Massive Fields with Arbitrary Integer Spin in Homogeneous Electromagnetic Field

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Abstract

We study the interaction of gauge fields of arbitrary integer spins with the constant electromagnetic field. We reduce the problem of obtaining the gauge-invariant Lagrangian of integer spin fields in the external field to purely algebraic problem of finding a set of operators with certain features using the representation of the high-spin fields in the form of vectors in a pseudo-Hilbert space. We consider such a construction up to the second order in the electromagnetic field strength and also present an explicit form of interaction Lagrangian for a massive particle of spin $s$ in terms of symmetrical tensor fields in linear approximation. The result obtained does not depend on dimensionality of space-time.
1 Introduction

In spite of the long history [1, 2, 3] the problem of obtaining the consistent interaction of high-spin fields is far from its completion now.

The problem of obtaining the consistent description of "minimal" interactions of high-spin fields with Abelian vector field plays a particular role as well as gravitational interactions of such fields. In a sense these interactions are the test ones since they allow one to connect the fields of high spins with the observable world.

It has been realized [4] that one cannot build the consistent "minimal" interaction with an Abelian vector field for the massless fields of spins \( s \geq \frac{3}{2} \) in an asymptotically flat space-time. The same is valid for the gravitational interaction of fields with spin \( s \geq 2 \) [5]. It is possible to argue the given statement as follows [4]: The free gauge-invariant Lagrangian for integer spin fields in the flat space has the structure \( \mathcal{L}_0 = \partial \Phi \partial \Phi \) with transformations \( \delta \Phi = \partial \xi \). The introduction of the "minimal" interaction means the replacement of the usual derivative with the covariant one \( \partial \rightarrow \mathcal{D} \). The gauge invariance fails and a residual of the type \([\mathcal{D}, \mathcal{D}] \mathcal{D}\Phi \xi = \mathcal{R}\mathcal{D}\Phi \xi\) appears, where \( \mathcal{R} \) is the strength tensor of the electromagnetic field or Riemann tensor. In the case of the electromagnetic interaction for the fields with spins \( s \geq \frac{3}{2} \), one cannot cancel the residual by any changes of the Lagrangian and the transformations in the linear approximation. Therefore, in such a case this approximation does not exist, but since linear approximation does not depend on the presence of any other fields in the system. This means that the whole theory of interaction does not exist either. The same is valid for the fermionic fields.

In the case of the gravitational interaction the residual for the field with spin \( \frac{3}{2} \) is proportional to the gravity equations of motion: \( \delta \mathcal{L}_0 \sim i(\bar{\psi} \gamma^\nu \eta)(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \). One can compensate such a residual by modifying the Lagrangian and the transformations. As a result, the theory of the supergravity appears. For fields with spins \( s > 2 \) the residual contains terms proportional to the Riemann tensor \( R_{\mu\nu} \). It is impossible to cancel such terms in an asymptotically flat space. Hence, the gravitational interaction does not exist for any massless fields with spin \( s > \frac{3}{2} \). So, even if the massless high-spin fields possess a nontrivial self-action, in any case they are "thing in itself".

These difficulties can be overcome in several ways. In case of the gravitational interaction, one can consider the fields in a constant curvature space. Then, the Lagrangian
for gravity would have an additional term $\Delta L \sim \sqrt{-g}\lambda$, where $\lambda$ is the cosmological constant. Modification of the Lagrangian and transformations leads to mixing of terms with different numbers of derivatives. This allows one to compensate the residual with terms proportional to $R_{\mu\nu\alpha\beta}$. The complete theory will be represented as series in the inverse value of cosmological constant $[6, 7]$. This means the non-analyticity of theory with respect to $\lambda$ at zero, i.e. impossibility of a smooth transition to the flat space. Such a theory was considered in Refs.[6, 7, 8].

It is also possible to avoid these difficulties if one considers massive high-spin fields $[9, 10]$. In literature the electromagnetic $[11]$ and gravitational $[12, 13]$ interactions of arbitrary spin fields were considered at the lowest order. Under consideration of the interactions, the authors start from the free theory of the massive fields in the classical form $[9]$. The ”minimal” introduction of the interaction leads to contradictions, therefore, it is necessary to consider non-minimal terms in the interaction Lagrangian. Since the massive Lagrangian for spin-$s$ fields $[9]$ is not gauge invariant, in such an approach there are no restrictions of the form of non-minimal interaction and it is necessary to introduce additional restrictions in order to build a consistent theory. For instance, when studying the electromagnetic interaction $[11]$, the authors have used the requirement that tree-level scattering amplitudes must possess a smooth $M \to 0$ fixed-charge limit in any theory describing the interaction of arbitrary-spin massive particles with photons. Under such requirement, the amplitudes do not violate unitarity up to center-of-mass energies $E \gg M/e$. This restriction leads to the gyromagnetic ratio $g = 2$ for massive particles of any spin. When investigating the gravitational interactions $[12, 13]$, the authors have required that tree-level amplitudes saturate the unitarity bounds only at Planck scale.

It seems to us that it is more suitable to use the gauge-invariant approach when one analyzes an interaction of massive fields $[14, 15, 16, 17, 18]$. Under this approach the interaction is considered as a deformation $[17, 18]$ of initial gauge algebra and Lagrangian$[4]$. Although, generally speaking, the gauge invariance does not ensure the complete consistency of massive theories but it allows one to narrow the search and provides the

$^1$Of course, one must consider only a non-trivial deformation of the free algebra and Lagrangian, which cannot be completely gauged away or removed by a redefinition of the fields.
appropriate number of physical degrees of freedom. Besides, this approach is pretty convenient and practical.

At present only the superstring theory claims to have the consistent description of interaction of the high-spin fields. But interacting strings describe the infinite set of fields and the question about interaction of finite number of fields is still open. In the case of the constant Abelian field one can obtain a gauge-invariant Lagrangian that describes the interaction for the fields of each string level. So, in Refs. [16, 18] the first and second massive levels of an open boson string were investigated. These levels contain the massive fields of spin 2 and 3. However, as was shown in Ref. [18], the presence of the constant electromagnetic field leads to mixing of states on each string level. Therefore, it is impossible to obtain an electromagnetic interaction for a single field of spin $s$ in such approach.

In this paper we consider the interaction of an arbitrary massive field of spin $s$ with homogeneous electromagnetic field up to the second order in the strength.

We represent a free state with an arbitrary integer spin $s$ as state $|\Phi^s\rangle$ in a Pseudo-Hilbert space. Tensor fields corresponding to the particle with spin $s$ are coefficient functions in the state $|\Phi^s\rangle$. We introduce a set of operators in the considered Fock space. We define the gauge transformations and necessary constraints for the state $|\Phi^s\rangle$ by means of these operators. The gauge-invariant Lagrangian has the form of expectation value of the Hermitian operator, which consists of these operators, in the state $|\Phi^s\rangle$. Using this construction, in section 2 we obtain the gauge-invariant Lagrangian describing the particles with arbitrary integer spins in the massless and massive case in terms of the coefficient functions.

In the considered approach the gauge invariance is a consequence of commutation relations of the introduced operators. Introduction of interaction by replacing usual derivatives with covariant ones leads to a change of algebraic features of the operators and as a consequence to the loss of the gauge invariance. The problem of restoring the invariance is reduced to an algebraic problem of finding such modified operators depending on the electromagnetic field strength, which satisfy the same commutation relations as initial operators in the absence of external field. It is possible to argue

\footnote{The representation of free fields with arbitrary integer spins in such a form was considered in Refs. [17, 19].}
existence of such operators from the consideration that the gauge transformation algebra in the free case and in the presence of an external field is the same (trivial)\footnote{But transformations for the massive fields are not trivial.}. However, we should note that in the massless case one cannot realize such a construction. For the massive theory (section 3) we construct a set of operators having the algebraic features of free ones up to the second order in strength. Besides, we give an explicit form of the interaction Lagrangian in terms of the tensor fields for a special case of the constructed linear approximation. In the Appendix we dwell on the case of massive spin-2 field.

\section{Free Field with Spin $s$}

\textbf{Massless fields.} Let us consider the Fock space generated by creation and annihilation operators $\bar{a}_\mu$ and $a_\mu$ which are vectors in the $D$-dimensional Minkowski space $\mathcal{M}_D$ and which satisfy the following algebra

$$[a_\mu, \bar{a}_\nu] = g_{\mu\nu}, \quad a_\mu^\dagger = \bar{a}_\mu,$$  \hspace{2cm} (1)

where $g_{\mu\nu}$ is the metric tensor with signature $\|g_{\mu\nu}\| = \text{diag}(-1, 1, 1, \ldots, 1)$. Since the metric is indefinite, the Fock space, which realizes the representation of the Heisenberg algebra (1), is a Pseudo-Hilbert space.

Let us consider the state in the introduced space of the following type:

$$|\Phi^s\rangle = \frac{1}{\sqrt{s}} \Phi_{\mu_1\ldots\mu_s}(x) \prod_{i=1}^s \bar{a}_{\mu_i} |0\rangle.$$  \hspace{2cm} (2)

Coefficient function $\Phi_{\mu_1\ldots\mu_s}(x)$ is a symmetrical tensor of rank $s$ in space $\mathcal{M}_D$. For this tensor field to describe the state with spin $s$ one has to imposes the condition:

$$\Phi_{\mu\nu\mu_1\ldots\mu_s} = 0.$$  \hspace{2cm} (3)

In terms of such fields Lagrangian $\cite{20, 21}$ has the form

$$\mathcal{L}_s = \frac{1}{2} (\partial_\mu \Phi^s) \cdot (\partial_\mu \Phi^s) - \frac{s}{2} (\partial \cdot \Phi^s) \cdot (\partial \cdot \Phi^s) - \frac{s(s - 1)}{4} (\partial_\mu \Phi^s) \cdot (\partial_\mu \Phi^s)$$

$$- \frac{s(s - 1)}{2} (\partial \cdot \partial \cdot \Phi^s) \cdot \Phi^s - \frac{1}{8} s(s - 1)(s - 2)(\partial \cdot \Phi^s) \cdot (\partial \cdot \Phi^s).$$  \hspace{2cm} (4)

\footnotetext[3]{But transformations for the massive fields are not trivial.}
\footnotetext[4]{We consider symmetric tensor fields only.}
The following notation $\Phi' = \Phi_{\mu\ldots}$ is used here while the point means the contraction of all indexes $\Phi^s \cdot \Phi^s \overset{def}{=} \Phi_{\mu_1\ldots\mu_s} \Phi^{\mu_1\ldots\mu_s}$. This Lagrangian is invariant under the transformation

$$\delta \Phi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1} \Lambda_{\mu_2\ldots\mu_{s-1})},$$
$$\Lambda_{\mu_\mu_\ldots\mu_\ldots\mu_{s-1}} = 0.$$  \hspace{1cm} (5)

Let us introduce the following operators in our pseudo-Hilbert space

$$L_1 = p \cdot a, \quad L_{-1} = L_1^\dagger, \quad L_2 = \frac{1}{2} a \cdot a, \quad L_{-2} = L_2^\dagger, \quad L_0 = p^2.$$  \hspace{1cm} (7)

Here $p_\mu = i\partial_\mu$ is the momentum operator that acts in the space of the coefficient functions.

Operators of such type appear as constraints of a two-particle system under quantization\textsuperscript{5} \cite{22}. Operators (7) satisfy the commutation relations:

\begin{align*}
[L_1, L_{-2}] &= L_{-1}, & [L_1, L_2] &= 0, \\
[L_2, L_{-2}] &= N + \frac{D_2}{2}, & [L_0, L_n] &= 0, \\
[L_1, L_{-1}] &= L_0, & [N, L_n] &= -nL_n, \quad n = 0, \pm 1, \pm 2.
\end{align*}

Here $N = \bar{a} \cdot a$ is a level operator that defines the spin of states. So, for instance, for the state (2) we have

$$N|\Phi^s\rangle = s|\Phi^s\rangle.$$  \hspace{1cm} (8)

In terms of operators (8) condition (3) can be written as

$$\langle L_2 \rangle^2 |\Phi^s\rangle = 0,$$

while gauge transformations (5) have the form

$$\delta |\Phi^s\rangle = L_{-1} |\Lambda^{s-1}\rangle.$$  \hspace{1cm} (10)

Here, the gauge state

$$|\Lambda^{s-1}\rangle = \Lambda_{\mu_1\ldots\mu_{s-1}} \prod_{i=1}^{s-1} \bar{a}_{\mu_i} |0\rangle$$

satisfies the condition

$$L_2 |\Lambda\rangle = 0.$$  \hspace{1cm} (11)

\textsuperscript{5}It is also possible to regard operators (7) as a truncation of the Virasoro algebra.
This condition is equivalent to (3) for the coefficient functions.

Lagrangian (4) can be written as an expectation value of a Hermitian operator in the state (2)

\[ L_s = \langle \Phi^s | L(L) | \Phi^s \rangle, \quad \langle \Phi^s | = | \Phi^s \rangle^\dagger, \]

where

\[ L(L) = L_0 - L_{-1}L_1 - 2L_{-2}L_0L_2 - L_{-2}L_{-1}L_1L_2 + \{L_{-2}L_1L_1 + h.c.\}. \] (13)

Lagrangian (12) is invariant under transformations (10) as a consequence of the relation

\[ L(L)L_{-1} = (...)L_2. \]

**Massive fields** Let us consider massive states of arbitrary spin \( s \) in a similar manner. For this we have to extend our Fock space by means of introduction of a scalar creation and annihilation operators \( \bar{b} \) and \( b \), which satisfy the usual commutation relations

\[ [b, \bar{b}] = 1, \quad \bar{b}^\dagger = \bar{b}. \] (14)

Operators (7) are modified as follows:

\[ L_1 = p \cdot a + mb, \quad L_2 = \frac{1}{2}(a \cdot a + b^2), \quad L_0 = p^2 + m^2. \] (15)

Here \( m \) is an arbitrary parameter that has dimensionality of mass. In non-interacting case one can consider such a transition as the dimensional reduction \( \mathcal{M}_{D+1} \rightarrow \mathcal{M}_D \otimes S^1 \) with the radius of sphere \( R \sim 1/m \) (refer also to[13, 19]).

We shall describe the massive field of spin \( s \) as the following vector in the extended Fock space:

\[ |\Phi^s\rangle = \sum_{n=0}^{s} \frac{1}{\sqrt{n!(s-n)!}} \Phi_{\mu_1...\mu_n}(x)\bar{b}^{s-n} \prod_{i=1}^{n} \bar{a}_{\mu_i} |0\rangle. \] (16)

The same as in the massless field case, this state satisfies condition (9) in terms of operators (13). The algebra of operators (8) changes weakly, the only commutator modified is

\[ [L_2, L_{-2}] = N + \frac{D+1}{2}. \] (17)
Here, as in the massless case, the operator \( N = \bar{a} \cdot a + \bar{b}b \) defines the spin of massive states. The Lagrangian describing the massive field of spin \( s \) also has the form (13), where the expectation value is taken in the state (16). Such Lagrangian is invariant under transformations (14) with the gauge Fock vector

\[
|\Lambda^{s-1}\rangle = \sum_{n=0}^{s-1} \frac{1}{\sqrt{(n + 1)!(s - n - 1)!}} \Lambda_{\mu_1 \ldots \mu_n} \bar{a}^{s-n-1} \prod_{i=1}^{n} \bar{a}_{\mu_i} |0\rangle,
\]

which satisfies condition (1).

Having calculated expectation (13) we obtain the explicit expression for the Lagrangian describing the massive state with arbitrary spin \( s \) in terms of the coefficient functions

\[
\mathcal{L}_0 = \sum_{n=0}^{s} \Phi^n \cdot p^2 \Phi^n \left( 1 - C_{s-n}^2 \right) - \sum_{n=2}^{s} \Phi^n \cdot p^2 \Phi^n C_n^2
- \frac{1}{2} \sum_{n=1}^{s} (\Phi^n \cdot p) \cdot (p \cdot \Phi^n) n (2 + C_{s-n}^2) - \frac{3}{2} \sum_{n=3}^{s} (\Phi^n \cdot p) \cdot (p \cdot \Phi^n) C_n^3
- \left\{ \frac{1}{2} \sum_{n=3}^{s} (\Phi^n \cdot p) \cdot (p \cdot \Phi^{n-2}) (n - 2) \sqrt{C_n^2 C_{s-n+2}^2} \right\}
+ \sum_{n=2}^{s} \bar{\Phi}^n \cdot p^2 \Phi^{n-2} \sqrt{C_n^2 C_{s-n+2}^2} - \sum_{n=2}^{s} \bar{\Phi}^n \cdot (p \cdot p \cdot \Phi^n) C_n^2
- \sum_{n=2}^{s} \bar{\Phi}^{n-2} \cdot (p \cdot p \cdot \Phi^n) \sqrt{C_n^2 C_{s-n+2}^2} + h.c. \}
- m \left\{ \frac{1}{2} \sum_{n=1}^{s} (\Phi^n \cdot p) \cdot \Phi^{n-1} (2 - C_{s-n}^1 + C_{s-n}^2) \sqrt{nC_{s-n+1}^1} \right\}
- \frac{1}{4} \sum_{n=2}^{s} \bar{\Phi}^n \cdot (p \cdot \Phi^{n-1}) (n - 1)(4 - C_{s-n}^1) \sqrt{nC_{s-n+1}^1} \right\}
+ \frac{3}{2} \sum_{n=3}^{s} (\Phi^n \cdot p) \cdot \Phi^{n-3} \sqrt{C_n^3 C_{s-n+3}^3} \}
+ \frac{1}{2} \sum_{n=3}^{s} (\Phi^n \cdot p) \cdot \Phi^{n-1} C_n^2 \sqrt{nC_{s-n+1}^1} + h.c. \}
+ \frac{m^2}{2} \left\{ \sum_{n=0}^{s} \Phi^n \cdot \Phi^n (2 - 2C_{s-n}^1 + 2C_{s-n}^2 - 3C_{s-n}^3) \right\}
- 2 \sum_{n=2}^{s} \left\{ \Phi^n \cdot \Phi^{n-2} C_{s-n}^1 \sqrt{C_n^2 C_{s-m+2}^2} + h.c. \right\}
- 2 \sum_{n=2}^{s} \bar{\Phi}^n \cdot \Phi^n C_n^2 (2 + C_{s-n}^1) \}
\]

Here \( \Phi^n \) denotes a symmetrical tensor field of rank \( n \) and we use the following notation
\( C_n^m = \frac{n(n-1)...(n-m+1)}{m!} \). Condition (9) for the fields has the following form in this case

\[
\sqrt{C_n^s C_2^2 n_{n-2} \Phi^{m-n}} + 2 \sqrt{C_n^{n-2} C_2^2 n_{n+2} \Phi^{m-2}} + \sqrt{C_n^{n-1} C_2^2 n_{n+4} C_2^2 s_{n+2} \Phi^{n-4}} = 0, \quad (19)
\]

where \( n = 4, ..., s \). Correspondingly, the gauge transformations have the form

\[
\delta_0 \Phi^{n} = \left\{ p A^{n-1} \right\}_s + m \sqrt{\frac{s-n}{n+1}} A^n. \quad (20)
\]

Here \( \{ ... \}_s \), denotes symmetrization over all indexes. At the same time condition (11) can be written as

\[
C_n^2 A^n + C_{s-n+1}^2 A^{n-2} = 0, \quad n = 2, ..., s - 1. \quad (21)
\]

Obviously, the dimensional parameter \( m \) has the sense of mass of the state. For convenience, hereinafter we assume \( m = 1 \).

One can derive the same result from (4) by means of the dimensional reduction of massless theory.

It is worth noting that the massive higher-spin fields represented by Lagrangian (18) are described by the first-class constraints only\( ^6 \) from point of view of the Hamilton formulation. As is well-known the ”minimal” coupling prescription breaks the right number of physical degrees of freedom. In the gauge manner of description this is represented as breaking the gauge invariance and, as a consequence, breaking the algebra of the first-class constraints. But if we can restore the gauge invariance by some deformation of the Lagrangian and the transformation, hence we can restore the algebra of the constraints and, as a consequence, the right number of physical degrees of freedom. A construction of such a type is considered in the next section.

### 3 Electromagnetic Interaction of Massive Spin \( s \) Field

In this section we consider the interaction of gauge massive fields of arbitrary integer spin \( s \) with constant electromagnetic field.

We introduce interaction by means of the ”minimal coupling”, i.e. we replace usual momentum operators with \( U(1) \)-covariant ones \( p_\mu \rightarrow \mathcal{P}_\mu \). The commutator of covariant

\^6The Hamilton formulation for the gauge-invariant description of the massive fields with spins 2 and 3 was considered in Ref. [25].
momenta defines the electromagnetic field strength

\[ [P_\mu, P_\nu] = F_{\mu\nu}. \]  

For convenience we included the imaginary unit and coupling constant into the definition of strength tensor.

In the definition of operators (15), we replace the usual momenta with the covariant ones as well. As a result the operators cease to obey algebra (8). Therefore, Lagrangian (13) loses the invariance under transformations (10).

To restore algebra (8), (17) let us represent operators (15) as normal ordered functions of creation and annihilation operators as well as of the electromagnetic field strength i.e.

\[ L_i = L_i \left( \bar{a}_\mu, \bar{b}, a_\mu, b, F_{\mu\nu} \right). \]

As a matter of fact, this means that these operators belong to extended universal enveloping algebra of Heisenberg algebra (1), (14). The particular form of operators \( L_i \) is defined from the condition of recovering of commutation relations (8) and (17) by these operators. We should note that it is enough to define the form of operators \( L_1 \) and \( L_2 \), since one can take following expressions:

\[ L_0 \overset{\text{def}}{=} [L_1, L_{-1}], \quad N \overset{\text{def}}{=} [L_2, L_{-2}] - \frac{D + 1}{2}. \]  

as definitions of operators \( L_0 \) and \( N \).

Since we have turned to the extended universal enveloping algebra, the arbitrariness in the definition of operators\(^7\) \( a \) and \( b \) appears. Besides, in the right-hand side of (1) and (14), we can admit the presence of the arbitrary operator functions depending on \( a, b, \) and \( F_{\mu\nu} \). Here, such a modification of the operators must not lead to breaking the Jacobi identity. Besides, these operators must restore the initial algebra in limit \( F_{\mu\nu} \to 0 \). However, one can make sure that using the arbitrariness in the definition of creation and annihilation operators, we can restore algebra (1), (14) at least up to the second order in the strength.

We shall search for operators \( L_1 \) and \( L_2 \) in the form of an expansion in the strength tensor which is equivalent to that in the coupling constant.

\(^7\)Such an arbitrariness has been also presented earlier as an internal automorphism of the Heisenberg-Weil algebra defining the Fock space \( \mathbb{F} \). But exactly the transformations depending on \( F_{\mu\nu} \) are important for us.
Let us consider the linear approximation.

Operator $L_1$ should be no higher than linear in operator $P_\mu$, since the presence of its higher number changes the type of gauge transformations and the number of physical degrees of freedom. Therefore, at this order we shall search for operators having the form

$$L_1^{(1)} = (\bar{a} F a) h_0(\bar{b}, b)b + (P F a) h_1(\bar{b}, b) + (\bar{a} F P) h_2(\bar{b}, b)b^2. \quad (24)$$

At the same time operator $L_2$ cannot depend on the momentum operators at all, since condition (3) defines purely algebraic constraints on the coefficient functions. Therefore, at this order we choose operator $L_2$ in the following form:

$$L_2^{(1)} = (\bar{a} F a) h_3(\bar{b}, b)b^2, \quad (25)$$

Here $h_i(\bar{b}, b)$ are normal ordered operator functions of the type

$$h_i(\bar{b}, b) = \sum_{n=0}^{\infty} H_i^n \bar{b}^n b^n,$$

where $H_i^n$ are arbitrary real coefficients. We consider only the real coefficients since the operators with purely imaginary coefficients do not give any contribution to the "minimal" interaction.

Let us define the particular form of functions $h_i$ from the condition recovering commutation relations (8) by operators (24) and (25). This algebra is entirely defined by (23) and by the following commutators

$$[L_2, L_1] = 0, \quad (26)$$
$$[L_2, L_{-1}] = L_1, \quad (27)$$
$$[L_0, L_1] = 0. \quad (28)$$

Having calculated (26) and passing to the normal symbols of creation and annihilation operators, we obtain a system of differential equations for the normal symbols of operator functions $h_i$. For the normal symbols of the operator functions we shall use the same notations. This does not lead to the mess since we consider the operator functions as the functions of two variables while their normal symbols as the functions of one variable. From (26) we have the equations

$$\frac{1}{2} h_{12}''(x) + h_{12}'(x) = 0,$$
\[
\frac{1}{2} h''_0(x) + h'_0(x) - h'_3(x) = 0, \\
\frac{1}{2} h''_1(x) + h'_1(x) - h_2(x) - h_3(x) = 0.
\] (29)

Here the prime denotes the derivative with respect to \( x \), while \( x = \bar{\beta} \beta \), where \( \bar{\beta} \) and \( \beta \) are the normal symbols of operators \( \bar{b} \) and \( b \), correspondingly.

Similarly, from (27) we derive another system of the equations:

\[
x^2 \left( \frac{1}{2} h''_2(x) + h'_2(x) \right) + 2x (h'_2(x) + h_2(x)) + h_2(x) - 2h_1(x) = 0,
\]

\[
\frac{1}{2} h''_1(x) + h'_1(x) - h_2(x) + h_3(x) = 0,
\] (30)

\[
x \left( \frac{1}{2} h''_0(x) + h'_0(x) + h'_3(x) \right) + h'_0(x) + 2h_3(x) = 0.
\]

Having solved the systems of equations (29) and (30) we obtain the particular form of functions \( h_1 \):

\[
\begin{align*}
 h_0(x) & = \text{const}, \\
 h_1(x) & = d_1 \left( \frac{1}{2} - x \right) e^{-2x} + d_2 \left( \frac{1}{2} + x \right), \\
 h_2(x) & = d_1 e^{-2x} + d_2, \\
 h_3(x) & = 0.
\end{align*}
\] (31)

Here \( d_1 \) and \( d_2 \) are arbitrary real parameters. Using (31) we obtain from (28)

\[
h_0(x) = 1 - d_2.
\]

The transition to the operator functions is realized in the conventional manner:

\[
h_i(\bar{b}, b) = \exp \left( \bar{b} \frac{\partial}{\partial \beta} \right) \exp \left( b \frac{\partial}{\partial \beta} \right) h_i(\bar{\beta} \beta) \bigg|_{\bar{\beta} \to 0} = :h_i(b \bar{b})::
\]

So, normal symobls of operators \( L_n \) has the following form in this approximation:

\[
L_1^{(1)} = (1 - d_2) (\bar{\alpha} \bar{F} \alpha) \beta + \left( e^{-2\beta} d_1 \left( \frac{1}{2} - \bar{\beta} \beta \right) + d_2 \left( \frac{1}{2} + \bar{\beta} \beta \right) \right) (\bar{F} \alpha)
\]

\[
+ \left( e^{-2\beta} d_1 + d_2 \right) (\bar{\alpha} \bar{F} \bar{P}) \beta^2,
\]

\[
L_0^{(1)} = (1 - 2d_2) (\bar{\alpha} \bar{F} \alpha) + \left\{ (1 + 2d_2) (\bar{F} \alpha) \bar{\beta} + h.c. \right\},
\]

\[
L_2^{(1)} = 0,
\]

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where $\tilde{a}_\mu$ and $\alpha_\mu$ are the normal symbols of operators $\bar{a}_\mu$ and $a_\mu$. Since operator $L_2$ has not changed in the linear approximation, hence from (13) it follows that operator $N$ and constraints (19), (21) have not changed either.

Thus, we have obtained the general form of operators $L_n$ which obey algebra (8) in the linear approximation. This means that Lagrangian (13) is an invariant under transformations (10) at this order.

From (32) it is clear that there exists the two-parametric arbitrariness in the linear approximation. But one of the arbitrary parameters $d_1$ and $d_2$ is determined in the second approximation In this, there are two solutions: when $d_1$ vanishes and $d_2$ is arbitrary, and vice versa, when $d_1$ is a free parameter and $d_2$ is equal to $\frac{1}{2}$. One can verify that the gyromagnetic ratio vanishes in the second case. Below we will consider the first solution only. Having set $d_1 = 0$ we calculate expectation value (13) and obtain the linear in $F_{\mu\nu}$ expression of the Lagrangian describing the e.m. interaction of massive field of spin $s$ in terms of the coefficient functions:

$$
\mathcal{L}^{(1)} = \sum_{n=1}^{s} \bar{\Phi}^n \cdot F \cdot \Phi^n n \left(1 - 2d_2 + 2(d_2 - 1) C_{s-n}^1 + (3 - 2d_2) C_{s-n}^2 \right) + 3(d_2 - 1) C_{s-n}^3
$$

$$
+ 3 \sum_{n=3}^{s} \bar{\Phi}^n \cdot F \cdot \Phi^n C_n \left(2d_2 - 1 + (d_2 - 1) C_{s-n}^1 \right)
$$

$$
+ \left\{ \sum_{n=3}^{s} \bar{\Phi}^n \cdot F \cdot \Phi^{n-2} \sqrt{C_n^2 C_{s-n+2}^2} (n - 2) \left(1 + (d_2 - 1) C_{s-n}^1 \right) \right\}
$$

$$
+ \frac{1}{2} \sum_{n=2}^{s} \left( \mathcal{P} \cdot \Phi^n \right) \cdot F \cdot \Phi^{n-2} \sqrt{n C_{s-n+1}^1} \left(n - 1 \right) (d_2 - 1) \left(2 - 2C_{s-n}^1 + C_{s-n}^2 \right)
$$

$$
+ \sum_{n=1}^{s} \left( \bar{\Phi}^n \cdot F \cdot \mathcal{P} \right) \cdot \Phi^{n-2} \sqrt{n C_{s-n+1}^1} \left(2 + 3d_2 - 3d_2 C_{s-n}^1 \right) - \left(2 - \frac{7}{2} d_2 \right) C_{s-n}^2
$$

$$
- 6d_2 C_{s-n}^3 + \frac{3}{2} \sum_{n=4}^{s} \left( \mathcal{P} \cdot \Phi^n \right) \cdot F \cdot \Phi^{n-1} \sqrt{n C_{s-n+1}^1 C_{n-1}^3} (d_2 - 1)
$$

$$
+ \frac{1}{2} \sum_{n=2}^{s} \bar{\Phi}^n \cdot \left( \mathcal{P} \cdot \Phi^{n-1} \right) \sqrt{n C_{s-n+1}^1} (n - 1) \left(d_2 - \frac{1 + d_2}{4} \right) C_{s-n}^1 - 8C_{s-n}^2
$$

$$
- \frac{1}{2} \sum_{n=3}^{s} \bar{\Phi}^n \cdot F \cdot \left( \mathcal{P} \cdot \Phi^{n-1} \right) \sqrt{n C_{s-n+1}^1 C_{n-1}^2} (d_2 - 1) \left(4 - C_{s-n}^1 \right)
$$

$$
- \sum_{n=3}^{s} \left( \bar{\Phi}^n \cdot F \cdot \mathcal{P} \right) \cdot \Phi^{n-1} \sqrt{n C_{s-n+1}^1 C_{n-2}^2} \left(1 + \frac{9}{4} d_2 - d_2 C_{s-n}^1 \right)
$$

$$
- 3 \sum_{n=3}^{s} \left( \bar{\Phi}^n \cdot F \cdot \mathcal{P} \right) \cdot \Phi^{n-3} \sqrt{C_n^3 C_{s-n+3}^3} \left(1 + \frac{d_2}{4} C_{s-n}^1 \right)
$$

13
+ \frac{3}{2} \sum_{n=4}^{s} \Phi^{n-3} \cdot F \cdot (\mathcal{P} \cdot \Phi^{n}) \sqrt{C_{n}^{3}C_{s-n+3}^{3}}(n-3)(d_{2}-1) \\
- \frac{d_{2}}{4} \sum_{n=1}^{s} \left( \Phi^{n} \cdot F \cdot \mathcal{P} \right) \cdot (\mathcal{P} \cdot \Phi^{n}) n \left( 2 + 4C_{s-n}^{1} - 7C_{s-n}^{2} + 6C_{s-n}^{3} \right) \\
+ d_{2} \sum_{n=2}^{s} \Phi^{n-1} \cdot (\mathcal{P} \cdot F \cdot (\mathcal{P} \cdot \Phi^{n})) C_{n}^{2} \left( 1 + 2C_{s-n}^{1} - C_{s-n}^{2} \right) \\
- \frac{3}{4} d_{2} \sum_{n=3}^{s} \left( \Phi^{n} \cdot F \cdot \mathcal{P} \right) \cdot (\mathcal{P} \cdot \Phi^{n}) C_{n}^{3} \left( 1 + 2C_{s-n}^{1} \right) \\
- d_{2} \sum_{n=2}^{s} \left( \left( \Phi^{n} \cdot \mathcal{P} \right) \cdot F \cdot \Phi^{n-2} \right) \sqrt{C_{n}^{2}C_{s-n+2}^{2}} \left( 1 - 2C_{s-n}^{1} + C_{s-n}^{2} \right) \\
+ \frac{d_{2}}{2} \sum_{n=4}^{s} \left( \left( \Phi^{n} \cdot \mathcal{P} \right) \cdot F \cdot \Phi^{n-2} \right) C_{n-2}^{2} \sqrt{C_{n}^{2}C_{s-n+2}^{2}} \\
- \frac{d_{2}}{2} \sum_{n=3}^{s} \left( \Phi^{n} \cdot \mathcal{P} \right) \cdot (\mathcal{P} \cdot F \cdot \Phi^{n-2}) \left( n-2 \right) \sqrt{C_{n}^{2}C_{s-n+2}^{2}} \left( 7 - 2C_{s-n}^{1} \right) \\
- \frac{d_{2}}{4} \sum_{n=3}^{s} \Phi^{n} \cdot (\mathcal{P} \cdot F \cdot \Phi^{n-2}) \left( n-2 \right) \sqrt{C_{n}^{2}C_{s-n+2}^{2}} \left( 1 + 2C_{s-n}^{1} \right) \\
- 3d_{2} \sum_{n=4}^{s} \left( \left( \Phi^{n} \cdot \mathcal{P} \right) \cdot F \cdot \mathcal{P} \right) \cdot \Phi^{n-4} \sqrt{C_{n}^{4}C_{s-n+4}^{4}} + h.c. \right) .

Correspondingly, in this approximation the gauge transformations have the form:

$$
\delta_{1} \Phi^{n} = \frac{d_{2}}{2} \left( 1 + 2(s-n) \right) \left\{ (F \cdot \mathcal{P}) \Lambda^{n-1} \right\}_{s} + 2d_{2} (n+1) C_{s-n}^{2} \sqrt{C_{n}^{2}C_{n+2}^{2}} \left( \mathcal{P} \cdot F \cdot \Lambda^{n+1} \right) + \left( 1 - d_{2} \right) n (s-n-1) \sqrt{\frac{s-n}{n+1}} \left( F \cdot \Lambda^{n} \right)_{s} .
$$

It is worth noting that the construction obtained is free from pathologies. Indeed, the gauge invariance ensures the appropriate number of physical degree of freedom. In this, the model is causal in linear approximation since by virtue of the antisymmetry and homogeneity of $F_{\mu \nu}$ the characteristic determinant for equations of motion of any massive state has the form $D(n) = (n^{2})^{p} + \mathcal{O}(F^{2})$, where $n_{\mu}$ is a normal vector to the characteristic surface and the integer constant $p$ depends on the spin of massive state. The equations of motion will be causal (hyperbolic) if the solutions $n^{0}$ to $D(n) = 0$ are real for any $\vec{n}$. In our case condition $D(n) = 0$ corresponds to the ordinary light cone at this order.

---

8The determinant is entirely determined by the coefficients of the highest derivatives in equations of motion after gauge fixing and resolving of all the constraints. [27].
Let us consider the quadratic approximation in the strength. The same as at preceding order, if one takes the general ansatz for operators $L_1, L_2$ and requires the recovering of relations (8), we will obtain a system of inhomogeneous differential equations at the second order. As was stated above, this system has the two solutions for parameters $d_1$ and $d_2$ and we choose the solution when $d_1$ vanishes and $d_2$ is arbitrary. According to our choice, operators $L_1$ and $L_2$ have the following form in this approximation:

$$L_1^{(2)} = (\bar{\alpha}F\bar{\alpha})\beta^2 \left(\frac{1}{4} - d_2^2 - d_2\right) - (\bar{\alpha}FF\bar{\alpha})\beta^3 \left(\frac{1}{3}e^{-2\bar{\beta}c_2} + \frac{1}{2}d_2 + \frac{1}{8}\right)$$

$$+ (\bar{\alpha}FF\bar{\alpha})\beta^2 \left(\frac{1}{3}e^{-2\bar{\beta}c_2} + 2c_1 - \frac{1}{2}d_2^2 - \frac{1}{2}d_2 + \frac{1}{8}\right)$$

$$+ (\bar{\alpha}F\bar{\alpha})^2 \beta \left(d_2^2 + \frac{1}{2}d_2 - \frac{5}{8}\right) + F^2 \beta \left(c_3 - \frac{1}{8}(1 + 4d_2)\bar{\beta}\beta\right)$$

$$+ (\bar{\alpha}FF\bar{\alpha})\beta \left(\left(d_2 + \frac{1}{4}\right)\bar{\beta}\beta + d_2^2 + \frac{3}{2}d_2 - \frac{3}{8}\right)$$

$$- (\alpha FF\alpha)\beta \left(\frac{1}{3}c_2 e^{-2\bar{\beta}c_2} \left(1 - \bar{\beta}\beta\right) + \frac{1}{8}(1 + 4d_2) \left(1 + \bar{\beta}\beta\right) + 2c_1\right)$$

$$+ (\bar{\alpha}F\bar{\alpha}) (PF\alpha) \left(\bar{\beta}\beta + \frac{1}{2}\right) \left(\frac{1}{4} - d_2 (d_2 + 1)\right)$$

$$- (PF\alpha) \left(c_4 + \frac{1}{8}(4d_2^2 + 4d_2 - 1)\bar{\beta}\beta\right),$$

$$L_2^{(2)} = - (\alpha FF\alpha) \left(\frac{1}{6}c_2 e^{-2\bar{\beta}c_2} \left(1 - 2\bar{\beta}\beta\right) + c_1 \left(1 + 2\bar{\beta}\beta\right)\right)$$

$$+ \left(2(\bar{\alpha}FF\alpha) - F^2\right)\beta^2 \left(c_1 + \frac{1}{6}e^{-2\bar{\beta}c_2}\right).$$

Here $c_1, c_2, c_3,$ and $c_4$ are arbitrary real parameters.

In contrast to the preceding order, operator $L_2$ and, correspondingly, $N$ and constraints (14), (21), depend on $F_{\mu\nu}$ in this approximation.

Operator $L_0$ is defined via the commutator of operators $L_1, L_{-1}$. The part that is proportional to $F^2$ has the form:

$$L_0^{(2)} = (PF\alpha) \left(\frac{1}{4}d_2^2 - 2c_4 + \left(d_2^2 - d_2 + \frac{1}{4}\right)\bar{\beta}\beta\right)$$

$$+ (\bar{\alpha}F\bar{\alpha}) (PF\alpha) \left(d_2^2 - d_2 + \frac{1}{4}\right) \left(1 + 2\bar{\beta}\beta\right)$$

$$+ (\bar{\alpha}F\bar{\alpha})^2 \left(3d_2^2 - d_2 - \frac{1}{4}\right) + (\bar{\alpha}FF\bar{\alpha}) \left((2d_2 + 1)\bar{\beta}\beta + 3d_2^2 + \frac{1}{4}\right)$$

$$- F^2 \left(\frac{1}{2}\bar{\beta}\beta - 2c_3 - \frac{1}{2}d_2\right) + \left\{ \left(\bar{\alpha}FF\bar{\alpha}\right)^2 \beta^2 \left(d_2^2 - d_2 + \frac{1}{4}\right) \right.$$

$$- (\alpha FF\alpha)\beta^2 \left(d_2 + \frac{1}{2}\right) - (\bar{\alpha}F\bar{\alpha}) (PF\alpha) \beta \left(4d_2^2 - d_2 + \frac{1}{2}\right)$$

$$- (\bar{\alpha}FF\alpha) \beta \left(2d_2^2 - \frac{1}{2}d_2 + \frac{1}{4}\right) + h.c.\}.}
Thereby, we have restored algebra (8) up to the second order in the electromagnetic field strength. It means that we restored the gauge invariance\(^9\) of Lagrangian (13) at the same order as well. Here we have not used the dimensionality of space-time anywhere explicitly, i.e. the obtained expressions do not depend on it.

4 Conclusion

In this paper we have constructed the Lagrangian describing the interaction of massive fields of arbitrary integer spins with the homogeneous electromagnetic field up to the second order in the strength. It is noteworthy that unlike the string approach\([16]\) our consideration does not depend on the space-time dimensionality, and, moreover, we have described the interaction of the single field with spin \(s\) while in the string approach the presence of constant electromagnetic field leads to the mixing of states with different spins\([18]\) and one cannot consider any states separately.

We should note that the case of a constant Abelian field can be easily extended to a non-Abelian case in linear approximation. In this case we have to consider the external field as the covariantly constant one. Here, one should take the whole vacuum as \(|0\rangle \otimes e^i\), where \(e^i\) are a basis vectors in space of linear representation of a non-Abelian group. The covariant derivative has the form \(\partial_\mu + A^a_\mu T^a\), where \(T^a\) are operators realizing the representation. In linear approximation such a modification does not change the algebraic features of our scheme and, therefore, all the derived results are valid in this case as well.

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\(^9\)Which ensures the right number of physical degree of freedom if we do not consider terms with higher number of derivatives.
A Massive spin-2 field.

Here we consider the propagation of the massive spin-2 field in the constant electromagnetic background. The following state in the Fock space corresponds to such field

\[ |2\rangle = \{(\bar{a} \cdot h \cdot a) + (v \cdot a) \bar{b} + \varphi b^2\}|0\rangle.\]

It is easy to see that this state trivially satisfies condition\(^{10}\) (9).

Having calculated the expectation value of operator (13) in this state we derive the following Lagrangian in linear approximation

\[
L = -2\bar{h}_{kl}D^2h_{kl} + 2\bar{h}D^2h + 4\bar{h}_{lm}D^2_{kl}h_{km} - 2\left(\bar{h}_{kl}D^2_{kl}h + \bar{h}_{kl}D^2_{kl}\varphi - \bar{h}D^2\varphi + h.c.\right)
- \bar{v}_kD^2v_k + \bar{v}_kD^2_{kl}v_l + 2i(\bar{v}_kD_kh - \bar{v}_lD_kh_{kl} - h.c.) + 2(\bar{h}_{kl}h_{kl} - \bar{h}h)
- d_2\left(2F_{km}\bar{h}D^2_{kl}h_{lm} + 2F_{km}\bar{h}_{mn}D^2_{kl}h_{tn} + 2F_{km}\bar{h}_{tm}D^2_{kl}\varphi + \frac{3}{2}F_{km}\bar{v}_mD^2_{kl}v_l + h.c.\right)
+ i\left(3d_2 + 2\right)F_{kl}\bar{v}_mD_kh_{tm} + 2(d_2 - 1)F_{im}\bar{h}_{kl}D_kv_m + (d_2 + 1)F_{kl}\bar{h}D_kv_l
+ 3F_{k\ell}\varphi D_{kl}v_l - h.c.\right) + 4(2d_2 - 1)F_{kl}\bar{h}_{tm}h_{km} + F_{kl}\bar{v}_tv_k,
\]

where \(h = g^{kl}h_{kl}\).

The gauge vector for the massive spin-2 state is

\[ |\Lambda, 2\rangle = \{(\xi \cdot a) + \eta b\}|0\rangle.\]

Condition (11) gives non-trivial constraints for gauge vectors of massive states with spin 3 and higher only.

From (11) we obtain the following gauge transformations for Lagrangian (32) in this order

\[
\delta h_{kl} = iD_{(k}\xi_{l)} + i\frac{d_2}{2}F_{(k|m}D_{m}\xi_{l)},
\]

\[
\delta v_k = iD_k\eta + \xi_k + i\frac{3}{2}d_2F_{km}D_m\eta + (1 - d_2)F_{km}\xi_m,
\]

\[
\delta \varphi = \eta + id_2F_{kl}D_k\xi_l.
\]

Let us fix the gauge invariance by means of the gauge condition

\[ v_{\mu} = 0, \quad \varphi = 0.\]

\(^{10}\) One can verify that condition (9) imposes non-trivial restrictions only on states with spin 4 and higher.
Now we have the following equations of motion

\[
0 = \frac{\delta \mathcal{L}_g}{\delta h_{kl}} = g_{kl} \left( D^2 h - D_{mn} h_{mn} - h - d_2 F_{mp} D^2_{mn} h_{np} \right) - D_{(kl)}^2 h + 2 D_{(kl)} h_{m|l|} \\
- D^2 h_{kl} + h_{kl} + d_2 F_{mp} D_{(kl)} h_{p|l|} + d_2 F_{m(k} D^2_{l)m} h \\
- d_2 F_{m(k} D_{mn} h_{n|l|} + 2(2d_2 - 1) F_{m(k} h_{l|m|},
\]

(33)

where \( \mathcal{L}_g \) is Lagrangian \([32]\) in the gauge under consideration. From these equations of motion we obtain the constraints:

\[
D_l h_{kl} + d_2 \left( \frac{3}{2} F_{tm} D_m h_{kl} - 2 F_{kl} D_m h_{tm} \right) = 0, \\
h = 0.
\]

(34)

Thereby, we have the appropriate number of the constraints and, respectively, the appropriate number of physical degree of freedom at this order.

Now we briefly consider the question of causality of the massive spin-2 state in linear approximation. Using relations (34) and equations of motion (33), one can infer that the characteristic determinant has the following form

\[
D(n) = \left( n^2 \right)^{D^2(D+1)} + O \left( F_{kl}^2 \right).
\]

Thereby, from the condition \( D(n) = 0 \), we obtain the usual light cone in this approximation, i.e. the massive spin-2 state has the causal propagation in the background under consideration.

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