DOUBLE YANGIANS OF CLASSICAL TYPES AND THEIR VERTEX REPRESENTATIONS

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Abstract. The Yangian double $DY(\mathfrak{g}_N)$ is introduced for the classical types of $\mathfrak{g}_N = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$. Via Gauss decomposition of the generator matrix, the Yangian double is given the Drinfeld presentation. In addition, bosonization of level 1 realizations for the Yangian double $DY(\mathfrak{g}_N)$ of non-simply-laced types are explicitly constructed.

1. Introduction

The quantum double is defined by gluing two isomorphic quantum Borel-type subalgebras [6] and provides conceptual foundation for the universal R-matrix of the quantum enveloping algebra [4]. The very construction not only solves the quantum Yang-Baxter equation at each irreducible representation of the quantum group (also see [5]), but also has important applications in representation theory [10, 7], the $q$-conformal field theory [8] and knot theory [26, 27].

The quantum doubles corresponding to finite dimensional simple Lie algebras provides a unified construction of the trigonometric solution of the Yang-Baxter equation [16, 24, 5]. The class of trigonometric R-matrices gives rise to the quantum enveloping algebras of Drinfeld and Jimbo. Besides quantum enveloping algebras, Drinfeld also introduced another class of quantum groups called the Yangians $Y_h(\mathfrak{g})$ corresponding to the rational R-matrices, and Yangians since then have quickly developed into one of the most important algebraic structures with numerous applications, see [25] for more information.

The quantum double $DY_h(\mathfrak{g})$ of the Yangian $Y_h(\mathfrak{g})$ was introduced by Iohara [15] for type $A$ with a central extension in the $RTT$ relation and its Drinfeld commutation relations were also stated in [15]. The general simply-laced types have recently been given by Guay et al [13] and the PBW theorem for the simply-laced affine types is obtained using the vertex representation, and there is also a recent uniform proof of PBW bases for several type $A$ algebras [28] (see cf. [31] for a PBW theorem in super type). The complete proof the isomorphism between the $RTT$ and Drinfeld presentations for the Yangian algebra $Y_h(\mathfrak{g})$ in type $A$ was obtained by Brundan and Kleshchev in [3] using the Gauss decomposition of the generator matrix $T(u)$.

Supported in part by National Natural Science Foundation of China grant no. 11531004 and Simons Foundation grant no. 523868.
The Drinfeld realization for the double Yangian algebras is particularly interesting as it is useful to classify finite dimensional representations via the Drinfeld polynomials and construct universal R-matrices in this case [27]. The Yangian algebra $Y_h(\mathfrak{g})$ in types B, C, and D was studied in terms of the RTT relations in [1] and a Poincaré-Birkhoff-Witt basis was proved therein. Recently the identification of Drinfeld realization and the RTT presentation of the Yangian algebras $Y_h(\mathfrak{g})$ for types B, C, and D has been proved in [21], which makes it possible to study the general non-simply-laced Drinfeld realization of the Yangian double algebra for all classical types. In this paper, we will construct a central extension of the Yangian double $DY(\mathfrak{g}_N)$ of type types $B, C$ and $D$ by using the RTT relation together with a unitarity condition. Then we will construct its Drinfeld generators using the Gauss decomposition and prove that these two constructions are isomorphic, moreover we also give a PBW theorem using the monomials in the generators $t_{ij}^{(r)}$ and $c$. Finally, we also construct level one modules of the Yangian double $DY(\mathfrak{g}_N)$ in terms of bosons. As a by-product our proof also contains a detailed demonstration for the double Yangian algebra in type $A$.

Vertex representations of quantum affine algebras of simply-laced types were constructed by Frenkel-Jing in [9] and other non-simply laced types are realized subsequently [2, 17, 18] in untwisted types. The Yangian analog of type $A$ was given in [15] and similar properties are generalized in [23]. Vertex operator representations and Drinfeld realization of quantum affine superalgebras in simply-laced types were recently given in [29]. Our construct of vertex representations for the central extension of the double Yangian algebras provide level one modules for other classical (non-simply laced) types in this paper, and the novelty is that our construction seems to be some different analogs from those in [2, 17, 18].

To explain our construction, let $\mathfrak{g}$ be the simple Lie algebra associated with the Cartan matrix $A = [a_{ij}]_{i,j=1}^n$. Let $\alpha_1, \ldots, \alpha_n$ be the corresponding simple roots. The Drinfeld Yangian double $DY^D(\mathfrak{g})$ is generated by elements $h_{i r}, \xi^+_i, \xi^-_i$ and the central element $c$ with $i = 1, \ldots, n$ and $r \in \mathbb{Z}$ subject to the defining relations

$$[H^\pm_i (u), H^\pm_j (v)] = 0,$$

$$(u_\pm - v_\mp - B_{ij})(u_\mp - v_\pm + B_{ij})H^\pm_i (u)H^\mp_j (v)$$

$$= (u_\pm - v_\mp + B_{ij})(u_\mp - v_\pm - B_{ij})H^\mp_j (v)H^\pm_i (u),$$

$$H^\pm_i (u)^{-1}E_j (v)H^\pm_i (u) = \frac{u_\mp - v + B_{ij}}{u_\mp - v - B_{ij}}E_j (v),$$

$$H^\pm_i (u)F_j (v)H^\pm_i (u)^{-1} = \frac{u_\pm - v + B_{ij}}{u_\pm - v - B_{ij}}F_j (v),$$

$$(u - v + B_{ij})E_i (u)E_j (v) = (u - v - B_{ij})E_j (v)E_i (u),$$

$$(u - v - B_{ij})F_i (u)F_j (v) = (u - v + B_{ij})F_j (v)F_i (u),$$
\[
\sum_{\sigma \in S_m} [E_i(u_{\sigma(1)}), [E_i(u_{\sigma(2)}), \ldots, [E_i(u_{\sigma(m)}), E_j(v)]] = 0,
\]
\[
\sum_{\sigma \in S_m} [F_i(u_{\sigma(1)}), [F_i(u_{\sigma(2)}), \ldots, [F_i(u_{\sigma(m)}), F_j(v)]] = 0,
\]
for \(i \neq j, m = 1 - a_{ij},\)
\[
[E_i(u), F_j(v)] = \delta_{ij} \{\delta(u_+ - v_+)H_i^-(v_+) - \delta(u_+ - v_-)H_i^+(u_+)\},
\]
where \(B_{ij} = \frac{1}{2}(\alpha_i, \alpha_j), u_+ = u \pm \frac{1}{2}c\) and
\[
H_i^+(u) = 1 + \sum_{r=0}^{\infty} h_{ir}^+ u^{-r-1}, \quad H_i^-(u) = 1 - \sum_{r=1}^{\infty} h_{ir}^- u^{r-1},
\]
\[
E_i(u) = \sum_{r \in \mathbb{Z}} \xi_{ir}^+ u^{-r-1}, \quad F_i(u) = \sum_{r \in \mathbb{Z}} \xi_{ir}^- u^{r-1}.
\]

If \(g = g_N\) is the orthogonal Lie algebra \(o_N\) (with \(N = 2n\) or \(N = 2n + 1\)) or symplectic Lie algebra \(sp_N\) (with even \(N = 2n\)) then the Yangian double \(DY^R(g_N)\) (the Yangian double in the RTT presentation) can be defined via the rational \(R\)-matrix. The defining relations take the form of the RTT relation
\[
(1.1) \quad R(u - v) T_i^+(u) T_j^-(v) = T_j^+(v) T_i^+(u) R(u - v),
\]
\[
R(u_+ - v_-) T_i^+(u) T_j^-(v) = T_j^-(v) T_i^+(u) R(u_+ - v_+),
\]
with the unitarity condition
\[
(1.2) \quad T_i^\pm(u + \kappa)' T_i^\pm(u) = 1,
\]
for the notation explained below in Section 2. Here \(T^\pm(u)\) is a square matrix of size \(N\) whose \((i, j)\) entry is the formal series
\[
t_{ij}^+(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r}, \quad t_{ij}^-(u) = \delta_{ij} - \sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1},
\]
so that the algebra \(DY^R(g_N)\) is generated by all coefficients \(t_{ij}^{(r)}\) and \(c\) subject to the defining relations (1.1) and (1.2). Our main result is the following.

**Main Theorem.** The algebra \(DY^R(g_N)\) is isomorphic to the algebra \(DY^D(g_N)\).

The paper is organized as follows. In section 2, we give definitions of the extended Yangian double \(DX(g_N)\) and the Yangian double \(DY(g_N)\). We also explain the notations which will be used in the rest of the paper. In section 4 and 5, we give the imbedding theorem which plays an important role in the proof of the main theorem. In section 6 and 7, we prove the main theorem by calculating the relations for the Gaussian generators. The surjectivity and injectivity is also proved step by step. As a by-product we also give a detailed proof of the isomorphism between the Drinfeld and RTT presentations in type \(A\), which was stated in [15]. Section 8 contains the construction of the level 1 modules for the Yangian double \(DY^D(g_N)\).
2. Notation and definitions

Let $\mathfrak{g}_N$ be the orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n + 1$ (type $B$), $N = 2n$ (type $D$), or the symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ (type $C$). The Lie algebra $\mathfrak{g}_N$ can be defined as the subalgebra of $\mathfrak{gl}_N$ spanned by the elements $F_{ij}$:

\[(2.1)\quad F_{ij} = E_{ij} - E_{ij}', \quad \text{and} \quad F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{ij}',\]

for $\mathfrak{o}_N$ and $\mathfrak{sp}_N$ respectively, and here the $E_{ij}$ denote the the standard matrix units of $\mathfrak{gl}_N$ with 1 sitting at the $(i, j)$ position and zero elsewhere. We have set that $i' = N - i + 1$, and $\varepsilon_i = -1$ for $i = n + 1, \ldots, 2n$ in the symplectic case and $\varepsilon_i = 1$ for other cases.

The elements $F_{ij}$ satisfy the relations

\[(2.2)\quad F_{ij} + \theta_{ij} F_{j'i'}, = 0,\]

for any $1 \leq i, j \leq N$ and

\[(2.3)\quad [F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \delta_{ki} \theta_{ij} F_{j'i'} + \delta_{lj} \theta_{ij} F_{ki}',\]

where $\theta_{ij} = \varepsilon_i \varepsilon_j$, so $\theta_{ij} = 1$ in the orthogonal case, and $\theta_{ij} = \varepsilon_i \varepsilon_j$ in the symplectic case.

The simple roots of $\mathfrak{g}_N$ can be uniformly defined as follows. Let $\varepsilon_1, \ldots, \varepsilon_n$ be an orthonormal basis of the $n$-dimensional Euclidian space with the bilinear form $(.,.)$, then the simple roots of $\mathfrak{g}_N$ are $\alpha_1, \ldots, \alpha_n$ with

\[(2.4)\quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \ldots, n - 1,\]

and

\[(2.5)\quad \alpha_n = \begin{cases} 
\varepsilon_n & \text{for } \mathfrak{o}_{2n+1} \\
2\varepsilon_n & \text{for } \mathfrak{sp}_{2n} \\
\varepsilon_{n-1} + \varepsilon_n & \text{for } \mathfrak{o}_{2n}
\end{cases}\]

To introduce the $R$-matrix presentation of the Yangian double, we will use the standard tensor notation. Let $e_1, \ldots, e_N$ be the canonical basis of $\mathbb{C}^N$ and the endomorphism algebra $\text{End } \mathbb{C}^N$ is identified with the algebra of $N \times N$ matrices with the matrix units $e_{ij}$. Consider the tensor product algebra of the form

\[(2.6)\quad \text{End } (\mathbb{C}^N)^{\otimes m} \otimes A = \text{End } \mathbb{C}^N \otimes \cdots \otimes \text{End } \mathbb{C}^N \otimes A,\]

where $A$ is a unital associative algebra. For any element

\[(2.7)\quad X = \sum_{i,j=1}^{N} e_{ij} \otimes X_{ij} \in \text{End } \mathbb{C}^N \otimes A\]

and any $a \in \{1, \ldots, m\}$ we denote by $X_a$ the following element

\[(2.8)\quad X_a = \sum_{i,j=1}^{N} 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (m-a)} \otimes X_{ij} \in \text{End } (\mathbb{C}^N)^{\otimes m} \otimes A,\]
where $1$ is the identity endomorphism. In the same spirit, for any operator
\[
C = \sum_{i,j,k,l=1}^N c_{ijkl} e_{ij} \otimes e_{kl} \in \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N
\]
and for each pair of indices $1 \leq a < b \leq m$, we denote by $C_{ab}$ the following endomorphism
\[
C_{ab} = \sum_{i,j,k,l=1}^N c_{ijkl} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)} \in \text{End} (\mathbb{C}^N)^{\otimes m}.
\]
In particular, one use $C_{ab}$ for the element $C_{ab} \otimes 1$ in the algebra (2.6).

Set
\[
\kappa = \begin{cases} 
N/2 - 1 & \text{in the orthogonal case,} \\
N/2 + 1 & \text{in the symplectic case.}
\end{cases}
\]
The $R$-matrix $R(u)$ associated with the defining representation of $\mathfrak{g}_N$ is the rational function in $u$ with values in the tensor product algebra $\text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N$ given by [30]
\[
R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa},
\]
where
\[
P = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}, \quad Q = P^{\tau_2}
\]
and the partial transpose $\tau_2$ is defined by $\tau(e_{ij}) = \varepsilon_i \varepsilon_j e_{j'i'}$, i.e. $Q$ is given by
\[
Q = \sum_{i,j=1}^N \varepsilon_i \varepsilon_j e_{ij} \otimes e_{i'j'},
\]
It is easy to see that $P^2 = 1$, $Q^2 = NQ$ and
\[
PQ = QP = \begin{cases} 
Q & \text{in the orthogonal case,} \\
-Q & \text{in the symplectic case.}
\end{cases}
\]
The rational function (2.10) satisfies the Yang–Baxter equation
\[
R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).
\]

We now define a central extension of the extended Yangian double $DX^R(\mathfrak{g}_N)$ and Yangian double $DY^R(\mathfrak{g}_N)$ corresponding to R-matrix $R(u)$.

**Definition 2.1.** The algebra $DX^R(\mathfrak{g}_N)$ is an associative algebra over $\mathbb{C}$ generated by $t_{ij}^{(r)}$ and $c$ where $1 \leq i, j \leq N$, $r \in \mathbb{Z}^\times$ and $t_{ij}^{(0)} = \delta_{ij}$ subject to the relations written in terms of generating series:
\[
[T^+(u), c] = 0,
\]
\[
R(u - v)T^+_1(v)T^+_2(v) = T^+_2(v)T^+_1(v)R(u - v),
\]
\begin{equation}
R(u_+ - v_-)T_1^+(u)T_2^-(v) = T_2^-(v)T_1^+(u)R(u_+ - v_+),
\end{equation}
where \( u_\pm = u \pm \frac{1}{4}c \), and the generating series \( T^\pm(u) \) are defined by
\begin{equation}
T^\pm(u) = \sum_{i,j=1}^{\infty} t_{ij}^\pm(u) \otimes E_{ij},
\end{equation}
with
\begin{equation}
t_{ij}^+(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r}, \quad t_{ij}^-(u) = \delta_{ij} - \sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1},
\end{equation}

In terms of the R-matrix the defining relations (2.15) and (2.16) can be written explicitly as
\begin{equation}
[t_{ij}^+(u), t_{kl}^+(v)] = \frac{1}{u - v} \left( t_{kj}^+(u) t_{il}^+(v) - t_{kj}^+(v) t_{il}^+(u) \right) - \frac{1}{u - v - \kappa} \left( \delta_{ki} \sum_{p=1}^{N} \theta_{ip} t_{pj}^+(u) t_{pj}^+(v) - \delta_{ij} \sum_{p=1}^{N} \theta_{jp} t_{kp}^+(v) t_{pj}^+(u) \right),
\end{equation}

\begin{equation}
[t_{ij}^+(u), t_{kl}^-(v)] = \frac{1}{u_+ - v_-} t_{kj}^+(u) t_{il}^-(v) - \frac{1}{u_- - v_+} t_{kj}^-(v) t_{il}^+(u) - \frac{1}{u_+ - v_- - \kappa} \delta_{ki} \sum_{p=1}^{N} \theta_{ip} t_{pj}^+(u) t_{p'l}^-(v) + \frac{1}{u_- - v_+ - \kappa} \delta_{ij} \sum_{p=1}^{N} \theta_{jp} t_{kp}^-(v) t_{ip}^+(u).
\end{equation}

Let \( f(u) \in \mathbb{C}[[u^{-1}]] \) and \( g(u) \in \mathbb{C}[[u]] \) be any non-zero formal series. It is easy to see that the map
\begin{equation}
\mu_{f,g} : \quad T^+(u) \mapsto f(u) T^+(u), \quad T^-(u) \mapsto g(u) T^-(u),
\end{equation}
\begin{equation}
c \mapsto c
\end{equation}
defines an algebra homomorphism of \( DX^R(\mathfrak{g}_N) \).

The Yangian double \( DY^R(\mathfrak{g}_N) \) is defined as the subalgebra of \( DX^R(\mathfrak{g}_N) \) consisting of the elements stable under all the automorphisms of the form (2.21).

**Proposition 2.2.** There exists formal series \( z_N^\pm(u) \in DX^R(\mathfrak{g}_N)[[u^\mp 1]] \) such that
\begin{equation}
T^\pm(u) T^\pm(u + \kappa) = T^\pm(u + \kappa) T^\pm(u) = z_N^\pm(u) I,
\end{equation}
where
\begin{equation}
z_N^\pm(u) = 1 + \sum_{r=1}^{\infty} z_N^{(r)} u^{-r}, \quad z_N^-(u) = 1 - \sum_{r=1}^{\infty} z_N^{(-r)} u^{r-1}.
\end{equation}
Let $ZDX(g_N)$ be the subalgebra of $DX^R(g_N)$ generated by all the elements $z_N^{(r)}$ with $r \in \mathbb{Z}^\times$. We have the following results similar to those in [1].

**Theorem 2.3.** One has the tensor product decomposition

\begin{equation}
DX^R(g_N) = ZDX(g_N) \otimes DY^R(g_N).
\end{equation}

Moreover, the Yangian double $DY^R(g_N)$ is isomorphic to the quotient of $DX^R(g_N)$ by the ideal generated by the coefficients $z_N^{(r)}$ of the series $z_N^{\pm}(u)$.

**Proof.** There exists a unique series $y_N^-(u)$ of the form

$$y_N^-(u) = 1 - y_N^{(-1)} - y_N^{(-2)}u - y_N^{(-3)}u^2 - \cdots, \quad y_N^{(i)} \in ZDX(g_N)$$

such that $y_N^-(u)y_N^-(u + \kappa) = z_N^-(u)$. By (2.23), the image of the series $z_N^-(u)$ under the automorphism (2.21) is $g(u)g(u + \kappa)z_N^-(u)$. Hence, the automorphism (2.21) takes $y_N^-(u)$ to $g(u)y_N^-(u)$. This implies that the series $\tau_{ij}^-(u)$ defined by

$$\tau_{ij}^-(u) = y_N^-(u)^{-1}t_{ij}^-(u)$$

are stable under all automorphisms (2.21). Therefore, the coefficients $\tau_{ij}^{(-r)}$ of $\tau_{ij}^-(u)$ belong to the subalgebra $DY^R(g_N)$. Similarly, there exists a unique series $y_N^+(u)$ of the form

$$y_N^+(u) = 1 + y_N^{(1)}u^{-1} + y_N^{(2)}u^{-2} + \cdots, \quad y_N^{(i)} \in ZDX(g_N)$$

such that $y_N^+(u)y_N^+(u + \kappa) = z_N^+(u)$. Define the series $\tau_{ij}^+(u)$ by

$$\tau_{ij}^+(u) = y_N^+(u)^{-1}t_{ij}^+(u).$$

Then the coefficients $\tau_{ij}^{(r)}$ of $\tau_{ij}^+(u)$ also belong to the subalgebra $DY^R(g_N)$. Now the decomposition $DX^R(g_N) = ZDX(g_N) \cdot DY^R(g_N)$ follows from the relation $t_{ij}^+(u) = y_N^+(u)\tau_{ij}^+(u)$.

It remains to demonstrate that the elements $z_N^{(r)}$ are algebraically independent over $DY^R(g_N)$. Due to the definition of $y_N^+(u)$, it suffices to do this for the elements $y_N^{(i)}$. Suppose on the contrary, that for some positive integer $m, n$ there exists a nonzero polynomial $B$ in $m + n$ variables with the coefficients in $DY^R(g_N)$ such that

\begin{equation}
B(y_N^{(1)}, \ldots, y_N^{(m)}, y_N^{(-1)}, \ldots, y_N^{(-n)}) = 0
\end{equation}

Take the minimal $m$ for with this property. The coefficients of $B$ are stable under any automorphism (2.21). Hence, applying the automorphism (2.21) with $f(u) = 1 + au^{-m}, g(u) = 1$ and $a \in \mathbb{C}$ to the equality (2.26) we get

\begin{equation}
B(y_N^{(1)}, \ldots, y_N^{(m)}, a, y_N^{(-1)}, \ldots, y_N^{(-n)}) = 0
\end{equation}

for any $a \in \mathbb{C}$. This means that the polynomial $B$ does not depend on its $m$-th variable, which contradicts the choice of $m$.

Let $I$ be the ideal generated by the elements $z_N^{(r)}$, it is clear that $DX^R(g_N) = I \oplus DY^R(g_N)$. It follows that $DY^R(g_N)$ is isomorphic to the quotient of $DX^R(g_N)$ by the ideal $I$. \qed
Next we introduce some notations for later discussion. Apply the Gauss decomposition to the matrix $T^±(u)$:

$$(2.28)\quad T^±(u) = F^±(u) H^±(u) E^±(u),$$

where $F^±(u)$, $H^±(u)$ and $± E(u)$ are uniquely determined matrices of the form

$$F^±(u) = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
f^±_{21}(u) & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
f^±_{N1}(u) & f^±_{N2}(u) & \cdots & 1
\end{bmatrix}, \quad E^±(u) = \begin{bmatrix}
1 & e^±_{12}(u) & \cdots & e^±_{1N}(u) \\
0 & 1 & \cdots & e^±_{2N}(u) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix},$$

and $H^±(u) = \text{diag} [k^±_1(u), \ldots, k^±_N(u)]$. Define the series with coefficients in $DX^R(\mathfrak{g}_N)$ by

$$H^±_n(u) = k^±_1(u - (i - 1)/2)^{-1} k^±_{i+1}(u - (i - 1)/2)$$

for $i = 1, \ldots, n - 1$, and

$$H^±_n(u) = \begin{cases}
\frac{k^±_n}{u - (n - 1)/2} \frac{k^±_{n+1}}{u - (n - 1)/2} & \text{for } \mathfrak{o}_{2n+1} \\
2k^±_n(u - n/2)^{-1} k^±_{n+1}(u - n/2) & \text{for } \mathfrak{sp}_{2n} \\
k^±_n(u - (n - 2)/2)^{-1} k^±_{n+1}(u - (n - 2)/2) & \text{for } \mathfrak{o}_{2n}
\end{cases}$$

Furthermore, we set

$$X^+_i(u) = e^+_i(u_+) - e^+_i(u_-), \quad X^-_i(u) = f^+_i(u_-) - f^-_i(u_+)$$

where

$$e^±_i(u) = e^±_{i+1}(u), \quad f^±_i(u) = f^±_{i+1}(u)$$

for $i = 1, \ldots, n - 1$,

$$e^±_n(u) = \begin{cases}
e^±_n(u) & \text{for } \mathfrak{o}_{2n+1} \text{ or } \mathfrak{sp}_{2n} \\
e^±_{n+1}(u) & \text{for } \mathfrak{o}_{2n}
\end{cases}$$

and

$$f^±_n(u) = \begin{cases}
f^±_{n+1}(u) & \text{for } \mathfrak{o}_{2n+1} \text{ or } \mathfrak{sp}_{2n} \\
f^±_{n+1}(u) & \text{for } \mathfrak{o}_{2n}
\end{cases}$$

Finally, we set

$$E^+_i(u) = X^+_i(u - (i - 1)/2), \quad F^+_i(u) = X^-_i(u - (i - 1)/2)$$

for $i = 1, \ldots, n - 1$,

$$E^+_n(u) = \begin{cases}
X^+_n(u - (n - 1)/2) & \text{for } \mathfrak{o}_{2n+1} \\
X^+_n(u - n/2) & \text{for } \mathfrak{sp}_{2n} \\
X^+_n(u - (n - 2)/2) & \text{for } \mathfrak{o}_{2n}
\end{cases}$$
and
\[
F_n(u) = \begin{cases} 
X_n^-(u - (n - 1)/2) & \text{for } \mathfrak{o}_{2n+1} \\
X_n^-(u - n/2) & \text{for } \mathfrak{sp}_{2n} \\
X_n^-(u - (n - 2)/2) & \text{for } \mathfrak{o}_{2n}
\end{cases}
\]

Now we have explicit formulas for the series \(z^\pm_N(u)\) in terms of the Gaussian generators \(k^\pm_i(u)\) (see [21]).

**Theorem 2.4.** The following identities hold in \(DX(\mathfrak{g}_N)\):
\[
z^\pm_N(u) = \prod_{i=1}^n k^\pm_i(u + \kappa - i)^{-1} \prod_{i=1}^n k^\pm_i(u + \kappa - i + 1) \cdot k^\pm_{n+1}(u) k^\pm_{n+1}(u - 1/2)
\]
for \(\mathfrak{g}_N = \mathfrak{o}_{2n+1},\)
\[
z^\pm_N(u) = \prod_{i=1}^{n-1} k^\pm_i(u + \kappa - i)^{-1} \prod_{i=1}^n k^\pm_i(u + \kappa - i + 1) \cdot k^\pm_{n+1}(u)
\]
for \(\mathfrak{g}_N = \mathfrak{o}_{2n} \text{ and } \mathfrak{g}_N = \mathfrak{sp}_{2n}.
\]

3. PBW THEOREMS

In this section, we will prove the Poincaré-Birkhoff-Witt theorem for the algebra \(DX(\mathfrak{g}_N)\) and \(DY(\mathfrak{g}_N)\). Firstly, we need a lemma which is implied by the proof of Theorem 2.3.

**Lemma 3.1.** The elements \(z^{(r)}_N\) \((r \geq 1)\) are algebraically independent over \(\mathbb{C}\), and the subalgebra \(ZDX^+(\mathfrak{g}_N)\) generated by \(z^{(r)}_N\) \((r \geq 1)\) is isomorphic to the algebra of polynomials in countably many variables. Similarly, the subalgebra \(ZDX^-(\mathfrak{g}_N)\) generated by \(z^{(-r)}_N\) \((r \geq 1)\) is also isomorphic to the algebra of polynomials in countably many variables.

By Theorem 2.3, the Yangian double \(DY^R(\mathfrak{g}_N)\) is generated by the elements \(\tau^{(r)}_{ij}\) and \(c\), \(1 \leq i, j \leq N, r \in \mathbb{Z}^+\) subject to the relations
\[
[\tau^\pm_{ij}(u), \tau^\pm_{kl}(v)] = \frac{1}{u - v} \left( \tau^\pm_{kj}(u) \tau^\pm_{il}(v) - \tau^\pm_{kj}(v) \tau^\pm_{il}(u) \right)
\]
\[-\frac{1}{u - v - \kappa} \left( \delta_{ki'} \sum_{p=1}^N \theta_{ip} \tau^\pm_{pj}(u) \tau^\pm_{pl}(v) - \delta_{lj} \sum_{p=1}^N \theta_{jp} \tau^\pm_{kp}(v) \tau^\pm_{ip}(u) \right),
\]
(3.2)
\[
[\tau^+_{ij}(u), \tau^-_{kl}(v)] = \frac{1}{u_+ - v_-} \tau^+_{kj}(u) \tau^-_{il}(v) - \frac{1}{u_- - v_+} \tau^-_{kj}(v) \tau^+_{il}(u)
\]
\[-\frac{1}{u_+ - v_- - \kappa} \delta_{ki'} \sum_{p=1}^N \theta_{ip} \tau^+_{pj}(u) \tau^-_{pl}(v) + \frac{1}{u_- - v_+ - \kappa} \delta_{lj} \sum_{p=1}^N \theta_{jp} \tau^-_{kp}(v) \tau^+_{ip}(u),
\]
and
\[(3.3) \sum_{p=1}^{N} \theta_{kp} \tau_{p'}^k (u + \kappa) \tau_{p''}^k (u) = \delta_{kl}, \]
where \(\tau_{ij}^+(u) = \delta_{ij} + \sum_{r=1}^{\infty} \tau_{ij}^{(r)} u^{-r}\) and \(\tau_{ij}^-(u) = \delta_{ij} - \sum_{r=1}^{\infty} \tau_{ij}^{(-r)} u^{-r}.\)

For simplicity we usually omit the superscript \(R\) and denote \(DY_R(\mathfrak{g}_N)\) (resp.\(DX_R(\mathfrak{g}_N)\)) by \(DY(\mathfrak{g}_N)\) (resp.\(DX(\mathfrak{g}_N)\)). Define a filtration on the Yangian double \(DX(\mathfrak{g}_N)\) by setting \(deg t_{ij}^{(r)} = r - 1\), \(deg t_{ij}^{(-r)} = -r\), for all \(r \geq 1\) and \(deg c = 0\). Denote by \(\bar{t}_{ij}^{(\pm r)}\) the image of \(t_{ij}^{(r)}\) in the \((r-1)\) (or \((-r)\)-th component of the associated graded algebra. \(\text{gr} DX(\mathfrak{g}_N)\) The filtration on the Yangian double \(DY(\mathfrak{g}_N)\) is induced by that of \(DX(\mathfrak{g}_N)\), where \(\text{gr} DY(\mathfrak{g}_N)\) is the associated graded algebra. Similar \(\bar{\tau}_{ij}^{(\pm r)}\) will denote the corresponding image in the associated graded algebra \(\text{gr} DY(\mathfrak{g}_N)\). One then has the following result for the graded algebras.

**Proposition 3.2.** The mapping
\[(3.4) \quad F_{ij} x^{-1} \mapsto \bar{\tau}_{ij}^{(r)}, \quad -F_{ij} x^{-r} \mapsto \bar{\tau}_{ij}^{(-r)}, \quad K \mapsto \bar{c},\]
defines a homomorphism
\[U(\mathfrak{g}_N[x, x^{-1}] \oplus \mathbb{C} K) \rightarrow \text{gr} DY(\mathfrak{g}_N),\]
where \(K\) is the central element.

**Proof.** It follows from (3.3) that \(\bar{\tau}_{ij}^{(r)} + \theta_{ij} \bar{\tau}_{ij}^{(r)} = 0\) for any \(1 \leq i, j \leq N\) and \(r \in \mathbb{Z}^\times\). Using the expansion
\[\frac{1}{u - v} = u^{-1} + u^{-2} v + u^{-3} v^2 + \cdots,\]
and taking the coefficient at \(u^{-r} v^s\) \((r, s \geq 1)\) and king the highest degree terms on both sides of the relation (3.2) gives that
\[(3.5) \quad [\bar{\tau}_{ij}^{(r)}, \bar{\tau}_{kl}^{(-s)}] = \begin{cases} \delta_{kj} \bar{\tau}_{il}^{(r-s-1)} - \delta_{il} \bar{\tau}_{kj}^{(r-s-1)} - \delta_{ki} \theta_{ij} \bar{\tau}_{kl}^{(r-s-1)}, & r \leq s \\ -\delta_{kj} \bar{\tau}_{il}^{(r-s)} + \delta_{il} \bar{\tau}_{kj}^{(r-s)} + \delta_{ki} \theta_{ij} \bar{\tau}_{kl}^{(r-s)}, & r > s \end{cases},\]
Similarly, the coefficients at \(u^{-r} v^{-s}\) \((r, s \geq 1)\) and \(u^r v^s\) \((r, s \geq 1)\) of (3.1) imply that
\[(3.6) \quad [\bar{\tau}_{ij}^{(r)}, \bar{\tau}_{kl}^{(s)}] = \delta_{kj} \bar{\tau}_{il}^{(r+s-2)} - \delta_{il} \bar{\tau}_{kj}^{(r+s-2)} - \delta_{ki} \theta_{ij} \bar{\tau}_{kl}^{(r+s-2)},\]
\[(3.7) \quad [\bar{\tau}_{ij}^{(-r)}, \bar{\tau}_{kl}^{(-s)}] = -\delta_{kj} \bar{\tau}_{il}^{(-r-s)} - \delta_{il} \bar{\tau}_{kj}^{(-r-s)} - \delta_{ki} \theta_{ij} \bar{\tau}_{kl}^{(-r-s)} + \delta_{il} \theta_{ij} \bar{\tau}_{kl}^{(-r-s)}).\]
The relations (2.3) then give the final result. \(\square\)

Let \(\rho\) be the vector representation of the Lie algebra \(\mathfrak{g}_N\) on the vector space \(\mathbb{C}^N:\)
\[\rho : F_{ij} \mapsto e_{ij} - \theta_{ij} e_{j'i'},\]
For any \(c \in \mathbb{C}\) let \(\rho_c\) be the evaluation representation of \(g_N[x, x^{-1}]\) defined by
\[\rho_c : F_{ij} x^s \mapsto a^s \rho(F_{ij}), \quad s \in \mathbb{Z}.\]
For any \( c_1, \ldots, c_l \in \mathbb{C} \) consider the tensor product of the evaluation representations of \( \mathfrak{g}_N[x, x^{-1}] \):
\[
\rho_{c_1, \ldots, c_l} = \rho_{c_1} \otimes \cdots \otimes \rho_{c_l}.
\]

**Lemma 3.3.** As the complex parameters \( c_1, \ldots, c_l \) and integer \( l \geq 0 \) vary, the intersection of all \( \ker \rho_{c_1, \ldots, c_l} \) in \( U(\mathfrak{g}_N[x, x^{-1}]) \) is trivial.

**Proof.** Choose a linear basis \( Y_1, \ldots, Y_M \) of \( \mathfrak{g}_N \), where \( M = \dim \mathfrak{g}_N \), and set \( y_i = \rho(Y_i) \).

Choose a total ordering on the set of basis elements \( Y_i x^s \) of \( \mathfrak{g}_N[x, x^{-1}] \) and write a nonzero \( A \in U(\mathfrak{g}_N[x, x^{-1}]) \) as a linear combination of ordered monomials in the basis elements. Suppose that
\[
(Y_{a_1} x^{s_1}) \cdots (Y_{a_m} x^{s_m}) \in U(\mathfrak{g}_N[x, x^{-1}])
\]
is a nonzero monomial occurring in \( A \) with maximal length \( m \), consider the corresponding symmetrization
\[
\sum_{q \in \mathcal{S}_m} (Y_{q_1} x^{s_{q_1}}) \cdots (Y_{q_{m}} x^{s_{q_{m}}}) \in (\mathfrak{g}_N[x, x^{-1}])^{\otimes m}.
\]
As vector spaces
\[
(\mathfrak{g}_N[x, x^{-1}])^{\otimes m} \simeq (\mathfrak{g}_N)^{\otimes m}[x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}],
\]
we can regard the sum (3.9) as a Laurent polynomial in \( m \) independent variables \( x_1, \ldots, x_m \) with values in the vector space \( (\mathfrak{g}_N)^{\otimes m} \),
\[
\sum_{q \in \mathcal{S}_m} x_1^{s_{q_1}} \cdots x_m^{s_{q_{m}}} Y_{q_1} \otimes \cdots \otimes Y_{q_{m}}.
\]
Note that the image of the monomial (3.8) under the representation \( \rho_{c_1, \ldots, c_m} \) is given by
\[
\sum_{k_1, \ldots, k_m = 1}^m c_{k_1}^{s_{k_1}} \cdots c_{k_m}^{s_{k_m}} y_{a_1}^{[k_1]} \cdots y_{a_m}^{[k_m]} \in \text{End}(\mathbb{C}^N)^{\otimes m},
\]
where \( y_a^{[k]} = 1^{\otimes (k-1)} \otimes y_a \otimes 1^{\otimes (m-k)} \). Let us complete the set of matrices \( y_1, \ldots, y_M \) to a basis \( y_1, \ldots, y_{N^2} \) of \( \text{End}(\mathbb{C}^N) \) in such a way that the identity matrix \( 1 \in \text{End}(\mathbb{C}^N) \) occurs as a basis vector \( y_i \) for some \( i \in \{M+1, \ldots, N^2\} \). Denote by \( V_m \) the subspace in \( \text{End}(\mathbb{C}^N)^{\otimes m} \) spanned by the basis elements \( y_{i_1} \otimes \cdots \otimes y_{i_m} \) with at least one factor being \( 1 \). Observe that the image under the representation \( \rho_{c_1, \ldots, c_m} \) of any monomial of length \( < m \) occurring in \( A \) is contained in \( V_m \). Furthermore, modulo elements belonging to \( V_m \) the sum (3.11) can be written as
\[
\sum_{q \in \mathcal{S}_m} c_{q_1}^{s_{q_1}} \cdots c_{q_m}^{s_{q_m}} y_{a_1}^{q_1} \otimes \cdots \otimes y_{a_m}^{q_m}.
\]
This sum is the value of (3.10) under the specialization \( x_i = c_i \) and replacement of \( Y_i \) with \( y_i \). However, since \( \rho \) is faithful and the elements (3.9) are linearly independent, there exist \( c_1, \ldots, c_m \) such that the corresponding sums (3.12) are linearly independent modulo the subspace \( V_m \) and this completes the proof. \( \square \)
We are now in a position to prove the Poincaré-Birkhoff-Witt theorem for the algebra $\text{DX}(\mathfrak{g}_N)$. We order the generators of $\text{DX}(\mathfrak{g}_N)$ as follows. Define $t_{ij}^+(u) \prec t_{kl}^+(u)$ for all $i, j, k, l$ and set $t_{ij}^{(r)} \prec t_{kl}^{(s)}$ (resp. $t_{ij}^{(-r)} \prec t_{kl}^{(-s)}$) when $(i, j, r) < (k, l, s)$ (resp. $(i, j, -r) < (k, l, -s)$) in the lexicographical order. This clearly defines a well-defined total ordering among the generators $t_{ij}^\pm, c$ of $\text{DX}(\mathfrak{g}_N)$ (with the central element $c$ included in the ordering in an arbitrary way).

**Theorem 3.4.** Any element of the algebra $\text{DX}(\mathfrak{g}_N)$ can be written uniquely as a linear combination of ordered monomials in the generators $t_{ij}^{(r)}, t_{ij}^{(-r)}$ and $c$.

**Proof.** It follows by an easy induction from the defining relations of $\text{DX}(\mathfrak{g}_N)$ that the ordered monomials span the algebra $\text{DX}(\mathfrak{g}_N)$.

To show that the ordered monomials are linearly independent, we consider the level zero subalgebra $\text{DX}_0(\mathfrak{g}_N)$ first, which is the quotient of $\text{DX}(\mathfrak{g}_N)$ by the ideal generated by $c$. For each nonzero $a \in \mathbb{C}$, we have the evaluation modules of $\text{DX}_0(\mathfrak{g}_N)$ defined by $\pi_a : \text{DX}_0(\mathfrak{g}_N) \rightarrow \text{End}\mathbb{C}^N, \quad t_{ij}^{(r)} \mapsto a^{-r}\rho(F_{ij}), \quad t_{ij}^{(-r)} \mapsto -a^{-r}\rho(F_{ij})$

for all $r \geq 1$. If there is a nontrivial linear combination of ordered monomials equal to zero, say $\sum_{i,j} A_{ij}(s_1,\ldots,s_p,t_1,\ldots,t_q) t_{i,j}^{(s_1)} t_{j,i}^{(-s_1)} \cdots t_{i,j}^{(t_1)} t_{j,i}^{(-t_1)} \cdots t_{i,j}^{(t_q)} t_{j,i}^{(-t_q)} = 0$, where the indices $s_1,\ldots,s_p,t_1,\ldots,t_q \geq 1$ and the number $p,q \geq 0$ may vary. Consider the image of this combination under the representation $\pi_{a_1} \otimes \cdots \otimes \pi_{a_l}$, which depends on $a_1,\ldots,a_l$ polynomially. Take the terms of this polynomial with maximal total degree in $a_1,\ldots,a_l$. Let $A$ the sum of these terms. Therefore $A \in (\text{End}\mathbb{C}^N)^{\otimes l}$ coincides with the image of the sum $B$

$$
\sum_{i,j} A_{ij}(s_1,\ldots,s_p,t_1,\ldots,t_q) (F_{i,j} x^{s_1-1}) \cdots (F_{j,i} x^{s_p-1}) (F_{k,l} x^{-t_1}) \cdots (F_{k,l} x^{-t_q}) \in U(\mathfrak{g}_N[x,x^{-1}])
$$

under the representation $\rho_{a_1,\ldots,a_l}$. By the Poincaré-Birkhoff-Witt theorem for $U(\mathfrak{g}_N[x,x^{-1}])$, $B \neq 0$. Due to Lemma 3.3, there exists $a_1,\ldots,a_l \in \mathbb{C}$ such that $\rho_{a_1,\ldots,a_l}(B) \neq 0$, i.e $A \neq 0$. But this linear combination equal to zero implies $A = 0$ for all $a_1,\ldots,a_l \in \mathbb{C}$. This is a contradiction. Hence, the ordered monomials are linearly independent for the level zero algebra $\text{DX}_0(\mathfrak{g}_N)$.

Then the similar argument of [19, Theorem 2.2] implies that ordered monomials are linearly independent in $\text{DX}(\mathfrak{g}_N)$. \hfill $\Box$

The following version of the Poincaré-Birkhoff-Witt theorem for the algebra $\text{DX}(\mathfrak{g}_N)$ is also useful.

**Corollary 3.5.** Given any total ordering on the set of elements $t_{ij}^{(r)}, t_{ij}^{(-r)}, z_1^{(r)}, z_1^{(-r)}$ and $c$ with

$$
i > j', \quad r \geq 1, \quad \text{in the orthogonal case},$$

and

$$
i \geq j', \quad r \geq 1, \quad \text{in the symplectic case},$$

the ordered monomials in these elements form a basis of $\text{DX}(\mathfrak{g}_N)$.\hfill $\Box$
Proof. It follows from Theorem 3.4 and the identity
\[ \sum_{i=1}^{N} \theta_{k_l} t_{\tau_{k_l}}^\pm (u + \kappa) t_{\tau_{ik_l}}^\pm (u) = z_N^\pm(u). \]
□

Corollary 3.6. Given any total ordering on the set of elements \( \tau_{ij}^{(r)}, \tau_{ij}^{(-r)} \) and \( c \) with

\[ i > j', \quad r \geq 1, \quad \text{in the orthogonal case}, \]
and

\[ i \geq j', \quad r \geq 1, \quad \text{in the symplectic case}, \]
the ordered monomials in these elements form a basis of \( \text{DY}(g_N) \).

Proof. It follows from Corollary 3.5 and the tensor product decomposition
\[ DX(g_N) = ZDX(g_N) \otimes DY(g_N). \]
□

Corollary 3.7. The mapping defined in Proposition 3.2 is an algebra isomorphism.

Proof. The injectivity follows from the Poincaré-Birkhoff-Witt theorem for the algebra \( \text{DY}(g_N) \).
□

Corollary 3.8. The mapping
\[ (3.13) \quad F_{ij} x^{-1} \mapsto \tilde{t}_{ij}^{(r)} - \frac{1}{2} \delta_{ij} z_N^{(r)}, \quad \zeta_\nu \mapsto \bar{z}_\nu^{(r)}, \quad -F_{ij} x^{-r} \mapsto \tilde{t}_{ij}^{(-r)} - \frac{1}{2} \delta_{ij} z_N^{(-r)}, \quad \bar{\zeta}_\nu \mapsto \bar{z}_\nu^{(-r)}, \quad K \mapsto \bar{c}, \]
defines an algebra isomorphism \( \psi : \)
\[ U(g_N[x, x^{-1}] \oplus CK) \otimes C[\zeta_1, \zeta_2, \ldots] \otimes C[\bar{\zeta}_1, \bar{\zeta}_2, \ldots] \to \text{gr} DX(g_N), \]
where \( C[\zeta_1, \zeta_2, \ldots] \) is the algebra of polynomials in variables \( \zeta_i \), \( C[\bar{\zeta}_1, \bar{\zeta}_2, \ldots] \) is the algebra of polynomials in variables \( \bar{\zeta}_i \), \( K \) is the central element.

Proof. It follows from Lemma 3.1, Corollary 3.7 and the tensor product decomposition
\[ DX(g_N) = ZDX(g_N) \otimes DY(g_N). \]
□

4. Embedding theorems

To begin with, we introduce the \( ij \)-th quasideterminant of a matrix. Let \( A = [a_{ij}] \) be an \( N \times N \) matrix over a ring with 1. Delete the \( i \)-th row and \( j \)-th column of \( A \), we obtain a submatrix of \( A \). We will denote it by \( A^{ij} \). If the matrix \( A^{ij} \) is invertible, we define the \( ij \)-th quasideterminant of \( A \) by the following formula
\[ |A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i, \]
(i) If $N_{N_{i-1}}$, $M_{i} \in \mathbb{N}$, then $\tau_{N_{i-1}}^{b_{i-1}b_{i}}(a_{i}a_{j}) \neq 0$.
(ii) If $b_{i} \neq b_{j}$ then $\tau_{N_{i-1}^{b_{i}b_{j}}}^{a_{i}a_{j}}(u) = -\tau_{N_{i-1}^{b_{i}b_{j}}}^{a_{i}a_{j}}(u)$.

Remark 4.3. The skew-symmetry properties still hold without the assumptions in the symplectic case.

Lemma 4.4. For any $2 \leq i, j \leq 2'$ we have

\[ s_{ij}^{\pm}(u) = t_{i1}^{\pm}(u + 1)^{-1} \tau_{ij}^{\pm 1i}(u + 1). \]

Moreover,

\[ [t_{i1}^{\pm}(u), \tau_{ij}^{\pm 1i}(v)] = 0, \]
Furthermore, we have Lemma 4.5.

\[ \left[ t^\pm_{i1}(u), s^\pm_{ij}(v) \right] = 0, \]

\[ \frac{u_+ - v_+ - 1}{u_+ - v_-} t^+_{i1}(u) \tau^{-1i}_{ij}(v) = \frac{u_- - v_+ - 1}{u_- - v_+} \tau^{-1i}_{ij}(v) t^+_{i1}(u), \]

\[ \frac{u_- - v_+ + 1}{u_- - v_+ + 2} t^-_{i1}(u) \tau^{+1i}_{ij}(v) = \frac{u_+ - v_- + 1}{u_+ - v_- + 2} \tau^{+1i}_{ij}(v) t^-_{i1}(u), \]

\[ \left[ t^+_{i1}(u), s^-_{ij}(v) \right] = 0, \]

\[ \frac{(u_- - v_+)^2}{(u_- - v_+)^2 - 1} t^-_{i1}(u) s^+_{ij}(v) = \frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - 1} s^+_{ij}(v) t^-_{i1}(u). \]

**Proof.** \((4.1)\) implies that

\[ t^+_{i1}(u + 1) s^+_{ij}(u) = t^+_{i1}(u + 1) t^+_{i1}(u) - t^+_{i1}(u + 1) t^+_{i1}(u) - t^+_{i1}(u + 1) t^+_i(u). \]

However, \( t^+_{i1}(u + 1) t^+_{i1}(u) = t^+_{i1}(u + 1) t^+_{i1}(u) \) by \((2.19)\), hence

\[ t^+_{i1}(u + 1) s^+_{ij}(u) = t^+_{i1}(u + 1) t^+_{i1}(u) - t^+_{i1}(u + 1) t^+_i(u). \]

The definition \((4.2)\) implies that this equals \( \tau^{+1i}_{ij}(u + 1) \) hence \((4.3)\) follows. Relation \((4.4)\) and \((4.5)\) is already proved in \([21]\). Relation \((4.6)\) follows easily from the defining relations of \( DX(g_N) \). Consider the following equation:

\[ R_{01}(u_+ - v_-) R_{02}(u_+ - v_- + 1) T^+_0(u) R_{12}(1) T^-_1(v) T^-_2(v - 1) = R_{12}(1) T^-_1(v) T^-_2(v - 1) T^+_0(u) R_{02}(u_- - v_+ + 1) R_{01}(u_- - v_+) \]

Furthermore, we have

\[ \langle 1, 1, i | R_{01}(u_+ - v_-) R_{02}(u_+ - v_- + 1) T^+_0(u) R_{12}(1) T^-_1(v) T^-_2(v - 1) | 1, 1, j \rangle = \langle 1, 1, i | R_{12}(1) T^-_1(v) T^-_2(v - 1) T^+_0(u) R_{02}(u_- - v_+ + 1) R_{01}(u_- - v_+) | 1, 1, j \rangle \]

which is just \((4.6)\). We can prove \((4.7)\) similarly by the following relation:

\[ \frac{(u_- - v_+)^2}{(u_- - v_+)^2 - 1} R(u_- - v_+) T^-_1(u) T^+_2(v) = \frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - 1} R(u_+ - v_-) T^+_2(v) T^-_1(u) R(u_- - v_+). \]

\((4.8)\) follows from \((4.3)\) and \((4.6)\). The proof of \((4.9)\) is similar. \(\square\)

Now we need a lemma to prove our first main result.

**Lemma 4.5.** Set

\[ \Gamma^\pm(u) = \sum_{ij=2}^{2'} E_{ij} \otimes \tau^\pm_{ij}(u). \]

The following relations hold:

\[ \left[ \Gamma^\pm(u), c \right] = 0, \]

\[ \left[ \Gamma^\pm(u), s^\pm_{ij}(v) \right] = 0, \]

\[ \left[ \Gamma^\pm(u), t^\pm_{i1}(u) \right] = 0, \]

\[ \left[ \Gamma^\pm(u), t^\pm_{i1}(u) s^\pm_{ij}(v) \right] = 0, \]

\[ \left[ \Gamma^\pm(u), t^\pm_{i1}(u) t^\pm_{i1}(u) \right] = 0. \]
Proof. We prove the relation (4.15) to show the method. Consider the tensor product algebra (2.6) with $m = 4$. Repeatedly using the Yang-Baxter equation (2.13) and the RTT relation (2.15), (2.16) imply immediately that

\begin{equation}
R_{23}(a_+ - 1)R_{13}(a_+)R_{24}(a_+)R_{14}(a_+ + 1)R_{12}(1)T^+_1(u)T^+_2(u - 1)R_{34}(1)T^-_1(v)T^-_2(v - 1) = R_{34}(1)T^-_3(v)T^-_4(v - 1)R_{12}(1)T^+_1(u)T^+_2(u - 1)R_{14}(a_+ + 1)R_{24}(a_-)R_{13}(a_-)R_{23}(a_- - 1),
\end{equation}

where $a_+ = u_+ - v_-$, $a_- = u_+ - v_+$. Let $V$ be the subspace of $(\mathbb{C}^N)^{\otimes 4}$ spanned by the basis vectors of the form $e_1 \otimes e_j \otimes e_1 \otimes e_l$ with $j, l \in \{2, \ldots, 2^*\}$. Apply the same calculation as in [21], we can conclude that the restriction of the operator on the right hand side of (4.16) to the subspace $V$ coincides with the operator

\begin{equation}
\frac{a_- - 2}{a_- - 1} R_{34}(1)T^-_3(v)T^-_4(v - 1)R_{12}(1)T^+_1(u)T^+_2(u - 1)\left(1 - \frac{P_{24}}{a_-} + \frac{Q_{24}}{a_- - \kappa + 1}\right).
\end{equation}

Similarly, the restriction of the operator on the left hand side of (4.16) to the subspace $V$ coincides with the operator

\begin{equation}
\frac{a_+ - 2}{a_+ - 1} \left(1 - \frac{P_{24}}{a_+} + \frac{Q_{24}}{a_+ - \kappa + 1}\right) R_{12}(1)T^+_1(u)T^+_2(u - 1)R_{34}(1)T^-_3(v)T^-_4(v - 1).
\end{equation}

Moreover, we have

\begin{equation}
R_{12}(1)T^+_1(u)T^+_2(u - 1)R_{34}(1)T^-_3(v)T^-_4(v - 1) (e_1 \otimes e_j \otimes e_1 \otimes e_l)
\equiv \sum_{c,d=1}^{1'} \tau_{1j}^{+1c}(u) \tau_{1l}^{-1d}(v) (e_1 \otimes e_c \otimes e_1 \otimes e_d),
\end{equation}

where we only keep the basis vectors which can give a nonzero contribution to the coefficient of $e_1 \otimes e_i \otimes e_1 \otimes e_k$ with $i, k \in \{2, \ldots, 2^*\}$ after the subsequent application of the operator $\left(1 - \frac{P_{24}}{a_+} + \frac{Q_{24}}{a_+ - \kappa + 1}\right)$. By Lemma 4.2(i), $\tau_{1j}^{+11}(u) = \tau_{1l}^{-11}(v) = 0$, hence the values $c = 1$ and $d = 1$ can be excluded from the range of the summation indices. The definition of $P, Q$ implies that the values $c = 1'$ and $d = 1'$ can also be excluded. Therefore, we can write an operator equality

\[ 1 - \frac{P_{24}}{a_+} + \frac{Q_{24}}{a_+ - \kappa + 1} = R_{24}(a_+), \]

where $R(u)$ is the $R$-matrix associated with the algebra $\mathfrak{dx}(\mathfrak{g}_{N-2})$. Similarly, we can prove the same property for the operator on the right hand side of (4.16) with the use of
Lemma 4.2(ii). So we have

\[
\frac{a_+ - 2}{a_+ - 1} R_{24}(a_+) R_{12}(1) T_1^+(u) T_2^+(u - 1) R_{34}(1) T_3^-(v) T_4^-(v - 1) = \frac{a_- - 2}{a_- - 1} R_{34}(1) T_3^-(v) T_4^-(v - 1) R_{12}(1) T_1^+(u) T_2^+(u - 1) R_{24}(a_-).
\]

By equating the matrix elements, this completes the proof of the equation (4.15). \(\square\)

Using Lemma 4.4 and 4.5, we can easily verify that the mapping (4.1) is an algebra homomorphism. Next we show that the homomorphism (4.1) is injective. For each \(N\),
define a filtration on the extended Yangian double \(DX^R(\mathfrak{g}_N)\) by setting \(\deg t_{ij}^{(r)} = r - 1\),
\(\deg t_{ij}^{(-r)} = -r\), for all \(r \geq 1\) and \(\deg c = 0\). Denote by \(\bar{t}_{ij}^{(r)}\) the image of \(t_{ij}^{(r)}\) in the \((r - 1)\)-th component of the associated graded algebra \(\text{gr} DX^R(\mathfrak{g}_N)\), \(\bar{t}_{ij}^{(-r)}\) the image of \(t_{ij}^{(-r)}\) in the \((-r)\)-th component of the associated graded algebra \(\text{gr} DX^R(\mathfrak{g}_N)\). The map (4.1) defines a homomorphism of the associated graded algebras \(\text{gr} DX^R(\mathfrak{g}_{N-2}) \to \text{gr} DX^R(\mathfrak{g}_N)\). It takes the generator \(\bar{t}_{ij}^{(r)}, \bar{t}_{ij}^{(-r)} \in \text{gr} DX^R(\mathfrak{g}_{N-2})\) to the element of \(\text{gr} DX^R(\mathfrak{g}_N)\) denoted by the same symbol. By Corollary 3.8, the mapping

\[
\bar{t}_{ij}^{(r)} \mapsto F_{ij} x^{r-1} + \frac{1}{2} \delta_{ij} \zeta_r, \quad \bar{t}_{ij}^{(-r)} \mapsto -F_{ij} x^{-r} + \frac{1}{2} \delta_{ij} \varsigma_r, \quad \bar{c} \mapsto K,
\]
defines an isomorphism

\[
\text{gr} DX^R(\mathfrak{g}_N) \cong U(\mathfrak{g}_N[x, x^{-1}] \oplus \mathbb{C}K) \otimes \mathbb{C}[\zeta_1, \zeta_2, \ldots] \otimes \mathbb{C}[\varsigma_1, \varsigma_2, \ldots].
\]
The variables \(\zeta_r, \varsigma_r\) correspond to the images of the elements \(z_N^{(r)}, z_N^{(-r)}\) defined in (2.24),

\[
z_N^{(r)} \mapsto \zeta_r, \quad z_N^{(-r)} \mapsto \varsigma_r.
\]
Therefore the homomorphism \(\text{gr} DX^R(\mathfrak{g}_{N-2}) \to \text{gr} DX^R(\mathfrak{g}_N)\) is injective and so is the homomorphism (4.1).

Fix a positive integer \(m\) such that \(m \leq n\) for type \(B\) and \(m \leq n - 1\) for types \(C\) and \(D\). Suppose that the generators \(t_{ij}^{(r)}\) of the algebra \(DX(\mathfrak{g}_{N-2m})\) are labelled by the indices \(m + 1 \leq i, j \leq (m + 1)\)' and \(r \in \mathbb{Z}^+\). We will give a generalization of Theorem 4.1.

**Theorem 4.6.** The mapping \(\psi_m:\)

\[
t_{ij}^{(r)}(u) \mapsto \begin{vmatrix} t_{11}^{(r)}(u) & \cdots & t_{1m}^{(r)}(u) & t_{1j}^{(r)}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{m1}^{(r)}(u) & \cdots & t_{mm}^{(r)}(u) & t_{mj}^{(r)}(u) \\ t_{11}^{(r)}(u) & \cdots & t_{1m}^{(r)}(u) & t_{1j}^{(r)}(u) \end{vmatrix}, \quad m + 1 \leq i, j \leq (m + 1)',
\]
is an injective homomorphism \(DX^R(\mathfrak{g}_{N-2m}) \to DX^R(\mathfrak{g}_N)\). Here \(DX^R(\mathfrak{g}_{N-2m})\) is the algebra corresponding to \(\bar{R}(u) = 1 - \frac{P}{u} + \frac{Q}{u-\kappa+m}\).
Proof. It follows from Theorem 4.1 and the well-known identity:

$$
\begin{bmatrix}
    t_{11}^+(u) & \ldots & t_{1m}^+(u) & t_{1j}^+(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{m1}^+(u) & \ldots & t_{mm}^+(u) & t_{mj}^+(u) \\
    t_{11}^-(u) & \ldots & t_{im}^-(u) & t_{ij}^-(u)
\end{bmatrix}
= \begin{bmatrix}
    s_{22}^+(u) & \ldots & s_{2m}^+(u) & s_{2j}^+(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    s_{m2}^+(u) & \ldots & s_{mm}^+(u) & s_{mj}^+(u) \\
    s_{12}^-(u) & \ldots & s_{im}^-(u) & s_{ij}^-(u)
\end{bmatrix}.
$$

(4.24)

For subsets \(\{a_1, \ldots, a_k\}\) and \(\{b_1, \ldots, b_k\}\) of \(\{1, \ldots, N\}\) we define A-type quantum minors \(t_{b_1 \ldots b_k}^{\pm a_1 \ldots a_k}(u)\) by the formula

$$
 t_{b_1 \ldots b_k}^{\pm a_1 \ldots a_k}(u) = \sum_{p \in \mathfrak{s}_k} \text{sgn} \, p \cdot t_{a_{p(1)}b_1}^{\pm}(u) \ldots t_{a_{p(k)}b_k}^{\pm}(u - k + 1).
$$

These are formal series in \(u^\pm\) with coefficients in \(DX(\mathfrak{g}_N)\). The following result follows from the same argument as [21].

**Proposition 4.7.** For all \(m + 1 \leq i, j \leq (m + 1)\)' we have the identity

$$
\begin{bmatrix}
    t_{11}^+(u) & \ldots & t_{1m}^+(u) & t_{1j}^+(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    t_{m1}^+(u) & \ldots & t_{mm}^+(u) & t_{mj}^+(u) \\
    t_{11}^-(u) & \ldots & t_{im}^-(u) & t_{ij}^-(u)
\end{bmatrix}
= t_{1 \ldots m}^{\pm 1}(u + m)^{-1} \cdot t_{1 \ldots m}^{\pm 1}(u + m).
$$

The following is a counterpart of the corresponding property of the Yangian for \(\mathfrak{g}_N\); see, e.g., [3].

**Lemma 4.8.** We have the relations

$$
[t_{ab}^{\pm}(u), t_{1 \ldots m}^{\pm 1}(v)] = 0,
$$

$$
\frac{u_+ - v_+ - 1}{u_+ - v_-} t_{ab}^{\pm}(u) t_{1 \ldots m}^{\pm 1}(v) = \frac{u_- - v_+ - 1}{u_- - v_-} t_{1 \ldots m}^{\pm 1}(v) t_{ab}^{\pm}(u),
$$

$$
\frac{u_+ - v_- - 1}{u_+ - v_-} t_{ab}^{\pm}(u) t_{1 \ldots m}^{\pm 1}(v) = \frac{u_- - v_+ - 1}{u_- - v_+} t_{1 \ldots m}^{\pm 1}(v) t_{ab}^{\pm}(u),
$$

for all \(1 \leq a, b \leq m\) and \(m + 1 \leq i, j \leq (m + 1)\).'</n

**Proposition 4.9.** We have the relations

$$
[t_{ab}^{\pm}(u), t_{ij}^{[m] \pm}(v)] = 0,
$$

$$
[t_{ab}^{\pm}(u), t_{ij}^{[m] -}(v)] = 0,
$$

$$
\frac{(u_- - v_+)^2}{(u_+ - v_-)^2 - 1} t_{ab}^{\pm}(u) t_{ij}^{[m] +}(v) = \frac{(u_- - v_-)^2}{(u_+ - v_-)^2 - 1} t_{ij}^{[m] +}(v) t_{ab}^{\pm}(u),
$$

for all \(1 \leq a, b \leq m\) and \(m + 1 \leq i, j \leq (m + 1)\).'</n

**Proof.** The relations for case \(m = 1\) follows from Lemma 4.4. The general case follows by applying the homomorphism \(\psi_m\). □
5. GAUSS DECOMPOSITION

The Gauss decomposition (2.28) has played an important role in constructing the Drinfeld generators of the Yangian algebra in type A \[6, 3\]. We will see that it also provides the fundamental connection between the Drinfeld generators and the L-operator formulism. For other classical types, with help of quasideterminants. Let \( T^\pm(u) \) be the generator matrix for the extended Yangian double \( DX(\mathfrak{g}_N) \) (that is, ignore relation (1.2)) and recall the well-known formulas for the entries of the matrices \( F^\pm(u), H^\pm(u) \) and \( E^\pm(u) \) in (2.28); see e.g. [25, Sec. 1.11]. We have

\[
(5.1) \quad k^\pm_i(u) = \begin{vmatrix} t^\pm_{i1}(u) & \ldots & t^\pm_{i-1,i}(u) & t^\pm_{ii}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t^\pm_{i-1,1}(u) & \ldots & t^\pm_{i-1,i-1}(u) & t^\pm_{i,i-1}(u) \\ t^\pm_{i1}(u) & \ldots & t^\pm_{i-1,i}(u) & t^\pm_{ii}(u) \end{vmatrix}, \quad i = 1, \ldots, N,
\]

whereas

\[
(5.2) \quad e^\pm_{ij}(u) = h^\pm_i(u)^{-1} \begin{vmatrix} t^\pm_{11}(u) & \ldots & t^\pm_{1,i-1}(u) & t^\pm_{i1}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t^\pm_{i-1,1}(u) & \ldots & t^\pm_{i-1,i-1}(u) & t^\pm_{i-1,i}(u) \\ t^\pm_{i1}(u) & \ldots & t^\pm_{i-1,i}(u) & t^\pm_{ii}(u) \end{vmatrix}
\]

and

\[
(5.3) \quad f^\pm_{ji}(u) = \begin{vmatrix} t^\pm_{11}(u) & \ldots & t^\pm_{1,i-1}(u) & t^\pm_{i1}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t^\pm_{i-1,1}(u) & \ldots & t^\pm_{i-1,i-1}(u) & t^\pm_{i-1,i}(u) \\ t^\pm_{j1}(u) & \ldots & t^\pm_{j,i-1}(u) & t^\pm_{ji}(u) \end{vmatrix} k^\pm_i(u)^{-1}
\]

for \( 1 \leq i < j \leq N \). Obviously, the algebra \( DX(\mathfrak{g}_N) \) is generated by the coefficients of the series \( f^\pm_{ji}(u), e^\pm_{ij}(u) \) and \( k^\pm_i(u) \), which will be referred as the Gaussian generators.

Introduce the coefficients of the series defined in (5.1), (5.2) and (5.3) by

\[
e^\pm_{ij}(u) = \sum_{r=1}^{\infty} e^{(r)}_{ij} u^{-r}, \quad f^\pm_{ji}(u) = \sum_{r=1}^{\infty} f^{(r)}_{ji} u^{-r}, \quad k^\pm_i(u) = 1 + \sum_{r=1}^{\infty} k^{(r)}_i u^{-r},
\]

\[
e^-_{ij}(u) = -\sum_{r=1}^{\infty} e^{(-r)}_{ij} u^{-r-1}, \quad f^-_{ji}(u) = -\sum_{r=1}^{\infty} f^{(-r)}_{ji} u^{-r-1}, \quad k^-_i(u) = 1 - \sum_{r=1}^{\infty} k^{(-r)}_i u^{-r-1}.
\]

Suppose that \( m \) is an integer such that \( 0 \leq m < n \) if \( N = 2n \) and \( 0 \leq m \leq n \) if \( N = 2n + 1 \), i.e. \( 0 \leq m < [(N + 1)/2] \). We will use the superscript \([m]\) to indicate square submatrices corresponding to rows and columns labelled by \( m + 1, m + 2, \ldots, (m + 1)' \). In
In particular, we set

\[
F^{[m] \pm}(u) = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
f_{m+2}^{\pm}(u) & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
f_{(m+1)^'}^{\pm}(u) & \cdots & f_{(m+1)^'+2}^{\pm}(u) & 1 \\
\end{bmatrix},
\]

and 

\[
H^{[m] \pm}(u) = \text{diag}\left[k_{m+1}^{\pm}(u), \ldots, k_{(m+1)^'}^{\pm}(u)\right].
\]

Furthermore, by Gauss decomposition of 

\[
T^{[m] \pm}(u) = F^{[m] \pm}(u) H^{[m] \pm}(u) E^{[m] \pm}(u).
\]

Accordingly, the entries of 

\[
T^{[m] \pm}(u)
\]

will be denoted by 

\[
t_{ij}^{[m] \pm}(u) \quad \text{with} \quad m+1 \leq i, j \leq (m+1)^'.
\]

We have a useful property about the series 

\[
t_{ij}^{[m] \pm}(u).
\]

**Proposition 5.1.** The series 

\[
t_{ij}^{[m] \pm}(u)
\]

coincides with the image of the generator series 

\[
t_{ij}^{\pm}(u)
\]

of the extended Yangian double 

\[DX^{\tilde{R}}(\mathfrak{g}_{N-2m})\]

under the embedding (4.23),

\[
t_{ij}^{[m] \pm}(u) = \psi_m(t_{ij}^{\pm}(u)), \quad m+1 \leq i, j \leq (m+1)^'.
\]

The following corollary is immediate from Proposition 5.1.

**Corollary 5.2.** The subalgebra 

\[DX^{[m]}(\mathfrak{g}_N)\]

generated by the coefficients of all series 

\[
t_{ij}^{[m] \pm}(u)
\]

with 

\[m+1 \leq i, j \leq (m+1)^'.\]

is isomorphic to the extended Yangian double 

\[DX^{\tilde{R}}(\mathfrak{g}_{N-2m})\].

Here 

\[DX^{\tilde{R}}(\mathfrak{g}_{N-2m})\]

is the algebra corresponding to 

\[
\tilde{R}(u) = 1 - \frac{E}{u} + \frac{Q}{u^2-\kappa+m}.
\]

In particular, we have the relations

\[R_{12}^{[m]}(u-v)T_1^{[m] \pm}(u)T_2^{[m] \pm}(v) = T_2^{[m] \pm}(v)T_1^{[m] \pm}(u)R_{12}^{[m]}(u-v),\]

\[R_{12}^{[m]}(u_+ - v_-)T_1^{[m] \pm}(u)T_2^{[m] \pm}(v) = T_2^{[m] \pm}(v)T_1^{[m] \pm}(u)R_{12}^{[m]}(u_+ - v_-),\]

where 

\[R^{[m]}(u) = 1 - \frac{E}{u} + \frac{Q}{u^2-\kappa+m}.\]

6. Drinfeld realization of extended Yangian doubles

Drinfeld realization provides a new set of generators for the extended Yangian double 

\[DX(\mathfrak{g}_N)\]

that facilitates the study of finite dimensional representations [15]. We first recall the isomorphism between Yangian algebras in low ranks [1].
6.1. Low rank isomorphisms. To distinguish two sets of generators, we use capital case letters to denote the generating series in the Gauss decomposition of the generator matrix of the Yangian double $DY(\mathfrak{gl}_N)$, i.e. $H^\pm_i(u), E^\pm_{ij}(u)$ and $F^\pm_{ji}(u)$ for the entries occurring in the $DY(\mathfrak{gl}_N)$ counterpart of (2.28), while keeping the lower case letters for the generating series for the corresponding factors in the Gauss decomposition for the extended Yangian double algebras. The next lemmas is immediate from [1, Sec. 4].

**Lemma 6.1.** In terms of the Gaussian generators, the isomorphism $DX(\mathfrak{sp}_2) \to DY(\mathfrak{gl}_2)$ is given by

$$
k_1^\pm(u) \mapsto K_1^\pm(u/2), \quad e_{12}^\pm(u) \mapsto E_{12}^\pm(u/2), \quad k_2^\pm(u) \mapsto K_2^\pm(u/2), \quad f_{21}^\pm(u) \mapsto F_{21}^\pm(u/2).
$$

Here the R-matrix for $DY(\mathfrak{gl}_2)$ is $\frac{2u-1}{u-2}(1 - \frac{P}{u})$.

**Lemma 6.2.** In terms of the Gaussian generators, the isomorphism $DX(\mathfrak{o}_3) \to DY(\mathfrak{gl}_2)$ is given by

$$
k_1^\pm(u) \mapsto K_1^\pm(2u)K_1^\pm(2u + 1), \quad e_{12}^\pm(u) \mapsto \sqrt{2}E_{12}^\pm(2u + 1), \quad f_{21}^\pm(u) \mapsto \sqrt{2}F_{21}^\pm(2u + 1),
$$

$$
k_2^\pm(u) \mapsto K_2^\pm(2u)K_2^\pm(2u + 1), \quad e_{23}^\pm(u) \mapsto -\sqrt{2}E_{23}^\pm(2u), \quad f_{32}^\pm(u) \mapsto -\sqrt{2}F_{32}^\pm(2u),
$$

$$
k_3^\pm(u) \mapsto K_3^\pm(2u)K_3^\pm(2u + 1), \quad e_{13}^\pm(u) \mapsto -E_{13}^\pm(2u + 1)^2, \quad f_{31}^\pm(u) \mapsto -F_{31}^\pm(2u + 1)^2.
$$

Here the R-matrix for $DY(\mathfrak{gl}_2)$ is $\frac{u+1}{u-1}(1 - \frac{P}{u})$.

**Lemma 6.3.** In terms of the Gaussian generators, the embedding $DX(\mathfrak{so}_4) \hookrightarrow DY(\mathfrak{gl}_2) \otimes DY(\mathfrak{gl}_2)$ is given by

$$
k_1^\pm(u) \mapsto K_1^\pm(u) \otimes K_1^\pm(u), \quad k_2^\pm(u) \mapsto K_2^\pm(u) \otimes K_1^\pm(u),
$$

$$
k_2^\pm(u) \mapsto K_1^\pm(u) \otimes K_2^\pm(u), \quad k_1^\pm(u) \mapsto K_2^\pm(u) \otimes K_2^\pm(u),
$$

together with

$$
e_{12}^\pm(u) \mapsto 1 \otimes E_{12}^\pm(u), \quad e_{12}'^\pm(u) \mapsto E_{12}^\pm(u) \otimes 1,
$$

$$
e_{11}^\pm(u) \mapsto -E_{12}^\pm(u) \otimes E_{12}^\pm(u), \quad e_{22}^\pm(u) \mapsto 0,
$$

$$
e_{21}^\pm(u) \mapsto -E_{12}^\pm(u) \otimes 1, \quad e_{21}'^\pm(u) \mapsto -1 \otimes E_{12}^\pm(u),
$$

and

$$
f_{21}^\pm(u) \mapsto 1 \otimes F_{21}^\pm(u), \quad f_{21}'^\pm(u) \mapsto F_{21}^\pm(u) \otimes 1,
$$

$$
f_{11}^\pm(u) \mapsto -F_{21}^\pm(u) \otimes F_{21}^\pm(u), \quad f_{22}^\pm(u) \mapsto 0,
$$

$$
f_{12}^\pm(u) \mapsto -F_{21}^\pm(u) \otimes 1, \quad f_{12}'^\pm(u) \mapsto -1 \otimes F_{21}^\pm(u).
$$

Here the R-matrix corresponding to $DY(\mathfrak{gl}_2)$ is $\frac{u}{u-1}(1 - \frac{P}{u})$. 
6.2. Type A relations. Since the extended Yangian $\text{DX}(g_N)$ contains a subalgebra isomorphic to the Yangian $\text{DY}(\mathfrak{gl}_n)$, some relations between the Gaussian generators of $\text{DX}(g_N)$ can be obtained from those of $\text{DY}(\mathfrak{gl}_n)$. We record them in the next proposition.

**Proposition 6.4.** The following relations hold in $\text{DX}(g_N)$, with the conditions on the indices $1 \leq i, j \leq n - 1$ and $1 \leq l, m \leq n$.

\[
\begin{align*}
&k_i^+(u)k_m^-(v) = k_m^-(v)k_i^+(u) \quad (l < m), \\
&\frac{u_+ - v_- - 1}{u_+ - v_-} k_i^+(u)k_i^-(v) = \frac{u_- - v_+ - 1}{u_- - v_+} k_i^-(v)k_i^+(u), \\
&\frac{(v_+ - u_-)^2}{(v_+ - u_-)^2 - 1} k_i^+(u)k_i^-(v) = \frac{(v_- - u_+)^2}{(v_- - u_+)^2 - 1} k_i^-(v)k_i^+(u) \quad (l < m),
\end{align*}
\[
\begin{align*}
&\frac{u_+ - v_\pm - 1}{u_+ - v_\pm} k_i^+(u)k_{i+1}^+(v)k_i^-(v)^{-1} = \frac{u_- - v_\pm - 1}{u_- - v_\pm} k_i^-(v)k_{i+1}^+(v)k_i^+(u) \quad (l \leq n - 1), \\
&\frac{u_+ - v_\pm + 1}{u_+ - v_\pm} k_{i+1}^+(u)k_i^+(v)k_i^-(v)^{-1} = \frac{u_- - v_\pm + 1}{u_- - v_\pm} k_i^-(v)k_{i+1}^+(v)k_i^+(u) \quad (l \leq n - 1), \\
&k_i^+(u)^{-1}X_i^+(v)k_i^+(u) = \frac{u_+ - v_- - 1}{u_+ - v_-} X_i^+(v), \quad k_i^+(u)X_i^-(v)k_i^+(u)^{-1} = \frac{u_- - v_+ - 1}{u_- - v_+} X_i^-(v), \\
&k_{i+1}^+(u)^{-1}X_i^+(v)k_{i+1}^+(u) = \frac{u_+ - v_- + 1}{u_+ - v_-} X_i^+(v), \quad k_{i+1}^+(u)X_i^-(v)k_{i+1}^+(u)^{-1} = \frac{u_- - v_+ + 1}{u_- - v_+} X_i^-(v), \\
&(u - v \pm 1)X_i^+(u)X_i^-(v) = (u - v \mp 1)X_i^-(v)X_i^+(u), \\
&(u - v - 1)X_i^+(u)X_{i+1}^+(v) = (u - v)X_{i+1}^+(v)X_i^+(u) \quad (i \leq n - 2), \\
&(u - v)X_{i+1}^+(u)X_i^-(v) = (u - v - 1)X_i^-(v)X_{i+1}^+(u) \quad (i \leq n - 2),
\end{align*}
\]

\[
\begin{align*}
X_i^+(u_1)X_{i+1}^+(u_2)X_j^+(v) - 2X_i^+(u_1)X_j^+(v)X_i^+(u_2) + X_j^+(v)X_i^+(u_1)X_i^+(u_2) + \{u_1 \leftrightarrow u_2\} = 0 \quad \text{if } |i - j| = 1, \\
X_i^+(u)X_j^+(v) - X_j^+(v)X_i^+(u) \quad \text{if } |i - j| > 1,
\end{align*}
\]

where $\delta(u - v) = \sum_{k \in \mathbb{Z}} u^{-k-1}v^k$.

**Proof.** The coefficients of the series $t_{ij}^\pm(u)$ with $i, j \in \{1, \ldots, n\}$ generate a subalgebra of $\text{DX}(g_N)$ isomorphic to the Yangian $\text{DY}(\mathfrak{gl}_n)$. Hence, the upper left $n \times n$ submatrices of the matrices $F^\pm(u)$, $H^\pm(u)$ and $E^\pm(u)$ defined by the Gauss decomposition (2.28) are given by the same formulas as the corresponding elements of $\text{DY}(\mathfrak{gl}_n)$. Therefore they satisfy the relations as described in [15, Section 3].

We also have the following proposition, which can be proved by the similar method as in [21, Section 5].
Proposition 6.5. All relations in $DX(g_N)$ given in Proposition 6.4 remain valid under the replacement of the indices of all series by $i \mapsto (n - i + 1)'$ for $1 \leq i \leq n$.

6.3. Relations for Gaussian generators.

Proposition 6.6. We have the relations in $DX(o_{2n+1})$,

\begin{align}
(6.1) \quad k_n^\pm(u)^{-1}X_n^+(v)k_n^\pm(u) &= \frac{u_\mp - v - 1}{u_\mp - v}X_n^+(v), \\
(6.2) \quad k_n^\pm(u)X_n^-(v)k_n^\pm(u)^{-1} &= \frac{u_\pm - v - 1}{u_\pm - v}X_n^-(v), \\
(6.3) \quad k_{n+1}^\pm(u)^{-1}X_n^+(v)k_{n+1}^\pm(u) &= \frac{(u_\mp - v - 1)(u_\mp - v + \frac{1}{2})}{(u_\mp - v)(u_\mp - v - \frac{1}{2})}X_n^+(v), \\
(6.4) \quad k_{n+1}^\pm(u)X_n^-(v)k_{n+1}^\pm(u)^{-1} &= \frac{(u_\pm - v - 1)(u_\pm - v + \frac{1}{2})}{(u_\pm - v)(u_\pm - v - \frac{1}{2})}X_n^-(v), \\
(6.5) \quad (u - v + \frac{1}{2})X_n^+(v)X_n^+(u) &= (u - v - \frac{1}{2})X_n^+(v)X_n^+(u), \\
(6.6) \quad (u - v - \frac{1}{2})X_n^-(v)X_n^-(u) &= (u - v + \frac{1}{2})X_n^-(v)X_n^-(u), \\
(6.7) \quad [X_n^+(u), X_n^-(v)] &= 2\delta(u_\mp - v_\mp)k_{n+1}^-(v_\pm)k_{n+1}^+(v_\pm)^{-1} \\
&\quad - \delta(u_\pm - v_\pm)k_{n+1}^+(v_\mp)k_{n+1}^-(v_\mp)^{-1}.
\end{align}

Proof. By Corollary 5.2, the subalgebra $DX^{[n-1]}(o_N)$ of $DX(o_N)$ is isomorphic to $DX(o_3)$. Hence the relations are implied by Lemma 6.2, and the Drinfeld presentation of the Yangian $DY(g_2)$; see [25, Sec. 3.1]. (Here we need to modify the $R$-matrix of $DY(g_2)$, the relations can also be calculated easily.) For instance, to verify the first relation in the proposition, use Lemma 6.2 to get

\begin{align*}
k_n^\pm(u)^{-1}e_n^\mp(v)k_n^\pm(u) &= k_1^\pm(2u + 1)^{-1}k_1^\pm(2u)^{-1}(\sqrt{2e_1^\mp(2v + 1)})k_1^\pm(2u)k_1^\pm(2u + 1) \\
&= \frac{u - v - 1 + \frac{1}{2}}{u - v + \frac{1}{2}}\sqrt{2e_1^\mp(2v + 1)} + \frac{1}{u - v + \frac{1}{2}}\sqrt{2e_1^\mp(2v + 1)} \\
&= \frac{u_\mp - v_\pm - 1}{u_\mp - v_\pm}e_n^\pm(v) + \frac{1}{u_\mp - v_\pm}e_n^\pm(u).
\end{align*}

Similarly, we have

\begin{align*}
k_n^\pm(u)^{-1}e_n^\pm(v)k_n^\pm(u) &= \frac{u - v - 1}{u - v}e_n^\pm(v) + \frac{1}{u - v}e_n^\pm(u).
\end{align*}

Since $X_n^+(v) = e_n^+(v_+) - e_n^-(v_-)$, then we have

\begin{align*}
k_n^\pm(u)^{-1}X_n^+(v)k_n^\pm(u) &= \frac{u_\mp - v - 1}{u_\mp - v}X_n^+(v).
\end{align*}
Proposition 6.7. We have the relations in $DX(\mathfrak{sp}_{2n})$,

\begin{align*}
(6.8) \quad k^\pm_n(u)^{-1}X^\pm_n(v)k^\pm_n(u) &= \frac{u_+ - v - 2}{u_+ - v} X^\pm_n(v), \\
(6.9) \quad k^\pm_n(u)X^\pm_n(v)k^\pm_n(u)^{-1} &= \frac{u_+ - v - 2}{u_+ - v} X^\pm_n(v), \\
(6.10) \quad k^\pm_{n+1}(u)^{-1}X^\pm_n(v)k^\pm_{n+1}(u) &= \frac{u_+ - v + 2}{u_+ - v} X^\pm_n(v), \\
(6.11) \quad k^\pm_{n+1}(u)X^\pm_n(v)k^\pm_{n+1}(u)^{-1} &= \frac{u_+ - v + 2}{u_+ - v} X^\pm_n(v), \\
(6.12) \quad (u - v + 2)X^+_n(u)X^+_n(v) &= (u - v - 2)X^+_n(v)X^+_n(u), \\
(6.13) \quad (u - v - 2)X^-_n(u)X^-_n(v) &= (u - v + 2)X^-_n(v)X^-_n(u), \\
(6.14) \quad [X^+_n(u), X^-_n(v)] &= 2\{\delta(u_+ - v_+)k^\pm_{n+1}(u_+)k^\pm_n(v_+)^{-1} \\
&\quad - \delta(u_+ - v_-)k^\pm_{n+1}(u_-)k^\pm_n(u_+)^{-1}\}.
\end{align*}

Proof. This proposition can be proved in a similar way as type B. \hfill \square

Proposition 6.8. We have the relations in $DX(\mathfrak{o}_{2n})$,

\begin{align*}
(6.15) \quad k^\pm_{n-1}(u)^{-1}X^\pm_n(v)k^\pm_{n-1}(u) &= \frac{u_+ - v - 1}{u_+ - v} X^\pm_n(v), \\
(6.16) \quad k^\pm_{n-1}(u)X^\pm_n(v)k^\pm_{n-1}(u)^{-1} &= \frac{u_+ - v - 1}{u_+ - v} X^\pm_n(v), \\
(6.17) \quad k^\pm_n(u)^{-1}X^\pm_n(v)k^\pm_n(u) &= \frac{u_+ - v - 1}{u_+ - v} X^\pm_n(v), \\
(6.18) \quad k^\pm_n(u)X^\pm_n(v)k^\pm_n(u)^{-1} &= \frac{u_+ - v - 1}{u_+ - v} X^\pm_n(v), \\
(6.19) \quad k^\pm_{n+1}(u)^{-1}X^\pm_n(v)k^\pm_{n+1}(u) &= \frac{u_+ - v + 1}{u_+ - v} X^\pm_n(v), \\
(6.20) \quad k^\pm_{n+1}(u)X^\pm_n(v)k^\pm_{n+1}(u)^{-1} &= \frac{u_+ - v + 1}{u_+ - v} X^\pm_n(v), \\
(6.21) \quad (u - v + 1)X^+_n(u)X^+_n(v) &= (u - v - 1)X^+_n(v)X^+_n(u), \\
(6.22) \quad (u - v - 1)X^-_n(u)X^-_n(v) &= (u - v + 1)X^-_n(v)X^-_n(u), \\
(6.23) \quad [X^\pm_{n-1}(u), X^\mp_n(v)] &= [X^\pm_n(u), X^\mp_{n-1}(v)] = 0,
\end{align*}
\[ X_{n-1}^+(u)X_n^+(v) = X_n^+(v)X_{n-1}^+(u), \]

\[ X_{n-1}^+(u)X_n^-(v) = X_n^-(v)X_{n-1}^+(u), \]

\[ [X_n^+(u), X_n^-(v)] = \{ \delta(u_- - v_+)k_{n+1}^-(v_+)k_{n-1}^-(v_+)^{-1} - \delta(u_+ - v_-)k_{n+1}^+(u_+)k_{n-1}^+(u_-)^{-1} \}. \]

**Proof.** This proposition can be proved in a similar way as type B. \(\square\)

**Lemma 6.9.** For nonnegative integer \(m < [(N+1)/2]\) and indices \(j, k, l\) such that \(m+1 \leq j, k, l \leq (m+1)\) and \(j \neq l\), the following relations hold in the extended Yangian double \(DX(\mathfrak{g}_N)\).

\[ [e_{m,j}^\pm(u), t_{k,l}^{[m] \pm}(v)] = \frac{1}{u-v} t_{k,j}^{[m] \pm}(v)(e_{m,l}^\pm(v) - e_{m,l}^\pm(u)), \]

\[ [f_{j,m}^\pm(u), t_{k,l}^{[m] \pm}(v)] = \frac{1}{u-v} (f_{k,m}^\pm(u) - f_{k,m}^\pm(v)) t_{j,l}^{[m] \pm}(v). \]

\[ [e_{m,j}^\pm(u), t_{k,l}^{[m] \pm}(v)] = \frac{1}{u_\pm - v_\pm} t_{k,j}^{[m] \pm}(v)(e_{m,l}^\pm(v) - e_{m,l}^\pm(u)). \]

\[ [f_{j,m}^\pm(u), t_{k,l}^{[m] \pm}(v)] = \frac{1}{u_\pm - v_\pm} (f_{k,m}^\pm(u) - f_{k,m}^\pm(v)) t_{j,l}^{[m] \pm}(v). \]

**Proof.** It suffices to verify the relations for \(m = 1\); the general case follows by applying the homomorphism \(\psi_m\), see Proposition 5.1. All relations follow by similar arguments so we only verify (6.30). Since

\[ k_1^+(v) = t_{11}^+(v), \quad f_1^-(v) = t_{11}^-(v)t_{11}^+(v)^{-1}, \quad e_1^+(v) = t_{11}^+(v)^{-1}t_{11}^-(v), \]

we have

\[ t_{k,l}^{[1] -}(v) = t_{kl}^-(v) - t_{k1}^-(v)t_{11}^+(v)^{-1}t_{1l}^-(v) = t_{kl}^-(v) - f_{k1}^-(v)k_1^-(v)e_{1l}^-(v). \]

The defining relations (2.20) imply

\[ t_{j1}^+(u), t_{k1}^-(v)] = \frac{1}{u_+ - v_-} t_{k1}^+(u)t_{j1}^-(v) - \frac{1}{u_- - v_+} t_{k1}^+(v)t_{j1}^-(u), \]

and so

\[ [t_{j1}^+(u), t_{k1}^{[1] -}(v)] + [t_{j1}^+(u), f_{k1}^-(v)k_1^-(v)e_{1l}^-(v)] \]

\[ = \frac{1}{u_+ - v_-} (t_{k1}^+(u)t_{j1}^{[1] -}(v) + t_{k1}^+(u)f_{j1}^-(u)k_1^-(v)e_{1l}^-(v)) - \frac{1}{u_- - v_+} t_{k1}^+(v)t_{j1}^-(u). \]

The second commutator on the left hand side can be transformed as

\[ [t_{j1}^+(u), f_{k1}^-(v)k_1^-(v)e_{1l}^-(v)] = [t_{j1}^+(u), f_{k1}^-(v)]t_{1l}^-(v) + f_{k1}^-(v)[t_{j1}^+(u), t_{1l}^-(v)] \]

\[ = [t_{j1}^+(u), t_{k1}^-(v)t_{11}^+(v)^{-1}]t_{1l}^-(v) + f_{k1}^-(v)[t_{j1}^+(u), t_{1l}^-(v)] \]
which equals
\[ [t^+_{j1}(u), f^-_{k1}(v)]e^-_{1i}(v) = f^-_{k1}(v)[t^+_{j1}(u), t^-_{11}(v)]e^-_{1i}(v) + f^-_{k1}(v)[t^+_{j1}(u), t^-_{11}(v)] \]
\[ = \frac{1}{u_- - v_-} f^+_k(u) t^-_{j1}(v)e^-_{1i}(v) - \frac{1}{u_- - v_+} f^-_{k1}(v) t^-_{j1}(u)e^-_{1i}(v) \]
\[ - \frac{1}{u_- - v_-} f^-_{k1}(v) t^-_{j1}(v)e^-_{11}(v) + \frac{1}{u_- - v_+} f^-_{k1}(v) t^-_{j1}(u)e^-_{11}(v) \]
\[ + \frac{1}{u_- - v_-} f^-_{k1}(v) t^-_{j1}(u)e^-_{1i}(v) - \frac{1}{u_- - v_+} f^-_{k1}(v) t^-_{j1}(u)e^-_{1i}(v). \]
Cancelling the common terms, we get that
\[ [t^+_{j1}(u), t^{[1]-}_{kl}(v)] = \frac{1}{u_- - v_-} (f^+_k(u) - f^-_{k1}(v)) k^+_i(u) t^{[1]-}_{j1}(v). \]
This yields
\[ [f^+_j(u), t^{[1]-}_{kl}(v)] = \frac{1}{u_- - v_-} (f^+_k(u) - f^-_{k1}(v)) t^{[1]-}_{j1}(v). \]
Similarly, we can prove
\[ [f^-_{j1}(u), t^{[1]+}_{kl}(v)] = \frac{1}{u_- - v_-} (f^-_{k1}(u) - f^+_k(v)) t^{[1]+}_{j1}(v). \]
So (6.30) with \( m = 1 \) follows.

**Proposition 6.10.** For \( DX(\mathfrak{o}_{2n+1}) \), we have
\[ (6.31) \]
\[ [e^\pm_{n-1}(u), e^\pm_{n}(v)] = \frac{1}{u - v}(e^\pm_{n-1,n+1}(v) - e^\pm_{n-1,n+1}(u) - e^\pm_{n-1}(v)e^\pm_{n}(v) + e^\pm_{n-1}(u)e^\pm_{n}(v)), \]
\[ (6.32) \]
\[ [e^\pm_{n-1}(u), e^\pm_{n}(v)] = \frac{1}{u_\pm - v_\pm}(e^\pm_{n-1,n+1}(v) - e^\pm_{n-1,n+1}(u) - e^\pm_{n-1}(v)e^\pm_{n}(v) + e^\pm_{n-1}(u)e^\pm_{n}(v)). \]

**Proof.** By Lemma 6.9, we get
\[ [e^\pm_{n-1}(u), t^{[n-1]+}_{n+1}(v)] = \frac{1}{u - v} t^{[n-1]+}_{n+1}(v)(e^\pm_{n-1,n+1}(v) - e^\pm_{n-1,n+1}(u)) \]
\[ = [e^\pm_{n-1}(u), k^\pm_{n}(v)e^\pm_{n,n+1}(v)] \]
\[ = [e^\pm_{n-1}(u), k^\pm_{n}(v)]e^\pm_{n,n+1}(v) + k^\pm_{n}(v)[e^\pm_{n-1}(u), e^\pm_{n,n+1}(v)] \]
\[ = \frac{1}{u - v} k^\pm_{n}(v)(e^\pm_{n-1}(v) - e^\pm_{n-1}(u))e^\pm_{n}(v) + k^\pm_{n}(v)[e^\pm_{n-1}(u), e^\pm_{n}(v)] \]
\[ = \frac{1}{u - v} k^\pm_{n}(v)(e^\pm_{n-1,n+1}(v) - e^\pm_{n-1,n+1}(u)). \]
Hence, \( [e^\pm_{n-1}(u), e^\pm_{n}(v)] = \frac{1}{u - v}(e^\pm_{n-1,n+1}(v) - e^\pm_{n-1,n+1}(u) - e^\pm_{n-1}(v)e^\pm_{n}(v) + e^\pm_{n-1}(u)e^\pm_{n}(v)). \) We can prove (6.32) similarly.

**Corollary 6.11.** For \( DX(\mathfrak{o}_{2n+1}) \), we have
\( (u-v-1)X^+_{n-1}(u)X^+_{n}(v) = (u-v)X^+_{n}(v)X^+_{n-1}(u). \)
Proposition 6.12. For $DX(sp_4)$, we have

\begin{equation}
[e^\pm_{12}(u), e^\pm_{23}(v)] = \frac{2}{u-v}(e^\pm_{13}(v) - e^\pm_{13}(u) - e^\pm_{12}(v)e^\pm_{23}(v) + e^\pm_{12}(u)e^\pm_{23}(v)),
\end{equation}

\begin{equation}
[e^\pm_{12}(u), e^\mp_{23}(v)] = \frac{2}{u_\mp-v_\pm}(e^\pm_{13}(v) - e^\pm_{13}(u) - e^\pm_{12}(v)e^\pm_{23}(v) + e^\mp_{12}(u)e^\pm_{23}(v)).
\end{equation}

**Proof.** The proof of (6.33) can be found in [21]. Next we will prove (6.34). From the defining relations, we have

\begin{equation}
[t^+_{12}(u), t^-_{23}(v)] = \frac{1}{u_+ - v_-}t^+_{22}(u)t^-_{13}(v) - \frac{1}{u_- - v_+}t^-_{22}(v)t^+_{13}(u)
+ \frac{1}{u_- - v_+ - 3}(t^-_{24}(v)t^+_{11}(u) + t^-_{23}(v)t^+_{12}(u) - t^-_{22}(v)t^+_{13}(u) - t^-_{21}(v)t^+_{14}(u)).
\end{equation}

The left hand side of (6.35) can be written as

\begin{align*}
k_1^+(u) & [e^+_{12}(u), k_2^-(v)e^\mp_{23}(v)] + [k_1^+(u), k_2^-(v)e^\mp_{23}(v)]e^\pm_{12}(u) \\
& + [k_1^+(u)e^\pm_{12}(u), f^+_{21}(v)k_1^-(v)]e^\pm_{13}(v) + f^+_{21}(v)k_1^-(v)[k_1^+(u)e^\pm_{12}(u), e^\mp_{13}(v)].
\end{align*}

It follows from the defining relations that

\begin{equation}
[k_1^+(u)e^\pm_{12}(u), f^+_{21}(v)k_1^-(v)] = [t^+_{12}(u), t^-_{21}(v)] = \frac{1}{u_+ - v_-}t^+_{22}(u)t^-_{11}(v) - \frac{1}{u_- - v_+}t^-_{22}(v)t^+_{11}(u)
= \frac{1}{u_+ - v_-}(k_2^+(u) + f^+_{21}(u)k_1^+(u)e^\pm_{12}(u))k_1^-(v) - \frac{1}{u_- - v_+}(k_2^-(v) + f^+_{21}(v)k_1^-(v)e^\pm_{12}(v))k_1^+(u).
\end{equation}

To compute the Lie bracket $[k_1^+(u)e^\pm_{12}(u), e^\pm_{13}(v)] = [t^+_{12}(u), t^-_{11}(v), t^-_{13}(v)]$, we have $[t^+_{12}(u), t^-_{11}(v)] = -t^-_{11}(v)^{-1}[t^+_{12}(u), t^-_{11}(v)]t^-_{11}(v)^{-1}$, which can be calculated easily by the defining relations. Also we have $[e^\pm_{12}(u), k_2^-(v)] = \frac{1}{u_- - v_+}k_2^-(v)(e^\pm_{13}(v) - e^\pm_{13}(u))$ by type A relations. Hence, the left hand side of (6.35) takes the form

\begin{align*}
k_1^+(u)k_2^-(v)[e^\pm_{12}(u), e^\mp_{23}(v)] & + \frac{1}{u_- - v_+}k_1^+(u)k_2^-(v)(e^\pm_{13}(v)e^\pm_{12}(v) - e^\pm_{13}(v)e^\pm_{12}(v)) \\
& + \frac{1}{u_+ - v_-}(k_2^+(u)k_1^+(v) + f^+_{21}(u)k_1^+(u)e^\pm_{12}(u))k_1^-(v)e^\pm_{13}(v) - \frac{1}{u_- - v_+}(k_2^-(v)k_1^+(u)e^\pm_{13}(v)) \\
& + f^+_{21}(v)k_1^-(v)e^\pm_{12}(v)k_1^+(v)e^\pm_{13}(v)) + \frac{1}{u_- - v_+}f^+_{21}(v)k_1^-(v)(e^\pm_{14}(v)k_1^+(u) + e^\pm_{13}(v)k_1^+(u)e^\pm_{12}(u)) \\
& - e^\pm_{12}(v)k_1^+(u)e^\pm_{13}(v) - k_1^+(u)e^\pm_{14}(u)) + \frac{1}{u_- - v_+}(f^+_{21}(v)k_1^-(v)e^\pm_{12}(v)k_1^+(u)e^\pm_{13}(v) \\
& - f^+_{21}(v)k_1^-(v)e^\pm_{12}(v)k_1^+(u)e^\pm_{13}(u)).
\end{align*}
The right hand side of (6.35) takes the form

\[
\begin{align*}
\frac{1}{u_+ - v_-} (f_{21}^+(u)k_1^+(u)e_{12}(u) + k_2^+(u))k_1^-(v)e_{13}(v) - \frac{1}{u_- - v_+} (f_{21}^-(v)k_1^-(v)e_{12}(v) + k_2^-(v)) \\
nk_1^+(u)e_{13}(u) + \frac{1}{u_- - v_+ - 3} \left[(f_{21}^+(v)k_1^+(v)e_{14}(v) + k_2^+(v)e_{24}(v)k_1^+(u) + (f_{21}^-(v)k_1^-(v)e_{13}(v)ight. \\
+ k_2^-(v)e_{23}(v))k_1^+(u)e_{12}(u) - (f_{21}^+(v)k_1^+(v)e_{12}(v) + k_2^+(v))k_1^+(u)e_{13}(u) - f_{21}^-(v)k_1^-(v)k_1^+(u)e_{14}(u)\right].
\end{align*}
\]

By cancelling the common terms, we have

\[
[e_{12}^+(u), e_{23}^-(v)] = \frac{1}{u_+ - v_-} (e_{13}(v) - e_{13}(u) + e_{12}^+(u)e_{23}(v) - e_{12}^-(v)e_{23}(v)) + \frac{1}{u_- - v_+ - 3} k_1^+(u)^{-1}
\]

\[
(e_{24}(v)k_1^+(u) + e_{23}(v)k_1^+(u)e_{12}(u) - k_1^+(u)e_{13}(u) = \frac{1}{u_+ - v_-} (e_{13}(v) - e_{13}(u) + e_{12}^+(u)e_{23}(v)
- e_{12}^-(v)e_{23}(v)) + \frac{1}{u_- - v_+ - 2} (e_{24}(v) + e_{2}^+(v)e_{1}^+(u) - e_{13}(u) = \frac{2}{u_+ - v_-} (e_{13}(v) - e_{13}(u)
- e_{12}^-(v) + e_{1}^+(u)e_{2}^+(v)) + \frac{1}{u_- - v_+ - 1} (e_{24}(v) + e_{1}^+(v)e_{2}^-(v) - e_{13}(v)).
\]

Therefore, \( e_{24}^-(v) + e_{1}^-(v)e_{2}^-(v) - e_{13}(v) = 0 \). Then

\[
[e_{12}^+(u), e_{23}^-(v)] = \frac{2}{u_+ - v_-} (e_{13}(v) - e_{13}(u) - e_{12}^-(v)e_{23}(v) + e_{12}^+(u)e_{23}(v)).
\]

Similarly, we can prove

\[
[e_{12}^-(u), e_{23}^+(v)] = \frac{2}{u_- - v_+} (e_{13}(v) - e_{13}(u) - e_{12}^+(v)e_{23}(v) + e_{12}^-(u)e_{23}(v)).
\]

This finishes the proof of (6.34).

The following result follows easily from Proposition 6.12.

**Corollary 6.13.** For \( DX(\mathfrak{sp}_{2n}) \), we have

\[
[e_{n-1}^\pm(u), e_{n}^\pm(v)] = \frac{2}{u - v} (e_{n-1,n+1}^\pm(v) - e_{n-1,n+1}^\pm(u) - e_{n-1}^\pm(v)e_{n}^\pm(v) + e_{n-1}^\pm(u)e_{n}^\pm(v)),
\]

\[
[e_{n-1}^\pm(u), e_{n}^\mp(v)] = \frac{2}{u - v} (e_{n-1,n+1}^\mp(v) - e_{n-1,n+1}^\mp(u) - e_{n-1}^\mp(v)e_{n}^\mp(v) + e_{n-1}^\pm(u)e_{n}^\pm(v)),
\]

\[
(u - v - 2)X_{n-1}^-(u)X_{n}^+(v) = (u - v)X_{n}^+(v)X_{n-1}^-(u).
\]

**Proposition 6.14.** For \( DX(\mathfrak{o}_{2n+1}) \), \( DX(\mathfrak{sp}_{2n}) \), we have \( [e_{n-1}^\pm(u), f_{n}^\mp(v)] = [e_{n-1}^\pm(u), f_{n}^\pm(v)] = [e_{n}^\pm(u), f_{n}^\mp(v)] = [e_{n}^\pm(u), f_{n}^\pm(v)] = 0 \).

**Proof.** As \( t_{n+1,1}^{[n-1]}(v) = f_{n}^\mp(v)k_{n}^\pm(v) \), it follows from Lemma 6.9 that

\[
[e_{n-1}^\pm(u), f_{n}^\pm(v)k_{n}^\pm(v)] = \frac{1}{u_+ - v_-} f_{n}^\pm(v)k_{n}^\pm(v)(e_{n-1}^\mp(v) - e_{n-1}^\pm(u)).
\]
On the other hand, by type A relations we have

\[(6.37) \quad [e_{n-1}^\pm(u), k_n^\pm(v)] = \frac{1}{u_\pm - v_\pm} k_n^\pm(v)(e_{n-1}^\pm(v) - e_{n-1}^\pm(u)),\]

which forces that \([e_{n-1}^\pm(u), f_n^\pm(v)] = 0\). The other relations are verified similarly. \(\square\)

**Corollary 6.15.** For \(DX(\mathfrak{o}_{2n+1}), DX(\mathfrak{sp}_{2n})\), we have \([X_n^+(u), X_n^-(v)] = [X_{n-1}^+(u), X_n^-(v)] = 0\).

**Proof.** As \(X_i^+(u) = e_i^+(u_+) - e_i^-(u_-)\) and \(X_i^-(u) = f_i^-(u_+) + f_i^+(u_-)\), the relations follows immediately from Prop. 6.14. \(\square\)

Now we recall a symmetry property [21] for the matrix elements in the Gauss decomposition, which will be useful for later discussion.

**Proposition 6.16.** The following symmetry relations hold in \(DX(\mathfrak{g}_N)\),

\[(6.38) \quad e_{(i+1)'}^\pm(u) = -e_i^\pm(u + \kappa - i) \quad \text{and} \quad f_{(i+1)'}^\pm(u) = -f_i^\pm(u + \kappa - i)\]

for \(i = 1, \ldots, n - 1\). In addition, for \(\mathfrak{g}_N = \mathfrak{sp}_{2n}\) one has that

\[e_{n+1}^\pm(u) = -e_{n-1}^\pm(u + 2) \quad \text{and} \quad f_{n+1}^\pm(u) = -f_{n-1}^\pm(u + 2),\]

and for \(\mathfrak{g}_N = \mathfrak{o}_{2n}\) one has that

\[e_{n+1}^\pm(u) = -e_{n-1}^\pm(u) \quad \text{and} \quad f_{n+1}^\pm(u) = -f_{n-1}^\pm(u).\]

**Proposition 6.17.** In \(DX(\mathfrak{g}_N)\), for \(\epsilon = \pm\) one has the following

\[\ [k_{n+1}^\pm(u), X_{n-1}^\epsilon(v)] = 0, \quad \text{for } \mathfrak{g}_N = \mathfrak{o}_{2n+1}\]

\[k_{n+1}^\pm(u)^{-1} X_{n-1}^\epsilon(v) k_{n+1}^\pm(u) = \frac{u_\pm - v + 1}{u_\pm - v + 2} X_{n-1}^\epsilon(v), \quad \text{for } \mathfrak{g}_N = \mathfrak{sp}_{2n}\]

\[k_{n+1}^\pm(u)^{-1} X_{n-1}^\epsilon(v) k_{n+1}^\pm(u) = \frac{u_\pm - v - 1}{u_\pm - v} X_{n-1}^\epsilon(v), \quad \text{for } \mathfrak{g}_N = \mathfrak{o}_{2n}\]

**Proof.** First consider \(\mathfrak{g}_N = \mathfrak{o}_{2n+1}\). Corollary 4.9 says that \(k_{n+1}^\pm(u)\) commutes with the elements of the subalgebra generated by the \(t_{ij}^\pm(u)\) with \(1 \leq i, j \leq n\), thus the relations follow. Now let \(\mathfrak{g}_N = \mathfrak{sp}_{2n}\) or \(\mathfrak{o}_{2n}\). By Corollary 5.2, the subalgebra \(DX^{[n-2]}(\mathfrak{g}_N)\) of \(DX(\mathfrak{g}_N)\) is isomorphic to \(DX(\mathfrak{g}_4)\). Applying Proposition 6.5 to this subalgebra, we get

\[k_{n+1}^\pm(u)^{-1} e_{n+1,n+2}^\pm(v) k_{n+1}^\pm(u) = \frac{u - v - 1}{u - v} e_{n+1,n+2}^\pm(u) + \frac{1}{u - v} e_{n+1,n+2}^\pm(u),\]

\[k_{n+1}^\pm(u)^{-1} e_{n+1,n+2}^\mp(v) k_{n+1}^\pm(u) = \frac{u - v - 1}{u - v} e_{n+1,n+2}^\mp(u) + \frac{1}{u - v} e_{n+1,n+2}^\mp(u),\]

\[k_{n+1}^\pm(u)^{-1} f_{n+2,n+1}^\pm(v) k_{n+1}^\pm(u)^{-1} = \frac{u - v - 1}{u - v} f_{n+2,n+1}^\pm(v) + \frac{1}{u - v} f_{n+2,n+1}^\pm(v),\]

\[k_{n+1}^\pm(u)^{-1} f_{n+2,n+1}^\mp(v) k_{n+1}^\pm(u)^{-1} = \frac{u - v - 1}{u - v} f_{n+2,n+1}^\mp(v) + \frac{1}{u - v} f_{n+2,n+1}^\mp(v).\]

Then the relations follow by applying Proposition 6.16. \(\square\)
Proposition 6.18. For $i=1,\ldots,n$ and $\epsilon=\pm$ we have in the algebra $\text{DX}(g_N)$

\begin{align}
(6.39) & \quad k_i^\pm(u)^{-1}X_n^\epsilon(v)k_i^\pm(u) = (\delta_{i\leftarrow n-1} + \delta_{i<n})X_n^\epsilon(v), \quad \text{for } g_N = \mathfrak{o}_{2n+1} \\
(6.40) & \quad k_i^\pm(u)^{-1}X_n^\epsilon(v)k_i^\pm(u) = (\delta_{i\leftarrow n-2} + \delta_{i<n-1})X_n^\epsilon(v), \quad \text{for } g_N = \mathfrak{sp}_{2n} \\
(6.41) & \quad k_i^\pm(u)^{-1}X_n^\epsilon(v)k_i^\pm(u) = (\delta_{i\leftarrow n-2} + \delta_{i<n-1})X_n^\epsilon(v), \quad \text{for } g_N = \mathfrak{o}_{2n+1} \\

Moreover in all three cases, we have that

\begin{align}
(6.39) & \quad [X_i^+(u),X_i^-(v)] = [X_i^+(u),X_i^-(v)] = 0 \quad (i < n) \\
(6.40) & \quad [k_{i+1}^\pm(u),X_i^+(v)] = [k_{i+1}^\pm(u),X_i^-(v)] = 0 \quad (i < n-1)
\end{align}

Here the notation $\delta_{i<n}$ means $\delta_{i<n} = 1$ if $i < n$, otherwise $\delta_{i<n} = 0$.

Proof. The relations (6.39)-(6.41) for $i = n$ were already given in Props. 6.6-6.8. Now suppose $g_N = \mathfrak{o}_{2n}$. By the defining relations (2.19) and (2.20), the subalgebra generated by the coefficients of the series $t_{ij}^\pm(u)$ with $i,j$ running over the set $J = \{1,\ldots,n-1,n+1\}$ is isomorphic to $\text{DY}(g_n)$. Moreover, Lemma 6.3 implies that

$$\epsilon_{n+1}^\pm(u) = f_{n+1}^\pm(u) = 0.$$ 

Therefore the following Gauss decomposition holds

\begin{align}
(6.44) & \quad T_{ij}^\pm(u) = F_{ij}^\pm(u) H_{ij}^\pm(u) E_{ij}^\pm(u),
\end{align}

where the subscript $J$ indicates the submatrices in (2.28) with rows and columns labelled by integers of $J$. This means that the Gaussian generators which occur as the entries of the matrices $F_{ij}^\pm(u)$, $H_{ij}^\pm(u)$ and $E_{ij}^\pm(u)$ satisfy the type $A$ relations as given in Prop. 6.4, and the case of type $D$ is done. Now let $g_N = \mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_{2n}$. Almost all the relations (6.39)-(6.40) with $i < n$, as well as (6.42) and (6.43) with $i < n-1$ follow from Cor. 4.9. For instance, to check the commutation relation between $k_i^-(u)$ and $f_n^+(v)$, we get

$$\frac{(u_- - v_+)^2}{(u_- - v_+)^2 - 1} t_{11}^-(u) t_{n+1}^{[n-1]+}(v) = \frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - 1} t_{n+1}^{[n-1]+}(v) t_{11}^-(u)$$

from Cor. 4.9. But $t_{11}^-(u) = k_i^-(u)$ and $t_{n+1}^{[n-1]+}(v) = f_n^+(v)$ for $k_i^+(v)$. We also have

$$\frac{(u_- - v_+)^2}{(u_- - v_+)^2 - 1} k_i^-(u) k_n^+(v) = \frac{(u_+ - v_-)^2}{(u_+ - v_-)^2 - 1} k_n^+(v) k_i^-(u).$$

It follows that $k_i^-(u) f_n^+(v) = f_n^+(v) k_i^-(u)$. As for the cases $i = n-1$ of (6.42), see Prop. 6.14.

Proposition 6.19. We have the relations in $\text{DX}(\mathfrak{o}_{2n})$:

\begin{align}
(6.45) & \quad (u-v-1)X_{n-2}^+(u)X_n^+(v) = (u-v)X_n^+(v)X_{n-2}^+(u), \\
(6.46) & \quad (u-v)X_{n-2}^+(u)X_n^+(v) = (u-v-1)X_n^-(v)X_{n-2}^-(u).
\end{align}
Proof. The Gauss generators occurring as the entries of $F_j^\pm(u), H_j^\pm(u)$ and $E_j^\pm(u)$ in (6.44) satisfy the Yangian relations of type $A$ (see the proof of Proposition 6.18), so (6.45)-(6.46) follow from the Drinfeld realization of the double Yangian $\text{DY}(\frak{g}_n)$.

\begin{proposition}
For $1 \leq i \leq n - 1$ and $(\alpha_i, \alpha_n) = 0$, we have
\begin{equation}
[X_i^\pm(u), X_n^\pm(v)] = 0.
\end{equation}
\end{proposition}

\begin{proof}
If $i \leq n - 2$ in types $B$ and $C$ or $i \leq n - 3$ in type $D$, the relation (6.47) follows from Corollary 4.9. If $i = n - 1$ in type $D$, (6.47) is given by Lemma 6.3 and Corollary 5.2.
\end{proof}

\begin{proposition}
For $\frak{g}_N = \frak{o}_{2n+1}$ we have
\begin{align}
(6.48) & & k_n^+(u)k_{n+1}^-(v) = k_{n+1}^-(v)k_n^+(u), \\
(6.49) & & \frac{(u_+ - v_+ - 1)(u_- - v_-)}{(u_+ - v_+ + 1)(u_- - v_-)}k_n^-(u)k_{n+1}^+(v) = \frac{(u_- - v_+ - 1)(u_+ - v_-)}{(u_+ - v_+ + 1)(u_- - v_-)}k_{n+1}^-(v)k_n^+(u), \\
(6.50) & & \frac{u_+ - v_+ + 1}{u_+ - v_-}k_n^+(u)k_{n+1}^-(v) = \frac{u_- - v_+ + 1}{u_- - v_-}k_{n+1}^-(v)k_n^+(u).
\end{align}

For $\frak{g}_N = \frak{sp}_{2n}$ we have
\begin{align}
(6.51) & & \frac{u_+ - v_+ - 2}{u_+ - v_+ - 1}k_n^+(u)k_{n+1}^-(v) = \frac{u_+ - v_+ - 2}{u_+ - v_+ - 1}k_{n+1}^-(v)k_n^+(u), \\
(6.52) & & \frac{(u_+ - v_- - 2)(u_- - v_+)}{(u_+ - v_+ + 1)(u_- - v_-)}k_n^-(u)k_{n+1}^+(v) = \frac{(u_- - v_+ - 2)(u_+ - v_-)}{(u_+ - v_+ + 1)(u_- - v_-)}k_{n+1}^+(v)k_n^-(u), \\
(6.53) & & \frac{u_+ - v_- + 1}{u_+ - v_-}k_n^+(u)k_{n+1}^-(v) = \frac{u_- - v_- + 1}{u_- - v_-}k_{n+1}^-(v)k_n^+(u).
\end{align}

For $\frak{g}_N = \frak{o}_{2n}$ we have
\begin{align}
(6.54) & & \frac{(u_+ - v_+)^2}{(u_+ - v_+ + 1)^2}k_n^+(u)k_{n+1}^-(v) = \frac{(u_+ - v_+)^2}{(u_+ - v_+ + 1)^2}k_{n+1}^-(v)k_n^+(u), \\
(6.55) & & \frac{u_+ - v_+ + 1}{u_+ - v_- - 1}k_n^-(u)k_{n+1}^+(v) = \frac{u_+ - v_+ + 1}{u_+ - v_- - 1}k_{n+1}^+(v)k_n^-(u), \\
(6.56) & & \frac{u_- - v_- + 1}{u_+ - v_-}k_n^+(u)k_{n+1}^-(v) = \frac{u_- - v_- + 1}{u_+ - v_-}k_{n+1}^-(v)k_n^+(u).
\end{align}

\begin{proof}
For $\frak{g}_N = \frak{o}_{2n+1}$, Proposition 6.4 implies that
\begin{align*}
\frac{u_- - v_- - 1}{u_+ - v_-}k_n^-(u)k_{n+1}^-(v)k_n^+(u) & = \frac{u_- - v_- - 1}{u_+ - v_-}k_{n+1}^-(v)k_n^+(u), \\
\frac{u_- - v_- - 1}{u_+ - v_-}k_n^+(u)k_n^-(v) & = \frac{u_- - v_- - 1}{u_+ - v_-}k_n^-(v)k_n^+(u).
\end{align*}
\end{proof}
It follows that $k^+_n(u)k^-_{n+1}(v) = k^-_{n+1}(v)k^+_n(u)$. Similarly, we have

$$\frac{u_+ - v_1 - 1}{u_+ - v_-} k_n^-(u)k^+_n(v)k^-_{n+1}(v) = \frac{u_- - v_+ - 1}{u_- - v_+} k_{n+1}^-(v)k^+_n(v)^{-1}k^-_n(u),$$

$$\frac{u_+ - v_- + 1}{u_+ - v_-} k_n^-(u)k^+_n(v) = \frac{u_- - v_+ + 1}{u_- - v_+} k_n^+(v)k^-_n(u).$$

It follows that

$$\frac{u_+ - v_- - 1}{u_+ - v_-} u_- - v_+ + 1 k^-_n(u)k^+_n(v)k^-_{n+1}(v) = \frac{u_- - v_+ - 1}{u_- - v_+} u_- - v_+ + 1 k^-_{n+1}(v)k^+_n(u).$$

Proposition 6.4 also implies that

$$\frac{u_- - v_+ - 1}{u_- - v_+} k^+_n(u)k^-_{n+1}(v)k^-_n(v) = \frac{u_- - v_+ - 1}{u_- - v_+} k^-_{n+1}(v)k^+_n(v)^{-1}k^-_n(u).$$

Therefore, we get

$$\frac{u_+ - v_- - 1}{u_+ - v_-} k^-_{n+1}(u)k^+_n(v) = \frac{u_- - v_+ - 1}{u_- - v_+} k^-_{n+1}(v)k^+_n(u).$$

This completes the proof for type $B$. The other relations can be checked similarly. \(\blacksquare\)

### 6.4. Drinfeld presentation

We now prove the Drinfeld presentation for $DX(\mathfrak{g}_N)$. The notation still follows that of Section 2.

The Drinfeld realization for the Yangian in type $B, C, D$ has been proved to be isomorphic to the $RTT$ presentation [21] via the Gauss decomposition of the generating matrix. The identification between the two realization can also be done through Drinfeld’s third presentation, recently given in [13]. Our method follows the recent work of [21] for the isomorphism between the Drinfeld realization and $RTT$ presentation for the Yangian algebra in types $B, C$ and $D$.

**Theorem 6.22.** The extended Yangian double $DX(\mathfrak{g}_N)$ is generated by the coefficients of the series $k^\pm_i(u)$ ($1 \leq i \leq n + 1$), $e^\pm_i(u)$ and $f^\pm_i(u)$ ($1 \leq i \leq n$), and $c$ subject to the following relations, where the indices run through all admissible values unless specified otherwise. The relations are

$$(6.57) \quad [k_i^\pm(u), k_j^\pm(v)] = 0.$$  

for $i < j \leq n$

$$(6.58) \quad k_i^+(u)k_j^-(v) = k_j^-(v)k_i^+(u),$$

$$(6.59) \quad \frac{(u_+ - v_1 - 1)}{(u_+ - v_-)^2} k_j^+(u)k_i^-(v) = \frac{(u_- - v_+ - 1)}{(u_- - v_+)^2} k_i^-(v)k_j^+(u).$$

For $i \leq n - 1$ we have

$$(6.60) \quad k_i^+(u)k_{n+1}^-(v) = k_{n+1}^-(v)k_i^+(u),$$

$$(6.61) \quad \frac{(u_+ - v_1 - 1)}{(u_+ - v_-)^2} k_{n+1}^+(u)k_i^-(v) = \frac{(u_- - v_+ - 1)}{(u_- - v_+)^2} k_i^-(v)k_{n+1}^+(u).$$
For \( i \leq n \) we have

\[
\frac{u_+ - v_+ + 1}{u_+ - v_-} k_i^+(u) k_i^+(v) = \frac{u_- - v_+ + 1}{u_- - v_+} k_i^+(v) k_i^+(v).
\]

For \( i \leq n - 1, j \leq n, j \neq i, i + 1, \) and \( \epsilon = \pm \) we have

\[
\begin{align*}
(6.63) \quad & k_i^+(u)^{-1} X_i^+(v) k_i^+(u) = \frac{u_\pm - v - 1}{u_\pm - v} X_i^+(v), \\
(6.64) \quad & k_i^+(u) X_i^-(v) k_i^+(u)^{-1} = \frac{u_\pm - v - 1}{u_\pm - v} X_i^-(v), \\
(6.65) \quad & k_{i+1}^+(u)^{-1} X_i^+(v) k_{i+1}^+(u) = \frac{u_\pm - v + 1}{u_\pm - v} X_i^+(v), \\
(6.66) \quad & k_{i+1}^+(u) X_i^-(v) k_{i+1}^+(u)^{-1} = \frac{u_\pm - v + 1}{u_\pm - v} X_i^-(v), \\
(6.67) \quad & k_j^+(u)^{-\epsilon} X_j^+(v) k_j^+(u) = X_j^+(v), \\
(6.68) \quad & (u - v \pm 1) X_i^+(u) X_i^+(v) = (u - v \mp 1) X_i^+(v) X_i^+(u), \\
(6.69) \quad & (u - v - 1) X_{i-1}^+(u) X_i^+(v) = (u - v) X_i^+(v) X_{i-1}^+(u), \\
(6.70) \quad & (u - v) X_{i-1}^-(u) X_i^-(v) = (u - v - 1) X_i^-(v) X_{i-1}^-(u).
\end{align*}
\]

For \( i, j \leq n - 1 \) we have

\[
[X_i^+(u), X_j^-(v)] = \delta_{ij} \left\{ \delta(u_+ - v_+) k_{i+1}^-(v_+) k_i^-(v_-)^{-1} - \delta(u_+ - v_-) k_{i+1}^+(u_+) k_i^+(u_-)^{-1} \right\}
\]

where \( \delta(u - v) = \Sigma_{k \in \mathbb{Z}} u^{-k-1} v^k \).

For \( \mathfrak{g}_N = \mathfrak{o}_{2n+1} \) we have

\[
\begin{align*}
(6.71) \quad & k_n^+(u) k_{n+1}^-(v) = k_{n+1}^-(v) k_n^+(u), \\
(6.72) \quad & \frac{u_+ - v_+ - 1}{u_- - v_+} k_n^-(u) k_{n+1}^+(v) = \frac{u_- - v_+ - 1}{u_+ - v_-} k_{n+1}^+(v) k_n^-(u), \\
(6.73) \quad & \frac{u_+ - v_+ + 1}{u_- - v_+} k_n^-(u) k_{n+1}^+(v) = \frac{u_- - v_+ + 1}{u_+ - v_-} k_{n+1}^+(v) k_n^-(u), \\
(6.74) \quad & k_i^+(u) X_i^+(v) = X_i^+(v) k_i^+(u), \quad k_i^-(u) X_i^-(v) = X_i^-(v) k_i^-(u), \quad (i \leq n - 1) \\
(6.75) \quad & k_n^+(u) X_1^+(v) = X_1^+(v) k_n^+(u), \quad k_n^-(u) X_1^-(v) = X_1^-(v) k_n^-(u), \\
(6.76) \quad & k_n^+(u) X_n^+(v) k_n^+(u)^{-1} = \frac{u_\pm - v - 1}{u_\pm - v} X_n^+(v), \\
(6.77) \quad & k_n^+(u) X_n^-(v) k_n^+(u)^{-1} = \frac{u_\pm - v - 1}{u_\pm - v} X_n^-(v).
\end{align*}
\]
\( k_{n+1}^+(u)^{-1} X_j^+(v) k_{n+1}^+(u) = X_j^+(v), \quad (j < n) \)

\( k_{n+1}^+(u) X_j^+(v) k_{n+1}^+(u)^{-1} = X_j^+(v), \quad (j < n) \)

\( k_{n+1}^+(u)^{-1} X_n^+(v) k_{n+1}^+(u) = \left( \frac{u_+ - v - 1}{(u_+ - v)(u_+ - v - 1)} \right) X_n^+(v), \)

\( k_{n+1}^+(u) X_n^+(v) k_{n+1}^+(u)^{-1} = \left( \frac{u_+ - v - 1}{(u_+ - v)(u_+ - v - 1)} \right) X_n^+(v), \)

\( (u - v + \frac{1}{2}) X_n^+(u) X_n^+(v) = (u - v - \frac{1}{2}) X_n^+(v) X_n^+(u), \)

\( (u - v - \frac{1}{2}) X_n^-(u) X_n^-(v) = (u - v + \frac{1}{2}) X_n^-(v) X_n^-(u), \)

\[ [X_{n-1}^+(u), X_n^+(v)] = [X_n^+(u), X_{n-1}^+(v)] = 0, \]

\[ [X_i^+(u), X_j^+(v)] = 0, \quad (i < n - 1) \]

\( (u - v - 1) X_{n-1}^+(u) X_n^+(v) = (u - v) X_n^+(v) X_{n-1}^+(u), \)

\( (u - v) X_{n-1}^+(u) X_n^-(v) = (u - v - 1) X_n^-(v) X_{n-1}^-(u), \)

\[ [X_n^+(u), X_n^-(v)] = \{ \delta(u_+ - v+) k_{n+1}^+(v_+) k_n^+(v_+) \} - \delta(u_+ - v_) k_{n+1}^+(u+) k_n^+(u_-) \}

For \( g_N = sp_{2n} \) we have

\( \frac{1}{u_- - v_+ - 1} k_n^+(u) k_{n+1}^-(v) k_{n+1}^+(u) = \frac{u_+ - v_- - 2}{u_+ - v_- - 1} k_{n+1}^+(v) k_n^+(u), \)

\( \frac{u_+ - v_- - 2}{u_+ - v_- + 1} k_n^-(u) k_{n+1}^+(v) k_{n+1}^+(u) = \frac{(u_+ - v_- - 2)(u_+ - v_-)}{(u_+ - v_- + 1)(u_+ - v_-)} k_{n+1}^+(v) k_n^-(u), \)

\( \frac{u_+ - v_- + 1}{u_+ - v_-} k_{n+1}^+(u) k_n^-(u) k_{n+1}^+(v) = \frac{u_+ - v_- + 1}{u_+ - v_-} k_{n+1}^+(v) k_n^+(u), \)

\( k_i^+(u) X_n^+(v) = X_n^+(v) k_i^+(u), \quad k_i^-(u) X_n^-(v) = X_n^-(v) k_i^+(u), \quad (i \leq n - 1) \)

\( k_i^+(u)^{-1} X_n^+(v) k_i^+(u) = \frac{u_+ - v_- - 2}{u_+ - v_-} X_n^+(v), \)

\( k_i^-(u) X_n^-(v) k_i^+(u)^{-1} = \frac{u_+ - v_- - 2}{u_+ - v_-} X_n^-(v), \)

\( k_{n+1}^+(u)^{-1} X_j^+(v) k_{n+1}^+(u) = X_j^+(v), \quad (j < n - 1) \)

\( k_{n+1}^+(u) X_j^-(v) k_{n+1}^+(u)^{-1} = X_j^-(v), \quad (j < n - 1) \)

\( k_{n+1}^+(u)^{-1} X_{n-1}^+(v) k_{n+1}^+(u) = \frac{u_+ - v_- + 1}{u_+ - v_- + 2} X_{n-1}^+(v), \)

\( k_{n+1}^+(u) X_{n-1}^-(v) k_{n+1}^+(u)^{-1} = \frac{u_+ - v_- + 1}{u_+ - v_- + 2} X_{n-1}^-(v), \)
\[(6.99) \quad k^\pm_{n+1}(u)^{-1}X^+_n(v)k^\pm_{n+1}(u) = \frac{u_\pm - v + 2}{u_\pm - v} X^+_n(v),\]

\[(6.100) \quad k^\pm_{n+1}(u)X^-_n(v)k^\pm_{n+1}(u)^{-1} = \frac{u_\pm - v + 2}{u_\pm - v} X^-_n(v),\]

\[(6.101) \quad (u - v + 2)X^+_n(u)X^+_n(v) = (u - v - 2)X^+_n(v)X^+_n(u),\]

\[(6.102) \quad (u - v - 2)X^-_n(u)X^-_n(v) = (u - v + 2)X^-_n(v)X^-_n(u),\]

\[(6.103) \quad [X^\pm_{n-1}(u), X^\mp_n(v)] = [X^\pm_n(u), X^\mp_{n-1}(v)] = 0,\]

\[(6.104) \quad [X^\mp_n(u), X^\pm_n(v)] = 0, \quad (i < n-1)\]

\[(6.105) \quad (u - v - 2)X^+_{n-1}(u)X^+_n(v) = (u - v)X^+_n(v)X^+_{n-1}(u),\]

\[(6.106) \quad (u - v)X^-_{n-1}(u)X^-_n(v) = (u - v - 2)X^-_n(v)X^-_{n-1}(u),\]

\[(6.107) \quad [X^+_n(u), X^-_n(v)] = 2\{\delta(u_+ - v_+)k^+_{n+1}(v_+)k^-_n(v_-)^{-1} - \delta(u_+ - v_-)k^+_n(u_+)k^+_n(u_+)^{-1}\}.\]

For \(\mathfrak{g}_N = \mathfrak{o}_{2n}\) we have

\[(6.108) \quad \frac{(u_- - v_+)^2}{(u_- - v_+ + 1)^2} k^+_n(u)k^-_{n+1}(v) = \frac{(u_+ - v_-)^2}{(u_+ - v_- + 1)^2} k^-_{n+1}(v)k^+_n(u),\]

\[(6.109) \quad \frac{u_- - v_+ + 1}{u_- - v_+ - 1} k^-_n(u)k^+_n(v) = \frac{u_+ - v_- + 1}{u_+ - v_- - 1} k^+_n(v)k^-_n(u),\]

\[(6.110) \quad \frac{u_- - v_+ + 1}{u_- - v_+ - 1} k^-_{n+1}(u)k^+_n(v) = \frac{u_+ - v_- + 1}{u_+ - v_- - 1} k^+_n(v)k^-_{n+1}(u),\]

\[(6.111) \quad k^+_n(u)X^+_n(v) = X^+_n(v)k^+_n(u), \quad k^+_n(u)X^-_n(v) = X^-_n(v)k^+_n(u), \quad (i \leq n - 2)\]

\[(6.112) \quad k^\pm_{n-1}(u)^{-1}X^+_n(v)k^\pm_{n-1}(u) = \frac{u_\pm - v - 1}{u_\mp - v} X^+_n(v),\]

\[(6.113) \quad k^\pm_{n-1}(u)X^-_n(v)k^\pm_{n-1}(u)^{-1} = \frac{u_\pm - v - 1}{u_\mp - v} X^-_n(v),\]

\[(6.114) \quad k^\pm_n(u)^{-1}X^+_n(v)k^\pm_n(u) = \frac{u_\pm - v - 1}{u_\mp - v} X^+_n(v),\]

\[(6.115) \quad k^\pm_n(u)X^-_n(v)k^\pm_n(u)^{-1} = \frac{u_\pm - v - 1}{u_\mp - v} X^-_n(v),\]

\[(6.116) \quad k^\pm_{n+1}(u)^{-1}X^+_j(v)k^\pm_{n+1}(u) = X^+_j(v), \quad (j < n - 1)\]

\[(6.117) \quad k^\pm_{n+1}(u)X^-_j(v)k^\pm_{n+1}(u)^{-1} = X^-_j(v), \quad (j < n - 1)\]

\[(6.118) \quad k^\pm_{n+1}(u)^{-1}X^+_n(v)k^\pm_{n+1}(u) = \frac{u_\pm - v - 1}{u_\mp - v} X^+_n(v),\]

\[(6.119) \quad k^\pm_{n+1}(u)X^-_n(v)k^\pm_{n+1}(u)^{-1} = \frac{u_\pm - v - 1}{u_\mp - v} X^-_n(v),\]
(6.120) \[ k_{n+1}^\pm(u)^{-1}X_n^+(v)k_{n+1}^\pm(u) = \frac{u_+ - v + 1}{u_+ - v}X_n^+(v), \]

(6.121) \[ k_{n+1}^\pm(u)X_n^-(v)k_{n+1}^\pm(u)^{-1} = \frac{u_- - v + 1}{u_- - v}X_n^-(v), \]

(6.122) \[ (u - v + 1)X_n^+(u)X_n^+(v) = (u - v - 1)X_n^+(v)X_n^+(u), \]

(6.123) \[ (u - v - 1)X_n^-(u)X_n^-(v) = (u - v + 1)X_n^-(v)X_n^-(u), \]

(6.124) \[ [X_{n-1}^\pm(u), X_n^+(v)] = [X_n^\pm(u), X_{n-1}^\pm(v)] = 0, \]

(6.125) \[ X_{n-1}^+(u)X_n^+(v) = X_n^+(v)X_{n-1}^+(u), \]

(6.126) \[ X_{n-1}^-(u)X_n^-(v) = X_n^-(v)X_{n-1}^-(u), \]

(6.127) \[ (u - v - 1)X_{n-2}^+(u)X_n^+(v) = (u - v)X_n^+(v)X_{n-2}^+(u), \]

(6.128) \[ (u - v)X_{n-2}^-(u)X_n^-(v) = (u - v - 1)X_n^-(v)X_{n-2}^-(u), \]

(6.129) \[ [X_i^+(u), X_i^-(v)] = 0, \quad (i < n - 2) \]

(6.130) \[ [X_i^+(u), X_i^-(v)] = \{\delta(u_- - v_+)k_{n+1}^-k_{n-1}^-(v_+)^{-1} - \delta(u_+ - v_-)k_{n+1}^+k_{n-1}^+(u_+)^{-1}\}. \]

In all three cases we have

(6.131) \[ [X_i^+(u), X_i^-(v)] = [X_n^+(u), X_i^-(v)] = 0. \quad (i < n) \]

and the Serre relations are

(6.132) \[ \sum_{\sigma \in S_m} [X_i^+(u_{\sigma(1)}), [X_i^+(u_{\sigma(2)}), \cdots, [X_i^+(u_{\sigma(m)}), X_j^\pm(v)]]] = 0 \quad i \neq j, m = 1 - a_{ij}. \]

Here \( A = (a_{ij}) \) is the Cartan matrix of the Lie algebra \( \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n} \) respectively.

**Proof.** Apart from the Serre relations, all the relations are satisfied in the algebra \( DX(\mathfrak{g}_N) \) due to Propositions 4.9, 6.6, 6.7, 6.8, 6.17, 6.18, 6.19, 6.20, 6.21 and Corollary 6.11, 6.13. To see the Serre relations, we need to write down the commutation relations between \( X_i^\pm(u) \) and \( X_j^\pm(v) \), which have been calculated in the previous section. Then it is not difficult to verify the Serre relations.

Now consider the algebra \( \widehat{DX}(\mathfrak{g}_N) \) with generators and relations as in the statement of the theorem. The above argument shows that there is a homomorphism

(6.133) \[ \widehat{DX}(\mathfrak{g}_N) \to DX(\mathfrak{g}_N) \]

which takes the generators \( k_{i}^{(r)}, e_{i}^{(r)}, f_{i}^{(r)} \) and \( c \) of \( \widehat{DX}(\mathfrak{g}_N) \) to the elements of \( DX(\mathfrak{g}_N) \) denoted by the same symbols. We need to demonstrate that this homomorphism is surjective and injective. To prove the surjectivity we need a lemma. \( \square \)
Lemma 6.23. Assume $r \in \mathbb{Z}, r \neq 0$. For all $1 \leq i < j \leq n$ in the algebra $\text{DX}(\mathfrak{o}_{2n+1})$ we have
\[ e^{(r)}_{ij+1} = [e^{(r)}_{ij}, e^{(1)}_j], \quad f^{(r)}_{j+1i} = [f^{(1)}_j, f^{(r)}_{ji}], \]
\[ e^{(r)}_{ij} = -[e^{(r)}_{j-1i}, e^{(1)}_j], \quad f^{(r)}_{ji} = -[f^{(1)}_j, f^{(r)}_{j-1i}]. \]
For all $1 \leq i < j \leq n - 1$ in the algebra $\text{DX}(\mathfrak{sp}_{2n})$ we have
\[ e^{(r)}_{ij+1} = [e^{(r)}_{ij}, e^{(1)}_j], \quad f^{(r)}_{j+1i} = [f^{(1)}_j, f^{(r)}_{ji}], \]
\[ e^{(r)}_{ij} = -[e^{(r)}_{j-1i}, e^{(1)}_j], \quad f^{(r)}_{ji} = -[f^{(1)}_j, f^{(r)}_{j-1i}]. \]
Moreover, for $1 \leq i \leq n - 1$ we have
\[ e^{(r)}_{in'} = \frac{1}{2} e^{(r)}_{i'n-1}, e^{(1)}_n], \quad f^{(r)}_{n'i} = \frac{1}{2} [f^{(1)}_n, f^{(r)}_{n'-1}], \]
\[ e^{(r)}_{i'i} = -[e^{(r)}_{i'-1i}, e^{(1)}_i], \quad f^{(r)}_{i'i} = -[f^{(1)}_i, f^{(r)}_{i'-1}]. \]
For all $1 \leq i < j \leq n - 1$ in the algebra $\text{DX}(\mathfrak{o}_{2n})$ we have
\[ e^{(r)}_{ij+1} = [e^{(r)}_{ij}, e^{(1)}_j], \quad f^{(r)}_{j+1i} = [f^{(1)}_j, f^{(r)}_{ji}], \]
\[ e^{(r)}_{ij} = -[e^{(r)}_{j-1i}, e^{(1)}_j], \quad f^{(r)}_{ji} = -[f^{(1)}_j, f^{(r)}_{j-1i}], \]
and for $1 \leq i \leq n - 2$ we have
\[ e^{(r)}_{i'n} = [e^{(r)}_{in-1}, e^{(1)}_n], \quad f^{(r)}_{n'i} = [f^{(1)}_n, f^{(r)}_{n'-1}]. \]

Proof. All relations follow easily from the Gauss decomposition (2.28) and defining relations (2.19)-(2.20). For example, consider the case $\mathfrak{g}_N = \mathfrak{sp}_{2n}, r < 0$. By taking the coefficients of $v^{-1}$ on both sides of (2.20), for $1 < j \leq n - 1$ we get $[t^{(1)}_{ij}(u), t^{(1)}_{j+1} = t^{(1)}_{j+1}(u)$. Writing this in terms of the Gaussian generators we come to the relation $k^{(1)}_1(u)[e^{(1)}_{1j}(u), e^{(1)}_j] = k^{(1)}_1(u)e^{(1)}_{j+1}(u)$, which gives $[e^{(1)}_{1j}(u), e^{(1)}_j] = e^{(1)}_{j+1}(u)$. So $e^{(r)}_{1j+1} = [e^{(r)}_{1j}, e^{(1)}_j]$ for $r < 0$.

By a similar argument, for $1 \leq j \leq n - 1$ we find that $e^{(r)}_{1j'} = [e^{(r)}_{1j'}, e^{(1)}_{j'+1}]$ if $r < 0$. Now apply Proposition 6.16 to write this as $e^{(r)}_{ij'} = -[e^{(r)}_{ij'}, e^{(1)}_j]$. The remaining cases with $i = 1$ are treated in the same way. The extension to arbitrary values of $i$ follows by the application of Corollary 5.2. \qed

By Lemma 6.23, all elements $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ with $r \in \mathbb{Z}, r \neq 0$ and the conditions $i < j$ and $i < j'$ in the orthogonal case, and $i < j$ and $i \leq j'$ in the symplectic case, belong to the subalgebra $\text{DX}(\mathfrak{gl}_N)$ of $\text{DX}(\mathfrak{g}_N)$ generated by the coefficients of the series $k_i^\pm(u)$ with $i = 1, \ldots, n+1$, and $e_i^\pm(u), f_i^\pm(u)$ with $i = 1, \ldots, n$ and $c$. Hence, the Gauss decomposition (2.28) implies that all coefficients of the series $t^{\pm}_{ij}(u)$ with the same respective conditions on the indices $i$ and $j$ also belong to the subalgebra $\text{DX}(\mathfrak{gl}_N)$. Furthermore, by Theorem 2.4, all coefficients of the series $z_i^\pm(u)$ are also in $\text{DX}(\mathfrak{gl}_N)$. Finally, taking the coefficients of $u^r$ for $r = -1, -2, \ldots$ or $r = 0, 1, \ldots$ in (2.23) and using induction on $r$, we conclude that
the coefficients of all series $t_{ij}^r(u)$ belong to $\widetilde{DX}(\mathfrak{gl}_N)$ so that $\widetilde{DX}(\mathfrak{gl}_N) = DX(\mathfrak{gl}_N)$. This proves that the homomorphism (6.133) is surjective.

Next we show that this homomorphism is injective. First, observe that the set of monomials in the generators $k_{ij}^{(r)}$, $k_{ji}^{(-r)}$ with $i = 1, \ldots, n + 1$ and $r \geq 1$, and $e_{ij}^{(r)}$, $e_{ij}^{(-r)}$ and $f_{ji}^{(r)}$, $f_{ji}^{(-r)}$ with $r \geq 1$ and the conditions $i < j$ and $i < j'$ in the orthogonal case, and $i < j$ and $i \leq j'$ in the symplectic case, and $c$, taken in a fixed order and is linearly independent in the extended Yangian double $DX(\mathfrak{g}_N)$. Indeed, under the isomorphism (4.21), the images of the elements $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ in the $(r - 1)$-th component of the graded algebra $gr DX(\mathfrak{g}_N)$ respectively correspond to $F_{ij}x^{r-1} + F_{ji}x^{r-1}$, the images of the elements $e_{ij}^{(-r)}$ and $f_{ji}^{(-r)}$ in the $(-r)$-th component of the graded algebra $gr DX(\mathfrak{g}_N)$ respectively correspond to $-F_{ij}x^{-r}$ and $-F_{ji}x^{-r}$. Similarly, the image of $\bar{h}_{ij}^{(r)}$ correspond to $F_{ii}x^{r-1} + \zeta_r/2$ for $i = 1, \ldots, n$, the image of $\bar{h}_{ij}^{(-r)}$ correspond to $-F_{ii}x^{-r} + \zeta_r/2$ for $i = 1, \ldots, n$, while for the image of $\bar{h}_{n+1,n+1}^{(r)}, \bar{h}_{n+1,n+1}^{(-r)}$ we have

$$
\bar{h}_{n+1}^{(r)} \mapsto \begin{cases} 
\zeta_r/2 & \text{for } \mathfrak{o}_{2n+1} \\
-F_{nn}x^{r-1} + \zeta_r/2 & \text{for } \mathfrak{sp}_{2n} \\
-F_{n-1,n}x^{r-1} - F_{nn}x^{r-1} + \zeta_r/2 & \text{for } \mathfrak{o}_{2n}, 
\end{cases}
$$

$$
\bar{h}_{n+1}^{(-r)} \mapsto \begin{cases} 
\zeta_r/2 & \text{for } \mathfrak{o}_{2n+1} \\
F_{nn}x^{-r} + \zeta_r/2 & \text{for } \mathfrak{sp}_{2n} \\
F_{n-1,n}x^{-r} - F_{nn}x^{-r} + \zeta_r/2 & \text{for } \mathfrak{o}_{2n}, 
\end{cases}
$$

which follows from (4.22) and Theorem 2.4. Hence the claim is implied by the Poincaré-Birkhoff-Witt theorem for $U(\mathfrak{g}_N[x, x^{-1}] \oplus \mathbb{C}K)$.

Now we assume $r \in \mathbb{Z}, r \neq 0$, define elements $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ of $\widetilde{DX}(\mathfrak{g}_N)$ inductively as follows. For $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ set $e_{i,i+1}^{(r)} = e_i^{(r)}$ and $f_{i,i+1}^{(r)} = f_i^{(r)}$, and

$$
e_{ij}^{(r)} = [e_{ij}^{(1)}, e_{ij}^{(1)}], \quad f_{ji}^{(r)} = [f_{ji}^{(1)}, f_{ji}^{(1)}], \quad e_{ij}^{(1)} = -[e_{ij}^{(1)}, e_{ij}^{(1)}], \quad f_{ji}^{(1)} = -[f_{ji}^{(1)}, f_{ji}^{(1)}], \quad \text{for } 1 \leq i < j \leq n.
$$

For $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ set $e_{i,i+1}^{(r)} = e_i^{(r)}$ and $f_{i,i+1}^{(r)} = f_i^{(r)}$, and

$$
e_{ij}^{(r)} = [e_{ij}^{(1)}, e_{ij}^{(1)}], \quad f_{ji}^{(r)} = [f_{ji}^{(1)}, f_{ji}^{(1)}], \quad e_{ij}^{(1)} = -[e_{ij}^{(1)}, e_{ij}^{(1)}], \quad f_{ji}^{(1)} = -[f_{ji}^{(1)}, f_{ji}^{(1)}], \quad \text{for } 1 \leq i < j \leq n-1.
$$

Furthermore, set $e_{nn}^{(r)} = e_n^{(r)}$ and $f_{nn}^{(r)} = f_n^{(r)}$, and

$$e_{in}^{(r)} = \frac{1}{2} [e_{i,n-1}^{(r)}, e_n^{(1)}], \quad f_{in}^{(r)} = \frac{1}{2} [f_n^{(1)}, f_{n-1}^{(r)}], \quad e_{in}^{(1)} = -[e_{i,n-1}^{(1)}, e_n^{(1)}], \quad f_{in}^{(1)} = -[f_n^{(1)}, f_{n-1}^{(1)}],$$
for $1 \leq i \leq n - 1$. For $\mathfrak{g}_N = \mathfrak{o}_{2n}$ set $e^{(r)}_{i_{i+1}} = e^{(r)}_i$ and $f^{(r)}_{i_{i+1}} = f^{(r)}_i$, and

$$
\begin{align*}
e^{(r)}_{i_{j+1}} &= [e^{(r)}_j, e^{(1)}_j], & f^{(r)}_{j_{i+1}} &= [f^{(r)}_j, f^{(1)}_j], \\
e^{(r)}_{i_{j'}} &= -[e^{(r)}_{j'-1}, e^{(1)}_j], & f^{(r)}_{j'_{i}} &= -[f^{(r)}_j, f^{(1)}_{j'-1}],
\end{align*}
$$

for $1 \leq i < j \leq n - 1$. Furthermore, set $e^{(r)}_{n-1n'} = e^{(r)}_n$ and $f^{(r)}_{n'n-1} = f^{(r)}_n$, and

$$
\begin{align*}
e^{(r)}_{i_{n'}} &= [e^{(r)}_{i_{n'-1}}, e^{(1)}_n], & f^{(r)}_{n'i} &= [f^{(r)}_n, f^{(r)}_{n'-1}],
\end{align*}
$$

for $1 \leq i \leq n - 2$.

By Lemma 6.23, these definitions are consistent with those of the elements of the algebra $DX(\mathfrak{g}_N)$ in the sense that the images of the elements $e^{(r)}_{ij}$ and $f^{(r)}_{ij}$ of the algebra $\widehat{DX}(\mathfrak{g}_N)$ under the homomorphism (6.133) coincide with the elements of $DX(\mathfrak{g}_N)$ with the same name.

The injectivity property of the homomorphism (6.133) will follow if we prove that the algebra $\widehat{DX}(\mathfrak{g}_N)$ is spanned by monomials in $k^{(r)}_i$, $e^{(r)}_{ij}$, $f^{(r)}_{ji}$ and $c$ taken in some fixed order. Denote by $\widehat{E}$, $\widehat{F}$ and $\widehat{H}$ the subalgebras of $\widehat{DX}(\mathfrak{g}_N)$ respectively generated by all elements of the form $e^{(r)}_i$, $f^{(r)}_i$ and $k^{(r)}_i$. Denote by $\widehat{E}^+$, $\widehat{F}^+$ and $\widehat{H}^+$ the subalgebras of $\widehat{DX}(\mathfrak{g}_N)$ respectively generated by all elements of the form $e^{(r)}_i$, $f^{(r)}_i$ and $k^{(r)}_i$ with $r > 0$. Denote by $\widehat{E}^-$, $\widehat{F}^-$ and $\widehat{H}^-$ the subalgebras of $\widehat{DX}(\mathfrak{g}_N)$ respectively generated by all elements of the form $e^{(r)}_i$, $f^{(r)}_i$ and $k^{(r)}_i$ with $r < 0$. Define a descending filtration on $\widehat{E}^-$ by setting $\deg e^{(-r)}_i = -r$. Denote by $\text{gr} \widehat{E}^-$ the corresponding graded algebra. Let $\widehat{e}^{(-r)}_{ij}$ be the image of $e^{(-r)}_{ij}$ in the $(-r)$-th component of the graded algebra $\text{gr} \widehat{E}^-$. Extend the range of subscripts of $\widehat{e}^{(-r)}_{ij}$ to all values $1 \leq i < j \leq 1'$ by using the skew-symmetry conditions

$$
\begin{align*}
\widehat{e}^{(-r)}_{ij} &= -\theta_{ij} \widehat{e}^{(-r')}_{j'i'}.
\end{align*}
$$

First we note that the algebra $\widehat{E}^+$ is spanned by the set of monomials in the elements $e^{(r)}_{ij}$ taken in some fixed order; see [21, Sec. 5.5]. Similarly, the desired spanning property of the monomials in the $e^{(-r)}_{ij}$ clearly follows from the relations

$$
\begin{align*}
[e^{(-r)}_{ij}, e^{(-s)}_{kl}] &= \delta_{kj} \widehat{e}^{(-r-s)}_{il} - \delta_{il} \widehat{e}^{(-r-s)}_{kj} - \theta_{ij} \delta_{k'v} \widehat{e}^{(-r-s)}_{j'i'l} + \theta_{ij} \delta_{j'i'l} \widehat{e}^{(-r-s)}_{k'v'},
\end{align*}
$$

We can use the same method in [21, Sec. 5.5] to prove (6.140). In addition, we can swap $e^{(r)}_{ij}$ and $e^{(-r)}_{ij}$ by defining relations in Theorem 6.22. Therefore, the algebra $\widehat{E}$ is spanned by the ordered monomials in the elements $e^{(r)}_{ij}$ and $e^{(-r)}_{ij}$. The same is true for $\widehat{F}$. Note also that the ordered monomials in $k^{(r)}_i$ span $\widehat{H}$. Furthermore, by the defining relations of $\widehat{DX}(\mathfrak{g}_N)$, the multiplication map

$$
\widehat{F} \otimes \widehat{H} \otimes \widehat{E} \otimes \mathbb{C} \rightarrow \widehat{DX}(\mathfrak{g}_N)
$$

is surjective. Thus, ordering the elements $k^{(r)}_i$, $e^{(r)}_{ij}$, $f^{(r)}_{ji}$ and $c$ in such a way that the elements of $\widehat{F}$ precede the elements of $\widehat{H}$, and the latter precede the elements of $\widehat{E}$, we can
conclude that the ordered monomials in these elements span \( \hat{D}X(\mathfrak{g}_N) \). This proves that (6.133) is an isomorphism.

7. Isomorphism theorem for the Drinfeld Yangian double

We will now prove the Main Theorem as stated in the Introduction. It is clear from the definition of the series \( H_i^\pm(u), E_i(u), F_i(u) \) that all the coefficients \( h_{ir}, \xi_{ir}^+, \xi_{ir}^- \) \((i = 1, \ldots, n, \ r \in \mathbb{Z})\) of \( H_i^\pm(u), E_i(u), F_i(u) \) belong to the subalgebra \( DY(\mathfrak{g}_N) \).

**Proposition 7.1.** The subalgebra \( DY(\mathfrak{g}_N) \) of \( DX(\mathfrak{g}_N) \) is generated by the coefficients \( h_{ir}, \xi_{ir}^+, \xi_{ir}^-, \ c \) with \( i = 1, \ldots, n \) and \( r \in \mathbb{Z} \).

**Proof.** Due to the tensor decomposition (2.25) of \( DX(\mathfrak{g}_N) \), it suffices to check that these elements together with the coefficients \( z_N(\gamma) \) of the series \( z_N^\pm(u) \) generate the algebra \( DX(\mathfrak{g}_N) \). Using the definition of the series \( H_i^\pm(u) \) it is straightforward to express \( k_i^\pm(u)k_{n+1}^\pm(u)^{-1} \) as a product of the series of the form \( H_i^\pm(u)^{-1} \) with some shifts of \( u \) by constants. On the other hand, Theorem 2.4 implies that \( k_i^\pm(u)^n \) equals \( z_N^\pm(u) \) times the same kind of product of the shifted series \( H_i^\pm(u)^{-1} \). Therefore, all coefficients of \( k_i^\pm(u) \) and hence all coefficients of the series \( k_i^\pm(u) \) with \( i = 1, \ldots, n + 1 \) belong to the subalgebra of \( DX(\mathfrak{g}_N) \) generated by the elements given in the proposition. Furthermore, for each \( i \), the elements \( e_i^{(r)} \) and \( f_i^{(r)} \) are found as linear combinations of the \( \xi_{is}^+ \) and \( \xi_{is}^- \), respectively. By Theorem 6.22, the coefficients of the series \( k_i^\pm(u) \) with \( i = 1, \ldots, n + 1 \), and \( c \) generate the algebra \( DX(\mathfrak{g}_N) \) thus completing the proof. \( \square \)

Now we will verify that the generators \( h_{ir}, \xi_{ir}^+, \xi_{ir}^- \) of the subalgebra \( DY(\mathfrak{g}_N) \) provided by Proposition 7.1 satisfy the defining relations of the Drinfeld Yangian double \( DYN(\mathfrak{g}_N) \) as given in the Introduction.

**Theorem 7.2.** The Yangian double \( DYN(\mathfrak{g}_N) \) is generated by the coefficients of the series \( H_i^\pm(u) \) with \( i = 1, \ldots, n \), and \( E_i(u) \) and \( F_i(u) \) with \( i = 1, \ldots, n \) and \( c \) subject only to the following sets of relations, where the indices take all admissible values unless specified otherwise.

\[
\begin{align*}
(7.1) \quad [H_i^\pm(u), H_j^\pm(v)] &= 0, \\
(7.2) \quad (u_\pm - v_\mp + B_{ij})(u_\mp - v_\pm + B_{ij})H_i^\pm(u)H_j^\pm(v) &= (u_\mp - v_\pm + B_{ij})(u_\mp - v_\pm - B_{ij})H_i^\pm(v)H_j^\pm(u), \\
(7.3) \quad H_i^\pm(u)^{-1}E_j(v)H_i^\pm(u) &= \frac{u_\mp - v_\pm + B_{ij}}{u_\mp - v_\pm - B_{ij}}E_j(v), \\
(7.4) \quad H_i^\pm(u)F_j(v)H_i^\pm(u)^{-1} &= \frac{u_\pm - v_\mp + B_{ij}}{u_\pm - v_\mp - B_{ij}}F_j(v), \\
(7.5) \quad (u - v + B_{ij})E_i(u)E_j(v) &= (u - v - B_{ij})E_j(v)E_i(u), \\
(7.6) \quad (u - v - B_{ij})F_i(u)F_j(v) &= (u - v + B_{ij})F_j(v)F_i(u),
\end{align*}
\]
In terms the series

\[
\sum_{\sigma \in \mathfrak{S}_m} [E_i(u_{\sigma(1)}), [E_i(u_{\sigma(2)}), \ldots, [E_i(u_{\sigma(m)}), E_j(v)] \ldots] = 0,
\]

\[
\sum_{\sigma \in \mathfrak{S}_m} [F_i(u_{\sigma(1)}), [F_i(u_{\sigma(2)}), \ldots, [F_i(u_{\sigma(m)}), F_j(v)] \ldots] = 0,
\]

\[i \neq j, m = 1 - a_{ij},\]

\[
[E_i(u), F_j(v)] = \delta_{ij} \{ \delta(u_+ - v_+)H_i^-(v_+) - \delta(u_+ - v_-)H_i^+(v_+) \}.
\]

Here we set \( B_{ij} = \frac{1}{2}(\alpha_i, \alpha_j). \)

**Proof.** The proof amounts to writing the relations of Theorem 6.22 in terms the series \( H_i^\pm(u), E_i(u) \) and \( F_i(u). \) The relations are immediate from the definition of the series \( H_i^\pm(u), E_i(u) \) and \( F_i(u). \) In the following we will demonstrate several relations for various types to show the idea.

Type \( B_n. \) To verify the \( i = j = n \) case of (7.3), we get \( H_n^\pm(u) = k_n^\pm(u-(n-1)/2)^{-1} k_{n+1}^\pm(u-(n-1)/2) \) and \( E_n(u) = X_n^+(u-(n-1)/2) \) from definition. By Theorem 6.22, we have

\[
k_n^\pm(u)^{-1} X_n^+(v) k_n^\pm(u) = \frac{u_+ - v - 1}{u_+ - v} X_n^+(v),
\]

\[
k_{n+1}^\pm(u)^{-1} X_n^+(v) k_{n+1}^\pm(u) = \frac{(u_+ - v - 1)(u_+ - v + \frac{1}{2})}{(u_+ - v)(u_+ - v - \frac{1}{2})} X_n^+(v).
\]

It follows that \( H_n^\pm(u)^{-1} E_n(v) H_n^\pm(u) = \frac{u_+ - v + B_{nn}}{u_+ - v - B_{nn}} E_n(v), \) where \( B_{nn} = \frac{1}{2}. \) To verify the \( i = j = n \) case of (7.9), we get \( E_n(u) = X_n^+(u-(n-1)/2) \) and \( F_n(u) = X_n^-(u-(n-1)/2) \) from definition. By Theorem 6.22, we have

\[
[X_n^+(v), X_n^-(v)] = \{ \delta(u_+ - v_+)k_{n+1}^-(v_+)k_n^-(v_+)^{-1} - \delta(u_+ - v_-)k_{n+1}^+(u_+)k_n^+(u_+)^{-1} \}.
\]

Since \( H_n^\pm(u) = k_n^\pm(u-(n-1)/2)^{-1} k_{n+1}^\pm(u-(n-1)/2), \) This yields \( [E_n(u), F_n(v)] = \{ \delta(u_+ - v_+)H_n^-(v_+) - \delta(u_+ - v_-)H_n^+(u_+) \}. \) By Proposition 6.4, we have

\[
\frac{u - v - \frac{1}{2} - \frac{\epsilon}{4}}{u - v - \frac{\epsilon}{4}} K_1^+(2u + 1) K_2^-(2v + 1) K_1^-(2v + 1)^{-1}
\]

\[
= \frac{u - v - \frac{1}{2} + \frac{\epsilon}{4}}{u - v + \frac{\epsilon}{4}} K_2^-(2v + 1) K_1^-(2v + 1)^{-1} K_1^+(2u + 1),
\]

\[
\frac{u - v - 1 - \frac{\epsilon}{4}}{u - v - 1 - \frac{\epsilon}{4}} K_1^+(2u) K_2^-(2v + 1) K_1^-(2v + 1)^{-1}
\]

\[
= \frac{u - v - 1 + \frac{\epsilon}{4}}{u - v - 1 + \frac{\epsilon}{4}} K_2^-(2v + 1) K_1^-(2v + 1)^{-1} K_1^+(2u).
\]

in \( \text{DY}(\mathfrak{gl}_2). \) Apply Lemma 6.2, we get

\[
\frac{u_+ - v_+ - 1}{u_+ - v_+} k_n^+(u) k_{n+1}^+(v) k_n^-(v)^{-1} = \frac{u_+ - v_+ - 1}{u_+ - v_-} k_{n+1}^-(v) k_n^+(v)^{-1} k_n^+(u).
\]
Corollary 4.9 implies \( k_{n-1}^+(u)^{-1}k_{n+1}^-(v) = k_{n+1}^-(v)k_{n-1}^+(u)^{-1} \). Since \( k_{n+1}^+(u)^{-1}k_{n}^-(v)^{-1} = k_{n}^-(v)^{-1}k_{n+1}^+(u)^{-1} \), then we have \( k_{n-1}^+(u)^{-1}k_{n+1}^-(v) = H_{n-1}^+(v)H_n^-(v) \). It follows that
\[
(u_--v_++\frac{1}{2})(u_+-v_-+\frac{1}{2})H_{n-1}^+(u)H_n^-(v) = (u_--v_++\frac{1}{2})(u_+-v_-+\frac{1}{2})H_n^+(v)H_{n-1}^+(u),
\]
which is just (7.2) when \( i = n-1, j = n \). All other remaining cases follow by similar calculations.

Type \( C_n \). To verify the \( i = j = n \) case of (7.3), we get \( H_n^+(u) = 2k_{n+1}^+(u-n/2)^{-1}k_{n+1}^+(u-n/2) \) and \( E_n(u) = X_n^+(u-n/2) \) from definition. By Theorem 6.22, we have
\[
k_n^+(u)^{-1}X_n^+(v)k_n^+(u) = \frac{u_--v_+ - 2}{u_--v_+}X_n^+(v),
k_{n+1}^+(u)^{-1}X_n^+(v)k_{n+1}^+(u) = \frac{u_--v_+ + 2}{u_--v_+}X_n^+(v).
\]
It follows that \( H_n^+(u)^{-1}E_n(v)H_n^+(u) = \frac{u_--v_+ + B_{nn}}{u_--v_+ - B_{nn}}E_n(v), \) where \( B_{nn} = 2 \). To verify the \( i = j = n \) case of (7.9), we get \( E_n(u) = X_n^+(u-n/2) \) and \( F_n(u) = X_n^+(u-n/2) \) from definition. By Theorem 6.22, we have
\[
[X_n^+(u), X_n^+(v)] = 2\{\delta(u_--v_+)k_{n+1}^+(v)k_n^+(v)^{-1} - \delta(u_+ - v_-)k_{n+1}^+(u_+)k_n^+(u_-)^{-1}\}.
\]
This yields \( [E_n(u), F_n(v)] = \{\delta(u_--v_+)H_n^+(v) - \delta(u_+ - v_-)H_n^+(u_-)\} \). Apply Lemma 6.1 and Proposition 6.4, we have
\[
\frac{u_--v_+ - 1}{u_--v_+ + 1}k_n^+(u-n/2)H_n^-(v) = \frac{u_+-v_- - 1}{u_+-v_- + 1}H_n^-(v)k_n^+(u-n/2).
\]
Corollary 4.9 implies \( k_{n-1}^+(u)^{-1}k_{n+1}^+(v) = k_{n+1}^+(v)k_{n-1}^+(u)^{-1} \). Since \( k_{n+1}^+(u)^{-1}k_{n}^-(v)^{-1} = k_{n}^-(v)^{-1}k_{n+1}^+(u)^{-1} \), then we have \( k_{n-1}^+(u)^{-1}H_n^-(v) = H_n^-(v)k_{n-1}^+(u)^{-1} \). It follows that
\[
(u_--v_+ - 1)(u_+-v_- + 1)H_{n-1}^+(u)H_n^-(v) = (u_--v_+ + 1)(u_+-v_- - 1)H_n^+(v)H_{n-1}^+(u),
\]
which is just (7.2) when \( i = n-1, j = n \). All other remaining cases follow by similar calculations.

Type \( D_n \). To verify the \( i = n-1, j = n \) case of (7.3), we get \( H_n^+(u) = k_{n+1}^+(u-(n-2)/2)^{-1}k_{n+1}^+(u-(n-2)/2) \) and \( E_n(u) = X_n^+(u-(n-2)/2) \) from definition. By Theorem 6.22, we have
\[
k_n^+(u)^{-1}X_n^+(v)k_n^+(u) = \frac{u_--v_+ - 1}{u_--v_+}X_n^+(v),
k_{n+1}^+(u)^{-1}X_n^+(v)k_{n+1}^+(u) = \frac{u_--v_+ + 1}{u_--v_+}X_n^+(v).
\]
It follows that \( H_n^+(u)^{-1}E_n(v)H_n^+(u) = E_n(v), \) which is just (7.3) when \( i = n-1, j = n \). To verify the \( i = j = n \) case of (7.9), we get \( E_n(u) = X_n^+(u-(n-2)/2) \) and \( X_n^+(u-(n-2)/2) \) from definition. By Theorem 6.22, we have
\[
[X_n^+(u), X_n^+(v)] = 2\{\delta(u_--v_+)k_{n+1}^+(v)k_{n-1}^+(v)^{-1} - \delta(u_+ - v_-)k_{n+1}^+(u_+)k_{n-1}^+(u_-)^{-1}\}.
\]
Since $H_{n+1}^+(u) = k_{n+1}^+(u - (n - 2)/2)^{-1} k_{n+1}^+(u - (n - 2)/2)$, this yields $[E_n(u), F_n(v)] = \{\delta(u_+ - v_+)H_n^-(v_+) - \delta(u_+ - v_-)H_n^+(u_+)\}$. Apply Lemma 6.3 and Proposition 6.4, we have

$$\frac{u_+ - v_+ - 1}{u_+ - v_+} k_{n+1}^+(u) k_{n+1}(v) k_{n+1}(v)^{-1} = \frac{u_+ - v_+ - 1}{u_+ - v_+} k_{n+1}(v) k_{n+1}(v)^{-1} k_{n+1}^+(u),$$

and

$$\frac{u_+ - v_+ - 1}{u_+ - v_+} k_{n+1}^+(u) k_{n+1}(v) k_{n+1}(v)^{-1} = \frac{u_+ - v_+ - 1}{u_+ - v_+} k_{n+1}(v) k_{n+1}(v)^{-1} k_{n+1}^+(u).$$

Since $H_{n-1}^+(u) = k_{n-1}^+(u - (n - 2)/2)^{-1} k_n^+(u - (n - 2)/2)$, then we get $H_{n-1}^+(u) H_n^-(v) = H_n^-(v) H_{n-1}^+(u)$, which is just (7.2) when $i = n - 1, j = n$. All other remaining cases follow by similar calculations.

Propositions 7.1 imply that the mapping $DY^D(g_N) \to DY(g_N)$ considered in the Main Theorem is a surjective homomorphism. Its injectivity follows from the decomposition

$$DY(g_N) = \mathcal{E} \otimes (DY(g_N) \cap \mathcal{H}) \otimes \mathcal{F} \otimes \mathbb{C}$$

and the corresponding arguments of the proof of Theorem 6.22. This completes the proof of the Main Theorem. □

8. Bosonization of level 1 modules

Here we construct level 1 $DY(g)$-module for $g = \mathfrak{o}_{2l+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ in terms of bosons. Introduce bosons $\{a_{i,k} \mid 1 \leq i \leq n + 4, k \in \mathbb{Z} \setminus \{0\}\}$ satisfying:

$$[a_{i,k}, a_{j,l}] = k \delta_{ij} \delta_{k+l,0}.$$

Define the standard bilinear form $(\epsilon_i, \epsilon_j) = \delta_{ij}$. For $g = \mathfrak{o}_{2l+1}$, We introduce the Fock space $\mathcal{F} = \mathbb{C}[a_{j,k}(1 \leq i \leq n + 3, k \in \mathbb{Z} \setminus \{0\})] \otimes \mathbb{C}[Q]$, where $Q = \bigoplus_{k=1}^{n+4} \mathbb{Z} \epsilon_i$, $\mathbb{C}[Q]$ is the group algebra of $Q$ over $\mathbb{C}$. On this space, we define the action of the operators $a_{i,k}, \partial \epsilon_j, \epsilon^{i,j}(1 \leq i \leq n + 3, 1 \leq j \leq n + 4)$ by

$$(8.1) \quad a_{i,k} \cdot f \otimes \epsilon^\beta = \begin{cases} a_{i,k} f \otimes \epsilon^\beta & \text{if } k < 0 \\ [a_{i,k}] f \otimes \epsilon^\beta & \text{if } k > 0 \end{cases}$$

$$\partial \epsilon_j \cdot f \otimes \epsilon^\beta = (\epsilon_j, \beta) f \otimes \epsilon^\beta,$$

$$\epsilon^{i,j} \cdot f \otimes \epsilon^\beta = f \otimes \epsilon^{i,j+\beta},$$

for $f \otimes \epsilon^\beta \in \mathcal{F}$.

**Theorem 8.1.** The following assignment defines a $DY(\mathfrak{o}_{2l+1})$-module structure on $\mathcal{F}$.

For $1 \leq j \leq n - 2$,

$$E_j(u) \mapsto e^{\exp} \sum_{k>0} \frac{a_{j,-k}}{k} \{ (u - \frac{1}{4})^k + (u + \frac{3}{4})^k \} - \sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} (u + \frac{1}{4})^k$$

$$e^{\exp} \{ \sum_{k>0} \frac{a_{j,k}}{k} (u - \frac{1}{4})^{-k} \} e^{\alpha_j} [(-1)^{j-1} (u - \frac{1}{4})]^{\partial \alpha_j},$$
\[ F_j(u) \mapsto \exp[-\sum_{k>0} \frac{a_{j-k}}{k}(u + \frac{1}{4})^k + (u - \frac{3}{4})^k] + \sum_{k>0} \frac{a_{j+1-k} + a_{j-1-k}}{k}(u - \frac{1}{4})^k \]

\[ \exp[\sum_{k>0} \frac{a_{j-k}}{k}(u + \frac{1}{4})^{-k}]e^{-\alpha j}(-1)^{j-1}(u + \frac{1}{4})^{-\beta_{\alpha j}}, \]

\[ H_j^+(u) \mapsto \exp[-\sum_{k>0} \frac{a_{j-k}}{k}(u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k}] \frac{u + \frac{1}{2}}{u - \frac{1}{2}}^{-\beta_{\alpha j}}, \]

\[ H_j^-(u) \mapsto \exp[-\sum_{k>0} \frac{a_{j-k}}{k}(u - 1)^k - (u + 1)^k] + \sum_{k>0} \frac{a_{j+1-k} + a_{j-1-k}}{k}(u - \frac{1}{2})^k - (u + \frac{1}{2})^k]. \]

\[ E_{n-1}(u) \mapsto \exp[\sum_{k>0} \frac{a_{n-1-k}}{k}(u - \frac{1}{4})^k + (u + \frac{3}{4})^k] - \sum_{k>0} \frac{a_{n-2-k} + a_{n-k}}{k}(u + \frac{1}{4})^k \]

\[ \exp[-\sum_{k>0} \frac{a_{n-1-k}}{k}(u - \frac{1}{4})^{-k}]e^{\alpha_{n-1}}(-1)^{n-2}(u - \frac{1}{4})^{-\beta_{\alpha_{n-1}+\beta_1}}, \]

\[ F_{n-1}(u) \mapsto \exp[-\sum_{k>0} \frac{a_{n-1-k}}{k}(u + \frac{1}{4})^k + (u - \frac{3}{4})^k] + \sum_{k>0} \frac{a_{n-2-k} + a_{n-k}}{k}(u - \frac{1}{4})^k \]

\[ \exp[\sum_{k>0} \frac{a_{n-1-k}}{k}(u + \frac{1}{4})^{-k}]e^{-\alpha_{n-1}}(-1)^{n-2}(u + \frac{1}{4})^{-\beta_{\alpha_{n-1}-\beta_1}}, \]

\[ E_n(u) \mapsto \exp[-\sum_{k>0} \frac{a_{n-k}}{k}(u + \frac{1}{4})^k + \sum_{k>0} \frac{a_{n+1-k}}{k}(u - \frac{1}{4})^k] \]

\[ + \sum_{k>0} \frac{a_{n+2-k}}{k}(u - \frac{3}{4})^k + (u + \frac{3}{4})^k] \exp[-\sum_{k>0} \frac{a_{n,k} + a_{n+1,k}}{k}(u - \frac{1}{4})^{-k} \]

\[ - \sum_{k>0} \frac{a_{n+2,k}}{k}(u + \frac{1}{4})^{-k}]e^{-\beta_3}(-1)^{n-1}(u - \frac{1}{4})^{-\beta_{\beta_2+\beta_3}}(u - \frac{1}{4})^{-\beta_{\beta_2}}(u + \frac{1}{4})^{-\beta_{\beta_3}}, \]

\[ F_n(u) \mapsto \exp[\sum_{k>0} \frac{a_{n-k}}{k}(u - \frac{1}{4})^k - \sum_{k>0} \frac{a_{n+1-k}}{k}(u + \frac{1}{4})^k + (u - \frac{3}{4})^k] \]

\[ - \sum_{k>0} \frac{a_{n+3-k}}{k}(u - \frac{3}{4})^k + (u + \frac{3}{4})^k] \exp[\sum_{k>0} \frac{a_{n,k} + a_{n+1,k}}{k}(u + \frac{1}{4})^{-k} \]

\[ + \sum_{k>0} \frac{a_{n+3,k}}{k}(u - \frac{1}{4})^{-k}]e^{\beta_3}(-1)^{n-1}(u + \frac{1}{4})^{-\beta_{\beta_2+\beta_3}}(u + \frac{1}{4})^{-\beta_{\beta_2}}(u - \frac{1}{4})^{-\beta_{\beta_3}}. \]
we obtain the following operator product expansion (OPE):

\[ H_{n-1}^+(u) \mapsto \exp\left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} ((u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k}) (u + \frac{1}{2}) - \partial a_{n-1} - \partial \beta_1, \right] \]

\[ H_{n-1}^-(u) \mapsto \exp\left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} ((u - 1)^k - (u + 1)^k) \right. \]

\[ + \left. \sum_{k>0} \frac{a_{n-2,k} + a_{n-1,k}}{k} ((u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k}) \right] \]

\[ \quad + \left. \sum_{k>0} \frac{a_{n-2,k} + a_{n-1,k}}{k} ((u - 1)^k - (u + 1)^k) \right] \]

\[ H_{n}^+(u) \mapsto \exp\left[ \sum_{k>0} \frac{a_{n+2,k}}{k} ((u - 1)^k + (u + 1)^k) - \sum_{k>0} \frac{a_{n+3,k}}{k} ((u + 1)^k + (u - \frac{1}{2})^k) \right] \]

\[ \exp\left[ \sum_{k>0} \frac{a_{n+1,k} + a_{n+1,k}}{k} ((u + \frac{1}{2})^{-k} - (u - \frac{1}{2})^{-k}) - \sum_{k>0} \frac{a_{n+3,k} - a_{n+2,k}}{k} (u - k) \right] \]

\[ \left( \frac{u + \frac{1}{2}}{u - \frac{1}{2}} \right)^{-\partial a_n - \partial \beta_2} e_{\beta_3 + \beta_4} u^{\partial \beta_3 + \partial \beta_4}, \]

\[ H_{n}^-(u) \mapsto \exp\left[ \sum_{k>0} \frac{a_{n-1,k}}{k} ((u - 1)^k - (u + \frac{1}{2})^k) - \sum_{k>0} \frac{a_{n+1,k}}{k} ((u - 1)^k - (u + 1)^k) \right] \]

\[ + \sum_{k>0} \frac{a_{n+2,k}}{k} ((u - 1)^k + (u + 1)^k) - \sum_{k>0} \frac{a_{n+3,k}}{k} ((u + 1)^k + (u - 1)^k) \]

\[ \exp\left[ - \sum_{k>0} \frac{a_{n+2,k}}{k} (u + \frac{1}{2})^{-k} + \sum_{k>0} \frac{a_{n+3,k}}{k} (u - \frac{1}{2})^{-k} \right] e_{\beta_3 + \beta_4} (u + \frac{1}{2})^{\partial \beta_3} (u - \frac{1}{2})^{\partial \beta_4}. \]

Here \( \alpha_j = \epsilon_j - \epsilon_{j+1} \) for \( j = 1, \cdots, n-1, \alpha_n = \epsilon_n, \beta_1 = \epsilon_{n+1}, \beta_i = \sqrt{2} \epsilon_{n+i} (i = 2, 3, 4). \)

**Proof.** We define the normal ordering \( : \cdot : \) of the fields by regarding \( a_{j,k} (k < 0), e^{\epsilon_j} \) as creation operators and \( a_{j,k} (k > 0), \partial e_j \) as annihilation operators. After some calculation, we obtain the following operator product expansion (OPE):

\[ (8.2) \quad H_{n}^+(u) E_n(v) =: H_{n}^+(u) E_n(v) : (u - v - \frac{5}{4})(u - v - \frac{3}{4}), \]

\[ (8.3) \quad E_n(v) H_{n}^+(u) =: E_n(v) H_{n}^+(u) : (u - v - \frac{5}{4})(u - v - \frac{1}{4}). \]

This implies \( H_{n}^+(u) E_n(v) H_{n}^+(u) = \frac{u - v + \frac{1}{4}}{u - v - \frac{3}{4}} E_n(v) \). The other relations can also be checked similarly. \( \square \)

For \( \mathfrak{g} = \mathfrak{sp}_n \), We introduce the Fock space \( \mathcal{F} = \mathbb{C}[a_{j,-k} (1 \leq i \leq n+4, k \in \mathbb{Z} \setminus \{0\})] \otimes \mathbb{C}[Q], \) where \( Q = \bigoplus_{i=1}^{n+5} \mathbb{Z} \epsilon_i, \mathbb{C}[Q] \) is the group algebra of \( Q \) over \( \mathbb{C} \). On this space, we define the action of the operators \( a_{i,k}, \partial \epsilon_j, e^{\epsilon_j} (1 \leq i \leq n+4, 1 \leq j \leq n + 5) \) by

\[ (8.4) \quad a_{i,k} \cdot f \otimes e^\beta = \begin{cases} a_{i,k} f \otimes e^\beta & \text{if } k < 0 \\ [a_{i,k}, f] \otimes e^\beta & \text{if } k > 0 \end{cases} \]
\[ \partial e_j \cdot f \otimes e^\beta = (e_j, \beta) f \otimes e^\beta, \]
\[ e^\epsilon_j \cdot f \otimes e^\beta = f \otimes e^{\epsilon_j + \beta}, \]
for \( f \otimes e^\beta \in \mathcal{F}. \)

**Theorem 8.2.** The following assignment defines a \( \text{DY}(\mathfrak{sp}_{2n}) \)-module structure on \( \mathcal{F}. \)
For \( 1 \leq j \leq n-2, \)
\[
E_j(u) \mapsto \exp\left[\sum_{k>0} \frac{a_{j,-k}}{k} \{(u - \frac{1}{4})^k + (u + \frac{3}{4})^k\} - \sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} (u + \frac{1}{4})^k\right]
\]

\[
e^{\alpha_j} \left[(-1)^{j-1}(u - \frac{1}{4})\right] \partial \alpha_j,
\]
\[
F_j(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,k}}{k} \{(u + \frac{1}{4})^k + (u - \frac{3}{4})^k\} + \sum_{k>0} \frac{a_{j+1,k} + a_{j-1,k}}{k} (u - \frac{1}{4})^k\right]
\]

\[
e^{-\alpha_j} \left[(-1)^{j-1}(u + \frac{1}{4})\right] \partial \alpha_j,
\]
\[
H_+^j(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,k}}{k} \{(u - \frac{1}{2})^k - (u + \frac{1}{2})^k\} \right]\frac{u + \frac{1}{2}}{u - \frac{1}{2}} \partial \alpha_j,
\]
\[
H_-(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,k}}{k} \{(u - 1)^k - (u + 1)^k\} \right]\frac{u + \frac{1}{2}}{u - \frac{1}{2}} \partial \alpha_j,
\]
\[
E_{n-1}(u) \mapsto \exp\left[\sum_{k>0} \frac{a_{n-1,-k}}{k} \{(u - \frac{1}{4})^k + (u + \frac{3}{4})^k\} - \sum_{k>0} \frac{a_{n-2,-k}}{k} (u + \frac{1}{4})^k\right]
\]

\[
e^{\alpha_{n-1}} \left[(-1)^{n-2}(u - \frac{1}{4})\right] \partial \alpha_{n-1},
\]
\[
F_{n-1}(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{n-1,k}}{k} \{(u + \frac{1}{4})^k + (u - \frac{3}{4})^k\} + \sum_{k>0} \frac{a_{n-2,k}}{k} (u - \frac{1}{4})^k\right]
\]

\[
e^{-\alpha_{n-1}} \left[(-1)^{n-2}(u + \frac{1}{4})\right] \partial \alpha_{n-1},
\]
\[
E_{n-1}(u) \mapsto \exp\left[\sum_{k>0} \frac{a_{n-1,k}}{k} \{(u + \frac{1}{4})^k + (u - \frac{3}{4})^k\} - \sum_{k>0} \frac{a_{n-2,k}}{k} (u - \frac{1}{4})^k\right]
\]

\[
e^{\alpha_{n-1}} \left[(-1)^{n-2}(u + \frac{1}{4})\right] \partial \alpha_{n-1},
\]
\[
F_{n-1}(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{n-1,k}}{k} \{(u - \frac{1}{4})^k - (u + \frac{1}{4})^k\} - \sum_{k>0} \frac{a_{n-2,k}}{k} (u + \frac{1}{4})^k\right]
\]

\[
e^{-\alpha_{n-1}} \left[(-1)^{n-2}(u - \frac{1}{4})\right] \partial \alpha_{n-1},
\]
\begin{align*}
E_n(u) & \rightarrow \exp\left[ \sum_{k>0} \frac{a_{n+1-k}}{k} (u + \frac{1}{2})^k \right] \exp\left[ \sum_{k>0} \frac{a_{n,k}}{k} (u - \frac{1}{2})^{-k} \right] e^{-\beta_2} \left[ (-1)^{n-1} (u - \frac{1}{2}) \right]^{-\beta_1} \\
F_n(u) & \rightarrow \exp\left[ \sum_{k>0} \frac{a_{n+1-k}}{k} (u - \frac{1}{2})^k \right] \exp\left[ \sum_{k>0} \frac{a_{n,k}}{k} (u + \frac{1}{2})^{-k} \right] e^{-\beta_2} \left[ (-1)^{n-1} (u + \frac{1}{2}) \right]^{-\beta_1} \\
H_{n-1}^+(u) & \rightarrow \exp\left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} \left( (u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k} \right) \right] \left( \frac{u + \frac{1}{2}}{u - \frac{3}{4}} \right)^{-\beta_2} \\
H_{n-1}^-(u) & \rightarrow \exp\left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} \left( (u - 1)^{k} - (u + 1)^{k} \right) \right] + \sum_{k>0} \frac{a_{n-2,k}}{k} \left( (u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k} \right) \\
& \exp\left[ \sum_{k>0} \frac{a_{n-1,k}}{k} \left( (u + \frac{1}{4})^{k} + (u - \frac{3}{4})^{k} \right) \right] \exp\left[ \sum_{k>0} \frac{a_{n+1,k}}{k} \left( (u - \frac{1}{4})^{-k} + (u + \frac{1}{4})^{-k} \right) \right] \\
& \left( \frac{u + \frac{1}{4}}{u - \frac{3}{4}} \right)^{-\beta_2},
\end{align*}
\[ H^+(u) \mapsto \exp \left[ \sum_{k>0} \frac{a_{n+1,-k}}{k} \{(u + \frac{1}{4})^k + (u - \frac{1}{4})^k\} \right] \]\[ \times \exp \left[ \sum_{k>0} \frac{a_{n,k}}{k} \{(u - \frac{3}{4})^k + (u + \frac{3}{4})^{-k}\} \right] \hspace{1cm}
\]
\[ e^{-2\beta_2}[-(1)^{n-1}(u - \frac{3}{4})]^{-\theta \beta_1}[-(1)^{n-1}(u + \frac{3}{4})]^{-\theta \beta_1} \]
\[ e^\beta \exp \sum_{k>0} \frac{a_{n+3,-k}}{k} \{(u - 1)^k + (u + 2)^k\} - \sum_{k>0} \frac{a_{n+4,-k}}{k} \{(u + 1)^k + (u - 2)^k\} \]
\[ e^{\beta_4 + \beta_5} \left( \frac{u + \frac{1}{2}}{u - \frac{1}{2}} \right)^{-\theta \beta_3} e^{\beta_4 + \theta \beta_5} \]

\[ H^-(u) \mapsto \exp \left[ \sum_{k>0} \frac{a_{n+1,-k}}{k} \{(u + \frac{3}{4})^k + (u - \frac{3}{4})^k\} \right] \]\[ \times \exp \left[ \sum_{k>0} \frac{a_{n,k}}{k} \{(u - \frac{1}{4})^{-k} + (u + \frac{1}{4})^{-k}\} \right] \hspace{1cm}
\]
\[ e^{-2\beta_2}[-(1)^{n-1}(u - \frac{1}{4})]^{-\theta \beta_1}[-(1)^{n-1}(u + \frac{1}{4})]^{-\theta \beta_1} \]
\[ e^\beta \exp \sum_{k>0} \frac{a_{n+3,-k}}{k} \{(u - 1)^k -(u + 1)^k\} + \sum_{k>0} \frac{a_{n+3,k}}{k} \{(u - \frac{5}{2})^k \}
- \sum_{k>0} \frac{a_{n+4,k}}{k} \{(u + \frac{1}{2})^k \}
\exp \left[ \sum_{k>0} \frac{a_{n+3,k}^*}{k} \{(u + \frac{1}{2})^{-k}\} \right] \exp \left[ \sum_{k>0} \frac{a_{n+4,k}^*}{k} \{(u + \frac{1}{2})^k\} \right] \exp \left[ \sum_{k>0} \frac{a_{n+4,k}}{k} \{(u - \frac{1}{2})^{-k}\} \right] \exp \left[ \sum_{k>0} \frac{a_{n+4,k}^*}{k} \{(u - \frac{1}{2})^k\} \right] \]
\[ e^{\beta_4 + \beta_5} (u + \frac{1}{2})^{\theta \beta_4} (u - \frac{1}{2})^{-\theta \beta_5} \]

Here \( \alpha_j = \epsilon_j - \epsilon_{j+1} \) for \( j = 1, \ldots, n - 1, \beta_i = \epsilon_{n+i} (i = 1, 2), \beta_j = \sqrt{2} \epsilon_{n+j} (j = 3, 4, 5) \).

**Proof.** The relations can be checked similarly as type B. \( \square \)

For \( \mathfrak{g} = \mathfrak{o}_{2n} \), We introduce the Fock space \( \mathcal{F} = \mathbb{C}[a_{j,-k} (1 \leq i \leq n, k \in \mathbb{Z}\setminus\{0\})] \otimes \mathbb{C}[Q], \)
where \( Q = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i, \mathbb{C}[Q] \) is the group algebra of \( Q \) over \( \mathbb{C} \). On this space, we define the action of the operators \( a_{j,k}, \partial \epsilon_j, \epsilon^{\epsilon_j} (1 \leq j \leq n) \) by

(8.5) \[ a_{j,k} \cdot f \otimes e^\beta = \begin{cases} a_{j,k} f \otimes e^\beta & \text{if } k < 0 \\ [a_{j,k}, f] \otimes e^\beta & \text{if } k > 0 \end{cases} \]

\[ \partial \epsilon_j \cdot f \otimes e^\beta = (\epsilon_j, \beta) f \otimes e^\beta, \]
\[ \epsilon^{\epsilon_j} \cdot f \otimes e^\beta = f \otimes e^{\epsilon_j + \beta}, \]

for \( f \otimes e^\beta \in \mathcal{F} \).
Theorem 8.3. The following assignment defines a $\text{DY}(\frak{o}_{2n})$-module structure on $\mathcal{F}$. For $1 \leq j \leq n - 3$,

$$E_j(u) \mapsto \exp\left[\sum_{k>0} \frac{a_{j,-k}}{k} \{(u - \frac{1}{4})^k + (u + \frac{3}{4})^k\} - \sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} (u + \frac{1}{4})^k\right]$$

$$\quad \cdot \exp\left[\sum_{k>0} \frac{a_{j,k}}{k} (u - \frac{1}{4})^{-k} \exp((-1)^{j-1}(u - \frac{1}{4})^j)\right]$$

$$\quad - \frac{\partial \alpha_j}{\partial \alpha_j}.$$

$$F_j(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,-k}}{k} \{(u + \frac{1}{4})^k + (u - \frac{3}{4})^k\} + \sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} (u - \frac{1}{4})^k\right]$$

$$\quad \cdot \exp\left[\sum_{k>0} \frac{a_{j,k}}{k} (u + \frac{1}{4})^{-k} \exp((-1)^{j-1}(u + \frac{1}{4})^j)\right]$$

$$\quad - \frac{\partial \alpha_j}{\partial \alpha_j}.$$

$$H_j^+(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,k}}{k} \{(u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k}\} \frac{u + \frac{1}{2}}{u - \frac{1}{2}}\right]$$

$$\quad + \sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} \{(u - \frac{1}{2})^k - (u + \frac{1}{2})^k\} \frac{u + \frac{1}{2}}{u - \frac{1}{2}}\right].$$

$$E_{n-2}(u) \mapsto \exp\left[\sum_{k>0} \frac{a_{n-2,-k}}{k} \{(u - \frac{1}{4})^k + (u + \frac{3}{4})^k\} - \sum_{k>0} \frac{a_{n-1,-k} + a_{n-3,-k} + a_{n,-k}}{k} (u + \frac{1}{4})^k\right]$$

$$\quad \cdot \exp\left[-\sum_{k>0} \frac{a_{n-2,k}}{k} (u - \frac{1}{4})^{-k} \exp((-1)^{n-3}(u - \frac{1}{4})^j)\right]$$

$$\quad \cdot \frac{\partial \alpha_{n-2}}{\partial \alpha_{n-2}}.$$

$$F_{n-2}(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{n-2,-k}}{k} \{(u + \frac{1}{4})^k + (u - \frac{3}{4})^k\} + \sum_{k>0} \frac{a_{n-1,-k} + a_{n-3,-k} + a_{n,-k}}{k} (u - \frac{1}{4})^k\right]$$

$$\quad \cdot \exp\left[\sum_{k>0} \frac{a_{n-2,k}}{k} (u + \frac{1}{4})^{-k} \exp((-1)^{n-3}(u + \frac{1}{4})^j)\right]$$

$$\quad \cdot \frac{\partial \alpha_{n-2}}{\partial \alpha_{n-2}}.$$

$$H_{n-2}^+(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{n-2,-k}}{k} \{(u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k}\} \frac{u + \frac{1}{2}}{u - \frac{1}{2}}\right]$$

$$\quad + \sum_{k>0} \frac{a_{n-1,-k} + a_{n-3,-k} + a_{n,-k}}{k} \{(u - \frac{1}{2})^k - (u + \frac{1}{2})^k\} \frac{u + \frac{1}{2}}{u - \frac{1}{2}}\right].$$
\[ E_{n-1}(u) \mapsto \exp \sum_{k>0} \frac{a_{n-1,k}}{k} ((u - \frac{1}{4})^k + (u + \frac{3}{4})^k) - \sum_{k>0} \frac{a_{n-2,k}}{k} (u + \frac{1}{4})^k \]

\[ \exp \left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} (u - \frac{1}{4})^{-k} \right] e^{\alpha_n - 1} \left[ (-1)^{n-2} (u - \frac{1}{4}) \right] \partial \alpha_n - 1, \]

\[ F_{n-1}(u) \mapsto \exp \left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} ((u + \frac{1}{4})^k - (u - \frac{3}{4})^k) + \sum_{k>0} \frac{a_{n-2,k}}{k} (u - \frac{1}{4})^k \right] \]

\[ \exp \sum_{k>0} \frac{a_{n-1,k}}{k} ((u - \frac{1}{2})^{-k} - (u + \frac{1}{2})^{-k}) (\frac{u + \frac{1}{2}}{u - \frac{1}{2}})^{\partial \alpha_n - 1}, \]

\[ H_{n-1}^+(u) \mapsto \exp \left[ - \sum_{k>0} \frac{a_{n-1,k}}{k} (u - 1)^k - (u + 1)^k \right] \]

\[ \sum_{k>0} \frac{a_{n-2,k}}{k} ((u - \frac{1}{2})^k - (u + \frac{1}{2})^k). \]

\[ E_n(u) \mapsto \exp \sum_{k>0} \frac{a_{n,k}}{k} ((u - \frac{1}{4})^k + (u + \frac{3}{4})^k) - \sum_{k>0} \frac{a_{n-2,k}}{k} (u + \frac{1}{4})^k \]

\[ \exp \left[ - \sum_{k>0} \frac{a_{n,k}}{k} (u - \frac{1}{4})^{-k} \right] e^{\alpha_n} \left[ (-1)^N (u - \frac{1}{4}) \right] \partial \alpha_n, \]

\[ F_n(u) \mapsto \exp \left[ - \sum_{k>0} \frac{a_{n,k}}{k} ((u + \frac{1}{4})^k - (u - \frac{3}{4})^k) + \sum_{k>0} \frac{a_{n-2,k}}{k} (u - \frac{1}{4})^k \right] \]

\[ \exp \sum_{k>0} \frac{a_{n,k}}{k} ((u + \frac{1}{4})^{-k} - (u - \frac{1}{2})^{-k}) (\frac{u + \frac{1}{4}}{u - \frac{1}{2}}) \partial \alpha_n \]

\[ H_n^+(u) \mapsto \exp \left[ - \sum_{k>0} \frac{a_{n,k}}{k} ((u - 1)^k - (u + 1)^k) \right] \]

\[ \sum_{k>0} \frac{a_{n-2,k}}{k} ((u - \frac{1}{2})^k - (u + \frac{1}{2})^k). \]

\[ H_n^-(u) \mapsto \exp \left[ - \sum_{k>0} \frac{a_{n,k}}{k} ((u - 1)^k - (u + 1)^k) \right] \]

\[ + \sum_{k>0} \frac{a_{n-2,k}}{k} ((u - \frac{1}{2})^k - (u + \frac{1}{2})^k). \]

Here \( \alpha_j = \epsilon_j - \epsilon_{j+1} \) for \( j = 1, \cdots, n-1 \), \( \alpha_n = \epsilon_{n-1} + \epsilon_n \).

**Proof.** The relations can be checked similarly as type B. \( \square \)

We remark that the analogous constructions of our level one modules in the affine Lie algebras are different from those in [2, 17, 18].
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