Total Mass-Momentum of Arbitrary Initial-Data Sets in General Relativity

Robert Geroch
Shyan-Ming Perng
Enrico Fermi Institute, University of Chicago,
5640 S. Ellis Avenue, Chicago, Illinois 60637

September 16, 2021

Abstract

For an asymptotically flat initial-data set in general relativity, the total mass-momentum may be interpreted as a Hermitian quadratic form on the complex, two-dimensional vector space of “asymptotic spinors”. We obtain a generalization to an arbitrary initial-data set. The mass-momentum is retained as a Hermitian quadratic form, but the space of “asymptotic spinors” on which it is a function is modified. Indeed, the dimension of this space may range from zero to infinity, depending on the initial data. There is given a variety of examples and general properties of this generalized mass-momentum.
1 Introduction

There is a well known procedure\(^1\) that assigns, to any space-time that is asymptotically flat in a suitable sense, a quantity representing the total mass-momentum of that space-time, measured at spatial infinity. To implement this procedure, first draw in that space-time a spacelike slice \(T\), and consider the induced initial data — consisting of the induced metric \(q_{ab}\) and the extrinsic curvature \(p_{ab}\) — on that slice. Next, impose on these initial data asymptotic flatness: that \(q_{ab}\) approach a flat metric at infinity, and \(p_{ab}\) approach zero, at suitable rates. Finally, define the components of the total mass-momentum of the initial data as the values of certain asymptotic integrals, whose integrands involve \(q_{ab}\), its first derivative, and \(p_{ab}\). This procedure requires for its success a detailed definition of asymptotic flatness, and the choice of definition is a rather delicate business. On the one hand, the definition must be weak enough that it permits \(q_{ab}\) and \(p_{ab}\) to convey, in their asymptotic behavior, information about the total mass-momentum. On the other hand, the definition must be strong enough that it permits recognition of “asymptotic directions” to serve as labels for the components of the total mass-momentum.

There have been a number of attempts, over the past ten years or so, to assign a suitable mass-momentum to a mere portion of space-time. Consider, to be specific, initial data for Einstein’s equation given on a closed 3-ball \(B\), with 2-sphere boundary \(K\). Can there be defined something that could reasonably be interpreted as “the mass-momentum within this region \(B\)”\(^2\)? Penrose\(^2\) introduced a complex mass-momentum-angular-momentum given as integrals over \(K\) of spinor solutions of a certain differential equation. Dougan and Mason\(^3\), also using spinor integrals, introduced a real mass-momentum, which turns out in addition to be timelike in an appropriate sense. Bartnik\(^4\) used a different method to introduce a total mass (only) associated with the region \(B\):

For each suitable extension of the local initial data on \(B\) to asymptotically flat, compute in the usual way the total mass, and then minimize it over extensions. There have also been proposed definitions involving time evolution\(^5\) of the data for short distance, and null evolution\(^6\) all the way to null infinity.

Returning now to the asymptotically flat case, Witten\(^7\) has obtained an elegant reformulation of this subject. Introduce spinor fields on \((T, q_{ab})\). Consider now a spinor field \(\lambda^A\) that satisfies the Witten equation — a certain first order, neutrino-like equation — and that approaches a constant asymptotically. Write down, for this \(\lambda^A\), a certain integral, \((12)\), over \(T\) with integrand quadratic in \(\lambda\) and its first derivative. It is known\(^7\) that this integral gives precisely the component of the total mass-momentum vector in the asymptotic null direction defined by the asymptotic behavior of \(\lambda^A\). Incidentally, the integrand for the mass integral is manifestly non-negative, an observation that proves the positive-mass theorem.

This paper is based on two key observations regarding the Witten reformulation: first, that minimization of the mass integral yields automatically the Witten equation; and, second, that finiteness of the mass integral yields automatically that \(\lambda^A\) approaches a constant asymptotically. Thus, this single integral is tied in to everything: the equation the spinor field \(\lambda^A\) is to satisfy, the asymptotic conditions on that spinor field, and the value of the mass-momentum component associated with that spinor field. Indeed, the construction of the total mass-momentum for an asymptotically flat initial-data set may
be reformulated as follows. First, introduce the space $\mathcal{S}$, essentially the quotient of the space of all spinor fields $\lambda^A$ on $(T, q_{ab})$ for which the mass integral converges by the subspace consisting of spinor fields of compact support. In the present instance, this $\mathcal{S}$ will be a two-dimensional vector space of “asymptotic spinors”, and so will provide a space of “directions along which to evaluate the components of the mass-momentum”. Next, introduce on this space $\mathcal{S}$ the function, $M$, whose value is given by the minimum of the mass integral. In the present instance, this $M$ will provide the appropriate mass-momentum components.

But note that the formulation of the previous paragraph nowhere uses asymptotic flatness. Thus quite generally — for any initial-data set whatever — one can introduce the quotient, $\mathcal{S}$, of spinor fields with finite mass integral by those of compact support, and then introduce the function $M$ on $\mathcal{S}$ by minimizing the mass integral. For asymptotically flat initial data, this construction yields the usual mass-momentum components. What happens for more exotic initial data — say, with $T$ compact, or consisting of a small patch from a large initial-data set? In general, the space $\mathcal{S}$ becomes modified in some way — it is no longer a simple two-dimensional vector space. Indeed, its dimension can range from zero to infinity. Thus, we retain the mass-momentum as a function on possible component-directions, but the space of such directions becomes more complicated. This, we suggest, is the natural notion of “total energy-momentum” for a general initial-data set.

This paper is organized as follows. Sect. 2 contains the basic definitions. We first introduce the space $\mathcal{S}$ representing the “asymptotic spinors”. It turns out that there are actually two natural mass functions on this space $\mathcal{S}$ — what we call the norm mass function $M_N$ and the asymptotic mass function $M_A$. We show that, in the case of asymptotically flat initial data, the space $\mathcal{S}$ reduces to a two-dimensional vector space, while the two mass functions coincide and yield the appropriate components of the total mass-momentum. Sect. 3 contains various examples and properties of the space $\mathcal{S}$ and the mass functions. The space $\mathcal{S}$ collapses to a single point — and the mass functions then necessarily to zero — when there is either “too much matter” or “too little room asymptotically”. By contrast, there is a large class of examples in which $\mathcal{S}$ is infinite-dimensional, with rather complicated mass functions. For virtually all complete initial data, the two mass functions are equal, while in the incomplete case the two can differ. Indeed, the asymptotic mass function can become negative in certain cases, while the norm mass function never can. The asymptotic mass function — but in general not the norm mass function — depends only on the “asymptotic behavior” of the initial data. Finally, we show that, for initial data with several “asymptotic regions”, there is a decomposition of the space $\mathcal{S}$ and of the mass functions into pieces associated with the individual asymptotic regions.

2 Basic definitions

Fix an initial-data set. That is, fix a connected, 3-dimensional manifold $T$, a smooth positive-definite metric $q_{ab}$ on $T$ and a smooth symmetric tensor field $p_{ab}$ on $T$. Given
such an initial-data set, we set

\[
\rho = \frac{1}{2} [R - p_{ab}p^{ab} + (p^m_m)^2],
\]

(1)

\[
\rho_a = D^b (p_{ab} - p^m_m q_{ab}),
\]

(2)

where \( R \) is the scalar curvature, and \( D_a \) the derivative operator, with respect to the metric \( q_{ab} \) on \( T \). These will be recognized\(^8\) as the mass and momentum density, respectively, of the matter source. The energy condition on this initial-data set is the condition that the mass-momentum vector be future-directed non-spacelike:

\[
\rho \geq (\rho^a \rho_a)^{1/2}.
\]

(3)

We shall here deal only with initial-data sets satisfying the energy condition.

It is necessary for what will follow to introduce the notion of spinor fields on such an initial-data set. To this end, fix a complex, two-dimensional vector space \( V \). By a spinor, we mean any element of a tensor product involving \( V \), its complex-conjugate space \( V^\star \), and their respective dual spaces, \( V^* \) and \( V^{\ast \ast} \). We designate spinors by upper-case Latin indices — unprimed superscripts for \( V \), primed superscripts for \( V^\star \), and corresponding subscripts for their corresponding duals. Thus, \( \alpha_{BD'} \) (an element of \( V \otimes V^* \otimes V^{\ast \ast} \otimes V^{\ast \ast} \otimes V^{\ast \ast} \)) is a typical spinor. By construction, we have on spinors the operations of outer product, contraction, complex conjugation (denoted by a bar), and, for spinors with identical index structure, addition. For example, if \( \alpha_A, \beta_B^C, \) and \( \gamma_D^E \) are spinors, then so is \( \alpha_A^B \beta^C_B^C \gamma_D^E \).

Now fix any antisymmetric spinor \( \epsilon_{AB} \), and any real, positive-definite spinor \( t_{AA^\prime} \) (i.e. \( t_{AA^\prime} \psi_A \psi_A^A \) is real and positive for any nonzero \( \psi_A \)), with these normalized with respect to each other by

\[
t_{AA^\prime} \epsilon_{A^\prime} = \frac{1}{2} \epsilon_{AB} \epsilon_{A^\prime B^\prime}.
\]

(4)

These two fixed spinors are incorporated into the notation in the following way. Use \( t_{AA^\prime} \) and its inverse \( t_{A^\prime A} \) (whose existence is guaranteed by positive-definiteness) to eliminate primed spinor indices in favor of unprimed ones. Thus, we need only work throughout with unprimed spinor indices. Then use \( \epsilon_{AB} \), and its inverse \( \epsilon_{AB} \) (whose existence is guaranteed by Eqn. (4)) to lower and raise these unprimed spinor indices in the usual way (i.e., \( \kappa_A = \epsilon_B^A \kappa_B, \) \( \kappa_A = \epsilon^{AB} \kappa_B \)).

The operation of complex conjugation on general spinors then translates to an adjoint operation on these unprimed spinors:

\[
\alpha_A^{\ast} \cdots C_{B \cdots D} = (-1)^r t_A^{A^r} \cdots t_C^{C^r} t_B^{B^r} \cdots t_D^{D^r} \overline{\alpha}_A^{\ast} \cdots C^r_{B^r \cdots D^r},
\]

(5)

where \( s \) is the number of superscripts of \( \alpha \). Thus, the adjoint of a scalar is its complex conjugate, while, \( \epsilon_{AB}, \epsilon^{AB} \), and the unit spinor \( \delta^A_B \) are self-adjoint. This adjoint operation commutes with outer product and contraction, and so with the raising and lowering of indices. Further, we have \( \dagger \dagger = (-1)^r \), where \( r \) is the number of indices of the spinor to which this equation is applied. It follows from positive-definiteness of \( t_{AA^\prime} \) that, for every spinor \( \alpha_A^{\ast} \cdots C_{B \cdots D} \), \( |\alpha|^2 = (\alpha_A^{\ast} \cdots C) \alpha_{A \cdots C} \geq 0 \), with equality if and only if \( \alpha = 0 \). But note, e.g., that \( \alpha_A^A \alpha_A \leq 0 \). The set of symmetric, self-adjoint spinors \( \lambda^{AB} \) forms a real 3-dimensional vector space, on which \( \lambda^{AB} \lambda_{AB} \) is a positive-definite norm. Now identify
this vector space with the tangent space at each point of $T$, in such a way that this positive-definite norm corresponds to the norm on the tangent space arising from $q_{ab}$. There results the notion of spinor fields on $T$. Thus, each tensor field on the manifold $T$ gives rise to a spinor field with twice as many (spinor) indices. Other examples of spinor fields include $\epsilon_{AB}$, $\epsilon^{AB}$ and $\delta^{A}_{B}$. The operations of outer product, contraction, addition, and taking of the adjoint extend immediately from spinors to spinor fields. The tensor field $q_{ab}$ on $T$ gives rise to the spinor field $q_{ABCD} = q_{(CD)(AB)} = -\epsilon_{A(C}\epsilon_{D)B}$ on $T$. Finally, the derivative operator $D_a$ on tensor fields on $T$ gives rise to a unique corresponding derivative operator $D_{AB}$ on spinor fields on $T$. This operator is symmetric in its indices, satisfies the Leibnitz rule under outer product, commutes with addition, contraction, and the adjoint operation, and satisfies $D_{AB}\epsilon_{CD} = 0$.

So far, we have used only the metric $q_{ab}$ of $T$, and not the symmetric tensor $p_{ab}$. We incorporate the latter by introducing a new operator, $D_{AB}$, on spinor fields, with action

$$D_{AB}\lambda^{C...}_{\sigma} = D_{AB}\lambda^{C...}_{\sigma} + \frac{i}{\sqrt{2}} p_{ABC}^{\quad M} \lambda^{D...}_{M} + \cdots$$

$$- \frac{i}{\sqrt{2}} p_{ABM}^{\quad D} \lambda^{C...}_{M} - \cdots.$$ (6)

Here, $p_{ABCD} = p_{(CD)(AB)} = p_{ABC}^{\dagger}$ is the spinor representation of $p_{ab}$. For example, we have

$$D_{AB}\lambda^{B} = D_{AB}\lambda^{B} + \frac{i}{2\sqrt{2}} p\lambda_{A}$$ (7)

$$D_{AB}\omega^{AB} = D_{AB}\omega^{AB}$$ (8)

where we have set $p = p_{AB}^{\dagger} = p_{m}^{\quad m}$. This operator $D_{AB}$ shares with $D_{AB}$ the Leibnitz rule, annihilation of $\epsilon_{AB}$, and commutation with addition, contraction and raising and lowering of indices. But $D_{AB}$, in contrast to $D_{AB}$, fails to be torsion-free and also fails to commute with the adjoint operation. Indeed, we have

$$(D_{AB}\lambda^{C}_{\sigma})^{\dagger} = D_{AB}\lambda^{C}_{\sigma} - \frac{2i}{\sqrt{2}} p_{ABC}^{\quad M} \lambda^{D}_{M} + \frac{2i}{\sqrt{2}} p_{ABM}^{\quad D} \lambda^{C}_{M}.$$ (9)

It is convenient to introduce the adjoint of $D_{AB}$, defined as follows:

$$(D_{AB}\lambda^{C...}_{\sigma})^{\dagger} = D_{AB}^{\dagger}\lambda^{C...}_{\sigma}. (10)$$

What makes the operator $D_{AB}$ so useful is the Witten-Sen identity, which we shall use repeatedly: For any spinor fields $\sigma^{A}$, $\lambda^{A}$ on $T$, we have

$$(D^{AB}\sigma^{C})^{\dagger}(D_{AB}\lambda_{C}) = \frac{1}{2}(\rho\sigma^{A}\lambda_{A} + i\sqrt{2}\rho_{AB}\sigma^{A}\lambda^{B})$$

$$= D^{AB}(\sigma^{C}\cdot D_{AB}\lambda_{C}) - 2\sigma^{C}D^{\dagger}_{AC}D_{AB}\lambda^{B}$$

$$= D^{AB}(\sigma^{C}\cdot D_{AB}\lambda^{C}) + 2\sigma^{B}_{B}D_{AC}\lambda^{C} + 2(D^{A}_{B}\sigma^{B})^{\dagger}(D_{AC}\lambda^{C}). (11)$$

To prove Eqn. (11), expand the right hand sides using the Leibnitz rule, eliminate $D_{AB}$ in favor of $D_{AB}$ using Eqn. (6), eliminate all second derivatives using $D_{M(A}D_{B)}^{\quad M}\lambda^{B} = \frac{R}{8}\lambda_{A}$, and finally eliminate the $p_{ab}$’s using Eqns. (1) – (2).
Thus, Eqn. (12) defines a quadratic, positive semi-definite norm on the completion of $H$ the collection of all smooth spinor fields $\lambda_A$ on $T$ for which the right side of

$$\| \lambda \|^2 \equiv \int_T \left\{ (D^{AB} \lambda^C \iota) (D_{AB} \lambda_C) + \frac{1}{2} (p \lambda^{\dagger A} \lambda_A + i \sqrt{2} p_{AB} \lambda^{\dagger A} \lambda_B) \right\}$$

(12)

converges. Note that this right side is nonnegative. Indeed, the first term in the integrand on the right is manifestly nonnegative, while the second term is nonnegative because of the energy condition and the fact that the norm of the (real) vector $i \lambda^{\dagger (A} \lambda_B)$ is $\frac{1}{\sqrt{2}} \lambda^{\dagger A} \lambda_A$. Thus, Eqn. (12) defines a quadratic, positive semi-definite, norm on $H$. It follows from this that $H$ is a (complex) vector space.

We next construct a certain completion, $\overline{H}$, of $H$. This is to be done essentially via the norm (12) — but we must exercise some care to accommodate the fact that this norm need not be strictly positive-definite. Suppose for a moment that there were some point $x$ of $T$ at which $\rho$ strictly exceeded $(\rho^2 \rho_a)^{1/2}$. Then, the norm (12) would already be strictly positive-definite. Indeed, $\| \lambda \|^2 = 0$ would imply, by the right side of (12), the vanishing of of $\lambda_A$ at $x$ and of $D_{AB} \lambda_C$ everywhere. But these two together imply the vanishing of $\lambda_A$ everywhere. Thus, in this case — when there is some point $x \in T$ at which $\rho > (\rho^2 \rho_a)^{1/2}$ — the norm (12) is already strictly positive-definite, and so we may simply take for $\overline{H}$ the completion of $H$ in this norm. But what if there exists no such point $x$? In this case, we introduce a new norm, $\| \cdot \|_f$, obtained by adding to $\rho$ in (12) any nonnegative function, $2f$, somewhere strictly positive, of compact support:

$$\| \lambda \|^2 = \| \lambda \|^2 + \int_T f \lambda^2.$$  

(13)

This norm is automatically positive-definite, and so we take for $\overline{H}$ the completion of $H$ in it. The result is independent of the choice of the function $f$.

**Theorem 1:** Let $f, \tilde{f}$ be two functions on $T$, each of which is nonnegative, somewhere strictly positive, and of compact support. Then each of the two norms $\| \cdot \|_f$ and $\| \cdot \|_{\tilde{f}}$ bounds some multiple of the other.

**Proof:** Fix $\lambda^A \in H$. Let $w^a$ be any smooth vector field on $T$ of compact support, denote by $\zeta_t$ $(t \in R)$ the corresponding one-parameter family of diffeomorphism on $T$, and set $\alpha(t) = \int_T (\zeta_t f) |\lambda|^2$. Then we have

$$\frac{d \alpha}{dt} = \int w^a [D_a (\zeta_t f)] |\lambda|^2 = \int [-(D_a w^a) (\zeta_t f) |\lambda|^2 - (\zeta_t f) w^a D_a |\lambda|^2] = \int [-(D_a w^a) (\zeta_t f) |\lambda|^2 - i \sqrt{2} (\zeta_t f) w^{AB} p_{ABCD} \lambda^C \lambda^{\dagger D} - (\zeta_t f) w^{AB} \lambda^C D_{AB} \lambda_C - \lambda^C (D_{AB} \lambda_C)^{\dagger}] \leq \int [-(D_a w^a + |p_{ab} w^b|) (\zeta_t f) |\lambda|^2] + 2 \int |w| (\zeta_t f) |\lambda|^2 \int |D_{AB} \lambda_C|^2 \leq b \alpha + ca^{1/2} \left( \int |D_{AB} \lambda_C|^2 \right)^{1/2},$$

(14)
where \( b \) and \( c \) are positive numbers independent of \( \lambda^A \). In the last step, we used the Schwarz inequality. Solving this differential inequality, we learn that \( \alpha(t) \) is bounded by a linear combination, with coefficients independent of \( \lambda^A \), of \( \alpha(0) \) and \( \int |D_{AB}\lambda_C|^2 \). Hence, some multiple of \( \| \lambda \|_f \) bounds \( \| \lambda \|_{Cf} \). The result now follows from the fact that \( \bar{f} \) is bounded by a finite linear combination of functions of the form \( \zeta_{gf} \).

It follows from Theorem 1 that any sequence Cauchy in the norm \( \| \cdot \|_f \) is also Cauchy in the norm \( \| \cdot \|_f \), and therefore that the completion, \( \overline{\mathcal{H}} \), of \( \mathcal{H} \) is indeed independent of the function \( f \) used to take that completion. This \( \overline{\mathcal{H}} \) is, by construction, a complete topological vector space with continuous, quadratic, positive semi-definite norm \( \| \cdot \| \).

An element of \( \overline{\mathcal{H}} \) is represented, via the construction above, by a sequence of spinor fields \( \{ \lambda_A^i \} \) on \( T \), Cauchy in the norm \( \| \cdot \|_f \). But there exists a more explicit representation. To obtain it, let \( \{ \lambda_A^i \} \) be any Cauchy sequence in \( \mathcal{H} \) and \( U \) any open subset of \( T \) with compact closure. Then the sequences \( \{ \lambda_A^i \} \) and \( \{ D_{AB}\lambda_C^i \} \) are both Cauchy in \( L^2(U) \), by Theorem 1 and Eqn. (12) respectively, and therefore converge in \( L^2(U) \) to some spinor fields \( \kappa_A^i \) and \( \omega_{ABC} \) respectively. Furthermore, this \( \omega_{ABC} \) is actually the weak derivative of \( \kappa_A^i \), i.e., we have, for every smooth \( \tau^{ABC} \) of compact support in \( U \),

\[
- \int_U (D_{AB}\tau^{ABC})\kappa_C = \int_U \tau^{ABC}\omega_{ABC}. \tag{15}
\]

To see this, note that

\[
\left| \int_U \left[ \tau^{ABC}\omega_{ABC} + (D_{AB}\tau^{ABC})\kappa_C \right] \right| \\
= \left| \int_U \left[ \tau^{ABC}(\omega_{ABC} - D_{AB}\lambda_C^i) + (D_{AB}\tau^{ABC})(\kappa_C - \lambda_C^i) \right] \right| \\
\leq \left[ \int_U |\tau|^2 \int_U |\omega_{ABC} - D_{AB}\lambda_C^i|^2 \right]^{1/2} + \left[ \int_U |D_{AB}\tau^{ABC}|^2 \int_U |\kappa_C - \lambda_C^i|^2 \right]^{1/2}, \tag{16}
\]

while the right side approaches zero as \( i \) approaches infinity. Since the subset \( U \), open with compact closure, is otherwise arbitrary, we conclude: Each element of \( \overline{\mathcal{H}} \) can be represented uniquely by a spinor field on \( T \), locally square integrable with locally square-integrable weak first derivative, for which the integral (12) converges. Then Eqn. (12) gives the continuous, positive semi-definite norm on this \( \overline{\mathcal{H}} \). In the “generic case” — when there is some point of \( T \) at which \( \rho > (\rho^a\rho^a)^{1/2} \) — this \( \overline{\mathcal{H}} \) is actually a Hilbert space under this norm.

Next, denote by \( \mathcal{C} \) the collection of all smooth spinor fields \( \lambda_A \) on \( T \) of compact support. Since every such spinor field is automatically in \( \mathcal{H} \), we have that \( \mathcal{C} \) is a complex vector subspace of \( \mathcal{H} \), and so also of its completion \( \overline{\mathcal{H}} \). Denote by \( \overline{\mathcal{C}} \) the closure of \( \mathcal{C} \) in \( \overline{\mathcal{H}} \), so \( \overline{\mathcal{C}} \) is a closed subspace of \( \overline{\mathcal{H}} \). Thus, an element of \( \overline{\mathcal{C}} \) is also represented by a Cauchy sequence of smooth spinor fields \( \lambda_A \) of compact support. But note that the corresponding locally square-integrable limiting spinor field, obtained as above, need not have compact support. Finally, denote by \( \mathcal{S} \) the quotient \( \overline{\mathcal{H}}/\overline{\mathcal{C}} \), so \( \mathcal{S} \) is itself a complete topological vector space. Thus, an element of \( \mathcal{S} \) is represented by a Cauchy sequence of smooth spinor fields \( \lambda_A \) on \( T \), where two such sequences define the same element of \( \mathcal{S} \) provided their difference converges to some element of \( \overline{\mathcal{C}} \). Alternatively, an element of \( \mathcal{S} \) may be
represented by a spinor field in $\mathcal{H}$, where two spinor fields define the same element of $\mathcal{S}$ provided their difference is in $\mathcal{C}$.

We now introduce two functions, $M_N$ and $M_A$, on $\mathcal{S}$ as follows. Fix $\alpha \in \mathcal{S}$, and consider

\begin{align*}
M_N(\alpha) &= \inf \| \lambda \|^2, \\ M_A(\alpha) &= \| \lambda \|^2 - 2 \int_T (\mathcal{D}^A_B \lambda^B)^\dagger (\mathcal{D}_A \lambda^C).
\end{align*}

In the first, the infimum on the right is over all $\lambda$ in the equivalent class $\alpha$, and so this right side indeed yields a function, $M_N$, on $\mathcal{S}$. For the second, first note that the right side is a continuous function on $\mathcal{H}$ (by $|\mathcal{D}^A_B \lambda^B|^2 \leq \frac{3}{2} |\mathcal{D}_A \lambda^C|^2$) that vanishes on $\mathcal{C}$ (by Eqn. (11)). Hence, that right side extends continuously to a function on $\mathcal{H}$ that vanishes on $\mathcal{C}$, thus yielding a function, $M_A$, on $\mathcal{S}$. Note that we may evaluate that right side of Eqn. (18) for any $\lambda$ in the equivalence class $\alpha$. Both of the functions are continuous and quadratic\(^\dagger\). For reasons that will emerge shortly, we call $M_N$ the \textit{norm mass function}, and $M_A$ the \textit{asymptotic mass function}.

The following example will motivate and illustrate these definitions. Let $T = R^3$, let $q_{ab}$ be the usual Euclidean metric on $R^3$, and let $p_{ab} = 0$. This is initial data for Minkowski spacetime. From Eqns. (1) – (2), these data have $\rho = 0$ and $\rho_a = 0$, and so satisfy the energy condition. We shall show that, for this initial-data set, $\mathcal{S}$ is a 2-dimensional complex vector space, which may be identified with the space of constant spinor fields on $T$, while both mass functions vanish.

Which smooth spinor fields $\lambda_A$ on $T$ have finite norm (12), i.e., which are in $\mathcal{H}$? We first show that every such $\lambda_A$ must, in a suitable sense, approach a constant asymptotically.

**Theorem 2:** Let $(T, q_{ab}, p_{ab})$ be the above initial data for Minkowski space-time, and let $\lambda_A \in \mathcal{H}$. Then there exists a constant spinor field $\tilde{\lambda}_A$ on $T$ such that

\begin{equation}
\int_T \frac{1}{r} (\lambda_A - \tilde{\lambda}_A)^2 \leq \frac{9}{2} \| \lambda \|^2,
\end{equation}

where $r$ denotes distance from some fixed origin on $T$.

**Proof:** In the norm (12) in this case, only the first term on the right survives. Taking one component at a time, it suffices to prove the result for a smooth scalar field $\lambda$, with

\begin{equation}
\| \lambda \|^2 = \int_T |D\lambda|^2
\end{equation}

finite. For each $0 \leq r < \infty$, set

\begin{equation}
g(r) = \int_{S_r} \lambda d\Omega,
\end{equation}

where $S_r$ denotes the sphere of radius $r$ centered at the fixed origin, and $d\Omega$ its unit surface-element. We have

\begin{align*}
r^2 \left( \frac{dg}{dr} \right)^2 &= r \int_{S_r} (D_a \lambda)(D^a r) d\Omega \\
&\lesssim \int_{S_r} r^2 (D^a r D_a \lambda) d\Omega \int_{S_r} (D^a r D_a r) d\Omega \\
&= \int_{S_r} |D\lambda|^2 r^2 d\Omega \cdot 4\pi.
\end{align*}

8
Integrating over $r$, we obtain
\[
\int_0^\infty r^2 \left( \frac{dg}{dr} \right)^2 dr \leq 4\pi \|\lambda\|^2 .
\]
(23)

It follows from (23), since $\|\lambda\|^2$ is finite, that $g(r)$ has a limit as $r$ approaches infinity. Subtract a constant, $\hat{\lambda}$, from $\lambda$ so that this limit becomes zero. Now expand $\lambda$ on $S_r$ in spherical harmonics to obtain
\[
\int_{S_r} \lambda^2 d\Omega \leq \frac{r^2}{2} \int_{S_r} |D\lambda|^2 d\Omega + \frac{1}{4\pi} \left[ \int_{S_r} \lambda d\Omega \right]^2 .
\]
(24)

Integrating this inequality over $r$, the first term on the right is bounded by $\frac{1}{2} \|\lambda\|^2$, and the second by $4 \|\lambda\|^2$, where we have used for the latter (23) and the following fact: If $g(r)$ approaches 0 as $r$ approaches infinity, then
\[
\int_0^\infty g^2 dr \leq 4 \int_0^\infty r^2 \left( \frac{dg}{dr} \right)^2 dr .
\]
(25)

The constant spinor field $\hat{\lambda}_A$ whose existence is guaranteed by the theorem is of course unique given $\lambda_A$. The theorem says, roughly speaking, that $\lambda_A$ approaches $\hat{\lambda}_A$ “faster than $r^{-1/2}$”. Thus, Theorem 2 gives the asymptotic behavior of the spinor fields with finite norm (i.e., those in $\mathcal{H}$). Which of these are limits of spinor fields of compact support, i.e., which are in $\overline{\mathcal{C}}$? The answer is provided by the following.

**Theorem 3:** Let, as in Theorem 2, $\lambda_A \in \mathcal{H}$. Then this $\lambda_A$ is in $\overline{\mathcal{C}}$ if and only if the constant field $\hat{\lambda}_A$ of that theorem vanishes.

**Proof:** For the “if” part, let this $\lambda_A$ have $\hat{\lambda}_A = 0$. Fix any number $r_0 > 0$, and any smooth nonnegative function $h(r)$ with $h(r) = 1$ for $r < r_0$, $h(r) = 0$ for $r > 2r_0$, and $|dh/dr| \leq 2/r$ for all $r$. Then $h(r)\lambda_A$ has compact support, while
\[
\|\lambda_A - h\lambda_A\|^2 = \int_T |D[(1-h)\lambda_A]|^2 \\
\leq 2 \int_T [(1-h)^2 |D\lambda_A|^2 + |\lambda_A|^2 |Dh|^2] \\
\leq 2 \int_{r \geq r_0} \left[ |D\lambda_A|^2 + 4 \frac{|\lambda_A|^2}{r^2} \right] .
\]
(26)

The last integral above converges, by $\lambda_A \in \mathcal{H}$ and Theorem 2 with $\hat{\lambda}_A = 0$, and the integrand is independent of $r_0$, so that integral approaches zero as $r_0$ approaches infinity. Repeating this argument for a succession of values of $r_0$, approaching infinity, we obtain a sequence of spinor fields of compact support, the corresponding $h\lambda_A$’s, which, by (26), converge to $\lambda_A$. For the “only if” part, fix any $r_0 > 0$ and any smooth vector field $w^a$ on $T$ equal to $-\frac{1}{4\pi} D^a(1/r)$ for $r > r_0$. Then, for any $\mu_A \in \mathcal{H}$ we have, again suppressing the spinor index,
\[
\hat{\mu} = \int_T D_a (\mu w^a) = \int_T w^a D_a \mu + \mu D_a w^a \\
\leq \left[ \int_T w^a w_a \int_T D^a \mu D_a \mu \right]^{1/2} + \left[ \int_{r \leq r_0} \mu^2 \int_{r \leq r_0} (D_a w^a)^2 \right]^{1/2} .
\]
(27)
But this formula shows that $\bar{\lambda}_A$, the asymptotic value of $\lambda_A$, is continuous in the topology of $\mathcal{H}$. The result follows. / 

Thus, two elements of $\mathcal{H}$ differ by an element of $\mathcal{C}$ when and only when they approach the same constant spinor field asymptotically. It follows that, in this example, $\mathcal{S}(=\bar{\mathcal{H}}/\mathcal{C})$ is a (complex) 2-dimensional vector space, which may be identified with the space of constant spinor fields on $T$. The operations of taking the adjoint and taking the $\epsilon$-inner product on constant spinor fields on $T$ extend, via this identification, to corresponding operations on $\mathcal{S}$. Thus $\mathcal{S}$ has all the structure of a spinor space. Both of the mass functions on $\mathcal{S}$ are zero, since the right sides of (17) and (18) vanish for constant $\lambda^A$. This, then, is $\mathcal{S}$ and the mass functions for these data for Minkowski space-time.

We turn now from flat initial data to asymptotically flat. We shall see that in this case the present framework yields the physically correct answer, an observation that serves as motivation for this framework. We first show that, for an initial-data set asymptotically flat in a suitable sense, the space $\mathcal{S}$ has a structure identical to that for Minkowski initial data.

**Theorem 4:** Let $(T, q_{ab}, p_{ab})$ be an initial-data set satisfying the energy condition, and having $T = R^3$. Let $\bar{q}_{ab}$ be a Euclidean metric on $T$ such that

1. $q_{ab} - \bar{q}_{ab}$ approaches zero asymptotically;
2. the fields $p_{ab}$ and $\bar{D}_a \bar{q}_{bc}$ are square-integrable, and the source $\rho$ integrable, over $T$; and
3. both $r p_{ab}$ and $r \bar{D}_a \bar{q}_{bc}$ approach zero asymptotically,

where $\bar{D}_a$ denotes the $\bar{q}$-derivative operator, and $r q$-distance from some origin. Then

i) the space $\mathcal{S}$ is 2-dimensional.

ii) each element, $\alpha$, of $\mathcal{S}$ has representative, $\lambda$, such that $\lambda^A$ is also in $\mathcal{H}$; and

iii) any two elements, $\alpha$ and $\beta$, of $\mathcal{S}$ have representatives, $\lambda$ and $\mu$, such that the function $\lambda^A \mu^A$ approaches a constant asymptotically.

**Proof:** Let $\bar{\lambda}^A$ be any constant spinor field on $(T, \bar{q}_{ab})$. Denote by $\lambda^A$ that spinor field (unique up to sign) on $(T, q_{ab})$ such that, at each point of $T$, the real part of the complex vector $\lambda^A \lambda^B$, as well as the 2-plane spanned by its real and imaginary parts, are identical with the corresponding vector and plane for $\bar{\lambda}^A \bar{\lambda}^B$. It is immediate from hypotheses $i)$ and $ii)$ that the $\lambda^A$ so constructed is in $\mathcal{H}$ (and so is the representative of some element of $\mathcal{S}$), and that the representatives so obtained themselves already satisfy conclusions $ii)$ and $iii)$ of the theorem. Thus, there remains only to show that every element of $\mathcal{S}$ is obtained via this construction, and that the zero element of $\mathcal{S}$ is obtained only via $\bar{\lambda}^A = 0$. These, in turn, are proven along the lines of Theorems 2 and 3, so modified to retain, and then bound via hypotheses $ii)$ and $iii)$, the additional terms involving $p_{ab}$ and $\bar{D}_a \bar{q}_{bc}$.

Fix $\mu^A \in \mathcal{H}$, denote by “$\lambda$” its components with respect to a basis constructed as in the paragraph above, and again define $g(r)$ by Eqn. (21). Then Eqn. (22) is replaced by

$$r^2 \left( \frac{dg}{dr} \right)^2 \leq \frac{32 \pi r^2}{\mathcal{C}} \left| \mathcal{D}_{AB} \mu_C \right|^2 d\Omega + r^2 V(r) \int_{S_r} \lambda^2 d\Omega,$$

where “$V(r)$” denotes the integral over $S_r$ of a certain expression quadratic in $p_{ab}$ and $\bar{q}_{ab}$. 

\[ \text{Eqn. (28)} \]
\( D_a q_{bc} \). So \( r^2 V(r) \) is bounded, by hypothesis (iii), and \( r \)-integrable to \( r = \infty \), by hypothesis (ii). Eqn. (24) is replaced by

\[
\int_{S_r} \lambda^2 d\Omega \leq 2r^2 \int_{S_r} |\mathcal{D}_{AB\mu C}|^2 d\Omega + \frac{1}{2\pi} g^2,
\]

for sufficiently large \( r \). A crucial step in the derivation uses of hypothesis (iii) to obtain

\[
r^2 \int_{S_r} (\text{quadratic in } p_{ab}, \ D_a q_{bc}) \lambda^2 d\Omega \leq 1/2 \int_{S_r} \lambda^2 d\Omega \text{ for sufficiently large } r. \]

Substituting (29) into (28), we obtain

\[
r^2 \left( \frac{dg}{dr} \right)^2 \leq r^2(8\pi + 2r^2 V(r)) \int_{S_r} |\mathcal{D}_{AB\mu C}|^2 d\Omega + \frac{r^2}{2\pi} V(r)g^2. \tag{30}
\]

Dividing both sides of this last inequality by \((1 + g^2)\), the right side is integrable to \( r = \infty \), and so therefore must be the left side. Thus, just as in the proof of Theorem 2, the function \( g \) must approach a limit as \( r \) approaches infinity, and so we may subtract from \( \lambda \) a constant to make this limit zero. Having done so, we have that \( r^2 (dg/dr)^2 \) is \( r \)-integrable (by Eqn. (30)), and so that \( g^2 \) is \( r \)-integrable (by Eqn. (25)), and so that \( \lambda^2 / r^2 \) is \( T \)-integrable (by Eqn. (29)). The appropriate modifications of the proof of Theorem 3 are similar but much simpler (requiring only hypotheses (i) and (ii)).

Thus, for any initial-data set that is asymptotically flat in the sense of the Theorem, the space \( \mathcal{S} \) has the structure of a spinor space. In more detail, \( \mathcal{S} \) is a complex, 2-dimensional vector space with an adjoint operation \( \dagger \) (obtained via conclusion (ii)) and an alternating tensor \( \epsilon \) (obtained via conclusion (iii)), with these two having the usual properties: \( \dagger \dagger = -1; \ \epsilon(\alpha, \beta) = -\epsilon(\beta, \alpha) = \epsilon(\alpha^\dagger, \beta^\dagger) \) for any \( \alpha, \beta \in \mathcal{S} \); and \( \epsilon(\alpha, \alpha^\dagger) > 0 \) for any nonzero \( \alpha \in \mathcal{S} \). Think of \( \mathcal{S} \) as the space of “asymptotic spinors”. The space of “asymptotic vectors” is now obtained from \( \mathcal{S} \) by the usual construction of vectors from spinors. Denote by \( \mathcal{V} \) the collection of all self-adjoint elements of the tensor product of \( \mathcal{S} \) with its complex-conjugate space \( \mathcal{\bar{S}} \), so \( \mathcal{V} \) is a real, 4-dimensional vector space. The alternating tensor \( \epsilon \) on the spinor space \( \mathcal{S} \) gives rise to a Lorentz metric \( \mathcal{g} \) on the vector space \( \mathcal{V} \); and then the adjoint operation \( \dagger \) on the spinor space \( \mathcal{S} \) gives rise to a unit timelike vector \( t \) on the vector space \( \mathcal{V} \). Think of \( t \) as the “asymptotic normal to the surface \( T \)”.

We next introduce, from their definitions (17) and (18), the two mass functions, \( M_N \) and \( M_A \), on the space \( \mathcal{S} \). It turns out (Theorem 6) that these two functions coincide in this case. The mass function is a Hermitian quadratic form on \( \mathcal{S} \), and so gives rise to a linear function on \( \mathcal{V} \), i.e., to an element of the dual space, \( \mathcal{V}^* \). This covector on \( \mathcal{V} \) is the total mass-momentum of our initial-data set. Since the original quadratic form was positive semi-definite, the mass-momentum is nonspacelike. The \( g \)-norm of the mass-momentum is the invariant mass of our initial-data set; its inner product with \( t \), the mass-component in the “direction normal to the surface \( T \)”. Thus, we have constructed the mass-momentum vector for any initial-data set asymptotically flat in the sense of Theorem 4.

The asymptotic conditions of Theorem 4 are weaker than those required in the standard (ADM) definition of mass-momentum. Let us now impose on our initial-data set stronger asymptotic conditions: that \( r|q_{ab} - \tilde{q}_{ab}| \), \( r^2|D_a q_{bc}| \), and \( r^2|p_{ab}| \) all be bounded.
Now the ADM mass-momentum is well defined, and so now comparison between it and the present mass-momentum is possible. The two agree\textsuperscript{14}. Thus, the present framework indeed represents a generalization of the ADM mass-momentum.

Bartnik\textsuperscript{15} has also generalized the ADM mass-momentum to weaker asymptotic conditions, which appear to be very slightly stronger than ours. Presumably, when both sets of conditions are satisfied, the two mass-momenta agree.

3 Properties

We now discuss some examples and some general properties of the space $S$ and the two mass functions, $M_N$ and $M_A$, thereon.

For certain initial-data sets, it can occur that $H = C$, i.e., that every spinor field $\lambda^A$ of finite norm (12) is a limit of spinor fields of compact support. When this occurs, we shall have the space $S$ zero-dimensional (with, e.g., representative $\lambda^A = 0$), and so, necessarily, the mass functions $M_N$ and $M_A$ vanishing. This circumstance can arise in at least two ways — because there is “too much matter” in the space-time, or “too little room at infinity”. These are illustrated in the following.

**Theorem 5:** Let $(T, g_{ab}, p_{ab})$ be an initial-data set satisfying the energy condition, and let $\kappa$ be a smooth positive function on $T$ such that, for any number $\kappa_0$, the set $B_{\kappa_0} = \{ x \in T | \kappa(x) \leq \kappa_0 \}$ is compact. Further, let there exist a compact subset of $T$ outside of which either

i) $\left( \frac{D_a \kappa}{\kappa} \right)^2 \leq \rho - |\rho^o| \quad$ everywhere, or

ii) $D^2 \kappa + |p_{ab} D^b \kappa| \leq 0 \quad$ everywhere.

Then $S = \{ 0 \}$.

**Proof:** Fix any $\lambda^A \in \mathcal{H}$, and any $\epsilon > 0$. Choose $\kappa_0$ sufficiently large that, first, $B_{\kappa_0} = \{ x \in T | \kappa(x) \leq \kappa_0 \}$ is compact. Further, let there exist a compact subset of $T$ outside of which either

- $\left( \frac{D_a \kappa}{\kappa} \right)^2 \leq \rho - |\rho^o| \quad$ everywhere, or
- $D^2 \kappa + |p_{ab} D^b \kappa| \leq 0 \quad$ everywhere.

Then $S = \{ 0 \}$.

**Proof:** Fix any $\lambda^A \in \mathcal{H}$, and any $\epsilon > 0$. Choose $\kappa_0$ sufficiently large that, first, $B_{\kappa_0}$ contains the compact subset of the theorem, and that, second,

$$\int_{(B_{\kappa_0})^c} \left[ |D_{AB} \lambda_C|^2 + \frac{1}{2}(\rho |\lambda|^2 + i\sqrt{2}\rho_{AB} \lambda^I\lambda^B) \right] \leq \epsilon. \quad (31)$$

Let $f$ be the function, of compact support, given by

$$f = \begin{cases} 
1, & \kappa \leq \kappa_0 \\
1 - s \log(\kappa/\kappa_0), & \kappa_0 < \kappa < \kappa_0 e^{1/s} \\
0, & \kappa_0 e^{1/s} \leq \kappa,
\end{cases} \quad (32)$$

where $s > 0$ is some number. We now have

$$|| \lambda - f \lambda ||^2 = \int_T \left[ |\lambda|^2 |Df|^2 + (1 - f)^2 \left[ |D_{AB} \lambda_C|^2 + \frac{1}{2}(\rho |\lambda|^2 + i\sqrt{2}\rho_{AB} \lambda^I\lambda^B) \right] \right] \leq \int_U |\lambda|^2 |Df|^2 + \epsilon. \quad (33)$$

where we have set $U = (B_{\kappa_0})^c \cap (B_{\kappa_0 e^{1/s}})$. In case i), substitute $|Df|^2 = s^2 |D\kappa/\kappa|^2$ into (33) and use condition i) to obtain $|| \lambda - f \lambda ||^2 \leq (2s^2 + 1) \epsilon$. In case ii), first choose $1 > t > 0$ sufficiently small that, setting $w^a = t(D^a \kappa)/\kappa$, we have $\int_{B_{\kappa_0}} |\lambda|^2 w \cdot dA \leq \epsilon$. 12
Next set $s^2 = t(1 - t)$, which, with condition $ii$), yields $D_a w^a + w^a u_a + |p_{ab} w^b| + |D f|^2 \leq 0$ in $U$. Then

$$0 \leq \int_{\partial (B_{q \epsilon} / \epsilon / s)} |\lambda|^2 w^a dA_a \leq \int_U D_a (w^a |\lambda|^2) + \epsilon$$

$$= \int_U \left\{ |\lambda|^2 D_a w^a + i \sqrt{2} w^{AB} p_{ABCD} \lambda^C \lambda^{D +} + w^{AB} [\lambda^C D_{AB} \lambda_C] \right\} + \epsilon$$

$$\leq \int_U \left\{ |\lambda|^2 (D_a w^a + |p_{ab} w^b| + w^2) - |D \lambda - w \lambda|^2 + |D \lambda|^2 \right\} + \epsilon$$

$$\leq \int_U -|D f|^2 |\lambda|^2 + 2\epsilon,$$

(34)

Now Eqn. (33) gives $|| \lambda - f \lambda ||^2 \leq 3\epsilon$. / 

We give a number of applications of this theorem.

Let $(T, q_{ab}, p_{ab})$ be an initial-data set with $(T, q_{ab})$ complete. Fix any origin $x$ on $T$ and let $r$ denote $q$-distance from that origin. Now suppose further that the data are such that, for some positive number $c$, $\rho - |p^a| \geq c^2 / r^2$ outside a compact set. Then $S = \{0\}$. (Proof: Set $\kappa = (1 + r^2)^{c/2}$ and apply Theorem 5, case $i$). This example renders in the present framework the physical statement “when the mass density falls off more slowly than $1/r^2$, the total mass is infinite”.

As a second example, let $T$ be an open ball in $R^3$ with radius $r_0$, let $q_{ab}$ the flat metric induced from that of $R^3$, and let $p_{ab} = q_{ab} / (r_0 - r)$. One can easily check that here $\rho - |p^a| = 1 / (r_0 - r)^2$, whence these initial data satisfy the energy condition. In this example, $S = \{0\}$. (Proof: Set $\kappa = (r_0 - r)^{-1}$ and apply Theorem 5, case $i$).

In these two examples of Theorem 5, case $i$), there is “too much matter in the space-time”. The total physical mass of these initial-data sets is actually “infinity”. But, since the present formalism permits only finite values for the mass functions, $M_N$ and $M_A$, the response of that formalism is to collapse $S$ to a point, i.e., to provide no asymptotic directions along which the components of the total mass-momentum can be evaluated.

For the next example, let $T = S^1 \times S^1 \times R$, $q_{ab}$ the natural flat metric on this product, and $p_{ab}$ zero. Thus, this is initial data for Minkowski space-time, but with two dimensions “suppressed by wrapping”. In this example, we have $S = \{0\}$. (Proof: Set $\kappa = r$ outside a compact set, where $r$ denotes distance in $R$ from an origin, and apply Theorem 5, case $ii$). Thus, the loss of two “asymptotic directions” suffices to collapse $S$ to a point. Indeed, even a single asymptotic direction suffices (but just barely): For $T = S^1 \times R^2$, $q_{ab}$ again the natural flat metric and $p_{ab}$ zero, we again have $S = \{0\}$. (Proof: Set $\kappa = \log r$ outside a compact set, where $r$ denotes distance in $R^2$ from an origin, and again apply Theorem 5, case $ii$).

In general, according to case $ii$) of Theorem 5, the space $S$ collapse to a point whenever there is “too little room at infinity”. This is illustrated in the examples of the previous paragraph, in which entire asymptotic dimensions are suppressed. A second class of examples involves “nearby” asymptotic regions. Let $(T, q_{ab}, p_{ab})$ be any initial-data set with $T$ compact. Then clearly $C = \mathcal{H}$, and so we have $S = \{0\}$. Let us now remove a single point $x$ from $T$. The resulting initial-data set again has $S = \{0\}$. (Proof: Set $\kappa = r^{-1/2}$ in a neighborhood of $x$, where $r$ denotes $q$-distance from $x$, and apply Theorem 5, case $ii$). Similarly, the removal of any finite number of points from an initially compact
initial-data set retains \( S = \{0\} \). Next, let us remove from the compact \( T \) above any closed curve. Then, again, we have \( S = \{0\} \). (Proof: Set \( \kappa = (- \log r)^{1/2} \) in a neighborhood of that curve, where \( r \) denotes \( q \)-distance from the curve, and apply Theorem 5, case ii).) Similarly, the removal of any finite number of closed curves and line segments retains \( S = \{0\} \).

The examples of the previous paragraph show that an otherwise compact initial-data set with a zero- or one-dimensional “edge” behaves as though it were compact: It retains \( S = \{0\} \) and, therefore, zero mass functions. What happens in the case of a two-dimensional “edge”? It turns out that, in this case, the space \( S \) is always infinite-dimensional, and the mass functions \( M_N \) and \( M_A \) are always nontrivial. This we now show. Let \( \overline{T} \) be a smooth, compact three-dimensional manifold with boundary, so \( K = \partial \overline{T} \) is a smooth, two-dimensional manifold. Fix smooth initial data, satisfying the energy condition, on \( \overline{T} \). Now consider the initial-data set \((T, q_{ab}, p_{ab})\), where \( T = \overline{T} - K \) is the interior of \( \overline{T} \), and \( q_{ab} \) and \( p_{ab} \) are the induced initial data. We wish to determine the space \( S \) and the mass functions \( M_N \) and \( M_A \) for this initial-data set.

Consider first any smooth spinor field \( \lambda^A \) on \( \overline{T} \). Then its restriction to \( T \) certainly defines an element of \( \mathcal{H} \). When are two such equivalent, i.e., when do they differ by an element of \( \overline{\mathcal{C}} \)? We claim: \( \lambda'^A - \lambda^A \in \overline{\mathcal{C}} \) if and only if \( \lambda'^A = \lambda^A \) on the boundary \( K \). To prove “if”, assume that \( \lambda'^A = \lambda^A \) on \( K \), and set, for \( n = 1, 2, \ldots \),

\[
\lambda^A_n = \begin{cases} 
0, & r \leq 1/n \\
(nr - 1)(\lambda'^A - \lambda^A), & 1/n < r < 2/n \\
\lambda^A - \lambda^A, & 2/n \leq r
\end{cases}
\]  

(35)

where \( r \) is \( q \)-distance from \( K \). Then verify that this sequence of spinor fields is in \( \mathcal{C} \), and that it converges in \( \overline{\mathcal{H}} \) to \( \lambda'^A - \lambda^A \). To prove “only if”, first note that, for any fields \( \mu^A \) and \( w^a_B \) on \( \overline{T} \), we have

\[
\int_K w^a_B \mu^B dS_a = \int_T D_a (w^a_B \mu^B) = \int_T (D_a w^a_B) \mu^B + \int_T w^a_B (D_a \mu^B) \\
\leq \left[ \int_T |D_a w^a_B|^2 \right]^{1/2} \left[ \int_T |\mu|^2 \right]^{1/2} + \left[ \int_T |w^a_B|^2 \right]^{1/2} \left[ \int_T |D_a \mu|^2 \right]^{1/2}.
\]  

(36)

But each term on the right is continuous in \( \mu^B \in \mathcal{H} \) (using Theorem 1 for the first term), and so the surface integral on the left is also continuous. The conclusion — that \( \lambda'^A_n \in \mathcal{C} \) converging in \( \overline{\mathcal{H}} \) to \( \lambda'^A - \lambda^A \) implies \( \lambda'^A = \lambda^A \) on \( K \) — follows.

Thus, each smooth spinor field specified on \( K \) gives rise to a point of \( S \), and distinct such fields give rise to distinct points of \( S \). So, in particular, the space \( S \) in this example is infinite-dimensional. It is easy to evaluate the asymptotic mass function at such points of \( S \). Indeed, for \( \lambda^A \) a spinor field specified on \( K \), giving rise to point \( \alpha \) of \( S \), we have\(^3\), from Eqn. (11),

\[
M_A(\alpha) = \int_K (\lambda^{\dagger C} D_{AB} \lambda_C + 2 \lambda^{\dagger (B} D_{A)C} \lambda^C) dS^{AB}.
\]  

(37)

Note that the right side does indeed depend only on \( \lambda^A \) at points of \( K \), i.e., it only involves derivatives of \( \lambda^A \) tangent to \( K \). Thus, the asymptotic mass function, at such points, is a simple surface integral. The norm mass function is of course more complicated.
The full space $S$ in this example is constructed essentially by “completing” the collection of special elements obtained above. The result\textsuperscript{17} is the Sobolev space $W^{1/2}(K)$. This consists, in more detail, of those spinor fields specified on $K$ such that the integral

\[
\int_K |\lambda|^2 + \int_K dS_x \int_K dS_y \frac{|\lambda(x) - \lambda(y)|^2}{d(x,y)^3}
\]

(which defines the Sobolev norm) converges. Here, $d(x,y)$ denotes the $q$-distance between points $x$ and $y$ of $K$, and the difference of spinors in the second integrand is to be taken using components in any basis. The spinor fields in $W^{1/2}(K)$ are better behaved than those in $W^0(K) = L^2(K)$, but not so well-behaved as those in $W^1(K)$ (whose norm is given by the integral over $K$ of the square of the spinor field plus the square of its derivative). The integral on the right in Eqn. (37) makes sense for spinor fields $\lambda^A$ in $W^{1/2}(K)$, and again yields the asymptotic mass function $M_A$.

Thus, the space $S$ in this example is infinite-dimensional, and the mass functions, $M_N$ and $M_A$, are complicated bounded, quadratic functions on $S$. Even in the example of a ball from Minkowski initial data, there are points of $S$ at which $M_N$ is positive (and so we shall not have $M_N = 0$), and points at which $M_A$ is negative (and so we shall not have $M_A = M_N$).

There is a simple inequality relating the two mass functions to each other:

\[
-2M_N(\alpha) \leq M_A(\alpha) \leq M_N(\alpha).
\]

To derive this, take the infimum of the right side of (18) and use $|DA^B\lambda^C| \leq \frac{3}{2} |DA^B\lambda^C|^2$. There are, as we shall see shortly, many examples for which the second inequality in (39) is an equality. Are there nontrivial examples for which the first inequality is an equality?

For which initial-data sets must the two mass functions be equal? There is, as it turns out, a large class for which we can assert that $M_A = M_N$, these consisting of “essentially all” complete initial-data sets.

**Theorem 6:** Let $(T, q_{ab}, p_{ab})$ be an initial-data set satisfying the energy condition. Assume that

i) $T$, $q_{ab}$ is complete, and

ii) $T$ admits no nonzero spinor field $\kappa^A \in L^2(T)$ with $\|\kappa\|^2 = 0$.

Then $M_A = M_N$.

**Proof:** Fix any $\alpha \in S$, and let $\lambda^A$ be any representative. So $D_A^B\lambda_B \in L^2(T)$. Denote by $\kappa^A$ the $L^2$-projection of $D_A^B\lambda_B$ orthogonal to the closed subspace of $L^2$ generated by elements of the form $D_A^B\sigma_B$ with $\sigma \in C$. Then $\kappa \in L^2(T)$, and (by orthogonality to the $D_A^B\sigma_B$) $D_A^B\kappa_B = 0$. We have only to show that $\kappa^A = 0$, for this implies, by Eqns. (17) and (18), that $M_A(\alpha) = M_N(\alpha)$.

For any function on $T$ of compact support, we have

\[
\int f^2[|D_A^B\kappa_C|^2 + \frac{1}{2}(\rho|\kappa|^2 + i\sqrt{2}\rho_{AB} \kappa^A \kappa_B)]^2 \leq \int |Df|^2|\kappa|^2 \left[ \int f^2[|D_A^B\kappa_C|^2 + \frac{1}{2}(\rho|\kappa|^2 + i\sqrt{2}\rho_{AB} \kappa^A \kappa_B)] \right]^{1/2},
\]

(40)
where we used (11) in the first step, and an integration by parts and the Schwarz inequality in the second. Now let \( r \) denote \( q \)-distance in \( T \) from some fixed point, and set, in Eqn. (40),

\[
f = \begin{cases} 
1, & r \leq r_0 \\
2 - r/r_0, & r_0 < r < 2r_0 \\
0, & 2r_0 \leq r
\end{cases}
\]  

(41)

where \( r_0 \) is some number. This \( f \) has compact support, by hypothesis \( i \). Letting \( r_0 \) approach infinity, the left side of Eqn. (40) approaches \( \| \kappa \|^2 \), while, since \( \kappa \in L^2 \) and \( |Df| \leq 1/r_0 \), the first factor on the right approaches zero. So, \( \| \kappa \|^2 = 0 \), and so, by hypothesis \( ii \), \( \kappa^A = 0 \).

Hypothesis \( ii \) of Theorem 6 serves only to rule out a few, very special, examples. Indeed, as we shall see shortly, the only initial-data sets admitting a nonzero \( \kappa^A \) with \( \| \kappa \|^2 = 0 \) are certain ones for flat space-times and certain ones for plane-wave space-times. So, for instance, hypothesis \( ii \) can be dropped entirely for any initial-data set having some point at which \( \rho > |\rho^a| \), or some point at which the Weyl tensor is other than type [-] or type [4].

But, nevertheless, Theorem 6 is actually false without hypothesis \( ii \). For example, fix constant spinor fields \( \kappa^A \) and \( \mu^A \) in Minkowski space-time, with these normalized by \( \kappa^A\mu_A = 1 \). Set \( \lambda^A = \kappa^A\mu_B\kappa^B x^{BB'} \), where \( x^{BB'} \) is a dilation vector field (i.e., one satisfying \( \nabla_a x^b = \delta^b_a \)). Then, for any slice \( T \) in this space-time, \( \kappa^A \) and \( \lambda^A \) become spinor fields on the corresponding initial-data set. Choosing the slice such that its unit normal, \( t^{AA'}/\sqrt{2} \), lies in the plane of \( \kappa^A\kappa^{A'} \) and \( \mu^A\mu^{A'} \), we have

\[
\mathcal{D}_{AB}\kappa_C = 0, \quad |\kappa|^2 = |\mathcal{D}_{AB}\lambda_C|^2 = i\kappa^A\mathcal{D}_A B\lambda_B = t_{AA'}\kappa^A\kappa^{A'}. \tag{42}
\]

Now choose Minkowskian coordinates, \((t, x, y, z)\), such that \( \kappa^A\kappa^{A'} \) and \( \mu^A\mu^{A'} \) have respective components \((1,1,0,0)\) and \((1,-1,0,0)\), let \( T \) be given by \( t = x(1 - (1 + x^2)^{-1/2}) \), and cyclically identify \( y \) and \( z \) (i.e., identify points \((t, x, y, z)\) and \((t, x, y + n, z + m)\), for \( n, m \) any integers). The resulting initial-data set has spinor fields \( \kappa^A \) and \( \lambda^A \) with (by Eqn. (42)) \( \mathcal{D}_{AB}\kappa_C = 0, \quad \kappa \in L^2, \quad \mathcal{D}_{AB}\lambda_C \in L^2, \quad f\kappa^A\mathcal{D}_A B\lambda_B \neq 0 \). It follows from the third of these that \( \lambda \in \mathcal{H} \), and so that this \( \lambda^A \) defines some element, \( \alpha \), of \( \mathcal{S} \). But the other three properties imply that \( \mathcal{D}_A B\lambda_B \) cannot be made arbitrarily small (in \( L^2 \)) by addition to \( \lambda^A \) of various \( \sigma^A \in \mathcal{C} \). This shows that \( M_A(\alpha) \neq M_N(\alpha) \), and, in particular, that \( \alpha \neq 0 \).

For incomplete initial data, there are much simpler examples in which \( M_A \) and \( M_N \) differ. For instance, fix \( r_0 > 0 \) and let \( (T, q_{ab}, p_{ab}) \) be the initial-data set obtained from the subset \( r_0 < r < 4r_0 \) of the standard flat initial-data set, where \( r \) is \( q \)-distance from some fixed origin. Choose spinor field \( \lambda^A \) on \( (T, q_{ab}) \) to be a constant field, \( \lambda_A^{(1)} \) on \( r_0 < r < 2r_0 \), and a different constant field, \( \lambda_A^{(2)} \) on \( 3r_0 < r < 4r_0 \). Denoting by \( \alpha \in \mathcal{S} \) the corresponding equivalence class, it is easy to check that \( M_A(\alpha) = 0 \) and \( M_N(\alpha) = \frac{16\pi}{3} r_0 |\lambda_A^{(1)} - \lambda_A^{(2)}|^2 \).

We now turn to the issue of the positivity of the two mass functions. The norm mass function, from its definition in (17), is manifestly nonnegative, while the asymptotic mass function, from its definition (18), is not. When the data are complete, the two mass functions are generally the same, and so nonnegativity also of the asymptotic mass function then follows.
Can the asymptotic mass function ever become negative? The answer is yes, as we see in the following example. Let \((T, q_{ab}, p_{ab})\) be the initial-data set with \(T\) any open subset, with compact closure, in \(R^3\), \(q_{ab}\) the metric on \(T\) inherited from the Euclidean metric of \(R^3\), and \(p_{ab} = 0\). Set

\[
\lambda_A = \beta_A^\dagger \beta_M \beta_P \beta_Q x^{MN} x^{PQ},
\]

where \(x^a\) is a dilation vector field on \(T\), and \(\beta_A\) any nonzero constant spinor field, normalized by \(\beta_A^\dagger \beta_A = 1\). Then for any spinor field \(\sigma_A\) on \(T\) of compact support we have

\[
\int_T |(D_{AB}(\lambda_C + \sigma_C)\|^2 = \int_T |D_{AB} \lambda_C|^2 + |D_{AB} \sigma_C|^2,
\]

for the cross term vanishes by virtue of \(D^\dagger_{AB} D_{BC} \lambda^C = 0\). Hence, this \(\lambda_A\) realizes the infimum in (18), i.e., we have

\[
M_N(\alpha) = \int_T |D_{AB} \lambda_C|^2 = \int_T |\beta_B x^{AB}|^2,
\]

where \(\alpha \in \mathcal{S}\) is the corresponding equivalence class. Substituting into (18), we now obtain \(M_A(\alpha) = -M_N(\alpha) < 0\). Is there a simple theorem guaranteeing \(M_A \geq 0\) for some large class of initial-data sets?

We have seen that the norm mass function, by its definition, can never be strictly negative. But it is sometimes possible for this mass function to attain the value zero, e.g., in the case of data for Minkowski space-time. When, more generally, can this occur? Fix an initial-data set \((T, q_{ab}, p_{ab})\) satisfying the energy condition, and denote by \(Z\) the collection of all spinor fields \(\lambda^A\) on \(T\) that are \(D\)-constant:

\[
D_{AB} \lambda_C = 0.
\]

Then \(Z\) is clearly a complex vector space, with (since two solutions of (46) agreeing at a point must agree everywhere) dimension at most two. By (11), \(\lambda^A \in Z\) if and only if \(\| \lambda \|^2 = 0\). Thus, each element of \(Z\) gives rise to a point of \(\mathcal{S}\) at which \(M_N = 0\). Are these the only points of \(\mathcal{S}\) at which the norm mass function vanishes? That they are (i.e., that \(\alpha \in \mathcal{S}\) with \(M_N(\alpha) = 0\) implies that there is a representative \(\lambda^A\) of \(\alpha\) with \(\lambda^A \in Z\) would follow from:

**Conjecture 7.** Let \((T, q_{ab}, p_{ab})\) be an initial-data set satisfying the energy condition, and \(x\) any point of \(T\). Then there exists a neighborhood \(U\) of \(x\) and a number \(c > 0\) with the following property: Given any \(\lambda^A \in \mathcal{H}\), there is a field \(\hat{\lambda}^A\) in \(U\), there satisfying (46), such that

\[
\int_U \left[ |D^{AB} \lambda_C|^2 + \frac{1}{2} (\rho \lambda^{A\Lambda} \lambda_A + i \sqrt{2} \rho_{AB} \lambda^{A\Lambda} \bar{\lambda}^B) \right] \geq c \int_U |\lambda - \hat{\lambda}|^2.
\]

This conjecture asserts, roughly speaking, that, locally and modulo fields of norm zero, the norm \(\| \cdot \|^2\) bounds the \(L^2\) norm. Note, e.g., that the conclusion of the conjecture holds automatically whenever \(\rho > |\rho^a|\) at the point \(x\). There follows from this conjecture, not only that all zeros of \(M_N\) arise from \(Z\), but an even stronger result, to the effect
that all “near zeros” of $M_N$ also arise from $Z$. More precisely, we have the following consequence of Conjecture 7: For $\alpha_1, \alpha_2, \ldots$ points of $S$, with $M_N(\alpha_i)$ approaching zero, there exist elements $\mu_1^A, \mu_2^A, \ldots$ of $Z$ such that the sequence $\alpha_i - \{\mu_i\}$ in $S$ approaches zero. This consequence would guarantee, e.g., that, whenever $Z = \{0\}$, $S$ is a Hilbert space under norm $M_N$. Thus, Conjecture 7 would provide good control, in terms of the simple vector space $Z$, of all the “zero behavior” of the norm mass function. It would be of interest to settle this conjecture.

To see what Eqn. (46) means geometrically, we proceed as follows. Fix an initial-data set satisfying the energy condition, and admitting a nonzero solution, $\lambda^A$, of Eqn. (46). It is convenient to embed this initial-data set in a full space-time satisfying the dominant energy condition: that $(R_{ab} - 1/2Rg_{ab})u^b$ is future-directed nonspacelike for every future-directed timelike $u$. (Such an embedding is always possible, e.g., by choosing for the matter source dust.) Thus, we obtain a full, 4-dimensional space-time, $(M, g_{ab})$, with a certain spacelike, 3-dimensional submanifold $T$. Then $\lambda^A$, the spinor field defined originally on the manifold $T$, becomes a spinor field in the full space-time, defined only at points of the submanifold $T$. Eqn. (46) becomes that $w^b\nabla_b \lambda^A = 0$ (48)
at each point of the submanifold $T$, where $w^a$ is any vector at that point tangent to $T$. Taking a second derivative tangent to $T$ and commuting, we find $t_{[a}R_{bc]de}t^e = 0, \quad t_{[a}R_{bc][de}l_f] = 0,$ (49) at all points of $T$, where $t^a$ is the unit normal to $T$, and $l^a$ the null-vector equivalent of the spinor $\lambda^A$. Contraction the second of these equations three times, we obtain $(R_{ab} - 1/2Rg_{ab})l^a t^b = 0$ (50)But this, along with the dominant energy condition, implies in turn that $R_{ab}$ is some multiple of $l_a l_b$, i.e., that the matter is null dust. Substituting this Ricci tensor into Eqns (49), we obtain these same equations on the Weyl tensor, $C_{abcd}$. But these in turn imply $C_{bcde}l^e = 0, \quad C_{bc[de}l_f] = 0,$ (51)i.e., that the Weyl tensor is type [4], with $l^a$ the repeated principal null direction.

To summarize, in the “generic” case, the vector space $Z$ is zero-dimensional, and the norm mass function is strictly positive on nonzero elements of $S$. In order that $Z$ be one-dimensional, the matter must be null dust, and the Weyl tensor type [4] with the dust velocity as its principal null direction. In order that $Z$ be two-dimensional, the initial-data must be flat. The space $Z$ can never be of dimension three or higher.

What happens under change in the initial data in some compact region? Will the space $S$ and the mass functions also change? To address this issue, consider two initial-data sets, $(T, q_{ab}, p_{ab})$ and $(T, \tilde{q}_{ab}, \tilde{p}_{ab})$, on the same underlying manifold $T$. Let both satisfy the energy condition, and let $q_{ab} = \tilde{q}_{ab}$ and $p_{ab} = \tilde{p}_{ab}$ outside some open subset $U$ of $T$ with compact closure. We first obtain a natural isomorphism $\mathfrak{I}$ between the corresponding $S$
and $\tilde{S}$, as follows. Given any element of $S$, with representative $\lambda_A$, map it via $\mathfrak{S}$ to that element of $\tilde{S}$ having some representative, $\tilde{\lambda}_A$, with $\tilde{\lambda}_A = \lambda_A$ outside $U$. This makes sense, since $q_{ab} = \tilde{q}_{ab}$ and $p_{ab} = \tilde{p}_{ab}$ outside $U$; and is independent of representative, since $U$ has compact closure. This map is indeed a continuous isomorphism between $S$ and $\tilde{S}$.

We first note that the asymptotic mass function $M_A$ is invariant under this isomorphism, for it can be written (by Eqns. (18) and (11)) as an integral whose integrand is a pure divergence. But what of the norm mass function? For complete initial data, the norm mass function is usually the same as the asymptotic mass function and therefore a pure divergence. But, for incomplete data, invariance can fail. For example, let $T$ be any open, bounded subset of $R^3$, $q_{ab}$ the flat metric on $T$ induced from that of $R^3$, and $p_{ab} = 2\sqrt{2}/3s_0q_{ab}$, where $s_0$ is any positive constant. This initial-data set satisfies the energy condition (indeed, with $\rho = \frac{2}{3}s_0^2$, $\rho_a = 0$). Set

$$\lambda^A = \exp(ik_a x^a)^\circ \lambda^A$$

where $\hat{\lambda}^A$ is any constant spinor field, normalized by $|\hat{\lambda}|^2 = 1$, $k_a$ is the constant vector field given by $k_{AB} = 2is_0 \hat{\lambda}^{\dagger}_{(A} \lambda_{B)}$, and $x^a$ is a dilation vector field. This $\lambda^A$ is in $\mathcal{H}$, and so defines an element, $\alpha$, of $S$. Using that $\mathcal{D}^A_B \mathcal{D}^B_C \lambda^C = 0$, it follows, from Eqn. (17) and (11) that

$$M_N(\alpha) = \|\lambda\|^2.$$  

(53)

Now change this initial data, to $\tilde{q}_{ab} = q_{ab}$ and $\tilde{p}_{ab} = p_{ab} - 2\sqrt{2}/3 hq_{ab}$, where $h$ is a smooth function on $T$ of compact support. For $h$ and its derivative sufficiently small, these will continue to satisfy the energy condition. Then from Eqn. (53) and the fact that $M_N$ is defined as an infimum, we have

$$M_N(\tilde{\alpha}) \leq M_N(\alpha) + 2 \int [ih(\lambda^A D_{AB} \lambda^B - \lambda^A (D_{AB} \lambda^B)^\dagger) + h^2 \lambda^A \lambda_A]$$

$$= M_N(\alpha) + 2 \int (h^2 - 2s_0 h)|\lambda|^2$$

(54)

Now choosing $h$ non-negative and sufficiently small, the last integral on the right becomes negative, yielding $M_N(\tilde{\alpha}) < M_N(\alpha)$. Thus, a change in the data, though restricted to a compact region, has nontheless changed the norm mass function. Is there any simple theorem that isolates a large class of initial-data sets having invariance of $M_N$ under compactly-supported changes in the data?

Some initial-data sets contain several “asymptotic regions”. A familiar example is that of a slice in the extended Schwarzschild spacetime that extends through the “wormhole-throat”. We consider now some properties of $S$ and the mass functions in such examples. Let $(T, q_{ab}, p_{ab})$ be an initial-data set satisfying the energy condition. Fix closed subsets, $T_1$ and $T_2$, of $T$ that have compact intersection and cover $T$. These represent the two “asymptotic regions”. We first show that, under this arrangement, the space $S$ splits as a direct sum of two subspaces. Denote by $\mathcal{H}_1$ the collection of all spinor fields $\lambda^A \in \mathcal{H}$ having $\text{supp}(\lambda) \subset T_1 \cup A$ for some compact set $A$, and similarly for $\mathcal{H}_2$. Then each of $\mathcal{H}_1$ and $\mathcal{H}_2$ is a vector subspace of $\mathcal{H}$, while these two have intersection $\mathcal{C}$ and together span $\mathcal{H}$. Furthermore, we have that any field $\lambda^A$ in $\overline{\mathcal{H}_1 \cap \mathcal{H}_2}$ is also in $\overline{\mathcal{C}}$. (Indeed, for
Denote by $M_{1A}$ and $M_{2A}$ the restrictions of the asymptotic mass function $M_A$ to the respective subspaces $S_1$ and $S_2$ of $S$. Similarly, $M_{1N}$ and $M_{2N}$ for the norm mass function. Thus, we introduce, for each of the two asymptotic regions, separate mass functions. In the case of the asymptotic mass, these separate mass functions add to give the total mass function, i.e., we have $M_A = M_{1A} + M_{2A}$. (This is easily seen by noting that every $\alpha \in S$ has a representative of the form $\lambda_1A + \lambda_2A$ with $\lambda_1A \in H_1$ and $\lambda_2A \in H_2$; and recalling that the right side of (18) is independent of representative.) It follows from this that also $M_N = M_{1N} + M_{2N}$ provided the initial-data set is one in which $M_N = M_A$, e.g., is one to which Theorem 6 applies. For example, in the case of the slice in the extended Schwarzschild space-time, $M_{1N}$ and $M_{2N}$ yield the usual mass-momenta corresponding to the respective asymptotic regions. Theorem 6 applies to this example, and so the total norm mass function is just the sum of these two.

We remark, however, that in general we need not have $M_N = M_{1N} + M_{2N}$. For example, fix numbers $0 < r_1 < r_2$, let $(T, q_{ab}, p_{ab})$ be the subset $r_1 < r < r_2$ of the standard initial data for Minkowski space-time, where $r$ is distance from some origin. Let $T_1$ and $T_2$ denote the subsets given by $r_1 < r \leq \frac{1}{2}(r_1 + r_2)$ and $\frac{1}{2}(r_1 + r_2) \leq r < r_2$, respectively. Fix a constant spinor field $\dot{\lambda}^A$ on $(T, q_{ab})$, and let $\alpha \in S$ denote its equivalence class. Then, by direct calculation, one verifies that $M_A(\alpha) = M_{1A}(\alpha) = M_{2A}(\alpha) = M_N(\alpha) = 0$, while $M_{1N}(\alpha) = M_{2N}(\alpha) = \frac{4\pi r_1 r_2}{r_2 - r_1} |\dot{\lambda}|^2$.

It follows in particular from the observations above, and the examples at the beginning of this section, that the removal of any finite number of points, closed curves, and closed line segments from an initial-data set changes neither the space $S$ nor the mass functions $M_N$ and $M_A$.

All the remarks above can be generalized, easily, to any finite number of asymptotic regions, and, somewhat less easily, to any infinite number.

### 4 Conclusion

We have constructed, for any initial-data set $(T, q_{ab}, p_{ab})$ satisfying the energy condition, a complex vector space, $S$, representing “asymptotic spinors”, and two functions, $M_N$ and $M_A$, on $S$, representing “components of total asymptotic mass-momentum”. For an asymptotically flat initial-data set, this framework reproduces the standard mass-momentum at spatial infinity. We have derived a number of general properties of these objects, and applied this construction to a number of examples. There follows a discussion of some open questions and outstanding issues.

Fix a space-time that is asymptotically flat at null infinity and consider space-like slices, approaching cross-sections of null infinity, in this space-time. The present
formalism can, of course, be applied to the initial-data sets so constructed. Can we thereby recover Bondi mass-momentum at null infinity? Consider first the flat case. Let \((T, q_{ab}, p_{ab})\) be the initial-data set that arises from the slice \(T\) in Minkowski spacetime given by the hyperboloid of points unit timelike distance from some origin. What is \(S\) for this example? Each nonzero constant spinor field on the Minkowski spacetime gives rise to a \(D\)-constant field on \(T\), and so to an element of the space \(S\). The elements of \(S\) so constructed are, presumably, nonzero, and so we obtain a 2-dimensional subspace of \(S\). Does this subspace exhaust \(S\)? Assuming that it does, we immediately acquire an alternating tensor on this \(S\), for the inner product of two spinor fields so constructed is constant. But note that we do not acquire an adjoint operation on \(S\), for the adjoint of a spinor field in \(H\) in this example is not in general in \(H\). Does there exist a result, similar to Theorem 4, asserting that any initial data that is “asymptotically hyperbolic” in an appropriate sense produces a space \(S\) of similar structure? Does the mass function \(M_N\) now reproduce the Bondi mass-momentum at null infinity? Can this be generalized to include slices that approach cross-sections of null infinity in ways different from those above?

The present framework is intended to describe total asymptotic mass-momentum. Is there anything analogous for angular momentum? It seems likely that, if there is, then there will be needed a new space to replace \(S\). Indeed, in the asymptotically flat case, angular momentum, because of its origin-dependent character, cannot be expressed as any structure over the space \(S\) of asymptotic spinors.

Let \((T, q_{ab}, p_{ab})\) and \((T, \tilde{q}_{ab}, \tilde{p}_{ab})\) be two initial-data sets, based on the same underlying manifold \(T\), with each satisfying the energy condition. Under what conditions can we construct a natural correspondence between their spaces \(S\) and \(\tilde{S}\)? We have seen in Sect. 3 that there is such a correspondence when the data are identical outside some compact subset of \(T\). Furthermore, Theorem 4 can be interpreted as providing just such a correspondence under the assumptions that \(T = R^3\), \(q_{ab}\) is Euclidean, \(p_{ab}\) is zero, and \((\tilde{q}_{ab}, \tilde{p}_{ab})\) approaches \((q_{ab}, p_{ab})\) sufficiently rapidly as \(r\) approaches infinity. Is there a generalization of these observations? Is there some simple theorem guaranteeing that \(S = \tilde{S}\) for general initial-data sets approaching each other asymptotically at an appropriate rate?

How much of the present formalism survives when there is no longer imposed the energy condition, Eqn. (3)? One might expect that everything will go through as before, with the sole exception that now the mass function \(M_N\) can become negative. But it turns out that, in the absence of the energy condition, the entire formalism disintegrates. We originally defined the space \(H\) as consisting of those spinor fields for which the integral on the right in Eqn. (12) converges. But in the absence of the energy condition we can no longer guarantee nonnegativity of the integrand: What, then, is “converge” to mean? One could, for example, require absolute convergence of the entire integral, or absolute convergence of the integral of each term. But neither of these, as it turns out, results in general in an \(H\) even having the structure of a vector space! It is possible to recover a vector-space structure for \(H\), e.g., by requiring convergence of the integrals of each of the first two terms in (12), and also of \(|\rho_a||\lambda|^2\) (which guarantees absolute convergence of the integral of the last term). But this version also appears to be unsatisfactory, for, when the energy condition is satisfied, the space \(H\) it produces is in some cases strictly smaller
than the space $\mathcal{H}$ as originally defined.

What is $\mathcal{S}$ for the general initial data for flat space-time? That this question may not be as simple as it appears is suggested by the following examples. We introduced at the beginning of this section the example of the hyperboloid in Minkowski space-time consisting of points unit timelike distance from some origin. While it appears likely that $\mathcal{S}$ in this case is 2-dimensional, corresponding to the constant spinor fields in Minkowski space-time, a proof in lacking. A more complicated example is that following Theorem 6 of Sect. 3. Again, constant spinor fields on the Minkowski space-time give rise to $\mathcal{D}$-constant spinor fields, $\mu^A$ and $\kappa^A$, on $T$, and so to points of $\mathcal{S}$. The point of $\mathcal{S}$ associated with the field $\kappa^A$ turns out to be zero (by the proof of Theorem 6), but we acquire a new point of $\mathcal{S}$ from the spinor field $\lambda^A$ given in that example. So, we end up with a 2-dimensional subspace of $\mathcal{S}$ — associated with the elements $\mu^A$ and $\lambda^A$. Does this subspace exhaust $\mathcal{S}$? Another example is that of a cosmic string: From the initial-data set with $(T, q_{ab})$ Euclidean space and $p_{ab}$ zero, remove a straight line and introduce a deficit angle. What is $\mathcal{S}$ for this example? We suggest that the most likely answer is that $\mathcal{S}=\{0\}$. Let $\lambda^A \in \mathcal{H}$. Then Theorem 5, case (ii) suggests that $\lambda^A$ can be approximated near the singular axis by fields of compact support. (Were the singular axis compact, then the theorem would apply directly.) Furthermore, Theorem 2 can be modified to show that $\lambda^A$ approaches a constant asymptotically within “any fixed, small solid angle”. But, because of the presence of the deficit angle, the only asymptotically constant spinor field in this example is zero. This suggests that $\lambda^A$ approaches zero asymptotically, and so can be approximated in the asymptotic region by fields of compact support. The above is only a plausibility argument that every $\lambda^A \in \mathcal{H}$ can be approximated in $\mathcal{H}$ by spinor fields of compact support, and so $\mathcal{S}=\{0\}$. We do not know for sure what the space $\mathcal{S}$ is in this example. This last example, by virtue of its deficit angle, carries no $\mathcal{D}$-constant spinors. There can also be constructed, by making identifications on ordinary flat initial data, examples that again carry no $\mathcal{D}$-constant spinor fields, but now definitely have nontrivial $\mathcal{S}$. In such examples, the mass function $M_N$ will be strictly positive. What is it? Is it true that, for every slice in Minkowski space-time, $\mathcal{S}$ is 2-dimensional? For every Cauchy surface?

Must the infimum in the definition, (17), of the norm mass function always be realized? That is, must there always exist, for any $\alpha \in \mathcal{S}$, representative $\lambda^A$ with $M_N(\alpha) = \|\lambda\|^2$? It is easy to show that the infima are always realized on any initial-data set that is “generic” in the sense that there is a point of it at which $\rho > |\rho^a|$. Furthermore, existence of an infimum would follow in every case from Conjecture 7. Nevertheless, realization of an infimum does not follow in general from elementary facts about operators on a Hilbert space: It is not hard to construct an example, on a Hilbert space, of a positive-semi-definite Hermitian quadratic form $\zeta$ and a translate $W$ of a closed subspace such that the infimum of $\zeta$ on $W$ is not realized.

We have seen in Theorem 4 that, for any asymptotically flat initial-data set, $\mathcal{S}$ has the structure of a spinor space: It is 2-dimensional, with an adjoint operation $\dagger$ and an alternating tensor $\epsilon$. So, we introduce the real, 4-dimensional vector space $\mathcal{V}$ with its Lorentz metric $g$ and preferred unit timelike vector $t$, on which the mass function $M_N$ becomes a real linear function. How much of all this can be carried over to more general initial-data sets? We can in every case introduce $\mathcal{V}$ as the self-adjoint elements of the
tensor product of $S$ and its complex-conjugate space. There results a real vector space (in infinite dimensions, a Hilbertable one, reflecting that $S$ is Hilbertable), on which the mass functions $M_N$ and $M_A$ in every case become real linear functions. But what of the remaining structure on $S$? We cannot guarantee an adjoint operation as in Theorem 4: It is false for a general initial-data set that every element of $S$ has representative $\lambda^A$ with $\lambda^{†A} \in \mathcal{H}$ (e.g., that of the hyperboloid in Minkowski space-time). However, there are always such representatives when $p_{\alpha\beta}$ has compact support, and so, presumably, under suitable asymptotic conditions on $p_{\alpha\beta}$. Is there a simple theorem to this effect? It seems to be more difficult to obtain an alternating tensor as in Theorem 4: It is not even close to being true, for a general initial-data set, that any two elements of $S$ have representatives with inner product asymptotically constant. We remark that an adjoint operation on $S$, with no alternating tensor, gives rise to a certain linear mapping on $\mathcal{V}$ (in the asymptotically flat case, a reflection about the $t$-axis); and that an alternating tensor on $S$, with no adjoint operation, gives rise to a metric on $\mathcal{V}$. Perhaps there is some other structure, combining parts of $\dagger$ and $\epsilon$, that can always be defined.

What happens to the space $S$ and the mass function $M_N$ and $M_A$ under time-evolution of an initial-data set? The simplest case is that in which evolution takes place only within a compact subset of $T$. Then, as we have seen in Sect. 3, neither the space $S$ nor the asymptotic mass function $M_A$ changes. The example of Eqn. (52) strongly suggests that, in general, the norm mass function will change under this evolution. What of the space $Z$ — the vector space of solutions of Eqn. (46)? It seems likely that the dimension of this space, at least, will not change under evolution. Indeed, if $Z$ is 2-dimensional, then the data is for a flat spacetime, and so, therefore, will be any evolution of those data. If $Z$ is 1-dimensional, then the spacetime has null Weyl tensor and matter a null fluid. This character is probably also preserved under time-evolution. Thus, we suggest, the dimensionality of $Z$ should be an evolution-invariant. Evolution within a compact subset of $T$ can proceed eventually to singular behavior. An example is that of a slice in the extended Schwarzschild spacetime, evolving to reach $“r = 0”$. Is it true that $S$ and $M_A$ are preserved even under this evolution? That is, does the present framework ignore “internally generated” singular behavior? To prove that it does would require good control over that singular behavior. What happens for evolution not restricted to compact sets? In the asymptotically flat case, we know that the space $S$ and the mass functions $M_N$ and $M_A$ all remain invariant. Is there any similar result using conditions significantly weaker than asymptotic flatness?

5 Acknowledgement

This work was supported in part by National Science Foundation Grant No. PHY-9220644 to the University of Chicago.

1 R. Arnowitt, S. Deser, and C. W. Misner, *Gravitation: An Introduction to Current Research*, ed Witten, L. (New York: Wiley, 1962)
2 R. Penrose, Proc. R. Soc. Lond. A381, 53–63 (1982).
3 A. J. Dougan, and L. J. Mason, Phys. Rev. Lett. 67, 2119–22 (1991); G. Bergqvist,
where the infimum is over all $\lambda^A$ in equivalence class $\alpha$, and where $\tau \geq -2$ (which guarantees that the infimum exist). Then $M_{-2} = M_A$ and $M_0 = M_N$. But this yields nothing new, for we have, using the fact that the right side of (18) is independent of representative, $M_\tau = -\tau/(2M_A + 1(1+\tau/2)M_N$.

11 By this, we mean that each of $M_A$ and $M_N$ satisfies the polarization relation: For every $\alpha$, $\beta \in \mathcal{S}$, $M(\alpha + \beta) + M(\alpha - \beta) = 2[M(\alpha) + M(\beta)]$. This is immediate for $M_A$. For $M_N$, it follows from the following fact: For any $\alpha \in \mathcal{S}$, and $\lambda^A$ any representative, we have $\|\lambda\|^2 - M_N(\alpha) \leq \epsilon^2$ if and only if $|<\lambda|s> - \epsilon \|s\|$ for every $\sigma \in \mathcal{C}$.

12 We note a few technical points regarding this theorem. We actually need less than that $T = R^3$, namely that $T$ outside a compact set be diffeomorphic with $R^3$ outside a compact set. And the Euclidean metric $\tilde{q}_{ab}$ need only be specified outside that compact set. Hypothesis $i)$ actually follows from hypotheses $ii)$ and $iii)$, together with the condition that $\tilde{q}_{ab}$ bound $q_{ab}$ below. Indeed, it even follows from these that $q_{ab} - \tilde{q}_{ab}$ approaches zero asymptotically to order $r^{-1/5}$. Hypotheses $i)$ and $ii)$ alone suffice to show that there is some 2-dimensional subspace of $\mathcal{S}$ satisfying conclusions $ii)$ and $iii)$. Hypotheses $iii)$ is then needed only to show that this subspace exhausts $\mathcal{S}$. Were we assuming power-law asymptotic behavior of $p_{ab}$ and $\tilde{D}_a q_{bc}$ — which we emphatically are not — then hypothesis $iii)$ would follow already from $ii)$. Hypothesis $iii)$ can in any case be replaced by the weaker hypothesis that $r|p_{ab}|$ and $r|\tilde{D}_a q_{bc}|$ are bounded asymptotically by sufficiently small — but nonzero — numbers. Conclusion $ii)$ actually holds for every representative $\lambda^A$ of $\alpha \in \mathcal{S}$. Furthermore, a weaker notion of “approach” can be incorporated into conclusion $iii)$ so that it, too, holds for all representatives.

13 A tensor field on $T$ is said to approach zero asymptotically provided that, given any $\epsilon > 0$, there exists a compact subset of $T$ outside which the $q$-norm of that field is everywhere less than $\epsilon$.

14 See Ref. 7 and O. Reula, J. Math. Phys. 23, 810-14 (1982).

15 R. Bartnik, Comm. Pure and Appl. Math. 39, 661–93 (1986).

16 In fact, there is a simple analog of condition $i)$ in Newtonian gravitation. The Newtonian mass density, $\rho$, is a function on Euclidean space $T$. Following case $i)$ above, let there exist a positive function $\kappa$ on Euclidean space such that the $B_{\kappa_0}$'s of Theorem 5 are all compact, and such that $|D\kappa/\kappa|^2$ bounds $\rho$ below. Then, we claim, the total Newtonian mass, $\int_T \rho dV$, must be infinite. Indeed, were $\rho$ integrable, then so would be
$|D \log \kappa|^2$, whence, by Theorem 2, $\log \kappa$ would approach a constant at infinity, violating compactness of the $B_{\kappa_0}$'s.

17 R. D. Adams, *Sobolev Spaces* (Academic Press, N. Y., 1975), theorems 4.28, 7.48 and 7.58.

18 It can occur that, even though $\lambda^A \in Z$ is nonzero, this $\lambda^A$ gives rise to the zero point of $S$. See, e.g., the earlier example of flat data on $S^1 \times S^1 \times R$.

19 There is in fact a simple Euclidean version of Conjecture 7. Conjecture: Let $\vec{v}$ be a smooth complex vector field, defined on the closed unit ball $B$ in Euclidean $R^3$. Then there exists a number $c > 0$ with the following property: Given any smooth complex function $f$ on $B$, there is a function $f_0$ on $B$, there satisfying $\vec{\nabla} f_0 = \vec{v} f_0$, such that

$$\int_B |\vec{\nabla} f - \vec{v} f|^2 \geq c \int_B |f - f_0|^2.$$  

20 By this, we mean that, for any $\alpha = \alpha_1 + \alpha_2$, with $\alpha \in S$, $\alpha_1 \in S_1$, $\alpha_2 \in S_2$, we have $M_A(\alpha) = M_1 A(\alpha_1) + M_2 A(\alpha_2)$.

21 R. Penrose, Proc. R. Soc. Lond. A284, 159 (1965), R. Geroch, in Asymptotic Structure of Space-time, F. P. Esposito, L. Witten, Ed. (Plenum, N. Y. 1977).

22 O. Reula and K. P. Tod, J. Math. Phys. 25, 1004–8 (1984).

23 Note that the Lorentz group acts as data-preserving diffeomorphisms on $T$, yielding an action of this group as mass-preserving linear maps on $S$. 

25