Signal analysis of gravitational waves

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1. Introduction

Signal analysis is an important component in the detection of gravitational waves from astrophysical sources. The primary goal of signal analysis is signal extraction or signal detection. There are several reasons why signal extraction will be a demanding exercise in searching for signatures of gravitational radiation in the output of an interferometric detector. Firstly, an interferometric detector is a wide band instrument, though in principle it can be tuned to gain a larger sensitivity in a narrow band. Typically the bandwidth is $\sim 1$ kHz. In addition to sampling useful data it is necessary to monitor outputs from several channels continuously to assess the performance of the detector. Consequently, the data output rate is expected to be high $\sim 1$ Mbyte s$^{-1}$. Due to the high rate of data output it is necessary to analyse the detector output online as otherwise data storage requirements would be very expensive. Secondly, gravitational wave antenna is more or less an isotropic detector and does not point towards any particular direction. This is an advantage in an important way since at once one is looking for sources almost all over the sky. However, this calls for formidable amount of data processing for sources that have to be integrated for a long duration, such as a pulsar emitting periodic gravitational waves, wherein it would be necessary to correct for Doppler modulations of the signal wave form caused by Earth’s motion. The Doppler correction depends both on the location of the source and frequency of the wave and current estimates show that it would be computationally impossible to search for periodic sources all over the sky and in a wide frequency band, in a data train that lasts a couple of months (Schutz 1996). Such Doppler corrections will be important even in the case of quasi-periodic sources, such as a compact binary consisting of two neutron stars or a neutron star and a white-dwarf, if the detector is sensitive at very low ($< 1$ Hz) frequencies. Thirdly, since there are many candidate astrophysics sources—supernovae, pulsars, coalescing binaries, to name a few—it is essential to synchronously search for radiation from each of these sources. Search algorithms for different sources are not all identical and hence this causes extra burden on data analysis systems. Fourthly, looking for stochastic background of gravitational radiation or for unknown sources, requires cross correlation of two or more detector outputs. Since the light travel time from one detector to another is significantly larger than the time interval between consecutive data samples, correlations will have to be carried out for several distinct overlaps of outputs from different detectors. Finally, even though, as in some cases, the signal wave form may be known accurately, the source parameters would be unknown and the data needs to be filtered through several thousand templates. (For a review on data analysis problems in gravitational wave detection see e.g. Schutz (1989).)

When the detectors are built and start operating, it may very well be that most of the analysis can be carried out just on a state-of-the-art workstation of that time. However, certain searches—notably all-sky, all-frequency search for pulsars which would be an important exercise towards understanding the pulsar population in our Galaxy—appear
to be computationally quite demanding even by the standard of computers expected to be available towards the turn of the century (Schutz 1996). It is therefore essential to work out efficient data analysis algorithms in each case; efficient not only in picking a weak signal from a noisy data but efficient in terms of computing requirements.

The second aspect in signal analysis is the estimation of parameters characterising a gravitational wave signal. After an initial detection has been made it is often possible to deduce the astrophysical quantities of the system emitting radiation, such as the masses and spin angular momenta of stars in a binary system, direction to a pulsar, etc. Such an estimation can ultimately be used in testing theories of gravitation, models of astrophysical systems and cosmology and so on.

In this lecture I will touch upon some aspects of signal analysis techniques that are currently favourite. I will indicate a couple of alternatives in the end but the goal of this lecture is to give you an idea of what is involved in signal processing. Signal analysis is expected to be a laborious exercise only for pulsar searches and, to some extent, coalescing binaries and stochastic background. Aspects related to pulsar searches can be found in lectures by Schutz and those related to stochastic background are dealt by Allen (1996); in this lecture I will only cover issues related to the detection and measurement of coalescing binaries. Throughout this article we shall work with geometrical units: $G = c = 1$.

2. Coalescing binary signal

Highly evolved systems of compact binaries, such as NS-NS, NS-BH \( \dagger \) are amongst promising sources of gravitational radiation for the planned laser interferometric gravitational wave detectors. As a binary system of stars inspirals, due to radiation reaction, the gravitational wave sweeps-up in amplitude and frequency. The resulting inspiral wave form has a characteristic power-law spectrum $f^{-7/3}$ (Thorne 1987) and is often called the chirp or the coalescing binary wave form. The wave form is worked out perturbatively using post-Newtonian (henceforth PN) or post-Minkowskian expansion and the lowest order expression is called the Newtonian or the quadrupole wave form. It turns out that for the purpose of detection of gravitational waves from inspiralling binaries it is sufficient to work with the so called restricted PN wave form (Cutler et al. 1993). In this approximation one incorporates the PN corrections only to the phase of the wave form working always with just the Newtonian amplitude. Going beyond the restricted PN approximation is not expected to change appreciably the magnitude of the statistical errors in the parameter extraction, so the restricted PN approximation can be used to estimate statistical errors as well. However, in the post-detection analysis it is necessary to employ more accurate templates since the use of just the restricted PN wave form would give rise to some systematic errors. This is an issue worth further exploration and we shall see one way of accomplishing this. In this lecture, however, we shall always deal with the restricted PN wave form.

As we shall see below all issues related to signal analysis can be addressed in the Fourier domain. In the stationary phase approximation the Fourier transform of the restricted 2-PN \( \dagger \) chirp wave form (Blanchet et al. 1995) for positive frequencies is given by

\[
\tilde{h}(f) = N f^{-7/6} \exp \left[ i \sum_{k=1}^{6} \psi_k(f) \lambda^k - i \frac{\pi}{4} \right].
\]

\( \dagger \) NS=Neutron Star, BH=Black Hole.

\( \dagger \) The notation used is that $n$-PN corresponds to a term $v^{2n}$ beyond the quadrupole term.
For $f < 0$ the Fourier transform is computed using the identity $\tilde{h}(-f) = \tilde{h}^*(f)$ obeyed by real functions $h(t)$. In the above expression $\psi$’s are functions only of $f$ and do not contain the parameters of the wave form $\lambda_k$,

$$\psi_1 = 2\pi f,$$  \hspace{1cm} (2.2)

$$\psi_2 = -1,$$  \hspace{1cm} (2.3)

$$\psi_3 = \frac{6\pi f_a}{5} \left( \frac{f}{f_a} \right)^{-5/3},$$  \hspace{1cm} (2.4)

$$\psi_4 = 2\pi f_a \left( \frac{f}{f_a} \right)^{-1},$$  \hspace{1cm} (2.5)

$$\psi_5 = -3\pi f_a \left( \frac{f}{f_a} \right)^{-2/3},$$  \hspace{1cm} (2.6)

$$\psi_6 = 6\pi f_a \left( \frac{f}{f_a} \right)^{-1/3},$$  \hspace{1cm} (2.7)

$N$ is a normalisation constant, $\lambda^k, k = 1, \ldots, 6$, represent the signal parameters

$$\lambda^k = \{ t_C, \Phi_C, \tau_0, \tau_1, \tau_{1.5}, \tau_2 \},$$  \hspace{1cm} (2.8)

$t_C$ and $\Phi_C$ are the instant of coalescence of the binary and the phase of the wave form at that instant and $\tau$’s are constants having dimensions of time and are called chirp times: In terms of the total mass $M$ of the two stars and the ratio of the total mass to reduced mass $\eta = \mu/M$ the chirp times are given by

$$\tau_0 = \frac{5}{256\eta M^{5/3} (\pi f_a)^{8/3}},$$  \hspace{1cm} (2.9)

$$\tau_1 = \frac{5}{192(\pi f_a)^2 \eta M \left( \frac{743}{336} + \frac{11}{4} \eta \right)},$$  \hspace{1cm} (2.10)

$$\tau_{1.5} = \frac{\pi}{8(\pi f_a)^{5/3} \eta M^{2/3}}$$  \hspace{1cm} (2.11)

and

$$\tau_2 = \frac{5}{128 (\pi f_a)^{4/3} \eta M^{1/3}} \left[ \frac{3058673}{1016064} + \frac{5429}{1008} \eta + \frac{617}{144} \eta^2 \right].$$  \hspace{1cm} (2.12)

The chirp times depend on the two masses of the stars and the lower frequency cutoff $f_a$ of the detector and hence all are not independent. The physical significance of $\tau$’s is that they contribute to the total inspiral time of the binary starting at a time when the gravitational wave frequency is $f_a$; $\tau_0$ is the Newtonian contribution and others are the various PN contributions, the total chirp time being $\tau_0 + \tau_1 - \tau_{1.5} + \tau_2$.

### 3. Detection

In this Section we cover aspects related to the detection of gravitational waves with the aid of matched filtering. First, we shall focus attention on the detection statistic and define the scalar product of two signal wave forms. The latter facilitates a quick derivation of an expression for the optimal Weiner filter and the notion of the ambiguity function which is a very powerful tool to deal with aspects of detection and estimation.
3.1. Detection statistic

Weiner filtering or matched filtering is a data analysis technique that efficiently searches for a signal of known shape [Helstrom 1968]. The method consists in correlating the raw output of a detector with a waveform, variously known as a template or a filter. Given a signal \( h(t; \lambda) \), where \( \lambda = \{ \lambda_k | k = 1, \ldots, N_s \} \) denotes the \( N_s \) signal parameters, buried in noisy data \( n(t) \), the task is to find an ‘optimal’ filter \( q(t; \bar{\mu}) \) that would produce, on the average, the best possible signal-to-noise ratio (SNR) to be defined below. Here \( \bar{\mu} = \{ \mu_k | k = 1, \ldots, N_f \} \) denotes the \( N_f \) filter parameters. Let us denote by \( o(t) \) the output of the detector:

\[
o(t) = h(t; \lambda) + n(t).
\]  

(3.1)

The correlation \( c \) of the filter with the detector output is defined as

\[
c(\tau; \bar{\mu}) \equiv \int_{-\infty}^{\infty} o(t)q(t + \tau; \bar{\mu})dt = \int_{-\infty}^{\infty} \tilde{a}(f)\tilde{q}^*(f; \bar{\mu})\exp(2\pi if\tau)df
\]

(3.2)

where \( \tau \) is the lag parameter, \( \ast \) denotes complex conjugation, \( \tilde{\ast} \) denotes the Fourier transform of the quantity underneath \( (\tilde{a}(f) \equiv \int_{-\infty}^{\infty} a(t)\exp(-2\pi if)dt) \) and the second equality follows from Parcevel’s theorem. Now, \( n(t) \) being a random variable, so is \( c(\tau; \bar{\mu}) \) and a decision about the presence or absence of a signal is made by setting a threshold on the detection statistic of matched filtering which is the SNR \( \rho \) defined as

\[
\rho \equiv \max_{\tau, \bar{\mu}} \frac{c(\tau; \bar{\mu})}{\sqrt{c(\tau; \bar{\mu}) - c(\tau; \bar{\mu})^2}}.
\]  

(3.3)

where an overbar denotes ensemble average of the quantity underneath it over different realisations of the detector noise. The noise is assumed to be a Gaussian random process with zero mean and real, symmetric, two-sided power spectral density \( S_h(f) \) defined by

\[
\hat{n}(f)\hat{n}^*(f') = S_h(f)\delta(f - f').
\]  

(3.4)

Before proceeding further let us define the scalar product of functions which plays a crucial role in signal analysis. Given two functions \( a(t) \) and \( b(t) \) their scalar product is defined as

\[
\langle a | b \rangle \equiv \int_{-\infty}^{\infty} \frac{df}{S_h(f)} \hat{a}(f)\hat{b}^*(f).
\]  

(3.5)

The scalar product is real and positive definite owing to the properties of the noise power spectral density and Fourier transform of real functions. A wave form is said to be normalised if its norm, computed using the above definition of the scalar product, is equal to unity: \( \langle h, h \rangle^{1/2} = 1 \). With the aid of this notation the statistic \( \rho \) takes the form

\[
\rho = \frac{\langle h, S_hq \rangle}{\langle S_hq, S_hq \rangle}.
\]  

(3.6)

From this it is clear that the template \( q \) that obtains the maximum value of \( \rho \) is simply \( \hat{q}(f) = \gamma \hat{h}(f)/S_h(f) \) where \( \gamma \) is an arbitrary constant. Thus, in the Fourier domain an optimal filter is nothing but the signal weighted down by the noise power spectral density. In order to decide whether or not a signal is present the detector output is filtered through a bank of templates which are chosen over the entire parameter space of the template and the optimal SNR is \( \rho = \langle h | h \rangle^{1/2} \). A template whose parameters are exactly matched with those of a signal enhances the SNR in proportion to the square-root of the number of cycles that the signal spends in the detector output, as opposed to the case when the shape of the wave form is not known a priori and all that can
be done is to pre-bandpass filter the detector output to the frequency band where the signal is assumed to lie, and to then look at the SNR for each data point in the time domain individually \cite{schutz1989}. For an interferometric detector, such as the LIGO (Abramovici et al. 1992) or the VIRGO (Bradaschia et al. 1990), operating with a lower frequency cutoff $\sim 40$ Hz and an upper cutoff $\sim 1$ kHz, this means an amplification in the SNR $\sim 30$-40 for a typical binary. This enhancement in the SNR not only increases the number of detectable events but, more importantly, it also allows a more accurate determination of signal parameters—the error in the estimation of a parameter being inversely proportional to the SNR.

### 3.2. Ambiguity function

In order to take full advantage of matched filtering it is essential that the inspiral wave form, and in particular the evolution of its phase, be known to a very high degree of accuracy \cite{cutler1993}. A mismatch in the phases of the template and the signal can severely reduce the SNR; even when the template and the signal go out of phase by one cycle in $10^4$ the SNR could reduce by as much as 10%. This is a positive aspect of Weiner filtering since the statistic will not pick out any spurious chirp-like signals present in the detector output. However, the problem is that we will not know a priori what the signal parameters are and consequently, the detector output needs to be filtered through a large number of templates each corresponding to a particular set of “test” parameters. Thus, we have to finely sample the parameter space in searching for a signal. The number of search templates needed to cover the astrophysically relevant range of the parameter space depends primarily on the effective dimensionality of the parameter space. Ambiguity function, well known in statistical theory of signal detection \cite{helstrom1968}, is a very powerful tool in signal analysis: It helps to assess the number of templates required to span the parameter space of the signal, to make estimates of variances and covariances involved in the measurement of various parameters, to compute biases introduced in using a wrong family of templates, etc. Later in this Section we will see how the ambiguity function can be used to compute the number of templates.

The ambiguity function is defined as the scalar product of two normalised wave forms $h(t; \vec{\lambda})$ and $g(t; \vec{\mu})$ which differ in all their parameter values, i.e., $\vec{\lambda}$ being in general different from $\vec{\mu}$:

$$A(\vec{\lambda}, \vec{\mu}) = \langle h(\vec{\lambda}), g(\vec{\mu}) \rangle. \quad (3.7)$$

Since the wave forms are of unit norm $A(\vec{\lambda}, \vec{\mu}) = 1$, if $\vec{\lambda} = \vec{\mu}$ and $A(\vec{\lambda}, \vec{\mu}) < 1$, if $\vec{\lambda} \neq \vec{\mu}$. Here $\vec{\lambda}$ can be thought of as the parameters of a signal while $\vec{\mu}$ those of a template. With this interpretation $A(\vec{\lambda}, \vec{\mu})$ is the SNR obtained using a template that is not necessarily matched on to the signal. Keeping the filter parameters $\vec{\mu}$ fixed if we vary the signal parameters $\vec{\lambda}$ the ambiguity function is a function of $N_s$ signal parameters giving the SNR obtained by the template for signals of different values of their parameters. The region in the signal parameter space for which a given template obtains SNRs larger than a certain value (sometimes called the minimal match \cite{owen1996}) is the span of that template and the templates should be so chosen that together they span the entire signal parameter space of interest with the least overlap of one other’s spans. One can equally well interpret the ambiguity function as the SNR obtained for a given signal by filters of different parameter values and in this case the ambiguity function is a function of $N_f$, rather than $N_s$, variables (see below the Section on biases for an application of this interpretation). Of course, in its entirety the ambiguity function is really a function of $N_s + N_f$ variables.
It is important to note that in the definition of the ambiguity function there is no need that the functional form of the template be that of a signal; the definition holds good for any signal-template pair of wave forms. Moreover, the set of $N_f$ template parameters $\hat{\mu}$ need not be identical (and usually aren’t) to the set of $N_s$ parameters $\hat{\lambda}$ characterising the signal. For instance, a binary can be characterised by a large number of parameters, such as the masses, spins, eccentricity of the orbit, etc., while we may take as a model wave form for the purpose of filtering the one only involving the masses. This is indeed an issue where substantial work is called for: What are all the physical effects to be considered so as not to miss out a binary wave form from our search?

It is thus possible to render a wider interpretation to the ambiguity function, the one that is conceptually more appropriate and useful in the context of gravitational waves. Such an interpretation is based on the geometrical viewpoint of signal analysis which is briefly the following: A wave form $h(t; \hat{\lambda})$ with a specific set of values for its parameters can be thought of as a signal vector. As its $N_s$ parameters are varied the signal vector spans a $N_s$-dimensional space in the underlying infinite-dimensional function vector space. The former has a manifold structure, the parameters of the signal constituting a coordinate system and the metric defined using the scalar product introduced earlier. A given template $g(t; \hat{\mu})$ is itself a vector but may or may not be a member of the signal space. The set of all template wave forms obtained by varying the $N_f$ filter parameters span an $N_f$-dimensional space which again has manifold structure. By choice the template manifold is of a dimension lower than the signal manifold. Now, the ambiguity function, viewed as a function of its arguments ($\hat{\lambda}$ and $\hat{\mu}$), gives the nearest distance between different points on the filter manifold from points on the signal manifold, distance being measured using the scalar product. In the context of gravitational waves $h(t; \hat{\lambda})$ is the exact general relativistic wave form emitted by a binary while the template family, as of now, is the 2-PN corrected chirp. Of course, in this case we cannot compute the ambiguity function since the exact wave form is not known (if it had been known, we would use that in our family of templates instead of the 2-PN wave form). However, it is possible to obtain ambiguity function similar to this but in the case of a test particle orbiting a Schwarzschild black hole. In the latter case the exact phase evolution of the wave form is known numerically and approximations to the phase evolution are known analytically up to 4-PN order. Such a study would then help us to make intelligent guesses about what will happen when both stars have comparable masses which in turn can be used to assess the effectiveness of a PN filter. The realisation that filters need not have shapes identical to a signal has led to the proposition that it might be worthwhile to consider templates that do not belong to any of the PN manifolds but lie outside of them (Balasubramanian et al. 1996).

To gauge the usefulness of various PN template families one can compute the ambiguity function by taking $h(t; \hat{\lambda})$ to be the 2-PN corrected signal and consider template families of different PN orders starting with the quadrupole wave form. Though this is not as good a method as suggested in the preceding paragraph it does address some of the issues. Such an attempt was first made by Balasubramanian & Dhurandhar (1994) and Kokkotas, Krolak & Tsegas (1994) and was investigated more formally and exhaustively by Apostolatos (1995). The result of all this analysis is that the 1-PN template families are inadequate in the detection of gravitational waves and that one has to work with at least 1.5-PN wave forms (Apostolatos 1995). A more reliable understanding is likely to come by studying the test particle case mentioned above.

Let us employ the ambiguity function to find how many templates are needed to span the range of astrophysical parameters using different families of templates. For brevity
we shall only deal with the Newtonian and the 1-PN template families and simply quote results in other cases. In general, as indicated by its arguments \( A(\lambda, \mu) \) depends on the individual values of the parameters both of the signal and the template. In what follows we will first see that in the case of restricted PN chirps to 1-PN order the ambiguity function only depends on the absolute difference in the parameter values \( |\lambda_k - \mu_k| \) provided the template and the filter are both chirps to the same PN order. Even at higher PN orders there is only a weak dependence on the absolute values of the parameters. Geometrically, this means that the signal manifold at restricted 1-PN order is flat and that higher PN order signal manifolds are only slightly curved. Secondly, we will see that a template of a given total chirp time obtains roughly the same SNR for signals of the same total chirp time though their Newtonian and PN chirp times may be different from that of the template. The former of these two results implies uniformity in the spacing of filters (Sathyaprakash & Dhurandhar 1991; Dhurandhar & Sathyaprakash 1994) while the latter result facilitates a massive reduction in the number of templates required in spanning the parameter space since instead of constructing filters separately for each of the Newtonian and PN chirp times we can construct filters simply for the total chirp time.

Making use of the expression for the Fourier transform given in eq. (2.1) when the signal and templates belong to the same PN order we have

\[
A(\lambda_k, \mu_k) = 2N^2 \int_0^\infty df \frac{f^{-7/3}}{S_n(f)} \cos \left[ \sum_{k=1}^n \psi_k(f) \delta_k \right] = A(\delta_k) \quad (3.8)
\]

where \( \delta_k = |\lambda_k - \mu_k| \) and \( n - 3 \geq 3 \) is the PN order of the wave forms. We see that the ambiguity function is independent of the individual chirp times of the signal and the template: For all signal-template pairs that have the same differences in times of arrival, phases, and chirp times one obtains the same value for the ambiguity function (Sathyaprakash 1994; Balasubramanian et al. 1996). Consequently, constancy of the distance, measured using the scalar product, between two nearest neighbour filters translates into the constancy of the distance, measured using the difference in their parameter values. As a concrete example let us take both the signal and filters to be the Newtonian wave form. In this case there is only one parameter for which filters need to be explicitly constructed, namely the Newtonian chirp time. The ambiguity function obtained by using the noise power spectral density expected in the initial LIGO interferometer (Finn & Chernoff 1993; Cutler and E. Flanagan 1994) and keeping the chirp time of the template constant at 4 s while varying the signal parameters in the range of 3.8 to 4.2 s, \( \nabla \) is plotted in Fig. 1 as solid line. Identical curves are obtained irrespective what value we choose for the chirp time of the template confirming the claim made above.

Now, if we want to search for all binaries with the mass of each star in the range \([0.2, 30] M_\odot\) then we need to span the range \(~ 0, 660\) s of the Newtonian chirp time. By drawing a horizontal line at, say \( A = 0.97 \), we get the span of a template to be 0.030 s which gives the total number of filters to be about \( 2.2 \times 10^4 \). Estimates of the number of filters required to span the parameter space of PN signals (for which the

\[\dag\] This is only an approximate property of the ambiguity function since the result relies on the accuracy of the stationary phase method employed in computing the Fourier transform of the chirp. Moreover, real wave forms are shut off after the signal has reached a frequency corresponding to the last stable circular orbit of the binary which occurs when the distance between the two stars is \( 6M \). However, results concerning the ambiguity function hold good only when all wave forms have the same upper frequency cutoff.

\[\ddag\] Newtonian chirp time 4 s corresponds to an equal mass binary of total mass 8.5 \( M_\odot\) assuming a lower frequency cutoff of 40 Hz as in the case of initial LIGO.
dimensionality of the parameter space is two—masses of the two stars, or any two of the
chirp times introduced earlier, or the total and the reduced mass of the system, any of
these pairs of parameters serve to characterise the signal) are only an order of magnitude
larger (\( \sim 2.4 \times 10^5 \)) \(^{1996}\) and not \(10^8\) as one would have expected from the
dimensionality of the parameter space. Even at 2-PN order the number of templates is
expected to remain the same, since there are no new parameters, except that the spacing
between templates will no more be uniform.

It is worth pointing out the relation between the number of filters and the bandwidth
of a detector. Keeping the sensitivity constant if we increase the bandwidth then we have
a wider frequency range to compare two wave forms and hence the correlation between
two signals of differing parameters would grow smaller. Decreasing the bandwidth has
just the opposite effect. Concerning parameter estimation, at any given SNR, greater
the bandwidth larger will be the accuracy with which parameters can be measured. It
seems from this that if we want a better parameter estimation then we have to cope with
a larger number of templates in our search (see Balasubramanian et al. \(1996\) for more
details on this and related issues).
4. Estimation

After a detection is made the next step in data analysis is estimation of parameters characterising an event and provide possible error bounds on the measured values. We will discuss aspects related to estimation in this section.

4.1. Covariance matrix and error bounds

In our search for a chirp signal in the output of an interferometric detector we use a discrete, rather than a continuous, family of templates. The spacing between templates could be quite large, as in the case of a hierarchical search, or small, as in the case of post-detection analysis. The parameters of the template that obtains the maximum SNR gives us a maximum likelihood estimate as such a template would also maximise what is called the maximum likelihood ratio. These estimates are most unlikely the actual parameters of the signal; the true parameters are expected to lie within an ellipsoid of $N_s$ dimensions at a certain level of confidence—the volume of the ellipsoid increasing with the level of confidence. The axes of the ellipsoid are the one sigma uncertainties in the estimation of parameters and the confidence level corresponding to a one sigma uncertainty is 67%. The errors (i.e. one sigma uncertainties) in the various parameters are given by the square root of the diagonal elements of the covariance matrix $C_{ij}$. The latter is the inverse of the Fisher information matrix $\Gamma_{ij}$ well known in statistical theory of signal detection (Helstrom 1968) given by

$$\Gamma_{ij} = \left\langle \frac{\partial h(\vec{\lambda})}{\partial \lambda_i}, \frac{\partial h(\vec{\lambda})}{\partial \lambda_j} \right\rangle, \quad C_{ij} = (\Gamma)^{-1}_{ij}. \quad (4.9)$$

Bounds on the estimation computed using the covariance matrix are called Cramer-Rao bounds. These are not very tight bounds on estimation; they are the minimum uncertainty one should expect for the various parameters. They are based on local analysis and do not take into consideration the effect of distant points in the parameter space on the error computed at a given point. The errors are typically much larger than that predicted by the covariance matrix and have to be computed by other, more involved, methods. We shall see one such numerical method in the next Section. Cramer-Rao bounds fall off as inverse of the SNR while actual errors do not necessarily follow this behaviour. One usefulness of the Cramer-Rao bounds is that they are asymptotically valid in the limit of high SNR and hence serve as a basis to test all other estimates of errors. Covariance matrix based errors in the estimation of the Newtonian and PN chirp times ($\tau_0$ and $\tau_1$) and the instant of coalescence $t_C$ are listed in Table I for different values of the SNR. Due to the fact that the scalar product of two chirps of the same PN family is independent of the absolute values of the parameters, it turns out that the errors in chirp times and the instant of coalescence are independent of which binary we are looking at. This is not so if we were to use as our parameters the masses of the stars or the reduced and total mass. Of course, the relative error in chirp times will be larger, for a given SNR, in the case of higher mass binaries since the latter last for a smaller duration. From Table I we see that a chirp detected at an SNR of 10 is expected to lie, according to covariance matrix estimates, within an ellipsoid of dimensions $45 \times 26 \times 0.26$ ms$^3$ at a confidence level of 67% and within an ellipsoid of 8 times larger volume at a confidence level of 95 %, and so on. (Estimates of errors for various PN signals can be found in (Finn & Chernoff 1993; Cutler and E. Flanagan 1994; Blanchet & Sathyaprakash 1994; Poisson & Will 1993; Balasubramanian et al. 1996) and a quick theory of estimation is given in (Finn 1992; Cutler and E. Flanagan 1994; Jaranowski & Krolak 1994).)

$\dagger$ For brevity we have not included the phase parameter $\Phi_C$. 

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| $\tau_0$ | $\tau_1$ | $t_C$ |
|----------|----------|-------|
| 0.01 ms  | 0.02 ms  | 0.03 ms |

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Table 1. Errors in parameter estimation computed using the covariance matrix \((\sigma_{\tau_0}, \sigma_{\tau_1}, \sigma_{t_C})\) are listed at various SNRs. Corresponding errors found with the aid of Monte Carlo simulations \((\Sigma_{\tau_0}, \Sigma_{\tau_1}, \Sigma_{t_C})\) are quoted in brackets. All values are quoted in ms.

| \(\rho\) | \(\sigma_{\tau_0} (\Sigma_{\tau_0})\) | \(\sigma_{\tau_1} (\Sigma_{\tau_1})\) | \(\sigma_{t_C} (\Sigma_{t_C})\) |
|---------|---------------------------------|---------------------------------|-----------------|
| 10.0    | 45 (135)                        | 26 (65)                         | 0.26 (0.54)     |
| 15.0    | 30 (65)                         | 17 (33)                         | 0.17 (0.30)     |
| 20.0    | 23 (30)                         | 13 (16)                         | 0.13 (0.17)     |
| 25.0    | 18 (19)                         | 10 (11)                         | 0.10 (0.10)     |

4.2. Biases in estimation

Ambiguity function can also be used to estimate biases. To do this we need the alternate interpretation of the ambiguity function namely, that it gives the SNR obtained for a given signal by different filters. Thus, we keep the parameters of the signal to be constant and find which amongst all filters obtains the maximum SNR. Let me illustrate this with a simple example. Suppose the filters are Newtonian wave forms and the signal is the 2-PN wave form. The ambiguity function obtained in this case, plotted as a dashed curve in Fig. 1, is not a symmetric curve and it does not any more possess the nice properties of being independent of the absolute values of the parameters. Moreover, the maximum has shifted: The maximum SNR is obtained not by a filter that matches the Newtonian chirp time of the signal but by some other filter whose chirp time is different from that of the signal. If the real world had consisted of 2-PN wave forms and we had used Newtonian wave forms we would not only miss a large number of events but our parameter estimation would also come out wrong. In the present case there is 15% loss in the SNR and \(-40\) ms bias in the estimation of Newtonian chirp time. In reality the wave form present in the detector is the fully general relativistic wave form with all the non-linearities but the template that we hope to use will be some high order PN approximation to it and consequently we are bound to make systematic errors in estimation. In the case when a lower order PN template family is used in detecting a higher PN signal, filters seem to match the instant of coalescence \(t_C\) to an accuracy better than any other parameter \((Balasubramanian et al. 1995)\). This is likely to be true even when PN templates are used to pick out the actual signal buried in our detector outputs. Based on this result we expect that \(t_C\) will be measured to a greater accuracy than any other parameter and it would be sensible to use this, or any other parameter that is determined best, in our tests of theories and models.

4.3. Monte Carlo estimation of parameters

As mentioned in the previous Section Cramer-Rao bounds are only a lower limit on the errors expected and realistic errors are much larger than this. One brute force way of estimating the errors is to carry out numerical simulations mimicking the actual detection process. An advantage of numerical simulations is that unlike in the derivation of the covariance matrix they make no assumption about the behaviour of the filtered noise distribution function. Ensemble averages that are required to make estimates are achieved by performing a large number of simulations corresponding to different realisations of the expected detector noise in each numerical experiment. This is equivalent to having a large number of detectors and upon filtering and maximising each detector output over
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the space of template wave forms gives a certain measured set of values of the parameters. The average of the measured values provides estimates of the parameters and variances in the measured values give typical errors involved in any one measurement. A simulation such as this was carried out to assess the degree of accuracy of the covariance matrix (Balasubramanian et al. 1995; Balasubramanian et al. 1996). Results from this study show that the covariance matrix underestimates the errors of various parameters by factors of 2–3 (cf. Table I) at an SNR of 10. However, at SNRs larger than \( \sim 25 \), errors obtained by Monte Carlo simulations agree with those predicted by covariance matrix calculations. There has been an effort to understand the discrepancy between the two results (Vecchio & Nicholson) which clearly indicates that the covariance matrix is a poor descriptor of errors and better greatest lower bounds on errors are in order.

Even though realistic errors are a factor of 2–3 larger than those predicted by the covariance matrix in absolute terms they are still quite small so that it would be possible to put theories of gravitation and models of cosmology to test and constrain them at high confidence levels. For instance, it has been shown that presence of non-linear tails of gravitational waves, occurring due to the scattering and subsequent re-emission of radiation off the curved Schwarzschild spacetime of the binary, can be detected in the case of BH-BH coalescences that have an SNR \( \sim 50 \) or larger (Blanchet & Sathyaprakash 1995). It has also been suggested that statistics based on catalogues of coalescing binary events are potential tools to measure cosmological parameters and test cosmological models (Finn 1996).

5. Summary and future

In this article we have discussed issues concerning the detection and measurement of gravitational radiation from coalescing compact binary systems. The fully general relativistic wave form that will be buried in the data stream is not known to us and we can only hope to use an approximate wave form that is presently known to an accuracy \((v/c)^4\) beyond the Newtonian order. Efforts are imminent to gauge the PN level up to which it is essential to know the wave form in the finite reduced mass case by studying the test particle case where both the exact and the approximate solutions are known (Damour et al. 1996). There are other effects such as the eccentricity-induced or spin-induced modulations which may affect the wave form so much as to make it inefficient in picking out a true coalescing binary signal. It is important to gauge these effects.

A considerable amount of work has been done in determining the number of templates required to span the signal parameter space and present compute resources seem adequate to filter chirp signals online. However, not all ideas concerning the choice of filters are explored yet (Balasubramanian et al. 1996). Moreover, one has to still take into account physical effects such as spin-orbit and spin-spin coupling, last stable orbit etc., in computing the ambiguity function.

We have seen how biases are introduced in the measurement of parameters and how the ambiguity function may be employed to compute these biases but much needs to be done to make concrete estimates of the biases.

All efforts so far have concentrated on the study of matched filtering as a tool to dig out chirps out of noise. However, there are a number of other detection algorithms—chirplets, adaptive filters, periodograms, Smith’s algorithm, etc.—some of which may be more ideally suited to the detection of chirps than Winer filtering is. Further study in this direction is needed to make our data analysis systems more efficient.

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