Proof. Given $\lambda \in [0, 1]$, let $\Sigma^T_\lambda$ be the Poincaré $T$-operator associated with the problem

$$\dot{u} = Au + \lambda F(t, u).$$

(66)

Observe that $\lambda F$ satisfies conditions implying that the set $\Sigma^T_\lambda(x)$, $x \in K$, of all solutions to (66) starting at $x$ is cell-like. If $x \in K$ and $x \in \Sigma^T_\lambda(x)$ then there is a solution $u : [0, T] \to K$ of (14) such that $x = u(0) = u(T)$, i.e.

$$x = S(T)x + \lambda y = S_0(T)x + \lambda y, \quad \text{where } y := \int_0^T S(T-s)w(s) \, ds \in E_0,$$

$w \in L^1(0, T; E)$ and $w(s) \in F(s, u(s))$ almost everywhere on $[0, T]$. Clearly $\|y\| \leq c_1$ for some constant.

In view of [48, Theorem 2.2.4], $1 \in \rho(S_0(T))$, the resolvent set of $S_0(T)$. Hence $x = (I - S_0(T))^{-1} y$ and

$$\|x\| \leq \| (I - S_0(T))^{-1} \| \|y\| \leq C := c_1 \| (I - S_0(T))^{-1} \|.$$  

This implies that for $x \in K$ with $\|x\| = R := C + 1$ and $\lambda \in [0, 1]$, $x \notin \Sigma^T_\lambda(x)$.

Let $U := B_K(0, R) := \{x \in K \mid \|x\| \leq R\}$ and $(p^\lambda_0, q^\lambda_0)$, $\lambda \in [0, 1]$, be an admissible homotopy joining $(p^T_0, q^T_0)$ to $(p^\lambda_0, q^\lambda_0)$ representing $S_0(T)$ (comp. Remark 6 (b)). Moreover, $x \notin \tilde{q}(\rho^{-1}(x, \lambda))$ for $x \in \partial K U$ and $\lambda \in [0, 1]$. Hence

$$\text{Ind}_K((p^T_0, q^T_0), U) = \text{Ind}_K(S_0(T), U),$$

in view of Remark 7 (i) and (ii). In order to compute $\text{Ind}_K(S_0(T), U)$, we see that $r(\tilde{S}_0(T)) = e^{T \omega_0(\tilde{A}_0)} < 1$. Thus, in view of the result due to Dancer [25, Theorem 10], $\text{Ind}_K(S_0(T), U) = \text{Ind}_{E_0}(S_0(t), U_0) = (-1)^m \neq 0$, where $U_0 = B_{E_0}(0, R)$ and $m$ stands for the number of eigenvalues of $S_0(T)$ greater than 1 counted with (algebraic) multiplicities.

This shows that $\text{Ind}_K((p^T_0, q^T_0), U) \neq 0$, i.e., $\Sigma_T = \Sigma^T_\lambda$ has a fixed point.

If $\omega(A_0) < 0$ then $\omega(\tilde{A}_0) < 0$ and $r(S_0(T)) < 1$, in view of (65) and (64); thus $1 \in \rho(S_0(T))$ and no additional assumptions on the spectrum of $A_0$ are necessary. Hence we have the following

Corollary 2. Suppose $0 \in K$, $F$ is bounded and $\omega_0(A) < 0$ then (14) has a periodic solution.

If $A$ satisfies (32) with $\omega < 0$ then $\omega(A) < 0$ and Corollary 2 holds true. But in this case we can dispense with boundedness of $F$ and obtain a result that corresponds well to the constrained version of Theorem 5.1.

Theorem 5.3. Assume (A), (32) with $\omega < 0$, (K), $0 \in K$ and let $F$ satisfy $(F_1) - (F_4)$. Assume that

$$\limsup_{\|x\| \to \infty, x \in K} \sup_{y \in F(t, x)} \frac{1}{\|y\|} = \gamma$$

(67)

uniformly with respect to $t \in [0, T]$. If $e^{\gamma T} < -\omega$, then (14) has a periodic solution.

Condition (67) is satisfied e.g. if in $(F_3)$, $a \in L^\infty([0, T])$ and then $\gamma \leq b$. 


Proof. There is $R > 0$ such that, for all $t \in [0, T]$ and $x \in K$ and $y \in F(t, x)$, if $\|x\| \geq R$ then $\|y\| \leq \gamma \|x\|$.

As before, we are going to establish \textit{a priori} bounds for fixed point of $\Sigma^T_k$ (see (66)). Assume to the contrary that there are no such bounds, i.e. for each $n \in \mathbb{N}$ there is $x_n \in K$, $x_n \in \Sigma T(x_n)$ with $\|x_n\| \geq n$; there also exists $u_n \in \Sigma (x_n)$ such that $x_n = u_n(0) = u_n(T)$.

We claim that for sufficiently large $n$, $n \geq n_0$ say, $\inf_{t \in [0, T]} \|u_n(t)\| \geq R$. If not then (up to a subsequence) for any $n$ there is $t_n \in (0, T)$ such that $\|u_n(t_n)\| = R$ and $\|u_n(t)\| \geq R$ for $t \in [t_n, T]$. For such $t$

$$u_n(t) = S(t - t_n)u_n(t_n) + \int_{t_n}^t S(t - s)w_n(s)\, ds,$$

where $w_n(s) \in F(s, u_n(s))$ for a.a. $s \in [t_n, t]$. Since $\|w_n(s)\| \leq \gamma \|u_n(s)\|$, the Gronwall inequality implies that $\|u_n(t)\| \leq \gamma T e^{\gamma T}$ for $t \in [t_n, T]$. In particular, $n \leq \|u_n(T)\| \leq Re^{\gamma T}$ for all $n$: a contradiction.

For $n \geq n_0$ let $y_n := \int_0^T S(T - s)w_n(s)\, ds$. As before, we show that

$$\|w_n(s)\| \leq \gamma \|u_n(s)\| \leq \gamma \|x_n\| e^{\gamma T}.$$

Hence

$$\|y_n\| \leq \gamma \|x_n\| e^{\gamma T} \frac{1 - e^{\omega T}}{-\omega}.$$

Therefore,

$$\|x_n\| \leq \|I - S(T)\|^{-1} \|y_n\| \leq (1 - e^{\omega T})^{-1} \gamma \|x_n\| e^{\gamma T} \frac{1 - e^{\omega T}}{-\omega} = \|x_n\| e^{\gamma T} \frac{\gamma e^{\omega T}}{-\omega} < \|x_n\|,$$

a contradiction. Now, we proceed as in the last part of the proof of Theorem 5.2. \hfill $\Box$

Example 3. If, for example, we are within the setting of Subsection 1.2 for (2) subject to the Dirichlet boundary condition then $\omega_0(A) < 0$. Assume that $0 \in C$. Then problem (7) has a periodic mild solution, i.e. (2) has a $T$-periodic strong solution provided $C$ is bounded. The same holds if $C$ is not bounded, but $\varphi$ is. If in $(\varphi_2)$ we have that $\alpha \in L^\infty([0, T] \times \Omega)$ and $e^{\beta} \beta < \lambda_1$, where $\lambda_1$ is the first eigenvalue of the Dirichlet Laplacian $-\Delta$, then the existence of periodic solutions to (2) follows from Theorem 5.3 since, by Remark 9 (2), the growth bound of $\Delta$ equals $-\lambda_1$. \hfill $\Box$

5.1. \textbf{The guiding function approach.} One of the most powerful tools to establish the existence of periodic solution to systems of ODE’s or differential inclusions in $\mathbb{R}^n$ relies on the use of the so-called \textit{guiding potential}. This method, started by Krasnosel’skii, is described in a recent monograph [45].

In this section we shall try to show how the guiding function approach may be applied in the context of partial differential equations and inclusions. We assume again that $E$ is a separable Hilbert space, conditions $(A_1, (K)$ hold true with $K$ unbounded and let $F : [0, T] \times K \to E$ satisfy conditions $(F_1), (F_4)$ and (33). In order to establish the existence of (strong) periodic solutions, we shall assume additionally that:

$(A_1)$ $-A$ is determined by the so-called \textit{elliptic quadratic form} $a(\cdot, \cdot)$ (see \cite[Thm. 4.3 and eq. (4.1)]{4});

$(V_1)$ there is a $C^1$-functional $V : E \to \mathbb{R}$ such that $V$ is bounded on bounded sets, $V(u) \to \infty$ as $\|u\| \to \infty$;
(V2) there is $R_0 > 0$ and $\varepsilon > 0$ such that for $u \in K \cap D(A)$, $\|u\| \geq R_0$ and $t \in [0,T]$, 
\[
\sup_{v \in F(t,u)} \langle \nabla V(u), Au + v \rangle \leq -\varepsilon
\]
Remark 10. (a) Assumption (A1) is fairly general. For instance, as discussed in Remark 2 (d), the Dirichlet (or Neumann) Laplacian satisfy this condition. Observe that in this case $A - \omega I$, for some $\omega$ is $m$-dissipative. Hence, in view of the Lumer theorem (see [28, Thm. II.3.15]), assumption (32) is satisfied. Therefore, in the setting of (2) given in section 1.2 all assumptions stated above are satisfied.

(b) Suppose (A1). If $u$ is a mild solution to (14) then it is a mild solution to $\dot{u} = (A - \omega I)u + w$, where $w(s) \in F(s,u(s)) + \omega u(s)$ on $[0,T]$. In view of (33), $w \in L^2(0,T;E)$. Since $A - \omega I$ is the subdifferential of the quadratic function determined by the form $a(\cdot, \cdot) + \omega(\cdot, \cdot)$ (see [4, Rem. 4.4]), by [16, Thm. 3.6] (see also [51, Thm.1.9.3]), we gather that $u$ is a strong solution to this problem, in particular $u(t) \in D(A)$ for $0 < t \leq T$, $u \in W^{1,1}(0,T;E)$ and $u'(t) \in Au(t) + F(t,u(t))$ a.e. on $(0,T)$.

(b) If $E$ is of finite dimension, $K = E$ and $A \equiv 0$, then (V2) means the $-V$ is a strict guiding function for (14) (see [45, Definition 2.3]). □

Theorem 5.4. Under the above assumptions, (14) has a periodic solution, i.e. there is a solution $u$ to (14) such that $u(0) = u(T)$ and $u(t) \in K$ for $t \in [0,T]$.

Proof. Let $\alpha := \sup \{ V(x) \mid \|x\| \leq R_0 \} + 1$, $r := \sup \{ \|x\| \mid V(x) \leq \alpha \}$ and $\beta = \sup \{ V(x) \mid \|x\| \leq R \}$. In view of (V1), $\alpha, r, \beta < \infty$. Let $R := r + 1$; then 
\[
B_E(0, R_0) \subset V^{-1}((-\infty, \alpha]) \subset D_E(0, r) \subset B_E(0, R) \subset V^{-1}((0, \beta]).
\]
We may assume that $R$ is large enough that $B_K(0, R) \neq \emptyset$.

We shall prove that 
\[
\text{Ind}_K((p_T, q_T), B_K(0, R)) \neq 0.
\]
(68)
This, of course, as in the proof of Theorem 5.1, will conclude the argument and show that (14) has a periodic solution.

If $x \in D(A) \cap K$ and $\|x\| = R$ then $0 \not\in Ax + F(0,x)$ in view of (V2). Hence, by Theorem 4.1, there is a small $t_0 \in (0,T)$ such that $\text{Ind}_K((p_{t_0}, q_{t_0}), B_K(0, R))$ is well-defined and equal to $\text{deg}_K(A + F(0, \cdot), B_K(0, R))$.

Consider a $c$-admissible pair 
\[
K \times [t_0,T] \xrightarrow{\xi} \Gamma \times [t_0,T] \xrightarrow{\xi} K,
\]
where, as before (see (26)), $\Gamma := \{(x, u) \in K \times C([0,T], K) \mid u \in \Sigma(x)\}$ is the graph of $\Sigma$, $p(x,u,s) := (p_{\Sigma}(x,u), s) = (x, s)$, $q(x,u,s) := c_s(q_{\Sigma}(x,u)) = u(s)$ for $(x,u) \in \Gamma$ and $s \in [t_0,T]$. In view of Remark 5 (1), (2), this pair is $c$-admissible and compact.

Suppose that there is $x_0 \in K$, $\|x_0\| = R$ and $u \in \Sigma(x_0)$ such that $u(\tau_0) = x_0$ for some $\tau_0 \in (t_0,T]$. Then either $\|u(s)\| \geq R_0$ for all $s \in [0,\tau_0]$ and, in view of (V2) and since $u$ is a strong solution, 
\[
0 = V(u(\tau_0)) - V(u(0)) = \int_0^{\tau_0} \langle \nabla V(u(s)), Au(s) + w(s) \rangle \, ds \leq -\varepsilon \tau_0,
\]
where $w(s) \in F(s,u(s))$ a.e. on $[0,T]$; or there is $s_0 \in [0,\tau_0)$ such that $V(u(s_0)) \leq \alpha$ and $R_0 \leq \|u(s)\|$ for all $s \in [s_0,\tau_0]$ and 
\[
0 < V(u(\tau_0)) - \alpha \leq V(u(\tau_0)) - V(u(s_0)) = \int_0^{\tau_0} \langle \nabla V(u(s)), Au(s) + w(s) \rangle \, ds \leq -\varepsilon \tau_0.
\]
In both cases we arrive at a contradiction. This shows that (69) provides a c-homotopy without fixed points on the boundary of \( B_K(0, R) \) showing that
\[
\text{Ind}_K((p_T, q_T), B_K(0, R)) = \text{Ind}_K((p_0, q_0), B_K(0, R)) = \text{deg}_K(A + F(0, \cdot), B_K(0, R)).
\]

Now it remains to show that \( \text{deg}_K(A + F(0, \cdot), B_K(0, R)) \neq 0 \). To this aim, let \( \Phi_t : K \to K \), where \( t \geq 0 \), denote the Poincaré \( t \)-operator associated with the problem
\[
\dot{u} = Au + F(0, u), \quad t \geq 0, \quad u \in K.
\]
Observe that \( \Phi_t \) is well-defined for all \( t \geq 0 \) since for \( x \in K \) solutions \( u \) to (70) starting at \( x \) and living in \( K \) are defined on the whole half-line \([0, +\infty)\), i.e. for \( x \in K \),
\[
\Phi_t(x) = \{ u(t) \mid u \text{ solves (70)}, u(s) \in K \text{ for all } s \geq 0, u(0) = x \}.
\]
Exactly as before, we see that \( \Phi_{t_2} \), \( t \geq 0 \), is a well-defined compact \( c \)-admissible map. Again in view of Theorem 4.1, there is \( t_1 > 0 \) such that
\[
\text{deg}(A + F(0, \cdot), B_K(0, R)) = \text{Ind}_K((\xi_t, \eta_t), B_K(0, R)),
\]
where \( (\xi_t, \eta_t), t \geq 0 \), is the \( c \)-admissible pair representing \( \Phi_t \) defined analogously as in (26) and (27).

Repeating the above argument we see that
\[
\text{Ind}_K((\xi_t, \eta_t), B_K(0, R)) = \text{Ind}_K((\xi_{t_1}, \eta_{t_1}), B_K(0, R)) = \text{deg}(A + F(0, \cdot), B_K(0, R))
\]
for any fixed \( t \geq t_1 \).

Let \( x \in K \), \( \|x\| = R \) and take a solution \( u \) to (70), \( u(0) = x \), i.e. \( V(x) \leq \beta \). The set
\[
X := \{ t \in [0, +\infty) \mid V(u(s)) > \alpha \text{ for all } s \in [0, t] \}
\]
is nonempty, open and for \( t \in X \)
\[
V(u(t)) - V(x) = \int_0^t \langle \nabla V(u(s)), Au(s) + w(s) \rangle \, ds \leq -t_\varepsilon,
\]
where \( w(s) \in F(0, u(s)) \) a.e. on \([0, +\infty), i.e.
\[
\alpha < V(u(t)) \leq V(x) - t_\varepsilon \leq \beta - t_\varepsilon
\]
and \( t < t_2 := \frac{\beta - \alpha}{\varepsilon} \). This implies that \( \Phi_{t_2}(x) \subset V^{-1}((\varepsilon, \alpha]) \subset D_E(0, r) \).

Let us consider a pair
\[
K \times [0, 1] \xrightarrow{\xi} Y \times [0, 1] \xrightarrow{\eta} K,
\]
where \( Y := \{ (x, u) \in K \times C([0, t_2], K) \mid u \text{ solves (70)}, u(0) = x \}, \xi(x, u, z) := (x, z), \eta(x, u, z) := zu(t_2) \text{ for } (x, u) \in Y \text{ and } z \in [0, 1]. \) It is clear that (71) provides a compact \( c \)-admissible homotopy joining \( (\xi_{t_2}, \eta_{t_2}) \) with the \( c \)-admissible pair representing the constant map equal to 0. It has no fixed point on the boundary of \( B(0, R) \) since for \( x \in K \), \( \|x\| = R \) and any solution to (70) such that \( u(0) = x \), we have \( \|u(t_2)\| \leq r < R \). This shows that
\[
\text{Ind}_K((\xi_{t_2}, \eta_{t_2}), B_K(0, R)) = 1.
\]
and ends the proof of (68). \( \square \)
Example 4. We shall study conditions concerning (2) subject to the Dirichlet boundary conditions implying that assumptions of Theorem 5.4 are met. Assume \((\varphi_1), (\varphi_3)\) and \((\varphi_2)\) with constant \(\alpha\). Let \(\xi \in C^2(\mathbb{R}^N, \mathbb{R})\). Assume that
\[
|\nabla \xi(y)| \leq a + b|y|, \quad y \in \mathbb{R}^N,
\]
for some \(a \geq 0\) and \(0 \leq b < \frac{1}{2}\). Then, for all \(y \in \mathbb{R}^N\),
\[
|\xi(y)| \leq c + b|y|^2,
\]
where \(c := |\xi(0)| + a\). Let
\[
h(y) := \frac{1}{2}|y|^2 - \xi(y), \quad y \in \mathbb{R}^N,
\]
\(E := L^2(\Omega, \mathbb{R}^N)\) and consider \(V : E \to \mathbb{R}\) given by
\[
V(u) := \int_{\Omega} h(u(x)) \, dx = \frac{1}{2}\|u\|^2 - \int_{\Omega} \xi(u(x)) \, dx, \quad u \in E.
\]
In view of (73) and (72), \(V\) is well-defined and \(V \in C^1(E, \mathbb{R})\); moreover
\[
\langle \nabla V(u), v \rangle = \int_{\Omega} \langle u(x), v(x) \rangle \, dx - \int_{\Omega} \langle \nabla \xi(u(x)), v(x) \rangle \, dx \quad \text{for} \ u, v \in E.
\]
By (73),
\[
|V(u)| \leq c\mu(\Omega) + \left(\frac{1}{2} + b\right)\|u\|^2, \quad V(u) \geq \left(\frac{1}{2} - b\right)\|u\|^2 - c\mu(\Omega)
\]
and, by (72),
\[
\|\nabla V(u)\| \geq (1 - b)\|u\| - a\sqrt{\mu(\Omega)}.
\]
This implies \(V\) is bounded on bounded sets, \(V(u) \to \infty\) as \(\|u\| \to \infty\) and \(\|\nabla V(u)\| \neq 0\) for all sufficiently large \(\|u\|\). Hence \(V\) satisfies \((V_1)\).

Now we suppose that
\[
\nabla \xi(0) = 0; \quad (\xi''(y)v, v) \leq |v|^2 \quad \text{for all} \ y, v \in \mathbb{R}^N,
\]
where \(\xi''(y)\) stands for the Hessian matrix of \(\xi\) at \(y\); and
\[
\limsup_{|y| \to \infty} \sup_{z \in \varphi(t,x,y)} \left| \langle \nabla h(y), z \rangle \right| = \ell < 0
\]
uniformly with respect to \(t \in [0, T]\) and \(x \in \Omega\).

Then, given \(A\) and \(F\) defined as in section 1.2, for any \(u \in D(A)\) and \(t \in [0, T]\)
\[
\sup_{v \in F(t,u)} \langle \nabla V(u), Au + v \rangle < -\varepsilon
\]
for some \(\varepsilon > 0\), provided \(\|u\|\) is sufficiently large.

Indeed, by (77), there is \(R > 0\) such that for all \(x \in \Omega\) and \(t \in [0, T]\)
\[
\sup_{z \in \varphi(t,x,y)} \langle \nabla h(y), z \rangle < \frac{\ell}{2}|y|^2
\]
provided \(|y| \geq R\). Let \(M = \sup\{\langle \nabla h(y), z \rangle - \ell|y|^2 \mid |y| \leq R, z \in \varphi(t,x,y), t \in [0, T], x \in \Omega\}\). In view of \((\varphi_2)\) and (72), \(M < \infty\). Let \(t \in [0, T], u \in E, v \in F(t,u)\)
and $\Omega_R := \{ x \in \Omega \mid |u(x)| \geq R \}$. Then $v \in E$ and
\[
\langle \nabla V(u), v \rangle - \ell \| u \|^2
= \int_{\Omega \setminus \Omega_R} \left( \langle \nabla h(u), v \rangle - \ell \| u \|^2 \right) dx + \int_{\Omega_R} \langle \nabla h(u), v \rangle - \ell \| u \|^2 \rangle dx \leq M_\mu(\Omega) - \frac{\ell}{2} \| u \|^2.
\]
Thus,
\[
\sup_{v \in F(t,u)} \langle \nabla V(u), v \rangle \leq M_\mu(\Omega) + \frac{\ell}{2} \| u \|^2 \to -\infty \text{ as } \| u \| \to \infty. \quad (79)
\]
To show that for $u \in D(A) \subset H^1_0(\Omega, \mathbb{R}^N)$ with large $\| u \|$, one has
\[
\langle \nabla V(u), Au \rangle \leq 0 \quad (80)
\]
take $u \in H^1_0(\Omega, \mathbb{R}^N)$ and note that, due to $(75)$, $\nabla h(u(\cdot)) \in H^1_0$. By the (generalized) Green formula, $(74)$ and $(76)$
\[
\langle \nabla V(u), Au \rangle = -\int_{\Omega} Du \cdot Du + \int_{\Omega} \xi''(u)Du \cdot Du dx = \int_{\Omega} (\xi''(u) - I)Du \cdot Du dx \leq 0.
\]
Clearly $(79)$ and $(80)$ imply $(78)$ and $(V_2)$.

Consequently if $\varphi$ in $(2)$ satisfies conditions $(\varphi_1) - (\varphi_3)$ ($\alpha$ in $(\varphi_2)$ is constant) and conditions $(72)$, $(73)$, $(75)$, $(76)$ and $(77)$ are met then $(2)$ admits a periodic solution.

**Remark 11.** (i) If $K = E$, then assumption $(V_2)$ may be relaxed. Namely one can show that $(2)$ has a periodic solution if instead of $(V_2)$
\[
\sup_{v \in F(t,u)} \langle \nabla V(u), Au + v \rangle \leq 0 \text{ and } \nabla V(u) \neq 0
\]
for $u \in D(A)$ with $\| u \| \geq R_0$.

(ii) Finally, let us observe that techniques discussed in this Section allow to study the existence of the so-called *antiperiodic* solutions to $(14)$ or $(2)$ yielding immediate generalizations of results in e.g. [20], or solutions $u \in C([0, T, K)$ to $(14)$ such that $u(0) = g(u(T))$, where $g : K \to K$ is a given continuous map. For this reason, it is sufficient to look for fixed points of the $c$-admissible and compact map $K \ni x \mapsto g \circ \Sigma(x)$.

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