Further results on uniform mixing on abelian Cayley graphs

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Abstract

In the past few decades, quantum algorithms have become a popular research area of both mathematicians and engineers. Among them, uniform mixing provides a uniform probability distribution of quantum information over time which attracts a special attention. However, there are only a few known examples of graphs which admit uniform mixing. In this paper, a characterization of abelian Cayley graphs having uniform mixing is presented. Some concrete constructions of such graphs are provided. Specifically, for cubelike graphs, it is shown that the Cayley graph Cay($\mathbb{F}_2^k$; $S$) has uniform mixing if the characteristic function of $S$ is bent. Moreover, a difference-balanced property of the eigenvalues of an abelian Cayley graph having uniform mixing is established. Some nonexistence results of uniform mixing on abelian Cayley graphs are presented also. Notably, for a linear abelian Cayley graph $\Gamma$ over $\mathbb{Z}_n$, it is proved that uniform mixing occurs on this graph only if $n = 2, 3, 4$ which confirms a long-standing conjecture.

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1. Introduction

Quantum algorithms stand in the central stage of quantum information processing and computation and are the research field of both mathematicians and engineers around a few decades. In 1998, Fahri and Gutmann [25] first introduced the concept of quantum walk (the definition will be given below). After that, Childs et al. [16] found a graph in which the continuous-time quantum walk spreads exponentially faster than any classical algorithm for a certain black-box problem. Childs also showed that the continuous-time quantum walk model is a universal computational model [17].

Let $\Gamma = (V, E)$ be an undirected simple graph where $V$ is the vertex set and $E$ is the edge set. Let $A$ be the adjacency matrix of $\Gamma$, i.e.,

$$A = (a_{uv})_{u, v \in V}, \quad a_{uv} = \begin{cases} 1, & \text{if } (u, v) \in E, \\ 0, & \text{otherwise}. \end{cases}$$

A continuous random walk on $\Gamma$ is determined by a family of matrices of the form $M(t)$, indexed by the vertices of $\Gamma$ and parameterized by a real positive time $t$. The $(u, v)$-entry of $M(t)$ represents...
the probability of starting at vertex \(u\) and reaching vertex \(v\) at time \(t\). Define a continuous random walk on \(\Gamma\) by setting

\[
M(t) = \exp(t(A - D)),
\]

where \(D\) is a diagonal matrix. Then each column of \(M(t)\) corresponds to a probability density of a walk whose initial state is the vertex indexing the column.

For quantum computations, Fahri and Gutmann [25] proposed an analogue continuous quantum walk. For a connected simple graph \(\Gamma\) with adjacency matrix \(A\), they defined the transfer matrix of \(\Gamma\) as the following \(n \times n\) matrix:

\[
H(t) = H_\Gamma(t) = \exp(\sqrt{-1} t A) = \sum_{s=0}^{\infty} (\sqrt{-1} t A)^s/s! = (H_{g,h}(t))_{g,h \in V}, \quad t \in \mathbb{R},
\]

where \(\sqrt{-1} = \sqrt{-1}\), \(\mathbb{R}\) is the field of real numbers, and \(n = |V(\Gamma)|\) is the number of vertices in \(\Gamma\).

**Definition 1.** Let \(\Gamma\) be a graph. For \(u, v \in V(\Gamma)\), we say that \(\Gamma\) has perfect state transfer (PST) from \(u\) to \(v\) at the time \(t > 0\) if the \((u,v)\)-entry of \(H(t)\), denoted by \(H_{uv}(t)\), has absolute value 1. Further, whenever \(|H_{uu}(t)| = 1\) we say that \(\Gamma\) is periodic at \(u\) with period \(t\). If \(\Gamma\) is periodic with period \(t\) at every point, then \(\Gamma\) is named periodic.

Since \(H(t)\) is a unitary matrix, if PST happens in a graph from \(u\) to \(v\), then the entries in the \(u\)-th row and the entries in the \(v\)-th column are all zero except for the \((u,v)\)-th entry. That is, the probability starting from \(u\) to \(v\) is absolutely 1 which is an ideal model of state transferring.

From the beginning of this century, the research on perfect state transfer on graphs becomes very active, see for example, [1, 2, 3, 4, 5, 8, 10, 13, 19, 20, 22, 23, 24, 32]. A graph is called integral if all its eigenvalues are integers. Angeles-Canul et al. [6] investigated PST in integral circulant graphs and the join of graphs. Godsil [27, 28, 29, 30] explained the close relationship between the existence of perfect state transfer on certain graphs and algebraic combinatorics such as the spectrum of the adjacency matrix, association schemes and so on. In [19, 20], Christandl et al. showed that perfect quantum state transfer occurs in some spin networks. Moreover, existence of PST has been obtained for distance-regular graphs [22, 24], complete bipartite graphs [47] (for discrete-time quantum walks), and Hadamard diagonalizable graphs [36]. Cayley graphs are good candidates for exhibiting PST due to their nice algebraic structure [10, 11, 12, 28, 29, 30, 52]. Among these results, Bašić et al. [7, 8, 9], Cheung and Godsil [18] presented a criterion on circulant graphs and cubelike graphs having PST. Remarkably, Coutinho et al. [23] showed that one can decide whether a graph admits PST in polynomial time with respect to the size of the graph. In a previous paper [51], we presented a characterization on connected simple Cayley graphs \(\Gamma = \text{Cay}(G,S)\) having PST, where \(G\) is an abelian group, and we gave a unified interpretation to many known results.

Contrary to PST on graphs in a sense, Moore and Russel introduced the notion of uniform mixing property of continuous-time quantum walks, which they named it as instantaneous uniform mixing [43].

**Definition 2.** We say that a complex matrix \(A\) is a flat matrix if each of its entries has the same modulus. A graph \(\Gamma\) admits uniform mixing at time \(t\) if its transfer matrix \(H(t)\) is flat at time \(t\). Equivalently, a graph \(\Gamma\) admits uniform mixing if and only if

\[
|H_{uv}(t)| = \frac{1}{\sqrt{n}}
\]

for each pair of vertices \(u\) and \(v\) in \(\Gamma\), where \(n\) is the size of the vertex set \(V\).
From the definition, a graph admitting uniform mixing confirms the uniform probability distribution of the walk at time $t$.

In [2], Chan found some graphs in the adjacency algebra of $(2^{k+2} - 8)$-cube that admit uniform mixing at time $\frac{\pi}{4}$ and graphs that have perfect state transfer at time $\frac{\pi}{2}$. She characterized the folded $n$-cubes, the halved $n$-cubes and the folded halved $n$-cubes whose adjacency algebra contains a complex Hadamard matrix. She obtained some conditions under which these graphs admitting instantaneous uniform mixing. Mullin [4] systematically studied uniform mixing on some graphs, he presented some algebraic and number theoretic properties of those graphs having uniform mixing. In [33], Godsil et al proved that if uniform mixing occurs on a bipartite graph with $n$ vertices, then $n$ is divisible by four; if the graph is also regular, then $n$ must be the sum of two integer squares.

Up to date, the known graphs having uniform mixing are rare, we list some of them as follows:

- the star $K_{1,3}$, [34];
- complete graphs $K_2$, $K_3$, $K_4$, [5];
- Hamming graphs $H(d, 2)$, $H(d, 3)$ and $H(d, 4)$, [15, 34, 43];
- the Paley graph of order nine, [33];
- some strongly regular graphs from regular symmetric Hadamard matrices, [33];
- some linear Cayley graphs over $\mathbb{Z}_2^d$, $\mathbb{Z}_3^d$ and $\mathbb{Z}_4^d$, [2, 34];
- the Cartesian product of graphs which admit uniform mixing at the same time, [34].

However, as far as we know, there is no general characterization on which graphs have uniform mixing. Even though there are some necessary and sufficient conditions for some specific graphs to have uniform mixing in [33, 34], no classification of graphs having uniform mixing is known in literature. Thus, a basic question for us is how to find explicit characterizations on graphs that have uniform mixing. Moreover, more concrete constructions or classification of graphs having uniform mixing are always desirable.

In this paper, we give a characterization of abelian Cayley graphs having uniform mixing. We show that uniform mixing on abelian Cayley graphs are closely related to bent functions over finite abelian groups (see Theorem 1, which provides a necessary and sufficient condition for an abelian Cayley graph to admit uniform mixing). We also present some concrete constructions of graphs having uniform mixing (see Example 1). Particularly, for cubelike graphs (i.e., the underlying group is $\mathbb{F}_2^n$, the $n$-dimensional vector space over the finite field $\mathbb{F}_2$), we show that every bent function leads to a cubelike graph having uniform mixing (see Theorem 8). Since there are plenty constructions of bent functions, we can obtain many (some infinite family) graphs having uniform mixing. We also discuss the integrality of graphs which have uniform mixing. Using a result of Kronecker, we prove that, with certain constrains, the time at which an integral abelian Cayley graph has uniform mixing corresponds to a root of unity (see Theorem 2). For integral $p$-graphs, we show that it has uniform mixing if its eigenvalues induce a difference-balanced mapping (see Theorem 3). We also present some nonexistence results, see Proposition 3, 4, and Theorem 4, 5, 6. We note that Theorem 4 is a well-known conjecture in the community, see for example [44]. We proved this conjecture by generalizing the MacWilliams Identity to
group codes, even though it is possibly known in the literature. In \[34\], Godsil et al. conjectured that if an abelian Cayley graph has uniform mixing, then the graph should be integral. Based on our results in this paper and a lot of numerical calculations, we would like to conjecture that if $G$ is an abelian group, then there is a suitable connection set $S \subset G$ such that $\text{Cay}(G, S)$ admits uniform mixing if and only if the underlying group $G$ is $\mathbb{Z}_2^r \times \mathbb{Z}_2^d$, $\mathbb{Z}_2^r$, or $\mathbb{Z}_2^d$, where $r \geq 1, d \geq 1$. Additionally, we also list some more open questions at the end of this paper.

2. Preliminaries

In this section, we give some notation and definitions which are needed in our discussion. Throughout this paper, we use $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ to stand for the set of non-negative integers, the ring of integers, the field of rational numbers, real numbers and complex numbers, respectively. $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ is the ring of integers modulo $n$. The elements of $\mathbb{Z}_n$ are sometimes viewed as integers if there is no confusion. We use $\{\ast \cdots \ast\}$ to stand for a multi-set. For any integer $n$ and a prime number $p$, if $p^s | n$ and $p^{s+1} \nmid n$, $s \geq 0$, then we write $v_p(n) = s$. We use $|S|$ to denote the cardinality of a set $S$.

2.1. Bent functions over finite abelian groups

The notation of Boolean bent functions was introduced by Rothaus in 1976. Some properties and constructions of such functions were presented in \[49\]. In \[42\], McFarland set up a connection between bent functions and some other combinatorial objects such as difference sets, strongly regular graphs and so on. Bent functions also find applications in coding theory, communications, and some other fields, see \[40, 53\] for example. The concept of bent function over finite abelian groups was introduced by Logachev et al. in \[39\].

Let $G$ be an abelian group with order $n$, the exponent of $G$ (the maximal order of elements of $G$) be $e$. Denote $\mathbb{C}^G = \{f : f$ is a complex-valued function on $G\}$. For $f_1, f_2 \in \mathbb{C}^G$, define

$$\langle f_1, f_2 \rangle = \sum_{x \in G} f_1(x) \overline{f_2(x)},$$

where $\overline{f_2(x)}$ is the complex conjugation of $f_2(x)$. Then $\mathbb{C}^G$ is a Hilbert space with the above scalar product. Let $\hat{G}$ be the character group of $G$ consisting of the homomorphisms $\chi : G \rightarrow \mathbb{T} = \{z \in \mathbb{C} : z^e = 1\}$. It is well-known that $G \cong \hat{G}$ as groups and $\chi_g(h) = \chi_h(g)$ for all $g, h \in G$. Write $\hat{G} = \{\chi_a : a \in G\}$. Then the characters of $G$ satisfy the following orthogonal relation:

$$\langle \chi_a, \chi_b \rangle = \begin{cases} 
  n, & \text{if } a = b, \\
  0, & \text{otherwise}.
\end{cases} \quad (1)$$

In other words, the set $\{\frac{1}{\sqrt{n}} \chi_a : a \in G\}$ forms an orthonormal basis of $\mathbb{C}^G$.

For any $f \in \mathbb{C}^G$, its Fourier transform is defined by

$$\hat{f}(\chi_a) = \sum_{x \in G} f(x) \chi_a(x) = \langle f, \chi_a \rangle.$$
The inverse transform is determined by
\[ f(x) = \frac{1}{n} \sum_{a \in G} \hat{f}(\chi_a) \chi_a(x) = \sum_{a \in G} \langle f, \frac{1}{\sqrt{n}} \chi_a \rangle \frac{1}{\sqrt{n}} \chi_a(x). \]

**Definition 3.** Let \( G \) be an abelian group of order \( n \). A function \( f \in \mathbb{C}^G \) is called bent on \( G \) if for every \( a \in G \), the absolute value of \( \hat{f}(\chi_a) \) satisfies
\[ |\hat{f}(\chi_a)| = \sqrt{n}. \]

For more properties on bent functions over abelian and non-abelian groups, we refer the reader to [26, 50].

### 2.2. Cayley graphs over abelian groups

Let \( G \) be a finite abelian group of order \( n \). The operation of \( G \) is addition, the identity element of \( G \) is 0. Let \( S \) be a subset of \( G \) with \( |S| = d \geq 1 \). The Cayley graph \( \Gamma = \text{Cay}(G, S) \) is defined by
\[
V(\Gamma) = G, \text{ the set of vertices,} \\
E(\Gamma) = \{(u, v) : u, v \in G, u - v \in S\}, \text{ the set of edges.}
\]

We assume that 0 \( \notin S \) and \( S = -S := \{-s : s \in S\} \) (which means that \( \Gamma \) is a simple graph) and \( G = \langle S \rangle \) (\( G \) is generated by \( S \) which means that \( \Gamma \) is connected). The adjacency matrix of \( \Gamma \) is defined by \( A = A(\Gamma) = (a_{g, h})_{g, h \in G} \), where
\[
a_{g, h} = \begin{cases} 
1, & \text{if } (g, h) \in E(\Gamma), \\
0, & \text{otherwise.}
\end{cases}
\]

For any symmetric real matrix \( A \) of \( n \) by \( n \), assume that its eigenvalues are \( \lambda_i, 1 \leq i \leq n \), not necessarily distinct. We can form an orthogonal matrix \( P = (p_1, \cdots, p_n) \), where \( p_i \) is an eigenvector of \( \lambda_i \) (\( 1 \leq i \leq n \)). So that we have the following spectral decomposition:
\[
A = \lambda_1 E_1 + \cdots + \lambda_n E_n, \tag{2}
\]
where \( E_i = p_i p_i^* \) for \( 1 \leq i \leq n \) satisfy
\[
E_i E_j = \begin{cases} 
E_i, & \text{if } i = j, \\
0, & \text{otherwise.}
\end{cases} \tag{3}
\]

Here we use the symbol superscript * to denote the conjugate transpose of a matrix. Therefore, we have the decomposition of the transfer matrix
\[ H(t) = \exp(t\lambda_1 E_1 + \cdots + t\lambda_n E_n). \tag{4} \]

Particularly, when \( A \) is the adjacency matrix of an abelian Cayley graph \( \Gamma = \text{Cay}(G, S) \), we can take \( P = \frac{1}{\sqrt{n}} (\chi_g(h))_{g, h \in G} \), where \( n = |G| \) and \( \chi_g \in \hat{G} \). Then \( P \) is a unitary matrix and it is easy to check that
\[ P^*AP = \text{diag}(\lambda_g; g \in G), \]
Lemma 1. Let notation be defined as above. Then we have

\begin{enumerate}[(1)]
  \item $M = \gcd(d - \lambda_g : 0 \neq g \in G)$,
  \item $M_h = \gcd(\lambda_{x+h} - \lambda_x : x \in G), 0 \neq h \in G$,
  \item If $G$ is an abelian $p$-group, then $D_G$ is a power of $p$.
\end{enumerate}

Proof. (1) Since $M = \gcd(d - \lambda_g : 0 \neq g \in G)$, we can write

$$d - \lambda_g = M t_g, g \in G \text{ and } t_g \in \mathbb{Z}.$$  
(9)

Then we have, for every $z \in S$, by (1),

$$\sum_{g \in G} \lambda_g \chi_g(z) = \sum_{g \in G} \sum_{s \in S} \chi_g(s) \chi_g(z) = \sum_{s \in S} \sum_{g \in G} \chi_g(s) \chi_g(z) = |G|. $$  
(10)

Meanwhile,

$$\sum_{g \in G} \lambda_g \overline{\chi_g(z)} = \sum_{g \in G} (d - M t_g) \overline{\chi_g(z)} = -M \sum_{g \in G} t_g \overline{\chi_g(z)}.$$  
(11)

Thus,

$$|G|/M = -\sum_{g \in G} t_g \overline{\chi_g(z)}.$$  
(12)

The left hand side of (12) is a rational number, while the right hand side of it is an algebraic integer. Thus both of them are integers.

(2) Write $\lambda_{x+h} - \lambda_x = M h \theta_h(x), x \in G, 0 \neq h \in G$. Then $\theta_h(x) \in \mathbb{Z}$ and for every $z \in S$, we have

$$\sum_{g \in G} (\lambda_{x+h} - \lambda_x) \chi_z(z) = M h \sum_{g \in G} \chi_z(z) \theta_h(x).$$

That is

$$\sum_{s \in S} \chi_h(s) \chi_z(z) \chi_s(z) - \sum_{s \in S} \chi_s(z) \chi_z(z) = M h \sum_{s \in S} \chi_z(z) \theta_h(x).$$

By the orthogonality of characters, we have

$$|G| (\chi_h(z) - 1) = M h \sum_{s \in G} \chi_z(z) \theta_h(x).$$

In the above equation, let $z$ run through $S$ and add them together. Then we obtain that

$$|G| (\lambda_h - d) = M h \sum_{s \in G} \theta_h(x) \lambda_s.$$  
(11)

(3) It is a direct consequence of (1) and (2). This completes the proof. 
\( \square \)
3. A characterization of abelian Cayley graphs having uniform mixing

Let $G$ be an abelian group with order $n$. Let $S$ be a subset of $G$ with $|S| = d \geq 1, 0 \notin S = -S$ and $G = \langle S \rangle$. Suppose that $\Gamma = \text{Cay}(G, S)$ is the Cayley graph with the connection set $S$. Take $P = \frac{1}{|G|} \chi_G(h)_{h \in G}$ and $E_x = p_x p_x^*$ as before, where $p_x$ is the $x$-th column of $P$. Then the adjacency matrix $A$ of $\Gamma$ has the following spectral decomposition:

$$A = \sum_{g \in G} \lambda_g E_g.$$ 

Meanwhile, the transfer matrix $H(t)$ has the following decomposition:

$$H(t) = \sum_{g \in G} \exp(i\lambda_g t)E_g.$$ 

Thus we have, for every pair $u, v \in G$,

$$H(t)_{u,v} = \sum_{g \in G} \exp(i\lambda_g t)(E_g)_{u,v} = \frac{1}{n} \sum_{g \in G} \exp(i\lambda_g t)\chi_g(u - v). \quad (13)$$

Therefore,

$$|H(t)_{u,v}| = \frac{1}{\sqrt{n}} \Leftrightarrow \left| \sum_{g \in G} \exp(i\lambda_g t)\chi_g(u - v) \right| = \sqrt{n}.$$ 

Hence, $\Gamma$ has uniform mixing at time $t$ if and only if for every $a \in G$, it holds that

$$\left| \sum_{g \in G} \exp(i\lambda_g t)\chi_g(a) \right| = \sqrt{n}. \quad (14)$$

Thus we have proved the following result.

**Theorem 1.** Let $\Gamma$ be a simple abelian Cayley graph. For each fixed real number $t$, define a complex-valued function on $G$ by

$$G_t(g) = \exp(i\lambda_g t).$$

Then $\Gamma$ has uniform mixing at time $t$ if and only if $G_t$ is a bent function on $G$.

We note that the result in Theorem 1 can also be deduced from Proposition 2.3 of [2]. Equivalently, we have

**Corollary 1.** Let notation be defined as above. Then $\Gamma$ has uniform mixing at time $t$ if and only if for every $h(\neq 0) \in G$, it holds that the correlation function

$$R_h(t) := \sum_{g \in G} \exp(i(\lambda_g + h) t) = 0. \quad (15)$$

**Corollary 2.** Define an $n$ by $n$ matrix

$$W = (\exp(i\lambda_{x+y} t))_{x,y \in G}.$$

Then $\Gamma$ has uniform mixing at time $t$ if and only if $W$ is a symmetric complex Hadamard matrix, namely, $W^T = W$ and $WW^* = nI_n$. 

7
Corollary 1 and Corollary 2 are equivalent, we just give the proof of the second one.

**Proof.** Denote

$$z_g := \frac{1}{\sqrt{n}} \sum_{x \in G} \exp(i \lambda x t) \chi_x(g), \forall g \in G. \quad (17)$$

Then by (14), we have that \( \Gamma \) has uniform mixing at time \( t \) if and only if \(|z_g| = 1\) for all \( g \in G \). From (17), we get

$$\exp(i \lambda x t) = \frac{1}{\sqrt{n}} \sum_{g \in G} z_g \chi_g(x), \forall x \in G. \quad (18)$$

Thus, if \(|z_g| = 1\) holds for all \( g \in G \), then

$$\sum_{g \in G} \exp(i \lambda x g t) \exp(i \lambda y g t) = \frac{1}{n} \sum_{a \in G} |z_a|^2 \chi_a(y - x) = \sum_{a \in G} \chi_a(y - x) = \begin{cases} 0, & \text{if } y \neq x, \\ n, & \text{if } y = x. \end{cases}$$

Thus \( W \) is a symmetric complex Hadamard matrix. Conversely, if \( W \) is a complex Hadamard matrix, then by the above computation, we have

$$\sum_{a \in G} |z_a|^2 \chi_a(a) = \begin{cases} 0, & \text{if } a \neq 0, \\ n, & \text{if } a = 0. \end{cases}$$

The desired result follows from the inverse Fourier transformation.

For a matrix \( A \) with all nonzero entries, the **Schur inverse** of \( A \), denoted by \( A^{(-1)} \), is given by

$$(A^{(-1)})_{uv} = \frac{1}{A_{uv}}.$$

An \( n \times n \) complex matrix \( A \) is called a **Type-II** matrix if

$$A \left( A^{(-1)} \right)^T = n I_n. \quad (19)$$

Let \( W \) be defined as (16). Since \( \lambda_g \) is a real number for all \( g \in G \), we know that \( W^* = W^{(-1)} \). Thus we have

**Corollary 3.** \( \Gamma \) has uniform mixing at time \( t \) if and only if \( W \) is a symmetric type-II matrix, where \( W \) is defined by (16).

4. **Classification of integral abelian Cayley graphs having uniform mixing**

In this section, we consider the integrality of \( \Gamma = \text{Cay}(G, S) \), where \( G \) is a finite abelian group. We divide this section into three subsections. The first subsection, Subsect. 4.1, studies the linear dependence of some transcendental numbers. The second subsection, Subsect. 4.2, investigates integral Cayley \( p \)-graphs having uniform mixing. In the third subsection, Subsect. 4.3, we will give some existence results of integral abelian Cayley graphs having uniform mixing.
4.1. Linear dependence

Before going to present our next results, we need a lemma due to Lindemann first.

**Lemma 2.** [43, Theorem 9.1] Given any \( m \) distinct algebraic numbers \( a_1, \ldots, a_m \), the values \( \exp(a_1), \ldots, \exp(a_m) \) are linearly independent over the field of algebraic numbers.

From Lemma 2, we have the following result.

**Proposition 1.** Let \( \Gamma = \text{Cay}(G, S) \) be an abelian Cayley graph. Then \( \Gamma \) cannot have uniform mixing at any algebraic number \( t \).

**Proof.** Suppose that \( \Gamma \) has uniform mixing at time \( t \), where \( t \) is an algebraic number. Then by Corollary 1, we have that
\[
R_h(t) = \sum_{g \in G} \exp(i(\lambda g + h - \lambda g)t) = 0
\]
holds true for any nonzero \( h \). Since \( i(\lambda g + h - \lambda g) \), \( g \in G \) are algebraic numbers, and not all of them are equal, we get a contradiction with Lemma 2.

We need further another lemma which is named Gelfond-Schneider Theorem.

**Lemma 3.** [14] If \( x \) and \( y \) are two non-zero complex numbers with \( x \) irrational, then at least one of the numbers \( x \), \( \exp(y) \), or \( \exp(xy) \) is transcendental.

A simple consequence of Lemma 3 is the following:

**Corollary 4.** If there are elements \( g, h \neq 0 \in G \) such that \( \lambda g + h - \lambda g \) is irrational, then \( \exp(i(\lambda g + h - \lambda g)\frac{2\pi}{N}) \) is transcendental for any positive integers \( r, N \geq 2 \).

**Proof.** Putting \( x = \lambda g + h - \lambda g, y = \frac{2\pi r}{N} \) and applying Lemma 3 since \( x \) and \( \exp(y) \) are algebraic numbers, we get the desired result.

Next, we consider integral abelian Cayley graphs that have uniform mixing at some time \( t \). Suppose that \( \Gamma = \text{Cay}(G, S) \) is an integral abelian Cayley graph having uniform mixing at time \( t \). Denote
\[
A_h(X) := \sum_{g \in G} X^{\lambda g + h - \lambda g} \in \mathbb{Z}[X, X^{-1}], \forall h(\neq 0) \in G.
\]

Let
\[
d_h = \max\{\lambda g + h - \lambda g : g \in G\}, \text{ and } B_h(X) = X^{d_h}A_h(X).
\]

Denote
\[
a(X) := \gcd(B_h(X) : 0 \neq h \in G).
\]

Based on the above preparations, we prove the following result:

**Theorem 2.** Let notation be defined as above. Suppose that \( \Gamma = \text{Cay}(G, S) \) is an integral abelian Cayley graph. Then the number of the time \( t \) in the interval \((0, 2\pi)\) such that \( \Gamma \) has uniform mixing at \( t \) is upper bounded by \( 2 \min_{0 \neq h \in G} d_h \). Moreover, assume that the \( a(X) \) in \((21)\) is monic and has all its roots on the unit circle. If \( \Gamma = \text{Cay}(G, S) \) has uniform mixing at a time \( t \), then \( \gamma = e^t \) is a root of unity.
In order to prove this result, we need the following lemma due to Kronecker.

**Lemma 4.** If an algebraic number whose conjugates all have absolute value 1, then the algebraic number is a root of unity.

**Proof.** Suppose that $\Gamma = \text{Cay}(G, S)$ is an integral abelian Cayley graph having uniform mixing at time $t$. Denote $$N_{h,u} = \|g \in G : \lambda_{g+h} - \lambda_g = u\|, 0 \neq h \in G, u \in \mathbb{Z}.$$ Since $\text{Cay}(G, S)$ is an integral abelian graph, we have $\lambda_g = \sum_{z \in S} \chi_g(z) = \sum_{z \in S} X^{-g} = \lambda_{-g}$ for all $g \in G$. Thus $$\lambda_{g+h} - \lambda_g = u \iff \lambda_{(g+h)+h} - \lambda_{(g+h)} = -u.$$ Hence $$N_{h,u} = N_{h,-u}.$$ Therefore, $$A_h(X) := \sum_{g \in G} X^{\lambda_{g+h} - \lambda_g} = \sum_{u \in \mathbb{N}} N_{h,u} (X^u + X^{-u})$$ and $B_h(X) = X^h A_h(X)$ is a self-reciprocal polynomial in $\mathbb{Z}[X]$.

Denote $\gamma = e^{it}$. Then by Corollary 3, $\gamma$ is a root of $B_h(X)$ which implies that $\gamma$ is an algebraic integer, where $d_h = \max\{d_h : 0 \neq h \in G\}$. Note that $B_h(X)$ is a monic polynomial in $\mathbb{Z}[X]$ of degree $2d_h$. Then $\gamma$ is a root of $a(X)$. Since $B_h(X)$ is a self-reciprocal polynomial for every $0 \neq h \in G$, so does $a(X)$. The first statement of Theorem 2 is proved.

Let $m(X)$ be the minimal polynomial of $\gamma$ over $\mathbb{Q}$. Then $m(X)/a(X)$. By assumption, all the roots of $m(X)$ are on the unit circle, thus all the conjugates of $\gamma$ have unit norm, and then by Lemma 4, $\gamma$ is a root of unity. \qed

**Remark 1.** We note that there are extensive researches on those self-reciprocal polynomials with all their roots on the unit circle, see 
[42] and the references therein.

In view of Proposition 1 and Corollary 4, we consider the possibility of integral abelian Cayley graphs having uniform mixing at time $t = \frac{2\pi}{N}$ for some integers $r$ and $N$ with $\gcd(r, N) = 1$. In this case, $\exp(it)$ is a $N$-th root of unity. In fact, our numerical experiments indicate that if an integral abelian Cayley graph has uniform mixing at a time $t$, then $\exp(it)$ is a root of unity.

We recall some basic facts about cyclotomic fields. Let $\omega_h = \exp(i \frac{2\pi}{n})$ and $K = \mathbb{Q}(\omega_h)$ be the corresponding cyclotomic field. The Galois group of the extension $K/\mathbb{Q}$ is $$\text{Gal}(K/\mathbb{Q}) = \{ \sigma_\ell : \ell \in \mathbb{Z}_n^*/n \} \cong \mathbb{Z}_n^*,$$ where $\mathbb{Z}_n^* = \{ \ell \in \mathbb{Z}_n : \gcd(\ell, n) = 1 \}$ is the unit group of the ring $\mathbb{Z}_n = \mathbb{Z}/(n)$ and $\sigma_\ell$ is defined by $\sigma_\ell(\omega_h) = \omega_h^\ell$.

Utilizing the cyclotomic field $\mathbb{Q}(\omega_h)$, we have the following criterion about the integrality of abelian Cayley graphs.

**Lemma 5.** Suppose that $G$ is an abelian group of order $n$ and $\Gamma = \text{Cay}(G, S)$ is a Cayley graph. Then the following statements are equivalent:

1. $\Gamma$ is an integral graph;
2. for every positive integer $\ell$, $1 \leq \ell \leq n$, and $\gcd(\ell, n) = 1$, it holds that $\lambda_{\ell g} = \lambda_g$;
3. for every positive integer $\ell$ with $\gcd(\ell, n) = 1$, it holds that $\ell S := \{ \ell s : s \in S \} = S$. 

10
\textbf{Proof.} (1) $\Leftrightarrow$ (2)

Since $\lambda_g$ is algebraic, it is rational if and only if it is integral. Thus, $\lambda_g$ is an integer if and only if for every $\sigma_t \in \text{Gal}(K/\mathbb{Q})$, it holds that $\sigma_t(\lambda_g) = \lambda_g$. Now, by \ref{5}, we have that

$$\sigma_t(\lambda_g) = \sigma_t \left( \sum_{s \in S} \chi_{s}(s) \right) = \sum_{s \in S} \chi_{s}(s)^{t} = \sum_{s \in S} \chi_{s}(\ell g) = \lambda_{\ell g}.$$ 

Note that in the above equation, we make use of the property that $\chi_{s}(b) = \chi_{s}(a)$ for every $a, b$ in an abelian group. Thus $\lambda_g$ is integral if and only if for every positive integer $\ell$ with $\gcd(\ell, n) = 1$, we have $\lambda_{\ell g} = \lambda_g$.

(1) $\Leftrightarrow$ (3) This fact is known to the community and has been proved in \cite{51,31}. \hfill \Box

4.2. Integral Cayley $p$-graphs

If $G$ is an $p$-group and $\Gamma = \text{Cay}(G, S)$ is an integral Cayley graph, we call $\Gamma$ an integral Cayley $p$-graph.

A $k$-term sum $x_1 + x_2 + \cdots + x_k$ is called a vanishing sum of order $k$ if its value is 0.

For vanishing sums of $N$-th roots of unity, Lam and Leung have the following result.

\textbf{Lemma 6.} \cite{33} Suppose that $p_1, \cdots, p_s$ are distinct prime numbers and $N = p_1^{e_1} \cdots p_s^{e_s}$. Then there is a vanishing sum of order $n$, composed from $N$-th roots of unity if and only if $n$ is a natural linear combination of the numbers $p_i$, i.e.

$$n = p_1\mathbb{N} + \cdots + p_s\mathbb{N} := \{ p_1a_1 + \cdots + p_s a_s : a_1, \cdots, a_s \in \mathbb{N} \}.$$ 

A direct consequence of Lemma 6 and Corollary 1 is the following:

\textbf{Corollary 5.} Let $\Gamma = \text{Cay}(G, S)$ be an integral abelian Cayley $p$-graph. If $\Gamma$ has uniform mixing at time $t = 2\pi r$ for some rational number $r$, then $v_p(r) \geq 0$ for any prime $p \neq p'$, i.e., $r$ should have the form $\frac{m}{p^e}$ for some integers $m$ and $e'$.

For any abelian group $G$ and an abelian group $H$, a mapping $f$ from $G$ to $H$ is called a difference balanced function if for every $a \neq 0 \in G$ and every $b \in H$, the number of solutions to the following equation

$$f(x + a) - f(x) = b$$

is independent on the choice of $a$ and $b$. Difference balanced functions find their applications in finite geometry, combinatorial theory, and coding theory etc, see for example, \cite{46}.

In words of difference balanced function, we have the following result.

\textbf{Theorem 3.} Let $G$ be an abelian group of an odd prime power order and $\Gamma = \text{Cay}(G, S)$ be an integral Cayley $p$-graph. Let $e'$ be a positive integer such that $p^{e'-1}(\lambda_{g+h} - \lambda_{g})$ for every $g, h \in G$. Let $H = \mathbb{Z}_{p^e} = \{ p^{e'-1} j : j = 0, 1, \cdots, p - 1 \}$. Then $\Gamma$ has uniform mixing at time $t = \frac{2\pi r}{p'}$ if and only if the mapping $\lambda : G \to H$, is a difference balanced function, where $r$ is a nonzero integer.

\textbf{Proof.} “ $\Rightarrow$ ” By Corollary 1 $\Gamma$ has uniform mixing at time $t$ if and only if $R_t(h) = 0$ for all $0 \neq h \in G$. Denote $N = p^{e'}$ and define a multi-set

$$\Omega_h = \{ \lambda_{g+h} - \lambda_{g} \pmod{N} : g \in G \}.$$
Then
\[ R_h(t) = \sum_{z \in \Omega_h} \omega_N^z. \]

We now show that if \( R_h(t) = 0 \), then \( \Omega_h = \{(p^d-1)^j(p^d-1), j \in \mathbb{Z}_p\} \), and \( \lambda \) is a difference balanced function from \( G \) to \( H \).

By the assumption, we can write \( \Omega_h = \{0, (p^d-1)^0, \cdots, (p^d-1)^{(p-1)}\} \).

Here we use \( j^{(t)} \) to indicate that the multiplicity of \( j \) in \( \Omega_h \) is \( t_j \) \((t_j \geq 0)\). Then
\[ R_h(t) = \sum_{j=0}^{p-1} t_j \omega_N^{p^d-1} = 0. \] (22)

Since \( \omega_N^{p^d-1} = \omega_p \) is a \( p \)th root of unity, we can rewrite (22) as
\[ R_h(t) = \sum_{j=0}^{p-1} t_j \omega_p = 0. \] (23)

It is obvious that
\[ \sum_{j=0}^{p-1} t_j = |G| = p^e. \] (24)

The minimal polynomial of \( \omega_p = \exp(\frac{2\pi}{p}) \) is the cyclotomic polynomial \( \phi_p(x) \), i.e.,
\[ \phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1. \]

Hence combing (23) and (24), we have
\[ t_0 = \cdots = t_{p-1} = p^e-1. \] (25)

" \( \Leftarrow \)" Obvious.

Remark 2. For an integral Cayley \( p \)-graph \( \text{Cay}(G, S) \), we denote integers \( M \) and \( M_h \) as in (6), (7), respectively. By \( D_G \) we denote \( \gcd(M_h : 0 \neq h \in G) \). Then Lemma \([7]\) indicates that both \( M \) and \( D_G \) are powers of \( p \). Hence the number \( e' \) in Theorem \([4]\) is then determined by \( D_G = p^{e'-1} \).

4.2.1. Uniform mixing on integral Cayley graphs over \( \mathbb{Z}_p^r \)

If the underlying group of an integral Cayley \( p \)-graph is an elementary group, then we have the following explicit characterization.

Proposition 2. Let \( p \) be an odd prime number and \( G = \mathbb{Z}_p^r \) the elementary commutative \( p \)-group. If \( r \leq 2 \) and \( \Gamma = \text{Cay}(G, S) \) is an integral graph, then \( \Gamma \) has uniform mixing at time \( t = 2\pi m/p^e \) for some integers \( m, e' \) and some subset \( S \subseteq G \) if and only if \( p = 3 \).
Proof. The “if” part is known to the community (see for example [27]). We only need to prove the “only if” part.

If \( r = 1 \), by the assumption that \( \Gamma \) is an integral graph, we know that \( \Gamma \) is the complete graph \( K_p \), the required result is well known.

Next we give the proof for \( r = 2 \).

Define the action of \( \mathbb{Z}_p^r = \{ \ell : 0 < \ell < p \} \) on \( G \) by

\[
\ell(i, j) = (\ell, \ell, j), \forall \ell \in \mathbb{Z}_p, (i, j) \in \mathbb{Z}_p \times \mathbb{Z}_p.
\]

The orbits of the action are

\[
[(1, 1)], [(1, x)](x \neq 1, 0), [(0, 1)], [(1, 0)], \text{ each one has length } p-1, \text{ and } [(0, 0)] \text{ of length one.}
\]

Where \( [(a, b)] = \{(\ell a, \ell b) : \ell \in \mathbb{Z}_p^r \} \). Let \( S \) be a symmetric subset of \( \mathbb{Z}_p^r \) and \( \Gamma = \text{Cay}(\mathbb{Z}_p^r, S) \) be an integral Cayley graph over \( \mathbb{Z}_p^r \). By Lemma 5, in order to guarantee the integrality of \( \Gamma \), \( S \) should be a union of some orbits listed above.

We can prove the result by a case-by-case study. Since \( \Gamma \) is connected, \( \mathbb{Z}_p^r = \langle S \rangle \), we have the following cases which are needed to deal with.

Case 1: \( S = [(1, 0)] \) or \( S = [(0, 1)] \) or \( S = [(1, x)] \) for some \( x \neq 0 \).

In these cases, \( \Gamma \) is not connected.

Case 2: \( S = [(1, 0)] \cup [(0, 1)] \).

If this case, \( \Gamma \) is a Hamming graph. It has uniform mixing if and only if \( p = 2 \) or \( 3 \) for prime \( p \), [2].

Case 3: For general case. By applying an automorphism of the \( \mathbb{Z}_p^2 \), without loss of generality, we may assume that \( S = \bigcup_{i \in I} [(1, x)] \cup [(1, 0)] \cup [(0, 1)] \), where the index set \( I \) is a subset of \( \{1, 2, \cdots, p-1\} \).

Suppose that \( I = \{i_1, \cdots, i_v\}, 1 \leq v \leq p-1 \). If \( v = p-1 \), then \( \Gamma \) is the complete graph and the desired result is known. Actually, we can assume that \( 1 \leq v \leq \frac{p-1}{2} \) since we can reply \( \Gamma \) with its complement by [13, Lemma 2.1] if necessary. Then we proceed to show that \( \Gamma = \text{Cay}(G, S) \) has uniform mixing only if \( p = 3 \). Firstly, note that

\[
d = |S| = 2(p-1) + (p - 1)v = (v + 2)(p - 1).
\]

And for every pair \( 0 \leq k, l \leq p-1 \),

\[
A_{(k,l)} = \sum_{i \in S} \chi_k(i) = \sum_{i=1}^{p-1} \chi_k(i) + \sum_{\ell=1}^{p-1} \sum_{1 \leq j \leq p-1} \chi_k(j) \chi_{\ell}(ji),
\]

where \( \chi_k, \chi_{\ell} \in \hat{\mathbb{Z}}_p \) are characters of \( \mathbb{Z}_p \).

Thus, if \( k = 0, l \neq 0 \),

\[
A_{(0,l)} = p - 1 + (-1) + \sum_{\ell=1}^{p-1} \sum_{1 \leq j \leq p-1} \chi_{\ell}(ji) = p - 2 - v.
\]

If \( k \neq 0, l = 0 \),

\[
A_{(k,0)} = \sum_{i=1}^{p-1} \chi_k(i) + (p-1) + \sum_{\ell=1}^{p-1} \sum_{1 \leq j \leq p-1} \chi_k(j) = p - 2 - v.
\]
If $kl \neq 0$,

$$\lambda_{(k,l)} = -2 + \sum_{i=1}^{p-1} \left( \sum_{j=1}^{v} \chi(j)\chi(ji) \right).$$

Thus when $kl \neq 0$,

$$\lambda_{(k,l)} = \begin{cases} 
p - v - 2, & \text{if } -k/\ell \in I, \\
- v - 2, & \text{otherwise.} 
\end{cases}$$

Therefore the spectra of $\Gamma = \text{Cay}(G,S)$ are

$$\text{Spec}(\Gamma) = \{(v + 2)(p - 1)^{11}, (p - v - 2)(v^2 - v^2 - 2), (-v - 2)(p^2 - (v + 2)p + v + 1)\}.$$ 

When $h = (u, 0), u \neq 0$, a direct computation shows that

$$\lambda_{(k,l)} - \lambda_{(k,l)} = \begin{cases} 
\text{value} & \text{condition} & \text{frequency} \\
-p(v + 1) & k = \ell = 0 & 1 \\
p(v + 1) & k = -u, \ell = 0 & 1 \\
0 & k \neq 0, -u, \ell = 0 & p - 2 \\
0 & k = 0, \ell \neq 0, -u/\ell \in I & v \\
-p & k = 0, \ell \neq 0, -u/\ell \notin I & p - 1 - v \\
0 & k = -u, \ell \neq 0, u/\ell \in I & v \\
p & k = -u, \ell \neq 0, u/\ell \notin I & p - 1 - v \\
0 & k \neq 0, -u, \ell \neq 0, -(k + u)/\ell \in I, -u/\ell \in I & ? \\
0 & k \neq 0, -u, \ell \neq 0, -(k + u)/\ell \notin I, -u/\ell \notin I & ? \\
p & k \neq 0, -u, \ell \neq 0, -(k + u)/\ell \in I, -u/\ell \notin I & ? \\
-p & k \neq 0, -u, \ell \neq 0, -(k + u)/\ell \notin I, -u/\ell \in I & ? 
\end{cases}$$

The symbol “?” in the above equation indicates that the corresponding number of frequency is unknown, anyway, we have

$$\lambda_{(k,l)} - \lambda_{(k,l)} : (k, \ell) \in \mathbb{Z}_p^2 \subseteq \{0, \pm p, \pm (v + 1)p\}.$$ 

By Theorem\textsuperscript{a} the distribution of $\lambda_{(k+l,u,\ell)} - \lambda_{(k,l)} : (k, \ell) \in \mathbb{Z}_p^2$ should be balanced, therefore, we know that $p = 3, 5$ (since the difference $\lambda_{(k+l,u,\ell)} - \lambda_{(k,l)} : (k, \ell) \in \mathbb{Z}_p^2$ is at most five-valued). However, when $p = 5$, the frequency of the number $(v+1)p$ is only once, but the values $0, \pm p, \pm 2p$ should appear equally times, thus the difference cannot be balance. Hence, we only have $p = 3$ as the possible candidate.

This completes the proof for $r = 2$.

\textbf{Remark 3.} Note that from the proof of Proposition\textsuperscript{a} we see that the integral Cayley graph in concern is a strongly regular graph, one can use \textsuperscript{b} Lemma 2.2 to get an alternative proof of this proposition for $r = 2$.

In order to consider the possibility of uniform mixing on $\mathbb{Z}_q^r$ with $r \geq 3$, we need to introduce the quotient of Hamming graphs.

Let $q, r$ be positive integers and $H(r,q)$ denote the Hamming graph. Let $M$ be a submodule of $\mathbb{Z}_q^r$. The quotient graph of $H(r,q)$ induced by $M$, denoted by $H(r,q)/M$, is a graph with a
vertex for each coset of $M$, such that two vertices are adjacent if there is a matching between the two associated cosets. It is shown that quotient graphs of $H(r,q)$ are exactly the linear Cayley graphs over $\mathbb{Z}_p^r$, that is, graphs $\text{Cay}(\mathbb{Z}_p^r, S)$ for which $S \cup \{0\}$ is closed under multiplication by $\mathbb{Z}_p$. Conversely, every linear Cayley graphs over $\mathbb{Z}_p^r$ is isomorphic to a quotient graph of $H(r,q)$ induced by some $M$ with minimum distance at least three [54, Theorem 2]. Obviously, if $q$ is a prime number, then the linear Cayley graphs over $\mathbb{Z}_p^r$ are exactly the integral Cayley graphs over the same group. The following lemma is crucial in the sequel.

**Lemma 7. [54, Lemma 7.4.2]** Let $p$ denote a prime, and let $M$ denote a subgroup of $\mathbb{Z}_p^r$ (viewed as a vector space over the finite field $\mathbb{Z}_p$) with minimum distance at least three such that $|M| = p^s$. The entries of the transition matrix of $H(r,p)/M$ are equal to

$$H(t)_{x,M} = \frac{1}{p^r} \sum_{u \in M} \chi_a(v) = \chi_a(v),$$

where $\chi_a(v)$ is the number of nonzero coordinates of $u$, and $\alpha = \exp(itp^{-1}), \beta = \exp(-it)$.

Below, let’s recall a generalization of the famous MacWilliams identity which is due to Zierler as follows.

**Lemma 8. [54, Theorem]** Let $A$ be a family of elements of $\mathbb{Z}_p^r$, that is, a collection in which each element may appear any finite number of times. Accordingly, we let $N(A,i)$ denote the number of appearances in $A$ of elements of weight $i$, including repetitions. Then

$$N(A,i)(x - y)^i = \sum_{v \in M} y^{\chi_a(v)} x^{-\chi(v)} \sum_{u \in A} \chi_a(v).$$

Particularly, when $A = a + M$ is a coset of a $s$-dimensional subspace $M$ in $\mathbb{Z}_p^r$, then

$$N(a + M, i)(x - y)^i = p^s \sum_{v \in M} y^{\chi_a(v)} x^{-\chi(v)} \chi_a(v).$$

As a consequence, one has an extension of Corollary 8.2.3 in [44].

**Corollary 6.** Let $p$ denote a prime, and let $M$ denote a subgroup of $\mathbb{Z}_p^r$ with minimum distance at least three such that $|M| = p^s$. If uniform mixing occurs on $H(r,p)/M$ at time $t$, then for every coset $a + M$,

$$\left| \sum_{u \in M} \exp(-itp^s) \chi_a(u) \right|^2 = p^{-r}.$$

**Proof.** By Lemma 7 the entries in the quotient $H(r,p)/M$ are

$$H(t)_{a + M} = \frac{1}{p^r} \sum_{u \in a + M} \chi_a(v) = \chi_a(v),$$

$$= \frac{1}{p^r} \sum_{i=0}^r N(a + M, i)(x - y)^i \chi_a(u) \chi_a(v) \quad \text{(Lemma 8)}$$

$$= \exp(-itp^{-1}r) \sum_{u \in M} \exp(-itp^s) \chi_a(u).$$
Thus, by definition, uniform mixing occurs on $H(r, p)/M$ at time $t$ if and only if (26) holds true.

Therefore, we have

**Corollary 7.** Let notations be defined as above. If uniform mixing occurs on $H(r, p)/M$ at time $t$, then for every nonzero $c \in M^*$,

$$Z(c) := \sum_{v \in M^*} \exp(ut(p(wt(v) + wt(c)) - wt(v))) = 0.$$  \hspace{1cm} (27)

**Proof.** If uniform mixing occurs on $H(r, p)/M$ at time $t$, then by (26), we have, for every coset $a + M$,

$$p^{r-s} = \left| \sum_{u \in M^*} \exp(-tp(wt(u)))\chi_a(u) \right|^2 = \sum_{u \in M^*} \exp(-tp(wt(u) - wt(v)))\chi_a(u - v) - 1 = \sum_{u \in M^*} \exp(-tp(wt(u) - wt(v)))\chi_a(u - v)

= p^{r-s} + \sum_{u \in M^*} \chi_a(u - v) - 1 = p^{r-s} + \sum_{0 \in M^*} Z(c)\chi_a(c).$$

Hence, $\sum_{0 \in M^*} Z(c)\chi_a(c) = 0$ for every coset $a + M$. Since $\{\chi_a(c) : \emptyset \neq c \in M^*\}$ is linear independent over the complex numbers field, we get the desired result.

An immediate fact which comes from Corollary 7 is the following

**Corollary 8.** Let notations be defined as above. Then uniform mixing cannot occur on $H(r, p)/M$ at time $t = \frac{2m}{p}$ for any positive integer $m$.

**Proof.** Indeed, by Corollary 7 If uniform mixing occurs on $H(r, p)/M$ at the time $t = \frac{2m}{p}$, then

$$0 = \sum_{0 \in M^*} \exp(ut(pwt(a + c) - wt(a))) = \sum_{0 \in M^*} 1 = |M^*| - 1 = p^{r-s} - 1,$$

which implies that $r = s$. A contradiction.

Moreover, an interesting result can be obtained as follows.

**Corollary 9.** If uniform mixing occurs on $H(r, p)/M$ at time $t$, then the Hamming graph $H(r - s, p)$ has uniform mixing at the same time $t$.

**Proof.** Since the Hamming graph is the Cayley graph over $G = \mathbb{Z}_p^{r-s}$ with connection set $S$ consisting of the orbits of the standard basis $\{e_1, \cdots, e_{r-s}\}$. For every $x = (x_1, \cdots, x_{r-s}) \in G$, the corresponding eigenvalue is

$$\lambda_x = \sum_{i=1}^{r-s} \sum_{l=1}^{p-1} e_p^{x_i} e_l = (p - 1)(r - s) - pwt(x).$$
Thus, by Corollary \[1\] \( H(r, p) \) has uniform mixing at time \( t \) if and only if
\[
R_h(t) = \sum_{x \in G} \exp(it(\lambda_x h - \lambda_h)) = \sum_{x \in G} \exp(-itp(wt(x) - wt(x))) = 0. \tag{28}
\]
Since \( M^t \cong G \), and \( (27) \) is identical with \( (28) \), we complete the proof. \[\square\]

Eventually, we get the following result.

**Proposition 3.** Let \( p \) be a prime number. If an integral Cayley graph \( \text{Cay}(\mathbb{Z}_p, S) \) has uniform mixing at some time \( t \), then \( p = 2, 3 \).

4.2.2. Uniform mixing on integral Cayley graphs over \( \mathbb{Z}_{p^r} \)

For cyclic \( p \)-groups, we have the following result.

**Proposition 4.** Let \( p \) be a prime number and \( r \) be a positive integer. Let \( \Gamma = \text{Cay}(\mathbb{Z}_{p^r}, S) \) be an integral Cayley graph. Then \( \Gamma \) admits uniform mixing at some time \( t \) if and only if \( \Gamma = K_2 \) or \( K_3 \), or \( p = 2 \) and \( r = 2 \).

**Proof.** The group \( \mathbb{Z}_{p^r} = \{\ell : 1 \leq \ell \leq p^r - 1, \gcd(\ell, p) = 1\} \) acts on \( \mathbb{Z}_{p^r} \) by the natural way. The orbits of this action are:
\[
[0], [1], [p], \ldots, [p^{r-1}],
\]
where \( [p^j] = \{j \leq p^j - 1, \gcd(j, p) = 1\} \). By Lemma \[6\], \( S \) should be a union of some orbits.

When \( r = 1 \), then \( \Gamma \) is the complete graph \( K_p \), since \( \Gamma \) is integral, \( S = [1] \). It is known that for a prime \( p \), \( K_p \) has uniform mixing at some time \( t \) if and only if \( p = 2, 3 \). When \( r = 2 \) and \( p = 2 \), we can take \( S = [1] \), then \( \Gamma \) is in fact the cycle \( C_4 \) and has uniform mixing at time \( t = \pi \). This proves the sufficiency.

For the necessity, we give an iterative proof of the desired result. Before going to do that, we recall a result on Ramanujan functions.

For any pair of integers \( k, n \), the Ramanujan function \( c(k, n) \) is defined by
\[
c(k, n) = \sum_{1 \leq n, \gcd(n, k) = 1} \exp(2\pi kx/n). \tag{29}
\]
For the computation of \( c(k, n) \), one has the following formula, see \[41\], Corollary 2.4,
\[
c(k, n) = \frac{\phi(n)\mu(m)}{\phi(m)}, \quad \text{where} \quad m = \frac{n}{\gcd(k, n)}. \tag{30}
\]
\( \phi \) is the Euler-phi function and \( \mu \) is the Mobius function.

Since \( \Gamma \) is integral and connected, the connection set \( S \) has the following form:
\[
S = [1] \cup \left( \bigcup_{\ell \in L} [p^\ell] \right),
\]
where \( L \) is a proper subset of \( \{j : 1 \leq j \leq r - 1\} \). Note that if \( [1] \) is not in \( S \), then the proof is almost the same as follows. So we omit it here. Thus, for every \( x \in G \), we have the corresponding eigenvalue
\[
\lambda_x = \sum_{\ell \in \mathbb{Z}_{p^r}} \xi_{p^\ell}^x + \sum_{d \in L} \sum_{\ell \in \mathbb{Z}_{p^\ell}} \zeta_{p^\ell}^x.
\]

17
If \( x \neq 0 \), then we can write \( x = p^{y(x)}x' \), where \( 1 \leq x' \leq p^{\ell - y(x)} - 1 \) and \( \gcd(x', p) = 1 \). Then \( x + p^{-1} = p^{y(x)}(x' + p^{\ell - y(x)}) \). If \( v_p(x) \leq r - 1 \), then \( \gcd(x' + p^{\ell - y(x)}, p) = 1 \) and there exists an \( \ell' \in \mathbb{Z}_p \) such that \( x + p^{-1} = \ell'x \). When \( v_p(x) = r - 1 \), then \( 1 \leq x' < p - 1 \). The relation \( x + p^{-1} = \ell'x \) still holds for some \( \ell' \in \mathbb{Z}_p^* \). Thus when \( x \neq 0 \), \( p' - p^{\ell - 1} \), there always exists \( \ell' \in \mathbb{Z}_p^* \) such that \( x + p^{-1} = \ell'x \). It follows that

\[
\lambda_{x+p^{-1}} = \sum_{\ell \in \mathbb{Z}_p^*} \lambda_{\ell x} + \sum_{d|\ell, \ell' \in \mathbb{Z}_p^*} \lambda_{\ell' x}.
\]

We then have

\[
\{ \lambda_{x+1} - \lambda_x : 0 \leq x \leq p' - 1 \} = \{(p'x(1), (-p'x(1), 0(p' - 2)) \}.
\]

Thus when \( r > 1 \), \( R_k(t) \neq 0 \).

In summary, when \( r \geq 1 \), \( \Gamma \) cannot have uniform mixing at a time \( t \) except for the case of \( r = 2 \) and \( p = 2 \).

4.2.3. Uniform mixing on linear abelian Cayley graphs over \( \mathbb{Z}_n^r \)

Recall that a linear abelian Cayley graph is a Cayley graph \( \text{Cay}(\mathbb{Z}_n^r, S) \) such that \( S \cup \{0\} \) is closed under the multiplication of the elements in \( \mathbb{Z}_n \). Moreover, a linear code \( M \) over \( \mathbb{Z}_n \) is a subgroup of \( \mathbb{Z}_n^r \) which is closed under the multiplication of the elements of \( \mathbb{Z}_n \). In fact it is a group code over \( \mathbb{Z}_n \), it can also be viewed as a module over \( \mathbb{Z}_n \). For \( x = (x_1, \ldots, x_r), y = (y_1, \ldots, y_r) \in \mathbb{Z}_n^r \), the Hamming distance of \( x, y \) is defined as \( d(x, y) = |\{i : 1 \leq i \leq r, x_i \neq y_i\}| \). And the weight of a vector \( x \) is the distance of \( x \) and the zero vector, i.e, \( d(x, 0) \). The inner product of \( x, y \) is defined by \( x \cdot y = \sum_{i=1}^r x_i y_i \). The dual of \( M \) is \( M^\perp = \{x \in \mathbb{Z}_n^r : x \cdot y = 0, \forall y \in M\} \). Now let \( A \) be a family of elements of \( \mathbb{Z}_n^r \), that is, a collection in which each element may appear any finite number of times. We still let \( N(A, t) \) denote the number of appearances in \( A \) of elements of weight \( t \), including repetitions. The next result is a generalization of [54, Theorem], see also Lemma 8.

**Lemma 9.** Let \( A \) be a family of elements of \( \mathbb{Z}_n^r \). Then

\[
\sum_{i=0}^r N(A, i)(x-y)^i(x+(n-1)y)^{r-i} = \sum_{v \in \mathbb{Z}_n} y^{w(v)} \sum_{u \in A} \chi_{w}^u(v) \sum_{v \in \mathbb{Z}_n} \chi_{w}^u(v)
\]

where \( \chi_{w}^u = \zeta_n^w \), and \( \zeta_n \) is a primitive \( n \)-th root of unity in \( \mathbb{C} \).

Specifically, if \( A = a + M \) is a coset of a submodule \( M \) of rank \( s \), then

\[
\sum_{i=0}^r N(A, i)(x-y)^i(x+(n-1)y)^{r-i} = n^s \sum_{v \in M} y^{w(v)} \chi_{w}^a(v).
\]

**Proof.** One can prove this lemma by a slight modification of [54, Theorem], we omit the details.
Moreover, Lemma 4 works for the case of $G = \mathbb{Z}_n^r$, see [14, Lemma 7.4.2]. Repeating the procedure of Section 5.2.1, we obtain the following main result.

**Theorem 4.** Let $n, r$ be two positive integers. If a linear Cayley graph $\text{Cay}(\mathbb{Z}_n^r, S)$ has uniform mixing at some time $t$, then $n = 2, 3, 4$.

**Remark 4.** The above Theorem 4 has the restriction that the graph is a linear abelian Cayley graph, we conjecture that the result holds true for all integral abelian Cayley graphs.

### 4.3. General integral abelian Cayley graphs

For any abelian group $G$, there are the following possibilities:

1. $G = \mathbb{Z}_2^n, \mathbb{Z}_3^d, \mathbb{Z}_4^d$ and direct product of these groups;
2. $\exp(G) \geq 5$, where $\exp(G)$ is the exponent of $G$, namely, the greatest order of the elements in $G$.

Firstly, if $G = \mathbb{Z}_2^n$, or $\mathbb{Z}_3^d$, or $\mathbb{Z}_4^d$, it is known that there are some suitable subset $S$ such that $\text{Cay}(G, S)$ has uniform mixing. Now we consider the case that $G$ is a direct product of these groups.

**Proposition 5.** Suppose that $G = \mathbb{Z}_2 \times \mathbb{Z}_4^d$, where $d \geq 1$. Then $\Gamma = \text{Cay}(G, S)$ exhibits uniform mixing at time $t = \frac{\pi}{4}$ for some subset $S$ in $G$.

**Proof.** By the Cartesian product construction, it is sufficient to prove that $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ admits uniform mixing at time $t = \frac{\pi}{4}$. To this end, we take $S = \{(10), (11), (13), (02)\} \subset G$. Then for every $(x, y) \in G$, the corresponding eigenvalue is

$$\lambda_{(x,y)} = (-1)^x + (-1)^y + (-1)^x(a^x + b^y).$$

Thus

$$\lambda_{(00)} = 4, \lambda_{(01)} = \lambda_{(02)} = \lambda_{(12)} = 0, \lambda_{(10)} = 2, \lambda_{(11)} = \lambda_{(13)} = -2.$$  

Thus, by (11), we have

$$H_{uv}\left(\frac{\pi}{4}\right) = \frac{1}{8} \sum_{(x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_4} \exp\left(\frac{\pi i}{4}\lambda_{(x,y)}\right)(-1)^u r^{xv} (\text{where } u - v = (a_1, a_2))$$

$$= \frac{1}{8} \left[ (r^2 + 1)((-1)^n r + (-1)^{n+1}) + (r^{2v} - 1)(1 - (-1)^{n+1}) \right]$$

$$= \pm \frac{1 \pm t}{4}.$$

Thus $|H_{uv}\left(\frac{\pi}{4}\right)| = 1/(2 \sqrt{2})$ for every $u, v \in G$ as required.

Using Cartesian product, we know that $G = \mathbb{Z}_2^r \times \mathbb{Z}_4^{2d}$ admits uniform mixing for some subset $S$, where $r \geq 2, d \geq 1$.

However, for $G = \mathbb{Z}_p \times H$, where $p$ is a prime (even or odd), $s \geq 3$ and $|H| \geq 1$, we have the following result.

**Theorem 5.** Suppose that $G = \mathbb{Z}_p \times H$, where $p = 2, s \geq 3$ or $p \geq 3, s \geq 2$, and $H$ is an abelian group of order $m$ with $\gcd(p, m) = 1$. Assume that $S$ is a symmetric subset of $G$ such that $\Gamma = \text{Cay}(G, S)$ is an integral graph. Then $\Gamma$ does not exhibit uniform mixing at any time $t$. 

19
Proof. Define the following set:

\[ L := \{(\ell_1, \ell_2) : 1 \leq \ell_1 \leq p^s, 1 \leq \ell_2 \leq \exp(H), \gcd(\ell_1, p) = 1, \gcd(\ell_2, \exp(H)) = 1 \}, \]

where \( \exp(H) \) is the largest order of the elements in \( H \). \( L \) forms a group with the operation defined by \((\ell_1, \ell_2)(\ell'_1, \ell'_2) = (\ell_1\ell'_1, \ell_2\ell'_2) \). \( L \) acts on \( G \) by setting \((\ell_1, \ell_2)(x, y) = (\ell_1x, \ell_2y) \) for \((x, y) \in G \). Since \( \exp(G) = p^s \exp(H) \), the eigenvalues of \( \Gamma \) belong to the cyclotomic field \( \mathbb{Q}(\omega_p) \), where \( e = \exp(G) \). It is easy to see that \( \text{Gal}(\mathbb{Q}(\omega_p)/\mathbb{Q}) \cong L \). Thus \( \Gamma \) is integral if and only if for every \((s_1, s_2) \in S \), we have \((\ell_1s_1, \ell_2s_2) \in S \), \( \forall (\ell_1, \ell_2) \in L \). In other words, \( \Gamma \) is integral if and only if \( S \) is a union of some orbits under the action of \( L \). Taking \( S \) as a set of the representatives of the orbits in \( S \), we obtain that the eigenvalues of \( \Gamma \) reads

\[ \lambda_{(x,y)} = \sum_{(\ell_1, \ell_2) \in L} \sum_{(s_1, s_2) \in S} \varepsilon_{p^s}^{\ell_1 s_1} \chi_\ell(\ell_2 s_2), (x, y) \in G. \]

Letting \( h = (p^{r-1}, 0) \in G \), we get

\[ \lambda_{(x+p^{r-1}y, y)} - \lambda_{(x,y)} = \sum_{(s_1, s_2) \in S} \sum_{\ell_2} \chi_\ell(\ell_2 s_2) \sum_{1 \leq \ell_1 \leq p^r \gcd(\ell_1, p) = 1} \left( \varepsilon_{p^s}^{\ell_1 (x+p^{r-1}) s_1} - \varepsilon_{p^s}^{\ell_1 (x s_1)} \right). \]

As in the proof of Proposition 4 one can show that when \((x, y) \) runs through \( \mathbb{Z}_{p^r} \times H \), there are \((p^{r-2})m \) many \((x, y) \)'s such that \( \lambda_{(x+p^{r-1}, y)} = \lambda_{(x,y)} \). Therefore, for any \( t \),

\[ R_0(t) = (p^{r-2})m + 2m \text{ terms each of which has norm } 1 \geq p^s m - 4m > 0. \]

This completes the proof. \( \square \)

Moreover, by analysing the orbits structure of the group \( \mathbb{Z}_{pq} \times H \) under the action of \( \mathbb{Z}_{pq} \times \mathbb{Z}_{\exp(H)}^* \), where \( p, q \) are distinct prime numbers and \( \mathbb{Z}_{n}^* \) for every positive integer \( n \), one can prove the following result.

**Theorem 6.** Assume that \( G = \mathbb{Z}_{pq} \times H \), where \( p, q \) are distinct prime numbers coprime with \( |H| \), and \( \Gamma = \text{Cay}(G, S) \) is an integral graph. Then \( \Gamma \) cannot have uniform mixing at any time \( t \).

**Remark 5.** (1) From the discussion above, it seems that for integer abelian Cayley graphs, uniform mixing appears only in those graphs whose underlying group is a \( p \)-group for \( p = 2, 3 \).

(2) It is actually conjectured in [34] that if an abelian Cayley graph has uniform mixing, then the graph should be integral. With the results we have obtained, we would like to conjecture that if \( G \) is an abelian group, then there is a suitable connection set \( S \subset G \) such that \( \text{Cay}(G, S) \) admits uniform mixing if and only if the underlying group \( G \) is \( \mathbb{Z}_2^d, \mathbb{Z}_3^d, \mathbb{Z}_4^d \) or \( \mathbb{Z}_2^r \times \mathbb{Z}_4^d \), where \( r \geq 2, d \geq 0 \).

5. Cubelike graphs

Let \( G \) be the additive group of \( \mathbb{F}_2^n \), here \( \mathbb{F}_2 = \mathbb{Z}_2 \) which is the finite field with two elements. For any subset \( S \) of \( \mathbb{F}_2^n \), \( \Gamma = \text{Cay}(G, S) \) is an integral graph and the order of any non-zero element in \( G \) is two. We note that Chan [2] found some examples of such graphs that admit uniform mixing. It is proved that for \( k \geq 2 \), there exist graphs in the Bose-Mesner algebra of the Hamming graph \( H(2^k+2, 2) \) that have uniform mixing at time \( \pi/2^k \). Best et al. [12] gave a
characterization of certain quotients of a Hamming graph that has uniform mixing. Mullin [44], Godsil et al. [3,4] also utilized graph quotients to find graphs that have uniform mixing.

In this section, we use Boolean functions to get cubelike graphs having uniform mixing. We show that when \( n = 2m \) is even, \( \Gamma = \text{Cay}(G,S) \) has uniform mixing at time \( t = \pi/2^m \) if \( f \) is a bent function, where \( f \) is the characteristic function of \( S \), that is, every bent function leads to a cubelike graph having uniform mixing. Since there are many constructions of bent functions (many infinite families), one can obtain some infinite families of cubelike graphs admitting uniform mixing. When \( n = 2m + 1 \) is odd, there is no bent function on \( \mathbb{F}_q^n \). In this case, we can use some special Boolean functions to get some cubelike graphs having uniform mixing also. An existence result of such graphs that admit uniform mixing is also presented.

Firstly, we recall some basic facts about finite fields.

The character group of \( G \) is
\[
\hat{G} = \mathbb{F}_2^n = \{ \chi : z \in \mathbb{F}_2^n \},
\]
where for \( g = (g_1, \ldots, g_n), z = (z_1, \ldots, z_n) \in \mathbb{F}_2^n \),
\[
\chi_z(g) = (-1)^{z \cdot g}, \quad z \cdot g = \sum_{j=1}^n z_j g_j \in \mathbb{F}_2.
\]

If we view \( \mathbb{F}_2^n \) as the additive group of the finite field \( \mathbb{F}_q \) with \( q = 2^n \), then
\[
\hat{G} = (\mathbb{F}_q, +) = \{ \chi_z : z \in \mathbb{F}_q \},
\]
where for \( g, z \in \mathbb{F}_q, \chi_z(g) = (-1)^{\text{Tr}(g z)}, \text{ and } \text{Tr} : \mathbb{F}_q \to \mathbb{F}_2 \) is the trace mapping.

Let \( f \) be the characteristic function of \( S \), namely,
\[
f(x) = \begin{cases} 
1, & \text{if } x \in S, \\
0, & \text{otherwise.}
\end{cases}
\]

In this case, \( S \) is also called the supporting set (or simply the support) of \( f \). For every \( g \neq 0 \in G \), the corresponding eigenvalue of \( \Gamma \) is
\[
\lambda_g = \sum_{x \in S} \chi_x(g) = \frac{1}{|G|} \sum_{x \in G} \frac{1 - (-1)^{f(x)}}{2} (-1)^{\text{Tr}(g x)} = -\frac{1}{2} \sum_{x \in G} (-1)^{f(x) + \text{Tr}(g x)} = -\frac{1}{2} \hat{F}(g),
\]
where \( F(x) = (-1)^{f(x)} \) and \( \hat{F} \) is the Fourier transform of \( F \). Note that \( \lambda_0 = |S| = d \). The sum \( \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + \text{Tr}(x)} \) is usually denoted by \( W_f(g) \) and termed as the Walsh-Hadamard Transformation of \( f \) at \( g \).

Recall the number \( M_b \) defined in (7) and \( D_G \) defined in (8). By Lemma 1 we have
\[
D_G = \gcd(M_h : 0 \neq h \in G) = 2^\nu.
\]

Theorem 7. Let \( G \) be an abelian 2-group and \( \Gamma = \text{Cay}(G,S) \) be an integral simple Cayley graph. Let \( \nu \) be defined as in (32). For every \( 0 \neq h \in G \), define the following sets:
\[
N_2(h) = \{ g \in G : v_2(A_{g+h} - A_g) > \nu + 1 \},
N_1(h) = \{ g \in G : v_2(A_{g+h} - A_g) = \nu + 1 \},
N_0(h) = \{ g \in G : v_2(A_{g+h} - A_g) = \nu \}.
\]

for some nonnegative integer \( \nu \).
and denote $n_i(h) = |N_i(h)|$, $i = 0, 1, 2$. Then $\Gamma$ has uniform mixing at time $t = \frac{\pi}{2^n}$ if and only if for every $0 \neq h \in \mathbb{Z}^n$, $n_2(h) = n_1(h)$.

**Proof.** It is obvious that, for every $g \in G$,

$$g \in N_3(h) \iff \frac{\lambda_{g+h} - \lambda_g}{2^{v+1}} \in \mathbb{Z},$$

$$g \in N_1(h) \iff \frac{\lambda_{g+h} - \lambda_g}{2^{v+1}} \text{ is an odd number},$$

$$g \in N_0(h) \iff \frac{\lambda_{g+h} - \lambda_g}{2^v} \text{ is an odd number}.$$  

Moreover, when $g \in N_0(h)$, the numbers $\frac{\lambda_{g+h} - \lambda_g}{2^v}$ appear in pairs with different signs. Thus

$$R_h(\frac{\pi}{2^{v+1}}) = \sum_{g \in G} \exp\left(i\frac{\pi}{2^{v+1}}(\lambda_{g+h} - \lambda_g)\right)$$

$$= \sum_{g \in \mathcal{N}_2(h)} \exp\left(i2\pi\frac{\lambda_{g+h} - \lambda_g}{2^{v+2}}\right) + \sum_{g \in \mathcal{N}_1(h)} \exp\left(i\pi\frac{\lambda_{g+h} - \lambda_g}{2^{v+1}}\right) + \sum_{g \in \mathcal{N}_0(h)} \exp\left(i\frac{\pi}{2}\frac{\lambda_{g+h} - \lambda_g}{2^v}\right)$$

$$= n_2(h) - n_1(h).$$

The desired result now follows with Corollary 1.

5.1. $n$ is even

Assume that $n = 2k$ and $v_2(d) \leq k - 1$. If $f(x)$ is a bent function, namely,

$$|\hat{F}(g)| = 2^k \text{ for every } g \in G.$$  

If $F(x) = (-1)^{f(x)}$ is bent, we also say $f$ is bent for convenience. It is well-known that bent function exists on $\mathbb{Z}_2^k$ for every $k$. See for example, [40], Chapter 4, pp.69-88].

In this subsection, we will show that the Cayley graph $\text{Cay}(\mathbb{Z}_2^k; S)$ has uniform mixing if $f$ is bent, where $f$ is the characteristic function of $S$.

When $f$ is bent, we can write

$$\tilde{F}(g) = 2^{k(-1)^{f(g)}}.$$  

The function $\tilde{f}$ is called the dual of $f$. It is well-known that $\tilde{f}$ is bent if and only if $f$ is bent (see for example, [40], line 12, page 72). Thus we have

$$\lambda_g = -2^{k(-1)^{f(g)}}, g \neq 0.$$  

Then

$$d - \lambda_g = d + 2^{k-1}(-1)^{f(g)} = 2^{v_2(d)}d' + 2^{k-v_2(d)-1}(-1)^{f(g)}.$$  

where $d = 2^{v_2(d)}d', d'$ is odd. Since $f$ is a bent function, we know that $d = 2^{v_2(d)}d' + 2^{k-1}$ (see [40], Theorem 4.3.1). Without loss of generality, we assume that $d = 2^{v_2(d)} = 2^{k-1}$, thus $v_2(d) = k - 1$ and $M = 2^k$, i.e., $\ell = k$. Taking $t = \frac{\pi}{2^k}$, then we have for every $h \neq 0$, the correlation function
\[ R_h(t) = \sum_{g \in G} \exp(t(A_g h - A_g)t) \]
\[ = \exp(t(A_h - d)t) + \exp(t(d - A_h)t) + \sum_{g \neq 0, h \neq g} \exp(t(A_g h - A_g)t) \]
\[ = \sum_{g \in G} \exp \left( \frac{t}{2}(-1)^{g(h) - (-1)^{g(0)}} \right). \]

Since
\[ \tilde{F}(0) = \sum_{x \in G} (-1)^{f(x)} = 2^n - 2|S| = 2^k, \]
we know that \( \tilde{f}(0) = 0 \). Noticing that \( (-1)^f = 1 - 2f \) for any boolean function \( f \), we get
\[ R_h(t) = \sum_{g \in G} \exp(-i\pi(\tilde{f}(g + h) - \tilde{f}(g))) = \sum_{g \in G} (-1)^{\tilde{f}(g) - \tilde{f}(0)}. \] (34)

Thus \( R_h(t) = 0 \) for all \( h \neq 0 \) if \( \tilde{f} \) is bent (see, for example [40], page 75). It is known that \( \tilde{f} \) is bent if and only if \( f \) is bent. Therefore, we have the following result:

**Theorem 8.** Let \( G = (\mathbb{P}^d_2, +) \) be an elementary 2-group and \( S \) be a subset of \( G \). Let \( f \) be the characteristic function of \( S \). Then \( \Gamma = \text{Cay}(G, S) \) has uniform mixing at time \( t = \frac{\pi}{2} \) if \( f \) is a bent function on \( G \).

Below we present an example.

**Example 1.** Let \( G = (\mathbb{P}^d_2, +) \) and
\[ f(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4. \]
Then the support of \( f \) is
\[ S = \{(1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 0, 1), (0, 0, 1, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}, \]
and \( \Gamma = \text{Cay}(G, S) \) has uniform mixing at time \( t = \frac{\pi}{2} \).

**Proof.** Let \( F(X) = (-1)^{f(X)}X = (x_1, x_2, x_3, x_4) \in \mathbb{P}^d_2 \). For every \( Y = (y_1, y_2, y_3, y_4) \in \mathbb{P}^d_2 \), we have
\[ \tilde{F}(Y) = \sum_{x_1, x_2, x_3, x_4 \in \mathbb{P}^d_2} (-1)^{x_1^1x_2^2x_3^3x_4^4 + x_1^1x_2^2x_3^2x_4^3} \sum_{x_4 \in \mathbb{P}^d_2} (-1)^{x_4} \]
\[ = 2(-1)^{y_3} \sum_{x_1, x_2, x_3 \in \mathbb{P}^d_2} (-1)^{y_1} \sum_{x_2 \in \mathbb{P}^d_2} (-1)^{x_1x_2} \]
\[ = 4(-1)^{y_3+y_2+y_1+y_0}. \]
Thus \( f \) is a self-dual bent function. For every \( Y = (y_1, y_2, y_3, y_4) \in \mathbb{P}^d_2 \), the corresponding eigenvalue of \( \Gamma \) is (see [5])
\[ \lambda_Y = (-1)^{y_1+y_2} + (-1)^{y_1+y_2+y_3} + (-1)^{y_1+y_2+y_4} + (-1)^{y_1+y_3} + (-1)^{y_1+y_4} + (-1)^{y_2+y_3} + (-1)^{y_2+y_4}. \]
Since both the functions $F(y_1, y_2, y_3, y_4) = (-1)^{y_1y_2 + y_3y_4}$ and $A(y_1, y_2, y_3, y_4)$ are invariant under the action of the Klein group $K = \{(1), (12)(34), (14)(23), (13)(24)\}$, one can check that

$$A_Y = A(y_1, y_2, y_3, y_4) = -2(-1)^{y_1y_2 + y_3y_4}, \quad Y = (y_1, y_2, y_3, y_4) \neq (0, 0, 0, 0).$$

One can also use (33) to get the above equation.

Thus by (13), the $(u, v)$-th entry of $H(t)$ at time $\pi/4$ is

$$H(t)_{uv} = \frac{1}{16} \sum_{g \in G} \exp(i \frac{\pi}{4} A_g) \chi_g(a) \quad (a = u - v)$$

$$= \frac{1}{16} \left[ \exp(i6 \cdot \frac{\pi}{4}) + \sum_{g \in G} \exp(i \frac{\pi}{4}(-2(-1)^{g_1g_2g_3g_4})(-1)^{g_k}) \right]$$

$$= \frac{1}{16} \left[ -1 - i \sum_{g \in G} (-1)^{g_1g_2g_3g_4}(-1)^{g_k} \right] \quad \text{(by } \exp(x) = \cos x + i \sin x)$$

$$= -\frac{i}{16} \tilde{F}(a)$$

$$= -\frac{1}{4}(-1)^{\tilde{f}(a)}.$$

Therefore, we get

$$|H(t)_{uv}| = \frac{1}{4}$$

as required. \hfill \Box

**Remark 6.** Godsil et al. [33] have a full characterization of strongly regular graphs that admit uniform mixing. It is obvious that if $f$ is bent on $\mathbb{F}_2^n$, then $\text{Cay}(\mathbb{F}_2^2, S)$ is a strongly regular graph, where $S = \text{supp}(f)$. It must be noted, however, that not all strong regular graphs exhibit uniform mixing. Our result (Theorem 5) shows that every bent function leads to a cubelike graph having uniform mixing.

5.2. $n$ is odd

When $n = 2m + 1$ is odd, it is well-known that there is no bent functions on $\mathbb{F}_2^n$. In this case, we choose a bent function $f$ on $\mathbb{F}_2^{2m}$ whose supporting set is $(0, x)S_1 \subseteq \mathbb{F}_2^{2m}$. We assume that, without loss of generality, $|S_1| = 2^{2m-1} - 2^{m-1}$. Define a subset $S \subseteq \mathbb{F}_2^n$ by

$$S = \{(1, z) : z \in S_1 \} \cup \{(0, x) : x \in S_1\}, \quad (35)$$

where $\overline{S_1} = \mathbb{F}_2^{2m} \setminus S_1$ is the complement of $S_1$. Let $\Gamma = \text{Cay}(\mathbb{F}_2^n, S)$ be the Cayley graph with connection set $S$. Then we have the following result.

**Theorem 9.** Assume that $n = 2m + 1$ and $f$ is a bent function on $\mathbb{F}_2^{2m}$ whose supporting set is $S_1$. Let $S$ be defined as in (35). Then $\Gamma = \text{Cay}(\mathbb{F}_2^n, S)$ has uniform mixing at time $t = \frac{\pi}{2}$ if and only if $m = 1$.

**Proof.** It is obvious that $|S| = d = 2^{2m}$. For every $g = (g_1, g_2) \in \mathbb{F}_2^n$, where $g_1 \in \mathbb{F}_2, g_2 \in \mathbb{F}_2^{2m}$, the corresponding eigenvalue is as follows:
Thus for every $h = (h_1, h_2) \neq 0$, we have

$$
\lambda_{g+h} = \begin{cases} 
2^m, & \text{if } g_1 = h_1, g_2 = h_2, \\
0, & \text{if } g_1 = h_1, g_2 \neq h_2, \\
-2^m(-1)^{\tilde{g}(h_1, h_2)}, & \text{if } g_1 \neq h_1, g_2 \neq h_2, \\
-2^m, & \text{if } g_1 \neq h_1, g_2 = h_2.
\end{cases}
$$

From (36), one can see that if $t = \frac{\pi}{2^n}$, $\ell \leq m - 1$, then $R_i(h) = 2^\ell$ for all $h \in G$. If $t = \frac{\pi}{2^n}$, then $R_i(h) = \pm 2^2m$ for all $h \neq 0$. For $t = \frac{\pi}{2^n}$, we have

$$
\sum_{g \in G} \exp \left( t(\lambda_{g+h} - \lambda_g) \right) 
= \delta_m \exp\left(-tL_{(h_1, h_2)}f \right) + \sum_{g \neq h} \exp\left(-tL_{(h_1, h_2)}f \right) 
- t \sum_{g \neq h} (-1)^{\tilde{g}(g, h_2)} \exp\left(-tL_{(1, h_1, h_2)}f \right) - t \exp\left(-tL_{(1, h_1, h_2)}f \right) 
= (\delta_m - 1) \exp\left(-tL_{(h_1, h_2)}f \right) + \sum_{g \neq h} \exp\left(-tL_{(h_1, h_2)}f \right) - t \sum_{g \neq h} (-1)^{\tilde{g}(g_2, h_2)} \exp\left(-tL_{(1, h_1, h_2)}f \right),
$$

where $\delta_m = -1$ if $m = 1$, and 1 if $m > 1$. Now, if $m > 1$, then

$$
\sum_{g \in G^n} \exp(-tL_{(0, 0, 0)}f) = \exp(-tL_{(0, 0, 0)}f) + \sum_{g \neq 0} \exp(tL_{(0, 0, 0)}f) 
= \exp(-2^m t) + \sum_{g \neq 0} \exp(0) 
= 1 + (2^{2m} - 1) 
= 2^{2m},
$$
and when $h_2 \neq 0$,

$$\sum_{g \in \mathbb{F}_2^m} (-1)^{\tilde{h}(g) + h_2} \exp(-it\lambda_{1,1}(g)t)$$

$$= (-1)^{\tilde{h}(h_2)} \exp(-it\lambda_{1,1}(0)t) + \sum_{g \neq 0} (-1)^{\tilde{h}(g) + h_2} \exp(-it\lambda_{1,1}(g)t)$$

$$= (-1)^{\tilde{h}(h_2)} \exp(-it(-2^m)t) + \sum_{g \neq 0} (-1)^{\tilde{h}(g) + h_2} \exp(-it(-2^m(-1)^{\tilde{h}(g)}t))$$

$$= t(-1)^{\tilde{h}(h_2)} + t \sum_{g \neq 0} (-1)^{\tilde{h}(g) + h_2} - \tilde{f}(g)$$

$$= t \sum_{g \neq 0} (-1)^{\tilde{h}(g) + h_2} - \tilde{f}(g)$$

$$= 0 \text{ (since } f \text{ is bent).}$$

Thus when $m > 1$ and $h = (0, h_2)$, $h_2 \neq 0$, we have $R_t(h) = 2^{2m}$, and $\Gamma$ cannot have uniform mixing at time $t = \frac{\pi}{4}$.

In fact, we can use Theorem 7 to give a simple proof of the above statement as follows:

Since it is obvious that $\nu = m$ and if we take $h = (0, h_2)$, $h_2 \neq 0$, then from (36), one can check that $n_2(h) \neq n_1(h)$. Thus $\Gamma$ cannot have uniform mixing at that time.

When $m = 1$, we take $f(x_1, x_2) = x_1x_2$, then $f$ is bent, its supporting set is $S_f = \{(11)\}$. A slight modification of the precede procedure, one can show that $R_{4\pi}(h) = 0$ for all $h \neq 0$, thus $\Gamma$ has uniform mixing at time $t = \frac{\pi}{4}$. Below, we provide a direct proof of this fact. Indeed, from the above computation, we have that the eigenvalues of $\Gamma$ are

$$\lambda_{(000)} = 4, \lambda_{(001)} = \lambda_{(010)} = \lambda_{(011)} = 0, \lambda_{(100)} = \lambda_{(101)} = \lambda_{(110)} = -2, \lambda_{(111)} = 2.$$  

The difference table of the eigenvalues are presented in the following Table 2.

| $h' \Gamma$ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 001        | -4  | 4   | 0   | 0   | 0   | 0   | 4   | -4  |
| 010        | -4  | 0   | 4   | 0   | 0   | 4   | 0   | -4  |
| 011        | -4  | 0   | 0   | 4   | 4   | 4   | 0   | -4  |
| 100        | -6  | -2  | -2  | 2   | 6   | 2   | 2   | -2  |
| 101        | -6  | -2  | 2   | -2  | 2   | 6   | 2   | -2  |
| 110        | -6  | 2   | -2  | -2  | 2   | 2   | 6   | -2  |
| 111        | -2  | -2  | -2  | -2  | 2   | 2   | 2   | 2   |

For each of the first three rows in Table 2, the corresponding correlation is

$$R_0(t) = 4 + 2\exp(it\frac{\pi}{4}) + 2\exp(-it\frac{\pi}{4}) = 0.$$

For each of the middle three rows, the corresponding correlation is

$$R_6(t) = \exp(-it\frac{6\pi}{4}) + \exp(it\frac{6\pi}{4}) + 3\exp(it\frac{2\pi}{4}) + 3\exp(-it\frac{2\pi}{4}) = 0.$$
For the last row,
\[ R_h(t) = 4 \exp(i \times \frac{\pi}{4}) + 4 \exp(-i \times \frac{\pi}{4}) = 0. \]

By Corollary 1, \( \Gamma \) has uniform mixing at \( t = \frac{\pi}{4} \).

Moreover, from Table 2, one can see that \( \Gamma \) cannot have uniform mixing at time less than \( \frac{\pi}{4} \).

We note that the case for \( m = 1 \) also illustrates Theorem 7. Indeed, from Table 2, it is easy to see that \( \gcd(\lambda_g + h - \lambda_g; g \in G, 0 \neq h \in G) = 2 \) and then the corresponding number \( \nu \) in Theorem 7 is \( \nu = 1 \). For the first three rows of Table 2, we have \( n_2(h) = n_1(h) = 4 \), and for the rest four rows, we have \( n_2(h) = n_1(h) = 0 \). Thus the conditions for Theorem 7 are satisfied and then \( \Gamma \) has uniform mixing at time \( t = \frac{\pi}{4} \).

As a consequence of Theorem 5, 9 and Example 1, we can construct more cubelike graphs having uniform mixing by using Cartesian product.

Concluding Remarks

In this paper, we mainly present the following results:

1. A necessary and sufficient condition for an abelian Cayley graph having uniform mixing is provided (see Theorem 1, Corollary 1, Corollary 2).
2. For an integral abelian Cayley graph \( \Gamma \) having uniform mixing at time \( t \), we show that \( \exp(it) \) is a root of unity under certain constrictions (see Theorem 2). We hope that some of the constrains can be removed.
3. For an abelian integral Cayley \( p \)-graph, we show that it has uniform mixing at a time of the form \( 2\pi r/p^r \) if and only if its eigenvalues induce a certain difference-balanced mapping (see Theorem 3).
4. It is shown that a linear Cayley graph \( \text{Cay}(\mathbb{Z}_n^r, S) \) admits uniform mixing only when \( n = 2, 3, \) or 4. See Theorem 4.
5. If \( G \) is an elementary \( p \)-group, where \( p = 2 \) or 3, or if \( G = \mathbb{Z}_p^r \times \mathbb{Z}_p^d \) for some integers \( r \geq 1, d \geq 1 \), it is known that \( \text{Cay}(G, S) \) has uniform mixing at some time \( t \) for some connection set \( S \). Conversely, for an abelian group \( G \), we conjecture that if \( \text{Cay}(G, S) \) is integral and has uniform mixing at some time \( t \), then \( G \) should be one of the groups mentioned above. If our conjecture is true, then we will get a complete classification of integral abelian Cayley graphs having uniform mixing.
6. For cubelike graphs, we show that \( \Gamma = \text{Cay}(\mathbb{F}_2^m; S) \) has uniform mixing at time \( t = \pi/2^m \) if \( f \) is a bent function, where \( f \) is the characteristic function of \( S \) (see Theorem 5).
7. Some concrete constructions of graphs having uniform mixing are presented (see Proposition 2, 4, Theorem 8, 19, and Theorem 7). We also present some nonexistence results, see Proposition 2, 4, and Theorem 7.

For further research, we list more open questions as follows:

**Open question 1:** Find more properties of the characteristic function of \( S \) such that the Cayley graph \( \text{Cay}(\mathbb{Z}_n^r; S) \) (or \( \text{Cay}(\mathbb{Z}_n^d; S) \)) has uniform mixing at some time \( t \).

**Open question 2:** Find some lower bounds on the time \( t \) at which \( \text{Cay}(\mathbb{Z}_n^r; S) \) (or \( \text{Cay}(\mathbb{Z}_n^d; S) \)) has uniform mixing.

**Open question 3:** If \( \Gamma = \text{Cay}(G, S) \) is an abelian Cayley graph having uniform mixing, prove or disprove that \( \Gamma \) is integral.

**Open question 4:** If \( \Gamma = \text{Cay}(G, S) \) is an abelian Cayley graph having uniform mixing at time \( t \), prove or disprove that \( \exp(it) \) is a root of unity.
Acknowledgements

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