ITERATED PARAPRODUCTS AND ITERATED COMMUTATOR ESTIMATES IN 
BESOV SPACES

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Abstract. We extend the results in [4] to Besov spaces $B^{\alpha,0}_{p,q}$ with $p, q \in [1, \infty]$ and $0 < \alpha < 1$.

1. Introduction

It is well known that, the definition of Besov space in Euclidean space $\mathbb{R}^d$ by Littlewood-Paley theory is equivalent to the one based on the estimate of Taylor remainder, when the regularity parameter is positive. See [1] Theorem 2.36 for instance. Especially, Besov space $B^{\alpha,0}_{\infty,\infty}(\mathbb{R}^d)$ is the same as Hölder space $C^{\alpha}(\mathbb{R}^d)$ if $\alpha$ is a positive noninteger.

In [4], we showed a similar equivalence result for Bony’s paraproduct and its iterated versions. For any distributions $f, g \in \mathcal{S}'(\mathbb{R}^d)$, the paraproduct $f \prec g$ is defined via Littlewood-Paley theory, so this is not a local operator. Nevertheless, in the case $f \in C^{\alpha}(\mathbb{R}^d)$ and $g \in C^{\beta}(\mathbb{R}^d)$ with $0 < \alpha, \beta$ and $\alpha + \beta < 1$, the previous result [4] Theorem 3.1 implies

\begin{equation}
(f \prec g)(y) = (f \prec g)(x) + f(x)(g(y) - g(x)) + O(|y - x|^\alpha \beta).
\end{equation}

Conversely, we can show that the function $h$ of such a local behavior is essentially the same as $f \prec g$. In [4], we studied a generalized version of (1.1) for the iterated paraproducts, and as a consequence, we also gave an algebraic proof of the commutator estimate [4] Lemma 2.4, which has an important role in the theory of paraprocontrolled distributions.

In this paper, we consider the Besov type extension of the results in [4]. First we show the estimate like (1.1), see Theorem 3.1 below. The result is no longer a uniform bound on $\mathbb{R}^d$, but an $L^p L^q$ type estimate of Taylor remainder. As a consequence, we also show the commutator estimate in Besov spaces, stated as below. Commutators discussed in this paper is defined as follows.

Definition 1.1. For any functions $\xi, f_1, f_2, \ldots$ in $\mathcal{S}(\mathbb{R}^d)$, define

\[
\mathcal{C}(f_1, \xi) := f_1 \prec \xi := f_1 \xi - f_1 \prec \xi),
\]

\[
\mathcal{C}(f_1, f_2, \xi) := \mathcal{C}(f_1 \prec f_2, \xi) - f_1 \mathcal{C}(f_2, \xi),
\]

\[
\mathcal{C}(f_1, \ldots, f_n, \xi) := \mathcal{C}(f_1 \prec f_2, \ldots, f_n, \xi) - f_1 \mathcal{C}(f_2, \ldots, f_n, \xi).
\]

We denote by $B^{\alpha,0}_{p,q}$ the closure of $\mathcal{S}(\mathbb{R}^d)$ in the space $B^{\alpha,0}_{p,q}(\mathbb{R}^d)$. The following theorem is a generalization of [4] Theorem 4.2 onto Besov norms.

Theorem 1.1. Let $\alpha_1, \ldots, \alpha_n \in (0, 1)$ and $\alpha_0 < 0$ be such that

\[
\alpha_1 + \cdots + \alpha_n < 1,
\]

\[
\alpha_2 + \cdots + \alpha_n + \alpha_0 < 0 < \alpha_1 + \cdots + \alpha_n + \alpha_0,
\]

and let $\alpha := \alpha_1 + \cdots + \alpha_n + \alpha_0$. Let $p_1, \ldots, p_n, p_0, q_1, \ldots, q_n, q_0 \in [1, \infty]$ be such that

\[
\frac{1}{p} := \frac{1}{p_1} + \cdots + \frac{1}{p_n} + \frac{1}{p_0} \leq 1, \quad \frac{1}{q} := \frac{1}{q_1} + \cdots + \frac{1}{q_n} + \frac{1}{q_0} \leq 1.
\]

Then there exists a unique multilinear continuous operator

\[
\tilde{\mathcal{C}} : B^{\alpha_1,0}_{p_1, q_1} \times \cdots \times B^{\alpha_n,0}_{p_n, q_n} \times B^{\alpha_0,0}_{p_0, q_0} \to B^{\alpha,0}_{p,q}
\]

such that,

\[
\tilde{\mathcal{C}}(f_1, \ldots, f_n, \xi) = \mathcal{C}(f_1, \ldots, f_n, \xi)
\]

for any smooth inputs $(f_1, \ldots, f_n, \xi)$. 

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To show the main theorems, we introduce a regularity structure suitable for our context. Note that, in [1] [8], the authors defined a Besov type modelled distribution and proved a generalized reconstruction theorem in their settings. Although they imposed $B_{p,q}$ type bounds on models, we also impose $B_{p,q}$ type bounds on models, not only on modelled distributions. In our case, each basis vector $\tau$ of the model space has three homogeneity parameters $(\alpha_\tau, p_\tau, q_\tau)$, which is a slightly different situation from the original one [3].

This paper is organized as follows. In Section 2, we define some important notations used in this paper; Besov type norms, paraproducts, and the word Hopf algebra. In Section 3, we show the Besov type estimates of Taylor remainders of iterated paraproducts. In Section 4, we show the Besov type commutator estimates.

2. Preliminaries

We introduce some important notions used through this paper.

2.1. Besov type norms. In this paper, we often use a sequence $\{a_j\}_{j=-1}^\infty$ of numbers, functions, or operators. We use simplifying notations for partial sums as follows.

$$a_{< j} := \sum_{i < j} a_i, \quad a_{\geq j} := \sum_{i \geq j} a_i.$$ 

Lemma 2.1. Let $q \in [1, \infty]$ and let $\{c_j\}_j$ be a sequence of nonnegative numbers. If $\alpha > 0$, then we have

$$(2.1) \quad \|\{2^j c_{\geq j}\}_j\|_{\ell^q} \lesssim \|\{2^j c_j\}_j\|_{\ell^q},$$

$$(2.2) \quad \|\{2^{-j} c_{\leq j}\}_j\|_{\ell^q} \lesssim \|\{2^{-j} c_j\}_j\|_{\ell^q}.$$ 

Proof. For (2.1), by Young’s inequality,

$$\|\{2^j c_{\geq j}\}_j\|_{\ell^q} = \left\{ \left( \sum_{j \geq j} 2^{(j-j)\alpha} 2^{j\alpha} c_j \right)_j \right\}_{\ell^q} \leq \sum_{i \geq 0} 2^{\alpha i} \|\{2^{j\alpha} c_j\}_j\|_{\ell^q} \lesssim \|\{2^{j\alpha} c_j\}_j\|_{\ell^q}.$$ 

The proof of (2.2) is just an analogue. \(\square\)

Denote by $S = S(\mathbb{R}^d)$ the space of Schwartz functions, and by $S'$ its dual space. Fix smooth radial functions $\chi$ and $\rho$ such that,

- $\text{supp}(\chi) \subset \{x; |x| < \frac{3}{4}\}$ and $\text{supp}(\rho) \subset \{x; \frac{1}{4} < |x| < \frac{5}{4}\}$,
- $\chi(x) + \sum_{j=0}^\infty \rho(2^{-j} x) = 1$ for any $x \in \mathbb{R}^d$.

Set $\rho_{-1} := \chi$ and $\rho_j := \rho(2^{-j} \cdot)$ for $j \geq 0$. We define the Littlewood-Paley blocks

$$\Delta_j f := \mathcal{F}^{-1}(\rho_j \mathcal{F} f)$$

for $f \in S'$, where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse. It is useful to write

$$\Delta_j f(x) = \int_{\mathbb{R}^d} Q_j(x, y) f(y) dy,$$

where $Q_j(x, y) = \mathcal{F}^{-1}(\rho_j)(x - y)$. We also write $Q_j(h) = \mathcal{F}^{-1}(\rho_j)(h)$.

Definition 2.1. For any $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$, we define the (nonhomogeneous) Besov space $B^\alpha_{p,q}$ by the space of all $f \in S'$ such that

$$\|f\|_{B^\alpha_{p,q}} := \left\| \left\{ 2^j \|\Delta_j f\|_{L^p} \right\}_{j \geq -1} \right\|_{\ell^q} < \infty.$$ 

As stated in [1 Theorem 2.36], it is possible to define Besov norms without Littlewood-Paley theory. The aim of this paper is to study the following norm for two parameter functions.

Definition 2.2. Let $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$. For any two parameter measurable function $\omega(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$, define

$$\|\omega\|_{D^\alpha_{p,q}} := \left\| |h|^{-\alpha} |\omega(x, x + h)|_{L^p(dx)} \right\|_{L^q(dh/|h|^d)}.$$
Proposition 2.2 ([1] Theorem 2.36). If \( \alpha \in (0, 1) \), then
\[
\|f\|_{B^\alpha_{p,q}} \lesssim \|f\|_{L^p} + \|\omega_f\|_{D^\alpha_{p,q}},
\]
where \( \omega_f(x,y) = f(y) - f(x) \).

2.2. Technical lemmas. We prove some technical lemmas used through this paper.

Definition 2.3. Let \( \alpha \in \mathbb{R} \) and \( p,q \in [1,\infty] \). For any sequence \( \{f_j(x)\}_j \) of measurable functions on \( \mathbb{R}^d \), define
\[
\|\{f_j\}_j\|_{\mathbb{Z}_{p,q}^\omega} := \left\{ \|2^{j\alpha}||f_j||_{L^p}\|_{\ell^q} \right\}.
\]

By definition, \( \|f\|_{B^\alpha_{p,q}} = \|\{\Delta_j f\}_j\|_{\mathbb{Z}_{p,q}^\omega} \). We often emphasize the variables \( j \) and \( x \) and write
\[
\|f(x)\|_{B^\alpha_{p,q}} = \left\{ \|2^{j\alpha}||f_j||_{L^p(dx)}\|_{\ell^q} \right\}.
\]

Lemma 2.3. Let \( \alpha \in \mathbb{R} \) and \( p,q \in [1,\infty] \). Let \( F \) be a nonnegative function on \( \mathbb{R}^d \) such that
\[
\|h^{-\alpha}F(h)\|_{L^q(|dh|/|h|^q)} \leq C.
\]
Then for any nonnegative function \( \varphi \in \mathcal{S} \), one has
\[
\left\|2^{j\alpha} \int_{\mathbb{R}^d} 2^{jd}\varphi(2^j h) F(h) dh \right\|_{\ell^q} \lesssim C.
\]

Proof. Essentially contained in the latter half part of the proof of [1] Theorem 2.36.

Lemma 2.4. For any \( \omega \in D^\alpha_{p,q} \), one has the bound
\[
\|\Delta_{<j} (\omega(x,\cdot))(x)\|_{\mathbb{Z}_{p,q}^\omega} \lesssim \|\omega\|_{D^\alpha_{p,q}}.
\]

Proof. Since
\[
\Delta_{<j} (\omega(x,\cdot))(x) = \int_{\mathbb{R}^d} Q_{<j}(x,y) \omega(x,y) dy = \int_{\mathbb{R}^d} Q_{<j}(-h) \omega(x,x+h) dh,
\]
by using Minkowski's inequality and Lemma 2.3 we have
\[
\|\Delta_{<j} (\omega(x,\cdot))(x)\|_{L^p} \lesssim \int_{\mathbb{R}^d} |Q_{<j}(-h)| \|\omega(x,x+h)\|_{L^p(dx)} dh
\lesssim \|\omega\|_{D^\alpha_{p,q}} 2^{-j\alpha} 1^q_j,
\]
where \( 1^q_j \) denotes a sequence belonging to the unit sphere of \( \ell^q \).

Through this paper, we often use the notation \( 1^q_j \) without notice.

Lemma 2.5. Let \( \{\omega_j(x,y)\}_{j \geq -1} \) be a sequence of two parameter functions. Assume that for some \( C > 0 \) and \( \alpha \in \mathbb{R} \), the bound
\[
(2.3) \quad \|\omega_j(x+h,x)\|_{\mathbb{Z}_{p,q}^\omega} := \left\{ \|2^{(\alpha-\theta)}\|\omega_j(x+h,x)\|_{L^p}\right\}_{j \geq -1} \|_{\ell^q} \leq C|h|^{\theta}.
\]

holds for any \( h \in \mathbb{R}^d \) and any \( \theta \) in a neighborhood of \( \alpha \). Then \( \omega = \sum_{j \geq -1} \omega_j \) converges in \( D^\alpha_{p,q} \) and one has the bound
\[
\|\omega\|_{D^\alpha_{p,q}} \lesssim C.
\]

Proof. We follow the proof of [1] Theorem 2.36. Since the case \( q = \infty \) is the same as [6] Lemma 3.7, we consider \( q < \infty \). Assume \( C \leq 1 \) without loss of generality. Let
\[
A_N = \{ h \in \mathbb{R}^d ; 2^{-N-1} \leq |h| < 2^{-N} \} \quad (N \geq 0),
\]
\[
A_{-1} = \{ h \in \mathbb{R}^d ; 1 \leq |h| \}.
\]
Fix a small \( \varepsilon > 0 \) such that (2.23) holds for \( \theta = \alpha \pm \varepsilon \). If \( h \in A_N \) with \( N \geq 0 \),
\[
|h|^{-\alpha} \|\omega_j(x+h,x)\|_{L^p(dx)} \lesssim \sum_{j \geq -1} 1^q_j |h|^{\varepsilon} + \sum_{j \geq N} 1^q_j 2^{-j\varepsilon} |h|^{-\varepsilon}.
\]
By Hölder’s inequality with the weight $2^{jε}$, we have
\[
\left( \sum_{j < N} 1^{q_j} |h|^{εq} \left( \sum_{j < N} 2^{jε} \right)^{q-1} \sum_{j < N} 1^{j} 2^{jε} \right)^{q} \lesssim |h|^{εq} \left( \sum_{j < N} 1^{j} 2^{jε} \right)^{q-1} \lesssim |h|^{εq} 2^{Nε(q-1)} \sum_{j < N} 1^{j} 2^{jε},
\]
and
\[
\left( \sum_{j \geq N} 1^{q_j} 2^{jε} \right)^{q} \lesssim |h|^{-εq} 2^{-Nε(q-1)} \sum_{j \geq N} 1^{j} 2^{-jε}.
\]

Since $|h| \sim 2^{-N}$ on $A_N$, we have
\[
\int_{A_N} \left( |h|^{-α} \|ω(x+h,x)\|_{L^p(dx)} \right)^q \frac{dh}{|h|^q} \lesssim 2^{-Nε} \sum_{j < N} 1^{j} 2^{jε} + 2^{Nε} \sum_{j \geq N} 1^{j} 2^{-jε} \lesssim \sum_{j \geq -1} 2^{-j+Nε} 1^j.
\]

Summing them over $N \geq 0$, by Young’s inequality we have
\[
\sum_{N \geq 0} \sum_{j \geq -1} 2^{-j-Nε} 1^j < \infty.
\]

If $h \in A_{-1}$, similarly to above,
\[
\int_{A_{-1}} \left( |h|^{-α} \|ω(x+h,x)\|_{L^p(dx)} \right)^q \frac{dh}{|h|^q} \lesssim \sum_{j \geq -1} 1^{j} 2^{-jε} \int_{|h| \geq 1} |h|^{-εq-d} dh \lesssim 1,
\]
which completes the proof. \(\square\)

Using above lemmas, we can prove Proposition 2.2.

**Proof of Proposition 2.2.** Note that $\|Δ_j f\|_{L^p} \lesssim \|f\|_{L^p}$. For $j \geq 0$, since $∫ Q_j = 0$ we have
\[
Δ_j f(x) = Δ_j (ωf(x,.))(x).
\]

Lemma 2.4 yields $\|f\|_{B^α_{p,q}} \lesssim \|f\|_{L^p} + \|ωf\|_{D^α_{p,q}}$.

To show the converse, let $ω_j(x,y) = Δ_j f(y) - Δ_j f(x)$ and apply Lemma 2.5. Obviously we have
\[
\|ω_j(x,x+h)\|_{L^p(dx)} \lesssim \|Δ_j f\|_{L^p} \lesssim 2^{-jα} 1^q \|f\|_{B^α_{p,q}}.
\]

By Minkowski’s inequality and the continuity of the differentiation $B^α_{p,q} \ni f \mapsto Δ_j f \in (B^α_{p,q})^d$ (see Proposition 2.7.8), we have
\[
\|ω_j(x,x+h)\|_{L^p(dx)} \leq \left\| h \cdot \int_0^1 \nabla Δ_j f(x+θh) dθ \right\|_{L^p(dx)} \leq |h|\|Δ_j (\nabla f)\|_{L^p} \lesssim |h| 2^{j(1-α)} 1^q \|f\|_{B^α_{p,q}}.
\]

By an interpolation, for any $θ \in [0,1]$ we have
\[
\|ω_j(x+h,x)\|_{L^p(dx)} \lesssim |h|^θ 2^{j(θ-α)} 1^q \|f\|_{B^α_{p,q}},
\]
so $\|ωf\|_{D^α_{p,q}} \lesssim \|f\|_{B^α_{p,q}}$ by Lemma 2.5. \(\square\)
2.3. Paraproduct. For any smooth functions $f, g$, we can decompose the product $fg$ as follows.

$$fg = \sum_{j, k \geq -1} \Delta_j f \Delta_k g$$

$$= \sum_{j < k - 1} \Delta_j f \Delta_k g + \sum_{|j - k| \leq 1} \Delta_j f \Delta_k g + \sum_{j + 1 < k} \Delta_j f \Delta_k g$$

$$=: f \prec g + f \circ g + f \succ g.$$ 

$f \prec g = g \succ f$ is called a paraproduct, and $f \circ g$ is called a resonant.

As in [2], we often use the two parameter extension of the paraproduct. For any measurable function $\omega(x, y)$, we define

$$P_j(\omega)(z) := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} Q_{<j-1}(z, x) Q_{j}(z, y) \omega(x, y) dx dy$$

and $P(\omega) := \sum_j P_j(\omega)$. Obviously, for the case $\omega(x, y) = f(x)g(y)$ we have $P(\omega) = f \prec g$.

Lemma 2.6. If $\omega \in D_{p,q}^\alpha$, then $P(\omega) \in B_{p,q}^\alpha$. The mapping $\omega \mapsto P(\omega)$ is continuous.

proof. In view of [2, Proposition 8],

$$\|P(\omega)\|_{D_{p,q}^\alpha} \lesssim \|P_j(\omega)\|_{D_{p,q}^\alpha}.$$ 

For the right hand side, exchanging variables $y = z + h$ and $x = z + h + k$ and using Minkowski’s inequality, we have

$$\|P_j(\omega)\|_{L^p}$$

$$\lesssim \int (Q_{<j-1}(-h)) \|Q_j(-h - k)\| \|\omega(z + h, z + h + k)\|_{L^p(dz)} dh dk$$

$$= \int 2^{j/2} \phi(2^{j/2} k) \|\omega(z, z + k)\|_{L^p(dz)} dk,$$

for some $\phi \in \mathcal{S}(\mathbb{R}^d)$. Thus Lemma 2.3 completes the proof. \hfill \Box

2.4. Word Hopf algebra. We introduce a kind of regularity structure. Fix an integer $n$. For any integers $1 \leq k \leq \ell \leq n$, denote by $(k \ldots \ell)$ the sequence from $k$ to $\ell$, which is called a word through this paper. We discuss the algebras made from the set $W$ of all such words.

Let $\text{Alg}(W)$ be the commutative algebra freely generated by $W$ with unit $1$. We regard $1$ as an empty word and consider the extended set $\overline{W} = W \cup \{\ell\}$ of words. For any nonempty words $\sigma = (k \ldots \ell)$ and $\eta = ((\ell + 1) \ldots m)$ in $W$, we define $\sigma \sqcup \eta = (k \ldots m)$. We also define $1 \sqcup \tau = \tau \sqcup 1 = \tau$.

Definition 2.4. Define the linear map $\Delta : \text{Alg}(W) \to \text{Alg}(W) \otimes \text{Alg}(W)$ by

$$\Delta \tau = \sum_{\sigma, \eta \in \overline{W}, \sigma \sqcup \eta = \tau} \sigma \otimes \eta$$

for any $\tau \in \overline{W}$.

The map $\Delta$ is coassociative; $(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta$. It is easy to show the existence of the algebra map $A : \text{Alg}(W) \to \text{Alg}(W)$ such that

$$A 1 = 1,$$

$$M(A \otimes \text{id}) \Delta \tau = M(\text{id} \otimes A) \Delta \tau = 0 \quad (\tau \in W),$$

where $M : \text{Alg}(W) \otimes \text{Alg}(W) \to \text{Alg}(W)$ is the product map. Such $A$ is called an antipode. In other words, $\text{Alg}(W)$ is a Hopf algebra. The existence of $A$ yields that, the set $G$ of all algebra maps $\gamma : \text{Alg}(W) \to \mathbb{R}$ forms a group by the product

$$(\gamma_1 \ast \gamma_2)(\tau) = (\gamma_1 \otimes \gamma_2) \Delta \tau.$$ 

The inverse of $\gamma \in G$ is given by $\gamma^{-1} = \gamma \circ A$.

In Section 3 we study the family $\{f_\tau = f_\tau(x)\}_{\tau \in W}$ of functions on $\mathbb{R}^d$, indexed by words. We regard $f(x) \in G$ by extending the map $\tau \mapsto f_\tau(x)$ algebraically. Then we define the $G$-valued two parameter function by

$$\omega(x, y) = f(x)^{-1} * f(y).$$
In other words, we have a family \( \{ \omega_\tau(x, y) := \omega(x, y)(\tau) \}_{\tau \in W} \) of two parameter functions, indexed by words. The following formulas are useful in Section 3.

**Lemma 2.7.** For any \( 1 \leq k \leq \ell \leq n \), one has

\[
\omega_{k...\ell}(x, y) = f_{k...\ell}(y) - f_{k...\ell}(x) - \sum_{m=k}^{\ell-1} f_{k...m}(x) \omega_{(m+1)...\ell}(x, y),
\]

\[
\omega_{k...\ell}(x, z) = \omega_{k...\ell}(x, y) + \omega_{k...\ell}(y, z)
\]

\[
+ \sum_{m=k}^{\ell-1} \omega_{k...m}(x,y) \omega_{(m+1)...\ell}(y,z).
\]

**proof.** Immediate consequences of the simple formulas. \((2.4): f(y) = f(x) * \omega(x, y), \quad (2.5): \omega(x, y) = \omega(x, y) * \omega(y, z). \)

3. **Taylor remainders of iterated paraproducts**

For a given sequence \( f_1, f_2, \ldots \) of functions, we define the **iterated paraproducts**

\[(f_1)^\prec := f_1, \quad (f_1, \ldots, f_n)^\prec := (f_1, \ldots, f_{n-1})^\prec \prec f_n.\]

The aim of this section is to show the following Besov type estimate, which is an extension of \([6, \text{Theorem 3.1}]\). We write \( f_{k...\ell} := (f_k, \ldots, f_\ell)^\prec \prec \).

**Theorem 3.1.** For any measurable functions \( f_1, \ldots, f_n \), we define the family

\[\{ \omega_{k...\ell}(x, y) \}_{1 \leq k \leq \ell \leq n}\]

of two parameter functions by the recursive formula \((2.4)\) with \( f_{k...\ell} \) replaced by \( f_{k...\ell} \). Let \( \alpha_1, \ldots, \alpha_n \in (0, 1), p_1, \ldots, p_n, q_1, \ldots, q_n \in [1, \infty] \), and \( f_i \in B_{p_i,q_i}^{\alpha_i} \) for each \( i \). If \( \alpha := \alpha_1 + \cdots + \alpha_n < 1, \quad \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} \leq 1, \) and \( \frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_n} \leq 1 \), then we have \( \omega_{1...n}^{\prec} \in D_{p,q}^\alpha \)

and

\[
\| \omega_{1...n}^{\prec} \|_{D_{p,q}^\alpha} \lesssim \| f_1 \|_{B_{p_1,q_1}^{\alpha_1}} \cdots \| f_n \|_{B_{p_n,q_n}^{\alpha_n}}.
\]

3.1. **Simplified iterated paraproducts.** Fix the parameters and the functions as in Theorem 3.1. For any \( 1 \leq k \leq \ell \leq n \), we use the following simplifying notations.

\[
\alpha_{k...\ell} = \alpha_k + \cdots + \alpha_\ell,
\]

\[
\frac{1}{p_{k...\ell}} = \frac{1}{p_k} + \cdots + \frac{1}{p_\ell}, \quad \frac{1}{q_{k...\ell}} = \frac{1}{q_k} + \cdots + \frac{1}{q_\ell}.
\]

First we show the existence of the family \( \{ \tilde{f}_{k...\ell} \}_{1 \leq k \leq \ell \leq n} \) such that the corresponding \( \{ \tilde{\omega}_{k...\ell} \}_{1 \leq k \leq \ell \leq n} \) satisfies the bound \(3.1\).

**Definition 3.1.** For any \( j \geq -1 \), we define

\[
(\tilde{f}_{k})_j := \Delta_j f_k, \quad (\tilde{f}_{k...\ell})_j := (\tilde{f}_{k...(\ell-1)})_{j-1}(\tilde{f}_\ell),
\]

(the latter definition has a meaning only if \( j \geq 1 \)) and set

\[
\tilde{f}_{k...\ell} = \sum_j (\tilde{f}_{k...\ell})_j.
\]

Moreover, we define the family \( \{ \tilde{\omega}_{k...\ell} \}_{1 \leq k \leq \ell \leq n} \) by the recursive formula \((2.4)\) with \( f_{k...\ell} \) replaced by \( \tilde{f}_{k...\ell} \).

We consider the decomposition \( \tilde{\omega}_{k...\ell} = \sum_j (\tilde{\omega}_{k...\ell})_j \) as follows. The proof of this lemma is left to the reader.

**Lemma 3.2.** Define \( (\tilde{\omega}_{k...\ell})_j \) recursively by

\[
(\tilde{\omega}_{k...\ell})_j(x, y) = (\tilde{f}_{k...\ell})_j(y) - (\tilde{f}_{k...\ell})_j(x) - \sum_{m=k}^{\ell-1} \tilde{f}_{k...m}(x)(\tilde{\omega}_{(m+1)...\ell})_j(x, y).
\]

Then one has the following formulas.

\[
(1) \quad (\tilde{\omega}_k)_j(x, y) = \Delta_j f_k(y) - \Delta_j f_k(x).
\]
(2) If $k < \ell$,\n\[
(\tilde{\omega}_{k...\ell})_j(x, y) = (\tilde{\omega}_{k...(\ell-1)})_{<j-1}(x, y)(\tilde{f}_\ell)_j(y)
- (C_{k...(\ell-1)})_{\geq j-1}(x)(\tilde{\omega}_{\ell})_j(x, y),
\]
where $(C_{k...\ell})_{1\leq k\leq \ell \leq n}$ is recursively defined by $(C_k)_j(x) = \Delta_j f_k(x)$ and
\[(C_{k...\ell})_j(x) = (\tilde{f}_{k...\ell})_j(x) - \sum_{m=k}^{\ell-1} \tilde{f}_{k...m}(x)(C_{(m+1)...\ell})_j(x).\n\]

(3) If $k < \ell$,
\[(C_{k...\ell})_j(x) = -(C_{k...(\ell-1)})_{\geq j-1}(x)(\tilde{f}_\ell)_j(x).\n\]

**Proof of the bound** (3.1) for $\tilde{\omega}$. Without loss of generality, we assume $\|f_1\|_{B_{p_1,q_1}^{\alpha_1}} \leq 1$ for any $i$. To apply Lemma 2.5, we show the bound
\[(3.2) \quad \|((\tilde{\omega}_{1...n})_j(x, x + h))\|_{B_{p_1,...,p_n,q_1,...,q_n}^{\alpha_1...\alpha_n}} \lesssim \|h\|^\theta
\]
uniformly over $\theta$ in a neighborhood of $\alpha_1...n < 1$. The case $n = 1$ is already proved in the proof of Proposition 2.3 in Section 2.2. Let $n \geq 2$. By Lemma 3.3 (3), we inductively have
\[
\|((C_{1...n})_j)_j(x, x + h))\|_{B_{p_1,...,p_n,q_1,...,q_n}^{\alpha_1...\alpha_n}} \lesssim \|h\|^\theta
\]
where we use Lemma 2.4 (2.1) for the bound of $(C_{1...(n-1)})_{\geq j-1}$. Assume (3.2) holds for the word $(1...n-1)$, uniformly over $\theta \in (\alpha_1...n-2, 1)$. Then by Lemma 2.4 (2.2), we have
\[
\|((\tilde{\omega}_{1...n-1})_{<j-1}(x, x + h))\|_{B_{p_1,...,p_n,q_1,...,q_n}^{\alpha_1...\alpha_n}} \lesssim \|h\|^\theta
\]
for any $\theta \in (\alpha_1...n-1, 1]$. For such $\theta$, by Lemma 3.2 (2),
\[
\|((\tilde{\omega}_{1...n})_j(x, x + h))\|_{B_{p_1,...,p_n,q_1,...,q_n}^{\alpha_1...\alpha_n}} \lesssim \|h\|^\theta
\]
Thus we have the required bound by an induction on $n$. \(\square\)

**3.2. Proof of Theorem 3.1.** We show the bound (3.1), which is really required. For any word $\tau = (k...\ell)$, denote by $\Pi(\tau)$ the set of all partitions of $\tau$, that is, we write
\[
\{\tau_1,...,\tau_m\} \in \Pi(\tau)
\]
if $\tau_1,\ldots,\tau_m$ are nonempty words of the form $\tau_j = (k_j...\ell_j)$ for each $j$, where $k_1 = k$, $\ell_m = \ell$, and $\ell_j + 1 = k_{j+1}$ for any $j$. Recall the definitions of $\alpha_\tau = \alpha_k...\ell$, $p_\tau$, and $q_\tau$ as above.

**Lemma 3.3.** For any word $\tau = (k...\ell)$, there exists a function $[\tilde{f}_\tau] \in B_{p_\tau,q_\tau}^{\alpha_\tau}$ continuously depending on $f_k,\ldots,f_\ell$, such that, one has the formula
\[(3.3) \quad \tilde{f}_\tau = \sum_{(\sigma,n) \in \Pi(\tau)} \tilde{f}_\sigma \ast [\tilde{f}_\tau]_n + [\tilde{f}_\tau]_\tau.
\]
Moreover, one has the atomic decomposition
\[(3.4) \quad \tilde{f}_\tau = \sum_{m=1}^{\infty} \sum_{(\tau_1,...,\tau_m) \in \Pi(\tau)} ([\tilde{f}_{\tau_1},...,\tilde{f}_{\tau_m}])^c.
\]

**proof.** Second formula (3.4) is an immediate consequence of the first one (3.3). The proof of (3.3) is essentially the same as [2] Proposition 12. The point is that we use Besov norms $B_{p,q}^\alpha$, while in [2] the particular case $p = q = \infty$ is considered.
Here we give a proof of (4.3). Write $\delta f(x, y) = f(y) - f(x)$ for simplicity. Expanding $\tilde{\omega}_\tau$ by repeating
(2.3), we have
\[
\tilde{\omega}_\tau(x, y) = \delta \tilde{f}_\tau(x, y) - \sum_{\{\tau_1, \tau_2\} \in \Pi(\cdot)} \tilde{f}_{\tau_1}(x) \tilde{\omega}_{\tau_2}(x, y)
\]
(3.5)
\[
\tilde{\omega}_\tau(x, y) = \cdots
= \delta \tilde{f}_\tau(x, y) - \sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m\} \in \Pi(\cdot)} (\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}})(x) \delta \tilde{f}_{\tau_m}(x, y).
\]
Applying the two parameter operator $P$ to both sides, we have
\[
P(\tilde{\omega}_\tau) = 1 - \tilde{f}_\tau - \sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m\} \in \Pi(\cdot)} (\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}}) < \tilde{f}_{\tau_m}.
\]
By Lemma 2.4, $P(\tilde{\omega}_\tau)$ belongs to $B^{\alpha_r}_{p_r, q_r}$ and continuously depends on $f_k, \ldots, f_\ell$. If $\tilde{f}_{\tau_m}$ has a decomposition (3.3),
\[
\sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m\} \in \Pi(\cdot)} (\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}}) < \tilde{f}_{\tau_m}
\]
\[
= \sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m\} \in \Pi(\cdot)} (\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}}) < [\tilde{f}]_{\tau_m}
\]
\[
+ \sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m, \tau_{m+1}\} \in \Pi(\cdot)} (\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}}) < (\tilde{f}_{\tau_m} < [\tilde{f}]_{\tau_{m+1}})
\]
\[
= \sum\limits_{\{\tau_1, \tau_2\} \in \Pi(\cdot)} \tilde{f}_{\tau_1} < [\tilde{f}]_{\tau_2}
\]
\[
+ \sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m, \tau_{m+1}\} \in \Pi(\cdot)} R(\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}}, \tilde{f}_{\tau_m}, [\tilde{f}]_{\tau_{m+1}}),
\]
where $R$ is the correcting operator defined by
\[
R(a, b, c) := a < (b < c) - (ab) < c.
\]
The sum of all $R$ terms belongs to $B^{\alpha_r}_{p_r, q_r}$ and continuously depends on $f_k, \ldots, f_\ell$. Its proof is left to Lemma 3.4 below. Then we obtain the formula (3.3) since
\[
\|\tilde{f}_\tau - 1 < \tilde{f}_{\tau}\|_{B^{\alpha_r}_{p_r, q_r}} = \|\Delta_{\leq 0} \tilde{f}_\tau\|_{B^{\alpha_r}_{p_r, q_r}} \lesssim \|\tilde{f}_\tau\|_{B^{\alpha_r}_{p_r, q_r}}
\]
for any $r > 0$.

**Lemma 3.4.** Let $\sigma = (k \ldots \ell)$, $\alpha' > 0$, $p', q' \in [1, \infty]$, and $g \in B^{\alpha'}_{p', q'}$. Assume that $\alpha = \alpha_{\sigma} + \alpha' < 1$, $1/p = 1/p_{\sigma} + 1/p' \leq 1$, and $1/q = 1/q_{\sigma} + 1/q' \leq 1$. Then one has the bound
\[
\left\| \sum_{m=2}^{\infty} (-1)^m \sum_{\{\tau_1, \ldots, \tau_m\} \in \Pi(\sigma)} R(\tilde{f}_{\tau_1} \cdots \tilde{f}_{\tau_{m-1}}, \tilde{f}_{\tau_m}, g) \right\|_{B^{\alpha}_{p, q}}
\]
\[
\lesssim \|f_k\|_{B^{\alpha_{\sigma} + \alpha'}_{p_{\sigma} + p', q_{\sigma} + q'}} \cdots \|f_\ell\|_{B^{\alpha'}_{p', q'}} \|g\|_{B^{\alpha'}_{p', q'}}.
\]
**proof.** Just an analogue of [2] Proposition 10, so see it for details. In view of the formula (3.5), it is sufficient to show that
\[
\|P(\tilde{\omega}_{\sigma}(x, \cdot) < g)(y)\|_{B^{\alpha}_{p, q}} < \infty,
\]
where we write $P(\tilde{\omega}_{\sigma}(x, \cdot) < g)(y)$ as an abuse of notation. Since the integral $\int Q_j(z, y)Q_{<i-1}(y, u)Q_i(y, v)dy$ vanishes if $|i-j| \geq N$ for some constant $N$,
\[
\int Q_j(z, y)((\tilde{\omega}_{\sigma}(x, \cdot) < g)(y)dy
\]
cardinality as the length of $\tau$

Summing (3.8) over all $\tau$

The bound for $\tilde{\omega}$

We prove the result by an induction on the number of the component $s$ of

1

By the formula (3.9), we see that $P_\ell ((\tilde{\omega}_s, \cdot) \cdot g)(y))(z)$ is a sum of the integrals of the form

(3.6)

where $A \in D^A_{p_A,q_A}$, $B \in B^B_{p_B,q_B}$, $C \in B^C_{p_C,q_C}$, and parameters are such that $\alpha = \alpha_A + \alpha_B + \alpha_C$, $1/p = 1/p_A + 1/p_B + 1/p_C$, and $1/q = 1/q_A + 1/q_B + 1/q_C$. Exchanging variables $y = z + h$ and $x = z + h + k$, we see that the $L^p(dx)$ bound of such an integral is as follows.

$$\sum_{i,j \in \mathbb{N}} \left| \int Q_{\tau_j}(x) Q_j(z,y) A(x,y) B_i(y) C_i(y) dy dz \right|$$

where $K \in S(\mathbb{R}^d)$ and $\{c_j\} \in \mathbb{R}^{qC}$ with $1/qBC = 1/q_B + 1/q_C$. By Lemma 2.3, we have that the above integral is bounded by $2^{-j(q_B + q_C)}c_j$, which completes the proof.

For any partition $\{\tau_1, \ldots, \tau_m\} \in \Pi(\tau)$, we define

(3.7)

$$[\tilde{\omega}]_{\tau_1 \ldots \tau_m} = ([\tilde{\omega}]_{\tau_1}, \ldots, [\tilde{\omega}]_{\tau_m})$$

Summing (3.8) over all $\{\tau_1, \ldots, \tau_m\} \in \Pi(\tau)$, we can inductively obtain

(3.9)

Proof of Theorem 3.7. We emphasize the dependence of $\omega^\kappa_{k_\ell}$ on $f_k \ldots, f_\ell$ by writing

$$\omega^\kappa_{k_\ell} = \omega^\kappa(f_k, \ldots, f_\ell)$$

We prove the result by an induction on the number of the components of $\omega^\kappa$.

By the formula (3.3), $[\tilde{f}]_{(k)} = f_k$ for a word with only one letter. Hence if $\Xi \in \Pi(\tau)$ has the same cardinality as the length of $\tau$ (denoted by $|\Xi| = |\tau|$), we have $[\tilde{f}]_{\Xi} = f_{\Xi}$ and $[\tilde{\omega}]_{\Xi} = \omega_{\Xi}$. Hence by (3.9),

$$\omega^\kappa_{k_\ell} = \omega^\kappa_{[\tilde{f}]_{(k_\ell)}} = \tilde{\omega}_{\tau} - \sum_{\Xi \in \Pi(\tau), |\Xi| < |\tau|} [\tilde{\omega}]_{\Xi}.$$
4. Besov type regularity structure and commutator estimates

We prove Theorem 1.1 in the rest of this paper. We show only the existence of the continuous map \( \tilde{\mathcal{C}} \). The uniqueness of \( \tilde{\mathcal{C}} \) and its multilinearity follows from the denseness argument.

4.1. Besov type regularity structure. We return to the Hopf algebra \( \text{Alg}(W) \) with the character group \( G \). We consider a subset

\[ V = \{ 1 \} \cup \{(k \ldots n) : k = 1, \ldots, n \}. \]

of \( W \) and a linear subspace \( T = \langle V \rangle \). Since \( \Delta T \subset \text{Alg}(W) \otimes T \), for any \( \gamma \in G \) we can define the linear map \( \Gamma_\gamma : T \to T \) by

\[ \Gamma_\gamma = (\gamma \otimes \text{id})\Delta. \]

The pair \((T, G)\) is an example of the regularity structure.

Remark 4.1. Note that the position of \( \gamma \) is opposite to the original definition \([3]\). Because of it, in the mapping \( \gamma \mapsto \Gamma_\gamma \), the order of multiplication is turned over as follows.

\[ (4.1) \]

We define a model \((\Pi, \Gamma)\) on the regularity structure \((T, G)\). Fix the parameters and the functions satisfying the assumptions in Theorem 4.1. For any \( 1 \leq k \leq n \), we define

\[ \alpha'_{k \ldots n} = \alpha_k + \ldots + \alpha_n + \alpha_0, \]

\[ \frac{1}{p'_k \ldots n} = \frac{1}{p_k} + \ldots + \frac{1}{p_n} + \frac{1}{p_0}, \]

\[ \frac{1}{q'_k \ldots n} = \frac{1}{q_k} + \ldots + \frac{1}{q_n} + \frac{1}{q_0}. \]

Moreover, set \( \alpha'_1 = \alpha_0 \), \( p'_1 = p_0 \), and \( q'_1 = q_0 \).

Definition 4.1. Let \( f_{k \ldots n} = (f_k, \ldots, f_n) \) be the iterated paraproduct. We regard \( f^{\prec}(x) \in G \) by extending the map \( \tau \mapsto f^{\prec}(x) \) algebraically, and define

\[ \omega^{\prec}(x, y) = f^{\prec}(x)^{-1} * f^{\prec}(y), \quad \Gamma_{xy} = \Gamma_{\omega^{\prec}(x, y)}. \]

Definition 4.2. For any linear map \( \Pi : T \to S' \), define

\[ \Pi_x \tau = (f^{\prec}(x)^{-1} \otimes \Pi)\Delta \tau. \]

Denote by \( \mathcal{M} \) the set of all maps \( \Pi \) such that

\[ \|\Pi\|_{\mathcal{M}} := \sup_{\tau \in V} \|\Delta_{\epsilon_f}(\Pi_x \tau)(x)\|_{\mathbb{R}^{\prec_p, \prec_q}} < \infty. \]

It is easy to show the following formulas.

\[ (4.2) \]

The pair \((\Pi, \Gamma)\) is called a model on the regularity structure \((T, G)\). Note that these formulas are slightly different from the original ones \([3]\), like the formula \((1.1)\).

As an analogue of \([3, 3]\), we can show that the space \( \mathcal{M} \) has a simple topological structure. Let

\[ V^- = \{ 1 \} \cup \{(k \ldots n) : k = 2, \ldots, n \}. \]

Note that \( \alpha'_1 < 0 \) for any \( \tau \in V^- \) by assumption.

Theorem 4.2. For any \( \Pi \in \mathcal{M} \), define the linear map \([\Pi] : T \to S' \) by

\[ [\Pi] \tau = \sum_{\sigma \wedge \eta = \tau, \sigma \neq 1} f_{\sigma}^{\prec} \otimes [\Pi]_{\eta} + [\Pi]_\tau. \]

Then \([\Pi] \tau \in B^{\alpha'_1}_{p'_1, q'_1} \) and the mapping

\[ (f_1, \ldots, f_n, \Pi) \mapsto [\Pi] \tau \in B^{\alpha'_1}_{p'_1, q'_1}, \]

is continuous.

Conversely, for any given family

\[ \{ [\Pi] \tau \}_{\tau \in V^-} \in \prod_{\tau \in V^-} B^{\alpha'_1}_{p'_1, q'_1}, \]

there exists a unique element \( \Pi \in \mathcal{M} \) satisfying \((4.3)\). Moreover, the map \( \{ [\Pi] \tau \}_{\tau \in V^-} \mapsto \Pi \) is continuous.
The above theorem is just an analogue of [2, Theorem 14 and Corollary 15], so we leave the details to the reader. The only modification is that we have to use the Besov type reconstruction theorem in Appendix.

4.2. Proof of Theorem 1.1. Now we show the iterated commutator estimates. This part is strictly an analogue of [6, Section 4].

Proof of Theorem 1.1. For any given $\xi \in B_{p,q}^{\alpha_0}$, we can define $\Pi^2 \in \mathcal{M}$ by

$$[\Pi^2] = \xi, \quad [\Pi^2](k \ldots n) = 0 \quad (2 \leq k \leq n).$$

Note that

$$\Pi^2 = f_{k,n}^{\alpha} \ll \xi \quad (2 \leq k \leq n),$$

by the formula (4.3). Then the map

$$(f_0, \ldots, f_n, \xi) \mapsto [\Pi^2](1 \ldots n)$$

is continuous, which turns out to be the required map $\tilde{C}$. It remains to show that

$$[\Pi^2](1 \ldots n) = C(f_1, \ldots, f_n, \xi)$$

if all inputs $(f_1, \ldots, f_n, \xi)$ are in $\mathcal{S}(\mathbb{R}^d)$. Since $\Pi^2 = (f^\omega(x) \otimes \Pi^2) \Delta$,

$$\Pi^2(k \ldots n)(x) = \Pi^2(k \ldots n)(x) + f_{k,n}^{\alpha}(x)(\xi(x))$$

$$+ \sum_{\ell=k}^{n-1} f_{\ell,n}^{\alpha}(x) \langle \Pi^2(\ell+1 \ldots n) \rangle(x),$$

for any $1 \leq k \leq n$. By using it and (4.3), we can inductively show that

$$\Pi^2(k \ldots n)(x) = -C(f_k, \ldots, f_n, \xi)(x)$$

for $2 \leq k \leq n$. Then letting $k = 1$ in (4.5) and using

$$\Pi^2(1 \ldots n)(x) = \lim_{j \to \infty} \Delta_{\ll j}(\Pi^2(1 \ldots n))(x) = 0$$

because $\Delta_{\ll j}(\Pi^2(1 \ldots n))(x) \in B_{p_1 \ldots p_n, q_1 \ldots q_n}^{\alpha_1 \ldots \alpha_n}$ and $\alpha_1 \ldots \alpha_n > 0$, we have (4.5) by the definition of $\mathcal{C}$. \qed

APPENDIX A. Besov Type Reconstruction Theorem

We define Besov type modelled distributions. Recall that $V = \{1\} \cup \{(k \ldots n) \mid k = 1, \ldots, n\}$, and $T = \{V\}$.

Definition A.1. For any function $g : \mathbb{R}^d \to T$, define

$$\omega^g(x,y) = g(y) - \Gamma_{xy} g(x)$$

and denote by $\omega^g(y,x)$ its $\tau$-component. Let $k$ be the smallest integer such that $\omega^g_{k \ldots n}(y,x)$ does not vanish, and let $\alpha > \alpha_{k \ldots n}$, $p \in [1,p_{k \ldots n}^{-}]$, and $q \in [1,q_{k \ldots n}^{\prime}]$. For such parameters, we define

$$\|g\|_{D_{p,q}^{\alpha}} := \sup_{\tau} \|\omega^g\|_{D_{p^{-\tau},q^{-\tau}}^{\alpha_{k \ldots n} \tau}},$$

where

$$\frac{1}{p} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{p'}, \quad \frac{1}{q} - \frac{1}{q'} = \frac{1}{q} - \frac{1}{q'}. $$

Let $D_{p,q}^{\alpha}$ be the set of functions $g : \mathbb{R}^d \to T$ such that $\|g\|_{D_{p,q}^{\alpha}} < \infty$.

Such $g$ is called a modelled distribution controlled by $\Gamma$. We show the Besov type reconstruction theorem.

Proposition A.1. For any $g \in D_{p,q}^{\alpha}$ and $\Pi \in \mathcal{M}$, we define

$$\mathcal{P} g(z) = \sum_j \int_{\mathbb{R}^d \times \mathbb{R}^d} P_j(z,x)Q_j(z,y) \Pi_\tau(g(x))(y) dx dy.$$
(1) If $\alpha > 0$, there exists a unique continuous bilinear map $Q : \mathcal{D}_{p,q}^\alpha \times \mathcal{M} \to B_{p,q}^\alpha$ such that
\[
\|\Delta_{<i}(P g + Q g - (\Pi_x g(x))(x))\|_{B_{p,q}^\alpha} < \infty.
\]
(2) If $\alpha < 0$,
\[
\|\Delta_{<i}(P g - (\Pi_x g(x))(x))\|_{B_{p,q}^\alpha} < \infty.
\]
(The operator defined by
\[
\mathcal{R} g = \begin{cases} 
P g + Q g, & \alpha > 0, \\
P g, & \alpha < 0.
\end{cases}
\]
is called a reconstruction operator.)

**proof.** The proof is almost the same as [2] Proposition 9. In view of it, here it is sufficient to show the bound
\[
\|\Delta_{<i} (P g - \Pi_x g(x))(x)\|_{B_{p,q}^\alpha} < \infty.
\]
We have only to consider $j \geq 1$. For such $j$, \[
\Delta_{j}(P g - \Pi_x g(x))(x) = \sum_{i\sim j} \int \int Q_j(x,y)Q_{<i-1}(y,u)Q_i(y,v)(\Pi_u g(u) - \Pi_x g(x))(v)dydudv,
\]
where $i \sim j$ means that $|i - j| \leq N$ for some constant $N$. By the formulas (4.2),
\[
\Pi_u g(u) - \Pi_x g(x) = \Pi_x (\Gamma_{ux} g(u) - g(x)) = -\sum_{\tau \in V} \omega^\tau_{g}(u,x)\Pi_x \tau.
\]
Hence the above integral is equal to
\[
-\sum_{i\sim j} \sum_{\tau} \int Q_j(x,y)\Delta_{<i-1}(\omega^\tau_{g}(\cdot,x))(y)\Delta_{i}(\Pi_x \tau)(y)dy.
\]
For the $\omega^g$ part, since
\[
\omega^g(u,x) = g(x) - \Gamma_{ux} g(u)
= g(x) - \Gamma_{yx} g(y) + \Gamma_{yx} g(y) - \Gamma_{uy} g(u)
= \omega^g(y,x) + \Gamma_{yx} \omega^g(u,y),
\]
we have
\[
\Delta_{<i-1}(\omega^\tau_{g}(\cdot,x))(y) = \omega^\tau_{g}(y,x) + \sum_{\sigma} \omega^\sigma_{g}(y,x)\Delta_{<i-1}(\omega^\sigma_{g}(\cdot,y))(y).
\]
Similarly, for the $\Pi$ part,
\[
\Delta_{i}(\Pi_x \tau)(y) = \Delta_{i}(\Pi_y \Gamma_{xy} \tau)(y) = \sum_{\sigma,\eta=\tau} \omega^\sigma_{g}(x,y)\Delta_{i}(\Pi_y \eta)(y).
\]
Hence it turns out that $\Delta_{j}(P g - \Pi_x g(x))(x)$ is a sum of the integrals of the form
\[
\sum_{i\sim j} \int \int Q_j(x,y)A(x,y)B_i(y)C_i(y)dydz,
\]
where $A \in \mathcal{D}_{p_A,q_A}^\alpha$, $B \in B_{p_B,q_B}^\alpha$, $C \in \mathcal{B}_{pc,qc}^{\alpha}$, and parameters are such that $\alpha = \alpha_A + \alpha_B + \alpha_C$, $1/p = 1/p_A + 1/p_B + 1/p_C$, and $1/q = 1/q_A + 1/q_B + 1/q_C$. Since we are in exactly the same situation as the previous one (3.6), we can complete the proof by a similar way to Lemma 3.4.

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