Petri Net and Max-Plus Algebra Model in Bank Queue with Two Servers in Each Service

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Abstract. Bank is one of the business entity used by most of the people in the world including Indonesia and play an important role of financial institutions in the economy of society. There are some of the bank services that are given to us such as customer service, teller, loan service, etc. The main services provided by every bank are customer services and tellers. Queuing problem that happened in bank services can be modeled using mathematics. Mathematics model used in this research is timed petri net. Petri net models the queue in the services system into a discrete form where in this research is used two main services in the bank with two counters in each service. Furthermore, from the flow of petri net that has been made is built the coverability tree to find out the liveness and deadlocks of each groove petri net. Next, the petri net model is constructed using max-plus algebra.

1 Introduction

These days, the world population grows rapidly that cause the bank consumer also grow, so the bank must give the best services to the customers and it must be developed in the better way. Enough and good services are needed so there is no stack queue happen. There are several services in the bank, such as customer services, teller services, loan services, etc. The main services provided by every bank are customer service and teller service.

Based on Gazovski [5], on the annual bases are spent more than 37 billion of hours on some of the queue strings types. According to Bunday [2], the theory of line queues represents and provides a mathematical analysis of several related processes, including arrival at the end of the line, waiting in line, and servicing the users by the repairers in order. The theory of queue lines allows distribution and calculation of several measures of performance, including the average time waiting in line or system, the expected number of customers waiting to be served or the one who already have been served, and the probability to find the system in certain situations, such as empty, full, available service at the moment, or taking some time waiting to be serviced. Mathematics model which can be used are petri net and max-plus algebra.

This research are constructed petri net groove form bank queue. Next, the net is constructed the coverability tree to analyze the liveness and deadlock. After knowing the state of the net, then max-plus algebra can be modeled.
2 Preliminaries

This section explains about petri net, max-plus algebra, max-plus algebra in timed petri net, and some related theory to them.

2.1 Petri Net

Definitions relating to petri net are taken from Subiono [10].

Definition 2.1. Petri net is 4-tuple \((P,T,A,w)\), where
- \(P\) : finite set of place, \(P = \{P_1, P_2, \ldots, P_n\}\),
- \(T\) : finite set of transition, \(T = \{T_1, T_2, \ldots, T_m\}\),
- \(A\) : arc set, \(A \subseteq (P \times T) \cup (T \times P)\),
- \(w\) : weight relation, \(w: A \rightarrow \text{finite } \mathbb{N}\).

Petri net is a directed graph (digraph) with two kinds of nodes. The circle one represents the places and the rectangular represents the transitions. The arcs are depicted by an arrow connecting the places and the transitions. The weight of the arc from place \(P_i\) to transition \(T_j\) is denoted by \(w(P_i, T_j) = k\) means that there are \(k\) arcs from place \(P_i\) to transition \(T_j\). A token is placed in a place \(P_i\) that state whether or not a condition is met. A transition is fired when an event stated by the transition is happened. The transition firing process can be happened when all the token in input place is reduced by the arc weight that connects the places. In petri net, token is depicted by a black dot in a place.

Definition 2.2. A marking \(x\) in a petri net is a function
\[ x : P \rightarrow \mathbb{N}, \quad x(P_i) \in \mathbb{N}. \]

Definition 2.3. Marked petri net is 5-tuple \((P,T,A,w,x_0)\) where \((P,T,A,w)\) is the petri net and \(x_0\) is the initial marking.

Definition 2.4. The state of the marked petri net is
\[ x = [x(P_1), x(P_2), \ldots, x(P_n)]^T. \]

Definition 2.5. Transition \(T_j \in T\) in marked petri net is said to be enable if
\[ x(P_i) \geq w(P_i, T_j), \quad \forall P_i \in I(T_j). \tag{1} \]

2.2 Matrix Representation from the Petri Net

Definition of two matrices in petri net, backward and forward matrix, given by Subiono [10].

Definition 2.6. Backward matrix (forward) incidence that represented petri net is \(n \times m\) matrix, where \(n \) and \(m\) is the number of places and transition consecutively, and the \(i\)th and \(j\)th element is \(M_b(i,j) = w(p_i, t_j)(M_f(i,j) = w(t_j, p_i))\).

One of the uses of the backward incidence matrix is to determine the enable transition. Equation (1) applies to input places. If \(P_i\) is not the input places from transition \(T_j\), then the weight of the arc from \(P_i\) to \(T_j\) is zero, \(w(P_i, T_j) = 0\). The formula to determine where the next token will be placed is
\[ x(k+1) = x(k) + Du, \tag{2} \]

where
- \(x(k+1)\) : the location of the token after a firing,
- \(x(k)\) : the location of the token before a firing,
- \(D\) : state incidence matrix \(D = D^+ - D^-\),
- \(u\) : enable transition that will be fired.
According to Sadiq et al. [9], a petri net is said to be deadlocks when there is a transition or a set of transition cannot be fired. This is also happen when all the places do not get token. Based on Murata [8], a petri net is said to be live for initial condition $x_0$, if the transition have no relation with the deadlocks. The term liveness is defined as the transition that maybe enable.

It is given initial condition $x_0$, Cassandras [3] classifies the liveness into
(i) dead or $L_0$-live, if the transition can never be firing sequence,
(ii) $L_1$-live, if transition can be fired at least once in some firing sequence,
(iii) $L_2$-live, for any positive integer $k$, transition can be fired at least $k$ times in some firing sequence,
(iv) $L_3$-live, if transition appears infinitely, often in some firing sequence, and
(v) live atau $L_4$-live, if transition is $L_1$-live for every possibility state from $x_0$.

Based on Dwina and Subiono [4], coverability tree is a technique that used to solve several analysis aspect in event discrete system. Each node stated the marking from the petri net and it is built from the initial marking $x(0)$. Nodes represent markings generated from $x(0)$ (the root) and its successors, and each arc represents a transition firing, which transforms one marking to another. If there is a loop, then it can be said that the petri net is not deadlock.

2.3 Max-Plus Algebra

In this section the concept of max-plus algebra is defined by Bacelli et al. [1].

**Notation.** $N$ is the set of natural numbers, $R$ is the set of real numbers, $\varepsilon = -\infty$, $e = 0$, $R_{\text{max}} = R \cup \{\varepsilon\}$, $n = 1, 2, \ldots, n$.

**Definition 2.7.** Max-plus algebra is the set of $R_{\text{max}}$ with the two operation $\oplus$ and $\otimes$ and is denoted by

$$ R_{\text{max}} = (R_{\text{max}}, \oplus, \otimes, \varepsilon, e) $$

where the operation $\oplus$ and $\otimes$ define as

$$ x \oplus y = \max(x, y) \quad \text{and} \quad x \otimes y = x + y $$

The following definition of semiring is given based on Königsberg [7].

**Definition 2.8.** A semiring is nonempty set $R$ endowed with two operations $\oplus_R$, $\otimes_R$, and two elements $\varepsilon_R$ andc $e_R$ such that:

(i) $\oplus_R$ is associative and commutative with zero element $\varepsilon_R$;
(ii) $\otimes_R$ is associative, distribute over $\oplus_R$, and has unit element $e_R$;
(iii) $\varepsilon_R$ is absorbing for $\otimes_R$ i.e., $x \otimes_R \varepsilon_R = \varepsilon_R \otimes x = x$

Such a semiring is denoted by $\mathcal{R} = (R, \oplus_R, \otimes_R, \varepsilon, e)$.

**Theorem 2.1.** The max-plus algebra $R_{\text{max}} = (R_{\text{max}}, \oplus, \otimes, \varepsilon, e)$ has the algebraic structure of a commutative and idempotent semiring.

2.4 Matrices in Max-Plus Algebra

The definition of the operations in matrices is given by Königsberg [7]. Let $R_{\text{max}}^{n \times n}$ is the set $n \times n$ matrices, it is defined:

(i) The sum of $M \in R_{\text{ax}}^{n \times l}$, $N \in R_{\text{max}}^{l \times n}$, denoted $M \otimes N$ is defined as

$$ (A \otimes B)_{ik} = \bigoplus_{j=1}^{l} m_{ij} \otimes n_{jk} = \max_{j \in \underline{l}}(m_{ij} + n_{jk}), \quad i, k \in \underline{n}. $$

(ii) The scalar multiplication for $\alpha \in R_{\text{max}}^{n}$ and $M \in R_{\text{max}}^{n \times n}$ denoted $\alpha \otimes M$ is defined as

$$ (\alpha \otimes M)_{ij} = \alpha \otimes m_{ij}, \quad i, j \in \underline{n}. $$
2.5 Max-Plus Algebra Model of Timed Petri Net

This section explains the max-plus algebra model from petri net that refers to Subiono [10]. Timed petri net in queueing system is given in Figure 1.

In Figure 1, $a(k)$ denotes time when the $k$th customer come, $v_{a,k}$ denotes time length of the $k$th customer arrive, $s(k)$ states time for the $k$th service begin, $d(k)$ states the time for the $k$th customer leave the service, and $v_{d,k}$ denotes time length for the $k$th customer leave the service, state $Q$ is the queue and $B$ is busy, it is obtained

$$
\begin{align*}
    a(k) &= v_{a,k} + a(k-1) \\
    s(k) &= \max\{a(k), d(k-1)\} \\
    d(k) &= v_{d,k} + s(k) \\
    &= v_{d,k} + \max\{a(k), d(k-1)\} \\
    &= \max\{(v_{d,k} + v_{a,k}) + a(k-1), v_{d,k} + d(k-1)\},
\end{align*}
$$

or it can be written as

$$
\begin{pmatrix}
    a(k) \\
    s(k) \\
    d(k)
\end{pmatrix} =
\begin{pmatrix}
    v_{a,k} & \varepsilon & \varepsilon \\
    v_{a,k} & \varepsilon & e \\
    v_{d,k} \otimes v_{a,k} & \varepsilon & v_{d,k}
\end{pmatrix} \otimes
\begin{pmatrix}
    a(k-1) \\
    s(k-1) \\
    d(k-1)
\end{pmatrix},
$$

where $c = \varepsilon k = 1, 2, 3...$ and initial condition $a(0) = d(0) = 0$.

3 Main Result

3.1 Flow of The Bank Queue

In general, the service flow in every bank is almost the same. Nonetheless, it depends on the privacy in every bank and what services is available there. This research is used two main services in every bank, customer service (CS) and teller service. The flow is given by the flowchart depicted in Figure 2. Customers who already have any purpose where they want to go will come or meet the security near the queue number machine. Then, the security will tell them what should they do, either fill the form or not. If they are told to fill the form, then they fill the form before getting the queue number. If they are not, they will get the queue number directly depends on their purpose. Afterward, the customers will enter the CS queue or teller queue depends on their number and wait until their number is called, then served in a counter. They will be served in counter 1 or 2 if their purpose is to teller and in counter 5 or 6 if their purpose is to CS. After the service is finished, they must choose whether still need other services or not.
3.2 Petri Net Model

After defining the queue bank flow, then it is constructed the petri net using definition given by Subiono [10]. It is showed in Figure 3, where

- $P_0$: to the security
- $P_1$: filling form
- $P_2$: enter the queue
- $P_3$: wait in teller queue
- $P_4$: wait in CS queue
- $P_5$: served in counter 1
- $P_6$: served in counter 2
- $P_7$: served in counter 5
- $P_8$: served in counter 6
- $P_9$: recheck
- $T_0$: customer some
- $T_1$: directed to fill the form
- $T_2$: get the queue number directly
- $T_3$: get the queue number after fill the form
- $T_4$: enter the teller queue
- $T_5$: enter the CS queue
- $T_6$: start to be served in counter 1
- $T_7$: start to be served in counter 2
- $T_8$: start to be served in counter 5
- $T_9$: start to be served in counter 6
- $T_{10}$: finished
- $T_{11}$: left the bank
- $T_{12}$: need more service

3.3 Liveness and Deadlock Analysis

It is defined the number of the transitions and places is 13 and 10, so the state matrix is sized $13 \times 10$. It is formed the state matrix is $D = [d_{ij}]$, $D = D^+ - D^-$
Figure 3. Petri net model

\[
\begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

where

\[
d_{ij} = \begin{cases} 
1, & \text{there is an arc from transition } T_j \text{ to place } P_i; \\
0, & \text{there is no arc from transition } T_j \text{ to place } P_i \text{ or vice versa;} \\
-1, & \text{there is an arc from place } P_i \text{ to transition } T_j. 
\end{cases}
\]

Furthermore, it is used the equation (2) to know if the petri net is not deadlock by building the coverability tree from several possibility groove, such as when transition \( T_0 \) is not fire, transition \( T_1 \) and \( T_4 \) is fire with the initial state \( x(0) = (1 0 0 0 0 0 0 0 0 0) \), which means there is one token in place \( P_0 \). It is obtained coverability tree as shown in Figure 4.

For the other possibility groove is also done in the same way and can be conclude that the petri net is not deadlock with the liveness classification as follows:

(i) transition \( T_0 \) is live because it always can be fired from \( x_0 \) at least once and in every state. 
(ii) transition \( T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, \) and \( T_{12} \) are \( L_1 \)-live, because they can be fired once after the previous transition is fired.

3.4 Max-Plus Algebra of the Petri Net

Timed petri net model is petri net in Figure 3 by adding the variable \( v_{T_j,k} \) which states the length of time the service process in a transition. While variable \( T_j(k) \) states the time when the customer start the service in the \( j \)th transition, where \( j = 0, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12 \). Furthermore, it is built the max-plus algebra model in four conditions, as follows
(i) condition when customers who fill the form first and go to teller has the max-plus algebra model as written below,

\[
\begin{pmatrix}
T_0(k) \\
T_1(k) \\
T_3(k) \\
T_4(k) \\
T_6(k) \\
T_7(k) \\
T_{10}(k) \\
T_{11}(k) \\
T_{12}(k)
\end{pmatrix} =
\begin{pmatrix}
T_0(k-1) \\
T_1(k-1) \\
T_3(k-1) \\
T_4(k-1) \\
T_6(k-1) \\
T_7(k-1) \\
T_{10}(k-1) \\
T_{11}(k-1) \\
T_{12}(k-1)
\end{pmatrix}
\]

where:

\[
S_1 = (v_{T_{11},k} \otimes v_{T_6,k} \otimes X_1) \oplus (v_{T_{11},k} \otimes v_{T_{7},k} \otimes X_1)
\]

\[
Z_1 = (v_{T_{12},k} \otimes v_{T_6,k} \otimes X_1) \oplus (v_{T_{12,k}} \otimes v_{T_{7},k} \otimes X_1)
\]

\[
J_1 = (v_{T_{11},k} \otimes v_{T_6,k} \otimes Y_1) \oplus (v_{T_{11,k}} \otimes v_{T_{7},k} \otimes Y_1)
\]

\[
W_1 = (v_{T_{12},k} \otimes v_{T_6,k} \otimes Y_1) \oplus (v_{T_{12,k}} \otimes v_{T_{7},k} \otimes Y_1)
\]

\[
X_1 = Y_1 \otimes v_{T_6,k} \\
Y_1 = v_{T_{11,k}} \otimes v_{T_3,k} \\
Q_1 = v_{T_{3,k}} \otimes v_{T_6,k}
\]

\[
R_1 = (v_{T_{6,k}} \otimes X_1) \oplus (v_{T_{7,k}} \otimes X_1) \\
U_1 = (v_{T_{6,k}} \otimes Y_1) \oplus (v_{T_{7,k}} \otimes Y_1)
\]

(ii) condition when customers who fill the form first and go to CS has the max-plus algebra modeled as (i), by changed \(T_4(k), T_6(k),\) and \(T_7(k)\) to \(T_5(k), T_8(k),\) and \(T_9(k),\) therefore \(v_{T_4(k)}, v_{T_6(k)}\) and \(v_{T_7(k)}\) change to \(v_{T_5(k)}, v_{T_8(k)}\), and \(v_{T_9(k)},\) hence

\[
\begin{pmatrix}
T_0(k) \\
T_1(k) \\
T_3(k) \\
T_5(k) \\
T_8(k) \\
T_9(k) \\
T_{10}(k) \\
T_{11}(k) \\
T_{12}(k)
\end{pmatrix} =
\begin{pmatrix}
v_{T_6,k} \\
v_{T_6,k} \\
v_{T_3,k} \otimes v_{T_6,k} \\
v_{T_3,k} \otimes v_{T_6,k} \\
X_2 \\
X_2 \\
R_2 \\
S_2 \\
Z_2
\end{pmatrix} \oplus
\begin{pmatrix}
v_{T_6,k} \\
v_{T_6,k} \\
v_{T_3,k} \\
v_{T_3,k} \\
Y_2 \\
Y_2 \\
U_2 \\
J_2 \\
W_2
\end{pmatrix}
\]

where

Figure 4. Coverability tree if \(T_0\) is not fire, \(T_1, T_4,\) and \(T_6\) is fire
\[ R_2 = (v_{T_7,k} \otimes v_{T_5,k} \otimes v_{T_3,k} \otimes v_{T_0,k}) \oplus (v_{T_9,k} \otimes v_{T_6,k} \otimes v_{T_3,k} \otimes v_{T_0,k}) \]
\[ S_2 = (v_{T_{11},k} \otimes v_{T_8,k} \otimes v_{T_5,k} \otimes v_{T_0,k}) \oplus (v_{T_{11},k} \otimes v_{T_9,k} \otimes v_{T_3,k} \otimes v_{T_0,k}) \]
\[ Z_2 = (v_{T_{12},k} \otimes v_{T_8,k} \otimes v_{T_5,k} \otimes v_{T_0,k}) \oplus (v_{T_{12},k} \otimes v_{T_9,k} \otimes v_{T_5,k} \otimes v_{T_0,k}) \]
\[ U_2 = (v_{T_8,k} \otimes v_{T_5,k} \otimes v_{T_3,k}) \oplus (v_{T_9,k} \otimes v_{T_5,k} \otimes v_{T_3,k}) \]
\[ J_2 = (v_{T_{11},k} \otimes v_{T_8,k} \otimes v_{T_5,k} \otimes v_{T_3,k}) \oplus (v_{T_{11},k} \otimes v_{T_9,k} \otimes v_{T_5,k} \otimes v_{T_3,k}) \]
\[ W_2 = (v_{T_{12},k} \otimes v_{T_8,k} \otimes v_{T_5,k} \otimes v_{T_3,k}) \oplus (v_{T_{12},k} \otimes v_{T_9,k} \otimes v_{T_5,k} \otimes v_{T_3,k}) \]
\[ X_2 = v_{T_5,k} \otimes v_{T_3,k} \otimes v_{T_0,k} \quad Y_2 = v_{T_3,k} \otimes v_{T_5,k} \]

(iii) condition when customers who get the queue number directly and go to teller has same model as (i) by eliminated \( T_1(k) \) and change \( T_3(k) \) to \( T_2(k) \).

(iv) condition when customers who get the queue number directly and go to CS has the same model as (iii) by changed \( T_4(k) \), \( T_6(k) \), and \( T_7(k) \) to \( T_5(k) \), \( T_6(k) \), and \( T_5(k) \), it made \( v_{T_5(k)}, v_{T_6(k)} \) and \( v_{T_7(k)} \) change to \( v_{T_5(k)}, v_{T_6(k)} \), and \( v_{T_6(k)} \).

4 Conclusion

This paper is constructed a petri net from the bank queue flow as in Figure 3. As it has been analyzed by using the state matrices and the coverability trees and one of them is shown in Figure 4. We can say the petri net is not deadlock with the classification of each transition as follows, transition \( T_0 \) is live because it can be fire from \( x_0 \) at least once and always can be fired in every state. While the other transitions are \( L_1 \)-live. Afterward, we built the max-plus algebra model from the timed petri net as written in 3.4(i)-3.4(iv).

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