Optimization of Clifford Circuits

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We study optimal synthesis of Clifford circuits, and apply the results to peep-hole optimization of quantum circuits. We report optimal circuits for all Clifford operations with up to four inputs. We perform peep-hole optimization of Clifford circuits with up to 40 inputs found in the literature, and demonstrate the reduction in the number of gates by about 50%. We extend our methods to the optimal synthesis of linear reversible circuits, partially specified Clifford functions, and optimal Clifford circuits with five inputs up to input/output permutation. The results find their application in randomized benchmarking protocols, quantum error correction, and quantum circuit optimization.

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I. INTRODUCTION

Randomized benchmarking protocols \cite{1} are a promising approach to the experimental assessment and evaluation of quantum information processing proposals. They are actively used to benchmark quantum information processing proposals \cite{2,3,4}. The advantages over other methods include the independence from the physical implementation details of those quantum information processing systems being tested \cite{2,3,4}, and scalability. A randomized benchmarking protocol may be described as a repeated application of a set of randomly chosen Clifford operations, followed by the measurement. Access to time optimal implementation of Clifford operations allows to reduce the time required to perform a given benchmarking experiment, and it is thus important for present practical purposes.

A goal of an experimentalist desiring to employ a randomized benchmarking protocol is to construct a complete set of physically implementable operations that can be used to generate any Clifford operation, and then be able to express any Clifford operation using the set of such implementable operations available. Those implementable operations are furthermore referred to as elementary operations. To illustrate, in \cite{2} the set of elementary operations consists of the two-qubit phase gate (controlled-Z) and all single qubit Clifford and Pauli gates. In \cite{4}, the two-qubit ZZ-interactions are provided by the driving Hamiltonian, single qubit gates in the X-Y plane are implemented as RF pulses, and single qubit gates in the Z-plane are implemented through a frame change, and require no physical action (as such, they are “free of charge”). The amount of physical resources required to implement each elementary gate, as well as the very set of gates that may be implemented directly, varies from one quantum information processing proposal to another. Unable to capture all possible elementary gate libraries and circuit cost metrics, we concentrated on the study of quantum circuits composed with Hadamard gate, Phase gate (and its inverse), and the two-qubit CNOT gate, and two simple metrics of circuit cost—the gate count and the depth. However, we designed our algorithms and implementation such that they may be modified to accommodate essentially any gate library, as well as more sophisticated metrics of the circuit cost.

In particular, we study the problem of the optimal synthesis of Clifford operations acting on a small number of qubits. We determine the cost of the overall Clifford operation based on the number of single and two-qubit elementary operations required to implement it. This constitutes a simple measure for estimating the difficulty of implementing Clifford operations in an experiment. We synthesize optimal Clifford circuits on two to four qubits, and optimal Clifford circuits on five qubits up to input/output permutation. We use the optimal implementations of Clifford operations acting on a small number of qubits in peep-hole optimization \cite{5} of larger Clifford circuits. The experiments reveal substantial practical improvement in large-scale designs of Clifford circuits. Finally, we apply the ideas developed in the paper to find an optimal encoding circuit for the five-qubit error correcting code. This method can be applied to synthesize encoding circuits for other error correcting codes that use a small number of qubits.

Stabilizer circuits have been studied well in the relevant literature: \cite{6} reports an 11-stage layered decomposition of the n-qubit Clifford operations using at most $O(n^2/\log_2 n)$ gates; \cite{7} develops linear depth implementations. Both papers report asymptotically optimal implementations, however, suboptimal in the absolute sense. As reported in \cite{2}, finding optimal implementations of Clifford circuits with up to two qubits is straightforward. In our paper, we report optimal Clifford circuits for up to four qubits, optimal Clifford circuits up to input/output permutation for up to five qubits, and optimize scalable implementations of the Clifford circuits by a factor of roughly two.
II. PRELIMINARIES

Clifford quantum circuits consist of Hadamard ($H$), Phase ($P$) and CNOT gates. The important property of these gates is that they map Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their tensor products to themselves by conjugation. More precisely:

$$HXH^\dagger = Z, \ HYH^\dagger = -Y, \ HZH^\dagger = X,$$

$$PX^p = Y, \ PY^p = -X, \ PZ^p = Z,$$

The CNOT gate acts on two qubits and its action is:

$$X \otimes I \mapsto X \otimes X, \ Z \otimes I \mapsto Z \otimes I,$$

$$I \otimes X \mapsto I \otimes X, \ I \otimes Z \mapsto Z \otimes Z.$$  

Compact representation of any unitary that can be computed by a Clifford circuit is a direct consequence of the Clifford gates’ property described above. Action of a circuit on any input is uniquely defined by this representation [8]. Taking into account the identity $Y = iXZ$, it suffices to know the action by conjugation of the $n$-qubit circuit on 2$n$ Pauli matrices. The result of the application of the circuit to each Pauli matrix can be encoded using 2$n$ + 1 bits [6]. Pauli matrices are encoded as follows:

$$I \sim (0,0), \ X \sim (1,0), \ Z \sim (0,1) \ Y \sim (1,1)$$

It is convenient to separate $X$ and $Z$ parts when encoding larger circuits:

$$I \sim (0,0), \ X \sim (1,0), \ -I \otimes X \sim (01,001)$$

One additional bit is used to encode the overall sign. For any unitary the sign can be adjusted by applying the round of Pauli gates at the end of the computation. In most of our applications this can be done for free. Further we will consider only the 2$n$ × 2$n$ part of the encoding matrix. Commutativity relations between Pauli matrices are preserved under conjugation and induce additional constraint on the encoding matrix—it must be symplectic. Furthermore, the canonical decomposition theorem [8] shows that any binary symplectic matrix encodes some Clifford circuit.

The tableau representation can be efficiently updated [6] when adding new gates to the end of an existing circuit. Adding a gate requires to update the one or two columns of the encoding matrix. The application of the Phase gate to qubit $k$ corresponds to the addition modulo 2 of column $k$ to column $n+k$, the Hadamard gate on qubit $k$ corresponds to exchanging columns $k$ and $n+k$, and the CNOT gate with control $k$ and target $j$ corresponds to the addition of column $k$ to column $j$ and the addition of column $n+j$ to column $n+k$. An empty Clifford circuit corresponds to the identity matrix. These rules suffice to determine the 2$n$ × 2$n$ binary symplectic matrix encoding the unitary computed by a given Clifford circuit.

For linear reversible circuits—those composed only with CNOT gates—it suffices to store only the top left $n \times n$ part of the binary symplectic matrix. The described procedure for updating columns immediately implies that the binary symplectic matrix for linear reversible circuit should be of the following form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$  

As the matrix must be symplectic we have $A^TB = I$, which uniquely determines $B$ given $A$. Therefore, we can store linear reversible unitaries more efficiently than a generic Clifford operation.

The two optimality measures that we consider are the minimal number of the Clifford gates required and the minimal depth of the circuit implementing the given unitary. For brevity, we call them the gate count and the depth of the unitary. Our ideas extend to other optimality measures, such as the number or the depth in terms of the CNOT gates.

III. ALGORITHMS

The main challenge in our approach to finding optimal circuits is a large search space (Table I). Our algorithm is based on the Breadth First Search. The number of distinct unitaries computed by Clifford circuits grows as $2^{\Theta(n^2)}$. We address the resulting challenge in several ways. First, each node of the search tree corresponds to an equivalence class of unitaries instead of the unitary itself. Second, we use meet in the middle technique to avoid building the full tree [9]. Finally, we use a special data structure to store the search tree in a compact way.

The equivalence relation we use to reduce the size of the search space is the following: two unitaries are equivalent if they can be computed by circuits that are the same up to simultaneous renaming of their inputs and outputs. Both gate count and depth of a unitary are

| $G$ | $n$ | $N_G$ | $\text{Size}_{G_n}$ (GB) |
|-----|-----|-------|--------------------------|
| Sp  | 4   | 47,377,612,800 | 14.71 |
| Gr  | 5   | 24,815,256,521,932,800 | 3.08 × 10^6 |
| GB  | 5   | 20,158,709,760 | 0.20 |
| GB  | 6   | 163,849,992,929,280 | 242.22 |

TABLE I: $G$ – group; $Sp$ – symplectic part of Clifford group; $Gr$ – group generated by linear reversible circuits; $n$ – number of qubits, $N_G$ – size of the corresponding group, $\text{Size}_{G_n}$ – lower bound on the size of the database taking into account input/output renaming (GB).
invariant with respect to such simultaneous renaming. During the search we store only a canonical representative of each class. For $n$ inputs this results in a reduction of the number of unitaries to be stored by a factor of approximately $n!$. The number of unitaries corresponding to the same canonical representative is not always $n!$, but this is the most common case. In particular, the fraction of four-qubit unitaries that have less than $24$ (= $4!$) elements in their equivalence class is less than $9.7 \times 10^{-5}$. To search for five-qubit optimal Clifford circuits we used the equivalence relation corresponding to the independent renaming of the inputs and outputs, in other words, we ignored SWAP gates. This further shrinks the search space, but the results are suboptimal in the scenario when SWAP has a non-zero cost.

The idea of the meet in the middle (MiM) technique is based on the optimality of subcircuits of any optimal circuit. Given a database $DB_c$ of all unitaries with the cost at most $c$, MiM allows to find optimal circuits for unitaries with the cost at most $2c$. Suppose we are looking for an optimal circuit computing a unitary $f$ with cost $c + d \leq 2c$. We can always split the optimal circuit into two optimal circuits with $d$ and $c$ gates. Therefore, there always exist a unitary $g$ with cost $d \leq c$ such that its composition with $f$ has cost $c$ and it is in our database. We can find $g$ by trying all unitaries from the database and checking if $g \circ f$ is also in the database. In the worst case, using meet in the middle increases the time required to find a circuit by a factor proportional to the size of $DB_c$, in comparison to using the database $DB_{2c}$. At the same time, meet in the middle significantly reduces the required memory. For example, in the case of four qubits the maximal number of gates required is 17 and the size of the database is 14.72 GB. Using the database with optimal circuits up to 9 gates reduces the required memory to just 108 MB. Meet in the middle is vital for the search of optimal five-qubit Clifford circuits up to input/output permutation. In this case, the size of the full database would have been about $3.08 \times 10^6$ GB.

A. Computing canonical representative

To find the canonical representative with respect to the simultaneous renaming of the inputs and outputs we compute all elements of the equivalence class, encode them as bit strings and find the minimum. We need to go through all possible permutations. This is accomplished by applying a single transposition at each step. Exchanging inputs $k$ and $j$ of an $n$-qubit Clifford circuit corresponds to swapping columns and rows of the binary symplectic matrix. The pair of columns $(k, k+n)$ must be swapped with $(j, j+n)$, pairs of rows with the same indexes must be swapped also. Internally we represent each binary matrix as an array of integers. Each integer corresponds to a column of the binary symplectic matrix. We precompute required transpositions of the bit strings of length $2n$ and use a lookup table to speed up the swapping of rows of the binary symplectic matrix.

When we allow an independent renaming of the inputs and outputs we apply a more efficient procedure for canonical representative computation. In most cases we have $(n!)^2$ representatives corresponding to the same equivalence class. First we find all $n!$ representatives corresponding to the different row permutations [10]. Then we store columns $k$ and $k+n$ together in one bit string and sort the resulting bit strings using a sorting network. This gives a canonical representative with respect to column permutation for a fixed row permutation. Finally, we encode the representative for each row permutation as a bit string and find the minimum.

For linear reversible circuits we apply the same idea. To exchange two inputs $k$ and $j$ we just need to swap columns $k$ and $j$ and rows $k$ and $j$ of the matrix encoding the circuit. This approach also extends to partially specified matrices.

B. Implementation details

The main bottleneck in our search is the amount of memory available. In addition to using canonical representation, we tried to minimize the memory overhead caused by the data structures. Here we describe the details of the gate count optimal search. The same ideas were adopted for depth optimal search and can be extended to more general cost functions. We did not target to study all possible optimizations in a systematic way. We present a set of solutions that allowed us to obtain the results in a reasonable amount of time and designed our software to be scalable enough to support different types of search.

Possible costs of unitaries belong to a short range of the integer values. For this reason, we introduced a separate data structure to store unitaries with the fixed cost. We call it a layer. We build layers one by one. To build the layer $k$ we pick an element of the layer $k-1$—we call it a parent unitary. Then we compose it with all possible gates and check if the resulting unitary was not found earlier. The only possible costs of the resulting unitary are $k$, $k-1$, or $k-2$. If we get cost less than $k-2$ this contradicts the knowledge that the cost of the parent unitary is indeed $k-1$. Therefore, during the search we need to keep only two previous layers in the memory. We repeat the procedure for all unitaries in the layer $k-1$. It can be executed in parallel for several parent unitaries. Only the addition of the unitaries with cost $k$ to the corresponding layer must be synchronized.

Finally, we describe how to find a circuit using the precomputed layers. If we find that a unitary belongs to the layer $k$ this means that there exists a circuit with $k$ gates computing the unitary. Therefore, by removing the last gate in the circuit we obtain an optimal circuit with $k-1$ gates which corresponds to a unitary with cost $k-1$. By composing the source unitary with all possible gates and checking cost of the result we identify the last
gate in the optimal circuit. We proceed further in a similar fashion, until we reach the canonical representative of the identity. In the case when we rename inputs and output simultaneously we always get an identity in the end. When renaming of inputs and outputs is independent we obtain a circuit that is composed entirely of SWAP gates that represents a permutation of the inputs.

IV. EXPERIMENTAL RESULTS

In this section we describe the results of our search together with the optimization experiments that rely on the databases of the optimal circuits we found. For the experiments that require more than 8 GB of RAM memory we used a high performance server with eight Quad-Core AMD Opteron 8356 (2.30 GHz) processors and 128 GB of RAM memory. These are the experiments involving 4- and 5-qubit Clifford unitaries. For all other experiments we used a machine with a single quad-core Intel Core i7-2600 (3.40 GHz) processor and 8 GB of RAM.

A. Distribution of the optimal circuits

We found optimal circuits for Clifford unitaries acting on 2–4 qubits (Figs. 1, 2) and optimal linear reversible circuits acting on up to 6 qubits. In both cases we found both circuits with the optimal gate count and those with the minimal depth. For the case of Clifford unitaries we also found circuits with optimal number of Controlled-Z gates.

Distributions reported in Figs. 1, 2 are interesting for the randomized benchmarking of quantum information processing systems. The benchmarking protocol [11] involves the application of a large number of randomly chosen Clifford unitaries. Knowledge of the distribution of the number of gates allows to estimate the average time required for each experiment, and evaluate its feasibility due to, e.g., the effects of the decoherence. Using optimized circuits minimizes the time required for an experiment. In addition, computation of the normalized quantities describing the quality of two qubit gates im-
TABLE II: The results of application of the peep-hole optimization to encoding circuits for Quantum Error Correcting codes. \([n,k,d]\) denotes the code that uses \(n\) physical qubits, encodes \(k\) logical qubits and has distance \(d\), \(c_k\) – number of gates in the circuit obtained using Algorithm \(k\), \(c_{k_0}\) – number of gates in the circuit after application of the peep-hole optimization using the database of 4-qubit optimal Clifford circuits, \(t_k\) – runtime of peep-hole optimization software (in seconds) as applied to the circuits produced by the Algorithm \(k\).

| Code        | \(c_1\) | \(c_{10}\) | \(c_2\) | \(c_{20}\) | \(t_{10}\) | \(t_{20}\) |
|-------------|---------|------------|---------|------------|-----------|-----------|
| \([25,1,9]\)| 440     | 285        | 387     | 205        | 22.2707   | 6.57012   |
| \([26,1,9]\)| 444     | 287        | 389     | 207        | 22.8359   | 7.97804   |
| \([26,4,8]\)| 500     | 336        | 528     | 250        | 30.073    | 18.6791   |
| \([27,1,9]\)| 592     | 396        | 479     | 241        | 31.8254   | 15.1547   |
| \([27,2,9]\)| 559     | 377        | 568     | 295        | 39.9342   | 18.5159   |
| \([27,3,9]\)| 566     | 373        | 566     | 274        | 38.629    | 10.6849   |
| \([27,4,8]\)| 504     | 335        | 530     | 252        | 28.0653   | 22.9666   |
| \([27,8,6]\)| 498     | 341        | 558     | 305        | 45.949    | 15.2595   |
| \([27,9,6]\)| 453     | 310        | 588     | 305        | 32.2992   | 17.1963   |
| \([27,10,5]\)| 428     | 293        | 563     | 293        | 59.7698   | 10.5993   |
| \([27,11,5]\)| 409     | 279        | 541     | 295        | 29.6078   | 12.5559   |
| \([28,0,10]\)| 652     | 446        | 526     | 248        | 45.3604   | 18.2336   |
| \([28,1,10]\)| 660     | 446        | 531     | 284        | 41.0448   | 13.5861   |
| \([28,2,10]\)| 666     | 427        | 592     | 285        | 44.143    | 16.4625   |
| \([28,3,9]\)| 570     | 378        | 568     | 276        | 60.009    | 10.2351   |

We used the database of the optimal 4-qubit Clifford circuits to perform peep-hole optimization described in detail in \([2]\). We applied it to encoding circuits for quantum error correcting codes (QECCs). To obtain an encoding circuit for QECC one starts with the stabilizer generators of the code and applies an algorithm that produces the encoding circuit. We implemented two algorithms. The first one is a version of the canonical decomposition theorem \([6]\) for stabilizers that produces layers of CNOT, H, and P gates (Algorithm 1). The second one (Algorithm 2), taken from \([15]\), produces circuits that do not have an expressed layered structure. Table \(\text{II}\) summarizes the results of our experiment with codes from \([14]\). Applying peep-hole optimization to the circuits produced by Algorithm 2 results in a reduction of the number of gates by 45-53%.

B. Five qubit Clifford functions

The search for five-qubit unitaries up to input/output order is challenging, but it is still tractable using modern computers. The number of the different unitaries on five qubits is about 2.4 x 10^{17} (Table\(\text{I}\)). We need 100 bits to store each group element. Factoring out simultaneous renaming of inputs and output allows us to reduce the size of the database by approximately 120 times. However, one still needs 3.08 x 10^{6} GB to store the full database in this case. To allow the search of any 5-qubit Clifford unitary up to input/output order we allowed the independent renaming of the inputs and outputs of the circuits and used meet in the middle approach. We synthesized all 5-qubit unitaries that use up to 11 gates which allowed us to search for unitaries that require up to 22 gates. It is unknown what is the maximum number of gates needed to implement any 5-qubit Clifford unitary. We ran an experiment to estimate the distribution of the number of gates required to implement a unitary. We used the algorithm described in \([12]\) to generate uniformly distributed random Clifford unitaries and found their gate count. The distribution of the number of gates for 5-qubit unitaries, shown in Fig. \(\text{I}\) was obtained using 20,000 samples. We used Hoeffding inequality \([13]\) to estimate errors for confidence level 0.999. Based on the above calculation, we believe that the 11-level database and the use of the meet in the middle allow to find optimal circuits for any 5-qubit Clifford unitary up to input/output order.

C. Peep-hole optimization

D. Optimal encoding circuit for five-qubit quantum error correcting code

Using a slightly modified version of our algorithm we found a depth optimal circuit for the five-qubit \([5,1,3]\) error correcting code. This code encodes one qubit and corrects any single qubit error. In this case only first
FIG. 4: Optimal encoding circuits for the five-qubit code: (left) depth optimal circuit, depth=5; (right) circuit with the minimal number of gates, being 11 gates. Input marked as $|\psi\rangle$ corresponds to the state that should be encoded.

FIG. 5: Encoding circuit for the five-qubit code used in [14]. The two-qubit gate corresponds to $e^{-iZZ\pi/4}$. Eight of them are required to implement the encoding circuit.

V. CONCLUSIONS

We explored the limitations of the brute force search for optimal circuits implementing Clifford and linear reversible unitaries. Using typical memory and processing power available today, it is possible to search for up to four-qubit optimal Clifford unitaries and six qubit linear reversible unitaries. We also demonstrated that additional assumptions allow to search for optimal Clifford unitaries up to input/output order. It is possible to make further assumptions resulting in greater sub-optimality, but reducing the size of the search space. For example, one may allow to apply Hadamard gates to each output in the end of the circuit for free. This will further reduce the size of search space by approximately $2^n$, where $n$ is the number of qubits. It is easy to come up with canonical form computation for this case. Of course, circuits produced by the algorithm will not be exactly optimal. However, the results will be very close to optimal if the cost of Hadamard gates is small. Using more restricted gate sets, such as those that allow only nearest neighbour or two nearest neighbour interactions has the opposite effect. In such case we do not have the symmetry between all qubits, which results in the growth of the search space.

Using lookup in our database as a part of the peep-hole optimization shows that this is an efficient and promising approach for the optimization of larger Clifford circuits.

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