Mathematical irrational numbers not so physically irrational

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Abstract

We investigate the topological structure of the decimal expansions of the three famous naturally occurring irrational numbers, π, e, and golden ratio, by explicitly calculating the diversity of the pair distributions of the ten digits ranging from 0 to 9. And we find that there is a universal two-phase behavior, which collapses into a single curve with a power law phenomenon. We further reveal that the two-phase behavior is closely related to general aspects of phase transitions in physical systems. It is then numerically shown that such characteristics originate from an intrinsic property of genuine random distribution of the digits in decimal expansions. Thus, mathematical irrational numbers are not so physically irrational as long as they have such an intrinsic property.

Keywords: two-phase behavior; irrational numbers; phase transition

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Diversity is an all-pervasive characteristic of the world. Literally, diversity in irrational numbers makes them so-called “irrational”. In this work, we investigate irrational numbers since numbers represent the reality. It is known that an irrational number is a number which cannot be expressed as a fraction of two integers, and that the decimal expansion of irrational numbers never repeats or terminates, unlike rational numbers. Here we focus on three naturally occurring irrational numbers, namely, $\pi$, $e$, and golden ratio. The $\pi$ is the ratio of a circle’s circumference to its diameter in Euclidean geometry. The $e$ is the unique irrational number such that the value of the derivative of $f(x) = e^x$ at the point $x = 0$ is exactly 1. Two quantities are seen to be in the golden ratio if the ratio between the sum of the two quantities and the larger one is the same as the ratio between the larger one and the smaller. The golden ratio is expressed as $(1 + \sqrt{5})/2$. The three naturally occurring irrational numbers have abundant of uses in physics, mathematics, and engineering. However, our question is if there is common interesting physical senses behind the three naturally occurring irrational numbers. Our answer is true, as to be addressed in this work.

Pair distributions are widely used, both experimentally and theoretically, to treat physical problems, as a major descriptor for various microstructures [1, 2, 3, 4, 5], in which a pair distribution function describes the density of inter-atomic or inter-particulate distances. We shall investigate pair distributions of the ten digits (namely, 0, 1, \ldots, 9) in the decimal expansions of the three irrational numbers, with a focus on their diversity, so that one could get insight into the connection between statistical physics and topological structures of numbers.

The information entropy of a set of pair distributions provides a measure of the diversity of the set. The greater the diversity of the pair distributions in a set, the greater the entropy, while a set with regular patterns has a small value for the entropy. In general, information entropy has the following characters [6]: (i) Changing the value of one of the probabilities by a very small amount should only change the entropy by a small amount (continuity); (ii) The result should keep unchanged if the probabilities are re-ordered (symmetry); (iii) If all the probabilities are equally likely, information entropy should be maximal (maximum); (iv) Adding or removing an event with probability zero does not contribute to the entropy; and (v) The amount of entropy should be the same independently of how the process is regarded as being divided into parts (additivity). Here (v) characterizes the information entropy of a system with sub-systems, and it demands that the information entropy of a system can be calculated from the information entropy of its sub-systems.
if we know how the sub-systems interact with each other.

For the first $10^4$ digits in the decimal expansions of the three irrational numbers, $\pi$, $e$ and golden ratio, we calculate their information entropy $H(N)$ of the pair distributions according to

$$H(N) = -\sum_{i=0}^{9} \sum_{j} (p_{i,j} \log p_{i,j}),$$

with $p_{i,j} = N_{i,j} / \sum_j N_{i,j}$, where $N_{i,j}$ denotes the number of pairs with step size $j$ (which is the distance between every pair of two $i$’s) for the digit $i$ ranging from 0 to 9.

Figure 1 displays the phase diagram for the existence of two distinct phases for the three naturally occurring irrational numbers, $\pi$, $e$, and golden ratio, by plotting order parameter $H$ (information entropy of the pair distributions of the ten digits, ranging from 0 to 9) as a function of the number $N$ of the first digits in the decimal expansions. The inset of Fig. 1 shows a log-log plot of information entropy $H$ versus $N$ for the same three irrational numbers, and a power-law relation appears, as indicated by a straight line. Such a single-curve collapse for the three irrational numbers suggests that they are governed by the same rule due to the same degree of disorder, which is, on the other hand, implied by Fig. 2. In fact, the two-phase behavior shown for the ten digits holds for each digit as well. In this work, we focus on the two-phase behavior for the ten digits as a whole.

We analyze the diversity of the pair distributions of the digits ranging from 0 to 9, which is quantitatively expressed by their information entropy $H$ [6], and discover not only a single-curve collapse but also the surprising existence of a sudden change region (Region B), as shown in Fig. 1. For $N$ within Region A, the value of $H$ is roughly zero; we interpret this as a uniformity phase in which the diversity of the pair distributions of the digits does not predominate. For Region C, $H$ increases significantly; we interpret this as an out-of-uniformity phase in which the diversity dominates and power-law phenomenon appears.

While phase transitions have been demonstrated to exist in various mathematical systems [7], our findings for the irrational-number problem are identical to what is known to occur in all phase-transition phenomena in physical systems. The distinguishing characteristic of a phase transition is an abrupt sudden change in one or more physical properties at a critical threshold $\kappa_c$ of some control parameter $\kappa$. The change in behavior at $\kappa_c$ can be quantified by an order parameter $\phi(\kappa)$. For the irrational-number problem, we find that the order parameter $\phi(\kappa)$ is given by the infor-
formation entropy $H(N)$. Region B bridges Regions A and C, and denotes a sudden change region, indicating a sudden increase of the diversity of the pair distributions. Next, we interpret these two irrational-number phases according to Regions A and C.

In Region A, the entropy is roughly zero; we interpret this to be the irrational-number uniformity phase, because the diversity of the pair distributions of the ten digits does not predominate. In the uniformity phase, there is almost no diversity, and information entropies fluctuate around their uniformity values, suggesting that most of the pair distributions do not have diversity, but behave as uniformity (or none).

In Region C, the entropy is large and increases step by step. We interpret this to be the out-of-uniformity phase, because increasing entropy suggests increasing diversity. So, in the out-of-uniformity phase, the prevalent uniformity has changed, and the diversity is now being driven to new degree, which is consistent with the fact that more pair distributions of the ten digits come to appear with longer step sizes as $N$ increases. A power law is any polynomial relationship that exhibits the property of scale invariance, and has been explicitly used to characterize a staggering number of natural patterns [8, 9, 10]. The observation of a power-law relation in the data for $\pi$, $e$, and golden ratio, points to a specific kind of mechanisms that underly the irrational-number phenomenon in question, and indicates a deep connection between the irrational numbers. Such a connection originates from the same degree of disorder, as shown in Fig. 2. Figure 2 presents the occurrence percentage of the distance between two consecutive digits in the decimal expansions of the three irrational numbers ($\pi$, $e$, and golden ratio) and a set of genuine random numbers. (Here “genuine random numbers” mean those random numbers or quasi-random numbers that have a very high degree of randomness. Throughout this work, the phrase “random numbers” is simply used to indicate “genuine random numbers”. Similarly, in this work “genuine random distribution” is used to represent the distribution with a very high degree of randomness.) These random numbers were generated by the commercial software Mathematica. The symbol lines for the three irrational numbers are almost overlapped, which are further nearly overlapped with that of the set of random numbers. This shows that the three irrational numbers possess the same degree of disorder as a group of random numbers. In other words, the above-mentioned characteristics related to the topological structure of the decimal expansions of $\pi$, $e$, and golden ratio actually originate from an intrinsic property of genuine random distribution of the digits in the decimal
expansions.

Our results suggest that there is a link between a mathematical system with many digits (the irrational number) and the ubiquitous phenomenon of phase transitions that occur in physical systems with many units.

We hope that our work will stimulate further studies of number physics. Here we have revealed a universal two-phase behavior for the three famous naturally occurring irrational numbers, $\pi$, $e$, and golden ratio. We should claim that the unique results obtained herein for the three irrational numbers (as shown in Fig. 1) also hold for many other irrational numbers like $\sqrt{2}$ and $\sqrt{3}$. It is also worth mentioning that the results obtained from Fig. 1 (e.g., the single-curve collapse) do not work for some other irrational numbers, e.g., $\sqrt{2.1}$. So far, there is no efficient method to sort such irrational numbers clearly, except for investigating their decimal expansions one by one. Nevertheless, we could safely conclude that our findings are a universal behavior for numerous irrational numbers as long as they have an intrinsic property of genuine random distribution of the digits in decimal expansions, and raise the possibility that the topological structure of irrational numbers are related to general aspects of phase transitions in physical systems. In this sense, such mathematical irrational numbers seem to be not so physically irrational.
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**Figure Captions**

Fig. 1. The diagram representing the existence of two distinct phases for the three naturally occurring irrational numbers, $\pi$, $e$, and golden ratio, by plotting order parameter $H$ (information entropy of the pair distributions of the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) as a function of the number $N$ of the first digits in the decimal expansions. Region A: A uniformity phase; Region B: An abrupt sudden change region; Region C: An out-of-uniformity phase. Inset: Log-log plot of information entropy $H$ versus $N$ for the same three irrational numbers, and a power-law relation is shown as indicated by the straight line. The symbol lines for the three irrational numbers are almost overlapped, as investigated for $N$ up to $10^4$. Such a single-curve collapse suggests that they are governed by the same rule, due to the same degree of disorder as presented by Fig. 2.

Fig. 2. The occurrence percentage of the distance between two consecutive digits in the decimal expansions of the three irrational numbers ($\pi$, $e$, and golden ratio) and a set of genuine random numbers (which were generated by Mathematica), for $N = 10^4$. The four symbol lines are almost overlapped. Such a single-curve collapse suggests that the three irrational numbers possess the same degree of disorder as the set of random numbers.
FIG. 1:
FIG. 2: