The Peter-Weyl Theorem for SU(1|1)

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Abstract

We study a generalization of the results in \cite{4} to the case of SU(1|1) interpreted as the supercircle $S^{1|2}$. We describe all of its finite dimensional complex irreducible representations, we give a reducibility result for representations not containing the trivial character, and we compute explicitly the corresponding matrix elements. In the end we give the Peter-Weyl theorem for $S^{1|2}$.

1 Introduction

The theory of representations of compact supergroups has not yet been fully understood and in particular there is neither a decisive classification result in this category, nor a thorough treatment of the fundamental results, as for example the Peter-Weyl theorem. In this paper we want to proceed and give another important example, beside the one already studied in \cite{4}, namely we want to fully discuss the case of SU(1|1). This is a natural generalization of the $S^{1|1}$ case studied in \cite{4} and \cite{14}: the reduced group is still $S^1$ in both cases, but here we are considering two odd variables, so we may very well call SU(1|1), $S^{1|2}$ the supercircle with two odd dimensions. This generalization is non trivial, because it is well known that as soon as the odd dimension becomes greater than one, new supersymmetric phenomena may appear to make the whole theory diverge significantly from the ordinary one, though
in this particular case, it does not happen. For example in [9], we see that, Aut([P^{1|1}]), the automorphism group of the projective superspace in one odd dimension coincides with the projective linear supergroup, but this isomorphism is lost when the odd dimension is greater than one. In case of odd dimension one, there is a strong connection between Aut([P^{1|1}]) and the theory of SUSY curves (see also [6], [8]). In [4], the theory of SUSY curves was linked with real forms of \((\mathbb{C}^{1|1})^\times\) and it was proven the remarkable result that the real forms reducing to \(S^1\) are actually all isomorphic to \(S^1\). When the odd dimension is greater than one, the theory departs significantly from the ordinary one, the automorphism group of \([P^{1|n}]\) is not the projective linear supergroup and the connection with SUSY curves and \(S^{1|n}\) is either lost or not immediately evident.

This is our main motivation for examining the case SU(1|1): we want to see if new phenomena arise and if, in the future, we can try to establish a connection with SUSY curves and shed light on this part of mathematics still very actively studied (see [6] and [16]).

Our paper is organized as follows. In Section 2 we recall very briefly our notation and few facts about real forms both in the sheaf theoretic and Super Harish-Chandra pair approach to supergroups. In Section 3 we completely classify the finite-dimensional irreducible complex representations of the supergroup \(S^{1|1}\), and we also give a reducibility result for representations not containing the trivial character. This section completes the study initiated in [4]. In Section 4 we classify the representations of SU(1|1) and we prove the super version of the Peter-Weyl Theorem.

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2 Supergeometry preliminaries

We would like to quickly summarize few definitions and key facts about supergeometry especially to establish our notation; for all the details see [2], [5], [14] and [10], besides our main reference [4].

Let us take our ground field \(k = \mathbb{R}\) or \(\mathbb{C}\).
Definition 2.1. A supspace $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ together with a sheaf of superalgebras $\mathcal{O}_S$, such that the stalk $\mathcal{O}_{S,x}$ is a local superalgebra, for $x \in |S|$. A morphism $\phi : S \rightarrow T$ of superspaces is given by $\phi = (|\phi|, \phi^*)$, where $\phi : |S| \rightarrow |T|$ is a map of topological spaces and $\phi^* : \mathcal{O}_T \rightarrow \phi_* \mathcal{O}_S$ is a local sheaf morphism. A differentiable (resp. analytic) supermanifold of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ where $|M|$ is manifold and $\mathcal{O}_M$ is a sheaf of superalgebras over $\mathbb{R}$ (resp. $\mathbb{C}$), which is locally isomorphic to $\mathbb{R}^{p|q}$ (resp. $\mathbb{C}^{p|q}$), where $\mathbb{R}^{p|q} = (\mathbb{R}^p, C^\infty_\mathbb{R} \otimes \Lambda(\xi_1, \ldots, \xi_q))$ and similarly for $\mathbb{C}^{p|q}$.

Let $M$ and $T$ be supermanifolds. A $T$-point of a supermanifold $M$ is a morphism $T \rightarrow M$ ($T \in \text{(smflds)}$). We denote the set of all $T$-points by $M(T)$. We define the functor of points of $M$:

$$M : \text{(smflds)}^0 \rightarrow \text{(sets)}, \quad T \mapsto M(T), \quad M(\phi)(\psi) = \psi \circ \phi, \quad (1)$$

where (smflds) denotes the category of supermanifolds and the index $0$ as usual refers to the opposite category. $\psi$ here is a $T$-point of $M$, i.e. $\psi : T \rightarrow M$. We shall write (smflds)$_\mathbb{R}$ or (smflds)$_\mathbb{C}$ whenever it is necessary to distinguish between real or complex supermanifolds.

We want to define the real supermanifold underlying a complex supermanifold and the concept of real form.

Definition 2.2. Let $M = (|M|, \mathcal{O}_M)$ be a complex supermanifold. We define the complex conjugate $\overline{M}$ of $M$ as the complex supermanifold $\overline{M} = (|M|, \mathcal{O}_M)$, where $\mathcal{O}_M$ is $\mathcal{O}_M$ with the $\mathbb{C}$-antilinear structure. We shall denote the map realizing the $\mathbb{C}$-antilinear isomorphism between $M$ and $\overline{M}$ as $\sigma : M \rightarrow \overline{M}$ and sometimes we shall write $(\sigma^*)^{-1}(f) = \overline{f}$, where $f$ is a section of the sheaf $\mathcal{O}_M$. We define a real structure on $M$ as an involutive isomorphism of ringed spaces $\rho : M \rightarrow \overline{M}$, which is $\mathbb{C}$-linear on the sheaves, and such that $\rho^* : \mathcal{O}_M \rightarrow \rho_* \mathcal{O}_M$ is a $\mathbb{C}$-linear sheaf isomorphism. We further define the isomorphism of ringed superspaces $\psi = \sigma^{-1} \circ \rho : M \rightarrow M$, which is $\mathbb{C}$-antilinear on the sheaves $\psi^* = \rho^* \circ (\sigma^*)^{-1} : \mathcal{O}_M \rightarrow \rho_* \mathcal{O}_M$.

The superspace $M^\rho = (|M|^\rho, \mathcal{O}_{M^\rho})$, where $|M|^\rho$ consists of the fixed points of $\rho$ and $\mathcal{O}_{M^\rho}$ consists of sections $f \in \mathcal{O}_M|_{M^\rho}$ such that $\psi^*(f) = f$, is the real form of $M$ defined by $\rho$.

If $G$ is a complex supergroup and $\rho$ is a supergroup morphism, then $G^\rho$ is a real supergroup.
Through the notion of real form it is possible to define the concept of real underlying supermanifold, which is mostly important for us.

**Definition 2.3.** We define on $M \times \overline{M}$ the real structure $\tau : M \times \overline{M} \rightarrow \overline{M} \times M$, $\tau((x, y)) = (y, x)$ and $\tau^* : \mathcal{O}_{M \times \overline{M}} \rightarrow \tau_* \mathcal{O}_{M \times \overline{M}}$ given by:

$$\tau(f \otimes g) = (-1)^{|f||g|} g \otimes f \quad f \in \mathcal{O}_{M}, \ g \in \mathcal{O}_{\overline{M}}.$$  \hspace{1cm} (2)

We call $M^r$ the **real supermanifold underlying $M$** and we denote it with $M_R$.

The next example is very instructive and essential for our treatment.

**Example 2.4.** We want to understand $\mathbb{C}^{m|n}_R$ the real supermanifold underlying $\mathbb{C}^{m|n}$. We define $|\tau| : |\mathbb{C}^{m|n}| \times |\mathbb{C}^{m|n}| \rightarrow |\mathbb{C}^{m|n}|$ as $|\tau|(p, q) = (q, p)$, while on the sheaves we define the $\mathbb{C}$-linear isomorphism $\tau^* : O_{\mathbb{C}^{m|n} \times \mathbb{C}^{m|n}} \rightarrow \tau_* O_{\mathbb{C}^{m|n} \times \mathbb{C}^{m|n}}$, $\tau^*(w_i) = z_i$, $\tau^*(\eta_j) = \zeta_j$, $\tau^*(z_i) = w_i$, $\tau^*(\zeta_j) = \eta_j$, where $(z_i, \zeta_j)$ and $(w_i, \eta_j)$ are global coordinates on $\mathbb{C}^{m|n}$ and $\overline{\mathbb{C}}^{m|n}$ respectively (with a common abuse of notation, we write $z_i \in O_{\mathbb{C}^{m|n} \times \mathbb{C}^{m|n}}$ in place of the more appropriate $1 \otimes z_i$ and similarly for the rest of the coordinates). We associate to $\tau^*$ (see (2)) the $\mathbb{C}$-antilinear isomorphism $\psi^* = \tau^* \circ ((\sigma^*)^{-1} \otimes \sigma^*)$:

$$\psi^*(w_i) = \overline{z}_i, \ \psi^*(\eta_j) = \overline{\zeta}_j, \ \psi^*(z_i) = \overline{w}_i, \ \psi^*(\zeta_j) = \overline{\eta}_j,$$ \hspace{1cm} (3)

where we write $\overline{z}_i$ instead of $(\sigma^*)^{-1}(z_i)$ etc. We can then write immediately global coordinates on $\mathbb{C}^{m|n}_R$:

$$x_i = (z_i + \overline{z}_i)/2, \ \ y_i = (z_i - \overline{z}_i)/2i, \ \ \mu_j = (\zeta_j + \overline{\zeta}_j)/2, \ \ \nu_j = (\zeta_j - \overline{\zeta}_j)/2i$$

As for the $T$-points, $T \in (\text{smflds})_R$, we have:

$$\mathbb{C}^{m|n}_R(T)_R = \text{Hom}_{(\text{smflds})_R}(T, \mathbb{C}^{m|n}_R) = \text{Hom}_{(\text{salg})_R}((O(\mathbb{C}^{m|n}_R), \mathcal{O}(T))) =$$

$$= \{ \phi : O(\mathbb{C}^{m|n}_R) \rightarrow \mathcal{O}(T) \} =$$

$$= \{ (t^0, t^1, \ldots, t^m, t^m, \theta^0, \theta^1, \ldots, \theta^n, \theta^n) \mid t^0, t^1, \theta^0, \theta^1 \in \mathcal{O}(T)_1 \}$$

Evidently $\mathbb{C}^{m|n}_R = \mathbb{R}^{2m|2n}$ as one expects.
It is customary to define:

\[ t^k := t^k_0 + i t^k_1 \quad \bar{t}^k := t^k_0 - i t^k_1, \quad \theta^j := \theta^j_0 + i \theta^j_1, \quad \bar{\theta}^j := \theta^j_0 - i \theta^j_1 \]

These are elements in \( \mathbb{C}^m_n(T) \otimes \mathbb{C} \).

Using the language of functor of points and the local coordinates \( t^i, \bar{t}^i, \theta^j, \bar{\theta}^j \), it is then very easy to give a real form of a given supermanifold, by giving a (\( \mathbb{C} \)-antilinear) involution of \( \mathbb{C}^m_n(T) \otimes \mathbb{C} \) functorial in \( T \). For example we can define:

\[ \sigma(t^k) = \bar{t}^k, \quad \sigma(\theta^j) = \bar{\theta}^j \]

(notice that since \( \sigma \) is an involution, it is enough to define the image of \( t^k \) and \( \theta^j \)). We have immediately that \( (\mathbb{C}^m_n(T) \otimes \mathbb{C})^\sigma = \mathbb{R}^m_n(T) \), so that \( \mathbb{R}^m_n(T) \) is the real form of \( \mathbb{C}^m_n(T) \) corresponding to the involution \( \sigma \). There are however deep questions on the foundation of the theory regarding the apparent simplicity of this definition. We invite the reader to consult [4]. For the case of \( \mathbb{C}^m_n(T) \) and its open sets, which is the only case regarding the present work, we do not need the full theory developed in [4] and we can give a real form simply by looking at the fixed points of a given involution on the \( T \)-points.

Another point of view on real forms of Lie supergroups is via the Super Harish-Chandra pairs (SHCP), that is viewing a supergroup \( G \) as a pair \( G = (G_0, \mathfrak{g}) \) consisting of the reduced group \( G_0 \) and the Lie superalgebra \( \mathfrak{g} = \text{Lie}(G) \) (see [2] Ch. 7, [15] and [3] for more details).

**Definition 2.5.** Let \((G_0, \mathfrak{g})\) be a complex analytic SHCP. \((r_0, \rho^r)\) is a **real structure** on \((G_0, \mathfrak{g})\) if

1. \( r_0 : G_0 \rightarrow \overline{G_0} \) is a real structure on the ordinary complex group \( G_0 \), \( G_0^r \) being the real Lie group of fixed points.

2. \( \rho^r : \mathfrak{g} \rightarrow \mathfrak{g} \) is a \( \mathbb{C} \)-antilinear involutive Lie superalgebra morphism, \( \mathfrak{g}^r \simeq \text{Lie}(G_0^r) \) its fixed points.

3. \((r_0, \rho^r)\) are compatible:

\[ \rho^r|_{\mathfrak{g}^{r_0}} \simeq d\psi_0 \quad \text{Ad}(\psi_0(g)) \circ \rho^r = \rho^r \circ \text{Ad}(g) \quad (4) \]

\((G_0^r, \mathfrak{g}^r)\) is called a **real form** of \((G_0, \mathfrak{g})\).

This definition is equivalent to [22] see [4] Sec. 2 for more details.
3 The supergroup $S^{1|1}$ and its representations

In [4] we have classified all finite dimensional complex irreducible representations not containing the trivial representation of the real Lie supergroup $S^{1|1}$. By definition $S^{1|1}$ is the real form of the complex analytic supergroup $(\mathbb{C}^{1|1})^\times$ with respect to the involution:

$$\rho : (\mathbb{C}^{1|1})^\times \to (\mathbb{C}^{1|1})^\times, \quad \rho(w, \eta) = (w^{-1}, iw^{-2}\eta)$$

(5)

We now want to complete the analysis of [4], and compute the representations of $S^{1|1}$, which correspond to the trivial representation of $S^{1|1}$. We recall that in the language of SHCP, in order to define a representation of $S^{1|1}$, we need to specify the action of the reduced group $S^1$ and then of $\text{Lie}(S^{1|1}) = \langle C, Z \rangle$ with $[C, C] = [C, Z] = 0$, $[Z, Z] = -2C$.

Let $W$ be a 1$|1$ dimensional super vector space, with homogeneous basis $w_0, w_1$. We define then a representation of $S^{1|1}$ on $W$ by letting $S^1$ act trivially on $W$, and letting $Z$ act by:

$$Z \cdot w_0 = 0, \quad Z \cdot w_1 = w_0.$$  

(6)

Obviously, we may define the parity-reverse of this representation, i.e. again letting $S^1$ act trivially on $W$, but this time setting:

$$Z \cdot w_0 = w_1, \quad Z \cdot w_1 = 0.$$  

(7)

The representations $W$ and $\Pi W$ are clearly indecomposable, but not semisimple. They are also non-isomorphic, since $\ker(Z)$ is 1$|0$-dimensional in the first case and 0$|1$-dimensional in the second one. Such representations will play a key role in the classification of weight zero representations. One can readily check they are $\text{Ad}(S^{1|1})$ or $\Pi \text{Ad}(S^{1|1})$.

**Proposition 3.1.** Let $(\pi, \rho, V)$ be a finite-dimensional $S^{1|1}$-representation on which the reduced group acts with weight 0. Then $(\pi, \rho, V)$ is isomorphic to $U \oplus \bigoplus_i W_i$, where $U$ is a trivial $S^{1|1}$ representation, and each $W_i$ is isomorphic to $\text{Ad}(S^{1|1})$ or $\Pi \text{Ad}(S^{1|1})$.

**Proof.** As before, the $S^1$-action commutes with the action of $Z$. So let us abuse notation and write $Z$ for $\rho(Z)$. We have $Z^2 = 0$. Let us view $Z$ as an ungraded matrix. Since the characteristic polynomial of $Z$ is quadratic,
the Jordan canonical form of $Z$ implies there is a basis of $V$ (not necessarily homogeneous) in which the matrix of $Z$ has the block form

$$
\begin{pmatrix}
0 & 0 \\
0 & J
\end{pmatrix}
$$

where $J$ is a block-diagonal matrix whose blocks are all of the form

$$
J_2 = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
$$

Hence we have a not-necessarily homogeneous basis $w_i, z_j$ of $V$ such that $Z(w_i) = z_i, i = 1, \ldots, k,$ and $z_j, j = 1, \ldots, l$ are a basis of ker($Z$), for some $k$ and $l$, $k \leq l$. Taking homogeneous components of the $w_i$, we find $Z((w_i)_{\alpha}) = (z_i)_{\alpha + 1}$ for each $i$. (Some of these homogeneous components may be zero, in which case we discard them).

Then each nonzero pair $(w_i)_{\alpha}, (z_i)_{\alpha + 1}$ is a basis of a 1|1 dimensional subspace $W_i$ such that $Z((w_i)_{\alpha}) = (z_i)_{\alpha + 1}$ and $Z((z_i)_{\alpha + 1}) = 0$, i.e. $W_i$ is a representation of the type discussed above (see (6) and (7)). The nonzero homogeneous components of the remaining $z_j$ (those not in the image of $Z$) are a basis for a subspace $U$ of ker($Z$). It is clear that $V = U \oplus \bigoplus_i W_i$ and that this splitting is $Z$-invariant, whence the theorem.

In order to complete the analysis of the irreducible finite dimensional complex representations of $S^{1|1}$ we recall the following definition from [4]. For each integer $m \neq 0$, define the super weight space $V_m$ to be a 1|1 dimensional super vector space, with homogeneous basis $v_0, v_1$ on which the reduced group $S^1$ acts through the character $t \mapsto t^m$, so that infinitesimally we have

$$
Z \cdot v_0 = \sqrt{-im}v_1, \quad \quad Z \cdot v_1 = \sqrt{-im}v_0.
$$

It is easily checked that the super weight spaces $V_m$ are irreducible.

Combining the previous proposition with the semisimplicity result for super weight spaces with nonzero $m$, (see [4] Theorem 6.1), we have a complete structure theorem for all $S^{1|1}$ representations for which the reduced weight spaces are finite-dimensional, showing that all such representations decompose into a direct sum of super weight spaces and the subspace corresponding to the zero eigenvalue of $C$ whose structure is described in Prop. 3.1.

More precisely, they split into a semisimple part (namely, $\bigoplus_i V_i \oplus U$) and a part which is a direct sum of the indecomposable, non-simple representations $\text{Ad}(S^{1|1})$ and $\Pi\text{Ad}(S^{1|1})$. 

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Theorem 3.2. Let \((\pi, \rho, V)\) be an \(S^{1|1}\) representation such that the weight spaces \(V_m := \{v \in V : t \cdot v = t^m v\}\) for the reduced group \(S^1\) are all finite-dimensional. Then \((\pi, \rho, V)\) is isomorphic to \(\bigoplus_i V_i \oplus U \oplus \bigoplus_j W_j\), where the \(V_i\) are super weight spaces (repeated according to their multiplicities), \(U\) is a trivial representation, and each \(W_j\) is isomorphic to \(\text{Ad}(S^{1|1})\) or \(\Pi\text{Ad}(S^{1|1})\).

As a corollary we are able to prove the Peter-Weyl theorem regarding the matrix elements of \(S^{1|1}\) representations. If \(\sigma : G \rightarrow \text{GL}(m|n)\) is a representation of the compact Lie supergroup \(G\), we define the matrix element as the section in \(\mathcal{O}(G)\) corresponding, via the Chart theorem, to the morphism

\[
a_{ij} : G(T) \mapsto C^{1|1}(T)
g \mapsto \sigma(g)_{ij}
\]

We have the following result (see [4] Theorem 6.3).

**Theorem 3.3.** The super Peter-Weyl theorem for \(S^{1|1}\). The complex linear span of the matrix coefficients of the representations \(V_i\)’s (see Theorem 3.2) and of the adjoint representation is dense in \(\mathcal{O}(S^{1|1}) \otimes \mathbb{C}\).

4 The supergroup \(SU(1|1)\), its representations and the Peter-Weyl theorem

In this section we study the special unitary supergroup \(SU(1|1)\) (see [7] for its definition and main properties). Since its reduced group is still \(S^1\), as in the case of \(S^{1|1}\) studied in the previous section, we also may refer to \(SU(1|1)\) as \(S^{1|2}\), that is the supercircle in two odd dimensions. We want to view \(SU(1|1)\) as a real form of the special linear supergroup \(\text{SL}(1|1) \cong (\mathbb{C}^{1|2})^\times\) and we shall achieve this, by providing an involution \(\sigma\) of \(\text{SL}(1|1)_{\mathbb{R}}(T)\) functorial in \(T \in (\text{smflds})_{\mathbb{R}}\) as we did in Example 2.4.

We define, following [7], \(\sigma : \text{SL}(1|1)_{\mathbb{R}}(T) \rightarrow \text{SL}(1|1)_{\mathbb{R}}(T)\)

\[
\sigma \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} d^{-1} & -i\alpha^{-1}\beta \\ -i\alpha^{-1}\beta & \alpha^{-1} \end{pmatrix}
\]

where \(\text{Ber} \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = d^{-1}(a - \beta d^{-1} \alpha) = 1\).

This yields immediately the supergroup:

\[
SU(1|1)(T) = \left\{ \begin{pmatrix} a & \beta \\ -i\beta a^2 & \alpha^{-1} \end{pmatrix} \mid a\beta(1 + i\beta^{-1}) = 1 \right\}
\]
Notice that the relation $a\overline{a}(1 + i\beta\overline{\beta}) = 1$ is effectively the condition of berezinian equal to 1.

SU(1|1) has dimension 1|2 and its Lie superalgebra is:

$$su(1|1) = \left\{ \begin{pmatrix} ix & z \\ -ix & i\overline{z} \end{pmatrix} \right\}$$

$$= \text{span}\left\{ C = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, U = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} \right\}$$

The commutation relations in su(1|1) are easily seen to be

$$[C, U] = [C, S] = [U, S] = 0, \quad [U, U] = [S, S] = -2C \quad (8)$$

We refer to [14] pg 112 for more details. In this work, Varadarajan classifies all real forms of sl(m|n) = Lie(SL(m|n)) and shows they are all of the type su(p, q|r, s)±. The ± refer to the fact, that for each (p, q|r, s), m = p + q, n = r + s, we have two different non isomorphic real forms su(p, q|r, s)±, called isomers, once we consider isomorphisms fixing a given real form of the even part. What is surprising is the fact that in the physical applications, these two non isomorphic real forms will give rise to different physical fields, which however are associated to the same physics. Thus for such applications it is irrelevant which form one actually chooses. For more details see [10] Ch. 4.

As far as we are concerned, we are considering the isomer su(1|1)+ in Varadarajan’s notation. As for the isomer su(1|1)−, the corresponding special unitary supergroup is

$$SU(1|1)_{-}(T) = \left\{ \begin{pmatrix} a & i\beta \\ \beta a^2 & \overline{\alpha} \end{pmatrix} \right\} \text{ such that } a\overline{a}(1 - i\beta\overline{\beta}) = 1$$

We are now interested to the theory of representations of such compact supergroups. For clarity of exposition we shall discuss just SU(1|1), with Lie superalgebra su(1|1) = su(1|1)+ leaving to the reader the easy modifications to obtain the representations of SU(1|1)−. To study such representations we shall resort to the theory of Super Harish-Chandra pairs (see [2], [4]). We start by noticing that, similarly to the situation of $S^{1}$, (see Sec. 3 and [4]), the reduced part of SU(1|1) is SU(1|1)$_{\text{red}} = S^{1}$, hence its irreducible representations are all one dimensional and parametrized by the integers.
In analogy with the $S^{1|1}$ case, we introduce the following super weights spaces. Let $m$ be a nonzero integer (the case $m = 0$ being similar to the $S^{1|1}$ setting). Let $\pi^+_m$ be the representation of $\text{SU}(1|1)$ acting in $W^+_m \simeq \mathbb{C}^{1|1}$, whose differentiated action is defined (with a slight abuse of notation) according to

$$
C = \begin{pmatrix} im & 0 \\ 0 & im \end{pmatrix}, \quad U = \begin{pmatrix} 0 & \sqrt{-im} \\ \sqrt{-im} & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & \frac{+m}{\sqrt{-im}} \\ \frac{+m}{\sqrt{-im}} & 0 \end{pmatrix}
$$

Observation 4.1. The representations $\pi^+_m$ and $\pi^-_k$ are inequivalent for each nonzero integer $m, k$. The case $m \neq k$ is obvious, so we only need consider $m = k$. Suppose $F : \pi^+_m \rightarrow \pi^-_m$ is an isomorphism of representations. Let $u^+, v^+$ (resp. $u^-, v^-$) be homogeneous bases of $\pi^+_m$ (resp. $\pi^-_m$) such that $C, S, U$ act by the above matrices. We abuse notation by denoting the matrix of $F$ with respect to these bases by $F$. Since the transformation $F$ is even, the matrix $F$ has the form $F = \text{diag}(a, b)$, for some invertible scalars $a, b$. One sees by direct calculation that the fact that $F$ intertwines the actions of $S$ implies $a = -b$, but the fact that $F$ intertwines the actions of $U$ implies $a = b$, whence $a = b = 0$. This contradicts the assumption that $F$ is an isomorphism.

We have the following result.

**Theorem 4.2.** Let $(\pi, \rho, V)$ be an $\text{SU}(1|1)$ representation such that the weight spaces $V_m := \{ v \in V : t \cdot v = t^m v \}$ for the reduced group $S^1$ are all finite-dimensional. Then $(\pi, \rho, V)$ is isomorphic to $V_0 \bigoplus_m W^+_m \bigoplus_k W^-_k$, where the $W^\pm_m$ are super weight spaces (repeated according to their multiplicities).

**Proof.** Since $S^1$ is compact we have

$$V = \bigoplus_{m \in \mathbb{Z}} V_m$$

where $m$ are the integers parametrizing the characters $\theta \mapsto e^{im\theta}$. Notice that the isotypic spaces $V_m$ are the eigenspaces of the operator $C$.

It follows from the commutation relations (8) that $U$ and $S$ preserve the isotypic decomposition of $V$. Hence, from now on, we can assume $V = V_m$, for some nonzero $m \in \mathbb{Z}$.

We first notice that, using the commutation relations (8),

$$U^2 = S^2 = -im$$

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Hence both $U$ and $S$ are diagonalizable with nonzero eigenvalues. Let $w$ be an eigenvector of $U$ with (nonzero) eigenvalue $\lambda (= \pm \sqrt{-im})$. Then, if $w = w_0 + w_1$ denotes its decomposition into homogeneous components, we must have

$$Uw_0 = \lambda w_1 \quad Uw_1 = \lambda w_0 \quad (9)$$

Applying this fact to a basis of eigenvectors of $U$, $u_j = z_j + \zeta_j$, $z_j$ even and $\zeta_j$ odd, we obtain an homogeneous basis of $V$

$$z_1, \ldots, z_n | \zeta_1, \ldots \zeta_n$$

Hence $V$ has dimension $n|n$, for a suitable $n$, corresponding to the multiplicity of the $V_m$ representation.

We now notice that

$$(US)^2 = -C^2 = m^2 \mathbb{I}$$

for some nonzero $m$. Hence $US$ is diagonalizable. Moreover, since it is even, we can assume that its eigenvectors are homogeneous. Let $\{f_j\}$ denote a basis of such eigenvectors for the even part of $V$ with eigenvalues $\lambda \in \{\pm m\}$. We want to show that: $Uf_j$ is an odd eigenvector for $US$, so that the subspace

$$W = \text{span}_C\{f_j, Uf_j\}$$

is su(1|1)-stable. Since (again from (8)) $U^2 = -im$, we have that $Uf_j$ is nonzero. Moreover

$$(US)Uf_j = -U(USf_j)$$

hence $Uf_j$ is an odd eigenvector for $US$. The claim that the subspace $W = \text{span}_C\{f_j, Uf_j\}$ is invariant follows again from (8) and in particular from:

$$U^2 = S^2 = -im, \quad (US)^2 = -C^2 \quad (10)$$

and

$$U^2Sf_j = \lambda Uf_j \quad \Leftrightarrow \quad -CSf_j = \lambda Uf_j \quad \Leftrightarrow \quad -imSf_j = \lambda Uf_j$$

which gives

$$Sf_j = \frac{i\lambda}{m}Uf_j$$

Using the basis for $W$:

$$f_j, \quad \phi_j = \frac{Uf_j}{\sqrt{-im}}$$
the explicit action of \( su(1|1) \) is obtained as follows. The action of \( C \) is
\[
C(f_j) = im f_j, \quad C(\phi_j) = im \phi_j
\]
The matrix representing \( C \) in the given basis is then:
\[
C = \begin{pmatrix}
im & 0 \\
0 & im
\end{pmatrix}
\]
The action of \( U \) is given by
\[
U(f_j) = \sqrt{-im} \frac{Uf_j}{\sqrt{-im}} = \sqrt{-im} \phi_j
\]
\[
U(\phi_j) = \frac{U^2 f_j}{\sqrt{-im}} = \frac{-im f_j}{\sqrt{-im}} = \sqrt{-im} f_j
\]
hence
\[
U = \begin{pmatrix}
0 & \sqrt{-im} \\
\sqrt{-im} & 0
\end{pmatrix}
\]
The action of \( S \) is given by
\[
S(f_j) = \frac{\lambda}{\sqrt{-im}} \phi_j
\]
\[
S(\phi_j) = -\frac{US f_j}{\sqrt{-im}} = -\frac{\lambda f_j}{\sqrt{-im}}
\]
hence
\[
S = \begin{pmatrix}
0 & -\frac{\lambda}{\sqrt{-im}} \\
\frac{\lambda}{\sqrt{-im}} & 0
\end{pmatrix}
\]
with \( \lambda \in \{ \pm m \} \).

Next we want to determine the matrix elements for such representations and prove the Peter-Weyl theorem.

**Lemma 4.3.** We can express uniquely any \( T \)-point \( g \in SU(1|1) \) as
\[
g = \text{diag}(t, \bar{t}^{-1})(1 + \theta U)(1 + \eta S),
\]
where \( t, \bar{t} \in \mathcal{O}(T)_0 \otimes \mathbb{C}, \theta, \eta \in \mathcal{O}(T)_1 \otimes \mathbb{C} \).
Proof. We need to show that there are unique $t, \theta, \eta$, such that
\[
\begin{pmatrix}
a & i\beta \\
b & -a^{-1}
\end{pmatrix} =
\begin{pmatrix}
t & 0 \\
0 & T^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & \theta \\
-i\theta & 1
\end{pmatrix}
\begin{pmatrix}
1 & i\eta \\
-\eta & 1
\end{pmatrix}
\]
A direct calculation shows that these are:
\[
t = a(1 - \frac{i}{2}\beta\beta), \quad \theta = \frac{1}{2}(\bar{\beta}a + \beta\bar{a}), \quad \eta = \frac{1}{2}(\bar{\beta}a - \beta\bar{a}).
\]

We finally define the adjoint representation $\text{Ad}(\text{SU}(1|1))$ as the representation acting on $\mathbb{C}^{1|2}$ according to
\[
C = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad U = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

We can now state and prove the Peter-Weyl Theorem for $\text{SU}(1|1)$.

**Theorem 4.4. The super Peter-Weyl theorem for $S^{1|2}$.** The complex linear span of the matrix coefficients of the $\text{SU}(1|1)$ representations $\{(\pi_m^+)^m \in \mathbb{Z}\}$ and of the adjoint representation is dense in $\mathcal{O}(\text{SU}(1|1)) \otimes \mathbb{C}$.

*Proof.* Let $r_m$ is the irreducible representation described in 4.2. We can view $r_m : \text{SU}(1|1) \rightarrow \text{GL}(1|1)$ or alternatively (as in 4.2) as a pair $(r_m^0, \rho_m)$, where $\rho_m$ is a representation of the Lie superalgebra $\text{su}(1|1)$. We observe that $r_m(I + \theta U) = I + \theta \rho_m(U)$ and $r_m(I + \eta S) = I + \eta \rho_m(S)$. Hence we can write using Lemma 4.3 for $g \in \text{SU}(1|1)(T)$:
\[
r_m(g) = r_m \left( \text{diag}(t, \bar{t}^{-1})(1 + \theta U)(1 + \eta S) \right) =
\]
\[
= r_m \begin{pmatrix}
t & 0 \\
0 & \bar{t}^{-1}
\end{pmatrix} (I + \theta \rho_m(U))(I + \eta \rho_m(S)) =
\]
\[
= \begin{pmatrix}
e^{int} & 0 \\
e^{int} & e^{int}
\end{pmatrix}
\begin{pmatrix}
1 & \sqrt{-im\theta} \\
\sqrt{-im\theta} & 1
\end{pmatrix}
\begin{pmatrix}
1 & -i\sqrt{-im\theta} \\
i\sqrt{-im\theta} & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
e^{int}(1 + m\theta \eta) & e^{int}\sqrt{-im(\theta - i\eta)} \\
e^{int}\sqrt{-im(\theta + i\eta)} & e^{int}(1 - m\theta \eta)
\end{pmatrix}
\]

It is then clear that the complex linear span of the matrix coefficients for $m$ arbitrary and the adjoint representation gives $\mathcal{O}(\text{SU}(1|1)) \otimes \mathbb{C}$. 

\[\square\]
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