FINITE MANY CRITICAL PROBLEMS INVOLVING FRACTIONAL LAPLACIANS IN $\mathbb{R}^N$

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Abstract. In this paper, we consider the nonlocal elliptic problems in $\mathbb{R}^N$, which involve finite many critical exponents. By using endpoint refined Hardy–Sobolev inequality, fractional Coulomb–Sobolev space and variational method, we establish the existence of nonnegative solution. Our results generalize some results obtained by Yang and Wu [Adv. Nonlinear Stud. (2017) 31].

1. Introduction

In this paper, we consider the following problems:

\[-\Delta^s u = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha_i}}}{|x-y|^{\alpha_i}} \, dy \right) |u|^{2^*_{\alpha_i}-2}u + \sum_{i=1}^{k} \frac{|u|^{2^*_{\alpha_i,\theta_i}-2}u}{|x|^{\theta_i}}, \text{ in } \mathbb{R}^N, \quad (P_1)\]

and

\[-\Delta^s u - \frac{\zeta u}{|x|^{2s}} = \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*}}{|x-y|^\alpha} \, dy \right) |u|^{2^*-2}u + \sum_{i=1}^{k} \frac{|u|^{2^*_{\alpha_i,\theta_i}-2}u}{|x|^{\theta_i}}, \text{ in } \mathbb{R}^N, \quad (P_2)\]

where $N \geq 3$, $s \in (0, 1)$, $\zeta \in \left[0, 4^s \frac{\Gamma(N+2s)}{\Gamma(\frac{N+2s}{s})}\right)$, $\alpha \in (0, N)$, $2^*_s = \frac{2N}{N-2s}$ is the critical Sobolev exponent, $2^*_{\alpha_i,\theta_i} = \frac{2(N-\theta_i)}{N-2s}$ are the critical Hardy–Sobolev exponents, $2^*_\alpha_i = \frac{2N-2\alpha_i}{N-2s}$ are the Hardy–Littlewood–Sobolev upper critical exponents, the parameters $\alpha_i$ and $\theta_i$ satisfy the assumptions:

\[(H_1) \ 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < N \ (k \in \mathbb{N}, 2 \leq k < \infty); \]

\[(H_2) \ 0 < \theta_1 < \cdots < \theta_k < 2s \ (k \in \mathbb{N}, 2 \leq k < \infty), \text{ and } 2\theta_k - \theta_1 \in (0, 2s).\]

The fractional Laplacian $(-\Delta)^s$ of a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ can be defined as

\[(-\Delta)^s u = F^{-1}( |\xi|^{2s} \mathcal{F}(u)(\xi) ), \quad \text{for all } \xi \in \mathbb{R}^N, \]

and for $u \in C_0^\infty(\mathbb{R}^N)$, where $\mathcal{F}(u)$ denotes the Fourier transform of $u$. The operator $(-\Delta)^s$ in $\mathbb{R}^N$ is a nonlocal pseudo–differential operator taking the form

\[(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,\]

where P.V. is the Cauchy principal value and $C_{N,s}$ is a normalization constant. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics,

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geophysical fluid dynamics, flames propagation, minimal surfaces and game theory (see [2, 6, 10]).

Problem \((P_1)\) and \((P_2)\) are related to the nonlinear Choquard equation as follows:

\begin{equation}
-\Delta u + V(x)u = (|x|^{\alpha} * |u|^q) |u|^{q-2} u, \quad \text{in } \mathbb{R}^N,
\end{equation}

where \(\frac{2N-\alpha}{N-2} \leq q \leq \frac{2N-\alpha}{N-2}\) and \(\alpha \in (0, N)\). For \(q = 2\) and \(\alpha = 1\), the problem \((1.1)\) goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 [21] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree–Fock theory of one–component plasma [22]. The existence and qualitative properties of solutions of Choquard type equations \((1.1)\) have been widely studied in the last decades (see [19]).

For Laplacian with nonlocal Hartree type nonlinearities, Gao and Yang [11] investigated the following critical Choquard equation:

\begin{equation}
-\Delta u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2s}_2}{|x-y|^{\alpha}} dy \right) |u|^{r-2} u + \lambda u, \quad \text{in } \Omega,
\end{equation}

where \(\Omega\) is a bounded domain of \(\mathbb{R}^N\), with lipschitz boundary, \(N \geq 3\), \(\alpha \in (0, N)\) and \(\lambda > 0\). By using variational methods, they established the existence, multiplicity and nonexistence of nontrivial solutions to equation \((1.2)\). For details and recent works we refer to [1, 12, 18] and the references therein.

For fractional Laplacian with nonlocal Hartree–type nonlinearities, D’Avenia, Siciliano and Squassina [8] considered the following fractional Choquard equation:

\begin{equation}
(-\Delta)^s u + \omega u = (K_{\alpha} * |u|^q) |u|^{q-2} u, \quad \text{in } \mathbb{R}^N,
\end{equation}

where \(N \geq 3\), \(s \in (0,1)\), \(\omega \geq 0\), \(\alpha \in (0, N)\) and \(q \in \left( \frac{2N-\alpha}{N} \right) \to \left( \frac{2N-\alpha}{N-2} \right)\). In particularly, when \(\omega = 0\), \(\alpha = 4s\) and \(q = 2\), then problem \((1.3)\) become a fractional Choquard equation with upper critical exponent in the sense of Hardy–Littlewood–Sobolev inequality as follows:

\begin{equation}
(-\Delta)^s u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2s}}{|x-y|^{\alpha}} dy \right) u, \quad \text{in } \mathbb{R}^N.
\end{equation}

D’Avenia, Siciliano and Squassina in [8] obtained regularity, existence, nonexistence of nontrivial solutions to problem \((1.3)\) and problem \((1.4)\). Mukherjee and Sreenadh [17] extended the study of problem \((1.2)\) to fractional Laplacian equation.

Recently, Yang and Wu [31] studied the following nonlocal elliptic problems:

\begin{equation}
(-\Delta)^s u - \frac{\zeta}{|x|^{2s}} u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2s}}{|x-y|^{\alpha}} dy \right) |u|^{2s-2} u + \left( \int_{\mathbb{R}^N} \frac{|u|^{2s}}{|x-y|^{\beta}} dy \right) |u|^{2s-2} u, \quad \text{in } \mathbb{R}^N,
\end{equation}

and

\begin{equation}
(-\Delta)^s u - \frac{\zeta}{|x|^{2s}} u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2s}}{|x-y|^{\alpha}} dy \right) |u|^{2s-2} u + \frac{|u|^{2s-2} u}{|x|^{\theta}}, \quad \text{in } \mathbb{R}^N,
\end{equation}

where \(N \geq 3\), \(s \in (0,1), \zeta \in \left[ 0, 4s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(\frac{N+2s}{4})} \right], \alpha, \beta \in (N-2s, N), \theta \in (0, 2s), 2s = \frac{2N-\alpha}{N-2s} \) and \(2s_\theta = \frac{2(N-\theta)}{N-2s}\). Using the refinement of the Sobolev inequality which is related to the Morrey space, they showed the existence of nontrivial solutions for problem \((1.5)\) and problem \((1.6)\). In [27], Wang, Zhang and Zhang
extended the study of problem (1.6) to the fractional Laplacian system. By using variational methods, they investigated the extremals of the corresponding best fractional Hardy–Sobolev constant and established the existence of solutions to the fractional Laplacian system.

Moreover, there are many other kinds of problem involving two critical nonlinearities, such as the Laplacian $-\Delta$ (see [15, 25, 32]), the $p$–Laplacian $-\Delta_p$ (see [9]), the biharmonic operator $\Delta^2$ (see [3]), and the fractional operator $(-\Delta)^s$ (see [13, 7]).

There are two questions arise:

Question 1: For $\zeta = 0$, can we extend the study of problem (1.5) in the finite many critical nonlinearities?

Question 2: Can we extend the studies of problem (1.6) and problem (1.3) in the finite many critical nonlinearities?

We answer above questions in this paper. To our knowledge, there are no results in these senses.

The variational approach that we adopt here, relies on the following inequalities:

**Lemma 1.1.** [14, Hardy-Littlewood-Sobolev inequality] Let $t, r > 1$ and $0 < \alpha < N$ with $\frac{1}{r} + \frac{1}{t} + \frac{\alpha}{N} = 2, f \in L^1(\mathbb{R}^N)$ and $h \in L^1(\mathbb{R}^N)$. There exists a sharp constant $C(N, \alpha, t, r) > 0$, independent of $f, g$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^\alpha} \, dx \, dy \leq C(N, \alpha, t, r) \|f\| \|h\|_r.$$

If $t = r = \frac{2N}{2N-\alpha}$, then

$$C(N, \alpha, t, r) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\alpha}{2}\right)}{\Gamma(N - \frac{\alpha}{2})} \left\{\frac{\Gamma(N)}{\pi\Gamma\left(\frac{N}{2}\right)}\right\} \frac{\alpha^{\frac{N-\alpha}{2}}}{N^{N-\alpha}}.$$

**Lemma 1.2.** [4, Endpoint refined Sobolev inequality] Let $s \in (0, \frac{N}{2})$ and $\alpha \in (0, N)$. Then there exists a constant $C_1 > 0$ such that the inequality

$$\|u\|_{L^{2^*_s}(\mathbb{R}^N)} \leq C_1 \|u\|_{L^2_D(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_s} |u(y)|^{2^*_s}}{|x-y|^\alpha} \, dx \, dy\right)^{\frac{s(N-2s)}{N(N-2s)-\alpha}}.$$

holds for all $u \in \mathcal{E}^{s,\alpha,2^*_s}(\mathbb{R}^N)$.

In particular, the Coulomb–Sobolev space and endpoint refined Sobolev inequality play the key roles in this paper. For $s = 1$, Mercuri, Moroz and Schaftingen [18] introduced the Coulomb–Sobolev space and a family of associated optimal interpolation inequalities (include endpoint refined Sobolev inequality). They studied the existence of solutions of the nonlocal Schrödinger–Poisson–Slater type equation by Coulomb–Sobolev space and endpoint refined Sobolev inequality. For $s \neq 1$, Bellazzini, Ghimenti, Mercuri, Moroz and Schaftingen [4] studied the fractional Coulomb–Sobolev space and endpoint refined Sobolev inequality.

The first result of this paper is as follows.

**Theorem 1.3.** Let $N \geq 3, s \in (0, 1)$ and $(H_1)$ hold. Then problem $(P_1)$ has a nonnegative solution $\bar{v}(x)$. Moreover, set

$$\bar{v}(x) = \frac{1}{|x|^{N-2s}} \bar{v}\left(\frac{x}{|x|^2}\right).$$
Then \( \tilde{v}(x) \) is a nonnegative solution of the problem
\[
(-\Delta)^s \tilde{v} = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \frac{|\tilde{v}|^{2N-2s}}{|x-y|^{N-s}} \, dy \right) |\tilde{v}|^{\frac{4s}{N-2s}} \tilde{v} + |\tilde{v}|^{\frac{4s}{4s-2s}} \tilde{v}, \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

**Remark 1.1.** Problem \((P_1)\) is invariant under the weighted dilation
\[ u \mapsto \tau^{\frac{N-2s}{2}} u(\tau x). \]

Therefore, it is well known that the mountain pass theorem does not yield critical points, but only the Palais–Smale sequences. In this type of situation, it is necessary to show the non–vanishing of Palais–Smale sequences. There are finite many Hardy–Littlewood–Sobolev critical exponents in problem \((P_1)\), it is difficult to show the non–vanishing of Palais–Smale sequences. By using fractional Coulomb–Sobolev space, endpoint refined Sobolev inequality and Lemma 3.2, we overcome this difficult in Lemma 5.3.

The second result of this paper is as follows.

**Theorem 1.4.** Let \( N \geq 3, \, \alpha \in (0, N), \, s \in (0, 1) \) and \((H_2)\) hold. Then problem \((P_2)\) has a nonnegative solution \( \tilde{u}(x) \). Moreover, set
\[
\tilde{u}(x) = \frac{1}{|x|^{N-2s}} \tilde{u} \left( \frac{x}{|x|^2} \right).
\]

Then \( \tilde{u}(x) \) is a nonnegative solution of the problem
\[
(-\Delta)^s \tilde{u} - \zeta \frac{\tilde{v}}{|x|^{2s}} = \left( \int_{\mathbb{R}^N} \frac{|\tilde{u}|^{2N}}{|x-y|^{N-s}} \, dy \right) |\tilde{u}|^{\frac{4s}{N-2s}} \tilde{u} + \sum_{i=1}^{k} \frac{|\tilde{u}|^{\frac{4s}{N-2s}}}{|x|^s} \tilde{u}, \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

**Remark 1.2.** This paper not only extends the studies of problem \((1.5)\) and problem \((1.6)\) in the finite many critical nonlinearities, but also extends \( \alpha \in (N-2s, N) \) to \( \alpha \in (0, N) \). In [27] and [31], the authors just studied the case of \( \alpha \in (N-2s, N) \). It is natural to ask the case of \( \alpha \in (0, N-2s) \). In order to overcome this difficult, we show the refinement of Hardy–Littlewood–Sobolev inequality for the case of \( \alpha \in (0, N) \) (see Lemma 3.1), and the endpoint refined Hardy–Sobolev inequality (see Lemma 3.3).

2. **Preliminaries**

The space \( H^s(\mathbb{R}^N) \) is defined as
\[
H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) | (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^N) \}.
\]

This space is endowed with the norm
\[
|\tilde{u}|^2_{H^s} = |(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 + |\tilde{u}|^2.
\]

The space \( D^{s,2}(\mathbb{R}^N) \) is the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm
\[
|\tilde{u}|^2_D = |(-\Delta)^{\frac{s}{2}} \tilde{u}|^2.
\]

It is well known that \( \Lambda = 4^s \tau^2 \frac{\Gamma\left(\frac{N}{2} + \frac{s}{2}\right)}{\Gamma\left(\frac{N}{2} - \frac{s}{2}\right)} \) is the best constant in the Hardy inequality
\[
\Lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} \, dx \leq |u|^2_D, \quad \text{for any } u \in D^{s,2}(\mathbb{R}^N).
\]
By Hardy inequality and $\zeta \in [0, \Lambda)$, we derive that
\[ \|u\|_\zeta^2 = \|u\|_D^2 - \zeta \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2\alpha}} \, dx, \]
is an equivalent norm in $D^{s,2}(\mathbb{R}^N)$, since the following inequalities hold:
\[ \left(1 - \frac{\zeta}{\Lambda}\right) \|u\|_D^2 \leq \|u\|_\zeta^2 \leq \|u\|_D^2. \]

For $\alpha \in (0, N)$ and $s \in (0, 1)$, the fractional Coulomb–Sobolev space [4] is defined by
\[ E^{s,\alpha,2^*_s}(\mathbb{R}^N) = \left\{ \|u\|_D < \infty \text{ and } \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_s} |u(y)|^{2^*_s}}{|x-y|^{N-2s\alpha}} \, dx \, dy < \infty \right\}. \]
We endow the space $E^{s,\alpha,2^*_s}(\mathbb{R}^N)$ with the norm
\[ \|u\|_{E^{s,\alpha}} = \|u\|_D^2 + \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_s} |u(y)|^{2^*_s}}{|x-y|^{N-2s\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_s}}. \]

For $\alpha \in [0, N)$, $\zeta \in [0, \Lambda)$ and $s \in (0, 1)$, we define the best constant:
\[ S_{\zeta,\alpha} := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_D^2 - \zeta \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2\alpha}} \, dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_s} |u(y)|^{2^*_s}}{|x-y|^{N-2s\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_s}}}. \]
We know that $S_{\zeta,\alpha}$ is attained in $\mathbb{R}^N$ (see [31]). For $s \in (0, 1)$ and $\theta \in (0, 2s)$, we define the best constant:
\[ H_{\zeta,\theta} := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_D^2 - \zeta \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2\alpha}} \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^{\theta}} \, dx \right)^{\frac{1}{2^*_s}}}. \]
where $H_{\theta}$ is attained in $\mathbb{R}^N$ (see [30]). A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\|u\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^N)}$ with $p \in [1, \infty)$ and $\varphi \in (0, N]$ if and only if
\[ \|u\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^N)}^p = \sup_{R > 0, x \in \mathbb{R}^N} R^{N-2s\alpha} \int_{B(x, R)} |u(y)|^p \, dy < \infty. \]

Lemma 2.1. [20, Theorem 1] For $s \in (0, \frac{N}{2\alpha})$, there exists $C_2 > 0$ such that for $i$ and $\theta$ satisfying $\frac{2}{2^*_s} \leq i < 1$, $1 \leq \theta < 2^*_s = \frac{2N}{N-2s\alpha}$, we have
\[ \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq C_2 \|u\|_D \|u\|_{\mathcal{L}^{2^*_s,\phi}(\mathbb{R}^N)}^{1-i} \|u\|_{\mathcal{L}^{2^*_s,\phi}(\mathbb{R}^N)}^{1-i}, \]
for any $u \in D^{s,2}(\mathbb{R}^N)$. 

We introduce the energy functionals associated to problems \((P_i)\) \((i = 1, 2, 3)\) by

\[
I_1(u) = \frac{1}{2} \|u\|_D^2 - \sum_{i=1}^k \frac{1}{2 \cdot 2^\alpha_i} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha_i}} |u(y)|^{2^*_{\alpha_i}}}{|x - y|^{\alpha_i}} \, dx\,dy - \frac{1}{2^*_{\alpha_i}} \int_{\mathbb{R}^N} |u|^{2^*_{\alpha_i}} \, dx,
\]

\[
I_2(u) = \frac{1}{2} \|u\|_D^2 - \frac{1}{2} \cdot 2^\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha}} |u(y)|^{2^*_{\alpha}}}{|x - y|^{\alpha}} \, dx\,dy - \sum_{i=1}^k \frac{1}{2^{s_i}} \int_{\mathbb{R}^N} |u|^{2^*_{s_i}} \, dx,
\]

\[
I_3(u) = \frac{1}{2} \|u\|_D^2 - \frac{1}{2} \cdot 2^\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha}} |u(y)|^{2^*_{\alpha}}}{|x - y|^{\alpha}} \, dx\,dy.
\]

The Nehari manifolds associated with problem \((P_i)\) \((i = 1, 2, 3)\), which are defined by

\[
\mathcal{N}_i = \{ u \in D^{s,2}(\mathbb{R}^N) | \langle I_i(u), u \rangle = 0, \ u \neq 0 \},
\]

and

\[
c_0 = \inf_{u \in \mathcal{N}_i} I_i(u), \ c_1 = \inf_{u \in D^{s,2}(\mathbb{R}^N)} \max_{t \geq 0} I_i(tu) \text{ and } c^i = \inf_{\Gamma_i} \max_{t \in [0,1]} I_i(tu),
\]

where \(\Gamma_i = \{ \Gamma_i \in C([0,1], D^{s,2}(\mathbb{R}^N)) : \Gamma_i(0) = 0, I_i(\Gamma_i(1)) < 0 \} \).

3. Some Key Lemmas

We show the refinement of Hardy-Littlewood-Sobolev inequality.

**Lemma 3.1.** For any \(s \in (0, \frac{N}{2})\) and \(\alpha \in (0, N)\), there exists \(C_3 > 0\) such that for \(\frac{1}{2^*_{\alpha}} \leq \iota < 1\), \(1 \leq \vartheta < 2^*_{s} = \frac{2N}{N - 2s}\), we have

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha}} |u(y)|^{2^*_{\alpha}}}{|x - y|^{\alpha}} \, dx\,dy \right)^{\frac{1}{2^*_{\alpha}}} \lesssim C_3 \|u\|_D \|u\|^{2(1 - \iota)}_{L^\vartheta \cdot \left(\frac{N - 2s}{2}\right)(\mathbb{R}^N)},
\]

for any \(u \in D^{s,2}(\mathbb{R}^N)\).

**Proof.** Let \(\frac{1}{2^*_{\alpha}} \leq \iota < 1\) and \(1 \leq \vartheta < 2^*_{s} = \frac{2N}{N - 2s}\). By Hardy-Littlewood-Sobolev inequality and Lemma 2.1, we obtain

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha}} |u(y)|^{2^*_{\alpha}}}{|x - y|^{\alpha}} \, dx\,dy \right)^{\frac{1}{2^*_{\alpha}}} \lesssim C(N, \alpha) \frac{\|u\|_{L^{2^*_{\alpha}}(\mathbb{R}^N)}}{\|u\|_{L^{\vartheta \cdot \left(\frac{N - 2s}{2}\right)}(\mathbb{R}^N)}} \lesssim C(N, \alpha) \frac{\|u\|_{L^{2^*_{\alpha}}(\mathbb{R}^N)}}{\|u\|_{L^{\vartheta \cdot \left(\frac{N - 2s}{2}\right)}(\mathbb{R}^N)}}.
\]

We show some properties of fractional Coulomb–Sobolev space \(E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N)\).

**Lemma 3.2.** Let \((H_1)\) hold. If \(u \in E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N)\) \((j = 1, \ldots, k)\), then

(i) \(\| \cdot \|_D\) is an equivalent norm in \(E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N)\);

(ii) \(u \in \bigcap_{i=1, i \neq j} E^{s,\alpha_i,2^*_{\alpha_i}}(\mathbb{R}^N)\);

(iii) \(\| \cdot \|_{E,\alpha_i}\) are equivalent norms in \(E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N)\), where \(i \neq j\) and \(i = 1, \ldots, k\).

**Proof.** (1). Set \(j = 1, \ldots, k\). For any \(u \in E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N)\), applying the definition of fractional Coulomb–Sobolev space, we know

\[
\|u\|_D^2 \lesssim \|u\|_{E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N)}^2 < \infty.
\]

This implies that \(E^{s,\alpha_j,2^*_{\alpha_j}}(\mathbb{R}^N) \subset D^{s,2}(\mathbb{R}^N)\).
According to $\mathcal{E}^{s,\alpha,2s^*}(\mathbb{R}^N) \subset D^{s,2}(\mathbb{R}^N)$ and (2.1), we have
\begin{equation}
\|u\|^2_{E,s,\alpha} \leq \left(1 + \frac{1}{S_{0,\alpha_j}}\right)\|u\|^2_D.
\end{equation}
Combining (3.1) and (3.2), we obtain
\begin{equation}
\|u\|^2_D \leq \|u\|^2_{E,s,\alpha} \leq \left(1 + \frac{1}{S_{0,\alpha}}\right)\|u\|^2_D.
\end{equation}
These imply that $\|\cdot\|_D$ is an equivalent norm in $\mathcal{E}^{s,\alpha,2s^*}(\mathbb{R}^N)$.

(2). For any $u \in \mathcal{E}^{s,\alpha,2s^*}(\mathbb{R}^N) \subset D^{s,2}(\mathbb{R}^N)$, by using (3.1) and (2.1), we know
\begin{equation}
S_{0,\alpha_i} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2s^*}|u(y)|^{2s^*}}{|x-y|^{\alpha_i}}\,dx\,dy\right)^{\frac{1}{2s^*}} \leq \|u\|^2_D \leq \|u\|^2_{E,s,\alpha} < \infty,
\end{equation}
where $i \neq j$ and $i = 1, \ldots, k$. The inequality (3.4) gives that
\begin{equation}
\|u\|^2_{E,s,\alpha} = \|u\|^2_D + \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2s^*}|u(y)|^{2s^*}}{|x-y|^{\alpha_i}}\,dx\,dy\right)^{\frac{1}{2s^*}} < \infty.
\end{equation}
This implies that $u \in \bigcap_{i=1}^k \mathcal{E}^{s,\alpha_i,2s^*}(\mathbb{R}^N)$.

(3). For any $u \in \mathcal{E}^{s,\alpha,2s^*}(\mathbb{R}^N)$, by using (3.2), we have
\begin{equation}
\|u\|^2_{E,s,\alpha} \leq \left(\frac{S_{0,\alpha} + 1}{S_{0,\alpha}}\right)\|u\|^2_D \leq \left(\frac{S_{0,\alpha} + 1}{S_{0,\alpha}}\right)\|u\|^2_{E,s,\alpha},
\end{equation}
which imply that
\begin{equation}
\left(\frac{S_{0,\alpha_i}}{S_{0,\alpha_i} + 1}\right)\|u\|^2_{E,s,\alpha} \leq \|u\|^2_{E,s,\alpha} \leq \left(\frac{S_{0,\alpha} + 1}{S_{0,\alpha}}\right)\|u\|^2_{E,s,\alpha},
\end{equation}
where $0 < \frac{S_{0,\alpha_i}}{S_{0,\alpha_i} + 1} < 1 < \frac{S_{0,\alpha} + 1}{S_{0,\alpha}} < \infty$. \hfill \Box

Lemma 3.3. [Endpoint refined Hardy–Sobolev inequality] Let $s \in (0,\frac{N}{2})$, $\theta \in (0,2s)$ and $\alpha \in (0, N)$. Then there exists a constant $C_4 > 0$ such that the inequality
\begin{equation}
\int_{\mathbb{R}^N} \frac{|u|^{2s^*,\theta}}{|x|^{\theta}}\,dx \leq C_4 \|u\|_{D^{s,2+s^*-\alpha}}^{\frac{2(N+s-\theta)}{2s^{*^2}} + \frac{2s^* - \alpha}{s^*}} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2s^*}|u(y)|^{2s^*}}{|x-y|^{\alpha}}\,dx\,dy\right)^{\frac{2s^* - \alpha}{s^*}},
\end{equation}
holds for all $u \in \mathcal{E}^{s,\alpha,2s^*}(\mathbb{R}^N)$.

Proof. For any $u \in \mathcal{E}^{s,\alpha,2s^*}(\mathbb{R}^N)$. By using H"older inequality and fractional Hardy inequality, we obtain
\begin{equation}
\int_{\mathbb{R}^N} \frac{|u|^{2s^*,\theta}}{|x|^\theta}\,dx = \int_{\mathbb{R}^N} \frac{|u|^{\theta}}{|x|^{\theta}} \cdot |u|^{\frac{2(N-\theta)}{2s^* - \theta}}\,dx \\
\leq \left(\int_{\mathbb{R}^N} \frac{|u|^{\theta}}{|x|^{\theta}}\,dx\right)^{\frac{\theta}{\theta}} \left(\int_{\mathbb{R}^N} \frac{|u|^{(2s-\theta)N}}{|x|^{2s^{*^2}}\,dx}\,dx\right)^{1-\frac{\theta}{\theta}} \\
= \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}\,dx\right)^{\frac{\theta}{2s}} \left(\int_{\mathbb{R}^N} \frac{|u|^{2N}}{|x|^{2s}}\,dx\right)^{\frac{2s^* - \alpha}{s^*}} \\
\leq \left(\frac{1}{\lambda}\right)^{\frac{\theta}{2s}} \|u\|_D \|u\|_{L^\infty(\mathbb{R}^N)}^{\frac{N(2s-\theta)}{(2s-\theta)2s}} \left(\int_{\mathbb{R}^N} \frac{|u|^{2N}}{|x|^{2s}}\,dx\right)^{\frac{2s^* - \alpha}{s^*}}.
\end{equation}
According to Lemma 1.2 and (3.6), we know
\[
\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\bar{\theta}}}}{|x|^{\theta}} \, dx \\
\leq C_4 \|u\|_{D}^{\frac{\theta}{2}} \|u\|_{D}^{\frac{(N(\frac{2}{s}\bar{\theta})-1)}{2(N-s)}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\bar{\theta}}}|u(y)|^{2^*_{\bar{\theta}}}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{2s-\theta}{N+2s-\alpha}} \\
= C_4 \|u\|_{D}^{\frac{2(N-\theta)}{N-2s}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\bar{\theta}}}|u(y)|^{2^*_{\bar{\theta}}}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{2s-\theta}{N+2s-\alpha}}.
\]

We study the refinement of Hardy–Sobolev inequality. In [30, 31], the authors also obtained the Refinement of Hardy–Sobolev inequality. However, their parameter $\bar{\theta}$ satisfying (see [30, Theorem 1])
\[
1 \leq \bar{\theta} < 2^*_{s,\theta}.
\]
It is easy to see that
\[
2^*_{s,\theta} = \frac{2(N-\theta)}{N-2s} < \frac{2N}{N-2s} = 2^*,
\]
for $s \in (0, \frac{N}{2})$ and $\theta \in (0, 2s)$. It is natural to ask the case of $\bar{\theta} \in [2^*_{s,\theta}, 2^*)$.

Our method extends the parameter $\bar{\theta}$ from $[1, 2^*_{s,\theta})$ to $[1, 2^*)$.

Lemma 3.4. [Refinement of Hardy–Sobolev inequality] For $s \in (0, \frac{N}{2})$ and $\theta \in (0, 2s)$, there exists $C_5 > 0$ such that for $\iota$ and $\bar{\theta}$ satisfying $\frac{2}{s} \leq \iota < 1$, $1 \leq \bar{\theta} < 2^*$, we have
\[
\left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_{\bar{\theta}}}}{|x|^{\theta}} \, dx \right)^{\frac{1}{2^*_{\bar{\theta}}}} \leq C_5 \|u\|_{D}^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{L^\infty}^{\frac{N(1-\iota)(2\theta-1)}{2s(N-\iota)}} \|u\|_{D}^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{L^\infty}^{\frac{N(1-\iota)(2\theta-1)}{2s(N-\iota)}} \|u\|_{L^\infty}^{\frac{1-\iota}{\iota}} \|u\|_{L^{2^*_{\bar{\theta}}}}^{\frac{N(2\theta-1)}{2s(N-\iota)}},
\]
for any $u \in D^{s,2}(\mathbb{R}^N)$.

Proof. Combining (3.6) and Lemma 2.1, we have
\[
\left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_{\bar{\theta}}}}{|x|^{\theta}} \, dx \right)^{\frac{1}{2^*_{\bar{\theta}}}} \leq \left( \frac{1}{A} \right)^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{D}^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{L^\infty}^{\frac{N(1-\iota)(2\theta-1)}{2s(N-\iota)}} \|u\|_{D}^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{L^\infty}^{\frac{1-\iota}{\iota}} \|u\|_{L^{2^*_{\bar{\theta}}}}^{\frac{N(2\theta-1)}{2s(N-\iota)}} \\
\leq \left( \frac{1}{A} \right)^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{D}^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \left( C_2 \|u\|_{L^\infty} \|u\|_{L^{2^*_{\bar{\theta}}}}^{1-\iota} \|u\|_{L^{2^*_{\bar{\theta}}}}^{\frac{(2\theta-1)}{2s(N-\iota)}} \right)^{\frac{N(2\theta-1)}{2s(N-\iota)}} \\
= C_5 \|u\|_{D}^{\frac{\theta(N-2\iota)}{2s(N-\iota)}} \|u\|_{L^\infty}^{\frac{N(1-\iota)(2\theta-1)}{2s(N-\iota)}} \|u\|_{L^{2^*_{\bar{\theta}}}}^{\frac{N(2\theta-1)}{2s(N-\iota)}}.
\]

Lemma 3.5. Let $s \in (0, \frac{N}{2})$ and $0 < \theta < \bar{\theta} < 2s$. Then the inequality
\[
\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\bar{\theta}}}}{|x|^{\theta}} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_{\bar{\theta}}}}{|x|^{\theta}} \, dx \right)^{\frac{\theta}{2}} \left( \int_{\mathbb{R}^N} |u|^{2^*_{\bar{\theta}} \bar{\theta}} \, dx \right)^{\frac{\theta}{2}},
\]
holds for all $u \in D^{s,2}(\mathbb{R}^N)$. 

Proof. For any $u \in D^{\ast,2}(\mathbb{R}^N)$. By using Hölder inequality and $0 < \theta < \bar{\theta} < 2s$, we obtain
\[
\int_{\mathbb{R}^N} \frac{|u|^{2s_\theta}}{|x|^\theta} \, dx = \int_{\mathbb{R}^N} \frac{|u|^{\frac{\theta}{2} + \frac{2(N-\theta)}{N-2s_\theta}}}{|x|^\theta} \cdot |u|^{\frac{2N}{N-2s_\theta} - \theta} \, dx \\
\leq \left( \int_{\mathbb{R}^N} \frac{|u|^{\frac{\theta}{2} + \frac{2(N-\theta)}{N-2s_\theta}}}{|x|^\theta} \, dx \right)^{\frac{\theta}{2}} \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s_\theta} - \theta} \, dx \right)^{1 - \frac{\theta}{2}} \\
= \left( \int_{\mathbb{R}^N} \frac{|u|^{2(N-\theta)}}{|x|^{2(N-\theta)}} \, dx \right)^{\frac{\theta}{2}} \left( \int_{\mathbb{R}^N} |u|^{2(N-\theta)} \, dx \right)^{\frac{1}{2}}.
\]

Lemma 3.6. Let $s \in \left(0, \frac{N}{2}\right)$, $0 < \bar{\theta} < \theta < 2s$ and $2\theta - \bar{\theta} < 2s$. Then the inequality
\[
\int_{\mathbb{R}^N} \frac{|u|^{2s_\theta}}{|x|^\theta} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{2s_\theta}}{|x|^\theta} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{2s_{2\theta - \bar{\theta}}} \, dx \right)^{\frac{1}{2}},
\]
holds for all $u \in D^{\ast,2}(\mathbb{R}^N)$.

Proof. For any $u \in D^{\ast,2}(\mathbb{R}^N)$. By using Hölder inequality and $0 < \theta < \theta < 2s$, we obtain
\[
\int_{\mathbb{R}^N} \frac{|u|^{2s_\theta}}{|x|^\theta} \, dx = \int_{\mathbb{R}^N} \frac{|u|^{\frac{N-\theta}{N-2s_\theta}}}{|x|^\theta} \cdot |u|^{\frac{N-(2\theta-\bar{\theta})}{N-2s_\theta}} \, dx \\
\leq \left( \int_{\mathbb{R}^N} \frac{|u|^{\frac{N-\theta}{N-2s_\theta}}}{|x|^\theta} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{\frac{N-(2\theta-\bar{\theta})}{N-2s_\theta}} \, dx \right)^{\frac{1}{2}}.
\]

Since $0 < 2\theta - \bar{\theta} < 2s$, we get
\[
\int_{\mathbb{R}^N} \frac{|u|^{2s_\theta}}{|x|^\theta} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{2s_\theta}}{|x|^\theta} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{2s_{2\theta - \bar{\theta}}} \, dx \right)^{\frac{1}{2}}.
\]

Lemma 3.7. Let $N \geq 3$, $\alpha \in (0, N)$, $s \in (0, 1)$ and $\zeta \in [0, \Lambda)$ hold. Then we have
\[
S_{0,\alpha} \geq S_{\zeta,\alpha} \geq \left(1 - \frac{\zeta}{\Lambda}\right) S_{0,\alpha},
\]
and
\[
H_{0,\theta} \geq H_{\zeta,\theta} \geq \left(1 - \frac{\zeta}{\Lambda}\right) H_{0,\theta}.
\]

Proof. For $\zeta \in [0, \Lambda)$ and $u \in D^{\ast,2}(\mathbb{R}^N)$, $u \not\equiv 0$. We set
\[
F_{\zeta}(u) := \frac{\|u\|_{L}^2}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2 |u(y)|^{2(\zeta)}}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{1}{2}}},
\]
clearly, for a fixed $u$, $F_{\zeta}(u)$ is decreasing with respect to $\zeta$.

Moreover, for any fixed $\zeta \in [0, \Lambda)$, we denote by $u_{\zeta} \in D^{\ast,2}(\mathbb{R}^N)$ a function such that (2.1) is achieved, that is $S_{\zeta,\alpha} = F_{\zeta}(u_{\zeta})$. 

\[\text{□} \]
Let \( 0 < \zeta_1 < \zeta_2 < \Lambda \). Then
\[
S_{\zeta_1, \alpha} = F_{\zeta_1}(u_{\zeta_1}) = \frac{\|u_{\zeta_1}\|_D^2 - \zeta_1 \int_{\mathbb{R}^N} \frac{|u_{\zeta_1}(x)|^{2^*_\alpha}}{|x-y|^\alpha} \, dx \, dy}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\zeta_1}(x)|^{2^*_\alpha}|u_{\zeta_1}(y)|^{2^*_\alpha}}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}}}
\]
\[
> \frac{\|u_{\zeta_1}\|_D^2 - \zeta_2 \int_{\mathbb{R}^N} \frac{|u_{\zeta_1}|^2}{|x-y|^\alpha} \, dx \, dy}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\zeta_1}(x)|^{2^*_\alpha}|u_{\zeta_1}(y)|^{2^*_\alpha}}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}}} \geq S_{\zeta_2, \alpha}.
\]

Let \( 0 < \zeta < \Lambda \). Since the best constant in the Hardy inequality is not achieved, we get
\[
S_{\zeta, \alpha} = F_{\zeta}(u_{\zeta}) > \left(1 - \frac{\zeta}{\Lambda}\right) \frac{\|u_{\zeta}\|_D^2}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\zeta}(x)|^{2^*_\alpha}|u_{\zeta}(y)|^{2^*_\alpha}}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}}} \geq \left(1 - \frac{\zeta}{\Lambda}\right) S_{0, \alpha}.
\]

\[\Box\]

4. The proof of Theorem 1.4

In this section, we show the existence of nonnegative solution of problem \((P_2)\).

In Lemma 4.1–Lemma 4.3, we will prove some properties of the Nehari manifolds associated with problems \((P_2)\) and \((P_4)\).

**Lemma 4.1.** Assume that the assumptions of Theorem 1.4 hold. Then
\[
c_0^2 = \inf_{u \in \mathcal{N}^2} I_2(u) > 0.
\]

**Proof.** **Step 1.** We claim that any limit point of a sequence in \(\mathcal{N}^2\) is different from zero. According to \((I_2', u) = 0\), \((2.1)\) and \((2.2)\), for any \(u \in \mathcal{N}^2\), we obtain
\[
0 = (I_2'(u), u) \geq \|u\|_{\zeta}^2 - \frac{1}{S_{2^*_\alpha, \alpha}^2} \|u\|_{2^*_\alpha}^{2^*_\alpha} - \sum_{i=1}^{k} \frac{1}{H_{\zeta, \theta_i}} \|u\|_{\zeta, \theta_i}^{2^*_\alpha, \theta_i}.
\]

From above expression, we have
\[
\|u\|_{\zeta}^2 \leq \frac{1}{S_{2^*_\alpha, \alpha}^2} \|u\|_{2^*_\alpha}^{2^*_\alpha} + \sum_{i=1}^{k} \frac{1}{H_{\zeta, \theta_i}} \|u\|_{\zeta, \theta_i}^{2^*_\alpha, \theta_i}.
\]

Set
\[
\kappa := \frac{1}{S_{2^*_\alpha, \alpha}^2} + \sum_{i=1}^{k} \frac{1}{H_{\zeta, \theta_i}}.
\]

Applying \((2.1)\) and \((2.2)\), we get
\[
0 < \kappa < \infty.
\]

From \((H_1)\), we know
\[
2^*_{\alpha, \theta_k} < \cdots < 2^*_{\alpha, \theta_1} < 2 \cdot 2^*_{\alpha}.
\]

Now the proof of Step 1 is divided into two cases: (i) \(\|u\|_{\zeta} \geq 1\); (ii) \(\|u\|_{\zeta} < 1\).
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Case (i) \( \|u\|_\zeta \geq 1 \). From (3.5), we have
\[
\|u\|^2_\zeta \leq \kappa \|u\|^{2s}_\zeta,
\]
which implies that
\[
(4.2) \quad \|u\|_\zeta \geq \kappa^\frac{2}{s} \cdot 2^* \alpha \zeta.
\]
Case (ii) \( \|u\|_\zeta < 1 \). From (4.1), we know
\[
(4.3) \quad \|u\|_\zeta \geq \kappa^{\frac{2}{s} - s, \theta_k}.
\]
Combining the Cases (i) and (ii), according to (4.2) and (4.3), we deduce that
\[
(4.4) \quad \|u\|_\zeta \geq \begin{cases} 
\kappa^\frac{2}{s} - s, \theta_k, & \kappa < 1, \\
\kappa^{\frac{2}{s} - s, \theta_k}, & \kappa \geq 1.
\end{cases}
\]
Hence, we know that any limit point of a sequence in \( \mathcal{N}^2 \) is different from zero.

**Step 2.** Now, we claim that \( I_2 \) is bounded from below on \( \mathcal{N}^2 \). For any \( u \in \mathcal{N}^2 \), by using (4.4), we get
\[
I_2(u) \geq \left( \frac{1}{2} - \frac{1}{2^* s, \theta_k} \right) \|u\|^2_\zeta \geq \begin{cases} 
\kappa^\frac{2}{s} - s, \theta_k, & \kappa \leq 1, \\
\kappa^{\frac{2}{s} - s, \theta_k}, & \kappa > 1.
\end{cases}
\]
Therefore, \( I_2 \) is bounded from below on \( \mathcal{N}^2 \), and \( c^3_0 > 0 \).

**Lemma 4.2.** Assume that the assumptions of Theorem 1.4 hold. Then
(i) for each \( u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\} \), there exists a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N}^2 \); (ii) \( c^3_0 = c^3_1 = c^3 > 0 \).

**Proof.** The proof is standard, so we sketch it. Further details can be derived as in the proofs of Theorem 4.1 and 4.2 in [28], we omit it.

Similar to Lemma 4.1 and Lemma 4.2, we also have the following result.

**Lemma 4.3.** Assume that the assumptions of Theorem 1.4 hold. For any \( u \in \mathcal{N}^3 \), we have
\[
\|u\|^2_D \geq S^{N-\alpha}_{\theta_k a},
\]
and
\[
c_0^3 = \inf_{u \in \mathcal{N}^3} I_3(u) > 0.
\]

We show that the functional \( I_2 \) satisfies the Mountain–Pass geometry, and estimate the Mountain–Pass levels.

**Lemma 4.4.** Assume that the assumptions of Theorem 1.4 hold. Then there exists a \( (PS)_{c^2} \) sequence of \( I_2 \) at level \( c^2 \), where
\[
0 < c^2 < c^{2,*} = \min \left\{ \frac{N + 2s - \alpha}{2(2N - \alpha)} S^{\frac{2N-\alpha}{2(2N-\alpha)}}_{\theta_k a}, \frac{2s - \theta_1}{2(N - \theta_1)} H^{\frac{N-\theta_1}{2(N - \theta_1)}}_{\theta_k \alpha}, \ldots, \frac{2s - \theta_k}{2(N - \theta_k)} H^{\frac{N-\theta_k}{2(N - \theta_k)}}_{\theta_k \alpha} \right\}.
\]

**Proof.** The proof is standard, so we sketch it. Further details can be derived as in the proofs of Theorem 2 in [9], we omit it.

The following result implies the non–vanishing of \( (PS)_{c^2} \) sequence.
Lemma 4.5. Assume that the assumptions of Theorem 1.4 hold. Let \( \{u_n\} \) be a \((PS)_{c^2} \) sequence of \( I_2 \) with \( c^2 \in (0, c^2_*) \), then
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_1} |u_n(y)|^{2^*_1}}{|x - y|^{\alpha}} \, dx \, dy > 0,
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_1, \alpha}}{|x|^{\theta_i}} \, dx > 0, \quad (i = 1, \ldots, k).
\]

Proof. The proof of this Lemma is divided into four cases:

1. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_1, \alpha}}{|x|^{\theta_1}} \, dx > 0; \)
2. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_1, \alpha_k}}{|x|^{\theta_k}} \, dx > 0; \)
3. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_j, \alpha_j}}{|x|^{\theta_j}} \, dx > 0, \quad (j = 2, \ldots, k - 1); \)
4. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\alpha} |u_n(y)|^{2^*_\alpha}}{|x - y|^{\alpha}} \, dx \, dy > 0. \)

Case 1. It is easy to see that \( \{u_n\} \) is uniformly bounded in \( D^{s, 2}(\mathbb{R}^N) \). Suppose on the contrary that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_1, \alpha_i}}{|x|^{\theta_i}} \, dx = 0.
\]

From \((H_2)\), we know
\[
0 < 2\theta_2 - \theta_1 < \cdots < 2\theta_k - \theta_1 < 2s.
\]

Since \( \{u_n\} \) is uniformly bounded in \( D^{s, 2}(\mathbb{R}^N) \), there exists a constant \( 0 < C < \infty \) such that \( \|u_n\|_D \leq C \). Applying \((4.6)\) and \((2.2)\), we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_1, 2\alpha_i - \alpha_i}}{|x|^{2\alpha_i - \theta_i}} \, dx \leq C, \quad (i = 2, \ldots, k).
\]

According to Lemma 3.6, \((4.5)\) and \((4.7)\), we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_1, \alpha_i}}{|x|^{\theta_i}} \, dx = 0, \quad (i = 2, \ldots, k).
\]

By using \((4.5), (4.8)\) and the definition of \((PS)_{c^2} \), we obtain
\[
c^2 + o(1) = \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \cdot 2^*_\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_1} |u_n(y)|^{2^*_1}}{|x - y|^{\alpha}} \, dx \, dy,
\]
and
\[
o(1) = \|u_n\|^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\alpha} |u_n(y)|^{2^*_\alpha}}{|x - y|^{\alpha}} \, dx \, dy.
\]

These yield
\[
c^2 + o(1) = \frac{N + 2s - \alpha}{2(2N - \alpha)} \|u_n\|^2.
\]
Moreover,
\[ S_{\zeta,\alpha} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\alpha} |u_n(y)|^{2^*_\alpha}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}} \leq \|u_n\|_{\zeta}^2, \]
which implies that
\[ S_{\zeta,\alpha}^{\frac{2N-\alpha}{N+2s-\alpha}} \leq \|u_n\|_{\zeta}^2. \]
Therefore, we obtain
\[ \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{\zeta,\alpha}^{\frac{2N-\alpha}{N+2s-\alpha}} \leq c^2. \]
This is a contradiction.

**Case 2.** Suppose on the contrary that
\[ (4.9) \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_\theta_k}}{|x|^{\theta_k}} \, dx = 0. \]
By using (2.1) and \( \|u_n\|_D \leq C \), we have
\[ (4.10) \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_\theta} \, dx \leq C. \]
Applying \((H_2)\), Lemma 3.5, (4.9) and (4.10), we obtain
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_\theta_i}}{|x|^\theta_i} \, dx \leq \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_\theta_k}}{|x|^\theta_k} \, dx \right)^{\frac{\theta_k - \theta_i}{\theta_k}} \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_\theta} \, dx \right)^{\frac{\theta_k - \theta_i}{\theta_k}} = 0 \quad (i = 1, \ldots, k - 1). \]
By using (4.9), (4.11) and the definition of \((PS)_{c^2}\) sequence, similar to Case 1, we get \( \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{\zeta,\alpha}^{\frac{2N-\alpha}{N+2s-\alpha}} \leq c^2 \). This is a contradiction.

**Case 3.** Set \( j \in [2, k - 1] \). Suppose on the contrary that
\[ (4.12) \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_\theta_j}}{|x|^\theta_j} \, dx = 0. \]
From \((H_2)\), we know
\[ 0 < 2\theta_{j+1} - \theta_j < \cdots < 2\theta_k - \theta_j < 2\theta_k - \theta_1 < 2s. \]
Similar to (4.8), we obtain
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_\theta_i}}{|x|^\theta_i} \, dx = 0 \quad (i = j + 1, \ldots, k). \]
Similar to (4.11), we have
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_\theta_i}}{|x|^\theta_i} \, dx = 0 \quad (i = 1, \ldots, j - 1). \]
Similar to Case 1, we get \( \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{\zeta,\alpha}^{\frac{2N-\alpha}{N+2s-\alpha}} \leq c^2 \). This is a contradiction.

**Case 4.** Suppose on the contrary that
\[ (4.13) \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\alpha} |u_n(y)|^{2^*_\alpha}}{|x-y|^{\alpha}} \, dx \, dy = 0. \]
By using \( \|u_n\|_D \leq C, (2.1) \) and the definition of fractional Coulomb–Sobolev space, we obtain \( u_n \in C^{s,\alpha,2}_\ast(\mathbb{R}^N) \). Applying Lemma 3.3 and (4.13), we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s}_{s,\alpha}}{|x|^\alpha} \, dx
\]

and

\[
\leq C \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_s}_{s,\alpha}|u_n(y)|^{2^*_s}_{s,\alpha}}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{2^*_s-\alpha}{N+2^*_s-\alpha}} = 0 \ (i = 1, \ldots, k).
\]

Applying (4.13) and (4.14), we get

\[
\epsilon^2 + o(1) = \frac{1}{2} \|u_n\|^2_{\zeta},
\]

and

\[
o(1) = \|u_n\|^2_{\zeta},
\]

which imply that \( \epsilon^2 = 0 \). This contradicts with \( \epsilon^2 > 0 \).

In next lemma, we show that \( c_0^3 > c_0^2 \). This result plays a key role in the proof of Theorem 1.4.

**Lemma 4.6.** Assume that the assumptions of Theorem 1.4 hold. Then \( c_0^3 > c_0^2 \).

**Proof.** Consider the family of functions \( \{U_\sigma\} \) defined as

\[
U_\sigma(x) = \sigma^{\frac{N-2\alpha}{N-2}} \left( \frac{u^*(\frac{x}{\sigma})}{\|u^*\|_{2^*_s}^2} \right),
\]

where

\[
u^*(x) = \frac{\varpi}{(1 + |x|^2)^{\frac{2^*_s-\alpha}{2}}}, \quad \varpi \in \mathbb{R} \setminus \{0\}.
\]

Then, for each \( \sigma > 0 \), \( U_\sigma \) (see [17]) satisfies

\[
(-\Delta)^s U_\sigma = |U_\sigma|^{2^*_s-2} U_\sigma, \quad \text{in } \mathbb{R}^N.
\]

Set

\[
U_{\sigma,\alpha}(x) = S_{0,0}^{\frac{N-\alpha(2+s)}{N-2}} C(N, \alpha)^{\frac{2^*_s-N}{N-2}} U_\sigma(x).
\]

Then, for each \( \sigma > 0 \), \( U_{\sigma,\alpha} \) satisfies

\[
(-\Delta)^s U_{\sigma,\alpha} = \left( \int_{\mathbb{R}^N} \frac{|U_{\sigma,\alpha}|^{2^*_s}_{s,\alpha}}{|x-y|^\alpha} \, dy \right) |U_{\sigma,\alpha}|^{2^*_s-2} U_{\sigma,\alpha}, \quad \text{in } \mathbb{R}^N.
\]

Hence, we know that \( U_{\sigma,\alpha} \in \mathbb{N}^3 \), and

\[
\|U_{\sigma,\alpha}\|^2_D = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_{\sigma,\alpha}(y)|^{2^*_s}_{s,\alpha}|U_{\sigma,\alpha}(x)|^{2^*_s}_{s,\alpha}}{|x-y|^\alpha} \, dx \, dy = S_{0,0}^{\frac{2N-\alpha}{2}}.
\]

Now, we show that

\[
c_0^3 = \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{0,0}^{\frac{2N-\alpha}{2}}.
\]

Suppose on the contrary that \( c_0^3 < \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{0,0}^{\frac{2N-\alpha}{2}} \). Then there exists \( \ddot{u} \) satisfies

\[
I_3(\ddot{u}) = c_0^3 \quad \text{and} \quad \ddot{u} \in \mathbb{N}^3. \quad \text{We get}
\]

\[
c_0^3 = I_3(\ddot{u}) - \frac{1}{2} \cdot 2^* \cdot (I_3'(\ddot{u}), \ddot{u}) = \left( \frac{1}{2^*} - \frac{1}{2 \cdot 2^*} \right) \|\ddot{u}\|_D.
\]
Combining (4.15) and (4.16), we know
\[
\left(\frac{1}{2^s} - \frac{1}{2 \cdot 2^s}\right) \|U_{\sigma,0}\|_D^2 = \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{0,\alpha}^{2N-\alpha} > c_0^3 = \left(\frac{1}{2^s} - \frac{1}{2 \cdot 2^s}\right) \|\tilde{u}\|_D^2,
\]
which implies that
\[
S_{0,\alpha}^{2N-\alpha} = \|U_{\sigma,0}\|_D^2 > \|\tilde{u}\|_D^2.
\]
This contradicts with \(\|\tilde{u}\|_D^2 \geq \frac{S_{0,\alpha}^{2N-\alpha}}{2(2N-\alpha)}\) (see Lemma 4.3). Hence, we know that
\[
c_0^3 = \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{0,\alpha}^{2N-\alpha} \geq \frac{N + 2s - \alpha}{2(2N - \alpha)} S_{0,\alpha}^{2N-\alpha} > c^2 = c_0^2.
\]
\[\square\]

**The proof of Theorem 1.4:** We divide our proof into five steps.

**Step 1.** Since \(\{u_n\}\) is a bounded sequence in \(D^{s,2}(\mathbb{R}^N)\), up to a subsequence, we can assume that
\[
u_n \to u, \text{ in } D^{s,2}(\mathbb{R}^N), \quad u_n \to u, \text{ a.e. in } \mathbb{R}^N,
\]
\[u_n \to u, \text{ in } L^{\infty}_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*_s).\]

According to Lemma 3.1, Lemma 3.4 and Lemma 4.5, there exists \(C > 0\) such that
\[\|u_n\|_{L^{2,2^*_s}(\mathbb{R}^N)} \geq C > 0.
\]

On the other hand, since the sequence is bounded in \(D^{s,2}(\mathbb{R}^N)\) and \(D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \hookrightarrow L^{2,2^*_s}(\mathbb{R}^N)\), we have
\[\|u_n\|_{L^{2,2^*_s}(\mathbb{R}^N)} \lesssim C,
\]
for some \(C > 0\) independent of \(n\). Hence, there exists a positive constant which we denote again by \(C\) such that for any \(n\) we obtain
\[C \leq \|u_n\|_{L^{2,2^*_s}(\mathbb{R}^N)} \leq C^{-1}.
\]

So we may find \(\sigma_n > 0\) and \(x_n \in \mathbb{R}^N\) such that
\[
\frac{1}{\sigma_n^{2s}} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geq \|u_n\|^2_{L^{2,2^*_s}(\mathbb{R}^N)} - \frac{C}{2n} \geq c_0 > 0.
\]

Let \(\bar{u}_n(x) = \sigma_n^{N-2s} u_n(x + \sigma_n x).\) We may readily verify that
\[\bar{I}_2(\bar{u}_n) = I_2(u_n) \to c^2 \text{ and } \bar{I}_2'(\bar{u}_n) \to 0 \text{ as } n \to \infty,
\]
where
\[
\bar{I}_2(\bar{u}_n) = \frac{1}{2} \|\bar{u}_n\|_D^2 - \frac{1}{2} \int_{\mathbb{R}^N} |\bar{u}_n|^2 \, dx
\]
\[- \frac{1}{2} \cdot 2^s \alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\bar{u}_n(x)|^{2^*_s} |\bar{u}_n(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy - \sum_{i=1}^{k} \frac{1}{2^*_s, \theta_i} \int_{\mathbb{R}^N} |\bar{u}_n|^{2^*_s, \theta_i} \, dx.
\]
Now, for all \(\varphi \in D^{s,2}(\mathbb{R}^N)\), we obtain
\[
|\langle \bar{I}_2'(\bar{u}_n), \varphi \rangle| = |\langle I_2'(u_n), \tilde{\varphi} \rangle|
\]
\[\leq \|I_2'(u_n)\|_{D^{-1}} \|\tilde{\varphi}\|_D
\]
\[= o(1) \|\tilde{\varphi}\|_D,
\]
where \( \tilde{\varphi} = \sigma_n^{\frac{N-2s}{s}} \varphi(\frac{x-x_n}{\sigma_n}) \). Since \( \|\tilde{\varphi}\|_D = \|\varphi\|_D \), we get

\[
\tilde{I}_2'(\tilde{u}_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus there exists \( \bar{u} \) such that

\[
\bar{u}_n \rightharpoonup \bar{u}, \quad \text{in} \quad D_s(\mathbb{R}^N),
\]

\[
\bar{u}_n \to \bar{u}, \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^N,
\]

\[
\bar{u}_n \to \bar{u}, \quad \text{in} \quad L^r_{loc}(\mathbb{R}^N) \quad \text{for all} \quad r \in [1, 2^*_s).
\]

Then

\[
\int_{B(0,1)} |\bar{u}_n(y)|^2 dy = \frac{1}{\sigma_n^2} \int_{B(x_n,\sigma_n)} |u_n(y)|^2 dy \geq C_0 > 0.
\]

As a result, \( \bar{u} \neq 0 \).

**Step 2.** Now, we claim that \( \{\frac{\bar{u}_n}{\sigma_n}\} \) is bounded. If \( \frac{\bar{u}_n}{\sigma_n} \to \infty \), then for any \( \varphi \in D^{s,2}(\mathbb{R}^N) \), we get

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{\bar{u}_n \varphi}{|x + \frac{x_n}{\sigma_n}|^{2s}} dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\bar{u}_n|_{2^*_s}^{2^*_s} - 2 \bar{u}_n \varphi}{|x + \frac{x_n}{\sigma_n}|^{2s}} dx = 0.
\]

We will show that

\[
\langle I_3(\bar{u}), \varphi \rangle = 0.
\]

Since \( \bar{u}_n \rightharpoonup \bar{u} \) weakly in \( D^{s,2}(\mathbb{R}^N) \), we know

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\bar{u}_n(x) - \bar{u}_n(y)|(|\varphi(x) - \varphi(y)|)}{|x-y|^{N+2s}} dx dy = 0.
\]

By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from \( L^{\frac{2N}{N-2s}}(\mathbb{R}^N) \) to \( L^{\frac{2N}{N}}(\mathbb{R}^N) \). Since \( |\bar{u}_n|_{2^*_s} \to |\bar{u}|_{2^*_s} \) weakly in \( L^{\frac{2N}{N}}(\mathbb{R}^N) \), it follows that as \( n \to \infty \),

\[
\int_{\mathbb{R}^N} \frac{|\bar{u}_n(y)|_{2^*_s}^{2^*_s}}{|x-y|^\alpha} dy \to \int_{\mathbb{R}^N} \frac{|\bar{u}(y)|_{2^*_s}^{2^*_s}}{|x-y|^\alpha} dy \quad \text{weakly in} \quad L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).
\]
Now, we show that $|\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \to |\tilde{u}|^{2^*_\alpha-2}\tilde{u}\varphi$ in $L^{\frac{2N}{2^*_\alpha-n}}(\mathbb{R}^N)$. For any $\varepsilon > 0$, there exists $R > 0$ large enough such that

$$
\lim_{n \to \infty} \int_{|x| > R} |\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \frac{2N}{2^*_\alpha-n} \ dx \\
\leq \lim_{n \to \infty} \int_{|x| > R} |\tilde{u}_n|^{(2^*_\alpha-1)\frac{2^*_\alpha}{n}} |\varphi| \frac{2^*_\alpha}{n} \ dx + \int_{|x| > R} |\tilde{u}|^{(2^*_\alpha-1)\frac{2^*_\alpha}{n}} |\varphi| \frac{2^*_\alpha}{n} \ dx \\
\leq \lim_{n \to \infty} \left( \int_{|x| > R} |\tilde{u}_n|^{2^*_\alpha} \ dx \right)^{1-\frac{1}{2^*_\alpha}} \left( \int_{|x| > R} |\varphi|^{2^*_\alpha} \ dx \right)^{\frac{1}{2^*_\alpha}} \\
+ \left( \int_{|x| > R} |\tilde{u}|^{2^*_\alpha} \ dx \right)^{1-\frac{1}{2^*_\alpha}} \left( \int_{|x| > R} |\varphi|^{2^*_\alpha} \ dx \right)^{\frac{1}{2^*_\alpha}} \\
\leq C \left( \int_{|x| > R} |\varphi|^{2^*_\alpha} \ dx \right)^{\frac{1}{2^*_\alpha}}
$$

(4.20)

On the other hand, by the boundedness of $\{\tilde{u}_n\}$, one has

$$
\left( \int_{|x| \leq R} |\tilde{u}_n|^{2^*_\alpha} \ dx \right)^{1-\frac{1}{2^*_\alpha}} \leq M.
$$

where $M > 0$ is a constant. Let $\Omega = \{x \in \mathbb{R}^N ||x| \leq R\}$. For any $\tilde{\varepsilon} > 0$, there exists $\delta > 0$, when $E \subseteq \Omega$ with $|E| < \delta$. We obtain

$$
\int_{\Omega} |\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \frac{2N}{2^*_\alpha-n} \ dx = \int_{\Omega} |\tilde{u}_n|^{(2^*_\alpha-1)\frac{2^*_\alpha}{n}} |\varphi| \frac{2^*_\alpha}{n} \ dx \\
\leq \left( \int_{\Omega} |\tilde{u}_n|^{2^*_\alpha} \ dx \right)^{1-\frac{1}{2^*_\alpha}} \left( \int_{\Omega} |\varphi|^{2^*_\alpha} \ dx \right)^{\frac{1}{2^*_\alpha}} \\
< M\tilde{\varepsilon},
$$

where the last inequality is from the absolutely continuity of $\int_{\Omega} |\varphi|^{2^*_\alpha} \ dx$. Moreover, $|\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \to |\tilde{u}|^{2^*_\alpha-2}\tilde{u}\varphi$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Thus, by the Vitali convergence Theorem, we get

(4.21) $\lim_{n \to \infty} \int_{|x| \leq R} |\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \frac{2N}{2^*_\alpha-n} \ dx = \int_{|x| \leq R} |\tilde{u}|^{2^*_\alpha-2}\tilde{u}\varphi \frac{2N}{2^*_\alpha-n} \ dx$.

It follows from (4.20) and (4.21) that

$$
\lim_{n \to \infty} \int_{|x| \leq R} |\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \frac{2N}{2^*_\alpha-n} \ dx \\
\leq \lim_{n \to \infty} \left| \int_{|x| \leq R} \right| |\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \frac{2N}{2^*_\alpha-n} \ dx \right| - \left| \int_{|x| \leq R} |\tilde{u}|^{2^*_\alpha-2}\tilde{u}\varphi \frac{2N}{2^*_\alpha-n} \ dx \right| \\
+ \lim_{n \to \infty} \left| \int_{|x| > R} \right| |\tilde{u}_n|^{2^*_\alpha-2}u_n\varphi \frac{2N}{2^*_\alpha-n} \ dx \right| - \left| \int_{|x| > R} |\tilde{u}|^{2^*_\alpha-2}\tilde{u}\varphi \frac{2N}{2^*_\alpha-n} \ dx \right| \\
< \varepsilon.
$$
This implies that

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \left| \tilde{u}_n \right|^{2^*_s} \, dx = \int_{\mathbb{R}^N} \left| \bar{u} \right|^{2^*_s} \, dx. \]  

Combining (4.19) and (4.22), we have

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\tilde{u}_n(y)^{2^*_s} \tilde{u}_n(x)^{2^*_s}}{|x - y|^\alpha} \, dy \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\bar{u}(y)^{2^*_s} \bar{u}(x)^{2^*_s}}{|x - y|^\alpha} \, dy \, dx. \]  

Applying \( \lim_{n \to \infty} (\tilde{I}_2'(\tilde{u}_n), \varphi) \to 0 \), (4.17), (4.18) and (4.23) we know

\[ \langle I_1'(\tilde{u}), \varphi \rangle = 0. \]  

Moreover, according to (4.24) and \( \tilde{u} \neq 0 \), we get that

\[ \tilde{u} \in \mathcal{N}^3. \]

By Brézis–Lieb lemma [11, Lemma 2.2], we have

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x)|^{2^*_s} |\tilde{u}_n(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x) - \tilde{u}(x)|^{2^*_s} |\tilde{u}_n(y) - \bar{u}(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy \]

\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x)|^{2^*_s} |\bar{u}(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy + o(1), \]

which implies that

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x)|^{2^*_s} |\tilde{u}_n(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x)|^{2^*_s} |\tilde{u}(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy + o(1). \]

Similarly, we get

\[ \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2^*_s, \gamma_i}}{|x|^\gamma_i} \, dx \geq \int_{\mathbb{R}^N} \frac{|\tilde{u}|^{2^*_s, \gamma_i}}{|x|^\gamma_i} \, dx + o(1). \]

Applying Lemma 4.6, Lemma 4.2, (4.25), (4.26), \( \tilde{u} \in \mathcal{N}^3 \) and Lemma 4.3, we obtain

\[ c_0^3 > c_0^2 = c^2 = I_2(\tilde{u}_n) - \frac{1}{2} \langle \tilde{I}_2'(\tilde{u}_n), \tilde{u}_n \rangle \]

\[ \geq \left( \frac{1}{2} - \frac{1}{2^{*}_s} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x)|^{2^*_s} |\tilde{u}(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy + o(1) \]

\[ = I_3(\tilde{u}) - \frac{1}{2} \langle \tilde{I}_3'(\tilde{u}), \tilde{u} \rangle = I_3(\tilde{u}) \geq c_0^3, \]

which yields a contradiction. Hence, \( \{ \tilde{u}_n \} \) is bounded.

**Step 3.** In this step, we study another \((PS)_{c^2}\) sequence of \( I_2 \). Let \( \tilde{u}_n(x) = \sigma_n^{-{\frac{N}{2^*_s}}} u_n(\sigma_n x) \). Then we can verify that

\[ I_2(\tilde{u}_n) = I_2(u_n) \to c^2, \quad I_2'(\tilde{u}_n) \to 0 \text{ as } n \to \infty. \]

Arguing as before, we have

\[ \tilde{u}_n \to \tilde{u} \text{ in } D^{1,2}(\mathbb{R}^N), \quad \tilde{u}_n \to \tilde{u} \text{ a.e. in } \mathbb{R}^N, \]

\[ \tilde{u}_n \to \tilde{u} \text{ in } L^r_{\text{loc}}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*_s). \]
By using \( \{ \frac{\epsilon_n}{n} \} \) is bounded, there exists \( \tilde{R} > 0 \) such that

\[
\int_{B(0, \tilde{R})} |\tilde{u}_n(y)|^2 \, dy > \int_{B(\frac{\tilde{R}}{2}, 1)} |\tilde{u}_n(y)|^2 \, dy = \frac{1}{\sigma_n^{2s}} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geq C_6 > 0.
\]

As a result, \( \tilde{u} \neq 0 \).

**Step 4.** In this step, we show \( \tilde{u}_n \to \tilde{u} \) strongly in \( D^{s,2}(\mathbb{R}^N) \). Set

\[
K(u) = \left( \frac{1}{2} - \frac{1}{2} \cdot 2^*_s \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2s} |u(y)|^{2s}}{|x-y|^s} \, dx \, dy + \sum_{i=1}^k \left( \frac{1}{2} - \frac{1}{2s_i} \right) \int_{\mathbb{R}^N} \frac{|u|^{2s_i}}{|x|^{\theta_i}} \, dx.
\]

Similar to Step 2, we know that

\[
(4.27) \quad \langle I_2'(\tilde{u}), \varphi \rangle = 0.
\]

Applying Lemma 4.1, Lemma 4.2, \( \tilde{u} \in \mathcal{N}^2 \) and (4.25) – (4.27), we obtain

\[
c_0^2 = c^2 = I_2(\tilde{u}_n) - \frac{1}{2} \langle I_2'(\tilde{u}_n), \tilde{u}_n \rangle = \lim_{n \to \infty} K(\tilde{u}_n) + o(1)
\]

\[
\geq K(\tilde{u}) + o(1)
\]

\[
= I_2(\tilde{u}) - \frac{1}{2} \langle I_2'(\tilde{u}), \tilde{u} \rangle = I_2(\tilde{u}) \geq c_0^2.
\]

Therefore, the inequalities above have to be equalities. We know

\[
\lim_{n \to \infty} K(\tilde{u}_n) = K(\tilde{u}).
\]

By using Brézis–Lieb lemma again, we have

\[
\lim_{n \to \infty} K(\tilde{u}_n) - \lim_{n \to \infty} K(\tilde{u}_n - \tilde{u}) = K(\tilde{u}) + o(1).
\]

Hence, we deduce that

\[
\lim_{n \to \infty} K(\tilde{u}_n - \tilde{u}) = 0,
\]

which implies that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x) - \tilde{u}(x)|^{2s} |\tilde{u}_n(y) - \tilde{u}(y)|^{2s}}{|x-y|^s} \, dx \, dy = 0,
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n - \tilde{u}|^{2s_i}}{|x|^{\theta_i}} \, dx = 0, \text{ for all } i = 1, \ldots, k.
\]

According to \( \langle I_2'(\tilde{u}_n), \tilde{u}_n \rangle = o(1) \), \( \langle I_2'(\tilde{u}), \tilde{u} \rangle = 0 \) and Brézis–Lieb lemma, we obtain

\[
o(1) = \langle I_2'(\tilde{u}_n), \tilde{u}_n \rangle - \langle I_2'(\tilde{u}), \tilde{u} \rangle
\]

\[
= \|\tilde{u}_n - \tilde{u}\|_c^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x) - \tilde{u}(x)|^{2s} |\tilde{u}_n(y) - \tilde{u}(y)|^{2s}}{|x-y|^s} \, dx \, dy
\]

\[
- \sum_{i=1}^k \frac{|\tilde{u}_n - \tilde{u}|^{2s_i}}{|x|^{\theta_i}} \, dx + o(1),
\]
which implies that
\[
\lim_{n \to \infty} \|\tilde{u}_n - \tilde{u}\|_\zeta^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x) - \tilde{u}(x)|^2 |\tilde{u}_n(y) - \tilde{u}(y)|^2}{|x - y|^\alpha} \, dx \, dy
\]
\[
+ \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n - \tilde{u}|^2}{|x|^{\theta_i}} \, dx = 0.
\]

Combining (4.29) and (4.30), we get
\[
\lim_{n \to \infty} \|\tilde{u}_n - \tilde{u}\|_\zeta^2 = 0.
\]

Since \( \tilde{u} \not\equiv 0 \), we know that \( \tilde{u}_n \to \tilde{u} \) strongly in \( D^{s,2}(\mathbb{R}^N) \).

**Step 5.** By using (4.28) again, we know that \( I_2(\tilde{u}) = c^2 \), which means that \( \tilde{u} \) is a nontrivial solution of problem \((P_2)\) at the energy level \( c^2 \). Then we have just to prove that we can choose \( \tilde{u} \geq 0 \). We know that
\[
0 = \langle I_2'(\tilde{u}), \tilde{u}^- \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x - y|^{N+2s}} \tilde{u}^-(x) \tilde{u}^-(y) \, dx \, dy - \zeta \int_{\mathbb{R}^N} \frac{\tilde{u}^-}{|x|^{2s}} \, dx
\]
\[
- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(y)|^2 |\tilde{u}(x)|^2 - 2 \tilde{u}(x) \tilde{u}^-(x) \tilde{u}^-}{|x - y|^\alpha} \, dx \, dy - \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|\tilde{u}|^2}{|x|^{\theta_i}} \, dx,
\]
where \( \tilde{u}^- = \max\{0, -\tilde{u}\} \). For a.e. \( x, y \in \mathbb{R}^N \), we have
\[
(\tilde{u}(x) - \tilde{u}(y))(\tilde{u}^-(x) - \tilde{u}^-(y)) \leq -|\tilde{u}^-| \tilde{u}^- (x) \tilde{u}^- (y)^2.
\]

Then, we get
\[
0 = -\|\tilde{u}^-\|_D^2 - \zeta \int_{\mathbb{R}^N} \frac{|\tilde{u}^-|^2}{|x|^{2s}} \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(y)|^2 |\tilde{u}^-(x)|^2}{|x - y|^\alpha} \, dx \, dy - \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|\tilde{u}|^2}{|x|^{\theta_i}} \, dx
\]
\[
\leq -\|\tilde{u}^-\|_D^2.
\]

Thus, \( \|\tilde{u}^-\|_D^2 = 0 \). Hence, we can choose \( \tilde{u} \geq 0 \). By using the fractional Kelvin transformation
\[
\tilde{u}(x) = \frac{1}{|x|^{N-2s}} \tilde{u} \left( \frac{x}{|x|^2} \right).
\]

It is well known that
\[
(-\Delta)^s \tilde{u}(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s \tilde{u} \left( \frac{x}{|x|^2} \right).
\]

The following identity is very useful. For \( \forall x, y \in \mathbb{R}^N \setminus \{0\} \), we get
\[
\frac{1}{|x|^\alpha} \frac{1}{|y|^\alpha} \left( \frac{x^2 - y^2}{(x+y)^2} \right)^\frac{\alpha}{\alpha - 1} \cdot \frac{1}{|x|^\alpha} = \frac{1}{|x|^\alpha} \frac{1}{|y|^\alpha} \left( \frac{x^2 - y^2}{xy} \right)^\frac{\alpha}{\alpha - 1}.
\]

(4.33)
Set \( z = \frac{|y|}{|x|} \). Applying (4.31) and (4.33), we have

\[
\int_{\mathbb{R}^N} |\tilde{u}(y)|^{2N-\alpha} \frac{2N-\alpha}{|x-y|^\alpha} dy = \int_{\mathbb{R}^N} \tilde{u} \left( \frac{y}{|y|^\alpha} \right)^{2N-\alpha} \cdot \frac{1}{|y|^{2N-\alpha}} dy \quad \text{(by (4.31))}
\]

\[
= \int_{\mathbb{R}^N} \tilde{u} \left( \frac{y}{|y|^\alpha} \right)^{2N-\alpha} \cdot \frac{1}{|x|^{\alpha} \cdot |y|^{2N-\alpha}} dy \quad \text{(by (4.33))}
\]

(4.34)

\[
= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^N} \tilde{u} \left( \frac{y}{|y|^\alpha} \right)^{2N-\alpha} \cdot \frac{1}{|y|^{2N}} dy
\]

\[
= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^N} \tilde{u} (z)^{2N-\alpha} |z|^{2s} dz \quad \text{(set } z = \frac{y}{|y|^\alpha}).
\]

By using (4.31), we get

(4.35)

\[
\int_{|x|^\beta_1} |\tilde{u}|^{4s-2\alpha} \tilde{u} \frac{4s-2\alpha}{|x|^{2s}} \tilde{u} = \int_{|x|^\beta_1} \tilde{u} \left( \frac{x}{|x|^\alpha} \right)^{4s-2\alpha} \tilde{u},
\]

and

(4.36)

\[
\frac{\tilde{u}(x)}{|x|^{2s}} = \frac{1}{|x|^{2s}} \tilde{u} \left( \frac{x}{|x|^\alpha} \right)^{2s}.
\]

Therefore, by using (4.32) and (4.34) – (4.36), we get

\[
(-\Delta)^s \tilde{u} - \zeta \frac{\tilde{u}}{|x|^{2s}} = \left( \int_{\mathbb{R}^N} \frac{|\tilde{u}|^{2N-\alpha}}{|x-y|^\alpha} dy \right) |\tilde{u}|^{4s-2\alpha} \tilde{u} + \sum_{i=1}^k \frac{\tilde{u}}{|x|^{\beta_i}}, \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

\[\square\]

5. The proof of Theorem 1.3

In this section, we study the existence of nonnegative solution of problem \((P_1)\).

Lemma 5.1. Assume that the assumptions of Theorem 1.3 hold. Then there exists a \((PS)_{c^1}\) sequence of \(I_1\) at level \(c^1\), where

\[
0 < c^1 < c^{1,*} = \min \left\{ \frac{N + 2s - \alpha_1}{2(2N - \alpha_1)} S_{0,\alpha_1}^{2N-\alpha_1}, \ldots, \frac{N + 2s - \alpha_k}{2(2N - \alpha_k)} S_{0,\alpha_k}^{2N-\alpha_k}, \frac{s N}{N}, S_{0,0}^{2N-\alpha_k} \right\}.
\]

Lemma 5.2. Assume that the assumptions of Theorem 1.3 hold. Then

\[
c^1 = c^1 = c^0 = \inf_{u \in \mathcal{N}^1} I_1(u) > 0.
\]

The following result implies the non–vanishing of \((PS)_{c^1}\) sequences.

Lemma 5.3. Assume that the assumptions of Theorem 1.3 hold. Let \(\{u_n\}\) be a \((PS)_{c^1}\) sequence of \(I_1\) with \(c^1 \in (0, c^{1,*})\), then

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx > 0,
\]
and
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy > 0, \quad (i = 1, \ldots, k). \]

**Proof.** Let \( \{u_n\} \) be a \((PS)_{c_1}\) sequence of \( I_1 \) with \( c_1 \in (0, c^1_*] \), it’s easy to see that \( \{u_n\} \) is uniformly bounded in \( D^{s,2}(\mathbb{R}^N) \). The proof of this Lemma is divided into three cases:

1. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy > 0; \)
2. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy > 0, \quad \text{for } j = 2, \ldots, k; \)
3. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s_i} \, dx > 0. \)

**Case 1.** Suppose that
\[ (5.1) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy = 0. \]

Since \( \{u_n\} \) is uniformly bounded in \( D^{s,2}(\mathbb{R}^N) \), there exists a constant \( 0 < C < \infty \) such that \( \|u_n\|_D \leq C \). By using (5.1) and the definition of fractional Coulomb–Sobolev space, we obtain \( u_n \in E^{s,\alpha_i;2s_i}(\mathbb{R}^N) \). Applying Lemma 1.2 and (5.1), we have
\[ (5.2) \quad \leq C \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy \right)^{\frac{i(\mathcal{N} - 2s)}{2N}} = 0. \]

Combining Hardy–Littlewood–Sobolev inequality and (5.2), for all \( i = 2, \ldots, k \), we know
\[ (5.3) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy \leq C \lim_{n \to \infty} \|u_n\|_{L^{2s_i}(\mathbb{R}^N)} = 0. \]

According to (5.1)–(5.3) and the definition of \((PS)_{c_1}\) sequence , we obtain
\[ c^1 + o(1) = \frac{1}{2} \|u_n\|_D^2, \]
and
\[ o(1) = \|u_n\|_D^2. \]

these imply that \( c^1 = 0 \), which contradicts as \( 0 < c^1 \).

**Case 2.** From Case 1, we have \( u_n \in E^{s,\alpha_i;2s_i}(\mathbb{R}^N) \). Applying the result of (ii) in Lemma 3.2, we know that \( u_n \in \bigcap_{i=2}^k E^{s,\alpha_i;2s_i}(\mathbb{R}^N) \). Similar to Case 1, for all \( i = 2, \ldots, k \), we prove that
\[ (5.4) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy > 0. \]

**Case 3.** Suppose that
\[ (5.5) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s_i} \, dx = 0, \]

By using Lemma 1.1 and (5.4), for all \( i = 1, \ldots, k \), we have
\[ (5.6) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2s_i}|u_n(y)|^{2s_i}}{|x-y|^{\alpha_i}} \, dx \, dy \leq C \lim_{n \to \infty} \|u_n\|_{L^{2s_i}(\mathbb{R}^N)} = 0. \]
Applying (5.4) and (5.5), we get
\[ c^1 + o(1) = \frac{1}{2} \|u_n\|_D^2, \]
and
\[ o(1) = \|u_n\|_D^2, \]
these imply that \( c^1 = 0 \), which contradicts as \( 0 < c^1 \).

We are now ready to prove the existence of nonnegative solution for problem \((P_1)\).

**Proof of Theorem 1.3:** **Step 1.** Since \( \{u_n\} \) is a bounded sequence in \( D^{s,2}(\mathbb{R}^N) \), up to a subsequence, we can assume that
\[ u_n \rightharpoonup u \quad \text{in} \quad D^{s,2}(\mathbb{R}^N), \quad u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N, \]
\[ u_n \to u \quad \text{in} \quad L^r_{\text{loc}}(\mathbb{R}^N) \quad \text{for all} \quad r \in [1,2^*_s). \]

According to Lemma 2.1, Lemma 3.1 and Lemma 5.3, there exists \( C > 0 \) such that
\[ \|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \geq C > 0. \]
On the other hand, since the sequence is bounded in \( D^{s,2}(\mathbb{R}^N) \) and \( D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \hookrightarrow L^{2,N-2s}(\mathbb{R}^N) \), we have
\[ \|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \leq C, \]
for some \( C > 0 \) independent of \( n \). Hence, there exists a positive constant which we denote again by \( C \) such that for any \( n \) we obtain
\[ C \leq \|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \leq C^{-1}. \]

So we may find \( \sigma_n > 0 \) and \( x_n \in \mathbb{R}^N \) such that
\[ \frac{1}{\sigma_n^N} \int_{B(x_n,\sigma_n)} |u_n(y)|^2 \, dy \geq \|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)}^2 - \frac{C}{2^n} \geq C_7 > 0. \]

Let \( \tilde{v}_n(x) = \sigma_n^{-\frac{N}{2s}} u_n(x + \sigma_n x) \). We could verify that
\[ I_1(\tilde{v}_n) = I(u_n) \to c^1, \quad \langle I'_1(\tilde{v}_n), \varphi \rangle \quad \text{as} \quad n \to \infty. \]
It’s obviously that \( \{\tilde{v}_n\} \) is uniformly bounded in \( D^{s,2}(\mathbb{R}^N) \). Thus there exists \( v \) such that
\[ \tilde{v}_n \rightharpoonup v \quad \text{in} \quad D^{s,2}(\mathbb{R}^N), \quad \tilde{v}_n \to v \quad \text{a.e. in} \quad \mathbb{R}^N, \]
\[ \tilde{v}_n \to v \quad \text{in} \quad L^r_{\text{loc}}(\mathbb{R}^N) \quad \text{for all} \quad r \in [1,2^*_s). \]

Then \( \int_{B(0,1)} |\tilde{v}_n(y)|^2 \, dy \geq C_7 > 0. \) As a result, \( \tilde{v} \not= 0. \)

**Step 2.** Similar to (4.24), we get
\[ \langle I'_1(\tilde{v}), \varphi \rangle = 0. \]

Similar to the proof of Theorem 1.4, we know that \( \tilde{v}_n \to \tilde{v} \) strongly in \( D^{s,2}(\mathbb{R}^N) \), and \( I_1(\tilde{v}) = c^1 \). Moreover, we can choose \( v \geq 0. \) By using the fractional Kelvin transformation
\[ \bar{v}(x) = \frac{1}{|x|^{N-2s}} \tilde{v} \left( \frac{x}{|x|^2} \right). \]

We obtain
\[ (-\Delta)^s \bar{v} = \sum_{i=1}^k \left( \int_{\mathbb{R}^N} \frac{|\bar{v}|^{2^*_s-1}}{|x-y|^{N-2s}} \, dy \right) |\bar{v}|^{\frac{2^*_s-1}{s}} \bar{v} + |\bar{v}|^{\frac{2^*_s}{s}} \bar{v}, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \]
Open Problem

During the preparation of the manuscript we faced one problem which is worth to be tackled in forthcoming investigation.

We want to generalize the study of problem (P$_1$) to the following problem:

$(-\Delta)^s u - \zeta \frac{u}{|x|^{2s}} = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \frac{|u|^{2_{(s)}}}{|x-y|^{\alpha_i}} \right) |u|^{2_{(s)}-2}u + |u|^{2_{(s)}-2}u, \text{ in } \mathbb{R}^N,$ (P$_4$)

where $N \geq 3$, $s \in (0,1)$, $\zeta \in \left( 0, 4^{\frac{s(N+2s)}{f(N-2s)}} \right)$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < N$ ($k \in \mathbb{N}$, $2 \leq k < \infty$). If $\zeta = 0$, then problem (P$_4$) goes back to problem (P$_1$). However, they are very different from each other.

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