Below the Breitenlohner-Freedman bound
in the nonrelativistic AdS/CFT correspondence

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Abstract

We propose that there is no analogue of the Breitenlohner-Freedman stability bound on the mass of a scalar field in the context of the nonrelativistic AdS/CFT correspondence. Our treatment is based on an equivalence between the field equation of a complex scalar in the AdS/CFT correspondence and the one-dimensional Schrödinger equation with an inverse square potential. We compute the two-point boundary correlation function for $m^2 < m^2_{BF}$ and discuss its relation to renormalization group limit cycles and the Efimov effect in quantum mechanics. The equivalence also helps to elucidate holographic renormalization group flows and calculations in the global coordinates for Schrödinger spacetime.
I. INTRODUCTION

The anti-de Sitter/ conformal field theory (AdS/CFT) correspondence [1] is a powerful technique which allows us to investigate gauge theories and develop some intuition about their behavior at strong coupling. Recently, Son [2] and Balasubramanian and McGreevy [3] extended the technique to the realm of nonrelativistic physics.\(^1\) Their work was mainly motivated by the rapid progress and strong interest in the theory of cold fermions at unitarity [5, 6] which is strongly coupled and described by the effective nonrelativistic conformal field theory [7]. The proposal in [2, 3] stimulated a considerable theoretical progress and led to a number of interesting insights (for review see [8]).

It often happens in physics that two apparently different physical problems have the same solution because they are described by the same mathematical equations. In this case it may be helpful for a better understanding of one of the problems to reformulate it in the language of the other one. In this paper we use one example of this equivalence, also recently mentioned in [9, 10], between the field equation of a complex scalar in the anti-de Sitter (or Schrödinger) background spacetime and the one-dimensional Schrödinger equation with an inverse square interaction potential defined on the real positive half-line. We argue that contrary to the known presence of a stability mass bound in the Minkowski and anti-de Sitter spacetimes, there is no restriction on the mass of a scalar field in the nonrelativistic holography. We arrive to this conclusion by performing calculations in the Poincaré and global coordinates. Additionally, the quantum mechanical analogy allows us to gain a simple understanding of the regime, where a single bulk theory describes two different conformal field theories on the boundary [11, 12]. We construct renormalization group flows between the two CFTs and find the quantum mechanical interpretation of the (in the nonrelativistic case spurious) Breitenlohner-Freedman (BF) mass bound \(m_{BF}^2\) [13]. Finally, using a standard AdS/CFT prescription, we compute the scalar two-point correlation function in momentum space for \(m^2 < m_{BF}^2\) in the nonrelativistic AdS/CFT, discuss its properties and comment on the connection to renormalization group limit cycles and the well-known Efimov effect in quantum mechanics. In two quantum mechanical problems, we also propose examples of local composite operators which might be dual to the scalar field with \(m^2 < m_{BF}^2\) in the

\(^1\) A related, but Galilean noninvariant version of the nonrelativistic holography was constructed in [4].
framework of AdS/CFT.

II. MASS STABILITY BOUND IN $Mink_d$, $AdS_{d+1}$ AND $Sch_{D+3}$ SPACETIMES

In this section we identify a stability bound on the mass of a free scalar field in different background spacetimes. As a warm-up we consider a free complex\(^2\) scalar field $\phi$ of mass $m$ in the Minkowski spacetime $Mink_d$, defined by the action

$$S[\phi, \phi^*] = -\int d^d x \left( \eta^\mu_\nu \partial_\mu \phi^* \partial_\nu \phi + m^2 \phi^* \phi \right),$$

where the metric signature convention is $\eta^\mu_\nu = \text{Diag}(-1, 1, 1, ...)$, and $\phi$ satisfies the field equation $\Box \phi = 0$. The simplest solutions of the field equation are the plane waves

$$\phi^{(p-w)} = \exp[-i q^0 t + i \vec{q} \cdot \vec{x}],$$

where $q^0$ is energy, $\vec{q}$ is momentum and $q^2 \equiv -(q^0)^2 + \vec{q}^2$. The energy and momentum must satisfy the on-shell condition $q^2 = -m^2$. We note that for $m^2 < 0$ we obtain a tachyonic solution, which is unstable. This is because in the range $q^2 < |m^2|$ the energy $q^0$ is pure imaginary and the solution $\phi^{(p-w)}$ can grow exponentially in time. The instability also manifests itself in the energy-momentum tensor which for the free complex scalar $\phi$ is generally given by

$$T_{\mu\nu} = [\partial_\mu \phi^* \partial_\nu \phi + \partial_\nu \phi^* \partial_\mu \phi] - g_{\mu\nu} [\partial \phi^* \cdot \partial \phi + m^2 |\phi|^2] + \chi [g_{\mu\nu} \Box - D_\mu \partial_\nu + R_{\mu\nu}] |\phi|^2,$$

where $g_{\mu\nu} = \eta_{\mu\nu}$, the covariant derivative $D_\mu = \partial_\mu$, and the Ricci tensor $R_{\mu\nu} = 0$ in the Minkowskian case. The energy-momentum tensor is conserved for an arbitrary value of $\chi$. For the plane-wave solution the energy-momentum tensor simplifies

$$T^{(p-w)}_{\mu\nu} = 2q_\mu q_\nu.$$ 

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\(^2\) The mass stability bounds presented in this section for complex fields are valid also for real scalars in $Mink_d$ and $AdS_{d+1}$ spacetimes. However, in the nonrelativistic version of AdS/CFT one must necessarily use complex fields in $Sch_{D+3}$ spacetime to describe massive nonrelativistic particles in the boundary field theory.

\(^3\) There is an ambiguity in the definition of the energy-momentum tensor $T_{\mu\nu}$ reflected in the presence of the last term in Eq. (3). This contribution originates from the coupling of the scalar field $\phi$ to the scalar curvature $R$ of the background spacetime, i.e. $\sim \chi \int d^d x \sqrt{-g} R |\phi|^2$. For the detailed discussion of the anti-de Sitter spacetime case see [13].
The energy density $\epsilon^{(p-w)} = -(T^{(p-w)})^0_0 = T^{(p-w)}_{00}$ of the tachyonic plane wave is negative for $q^2 < q^2$ suggesting the presence of the instability. In general, a field theory with $m^2 < 0$ in $Mink_d$ is stabilized by an addition of repulsive interactions. Specifically, in the quantum field theory an effective potential becomes bounded from below and the condensate $\langle \phi \rangle \neq 0$ is formed. This defines a new vacuum, around which solutions of the theory must be expanded.

In the context of the AdS/CFT correspondence consider a free complex scalar in the anti-de Sitter spacetime $AdS_{d+1}$ with the action

$$S[\phi, \phi^*] = -\int dz dx \sqrt{-g} \left( g^{\mu \nu} \partial_\mu \phi^* \partial_\nu \phi + m^2 \phi^* \phi \right), \quad (5)$$

where the $AdS_{d+1}$ metric in the Poincaré patch is given by

$$ds^2 = \frac{dz^2 + \eta_{\mu \nu} dx_\mu dx_\nu}{z^2} \quad (6)$$

with $z \in [0, \infty)$ denoting the radial $AdS$ coordinate. If we perform a Fourier transform to the momentum space $x^\mu \rightarrow q^\mu$ on the boundary, the field equation can be written as

$$\partial_z^2 \phi - \frac{d-1}{z} \partial_z \phi - \frac{m^2}{z^2} \phi - q^2 \phi = 0, \quad q^2 = -(q^0)^2 + \vec{q}^2, \quad (7)$$

which after the change of variables $\phi = z^{(d-1)/2} \psi$ can be expressed as

$$-\partial_z^2 \psi + \frac{m^2 + \frac{d-1}{z^2}}{z^2} \psi = -q^2 \psi. \quad (8)$$

This is a one-dimensional Schrödinger equation, defined on the real positive half-line, with the classically scale invariant inverse square potential of the strength $\kappa = -m^2 - \frac{d-1}{4}$ and the energy $E = -q^2$. This quantum mechanical problem was studied extensively and by now is well understood: The inverse square potential is on the boundary between regular and singular potentials and must be regularized near the origin $z = 0$. Depending on the value of the coupling constant $\kappa$, the solution of the Schrödinger equation has two qualitatively different regimes. While for $\kappa < \kappa_{cr} = \frac{1}{4}$ there are no bound states and the energy spectrum is continuous and positive, for $\kappa > \kappa_{cr}$ an infinite bound state spectrum develops. The bound state spectrum is geometric, i.e. the ratios of energies of the adjacent levels are constant, with the accumulation point at $E = 0$. In analogy with the Minkowski

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4 Henceforth we take the radius of the AdS spacetime to be $R = 1$.

5 In our convention $\hbar = 2M = 1$ with the particle mass $M$. 

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case the instability in Eq. (7) can appear only when \( q^2 > 0 \). In the quantum mechanical language this corresponds to \( E < 0 \). For instability actually to appear we additionally require that the corresponding bound state wave function \( \psi = z^{(1-d)/2} \phi \) is physically acceptable due to Calogero [28], i.e. that both \(|\psi|^2\) and \(\psi \partial_z \psi\) are continuous functions regular at the origin. Both conditions for instability are fulfilled only for \( \kappa > \kappa_{cr} \). From Eq. (8) this gives rise to a bound on the possible mass \( m^2 \) of a scalar in \( AdS_{d+1} \)

\[
m^2 \geq m^2_{BF} = -\frac{d^2}{4},
\]

which is the BF bound in the anti-de Sitter spacetime. It was first derived in [13] by demanding positivity of the conserved energy functional for scalar fluctuations which vanish sufficiently fast at spatial infinity. Below the BF bound the \( AdS_{d+1} \) background becomes unstable. To stabilize the theory bulk interactions must be introduced, which deform the AdS metric. This often leads to a formation of an IR wall [29] at some \( z = z_{IR} \) (see also [10] for a simple realization of this kind of deformation). In the boundary theory the IR momentum scale \( \Lambda_{IR} = z_{IR}^{-1} \) is dynamically generated and the boundary operator \( O \) dual to the bulk field \( \phi \) acquires a nonzero expectation value even in the absence of an external source \( J \).

Recently, the concept of holography was extended to nonrelativistic physics [2, 3]. The key idea is to investigate Einstein gravity (and its extensions) on the Schrödinger spacetime \( Sch_{D+3} \) background with the metric in the Poincaré coordinates given by

\[
ds^2 = -\frac{dt^2}{z^4} + \frac{-2dtd\xi + dx^i dx^i + d\xi^2}{z^2}, \quad i = 1, \ldots, D.
\]

(10)

The isometries of the metric (10) form the so called Schrödinger group [30]. The dual nonrelativistic field theory is defined on the \( D + 2 \)-dimensional anisotropic conformal boundary [31] with the metric

\[
\tilde{ds}^2 = -dt^2 - 2dtd\xi + dx^i dx^i.
\]

(11)

Consider a free complex scalar of mass \( m_0 \) on the \( Sch_{D+3} \) background defined by

\[
S[\phi, \phi^*] = -\int dzdtd\xi d^D x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m_0^2 \phi^* \phi).
\]

(12)

After transforming to the momentum space \( (t, \xi, \vec{x}) \rightarrow (\omega, M, \vec{q}) \), the field equation reads [2]

\[
\partial^2_\omega \phi - \frac{D+1}{z} \partial_z \phi - \frac{m^2}{z^2} \phi - \tilde{q}^2 \phi = 0, \quad \tilde{q}^2 \equiv -2M \omega + \vec{q}^2,
\]

(13)
with \( m^2 = m_0^2 + M^2 \), where \( M \) denotes the mass (particle number) of a particle in the nonrelativistic boundary field theory. It is assumed to be a positive integer.\(^6\) Eq. (13) can be also casted in the form of the one-dimensional Schrödinger equation

\[
- \partial_x^2 \psi + \frac{m^2 + (D+2)^2 - 1}{z^2} \psi = -\tilde{q}^2 \psi
\]

with the help of the substitution \( \phi = z^\frac{D+1}{2} \psi \). We note, however, that in this case there is no lower bound on the scalar mass \( m^2 \). The reason is simple: due to the nonrelativistic form of the dispersion relation the condition \( \tilde{q}^2 > 0 \) leads not to the imaginary boundary energy \( \omega \), as was valid in the preceding two examples, but only to \( \omega < 0 \). The nonrelativistic boundary plane-wave excitations remain oscillatory producing no instabilities. No condensate can be formed in the nonrelativistic vacuum, and hence there is no dynamical IR scale generation in the boundary theory. For this reason the \( Sch_{D+3} \) spacetime is a reliable background for any scalar mass, and one can use the AdS/CFT correspondence for \( m^2 < m_{BF}^2 = -\frac{(D+2)^2}{4} \).

In the next section we draw a similar conclusion by examining the scalar field equation in the global coordinates.

### III. COMPLEX SCALAR FIELD IN GLOBAL COORDINATES FOR \( Sch_{D+3} \) SPACETIME

A global coordinate system for \( Sch_{D+3} \) spacetime was recently constructed and discussed in [32]. Consider a coordinate system \((T, V, R, \vec{X})\) for \( Sch_{D+3} \), in terms of which the metric reads

\[
ds^2 = -\frac{dT^2}{R^4} + \frac{1}{R^2} \left( -2dTdV - \omega^2 (R^2 + \vec{X})dT^2 + dR^2 + d\vec{X}^2 \right) ,
\]

where \( \omega \) is an interpolating frequency parameter and \( R \in [0, \infty) \) is the radial coordinate. The metric interpolates smoothly between the Poincaré metric \((\omega = 0)\) and the global metric \((\omega = 1)\). In what follows we work with a general frequency \( \omega \), keeping in mind that the solution for the global coordinates is recovered only after one fixes \( \omega = 1 \).

In this section, we solve following [32] the Klein-Gordon equation

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - m_0^2 \phi = 0
\]

\(^6\) This discreteness originates from the assumption that \( \xi \) is a compact coordinate in the Schrödinger spacetime \( Sch_{D+3} \).
for the complex scalar field $\phi$ of mass $m_0$ in the global coordinates. As the coefficients of the metric (15) are independent of the coordinates $T$ and $V$, there are two obvious Killing vector fields $\partial_T$ and $\partial_V$ in the $Sch_{D+3}$ spacetime. Additionally, the metric is symmetric under the rotations in the $\vec{X}$-space. Hence, the scalar eigenmodes with a definite energy $E$, particle number $M$ (assumed to be positive integer) and angular quantum number $L$ can be written as

$$\phi_{E,M,L} = e^{-iET}e^{-iMV}Y_L(\Omega_{D-1})\varphi(X)\Phi(R).$$  \hspace{1cm} (17)

Here we introduced hyperspherical coordinates in the $\vec{X}$-space, i.e.

$$d\vec{X}^2 = dX^2 + X^2d\Omega^2_{D-1}, \quad X \in [0, \infty)$$  \hspace{1cm} (18)

and $Y_L(\Omega_{D-1})$ are spherical harmonics defined on the sphere $S^{D-1}$. The ansatz (17) can now be substituted into the field equation (16), which gives us two separate differential equations for the functions $\varphi(X)$ and $\Phi(R)$

$$\partial_X^2 \varphi + \frac{D+1}{X} \partial_X \varphi - M^2 \omega^2 X^2 \varphi - \frac{L(L + D - 2)}{X^2} \varphi = -\delta \varphi, \hspace{1cm} (19)$$

$$\partial_R^2 \Phi - \frac{D+1}{R} \partial_R \Phi - M^2 \omega^2 R^2 \Phi - \frac{m^2}{R^2} \Phi = (\delta - 2ME)\Phi, \hspace{1cm} (20)$$

where $\delta$ is a so far undetermined constant and $m^2 = m_0^2 + M^2$. The equations (19, 20) can be rewritten in the form of one-dimensional Schrödinger equations by employing the redefinitions $\varphi = X^{-\frac{D}{2}} \psi$ and $\Phi = R^{\frac{D+1}{2}} \Psi$ yielding the result

$$-\partial_X^2 \psi + M^2 \omega^2 X^2 \psi + (L(L + D - 2) + [(D - 2)^2 - 1]/4) \psi_X^2 = \delta \psi, \hspace{1cm} (21)$$

$$-\partial_R^2 \Psi + M^2 \omega^2 R^2 \Psi + (m^2 + [(D + 2)^2 - 1]/4) \Psi_X^2 = (2ME - \delta) \Psi. \hspace{1cm} (22)$$

Remarkably, both equations define the quantum-mechanical problem of a particle (constrained to a real positive half-line) in a combined inverse square and harmonic potential.

We first consider Eq. (21) with the inverse square potential coupling $\kappa_X < 0$, which corresponds to a repulsion. This problem was solved by Calogero [28] with the result

$$\delta_n^\pm = 2M\omega(2n \pm a + 1), \quad a = L + \frac{D}{2} - 1, \quad n = 0, 1, 2, \ldots,$$

$$\psi_n^\pm = X^{\pm a + \frac{1}{2}} \exp(-\frac{1}{2}M\omega X^2)L_n^{\pm a}(M\omega X^2),$$  \hspace{1cm} (23)
where \( L_n^{\pm a} \) denotes a generalized Laguerre polynomial. Due to the harmonic part of the potential the spectrum is discrete and equidistant. Following Calogero, we consider only physically acceptable wave functions, i.e. we require both \( |\psi(X)|^2 \) and \( \psi(X)\psi'(X) \) to be continuous. This condition picks out the \( \psi_n^+ \) wavefunctions and the corresponding \( \mathcal{E}_n^+ \) branch of the spectrum. The original differential equation (19) thus has the solution

\[
\varphi_n^+ = X^{1/2} \psi_n^+ = X^L \exp\left(\frac{-1}{2} M\omega X^2\right)L_n^{\pm \frac{D}{2}-1}(M\omega X^2)
\]

in agreement with [32].

Now we turn our attention to the radial equation (22). As expected, up to the harmonic term it is identical with Eq. (14), derived in Sec. II in the Poincaré coordinates. The form of the solution of the differential equation is determined by the value of the inverse square potential coupling \( \kappa_R \). In particular, for a repulsion and weak attraction \( \kappa_R < \frac{1}{4} \), which corresponds to \( m^2 > m_{BF}^2 = -\frac{(D+2)^2}{4} \), the original Calogero solution holds

\[
\bar{E}_l^\pm = 2M\omega(2l \pm a + 1), \quad a = \sqrt{\frac{1}{4} - \kappa_R} = \nu, \quad l = 0, 1, 2, ..., \quad \Psi_l^\pm = R^{\pm a + \frac{1}{2}} \exp\left(-\frac{1}{2} M\omega R^2\right)L_{l/2}^{\pm a}(M\omega R^2).
\]

In this case we do not restrict ourselves to the physically acceptable wave functions and consider both branches of Eq. (25). Hence, the radial function \( \Phi_l \) in the original AdS/CFT problem has two branches and reads

\[
\Phi_l^\pm = R^{\pm a + \frac{1}{2}} \Psi_l^\pm = R^{\Delta \pm \nu} \exp\left(-\frac{1}{2} M\omega R^2\right)L_{l/2}^{\pm \nu}(M\omega R^2).
\]

The asymptotic behavior of \( \Phi_l^\pm \) for \( R \to 0 \) and \( R \to \infty \) agrees with findings in the Poincaré coordinates. The global energy spectrum is given by

\[
E_{n,l}^\pm = \frac{\mathcal{E}_n + \bar{E}_l}{2M} = \omega \left(2n + 2l \pm \nu + L + \frac{D}{2} + 1\right)
\]

and has two discrete quantum numbers. On the other hand, for strong attraction \( \kappa_R > \frac{1}{4} \) equivalent to \( m^2 < m_{BF}^2 \) the potential in Eq. (22) is truly singular and must be regularized. This case was treated in [33], where a cutoff radius \( R_0 \ll (M\omega)^{-\frac{1}{2}} \) was imposed leading to a boundary condition \( \Psi(R_0) = 0 \) for the wave function \( \Psi \). The Hamiltonian of the regularized problem is bounded from below by \( \bar{E}_{\text{min}} \approx -\frac{a}{\kappa_R R_0} \). As shown in [33], the energy spectrum \( \bar{E}_l \) can be determined from the transcendental equation

\[
u \left(1 - i|\nu|\right) = \frac{\Gamma\left(1 - i|\nu|\right) \Gamma\left(\frac{1+|\nu|}{2} - \frac{\bar{E}}{4\omega M}\right)}{\Gamma\left(1 + i|\nu|\right) \Gamma\left(\frac{1-|\nu|}{2} - \frac{\bar{E}}{4\omega M}\right)}, \quad \bar{E} > \bar{E}_{\text{min}}.
\]
Figure 1: Graphical solution of Eq. (28) for $u_0 = 10^{-4}$, $|\nu| = 10$ and $M\omega = \frac{1}{4}$. The blue (red) line corresponds to the argument function of the left (right) hand side of Eq. (28). Points of intersection determine the discrete dimensionless eigenenergies $\epsilon_l = \frac{E_l}{4\omega M}$.

where $u_0 \equiv M\omega R_0^2$ and $\nu = i|\nu| = \sqrt{\frac{1}{4} - \kappa R}$. We plot a graphical solution of Eq. (28) in Fig. 1.

The qualitative features of the energy spectrum can be understood by studying two limits of Eq. (28). In particular, for energies $\bar{E}_{\text{min}} < \bar{E} \ll 0$ the harmonic term can be neglected in the original Schrödinger equation (22). One is allowed to do so because in this regime the bound state wave functions are well-localized around the origin, where the $1/r^2$ potential dominates over the harmonic potential. Hence, for $\bar{E}_{\text{min}} < \bar{E} \ll 0$ the bound state energies form an almost geometric discrete spectrum, characteristic for the inverse square problem. In fact, by formally expanding Eq. (28) around $\bar{E} = -\infty$ one obtains the exact geometric scaling of energies $\frac{|E_{l+1}|}{|E_l|} = e^{-\frac{2\pi}{|\nu|}}$. For $\bar{E} \gg 0$ the bound states wave functions are more sensitive to the large distances where the harmonic potential dominates. In this case the inverse square potential determines only the near-origin behavior of the wave functions. In the limit $\bar{E} \to +\infty$ in Eq. (28) we get an infinite equidistant spectrum with

\[7\] We note that this finding is in a disagreement with [33], where only one negative energy state in the spectrum was identified.
$E_{l+1} - E_l = 4\omega M$. For the moderate values of $E$ the spectrum changes its behavior from geometric to equidistant (see Fig. 1). The arguments presented in this paragraph are close in spirit to [20], where a somewhat similar problem was studied.

Most importantly, in the nonrelativistic AdS/CFT correspondence for $m^2 < m_{BF}^2$ the spectrum of global energies

$$E_{n,l} = \frac{E_n + \bar{E}_l}{2M} = \omega(2n + L + \frac{D}{2}) + \frac{\bar{E}_l}{2M}$$

(29)

remains real. For this reason we conclude that there appears no instability in the solution as one crosses the BF bound. This is in a stark contrast with the solution of the scalar field equation in the global coordinates for $AdS_{d+1}$ [13]. In this case the global energy is actually conserved only if its flux at infinity vanishes. This leads to the quantization condition

$$E_n = 2n + \Delta_{\pm} + L, \quad n = 0, 1, 2, ...$$

(30)

According to our findings in the Poincaré coordinates, the scaling dimension $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$ becomes complex below the relativistic BF bound and the global energy $E_n$ acquires an imaginary part signaling a presence of instability.

IV. INVERSE SQUARE POTENTIAL AND HOLOGRAPHIC RG FLOWS

In the context of the relativistic AdS/CFT correspondence it was shown in [12] that, if the mass of a (complex) scalar bulk field $\phi$ lies in the interval

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1,$$

(31)

then a single gravity theory in the bulk describes two different conformal field theories on the boundary which we call $CFT_+$ and $CFT_-$. The CFT operator $O$ dual to the scalar field $\phi$ has the scaling dimension

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$$

(32)

in these two conformal field theories. The near-boundary asymptotic of the field $\phi$ reads

$$\phi(z) = c_- z^{\Delta_-} + c_+ z^{\Delta_+},$$

(33)

where $c_-$ and $c_+$ are proportional to the source $J$ coupled to $O$ and the condensate $\langle O \rangle$ in $CFT_+$ perturbed by $\int d^d x (\bar{J} O + J \bar{O})$. In the perturbed $CFT_-$ the roles of the coefficients
$c_-$ and $c_+$ are interchanged: $c_-$ is identified with the condensate, while $c_+$ with the source.

In this section we demonstrate how the two CFTs arise in the equivalent one-dimensional problem of quantum mechanics with an inverse square potential (a similar construction was made recently in [10]) and construct RG flows between CFT$_+$ to CFT$_-$. Additionally, we discuss the case of the nonrelativistic version of AdS/CFT, where one can go below the BF bound without experiencing any instability. In particular, we construct RG flows and compute the two-point scalar correlation function for $m^2 < m^2_{BF}$ in the nonrelativistic holography.

A. Relativistic holography

We consider the regime of $m^2 > m^2_{BF}$ and $q^2 > 0$, where the solution of Eq. (7) regular in the bulk is given by

$$
\phi_q(z) = \frac{z^{d/2} K_\nu(qz)}{z_0^{d/2} K_\nu(qz_0)}, \quad \nu = \sqrt{\frac{d^2}{4} + m^2}. \quad (34)
$$

Here we introduced the IR bulk cutoff $z_0$, which corresponds to the UV momentum cutoff in the boundary field theory, and normalized the solution so that $\phi_q(z_0) = 1$. Albeit the first condition for instability $q^2 > 0$ from Sec. II is fulfilled by the solution (34), the second condition due to Calogero is not satisfied by it. For $m^2 > m^2_{BF}$ the associated $\psi_q = z^{(1-d)/2} \phi_q$ is not a physically acceptable bound state wave function in the inverse square potential problem. Hence, one is in the stable regime.

In the interval $m^2 > m_{BF}$ the single bulk solution (34) describes two different conformal boundary field theories. The RG flow from CFT$_+$ to CFT$_-$ can be obtained by turning $O^\dagger O$ interaction in the boundary field theory [34] (for the AdS/CFT treatment see [35]). Some insight into this subtlety can be gained by considering the equivalent Schrödinger equation (8) in the regime $y \equiv zq \ll 1$ and $\kappa < \kappa_{cr}$. In this domain the general complex solution of Eq. (8) reads

$$
\psi(z) = c_- z^{1/2-\nu} + c_+ z^{1/2+\nu} = \tilde{c}_- y^{1/2-\nu} + \tilde{c}_+ y^{1/2+\nu}, \quad \nu = \sqrt{\frac{1}{4} - \kappa} \quad (35)
$$

with $c_-, c_+ \in \mathbb{C}$ and $\tilde{c}_\pm \equiv \frac{c_\pm}{q^{1/2+\nu}}$. Notably, only in the interval $m^2 > m_{BF}$ the general solution (35) is square integrable around the origin. Thus, in the quantum mechanical language
it is the property of square integrability around the origin which determines whether the solution (34) is normalizable or not. This point of view complements the original arguments of Breitenlohner and Freedman [13] based on the finiteness of energy in the AdS spacetime or the argument of Klebanov and Witten [12] based on the finiteness of the action in the Euclidean AdS space.

The inverse square potential is singular at $z = 0$ and in its own gives an ill-defined quantum mechanical problem. It must be regularized at the origin and this can be achieved in various ways [18–25, 36]. We choose to regularize the problem by extending it to the full real $z$-line, cutting the potential off for $|z| < z_0$ and introducing a localized $\delta$-function potential (counterterm) at the origin. As will be illustrated in the rest of this subsection, this procedure corresponds to the inclusion of the double-trace $O^1O$ boundary contact term in the AdS/CFT correspondence [35].

The modified regularized potential reads

\[ V(z) = \begin{cases} 
-\frac{\lambda}{z_0} \delta(z), & |z| < z_0, \\
-\frac{\kappa}{z^2}, & |z| > z_0,
\end{cases} \]  

(36)

where the coordinate $z \in \mathbb{R}$ in the regularized quantum mechanical one-dimensional problem, and $z_0$ serves as an IR position cutoff. $\lambda$ is a dimensionless coupling constant. Here we find a general solution of the regularized problem with the negative energy $E = -q^2 < 0$. We concentrate our attention on the domain $z > 0$, $y \equiv zq \ll 1$, where the quantum mechanical wave function is given by

\[ \psi(y) = \begin{cases} 
 e^{-y} + De^{y}, & 0 < y < y_0 \equiv qz_0, \\
 \mathcal{N}(\tilde{c}_- y^{1/2-\nu} + \tilde{c}_+ y^{1/2+\nu}), & y > y_0.
\end{cases} \]  

(37)

Note that $\psi(y)$ must be an even function because an odd $\psi(y)$, which is also allowed by the even potential (36), yields necessarily a vanishing counterterm $\lambda$. Now the coefficient $D$ can be easily expressed from the continuity of the wave function $\psi(y)$ and the known discontinuity of its first derivative at $y = 0$. Specifically, $D$ is related to the coupling constant $\lambda$ by

\[ \lambda = 2y \frac{1 - D}{1 + D}. \]  

(38)

The normalization constant $\mathcal{N}$ can be determined from the continuity condition $\psi(y)|_{y \to y_0^-} = \psi(y)|_{y \to y_0^+}$

\[ e^{-y_0} + De^{y_0} \approx 1 + D = \mathcal{N}(\tilde{c}_- y_0^{1/2-\nu} + \tilde{c}_+ y_0^{1/2+\nu}). \]  

(39)
Additionally, by matching the derivatives \( \frac{d}{dy} \psi \big|_{y \to y_0} = \frac{d}{dy} \psi \big|_{y \to y_0 + 0} \) we obtain

\[
- e^{-y_0} + De^{y_0} \approx -1 + D = \mathcal{N} \left[ \tilde{c}_-(1/2 - \nu) y_0^{-1/2 - \nu} + \tilde{c}_+(1/2 + \nu) y_0^{-1/2 + \nu} \right].
\]  

(40)

Finally, substituting Eqs. (39) and (40) into Eq. (38) one arrives at

\[
\lambda(t) = -1 + 2 \nu \frac{e^{\nu t} - C e^{-\nu t}}{e^{\nu t} + C e^{-\nu t}}, \quad C \equiv \frac{\tilde{c}_+}{\tilde{c}_-},
\]  

(41)

where we introduced \( t \equiv -\ln(qz_0) \). This expression can be interpreted as the RG flow of the contact coupling \( \lambda \) as a function of the logarithmic RG scale \( t \). The dimensionless parameter \( C \) which we allow to be complex is considered to be a fixed constant during the RG evolution. It determines in general complex initial condition \( \lambda(t = 0) \) via the Möbius (linear fractional) transformation. The RG flow (41) solves the inhomogeneous Riccati differential equation

\[
\partial_t \lambda = -\frac{\lambda^2}{2} - \lambda - 2\kappa = -\frac{1}{2} \left( \lambda + 1 + 2\nu \right) \left( \lambda + 1 - 2\nu \right)
\]  

(42)

and possesses the UV fixed point \( \lambda_{UV} = -1 + 2\nu \) (corresponds to the boundary CFT\(_-\) in AdS/CFT) and the IR fixed point \( \lambda_{IR} = -1 - 2\nu \) (corresponds to the CFT\(_+\) in AdS/CFT). The phase portrait of the RG evolution in the complex \( \lambda \) plane can be found in Fig. 2(A).

The RG trajectories are arcs of circles of radius \( R = 2\nu |\frac{C}{\tilde{c}_-}| \).

The extension to a complex coupling \( \lambda \) provides a deeper mathematical understanding of renormalization in the nonrelativistic quantum mechanical problem with an inverse square potential. It was introduced and motivated in [36]. Physically, the generalization to the complex \( \lambda \) opens an inelastic channel in the quantum mechanical scattering and converts bound states to metastable resonances. Here we note that one can further investigate this extension using holography by allowing bulk solutions with generally complex \( c_- \) and \( c_+ \) in the asymptotic expression (33). Complex generalization also plays an important role in the next subsection, where the nonrelativistic case is discussed.

Of special physical interest, however, is the real domain of the coupling \( \lambda \), which corresponds to \( C \in \mathbb{R} \) in Eq. (41). Notably, both fixed points \( \lambda_{UV} \) and \( \lambda_{IR} \) lie on the real axis. If one tunes the initial condition \( \lambda(t = 0) \) to the real line, then the RG flows remain on the real line. In the AdS/CFT correspondence, information about both conformal field theories is contained in the solution (34). The detailed procedure of how the values of the source \( J \) coupled to the operator \( O \) and the condensate \( \langle O \rangle \) are extracted from the asymptotic form \( (33) \) can be found in [10, 12]. For sake of comparison with the nonrelativistic result which we
Figure 2: The phase portraits of the RG flows of the contact complex coupling $\lambda = \lambda_1 + i\lambda_2$ in the one-dimensional inverse square potential problem: (A) undercritical regime $\kappa = \frac{1}{8} < \kappa_{cr}$, (B) overcritical regime $\kappa = \frac{1}{2} > \kappa_{cr}$. Arrows denote the direction towards the UV. In AdS/CFT the case (A) corresponds to $m^2 > m_{BF}^2$, while (B) appears for $m^2 < m_{BF}^2$.

derive in the next subsection, note that the two-point function $\langle OO^\dagger \rangle$ in momentum space is proportional to $q^{2\nu}$ in the boundary CFT$_+$ and $q^{-2\nu}$ in the boundary CFT$_-$.

B. Nonrelativistic holography

The ideas presented in the last subsection can be straightforwardly applied to the case of the nonrelativistic AdS/CFT correspondence. All the derived results still hold provided we substitute $d \to D + 2$ and $q \to \tilde{q}$. Moreover, as was pointed out in Sec. III we can go below the nonrelativistic (spurious) BF bound $m_{BF}^2 = -\frac{(D+2)^2}{4}$ without any instability in the $Sch_{D+3}$ spacetime.

In this subsection we concentrate our attention to the interesting regime $m^2 < m_{BF}^2$ for $\tilde{q}^2 > 0$, where the regular bulk solution of Eq. (13) is given by

$$\phi_{\tilde{q}}(z) = z^{(D+2)/2}K_{(D+2)/2}(\tilde{q}z)\frac{z_0^{(D+2)/2}K_{(D+2)/2}(\tilde{q}z_0)}{z_0^{(D+2)/2}K_{(D+2)/2}(\tilde{q}z_0)}, \quad \nu = \sqrt{\frac{(D + 2)^2}{4} + m^2}. \quad (43)$$

This solution is real and normalized as $\phi_{\tilde{q}}(z_0) = 1$.

In order to gain some intuition we first solve the equivalent regularized one-dimensional
Schrödinger equation (14) in the regime $\kappa > \kappa_{cr} = \frac{1}{4}$. For $z > 0$, $y \equiv \tilde{q} z \ll 1$ the general solution reads

$$\psi(z) = c_{-} z^{1/2 - i|\nu|} + c_{+} z^{1/2 + i|\nu|} = \tilde{c}_{-} y^{1/2 - i|\nu|} + \tilde{c}_{+} y^{1/2 + i|\nu|}, \quad \nu = \sqrt{\frac{1}{4} - \kappa},$$

(44)

where $\tilde{c}_{\pm} \equiv \frac{c_{\pm}}{q^{1/2 + i|\nu|}}$. Here $\psi(z)$ is square integrable around the origin for any $c_{-}, c_{+} \in \mathbb{C}$, hence the general solution (43) is normalizable. Following the same steps as in the last subsection, we construct the RG trajectories of the contact emergent coupling $\lambda$ which take the form

$$\lambda(t) = -1 + 2i|\nu| \frac{e^{i|\nu|\tau} - Ce^{-i|\nu|\tau}}{e^{i|\nu|\tau} + Ce^{-i|\nu|\tau}}, \quad C = \frac{\tilde{c}_{+}}{\tilde{c}_{-}}.$$  

(45)

The RG flow solves Eq. (42) and has two complex fixed points $\lambda_{\pm} = -1 \pm 2i|\nu|$. Its phase portrait is depicted in Fig. 2(B). In the complex plane the RG trajectories form closed circles of radius $R = 4 \left| \frac{C e^{i|\nu|\tau}}{1 + |C e^{i|\nu|\tau}|} \right|$. The real $\lambda$-line is a separatrix of the two complex fixed points, and for the real initial condition\(^8\) $\lambda(t = 0) \in \mathbb{R}$ the RG flow remains on the real line. In this regime the renormalization of the coupling $\lambda$ exhibits an infinite (unbounded) limit cycle\(^9\) and periodically traverses the real $\lambda$-line. The continuous scale symmetry of the classical inverse square potential is broken to the discrete subgroup $\mathbb{Z}$ by a quantum anomaly.

In the rest of this subsection we demonstrate that for $m^2 < m_{BF}^2$ the real gravity solution (43) is dual to a nonrelativistic boundary field theory with an unbounded limit cycle. To this end, first, we find from the near-boundary asymptotic form (33) of the solution (43) that

$$C = \frac{\tilde{c}_{+}}{\tilde{c}_{-}} = \Gamma(-i|\nu|) \frac{\Gamma(i|\nu|)}{\Gamma(i|\nu|)} \left( \frac{1}{2} \right)^{2|\nu|},$$

(46)

which is a pure complex phase. This gives rise to the real initial condition $\lambda(t = 0)$ in the equivalent inverse square potential problem and tunes the RG flow to the real limit cycle. Second, using the standard AdS/CFT machinery, we calculate the two-point function $\langle OO^{\dagger} \rangle$ of the operator $O$ dual to the scalar field $\phi$ with $m^2 < m_{BF}^2$ in the nonrelativistic holography. The two-point correlator can be extracted from the quadratic part of the on-
shell action $S_{\text{on-shell}}$, which can be written as the boundary integral

$$S_{\text{on-shell}}[\phi_0, \phi_0^*] = -\int dX \sqrt{-g} g^{zz} \phi^*(X, z) \partial_z \phi(X, z)|_{z=z_0}$$  \hspace{1cm} (47)

with $X = \{t, \xi, \vec{x}\}$. The general on-shell field $\phi(X, z)$ can be now decomposed into the Fourier modes

$$\phi(X, z) = \int d^{D+2}Q \phi_0(Q, z_0) \phi_\bar{q}(z)e^{iQ \cdot X}, \quad Q = \{\omega, M, \vec{q}\}. \hspace{1cm} (48)$$

Using this representation, the on-shell action can be conveniently rewritten as

$$S_{\text{on-shell}}[\phi_0, \phi_0^*] = -\int d^{D+2}Q \phi_0^*(Q, z_0) \mathcal{F}(\vec{q}, z_0) \phi_\bar{q}(Q, z_0)$$  \hspace{1cm} (49)

with the flux factor

$$\mathcal{F}(\vec{q}, z_0) = \lim_{z \rightarrow z_0} \sqrt{-g} g^{zz} \phi_\bar{q}^*(z) \partial_z \phi_\bar{q}(z).$$ \hspace{1cm} (50)

The two-point function can now easily be expressed in terms of the flux factor

$$\langle O(Q_1) O^\dagger(Q_2) \rangle = -\frac{\delta}{\delta \phi_\bar{q}^*(Q_1)} \frac{\delta}{\delta \phi_\bar{q}(Q_2)} S_{\text{on-shell}}[\phi_0, \phi_0^*] = (2\pi)^{D+2} \delta(Q_1 - Q_2) \mathcal{F}(\vec{q}, z_0). \hspace{1cm} (51)$$

Substituting Eq. (43) into Eq. (50) we obtain

$$\mathcal{F}(\vec{q}, z_0) = z_0^{-D-1} \partial_z \frac{z^{(D+2)/2} K_{i|\nu|}(\vec{q}z)}{z_{0}^{(D+2)/2} K_{i|\nu|}(\vec{q}z_0)}|_{z=z_0}. \hspace{1cm} (52)$$

This can be evaluated by introducing the near-boundary asymptotic of the Bessel function

$$K_{i|\nu|}(\vec{q}z) = a_-(\vec{q}z)^{-i|\nu|} + a_+(\vec{q}z)^{i|\nu|}, \quad a_+ = a_- = 2^{-1-i|\nu|} \Gamma(-i|\nu|) \equiv |a_+| e^{i\alpha} \hspace{1cm} (53)$$

into Eq. (52)

$$\mathcal{F}(\vec{q}, z_0) = z_0^{-D-1} \partial_z \frac{a_-(\vec{q}z)^{-i|\nu|+D/2} + a_+(\vec{q}z)^{i|\nu|+D/2}}{a_-(\vec{q}z_0)^{-i|\nu|+D/2} + a_+(\vec{q}z_0)^{i|\nu|+D/2}}|_{z=z_0} \hspace{1cm} (54)$$

$$= z_0^{-D-2}(D/2 + 1) - z_0^{-D-2}|\nu| \tan \{ |\nu| \ln(\vec{q}z_0) + \alpha \}.$$  

The first term is a contact contribution and can be subtracted by a proper boundary counterterm. Thus, the two-point function in momentum space is given by

$$\langle OO^\dagger \rangle \sim \tan \left\{ |\nu| \ln \vec{q} + |\nu| \ln z_0 + \alpha \right\}_{\gamma(z_0)} \hspace{1cm} (55)$$

and has the following properties:
• $\langle OO^\dagger \rangle$ is log-periodic in $\tilde{q}$ with the period $T = \frac{\pi}{|\nu|}$.

• The infinite series of simple pole divergences of the two-point function indicates that the boundary field $O$ describes infinitely many stable particles with energies

$$\omega_n = -\frac{1}{2M} \exp \left( -\frac{2\pi n}{|\nu|} + \frac{\pi - 2\gamma(z_0)}{|\nu|} \right), \quad n \in \mathbb{Z}. \quad (56)$$

The spectrum is infinite with the accumulation point at $\omega = 0$ as $n \to \infty$. It exhibits the geometric behavior

$$\frac{\omega_{n+1}}{\omega_n} = \exp \left( -\frac{2\pi}{|\nu|} \right). \quad (57)$$

From the form of the energy spectrum one can infer, that the continuous scale symmetry is broken to the discrete subgroup $Z$, and we are dealing with a limit cycle solution of the renormalization group.

• As every limit cycle solution has to be defined with a physical UV momentum cutoff [38], the two-point function (55) explicitly depends on $z_0$ through the angle $\gamma(z_0)$. In the RG language $\gamma(z_0)$ determines the ultraviolet initial condition on the limit cycle trajectory.

There is a well-known subtlety in the calculation of the two-point function in the relativistic AdS/CFT correspondence. Only by using the normalized solution [34] and expanding both the numerator and the denominator of the relativistic version of the flux factor formula (52) one obtains the correct normalized two-point function, which is consistent with the Ward identity [37]. We stress that in the nonrelativistic calculation presented above it is absolutely crucial to use the normalized bulk solution (43) and follow the correct prescription. Without proper normalization one would obtain $\langle OO^\dagger \rangle \sim \sin \{ |\nu| \ln(\tilde{q}z_0) + \alpha \}$.

The limit cycle solution appears in different nonrelativistic quantum mechanical problems [38]. One prominent example is the Efimov effect [39] for three identical bosons interacting through a pointlike potential tuned to the unitarity point. Remarkably, an infinite geometric three-body spectrum is developed in this system signaling the limit cycle RG behavior, i.e. the nonrelativistic quantum scale anomaly [21]. The RG period of this limit

\footnote{At the unitarity point the quantum two-body problem has a zero-energy shallow bound state and a scattering cross section saturates the unitarity bound [3, 6].}
cycle is $T = \frac{\pi}{s_0}$, where the so-called Efimov parameter is $s_0 \approx 1.0062$. The first experimental signatures of the Efimov effect were recently observed in experiments with cold bosonic atoms [40]. We speculate that the Efimov effect can be studied with the nonrelativistic AdS/CFT correspondence by incorporating bulk scalars with $m^2 < m_{BF}^2$. In particular, the local atom-dimer composite scalar operator $O = \psi \phi$ has the complex scaling dimension

$$\Delta_\pm = \frac{5}{2} \pm is_0$$

in the three-dimensional Efimov problem [41]. In the light of our proposal, this operator should be dual to the bulk scalar with $\nu = \pm is_0$, i.e. with $m^2 < m_{BF}^2$. It would be interesting to calculate different n-point correlation functions involving this bulk scalar in $Sch_{D+3}$ for $D = 3$ and to compare the result with the known field-theoretical calculations of the scattering amplitudes in the Efimov physics [38].

Another, more simple example, where limit cycles appear, is quantum mechanics in general $D$ spatial dimensions with an inverse square potential [14–20, 22–26, 36]. In the nonconformal phase, i.e. for $\kappa > \kappa_{cr} = \frac{(D-2)^2}{4}$, the composite operator $O = \psi \psi$ acquires a complex scaling dimension. If a gravity dual of this problem can be constructed, the field $O$ should be dual to a bulk scalar with $m^2 < m_{BF}^2$.

We are not aware of the field-theoretical calculations of the two-point function of the composite operators introduced in the preceding two paragraphs. However, it is reassuring that the renormalization group studies [36, 38] of the relevant couplings in both quantum theories reveal periodic dependence on the logarithmic RG scale $t$ of the form\(^{12} \sim \tan(|\nu|t)$, which is consistent with our result (55).

V. CONCLUSION AND OUTLOOK

In this work we revisited the problem of a free complex scalar field in the Poincaré coordinates for the anti-de Sitter and Schrödinger background spacetimes by exploiting its mathematical equivalence to the well-understood problem of quantum mechanics with an inverse square potential in one spatial dimension. With the help of this equivalence it was

\(^{11}\) The prescription for calculation of the n-point correlators in the nonrelativistic AdS/CFT correspondence was given recently in [42, 43].

\(^{12}\) Specifically, $|\nu| = s_0$ for the Efimov effect and $|\nu| = \sqrt{\kappa - \frac{(D-2)^2}{4}}$ in the inverse square potential problem.
demonstrated that, due to the nonrelativistic form of the boundary dispersion relation, there is no need for the mass stability bound in the nonrelativistic AdS/CFT correspondence. We arrived to the same conclusion by solving the problem in the global coordinates for $Sch_{D+3}$.

In the domain where a single bulk theory describes two different conformal field theories we related the RG flows between the two CFTs to the RG evolution of the emergent contact coupling constant in the inverse square potential problem. We argued that for a deeper mathematical understanding the RG flows can be extended to the complex values of the coupling constant and we motivated this generalization.

Finally, the scalar two-point correlator for $m^2 < m^2_{BF}$ was computed in the nonrelativistic holography. Most importantly, the two-point function turned out to depend explicitly on the momentum cut-off, thus violating nonrelativistic continuous scale symmetry. It was demonstrated, however, that it is symmetric under the discrete scale symmetry subgroup. For this reason we concluded that for $m^2 < m^2_{BF}$ the nonrelativistic holography describes a quantum field theory with a quantum mechanical scale anomaly, manifested by the RG limit cycle scaling. As the well-known quantum-mechanical Efimov effect for three equivalent bosons provides a paradigmatic realization of limit cycles in atomic and nuclear physics, we propose that cold bosons at unitarity and the Efimov effect in particular can be studied in the framework of the nonrelativistic AdS/CFT correspondence.

In this paper we argued that there is no mass stability bound for the free scalar field in the nonrelativistic AdS/CFT correspondence. It is natural to ask the question whether our finding still holds even for the interacting scalar theory in the bulk. As an example, let us modify the free action (12) in $Sch_{D+3}$ spacetime by adding the interaction part

$$S_{\text{int}}[\phi, \phi^*] = -\frac{\alpha}{2} \int dz dt d\xi dx \sqrt{-g}(\phi^* \phi)^2.$$  \hspace{1cm} (59)

As in Sec. II the scalar field equation can still be mapped onto the one-dimensional quantum mechanical "Schrödinger" equation

$$-\partial_z^2 \psi(Q, z) + \frac{m^2 + (D+2)^2 - 1}{4} \frac{1}{z^2} \psi(Q, z) + \alpha z^{D-1} \psi^* \psi^2(X, z) = -\tilde{q}^2 \psi(Q, z),$$  \hspace{1cm} (60)

where $X \equiv \{t, \xi, \vec{x}\}$ and $Q \equiv \{\omega, M, \vec{q}\}$ and the interaction term is expressed in the position space, where it is local. The equation is nonlinear, and the analytic solution is difficult to obtain. Physically, we expect the nonlinear term to modify the spectrum, but not to change its reality property. If the energy spectrum of Eq. (60) is real, the argument presented in Sec.
still holds, and there is no mass stability bound even in the presence of interactions. The proper inclusion of interactions can be done perturbatively using diagrammatic techniques of AdS/CFT, and we defer this problem to future.

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