CONE MONOTONE MAPPINGS: CONTINUITY AND DIFFERENTIABILITY

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ABSTRACT. We generalize some results of Borwein, Burke, Lewis, and Wang to mappings with values in metric (resp. ordered normed linear) spaces. We define two classes of monotone mappings between an ordered linear space and a metric space (resp. ordered linear space): $K$-monotone dominated and cone-to-cone monotone mappings. $K$-monotone dominated mappings naturally generalize mappings with finite variation (in the classical sense) and $K$-monotone functions defined by Borwein, Burke and Lewis, to mappings with domains and ranges of higher dimensions. First, using results of Veselý and Zajíček, we show some relationships between these classes. Then, we show that every $K$-monotone function $f : X \to \mathbb{R}$, where $X$ is any Banach space, is continuous outside of a set which can be covered by countably many Lipschitz hypersurfaces. This sharpens a result due to Borwein and Wang. As a consequence, we obtain a similar result for $K$-monotone dominated and cone-to-cone monotone mappings. Finally, we prove several results concerning almost everywhere differentiability (also in metric and $w^*$-senses) of these mappings.

1. INTRODUCTION

Let $X$ be a normed linear space, $K \subset X$ be a cone. The cone induces ordering $\leq_K$ on $X$ as follows: $x \leq_K y$ if $y - x \in K$. Borwein, Burke and Lewis [4] defined the $K$-increasing functions $f : X \to \mathbb{R}$ as those functions that satisfy $f(x) \leq f(y)$ whenever $x \leq_K y$. They say that a function $f : X \to \mathbb{R}$ is $K$-monotone provided $f$ or $-f$ is $K$-increasing. The continuity and differentiability of $K$-monotone functions was studied by Chabrillac, Crouzeix [7] (in case of $\mathbb{R}^n$ with the standard coordinate ordering), Borwein, Burke, Lewis [4], Borwein, Wang [6], and others. Borwein and Goebel [5] provided examples showing the necessity of certain assumptions on the cone $K$ in [4, 6]. The authors of [4] mention on page 1075 that the Rademacher’s theorem holds when the range of the function is a Banach space with RNP, but that “it is not clear what is true for cone-monotone operators”. We address this issue in the current paper.

We define two classes of monotone mappings with values in metric (resp. ordered normed linear) spaces: $K$-monotone dominated mappings and cone-to-cone monotone mappings. The idea of a dominating function in the definition of $K$-monotone dominated mappings is similar to the idea of a control function in the definition of d.c. mappings (see [16]). However, the analogy should not be taken too far as simple examples show that for instance the $K$-monotone dominated mappings do...
not inherit differentiability properties from their dominating functions (on the other hand, d.c. mappings do inherit certain differentiability properties of their control functions – see [16]). Still, continuity (resp. pointwise-Lipschitzness) of the dominating function forces the dominated mapping to be continuous (resp. pointwise-Lipschitz); see Lemma 2.4. Similarly as in the case of d.c. mappings, there usually does not exist a canonical dominating function for a $K$-monotone dominated mapping $F : X \to Y$ (except, perhaps, in the case $X = \mathbb{R}$). A mapping $F : \mathbb{R} \to Y$ is $K$-monotone dominated if and only if $F$ has locally bounded variation and thus our class generalizes mappings with bounded variation.

Let us describe the structure of the present paper. Section 2 contains basic definitions and facts. Section 3 shows some relationships between $K$-monotone dominated and cone-to-cone monotone mappings. This section is motivated by results of Veselý and Zajíček [17]. Section 4 contains results about continuity of monotone dominated and cone-to-cone monotone mappings. We prove that every $K$-monotone function $f : X \to \mathbb{R}$, where $X$ is an arbitrary Banach space and $K$ is a convex cone with non-empty interior, is continuous outside of a set which can be covered by countably many Lipschitz hypersurfaces. This sharpens [6, Proposition 6] of Borwein and Wang and seems to be new even in the case $X = \mathbb{R}^n$. Then we obtain similar results for $K$-monotone dominated and cone-to-cone monotone mappings as a consequence. Section 5 contains results about a.e. Gâteaux differentiability of monotone dominated and cone-to-cone monotone maps, and section 6 contains results about metric and $w^*$-differentiability of $K$-monotone dominated maps with values in metric spaces.

Most results are formulated for mappings defined on the whole space, however all of them can be localized; i.e. they hold also for maps defined only on open sets. We leave the details to the interested reader. Also, a simple example based on Lemma 2.2 and the well known nowhere Fréchet differentiable map $f : \ell_2 \to \ell_2$, $f((x_n)_n) = (|x_n|)_n$ shows that there is no hope to establish Fréchet differentiability of $K$-monotone mappings between infinite-dimensional spaces. Every Lipschitz $f : X \to Y$ (such that $X,Y$ are Banach spaces) is also $K$-monotone dominated (for some convex cone $K \subset X$ with non-empty interior) by Lemma 2.2, and thus the Lipschitz theory shows that we can only expect Gâteaux differentiability a.e. (in the appropriate sense) of $K$-monotone dominated mappings between Banach spaces provided $X$ is separable, and $Y$ has RNP; see e.g. [2, Chapter 6]. Since all $K$-monotone functions are $K$-monotone dominated (the dominating function is the function itself), we cannot expect that we will be able to prove any results provided the cone $K$ is too small; see [4, Section 6] and [5]. Example 5.5 shows that if the cone in the target space is not properly positioned, then there might be cone-to-cone monotone mappings which are nowhere differentiable. It remains open whether in Theorem 5.1 and Corollaries 5.3 and 5.4 we can replace the family $\tilde{C}$ by $\tilde{A}$.

2. Preliminaries

All normed (and Banach) spaces are real. Let $X$ be a normed linear space. By $B(x, r) = B_X(x, r)$ we denote the closed ball of $X$ (with center $x$ and radius $r$) and by $S(x, r) = S_X(x, r)$ the corresponding sphere (omitting the subscript where no confusion is possible). We will write $S_X = S_X(0, 1)$. 
We say that $X$ is an ordered normed linear space provided it is a normed linear space equipped with an (antisymmetric) partial ordering $\le$ such that for $x, y, z \in X$ and $\lambda \ge 0$, the implications $x \le y \implies x + z \le y + z$, and $x \le y \implies \lambda x \le \lambda y$, hold. Then the corresponding cone $X_+ := \{ x \in X : x \ge 0 \}$ is called the positive cone of $X$. We say that $M \subset X$ has an upper bound provided there exists $e \in X_+$ such that $m \le e$ for each $m \in M$. It is easy to see that if $X_+$ is convex, then we have $X = X_+ - X_+$ if and only if every pair of $x, y \in X$ has an upper bound in $X_+$.

We say that an ordered normed linear space $Y$ is a Banach lattice if it is a Banach space, each pair of elements of $Y$ has a supremum and an infimum, and $0 \le |x| \le |y|$ implies $||x|| \le ||y||$ (where $|y| := \sup(x, -x)$). Let $X$ be a Banach lattice. We say that $X$ has the $\sigma$-Levi property if each norm-bounded non-decreasing sequence in $X_+$ has the least upper bound (see [1]).

Suppose that $X$ is an ordered normed linear space. We say that a convex subset $B \subset X$ is a base for the cone $X_+$ if for each $y \in K \setminus \{0\}$ there exists a unique $\lambda > 0$ such that $\lambda y \in B$. Following [12] (p. 120), we say that $X_+$ is well-based if it has a bounded base $B$ such that $0 \not\in \overline{B}$. By [12, 3.8.12], $X_+$ is well-based if and only if there exists $\varphi \in X^*$ such that $\varphi(u) \ge ||u||$ for each $u \in X_+$.

If $(Y, \rho)$ is a metric space, and $F : [a, b] \to Y$, then we define the variation $\int_a^b f$ as a supremum of the sums

$$\sum_{i=1}^n \rho(f(x_i), f(x_{i-1})), $$

taken over all partitions $\{ a = x_0 < \cdots < x_n = b \}$ of $[a, b]$. We say that $F : \mathbb{R} \to Y$ has locally finite variation provided $\int_a^b f < \infty$ for all $-\infty < a < b < \infty$.

**Definition 2.1.** Let $X$ be a normed linear space, $K \subset X$ be a non-empty cone, let $(Y, \rho)$ be a metric space. We say that $F : X \to Y$ is $K$-monotone dominated provided there exists $h : X \to \mathbb{R}$ such that

$$\rho(H(x + k), H(x)) \le h(x + k) - h(x),$$

whenever $x \in X$ and $k \in K$. Then we say that $h$ is the dominating function for $H$.

Trivially, every dominating function is $K$-monotone. If we fix $K, X, Y$, and assume that $Y$ is a normed linear space, then it is easy to see that $K$-monotone mappings $F : X \to Y$ form a vector space. Since every metric space embeds isometrically into some $\ell_\infty(\Gamma)$ (for some $\Gamma$), there would be no loss of generality in assuming that the space $Y$ in the definition of $K$-monotone dominated mappings is a normed linear space.

**Lemma 2.2.** Let $X$ be a normed linear space, $(Y, d)$ be a metric space. If $f : X \to Y$ is Lipschitz, then $f$ is $K$-monotone dominated for some convex cone $K \subset X$ with non-empty interior.

**Proof.** Take $0 \neq x \in X$ with $||x|| = 1$. Find $x^* \in X^*$ such that $||x^*|| = \langle x^*, x \rangle = 1$. Let $\alpha \in (0, 1)$, and put $K := \{ y \in X : \alpha ||y|| \le x^*(y) \}$. Then $K$ is a convex cone with non-empty interior. If $y, z \in X$ are such that $y - z \in K$, then $d(f(y), f(z)) \le L ||y - z|| \le \frac{1}{\alpha} \cdot x^*(y - z)$, and thus $\frac{1}{\alpha} x^*$ is a dominating function for $f$, and $f$ is $K$-monotone dominated. \qed
If $X = \mathbb{R}$, then $\mathbb{R}^n$-monotone dominated mappings are exactly those mappings that have locally finite variation; see Section 3 for more details.

**Definition 2.3.** Let $X, Y$ be ordered normed linear spaces, $F : X \to Y$. We say that $F$ is $(X_+, Y_+)$-increasing provided $x \leq y \implies F(x) \leq F(y)$, whenever $x, y \in X$. We say that $F$ is $(X_+, Y_+)$-decreasing provided $-F$ is $(X_+, Y_+)$-increasing. Finally, we say that $F$ is $(X_+, Y_+)$-monotone provided it is either $(X_+, Y_+)$-increasing or it is $(X_+, Y_+)$-decreasing.

We shall also refer to $(X_+, Y_+)$-monotone mappings as cone-to-cone monotone mappings.

We say that $F : X \to Y$ is pointwise-Lipschitz at $x \in X$ provided $\text{Lip}(F, x) := \limsup_{t \to 0} \frac{\|F(t) - F(x)\|}{|t|} < \infty$. We say that $F : X \to Y$ is Lipschitz provided there exists $C > 0$ such that $\|F(x) - F(y)\| \leq C\|x - y\|$ whenever $x, y \in X$. We say that $F : X \to Y$ is Gâteaux differentiable at $x \in X$ provided $T(h) := \lim_{t \to 0} \frac{F(x + th) - F(x)}{t}$ exists for each $h \in X$, and $T : X \to Y$ is a bounded linear operator. For basic facts about Gâteaux derivatives, see [2]. Given $f : X \to \mathbb{R}$, we use $\overline{f}$ (resp. $\underline{f}$) for the upper (resp. lower) semi-continuous envelope of $f$.

Let $X, Y$ be (real) normed linear spaces. For $f : X \to Y$ we shall denote

$$MD(f, x)(u) = \lim_{r \to 0} \frac{\|f(x + ru) - f(x)\|}{|r|} \quad \text{for } x, u \in X.$$  

It was defined in [13]. If $MD(f, x)(u)$ exists for all $u \in X$, we say that $f$ is directionally metrically differentiable at $x$. We will say that $f$ is metrically Gâteaux differentiable at $x$ provided $f$ is directionally metrically differentiable at $x$, and $MD(f, x)(\cdot)$ is a continuous seminorm. We say that $f$ is metrically differentiable at $x$, provided $f$ is metrically Gâteaux differentiable, and

$$(1) \quad \|f(z) - f(y)\| - MD(f, x)(z - y) = o(\|z - x\| + \|y - x\|), \quad \text{when } (y, z) \to (x, x).$$

We will also need the notion $w^*$-Gâteaux derivatives. It goes back to [11]. Let $X, Y$ be separable Banach spaces, $f : X \to Y^*$ be a mapping. For $v \in X$ we say that $w^d(f, x)(v)$ exists provided $w^d(f, x)(v) = w^* - \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$ exists. We say that $f$ is $w^*$-Gâteaux differentiable at $x$ provided $w^d(f, x)(v)$ exists for all $v \in X$, and $w^d(f, x)(\cdot)$ is a bounded linear map. We say that $f$ is $w^*$-Fréchet differentiable at $x$ provided $f$ is $w^*$-Gâteaux differentiable at $x$, and

$$(2) \quad w^* - \lim_{y \to x} \frac{f(y) - f(x) - w^d(f, x)(y - x)}{\|y - x\|} = 0.$$  

$w^*$-Gâteaux (and $w^*$-Fréchet) differentiability of pointwise Lipschitz mappings was studied in [9], where we introduced these notions.

Suppose that $X$ is a normed linear space. We say that $M \subset X$ is a Lipschitz hypersurface provided there exists $x^* \in X^*$, a Lipschitz function $f : Y = \{x^* = 0\} \to \mathbb{R}$, and $0 \neq v \subset X$ such that $M = \{y + f(y)v : y \in Y\}$. We say that $A \subset X$ can be covered by countably many Lipschitz hypersurfaces provided there exist Lipschitz hypersurfaces $L_n \subset X$ such that $A \subset \bigcup_n L_n$.

Preiss and Zajíček introduced the $\sigma$-directionally porous sets in [15]. Note that every Lipschitz hypersurface is a directionally porous set, and thus every set which can be covered by countably many Lipschitz hypersurfaces is $\sigma$-directionally porous.

For a recent survey of use of negligible sets in similar contexts, see [20].
Let \( X \) be a separable Banach space, \( A \subset X \), and \( 0 \neq u \in X \). We say that \( A \in \mathcal{A}(u) \) provided \( A \) is Borel and \( L^1(\{y \in \mathbb{R} : x + \lambda u \in A\}) = 0 \), for all \( x \in X \). For a sequence \( \{u_n\} \subset X \) we define

\[
\mathcal{A}(\{u_n\}) = \left\{ E \subset X : E = \bigcup_n E_n \text{ with } E_n \in \mathcal{A}(u_n) \right\}.
\]

Finally, we say that \( A \) is Aronszajn null provided \( A \) is Borel and for each complete sequence \( \{u_n\} \subset X \) we have \( A \in \mathcal{A}(\{u_n\}) \) (a sequence \( \{u_n\} \) is complete provided \( X = \bigcup_n \mathcal{A}(\{u_n\}) \)). For more information about Aronszajn null sets, see [2].

Let \( X \) be a separable Banach space. If \( \emptyset \neq v \in X \), then let \( \mathcal{A}(v, \varepsilon) \) be the system of all Borel sets \( B \subset X \) such that \( \{t : \varphi(t) \in B\} \) is Lebesgue null whenever \( \varphi : \mathbb{R} \to X \) is such that the function \( t \to \varphi(t) - tv \) has Lipschitz constant at most \( \varepsilon \), and \( \mathcal{A}(v) \) is the system of all sets \( B \) such that \( B = \bigcup_{k=1}^{\infty} B_k \), where \( B_k \in \mathcal{A}(v, \varepsilon_k) \) for some \( \varepsilon_k > 0 \).

We define \( \tilde{\mathcal{A}} \) (resp. \( \check{\mathcal{A}} \)) as the system of those \( B \subset X \) that can be, for all complete sequences \( \{v_n\} \subset X \) (resp. for some sequence \( \{v_n\} \subset X \)), written as \( B = \bigcup_{n=1}^{\infty} B_n \), where each \( B_n \) belongs to \( \tilde{\mathcal{A}}(v_n) \). These families of sets were defined in [15] as proper subfamilies of Aronszajn null sets.

We will also use the following notation: The symbol \( C_b(T) \) (resp. \( C(K) \)) denotes the space of all continuous bounded functions on an arbitrary topological space \( T \) (resp. of all continuous functions on an arbitrary compact space \( K \)) equipped with the supremum norm. When we deal with the spaces \( L^p(\mu) \), we allow an arbitrary measure \( \mu \).

The following lemma shows that the cone monotone mappings do inherit some properties of their dominating functions (see also Introduction).

**Lemma 2.4.** Let \( X, Y \) be a normed linear spaces, let \( K \subset X \) be a convex cone with non-empty interior. Suppose that \( H : X \to Y \) is \( K \)-monotone dominated with a dominating function \( h : X \to \mathbb{R} \).

(i) If \( h \) is continuous at \( x \in X \), then \( H \) is continuous at \( x \).

(ii) If \( h \) is pointwise-Lipschitz at \( x \in X \), then \( H \) is pointwise Lipschitz at \( x \).

**Proof.** We will only prove (ii), as the proof of (i) is similar. Without any loss of generality, we can assume that \( x = 0 \). Because \( K \) has a non-empty interior, choose \( k \in K \) such that

\[
k + y \in K \quad \text{whenever } y \in B(0, 1).
\]

Put \( C := 2\|k\| \). Then it is easy to see that each \( y \in X \) can be written as \( y = k_1 - k_2 \), where \( k_1, k_2 \in K \cap B(0, C\|y\|) \). For any \( \varepsilon > 0 \) find \( \delta > 0 \) such that \( |h(t) - h(0)| \leq (1 + \varepsilon)\text{Lip}(h, 0)||t|| \) for any \( t \in B(0, \delta) \). Suppose that \( y \in B(0, \delta/C) \). Then there exist \( k_1, k_2 \in K \cap B(0, \delta) \) such that \( y = k_1 - k_2 \), and \( \max(||k_1||, ||k_2||) \leq C\|y\| \).

Now,

\[
\|H(y) - H(0)\| = \|H(k_1 - k_2) - H(0)\|
\]

\[
\leq \|H(k_1 - k_2) - H(-k_2)\| + \|H(-k_2) - H(0)\|
\]

\[
\leq h(k_1 - k_2) - h(-k_2) + h(0) - h(-k_2),
\]

Therefore, \( H \) is pointwise Lipschitz at \( x \).
Suppose that function $h : Y \rightarrow X$. Without any loss of generality, we can assume that $f$ is pointwise-Lipschitz at $x$.

**Proof.**

**Lemma 2.5.** Let $X$ be a normed linear space, $K \subset X$ a closed convex cone with non-empty interior, and $f : X \rightarrow \mathbb{R}$ be $K$-monotone. If $f$ is Gâteaux differentiable at $x$, then $f$ is pointwise-Lipschitz at $x$.

**Proof.**

Let $k \in K$ be as in (3). Let $C > 0$ be such that each $y \in X$ can be written as $y = k_1 - k_2$ with $k_1, k_2 \in K \cap B(0, C \|y\|)$ (see the text after (3)). Without any loss of generality, we can assume that $x = 0$. Let $\delta > 0$ be such that $|f(\lambda k) - f(0)| \leq M|\lambda|$ whenever $|\lambda| < \delta$. Let $y \in B(0, \delta/(C\|k\|))$. There exist $k_1, k_2 \in K \cap B(0, \delta/\|k\|)$ such that $y = k_1 - k_2$, and $\max(\|k_1\|, \|k_2\|) \leq C\|y\|$. Since

$$|f(y) - f(0)| \leq f(k_1) - f(k_1 - k_2) + f(k_1) - f(0),$$

and

$$|f(y) - f(0)| \leq f(k_1 - k_2) - f(-k_2) + f(0) - f(-k_2),$$

by adding (6) to (7), we obtain

$$|f(y) - f(0)| \leq f(k_1 - k_2) - f(-k_2) + f(0) - f(-k_2).$$

By (3), we have that $-k_i + \|k_i\|k \in K$ for $i = 1, 2$. Thus

$$f(k_1) - f(0) \leq f(\|k_1\|k) - f(0) \leq f(C\|y\|k) - f(0) \leq CM\|y\|.$$  

Similarly,

$$-f(-k_2) + f(0) \leq -f(\|k_2\|k) + f(0) \leq -f(C\|y\|k) + f(0) \leq CM\|y\|.$$ 

Putting (8), (9), and (10) together, we obtain that $f$ is pointwise Lipschitz at $x$. 

The following auxiliary proposition tells us that the composition of a cone-to-cone monotone mapping with a $K$-monotone dominated mapping is again $K$-monotone dominated provided the intermediate cone is the same.

**Proposition 2.6.** Let $X$, $Y$ be ordered normed linear spaces, let $Z$ be a linear space. Suppose that $G : X \rightarrow Y$ is $(X_+, Y_+)$-monotone, and that $H : Y \rightarrow Z$ is $Y_+$-monotone dominated. Then $H \circ G$ is $X_+$-monotone dominated.

**Proof.** Without any loss of generality, we can assume that $G$ is $(X_+, Y_+)$-increasing. Suppose that $x \leq y$ for some $x, y \in X$. Then $G(x) \leq G(y)$, and thus $h(G(x)) \leq h(G(y))$. We have that $H \circ G$ is $X_+$-monotone dominated with the dominating function $h \circ G$. 

\[ \square \]
3. Relationships between the classes of cone monotone mappings

If \( Y \) is a normed linear space, and \( F : \mathbb{R} \to Y \), then it is easy to see that \( F \) is \( \mathbb{R}_+ \)-monotone dominated mapping if and only if \( F \) has locally finite variation. These mappings were studied before; see e.g. [17] for references.

The following theorem is proved in [17, Theorem 2.7]:

**Theorem 3.1.** Let \( I \subset \mathbb{R} \) be an open (or closed) interval, \( Y \) be a Banach lattice with the \( \sigma \)-Levi property, and \( f : I \to Y \) be a mapping having locally finite variation.

Then there exist nondecreasing mappings \( g, h : I \to Y \) such that \( f = g - h \) and \( g, h \) have locally finite variation. Moreover, the decomposition \( f = g - h \) is minimal in the class of all representations of \( f \) as the difference of nondecreasing mappings, i.e.: if \( f = g^* - h^* \) is such a representation then \( g(\beta) - g(\alpha) \leq g^*(\beta) - g^*(\alpha) \) for all \( \alpha < \beta, \alpha, \beta \in I \).

It has the following corollary:

**Corollary 3.2.** Let \( Y \) be a Banach lattice with the \( \sigma \)-Levi property, and \( f : \mathbb{R} \to Y \) be \( \mathbb{R}_+ \)-monotone dominated. Then \( f \) can be written as a difference of two \((\mathbb{R}_+, Y_+)\)-monotone mappings.

The following is a direct consequence of Corollary 3.2 and [17, Remark 2.8]; we formulate it for the reader’s convenience:

**Corollary 3.3.** Let \( Y \) be any dual Banach lattice (in particular \( L_p(\mu), 1 < p < \infty, \ell_\infty(\Gamma) \) for any \( \Gamma \), or \( L_\infty \) when \( \mu \) is \( \sigma \)-finite). If \( F : \mathbb{R} \to Y \) is \( \mathbb{R}_+ \)-monotone dominated, then \( F \) can be written as a difference of two \((\mathbb{R}_+, Y_+)\)-monotone mappings.

The following proposition gives a general condition when \( K \)-monotone dominated mappings between two ordered normed linear spaces can be written as differences of cone-to-cone monotone mappings.

**Proposition 3.4.** Let \( Y \) be an ordered normed linear space whose unit ball has an upper bound, let \( X \) be an ordered normed linear space. Then each for each \( X_+ \)-monotone dominated mapping \( F : X \to Y \) can be written as a difference of two \((X_+, Y_+)\)-monotone mappings.

**Proof.** Let \( e \in Y_+ \) be the upper bound of \( B_Y \), i.e. \( y \leq \|y\|e \) for each \( y \in Y \), and let \( f : X \to \mathbb{R} \) be the dominating function for \( F \). Then for \( x, y \in X \) with \( x \leq y \) we have \( F(x) - F(y) \leq \|F(y) - F(x)\|e \leq f(y)e - f(x)e \). If we put \( H = F + f \cdot e \), then \( H \) is \((X_+, Y_+)\)-monotone by the above, and \( F = H - f \cdot e \). This gives us the conclusion since \( f \cdot e \) is trivially \((X_+, Y_+)\)-monotone.

**Corollary 3.5.** Let \((Y, K)\) be any of the spaces \( C_b(T), L^\infty(\mu), C_0(T)^{**}, \) or \( L^\infty(\mu)^{**} \) with its canonical cone. Let \( X \) be an ordered normed linear space. Then every \( X_+ \)-monotone dominated mapping \( F : X \to Y \) can be written as a difference of two \((X_+, Y_+)\)-monotone mappings.

Now we will show that in some cases cone-to-cone monotone mappings are, in fact, \( K \)-monotone dominated.

**Proposition 3.6.** Let \( Y \) be an ordered normed linear space whose cone \( Y_+ \) is well-based, let \( X \) be an ordered linear space. Then every \((X_+, Y_+)\)-monotone mapping \( F : X \to Y \) is \( X_+ \)-monotone dominated.
Proposition 4.3. To see that every Lipschitz hypersurface is a directionally porous set.

Lemma 4.1 (resp. Remark 4.2), we can improve [6, Proposition 12], since it is easy to see that every Lipschitz hypersurface is a directionally porous set.

Proof. Without any loss of generality, we can assume that $F$ is $(X_+, Y_+)$-increasing. Since $Y_+$ is well-based, there exists $\varphi \in Y^*$ such that $\varphi(u) \geq \|u\|$ for each $u \in Y_+$. This shows that $Id_Y : Y \to Y$ is $Y_+$-monotone dominated. Thus Proposition 2.6 implies that $F = Id_Y \circ F$ is $X_+$-monotone dominated. □

Corollary 3.7. Let $Y$ be any of the spaces $L_1(\mu)$, $L^\infty(\mu)^*$, $C_b(T)^*$ with its canonical cone. Let $X$ be an ordered normed linear space. Then each $(X_+, Y_+)$-monotone mapping is $X_+$-monotone dominated.

Proof. In the proof of [17, Corollary 3.2] it is shown that the canonical cone of each of the spaces is well-based. Thus Proposition 3.6 gives the conclusion. □

4. CONTINUITY OF CONE MONOTONE MAPPINGS

Let $(X, \| \cdot \|)$ be a normed linear space. We say that $\| \cdot \|$ is LUR at $x \in S_X$ provided $x_n \to x$ whenever $\|x_n\| = 1$, and $\|x_n + x\| \to 2$. We say that $\| \cdot \|$ is LUR (or locally uniformly rotund) provided it is LUR at each point $x \in S_X$. For more information about rotundity and renormings, see [8].

We will need the following renorming result, which is proved e.g. in [8, Lemma II.8.1].

Lemma 4.1. Let $Y$ be a subspace of a Banach space $X$ and let $| \cdot |$ be an equivalent norm on $Y$. Then $| \cdot |$ can be extended to an equivalent norm on $X$. If $| \cdot |$ is an equivalent locally uniformly rotund norm on $Y$ then $| \cdot |$ can be extended to an equivalent norm on $X$ which is locally uniformly rotund at each point of $Y$.

Remark 4.2. Let $(X, \| \cdot \|_X)$ be a normed linear space, $y \in S_X$. By $\hat{X}$ denote the Banach space completion of $X$ (i.e. $\hat{X} = \overline{X}$, $\|x\|_{\hat{X}} = \|x\|_X$ for each $x \in X$, and $\hat{X}$ is a Banach space). Then we can apply Lemma 4.1 to $\hat{X}$ and $Y = \text{span}\{y\}$ to obtain an equivalent norm $\| \cdot \|_1$ on $\hat{X}$ which is LUR at each point of $Y$, and with the property that $\|\lambda y\|_1 = \|\lambda y\|_X$ for each $\lambda \in \mathbb{R}$. It follows that $\| \cdot \|_1$ (restricted to $X$) is LUR at each point of $Y$ on $X$. Thus Lemma 4.1 also holds for any normed linear space $X$ and $Y = \text{span}\{y\}$, where $y \in S_X$.

Using similar reasoning as in Zajíček [18, Lemma 1] together with a renorming Lemma 4.1 (resp. Remark 4.2), we can improve [6, Proposition 12], since it is easy to see that every Lipschitz hypersurface is a directionally porous set.

Proposition 4.3. Let $X$ be a normed linear space. Assume that $K \subset X$ is a convex cone with non-empty interior and $f : X \to \mathbb{R}$ is $K$-monotone. Then $D := \{x \in X : f$ is discontinuous at $x\}$ can be covered by countably many Lipschitz hypersurfaces.

Proof. Following the proof of [6, Proposition 12] we have $D = \{x \in X : f(x) < \mathcal{L}(x)\}$. Write $S_1 := \{x \in X : f(x) < f(x)\}$, and $S_2 := \{x \in X : f(x) < \mathcal{L}(x)\}$. We will only prove that $S_2$ can be covered by countably many Lipschitz hypersurfaces (the proof for $S_1$ is similar). Write $S_2 = \bigcup_{p \in \mathbb{Q}} D_p$, where $D_p := \{x \in X : f(x) < p < \mathcal{L}(x)\}$. As in the proof of [6, Proposition 12], we see that

$$(11) \quad \{x - \text{int}(K)\} \cap D_p = \emptyset,$$

for each $x \in D_p$.

We will prove that each $D_p$ is contained in a Lipschitz hypersurface. Fix $p \in \mathbb{Q}$. Choose $k \in \text{int}(K)$. We can assume that $\|k\| = 1$. By Remark 4.2, find an equivalent norm $\| \cdot \|_1$ on $X$ such that $\| \cdot \|_1$ is LUR at $k$ (and with $\|k\|_1 = 1$). From
now on, we consider $X$ equipped with $\|\cdot\|_1$. Then $B(k,\varepsilon) \subset K$ for some $\varepsilon > 0$. Choose $x^* \in X^* = (X,\|\cdot\|_1)^*$ such that $\|x^*\|^* = (x^*,k) = 1$. Since $\|\cdot\|_1$ is LUR at $k$, we have that if $x_n \in X$ are such that $\|k + x_n\|_1 \to 2$, and $\|x_n\|_1 = 1$, then $x_n \to k$. Let $0 < \alpha < 1$. Then $K_\alpha = \{x \in X : \alpha\|x\|_1 < x^*(x)\}$ is a convex cone with non-empty interior. If $x \in K_\alpha \cap S_X$, then $1 + \alpha = x^*(k + x) \leq \|k + x\|_1$, and by the LUR property of $\|\cdot\|_1$ at $k$, we see that there exists $0 < \alpha < 1$ such that $K_\alpha \cap S_X \subset B(k,\varepsilon/2) \subset \text{int}(K)$ (and thus $K_\alpha \subset \text{int}(K)$).

By (11), we have $[x - \text{int}(K)] \cap D_p = \emptyset$, for any $x \in D_p$, and thus $[x - K_\alpha] \cap D_p = \emptyset$, for any $x \in D_p$. Let $Y := \{x^* = 0\}$. We have that if $x, y \in D_p$, then $|x^*(x - y)| \leq \alpha\|x - y\|_1$. This implies that if $x, y \in D_p$, then

$$\begin{align*}
(1 - \alpha)|x^*(y) - x^*(x)| &\leq (1 - \alpha)|y - x|_1 \\
&\leq \|x - y\|_1 - \alpha|x^*(y - x)| \\
&\leq \|x - y\|_1 - \|x^*(y - x)\|_1 \\
&\leq \|x - y - x^*(x - y)\|_1.
\end{align*}$$

(12)

It is easy to see that the mapping $\pi : D_p \to Y$ defined as $\pi(x) := x - x^*(x)k$ is one-to-one. Suppose that $x', y' \in \pi(D_p)$. Then there exist $x, y \in D_p$ such that $\pi(x) = x'$, and $\pi(y) = y'$. The inequality (12) shows that the function $f : \pi(D_p) \to \mathbb{R}$ defined as $f(x') = x^*(x)$, where $x$ is the (unique) vector in $D_p$ such that $\pi(x) = x'$, is Lipschitz. Thus $D_p = \{y + f(y)k : y \in \pi(D_p)\}$. Since $f : \pi(D_p) \to \mathbb{R}$ is Lipschitz, it can be extended to a Lipschitz function (call it again $f$) on $Y$. Define $L := \{y + f(y)k : y \in Y\}$. Then $L$ is a Lipschitz hypersurface, and $D_p \subset L$. □

**Theorem 4.4.** Suppose that $X$ is an ordered normed linear space such that $X_+$ is convex with non-empty interior, $(Y,d)$ is a metric space, and $F : X \to Y$ is $X_+$-monotone dominated. Then the set $D$ of points where $F$ is not continuous can be covered by countably many Lipschitz hypersurfaces.

**Proof.** We can isometrically embed $Y$ into $\ell_\infty(\Gamma)$ for some $\Gamma$, and thus without any loss of generality, we can assume that $Y$ is a normed linear space. Let $f : X \to \mathbb{R}$ be the dominating function for $F$. By Proposition 4.3, the set of points $D' \subset X$ where $f$ is discontinuous can be covered by countably many Lipschitz hypersurfaces. Lemma 2.4(i) implies that $D \subset D'$.

**Corollary 4.5.** Suppose that $X, Y$ are ordered normed linear spaces such that $X_+$ is convex with non-empty interior, and $Y_+$ is well-based. If $F : X \to Y$ is $(X_+, Y_+)$-monotone, then $D := \{x \in X : F$ is discontinuous at $x\}$ can be covered by countably many Lipschitz hypersurfaces.

**Proof.** Proposition 3.6 shows that $F$ is $X_+$-monotone dominated since $Y_+$ is well-based. The conclusion now follows from Theorem 4.4. □

5. Differentiability of cone monotone mappings

We have the following theorem concerning Gâteaux differentiability of monotone dominated mappings. It is a corollary of a general version of Stepanoff’s theorem which was proved in [10] (as a strengthening of a result due to Bongiorno [3]).

**Theorem 5.1.** Suppose that $X$ is a separable Banach space, $K \subset X$ is a convex cone with non-empty interior, $Y$ is a Banach space with RNP, and $F : X \to Y$ is $K$-monotone dominated. Then there exists a set $A \in \mathcal{C}$ such that $F$ is Gâteaux differentiable at all $x \in X \setminus A$.
Proof. Let \( f : X \to \mathbb{R} \) be a dominating function for \( F \). Then by [10, Theorem 15], there exists a set \( A_1 \in \mathcal{C} \) such that \( f \) is Gâteaux differentiable at all \( x \in X \setminus A_1 \). Lemma 2.5 implies that \( h \) is pointwise-Lipschitz at each \( x \in X \setminus A_1 \). By Lemma 2.4 it follows that \( F \) is pointwise-Lipschitz at all \( x \in X \setminus A_1 \). By [10, Theorem 10] there exists a set \( A_2 \in \mathcal{A} \) such that \( F \) is Gâteaux differentiable at all \( x \in X \setminus (A_1 \cup A_2) \). Putting \( A := A_1 \cup A_2 \) finishes the proof of the theorem.

Remark 5.2. Using the methods of [10], we can actually obtain a version of [10, Theorem 10] where the notion of “Gâteaux differentiability” is replaced by “Hadamard differentiability” and thus Theorem 5.1 also holds for this notion. We do not enter this subject here.

If \( X = \mathbb{R}^n \) in the previous theorem, then we can conclude that \( F \) is even almost everywhere Fréchet differentiable (the proof is analogous to the proof of Theorem 5.1; it uses [9, Theorem 2.10] and the fact that \( K \)-monotone functions on \( \mathbb{R}^n \) are (Fréchet) differentiable almost everywhere; see e.g. [4]).

Using the same reasoning as in [10, Corollary 11], we can obtain the following corollary. It is a consequence of a recent result of Zajíček [19] who proved that the sets in \( \mathcal{C} \) (in a separable Banach space) are \( \Gamma \)-null (in the sense of [14]) and the results of [14].

Corollary 5.3. Let \( X \) be a Banach space such that \( X^* \) is separable, \( K \subset X \) be a convex cone with non-empty interior, let \( Y \) be a Banach space with RNP, \( f : X \to Y \) be \( K \)-monotone dominated, \( g : X \to \mathbb{R} \) continuous convex. Then there exists \( x \in X \) such that \( f \) is Gâteaux differentiable at \( x \) and \( g \) is Fréchet differentiable at \( x \).

The following corollary tells us that if the cone in the target space is not too big, then cone-to-cone monotone mappings are Gâteaux differentiable almost everywhere.

Corollary 5.4. Suppose that \( X \) be an ordered separable Banach space such that \( X_+ \) is convex with non-empty interior, \( Y \) be an ordered Banach space with RNP such that \( Y_+ \) is well-based, and \( F : X \to Y \) be \((X_+, Y_+)\)-monotone. Then there exists a set \( A \in \mathcal{C} \) such that \( F \) is Gâteaux differentiable at all \( x \in X \setminus A \).

Proof. Proposition 3.6 implies that \( F \) is \( X_+ \)-monotone dominated. Theorem 5.1 now implies that there exists a set \( A \in \mathcal{C} \) such that \( F \) is Gâteaux differentiable at all \( x \in X \setminus A \). □

The following example shows that if the cone in the target space is not properly positioned, then Corollary 5.4 does not hold.

Example 5.5. Let \( K \) be the non-negative cone in \( \ell_2 \) (i.e. \( x = (x_n)_n \in K \) iff \( x_n \geq 0 \) for all \( n \in \mathbb{N} \)). Then there exists a mapping \( f : \mathbb{R} \to \ell_2 \), which is \((\mathbb{R}_+, K)\)-monotone, but nowhere differentiable.

Remark 5.6. In our example, the domain of \( f \) is \( \mathbb{R} \) and thus the notions of Gâteaux and Fréchet differentiability coincide.

Proof. Let \((e_n)_n\) be the canonical orthonormal basis of \( \ell_2 \). We will use the following easy observation: if \( f : \mathbb{R} \to \ell_2 \) is differentiable at \( x \), then each \( f_j \) is differentiable at \( x \) (where \( f(x) = \sum_j f_j(x) \cdot e_j \)), and \( \max_j |f_j(x)| \leq |f'(x)| \).

It is easy to see that there exist sequences \((a_n)_{n \in \mathbb{N}} \subset \mathbb{R} \) and \((m_n)_{n \in \mathbb{N}} \subset \mathbb{N} \) such that
• $a_n > 0$ for all $n \in \mathbb{N}$,
• $(m_n)_{n}$ is increasing,
• $\sum_n a_n^2 < \infty$,
• $\sum_{j=m_n}^{m_{n+1}-1} a_j = 2n$ (for $n = 1, 2, \ldots$).

Find $k_n > 0$ such that $\sum_n (k_n \cdot a_n)^2 < \infty$ and $\lim_{n \to \infty} k_n = \infty$. Define $f_j : \mathbb{R} \to \mathbb{R}$ as

$$f_j(x) = k_j \cdot \max \left( 0, \min \left( a_j, x + n - \sum_{k=m_n}^{j-1} a_k \right) \right),$$

for $j = m_n + 1, \ldots, m_{n+1} - 1$, and $f_j(x) = k_j \cdot \max(0, \min(x + n, a_j))$ for $j = m_n$. Then we have $0 \leq f_j(x) \leq k_j \cdot a_j$ for each $x \in \mathbb{R}$. For $x \in \mathbb{R}$ define $f(x) := \sum_j f_j(x) \cdot e_j$. It is easy to see that $f$ is well-defined, continuous, and $(\mathbb{R}_+, K)$-increasing (since each $f_j$ is increasing). Assume that $f$ is differentiable at some point $x \in \mathbb{R}$. Take any $n \in \mathbb{N}$ such that $|x| < n$. If $x = -n + \sum_{j=m_n}^{j-1} a_j$ for some $k \in \{m_n, \ldots, m_{n+1} - 2\}$, then $f_k$ is not differentiable at $x$ (and thus $f$ cannot be differentiable at $x$). Thus $x \in (-n, -n + a_{m_n})$ or $x \in (-n + \sum_{j=m_n}^{j+1} a_j, -n + \sum_{j=m_n}^{j+1} a_j)$ for some $l \in \{m_n, \ldots, m_{n+1} - 2\}$. But then $f_{m_n}'(x) = k_{m_n}$ or $f_l'(x) = k_l$, and since $k_j \to \infty$, we have that $f$ is not differentiable at $x$.

6. Metric differentiability of cone monotone mappings

**Theorem 6.1.** Suppose that $K \subset \mathbb{R}^n$ is a convex cone with non-empty interior. Let $(M, \rho)$ be a metric space, and let $F : \mathbb{R}^n \to M$ be $K$-monotone dominated. Then for almost every $x \in \mathbb{R}^n$ we have that $F$ is metrically differentiable at $x$.

If $M = Y^*$, where $Y$ is a separable Banach space, then for almost every $x \in \mathbb{R}^n$ we have that $F$ is metrically differentiable at $x$, $F$ is $w^*$-Fréchet differentiable at $x$, and $MD(F, x)(w) = \|wd(F, x)(w)\|$ for all $w \in \mathbb{R}^n$.

**Proof.** If $M$ is a metric space, then we can embed $(M, \rho)$ isometrically into $\ell_\infty(G)$ for some $G$, and so we can assume that $F : \mathbb{R}^n \to \ell_\infty(G)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a dominating function for $F$. Then [4, Theorem 6] implies that $f$ is differentiable (and thus pointwise Lipschitz) at all $x \in \mathbb{R}^n \setminus A$ with $\mathcal{L}^n(A) = 0$. Lemma 2.4 implies that $F$ is pointwise-Lipschitz at all $x \in \mathbb{R}^n \setminus A$. [9, Theorem 2.6] implies that there exists $B \subset \mathbb{R}^n$ with $\mathcal{L}^n(B) = 0$ such that $F$ is metrically differentiable at all $x \in \mathbb{R}^n \setminus (A \cup B)$.

If $M = Y^*$, then the result follows similarly from [9, Corollary 2.8].

**Theorem 6.2.** Let $X$ be an separable Banach space, $K \subset X$ be a closed convex cone with non-empty interior, and let $(M, \rho)$ be a metric space. Then for every $K$-monotone dominated $F : X \to M$ there exists an Aronszajn null set $A \subset X$ such that $F$ is metrically Gâteaux differentiable at all points of $X \setminus A$.

**Proof.** First, we can embed $(M, \rho)$ isometrically into some $\ell_\infty(G)$, and so we can assume that $F : \mathbb{R}^n \to \ell_\infty(G)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a dominating function for $F$. Then [6, Theorem 9] together with [4, Proposition 4] implies that there exists an Aronszajn null set $A_1 \subset X$ such that $f$ is Gâteaux differentiable at all $x \in X \setminus A_1$. Lemma 2.5 implies that $f$ is pointwise-Lipschitz at all $x \in X \setminus A_1$, and thus Lemma 2.4 implies that $F$ is pointwise-Lipschitz at all $x \in X \setminus A_1$. [9, Theorem 5.4] implies that there exists an Aronszajn null set $A_2 \subset X$ such that $F$ is metrically Gâteaux differentiable at all $x \in X \setminus (A_1 \cup A_2)$. 


We have the following corollary, which follows from Theorem 6.2 in the same way as Corollary 5.4 follows from Theorem 5.1.

**Corollary 6.3.** Let $X$ be an separable Banach space, $Y$ be an ordered normed linear space such that $Y_+$ is well-based, $K \subset X$ be a closed convex cone with non-empty interior. Then for every $(K,Y_+)$-monotone mapping $F : X \to Y$ there exists an Aronszajn null set $A \subset X$ such that $F$ is metrically Gâteaux differentiable at all points of $X \setminus A$.

Using [9, Theorem 5.3] we can prove the following result (we leave the details to the reader as the proof is similar to the proof of Theorem 6.2).

**Theorem 6.4.** Let $X$ be an separable Banach space, $K \subset X$ be a closed convex cone with non-empty interior, and let $Y$ be a separable Banach space. Then for every $K$-monotone dominated $F : X \to Y^*$ there exists an Aronszajn null set $A \subset X$ such that for each $x \in X \setminus A$ we have that $F$ is metrically Gâteaux differentiable at $x$, $F$ is $w^*$-Gâteaux differentiable at $x$, and $MD(F,x)(w) = \|w d(F,x)(w)\|$ for all $w \in X$.

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