Gravitational Collapse of Massless Scalar Field with Negative Cosmological Constant in (2+1) Dimensions

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The 2 + 1-dimensional geodesic circularly symmetric solutions of Einstein-massless-scalar field equations with negative cosmological constant are found and their local and global properties are studied. It is found that one of them represents gravitational collapse where black holes are always formed.

I. INTRODUCTION

One of the most outstanding problems in gravitation theory is the evolution of a collapsing massive star, after it has exhausted its nuclear fuel. However, in order to obtain models for the collapse, we need to solve complicated systems of nonlinear differential equations. This mathematical complexity is indirectly revealed by both the relatively few analytic solutions available for study and the challenges associated with the construction of a good numerical code to simulate the collapse process. Nonetheless, work over the last decade has revealed surprisingly rich behaviour in even nominally simple (e.g. spherically symmetric) systems [1]. These efforts have led to important advances in our understanding of the process of black hole formation and the presence of “critical” behaviour in these dynamical, gravitating systems. Indeed, this behaviour has been shown to be present in a great variety of systems [2].

The studies of non-linearity of the Einstein field equations near the threshold of black hole formation reveal very rich phenomena [1], which are quite similar to critical phenomena in Statistical Mechanics and Quantum Field Theory [3]. In particular, by numerically studying the gravitational collapse of a massless scalar field in 3 + 1-dimensional spherically symmetric spacetimes, Choptuik found that the mass of such formed black holes takes a scaling form,

\[ M_{BH} = C(p) (p - p^*)^\gamma, \]

where \( C(p) \) is a constant and depends on the initial data, and \( p \) parameterises a family of initial data in such a way that when \( p > p^* \) black holes are formed, and when \( p < p^* \) no black holes are formed. It was shown that, in contrast to \( C(p) \), the exponent \( \gamma \) is universal to all the families of initial data studied. Numerically it was determined as \( \gamma \approx 0.37 \).

The solution with \( p = p^* \), usually called the critical solution, is found also universal. Moreover, for the massless scalar field it is periodic, too. Universality of the critical solution and exponent, as well as the power-law scaling of the black hole mass all have given rise to the name Critical Phenomena in Gravitational Collapse. Choptuik’s studies were soon generalised to other matter fields [2,4], and now the following seems clear: (a) There are two types of critical collapse, depending on whether the black hole mass takes the scaling form (1) or not. When it takes the scaling form, the corresponding collapse is called Type \( \text{II} \) collapse, and when it does not it is called Type \( \text{I} \) collapse. In the type \( \text{II} \) collapse, all the critical solutions found so far have either discrete self-similarity (DSS) or homothetic self-similarity (HSS), depending on the matter fields. In the type \( \text{I} \) collapse, the critical solutions have neither DSS nor HSS. For certain matter fields, these two types of collapse can co-exist. (b) For Type \( \text{II} \) collapse, the corresponding exponent is universal only with respect to certain matter fields. Usually, different matter fields have different critical solutions and, in the sequel, different exponents. But for a given matter field the critical solution and the exponent are universal. So far, the studies have been mainly restricted to spherically symmetric case and their non-spherical linear perturbations. Therefore, it is not really clear whether or not the critical solution and exponent are universal with respect to different symmetries of the spacetimes [5,6]. (c) A critical solution for both of the two types has one and only one unstable mode. This now is considered as one of the main criteria for a solution to be critical. (d) The
universality of the exponent is closely related to the last property. In fact, using dimensional analysis \[7\] one can show that
\[
\gamma = \frac{1}{|k_1|},
\]
where \(k_1\) denotes the unstable mode.

Some more recent work has been taken the consideration of the gravitational collapse of a minimally coupled scalar field in the presence of a cosmological constant but in a lower dimensional spacetime, namely \(2+1\). There are several motivations for studying such a model beyond the intrinsic interest in examining critical behaviour in another system. Among these is the recent flurry of work on anti-de Sitter (AdS) spacetimes stemming from the AdS/CFT conjecture. This conjecture assumes a correspondence between the gravitational physics in an AdS spacetime and the physics of a conformal field theory on the boundary of AdS. Hence, understanding AdS spacetimes can potentially yield insight into Super-Yang-Mills theory (and vice versa). Another motivation for studying \(2+1\) scalar field collapse is partly the relative simplification that results in going from \(3+1\) to \(2+1\) dimensional gravity. By itself, this would not necessarily be that compelling, but there are, of course, some intriguing solutions in \(2+1\) such as the BTZ black hole \[8\] that closely parallel the black hole solutions of \(3+1\) gravity. Earlier work has considered the question of gravitational collapse to a BTZ black hole, but using either null fluid or dust as the collapsing matter \[9,10\].

Lately, Pretorius and Choptuik (PC) \[11\] studied gravitational collapse of a massless scalar field in an anti-de Sitter background in \(2+1\)-dimensional spacetimes with circular symmetry, and found that the collapse exhibits critical phenomena and the mass of such formed black holes takes the scaling form of equation (2) with \(\gamma = 1.2 \pm 0.02\), which is different from that of the corresponding \(3+1\)-dimensional case. In addition, the critical solution is also different, and, instead of having DSS, now has HSS. The above results were confirmed by independent numerical studies of Husain and Olivier \[12\]. However, their exponent, \(\gamma \sim 0.81\), is quite different from the one obtained by PC. It is not clear whether the difference is due to numerical errors or to some unknown physics.

In \(2+1\) dimensional spacetimes, the cosmological constant plays a fundamental role in black holes structure. The BTZ solution itself needs a negative cosmological constant and Ida \[13\] demonstrated that, in fact, black holes in this class of spacetimes are formed only in the presence of negative cosmological constant. To contribute to the above open problems, in this paper we shall present the general geodesic solution of the Einstein-massless scalar field equations in \(2+1\) dimensional circularly symmetric spacetimes with negative cosmological constant, and then study their local and global properties. The study of their linear perturbations will be considered somewhere else.

II. THE FIELD EQUATIONS

The Einstein’s equation for a massless scalar field with cosmological constant can be written as
\[
R_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - 2g_{\mu\nu}\Lambda,
\]
where the comma denotes partial differentiation.

The general metric for circularly symmetric spacetimes is given by
\[
ds^2 = A^2(r,t)dt^2 - B^2(r,t)dr^2 - C^2(r,t)d\theta^2.
\]

Thus, the non-null components of the Ricci tensor are
\[
R_{00} = \frac{A^2}{B^2} \left( \frac{A'}{A} - \frac{A''}{A} \right) + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A} + \dot{B} + \dot{C}}{AB},
\]
(5)
\[
R_{10} = \frac{\dot{C}}{C} - \frac{\dot{B} C'}{B C} - \frac{\dot{C} A'}{C A},
\]
(6)
\[
R_{11} = \frac{C''}{C} + \frac{A''}{A} - \frac{A' B'}{AB} - \frac{B' C'}{BC} - \frac{B^2}{A^2} \left( \frac{\dot{B}}{B} - \frac{\dot{A} + \dot{B} + \dot{C}}{AB} + \frac{\dot{B} C}{BC} \right),
\]
(7)
\[
R_{22} = \frac{C^2}{B^2} \left( \frac{C''}{C} + \frac{A' C'}{A C} - \frac{B' C'}{B C} \right) - \frac{C^2}{A^2} \left( \frac{\dot{C}}{C} - \frac{\dot{A} + \dot{B} + \dot{C}}{AB} + \frac{\dot{B} C}{BC} \right).
\]
(8)

In the following we will study the geodesic solutions. In this case, these are given in a referential where the fluid has null acceleration. This condition implies that
The non-null components of the Ricci tensor for the metric (4) with the condition (9) are

$$R_{00}^0 = \dddot{B} + \dddot{C},$$

$$R_{10}^0 = \dddot{C} - \dddot{B} \frac{C'}{B},$$

$$R_{11}^0 = \dddot{B} + \dddot{C} - \frac{1}{B^2} \left( C'' - \frac{B' C'}{B} \right),$$

$$R_{22}^0 = \dddot{C} + \dddot{B} \frac{C'}{B} - \frac{1}{B^2} \left( C'' - \frac{B' C'}{B} \right).$$

Above, the prime means partial differentiation in relation to the radial coordinate, and the symbol dot means time differentiation.

Considering now equation (3), the explicit form of the field equations is

$$R_{00}^0 + 2\Lambda = -\dot{\phi}^2,$$

$$R_{11}^0 + R_{22}^0 + 4\Lambda = 0,$$

$$R_{10}^0 = 0,$$

$$R_{11}^0 - R_{22}^0 = 0.$$  

From (10) and (14) we have

$$\dot{\phi}^2 = \dddot{B} + \dddot{C} + 2\Lambda.$$  

On the other hand, equations (12)-(13) and (17) furnish

$$\dddot{B} - \dddot{C} = 0.$$  

Equations (12)-(13) and (15) yield

$$-\frac{2}{B^2} \left( \frac{C''}{C} - \frac{B' C'}{B C} \right) + \left( \frac{\dddot{C} + \dddot{B} + 2 \dddot{B} \dddot{C}}{B} \right) + 4\Lambda = 0.$$  

Finally equations (11) and (16) give

$$\frac{C'}{C} \left( \frac{C'}{C'} - \dddot{B} B \right) = 0.$$  

The above equation is satisfied for two different relations for the function $C$

$$C' = 0,$$

$$\frac{C'}{C'} = \frac{B \dddot{C}}{B C}.$$  

Henceforth, we will study them separately.

III. SOLUTIONS FOR THE FIELD EQUATIONS

As we will see below, equation (22) yields only one family of solutions, while equation (23) ramifies into two branches.

Case i) Solutions for $C'=0$
From (22) we readily have

\[ C = C(t). \] (24)

Substituting (22) into (20) we obtain

\[ \frac{\ddot{B}}{B} + \left( \frac{\ddot{C}}{C} + 2 \frac{\dot{B} \dot{C}}{B C} \right) + 4\Lambda = 0, \] (25)

that can be rewritten as

\[ (BC)^\prime + 4\Lambda BC = 0, \] (26)

which furnishes

\[ BC = c_1(r)e^{2\sqrt{-\Lambda}t} + c_2(r)e^{-2\sqrt{-\Lambda}t}. \] (27)

On the other hand, equations (20) and (19) leads to

\[ \ddot{C} C + \dot{B} \dot{B} \dot{C} C = -2\Lambda, \] (28)

From equation (22) we can also readily write

\[ \frac{\dot{C}}{C} = f_1(t), \] (29)

\[ \frac{\ddot{C}}{C} = f_2(t). \] (30)

Setting the above definitions into equation (28) we have

\[ \frac{\ddot{B}}{B} = -2\Lambda + \frac{f_2(t)}{f_1(t)}, \] (31)

where \( f_2(t) = \dot{f}_1(t) + f_1(t)^2 \), whose solution is

\[ B = e^{-2\Lambda} \int \frac{dt}{f_1(t)} - \int f_1(t) \ln f_1(t) + g(r), \] (32)

where \( g(r) \) is an arbitrary function of \( r \). Moreover, the integration of equation (29) furnishes

\[ C = e^{\int f_1(t) dt}. \] (33)

The relation (14) for the scalar field is now

\[ \ddot{\phi}^2 = 2[\dot{f}_1(t) + f_1(t)^2 + \Lambda]. \] (34)

Considering equations (32)-(33) into equation (25) we can show that \( f_1(t) \) must obey

\[ 4\Lambda^2 + 4\Lambda f_1(t)^2 + 6\Lambda \ddot{f}_1(t) - f_1(t) \dddot{f}_1(t) + 2\dot{f}_1(t)^2 = 0. \] (35)

Below we will use it as an integration condition.

From equations (27), (32) and (33) we must have \( c_1(r) = ac_2(r) \), where \( a \) is a constant. Then

\[ \dot{f}_1 + \frac{2\sqrt{-\Lambda} \left( e^{2\sqrt{-\Lambda}t} - ae^{-2\sqrt{-\Lambda}t} \right)}{e^{2\sqrt{-\Lambda}t} + ae^{-2\sqrt{-\Lambda}t}} f_1 + 2\Lambda = 0, \] (36)

which can be solved giving us

\[ f_1(t) = \frac{a\Lambda + b\sqrt{-\Lambda} e^{2\sqrt{-\Lambda}t} - \Lambda e^{4\sqrt{-\Lambda}t} - \Lambda}{\sqrt{-\Lambda} (e^{4\sqrt{-\Lambda}t} + a)}, \] (37)
where $b$ is an integration constant. This solution satisfies readily the condition (35). Then, the resulting spacetime can be written as

$$ds^2 = dt^2 - \left( e^{2\sqrt{-\Lambda}t} + ae^{-2\sqrt{-\Lambda}t} \right)^2 \times \left[ -\Lambda e^{-\frac{b}{2\sqrt{-\Lambda}} \arctan \left( \frac{1}{\sqrt{a}} e^{2\sqrt{-\Lambda}t} \right)} dr^2 + e^{2\sqrt{-\Lambda}t} \right]$$

with $a > 0$. The associated scalar field is given by

$$\phi(t) = \pm \frac{\sqrt{2}}{4} \sqrt{-\frac{16a\Lambda - b^2}{a\Lambda}} \arctan \left( \sqrt{ae^{2\sqrt{-\Lambda}t}} \right) + \phi_0,$$

Thus, we have an imaginary scalar field and besides we can easily show that we do not have singularity formation.

**Case ii) Solutions for $C' \neq 0$**

From equation (23) we have

$$\left( \frac{C'}{B} \right) = 0,$$

that can be readily solved giving

$$C' = Bh(r).$$

Differentiating twice in relation to $t$ we get

$$\frac{\ddot{B}}{B} = \frac{\ddot{C'}}{C'},$$

Taking this last relation into equation (19) we find

$$\frac{\ddot{C}}{C} - \frac{\ddot{C'}}{C'} = 0,$$

which can be rewritten as

$$\frac{\ddot{C}}{C'} \left( \frac{C'}{C} - \frac{\ddot{C'}}{C} \right) = 0.$$  

Thus, we have two possibilities:

**ii.a)** $\frac{\ddot{C}}{C} = 0$  \hspace{1cm} (45)

**ii.b)** $\frac{C'}{C} - \frac{\ddot{C'}}{C} = 0$.  \hspace{1cm} (46)

Below we will analyse each of them separately.

**Case ii.a) Solutions for $C' \neq 0$ and $\ddot{C} = 0$**

From equation (45) we have

$$C = h_1(r)t + h_2(r).$$

Using equation (41) we get

$$B = \frac{h_1' t + h_2'}{h}.$$  

5
Substituting equations (48)-(47) into (20) we get

\[ h'_1 h_1 - h h' + 2\Lambda (h'_1 h_1 t^2 + h'_1 h_2 t + h'_2 h_1 t + h'_2 h_2) = 0, \]  \hspace{1cm} (49)

That yields the following system of equations

\[ h'_1 h_1 - h h' + 2\Lambda h'_2 h_2 = 0, \]  \hspace{1cm} (50)
\[ 2\Lambda h'_1 h_1 = 0, \]  \hspace{1cm} (51)
\[ 2\Lambda (h'_1 h_2 + h'_2 h_1) = 0. \]  \hspace{1cm} (52)

This system presents two solutions that are \( h_1 = 0 \) and \( h'_1 = 0 \).

**Case ii.a.1 Solution for \( h_1 = 0 \)**

This implies \( h = \pm \sqrt{2\Lambda h_2 + h_0} \) which furnishes \( \dot{\phi} \pm \sqrt{2\Lambda} \) and thus

\[ C = h_2(r), \]
\[ B = \frac{h'_2(r)}{g(r)}, \]
\[ \phi = \pm t\sqrt{2\Lambda} + \phi_0, \]
\[ R = -4\Lambda, \]  \hspace{1cm} (53)

where \( R \) is the Ricci scalar. We have then a static metric and an imaginary scalar field.

**Case ii.a.2) Solution for \( h'_1 = 0 \)**

This implies \( h_2 = \text{constant} \) and thus \( B = 0 \), which is also unacceptable for obvious reasons.

**Case ii.b) Solutions for \( C' \neq 0 \) and \( \ddot{C} \neq 0 \)**

Substituting equation (41) into equation (19) we can write

\[ \ddot{C}' = C', \]  \hspace{1cm} (54)

then

\[ \ddot{C} = q(t)C. \]  \hspace{1cm} (55)

Considering equations (41)-(19) into equation (15) we have

\[ 2 \left[ q(t) + 2\Lambda \right] C'C + 2\dot{C}'\dot{C} = 2hh', \]  \hspace{1cm} (56)

whose integration in \( r \) furnishes

\[ [q(t) + 2\Lambda] C'^2 + \dot{C}'^2 - h^2 = q_1(t). \]  \hspace{1cm} (57)

Now taking equation (41) into equation (18), we obtain

\[ \dot{\phi}^2 = \frac{\ddot{C}'}{C'} + \frac{\ddot{C}}{C} + 2\Lambda, \]  \hspace{1cm} (58)

and, with equation (55), we can write

\[ \dot{\phi}^2 = 2[q(t) + \Lambda], \]  \hspace{1cm} (59)

Putting \( q(t) \) from equation (55) into equation (57) we have now

\[ (C'^2)' + 4\Lambda C^2 = 2h^2 + 2q_1. \]  \hspace{1cm} (60)
the integration of the above equation holds

\[ C^2 = e^{2\sqrt{-\Lambda}} \left[ \frac{1}{2\sqrt{-\Lambda}} \int q_1(t)e^{-2\sqrt{-\Lambda}t}dt + h_1(r) \right] \]

\[ -e^{-2\sqrt{-\Lambda}} \left[ \frac{1}{2\sqrt{-\Lambda}} \int q_1(t)e^{2\sqrt{-\Lambda}t}dt + h_2(r) \right] + \frac{h(r)^2}{2\Lambda}. \]  

(61)

Finally, substituting equation (61) into equation (55) we can find

\[ q = \frac{2(-\Lambda) \left[ e^{2\sqrt{-\Lambda}\alpha(t,r)} - e^{-2\sqrt{-\Lambda}\beta(t,r)} \right] + q_1(t)}{e^{2\sqrt{-\Lambda}\alpha(t,r)} - e^{-2\sqrt{-\Lambda}\beta(t,r)} + \frac{h(r)^2}{2\Lambda}} \]

\[ \left[ e^{2\sqrt{-\Lambda}\alpha(t,r)} - e^{-2\sqrt{-\Lambda}\beta(t,r)} + \frac{h(r)^2}{2\Lambda} \right]^2, \]

(62)

where

\[ \alpha(t,r) = \frac{1}{2\sqrt{-\Lambda}} F_1(t) + h_1(r), \]

(63)

\[ \beta(t,r) = \frac{1}{2\sqrt{-\Lambda}} F_2(t) + h_2(r), \]

(64)

\[ F_1(t) = \int q_1(t)e^{-2\sqrt{-\Lambda}t}dt, \]

(65)

and

\[ F_2(t) = \int q_1(t)e^{2\sqrt{-\Lambda}t}dt. \]

(66)

Since \( h_1 = h_2 = h = \text{constant}, \) we have \( C' = 0, \) the unique solution for this case is

\[ d_0h^2 = h_1d_1 = h_2d_2 \text{ and } q_1(t) = 0, \]

(67)

which give us the general solution for the Case ii,

\[ C^2 = h(r)^2 \left( \frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{-\Lambda}t} + d_2 e^{-2\sqrt{-\Lambda}t} \right), \]

(68)

\[ B^2 = \frac{h'(r)^2}{\sqrt{d_0h(r)^2} + d_3} \left( \frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{-\Lambda}t} + d_2 e^{-2\sqrt{-\Lambda}t} \right), \]

(69)

\[ \phi = -\frac{1}{\sqrt{-\Lambda}} \arctan \left( \frac{4d_1e^{2\sqrt{-\Lambda}t} + d_0}{\sqrt{16d_1d_2^2 - d_0^2}} \right) + \phi_0, \]

(70)

where \( d_0, d_1, d_2, d_3 \) and \( \phi_0 \) are constants.

Doing a coordinate transformation, such as

\[ \bar{r} = \frac{1}{\sqrt{d_0}} \ln \left( \sqrt{d_0} h(r) + \sqrt{d_0h(r)^2} + \sqrt{d_3} \right) + \text{constant} \]

(71)

we can rewrite the metric, dropping the bar notation for the new coordinate, as

\[ ds^2 = dt^2 - \left( \frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{-\Lambda}t} + d_2 e^{-2\sqrt{-\Lambda}t} \right) \]

\[ \times \left[ dr^2 + \frac{1}{4d_0} (e^{\sqrt{d_0}r} - d_3 e^{\sqrt{-\Lambda}r})^2 d\theta^2 \right]. \]

(72)

This is then the unique physical solution. Below we will study it in more details.
IV. STUDY OF THE GLOBAL STRUCTURE OF THE PHYSICAL SOLUTION

For gravitational collapse, we impose the following conditions at \( r = 0 \):

(i) There must exist a symmetry axis, which can be expressed as

\[
X = \left| \xi^\mu(\theta) \xi^\nu(\theta) g_{\mu\nu} \right| \to 0, \tag{73}
\]

as \( r \to 0 \), we have chosen the radial coordinate such that the axis is located at \( r = 0 \), and \( \xi^\mu(\theta) \) is the Killing vector with a close orbit, and given by \( \xi^\alpha(\theta) \partial_\alpha = \partial_\theta \).

(ii) The spacetime near the symmetry axis is locally flat, which can be written as \(^{14}\)

\[
\frac{X_\alpha X_\beta g^{\alpha\beta}}{4X} \to -1, \tag{74}
\]

as \( r \to 0 \). Note that solutions failing to satisfy this condition sometimes are also acceptable. For example, when the left-hand side of the above equation approaches a finite constant, the singularity at \( r = 0 \) may be related to a point-like particle \(^{15}\).

(iii) No closed timelike curves (CTC’s). In spacetimes with circular symmetry, CTC’s can be easily introduced. To ensure their absence, we assume that the condition

\[
\xi^\mu(\theta) \xi^\nu(\theta) g_{\mu\nu} < 0, \tag{75}
\]

holds in the whole spacetime.

Applying these conditions we can easily see from the metric (72) that the regularity conditions on the axis \( r = 0 \) requires \( d_3 = 1 \).

The geometrical radius is given by

\[
R_g \equiv \sqrt{g_{\theta\theta}} = \frac{1}{2\sqrt{d_0}} \sqrt{\frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{\Lambda}t} + d_2 e^{-2\sqrt{\Lambda}t}} \left( e^{r\sqrt{d_0}} - e^{-r\sqrt{d_0}} \right). \tag{76}
\]

In order to have a real positive geometrical radius we may have two possibilities:

a) \( d_0 > 0, \frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{\Lambda}t} + d_2 e^{-2\sqrt{\Lambda}t} > 0, \quad d_3 \leq 1, \)

b) \( d_0 < 0, \frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{\Lambda}t} + d_2 e^{-2\sqrt{\Lambda}t} < 0, \quad d_3 = -1, \)

but this last condition violates the regularity condition.

The condition for a collapse is given by

\[
\dot{R}_g = \frac{\sqrt{-\Lambda}}{4\sqrt{d_0}} \frac{d_1 e^{2\sqrt{\Lambda}t} - d_2 e^{-2\sqrt{\Lambda}t}}{\sqrt{\frac{d_0}{2\Lambda} + d_1 e^{2\sqrt{\Lambda}t} + d_2 e^{-2\sqrt{\Lambda}t}}} \left( e^{r\sqrt{d_0}} - e^{-r\sqrt{d_0}} \right) < 0. \tag{77}
\]

Thus, we have

\[
d_0 > 0, \quad d_1 < 0, \quad d_2 > 0, \tag{78}
\]

without any restriction in the time coordinate.

The condition to have a real geometrical radius gives an additional constrain written as

a) for \( d_0 > 0, \quad d_2 > 0 \) and \( d_1 < 0, \quad \frac{d_0}{2\Lambda} - |d_1| + |d_2| \geq 0, \)

b) for \( d_0 < 0, \quad d_2 < 0 \) and \( d_1 > 0, \quad -\frac{|d_0|}{2\Lambda} + |d_1| - |d_2| \geq 0. \)
The second conditions represent an expansion instead of a collapse, so we consider hereinafter only the first ones.

The energy density of the fluid is given by

\[ \rho = \frac{-\Lambda}{(\frac{d^2}{dt^2} + d_1 e^{2\sqrt{-\Lambda}} + d_2 e^{-2\sqrt{-\Lambda}})^2} \]

(79)

and it can be shown that \( \rho \) is positive for \( \Lambda < 0 \), \( d_1 < 0 \) and \( d_2 > 0 \), and whose denominator can vanish at

\[ t_{\text{sing}} = \frac{1}{2\sqrt{-\Lambda}} \ln \left( \frac{-d_0 \pm \sqrt{d_0^2 - 16d_1d_2\Lambda^2}}{4d_1\Lambda} \right) \]

(80)

which represents the instant of the singularity formation. Here the positive sign is for \( d_0 > 0 \) and the negative sign is for \( d_0 < 0 \).

It is easily shown that \( R_g \) vanishes for the singularity, i.e., \( R_g(t = t_{\text{sing}}) = 0 \).

The pressure is given by

\[ p = \frac{-\Lambda}{\frac{d^2}{dt^2} + d_1 e^{2\sqrt{-\Lambda}} + d_2 e^{-2\sqrt{-\Lambda}})^2} \]

(81)

The expansion of the congruence of null outgoing and ingoing geodesics can be written, respectively, as

\[ \theta_t = \frac{F}{R_g} \left( \frac{1}{A} R_{g\,tt} + \frac{1}{B} R_{g\,rr} \right) \]

\[ = \frac{\sqrt{-\Lambda}}{\sqrt{2d_0 + d_1 e^{2\sqrt{-\Lambda}} + d_2 e^{-2\sqrt{-\Lambda}}}^2} \left( d_1 e^{2\sqrt{-\Lambda}t} - d_2 e^{-2\sqrt{-\Lambda}t} \right) \]

\[ + \frac{1}{2} \left( e^{r\sqrt{d_0}} + e^{-r\sqrt{d_0}} \right) \]

(82)

\[ \theta_n = \frac{G}{R_g} \left( \frac{1}{A} R_{g\,tt} - \frac{1}{B} R_{g\,rr} \right) \]

\[ = \frac{\sqrt{-\Lambda}}{\sqrt{2d_0 + d_1 e^{2\sqrt{-\Lambda}} + d_2 e^{-2\sqrt{-\Lambda}}}^2} \left( d_1 e^{2\sqrt{-\Lambda}t} - d_2 e^{-2\sqrt{-\Lambda}t} \right) \]

\[ - \frac{1}{2} \left( e^{r\sqrt{d_0}} + e^{-r\sqrt{d_0}} \right) \]

(83)

where \( F \) and \( G \) are always positive [16]. The apparent horizon \( r_{AH} \) is located at the hypersurface \( \theta_n \theta_t = 0 \) [17].

Besides, equation (83) shows us that \( \theta_n \) is always negative, as it would be expected in a collapse process. It can be also shown that for \( r > r_{AH} \), we have \( \theta_t \) positive, while in the region where \( r < r_{AH} \), the expansion \( \theta_t \) is always negative. This means that the collapse always forms black holes.

Summarizing, the solution obtained here satisfies the regularity conditions at the origin and presents a real and positive geometric radius, which always decreases with the time. A singularity is formed in a finite time, where the geometric radius vanishes.

V. CONCLUSION

In this paper, we found all the geodesic solutions of the Einstein-massless-scalar field equations with negative cosmological constant in the \( (2 + 1) \)-dimensional spacetimes with circular symmetry. From these, we have shown that only one of the solutions, given by the equations (70) and (72), can represent gravitational collapse of the scalar field. The collapse always forms black holes. We intend to analyse the perturbations of this solution in order to investigate the possibility of a critical type I collapse and present the results as soon as possible.
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