Radial solutions to a chemotaxis-consumption model involving prescribed signal concentrations on the boundary

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Abstract

The chemotaxis system $u_t = \Delta u - \nabla \cdot (u \nabla v)$, $v_t = \Delta v - uv$, is considered under the boundary conditions $\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0$ and $v = v_\ast$ on $\partial \Omega$, where $\Omega \subset \mathbb{R}^n$ is a ball and $v_\ast$ is a given positive constant. In the setting of radially symmetric and suitably regular initial data, a result on global existence of bounded classical solutions is derived in the case $n = 2$, while global weak solutions are constructed when $n \in \{3, 4, 5\}$. This is achieved by analyzing an energy-type inequality reminiscent of global structures previously observed in related homogeneous Neumann problems. Ill-signed boundary integrals newly appearing therein are controlled by means of spatially localized smoothing arguments revealing higher order regularity features outside the spatial origin. Additionally, unique classical solvability in the corresponding stationary problem is asserted, even in nonradial frameworks.

Keywords: chemotaxis, signal consumption, global existence, stationary states, inhomogeneous boundary conditions

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1. Introduction

Chemotaxis systems, if posed in bounded domains, are usually studied with homogeneous Neumann boundary conditions. Especially where the chemotactic agents partially direct their motion toward higher concentrations of a signal which they consume instead of produce, however, other boundary conditions may become relevant.

In this article, we consider the chemotaxis consumption system

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla v), \\
    v_t &= \Delta v - uv,
\end{align*}
\]

posing Dirichlet boundary conditions for the signal concentration \(v\) and no-flux conditions for the bacterial population density \(u\).

Arising from a line of investigations concerned with pattern formation in colonies of \(B.\ subtilis\) in a fluid environment \([30]\), such chemotaxis systems with signal consumption, additionally coupled to a Stokes- or Navier–Stokes fluid have been studied extensively over the last decade (for pointers to the literature see e.g. section 4.1 of the survey \([2]\) or the introduction of \([8]\); for the model in the context of coral spawning, see e.g. \([42]\)). While most works prescribed no-flux, homogeneous Neumann and homogeneous Dirichlet conditions for bacterial population density, signal concentration and fluid velocity, it turned out that these failed to adequately capture the colourful dynamics observed in the form of the patterns previously alluded to.

In particular, a common result for the long-term behaviour was convergence to a constant state (i.e. a state without any patterns), as e.g. in \([27, 35, 38, 41]\), in \([19, 20, 37]\) in a system additionally including population growth, or also \([11]\) in a related system with nonlinear diffusion.

Not least because of this, it has been suggested to use different, more realistic, inhomogeneous boundary conditions for the chemical signal (see \([5, 6]\), but also \([30]\)), namely either Robin type boundary conditions where the rate of oxygen influx is controlled by the local oxygen concentration at the boundary or nonzero Dirichlet conditions directly prescribing the latter. (It has been confirmed \([6, \text{proposition 5.1}]\) that the latter kind of conditions arises as a limit case of the former.)

In general, altering boundary conditions can have a profound impact on the solution behaviour in chemotaxis systems, see the appearance of a second critical mass in the Keller–Segel type system with Dirichlet conditions for \(v\) studied in \([13]\) if compared to the same system with Neumann conditions; in both cases homogeneous. While—in these particular settings related to \((1.1)\)—more realistic from a modelling perspective, the change of boundary conditions to inhomogeneous conditions brings about additional mathematical challenges.

In particular, a large part of the analysis of \((1.1)\) and its relatives relies on certain energy-like structures such as that expressed in the inequality

\[
\frac{d}{dr} \left( \int_\Omega u \ln u + 2 \int_\Omega \sqrt{\nabla v}^2 \right) + \int_\Omega \frac{\nabla u^2}{u} + \int_\Omega v |D^2 \ln v|^2 + \frac{1}{2} \int_\Omega \frac{u |\nabla v|^2}{v} \leq 0, \quad (1.2)
\]
as documented for the Neumann problem for (1.1) in [27, (3.1)]. Replacing homogeneous Neumann by inhomogeneous Dirichlet boundary conditions for \( v \), in the derivation of (1.2) additional boundary integrals arise, destroying the (quasi-)Lyapunov structure of (1.2) or relatives thereof on which existence results in, e.g., [4, 19, 27, 28, 34, 36, 41] or also [16] essentially relied.

Accordingly, only few results for chemotaxis-consumption models with boundary conditions different from homogeneous Neumann conditions are available: those concerning the related system with slightly different chemotaxis and energy consumption studied in [17, 18] with inhomogeneous Neumann and Dirichlet conditions are restricted to spatially one-dimensional domains. For Robin-type conditions of the form introduced in [5], also in higher dimensions, the stationary problem of (1.1) has been shown to be uniquely solvable (for any prescribed total mass \( \int_\Omega u \) of the first component, see [6]), and (under a moderate smallness condition) features as the limit of a parabolic-elliptic simplification of (1.1) (cf [12]). Also in fluid-coupled systems solutions have been found in the presence of logistic source terms ([5]), or superlinear diffusion ([29, 40]), or without both ([7]).

In the Dirichlet setting this article is concerned with we mention [21] where an asymptotic analysis of the vanishing diffusivity limit for \( v \) in the stationary system seems to confirm the potential of (1.1) to capture pattern dynamics. In the time-dependent problem (including fluid flow), solutions in \( \mathbb{R}^2 \times [0, 1] \) were constructed in [23] if the signal consumption was strong, at least quadratic with respect to \( u \), and in \( \Omega \subset \mathbb{R}^2 \) in [33], in both cases under a smallness condition on the initial data. Without smallness conditions, solutions to (1.1) coupled to a fluid flow governed by the Stokes equations were constructed in [31] (\( \Omega \subset \mathbb{R}^N \) with linear diffusion for \( N = 2 \) and porous medium type diffusion in higher dimensions) and in [32] (\( \Omega \subset \mathbb{R}^3 \)). Nevertheless, the solution concepts pursued in these works are rather weak and do not yield comparable regularity as [26, 34] for solutions to the system with homogeneous Neumann conditions.

As to the above-mentioned difficulties concerning (1.2), different strategies have been employed: exploiting the Robin condition in their systems, the works [5] and subsequently [29, 40] rely on a Lions–Magenest type transformation converting \( v \) to a function with homogeneous Neumann boundary conditions. The energy functional is enhanced by additional ‘boundary energy’ terms in [7]. In [12, 23], a trace theorem is used to control the boundary integrals by integrals over the domain involving higher derivatives, which are available either due to the simpler elliptic form of the second equation (in [12]) or due to a smallness condition ([23], also in the result on long-term limit in [12]). In [32], a localized modification of the energy functional was investigated, the localization being detrimental to the regularity information near the boundary. Leaving (1.2) behind, the approaches in [31, 33] used different energy functionals (or small-data energy functionals), giving rise to less potent \textit{a priori} estimates, as reflected in the generalized sense of solvability obtained.

\textbf{Regularity control on the boundary for radial solutions. Main results.} In this article, we plan to use radial symmetry as a means to unravel difficulties related to possible effects that the change from Neumann to Dirichlet boundary conditions for the signal may have on boundary regularity of solutions. Specifically, in a ball \( \Omega = B_R(0) \subset \mathbb{R}^n \) with \( R > 0 \) and \( n \geq 2 \), and with a given positive constant \( v_\ast \), we shall consider the initial-boundary value problem
\[
\begin{aligned}
\begin{cases}
  u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\
v_t = \Delta v - uv, & x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u^0(x), \ v(x, 0) = v^0(x), & x \in \Omega,
\end{cases}
\end{aligned}
\] (1.3)
assuming that
\[
\begin{aligned}
\begin{cases}
  u^0 \in W^{1,\infty}(\Omega) \text{ is nonnegative in } \Omega \text{ and radially symmetric with } u^0 \not\equiv 0, \text{ and} \\
v^0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega} \text{ and radially symmetric with } v^0 = v^* \text{ on } \partial \Omega,
\end{cases}
\end{aligned}
\] (1.4)
where, as throughout the sequel, radial symmetry of a function \(\varphi\) on \(\Omega\) is to be understood as referring to the spatial origin.

To make appropriate use of these symmetry assumptions, at a first stage of our analysis we shall rely on the essentially one-dimensional framework thereby generated in order to step by step turn the basic properties of mass conservation in the first component, and uniform \(L^\infty\) boundedness in the second, into knowledge on higher order regularity features locally outside the spatial origin (see section 3 and especially corollary 3.7). This will particularly enable us to appropriately control boundary integrals which due to the presence of possibly nonzero normal derivatives arise in a spatially global energy analysis related to that in (1.2) (section 4).

In the spatially planar case, this will be found to entail \textit{a priori} bounds actually in \(L^\infty \times W^{1,\infty}\), and to thus imply the following statement on global classical solvability and boundedness in (1.3):

**Theorem 1.1.** Let \(R > 0\) and \(\Omega = B_R(0) \subset \mathbb{R}^2\), and suppose that \(v^* \geq 0\), and that \(u^0\) and \(v^0\) satisfy (1.4). Then there exist unique functions

\[
\begin{aligned}
\begin{cases}
  u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \text{ and} \\
v \in \bigcap_{q>2} C^0([0, \infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{cases}
\end{aligned}
\]

which are such that \(u > 0\) and \(v > 0\) in \(\overline{\Omega} \times [0, \infty)\), that \((u(\cdot, t), v(\cdot, t))\) is radially symmetric for all \(t > 0\), and that \((u, v)\) solves (1.3) in the classical sense in \(\overline{\Omega} \times (0, \infty)\). Moreover, there exists \(C > 0\) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0.
\] (1.5)

But also in some higher-dimensional situations, the regularity information gained from our energy analysis can be used to establish a result on global existence, albeit in a slightly weaker framework of solvability which, as to be found specified in definition 6.1 below, can essentially be viewed as reformulating both sub-problems in (1.3) in a form tested by smooth functions on the basis of first order integration by parts wherever applicable:

**Theorem 1.2.** Let \(n \in \{3, 4, 5\}, R > 0\) and \(\Omega = B_R(0) \subset \mathbb{R}^n\), let \(v^* > 0\), and assume (1.4). Then one can find nonnegative functions
\[
\begin{align*}
\left\{ \begin{array}{ll}
u \in L^\infty((0, \infty); L^1(\Omega)) & \cap L_\text{loc}^{\frac{n+2}{n-2}}(\Omega \times [0, \infty)) & \cap L_\text{loc}^{\frac{n+2}{n-2}}([0, \infty); W^{1,\frac{n+2}{n-2}}(\Omega)) \quad \text{and} \\
v \in L^\infty(\Omega \times (0, \infty)) & \quad \text{with } v - v_* \in L^\infty((0, \infty); W_0^{1,2}(\Omega)) & \cap L_\text{loc}^{\frac{n+2}{n-2}}([0, \infty); W^{1,\frac{n+2}{n-2}}(\Omega))
\end{array} \right.
\end{align*}
\]

such that \(u(\cdot, t)\) and \(v(\cdot, t)\) are radially symmetric for a.e. \(t > 0\), and that \((u, v)\) forms a global weak solution of (1.3) in the sense of definition 6.1 below. Furthermore, there exists \(C > 0\) such that

\begin{align*}
\int_\Omega u(\cdot, t) \ln u(\cdot, t) & \leq C \quad \text{for almost all } t > 0 \\
\int_\Omega |\nabla v(\cdot, t)|^2 & \leq C \quad \text{for almost all } t > 0
\end{align*}

as well as

\[
\int_t^{t+1} \int_\Omega \left\{ u^\frac{n}{n-1} + |\nabla u|^\frac{n+2}{n-2} + |\nabla v|^4 \right\} \leq C \quad \text{for all } t > 0.
\]

We have to leave open here the question how far information on the large time behaviour of the above solutions that goes beyond the boundedness features in (1.5) and in (1.7)–(1.9) can be derived, especially in the presence of large initial data. After all, a steady state analysis guided by the approach developed in [6] provides the following result which may be viewed as an indication for nontrivial dynamics involving structured states in (1.3):

**Theorem 1.3.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary, and suppose that \(v^* \in \bigcup_{\beta(0,1)} C^{2+\beta}(\partial \Omega)\) is nonnegative. Then for every \(m \geq 0\), the stationary problem (7.1) has a unique solution \((u, v) \in (C^2(\overline{\Omega}))^2\) satisfying \(\int_\Omega u = m\). If \(\Omega = B_\rho(0)\) and \(v^*\) is constant, then this solution is radially symmetric, and both \(u\) and \(v\) are convex.

Global existence of solutions to (1.3) in one-dimensional domains for initial data close to steady states has been investigated in [15], where also local asymptotic stability of the stationary solutions has been shown. The methods of the proofs in [15], which rely on a study of the first primitives of \(u\) and \(v\), unfortunately seem inherently restricted to the one-dimensional case.

### 2. Local solvability, approximation and basic properties

In order to simultaneously address, throughout large parts of our analysis, the case \(n = 2\) in which classical solvability is strived for, and the case \(n \in \{3, 4, 5\}\) in which we intend to construct a solution via approximation, for \(\varepsilon \in [0, 1]\) let us consider the variants of (1.3) given by

\[
\begin{align*}
u_{\varepsilon, t} & = \Delta u_{\varepsilon} - \nabla \cdot \left( u_{\varepsilon} F'_\varepsilon(u_{\varepsilon}) \nabla v_{\varepsilon} \right), & x & \in \Omega, \ t > 0, \\
v_{\varepsilon, t} & = \Delta v_{\varepsilon} - F'_\varepsilon(u_{\varepsilon}) v_{\varepsilon}, & x & \in \Omega, \ t > 0, \\
\frac{\partial u_{\varepsilon}}{\partial \nu} - u_{\varepsilon} F'_\varepsilon(u_{\varepsilon}) \frac{\partial v_{\varepsilon}}{\partial \nu} & = 0, & v_{\varepsilon} & = v_*, & x & \in \partial \Omega^0, \ t > 0, \\
u_{\varepsilon}(x, 0) & = u^{(0)}(x), & v_{\varepsilon}(x, 0) & = v^{(0)}(x), & x & \in \Omega,
\end{align*}
\]
satisfies
\[ 0 \leq F_\varepsilon(\xi) \leq \xi \quad \text{and} \quad 0 \leq F'_\varepsilon(\xi) = \frac{1}{(1 + \varepsilon \xi)^2} \leq 1 \quad \text{for all} \, \xi \geq 0 \, \text{and} \, \varepsilon \in [0, 1); \] (2.3)

indeed, these choices ensure that (2.1) coincides with (1.3) when \( \varepsilon = 0 \).

Local solvability and a handy extensibility criterion can be obtained by resorting to standard literature:

**Lemma 2.1.** Let \( \varepsilon \in [0, 1) \). Then there exist \( T_{\text{max},\varepsilon} \in (0, \infty) \) and uniquely determined functions
\[
\begin{align*}
 & u_\varepsilon \in C^0([\Omega \times [0, T_{\text{max},\varepsilon}])) \cap C^{1,1}(\Omega \times (0, T_{\text{max},\varepsilon})) \quad \text{and} \\
 & v_\varepsilon \in \bigcap_{q>n} C^0([0, T_{\text{max},\varepsilon}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max},\varepsilon}))
\end{align*}
\]
such that \( u_\varepsilon > 0 \) and \( v_\varepsilon > 0 \) in \( \Omega \times [0, T_{\text{max},\varepsilon}) \), that \( u_\varepsilon(\cdot, t) \) and \( v_\varepsilon(\cdot, t) \) are radially symmetric for all \( t \in (0, T_{\text{max},\varepsilon}) \), that \( (u_\varepsilon, v_\varepsilon) \) solves (2.1) classically in \( \Omega \times (0, T_{\text{max},\varepsilon}) \), and that

\[ \limsup_{t \nearrow T_{\text{max},\varepsilon}} \{ \| u_\varepsilon(\cdot, t) \|_{W^{1,q}(\Omega)} + \| v_\varepsilon(\cdot, t) \|_{W^{1,q}(\Omega)} \} = \infty \quad \text{for all} \, q > n. \] (2.4)

**Proof.** This results from [1, theorems 14.4 and 14.6] when, for \( U = \begin{pmatrix} u \\ v - v_\ast \end{pmatrix} \), applied to the evolution problem given by
\[
U_t = \nabla \cdot (A(U) \nabla U) + f(U),
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
A(U)\nu +
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
U |_{\partial \Omega} = 0,
\]
\[
U(0) = \begin{pmatrix} u_\ast \\ v_\ast - v_\ast \end{pmatrix},
\]

with \( A(U) = \begin{pmatrix} 1 - U_1F_1(U_1) \\ 0 \\
0 & 1
\end{pmatrix}, f(U) = \begin{pmatrix} 0 \\ -F(U_1)(U_2 + v_\ast) \end{pmatrix} \). \( \square \)

These solutions clearly preserve mass in their first component and are bounded in the second.

**Lemma 2.2.** Let \( \varepsilon \in [0, 1) \). Then
\[
\int_{\Omega} u_\varepsilon(\cdot, t) = \int_{\Omega} u_\ast \quad \text{for all} \, t \in (0, T_{\text{max},\varepsilon})
\] (2.5)

and
\[
\| v_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq \| v_\ast \|_{L^\infty(\Omega)} \quad \text{for all} \, t \in (0, T_{\text{max},\varepsilon}).
\] (2.6)

**Proof.** While (2.5) can directly be seen upon integrating the first equation in (2.1), the inequality in (2.6) can be verified by means of the comparison principle applied to the second equation in (2.1), because \( \bar{\Pi}(x, t) := \| v_\ast \|_{L^\infty(\Omega)}, (x, t) \in \overline{\Omega} \times [0, \infty) \), satisfies \( \bar{\Pi}_t - \Delta \bar{\Pi} +
\( F_\varepsilon(u_\varepsilon)v = F_\varepsilon(u_\varepsilon) v \geq 0 \) in \( \Omega \times (0, T_{\max, \varepsilon}) \) for all \( \varepsilon \in (0, 1) \) as well as \( \| \nabla \|_{(r=\varepsilon)} \geq \| v^0 \| \) and \( \| \nabla \|_{\partial \Omega} \geq v_* \), the latter due to the fact that (1.4) necessarily requires that \( v_* \leq \| v^0 \|_{L^\infty(\Omega)} \).

Also for the gradient of the second solution component some first \textit{a priori} estimates are available.

**Lemma 2.3.** There exists \( C > 0 \) such that
\[
\| \nabla v_\varepsilon(\cdot, t) \|_{L^1(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}) \quad \text{and} \quad \varepsilon \in [0, 1].
\]  
\( \Box \)

**Proof.** According to well-known smoothing estimates for the Dirichlet heat semigroup \( (e^{\varepsilon \Delta})_{\varepsilon \geq 0} \) on \( \Omega \) ([10], [24, section 48.2]), there exist \( \lambda > 0 \), \( c_1 > 0 \) and \( c_2 > 0 \) such that for all \( t > 0 \),
\[
\| \nabla e^{\varepsilon \Delta} \varphi \|_{L^1(\Omega)} \leq c_1 \| \varphi \|_{W^{1,\infty}(\Omega)} \quad \text{for all} \quad \varphi \in W^{1,\infty}(\Omega) \quad \text{such that} \quad \varphi = v_* \text{ on } \partial \Omega,
\]
and
\[
\| \nabla e^{\varepsilon \Delta} \varphi \|_{L^1(\Omega)} \leq c_2 \cdot \left( 1 + t^{-\frac{1}{2}} \right) e^{-\lambda t} \| \varphi \|_{L^1(\Omega)} \quad \text{for all} \quad \varphi \in C^0(\overline{\Omega}) \quad \text{such that} \quad \varphi = v_* \text{ on } \partial \Omega.
\]
Since (2.3) together with (2.5) and (2.6) ensures that for all \( \varepsilon \in [0, 1] \) we have \( 0 \leq F_\varepsilon(u_\varepsilon) \leq u_\varepsilon \) and hence
\[
\| F_\varepsilon(u_\varepsilon)v_\varepsilon \|_{L^1(\Omega)} \leq \| F_\varepsilon(u_\varepsilon) \|_{L^1(\Omega)}\| v_\varepsilon \|_{L^\infty(\Omega)} \leq \| u_\varepsilon \|_{L^1(\Omega)}\| v_\varepsilon \|_{L^\infty(\Omega)} \leq c_3 := \| u^0 \|_{L^1(\Omega)}\| v^0 \|_{L^\infty(\Omega)}
\]
for all \( t \in (0, T_{\max, \varepsilon}) \), in view of (2.1) this implies that for any such \( \varepsilon \),
\[
\| \nabla v_\varepsilon(\cdot, t) \|_{L^1(\Omega)} = \| \nabla (v_\varepsilon(\cdot, t) - v_*) \|_{L^1(\Omega)}
\]
\[
= \left\| \nabla e^{\varepsilon \Delta} (v^0 - v_*) - \int_0^t \nabla e^{\varepsilon \Delta \{ F_\varepsilon(u_\varepsilon(\cdot, s)) v_\varepsilon(\cdot, s) \} ds \right\|_{L^1(\Omega)}
\]
\[
\leq c_1 \| v^0 - v_* \|_{W^{1,\infty}(\Omega)} + c_2 c_3 \int_0^t \left( 1 + (t - s)^{-\frac{1}{2}} \right) e^{-\lambda (t - s)} ds
\]
\[
\leq c_1 \| v^0 - v_* \|_{W^{1,\infty}(\Omega)} + c_2 c_3 \int_0^\infty \left( 1 + \sigma^{-\frac{1}{2}} \right) e^{-\lambda \sigma} d\sigma
\]
for all \( t \in (0, T_{\max, \varepsilon}) \), which establishes (2.7). \( \Box \)

**3. Local estimates outside the origin**

In this section, we strive for estimates outside the origin only. Its final outcome will be the estimates on boundary terms listed in corollary 3.7. Only these final estimates will later (in the proof of lemma 4.3) be employed to deal with boundary integrals arising in the study of a (spatially global) quasi-energy functional, thus helping to obtain global estimates. Note, however, that bounds extending to \( r = 0 \) stem from the functional introduced in lemma 4.1 and not directly from the local bounds in the present section.

While throughout sections 3–6 assuming that \( \Omega = B_R \), as required by theorems 1.1 and 1.2, in line with common abuse of notation, we occasionally write \( u_\varepsilon(r, t) \) and \( v_\varepsilon(r, t) \), instead of
Lemma 3.1. Let \( q \in (1, \infty) \) and \( \delta \in (0, R) \). Then there exists \( C(q, \delta) > 0 \) with the property that
\[
\| v_{\delta r} (\cdot, t) \|_{L^q (0, R)} \leq C(q, \delta) \quad \text{for all } t \in (0, T_{\max, \varepsilon}) \text{ and } \varepsilon \in (0, 1) .
\]  

Proof. Given \( \delta \in (0, R) \), we fix \( \chi \in \mathcal{C}_c^\infty ([0, R]) \) such that \( 0 \leq \chi \leq 1 \), that \( \chi \equiv 0 \) in \( [0, \frac{\delta}{2}] \), and that \( \chi \equiv 1 \) in \( [\delta, R] \), and let \( (e^{-tA})_{t \geq 0} \) denote the one-dimensional heat semigroup generated by the operator \( A := - (\cdot)_r \) under homogeneous Dirichlet boundary conditions on \( (\frac{\delta}{2}, R) \). Then known regularization features of the latter ([10, 24]) ensure that if we fix \( q \in (1, \infty) \), then we can find \( \lambda = \lambda(q, \delta) > 0 \), \( c_1 = c_1(q, \delta) > 0 \) and \( c_2 = c_2(q, \delta) > 0 \) such that whenever \( t > 0 \),
\[
\| \partial_r e^{-tA} \varphi \|_{L^q (0, R)} \leq c_1 \| \varphi \|_{W^{1, \infty}((\frac{\delta}{2}, R))}
\]
for all \( \varphi \in W^{1, \infty} \left( \left( \frac{\delta}{2}, R \right) \right) \) such that \( \varphi \left( \frac{\delta}{2} \right) = \varphi(R) = 0 \)
and
\[
\| \partial_r e^{-tA} \varphi \|_{L^2 (0, R)} \leq c_2 \cdot \left( 1 + t^{-1 + \frac{1}{q_0}} \right) e^{-\lambda \varepsilon} \| \varphi \|_{L^1 (0, R)}
\]
for all \( \varphi \in C^0 \left( \left( \frac{\delta}{2}, R \right) \right) \) with \( \varphi \left( \frac{\delta}{2} \right) = \varphi(R) = 0 \).

Apart from that, a combination of lemma 2.3 with (2.5) and (2.6) shows that since \( \text{supp} \chi \subset [\frac{\delta}{2}, R] \), we can pick \( c_3 = c_3(\delta) > 0 \) in such a way that for any \( \varepsilon \in (0, 1) \), the function \( b_\varepsilon = b_\varepsilon^{(1)} \) defined in (3.2) satisfies
\[
\| b_\varepsilon (\cdot, t) \|_{L^1 (0, R)} \leq c_3 \quad \text{for all } t \in (0, T_{\max, \varepsilon}) .
\]

On the basis of (3.1) and the fact that \( \chi \cdot (v_\varepsilon - v_\ast) = 0 \) on \( (\frac{\delta}{2}, R) \times (0, T_{\max, \varepsilon}) \), we can therefore utilize (3.4) and (3.5) to estimate
\[
\| \partial_r \left\{ \chi \cdot (v_\varepsilon (\cdot, t) - v_\ast) \right\} \|_{L^q (0, R)} = \left\| \partial_r e^{-tA} \left\{ \chi \cdot (v_\varepsilon^0 - v_\ast) \right\} + \int_0^t \partial_r e^{-sA} b_\varepsilon (\cdot, s) ds \right\|_{L^q (0, R)} \nu \leq c_1 \left\| \chi \cdot (v_\varepsilon^0 - v_\ast) \right\|_{W^{1, \infty}((\frac{\delta}{2}, R))}
\]
\[
\frac{1}{p} \int \frac{d}{dt} \int_{\Omega} \left( \xi u^p \right)^2 = \int \xi^2 u^p \nabla \cdot \{ \nabla u - uF'(u)\nabla v \} \\
= (1 - p) \int \xi^2 u^{p-2} |\nabla u|^2 - (1 - p) \int \xi^2 u^{p-1} F'(u)\nabla u \cdot \nabla v \\
- 2 \int \xi u^{p-1} \nabla u \cdot \nabla \xi + 2 \int \xi u^p F'(u)\nabla v \cdot \nabla \xi \\
\geq \frac{1 - p}{2} \int \xi^2 u^{p-2} |\nabla u|^2 - (2 - p) \int \xi^2 u^p |\nabla v|^2 \\
- \frac{5}{1 - p} \int |\nabla \xi|^2 u^p 
\]
for all \( t \in (0, T_{\max, \varepsilon}) \) and \( \varepsilon \in [0, 1) \). Here by the H"older inequality,

\[
\int \xi^2 u^p |\nabla v|^2 \leq \left\{ \int \xi u^p \right\}^p \left\{ \int \xi^{\frac{2}{p}} |\nabla v|^2 \right\}^{\frac{1}{2}} 
\]
and

\[
\int |\nabla \xi|^2 u^p \leq \left\{ \int u^p \right\}^p \left\{ \int L^{1-p}(\Omega) \cdot |\nabla \xi|^2 \right\}^{\frac{1}{2}} 
\]
for all \( t \in (0, T_{\max, \varepsilon}) \) and \( \varepsilon \in [0, 1) \),
so that since supp $\zeta \subset \overline{\Omega} \setminus B_{\frac{3}{2}}(0)$ we may apply lemma 3.1 to $q := \frac{2}{p}$ to see that thanks to (2.5), with some $c_1 = c_1(p, \delta) > 0$ we have

$$(2 - p) \int_\Omega \zeta^2 u_0^p (\nabla v) + \frac{5 - p}{1 - p} \int_\Omega \nabla^2 \zeta^2 u_0^p \leq c_1 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \text{ and } \varepsilon \in [0, 1).$$

Therefore, an integration in (3.9) shows that again due to the Hölder inequality and (2.5),

$$\frac{1 - p}{2} \int_0^{t+\tau_\varepsilon} \int_{\Omega \setminus B_0(0)} u_0^{p-2} |\nabla u|^2 \leq \frac{1 - p}{2} \int_0^{t+\tau_\varepsilon} \int_{\Omega} \zeta^2 u_0^{p-2} |\nabla u|^2$$

$$\leq \frac{1}{p} \int_\Omega \zeta^2 u_0^p (\cdot, t - \tau_\varepsilon) - \frac{1}{p} \int_\Omega \zeta^2 u_0^p (\cdot) + c_1 \tau_\varepsilon$$

$$\leq \frac{1}{p} \left\{ \int_{\Omega} u_0^{p(0)} \right\} + c_1 \text{ for all } t \in [0, T_{\text{max}, \varepsilon} - \tau_\varepsilon) \text{ and } \varepsilon \in [0, 1),$$

because $\tau_\varepsilon \leq 1$. This implies (3.7), whereupon (3.8) readily results from (3.7) according to the fact that the Gagliardo–Nirenberg inequality provides $c_2 = c_2(p, \delta) > 0$ fulfilling

$$\int_0^R u_0^{p+2} = \|u_0^{p(\cdot)}\|_{L^{p+2}(0, R)} \leq c_2 \left( \int_{\Omega} \left( u_0^{p(\cdot)} \right)^2 \right)^{\frac{2}{p}} + c_2 \|u_0\|_{L^{p+2}(0, R)}$$

for all $t \in (0, T_{\text{max}, \varepsilon})$ and $\varepsilon \in [0, 1)$,

and because for all $t \in (0, T_{\text{max}, \varepsilon})$ and $\varepsilon \in [0, 1)$,

$$\left\| \frac{\partial}{\partial \varepsilon} \right\|_{L^2((0, R))} = \int_0^R u_0(r, t) dr \leq \delta^{1-n} \int_0^R \int_0^R r^{n-1} u_0(r, t) dr = \delta^{1-n} \int_0^R r^{n-1} u_0(r) dr$$

by (2.5).

In contrast to settings with homogeneous boundary conditions, in the present situation it will become necessary to deal with non-vanishing boundary terms. While this section will culminate in corresponding estimates, a key to these becomes visible in the following corollary already.

**Corollary 3.3.** There exists $C > 0$ such that

$$\int_0^{t+\tau_\varepsilon} u_0 (R, s) ds \leq C \quad \text{for all } t \in [0, T_{\text{max}, \varepsilon} - \tau_\varepsilon),$$

(3.10)

where again $\tau_\varepsilon = \min \{ 1, \frac{1}{2} T_{\text{max}, \varepsilon} \}$ for $\varepsilon \in [0, 1)$.

**Proof.** By means of the Gagliardo–Nirenberg inequality, we can pick $c_1 > 0$ with the property that

$$\|\varphi\|_{L^\infty((\frac{R}{2}, R))} \leq c_1 \|\varphi\|_{L^2((\frac{R}{2}, R))}^2 + c_1 \|\varphi\|_{L^4((\frac{R}{2}, R))}^4 + c_1 \|\varphi\|_{L^6((\frac{R}{2}, R))}^6$$

for all $\varphi \in W^{1,2} \left( \left( \frac{R}{2}, R \right) \right)$.
whence
\[
\int_t^{t+\tau} \frac{1}{2} u_s^2 (R, s) ds \leq \int_t^{t+\tau} \left\| u_s^\frac{1}{2} (\cdot, s) \right\|_{L^\infty((\frac{\Omega}{2}, R))}^6 ds
\]
\[
\leq c_1 \int_t^{t+\tau} \left( \frac{1}{r} \int \left( \frac{1}{2} u_s^\frac{1}{2} (\cdot, s) \right) \right)_{L^2((\frac{\Omega}{2}, R))}^2 \left\| u_s^\frac{1}{2} (\cdot, s) \right\|_{L^4((\frac{\Omega}{2}, R))}^4 ds
\]
\[+ c_1 \int_t^{t+\tau} \left\| u_s^\frac{1}{2} (\cdot, s) \right\|_{L^6((\frac{\Omega}{2}, R))}^6 ds
\]
for all \( t \in [0, T_{\text{max}, \varepsilon} - \tau \varepsilon) \) and \( \varepsilon \in [0, 1) \).

Combining (2.5) with an application of lemma 3.2 to \( p = \frac{1}{2} \) thus shows that with some \( c_2 > 0 \) we have
\[
\int_t^{t+\tau} \frac{1}{2} u_s^2 (R, s) ds \leq c_2 \quad \text{for all } t \in [0, T_{\text{max}, \varepsilon} - \tau \varepsilon) \text{ and } \varepsilon \in [0, 1),
\]
from which (3.10) follows upon employing the Hölder inequality. \( \square \)

The following elementary observation, a proof of which can be found in [39, lemma 3.4], will be referred to in lemmata 3.5, 4.3 and 5.1.

**Lemma 3.4.** Let \( T \in (0, \infty) \) and \( \tau \in (0, T) \), and let \( h \in L^1_{\text{loc}}((0, T)) \) be nonnegative and such that
\[
\int_t^{t+\tau} h(s) ds \leq b \quad \text{for all } t \in (0, T - \tau)
\]
with some \( b > 0 \). Then
\[
\int_0^t e^{-\lambda(t-s)} h(s) ds \leq \frac{b \tau}{1 - e^{-\lambda T}} \quad \text{for all } t \in (0, T) \text{ and any } \lambda > 0.
\]

Whereas the previous estimates for \( u_s \) were concerned with temporally integrated quantities, the following lemma provides a temporally uniform bound.

**Lemma 3.5.** Let \( p \in (1, 3) \) and \( \delta \in (0, R) \). Then there exists \( C(p, \delta) > 0 \) such that
\[
\int_\delta^R u_s^p (r, t) dr \leq C(p, \delta) \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \text{ and } \varepsilon \in [0, 1).
\]

**Proof.** We again take a function \( \zeta \in C^\infty(\Omega) \) fulfilling \( 0 \leq \zeta \leq 1 \) and \( \zeta|_{\partial \Omega \times (0)} \equiv 0 \) as well as \( \zeta|_{\Omega \setminus (\delta, \nu)} \equiv 1 \), and once more rely on (2.1) to see by means of Young’s inequality and (2.3) that
\[
\frac{d}{dr} \int_\Omega \zeta^2 u_s^p + \int_\Omega \zeta^2 u_s^p - p(p-1) \int_\Omega \zeta^2 u_s^{p-2} |\nabla u_s|^2
\]
\[+ p(p-1) \int_\Omega \zeta^2 u_s^{p-1} F_s'(u_s) \nabla u_s \cdot \nabla \zeta - 2p \int_\Omega \zeta u_s^{p-2} \nabla u_s \cdot \nabla \zeta
\]
\[+ 2p \int_\Omega u_s^p F_s'(u_s) \nabla \zeta \cdot \nabla \zeta + \int_\Omega \zeta^2 u_s^p
\]
Here, taking any $p_0 = p_0(p) > p \in (1, 3)$ such that $p_0 < 3$, we may again draw on Young’s inequality to estimate

$$\int_\Omega \frac{1}{p} |\nabla v_\varepsilon|^p \leq \int_\Omega \frac{1}{p} |\nabla v_\varepsilon|^p + \int_\Omega |\nabla \xi|^2 u_\varepsilon^p$$

and

$$\int_\Omega |\nabla \xi|^2 u_\varepsilon^p \leq \int_\Omega |\nabla \xi|^2 u_\varepsilon^p + \int_\Omega |\nabla \xi|^2 u_\varepsilon^p$$

as well as

$$\int_\Omega \frac{1}{p} |\nabla v_\varepsilon|^p \leq \int_\Omega |\nabla v_\varepsilon|^p + |\Omega|$$

for all $t \in (0, T_{\text{max}, e})$ and $\varepsilon \in [0, 1)$.

so that invoking lemma 3.1 we find $c_1 = c_1(p, \delta) > 0$ fulfilling

$$\frac{p(p + 1)}{2} \int_\Omega \nabla v_\varepsilon^p \nabla v_\varepsilon^p + \frac{p(p + 1)}{p - 1} \int_\Omega |\nabla \xi|^2 u_\varepsilon^p + \int_\Omega |\nabla \xi|^2 u_\varepsilon^p \leq h_\varepsilon(t) := c_1 \int_\Omega u_\varepsilon^p + c_1$$

for all $t \in (0, T_{\text{max}, e})$ and $\varepsilon \in [0, 1)$.

From (3.12) we therefore obtain that

$$\int_\Omega \frac{1}{p} |\nabla v_\varepsilon|^p + \int_\Omega |\nabla \xi|^2 u_\varepsilon^p \leq h_\varepsilon(t)$$

for all $t \in (0, T_{\text{max}, e})$ and $\varepsilon \in [0, 1)$,

so that since $c_2 = c_2(p, \delta) := \sup_{\varepsilon \in [0, 1)} \sup_{t \in (0, T_{\text{max}, e})} \int_\Omega (t + \tau_\varepsilon) h_\varepsilon(s) ds$ with $\tau_\varepsilon = \min \{1, \frac{1}{4} T_{\text{max}, e} \}$ is finite according to (3.8) in lemma 3.2 and the fact that $p_0 < 3$, by using an ODE comparison argument along with lemma 3.4 we infer that

$$\int_\Omega \frac{1}{p} |\nabla v_\varepsilon|^p \leq e^{-\tau} \int_\Omega \frac{1}{p} |\nabla v_\varepsilon|^p + \int_0^t e^{-u_\varepsilon(s)} h_\varepsilon(s) ds$$

$$\leq \int_\Omega (u_\varepsilon^{(0)})^p + \frac{c_2 \tau_\varepsilon}{1 - e^{-\tau_\varepsilon}}$$

for all $t \in (0, T_{\text{max}, e})$ and $\varepsilon \in [0, 1)$,

and hence conclude as intended, because $\frac{1}{1 - e^{-\tau}} \leq \frac{1}{\varepsilon}$ for all $\tau \in (0, 1)$, and because $\xi \equiv 1$ in $\Omega \setminus B_\delta(0)$.

With these bounds at hand, we can even estimate the derivative of the second component uniformly, again outside a neighbourhood of the origin.

**Lemma 3.6.** For each $\delta \in (0, R)$ there exists $C(\delta) > 0$ satisfying

$$|v_{\gamma_\varepsilon}(r, t)| \leq C(\delta) \text{ for all } r \in [\delta, R], t \in (0, T_{\text{max}, e}) \text{ and } \varepsilon \in [0, 1).$$

(3.13)
Proof. We once more take \( \chi \in C^\infty([0,1]) \) such that \( 0 \leq \chi \leq 1 \) and \( \chi|_{[0,\frac{1}{2}]} \equiv 0 \), as well as \( \chi|_{[\frac{1}{2},1]} \equiv 1 \), and then infer from lemma 3.5 in conjunction with (2.3), (2.6) and lemma 3.1 that there exists \( c_1 = c_1(\delta) > 0 \) such that with \( (b_\delta)_e \in \mathbb{R} \), as defined in (3.2) we have

\[
\| b_\delta(\cdot,t) \|_2(\frac{\delta}{2},R) \leq c_1 \quad \text{for all } t \in (0,T_{\max,e}) \text{ and } \varepsilon \in [0,1).
\]

As for the Dirichlet heat semigroup \((e^{-\lambda t})_{t \geq 0} \text{ on } (\frac{\delta}{2},R)\) it is known \((10,24)\) that there exist \( \lambda = \lambda(\delta) > 0 \), \( c_2 = c_2(\delta) > 0 \) and \( c_3 = c_3(\delta) > 0 \) with the property that for all \( t > 0 \),

\[
\| \partial_t e^{-\lambda t} \|_{L^\infty}\left(\frac{\delta}{2},R\right) \leq c_2 \| \varphi \|_{W^{1,\infty}\left(\frac{\delta}{2},R\right)} \quad \text{for all } \varphi \in W^{1,\infty}\left(\left(\frac{\delta}{2},R\right)\right) \text{ such that } \varphi\left(\frac{\delta}{2}\right) = \varphi(R) = 0,
\]

as well as

\[
\| \partial_t e^{-\lambda t} \|_{L^\infty}\left(\frac{\delta}{2},R\right) \leq c_3 \cdot \left(1 + t^{-\frac{3}{4}}\right) e^{-\lambda t} \| \varphi \|_{L^2}\left(\frac{\delta}{2},R\right) \quad \text{for all } \varphi \in C^0\left(\left(\frac{\delta}{2},R\right)\right) \text{ with } \varphi\left(\frac{\delta}{2}\right) = \varphi(R) = 0,
\]

from (3.1) we obtain that

\[
\| v_{\tau e}(\cdot,t) \|_{L^\infty(\partial\Omega_e)} \leq \| \partial_t \left( \chi \cdot (v_{\tau e}(\cdot,t) - v_{\tau}) \right) \|_{L^\infty}\left(\frac{\delta}{2},R\right) \leq c_2 \| \chi \cdot (v(0) - v_{\tau}) \|_{W^{1,\infty}\left(\frac{\delta}{2},R\right)} + c_1 c_3 \int_0^\infty \left(1 + \sigma^{-\frac{3}{4}}\right) e^{-\lambda \sigma} \, d\sigma
\]

for all \( t \in (0,T_{\max,e}) \) and any \( \varepsilon \in [0,1) \).

In conclusion, this allows for appropriately controlling all the boundary integrals that will turn out to appear in the course of our subsequent energy analysis:

**Corollary 3.7.** There exists \( C > 0 \) such that

\[
\int_{t}^{t+\tau_e} \int_{\partial\Omega_e} \frac{1}{v_{\tau e}} \frac{\partial |\nabla v_{\tau e}|^2}{\partial \nu} \leq C \quad \text{for all } t \in [0,T_{\max,e} - \tau_e)
\]

and

\[
\int_{t}^{t+\tau_e} \int_{\partial\Omega_e} |\nabla v_{\tau e}|^2 \frac{\partial |\nabla v_{\tau e}|^2}{\partial \nu} \leq C \quad \text{for all } t \in [0,T_{\max,e} - \tau_e)
\]

as well as

\[
\left| \int_{\partial\Omega_e} \frac{|\nabla v_{\tau e}|^2}{v_{\tau e}^2} \frac{\partial v_{\tau e}}{\partial \nu} \right| \leq C \quad \text{for all } t \in (0,T_{\max,e}) \text{ and } \varepsilon \in [0,1),
\]

where, as before, \( \tau_e = \min\{1,\frac{1}{2}T_{\max,e}\} \) for \( \varepsilon \in [0,1) \).

**Proof.** Once more explicitly relying on radial symmetry, we may use that according to lemma 2.1 the second equation in (2.1) holds up to \( \partial\Omega \) throughout \( (0,T_{\max,e}) \), which namely ensures that on \( \partial\Omega \) we have the one-sided inequality
\[ \frac{1}{v_\varepsilon} \cdot \frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} = \frac{2}{v_\varepsilon} v_{\varepsilon r} v_{\varepsilon r} = \frac{2}{v_\varepsilon} v_{\varepsilon r} \cdot \left( v_{\varepsilon r} + \frac{n-1}{R} v_{\varepsilon r} \right) - \frac{2(n-1)}{Rv_\varepsilon} v_{\varepsilon r}^2 = \frac{2}{v_\varepsilon} F'_r(u_\varepsilon) v_{\varepsilon r} - \frac{2(n-1)}{Rv_\varepsilon} v_{\varepsilon r}^2 \leq 2F'_r(u_\varepsilon) v_{\varepsilon r} \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}) \text{ and } \varepsilon \in [0, 1). \]

Again thanks to (2.3), this implies that
\[
\frac{1}{2} \int_{t - \tau_\varepsilon}^{t + \tau_\varepsilon} \int_{\partial \Omega} \frac{1}{v_\varepsilon} |\nabla v_\varepsilon|^2 \cdot |v_{\varepsilon r}| ds \leq |\partial \Omega| \cdot \int_{t - \tau_\varepsilon}^{t + \tau_\varepsilon} |u_\varepsilon(R, \cdot) - |v_{\varepsilon r}(0, \cdot)||L^\infty((\frac{3}{4}, R))| ds \leq |\partial \Omega| \cdot \|v_{\varepsilon r}\|_{L^\infty((\frac{3}{4}, R))} \cdot \int_{t - \tau_\varepsilon}^{t + \tau_\varepsilon} |u_\varepsilon(R, \cdot)| ds
\]
for all \( t \in [0, T_{\text{max},\varepsilon} - \tau_\varepsilon) \) and \( \varepsilon \in [0, 1) \),
and that, similarly,
\[
\int_{t - \tau_\varepsilon}^{t + \tau_\varepsilon} \int_{\partial \Omega} |\nabla v_\varepsilon|^2 \cdot |v_{\varepsilon r}| ds \leq 2|\partial \Omega| \|v_{\varepsilon r}\|_{L^\infty((\frac{3}{4}, R))} \cdot \int_{t - \tau_\varepsilon}^{t + \tau_\varepsilon} |u_\varepsilon(R, \cdot)| ds
\]
for all \( t \in [0, T_{\text{max},\varepsilon} - \tau_\varepsilon) \) and \( \varepsilon \in [0, 1) \),
so that since furthermore
\[
\left| \int_{\partial \Omega} |\nabla v_\varepsilon|^2 \cdot \frac{\partial v_\varepsilon}{\partial \nu} \right| \leq \frac{|\partial \Omega|}{2v_\varepsilon} \|v_{\varepsilon r}\|_{L^\infty((\frac{3}{4}, R))} \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}) \text{ and } \varepsilon \in [0, 1),
\]
the claim results from lemma 3.6 when combined with corollary 3.3.

\[ \Box \]

4. Energy analysis

Our approach toward deriving a suitable relative of (1.2) is now launched by the following observation.

**Lemma 4.1.** Let \( \varepsilon \in [0, 1) \). Then
\[
\frac{d}{dt} \left( \int_{\Omega} u_\varepsilon \ln u_\varepsilon + \frac{1}{2} \int_{\Omega} |\nabla v_\varepsilon|^2 \right) + \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_{\partial \Omega} \frac{v_{\varepsilon r}|D \ln v_{\varepsilon}|^2}{v_\varepsilon} = -\frac{1}{2} \int_{\Omega} F'_r(u_\varepsilon) v_{\varepsilon r}^2 + \frac{1}{2} \int_{\partial \Omega} \frac{1}{v_\varepsilon} |\nabla v_{\varepsilon r}|^2 \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]

**Proof.** According to the no-flux boundary condition accompanying the first equation in (2.1),
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon \ln u_\varepsilon + \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} = \int_{\Omega} F'_r(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_{\varepsilon r} \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]
while on the basis of the second equation in (2.1) we first compute
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{v_e} |\nabla v_e|^2 = \int_{\Omega} \frac{1}{v_e} \nabla v_e \cdot \nabla \{\Delta v_e - F_e(u_e)v_e\} \\
= \frac{1}{2} \int_{\Omega} \frac{1}{v_e} |\nabla v_e|^2 \cdot \{\Delta v_e - F_e(u_e)v_e\} \\
= \frac{1}{2} \int_{\Omega} \frac{1}{v_e} \Delta |\nabla v_e|^2 = \frac{1}{2} \int_{\Omega} \frac{1}{v_e} |D^2 v_e|^2 - \frac{1}{2} \int_{\Omega} \frac{1}{v_e^2} \Delta v_e \\
- \frac{1}{2} \int_{\Omega} \frac{1}{v_e} \frac{\partial |\nabla v_e|^2}{\partial \nu}\text{ for all } t \in (0, T_{\text{max},e}),
\]
(4.3)
because \(\nabla v_e \cdot \nabla \Delta v_e = \frac{1}{2} \Delta |\nabla v_e|^2 - |D^2 v_e|^2\). Here two integrations by parts show that
\[
\frac{1}{2} \int_{\Omega} \frac{1}{v_e} \Delta |\nabla v_e|^2 = \frac{1}{2} \int_{\Omega} \frac{1}{v_e} \nabla v_e \cdot \nabla |\nabla v_e|^2 + \frac{1}{2} \int_{\partial \Omega} \frac{1}{v_e} \frac{\partial |\nabla v_e|^2}{\partial \nu}\text{ for all } t \in (0, T_{\text{max},e}),
\]
and that
\[
- \frac{1}{2} \int_{\Omega} \frac{1}{v_e} \frac{\partial |\nabla v_e|^2}{\partial \nu} = \frac{1}{2} \int_{\Omega} \frac{1}{v_e} \nabla v_e \cdot \nabla |\nabla v_e|^2 - \int_{\partial \Omega} \frac{1}{v_e^3} |\nabla v_e|^4 \\
- \frac{1}{2} \int_{\partial \Omega} \frac{1}{v_e} \frac{\partial |\nabla v_e|^2}{\partial \nu}\text{ for all } t \in (0, T_{\text{max},e}),
\]
so that since \(\nabla |\nabla v_e|^2 = 2D^2 v_e \cdot \nabla v_e\), we obtain that
\[
\frac{1}{2} \int_{\Omega} \frac{1}{v_e} \Delta |\nabla v_e|^2 - \frac{1}{2} \int_{\Omega} \frac{1}{v_e} |D^2 v_e|^2 - \frac{1}{2} \int_{\Omega} \frac{1}{v_e^2} \Delta v_e - \frac{1}{2} \int_{\partial \Omega} \frac{1}{v_e} \frac{\partial |\nabla v_e|^2}{\partial \nu} \\
+ \frac{1}{2} \int_{\partial \Omega} \frac{\nabla v_e \cdot \nabla |\nabla v_e|^2}{v_e} \\
= - \int_{\Omega} \frac{1}{v_e} |D^2 v_e|^2 + 2 \int_{\Omega} \frac{1}{v_e} \nabla v_e \cdot (D^2 v_e \cdot \nabla v_e) - \int_{\partial \Omega} \frac{1}{v_e^3} |\nabla v_e|^4 \\
= - \sum_{i,j=1}^n \int_{\Omega} v_e \left| \frac{\partial x_i v_e}{v_e^2} \right| - \frac{\partial x_i v_e \partial x_j v_e}{v_e^2} \\
= - \sum_{i,j=1}^n \int_{\Omega} v_e |\partial x_i | \ln v_e | \\
= - \int_{\Omega} v_e |D^2 \ln v_e|^2\text{ for all } t \in (0, T_{\text{max},e}).
\]
Therefore, (4.3) is equivalent to (4.1). \(\square\)
In order to make use of the last term on the left of (4.1), we will employ the following variant of a functional inequality which for functions with vanishing normal derivative on \( \partial \Omega \) has been documented in [34, lemma 3.3].

**Lemma 4.2.** Let \( \varphi \in C^2(\overline{\Omega}) \) be such that \( \varphi > 0 \) in \( \overline{\Omega} \). Then

\[
\int_{\Omega} \frac{\left| \nabla \varphi \right|^4}{\varphi^3} \leq (2 + \sqrt{n})^2 \int_{\Omega} |D^2 \ln \varphi|^2 + 2 \int_{\partial \Omega} \frac{\left| \nabla \varphi \right|^2}{\varphi^2} \frac{\partial \varphi}{\partial \nu}.
\]

(4.4)

**Proof.** We integrate by parts to see that

\[
\int_{\Omega} \frac{\left| \nabla \varphi \right|^4}{\varphi^3} = \int_{\Omega} \left| \nabla \ln \varphi \right|^2 \nabla \ln \varphi \cdot \nabla \varphi
\]

\[
= -\int_{\Omega} \varphi \nabla \ln \varphi \cdot \nabla |\nabla \ln \varphi|^2 - \int_{\Omega} \varphi |\nabla \ln \varphi|^2 \Delta \ln \varphi
\]

\[
+ \int_{\partial \Omega} \varphi |\nabla \ln \varphi|^2 \frac{\partial \ln \varphi}{\partial \nu}
\]

\[
= -2 \int_{\Omega} \frac{1}{\varphi} \nabla \varphi \cdot (D^2 \ln \varphi \cdot \nabla \varphi) - \int_{\Omega} \frac{\left| \nabla \varphi \right|^2}{\varphi} \Delta \ln \varphi + \int_{\partial \Omega} \frac{\left| \nabla \varphi \right|^2}{\varphi^2} \frac{\partial \varphi}{\partial \nu}.
\]

As

\[
-2 \int_{\Omega} \frac{1}{\varphi} \nabla \varphi \cdot (D^2 \ln \varphi \cdot \nabla \varphi) - \int_{\Omega} \frac{\left| \nabla \varphi \right|^2}{\varphi} \Delta \ln \varphi
\]

\[
\leq (2 + \sqrt{n})^2 \int_{\Omega} \frac{\left| \nabla \varphi \right|^2}{\varphi} |D^2 \ln \varphi|
\]

\[
\leq \frac{1}{2} \int_{\Omega} \frac{\left| \nabla \varphi \right|^4}{\varphi^3} + \frac{(2 + \sqrt{n})^2}{2} \int_{\Omega} \varphi |D^2 \ln \varphi|^2
\]

by Young’s inequality, this implies (4.4). \( \square \)

Now an exploitation of the latter in the context of (4.1) shows that the boundary regularity features obtained in corollary 3.7 imply the following spatially global estimates.

**Lemma 4.3.** There exists \( C > 0 \) such that for each \( \varepsilon \in [0, 1) \), writing \( \tau_\varepsilon = \min \left\{ 1, \frac{T}{\varepsilon} \right\} \), we have

\[
\int_{\Omega} u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\max, \varepsilon})
\]

(4.5)

and

\[
\int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\max, \varepsilon})
\]

(4.6)

as well as

\[
\int_{t}^{t + \tau_\varepsilon} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \leq C \quad \text{for all } t \in [0, T_{\max, \varepsilon} - \tau_\varepsilon)
\]

(4.7)

and

\[
\int_{t}^{t + \tau_\varepsilon} \int_{\Omega} |\nabla v_\varepsilon|^4 \leq C \quad \text{for all } t \in [0, T_{\max, \varepsilon} - \tau_\varepsilon).
\]

(4.8)

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Proof. We first employ the Gagliardo–Nirenberg inequality to pick \( c_1 > 0 \) such that
\[
\| \varphi \|_{L^{2(n+2)\over n}(\Omega)}^{2(n+2)\over n} \leq c_1 \| \nabla \varphi \|_{L^2(\Omega)}^2 \| \varphi \|_{L^2(\Omega)}^2 + c_1 \| \varphi \|_{L^2(\Omega)}^{2(n+2)\over n} \quad \text{for all } \varphi \in W^{1,2}(\Omega),
\]  
(4.9)
and use that \( {\ln \xi \over \xi} \to 0 \) as \( \xi \to +\infty \) in choosing \( c_2 > 0 \) such that abbreviating \( c_3 := \int_{\Omega} u^{(0)} \) we have
\[
\xi \ln \xi \leq \frac{2}{c_1c_3} \xi^{1\over 2(n+2)} + c_2 \quad \text{for all } \xi > 0.
\]  
(4.10)
Then writing \( c_4 := {1 \over (2 + \sqrt{n})} \) and \( c_5 := \frac{(n+1)\| u^{(0)} \|_{L^\infty(\Omega)}}{8c_4} \), by means of (4.10), Young’s inequality, (4.9), (2.5) and (2.6) we see that for each \( \varepsilon \in [0, 1) \),
\[
y_\varepsilon(t) := \int_{\Omega} u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) + \frac{1}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon(\cdot, t)|^2}{v_\varepsilon(\cdot, t)} , \quad t \in [0, T_{\max,\varepsilon}),
\]
has the property that
\[
y_\varepsilon(t) \leq \frac{2}{c_1c_3} \int_{\Omega} u_\varepsilon^\varepsilon + c_2|\Omega| + \frac{c_4}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{8c_4} \int_{\Omega} v_\varepsilon
\]
\[
= \frac{2}{c_1c_3} \| \sqrt{u_\varepsilon} \|_{L^{2n\over n}(\Omega)}^{2n\over n} + c_2|\Omega| + \frac{c_4}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{8c_4} \int_{\Omega} v_\varepsilon
\]
\[
\leq \frac{2}{c_3^2} \| \sqrt{u_\varepsilon} \|_{L^{2n\over n}(\Omega)}^2 \| \sqrt{v_\varepsilon} \|_{L^2(\Omega)}^2 + \frac{2}{c_3^2} \| \sqrt{u_\varepsilon} \|_{L^{2n\over n}(\Omega)}^{2n\over n} + c_2|\Omega| + \frac{c_4}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_5
\]
\[
= \frac{1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + 2c_3 + c_2|\Omega| + \frac{c_4}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_5 \quad \text{for all } t \in (0, T_{\max,\varepsilon}),
\]
so that since lemma 4.2 warrants that
\[
c_4 \int_{0}^{\varepsilon} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq \int_{\Omega} v_\varepsilon D^2 \ln v_\varepsilon|^2 + 2c_4 \int_{\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \frac{\partial v_\varepsilon}{\partial \nu} \quad \text{for all } t \in (0, T_{\max,\varepsilon}),
\]
it follows that
\[
y_\varepsilon(t) + \frac{1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{c_4}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_{\Omega} v_\varepsilon D^2 \ln v_\varepsilon|^2
\]
\[
+ 2c_4 \int_{\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \frac{\partial v_\varepsilon}{\partial \nu} + c_6 \quad \text{for all } t \in (0, T_{\max,\varepsilon}),
\]
with \( c_6 := 2c_3 + c_2|\Omega| + c_5 \). Accordingly, from (4.1) we infer upon dropping a favorably signed summand therein that
\[
y_\varepsilon'(t) + y_\varepsilon(t) + \frac{1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{c_4}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}
\]
\[
\leq h_\varepsilon(t) := \frac{1}{2} \int_{\partial \Omega} \frac{1}{v_\varepsilon} \frac{\partial v_\varepsilon}{\partial \nu} + \left( 2c_4 - \frac{1}{2} \right) \int_{\partial \Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \frac{\partial v_\varepsilon}{\partial \nu} + c_6
\]
for all \( t \in (0, T_{\max,\varepsilon}) \).  
(4.12)
As corollary 3.7 provides $c_7 > 0$ such that

$$\int_t^{t+\tau_\varepsilon} h_\varepsilon(s)ds \leq c_7 \quad \text{for all } t \in [0, T_{\max,\varepsilon} - \tau_\varepsilon) \text{ and } \varepsilon \in [0, 1), \quad (4.13)$$

through lemma 3.4 this firstly ensures that for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in [0, 1),

$$y_\varepsilon(t) \leq y_\varepsilon(0)e^{-t} + \int_0^t e^{-(t-s)}h_\varepsilon(s)ds$$

$$\leq |y_\varepsilon(0)| + c_7 \varepsilon \tau_\varepsilon$$

$$\leq c_8 := \int_\Omega u_0^{(0)}|\ln u_0^{(0)}| + \frac{1}{2} \int_\Omega |\nabla u_0^{(0)}|^2 + \frac{c_7}{1 - e^{-1}} \quad (4.14)$$

again because $\frac{\tau_\varepsilon}{1 - e^{-\tau_\varepsilon}} \leq \frac{1}{1 - e^{-1}}$ for all $\varepsilon \in [0, 1)$. Going back to (4.12), from this we thereupon infer that

$$\frac{1}{2} \int_t^{t+\tau_\varepsilon} \int_\Omega \frac{|
abla u_\varepsilon|^2}{u_\varepsilon} + \frac{c_4}{2} \int_t^{t+\tau_\varepsilon} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}$$

$$\leq y_\varepsilon(t) - y_\varepsilon(t + \tau_\varepsilon) - \int_t^{t+\tau_\varepsilon} y_\varepsilon(s)ds + \int_t^{t+\tau_\varepsilon} h_\varepsilon(s)ds$$

$$\leq c_8 + \frac{2|\Omega|}{e} + c_7 \quad \text{for all } t \in [0, T_{\max,\varepsilon} - \tau_\varepsilon) \text{ and } \varepsilon \in [0, 1), \quad (4.15)$$

since evidently

$$y_\varepsilon(t) \geq \int_\Omega u_\varepsilon \ln u_\varepsilon \geq -\frac{|\Omega|}{e} \quad \text{for all } t \in [0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in [0, 1). \quad (4.16)$$

Once more relying on (2.6), from (4.15) we obtain both (4.7) and (4.8), whereas (4.5) and (4.6) similarly result from (4.14) due to the second inequality in (4.16).

5. The two-dimensional case. Proof of theorem 1.1

In this section we concentrate on the two-dimensional setting of theorem 1.1. Since the solutions there will already turn out to be bounded and classical, it is not necessary to resort to an approximation by means of (2.1) for $\varepsilon > 0$. Throughout this section, we will therefore directly address the solutions $(u, v) := (u_0, v_0)$ of (2.1) obtained for $\varepsilon = 0$.

Based on the information provided by lemma 4.3, we can combine the outcomes of two further testing procedures applied to (2.1) in quite a standard manner, and thereby achieve the following key toward higher order bounds:

**Lemma 5.1.** Let $n = 2$. Then there exists $C > 0$ such that the solution $(u, v) \equiv (u_0, v_0)$ of (2.1), as corresponding to the choice $\varepsilon = 0$, satisfies

$$\int_\Omega |\nabla v(\cdot, t)|^4 \leq C \quad \text{for all } t \in (0, T_{\max}), \quad (5.1)$$

where $T_{\max} := T_{\max,0}$ is as accordingly provided by lemma 2.1.
Proof. On the basis of (2.1) when restricted to $\varepsilon = 0$, by means of Young’s inequality we see that
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 = - \int_{\Omega} |\nabla u|^2 + 2 \int_{\Omega} u \nabla u \cdot \nabla v \leq \int_{\Omega} u^2 |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$, and that for all $t \in (0, T_{\text{max}})$,
\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 = \int_{\Omega} |\nabla v|^2 \nabla v \cdot \nabla \{\Delta v - u v\}
\]
which is
\[
= \frac{1}{2} \int_{\Omega} |\nabla v|^2 \Delta |\nabla v|^2 - \int_{\Omega} |\nabla v|^2 |D^2 v|^2 - \int_{\Omega} u |\nabla v|^4
\]
\[
- \int_{\Omega} v |\nabla v|^2 \nabla u \cdot \nabla v
\]
\[
= - \frac{1}{2} \int_{\Omega} |\nabla v|^2 |\nabla v|^2 - \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 \partial |\nabla v|^2
\]
\[
- \int_{\Omega} |\nabla v|^2 |D^2 v|^2 - \int_{\Omega} u |\nabla v|^4 - \int_{\Omega} v |\nabla v|^2 \nabla u \cdot \nabla v
\]
\[
\leq - \frac{1}{2} \int_{\Omega} |\nabla v|^2 |\nabla v|^2 + \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 \partial |\nabla v|^2 - \int_{\Omega} v |\nabla v|^2 \nabla u \cdot \nabla v
\]
\[
\leq - \frac{1}{2} \int_{\Omega} |\nabla v|^2 |\nabla v|^2 + \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 \partial |\nabla v|^2
\]
\[
+ \|v\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla u| |\nabla v|^3
\]
(5.3)
because of (2.6). To proceed from this, we employ the Gagliardo–Nirenberg inequality to find $c_1 > 0$ such that
\[
\int_{\Omega} |\nabla v|^6 = \|\nabla v|^3\|^3_{L^2(\Omega)}
\]
\[
\leq c_1 \|\nabla v|^2\|_{L^2(\Omega)}^2 \|\nabla v|^2\|_{L^1(\Omega)} + c_1 \|\nabla v|^2\|^3_{L^2(\Omega)}
\]
\[
\leq c_1 c_2 \|\nabla v|^2\|_{L^2(\Omega)}^2 + c_1 c_2^3 \text{ for all } t \in (0, T_{\text{max}}),
\]
(5.4)
with finiteness of $c_2 := \sup_{t \in (0, T_{\text{max}})} \|\nabla v(\cdot, t)\|^2$ being asserted by lemma 4.3. We then fix $a > 0$ suitably small such that
\[
8a^2 \|v(0, t)^2\|_{L^\infty(\Omega)}^2 \leq \frac{a}{c_1 c_2},
\]
(5.5)
and take $\eta > 0$ small enough fulfilling
\[
\frac{1}{2} \eta \geq 1 + \left( \frac{2c_1 c_2}{a} \right)^2,
\]
(5.6)
whereupon an application of a well-known variant of the Gagliardo–Nirenberg inequality ([3]) shows that since lemma 4.3 warrants boundedness of $(u(\cdot, t) \in u(\cdot, t)_{t \in (0, T_{\text{max}})}$ in $L^1(\Omega)$, there
exists $c_3 > 0$ such that
\[
\int_\Omega u^3 \leq \eta \int_\Omega |\nabla u|^2 + c_3 \quad \text{for all } t \in (0, T_{\max}).
\] (5.7)

We now let
\[
y(t) := \int_\Omega u^2(\cdot, t) + a \int_\Omega |\nabla v(\cdot, t)|^4, \quad t \in [0, T_{\max}),
\]
and combine (5.2) with (5.3) to obtain that since due to Young’s inequality,
\[
\int_\Omega |\nabla u|^2 \geq \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\eta} \int_\Omega u^3 - \frac{c_3}{2\eta} \quad \text{for all } t \in (0, T_{\max})
\]
and
\[
2 \int_\Omega |\nabla|\nabla v|^2|^2 \geq \frac{2}{c_1 c_2} \int_\Omega |\nabla v|^6 - 2c_3^2 \quad \text{for all } t \in (0, T_{\max})
\]
by (5.7) and (5.4), we have
\[
y'(t) + y(t) + \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega u^3 + \frac{2a}{c_1 c_2} \int_\Omega |\nabla v|^6
\leq \int_\Omega u^2 + a \int_\Omega |\nabla v|^4 + \int_\Omega u^2 |\nabla u|^2 + \frac{1}{2} \int_\Omega |\nabla u|^2 + 8a^2 |v^{(0)}|_{L^\infty(\Omega)} \int_\Omega |\nabla v|^6
\]
\[
= \int_\Omega u^2 + \int_\Omega \left\{ \frac{a}{2c_1 c_2} |\nabla v|^6 \right\}^{\frac{1}{2}} \cdot (2c_1 c_2)^{\frac{1}{2}} a^{\frac{3}{2}} + \int_\Omega \left\{ \frac{a}{2c_1 c_2} |\nabla v|^6 \right\}^{\frac{1}{2}} \cdot \left( \frac{2c_1 c_2}{a} \right)^{\frac{1}{2}} u^2
\]
\[
+ \frac{1}{2} \int_\Omega |\nabla u|^2 + 8a^2 |v^{(0)}|_{L^\infty(\Omega)} \int_\Omega |\nabla v|^6
\]
\[
\leq \int_\Omega u^3 + |\Omega| + \frac{a}{2c_1 c_2} \int_\Omega |\nabla v|^6 + (2c_1 c_2)^3 a |\Omega| + \frac{a}{2c_1 c_2} \int_\Omega |\nabla v|^6
\]
\[
+ \left( \frac{2c_1 c_2}{a} \right)^{\frac{1}{2}} \int_\Omega u^3 + \frac{1}{2} \int_\Omega |\nabla u|^2 + 8a^2 |v^{(0)}|_{L^\infty(\Omega)} \int_\Omega |\nabla v|^6
\]
\[
= \left\{ 1 + \left( \frac{2c_1 c_2}{a} \right)^{\frac{1}{2}} \right\} \left\{ \int_\Omega u^3 + \left\{ \frac{a}{c_1 c_2} + 8a^2 |v^{(0)}|_{L^\infty(\Omega)} \right\} \cdot \int_\Omega |\nabla v|^6 + |\Omega|
\]
\[
+ (2c_1 c_2)^3 a |\Omega| + \frac{1}{2} \int_\Omega |\nabla u|^2 \right\} \quad \text{for all } t \in (0, T_{\max}).
\]
whence drawing on (5.6) and (5.5) we infer from (5.8) that
\[
y'(t) + y(t) \leq h(t) := \frac{c_1}{2\eta} + 2ac_2^2 + |\Omega| + (2c_1c_2)^2 a|\Omega| + 2a \int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} \quad \text{for all } t \in (0, T_{\max}).
\]

Since corollary 3.7 ensures that \(\sup_{t \in (0,T_{max} - \tau_0)} \int^{t+\tau_0} h(s)ds\), with \(\tau_0 = \min\{1, \frac{1}{2}T_{\max}\}\), is finite, by means of lemma 3.4 this entails that \(y\) is bounded in \((0, T_{\max})\), which in particular implies (5.1) with some suitably large \(C > 0\).

Indeed, this implies boundedness in the respective first solution components.

**Lemma 5.2.** Let \(n = 2\). Then there exists \(C > 0\) such that with \((u, v) \equiv (u_0, v_0)\) and \(T_{\max} = T_{\max,0}\) taken from lemma 2.1 we have
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).
\]  

**Proof.** We write the first equation in (2.1) for \(\varepsilon = 0\) in the form \(u_t = \Delta u + \nabla \cdot (b(x, t)u), (x, t) \in \Omega \times (0, T_{\max})\), and note that according to lemma 5.1, \(b := -\nabla v\) belongs to \(L^\infty((0,T_{\max}); L^q(\Omega))\) with \(q := 4\) exceeding the considered spatial dimension. Since \((\nabla u + b(x, t)u) \cdot \nu = 0\) on \(\partial\Omega \times (0, T_{\max})\), we may therefore refer to a boundedness statement derived by means of a straightforward Moser-type iteration ([9]) to directly obtain (5.9). □

For the proof of theorem 1.1, we are merely lacking a transfer of the boundedness properties we have just obtained to the spaces that actually occur in the extensibility criterion (2.4):

**Proof of theorem 1.1** Based on the outcome of lemma 5.2, we may again utilize known smoothing properties of the Dirichlet heat semigroup on \(\Omega\), and additionally employ a standard result on gradient Hölder regularity in scalar parabolic equations ([22]), to find \(c_1 > 0, c_2 > 0\) and \(\theta_1 \in (0, 1)\) such that
\[
\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq c_1 \quad \text{for all } t \in (0, T_{\max})
\]  

and
\[
\|\nabla v\|_{C^{\theta_2,h_2}(\Omega \times (t+\tau_0))} \leq c_2 \quad \text{for all } t \in \left(\frac{T_0}{4}, T_{\max} - \tau_0\right),
\]

where again \(\tau_0 = \min\{1, \frac{1}{2}T_{\max}\}\). According to (5.10), we may thereafter rely on the latter token once again to infer from lemma 5.2 and the first sub-problem in (2.1) that with some \(c_3 > 0\) and \(\theta_2 \in (0, 1)\) we have
\[
\|u\|_{C^{\theta_1,h_1}(\Omega \times (t+\tau_0))} \leq c_3 \quad \text{for all } t \in \left(\frac{T_0}{2}, T_{\max} - \tau_0\right),
\]

which combined with (5.10) and (2.4) shows that lemma 2.1 indeed asserts that \(T_{\max} = \infty\), whereupon (1.5) becomes a consequence of lemma 5.2 and (5.10). □
6. The case \( n \geq 3 \). Proof of theorem 1.2

The solution concept to be pursued in higher-dimensional cases appears to be quite natural.

**Definition 6.1.** Let

\[
\begin{align*}
    u &\in L^1_{\text{loc}}((0, \infty); W^{1,1} \Omega) \quad \text{and} \\
    v &\in L^1_{\text{loc}}((0, \infty); W^{1,1} \Omega)
\end{align*}
\]

be nonnegative and such that \( v(\cdot, t) - v_\varepsilon \in W^{1,1}_0 \Omega \) for a.e. \( t > 0 \), and that

\[
    u \nabla v \in L^1_{\text{loc}} \left( \Omega \times [0, \infty); \mathbb{R}^n \right) \quad \text{and} \quad uv \in L^1_{\text{loc}} \left( \Omega \times [0, \infty) \right).
\]

Then \((u, v)\) will be called a global weak solution of (1.3) if

\[
    - \int_0^\infty \int_\Omega u \phi_t - \int_\Omega u(0) \phi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \phi + \int_0^\infty \int_\Omega u \nabla v \cdot \nabla \phi \tag{6.3}
\]

for all \( \phi \in C_0^\infty \left( \Omega \times [0, \infty) \right) \), and if

\[
    - \int_0^\infty \int_\Omega v \phi_t - \int_\Omega v(0) \phi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla v \cdot \nabla \phi - \int_0^\infty uv \phi \tag{6.4}
\]

for all \( \phi \in C_0^\infty \left( \Omega \times [0, \infty) \right) \).

In order to construct such solutions, we now utilize solutions of the approximate versions of (2.1), that is, those corresponding to positive values of \( \varepsilon \). As can easily be seen, the strength of the accordingly regularizing features \( F_\varepsilon \) is sufficient to ensure that each of these solutions is global in time:

**Lemma 6.2.** Let \( n \geq 3 \) and \( \varepsilon \in (0, 1) \). Then \( T_{\text{max}, \varepsilon} = \infty \).

**Proof.** Supposing for contradiction that \( T_{\text{max}, \varepsilon} \) be finite for some \( \varepsilon \in (0, 1) \), we note that since \( 0 \leq F_\varepsilon \leq \frac{1}{4} \) and hence \( F_\varepsilon(u_\varepsilon)v_\varepsilon \) belongs to \( L^\infty(\Omega \times (0, T_{\text{max}, \varepsilon})) \) by (2.6), the standard result on parabolic \( C^{1+\theta} \) regularity from [22] would become applicable so as to ensure that

\[
    v_\varepsilon \in C^{1+\theta_1, \frac{1}{4}} \left( \Omega \times \left[ \frac{1}{4} T_{\text{max}, \varepsilon}, T_{\text{max}, \varepsilon} \right] \right)
\]

for some \( \theta_1 \in (0, 1) \). As \( 0 \leq \xi F_\varepsilon(\xi) \leq \frac{1}{4} \) for all \( \xi \geq 0 \), this would especially assert boundedness of \( u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \) in \( \Omega \times (\frac{1}{4} T_{\text{max}, \varepsilon}, T_{\text{max}, \varepsilon}) \) and hence, firstly, warrant the inclusion \( u_\varepsilon \in L^\infty(\Omega \times (\frac{1}{4} T_{\text{max}, \varepsilon}, T_{\text{max}, \varepsilon})) \) through the Moser iteration result from [9]. Thereafter, another application of standard parabolic regularity theory ([22]) would entail that, in fact, \( u_\varepsilon \in C^{1+\theta_2, \frac{1}{2}}(\Omega \times [\frac{1}{4} T_{\text{max}, \varepsilon}, T_{\text{max}, \varepsilon}]) \) for some \( \theta_2 \in (0, 1) \), which together with (6.5) would clearly contradict (2.4). \( \square \)

In passing to the limit \( \varepsilon \searrow 0 \), we will use the following consequences of lemma 4.3 on further spatio-temporal bounds.

**Lemma 6.3.** Let \( n \geq 3 \). Then there exists \( C > 0 \) such that

\[
    \int_t^{t+1} \int_\Omega u_{\varepsilon}^{\frac{1}{2}+\varepsilon} \leq C \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in (0, 1) \tag{6.6}
\]
\[
\int_t^{t+1} \int_\Omega |\nabla u_\varepsilon|^\frac{p+2}{p+1} \leq C \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in (0,1).
\] (6.7)

**Proof.** From the Gagliardo–Nirenberg inequality and (2.5) we obtain \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
\int_\Omega \frac{\varepsilon + 2}{\varepsilon} u_\varepsilon = \|\nabla u_\varepsilon\|^\frac{2(n+3)}{n+2} L^\frac{n+2}{2} (\Omega)
\]
\[
\leq c_1 \|\nabla u_\varepsilon\|^2 L^\frac{4}{2} (\Omega) \|\sqrt{u_\varepsilon}\|^\frac{2}{2} L^\frac{2}{2} (\Omega) + c_1 \|\sqrt{u_\varepsilon}\|^\frac{2(n+3)}{2} L^\frac{2}{2} (\Omega)
\]
\[
\leq c_2 \int_\Omega \frac{\|\nabla u_\varepsilon\|^2}{u_\varepsilon} + c_2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),
\]
and an application of Young’s inequality shows that
\[
\int_\Omega |\nabla u_\varepsilon|^\frac{p+2}{p+1} = \int_\Omega \left( \left\{ \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \right\}^{\varepsilon + 2} \right)^{\frac{\varepsilon + 2}{\varepsilon}} \leq \int_\Omega \frac{\|\nabla u_\varepsilon\|^2}{u_\varepsilon} + \int_\Omega \frac{\varepsilon + 2}{\varepsilon} u_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\]
Therefore, both (6.6) and (6.7) result from (4.7).

In preparation of an Aubin–Lions type argument, we additionally require some weak estimates for time derivatives.

**Lemma 6.4.** Let \( n \in \{3, 4, 5\} \). Then for all \( T > 0 \) one can find \( C(T) > 0 \) such that
\[
\int_0^T \|u_\varepsilon(\cdot,t)\|^{1 + \frac{4(n+2)}{5n+2}} W^{1,1 + \frac{4(n+2)}{5n+2}} (\Omega)^e \, dt \leq C(T) \quad \text{for all } \varepsilon \in (0,1)
\] (6.8)

and
\[
\int_0^T \|v_\varepsilon(\cdot,t)\|^{\frac{n+1}{n+2}} W^{1,1 + \frac{n+1}{n+2}} (\Omega)^e \leq C(T) \quad \text{for all } \varepsilon \in (0,1).
\] (6.9)

**Proof.** We note that since \( 2 \leq n < 6 \), we have \( 1 + \frac{5n+6}{6n} = \frac{4n+2}{6n} \geq n + 2 \), so that we can fix \( c_1 > 0 \) with the property that \( \|\nabla \psi\| L^{n+2} (\Omega) + \|\nabla \psi\| L^{\frac{4n+2}{6n}} (\Omega) \leq c_1 \) for all \( \psi \in C^1(\overline{\Omega}) \) such that \( \|\psi\| W^{1,1 + \frac{4(n+2)}{5n+2}} (\Omega) \leq 1 \). Given any such \( \psi \), using (2.1) along with the Hölder inequality, we thus obtain that since \( |F'_{\varepsilon}| \leq 1 \) for all \( \varepsilon \in (0,1) \) by (2.3),
\[
\left| \int_\Omega u_\varepsilon \psi \right| = -\int_\Omega \nabla u_\varepsilon \cdot \nabla \psi + \int_\Omega u_\varepsilon F'_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \psi
\]
\[
\leq \|\nabla u_\varepsilon\| L^{\frac{n+2}{n+1}} (\Omega) \|\nabla \psi\| L^{n+2} (\Omega) + \|u_\varepsilon\| L^{\frac{n+2}{n+1}} (\Omega) \|\nabla v_\varepsilon\| L^1 (\Omega) \|\nabla \psi\| L^{\frac{4n+2}{6n}} (\Omega)
\]
\[
\leq c_1 \|\nabla u_\varepsilon\| L^{\frac{n+2}{n+1}} (\Omega) + c_1 \|u_\varepsilon\| L^{\frac{n+2}{n+1}} (\Omega) \|\nabla v_\varepsilon\| L^1 (\Omega) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)
\]
For all \( t > 0 \) and \( \varepsilon \in (0, 1) \), we therefore have

\[
\|u(t)\|_{W^{1,1+\frac{n+\frac{4}{n+2}}{3n+2}+\frac{1}{n+2}}(\Omega)^*} \leq (2c_1)^{\frac{4n+8}{n+2}} \cdot \left\{ \|\nabla u(t)\|_{L^{1+\frac{4}{n+2}}(\Omega)} + \|u(t)\|_{L^{1+\frac{4}{n+2}}(\Omega)} \right\} 
\leq \left(2c_1\right)^{\frac{4n+8}{n+2}} \cdot \left\{ \|\nabla u(t)\|_{L^{1+\frac{4}{n+2}}(\Omega)} + 1 + \|u(t)\|_{L^{1+\frac{4}{n+2}}(\Omega)} + \|\nabla v_2\|_{L^1(\Omega)} \right\} 
\tag{6.10}
\]

by Young’s inequality, because \( 1 + \frac{6-n}{3n+2} = \frac{4n+8}{n+2} \leq \frac{n+2}{n} \), again due to the fact that \( n \geq 2 \).

Integrating (6.10) in time shows that (6.8) is implied by lemmata 4.3 and 6.3.

Likewise, to derive (6.9) we observe that since the inequality \( n \geq 3 \) warrants that \( 1 + \frac{n+10(n-1)}{3n+2} = \frac{n(n+2)}{n+2} \geq \frac{4}{3} \), and since \( W^{1,\frac{n(n+2)}{n+2}}(\Omega) \hookrightarrow L^{\frac{n(n+2)}{n+2}}(\Omega) \) due to the fact that \( 1 - \frac{4n+2}{n+2} = -\frac{2n}{n+2} \), with some \( c_2 > 0 \) we have \( \|\nabla \psi\|_{L^2(\Omega)} + \|\psi\|_{L^{\frac{2(n+2)}{n+2}}(\Omega)} \leq c_2 \) for all \( \psi \in \mathcal{B} := \left\{ \tilde{\psi} \in C^0(\Omega) : \|\tilde{\psi}\|_{W^{1,\frac{n(n+2)}{n+2}}(\Omega)} \leq 1 \right\} \). Therefore, the second equation in (2.1) shows that due to the Hölder inequality, (2.3) and Young’s inequality,

\[
\|v_{\varepsilon(t)}\|_{\left(\frac{n+2}{n+2+\frac{4}{n+2}}\right)^*} = \sup_{\psi \in \mathcal{B}} \left\| \int_\Omega v_{\varepsilon(t)} \psi \right\|^{\frac{4}{n+2}} \\
= \sup_{\psi \in \mathcal{B}} \left\| \int_\Omega \nabla v_2 \cdot \nabla \psi - \int_\Omega F_\varepsilon(u_{\varepsilon(t)}) v_2 \psi \right\|^{\frac{4}{n+2}} \\
\leq \sup_{\psi \in \mathcal{B}} \left\{ \left\| \nabla v_2 \right\|_{L^2(\Omega)} \left\| \nabla \psi \right\|_{L^{\frac{2(n+2)}{n+2}}(\Omega)} \right\}^{\frac{4}{n+2}} \\
\leq \left(2c_2\right)^{\frac{4}{n+2}} \cdot \left\{ \left\| \nabla v_2 \right\|_{L^2(\Omega)} + \left\| u_{\varepsilon(t)} \right\|_{L^{\frac{2(n+2)}{n+2}}(\Omega)} \right\}^{\frac{4}{n+2}} \\
\leq \left(2c_2\right)^{\frac{4}{n+2}} \cdot \left\{ \left\| \nabla v_2 \right\|_{L^2(\Omega)} + 1 + \left\| u_{\varepsilon(t)} \right\|_{L^{\frac{2(n+2)}{n+2}}(\Omega)} \right\}^{\frac{4}{n+2}}
\]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \), as clearly \( \frac{4}{n+2} \leq 4 \). In view of lemmata 4.3, 6.3 and (2.6), the inequality in (6.9) thus results upon an integration.

Our limit passage has thereby been prepared:

**Lemma 6.5.** Let \( n \in \{3, 4, 5\} \). Then there exist \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \), fulfilling \( \varepsilon_j \searrow 0 \) as \( j \to \infty \), as well as nonnegative functions \( u \) and \( v \) on \( \Omega \times (0, \infty) \) which satisfy (1.6), for which \( (u(\cdot, t), v(\cdot, t)) \) is radially symmetric for a.e. \( t > 0 \), for which as \( \varepsilon = \varepsilon_j \searrow 0 \) we have

\[
u_j \to u \quad \text{in} \quad \bigcap_{\rho \in (1, \frac{n+2}{2})} L^p_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \quad \text{and a.e. in} \ \Omega \times (0, \infty),
\tag{6.11}
\]

\[
u_j \to u \quad \text{in} \quad L^\infty_{\text{loc}}(\overline{\Omega} \times [0, \infty)),
\tag{6.12}
\]

\[
\nabla u_j \to \nabla u \quad \text{in} \quad L^\infty_{\text{loc}}(\overline{\Omega} \times [0, \infty)),
\tag{6.13}
\]
\[ v_\varepsilon \to v \quad \text{in} \quad \bigcap_{p \in (1, \infty)} L^p_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \] and a.e. in \( \Omega \times (0, \infty) \) and \( (6.14) \)

\[ \nabla v_\varepsilon \rightharpoonup \nabla v \quad \text{in} \quad L^4_{\text{loc}}(\overline{\Omega} \times [0, \infty)) , \] \( (6.15) \)

\[ \nabla v_\varepsilon \rightharpoonup \nabla v \quad \text{in} \quad L^\infty_{\text{loc}}((0, \infty); L^2(\Omega)) , \] \( (6.16) \)

and such that \((u, v)\) is a global weak solution of \((1.3)\) in the sense of definition 6.1.

**Proof.** Given \( T > 0 \), from lemmata 6.3 and 6.4 we know that

\( (u_\varepsilon)_{\varepsilon \in (0,1)} \) is bounded in \( L^{\frac{4+2}{n+2}}((0, T); W^{1,\frac{4+2}{n+2}}(\Omega)) \)

and that

\( (u_\varepsilon)_{\varepsilon \in (0,1)} \) is bounded in \( L^{1+\frac{6+2}{n+2}}((0, T); \left( W^{1,1+\frac{6+2}{n+2}}(\Omega) \right)^*) \),

while according to lemma 4.3, (2.6) and lemma 6.4,

\( (v_\varepsilon - v_\ast)_{\varepsilon \in (0,1)} \) is bounded in \( L^4((0, T); W^{1,4}(\Omega)) \)

and

\( (\partial_t(v_\varepsilon - v_\ast))_{\varepsilon \in (0,1)} \) is bounded in \( L^{\frac{n+2}{n}}((0, T); \left( W_0^{1,1+\frac{n-2}{n+2}}(\Omega) \right)^*) \),

because clearly \( \partial_t(v_\varepsilon - v_\ast) = v_{\varepsilon t} \). Therefore, two applications of an Aubin–Lions lemma \(([25])\) provide \((\varepsilon_j)_{\varepsilon \in \mathbb{N}} \subset (0, 1)\) as well as nonnegative radially symmetric functions \( u \in L^{\frac{4+2}{n+2}}(0, \infty); W^{1,\frac{4+2}{n+2}}(\Omega) \) and \( (v - v_\ast) \in L^4(0, \infty); W_0^{1,4}(\Omega) \) such that \( \varepsilon_j \searrow 0 \) as \( j \to \infty \), that \((6.13)\) and \((6.15)\) hold, and that \((u_\varepsilon, v_\varepsilon) \to (u, v)\) a.e. in \( \Omega \times (0, \infty) \) as \( \varepsilon = \varepsilon_j \searrow 0 \).

Since furthermore \((u_\varepsilon)_{\varepsilon \in (0,1)} \) is bounded in \( L^\infty((0, \infty); L^1(\Omega)) \) and in \( L^{\frac{n+2}{n}}(\Omega \times (0, T)) \) for all \( T > 0 \) by \((2.5)\) and lemma 4.2, and since \((v_\varepsilon)_{\varepsilon \in (0,1)} \) is bounded in \( L^\infty(\Omega \times (0, \infty)) \) and in \( L^\infty((0, \infty); W^{1,2}(\Omega)) \) according to \((2.6)\) and lemma 4.3, it is clear that actually \((6.12)\) and \((6.16)\) hold and \( u \) and \( v \) have all the regularity features in \((1.6)\), and that the Vitali convergence theorem along with \((2.2)\) and \((2.3)\) ensures that

\[ u_\varepsilon \to u, \quad F_s(u_\varepsilon) \to u \quad \text{and} \quad u, F_s'(u_\varepsilon) \to u \quad \text{in} \quad \bigcap_{p \in (1, \frac{n+2}{n})} L^p_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \]

as well as

\[ v_\varepsilon \to v \quad \text{in} \quad \bigcap_{p \in (1, \infty)} L^p_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \]

as \( \varepsilon = \varepsilon_j \searrow 0 \). Besides especially completing the verification of \((6.11)\) and \((6.14)\), and hence especially also of all the regularity requirements in definition 6.1, together with \((6.15)\) these two latter properties guarantee that

\[ u, F_s'(u_\varepsilon) \nabla v_\varepsilon \rightharpoonup u \nabla v \quad \text{in} \quad L^1_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \] \( (6.17) \)

and

\[ F_s(u_\varepsilon) v_\varepsilon \rightharpoonup uv \quad \text{in} \quad L^1_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \] \( (6.18) \).
as \( \varepsilon = \varepsilon_j \searrow 0 \), because
\[
\lim_{p \to \frac{n+2}{4n+2}} \frac{1}{p} + \frac{1}{4} = \frac{5n+2}{4(n+2)} = 1 - \frac{6-n}{4(n+2)} < 1
\]
and
\[
\lim_{p \to \frac{n+2}{4n+2}} \frac{1}{p} + \lim_{p \to \infty} \frac{1}{p} = \frac{n}{n+2} < 1.
\]
Since for each \( \varepsilon \in (0, 1) \) we have
\[
- \int_0^\infty \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u^{(0)} \varphi(:, 0) = - \int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi + \int_0^\infty \int_\Omega u_\varepsilon F'_\varepsilon (u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi
\]
for all \( \varphi \in C^\infty_0 (\bar{\Omega} \times [0, \infty)) \) and
\[
- \int_0^\infty \int_\Omega v_\varepsilon \varphi_t - \int_\Omega v^{(0)} \varphi(:, 0) = - \int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi - \int_0^\infty \int_\Omega F'_\varepsilon (u_\varepsilon) v_\varepsilon \varphi
\]
for all \( \varphi \in C^\infty_0 (\bar{\Omega} \times [0, \infty)) \), taking \( \varepsilon = \varepsilon_j \searrow 0 \) we therefore readily obtain (6.3) from (6.11), (6.13) and (6.17), whereas (6.4) results from (6.14), (6.15) and (6.18). Consequently, it follows that \((u, v)\) indeed forms a global weak solution of (1.3) in the sense of definition 6.1. \( \square \)

This essentially establishes our main result on global weak solvability in (1.3) already.

**Proof of theorem 1.2** The statement on existence of a global weak solution fulfilling (1.6) directly results from lemma 6.5, whereas the boundedness properties in (1.7)–(1.9) can readily be obtained upon combining (4.5), (4.6), (6.6), (6.7) and (4.8) with (6.11), (6.16), (6.12), (6.13) and (6.15). \( \square \)

### 7. Stationary states

We finally consider the stationary problem associated with (1.3), that is, the boundary value problem
\[
\begin{align*}
0 &= \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega \\
0 &= \Delta v - uv & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= u \frac{\partial v}{\partial \nu}, \quad v = v^* & \text{on } \partial \Omega,
\end{align*}
\]
and our arguments in this regard will be closely related to those in [6], where the second equation was instead supplemented by Robin-type boundary conditions. Dropping the requirement of radial symmetry here, we will assume that \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary, and that with some \( \beta \in (0, 1) \), \( v^* \) belongs to \( C^{2+\beta}(\partial \Omega) \) and is nonnegative on \( \partial \Omega \).

A first essential observation is that we can eliminate \( u \) from the stationary system and subsequently deal with a single equation only. Especially regarding the question of uniqueness, the appearance of a constant parameter \( \alpha \) in said equation could turn out to be unfortunate. However, we will later show that \( \alpha \) is in one-to-one correspondence with \( \int_\Omega u \). (An alternative
would be to compute $\alpha = \frac{m}{\int_{\Omega} e^v}$ and work with the nonlocal equation for $v$, see [21], where this approach was used for the special case of constant boundary value.)

**Lemma 7.1.** Let $v \in C^2(\Omega)$. If $u \in C^2(\Omega)$ satisfies

$$
\begin{aligned}
0 &= \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= u \frac{\partial v}{\partial \nu} & \text{on } \partial \Omega 
\end{aligned}
$$

then there is $\alpha \in \mathbb{R}$ such that

$$
u = \alpha e^v. \tag{7.3}$$

If, on the other hand, (7.3) holds for some $v \in C^2(\Omega)$ and $\alpha \in \mathbb{R}$, then (7.2) is satisfied. Furthermore, the signs of $\int_{\Omega} u$ and $\alpha$ coincide.

**Proof.** While the second part of the statement directly follows from the chain rule and the last part is obvious after integration of (7.3), the first is identical to [6, lemma 4.1]. □

We now take care of solvability and some a priori estimates for solutions of the second equation of (7.1) if we insert (7.3), firstly in a related linear problem.

**Lemma 7.2.** Let $\alpha \geq 0, v^* \in C^{2+\beta}(\partial \Omega), v^* \geq 0$ and $v \in C^\beta(\Omega)$. Then

$$
\begin{aligned}
\Delta \tilde{v} &= \alpha \tilde{v} e^v & \text{in } \Omega \\
\tilde{v} &= v^* & \text{on } \partial \Omega 
\end{aligned} \tag{7.4}
$$

has a unique solution $\tilde{v} \in C^{2+\beta}(\Omega)$. This solution satisfies

$$0 \leq \tilde{v} \leq \gamma := \max_{\partial \Omega} v^*. \tag{7.5}$$

Moreover, for every $\alpha$ and $v^*$ as above, there is $C > 0$ such that for every $v \in C^1(\Omega)$ with $0 \leq v \leq \gamma$ the corresponding solution $\tilde{v}$ satisfies

$$
\| \tilde{v} \|_{C^\beta(\Omega)} \leq C. \tag{7.6}
$$

**Proof.** Unique solvability results from [14, theorem 6.14]. If $\alpha = 0$, then (7.5) follows from the classical maximum principle ([14, theorem 3.1]) for the harmonic function $\tilde{v}$ and the assumptions on $v^*$. For $\alpha > 0$, we note that if $\tilde{v}$ is minimal at some $x_0 \in \Omega$, then $0 \leq \Delta \tilde{v}(x_0) = \alpha \tilde{v}(x_0) e^{v(x_0)}$, due to the positivity of $\alpha e^{v(x_0)}$, entails that $\tilde{v} \geq \min \tilde{v} = \tilde{v}(x_0) \geq 0$. If $\tilde{v}$, however, is minimal at some $x_0 \in \partial \Omega$, then, again, $\tilde{v}(x_0) = v^*(x_0) \geq 0$. With nonnegativity of $\tilde{v}$ thus ensured and hence $\Delta \tilde{v} \geq 0$ in $\Omega$, the second part of (7.5) follows from the maximum principle ([14, corollary 3.2]). The Hölder bound in (7.6) thereby can be concluded from the boundedness of the right-hand side in (7.4) and elliptic regularity [14, theorem 8.29]. □

With this, solving the second equation of (7.1) with (7.3) is possible:

**Lemma 7.3.** For every $\alpha \geq 0$, the boundary value problem

$$
\begin{aligned}
\Delta v &= \alpha v e^v & \text{in } \Omega \\
v &= v^* & \text{on } \partial \Omega
\end{aligned} \tag{7.7}
$$

has a solution $v \in C^2(\Omega)$, and $v$ satisfies (7.5) and (7.6).
Lemma 7.4. Let $v_1$ and $v_2$ be two solutions of (7.7) with $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, respectively. If $\alpha_1 \geq \alpha_2$, then $v_1 \leq v_2$.

Proof. Both $v_1$ and $v_2$—being solutions to (7.4) with $v = v_1$ or $v = v_2$, respectively—are nonnegative. We let $\Omega_1 := \{ x \in \Omega | v_1(x) > v_2(x) \}$ and $\bar{v} = v_1 - v_2$. As $x \mapsto x e^x$ is monotone for $x \in [0, \infty)$,

$$\Delta \bar{v} = \alpha_1 v_1 e^{\alpha_1} - \alpha_2 v_2 e^{\alpha_2} \geq \alpha_2 (v_1 e^{\alpha_1} - v_2 e^{\alpha_2}) \geq 0 \quad \text{in } \Omega_1$$

$$\bar{v} = 0 \quad \text{on } \partial \Omega_1.$$

By the maximum principle therefore $\max_{\Omega_1} \bar{v} = \max_{\partial \Omega_1} \bar{v} = 0$ and thus $\Omega_1 = \emptyset$, so that $v_1 \leq v_2$ in $\Omega \setminus \Omega_1 = \Omega$.  

As a particular consequence of lemma 7.4, for every $\alpha \geq 0$ the solution to (7.7) is unique. From now on, we will denote it by $v_\alpha$.

That, according to lemma 7.4, $v_\alpha$ is decreasing with respect to $\alpha$ is of little help with regard to the monotonicity of $\alpha e^{v_\alpha}$ (or rather $\alpha \int_\Omega e^{v_\alpha}$). For further information we study the derivative of $v_\alpha$ w.r.t. $\alpha$.

Lemma 7.5. For every $\alpha_1 > 0$, the function

$$v_\alpha' = \frac{d}{d\alpha} v_\alpha \bigg|_{\alpha = \alpha_1} = \lim_{\alpha_2 \rightarrow \alpha_1} \frac{v_{\alpha_2} - v_{\alpha_1}}{\alpha_2 - \alpha_1} \tag{7.8}$$

exists (with the limit taken in $C^2(\Omega)$) and satisfies

$$\begin{cases}
\Delta v_\alpha' = v_{\alpha_1} e^{v_{\alpha_1}} + (\alpha_1 e^{v_{\alpha_1}} + \alpha_1 v_{\alpha_1} e^{v_{\alpha_1}}) v_\alpha' & \text{in } \Omega, \\
\frac{v_\alpha'}{v_{\alpha_1}} = 0 & \text{on } \partial \Omega. 
\end{cases} \tag{7.9}$$

Proof. For $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, we let $w_{\alpha_2, \alpha_1} = \frac{v_{\alpha_2} - v_{\alpha_1}}{\alpha_2 - \alpha_1}$ and note that $w = w_{\alpha_2, \alpha_1}$ solves

$$\begin{cases}
\Delta w = f_{\alpha_1} \alpha_2 + f_{\alpha_2} \alpha_1 \quad & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega \tag{7.10}
\end{cases}$$

with $f_{\alpha_1} = v_{\alpha_2} e^{v_{\alpha_2}}, f_{\alpha_2, \alpha_1} = \alpha_1 e^{v_{\alpha_1}} + \alpha_1 v_{\alpha_1} e^{v_{\alpha_1}} F(v_{\alpha_2} - v_{\alpha_1}), F(z) = \frac{2 \gamma}{z}, \gamma \neq 0, F(0) = 1$. Hölder bounds and elliptic regularity theory (lemma 7.3, theorem 6.6) in combination with the Arzelà–Ascoli theorem make it possible to pass to the limit $\alpha_2 \rightarrow \alpha_1$ in $C^2(\Omega)$, where we do not have to restrict the reasoning to subsequences, since the limit problem is uniquely solvable (due to $-\frac{1}{2} |\nabla (w_1 - w_2)|^2 \geq f_{\alpha_1} f_{\alpha_2} (w_1 - w_2)^2 \geq 0$ whenever $w_1$ and $w_2$ solve (7.10) for $\alpha_2 = \alpha_1$).  

\[ \square \]
**Lemma 7.6.** For every $\alpha > 0$,

$$0 \geq v'_\alpha > -\frac{1}{\alpha} \quad \text{in } \Omega.$$

**Proof.** We abbreviate $v' = v'_\alpha$ and $v = v_\alpha$. From lemma 7.4, we obtain that $0 \geq v'$. We let $x_0 \in \overline{\Omega}$ be such that $v'(x_0) = \min_{\partial \Omega} v'$. Then either $v'(x_0) = 0$, meaning that the nonpositive function $v'$ would have to satisfy $v' \equiv 0$ (which would finish the proof), or, on account of $v'$ vanishing on $\partial \Omega$, $x_0 \in \Omega$ and due to (7.9)

$$0 \leq \Delta v' = ve^v + (\alpha e^v + \alpha ve')v' \quad \text{at } x_0,$$

so that by positivity of $e^{v_0}$

$$0 \leq v + \alpha(1 + v)v' \quad \text{at } x_0,$$

which yields

$$v'(x_0) \geq -\frac{1}{\alpha} \cdot \frac{v(x_0)}{1 + v(x_0)} > -\frac{1}{\alpha}.$$

□

Consequences of lemma 7.6 on the desired relation between $\alpha$ and $m = \int_\Omega u$ are as follows:

**Lemma 7.7.** The map

$$m : \begin{cases} [0, \infty) \to [0, \infty) \\ \alpha \mapsto \int_\Omega e^{\alpha u} \end{cases} \quad (7.11)$$

is bijective.

**Proof.** Computing the derivative of $m$ (which uses lemma 7.5), like in [6, lemma 3.15], we obtain

$$m'(\alpha) = \int_\Omega e^{\alpha u} + \int_\Omega \alpha e^{\alpha u} v'_\alpha = \int_\Omega e^{\alpha u} (1 + \alpha v'_\alpha),$$

so that $m'(\alpha) > 0$ by lemma 7.6, ensuring injectivity. Since $m(0) = 0$ and $m(\alpha) = \int_\Omega \alpha e^{\alpha u} \geq \alpha |\Omega| \to \infty$ as $\alpha \to \infty$, surjectivity is obvious. □

**Proof of theorem 1.3** Combining lemma 7.1 with (7.11) shows that $(u, v) \in (C^2(\overline{\Omega}))^2$ solves (7.1) with $\int_\Omega u = m_0$ if and only if $m_0 = m(\alpha)$, $u = \alpha e^v$ and $v = v_\alpha$ solves (7.7). Bijectivity of $m$ (lemma 7.7), existence and uniqueness of $v_\alpha$ (lemmata 7.3 and 7.4) therefore imply the first part of theorem 1.3.

In the case when $\Omega$ is a ball and $v^*$ is constant, radial symmetry follows from the above uniqueness statement. Since $u = \alpha \exp(v)$ by lemma 7.1 and exp is monotone and convex, to complete the proof it is sufficient to show convexity of $v$, that is of the solution to (7.7). But when written in radial coordinates, (7.7) turns into

$$(r^{\alpha - 1} v_r)_r = \alpha r^{\alpha - 1} v e^v, \quad r \in (0, R), \quad v(R) = v^*.$$
with \( v_r(0) = 0 \) due to radial symmetry and differentiability of \( v \). Hence,

\[
v_r(r) = r^{1-n} \int_0^r \alpha s^{n-1} v(s) e^{v(s)} \, ds = \alpha r \int_0^1 r^{n-1} v(r) e^{v(r)} \, dr.
\] (7.12)

Nonnegativity of \( v \) (cf (7.5)) shows that hence \( v_r \geq 0 \); thus the rightmost expression in (7.12) is clearly increasing with respect to \( r \), which shows monotonicity of \( v_r \) and therefore convexity of \( v \).

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