Microwave Scattering in the Subohmic Spin-Boson Systems in Superconducting Circuits

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We study quantum critical phenomena in microwave scattering of the subohmic spin-boson system, which exhibits quantum phase transition at a critical system-reservoir coupling. By relating the reflection coefficient of a microwave with a dynamic susceptibility of the subohmic spin-boson system, we clarify how quantum critical phenomena appear in a microwave scattering. We also propose experimental setups to realize the sub-ohmic spin-boson system in a superconducting circuit composed of a charge qubit and a dissipative transmission line.

1. Introduction

Quantum critical phenomena (QCP) are ubiquitous in strongly correlated systems, and have been investigated in a number of studies. It is, however, left challenging to observe QCP in well-controllable systems by fine adjustment of experimental parameters. Recent development in nanostructure fabrication and measurement has enabled us to design nanoscale devices for study of novel quantum phenomena in a system strongly interacting with an environment. For example, QCP in the two-channel Kondo model have been studied in detail in quantum dot systems, which is effectively described by a local spin interacting antiferromagnetically with an electronic environment. It is natural to ask whether one can realize a counterpart in controllable bosonic systems which display quantum phase transitions (QPTs).

The spin-boson model has been studied as a minimal model of a quantum two-state system interacting with a bosonic environment for long time. This model has a variety of applications, including superconducting circuits, photon waveguides, and molecular junctions. Properties of the bosonic reservoir are characterized by a spectral density, \( I(\omega) \propto \omega^s \). The ohmic spin-boson system \( (s = 1) \) has been studied in a number of theoretical works in various contexts, since it displays remarkable phenomena such as the Kosterlitz-Thouless-type phase transition and the Kondo effect. On the other hand, the subohmic spin-boson system \( (s < 1) \) induces a second-order phase transition at zero temperature by tuning a system-reservoir coupling to a critical value. In the previous work by the present authors, QCP have been studied in heat transport by evaluating temperature dependence of a thermal conductance. It is, however, preferable to study QCP using standard techniques of measurement, since measurement of a thermal conductance is expected to be difficult.

Over the past decades, experimental techniques on light-matter interaction systems have greatly improved. Recently, the ohmic spin-boson model has been realized in a flux qubit coupled to a \( LC \) transmission line, and there, microwave scattering measurement has been performed experimentally. This development suggests a possibility to probe QCP in the subohmic spin-boson system by microwave scattering in a transmission line weakly coupled to it. Realization of the subohmic reservoir in a circuit model has first been discussed for the special case of \( s = 1/2 \) in Ref. 46, and recently for arbitrary values of \( s \) in the range \( 0 < s < 1 \) in Ref. 43 by the present authors. In the latter paper, we have shown that a flux qubit coupled to a \( RLC \) series circuit is effectively described by the subohmic spin-boson model.

In this paper, we study QCP in a charge qubit coupled to the subohmic reservoir and consider microwave scattering using an additional transmission line weakly coupled to the charge qubit. By using the input-output theory, we derive the relation between a reflection coefficient and a dynamic susceptibility of the subohmic spin-boson system. We clarify how quantum critical behavior appears in the frequency dependence of the dynamic susceptibility by analytic discussion as well as numerical calculation based on the continuous-time quantum Monte Carlo (CTQMC) method. We show that in the quantum critical regime, the dynamic susceptibility exhibits distinctive power-law frequency dependence reflected from the nature of QPT. Moreover, we propose three experimental setups to realize the subohmic reservoir using the charge qubit and the \( RLC \) transmission line. One of the proposed setups has advantage that a number of circuit elements is less than the one proposed in the previous works.

This paper is organized as follows. In Sect. 2, we introduce our model composed of a charge qubit, a subohmic reservoir, and a transmission line. In Sect. 3, a reflection coefficient of microwave scattering in a transmission line is related with a dynamic susceptibility of the subohmic spin-boson system. After we briefly summarize analytic results on the dynamic susceptibility in Sect. 4, we numerically evaluate it as well as the reflection amplitude by the CTQMC method in Sect. 5. In Sect. 6, we propose circuit models to realize the subohmic spin-boson system. Finally, we summarize our results in Sect. 7.

2. Model

We prepare a target system composed of a two-state system and a subohmic reservoir, and consider microwave scattering in the target system within the linear response theory (see Fig. 1 (a)). An incident microwave enters from a transmission line into the charge qubit, and a reflected microwave propagates in the same transmission line in the opposite direction. The frequency dependence of the microwave loss at the target system can be used to probe quantum critical behavior of the

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target system, i.e., the subohmic spin-boson system.

2.1 Charge qubit

We consider a superconducting circuit shown in Fig. 1 (b) for a charge qubit composed of two Josephson junctions with Josephson energies $E_{J,v}$ and capacitances $C_{J,v}$ ($v = L, R$). The charge state of the qubit is controlled by a gate voltage $V_g$ via a gate capacitance $C_g$. If there is no excitation in both the subohmic reservoir and the transmission line, the output voltages of them, $V_{L,0}$ and $V_{R,0}$, are zero. Then, the Hamiltonian of the charge qubit is given as

$$H_S = E_C(n - n_s)^2 - \sum_{v=L,R} E_{J,v} \cos(\phi_v/\phi_0),$$

where $E_C = 2e^2/C_g$ ($C_g = C_L + C_R$: the total capacitance) is the Coulomb energy, $n_s$ is the offset Cooper-pair number defined by $n_s \equiv Q_g/2e$ ($Q_g$: the charge accumulated at the gate capacitance), and $\phi_0 = h/2e$ is a flux quantum. The operator $n$ describes the number of excess Cooper pairs at the superconducting island ($Q = 2en$). Considering small Josephson junctions ($E_{J,L} \ll E_C$) and setting the offset charge as $n_s = 0.5$, the system is effectively described only by the two charge states ($n = 0$ and $n = 1$), whose charging energies are well separated from others. After truncating the system into a two-state system, the Hamiltonian is given as

$$H_S = -\frac{\hbar\Delta}{2}\sigma_z - e\sigma_x,$$

where the first term describes the Josephson energy, and the second term corresponds to detuning from the degenerate gate voltage $V_g = 0.5$. Here, $\Delta \equiv E_J/h$ is a tunneling frequency and $\sigma_i$ ($i = x, y, z$) is the Pauli matrix. We have rewritten the two charge states, $|n = 1\rangle$ and $|n = 0\rangle$, with $|1\rangle = |0\rangle$ respectively, and have replaced the number operator $n$ with $\sigma_z = 2n - 1$. In this paper, we consider the symmetric case ($\epsilon = 0$) and use only the detuning energy to define the static susceptibility.

2.2 Subohmic reservoir

The Hamiltonian of the target system (the charge qubit coupled to the subohmic reservoir) is given as

$$H_{sub} = H_S + H_{R,sub} + H_{I,sub}.$$  

The second and third terms describe the reservoir and the system-reservoir interaction, respectively. They are in general written as

$$H_{R,sub} = \sum_{k} h_{0k}b_{k}^\dagger b_k,$$

$$H_{I,sub} = -\frac{\sigma_z}{2}\sum_{k} h_{k}\left(b_k + b_k^\dagger\right),$$

respectively. This Hamiltonian is called as the spin-boson Hamiltonian. Here, $b_k (b_k^\dagger)$ is an annihilation (a creation) operator of excitation with a frequency $\omega_k$ and a system-reservoir coupling constant $\lambda_k$ in the subohmic reservoir. The properties of the reservoir are characterized by the spectral density:

$$I_{sub}(\omega) = \sum_{k}\lambda_k^2\delta(\omega - \omega_k).$$

In numerical calculation shown in Sect. 5, we assume that the spectral density takes a form

$$I_{sub}(\omega) = 2\alpha\omega\left(\frac{\omega}{\omega_c}\right)^{s-1}e^{-\omega/\omega_c},$$

for simplicity, where $\alpha$ is a dimensionless system-reservoir coupling strength, and $\omega_c$ is a cutoff frequency. The exponent of spectral density is taken as $s < 1$ (the subohmic reservoir).

The two-state system coupled to the subohmic reservoir displays a quantum phase transition when the coupling strength is tuned to the critical point $\alpha_c$, which is a function of $s$ and $\Delta/\omega_c$.\cite{11,30,31,38,43,49} For $\alpha > \alpha_c$, the ground state is expressed by a coherent superposition of two $\sigma_z$-basis states, $|\pm\rangle = \pm 1). This phase is called a ‘delocalized phase’. For $\alpha < \alpha_c$, the ground state becomes two-fold degenerate since quantum coherence is completely broken. This phase is called a ‘localized phase’.

In Sect. 6, we explicitly construct the subohmic reservoir using a RLC transmission line. Then, the interaction between the charge qubit and the subohmic reservoir is written as

$$H_{I,sub} = -\frac{\epsilon C_{J,L}}{C_g}\sigma_z V_{L,0},$$

$$V_{L,0} = \frac{C_g}{2\epsilon|C_{J,L}|}\sum_{k}\hbar\lambda_k(b_k + b_k^\dagger),$$

where $V_{L,0}$ is an output voltage of the subohmic reservoir (see Fig. 1 (b)). We introduce the retarded voltage-voltage correlation function defined by

$$G^R_V(t) = \frac{1}{\hbar}\theta(t)\langle V_{L,0}(t), V_{L,0}(0)\rangle_0,$$

where $\theta(t)$ is the Heaviside step function, $V_{L,0}(t) = e^{iH_{res,t}/\hbar}V_{L,0}e^{-iH_{res,t}/\hbar}$ and $\langle \cdot \rangle_0$ indicates an ensemble average with respect to $H_{R,sub}$. Using Eq. (6), the imaginary part of the Fourier transformation of $G^R_V(t)$ can be expressed in terms of the spectral density:

$$\text{Im}\left[G^R_V(\omega)\right] = -\pi\hbar\left(C_g/2\epsilon|C_{J,L}|\right)^2I_{sub}(\omega).$$
Using the linear response theory, the voltage-voltage correlation function $G^R_{x}(t)$ is related with the impedance $Z_{sub}(\omega)$ of the subohmic reservoir as
\[ G^R_{x}(\omega) = -i\omega Z_{sub}(\omega). \] (12)

Therefore, the spectral function can be related with the impedance of the circuit as
\[ I_{sub}(\omega) = \frac{1}{\pi\hbar} \left( \frac{2|e|C_{LR}}{C_S} \right)^2 \omega \text{Re}[Z_{sub}(\omega)]. \] (13)

2.3 Transmission line

To prove critical behavior near the quantum phase transition, we prepare for a transmission line weakly coupled to the target system (see Fig. 1). The transmission line is described by a continuum limit of the LC circuit shown in Fig. 2. We define a phase operator as
\[ \phi(x,t) \equiv \int_{-\infty}^{t} dt' V(x,t'), \] (14)
where $V(x,t)$ is the voltage on the LC transmission line at position $x$ at time $t$. Classical partial differential equations for the phase operator $\phi(x,t)$ can be derived from the Hamiltonian defined by \[ H_{\text{R.line}} = \int_{0}^{d} dx \left\{ \frac{1}{2c} \phi(x)^2 + \frac{1}{2} \left( \partial_t \phi(x) \right)^2 \right\}, \] (15)
where $c$ and $l$ is the capacitance and the inductance per unit length, respectively, $d$ is the length of the transmission line, and $q(x)$ is a charge density operator conjugate to $\phi(x)$. For description of quantum dynamics, we impose an exchange relation
\[ [\phi(x), q(x')] = i\hbar \delta(x - x'). \] (16)

Using a Bosonic operator
\[ B_{k} = \frac{1}{\sqrt{\hbar k_c d}} \int_{0}^{d} dx \; e^{-ikx} q(x), \] (17)
the Hamiltonian of the transmission line is diagonalized as
\[ H_{\text{R.line}} = \sum_{k} \hbar \Omega_{k} B_{k}^\dagger B_{k}, \] (18)
where $\Omega_{k} = |k|/\sqrt{\hbar c}$. Here we note that the commutation relation $[B_{k}, B_{k'}^\dagger] = \delta_{kk'}$ holds from Eq. (16).

We consider a capacitive coupling between the charge qubit and the transmission line. The voltage is written by the field operators, $B_{k}$ and $B_{k'}^\dagger$, as
\[ V(x) = \sum_{k} \frac{\hbar \Omega_{k}}{dc} \left( B_{k} e^{ikx} + B_{k'}^\dagger e^{-ikx} \right), \] (19)
and the output voltage at the qubit is written as $V_{R,0} = V(x = 0)$. The Hamiltonian of the interaction between the charge qubit and the transmission line is given as
\[ H_{\text{line}} = -\frac{|e|}{2} \frac{C_{LR}}{C_S} \sigma_{x} V_{R,0} - \frac{\sigma_{x}}{2} \sum_{k} \hbar \omega \left( B_{k} + B_{k}' \right), \] (20)
\[ \Lambda_{k} = \frac{2|e|C_{LR}}{C_S} \frac{\sqrt{\Omega_{k}}}{\hbar dc}. \] (21)

The spectral density for the transmission line is obtained as
\[ I_{\text{line}}(\omega) = \sum_{k} \Lambda_{k}^2 \delta (\omega - \Omega_{k}) = \frac{1}{\pi \hbar} \sqrt{\frac{l}{c}} \left( \frac{2|e|C_{LR}}{C_S} \right)^2 \omega \equiv \alpha_{\text{line}} \omega. \] (22)

3. Input-output theory

In this section, we relate a reflection coefficient in microwave scattering with a dynamic susceptibility of the target system by combining the input-output theory \cite{9,47,48} and the linear response theory. We begin with the total Hamiltonian
\[ H = H_{\text{sub}} + H_{\text{R.line}} + H_{\text{line}}, \] (23)
respectively. Fourier transformation of the voltages of input and output modes gives
\[ 
\langle V_{\text{out}}(\omega) \rangle = \langle V_{\text{in}}(\omega) \rangle + i\frac{\pi nC_\Sigma}{2|e|C_{\text{LR}}} \text{Im} \langle \sigma_\text{c}^{(\omega)} \rangle, \tag{29} 
\]
where \( \text{Im} \langle \sigma_\text{c}^{(\omega)} \rangle \propto \omega \) is the spectral density of the transmission line. \( \langle \cdots \rangle = \text{tr} \{ e^{-\beta H_{\text{sub}}} \cdots \} / \text{tr} \{ e^{-\beta H_{\text{sub}}} \} \), and \( \beta = 1/k_B T \).

Assuming a weak coupling between the two-state system and the ohmic reservoir \( (\omega_{\text{line}} = 1) \), the population \( \langle \sigma_\text{c}^{(\omega)} \rangle \) in Eq. (29) can be calculated by using the linear response theory as follows:
\[ 
\langle \sigma_\text{c}^{(\omega)} \rangle = \frac{|e|C_{\text{LR}}}{C_\Sigma} \chi_{\text{sub}}(\omega) \langle V_{\text{in}}(\omega) \rangle. \tag{30} 
\]
Here \( \chi_{\text{sub}}(\omega) \) is the Fourier transformation of the dynamic susceptibility of the two-state system coupled to the subohmic reservoir defined by
\[ 
\chi_{\text{sub}}(\omega) \equiv \frac{\omega}{\text{Im} \chi_{\text{sub}}(\omega)} \langle V_{\text{in}}(\omega) \rangle, \tag{31} 
\]
where \( \langle \cdots \rangle_{\text{sub}} = \text{tr} \{ e^{-\beta H_{\text{sub}}} \cdots \} \). From Eqs. (29) and (30), we obtain the reflection coefficient as
\[ 
r(\omega) \equiv \frac{\langle V_{\text{out}}(\omega) \rangle}{\langle V_{\text{in}}(\omega) \rangle} = 1 + i\frac{\pi n}{2} \text{Im} \langle V_{\text{in}}(\omega) \rangle \chi_{\text{sub}}(\omega). \tag{32} 
\]

4. **Dynamic Susceptibility**

In this section, we briefly summarize features of the dynamic susceptibility \( \chi(\omega) \) at the critical point \( (\alpha = \alpha_c) \) and in the delocalized regime \( (\alpha > \alpha_c) \).

4.1 **Quantum critical regime**

When the system-reservoir coupling is tuned at QPT \( (\alpha = \alpha_c) \), the dynamic susceptibility exhibits the distinctive frequency dependence reflecting the nature of the QPT:
\[ 
\chi_{\text{sub}}(\omega) \sim \omega^{-\eta}, \quad (\alpha = \alpha_c). \tag{33} 
\]
The critical exponent \( \eta \) is related to the critical exponent \( \eta \) of the imaginary-time spin-spin correlation function:
\[ 
C(\tau) \equiv \langle \sigma^{(\tau)} \sigma^{(0)} \rangle_{\text{sub}} \sim \tau^{-\eta+1}, \tag{34} 
\]
\[ 
y = 2 - \eta, \tag{35} 
\]
where \( \sigma^{(\tau)} = e^{iH_{\text{sub}}\tau} \sigma_x e^{-iH_{\text{sub}}\tau} \). For \( 0 < s \leq 0.5 \), the QPT belongs to the mean-field universality class \( [8,41,43,52-54] \) and we obtain \( \eta = 2 - s \). This leads to
\[ 
y = s, \quad (0 < s \leq 0.5), \tag{36} 
\]
In contrast, the critical exponent \( y \) becomes a complex function of \( s \) for \( 0.5 < s < 1 \), since the QPT belongs to a nontrivial universality class \( [54] \).

4.2 **Delocalized regime**

If the system-reservoir coupling is set sufficiently below the critical value \( (\alpha < \alpha_c) \), the ground state is the coherent superposition of the two charge states. At low temperatures, we can apply the generalized Shiba relation \( [49,55] \)
\[ 
\lim_{\omega \to 0} \frac{\text{Im} \chi_{\text{sub}}(\omega)}{\omega} = 2\pi \alpha_\omega \left( \frac{\chi_{\text{sub}}(0)}{2} \right)^2, \tag{37} 
\]
where the static susceptibility \( \chi_{\text{sub}}(0) \) is defined as
\[ 
\chi_{\text{sub}}(0) \equiv \lim_{\omega \to 0} \langle \sigma^{(\omega)} \rangle_{\text{sub}}. \tag{38} 
\]

Using this generalized Shiba relation, the low-frequency dynamic susceptibility is obtained as
\[ 
\text{Im} \chi_{\text{sub}}(\omega) \sim \omega^y. \tag{39} 
\]

5. **Numerical Calculation**

To demonstrate critical behavior of the dynamic susceptibility, it is convenient to perform the CTQMC simulation \( [38,49] \) by which the correlation function \( C(\tau) = \langle \sigma^{(\tau)} \sigma^{(0)} \rangle_{\text{sub}} \) can be evaluated numerically (for details, see Ref. 49). We define the Fourier transformation of the imaginary-time spin-spin correlation function as
\[ 
C(i\omega_n) = \int_0^\theta d\tau e^{i\omega_n\tau} C(\tau), \tag{40} 
\]
where \( \omega_n = 2m^2/\hbar \) is the Matsubara frequency. The Monte Carlo data presented below represent averages over \( 10^5 - 10^6 \) cluster updates. The dynamic susceptibility is obtained from \( C(i\omega_n) \) by the analytic continuation as follows:
\[ 
\chi_{\text{sub}}(\omega) = C(i\omega_n \to \omega + i\delta). \tag{41} 
\]
For numerical analytic continuation, we employ the Padé approximation \( [56,57] \). From the dynamic susceptibility \( \chi_{\text{sub}}(\omega) \), the reflection coefficient \( r(\omega) \) is evaluated by using Eq. (32). We first discuss the dynamic susceptibility in the quantum
critical regime ($\alpha = \alpha_c$). For demonstration, we consider the case of $s = 0.5$ and $\Delta/\omega_c = 0.1$, for which the critical point is determined as $\alpha_c = 0.1074$ by the Binder analysis (see Ref. 49). In Fig. 3 (a) and (b), we plot the real and imaginary parts of the dynamic susceptibility obtained by the CTQMC simulations as a function of $\alpha = \alpha_c$ for three different temperatures ($\beta h \omega_c = 2^{12}$, $2^{13}$, and $2^{14}$), respectively. The dynamic susceptibility exhibits the quantum critical behavior, $\chi(\omega) \sim \omega^{-\gamma}$, as discussed in Sect. 4.1. As shown in Fig. 3, the dynamic susceptibility is proportional to $\omega^{-0.5}$ for a frequency range, $k_B T / h \ll \omega \ll \Delta$, where $\Delta \lesssim \Lambda$ is a renormalized tunneling frequency, which a function of $s$, $\Delta/\omega_c$, and $\alpha$. We note that our result is consistent with the previous works.\(^{32,35}\) The result for the low-frequency region is in good agreement with the critical exponent in Eq. (36) for $s = 0.5$.

Next, we consider the delocalized regime ($\alpha < \alpha_c$) in the case of $s = 0.5$ and $\Delta/\omega_c = 0.1$. In Fig. 3 (a) and (b), we also plot the dynamic susceptibility for the coupling strength $\alpha = 0.05$, which is sufficiently smaller than the critical point $\alpha_c = 0.1074$. The inverse temperature is set as $\beta h \omega_c = 2^{14}$. The real (imaginary) part of the dynamic susceptibility has a shoulder (a peak) at the slightly higher frequency than $\omega = \Delta$ for the quantum critical regime. This indicates that the renormalization effect of $\Delta$ due to the subohmic reservoir becomes stronger as the coupling strength increases. For $\omega \ll \Delta$, the numerical result for the imaginary part of the dynamic susceptibility is consistent with the power-law frequency behavior obtained from the generalized Shiba relation (37), which is proportional to $\omega^{0.5}$ for $s = 0.5$.

In Fig. 4, we show the reflection loss, $1 - |r(\omega)|^2$, for $s = 0.5$ and $\Delta/\omega_c = 0.1$. The reflection has a peak for both $\alpha = \alpha_c$ and $\alpha < \alpha_c$ reflecting the renormalized tunneling amplitude $\Delta$. For $\omega < \Delta$, the reflection loss approaches zero more slowly at the quantum critical point as $\omega$ decreases. From Eq. (32), the reflection loss is written as

$$1 - |r(\omega)|^2 = \pi h I_{\text{line}}(\omega)|\chi_{\text{sub}}(\omega)| + O(\alpha_c^2) $$

Since $I_{\text{line}}(\omega) \propto \omega$, the frequency dependences of the dynamic susceptibility given in Eqs. (33) and (39) lead that the reflection loss is proportional to $\omega^{-1}$ in the quantum critical regime for $k_B T / h \ll \omega \ll \Delta$, and to $\omega^{1.5}$ in the delocalized regime for $\omega \ll \Delta$. These frequency dependences are consistent with the numerical result for $s = 0.5$ (see Fig. 4).

6. Experimental Realization

In this section, we propose superconducting circuits including RLC transmission lines to realize the subohmic spin-boson model. As shown in Sect. 2.2, the spectral density of the superconducting circuit is related to the total impedance as

$$Z_{\text{sub}}(\omega) = \frac{1}{Z_L(\omega)|1 + i\omega C_0|} $$

where $Z_L(\omega)$ is the impedance of the RLC circuit and $C_0$ is the capacitance connected to the output terminals in parallel (see Fig. 1). Using Eq. (13), the spectral density of the subohmic reservoir is given as

$$I_{\text{sub}}(\omega) = \frac{1}{\pi R} \left( \frac{2e |C_L|^2}{C_L} \right)^2 I_{\text{sub}}(\omega), \quad (44)$$

$$I_{\text{sub}}(\omega) = \omega \text{Re}[Z_{\text{sub}}(\omega)]. \quad (45)$$

To obtain the impedance of the RLC transmission line, $Z_L(\omega)$, we use the recurrence relation:

$$Z_L(\omega) = 1 + i\omega L + \frac{1}{Z_{L+1}(\omega)^{-1} + i\omega C_{L+1}}, \quad (46)$$

where $Z_{L,N+1}(\omega)^{-1} = 0$ and $N$ is a number of the repeated structures of circuit elements.

We first consider a simple circuit to realize the subohmic reservoir with $s = 0.5$ in Sect. 6.1. Next, we expand a circuit model for arbitrary value of $s$ smaller than 0.5 in Sect. 6.2. Finally, we mention a circuit model for arbitrary $s$ in the range $0 < s < 1$.

6.1 Subohmic reservoir of $s = 0.5$

Let us consider a uniform RLC circuit:

$$R_j = R, \quad L_j = L, \quad C_j = C. \quad (48)$$

In Fig. 6, we show the spectral density, $I_{\text{sub}}(\omega)$, of the RLC circuit for $N = 10^2$, $10^3$, $10^4$, and $\infty$. We note that the case of $N = \infty$ has been discussed in Ref. 46. Here, the circuit parameters are set as $R = 1 \Omega$, $L = 1 \mu H$, $C = 1 \text{ pF}$, and $C_0 = 1 \text{ pF}$. To see effect of inhomogeneity in the circuit, we have also added 10% relative randomness to circuit parameters. As shown in the figure, inhomogeneity in the circuit does not make any visible influence to the spectral density except for the high-frequency region of $N = 10^4$. Fig. 6 indicates that this circuit realizes the subohmic reservoir with $s = 0.5$ for a frequency range, $\omega^* \ll \omega \ll \omega_c$, where $\omega^*$ and $\omega_c$ are low- and high-frequency cutoffs, respectively. From the recurrence relation, Eq. (47), these cutoffs are given as
Fig. 6. (Color online) The spectral density of the reservoir realized by the RLC transmission line to realize the subohmic spin-boson system with $s = 0.5$. The elements of the circuit are set as $R = 1\, \Omega$, $L = 1\, \text{pH}$, $C = 1\, \text{pF}$, $C_0 = 1\, \text{pF}$.

Fig. 7. (Color online) The spectral density of the reservoir realized by the $RC$ transmission line to realize the subohmic spin-boson system with $s = 0.25$ ($n = 2$). The circuit parameters are $R = 50\, \text{m}\Omega$, $L = 0\, \text{H}$, $C = 0.2\, \mu\text{F}$, $C_0 = 0\, \mu\text{F}$, $N = 10^5$.

We consider a system composed of a charge qubit and a $RLC$ circuit. We investigate quantum critical phenomena in microwave scattering of a superconducting circuit theoretically. We considered a system composed of a charge qubit and a $RLC$ circuit whose elements have spatial dependence as follows:

$$R_j = R \left( \frac{j}{N} \right)^n, \quad L_j = 0, \quad C_j = C.$$  (53)

Here, $n$ is a positive real number. In Fig. 7, we show the spectral density, $I_{\text{sub}}(\omega)$, for $n = 2$ and $N = 10^5$. The circuit parameters are set as $R = 50\, \text{m}\Omega$, $L = 0\, \text{H}$, $C = 0.2\, \mu\text{F}$, and $C_0 = 0\, \mu\text{F}$, to which we have added $10\%$ relative randomness (the blue line). As seen in the figure, the present circuit realizes the subohmic reservoir for the frequency range of $\omega^* \ll \omega \ll \omega_c$, where $\omega^*$ and $\omega_c$ are the low- and high-frequency cutoffs, respectively. The randomness in the circuit produces small fluctuations, but does not change the overall feature of the spectral density for the case of no randomness (indicated by the red line).

Detailed derivation is given in Appendix. This formula indicates that the present $RC$ transmission line can realize the subohmic reservoir of $s < 0.5$, because $n$ is a real positive number. This formula is also consistent with Fig. 7, in which the $RC$ circuit of $n = 2$ realizes the subohmic reservoir of $s = 1/(n + 2) = 0.25$. The low- and high-frequency cutoffs are given by

$$\omega^* = \frac{1}{RC} \left( \frac{n}{2N} \right)^2 \left( 1 + \frac{2\sqrt{n}}{n} \right)^{n/2},$$  (51)

$$\omega_c = \frac{1}{RC} \left( \frac{2N}{n} \right)^{n},$$  (52)

respectively, and are shown in Fig. 7. We note that as $n$ decreases, the frequency range in which the spectral density behaves like the subohmic reservoir becomes narrower.

6.3 Subohmic reservoir of $0 < s < 1$

Finally, we briefly discuss how to realize the subohmic spin-boson model with $0 < s < 1$, which includes the region of $0.5 < s < 1$ where the dynamic susceptibility, $\chi(\omega)$, has the nontrivial critical exponent in the quantum critical regime (36). We assume that resistances and inductances depend on the position as follows:

$$R_j = R \left( \frac{j}{N} \right)^n, \quad L_j = L \left( \frac{j}{N} \right)^p, \quad C_j = C.$$  (53)

where $n$ and $p$ are non-negative real numbers. We can derive an analytic expression of the spectral density realized by this circuit as

$$I_{\text{sub}}(\omega) \propto \omega^{2/(p+2)}, \quad (\omega^* \ll \omega \ll \omega_c),$$  (54)

where the low-frequency cutoff, $\omega^*$, is given by

$$\omega^* = \left( \frac{P}{2N} \right)^{2n/(2n+p)} \frac{R^{p+2}}{L^{(p+2)(n-p)}} \left( \frac{1}{(2n+p)} \right)^{(2n+p)/(2n+p)}.$$  (55)

and the high-frequency cutoff, $\omega_c$, is a complex function of the element parameters in the $RLC$ transmission line. Although this circuit can realize a subohmic reservoir of an arbitrary value of $s$ in the range of $0 < s < 1$, we need a very large number of circuit elements, $N \geq 10^5$, to get enough frequency range where the spectral density behaves like the subohmic one.

7. Summary

We investigated quantum critical phenomena in microwave scattering of a superconducting circuit theoretically. We considered a system composed of a charge qubit and a $RLC$ cir-
cuit, which is effectively described by the subohmic spin-boson model, and studied a reflection coefficient of a microwave entered from a transmission line. By performing numerical calculation with the continuous-time Monte Carlo method, we clarified the frequency dependence of the dynamic susceptibility at and far from quantum phase transition. We also clarified how quantum critical behavior appears in the frequency dependence of a microwave loss. We proposed three types of superconducting circuits to realize the subohmic spin-boson model, and derived detailed conditions for observing quantum critical phenomena. Our study is expected to provide a hopeful experimental platform for investigating quantum critical phenomena in a controlled way.

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Appendix: Analytic Expression of the Spectral Density

In this appendix, we derive the analytic form of the spectral density, Eq. (50), in the circuit model discussed in Sect. 6.2. Assuming $|\omega C_1 Z_{c2}|(\omega) \ll 1$, the recurrence relation (47) is rewritten as the following differential equation in the limit $N \to \infty$:

$$\frac{\partial Z(\omega, x)}{\partial x} = -r(x) + i\omega l(x) + i\omega c(x) Z(\omega, x)^2,$$

(A-1)

where $Z(\omega, x = j/N) \equiv Z_{c1}(\omega)$, and $r(x), l(x)$, and $c(x)$ are the resistance, capacitance and inductance per unit length at $x = j/N$, respectively. The total impedance of the RLC circuit is obtained by $Z_c(\omega) = Z(\omega, x \to 0)$. The spatial dependence of the circuit elements given in Eq. (49) is rewritten as

$$r(x) = rB^n, \quad l(x) = 0, \quad c(x) = c.$$  \hspace{1cm} (A-2)

For $x \gg x' \equiv (n^2/4\omega c r)_{1/(\nu+2)}$, the impedance is given by

$$Z_A(\omega, x) = \sqrt{r \omega c x^{\nu/2}},$$  \hspace{1cm} (A-3)

because $\partial_x Z(\omega, x)$ is sufficiently small in comparison with other terms. In contrast, for $x \ll x'$, since the first term in the right side of Eq. (A-1), $rB^n$, can be neglected, the impedance is expressed as

$$Z_B(\omega, x) = \frac{1}{-i\omega c x + A(\omega)},$$  \hspace{1cm} (A-4)

where $A(\omega)$ is the constant of integration determined from the condition $Z_A(\omega, x') = Z_B(\omega, x')$. Finally, we obtain the impedance of the RLC circuit as

$$Z_L(\omega) \sim Z_B(\omega, x \to 0) = \frac{n}{\sqrt{2\omega c}} \left(\frac{4\omega c r}{n^2}\right)^{1/(\nu+2)} \left(1 + i\frac{n + \sqrt{2}}{\sqrt{2}}\right)^{-1}.$$  \hspace{1cm} (A-5)

for $\omega^* \ll \omega \ll \omega_c$. Therefore, the spectral density is obtained as

$$I_{\text{sub}}(\omega) \propto \omega \text{Re}[Z_B(\omega)] \propto \omega \text{Re}[Z_c(\omega)] \propto \omega^{1/(\nu+2)}. $$  \hspace{1cm} (A-6)

This expression shows the subohmic spectral density with $s < 0.5$, which corresponds to Eq. (49). Note that the high-frequency cutoff, $\omega_c$, is obtained by the condition, $\omega C[Z_c(\omega, x^*)] \ll 1$, where the recurrence relation (47) can be reduced to the differential equation (A-1) for $x < x^*$. The imaginary part of the impedance shows the sharp peak for $x \approx 1$. In the above analysis, we have neglected it for $x > x^*$, leading to the condition $\omega(1 - x^*)|Z_c(\omega, x^*)| \gg 1$, by which the low-frequency cutoff $\omega^*$ is determined.
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