On the Ext groups between Weyl modules for $\text{GL}_n$

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Abstract

This paper studies extension groups between certain Weyl modules for the algebraic group $\text{GL}_n$ over the integers. Main results include: (1) A complete determination of Ext groups between Weyl modules whose highest weights differ by a single root and (2) Determination of Ext$^1$ between an exterior power of the defining representation and any Weyl module. The significance of these results for modular representation theory of $\text{GL}_n$ is discussed in several Remarks. Notably the first result leads to a calculation of Ext groups between neighboring Weyl modules for $\text{GL}_n$ and also recovers the $\text{GL}_n$ case of a recent result of Andersen. Some generalities about Ext groups between Weyl modules and a brief overview of known results about these groups are also included.

INTRODUCTION

This paper studies some homological aspects of the representation theory of the algebraic group $\text{GL}_n$ over the integers. More specifically we will investigate Ext groups between certain pairs of Weyl modules for $\text{GL}_n(\mathbb{Z})$ and also discuss the significance of these results in modular representation theory of $\text{GL}_n$. Weyl modules are universal highest weight modules in the representation category of split reductive algebraic groups. Extensions between these modules are of interest in representation theory. Let us outline and briefly discuss the main results. For a highest weight $\lambda$, let $K_\lambda$ denote the corresponding Weyl module.

(1) Theorem 2.1 gives a complete determination of the groups Ext$^i_{\text{GL}_n(\mathbb{Z})}(K_\lambda, K_\mu)$ where $\mu - \lambda$ is a positive root $\alpha$ of $\text{GL}_n$. Here Ext$^1$ is cyclic of order $\langle \lambda + \rho, \alpha \rangle + 1$, where $\rho$ is half the sum of positive roots and all other Ext groups vanish. By the Universal Coefficient Theorem, one then gets all the modular Ext groups as well for this class of examples. Moreover, in the modular case and also over the $p$-adic integers, one can calculate Ext groups between any two neighboring Weyl modules for $\text{GL}_n$ using Theorem 2.1 along with the translation principle. This recovers the $\text{GL}_n$ case of a recent result of [Andersen3] obtained independently around the same time as an equivalent version of Theorem 2.1. Neighboring Weyl modules are defined only if regular weights exist, which happens only if the characteristic $p$ is at least $n$. Theorem 2.1 is of additional interest because it gives results in the modular case even for small primes. See the Remarks after the proof of Theorem 2.1 for a discussion. In view of Andersen’s result, it seems natural to hope
that Theorem 2.1, as stated at the beginning of this paragraph, remains true for all split reductive algebraic groups over $\mathbb{Z}$.

(2) Theorem 2.2 determines $\text{Ext}^1$ between an exterior power of the defining representation and any Weyl module $K_\lambda$. This group is cyclic and its order is the $\gcd$ of several integers that can be described explicitly in terms of the weight $\lambda$. By contravariant duality and conjugate symmetry of Ext groups, this result also leads to the calculation of certain other $\text{Ext}^1$ groups, e.g., $\text{Ext}^1$ between a symmetric power of the defining representation and a dual Weyl module. It is interesting to compare Theorem 2.2 with the known enumeration of the composition factors of symmetric powers in characteristic $p$. See Remark 3 after the proof of Theorem 2.2.

Let us review the previously known information about the groups $\text{Ext}^i(K_\lambda, K_\mu)$, in the modular as well as the integral setting. (Note that the Ext is always taken in the appropriate category of representations. But, as in the previous sentence, the notation will often omit this fact relying on the context to convey the intended meaning.) Even though the work in this paper deals directly only with the integral case, the two cases are intimately connected via the Universal Coefficient Theorem.

Let us begin with the modular case, i.e., that of reductive algebraic groups over an algebraically closed field of characteristic $p > 0$. An important result here is the vanishing theorem of Cline-Parshall-Scott-van der Kallen. [CPSvdK] proves that for any such group $\text{Ext}^i(K_\lambda, K_\mu)$ vanishes unless $\lambda \leq \mu$ under the dominance partial order, i.e., unless $\mu - \lambda$ is a sum of positive roots. This vanishing result was strengthened after the proof of the strong linkage principle (see, e.g., [Jantzen]). As a result, one can replace the dominance relation $\leq$ in the vanishing theorem above by a finer relation $\uparrow$, where $\uparrow$ is the “linkage” relation defined using the dot action of the affine Weyl group $W_p$ on weights. When $\lambda \uparrow \mu$, the calculation of the groups $\text{Ext}^i(K_\lambda, K_\mu)$ is not known in general and is likely to be difficult. But we do know the answer in the important case of “neighboring Weyl modules.” This refers to the situation when $\lambda$ and $\mu$ are regular (i.e., have a trivial stabilizer in $W_p$) and $\lambda < s.\lambda = \mu$, where $s$ is the reflection in a wall of the alcove containing $\lambda$. In this case [Jantzen, II.7.19] proves that $\text{Ext}^i(K_\lambda, K_{s.\lambda})$ is one-dimensional if $i = 0$ or 1 and vanishes otherwise. (The relevance of this result for us will be discussed in the Remarks after the proof of Theorem 2.1.) Beyond this there are several results in more or less special cases, in which the answers as well as the needed arguments are often involved. See [Jantzen, II.6.25] for a discussion of results regarding homomorphisms between Weyl modules and [Wen] for some further cases. [O-M] calculates $\text{Hom}$ between certain hook representations for $GL_n$. [Erdmann] and [CE] respectively calculate $\text{Ext}^1$ and $\text{Ext}^2$ between modular Weyl modules for $SL_2$.

Before turning to the integral case, let us digress to comment on the case of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of representations of complex semisimple Lie algebras. This situation is somewhat parallel to modular representations of reductive algebraic groups. The universal highest weight modules here (i.e., analogues of Weyl modules) are Verma modules. The extent of our knowledge about the Ext groups between these is simi-
lar to that for Weyl modules. Similar vanishing properties hold, and Hom groups between neighboring Verma modules are one-dimensional. But additionally, unlike in the case of Weyl modules, there is a well-known calculation of Hom groups between arbitrary Verma modules. The answer involves the Bruhat order on the associated Weyl group. A guess was made in [GJ] expressing all Ext groups between arbitrary Verma modules in terms of $R$-polynomials for the associated Weyl group. But this guess was found to be incorrect in [Boe], underscoring the seeming difficulty of calculating these Ext groups.

Let us now survey the integral case, i.e., that of Ext groups between integral Weyl modules. First of all, the vanishing result of [CPSvdK] immediately carries over to this case. The strengthening due to the linkage principle is not available over $\mathbb{Z}$. But the Universal Coefficient Theorem does allow us to translate the characteristic $p$ results mentioned above into information about $p$-torsion of integral Ext groups between Weyl modules. So we know that there will be no $p$-torsion in $\operatorname{Ext}^1_{\mathbb{Z}}(K_\lambda, K_\mu)$ unless $\lambda \uparrow \mu$ under the action of $W_p$. Similarly a nonzero homomorphism (respectively, a one-dimensional Hom group) between two Weyl modules in characteristic $p$ translates into nonvanishing (respectively cyclic) $p$-torsion in the corresponding integral Ext$^1$. Beyond this what we have are mainly special case results for the Ext groups between Weyl modules for $GL_n$ due to various authors. These results are obtained following the basic approach in [AB2], which relies on constructing explicit projective resolutions of Weyl modules. See [AB2], [F], [BF] for Ext$^1$ between special pairs of representations of $GL_2$ and $GL_3$. See [Akin] and [Maliakas] for Ext$^1$ in some other special cases involving hooks. (The calculation in [Maliakas] implies the earlier modular result in [O-M].) See [R-G] for Ext$^2$ for $GL_2$. (The results of [AB2] and [R-G] together recover the modular calculation of Ext$^1$ for $SL_2$ cited above. [Erdmann, p. 456] describes this calculation in terms of a set $\psi(r)$ which the author finds convenient to define as a union of two subsets. It turns out that via the Universal Coefficient Theorem, the first of these subsets is accounted for by integral Ext$^2$ [R-G] and the second by integral Ext$^1$ [AB2].) Theorem 2.1 in this paper generalizes a result of [Maliakas] and intersects to various degrees with the results of other papers. Theorem 2.2 generalizes certain Ext$^1$ calculations in [AB2] and [Akin].

Unlike [AB2], the approach here does not use resolutions directly. Instead a key tool will be the Skew Representative Theorem from [Kulkarni1]. Some other notions and methods that we will use are as follows. (These topics are discussed in Section 1.) The explicit combinatorial/multilinear-algebraic descriptions of Weyl modules for $GL_n$, ordinary as well as “skew,” due to Akin-Buchsbaum-Weyman will be very useful for us in proofs. Also very useful will be Pieri-type rules giving filtrations of certain special skew Weyl modules. The Schur algebras $S(n, r)$ will play a role, though mostly in the background. These algebras were first treated systematically in [Green], where it is proved that polynomial representations of $GL_n$ of degree $r$ are equivalent to the representations of the Schur algebra $S(n, r)$. Then the main new idea can be described as a way to reduce questions about a Schur algebra to questions about another Schur algebra of smaller degree and then to use recursion. The Skew Representative Theorem is the vehicle that allows one to do this.

Finally, let me indicate why this paper deals only with $GL_n$ and comment on possible
generalizations. (Incidentally the main results here stay valid for SL\(_n\). We will use GL\(_n\) as it will be more natural to for us work in that setting.) Many of the generalities used in this paper for GL\(_n\) hold for other reductive algebraic groups too. Donkin has defined algebras generalizing \(S(n, r)\) for all split reductive algebraic groups, now known as Schur algebras. These more general Schur algebras are examples of the quasihereditary algebras of Cline-Parshall-Scott. Several of the properties relevant to this paper (existence of suitable filtrations, triangular Ext-vanishing properties with respect to a suitable partial order on weights) hold in the broader setting of highest weight categories (i.e., representations of quasihereditary algebras) of Cline-Parshall-Scott or BGG categories of Irving. But after the generalities, what allows one to push through with the calculations is the availability for GL\(_n\) of some very explicit characteristic-free constructions. To carry out a similar method for other reductive groups, one should look for a good analogue of skew Weyl modules for which a version of the Skew Representative Theorem holds. Donkin has previously constructed “skew modules” for reductive groups [Donkin2]. More recently, he has also proved [Donkin3] a version of the Skew Representative Theorem for these general skew modules. Since all his constructions are abstract, it is not immediately clear how one can use them to get analogues of results for GL\(_n\) that are obtained in this paper by concrete calculations.

In another direction, natural analogues of all the results in this paper should hold for quantum GL\(_n\) as well. One just has to replace the [ABW] constructions by the constructions of Hashimoto-Hayashi for quantum Schur and Weyl modules. But we will not pursue this here.

Since the first version of this paper was written, a result similar to Theorem 2.1 and obtained independently around the same time has been published by Andersen. For any reductive algebraic group over an algebraically closed field of characteristic \(p > 0\), [Andersen3] calculates Ext groups between neighboring Weyl modules for the corresponding Chevalley group over the \(p\)-adic integers. The connection between Theorem 2.1 and Andersen’s result is discussed after the proof of Theorem 2.1. One upshot is that it seems natural to hope for the validity of Theorem 2.1 for any split reductive algebraic group over \(\mathbb{Z}\). A possible way to approach the expected generalization is to use Donkin’s work mentioned above. Another possibility is to try to extend Andersen’s proof, namely work with one prime at a time and use translation functors, keeping careful track of the modules that arise for small primes.

1. BACKGROUND AND NOTATION

This section discusses, in a little more detail than is strictly necessary, the following two topics. First, the definition and some properties of Weyl and Schur modules (both ordinary and “skew”) following Akin-Buchsbaum-Weyman, and second, some results of a general nature about certain Ext groups of interest. Proofs are given for a couple of easy results apparently unrecorded elsewhere. Along the way we will also discuss the role of the
Schur algebras $S(n, r)$ and then use these algebras while discussing Ext groups.

Throughout the rest of this paper, the ground ring will be $\mathbb{Z}$ unless otherwise indicated. The exceptions will mainly occur in some of the Remarks after the results, where the significance of the results in modular representation theory is discussed. In any event all discussions where we will need to consider a ground ring other than $\mathbb{Z}$ will take place in a separate paragraph, with an appropriate notice to that effect at the beginning of that paragraph.

We will study some homological aspects of the representation theory of the reductive algebraic group scheme $GL(F) = GL_n$, where $F$ is a free abelian group of rank $n$. See, e.g., [Jantzen]. Weights for $GL_n$ are multiplicative characters of a maximal toral subgroup scheme. We will follow the common practice of taking this to be the diagonal subgroup $Diag(\mathbb{Z}^n)$ and identifying weights for $GL_n$ with $n$-tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$. A weight $\lambda$ is a polynomial weight if $\lambda_i$ are all nonnegative and then the degree of $\lambda$ is $|\lambda| = \sum \lambda_i$. Dominant weights are the ones with $\lambda_1 \geq \ldots \geq \lambda_n$. Dominant polynomial weights are thus just partitions (with at most $n$ parts), which we will frequently identify with their Young diagrams (with at most $n$ rows). The usual dominance partial order on weights is the one generated by stipulating that $(\ldots, \lambda_i, \lambda_{i+1}, \ldots) < (\ldots, \lambda_i + 1, \lambda_{i+1} - 1, \ldots)$. For Young diagrams this means that moving boxes upwards gives a bigger partition. Note that a given Young diagram $\lambda$ can be considered a weight for all $GL_n$ with $n \geq$ the number of rows of $\lambda$. In the main results it will be convenient for us to adopt this point of view by fixing the partitions and dealing with all such $n$ simultaneously.

Given any partition $\lambda$, its conjugate $\tilde{\lambda}$ is the partition whose diagram is obtained by transposing (i.e., by exchanging rows and columns of) the diagram of $\lambda$. In the literature the notation $\lambda'$ is often used instead of $\tilde{\lambda}$, but we will follow the notation in [ABW]. It should be noted that even if $\lambda$ has at most $n$ nonzero parts (and so is a weight for $GL_n$), $\tilde{\lambda}$ may have more nonzero parts and so need not be a weight for $GL_n$. Thus conjugation is really an operation on “stable” weights.

**Weyl and Schur modules.** Let $\lambda$ be a partition having number of rows $\leq$ the rank of $F$. Following [ABW], we will use $K_{\lambda}(F)$ to denote the Weyl module of highest weight $\lambda$. When there is no danger of confusion, we will often drop the $F$ and simply write $K_{\lambda}$. The notation in this paper is different from the standard one! The standard notation for the Weyl module of highest weight $\lambda$ is $V(\lambda)$. See, e.g., [Jantzen]. But we will need to use heavily the constructions in [ABW] of Weyl modules as well as of certain generalizations of Weyl modules defined in the same paper (see below). So we will sacrifice the standard notation for the sake of consistency with this reference. Note further that in [ABW] Weyl modules are called coSchur modules, a term that will not be used in this paper (and which was abandoned later by its authors as well).

Some remarks are in order before introducing the “skew Weyl modules.” First of all, as defined in [ABW], $K_{\lambda}$ is a functor that assigns to any finitely generated free abelian group $F$ an abelian group $K_{\lambda}(F)$. That $K_{\lambda}$ is a functor means in particular that $K_{\lambda}(F)$ is a representation of $GL(F)$. Also note that the [ABW] definition of $K_{\lambda}$ works even when
the number of rows in $\lambda$ is greater than the rank of $F$, i.e., when $\lambda$ is not a weight of $\text{GL}(F)$. But in that case $K_{\lambda}(F) = 0$. While speaking of Weyl modules, we will always tacitly assume that the rank of $F$ is big enough so that the involved partitions are weights for $\text{GL}(F)$ and hence the Weyl modules in question are nonzero. With this proviso, the rank of $F$ will not really matter to us. See the discussion after Proposition 1.1 below.

Let us introduce the “skew Weyl modules” defined in [ABW]. For an arbitrary skew partition $\lambda/\mu$, [ABW] defines a functor $K_{\lambda/\mu}$ assigning to any finitely generated free abelian group $F$ the skew Weyl module $K_{\lambda/\mu}(F)$. (Again we will usually drop the $F$ in the notation.) Here $\lambda$ is a partition and $\mu$ is any partition whose Young diagram is contained in that of $\lambda$. The skew partition $\lambda/\mu$ is best visualized as the diagram obtained by removing the diagram of $\mu$ from the diagram of $\lambda$. Thus any partition $\lambda$ is also a skew partition $\lambda/\mu$ with $\mu$ empty and in fact ordinary Weyl modules are special cases of skew Weyl modules.

Given the importance of skew Weyl modules in our proofs, let us describe briefly the nature of their definition and the fundamental results about them in [ABW]. Only a bare sketch is given here. A thorough discussion can be found in [ABW, section II] to which we will refer freely. Let $D_t$ and $\Lambda_t$ respectively denote the $t$-fold divided and exterior power functors. So for example, $D_t(F)$ is the $t$-fold divided power of the defining representation $F$ of $\text{GL}(F)$. For a skew partition $\lambda/\mu$, let

\[ D_{\lambda/\mu} = D_{\lambda_1-\mu_1} \otimes D_{\lambda_2-\mu_2} \otimes \ldots \]

and similarly for exterior powers. Then [ABW] defines $K_{\lambda/\mu}$ as the image of a generalized symmetrizer map

\[ d'_{\lambda/\mu} : D_{\lambda/\mu} \rightarrow \Lambda_{\lambda/\mu} \cdot \]

A concise (though informal and somewhat imprecise) way to think of the map $d'_{\lambda/\mu}$ is that it is “comultiplication in the divided power algebra along the rows of $\lambda/\mu$ followed by multiplication in the exterior algebra along the columns of $\lambda/\mu$.”

[ABW, Theorem II.3.16] proves two important results that permit us to think about the skew Weyl modules in two different ways. First, this theorem explicitly describes a “standard basis” \( \{d'_{\lambda/\mu}(X_T)\} \) for $K_{\lambda/\mu}(F)$ in terms of standard tableaux $T$ of shape $\lambda/\mu$. (For us, entries in a standard tableau—taken from a basis of the defining representation $F$—will increase weakly along rows and strictly along columns. Note that [ABW] calls such tableaux “co-standard.”) The second important result is a description of $K_{\lambda/\mu}$ by generators and relations in $D_{\lambda/\mu}$. See [ABW, pp. 234-5 and pp. 226-9] for details. Both these results will be crucial in proving the main results in this paper.

Now let us record an important general fact about the existence of certain filtrations of skew Weyl modules proved independently in [Kouwenhoven] and [Boffi].

**Filtration Theorem** Any skew Weyl module $K_{\lambda/\mu}$ has a characteristic-free filtration such that the filtration factors are isomorphic to ordinary Weyl modules, i.e., those corresponding to partitions.
Such a filtration is called a Weyl filtration. Note that tensor products of Weyl modules can also be regarded as skew Weyl modules, so the theorem applies to them too. In the proofs of the main results we will only need some special cases of the Filtration Theorem, in which the necessary filtrations are constructed explicitly in [AB1, Section 3].

**Stability of** $K_{\lambda/\mu}(F)$ **under variation of rank of** $F$. The [ABW] construction of the modules $K_{\lambda/\mu}(F)$ “does not depend” on the rank $n$ of $F$. To make this precise let $E$ be a free abelian group of finite rank $N > n$ and recall the functor $d_{N,n}$ of [Green, Section 6.5]. $d_{N,n}$ not only takes $K_{\lambda/\mu}(E)$ to $K_{\lambda/\mu}(F)$ but it also takes the entire apparatus involved in the definition of $K_{\lambda/\mu}(E)$ (i.e., the modules and the maps) to that involved in defining $K_{\lambda/\mu}(F)$. Further, $d_{N,n}$ takes a Weyl filtration of $K_{\lambda/\mu}(E)$ to one of $K_{\lambda/\mu}(F)$. (Note that a filtration factor $K_{\nu}(E)$ will become 0 under $d_{N,n}$ if $n$ is smaller than the number of rows in $\nu$.) Perhaps a better way to express the “irrelevance of $F$” is to note the following. All the constructions in [ABW] and [Boffi] are functorial in $F$, i.e., all the involved objects are functors and all the involved maps are natural transformations between functors. So these constructions are the “same” regardless of which $F$ (free abelian of finite rank) they are applied to.

We will occasionally need to mention linear duals of Weyl modules (“dual Weyl modules” for short). The standard notation for the dual Weyl module with largest weight $\lambda$ is $\text{Ind}_B^G \lambda$. The reason is that this module is also obtained by first extending the character $\lambda$ to a Borel subgroup $B$, and then inducing the character from $B$ to the whole group $G$, which for us is $\text{GL}_n$. See, e.g., [Jantzen]. (Note that Jantzen uses the short notation $H^0(\lambda)$, motivated by yet another construction of a dual Weyl module as the space of sections of a line bundle on the flag variety.) Again we will deviate from the standard notation and follow [ABW] by using $L_\lambda^\circ(F)$, or just $L_\lambda$, to denote $\text{Ind}_B^G \lambda$. Note that [ABW] calls these Schur modules. A discussion parallel to the one above for Weyl modules is valid for these Schur modules (functoriality, definition and filtrations of skew Schur modules, stability under variation of rank of $F$, etc.). We will only need to make limited use of ordinary dual Weyl modules. One fact we will use is that $K_\lambda(F)$ and $L_\lambda^\circ(F)$ are contravariant duals of each other. For example, combine [ABW, Proposition II.4.1], with an isomorphism $F \simeq F^\ast$.

**Notes.** (1) Given a representation $V$ of a groups $G$, the linear dual $V^\ast$ also becomes a left $G$-module, called the contravariant dual of $V$, by composing the natural (right) action of $G$ on $V^\ast$ with an antiautomorphism of $G$. Usually this antiautomorphism is taken to be the group inverse, but we will take it to be the transpose of a matrix in $G = \text{GL}_n$. This choice ensures that the contravariant dual of any polynomial representation of a certain degree is also a polynomial representation of the same degree. The notion of a polynomial representation is recalled below. (2) Taking the transpose of a matrix requires choosing an isomorphism $F \simeq F^\ast$, as we did above, so the contravariant duality we use is not functorial in $F$, unlike the formulation in [ABW, Proposition II.4.1]

Let us now discuss the Schur algebra $S(n, r)$ and its connection with the representation theory of $\text{GL}_n = \text{GL}(F)$. A representation $V$ of $\text{GL}_n$ is called a polynomial representation (of degree $r$) if the entries of the matrix by which an arbitrary $g \in \text{GL}_n$ acts on $V$ are
polynomial functions (of degree \(r\)) of the entries of the matrix \(g\). \(K_{\lambda/\mu}(F)\) is a polynomial representation of \(\text{GL}(F)\) of degree \(|\lambda| - |\mu|\) (similarly for Schur modules). The full subcategory of polynomial representations of \(\text{GL}_n\) of degree \(r\) is equivalent to the category of left modules of the Schur algebra \(S(n, r)\). See [Green, Chapter 2] for a discussion. So Weyl and Schur modules can be regarded as modules over Schur algebras \(S(n, r)\) for appropriate \(r\). The main use of this equivalence for us will be in analyzing \(\text{Ext}\) groups, which will be our next topic.

**Ext groups.** Our primary interest will be in the groups \(\text{Ext}^i_{\text{GL}_n(\mathbb{Z})}(K_{\lambda}, K_{\mu})\) for dominant weights \(\lambda\) and \(\mu\). In the course of arguments, we will more generally need to consider groups of the type \(\text{Ext}^i_{\text{GL}_n(\mathbb{Z})}(K_1, K_2)\), where \(K_1\) and \(K_2\) are representations having Weyl filtrations. When no confusion is likely, the subscript \(\text{GL}_n(\mathbb{Z})\) will be dropped from the notation.

Here is how the Schur algebras \(S(n, r)\) enter the picture. While computing the groups \(\text{Ext}^i(K_1, K_2)\), we can (and will in all the main proofs) assume the following without loss of generality. We can take \(\lambda\) and \(\mu\) to be partitions (by tensoring with a suitable power of the determinant) with the same number of boxes (since the action of the center of \(\text{GL}_n\) breaks up the category of \(\text{GL}_n\)-modules into a direct sum by degree). For the more general \(\text{Ext}^i(K_1, K_2)\) that we need to consider, the same reasoning allows us to assume that \(K_1\) and \(K_2\) are polynomial representations of the same degree \(r\). Since such \(K_1\) and \(K_2\) can be regarded as modules over the algebra \(S(n, r)\), we may also contemplate the group \(\text{Ext}^i_{S(n, r)}(K_1, K_2)\). Happily [Donkin1] proves that the two kinds of \(\text{Ext}\) groups coincide. In fact Donkin proves this result for any Schur algebra, not just \(S(n, r)\), and for any representations \(K_1\) and \(K_2\) of that Schur algebra. Note further that even though [Donkin1] proves the result over algebraically closed fields, the result extends easily to the integral setting by universal coefficients, as made explicit in [Kulkarni1]. We will use Donkin’s theorem without further comment. In particular note that any argument about \(\text{Ext}\) groups for \(\text{GL}_n\) that requires considering the involved representations as modules over a Schur algebra will need to use Donkin’s theorem. Let us now gather several results of a general nature about the \(\text{Ext}\) groups of interest.

To start with let us remark that for finitely generated representations \(M\) and \(N\), the groups \(\text{Ext}^i(M, N)\) for \(i > 0\) are all finite. Proving this will involve a comparison with extension of scalars to \(\mathbb{Q}\). The stated result follows from Donkin’s theorem (which shows that the \(\text{Ext}\) groups are finitely generated, e.g., on account of the appropriate Schur algebra being free of finite rank over \(\mathbb{Z}\)) combined with semisimplicity of representations over \(\mathbb{Q}\) and the Universal Coefficient Theorem. For a statement of this theorem for group schemes see [Jantzen, I.4.18a] and for a Schur algebra formulation, see [AB2, Theorem 5.3]. (Applying the Universal Coefficient Theorem requires \(M\) and \(N\) to be \(\mathbb{Z}\)-free, but one easily reduces to this case by taking the quotient of \(M\) and \(N\) by their respective torsion submodules \(M_{\text{tor}}\) and \(N_{\text{tor}}\), and then considering the appropriate long exact sequences of \(\text{Ext}\) groups.) We will also need to use the Universal Coefficient Theorem in some other places to relate integral \(\text{Ext}\) groups with modular ones.
Let us now record an important result due to Cline-Parshall-Scott-van der Kallen [CPSvdK, Corollaries 3.2 and 3.3].

**Vanishing Theorem.** (Cline-Parshall-Scott-van der Kallen) $\text{Ext}^i(K_\lambda, K_\mu) = 0$ unless $\lambda < \mu$ or $(\lambda = \mu$ and $i = 0)$.

**Remarks.** (1) The importance of this result for the knowledge of Ext groups is clear. In this paper we will need to use the result only in the case $\lambda = \mu$. (2) The integral result as formulated above is not stated explicitly in [CPSvdK], but it is immediate from the modular result stated and proved there. Alternatively, by imitating later arguments of Cline-Parshall-Scott or Irving, one can give an easy direct proof of the result over $\mathbb{Z}$ using the Schur algebra setting. These authors prove similar vanishing results (over fields) respectively in the general setting of “highest weight categories” [CPS] and “BGG categories” [Irving]. See [CPS, Lemma 3.8b] and [Irving, Proposition 4.4].

A key technical tool used to handle Ext groups in this paper will be the following theorem from [Kulkarni1].

**Skew Representative Theorem.** For any polynomial representation $X$ and for a partition $\lambda$ containing a partition $\mu$,

$$\text{Ext}^i(K_\lambda, L_\mu \otimes X) \simeq \text{Ext}^i(K_{\lambda/\mu}, X).$$

**Remark.** Note the necessity of the tacit assumption on $n$ mentioned earlier, namely $n \geq$ number of rows in $\lambda$. If this assumption were false, $K_\lambda$ would vanish, whereas the right hand side need not. This is because $\lambda/\mu$ may well have fewer rows than $\lambda$, possibly leading to nonzero Ext groups on the right hand side. (The necessary assumption on $n$ is made for the whole paper at the end of the second paragraph on p. 253 in [Kulkarni1], but it should have been included in the statements of the theorems there as well.) The “irrelevance of $n$” for the results in this paper is discussed after the proof of the next proposition.

The next proposition is a slight strengthening of a special case of the following result. [Green, Section 6.5] proves that if $N > n \geq r$, then the functor $d_{N,n}$ is an equivalence of categories of modules over the Schur algebras $S(N, r)$ and $S(n, r)$. In particular it preserves Ext groups over these Schur algebras and by Donkin’s theorem also over the corresponding general linear groups.

**Proposition 1.1.** For partitions $\lambda$ and $\mu$, let $N > n \geq$ the number of rows in $\lambda$ and number of rows in $\mu$. Let $E$ and $F$ be free abelian groups of rank $N$ and $n$ respectively. Then we get the following isomorphisms via the functor $d_{N,n}$ of [Green, Section 6.5].

$$\text{Ext}^i_{GL(E)}(K_\lambda(E), K_\mu(E)) \simeq \text{Ext}^i_{GL(F)}(K_\lambda(F), K_\mu(F)).$$

**Proof.** [AB2, Section 4] gives a resolution of $K_\lambda(E)$. The terms of this resolution are direct sums of modules of the form $D_\nu(E)$ (i.e., direct sums of tensor products of divided powers of $E$), where $\nu$ has at most as many nonzero parts as $\lambda$. The construction of this resolution is functorial. So the preceding sentences in this proof remain valid after replacing
by $F$ and in fact the two resolutions are the “same.” Speaking more precisely, the functor $d_{N,n}$ applied to the [AB2] resolution of $K_\lambda(E)$ gives the [AB2] resolution of $K_\lambda(F)$. In particular $D_\nu$ for the same $\nu$ and the same direct sums appear in either resolution.

Note that by the hypothesis on $N$ and $n$, all the $\nu$ that appear in the terms of either resolution are weights for $\text{GL}(E)$ as well as for $\text{GL}(F)$. (In fact we can arrange all $\nu$ to be $\geq \lambda$ under the dominance order, but we don’t need this stronger fact.) This ensures that all the $D_\nu$ are projective modules over the appropriate Schur algebra, see [AB2]. Thus we may compute the Ext groups under consideration by applying the appropriate Hom to these two resolutions. Now the functor $d_{N,n}$ gives a chain map from the Hom complex involving $E$ to that involving $F$. We can see as follows that this map is an isomorphism. First, by [AB2, Section 2], $\text{Hom}(D_\nu(F), K_\mu(F))$ is isomorphic to the $\nu$-weight submodule of $K_\mu(F)$; in fact this isomorphism is natural so in particular the statement holds if we replace every $F$ by $E$. Secondly the functor $d_{N,n}$ does not change a $\nu$-weight submodule if $\nu$ is a weight for $\text{GL}_n$ (and annihilates it otherwise, but this does not happen in our situation). This completes the proof of Proposition 1.1.

Remark. The above result is true by the same argument if one replaces $K_\lambda$ and $K_\mu$ by skew Weyl modules (or even more generally, functors with finite Weyl filtrations). $n$ should be taken at least as much as the number of nonezero rows in the corresponding skew partitions (for the more general version, at least as big as the number of rows in the partitions corresponding to all Weyl modules that occur as filtration factors in the two modules). One just has to construct suitable projective resolutions for these modules from such resolutions for the individual filtration factors. (Or, in case of skew Weyl modules, use the projective resolutions already constructed in [AB2].)

Beyond its use in proving Proposition 1.2, Proposition 1.1 is not really necessary for the rest of the paper. But let us use the logical opportunity it provides to discuss the minimal relevance of the $n$ in $\text{GL}_n$ for the main results in this paper regarding Ext groups between Weyl modules. Proposition 1.1 is not enough by itself to address this issue because the proofs will more generally employ Ext groups involving modules with Weyl filtrations. In view of Green’s result quoted before Proposition 1.1, we can a priori take $n$ to be large enough. After a “stable” result about Ext groups between two Weyl modules is proved, Proposition 1.1 then extends such a result to all $n$ for which the partitions involved in the statement of the result are weights for $\text{GL}_n$. But a better reasoning is to simply note that the proofs of the main results stay valid for any such “appropriate” value of $n$. (For details on how various ingredients involved in the proofs behave vis a vis the value of $n$, see the Remark after Proposition 1.1, the Remark after the statement of the Skew Representative Theorem above, and the paragraph discussing stability of skew Weyl modules under variation of rank of $F$.)

Before turning to Proposition 1.2 we need to record the following isomorphisms of Ext groups. First, by contravariant duality, $\text{Ext}^i(K_\lambda, K_\mu) \simeq \text{Ext}^i(L_\mu, L_\lambda)$. Combining this with the Ext-preserving functor $\Omega$ (a weak form of Howe duality) from [AB2, Section 7], one gets the following conjugate symmetry of Ext groups. $\text{Ext}^i(K_\lambda, K_\mu) \simeq \text{Ext}^i(K_\mu, K_\lambda)$. 

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Note that one may need to increase $n$ (valid by Proposition 1.1) so that the conjugate partitions are weights for $GL_n$.

**Proposition 1.2.** (column and row removal principles) Suppose the lengths of the first columns (rows) of partitions $\lambda$ and $\mu$ are equal. Let $\lambda'$ and $\mu'$ be the partitions obtained by removing the first columns (rows) of $\lambda$ and $\mu$ respectively. Then $\text{Ext}^i(K_\lambda, K_\mu) \simeq \text{Ext}^i(K_{\lambda'}, K_{\mu'})$.

**Proof.** Let us first prove the column removal principle. Call the common length of the first columns $m$. Evidently $m \leq n$. By Proposition 1.1 we can change $n$ to $m$. Then $K_\lambda = K_{\lambda'} \otimes \text{det}, K_\mu = K_{\mu'} \otimes \text{det}$, where $\text{det}$ is the one-dimensional determinant representation. The claim is immediate by canceling the determinant. Row removal now follows by conjugate symmetry of Ext groups. (It may be necessary to increase and then restore $n$, again using Proposition 1.1.)

**Remark.** Results similar to the two Propositions above, but about decomposition numbers in characteristic $p$, have been proved in [Green] and [James] respectively. Let us outline an approach to these earlier results using certain Ext groups, namely those between Weyl modules and simple modules in characteristic $p$. The analogues of Proposition 1.1 and the column removal principle for these “mixed” Ext groups are easily obtained by repeating the arguments above. Combining this with the well-known connection between Euler characteristics built out of such Ext groups and decomposition numbers, one gets [Green, Theorem 6.6e] and [James, Theorem 1]. For [James, Theorem 2] (row removal principle for decomposition numbers), additional arguments seem necessary, as the functor $\Omega$ is not available in this situation. For instance one can use the connection with tilting multiplicities obtained by Donkin to get the desired result. Let us skip the details.

### 2. ON THE EXTENSIONS BETWEEN WEYL MODULES

This section contains the main results of this paper concerning Ext groups between certain Weyl modules for $GL_n$ over the integers. The significance of these results for modular representation theory is discussed in the Remarks after each result.

Let us outline the general strategy that we will use to study Ext groups between Weyl modules. As remarked before, one can without loss of generality take the corresponding two dominant weights to be partitions with the same number of boxes. Now the Skew Representative Theorem [Kulkarni1] gives the following isomorphism

$$\text{Ext}^i(K_\lambda, L_{\bar{\mu}} \otimes K_{\nu}) \simeq \text{Ext}^i(K_{\lambda/\mu}, K_{\nu}),$$

where $\lambda, \mu, \nu$ are partitions with the diagram of $\mu$ contained in that of $\lambda$ and $|\nu| = |\lambda| - |\mu|$. (Let me again mention that following [ABW], $L_{\bar{\mu}}$ denotes the dual Weyl module with highest weight $\mu$.) Notice that the degree of representations occuring on the LHS is $|\lambda|$ whereas that on the RHS is $|\lambda| - |\mu|$. This leads to the hope of somehow analyzing Ext groups recursively, using the just mentioned reduction in degree. A difficulty involved
in this approach is that one has to deal with more complicated representations, namely $K_{\lambda/\mu}$ and $L_{\mu} \otimes K_{\nu}$. One can try to deal with these two objects piece by piece by working with their filtrations. One does have a Weyl filtration of $K_{\lambda/\mu}$ which, however, is very complicated in general. Worse still, no information in general is known about the filtrations of the latter.

One can get around both these difficulties by choosing $L_{\mu}$ to be an exterior power $\Lambda^t$ of the defining representation, i.e., by taking $\mu = 1^t$, a single column of length $t$. Since $\Lambda^t$ is a Weyl module as well, $\Lambda^t \otimes K_{\nu}$ also has a Weyl filtration. Weyl filtrations of both these modules (i.e., of $K_{\lambda/1^t}$, and $\Lambda^t \otimes K_{\nu}$) are constructed explicitly in [AB1, Section 3]. These filtrations are fairly simple to describe and one has a very easy description of the Weyl modules that occur as filtration factors by Pieri-type rules. In the simplest new case (treated in Theorem 2.1) one can inductively control the maps induced on Ext groups as we patch together the Weyl modules in these filtrations. In order to seriously entertain the idea of computing all Ext groups between Weyl modules using this technique, one must be able to grapple successfully with spectral sequences arising from Ext groups applied to filtered complexes whose terms have Weyl filtrations. This seems difficult in general. Here is what one can do at present using the approach outlined above.

(1) Theorem 2.1 gives a complete determination of the groups $\text{Ext}^i(K_{\lambda}, K_{\mu})$ where $\mu - \lambda$ is a positive root $\alpha$ of $\text{GL}_n$. This is the simplest case left open after the [CPSvdK] vanishing theorem. (2) For general $\lambda < \mu$, one can circumvent the difficulty involved in dealing with filtrations that was mentioned above by attempting a less ambitious task. One can compute a “multiplicative Euler characteristic” $\chi$ defined as the alternating product of cardinalities of Ext groups between a given pair of Weyl modules. One first gets a recursive algorithm to do this computation which in turn leads to a simple formula for $\chi$. This is done in [Kulkarni2]. The base of the recursion is precisely the case treated by Theorem 2.1. (3) The Skew Representative Theorem takes a particularly simple form if the first partition is a single column. Using this, one can show that $\text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda})$ is always cyclic. A little more reasoning enables one to explicitly calculate the order of this Ext group in terms of $\lambda$. By contravariant duality and conjugate symmetry of Ext groups, one also determines three other types of $\text{Ext}^1$ groups. This is the content of Theorem 2.2.

Let us tackle Theorem 2.2 first as its proof is much shorter and gives a quick demonstration of the use of two tools that will be used much more substantially while proving Theorem 2.1: (1) the Skew Representative Theorem and (2) the [ABW] description of Weyl modules by generators and relations. Theorem 2.2 computes the group $\text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda})$. Note that if the partition $\lambda$ has only one column, then $K_{\lambda} = \Lambda_{|\lambda|}$ and this Ext group is trivial by the [CPSvdK] vanishing theorem. So let us assume that $\lambda$ has at least two columns.

**Theorem 2.2.** The group $\text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda})$ between an exterior power of the defining representation and another Weyl module is always cyclic and its order is the gcd $g$ of integers

$$\frac{a + 1}{\gcd(a + 1, \text{lcm}(1, 2, 3, \ldots, b))}$$
where \(a, b\) are the lengths of any two consecutive columns of \(\lambda\) with \(a \geq b\).

**Corollary.** The statement of Theorem 2.2 stays valid if we replace the Ext group there by any of the following: \(\text{Ext}^1(L_{\tilde{\lambda}}, \Lambda_{|\lambda|})\), \(\text{Ext}^1(K_{\tilde{\lambda}}, D_{|\lambda|})\) and \(\text{Ext}^1(S_{|\lambda|}, L_{\lambda})\), where \(D\) and \(S\) denote respectively the divided and symmetric powers of the defining representation. (This follows from the fact that the four Ext groups in question are isomorphic by contravariant duality and conjugate symmetry.)

**Proof of Theorem 2.2.** If \(\lambda\) has exactly two columns of lengths \(a \geq b\), then by [AB2, Section 9] \(\text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda})\) is cyclic and its order is as given above. If \(\lambda\) has more than two columns, pick one of them, say of length \(\ell\), and call the partition left by erasing that column \(\lambda'\). Take the exact sequence \(0 \rightarrow K_{\lambda} \rightarrow K_{\lambda'} \otimes \Lambda_{\ell} \rightarrow X \rightarrow 0\) obtained from the Weyl filtration of \(K_{\lambda'} \otimes \Lambda_{\ell}\) given in [AB1, Section 3]. Apply \(\text{Hom}(\Lambda_{|\lambda|}, -)\) and take the associated long exact sequence. It begins as follows. \(0 \rightarrow \text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda}) \rightarrow \text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda'} \otimes \Lambda_{\ell})\). By the Skew Representative Theorem, \(\text{Ext}^1(\Lambda_{|\lambda|}, K_{\lambda'} \otimes \Lambda_{\ell}) \simeq \text{Ext}^1(\Lambda_{|\lambda'|}, K_{\lambda'}).\) Now we can repeat the process until only two columns are left. Clearly these can be arranged to be any two columns of the original partition \(\lambda\). So we have proved that the desired \(\text{Ext}^1\) is cyclic and that its order divides \(g\). To show that the order equals \(g\), we will instead use \(\text{Ext}^1(K_{\tilde{\lambda}}, D_{|\lambda|})\), which is isomorphic to the desired \(\text{Ext}^1\).

Let us recall some facts from [ABW, Section II] about \(K_{\tilde{\lambda}}\) (see also Section 3, specifically the beginning of the proof of Lemma C and Note 2 at the end of Example 2 in the same proof). We have \(d'_{\tilde{\lambda}} : D_{\tilde{\lambda}} \rightarrow K_{\tilde{\lambda}}\). Label a basis of the defining representation as \(e_1, e_2, \ldots\). Then we have a cyclic generator \(e_1^{(\tilde{\lambda}_1)} \otimes e_2^{(\tilde{\lambda}_2)} \otimes \ldots\) for \(D_{\tilde{\lambda}}\) (and hence one for \(K_{\tilde{\lambda}}\) and the kernel of \(d'_{\tilde{\lambda}}\) is generated by elements

\[
\ldots \otimes e_k^{(a)} \otimes e_k^{(t)} e_{k+1}^{(b-t)} \otimes \ldots, \quad 1 \leq t \leq b,
\]

where \(k, k+1\) are any two consecutive rows of the partition \(\tilde{\lambda}\) of lengths \(a \geq b\) respectively and “…” indicates tensor factors \(e_i^{(\tilde{\lambda}_i)}\) in all positions \(i\) other than \(k\) and \(k+1\).

By the Universal Coefficient Theorem it suffices to show that \(g\) is the largest integer modulo which there is a nonzero equivariant map \(K_{\tilde{\lambda}} \rightarrow D_{|\lambda|}\). So let us work over \(\mathbb{Z}/m\mathbb{Z}\) and characterize the integers \(m\) for which such a map exists. Since all weight spaces of \(D_{|\lambda|}\) are one-dimensional, \(\text{Hom}(D_{\tilde{\lambda}}, D_{|\lambda|})\) is generated by the map taking the cyclic generator \(e_1^{(\tilde{\lambda}_1)} \otimes e_2^{(\tilde{\lambda}_2)} \otimes \ldots\) to the basis element \(e_1^{(\tilde{\lambda}_1)} e_2^{(\tilde{\lambda}_2)} \ldots\) of the relevant weight space. Clearly this map is just multiplication in the divided power algebra. This map will descend to \(K_{\tilde{\lambda}}\) exactly when it kills the generators of the kernel of \(d'_{\tilde{\lambda}}\) listed above. Recalling how to multiply in the divided power algebra, this in turn will happen exactly when \(\binom{a+t}{t} = 0\) in \(\mathbb{Z}/m\mathbb{Z}\) for all \(1 \leq t \leq b\), where \(a \geq b\) are lengths of any two consecutive rows of \(\tilde{\lambda}\), i.e., consecutive columns of \(\lambda\). Fixing \(a\) and \(b\), the \(gcd\) of the resulting \(b\) binomial coefficients is easily seen to be the number displayed in the statement of Theorem 2.2, thus completing the proof.
Remarks. (1) It should be clear that the argument in the previous paragraph can prove all of Theorem 2.2, making the first paragraph of the proof redundant. (The first paragraph was nevertheless included to illustrate how the Skew Representative Theorem may be used.) Such a streamlined proof would also recover as a special case the result from [AB2, Section 9] quoted in the first paragraph. Actually the argument in the previous paragraph is close to the one in [AB2], just formulated differently and applied in a more general situation.

(2) In fact one can get an even more general result from the argument in the last paragraph of the proof. Since skew Weyl modules are also cyclic and have a description by generators and relations, one can easily calculate by the same reasoning Ext¹ between any skew Weyl module and a divided power of the defining representation (and three other types of Ext groups by the argument in the Corollary). This group is also cyclic and its order is equal to the gcd of several binomial coefficients determined by lengths of and overlap between successive pairs of adjacent rows of the skew partition in question. (The simplification contained in the last sentence of the proof will generally not be available here.)

(3) Let us work in prime characteristic \( p \) in this and the next paragraph. Again using the Universal Coefficient Theorem, the Corollary to Theorem 2.2 shows that Hom\((S_{[\lambda]}, L_{\lambda})\) always has dimension 0 or 1, and that it is one-dimensional exactly when \( p \) divides the number appearing in Theorem 2.2 for each pair of successive column lengths \( a, b \) of the partition \( \lambda \). Thus we get a necessary and sufficient condition for the existence of a homomorphism from a symmetric power of the defining representation to a dual Weyl module \( L_{\lambda} \) of largest weight \( \tilde{\lambda} \). This condition is easier to test by writing the length of each row of weight \( \tilde{\lambda} \) (we are expressing everything in terms of the highest weight \( \tilde{\lambda} \)) \( p \)-adically and displaying the digits for successive rows of \( \lambda \) in a kind of tableau form. Thus the \( i \)-th column from right of this “digit tableau” displays the \( p^i \)-place digits of successive row lengths of \( \tilde{\lambda} \). The condition obtained from Theorem 2.2 then amounts to the requirement that in the digit tableau each digit strictly above and weakly to the right of any nonzero digit must be \( p - 1 \).

Note that the existence of a homomorphism from a symmetric power of the defining representation to a dual Weyl module \( L_{\lambda} \) implies in particular that the simple module with highest weight \( \tilde{\lambda} \) is a composition factor of that symmetric power, though of course the former is a significantly stronger requirement. It is interesting to compare the composition factors of \( S_{[\lambda]} \) – known as a result of the known GLₙ-submodule structure of symmetric powers [Doty] – with the condition in Theorem 2.2. This is easier to state using the \( p \)-adic digit tableau above. (This language is modelled on that in [Krop], which also proves the same submodule structure for symmetric powers under the action of the full matrix semigroup. The idea of exploiting one-dimensionality of weight spaces in symmetric/divided powers too was taken from the independent works of Doty and Krop.) The highest weights of the composition factors of \( S_{[\lambda]} \) are characterized by each column of digits in the digit tableau of the highest weight having the form of a string of \((p - 1)\)'s possibly ending with a single different digit. Clearly this is a weaker condition than the one in the previous
Theorem 2.1. Suppose \( \lambda \) and \( \mu \) are dominant weights for \( \text{GL}_n \) such that \( \mu - \lambda \) is a positive root \( \alpha \). Let \( \rho \) be half the sum of positive roots. Then \( \text{Ext}^i(K_\lambda, K_\mu) \) is cyclic of order \( \langle \lambda + \rho, \alpha \rangle + 1 \) for \( i = 1 \) and vanishes for all other \( i \).

To facilitate the proof of Theorem 2.1, let us translate this statement into combinatorial language and make some reductions. We have \( \lambda = \sum \lambda_i \epsilon_i \), where \( \epsilon_i \) is the weight consisting of 1 in the \( i \)-th position and zeroes elsewhere. Let \( \alpha \) be the positive root \( \epsilon_r - \epsilon_s \) for \( r < s \). Then the claimed order of \( \text{Ext}^1(K_\lambda, K_\mu) \) is \( \lambda_r - \lambda_s + s - r + 1 \). As noted before, without loss of generality we can take \( \lambda \) and \( \mu \) to be partitions with the same number of boxes.

Then the Young diagram of \( \mu \) is obtained from that of \( \lambda \) by raising a single box from the end of the \( s \)-th row up to the end of the \( r \)-th row. The number \( \lambda_s - \lambda_s + s - r + 1 \) is easily seen to be the hook length of the box in the \( r \)-th row and \( \lambda_s \)-th column of the diagram of \( \lambda \) (or the diagram of \( \mu \)). (Recall that the hook length of a box in the Young diagram of a partition is the total number of boxes to the right and below it, including itself.) Now by appealing to the row and column removal principles, we may further strip off the \( r - 1 \) identical first rows and \( \lambda_s - 1 \) identical first columns from \( \lambda \) and \( \mu \) without changing the Ext groups. Then the assertion about \( \text{Ext}^1(K_\lambda, K_\mu) \) amounts to saying that this group is cyclic of order equal to the hook length of the box in the top left corner of the diagram of \( \lambda \) (or \( \mu \)). Let us set up some notation so we can precisely state and prove Theorem 2.1 in this equivalent combinatorial form.

Let \( a_i, p_i \) for \( i = 1, \ldots, k \) be positive integers with \( a_1 > a_2 > \ldots > a_k \). Let \( \lambda = a_1^{p_1} \ldots a_k^{p_k} 1 \), \( \mu = (a_1 + 1)a_1^{p_1-1}a_2^{p_2} \ldots a_k^{p_k} \) and \( \nu = a_1^{p_1} \ldots a_k^{p_k} \). In words, \( \nu \) is a partition whose diagram consists of \( k \) rectangular blocks of rows; \( \lambda \) and \( \mu \) are obtained by adding a single box to the first column and the first row of \( \nu \) respectively. Let \( h_i \) be the hook length of the \((p_1 + \ldots + p_{i-1} + 1)\)-th box from top in the first column of \( \lambda \), i.e., the top left box in the \( i \)-th rectangular block in the diagram of \( \lambda \). Let \( \ell_i \) be the hook length of the \((p_1 + \ldots + p_i)\)-th box from top in the first column of \( \nu \), i.e., the bottom left box in the \( i \)-th rectangular block in the diagram of \( \nu \). For future use, note that \( \ell_j = h_j - p_j = a_j + p_{j+1} + p_{j+2} + \ldots + p_k \). Now we can state and prove an equivalent form of Theorem 2.1 using this notation. For technical convenience we will exclude the Hom case from the statement. (See the parenthetical note at the end of the first paragraph of the proof.) We may do so since it is already known that \( \text{Hom}(K_\lambda, K_\mu) = 0 \) for \( \lambda \neq \mu \).

Theorem 2.1. (Combinatorial version) For \( \lambda, \mu \) as in the preceding paragraph,

\[
\text{Ext}^1(K_\lambda, K_\mu) = \mathbb{Z}/h_1 \mathbb{Z}, \quad \text{Ext}^i(K_\lambda, K_\mu) = 0 \text{ for } i > 1.
\]

Proof. Note that \( \mu \) is obtained by removing the single box from the last row of \( \lambda \) and placing it at the end of the first row. By induction we assume the result for all such pairs \((\lambda, \mu)\) of smaller degree. This induction starts in degree 1. Here \( \nu \) must be empty, \( \lambda = \mu = \) a single box and \( h_1 = 1 \). We need to show that all \( \text{Ext}^i \) vanish for \( i > 0 \), which is immediate.
by the [CPSvdK] vanishing theorem. (This case does not exist in the original formulation of Theorem 2.1, but does make sense for the combinatorial version we are proving. Also note that in this case Hom is nonzero, which is why the Hom case was excluded in the statement of the theorem. Alternatively we could have started the induction in degree 2, where again there is just one case, namely when \( \nu \) consists of a single box. Then one needs to show that \( \text{Ext}^i(\Lambda_2, D_2) \) is \( \mathbb{Z}/2\mathbb{Z} \) for \( i = 1 \) and 0 otherwise. This is immediate, e.g., from the projective resolution \( 0 \to D_2 \to F \otimes F \to \Lambda_2 \to 0 \). But in fact this case is subsumed in the inductive step below.)

We will use the explicit Weyl filtrations for \( K_{\lambda/1} \) and \( K_{\nu} \otimes F \) constructed in [AB1, Section 3]. The filtration factors are described by Pieri-type rules. The factors for \( K_{\lambda/1} \) correspond to partitions obtained by deleting one box from the diagram of \( \lambda \). The factors for \( K_{\nu} \otimes F \) correspond to partitions obtained by adding one box to the diagram of \( \nu \).

The Skew Representative Theorem implies that \( \text{Ext}^i(K_{\lambda}, F \otimes K_{\nu}) \simeq \text{Ext}^i(K_{\lambda/1}, K_{\nu}). \) So let us consider the [AB1] Weyl filtrations of \( K_{\lambda/1} \) and \( F \otimes K_{\nu}. \) These give us the following exact sequences: \( 0 \to K_{\nu} \to K_{\lambda/1} \to M \to 0, 0 \to K_{\mu} \to F \otimes K_{\nu} \to N \to 0 \) and \( 0 \to P \to N \to K_{\lambda} \to 0 \), where the Weyl modules occurring in \( grM, grN \) and \( grP \) are prescribed by Pieri’s rules. We will analyze certain long exact sequences associated to these short exact sequences.

First apply \( \text{Hom}(-, K_{\nu}) \) to \( 0 \to K_{\nu} \to K_{\lambda/1} \to M \to 0 \) and take the corresponding long exact sequence. By [CPSvdK], \( \text{Ext}^i(K_{\nu}, K_{\nu}) = 0 \) for \( i \neq 0 \). Our inductive hypothesis combined with the row removal principle applies as we patch together the Weyl modules occurring in \( grM \) and take the corresponding long exact sequences of Ext groups with \( K_{\nu} \). We get as a result that \( \text{Ext}^i(M, K_{\nu}) = 0 \) for \( i \neq 1 \) and that the cardinality of \( \text{Ext}^1(M, K_{\nu}) \) equals \( \ell_1 \ldots \ell_k \). Known information now forces \( \text{Ext}^i(K_{\lambda/1}, K_{\nu}) = 0 \) for \( i > 1 \). Therefore the long exact sequence reduces to

\[
0 \longrightarrow \text{Hom}(K_{\lambda/1}, K_{\nu}) \longrightarrow \text{Hom}(K_{\nu}, K_{\nu}) \longrightarrow \text{Ext}^1(M, K_{\nu}) \longrightarrow \text{Ext}^1(K_{\lambda/1}, K_{\nu}) \longrightarrow 0.
\]

As \( K_{\nu} \) occurs once in \( gr(K_{\lambda/1}) \), the first two Hom terms are both isomorphic to \( \mathbb{Z} \). Therefore the map between them is given by an integer. Now we state

**Lemma A.** The map \( \text{Hom}(K_{\lambda/1}, K_{\nu}) \to \text{Hom}(K_{\nu}, K_{\nu}) \) induced by the inclusion \( K_{\nu} \hookrightarrow K_{\lambda/1} \) arising from the [AB1] Weyl filtration of \( K_{\lambda/1} \) is given by the integer \( \ell_1 \ldots \ell_k \).

Assuming Lemma A, \( \text{Ext}^1(M, K_{\nu}) \) is forced to be isomorphic to \( \mathbb{Z}/\ell_1 \ldots \ell_k \mathbb{Z} \) and \( \text{Ext}^1(K_{\lambda/1}, K_{\nu}) \) is seen to vanish. To sum up, modulo Lemma A, we have proved that \( \text{Ext}^i(K_{\lambda/1}, K_{\nu}) = 0 \) for \( i \neq 0 \). Therefore \( \text{Ext}^i(K_{\lambda}, F \otimes K_{\nu}) = 0 \) for \( i \neq 0 \) by the Skew Representative Theorem.

Next we apply \( \text{Hom}(K_{\lambda}, -) \) to \( 0 \to P \to N \to K_{\lambda} \to 0 \) and take the corresponding long exact sequence. \( \text{Ext}^i(K_{\lambda}, K_{\lambda}) = 0 \) for \( i \neq 0 \) by the [CPSvdK] vanishing theorem. Again our inductive hypothesis combined with row removal principle applies as we patch together the Weyl modules occurring in \( grP \) and take the corresponding long exact sequences.
of Ext groups of $K_{\lambda}$ with the resulting modules. We get as a result that $\text{Ext}^i(K_{\lambda}, P) = 0$ for $i \neq 1$ and that the cardinality of $\text{Ext}^1(K_{\lambda}, P)$ equals $h_2 \ldots h_k$. Known information now forces $\text{Ext}^i(K_{\lambda}, N) = 0$ for $i > 1$. Therefore the long exact sequence reduces to

$$0 \rightarrow \text{Hom}(K_{\lambda}, N) \rightarrow \text{Hom}(K_{\lambda}, K_{\lambda}) \rightarrow \text{Ext}^1(K_{\lambda}, P) \rightarrow \text{Ext}^1(K_{\lambda}, N) \rightarrow 0.$$ 

As $K_{\lambda}$ occurs once in $\text{gr}N$, the first two Hom terms are both isomorphic to $\mathbb{Z}$. Therefore the map between them is given by an integer. Now we state

**Lemma B.** *The map* $\text{Hom}(K_{\lambda}, N) \rightarrow \text{Hom}(K_{\lambda}, K_{\lambda})$ *induced by surjection* $N \rightarrow K_{\lambda}$ *arising from the* $[AB1]$ *Weyl filtration of* $F \otimes K_\nu$ *is given by the integer* $h_2 \ldots h_k$.

Assuming Lemma B, $\text{Ext}^1(K_{\lambda}, P)$ is forced to be isomorphic to $\mathbb{Z}/h_2 \ldots h_k \mathbb{Z}$ and $\text{Ext}^1(K_{\lambda}, N)$ is seen to vanish. So modulo Lemma B, we have proved that $\text{Ext}^i(K_{\lambda}, N) = 0$ for $i \neq 0$.

Now we are ready to apply $\text{Hom}(K_{\lambda}, \_)$ to $0 \rightarrow K_\mu \rightarrow F \otimes K_\nu \rightarrow N \rightarrow 0$ and analyze the resulting long exact sequence. Clearly $\text{Hom}(K_{\lambda}, K_\mu) = 0$. We already know that $\text{Ext}^i(K_{\lambda}, N) = 0$ and $\text{Ext}^i(K_{\lambda}, F \otimes K_\nu) = 0$ for $i \neq 0$. Therefore $\text{Ext}^i(K_{\lambda}, K_\mu)$ is forced to vanish for $i > 1$ and the exact sequence reduces to

$$0 \rightarrow \text{Hom}(K_{\lambda}, F \otimes K_\nu) \rightarrow \text{Hom}(K_{\lambda}, N) \rightarrow \text{Ext}^1(K_{\lambda}, K_\mu) \rightarrow 0.$$ 

As $K_{\lambda}$ occurs once as a factor in the filtrations of $F \otimes K_\nu$ as well as $N$, the two Hom terms are both isomorphic to $\mathbb{Z}$. Therefore the map between them is given by an integer. Now we state

**Lemma C.** *The map* $\text{Hom}(K_{\lambda}, F \otimes K_\nu) \rightarrow \text{Hom}(K_{\lambda}, K_{\lambda})$ *induced by the surjection* $F \otimes K_\nu \rightarrow K_{\lambda}$ *arising from the* $[AB1]$ *Weyl filtration of* $F \otimes K_\nu$ *is given by the integer* $h_1 \ldots h_k$.

Now Lemma B and Lemma C together imply easily that the map $\text{Hom}(K_{\lambda}, F \otimes K_\nu) \rightarrow \text{Hom}(K_{\lambda}, N)$ is given by the integer $h_1$. Therefore $\text{Ext}^1(K_{\lambda}, K_\mu) \simeq \mathbb{Z}/h_1 \mathbb{Z}$. Thus Theorem 2.1 is proved modulo Lemmas A, B and C. The proofs of these lemmas are similar to each other and somewhat intricate. They will be presented in the next section.

**Remarks.** (1) Let us discuss the significance of Theorem 2.1 for modular representation theory. So we will work over a field of characteristic $p$ until further notice. Consider the dot action of the affine Weyl group $W_p$ on weights. Let $\lambda$ be a dominant regular weight in an alcove $C$ and $s$ the reflection in a wall of $C$ such that $\lambda < s.\lambda$ (and so $s.\lambda$ is also dominant). Then $K_{\lambda}$ and $K_{s.\lambda}$ are called neighboring Weyl modules. It is known that for any split reductive algebraic group $\text{Ext}^i(K_{\lambda}, K_{s.\lambda})$ is one dimensional for $i = 0$ or 1 and vanishes otherwise. See, e.g., [Jantzen, II.7.19d]. The nonzero homomorphism is of interest because, for instance, it is involved in an important conjecture of Jantzen. See [Andersen1]. Here is how Theorem 2.1 combined with the translation principle can be used to recover this Ext calculation for $\text{GL}_n$. (This strategy is admittedly strange, since the stated result was obtained using just translation functors, albeit in a little more substantial way. The
point here is to observe that the case in Theorem 2.1 is enough to realize neighboring modules between any two given adjacent dominant alcoves.) We have $s.\lambda - \lambda = k\alpha$, a positive multiple of a positive root $\alpha$. Consider first the case $k = 1$, so the two weights are as in Theorem 2.1. It is easy to see that $p$ divides $\langle \lambda + \rho, \alpha^\vee \rangle + 1$ and then the Universal Coefficient Theorem gives the desired modular Ext groups. Now suppose $k > 1$. Note that by the translation principle $\text{Ext}^i(K_\lambda, K_{s.\lambda}) \simeq \text{Ext}^i(K_\eta, K_{s.\eta})$ for any dominant regular weight $\eta$ in $C$. We will choose $\eta$ close to the wall separating $C$ and $s.C$ so as to reduce to the case $k = 1$. For this first note that all weights between the two regular weights $\lambda$ and $s.\lambda$ (on the straight line connecting them) except $\lambda + (k/2)\alpha$ are regular and lie either in $C$ or $s.C$. The exception, not necessarily an integral weight, is where the line intersects the wall separating $C$ and $s.C$. If $k = 2t + 1$ is odd, then let $\eta = \lambda + t\alpha$ and then $s.\eta = \lambda + (t + 1)\alpha$, reducing to the case $k = 1$. If $k = 2t$ is even, then we may first replace $\lambda$ by $\lambda + (t - 1)\alpha$ and so $s.\lambda$ by $\lambda + (t + 1)\alpha$. In other words we may assume that $k = 2$. Let $\alpha = \epsilon_q - \epsilon_r$ with $q < r$. Now take $\eta = \lambda - \epsilon_r$. Then $s.\eta = s.\lambda - \epsilon_q$. (If $\lambda$ and $s.\lambda$ are partitions, $k = 2$ means $s.\lambda$ is obtained by moving two boxes from a lower row of $\lambda$ to a higher row. Now $\eta$ as above is simply obtained by erasing one of these two boxes from $\lambda$ and then $s.\eta$ similarly has one box less than $s.\lambda$.) It is easy to check that $\eta$ is in $C$ and that we are again in the case $k = 1$.

A reason why Theorem 2.1 is of additional interest is as follows. The definition of neighboring Weyl modules makes sense only if there are regular weights, which happens iff $p \geq n$, the Coxeter number for $\text{GL}_n$. But even for small $p$, Theorem 2.1 still gives results about $\text{Ext}$ groups between Weyl modules whose dominant weights differ by a single root.

(2) Continue with neighboring Weyl modules $K_\lambda$ and $K_{s.\lambda}$ as in Remark 1 but work over $\mathbb{Z}_p$, the ring of $p$-adic integers. Let $v$ be the largest power of $p$ dividing $\langle \mu + \rho, \alpha^\vee \rangle$, where $\mu$ is a weight on the wall separating the alcoves containing $\lambda$ and $s.\lambda$. About the same time when the combinatorial version of Theorem 2.1 was first proved, Andersen proved that for any split reductive algebraic group $\text{Ext}^1(K_\lambda, K_{s.\lambda})$ is cyclic of order $p^v$ and all other $\text{Ext}^i(K_\lambda, K_{s.\lambda})$ vanish. See [Andersen3]. For $\text{GL}_n$, Andersen’s result can be recovered from Theorem 2.1 and base change to $\mathbb{Z}_p$ using the same argument that was used in Remark 1 to recover the weaker result on modular Ext. (As shown in [Andersen1] translation functors work over $\mathbb{Z}_p$ too.) Again the result over $\mathbb{Z}_p$ cannot apply for small $p$. But Andersen’s result and Theorem 2.1 together lead one to hope that the statement of Theorem 2.1 should stay valid for any reductive algebraic group over $\mathbb{Z}$. Granting the use of translation, such a result (if true) can be thought of as a natural common generalization of Theorem 2.1 and Andersen’s result. See the Introduction for an indication of some possible approaches to proving such a result.

(3) Now let $\lambda$ and $\mu$ be as in the statement of Theorem 2.1, but work over a field of characteristic $p$. Theorem 2.1 contains more information than is registered in the modular calculation of the groups $\text{Ext}^i(K_\lambda, K_\mu)$ indicated in Remark 1. The extra information is the power of $p$ dividing $\langle \lambda + \rho, \alpha^\vee \rangle + 1$. This power turns out to be the coefficient of $\chi(K_\lambda)$ in the Jantzen sum formula for $K_\mu$ obtained from the determinant of the contravariant form on integral $K_\mu$. Actually in general this coefficient is encoded in the Euler characteristic $\chi$. 

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defined in the discussion near the beginning of this section, but for the case in Theorem 2.1 \( \chi \) amounts to \( \text{Ext}^1 \) due to the vanishing of other Exts. This connection with \( \chi \) is explained in [Kulkarni2]. It is unclear if the vanishing of higher Exts in Theorem 2.1 has any direct representation theoretic significance.

(4) The proof of Theorem 2.1 illustrates the lines along which one may hope to calculate Ext groups in general using the Skew Representative Theorem. In view of the discussion on neighboring Weyl modules, one case of interest is the generalization when \( \mu - \lambda = k\alpha \) with \( k > 1 \). (This is a generalization because for neighboring Weyl modules the \( k \) depends on the weight \( \lambda \) and the prime \( p \).) The difficulties here are formidable, as can be seen from the complexity of the answer calculated in [BF] for \( \text{Ext}^1 \) in this case for \( \text{GL}_3 \). Unlike the simple answer given by Theorem 2.1, the answer in [BF] involves taking the \( \gcd \) of many numbers (this is true even for the \( \text{GL}_2 \) case in [AB2] that was used in Theorem 2.2) and uses the lengths of all rows of \( \lambda \).

In the light of quickly mounting and seemingly inherent difficulties that one faces when \( k > 1 \), the fact that one has such a simple answer over \( \mathbb{Z} \) for \( k = 1 \) (and possibly valid for other reductive groups too) acquires added interest. It is tempting to wonder if the \( k = 1 \) case can at least partially serve as a reasonable \( \mathbb{Z} \) analogue of the notion of neighboring Weyl modules.

3. PROOFS OF LEMMAS A, B AND C

The strategy behind the proofs of the three lemmas is quite simple, but to carry it out requires some notational and computational effort. Each lemma identifies the integer giving a specific map between two Hom groups each of which is isomorphic to \( \mathbb{Z} \). The proofs explicitly find a generator of the source Hom group (the hardest step), apply the map to this generator, and inspect which multiple of the generator one gets in the target Hom group. The hardest step comes down to explicitly finding and solving a linear system of integer equations. What is noteworthy is that beyond keeping track of all the paraphernalia required for bookkeeping, the lemmas offer no further difficulty. The computations stay reasonable at all stages, in particular one does not have to resort to taking \( \gcd \)'s at any point. This should be contrasted with what usually happens in this type of setting, e.g., see Remark 4 after the proof of Theorem 2.1. Further, one can give very similar (and equally long) proofs for all three lemmas and all features occurring in such proofs (except some straightening required in one step for Lemma A) can be already seen in the much simpler special cases of Lemma C worked out below. Moreover, here we will be able to get away with doing the hardest step only once (for Lemma C) and thus substantially shorten the proofs of the remaining two lemmas.

These facts suggest that there may be a more efficient way to organize these computations, and perhaps even existence of a more conceptual explanation. The following considerations might be relevant in this regard. Just like the answers claimed in the three lemmas, several objects in the proofs are naturally indexed by \( k \)-tuples, where \( k \) is the
number of rectangular blocks in the partition $\nu$. It seems plausible that there should be a simpler approach to proving these lemmas that uses induction on $k$. Such induction does make an appearance in the proofs below, but only at the very end when one has to simplify a laboriously obtained algebraic expression. In another direction, since the order of the Ext$^1$ in Theorem 2.1 arises ultimately from the three lemmas, it is interesting to look for the origin of Andersen’s $p$-adic answer [Andersen3] in his proof. In his computation, a role analogous to that of the three lemmas is played by [Andersen2, Lemmas 2.2 and 2.4]. To prove these lemmas Andersen gives a short and slick argument using adjointness properties of translation functors. He then brings in the Weyl character formula to get the final numerical form of the answer. It would be nice to be able to adapt his approach to the integral situation (perhaps still working with one prime at a time, but letting the prime to be arbitrary, i.e., allowing nonregular weights and thereby allowing more complicated modules with Weyl filtration to arise upon translation).

After these speculations let us turn to the proofs of the lemmas, beginning with Lemma C. Throughout we will use the notation set up before the combinatorial version of Theorem 2.1. Additionally, let us make two notes about terminology used throughout the three lemmas. First, by “natural” maps, we will mean certain maps derived from the Weyl filtrations analogous to Pieri rules in [AB1]. These are the maps with which we have to compose to get the appropriate maps between Hom groups as indicated in the statements of the lemmas. Secondly, note the following abuse of notation. $K_\nu \otimes F$ is isomorphic to the skew Weyl module corresponding to the skew partition

$$(a_1 + 1)^{p_1} (a_2 + 1)^{p_2} \ldots (a_k + 1)^{p_k} 1/1^{p_1} + \ldots + p_k,$$

so for simplicity we will call this skew partition $\nu \otimes 1$. For example we will speak of tableaux of shape $\nu \otimes 1$, the generalized symmetrizer map $d'_\nu \otimes 1$, etc. In the proof of Lemma C, often we will even write $K_\nu \otimes 1$ meaning, of course, $K_\nu \otimes F$.

**Proof of Lemma C.** We will explicitly find a generator $f$ of Hom($K_\lambda$, $K_\nu \otimes 1$), follow the action of $f$ with the natural surjection $K_\nu \otimes 1 \rightarrow K_\lambda$, and see which multiple of the identity we get in Hom($K_\lambda$, $K_\lambda$).

Recall the results regarding skew Weyl modules summarized in [ABW, Theorem II.3.16]. Following the notation there we have $d'_\lambda : D_\lambda \rightarrow K_\lambda$ with relations in $D_\lambda$ corresponding to each pair of adjacent rows in $\lambda$. So $f$ (i.e., the map we seek) comes from a map $g : D_\lambda \rightarrow K_\nu \otimes 1$ that sends these relations to 0. We will write a formula for a general map $g : D_\lambda \rightarrow K_\nu \otimes 1$ and solve the resulting constraints on $g$ to find $f$. In these calculations, we will have to rely very heavily on the material sketched on pp.234-6 in [ABW]. More specifically, we will need (1) the definition of the “box” map to get relations defining $K_\lambda$ (these are described in general in Note 2 at the end of Example 2 below), and (2) the procedure to “straighten tableaux” in $K_\nu \otimes 1$, which in turn is based on the box map associated to $K_\nu \otimes 1$. (The second item is necessary as a systematic way to find the answers but not necessary to verify them. However, straightening will play a more essential role in the proof of Lemma A.) To understand this material one should additionally look at the careful treatment of the same topics in the dual case of “Schur modules” on pp.226-232.
of the same paper (statements of II.2.7 through II.2.16 and the proof of II.2.15). While following all subsequent calculations involving tableaux it will be helpful to write the algebraic expressions pictorially in tableau form.

Label an ordered basis of $F$ as $e_1, e_2, \ldots$. Recall that for us standard tableaux (called co-standard in [ABW]) will be fillings of Young diagrams with entries from the chosen basis of $F$ such that the entries increase weakly along rows and strictly along columns. In what follows, we will for convenience identify any standard tableau with the corresponding element of the appropriate tensor product of divided powers of $F$. For example, the canonical tableau (i.e., the tableau with all $e_1$’s in the first row, all $e_2$’s in the second row and so on) $C_\lambda$ of shape $\lambda$ is identified with $e_1^{(\lambda_1)} \otimes e_2^{(\lambda_2)} \otimes \ldots \in D_\lambda$. By the method of weight submodules [AB2, pp.177-8], any map from $D_\lambda$ is specified completely by its action on $C_\lambda$ (since $C_\lambda$ generates $D_\lambda$ as a GL($F$)-module) and such a map may send $C_\lambda$ arbitrarily into the $\lambda$-weight submodule of the target module. Together with the “standard basis” for $K_{\nu \otimes 1}$ obtained from [ABW, Theorem II.3.16], this gives us the following basis for $\text{Hom}(D_\lambda, K_{\nu \otimes 1})$.

$$\left\{ C_\lambda \mapsto d'_{\nu \otimes 1}(T_\lambda) \mid T_\lambda \text{ a standard tableau of shape } \nu \otimes 1 \text{ and weight } \lambda \right\}.$$ 

Thus a general map $g : D_\lambda \to K_{\nu \otimes 1}$ may be written as $C_\lambda \mapsto \sum \pm c_\lambda d'_{\nu \otimes 1}(T_\lambda)$ with arbitrary integers $c_\lambda$. It is not very hard to explicitly describe all the tableaux $T_\lambda$. After that one can explicitly write formulas for the maps $\{ C_\lambda \mapsto C_{\nu \otimes 1}(T_\lambda) \}$ in the above basis by using appropriate polarizations (i.e., comultiplication followed by multiplication) in the divided power algebra of $F$ (see examples below). It is now straightforward but tedious to get $f$ following the strategy explained near the beginning of the previous paragraph. Let us first work out three simple examples that will simultaneously illustrate all the main features of the cases that we will have to consider in the general proof.

**Example 1.** $\nu = a$, a partition with a single row. So $\lambda = a1$ and $K_{\nu \otimes 1} = D_a \otimes F$. We have to find the smallest multiple of identity in $\text{Hom}(K_{a1}, K_{a1})$ that factors through $D_a \otimes F$. Note that here $d'_{\nu_1} : D_a \to K_{a1}$ with a single relation in $D_{a1} = D_a \otimes F$ defining $K_{a1}$ (see below). To find a generator $f$ of $\text{Hom}(K_{a1}, D_a \otimes F)$, let us look at the general map $g$ in $\text{Hom}(D_a \otimes F, D_a \otimes F) \simeq a1$-weight submodule of $D_a \otimes F$. The following two standard tableaux of shape $a \otimes 1$ form a basis for this weight submodule: $e_1^{(a)} \otimes e_2$ (i.e., the canonical tableau $C_{a \otimes 1}$) and $e_1^{(a-1)} e_2 \otimes e_1$. These correspond respectively to the maps $g_0 = \text{identity}$ and $g_1 = \text{the polarization defined by the composite map}$

$$D_a \otimes F \xrightarrow{\Delta} D_{a-1} \otimes F \otimes F \xrightarrow{m_{13}} D_a \otimes F,$$

where $\Delta = \text{appropriate component of comultiplication on the first tensor factor and } m_{13} = \text{multiplication of the first and third tensor factors tensored with identity on the second tensor factor}$. Now the map $g = c_0 g_0 + c_1 g_1 : D_a \otimes F \to D_a \otimes F$ descends to a map from $K_{a1}$ exactly when it sends the kernel of $d'_{a1} : D_a \otimes F \to K_{a1}$ to zero. This kernel is the image of the comultiplication map $\Delta : D_{a+1} \to D_a \otimes F$ and so is generated by
\( \Delta \) (the canonical tableau in \( D_{a+1} \)) = \( \Delta(e_1^{(a+1)}) = e_1^{(a)} \otimes e_1 \). Evaluating \( g \) on this, we get 
\( g(e_1^{(a)} \otimes e_1) = (c_0 + ac_1)(e_1^{(a)} \otimes e_1) \). Note the coefficient \( a \). It arises while calculating \( m_{13} \) when we multiply \( e_1^{(a-1)} \) and \( e_1 \) in the divided power algebra of \( F \). We now conclude that \( f \) is obtained by taking \( c_0 = a \) and \( c_1 = -1 \). It remains to calculate \( d'_a \circ f \) and see which 
multiple of the identity we get in \( \text{Hom}(K_{a1}, K_{a1}) \). This is easiest to do by tracing what 
happens to the canonical tableau. Applying \( d'_a \) to the images of the canonical tableau \( C_{a1} \) 
under \( g_0 \) and \( g_1 \) (i.e., to \( e_1^{(a)} \otimes e_2 \) and \( e_1^{(a-1)} e_2 \otimes e_1 \)), the desired integer is easily checked 
to be \( a + 1 \). The only thing to note is that \( d'_a(e_1^{(a-1)} e_2 \otimes e_1) = -d'_a(e_1^{(a)} \otimes e_2) \) by direct 
calculation.

**Example 2.** \( \nu = a^2 \), a partition with two rows of equal length \( a \). So \( \lambda = a^21 \) and 
\( K_{\nu \otimes 1} = K_{a^2 \otimes F} \). We have to find the smallest multiple of identity in \( \text{Hom}(K_{a^21}, K_{a^21}) \) 
that factors through \( K_{a^2 \otimes F} \). To find a generator \( f \) of \( \text{Hom}(K_{a^21}, K_{a^2 \otimes F}) \), let us look at 
the general map \( g \) in \( \text{Hom}(D_{a^21}, K_{a^2 \otimes F}) \simeq a^21 \)-weight submodule of \( K_{a^2 \otimes F} \). A basis 
for this weight submodule is given by images under \( d'_{a^2 \otimes 1} \) of the three standard tableaux 
of shape \( a^2 \otimes 1 \) and weight \( a^21 \). Let us explicitly write this basis and the corresponding 
maps.

\[
d'_{a^2 \otimes 1}(e_1^{(a)} \otimes e_2^{(a)} \otimes e_3) \text{ corresponds to } g_0 = d'_{a^2 \otimes 1}.
\]

\[
d'_{a^2 \otimes 1}(e_1^{(a-1)} e_2 \otimes e_2^{(a-1)} e_3 \otimes e_1) \text{ corresponds to } g_1 = \text{the composite map}
\]

\[
D_a \otimes D_a \otimes F \xrightarrow{\Delta} D_{a-1} \otimes F \otimes D_{a-1} \otimes F \otimes F \longrightarrow D_a \otimes D_a \otimes F \xrightarrow{d'_{a^2 \otimes 1}} K_{a^2 \otimes 1},
\]

where \( \Delta \) is the appropriate comultiplication on the first and second factors and the second 
map is \( m_{14} \otimes m_{35} \otimes \text{id} \), i.e., multiplication on the indicated factors tensored with identity 
on the second factor.

\[
d'_{a^2 \otimes 1}(e_1^{(a)} \otimes e_2^{(a-1)} e_3 \otimes e_2) \text{ corresponds to } g_2 = \text{the composite map}
\]

\[
D_a \otimes D_a \otimes F \xrightarrow{\Delta} D_a \otimes D_{a-1} \otimes F \otimes F \longrightarrow D_a \otimes D_a \otimes F \xrightarrow{d'_{a^2 \otimes 1}} K_{a^2 \otimes 1},
\]

where \( \Delta \) is the appropriate comultiplication on the second factor and the second map is 
\( \text{id} \otimes m_{24} \otimes \text{id} \), i.e., multiplication on the indicated factors tensored with identity on the 
first and third factors.

Now the map \( g = c_0 g_0 + c_1 g_1 + c_2 g_2 : D_{a^21} \rightarrow K_{a^2 \otimes F} \) descends to a map from \( K_{a^21} \) 
exactly when it sends the kernel of \( d'_{a^21} : D_{a^21} \rightarrow K_{a^21} \) to zero. This kernel is generated 
by \( a \) relations derived from the first two rows and a single relation derived from the last 
two rows of the partition \( a^21 \). (See Note 2 at the end of this example.) The former are 
images of the polarizations involving first two rows

\[
D_{a+t} \otimes D_{a-t} \otimes F \longrightarrow D_a \otimes D_a \otimes F
\]

for \( 1 \leq t \leq a \), and so are generated by images of the canonical tableau \( e_1^{(a+t)} \otimes e_2^{(a-t)} \otimes e_3 \), 
i.e., \( e_1^{(a)} \otimes e_1^{(t)} e_2^{(a-t)} \otimes e_3 \) with \( 1 \leq t \leq a \). The latter relation is similarly obtained by a
polarization involving the last two rows and is generated by \( e_1^{(a)} \otimes e_2^{(a)} \otimes e_2 \). Let us first use the last relation. By a calculation very similar to the one in Example 1 and noting by direct calculation that the map \( g_1 \) already kills \( e_1^{(a)} \otimes e_2^{(a)} \otimes e_2 \), we have

\[
g\left( e_1^{(a)} \otimes e_2^{(a)} \otimes e_2 \right) = (c_0 + ac_2) d'_{a^2 \otimes 1} \left( e_1^{(a)} \otimes e_2^{(a)} \otimes e_2 \right).
\]

Now from the remaining \( a \) relations, look at the one with \( t = 1 \). Note that \( g_0 \) already kills \( e_1^{(a)} \otimes e_2^{(a-1)} \otimes e_3 \). So

\[
g\left( e_1^{(a)} \otimes e_1 e_2^{(a-1)} \otimes e_3 \right) = c_1 d''_{a^2 \otimes 1} \left( e_1^{(a-1)} e_2 \otimes e_1 e_2^{(a-2)} e_3 \otimes e_1 + a e_1^{(a)} \otimes e_2^{(a-1)} e_3 \otimes e_1 \right) + c_2 d''_{a^2 \otimes 1} \left( e_1^{(a)} \otimes e_2^{(a-1)} e_3 \otimes e_1 + e_1^{(a)} \otimes e_1 e_2^{(a-2)} e_3 \otimes e_2 \right).
\]

Looking at the four terms inside two sets of parentheses on the right hand side, the last one is killed by \( d''_{a^2 \otimes 1} \) and the tableaux in middle two terms are standard (in fact the same). For the first term, either by inspection using the definition of \( d''_{a^2 \otimes 1} \) or by using the “straightening” procedure in [ABW], we have

\[
d''_{a^2 \otimes 1} \left( e_1^{(a-1)} e_2 \otimes e_1 e_2^{(a-2)} e_3 \otimes e_1 \right) = -(a - 1) d''_{a^2 \otimes 1} \left( e_1^{(a)} \otimes e_2^{(a-1)} e_3 \otimes e_1 \right).
\]

So we get

\[
g\left( e_1^{(a)} \otimes e_1 e_2^{(a-1)} \otimes e_3 \right) = (c_2 + c_1) d''_{a^2 \otimes 1} \left( e_1^{(a)} \otimes e_2^{(a-1)} e_3 \otimes e_1 \right).
\]

Equating the two relations treated above to zero, we get a one parameter family of solutions for the integer coefficients \( c_i \), namely \( c_2 = -1, c_1 = 1 \) and \( c_0 = a \). Since we are assured of the existence of \( f \), and since \( GL(F) \)-equivariance of an integral multiple of any map clearly guarantees \( GL(F) \)-equivariance of the original map, the solution we found must give us the desired map \( f \) generating \( \text{Hom}(K_{a^2 1}, K_{a^2} \otimes F) \). (So the remaining relations must be automatically satisfied and were not necessary for our purpose. This feature will repeat in the general calculation and can be understood more conceptually as explained in Note 2 below.)

It remains to follow the action of \( f \) with the natural surjection \( K_{a^2} \otimes F \to K_{a^2 1} \) and see which multiple of the identity we get in \( \text{Hom}(K_{a^2 1}, K_{a^2} \otimes F) \). Applying \( d'_{a^2 1} \) to the images of the canonical tableau \( C_{a^2 1} \) under \( g_0, g_1 \) and \( g_2 \) the desired integer is easily checked by direct calculation to be \( c_0 + c_1 - c_2 = a + 2 \).

**Notes.** (1) If \( a = 1 \), the above calculation has to be modified since the terms where \( a - 2 \) occurs no longer make sense and have to be replaced by 0. But it is easily checked that the final answer \( a + 2 = 3 \) is still valid.

(2) In general, to define any skew Weyl module by generators and relations, we need for every pair of consecutive rows of the corresponding skew partition as many relations
as the length of the overlap between the two rows. Calling the lengths of two such rows \( p \) and \( q \) and the length of their overlap \( r \), the relations corresponding to these rows are the images of the following \( r \) polarizations involving these two rows

\[
\ldots D_{p+t} \otimes D_{q-t} \ldots \rightarrow \ldots D_p \otimes D_q \ldots,
\]

where \( q-r+1 \leq t \leq q \), “…” indicates tensoring by the divided powers corresponding to the remaining rows and the above maps are identity on “…” (Tracing the canonical tableau, it is easily seen from the definition of skew Weyl modules that these are indeed relations, i.e., that \( d' \) annihilates the images of these maps. [ABW, Theorem II.3.16] shows, among other things, that these relations suffice to define the skew Weyl module in question.) In fact it turns out that the single relation with \( t = q - r + 1 \) generates all the others up to a multiple (an argument for this is sketched below) and so any equivariant map from the appropriate tensor product of divided powers of \( F \) satisfying this one relation must automatically satisfy all the others.

Sketch for the claim in the previous sentence: letting the two rows be those numbered \( i-1 \) and \( i \), consider the map in \( \text{GL}(F) \) that takes \( e_i \) to \( e_{i-1} + e_i \) and acts as the identity on other \( e_j \). Apply the map induced on the divided power algebra of \( F \) to

\[
\ldots e_{i-1}^{(p)} \otimes e_{i-1}^{(q-r+1)}(e_{i-1} + e_i)^{(r-1)} \ldots,
\]
i.e., to the image of the canonical tableau under the polarization we picked out. Expand the last divided power and notice that all the summands have different weights and individually give all \( r \) relations up to a multiple. (Also see lines 12-13 on [ABW, p.209] but note the typo on line 13: \( \lambda_2 - \mu_2 + 1 \) should instead be \( \mu_1 - \mu_2 + 1 \).)

**Example 3.** \( \nu = ab \), a partition with two rows of unequal lengths \( a > b \). So \( \lambda = ab1 \) and \( K_{\nu \otimes 1} = K_{ab} \otimes F \). We have to find the smallest multiple of identity in \( \text{Hom}(K_{ab1}, K_{ab1}) \) that factors through \( K_{ab} \otimes F \). The outline of this calculation is very similar to that in Example 2 with the following crucial change. The relevant weight submodule now has dimension four instead of three, resulting in significantly different constraints on the constants. To find a generator \( f \) of \( \text{Hom}(K_{ab1}, K_{ab} \otimes F) \), let us look at the general map \( g \) in \( \text{Hom}(D_{ab1}, K_{ab} \otimes F) \simeq ab1 \)-weight submodule of \( K_{ab} \otimes F \). A basis for this weight submodule is given by images under \( d'_{ab \otimes 1} \) of the four standard tableaux of shape \( ab \otimes 1 \) and weight \( ab1 \). Let us explicitly write this basis and the corresponding maps. (The labeling of the maps in all three examples is chosen so as to be consistent with the general case treated below.)

\[
\begin{align*}
d'_{ab \otimes 1}(e_1^{(a)} \otimes e_2^{(b)} \otimes e_3) & \text{ corresponds to } g_{00} = d'_{ab \otimes 1}. \\
d'_{ab \otimes 1}(e_1^{(a)} \otimes e_2^{(b-1)} \otimes e_3 \otimes e_2) & \text{ corresponds to } g_{01} = \text{ the composite map} \\
D_a \otimes D_b \otimes F & \overset{\Delta}{\longrightarrow} D_a \otimes D_{b-1} \otimes F \otimes F \longrightarrow D_a \otimes D_b \otimes F \overset{d'_{ab \otimes 1}}{\longrightarrow} K_{ab \otimes 1},
\end{align*}
\]

where \( \Delta \) is the appropriate comultiplication on the second factor and the second map is \( \text{id} \otimes m_{24} \otimes \text{id} \), i.e., multiplication on the indicated factors tensored with identity on the first and third factors.
\[d'_{ab\otimes 1}(e_1^{(a-1)}e_3 \otimes e_2^{(b)} \otimes e_1)\] corresponds to \(g_{10} = \text{the composite map}\)

\[D_a \otimes D_b \otimes F \xrightarrow{\Delta} D_{a-1} \otimes F \otimes D_{b-1} \otimes F \xrightarrow{\Delta} D_a \otimes D_b \otimes F \xrightarrow{d'_{ab\otimes 1}} K_{ab\otimes 1},\]

where \(\Delta\) is the appropriate comultiplication on the first factor and the second map is 
\(m_{14} \otimes s_{23}, \text{i.e., multiplication on the indicated factors tensored with switching the middle two factors.} \) (This is the new case compared to Example 2, made possible by the fact that the first row is longer than the second.)

\[d'_{ab\otimes 1}(e_1^{(a-1)}e_2 \otimes e_2^{(b-1)}e_3 \otimes e_1)\] corresponds to \(g_{11} = \text{the composite map}\)

\[D_a \otimes D_b \otimes F \xrightarrow{\Delta} D_{a-1} \otimes F \otimes D_{b-1} \otimes F \xrightarrow{\Delta} D_a \otimes D_b \otimes F \xrightarrow{d'_{ab\otimes 1}} K_{ab\otimes 1},\]

where \(\Delta\) is the appropriate comultiplication on the first and second factors and the second map is 
\(m_{14} \otimes m_{35} \otimes \text{id}, \text{i.e., multiplication on the indicated factors tensored with identity on the second factor.}\)

Now the map \(g = c_{00}g_{00} + c_{01}g_{01} + c_{10}g_{10} + c_{11}g_{11} : D_{ab1} \to K_{ab1} \otimes F\) descends to a map from \(K_{ab1}\) exactly when it sends the kernel of \(d'_{ab1} : D_{ab1} \to K_{ab1}\) to zero. This kernel is generated by \(b\) relations derived from the first two rows and a single relation derived from the last two rows of the partition \(ab1\). The former are generated by 
\(e_1^{(a)} \otimes e_1^{(t)}e_2^{(b-t)} \otimes e_3\) with \(1 \leq t \leq b\) and the latter by \(e_1^{(a)} \otimes e_2^{(b)} \otimes e_2\). Let us first use the last relation. By a calculation similar to the one in Examples 1 and 2, we have

\[g(e_1^{(a)} \otimes e_2^{(b)} \otimes e_2) = (c_{00} + bc_{01})d'_{ab\otimes 1}(e_1^{(a)} \otimes e_2^{(b)} \otimes e_2) + (c_{10} + bc_{11})d'_{ab\otimes 1}(e_1^{(a-1)}e_2 \otimes e_2 \otimes e_1).\]

Now from the remaining \(a\) relations, look at the one with \(t = 1\). Note that \(g_{00}\) already kills \(e_1^{(a)} \otimes e_1^{(2-(b-1))} \otimes e_3\). So

\[g(e_1^{(a)} \otimes e_1e_2^{(b-1)} \otimes e_3) = c_{01}d'_{ab\otimes 1}(e_1^{(a)} \otimes e_2^{(b-1)}e_3 \otimes e_1 + e_1^{(a)} \otimes e_1e_2^{(b-2)}e_3 \otimes e_2) + c_{10}d'_{ab\otimes 1}(e_1^{(a-1)}e_3 \otimes e_1e_2^{(b-1)} \otimes e_1) + c_{11}d'_{ab\otimes 1}(e_1^{(a-1)}e_2 \otimes e_1e_2^{(b-2)}e_3 \otimes e_1 + ae_1^{(a)} \otimes e_2^{(b-1)}e_3 \otimes e_1).\]

Looking at the five terms inside three sets of parentheses on the right hand side, the second one is killed by \(d'_{ab \otimes 1}\), the tableaux in the first and the last terms are standard (in fact the same), and those in the third and fourth need to be “straightened.” Either by inspection using the definition of \(d'_{ab \otimes 1}\) or by using the straightening procedure in \([ABW]\), for the third term we get

\[d'_{ab \otimes 1}(e_1^{(a-1)}e_3 \otimes e_1e_2^{(b-2)} \otimes e_1) = -d'_{ab \otimes 1}(e_1^{(a)} \otimes e_2^{(b-1)}e_3 \otimes e_1).\]

As for the fourth term, exactly as in Example 2, we have

\[d'_{ab \otimes 1}(e_1^{(a-1)}e_2 \otimes e_1e_2^{(b-2)}e_3 \otimes e_1) = -(b-1)d'_{ab \otimes 1}(e_1^{(a)} \otimes e_2^{(b)}e_3 \otimes e_1).\]
So we get

\[ g\left(e_1^{(a)} \otimes e_1 e_2^{(b-1)} \otimes e_3\right) = (c_{01} - c_{10} + c_{11}(a - b + 1)) d'_{ab \otimes 1}\left(e_1^{(a)} \otimes e_2^{(b-1)} e_3 \otimes e_1\right). \]

Equating the two relations treated above to zero, we get a one parameter family of solutions for the integer coefficients, namely \(c_{11} = 1, c_{01} = -(a+1), c_{10} = -b\) and \(c_{00} = (a+1)b\). As explained in Example 2, this must give us the desired map \(f\).

It remains to follow the action of \(f\) with the natural surjection \(K_{ab} \otimes F \to K_{ab1}\) and see which multiple of the identity we get in \(\text{Hom}(K_{ab1}, K_{ab1})\). Applying \(d'_{ab1}\) to the images of the canonical tableau \(C_{ab1}\) under \(g_{00}, g_{01}, g_{10}\) and \(g_{11}\) the desired integer is easily checked by direct calculation to be \(c_{00} - c_{01} - c_{10} + c_{11} = (a+1)b + (a+1) + b + 1 = (a+2)(b+1)\).

Returning to the general case, the diagram of \(\nu \otimes 1\) consists of \(k\) rectangular blocks of rows (with the \(j\)-th block being \(a_j^{p_j}\)) plus a lone box in the last row. Let us first describe the set of standard tableaux of shape \(\nu \otimes 1\) and weight \(\lambda\). Such tableaux are in bijective correspondence with \(k\)-tuples of integers \(i = i_1 \ldots i_k\) with \(0 \leq i_j \leq p_j\). For example, for \(\nu = a_1^5a_2^3a_3^2\), the index \(i = 302\) corresponds to the tableau with shape \(\nu \otimes 1\), whose rows have rightmost entries \(e_1 e_2 e_4 e_5 e_9 e_6 e_7 e_8 e_{10} e_{11} e_3\) from top to bottom and whose all other entries match with the corresponding entries in the canonical tableau \(C_{\nu \otimes 1}\). The example will be clarified by the general description given next.

Let us describe \(T_i\) in general. It has the same entries as the canonical tableau \(C_{\nu \otimes 1}\) except possibly in the rightmost border strip, in which the entries of \(C_{\nu \otimes 1}\) undergo a cyclic permutation as follows. When \(i_j = 0\), the \(j\)-th block is unaffected, e.g., the second block consisting of rows 6 and 7 in the above example. When the first nonzero \(i_j\) from the left appears, say \(i_{j_1}\), the \(i_{j_1}\)-th entry in the rightmost column of the \(j_1\)-th block is removed and the entries below it in the same block are moved upward by one slot each. Now we find the next nonzero entry in \(i\), say \(i_{j_2}\), remove the \(i_{j_2}\)-th entry in the rightmost column of the \(j_2\)-th block and place it in the empty space created at the bottom of the \(j_1\) th block. Again the entries below the removed one in the \(j_2\)-th block move one slot upward and so on, until the last nonzero \(i_{j_t}\) is used up. Now we remove the number in the last row of \(\nu \otimes 1\) and place it in the empty slot at the bottom of the \(j_i\)-th block. Finally we place the very first entry we removed, i.e., the \(i_{j_1}\)-th entry in the rightmost column of the \(j_1\)-th block, in the lone box in the last row of \(\nu \otimes 1\). Note for instance that \(T_0 = \text{the canonical tableau } C_{\nu \otimes 1}\). (It is a combinatorial exercise to prove that \(T_i\) are indeed all the standard tableaux of shape \(\nu \otimes 1\) and weight \(\lambda\). But it is not really necessary to check this as long as the listed tableaux suffice to produce a nontrivial map via our procedure.)

The map \(g_i : D_{\lambda} \to K_{\nu \otimes 1}\) corresponding to \(T_i\) is obtained in a completely analogous manner to the maps seen in the examples above. More precisely, in the first step we split off via comultiplication a degree one piece from every row in which \(T_i\) differs from the canonical tableau \(C_{\nu \otimes 1}\). Then we multiply what is left for each such row (numbered, say,
by the degree one piece split off from $s$-th row where $e_s$ is the the last entry of $r$-th row in $T_\lambda$. Finally we apply $d'_\nu \otimes 1$. Note for instance that $g_{\emptyset}$ is just $d'_\nu \otimes 1$.

Let us now turn to the relations in $D_\lambda$ defining $K_\lambda$. These will help us solve for the unknowns $c_{i_1...i_k}$. In what follows, sometimes we will have to temporarily fix values of some of the components in $i_1...i_k$. For ease of notation, after explaining such a choice, we will often denote such fixed components simply by writing “...”. The intended meaning will be clear from the context. The relations defining $K_\lambda$ can be divided into three types as follows.

Type 1. The relation involving the last two rows of $\lambda$, i.e. the last row of the partition $\nu$ and the last row of $\lambda$ consisting of exactly one box. Suppose these rows are numbered $x$ and $x+1$ (so $x = p_1+...+p_k$). Then the relation in question is generated by

$R_x = \ldots \otimes e_x^{(a_k)} \otimes e_x$, where $\ldots$ indicates $x-1$ factors matching the first $x-1$ factors of the canonical tableau $C_\lambda$. Applying $g$ to this relation, it is clear by explicit calculation that unless $i_k = 0$ or $p_k$, $g_\nu(R_x) = 0$, essentially because otherwise all $a_k + 1$ occurrences of $e_x$ in $R_x$ are sent inside the last block (which has only $a_k$ columns) by $g_\nu$. (In Example 2 this was manifested in the fact that the $g_1$ there already killed the relation in question.) So let $i_k = 0$ or $p_k$ henceforth in this paragraph. For the moment arbitrarily fix values of all other $i$’s so that we are considering only two of the maps $g_i$. Applying the corresponding two terms in $g$ to $R_x$, we get exactly as in Examples 1 and 2 the expression $(c_{...0} + a_k c_{...p_k})d'_\nu \otimes 1(S)$, where $S$ is the standard tableau of shape $\nu \otimes 1$ obtained as follows. Take $T_i$ corresponding to either of the two $g_i$ being evaluated and obtain $S$ from $T_i$ by replacing the single occurrence of the entry $e_{x+1}$ with $e_x$. Now allowing all possible choices of $i_1...i_{k-1}$, it is easy to see that the standard tableaux $S$ that arise in the way just explained are all distinct. (Compare Example 3.) Altogether, the relation under consideration gives us the following constraints.

$$c_{i_1...i_{k-1}0} + a_k c_{i_1...i_{k-1}p_k} = 0.$$ 

Type 2. Relations involving consecutive rows in the same block. Let us see that this case leads to a calculation virtually identical to the one in Example 2 where a relation involving the first two rows was treated. Suppose we are dealing with relations involving rows numbered $x$ and $x+1$ and that these are respectively the $y$-th and $(y+1)$-th rows of the $j$-th block (and so each is of length $a_j$). Then one of the relevant relations is generated by

$R_x = \ldots \otimes e_x^{(a_j)} \otimes e_x e_{x+1}^{(a_j-1)} \otimes \ldots$, where $\ldots$ indicates expressions identical to the corresponding factors of the canonical tableau $C_\lambda$. Applying $g$ to this relation results in the following. For $i_j$ other than $y$ and $y+1$, $g_\nu(R_x) = 0$ by explicit calculation, essentially because all $a_j + 1$ occurrences of $e_x$ in $R_x$ are kept within the $j$-th block of $\lambda$ (which has only $a_j$ columns) by such $g_\nu$. (In Example 2 this was manifested in the fact that the $g_0$ there already killed the relation in question.) So let $i_j = y$ or $y+1$ henceforth in this paragraph. For the moment arbitrarily fix values of all other $i$’s so that we are considering only two of the $g_i$. Then exactly as in Example 2, we get an expression with four terms inside two sets of parantheses with obvious changes in subscripts and placement of $e$’s. By a very similar calculation, this expression simplifies to $(c_{...y} + c_{...y+1})d'_\nu \otimes 1(T)$, where $y$ and $y+1$ appear in the $j$-th slot and $T$ is the following standard tableau of shape $\nu \otimes 1$. 

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From the two \( g_k \) being evaluated, consider the one with \( i_j = y \) and take the corresponding tableau \( T_k \). Obtain \( T \) from this \( T_k \) by replacing the single \( e_{x+1} \) in the \( x \)-th row with \( e_x \). It is now easy to see that the tableaux \( T \) that arise in this fashion are all distinct for distinct choices of \( i \). (Of course only \( k-1 \) components of \( i \) are being chosen here, in view of our earlier reasoning for \( i_j \).) So just as we got \( c_1 + c_2 = 0 \) in Example 2, we get the constraint \( c_0 \ldots y \ldots c_{(y+1)} = 0 \), where \( y \) and \( y+1 \) appear in the \( j \)-th slot. It should be clear now that in general such relations give us the following equations.

\[
c_i_1 \ldots i_j \ldots i_k + c_{i_1} \ldots (i_j + 1) \ldots i_k = 0, \quad 1 \leq j \leq k, \quad 1 \leq i_j \leq p - 1.
\]

In view of these relations it is enough to determine \( c_k \) with \( i_j = 0 \) or \( p_j \) for each \( j \).

**Type 3.** Relations involving the last row of one block and the first in the next. This case leads to calculation very similar to the corresponding calculation in Example 3. Suppose we are dealing with relations involving rows numbered \( x \) and \( x+1 \) and that these are respectively the last row of the \( j \)-th block and the first row of the \((j+1)\)-th block. So row \( x \) is of length \( a_j \) and row \( x+1 \) of smaller length \( a_{j+1} \). Then one of the relevant relations is generated by \( g_x = \ldots \otimes e^{(a_j)} \otimes e_x e^{(a_{j+1} + 1)} \otimes \ldots \), where \( \ldots \) indicates expressions identical to the corresponding factors of the canonical tableau \( C_x \). Consider the calculation of \( g(R_x) \).

Note that all \( a_j + 1 \) occurrences of \( e_x \) in \( R_x \) are contained within the \( j \)-th and \((j+1)\)-th blocks, where the number of columns is never more than \( a_j \). So for any \( g_k \) that leaves this property unchanged, clearly \( g_k(R_x) = 0 \). Using this it is easy to see that for \( i_j \) other than 0 and \( p_j \), \( g_k(R_x) = 0 \) and furthermore, if \( i_j = 0 \) then again \( g_k(R_x) = 0 \) unless \( i_{j+1} = 1 \). Thus the terms involving only those \( g_k \) where either \( \{i_j = 0 \text{ and } i_{j+1} = 1 \} \) or \( \{i_j = p_j \text{ and } i_{j+1} = 0, \ldots, p_{j+1} \} \) can possibly survive. Until further notice arbitrarily fix values of all \( i \)'s other than \( i_j \) and \( i_{j+1} \). So in view of the preceding discussion, we are considering only \( p_{j+1} + 2 \) terms. Let us consider these one by one after setting up some more notation. \( T_x \) will denote the (nonstandard) tableau obtained from \( T_k \) by replacing the first appearance of \( e_{x+1} \) in row \( x+1 \) with \( e_x \). (See Remark 1 at the end of this proof if there is no such appearance.)

Define a standard tableau \( U \) as follows. Consider the \( T_k \) with \( i_j = 0 \) and \( i_{j+1} = 1 \). Obtain \( U \) by replacing the only occurrence of \( e_{x+1} \) outside row \( x+1 \) with \( e_x \).

(i) For the \( i \) with \( i_j = 0 \) and \( i_{j+1} = 1 \), \( g_k(R_x) = d'_{u \otimes 1}(T^x \mid U) = d'_{u \otimes 1}(U) \) by noting that \( d'_{u \otimes 1}(T^x) = 0 \). (Compare the very similar evaluation of \( g_{01} \) in Example 3 on the relevant relation.)

(ii) For the \( i \) with \( i_j = p_j \) and \( i_{j+1} = 0 \), \( g_k(R_x) = d'_{u \otimes 1}(T^x \mid U) = (-1)^{p_{j+1}} d'_{u \otimes 1}(U) \). The second of these equalities is obtained by easy application of the “straightening” procedure in [ABW] or checked even more easily by explicit calculation using the definition of \( d'_{u \otimes 1} \). (Compare the very similar evaluation of \( g_{10} \) in Example 3 on the relevant relation.)

(iii) For the \( i \) with \( i_j = p_j \) and \( i_{j+1} = 1 \),

\[
g_k(R_x) = d'_{u \otimes 1}(T^x \mid U) = (a_j - a_{j+1} + 1) d'_{u \otimes 1}(U)
\]

since by straightening or explicit calculation, \( d'_{u \otimes 1}(T^x) = -(a_j + 1 - 1) d'_{u \otimes 1}(U) \). (Compare the very similar evaluation of \( g_{11} \) in Example 3 on the relevant relation.)
(iv) For the $\hat{i}$ with $i_j = p_j$ and $i_{j+1} = r, 2 \leq r \leq p_{j+1}$, $g_{\hat{i}}(R_x) = d'_{\nu \otimes 1}(T^r_{\hat{i}}) = (-1)^{r-1}d'_{\nu \otimes 1}(U)$ by straightening or explicit calculation. (This case does not have a counterpart in Example 3, since there $p_{j+1}$ was 1. But the necessary simplification follows the same pattern as in item (ii) above.)

Collecting the information in items (i) through (iv) and using the constraints obtained from relations of type 2 to convert all $i_{j+1}$ into 0 or $p_{j+1}$, we get

$$g(R_x) = (-1)^{p_{j+1}-1} \left( c \ldots c_{p_{j+1}} \ldots - c \ldots c_{p_0} \ldots + (a_j - a_{j+1} + p_{j+1})c \ldots c_{p_j p_{j+1}} \ldots \right) d'_{\nu \otimes 1}(U),$$

where only the $j$-th and $(j + 1)$-th components of $\hat{i}$ are written. Now allowing $\hat{i}$ to vary, it is easy to see that the standard tableaux $U$ that arise as explained above are all distinct. So we get the following constraints, where again only the $j$-th and $(j + 1)$-th components in $\hat{i}$ are written, all others being assumed to be the same in each term of the equation. For $1 \leq j \leq k - 1$,

$$c_0 p_{j+1} - c_{p_j 0} + (a_j - a_{j+1} + p_{j+1})c_{p_j p_{j+1}} = 0.$$ 

Let us now solve all the constraints on $c\hat{i}$ obtained above. Order all $c\hat{i}$ lexicographically with respect to the indices $\hat{i}$. (In each constraint above the variables are written in lexicographic order.) It is easily seen that the constraints give us a homogeneous linear system of integer equations whose coefficient matrix is triangular with all 1’s on the diagonal and that $c_{p_1 \ldots p_k}$ is the only free variable. So we have a one-parameter family of solutions for $c\hat{i}$ and each solution is uniquely specified by assigning an arbitrary integer value to $c_{p_1 \ldots p_k}$. We will verify that

$$b_{\hat{i}} = \epsilon_{\hat{i}} \prod_{i_j = 0} (a_j + p_{j+1} + p_{j+2} + \ldots + p_k) = \epsilon_{\hat{i}} \prod_{i_j = 0} (h_j - p_j)$$

is a generator of this family, where $\epsilon_{\hat{i}}$ is the sign of the cyclic permutation in the rightmost border strip of $T_{\hat{i}}$ with reference to the canonical tableau $C_{\nu \otimes 1}$. Since $\epsilon_{i_1 \ldots i_{k-1} 0} = -\epsilon_{i_1 \ldots i_{k-1} p_k}$ and $b_{i_1 \ldots i_{k-1} 0} = a_k b_{i_1 \ldots i_{k-1} p_k}$ the constraints of type 1 are satisfied. The sign $\epsilon_{\hat{i}}$ ensures that the constraints of type 2 are satisfied. As for the constraints of type 3, noting that the sign of the $b_{\hat{i}}$ corresponding to the middle term of this constraint is opposite to that for the first and the third term, the verification boils down to the simple fact that

$$(a_j + p_{j+1} + p_{j+2} + \ldots + p_k) - (a_j - a_{j+1} + p_{j+1}) - (a_{j+1} + p_{j+2} + \ldots + p_k) = 0.$$ 

Since $|b_{p_1 \ldots p_k}| = 1$, we have the desired map $f$. After having computed $f$, we need to follow its action with the natural surjection $K_{\nu \otimes 1} \to K_\lambda$. It is easy to see that under this surjection $d'_{\nu \otimes 1}(T_{\hat{i}})$ is sent to $\epsilon_{\hat{i}} d'_{\lambda}(C_{\lambda})$. Therefore the map $\text{Hom}(K_\lambda, K_{\nu \otimes 1}) \to \text{Hom}(K_\lambda, K_\lambda)$ is given by the integer $\sum_{\hat{i}} |b_{\hat{i}}|$. Call this expression $E_\nu$. We will show by induction on the number of blocks $k$ in the partition $\nu$ that $E_{\nu} = h_1 \ldots h_k$. The base case $k = 1$ is an easy check (or even subsumed in the induction by defining $E_{\phi} = 1$ and seeing that this makes obvious sense in the context of the lemma). Let $\nu' = a^{p_2}_{\lambda 2} \ldots a^{p_k}_{\lambda k}$ be the partition obtained from $\nu$ by deleting the first block $a^{p_1}_{\lambda 1}$. The terms in $E_\nu$ with $i_1 = 0$
give \((h_1 - p_1)E_{\nu'}\). The remaining terms in \(E_{\nu}\), by fixing \(i_1 = 1, \ldots, p_1\) at a time, can be partitioned into \(p_1\) groups each of which is \(E_{\nu'}\). This completes the proof of Lemma C.

Remarks. (1) If \(a_k = 1\), one has to modify a few of the details in the calculations required to find the constraints, but the same constraints stay valid. Specifically, while dealing with relations of type 2 for rows in the last block, two of the four terms within two sets of parentheses have to replaced by 0 (see Note 1 at the end of Example 2). While dealing with relations of type 3, the definition of tableaux \(T_{\nu}^x\) no longer makes sense when \(j + 1 = k\) and \(i_k = 1\) because then the single entry in row \(x + 1\) of \(T_{\nu}\) is not \(e_{x+1}\). But this simply means that the \(T_{\nu}^x\) in items (i) and (iii) has to be replaced by 0, leaving the end result in these items unchanged for the case under consideration (see the Note at the end of Example 3).

(2) The map \(f\) may be of independent interest because in a sense it gives us an explicit formula for the intertwining homomorphism between neighboring Weyl modules for \(GL_n\), see [Andersen1, Section 6].

In proofs of the remaining lemmas, we will need several items from the proof of Lemma C. These include the ordered basis \(e_1, e_2, \ldots\) for \(F\), the indices \(\underline{i}\), the tableaux \(T_{\underline{i}}\) and the associated maps \(g_{\underline{i}}\), the numbers \(b_{\underline{i}}\) and the map \(f\).

Proof of Lemma B. Let us first identify the module \(N\) in the statement of the lemma as a skew Weyl module. Using [AB1, 6.6] and contravariant duality, we have the short exact sequence \(0 \rightarrow K_{\mu} \rightarrow K_{\nu} \otimes F \overset{\pi}{\twoheadrightarrow} K_{\xi} \rightarrow 0\), where \(K_{\xi}\) is the skew Weyl module corresponding to the skew partition \(\xi = a_{1}^{p_1 + 1}a_{2}^{p_2} \ldots a_{k}^{p_k}/a_{1} - 1\). (In words, the diagram of \(\xi\) is obtained by placing a single box immediately above the last box in the first row of \(\nu\).) So \(N = K_{\xi}\) and our task is to find the smallest multiple of identity in \(\text{Hom}(K_{\lambda}, K_{\lambda})\) that factors through \(K_{\xi}\). For this we will find a generator \(f'\) of \(\text{Hom}(K_{\lambda}, K_{\xi})\) and follow its action with the natural surjection \(K_{\xi} \rightarrow K_{\lambda}\) (obtained from the natural surjection \(K_{\nu} \otimes F \rightarrow K_{\lambda}\) used in Lemma C, which kills the submodule \(K_{\mu}\)). See the exact sequence above.) The generator \(f'\) can be found exactly as \(f\) was found in Lemma C. But instead we will use the work already done to find \(f\) and finish the proof as follows. Consider the composite map \(\pi \circ f : K_{\lambda} \rightarrow K_{\xi}\) where \(\pi\) is the surjection in the exact sequence above. Using the standard basis for \(K_{\xi}\), we will write an explicit expression for \(\pi \circ f(d_{\lambda}(C_{\lambda}))\). It will be obvious that this expression is divisible by \(h_1\) and by no larger integer. So \(f' = \pi \circ f/h_1\) is a generator for \(\text{Hom}(K_{\lambda}, K_{\xi})\) and composing it with the natural surjection \(K_{\xi} \rightarrow K_{\lambda}\) gives the desired result in view of Lemma C.

Let us lay some groundwork for the calculation of \(\pi \circ f(d_{\lambda}(C_{\lambda}))\). First note that removing the lone box in the last row of \(\nu \otimes 1\) and placing it directly above the last box in the first row of \(\nu\) gives us the diagram of the skew partition \(\xi\). We will need this relocation on two occasions. In the first instance it is involved in the surjection \(\pi : K_{\nu} \otimes F \rightarrow K_{\xi}\). Explicitly \(\pi\) maps \(d_{\nu \otimes 1}(C_{\nu \otimes 1})\) to \(d_{\nu'}(C_{\nu'}) \otimes (e_m \wedge e_1 \wedge \ldots \wedge e_{p_1})\), where \(\nu'\) is the partition obtained by stripping off the last column of \(\nu\) and \(e_m\) is the entry in the last row of \(C_{\nu \otimes 1}\). (All this is only a notational issue arising from the simple fact that the diagram of \(\nu\) and
Lemma C by employing an idea from [AB1] as follows. Consider an extra copy of \( \mathbb{Z} \) (which we will regard as the trivial representation of \( \text{GL}(F) \)) with basis \( e_0 \) and use the ordered basis \( e_0, e_1, \ldots \) for \( \mathbb{Z} \oplus F \). Now the module \( K_\lambda(\mathbb{Z} \oplus F) \) considered as a representation of \( \text{GL}(F) \) splits into a direct sum by the content of \( e_0 \). Using the standard basis theorem, the summand with \( e_0 \)-content one is spanned by standard tableaux with a single \( e_0 \) entry in the top left corner of \( \lambda \) and is clearly isomorphic to \( K_{\lambda/1}(F) \). Consider the following composite map of representations of \( \text{GL}(F) \) beginning with the containment just explained.

\[
K_{\lambda/1}(F) \to K_\lambda(\mathbb{Z} \oplus F) \xrightarrow{f} K_\nu(\mathbb{Z} \oplus F) \otimes (\mathbb{Z} \oplus F) \xrightarrow{pr} K_\nu(\mathbb{Z} \oplus F),
\]

where the second map is the map \( f \) in the proof of Lemma C (now between representations of \( \text{GL}(\mathbb{Z} \oplus F) \)) and the third map is the projection \( pr \) derived from the direct sum decomposition \( K_\nu(\mathbb{Z} \oplus F) \otimes (\mathbb{Z} \oplus F) \simeq K_\nu(\mathbb{Z} \oplus F) \oplus (K_\nu(\mathbb{Z} \oplus F) \otimes F) \). Let us trace the action of this composite map on the canonical tableau \( C_{\lambda/1} \) of shape \( \lambda/1 \). The first map takes \( d'_\lambda(e_1^{(\lambda_1-1)} \otimes e_2^{(\lambda_2)} \otimes \ldots) \) to \( d'_\lambda(e_0 e_1^{(\lambda_1-1)} \otimes e_2^{(\lambda_2)} \otimes \ldots) \). Now using the work done for Lemma C,

\[
f \left( d'_\lambda(e_0 e_1^{(\lambda_1-1)} \otimes e_2^{(\lambda_2)} \otimes \ldots) \right) = \sum_i b_ig_i \left( e_0 e_1^{(\lambda_1-1)} \otimes e_2^{(\lambda_2)} \otimes \ldots \right).
\]
Applying the projection $pr$ to this element, we see that the terms involving $g_{\underline{i}}$ with $i_1 \neq 1$ are killed. This is because applying such $g_{\underline{i}}$ would result in a tableau having an entry other than $e_0$ in the lone box in the last row of $\nu \otimes 1$, so this tableau would be killed by $pr$. As for $g_{1i_2...i_k}(e_0e^{(\lambda_1-1)}_1 \otimes e^{(\lambda_2)}_2 \otimes \ldots)$, what survives upon projection is exactly $d'_\nu(T''_{i_2...i_k})$, where the standard tableau $T''_{i_2...i_k}$ of shape $\nu$ is the same as the tableau $T_{1i_2...i_k}$ in the proof of Lemma C (or $T_{i_2...i_k}$ in the proof of Lemma B) with the extra box containing the entry $e_1$ removed. (Again it is an optional combinatorial check – or immediate from the corresponding fact in proofs of Lemma C or Lemma B – that $T''_{i_2...i_k}$ exhaust the standard tableaux of shape $\nu$ and weight $e^{\lambda_1-1}_1e^{\lambda_2}_2e^{\lambda_3}_3\ldots$). It is also clear from the calculation (or even a priori) that the image of the composite map above lies in the submodule $K_\nu(F)$ of $K_\nu(Z \oplus F)$. All in all, we have constructed a map $f'' : K_{\lambda/1}(F) \rightarrow K_\nu(F)$ such that

$$f''(d'_{\lambda/1}(e^{(\lambda_1-1)}_1 \otimes e^{(\lambda_2)}_2 \otimes \ldots)) = \sum_{i=1}^{i_k} b_{i_1i_2...i_k}d'_\nu(T''_{i_2...i_k}).$$

Recall how in the proof of Lemma C a formula for $f$ was written using maps $g_i$. In exactly the same fashion from the preceding equation we can say that $f''$ descends from the map $\sum_{i=1}^{i_k} b_{1i_2...i_k}g''_{i_2...i_k} : D_{\lambda/1}(F) \rightarrow K_\nu(F)$, where $g''_{i_2...i_k}$ is the map “corresponding” to the tableau $T''_{i_2...i_k}$. (In other words $g''_{i_2...i_k}$ can be built from the tableau $T''_{i_2...i_k}$ using polarizations in the divided power algebra of $F$ exactly the way $g_{\underline{i}}$ was built from the tableau $T_{\underline{i}}$ in the proof of Lemma C, see the paragraph immediately after the one describing $T_{\underline{i}}$.) Further, since $|b_{1p_1...p_k}| = 1$, $f''$ is indivisible and hence a generator of $\text{Hom}(K_{\lambda/1}(F), K_\nu(F))$.

Let us now compose the the map $f''$ in the previous paragraph with the natural injection $\iota : K_\nu \hookrightarrow K_{\lambda/1}$. $\iota$ descends from a map that “polarizes a degree one piece” from each row into the next row. Recall that this means splitting off a degree one component from one row via diagonalization and then multiplying this component into another row, both operations being done in the divided power algebra of $F$. Let us make this explicit using the canonical tableau $C_\nu = e^{(\nu_1)}_1 \otimes e^{(\nu_2)}_2 \otimes \ldots$. We have $\iota(d'_\nu(C_\nu)) = d'_{\lambda/1}(e^{(\lambda_1-1)}_1 \otimes e^{(\lambda_2)}_2 \otimes \ldots)$. (Note that $\lambda$ and $\nu$ are identical with the exception that $\lambda$ has an extra row consisting of a single box.) We will find the image of this element under $f''$ by computing the individual terms $g''_{i_2...i_k}(e^{(\lambda_1-1)}_1 \otimes e^{(\lambda_2-1)}_2 \otimes e^{(\lambda_3-1)}_3 \otimes \ldots)$ of the computation involves two steps. (1) Polarize degree one pieces from several rows into previous rows as dictated by the entries in the rightmost border strip of the tableau $T''_{i_2...i_k}$. (2) Apply $d'_\nu$ to the result of the first step. Since the result of the first step is a linear combination of tableaux that are in general not standard, in the second step one has to straighten these tableaux using the procedure in [ABW]. To keep control of the calculation it will be convenient for us to mix the order of operations involved in steps 1 and 2 as follows. Proceeding from top row to the bottom row, we will follow each single polarization immediately by straightening. Each such straightening will involve only a fragment of a tableau up to the rows involved the preceding polarization. It will be clear that the end result is unaffected by such interlacing of steps 1 and 2 used for one polarization at a time.

Let us illustrate the above discussion by first treating an extreme case. Consider
\( g''_{1\ldots 1}(e_1^{(\lambda_1-1)} \otimes e_1 e_2^{(\lambda_2-1)} \otimes e_2 e_3^{(\lambda_3-1)} \otimes \ldots) \). Note that \( g''_{1\ldots 1} \) involves, for each pair of consecutive nonzero rows of \( \lambda \), polarization of a degree one component from the lower row into the immediately preceding row. For now look at what happens after doing only the polarization from the second row into the first row, which affects only the first two tensor factors. This gives

\[
e_1^{(\lambda_1-1)} \otimes e_1 e_2^{(\lambda_2-1)} \otimes \ldots \mapsto \lambda_1 \left( e_1^{(\lambda_1)} \otimes e_2^{(\lambda_2-1)} \otimes \ldots \right) + \left( e_1^{(\lambda_1-1)} e_2 \otimes e_1 e_2^{(\lambda_2-2)} \otimes \ldots \right).
\]

Notice that the tableau fragment displayed in second term on the right hand side is nonstandard due to a violation in the very first column. It is clear by looking that this non-standardness will persist after subsequent polarizations. So after the application of \( d'_\nu \), we will need to apply the straightening procedure to the first two rows. (And lower rows too, but we will deal with that later. Here “straightening” means replacing a tableau by a linear combination of tableaux as prescribed in [ABW] that will give the same result upon applying the appropriate generalized symmetrizer map \( d' \).) Moreover, we may do this straightening before applying the rest of the polarizations involved in \( g''_{1\ldots 1} \) without affecting the overall result. This is because subsequent polarizations will only involve rows numbered two and below. Rows below the second are entirely unaffected by the results of the proposed straightening. As for the second row, the next polarization will result in a degree one piece being multiplied into it, but this multiplication and the proposed straightening together give the same result for the second row regardless of the order in which they are performed. This is simply from associativity of multiplication. To carry out the proposed straightening, consider the diagonalization \( \Delta(e_1^{(\lambda_1)} e_2) = e_1^{(\lambda_1)} \otimes e_2 + e_1^{(\lambda_1-1)} e_2 \otimes e_1 \). Using this the nonstandard term straightens to \(- (\lambda_2 - 1)(e_1^{(\lambda_1)} \otimes e_2^{(\lambda_2-1)} \otimes \ldots)\), where the constant \((\lambda_2 - 1)\) is due to the multiplication of \( e_2 \) and \( e_2^{(\lambda_2-2)} \) in the divided power algebra. Combining, the result so far in calculating \( g''_{1\ldots 1} \) can be shown as follows.

\[
e_1^{(\lambda_1-1)} \otimes e_1 e_2^{(\lambda_2-1)} \otimes \ldots \mapsto (\lambda_1 - \lambda_2 + 1) \left( e_1^{(\lambda_1)} \otimes e_2^{(\lambda_2-1)} \otimes \ldots \right).
\]

(Note that this is just a schematic representation of what happens after applying steps 1 and 2 for a single polarization. In particular we cannot write \( d'_\nu \) on the right hand side until all polarizations are applied.) Since the fragment obtained so far matches that for the standard tableau \( C_\nu \), clearly subsequent polarizations in the calculation of \( g''_{1\ldots 1} \) will not result in terms that are nonstandard in the first two rows. So evidently we may use the same logic on successive pairs of rows of \( \lambda \). Inductively we get the end result to be \( d'_\nu(C_\nu) \) times the product of \( \lambda_{t-1} - \lambda_t + 1 \) over successive pairs of nonzero rows of \( \lambda \).

By an extension of the above argument, we will show that in general,

\[
(*) \quad g''_{t_2 \ldots t_k} \left( e_1^{(\lambda_1-1)} \otimes e_1 e_2^{(\lambda_2-1)} \otimes \ldots \right) = (-1)^{p_1 + \ldots + p_k} e_{1t_2 \ldots t_k} \prod_{t \in S} (\lambda_{t-1} - \lambda_t + 1) d'_\nu(C_\nu),
\]

where the product is taken over the set \( S \) of nonzero rows numbered \( t \) such that \( g''_{t_2 \ldots t_k} \) involves a polarization of the \( t \)-th row of \( \lambda \) into a previous row of \( \lambda \). (Recall from Lemma
C that \( \epsilon_i \) is the sign of the cyclic permutation in the rightmost border strip of \( T_i \) with reference to the canonical tableau \( C_{\nu \otimes 1} \) and that \( p_1 + \ldots + p_k \) is the number of rows in \( \nu \).)

To prove the claim, fix an arbitrary \( g_{t_1 \ldots i_k} \) and as before, let us consider just the first polarization involved in calculating it. Let us suppose that the first row that it polarizes (necessarily into the first row) is that numbered \( t \). Then we have

\[
e_{1}^{(\lambda_1-1)} \otimes e_{2}^{(\lambda_2-1)} \otimes \ldots \otimes e_{t-1}^{(\lambda_{t-1})} \otimes e_{t}^{(\lambda_{t})} \otimes \ldots \quad \leftrightarrow \quad e_{1}^{(\lambda_1-1)} e_{t-1} \otimes e_{2}^{(\lambda_2-1)} \otimes \ldots \otimes e_{t}^{(\lambda_{t})} + e_{1}^{(\lambda_1-1)} e_{t} \otimes e_{2}^{(\lambda_2-1)} \otimes \ldots \otimes e_{t-1} e_{t}^{(\lambda_{t-2})} \otimes \ldots .
\]

Now in general, after applying \( d'_\nu \), both fragments displayed on the right hand side will need to be straightened. To straighten the first term (needed if \( t > 2 \), we use \( \Delta(e_{1}^{(\lambda_1)} e_{t-1}) = e_{1}^{(\lambda_1)} \otimes e_{t-1} \otimes e_{1} \). This results in \( -e_{1}^{(\lambda_1)} e_{2}^{(\lambda_2-1)} e_{t-1} \otimes \ldots \otimes e_{t}^{(\lambda_{t})} \otimes \ldots \), i.e., the first term undergoes an exchange of \( e_{t-1} \) and \( e_{1} \) between the first and second rows and picks up a negative sign. Now standardness is violated between second and third rows (unless \( t = 3 \)). So repeat the same procedure using the second and third rows, and so on until \( e_{t-1} \) moves into the \( (t-1) \)-th row. Thus we need to perform in all \( t-2 \) straightening operations, the last one resulting in a multiple of \( \lambda_{t-1} \) as we have to multiply \( e_{t-1} \) and \( e_{t}^{(\lambda_{t-1})} \) while moving \( e_{t-1} \) into the \( (t-1) \)-th row. Each straightening also results in a negative sign. Altogether, the first term after straightening gives

\[
(-1)^{t-2} \lambda_{t-1} \left( e_{1}^{(\lambda_1)} \otimes e_{2}^{(\lambda_2)} \otimes \ldots \otimes e_{t-1}^{(\lambda_{t-1})} \otimes e_{t}^{(\lambda_{t-1})} \ldots \right).
\]

By the exact same procedure, the second term, after \( t-1 \) straightening operations, gives

\[
(-1)^{t-1} (\lambda_{t} - 1) \left( e_{1}^{(\lambda_1)} \otimes e_{2}^{(\lambda_2)} \otimes \ldots \otimes e_{t-1}^{(\lambda_{t-1})} \otimes e_{t}^{(\lambda_{t-1})} \ldots \right).
\]

 Altogether we get \( (-1)^{t-2}(\lambda_{t-1} - \lambda_{t} + 1) \) times a fragment that matches the corresponding tableau fragment in the canonical tableau \( C_{\nu} \). Evidently the same pattern will continue as we apply further polarizations. For example after applying the next polarization, say from the \( s \)-th row into the \( t \)-th row, and straightening we will get an additional multiple of \( (-1)^{s-t-1}(\lambda_{s-1} - \lambda_{s} + 1) \) and the resulting tableau fragment with \( s \) rows will match the corresponding fragment of the canonical tableau \( C_{\nu} \). The claimed expression follows after checking easily that the resulting product of signs matches the claimed sign.

Using the work done so far, we can lay out the whole calculation as follows.

\[
f''(\iota(d'_{\nu}(C_{\nu}))) = f'' \left( d'_{\lambda/1} \left( e_{1}^{(\lambda_1-1)} \otimes e_{2}^{(\lambda_2-1)} \otimes \ldots \right) \right)
\]

\[
= \sum_{1i_2 \ldots i_k} b_{1i_2 \ldots i_k} g_{1i_2 \ldots i_k}'' \left( e_{1}^{(\lambda_1-1)} \otimes e_{2}^{(\lambda_2-1)} \otimes \ldots \right)
\]

\[
= (-1)^{p_1 + \ldots + p_k} D_{\nu} d'_{\nu}(C_{\nu}),
\]

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where $D_{\nu}$ is the appropriate integer obtained using (*) above and the values of $b_i$ obtained near the end of the proof of Lemma C. To finish the proof we will write an explicit expression for $D_{\nu}$ and simplify it. First observe that the product $\prod_{t \in S}(\lambda_{t-1} - \lambda_t + 1)$ in (*) immediately boils down to $a_k \prod_{i,j=1}^{k-1} (a_j - 1 - a_j + 1)$, where $2 \leq j \leq k$. (The factor $a_k$ is present because the last row of $\lambda$, consisting of exactly one box, will always satisfy the condition defining the set $S$ for all $g''_{i_2 \ldots i_k}$.) So we have

$$D_{\nu} = \sum_{i=1}^{i_2 \ldots i_k} \left( \prod_{i,j=0} (h_j - p_j) \right) \left( a_k \prod_{i,j=1}^{i_2 \ldots i_k} (a_{i-1} - a_j + 1) \right),$$

where $2 \leq j \leq k$. We will show by induction on the number of blocks $k$ in the partition $\nu$ that $D_{\nu} = \ell_1 \ldots \ell_k$. The case $k = 1$ is immediate, since then the whole expression collapses to just $a_k = a_1 = \ell_1$. When $k > 1$, let $\nu' = \text{the partition obtained from } \nu \text{ by deleting the first block } a_{i_2}^{p_1}$. Then the terms in $D_{\nu}$ with $i_2 = 0$ add up to $(h_2 - p_2)D_{\nu'}$, the terms with $i_2 = 1$ add up to $(a_1 - a_2 + 1)D_{\nu'}$ and the terms with any other fixed value of $i_2$ (i.e., $2 \leq i_2 \leq p_2$) add up to $D_{\nu'}$. Since

$$(h_2 - p_2) + (a_1 - a_2 + 1) + (p_2 - 1) = (a_2 + p_3 + \ldots + p_k) + a_1 - a_2 + p_2 = a_1 + p_2 + \ldots + p_k = \ell_1$$

the proof of Lemma A is complete. This also completes the proof of Theorem 2.1.

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