Sobolev mappings and moduli inequalities on Carnot groups

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Abstract. We study the mappings that satisfy moduli inequalities on Carnot groups. We prove that the homeomorphisms satisfying the moduli inequalities ($Q$-homeomorphisms) with a locally integrable function $Q$ are Sobolev mappings. On this base in the frameworks of the weak inverse mapping theorem, we prove that, on the Carnot groups $G$, the mappings inverse to Sobolev homeomorphisms of finite distortion of the class $W^{1}_{
u,\text{loc}}(\Omega;\Omega')$ belong to the Sobolev class $W^{1}_{1,\text{loc}}(\Omega';\Omega)$.

Keywords. Sobolev spaces, moduli inequalities, Carnot group.

1. Introduction

It is known that the Sobolev mappings on Carnot groups $G$ cannot be characterized only in terms of its coordinate functions. The basic approach to the theory of Sobolev mappings on Carnot groups is based on the notion of absolutely continuity on almost all horizontal lines, which allows one to define a weak upper gradient of mappings. In the present work, we prove that the homeomorphisms satisfying the moduli inequalities on Carnot groups are Sobolev mappings. On this base, we obtain a weak version of the inverse mapping theorem on Carnot groups. Namely, we prove that the mappings inverse to Sobolev homeomorphisms of finite distortion of the class $W^{1}_{\nu,\text{loc}}(\Omega;\Omega')$ are Sobolev mappings of the class $W^{1}_{1,\text{loc}}(\Omega';\Omega)$. The problem of regularity of the mappings inverse to Sobolev homeomorphisms represents a significant part of the weak inverse mapping theorem and was studied in [50] for a bi-measurable Sobolev homeomorphism $\varphi: \Omega \rightarrow \Omega'$, $\Omega, \Omega' \subset \mathbb{R}^n$ of the class $W^{1}_{p}(\Omega;\Omega')$, $p > n-1$. In [38], it was proved that the homeomorphism inverse to $\varphi \in L^{1}_{p}(\Omega;\Omega')$, $p > n-1$, satisfies $\varphi^{-1} \in BV_{\text{loc}}(\Omega';\Omega)$.

In the last decades, the regularity of mappings inverse to Sobolev homeomorphisms was intensively studied in the frameworks of the non-linear elasticity theory [1], see, for example, [8, 14, 17, 18, 29].

The suggested approach on Carnot groups is based on the moduli inequalities, namely on the notion of $Q$-mappings introduced in [24] (see also [25–26]). Recall that a homeomorphism $\varphi: \Omega \rightarrow \Omega'$ of domains $\Omega, \Omega' \subset G$ is called a $Q$-homeomorphism with a non-negative measurable function $Q$, if the inequality

$$M(\varphi\Gamma) \leq \int_{\Omega} Q(x) \cdot \rho'(x) dx$$

holds for every family $\Gamma$ of rectifiable paths in $\Omega$ and every admissible function $\rho$ for $\Gamma$.

For the Euclidean space $\mathbb{R}^n$, it was proved [25] that a homeomorphism $\varphi \in W^{1}_{\nu,\text{loc}}(\Omega)$ such that $\varphi^{-1} \in W^{1}_{n,\text{loc}}$ is a $Q$-mapping with $Q = K_{f}(x, \varphi)$, where $K_{f}(x, \varphi)$ is the inner quasiconformal dilatation of $\varphi$. The systematic applications of the moduli theory to the geometric mapping theory can be found in [27].

The main result of the present work concerns the weak differentiability of mappings satisfying the moduli inequalities on Carnot groups (Theorem 5.1). The proof is based on the capacity estimates.
and the Fubini-type decomposition of measures associated with horizontal foliations defined by left-invariant vector fields and moduli (capacity) inequalities on Carnot groups.

Using the property of weak differentiability and connection between Sobolev mappings and moduli inequalities, we prove the weak regularity of Sobolev homeomorphisms on Carnot groups: if \( \varphi : \Omega \to \Omega' \) is a Sobolev homeomorphism of finite distortion of the class \( W^{1,loc}_V(\Omega; \Omega') \), then the inverse mapping \( \varphi^{-1} \in W^{1,loc}_V(\Omega'; \Omega) \).

The weak differentiability is a part of the analytic definition of quasiconformal mappings and mappings of bounded distortion (see, e.g., [31] and [23]). The ACL-property of \( Q \)-mappings defined on planar domains of the Euclidean space \( \mathbb{R}^2 \) was considered by Brakalova and Jenkins, who proved this property for solutions of the Beltrami equations in a plane (see [3, Lemma 3]). Under the assumption that \( Q \in L^{1}_{loc} \), the ACL-property was proved in \( \mathbb{R}^n \) for the \( Q \)-homeomorphisms (see [32]) and for the mappings with branching (see, e.g., [33,34]).

\( Q \)-homeomorphisms are closely connected with the mappings that generate bounded composition operators on Sobolev spaces \( p,q \)-quasiconformal mappings) [12,36,47,48] which were studied on Carnot groups in [39,40,47,49]. In the recent decade, the geometric theory of composition operators on Sobolev spaces was applied to spectral estimates of the Laplace operator in Euclidean non-convex domains (see, e.g., [5,6,11,13,15,16]). So, the results of this article have applications to the Sobolev mappings theory, to the spectral theory of (sub)elliptic operators, and to the non-linear elasticity problems associated with vector fields that satisfy the Hörmander’s hypoellipticity condition.

## 2. Sobolev mappings on Carnot groups

### 2.1 Carnot groups

Recall that a stratified homogeneous group [10] or, in another terminology, a Carnot group [30] is a connected (simply connected) nilpotent Lie group \( G \) whose Lie algebra \( V \) is decomposed into the direct sum \( V_1 \oplus \cdots \oplus V_m \) of vector spaces such that \( \dim V_i \geq 2 \), \( [V_i, V_{i+1}] = V_{i+1} \) for \( 1 \leq i \leq m-1 \) and \( [V_1, V_m] = \{0\} \). Let \( X_{11}, \ldots, X_{1n_1} \) be left-invariant basis vector fields of \( V_1 \). Since they generate \( V \) for each \( i \), \( 1 < i \leq m \), one can choose a basis \( X_{ik} \) in \( V_i \), \( 1 \leq k \leq n_i \). Each \( X_{ik} \) is generated by the exponential map

\[
X_{ik} = \exp(t \sum_{j=1}^{m} X_{ij} X_{ik}),
\]

are automorphisms of \( G \) for each \( t > 0 \). The Lebesgue measure \( dx \) on \( \mathbb{R}^N \) is a bi-invariant Haar measure on \( G \) (which is generated by the Lebesgue measure by means of the exponential map), and \( d(\delta_t x) = t^\nu \, dx \), where the number \( \nu = \sum_{i=1}^{m} in_i \) is called the homogeneous dimension of the group \( G \).

The measure \( |E| \) of a measurable subset \( E \) of \( G \) is defined by

\[
|E| = \int_E \, dx.
\]

The system of basis vectors \( X_1, X_2, \ldots, X_n \) of the space \( V_1 \) (here and below, we set \( n_1 = n \) and \( X_{i1} = X_i \), where \( i = 1, \ldots, n \)) satisfies the Hörmander hypoellipticity condition.

The Euclidean space \( \mathbb{R}^n \) with the standard structure is an example of an Abelian group: the vector fields \( \partial / \partial x_i, i = 1, \ldots, n \), have no non-trivial commutation relations and form the basis of the

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corresponding Lie algebra. One example of a non-Abelian stratified group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$hh' = (x, y, z)(x', y', z') = (x + x', y + y', z + z' - 2xy' + 2yx'),$$

where $x, x', y, y', z, z' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}$. The left translation $L_h(\cdot)$ is defined as $L_h(h') = hh'$. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial z}, \quad i = 1, ..., n, \quad Z = \frac{\partial}{\partial z},$$

constitute the basis of the Lie algebra $V$ of the Heisenberg group $\mathbb{H}^n$. All non-trivial relations are only of the form $[X_i, Y_i] = -4Z$, $i = 1, ..., n$, and all other commutators vanish.

The Lie algebra of the Heisenberg group $\mathbb{H}^n$ has dimension $2n + 1$ and splits into the direct sum $V = V_1 \oplus V_2$. The vector space $V_1$ is generated by the vector fields $X_i, Y_i, i = 1, ..., n$, and the space $V_2$ is the one-dimensional center which is spanned by the vector field $Z$.

Recall that a homogeneous norm on the group $G$ is a continuous function $| \cdot | : G \to [0, \infty)$ that is $C^\infty$-smooth on $G \setminus \{0\}$ and has the following properties:

(a) $|x| = |x^{-1}|$ and $|\delta_t(x)| = t|x|$

(b) $|x| = 0$, if and only if $x = 0$

(c) there exists a constant $\tau_0 > 0$ such that $|x_1x_2| \leq \tau_0(|x_1| + |x_2|)$ for all $x_1, x_2 \in G$.

The homogeneous norm on the group $G$ defines a homogeneous (quasi)metric

$$\rho(x, y) = |y^{-1}x|.$$

Note that a continuous map $\gamma : [a, b] \to G$ is called a continuous curve on $G$. This continuous curve is rectifiable, if

$$\sup \left\{ \sum_{k=1}^{m} |(\gamma(t_k))^{-1} \gamma(t_{k+1})| \right\} < \infty,$$

where the supremum is taken over all partitions $a = t_1 < t_2 < ... < t_m = b$ of the segment $[a, b]$.

In [30], it was proved that any rectifiable curve is differentiable almost everywhere, and $\dot{\gamma}(t) \in V_1$:

$$\dot{\gamma}(t) = \sum_{i=1}^{n} a_i(t)X_i(\gamma(t))$$

for all $t \in (a, b)$. The length $l(\gamma)$ of a rectifiable curve $\gamma : [a, b] \to G$ can be calculated by the formula

$$l(\gamma) = \int_{a}^{b} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_0^{\frac{1}{2}} dt = \int_{a}^{b} \left( \sum_{i=1}^{n} |a_i(t)|^2 \right)^{\frac{1}{2}} dt,$$

where $\langle \cdot, \cdot \rangle_0$ is the inner product on $V_1$. The result of [7] implies that one can connect two arbitrary points $x, y \in G$ by a rectifiable curve. The Carnot–Carathéodory distance $d(x, y)$ is the infimum of the lengths over all rectifiable curves with endpoints $x$ and $y$ in $G$. The Hausdorff dimension of the metric space $(G, d)$ coincides with the homogeneous dimension $\nu$ of the group $G$. 756
2.2 Sobolev spaces on Carnot groups

Let $\mathbb{G}$ be a Carnot group with one-parameter dilatation group $\delta_t$, $t > 0$, and a homogeneous norm $\rho$, and let $E$ be a measurable subset of $\mathbb{G}$. The Lebesgue space $L_p(E), p \in [1, \infty]$, is the space of measurable $p$th-power integrable functions $f : E \to \mathbb{R}$ with the standard norm:

$$
\|f\|_{L_p(E)} = \left( \int_E |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

and $\|f\|_{L_\infty(E)} = \text{ess sup}_E |f(x)|$ for $p = \infty$. We denote, by $L_{p,\text{loc}}(E)$, the space of functions $f : E \to \mathbb{R}$ such that $f \in L_p(F)$ for each compact subset $F$ of $E$.

Let $\Omega$ be an open set in $\mathbb{G}$. The (horizontal) Sobolev space $W^1_p(\Omega), 1 \leq p \leq \infty, (L^1_p(\Omega), 1 \leq p \leq \infty)$ consists of the functions $f : \Omega \to \mathbb{R}$ locally integrable in $\Omega$, having a weak derivatives $X_i f$ along the horizontal vector fields $X_i, i = 1, \ldots, n$, and a finite (semi)norn

$$
\|f\|_{W^1_p(\Omega)} = \|f\|_{L_p(\Omega)} + \|\nabla_H f\|_{L_p(\Omega)},
$$

where $\nabla_H f = (X_1 f, \ldots, X_n f)$ is the horizontal subgradient of $f$. If $f \in W^1_p(U)$ for each bounded open set $U$ such that $\overline{U} \subset \Omega$, then we say that $f$ belongs to the class $W^1_{p,\text{loc}}(\Omega)$.

Let $\varphi : \Omega \to \mathbb{G}$ be a mapping defined on the open set $\Omega \subset \mathbb{G}$. A Lie group homomorphism $\psi : \mathbb{G} \to \mathbb{G}$ such that $\exp^{-1} \circ \psi \circ \exp(V_1) \subset V_1$ is called the $P$-differential of $\varphi$ at the point $a$ of the set $\Omega$, if the set

$$
A_a = \{z \in E : d(\psi(a^{-1}x)^{-1}\varphi(a^{-1}\varphi(x))) < \varepsilon d(a^{-1}x)\}
$$

is a neighborhood of $a$ (relative to $\Omega$) for every $\varepsilon > 0$. The notion of $P$-differentiability was introduced in [30], where it was proved that the Lipschitz mappings defined on open subsets of Carnot groups are $P$-differentiable almost everywhere. The Stepanov-type theorem on Carnot groups was obtained in [46] (see also [42]), where it was proved that the Lipschitz mappings defined on measurable subsets of Carnot groups are (approximately) $P$-differentiable almost everywhere.

We say that a mapping $\varphi : \Omega \to \mathbb{G}$ is absolutely continuous on lines ($\varphi \in \text{ACL}(\Omega; \mathbb{G})$), if, for each domain $U$ such that $\overline{U} \subset \Omega$ and each foliation $\Gamma_i$ defined by a left-invariant vector field $X_i, i = 1, \ldots, n$, $\varphi$ is absolutely continuous on $\gamma \cap U$ with respect to the one-dimensional Hausdorff measure for $d\gamma$-almost every curve $\gamma \in \Gamma_i$. Recall that the measure $d\gamma$ on a foliation $\Gamma_i$ equals the inner product $i(X_i)dx$ of the vector field $X_i$ and the bi-invariant volume $dx$ (see, e.g., [9, 46]).

Since $X_i \varphi(x) \in V_1$ for almost all $x \in \Omega$ [30], $i = 1, \ldots, n$, the linear mapping $D_H \varphi(x)$ with the matrix $(X_i \varphi_j(x)), i, j = 1, \ldots, n$, takes the horizontal subspace $V_1$ to $V_1$ and is called the formal horizontal differential of the mapping $\varphi$ at $x$. Let $|D_H \varphi(x)|$ be its norm:

$$
|D_H \varphi(x)| = \sup_{\xi \in V_1 : \xi = 1} |D_H \varphi(x)(\xi)|.
$$

We say that a mapping $\varphi : \Omega \to \mathbb{G}$ belongs to $\text{ACL}_p(\Omega; \mathbb{G})$ ($\text{ACL}_{p,\text{loc}}(\Omega; \mathbb{G})$), if $\varphi \in \text{ACL}(\Omega; \mathbb{G})$ and $|D_H \varphi| \in L_p(\Omega)$ ($|D_H \varphi| \in L_{p,\text{loc}}(\Omega)$).

Smooth mappings with differentials respecting the horizontal structure are said to be contact. For this reason, one could say that mappings in the class $\text{ACL}(\Omega; \mathbb{G})$ are (weakly) contact. It was proved in [42, 46] that a formal horizontal differential $D_H : V_1 \to V_1$ induces a homomorphism $D \varphi : V \to V$ of Lie algebras which is called the formal differential. The determinant of the matrix $D \varphi(x)$ is called the (formal) Jacobian of the mapping $\varphi$, and it is denoted by $J(x, \varphi)$.
The definition of Sobolev mappings in terms of Lipschitz functions was introduced in [37, 42]. Let \( \Omega \) be a domain in the stratified group \( G \). The mapping \( \varphi : \Omega \to G \) belongs to \( W^{1,\text{loc}}_p(\Omega; G) \), if, for each function \( f \in \text{Lip}(G) \), the composition \( f \circ \varphi \) belongs to \( W^{1,\text{loc}}_p(\Omega) \) and \( |\nabla_H (f \circ \varphi)|(x) \leq \text{Lip} \cdot g(x) \), where \( g \in L^p_{\text{loc}}(\Omega) \) is independent of \( f \). The function \( g \) is called the upper gradient of the mapping \( \varphi \).

3. Foliations and Set Functions

3.1 The Fubini-type decomposition

We consider the families \( \Gamma_k \) of orbits of horizontal vector fields \( X_{1k} \in V_1 \), \( 1 \leq k \leq n_1 \), generating smooth foliations of a domain \( \Omega \subset G \). Denote the flow corresponding to the vector field \( X_{1k} \) by the symbol \( f_t \). Then each fiber has the form \( \gamma(t) = f_t(s) \), where \( s \) belongs to the surface \( S_k \) transversal to \( X_{1k} \) and a parameter \( t \in \mathbb{R} \).

We suppose that the foliation \( \Gamma_k \) of \( \Omega \) is furnished with a measure \( d\gamma \) satisfying the inequality

\[
\int_{\gamma \in \Gamma, \gamma \cap B(x,r) \neq \emptyset} d\gamma \leq c_2 |B(x,r)|^{\frac{n-1}{n}}
\]

(3.1)

for sufficiently small balls \( B(x,r) \subset \Omega \), where constants \( c_1 \) and \( c_2 \) are independent on balls \( B(x,r) \).

The measure \( d\gamma \) can be obtained [46] as the interior multiplication \( i(X_{1k}) \) of the vector field \( X_{1k} \) with the bi-invariant volume form \( dx \). Let \( J_{f_t} \) be a Jacobian of the flow \( f_t \). Then

\[
f_t^* i(X_{1k}) dx = J_{f_t} i(X_{1k}) dx \text{ or } f_t^* (J_{f_{-t}} i(X_{1k}) dx) = i(X_{1k}) dx.
\]

The tangent vector to a one-parameter family of curves \( \gamma_t \) passing through points \( s \exp t X_{1k} \) can be identified with the tangent vector \( X_{1k} \) at the point \( s \in S \). The flow \( f_t \) takes the vector \( X_{1k} \) to \( (f_t)_* X_{1k} \). Consequently, the form \( J_{f_{-t}} i(X_{1k}) \) determines the measure \( d\gamma \) on the foliation \( \Gamma_k \).

Note that, by inequality (3.1), the measure \( d\gamma \) is a locally doubling measure,

\[
\int_{\gamma \in \Gamma_k, \gamma \cap B(x,2r) \neq \emptyset} d\gamma \leq c_d \int_{\gamma \in \Gamma_k, \gamma \cap B(x,r) \neq \emptyset} d\gamma,
\]

(3.2)

for sufficiently small balls \( B = B(x,r) \subset \Omega \).

Because \( X_{1k} \) is a left-invariant vector field, the flow \( f_t \) is the right translation on \( \exp t X_{1k} \). Since \( dx \) is a bi-invariant form, we have \( J_{f_t} = c_m \), where the constant \( c_m \) can be calculated exactly. Using the left invariance and homogeneity under dilatations, we obtain

\[
\int_{\gamma \in \Gamma_k, \gamma \cap B(x,r) \neq \emptyset} d\gamma = c_m |B(x,r)|^{\frac{n-1}{n}} \|X_{1k}\|,
\]

(3.3)

where \( \|X_{1k}\| \) is the length of the tangent vector \( X_{1k} \).

3.2 Additive set functions

Recall that a mapping \( \Phi \) defined on open subsets from \( \Omega \subset G \) and taking nonnegative values is called a finitely quasiadditive set function [49], if

1) for any point \( x \in \Omega \), there exists \( \delta \), \( 0 < \delta < \text{dist}(x, \partial \Omega) \), such that \( 0 \leq \Phi(B(x, \delta)) < \infty \) (here and in what follows \( B(x, \delta) = \{ y \in G : \rho(x, y) < \delta \} \)).
2) for any finite collection \( U_i \subset U \subset \Omega, \, i = 1, \ldots, k \), of mutually disjoint open sets, the inequality
\[ \sum_{i=1}^{k} \Phi(U_i) \leq \Phi(U) \]
takes place.

Obviously, the inequality in the second condition of this definition can be extended to a countable collection of mutually disjoint open sets from \( \Omega \). So, a finitely quasiadditive set function is also countable quasiadditive.

If, instead of the second condition, we suppose that, for any finite collection \( U_i \subset \Omega, \, i = 1, \ldots, k \), of mutually disjoint open sets, the equality
\[ \sum_{i=1}^{k} \Phi(U_i) = \Phi(U) \]
takes place, then such a function is said to be finitely additive. If the equality in this condition can be extended to a countable collection of mutually disjoint open sets from \( \Omega \), then such a function is said to be countably additive.

A mapping \( \Phi \) defined on open subsets of \( \Omega \) and taking nonnegative values is called a monotone set function \([49]\) if \( \Phi(U_1) \leq \Phi(U_2) \) under the condition that \( U_1 \subset U_2 \subset \Omega \) are open sets.

Let us formulate a result from \([49]\) in a form convenient for us.

**Theorem 3.1.** \([49]\) Let a finitely quasiadditive set function \( \Phi \) be defined on open subsets of the domain \( \Omega \subset \mathbb{G} \). Then, for almost all points \( x \in \Omega \), the finite derivative
\[ \Phi'(x) = \lim_{\delta \to 0, B_\delta \ni x} \frac{\Phi(B_\delta)}{|B_\delta|} \]
exists, and, for any open set \( U \subset \Omega \), the inequality
\[ \int_U \Phi'(x) \, dx \leq \Phi(U) \]
holds.

We consider the cube \( P = S_k \exp tX_{1k} \), where \(|t| \leq M\) and \( S_k \) is the transversal hyperplane to \( X_{1k} \):
\[ S_k = \{(x_{ij}), 1 \leq i \leq m, 1 \leq j \leq n_1 : x_{1k} = 0 \text{ and } |x_{ij}| \leq M\} . \]

Given a point \( s \in S_k \), we denote, by \( \gamma_s \), the element \( s \exp tX_{1k} \) of the horizontal fibration which starts at the point \( s \). Thus, \( P \) is the union of all such intervals of integral lines. Consider the following tubular neighborhood of the fiber \( \gamma_s \) with radius \( r \):
\[ E(s, r) = \gamma_s B(e, r) \cap P = \left( \bigcup_{x \in \gamma_s} B(x, r) \right) \cap P. \]

The following lemma is valid (see \([45]\)):

**Lemma 3.1.** Let \( \Phi \) be a quasiadditive set function on \( \mathbb{G} \). Then
\[ \lim_{r \to 0} \frac{\Phi(E(s, r))}{r^{n-1}} < \infty \]
for \( d\gamma \)-almost all \( s \in S_k \).
4. Capacity and Modules

4.1 The basic definitions

A well-ordered triple \((F_0, F_1; \Omega)\) of nonempty sets, where \(\Omega\) is an open set in \(\mathbb{G}\), and \(F_0, F_1\) are compact subsets of \(\Omega\), is called a condenser in the group \(\mathbb{G}\).

The quantity
\[
\text{cap}_p(F_0, F_1; \Omega) = \inf \int_{\Omega} |\nabla H v|^p \, dx,
\]
where the infimum is taken over all nonnegative functions \(v \in C(\Omega) \cap L^1_p(\Omega)\) such that \(v = 0\) in a neighborhood of the set \(F_0\) and \(v \geq 1\) in a neighborhood of the set \(F_1\), is called the \(p\)-capacity of the condenser \((F_0, F_1; \Omega)\). If \(G \subset \mathbb{G}\) is an open set, and \(E\) is a compact subset in \(G\), then the condenser \((\partial G, E; G)\) will be denoted by \((E, G)\). Properties of \(p\)-capacity in the geometry of vector fields satisfying the Hörmander hypoellipticity condition can be found in [43, 44].

The linear integral is denoted by
\[
\int_{\gamma} \rho \, ds = \sup_{\gamma'} \int_{\gamma'} \rho \, ds = \sup_{0} \int_{0} \rho(\gamma'(s)) \, ds,
\]
where the supremum is taken over all closed parts \(\gamma'\) of \(\gamma\), and \(l(\gamma')\) is the length of \(\gamma'\). Let \(\Gamma\) be a family of curves in \(\mathbb{G}\). Denote, by \(\text{adm}(\Gamma)\), the set of Borel functions (admissible functions) \(\rho : \mathbb{G} \to [0, \infty]\) such that the inequality
\[
\int_{\gamma} \rho \, ds \geq 1
\]
holds for locally rectifiable curves \(\gamma \in \Gamma\).

Let \(\Gamma\) be a family of curves in \(\overline{\mathbb{G}}\), where \(\overline{\mathbb{G}}\) is a one point compactification of a Carnot group \(\mathbb{G}\). The quantity
\[
M(\Gamma) = \inf \int_{\mathbb{G}} \rho^\nu \, dx
\]
is called the module of the family of curves \(\Gamma\) [20]. The infimum is taken over all admissible functions \(\rho \in \text{adm}(\Gamma)\).

Let \(\Omega\) be a bounded domain on \(\mathbb{G}\), and let \(F_0, F_1\) be disjoint non-empty compact sets in the closure of \(\Omega\). Let \(M(\Gamma(F_0, F_1; \Omega))\) stand for the module of a family of curves which connect \(F_0\) and \(F_1\) in \(\Omega\). Then [21]
\[
M(\Gamma(F_0, F_1; \Omega)) = \text{cap}_p(F_0, F_1; \Omega).
\]

4.2 The lower estimate of the \(p\)-capacity

The following lower estimate of the \(p\)-capacity was proved in [47, Lemma 5]. For readers' convenience, we reproduce here the detailed proof of this lemma.

**Lemma 4.1.** Let \(\nu - 1 < p < \infty\). Suppose that \(E\) is a compact connected set and \(G \subset \{x \in \mathbb{G} : \rho(x, E) \leq c_0 \text{diam } E\}\), where \(c_0\) is a small number depending on the constant in the generalized triangle inequality. Then
\[
\text{cap}_p^{\nu - 1}(E, G) \geq c(\nu, p) \frac{(\text{diam } E)^p}{|G|^{p-(\nu-1)}},
\]
(4.2)
where a constant $c(\nu, p)$ depends only on $\nu$ and $p$.

**Proof.** Since inequality (4.2) is invariant under left translations and has the same degree of homogeneity under dilatations, we can suppose, without loss of generality, that $0 \in E$ and $\text{diam } E = \rho(0, \sigma) = 1$ for some point $\sigma \in E$.

Consider a point $\sigma^{-1} \in S(0, 1)$. Then there exists a constant $c_1$ such that

\[
\text{diam } E = 1 \leq c_1(r_2 - r_1),
\]

where $r_1 = |\sigma^{-1}| = 1$ and $r_2 = \rho(\sigma^{-1}, \sigma) = |\sigma^2|$.

Since $c_0$ was chosen so that $G \subset \{ x \in \mathbb{G} : \rho(x, E) \leq c_0 \text{ diam } E \}$, we have in view of the generalized triangle inequality that

\[
S(\sigma^{-1}, r) \cap (G \setminus G) \neq \emptyset \text{ for all } r_1 \leq r \leq r_2.
\]

Let $r_1 \leq r \leq r_2$. We choose some point $x_r \in E$ such that $\rho(\sigma^{-1}, x_r) = r$ and denote

\[
P(r) = \{ s \in S(\sigma^{-1}, r) : \rho(x_r, s) \leq \rho(x_r, \{(G \setminus G) \cap S(\sigma^{-1}, r)\}) \}.
\]

Consider an arbitrary function $u \in \dot{L}^1_p(G) \cap C(G)$ such that $u \geq 1$ on $E$. Then the function $u$ takes the value 0 on the sphere $S(x, r)$, $r_1 < r < r_2$. Therefore, the following inequality is valid for almost all $r_1 < r < r_2$ [41, Theorem 1]:

\[
\int_{S(x, r) \cap G} M_{\gamma r}(|\nabla Hu|)^p(\xi) d\sigma_r(\xi) \geq c_2 \omega_r(P(r))^{\frac{\nu-1}{\nu-1}}.
\]

where $\omega_r$ is the measure on $S(x, r)$ associated with the “spherical” coordinate system [41]. (Here, $\gamma > 1$ is some constant, and $M_g$ denotes the maximal function defined for every locally summable function $g$ as

\[
M_g(x) = \sup \left\{ |B(x, r)|^{-1} \int_{B(x, r)} |g| dx : r \leq \delta \right\} ,
\]

where $B(x, r) = \{ y \in \mathbb{G} : \rho(x, y) < r \}$ is the ball of radius $r$ centered at $x \in \mathbb{G}$.) Consequently,

\[
\int_G M_{\gamma r}(|\nabla Hu|^p) dx \geq c_2 \int_{r_1}^{r_2} \omega_r(P(r))^{\frac{\nu-1}{\nu-1}} dr .
\]

Now,

\[
(d\text{iam } E)^p \leq \left( c_1 \int_{r_1}^{r_2} dr \right)^p \leq c_1^\nu \left( \int_{r_1}^{r_2} \omega_r(P(r)) dr \right)^{p-(\nu-1)} \left( \int_{r_1}^{r_2} \omega_r^{\frac{\nu-1}{\nu-1}}(P(r)) dr \right)^{\nu-1} \leq \frac{c_1^\nu}{c_2} |G|^{p-(\nu-1)} \left( \int_G M_{\gamma r}(|\nabla Hu|^p) dx \right)^{\nu-1} .
\]

By the maximal function theorem, we obtain

\[
\left( \int_G |\nabla Hu|^p dx \right)^{\nu-1} \geq c(\nu, p) \frac{(d\text{iam } E)^p}{|G|^{p-(\nu-1)}} .
\]
for arbitrary function \( u \in \dot{L}^1_p(G) \cap C(G) \) admissible for the condenser \((E, G)\). Hence,
\[
\text{cap}_p^{\nu-1}(E, G) \geq c(\nu, p) \left( \frac{\text{diam } E}{|G|^{p-(\nu-1)}} \right).
\]

\[\square\]

5. Sobolev spaces and \(Q\)-Homeomorphisms

In this section, we consider the connection between Sobolev mappings and \(Q\)-homeomorphisms on Carnot groups.

5.1 ACL-property of \(Q\)-homeomorphisms

We prove the ACL-property of \(Q\)-homeomorphisms with a locally integrable function \(Q\).

**Theorem 5.1.** Let \( \varphi : \Omega \to \Omega' \) be a \(Q\)-homeomorphism of domains \( \Omega, \Omega' \subset \mathbb{G} \) with \( Q \in L^1_{\text{loc}}(\Omega) \). Then \( \varphi \in W^{1,1}_{\text{loc}}(\Omega; \Omega') \).

**Proof.** Fix some field \( X^1_k \), \( 6 \leq k \leq n \), and let \( \Gamma_k \) be the fibration generated by this field. Take the cube \( P = S_k \exp t X^1_k \), where \( |t| \leq M \), and \( S_k \) is the transversal hyperplane to \( X^1_k \):
\[
S_k = \{ (x_{ij}) \mid 1 \leq i \leq m, 1 \leq j \leq n \_i : x_{1k} = 0 \text{ and } |x_{ij}| \leq M \}.
\]

Given a point \( s \in S_k \), we denote, by \( \gamma_s \), the element \( s \exp t X^1_k \) of the fibration which starts at \( s \). Thus, \( P \) is the union of all such intervals of integral lines. Consider the following tubular neighborhood of the fiber \( \gamma_s \) with radius \( r \):
\[
E(s, r) = \gamma_s B(e, r) \cap P = \left( \bigcup_{x \in \gamma_s} B(x, r) \right) \cap P.
\]

Take a point \( s \in S_k \) so that the assertion of Lemma 3.1 holds for \( \gamma_s \). On \( \gamma_s \), we take arbitrary pairwise disjoint closed segments \( \gamma_{s1}, \ldots, \gamma_{sk} \) of lengths \( \delta_1, \ldots, \delta_k \). Denote, by \( R_i \), the open set of points at a distance less than a given \( r > 0 \) from \( \gamma_{si} \), \( i = 1, \ldots, k \), and consider the condensers \( (\gamma_{si}, R_i) \), \( i = 1, \ldots, k \). Suppose that \( r > 0 \) is chosen so small that the sets \( R_1, \ldots, R_k \) are pairwise disjoint, and the condenser \( (\varphi(\gamma_{si}), \varphi(R_i)) \) satisfies the conditions of Lemma 4.1. Let \( \Gamma \) be a family of curves connecting \( \varphi(\gamma_{si}) \) and \( \partial \varphi(R_i) \) in \( \Omega \). Now, by (4.1)
\[
M(\varphi(\Gamma)) = \text{cap}_\nu(\varphi(\gamma_{si})), \varphi(R_i)).
\]

Observe that the function \( \rho(x) = \begin{cases} \frac{1}{r}, & x \in R_i, \\ 0, & x \in \mathbb{G} \setminus R_i \end{cases} \) is admissible for \( \Gamma \). Now, by (5.1),
\[
\text{cap}_\nu(\varphi(\gamma_{si})), \varphi(R_i)) \leq \frac{1}{r^p} \int_{R_i} Q(x) \, dx.
\]
On the other hand, by Lemma 4.1,
\[
\text{cap}_\nu(\varphi(\gamma_{si})), \varphi(R_i)) \geq c \left( \frac{(\text{diam } \varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|} \right)^{1/(\nu-1)}. 
\] (5.3)

Combining (5.2) and (5.3), we have the inequalities
\[
\left( \frac{(\text{diam } \varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|} \right)^{1/(\nu-1)} \leq \frac{c_\nu}{r^\nu} \int_{R_i} Q(x) \, dx, \quad i = 1, \ldots, k, 
\] (5.4)
where the constant \( c_\nu \) depends only on \( \nu \).

By the discrete Hölder inequality (see, e.g., (17.3) in [2]), we obtain
\[
\sum_{i=1}^k \text{diam } \varphi(\gamma_{si}) \leq \left( \sum_{i=1}^k \left( \frac{(\text{diam } \varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|} \right)^{1/(\nu-1)} \right)^{\nu/(\nu-1)} \left( \sum_{i=1}^k |\varphi(R_i)| \right)^{1/(\nu-1)}, 
\] (5.5)
i.e.,
\[
\left( \sum_{i=1}^k \text{diam } \varphi(\gamma_{si}) \right)^\nu \leq \left( \sum_{i=1}^k \left( \frac{(\text{diam } \varphi(\gamma_{si}))^\nu}{|\varphi(R_i)|} \right)^{1/(\nu-1)} \right)^{\nu/(\nu-1)} |\varphi(E(s, r))|, 
\] (5.6)
and, in view of (5.4),
\[
\left( \sum_{i=1}^k \text{diam } \varphi(\gamma_{si}) \right)^\nu \leq c_\nu \frac{|\varphi(E(s, r))|}{r^{\nu-1}} \left( \sum_{i=1}^k \frac{\int_{R_i} Q(x) \, dx}{r^{\nu-1}} \right)^{\nu/(\nu-1)} 
\] (5.7)
where a constant \( c_\nu \) depends only on \( \nu \).

By [47, Lemma 4]),
\[
\lim_{r \to 0} \frac{|\varphi(E(s, r))|}{r^{\nu-1}} := \omega(s) < \infty.
\]

Denote
\[
\omega_i(s) = \int_{\delta_i} Q(s, t) \, dt, \quad s \in S_k.
\]

Because \( Q \) is a locally integrable function, there exists, by the Fubini theorem for any \( \varepsilon > 0 \), a number \( \delta > 0 \) such that \( \omega_i(s) < \varepsilon \) if \( \delta_i < \delta \), \( i = 1, \ldots, k \).

By the Fubini-type decomposition (3.3), we have
\[
\lim_{r \to 0} \frac{\int_{R_i} Q(x) \, dx}{r^{\nu-1}} = \frac{\omega_i(s)}{c_m \|X_{1k}\|} < \infty.
\]

Passing in (5.7) to the limit as \( r \to 0 \), we get
\[
\left( \sum_{i=1}^k \text{diam } \varphi(\gamma_{si}) \right)^\nu \leq \frac{c_\nu \omega(s)}{c_m \|X_{1k}\|} \left( \sum_{i=1}^k \omega_i(s) \right)^{\nu/(\nu-1)}. 
\] (5.8)

Hence, \( \varphi \) is absolutely continuous on \( \gamma \cap P \) with respect to the one-dimensional Hausdorff measure for \( d\gamma \)-almost every curve \( \gamma \in \Gamma_k \). Hence, \( \varphi \in W^{1,1}_{1,\text{loc}}(\Omega; \Omega') \).

\( \Box \)
5.2 Mappings of integrable distortion

Let a homeomorphism $\varphi : \Omega \to \Omega'$ belong to the Sobolev space $W^{1,1}_{1,\text{loc}}(\Omega; \Omega')$. Recall that a weakly differentiable mapping $\varphi : \Omega \to \Omega'$ is called a mapping of finite distortion, if $|D_H\varphi(x)| = 0$ for almost all $x \in Z = \{x \in \Omega : J(x, \varphi) = 0\}$. We say that a homeomorphism $\varphi : \Omega \to \Omega'$ has the Luzin $N$-property, if the image of a set of measure zero has measure zero.

The outer dilatation of the mapping of finite distortion $\varphi$ at $x$ is defined by

$$K_O(x) = K_O(x, \varphi) = \begin{cases} \frac{|D_H\varphi(x)|^\nu}{|J(x, \varphi)|}, & \text{if } J(x, \varphi) \neq 0, \\ 0, & \text{if } D_H\varphi(x) = 0. \end{cases}$$

**Theorem 5.2.** Let $\varphi : \Omega \to \Omega'$ be a homeomorphism of finite distortion of the Sobolev class $W^{1,1}_{1,\text{loc}}(\Omega; \Omega')$. Then, for every family $\Gamma$ of rectifiable paths in $\Omega$ and every $\rho \in \text{adm}(\Gamma)$,

$$M(\varphi^{-1}(\Gamma)) \leq \int K_O(\varphi^{-1}(y), \varphi) \rho'(y) \, dy,$$

i.e., $\varphi^{-1}$ is a $Q$-homeomorphism with $Q(y) = K_O(\varphi^{-1}(y), \varphi) \in L_{1,\text{loc}}(\Omega').$

**Proof.** Let $F$ be a compact subdomain of $\Omega$, $F' = \varphi(F)$. Denote

$$Z = \{x \in \Omega : J(x, \varphi) = 0\}.$$

Because $\varphi$ is a mapping of finite distortion, $|D_H\varphi| = 0$ a.e. on $Z$, and $K_O(x, \varphi)$ is well defined for almost all $x \in \Omega$. Since $\varphi \in W^{1,1}_{1,\text{loc}}(\Omega)$, $\varphi$ possesses the Luzin $N$-property, and the outer distortion $K_O(\varphi^{-1}(y), \varphi)$ is well defined for almost all $y \in \Omega'$. Then

$$\int_{F'} K_O(\varphi^{-1}(y), \varphi) \, dy = \int_{F \setminus \varphi(Z)} \frac{|D_H\varphi(\varphi^{-1}(y))|^\nu}{|J(\varphi^{-1}(y), \varphi)|} \, dy + \int_{F \cap \varphi(Z)} \, dy$$

$$= \int_{F \setminus Z} \frac{|D_H\varphi(x)|^\nu}{|J(x, \varphi)|} \, dx + \int_{F \cap Z} |J(x, \varphi)| \, dx = \int_{F \setminus Z} |D_H\varphi(x)|^\nu \, dx + \int_{F \cap Z} |J(x, \varphi)| \, dx < \infty.$$ 

Since $\varphi : \Omega \to \Omega'$ belongs to $W^{1,1}_{1,\text{loc}}(\Omega; \Omega')$, $\varphi$ is a (weakly) contact mapping differentiable almost everywhere in $\Omega$ and absolutely continuous on almost all horizontal curves. By the generalized Fuglede theorem (see [22,35]), we have that if $\tilde{\Gamma}$ is the family of all paths $\gamma \in \varphi^{-1}(\Gamma)$ such that $\varphi$ is absolutely continuous on all closed subpaths of $\gamma$, then $M(\varphi^{-1}(\Gamma)) = M(\tilde{\Gamma})$.

For a given function $\rho \in \text{adm}\Gamma$, we define

$$\tilde{\rho}(x) = \begin{cases} \rho(\varphi(x)) |D_H\varphi(x)| & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (5.9)$$

Then, for almost all $\tilde{\gamma} \in \tilde{\Gamma}$,

$$\int_{\tilde{\gamma}} \tilde{\rho} \, ds \geq \int_{\varphi\circ\tilde{\gamma}} \rho \, ds \geq 1$$

and, consequently, $\tilde{\rho} \in \text{adm}\tilde{\Gamma}$.  

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Therefore, making the change of a variable [46], we obtain

\[ M(\varphi^{-1}(\Gamma)) = M(\tilde{\Gamma}) \leq \int_{\Omega} \rho'(x) |D_H \varphi(x)|^\nu \, dx \]
\[ = \int_{\Omega \setminus \tilde{Z}} \rho'(\varphi(x)) |D_H \varphi(x)|^\nu \, dx = \int_{\Omega \setminus \tilde{Z}} \rho'(\varphi(x)) \frac{|D_H \varphi(x)|^\nu}{|J(x, \varphi)|} |J(x, \varphi)| \, dx \]
\[ = \int_{\Omega \setminus \varphi(Z)} \rho'(y) \frac{|D_H \varphi^{-1}(y)|}{|J(\varphi^{-1}(y), \varphi)|} \, dy = \int_{\Omega'} K_0 \left( \varphi^{-1}(y), \varphi \right) \rho'(y) \, dy. \quad (5.10) \]

Hence, \( \varphi^{-1} \) is a \( Q \)-homeomorphism with \( Q(y) = K_0 \left( \varphi^{-1}(y), \varphi \right) \in L_{1,\text{loc}}(\Omega') \).

\section{5.3 The weak inverse mapping theorem on Carnot groups}

In this section, we prove that the mappings inverse to Sobolev homeomorphisms of finite distortion of the class \( W_{1,\text{loc}}^1(\Omega; \Omega') \) are Sobolev mappings.

**Theorem 5.3.** Let \( \varphi : \Omega \to \Omega' \) be a Sobolev homeomorphism of finite distortion of the class \( W_{1,\text{loc}}^1(\Omega; \Omega') \). Then \( \varphi^{-1} \in W_{1,\text{loc}}^1(\Omega'; \Omega) \).

**Proof.** By Theorem 5.2, we obtain that the inverse mapping \( \varphi^{-1} : \Omega' \to \Omega \) is a \( Q \)-homeomorphism with \( Q \in L_{1,\text{loc}}(\Omega') \). Hence, using Theorem 5.1, we conclude that the inverse mapping \( \varphi^{-1} \in W_{1,\text{loc}}^1(\Omega'; \Omega) \). \( \square \)

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