ANOTHER SIMPLE PROOF OF A THEOREM OF CHANDLER DAVIS

IGOR RIVIN

Abstract. In 1957, Chandler Davis proved that unitarily invariant convex functions on the space of hermitian matrices are precisely those which are convex and symmetrically invariant on the set of diagonal matrices. We give a simple perturbation theoretic proof of this result. (Davis’ argument was also very short, though based on completely different ideas).

Consider an orthogonally invariant function $f$ defined on the set of $n \times n$ symmetric matrices. Such a function has to factor through the spectrum:

$$f(M) = g \circ \lambda(M),$$

where $g$ is a symmetric function:

$$g(\lambda_1, \ldots, \lambda_n) = g(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}),$$

for any permutation $\sigma$.

In the sequel we shall further assume that $f$ is a $C^2$ convex function, and under this assumption we shall show that such functions are precisely those decomposing as per Eq. (1), with convex $g$. The argument and the statement are identical for unitarily invariant functions of Hermitian matrices; in that setting the theorem was proved in [Davis57], by a completely different argument (Davis made no regularity assumption, but this is easily dispensed with (see Remark 1.2)).

To show this, let $M = P + tQ$, and let $\tilde{f}_{P,Q}(t) = f(M)$. It is enough to show that for any symmetric $P, Q$,

$$\frac{d^2 \tilde{f}_{P,Q}(0)}{dt^2} > 0.$$

We compute (dropping the subscript, since from now on $P, Q$ do not vary):

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\[
\frac{d\tilde{f}}{dt} = \sum_{i=1}^{n} \frac{\partial g}{\partial \lambda_i} \dot{\lambda}_i.
\]

(3)

\[
\frac{d^2\tilde{f}}{dt^2} = \sum_{1 \leq i,j \leq n} \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \dot{\lambda}_i \dot{\lambda}_j + \sum_{i=1}^{n} \frac{\partial g}{\partial \lambda_i} \ddot{\lambda}_i.
\]

(4)

The first sum is positive, since it equals \( \dot{\lambda}^t H(g) \dot{\lambda} \), and the Hessian \( H(g) \) is positive definite by assumption. It now suffices to show that the second sum (which can be written as \( \nabla g \cdot \ddot{\lambda} \)) is non-negative.

By continuity, it is sufficient to prove this for \( P \) whose spectrum is simple. By orthogonal invariance, we can compute in a basis where \( P \) is diagonal. According to [Kato80, page 81], in that case

\[
\ddot{\lambda}_i = -\sum_{j \neq i} (\lambda_j - \lambda_i)^{-1} Q_{ij}^2,
\]

and so

\[
\nabla f \cdot \ddot{\lambda} = \sum_{j > i} Q_{ij} Q_{ij} \frac{\partial g}{\partial \lambda_j} \frac{\partial g}{\partial \lambda_i} \frac{\partial g}{\partial \lambda_j} - \frac{\partial g}{\partial \lambda_i} \lambda_j - \lambda_i.
\]

The result then follows from the Lemma below.

**Lemma 0.1.** Let \( f \) be a convex function such that \( f(x, y) = f(y, x) \), where \( y \neq x \). Then

\[
\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} > 0.
\]

**Proof.** The conclusion of the lemma is obviously equivalent to:

\[
\left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) (x - y) > 0.
\]

To show this, consider

\[
h(t) = f((1 - t)x + ty, tx + (1 - t)y).
\]

The function \( h(t) \) is convex, and \( h(0) = h(1) \). This obviously implies that \( \dot{h}(0) \leq 0 \). Since

\[
\dot{h}(0) = \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) (y - x),
\]

the conclusion follows.
Remark 0.2. Since smooth symmetric convex functions are dense in the set of all symmetric functions, we have actually shown the result for all convex symmetric functions, as follows:

- We start with a symmetric locally integrable function $g$ on the set of diagonal matrices.
- The function $g$ can be extended (by orthogonal invariance) to the set of all symmetric matrices, to obtain a function $f$.
- Convolving $f$ with a $C^\infty$ Gaussian approximation $D_n$ to the delta function, then averaging over the orthogonal orbits we obtain a convex orthogonally invariant function $f_n$, such that the $\|f - f_n\| \leq 1/n$, where $\| \|$ denotes the sup-norm. The restriction of $f_n$ to the diagonal matrices is smooth and convex (since convolving with a positive kernel and averaging both preserve convexity) function $g_n$.
- We apply the smooth argument above to $g_n$ and $f_n$, to show that $f_n$ is a convex function.
- Since $f$ is a sup-norm limit of convex functions $f_n$, the result follows.

Remark 0.3. The computation above can also be extended to certain infinite dimensional situations. In particular, it is not hard to see that if $P$ has a compact resolvent and $Q$ is bounded then if the function $g$ is the spectral zeta function or the logarithm of the determinant of the operator. In particular, the regularized log det of the Laplacian on a Riemann surface is convex with respect to bounded perturbations, as are special values of the spectral zeta function. This is very much not true for arbitrary perturbations: in particular, when $P = Q = \Delta$, the convexity log det depends on the sign of the spectral zeta function at 0, which is the same as the sign of the Euler characteristic, so that log det is not convex with respect to the “diagonal” perturbation of the Laplacian whenever the Riemann surface has negative Euler characteristic.

References

[Davis57] C. Davis (1957) On invariant convex functions, Arch. Math, 1957.
[Kato80] T. Kato (1980). Perturbation Theory for Linear Operators, Springer-Verlag Berlin-Heidelberg-New York.

Mathematics Department, Temple University, 1805 N Broad St, Philadelphia, PA 19122

Mathematics Department, Princeton University, Princeton, NJ 08544
E-mail address: rivin@math.temple.edu