MINIMAX AND ADAPTIVE ESTIMATION OF THE WIGNER FUNCTION
IN QUANTUM HOMODYNE TOMOGRAPHY WITH NOISY DATA

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Abstract. In quantum optics, the quantum state of a light beam is represented through the
Wigner function, a density on $\mathbb{R}^2$ which may take negative values but must respect intrinsic
positivity constraints imposed by quantum physics. In the framework of noisy quantum homo-
dyne tomography with efficiency parameter $1/2 < \eta \leq 1$, we study the theoretical performance
of a kernel estimator of the Wigner function. We prove that it is minimax efficient, up to a
logarithmic factor in the sample size, for the $L^\infty$-risk over a class of infinitely differentiable. We
compute also the lower bound for the $L^2$-risk. We construct adaptive estimator, i.e. which does
not depend on the smoothness parameters, and prove that it attains the minimax rates for the
corresponding smoothness class functions. Finite sample behaviour of our adaptive procedure
are explored through numerical experiments.

Keyword : Non-parametric minimax estimation Adaptive estimation Inverse problem $L_2$ and
sup-norm Risk Quantum homodyne tomography Wigner function Radon transform Quantum
state

This paper deals with a severely ill-posed inverse problem which comes from quantum optics. Quantum optics is a branch of quantum mechanics which studies physical systems at the atomic
and subatomic scales. Unlike classical mechanics, the result of a physical measurement is gener-
ally random. Quantum mechanics does not predict a deterministic course of events, but rather
the probabilities of various alternative possible events. It provides predictions on the outcome
measures, therefore explore measurements involve non-trivial statistical methods and inference on
the result of a measurement should be done on identically prepared quantum system.

To understand our statistical model, we start in Section 1 with a short introduction to the needed
quantum notions. Section 2 introduces the statistical model by making the link with quantum
theory. Interested reader can get further acquaintance with quantum concepts through the text-
books or the review articles of Helstrom (1976); Holevo (1982); Barndorff-Nielsen, Gill and Jupp
(2003) and Leonhardt (1997).

1. Physical background

In quantum mechanics, the measurable properties (ex: spin, energy, position, ...) of a quantum
system are called "observables". The probability of obtaining each of the possible outcomes when
measuring an observable is encoded in the quantum state of the considered physical system.

1.1. Quantum state and observable. The mathematical description of the quantum state of
a system is given in form of a density operator $\rho$ on a complex Hilbert space $\mathcal{H}$ (called the space
of states) satisfying the three following conditions:

\begin{enumerate}
\item Self adjoint: $\rho = \rho^*$, where $\rho^*$ is the adjoint of $\rho$.
\item Positive: $\rho \geq 0$, or equivalently $\langle \psi, \rho \psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$.
\item Trace one: $\text{Tr}(\rho) = 1$.
\end{enumerate}

Notice that $\mathcal{D}(\mathcal{H})$ the set of density operator $\rho$ on $\mathcal{H}$ is a convex set. The extreme points of the
convex set $\mathcal{D}(\mathcal{H})$ are called pure states and all others states are called mixed states.


In this paper, the quantum system we are interested in is a monochromatic light in a cavity. In this setting of quantum optics, the space of states $\mathcal{H}$ we are dealing with is the space of square integrable complex valued functions on the real line. A particular orthonormal basis comes with this Hilbert space is the Fock basis $\{\psi_j\}_{j \in \mathbb{N}}$:

$$\psi_j(x) := \frac{1}{\sqrt{\sqrt{\pi}2^j j!}} H_j(x)e^{-x^2/2},$$

where $H_j(x) := (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}$ denote the $j$-th Hermite polynomial. In this basis, a quantum state is described by an infinite density matrix $\rho = [\rho_{j,k}]_{j,k \in \mathbb{N}}$ whose entries are equal to

$$\rho_{j,k} = \langle \psi_j, \rho \psi_k \rangle,$$

with $\langle \cdot, \cdot \rangle$ the inner product. The quantum states which can be created at this moment in laboratory are matrices whose entries are decreasing exponentially to 0, i.e., belong to the natural class $\mathcal{R}(C, B, r)$ defined below, with $r = 2$. Let us define for $C \geq 1$, $B > 0$ and $0 < r \leq 2$, the class $\mathcal{R}(C, B, r)$ is defined as follow

$$\mathcal{R}(C, B, r) := \{ \rho \text{ quantum state} : |\rho_{m,n}| \leq C \exp(-B(m + n)^r/2) \}.$$

An example of density matrix of a pure state whose entries are real is given in Figure 1.

![Figure 1. The density matrix $\rho$ of a coherent-3 state.](image-url)

In order to describe mathematically a measurement performed on an observable of a quantum system prepared in state $\rho$, we give the mathematical description of an observable. An observable $X$ is a self adjoint operator on the same space of states $\mathcal{H}$ and

$$X = \sum_{a} x_a P_a,$$

where the eigenvalues $\{x_a\}_a$ of the observable $X$ are real and $P_a$ is the projection onto the one dimensional space generated by the eigenvector of $X$ corresponding to the eigenvalue $x_a$.

As a quantum state $\rho$ encompasses all the probabilities of the observables of the considered quantum system, when performing a measurement of the observable $X$ of a quantum state $\rho$, the result is a random variable $X$ with values in the set of the eigenvalues of the observable $X$. For a quantum system prepared in state $\rho$, $X$ has the following probability distribution and expectation function

$$P_{\rho}(X = x_a) = \text{Tr}(P_a \rho) \quad \text{and} \quad E_{\rho}(X) = \text{Tr}(X \rho).$$

Note that the conditions defining the density matrix $\rho$ insure that $P_{\rho}$ is a probability distribution. In particular, the characteristic function is given by

$$E_{\rho}(e^{itX}) = \text{Tr}(\rho e^{itX}).$$
1.2. Quantum homodyne tomography and Wigner function. In quantum optics, a monochromatic light in a cavity is described by a quantum harmonic oscillator. In this setting, the observables of interest are usually $Q$ and $P$ (resp. the electric and magnetic fields). But according to Heisenberg’s uncertainty principle, $Q$ and $P$ are non-commuting observables, they may not be simultaneously measurable. Therefore, by performing measurements on $(Q, P)$, we cannot get a probability density of the result $(Q, P)$. However, for all phase $\phi \in [0, \pi]$ we can measure the quadrature observables $X_\phi := Q \cos \phi + P \sin \phi$.

Each of these quadratures could be measured on a laser beam by a technique put in practice for the first time by Smithey and called Quantum Homodyne Tomography (QHT). The theoretical foundation of quantum homodyne tomography was outlined by Vogel and Risken (1989).

**Figure 2.** QHT measurement scheme.

The experimental set-up, described in Figure 2, consists of mixing the signal field with a local oscillator field (LO) of high intensity $|z| >> 1$. The phase $\Phi$ of the LO is chosen s.t. $\Phi \sim U[0, \pi]$. The resulting beam is split by a 50-50 beam splitter, and the photodetectors count the photons in the two output beams by giving integrated currents $I_1$ and $I_2$ proportional to the number of photons. The result of the measurement is produced by taking the difference of the two currents and rescaling it by the intensity $|z|$. In the case of noiseless measurement and for a phase $\Phi = \phi$, the result $X_\phi = \frac{I_2 - I_1}{|z|}$ has density $p_{\rho}(|\cdot|\phi)$ corresponding to measuring $X_\phi$.

In other words, when performing a QHT measurement of the observable $X_\phi$ of the quantum state $\rho$, the result is a random variable $X_\phi$ whose density conditionally to $\Phi = \phi$ is denoted by $p_{\rho}(|\cdot|\phi)$. Its characteristic function is given by

$$E_\rho(e^{itX_\phi}) = \text{Tr}(\rho e^{itX_\phi}) = \text{Tr}(\rho e^{it(Q \cos \phi + P \sin \phi)}) = F_1[p_{\rho}(|\cdot|\phi)](t),$$

where $F_1[p_{\rho}(|\cdot|\phi)](t) = \int e^{itx}p_{\rho}(|\cdot|\phi)dx$ denotes the Fourier transform with respect to the first variable. Moreover if $\Phi$ is chosen uniformly on $[0, \pi]$, the joint density probability of $(X_\phi, \Phi)$ with respect to the Lebesgue measure on $\mathbb{R} \times [0, \pi]$ is

$$p_{\rho}(x, \phi) = \frac{1}{\pi}p_{\rho}(x|\phi)1_{[0,\pi]}(\phi).$$

An equivalent representation for a quantum state $\rho$ is the function $W_\rho : \mathbb{R}^2 \rightarrow \mathbb{R}$ called the Wigner function, introduced for the first time by Wigner (1932). The Wigner function may be obtained from the momentum representation

$$(3) \quad \tilde{W}_\rho(u, v) := F_2[W_\rho](u, v) = \text{Tr} \left( e^{i(uQ + vP)} \right),$$
\( F_2 \) is its Fourier transform with respect to both variables. By applying a change of variables 
\((u, v) \) into \((t \cos \phi, t \sin \phi)\), we get

\[
\tilde{W}_{\rho}(t \cos \phi, t \sin \phi) = F_1[p_{\rho}(\phi)](t) = \text{Tr}(\rho e^{itX_\phi}).
\]  

The origin of the appellation quantum homodyne tomography comes from the fact that the procedure described above is similar to positron emission tomography (PET), where the density of the observations is the Radon transform of the underlying distribution

\[
p_{\rho}(x | \phi) = \mathcal{R}[W_{\rho}](x, \phi) = \int W_{\rho}(x \cos \phi + t \sin \phi, x \sin \phi - t \cos \phi) dt,
\]

where \( \mathcal{R}[W_{\rho}] \) denotes the Radon transform of \( W_{\rho} \). The main difference with PET is that the role of the unknown distribution is played by the Wigner function which can be negative.

The physicists consider the Wigner function as a quasi-probability density of \((Q, P)\) if one can measure simultaneously \((Q, P)\). Nevertheless, the Wigner function does not satisfy all the properties of a conventional probability density but satisfies boundedness properties unavailable for classical densities. For instance, the Wigner function can and normally does go negative for states which have no classical model. The Wigner function is such that

\[
W_{\rho} : \mathbb{R}^2 \to \mathbb{R}, \quad \iint W_{\rho}(q, p) dq dp = 1.
\]

Therefore, the negative part of the Wigner function makes the interpretation in term of density of probability in space phases less intuitive. However, the Radon transform of the Wigner function is always a probability density. Indeed, conditionally to \( \Phi = \phi \) and by applying the change of variables \((q, p)\) into \((x \cos \phi + t \sin \phi, x \sin \phi - t \cos \phi)\), it comes

\[
1 = \iint W_{\rho}(q, p) dq dp
= \iint W_{\rho}(x \cos \phi + t \sin \phi, x \sin \phi - t \cos \phi) dt dx
= \int \mathcal{R}[W_{\rho}](x, \phi) dx = \int p_{\rho}(x | \phi) dx.
\]

Note that the existence of negative values in the function of Wigner can be precisely taken like criterion to discriminate nonclassical states of the field. Figure 3 is the representation of the Wigner function of the vacuum state and the nonclassical one-photon state.

**Figure 3.** The Wigner function of the single photon state (left) and the Wigner function of the vacuum state (right).
In the Fock basis, we can write $W_{\rho}$ in terms of the density matrix $[\rho_{jk}]$ as follows (see Leonhardt (1997) for the details).

$$W_{\rho}(q, p) = \sum_{j,k} \rho_{jk} W_{j,k}(q, p)$$

where for $j \geq k$,

$$W_{j,k}(q, p) = \frac{(-1)^j}{\pi} \left( \frac{k!}{j!} \right)^{1/2} \left( \sqrt{2}(ip - q) \right)^{j-k} e^{-(q^2 + p^2)} L_k^{j-k}(2q^2 + 2p^2).$$

and $L_k^\alpha(x)$ the Laguerre polynomial of degree $k$ and order $\alpha$.

### 1.3. Pattern functions.

The ideal result of the QHT measurement provides $(X_\phi, \Phi)$ of joint probability density with respect to the Lebesgue measure on $\mathbb{R} \times [0, \pi]$ equals to

$$p_\rho(x, \phi) = \frac{1}{\pi} p_\rho(x|\phi) 1_{[0,\pi]}(\phi) = \frac{1}{\pi} \mathcal{R}[W_{\rho}](x, \phi) 1_{[0,\pi]}(\phi)$$

The density $p_\rho(\cdot, \cdot)$ can be written in terms of the entries of the density matrix $\rho$ (see Leonhardt (1997))

$$p_\rho(x, \phi) = \sum_{j,k=0}^{\infty} \rho_{jk} \psi_j(x) \psi_k(x) e^{-(j-k)\phi},$$

where $\{\psi_j\}_{j \in \mathbb{N}}$ is the Fock basis defined in (1). Inversely (see D’Ariano, Macchiavello and Paris (1994); Leonhardt (1997) for details), we can write

$$\rho_{jk} = \int_{0}^{\pi} \int_{0}^{\pi} p_\rho(x, \phi) f_{j,k}(x) e^{(j-k)\phi} dx d\phi,$$

where the functions $f_{j,k}: \mathbb{R} \rightarrow \mathbb{R}$ introduced by Leonhardt, Paul and D’Ariano (1995) are called the "pattern functions". A explicit form of $f_{j,k}(\cdot)$ is given by its Fourier transform by Richter (2000): for all $j \geq k$

$$\tilde{f}_{j,k}(t) = \int_{0}^{\pi} \int_{0}^{\pi} p_\rho(x, \phi) f_{j,k}(x) e^{(j-k)\phi} dx d\phi,$$

where $L_k^\alpha(x)$ denotes the generalized Laguerre polynomial of degree $k$ and order $\alpha$. Note that by Writing $t = ||w|| = ||(q, p)|| = \sqrt{q^2 + p^2}$ in the equation (7), we can define

$$l_{j,k}(t) := ||W_{j,k}(q, p)|| = \frac{2^{j+k}}{\pi} \left( \frac{k!}{j!} \right)^{1/2} t^{j-k} e^{-t^2} \left| L_k^{j-k}(2t^2) \right|.$$ 

Therefore, there exists an useful relation, for all $j \geq k$

$$\left| \tilde{f}_{j,k}(t) \right| = \pi^2 |t| l_{j,k}(t/2).$$

Moreover Aubry, Butucea and Meziani (2009) have given the following Lemma which will be useful to prove our main results.

**Lemma 1** (Aubry, Butucea and Meziani (2009)).

For all $j, k \in \mathbb{N}$ and $J := j + k + 1$, for all $t \geq 0$,

$$l_{j,k}(t) \leq \frac{1}{\pi} \left\{ \begin{array}{ll}
1 - e^{-(t-\sqrt{J})^2} & \text{if } 0 \leq t \leq \sqrt{J}, \\
1 - e^{-(t-\sqrt{J})^2} & \text{if } t \geq \sqrt{J}.
\end{array} \right.$$
2. Statistical model

In practice, when one performs a QHT measurement (see Figure 2), a number of photons fails to be detected. These losses may be quantified by one single coefficient \( \eta \in [0, 1] \), such that \( \eta = 0 \) when there is no detection and \( \eta = 1 \) corresponds to the ideal case (no loss). The quantity \( (1 - \eta) \) represents the proportion of photons which are not detected due to various losses in the measurement process. The parameter \( \eta \) is supposed to be known, as physicists argue, that their machines actually have high detection efficiency, around \( 0.9 = \eta \). In this paper we consider \( \eta \in [1/2, 1] \). Moreover, as the detection process is inefficient, an independent gaussian noise interferes additively with the ideal data \( X_\phi \). Note that the gaussian nature of the noise is imposed by the gaussian nature of the vacuum state which interferes additively (see figure 3).

To resume, for \( \Phi = \phi \), the effective result of the QHT measurement is for a known efficiency \( \eta \in [1/2, 1] \),

\[
Y = \sqrt{\eta} X_\phi + \sqrt{(1 - \eta)/2} \xi
\]

where \( \xi \) is a standard Gaussian random variable, independent of the random variable \( X_\phi \), having density, with respect to the Lebesgue measure on \( \mathbb{R} \times [0, \pi] \), equal to \( p_\rho (\cdot, \cdot) \) defined in equation (8). For the sake of simplicity, we re-parametrize (15) as follow

\[
Z := Y/\sqrt{\eta} = X_\phi + \sqrt{(1 - \eta)/2\eta} \xi := X_\phi + \sqrt{2\gamma} \xi,
\]

where \( \gamma = (1 - \eta)/(4\eta) \) is known and \( \gamma \in [0, 1/4] \). Note that \( \gamma = 0 \) corresponds to the ideal case.

Let us denote by \( p_\rho^\gamma (\cdot, \cdot) \) the density of \( (Z, \Phi) \) which is the convolution of the density of \( X_\phi \) with \( N^\gamma (\cdot) \) the density of a centered Gaussian distribution having variance 2\( \gamma \), that is

\[
p_\rho^\gamma (z, \phi) = \frac{1}{\pi} R[W_\rho] (\cdot, \phi) 1_{[0,\pi]} (\phi) * N^\gamma (z) = p_\rho (\cdot, \phi) * N^\gamma (z)
\]

\[
= \int p_\rho (z - x, \phi) N^\gamma (x) dx.
\]

For \( \Phi = \phi \), a useful equation in the Fourier domain, deduced by the previous relation (17) and equation (4) is

\[
F_1[p_\rho^\gamma (\cdot, \cdot)](t) = F_1[p_\rho (\cdot, \cdot)](t) \tilde{N}^\gamma (t) = \tilde{W}_\rho (t \cos(\phi), t \sin(\phi)) \tilde{N}^\gamma (t),
\]

where \( F_1 \) denotes the Fourier transform with respect to the first variable and the Fourier transform of \( N^\gamma (\cdot) \) is \( \tilde{N}^\gamma (t) = e^{-t^2} \).

This paper aims at reconstructing the Wigner function \( W_\rho \) of a monochromatic light in a cavity prepared in state \( \rho \) from \( n \) observations. As we cannot measure precisely the quantum state in a single experiment, we perform measurements on \( n \) independent identically prepared quantum systems. The measurement carried out on each of the \( n \) systems in state \( \rho \) is done by QHT as described in Section 1. In practice, the results of such experiments would be \( n \) independent identically distributed random variables \( (Z_1, \Phi_1), \ldots, (Z_n, \Phi_n) \) such that

\[
Z_\ell := X_\ell + \sqrt{2\gamma} \xi_\ell.
\]

with values in \( \mathbb{R} \times [0, \pi] \) and distribution \( \mathbb{P}_\rho^\gamma \) having density with respect to the Lebesgue measure on \( \mathbb{R} \times [0, \pi] \) equal to \( p_\rho^\gamma (\cdot, \cdot) \) defined in (17). For all \( \ell = 1, \ldots, n \), the \( \xi_\ell \)'s are independent standard Gaussian random variables, independent of all \( (X_\ell, \Phi_\ell) \).

In order to study the theoretical performance of our different procedures, we use the fact that the unknown Wigner function belong to the class of very smooth functions \( A(\beta, r, L) \) (similar to those of Butucea, Guţă and Artiles (2007); Aubry, Butucea and Meziani (2009)) described via its Fourier transform:

\[
A(\beta, r, L) := \left\{ f : \mathbb{R}^2 \to \mathbb{R}, \int \int |f(u, v)|^2 e^{2\beta ||(u,v)||^r} du dv \leq (2\pi)^2 L \right\},
\]
where $\tilde{f}(\cdot, \cdot)$ denotes the Fourier transform with respect to both variables and $\| (u, v) \| = \sqrt{u^2 + v^2}$ denote the usual Euclidean scalar norm. Note that this class is reasonable from a physical point of view as the class realistic $R(C, B, r)$ of density matrix defined in (2) has been translated in terms of Wigner functions by Aubry, Butucea and Meziani (2009). They prove that the fast decay of the elements of the density matrix implies both rapid decay of the Wigner function and of its Fourier transform.

Outline of the results. The problem of reconstructing the quantum state of a light beam has been extensively studied in physical literature and in quantum statistics. We mention only papers with theoretical analysis of the performance of their estimation procedure. Many other physical papers references can be found therein. Methods for reconstructing a quantum state are based on the estimation of either the density matrix $\rho$ or the Wigner function $W_\rho$. In order to compute the performance of a procedure, a realistic class of quantum states $R(C, B, r)$ has defined in many papers as in (2) in which the elements of the density matrix decrease rapidly. From the physical point of view, all the states which have been produced in the laboratory up to date belong to such a class with $r = 2$, and a more detailed argument can be found in the paper of Butucea, Guţă and Artiles (2007).

The estimation of the density matrix from averages of data has been considered in the framework of ideal detection ($\eta = 1$ i.e. $\gamma = 0$) by Artiles, Gill and Guţă (2005) while the noisy setting as investigated by Aubry, Butucea and Meziani (2009) for the Frobenius - norm risk. More recently in the noisy setting, an adaptive estimation procedure over the classes of quantum states $R(C, B, r)$, i.e. without assuming the knowledge of the regularity parameters, has been proposed by Alquier, Meziani and Peyré (2013) and an upper bound for Frobenius - norm risk has been given. The problem of goodness-of-fit testing in quantum statistics has been considered in Meziani (2008). In this noisy setting, the latter paper derived a testing procedure from a projection-type estimator where the projection is done in $L_2$ distance on some suitably chosen pattern functions.

Note that we may capture some features of the quantum states more easily on the Wigner function $W_\rho$, for instance when this function has significant negative parts, the fact that the quantum state is non classical. Aubry, Butucea and Meziani (2009) translate the class $R(C, B, r)$ in terms of rapid decay of the Fourier transform of its associated Wigner functions as defined in (20) by the class $A(\beta, r, L)$. Over this class with $r = 1$ and for the problem of pointwise estimation of the Wigner function, when no noise is present, we mention the work of Guţă and Artiles (2007). They propose a kernel estimator and derive sharp minimax results over this class. The estimation of a quadratic functional of the Wigner function, as an estimator of the purity, was explored in Meziani (2007).

This paper deals with the problem of reconstruction the Wigner function $W_\rho$ in the context of QHT when taking into account the detection losses occurring in the measurement, leading to an additional Gaussian noise in the measurement data ($\eta \in [1/2, 1]$). The same problem in the noisy setting was treated by Butucea, Guţă and Artiles (2007), they obtain minimax rates for the pointwise risk over the class $A(\beta, r, L)$ for the procedure defined in (21). Moreover, a truncated version of their estimator is proposed by Aubry, Butucea and Meziani (2009) where a upper bounds is computed for the $L_2$ risk over the class $A(\beta, r, L)$. The estimation of a quadratic functional of the Wigner function, as an estimator of the purity, was explored in Meziani (2007).

The remainder of the article is organized as follows. In Section 3, we establish in Theorem 1 the first sup-norm risk upper bound for the estimation procedure (21) of the Wigner function while in Theorem 2 we establish the first minimax lower bounds for the estimation of the Wigner function for the quadratic and the sup-norm risks. These results match our sup-norm upper bounds results up to a logarithmic factor in the sample size $n$.

We propose in Section 4 a Lepski-type procedure that adapts to the unknown smoothness parameters $\beta > 0$ and $r \in [0, 2]$ of the Wigner function of interest. The only previous result on adaptation is due to Butucea, Guţă and Artiles (2007) but concerns the simplest case $r \in [0, 1]$ where the
estimation procedure (21) with a proper choice of the parameter $h$ independent of $\beta, r$ is naturally minimax adaptive up to a logarithmic factor in the sample size $n$. Theoretical investigations are complemented by numerical experiments reported in Section 5. The proofs of the main results are deferred to the Appendix.

3. Wigner function estimation and minimax risk

From now, we work in the practice framework and we assume that $n$ independent identically distributed random pairs $(Z_i, \Phi_i)_{i=1,\ldots,n}$ are observed, where $\Phi_i$ is uniformly distributed in $[0, \pi]$ and the joint density of $(Z_i, \Phi_i)$ is $p^Z_\rho(z, \cdot)$ (see (17)). As Butucea, Guţă and Artiles (2007), we use the modified the usual tomography kernel in order to take into account the additive noise on the observations and construct a kernel $K^\gamma_h$ which performs both deconvolution and inverse Radon transform on our data, asymptotically such that our estimation procedure is

$$
\hat{W}^\gamma_h(q, p) = \frac{1}{2\pi n} \sum_{i=1}^n K^\gamma_h ([z, \Phi_i] - Z_i),
$$

where $0 \leq \gamma < 1/4$ is a fixed parameter $h > 0$ tends to 0 when $n \to \infty$ in a proper way to be chosen later. The kernel is defined by

$$
K^\gamma_h(t) = |t|e^{\gamma t^2}1_{|t| \leq 1/h},
$$

where $z = (q, p)$ and $[z, \phi] = q \cos \phi + p \sin \phi$. From now, $\| \cdot \|_\infty$ and $\| \cdot \|_2$ and $\| \cdot \|_1$ will denote respectively the sup-norm, the $L_2$-norm and the $L_1$-norm. As the sup-norm risk can be trivially bounded as follow

$$
\|\hat{W}^\gamma_h - W_\rho\|_\infty \leq \|\hat{W}^\gamma_h - E[\hat{W}^\gamma_h]\|_\infty + \|E[\hat{W}^\gamma_h] - W_\rho\|_\infty,
$$

and in order to study the sup-norm risk of our procedure $\hat{W}^\gamma_h$, we study in Proposition 1 and 2, respectively the bias term and the stochastic term.

**Proposition 1.** Let $\hat{W}^\gamma_h$ be the estimator of $W_\rho$ defined in (21) and $h > 0$ tends to 0 when $n \to \infty$. Then,

$$
\|E[\hat{W}^\gamma_h] - W_\rho\|_\infty \leq \sqrt{\frac{L}{(2\pi)^2 \beta r}} \frac{h^{(r-2)/2}e^{-\beta h^{-r}}(1 + o(1))},
$$

where $W_\rho \in \mathcal{A}(\beta, r, L)$ defined in (20) and $r \in [0, 2]$.

The proof is deferred to Appendix A.1.

**Proposition 2.** Let $\hat{W}^\gamma_h$ be the estimator of $W_\rho$ defined in (21) and $0 < h < 1$. Then, there exists a constant $C_1$, depending only on $\gamma$ such that

$$
E[\|\hat{W}^\gamma_h - E[\hat{W}^\gamma_h]\|_\infty] \leq C_1 e^{\gamma h^{-2}} \left( \sqrt{\frac{1}{n}} + \frac{1}{n} \right).
$$

Moreover, for any $x > 0$, we have with probability at least $1 - e^{-x}$ that

$$
\|\hat{W}^\gamma_h - E[\hat{W}^\gamma_h]\|_\infty \leq C_2 e^{\gamma h^{-2}} \max \left\{ \sqrt{\frac{1+x}{n}}, \frac{1+x}{n} \right\},
$$

where $C_2 > 0$ depends only on $\gamma$.

The proof is deferred to Appendix A.2. The following Theorem establishes the upper bound of the sup-norm risk.

**Theorem 1.** Assume that $W_\rho$ belongs to the class $\mathcal{A}(\beta, r, L)$ defined in (20) for some $r \in [0, 2]$ and $\beta, L > 0$. Consider the estimator (21) with $h^* = h^*(r)$ such that

$$
\left\{ \begin{array}{ll}
\frac{\gamma}{(\pi n)^2} + \frac{\beta}{(\pi n)^2} = \frac{1}{2} \log(n) & \text{if } 0 < r < 2, \\
\frac{\beta}{(2\log n)^{1/2}} & \text{if } r = 2.
\end{array} \right.
$$

Then,
Then we have
\[
\mathbb{E}\left[\|\hat{W}_h^\gamma - W_p\|_\infty\right] \leq C v_n(r),
\]
where \( C > 0 \) can depend only on \( \gamma, \beta, r, L \) and the rate of convergence \( v_n \) is such that
\[
v_n(r) = \begin{cases} 
(h^*)^{(r-2)/2}e^{-\beta(h^*)^{-r}} & \text{if } 0 < r < 2, \\
n^{-\frac{\beta}{2(r-1)}} & \text{if } r = 2. 
\end{cases}
\]
Note that for \( r \in [0, 2) \) the rate of convergence \( v_n \) is faster than any logarithmic rate in the sample size but slower than any polynomial rate. For \( r = 2 \), the rate of convergence is polynomial in the sample size.

**Proof of Theorem 1:** Taking the expectation in (23) and using Propositions 1 and 2, we get for all \( 0 < h < 1 \)
\[
\mathbb{E}\left[\|\hat{W}_h^\gamma - W_p\|_\infty\right] \leq \mathbb{E}\left[\|\hat{W}_h^\gamma - \mathbb{E}[\hat{W}_h^\gamma]\|_\infty\right] + \|\mathbb{E}[\hat{W}_h^\gamma] - W_p\|_\infty
\]
\[
\leq C e^{\gamma h^2} \sqrt{\frac{1}{n}} (1 + o(1)) + C_B h^{(r-2)/2} e^{-\beta h^{-r}} (1 + o(1))
\]
where \( C_B = \sqrt{\frac{1}{(2\pi)^{2\beta}}}, h \to 0 \) as \( n \to \infty \) and \( W_p \in \mathcal{A}(\beta, r, L) \). The optimal bandwidth parameter \( h^*(r) := h^* \) is such that
\[
h^* = \arg \inf_{h > 0} \left\{ C_B h^{(r-2)/2} e^{-\beta h^{-r}} + C e^{\gamma h^2} \sqrt{\frac{1}{n}} \right\}.
\]
Therefore, by taking derivative, we get
\[
\frac{\gamma}{(h^*)^2} + \frac{\beta}{(h^*)^r} = \frac{1}{2} \log(n) + C(1 + o(1)).
\]
By plugging the result in (28) for \( 0 < r < 2 \) we have
\[
(h^*)^{(r-2)/2} e^{-\beta(h^*)^{-r}} = (h^*)^{(r-2)/2} \frac{1}{\sqrt{n}} e^{\gamma(h^*)^{-2}}.
\]
It comes that the bias term is much larger than the stochastic term for \( 0 < r < 2 \). It is easy to see that for \( r = 2 \), we have \( h^* = \left(\frac{2(\beta + \gamma)}{\log n}\right)^{1/2} \) and that the the bias term and the stochastic term are of the same order.

We derive now a minimax lower bound. We consider specifically the case \( r = 2 \) since it is relevant with quantum physic applications, but our results can easily be generalized to the case \( r \in [0, 2] \). However, similar arguments can be applied to the case \( 0 < r < 2 \). The only known lower bound result for the estimation of a Wigner function is due to Butucea, Guţă and Artiles (2007) and concerns the pointwise risk. In Theorem 2 below, we obtain the first minimax lower bounds for the estimation of a Wigner function \( W_p \in \mathcal{A}(\beta, 2, L) \) with the quadratic and sup-norm risks.

**Theorem 2.** Assume that \((Z_1, \Phi_1), \ldots, (Z_n, \Phi_n)\) coming from the model (16) with \( \gamma \in [0, 1/4] \).
Then, for any \( \beta, L > 0 \) and \( r = 2 \) there exists a constant \( c := c(\beta, L, \gamma) > 0 \) such that for \( n \) large enough
\[
\inf_{\hat{W}_n} \sup_{W_p \in \mathcal{A}(\beta, 2, L)} \mathbb{E}\|\hat{W}_n - W_p\|^p \geq \begin{cases} 
\frac{c n^{-\frac{d}{p+1}} \log^{3/2}(n)}{n^{\frac{\gamma}{2}}} & \text{if } p = \infty, \\
\frac{c n^{-\frac{d}{p+1}} \log^{3/2}(n)}{n^{\frac{\gamma}{2}}} & \text{if } p = 2.
\end{cases}
\]
where the infimum is taken over all possible estimators \( \hat{W}_n \) based on the i.i.d. sample \( \{(Z_i, \Phi_i)\}_{i=1}^n \).

The proof is defered to Appendix B. This theorem guarantees that the sup-norm upper bound derived in Theorem 1 and the quadratic risk upper bound in the paper of Aubry, Butucea and Meziani (2009) are minimax optimal up to a logarithmic factor in the sample size. We believe that the logarithmic factors for both cases are artefact of the proofs.
4. Adaptation to the smoothness

As we see in (28), the optimal choice of the bandwidth $h^*$ depends on the unknown smoothness $\beta$. For any $0 < r < L = 2$, we propose here to implement a Lepski type procedure to select an adaptive bandwidth $h$. We will show that the estimator obtained with this bandwidth achieves the optimal minimax rate up to a logarithmic factor. Our adaptive procedure is implemented in Section 5.

Let $M \geq 2$, and $0 < h_M < \cdots < h_1 < 1$ a grid of $[0, 1]$, we build estimators $\hat{W}_{h_m}^\gamma$ associated to bandwidth $h_m$ for any $1 \leq m \leq M$. For any fixed $x > 0$, let us define $r_n(x) = \max(\sqrt{\frac{1+x}{n}}, \frac{1+x}{n})$.

We denote by $\mathcal{L}_n(\cdot)$, the Lepski functional such that

$$
\mathcal{L}_n(m) = \max_{j > m} \left\{ \| \hat{W}_{h_j} - \hat{W}_{h_m} \|_\infty - 2ke^{\gamma h_{j-2}} r_n(x + \log M) \right\}
$$

(29)

where $\kappa > 0$ is a fixed constant. Therefore, our final adaptive estimator denoted by $\hat{W}_{h_m}^\gamma$ will be the estimator defined in (21) for the bandwidth $h_m$. The bandwidth $h_m$ is such that

$$
m = \arg\min_{1 \leq m \leq M} \mathcal{L}_n(m).
$$

Theorem 3. Assume that $W_\rho \in \mathcal{A}(\beta, r, L)$. Take $\kappa > 0$ sufficiently large and $M \geq 2$. Choose $0 < h_M < \cdots < h_1 < 1$. Then, for the bandwidth $h_m$ with $m$ defined in (30) and for any $x > 0$, we have with probability at least $1 - e^{-x}$

$$
\| \hat{W}_{h_m}^\gamma - W_\rho \|_\infty \leq C \min_{1 \leq m \leq M} \left\{ h_m^{r/2-1} e^{-\frac{\rho e}{h_m}} + e^{\gamma h_m^{-2}} r_n(x + \log M) \right\},
$$

(31)

where $C > 0$ is a constant depending only on $\gamma, \beta, r, L$.

In addition, we have in expectation

$$
\mathbb{E} \left[ \| \hat{W}_{h_m}^\gamma - W_\rho \|_\infty \right] \leq C' \min_{1 \leq m \leq M} \left\{ h_m^{r/2-1} e^{-\frac{\rho e}{h_m}} + e^{\gamma h_m^{-2}} r_n(\log M) \right\},
$$

(32)

where $C' > 0$ is a constant depending only on $\gamma, r, \beta, L$.

The proof is deferred to the Appendix C.

The idea is now to build a sufficiently fine grid $0 < h_M < \cdots < h_1 < 1$ to achieve the optimal rate of convergence simultaneously over $r \in (0, 2]$ and $\beta > 0$. Take $M = \lfloor \sqrt{\log n/(2\gamma)} \rfloor$. We consider the following grid for the bandwidth parameter $h$:

$$
h_1 = 1/2, \quad h_m = \frac{1}{2} \left( 1 - (m - 1) \sqrt{\frac{2\gamma}{\log n}} \right), \quad 1 \leq m \leq M.
$$

(33)

We build the corresponding estimators $\hat{W}_{h_m}^\gamma$ and apply the Lepski procedure (29)-(30) to obtain the estimator $\hat{W}_{h_m}^\gamma$. The next result guarantees that this estimator is minimax adaptive over the class

$$
\Omega := \{ (\beta, r, L), \beta > 0, 0 < r \leq 2, L > 0 \}.
$$

Corollary 1. Let the conditions Theorem 3 be satisfied. Then the estimator $\hat{W}_{h_m}^\gamma$ for the bandwidth $h_m$ with $m$ defined in (30) and for any $(\beta, r, L) \in \Omega$ satisfies

$$
\limsup_{n \to \infty} \sup_{W_\rho \in \mathcal{A}(\beta, r, L)} \mathbb{E} \left[ \| \hat{W}_{h_m}^\gamma - W_\rho \|_\infty \right] \leq C_v_n(r),
$$

where $v_n(r)$ is the rate defined in (27) and $C_v$ is a positive constant depending only on $r, L, \beta$ and $\gamma$.

Proof of Corollary 1: First note that for all $m = 1, \cdots, M$ and as

$$
h_m \in [\gamma/(2 \log n)]^{1/2}, 1/2],
$$

we have with probability at least $1 - e^{-x}$
the bias term $h_m^{r/2 - 1} e^{-\frac{\gamma}{2m}}$ is larger than the stochastic term $e^{\gamma h_m^{-2}} r_n (\log M)$ up to a numerical constant. Let define
\[ \tilde{m} := \arg \max_{1 \leq m \leq M} \{ [h_m - h^*] : h_m \leq h^* \}, \]
where $\tilde{m}$ is well defined as
\[ \frac{h_M}{h^*} = \left( \frac{1}{2} \right) \left( 1 - M(2\gamma / \log n)^{1/2} + (2\gamma / \log n)^{1/2} \right) \left( \frac{\log n}{(2\gamma)} - (\beta / \gamma)(h^* - r)^{-1/2} \right) \]
\[ = \frac{1}{2} \left( 1 + \left( \frac{\log n}{(2\gamma)} \right) \right) \left( 1 - (2\beta / (\log n))(h^* - r)^{-1/2} \right). \]
Moreover, as $0 \leq \left( \frac{\log n}{(2\gamma)} \right)^{1/2} - M \leq 1$ we get
\[ \frac{h_M}{h^*} \leq \left( 1 - (2\beta / (\log n))(h^* - r)^{-1/2} \right) \leq 1. \]
Therefore, from (32),
\[ E \left[ \left\| \hat{W}_{h_m}^\gamma - W_\rho \right\|_\infty \right] \leq C h_m^{r/2 - 1} e^{-\frac{\gamma}{2m}} \leq C h_m^{r/2 - 1} e^{-\frac{\gamma}{2m}} v_n(r) v_n(r)^{-1} \]
\[ = C \left( \frac{h_m}{h^*} \right)^{r/2 - 1} e^{-\beta (h_m - h^*)^{-1/2} + 1} v_n(r). \]
By the definition of $\tilde{m}$, it comes that $h_m \geq (h^*)^{-1}$, then
\[ E \left[ \left\| \hat{W}_{h_m}^\gamma - W_\rho \right\|_\infty \right] \leq C \left( \frac{h_m}{h^*} \right)^{r/2 - 1} v_n(r) = C \left( \frac{h_m}{h^*} + 1 \right)^{r/2 - 1} v_n(r). \]
By construction $|h_m - h^*| \leq (\gamma / (2 \log n))^{1/2}$, then we have
\[ E \left[ \left\| \hat{W}_{h_m}^\gamma - W_\rho \right\|_\infty \right] \leq C \left( 1 - \frac{(\gamma / (2 \log n))^{1/2}}{h^*} \right)^{r/2 - 1} v_n(r). \]
As $(h^*)^{-1} \leq (\log n / (2\gamma))^{1/2}$, it holds $1 - \frac{(\gamma / (2 \log n))^{1/2}}{h^*} \geq 1/2$. Therefore as $r/2 - 1 < 0$, the result follow
\[ E \left[ \left\| \hat{W}_{h_m}^\gamma - W_\rho \right\|_\infty \right] \leq C v_n(r). \]

5. Experimental evaluation

We test our method on two examples of Wigner functions, corresponding to the single-photon and the Schrödinger’s cat states, and that are respectively defined as
\[ W_\rho(q, p) = -(1 - 2(q^2 + p^2)) e^{-q^2 - p^2}, \]
\[ W_\rho(q, p) = \frac{1}{2} e^{-(q - q_0)^2 - p^2} + \frac{1}{2} e^{-(q + q_0)^2 - p^2} + \cos(2q_0p) e^{-q^2 - p^2}. \]
We used $q_0 = 3$ in our numerical tests. The toolbox to reproduce the numerical results of this article is available online\(^1\). Following the paper of Butucea, Guţă and Artiles (2007) and in order to obtain a fast numerical procedure, we implemented the estimator $\hat{W}_n^\gamma$ defined in (21) on a regular grid. More precisely, 2-D functions such as $W_\rho$ are discretized on a fine 2-D grid of 256 × 256 points. We use the Fast Slant Stack Radon transform of Averbuch et al. (2008), which is both fast and faithful to the continuous Radon transform $R$. It also implements a fast pseudo-inverse which accounts for the filtered back projection formula (21). The filtering against the 1-D kernel (22) is computed along the radial rays in the Radon domain using Fast Fourier transforms. We computed the Lepski functional (29) using the values $x = \log(M)$ and $\kappa = 1$.

\(^1\)https://github.com/gpeyre/2015-AOS-AdaptiveWigner
Figure 4. Single photon cat state estimation, with $\eta = 0.9$, $n = 100 \times 10^3$. Left, top: display of $\|\hat{W}^\gamma_h - W_\rho\|_\infty/\|W_\rho\|_\infty$ as a function of $1/\hbar$. The central curve is the mean of this quantity, while the shaded area displays the $\pm 2\times$ standard deviation of this quantity. Left, right: histogram of the empirical repartition of $h_m$ computed by the Lepski procedure (30). Center: display as a 2-D image using level sets of $W_\rho$ (top) and $\hat{W}^\gamma_{h_m}$ (bottom). Right: same, but displayed as an elevation surface.

Figure 5. Schrödinger’s cat state estimation, with $\eta = 0.9$, $n = 500 \times 10^3$. We refer to Figure 4 for the description of the plots.
Figures 4 and 5 reports the numerical results of our method on both test cases. The left part compares the error \( \| \tilde{W}_h^\gamma - W_\rho \|_\infty \) (displayed as a function of \( h \)) to the parameters \( h_{\hat{m}} \) selected by the Lepski procedure (30). The error \( \| \tilde{W}_h^\gamma - W_\rho \|_\infty \) (its empirical mean and its standard deviation) is computed in an “oracle” manner (since for these examples, the Wigner function to estimate \( W_\rho \) is known) using 20 realizations of the sampling for each tested value \( (h_i)^M_{i=1} \). The histogram of values \( h_{\hat{m}} \) is computed by solving (29) for 20 realizations of the sampling. This comparison shows, on both test cases, that the method is able to select a parameter value \( h_{\hat{m}} \) which lies around the optimal parameter value (as indicated by the minimum of the \( L_\infty \) error).

The central and right parts show graphical displays of \( \tilde{W}_{h_{\hat{m}}}^\gamma \), where \( \hat{m} \) is selected using the Lepski procedure (30), for a given sampling realization.

**Appendix A. Proof of Propositions**

A.1. **Proof of Proposition 1.** First remark that by the Fourier transform formula for \( w = (q, p) \in \mathbb{R}^2 \) and \( x = (x_1, x_2) \)

\[
W_\rho(w) = \frac{1}{(2\pi)^2} \iint \tilde{W}_\rho(x)e^{-i(qx_1+px_2)}dx.
\]

(34)

Let \( \tilde{W}_h^\gamma \) be the estimator of \( W_\rho \) defined in (21), then

\[
\mathbb{E} \left[ \tilde{W}_h^\gamma(w) \right] = \frac{1}{2\pi} \mathbb{E} \left[ K_h^\gamma([w, \Phi_1] - Z_1) \right] = \frac{1}{2\pi} \int_0^\pi \int_0^\pi K_h^\gamma([w, \phi] - z)p_h^\gamma(z, \phi)dzd\phi = \frac{1}{2\pi} \int_0^\pi K_h^\gamma * p_h^\gamma(*, \phi)([w, \phi])d\phi.
\]

In the fourier domain, the convolution becomes a product, combining with (18), we obtain

\[
\mathbb{E} \left[ \tilde{W}_h^\gamma(w) \right] = \int_0^\pi \frac{1}{(2\pi)^2} \int \tilde{K}_h^\gamma(t)\mathcal{F}_1[p_h^\gamma(*, \phi)](t)e^{-it[w, \phi]}dt d\phi.
\]

As \( \tilde{K}_h^\gamma(t) = e^{-\gamma^2 t^2} \), the definition (22) of the kernel combining with (18) gives

\[
\mathbb{E} \left[ \tilde{W}_h^\gamma(w) \right] = \int_0^\pi \frac{1}{(2\pi)^2} \int \tilde{K}_h^\gamma(t)\tilde{W}_\rho(t \cos(\phi), t \sin(\phi))\tilde{N}_h^\gamma(t)e^{-it[w, \phi]}dt d\phi = \int_0^\pi \frac{1}{(2\pi)^2} \int_{|t| \leq 1/h} |t|\tilde{W}_\rho(t \cos(\phi), t \sin(\phi))e^{-it[w, \phi]}dt d\phi.
\]

Therefore, by the change of variable \( x = (t \cos(\phi), t \sin(\phi)) \), it comes

\[
\mathbb{E} \left[ \tilde{W}_h^\gamma(w) \right] = \frac{1}{(2\pi)^2} \int_{||x||_2 \leq 1/h} \tilde{W}_\rho(x)e^{-i(qx_1+px_2)}dx.
\]

(35)

From equations (34) and (35), we have

\[
\left| \mathbb{E} \left[ \tilde{W}_h^\gamma(w) \right] - W_\rho(w) \right| \leq \frac{1}{(2\pi)^2} \int_{||x||_2 > 1/h} \left| \tilde{W}_\rho(x) \right| dx \leq \frac{1}{(2\pi)^2} \left[ \int \left| \tilde{W}_\rho(x) \right|^2 e^{2\beta ||x||_2^2} dx \right]^{1/2} \left[ \int_{||x||_2 > 1/h} e^{-2\beta ||x||_2^2} dx \right]^{1/2} \leq \sqrt{\frac{L}{(2\pi)^2}} \frac{h^{(r-2)/2}}{\beta^{r/2}} e^{-\beta h^{-r}}(1 + o(1)), \quad h \to 0
\]

as \( W_\rho \in \mathcal{A}(\beta, r, L) \) the class defined in (20).
A.2. Proof of Proposition 2. The following Lemma is needed to prove the Proposition 2.

**Lemma 2.** Let \( \delta_h := h^{-1} e^{\frac{\gamma}{2}} > 0 \) for any \( 0 < h \leq 1 \), then the class

\[
\mathcal{H}_h = \{ \delta_h^{-1} K_h^n(\cdot - t), t \in \mathbb{R} \}, \quad h > 0
\]

is uniformly bounded by \( U := \frac{h}{2\sqrt{\pi}} \). Moreover, for every \( 0 < \epsilon < A \) and for finite positive constants \( A, \epsilon \) depending only on \( \gamma \),

\[
\sup_Q N(\epsilon, \mathcal{H}_h, L^2(Q)) \leq (A/\epsilon)\epsilon^n,
\]

where the supremum extends over all probability measures \( Q \) on \( \mathbb{R} \).

The proof of this Lemma can be found in D.1. To prove (24), we have to bound the following quantity:

\[
\mathbb{E}[|\tilde{W}_h^n([z, \phi] - Y/\sqrt{n})|^2] \leq \| K_h^n([z, \phi] - Z_l) - \mathbb{E}[K_h^n([z, \phi] - Z_l)] \|^2 \leq \frac{1}{\gamma^2} e^{2\gamma h^2}.
\]

Moreover for \( \delta_h = h^{-1} e^{\gamma h^2} \), we have

\[
\delta_h^{-2} \mathbb{E}[|K_h^n([z, \phi] - Y/\sqrt{n})|^2] \leq \frac{h^2}{\gamma^2}.
\]

By Lemma 2, it comes that the class \( \mathcal{H}_h \) is VC. Hence, we can apply (57) in the paper of Giné and Nickl (2009) to get

\[
\mathbb{E} \left[ \int_{\mathbb{R}} \tilde{W}_h^n(z, \phi) \right] \leq C(\gamma) \frac{\delta_h}{2\pi n} \left( \sqrt{n \log \frac{AU}{\sigma}} + U \log \frac{AU}{\sigma} \right),
\]

where \( U = \frac{h}{2\sqrt{\pi}} \) is the envelop of the class \( \mathcal{H}_h \) defined in Lemma 2. By choosing

\[
\sigma^2 := \frac{h}{\gamma} \geq \sup_{z \in \mathbb{R}^2} \left( \delta_h^{-1} K_h^n([z, \phi] - Y/\sqrt{n}) \right)^2
\]

in (40) we get the result in expectation (24).

To prove the result in probability (25), we use Talagrand’s inequality as in Theorem 2.3 of Bousquet (2002). Let us define

\[
Z := \frac{n\gamma}{h \delta_h} \||\tilde{W}_h^n - \mathbb{E}[\tilde{W}_h^n]||_\infty.
\]

In view of the previous display (38), we have

\[
\text{Var} \left( \gamma(h\delta_h)^{-1} K_h^n(\cdot - Y/\sqrt{n}) - \mathbb{E} \left[ K_h^n(\cdot - Y/\sqrt{n}) \right] \right) \leq \gamma^2 (h\delta_h)^{-2} \mathbb{E} \left[ K_h^n(\cdot - Y/\sqrt{n}) \right]^2 \leq \gamma^2 (h\delta_h)^{-2} \frac{1}{\gamma^2} 2\gamma h^{-2} = 1.
\]

As \( U = \frac{h}{2\sqrt{\pi}} \) and by (D.1), it comes

\[
\gamma(h\delta_h)^{-1} \left\| K_h^n(\cdot - Y/\sqrt{n}) - \mathbb{E}[K_h^n(\cdot - Y/\sqrt{n})] \right\|_\infty \leq \gamma(h\delta_h)^{-1} ||K_h^n||_\infty \leq \gamma h^{-1} U \leq 1.
\]
Then, for any \( x > 0 \) and with probability at least \( 1 - e^{-x} \), we obtain
\[
Z \leq \mathbb{E}[Z] + \sqrt{2x n} + 4x \mathbb{E}[Z] + \frac{x}{3} \leq \mathbb{E}[Z] + \sqrt{2x n} + 2\sqrt{x \mathbb{E}[Z]} + \frac{x}{3}
\]
where we have used the decoupling inequality \( 2ab \leq a^2 + b^2 \) with \( a = \sqrt{x} \) and \( b = \sqrt{\mathbb{E}[Z]} \). Thus, with probability at least \( 1 - e^{-x} \), we get
\[
\| \hat{W}_h^\gamma - \mathbb{E}[\hat{W}_h^\gamma] \|_{\infty} = \frac{\delta h}{n^{\gamma}} Z \leq 2\mathbb{E} \left[ \| \hat{W}_h^\gamma - \mathbb{E}[\hat{W}_h^\gamma] \|_{\infty} \right] + \frac{e^{h^{-2}}}{h} \left( \sqrt{\frac{2}{\pi}} + \frac{4}{3n} \right).
\]
Plugging our control (24) on \( \mathbb{E}[\| \hat{W}_h^\gamma - \mathbb{E}[\hat{W}_h^\gamma] \|_{\infty}] \), the result in probability follows.

**APPENDIX B. PROOF OF THEOREM 2 - LOWER BOUNDS**

**B.1. Proof of Theorem 2 - Lower bounds for the \( \mathbb{L}_2 \)-norm.** The proof for the minimax lower bounds follows a standard scheme for deconvolution problem as in the paper of Butucea, Guţă and Artiles (2007); Lounici, K. and Nickl (2001). However, additional technicalities arise to build a proper set of Wigner functions and then to derive a lower bound. From now on, for the sake of brevity, we will denote \( A(\beta, L) \) by \( A(\beta, L) \) as we consider the practice case \( r = 2 \).

Let \( W_0 \in A(\beta, L) \) be a Wigner function. Its associated density function will be denoted by \( p_0(x, \phi) = \frac{1}{\pi} \mathcal{R}[W_0](x, \phi)1_{[0, \pi]}(\phi) \).

Let \( M = \lfloor \sqrt{\log n} \rfloor \) be the integer part of \( \log n \), and
\[\delta := \log^{-1}(n).\]

We suggest the construction of a family of \( M \) Wigner functions such that for all \( m = 1, \ldots, M \) and \( w \in \mathbb{R}^2 \):
\[W_{m,h}(w) = W_0(w) + V_{m,h}(w), \quad 1 \leq m \leq M,\]
depending on a parameter \( h = h(n) \to 0 \) as \( n \to \infty \). The construction of \( W_0 \) and \( V_{m,h} \) are discussed in Appendix B.1.1 and B.1.2. We denote by
\[p_{m,h}(x, \phi) = \frac{1}{\pi} \mathcal{R}[W_{m,h}](x, \phi)1_{[0, \pi]}(\phi)\]
the associated density function of the Wigner function \( W_{m,h} \). As we consider the noisy framework (16) and in view of (17), we set for all \( 1 \leq m \leq M \)
\[p_{m,h}^\gamma(z, \phi) = [p_{m,h}(z, \phi) + N^\gamma](z) \quad \text{and} \quad p_0^\gamma(z, \phi) = [p_0(\cdot, \phi) + N^\gamma](z).\]

If the following conditions (C1) to (C3) are satisfied, then Theorem 2.6 in the book of Tsybakov (2009) gives the lower bound.

(C1) For all \( m = 1 \cdots M \), \( W_{m,h} \in A(\beta, L) \).

(C2) For any \( 1 \leq k \neq m \leq M \), we have for \( \| W_{k,h} - W_{m,h} \|_2^2 \geq 4 \varphi_n^2 \), with \( \varphi_n^2 = \mathcal{O}(n^{-\frac{\beta}{1+\beta}}) \).

(C3) For all \( 1 \leq m \leq M \),
\[n \mathcal{A}^2(p_{m,h}^\gamma, p_0^\gamma) := n \int_0^{\pi} \frac{(p_{m,h}^\gamma(z, \phi) - p_0^\gamma(z, \phi))^2}{p_0^\gamma(z, \phi)} dz d\phi \leq \frac{M}{4}.
\]

Proofs of these three conditions are done in Appendix B.1.3 to B.1.5.

**B.1.1. Construction of \( W_0 \).** The Wigner function \( W_0 \) is the same as in the paper of Butucea, Guţă and Artiles (2007). For the sake of completeness, we recall its construction here. The probability density function associated to any density matrix \( \rho \) in the ideal noiseless setting is given by equation (9). In particular, for diagonal density matrix \( \rho \), the associated probability density function is
\[p_\rho(x, \phi) = \sum_{k=0}^{\infty} \rho_{kk} \psi_k^2(x).\]
For all $0 < \alpha, \lambda < 1$, we introduce a family of diagonal density matrix $\rho^{\alpha, \lambda}$ such that for all $k \in \mathbb{N}$

\begin{equation}
\rho_{kk}^{\alpha, \lambda} = \int_0^1 z^k \alpha \frac{(1-z)^\alpha}{(1-\lambda)^\alpha} 1_{\lambda \leq z \leq 1} dz.
\end{equation}

Therefore the probability density associated to this diagonal density matrix $\rho^{\alpha, \lambda}$ can be written as follow

\begin{equation}
p_{\alpha, \lambda}(x, \phi) = \sum_{k=0}^{\infty} \rho_{kk}^{\alpha, \lambda} = \sum_{k=0}^{\infty} \psi_k^2(x) \int_0^1 z^k \alpha \frac{(1-z)^\alpha}{(1-\lambda)^\alpha} 1_{\lambda \leq z \leq 1} dz.
\end{equation}

Moreover by the well known Mehler formula (see Erdélyi, Magnus, Oberhettinger and Tricomi (1953)), we have

\[ \sum_{k=0}^{\infty} z^k \psi_k^2(x) = \frac{1}{\sqrt{\pi(1-z^2)}} \exp \left( -x^2 \frac{1-z}{1+z} \right). \]

Then, it comes

\[ p_{\alpha, \lambda}(x, \phi) = \frac{\alpha}{(1-\lambda)^\alpha} \int_0^1 \frac{(1-z)^\alpha}{\sqrt{\pi(1-z^2)}} \exp \left( -x^2 \frac{1-z}{1+z} \right) 1_{\lambda \leq z \leq 1} dz. \]

The following Lemma, proved in the paper of Butucea, Guţă and Artiles (2007), gives a control on the tails of the associated density $p_{\alpha, \lambda}(x, \phi) = p_{\alpha, \lambda}(x)$ as it doesn’t depend on $\phi$.

**Lemma 3 (Butucea, Guţă and Artiles (2007)).** For all $\phi \in [0,1]$ and all $0 < \alpha, \lambda < 1$ and $|x| > 1$ there exist constants $c, C$ depending on $\alpha$ and $\lambda$ such that

\[ c|x|^{-(1+2\alpha)} \leq p_{\alpha, \lambda}(x) \leq C|x|^{-(1+2\alpha)}. \]

In view of Lemma 3 of Butucea, Guţă and Artiles (2007), the Wigner function $W_0$ will be chosen in the set

\[ W^{\alpha, \lambda} = \{ W^{\alpha, \lambda} = W_{\rho_{\alpha, \lambda}} ; \text{Wigner function associated to } \rho_{\alpha, \lambda} : 0 < \alpha, \lambda < 1 \}, \]

with $\lambda$ close enough to 1 so that $W_0 \in A(\beta, L)$ (see Butucea, Guţă and Artiles (2007) for the proof and details).

**B.1.2. Construction of the set of Wigner functions $W_{\alpha, \lambda}$ for the $L_2$-norm.**

We define $M+1$ infinitely differentiable functions such that:

- For all $m = 1, \ldots, M$, $g_m : \mathbb{R} \to [0,1]$.
- The support of $g_m$ is $\text{Supp}(g_m) = [m\delta, (m+1)\delta]$.
- And $\forall t \in [(m+1/3)\delta, (m+2/3)\delta]$, $g_m(t) = 1$.
- An odd function $g : \mathbb{R} \to [-1,1]$, such that for some fixed $\epsilon > 0$, $g(x) = 1$ for any $x \geq \epsilon$.

Define also the following constants:

\begin{align}
a_m &:= (h^{-2} + m\delta)^{1/2}, \quad b_m := (h^{-2} + (m + 1)\delta)^{1/2}, \quad \forall m = 1, \ldots, M. \\
b_m &:= (h^{-2} + (m + 1/3)\delta)^{1/2}, \quad \bar{b}_m := (h^{-2} + (m + 2/3)\delta)^{1/2}, \quad \forall m = 1, \ldots, M. \\
C_0 &:= \sqrt{\pi L(\beta + \gamma)}.
\end{align}

We also introduce $M$ infinitely differentiable functions such that:

- For all $m = 1, \ldots, M$, $V_{m,h} : \mathbb{R}^2 \to \mathbb{R}$ is an odd real-valued function.
- Set $t = \sqrt{w_1^2 + w_2^2}$, then the function $V_{m,h}$ admitting Fourier transform with respect to both variable equals to

\begin{equation}
\tilde{V}_{m,h}(w) := \mathcal{F}_2[V_{m,h}](w) := iaC_0h^{-1}e^{\beta h^2}e^{-2\beta |t|^2}g_m(|t|^2 - h^{-2})g(w),
\end{equation}

where $a > 0$ is a numerical constant chosen sufficiently small. The bandwidth is such that

\begin{equation}
h = \left( \frac{\log n}{2(\beta + \gamma)} \right)^{-1/2}.
\end{equation}
Note that $\tilde{V}_{m,h}(w)$ is infinitely differentiable and compactly supported, thus it belongs to the Schwartz class $S(\mathbb{R}^2)$ of fast decreasing functions on $\mathbb{R}^2$. The Fourier transform being a continuous mapping of the Schwartz class onto itself, this implies that $V_{m,h}$ is also in the Schwartz class $S(\mathbb{R}^2)$. Moreover, $\tilde{V}_{m,h}(w)$ is an odd function with purely imaginary values. Consequently, $V_{m,h}$ is an odd real-valued function. Consequently, we get

\begin{equation}
\int \int V_{m,h}(p,q)dpdq = \int \mathcal{R}[V_{m,h}](x,\phi)dx = 0,
\end{equation}

for all $\phi \in [0, \pi]$ and $\mathcal{R}[V_{m,h}]$ the Radon transform of $V_{m,h}$. As in (8), we define

\begin{equation}
p_{m,h}(x,\phi) = \frac{1}{\pi} \mathcal{R}[W_{m,h}](x,\phi)1_{(0,\pi]}(\phi),
\end{equation}

and $\rho^{(m,h)} = \int_0^\pi p_{m,h}(x,\phi)f_j(x)e^{(j-k)\phi}d\phi$. By Lemma 6 in Appendix D.4, the matrix $\rho^{(m,h)}$ is proved to be a density matrix. Therefore, in view of (9) and (49), the function $W_{m,h}$ is a Wigner function. Now, we can define our set of Wigner functions

\begin{equation}
W_{0,h} = \{W_{m,h} : \mathbb{R}^2 \to \mathbb{R}, W_{m,h}(z) = W_0(z) + V_{m,h}(z), m = 1, \ldots , M\},
\end{equation}

where $W_0$ is the Wigner function associated to the density $p_0$ defined in (42).

**B.1.3. Condition (C1).** By the triangle inequality and for any $1 \leq m \leq M$, we have

\begin{equation}
\|\tilde{W}_{m,h}e^{\beta \cdot \|_2} \|_2 \leq \|W_{0,h}\|_2^2 + \|\tilde{V}_{m,h}\|_2^2.
\end{equation}

The first term in the above sum has been bounded in Lemma 3 of Butucea, Guţă and Artiles (2007) as follow

\begin{equation}
\|\tilde{W}_{0,h}e^{\beta \cdot \|_2} \|_2^2 \leq \pi^2 L.
\end{equation}

To study the second term in the sum above, we consider the change of variables $w = (t \cos \phi, t \sin \phi)$ and as $g$ is bounded by 1, we get since (41), (44) and (46) that

\begin{equation}
\|\tilde{V}_{m,h}e^{\beta \cdot \|_2} \|_2^2 \leq \int \int \left[aC_0h^{-1}e^{3h^{-2}}\right]^2 e^{-2\beta \|w\|^2} g_m^2(\|w\|^2 - h^{-2})dw \\
\leq a^2C_0^2h^{-2}e^{3h^{-2}} \int_0^\pi \int_{\mathcal{A}_m} |t|e^{-2\beta |t|^2}dt \\
\leq \frac{\pi}{3}a^2C_0^2h^{-2}e^{3/2}e^{-2\beta \mathcal{A}_m} \int_{\mathcal{A}_m} tdt \leq \frac{\pi}{2}a^2C_0^2h^{-2}e^{-2\beta \mathcal{A}_m} \left[g_m^2 - a^2_m\right] \\
\leq \frac{\pi}{3}a^2C_0^2h^{-2}\delta e^{-2\beta \mathcal{A}_m} \leq \pi^2 L,
\end{equation}

for $\delta$ small enough. Combining (52) and (53), it comes $W_{m,h} \in \mathcal{A}(\beta, L)$ for any $1 \leq m \leq M$.

**B.1.4. Condition (C2).** By applying Plancherel Theorem and the change of variables $w = (t \cos \phi, t \sin \phi)$, we have since the supports $\text{Supp}(g_k)$ and $\text{Supp}(g_m)$ are disjoints for any $k \neq m$ that

\begin{equation}
\|W_{k,h} - W_{m,h}\|_2^2 = \|V_{k,h} - V_{m,h}\|_2^2 = \frac{1}{4\pi^2} \int_0^\pi \int_0^\pi \left|\tilde{V}_{k,h}(t,\phi) - \tilde{V}_{m,h}(t,\phi)\right|^2 dt d\phi \\
= \frac{a^2C_0^2}{4\pi^2}h^{-2}e^{3h^{-2}} \int_0^\pi \int_0^\pi |t|e^{-4\beta t^2}g^2(t \sin \phi) \left[g_m^2(t^2 - h^{-2}) + g_m^2(t^2 - h^{-2})\right] dt d\phi.
\end{equation}

Note that for a fixed $\mu \in [0, \pi /4]$, there exists a numerical constant $c > 0$ such that $\sin(\phi) > c$ on $|\mu, \pi - \mu|$. From now, we denote by $\tilde{A}_m$ the set

\begin{equation}
\tilde{A}_m := \{w \in \mathbb{R}^2 : (m + 1/3)\delta \leq \|w\|^2 \leq (m + 2/3)\delta\}, \quad \forall m = 1, \ldots , M.
\end{equation}
By definition of $g$ and for a large enough $n$, we have for any $(t, \phi) \in (\tilde{A}_k \cup \tilde{A}_m) \times [\mu, \pi - \mu]$ that $g^2(t \sin(\phi)) = 1$ with $t^2 = \|w\|^2$. Therefore, (54) can be lower bounded as follows

$$
\|W_{k,h} - W_{m,h}\|^2 \geq \frac{\pi - 2\mu}{4\pi} a^2 C_0^2 h^{-2} e^{2\beta h^{-2}} \int \frac{\pi - 2\mu}{4\pi} a^2 C_0^2 h^{-2} e^{2\beta h^{-2}} \int |t e^{-4\beta t^2} [g_k^2(t^2 - h^{-2}) + g_m^2(t^2 - h^{-2})] dt d\phi
$$

(56)

On $\tilde{A}_m$ and by construction of the function $g_m$, we have

$$
g_m^2(t^2 - h^{-2}) = 1, \quad 1 \leq m \leq M.
$$

Constants defined in (45) are such that for $k > m$, we have $\tilde{a}_m < b_m < \tilde{a}_k < b_k$. Whence, since $\tilde{A}_m$ and $\tilde{A}_k$ are disjoint sets for any $k > m$, it results

$$
I := \int_{\tilde{A}_k \cup \tilde{A}_m} |t| e^{-4\beta t^2} [g_k^2(t^2 - h^{-2}) + g_m^2(t^2 - h^{-2})] dt \\
\geq e^{-4\beta \delta_k^2} \int_{\tilde{A}_k \cup \tilde{A}_m} |t| [g_k^2(t^2 - h^{-2}) + g_m^2(t^2 - h^{-2})] dt \\
\geq 2e^{-4\beta \delta_k^2} \int_{\tilde{a}_k}^{b_k} t dt \geq e^{-4\beta \delta_k^2} (\tilde{b}_k - \tilde{a}_k) t \geq \frac{1}{3} \delta e^{-4\beta \delta_k^2}.
$$

(57)

Combining (56) and (57), we get since $C_0^2 h^{-2} \delta = \pi L/2$

$$
\|W_{k,h} - W_{m,h}\|^2 \geq \frac{\pi - 2\mu}{12\pi^2} a^2 C_0^2 h^{-2} e^{2\beta h^{-2}} \delta e^{-4\beta \delta_k^2} = \frac{\pi - 2\mu}{24\pi} a^2 L e^{2\beta h^{-2}} e^{-4\beta \delta_k^2}.
$$

Since $1 \leq k \leq M \leq 1/\delta$ and (48), it comes

$$
\|W_{k,h} - W_{m,h}\|^2 \geq \frac{\pi - 2\mu}{24\pi} a^2 L e^{-2\beta h^{-2}} e^{-4\beta(k+2/3)\delta}.
$$

B.1.5. **Condition (C3).** Denote by $\tilde{C} > 0$ a constant whose value may change from line to line and recall that $N^\gamma$ is the density of the Gaussian distribution with zero mean and variance $2\gamma$. Note that $p_0$ and $N^\gamma$ do not depend on $\phi$. Consequently, in the framework of noisy data defined in (16), $p_0(z, \phi) = p_0(z) \chi_{1(0,\pi)}(\phi)$.

**Lemma 4.** There exists numerical constants $c' > 0$ and $c'' > 0$ such that

$$
p_0^\gamma(z) \geq c' z^{-2}, \quad \forall |z| \geq 1 + \sqrt{2\gamma},
$$

(58)

and

$$
p_0^\gamma(z) \geq c'', \quad \forall |z| \leq 1 + \sqrt{2\gamma}.
$$

(59)

The proof of this Lemma is done in Appendix D.3. Using Lemma 4, the $\chi^2$-divergence can be upper bounded as follow

$$
n \chi^2(p_{m,h}, p_0) = n \int_0^\pi \frac{(p_{m,h}(z, \phi) - p_0(z, \phi))^2}{p_0(z, \phi)} dz d\phi \\
\leq \frac{n}{c^2} \int_0^\pi \int_{-(1+\sqrt{2\gamma})}^{1+\sqrt{2\gamma}} \left( p_{m,h}(z, \phi) - p_0(z, \phi) \right)^2 dz d\phi \\
+ \frac{n}{c} \int_0^\pi \int_{(1+\sqrt{2\gamma)}(1+\sqrt{2\gamma})} z^2 \left( p_{m,h}(z, \phi) - p_0(z, \phi) \right)^2 dz d\phi
$$

(60)

$$
=: \frac{n}{c^2} l_1 + \frac{n}{c} l_2.
$$
First underline, as in (18) the Fourier transforms of \( p_{m,h}^\gamma \) and \( p_0^\gamma \) with respect to the first variable are equal respectively to
\[
\mathcal{F}_t[p_{m,h}^\gamma(\cdot, \phi)](t) = \bar{V}_{m,h}(t \cos \phi, t \sin \phi) \tilde{N}^\gamma(t)
\]
\[
\mathcal{F}_t[p_0^\gamma(\cdot, \phi)](t) = \bar{W}_0(t \cos \phi, t \sin \phi) e^{-\gamma t^2}.
\]
(61)
(62)
since \( \tilde{N}^\gamma(t) = e^{-\gamma t^2} \). Using Plancherel Theorem and (47), equations (61) and (62), the first integral \( I_1 \) in the sum (60) is bounded by
\[
I_1 \leq \int_0^\pi \int_0^{2\pi} \left( p_{m,h}^\gamma(z, \phi) - p_0^\gamma(z, \phi) \right)^2 \, dz \, d\phi = \frac{1}{4\pi^2} \int_0^\pi \int_0^{2\pi} \left| \mathcal{F}_t[p_{m,h}^\gamma(\cdot, \phi)](t) - \mathcal{F}_t[p_0^\gamma(\cdot, \phi)](t) \right|^2 \, dt \, d\phi
\]
\[
= \frac{1}{4\pi^2} \int_0^\pi \int_0^{2\pi} \left| \bar{V}_{m,h}(t \cos \phi, t \sin \phi) \right|^2 e^{-2\gamma t^2} \, dt \, d\phi
\]
\[
= \frac{a^2 C_0^2}{4\pi^2} h^{-2} e^{2\beta h^2} \int_0^\pi \int_0^{2\pi} e^{-4\beta t^2-2\gamma t^2} g_m(t^2 - h^{-2}) \, dt \, d\phi.
\]
By construction, the function \( g \) is bounded by 1 and the function \( g_m \) admits as support \( \text{Supp}(g_m) = [m\delta, (m+1)\delta] \) for all \( m = 1, \cdots, M \). Thus,
\[
I_1 \leq \frac{a^2 C_0^2}{4\pi^2} e^{2\beta h^2} \int_0^\pi \int_0^{2\pi} e^{-4\beta t^2-2\gamma t^2} g_m(t^2 - h^{-2}) \, dt \, d\phi \leq \frac{a^2 C_0^2}{4\pi^2} h^{-2} e^{2\beta h^2} \int_{a_m}^{b_m} \int_0^\pi \int_0^{2\pi} e^{-4\beta t^2-2\gamma t^2} \, dt \, d\phi
\]
\[
\leq \frac{a^2 C_0^2}{4\pi^2} (b_m - a_m) h^{-2} e^{2\beta h^2} e^{-4\beta a_m^2 - 2\gamma a_m^2} \leq \frac{a^2 C_0^2}{4\pi^2} \frac{b_m^2 - a_m^2}{2a_m} h^{-2} e^{2\beta h^2 - 4\beta a_m^2 - 2\gamma a_m^2}.
\]
Some basic algebra, (41), (44), (46) and (48) yield
\[
\frac{n}{c''} I_1 \leq \frac{a^2 \tilde{C}}{\sqrt{\log n}},
\]
for some a constant \( \tilde{C} > 0 \) which may depend on \( \beta, \gamma, L \) and \( c'' \). For the second term \( I_2 \) in the sum (60), with the same tools we obtain using in addition the spectral representation of the differential operator, that
\[
I_2 \leq \int_0^\pi \int_0^{2\pi} z^2 \left( p_{m,h}^\gamma(z, \phi) - p_0^\gamma(z, \phi) \right)^2 \, dz \, d\phi
\]
\[
= \int_0^\pi \int_0^{2\pi} \left| \frac{\partial}{\partial t} \left( \mathcal{F}_t[p_{m,h}^\gamma(\cdot, \phi)](t) - \mathcal{F}_t[p_0^\gamma(\cdot, \phi)](t) \right) \right|^2 \, dt \, d\phi
\]
\[
= \int_0^\pi \int_0^{2\pi} e^{-\gamma t^2} \left| \frac{\partial}{\partial t} \bar{V}_{m,h}(t \cos \phi, t \sin \phi) e^{-\gamma t^2} \right|^2 \, dt \, d\phi
\]
\[
= \int_0^\pi \int_0^{2\pi} e^{-\gamma t^2} \left| \frac{\partial}{\partial t} \bar{V}_{m,h}(t \cos \phi, t \sin \phi) - 2\gamma te^{-\gamma t^2} \bar{V}_{m,h}(t \cos \phi, t \sin \phi) \right|^2 \, dt \, d\phi
\]
\[
\leq 2 \int_0^\pi \int_0^{2\pi} e^{-2\gamma t^2} |I_{2,1}|^2 \, dt \, d\phi + 16\gamma^2 \int_0^\pi \int_0^{2\pi} t^2 e^{-2\gamma t^2} |I_{2,1}|^2 \, dt \, d\phi,
\]
(64)
where \( I_{2,2} = \bar{V}_{m,h}(t \cos \phi, t \sin \phi) \) and \( I_{2,1} \), the partial derivative \( \frac{\partial}{\partial t} \bar{V}_{m,h}(t \cos \phi, t \sin \phi) \), is equal to
\[
\left. i a_0 C_0 h^{-1} e^{\beta h^2 - 2\beta t^2} \left[ g_m(t^2 - h^{-2}) (-4\beta t g(t \sin \phi) + g'(t \sin \phi) \sin \phi) + 2tg_m'(t^2 - h^{-2}) g(t \sin \phi) \right] \right|_{t = 0}^t.
\]
Since \( g_m \) and \( g \) belong to the Schwartz class, there exists a numerical constant \( c_S > 0 \) such that
\[
\max \|g_m\|_{\infty}, \|g'_m\|_{\infty}, \|g\|_{\infty}, \|g'\|_{\infty} \leq c_S.
\]
Furthermore, for all \( m = 1, \cdots, M \), the support of the function \( g_m \) is \( \text{Supp}(g_m) = [m\delta, (m+1)\delta] \), then
\[
|I_{2,1}|^2 \leq a^2 C_0^2 h^{-2} e^{2\beta h^2 - 4\beta a_m^2} (4\beta + 2) |t| + 1)^2 1_{(a_m, b_m)}(t),
\]
(65)
with $a_m$ and $b_m$ defined in (44). Similarly, we have

$$|I_2|^2 = \left| aC_0^{\delta,h} e^{-\frac{1}{2}h^2} e^{-2B\gamma^2} g_m(t^2 - h^{-2}) g(t \sin \phi) \right|^2 \leq a^2 C_0^{\delta,h} e^{-2\beta h^{-2}} e^{-8\gamma^2} 1_{(a_m,b_m)}(t).$$

(66)

Combining (65) and (66) with (64), as $0 \leq m\delta \leq 1$

$$I_2 \leq 2a^2 C_0^{\delta,h} e^{-2\beta h^{-2}} e^{-4\gamma^2} \int_0^{b_m} e^{-2\gamma t^2} e^{-4\gamma t^2} \left[ ((4\beta + 2)\delta + 1)^2 + 8\gamma^2 t^2 \right] dt d\phi$$

$$\leq 2\pi a^2 C_0^{\delta,h} e^{-2\beta h^{-2}} e^{-\frac{1}{2}(\beta + \gamma)^2} \left[ ((4\beta + 2)b_m + 1)^2 + 8\gamma^2 b_m^2 \right] \frac{b_m^2 - a_m^2}{2a_m} \delta.$$  

Some basic algebra, (41), (44), (46) and (48) yield

$$n \frac{\partial}{\partial \rho} I_2 \leq a^2 C \sqrt{\log n},$$

for some a constant $\tilde{C} > 0$ whose may depend on $\beta$, $\gamma$, $Lc_S$ and $c'$. Combining (67) and (63) with (60), we get for $n$ large enough

$$n\lambda^2(p_{k,h}^{\gamma}, p_{0}^{\gamma}) := n \left[ \int_0^{\pi} \frac{(p_{k,h}^{\gamma}(z,\phi) - p_{0}^{\gamma}(z,\phi))^2}{p_{0}^{\gamma}(z,\phi)} d\phi \right] \leq a^2 \tilde{C} \sqrt{\log n},$$

where $\tilde{C} > 0$ is a constant whose may depend on $\beta$, $\gamma$, $Lc_S$, $c'$ and $c'$. Taking the numerical constant $a > 0$ small enough, we deduce from the previous display that

$$n\lambda^2(p_{k,h}^{\gamma}, p_{0}^{\gamma}) \leq M,$$

since $M = \lfloor \sqrt{\log n} \rfloor$.

B.2. Proof of Theorem 2 - Lower bounds for the sup-norm. To prove the lower bound for the sup-norm, we need to slightly modify the construction of the Wigner classe $\mathcal{W}_{k,h}$ defined in (51) into

$$\mathcal{W}_{k,h} = \{ W_{m,h,c} : \mathbb{R}^2 \to \mathbb{R}, W_{m,h,c}(z) = W_0(z) + V_{m,h,c}(z), m = 1, \cdots, M \},$$

where $W_0$ is the Wigner function associated to the density $p_0$ defined in (42) stay unchanged as compared to the $L_2$ case. However, the construction of the $\{V_{m,h}\}$-functions defined in (47) only changed through modification of the functions $g_m$ and $g$ respectively into $g_{m,c}$ and $g_c$, for $0 < \epsilon < 1$.

We define $M + 1$ infinitely differentiable functions such that:

- For all $m = 1, \cdots, M$, $g_{m,c} : \mathbb{R} \to [0,1]$.
- The support of $g_{m,c}$ is $\text{Supp}(g_{m,c}) = [m\delta, (m+1)\delta]$.
- Using a similar construction as for function $g_m$, we can also assume that

$$g_{m,c}(t) = 1, \quad \forall t \in B_{m,c} := [(m+\epsilon)\delta, (m+1-\epsilon)\delta],$$

and

$$\|g_{m,c}\|_\infty \leq \frac{c}{\epsilon \delta},$$

for some numerical constant $c > 0$.  

• An odd function \( g_z : \mathbb{R} \to [-1, 1] \) satisfies the same conditions as \( g \) above but we assume in addition that

\[
\|g'_z\|_\infty \leq \frac{c}{\epsilon},
\]

for some numerical constant \( c > 0 \).

The condition (71) will be needed to check Condition (C3). Such a function can be easily constructed. Consider for instance a function \( g_z \) such that its derivative satisfies

\[
g'_z(t) = \left[ \psi \ast \frac{1}{\epsilon} 1_{(0,\epsilon)} \right](t),
\]

for any \( t \in (0, \epsilon) \) where \( \psi \) is a mollifier. Integrate this function and renormalize it properly so that \( g_z(t) = 1 \) for any \( t \geq \epsilon \). Complete the function by symmetry to obtain an odd function defined on the whole real line. Such a construction satisfies condition (71).

It is easy to see that Condition (C1) is always satisfied by the new test functions \( \{W_{m,h,\epsilon}\}_m \). To check Condition (C2) set \( C_h = i a C_0 h^{-1} e^{\beta h^{-2}} \) and then we have

\[
W_{k,h,\epsilon}(z) - W_{m,h,\epsilon}(z) = \frac{1}{4\pi^2} \int e^{-i z \cdot w} \left( \tilde{W}_{k,h,\epsilon}(w) - \tilde{W}_{m,h,\epsilon}(w) \right) dw
\]

\[
= \frac{1}{4\pi^2} \int_0^\pi e^{-i t |z|} t \left( \tilde{W}_{k,h,\epsilon}(t \cos \phi, t \sin \phi) - \tilde{W}_{m,h,\epsilon}(t \cos \phi, t \sin \phi) \right) dtd\phi
\]

\[
= \frac{1}{4\pi^2} \int_0^\pi e^{-i t |z|} t |t| C_h e^{-2\beta t^2} (g_{k,\epsilon} - g_{m,\epsilon}) (t^2 - h^2) g_z(t)dtd\phi.
\]

For all \( z \in \mathbb{R}^2 \) and \( B_{m,0} = \lim_{\epsilon \to 0} B_{m,\epsilon} \) defined in (69), we define the following quantity

\[
I(z) := \int_0^\pi e^{-i t |z|} t |t| C_h e^{-2\beta t^2} [1_{B_{k,0}} - 1_{B_{m,0}}] (t^2 - h^2) \left[ 1_{(0,\infty)}(t) - 1_{(-\infty,0)}(t) \right] dtd\phi.
\]

Lebesgue dominated convergence Theorem guarantees that

\[
\lim_{\epsilon \to 0} \left( \int_0^\pi e^{-i t |z|} t |t| C_h e^{-2\beta t^2} (g_{k,\epsilon} - g_{m,\epsilon}) (t^2 - h^2) g_z(t)dtd\phi \right) = I(z).
\]

Therefore, there exists an \( \epsilon > 0 \) (possibly depending on \( n, z \)) such that

\[
|W_{k,h,\epsilon}(z) - W_{m,h,\epsilon}(z)| \geq \frac{1}{2} |I(z)|.
\]

Taking \( z = (0, 2h) \), Fubini’s Theorem gives

\[
I(z) = \frac{1}{4\pi^2} \int_0^\pi e^{-i t 2h \sin \phi} t |t| C_h e^{-2\beta t^2} [1_{A_{k,0}} - 1_{A_{m,0}}] (t^2 - h^2)
\]

\[
\times \left[ 1_{(0,\infty)}(t) - 1_{(-\infty,0)}(t) \right] dtd\phi
\]

\[
= \frac{1}{4\pi^2} \int_0^\pi \left( \int_0^\pi e^{-i t 2h \sin \phi} d\phi \right) t |t| C_h e^{-2\beta t^2} [1_{A_{k,0}} - 1_{A_{m,0}}] (t^2 - h^2)
\]

\[
\times \left[ 1_{(0,\infty)}(t) - 1_{(-\infty,0)}(t) \right] dt.
\]

Note that

\[
\int_0^\pi e^{-i t 2h \sin \phi} d\phi = \pi (i H_0(2ht) + J_0(2ht)),
\]

where \( H_0 \) and \( J_0 \) denote respectively the Struve and Bessel functions of order 0. By definition, \( H_0 \) is an odd function while \( J_0 \) and \( t \to |t| C_h e^{-2\beta t^2} [1_{A_{k,0}} - 1_{A_{m,0}}] (t^2 - h^2) \) are even functions.
Consequently, we get
\[
I(z) = \frac{1}{4\pi} iC_h \int |t| H_0(2ht)e^{-2\beta t^2} [1_{A_{k,0}} - 1_{A_{m,0}}] (t^2 - h^2) \left[ 1_{(0,\infty)}(2ht) - 1_{(-\infty,0)}(t) \right] dt
\]
\[
= \frac{1}{2\pi} iC_h \int_0^\infty t H_0(2ht)e^{-2\beta t^2} [1_{A_{k,0}} - 1_{A_{m,0}}] (t^2 - h^2) dt
\]
\[
= iC_h \left( \int_{a_k}^{b_k} t H_0(2ht)e^{-2\beta t^2} dt - \int_{a_m}^{b_m} t H_0(2ht)e^{-2\beta t^2} dt \right)
\]
with \(a_k\) and \(b_k\) defined in (44). For some numerical constant \(c > 0\),
\[
|a_m, b_m| \leq [h^{-1}, h^{-1} + \epsilon]\).
\]
On \([h^{-1}, h^{-1} + \epsilon]\) and for a large enough \(n\), functions \(t \to H_0(2ht)\) and \(t \to te^{-2\beta t^2}\) are decreasing and
\[
\min_{t \in [h^{-1}, h^{-1} + \epsilon]} \{H_0(2ht)\} \geq 1/2.
\]
Assume without loss of generality that \(k < m\). We easily deduce from the previous observations that
\[
|I(z)| \geq \frac{|C_h|}{4\pi} \left( \int_{a_k}^{b_k} te^{-2\beta t^2} dt - \int_{a_m}^{b_m} te^{-2\beta t^2} dt \right)
\]
\[
\geq \frac{|C_h|}{16\pi \beta} e^{-2\beta a_k^2} - e^{-2\beta b_k^2} + e^{-2\beta b_m^2} - e^{-2\beta a_m^2}
\]
\[
\geq \frac{|C_h|}{16\pi \beta} e^{-2\beta h^2} - e^{-2\beta h \delta} (1 - e^{-2\beta h (m-k) \delta}) (1 - e^{-2\beta \delta}).
\]
Therefore, some simple algebra gives
\[
|I(z)| \geq c \delta^2 |C_h| n^{-\frac{1}{\beta}} \geq a c' n^{-\frac{1}{\beta}} \log^{-3/2}(n),
\]
for some numerical constants \(c, c' > 0\) depending only \(\beta\). Taking the numerical constant \(a > 0\) small enough independently of \(n, \beta, \gamma\), we get that Condition (C2) is satisfied with \(\varphi_n = cn^{-\frac{1}{\beta}} \log^{-3/2}(n)\).
Concerning Condition (C3), we proceed similarly as above for the quadratic risk. The only modification appears in (65)-(66) where we now use (69)-(70) combined with the fact that
\[
|\text{Supp}(g'_n)| \leq 2\epsilon \text{ and } |\text{Supp}(g_{m',\epsilon'})| \leq 2\delta \epsilon
\]
by construction of these functions. Therefore, the details will be omitted here.

**APPENDIX C. PROOF OF THEOREM 3 - ADAPTATION**

The following Lemma is needed to prove the Theorem 3.

**Lemma 5.** For \(\kappa > 0\), a constant, let \(\mathcal{E}_\kappa\) be the event defined such that
\[
\mathcal{E}_\kappa = \bigcap_{m=1}^M \left\{ \|\hat{W}_{h_m}^\gamma - E[\hat{W}_{h_m}^\gamma]\|_\infty \leq \kappa \epsilon h_m^{-2} r_n(x + \log M) \right\}.
\]

Therefore, on the event \(\mathcal{E}_\kappa\)
\[
\|\hat{W}_{h_m}^\gamma - W_\rho\|_\infty \leq C \min_{1 \leq m \leq M} \left\{ h_m^{r/2-1} e^{-\beta h_m^{-r}} + \epsilon h_m^{-2} r_n(x + \log M) \right\},
\]
where \(C > 0\) is a constant depending only on \(\gamma, \beta, L, r, \kappa\) and \(\hat{W}_{h_m}^\gamma\) is the adaptive estimator with the bandwidth \(h_m\) defined in (30).

The proof of the previous Lemma is done in D.2. For any fixed \(m \in \{1, \cdots, M\}\), we have in view of Proposition 2 that
\[
P \left( \|\hat{W}_{h_m}^\gamma - E[\hat{W}_{h_m}^\gamma]\|_\infty \leq C \epsilon h_m^{-2} r_n(x) \right) \geq 1 - e^{-x},
\]
where \( r_n(x) = \max \left( \sqrt{\frac{1+x}{n}}, \sqrt{\frac{1-x}{n}} \right) \). By a simple union bound, we get

\[
P \left( \bigcap_{1 \leq m \leq M} \left\{ \left\| \hat{W}_{h_m}^\gamma - E[\hat{W}_{h_m}^\gamma] \right\|_\infty \leq C_2 e^{x h_m - 2} r_n(x) \right\} \right) \geq 1 - M e^{-x}.
\]

Replacing \( x \) by \( (x + \log M) \), implies

\[
P \left( \bigcap_{1 \leq m \leq M} \left\{ \left\| \hat{W}_{h_m}^\gamma - E[\hat{W}_{h_m}^\gamma] \right\|_\infty \leq C_2 e^{x h_m - 2} r_n(x + \log M) \right\} \right) \geq 1 - e^{-x}.
\]

For \( \kappa > C_2 \), we immediately get that \( \mathbb{P}(\mathcal{E}_n) \geq 1 - e^{-x} \) and the result in probability (31) follows by Lemma 5. To prove the result in expectation (32), we use the property \( \mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z \geq t) dt \), where \( Z \) is any positive random variable. We have indeed for any \( 1 \leq m \leq M \) that

\[
\mathbb{P} \left( \left\| \hat{W}_{h_m}^\gamma - W \right\|_\infty \geq C \left( h_m^{r/2 - 1} e^{-\frac{n}{h_m}} + e^{h_m - 2} r_n(x + \log M) \right) \right) \leq e^{-x}, \quad \forall x > 0.
\]

Note that

\[
r_n(x + \log M) = \max \left\{ \sqrt{\frac{x + \log(eM)}{n}}, \sqrt{\frac{x}{n}} \right\},
\]

\[
\leq \max \left\{ \sqrt{\frac{\log(eM)}{n}}, \frac{\log(eM)}{n} \right\} + \max \left\{ \sqrt{\frac{x}{n}}, \frac{x}{n} \right\},
\]

\[
\leq r_n(\log M) + r_n(x - 1).
\]

Combining the two previous displays, we get \( \forall x > 0 \)

\[
P \left( \left\| \hat{W}_{h_m}^\gamma - W \right\|_\infty \geq C \left( h_m^{r/2 - 1} e^{-\frac{n}{h_m}} + e^{h_m - 2} [r_n(\log M) + r_n(x - 1)] \right) \right) \leq e^{-x}.
\]

Set \( Y = \left\| \hat{W}_{h_m}^\gamma - W \right\|_\infty / C, a = h_m^{r/2 - 1} e^{-\frac{n}{h_m}} + e^{h_m - 2} [r_n(\log M) + r_n(x - 1)] \) and \( b = e^{h_m - 2} \). We have

\[
\mathbb{E}[Y] = a + \mathbb{E}[Y - a] = a + \int_0^\infty \mathbb{P}(Y - a \geq u) du = a + b \int_0^\infty \mathbb{P}(Y - a \geq bt) dt.
\]

Set now \( t = r_n(x - 1) \). If \( 0 < t < 1 \), then we have \( t = \sqrt{\frac{x}{n}} \). If \( t \geq 1 \) then we have \( t = \frac{x}{n} \). Thus we get by the change of variable \( t = \sqrt{\frac{x}{n}} \) that

\[
\int_0^1 \mathbb{P}(Y - a \geq bt) dt = \int_0^n \mathbb{P} \left( Y - a \geq \sqrt{\frac{x}{n}} \right) \frac{1}{2\sqrt{2\pi n}} dx \leq \frac{1}{2\sqrt{2\pi n}} \int_0^n \frac{e^{-x}}{\sqrt{x}} dx \leq \frac{c}{\sqrt{n}},
\]

where \( c > 0 \) is a numerical constant. Similarly, we get by change of variable \( t = \frac{x}{n} \)

\[
\int_1^\infty \mathbb{P}(Y - a \geq bt) dt = \int_1^n \mathbb{P} \left( Y - a \geq \frac{x}{n} \right) \frac{1}{n} dx \leq \frac{1}{n} \int_1^\infty e^{-x} dx \leq \frac{c'}{n},
\]

where \( c' > 0 \) is a numerical constant. Combining the last three displays, we obtain the result in expectation.

**Appendix D. Proof of Auxiliary Lemmas**

**D.1. Proof of Lemma 2.** To prove the uniform bound of (36), we define

\[
d_h = \max_{|t| \leq h^{-1}} \left\{ |t| e^{\gamma t^2} \right\}.
\]
Then, by definition of $K_h^\gamma$ and by using the inverse Fourier transform formula, we have
\[
\delta_h^{-1}\|K_h^\gamma\|_\infty = \frac{1}{2\pi} \delta_h^{-1} \sup_{x \in \mathbb{R}} \left| \int e^{-ix\widetilde{K}_h^\gamma(t)} dt \right| \leq \frac{1}{2\pi} \delta_h^{-1} \int_{-h^{-1}}^{h^{-1}} |t| e^{\gamma t^2} dt
\]
\[
\leq \frac{1}{\pi} \delta_h^{-1} \int_{0}^{h^{-1}} t e^{\gamma t^2} dt \leq \frac{1}{2\gamma \pi} \delta_h^{-1} \int_{0}^{h^{-1}} 2\gamma t e^{\gamma t^2} dt
\]
(73)
\[
\leq \frac{1}{2\gamma \pi} \delta_h^{-1} (e^{\gamma h^{-2}} - 1) \leq \frac{1}{2\gamma \pi} \delta_h^{-1} (e^{h^{-2}} - 1) \leq \frac{h}{2\gamma \pi} := U.
\]
For the entropy bound (37), we need to prove that $K_h^\gamma(\cdot)$ admits finite quadratic variation, i.e. \(K_h^\gamma \in V_2(\mathbb{R})\), where \(V_2(\mathbb{R})\) is the set of functions with finite quadratic variation (see Theorem 5 of Bourdaud, Lanza de Cristoforis and Sickel (2006)). To do this, it is enough to verify that $K_h^\gamma \in B_{2,1}^{1/2}(\mathbb{R})$ and the result is a consequence of the embedding $B_{2,1}^{1/2}(\mathbb{R}) \subset V_2(\mathbb{R})$.

Let us define the Littlewood-Paley characterization of the seminorm $\| \cdot \|_{1/2,2,1}$ as follow
\[
\|g\|_{1/2,2,1} := \sum_{l \in \mathbb{Z}} 2^{l/2} \|F_1^{-1}[\alpha_l F_1[g]]\|_2,
\]
where $\alpha_l(\cdot)$ is a dyadic partition of unity with $\alpha_l$ symmetric w.r.t to 0, supported in
\[-2^{l+1}, -2^l \] \cup \[2^l, 2^{l+1}\]
and $0 \leq \alpha_l \leq 1$ (see e.g. Theorem 6.3.1 and Lemma 6.1.7 in the paper of Bergh and Löfström (1976)). Then, $K_h^\gamma \in B_{2,1}^{1/2}(\mathbb{R})$, if and only if $\|K_h^\gamma\|_{1/2,2,1}$ is bounded by a fixed constant. By isometry of the Fourier transform combining with definition of $\alpha_l$ and $K_h^\gamma$, we get that
\[
\|F_1^{-1}[\alpha_l F_1[K_h^\gamma]]\|_2 = \|\alpha_l F_1[K_h^\gamma]\|_2 = \|\alpha_l K_h^\gamma\|_2
\]
\[
= \sqrt{2} \int_{[0,h^{-1}]} |t|^2 e^{\gamma t^2} dt
\]
\[
\leq \sqrt{2} \int_{[0,h^{-1}]} t^2 e^{\gamma t^2} dt.
\]
A primitive of $t \to t^2 e^{\gamma t^2}$ is $\frac{1}{2\gamma^2} e^{\gamma t^2} - 1$, thus, we get that
\[
\|F_1^{-1}[\alpha_l F_1[K_h^\gamma]]\|_2 \leq \sqrt{2} h^{-1/2} e^{\gamma h^{-2}}, \quad \forall l \in \mathbb{Z},
\]
and
\[
\|K_h^\gamma\|_{1/2,2,1} \leq \sqrt{\frac{1}{\gamma}} h^{-1/2} e^{\gamma h^{-2}} \sum_{l=-\infty}^{L_h} 2^{l/2},
\]
where $L_h = \lfloor \log_2(h^{-1}) + 1 \rfloor$. A simple computation gives that
\[
\sum_{l=-\infty}^{L_h} 2^{l/2} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} + \frac{2(L_h+1)/2 - 1}{\sqrt{2} - 1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} + \frac{2}{\sqrt{2} - 1} h^{-1/2}.
\]
Combining the last two displays and since $h^{-1} \geq 1$, we get
\[
\|K_h^\gamma\|_{1/2,2,1} \leq c \sqrt{\frac{1}{\gamma}} h^{-1} e^{\gamma h^{-2}},
\]
where $c > 0$ is a numerical constant. This shows that $\delta_h^{-1} \|K_h^\gamma\|_{1/2,2,1}$ is bounded by a fixed constant depending only on $\gamma$. Therefore $K_h^\gamma \in V_2(\mathbb{R})$ and the entropy bound (37) is obtained by applying Lemma 1 of Giné and Nickl (2009).
D.2. Proof of Lemma 5. We recall that the bandwidth \( h_{\hat{m}} \) with \( \hat{m} \) is defined in (30). Let \( r_n(x) = \max \left( \frac{1}{\sqrt{n}}, \frac{1+x}{n} \right) \) and define

\[
    m^* := \arg\min_{1 \leq m \leq M} \left\{ h_m^{\gamma/2-1} e^{-\frac{\gamma}{h_m}} + e^{\gamma h_m^2} r_n(x + \log M) \right\},
\]

and

\[
    B(m) = \max_{j > m} \left\{ \| \hat{W}_{h_m}^\gamma - \hat{W}_{h_j}^\gamma \|_\infty - 2k e^{\gamma h_j^2} r_n(x + \log M) \right\}.
\]

In one hand, we have

\[
    \| \hat{W}_{h_m}^\gamma - \hat{W}_{h_m}^\gamma \|_\infty 1_{m > m^*} \leq \left( B(m^*) + 2k e^{\gamma h_m^2} r_n(x + \log M) \right) 1_{m > m^*}.
\]

In the other hand, similarly, we have

\[
    \| \hat{W}_{h_m}^\gamma - \hat{W}_{h_m}^\gamma \|_\infty 1_{m \leq m^*} \leq \left( B(\hat{m}) + 2k e^{\gamma h_m^2} r_n(x + \log M) \right) 1_{m \leq m^*}.
\]

Combining the last two displays, and by definition of \( \mathcal{L}_e(\cdot) \) in (29), we get

\[
    \| \hat{W}_{h_m}^\gamma - \hat{W}_{h_m}^\gamma \|_\infty \leq \left( B(m^*) + 2k e^{\gamma h_m^2} r_n(x + \log M) \right) 1_{m > m^*} + \left( B(\hat{m}) + 2k e^{\gamma h_m^2} r_n(x + \log M) \right) 1_{m \leq m^*}.
\]

where the last inequality follows from the definition of \( \hat{m} \) in (30). By the definition of \( B(\cdot) \), it comes

\[
    \mathcal{L}(m^*) = B(m^*) + 2k e^{\gamma h_m^2} r_n(x + \log M)
\]

\[
    = \max_{j > m^*} \left\{ \| \hat{W}_{h_m}^\gamma - \hat{W}_{h_j}^\gamma \|_\infty - 2k e^{\gamma h_j^2} r_n(x + \log M) \right\}
\]

\[
    \leq \max_{j > m^*} \left\{ \| \hat{W}_{h_m}^\gamma - \hat{E}[\hat{W}_{h_m}^\gamma] \|_\infty + \| \hat{E}[\hat{W}_{h_m}^\gamma] - W_{\rho} \|_\infty + \| W_{\rho} - \hat{E}[\hat{W}_{h_m}^\gamma] \|_\infty 
\]

\[
    + \| \hat{E}[\hat{W}_{h_m}^\gamma] - \hat{W}_{h_j}^\gamma \|_\infty - 2k e^{\gamma h_j^2} r_n(x + \log M) \right\} + 2k e^{\gamma h_m^2} r_n(x + \log M).
\]

On the event \( \mathcal{E}_r \), it follows that

\[
    \mathcal{L}(m^*) \leq \max_{j > m^*} \left\{ \| \hat{W}_{h_m}^\gamma - \hat{E}[\hat{W}_{h_m}^\gamma] \|_\infty + \| \hat{E}[\hat{W}_{h_m}^\gamma] - W_{\rho} \|_\infty + \| W_{\rho} - \hat{E}[\hat{W}_{h_m}^\gamma] \|_\infty 
\]

\[
    - e^{\gamma h_j^2} r_n(x + \log M) \right\} + 2k e^{\gamma h_m^2} r_n(x + \log M).
\]

As \( h_m^* > h_j \) for all \( j > m^* \), we have \(-e^{\gamma h_j^2} < -e^{\gamma h_m^2} \). Therefore, on the event \( \mathcal{E}_r \), we get

\[
    \mathcal{L}(m^*) \leq \| \hat{W}_{h_m}^\gamma - W_{\rho} \|_\infty + \max_{j > m^*} \left\{ \| \hat{E}[\hat{W}_{h_j}^\gamma] - W_{\rho} \|_\infty \right\} + 2k e^{\gamma h_m^2} r_n(x + \log M).
\]

From (75) and on the event \( \mathcal{E}_r \), we have

\[
    \| \hat{W}_{h_m}^\gamma - W_{\rho} \|_\infty \leq \| \hat{W}_{h_m}^\gamma - \hat{W}_{h_m}^\gamma \|_\infty + \| \hat{W}_{h_m}^\gamma - W_{\rho} \|_\infty \leq \| \hat{W}_{h_m}^\gamma - W_{\rho} \|_\infty + 2L(m^*)
\]

\[
    \leq \| \hat{W}_{h_m}^\gamma - \hat{E}[\hat{W}_{h_m}^\gamma] \|_\infty + \| \hat{E}[\hat{W}_{h_m}^\gamma] - W_{\rho} \|_\infty + 2L(m^*)
\]

\[
    \leq k e^{\gamma h_m^2} r_n(x + \log M) + \| \hat{E}[\hat{W}_{h_m}^\gamma] - W_{\rho} \|_\infty + 2L(m^*).
\]

Combining the last inequality with (76)

\[
    \| \hat{W}_{h_m}^\gamma - W_{\rho} \|_\infty \leq 5k e^{\gamma h_m^2} r_n(x + \log M) + 3\| \hat{E}[\hat{W}_{h_m}^\gamma] - W_{\rho} \|_\infty + 2 \max_{j > m^*} \left\{ \| \hat{E}[\hat{W}_{h_j}^\gamma] - W_{\rho} \|_\infty \right\}.
\]

From Proposition 1, the bias is bounded by \( t \rightarrow t^{r/2-1}e^{-\beta t^{-r}} \), an increasing function for sufficiently small \( t > 0 \), and as \( s \geq h_m \), \( h_{j} \) for all \( j > m^{*} \), we can write

\[
\|W_{h_{m}} - W_{p}\| \leq C\left(ke^{\frac{n^{2}}{m^{2}}r_n(x + \log M + h_{m}^{r/2-1}e^{-\beta h_{m}^{-r}})}\right).
\]

The result comes from (74), the definition of \( m^{*} \).

D.3. Proof of Lemma 4. In view of Fatou’s Lemma, we have

\[
\liminf_{|z| \to \infty} \int \rho_{h_{m}}(z) \geq \int \liminf_{|z| \to \infty} \rho_{0}(z - x)N^{\gamma}(x)dx \\
\geq \int_{-\sqrt{2\gamma}}^{\sqrt{2\gamma}} \liminf_{|z| \to \infty} \int \rho_{0}(z - x)N^{\gamma}(x)dx.
\]

Recall that \( \gamma = \frac{1}{4} < 1/4 \), then for \( |z| \geq \sqrt{2\gamma} + 1 \) and any \( x \in (-\sqrt{2\gamma}, \sqrt{2\gamma}) \), it comes by Lemma 3 that \( \rho_{0}(x) \geq c(z - x)^{-2} \). Thus,

\[
\liminf_{|z| \to \infty} \int \rho_{0}(z) \geq c \int_{-\sqrt{2\gamma}}^{\sqrt{2\gamma}} N^{\gamma}(x)dx = c \int_{-1}^{1} \frac{1}{2\pi} e^{-x^2} dx \geq c' > 0,
\]

where \( c' > 0 \) is a numerical constant. Choose now a numerical constant \( \tilde{c} \geq 0 \) such that \( \int_{\tilde{c}}^{\infty} p_{0}(x)dx \geq 1/2 \), therefore, for any \( |z| \leq 1 + \sqrt{2\gamma} \) and some numerical constant \( c'' > 0 \) we get

\[
p_{0}(z) \geq \int_{-\tilde{c}}^{\tilde{c}} p_{0}(x)N^{\gamma}(z - x)dx \geq \min_{|y| \leq M + 1 + \sqrt{2\gamma}}\{N^{\gamma}(y)\} \int_{-\tilde{c}}^{\tilde{c}} p_{0}(x)dx \\
\geq \frac{1}{2} \min_{|y| \leq M + 1 + \sqrt{2\gamma}}\{N^{\gamma}(y)\} \geq c'' > 0.
\]

D.4. Lemma 6. The density matrix \( \rho^{(m,h)} \) defined in (50) satisfies the following conditions are satisfied:

(i) Self adjoint: \( \rho^{(m,h)} = (\rho^{(m,h)})^{*} \).
(ii) Positive semi-definite: \( \rho^{(m,h)} \geq 0 \).
(iii) Trace one: \( \text{Tr}(\rho^{(m,h)}) = 1 \).

Proof:
• Note first that \( V_{m,h} \) is not a Wigner function, however it belongs to the linear spans of Wigner functions. Consequently, it admits the following representation

\[
\frac{1}{\pi} \mathcal{R}[V_{m,h}](x, \phi) = \sum_{j,k=0}^{\infty} \tau_{j,k}^{(m,h)} \psi_{j}(x)\psi_{k}(x)e^{-i(j-k)\phi},
\]

where

\[
\tau_{j,k}^{(m,h)} = \int_{-\pi}^{\pi} \frac{1}{\pi} \mathcal{R}[V_{m,h}](x, \phi)\psi_{j}(x)\psi_{k}(x)e^{-i(j-k)\phi}dx d\phi.
\]

For the sake of brevity, we set from now on \( \tau = \tau^{(m,h)} \). Note that the matrix \( \rho^{(m,h)} \) satisfies \( \rho_{j,k}^{(m,h)} = \rho_{j,k}^{(0)} + \tau_{j,k}^{(m,h)} \). Exploiting the above representation of \( \tau_{j,k}^{(m,h)} \), it is easy to see that \( \tau_{j,k}^{(m,h)} = \tau_{k,j}^{(m,h)} \) for any \( j, k \geq 0 \). On the other hand, \( \rho^{(0)} \) is a diagonal matrix with real-valued entries. This gives (i) immediately.

• We consider now (iii). First, note that \( \mathcal{R}[V_{m,h}](\cdot, \phi) \) is an odd function for any fixed \( \phi \). Indeed, its Fourier transform with respect of the first variable

\[
\mathcal{F}_{1} [\mathcal{R}[V_{m,h}](\cdot, \phi)](t) = \tilde{V}_{m,h}(t \cos \phi, t \sin \phi),
\]
Recall that \( k \) is an odd function of \( t \) for any fixed \( \phi \). Thus, it is easy to see that \( \tau_{j,j} = 0 \), for any \( j \geq 0 \). Since \( \rho^{(0)} \) is already known to be a density matrix, this implies that
\[
\text{Tr}(\rho^{(m,k)}) = \text{Tr}(\rho^{(0)}) + \text{Tr}(\tau) = 1.
\]

• Now prove (ii). From (13), we have
\[
|\tilde{f}_{k,j}(t)| = \pi^2 |t| |l_{j,k}(t)|/2, \quad j \geq k.
\]

Moreover by Lemma 1, we have
\[
l_{j,k}(x) \leq \frac{1}{\pi} \left\{ \begin{array}{ll}
1 & \text{if } 0 \leq x \leq \sqrt{j+k+1}, \\
e^{-\left(x-\sqrt{j+k+1}\right)^2} & \text{if } x > \sqrt{j+k+1}.
\end{array} \right.
\]
Then, by the change of variable \((t, \phi)\) into \(w = (w_1, w_2)\), (77) is such that
\[
|\tau_{j,k}| \leq \frac{1}{\pi} \int_0^\pi \int \tilde{V}(t \cos \phi, t \sin \phi) |\tilde{f}(t)| dt \int \tilde{V}(w) |l_{j,k}(||w||)/2| dw
\]
\[
\leq \int_{||w|| \leq \sqrt{J}} |\tilde{V}(w)| dw + \int_{||w|| > \sqrt{J}} |\tilde{V}(w)| e^{-(||w||-J)^2} dw = I_1 + I_2,
\]
where \( J = j + k + 1 \). The term \( I_1 \) can be bounded as follow
\[
I_1 = aC_0 h^{-1} e^\beta h^{-2} \int e^{-2\beta ||w||^2} g_m(||w||^2 - h^{-2}) g(w_2)/dw
\]
\[
\leq aC_0 h^{-1} e^\beta h^{-2} \int e^{-2\beta ||w||^2} g_m(||w||^2 - h^{-2}) dw,
\]
where \( C_0 = \sqrt{\pi L(\beta + \gamma)} \).
If \( k + j + 1 < a_2^2 / m \), then \( I_1 = 0 \). If \( k + j + 1 \geq a_2^2 / m \), then
\[
I_1 \leq aC_0 h^{-1} e^\beta h^{-2} \int e^{-2\beta ||w||^2} g_m(||w||^2 - h^{-2}) dw \leq aC_0 h^{-1} \delta^2 e^\beta h^{-2} e^{-2\beta a_m^2},
\]
(79)
\[
\leq aC_0 \delta^2 h^{-1} \delta^2 e^{-\beta a_m^2} \leq aC_1 \delta e^{-\beta J},
\]
where \( C_1 > 0 \) is a constant depending only on \( L, \beta, \gamma \). Similarly for \( I_2 \), we get
\[
I_2 = aC_0 h^{-1} e^\beta h^{-2} \int e^{-2\beta ||w||^2} g_m(||w||^2 - h^{-2}) c(||w|| - \sqrt{J})^2 dw
\]
\[
\leq aC_0 h^{-1} e^\beta h^{-2} \int e^{-2\beta ||w||^2} g_m(||w||^2 - h^{-2}) c(||w|| - \sqrt{J})^2 dw.
\]
If \( k + j + 1 \geq b_2^2 / m \), then \( I_2 = 0 \), otherwise if \( k + j + 1 \leq b_2^2 / m \), we have
\[
I_2 \leq aC_0 h^{-1} e^\beta h^{-2} \int e^{-2\beta ||w||^2} g_m(||w||^2 - h^{-2}) c(||w|| - \sqrt{J})^2 dw
\]
\[
\leq aC_0 h^{-1} \delta^2 e^\beta h^{-2} e^{-2\beta a_m^2} \leq aC_0 h^{-1} \delta^2 e^{-\beta a_m^2},
\]
\[
\leq aC_1 h^{-1} \delta^2 e^{-\beta a_m^2} \leq aC_1 \delta e^{-\beta J}.
\]
Combining (78), (79) and (80), we get for any \( j \neq k \) that
\[
|\tau_{j,k}| \leq c a\delta e^{-\beta(j+k+1)},
\]
for some numerical constant \( c > 0 \). Since \( \rho \) is an Hermitian matrix (iii), it admits real eigenvalues. For any eigenvalue \( \lambda \) of \( \rho \), in view of Theorem 4 below, there exists an integer \( j \geq 1 \) such that
\[
|\lambda - \rho_{j,j}^{(0)}| \leq \sum_{k=1 \atop k \neq j}^\infty |\tau_{j,k}| \leq c a\delta e^{-\beta j} =: r_j.
\]
Recall that \( \rho^{(0)} = \rho^{\alpha,\lambda} \) for some \( 0 < \alpha, \lambda < 1 \) where \( \rho^{\alpha,\lambda} \) is defined in (42). Lemme 2 in the paper of Butucea, Gută and Artiles (2007) guarantees that
\[
\rho^{\alpha,\lambda}_{j,j} = \frac{\alpha}{(1-\lambda)^\alpha} \Gamma(\alpha+1) j^{-(1+\alpha)} (1 + o(1)),
\]
as $n \to \infty$. We note that $\rho_{jj}^{(0)} > 0$ decreases polynomially with $j$ whereas $r_j$ decreases exponentially. Taking the numerical constant $a > 0$ small enough in (47) independently of $j$, we get $\rho_{jj} \geq r_j^2 \geq 0$. Thus $\rho$ is positive semi-definite.

**Theorem 4** (Gershgorin Disk Theorem). Let $A$ be an infinite square matrix and let $\mu$ be any eigenvalue of $A$. Then, for some $j \geq 1$, we have

$$|\mu - A_{jj}| \leq r_j(A),$$

where $r_j(A) = \sum_{k \neq j} |A_{jk}|$.

**Proof:** Let $\mu$ be an eigenvalue of $A$ with associated unit eigenvector $v = (v_1, v_2, \ldots)$. We have

$$\lambda v_k = [Av]_k = \sum_{l \geq 1} A_{kl} v_l.$$

We set $\tilde{k} = \arg\max_{k \geq 1} |v_k|$. Then

$$(\mu - A_{\tilde{k} \tilde{k}}) v_{\tilde{k}} = \sum_{l : l \neq \tilde{k}} A_{\tilde{k}l} v_l.$$

Consequently

$$|\mu - A_{\tilde{k} \tilde{k}}| \leq \sum_{l : l \neq \tilde{k}} |A_{\tilde{k}l}| |v_l| / |v_{\tilde{k}}| \leq \sum_{l : l \neq \tilde{k}} |A_{\tilde{k}l}| := r_{\tilde{k}}(A).$$

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**References**

Alquier, P., Meziani, K. and Peyré, G., Adaptive Estimation of the Density Matrix in Quantum Homodyne Tomography with Noisy Data. Inverse Problems, 29, 7, 075017, 2013.

Aubry, J.-M. and Butucea, C. and Meziani, K., State estimation in quantum homodyne tomography with noisy data. Inverse Problems, 25, 1, 2009.

Artiles, L. and Gill, R. and Guţă, M., An invitation to quantum tomography. J. Royal Statist. Soc. B (Methodological), 67,109–134, 2005.

Averbuch, A., Coifman, R.R., Donoho, D.L., Israeli, M., Shkolnisky, Y. and Seidelnikov, I., A framework for discrete integral transformations: II. The 2D discrete Radon transform. SIAM J. Sci. Comput., 30(2), 785–803, 2008.

Barndorff-Nielsen, O. E. and Gill, R. and Jupp, P. E., On quantum statistical inference (with discussion). J. Royal Stat. Soc. B, 65, 775–816, 2003.

Bergh, J. and Löfström, J., Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin, 1976.

Bousquet, O., A Bennett concentration inequality and its application to suprema of empirical processes. C. R. Math. Acad. Sci. Paris, 334, 6, 495–500, 2002.

Bourdaud, G. and Lanza de Cristoforis, M. and Sickel, W., Superposition operators and functions of bounded $p$-variation. Rev. Math. Iberoam., 2, 455–487, 2006.

Butucea, C. and Guţă, M. and Artiles, L., Minimax and adaptive estimation of the Wigner function in quantum homodyne tomography with noisy data. Ann. Statist., 2, 35, 465–494, 2007.

D'Ariano, G. M. and Macchiavello, C. and Paris, M. G. A., Detection of the density matrix through optical homodyne tomography without filtered back projection. Phys. Rev. A, 50, 4298–4302, 1994.

Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G., Higher transcendental functions. McGraw-Hill Book Company, Inc., New York-Toronto-London, Vols. I, II, 1953.

Giné, E. and Nickl, R., Uniform limit Theorems for wavelet density estimators. Ann. Probab., 37, 4, 1605–1646, 2009.
Guţă, M. and Artiles, L., Minimax estimation of the Wigner in quantum homodyne tomography with ideal detectors. Math. Methods Statist., 16, 1,1–15, 2007.

Helstrom, C. W., Quantum Detection and Estimation Theory. Academic Press, New York, 1976.

Holevo, A. S., Probabilistic and Statistical Aspects of Quantum Theory. North-Holland, 1982.

Leonhardt, U., Measuring the Quantum State of Light. Cambridge University Press, 1997.

Leonhardt, U. and Paul, H. and D’Ariano, G. M., Tomographic reconstruction of the density matrix via pattern functions. Phys. Rev. A, 52, 4899–4907, 1995.

Lounici, K. and Nickl, R., Global uniform risk bounds for wavelet deconvolution estimators. Ann. Statist., 39, 1, 201–231, 2001.

Meziani, K., Nonparametric goodness-of fit testing in quantum homodyne tomography with noisy data. Electron. J. Stat., 2, 1195–1223, 2008.

Meziani, K. Nonparametric Estimation of the Purity of a Quantum State in Quantum Homodyne Tomography with Noisy Data. Math. Meth. of Stat., 4, 16, 1–15, 2007.

Richter, T., Realistic pattern functions for optical homodyne tomography and determination of specific expectation values. Phys. Rev. A, 61, 2000.

Vogel, K. and Risken, H., Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase. Phys. Rev. A, 40, 2847–2849, 1989.

Wigner, E., On the quantum correction for thermodynamic equations. Phys. Rev., 40, 749–759, 1932.

Tsybakov, A.B., Introduction to Nonparametric Estimation. Springer Series in Statistics, New York, 2009.