A Note on Quantum Cloning in $d$ dimensions

Paolo Zanardi

Institute for Scientific Interchange Foundation, Villa Gualino
Viale Settimo Severo 65, I-10133 Torino, Italy
and Unità INFN, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

The quantum state space $\mathcal{S}$ over a $d$-dimensional Hilbert space is represented as a convex subset of a $D-1$-dimensional sphere $S_{D-1} \subset \mathbb{R}^D$, where $D = d^2 - 1$. Quantum transformations (CP-maps) are then associated with the affine transformations of $\mathbb{R}^D$, and $N \mapsto M$ cloners induce polynomial mappings. In this geometrical setting it is shown that an optimal cloner can be chosen covariant and induces a map between reduced density matrices given by a simple contraction of the associated $D$-dimensional Bloch vectors.

I. INTRODUCTION

The quantum no-cloning theorem [1] represents the most basic difference between quantum and classical information theory. It stems simply from the unitary character of any allowable evolution for a closed quantum system. Since perfect copying of quantum information is forbidden it is a relevant (conceptually as well as practically) question to ask how close one can get to that ideal (unphysical) process, and in what way. More formally one has to face a complex optimization problem involving all allowed quantum transformations between multipartite Hilbert spaces (CP maps, [2]).

Several papers, addressing this issue, have appeared recently. Optimal fidelities and explicit forms for the cloning problem for an arbitrary quantum system will be discussed. In particular the optimization problem of imperfect cloning will be formulated in geometric fashion.

A. The GB Representation

Let $\mathcal{H}$ be a $d$-dimensional Hilbert space. The set $End(\mathcal{H})$ of linear operators over $\mathcal{H}$ can be endowed with a metric structure in several ways. In view of its direct connection with the geometrical framework of this paper, we shall consider $End(\mathcal{H})$ as a metric space with distance

$$d(A, B) = 2^{-1/2} \sqrt{(A - B, A^\dagger - B^\dagger)}$$

induced by the Hilbert-Schmidt scalar product $(A, B) \equiv \text{tr} A B^\dagger$ (the normalization has been chosen for later convenience). The Lie algebra of hermitian $d \times d$ traceless matrices, $su(d)$, is a $D$-dimensional real subspace of $End(\mathcal{H})$, where $D = d^2 - 1$. One can choose a basis $\{\tau_i\}_{i=1}^D$ of $su(d)$ satisfying the relations $[\tau_i, \tau_j] = 2 \delta_{ij}$. The set $\mathcal{B}_1$ of the unit-trace Hermitian operators is a $D$-dimensional hyperplane of $End(\mathcal{H})$. Any element of $\mathcal{B}_1$ can be written as

$$\rho(\lambda) = \frac{1}{d} \mathbb{I} + \frac{1}{2} \sum_{i=1}^D \lambda_i \tau_i$$

The vector $\lambda \equiv (\lambda_1, \ldots, \lambda_D) \in \mathbb{R}^D$ will be referred to as the Generalized Bloch Representation [GBR] of $\rho$. Equation (2) defines a mapping $m: \mathcal{B}_1 \rightarrow \mathbb{R}^D$ that associates to any $\rho \in \mathcal{B}_1$ its GBR vector, such that $\rho = m(\rho)$. Let $\mathcal{P} \subset \mathcal{B}_1$ the set of pure states on $\mathcal{H}$, and $\mathcal{S} = \text{hull}(\mathcal{P})$ its convex hull (the state space). The corresponding objects over $\mathcal{H}^\otimes N$ will be denoted by same notation with an extra index $N$.

In the following $\mathbb{R}^D$ will be considered endowed with the geometrical structure associated with the euclidean scalar...
Proposition 2 Let \( T : B_1 \mapsto B_1 \) be a trace-preserving CP-map. Then: i) \( T(I/d) = I/d + \sum_j c_j \tau_j \); ii) \( T(\tau_i) = \sum_{j=1}^D M_{ji} \tau_j \).

Proof i) \( T(I/d) \) must be a trace one hermitian operator by definition of CP-map. ii) \( T(\tau_i) \) must be traceless and hermitian; the statement of the fact that \( \tau_i \) are a \( su(d) \) basis.

Therefore, if \( \rho \) has the form \( \rho = M + c \), \( M = (M_{ij}) \in \text{End}(R^D) \), \( c \in R^D \).

\[ T(\rho) = \frac{1}{d} I + \frac{1}{d} \sum_{j=1}^D \lambda_j M_{ji} \tau_j \]

(4)

where \( \lambda' = M(\lambda) + c \), \( M = (M_{ij}) \in \text{End}(R^D) \), \( c \in R^D \).

This realizes an (affine) mapping \( M \) between the trace-preserving CP-maps on \( B_1 \) and the affine transformations of \( R^D \) in itself. \( M : CP(B_1) \mapsto A(\text{aff}(R^D)) : T \mapsto M(T) = m \circ T \circ m^{-1} \)

A particularly relevant class of CP-maps is given by the unitary transformations. Any \( X \in SU(d) \) defines, via the adjoint action, a CP-map on \( B_1 \), \( \rho \mapsto AdX(\rho) = X \rho X^\dagger \).

The following proposition shows that unitary transformations correspond, in the GBR, to rotations.

Proposition 3 \( \varphi \equiv M \circ Ad \) is a homomorphism of \( SU(d) \) in \( SO(D) \).

Proof First observe that from \( AdX(I) = I \) (\( \forall X \in SU(d) \)) there follows that \( \varphi : M(T) = \text{linear} \). Since obviously \( M(T_1 T_2) = M(T_1) M(T_2) \), one has just to check that, for any \( X \in SU(d) \), the mapping \( \lambda \mapsto m(AdX(m^{-1}(\lambda))) \) preserves scalar product (and then the norm) on \( R^D \). Indeed \( (\mu, \lambda) = 2(tr[\rho(\lambda)\rho(\mu)] - 1/d) \), and the trace is \( Ad \)-invariant. Since \( X \tau_i = \sum_j X_{ji} \tau_j \), one has that the induced \( R^D \) mapping has the form \( \lambda \mapsto X(\lambda) \) where the matrices \( X = (X_{ji}) \) are the adjoint representatives of \( SU(d) \).

Since \( SU(d) \) acts (via \( Ad \)) transitively on \( P \), it follows immediately that the subgroup \( \varphi(SU(d)) \) acts transitively over \( m(P) \).

Once again it is worth emphasizing that, for \( d > 2 \), \( \varphi(SU(d)) \) is a proper subset of \( SO(D) \). This can be easily understood observing that any pair \( \lambda, \mu \) of points of \( S_{D-1} \) are connected by an orthogonal transformation \( R_{\lambda,\mu} \), in particular one can have \( \lambda \in m(P) \) and \( \mu \notin m(P) \).

Since \( m(P) \) is \( SU(d) \)-invariant, \( R_{\lambda,\mu} \notin \varphi(SU(d)) \).

B. Optimality

The metric structure over \( S \) allows us to introduce several natural ’figures of merit’ for cloning. For example let us consider, for given \( T \in M(B_1) \), the functional \( F_1 : S \mapsto [0, 1] \) given by
\[
F_1(T, \rho) = 1 - |d(\rho, T(\rho))|^2 \\
= 2^{-1} \delta(T(\rho)) + F(T, \rho),
\]
where \(\delta(T(\rho)) = 1 - \text{tr} T^2(\rho)\) is the idempotency deficit (or linear entropy) of \(T(\rho)\) and \(F(T, \rho) \equiv \langle T(\rho), \rho \rangle\), is the pure state fidelity \([1]\).

The naturality of \(F_1\) as (state dependent) measure of cloning goodness should be clear: it is maximum (equal to 1) when \(\rho = T(\rho)\) and minimum (0) when \(\rho\) and \(T(\rho)\) have disjoint supports. Moreover both contributions \(\delta\) and \(F\) to the 'merit' function \(F_1\) have a clear geometrical meaning in \(\mathbb{R}^D\). Indeed, by using the GBR one finds (from Proposition 1)
\[
\delta(T(\rho)) = \frac{1}{2} (R_2^2 - \|T(\lambda)\|^2), \\
F(T, \rho) = \frac{1}{d} + \frac{1}{2} \langle T(\lambda), \lambda \rangle.
\]

It is interesting to consider a special class of transformations for which the quality of the cloning process is independent on the (pure) input state \([\mathcal{F}]\). This motivates the following

**Definition 1** A map \(T \in \mathcal{M}(\mathcal{S})\) is universal if \(F_1(T, \rho)\) is independent on \(\rho \in \mathcal{P}\).

For general maps (i.e. non universal) one can be interested in optimizing the worst case, with pure initial input. Therefore it is natural to introduce the quantity
\[
\tilde{F}_1(T) = \min_{\rho \in \mathcal{P}} F_1(T, \rho).
\]

The following proposition will turn to be useful:

**Proposition 4** i) \(\tilde{F}_1\) is a concave functional over \(\mathcal{M}(\mathcal{S})\).

ii) If \(U \in SU(d)\) and \(T_U \in \mathcal{M}(\mathcal{S})\) is defined by \(T_U(\rho) = U^\dagger T(U \rho U^\dagger) U\), one has \(\tilde{F}_1(T_U) = \tilde{F}_1(T)\).

**Proof**

i) Let \(T_1, T_2 \in \mathcal{M}(\mathcal{S}), \mu \in \mathbb{R}_+^2\). In view of the concavity of \(\delta\) one has \(\tilde{F}_1(\mu T_1 + (1 - \mu) T_2) \geq 2^{-1} \mu \tilde{F}_1(T_1(\rho)) + 2^{-1} (1 - \mu) \tilde{F}_1(T_2(\rho)) + \mu F_1(T_1(\rho)) + (1 - \mu) F_1(T_2(\rho))\). Then, by the superadditivity of the infimum one gets
\[
\tilde{F}_1(\mu T_1 + (1 - \mu) T_2) \geq \mu \tilde{F}_1(T_1) + (1 - \mu) \tilde{F}_1(T_2).
\]

ii) Explicitly using \(SU(d)\)-invariance of the metric, and transitivity of \(SU(d)\)-action over \(\mathcal{P}\),
\[
\tilde{F}_1(T_U) = \inf_{\pi \in \mathcal{S}} F(T_U, \rho) = \inf_{\pi \in \mathcal{S}} \inf_{\sigma \in \mathcal{S}} F(1 - d^2(U^\dagger \sigma U, U^\dagger T(\sigma) U)) \]
\[
= \inf_{\pi \in \mathcal{S}} \inf_{\sigma \in \mathcal{S}} F(1 - d^2(\sigma, T(\sigma))) = \tilde{F}_1(T).
\]

The mapping \(T \to T_U\) defines a \(SU(d)\)-action \(\Phi\) such that \(T_U \equiv \Phi(U, T)\) over \(\mathcal{M}(\mathcal{S})\). Point ii) of the previous proposition simply states that the quality of cloning is constant along the orbits of \(\Phi\). The fixed points of \(\Phi\) therefore play a special role.

**Definition 2** A map \(T \in \mathcal{M}(\mathcal{S})\) is covariant iff \(T_X = T, \forall X \in SU(d)\).

Next proposition shows that covariance implies universality and imposes strong geometrical constraints to the GBR.

**Proposition 5** Suppose \(T \in \mathcal{M}(\mathcal{S})\) covariant. Then: i) \(T\) is universal, ii) \(U \tilde{F}_1(\lambda) = T_U(\tilde{U}(\lambda))\), \(\forall \lambda \in \mathbb{R}_+^D\), \(U \in \varphi(SU(d))\); iii) \(||T(\lambda)||\) and \(||T(\lambda, \lambda)||\) are constant over \(m(\mathcal{P})\)

**Proof** i) Since \(\text{Ad SU}(d)\) is transitive over \(\mathcal{P}\), it suffices to show that \(F_1(T, \rho) = F_1(T, \text{Ad X}(\rho))\) (\(\forall X \in SU(d), \rho \in \mathcal{P}\)). Indeed, \(\delta(T(\text{Ad X}(\rho))) = \delta(\text{Ad X}T(\rho)) = \delta(T(\rho))\), and
\[
F(T, \rho) = \text{tr} (\rho T(\rho)) = \text{tr} (X^\dagger \rho T(\rho) X) = \text{tr} (X^\dagger \rho X T(X^\dagger \rho X)) = \text{tr} (\text{Ad X}(\rho) T(\text{Ad X}(\rho))).
\]

ii) This point requires just an explicit check. iii) If \(\lambda, \lambda' \in m(\mathcal{P})\) then \(\forall \lambda', \lambda' \in SU(d)\), \(U\lambda = \lambda'\). Then \(||T(\lambda')|| = ||T(U(\lambda))|| = ||U T(\lambda)|| = ||T(\lambda)||\). Moreover, \(\forall \lambda, \lambda' \in m(\mathcal{P})\),
\[
\langle T(\lambda), \lambda \rangle = \langle U T(\lambda), U \lambda \rangle = \langle T(U \lambda), U \lambda \rangle = \langle T(\lambda'), \lambda' \rangle.
\]

Mappings satisfying relation ii), for \(U\) belonging to some group \(\mathcal{G}\), are known as \(\mathcal{G}\)-automorphic functions. Therefore point ii) of the previous proposition can be rephrased saying that \(\text{GBR of covariant maps of } \mathcal{M}(\mathcal{S})\) are \(\mathcal{G}\)-automorphic functions of \(\mathbb{R}^D\) itself, where \(\mathcal{G} \equiv \varphi(SU(d))\).

Of course any linear mapping \(M \in \text{End}(\mathbb{R}^D)\) is \(\mathcal{G}\)-automorphic for any subgroup \(\mathcal{G} \subset GL(D, \mathbb{R})\) such that \([M, \mathcal{G}] = 0\) (\(M\) belongs to the centralizer of \(\mathcal{G}\)). An example of \(SO(D)\)-automorphic functions is given by \(T(\lambda) = f(||\lambda||)\lambda\), with \(f: \mathbb{R} \rightarrow (0, 1)\). Notice, that, for these mappings, the relations \([\mathcal{F}]\) are constants over \(D - 1\)-dimensional spheres.

Let \(\mathcal{M}'\) a convex \(\Phi\)-invariant subset of \(\mathcal{M}(\mathcal{S})\). The notion of optimality used in this paper is given by

**Definition 3** Let \(\mathcal{M}'\) a convex \(\Phi\)-invariant subset of \(\mathcal{M}(\mathcal{S})\). A map \(T^* \in \mathcal{M}'\) is optimal (in \(\mathcal{M}'\)) if
\[
\tilde{F}_1(T^*) = \sup_{T \in \mathcal{M}'} \tilde{F}_1(T).
\]

Now we show that, as far as optimality is concerned, one can restrict oneself to covariant transformations without loss of generality. The basic idea is very simple: since our 'merit' functional \(F_1\) is concave and \(SU(d)\)-invariant one can be build, for any given \(SU(d)\)-orbit, a convex average transformation \(T^*\) non-decreasing the cloning quality (i.e. \(\tilde{F}_1(T^*) \geq \tilde{F}_1(T)\)). \(T^*\) will be, by construction, covariant and it is clear that the one associated with an optimal cloner will turn out to be optimal as well.
Proposition 6 The optimal map can be chosen to be covariant.

Proof
Given the $SU(d)$ action $\Phi$ over $\mathcal{M}'$, the element of $\mathcal{M}'$, $T^* = \int_{SU(d)} d\mu(X) T_X$ (where $T_X = \Phi(X, T)$, for a given $T \in \mathcal{M}'$) is a covariant map. Indeed, for any $Y \in SU(d)$, $T_Y^* = \int_{SU(d)} d\mu(X) T_X Y = \int_{SU(d)} d\mu(X) T_{XY} = T_Y^*$. Since this equality holds for any $T \in \mathcal{M}'$ is a covariant map. Indeed, for any $Y \in SU(d)$, $T_Y^* = \int_{SU(d)} d\mu(X) T_X Y = \int_{SU(d)} d\mu(X) T_{XY} = T_Y^*$, where the invariance of the Haar measure $d\mu$ was used [2]. If $T$ is optimal $\tilde{F}_1(T) \geq \int_{SU(d)} d\mu(X) \tilde{F}_1(T_X) = \tilde{F}_1(T)$; thus $\tilde{F}_1(T^*) = \tilde{F}_1(T)$ (where we used the concavity of $\tilde{F}_1$, Proposition 4, and the normalization $\int_{SU(d)} d\mu(X) = 1$ of the measure $d\mu$).

Notice that if $T$ is universal, then the mapping $T^*$ introduced in Proposition 6 has the same value of the merit functional. Indeed $\forall \rho \in \mathcal{P}$ one has

$$\tilde{F}_1(T) = F_1(T, \rho) = \int d\mu(X) F_1(T, X \rho X^\dagger)$$

$$= \int d\mu(X) (1 - d^2(X \rho X^\dagger, T(X \rho X^\dagger)))$$

$$= \int d\mu(X) (1 - d^2(\rho, T^* X(X \rho X^\dagger) X))$$

$$= 1 - d^2(\rho, T^*(\rho)) = \tilde{F}_1(T^*),$$

where we have used linearity in $T$ of $F_1$, normalization of $d\mu$ and $SU(d)$-invariance of the metric.

This observation makes clear that for optimization purposes $t$ one can identify the notion of covariance and universality: a covariant map is universal and for any universal map there exists a covariant map with same cloning quality.

The next theorem shows that the structure of affine covariant maps is very simple.

Proposition 7 $T \in CP(\mathcal{S})$ is covariant iff $T = \xi I$ with $\xi \in (0, 1)$.

Proof
a) If $T = \xi I$, it is trivial to check that $T$ is covariant.

b) The components of the GBR of $T$ are given by $T_{i}(\tau) = (\tau_i, T(\rho)) = (F_i, \rho)$, where $F_i = T^*(\tau_i)$ are traceless operators ($T^*$ is the transpose map of $T$ with respect to the Hilbert-Schmidt scalar product). On has to show that $F_i = \xi \tau_i$. If $T$ is covariant, $T(X \rho X^\dagger) = X T(\rho) X^\dagger$; therefore

$$(\tau_i, X T(\rho) X^\dagger) = (X^\dagger F_i X, \rho) = (\tau_i, X \rho X^\dagger)$$

$$\sum_j X_{ji} (\tau_j, T(\rho)) = \sum_j X_{ji} (F_j, \rho).$$

Since this equality holds for any $\rho \in \mathcal{S}$, one gets

$$X^\dagger F_i X = \sum_j X_{ji} F_j,$$

namely the $F_i$’s transform under the adjoint action of $SU(d)$ as the $\tau_i$’s. As such action is irreducible, this implies that $F_i = \xi \tau_i$, $(i = 1, \ldots, D)$. Indeed let $F_i = \sum_j M_{ij} \tau_j$: from equation (13) one finds $[X, M] = 0, \forall X \in SU(d)$, then – by Schur’s lemma $-M = \xi I$. Moreover $\xi \in [0, 1]$ due to positivity requirement. □

A covariant map $T \in CP(\mathcal{S})$ has the form $\Phi$ given by

$$T(\rho) = (1 - \xi) d^{-1} I + \xi \rho.$$  (14)

The following example shows that one can have a whole family of covariant, positive, trace-preserving, non-linear maps of $\mathcal{S}$ in itself $\Gamma(\rho) = (1 - \Gamma[\rho]) d^{-1} I + \Gamma[\rho] \rho$, where $\Gamma: \mathcal{S} \rightarrow (0, 1)$ is a $SU(d)$-invariant non-linear functional. Such functionals can be built, for example, given any non-linear map $\gamma: (0, 1) \rightarrow (0, 1)$ by any convex superposition of the maps $\Gamma_n(\rho) = \gamma(\text{tr} \rho^n)$.

These maps, restricted to $\mathcal{P}$, amount to a simple (state independent) shrinking of the generalized Bloch vector $m(\rho)$. Nevertheless, since they are not affine, the property cannot be extended to the whole $\mathcal{S}$.

III. CLONERS $N \rightarrow M$

Now we turn the $N \rightarrow M$ cloning. In this section we shall set $\tau_0 \equiv I$, $\lambda_0 \equiv d^{-1}$, and $\lambda_i \equiv \lambda_i/2$. Let us consider the $N$-system state $\rho_N \equiv \rho(\lambda_0 \otimes \cdots \otimes \lambda_N)$,

$$\rho_N = \sum_{i_1, \ldots, i_N=0}^{D} \lambda_{i_1} \cdots \lambda_{i_N} \otimes \sum_{k=0}^{N} \lambda_k \tau_k,$$  (15)

where $F_{N,D}$ is the set of the maps from $\{0, \ldots, N\}$ to $\{0, \ldots, D\}$, and $\lambda_i \equiv \prod_{k=0}^{N} \lambda_k \tau_k$. Notice that in equation (15) the only non trace-less term is $\lambda_0 \tau_0 \equiv d^{-N} I \otimes N$.

The set of trace-preserving $CP$-maps from $\mathcal{S}^N$ to $\mathcal{S}^M$ will be denoted as $CP_{M,N}$.

The problem is now to find the optimal (with respect to some criterion) transformations of $CP_{M,N}$.

Since $X \in SU(d)$ acts naturally over $CP_{M,N}$ by $\Phi^N: (X, T) \rightarrow T_X$ in which

$$T_X(\rho) = \text{Ad}^\otimes M X^\dagger (T(\text{Ad}^\otimes N X(\rho))),$$  (16)

the notion of covariance is immediately extended to $CP_{M,N}$. It means that $T$ ‘intertwines’ between the $N$ and $M$-fold tensor representations of $SU(d)$. This can be pictorially described by the following commutative diagram

$$\begin{array}{ccc}
\mathcal{S}^N & \xrightarrow{\text{Ad}^\otimes N} & \mathcal{S}^N \\
\downarrow T & & \downarrow T \\
\mathcal{S}^M & \xrightarrow{\text{Ad}^\otimes M} & \mathcal{S}^M
\end{array}$$

4
To grasp what covariance means consider a set of operators \( \{ \phi_i \} \) in the domain of \( T \in CP_{M,N} \), that under the \( \text{Ad}^{\otimes N} \)-action of \( SU(d) \) transform according to an irreducible representations \( \mathbf{R} \) (i.e. \( \text{Ad}^{\otimes N} X(\phi_i) = \sum_j R_{ji}(X) \phi_j \)). If \( T \) is covariant then \( \phi_i \equiv T(\phi_i) \) transform under \( \text{Ad}^{\otimes M} \) according the same irrep. In other words a covariant mapping conserves the \( SU(d) \) symmetry content of the states. For example, \( X \otimes N, \rho = 0 \Rightarrow X \otimes M, T(\rho) = 0 \), in particular if \( \rho = d^{-1} I \) one has that \( T(\rho) \) belongs to the centralizer \( \mathcal{A}_{d,M} \) of the \( n \)-fold tensor representation of \( SU(d) \). \( \mathcal{A}_{d,M} \) is an algebra generated by the representatives of the symmetric group \( S_N \) acting in the natural way. Of course for \( \mathcal{A}_{d,1} \propto I \).

In the multi-system case now under consideration, one also has the natural action of the symmetric group \( S_M \) over \( S_M \) \( \text{if} \ \sigma \in S_M \) and \( \rho = |\Psi\rangle\langle\Psi|, \sigma \cdot \rho = U_\sigma |\Psi\rangle\langle\Psi| U_\sigma^\dagger \), where \( U_\sigma \otimes M \equiv \otimes \{ |\psi_j\rangle \}_{j \in (\sigma(j))} \) therefore one can consider the maps \( T_\sigma(\rho) = \sigma \cdot T(\rho) (\sigma \in S_M) \).

**Definition 4** A map \( T \in CP_{M,N} \) such that \( T = T_\sigma \), \( \forall \sigma \in S_M \) will be referred to as symmetric.

**Remark** For symmetric maps \( T(\rho) \) is totally symmetric operator. Let us denote with \( \text{tr} \) the trace over all because the \( k \)-th factor of the tensor product \( \mathcal{H}^{\otimes M} \). One can associate, to any element \( T \in CP_{M,N} \), \( M \)-reduced maps of \( \mathcal{M}(\mathcal{S}) \) defined by the rule \( \text{tr}_k: \rho \rightarrow \text{tr}_k T(\rho^{\otimes M}) \).

**Proposition 8** The maps \( \{ T_k \}_{k=1}^M \) fulfill the following

i) \( T_k \in \mathcal{M}(\mathcal{S}) \).

ii) The GBR of the \( T_k \)’s have components that are polynomials of order \( N \).

iii) If \( T \) is symmetric the \( T_k \)’s are identical.

iv) If \( T \) is covariant so are the \( T_k \)’s.

**Proof**

i) The \( T_k \)’s are positive in that they are compositions of the positive maps. ii) One has \( T(\tau_l) = \sum_{j \in \mathcal{F}_{M,D}} M_{jl} \tau_j \), therefore \( T(\rho_N) = \sum_{j \in \mathcal{F}_{M,D}} \lambda_j^M \tau_j \), where \( \lambda_j^M = \sum_{i \in \mathcal{F}_{N,D}} M_{ji} \lambda_i \). In particular \( T(\tau_l) = d^{-M} I^{\otimes M} + \sum_{j \neq l} \delta_{jl,0} \tau_j \), and \( \lambda_j^M = 0 \Rightarrow \text{tr}_l(\tau_l) = 0 \Rightarrow M_{jl} = 0 \). Moreover \( \text{tr}_k \tau_j = \tau_j d^{-M-1} \sum_{i \neq l} \delta_{jl,0} \). Therefore \( T_k(\rho_N) = d^{-1} I + \sum_{j=1}^M T_k^j(\lambda) \tau_j \), where

\[
T_k^j(\lambda) = d^{-M-1} \sum_{i \neq 0} (M_{jk,i} \lambda_i + c_{jk,i}).
\]  

(17)

Here \( j_k \) is a \( M \)-component column with \( j \) in the \( k \)-th entry and zero elsewhere. iii) If \( T \) is symmetric, it is simple to check that \( M_{jl,0} = M_{jk,0} \), and \( c_{jl,0} = c_{jk,0} \), \( \forall \sigma \in S_M \), \( l \in \mathcal{F}_{N,D}, j \in \mathcal{F}_{M,D} \). In particular, if \( l, k \in \{ 1, \ldots, M \} \), by applying the transposition \( \sigma_{lk} = (k, l) \) one finds \( \tau(\lambda)_{lk} = \tau(\lambda)_{kl} \). iv) One proves, by direct calculation, that

\[
T_k(X \rho X^\dagger) = \text{tr}_k \tau(T(\rho)^{\otimes N} X),
\]

(18)

\[
\text{tr}_k \tau(T(\rho)^{\otimes N} X) = \text{tr}_k \tau(T(\rho)^{\otimes N} X^{\otimes M}) = \text{tr}(\rho^{\otimes N} X^{\otimes M} X^{\otimes M} T(X^{\otimes M} X^{\otimes M} T(R^{\otimes N}) X)).
\]

**Definition 5** We introduce, for the elements of \( CP_{M,N} \), the (global) figures of merit based on the quality of the reduced clones

\[
F_1^{MN}(T, \rho) = \min_{k} F_1(T_k, \rho) (\rho \in \mathcal{P}),
\]

(19)

the notion of optimality being given as for the reduced maps for a convex, \( \Phi_N \)-invariant \( \mathcal{M}_{M,N} \subset CP_{M,N} \).

The next proposition is an extension of Proposition 6 to the \( N \rightarrow M \) case.

**Proposition 9** An optimal \( T \in \mathcal{M}_{M,N} \) can be chosen covariant and symmetric.

**Proof**

Let us first observe that the functional \( \tilde{F}_1^{MN} \) is constant over the orbits of both the \( SU(d) \) and \( S_M \) actions. Indeed for \( k = 1, \ldots, M, U \in SU(d), \sigma \in S_M \), one has : i) \( (T_k)_{U} = (T_k)_{U} \), from which \( \tilde{F}_1^{MN}(T_k) = F_1^{MN}(T) \) and ii) \( (T_k)_{\sigma} = T_k^{-1}(k) \), from which \( \tilde{F}_1^{MN}(T_k) = F_1^{MN}(T) \). Furthermore, it follows from linearity of the mapping \( T \rightarrow T_k \), the properties of \( F_1 \), and \( \min \) that \( \tilde{F}_1^{MN} \) is a concave functional over \( CP_{M,N} \). Now one can proceed as in Proposition 1, by introducing the ‘covariantized’ maps \( T^*_G \equiv \int d \mu(\rho) T_{G} (\mathcal{G} = SU(d), S_M) \). For the symmetric group the covariant map associated to \( T \) is

\[
T^* = (M!)^{-1} \sum_{\sigma \in S_M} T_k(\sigma).
\]

A. Universal Cloners

Let us suppose that the map \( T^{MN} \in CP_{M,N} \) is defined over the input set

\[
S_m = \{ \rho^{\otimes N}, \rho \in \mathcal{P} \}.
\]  

(20)

According to Proposition 9, such a map can be assumed – for optimality purposes – covariant and symmetric. The associated (reduced) pure-state fidelity, that has to be minimized over \( m(\rho) \), is given by equation \([\mathbb{E}]\) for a symmetric cloner \( T^{MN} \in CP_{M,N} \) we put \( T_k^{MN} = T (k = 1, \ldots, M) \) whereby \( T: m(\rho) \rightarrow m(S) \).

The next theorem shows how the deep geometrical meaning of covariance allows us to easily characterize the solutions of the optimization problem.

**Theorem 1** An optimal cloner \( \rho \rightarrow T^{MN}(\rho) (\rho \in S^N \cap \text{span} S_m) \), can be chosen in such a way that the associated reduced map is given by a shrinking of the generalized Bloch vector.

**Proof**

Due to the compacteness of \( m(\rho) \), there exists a \( \lambda^* \in m(\rho) \) such that \( \mathcal{F}(T) = 1/4 (R^2 - ||T(\lambda^*)||^2) + d^{-1} + 1/2 ||T(\lambda^*)||^2 \). Then

\[
\mathcal{F}(T) = 1/4 (R^2 - ||T(\lambda^*)||^2) + d^{-1} + 1/2 ||\lambda^*|| ||T(\lambda^*)||.
\]  

(21)

First notice that, since \( T \) can be chosen to be covariant, one has, from Proposition 5, that \( ||T(\lambda)|| \) is a constant
over \(m(\mathcal{P})\). Therefore: i) the first contribution to the fidelity does not depend on \(\lambda\), ii) the upper bound can be achieved if \(T(\lambda^*) = \xi \lambda^*\). Now we observe that, as the scalar product \(\langle T(\lambda), \lambda \rangle\) is constant over \(m(\mathcal{P})\) (Proposition 5), then \(T(\lambda) = \xi(\lambda)\lambda\). But the automorphic constraint implies \(\xi(\lambda) = \xi(U\lambda), \forall U \in \varphi(SU(d))\) whence – by transitivity of the \(SU(d)\)-action over \(m(\mathcal{P})\) – it must be \(\xi|_{m(\mathcal{P})} = \text{const.}\) The optimal (reduced) map has the form \(\lambda_\mu\). Since this map is \textit{affine}, it can be extended to the whole set of states belonging to the linear span of \(S_m\).

Remark. One must have \(\mu_j' = \xi \lambda_j\ (j = 1, \ldots, D)\). Therefore \(M_{j0\ldots0,i} = 0\) unless \(\exists l \in \{0, \ldots, N\}\) such that \(i_m = 0\) (\(m \neq l\)) and \(i_l = j\). In this case one finds

\[
\xi = \sum_{k=1}^{N} M(T)_{jl,k}, \ (l = 1, \ldots, M).
\]  

Therefore

\[
T(\rho_{in}) = d^{-M} I \otimes M + N \xi \sum_{j=1}^{D} \lambda_j \Delta_M(\tau_j) + R(\lambda),
\]  

where \(\Delta_M(\tau_i) \equiv M^{-1} \sum_{j=1}^{M} \tau_j^{(i)}\) is the coproduct of \(\tau_i\) [i.e. \(\tau_j^{(i)}\) acts as \(\tau_i\) in the \(l\)-th factor of the tensor product and trivially in the others] and \(R(\lambda)\) contains all the tensor products in which a factor \(\tau_j \neq I\) appears at least twice.

B. Algebraic approach

In this section we shall show that the shrinking property \([14]\) follows from covariance alone. To this aim it is convenient to turn to a more algebraic approach in that the notion of covariance is naturally related to representation-theoretic concepts. We consider now general \(\rho \in S\).

\textbf{Proposition 10} The components of the map \(T\) are given by \(T_i(\lambda) = (F_i, \rho_\lambda^{(N)}, \rho_\lambda^{(N)}, (i = 1, \ldots, D)\) where \(F_i \in \text{End}(\mathcal{H}^{\otimes N})\) are \(S_N\)-invariant, traceless hermitian operators.

\textbf{Proof} By using \(S_M\)-invariance of \(T^{MN}\) one checks directly that the components of the map \(T\) are

\[
T_i(\lambda) = (\tau_i, T(\rho)) = (\tau_i, \text{tr}_1(T^{MN}(\rho_\lambda^{(N)})))
\]

\[
= (\tau_i \otimes I \otimes (M-1), T^{MN}(\rho_\lambda^{(N)}))
\]

\[
= (\Delta_M(\tau_i), T^{MN}(\rho_\lambda^{(N)})) = (F_i, \rho_\lambda^{(N)}),
\]

where \(F_i \equiv T(\rho_\lambda^{(N)}(\Delta_M(\tau_i))).\) Now we observe that, since \(\rho_\lambda^{(N)}\) is \(S_N\)-invariant, the \(F_i\)'s can be chosen to symmetric

\[
(F_i, \rho_\lambda^{(N)}) = \frac{1}{N!} \sum_{\sigma \in S_N} (F_i, U_{\sigma} \rho_\lambda^{(N)} U_{\sigma}^*)
\]

\[
= \frac{1}{N!} \sum_{\sigma \in S_N} (U_{\sigma} F_i U_{\sigma}, \rho_\lambda^{(N)}) = (\tilde{F}_i, \rho_\lambda^{(N)}),
\]

where \(\tilde{F}_i \equiv 1/M! \sum_{\sigma \in S_N} U_{\sigma} F_i U_{\sigma}\) is manifestly symmetric. Tracelessness and hermiticity follow form the general properties of \(C^\ast\)-maps.

From the covariance constraint it follows that

\[
(U \otimes F_i, \rho_\lambda^{(N)}) = \sum_{j=1}^{D} \mathbf{X}_{ji}(U) (F_j, \rho_\lambda^{(N)}).
\]

By introducing the functionals \(\Lambda_{\rho}\) to \(\text{End}(\mathcal{H}^{\otimes N})\), \(\Lambda_{\rho}: A \mapsto (\rho, A)\), equation \((24)\) can be rewritten as

\[
\Lambda_{\rho N}(A_{\lambda}^{(N)}) = 0 \quad (\forall \rho \in S, U \in SU(d), i = 1, \ldots, D),
\]

\[
A_{\lambda}^{(U)} \equiv U^{\otimes N} F_i U^{\otimes N} - \sum_{j=1}^{D} \mathbf{X}_{ji}(U) F_j.
\]

Notice that, for \(N = 1\), from (functional) equation \(\Lambda_{\rho N}(A_{\lambda}^{(1)}) = 0\) follows the \textit{operatorial} equation \([13]\). Let us consider now the pure state case \(\rho = |\psi\rangle \langle \psi|, |\psi\rangle \in \mathcal{H}\). Let \(\mathcal{H}_{sym}^{N}\) the totally symmetric subspace of \(\mathcal{H}^{\otimes N}\). One has: i) \(\mathcal{H}_{sym}^{N}\) is the space associated to the identity representation of \(S_N\); ii) it is also the space of a totally symmetric (irreducible) representation \(\phi_s\) of \(SU(d)\); iii) \(\mathcal{H}_{sym}^{N} = \text{span}\{|\psi\rangle \otimes N : |\psi\rangle \in \mathcal{H}\}\).

\textbf{Theorem 2A} covariant cloner over \(\mathcal{H}_{sym}^{N}\) induces a mapping between reduced states given by a simple shrinking of the generalized Bloch vector.

\textbf{Proof} It follows from i)-iii) that the linear span of the operators \(\rho_\lambda^{(N)} (\rho \in \mathcal{P})\) is the space of states with support in \(\mathcal{H}_{sym}^{N}\). In this case – since a symmetric operator leaves \(\mathcal{H}_{sym}^{N}\) invariant – the \(A_{\lambda}^{(U)}\)'s can be considered as belonging to \(\text{End}(\mathcal{H}_{sym}^{N})\). The functional equations \(\Lambda_{\rho N}(A_{\lambda}^{(N)}) = 0\) then imply the operatorial equations \(A_{\lambda}^{(U)} = 0\) over \(\mathcal{H}_{sym}^{N}\). Let \(\Phi\) be the representation over \(\text{End}(\mathcal{H}_{sym}^{N})\) associated with \(\phi_s\). Since \(\text{End}(\mathcal{H}_{sym}^{N}) \cong \mathcal{H}_{sym}^{N} \otimes \mathcal{H}_{sym}^{N}\) one has \(\Phi \cong \phi_s \otimes \phi_s^*\), the tensor product or two totally symmetric \(SU(d)-\)irreps, therefore in the decomposition of \(\Phi\) each \(SU(d)-\)irrep appears once \([12]\). As \(A_{\lambda}^{(U)} = 0\) simply means that the \(F_i\)’s transform according to the adjoint representation, one must have \(F_i \equiv T^{MN}(\Delta_M(\tau_i)) = \xi \Delta_N(\tau_i).

Form this relation it follows (see equation \((24)\)) that \(T_i(\lambda) = \xi \lambda_i (i = 1, \ldots, D)\).

This proof helps to shed some light on the basic difference between the pure and the general (mixed) state problem. Let \(\mathcal{H}^{\otimes N} = \oplus_{j \in J} \mathcal{H}^{(j)}\) denote the decomposition of the input Hilbert space into \(S_N\)-isotopically components (i.e. \(\mathcal{H}^{(j)}\) is the subspace of vectors transforming according a given \(S_N\)-irrep labelled by \(j\)). If \(\Pi_j\) denotes the projector over \(\mathcal{H}^{(j)}\) one has, for general \(\rho\) that \(\rho^{\otimes N} = \sum_{j \in J} \lambda_j \rho^{(j)}_N\),

6
where $\rho^{(j)}_N \equiv \lambda^{-1}_j \Pi_j \rho^\otimes N \Pi_j$, $\lambda_j \equiv \text{tr}(\rho^\otimes N \Pi_j)$. In this case the relevant functional equations from covariance are

$$
\sum_{j \in J} \text{tr}(\rho^{(j)}_N A^U_j) = 0,
$$

(28)
i = 1, \ldots, D, U \in SU(d)$, where in each term $A^U_j$ can be considered as belonging to $\text{End}(\mathcal{H}^U)$. When $\rho \in \mathcal{P}$ only the $j = 0$ ($\mathcal{H}^{(0)} \equiv H^\otimes N_{\text{sym}}$) term survives, and one succeeds in getting an operatorial equation. In general one has to deal directly with equations (28), that represent a much weaker constraint on the cloner structure.

The next (almost obvious) corollary shows that concatenating optimal cloners the shrinking factors multiply

**Corollary 1** Let $T_1 \in CP_{M,N}$ and $T_2 \in CP_{R,M}$ be symmetric and covariant maps. Then: i) $T_2 \circ T_1$ is a covariant and symmetric map. ii) Let $r(T)$ denote the unique map of $\mathcal{M}(S)$ associated to a symmetric $T \in \mathcal{C}P_{M,N}$. If $r(T)$ is affine then $r(T_2 \circ T_1) = r(T_2) \circ r(T_1)$.

**Proof**
i) Requires a simple check. ii) From previous Theorem $r(T_2 \circ T_1)$ and $r(T_2) \circ r(T_1)$ are covariant maps of $CP(S_1)$.

Let $\xi, \xi_2, \xi_1$ be the associated scale factors One has to show that $\xi = \xi_2 \xi_1$. From equation (22) one finds indeed

$$
\xi = \sum_{k=1}^{N} M(T_2 \circ T_1)_{j_1,i_k} = \sum_{l=1}^{M} \sum_{k=1}^{N} M(T_2)_{j_1,i_k} M(T_1)_{h_l,i_k} = (\sum_{l=1}^{M} M(T_2)_{j_1,i_k}) (\sum_{k=1}^{N} M(T_1)_{h_l,i_k}) = \xi_2 \xi_1,
$$

(29)

where we used the independence of $M(T_2)_{h_l,i_k}$ on $l$. $\square$

We conclude the section by a simple explicit computation, that shows the power of the notion of covariance. Let us consider the case $d = 2$, $N = 1$, $M = 2$, with initial state $\rho = 2^{-1}(I + \sum_{\alpha=x,y,z} \lambda_\alpha \sigma_\alpha)$ (the $\sigma$'s are the Pauli matrices). If $T \in CP_{Q,2}$ is covariant and symmetric one must have $T(I) \in A_{2,2} = \text{span}\{I_2, C_2\}$ where

$$
C_2 = \sum_{\alpha=x,y,z} \sigma_\alpha \otimes \sigma_\alpha = 2P - I,
$$

(30)
is a traceless combination of the identity and the transposition $P|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Moreover the $T(\sigma_\alpha)$'s must be totally symmetric operators that transform according the adjoint ($j = 1$) representation of $SU(2)$. The totally symmetric sector of $\text{End}(\mathcal{C}^4)$ is ten dimensional. It is spanned by the elements of $A_{2,2}$, ($j = 0$) five operators realizing a $j = 2$ multiplet of $SU(2)$, and by the $S_\alpha = 2\Delta_2(\sigma_\alpha)$, ($\alpha = x, y, z$) corresponding to $j = 1$. Therefore, from Theorem 2 [notice that trivially $H^\otimes N_{\text{sym}} = \mathcal{H}$] one has, $T(\sigma_\alpha) = \xi S_\alpha$. Putting all together $T(\rho) = 4^{-1}(I + t C_2 + \xi \sum_\alpha \lambda_\alpha S_\alpha)$, one has

$$
\text{spec} T(\rho) = \{\frac{1}{4}(1 \pm 2\xi + t), \frac{1}{4}(1 - 3t)\}.
$$

IV. SUMMARY

In this note it has been rigorously shown that the optimal (with respect to a metric criterion) $N \rightarrow M$ pure state cloner of a general $d$-dimensional quantum system can be described by a simple state-independent shrinking of the generalized Bloch vectors associated to the reduced density matrices. The structure of the proof can be summarized as follows. Over the space $CP_{M,N}$ of $N \rightarrow M$ cloners a ‘merit’ functional is introduced in terms of the induced (non linear) maps of reduced (one-system) states. This functional –which has a clear geometrical meaning in the setting of the generalized Bloch representation (GBR) – is concave and invariant under the natural actions of the groups $S_M$, $SU(d)$. This allows us to restrict our attention to covariant (i.e. invariant respect to the group action) cloners: given a group orbit, by ‘averaging’ and using concavity, one can build a covariant cloner with no worse quality. This cloner results to be universal (cloning quality independent on the input state), and the components of the associated GBR map satisfy an automorphicity constraint. Allowing only for pure inputs and resorting to the intimate connection between representation theory of unitary and symmetric groups, one obtains the final result, that by linearity extends to the whole space of states over the totally symmetric subspace of $H^\otimes N$.

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