COMPACT HOMOMORPHISMS BETWEEN ALGEBRAS OF
C(K)-VALUED LIPSCHITZ FUNCTIONS

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ABSTRACT. We give a complete description of homomorphisms between two Banach algebras of Lipschitz functions with values in continuous functions. We also characterize the compactness of those homomorphisms.

1. Introduction

Let $X$ be a compact metric space with metric $d_X$ and $A$ a commutative Banach algebra with norm $\| \cdot \|_A$. By $C(X, A)$, we denote the Banach algebra of all $A$-valued continuous functions on $X$, with norm

$$\| f \|_{C(X, A)} = \sup \{ \| f(x) \|_A : x \in X \}.$$ 

If an $A$-valued function $f$ on $X$ satisfies

$$\mathcal{L}_{X, A}(f) = \sup_{x, x' \in X, x \neq x'} \frac{\| f(x) - f(x') \|_A}{d_X(x, x')} < \infty,$$

then we say that $f$ is a Lipschitz function. By $\text{Lip}(X, A)$, we denote the set of all $A$-valued Lipschitz functions on $X$. Clearly, $\text{Lip}(X, A) \subset C(X, A)$ and $\text{Lip}(X, A)$ is a Banach algebra with norm

$$\| f \|_{\text{Lip}(X, A)} = \| f \|_{C(X, A)} + \mathcal{L}_{X, A}(f).$$

In case that $A = \mathbb{C}$, we write $C(X) = C(X, \mathbb{C})$ and $\text{Lip}(X) = \text{Lip}(X, \mathbb{C})$. The Lipschitz algebra $\text{Lip}(X)$ has been well studied. The researches on this subject may be found in the book [7]. Here a mapping between two Banach algebras is said to be a homomorphism if it preserving addition, scalar multiplication and multiplication. Moreover if it maps unit to unit, then we say that it is unital. One of the results is the description of homomorphisms between Lipschitz algebras.

Theorem A (Sherbert, [6], Proposition 2.1). Suppose that $X$ and $Y$ are compact metric spaces with metrics $d_X$ and $d_Y$ respectively. Then $T$ is a unital homomorphism from $\text{Lip}(X)$ into $\text{Lip}(Y)$, if and only if there exists a mapping $\varphi : Y \rightarrow X$
with
\[ \sup_{y, y' \in Y \atop y \neq y'} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} < \infty, \]
such that
\[ (Tf)(y) = f(\varphi(y)) \quad (y \in Y) \]
for all \( f \in \text{Lip}(X) \).

This theorem has been developed in several directions. In [1], F. Botelho and J. Jamison replaced \( \text{Lip}(X) \) by \( \text{Lip}(X, \mathcal{A}) \), where \( \mathcal{A} \) is the Banach algebra \( \ell^\infty \) of bounded sequences. They determined the unital homomorphisms from \( \text{Lip}(X, \mathcal{A}) \) into \( \text{Lip}(Y, \mathcal{A}) \) and those from \( \text{Lip}(X, \ell^\infty) \) into \( \text{Lip}(Y, \ell^\infty) \), where \( X \) and \( Y \) are compact metric spaces.

In general, if \( K \) is a compact Hausdorff space, then \( C(K) \) denotes the Banach algebras of all complex-valued continuous functions on \( K \), with norm \( \|f\|_{C(K)} = \sup_{\xi \in K} |f(\xi)| \). In [4], S. Oi took up the algebra \( \text{Lip}(X, C(K)) \) and proved the following theorem:

**Theorem B** ([4, Oi]). Suppose that \( X \) and \( Y \) are as in Theorem A, and that \( K \) and \( M \) are compact Hausdorff spaces. Assume that \( Y \) is connected. Then \( T \) is a unital homomorphism from \( \text{Lip}(X, C(K)) \) into \( \text{Lip}(Y, C(M)) \) if and only if there exist a class \( \{\varphi_\eta\}_{\eta \in M} \) of mappings from \( Y \) to \( X \) with the properties (a) and (b) and a continuous mapping \( \psi : M \to K \) such that
\[
[(Tf)(y)](\eta) = [f(\varphi_\eta(y))] (\psi(\eta)) \quad (y \in Y, \eta \in M)
\]
for all \( f \in \text{Lip}(X, C(K)) \).

1. For each \( y \in Y \), the mapping \( \eta \mapsto \varphi_\eta(y) \) from \( M \) to \( X \) is continuous.
2. \[ \sup_{\eta \in M} \sup_{y, y' \in Y \atop y \neq y'} \frac{d_X(\varphi_\eta(y), \varphi_\eta(y'))}{d_Y(y, y')} < \infty. \]

This theorem leads to the result of Botelho and Jamison mentioned above. Here we turn our attention to two assumptions in Theorem B. One is that \( Y \) is connected and the other is that \( T \) is unital. These assumptions seem to be inessential but they simplify the statement of theorem. In order to remove these assumptions and to state a general result, we consider a function \( f \) in \( \text{Lip}(X, C(K)) \) as a function of two variables \( x \in X \) and \( \xi \in K \). So we write \( f(x, \xi) \) instead of \([f(x)](\xi)\). Let \( f \) be a function on \( X \times K \). With \( x \in X \) we associate a function \( f_x \) defined on \( K \) by \( f_x(\xi) = f(x, \xi) \). Similarly, if \( \xi \in K \), \( f^\xi \) is the function defined on \( X \) by \( f^\xi(x) = f(x, \xi) \). In general, for any mapping of two variables, we use the same expression: For example, if \( \psi : Y \times M \to K \), then \( \psi^\eta : Y \to K \) and \( \psi_y : M \to K \) are defined by \( \psi^\eta(y) = \psi(y, \eta) \) and \( \psi_y(\eta) = \psi(y, \eta) \).
A subset \( A \) of a topological space is said to be \textit{clopen}, if \( A \) is both open and closed. We do not exclude the possibility that a clopen set is empty. We understand that the statement about an empty set is true.

**Theorem 1.** Suppose that \( X \) and \( Y \) are compact metric spaces with metrics \( d_X \) and \( d_Y \) respectively, and that \( K \) and \( M \) are compact Hausdorff spaces. If \( T \) is a homomorphism from \( \text{Lip}(X,C(K)) \) into \( \text{Lip}(Y,C(M)) \), then there exist a clopen subset \( D \) of \( Y \times M \) and two continuous mappings \( \varphi : D \rightarrow X \) and \( \psi : D \rightarrow K \) with (i) and (ii) such that \( T \) has the form:

\[
(Tf)(y,\eta) = \begin{cases} 
  f(\varphi(y,\eta),\psi(y,\eta)) & (y,\eta) \in D \\
  0 & (y,\eta) \in (Y \times M) \setminus D 
\end{cases}
\]

for all \( f \in \text{Lip}(X,C(K)) \).

(1) There exists a bound \( L \geq 0 \) such that

\[
d_X(\varphi(y,\eta),\varphi(y',\eta))/d_Y(y,y') \leq L.
\]

(2) For any \( \eta \in M \), the set \( D^\eta = \{ y \in Y : (y,\eta) \in D \} \) is a union of finitely many disjoint clopen subsets \( V_1^\eta, \ldots, V_n^\eta \) of \( Y \) such that \( \psi^\eta \) is constant on \( V_i^\eta \) for \( i = 1, \ldots, n_\eta \),

and

\[
d_Y(V_i^\eta, V_j^\eta) \geq r \quad (i \neq j).
\]

Here \( r \) is a positive constant independent of \( \eta \).

Conversely, if \( D, \varphi, \psi \) are given as above, then \( T \) defined by (1) is a homomorphism from \( \text{Lip}(X,C(K)) \) into \( \text{Lip}(Y,C(M)) \). Moreover, \( T \) is unital if and only if \( D = Y \times M \).

In (3), \( d_Y(A,B) \) denotes the usual distance between two sets \( A, B \subset Y \), that is, \( d_Y(A,B) = \inf \{ d_Y(y,y') : y \in A, y' \in B \} \). (If \( A = \emptyset \) or if \( B = \emptyset \), then we set \( d_Y(A,B) = \infty \)).

Next we consider the following problem:

When is a homomorphism between Lipschitz algebras compact?

In [3], H. Kamowitz and S. Scheinberg answered to this problem as follows:

**Theorem C** (Kamowitz and Scheinberg, [3]). Let \( T \) be a unital homomorphism from \( \text{Lip}(X) \) into \( \text{Lip}(Y) \) described in Theorem A. Then \( T \) is compact if and only if

\[
\lim_{d_Y(y,y') \to 0} \frac{d_X(\varphi(y),\varphi(y'))}{d_Y(y,y')} = 0.
\]
In this paper, we give a necessary and sufficient condition for $T$ in Theorem 1 to be compact.

**Theorem 2.** Let $X, Y, K, M$ be as in Theorem 1. Suppose that $T$ is a homomorphism from $\text{Lip}(X, C(K))$ into $\text{Lip}(Y, C(M))$ with form (1) in Theorem 1. Then $T$ is compact if and only if (iii) and (iv) hold.

(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
(y, \eta), (y', \eta) \in D \text{ and } 0 < d_Y(y, y') < \delta \implies \frac{d_X(\varphi(y, \eta), \varphi(y', \eta))}{d_Y(y, y')} < \varepsilon.
$$

(iv) For any $y \in Y$, the set $D_y = \{\eta \in M : (y, \eta) \in D\}$ is a union of finitely many disjoint clopen sets $\Omega^y_1, \ldots, \Omega^y_{n_y}$ such that

$$
\psi_y \text{ is constant on } \Omega^y_i \text{ for } i = 1, \ldots, n_y.
$$

2. Preliminaries

As mentioned in Introduction, we consider a function $f$ in $\text{Lip}(X, C(K))$ as a function on $X \times K$.

**Proposition 2.1.** Let $f$ be a complex-valued function on $X \times K$. Then $f \in \text{Lip}(X, C(K))$ if and only if $f \in C(X \times K)$ and

$$
\mathcal{L}_{X, C(K)}(f) = \sup_{x, x' \in X} \frac{\|f_x - f_{x'}\|_{C(K)}}{d_X(x, x')} < \infty.
$$

Moreover, $\|f\|_{C(X \times K)} = \|f\|_{C(X \times K)}$.

*Proof.* Straightforward. □

The next proposition implies that $\text{Lip}(X, C(K))$ separates the points of $X \times K$.

**Proposition 2.2.** For any $(x_0, \xi_0) \in X \times K$ and for any open neighborhood $U$ of $(x_0, \xi_0)$, there exists a $f \in \text{Lip}(X, C(K))$ such that $0 \leq f \leq 1$, $f(x_0, \xi_0) = 1$ and $f(x, \xi) \leq m < 1$ for all $(x, \xi) \in (X \times K) \setminus U$.

*Proof.* Let $(x_0, \xi_0) \in X \times K$ and let $U$ be an open neighborhood of $(x_0, \xi_0)$. Then there exist an open neighborhood $U$ of $x_0$ in $X$ and an open neighborhood $\Theta$ of $\xi_0$ in $K$ such that $(x_0, \xi_0) \in U \cap \Theta \subset U$.

Let $h$ be a function on $X$ defined by

$$h(x) = 1 - \frac{d_X(x, x_0)}{\text{diam}(X)} \quad (x \in X),$$

where $\text{diam}(X) = \sup\{d_X(x, x') : x, x' \in X\}$. We easily see that $h \in \text{Lip}(X)$, $0 \leq h \leq 1$, $h(x_0) = 1$ and $h(x) < 1$ for all $x \in X \setminus \{x_0\}$. By Urysohn’s lemma, there is a $u \in C(K)$ such that $0 \leq u \leq 1$, $u(\xi_0) = 1$ and $u(\xi) = 0$ for all $\xi \in K \setminus \Theta$. Now, put $f(x, \xi) = h(x)u(\xi)$ for $(x, \xi) \in X \times K$. Then we can verify that $f$ has the
desired properties. Here \( m \) may be taken as the maximum of \( f \) on the compact set \((X \times K) \setminus U\).

Here we summarize a fundamental fact on the Banach algebra \( \text{Lip}(X, C(K)) \).

**Proposition 2.3.** \( \text{Lip}(X, C(K)) \) is a semi-simple unital commutative Banach algebra and its maximal ideal space is identified with \( X \times K \). In fact, for any multiplicative linear functional \( \Psi \) on \( \text{Lip}(X, C(K)) \), there exists a unique point \((x, \xi) \in X \times K\) such that \( \Psi(f) = f(x, \xi) \) for all \( f \in \text{Lip}(X, C(K)) \).

We can prove this proposition by the well-known argument in theory of Banach algebras. The details may be found in [4, Propositions 11 and 12].

### 3. Proof of Theorem 1

In this section we prove Theorem 1.

#### 3.1. Proof of Sufficiency

We first settle the converse statement. Suppose that \( \mathcal{D} \) is a clopen subset of \( Y \times M \), that \( \varphi: \mathcal{D} \to X \) and \( \psi: \mathcal{D} \to K \) are continuous mappings with (i) and (ii), and that \( T \) is defined by (1).

**Lemma 3.1.** If

\[
\rho = \inf \{ d_Y(y, y') : (y, \eta) \in \mathcal{D} \text{ and } (y', \eta) \in (Y \times M) \setminus \mathcal{D} \text{ for some } \eta \in M \},
\]

then \( \rho > 0 \).

If there is no pair \((y, y') \in Y \times Y\) such that \((y, \eta) \in \mathcal{D}\) and \((y', \eta) \in (Y \times M) \setminus \mathcal{D}\) for some \( \eta \in K \), then we understand that \( \rho = \infty \).

**Proof.** Conversely, assume that \( \rho = 0 \). Then for each \( n = 1, 2, \ldots \), there exist \((y_n, \eta_n) \in \mathcal{D}\) and \((y'_n, \eta_n) \in (Y \times M) \setminus \mathcal{D}\) such that \( d_Y(y_n, y'_n) < 1/n \). Since \( \mathcal{D} \) is compact, there exist a net \( \{\eta_n\} \) and a point \((y, \eta) \in \mathcal{D}\) such that \( y_n \to y \) and \( \eta_n \to \eta \). Then \( d_Y(y_n, y'_n) < 1/n \to 0 \). Hence \( y'_n \to y \) and so \((y'_n, \eta_n) \to (y, \eta) \) for all \( n \in \mathbb{N} \).

Since \((Y \times M) \setminus \mathcal{D}\) is closed, we get \((y, \eta) \in (Y \times M) \setminus \mathcal{D}\). This contradicts the fact that \((y, \eta) \in \mathcal{D}\). Consequently, we have \( \rho > 0 \). \( \square \)

**Lemma 3.2.** For any \( f \in \text{Lip}(X, C(K)) \), \( Tf \in \text{Lip}(Y, C(M)) \).

**Proof.** Let \( f \in \text{Lip}(X, C(K)) \). By Proposition 2.1, we have \( f \in C(X \times K) \) and (5).

We first show that \( Tf \in C(Y \times M) \). Since \( \varphi: \mathcal{D} \to X \) and \( \psi: \mathcal{D} \to K \) are continuous and since \( f \in C(X \times K) \), the first line in (1) implies that \( Tf \) is continuous on \( \mathcal{D} \). Of course, the second one implies that it is so on \((Y \times M) \setminus \mathcal{D}\). Noting that \( \mathcal{D} \) is clopen, we see that \( Tf \) is continuous on \( Y \times M \).

To see that \( Tf \in \text{Lip}(Y, C(M)) \), it suffices to show that

\[
(6) \quad \mathcal{L}_{Y,C(M)}(Tf) = \sup_{y, y' \in Y, y \neq y'} \frac{||(Tf)_y - (Tf)_{y'}||_{C(M)}}{d_Y(y, y')} < \infty.
\]
For this end, choose $y, y' \in Y$ so that $y \neq y'$ and let $\eta \in M$. We consider three cases.

**Case 1** $(y, \eta), (y', \eta) \in \mathcal{D}$: By (ii), $y \in V_i^n$ and $y' \in V_{i'}^n$ for some $i, i' \in \{1, \ldots, n\}$. We first consider the case $i = i'$. Then $y, y' \in V_i^n$. Since $\psi^n$ is constant on $V_i^n$, $\psi(y, \eta) = \psi^n(y) = \psi^n(y') = \psi(y', \eta)$. Put $x = \varphi(y, \eta)$, $x' = \varphi(y', \eta)$ and $\xi = \psi(y, \eta) = \psi(y', \eta)$. Using (1), we compute

\[
\left| (Tf)(y, \eta) - (Tf)(y', \eta) \right| = \left| f(\varphi(y, \eta), \psi(y, \eta)) - f(\varphi(y', \eta), \psi(y', \eta)) \right| \\
= \left| f(x, \xi) - f(x', \xi) \right| = \left| f_x(\xi) - f_{x'}(\xi) \right| \\
\leq \|f_x - f_{x'}\|_{C(K)} \\
\leq \mathcal{L}_{X,K}(f) \left( x, x' \right) \\
= \mathcal{L}_{X,K}(f) \left( \varphi(y, \eta), \varphi(y', \eta) \right) \\
\leq \mathcal{L}_{X,K}(f) \left( L \right) d_Y(y, y'),
\]

where the fourth and last lines follow from (2) and (3), respectively.

On the other hand, if $i \neq i'$, then (3) yields $d_Y(y, y') \geq d_Y(V_i^n, V_{i'}^n) \geq r$. Hence

\[
\frac{|(Tf)(y, \eta) - (Tf)(y', \eta)|}{d_Y(y, y')} \leq \frac{|(Tf)(y, \eta)| + |(Tf)(y', \eta)|}{r} \leq \frac{2 \|f\|_{C(X \times K)}}{r}.
\]

**Case 2** $(y, \eta), (y', \eta) \in (Y \times M) \setminus \mathcal{D}$: Then Lemma 3.1 says that $d_Y(y, y') \geq \rho > 0$. By (1), we get

\[
\frac{|(Tf)(y, \eta) - (Tf)(y', \eta)|}{d_Y(y, y')} \leq \frac{|f(\varphi(y, \eta), \psi(y, \eta)) - 0|}{\rho} \leq \frac{\|f\|_{C(X \times K)}}{\rho}.
\]

**Case 3** $(y, \eta), (y', \eta) \in (Y \times M) \setminus \mathcal{D}$: By (1),

\[
(Tf)(y, \eta) - (Tf)(y', \eta) = 0.
\]

Combining (7)–(10), we can arrive at (6). Indeed, if we put $C = \max\{L, 2/r, 1/\rho\}$, then we have

\[
\mathcal{L}_{Y,C(M)}(Tf) = \sup_{y', \eta} \sup_{y \neq y'} \frac{|(Tf)(y, \eta) - (Tf)(y', \eta)|}{d_Y(y, y')} \leq C \|f\|_{\text{Lip}(C(K))},
\]

because $\mathcal{L}_{X,K}(f) \leq \|f\|_{\text{Lip}(C(K))}$ and $\|f\|_{C(X \times K)} \leq \|f\|_{\text{Lip}(C(K))}$. \hfill $\square$

Lemma 3.2 says that $T$ maps $\text{Lip}(X, C(K))$ into $\text{Lip}(Y, C(M))$. While the form (1) shows that $T$ is a homomorphism. Thus we obtain the converse statement of Theorem 1.

**Remark.** From (1), we see that $\|Tf\|_{C(Y \times M)} \leq \|f\|_{C(X \times K)}$. Using this and (11), we obtain the norm estimate

\[
\|T\| = \sup_{\|f\|_{\text{Lip}(C(K))} \leq 1} \|Tf\|_{\text{Lip}(Y,C(M))} \leq C + 1.
\]
This estimate is not sharp, but it seems to be difficult to give an exact expression of \( \|T\| \).

### 3.2. Proof of Necessity

We turn to the proof of the main statement of Theorem 1. Suppose that \( T \) is an arbitrary homomorphism from \( \text{Lip}(X, C(K)) \) into \( \text{Lip}(Y, C(M)) \). Since \( \text{Lip}(Y, C(M)) \) is semi-simple, we know from [5, Theorem 11.10] that \( T \) is continuous. Thus the norm \( \|T\| \) is determined as a bounded linear operator \( T \).

If \( T = 0 \), then we only take \( D = \emptyset \). So, we assume that \( T \neq 0 \).

**Lemma 3.3.** There exist a clopen subset \( D \) of \( Y \times M \) and two mapping \( \varphi : D \to X \) and \( \psi : D \to K \) such that (1) holds.

**Proof.** Let \( 1 \) denote the unit of \( \text{Lip}(X, C(K)) \), namely, the constant 1 function on \( X \times K \). Since \( (T 1)^2 = T(1^2) = T 1 \), we have \( (T 1)(y, \eta) \in \{1, 0\} \) for all \( (y, \eta) \in Y \times M \).

Put
\[
D = \{ (y, \eta) \in Y \times M : (T 1)(y, \eta) = 1 \}.
\]

Then
\[
(Y \times M) \setminus D = \{ (y, \eta) \in Y \times M : (T 1)(y, \eta) = 0 \}.
\]

Since \( T 1 \) is continuous on \( Y \times M \), both \( D \) and \( (Y \times M) \setminus D \) are closed. Hence \( D \) is clopen.

To determine the mappings \( \varphi : D \to X \) and \( \psi : D \to K \), fix any \( (y, \eta) \in D \).

Define a functional \( \Psi_{(y, \eta)} \) on \( \text{Lip}(X, C(K)) \) by
\[
\Psi_{(y, \eta)}(f) = (T f)(y, \eta) \quad (f \in \text{Lip}(X, C(K))).
\]

Then \( \Psi_{(y, \eta)} \) is a homomorphism from \( \text{Lip}(X, C(K)) \) into \( \mathbb{C} \). Moreover, (12) yields \( \Psi_{(y, \eta)}(1) = (T 1)(y, \eta) = 1 \). Hence \( \Psi_{(y, \eta)} \) is a multiplicative linear functional on \( \text{Lip}(X, C(K)) \). Thus Proposition 2.3 gives a unique point \( (x, \xi) \in X \times K \) such that
\[
\Psi_{(y, \eta)}(f) = f(x, \xi) \quad (f \in \text{Lip}(X, C(K))).
\]

By putting \( \varphi(y, \eta) = x \) and \( \psi(y, \eta) = \xi \), we determine the mappings \( \varphi : D \to X \) and \( \psi : D \to K \). Then, for any \( f \in \text{Lip}(X, C(K)) \),
\[
(T f)(y, \eta) = \Psi_{(y, \eta)}(f) = f(x, \xi) = f(\varphi(y, \eta), \psi(y, \eta)).
\]

Finally, if \( (y, \eta) \in (Y \times M) \setminus D \), then \( (T 1)(y, \eta) = 0 \) and so for any \( f \in \text{Lip}(X, C(K)) \), \( T f = T(f 1) = (T f)(T 1) \) and so
\[
(T f)(y, \eta) = (T f)(y, \eta) (T 1)(y, \eta) = 0.
\]

Together with (13), we establish (1). \( \square \)

**Lemma 3.4.** The mappings \( \varphi : D \to X \) and \( \psi : D \to K \) are continuous.
Proof. Define a mapping \( \Phi : \mathcal{D} \to X \times K \) by
\[
\Phi(y, \eta) = (\varphi(y, \eta), \psi(y, \eta)) \quad ((y, \eta) \in \mathcal{D}).
\]
We prove the lemma by verifying that \( \Phi \) is continuous at each point \((y_0, \eta_0) \in \mathcal{D}\). Let \( \mathcal{U} \) be an arbitrary open neighborhood of \( \Phi(y_0, \eta_0) \) in \( X \times K \). By Proposition 2.2, there exists an \( f \in \text{Lip}(X, C(K)) \) such that \( 0 \leq f \leq 1 \), \( f(\Phi(y_0, \eta_0)) = 1 \) and
\[
0 \leq f(x, \xi) \leq m < 1 \quad ((x, \xi) \in (X \times K) \setminus \mathcal{U}).
\]
Put \( \mathcal{V} = \{(y, \eta) \in \mathcal{D} : |(Tf)(y, \eta)| > m\} \). Since \( Tf \) is continuous on \( Y \times M \), \( \mathcal{V} \) is open. Also, \((y_0, \eta_0) \in \mathcal{V}\) because
\[
(Tf)(y_0, \eta_0) = f(\varphi(y_0, \eta_0), \psi(y_0, \eta_0)) = f(\Phi(y_0, \eta_0)) = 1 > m.
\]
Moreover, if \((y, \eta) \in \mathcal{V}\), then
\[
|f(\Phi(y, \eta))| = |f(\varphi(y, \eta), \psi(y, \eta))| = |(Tf)(y, \eta)| > m
\]
and \((14)\) forces that \( \Phi(y, \eta) \in \mathcal{U} \). Hence \( \Phi(\mathcal{V}) \subset \mathcal{U} \). Thus \( \Phi \) is continuous at \((y_0, \eta_0)\), as desired. \( \square \)

Lemma 3.5. \( \varphi \) satisfies (i).

Proof. Let \((y, \eta), (y', \eta) \in \mathcal{D}\) with \( y \neq y' \). Put \( x_0 = \varphi(y', \eta) \) and
\[
f(x) = d_X(x, x_0) \quad (x \in X).
\]
Then \( f \in \text{Lip}(X) \) and \( \|f\|_{\text{Lip}(X)} \leq \text{diam}(X) + 1 \). Extend \( f \) to \( X \times K \) by \( \hat{f}(x, \xi) = f(x) \) for all \((x, \xi) \in X \times K \). Clearly \( \hat{f} \in \text{Lip}(X, C(K)) \) and \( \|\hat{f}\|_{\text{Lip}(X, C(K))} = \|f\|_{\text{Lip}(X)} \).
Moreover, we have
\[
d_X(\varphi(y, \eta), \varphi(y', \eta)) = |d_X(\varphi(y, \eta), x_0) - d_X(\varphi(y', \eta), x_0)|
\]
\[
= |f(\varphi(y, \eta)) - f(\varphi(y', \eta))|
\]
\[
= |\hat{f}(\varphi(y, \eta), \psi(y, \eta)) - \hat{f}(\varphi(y', \eta), \psi(y', \eta))|
\]
\[
= \|(T\hat{f})(y, \eta) - (T\hat{f})(y', \eta)\| = \|(T\hat{f})_y(\eta) - (T\hat{f})_{y'}(\eta)\|
\]
\[
\leq \|\|(T\hat{f})_{y'} - (T\hat{f})_y\|_{C(M)}
\]
\[
\leq \mathcal{L}_{Y,C(M)}(T\hat{f}) d_Y(y, y').
\]
Since \( \mathcal{L}_{Y,C(M)}(T\hat{f}) \leq \|T\hat{f}\|_{\text{Lip}(Y,C(M))} \leq \|T\| \|\hat{f}\|_{\text{Lip}(X,C(K))} \leq \|T\| \text{diam}(X) + 1 \), we obtain
\[
\frac{d_X(\varphi(y, \eta), \varphi(y', \eta))}{d_Y(y, y')} \leq \|T\| \text{diam}(X) + 1,
\]
which is (i). \( \square \)

Lemma 3.6. There exists an \( r > 0 \) such that
\[
(y, \eta), (y', \eta) \in \mathcal{D} \text{ and } d_Y(y, y') < r \text{ imply } \psi^n(y) = \psi^n(y').
\]
Proof. Take $r$ so that $0 < r < 1/\|T\|$. Choose $(y, \eta), (y', \eta) \in D$ with $d_Y(y, y') < r$ and assume that $\psi^n(y) \neq \psi^n(y')$. By Urysohn’s lemma, we find a $u \in C(K)$ such that $0 \leq u \leq 1$, $u(\psi^n(y)) = 1$ and $u(\psi^n(y')) = 0$. Define a function on $X \times K$ as $\tilde{u}(x, \xi) = u(\xi)$ for all $(x, \xi) \in X \times K$. Then $\tilde{u} \in \text{Lip}(X, C(K))$ and $\|\tilde{u}\|_{\text{Lip}(X, C(K))} = \|u\|_{C(K)} = 1$. Moreover we have

$$1 = \|u(\psi^n(y)) - u(\psi^n(y'))\| = \|u(\psi(y, \eta)) - u(\psi(y', \eta))\|$$

$$= \|\tilde{u} (\varphi(y, \eta), \psi(y, \eta)) - \tilde{u} (\varphi(y', \eta), \psi(y', \eta))\|$$

$$= \|(T\tilde{u})(y, \eta) - (T\tilde{u})(y', \eta)\|$$

$$\leq \|\psi^n(y) - \psi^n(y')\|_{C(M)}$$

$$\leq \mathcal{L}_{Y,C(M)}(T\tilde{u}) d_Y(y, y')$$

$$< \|T\tilde{u}\|_{\text{Lip}(Y,C(M))} r \leq \|T\| \|\tilde{u}\|_{\text{Lip}(X, C(K))} r = \|T\| r < 1,$$

a contradiction. Hence $\psi^n(y) = \psi^n(y')$. \hfill \Box

Lemma 3.7. $\psi$ satisfies (ii).

Proof. Fix any $\eta \in M$ and put $D^n = \{y \in Y : (y, \eta) \in D\}$. Since $D$ is clopen, $D^n$ is a clopen subset of $Y$. For any $y \in D^n$, put

$$(15) \quad V_y = \{z \in D^n : \psi^n(z) = \psi^n(y)\}.$$ 

Clearly, $\psi^n$ is constant on $V_y$. Also, we have

$$V_y \cap V_{y'} \neq \emptyset \implies V_y = V_{y'}.$$ 

Since $\psi^n$ is continuous by Lemma 3.4, $V_y$ is a closed subset of $D^n$. To see that $V_y$ is an open subset of $D^n$, let $z \in V_y$ and consider an $r$-ball $B(z; r) = \{w \in D^n : d_Y(w, z) < r\}$, where $r$ is given in Lemma 3.6. If $w \in B(z; r)$, then $(w, \eta), (z, \eta) \in D$ and $d_Y(w, z) < r$. Hence Lemma 3.6 implies that $\psi^n(w) = \psi^n(z) = \psi^n(y)$, and so $w \in V_y$. Therefore $B(z; r) \subset V_y$. Thus $V_y$ is an open subset of $D^n$. Consequently, $V_y$ is a clopen subset of $Y$.

Note that

$$D^n = \bigcup_{y \in D^n} V_y.$$ 

Since $D^n$ is compact, we can select finitely many $y_1, \ldots, y_n \in D^n$ such that

$$D^n = \bigcup_{i=1}^n V_{y_i}.$$ 

By (16), we may assume that $V_{y_1}, \ldots, V_{y_n}$ are disjoint.

Finally we show that $d_Y(V_{y_i}, V_{y_j}) \geq r$ $(i \neq j)$. Assume that $d_Y(V_{y_i}, V_{y_j}) < r$. Then there exist $z_i \in V_{y_i}$ and $z_j \in V_{y_j}$ such that $d_Y(z_i, z_j) < r$. By Lemma 3.6 $\psi^n(z_i) = \psi^n(z_j)$, and hence (15) and (16) yield $V_{y_i} = V_{y_j}$. Since $V_{y_1}, \ldots, V_{y_n}$ is disjoint, we must have $d_Y(V_{y_i}, V_{y_j}) \geq r$ $(i \neq j)$. 


Putting \( n_\eta = n \) and writing \( V^\eta_i \) for \( V_{yi} \) \((i = 1, \ldots, n_\eta)\), we obtain (ii). \( \square \)

Thus the proof of Theorem 1 is completed.

4. PROOF OF THEOREM 2

In this section, we prove Theorem 2. Throughout this section, \( T \) is a homomorphism from \( \text{Lip}(X, C(K)) \) into \( \text{Lip}(Y, C(M)) \) with the form (11) in Theorem 1. Of course, the set \( D \) and the mappings \( \varphi \) and \( \psi \) are as in Theorem 1. Since \( T \) is bounded, we use its norm \( \|T\| \) again. Let \( B_{\text{Lip}(X, C(K))} \) be the unit ball of \( \text{Lip}(X, C(K)) \), that is,

\[
B_{\text{Lip}(X, C(K))} = \{ f \in \text{Lip}(X, C(K)) : \| f \|_{\text{Lip}(X, C(K))} \leq 1 \}.
\]

4.1. PROOF OF SUFFICIENCY. We first show the “if” part in Theorem 2.

Suppose that \( \varphi \) and \( \psi \) satisfy (iii) and (iv) respectively. We prove that \( T \) is compact. Here we may assume that \( T \neq O \), otherwise there is nothing to prove.

**Lemma 4.1.** Let \((y_0, \eta_0) \in D\). For any \( \varepsilon > 0 \), there exists an open neighborhood \( \Theta \) of \( \eta_0 \) in \( M \) such that

\[
(17) \quad \eta \in \Theta \implies \sup_{f \in B_{\text{Lip}(X, C(K))}} |(Tf)(y_0, \eta) - (Tf)(y_0, \eta_0)| < \varepsilon.
\]

**Proof.** Put \( D_{y_0} = \{ \eta \in M : (y_0, \eta) \in D \} \). Since \( \eta_0 \in D_{y_0} \), by (iv), there exists \( j \in \{1, \ldots, n_{y_0}\} \) such that \( \eta_0 \in \Omega^j_{y_0} \). Then \( \Omega^j_{y_0} \) is a clopen subset on which \( \psi_{y_0} \) is constant. Hence if \( \eta \in \Omega^j_{y_0} \), then \( \psi(y_0, \eta) = \psi_{y_0}(\eta) = \psi_{y_0}(\eta_0) = \psi(y_0, \eta_0) \). Let \( \varepsilon > 0 \) and put

\[
\Theta = \{ \eta \in \Omega^j_{y_0} : d_X (\varphi_{y_0}(\eta), \varphi_{y_0}(\eta_0)) < \varepsilon \}.
\]

Since \( \varphi_{y_0} : D_{y_0} \to X \) is continuous, \( \Theta \) is an open neighborhood of \( \eta_0 \) in \( D_{y_0} \). For any \( \eta \in \Theta \), put \( x = \varphi(y_0, \eta) \), \( x_0 = \varphi(y_0, \eta_0) \) and \( \xi = \psi(y_0, \eta) = \psi(y_0, \eta_0) \). Then, for any \( f \in B_{\text{Lip}(X, C(K))} \), we have

\[
\|(Tf)(y_0, \eta) - (Tf)(y_0, \eta_0)\| = \|f(\varphi(y_0, \eta), \psi(y_0, \eta)) - f(\varphi(y_0, \eta_0), \psi(y_0, \eta_0))\|
\]
\[
= \|f(x, \xi) - f(x_0, \xi)\| = \|f_x(\xi) - f_{x_0}(\xi)\|
\]
\[
\leq \|f_x - f_{x_0}\|_{C(K)}
\]
\[
\leq \mathcal{L}_{X, C(K)}(f) d_X(x, x_0)
\]
\[
= \mathcal{L}_{X, C(K)}(f) d_X(\varphi(y_0, \eta), \varphi(y_0, \eta_0))
\]
\[
= \mathcal{L}_{X, C(K)}(f) d_X(\varphi_{y_0}(\eta), \varphi_{y_0}(\eta_0))
\]
\[
\leq \|f\|_{\text{Lip}(X, C(K))} \varepsilon \leq \varepsilon.
\]

Hence we obtain (17). \( \square \)

**Lemma 4.2.** In \( C(Y \times M) \), the closure of \( T(B_{\text{Lip}(X, C(K))}) \) is compact.
Proof. According to Arzelá-Ascoli theorem ([2, Theorem IV.6.7]), we show that $T(\mathcal{B}_{\text{Lip}(X,C(K))})$ is bounded and equicontinuous on $Y \times M$.

The boundedness follows from an easy computation:

$$\|(Tf)(y, \eta)\| \leq \|Tf\|_{C(Y \times M)} \leq \|Tf\|_{\text{Lip}(Y,C(M))} \leq \|T\| \|f\|_{\text{Lip}(X,C(K))} \leq \|T\|$$

for all $(y, \eta) \in Y \times M$ and all $f \in \mathcal{B}_{\text{Lip}(X,C(K))}$.

The equicontinuity will be shown as follows: Clearly, $T(\mathcal{B}_{\text{Lip}(X,C(K))})$ is equicontinuous on the clopen set $(Y \times M) \setminus \mathcal{D}$, because $Tf = 0$ on $(Y \times M) \setminus \mathcal{D}$ for all $f \in \text{Lip}(X,C(K))$, by (1). To show that $T(\mathcal{B}_{\text{Lip}(X,C(K))})$ is equicontinuous at each $(y_0, \eta_0) \in \mathcal{D}$, let $\varepsilon > 0$. Take an open neighborhood $\Theta$ of $\eta_0$ in $M$ as in Lemma 4.1 and put $V = \{y \in Y : d_Y(y, y_0) < \varepsilon/\|T\|\}$. Define an open neighborhood $\mathcal{W}$ of $(y_0, \eta_0)$ in $Y \times M$ as

$$\mathcal{W} = (V \times \Theta) \cap \mathcal{D}.$$ 

Then, for any $(y, \eta) \in \mathcal{W}$ and $f \in \mathcal{B}_{\text{Lip}(X,C(K))}$, we have

$$\|(Tf)(y, \eta) - (Tf)(y_0, \eta_0)\| \leq \|(Tf)y - (Tf)y_0\|_{C(M)} \leq \mathcal{L}_{Y,C(M)}(Tf) d_Y(y, y_0)$$

$$\leq \|Tf\|_{\text{Lip}(Y,C(M))} (\varepsilon/\|T\|) \leq \|T\| \|f\|_{\text{Lip}(X,C(K))} (\varepsilon/\|T\|) \leq \varepsilon,$$

because $y \in V$, while Lemma 4.1 implies

$$\|(Tf)(y_0, \eta) - (Tf)(y_0, \eta_0)\| < \varepsilon,$$

because $\eta \in \Theta$. Hence the triangle inequality shows that

$$(y, \eta) \in \mathcal{W} \implies \sup_{f \in \mathcal{B}_{\text{Lip}(X,C(K))}} \|(Tf)(y, \eta) - (Tf)(y_0, \eta_0)\| < 2\varepsilon.$$

Thus we conclude that $T(\mathcal{B}_{\text{Lip}(X,C(K))})$ is equicontinuous on $Y \times M$.  

\textbf{Lemma 4.3.} For any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that

$$\|(Tf)\|_{\text{Lip}(Y,C(M))} \leq \varepsilon + c_\varepsilon \|Tf\|_{C(Y \times M)}$$

for all $f \in \mathcal{B}_{\text{Lip}(X,C(K))}$.

\textit{Proof.} Fix $\varepsilon > 0$. By (iii), there exists a $\delta_\varepsilon > 0$ such that

$$d_Y(y, y') < \delta_\varepsilon \implies \frac{d_X(\varphi(y, \eta), \varphi(y', \eta))}{d_Y(y, y')} < \varepsilon.$$

Let $f \in \mathcal{B}_{\text{Lip}(X,C(K))}$, and choose $(y, \eta), (y', \eta) \in Y \times M$ with $y \neq y'$. We consider three cases.

\textbf{Case 1} $(y, \eta), (y', \eta) \in \mathcal{D}$ : By (ii) in Theorem 1, $y \in V_i^\eta$ and $y' \in V_{i'}^\eta$ for some $i, i' \in \{1, \ldots, n_\eta\}$. We first consider the case $i = i'$. If $d_Y(y, y') < \delta_\varepsilon$, then the computation (7) using (19) instead of (2) gives

$$\|(Tf)(y, \eta) - (Tf)(y', \eta)\| \leq \mathcal{L}_{X,C(K)}(f) d_X(\varphi(y, \eta), \varphi(y', \eta))$$

$$\leq \mathcal{L}_{X,C(K)}(f) \varepsilon d_Y(y, y')$$

$$\leq \|f\|_{\text{Lip}(X,C(K))} \varepsilon d_Y(y, y') \leq \varepsilon d_Y(y, y').$$
On the other hand, if \( d_{Y}(y, y') \geq \delta_{\varepsilon} \), then

\[
(21) \quad \left| \frac{(Tf)(y, \eta) - (Tf)(y', \eta)}{d_{Y}(y, y')} \right| \leq (Tf)(y, \eta) + |(Tf)(y', \eta)| \leq \frac{2\|Tf\|_{\mathcal{C}(Y \times M)}}{\delta_{\varepsilon}}.
\]

In case that \( i \neq i' \), we have \( d_{Y}(y, y') \geq r \) by \((3)\), and so

\[
(22) \quad \left| \frac{(Tf)(y, \eta) - (Tf)(y', \eta)}{d_{Y}(y, y')} \right| \leq \frac{2\|Tf\|_{\mathcal{C}(Y \times M)}}{r}.
\]

**Case 2** \((y, \eta) \in \mathcal{D} \) and \((y', \eta) \in (Y \times M) \setminus \mathcal{D} : \) Then Lemma 3.1 says that \( d_{Y}(y, y') \geq \rho \) and so

\[
(23) \quad \left| \frac{(Tf)(y, \eta) - (Tf)(y', \eta)}{d_{Y}(y, y')} \right| \leq \frac{|(Tf)(y, \eta)|}{\rho} \leq \frac{\|Tf\|_{\mathcal{C}(Y \times M)}}{\rho}.
\]

**Case 3** \((y, \eta), (y', \eta) \in (Y \times M) \setminus \mathcal{D} : \) By \((1)\),

\[
(24) \quad (Tf)(y, \eta) - (Tf)(y', \eta) = 0.
\]

Now, put \( \bar{\varepsilon} = \max\{2/\delta_{\varepsilon}, 2/r, 1/\rho\} \). We combine \((20)-(24)\) to get

\[
\mathcal{L}_{Y,C(M)}(Tf) = \sup_{y, y' \in Y} \sup_{\eta \in M} \left| (Tf)(y, \eta) - (Tf)(y', \eta) \right|_{d_{Y}(y, y')} \leq \max\{\varepsilon, \bar{\varepsilon}\|Tf\|_{\mathcal{C}(Y \times M)}\}.
\]

Hence

\[
\|Tf\|_{\operatorname{Lip}(Y,C(M))} \leq \varepsilon + (\bar{\varepsilon} + 1)\|Tf\|_{\mathcal{C}(Y \times M)},
\]

which is \((18)\). \(\square\)

**Lemma 4.4.** In \( \operatorname{Lip}(Y, C(M)) \), the closure of \( T\left(\overline{\mathcal{B}}_{\operatorname{Lip}(X,C(K))}\right) \) is compact.

**Proof.** Let \( \{f_{n}\} \) be an arbitrary sequence in \( \overline{\mathcal{B}}_{\operatorname{Lip}(X,C(K))} \). By Lemma 4.3, there exist a subsequence \( \{f_{n_{i}}\} \) and a function \( g \in C(Y \times M) \) such that \( \|Tf_{n_{i}} - g\|_{\mathcal{C}(Y \times M)} \rightarrow 0 \). To see that \( \{Tf_{n_{i}}\} \) is a Cauchy sequence in \( \operatorname{Lip}(Y, C(M)) \), let \( \varepsilon > 0 \). Since \( \{Tf_{n_{i}}\} \) is a Cauchy sequence in \( C(Y \times M) \), there exists an \( N \) such that \( i, j \geq N \) implies \( \|Tf_{n_{i}} - Tf_{n_{j}}\|_{\mathcal{C}(Y \times M)} < \varepsilon/c_{\varepsilon} \). Substituting \( f = (f_{n_{i}} - f_{n_{j}})/2 \) in \((18)\), we see

\[
i, j \geq N \text{ implies } \|Tf_{n_{i}} - Tf_{n_{j}}\|_{\operatorname{Lip}(Y,C(M))} \leq 2\varepsilon + c_{\varepsilon} \|Tf_{n_{i}} - Tf_{n_{j}}\|_{\mathcal{C}(Y \times M)} < 3\varepsilon.
\]

Hence \( \{Tf_{n_{i}}\} \) is a Cauchy sequence in \( \operatorname{Lip}(Y, C(M)) \), and so it converges to some function in \( \operatorname{Lip}(Y, C(M)) \). Thus we conclude that the closure of \( T\left(\overline{\mathcal{B}}_{\operatorname{Lip}(X,C(K))}\right) \) is compact in \( \operatorname{Lip}(Y, C(M)) \). \(\square\)

Lemma 4.4 says that \( T \) is a compact operator from \( \operatorname{Lip}(X, C(K)) \) into \( \operatorname{Lip}(Y, C(M)) \), and the “if” part was proved.
4.2. Proof of Necessity. In the sequel, we suppose that \( T \) is compact.

**Lemma 4.5.** \( \varphi \) satisfies (iii).

*Proof*. Assume, to reach a contradiction, that \( \varphi \) does not satisfy (iii). Then there exist an \( \varepsilon_0 > 0 \) and two sequences \( \{(y_n, \eta_n)\} \) and \( \{(y'_n, \eta_n)\} \) in \( \mathcal{D} \) such that

\[
0 < d_Y(y_n, y'_n) < \frac{1}{n^2} \quad \text{and} \quad \frac{d_X(\varphi(y_n, \eta_n), \varphi(y'_n, \eta_n))}{d_Y(y_n, y'_n)} \geq \varepsilon_0.
\]

Put \( z_n = \varphi(y_n, \eta_n) \) and \( z'_n = \varphi(y'_n, \eta_n) \) for \( n = 1, 2, \ldots \). In order to arrange the distance \( d_X \), we here introduce a function \( \chi_n : \)

\[
\chi_n(t) = \frac{1}{2n}(1 - e^{-nt}) \quad (t \in [0, \infty)).
\]

Clearly, \( 0 \leq \chi_n \leq 1/2 \) and \( \chi_n \) is differentiable and \( \chi'_n(t) = e^{-nt}/2 \). Define

\[
f_n(x) = \chi_n(d_X(x, z'_n)) \quad (x \in X).
\]

For any \( x, x' \in X \) with \( x \neq x' \), the mean value theorem gives a point \( s_n \) between \( d_X(x, z'_n) \) and \( d_X(x', z'_n) \) such that

\[
\chi_n(d_X(x, z'_n)) - \chi_n(d_X(x', z'_n)) = \chi'_n(s_n)(d_X(x, z'_n) - d_X(x', z'_n)),
\]

and so

\[
|f_n(x) - f_n(x')| = |\chi'_n(s_n)| |d_X(x, z'_n) - d_X(x', z'_n)| \leq \frac{e^{-ns_n}}{2} d_X(x, x').
\]

Hence \( f_n \in \text{Lip}(X) \) and \( \|f_n\|_{\text{Lip}(X)} = \|f_n\|_{C(X)} + L_X, \mathcal{C}(f_n) \leq \frac{1}{2n} + \frac{e^{-ns_n}}{2} \leq 1 \).

Now put \( \hat{f}_n(x, \xi) = f_n(x) \) for all \( (x, \xi) \in X \times K \). Then \( \hat{f}_n \in \text{Lip}(X, C(K)) \) and \( \|\hat{f}_n\|_{\text{Lip}(X, C(K))} \leq 1 \), that is, \( \hat{f}_n \in \mathcal{B}_{\text{Lip}(X, C(K))} \).

Next we estimate the norm \( \|T \hat{f}_n\|_{\text{Lip}(Y, C(M))} \). We use the mean value theorem again, we compute as follows:

\[
|(T \hat{f}_n)(y_n, \eta_n) - (T \hat{f}_n)(y'_n, \eta_n)| = |\hat{f}_n(\varphi(y_n, \eta_n), \psi(y'_n, \eta_n)) - \hat{f}_n(\varphi(y'_n, \eta_n), \psi(y_n, \eta_n))|
\]

\[
= |\hat{f}_n(z_n) - \hat{f}_n(z'_n)| = |\chi_n(d_X(z_n, z'_n)) - \chi_n(0)|
\]

\[
= |\chi'_n(\sigma_n)| |d_X(z_n, z'_n) - 0|
\]

\[
= \frac{e^{-n\sigma_n}}{2} d_X(\varphi(y_n, \eta_n), \varphi(y'_n, \eta_n))
\]

\[
\geq \frac{e^{-n\sigma_n}}{2} \varepsilon_0 d_Y(y_n, y'_n),
\]

where \( 0 \leq \sigma_n \leq d_X(z_n, z'_n) \). Hence

\[
(25) \quad \|T \hat{f}_n\|_{\text{Lip}(Y, C(M))} \geq L_{Y, C(M)}(T \hat{f}_n) \geq \frac{|(T \hat{f}_n)_{y_n} - (T \hat{f}_n)_{y'_n}|}{d_Y(y_n, y'_n)} \geq \frac{e^{-n\sigma_n}}{2} \varepsilon_0.
\]
While (2) in Theorem 1 implies
\[ 0 \leq \sigma_n \leq d_X(z_n, z'_n) = d_X(\varphi(y_n, \eta_n), \varphi(y'_n, \eta_n)) \leq L d_Y(y_n, y'_n) \leq \frac{1}{n^2}, \]
and so \( n\sigma_n \to 0 \). Thus (25) implies
\[
\liminf_{n \to \infty} \| T\hat{f}_n \|_{\text{Lip}(Y,C(M))} \geq \frac{\varepsilon_0}{2}.
\]

Recall that \( T \) is compact. Since \( \{ \hat{f}_n \} \subset \mathbb{B}_{\text{Lip}(X,C(K))} \), there exist a subsequence \( \{ \hat{f}_{n_i} \} \) and a function \( g \in \text{Lip}(Y,C(M)) \) such that \( \| T\hat{f}_{n_i} - g \|_{\text{Lip}(Y,C(M))} \to 0 \). Since
\[
\| T\hat{f}_{n_i} - g \|_{C(Y \times M)} \leq \| T\hat{f}_{n_i} - g \|_{\text{Lip}(Y,C(M))},
\]
we have \( (T\hat{f}_{n_i})(y, \eta) \to g(y, \eta) \) for each \( (y, \eta) \in Y \times M \). If \( (y, \eta) \in \mathcal{D} \), then
\[
|(T\hat{f}_{n_i})(y, \eta)| = |\hat{f}_{n_i}(\varphi(y, \eta), \psi(y, \eta))| = |f_{n_i}(\varphi(y, \eta))| \leq \frac{1}{2n_i} \to 0,
\]
while if \( (y, \eta) \in (Y \times M) \setminus \mathcal{D} \), then \( (T\hat{f}_{n_i})(y, \eta) = 0 \). As a result, we have \( g(y, \eta) = 0 \) for all \( (y, \eta) \in Y \times M \), and so
\[
\| T\hat{f}_{n_i} \|_{\text{Lip}(Y,C(M))} \to 0.
\]
This contradicts (26).

Fix \( y \in Y \) and put \( \mathcal{D}_y = \{ \eta \in M : (y, \eta) \in \mathcal{D} \} \).

**Lemma 4.6.** For any \( \eta_0 \in \mathcal{D}_y \), there exists an open neighborhood of \( \eta_0 \) in \( \mathcal{D}_y \) on which \( \psi_y \) is constant.

**Proof.** Since \( \mathcal{D}_y \) is a compact subset of \( M \), we can treat the Banach algebra \( C(\mathcal{D}_y) \) and a projection \( P \) from \( \text{Lip}(Y,C(M)) \) into \( C(\mathcal{D}_y) : \)
\[
(Pg)(\eta) = g(y, \eta) \quad (\eta \in \mathcal{D}_y, \ g \in \text{Lip}(Y,C(M))).
\]
Clearly \( P \) is a bounded linear operator from \( \text{Lip}(Y,C(M)) \) into \( C(\mathcal{D}_y) \).

Now put \( S = PT \). Since \( T \) is compact, \( S \) is a compact operator from \( \text{Lip}(X,C(K)) \) into \( C(\mathcal{D}_y) \). Hence Arzelá-Ascoli theorem says that \( S(\mathbb{B}_{\text{Lip}(X,C(K))}) \) is equicontinuous on \( \mathcal{D}_y \). Hence there exists an open neighborhood \( \Theta \) of \( \eta_0 \) such that
\[
\eta \in \Theta \implies \sup_{f \in \mathbb{B}_{\text{Lip}(X,C(K))}} \left| (Sf)(\eta) - (Sf)(\eta_0) \right| < \frac{1}{2}.
\]

Conversely, assume that there exist \( \eta_1 \in \Theta \) such that \( \psi_y(\eta_1) \neq \psi_y(\eta_0) \). By Urysohn’s lemma, there exists a \( u \in C(M) \) such that \( 0 \leq u \leq 1 \), \( u(\psi_y(\eta_1)) = 1 \) and \( \psi(\psi_y(\eta_0)) = 0 \). Put \( \tilde{u}(x, \xi) = u(\xi) \) for all \( (x, \xi) \in X \times K \). Then \( \tilde{u} \in \mathbb{B}_{\text{Lip}(X,C(K))} \). Hence (27) implies
\[
\left| (S\tilde{u})(\eta_1) - (S\tilde{u})(\eta_0) \right| < \frac{1}{2}.
\]
But
\[ \left| (S \tilde{u})(\eta_1) - (S \tilde{u})(\eta_0) \right| = \left| (P T \tilde{u})(\eta_1) - (P T \tilde{u})(\eta_0) \right| \]
\[ = \left| (T \tilde{u})(y, \eta_1) - (T \tilde{u})(y, \eta_0) \right| \]
\[ = \left| u(\varphi(y, \eta_1), \psi(y, \eta_1)) - \tilde{u}(\varphi(y, \eta_0), \psi(y, \eta_0)) \right| \]
\[ = \left| u(\psi(y, \eta_1)) - u(\psi(y, \eta_0)) \right| = \left| u(\psi_y(\eta_1)) - u(\psi_y(\eta_0)) \right| = 1 \]
a contradiction. Thus we conclude that \( \psi_y \) is constant on \( \Theta \). \( \square \)

**Lemma 4.7.** \( \psi \) satisfies (iv).

**Proof.** For any \( \eta \in \mathcal{D}_y \), put
\[ \Omega^\eta = \{ \zeta \in \mathcal{D}_y : \psi_y(\zeta) = \psi_y(\eta) \}. \]
Clearly, \( \psi_y \) is constant on \( \Omega^\eta \). Also, we have
\[ \Omega^\eta \cap \Omega^{\eta'} = \emptyset \implies \Omega^\eta = \Omega^{\eta'}. \]
Since \( \psi_y \) is continuous, \( \Omega^\eta \) is a closed subset of \( \mathcal{D}_y \). Also we can easily see that Lemma 4.6 implies that \( \Omega^\eta \) is an open subset of \( \mathcal{D}_y \). Thus \( \Omega^\eta \) is a clopen subset of \( \mathcal{M} \).

Note that
\[ \mathcal{D}_y = \bigcup_{\eta \in \mathcal{D}_y} \Omega^\eta. \]
Since \( \mathcal{D}_y \) is compact, we can select finitely many \( \eta_1, \ldots, \eta_n \in \mathcal{D}_y \) such that
\[ \mathcal{D}_y = \bigcup_{i=1}^n \Omega^{\eta_i}. \]
By (28), we may assume that \( \Omega^{\eta_1}, \ldots, \Omega^{\eta_m} \) are disjoint. Putting \( n_y = n \) and writing \( \Omega_i^y = \Omega^{\eta_i} (i = 1, \ldots, n_y) \), we obtain (iv). \( \square \)

5. **Applications**

we consider the following three conditions:

Consider the case that \( K \) is a one-point set. Then Lip\( (X, C(K)) \) is isometrically isomorphic to Lip\( (X) \). On the other hand, if \( X \) is a one-point set, Lip\( (X, C(K)) \) is isometrically isomorphic to \( C(K) \).

**Corollary 1.** Suppose that \( X \) and \( Y \) are compact metric spaces with metrics \( d_X \) and \( d_Y \) respectively.

(I) If \( T \) is a homomorphism from Lip\( (X) \) into Lip\( (Y) \), then there exist a clopen subset \( Y_0 \) of \( Y \) and a continuous mapping \( \varphi : Y_0 \to X \) with
\[ \sup_{y, y' \in Y_0, y \neq y'} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} < \infty \]
such that $T$ has the form:

\begin{equation}
(Tf)(y) = \begin{cases} f(\varphi(y)) & (y \in Y_0) \\ 0 & (y \in Y \setminus Y_0) \end{cases}
\end{equation}

for all $f \in \text{Lip}(X)$. Conversely, if $Y_0, \varphi$ are given as above, then $T$ defined by (29) is a homomorphism from $\text{Lip}(X)$ into $\text{Lip}(Y)$. Moreover, $T$ is unital if and only if $Y_0 = Y$.

(II) Suppose that $T$ is a homomorphism from $\text{Lip}(X)$ into $\text{Lip}(Y)$ with the form (29). Then $T$ is compact if and only if

\[
\lim_{y, y' \in Y_0 \atop d_Y(y, y') \to 0} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} = 0.
\]

Now we turn to another setting.

**Corollary 2.** Suppose that $K$ and $M$ are compact Hausdorff spaces.

(I) If $T$ is a homomorphism from $C(K)$ into $C(M)$, then there exist a clopen subset $M_0$ of $M$ and a continuous mapping $\psi : M_0 \to K$ such that $T$ has the form:

\begin{equation}
(Tf)(\eta) = \begin{cases} f(\psi(\eta)) & (\eta \in M_0) \\ 0 & (\eta \in M \setminus M_0) \end{cases}
\end{equation}

for all $f \in C(K)$. Conversely, if $M_0, \psi$ are given as above, then $T$ defined by (30) is a homomorphism from $C(K)$ into $C(M)$. Moreover, $T$ is unital if and only if $M_0 = M$.

(II) Suppose that $T$ is a homomorphism from $C(K)$ into $C(M)$ with the form (30). Then $T$ is compact if and only if $M_0$ is a union of finitely many clopen subset $M_1, \ldots, M_n$ such that $\psi$ is constant on each $M_i$ for $i = 1, \ldots, n$. Moreover, $T$ is compact if and only if $T$ has a finite rank,

**Corollary 3.** Suppose that $X$ is a compact metric space with metric $d_X$, and that $M$ is a compact Hausdorff space.

(I) If $T$ is a homomorphism from $\text{Lip}(X)$ into $C(M)$, then there exist a clopen subset $M_0$ of $M$ and a continuous mapping $\varphi : M_0 \to X$ such that $T$ has the form:

\begin{equation}
(Tf)(\eta) = \begin{cases} f(\varphi(\eta)) & (\eta \in M_0) \\ 0 & (\eta \in M \setminus M_0) \end{cases}
\end{equation}

for all $f \in \text{Lip}(X)$. Conversely, if $M_0, \varphi$ are given as above, then $T$ defined by (31) is a homomorphism from $\text{Lip}(X)$ into $C(M)$. Moreover, $T$ is unital if and only if $M_0 = M$.

(II) Every homomorphism from $\text{Lip}(X)$ into $C(M)$ is compact.
Corollary 4. Suppose that $Y$ is a compact metric space with metric $d_Y$, and that $K$ is a compact Hausdorff space.

(I) If $T$ is a homomorphism from $C(K)$ into $\text{Lip}(Y)$, then $Y$ is a union of finitely many disjoint clopen subset $Y_0, Y_1, \ldots, Y_n$ and there exist constant mappings $\psi_i : Y_i \to K$ ($i = 1, \ldots, n$) such that $T$ has the form:

\begin{equation}
(Tf)(y) = \begin{cases} 
f(\psi_i(y)) & (y \in Y_i, i = 1, \ldots, n) \\
0 & (y \in Y_0) 
\end{cases}
\end{equation}

for all $f \in C(K)$. Conversely, if $Y_0, Y_1, \ldots, Y_n, \psi_1, \ldots, \psi_n$ are given as above, then $T$ defined by (32) is a homomorphism from $C(K)$ into $\text{Lip}(Y)$. Moreover, $T$ is unital if and only if $Y_0 = \emptyset$.

(II) Every homomorphism from $C(K)$ into $\text{Lip}(Y)$ has a finite rank.

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