IMPLEMENTATION OF THE VEHICULAR OCCUPANCY-EMISSION RELATION USING A CUBIC B-SPLINES COLLOCATION METHOD

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Abstract. The complexity and non-linearity of flow phenomena are explained by numerous criteria, including the interactions of the large number of vehicles occupying the road, which influence the road density. This density under certain conditions, leads to traffic congestion which has dangerous effects on the environment such as; resources consumption; noise and the effect caused by greenhouse gas emissions of the $\text{CO}_2$ and other pollutants. In this paper we consider working in an uniform, homogeneous road where the traffic is described by the Lighthill Whitham-Richard (LWR) model resolved using a cubic B-spline collocation scheme in space and an implicit Runge Kutta scheme in time. We also shed light on the relation between vehicle occupancy and vehicle emissions.

1. Introduction. Road traffic is a complex phenomenon, due to the high number of actors involved and to the highly meshed nature of the network on which it operates (highways, arterial roads, surfaces treets, and other kinds of roadways). However, traffic conditions are becoming increasingly congested. Therefore, the performance of road networks demines and affects economic, social and security activities [6].

To understand the congestion, it is necessary to keep in mind that it is a phenomenon which occurs when the demand (the number of vehicles seeking to use a given infrastructure) exceeds the capacity of an infrastructure. If the demand exceeds the capacity, then vehicles slow down at the entrance of the infrastructure and forme a cork. These surplus vehicles become more numerous at every moment than at the previous one. Since each vehicle occupies a certain length of the road, the length of the queue increases in proportion to the number of vehicles in that queue. Therefore, we conclude that congestion is an evolutionary phenomenon,

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both in time and space. In order to increase the capacity of road networks or reduce their load, many strategies have been pursued. One of them consists on expanding existing road facilities by adding new roads or lanes, rebuilding key network components, and enhancing the physical condition of roadways. While, another one consists on applying traffic management and operation technologies, including automatic highways, travel demand management, freeway management, incident management, emergency response management, weather response management, arterial signal control and on-ramp metering. According to the literature the second strategy is more preferable compared to the first because of its limits like the huge construction costs and the difficulty of addressing public and environmental concerns. Even that the two strategies aim to improve the mobility, their success relies on a better understanding of the properties of congestion based on some conditions concluded based on field data collection. However, this collection is not an easy job, so to accurately estimate traffic conditions in a road network during certain time periods and understand the evolution pattern of traffic conditions, we use what called traffic models.

Traffic models are defined as functions which relate the movement of a vehicle to the driver’s behavior, vehicle type, network characteristics, weather conditions, traffic signals, guidance information and the interactions with other vehicles. Their role is to help to understand well; the phenomena related to traffic, better find out traffic dynamics, or to serve as a simulation platform [38]. Also they are classified according to many criteria like: the level of details adapted, traffic conditions, roadway conditions and traffic network structure. Generally we find three types of models or three approaches: macroscopic, mesoscopic and microscopic.

The macroscopic approach treats the traffic as a gas of interacting particles, in which, each particle represents a vehicle, resulting a model called a kinetic model. Or as a compressible fluid and the resulting model in this case is called continuum model, where the basic characteristics are, the traffic flow $q$, the traffic density $\rho$ and the travel speed $v$, which are all functions in time and space. While the microscopic approach studies the movement of each vehicle and the interactions between the vehicle pairs. Since 1930, several studies have been made to describe traffic dynamics using models. Among these studies, we find the Greenshields who studied volume models and speed, one year later a study made [1] applying the probability theory to describe the traffic. Thus, during the 1950s, several studies were carried out based on different theoretical approaches, such as queueing [37], vehicle tracking [33, 9], also the theory of the road flow [26, 35]. Besides others appeared during the 1970s. Recently Catalin et al [7] inspired from the hydrodynamic theory, proposed a new model to describe traffic dynamics, while in [8] he proposed in collaboration with other authors, another model based on fuzzy logic. Until today the traffic knew an interesting development by improving the existing models, or proposing new ones. For more details we refer the reader to [11].

Traffic congestion is an intuitively simple concept, but its analysis is actually rather complex. Congestion becomes heavy at low speeds and contributes to $CO_2$ emissions, it is also known that $CO_2$ is proportional to fuel consumption and vehicle speed, that’s why a large range of models has been proposed to predict fuel consumption and emission rates. In particular, an obvious consideration is given to reduce levels of $CO_2$ through better operational level planning. Measuring and reducing emissions requires good estimations to be fed into planning activities, which in turn require estimation models to be incorporated into the planning methods.
There exist many analytical emission models, which differ in the way they estimate fuel consumption or emissions, and which can be divided based on the network type [2] or based on emission factors.

The rate of emissions is affected by many factors categorized according to, vehicle characteristics (vehicle mass, engine size, engine type, transmission type, tire type and size, tire pressure, the status of brake and carburetion systems, engine temperature, oil viscosity, gasoline type, vehicle shape and the degree of use of auxiliary electric devices such as air-conditioning), environment conditions (roadway gradient, wind conditions, ambient temperature, altitude and pavement type), driver behavior (the degree of driver aggressiveness) and traffic conditions (speed, number of stops, speed noise and acceleration noise). In the context of reducing the environmental consequences of traffic, some studies have treated, for example, the case of vehicle routing, where Palmer [32] has proposed an integrated routing and emissions model in which he has investigated the role of speed in reducing CO₂. Maden et al [27] have considered a vehicle routing and scheduling problem with time windows in which speed depends on the travel time, whereas Jabali et al [22] also have considered a similar problem as Maden to estimate emissions based on a nonlinear function of speed among other vehicle specific parameters, then Fagerholt et al [15] have treated the problem of reducing emissions in shipping for a given road. A new approach was proposed by Bektas et al [3] combining energy with emissions and based on the pollution routing problem (PRP) with time windows, while Franceschetti et al [16] have suggested a time-dependent pollution routing problem model which extends the PRP in order to take into account traffic congestion.

For one dimensional flow models, the LWR model is the most seminal work in macroscopic traffic flow modeling and it is widely used since its appearance in 1950 and many numerical schemes have been used to compute numerical solutions to it, such as finite difference methods (FMD) and others based on FMDs, such as the Godunov scheme [24], the upwind scheme [25] and Lax-Friedrichs scheme [19, 40]. For more details on how to solve LWR model numerically, we refer the reader to [39].

In this paper, we are interested with two notions, occupancy (which is related to congestion) and emissions, which according to the literature they are very related. Our aim is to shed light on this relation using a B-spline collocation method and other numerical methods. The calculation of the vehicle occupancy is based on the computation of the density in time and space. To do this we have opted for a B-spline collocation method to calculate density in space and an implicit Runge-Kutta method in time. Knowing the density helps also to know the Aggregate emission rate according to a predefined relationship that relates them. In addition, this paper is not a comparative analysis of our used method with the other methods used in the literature, our aim is to use a new scheme never used to resolve the LWR method and to have some analysis based on the results found and to see how it can be accurate.

The work presented herein is organized as: section 2, is dedicated to present the Lighthill, Whitham and Richards model (LWR) as a model used to describe and predict the evolution of density throughout a given section of road. Given that B-spline is an approximate method, section 3, presents a summary about approximation and interpolation methods and a detailed description of B-spline and the collocation method. In section 4, we present the results of our methods and their interpretation, while section 5 provides some concluding remarks and prospects.
2. **LWR model.**

2.1. **Nomenclature.** The notations considered in this paper are shown in the table below.

| Notation | Signification |
|----------|----------------|
| ρ        | traffic density (veh/m) |
| v        | travel speed (s) |
| q        | traffic flow (veh/s) |
| t        | temporal variable |
| x        | spatial variable |
| [t₀, T]  | temporal domain |
| [a, b]   | spatial domain |
| L        | link length (m) |
| ∂ₜ      | partial derivative with respect to time t |
| ∂ₓ      | partial derivative with respect to location x |
| ρ_jam   | jam density, the density when traffic is so heavy that it is at a complete standstill (veh/m) |
| ρ₀(x)    | initial value (veh/m) |
| g₁(t), g₂(t) | boundaries values |
| ξ₀, ξ₁, ..., ξₘ | control points |
| h        | spatial step |
| Δt       | temporal step |
| N(t)     | link occupancy (veh) |
| AER      | aggregate emission rate (gram/s) |
| M        | vehicle mass (kg) |
| ac       | vehicle acceleration (m/s²) |
| θ        | road grade angle |
| r(t)     | hydrocarbon emission rate (gram/s) |
| Z        | overall instantaneous total power demand (kilowatt) |

2.2. **Considered model.** As already mentioned, the macroscopic approach adopted to develop models consists on appealing to analogies, like the first model for the atom which was based on an analogy of the planetary motion of the solar system. As well as for traffic flow, it can be modeled by analogy of a compressible gas or by using the statistical tools developed for kinetic theory. The LWR is a macroscopic model, also called hydrodynamic model. This model relies on the assumption that there exists an equilibrium speed-density relationship. Like other dynamic continuum flow models, the LWR is based on the mass conservation, describes the traffic as a compressible fluid by a system of a first-order, nonlinear hyperbolic PDE:

\[
\partial_t \rho(x, t) + \partial_x f(\rho(x, t)) = 0 \tag{1}
\]

LWR describes traffic dynamics in terms of the formation, propagation and interaction of kinematic waves. Compared to other models, it has a simple mathematical structure and the capability of capturing realistic traffic network phenomena such as shock waves and vehicle spillback, also describes the temporal-spatial evolution of vehicle density on an homogeneous link.

In equation 1, the fundamental diagram \( f(.) \) is a continuous and concave function defined in \([0, \rho_{jam}]\). Classical mathematical results on first-order PDEs of the form 1 can be found in [5]. While \( t \) and \( x \) are successively defined in \([t₀, T]\) and \([a, b]\).

The fundamental diagram relates each two of the three variables: travel speed, flow and density to each other. If two of these variables are known, the third can be derived using the relation \( q = \rho v \). Since their appearance with Greenshields [20], many theories have been developed to work with the fundamental diagram, including the often used LWR model. Many shapes have been proposed for the
fundamental diagram. The Greenshields one is called a univariate model. Another well known univariate model is the Drake’s model [13], where the speed is an exponentially decreasing function of the density. The simplest variate model is the triangular fundamental diagram, which describes both congested and uncongested regime using a straight relationship. Daganzo [10] introduced the truncated triangular fundamental diagram. Koshi et al [23] and Wu [41] introduced an inverse lambda shaped fundamental diagram. In this paper the flow is defined according to the relation

\[ f(\rho) = \gamma \rho (\rho_{jam} - \rho) \]

We suppose that the road is assumed to be uniform, homogeneous, without intersections, without red lights or other embellishments that could be taken into account in a more sophisticated approach. The approach taken here is that of continuous modeling: that is, road traffic will not be represented by the individual position of each car. We are rather interested in average quantities such as, traffic density which correspond to the number of cars per unit of road length. As already mentioned, the objective of this paper is to estimate the aggregate emission rate emitted by vehicles as a function of traffic density, so we will need to calculate the two rates. Concerning the density rate will be calculated via the resolution of the equation 1, while the emission rate will be discussed in what follows.

3. Interpolation and approximation.

3.1. Principle. Physical phenomena modeling is based on solving partial differential equations. In the vast majority of cases, the partial differential equations can be nonlinear, in this case a computer must be used to solve them. While in the linear case, it is more interesting to solve the equations analytically. One of the first powerful tools for solving differential equations was the differential analyzer. It was invented in 1876 by James Thomson, but with the evolution of the computer, digital methods can emerge. There are three main ones. One of them is the finite differences [21] which is based on the Euler method, a method published by Leonard Euler in 1768. Many years after, the finite elements method for solving PDE appeared based on Galerkin method. Thereafter the finite volumes method was invented in 1971 by McDonald mainly motivated by the mechanics of fluids and the study of conservation laws.

Among the objectives of this paper is to solve the partial differential equation 1, which is hyperbolic type and which can be solved using the mentioned methods above. In this order, we resorted to the interpolation and approximation methods.

There exist different types of interpolation, the simplest one is the linear interpolation, which consists in joining the given points. While the quadratic interpolation aims to connect the points by second degree curves. The problem with this type of interpolation is that; some breaks appear at the joins of the curve. This method therefore has no advantage over linear interpolation. To deal with this problem, we impose the curve to pass through two imposed points and at each given point, the slope must be equal to the slope of the end of the preceding parabola, but this solution fails, because the choice of initial slope is very important and the curve can have a completely different appearance according to this choice. Cubic interpolation is based on the same principle as quadratic interpolation, with the conditions that; the curve must pass through two imposed points and impose the tangents at these
two points. The shortcomings of this cubic interpolation is that, if the slope does not have a break, the radius of curvature has breaks and this affects the fluidity of the curve [18].

As an alternative, instead of interpolating points by arcs of parabolas or cubics, we use a single polynomial function which connects all points (Lagrange interpolation). The disadvantage of this method is that, more the number of points to manipulate increases, more the polynomial degree increases, so the manipulations become heavy.

3.2. Spline and B-splines.

3.2.1. Generality. Spline interpolation was developed by the automotive industry at General Motors around 1950. Cubic spline interpolation is based on the same principle as cubic interpolation. The difference is that not only the continuity of the slope, but also the radius of curvature were looked for. Therefore the continuity of the derivative is not sufficient, we also impose that of the second derivative. Cubic spline is a good way for drawing on computer, despite that the curve is much less robust than the cubic interpolation since all the points are linked, therefore, the change of a point influences all the system of equations. Spline functions can be integrated and differentiated due to being piecewise polynomials and since they have basis with small support. Thus, spline functions are adapted to numerical methods to get the solution of the differential equations.

The revolutionary idea of the Bezier curves is the use of the control points and not the interpolation points. This means that the curve does not pass through the given points but approaches them. The Bezier curves are therefore not interpolation but approximation. The Bezier method is based on the deformation of space, starting from a simple curve on a system of axes, then deforming the space, which changes the curve. The advantages of this, is that the curve is stable, it is easy to deform the curve without unexpected results, easy to modify the curve by modifying only the control points and the placement of the control points is relatively straightforward. The problem with Bezier curves is that, since the curve does not go through the control points, it can be difficult to control the curve, although, the use of the control points often facilitates the design. Thus, Bezier curves would not be very effective in drawing a trend line based on measurements. For this type of problems, we turn more towards cubic interpolation and cubic splines. Also, changing a point moves the entire curve and the degree of the curves. Indeed, for a complex form, we have to use a lot of control points so the degree of the curve become higher [12].

It was necessary to think about constructing a curve which possesses all the advantages of Bezier curve but without its disadvantages. Thus, the curve should approximate the control points; be simple to handle; have the same properties as Bezier curve; the degree of the curve should not be proportional to the number of control points and has a local propagation (the modification of a control point should not affects the rest of the control points). B-splines are extension of Bezier curves, their main idea is to replace the Bernstein polynomials used for the construction of Bezier curves by functions, then summon these functions with the control points to get the curve. A B-spline depends not only on the control points but also on a knot vector.

3.2.2. Construction of a B-spline. A spline is a piecewise polynomial function defined in a region $D$, such that there exists a decomposition of $D$ into subregions in
each of which the function is a polynomial of some degree $k$. The piecewise polynomial is considered, and $[a, b] \subset \mathbb{R}$ is a finite interval. We introduce a set of partition $\Delta_m = x_0, ..., x_m$ of $[a, b]$. The set of piecewise polynomial of degree $k$ defined on a partition $\Delta_m$ is denoted by $S_k(\Delta_m)$ in each subinterval; $I_i = [x_{i-1}, x_i]$ is a $k$th degree polynomial. Here we consider an equidistance partition. Moreover, we take $h = \frac{b-a}{m}$, $x_0 = a$ and $x_i = x_0 + ih$, $i \in \{0, 1, ..., m\}$.

The B-spline of degree $k$ is denoted by $B^k_i(x)$ and then we have the following properties:

- recurrence relation: for $k \geq 1$; $B^k_i(x) = \omega^k_i B^k_{i-1}(x) + (1 - \omega^k_{i+1}) B^k_{i+1}(x)$
- support and positivity: $B^k_i(x) > 0$, $x_i < x < x_{i+k}$ while $x_i = x_{i+k} \Rightarrow B^k_i(x) = 0$
- positive and local partition of unity: each $B^k_i$ is positive on $(x_i, x_{i+k})$, is zero off $[x_i, x_{i+k}]$ and $\sum_i B^k_i(x) = 1, \forall x \in \mathbb{R}$
- local linear independence
- differentiation

The B-spline of degree 0 is defined by $B^0_i(x) = \begin{cases} 1 & x_i \leq x < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$

If we generate, the B-spline of order $k$ is defined as follows:

$$B^k_i(x) = \left( \frac{x - x_i}{x_{i+1} - x_i} \right) B^{k-1}_i(x) + \left( \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} \right) B^{k-1}_{i+1}(x); \ k \geq 1$$ (2)

With the function $B^0_i$ as a starting point and the formula 2, we obtain the higher degree B-splines. Hence, we can obtain the B-spline of various orders by taking various values of $k$.

One quality of B-splines is that the continuity conditions are inherent. Hence, the B-splines are the smoothest interpolating function compared with other piecewise polynomial interpolating functions. The other quality is that they have small local support, i.e. each B-spline function is only non-zero over a few mesh subintervals. Due to their smoothness and capability to handle local phenomena, B-splines offer distinct advantages.

3.3. Collocation method. To solve partial differential equations (PDEs), finite element method is the most widely used to the weak form, which is derived by multiplying the strong form of the governing differential equation with a weight function and integrate over the computational domain, which leads to the so-called weighted residual formulation. Collocation methods are an alternative approach in which the strong form of the PDE is enforced at a set of sites called collocation points.

A collocation method is a method for numerical solution of ordinary differential equations, partial differential equations and integral equations. The idea is to choose a finite-dimensional space of candidate solutions and a number of points in the domain (called collocation points), and to select that solution which satisfies the given equation at the collocation points.
In recent years, B-splines collocation method has been studied extensively for the solution of various problems involving differential equations. For example, used for the resolution of the hyperbolic telegraph equation [28], diffusion problems [4], convection-diffusion equations [29], nonlinear parabolic partial differential equations [30], Burgers’ equation [36], time fractional gas dynamics equation [14], generalized Black-Scholes equation governing option pricing [31] and class of partial integro-differential equation [17].

In this paper, we have developed a collocation method which is based on cubic B-splines basis functions for solving the equation 1.

We suppose having the equally-spaced points \( x_0 \leq \ldots \leq x_m \) called Knots. Our numerical treatment for solving equation 1 using the collocation method with cubic B-spline is to find an approximate solution \( U_{\text{app}}(x,t) \) to the exact solution \( u(x,t) \) in the form:

\[
U_{\text{app}}(x,t) = \sum_{j=-1}^{m+1} \alpha_j(t) B_j(x)
\]

Where \( \alpha_j(t) \) are unknown time dependent quantities to be determined from the boundary conditions and collocation from the differential equation. \( B_j(x) \) is the cubic B-splines at the knots given by:

\[
B_j(x) = \begin{cases} 
(x_{j+2} - x)^3 - 4(x_{j+1} - x)^3 + 6(x_j - x)^3 - 4(x_{j-1} - x)^3 & x_{j-2} < x \leq x_{j-1} \\
(x_{j+2} - x)^3 - 4(x_{j+1} - x)^3 + 6(x_j - x)^3 & x_{j-1} < x \leq x_j \\
(x_{j+2} - x)^3 - 4(x_{j+1} - x)^3 & x_j < x \leq x_{j+1} \\
(x_{j+2} - x)^3 & x_{j+1} < x \leq x_{j+2} \\
0 & \text{otherwise}
\end{cases}
\]

The coefficients of \( B_j(x) \) and its derivatives are given in Table 1

| \( j \) | \( x_{j-1} \) | \( x_j \) | \( x_{j+1} \) |
|---|---|---|---|
| \( \frac{1}{h} \) | \( \frac{1}{h} \) | \( \frac{1}{h} \) | \( \frac{1}{h} \) |
| \( \frac{1}{h^2} \) | 0 | \( \frac{1}{h^2} \) | \( \frac{1}{h^2} \) |
| \( \frac{1}{h^3} \) | \( \frac{1}{h^3} \) | \( \frac{1}{h^3} \) | \( \frac{1}{h^3} \) |

Table 1. Coefficients of \( B_j(x) \) and its derivatives

To construct a numerical solution, consider nodal points \((x_i, t_l)\) defined in the region \([a, b] \times [t, T]\) where:

We consider the same mesh already mentioned:

\( a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b \), \( x_{i-1} - x_i = h \) for \( i = 0, \ldots, m \)

and

\( t = t_0 < t_1 < \ldots < t_{n-1} < t_n = T \), \( t_{l-1} - t_l = \Delta t \) for \( l = 0, \ldots, n \)

The values of the approximate solution \( U_{\text{app}}(x,t) \) are derived from the equation 3
and the Table 1 in the following form

\[ U_{app}(x_i, t) = \sum_{j=-1}^{m+1} \alpha_j(t) B_j(x_i) = \frac{1}{6}(\alpha_{j-1}(t) + 4\alpha_j(t) + \alpha_{j+1}(t)) \]  

Where the knots \( x_i \) are used as collocation points.

4. **Implementation and discussion results.**

4.1. **Density approximation.** The LWR equation is given by

\[ \begin{align*}
\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} f(\rho(x, t)) &= 0 \quad \forall x \in [a, b], \forall t \in [t_0, T] \\
\rho(a, t) &= g_1(t); \quad \rho(b, t) = g_2(t) \quad \forall t \in [t_0, T] \\
\rho(x, t_0) &= \rho_0(x) \quad \forall x \in [a, b].
\end{align*} \]  

where the flow is defined as

\[ f(\rho) = \gamma \rho (\rho_{jam} - \rho) \]

Cubic B-splines used for setting up a trial function over the domain \([a, b]\). So an approximation \( \rho_{app}(x, t) \) to the exact solution \( \rho(x, t) \) is sought by

\[ \rho_{app}(x_i, t) = \sum_{j=-1}^{m+1} \alpha_j(t) B_j(x_i) \]  

where the \( \alpha_j \) to be determined from the cubic B-spline collocation form of the LWR equation. Using the equations above, the values of \( \rho_{app} \) and \( \rho_{app}' \) at the knots \( x_i \) are obtained in terms of the element parameters by

\[ \begin{align*}
\rho_{app} &= \frac{1}{6}(\alpha_{j-1}(t) + 4\alpha_j(t) + \alpha_{j+1}(t)) \\
\partial_t \rho_{app} &= \frac{1}{6}(\alpha'_{j-1}(t) + 4\alpha'_j(t) + \alpha'_{j+1}(t)) \\
\partial_x \rho_{app} &= \frac{1}{2h}(\alpha_{j-1}(t) - \alpha_{j+1}(t))
\end{align*} \]  

Substituting nodal values \( \rho_{app} \) and first derivative into the formula 6 yields the following matrix system of the first order ordinary differential equations

\[ \frac{1}{6}(\alpha'_{j-1}(t) + 4\alpha'_j(t) + \alpha'_{j+1}(t)) = -\frac{\gamma(x_i, t)}{2h} (\alpha_{j-1}(t) - \alpha_{j+1}(t)) + F(x_i, t) + \varphi(\alpha_j(t), t)) \]

With \( F(x, t) \) in the formula 9 is the source term (second member) and \( \varphi \) is the non-linearity term. And

\[ \alpha_{-1} = 2\alpha_0 - \alpha_2 \text{ and } \alpha_{m-1} = 2\alpha_m - \alpha_{m+1} \]  

Putting expressions 10 in the matrix system leads to the relation between the element parameters.

For \( x_0 \) according to the boundary conditions

\[ \rho(x_0, t) = g_1(t) = \frac{1}{6}(\alpha_{-1}(t) + 4\alpha_0(t) + \alpha_1(t)) \]

so

\[ \alpha_0(t) = g_1(t) \]

while
The above system consists of \( m \) unknown parameters. Always according to the boundary conditions

\[
\rho(x_m, t) = g_2(t) = \frac{h}{6} (\alpha_{m-1}(t) + 4\alpha_m(t) + \alpha_{m+1}(t))
\]

so

\[
\alpha_m(t) = g_2(t)
\]

while

\[
\alpha'_m(t) = -\frac{\gamma(x_m, t)}{h} (\alpha_{m-1}(t) - g_2(t)) + F(x_m, t)
\]

For \( x_m \)

The boundary conditions \( \rho(a, t) = g_1(t), \rho(b, t) = g_2(t) \) helps to eliminate the parameters \( \alpha_{-1}, \alpha_0, \alpha_m \) and \( \alpha_{m+1} \) so that we reach the solvable \( (m-1)x(m-1) \) matrix system, where.

\[
A = \begin{bmatrix}
4 & 1 & 0 & \ldots & 0 \\
1 & 4 & 1 & 0 & 0 \\
0 & 1 & 4 & 1 & 0 \\
\vdots & 0 & 1 & 4 & 1 \\
\vdots & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and

\[
B = \frac{1}{h} \begin{bmatrix}
\gamma(x_0, t) & -\frac{3}{2} \gamma(x_1, t) & 0 & \ldots & 0 \\
\frac{3}{2} \gamma(x_2, t) & 0 & 0 & \ldots & 0 \\
0 & \frac{3}{2} \gamma(x_3, t) & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & \frac{3}{2} \gamma(x_{m-1}, t)
\end{bmatrix}
\]

For the matrix \( C \) of the constants, we have
\[ C_1(t) = \frac{1}{h}(\gamma(x_0, t) - 3 \gamma(x_1, t))g_1(t) + 6F(x_1, t) - F(x_0, t) \]
\[ C_i(t) = 6F(x_1, t) \quad i = 2, \ldots, m - 2 \]
\[ C_{m-1}(t) = \frac{1}{h}(\gamma(x_m, t) - 3 \gamma(x_{m-1}, t))g_2(t) + 6F(x_m, t) - F(x_{m-1}, t) \]

And for that of non-linearity, we have
\[ \varphi(\alpha_1(t), t) = -\frac{\gamma}{h}(4\alpha_1(t) + \alpha_2(t))\alpha_2(t) - \frac{2\gamma \rho_i^{am}}{h} g_1(t) - \frac{\gamma}{h} g_1(t)^2 \]
\[ \varphi(\alpha_i(t), t) = \frac{\gamma}{h}(\alpha_{i-1}(t) - \alpha_{i+1}(t)(\alpha_{i-1}(t) + 4\alpha_i(t) + \alpha_{i+1}(t)) \]
\[ \varphi(\alpha_{m-1}(t), t) = \frac{\gamma}{h}(4\alpha_{m-1}(t) + \alpha_{m-2}(t)\alpha_{m-2}(t) + \frac{2\gamma \rho_i^{am}}{h} g_2(t) + \frac{\gamma}{h} g_2(t)^2 \]

With
\[
y_0 = \begin{pmatrix}
6 \rho_0(x_1) - \rho_0(x_0) \\
6 \rho_0(x_2) \\
6 \rho_0(x_3) \\
\vdots \\
6 \rho_0(x_{m-2}) \\
6 \rho_0(x_{m-1}) - \rho_0(x_m)
\end{pmatrix}
\]

So we obtain a system of a first order ordinary differential equation of the form
\[
\begin{cases}
Ay'(t) = B(t)y(t) + C(t) + \varphi(y(t), t) \\
y(t_0) = y_0
\end{cases}
\]  

(11)

This paper investigates the application of the B-spline collocation method to find and represent the solution of the LWR equation 6 with boundary conditions by reducing it into a coupled system of equations, which produces a system of first order ordinary differential equations. Adopting such approach was already subject of other works which deal with other equations besides LWR. The difference is in the time scheme used. Working on the telegraph equation, Mittal and Bhatia [28] in their paper, chosed a scheme SSP-RK45, while Mohammadi [31] used a Crank–Nicolson finite-difference scheme for the traitement of the generalized Black–Scholes equation. Gholamian and Saberi [17] adapted a backward Euler method to estimate solutions for a class of partial integro-differential equation. In our paper we vote for an implicit runge kutta of order 5 method. The choice of the implicit method was not random, but because it is more stable than the explicit one especially for being applied for such equation said to be stiff. Runge-Kutta (RK) methods are a popular class of methods for the numerical solution of ODEs.

The RK method is used in a stepwise fashion, starting from the initial point \( t_0 \), and passing through a sequence of points \( t_i \), with computed solution approximations \( y_i \), and ending at the required end point \( t_{end} \). Our goal is to compute an approximate solution \( y_{n+1} \) at \( t_{n+1} = t_n + h \) and each step is almost independent of the previous steps.
Figure 1 shows the values of density in time and space generated using our approach, while Figure 2 compares the computed results with the exact solution. In our case the exact solution equals $e^{-tx^2}$. The same figure shows also how our method is capable of producing highly accurate solution with an estimated error of $10^{-4}$.

![Figure 1. Approximate density (veh/m)](image)

**Figure 1.** Approximate density (veh/m)

![Figure 2. Approximate density Vs exact density (veh/m)](image)

**Figure 2.** Approximate density Vs exact density (veh/m)

4.2. **Link occupancy.** Once vehicle densities are available, we apply the emission model to compute the Aggregate emission rate and the link occupancy $N(t)$:

$$N(t) = \int_a^b \rho(t, x)dx$$  \hspace{1cm} (12)
where the link of interest is expressed as the spatial interval \([a, b]\), \(\rho(x, t)\) is the solution of the LWR PDE.

Replacing \(\rho\) by its value in the equation 12, we get

\[
N(t) = \sum_{j=-1}^{m+1} \alpha_j(t) \int_a^b B_j(x) dx
\]

\[
= \sum_{j=-1}^{m+1} \alpha_j(t) \int_a^b B_0(\frac{x - jh}{h}) dx
\]

\[
= \frac{1}{h} \sum_{j=-1}^{m+1} \alpha_j(t) \int_{-j}^{m-j} B_0(x) dx
\]

The values of the Link occupancy are presented in Figure 3.

**Figure 3.** The variations of Link occupancy in time

**Proposition 1.** Let \(N(t_i) = \sum_{j=-1}^{m+1} \alpha_j(t_i) B_j(x)\) be an approximation of the exact solution \(N(t)\) as given above. We then obtain the following result,

\[
N(t_i) = \frac{1}{24h} \left[ \alpha_{-1}(t_i) + 12\alpha_0(t_i) + 23\alpha_1(t_i) + 24 \sum_{j=2}^{m-2} \alpha_j(t_i) + 23\alpha_{m-1}(t_i) \right.
\]

\[
+ 12\alpha_m(t_i) + \alpha_{m+1}(t_i) \right]
\]

**Proof.** We have \(N(t) = \int_a^b \rho(t, x) dx\) and at points \(t_i\) we obtain the following approximation \(N(t_i) = \int_a^b \rho(t_i, x) dx\). Therefore,

\[
N(t_i) = \frac{1}{h} \sum_{j=-1}^{m+1} \alpha_j \int_j^{m-j} B(x) dx
\]

\[
= \frac{1}{h} \left[ \alpha_{-1}(t_i)I_1 + \alpha_0(t_i)(I_0 + I_1) + \alpha_1(t_i)(I_{-1} + I_0 + I_1) + \sum_{j=2}^{m-2} \alpha_j(t_i) \right.
\]

\[
+ \alpha_{m-1}(t_i)(I_{-2} + I_{-1} + I_0) + \alpha_m(t_i)(I_{-2} + I_{-1} + I_0) + \alpha_{m+1}(t_i)I_{-2} \right]
\]

where \(I_{-2} = \frac{1}{24}, I_{-1} = \frac{11}{24}, I_0 = \frac{11}{24}\) and \(I_1 = \frac{1}{24}\) are obtained by integral computing of B-spline \(B_j(x)\) at this points in equation 4. We then obtain the desired result. \(\square\)
4.3. Aggregate emission rate. In order to estimate emission rate, we investigate a vehicle emission model which is associated with modal events called modal model, consists of the following:

\[ r(t) = \begin{cases} 
52.8 + 4.2Z & Z > 0 \\
52.8 & Z \leq 0 
\end{cases} \]  

(13)

According to [34], the equation 13 calculates the hydrocarbon emission rate (Figure 4), while \( Z \) is the overall instantaneous total power demand \( Z \) for a vehicle given by

\[ Z = (0.04v + 0.5 \times 10^{-3}v^2 + 10.8 \times 10^{-6}v^3) + \frac{M}{1000} \times \frac{v}{3.6} + 9.81 \sin(\theta) \]  

(14)

Where

\[ v(x,t) = \frac{f(\rho(x,t))}{\rho(x,t)} = \gamma (\rho_{jam} - \rho(x,t)) \]  

(15)

While the acceleration \( ac \), viewed as the derivative of the speed along the trajectories of moving vehicle

\[ ac(x,t) = \partial_t v(x,t) + v(x,t) \partial_x v(x,t) \]  

(16)

Then we have

\[ ac(x_i,t_t) = \frac{v(x_i,t_{t+1}) - v(x_i,t_{t-1})}{2 \Delta t} + v(x_i,t_t) \frac{v(x_{i+1},t_t) - v(x_{i-1},t_t)}{2h} \]  

(17)
Following the previous discussion, we compute the Aggregate emission rate (AER) in discrete time as

\[ AER(t_l) = h \sum_i \rho(x_i, t_l) r(x_i, t_l) \] (18)

with

\[ \rho(x_i, t_i) \approx \sum_{k=-1}^{m+1} \alpha_k(t_i) B_k(x_j) \] (19)

By using B-spline we obtain,

\[ \rho(x_i, t_i) \approx \frac{1}{6} (\alpha_{j-1}(t_i) + 4\alpha_j(t_i) + \alpha_{j+1}(t_i)) \] (20)

and we use the expression of emission rate as mentioned in 13

\[ r(x_i, t_i) \approx \begin{cases} 52.8 + 4.2 Z(x_i, t_i) & Z(x_i, t_i) > 0 \\ 52.8 & Z(x_i, t_i) \leq 0 \end{cases} \] (21)

The resulting scatter plot of the link occupancy (LO) vs. the Aggregate emission rate (AER) is shown in Figure 5. Which shows the positive correlation between the Link occupancy and the Aggregate emission rate. It is observed that there existe a well-defined macroscopic relationship between the two quantities. The results in the figure comply with what is expected, so that the total emission rate at the link should generally increases with an increased number of vehicles on this link. The observed relationship between LO and AER may be approximated in a number of ways, e.g. using linear [34], piecewise linear, or piecewise smooth curve fittings and can be quite interesting from a robust optimization perspective.

According to the fundamental diagram, the flow vanishes when the density equals zero or when it achieves it’s maximum and it become maximum when the density is equal to the critical density and as it is illustrated in the Figure 6, we see that this principle is respected also by our approach. Another principle of the fundamental diagram is that when two of the three basic variables (speed, flow, density) are known so we can calculate the third, in our exemple once the density and the flow
are calculated using the cubic B-spline collocation method then we can calculate the values of speed in time and space Figure 7.

![Figure 7. Travel speed](image)

5. **Conclusion.** In this paper, a new numerical resolution for the LWR equation is proposed using a collocation method with the cubic B-spline functions as alternative of the others methods already used like the Godunov method, supply-demand method and the variational method. To this end, the cubic B-spline collocation scheme in space and the implicit RK5 scheme in time have been combined. The accuracy and efficiency of the proposed method was examined. Comparisons of the computed results with exact solutions showed that the scheme is capable of solving
the equation and is also capable of producing highly accurate solutions with minimal computational effort for both time and space. The produced results also seem to be more accurate than some available results given in the literature.

Also among the objectives of this paper, is to implement the relationship relating the occupancy and the aggregate emission rate. The interest of such a relationship is that it can be integrated into optimization problems as objective and/or constraint functions. Optimization problems such as the optimization of the emission rate, which is part of our perspective. Among our perspective is to make comparisons between the results obtained in this paper with other works dealing with the same subject, also to study other variances of the model used.

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