A NOTE ON $N = (2, 2)$ SUPERFIELDS IN TWO DIMENSIONS

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ABSTRACT

Motivated by the results in \texttt{hep-th/0508228}, we perform a careful analysis of the allowed linear constraints on $N = (2, 2)$ scalar superfields. We show that only chiral, twisted-chiral and semi-chiral superfields are possible. Various subtleties are discussed.

Non-linear $\sigma$-models in two dimensions with an $N = (2, 2)$ supersymmetry play a central role in the description of type II superstrings in the absence of R-R fluxes. The interest in these models was recently rekindled as well in the physics as in the mathematics community. For physicists, these models allow for the study of compactifications in the presence of non-trivial NS-NS fluxes while for mathematicians the models provide a concrete realization of generalized complex geometries. The ideal setting for studying them is provided by $N = (2, 2)$ superspace where the whole (local) geometry gets encoded in a single scalar function, the Lagrange density. The $N = (2, 2)$ superfields and as a direct consequence the geometry of the resulting $\sigma$-model as well, are fully characterized by their constraints. The analysis of auxiliary field configurations made in [1] raised the suspicion that more general $N = (2, 2)$ superfields than those known up till now might exist. This point will be explored in the present note.

The target space geometry of a bosonic non-linear $\sigma$-model in two dimensions\footnote{We consider here only $\sigma$-models without boundaries.} is characterized by the metric $g_{ab}$ and a closed 3-form (the torsion) $T_{abc}$ on it. We denote the local coordinates on the target manifold by $X^a$. The indices $a, b, \ldots$ run from 1 to $D$, with $D$ the dimension of the target manifold. Such a model can be lifted to an $N = (1, 1)$ supersymmetric $\sigma$-model without any further restrictions on the geometry. However, passing from $N = (1, 1)$ to $N = (2, 2)$ supersymmetry introduces additional geometric structure [2]–[5].

Indeed $N = (2, 2)$ requires the existence of two $(1, 1)$ tensors, $J^a_{\dot{a}b}(X)$ and $J^a_{\dot{b}b}(X)$, on the target manifold. \textit{On-shell} closure of the $N = (2, 2)$ supersymmetry algebra is realized provided both $J_+$ and $J_-$ are complex structures. I.e. they square to $-1$,
One finds that the off-shell non-closing terms in the algebra are proportional to the commutator of the two complex structures, $[J_+, J_-]$. Decomposing the tangent target space as $\text{ker}[J_+, J_-] \oplus \text{coker}[J_+, J_-]$, one anticipates that the description of $\text{ker}[J_+, J_-]$ will be possible with the fields at hand while the description of $\text{coker}[J_+, J_-]$ will require the introduction of additional auxiliary fields. 

Realizing the $N = (2, 2)$ supersymmetry algebra is obviously not sufficient, the $N = (1, 1)$ supersymmetric non-linear $\sigma$-model has to be invariant under the additional supersymmetry transformations as well. One finds that this is indeed so provided the metric is hermitean with respect to both complex structures.

Furthermore, both complex structures have to be covariantly constant,

$$0 = \nabla^\pm_c J^a_{\pm b} \equiv \partial_c J^a_{\pm b} + \Gamma^a_{\pm dc}J^d_{\pm b} - \Gamma^d_{\pm bc}J^a_{\pm d},$$

with the connections $\Gamma^a_{\pm}$ given by,

$$\Gamma^a_{\pm bc} \equiv \{a^\prime_{bc}\} \pm T_{\pm}^{ab},$$

where the first term at the right hand side is the standard Christoffel symbol and $T_{\pm}^{ab} = g^{ad}T_{dbc}$. 

Finding an off-shell description of a general $N = (2, 2)$ supersymmetric non-linear $\sigma$-model remained for more than two decades an open problem. Recently this has been solved in [6] (papers preparing the road to this result are e.g. [7]–[10]). There it was shown that chiral, twisted-chiral [3] and semi-chiral [7] multiplets are sufficient to describe any $N = (2, 2)$ non-linear $\sigma$-model. Roughly speaking one gets that when writing $\ker[J_+, J_-] = \ker(J_+ + J_-) \oplus \ker(J_+ - J_-)$ resp. can be integrated to chiral and twisted chiral multiplets resp. [8]. Semi-chiral multiplets allow then for a description of $\text{coker}[J_+, J_-]$ [6], [9], [10]. 

However, chiral, twisted-chiral and semi-chiral multiplets are by no means the only representations of $d = 2, N = (2, 2)$ supersymmetry. Other representations are known such as linear [11]–[13] and twisted linear multiplets [14]. Having other multiplets at hand allow e.g. for dual formulations of a model (see e.g. [15] and references therein). In [1], a detailed analysis of potential auxiliary field configurations in $d = 2, N = (2, 2)$ $\sigma$-models was performed. The resulting expressions were very involved and raise the

\[ J^a_{\pm c} J^b_{\pm d} = -\delta^a_b \] and their Nijenhuis tensors vanish, $N[J_+, J_+]^a_{bc} = 0$. One finds that

$$J^c_{\pm a} J^d_{\pm b} g_{cd} = g_{ab}. \quad (1)$$

This implies the existence of two two-forms $\omega_{\pm ab} = -\omega_{\pm ba} = g_{ac} J^c_{\pm b}$. In general they are not closed. Using eq. (2), one shows that $\omega_{\pm ab} = \mp 2 J^2_{\pm [a T_{bc]d} = \mp (2/3) J^a_{\pm a} J^b_{\pm b} J^c_{\pm c} T_{def}$, where for the last step we used the fact that the Nijenhuis tensors vanish.
pertinent question whether other solutions besides semi-chiral multiplets exist. This motivates the present note. We analyze constraints linear in derivatives on $N = (2, 2)$ superfields in order to clarify this.

The $d = 2$, $N = (1, 1)$ superspace has two bosonic (lightcone) coordinates $\sigma^\pm \equiv \tau + \sigma$, $\sigma^- \equiv \tau - \sigma$ and two (chiral, real) fermionic coordinates $\theta^+$ and $\theta^-$. Passing to $N = (2, 2)$ superspace requires the introduction of two additional (chiral, real) fermionic coordinates $\hat{\theta}^+$ and $\hat{\theta}^-$. We introduce the fermionic derivatives w.r.t. $\theta^\pm$, $D^\pm$, and those w.r.t. $\hat{\theta}^\pm$, $\hat{D}^\pm$. They are defined by,

$$D^+_2 = \hat{D}^+_2 = -\frac{i}{2} \partial_\hat{\tau}, \quad D^-_2 = \hat{D}^-_2 = -\frac{i}{2} \partial_\tau,$$

and all other anti-commutators vanish.

The action $S$ in $N = (2, 2)$ superspace is given by,

$$S = \int d^2 \sigma d^2 \theta d^2 \hat{\theta} V.$$

As the measure has dimension zero, the Lagrange density $V$ can only be some function of scalar $N = (2, 2)$ superfields. It is clear that in order to generate dynamics, one will have to judiciously constrain the $N = (2, 2)$ superfields.

Consider a set of bosonic, scalar $N = (2, 2)$ superfields $X^a$, $a \in \{1, \cdots D\}$. Expanding them in powers of $\theta^+$ and $\theta^-$, one finds that each general $N = (2, 2)$ superfield consists of 4 $N = (1, 1)$ superfields. As a warming up exercise, let us first study those constraints linear in the derivatives that reduce the number of $N = (1, 1)$ components in a general $N = (2, 2)$ superfield to one. Fixing both $\hat{D}^+_a X^a$ and $\hat{D}^- X^a$ simultaneously does the job. The most general constraints consistent with dimensions and Lorentz covariance are then given by,

$$\hat{D}^+_a X^a = J^a_{+ b}(X) D^+_b X^b, \quad (6)$$

where $J_\pm(X)$ are at this point two arbitrary $(1, 1)$ tensors. From this, one gets immediately,

$$\hat{D}^2_a X^a = + \frac{i}{2} (J^2_+)^a_{b \partial_\pm} X^b + \frac{1}{2} \mathcal{N}[J_+, J_+]^a_{bc} D^+_b D^+_c X^c,$$

$$\hat{D}^2_a X^a = - \frac{i}{2} (J^2_+)^a_{b \partial_-} X^b + \frac{1}{2} \mathcal{N}[J_-, J_-]^a_{bc} D^-_b D^-_c X^c,$$

and

$$\{ \hat{D}_+, \hat{D}_- \} X^a = [J_+, J_-]^a_{b \partial} D_- D^+_b X^b + 2 \mathcal{M}[J_-, J_+]^a_{bc} D^+_b X^b D_- X^c.$$

$^4$Out of two commuting $(1, 1)$ tensors $R^a_{\pm b}$ and $S^a_{\pm b}$, one contracts a $(1, 2)$ tensor, $\mathcal{M}[R, S]^a_{bc} = \frac{(R^a d S^d_{\pm b c} - S^a d R^d_{\pm c b} + \hat{S}^d_{\pm b R^a_{\pm c d}} - R^c S^a_{\pm d})}{2}$. One has that $\mathcal{M}[R, S]^a_{bc} = - \mathcal{M}[S, R]^a_{cb}$ and $\mathcal{N}[R, S]^a_{bc} = \mathcal{M}[R, S]^a_{bc} + \mathcal{M}[S, R]^a_{bc}$. 

3
Requiring that these constraints are consistent with \( \hat{D}_+^2 = -(i/2)\partial_+ \), \( \hat{D}_-^2 = -(i/2)\partial_- \) and \( \{ \hat{D}_+, \hat{D}_- \} = 0 \), we get the following conditions,

\[
\begin{align*}
J_\pm^2 &= -1, \quad [J_+, J_-] = 0, \\
\mathcal{N}[J_\pm, J_\pm] &= \mathcal{M}[J_+, J_-] = 0.
\end{align*}
\] (9)

All conditions in eq. (9) are necessary. Indeed, in the literature it is often erroneously stated that two complex structures are simultaneously integrable if they commute\(^5\).

In other words, \( J_\pm^2 = -1 \), \( \mathcal{N}[J_\pm, J_\pm] = 0 \) and \( [J_+, J_-] = 0 \) would imply that \( \mathcal{N}[J_+, J_-] = 0 \) holds. However this is wrong! A very nice and explicit counter example was provided in [16]. Consider a six-dimensional target manifold with the following two complex structures,

\[
J_+ = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-X^1 & -X^1 & X^3 & -X^3 & 0 & -1 \\
-X^1 & X^1 & X^3 & X^3 & 1 & 0 \\
\end{pmatrix}, \quad \text{and} \quad (10)
\]

\[
J_- = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-X^1 & X^1 & X^3 & X^3 & 0 & 1 \\
-X^1 & -X^1 & X^3 & -X^3 & -1 & 0 \\
\end{pmatrix}. \quad (11)
\]

One readily verifies that they are complex structures, i.e. \( J_\pm^2 = -1 \) and \( \mathcal{N}[J_\pm, J_\pm] = 0 \). They commute, \( [J_+, J_-] = 0 \), as well. However, when calculating \( \mathcal{M}[J_+, J_-] \), one finds that all components vanish except for,

\[
\begin{align*}
\mathcal{M}[J_+, J_-]_{11}^{5} &= \mathcal{M}[J_+, J_-]_{21}^{5} = \mathcal{M}[J_+, J_-]_{22}^{5} = \mathcal{M}[J_+, J_-]_{43}^{5} = \mathcal{M}[J_+, J_-]_{21}^{6} = \\
\mathcal{M}[J_+, J_-]_{33}^{6} &= \mathcal{M}[J_+, J_-]_{43}^{6} = \mathcal{M}[J_+, J_-]_{44}^{6} = +1, \\
\mathcal{M}[J_+, J_-]_{12}^{5} &= \mathcal{M}[J_+, J_-]_{33}^{5} = \mathcal{M}[J_+, J_-]_{34}^{5} = \mathcal{M}[J_+, J_-]_{44}^{5} = \mathcal{M}[J_+, J_-]_{11}^{6} = \\
\mathcal{M}[J_+, J_-]_{12}^{6} &= \mathcal{M}[J_+, J_-]_{22}^{6} = \mathcal{M}[J_+, J_-]_{34}^{6} = -1. \quad (12)
\end{align*}
\]

Consequently, the mixed Nijenhuis tensor \( \mathcal{N}[J_+, J_-] = \mathcal{M}[J_+, J_-] + \mathcal{M}[J_- J_+] \) does not vanish! In other words, all four conditions in eq. (9) have to be imposed independently. The non-linear σ-model constructed out of superfields constrained as in

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\(^5\)While this statement is made in numerous papers, we just mention one paper for which one of the present authors is responsible [9]. In that paper a multiplicative factor of two is lacking in front of the last term in eq. (A.6), which invalidates the proof of lemma 1.
eq. (6) so that eq. (9) holds, will be such that eqs. (1) and (2) are automatically satisfied\(^6\).

Let us now come back to eq. (9) which does imply that both \(J_+\) and \(J_-\) are simultaneously integrable. This means that we can make a coordinate transformation such that both \(J_+\) and \(J_-\) are diagonal with eigenvalues \(\pm i\). If the eigenvalue of \(J_+\) and \(J_-\) have the same (opposite) sign, we are dealing with a (twisted) chiral field. Indeed, a chiral field \(Z\), and its hermitean conjugate \(\bar{Z}\), satisfy,

\[
\hat{D}_\pm Z = +i D_\pm Z, \quad \hat{D}_\pm \bar{Z} = -i D_\pm \bar{Z},
\]

while for a twisted chiral field \(Y\) (and its hermitean conjugate \(\bar{Y}\)) we get,

\[
\hat{D}_\pm Y = +i D_\pm Y, \quad \hat{D}_\mp Y = -i D_\mp Y, \quad \hat{D}_\pm \bar{Y} = -i D_\pm \bar{Y}, \quad \hat{D}_\mp \bar{Y} = +i D_\mp \bar{Y}.
\]

The first explicit example of a non-linear \(\sigma\)-model which requires both chiral and twisted chiral superfields – the \(S^3 \times S^1\) WZW model – was given in [17].

We now generalize the constraints by allowing for additional auxiliary fields. We start from a set of \(m\) general \(N = (2,2)\) superfields which we combine in an \(m \times 1\) matrix \(X\). The most general right handed constraint linear in the derivatives which we can impose is,

\[
\hat{D}_+ X = J_+ D_+ X + K_+ \Psi_+,
\]

where \(\Psi_+\) is a \(m \times 1\) column matrix of fermionic \(N = (2,2)\) superfields and \(J_+\) and \(K_+\) are constant \(m \times m\) matrices\(^7\). Through a linear transformation we can bring \(K_+\) in its Jordan normal form. Making an appropriate linear combination of the components of \(\Psi_+\) allows one to reduce \(K_+\) to the form of a projection operator,

\[
K_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1_{k \times k} \end{pmatrix},
\]

where \(1_{k \times k}\) is the \(k \times k\) \((k \leq m)\) unit matrix. It is clear that \(k\) determines the number of fermionic (auxiliary) superfields which will appear in eq. (15). We now split \(X\) into a \(k \times 1\) matrix \(\tilde{X}\) and an \((m - k) \times 1\) matrix \(\check{X}\) of superfields. Appropriately

\(^6\)One can also look at things from the point of view of the \(N = (1,1)\) non-linear \(\sigma\)-model without referring to \(N = (2,2)\) superspace. Using the covariantly constancy of the complex structures, one shows \(2\mathcal{M}_{bc}[J_+, J_-] = [J_+, J_-]^{\alpha}_{\beta} F_{\alpha \beta}^{d} \). Put differently: the fact that two complex structures commute and that they are covariantly constant does imply that they are simultaneously integrable. However, we reiterate that in an \(N = (2,2)\) superspace treatment one only deals with eq. (9) which will imply – at least for the case of commuting complex structures – eqs. (1) and (2).

\(^7\)In order to keep the analysis feasible, we only consider constraints linear in the superfields. This is a reasonable simplification as non-linear constraints would significantly complicate – if not make it impossible – the quantization of the resulting non-linear \(\sigma\)-model.
redefining the fermionic superfields in $\Psi^+$, we can rewrite eq. (15) without any loss of generality as,

$$\hat{D}_+ \check{X} = J_+ D_+ \check{X} + K_+ D_+ \check{X},$$
$$\check{D}_+ \check{X} = \check{\psi}_+, \quad (17)$$

where $J_+$ and $K_+$ resp. are $(m-k) \times (m-k)$ and $(m-k) \times k$ constant matrices resp. and $\check{\psi}_+$ is a $k \times 1$ column matrix of fermionic (auxiliary) superfields. Implementing $\hat{D}_+^2 = -(i/2)\partial_\pm$ in eq. (17) yields,

$$K_+ = 0, \quad J_+^2 = -1_{(m-k) \times (m-k)}. \quad (18)$$

This implies that $m - k$ should be even and we put $2n = m - k$ with $n \in \mathbb{N}$. At this point we can, through an appropriate linear transformation on $\check{X}$, diagonalize $J_+$ with eigenvalues $\pm i$. Summarizing, we get – without any loss of generality – the following right handed constraints,

$$\hat{D}_+ \check{X} = i\check{\mathbb{P}} D_+ \check{X},$$
$$\check{D}_+ \check{X} = \check{\psi}_+, \quad (19)$$

where $\check{\mathbb{P}}$ is given by,

$$\check{\mathbb{P}} = \begin{pmatrix} 1_{n \times n} & 0 \\ 0 & -1_{n \times n} \end{pmatrix}. \quad (20)$$

At this point we can still make arbitrary linear transformations on $\check{X}$. On $\check{X}$, linear transformations which commute with $\check{\mathbb{P}}$ are allowed as well. This freedom will be used later on.

The most general left handed constraints are of a form similar to the one in eq. (15),

$$\hat{D}_- \check{X} = \check{\mathbb{J}}_- D_- \check{X} + \check{\mathbb{K}}_- \Psi_. \quad (21)$$

This can be further simplified by looking at the non-linear $\sigma$-model we ultimately want to describe. We start with a Lagrange density $\mathcal{V}(X)$ which is some scalar function of the superfields $X$. Integrating over $\check{\theta}^\pm$, we get schematically the following dependence on $\Psi_+$ and $\Psi_-$ of the action in $N = (1, 1)$ superspace,

$$S = \int d^2 \sigma d^2 \theta d^2 \check{\theta} \mathcal{V}(X) = \int d^2 \sigma d^2 \theta \left( A_1 + \Psi_+^T A_2 + A_3 \Psi_- + \Psi_+^T A_4 \Psi_- \right), \quad (22)$$

where $A_\alpha$, $\alpha = 1, \cdots, 4$ are matrices which depend on the remainder of the $N = (1, 1)$ superfields. It is clear from this expression that the components of $\Psi_\pm$ appear as auxiliary fields. In order to solve for them, we have to require that the number of
components of $\Psi_+$ which appear in eq. (15) equals the number of components of $\Psi_-$ appearing in eq. (21).

A further exploration of the structure of the $N = (1, 1)$ action will show that the right hand side of $\hat{D}_- \tilde{X}$ cannot contain any components of $\Psi_-$. Indeed, assume that some components of $\Psi_-$ do appear at the right hand side of one or more components of $\hat{D}_- \tilde{X}$. Call these components $\tilde{X}^r$, $r \in \{1, \cdots, l \leq k\}$. Making an appropriate field redefinition on those components of $\Psi_-$, we would get the following schematical structure for the constraints,

$$\hat{D}_+ \tilde{X}^r = \tilde{\psi}_+^r, \quad \hat{D}_- \tilde{X}^r = \tilde{\psi}_-^r. \quad (23)$$

Because of $\{\hat{D}_+, \hat{D}_-\} = 0$, we get,

$$\hat{D}_+ \tilde{\psi}_+^r = -\hat{D}_- \tilde{\psi}_-^r \equiv F_{+ -}^r, \quad (24)$$

where when reducing to $N = (1, 1)$ superspace, $F_{+ -}^r$ will survive as new $N = (1, 1)$ superfields. They will appear in the action as,

$$S = \int d^2\sigma d^2\theta d^2\bar{\theta} \mathcal{V}(X) = \int d^2\sigma d^2\theta \left( \sum_{r=1}^i \partial_r \mathcal{V}(X) F_{+ -}^r + \cdots \right). \quad (25)$$

The equations of motion for $F_{+ -}^r$ force the potential $\mathcal{V}$ to be independent of $X^r$. Consequently we can impose that $\hat{D}_- \tilde{X}$ does not depend on any of the components of $\Psi_-$. These considerations lead us to the conclusion that $2n \geq k$. Decomposing $\tilde{X}$ as a $k \times 1$ column matrix $X$ and a $2n - k$ column matrix $\hat{X}$, we arrive – still without any loss of generality (but using input from the goal, the $\sigma$-model, we aim for) – at the following form for the constraints,

$$\begin{align*}
\hat{D}_+ X &= i \mathbb{P} D_+ X, \quad \hat{D}_- X = \psi_-, \\
\hat{D}_+ \hat{X} &= \bar{\psi}_+, \quad \hat{D}_- \hat{X} = \hat{J}_- D_- \hat{X} + \hat{K}_- D_- \hat{X} + \hat{L}_- D_- \hat{X} \\
\hat{D}_+ \hat{\psi}_+ &= i \hat{\mathbb{P}} D_+ \hat{\psi}_+, \quad \hat{D}_- \hat{\psi}_+ = \hat{J}_- D_- \hat{\psi}_+ + \hat{K}_- D_- \hat{\psi}_+ + \hat{L}_- D_- \hat{\psi}_+ + \hat{M}_- \psi_-, \quad (26)
\end{align*}$$

where $\psi_-$ is a $k \times 1$ column matrix of fermionic superfields which are related to the non-vanishing components of $\Psi_-$ through an appropriate coordinate transformation. Furthermore, we used the notation,

$$\begin{align*}
\mathbb{P} &= \begin{pmatrix} 1_{k/2 \times k/2} & 0 \\ 0 & -1_{k/2 \times k/2} \end{pmatrix}, \quad \hat{\mathbb{P}} = \begin{pmatrix} 1_{(n-k/2) \times (n-k/2)} & 0 \\ 0 & -1_{(n-k/2) \times (n-k/2)} \end{pmatrix}, \quad (27)
\end{align*}$$

where the reality properties of the fields force us to take $k \in 2\mathbb{N}$.

Rests us to impose the integrability conditions. One verifies that $0 = \{\hat{D}_+, \hat{D}_-\} \tilde{X}$, $0 = \{\hat{D}_+, \hat{D}_-\} X$ and $\hat{D}_-^2 X = -(i/2)\partial_- X$ resp. give us expressions for $\hat{D}_- \tilde{\psi}_+$, $\hat{D}_+ \tilde{\psi}_-$ and $\hat{D}_- \psi_-$. 7
Implementing the integrability conditions which follow from $0 = \{\hat{D}_+, \hat{D}_-\} \hat{X}$, \(\hat{D}_-^2 \hat{X} = -(i/2) \partial_- \hat{X}\) and \(\hat{D}_-^2 \hat{X} = -(i/2) \partial_- \hat{X}\) reduce eq. (26) to,
\[
\hat{D}_+ X = i \mathbb{P} D_+ X, \quad \hat{D}_- X = \psi_-, \quad \hat{D}_+ \hat{X} = \tilde{\psi}_+, \quad \hat{D}_- \hat{X} = \tilde{J}_- D_- \hat{X} + \tilde{K}_- D_- \hat{X}, \quad \hat{D}_+ \hat{X} = i \mathbb{P} D_+ \hat{X}, \quad \hat{D}_- \hat{X} = \tilde{J}_- D_- \hat{X} + \tilde{L}_- D_- \hat{X} + \tilde{J}_- \tilde{L}_- \psi_-, 
\]  
(28)

with,
\[
\tilde{J}_-^2 = -\mathbf{1}_{(2n-k) \times (2n-k)}, \quad \tilde{J}_-^2 = -\mathbf{1}_{k \times k},
\]
\[
[\tilde{J}_-, \mathbb{P}] = 0, \quad \hat{L}_- \mathbb{P} = \mathbb{P} \hat{L}_-, \quad \tilde{J}_- \tilde{K}_- = -\tilde{K}_- \tilde{J}_-. 
\]
(29)

Combining the second of these equations with the freedom to make an arbitrary linear transformation on \(\tilde{\psi}_+\) (making simultaneously the same transformation on \(\tilde{\psi}_+\)) allows one to put,
\[
\tilde{J}_- = i \mathbb{P},
\]
(30)

where \(\mathbb{P}\) was defined in eq. (27). A final simplification is achieved by making the following field redefinitions,
\[
\hat{X} \rightarrow \hat{X}' = \hat{X} - \tilde{J}_- \tilde{L}_- X, \quad \tilde{X} \rightarrow \tilde{X}' = \tilde{X} + \frac{1}{2} \tilde{K}_- \tilde{J}_- \hat{X} + \frac{1}{2} \hat{K}_- \hat{L}_- X, \quad \tilde{\psi}_+ \rightarrow \tilde{\psi}'_+ = \tilde{\psi}_+ + \frac{i}{2} \tilde{K}_- \tilde{J}_- \mathbb{P} D_+ \hat{X}'.
\]
(31)

This reduces eq. (28) to,
\[
\hat{D}_+ X = i \mathbb{P} D_+ X, \quad \hat{D}_- X = \psi_- , \quad \hat{D}_+ \hat{X}' = \tilde{\psi}'_+, \quad \hat{D}_- \hat{X}' = i \mathbb{P} D_- \hat{X}', \quad \hat{D}_+ \hat{X}' = i \mathbb{P} D_+ \hat{X}', \quad \hat{D}_- \hat{X}' = \tilde{J}_- D_- \hat{X}', 
\]
(32)

where,
\[
\tilde{J}_-^2 = -\mathbf{1}, \quad [\tilde{J}_-, \mathbb{P}] = 0.
\]
(33)

We still have the freedom to make a linear transformation on \(\hat{X}'\) provided it commutes with \(\mathbb{P}\). We use this freedom to diagonalize \(\tilde{J}_-\) with eigenvalues \(\pm i\). Pending upon the sign of the eigenvalues, \(\hat{X}'\) constitutes of chiral and twisted chiral superfields [3]. The remaining superfields \((X, \tilde{X}')\) are recognized as semi-chiral superfields [7].
While the analysis of [1] raised the hope that other auxiliary field structures beyond semi-chiral superfields might exist, we showed in the present note that this is not so. We made two important restrictions: we limited ourselves to constraints which were both linear in the derivatives as well as linear in the superfields (and it might very well be that using an appropriate coordinate transformation any constraint non-linear in the superfields can be brought to a constraint linear in the fields). If one wants to quantize the model, the latter restriction is essentially unavoidable. This note strengthens the quite unique role of semi-chiral superfields. When studying $d = 2, N = (2, 2)$ non-linear $\sigma$-models in the presence of NS-NS fluxes, semi-chiral superfields will be very generic. Indeed, consider e.g. a very large but quite simple class of integrable $\sigma$-models, the WZW models. In [18] it was shown that any even-dimensional WZW model allowed for an $N = (2, 2)$ supersymmetry. However, as argued in [17], only the $SU(2) \times U(1)$ model can be described solely in terms of chiral and twisted chiral superfields. All other WZW models on even dimensional group manifolds will require semi-chiral superfields as well (some explicit examples can be found in [8] and [9]). Up till now, the literature on semi-chiral superfields is rather limited. We hope that the present note will raise the interest in them.

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