η- Ricci solitons in α- para kenmotsu manifolds

Prabhavati G Angadi¹, G S Shivaprasanna² and G Somashekhara³
¹Department of Mathematics, Dr.Ambedkar institute of technology, Bengaluru-560056,India
²Department of Mathematics, Dr.Ambedkar institute of technology, Bengaluru-560056,India
³M.S.Ramaiah University of Applied Science, Bengaluru-560058, India
E-mail: ¹prabhavatiangadi71@gmail.com, ²shivaprasanna28@gmail.com, ³someshekhar96@gmail.com

Abstract. The motto of the paper is to examine “η-Ricci solitons” in “α-para Kenmotsu manifolds” with second order parallel tensor. Also study “η-Ricci solitons” are considered manifolds fulfilling some particular curvature conditions: \( S(\zeta, X_1) \cdot R = 0 \), \( R \cdot S = 0 \). Further we show that Pseudo quasi conformally flat is an Einstein with scalar curvature is constant.

keywords: “η-Ricci solitons”, Second order parallel tensor, Einstein manifold and “η- Einstein manifold”.

1. Introduction
In 1972 Kenmotsu have introduced Kenmotsu manifolds [12], and “almost Kenmotsu manifolds” to investigated in many angles. Maximum outcomes carried in [9],[10] would be simply generalized to the “class of almost α-Kenmotsu manifolds” [8]. The properties of “α-para Kenmotsu manifolds” have studied by Srivastava and Srivastava [20]. In the examination of Ricci flow, “Ricci Solitons” are the major things, since they are self-similar solutions of the flow. A “η-Ricci solitons” is a triple \( (g, \varphi, \lambda) \) with \( g \) a “Riemannian metric”, \( \varphi \) a “vector field” generated by \( \varphi_t \in \mathbb{R} \) and \( \lambda \) a real scalar” such that \( L_V g + 2S + 2\lambda g = 0 \). “Ricci soliton” is a generalization of Einstein metric. Where \( S \) is a Ricci tensor of \( M, L_V \) indicates the “Lie derivative” operator with the “vector field” \( V \). The “Ricci soliton” is noted “shrinking, steady and expanding when \( \lambda \) is negative, zero and positive” respectively.

Motivated by the above results we studied, in section 3 the η-Ricci Solitons in \( (\alpha - pkm)_3 \), in the section 4 we discuss η-Ricci Solitons in \( (\alpha - pkm)_3 \) Satisfying \( S(\zeta, B_1) \cdot R = 0 \). Section 5 is devoted to the study of η-Ricci Solitons on \( (\alpha - pkm)_3 \) satisfying \( R \cdot S = 0 \). In section 6, we establish η-Ricci Solitons in Pseudo Quasi Conformally flat \( (\alpha - pkm)_3 \). In section 7, we discuss the η-Ricci Solitons in Partially Ricci Pseudo Symmetric \( (\alpha - pkm)_3 \).

2. Preliminaries
A “smooth manifold \( M \)” of dim. \( 2n+1 \) is called an “almost paracontact manifold” [13],[15] provided with the structure \( (\varphi, \zeta, \eta) \) where \( \varphi \) is a tensor field of type \((1,1)\), a “vector field” \( \zeta \) and a 1-form \( \eta \) fulfilling

\[ \varphi^2 = I - \eta \otimes \zeta, \]

\[ \eta(\zeta) = 1, \]

\[ \varphi \zeta = 0, \eta \circ \varphi = 0, rank(\varphi) = 2n. \]
If an “almost paracontact manifold” $M$ acknowledges a “pseudo-Riemannian metric” $g$ fulfilling

$$\eta(B_1) = g(B_1, \zeta)$$

$$g(\varphi B_1, \varphi B_2) = -g(B_1, B_2) + \eta(B_1)\eta(B_2)$$

$$-g(\varphi B_1, B_2) = g(B_1, \varphi B_2),$$

A 3-dim. “normal almost paracontact metric manifold” of type $(\alpha, \beta)$ are Paracosymplectic, “$\alpha$-para Kenmotsu” and “para Kenmotsu” respectively [3],[21],[7].

In a 3-dim. “$\alpha$-Para Kenmotsu manifold”, the mentioned result holds [20]

$$R(B_1, B_2)B_3 = \left(\frac{r}{2} + 2\alpha^2\right)[g(B_2, B_3)B_1 - g(B_1, B_3)B_2] - \left(\frac{r}{2} + 3\alpha^2\right)[g(B_2, B_3)\eta(B_1) - g(B_1, B_3)\eta(B_2)]\zeta$$

$$+\left(\frac{r}{2} + 3\alpha^2\right)[\eta(B_1)B_2 - \eta(B_2)B_1]\eta(B_3),$$

$$S(B_1, B_2) = \left(\frac{r}{2} + \alpha^2\right)g(B_1, B_2) - \left(\frac{r}{2} + 3\alpha^2\right)\eta(B_1)\eta(B_2),$$

$$S(B_1, \zeta) = -2\alpha^2\eta(B_1),$$

$$R(B_1, B_2)\zeta = -\alpha^2\{\eta(B_2)B_1 - \eta(B_1)B_2\},$$

$$(D_{B_1}\eta)B_2 = \alpha\{g(B_1, B_2) - \eta(B_1)\eta(B_2)\},$$

$$(D_{B_1}\varphi)B_2 = \alpha\{g(\varphi B_1, B_2)\zeta - \eta(B_2)\varphi B_1\},$$

$$D_{B_1}\zeta = \alpha\{B_1 - \eta(B_1)\zeta\},$$

for all “vector fields” $B_1, B_2, B_3$ and $B_4 \in \chi(M)$. We assume 3- dimensional “$\alpha$-Para Kenmotsu Manifold” to be $(\alpha-pkm)_3$.

3. $\eta$-Ricci Solitons in $(\alpha-pkm)_3$

Let $(M, \varphi, \zeta, \eta, g)$ be an “almost paracontact metric manifold”, By examine the equation

$$L_{\zeta}g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0$$

where $L_{\zeta}$ is the “Lie derivative operator” with the “vector field” $\zeta$, $S$ is the Ricci curvature tensor field of the metric $g$, “$\lambda$ and $\mu$” are real constants. Creating $L_{\zeta}g$ in terms of the “Levi-Civita connection $D$", we get:

$$2S(B_1, B_2) = -g(D_{B_1}\zeta, B_2) - g(B_1, D_{B_2}\zeta) - 2\lambda g(B_1, B_2) - 2\mu\eta(B_1)\eta(B_2),$$

for any $B_1, B_2 \in \chi(M)$.

$$S(B_1, B_2) = -(\alpha + \lambda)g(B_1, B_2) + (\alpha - \mu)\eta(B_1)\eta(B_2)$$

The content $(g, \zeta, \lambda, \mu)$ it delivers the equation (15) is called as “$\eta$-Ricci solitons” on $M$ [5]; in some specific, it is labeled “Ricci soliton” when $\mu = 0$ and it is noted to be a shrinking, steady or expanding based on $\lambda$ is negative, zero or positive respectively [6].

**Definition 1** A tensor $\vartheta$ of second order is said to be a parallel tensor if $D\vartheta = 0$, where $D$ indicates the operator of covariant differentiation with respect to the metric tensor $g$. 

Let $\vartheta$ be a $(0,2)$-symmetric tensor field on a $(\alpha - pkm)_3 M$ such that $D\vartheta = 0$. Using Ricci identity [18] we get

\[
D^2\vartheta(B_1, B_2; B_3, B_4) - D^2\vartheta(B_1, B_2; B_4, B_3) = 0
\]

(18)

and (11) in the above equation, we get

\[
\vartheta(R(B_1, B_1)B_2, B_3) + \vartheta(B_2, R(B_1, B_1)B_3) = 0
\]

(19)

for arbitrary vector field $B_4, B_1, B_2, B_3$ on $M$. The substitution of $B_2 = B_3 = \zeta$ in (19) gives

\[
\vartheta(\zeta, R(B_4, B_1)\zeta) = 0
\]

(20)

Since $\vartheta$ is symmetric. By using the expression (8) for $(\alpha - pkm)_3$ and (11) in the above equation, we get

\[
\alpha^2[g(B_4, \zeta)\vartheta(B_1, \zeta) - g(B_1, \zeta)\vartheta(B_4, \zeta)] = 0
\]

(21)

\textbf{Definition 2} If $(\alpha^2 \neq 0)$ then $M^{2n+1}(\zeta)$ its known as regular.

In the sense of getting a characterisation of such manifolds we consider:

\textbf{Definition 3} [16] $\zeta$ is known as semi-torse forming “vector field for $(M, g)$”, for all “vector fields” $B_1$:

\[
R(B_1, \zeta)\zeta = 0
\]

(22)

From (8) we get

\[
R(B_1, \zeta)\zeta = -\alpha^2 B_1 - \eta(B_1)\zeta
\]

(23)

and therefore, if $B_1\eta = \zeta^\perp$, then $R(B_1, \zeta)\zeta = -\alpha^2 B_1$ and we obtain:

\textbf{Proposition 1} For $M^{2n+1}(\zeta)$ the followig are equivalent:

\begin{enumerate}
  \item[i)] $\zeta$ is regular,
  \item[ii)] $\zeta$ is not semi-torse forming,
  \item[iii)] $S(\zeta, \zeta) \neq 0$ i.e., $\zeta$ is non-degenetate with respect to $S$,
  \item[iv)] $Q(\zeta) \neq 0$ i.e., $\zeta$ does not belong to the kernel of $Q$. In particular, if $\zeta$ is parallel $(D\zeta = 0)$ then $M$ is not regular.
\end{enumerate}

Regards to the above we restrict to the regular case. Returning to (21), with $B_1 = \zeta$ then we obtain:

\[
\alpha^2\{\eta(B_1)\vartheta(\zeta, \zeta) - \vartheta(B_1, \zeta)\} = 0
\]

(24)

By differentiating (24) covariantly along $B_2$, we get

\[
\alpha^2\{[g(DB_2B_1, \zeta) + g(B_1, DB_2\zeta)]\vartheta(\zeta, \zeta)
+ 2g(B_1, \zeta)\vartheta(DB_2\zeta, \zeta) - [\vartheta(DB_2B_1, \zeta) + \vartheta(B_1, DB_2\zeta)]\} = 0
\]

(25)

put $B_1 = DB_2B_1$ in (21)

\[
\alpha^2\{g(DB_2B_1, \zeta)\vartheta(\zeta, \zeta) - \vartheta(DB_2B_1, \zeta)\} = 0
\]

(26)

From (25) and (26), we get

\[
-\alpha^3[g(B_1, B_2) - \eta(B_2)\eta(B_1)]\vartheta(\zeta, \zeta) - 2\alpha^3\eta(B_1)[\vartheta(B_2, \zeta) - \eta(B_2)\vartheta(\zeta, \zeta)]
+ \alpha^3[\vartheta(B_1, B_2) - \eta(B_2)\vartheta(B_1, \zeta)] = 0
\]

(27)

Replace $B_1$ by $\varphi B_2$ in (24), we have

\[
\alpha^2\vartheta(\varphi B_2, \zeta) = 0
\]

(28)
Replace $B_2$ by $\varphi B_2$ in (27) and using (28) we get
\[
\alpha^3 \{g(B_1, \varphi B_2) \vartheta(\zeta, \zeta) - \vartheta(B_1, \varphi B_2)\} = 0 \quad (29)
\]
Again replace $B_2$ by $\varphi B_2$ in (29) and using (2) and (24), we get
\[
\alpha^3 [g(B_1, B_2) \vartheta(\zeta, \zeta) - \vartheta(B_1, B_2)] = 0 \quad (30)
\]
By differentiating (30) invariantly along any “vector field on $M$”, it can be easily seen that $\vartheta(\zeta, \zeta)$ is constant when $\alpha^3 \neq 0$. Hence we can state the following theorem:

**Theorem 1** Let $M$ be a $(\alpha - pkm)_3$ with non vanishing $\zeta$ sectional curvature and admit with a tensor field $\vartheta$ which is symmetric. If $\vartheta$ is parallel with respect to $D$ then it is a constant multiple of metric tensor $g$ when $\alpha^3 \neq 0$.

Now,
\[
\vartheta(B_1, B_2) = L_\zeta g(B_1, B_2) + 2S(B_1, B_2) + 2\mu \eta(B_1) \eta(B_2) \quad (31)
\]
Put $B_1 = B_2 = \zeta$
\[
\vartheta(\zeta, \zeta) = -2\lambda \quad (32)
\]
which implies
\[
\vartheta(B_1, B_2) = -2\lambda g(B_1, B_2) \quad (33)
\]
for any $B_1, B_2 \in \chi(M)$ (31) becomes
\[
L_\zeta g + 2S + 2\mu \eta \otimes \eta = -2\lambda g \quad (34)
\]
we conclude that

**Theorem 2** On $(\alpha - pkm)_3$ with the property that the symmetric tensor field $\vartheta(B_1, B_2) = L_\zeta g(B_1, B_2) + 2S(B_1, B_2)$ is parallel with respect to $D$ associated to $g$, the “Ricci soliton” on $M$ defines $\lambda = 2\alpha^2$ when $\mu = 0$.

**Proposition 2** Under this hypothesis if $\alpha$ is positive or negative then Ricci soliton is expanding.

4. **$\eta$-Ricci Solitons in $(\alpha - pkm)_3$ Satisfying** $S(\zeta, B_1) \cdot R = 0$

The condition that must be fulfilled as $S(\zeta, B_1) \cdot R = 0$
\[
S(B_1, R(B_2, B_3)B_4)\zeta - S(\zeta, R(B_2, B_3)B_4)B_1 + S(B_1, B_2)R(\zeta, B_3)B_4 \\
- S(\zeta, B_2)R(B_1, B_3)B_4 + S(B_1, B_3)R(B_2, \zeta)B_4 - S(\zeta, B_3)R(B_2, B_1)B_4 \\
+ S(B_1, B_4)R(B_2, B_3)\zeta - S(\zeta, B_4)R(B_2, B_3)B_1 = 0
\]
For any $B_1, B_2, B_3, B_4 \in \chi(M)$
Taking inner product with $\zeta$ to the relation (35) and by virtue of (8),(10).
\[
\begin{align*}
\frac{r}{2} + 2\alpha^2 & [g(B_3, B_4)S(B_1, B_2) - g(B_2, B_4)S(B_1, B_3)] \\
- \frac{r}{2} + 3\alpha^2 & [g(B_3, B_4)\eta(B_2) - g(B_2, B_4)\eta(B_3)]S(B_1, \zeta) \\
+ \frac{r}{2} + 3\alpha^2 & [\eta(B_2)S(B_1, B_3) - \eta(B_3)S(B_1, B_2)]\eta(B_4) \\
- \frac{r}{2} + 2\alpha^2 & [g(B_3, B_4)S(\zeta, B_2) - g(B_2, B_4)S(\zeta, B_3)] \\
- \frac{r}{2} + 3\alpha^2 & [g(B_3, B_4)\eta(B_2) - g(B_2, B_4)\eta(B_3)]S(\zeta, \zeta) + \left( \frac{r}{2} + 3\alpha^2 \right)[\eta(B_2)S(\zeta, B_3) \\
- \eta(B_3)S(\zeta, B_2)[\eta(B_1) + S(B_1, B_2) - \alpha^2[g(B_3, B_4) - \eta(B_3)\eta(B_4)] \\
- S(\zeta, B_2) - \alpha^2[g(B_3, B_4)\eta(B_1) - g(B_1, B_4)\eta(B_3)] \\
+ S(B_1, B_3) - \alpha^2[\eta(B_4)\eta(B_2) - g(B_2, B_4)] - S(\zeta, B_3) - \alpha^2[g(B_1, B_4)\eta(B_2) - g(B_2, B_4)\eta(B_1)] \\
- S(\zeta, B_4) - \alpha^2[g(B_3, B_1)\eta(B_2) - g(B_2, B_1)\eta(B_3)] = 0
\end{align*}
\]
Taking $B_4 = B_3 = \zeta$ to (36) Which implies
\[
S(B_1, B_2) = 2\alpha^2 g(B_1, B_2) - 4\alpha^2 \eta(B_1)\eta(B_2)
\]

Lemma 1 Let $M$ be a $(\alpha - pkm)_3$ fulfilled $S(\zeta, B_1) \cdot R = 0$ then $M$ is an “$\eta$- Einstein manifold” with scalar curvature $2\alpha^2$.

Then from equations (17) and (37), we get
\[
2\alpha^2 g(B_1, B_2) - 4\alpha^2 \eta(B_1)\eta(B_2) = -(\alpha + \lambda)g(B_1, B_2) + (\alpha - \mu)\eta(B_1)\eta(B_2)
\]
Substitution of $B_1 = \zeta$ in (38), we get the relation
\[
\lambda + \mu = 2\alpha^2
\]

Theorem 3 Let $M$ be $(\alpha - pkm)_3$ admitting the “$\eta$-Ricci solitons” with satisfying $S(\zeta, B_1) = 0$ then $\lambda + \mu = 2\alpha^2$.

Proposition 3 i) A $(\alpha - pkm)_3$ satisfying $S(\zeta, B_1) \cdot R = 0$ then a “Ricci soliton” in $M$ is expanding when $\alpha$ is positive or negative.

ii) Let $(\alpha - pkm)_3$ be Paracosympletic manifold ($\alpha = 0$) satisfying $S(\zeta, B_1) \cdot R = 0$ then “Ricci soliton” in $M$ is steady.

5. $\eta$-Ricci Solitons on $(\alpha - pkm)_3$ satisfying $R \cdot S = 0$

A $(\alpha - pkm)_3$ are equivalent by comparing to the followed statements. $M$ is (1) Einstein (2) $DS = 0$ (3) $R \cdot S = 0$.

The imputation “(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial”. Now we establish the imputation “(3) $\Longrightarrow$ (1)” and $R \cdot S = 0$ means exactly (19) with substituted $\vartheta$ by $S$. Note that, the process that must be fulfilled by $S$ is:
\[
S(R(\zeta, B_1)B_2, B_3) + S(B_2, R(\zeta, B_1)B_3) = 0
\]
For any $B_1, B_2, B_3 \in \chi(M)$.

In view of (9) and (40)
\[
2\alpha^4 g(B_1, B_2)\eta(B_3) + \alpha^2 \eta(B_2)S(B_1, B_3) + 2\alpha^4 g(B_1, B_3)\eta(B_2) + \alpha^2 \eta(B_3)S(B_1, B_2) = 0
\]
For $B_3 = \zeta$ we have
\[
S(B_1, B_2) = -2\alpha^2 g(B_1, B_2)
\]
Lemma 2 A locally Ricci symmetric $(D S=0)$ $(\alpha - pkm)_3$ is an Einstein manifolds.

Theorem 4 On $(\alpha - pkm)_3$ there is no non-zero second order skew symmetric parallel tensor

Proposition 4 i) Let $M$ be a $(\alpha - pkm)_3$ satisfying $R \cdot S = 0$. Then “Ricci soliton” in $M$ is
a) Shrinking provided $\alpha > 0$ and $\alpha = 1$ (para Kenmotsu manifold).
b) Expanding provided $\alpha < 0$.
c) Steady provided $\alpha = 0$ (Paracosympletic manifold).

6. $\eta$-Ricci Solitons in Pseudo Quasi Conformally flat $(\alpha - pkm)_3$

Definition 4 The Pseudo quasi conformal curvature tensor [17] $L$ on a $(\alpha - pkm)_3$ is defined by

$$L(B_1, B_2)B_3 = (p + d)R(B_1, B_2)B_3 + \left( q - \frac{d}{n - 1} \right) [S(B_2, B_3)B_1 - S(B_1, B_3)B_2]$$
$$+ q[g(B_2, B_3)QB_1 - g(B_1, B_3)QB_2] - \frac{r}{n(n - 1)} \{ p + 2(n - 1)q \} [g(B_2, B_3)B_1 - g(B_1, B_3)B_2]$$

(43)

for all vector fields $B_1, B_2, B_3$ where $p, q, d$ are arbitrary constants not simultaneously zero, $S$ is the "Ricci tensor", $Q$ is the Ricci operator, $R$ is the Reimannian curvature tensor and $r$ is the scalar curvature tensor of the manifold $M$.

We consider $(\alpha - pkm)_3$ $M$ which is Pseudo-Quasi conformally flat. Then from definition (2) and (43) we have

$$(p + d)R(B_1, B_2)B_3 + \left( q - \frac{d}{2} \right) [S(B_2, B_3)B_1 - S(B_1, B_3)B_2]$$
$$+ q[g(B_2, B_3)QB_1 - g(B_1, B_3)QB_2] - \frac{r}{6} \{ p + 4q \} [g(B_2, B_3)B_1 - g(B_1, B_3)B_2] = 0$$

(44)

Contracting with $B_1$ and $B_2$ in (44), we obtain

$$(p + d)S(B_2, B_3) + \left( q - \frac{d}{2} \right) [3S(B_2, B_3) - S(B_1, B_3)] + q [rg(B_2, B_3) - g(QB_2, B_3)]$$
$$- \frac{r}{6} \{ p + 4q \} [2g(B_2, B_3)] = 0$$

(45)

Hence we get

$$S(B_2, B_3) = -2g(B_2, B_3)$$

(46)

Lemma 3 A pseudo quasi conformally flat $(\alpha - pkm)_3$ is an Einstein with constant scalar curvature.

In view of (17) and (46), we state that

Theorem 5 Let $M$ be a pseudo quasi conformally flat $(\alpha - pkm)_3$ then $\eta$-Ricci soliton in $M$ is $\lambda + \mu = 2$.

Proposition 5 i) A Ricci soliton in pseudo quasi conformally flat para Kenmotsu manifold $(\alpha = 1)$ is expanding.
ii) A Ricci soliton in pseudo quasi conformally flat paracosympletic $(\alpha = 0)$ is expanding.
7. $\eta$-Ricci Solitons in Partially Ricci Pseudo Symmetric $(\alpha - pkm)_3$

**Definition 5** An $(\alpha - pkm)_3$ is called partially Ricci-pseudo symmetric if and only if the relation

$$R \cdot S = j(p)Q(g, S)$$  \hspace{1cm} (47)

hold on the set $A = \{ B \in M : Q(g, S) \neq 0 \text{ at } B \}$. where $j \in C^\infty(M)$ for $p \in A$. $R \cdot S$, $Q(g, S)$ and $(B_1 \land B_2)$ are respectively defined as

$$(R(B_1, B_2)S)(U, V) = -S(R(B_1, B_2)U, V) - S(U, R(B_1, B_2)V)$$  \hspace{1cm} (48)

$$Q(g, S) = ((B_1 \land B_2) \cdot S)(U, V)$$  \hspace{1cm} (49)

$$(B_1 \land B_2)B_3 = g(B_2, B_3)B_1 - g(B_1, B_3)B_2$$  \hspace{1cm} (50)

for all $B_1, B_2, U$ and $V \in TM^p$.

Let us consider partially Ricci-pseudo symmetric $(\alpha - pkm)_3$. Then from definition (14), we have

$$(R(B_1, B_2) \cdot S)(B_3, U) = j(p)[(B_1 \land B_2) \cdot S](B_3, U)$$  \hspace{1cm} (51)

From equations (8) and (49), it follows that

$$S(R(B_1, B_2)B_3, U) + S(B_3, R(B_1, B_2)U) = j(p)[S((B_1 \land B_2)B_3, U) + S(B_3, (B_1 \land B_2)U]$$  \hspace{1cm} (52)

Taking $B_2 = U = \zeta$ in Applying (8), (10) in (52), we have

$$\alpha^2g(B_1, B_3)S(\zeta, \zeta) - \eta(B_3)S(B_1, \zeta) - \alpha^2[S(B_1, B_3) - \eta(B_3)S(B_3, \zeta)]$$

$$= j(p)[\eta(B_3)S(B_1, \zeta) - g(B_1, B_3)S(\zeta, \zeta) + S(B_1, B_3) - \eta(B_1)S(B_3, \zeta)],$$

This can be written as

$$-2\alpha^2[j(p) + \alpha^2]g(B_1, B_3) - S(B_1, B_3)[j(p) + \alpha^2] = 0$$  \hspace{1cm} (54)

Thus, we have which gives

$$S(B_1, B_3) = -2\alpha^2g(B_1, B_3) \text{ provided } j(p) \neq -\alpha^2$$  \hspace{1cm} (55)

Hence, we state the following lemma:

**Lemma 4** : A partially Ricci pseudo symmetric $(\alpha - pkm)_3$ is an Einstein manifold provided $j(p) \neq -\alpha^2$.

Let a partially Ricci pseudo symmetric $(\alpha - pkm)_3$ admits “$\eta$-Ricci solitons” on $M$.

Then from (17) and (55), we get

$$(\alpha + \lambda - 2\alpha^2)g(B_1, B_3) - (\alpha - \mu)\eta(B_1)\eta(B_3) = 0$$  \hspace{1cm} (56)

Take $B_1 = \zeta$ in (56), we obtain

$$\lambda + \mu = 2\alpha^2$$  \hspace{1cm} (57)

Hence we state that the following theorem:

**Theorem 6** A partially Ricci pseudo symmetric $(\alpha - pkm)_3$ $M$ is in “$\eta$-Ricci Solitons” on $M$ then “$\lambda + \mu = 2\alpha^2$”.

**Proposition 6** A “Ricci soliton” in $(g, \zeta, \lambda)$ in partial Ricci Pseudo symmetric $(\alpha - pkm)_3$ is expanding provided $\alpha$ is positive or negative.

**Proposition 7** A partially Ricci pseudo symmetric $(\alpha - pkm)_3$ is not an Einstein manifold provided $j(p) = -\alpha^2$. 
References

[1] Bagewadi CS, Ingalahalli G and Ashoka SR 2013 A Study on Ricci Solitons in Kenmotsu manifolds ISRN Geometry vol.2013, Article ID 412593, 6 pages.

[2] Bejan CL 1988 Almost parahermitian structures on the tangent bundle of an almost para-coHermitian manifold In: The proceedings of the Fifth National seminar of Finsler and Lagrange Space Soc. vert, Stine te Mat. R. S. Romania, Bucharest, (1989) 105-109.

[3] Blaga AM η-Ricci solitons on para-Kenmotsu manifolds 2014 reprint arXiv 0223v3 1402.

[4] Calvaruso G and Perron D 2014 η-Ricci solitons on para-Kenmotsu manifolds arXiv 0223v3 1402.

[5] Cho JT and Kimura M 2009 Ricci solitons and real hyper surfaces in a conleb space form Tohoku Math. J 61 205-212.

[6] Chow B, Lu P, Ni L and Hamilton’s 2006 Ricci flow, Graduate Studies in Mathematics AMS, Providence, RI, USA 77.

[7] Dacko P 2004 On almost para-cosymplectic manifolds Tsukuba J. Math 28 193-213.

[8] Dileo G 2011 On the Geometry of almost contact metric manifolds of Kenmotsu type Differential Geom. Appl 29 558-564.

[9] Dileo G and Pastore AM 2007 Almost Kenmotsu manifolds and local symmetry Bull. Belg. Math. Soc. Simon Stevin 14 343-354.

[10] Dileo G and Pastore AM 2009 Almost Kenmotsu manifolds and nullity distribution J. Geom 93 46-61.

[11] Hamilton RS 1986 The Ricci flow on surfaces Math. and general relativity (Santa Cruz, CA, Contemp. Math AMS (1988) 71 237-262.

[12] Kenmotsu K 1971 A class of almost contact Riemannian manifolds T^ω ohoku Math. J 24 93-103.

[13] Manev M and Staikova M 2001 On almost paracontact Riemannian manifolds of type (n, n) J. Geom 72 108-114.

[14] Maralabhavi YB and Shivaparasanna G S 2012 Second order parallel tensors on generalized sasakian spaceforms ISSN 1 Issue 10 2277-6982.

[15] Nakova G and Zamkovoy S 2009 Almost paracomplex manifolds arXiv 3859v2 0806.

[16] Rachunek L and Mikes J 2005 On tensor fields semiconjugated with torseforming vector fields Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. 44 151-160 MR 2218574 (2007b:53038).

[17] Shaik AA and Sanjib Kumar Jana 2005 A pseudo Quasi-conformalcurvature tensor on a Riemannian manifolds South East Asian J. Math. Math. Sci. 4(1). pp.15-20.

[18] Sharma R 1989 Second order parallel tensor in real and compleB space forms International. J. Math. Math. Sci 12 787-790.

[19] Shivaparasanna GS and Maralabhavi YB 2014 Ricci soliton in 3-dimensional (ε, δ)-Trans-sasakian structure ISSN 2229-5046 5(4) 258-265.

[20] Srivastava K and Srivastava SK 2014 On a class of α-Para Kenmotsu Manifolds Mediterr. J. Math. DOI 10.1007/s00009 014-0496-9.

[21] Welyczko J 2014 Slant curves in 3-dimensional normal almost paracontact metric manifolds Mediterr. J. Math. 11 965-978.