An improvement to the vertex-splitting conjecture

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Abstract

For a simple graph $G$, denote by $n$, $\Delta(G)$, and $\chi'(G)$ its order, maximum degree, and chromatic index, respectively. A connected class 2 graph $G$ is edge-chromatic critical if $\chi'(G - e) < \Delta(G) + 1$ for every edge $e$ of $G$. Define $G$ to be overfull if $|E(G)| > \Delta(G)\lfloor n/2 \rfloor$. Clearly, overfull graphs are class 2 and any graph obtained from a regular graph of even order by splitting a vertex is overfull. Let $G$ be an $n$-vertex connected regular class 1 graph with $\Delta(G) > n/3$. Hilton and Zhao in 1997 conjectured that if $G^*$ is obtained from $G$ by splitting one vertex of $G$ into two vertices, then $G^*$ is edge-chromatic critical, and they verified the conjecture for graphs $G$ with $\Delta(G) \geq n^2/(\sqrt{7} - 1) \approx 0.82n$. The graph $G^*$ is easily verified to be overfull, and so the hardness of the conjecture lies in showing that the deletion of every of its edge decreases the chromatic index. Except in 2002, Song showed that the conjecture is true for a special class of graphs $G$ with $\Delta(G) \geq n/2$, no other progress on this conjecture had been made. In this paper, we confirm the conjecture for graphs $G$ with $\Delta(G) \geq 0.75n$.

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1 Introduction

We consider only simple graphs. Let $G$ be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively. An edge $k$-coloring of $G$ is a mapping $\varphi$ from $E(G)$ to

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the set of integers \([1, k] := \{1, \ldots, k\}\), called \textit{colors}, such that no two adjacent edges receive the same color with respect to \(\varphi\). The \textit{chromatic index} of \(G\), denoted \(\chi'(G)\), is defined to be the smallest integer \(k\) so that \(G\) has an edge \(k\)-coloring. We denote by \(\mathcal{C}^k(G)\) the set of all edge \(k\)-colorings of \(G\). In 1960’s, Vizing [10] showed that every simple graph \(G\) has chromatic index either \(\Delta(G)\) or \(\Delta(G)+1\). If \(\chi'(G) = \Delta(G)\), then \(G\) is said to be of \textit{class 1}; otherwise, it is said to be of \textit{class 2}. Holyer [4] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. However, if a graph \(G\) has too many edges, i.e., \(|E(G)| > \Delta(G)|V(G)|/2\), then \(G\) is class 2. Such graphs are called \textit{overfull}.

Easily implied by its definition, overfull graphs are of odd order.

We call \(G\) \textit{edge-chromatic critical} or \(\Delta\)-critical if \(\chi'(G) = \Delta(G)+1\) and \(\chi'(H) < \Delta(G)+1\) for every proper subgraph \(H\) of \(G\). For example, odd cycles and the graph obtained from the Petersen graph by deleting one vertex are edge-chromatic critical. We study sufficient conditions for a class 2 graph to be edge-chromatic critical. A \textit{vertex-splitting} in \(G\) at a vertex \(v\) is obtained by replacing \(v\) with two new adjacent vertices \(v_1\) and \(v_2\) and partition the neighborhood \(N_G(v)\) into two nonempty subsets that serve as the neighborhoods of \(v_1\) and \(v_2\) in \(G'\), respectively. We say \(G'\) is obtained from \(G\) by a vertex-splitting. A vertex-splitting was formulated in terms of the “Möbius-type gluing technique” in [1] and [7].

Hilton and Zhao [3] in 1997 propsed the following conjecture.

\textbf{Conjecture 1} (Vertex-splitting conjecture). \textit{Let \(G\) be an \(n\)-vertex class 1 \(\Delta\)-regular graph with \(\Delta > \frac{n}{2}\). If \(G^*\) is obtained from \(G\) by a vertex-splitting, then \(G^*\) is \(\Delta\)-critical.}

Since the graph \(G^*\) above is overfull and so is class 2, the difficulty of the vertex-splitting conjecture lies in checking every edge of \(G^*\) is critical, i.e., whose deletion decreases the chromatic index of \(G^*\). Hilton and Zhao [3] in the same paper verified the conjecture for graphs \(G\) with \(\Delta(G) \geq \frac{n}{2}(\sqrt{7} - 1) \approx 0.82n\). Song [5] in 2002 showed that the conjecture is true for a special class of graphs \(G\) with \(\Delta(G) \geq \frac{n}{2}\). Except this result, to our best knowledge, we are not aware of any other progress on the conjecture. In this paper, we verify the conjecture for graphs \(G\) with \(\Delta(G) \geq 0.75n\) as below.

\textbf{Theorem 1.} \textit{Let \(n\) and \(\Delta\) be positive integers such that \(\Delta \geq \frac{3(n-1)}{4}\). If \(G\) is obtained from an \((n-1)\)-vertex \(\Delta\)-regular class 1 graph by a vertex-splitting, then \(G\) is \(\Delta\)-critical.}

The reminder of this paper is organized as follows. In Section 2, we introduce some definitions and preliminary results. In Section 3, we prove Theorem 1. In the last Section, we prove one newly developed adjacency lemma.

### 2 Definitions and Preliminary Results

Let \(G\) be a graph. For \(e \in E(G)\), \(G - e\) denotes the graph obtained from \(G\) by deleting the edge \(e\). The symbol \(\Delta\) is reserved for \(\Delta(G)\), the maximum degree of \(G\) throughout this
paper. A $k$-vertex in $G$ is a vertex of degree $k$ in $G$, and a $k$-neighbor of a vertex $v$ is a neighbor of $v$ that is a $k$-vertex in $G$. For $u, v \in V(G)$, we use $\text{dist}_G(u, v)$ to denote the distance between $u$ and $v$, which is the length of a shortest path connecting $u$ and $v$ in $G$. For $S \subseteq V(G)$, define $\text{dist}_G(u, S) = \min_{v \in S} \text{dist}_G(u, v)$.

An edge $e \in E(G)$ is a critical edge of $G$ if $\chi'(G - e) < \chi'(G)$. It is not hard to see that for a connected class 2 graph, if every of its edge is critical, then $G$ is $\Delta$-critical. Edge-chromatic critical graphs provide more information about the structure around a vertex than general class 2 graphs. For example, Vizing’s Adjacency Lemma (VAL) from 1965 [10] is a useful tool that reveals certain structure at a vertex by assuming the criticality of an edge.

**Lemma 2** (Vizing’s Adjacency Lemma (VAL),[10]). Let $G$ be a class 2 graph with maximum degree $\Delta$. If $e = xy$ is a critical edge of $G$, then $x$ has at least $\Delta - d_G(y) + 1$ $\Delta$-neighbors in $V(G) \setminus \{y\}$.

Let $G$ be a graph and $\varphi \in \mathcal{C}^k(G - e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For any $v \in V(G)$, the set of colors present at $v$ is $\varphi(v) = \{\varphi(f) : f \text{ is incident to } v\}$, and the set of colors missing at $v$ is $\overline{\varphi}(v) = [1, k] \setminus \varphi(v)$. For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v)$. We call $X$ elementary with respect to $\varphi$ or simply $\varphi$-elementary if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. Sometimes, we just say that $X$ is elementary if the edge coloring is understood.

For two distinct colors $\alpha, \beta \in [1, k]$, let $H$ be the spanning subgraph of $G$ with its edges from $E(G)$ that are colored by $\alpha$ or $\beta$ with respect to $\varphi$. Each component of $H$ is either an even cycle or a path, which is called an $(\alpha, \beta)$-chain of $G$ with respect to $\varphi$. If we interchange the colors $\alpha$ and $\beta$ on an $(\alpha, \beta)$-chain $C$ of $G$, we get a new edge $k$-coloring of $G$, and we write

$$\varphi' = \varphi/C.$$ 

This operation is called a Kempe change.

Let $x, y \in V(G)$, and $\alpha, \beta \in [1, k]$ be two distinct colors. If $x$ and $y$ are contained in a same $(\alpha, \beta)$-chain of $G$ with respect to $\varphi$, we say $x$ and $y$ are $(\alpha, \beta)$-linked with respect to $\varphi$. Otherwise, $x$ and $y$ are $(\alpha, \beta)$-unlinked with respect to $\varphi$. For a vertex-edge sequence $S$, we use $V(S)$ to denote the set of all vertices contained in the sequence.

### 2.1 Multifan and Kierstead Path

The fan argument was introduced by Vizing [8, 9] in his classical results on the upper bounds of chromatic indices for simple graphs and multigraphs. We will use multifan, a generalized version of Vizing fan, given by Stiebitz et al. [6], in our proof.

Let $G$ be a graph, $e = rs_1 \in E(G)$ and $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$. A multifan centered at $r$ with respect to $e$ and $\varphi$ is a sequence $F_{\varphi}(r, s_1 : s_p) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_p, s_p)$.
with $p \geq 1$ consisting of distinct vertices $r, s_1, s_2, \ldots, s_p$ and distinct edges $rs_1, rs_2, \ldots, rs_p$ satisfying the following condition:

(F1) For every edge $rs_i$ with $i \in [2, p]$, there exists $j \in [1, i - 1]$ such that $\varphi(rs_i) \in \overline{\varphi(s_j)}$.

We will simply denote a multifan $F_{\varphi}(r, s_1 : s_p)$ by $F$ if $\varphi$ and the vertices and edges in $F_{\varphi}(r, s_1 : s_p)$ are clear. The following result regarding a multifan can be found in [6, Theorem 2.1].

**Lemma 3.** Let $G$ be a class 2 graph and $F_{\varphi}(r, s_1 : s_p)$ be a multifan with respect to a critical edge $e = rs_1$ and a coloring $\varphi \in C^{\Delta}(G - e)$. Then the following statements hold.

(a) $V(F)$ is $\varphi$-elementary.

(b) Let $\alpha \in \overline{\varphi(r)}$. Then for every $i \in [1, p]$ and $\beta \in \overline{\varphi(s_i)}$, $r$ and $s_i$ are $(\alpha, \beta)$-linked with respect to $\varphi$.

Let $G$ be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in C^k(G - e)$ for some integer $k \geq 0$. A Kierstead path with respect to $e$ and $\varphi$ is a sequence $K = (v_0, v_0v_1, v_1v_2, v_2, \ldots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices $v_0, v_1, \ldots, v_p$ and distinct edges $v_0v_1, v_1v_2, \ldots, v_{p-1}v_p$ satisfying the following condition:

(K1) For every edge $v_iv_{i+1}$ with $i \in [1, p-1]$, there exists $j \in [0, i - 1]$ such that $\varphi(v_iv_{i+1}) \in \overline{\varphi(v_j)}$.

Clearly a Kierstead path with at most 3 vertices is a multifan. We consider Kierstead paths with 4 vertices. The result below was proved in Theorem 3.3 from [6].

**Lemma 4.** Let $G$ be a class 2 graph, $e = v_0v_1 \in E(G)$ be a critical edge, and $\varphi \in C^{\Delta}(G - e)$. If $K = (v_0, v_0v_1, v_1v_2, v_2, v_2v_3, v_3)$ is a Kierstead path with respect to $e$ and $\varphi$, then the following statements hold.

(a) If $\min\{d_G(v_1), d_G(v_2)\} < \Delta$, then $V(K)$ is $\varphi$-elementary.

(b) $|\overline{\varphi(v_3)} \cap (\overline{\varphi(v_0)} \cup \overline{\varphi(v_1)})| \leq 1$.

### 3 Proof of Theorem 1

The proof of Theorems 1 is mainly an application of a new adjacency lemma–Lemma 5 below. We define a short-kite to be a 6-vertex graph consisting of a 4-cycle $abuca$ and two additional edges $ux$ and $uy$. The truth of the vertex-splitting conjecture would be evident when $\Delta \geq \frac{n}{2}$ if the vertex set of every Kierstead path on four vertices is elementary. Unfortunately, the statement is not true and a counterexample has been found, see Figure 1, where $K = (x, xy, y, yz, z, zw, w)$ and $V(K)$ is not elementary with respect to the given coloring, and $P^*$ is obtained from the Petersen graph by deleting one vertex.
The new adjacency lemma below is an attempt to reveal some elementary properties of a Kierstead path on four vertices by incorporating some additional structure to the path.

**Lemma 5.** Let $G$ be a class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose $K = (a, ab, b, bu, u, ux, x)$ and $K^* = (b, ab, a, ac, c, cu, u, uy, y)$ are two Kierstead path with respect to $ab$ and $\varphi$. If $\varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b)$, then $\max\{d_G(x), d_G(y)\} = \Delta$.

The proof of Lemma 5 will be given in the last section. Since all vertices not missing a given color $\alpha$ are saturated by the matching that consists of all edges colored by $\alpha$ in $G$, we have the Parity Lemma below, which has appeared in many papers, for example, see [2, Lemma 2.1].

**Lemma 6 (Parity Lemma).** Let $G$ be an $n$-vertex graph and $\varphi \in \mathcal{C}^\Delta(G)$. Then for any color $\alpha \in [1, \Delta]$, $|\{v \in V(G) : \alpha \in \varphi(v)\}| \equiv n \pmod{2}$.

Let $G$ be a graph and $u, v \in V(G)$ be adjacent. We call $(u, v)$ a full-deficiency pair of $G$ if $d(u) + d(v) = \Delta(G) + 2$. If $G$ is $\Delta$-critical, then a full-deficiency pair $(u, v)$ of $G$ is called a saturating pair of $G - uv$ in [1].

**Lemma 7.** If $G$ is an $n$-vertex class 2 graph with a full-deficiency pair $(a, b)$ such that $ab$ is a critical edge of $G$, then $G$ satisfies the following properties.

(i) For every $x \in (N_G(a) \cup N_G(b)) \setminus \{a, b\}$, $d_G(x) = \Delta$;

(ii) For every $x \in V(G) \setminus \{a, b\}$, if $\text{dist}_G(x, \{a, b\}) = 2$, then $d_G(x) \geq \Delta - 1$. Furthermore, if $d_G(a) < \Delta$ and $d_G(b) < \Delta$, then $d_G(x) = \Delta$;

(iii) For every $x \in V(G) \setminus \{a, b\}$, if $d_G(x) \geq n - |N_G(b) \cup N_G(a)|$, then $d_G(x) \geq \Delta - 1$. Furthermore, if $d_G(a) < \Delta$ and $d_G(b) < \Delta$, then $d_G(x) = \Delta$;

(iv) Suppose that $n$ is odd. If there exists $x \in V(G) \setminus \{a, b\}$ such that $d_G(x) < \Delta$, then there exists $y \in V(G) \setminus \{a, b, x\}$ such that $d_G(y) < \Delta$. 

Figure 1: A Kierstead path with non-elementary vertex set in a 3-coloring of $P^* - xy$
Proof. We let \( \varphi \in \mathcal{C}^\Delta(G - ab) \) and \( F = (b, ba, a) \) be the multifan with respect to \( ab \) and \( \varphi \). By Lemma 3 (a),
\[
|\varphi(V(F))| = 2\Delta - (d_G(a) + d_G(b) - 2) = 2\Delta + 2(\Delta + 2) = \Delta.
\] (1)

By Lemma 3, for every \( \varphi' \in \mathcal{C}^\Delta(G - ab) \), \( \{a, b\} \) is \( \varphi' \)-elementary and for every \( i \in \varphi'(a) \) and \( j \in \varphi'(b) \), \( a \) and \( b \) are \( (i, j) \)-linked with respect to \( \varphi' \). We will use this fact very often.

Since \( \varphi(a) \cap \varphi(b) = \emptyset \) and \( \varphi(a) \cup \varphi(b) = [1, \Delta] \), it follows that \( \varphi(a) = \varphi(b) \). Thus for any \( x \in N_G(a) \setminus \{b\} \), \( (a, ab, ba, ax, x) \) is a multifan with respect to \( ab \) and \( \varphi \) and so \( \{a, b, x\} \) is \( \varphi \)-elementary by Lemma 3 (a). It follows from (1) that \( d_G(x) = \Delta \). Symmetrically, for each \( x \in N_G(b) \setminus \{a\} \), \( d_G(x) = \Delta \). This proves (i).

For (ii), let \( x \in V(G) \setminus \{a, b\} \) such that \( \text{dist}_G(x, \{a, b\}) = 2 \). We assume that \( \text{dist}_G(x, \{a, b\}) = 2 \) and let \( u \in (N_G(b) \setminus \{a\}) \cap N_G(x) \). Then by (1), \( K = (a, ab, bu, u, ux, x) \) is a Kierstead path with respect to \( ab \) and \( \varphi \). By (1) and Lemma 4 (b), it follows that \( d_G(x) \geq \Delta - 1 \). If \( d_G(a) < \Delta \) and \( d_G(b) < \Delta \), then \( V(K) \) is \( \varphi \)-elementary by Lemma 4 (a). Since \( \varphi(a) \cup \varphi(b) = [1, \Delta] \) by (1), it follows that \( d_G(x) = \Delta \).

For (iii), let \( x \in V(G) \setminus \{a, b\} \) such that \( d_G(x) \geq n - |N_G(b) \cup N_G(a)| \). By (i), we may assume that \( x \notin (N_G(a) \cup N_G(b)) \setminus \{a, b\} \). Thus \( d_G(x) \geq n - |N_G(b) \cup N_G(a)| \) implies that there exists \( u \in ((N_G(a) \cup N_G(b))) \cap N_G(x) \). Therefore, \( \text{dist}_G(x, \{a, b\}) = 2 \). Now Statement (ii) yields the conclusion. For any color \( \alpha \in \varphi(x) \), either \( \alpha \in \varphi(a) \) or \( \alpha \in \varphi(b) \), since \( \{a, b\} \) is \( \varphi \)-elementary and \( \varphi(a) \cup \varphi(b) = [1, \Delta] \). Since \( n \) is odd, by the Parity Lemma, there exists \( y \in V(G) \setminus \{a, b, x\} \) such that \( \alpha \in \varphi(y) \), and so \( d_G(y) < \Delta \), proving (iv).

Corollary 8. Let \( G \) be an \( n \)-vertex class 2 graph with a full-deficiency pair \( (a, b) \) such that \( ab \) is a critical edge of \( G \). If \( \Delta \geq \frac{3(a-1)}{4} \), then there exists at most one vertex \( x \in V(G) \setminus \{a, b\} \) such that \( d_G(x) = \Delta - 1 \).

Proof. Assume to the contrary that there exist distinct \( x, y \in V(G) \setminus \{a, b\} \) such that \( d_G(x) = d_G(y) = \Delta - 1 \). By Lemma 7 (i), \( x, y \notin (N_G(a) \cup N_G(b)) \setminus \{a, b\} \). By Lemma 7 (iii), we may assume that \( d_G(b) = \Delta \). Thus \( d_G(a) = 2 \) as \( d_G(a) + d_G(b) = \Delta + 2 \). Let \( c \) be the other neighbor of \( a \) in \( G \). Since \( (a, c) \) is a full-deficiency pair of \( G \) as well, we may assume \( x, y \notin N_G(c) \).

Since \( d_G(b) = d_G(c) = \Delta \) and \( d_G(x) = d_G(y) = \Delta - 1 \), we get \( |N_G(b) \cap N_G(c)| \geq \frac{n}{2} - 1 \) and \( |N_G(x) \cap N_G(y)| \geq \frac{n}{2} - 2 \). Since \( b, c, x, y \notin N_G(b) \cap N_G(c) \) and \( b, c, x, y \notin N_G(x) \cap N_G(y) \), we get \( |N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)| \geq 1 \). Let \( u \in N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y) \), \( H \) be the short-kite with \( V(H) = \{a, b, c, u, x, y\} \), and \( \varphi \in \mathcal{C}^\Delta(G - ab) \). As \( \{a, b\} \) is \( \varphi \)-elementary, \( |\varphi(a) \cup \varphi(b)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = \Delta \) and so \( \varphi(a) \cup \varphi(b) = [1, \Delta] \). Thus \( K = (a, ab, bu, u, ax, x) \) and \( K^* = (b, ab, a, ac, cu, u, uy) \) are two Kierstead paths with respect to \( ab \) and \( \varphi \), and \( \varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b) \). However, \( d_G(x) = d_G(y) = \Delta - 1 \), contradicting Lemma 5.

\( \square \)
**Proof of Theorem 1.** Since $G$ is overfull, it is class 2. We only need to show that every edge of $G$ is critical. Suppose to the contrary that there exists $xy \in E(G)$ such that $xy$ is not a critical edge of $G$. Let $G^* = G - xy$. Then $\chi'(G^*) = \Delta + 1$.

Since $ab$ is a critical edge of $G$, $ab \neq xy$. Also, since $ab$ is a critical edge of $G$, and any $\Delta$-coloring of $G - ab$ gives a $\Delta$-coloring of $G^* - ab$, $ab$ is also a critical edge of $G^*$. Now $d_{G^*}(x) = d_{G^*}(y) = \Delta - 1$, reaching a contradiction to Corollary 8.

4 Proof of Lemma 5

We start with some notation. Let $G$ be a graph and $\varphi \in C^k(G - e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For all the concepts below, when we use them later on, if we skip $\varphi$, we mean the concept is defined with respect to the current edge coloring.

Let $x, y \in V(G)$, and $\alpha, \beta, \gamma \in [1, k]$ be three colors. Let $P$ be an $(\alpha, \beta)$-chain of $G$ with respect to $\varphi$ that contains both $x$ and $y$. If $P$ is a path, denote by $P_{[x,y]}(\alpha, \beta, \varphi)$ the subchain of $P$ that has endvertices $x$ and $y$. By **swapping colors** along $P_{[x,y]}(\alpha, \beta, \varphi)$, we mean exchanging the two colors $\alpha$ and $\beta$ on the path $P_{[x,y]}(\alpha, \beta, \varphi)$.

Define $P_x(\alpha, \beta, \varphi)$ to be an $(\alpha, \beta)$-chain or an $(\alpha, \beta)$-subchain of $G$ with respect to $\varphi$ that starts at $x$ and ends at a different vertex missing exactly one of $\alpha$ and $\beta$. If $x$ is an endvertex of the $(\alpha, \beta)$-chain that contains $x$, then $P_x(\alpha, \beta, \varphi)$ is unique. Otherwise, we take one segment of the whole chain to be $P_x(\alpha, \beta, \varphi)$. We will specify the segment when it is used.

If $u$ is a vertex on $P_x(\alpha, \beta, \varphi)$, we write $u \in P_x(\alpha, \beta, \varphi)$; and if $uv$ is an edge on $P_x(\alpha, \beta, \varphi)$, we write $uv \in P_x(\alpha, \beta, \varphi)$. If $u, v \in P_x(\alpha, \beta, \varphi)$ such that $u$ lies between $x$ and $v$ on the path, then we say that $P_x(\alpha, \beta, \varphi)$ meets $u$ before $v$. Suppose the current color of an edge $uv$ of $G$ is $\alpha$, the notation $uv : \alpha \rightarrow \beta$ means to recolor the edge $uv$ using the color $\beta$. If $|\varphi(x)| = 1$, we will also use $\varphi(x)$ to denote the color that is missing at $x$.

Let $\alpha, \beta, \gamma, \tau, \eta \in [1, k]$. We will use a matrix with two rows to denote a sequence of operations taken on $\varphi$. Each entry in the first row represents a path or a sequence of vertices. Each entry in the second row, indicates the action taken on the object above this entry. We require the operations to be taken to follow the “left to right” order as they appear in the matrix. For example, the matrix below indicates three sequential operations taken on the graph based on the coloring from the previous step:

$$
\begin{bmatrix}
P_{[a,b]}(\alpha, \beta) & rs & ab \\
\alpha/\beta & \gamma \rightarrow \tau & \eta
\end{bmatrix}.
$$

Step 1 Swap colors on the $(\alpha, \beta)$-subchain $P_{[a,b]}(\alpha, \beta, \varphi)$.

Step 2 Do $rs : \gamma \rightarrow \tau$.

Step 3 Color the edge $ab$ using color $\eta$. 

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Lemma 5. Let $G$ be a class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in C_\Delta(G - ab)$. Suppose
\[
K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, cu, u, uy, y)
\]
are two Kierstead paths with respect to $ab$ and $\varphi$. If $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then $\max\{d_G(x), d_G(y)\} = \Delta$.

Proof. Assume to the contrary that $\max\{d_G(x), d_G(y)\} \leq \Delta - 1$. Since both $K$ and $K^*$ are Kierstead paths and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, Lemma 4 (a) and (b) implies that $d_G(b) = d_G(u) = \Delta$ and $d_G(x) = d_G(y) = \Delta - 1$.

Let $\overline{\varphi}(b) = \{1\}$. Then $\varphi(ac) = 1$. We may assume $\varphi(uy) = 1$. The reasoning is below. Since $a$ and $b$ are $(1, \alpha)$-linked for every $\alpha \in \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, we may assume $\overline{\varphi}(y) = 1$. Then a $(1, \varphi(uy))$-swap at $y$ gives a coloring, call it still $\varphi$, such that $\varphi(uy) = 1$. We consider now two cases.

Case 1: $\overline{\varphi}(x) = \overline{\varphi}(y)$.

Let $\varphi ux = \gamma$, and $\overline{\varphi}(x) = \overline{\varphi}(y) = \eta$. As $\varphi uy = \overline{\varphi}(b) = 1$, $1 \notin \{\gamma, \eta\}$. As both $K$ and $K^*$ are Kierstead paths and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, $\gamma, \eta \in \overline{\varphi}(a)$. Denote by $P_u(1, \gamma)$ the $(1, \gamma)$-subchain starting at $u$ that does not include the edge $ux$.

Claim 2. We may assume that $P_u(1, \gamma)$ ends at $x$, some vertex $z \in V(G) \setminus \{a, b, c, u, x, y\}$, or passing $c$ ends at $a$.

Proof. Note that $P_u(1, \gamma) = P_b(1, \gamma)$. If $u \notin P_a(1, \gamma)$, then the $(1, \gamma)$-chain containing $u$ is a cycle or a path with endvertices contained in $V(G) \setminus \{a, b, c, u, x, y\}$. Thus $P_u(1, \gamma)$ ends at $x$ or some $z \in V(G) \setminus \{a, b, c, u, x, y\}$. Hence we assume $u \in P_a(1, \gamma)$. As a consequence, $P_u(1, \gamma)$ ends at either $b$ or $a$. If $P_x(1, \gamma)$ ends at $b$, we color $ab$ by 1, uncolor $ac$, and exchange the vertex labels $b$ and $c$. This gives an edge $\Delta$-coloring of $G - ab$ such that $P_u(1, \gamma)$ ends at $a$. Thus, if $u \in P_a(1, \gamma)$, we may always assume that $P_u(1, \gamma)$ ends at $a$.

Let $\varphi (bu) = \delta$. Again, $\delta \in \overline{\varphi}(a)$. Figure 2 depicts the colors and missing colors on these specified edges and vertices, respectively. Clearly, $\delta \neq 1, \gamma$. Since $a$ and $b$ are $(1, \delta)$-linked with respect to $\varphi$, $\eta \neq \delta$. Otherwise, $b$ and $y$ would be $(1, \delta)$-linked. Thus, $\gamma, \delta$ and $\eta$ are pairwise distinct.

The claim below is simple though it is actually the breakthrough for showing Lemma 5. Without using the symmetry of the 4-cycle $abuca$, it seems extremely difficult to prove the same result.

Claim 3. It holds that $ub \in P_y(\eta, \delta)$ and $P_y(\eta, \delta)$ meets $u$ before $b$.

Proof. Let $\varphi'$ be obtained from $\varphi$ by coloring $ab$ by $\delta$ and uncoloring $bu$. Note that $\overline{\varphi'}(b) = 1, \overline{\varphi'}(u) = \delta$ and $\varphi' uy = 1$. Thus $F^* = (u, ub, b, uy, y)$ is a multifan and so $u$ and $y$ are $(\eta, \delta)$-linked by Lemma 3 (b). By uncoloring $ab$ and coloring $bu$ by $\delta$, we get back
the original coloring $\varphi$. Therefore, under the coloring $\varphi$, $u \in \mathcal{P}_y(\eta, \delta)$ and $\mathcal{P}_y(\eta, \delta)$ meets $u$ before $b$. \hfill \Box$

We apply the following operations based on $\varphi$:

$$
\begin{bmatrix}
ux & \mathcal{P}_{[u,y]}(\eta, \delta) & ub & \mathcal{P}_u(1, \gamma) & ab \\
\gamma \to \eta & \delta/\eta & \delta \to 1 & 1/\gamma & \delta
\end{bmatrix}.
$$

By Claim 2, $\mathcal{P}_u(1, \gamma)$ does not end at $b$. In any case, the above operations give an edge $\Delta$-coloring of $G$. This contradicts the earlier assumption that $\chi'(G) = \Delta + 1$.

**Case 2**: $\varphi(x) \neq \varphi(y)$.

Let

$$
\varphi(bu) = \alpha, \quad \varphi(ux) = \beta, \quad \varphi(x) = \tau, \quad \text{and} \quad \varphi(y) = \gamma.
$$

As $\varphi(uy) = \varphi(b) = 1$, $1 \notin \{\alpha, \beta, \gamma\}$. Also, since $a$ and $b$ are $(1, \alpha)$-linked, $\gamma \neq \alpha$. Otherwise, $b$ and $y$ would be $(1, \alpha)$-linked. Since both $K$ and $K^*$ are Kierstead paths and $\mathcal{F}(x) \cup \mathcal{F}(y) \subseteq \mathcal{F}(a)$, we have $\alpha, \beta, \tau, \gamma \in \mathcal{F}(a)$.

**Claim 4.** We may assume $\mathcal{F}(x) = \tau = 1$.

**Proof.** If $uy \notin \mathcal{P}_x(1, \tau)$, we simply do a $(1, \tau)$-swap at $x$. Thus, we assume that $u \in \mathcal{P}_x(1, \tau)$. We first do a $(1, \tau)$-swap at $b$, then an $(\alpha, \tau)$-swap at $x$. Then we do a $(\gamma, \tau)$-swap at $b$. Finally, a $(1, \gamma)$-swap at $b$ and a $(1, \alpha)$-swap at $x$ give the desired coloring. \hfill \Box

Since $ux \in \mathcal{P}_x(1, \beta)$, and $a$ and $b$ are $(1, \beta)$-linked, we do a $(1, \beta)$-swap at $b$. Now we color $ab$ by $\alpha$, recolor $bu$ by $\beta$ and uncolor $ux$, see Figure 3 for a depiction.

Note that

$$F^* = (u, ux, x, uy, y), \quad K^* = (x, xu, u, ub, b, ba, a)$$

are, respectively, a multifan and a Kierstead path. By Lemma 3 (b), $u$ and $y$ are $(\alpha, \gamma)$-linked, and $u$ and $x$ are $(\alpha, \beta)$-linked and $(1, \alpha)$-linked. Thus, we do an $(\alpha, \gamma)$-swap at $a$,
an \((\alpha, \beta)\)-swap at \(a\), a \((1, \alpha)\)-swap at \(a\), and then an \((\alpha, \gamma)\)-swap at \(a\). Now \(P_u(\alpha, \beta) = uba\), contradicting Lemma 3 (b) that \(u\) and \(x\) are \((\alpha, \beta)\)-linked. The proof is now completed.

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