Toeplitz matrices for the study of the fractional Laplacian on a bounded interval.

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Abstract

Toeplitz matrices for the study of the fractional Laplacian on a bounded interval, an application to fractional equations.

In this work we get a deep link between \((-\Delta)^{\alpha}_{[0,1]} \) the fractional Laplacian on the interval \([0,1]\) and \(T_N^\alpha(\varphi_\alpha)\) the Toeplitz matrices of symbol \(\varphi_\alpha : \theta \mapsto |1 - e^{i\theta}|^{2\alpha}\) when \(N\) goes to the infinity and for \(\alpha \in [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]\). In the second part of the paper we provide a Green function for the fractional equation \((-\Delta)^{\alpha}_{[0,1]}(\psi) = f\) for \(\alpha \in [0, \frac{1}{2}]\) and \(f\) a sufficiently smooth function on \([0,1]\). The interest is that this Green’s function is the same as the Laplacian operator of order \(2n, n \in \mathbb{N}\).

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1 Introduction and statement of the main results

For \(\alpha \in [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]\) we recall the definition of the one-dimensional fractional Laplace operator \((-\Delta)^{\alpha}\) (with \(\Delta = -\frac{d^2}{dx^2}\)). It is defined pointwise by the principal value of the integral, if convergent, (see [12])

\[
(-\Delta)^{\alpha}(u)(x) = C_1(\alpha)\text{PV.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2\alpha}} dy \quad \text{with} \quad C_1(\alpha) = \frac{2^{2\alpha}\Gamma(\frac{1+2\alpha}{2})}{\sqrt{\pi}\Gamma(-\alpha)} \quad x \in \mathbb{R}.
\]

\((-\Delta)^{\alpha}(u)(x)\) is convergent if, for instance, \(f\) is smooth in a neighborhood of \(x\) and bounded on \(\mathbb{R}\).

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More generally we can refer to \[13\] for the different equivalent definitions of the fractional Laplace operator on the real line. Up a constant, this operator is the left inverse of the Riesz operator on the real line, often denoted by $I^{-2\alpha}(\alpha \in [0, \frac{1}{2}])$, and defined by $I^{-2\alpha}(\psi)(x) = \frac{1}{2(2\alpha)\cos(\pi \alpha)} \int_{-\infty}^{+\infty} \frac{\psi(t)}{|t-x|^{1+2\alpha}} dt,$ for $x \in \mathbb{R}$ and $\psi \in L^p(\mathbb{R})$, with $1 \leq p < \frac{1}{\alpha}$ (see \[21\]).

For $f \in \mathcal{C}_c^\infty([0, 1])$, the fractional Laplacian operator on $[0, 1]$, denoted by $(-\Delta)^\alpha_{[0, 1]}$, is defined to be the restriction of $(-\Delta)^\alpha(f)$ to $[0, 1]$ (see \[13\]). Again $(-\Delta)^\alpha_{[0, 1]}$ extends to an unbounded operator on $L^2([0, 1])$.

We can easily obtain that for $f \in \mathcal{C}_c^\infty([0, 1])$ and for $x \in [0, 1]$}

$$\left((-\Delta)^\alpha_{[0, 1]}(f)\right)(x) = C_1(\alpha) \left(\mathrm{PV.} \int_0^1 \frac{f(x) - f(y)}{|x-y|^{1+2\alpha}} dy + \frac{f(x)}{2\alpha} \left(x^{-2\alpha} + (1 - x)^{-2\alpha}\right)\right).$$ \hfill (1)

More generally for $f$ a function defined on $[0, 1]$ and $x \in [0, 1]$ we denote by $\left((-\Delta)^\alpha_{[0, 1]}(f)\right)(x)$ the quantity (1) if it exists. In the first part of the article we get a link between this operator and the Toeplitz matrices of order $N + 1$ with symbol $\varphi_{\alpha} = |1 - \chi|^2 \alpha$ where $\chi$ is defined on $[0, 2\pi]$ by $\theta \mapsto e^{i\theta}$. We recall that a Toeplitz matrix of order $N$ with symbol $h \in L^1([0, 2\pi])$ is the $(N+1) \times (N+1)$ matrix $T_N(h)$ defined by $(T_N(h))_{i+1,j+1} = \hat{h}(i-j)$ where $\hat{h}(s)$ denote the Fourier coefficient of order $s$ of the function $h$ (see \[11\], \[12\]). The Toeplitz matrices of order $N \times N$ are decisive here because they have the property to make the link between the discrete and the continuous when $N$ goes to infinity, and they are useful to obtain a good discretization of the problem. With these tools, taking the limit at infinity, we can obtain operators $D_{\alpha}$ for $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right] \cup \left]\frac{1}{2}, , 1\right[ \right]$ that can be interpreted as fractional derivatives. For a function $f$ defined on $[0, 1]$, we define these operators as the following limit (if it exists)

$$D_{\alpha}f(x) = \lim_{N \to +\infty} N^{2\alpha} \sum_{l=0}^{N} (T_N(\varphi_{\alpha}))_{[Nx]+1,l+1} f\left(\frac{l}{N}\right)$$ \hfill (2)

for $x \in [0, 1]$ and where $[Nx] = \max\{k \in \mathbb{Z} | k \leq Nx\}$.

For $\alpha \in \left[\frac{1}{2}, 1\right[ \cup \left]\frac{1}{2}, , 1\right[ \right]$, it is the framework of the theorem \[1\] to show that this limit exists and is $\left((-\Delta)^\alpha_{[0, 1]}(f)\right)(x)$ for $f$ belonging to certain classes of functions.

Remark 1 The operator definition $D_{\alpha}$ can be easily transported over any interval $]a, b[$ as follows. If $a < b$ are two reals and $h$ is a function defined on $]a, b[$, one defines for $x$ in $[0, 1]$ $h_{a,b}(x) = h(a + (b - a)x)$. Then for $\alpha \in \left(-\frac{1}{2}, 0]\cup\left]\frac{1}{2}, , 1\right[ \right]$ and $u \in ]a, b[$, we have $D_{\alpha,a,b}(u)(\frac{u-a}{b-a})$.

The second point of this paper is to invert the fractional Laplacian $(-\Delta)^\alpha$ on the open interval $]0, 1[$ for $\alpha \in \left[0, \frac{1}{2}\right[$. For $f$ a contracting function on $[0, 1]$ such that $\text{Supp}(f) \subset ]0, 1[$ we solve the equation in $\phi$:

$$(-\Delta)^\alpha_{[0, 1]}(\phi) = f$$ \hfill (3)

that is also (see \[1\] and \[12\])

$$\begin{cases} (-\Delta)^\alpha(\phi) = f & \text{in } ]0, 1[ \\ \phi = 0 & \text{in } ]-\infty, 0] \cup [1, +\infty[. \end{cases}$$ \hfill (4)
We recall that for $\mu > 0$ we denote by $C^0,\mu([a, b])$ is defined as the set of the functions $\psi$ such that for all interval $[c, d] \subset [a, b]$ with $d - c$ sufficiently small there a real $K_{[c,d]} > 0$ such that the inequality
$$|\psi(x) - \psi(x')| \leq K_{[c,d]} |x - x'|^\mu$$
is true for all $x, x'$ in $[c, d]$.
In all this work we say that a function $f$ is contracting on an interval $I$ if
$$|f(x) - f(y)| \leq |x - y| \quad \forall x, y \in I.$$
Finally we will say that a function is locally contracting over an interval $[a, b]$ if it belongs to $C^{0,1}([a, b])$.

If the rest of the paper we denote by $C_\alpha$ the constant $-\frac{\Gamma(2\alpha+1)\sin(\pi\alpha)}{\pi}$. We check that $C(\alpha) = -C_1(\alpha)$. Then we can write the following statements :

**Theorem 1** We have :

1. If $-\frac{1}{2} < \alpha < 0$ and $h \in L^1([0, 1])$ then for all $x \in [0, 1]$ in $L^1([0, 1])$
$$\left(D_\alpha h\right)(u) = C_\alpha \int_0^1 \frac{h(t)}{|t - x|^{1+2\alpha}} dt.$$

2. For $0 < \alpha < \frac{1}{2}$ and $h$ a function in $C^{0,\mu}([0, 1])$ with $2\alpha < \mu < 1$
$$\left(D_\alpha h\right)(x) = \left(\frac{\Delta}{2}\right)_0^1 (h) (x),$$
uniformly in $x \in [\delta_1, \delta_2]$ for $[\delta_1, \delta_2] \subset [0, 1]$.

3. If $\frac{1}{2} < \alpha < 1$ and $h \in C^2([0, 1])$ then
$$\left(D_\alpha h\right)(x) = \left(\frac{\Delta}{2}\right)_0^1 (h) (x),$$
uniformly in $x \in [\delta_1, \delta_2]$ for $[\delta_1, \delta_2] \subset [0, 1]$.

**Theorem 2** Let $0 < \alpha < \frac{1}{2}$ and let $f$ be a real function contracting on $[0, 1]$ such that $\text{Supp}(f) = [a, b] \subset [0, 1]$. Then the differential equation
$$\left(\frac{\Delta}{2}\right)_0^1 (g) = f$$
has only one solution locally contracting on $[0, 1]$. This solution is defined for $z \in [0, 1]$ by
$$g(z) = \left(D_{-\alpha}(f)\right)(z) - \int_0^1 K_\alpha(z, y) f(y) dy,$$
where
$$K_\alpha(u, y) = \frac{1}{\Gamma^2(\alpha)} u^\alpha y^\alpha \left(\int_1^{+\infty} \frac{(t - u)^{\alpha-1}(t - y)^{\alpha-1}}{t^{2\alpha}} dt + \int_0^{+\infty} \frac{(t + u)^{\alpha-1}(t + y)^{\alpha-1}}{t^{2\alpha}} dt\right).$$
Remark 2 The consistency of the statement of the theorem 2 (i.e. if \( \alpha \in ]0, \frac{1}{2} [\) the function \( (D_{-\alpha}(f))(z) - (K_{\alpha}(f)) \) is locally contracting on \( [0, 1] \) is specified in the theorem demonstration). On the other hand in [17] we have obtained \( K_{\alpha}(x, y) = O(|x - y|^{\alpha - 1}) \) for \( 0 < x \neq y < 1 \). Hence our solution is well defined.

The theorem 2 also gives us that the solution of the equation \((-\Delta)^{\alpha}(\psi) = f\) found for the fractional Laplacian of order \( \alpha \) defined on an interval is not up a constant the Riesz operator of order \(-\alpha\) on the same interval, there is a perturbation, unlike the result on the real line (see [21]).

Our calculation methods can also invert the Riesz operator over a bounded interval. It is an alternative to the results given for example in [21].

Using the results of [19] we can also write the following equivalent statement.

**Theorem 3** Let \( 0 < \alpha < \frac{1}{2} \) and let \( h \) be a real function contracting on \( [0, 1] \) such that \( \text{Supp}(f) = [a, b] \subset [0, 1] \). Then the differential equation

\[
((-\Delta)^{\alpha})_{[0,1]}(\psi) = h
\]

has only one solution locally contracting on \( [0, 1] \) (hence in \( C^0([0, 1]) \)) with \( g(0) = g(1) = 0 \). This solution is defined by

1. \( g(x) = \int_0^1 G_{\alpha}(x, y)h(y)dy \) with

\[
G_{\alpha}(x, y) = \frac{1}{\Gamma(2(\alpha))}(x)^{\alpha}(y)^{\alpha}\int_{\max(x,y)}^1 \frac{(t-x)^{\alpha-1}(t-y)^{\alpha-1}}{(t-1)^{2\alpha}}dt,
\]

for \( 0 < x \neq y < 1 \)

and

\[
G_{\alpha}(0, 0) = 0.
\]

2. \( g(z) = 0 \) for \( z \leq 0 \) or \( z \geq 1 \).

Remark 3 The expression of the Green kernel \( G_{\alpha} \) makes it easy to verify that the solution proposed in the theorem 2 is extendable by zero on \( \mathbb{R}\setminus[0, 1] \). This solution is well defined. In [17] we have obtained \( G_{\alpha}(x, y) = O(|x - y|^{2\alpha - 1}) \) for \( 0 < x \neq y < 1 \).

In fact the theorem 3 is a generalization of the well known case where \( \alpha \in \mathbb{N}^* \) and \([a, b] = [0, 1]\) (see [3] [22] [13]). In this case we have a Green function \( G_{\alpha}(x, y) = \frac{x^\alpha y^\alpha}{\Gamma(2(\alpha))} \int_{\max(x,y)}^1 \frac{(t-x)^{\alpha-1}(t-y)^{\alpha-1}}{t^{2\alpha}}dt \) for \( 0 < \max(x, y) \leq 1 \) and \( G(0, 0) = 0 \) such that for all function \( f \in L^1([0, 1]) \) the function \( g \) defined on \([0, 1]\) by \( g(x) = \int_0^1 G_{\alpha}(x, y)f(y)dy \) is the solution of the equation 3 with the bound condition \( g(0) = \cdots = g^{\alpha-1}(0) = 0 \) and \( g(1) = \cdots = g^{\alpha-1}(1) = 0 \). It is important to remark that the expression of the Green function is finally the same in the case of \( \alpha \in \mathbb{N} \) and for the case \( \alpha \in ]0, \frac{1}{2} [\).

To get the solution to the equation 3 we use the fine knowledge of the matrices \( (T_N(1 - \chi^{2\alpha}))^{-1} \) for \( \alpha \in ]0, \frac{1}{2} [\) that we acquired in our previous works. More precisely the theorems 2 and 3 may be related to the following results obtained respectively in [17] and [19], where we have obtained two alternative asymptotic expansions when \( N \) goes to the infinity of \( (T_N(\varphi_{\alpha}))_{k+l+1}^{-1} \) for \( k, l \) sufficiently larges and \( \alpha \in ]0, \frac{1}{2} [\). These expressions are specified in the two following theorems.
Theorem 4 For $\alpha \in ]0, \frac{1}{2}[$ we have
\[
(T_N(\varphi_\alpha))^{-1}_{[Nx]+1,[Ny]+1} = \hat{\varphi}_\alpha(\lfloor Nx \rfloor - \lfloor Ny \rfloor) - N^{2\alpha - 1}K_\alpha(z,y) + o(N^{2\alpha - 1})
\] (5)
uniformly in $x, y$ for $x, y \in [\delta_1, \delta_2]$.

and

Theorem 5 For $\alpha \in ]0, \frac{1}{2}[$ we have
\[
(T_N(\varphi_\alpha))^{-1}_{[Nx]+1,[Ny]+1} = N^{2\alpha - 1}G_\alpha(x,y) + o(N^{2\alpha - 1})
\] (6)
uniformly in $x, y$ for $0 < \delta_1 < x \neq y < \delta_2 < 1$.

The references [5], [21], [10] are good introductions to fractional integrals and derivatives, fractional Laplacian, and fractional differential equations.

The discretization methods used here can be extended to the study of other fractional differential operators. Thus in an other work [20] we found known results concerning other fractional derivatives by the same discretization process using an $N + 1 \times N + 1$ Toeplitz matrix of symbol $h_\alpha = \lim_{R \to 1} - h_{\alpha,R}$ with $1 > \alpha > 0$ and where $h_{\alpha,R}$ is the function defined by $\theta \mapsto (1 - Re^{i\theta})^\alpha(1 + Re^{-i\theta})^\alpha$, for $R \in ]0, 1[$ and $\theta \in [0, 2\pi[$. For $f$ a function defined on $[0, 1]$ and $0 \leq x \leq 1$ we then study the limit
\[
\lim_{N \to +\infty} N^{\alpha} \left( \sum_{t=0}^{N} T_N(\varphi_\alpha)_{k+1,l+1}(X_N)_l \right) = \left( \tilde{D}_\alpha(f) \right)(x), \quad \text{with } k = \lfloor Nx \rfloor.
\]
We show in [20] that for $f$ a locally locally contracting on $]0, 1[$ this limit is
\[
\frac{2^\alpha}{\Gamma(-\alpha)} \left( \int_0^x \frac{f(t) - f(x)}{|x-t|^{\alpha-1}} dt - f(x) \frac{(x)^{-\alpha}}{\alpha} \right)
\]
which is nothing more than the inferior fractional Marchaud derivative of order $\alpha$ on $[0, 1]$.

We can also verify that if we choose as a symbol the function $\tilde{h}_\alpha$ this same limit gives us the superior fractional Marchaud derivative of order $\alpha$ on $[0, 1]$. Still in [20] we find, by methods similar to those used here, the inverse of these fractional derivatives.

Another approach to fractional differential equations different from the classical approach can be found in [9] where the authors use Hankel’s operators to solve on $S^1$, the torus of dimension 1, the equation $i\partial_t u = \pi(|u|^2u)$ where $\pi$ is the usual orthogonal projection from $L^2(S^1)$ on the subspace $H^2(S^1)$ defined by $h \in H^2(S^1) \iff \hat{h}(s) = 0 \forall s < 0$.

Integrals and fractional derivatives are currently the focus of much mathematical works. For example, one could consult [3, 4, 5, 8, 16, 7].

2 Proof of the theorem

1. It is clear for $\alpha \in ]-\frac{1}{2}, 0[$.
2. Now we have to obtain the result for \( \alpha \in ]0, \frac{1}{2}[. \)
Assume that \( x \) belongs to an interval \([\delta_1, \delta_2]\) include in \([0, 1]\). We assume that \( N \) is a fixed integer and we put \( k = \lfloor Nx \rfloor \). In the following we denote by \( K \) the positive constant such that for \( x, y \in [\delta_1, \delta_2] \)
\[
|f(x) - f(y)| \leq K|x - y|^{\mu}. \tag{7}
\]
We will also use that \( \hat{\varphi}_\alpha(u) = C_\alpha |u|^{-2\alpha - 1} \left( 1 + o(1) \right) \) for \( u \) an integer with a sufficiently large absolute value. Let \( \delta = N^{-\beta} \) with \( 0 < \beta < 1 \). We can write
\[
N \sum_{l=0}^{N} (T_N \varphi_\alpha)_{k+1,l+1} f\left( \frac{l}{N} \right) = \sum_{l=0}^{k-[N\delta]-1} (T_N \varphi_\alpha)_{k+1,l+1} f\left( \frac{l}{N} \right) + \sum_{l=[k+N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} f\left( \frac{l}{N} \right) + \sum_{l=[k+N\delta]+1}^{N} (T_N \varphi_\alpha)_{k+1,l+1} f\left( \frac{l}{N} \right).
\]
We can now observe that
\[
\sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} f\left( \frac{l}{N} \right) = \sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} \left( f\left( \frac{l}{N} \right) - f\left( \frac{k}{N} \right) + f\left( \frac{k}{N} \right) \right).
\]
Using (7) we have
\[
\left| \sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} \left( f\left( \frac{l}{N} \right) - f\left( \frac{k}{N} \right) \right) \right| \leq |K| \sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} \left( \frac{l}{N} - \frac{k}{N} \right)^{\mu}
\]
that implies
\[
\left| \sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} \left( f\left( \frac{l}{N} \right) - f\left( \frac{k}{N} \right) \right) \right| = O(\delta^{\mu}).
\]
Lastly for \( \frac{2\alpha}{\mu} \leq \beta < 1 \) we obtain
\[
N^{2\alpha} \left( \sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} \left( f\left( \frac{l}{N} \right) - f\left( \frac{k}{N} \right) \right) \right) = o(1).
\]
On the other, since \( \sum_{n \in \mathbb{Z}} \hat{\varphi}_\alpha(n) = 0 \), we have clearly
\[
\sum_{l=k-[N\delta]}^{k+[N\delta]} (T_N \varphi_\alpha)_{k+1,l+1} f\left( \frac{k}{N} \right) = - \sum_{l<k-[N\delta]} \hat{\varphi}_\alpha(k-l) f\left( \frac{k}{N} \right) - \sum_{l>k+[N\delta]} \hat{\varphi}_\alpha(k-l) f\left( \frac{k}{N} \right)
\]
with

\[ N^{2\alpha} \sum_{t < k - [N\delta]} \hat{\varphi}_\alpha(k - l) f\left(\frac{k}{N}\right) = \frac{C_\alpha}{N} \left( \sum_{l=0}^{k-[N\delta]-1} \frac{|k-l|}{N} - 2\alpha - 1 f\left(\frac{k}{N}\right) - \frac{x^{-2\alpha}}{2\alpha} f(x) \right) + o(1), \]

and

\[ N^{2\alpha} \sum_{t > k + [N\delta]} \hat{\varphi}_\alpha(k - l) f\left(\frac{k}{N}\right) = \frac{C_\alpha}{N} \left( \sum_{l=k+[N\delta]+1}^{N} \frac{|k-l|}{N} - 2\alpha - 1 f\left(\frac{k}{N}\right) - \frac{(1-x)^{-2\alpha}}{2\alpha} f(x) \right) + o(1). \]

Since \( f \in C^{0,\mu}(0, 1] \) with \( \mu > 2\alpha \) we know that \( P.V. \int_0^1 f(t) - f(x) \frac{dt}{|t-x|^{2\alpha+1}} \) is convergent and

\[ \lim_{N \to +\infty} \frac{C_\alpha}{N} \left( \sum_{l=0}^{k-[N\delta]-1} \frac{|k-l|}{N} - 2\alpha - 1 \left( f\left(\frac{l}{N}\right) - f\left(\frac{k}{N}\right) \right) \right) + \sum_{l=k+[N\delta]+1}^{N} \frac{|k-l|}{N} - 2\alpha - 1 \left( f\left(\frac{l}{N}\right) - f\left(\frac{k}{N}\right) \right) = C_\alpha P.V. \int_0^1 f(t) - f(x) \frac{dt}{|t-x|^{2\alpha+1}} = C_1(\alpha) P.V. \int_0^1 f(x) - f(t) \frac{dt}{|x-t|^{2\alpha+1}}. \]

that ends the proof for \( \alpha \in ]0, \frac{1}{2}[. \)

3. Lastly we have to assume that \( \alpha \in ]\frac{1}{2}, 1[. \) Always with \( \delta = N^{-\gamma}, 0 < \gamma < 1 \) we have

\[ N^{2\alpha} \sum_{l=0}^{N} \hat{\varphi}_\alpha(k - l) f\left(\frac{l}{N}\right) = N^{2\alpha} \left( \sum_{l=0}^{k-[N\delta]-1} \hat{\varphi}_\alpha(k - l) f\left(\frac{l}{N}\right) + \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) f\left(\frac{l}{N}\right) + \sum_{l=k+[N\delta]+1}^{N} \hat{\varphi}_\alpha(k - l) f\left(\frac{l}{N}\right) \right) \]

We can write

\[ \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) f\left(\frac{l}{N}\right) = \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \left( f\left(\frac{l}{N}\right) - f\left(\frac{k}{N}\right) \right) + \left( \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \right) f\left(\frac{k}{N}\right). \]

By Taylor expansion of \( f \) we obtain:

\[ \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \left( f\left(\frac{l}{N}\right) - f\left(\frac{k}{N}\right) \right) = \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \frac{l-k}{N} f'(\frac{k}{N}) \]

\[ + \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \left( \frac{l-k}{N} \right)^2 f''(c_{k,l,N}) \]

\[ + \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \frac{l-k}{N} f'(\frac{k}{N}) \]

\[ + \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \left( \frac{l-k}{N} \right)^2 f''(c_{k,l,N}) + \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \frac{l-k}{N} f'(\frac{k}{N}) \]

\[ + \sum_{l=k-[N\delta]}^{k+[N\delta]} \hat{\varphi}_\alpha(k - l) \left( \frac{l-k}{N} \right)^2 f''(c_{k,l,N}). \]
where \( c_{k,l,N} \in ]\frac{k}{N}, \frac{k+1}{N}[ \) or \( c_{k,l,N} \in ]\frac{k}{N}, \frac{k}{N}[ \). Since \( \varphi'_\alpha(0) = 0 \) and with the additional remark that \( \varphi'_\alpha \) is an odd function we can write

\[
N^{2\alpha} \left( \sum_{l=k-[N\delta]}^{k+[N\delta]} \varphi'_\alpha(k-l) \frac{l-k}{N} \right) = (-i)N^{2\alpha-1} \left( \sum_{l=k-[N\delta]}^{k+[N\delta]} \varphi''_\alpha(k-l) \right)
\]

\[
= iN^{2\alpha-1} \left( \sum_{l<k-[N\delta]} \varphi'_\alpha(k-l) + \sum_{l>k+[N\delta]} \varphi'_\alpha(k-l) \right) = 0
\]

On the other hand for a good choice of \( \gamma \):

\[
N^{2\alpha} \left( \sum_{l=k-[N\delta]}^{k+[N\delta]} \varphi'_\alpha(k-l) \left( \frac{l-k}{N} \right)^2 f''(c_{k,l,N}) \right) \leq K'N^{2\alpha-2}(N\delta)^2 \|f''\|_\infty = O(N^{2\alpha-2\gamma}) = o(1).
\]

Always since \( \varphi_\alpha(0) = 0 \) we obtain as for the case \( 0 < \alpha < \frac{1}{2} \)

\[
N^{2\alpha} \sum_{l=0}^{N} \varphi_\alpha(k-l) f \left( \frac{l}{N} \right) = \frac{C_\alpha}{N} \left( \sum_{l=0}^{k-[N\delta]-1} \varphi_\alpha(k-l) \left( f \left( \frac{l}{N} \right) - f \left( \frac{k}{N} \right) \right) \right)
\]

\[
+ \sum_{k+[N\delta]+1}^{N} \varphi_\alpha(k-l) \left( f \left( \frac{l}{N} \right) - f \left( \frac{k}{N} \right) \right)
\]

\[
- \left( \frac{x^{-2\alpha}}{2\alpha} + \frac{(1-x)^{-2\alpha}}{2\alpha} \right) f(x) + o(1).
\]

Since \( P.V \int_0^1 \frac{f(t)-f(x)}{|t-x|^{2\alpha+1}} dt \) is convergent for \( f \in C^2([0,1]) \) we obtain

\[
N^{2\alpha} \sum_{l=0}^{N} \varphi_\alpha(k-l) f \left( \frac{l}{N} \right) = C_\alpha \left( P.V \int_0^1 \frac{f(t)-f(x)}{|t-x|^{2\alpha+1}} dt - \left( \frac{x^{-2\alpha}}{2\alpha} + \frac{(1-x)^{-2\alpha}}{2\alpha} \right) f(x) \right)
\]

\[
= C_\alpha \left( P.V \int_0^1 \frac{f(x)-f(t)}{|t-x|^{2\alpha+1}} dt + \left( \frac{x^{-2\alpha}}{2\alpha} + \frac{(1-x)^{-2\alpha}}{2\alpha} \right) f(x) \right)
\]

that is the expected result.

3 Demonstration of the theorems 2 and 3

In the following we denote by \( g_\alpha \) the function defined for \( \theta \in [0,2\pi] \) by \( \theta \mapsto (1-e^{i\theta})^\alpha \), \( 0 < \alpha < \frac{1}{2} \) and \( \beta_\alpha^u \) will be the Fourier coefficient \( g_\alpha^{-1}(u) \) for \( u \in \mathbb{N} \). It is known that for a sufficiently large integer \( u \) we have \( \beta_\alpha^u = \frac{u^{\alpha-1}}{1^{(\alpha)}} + o(u^{\alpha-1}) \). In the demonstration we use the predictor polynomial of the functions \( \varphi_\alpha \), \( \alpha \in ]0, \frac{1}{2}[ \) and an expression of their coefficients which has been obtained in a previous work. In the following section the reader will find some reminders about these results.
3.1 Predictor polynomials of $\varphi_\alpha$

First we have to recall the definition of a predictor polynomial of the function $f$.

**Définition 1** If $h$ is an integrable positive function with have only a finite number of zeros on $[0,2\pi[$ the predictor polynomial of degree $M$ of $h$ is the trigonometric polynomial defined by

$$P_M = \frac{1}{\sqrt{(T_N(h))^{-1}_{(1,1)}}} \sum_{u=0}^{M} (T_N(h))^{-1}_{(u+1,1)} \chi^u.$$ 

These predictor polynomials are closely related to the orthogonal polynomials $\Phi_{M_1,M_2} \in \mathbb{N}$ with respect to the weight $h$ by the relation $\Phi_{M_1}(z) = z^M P_M(z)$, for $|z| = 1$. This relation and the classic results on the orthogonal polynomial imply that $P_M(e^{i\theta}) \neq 0$ for all real $\theta$. In the proof we will also need to use the fundamental property:

**Theorem 6** If $P_M$ is the predictor polynomial of a function $h$ then

$$\frac{1}{|P_M|^2}(s) = \hat{h}(s) \quad \forall s \quad -M \leq s \leq M.$$ 

That provides

$$T_M \left( \frac{1}{|P_M|^2} \right) = T_M(h).$$ 

[15] is a good reference about the predictor polynomials. In the proof of the lemma[11] we use the coefficients of the predictor polynomials of the functions $\varphi_\alpha$. The expression we need and use is an exact expression and has been obtained in[19]. Than k of the results of this last paper we can write,

$$\forall k,l \in [0,N] \quad (T_N \varphi_\alpha)^{-1}_{k+1,1} = \beta_k^\alpha - \frac{1}{N} \sum_{u=0}^{k} \beta_{k-u}^\alpha F_{\alpha,N}(\frac{u}{N}).$$ (8)

For $z \in [0,1]$ the quantity $F_{\alpha,N}(z)$ is defined by

$$F_{\alpha,N}(z) = \sum_{m=0}^{+\infty} F_{m,N,\alpha}(z) \left( \frac{\sin \pi \alpha}{\pi} \right)^{2m+2}$$ (9)

where

$$F_{m,N,\alpha}(z) = \sum_{w_0=0}^{+\infty} \frac{1}{1 + w_0 + \frac{1+\alpha}{N}} \sum_{w_1=0}^{+\infty} \frac{1}{w_0 + w_1 + N + 1 + \alpha} \times \ldots$$

$$\ldots \sum_{w_{2m-1}=0}^{+\infty} \frac{1}{w_{2m-2} + w_{2m-1} + N + 1 + \alpha}$$

$$\sum_{w_{2m}=0}^{+\infty} \frac{1}{w_{2m-1} + w_{2m} + N + 1 + \alpha} \frac{1}{1 + \frac{w_{2m}}{N} + \frac{1+\alpha}{N} - z}.$$
Using integrals we can bounded these sums by
\[
\tilde{F}_{m,N,\alpha}(z,z') = \int_0^{+\infty} \int_0^{+\infty} \frac{1}{1 + t_0 + \frac{1 + \alpha}{N}} dt_0 \frac{1}{1 + t_0 + t_1} \times \cdots \\
\int_0^{+\infty} \int_0^{+\infty} \frac{1}{1 + t_{2m-2} + t_{2m-1}} dt_{2m-2} \frac{1}{1 + t_{2m-1} + t_{2m}} dt_{2m-1} \cdots dt_2 dt_1 \\
\int_0^{+\infty} \frac{1}{1 + t_2 m - 2} dt_2 dt_1 \cdots dt_2 m dt_2 m.
\]
and we obtain the following upper bound, that we use the demonstration
\[
\forall z \in [0, 1] \quad |F_{N,\alpha}(z)| \leq K_0 \left( 1 + \left| \ln \left( 1 - z + \frac{1 + \alpha}{N} \right) \right| \right).
\]
(10)

3.2 Existence of the solution

Let us recall the following formula which is an adaptation of the Gohberg-Semencul formula.

**Property 1** Let \( K_N = \sum_{u=0}^{N} \omega_u \chi^u \) be a trigonometric polynomial of degree \( N \) such that \( K_N(e^{i\theta}) \neq 0 \) for all \( \theta \in \mathbb{R} \). We have, for \( 0 \leq k \leq l \leq N \)
\[
T_N \left( \frac{1}{|K_N|^2} \right)^{-1} = \sum_{u=0}^{k} \omega_{k-u} \bar{\omega}_{l-u} - \sum_{u=0}^{k} \omega_{u+N-l} \bar{\omega}_{u+N-k}.
\]

In the next of the proof we apply this property to \( P_{N,\alpha} \) the predictor polynomial of \( \varphi_\alpha \), and we denote by \( \gamma_a^\alpha \) the coefficients of \( P_{N,\alpha} \).

The demonstration of the lemma is based on the remark that for \( k \) such that \( k = \lfloor Nx \rfloor, x \in [0, 1[ \),
\[
\sum_{l=0}^{N} (T_N(\varphi_\alpha))_{k+1,l+1} \sum_{m=0}^{N} (T_N^{-1}(\varphi_\alpha))_{l+1,m+1} f\left( \frac{m}{N} \right) = f\left( \frac{k}{N} \right).
\]
(11)

We consider now the term
\[
\Phi_N(l) = \sum_{m=0}^{N} (T_N^{-1}(\varphi_\alpha))_{l+1,m+1} f\left( \frac{m}{N} \right) = \sum_{m=\lfloor Nl \rfloor}^{\lfloor Nl' \rfloor} (T_N^{-1}(\varphi_\alpha))_{l+1,m+1} f\left( \frac{m}{N} \right).
\]

Then (11) can be written
\[
\sum_{l=0}^{N} (T_N(\varphi_\alpha))_{k+1,l+1} \Phi_N(l) = f\left( \frac{k}{N} \right).
\]
(12)

We have now to use the following lemma

**Lemma 1** Let \( \delta_1, \delta_2 \) be two reals in \( [0, 1[ \), and \( f \) a locally contracting function on \( [0, 1[ \). Then for all reals \( \delta_1, \delta_2 \) such that \( 0 < \delta_1 < \delta_2 < 1 \) there is a positive constant \( M \) such that \( \frac{1}{N}, \frac{l'}{N} \in [\delta_1, \delta_2] \) we have
\[
|\Phi_N(l) - \Phi_N(l')| \leq M \left| \frac{l}{N} - \frac{l'}{N} \right| N^{2\alpha}.
\]
uniformly in \( l, l' \) in \([N\delta_1, N\delta_2]\).
This lemma is shown in the appendix of this article. By the same methods as in the proof of the theorem 1, for \( \alpha \in [0, \frac{1}{2}] \) we get, for a good choice of the real \( \delta = N^{-\beta} \)

\[
\sum_{l=0}^{N} \tilde{\phi}_\alpha(k-l) \tilde{\Phi}_N(l) = -C_\alpha \sum_{l=0}^{k-N\delta+1} |k-l|^{-2\alpha-1} \left( \tilde{\Phi}_N(l) - \tilde{\Phi}_N(k) \right) + C_\alpha \tilde{\Phi}_N(k) \left( \sum_{l<0} |k-l|^{-2\alpha-1} + \sum_{l>N} |k-l|^{-2\alpha-1} \right) + o(1).
\]

\( \forall y \in [0,1] \) we put

\[
\tilde{\Phi}(y) = \int_0^1 H_\alpha(y,z) f(z) dz
\]

with \( H_\alpha(u,v) = C_\alpha |u-v|^{2\alpha-1} - K_\alpha(u,v) \) for \( u, v \in [0,1] \). We have now to prove the following property

**Property 2**

1. For a good choice of \( \delta \) and for \( k = \lfloor Nx \rfloor \), \( 0 < x < 1 \) we have,

   \( (a) \)

   \[
   \sum_{l=0}^{k-N\delta} |k-l|^{-2\alpha-1} \left( \tilde{\Phi}_N(l) - \tilde{\Phi}_N(k) \right) = \int_0^{x-\delta} \frac{\tilde{\Phi}(y) - \tilde{\Phi}(x)}{|x-y|^{2\alpha+1}} dy + o(1).
   \]

   \( (b) \)

   \[
   \sum_{l=N+1}^{N+1} |k-l|^{-2\alpha-1} \left( \tilde{\Phi}_N(l) - \tilde{\Phi}_N(k) \right) = \int_{x+\delta}^{1} \frac{\tilde{\Phi}(y) - \tilde{\Phi}(x)}{|x-y|^{2\alpha+1}} dy + o(1).
   \]

2. For \( k = \lfloor Nx \rfloor \) \( 0 < x < 1 \) we can write uniformly in \( x \in [\delta_1, \delta_2] \subset [0,1] \),

\[
\tilde{\Phi}_N(k) \left( \sum_{l<0} |k-l|^{-2\alpha-1} + \sum_{l>N} |k-l|^{-2\alpha-1} \right) = \tilde{\Phi}(x) \left( \int_{-\infty}^{0} |x-y|^{-2\alpha-1} dy + \int_{1}^{+\infty} |x-y|^{-2\alpha-1} dy \right) + o(1)
\]

We have now to prove the property 2.

**Proof of the property 2:** First we consider an interval \([\delta_1, 1 - \delta_1]\) which contains \([a, b] = \text{Supp}f\) and such that \( k \in [\delta_1, 1 - \delta_1]\). For bound \( \left| (T_N^{-1}(\varphi_\alpha))_{l+1,m+1} \right| \) when \( l \leq N\delta_1 \) and \( m \in [Na, Nb] \) we will use the following lemma that we have demonstrated in [17].

**Lemma 2** Let \( \delta_0 \) be a positive real and \( \alpha \in (0, \frac{1}{2}]. \) Then we have a constant \( K_{1,\alpha} \) depending only from \( \alpha \) such that, for a sufficiently large \( N \),

\[
(T_N^{-1}(\varphi_\alpha))_{k+l+1} \leq K_{1,\alpha} |l - k|^{-\alpha-1} N^\alpha \delta_0^{\alpha/2}
\]

for all \( (k,l) \in \mathbb{N}^2 \) with \( 0 \leq \min(k,l) < N\delta \) and \( 2N\delta_0 \leq \max(k,l) < N - 2N\delta_0. \)
This lemma gives us, for $0 \leq l \leq N\delta_1$ and $m \in [Na, Nb]$,
\[
\left| (T_N(\varphi_{\alpha}))^{-1}_{l+1,m+1} \right| = O \left( N^{2\alpha-1}\delta_1^{\alpha/2} \right),
\]
and, by the symmetries of the matrix $(T_N(\varphi_{\alpha}))^{-1}$ we have also, for $N(1 - \delta_1) < l \leq N$ and $m \in [Na, Nb]$,
\[
\left| (T_N(\varphi_{\alpha}))^{-1}_{l+1,m+1} \right| = O \left( N^{2\alpha-1}\delta_1^{\alpha/2} \right),
\]
On the other hand the theorems 4 or 5 provide, for all $m \in [Na, Nb]$,
\[
\left| (T_N^{-1}(\varphi_{\alpha}))_{k+1,m+1} \right| = O \left( N^{2\alpha-1} \right).
\]
Hence we can say that for $0 \leq l \leq N\delta_1 \leq N$ or $N(1 - \delta_1) \leq l \leq N$ we have $|\tilde{\Phi}_N(l)| \leq O(N^{2\alpha}\delta_1^{\alpha/2})$.

Then
1. 
\[
\sum_{l=0}^{N\delta_1} \left| k - l \right|^{-2\alpha-1}\tilde{\Phi}_N(l) \leq O(\delta_1^{3\alpha/2}) \quad \text{and} \quad \sum_{l=N(1-\delta_1)}^{N} \left| k - l \right|^{-2\alpha-1}\tilde{\Phi}_N(l) \leq O(\delta_1^{3\alpha/2}).
\]
2. 
\[
\sum_{l=0}^{N\delta_1} \left| k - l \right|^{-2\alpha-1}\tilde{\Phi}_N(k) \leq O(\delta_1).
\]

Hence we can write, for $\delta_1$ sufficiently small,
\[
\sum_{l=0}^{k-N\delta} \left| k - l \right|^{-2\alpha-1}(\tilde{\Phi}_N(l) - \tilde{\Phi}_N(k)) = \sum_{l=N\delta_1+1}^{k-N\delta} \left| k - l \right|^{-2\alpha-1}(\tilde{\Phi}_N(l) - \tilde{\Phi}_N(k)) + O(\delta_1^{\alpha/2})
\]
and
\[
\sum_{l=k+N\delta}^{N} \left| k - l \right|^{-2\alpha-1}(\tilde{\Phi}_N(l) - \tilde{\Phi}_N(k)) = \sum_{l=k+N\delta}^{N(1-\delta_1)} \left| k - l \right|^{-2\alpha-1}(\tilde{\Phi}_N(l) - \tilde{\Phi}_N(k)) + O(\delta_1^{\alpha/2})
\]

With the uniformity in the theorem 4 we obtain
\[
\sum_{l=N\delta_1+1}^{k-N\delta} \left| k - l \right|^{-2\alpha-1}(\tilde{\Phi}_N(l) - \tilde{\Phi}_N(k)) = \sum_{l=N\delta_1+1}^{k-N\delta} \left| k - l \right|^{-2\alpha-1}N^{2\alpha} \left( \int_{a}^{b} \left( H_{\alpha}(t, \frac{l}{N}) - H_{\alpha}(t, \frac{k}{N}) \right) f(t)dt \right) + o(1)
\]
and
\[
\sum_{l=k+N\delta}^{N(1-\delta_1)} \left| k - l \right|^{-2\alpha-1}(\tilde{\Phi}_N(l) - \tilde{\Phi}_N(k)) = \sum_{l=k+N\delta}^{N(1-\delta_1)} \left| k - l \right|^{-2\alpha-1}N^{2\alpha} \left( \int_{a}^{b} \left( H_{\alpha}(t, \frac{l}{N}) - H_{\alpha}(t, \frac{k}{N}) \right) f(t)dt \right) + o(1)
\]
To conclude we obtain
\[
\sum_{l=N\delta+1}^{k-N\delta} |k-l|^{-2\alpha-1} N^{2\alpha} \int_{a}^{b} \left( H_{\alpha}(t, l/N) - H_{\alpha}(t, k/N) \right) f(t) dt = \int_{\delta_1}^{x-\delta} (x-y)^{-2\alpha-1} \int_{a}^{b} (H_{\alpha}(t, y) - H_{\alpha}(t, x)) f(t) dt dy + o(1).
\]

that is also,
\[
\sum_{l=0}^{k-N\delta} |k-l|^{-2\alpha-1} N^{2\alpha} \int_{a}^{b} \left( H_{\alpha}(t, l/N) - H_{\alpha}(t, k/N) \right) f(t) dt = \int_{0}^{x-\delta} (x-y)^{-2\alpha-1} \int_{0}^{1} (H_{\alpha}(t, y) - H_{\alpha}(t, x)) f(t) dt dy + o(1).
\]

Since the function \( t \to H(t, z) \) is in \( L^1[0,1] \) for all \( z \in [0,1] \) the beginning of the proof implies that for \( \delta_1 \to 0 \),
\[
\sum_{l=0}^{k-N\delta} |k-l|^{-2\alpha-1} \left( \Phi_N(l) - \Phi_N(k) \right) = \int_{0}^{x-\delta} (x-y)^{-2\alpha-1} \int_{0}^{1} (H_{\alpha}(t, y) - H_{\alpha}(t, x)) f(t) dt dy + o(1).
\]

and identically
\[
\sum_{l=k+N\delta}^{N} |k-l|^{-2\alpha-1} \left( \Phi_N(l) - \Phi_N(k) \right) = \int_{x+\delta}^{1} (x-y)^{-2\alpha-1} \int_{0}^{1} (H_{\alpha}(t, y) - H_{\alpha}(t, x)) f(t) dt dy + o(1).
\]

The second point of the property is clear.

It is now a direct consequence of the property (2) and of the equality (5) is that for \( x \in ]0,1[ \)
\[
f(x) = -C_{\alpha} \left( P.V. \int_{0}^{1} \frac{\Phi(y) - \Phi(x)}{|x-y|^{2\alpha-1}} dy - \frac{\Phi(x)}{-2\alpha} (x^{-2\alpha} + (1-x)^{-2\alpha}) \right), \quad (13)
\]

A direct consequence of the lemma (1) and is that the function \( \tilde{\Phi} \) is locally contracting the equation (13) implies
\[
f(x) = \left( (-\Delta)^{\alpha}(\tilde{\Phi}) \right)(x). \quad (14)
\]

To ends the demonstration we have to notice that for \( y \in ]0,1[ \) we have the following relation
\[
\tilde{\Phi}(y) = C_{-\alpha} \int_{0}^{1} \frac{f(z)}{|y-z|^{2\alpha-1}} dz - \int_{0}^{1} K_{\alpha}(y, z)dz \text{ that is also } \tilde{\Phi}(y) = (D_{-\alpha}(f))(y) - \int_{0}^{1} K_{\alpha}(y, z)dz.
\]
3.3 Proof of the theorem 3

Now we have to involve the generalized Green kernel \( G_\alpha \) to obtain the theorem (3). For \( x, y \in [\delta_1, \delta_2] \subset ]0,1[ \) the theorems (4) provides the equality

\[
N^{2\alpha - 1} G_\alpha (x, y) f(y) + o(N^{2\alpha - 1}) = N^{2\alpha - 1} C_{-\alpha} |x - y|^{2\alpha - 1} f(y) - N^{2\alpha - 1} K_\alpha (x, y) f(y)
\]

for \( x, y \in [\delta_1, \delta_2] \subset ]0,1[, x \neq y \). That means

\[
G_\alpha (x, y) f(y) + o(1) = |x - y|^{2\alpha - 1} f(y) - K_\alpha (x, y) f(y).
\]

Since for a fixed \( x \) the functions \( y \mapsto G(x, y) \), and \( y \mapsto K_\alpha (x, y) \) are in \( L^1([0,1]) \) we can write, for \( x, y \in [\delta_1, \delta_2] \subset ]0,1[, x \neq y \),

\[
\int_0^1 G_\alpha (x, y) f(y) dy = \int_0^1 C_{-\alpha} \frac{f(y)}{|x - y|^{1 - 2\alpha}} dy - \int_0^1 K_\alpha (x, y) f(y) dy
\]

That is also

\[
\int_0^1 G_\alpha (x, y) f(y) dy = (D_\alpha (f)) (x) - \int_0^1 K_\alpha (x, y) f(y) dy.
\]

That ends the proof.

3.4 Unicity of the solution

Let \( f \) and \( \phi \) be two functions defined on \([0,1]\), with \( \phi \) a locally contracting on \([0,1]\) such that \( \Phi(0) = \Phi(1) = 0 \). Moreover we assume than \( f \) is a contracting function on \([0,1]\) with \( \text{Supp}(f) = [a, b] \subset ]0,1[ \). We make the hypotheses \( (-\Delta)^\alpha \phi = f \). The aim of this part is to prove that the function \( \phi \) checks the theorem assumptions on \([0,1]\).

For fixed \( N \) we define the vector \( X_N \) (resp. \( Y_N \)) of length \( N + 1 \) by \( (X_N)_k = \phi (\frac{k}{N}) \) (resp. \( (Y_N)_k = f (\frac{k}{N}) \)). For a sufficiently large \( N \) we can write \( N^{2\alpha} (T_N (\varphi_\alpha) (X_N))_k = (Y_N)_k + (R_N)_k \), that gives us, for \( 0 \leq m \leq N \)

\[
(X_N)_m = N^{-2\alpha} (T_N (\varphi_\alpha)^{-1} (Y_N))_m + (T_N (\varphi_\alpha)^{-1} (R_N))_m.
\]

First we have to compute \( N^{2\alpha} (T_N (\varphi_\alpha) (X_N))_k \) to evaluate precisely the order of \( |R_N(k)| \).

To do this we have to recall that for all positive real \( \epsilon \) we have an integer \( M_0 \) such that for \( |u| \geq M_0 \) \( \varphi_\alpha (u) = C_\alpha u^{-2\alpha - 1} + C_\alpha' u^{-2\alpha - 2} (1 + r_u) \) with \( |r_u| \leq \epsilon \). In the following of the proof we denote by \( M \) an integer \( M = N \epsilon \) with \( \epsilon = N^{-\beta}, 2\alpha < \beta < 1 \), these conditions being set to be consistent with the demonstration of the theorem 4. Now let \( k \) such that \( \frac{k}{N} \in [\delta_1, 1 - \delta_1] \), with \( 0 < \delta_1 < 1 - \delta_1 < 1 \). According to the proof of the theorem 4 we have to consider the seven following quantities.

1. The sum

\[
N^{2\alpha} \left( \sum_{l=k-M}^{k+M} (T_N \varphi_\alpha)_{k+1,l+1} \left( \phi (l) - \phi (k) \right) \right)
\]

which is \( O(N^{-\beta + 2\alpha}) = o(1) \).
2. The difference

\[ N^{2\alpha} \sum_{l=0}^{k-M-1} (T_N \varphi_\alpha)_{k+1,l+1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) - N^{2\alpha} C_\alpha \sum_{l=0}^{k-M-1} |k-l|^{-2\alpha-1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) \]

which is bounded by

\[ N^{2\alpha} \sum_{l=0}^{k-M-1} O(|k-l|^{-2\alpha-2}) \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) = O \left( N^{-1} \int_{0}^{x} \frac{\Phi(t) - \Phi(x)}{|t-x|^{2\alpha+1}} \right) = o(1). \]

3. Identically

\[ N^{2\alpha} \sum_{l=k+M}^{N} (T_N \varphi_\alpha)_{k+1,l+1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) - N^{2\alpha} C_\alpha \sum_{l=k+M}^{N} |k-l|^{-2\alpha-1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) = O(N^{-1}) = o(1). \]

4. The difference between the quantity

\[ N^{2\alpha} C_\alpha \sum_{0 \leq l \leq k-M,N \geq l \geq k+M} |k-l|^{-2\alpha-1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) \]

and the quantity

\[ N^{2\alpha} \left( \int_{0}^{k-M} |k-l|^{-2\alpha-1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) dl + \int_{k+M}^{N} |k-l|^{-2\alpha-1} \left( \phi \left( \frac{l}{N} \right) - \phi \left( \frac{k}{N} \right) \right) dl \right) \]

is in \( O(k^{-2\alpha-1} |\Phi(x)| + N^{-1+2\alpha} \beta) = O(N^{-2\alpha-1}) = o(1) \) (residual term of the Euler and Mac-Laurin formula).

5. The difference between \( N^{2\alpha} \sum_{l<0} \hat{\varphi}_\alpha(k-l)f \left( \frac{k}{N} \right) \) and \( N^{2\alpha} \sum_{l<0} |k-l|^{-2\alpha-1} f \left( \frac{k}{N} \right) \) is in \( O(N^{-1}) \).

Idem for the difference between \( N^{2\alpha} \sum_{l>0} \hat{\varphi}_\alpha(k-l)f \left( \frac{k}{N} \right) \) and \( N^{2\alpha} \sum_{l>0} |k-l|^{-2\alpha-1} f \left( \frac{k}{N} \right) \)

6. The difference between \( N^{2\alpha} \sum_{l<0} |k-l|^{-2\alpha-1} f \left( \frac{k}{N} \right) \) and \( \int_{0}^{x} |x-t|^{-2\alpha-1} f(x)dt \) is in \( O(N^{-1}) \)

(residual term of the Euler Mac-Laurin formula). Idem for \( N^{2\alpha} \sum_{l>0} |k-l|^{-2\alpha-1} f \left( \frac{k}{N} \right) \) and \( \int_{1}^{\infty} |x-t|^{-2\alpha-1} f(x)dt \)

7. Finally the difference between the integrals of the previous point and \( \int_{0}^{1} \frac{\phi(t) - \phi(x)}{|t-x|^{2\alpha+1}} dt \) is \( O \left( N^{\beta(2\alpha-1)} \right) = o(1) \).

Thanks to these results we can affirm the property

**Property 3** for all integer \( k, M \leq k \leq N - M \) \( R_N(k) = o(1) \), uniformly in \( k \).
On the other hand if $0 \leq k \leq N\delta_1$ or $N - N\delta_1 \leq k \leq N$

$$R_N(k) = f\left(\frac{k}{N}\right) - \sum_{l=0}^{N} (T_N(\varphi_\alpha))_{k+1,l+1} \phi\left(\frac{l}{N}\right)$$

which is bounded by a constant $K_0$ no depending from $N$.

Now we have to study

$$S_m = N^{-2\alpha} \sum_{k=0}^{N} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} (R_N)_k$$

for $m \in \mathbb{N}$ such that $\lim_{N \to \infty} \frac{m}{N} = x$ and $x \in [0, 1[$. To evaluate $S_m$ we will use the lemma $[2]$ the property $[3]$ and the theorem $[4]$. Then we split the sum $S_m$ as follows

$$S_m = N^{-2\alpha} \left( \sum_{k=0}^{N-N\delta_2} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} R_N(k) ight) + \sum_{k=N-N\delta_2}^{N-N\delta_2} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} R_N(k)$$

$$+ \sum_{k=N-N\delta_2}^{N-N\delta_2} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} R_N(k)$$

where $0 < \delta_2 < \frac{m}{N} < 1 - \delta_2 < 1$.

With the theorem $[4]$ and the property $[3]$ we get

$$N^{-2\alpha} \sum_{k=N-N\delta_2}^{N-N\delta_2} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} R_N(k) = O(N^{-\tau}) = o(1).$$

Then the lemma $[2]$ and the symmetries of the matrix $(T_N(\varphi_\alpha))^{-1}$ provides us that the sums

$$N^{-2\alpha} \sum_{k=0}^{N-N\delta_2-1} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} R_N(k),$$

and

$$N^{-2\alpha} \sum_{k=N-N\delta_2+1}^{N} (T_N(\varphi_\alpha))_{m+1,k+1}^{-1} R_N(k),$$

are respectively of same order that the quantities

$$N^{-2\alpha} \sum_{k=0}^{N-N\delta_2-1} |m - k|^{\alpha-1} N^\alpha \delta_2^{\alpha/2} = O(\delta_2^{\alpha/2})$$

and

$$N^{-2\alpha} \sum_{k=N-N\delta_2+1}^{N} |m - k|^{\alpha-1} N^\alpha \delta_2^{\alpha/2} = O(\delta_2^{\alpha/2}).$$

Hence for $\delta_2$ sufficiently close to zero we can bound $S_m$ by $\epsilon \to 0$. That means that, for considered $m$ such that $\lim_{N \to \infty} \frac{m}{N} = x, x \in [0, 1[$ we have

$$\phi(x) = \lim_{N \to +\infty} N^{-2\alpha} \left( (T_N(\varphi_\alpha))^{-1} Y_N \right)_m = \lim_{N \to +\infty} N^{-2\alpha} \sum_{k=0}^{N} \sum_{k=0}^{N} (T_N(\varphi_\alpha))^{-1}_{m+1,k+1} f\left(\frac{k}{N}\right).$$

And, by repeating a calculation already done, with the notations used in the proof of the existence of a solution

$$\lim_{N \to +\infty} N^{-2\alpha} \sum_{k=0}^{N} (T_N(\varphi_\alpha))^{-1}_{m+1,k+1} f\left(\frac{k}{N}\right) = \frac{1}{\Gamma^2(\alpha)} \int_0^1 H_\alpha(x, z) f(z) dz.$$

That gives us the announced unicity.
4 Appendix : proof of the lemmas

4.1 Proof of the lemma

According to (1) we have to prove that the both functions $l \mapsto \sum_{m=0}^{N} \sum_{u=0}^{\min(l,m)} \gamma_{\min(l,m)-u}^{\alpha} \gamma_{\max(l,m)-u}^{\alpha}$

and $l \mapsto \sum_{m=0}^{N} \sum_{u=0}^{\min(l,m)} \gamma_{\min(l,m)-u}^{\alpha} \gamma_{\max(l,m)-u}^{\alpha}$ satisfy the property of the lemma.

First we prove the property for the first of these functions. To do this it is clear with (8) that we have to state the three following inequalities

$$|\psi_{1,N}(l) - \psi_{1,N}(l')| \leq M_1|\frac{l}{N} - \frac{l'}{N}|N^{2\alpha},$$

$$|\psi_{2,N}(l) - \psi_{2,N}(l')| \leq M_2|\frac{l}{N} - \frac{l'}{N}|N^{2\alpha},$$

$$|\psi_{3,N}(l) - \psi_{3,N}(l')| \leq M_3|\frac{l}{N} - \frac{l'}{N}|N^{2\alpha},$$

$$|\psi_{4,N}(l) - \psi_{4,N}(l')| \leq M_4|\frac{l}{N} - \frac{l'}{N}|N^{2\alpha}.$$
First we have to prove the lemma for $\psi_{1,N}$. We can write $\psi_{1,N}(l) = \psi_{1,1,N}(l) + \psi_{1,2,N}(l)$ with

$$\psi_{1,1,N}(l) = \sum_{m=0}^{l} \left( \sum_{u=0}^{m} \beta_{m-u} \beta_{l-u} \right) f\left( \frac{m}{N} \right).$$

$$\psi_{1,2,N}(l) = \sum_{m=l}^{N} \left( \sum_{u=0}^{l} \beta_{l-u} \beta_{m-u} \right) f\left( \frac{m}{N} \right).$$

Let be $l' \leq l$ two integers with $l' < l$. We have to bound the difference

$$|\psi_{1,1,N}(l) - \psi_{1,1,N}(l')| \quad \text{for} \quad \frac{l}{N}, \frac{l'}{N} \in [\delta_1, \delta_2].$$

We can write

$$\psi_{1,1,N}(l) = \sum_{m=0}^{l'} \left( \sum_{u=0}^{l'} \beta_{u} \beta_{l-u} \right) f\left( \frac{l-m}{N} \right) + \sum_{m=0}^{l'} \left( \sum_{u=l'-m}^{l-m} \beta_{u} \beta_{l-u} \right) f\left( \frac{l-m}{N} \right)$$

$$+ \sum_{m=l'}^{l} \left( \sum_{u=0}^{l-m} \beta_{u} \beta_{l-u} \right) f\left( \frac{l-m}{N} \right).$$

Then using the asymptotic expansion of the coefficients $\beta_{u}$ and the hypotheses on $f$ we obtain

1.

$$\left| \sum_{m=0}^{l'} \left( \sum_{u=0}^{l'} \beta_{u} \beta_{l-u} \right) f\left( \frac{l-m}{N} \right) - f\left( \frac{l-m}{N} \right) \right|$$

$$\leq K_1 \frac{l-l'}{N} N^{2\alpha} \left( \frac{1}{T^{2}(\alpha)} \int_{0}^{l'/N} \int_{0}^{t'/N-z} t^{\alpha-1}(z+t)^{\alpha-1} dt dz + o(1) \right)$$

$$\leq K_2 \frac{l-l'}{N} N^{2\alpha} \left( \frac{1}{T^{2}(\alpha)} \int_{0}^{1} \int_{0}^{1} t^{\alpha-1}(z+t)^{\alpha-1} dt dz + o(1) \right)$$

2.

$$\left| \sum_{m=0}^{l'} \left( \sum_{u=l'-m}^{l-m} \beta_{u} \beta_{l-u} \right) f\left( \frac{l-m}{N} \right) \right| \leq K_3 N^{2\alpha-1} \frac{l-l'}{N} \sum_{m=0}^{l} \left( \frac{l-m}{N} \right)^{\alpha-1} f\left( \frac{l-m}{N} \right)$$

$$\leq K_4 N^{2\alpha} \frac{l-l'}{N} \|f\|_{\infty}$$

3. Lastly since the function $f$ is contracting we obtain

$$\left| \sum_{m=l'}^{l} \left( \sum_{u=0}^{l-m} \beta_{u} \beta_{l-u} \right) f\left( \frac{l-m}{N} \right) \right| \leq K_5 \frac{l-l'}{N} \sum_{m=l'}^{l} \sum_{u=0}^{l-m} \beta_{u} \beta_{l-u} \|f\|_{\infty}$$

$$\leq K_6 \frac{l-l'}{N} N^{2\alpha}$$
with $K_1, \ldots, K_6$ which are not dependent on $N$. So we can conclude that we have a positive constant $M$ such that $|\psi_{1,1,N}(l) - \psi_{1,1,N}(l')| \leq M \frac{l - l'}{N} N^{2\alpha}$ for $\frac{1}{N}, \frac{l'}{N} \in [\delta_1, \delta_2]$. To get the same result for $\psi_{1,2,N}$ we introduce decomposition

$$\psi_{1,2,N}(l) = \sum_{m=0}^{N-l'} \sum_{\nu=0}^{l'} \beta_{\nu}^\alpha \beta_{m+\nu}^\alpha f\left(\frac{m-l}{N}\right) - \sum_{m=N-l}^{N-l'} \sum_{\nu=0}^{l'} \beta_{\nu}^\alpha \beta_{m+\nu}^\alpha f\left(\frac{m-l}{N}\right)$$

$$+ \sum_{m=0}^{N-l} \sum_{\nu=\nu'+1}^{l} \beta_{\nu}^\alpha \beta_{m+\nu}^\alpha f\left(\frac{m-l}{N}\right)$$

Then the same methods as above allow us to conclude.

Let’s now show the lemma for the difference $|\psi_{2,N}(l) - \psi_{2,N}(l')|$. As previously we can split $\psi_{2,N}(l)$ in $\psi_{2,1,N}(l) + \psi_{2,2,N}(l)$ with

$$\psi_{2,1,N}(l) = \sum_{m=0}^{l} \left(\sum_{u=0}^{l-m} \beta_{m-u}^\alpha \sum_{v=0}^{l-u} \beta_{l-u-v}^\alpha F_{N,\alpha}(\frac{v}{N})\right) f\left(\frac{m}{N}\right)$$

and

$$\psi_{2,2,N}(l) = \sum_{m=l}^{N} \left(\sum_{u=0}^{l-m} \beta_{m-u}^\alpha \sum_{v=0}^{l-u} \beta_{l-u-v}^\alpha F_{N,\alpha}(\frac{v}{N})\right) f\left(\frac{m}{N}\right).$$

First $\psi_{2,1,N}(l)$ is also

$$\psi_{2,1,N}(l) = \sum_{m=0}^{l} \left(\sum_{u=0}^{l-m} \beta_{m-u}^\alpha \sum_{v=0}^{l-u} \beta_{l-u-v}^\alpha F_{N,\alpha}(\frac{m+u-v}{N})\right) f\left(\frac{l-m}{N}\right).$$

Finally we have (always for $l' < l$)

$$\psi_{2,1,N}(l) = \sum_{m=0}^{l'} f\left(\frac{l-m}{N}\right) \left(\sum_{u=0}^{l'-m} \beta_{m-u}^\alpha \sum_{v=0}^{l'-u} \beta_{l'-u-v}^\alpha F_{N,\alpha}(\frac{m+u-v}{N})\right)$$

$$+ \sum_{m=0}^{l} f\left(\frac{l-m}{N}\right) \sum_{u=0}^{l-m} \beta_{m+u}^\alpha \left(\sum_{v=0}^{l-m-u} \beta_{l-m-u-v}^\alpha F_{N,\alpha}(\frac{m+u-v}{N})\right)$$

$$+ \sum_{m=l'}^{l} f\left(\frac{l-m}{N}\right) \sum_{u=0}^{l-m} \beta_{m+u}^\alpha \sum_{v=0}^{l-m-u} \beta_{l-m-u-v}^\alpha F_{N,\alpha}(\frac{m+u-v}{N})$$

Hence to bound the difference $|\psi_{2,1,N}(l) - \psi_{2,1,N}(l')|$ with $l' < l$ we have three terms to consider.

1. First

$$\sum_{m=0}^{l'} \left(f\left(\frac{l-m}{N}\right) - f\left(\frac{l'-m}{N}\right)\right) \left(\sum_{u=0}^{l'-m} \beta_{m-u}^\alpha \sum_{v=0}^{l'-u} \beta_{l'-u-v}^\alpha F_{N,\alpha}(\frac{m+u-v}{N})\right)$$

using (10) we can bound this term by

$$K_7 \frac{l-l'}{N} N^{2\alpha} \frac{1}{N} \sum_{m=0}^{l'} \left(\frac{l'-m}{N}\right)^\alpha \left(\frac{l'}{N}\right)^\alpha |\ln(1 - \delta_2)| = K_8 \frac{l-l'}{N} N^{2\alpha}. $$
2. Then we have to study

\[
\sum_{m=0}^{l'} f\left(\frac{l-m}{N}\right) \sum_{u=0}^{l'-m} \frac{\beta_u}{N} \left(\sum_{v=0}^{m+u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{m+u-v}{N})\right)
\]

Always with (10) we can bound this quantity by

\[
K_0 \frac{l-l'}{N} \sum_{m=0}^{l'} f\left(\frac{l-m}{N}\right) N^{2\alpha-1} \left(\frac{l-m}{N}\right)^{\alpha-1} \left(\frac{l}{N}\right)^{\alpha} |\ln(1-\delta_2)| = K_{10} \frac{l-l'}{N} N^{2\alpha}
\]

Lastly we have to consider

\[
\sum_{m=l'}^{l} f\left(\frac{l-m}{N}\right) \sum_{u=0}^{l-m} \frac{\beta_u}{N} \sum_{v=0}^{m+u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{m+u-v}{N}).
\]

Always with the additional remark that \( f \) is contracting on \([0,1]\) we can bound this quantity by

\[
K_{11}\left(\frac{l-l'}{N}\right)^2 N^{2\alpha} \left(\int_0^1 t^{\alpha-1} dt\right)^2 |\ln(1-\delta_2)| = K_{12} N^{2\alpha}\left(\frac{l-l'}{N}\right).
\]

Since the quantities \( K_7, \cdots, K_{12} \) are not dependent on \( N \) we have got the property for this case. To treat the difference \(|\psi_{2,2,N}(l) - \psi_{2,2,N}(l')|\) we write, always for \( l' < l \):

\[
\psi_{2,2,N}(l) = \sum_{m=0}^{N-l} f\left(\frac{m-l}{N}\right) \left(\sum_{u=0}^{l} \frac{\beta_u}{N} \sum_{v=0}^{m+u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{v}{N})\right)
\]

\[
- \sum_{m=N-l}^{N-l'} f\left(\frac{m-l}{N}\right) \left(\sum_{u=0}^{l} \sum_{v=0}^{m+u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{v}{N})\right)
\]

\[
+ \sum_{m=0}^{N} f\left(\frac{m-l'}{N}\right) \left(\sum_{u=0}^{l'} \sum_{v=0}^{m+u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{v}{N})\right)
\]

And the same methods as for \( \psi_{2,1,N} \) provides that

\[
\left|\sum_{m=l}^{N} \left(\sum_{u=0}^{l} \frac{\beta_u}{N} \sum_{v=0}^{m+l-u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{v}{N})\right) f\left(\frac{m}{N}\right)\right|
\]

\[
- \sum_{m'=l'}^{N} \left(\sum_{u=0}^{l'} \frac{\beta_u}{N} \sum_{v=0}^{m'+l-u} \beta_{m+l-u-v} F_{N,\alpha}(\frac{v}{N})\right) f\left(\frac{m'}{N}\right)\right| \leq O(N^{2\alpha}\frac{l-l'}{N})
\]

For end the proof of the lemma we have to get the lemma for the difference \(|\psi_{3,2,N}(l) - \psi_{3,2,N}(l')|\) and \(|\psi_{4,2,N}(l) - \psi_{4,2,N}(l')|\). These are the same ideas and methods used in the two proofs of the inequalities \(|\psi_{1,2,N}(l) - \psi_{1,2,N}(l')|\) and \(|\psi_{2,2,N}(l) - \psi_{2,2,N}(l')|\). These same methods also make it possible to treat the function \( l \mapsto \sum_{m=0}^{N} \sum_{u=0}^{\min(l,m)} \gamma_{N-l,m-u} F_{N,\alpha}(\frac{m+u-l}{N})\).
5 Declarations

5.1 Availability of data and material
Not applicable

5.2 Competing interests
Not applicable

5.3 Funding
Not applicable

5.4 Authors’ contributions
Not applicable

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