John and Uniform Domains in Generalized Siegel Boundaries

Roberto Monti1 · Daniele Morbidelli2

Received: 17 May 2018 / Accepted: 7 July 2019 / Published online: 9 August 2019 © Springer Nature B.V. 2019

Abstract
Given the pair of vector fields $X = \partial_x + |z|^{2m} y \partial_t$ and $Y = \partial_y - |z|^{2m} x \partial_t$, where $(x, y, t) = (z, t) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, we give a condition on a bounded domain $\Omega \subset \mathbb{R}^3$ which ensures that $\Omega$ is an $(\varepsilon, \delta)$-domain for the Carnot-Carathéodory metric. We also analyze the Ahlfors regularity of the natural surface measure induced on $\partial \Omega$ by the vector fields.

Keywords SubRiemannian distance · John domains · $(\varepsilon, \delta)$ domains

Mathematics Subject Classification (2010) Primary 53C17; Secondary 49J15

1 Introduction
In $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ we consider the vector fields
$$X = \partial_x + |z|^{2m} y \partial_t \quad \text{and} \quad Y = \partial_y - |z|^{2m} x \partial_t,$$
where $(x, y, t) = (z, t) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and $m \in [1, +\infty[$ is a real parameter. The vector fields $X$ and $Y$ naturally arise as the real and imaginary part of the holomorphic vector field tangent to the boundary of the generalized Siegel domain $\{(z_1, z_2) \in \mathbb{C}^2 : \text{Im} z_2 > \frac{1}{2m+2} \Re z_1 |z_1|^{2m+2}\}$.

We study the interaction of the Carnot-Carathéodory (CC) distance $d$ induced by $X$ and $Y$ with the geometry of a surface embedded in $\mathbb{R}^3$. Namely, we give conditions on the boundary $\partial \Omega$ such that an open set $\Omega \subset \mathbb{R}^3$ is a John domain, a uniform domain and such that the natural surface measure induced on $\partial \Omega$ by $X$ and $Y$ is Ahlfors regular, see Definition 1.4.

John domains are also known as domains with the twisted cone property, see Definition 5.1. When the distance is induced by Hörmander vector fields in $\mathbb{R}^n$, several authors proved

1 Dipartimento di Matematica “Tullio Levi-Civita”, Università degli Studi di Padova, Padova, Italy
2 Dipartimento di Matematica, Università di Bologna, Bologna, Italy
that a bounded John domain supports a global Sobolev-Poincaré inequality, see [4, 5, 12, 22] and the discussion for a general metric space in [11]. The exterior twisted cone property is also relevant in classical potential theory because it implies the subelliptic Wiener criterion (see [19]).

Uniform domains are also known as \((\varepsilon, \delta)\)-domains, see Definition 6.1. They form a subset of John domains. In the global theory of Sobolev spaces for Hörmander vector fields, Garofalo and Nhieu proved in [6] that subelliptic Sobolev functions in a uniform domain can be extended to the whole space. In [3] it is also shown that the trace of a Sobolev function in a uniform domain with Ahlfors regular boundary belongs to a suitable Besov space of the boundary. Also for this reason, we shall study the Ahlfors property very carefully. The trace problem was analyzed in [13] in the non-characteristic case and in a two-dimensional model. In [7], we study by a direct approach the trace problem at the boundary of the characteristic half plane \(t > 0\) for vector fields of Martinet type \(X = \partial_x, Y = \partial_y + |x|^\alpha \partial_t\) in \(\mathbb{R}^3\).

In spite of the previous results, there are not many examples of John and uniform domains in Carnot-Carathéodory spaces. In fact, the subRiemannian case is more delicate than the Euclidean one because of the presence of characteristic points, i.e., points where the Hörmander vector fields are all tangent to the boundary. Such points make the construction of the inner cone more difficult. Sometimes the inner cone does not exist at all, even for analytic boundaries, see e.g. [16, Theorem 1.2].

A well known general fact is that small CC-balls are John domains. We also mention some contributions by the russian school. See [9, 10, 23], where the authors study the uniformity of subRiemannian balls in Heisenberg groups. See also [8] and [21], for further examples. In [16] it is proved that \(C^2\) domains in Carnot groups of step two are uniform. The case of cylindrically symmetric domains was already considered in [1] in the Heisenberg group, that is the model (1.1) with \(m = 0\). In [14] and [15], the authors studied the case of diagonal vector fields.

In this paper, we study uniform domains in \(\mathbb{R}^3\) for the CC distance of the vector fields (1.1). Our sufficient condition for a domain to be uniform requires the boundary to be “flat” near characteristic points on the \(t\)-axis.

Let \(\Omega \subset \mathbb{R}^3\) be an open set with \(C^\infty\) boundary. If both \(X\) and \(Y\) are tangent to \(\partial \Omega\) at the point \(p \in \partial \Omega\), then there is a neighborhood \(U_p\) of \(p\) such that \(U_p \cap \partial \Omega\) is a graph of the form \(t = \varphi(z)\). So we start from the following definition.

Let \(A \subset \mathbb{R}^2\) be an open set and \(\varphi \in C^\infty(A)\). We say that \(\Sigma = \text{gr}(\varphi) = \{(z, \varphi(z)) \in \mathbb{R}^3 : z \in A\}\) is an \(m\)-admissible graph if there exists a constant \(C > 0\) such that for all \(z \in A\)

\[
|D^3\varphi(z)| \leq C|z|^{2m-1}, \quad |D^2\varphi(z)| \leq C|z|^{2m} \quad \text{and} \quad |D\varphi(z)| \leq C|z|^{2m+1}.
\]

When \(0 \not\in A\), the three conditions (1.2) are trivially satisfied in a compact subset of \(A\). The conditions are instead restrictive when \(0 \in A\). In Eq. 1.2 we adopted the notation \(|D^k\varphi(z)| := \max_{j_1,\ldots,j_k} |\frac{\partial^k\varphi}{\partial x_{j_1} \cdots \partial x_{j_k}}(z)|\) to denote the largest \(k\)-th order derivative of \(\varphi\).

**Definition 1.1** Let \(m \in [1, +\infty[\). We say that a bounded domain \(\Omega \subset \mathbb{R}^3\) with smooth boundary is \(m\)-admissible if for any characteristic point \(p \in \partial \Omega\) there exists a neighborhood \(U_p\) of \(p\) in \(\mathbb{R}^3\) such that \(\partial \Omega \cap U_p\) is an \(m\)-admissible graph.

It is easy to construct simple examples of admissible sets. Consider for instance the bounded domain

\[
\Omega = \{(z, t) \in \mathbb{R}^3 : |z|^{2(m+1)} + t^2 = 1\}.
\]
The boundary $\partial \Omega$ has two characteristic points, namely $(0, 0, -1)$ and $(0, 0, 1)$, and the functions $\varphi(z) = \pm \sqrt{1 - |z|^{2m+1}}$ satisfy condition (1.2). Small perturbations of the boundary, compactly supported outside the characteristic set, give nonradial examples.

Our main result is the following:

**Theorem 1.2** Let $m \in \mathbb{N}$. Any $m$-admissible domain $\Omega \subset \mathbb{R}^3$ is uniform and, in particular, is a John domain in $(\mathbb{R}^3, d)$.

In fact, our proof shows that admissible domains are also non-tangentially accessible (NTA). Concerning our requirements on the rate of growth (1.2) for the function $\varphi$, it is easy to check that any open set which agrees in a neighborhood of the origin with the epigraph \{ $t > |z|^\alpha$ \} with $\alpha < 2m + 2$ is not a John domain.

On the other hand, let us consider the epigraph \{ $t > -x^{2m+1}y$ \} of Example 3.1. All the points $(x, 0, 0)$ of the $x$-axis are characteristic points of the boundary. However, the “order of degeneration” of such points is 2 when $x \neq 0$, while it is $2m + 2$ when $x = 0$. The difficulty of our work in Section 5 is due to the fact that we need to construct a family of inner cones of constant aperture contained in \{ $t > -x^{2m+1}y$ \} and with vertex at points arbitrarily close to the characteristic set. Furthermore, in order to prove the $(\varepsilon, \delta)$-property, in Section 6 we also need to show that cones with close vertices have quantitatively close axes.

Theorem 1.2 is proved in Sections 5, 6 and 7. We first show that global (i.e., with $A = \mathbb{R}^2$) admissible graphs have the global cone property and then that they satisfy the $(\varepsilon, \delta)$-condition. Finally, we deal with the case of bounded domains. The proofs rely on a precise description of the distance $d$, which will be discussed in Section 2, and on some preliminary results proved in Section 3.

The natural surface area on $\partial \Omega$ is the perimeter measure of $\Omega$ induced by the vector fields (1.1). This is the measure

$$\mu = \sqrt{\langle N, X \rangle^2 + \langle N, Y \rangle^2} \ \mathcal{H}^2 \mathcal{L} \partial \Omega,$$

where $N$ is the unit Euclidean normal to $\partial \Omega$, $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{R}^3$, and $\mathcal{H}^2 \mathcal{L} \partial \Omega$ is the standard surface measure, i.e., the restriction of the 2-dimensional Hausdorff measure to $\partial \Omega$. This is a special case of the variational definition of perimeter measure in CC-spaces, see [6] and [18]. For admissible domains, the measure $\mu$ is codimension 1 Ahlfors regular in the following sense.

**Theorem 1.3** Let $m \in \mathbb{N}$ and denote by $B(p, r)$ the CC-balls. For any $m$-admissible domain $\Omega \subset \mathbb{R}^3$ there exist constants $C > 0$ and $r_0 > 0$ such that for all $p \in \partial \Omega$ and $0 < r \leq r_0$

$$C^{-1} \frac{|B(p, r)|}{r} \leq \mu(B(p, r)) \leq C \frac{|B(p, r)|}{r}.$$

(1.5)

Above, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^3$. This theorem is proved in Section 7 and relies on the delicate analysis of global admissible graphs tackled in Section 4. Our analysis will require the study of several situations, depending on how CC-balls intersect the graph near the characteristic set.

The ball-box theorem for the distance $d$ is proved in the first part of the paper. For any $(z, t), (\zeta, \tau) \in \mathbb{R}^3$, we define the function

$$\delta((z, t), (\zeta, \tau)) = |z - \zeta| + \min \left\{ |v|^{1/2}, \frac{|v|^{1/2}}{|z|^{m}} \right\},$$

(1.6)

where $v = \tau - t + |z|^{2m} \omega(z, \zeta)$, and $\omega(z, \zeta) = x \eta - y \xi$ with $z = (x, y)$ and $\zeta = (\xi, \eta)$.
Theorem 1.4 Let $m \in [1, +\infty]$. There is a constant $C > 1$ such that for all $p = (z, t), q = (\zeta, \tau) \in \mathbb{R}^3$

$$C^{-1}\delta(p, q) \leq d(p, q) \leq C\delta(p, q).$$

This theorem is proved in Section 2. Our proof is completely self-contained and works for any $m \geq 1$, also noninteger. Note that when $m \in [1, +\infty]\setminus\mathbb{N}$, the well known ball-box theorem in [20] cannot be applied, because the vector fields (1.1) are not smooth at $z = 0$.

In the case $m \in \mathbb{N}$, a local version of Theorem 1.4 can be obtained from the classical results in [20]. The statement can in principle be globalized by some dilation argument, but this requires some care. Here, we give an independent self-contained proof of Theorem 1.4. In particular, Step 2 and Step 3 of the proof of this theorem give a constructive and quantitative explanation of the fact that any pair of points can be connected with a horizontal path.

2 Ball-box Estimate

In this section, we prove Theorem 1.4 and, in Corollary 2.2 below, we rephrase it as a ball-box estimate.

An absolutely continuous curve $\gamma : [0, 1] \to \mathbb{R}^3$ is horizontal for the vector fields (1.1), if it satisfies $\dot{\gamma} = \alpha(s)X(\gamma) + \beta(s)Y(\gamma)$ for a.e. $s \in [0, 1]$. The length of $\gamma$ is defined as

$$\text{length}(\gamma) = \int_0^1 |(\alpha(s), \beta(s))|ds.$$ 

Given points $(z, t), (\zeta, \tau) \in \mathbb{R}^3$, the CC distance $d((z, t), (\zeta, \tau))$ is defined as the infimum (the minimum, in fact) of the length of all absolutely continuous curves $\gamma : [0, 1] \to \mathbb{R}^3$ connecting them.

We will use the following invariance properties of $d$. For all $(z, t), (\zeta, \tau) \in \mathbb{C} \times \mathbb{R}$, $s, \theta \in \mathbb{R}$, and $r > 0$ we have:

$$d((z, t), (\zeta, \tau)) = d(e^{i\theta}z, t), (e^{i\theta}\zeta, \tau));$$

$$d((z, t), (\zeta, t)) = d((z, t + s), (\zeta, \tau + s));$$

$$d((rz, r^{2m+2}t), (r\zeta, r^{2m+2}\tau)) = rd((z, t), (\zeta, \tau)).$$

We will also use the following elementary estimate, holding for any $x, y \in \mathbb{R}$ and $m \geq 1$:

$$C_m^{-1}(|x|^{m-1} + |y|^{m-1}) \geq |x| - |y| \leq |x|^m - |y|^m \leq C_m(|x|^{m-1} + |y|^{m-1})|y - x|.$$ 

Proof of Theorem 1.4 Step 1. We claim that there exists a constant $C > 0$, depending on $m$, such that $\delta((z_0, t_0), (\zeta, \tau)) \leq Cd((z_0, t_0), (\zeta, \tau))$ for all points $(z_0, t_0), (\zeta, \tau) \in \mathbb{R}^3$.

By Eqs. 2.1-2.2, we can assume that $z_0 = (x_0, 0)$ with $x_0 \geq 0$ and $t_0 = 0$. In this case, we have $\omega(z_0, \zeta) = x_0\eta$, with $\zeta = (\xi, \eta)$, and the definition in Eq. 1.6 for $\delta$ reads, with $v = \tau + x_0^{2m+1}\eta$,

$$\delta((z_0, 0), (\zeta, \tau)) = |z_0 - \zeta| + \min\left\{|v|^{1/2}, |v|^{1/2m+2}\right\}.$$ 

Let $\gamma = (z, t) : [0, T] \to \mathbb{R}^3, T > 0$, be a unit-speed horizontal curve connecting $(z_0, 0)$ and $(\zeta, \tau)$. We let $z = z(s) = (x(s), y(s)) = (x, y)$. From the unit-speed condition $|\dot{z}| \leq 1$, we deduce that

$$|z_0 - \zeta| = \int_0^T |\dot{z}|ds \leq T.$$
We estimate the quantity
\[ v = \tau + x_0^{2m+1} \eta = \int_0^T \left\{ |z|^{2m} y \dot{x} + (x_0^{2m+1} - |z|^{2m} x) \dot{y} \right\} ds. \]

We claim that there exists a constant \( C > 0 \) such that for all \( s \in [0, T] \) we have
\[ |z|^{2m}|y| + |x_0^{2m+1} - |z|^{2m} x| \leq C(x_0^{2m}s + s^{2m+1}). \]

The left-hand side is evaluated at \( s \in [0, T] \). From \( |z| \leq x_0 + s \) and \(|y| \leq s \) we deduce that \(|z|^{2m}|y| \leq C(x_0^{2m}s + s^{2m+1})\). By the triangle inequality and Eq. 2.4, we have
\[ |x_0^{2m+1} - |z|^{2m} x| \leq |x_0^{2m} - |z|^{2m} x| + |x_0 - x_0^{2m} x| \leq C_m(|z|^{2m-1} + x_0^{2m-1})|z| - x_0| |x| + x_0^{2m}|x - x_0|. \]

Using \(|x| \leq |z| \leq x_0 + s, |z| - x_0| \leq s\) and \(|x - x_0| \leq s\) we obtain \(|x_0^{2m+1} - |z|^{2m} x| \leq C(x_0^{2m}s + s^{2m+1})\). This finishes the proof of Eq. 2.6.

Now, Eq. 2.6 implies that \(|\tau + x_0^{2m+1} \eta| \leq C(x_0^{2m} T^2 + T^{2m+2})\), which is equivalent to
\[ T \geq C^{-1} \min \left\{ \frac{|\tau + x_0^{2m+1} \eta|^{1/2}}{x_0^m}, |\tau + x_0^{2m+1} \eta|^{1/2} \right\}. \]

The inequalities (2.7) and (2.5) imply \( \delta((z_0, 0), (\zeta, \tau)) \leq CT \) and minimizing on \( T \) we get the claim made in the Step 1.

**Step 2.** We claim that there exists a constant \( C > 0 \) such that \( d((z,t), (z,\tau)) \leq C\delta((z,t), (z,\tau)) \) for all \( z \in \mathbb{C} \) and \( t, \tau \in \mathbb{R} \), i.e.,
\[ d((z,t), (z,\tau)) \leq C \min \left\{ |\tau - t|^{1/2} \right\}. \]

By Eqs. 2.1–2.2, we can without loss of generality assume that \( z = (x, 0) \) with \( x \geq 0 \), \( t = 0 \) and \( \tau \geq 0 \).

For each \( u \geq 0 \) consider the unit-speed path \( [0, 4u] \ni s \mapsto \zeta(s) \in \mathbb{R}^2 \) that linearly connects the points in the plane \((x, 0), (x, u), (x + u, u), (x + u, 0)\) and \((x, 0)\). Let \( R_u \) be the square enclosed by \( \zeta \). The path \( \zeta \) has length \( 4u \) and its unique absolutely continuous horizontal lift \( s \mapsto \gamma(s) = (\zeta(s), \tau(s)) \) satisfying \( \tau(0) = 0 \) has final point \( \tau(4u) = \int_{\zeta} |\zeta|^{2m}(\eta d\xi - \xi d\eta) = 2(m + 1) \int_{R_u} |\zeta|^{2m} d\xi d\eta \geq C_0^{-1} u^2 (x^{2m} + u^{2m}). \)

We used Stokes’ theorem with the counterclockwise orientation of \( \zeta \). The function \( u \mapsto \int_{R_u} |\zeta|^{2m} d\xi d\eta \) is a strictly increasing bijection of \([0, +\infty[ \) onto itself.

Let \( \overline{u} \) be the unique number such that \( \tau(4\overline{u}) = \tau \). By the definition of the distance \( d \) and by Eq. 2.9, we have
\[ d((x, 0, 0), (x, 0, \tau)) \leq 4\overline{u} = \min \left\{ 4u > 0 : \tau(4u) \geq \tau \right\} \leq \min \left\{ 4u > 0 : \tau \leq C_0^{-1} u^2 (x^{2m} + u^{2m}) \right\} \leq C \min \left\{ \frac{\tau^{1/2}}{x^m}, \tau^{1/2} \right\}. \]

This concludes the proof of the Step 2.

**Step 3.** We claim that the inequality \( d((z,t), (\zeta,\tau)) \leq C\delta((z,t), (\zeta,\tau)) \) holds for all points \((z,t), (\zeta,\tau) \in \mathbb{C} \times \mathbb{R} \).
We preliminarily observe that, given \((u, v) = w \in \mathbb{C}\), for any point \((z, t) = (x, y, t)\) we have
\[
e^{uX+vY}(z, t) = \left(z + w, t + \omega(w, z) \int_0^1 |z + sw|^{2m} ds\right),
\]
where \(\omega(w, z) = uy - vx\) and \(e^{uX+vY}(z, t)\) denotes the value at time 1 of the integral curve of \(uX + vY\) starting from \((z, t)\) at time 0.

By the triangle inequality, it follows that
\[
d((z, t), (\xi, \tau)) \leq d((z, t), e^{(\xi-x)X+(\eta-y)Y}(z, t)) + d(e^{(\xi-x)X+(\eta-y)Y}(z, t), (\xi, \tau)).
\]
In the last distance, the points are one above each other and so, by Eq. 2.8, we get
\[
d((z, t), (\xi, \tau)) \leq C \left(|z - \xi| + \min\left\{|t - \tau + \lambda|^{1/2m^2}, |t - \tau + \lambda|^{1/2m} \right\}\right),
\]
where
\[
\lambda = \omega(\xi, z) \int_0^1 |z + s(\xi - z)|^{2m} ds.
\]
We used \(\omega(\xi, z) = \omega(\xi, z)\).

In order to prove the claim in the Step 3, we have to show that the right-hand side in Eq. 2.10 is less than \(C\delta((z, t), (\xi, \tau))\). By Eqs. 2.1–2.2, it is enough to prove this estimate in the case \(z = (x, 0)\) with \(x \geq 0\) and \(t = 0\). In this case, the distance \(\delta\) is
\[
\delta((z, 0), (\xi, \tau)) = |z - \xi| + \min\left\{|v|^{1/2m^2}, |v|^{1/2m} \right\}, \quad v = \tau + x^{2m+1}\eta.
\]

We distinguish two cases:

**Case G1:** \(|v|^{1/2m^2} \leq |v|^{1/2m}, i.e., x \leq |v|^{1/2m^2};

**Case G2:** \(|v|^{1/2m^2} \geq |v|^{1/2m}, i.e., x \geq |v|^{1/2m^2}.

**Case G1.** When \(z = (x, 0)\) and \(t = 0\) we have \(\omega(\xi, z) = -x\eta\) and, see Eq. 2.10,
\[
|t - \tau + \lambda| = |\tau + x\eta \int_0^1 |z + s(\xi - z)|^{2m} ds|.
\]

We claim that in the Case G1 we have
\[
|\tau + x\eta \int_0^1 |z + s(\xi - z)|^{2m} ds|^{1/2m^2} \leq C \left(|z - \xi| + |v|^{1/2m^2}\right).
\]

This and Eq. 2.10 finish the proof of the the Step 3 in the Case G1, because the right-hand side in Eq. 2.11 is \(C\delta((z, 0), (\xi, \tau))\).

We prove (2.11). By the triangle inequality, we have
\[
\left|\tau + x\eta \int_0^1 |z + s(\xi - z)|^{2m} ds\right| \leq |v| + x|\eta| - x^{2m} + \int_0^1 |z + s(\xi - z)|^{2m} ds
\]
\[
= |v| + x|\eta| \int_0^1 \int_0^s \frac{d}{d\xi} |z + \phi(\xi - z)|^{2m} d\phi ds
\]
\[
= |v| + 2mx |\eta| \int_0^1 \int_0^s |z + \phi(\xi - z)|^{2m-2} (z, \xi - z) + \phi|z - \xi|^2 d\phi ds
\]
\[
\leq C(|v| + \Theta),
\]
where we let
\[
\Theta = x|\eta|(x + |\xi - z|)^{2m-2}(x|\xi - x| + |\xi - z|^2).
\]
By the Hölder inequality and by the Case G1 we have
\[ \Theta \leq C |\zeta - z| (x + |\zeta - z|)^{2m} \leq C (x + |\zeta - z|)^{2m+2} \leq C (|v| + |\zeta - z|^{2m+2}). \]
This and Eq. 2.12 finish the proof of Eq. 2.11.

**Case G2.** In this case we have \( x \geq |v|^{1/(m+2)} \), and thus
\[ \delta((z, 0), (\zeta, \tau)) = |z - \zeta| + \frac{|v|^{1/2}}{x^m}, \quad v = \tau + x^{2m+1} \eta. \]
We distinguish the following three subcases:
\[
\begin{align*}
|\xi - x| &\leq \frac{1}{2} x; \\
\max\{|\xi|, |\eta|\} &\leq \frac{1}{2} x; \\
\max\{|\xi|, |\eta|\} &\geq \frac{1}{2} x \quad \text{and} \quad |\xi - x| \geq \frac{1}{2} x.
\end{align*}
\]

In the Case G2a, the quantity \( \Theta \) in Eq. 2.13 can be estimated as follows
\[ \Theta \leq C x|\eta|(x + |\eta|)^{2m-2}(x|x - \xi| + \eta^2) \]
and from \( x \leq x + |\eta| \leq C |\xi| \) we deduce that
\[ \frac{1}{|\xi|^{2m}} (|v| + \Theta) \leq C \left( \frac{|v|}{x^{2m}} + \frac{x|\eta|}{(x + |\eta|)^2} (x|x - \xi| + \eta^2) \right) \leq C \left( \frac{|v|}{x^{2m}} + |\zeta - z|^2 \right). \]
This along with Eqs. 2.12 and 2.10 finish the proof of the Step 3 in the Case G2a.

In the Case G2b, the quantities \( x, x + |\xi|, x + |\eta|, |\xi - x| \) are mutually comparable with absolute constants and therefore, also using the Hölder inequality, the quantity \( \Theta \) in Eq. 2.13 can be estimated as follows
\[ \Theta \leq C x^{2m+1}|\eta| \leq C |\zeta - z|^{2m+2}. \]
On the other hand, we have
\[ |v|^{1/(m+2)} \leq C \left( x + \frac{|v|^{1/2}}{x^m} \right) \leq C \left( |\zeta - z| + \frac{|v|^{1/2}}{x^m} \right). \]
These two inequalities imply, via (2.12), that the claim (2.11) holds also in the Case G2b.

In the Case G2c, we have \( x \leq C |\xi| \) and from Eq. 2.13 we estimate
\[ \Theta \leq C |\xi| |\eta|(|\xi| + |\zeta - z|)^{2m-2}(|\xi||\zeta - z| + |\zeta - z|^2) \leq C |\xi|^{2m}|\zeta - z|^2, \]
and we conclude that
\[ \frac{1}{|\xi|^{m}} (\Theta + |v|)^{1/2} \leq C \left( |\zeta - z| + \frac{|v|^{1/2}}{x^m} \right). \]
The proof is concluded also in this case.

**Remark 2.1** The argument of the proof of Theorem 1.4 shows in fact the global equivalence
\[ |(u, v)| \leq d((z, t), e^{uX+vY}(z, t)) \leq C_0 |(u, v)|, \quad u, v, t \in \mathbb{R}, \quad z \in \mathbb{R}^2. \quad (2.14) \]

Next we describe the \( d \)-balls as suitable boxes. For any \( \beta > 0 \) we define the weighted norm of \( u = (u_1, u_2, u_3) \in \mathbb{R}^3 \)
\[ \|u\|_{1,1,\beta} = \max\{|u_1|, |u_2|, |u_3|^{1/\beta}\}, \]
and for any $p = (z, t) = (x, y, t) ∈ \mathbb{R}^3$ and $r > 0$ we define the boxes

$$\text{Box}_I(p, r) = \left\{ (x + u_1, y + u_2, t + |z|^{2m}(u_3 + yu_1 - xu_2)) : \|u\|_{1,1,2} < r \right\},$$

$$\text{Box}_J(p, r) = \left\{ (x + u_1, y + u_2, t + u_3 + |z|^{2m}(yu_1 - xu_2)) : \|u\|_{1,1,2m+2} < r \right\}.$$

**Corollary 2.2** Let $m ∈ [1, +∞[. For any $α > 0$ there exist constants $b_1, b_2, δ_0 > 0$ such that for all $p = (z, t) ∈ \mathbb{R}^3$ and $r > 0$ we have:

(i) if $|z| ≥ αr$, then

$$\text{Box}_I(p, δ_0r) ⊂ B(p, r) ⊂ \text{Box}_I(p, b_1r); \quad (2.15)$$

(ii) if $r ≥ α|z|$, then

$$\text{Box}_J(p, δ_0r) ⊂ B(p, r) ⊂ \text{Box}_J(p, b_2r). \quad (2.16)$$

**Proof** Step 1. We claim that for a suitable $δ_0 > 0$ we have

$$\text{Box}_I(p, δ_0r) ∪ \text{Box}_J(p, δ_0r) ⊂ B(p, r) ⊂ \text{Box}_I(p, r/δ_0) ∪ \text{Box}_J(p, r/δ_0).$$

To prove these inclusions, we observe that, letting $v = τ - t + |z|^{2m}(xη - yξ)$,

$$(ζ, τ) ∈ \text{Box}_I(p, r) ⇔ \max \left\{ |ξ - x|, |η - y|, \frac{|v|^{1/2}}{|z|^m} \right\} < r, \quad (2.17)$$

$$(ζ, τ) ∈ \text{Box}_J(p, r) ⇔ \max \left\{ |ξ - x|, |η - y|, |v|^{1/2m+2} \right\} < r. \quad (2.18)$$

Thus the point $(ζ, τ)$ belongs to the union of the boxes if and only if $|ξ - x| < r$, $|η - y| < r$ and

$$\min \left\{ |v|^{1/2m+2}, \frac{|v|^{1/2}}{|z|^m} \right\} < r.$$

Now the claim follows from Theorem 1.4. We also proved both the inclusions in the left-hand side of Eqs. 2.15 and 2.16.

Step 2. We prove the inclusion in the right-hand side of Eq. 2.15. Let $|z| ≥ αr$ and let $(ζ, τ) ∈ B(p, r)$. By the Step 1 we know that $(ζ, τ) ∈ (B_I ∪ B_J)(p, r/δ_0)$. Then, we are left to show that

$$|z| ≥ αr \quad \text{and} \quad \min \left\{ |v|^{1/2m+2}, \frac{|v|^{1/2}}{|z|^m} \right\} < r/δ_0 \quad ⇒ \quad \frac{|v|^{1/2}}{|z|^m} < b_1r.$$ 

If the minimum is $|v|^{1/2}/|z|^m$, there is nothing to prove. Otherwise we have $|v|^{1/(2m+2)} < r/δ_0$, i.e., $|v|^{1/2} < (r/δ_0)^{m+1}$. This and $|z| ≥ αr$ imply

$$\frac{|v|^{1/2}}{|z|^m} ≤ \frac{|v|^{1/2}}{(αr)^m} ≤ \frac{(r/δ_0)^{m+1}}{(αr)^m} = \frac{1}{α^m δ_0^{m+1} r}.$$ 

Step 3. We prove the inclusion (2.16). Arguing as in the Step 2, it suffices to prove that

$$r ≥ α|z| \quad \text{and} \quad \min \left\{ |v|^{1/2m+2}, \frac{|v|^{1/2}}{|z|^m} \right\} < \frac{r}{δ_0} \quad ⇒ \quad |v|^{1/2m+2} ≤ b_2r.$$ 

If the minimum is $|v|^{1/2m+2}$, there is nothing to prove. Otherwise we have

$$\frac{r}{δ_0} ≥ \frac{|v|^{1/2}}{|z|^m} ≥ |v|^{1/2} \left( \frac{α}{r} \right)^m,$$

that is equivalent to $|v|^{1/2} ≤ r^{m+1}/δ_0α^m$. This is the claim. \qed
3 Geometry of Admissible Graphs

Let \( \varphi \in C^\infty(\mathbb{R}^2) \) be a smooth function satisfying the flatness conditions (1.2) at any point \( z \in \mathbb{R}^2 \). A defining function for the graph of \( \varphi \) is the function \( F \in C^\infty(\mathbb{R}^3) \) given by \( F(z, t) = \varphi(z) - t \). The derivatives

\[
XF(z, t) = XF(z) = \frac{\varphi_x(z)}{\mu_{z, \zeta}} z + |z|^{2m+1} y,
YF(z, t) = YF(z) = \frac{\varphi_y(z)}{\mu_{z, \zeta}} + |z|^{2m} x
\]
do not depend on \( t \) and we let \( ZF(z) = (XF(z), YF(z)) \). A point \((z, \varphi(z)) \in \Sigma = \text{gr}(\varphi)\) is characteristic if and only if \( ZF(z) = 0 \). By Eq. 1.2, the function \( ZF \) satisfies

\[
|ZF(z)| \leq C|z|^{2m+1}, \quad z \in \mathbb{R}^2.
\]

**Example 3.1** Let \( m \in \mathbb{N} \). The graph of the function \( \varphi(z) = -x^{2m+1} y \) is \( m \)-admissible and each point of the \( x \)-axis is characteristic.

The next proposition describes the restriction of the distance \( d \) to an admissible graph.

**Lemma 3.2** Let \( \varphi \in C^\infty(\mathbb{R}^2) \) satisfy the conditions (1.2). Then there exist a constant \( C_0 > 0 \) such that for all \( p = (z, \varphi(z)), q = (\zeta, \varphi(\zeta)) \in \Sigma \)

\[
C_0^{-1} d(p, q) \leq |z - \zeta| + \left| \frac{\varphi(\zeta) - \varphi(z)}{\mu_{z, \zeta}} + \omega(z, \zeta) \right|^{1/2} \leq C_0 d(p, q),
\]

where \( \mu_{z, \zeta} = \max(|z|, |\zeta|) \) and \( \omega(z, \zeta) = x \eta - y \xi \).

**Proof** Without loss of generality, we prove the lemma in the case \(|z| \geq |\zeta|\). We claim that, letting \( v = \varphi(\zeta) - \varphi(z) + |z|^{2m} \omega(z, \zeta) \), we have

\[
\frac{|v|^{1/2}}{|z|^m} \leq C(|\zeta - z| + |v|^{1/m})
\]

Taking (3.3) for granted and starting from Theorem 1.4, by Eq. 3.3 it follows that

\[
d(p, q) \simeq |z - \zeta| + \min\left\{ |v|^{1/m} + \frac{|v|^{1/2}}{|z|^m}, |z - \zeta| + \frac{|v|^{1/2}}{|z|^m} \right\} \simeq |z - \zeta| + \frac{|v|^{1/2}}{|z|^m},
\]

and this is Eq. 3.2.

We prove (3.3). From the elementary inequality \( a^{1/2} b^{1/2} \leq C(a^{2m+1} b^{-1/2m+1} + b) \) for \( a, b \geq 0 \), we obtain

\[
\frac{|v|^{1/2}}{|z|^m} = |z|^{1/2} \left| \frac{\varphi(\zeta) - \varphi(z)}{|z|^{2m+1}} + \frac{\omega(z, \zeta)}{|z|^{2m+1}} \right|^{1/2} \leq C \left\{ |z|^{2m+1} + \left| \frac{\varphi(\zeta) - \varphi(z)}{|z|^{2m+1}} \right|^{2m+1} + \left| \frac{\omega(z, \zeta)}{|z|^{2m+1}} \right| \right\}^{1/2} \leq C \left\{ |v|^{1/2m} + \frac{|v|^{1/2}}{|z|^{2m+1}} + \frac{|\omega(z, \zeta)|}{|z|^{2m+1}} \right\} \leq C \left\{ |v|^{1/2m} + |z - \zeta| \right\},
\]

because by Eq. 1.2 and \(|\zeta| \leq |z|\) we have

\[
|\varphi(\zeta) - \varphi(z)| \leq C(|z|^{2m+1} + |\zeta|^{2m+1})|\zeta - z| \leq C|z|^{2m+1}|\zeta - z|,
\]

and, moreover, \(|\omega(z, \zeta)| = |\omega(z, \zeta - z)| \leq |z||\zeta - z|\). \( \square \)

In the next propositions, we discuss other consequences of the conditions (1.2).
Proposition 3.3 Let \( \varphi \in C^\infty(\mathbb{R}^2) \) satisfy (1.2). For all \( z, \xi \in \mathbb{R}^2 \) we have

\[
\varphi(\xi) - \varphi(z) + \mu^2_{\xi,z} \omega(z, \xi) = (ZF(z), \xi - z) + \mu^2_{\xi,z} O(|\xi - z|^2),
\]

(3.4)

where \( \mu_{z,\xi} = \max\{|z|, |\xi|\} \) and the remainder satisfies \( |O(|\xi - z|^2)| \leq C|z - \xi|^2 \) for a constant \( C > 0 \).

Proof Expanding \( \varphi \) at the second order at a point \( z \in \mathbb{R}^2 \), we obtain for any \( \xi \in \mathbb{R}^2 \)

\[
\varphi(\xi) - \varphi(z) + |z|^{2m} \omega(z, \xi) = (ZF(z), \xi - z) + \mu^2_{\xi,z} O(|\xi - z|^2).
\]

(3.5)

By Eq. 1.2, the remainder satisfies the uniform estimate \( |O(|\xi - z|^2)| \leq C|\xi - z|^2 \) for all \( z, \xi \in \mathbb{C} \). If \( \mu_{z,\xi} = |z|, \) this is our claim (3.4).

If \( |z| \leq |\xi| \), starting from Eq. 3.5 it suffices to use the estimates \( |\omega(z, \xi)| \leq |\xi||\xi - z| \) and

\[
||\xi||^2 - |z|^2m \leq C\mu^{2m-1}_{\xi,z}|\xi - z| \leq C|\xi|^2|z|^m - |\xi - z|^2.
\]

and the proof is concluded. \( \Box \)

Next we get a Taylor expansion of \( ZF(\xi) \) with a remainder \( O(|\xi - z|^2) \). This is the only point where we use the assumption \( |D^3 \varphi(z)| \leq C|z|^{2m-1} \).

Let \( z, \xi \in \mathbb{R}^2 \) be points with \( |\xi| \leq 2|z| \). There exist points \( z' = z'(z, \xi) \) and \( z'' = z''(z, \xi) \) in the line segment \([z, \xi]\) such that:

\[
\begin{align*}
\varphi_x(\xi) &= \varphi_x(z) + \langle DF_x(z'), \xi - z \rangle + |z|^{2m-1}O(|\xi - z|^2), \\
\varphi_y(\xi) &= \varphi_y(z) + \langle DF_y(z''), \xi - z \rangle + |z|^{2m-1}O(|\xi - z|^2).
\end{align*}
\]

(3.6)

On the other hand, we have

\[
\begin{align*}
|\xi|^2m \eta &= |z|^{2m} + |z|^{2m-2}(2mxy, |z|^2 + 2my^2), \xi - z) + |z|^{2m-1}O(|\xi - z|^2), \\
|\xi|^2m \xi &= |z|^{2m}x + |z|^{2m-2}(|z|^2 + 2mx^2, 2mxy), \xi - z) + |z|^{2m-1}O(|\xi - z|^2).
\end{align*}
\]

With these estimates, we have proved the following:

Proposition 3.4 Let \( \varphi \in C^\infty(\mathbb{R}^2) \) satisfy (1.2). For any \( z, \xi \in \mathbb{R}^2 \) with \( |\xi| \leq 2|z| \) we have

\[
ZF(\xi) = ZF(z) + M(z, \xi)((\xi - z) + |z|^{2m-1}O(|\xi - z|^2)
\]

where \( ZF = \begin{bmatrix} X \bar{F} & Y \bar{F} \end{bmatrix} \), \( \xi - z = \begin{bmatrix} \xi - x \\ \eta - y \end{bmatrix} \) and \( M(z, \xi) \) is the \( 2 \times 2 \) matrix

\[
M = \begin{bmatrix} \varphi_{xx}(z') - 2m|z|^{2m-2}x^2 & \varphi_{xy}(z') - |z|^{2m-2}(|z|^2 + 2my^2) \\
\varphi_{xy}(z'') + |z|^{2m-2}(|z|^2 + 2mx^2) & \varphi_{yy}(z'') + 2m|z|^{2m-2}xy \end{bmatrix}.
\]

(3.7)

If \( \varphi = 0 \), then \( \det M = (1 + 2m)|z|^{4m} \). In this case, the matrix \( M \) is nonsingular for all \( z \neq 0 \). Example 3.1 shows that, for some admissible functions, nonsingularity may fail also at points \( z \neq 0 \). However, we are able to show that the matrix \( M(z, \xi) \) has always rank at least one and that it satisfies the following quantitative nondegeneration property. This property is needed to get an Ahlfors lower bound in the noncharacteristic case, the Case 1c in next section.

Proposition 3.5 Let \( \varphi \in C^\infty(\mathbb{R}^2) \) satisfy (1.2). There exist constants \( C_2 > 1, \varepsilon_0, \varepsilon_2 \in ]0,1[ \) such that for all \( z \neq 0 \) there is a unit vector \( u \in S^1 \subset \mathbb{R}^2 \) such that for all \( r \in ]0, \varepsilon_0 |z| \) we have

\[
|M(z, \xi)((\xi - z)| \geq C_2^{-1}|z|^{2m}r
\]

(3.8)
for all $\zeta \in B_{\text{Euc}}(z + \varepsilon_1u, \varepsilon_2r) \subset B_{\text{Euc}}(z, r) \subset \mathbb{R}^2$.

**Proof** Denote by $e_1, e_2$ the coordinate versors of $\mathbb{R}^2$. Then, letting $M = M(z, \zeta)$, we have

$$|Me_1| + |Me_2| \geq |\varphi_{xy}(z'')| + |z|^{2m-2}(|z|^2 + 2mx^2) + |\varphi_{xy}(z') - |z|^{2m-2}(|z|^2 + 2my^2)|$$

$$\geq (2m + 2)|z|^{2m} - |\varphi_{xy}(z') - \varphi_{xy}(z'')|.$$

From Eq. 1.2 and $|z'| - |z''| \leq |\zeta - z| \leq \varepsilon_0|z|$, we deduce that, if $\varepsilon_0$ is conveniently small, then we have the inequality $|\varphi_{xy}(z') - \varphi_{xy}(z'')| \leq |z|^{2m}$. This implies that $|Me_1| + |Me_2| \geq 2m|z|^{2m}$. Thus, given $\zeta \neq 0$, at least one of the choices $u = e_1$ or $u = e_2$ ensures that $|M(z, \zeta)u| \geq |z|^{2m}$ for all $\zeta$ such that $|\zeta - z| < \varepsilon_0|z|$. Therefore, for any $v$ with $|v| \leq 1$ and $\varepsilon_2 > 0$ we have

$$|M(z, \zeta)(u + \varepsilon_2 v)| \geq |z|^{2m} - \varepsilon_2 |M(z, \zeta)| \geq |z|^{2m} - \varepsilon_2 C|z|^{2m}$$

where $|M(z, \zeta)|$ denotes the operatorial norm, which under our assumptions satisfies $|M(z, \zeta)| \leq C|z|^{2m}$. Thus, taking $\varepsilon_2$ small enough we get a lower estimate with $\frac{1}{2}|z|^{2m}$.

The claim (3.8) follows by multiplying the last inequality by $r/2$. \qed

## 4 Ahlfors Property for Entire Admissible Graphs

In this section we prove Theorem 1.3 in the case when the boundary of the domain is an entire admissible graph. The case of a bounded domain is in Section 7. A discussion of the problem in a translation invariant setting of step two is contained in [2].

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $\pi(z, t) = z$ and denote by $\Sigma$ the graph of a function $\varphi \in C^\infty(\mathbb{R}^2)$ satisfying (1.2). By the area formula, the measure $\mu$ defined in Eq. 1.4 satisfies, for any $p \in \Sigma$ and $r > 0$,

$$\mu(B(p, r) \cap \Sigma) = \int_{\pi(B(p, r) \cap \Sigma)} |ZF(\zeta)| \, d\xi \, d\eta. \quad (4.1)$$

The integration domain can be estimated using Lemma 3.2. For any $z \in \mathbb{R}^2$ and $r > 0$ we define the “disks”

$$D(z, r) = \{ \zeta \in \mathbb{R}^2 : |\zeta - z| \leq r, |\varphi(\zeta) - \varphi(z) + \mu_{\zeta, z}^2 \omega(z, \zeta)| \leq \mu_{\zeta, z}^2 r^2 \}, \quad (4.2)$$

where $\mu_{\zeta, z} = \max(|z|, |\zeta|)$. By Lemma 3.2, there exists a constant $C_0 > 0$ such that

$$D(z, r/C_0) \subset \pi(B(p, r) \cap \Sigma) \subset D(z, C_0 r), \quad \text{for all } z \in \mathbb{C} \text{ and } r \in [0, +\infty[. \quad (4.3)$$

The Lebesgue measure of the ball $B(p, r)$, $p = (z, \varphi(z)) \in \mathbb{R}^3$, can be computed using Corollary 2.2. Let $\varepsilon_0 \in [0, 1]$ be the constant given by Proposition 3.5. From Eqs. 2.15–2.17 and 2.16–2.18, using Fubini-Tonelli Theorem we obtain

$$|B(p, r)| \simeq |z|^{2m} r^4, \quad \text{if } r \leq \varepsilon_0|z|,$$

$$|B(p, r)| \simeq r^{2m+4}, \quad \text{if } r \geq \varepsilon_0|z|.$$ 

The equivalence constants depend on the parameter $\varepsilon_0$.

### 4.1 Proof in the Case 1: $r \leq \varepsilon_0|z|$.

We claim that for any point $p = (z, \varphi(z)) \in \Sigma$ and $r > 0$ such that $r \leq \varepsilon_0|z|$ we have:

$$\mu(B(p, r) \cap \Sigma) \simeq |z|^{2m} r^3. \quad (4.4)$$
Observe that in Case 1 we have the obvious equivalence $\frac{1}{2}|z| \leq |\xi| \leq \frac{3}{2}|z|$. Let $\beta > 0$ be a parameter that will be fixed after (4.10). We distinguish two subcases:

Case 1c: $|z|^{2m}r \geq \beta |ZF(z)|$. This is the characteristic case.

Case Inc: $|z|^{2m}r \leq \beta |ZF(z)|$. This is the non-characteristic case.

In the Case 1c, points are in a quantitative way near the characteristic set of $\Sigma$, where $|ZF(z)| = 0$.

Case 1c – upper bound. We start from the elementary inclusion $\pi(B(p, r) \cap \Sigma) \subset \{\xi \in \mathbb{R}^2 : |\xi - z| \leq r\}$. Thus, using the expansion (3.4) and the trivial estimate $|M(\xi - z)| \leq C|z|^{2m} |\xi - z|$ we obtain

$$
\int_{|\xi - z| \leq r} |ZF(\xi)|d\xi d\eta = \int_{|\xi - z| \leq r} \left|ZF(z) + M(\xi - z) + |z|^{2m-1}O(|\xi - z|^2)\right|d\xi d\eta
\leq C \int_{|\xi - z| \leq r} \left(\frac{1}{\beta}|z|^{2m}r + C|z|^{2m} |\xi - z| + C|z|^{2m-1} |\xi - z|^2\right)d\xi d\eta
\leq C \left(\frac{1}{\beta} + C\right)|z|^{2m}r^3.
$$

We also used $r \leq \varepsilon_0|z|$ to estimate the third term.

Case 1c – lower bound. We claim that there exist constants $\varepsilon_1 > 0$ and $\beta > 0$ such that

$$\{\xi \in \mathbb{R}^2 : |\xi - z| \leq \varepsilon_1r/C_0\} \subset \pi(B(p, r) \cap \Sigma).$$

The constant $C_0$ is the one given by Lemma 3.2.

In view of the expansion (3.4), the set $D(z, r/C_0)$ introduced in Eq. 4.2 satisfies

$$D(z, r/C_0) = \left\{\xi \in \mathbb{R}^2 : |\xi - z| \leq r/C_0, \ |(ZF(z), \xi - z) + \mu_{z, \xi} O(|\xi - z|^2)| \leq C_0^{-2} \mu_{z, \xi} r^2\right\},$$

where, for some absolute constant $C_1$, we have $O(|\xi - z|^2) \leq C_1 |\xi - z|^2 \leq C_1 C_0^{-2} \varepsilon_1^2 r^2$, provided that $|\xi - z| \leq \varepsilon_1r/C_0$. If $\varepsilon_1$ satisfies

$$C_1 \varepsilon_1^2 \leq \frac{1}{2},$$

then we have the inclusion

$$\{\xi \in \mathbb{R}^2 : |\xi - z| \leq \varepsilon_1r/C_0, \ |(ZF(z), \xi - z)| \leq \frac{1}{2} C_0^{-2} \mu_{z, \xi}^{2m} r^2\} \subset D(z, r/C_0).$$

(4.7)

By the Case 1c, we have

$$|(ZF(z), \xi - z)| \leq \frac{1}{\beta} |z|^{2m}r|\xi - z| \leq \frac{1}{\beta} \varepsilon_1 C_0^{-1} \mu_{z, \xi}^{2m} r^2.$$

Thus, if $\varepsilon_1$ and $\beta$ are such that

$$\frac{\varepsilon_1 C_0^{-1}}{\beta} \leq \frac{C_0^{-2}}{2},$$

the inclusion (4.5) holds, as we claimed.

Next we use Proposition 3.5 to estimate $|ZF(\xi)|$ from below at all points $\xi \in B_{Euc}(z + qu/2, \varepsilon_2 \Phi)$, where $\Phi = \varepsilon_1 C_0^{-1}r$ and $u \in S^1$ is such that Eq. 3.8 holds. Namely, we have

$$|M(\xi - z)| \geq C_2^{-1} |z|^{2m} \varepsilon_1 C_0^{-1} r,$$
where $M = M(z, \zeta)$ is the matrix (3.7). For such a point $\zeta$ we have
\[
|ZF(\zeta)| = |ZF(z) + M(\zeta - z) + |z|^{2m-1}O(|\zeta - z|^2)|
\geq C_2^{-1}|z|^{2m} \varepsilon_1 C_0^{-1} r - |ZF(z)| - |z|^{2m-1}O(|\zeta - z|^2)
\geq C_2^{-1}|z|^{2m} \varepsilon_1 C_0^{-1} r - \frac{1}{\beta}|z|^{2m} - C_1 |z|^{2m-1} \varepsilon_1 r^2
\geq C_2^{-1}|z|^{2m} \varepsilon_1 C_0^{-1} r - \frac{1}{\beta}|z|^{2m} - C_1 |z|^{2m} \varepsilon_1 r^2 \geq \frac{1}{2} C_2^{-1}|z|^{2m} \varepsilon_1 C_0^{-1} r. \quad (4.9)
\]
provided that
\[
\frac{1}{\beta} \leq \frac{1}{4} C_2^{-1} \varepsilon_1 C_0^{-1} \quad \text{and} \quad C_1 \varepsilon_1 \leq \frac{1}{4} C_2^{-1} \varepsilon_1 C_0^{-1}. \quad (4.10)
\]
We may choose $\varepsilon_1 > 0$ such that the inequalities in the right-hand side of Eqs. 4.10 and in Eq. 4.6 both hold. Then we fix $\beta > 0$ such that the inequalities in the left-hand side of Eqs. 4.10 and in 4.8 both hold.

By Eqs. 4.5 and 4.9, we finally obtain
\[
\int_{\pi(B(p,r) \cap \Sigma)} |ZF(\zeta)| d\xi d\eta \geq C_3^{-1}|z|^{2m} r^3,
\]
where, after fixing $\varepsilon_1$ and $\beta$, the constant $C_3$ is absolute.

**Case 1nc – upper bound.** In order to evaluate from above the integral in Eq. 4.1, we start from the estimates $|M(\zeta - z)| \leq C |z|^{2m} |\zeta - z| \leq C |z|^{2m} r \leq C|ZF(z)|$, where $C$ is an absolute constant. Here we used the fact that $|\zeta - z| < r$ for all points $\zeta$ in the integration set. Therefore, the weight in the integral (4.1) satisfies
\[
|ZF(\zeta)| = |ZF(z) + M(\zeta - z) + |z|^{2m-1}O(|\zeta - z|^2)| \leq C(1 + \beta)|ZF(z)|. \quad (4.11)
\]
In order to get the required estimate, the obvious inclusion $\pi(B(p, r) \cap \Sigma) \subset B_{\text{Eucl}}(x, r)$ does not suffice. We need the stronger condition (3.2), which tells that for some absolute constant $C_0 > 0$ we have $\pi(B(p, r) \cap \Sigma) \subset D(z, C_0 r)$. Using the definition (4.2) of $D(z, C_0 r)$, the expansion (3.4) and also using $|z| \simeq |\zeta|$, that follows from Case 1, we discover that
\[
\left| \frac{ZF(z)}{|ZF(z)|} \right| \cdot |\zeta - z| \leq C \frac{|z|^{2m}}{|ZF(z)|} r^2, \quad \text{for all} \quad \zeta \in D(z, C_0 r).
\]
This tells that the projection of the set $D(z, C_0 r)$ along the unit direction $\frac{ZF(\zeta)}{|ZF(z)|}$ has size $|z|^{2m} r^2$. Therefore the Lebesgue measure of $\pi(B(p, r) \cap \Sigma)$ satisfies the inequalities
\[
|\pi(B(p, r) \cap \Sigma)| \leq |D(z, C_0 r)| \leq C \frac{|z|^{2m}}{|ZF(z)|} r^3, \quad (4.12)
\]
for an absolute constant $C > 0$. Ultimately, from Eqs. 4.11 and 4.12, we obtain the upper-bound:
\[
\mu(B(p, r) \cap \Sigma) = \int_{\pi(B(p,r) \cap \Sigma)} |ZF(\zeta)| d\xi d\eta \leq C(1 + \beta)|z|^{2m} r^3.
\]

**Case 1nc – lower bound.** Observe first that Eq. 4.7 holds also in this case. Then, under our choice of $\varepsilon_1$ and $\beta$, we have
\[
\left\{ \zeta \in \mathbb{R}^2 : |\zeta - z| \leq \varepsilon_1 C_0^{-1} r \quad \text{and} \quad \langle ZF(z), \zeta - z \rangle \leq \frac{1}{2} \mu_{\xi, \zeta} C_0^{-2} r^2 \right\} \subset \pi(B(p, r) \cap \Sigma).
\]
Let $\varepsilon_3 \leq \varepsilon_1$ be a small positive constant to be fixed below. Then we have

$$\begin{cases} \zeta \in \mathbb{R}^2 : |\zeta - z| \leq \varepsilon_3 C_0^{-1} r \text{ and } |(ZF(z), \zeta - z)| \leq |z|^{2m} C_{-1}^{-1} r^2 \end{cases} \subset \pi(B(p, r) \cap \Sigma),$$

The set in the left-hand side has size $|z|^{2m} r^3 C_{-1}$ along the unit direction $ZF(z)/|ZF(z)|$. Then we have the estimate from below for the Lebesgue measure of the integration set.

$$|\pi(B(p, r) \cap \Sigma)| \geq C |z|^{2m} |ZF(z)| r^3. \quad (4.13)$$

To conclude the proof, we get a lower estimate for the function in the integral.

$$|ZF(\zeta)| = |ZF(z) + M(\zeta - z) + |z|^{2m-1} O(|\zeta - z|^2)| \geq |ZF(z)| - C_4 |z|^{2m} |\zeta - z|$$

$$\geq |ZF(z)| - C_4 \varepsilon_3 C_0^{-1} \beta |ZF(z)| \geq \frac{1}{2} |ZF(z)|, \quad (4.14)$$

provided that $\varepsilon_3$ is so small that $C_4 \beta \varepsilon_3 C_0^{-1} \leq \frac{1}{2}$. The lower bound follows from Eqs. 4.13 and 4.14. This ends the proof of Eq. 4.4.

4.2 Proof in the Case 2: $r \geq \varepsilon_0 |z|$.

We claim that for any point $p = (z, \varphi(z)) \in \Sigma$ and $r > 0$ such that $r \geq \varepsilon_0 |z|$ we have:

$$\mu(B(p, r) \cap \Sigma) \simeq r^{2m+3}. \quad (4.15)$$

We can without loss of generality assume that $\varphi(0) = 0$. By Lemma 3.2 and $|\varphi(z)| \leq C|z|^{2m+2}$, there exists a constant $C_5 > 1$ such that

$$C_5^{-1} |z| \leq d((0, 0), (z, \varphi(z))) \leq C_5 |z|. \quad (4.16)$$

Case 2 – upper bound. For $z \in \mathbb{R}^2$ and $r \geq \varepsilon_0 |z|$, we have the inclusions

$$B((z, \varphi(z)), r) \subset B((0, 0), C_5 |z| + r) \subset B((0, 0), (\varepsilon_0^{-1} + C_5) r).$$

Thus, we obtain

$$\int_{\pi(B(p, r) \cap \Sigma)} |ZF(\zeta)| d\xi d\eta \leq \int_{|\zeta| \leq C r} C|\zeta|^{2m+1} d\xi d\eta \leq C r^{2m+3}.$$ 

To check the lower bound, we distinguish two cases.

Case 2a: $\varepsilon_0 |z| \leq r \leq 2C_5 |z|$. It suffices to start from inclusion $B((z, \varphi(z)), r) \supset B((0, 0), (\varepsilon_0^{-1} + C_5) r)$. Then the perimeter measure of the smaller ball can be estimated as in Case 1. To conclude observe that $|z| \simeq r$.

Case 2b: $2C_5 |z| \leq r < \infty$. In such case we have

$$B(0, r/(2C_5^2)) \cap \Sigma \subset B(p, r) \cap \Sigma. \quad (4.17)$$

Indeed, for any $q = (\zeta, \varphi(\zeta)) \in B(0, r/(2C_5^2))$ we have $|\zeta| \leq r/(2C_5)$ and

$$d(p, q) \leq d(0, p) + d(0, q) \leq C_5 |z| + C_5 |\zeta| \leq r,$$

as claimed.

By Eq. 4.17, it is enough to prove the lower-bound estimate in the case $z = 0$. 

---

\[ \mathbb{S} \text{ Springer} \]
For $\varrho > 0$, we calculate by Stokes’ theorem the following integral on the curve $\gamma_\varrho(s) = \varrho e^{-is}$, with $s \in [0, 2\pi]$,
\[
\int_{\gamma_\varrho} (XF d\xi + YF d\eta) = \int_{|\xi| < \varrho} \left( -\partial_\eta (\partial_\xi \varphi(\xi) - |\xi|^{2m} \eta) + \partial_\xi (\partial_\eta \varphi(\xi) + |\xi|^{2m} \xi) \right) d\xi d\eta
\]
\[
= (2 + 2m) \int_{|\xi| < \varrho} |\xi|^{2m} d\xi d\eta = C^{-1} \varrho^{2m+2},
\]
that implies $\int_{|\xi| = \varrho} |ZF(\xi)|dH^1(\xi) \geq C^{-1} \varrho^{2m+2}$. Thus we get the estimate
\[
\int_{|\xi| < r} |ZF(\xi)|dH^1(\xi) \geq \int_0^r \int_{|\xi| = \varrho} |ZF(\xi)|dH^1(\xi)d\varrho \geq C^{-1} r^{2m+3}.
\]
This inequality ends the proof of Eq. 4.15 and thus of Theorem 1.3 in the case of entire admissible graphs.

## 5 Cone property for admissible entire epigraphs

We recall the definition of a John domain, specialized to the metric space $(\mathbb{R}^3, d)$.

**Definition 5.1** A bounded open set $\Omega \subset \mathbb{R}^3$ is a $\lambda$-John domain, with $\lambda > 0$, if there exists a point $p_0 \in \Omega$ such that for all $p \in \Omega$ there is a continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = p$, $\gamma(1) = p_0$, and
\[
B(\gamma(t), \lambda \text{diam}(\gamma|_{[0,t]})) \subset \Omega \quad \text{for all } t \in ]0, 1[. \tag{5.1}
\]
A curve $\gamma$ satisfying (5.1) is called a John curve in $\Omega$ with parameter $\lambda > 0$.

In our definition, the curve $\gamma$ is not required to be rectifiable. By the results of [17], Definition 5.1 is equivalent to the more standard one with rectifiable curves and with diameters replaced by lengths. John domains are also known as domains with the twisted interior cone property.

In this section, we consider an unbounded domain of the epigraph type $\Omega = \text{epi}(\varphi) = \{(z, t) \in \mathbb{R}^3 : t > \varphi(z)\}$, where $\varphi \in C^\infty(\mathbb{R}^2)$ is an $m$-admissible function, and we construct a nontrivial John curve starting from any point $p = (z, t) \in \Omega$. The case of a bounded domain is discussed in Section 7.

For any point $z \in \mathbb{R}^2$ with $|ZF(z)| \neq 0$ there exists a unit vector $(u, v) \in S^1 \subset \mathbb{R}^2$ such that
\[
-(uX + vY)F(z) \geq \frac{1}{2} |ZF(z)|, \tag{5.2}
\]
where $F(z, t) = \varphi(z) - t$. Our John curve starting from $(z, t) \in \text{epi}(\varphi)$ is the integral curve of $-(uX + vY)$ for times $s \in [0, \tilde{s}]$, where the time $\tilde{s} = \tilde{s}(z)$ is
\[
\tilde{s} = \epsilon_0 \frac{|ZF(z)|}{|z|^{2m}}, \tag{5.3}
\]
and $\epsilon_0 > 0$ is a suitable constant. For $s \geq \tilde{s}$, the John curve is an integral curve of $\partial / \partial t$. This piece of curve is nonrectifiable. When $(z, \varphi(z)) \in \text{gr}(\varphi)$ is a characteristic point of $\text{gr}(\varphi)$, i.e. $|ZF(z)| = 0$, we have $\tilde{s} = 0$ and the first piece of the curve disappears.

By the rotational invariance (2.1) of the metric $d$, in Eq. 5.2 we can assume that $u = 1$ and $v = 0$.
Theorem 5.2 Let \( \varphi \in C^\infty (\mathbb{R}^2) \) be a function satisfying (1.2). There exist constants \( \varepsilon_0 > 0 \) and \( \lambda > 0 \) such that for any \( z \in \mathbb{R}^2 \) with
\[
-XF(z) > \frac{1}{2} |ZF(z)|,
\]the curve \( \gamma : [0, \infty) \to \mathbb{R}^3 \)
\[
\gamma(s) = \begin{cases} 
e^{sX}(z, t), & \text{if } 0 \leq s \leq \tilde{s} = \varepsilon_0 \frac{|ZF(z)|}{|z|^{2m}} \\ \gamma(\tilde{s}) + (0, s - \tilde{s}), & \text{if } \tilde{s} \leq s < \infty \end{cases}
\]
is a John curve in epi(\( \varphi \)) with parameter \( \lambda \) starting from \( (z, t) \in \text{epi}(\varphi) \).

Let \( \lambda > 0 \) be the parameter of our John curves. In the proof of the theorem and in the following sections, we denote by \( \sigma_\lambda \) any constant of the form \( C\lambda^\beta \), where \( C \) is an absolute constant and \( \beta > 0 \) is a positive power.

Proof Without loss of generality, we prove the claim for \( t = \varphi(z) \), i.e., we construct a John curve in the epigraph of \( \varphi \) starting from a boundary point. In the following we let \( z_s = z + se_1 \). When \( s \in [0, \tilde{s}] \), an explicit formula for \( \gamma(s) \) is
\[
\gamma(s) = e^{sX}(z, \varphi(z)) = \left(z_s, \varphi(z) + y \int_0^s |z|^{2m} d\varrho\right).
\]
The definition in Eq. 5.3 for \( \tilde{s} \) implies that \( \tilde{s} \leq \varepsilon_0 C|z| \), where \( C > 0 \) is the constant appearing in Eq. 3.1. We can choose \( \varepsilon_0 > 0 \) such that \( \varepsilon_0 C < \frac{1}{2} \). Then, for any \( s \in [0, \tilde{s}] \) we have
\[
\frac{|z|}{2} \leq |z_s| \leq \frac{3}{2}|z|.
\]Further conditions on \( \varepsilon_0 \) will be required below.

Step 1. We claim that for any sufficiently small \( \lambda > 0 \) we have \( B(\gamma(s), \lambda s) \subset \text{epi}(\varphi) \) for all \( 0 < s \leq \tilde{s} \).

From \( s \leq \tilde{s} \leq \frac{1}{2}|z| \) and Corollary 2.2, we have the inclusion \( B(\gamma(s), \lambda s) \subset \text{Box}_I(\gamma(s), \sigma_\lambda s) \) for any \( 0 < \lambda \leq 1 \), where \( \sigma_\lambda = b_1 \lambda \) and \( b_1 \) is the constant given by Corollary 2.2. So our claim is implied by \( \text{Box}_I(\gamma(s), \sigma_\lambda s) \subset \text{epi}(\varphi) \). We have \( p \in \text{Box}_I(\gamma(s), \sigma_\lambda s) \) if and only if
\[
p = \left(z_s + v, \varphi(z) + y \int_0^s |z|^{2m} d\varrho + |z_s|^{2m}(u_3 + \omega(v, z_s))\right),
\]
with \( \|u\|_{1,2} \leq \sigma_\lambda s \) and \( u = (u_1, u_2, u_3) = (v, u_3) \). Then the claim in Step 1 is implied by the inequality
\[
\varphi(z_s + v) - \varphi(z) < y \int_0^s |z|^{2m} d\varrho + |z_s|^{2m}(u_3 + \omega(v, z_s)) =: H,
\]
for \( \|u\|_{1,2} < \sigma_\lambda s \) and \( s \leq \tilde{s} \). The left-hand side of Eq. 5.6 can be expanded using Eq. 3.5:
\[
\varphi(z_s + v) - \varphi(z) = (ZF(z), v_s) + |z|^{2m} \omega(v_s, z) + \mu_{z, z_s}^{2m} O(|v_s|^2).
\]
By Eq. 5.4, \( |z_s| \leq C|z| \), and \( u_1 + s \geq 0 \) we get
\[
\varphi(z_s + v) - \varphi(z) \leq |ZF(z)|\left(-\frac{1}{2}(u_1 + s) + |u_2|\right) + |z|^{2m} \omega(v_s, z) + C|z|^{2m}|v_s|^2.
\]
Since $|u_2| \leq \sigma_\lambda s$ and $|u_1| \leq \sigma_\lambda s$, we have $|v_3| \leq Cs$ and for small $\lambda > 0$, we get
\[ \varphi(z_3 + v) - \varphi(z) \leq -}\frac{s}{4}|ZF(z)| + |z|^{2m}(\omega(v, z) + sy + C_0s^2). \]  
(5.7)

Now we estimate the quantity $H$ in the right-hand side of Eq. 5.6. Since $\varphi \leq s \leq \bar{s} \leq \frac{1}{2}|z|$, and $u_3 \geq -\sigma_s s^2$ we have the elementary inequalities
\[ |z_3|^{2m} - |z|^{2m} \leq C|z|^{2m-1} \varphi, \]
\[ |z|^{2m} u_3 \geq -C|z|^{2m} \sigma_s^2, \]
and
\[ |z|^{2m} \omega(v, z_s) \geq |z|^{2m} \omega(v, z_s) - C|z|^{2m}s^2. \]
(5.8)

Thus, we obtain
\[ H \geq |z|^{2m}(\omega(v, z) + ys - C_1s^2) \]  
(5.9)

for a suitable absolute constant $C_1$.

By Eqs. 5.7 and 5.9, the claim (5.6) is then implied by the inequality
\[ (C_0 + C_1)|z|^{2m}s < \frac{1}{4}|ZF(z)|, \]
that holds for all $s \leq \bar{s} = \varepsilon_0 |ZF(z)|$ as soon as $\varepsilon_0 \leq \frac{1}{4(C_0 + C_1)}$.

Step 2. For $s \geq \bar{s}$, the curve $\gamma$ is defined by the formula
\[ \gamma(s) = (z_s, \varphi(z)) + \int_{0}^{s} |z_\omega|^{2m} d\varphi + s - \bar{s}. \]

In the following we let $s - \bar{s} = \tau^{2m+2}$. As a function of $\tau$, the diameter of $\gamma$ restricted to $[0, s]$ satisfies
\[ \Delta_\tau = \text{diam}(\gamma|_{[0, s + \tau^{2m+2}]} \leq C(\bar{s} + \min \{\tau, \frac{\tau^{m+1}}{|z|^{2m}}\}) =: \bar{s} + \sigma_\tau. \]

The proof of Theorem 1.4 shows in fact that the inequality above is a global equivalence for $\bar{s}$, $\tau \in [0, +\infty[$.

We claim that for any sufficiently small $\lambda > 0$ we have $B(\gamma(s), \lambda \Delta_\tau) \subset \text{epi}(\varphi)$ for all $\tau \geq 0$. By Corollary 2.2, this claim is equivalent to
\[ \text{Box}_J(\gamma(s), \sigma_\lambda \Delta_\tau) \subset \text{epi}(\varphi) \quad \text{and} \quad \text{Box}_J(\gamma(s), \sigma_\lambda \Delta_\tau) \subset \text{epi}(\varphi). \]  
(5.10)

We prove the inclusion in the left-hand side of Eq. 5.10. We have $p \in \text{Box}_J(\gamma(s), \sigma_\lambda \Delta_\tau)$ if and only if
\[ p = (z_\bar{s} + v, \varphi(z)) + y \int_{0}^{\bar{s}} |z_\omega|^{2m} d\varphi + |z_\bar{s}|^{2m}(u_3 + \omega(v, z_\bar{s})) + \tau^{2m+2}. \]  
(5.11)

with $\|u\|_{1, 2} \leq \sigma_\lambda \Delta_\tau$ and $u = (u_1, u_2, u_3) = (v, u_3)$. The point $p$ belongs to $\text{epi}(\varphi)$ if
\[ L := \varphi(z_\bar{s} + v) - \varphi(z) < y \int_{0}^{\bar{s}} |z_\omega|^{2m} d\varphi + |z_\bar{s}|^{2m}(u_3 + \omega(v, z_\bar{s})) + \tau^{2m+2} =: R. \]

First we get an upper bound for $L$. Observe first that assumption (5.4) and the inequalities $\bar{s} \leq \frac{1}{2}|z|$ and $|v| \leq \sigma_\lambda (\bar{s} + \sigma_\mu \tau)$ give
\[ \langle ZF(z), v_\bar{s} \rangle \leq |ZF(z)|\left( -\frac{1}{4} \bar{s} + \sigma_\lambda \sigma_\mu \tau \right). \]
Thus, using formula (3.5) we obtain

\[
L = \varphi(z + v\bar{s}) - \varphi(z) = (ZF(z), v\bar{s}) + |z|^{2m} \omega(v, z) + \max \left\{ |z|^{2m}, |v\bar{s}|^{2m} \right\} O(|v\bar{s}|^2)
\]

\[
\leq |ZF(z)| \left( - \frac{1}{4} \bar{s} + \sigma_\lambda m_\tau \right) + |z|^{2m} \omega(v, z) + C \left( |z|^{2m} + \sigma_\lambda m_\tau^{2m} \right) (\bar{s}^2 + \sigma_\lambda^2 m_\tau^2)
\]

\[
\leq |ZF(z)| \left( - \frac{1}{4} \bar{s} + \sigma_\lambda m_\tau \right) + |z|^{2m} \left( \omega(v, z) + \bar{s} y + C_0 \bar{s}^2 + \sigma_\lambda^2 m_\tau^2 \right) + \sigma_\lambda m_\tau^{2m+2}.
\]

(5.12)

We compute a lower bound for the right-hand side $R$. Using Eq. 5.8 we get

\[
R = y \int_0^{\bar{s}} |z_{\varphi}|^{2m} d\varphi + |z\bar{s}|^{2m} (u_3 + \omega(v, z\bar{s})) + \tau^{2m+2}
\]

\[
\geq y |z|^{2m} \bar{s} - C |z|^{2m} \bar{s}^2 + \tau^{2m+2} - \sigma_\lambda |z|^{2m} m_\tau^2 + |z|^{2m} (\omega(v, z) - u_2 \bar{s})
\]

\[
\geq \tau^{2m+2} + |z|^{2m} \left( \omega(v, z) + \bar{s} y - C_1 \bar{s}^2 - \sigma_\lambda m_\tau^2 \right).
\]

(5.13)

Then, the inequality $L < R$ follows from

\[
\sigma_\lambda m_\tau |ZF(z)| + |z|^{2m} \left( C_2 \bar{s}^2 + \sigma_\lambda m_\tau^2 \right) + \sigma_\lambda m_\tau^{2m+2} < \frac{1}{4} \bar{s} |ZF(z)| + \tau^{2m+2}.
\]

(5.14)

To prove (5.14), we start from the second term. By the definition of $\bar{s}$, we have

\[
C_2 \bar{s}^2 |z|^{2m} = \varepsilon_0 C_2 |ZF(z)| \bar{s} \leq \frac{1}{4} |ZF(z)| \bar{s},
\]

as soon as $\varepsilon_0$ satisfies $\varepsilon_0 C_2 < 1/4$. This is the last time we modify the choice of $\varepsilon_0$.

Next we look at the first term. Observe that

\[
\sigma_\lambda |ZF(z)| m_\tau = \sigma_\lambda |ZF(z)| \min \left\{ \tau, \frac{\tau^{m+1}}{|z|^m} \right\} \geq \sigma_\lambda |ZF(z)| \frac{\tau^{m+1}}{|z|^m} \leq \sigma_\lambda \left( \frac{|ZF(z)|^2}{|z|^{2m}} + \tau^{2m+2} \right).
\]

Then, since $\frac{1}{4} |ZF(z)| \bar{s} + \tau^{2m+2} = \frac{\varepsilon_0}{4} \frac{|ZF(z)|^2}{|z|^{2m}} + \tau^{2m+2}$, we can finish the estimate as soon as $\sigma_\lambda$ is small with respect to absolute constants (which include $\varepsilon_0$, now).

The estimate of the third term is easy:

\[
\sigma_\lambda |z|^{2m} m_\tau^2 = \sigma_\lambda |z|^{2m} \left( \min \left\{ \tau, \frac{\tau^{m+1}}{|z|^m} \right\} \right)^2 \leq \sigma_\lambda \tau^{2m+2},
\]

which is correctly estimated, provided that $\sigma_\lambda$ is small enough. Finally, we have

\[
\sigma_\lambda m_\tau^{2m+2} = \sigma_\lambda \left( \min \left\{ \tau, \frac{\tau^{m+1}}{|z|^m} \right\} \right)^{2m+2} \leq \sigma_\lambda \tau^{2m+2},
\]

which again satisfies the required estimate.

To conclude the proof, we have to check the inclusion in the right-hand side of Eq. 5.10. In this case the box Box $J(s, \sigma_\lambda (\bar{s} + m_\tau))$ is made of points of the form

\[
(z\bar{s} + v, \varphi(z) + y \int_0^{\bar{s}} |z_{\varphi}|^{2m} d\varphi + \tau^{2m+2} + u_3 + |z\bar{s}|^{2m} \omega(v, z\bar{s})).
\]

The unique difference with Eq. 5.11 is that the term $u_3$ replaces the term $|z\bar{s}|^{2m} u_3$, and now $|u_3| \leq \sigma_\lambda (\bar{s} + m_\tau)^{2m+2}$.

The estimate from above for $L$ remains unchanged, because it does not involve $u_3$. In the estimate from below for $R$, we need the following evaluation for the term $u_3$:

\[
u_3 \geq -\sigma_\lambda (\bar{s} + m_\tau)^{2m+2} \geq -\sigma_\lambda |z|^{2m} \bar{s}^2 - \sigma_\lambda m_\tau^{2m+2}.
\]
Therefore, the inequality (5.14) remains unchanged and the proof can be concluded arguing as in the previous case.

6 Uniform Property of Entire Admissible epigraphs

We recall the definition of a uniform domain, specialized to the metric space \((\mathbb{R}^3, d)\).

**Definition 6.1** An open set \(\Omega \subset \mathbb{R}^3\) is a uniform domain if there exist \(\varepsilon > 0\) and \(\delta > 0\) with the following property. For any pair of points \(x, y \in \Omega\) there is a continuous curve \(\gamma : [0, 1] \rightarrow \Omega\) such that \(\gamma(0) = x, \gamma(1) = y,\)

\[
\text{diam}(\gamma) \leq \delta^{-1} d(x, y),
\]

and, letting \(\Delta_t = \min\{\text{diam}(\gamma|_{[0, t]}), \text{diam}(\gamma|_{[t, 1]})\}\), for any \(t \in [0, 1]\) we have

\[
B(\gamma(t), \varepsilon/\Delta_t) \subset \Omega.
\]

Uniform domains are also known as \((\varepsilon, \delta)\)-domains. As for John domains, the curves in our definition are not required to be rectifiable. By the results of [17], this is equivalent to the more standard definition which requires rectifiability.

We consider an unbounded domain of the epigraph type \(\Omega = \text{epi}(\varphi) = \{(z, t) \in \mathbb{R}^3 : t > \varphi(z)\}\), where \(\varphi \in C^\infty(\mathbb{R}^2)\) is an \(m\)-admissible function. For any pair of points \(p, q \in \Omega\), we construct a curve connecting them and satisfying the conditions (6.1) and (6.2) with uniform constants \(\delta\) and \(\varepsilon\). The case of a bounded domain is discussed in Section 7.

**Theorem 6.2** Let \(\varphi \in C^\infty(\mathbb{R}^2)\) be a function satisfying (1.2). Then, the epigraph \(\Omega = \text{epi}(\varphi)\) is a uniform domain.

**Proof** Let \(p = (z, \varphi(z) + b)\) and \(q = (\zeta, \varphi(\zeta) + \beta)\), with \(b, \beta > 0\), be points in the epigraph of \(\varphi\). We can without loss of generality assume that

\[
\frac{|ZF(z)|}{|z|^{2m}} = \max \left\{ \frac{|ZF(z)|}{|z|^{2m}}, \frac{|ZF(\zeta)|}{|\zeta|^{2m}} \right\} \geq 0.
\]

The maximum can be 0, even for arbitrarily close points. This happens for instance in Example 3.1. By assumption (1.2) we can define continuously \(\frac{|ZF(z)|}{|z|^{2m}} = 0\) for \(z = 0\).

Let \(\mu > 0\) be a parameter that will be fixed along the proof. We distinguish two cases:

\[
d(p, q) < \mu \max \left\{ \frac{|ZF(z)|}{|z|^{2m}}, \frac{|ZF(\zeta)|}{|\zeta|^{2m}} \right\} \quad \text{(Case A)};
\]

\[
d(p, q) \geq \mu \max \left\{ \frac{|ZF(z)|}{|z|^{2m}}, \frac{|ZF(\zeta)|}{|\zeta|^{2m}} \right\} \quad \text{(Case B)}.
\]

Roughly speaking, the maximum appearing in the right-hand side describes quantitatively “how much” the involved points are close to the characteristic set. In case A, where \(d(p, q)\) is much smaller than such maximum, the first pieces of the John curves from \(p\) and \(q\) are “parallel” and the curve realizing the uniform condition is constructed using the first pieces of \(\gamma_z\) and \(\gamma_\zeta\). In Case B, where the distance among \(p\) and \(q\) is large, the curve realizing the uniform condition is constructed using both pieces of the John curves starting from \(p\) and \(q\).
We can without loss of generality assume that Eq. 5.4 holds at the point $z$, i.e.: $-XF(z) = |XF(z)| > \frac{1}{2}|ZF(z)|$. Then, if we denote by $\varepsilon_0$ and $\lambda > 0$ the parameters fixed in Section 5, we know that the curve $\gamma_H > a$ suitable diam we know that the curve $z \mapsto |z|^2$, is a John curve with parameter $\lambda$.

**Analysis of Case A.** We claim that there exists $\mu > 0$ such that the curve

$$
\gamma_{\varepsilon}(s) = \begin{cases} (z_s, \varphi(z) + b + y \int_0^s |z|^{2m} \, dq), & \text{if } s \leq \bar{s} = \varepsilon_0 \frac{|ZF(s)|}{|z|^2} \\
(z_{\bar{s}}, \varphi(z) + b + y \int_0^{\bar{s}} |z|^{2m} \, dq + s - \bar{s}), & \text{if } s \geq \bar{s},
\end{cases}
$$

is a John curve with parameter $\lambda$. To prove this claim, it suffices to show that $-XF(\xi) > \frac{1}{2}|ZF(\xi)|$ if Case A holds and $\mu$ is small enough.

From Eq. 1.2 it follows that $|VZF(\xi)| \leq C|z|^{2m}$ for all $z \in \mathbb{R}^2$ and thus the function $z \mapsto |ZF(z)|/|z|^{2m}$ is globally Lipschitz continuous on $\mathbb{R}^2$. Let $L$ be the Lipschitz constant. By Eq. 6.3 and by the Case A with sufficiently small $\mu$, we have

$$
-\frac{XF(\xi)}{|\xi|^{2m}} \geq -\frac{XF(z)}{|z|^{2m}} - L|\xi - z| \geq \frac{1}{2} \frac{|ZF(z)|}{|z|^{2m}} - Ld(p, q) \geq \frac{1}{2} \frac{|ZF(z)|}{|z|^{2m}} - L\mu \frac{|ZF(z)|}{|z|^{2m}}
$$

$$
\geq \frac{1}{4} \frac{|ZF(z)|}{|z|^{2m}} \geq \frac{1}{4} \frac{|ZF(\xi)|}{|\xi|^{2m}}.
$$

Also the mapping $z \mapsto \tilde{s}(z)$ in Eq. 5.3 is Lipschitz continuous. Then, for $\mu$ small enough, in the Case A the times $\bar{s} = \tilde{s}(z)$ and $\tilde{s} = \tilde{s}(\xi)$ satisfy

$$
\frac{1}{2} \bar{s} \leq \tilde{s} \leq \frac{3}{2} \bar{s}.
$$

(6.4) Finally, we also have $|\xi - z| \leq d(p, q) \leq \mu \frac{|ZF(z)|}{|z|^{2m}} \leq C\mu |z| \leq \frac{1}{2} |z|$, for $\mu$ sufficiently small.

We are now ready to define the curve joining $p$ and $q$ and satisfying (6.1) and (6.2). For a suitable $H > 0$, let

$$
\hat{s} = H d(p, q).
$$

(6.5) Then, the curve $\gamma$ is the concatenation of $\gamma_{\varepsilon}|_{[0, \bar{s}]}$, a length-minimizing path $\gamma$ joining $\hat{y}(0) = \gamma_{\varepsilon}(\bar{s})$ and $\hat{y}(1) = \gamma_{\varepsilon}(\tilde{s})$, and the opposite of $\gamma_{\varepsilon}|_{[0, \bar{s}]}$.

We claim that there exist $H > 0$ and $\mu > 0$ such that the curve $\gamma$ satisfies (6.1) and (6.2). We preliminarily show that:

(i) $\hat{s} \leq \min\{\bar{s}, \tilde{s}\}$, i.e., the points $\gamma_{\varepsilon}(\bar{s})$ and $\gamma_{\varepsilon}(\tilde{s})$ belong to the first piece of the curves $\gamma_{\varepsilon}$ and $\gamma_{\varepsilon}$, respectively;

(ii) $d(\gamma_{\varepsilon}(\bar{s}), \gamma_{\varepsilon}(\tilde{s})) \leq \frac{1}{2} \lambda \hat{s}$, where $\lambda$ is the John constant of $\gamma_{\varepsilon}$ and $\gamma_{\varepsilon}$;

(iii) $\text{diam}(\gamma) \leq C d(p, q)$.

Condition (6.1) is (iii). We show that (i)–(iii) imply (6.2). For $s \leq \hat{s}$, by $\Delta_\varepsilon \leq \text{diam}(\gamma_{\varepsilon}|_{[0, \varepsilon]})$ and by the cone property (5.1) we have

$$
B(\gamma(s), \lambda \Delta_\varepsilon) \subset B(\gamma_{\varepsilon}(s), \lambda \text{diam}(\gamma_{\varepsilon}|_{[0, \varepsilon]})) \subset \text{epi}(\varphi).
$$
Then Eq. 6.2 holds with $\epsilon = \lambda$. The same happens for points $y_\xi(s)$ with $s \leq \hat{s}$. Finally, for a point $\hat{\gamma}(s^\ast)$ in the intermediate part, by (ii) we have

$$
\text{dist}(\hat{\gamma}(s^\ast), \text{gr}(\varphi)) \geq \text{dist}(y_\xi(\hat{s}), \text{gr}(\varphi)) - \frac{\lambda}{2}\hat{s} \\
\geq \lambda \text{diam}(y_\xi\vert_{[0,\hat{s}]}) - \frac{\lambda}{2}\text{diam}(y_\xi\vert_{[0,\hat{s}]}) = \frac{\lambda}{2}\text{diam}(y_\xi\vert_{[0,\hat{s}]}). \quad (6.6)
$$

In order to get a lower bound for the last diameter, we use the length-minimizing property of $\hat{\gamma}$ and property (ii), which give

$$
\text{diam}(\hat{\gamma}[0,s^\ast]) \leq d(\hat{\gamma}(0), \hat{\gamma}(1)) \leq \frac{\lambda}{2}\hat{s} \leq \frac{\lambda}{2}\text{diam}(y_\xi\vert_{[0,\hat{s}]}).
$$

Therefore, we have $\text{diam}(y_\xi\vert_{[0,\hat{s}]} + \hat{\gamma}\vert_{[0,s^\ast]}) \leq 2\text{diam}(y_\xi\vert_{[0,\hat{s}]})$ and then it is easy to conclude that Eq. 6.2 holds with $\epsilon = \frac{\lambda}{2}\hat{s}$.

Now we prove (i). By Eq. 6.4 this is implied by $\hat{s} \leq \frac{1}{2}\hat{s}$. By Eq. 6.5, Case A, Eq. 6.3, we have $\hat{s} \leq H\mu \frac{|ZF(z)|}{|z|^{2m}} = H\mu \frac{\hat{s}}{\hat{s}_0}$. Thus, we deduce that (i) holds provided that

$$
H\mu \leq \frac{1}{2}\epsilon_0. \quad (6.7)
$$

This is the first requirement on $H$ and $\mu$. This restriction is compatible with further conditions made below.

Next we prove (ii). Theorem 1.4 gives

$$
d(y_\xi(\hat{s}), y_\xi(\hat{s})) \leq C_0|\xi - z| + C_0 \min \left\{ \frac{|\Theta|^{1/2}}{|z|^{m}}, \frac{|\Theta|^{1/2}}{|z|^{m/2}} \right\} \quad (6.8)
$$

where

$$
\Theta = \varphi(\xi) - \varphi(z) + b - \int_0^{\hat{s}} (\eta|\xi_0|^2 - y|z_0|^2) d\eta + |\hat{s}|^2 \omega(z, \xi, \hat{s}).
$$

Let $\Theta = \Theta_1 + \Theta_2 + \Theta_3$, with

$$
\Theta_1 = \varphi(\xi) - \varphi(z) + b - |z|^{2m} \omega(z, \xi),
\Theta_2 = |z|^{2m} \omega(z, \xi) - |z|^{2m} \omega(z, \xi),
\Theta_3 = \int_0^{\hat{s}} (\eta|\xi_0|^2 - y|z_0|^2) d\eta.
$$

The first term in the right-hand side of Eq. 6.8 can be estimated as follows

$$
C_0|\xi - z| \leq C_0 d(p, q) \leq \frac{\lambda}{8} H d(p, q) = \frac{\lambda}{8}\hat{s},
$$

as soon as $H$ is large enough to ensure that

$$
C_0 \leq \frac{\lambda}{8} H, \quad (6.9)
$$

where $C_0$ is the absolute constant in Eq. 6.8. We used definition (6.5) of $\hat{s}$.

Concerning the second term in the right-hand side of Eq. 6.8, we claim that for all $j = 1, 2, 3$ we have

$$
C_0 \min \left\{ \frac{|\Theta_j|^{1/2}}{|z|^{m/2}}, \frac{|\Theta_j|^{1/2}}{|z|^{m}} \right\} \leq \frac{\lambda}{8} H d(p, q). \quad (6.10)
$$

By Theorem 1.4, we have

$$
C_0 \min \left\{ \frac{|\Theta_1|^{1/2}}{|z|^{m/2}}, \frac{|\Theta_1|^{1/2}}{|z|^{m}} \right\} \leq C_0 \delta(p, q) \leq C C_0 d(p, q) \leq \frac{\lambda}{8} H d(p, q),
$$

$$
\text{Springer}
$$
as soon as $H$ is large enough so that
\[ CC_0 \leq \frac{\lambda}{8} H. \]  
(6.11)

To evaluate the term with $\Theta_2$, we apply the inequalities
\[
|\Theta_2| = \left| \left( |z|^{2m} - |\zeta|^{2m} \right) \omega(z, \zeta) + |z|^{2m} \hat{s}(\eta - y) \right| \leq C |z|^{2m} |\zeta - z| \hat{s}
\]
\[
\leq C |z|^{2m} d(p, q) \hat{s} = C |z|^{2m} H d(p, q)^2.
\]
Thus, we deduce that for some absolute constant $C_2 > 0$ we have
\[
\frac{|\Theta_2|^{1/2}}{|z|^m} \leq C_2 d(p, q) \sqrt{H} \leq C_0^{-1} \frac{\lambda}{8} H d(p, q)
\]
as soon as
\[ C_2 \leq C_0^{-1} \frac{\lambda}{8} \sqrt{H}. \]  
(6.12)

Finally, we estimate $\Theta_3$:
\[
|\Theta_3| \leq |y - \eta| \int_0^{\hat{s}} |z|^{2m} d\varrho + |\eta| \int_0^{\hat{s}} (|z|^{2m} - |\zeta|^{2m}) d\varrho
\]
\[
\leq C\hat{s}|z|^{2m} \delta(p, q) + C\hat{s}|\eta||z|^{2m-1} |z - \zeta|
\]
\[
\leq C\hat{s}|z|^{2m} d(p, q) = C |z|^{2m} d(p, q)^2 H,
\]
and we end up with again with the requirement (6.12).

To conclude the proof, we choose $H > 0$ large enough so that Eqs. 6.9, 6.11 and 6.12 hold. This implies (ii). Then we choose $\mu > 0$ such that Eq. 6.7 holds. This implies (i). The diameter estimate in (iii) holds in terms of such constants and the proof of Case A is concluded.

**Analysis of Case B.** Let us consider the second piece of the curve from $(z, \varphi(z) + b)$,
\[
\gamma_z(s) = \left( z \bar{t} + b + y \int_0^{\bar{s}} |z|^{2m} d\varrho + s - \bar{s} \right) \text{ for } s \geq \bar{s} = \varepsilon_0 \frac{|ZF(z)|}{|z|^{2m}}.
\]
Let also $(\zeta, \varphi(\zeta) + \beta)$ be such that Case B holds. Then, there is a unit vector $w = (u, v) \in \mathbb{R}^2$ such that the curve $\gamma_z$ is a $\lambda$-John curve in $\text{epi}(\varphi)$ starting from $(\zeta, \varphi(\zeta) + \beta)$. When $s \geq \hat{s} = \varepsilon_0 |ZF(\zeta)|/|\zeta|^{2m}$, the curve is
\[
\gamma_z(s) = \left( \zeta + \hat{s}w, \varphi(\zeta) + \beta + \omega(w, \zeta) \int_0^{\hat{s}} |\zeta + \varphi w|^{2m} d\varrho + s - \hat{s} \right).
\]
Note that the numbers $\bar{s}$ and $\hat{s}$ could both vanish. Furthermore, we will assume without loss of generality that $\text{diam} \left( \gamma_z|_{[0, \bar{s}]} \right) \geq \text{diam} \left( \gamma_z|_{[0, \hat{s}]} \right)$.

For $\tau \geq 0$ consider the points $\gamma_z(\bar{s}_z)$ and $\gamma_z(\hat{s}_z)$, where
\[
\bar{s}_z = \bar{s} + \tau^{2m+2}, \quad \hat{s}_z = \hat{s} + \tau^{2m+2}.
\]

We claim that there exists $M > 0$ such that for all $p = (z, \varphi(z)+b)$ and $q = (\zeta, \varphi(\zeta)+\beta)$ for which Case B holds, if $\tau \geq 0$ satisfies
\[
\text{diam} \left( \gamma_z|_{[0, \bar{s}_z]} \right) = \max \left\{ \text{diam} \left( \gamma_z|_{[0, \bar{s}]} \right), \text{diam} \left( \gamma_z|_{[0, \hat{s}]} \right) \right\} = Md(p, q),
\]  
(6.13)
then we have
\[
d \left( \gamma_z(\bar{s}_z), \gamma_z(\hat{s}_z) \right) \leq \frac{\lambda}{2} \text{diam} \left( \gamma_z|_{[0, \bar{s}_z]} \right),
\]  
(6.14)
where $\lambda$ is the John constant of the curves.
Notice that for any $M, p, q$ there is always a $\tau$ such that Eq. 6.13 holds because the left-hand side of Eq. 6.13 is increasing in $\tau$ and tends to $+\infty$, as $\tau \to +\infty$.

We prove the claim. By the invariance of the distance with respect to vertical translations we have

$$d\left(\gamma_\tau(s + \tau^{2m+2}), \gamma_\tau(s) + \tau^{2m+2}\right) = d(\gamma_\tau(s), \gamma_\tau(s)) \leq d(\gamma_\tau(s), \gamma_\tau(0)) + d(\gamma_\tau(0), \gamma_\tau(0))$$

$$\leq \varepsilon_0 \frac{|ZF(z)|}{|\xi|^{2m}} + d(p, q) + \varepsilon_0 \frac{|ZF(\xi)|}{|\xi|^{2m}} \leq \frac{2\varepsilon_0}{\mu} d(p, q) + d(p, q)$$

$$= \frac{1}{M} \left( \frac{2\varepsilon_0}{\mu} + 1 \right) \max\{\operatorname{diam}(\gamma_\tau[0,\xi]), \operatorname{diam}(\gamma_\tau[0,\xi])\},$$

by Eq. 6.13. Thus Eq. 6.14 holds if $M$ is large enough, and the claim is proved.

To conclude the proof, we show that the path $\gamma_\tau$, given by the concatenation of $\gamma_\tau[0,\xi]$, a length minimizing path $\hat{\gamma}$ connecting $\hat{\gamma}(0) = \gamma_\tau(\xi)$ and $\hat{\gamma}(1) = \gamma_\tau(\xi)$ and the reverse of $\gamma_\tau[0,\xi]$ satisfies the $(\varepsilon, \delta)$-condition. Since the diameter estimate (6.1) is contained in the claim above, we are left with the proof of Eq. 6.2.

Let $q$ be a point of $\gamma$. If $q = \gamma_\tau(s)$ with $s \leq \xi$, or $q = \gamma_\tau(s)$ with $s \leq \xi$, then Eq. 6.2 follows with $\varepsilon = \lambda$ from the John property (5.1). If $q = \gamma(s^*)$, then we argue as in Eq. 6.6. precisely

$$\operatorname{dist}(\hat{\gamma}(s^*), \operatorname{gr} \varphi) \geq \operatorname{dist}(\gamma_\tau(\xi), \operatorname{gr} \varphi) - d(\hat{\gamma}(s^*), \gamma_\tau(\xi)) \geq \lambda \operatorname{diam}(\gamma_\tau[0,\xi]) - \operatorname{diam}(\hat{\gamma}) \geq \frac{\lambda}{2} \operatorname{diam}(\gamma_\tau[0,\xi]),$$

by Eq. 6.14. Finally, to get a lower estimate of the latter diameter with $\operatorname{diam}(\gamma_\tau[0,\xi] + \gamma_\tau[0,\xi^*])$, which will give the John property, it suffices to use the length minimizing property of $\hat{\gamma}$

$$\operatorname{diam}(\hat{\gamma}(0), \hat{\gamma}(1)) \leq \frac{\lambda}{2} \operatorname{diam}(\gamma_\tau[0,\xi]) + \operatorname{diam}(\gamma_\tau[0,\xi^*]).$$

Thus, as in Case A, we get the correct lower bound for the last line of Eq. 6.15 and the proof is easily concluded.

\section{7 Bounded Admissible Domains are Uniform}

In this section we prove Theorems 1.2 and 1.3 in the case of a bounded $m$-admissible domain. Now we assume that $m \in \mathbb{N}$ is an integer.

\textbf{Proof of Theorem 1.2} Let $\Omega \subset \mathbb{R}^3$ be an $m$-admissible domain. By a standard localization argument (see e.g. [15, Proposition 2.5]), it suffices to show that for all $p_0 \in \partial \Omega$ there is a neighborhood $A_{p_0}$ of $p_0$ in $\mathbb{R}^3$ such that for all $p, q \in A_{p_0}$ there is a continuous curve $\gamma : [0, 1] \to \Omega \cap A_{p_0}$ satisfying $\gamma(0) = p$ and $\gamma(1) = q$ and such that Eqs. 6.1 and 6.2 hold.

There are two cases:

1. $p_0$ is a noncharacteristic point, i.e., $\operatorname{span}\{X(p_0), Y(p_0)\}$ is not contained in $T_{p_0} \partial \Omega$.
2. $p_0$ is a characteristic point of $\partial \Omega$. 

Springer
In Case 1, the claim is proved in [15, Theorem 1.1]. To use this result, we need a $C^\infty$ boundary and smooth vector fields. For this reason we require $m \in \mathbb{N}$.

In the Case 2, in a neighborhood of $p_0$ the boundary of $\Omega$ is a graph of the type $t = \phi(z)$ for an $m$-admissible function $\phi \in C^\infty(D)$ for some open set $D \subset \mathbb{R}^2$. The claim is proved in Sections 5 and 6.

**Proof of Theorem 1.3** By compactness, we can cover $\partial \Omega$ with a finite union of $m$-admissible graphs, together with a compact subset $K \subset \partial \Omega$ containing only noncharacteristic points.

At points $p \in K$, the Ahlfors estimates (1.5) is proved in [13, Corollary 1]. To use this result, we need a smooth boundary and smooth vector fields ($m \in \mathbb{N}$).

On $m$-admissible graphs, the Ahlfors estimate is proved in Section 4.

**References**

1. Capogna, L., Garofalo, N.: Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics. J. Fourier Anal. Appl. **4**(4-5), 403–432 (1998)
2. Capogna, L., Garofalo, N.: Ahlfors type estimates for perimeter measures in Carnot-Carathéodory spaces. J. Geom. Anal. **3**, 455–497 (2006)
3. Danielli, D., Garofalo, N., Nhieu, D.-M.: Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot-Carathéodory spaces. Mem. Amer. Math. Soc. **182**(857), x+119 (2006)
4. Franchi, B., Lu, G., Wheeden, R.L.: A relationship between Poincaré-type inequalities and representation formulas in spaces of homogeneous type. Internat. Math. Res. Notices **1**, 1–14 (1996)
5. Garofalo, N., Nhieu, D.-M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. Comm. Pure Appl. Math. **49**(10), 1081–1144 (1996). MR 1404326 (97i:58032)
6. Garofalo, N., Nhieu, D.M.: Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces. J. Anal. Math. **74**, 67–97 (1998)
7. Gerossi, D., Monti, R., Morbidelli, D.: Trace theorems for Martinet-type vector fields in the characteristic case. Commun. Contemp. Math. https://doi.org/10.1142/S0219199719500664
8. Gresson, A.V.: On uniform and NTA-domains on Carnot groups. Siberian Math. J. **42**(5), 851–864 (2001)
9. Gresson, A.V.: The geometry of cc-balls and constants in the ball-box theorem on Heisenberg group algebras. Sib. Math. J. **55**(5), 849–865 (2014)
10. Gresson, A.V.: The uniformity of cc-balls on a class of 2-step Carnot groups. (Russian. English Summ.) Sib. Elektron. Mat. Izv. **15**, 1182–1197 (2018)
11. Hajlasz, P., Koskela, P.: Sobolev met Poincaré. Mem. Amer. Math. Soc. **145**(688), x+101 (2000)
12. Jerison, D.a.: The geometry of cc-balls and constants in the ball-box theorem on Heisenberg group algebras. Sib. Math. J. **55**(5), 849–865 (2014)
13. Monti, R., Morbidelli, D.: Trace theorems for vector fields. Math. Z. **239**(4), 747–776 (2002)
14. Monti, R., Morbidelli, D.: John domains for the control distance of diagonal vector fields. J. Anal. Math. **92**, 259–284 (2004)
15. Monti, R., Morbidelli, D.: Non-tangentially accessible domains for vector fields. Indiana Univ. Math. J. **54**(2), 473–498 (2005). MR 2136818
16. Monti, R., Morbidelli, D.: Regular domains in homogeneous groups. Trans. Amer. Math. Soc. **357**(8), 2975–3011 (2005)
17. Martio, O., Sarvas, J.: Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A I Math. **4**(2), 383–401 (1979)
18. Monti, R., Serra Cassano, F.: Surface measures in Carnot-Carathéodory spaces. Calc. Var. Partial Differ. Equ. **13**(3), 339–376 (2001)
19. Negrini, P., Scornazzani, V.: Wiener criterion for a class of degenerate elliptic operators. J. Differ. Equ. **66**(2), 151–164 (1987)

---

944

R. Monti, D. Morbidelli

---

Springer
20. Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields. I. Basic properties. Acta Math. 155(1-2), 103–147 (1985)

21. Romanovskii, N.N.: Integral representations and embedding theorems for functions defined on the Heisenberg groups $H^n$. St. Petersburg Math. J. 16(2), 349–375 (2005)

22. Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities. Internat. Math. Res. Notices 2, 27–38 (1992)

23. Vodop’yanov, S.K., Greshnov, A.V.: On the continuation of functions of bounded mean oscillation on spaces of homogeneous type with intrinsic metric. Siberian Math. J. 36(5), 873–901 (1995)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.