Practical variational tomography for critical 1D systems

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We improve upon a recently introduced efficient quantum state reconstruction procedure targeted to states well-approximated by the multi-scale entanglement renormalization ansatz (MERA), e.g., ground states of critical models. We show how to numerically select a subset of experimentally accessible measurements which maximizes information extraction about renormalized particles, thus dramatically reducing the required number of physical measurements. We numerically estimate the number of measurements required to characterize the ground state of the critical 1D Ising (resp. XX) model and find that MERA tomography on 16-qubit (resp. 24-qubit) systems requires the same experimental effort than brute-force tomography on 8 qubits. We derive a bound computable from experimental data which certifies the distance between the experimental and reconstructed states.

The understanding of many-body quantum systems has dramatically progressed recently, theoretically and experimentally. New efficient numerical methods use the properties of entanglement in many-body states, such as the area law of entanglement entropy [1], to describe efficiently the many-body wave function of physical systems [2]. In parallel, experimentalists achieve a very high degree of control over larger and larger systems [3–4]. However, efficient methods to quantitatively compare theoretical predictions to experimental realizations are few.

Quantum state tomography [5] is a paradigm that aims to reconstruct the quantum state of a system by performing multiple measurements on identically prepared copies of the system. Since measurements perturb a quantum system, many copies of the system are needed to extract information about the many-body wave function. Given the experimental data, a numerical procedure determines which density matrix fits best the measurements. This quantum state reconstruction can be performed using different approaches, the most used being maximum likelihood estimation [6].

Generally, both the number of measurements and the post-processing time of quantum state reconstruction grows exponentially with the system size. This is not surprising since the dimension of the Hilbert space of \( n \)-particles grows exponentially in \( n \). Thus, there is an exponential number of coefficients to estimate. Furthermore, for a (Haar)-random quantum state, most coefficients have exponentially small amplitudes in a local basis, so to distinguish any one of those amplitudes from zero, one must take an exponential number of samples. This simple reasoning hints towards an experimental and numerical efforts that scales doubly exponentially with system size.

However, physical quantum states, for instance ground states of local Hamiltonians, only constitute a very small subset of all states in the Hilbert space [7]. A general and very fruitful idea is to approximate those states of our interest by a suitable variational family of states. An efficient family of states not only allows for a succinct description of states (with the polynomial number of parameters) but also allows to efficiently compute physical quantities, such as expectation values of local observables in polynomial time.

Tensor network (TN) states are variational families of states which are strong candidates to parametrize the physical part of the Hilbert space [2]. TN states are built to accommodate the different scaling characteristic of entanglement for various physical states. For instance, matrix product states (MPS) representation is based on the property that the entanglement of a block of particles grows with the boundary of the block. Since ground states of 1D gapped systems follow area laws, they are well-approximated [8] with matrix product states [9–12] (MPS) and there exist convenient numerical methods, such as the density matrix renormalization group (DMRG) method [13, 14] to find such a MPS approximation. However, this area law is violated by critical systems, i.e. ground states of quantum systems near a quantum phase transition [15]. Indeed, in 1D critical systems, the entanglement of a block of \( n \) particles diverges as \( \log(n) \). To reproduce this entanglement scaling, Multi-Scale Entanglement Renormalization Ansatz (MERA) was introduced in [16]. A MERA state is the output of a specific type of quantum circuit whose gates arrangement generates a logarithmic amount of entanglement.

Recently, the use of variational states has been applied to tomography [17, 19] and explicit state reconstruction methods have been given. The pioneering work on MPS tomography [18] provided the first demonstration that variational tomography could be performed efficiently. Subsequent work demonstrated that variational tomography was also possible for 1D critical systems described by MERA states [19]. MERA tomography offers the perspective to be an extremely valuable tool in the experimental characterization of quantum simulators, finely controlled systems which experimentalists can tune to reproduce the dynamics of a model Hamiltonian. Indeed, the MERA can be straightforwardly extended to study critical 2D models which are precisely the Hamiltonians that quantum simulators [20, 21] offer to probe experimentally.

Here, we revisit the idea of MERA tomography in 1D and explicitly investigate some of the challenges left open...
in the original proposal [19]. The original article gave a proof of principle that the tomography protocol only required a numerical and experimental effort scaling polynomially with the size of the system. While this result is crucial theoretically, it does not guarantee that the number of measurement and the processing time is reasonable in practice for moderately small systems within experimental reach. Here, we

1. Improve the optimization techniques of the original MERA tomography proposal.

2. Assess the scaling of the number of repeated measurements needed to reconstruct the ground state of critical models in the case of the Ising and XX models.

3. Provide an error analysis which yields an upper bound on the distance between the reconstructed and experimental states.

The article is organized as follows. In section I, we recall the idea of variational tomography focusing on MERA tomography. In section II, we improve the numerical technique used to optimize the quantum gates of the MERA circuit by using a linearization scheme and SVD approach. In section III, we investigate the scaling of the number of experimental measurements that have to be performed to acquire information about longer scale properties of the state. First, we show that the naive approach of [19] requires an unreasonable (yet polynomial) amount of measurements, and then demonstrate numerically that the combinations of two solutions significantly reduces the number of experimental measurements. In section A, we provide a detailed analysis of the source of errors in our tomography scheme and infer the bound of the distance between experimental and reconstructed states, based on a detailed analysis provided in the Appendix.

I. MERA TOMOGRAPHY

A. Variational tomography

The core idea of variational tomography is to take advantage of the succinct description of variational states in order to devise an efficient learning method. A learning method consists of three parts: i) the measurement prescription which identifies the measurements to perform, ii) the data acquisition when the measurements are performed and iii) the state reconstruction that infers the compatible quantum state via post-processing. Note that the measurement prescription can be performed in parallel to data acquisition as preliminary data can improve the choice of measurements. This is the case for MERA tomography.

As mentioned in the introduction, the idea of variational tomography has been demonstrated on two variational class of states: MPS and MERA. In both cases, quantum state tomography is performed on small systems and numerical processing is used to stitch the density matrices of those small systems into a global state. While this stitching is efficient for both MPS and MERA, this procedure is expected to be very hard for arbitrary state in the Hilbert space. Recent progress has been made to understand the structure of quantum states for which local measurements are informationally-complete [22].

In MPS tomography, reduced density matrices $\sigma_i$ on all blocks of a constant number (independent of system size) of particles are estimated. Then, a classical algorithm, inspired from ideas of compressed sensing, is used to reconstruct the global state. Alternatively, one could learn the quantum circuit preparing the state. Indeed, any MPS can be prepared using a staircase circuit with linear depth (see Fig. 1).

One can learn the rightmost quantum gate of Fig. 1 by performing tomography on a small number of particles. To learn the second gate rightmost, the original proposal of [18] was to experimentally apply the gate. However, one can use the knowledge of the first gate to see how it modifies the physical observables. Indeed, each physical observable will become a renormalized observable. As long as renormalized observables span the support of the density matrix, they are informationally complete. The power of renormalized observables was not immediately realized in [18], but became apparent when MERA tomography was devised [19]. We now describe this procedure in great details as our work builds upon it.

B. Learning MERA states

1. Quantum circuit for MERA states

The MERA is a variational family of states [10] arising from a real-space renormalization group approach called entanglement renormalization [23]. Entanglement renormalization creates a sequence of quantum states
{(ρτ)}τ=0...τ where ρ0 is the physical state (which we will also refer to as the experimental state in the context of tomography) and ρτ>0 are coarse-grained version of the physical state which encode entanglement on a larger scale. The crucial insight of MERA is that for critical states, it is important to get rid of short scale entanglement before each renormalization step. Otherwise, the short scale entanglement accumulates and the renormalization cannot be carried anymore. This renormalization approach translates into a quantum circuit, depicted on Fig. 2 that turns the physical state ρ0 on n qudits into the all zero state |0⟩⊗n (in the case of a pure state).

This MERA consists of two families of quantum gates. The disentanglers are unitary transformations, depicted by squares on Fig. 2 and denoted u, whose goal is to remove short scale entanglement. The isometries, depicted by triangles on Fig. 2 and denoted w, map several qudits into a single qudit by applying a unitary transformation v followed by projection operator P. For instance, in binary MERA, two particles of quantum dimension χ2 are mapped into a single particle of quantum dimension χ (taking χ = 2 in all numerical discussions). Note that this transformation is only possible if the density matrix before the isometry is (approximately) supported on a space of dimension χ rather than having full rank χ2. In other words, the purpose of the disentanglers is precisely to locally rotate the Hilbert space to concentrate the support of the density matrices. This remark is at the heart of the numerical method to identify disentanglers.

Another important notion of MERA circuit is the past causal cone of a quantum gate and the future causal cone of particles. Imagine that time flows from the bottom of Fig. 2 to the top. In other words, the level index τ plays the role of time. For any given quantum gate of the circuit (disentangler or isometry), its past causal cone is the set of physical particles whose state can be affected by changing the quantum gate. For any set of physical particles, its future causal cone is the set of quantum gates such that a particle belong to the past causal cone of at least one of the gates in the set.

2. MERA tomography procedure

Let us briefly describe the MERA tomography procedure, taking the binary MERA geometry (see Fig. 2) as example. The goal is to find a MERA circuit representing a given experimental state. To do this, MERA tomography repeatedly measures local observables to obtain reduced density matrices of 4 sites in lattice Lτ which are the past causal cone of each isometry mapping Lτ+1 to Lτ+1. For the physical level, i.e., τ = 0, those reduced density matrices are obtained by brute-force quantum state tomography. For higher layers, the density matrices can be inferred from physical measurements and the knowledge of the quantum gates in the previous layers. We will describe the procedure in more details in Sec. III.

For the moment, let us assume that we know every 4-site density matrix ρi corresponding to the past causal cones of every isometry i in layer τ + 1.

Given {ρτi}, the goal is to find the disentanglers in layer τ. Let us focus on a single disentangler u = ui, supposing that all other disentanglers in the layer are fixed. The choice of u will affect the isometries, wi and wi+1, respectively to the left and the right of u, see Fig. 3.

Thus, the objective function g splits into two parts

\[ g(u, ρ_i^\tau, ρ_{i+1}^\tau) = f_L(u, ρ_i^\tau) + f_R(u, ρ_{i+1}^\tau) \]  

(1)

where \( f_L(u, ρ_i) \) corresponds to minimizing the rank of the 2-site reduced density matrix that is the input of \( w_i \)
and similarly for $f_R$ with respect to $w_{i+1}$.

After applying the optimal disentanglers, the two-site reduced density matrix at the input of the isometry $w_i$ should have a rank at most $\chi$ so that the isometry keeps the $\chi$ eigenvectors with largest eigenvalues. In other words, we want the probability weight to be supported on the $\chi$ largest eigenvalues. Thus, we maximize the objective function,

$$f_{L,R}(u, \rho_i^\tau) = \sum_{k \leq \chi} \lambda_k$$

where $\rho_i^\tau$ is the reduced density matrix for the $i$-th block at level $\tau$ and $\lambda_k$ is $k$-th eigenvalue of the reduced density matrix after the disentangler $u$ has been applied.

Once all disentanglers have been obtained, the isometries $w_i$ are obtained by diagonalizing the reduced density matrices $\sigma_i = \text{tr}_{14}[\rho_i]$ at the input of the isometries (See Fig. 4). $\text{tr}_{14}$ implies tracing over site $i_1$ and $i_4$ after disentanglers). Indeed, one can decompose the isometry $w_i$ as an unitary transformation $v_i$ followed by a projector $P$ of rank $\chi$. Given the diagonalization

$$\sigma_i = \sum_{k \leq \chi} \lambda_k |\phi_k\rangle\langle\phi_k| + \sum_{k > \chi} \lambda_k |\phi_k\rangle\langle\phi_k|$$

the unitary $v_i$ maps the first $k$ eigenvectors $|\phi_k\rangle$ to $|k\rangle \otimes |0\rangle$. The way it acts on the other eigenvectors is arbitrary, as long as $v_i$ is unitary. Afterwards, the projector $P = I_\chi \otimes |0\rangle\langle0|$ throws away the irrelevant eigenvectors. This procedure is repeated over each layer of the MERA circuit.

In the original MERA tomography procedure described in [19], a conjugate gradient method was used to maximize the objective function given by Eq. (2). We now present an alternative approach to perform this maximization.

II. NUMERICAL TECHNIQUE

The conjugate gradient approach used in [19] to find a disentangler and an isometry from a set of measurements data is a standard method for such optimization problem. However, the method is prone to getting stuck in local minima and is hard to extend for large bond dimension $\chi$. Here, we improve the algorithm for optimization by borrowing an algorithm developed for efficient MERA energy minimization procedure in [24]. In this new numerical technique, the objective function remains the same but we interpret it as a tensor contraction

$$f(u, \rho_i) = \text{tr}(u \Gamma^R_u + u \Gamma^L_u)$$

where $\Gamma_u$ is the environment of disentangler $u$, shown on Fig. 4. Since an isometry $W$ keeps only first $\chi$ components, contraction of this small part of a MERA circuit gives the sum of first $\chi$ singular values of a reduced density matrix for three sites.

In fact, optimization of $\text{tr}(u \Gamma_u)$ is analytically hard as $\Gamma_u$ also depends on disentangler through $u^\dagger$. It is a quadratic optimization problem, which is generally hard. Here, we will linearize the function. We first linearize the objective function by fixing $u^\dagger$ and thus $\Gamma_u$ and only varying the disentangler $u$. For a fixed $\Gamma_u$, the optimal $u$ can be found by standard singular value decomposition (SVD) technique. One finds the SVD decomposition of the environment, $\Gamma_u = NSM^\dagger$. The trace in Eq. (5) is extremized by the choice of $u = MN^\dagger$. Let $u_0$ be an initial guess (random) unitary transformation. Then, starting with $k = 1$ and for increasing values $k = 2, 3, ..., u_k$ from $u_{k-1}$ by optimizing $\text{tr}(u_k \Gamma_{u_{k-1}})$.

In practice, optimizing a disentangler depends on the neighboring disentanglers. Thus, the optimization not only iterates the above process to optimize the disentanglers, but also sweeps over all the disentanglers of a given layer. An optimization algorithm can choose to balance the number of iterations and the number of sweeps in different ways. In practice, we find that a single iteration and multiple sweeps gave satisfactory results. Although the objective function is not guaranteed to improve at each step, nor to converge, we find that this method typically converges faster than the previous method based on a conjugate gradient technique.

To check the validity of our algorithm, we generated random MERA states by choosing the quantum gates of the MERA circuit at random according to the Haar measure. We then performed our quantum tomography algorithm to find MERA circuits for the states. The algorithm was tested on the binary MERA geometry with 24 qubits. In order to check the numerical optimization algorithm, we assumed brute-force tomography to be perfect, implying that the 4-sites reduced density matrices $\rho_i$ are accurate. At the end, the fidelity

$$F(\rho_0, \rho_r) \equiv \left( \frac{\text{tr}(\sqrt{\rho_0} \rho_r \sqrt{\rho_0})}{\sqrt{\text{tr} \rho_0 \text{tr} \rho_r}} \right)^2$$

between the original state $\rho_0$ and the reconstructed state $\rho_r$ was computed. The infidelity $1 - F(\rho_0, \rho_r)$ obtained on average was $10^{-13}$. There were few cases which took more than hundred seconds to reach that fidelity, but it
FIG. 5: Convergence of the average value of the objective function $f(u)$ for different layers using MERA tomography on 24 qubits and a binary geometry, as a function of the number of sweeps. (up) Results for 10 random MERA states. (down) Results for 10 superpositions of a MERA state and a Haar-random state of magnitude $\epsilon = 0.1$ (see Eq. 6).

was only one out of twenty. In Fig. 5, we can see that 20 runs all gave convergence in 100 iterations, which take only ten seconds. We conclude that, for states with an exact MERA representation, our method requires reasonable processing time and yields a very accurate reconstruction result.

Also, we examined how our algorithm performed if the experimental state does not admit an exact MERA representation. For this, we considered the experimental state $|\psi\rangle$ to be the superposition of a state $|\psi_{\text{MERA}}\rangle$ generated by a MERA circuit and a Haar-random state $|\psi_e\rangle$, i.e.,

$$|\psi\rangle = \sqrt{1-c^2}|\psi_{\text{MERA}}\rangle + c|\psi_e\rangle \quad (6)$$

Results for that case are given in the bottom plot of Fig. 5. In that case, the optimization performs poorly on the first layer, converging around $10^{-2}$ and improves for the second layer (around $10^{-6}$) and the third (around $10^{-8}$). Infidelity between the given state and reconstructed state was around $10^{-4}$, which tells us that the reconstructed state is exactly the MERA part of the given state. Our interpretation is that the isometries of the layers progressively filter out the non-MERA part of the state.

III. SCALING OF THE NUMBER OF EXPERIMENTAL MEASUREMENTS

A. Ascending superoperator

As explained in section II, MERA tomography infers the quantum circuit preparing the experimental state. To identify each gate, the numerical procedure takes as input the reduced density matrix on a small block of particles. For the physical layer, denoted $L_0$ on Fig. 2, those particles correspond to experimentally measurable particles. However, this is not the case, for higher renormalized levels, $L_\tau$ for $\tau > 0$. To get access to the density matrices on block of renormalized particles, we will assess how physical measurements will be mapped into effective measurements at higher layers. This mapping depends on the disentanglers and isometries between the physical level $L_0$ and the current level $\tau$. Thus, it depends on the information acquired by tomography on the previous layers.

1. First layer of renormalization

Let us consider the first layer of renormalization. Let’s define $U_0^\dagger$ as the product of all the disentanglers $u$ and all unitary transformations $v$ (see Fig. 2). Note that $U_0^\dagger$ is a unitary transformation since it does not contain $P$, the projection part of isometries, which reduces the dimension of the Hilbert space. Thus, before truncation, the observable $O_0$ at the physical level is mapped to the semi-renormalized observables $U_0^\dagger O_0 U_0^\dagger \dagger$ since

$$\text{tr} (\rho_0 O_0) = \text{tr} (U_0^\dagger \rho_0 (U_0^\dagger)^\dagger U_0^\dagger O_0 (U_0^\dagger)^\dagger) \quad (7)$$

where $\rho_0$ is a density matrix at the physical level.

However, the crucial step of the renormalization scheme is to reduce the dimension of the Hilbert space. Formally, the idea is that $\tilde{\rho}_1 = U_0^\dagger \rho_0 (U_0^\dagger)^\dagger$ is not full rank but has the form $\tilde{\rho}_1 = \rho_1 \otimes |00\ldots0\rangle\langle00\ldots0|$. Thus, one can keep only the relevant degrees of freedom by applying an projector part of isometry $P$ which removes the superfluous degrees of freedom (see Fig. 2), i.e.,

$$P \tilde{\rho}_1 P^\dagger = \rho_1 \quad (8)$$

Hence, the expectation value of the physical operator $O_0$ can be written as

$$\text{tr} (\rho_0 O_0) = \text{tr} (P \tilde{\rho}_1 P^\dagger P U_0^\dagger O_0 (U_0^\dagger)^\dagger P) = \text{tr} (\rho_1 \mathcal{A}_0^1 [O_0]) \quad (9)$$

where $\mathcal{A}_0^1 [O_0]$ is a renormalized observable. The action of an ascending superoperator $\mathcal{A}_0^1$ is defined by

$$\mathcal{A}_0^1 […] = PU_0^\dagger […] (U_0^\dagger)^\dagger P \quad (11)$$
2. Multiple layers of renormalization

The reasoning to go from the physical level to the first renormalized level can be iterated. In that way, one defines an ascending superoperator from level 0 to level \( m \)
\( \mathcal{A}_0^m = \prod_{k=1}^{m} \mathcal{A}_k \), which maps operators at the physical level \( O_0 \) to operators acting at level \( m \).

We can express the superoperator \( \mathcal{A}_0^m \) as a matrix \( M_{ij} \) by choosing bases of observables \( \{ O_m \} \) at the physical level and \( \{ O'_m \} \) at level \( m \). Inferring the physical measurement corresponding to an effective measurement on renormalized particles then reduces to inverting this matrix to get \( M^{-1} \),

\[
M_{ij}^{-1} = \sum_i M_{ij} O_m^i
\]

\[
\text{tr}(\rho_m O'_m) = \sum_i M_{ij} \text{tr}(\rho_0 O'_m)
\]

B. Overhead on the number of physical measurements

The strategy to infer information about the renormalized state at level \( m \) is now clear: one performs measurements \( O_0 \) at the physical level and then use the knowledge of the gates in the circuit to compute the ascending superoperator \( \mathcal{A}_0^m \) and thus the renormalized observables \( \mathcal{A}_0^m [O_0] \).

Let us illustrate this approach for scale-invariant MERA in critical systems. In that case, translation-invariance and scale-invariance guarantee that isometries and disentanglers at all levels and sites are the same, which means scaling behavior of operators at all levels is invariant. Moreover, in ternary 1D MERA, we have a lattice site whose operator is mapped into one site operator (see Fig. 6). If \( O_0 \) has a support in that site, then the tensor network contraction for \( \mathcal{A}(O_0) \) can be simplified as in Fig. 6. This simplifies the tomography procedure. Indeed, to calculate the scaling, we only need information about the isometry \( w \).

We studied a few 1D critical models including Ising, XX, and Potts using ternary MERA code to study the scaling behaviors of observables. Let us focus on the case of the critical Ising model. Choosing the Pauli basis, \( \{ O^v \} = \{ I, \sigma_x, \sigma_y, \sigma_z \} \) for observables, the matrix representation of the descending superoperator \( M^{-1} \) reads

\[
(M_{ij}^{-1}) = \begin{pmatrix}
1 & 1.1 & 0 & 1.7 \\
0 & 2.01 & 0 & 1.55 \\
0 & 0 & 2.41 & 0 \\
0 & 0.3 & 0 & 2.41
\end{pmatrix}
\]

FIG. 6: Ascending super-operator and renormalized observable for a ternary MERA. The tensor network contraction turns a single-site operator \( O_i \) at level \( i \) into a single-site operator \( \mathcal{A}(O_i) \) at level \( i+1 \).

Let’s focus on the observable \( \sigma^y \) which is an eigenvector of the ascending superoperator since

\[
\mathcal{A}_0^1 [\sigma^y_0] = \frac{1}{\sqrt{\lambda_y}} \sigma^y_1
\]

where \( \sqrt{\lambda_y} = 2.41 \).

Using Eq. (12) one gets that

\[
\text{tr}(\rho_1 \sigma^y_1) = \sqrt{\lambda_y} \text{tr}(\rho_0 \sigma^y_0)
\]

One crucial point that we will now discuss is the statistical error on the expectation values due to the finite number of measurements. In practice, the measured expectation value \( \langle \sigma^y_0 \rangle_{\rho_0} \) will be equal to the proper expectation value \( \text{tr}(\rho_0 \sigma^y_0) \) up to some error \( \epsilon_0 \) which scales like \( N^{-1/2} \) where \( N_0 \) is the number of repeated measurements, i.e.,

\[
\langle \sigma^y_0 \rangle_{\rho_0} = \text{tr}(\rho_0 \sigma^y_0) \pm \epsilon_0
\]

When inferring the expectation value \( \text{tr}(\rho_1 \sigma^y_1) \), the uncertainty will also be multiplied, i.e.

\[
\langle \sigma^y_1 \rangle_{\rho_1} = \sqrt{\lambda_y} \langle \sigma^y_0 \rangle_{\rho_0}
\]

\[
= \sqrt{\lambda_y} \text{tr}(\rho_0 \sigma^y_0) \pm \sqrt{\lambda_y} \epsilon
\]

\[
= \text{tr}(\rho_1 \sigma^y_1) \pm \lambda_y \epsilon
\]

Thus, to maintain the accuracy \( \epsilon_0 \) at the renormalized level, one needs to perform a number of measurements

\[
N = \lambda_y N_0
\]

More generally, if \( O^v_i \) is not an eigenvector of the ascending super-operator, the total number of measurements need to be multiplied by \( \sum_i |M_{ij}^{-1}|^2 \).

This overhead in the number of measurements (i) will multiply with the number of particles if the observables is on many sites and (ii) will multiply between each layers. From a theoretical point of view, points (i) and (ii) are not catastrophic since they only correspond to a polynomial overhead. Indeed, for point (i), the number of particles in a tomography block is a constant, independent
of system size. For point (ii), the overhead depends on system size but is only polynomial. To go from level 0 to level \( n \), the multiplicative factor will be \( \lambda_m^n = \prod_{k=1}^{m} \lambda_{k-1}^k \) but there is only a logarithmic number of layers in the MERA circuit. Thus, from an asymptotic scaling point of view, the method works. However, for finite size system of interest, this overhead on the number of physical measurements can be dramatic. We will now see that the naive approach outlined here leads to a huge overhead in practice, before suggesting two improvements that will keep the total number of measurements reasonable.

1. Prohibitive experimental cost for 1-site observables

Returning to the example of the Ising model at criticality, we see from Eq. (15) that maintaining the accuracy at the renormalized level requires \( \lambda \approx 6 \) times the number of measurements than the one at the physical level. However, this analysis is appropriate only for one-site observable. This fact, which had not been appreciated in [19] has dramatic consequences.

Indeed, to identify five-sites reduced density matrix, we also need operators on two-sites, three-sites, and so on. If the multiplier for the number of measurement for one-site observable is \( \lambda \), then to obtain five-sites reduced density matrix with same precision, the total multiplier for the number of measurements for a block of 5 sites is

\[
\lambda_{\text{block}} = \frac{\sum_{n=1}^{5} \lambda^n 3^n \binom{5}{n}}{\sum_{n=1}^{5} 3^n \binom{5}{n}} = \left( \frac{3\lambda + 1}{4} \right)^5 \tag{23}
\]

i.e., it scales roughly like \( \lambda^5 \). Furthermore, for a block of renormalized operators at the third layer, the multiplicative overhead is \( (\lambda^2)^5 \). For \( \lambda = 6 \) as in our case, this amounts to \( 6^{10} \approx 6 \times 10^7 \). This is an impractical number of measurements for experimentalists. Therefore, we suggest two improvements to limit the overhead on the number of measurements.

2. Improvement 1: optimizing the choice of physical observables

Using tensor product of physical observables which are eigenvectors of the ascending superoperator was appealing from a theoretical point of view but leads to a huge number of measurements. Instead, one can vary over the physical observables and select a subset of them which maximizes information extraction. From a tomography point of view, any set of physical observables whose renormalized versions span the space of density matrices on the renormalized block is admissible. We expect that there exist many sets of admissible physical observables since the number of physical particles in the past light cone of a renormalized block is much larger than the number of renormalized particles in the block.

Of course, one needs to vary over experimentally accessible physical observables. In the case of qubits, we restrict ourselves to Pauli observables, i.e., tensor product of Pauli operators. When we map those Pauli observables to renormalized operators, they will become a set of non-orthogonal operators, each of which has different length and direction in the operator space. Among those renormalized operators, we can find a set of operators that give maximum information about the renormalized layer. In Sec. [IV] we will introduce a heuristic to perform this task and show that this approach significantly reduces the overhead on the number of physical measurements.

3. Improvement 2: changing the MERA geometry

Another suggestion is to use the binary MERA geometry rather than the ternary MERA geometry. The ternary MERA geometry is unfavorable since it requires to identify the 5-site reduced density matrix in the past light cone of each isometry, while for the binary MERA, one needs to identify only 4-site reduced density matrix. This can make a significant difference for finite size systems.

Moreover, the binary MERA geometry is practical to apply the algorithm to select the observables which give the most information about the renormalized density matrix. Indeed, the past light cone of the 4-site reduced density matrix at level \( \tau \) is 10 sites at level \( \tau - 1 \) for the binary MERA geometry, much fewer than the 17 sites required in the ternary MERA geometry. Thus, we chose to select among the \( 4^{10} \) Pauli observables on 10 qubit a subset of \( 4^4 \) which give maximal information about the 4-qubit density matrix at the next level using a heuristic which will be presented in details in Sec. [IV A]. Numerically, we found that restricting the Pauli observables to act only on the 8 qubits indicated on Fig. [7] gave satisfactory results and made the running time and memory requirements of the heuristic more reasonable.

IV. NUMERICAL RESULTS

A. Optimizing the choice of physical observables

To optimize the choice of physical observables, we used a heuristic approach. We tested on a 24-qubit ground state of the critical Ising and XX models. The state used to represent the experimental state is a \( \chi = 2 \) binary MERA approximation to the ground state, which is obtained by a MERA energy minimization program.
The observable $O_{\tau-1}$ acting on 8 sites at level $\tau-1$ is renormalized into a four-sites operator $O_{\tau} = A(O_{\tau-1})$.

1. Greedy algorithm to maximize the determinant

Given the disentanglers and isometries between levels $\tau-1$ and $\tau$ (which would have been identified thanks to tomography procedures), we calculated the 4$^4$ renormalized operators corresponding to Fig. 7. The task is now to choose a subset of 4$^4$ renormalized operators that (i) span the space of the 4-qubit density matrix and (ii) span it in a way that maximizes the information acquisition (and thus minimize the number of repeated measurements). Criterion (ii) would be interesting to investigate from a theoretical point of view. In our work, we chose to maximize the absolute value of the determinant of the set of renormalized operators as a proxy to maximizing the information acquisition. The intuition is that a large determinant will correspond to a set of renormalized operators that span well the space of the 4-qubit density matrix.

To maximize the determinant, we used the following heuristic. We first chose the renormalized observable with the largest norm (choosing the norm induced by the Hilbert Schmidt inner product). Then, we vary over the remaining renormalized observables to find one that maximizes the determinant with the first one. We repeat this procedure over and over, obtaining a greedy algorithm to select the 4$^4$ renormalized operators. This algorithm, named ‘Longest residual vector selection (LRV)’, in [25], is one of approaches for the problem found in classic signal processing problem, called matching pursuit.

The LRV algorithm is a heuristic which can be sub-optimal. Let’s illustrate such a situation by considering a simple two-dimensional space spanned by the orthonormal vectors $\hat{e}_1$ and $\hat{e}_2$. Consider the candidate set $\{\hat{e}_1, \frac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2), \frac{1}{\sqrt{2}}(\hat{e}_1 - \hat{e}_2)\}$ where $\epsilon > 0$ is small. We want to choose 2 vectors which maximize the absolute value of the determinant. By inspection, the best choice is $\{\frac{1-\epsilon}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2), \frac{1-\epsilon}{\sqrt{2}}(\hat{e}_1 - \hat{e}_2)\}$ which has determinant $(1 - \epsilon^2)$. However, the LRV algorithm will first select $\hat{e}_1$ which has maximal norm and then select either one of the two remaining vector resulting in the choice $\{\hat{e}_1, \frac{1-\epsilon}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2)\}$ which has determinant $\frac{1-\epsilon}{\sqrt{2}}$. For a non-zero small $\epsilon$, the choice made by the LRV algorithm is dramatically worse than the optimal choice. In [25], an algorithm called ‘one by one replacement’ is introduced to improve a (suboptimal) set of vectors by iteratively identifying bad choices in the current set and replacing it by a better vector from the candidate set of vectors.

In our work, we first use the LRV algorithm to select 4$^4$ renormalized operators from the candidate set made of 4$^8$ operators $A(O_{\tau-1})$ of Fig. 7. We then use the ‘one by one replacement’ algorithm to improve this initial choice. Remember that maximizing the determinant is only a proxy to minimize the total number of measurements required to maintain a desired level of accuracy on the tomographic reconstruction. We now discuss how the choice of renormalized operators impacts the number of repeated measurements of physical observables needed to maintain accuracy.

2. Maintaining the accuracy level using renormalized operators

From now on, let us consider the set of chosen renormalized operators $\{O_i^1\} = \{A(O_0^i)\}$. Since the renormalized operators $O_i^1$ are not orthogonal, it is convenient to construct a set of orthogonal operators, following the approach introduced in [19]. We first define the Gram-matrix $G_{ij} = \text{tr}[O_i^1(O_j^1)\dagger]$, and diagonalize it to obtain the matrices $Z$ and $D$ such that $G = ZDZ^\dagger$. Then, we obtain a set of orthogonal operators $\{R_i, i = 1, 2, 3, \ldots\}$, which are eigenvectors of $G$, i.e.,

$$R_i^1 = \frac{4}{\sqrt{D_{ii}}} \sum_j Z_{ij}^1 O_i^1 = \sum_j \beta_{ij} O_i^1$$  \hspace{1cm} (24)$$

We introduced a factor 4 to normalize the operators $R_i^1$ to have the same trace norm as 4-site Pauli observables. Thus, we can relate the expectation value of the orthonormal operators $R_i^1$ to the expectation values of the physical observables by

$$\text{tr}(\rho_i R_i^1) = \sum_j \beta_{ij} \text{tr}(\rho_i O_i^1) = \sum_j \beta_{ij} \text{tr}(\rho_0 O_0^i)$$  \hspace{1cm} (25)$$

To assess how the number of physical measurements $N_j$ on $O_0^j$ is increased, consider that the measurement of $R_i^1$ is a random variable whose variance is $\nabla(R_i^1)$. Since physical measurements are performed on different copies of the states, the physical measurements correspond to independent variables and

$$\nabla(R_i^1) = \sum_j |\beta_{ij}|^2 \nabla(O_0^j)$$  \hspace{1cm} (26)$$

Let $M_i(\epsilon)$ be the number of measurements needed to achieve a desired variance $\epsilon$ for $i$th renormalized observable. Since variance $\nabla$ is proportional to the inverse of the number of physical measurements, $N_j$ of $O_0^j$ is related
to $M_i(\epsilon)$ by

$$\forall i \quad (M_i(\epsilon))^{-1} = \sum_j |\beta_{ij}|^2 N_j^{-1} \quad (27)$$

Let’s define the matrix $B_{ij} = |\beta_{ij}|^2$. We want to minimize the total number of measurements $N = \sum_j N_j$ while maintaining the same minimum precision $1/M_0$. We thus want minimize $N$ under the condition

$$\forall i \quad \sum_j B_{ij} N_j^{-1} \leq \frac{1}{M_0} \quad N_j \geq 0 \quad (28)$$

Note that we cannot simply choose to minimize the $N_j$ independently since the precision level of different operators are not independent under the condition $N_i \geq 0$. We found numerically that in most instances we looked at, we cannot avoid the situation in which some observables have better precisions than the others.

Introducing the normalized variables $\tilde{N}_j = N_j/M_0$, we are faced with the optimization problem of minimizing $\sum_j \tilde{N}_j$ under the constraint

$$\forall i \quad \sum_j B_{ij} \tilde{N}_j^{-1} \leq 1 \quad \tilde{N}_j \geq 0 \quad (29)$$

For the $4^4 \times 4^4$ matrix $B_{ij}$, considering the maximal element $\gamma_j = \max_i B_{ij}$ for every column, we know that

$$\forall j \quad \sum_i B_{ij} (\gamma_j)^{-1} \leq 4^4 \quad (30)$$

Thus, a naive choice of $\tilde{N}_j$ would be to choose $4^4 \gamma_j$. Alternatively, we calculated $K_j = \sum_i B_{ij} (\gamma_j)^{-1}$ which is guaranteed to be smaller than $4^4$ and consider the biggest of them $K = \max_j K_j$. We can then take $\tilde{N}_j = K \gamma_j$ to guarantee that Eq. (29) is satisfied.

The total number of measurements $N$ is

$$N = \sum_i \tilde{N}_i M_0 \quad (31)$$

where $\tilde{N}_i$ can be interpreted as a multiplicative factor which ensures that the estimation of the expectation value using renormalized operators has the same precision as the one obtained using $M_0$ measurements on physical Pauli measurements. To report a single number, we introduce the conditioning factor $S$, defined as the average multiplier in the number of measurements

$$S \equiv \frac{\sum_i \tilde{N}_i}{4^4} \quad (32)$$

3. Estimation of the conditioning factor

We wrote a simulation code to estimate the conditioning factor $S_{k\rightarrow k+1}$ corresponding to the multiplicative factor needed to estimate the 4-site density matrix at level $k + 1$ using Pauli measurement at level $k$.

![FIG. 8: The tensor product of $O_A \otimes O_B$ renormalize under the superoperator $A$ into the operator resulting from the tensor network contraction. In the upper figure, the distributive law (34) holds so that the renormalized operators $A(O_A)$ and $A(O_B)$ can be computed independently and then multiplied. For the figure below, disentanglers within the blue box mixes the renormalized operators and Eq. (34) does not hold.](image)

Note that, experimentally, we are interested in the multiplicative factor $S_{0\rightarrow \tau}$ between physical Pauli measurement, i.e. measurement at level 0, and renormalized operator $\{R_i^1\}$ at level $\tau > 0$, defined by Eq. (24). Let’s consider $\tau = 2$ for concreteness.

We would like to argue that

$$S_{0\rightarrow 2} \simeq S_{0\rightarrow 1} \times S_{1\rightarrow 2} \quad (33)$$

where the approximation comes the fact the ascending superoperator is not distributive, as we now explain.

Since the $\{R_i^1\}$ are orthogonal operators with the same normalization as Pauli operators $\{\Sigma_i^1\}$, there is a unitary transformation mapping between those two set of operators. Thus, mapping $\{R_i^1\}$ or Pauli operators at level 1 $\{\Sigma_i^1\}$ to $\{R_i^2\}$ will have the same overhead because of unitarity. However, when mapping Pauli observables at level 1 to renormalized operators at level 2, we take the tensor product of Pauli operators on two blocks to compute $S_{1\rightarrow 2}$. The renormalized operators $\{R_i^1\}$ on two neighboring block do not obey this tensor product structure. This is illustrated in the bottom figure of Fig. 8.

If the ascending superoperator is distributive, i.e.,

$$A(O_A \otimes O_B) = A(O_A) \otimes A(O_B) \quad (34)$$

for physical operators $O_A$ and $O_B$, the approximation (33) is exact. However, this is not true for if $O_A$ and $O_B$ are 8-sites Pauli operators and there will be a deviation $\mathcal{E}$ from the distributive law

$$A(O_A \otimes O_B) = A(O_A) \otimes A(O_B) + \mathcal{E} \quad (35)$$
resulting from the mixing of operators at the blue box in Fig. 8. However, since the contribution of 2-sites on to the whole 16-sites would be small, we expect the approximation \((33)\) to be good.

To test the quality of the approximation \((33)\), we performed a simulation to get the exact scaling factor \(S_{0 \to 2}\) between the physical level and the second renormalized level and compared it to the product \(S_{0 \to 1} \times S_{1 \to 2}\) using a 16-qubit MERA approximation to the groundstate of the critical Ising model. The scaling factor \(S_{0 \to 2}\) was obtained by following procedure (see Fig. 8): for each layer and compared it to the product \(S_{0 \to 1} \times S_{1 \to 2}\) of the physical level and the second renormalized counterparts \(A(\mathcal{O}_A^i)\) (resp. \(A(\mathcal{O}_B^i)\)) to find the optimal basis \((4^4)\) maximizing determinant. Then, we have two basis sets with \(4^4\) operators \(\{A(\mathcal{O}_A^i), i = 1, 2, ..., 4^4\}\) and \(\{A(\mathcal{O}_B^i), i = 1, 2, ..., 4^4\}\). Now, to estimate the reduced density matrix on 4 sites at the second renormalized level, we need to find the best \(4^4\) operators out of \(\{A(\mathcal{O}_A^i \otimes \mathcal{O}_B^j)\}\). The distributive law \((34)\) holds for \(\mathcal{O}_A^i\) and \(\mathcal{O}_B^j\) since they were based on non-interfering six-site operators at the physical level, we can easily calculate the 4\(^4\) operators \(A(\mathcal{O}_A^i \otimes \mathcal{O}_B^j) = A(\mathcal{O}_A^i) \otimes A(\mathcal{O}_B^j)\). Now, out of these \(4^4\) operators, we renormalize them again using the second renormalized layer, and then find the \(4^4\) optimal operators.

We ran the simulation several times and obtained values for the scaling factor \(S_{0 \to 2}\) ranging between 23 and 27, which is comparable to \(S_{0 \to 1} \times S_{1 \to 2}\) \((25-36)\). Therefore, we consider the approximation \((33)\) to be valid. In fact, the method used to test this assumption gives a scalable way to obtain the optimal set of physical Pauli operators to estimate reduced density matrix at higher renormalized level.

Now that we assessed the quality of the approximation \((33)\), we will use the formula

\[
S_{0 \to \ell} \simeq \prod_{k=0}^{\ell-1} S_{k \to k+1} \tag{36}
\]

to approximate the total number of measurements needed for MERA tomography at level \(\ell\).

We estimated the conditioning factor for MERA tomography on a 24-qubit translation-invariant binary MERA approximation of the ground state of the critical Ising model with periodic boundary condition. Note that the finite system size is too small to reach scale-invariance, which we expect in the thermodynamic limit. However, translation invariance guarantees that disentangler and isometries are the same in a given layer of the MERA circuit. Since a 24-qubit binary MERA circuit contains 3 renormalization layers, we obtained three conditioning factor \(S_{k \to k+1}\) for \(k = 0, 1, 2\).

Results for \(S_{0 \to \ell}\) for \(\ell = 1, 2, 3\) are presented on Fig. 9 for 10 different MERA approximation of the ground state of the critical Ising model for 24 qubit. Since the disentanglers and isometries are different for every energy minimization, the condition factors also vary.

B. Estimates of the total number of measurements required for MERA tomography

We are now in position to give an estimate of the total number of physical measurements needed to perform MERA tomography. We estimate these numbers by using the conditioning factor and by choosing the reference number of measurements to be \(M_0 = 100\). This is the number of measurements used to estimate the expectation value of every Pauli operators in the tomography of an 8-qubit W state on cold atoms \([3]\).

In Fig. 10, we compare the total number of measurements \(N\) for binary and ternary MERA, in both cases for the groundstate of the critical 1D Ising and critical 1D XX models, as a function of the size of a quantum system \(n\), i.e., the number of qubits. For ternary MERA, we use the naive approach based on observables which are eigenvectors of the ascending superoperator of Fig. 6. For binary MERA, we used the heuristic choice of observables which maximizes the determinant, obtaining a condition factor \(\lambda\) varying between 5 and 6 for critical Ising and between 3 and 3.5 for XX model. The total number of measurements is related to the scaling factor
Ising, Binary
XX, Binary
Ising, Ternary
XX, Ternary
Brute Force

FIG. 10: The number of measurements required versus the number of qubits in MERA tomography. For binary MERA, we selected the renormalized operators using the heuristic described in Sec. IV A I and for ternary MERA, we used the naive approach of taking one-site operators. The error bars account for the uncertainty in the condition factor. We used $5 < S_{\text{Ising}} < 6$ and $3 < S_{\text{XX}} < 3.5$

through the formula

$$N = 100 \times 4^4 \sum_{\tau=0}^{m-3} 2^{m-\tau+1} S^\tau + 4^D S^{m-2}$$

(37)

where $m$ is the total number of layers, $2^{m-\tau+1}$ is the number of isometries between level $\tau$ and $\tau + 1$ and the last term comes from the fact that at level $m - 2$ there are $D \leq 4$ renormalized particles.

We also indicated the scaling of brute-force quantum state tomography using the $3^n$ approach of [6]. The figure confirms the asymptotic advantage of MERA tomography whose polynomial scaling $N \propto n^{\log_2 D}$ outperforms the exponential cost of brute-force tomography, even for small system size.

To better appreciate the performance of MERA tomography for system size relevant to experiments, we plotted the total number of measurements needed for binary MERA of the critical Ising and XX models on Fig. 11. We compared the number of measurements to the 656,000 measurements used in the largest tomography experiment performed to date, on a 8-qubit system [3].

Fig. 11 shows that MERA tomography outperforms brute-force tomography for system sizes that are accessible experimentally, and requires a reasonable experimental effort. More specifically, using our scheme we can perform MERA tomography on a 16-qubit ground state of the critical Ising model and a 24-qubit ground state of the critical XX model with at most twice the number of measurements of the qubyte experiment. Hence, MERA tomography, for a comparable experimental effort, allows to probe quantum systems twice to three times larger than brute-force quantum tomography. Moreover, the numerical processing required by MERA tomography is very simple and requires at most a few hours of running time, which is a dramatic improvement over the running time of the Maximum Likelihood Estimation (MLE) used to infer the quantum state compatible with the experimental data.

V. PROPAGATION OF ERRORS

The MERA tomography procedure aims to reconstruct a MERA state $\rho_{\text{tomo}}$ that approximates the experimental state, defined by

$$\rho_{\text{tomo}} = U_0^\dagger \rho_{\text{trunc}} U_0$$

(38)

where $U_0 = \prod_{j=1}^{m-1} U_{j \to j+1}$ is the global MERA circuit and $\rho_{\text{trunc}}$ is the output after the final $m$-th layer. However, our reconstructed state will deviate from the experimental state $\rho_0$ due to (1) imperfect estimation of expectation values of physical observables and (2) MERA tomography introduces truncation errors since each isometry throws out part of the Hilbert space.

A detailed analysis of the impact of truncation errors is presented in the appendix. We will highlight the key results in this Section and refer to interested reader to the Appendix for the technical proofs.

Loss of information in the MERA circuit is due to truncation errors. For every level $\tau \geq 1$, consider $\rho_\tau$ to be the state obtained from the experimental state $\rho_0$ by applying every gates of the quantum circuit before truncation at level $\tau$. Define $\rho_{\tau\text{trunc}}$ to be the normalized state obtained from keeping the eigenvectors corresponding to the $\chi$ largest eigenvalues of $\rho_\tau$. Those different states are represented on Fig. 12.
We prove in the Appendix (see Lemma A.1) that

\[ D(\rho_0, \rho_{\text{tomo}}) \leq \sum_{\tau=1}^{m} D(\rho_\tau, \rho^\text{trunc}_\tau) \] (39)

where we use the trace distance \( D(\rho, \sigma) = \frac{1}{2} \| \rho - \sigma \|_1 \) and \( \| A \|_1 \) is the sum of the singular values.

However, the disentanglers and isometries are computed on blocks of the state \( \rho^\text{rec}_\tau \), which is reconstructed for evaluating how expectation values physical observables relate to expectation values of ascended operators on renormalized particles. Because of truncation errors, the ascended operators will be erroneous and the estimated \( \rho^\text{rec} \) will differ from \( \rho_\tau \). We can thus use the triangle inequality to get

\[ D(\rho_0, \rho_{\text{tomo}}) \leq \sum_{\tau=1}^{m} D(\rho_\tau, \rho^\text{rec}_\tau) + \sum_{\tau=1}^{m} D(\rho^\text{rec}_\tau, \rho^\text{trunc}_\tau) \] (40)

The second term of Eq. 40 is straightforward. It is the intrinsic error introduced by truncation errors and we prove that

\[ \sum_{\tau=1}^{m} D(\rho^\text{rec}_\tau, \rho^\text{trunc}_\tau) \leq \sum_{\tau=1}^{m} \sum_{k=1}^{\ell} \epsilon^\tau_k \] (41)

where \( k \) indexes the different isometries at level \( \tau \) and \( \epsilon^\tau_k \) is the probability weight being removed by the truncation. Note that this term is simply the sum of all truncation errors and will scale linearly with the size of the system.

The first term of Eq. 40 is due to the relations between renormalized operators and physical observables. It will be related not only to truncation errors at level \( \tau \) but also to all truncation errors at previous levels. We prove that

\[ \sum_{\tau=1}^{m} D(\rho_\tau, \rho^\text{rec}_\tau) \leq \frac{1}{2} \sum_{\tau=1}^{m} \sum_{\ell=1}^{\tau} \sum_{k=1}^{\ell} \epsilon^\ell_k \| \sum_{i,j} \beta_{ij} R_\ell \|_1 \] (42)

where the matrix \( \beta_{ij} \) and the \( R_\ell \) are defined in Sec. I. Note that this term will scale quadratically with system size since truncation errors in previous levels influence truncation errors in subsequent levels. However, since the truncation error \( \epsilon \) gets dramatically smaller for higher layer (see Fig. 5), and we observed numerically that the summation \( \| \sum_{ij} \beta_{ij} R_\ell \|_1 \) has the order of unity for critical Ising, we expect this contribution to be comparable to the Eq. 11.

The final bound will simply be the sum of Eq. 42 and Eq. 41. Crucially, every term that appears in this bound can be estimated during the tomographic procedure. Thus, it can be used as a certificate to check a posteriori if the reconstruct MERA state is indeed close to the experimental state. We expect this to give reasonable bounds in the limit where the truncation errors are very small. However, this might not be the case. Alternatively, one can directly estimate fidelity to assess the closeness between the experimental state and the state obtained by MERA tomography using Monte Carlo fidelity estimation [26, 27].

**VI. CONCLUSION**

In this work, we investigated and improved upon the original MERA tomography method introduced in [19]. First, we proposed and tested a new optimization technique for MERA tomography. Our new optimization technique easily gives exact reconstruction of original states for any random MERA states for both binary and ternary MERA geometries. We also studied the performance of MERA tomography when the prepared state is not exactly a MERA state, and data indicates that the procedure was robust to noise.

Second, we studied the way to identify density matrices of renormalized particles in detail. We showed that the scaling of the number of measurements required to maintain accuracy presented in [19] was only valid for single site observable and that its straightforward application to multi-site observables led to a very large overhead. To circumvent this issue, we suggested to use a different MERA geometry, namely binary MERA, which required performing brute-force tomography on block of 4 renormalized sites (instead of 5 for the ternary MERA case). Furthermore, we introduced a heuristic to identify the physical measurement which give the most information about renormalized particles, in order to minimize the number of physical measurement required. We tested this approach numerically and found that the total number of physical measurements needed to perform MERA tomography on moderate size system is reasonable for experimentalist. For instance, performing MERA tomography on the ground state of the critical Ising model on 16 qubits requires only twice the number of physical measurements than brute-force quantum state tomography on 8 qubits.

Third, we gave a deeper understanding of propagation of error in MERA tomography. We bounded the distance between the experimental state and the state reconstructed by MERA tomography in terms of quantities that are estimated locally throughout the tomography procedure. In particular, the propagation of error when using renormalized observables was quantified, and
turned out to be closely related to the scaling factor $S$. Since the deviation of the reconstructed state from the experimental state is bounded by a quantity which can be estimated during the tomography procedure, this bound can be used as a certificate to justify a posteriori that the experimental state was close to a MERA state.

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Appendix A: Error analysis

We want to assess how the errors accumulate throughout the tomography procedure. One source of errors is the imperfect estimation of expectation values of physical observables due to the finite number of repeated measurements, fluctuations in the state preparation and measurement errors. This source of errors is common to all tomography schemes and putting meaningful error bars on brute-force tomography is a complex issue which is an active area of research [28, 29]. We will not address this issue here, assuming that expectation values of physical observables are perfect. However, introducing those errors would be straightforward in our analysis.

Instead, we will focus on the errors which are introduced by the idea of MERA tomography itself. More precisely, the truncation errors at the level of every isometry will introduce i) intrinsic errors since part of the information on the state is thrown away and ii) reconstruction errors since the relationship between renormalized operators and physical observables is not exact.

We will first focus on intrinsic errors by assuming that any reduced density matrix in the circuit can be obtained exactly in Sec. A1. In a second step, we will estimate the error introduced by using renormalized operators to extract information about density matrices in higher level of the MERA circuit in Sec. A2.
the renormalized observable also as support on the virtual space that is truncated by the MERA circuit

\[ O_1 = u_1 O_0 u_1^\dagger = A(O_0) + E \quad (A2) \]

For convenience, let’s consider the normalized states $\rho^{\text{trunc}} = \rho_r/(1 - \epsilon)$ and $\hat{\rho}_e = \rho_e/\epsilon$. Then, in the imperfect MERA setting, Eq.12 is modified and reads

\[ \text{tr}(\rho_0 O_0) = (1 - \epsilon) \text{tr}(\rho^{\text{trunc}} A(O_0)) + \text{ctr}(\hat{\rho}_e E) \quad (A3) \]

Because we truncate out $\rho_e$ and measure $\rho^{\text{trunc}}$ in next layer for actual MERA tomography, there will be a discrepancy $|\text{tr}(\hat{\rho}_e E)|$ between the measured quantity $\text{tr}(\rho_0 O_0)$ and the quantity of interest $\text{tr}(\rho^{\text{trunc}} A(O_0))$.

In the rest of the discussion, we consider $\text{ctr}(\hat{\rho}_e E)$ as a random error about which only $\epsilon$ is known.

b. Error analysis for a layer

Having understood how an error $\epsilon$ affects a single isometry, we will now assess how the errors for each isometry combine inside a layer. Let’s consider the global state after it goes through the isometries in figure 12. As we discussed above, if the MERA circuit is not perfect, the global state $\rho$ would have support over the virtual subspace represented by red lines and let

\[ \rho = \rho_r + \rho_e \quad (A4) \]

Since only $\rho_r$ is passed through the MERA circuit and $\rho_e$ is thrown away, $\rho_e$ is informationally inaccessible in the later step of tomography. However, we want to quantify $\rho_e$ in order to bound the distance between $\rho$ and $\rho_r$.

To achieve our purpose, we will divide $\rho_e$ into smaller parts. We will label the isometry with an index $i = 1, \ldots, \ell$. Let $\mathbb{C}_i = \{0\}^{\otimes k-1} \otimes \mathbb{V}$ be the correct $\chi$-dimensional subspace kept by the MERA circuit, and $\mathbb{E}_i = \mathbb{V}^{\otimes k} \setminus \mathbb{C}_i$ be the incorrect subspace, indicated by red lines on Fig. 12. Now let $\rho_i$ be the reduced density matrix before the $i$-th isometry. Then, the density matrix at the input of the $i$-th isometry is

\[ \rho_i = \rho_i^r + \rho_i^e = \text{tr}_i(\rho) \quad (A5) \]

where $\text{tr}_i$ represent the partial trace on all sites except the input of the $i$-th isometry and $\rho_i^r \in \mathbb{C}_i$ and $\rho_i^e \in \mathbb{E}_i$ (cf. Fig. 13).

While the globally correct state $\rho_i$ is locally in the correct subspace $\mathbb{C}_i$ for all $i$, the globally incorrect state $\rho_e$ contains part which are locally correct for some $i$ and locally incorrect for some non-empty set $I \subset [1; \ell]$. We will denote $\rho_{eI}$ the part of $\rho_e$ having support on the subspace $(\otimes_{i \in I} \mathbb{E}_i)(\otimes_{i \notin I} \mathbb{C}_i)$. Thus,

\[ \rho_e = \sum_I \rho_{eI} \quad (A6) \]
Note however, that $\rho_{c, j}$ is locally correct for $i \notin I$. The terms of the density matrix at the input of the $i$-th isometry of Eq. A5 decomposes into

$$\rho^i_c = \text{tr}_i(\rho_r + \sum_{i \notin I} \rho_{c, i}) \quad (A7)$$

$$\rho^i_c = \text{tr}_i(\sum_{i \in I} \rho_{c, i}) \quad (A8)$$

The density matrices $\rho^i_c$ are estimated by physical measurements and the numerical optimization gives the trace of the locally incorrect state $\text{tr}(\rho^i_c) = 1 - f(U) = \epsilon_i$. Knowing this error for each isometry on the layer allows us to estimate the weight of the globally incorrect state

$$\epsilon_i \equiv \text{tr}(\rho_{c, i}) = \sum_{i \in I} \text{tr}(\rho_{c, i}) \quad (A9)$$

$$\leq \sum_{i=1}^{\ell} \sum_{i \in I} \text{tr}(\rho_{c, i}) \quad (A10)$$

$$= \sum_{i=1}^{\ell} \sum_{i \in I} \rho_{c, i} = \sum_{i} \epsilon_i \quad (A11)$$

The normalized truncated state $\rho^{\text{trunc}} = \rho_r / (1 - \epsilon_i)$ is the one we are interested in learning in the next step of variational tomography. The distance between the state before truncation $\rho$ and the normalized truncated state $\rho^{\text{trunc}}$ can be bounded in fidelity and in trace distance.

For the fidelity, one can observe that

$$F(\rho, \rho^{\text{trunc}}) = \|\sqrt{\rho^{\text{trunc}}} \sqrt{\rho}\|_1 = \|\sqrt{\rho^{\text{trunc}}} \sqrt{\rho_r}\|_1 = \sqrt{1 - \epsilon_i} \quad (A12)$$

and for the trace distance

$$\|\rho - \rho^{\text{trunc}}\|_1 \leq \frac{\epsilon_i}{1 - \epsilon_i} \|\rho_r\|_1 + \|\rho_{c, i}\|_1 \leq 2\epsilon_i \quad (A13)$$

Using those relations and Eq. A11 we get

$$1 - F(\rho, \rho^{\text{trunc}}) \leq \frac{1}{2} \sum_{i} \epsilon_i \quad (A14)$$

$$\frac{1}{2} \|\rho - \rho^{\text{trunc}}\|_1 \leq \sum_{i} \epsilon_i \quad (A15)$$

### c. Error analysis of the global circuit

The final goal is to estimate the distance between the physical state $\rho_0$ and the reconstructed MERA state

$$\rho_{\text{tomo}} = U^+_0 \rho^{\text{trunc}}_m U_0 \quad (A16)$$

where $U_0 = \prod_{j=0}^{m-1} U_j$ is the global MERA circuit and $\rho^{\text{trunc}}_m$ is the output after the final $m$-th layer. The idea is to relate this distance to the truncation error $d(\rho^i_c, \rho^{\text{rec}}_j)$ introduced after each layer $j$ of the MERA circuit when the state $\rho_j$ is truncated to $\rho^{\text{trunc}}_j$. We will use the following lemma.

**Lemma A.1.** For any distance $d(\sigma, \rho)$ which obeys the property $d(U\sigma U^+ , \rho) = d(\sigma, U^+ \rho U^+) \equiv d(\sigma, \rho)$ and the triangle inequality, the following inequality holds

$$d(\rho_0, \rho^{\text{tomo}}) \leq \sum_{r=1}^{m} d(\rho_r, \rho_j^{\text{trunc}}) \quad (A17)$$

**Proof.** For $m=1$, we have

$$d(\rho_0, U_0^+ \rho U_0 \rho^{\text{rec}}_0 U_0) = d(U_0^+ \rho_0 U_0 \rho^{\text{rec}}_0 U_0, \rho^{\text{rec}}_0) \quad (A18)$$

$$= d(\rho_1, \rho_1^{\text{rec}}) \quad (A19)$$

For arbitrary $m > 1$, we have

$$d(\rho_0, U_0^+ m \rho^{\text{trunc}}_m U_{0-m} \rho^{\text{rec}}_0 U_{0-m}) \leq d(\rho_0, U_0^+ m \rho^{\text{trunc}}_m U_{0-m-1} \rho_0 U_{0-m-1} U_0^+ m \rho^{\text{trunc}}_m U_{0-m}) \quad (A20)$$

The last term can be rewritten as

$$d(U_{m-1} \rho^{\text{trunc}}_m U_{m-1} \rho^{\text{tomo}}_{m-1} \rho^{\text{tomo}}_m) = d(\rho_m, \rho^{\text{tomo}}_m) \quad (A21)$$

Recursively applying this inequality proves the lemma.

$$\Box$$

We will now apply this lemma to the distance corresponding to the fidelity and the trace distance.

The fidelity can be used to define the distance

$$\theta(\rho, \sigma) = \arccos F(\rho, \sigma) \quad (A22)$$

For every layer, we assume that the error is small enough for the approximation $\cos(\theta) \approx 1 - \theta^2 / 2$ to hold. Combined with Eq. A13 we get

$$\forall j \quad \theta(\rho_j, \rho_j^{\text{tomo}}) \approx \left( \sum_i \epsilon_i^j \right)^{1/2} \quad (A23)$$

where $\epsilon_i^j$ is the error for the $i$-th isometry in layer $j$.

Applying the lemma, we get

$$\theta(\rho, \rho^{\text{tomo}}) \leq \sum_{j=1}^{m} \theta(\rho_j, \rho_j^{\text{tomo}}) \approx \sum_{j=1}^{m} \left( \sum_i \epsilon_i^j \right)^{1/2} \quad (A24)$$

To relate the distance to the fidelity, we use the inequality $\cos(\theta) \geq 1 - \theta^2 / 2$ which implies

$$1 - F(\rho, \rho^{\text{tomo}}) \leq \frac{1}{2} \left[ \sum_{j=1}^{m} \left( \sum_i \epsilon_i^j \right)^{1/2} \right]^2 \quad (A25)$$

$$= \frac{1}{2} \sum_{ij} \epsilon_i^j + \sum_{i,j} \sqrt{\sum_i \epsilon_i^j \epsilon_i^{j'}} \quad (A26)$$

where the first term in Eq. A25 is the incoherent sum of all truncation errors whereas the second term of
Eq. (A26) are due to coherent interference of errors in different layers and isometries.

We can also directly apply the lemma to the trace distance, applying Eq. [A15] to relate the terms to the truncation errors to obtain:

\[ D(\rho, \rho^{\text{tomo}}) \leq \sum_{j=1}^{m} \sum_{i} \epsilon_{i}^j \]  \hspace{2cm} (A27)

Figure 14 compares the upper bound obtained by Eq. (A25) to the (in)fidelity between simulated experimental states which are the superposition of a random MERA state with a Haar-random state of magnitude \( \epsilon \). The results on the figure show that the theoretical upper bound for \( 1 - F(\rho, \rho^{\text{tomo}}) \) is a useful proxy.

We can also directly apply the lemma to the trace distance, applying Eq. [A15] to relate the terms to the truncation errors to obtain:

\[ D(\rho, \rho^{\text{tomo}}) \leq \sum_{j=1}^{m} \sum_{i} \epsilon_{i}^j \]  \hspace{2cm} (A27)

Note that the magnitude of the Haar-random state \( \epsilon \) sets the value of the (in)fidelity between the experimental state and the tomographically reconstructed state since the MERA tomography seems to reconstruct the MERA part of the experimental state, leading to \( 1 - F \approx \epsilon^2 \), which appears clearly on Fig. 14.

The upper bounds of Eqs. (A25) and (A27) can be estimated directly from tomographic data obtained during the reconstruction. Thus, they are a certificate on the distance between the experimental state and the one reconstructed by MERA tomography.

The error analysis until now assumed that we had access to perfect tomographic estimate of the reduced density matrices on small blocks of particles. However, when we use the structure of the MERA circuit to relate physical measurements to renormalized observables, the truncation error will inevitably introduce errors on the tomographic estimates. We now discuss those type of errors and see how they modify our error bounds.

2. Error introduced by renormalizing physical measurements

As described in Sec. A1.2, the truncation errors will not only introduce an intrinsic error, but also lead to an erroneous reconstruction of the reduced density matrix in renormalized layer. Indeed, Eq. (A3) which relates the expectation value of the physical observable \( \text{tr} \left[ \rho_{0} O_{0}^{j} \right] \) to the expectation value of \( \mathcal{A}^j(O_{0}^{j}) \), the renormalized observable at level \( \tau \), on the state we want to reconstruct \( \rho^{\text{trunc}} \) contains a random error term

\[ \Delta^j_{\tau} = \epsilon^j_{\tau} \text{tr} \left[ \rho^{\tau}_{\text{rec}} E^j \right] \]  \hspace{2cm} (A28)

where \( \epsilon^j_{\tau} \) is bounded thanks to Eq. (A11). This erroneous reconstruction will introduce an additional term in the error bound (A27), which we now analyze.

The reduced density matrix at a renormalized level \( \rho^{\tau} \) will be reconstructed using the the orthonormal operators \( R_i \), see Eq. (24), which span the entire Hilbert space for density operator. Due to the erroneous terms \( \Delta^j_{\tau} \), we have

\[ \rho_{\tau} = \sum_{i} \text{tr} [\rho^{\tau} R_{i}] R_{i} = \sum_{i,j} \beta_{ij} \text{tr} [\rho^{\tau} O_{0}^{j}] R_{i} \]  \hspace{2cm} (A29)

\[ = \sum_{i,j} \beta_{ij} \left( \text{tr} [\rho_{\tau-1} O_{0}^{j}] \Delta^j_{\tau-1} \right) R_{i} \]  \hspace{2cm} (A30)

\[ = \sum_{i,j} \beta_{ij} \left( \text{tr} [\rho_{0} O_{0}^{j}] - \sum_{\ell=1}^{\tau} \Delta^j_{\ell} \right) R_{i} \]  \hspace{2cm} (A31)

\[ = \sum_{i,j} \beta_{ij} \text{tr} [\rho_{0} O_{0}^{j}] R_{i} - \sum_{i,j} \sum_{\ell=1}^{\tau} \Delta^j_{\ell} \beta_{ij} R_{i} \]  \hspace{2cm} (A32)

Thus, the density matrix reconstructed by our method \( \rho^{\text{trunc}}_{\tau} \) will be

\[ \rho^{\text{rec}}_{\tau} = \sum_{i,j} \beta_{ij} \text{tr} [\rho_{0} O_{0}^{j}] R_{i} = \rho_{\tau} + \sum_{i,j} \sum_{\ell=1}^{\tau} \Delta^j_{\ell} \beta_{ij} R_{i} \]  \hspace{2cm} (A33)

where the last term quantifies the error due to the erroneous reconstruction, leading to the inequality

\[ D(\rho_{\tau}, \rho^{\text{trunc}}_{\tau}) = \frac{1}{2} \| \sum_{i,j} \sum_{\ell=1}^{\tau} \Delta^j_{\ell} \beta_{ij} R_{i} \|_1 \]  \hspace{2cm} (A34)

\[ \leq \frac{1}{2} \sum_{\ell=1}^{\tau} \sum_{k} \epsilon_{\ell,k} \| \sum_{i,j} \beta_{ij} R_{i} \|_1 \]  \hspace{2cm} (A35)

Note that this error term depends not only on the truncation errors at level \( \tau \), but also depends on all the truncation errors in previous levels. Thus, this term will in general scale quadratically with the size of the system. However, it appears to be well-behaved numerically.