Bifurcation of gap solitons through catastrophe theory

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In the theory of optical gap solitons, slowly-moving finite-amplitude Lorentzian solutions are found to mediate the transition from bright to coexistent dark-antidark solitary wave pairs when the laser frequency is detuned out of the proper edge of a dynamical photonic bandgap. Catastrophe theory is applied to give a geometrical description of this strongly asymmetrical ‘morphing’ process.

The confinement of optical radiation in periodic media (gratings) with nonlinear response occurs in the form of gap solitons (GS), or more properly solitary waves, as first predicted by Chen and Mills [1], and studied extensively afterwards [2–6]. In Kerr media, the prototype model for GS is the following system of hyperbolic PDEs with Hamiltonian (conservative) structure [7,8], which couples the forward $u_+(z,t)$ and backward $u_-(z,t)$ propagating envelopes at Bragg carrier frequency $\omega_B$

$$i(\partial_t + \partial_z) u_+ + u_+ + (X \ |u_+|^2 + S \ |u_-|^2) u_+ = 0,$$

$$i(\partial_t - \partial_z) u_- + u_+ + (X \ |u_-|^2 + S \ |u_+|^2) u_- = 0.$$  \hspace{1cm} (1)

Equations (1) have been conveniently written in usual dimensionless units $z = \Gamma Z$ and $t = \Gamma V_B T$, where $Z$ and $T$ are the real-world propagation distance and time, $\Gamma$ is the Bragg coupling coefficient, and $V_B$ is the group-velocity at Bragg frequency. Moreover $S$ and $X$ are coefficients which specify the relative weight of self- and cross-phase modulation, and $u_\pm$ are proportional to real-world envelope amplitudes. Equations (1) are usually analyzed with $S = 1$ and $X = 2$ (one of the two coefficients can be always set to have unitary modulus by a suitable rescaling of the field amplitudes) which describes scalar mode coupling, e.g., in optical fiber gratings [2–5]. Conversely we find convenient to leave them as generic coefficients in order to describe both the cases of focusing $(S, X > 0 \ [2–6])$ and defocusing $(S, X < 0 \ [3])$ nonlinearity, as well as the two limit cases $X = 0$ and $S = 0$, which arise, e.g., when the cubic nonlinearity originates from cascading in quadratic media [3,4,2]. Cascading adds improved flexibility since it permits to control the sign of the effective Kerr nonlinearity by tuning the wavevector mismatch. We also emphasize that, in the case $S = 0$, Eqs. (1) reduce to the integrable (by means of the inverse scattering method) massive Thirring model, and hence the localized waves are strictly speaking solitons.

Despite the fact that GS arise in a variety of physical settings and models, the importance of Eqs. (1) is threefold: (i) they describe with reasonable accuracy optical GS experimentally investigated in fiber Bragg gratings [3] and in corrugated GaAs waveguides [8]; (ii) though not integrable by inverse scattering method (except for the massive Thirring case), the model allows to construct the whole family of solitary waves; (iii) Eqs. (1) have allowed to assess the occurrence of peculiar effects such as the onset of oscillatory instabilities [2,4], ultimately related with the absence of material dispersion (i.e., second-order derivatives) which distinguishes Eqs. (1) from other soliton-bearing dispersive models (e.g., those of the nonlinear Schrödinger type). GS of Eqs. (1) have been studied for more than a decade, and both bright [2,3] and dark [4] solutions were reported. Yet, the existence of such GS solutions and their bifurcations (how they change qualitatively against changes of parameters) were never investigated to full extent. Here we unveil the bifurcation structure of GS, restricting ourselves to subluminal solutions for physical reasons. We show that moving Lorentzian GS mark the transition between in-gap bright GS and dark-antidark GS pairs which coexist either below or above (depending on the focusing or defocusing nature of the nonlinearity, respectively) the edge of a suitably defined dynamical gap.

Following the notation of Ref. [3], the entire family of solitary waves of Eqs. (1) can be characterized by seeking solutions in the following form

$$u_+(z,t) = U_+ \sqrt{\eta(\zeta)} \ \exp\{-i\Delta t + i [\beta \zeta + \phi_+(\zeta)]\}$$

$$u_-(z,t) = U_- \sqrt{\eta(\zeta)} \ \exp\{-i\Delta t + i [\beta \zeta + \phi_-(\zeta)]\}.$$  \hspace{1cm} (2)
where $\beta \equiv \gamma v \Delta$ plays the role of GS propagation constant, and the intensity $\eta$ and the chirp (nonlinear phase) $\phi_{\pm}$ profiles depend on $\zeta \equiv \gamma (z - vt)$, with $\gamma = (1 - v^2)^{-1/2}$ being the (subluminal, $|v| < 1$) Lorentz factor. Furthermore $U_+ = A \sqrt{1 + \frac{v}{c}}$ and $U_- = -sA \sqrt{1 + \frac{v}{c}}$ account for the velocity induced asymmetry between the foward and backward modes, whereas $s \equiv \text{sign} \left[ X(1 - v^2) + S(1 + v^2) \right]$ is the sign of the nonlinearity which appears only through the common overall coefficient $A = \gamma^{-1} \left| 2X (1 - v^2) + 2S (1 + v^2) \right|^{-1/2}$. Importantly, the entire family of GS is characterized by two independent parameters, namely the normalized detuning $\Delta$ and soliton velocity $v$ (corresponding, in real-world units, to $\delta \omega = \Gamma V_B \Delta$ and $V = V_B v$), which are related to rotational and translational group symmetries of Eqs. (1), respectively [10]. Note that $\Delta = 0$ corresponds to the Bragg frequency, and $v = 0$ yields still GS, i.e. confinement of light with zero velocity in the lab frame.

By direct substitution of Eqs. (2) into Eqs. (1), it is readily verified that the intensity $\eta$ and overall phase $\theta \equiv \phi_+ - \phi_-$ obey the following one-dimensional (thus integrable) Hamiltonian system of equations

$$\begin{align*}
\dot{\eta} &= 2\eta \sin \theta = -\frac{\partial H}{\partial \theta}, \\
\dot{\theta} &= 2\delta + 2 \cos \theta - \eta = \frac{\partial H}{\partial \eta},
\end{align*}$$

(3)

where the dot stands for $d/d\zeta$, $H = H(\eta, \theta) = 2\eta \cos \theta + 2\eta \eta - \eta^2/2$ is the reduced conservative ($\dot{H} = 0$) Hamiltonian, which now depends on the single parameter $\delta \equiv \gamma \Delta$. In Eq. (1) and hereafter we implicitly assume to deal with the self-defocuser nonlinearity $s = -1$ originally considered in Ref. [1], including the case of vanishing $S$ or $X$. The results can be readily extended to the self-focusing case $s = 1$ with the substitution $\theta \to 2\pi - \theta$ and $\delta \to -\delta$. Importantly, the latter condition means that the role of frequencies below and above Bragg frequency must be simply interchanged.

The reduced system (3) permits to find the solitary waves of Eqs. (1) as the separatrix trajectories which are homoclinic to (i.e., emanate from and return to) the unstable fixed points $\eta_s, \theta_s$ of Eqs. (3). Given the constraint $\eta > 0$ in Eq. (3), these are easily found to be of two kinds: (i) $(\eta_s, \cos \theta_s) = (0, -\delta)$ for $\delta^2 < 1$, which is associated with solutions of the bright type since $\eta(\zeta = \pm \infty) = \eta_s = 0$; (ii) $(\eta_s, \theta_s) = (2(\delta - 1), \pi)$ for $\delta > 1$, which describe GS with non-vanishing background or pedestal $\eta(\zeta = \pm \infty) = \eta_s = 2(\delta - 1)$. For any fixed value of the parameter $\delta$, the solitary waves correspond, in both cases, to local maxima of the Hamiltonian $H_s = H(\eta_s, \theta_s)$.

Let us first clarify the relation between the existence domain of these two families of GS and the stopband or forbidden gap of frequencies exhibited by the grating in the linear operation regime. Bright solutions exist for $|\delta| > 1$, which is mapped into the inner domain $\Delta^2 + v^2 < 1$ of the parameter plane $\Delta, v$ (such representation was already adopted in Refs. [10][13]). This unitary circle can be regarded as a dynamical gap, since in this domain the linear problem (i.e., Eqs. (1) with $S, X = 0$) yields exponentially damped traveling-wave solutions ($u^{\pm} \propto \exp(-i\Delta t + iQ_\pm)$, with $Q^2 < 0$). On the other hand, solutions with nonzero pedestal exist only outside this dynamical gap or unitary circle.

To clarify further the role of the soliton velocity $v$, it is important to link the dynamical gap $\Delta^2 + v^2 < 1$ with the bandgap (henceforth termed rest gap to rule out any possible source of misunderstanding) of the stationary linear coupled-mode problem. Such rest gap is well known to be related to the reflectivity bandwidth of the grating, in turn measurable by means of a tunable cw laser in the laboratory (or rest) frame $(z, t)$. Quantitatively, the rest gap is given by the frequency range $|\delta_1| < 1$ (i.e., $|\omega_0| < \Gamma V_B$ in real-world units) where solutions $u_{\pm}(z, t) = u_{\pm}(z) \exp(-i\delta t)$ with frequency detuning $\delta_1$ (real-world frequency $\omega_B + \delta_0$) are exponentially damped in space, at variance with the out-gap case $|\delta| > 1$ where they become oscillatory [1]. In the velocity-frequency plane $(v, \delta)$ the rest gap is the ($v$-dependent) lighter shaded domain shown in Fig. 1(a), which has a clear one-to-one correspondence with the reflectivity bandwidth reported for comparison in Fig. 1(b). To map the dynamical bandgap in the same parameter plane, we need to know what is the actual normalized GS frequency detuning in the rest frame, which is readily found to be $\delta_1 \equiv \gamma^2 \Delta$ by grouping in Eqs. (2) phase terms proportional to $t$. In other words the excitation of a GS characterized by the parameters $v$ and $\Delta$ requires a source with detuning $\delta_1 = \gamma^2 \Delta$ from Bragg frequency. As a consequence the dynamical gap where bright GS exist, can be mapped onto the whole shaded domain $\delta_0^2 < \gamma^2 (1 - v^2)$ or equivalently $|\delta_1| < \gamma$ of Fig. 1. It is clear that, for any given velocity $v$, this entails a frequency range which is wider than the rest gap $|\delta_1| < 1$ and reduces to it only in the $v = 0$ limit. As a consequence bright GS exist for frequencies ranging from the rest gap (reflectivity bandwidth) in the $v = 0$ case, to the whole frequency axis as $|v| \to 1$ (i.e., as the soliton velocity $V$ approaches the linear group-velocity $V_B$ in the forward or backward direction).
The first one represents a dark soliton Eq. (5) holds valid [exploit \( \cosh(2\delta) \)] where \( \delta \) → 0, which is characteristic of the one-soliton solution of the focusing nonlinear Schrödinger equation, which provides a reasonable description of GS in this region of the gap \[ \delta > 1 \]. Therefore the rest gap condition \( |\delta| < 1 \) must be replaced in moving soliton coordinates by the condition \( |\delta| = |\delta|/\gamma < 1 \) which coincides with the dynamical gap.

Equations (3) are equivalent to the motion of an ideal particle of unitary mass and total energy \( E \) in a quartic potential well \( U(\eta) \), ruled by the equation \( \ddot{\eta} = -\partial U/\partial \eta \). The kinetic energy is easily obtained in the standard form from the first of Eqs. (3) by eliminating \( \sin \theta \) through \( H \), which yields

\[
\ddot{\eta} = \sqrt{2[E-U(\eta)]}, \tag{4}
\]

where \( U = -2\delta H \eta - \frac{1}{2} \left( 4 - 4\delta^2 - H \right) \eta^2 - \delta \eta^3 + \frac{1}{2}\eta^4 \), and \( E = -H^2/2 \). The GS solutions can be worked out explicitly by inverting the quadrature integral obtained from Eq. (3) with the energy \( E = -H^2/2 \) pertaining to the unstable fixed point. We found the following expressions for the case (i) \( |\delta| < 1 \), entailing bright GS

\[
\eta_B = \frac{4 \left( 1 - \delta^2 \right)}{\cosh \left( 2\sqrt{1 - \delta^2} \right) - \delta}, \quad \tag{5}
\]

\[
\theta_B = \tan^{-1} \left[ \frac{\sqrt{1 - \delta^2} \sinh(2\sqrt{1 - \delta^2} \zeta)}{-1 + \delta \cosh(2\sqrt{1 - \delta^2} \zeta)} \right]. \tag{6}
\]

Close to the low-frequency edge of the dynamical gap \( \delta \sim -1 \), the following approximation of the intensity profile in Eq. (3) holds valid [exploit \( \cosh(2x) = 2 \cosh^2(x) - 1 \) and \( 1 - \delta^2 \sim 2(1 + \delta) \)]

\[
\eta_B = 4(1 + \delta) \text{sech}^2 \left( \sqrt{2(1 + \delta)} \zeta \right), \tag{7}
\]

which is characteristic of the one-soliton solution of the focusing nonlinear Schrödinger equation, which provides a reasonable description of GS in this region of the gap \[ \delta > 1 \].

In the case (ii), i.e. for \( \delta > 1 \) two solutions coexist, being associated to two branches of a double-loop separatrix. The first one represents a dark soliton

\[
\eta_{DK} = 2(\delta - 1) \frac{\cosh(2\sqrt{\delta - 1} \zeta) - \sqrt{\delta}}{\sqrt{\delta} \cosh(2\sqrt{\delta - 1} \zeta) + 1}, \tag{8}
\]
\[ \theta_{DK} = \tan^{-1} \left[ \frac{4\sqrt{\delta - 1} \sinh(2\sqrt{\delta - 1})}{3\delta^2 + 2\delta^2 - 3 + 4(\delta - 1)\delta \cosh(2\sqrt{\delta - 1}) + (\delta^2 - 2\delta^2 - 1)\cosh(4\sqrt{\delta - 1})} \right], \]  

whereas the second one is a bright on pedestal or so-called antidark solution \[ \eta_{AK} = 2(\delta - 1) \frac{\sqrt{\delta} \cosh \left( 2\sqrt{\delta - 1} \right) + 1}{\sqrt{\delta} \cosh \left( 2\sqrt{\delta - 1} \right) - 1}, \] 

\[ \theta_{AK} = \tan^{-1} \left[ \frac{4\sqrt{\delta - 1} \sinh(2\sqrt{\delta - 1})}{2 - 3\delta + \delta \cosh(4\sqrt{\delta - 1})} \right]. \]

These dark and antidark solutions specialized to the zero-velocity case \( (v = 0) \) have important implications in terms of the stationary \( (\partial_t = 0) \) response of the grating, inducing limiting or frustrated bistability, as discussed in Ref. [18].

Right on the high-frequency edge of the dynamical gap, i.e. for \( \delta \to 1 \), the dark GS vanishes \( (\eta_{DK} \to 0) \), while the bright and the antidark solutions have the following common limit

\[ \eta_{LZ} = \frac{8}{1 + (2\zeta)^2}, \]

\[ \theta_{LZ} = \tan^{-1} \left[ \frac{4\zeta}{(2\zeta)^2 - 1} \right], \]

which represents a finite-amplitude moving Lorentzian soliton, i.e., a GS with Lorentzian intensity profile. The existence of an exact solution of Eqs. (1) with non-exponentially decaying tails can be understood from the dynamical system \[ \frac{d\Delta}{dt} = (\Delta^2 + v^2 - 1) \] as being associated with a degenerate fixed point at the origin with zero eigenvalues. The fact that the Lorentzian shape approximates well bright GS close to the upper bound of the gap was noticed earlier under strictly stationary conditions \( (v = 0) \).

Vice versa, for \( \delta \to -1 \), i.e., close to the low-frequency edge of the dynamical gap, the intensity of the bright GS in Eq. (5) \[ \text{or, consistently, of its nonlinear Schrödinger approximation (7)} \] reduces to the following Lorentzian profile

\[ \eta_{LZ} = \frac{4(1 + \delta)}{1 + 4(1 + \delta)\zeta^2} \approx \frac{4(1 + \delta)}{1 + 2(1 + \delta)\zeta^2}. \]

In this case, however, as the stopband edge is approached the Lorentzian GS \[ \text{becomes broader and smaller, and eventually vanishes identically (} \eta_{B} \to 0 \text{ for} \delta = -1 \). The difference between the finite-amplitude [Eq.(13)] and the vanishing-amplitude [Eq.(14)] Lorentzian GS accounts for an intrinsic asymmetry of the nonlinear grating operation with respect to interchange of frequencies below and above the Bragg frequency, respectively.

**FIG. 2.** Existence diagram (defocusing case) for subluminal GS in the region of detuning-velocity \( (\Delta, v) \) plane bounded by the thin lines at \( v = \pm 1 \). The insets show GS intensity profiles \( |u_\pm|^2 \) sampled at the nearest marker (filled circle). Bright GS of low-amplitude (LA), high-amplitude (HA), and asymmetric (AS) types exist inside the dynamical gap (unitary circle \( \Delta^2 + v^2 = 1 \)). HA solitons become finite-amplitude Lorentzian (LZ) over the right semi-circle, and then bifurcate into dark (DK) and antidark (AK) pairs which coexist for frequencies above the upper edge of the dynamical bandgap. In the focusing case the same picture holds with \( \Delta \to -\Delta \).
Figure 2 summarizes the nature of the different GS solutions by reporting the intensity profiles \(|u_{\pm}|^2\) of qualitatively different GS solutions sampled in the parameter plane \((\Delta, v)\). First, it must be noticed that still \((v = 0)\) GS have equal intensities \(|u_+|^2 = |u_-|^2\) as a consequence of the fact that the net photon flux is zero, for both in-gap (bright) and out-gap (dark-antidark) solutions. Moreover, let us recall that the amplitude of the in-gap (bright) GS increases and their width decreases by spanning the gap from left to right. The symmetry \(|u_+|^2 = |u_-|^2\) is broken for moving GS, which have a stronger component in the direction of motion (i.e., \(|u_+| > |u_-|\) for \(v > 0\) and \(|u_-| > |u_+|\) for \(v < 0\)). Importantly, bright GS which have high-amplitude (see HA inset in Fig. 2), close to the upper edge of the dynamical gap, become Lorentzian GS (LZ inset in Fig. 2) over the edge \(\Delta + v^2 = 1\), and then bifurcate into dark-antidark pairs (DK and AK insets in Fig. 3) outside the dynamical bandgap. Vice versa, as explained above, the low-amplitude (LA inset) GS which exist close to the lower edge of the gap vanishes in the limit \(\delta \to -1\), and no solutions exist below the bottom of the stopband (i.e., outside the circle for \(\Delta < 0\)). For a focusing nonlinearity \((s = 1)\) an identical picture with \(\Delta \to -\Delta\) holds true, meaning that the dark-antidark pairs originate always from the high-amplitude bright GS, though, in this case, they now exist below the low-frequency edge of the dynamical gap.

The bifurcation of GS can be effectively explained in term of the catastrophe theory \([19]\), and the underlying classification of the singularities of smooth functions. The quartic potential \(U(\eta)\) belongs to the so called cusp \(A_{1}3\) family \([19]\). It can be recast in the following standard form by means of the change of variable \(\eta = \sqrt[2]{x} + 2\delta\), thus obtaining

\[
\hat{U}(x) = \frac{1}{4}x^4 + \frac{a}{2}x^2 + bx,
\]

where \(a \equiv \sqrt{2} \left(-4 - 2\delta^2 + H\right)\) and \(b \equiv -8\sqrt{2}\delta\). A necessary condition for the solitary waves to exist is that the potential \(\hat{U}(x)\) has three critical points (i.e., such that \(\partial\hat{U}/\partial x = 0\)), thus being of a double-well type. In the parameter plane \((a, b)\) this occurs in a domain bounded by the following curve, so-called ”bifurcation set” \([19]\),

\[
\left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2 = 0,
\]

which is reported as a thin solid line in Fig. 3. As shown this curve has the characteristic shape of a cusp. In the spirit of the catastrophe theory, we also report in Fig. 3 the so-called control line, i.e., how the parameters \((a, b)\), and as a consequence the potential \(\hat{U}\), vary by changing the single control parameter \(\delta\) from large negative values to large positive values (indicated in Fig. 3 by the limit \(\delta = -\infty\) and \(\delta = \infty\), respectively). In doing so we calculate \(a\) with \(H = H_s\), i.e. the value of the Hamiltonian pertaining to the solitons.

![Fig. 3](image)

FIG. 3. Plot of the bifurcation set (thin line) from Eq. (15) and the control line (thick line) which gives the parametric dependence of the coefficients \(a\) and \(b\) of the potential \(\hat{U}(x)\) in Eq. (14) on the control parameter \(\delta\), when this is varied from large negative values \((\delta = -\infty)\) to large positive values \((\delta = \infty)\). The dashed portion of the control line correspond to unphysical solutions \((\eta < 0)\). Two catastrophes occur at the points \(\delta = \pm 1\).

According to our analysis, by varying \(\delta\) in the range \((-\infty, \infty)\) the control line remains always inside the cusp, indicating the possibility to have solitary waves. Dramatic qualitative changes of the solutions must be expected at the catastrophe points \(\delta = \pm 1\) where the control line becomes tangent to the bifurcation set. In spite of the apparent symmetry of Fig. 3, the two catastrophes are the signature of the strongly asymmetrical behavior of GS against interchange of frequencies below \((\delta < 0)\) and above \((\delta > 0)\) the Bragg frequency. Indeed, the \(\delta = 1\) catastrophe marks the point where the maximum of the potential \(\hat{U}(x)\) moves from a finite positive value for \(\delta > 1\) (which makes
accessible two distinct asymptotic evolutions inside the two wells, in turn corresponding to the dark and antidark GS),
to the origin for $-1 < \delta < 1$ (where only one well is accessible for the asymptotic motion towards the origin which
describes the bright GS). Conversely, the other catastrophe at $\delta = -1$ marks the point where these bright GS simply
cease to exist because the maximum of the potential moves towards negative values. In fact, though the potential still
has a double-well shape even for $-\infty < \delta < -1$, the possibility to have solitary waves is ruled out by the fact that the
two wells become accessible only with $\eta < 0$, and hence the solutions are unphysical (recall that $\eta$ is an intensity).
The dashed line in Fig. 3 displays that portion of the control line where the solutions are unphysical.

In summary we have shown that GS solutions of a well-known standard coupled-mode model with Kerr or Kerr-
equivalent nonlinearity undergo a bifurcation which is strongly asymmetrical with respect to the Bragg frequency.
The qualitative change of the solutions is explained with a geometrical picture based on the application of the catastrophe
theory. The bifurcation is marked by the existence of finite-amplitude Lorentzian GS. In this sense it is reminiscent
of the recently investigated case of localized waves sustained by a gap of full nonlinear origin [20]. In spite of the
diversity between the bifurcation discussed here and that of Ref. [20], this suggests that Lorentzian solitons can play
a universal role in the localization of light in periodic media.

ACKNOWLEDGMENTS

We thank Yuri Kivshar for fruitful discussions concerning Lorentzian solitons.

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