We show that the computational power of the non-causal circuit model, i.e., the circuit model where the assumption of a global causal order is replaced by the assumption of logical consistency, is completely characterized by the complexity class \( \text{UP} \cap \text{coUP} \). An example of a problem in that class is factorization. Our result implies that the non-causal model of classical closed timelike curves (CTCs) cannot solve problems that lie outside of that class. Thus, in stark contrast to other CTC models, these CTCs cannot solve \( \text{NP-complete} \) problems, unless \( \text{NP} = \text{UP} \cap \text{coUP} \), which lets their existence in nature appear less implausible. This result gives a new characterization of \( \text{UP} \cap \text{coUP} \) in terms of fixed points.

**MOTIVATION AND RESULTS**

The *acyclic* feature of “causality” [1], that an effect cannot be the cause of its cause, plays a central role in everyday life, physical theories, and models of computation. A *cyclic* causal structure is — in the classical meaning\(^1\) of the following adjective — paradoxical. That may be a reason for why an *acyclic* notion is not only preferred but also a hidden assumption for many theories. Objections against cyclic causal structures are the *grandfather* and the *information antinomies* (see e.g. Refs [2, 3]). The former reads: By travelling to the past and killing his or her own grandfather, one could never have been born to travel to the past to kill his or her own grandfather. The latter is *ex nihilo* appearance of information, as illustrated in the following example. Assume one morning you wake up to find a proof of \( P = \text{NP} \) on your desk. You decide to publish it and, after publication, you travel back in time to the night before you found the proof to place the original copy on your desk, while your younger self is asleep. Who wrote the proof?\(^2\) However, if the proof you find on your desk is *uniquely* determined by a process, then the proof does not appear *ex nihilo*, but is the result of that process [5].

Closed timelike curves (CTCs) are loops in spacetime. That is, by traveling on such a curve, one would bump into oneself on the same position in space and time. Interestingly, CTCs appear as solutions to Einstein’s equations [6, 7], yet they have been or still are believed to be unphysical; their underlying structure is cyclic. For over twenty years people have studied different models of CTCs and their implications. Echeverria, Klinkhammer, and Thorne [8] analyzed CTCs in the gravitational setting and showed that in some scenarios the grandfather antinomy is avoided. By Novikov’s principle [9], the scenarios where the grandfather antinomy arises are simply excluded. Deutsch [4] analyzed CTCs in the quantum information realm and showed that there, the grandfather antinomy *never* occurs. Because *multiple* consistent states to some initial conditions exist, Deutsch singles out the mixture of all consistent states that maximize the entropy as solution; by this he avoids the information antinomy. However, this maximum-entropy strategy has a price: The evolution becomes non-linear. Pegg [10] and others [11–15] designed a different model of CTCs, in which states are sent with the help of quantum teleportation to the past. That model, however, also leads to a non-linear evolution. Recently, Oreshkov, Costa, and Brukner [16] came up with a framework for quantum correlations without global causal order. There, the main assumptions are *linearity* and *locality* of ordinary quantum theory. Interestingly, the framework describes correlations that cannot be simulated with a global causal order [16–19], and allows for advantages in query [20–24], as well as communication complexity [25, 26]. It was shown [19] that the classical spacial case [27] of that framework describes CTCs that avoid the grandfather and the information antinomies (we call such CTCs *logically consistent*). Furthermore, it can be used to define a *non-causal* circuit model of computation [28] that also avoids both antinomies.

Even though we do not know whether CTCs exist in nature or not, we can study their consequences. As Aaronson [29] put it, one could assume that nature *cannot* solve certain tasks (e.g., \( \text{NP-hard} \) problems), in the same spirit as nature cannot signal faster than at the speed of light, and conclude that certain theories are unphysical. The same idea is used in reconstructions of quantum theory where the standard, unintuitive axioms are replaced by “more natural” ones (see e.g., Refs. [30–49]). As it turns out [50], the class \( \text{P}_{\text{CTC}} \) of all problems solvable in polynomial time by classical Deutschian CTCs is equal to

\(^1\) The noun “paradox” means a *seeming* contradiction as opposed to an *actual* contradiction. It originates from the Greek word *paradoxon* which is composed out of *para* (against) and *doxa* (opinion). We use the term *antinomy* for actual contradictions.

\(^2\) Because it “contradict[s] the philosophy of science,” Deutsch [4] views this antinomy as by “far more serious” when compared to the grandfather antinomy. According to Deutsch, solutions to problems need to emerge through evolutionary or rational processes — otherwise the underlying theory would follow the doctrine of creationism.
its quantum analog $\text{BQP}_{\text{CTC}}$, and furthermore, equal to $\text{PSPACE}$\textsuperscript{3}. Most recently, Aaronson, Bavarian, and Guerlin [52] showed that the Deutsch model can even solve the halting problem. The model of CTCs where the loops are generated through quantum teleportation to the past can solve all problems in the class $\text{PostBQP} = \text{PP}$ [13, 53, 54]. The classical analogue thereof can solve problems in $\text{BPP}_{\text{path}}$ [13] — the classical analogue of $\text{PostBQP}$ [55]. The inclusion relations between these classes are $\text{NP} \subseteq \text{PostBQP} \subseteq \text{PSPACE}$ class where the loops are generated through quantum teleportation to the past can solve all problems in the class $\text{PostBQP} = \text{PP}$ [13, 53, 54]. The classical analogue thereof can solve problems in $\text{BPP}_{\text{path}}$ [13] — the classical analogue of $\text{PostBQP}$ [55]. The inclusion relations between these classes are $\text{NP} \subseteq \text{PostBQP} \subseteq \text{PSPACE}$, and furthermore, equal to $\text{PSPACE}$\textsuperscript{3}. Most recently, Aaronson, Bavarian, and Guerlin [52] showed that the Deutsch model can even solve the halting problem. The model of CTCs where the loops are generated through quantum teleportation to the past can solve all problems in the class $\text{PostBQP} = \text{PP}$ [13, 53, 54]. The classical analogue thereof can solve problems in $\text{BPP}_{\text{path}}$ [13] — the classical analogue of $\text{PostBQP}$ [55]. The inclusion relations between these classes are $\text{NP} \subseteq \text{PostBQP} \subseteq \text{PSPACE}$, and furthermore, equal to $\text{PSPACE}$\textsuperscript{3}. Most recently, Aaronson, Bavarian, and Guerlin [52] showed that the Deutsch model can even solve the halting problem. The model of CTCs where the loops are generated through quantum teleportation to the past can solve all problems in the class $\text{PostBQP} = \text{PP}$ [13, 53, 54]. The classical analogue thereof can solve problems in $\text{BPP}_{\text{path}}$ [13] — the classical analogue of $\text{PostBQP}$ [55].

\[ \mathcal{P}_{\text{NCCirc}} = \text{UP} \cap \text{coUP}, \]

\textit{i.e.}, that the class $\mathcal{P}_{\text{NCCirc}}$ (\text{NCCirc} standing for “non-causal circuit”) of decision problems solvable in polynomial time with the non-causal circuit model is equal to $\text{UP} \cap \text{coUP}$. This class contains all decision problems where every answer (“yes” or “no”) has a unique witness. Examples of such problems are integer factorization [56] and parity games [57], casted as decision problems. Thus, the class $\text{UP} \cap \text{coUP}$ is of great importance to the field of cryptography. Our result implies that the logically consistent CTC model [19] cannot solve decision problems outside of $\text{UP} \cap \text{coUP}$. If we denote by $\mathcal{P}_{\text{LCCTC}}$ (\text{LCCTC} standing for “logically consistent CTC”) the class of decision problems solvable by the latter framework, then we have $\mathcal{P} \subseteq \mathcal{P}_{\text{LCCTC}} \subseteq \mathcal{P}_{\text{NCCirc}} \subseteq \text{NP} \subseteq \text{PostBQP} \subseteq \text{PCTC}$. This means that the logically consistent CTC model [19] is the weakest of all known CTC models in terms of computation, and is unable to solve \text{NP-complete} problems (unless $\text{NP} = \text{UP} \cap \text{coUP}$). We also show the analogous statement for search problems: $\mathcal{F}_{\text{NCCirc}} = \mathcal{F}(\text{UP} \cap \text{coUP}) = \text{TFUP}$, where $\text{TFUP}$ is the class of all search problems with unique solutions. Furthermore, these results give an interpretation of the classes $\text{UP} \cap \text{coUP}$ and $\text{TFUP}$ in terms of \textit{fixed points}: Every instance of such a problem can be solved by finding a unique fixed point.

This work is organized as follows. First, we describe the computational model, and after that, we define some complexity classes and present our results. Then, we present an example on how to factorize integers by using that model, give conclusions, and state some open problems.

**MODEL OF COMPUTATION**

The non-causal circuit model, whose computational complexity we study, is based on the framework for cor-

\[ e : O_R \times O_S \times O_T \times \cdots \rightarrow I_R \times I_S \times I_T \times \cdots \]

be the function that represents the deterministic $E$, where $O_V$ and $I_V$, for $V \in \{R, S, T, \ldots\}$, are the sets of values the random variables $O_V$ and $I_V$ can take. Furthermore, we define the set $\mathcal{F}_V$ as the set of all functions from $I_V$ to $O_V$, i.e., $\mathcal{F}_V = \{f_v : I_V \rightarrow O_V\}$. A deterministic $E$ is called logically consistent if and only if

\[ \forall v \in \mathcal{F}_R, s \in \mathcal{F}_S, t \in \mathcal{F}_T, \exists ! (x_R, x_S, x_T, \ldots) : \\
(x_R, x_S, x_T, \ldots) = e(r(x_R), s(x_S), t(x_T), \ldots), \]

where $\exists !$ is the uniqueness quantifier. The condition of a unique fixed point ensures that the grandfather (no fixed point) and the information antinomies (multiple fixed points) are avoided.

The non-causal circuit model [28], then again, is formulated in terms of \textit{gates} as opposed to \textit{parties} and \textit{process matrices}. A \textit{circuit} is a collection of gates that are connected in an acyclic fashion, and where the input and

\[ \text{Figure 1. The parties } R, S, T, \ldots \text{ are modeled by stochastic operations} P_{X_V, O_V | A_V, I_V} \text{ for } V \in \{R, S, T, \ldots\}. \]

The process matrix $E$ (environment) is a channel (stochastic operation) $P_{I_R, I_S, I_T | O_R, O_S, O_T, \ldots}$ that connects the outputs of the parties with their inputs.

\[ \text{Figure 1. The parties } R, S, T, \ldots \text{ are modeled by stochastic operations} P_{X_V, O_V | A_V, I_V} \text{ for } V \in \{R, S, T, \ldots\}. \]

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The process matrix $E$ (environment) is a channel (stochastic operation) $P_{I_R, I_S, I_T | O_R, O_S, O_T, \ldots}$ that connects the outputs of the parties with their inputs.
The difference between this model and the framework discussed above is that here, the circuit is fixed whereas in the framework, every party can arbitrarily choose its local operation. Thus we omit the all quantifier in the logical-consistency condition for circuits (compare the above Equation with Equation (1)). Logically consistent closed circuits can be used to find the unique fixed, which is exploited in the following.

**Complexity Classes**

A decision problem \( \Pi \) is often casted as the membership problem of a language \( L \subseteq \Sigma^* \) with alphabet \( \Sigma \). For simplicity, and without loss of generality, we choose \( \Sigma = \{0,1\} \). An instance of \( \Pi \) is a string \( x \in \Sigma^* \), and the question is: Is \( x \) a word of \( L \), i.e., does \( x \in L \) hold? An algorithm that solves a decision problem outputs either “yes” or “no.”

Search problems, then again, are mostly defined via binary relations. A problem \( \Pi \) is associated with a binary relation \( R \subseteq \Sigma^* \times \Sigma^* \). An instance of \( \Pi \) is some \( x \in \Sigma^* \), and the question is: What (if there exists one) is \( y \in \Sigma^* \) such that \((x, y) \in R\)? An algorithm that solves a search problem outputs \( y \) if there exists a \( y \) satisfying \((x, y) \in R\), and returns “no” otherwise.

We use \(|x|\) to denote the length of some string \( x \in \Sigma^* \). A binary relation \( R \) is called polynomially decidable if there exists a deterministic Turing machine deciding the language \( \{(x, y) \in R\} \) in polynomial time, and \( R \) is called polynomially balanced if \((x, y) \in R\) implies the existence of some polynomial \( q \) such that \(|y| \leq q(|x|)\).

In the following definitions of complexity classes, we require that for every problem \( \Pi \) and given a string \( x \in \Sigma^* \), we can check in polynomial time whether \( x \) is an instance of \( \Pi \) or not. If \( x \) is not an instance of \( \Pi \), then we abort. We refer the reader to Refs. [59, 60] for common concepts in complexity theory.

**Definition 1** (Deterministic NCCirc algorithm). A deterministic NCCirc algorithm \( A \) is a polynomial time deterministic algorithm that takes as input some bit string \( x \in \{0,1\}^* \) and outputs a Boolean circuit \( C_x \) over AND, OR, and NOT, such that for every \( x \) the closed circuit \( C'_x \) is logically consistent, i.e.,

\[
\forall x \in \{0,1\}^*, \exists y : c_x(y) = y.
\]

If the fixed point \( y \) has the form \( y = 1z \) for some \( z \), then we say \( A \) accepts \( x \), otherwise, \( A \) rejects \( x \). The algorithm \( A \) decides a language \( L \subseteq \{0,1\}^* \) if \( A \) accepts every \( x \in L \) and rejects every \( x \notin L \). Furthermore, the algorithm \( A \) decides a binary relation \( R \subseteq \{0,1\}^* \times \{0,1\}^* \) if for every \( x \in \{0,1\}^* \) the pair \((x, y)\), with \( c_x(y) = y \), is in \( R \).

Based on the above definition, we define the complexity classes \( P_{NCCirc} \) and \( FP_{NCCirc} \).

**Definition 2** (\( P_{NCCirc} \) and \( FP_{NCCirc} \)). The class \( P_{NCCirc} \) contains all languages decidable by some deterministic NCCirc algorithm. The class \( FP_{NCCirc} \) contains all binary relations decidable by some deterministic NCCirc algorithm.

We will relate \( P_{NCCirc} \) to the following complexity class.

**Definition 3** (UP). The class UP (Unambiguous Polynomial-time) contains all languages \( L \) for which a polynomial-time verifier \( V : \{0,1\}^* \rightarrow \{0,1\} \) exists such that for every \( x \), if \( x \in L \) then \( \exists y : V(x,y) = 1 \), and if \( x \notin L \) then \( \forall y : V(x,y) = 0 \).

The complexity class UP was first defined by Valiant [61]. The only difference between the classes NP and UP is that in the former, multiple witnesses are allowed. The class coUP contains all languages \( L \) where the complement of \( L \) is in UP.

We are now ready to state our first theorem.
Let TFUP

We start with a circuit \( C \) such that for every input \( x \), if \( x \in L \), then

\[
E x \in L \quad \exists w : \forall w' : V_\text{yes}(x, w) = 1 \wedge V_\text{no}(x, w') = 0,
\]

and otherwise

\[
\forall w : V_\text{yes}(x, w) = 0 \land \exists w' : V_\text{no}(w, w') = 1.
\]

The following deterministic NCCirc algorithm \( A \) decides the language \( L \). Upon receiving \( x \in \{0, 1\}^* \), \( A \) generates the circuit \( C_x \) as shown in Figure 3. The subcircuits \( V_\text{yes}, V_\text{no} \) implement the verifiers \( V_\text{yes}, V_\text{no} \), and can be constructed in polynomial time, because \( L \) is assumed to be in \( \text{UP} \cap \text{coUP} \). The circuit acts in the following way:

\[
c_x : \{0, 1\} \times \{0, 1\}^{|x|} \to \{0, 1\} \times \{0, 1\}^{|x|},
\]

\[
: (b, w) \mapsto \begin{cases} (0, w) & V_\text{no}(x, w) = 1, \\ (1, w) & V_\text{yes}(x, w) = 1, \\ (b \oplus 1, w) & \text{otherwise}, \end{cases}
\]

where \( q \) is a polynomial. The function \( c_x \) has a unique fixed point. If \( x \in L \), then there exists a unique \( w \) with \( V_\text{yes}(x, w) = 1 \), and \( c_x(1w) = 1w \). Otherwise, there exists a unique \( w \) with \( V_\text{no}(x, w) = 1 \), and \( c_x(0w) = 0w \).

The converse \( \text{P}_{\text{NCCirc}} \subseteq \text{UP} \cap \text{coUP} \) holds for the following reason. First, assume \( L \) is in \( \text{P}_{\text{NCCirc}} \). This means that for every \( x \) we have some logically consistent circuit \( C'_x \). We design both verifiers \( V_\text{yes} \) and \( V_\text{no} \) to act as

\[
V_\text{yes} : (x, z) \mapsto c_x(z) = z \land z = 1w,
V_\text{no} : (x, z) \mapsto c_x(z) = z \land z = 0w.
\]

That is, both verifiers just check whether \( z \) is a fixed point of \( C_x \), and additionally check for the first bit.

Finally, we discuss the respective search problems.

**Definition 4 (FUP).** A binary relation \( R \) is in \( \text{FUP} \) (Function UP) if and only if \( R \) is polynomially decidable, polynomially balanced, and \( \forall x : |\{y \mid (x, y) \in R\}| \leq 1 \).

Informally, a problem is in \( \text{FUP} \) if for every instance there exists at most one solution.

**Definition 5 (F(UP \cap \text{coUP})).** A pair \( (R_1, R_2) \) of relations is in \( \text{F(UP \cap \text{coUP})} \) if and only if both relations are polynomially decidable, polynomially balanced, and for every instance \( x \)

\[
(\exists y : (x, y) \in R_1 \land \forall z : (x, z) \notin R_2) \lor
(\forall y : (x, y) \notin R_1 \land \exists z : (x, z) \in R_2)
\]

holds. The exclusive or (\( \lor \)) asks for either yet not both expressions to be true.

Note that the output of a search problem in \( \text{F(UP \cap \text{coUP})} \) is some string \( w \) that satisfies either \( (x, w) \in R_1 \) or (exclusively) \( (x, w) \in R_2 \), but, as we formulated it, does not tell us in which relation the pair \( (x, y) \) appears. However, since both relations are polynomially decidable, we can check in polynomial time whether \( y \) is a solution of \( R_1 \) or \( R_2 \). This brings us to the following class, which is equal.

**Definition 6 (TFUP).** A binary relation \( R \) is in \( \text{TFUP} \) (Totally FUP) if and only if \( R \) is polynomially decidable, polynomially balanced, and \( \forall x, \exists y : (x, y) \in R \).

**Theorem 2.** \( \text{TFUP} = \text{F(UP \cap \text{coUP})} \).

**Proof.** Let \( R \) be a relation in \( \text{TFUP} \) and \( R_1, R_2 \) two relations such that for every \( x \):

\[
(\exists y : (x, y) \in R_1 \land \forall z : (x, z) \notin R_2) \lor
(\forall y : (x, y) \notin R_1 \land \exists z : (x, z) \in R_2).
\]

To show \( \text{TFUP} \subseteq \text{F(UP \cap \text{coUP})} \), set \( R_1 = R \) and \( R_2 = \emptyset \), and to show \( \text{F(UP \cap \text{coUP})} \subseteq \text{TFUP} \), set \( R = R_1 \cup R_2 \).

A similar statement \( \text{TFNP} = \text{F(NP \cap \text{coNP})} \) can also be made [62]. The complexity class \( \text{TFNP} \) is the class of all total relations that are polynomially decidable and polynomially balanced.

We now state and prove the final theorem:

**Theorem 3.** \( \text{FP}_{\text{NCCirc}} = \text{TFUP} \).

**Proof.** We start with \( \text{TFUP} \subseteq \text{FP}_{\text{NCCirc}} \). A binary relation \( R \) in \( \text{TFUP} \) is polynomially decidable and polynomially balanced. Therefore, there exists an algorithm \( D \) that takes two inputs \( x, y \), runs in polynomial time in \(|x|\), and if \((x, y) \in R \) then \( D \) outputs “yes,” otherwise, \( D \) outputs “no.” Furthermore, for every instance \( x \) there exists a unique \( y \) with \((x, y) \in R \). The deterministic NCCirc algorithm \( A \), upon receiving \( x \), generates the circuit \( C_x \) that acts as

\[
c_x : y \mapsto \begin{cases} y & (x, y) \in R, \\ y' & \text{otherwise}, \end{cases}
\]

Figure 3. Circuit \( C_x \) used to reduce a problem from \( \text{UP} \cap \text{coUP} \) to \( \text{P}_{\text{NCCirc}} \). The wire that carries \( w \) consists of \( q(|x|) \) bits.
where, if \( y = bz \) with \( b \in \{0, 1\} \), then \( y' = (b \oplus 1)z \). Thus, for every \( x \) we have a circuit \( C_x \) with a unique fixed point that equals the solution, i.e., \( c_x(y) = y \rightarrow (x, y) \in R \). The converse inclusion relation \( FP_{\text{NCCirc}} \subseteq TFUP \) is shown as follows. Suppose we are given a relation \( R \) that is decidable by a deterministic NCCirc algorithm \( \mathcal{A} \).

We now need to show that \( R \) is polynomially decidable, polynomially balanced, and that every \( x \) has a unique solution. Indeed, \( R \) is polynomially decidable and polynomially balanced because \( C_x \) is generated in polynomial time, and \( C_x \) upon input \( y \) is computed in polynomial time in \( |x| \). Furthermore, \( C_x \) has a unique fixed point. The algorithm \( D \) to decide \( R \) on input \( x \) returns the truth value of \( c_x(y) = y \).

**EXAMPLE: INTEGER FACTORIZATION**

We give an example of an algorithm to factorize integers. The NCCirc algorithm \( \mathcal{A} \) outputs, on input \( N \in \mathbb{Z} \), a circuit \( C_N \) with which \( N = p_1^{e_1}p_2^{e_2} \cdots \) can be decomposed into its prime factors \( p_1, p_2, \ldots \) along with its exponents \( e_1, e_2, \ldots \). We give a description of \( C_N \) as an algorithm. Clearly, this algorithm can be transformed into a circuit. The following algorithm runs in a time polynomial in \( n = \lceil \log N \rceil \).

**Algorithm 1** Factoring \( N \)

**Input:** \( b \in \{0, 1\}, a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n \in K \)

**Output:** \( b' \in \{0, 1\}, a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n \in K \)

```plaintext
1: w ← ¬b, a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n
2: for i = 1 to n - 1 do
3:   if (a_i < a_{i+1}) ∨ (a_i ≠ 1 ∧ a_i = a_{i+1}) then
4:     return w
5:   end if
6: end for
7: for i = 1 to n do
8:   if (a_i = 1 ∧ e_i > 1) ∨ a_i ∉ PRIME \cup \{1\} then
9:     return w
10: end if
11: end for
12: if a_1^{e_1}a_2^{e_2} \cdots a_n^{e_n} ≠ N then
13:   return w
14: end if
15: return 0, a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n
```

Algorithm 1 takes as input 1 bit and 2n numbers in \( K = \{1, 2, \ldots, N - 1\} \), where every number is represented as an \( n \)-bit string. On line 3 we check whether the first \( n \) numbers are ordered. On line 8 we check whether \( e_i \) is 1 whenever \( a_i = 1 \), and whether \( a_i \) is indeed prime (or 1). A deterministic primality test can be performed in polynomial time as was recently shown [63]. Finally, on line 12 we check whether the decomposition is correct. If all tests are passed, then the algorithm returns \( 0, a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n \) where \( \prod_{i=1}^{n} a_i^{e_i} = N \), otherwise, the algorithm flips the first input bit. This algorithm and, therefore, the circuit \( C_N \), has a unique fixed point \( 0, p_1, p_2, \ldots, p_m, 1^{n-m}, e_1, e_2, \ldots, e_m, 1^{n-m} \), where \( p_1 > p_2 > \cdots > p_m \) are primes and \( \prod_{i=1}^{m} p_i^{e_i} = N \). Intuitively, one can understand this algorithm as “killing the grandfather” whenever a wrong factorization is given.

**CONCLUSION AND OPEN QUESTIONS**

The non-causal circuit model describes circuits where the assumption of a global causal order is replaced by the assumption of logical consistency (i.e., no grandfather and no information antinomies). The problems that are solvable in polynomial time by such circuits form the complexity class \( P_{\text{NCCirc}} \). We show that this class equals \( \text{UP} \cap \text{coUP} \), where \( \text{UP} \) consists of all problems in \( \text{NP} \) which have an unambiguous accepting path. Notable problems within \( \text{UP} \cap \text{coUP} \) are integer factorization and parity games. Intuitively, the class \( P_{\text{NCCirc}} \) contains all search problems that can be solved by determining the unique fixed point of a specific reformulation of the problem. This gives a new interpretation of the class \( \text{UP} \cap \text{coUP} \). The uniqueness requirement can be understood as arising from the assumption of no overdetermination (grandfather antinomy) and of no underdetermination (information antinomy). Similar complexity classes to \( FP_{\text{NCCirc}} \) (the functional equivalent of \( P_{\text{NCCirc}} \)) are \( \text{FIXP} \) and \( \text{linear-FIXP} = \text{PPAD} [64] \). These classes can solve problems where multiple fixed points might exist, and in \( \text{FIXP} \), the fixed point are even allowed to be irrational. Finding a Nash equilibrium for two parties is \( \text{linear-FIXP-complete} \), and the same problem for three parties or more is \( \text{FIXP-complete} [64] \). The class \( P_{\text{NCCirc}} = \text{UP} \cap \text{coUP} \) is not believed to contain complete problems [65].

This result leads us to conclude that closed timelike curves, based on the classical framework for correlations without global causal order, cannot solve problems outside of \( \text{UP} \cap \text{coUP} \), i.e., \( \text{P}_{\text{LCCFTC}} \subseteq \text{P}_{\text{NCCirc}} \). The reason for this is that in the CTC model we require the composed map of the parties with the environment to have a unique fixed point for any choice of local operations of the parties. This assumption was dropped when defining the non-causal circuit model. Thus, the CTC model is equal to the circuit model up to this additional condition, which can only lower the computational power. However, note the subtlety that the framework for classical correlations without causal order (as opposed to the CTC model) could, then again, solve problems not solvable by the CTC model. The reason for this is that in the correlations framework, fine-tuned process matrices are allowed [27] which are inherently probabilistic.

When we compare this result to the computational power of the Deutsch CTC model, we note that the CTC model studied here is dramatically weaker. This (possibly extreme) drop of computational power could be ex-
plained by the assumption of linearity which, in contrast to Deutsch’s model, is present in the model studied here. It is known that non-linearity can lead to astonishing results [66–68]. Put differently, the absence of the grandfather antinomy allows to solve problems in PSPACE, yet, if we additionally ask for the absence of the information antinomy, the complexity drops down to $UP \cap coUP$. As a side remark: The Deutsch version of CTCS restricted to deterministic fixed points gives a power of at most $NP \cap coNP$ [52].

One can put this result in the following perspective: Previous results on closed timelike curves show that they are not problematic from a general relativity theory point of view, from a logic point of view, and now we show their relative innocence from a computing of view.

Some of the main open questions that remain are: Does $P_{LCCTC} \supseteq P_{NCCirc}$ hold or not, what are the quantum versions $BQP_{NCCirc}$ and $BQP_{LCCTC}$ of the complexity classes defined here, and how does $BQP$ relate to $P_{NCCirc}$?

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