Noncommutative Gauge Theory on Fuzzy Sphere from Matrix Model

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Abstract

We derive a noncommutative $U(1)$ and $U(n)$ gauge theory on the fuzzy sphere from a three dimensional matrix model by expanding the model around a classical solution of the fuzzy sphere. Chern-Simons term is added in the matrix model to make the fuzzy sphere as a classical solution of the model. Majorana mass term is also added to make it supersymmetric. We consider two large $N$ limits, one corresponding to a gauge theory on a commutative sphere and the other to that on a noncommutative plane. We also investigate stability of the fuzzy sphere by calculating one-loop effective action around classical solutions. In the final part of this paper, we consider another matrix model which gives a supersymmetric gauge theory on the fuzzy sphere. In this matrix model, only Chern-Simons term is added and supersymmetry transformation is modified.

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1 Introduction

To formulate nonperturbative aspects of string theory or M theory, several kinds of matrix models have been proposed\cite{1, 2}. These proposals are based on the developments of D-brane physics\cite{3, 4}. IIB matrix model is one of these proposals\cite{1}. It is a large $N$ reduced model\cite{3} of ten-dimensional supersymmetric Yang-Mills theory and the action has a matrix regularized form of the Green-Schwarz action of IIB superstring. It is postulated that it gives a constructive definition of type IIB superstring theory.

In the matrix model, matter and even spacetime are dynamically emerged out of matrices\cite{6, 7}. Spacetime coordinates are represented by matrices and therefore noncommutative geometry appears naturally. The idea of the noncommutative geometry is to modify the microscopic structure of the spacetime. This modification is implemented by replacing fields on the spacetime by matrices. It was shown\cite{8, 9, 10, 11} that noncommutative Yang-Mills theories in a flat background are obtained by expanding the matrix model around a flat noncommutative background. The noncommutative background is a D-brane-like background which is a solution of the equation of motion and preserves a part of supersymmetry. Various properties of noncommutative Yang-Mills have been studied from the matrix model point of view \cite{12}. In string theory, it is discussed that the world volume theory on D-branes with NS-NS two-form background is described by noncommutative Yang-Mills theory\cite{13}.

A different kind of noncommutative backgrounds, a noncommutative sphere, or a fuzzy sphere is also studied in many contexts. In \cite{14}, it is discussed in the framework of matrix regularization of a membrane. In the light-cone gauge, they gave a map between functions on spherical membrane and hermitian matrices. In BFSS matrix model\cite{2}, membranes of spherical topology are considered in \cite{23, 24, 25}. A noncommutative gauge theory on a fuzzy sphere in string theory context is discussed in \cite{26, 27}. The approach to construct a gauge theory on the fuzzy sphere were pursued in \cite{13, 16, 17, 18, 19, 20, 21, 22}.

The fuzzy sphere\cite{15} can be constructed by introducing a cut off parameter $N$ for angular momentum of the spherical harmonics. The number of independent functions is $\sum_{l=0}^{N} (2l + 1) = (N + 1)^2$. Therefore, we can replace the functions by $(N + 1) \times (N + 1)$ hermitian matrices on the fuzzy sphere. Thus, the algebra on the fuzzy sphere becomes noncommutative.

In this paper, to construct a noncommutative gauge theory on the fuzzy sphere, we first consider a three dimensional supersymmetric reduced model. We add Chern-Simons term and Majorana spinor mass term to the action of original supersymmetric matrix model. An action with both of Yang-Mills and Chern-Simons terms are also considered in \cite{27}. Although an ordinary matrix model has only a flat background as a classical solution, our matrix model can describe a curved background owing to these terms.

We also study another type of action containing Chern-Simons term but not Majorana mass term. Although this action is not invariant under the original supersymmetry, it is
invariant under a modified supersymmetry. It is also invariant under a constant shift of the
fermion and hence it has $\mathcal{N} = 2$ supersymmetry.

This paper is organized as follows. In section 2, we study a matrix model which gives
a noncommutative gauge theory on a fuzzy sphere. We derive a noncommutative gauge
theory by expanding the model around a classical solution which represents a fuzzy sphere.
It is shown that in a commutative limit, it gives a standard theory on a commutative
sphere. In the latter part of this section, another large $N$ limit corresponding to a sphere
with a large radius but the noncommutativity fixed is considered. By taking the limit, we
can obtain a noncommutative gauge theory in a flat noncommutative background, which is
considered in [8]. In section 3, some properties of Dirac operator and chirality operator in
this model are considered. These operators become the correct operators on a commutative
sphere in a commutative limit. In section 4, stability of two classical solutions, diagonal
matrices and the fuzzy sphere, is examined. One-loop effective action is calculated for the
classical solutions. For diagonal matrices, all supersymmetry is preserved. Hence, one-
loop effective action vanishes and eigenvalues can move freely. On the other hand, all
supersymmetry is broken for the fuzzy sphere. Therefore one-loop effective action does
not vanish. It is shown that the value of the action including the one-loop corrections
of the fuzzy sphere is lower than that of the commuting matrices when $N$ is sufficiently
large for fixed $g_{YM}$. Another matrix model is analyzed by the same manner as the case
of a former model in section 5. While in the former model the fuzzy sphere does not
preserve supersymmetry, in this model it is a BPS solution. Therefore we can obtain a
supersymmetric noncommutative gauge theory on a fuzzy sphere by expanding this model
around a fuzzy sphere. Section 6 is devoted to conclusions and discussions. In appendix A,
a noncommutative product on a fuzzy sphere is constructed by following [14]. In appendix
B, by projecting a fuzzy sphere to a fuzzy complex plane stereographically using a coherent
state approach developed by [31, 32, 33], a noncommutative product which corresponds to
the normal ordered product is considered. We also give mapping rules from matrix models
to field theories on the projected plane. This construction leads to field theories with the
Berezin type noncommutative product instead of the Moyal type.

# 2 Noncommutative Gauge Theory on Fuzzy Sphere

We start with a three-dimensional reduced model which is defined by the following
action,

$$S = \frac{1}{g^2} Tr\left(-\frac{1}{4}[A_i, A_j][A^i, A^j] + \frac{2}{3} i\alpha\epsilon_{ijk} A^i A^j A^k + \frac{1}{2} \bar{\psi} \sigma^i [A_i, \psi] + \alpha \bar{\psi} \psi\right).$$

Reduced models are obtained by reducing the spacetime volume to a single point [3]. We
have added Chern-Simons term and Majorana mass term to the reduced model of super-
symmetric Yang-Mills. $A_i$ and $\psi$ are $(N+1) \times (N+1)$ hermitian matrices, and $\psi$ is a three-dimensional Majorana spinor field. $\sigma_i(i=1,2,3)$ denote Pauli matrices. $\alpha$ is a dimensionful parameter which depends on $N$.

This model possesses $SO(3)$ symmetry and the following translation symmetry

$$A_i \rightarrow A_i + \alpha 1. \quad (2)$$

Gauge symmetry of this model is expressed by the unitary transformations,

$$A_i \rightarrow UA_i U^\dagger, \quad \psi \rightarrow U\psi U^\dagger. \quad (3)$$

In addition to the above symmetries, this model has $\mathcal{N}=1$ supersymmetry:

$$\delta A_i = i\bar{\epsilon}\sigma_i \psi$$

$$\delta \psi = \frac{i}{2}[A_i, A_j] \sigma^{ij} \epsilon. \quad (4)$$

Because of the fermionic mass term, translation symmetry of $\psi$ is not present.

We next consider the equation of motion of (1). When $\psi = 0$, it is

$$[A_i, [A_i, A_j]] = -i\alpha \epsilon_{jkl}[A_k, A_l]. \quad (5)$$

The simplest solution is realized by commuting diagonal matrices:

$$A_i = \text{diag}(x_1^{(N+1)}, x_2^{(N)}, \cdots, x_3^{(1)}). \quad (6)$$

Another solution represents an algebra of the fuzzy sphere:

$$[\hat{x}_i, \hat{x}_j] = i\alpha \epsilon_{ijk} \hat{x}_k. \quad (7)$$

We impose the following condition for $\hat{x}_i$,

$$\hat{x}_1\hat{x}_1 + \hat{x}_2\hat{x}_2 + \hat{x}_3\hat{x}_3 = \rho^2. \quad (8)$$

In the $\alpha \rightarrow 0$ limit, $\hat{x}_i$ becomes commutative coordinates $x_i$:

$$x_1 = \rho \sin \theta \cos \phi$$

$$x_2 = \rho \sin \theta \sin \phi$$

$$x_3 = \rho \cos \theta, \quad (9)$$

where $\rho$ denotes the radius of the sphere. Although the commuting matrices preserve the supersymmetry, the solution representing the fuzzy sphere breaks it. The noncommutative coordinates of (7) can be constructed by the generators of the $(N+1)$-dimensional irreducible representation of $SU(2)$

$$\hat{x}_i = \alpha \hat{L}_i. \quad (10)$$
where
\[ [\hat{L}_i, \hat{L}_j] = i \epsilon_{ijk} \hat{L}_k. \] (11)

\( \alpha \) and \( \rho \) are related by the following relation
\[ \rho^2 = \alpha^2 \frac{N(N+2)}{4}. \] (12)

where we have used the fact that the quadratic Casimir of \( SU(2) \) in the \((N+1)\)-dimensional irreducible representation is given by \( N(N+2)/4 \). The Plank constant, which represents the area occupied by the unit quantum on the fuzzy sphere, is given by
\[ \frac{4\pi \rho^2}{N+1} = \frac{N(N+2)}{N+1} \pi \alpha^2. \] (13)

Commutative limit is realized by
\[ \rho = \text{fixed}, \quad N \to \infty (\alpha \to 0). \] (14)

From now on, we set \( \rho = \text{fixed} \).

Now we show that, expanding the model around the classical backgrounds \( (7) \) by the similar procedure as in [8, 9] it leads to a noncommutative Yang-Mills on the fuzzy sphere. We first consider \( U(1) \) noncommutative gauge theory on the fuzzy sphere. We expand the bosonic matrices around the classical solution \( (7) \):
\[ A_i = \hat{x}_i + \alpha \rho \hat{a}_i = \alpha \rho \left( \frac{\hat{L}_i}{\rho} + \hat{a}_i \right). \] (15)

A correspondence between matrices and functions on a sphere is given as follow. Ordinary functions on the sphere can be expanded by the spherical harmonics,
\[ a(\Omega) = \sum_{l=0}^\infty \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega), \]
\[ \psi(\Omega) = \sum_{l=0}^\infty \sum_{m=-l}^{l} \psi_{lm} Y_{lm}(\Omega), \] (16)

where
\[ Y_{lm} = \rho^{-l} \sum_{a} f_{a_{1},a_{2},\ldots,a_{l}}^{(lm)} x^{a_{1}} \cdots x^{a_{l}} \] (17)
is a spherical harmonics and \( f_{a_{1},a_{2},\ldots,a_{l}}^{(lm)} \) is a traceless and symmetric tensor. The traceless condition comes from \( x_i x_i = \rho^2 \). The normalization of the spherical harmonics is fixed by
\[ \int \frac{d\Omega}{4\pi} Y_{lm}^{*} Y_{lm} = \frac{1}{4\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \sin \theta d\theta d\phi \int_{0}^{\pi} \sin \theta d\theta Y_{l'm'}^{*} Y_{lm} = \delta_{ll'} \delta_{mm'}. \] (18)
Matrices on the fuzzy sphere, on the other hand, can be expanded by the noncommutative spherical harmonics $\hat{Y}_{lm}$ as

\[
\hat{a} = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}, \\
\hat{\psi} = \sum_{l=0}^{N} \sum_{m=-l}^{l} \psi_{lm} \hat{Y}_{lm}.
\]  

(19)

$\hat{Y}_{lm}$ is a $(N+1) \times (N+1)$ matrix and defined by

\[
\hat{Y}_{lm} = \rho^{-l} \sum_{a} f_{a_1}^{l} f_{a_2}^{m} \cdots f_{a_l}^{m} \hat{x}_{a_1} \cdots \hat{x}_{a_l},
\]  

(20)

where the same coefficients as (17) are used. Angular momentum $l$ is bounded at $l = N$ and these $\hat{Y}_{lm}$’s form a complete basis of $(N+1) \times (N+1)$ hermitian matrices. From the symmetry of the indices, the ordering of $\hat{x}$ corresponds to the Weyl type ordering. A hermiticity condition requires that $a_{lm}^* = a_{l-m}$. Normalization of the noncommutative spherical harmonics is given by

\[
\frac{1}{N+1} Tr(\hat{Y}_{lm}^{\dagger} \hat{Y}_{lm}) = \delta_{ll'} \delta_{m'm}.
\]  

(21)

A map from matrices to functions is given by

\[
\hat{a} = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \rightarrow a(\Omega) = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega),
\]  

(22)

and correspondingly a product of matrices is mapped to the star product on the fuzzy sphere:

\[
\hat{a} \hat{b} \rightarrow a \ast b.
\]  

(23)

Detailed structures of the star product are summarized in Appendix A.

An action of $Ad(\hat{L}_3)$ is obtained by

\[
Ad(\hat{L}_3)\hat{a} = \sum_{lm} a_{lm} [\hat{L}_3, \hat{Y}_{lm}] = \sum_{lm} a_{lm} m \hat{Y}_{lm}.
\]  

(24)

This property and $SO(3)$ symmetry gives the following correspondence:

\[
Ad(\hat{L}_i) \rightarrow L_i \equiv \frac{1}{i} \epsilon_{ijk} x_j \partial_k.
\]  

(25)

The laplacian on the fuzzy sphere is given by

\[
\frac{1}{\rho^2} Ad(\hat{L})^2 \hat{a} = \frac{1}{\rho^2} \sum_{lm} a_{lm} [\hat{L}_i, [\hat{L}_i, \hat{Y}_{lm}]] = \sum_{lm} \frac{l(l+1)}{\rho^2} a_{lm} \hat{Y}_{lm}.
\]  

(26)
\[
\text{Tr over matrices can be mapped to the integration over functions as}
\]
\[
\frac{1}{N+1} \text{Tr} \rightarrow \int \frac{d\Omega}{4\pi}.
\] (27)

Let us expand the action (1) around the classical solution (7) and apply these mapping rules. The bosonic part of the action (1) becomes
\[
S_B = -\frac{\alpha^4}{4g^2} \text{Tr}(\hat{F}_{ij}\hat{F}_{ij})
\]
\[
-\frac{i}{2g^2}\alpha^4 \rho^3 \epsilon^{ijk} \text{Tr}(\frac{1}{\rho} [\hat{L}_i, \hat{a}_j] \hat{a}_k + \frac{1}{3} \hat{a}_i [\hat{a}_j, \hat{a}_k] - \frac{i}{2\rho} \epsilon_{ijm} \hat{a}^m \hat{a}_k) - \frac{\alpha^2}{6g^2} \text{Tr}(\hat{x}_i \hat{x}_i)
\]
\[
= -\frac{\alpha^4}{4g^2} (N+1) \int \frac{d\Omega}{4\pi} (F_{ij} F_{ij})_*,
\]
\[
-\frac{i}{2g^2}\alpha^4 \rho^3 \epsilon^{ijk} (N+1) \int \frac{d\Omega}{4\pi} (\frac{1}{\rho} (L_i a_j) a_k + \frac{1}{3} a_i [a_j, a_k] - \frac{i}{2\rho} \epsilon_{ijm} a^m a_k)_*,
\]
\[
-\frac{\alpha^4}{24g^2} N(N+1)(N+2),
\] (28)

where \(\hat{F}_{ij}\) is defined as follows
\[
\hat{F}_{ij} = \frac{1}{\alpha^2 \rho^2} ([A_i, A_j] - i\alpha \epsilon_{ijk} A_k)
\]
\[
= \frac{1}{\rho} [\hat{L}_i, \hat{a}_j] - \frac{1}{\rho} [\hat{L}_j, \hat{a}_i] + [\hat{a}_i, \hat{a}_j] - \frac{1}{\rho} i\epsilon_{ijk} \hat{a}_k,
\] (29)

and the corresponding function \(F_{ij}(\Omega)\) becomes
\[
F_{ij}(\Omega) = \frac{1}{\rho} L_i a_j(\Omega) - \frac{1}{\rho} L_j a_i(\Omega) + [a_i(\Omega), a_j(\Omega)]_* - \frac{1}{\rho} i\epsilon_{ijk} a_k(\Omega).
\] (30)

This quantity is gauge covariant and becomes zero when the fluctuation is set to zero. The Yang-Mills coupling is found to be
\[
g_{YM}^2 = 4\pi g^2/(N+1)\alpha^4 \rho^2.
\]
( )_* means that the products should be taken as the star product. Hence commutators do not vanish even in the case of \(U(1)\) gauge group.

The fermionic part of the action becomes
\[
S_F = \frac{\alpha}{2g^2} \text{Tr}(\bar{\psi}\sigma^i [\hat{L}_i + \rho \hat{a}_i, \hat{\psi}] + 2\bar{\psi}\hat{\psi})
\]
\[
= \frac{\alpha \rho}{2g^2} (N+1) \int \frac{d\Omega}{4\pi} (\frac{1}{\rho} \bar{\psi}\sigma^i L_i \psi + \bar{\psi}\sigma^i [a_i, \psi] + \frac{2}{\rho} \bar{\psi}\psi)_*.
\] (31)
We next focus on the gauge symmetry of this action. The action (1) is invariant under the unitary transformation (3). For an infinitesimal transformation $U = \exp(i\hat{\lambda}) \sim 1 + i\hat{\lambda}$ in (3) where $\hat{\lambda} = \sum_{lm} \lambda_{lm}\hat{Y}_{lm}$, the fluctuation around the fixed background transforms as

$$\hat{a}_i \rightarrow \hat{a}_i - \frac{i}{\rho}[\hat{L}_i, \hat{\lambda}] + i[\hat{\lambda}, \hat{a}_i].$$

(32)

After mapping to functions, we have local gauge symmetry

$$a_i \rightarrow a_i - \frac{i}{\rho}L_i\lambda + i[\lambda, a_i].$$

(33)

Let us discuss a scalar field which is defined by

$$\hat{\phi} \equiv \frac{1}{2\alpha\rho}(A_i A_i - \hat{x}_i \hat{x}_i) = \frac{1}{2}(\hat{x}_i \hat{a}_i + \hat{a}_i \hat{x}_i + \alpha\rho\hat{a}_i \hat{a}_i).$$

(34)

It transforms covariantly as an adjoint representation

$$\hat{\phi} \rightarrow \hat{\phi} + i[\hat{\lambda}, \hat{\phi}].$$

(35)

Since the scalar field should become the radial component of $\hat{a}_i$ in the commutative limit, a naive choice is $\hat{\phi}_0 = (\hat{x}_i \hat{a}_i + \hat{a}_i \hat{x}_i)/2$. For small fluctuations this field is the correct component of the field $\hat{a}_i$ but large fluctuations of $\hat{a}_i$ deform the shape of the sphere and $\hat{\phi}_0$ can be no longer interpreted as the radial component of $\hat{a}_i$. This is a manifestation of the fact that matrix models or noncommutative gauge theories naturally unify spacetime and matter on the same footing. An addition of the non-linear term $\hat{a}_i \hat{a}_i$ makes $\hat{\phi}$ transform correctly as the scalar field in the adjoint representation.

We have so far discussed the $U(1)$ noncommutative gauge theory on the fuzzy sphere. A generalization to $U(m)$ gauge group is realized by the following replacement:

$$\hat{x}_i \rightarrow \hat{x}_i \otimes 1_m.$$  

(36)

$\hat{a}$ is also replaced as follows:

$$\hat{a} \rightarrow \sum_{a=1}^{m^2} \hat{a}^a \otimes T^a,$$  

(37)

where $T^a (a = 1, \cdots, m^2)$ denote the generators of $U(m)$. Then we obtain $U(m)$ noncommutative gauge theory by the same procedure as the $U(1)$ case:

$$S_B = -\frac{\alpha^4\rho^4}{4g^2}(N + 1)tr \int \frac{d4\Omega}{4\pi}(F_{ij}F_{ij})_*.$$
\[ -\frac{i}{2g^2}\alpha^4\rho^3\epsilon^{ijk}(N + 1)\text{tr}\int\frac{d\Omega}{4\pi}\left(\frac{1}{\rho}(L_ia_j)a_k + \frac{1}{3}a_i[a_j, a_k] - \frac{i}{2\rho}\epsilon_{ijm}a^ma_k\right) \]

\[ -\frac{\alpha^4}{24g^2}mN(N + 1)(N + 2) \]

\[ S_F = \frac{\alpha^2}{2g^2}(N + 1)\text{tr}\int\frac{d\Omega}{4\pi}\left(\frac{1}{\rho}\bar{\psi}\sigma^iL_i\psi + \bar{\psi}\sigma^i[a_i, \psi] + \frac{2}{\rho}\bar{\psi}\psi\right), \]  

(38)

where \text{tr} is taken over \( m \times m \) matrices.

Let us consider a commutative limit. It is realized by the large \( N \) limit with fixed \( \rho \).

The following relations are satisfied in the commutative limit:

\[ \rho a_i(x) = K_i^a(x)b_a(x) + \frac{x_i}{\rho} \]

(39)

where \( i = 1, 2, 3, a = \theta, \phi \) and \( b_a \) is a gauge field on the sphere. Two fields \( b_a \) are defined on the local coordinates \( \theta, \phi \). \( K_i^a \) are two Killing vectors and defined by

\[ L_i = -iK_i^a(x)\partial_a. \]

(40)

The explicit forms of these Killing vectors are given as follows

\[ K_1^\theta = -\sin \phi \quad K_1^\phi = -\cot \theta \cos \phi \]
\[ K_2^\theta = \cos \phi \quad K_2^\phi = -\cot \theta \sin \phi \]
\[ K_3^\theta = 0 \quad K_3^\phi = 1. \]

(41)

From these Killing vectors, we obtain the metric tensor on \( S^2 \) as

\[ g^{ab} = K_i^aK_i^b. \]

(42)

Three fields \( a_i \) are defined in three dimensional space \( \mathbb{R}^3 \) and contain a gauge field \( b_a \) on \( S^2 \) and a scalar field \( \phi \) as well. In the commutative limit, we can separate the gauge field \( b_a \) from \( a_i \) using the Killing vectors as in (39). The bosonic part of the action is rewritten as

\[ S_B = -\frac{1}{4g_Y^2M^2}\int d\Omega(K_i^aK_j^bK_i^cK_j^dF_{ab}F_{cd} + 2iK_i^aK_j^bF_{ab}\epsilon_{ijk}\frac{x_k}{\rho}\phi)
+2K_i^aK_i^b(D_a\phi)(D_b\phi) - 2\phi^2)
-\frac{1}{2g_Y^2M^2}\int d\Omega(i\epsilon_{ijk}K_i^aK_j^bF_{ab}\frac{x_k}{\rho}\phi - \phi^2)
-\frac{1}{2g_Y^2M^2}\int d^2x\sqrt{g}(F_{ab}F^{ab} + \frac{2i}{\sqrt{g}}F_{ab}\phi + (D_a\phi)(D_a\phi) - 2\phi^2) \]

(43)

where \( F_{ab} = \frac{1}{i}\partial_a b_b - \frac{1}{i}\partial_b b_a + [b_a, b_b], D_a = \frac{1}{i}\partial_a + [b_a, \cdot] \) and \( \epsilon^{\theta\phi} = -\epsilon^{\phi\theta} = 1. \) The fermionic part is also rewritten as

\[ S_F = \frac{1}{2g_Y^2M\alpha^3\rho^2}\int d\Omega(\bar{\psi}\gamma^aD_a\psi + \bar{\psi}\gamma_3[\phi, \psi] + 2\bar{\psi}\psi) \]

(44)
where $\gamma_3 = \sigma_i x_i / \rho$ (See (31)) and $\gamma^a = \sigma^i K_i^a$. We thus obtained the action of a field theory with a gauge field, an adjoint scalar and a gaugino field on a sphere.

In the commutative limit, the gauge transformation becomes

$$b_a \to b_a - \partial_a \lambda$$

for $U(1)$ gauge group and

$$b_a \to b_a - \partial_a \lambda + i[\lambda, b_a]$$

for $U(m)$ gauge group.

In the latter part of this section, we investigate the relation to a noncommutative gauge theory in a flat background by taking $\rho \to \infty$ limit while fixing $\alpha$. By virtue of the $SO(3)$ symmetry, we may consider the theory around the north pole without loss of generality.

Around the north pole, $\hat{L}_3$ can be approximated as $\hat{L}_3 \sim N/2$. By defining $\hat{x}_i = \sqrt{2N} \hat{L}_i$, the commutation relation (11) becomes

$$[\hat{x}_1', \hat{x}_2'] \sim i.$$

By further defining $\hat{x}_i = \alpha \hat{L}_i$ and $\hat{p}_i = \alpha^{-1} \varepsilon_{ij} \hat{L}_j$ ($i, j = 1, 2$), we have

$$[\hat{x}_1', \hat{x}_2'] = i\alpha^2, \quad [\hat{p}_1', \hat{p}_2'] = i\alpha^{-2}, \quad [\hat{x}_i', \hat{p}_j'] = i\delta_{ij}.$$

We take the following limit to decompactify the sphere,

$$\alpha = \text{fixed}, \quad \rho' \gg 1 \ (N \gg 1),$$

where $\rho^2 = \hat{x}_i' \hat{x}_i = \frac{2}{N} \rho^2 = \frac{N+2}{2} \alpha^2 \sim \frac{N}{2} \alpha^2$. In the coordinates of $\hat{x}_i'$, the Plank constant is given by

$$\frac{4\pi \rho^2}{N + 1} \sim 2\pi \alpha^2.$$

$a \ast b$ which is defined in (23) becomes the Moyal product $a \ast_M b$ because of (17) and the Weyl type ordering property in (20). The following replacements hold in this limit,

$$\frac{1}{N + 1} Tr \to \int \frac{d\Omega}{4\pi} = \int \frac{d^2x}{4\pi \rho^2} = \int \frac{d^2x'}{4\pi \rho'^2}$$

$$Ad(\hat{p}_i') = \frac{1}{\rho'} \varepsilon_{ij} Ad(\hat{L}_j) \to \frac{1}{i} \partial_i' \quad (i = 1, 2)$$

We can regard $\hat{a}_3$ as the scalar field $\hat{\phi}$ around the north pole. For simplicity, only $U(1)$ case is treated in the present discussions. The bosonic part of the action (4) becomes

$$S_B = -\frac{\alpha^4}{2g^2} Tr([\hat{L}_1 + \rho \hat{a}_1, \hat{L}_2 + \rho \hat{a}_2]^2) + \frac{\alpha^4}{2g^2} Tr(\rho \hat{\phi} [\hat{L}_i + \rho \hat{a}_i, [\hat{L}_i + \rho \hat{a}_i, \rho \hat{\phi}]]).$$
The gauge transformation (32) becomes

\[ + \frac{2i\alpha^4}{g^2} Tr([\hat{L}_1 + \rho\hat{a}_1, \hat{L}_2 + \rho\hat{a}_2] \rho \hat{\phi}) \]

\[ = \frac{\alpha^4}{2g^2} (\rho')^4 \{- Tr([-\hat{p}_2 - \hat{a}_2, \hat{p}_1' + \hat{a}_1']^2) + Tr(\hat{\phi}'[\hat{p}_1 + \hat{a}_1'][\hat{p}_1 + \hat{a}_1', \hat{\phi}']) \} \\
+ \frac{4i}{\rho'} Tr([-\hat{p}_2 - \hat{a}_2', \hat{p}_1' + \hat{a}_1'] \hat{\phi}'), \] (53)

where we have defined \( \hat{a}_i = \sqrt{\frac{2}{\pi}} \hat{a}_j \epsilon_{ji} \) (\( i, j = 1, 2 \)) and \( \hat{\phi} = \sqrt{\frac{2}{\pi}} \hat{\phi}' \). This action can be mapped to the following action,

\[ S_B = \frac{\alpha^6 N^2}{16\pi g^2} \{- \int d^2 x' F_{12}(x) \ast F_{12}(x) + \int d^2 x' \phi'(x)(Ad(D))^2 \phi'(x) \]
\[ + \frac{4i}{\rho'} \int d^2 x' F_{12}(x) \ast \phi'(x) \}, \] (54)

where \( Ad(D) = \left[ \frac{1}{4} \partial'_i a'_2 - \frac{1}{4} \partial'_2 a'_1 + [a'_1, a'_2]_s \right]. \) It is found that the Yang-Mills coupling is \( g_{YM}^2 = 4\pi g^2/N^2\alpha^6 \). Similarly the fermionic part of the action (II) becomes

\[ S_F = \frac{\alpha}{2g^2} Tr(\bar{\psi} \sigma^i [\hat{L}_i + \rho\hat{a}_i, \hat{\psi}] + 2\bar{\psi} \hat{\psi}) \]
\[ = \frac{\alpha' \rho}{2g^2} Tr(\bar{\psi} \bar{\sigma}^i [\hat{p}_1' + \hat{a}_1', \hat{\psi}] + \bar{\psi} \sigma^3 [\hat{\phi}', \hat{\psi}] + \frac{2}{\rho} \bar{\psi} \hat{\psi}) \] (55)

where we have defined \( \bar{\sigma}_j = \epsilon_{ji} \sigma_i \) (\( i, j = 1, 2 \)). This is also mapped as follows

\[ S_F = \frac{\alpha' \rho'}{2g^2} (N + 1) \int \frac{d^2 x'}{4\pi \rho^2} \left( \frac{1}{i} \bar{\psi}(x) \bar{\sigma}^i \partial'_i \psi(x) \right. \\
+ \bar{\psi}(x) \bar{\sigma}^i [a'_1(x), \psi(x)]_s + \bar{\psi}(x) \sigma^3 [\phi'(x), \psi(x)]_s + \frac{2}{\rho} \bar{\psi}(x) \psi(x)). \] (56)

The gauge transformation (32) becomes

\[ a'_i \rightarrow a'_i - \partial'_i \lambda + i[\lambda, a'_i]_s \]. (57)

We have seen that in the large radius and \( \alpha = \) fixed limit (49), the noncommutative Yang-Mills theory in the flat backgrounds can be obtained. The last term in (54) and the last term in (56) become small in the \( \rho' \rightarrow \infty \). To discuss the meaning of these terms, we integrate out the scalar field \( \phi \). The action (54) becomes

\[ S = \frac{1}{4g_{YM}^2} \{- \int d^2 x' F_{12}(x) \ast F_{12}(x) + \int d^2 x' \phi'(x)(Ad(D))^2 \phi'(x) \]

does not hold. The term \( \int d^2 x' F_{12}(x) \ast F_{12}(x) \) is not a gauge invariant action. The term in (54) is a gauge invariant action because it is a term in the noncommutative Yang-Mills theory in the flat backgrounds. The term in (54) is not a gauge invariant action because it is a term in the noncommutative Yang-Mills theory in the flat backgrounds.
\[
\begin{align*}
&+ \frac{4i}{\rho'} \int d^2 x' F_{12}(x) \ast \phi'(x)) \\
&\rightarrow \frac{1}{4g^2_Y M} \{- \int d^2 x' F^2_{12}(x) + \frac{4}{\rho'^2} \int d^2 x' F_{12}(x)(-\frac{1}{\rho^2})F_{12}(x) + O(a^3)\} \\
&= \frac{1}{4g^2_Y M} \{- \int d^2 x' F^2_{12}(x) + \frac{4}{\rho'^2} \int d^2 x' d^2 y' F_{12}(x)\Delta(x - y)F_{12}(y) + O(a^3)\} \\
&= \frac{1}{4g^2_Y M} \int d^2 p(-1 + \frac{4}{\rho'^2 p^2})F_{12}(-p)F_{12}(p) + O(a^3),
\end{align*}
\]

where \(-1/\partial^2 = \Delta(x - y) = \int d^2 k e^{ik(x' - y')}/k^2\). The mass of the gauge field is found to be

\[
M_{\text{gauge}} = \frac{2}{\rho'}.
\]

We found that the gauge field has acquired the mass by absorbing the degree of freedom of the scalar field. From (56), the mass of the gaugino field is \(2/\rho'\) which is degenerate with the mass of the gauge field. In the \(\rho' \to \infty\) limit, the gauge field and the gaugino field become massless.

### 3 Some Properties of Dirac Operator

In this section, we investigate some properties of Dirac operator. Dirac operator in our model is given by

\[
\hat{D} = \frac{1}{\rho} (\sigma_i (Ad(\hat{L}_i) + 1)).
\]

Other Dirac operators are also constructed in \([17, 19, 20]\). In the commutative limit, this operator becomes

\[
D = \frac{1}{\rho} (\sigma_i L_i + 1),
\]

which is the standard massless Dirac operator on the sphere \([28]\). The second term of this operator comes from the contribution of the spin connection on the sphere. Around the north pole, the operator behaves as

\[
D = \sqrt{\frac{2}{N}} (\tilde{\sigma}_i p'_i + \frac{1}{\rho'}),
\]

and in the \(\rho' \to \infty\) limit, it approaches the massless Dirac operator in the flat background.

We consider the following chirality operator

\[
\hat{\gamma}_3 = \frac{1}{\rho} \sigma_i \hat{x}_i,
\]
which becomes, in the commutative limit,
\[ \gamma_3 = \frac{1}{\rho} \sigma_i x_i. \] (64)

It is the standard chirality operator on the sphere [28]. This chirality operator obeys \[ \hat{\gamma}_3^2 = 1 - 2 \hat{\gamma}_3 / \sqrt{N(N + 2)} \] and does not satisfy the usual condition for the chirality operator before taking the commutative limit. A chirality operator which satisfies the condition \[ \hat{\gamma}_3^2 = 1 \] and becomes (64) in the commutative limit is given [17] by
\[ \hat{\gamma}_3' = \frac{1}{M} (\sigma_i \hat{x}_i + \frac{\alpha}{2}), \] (65)
where \( M = \alpha (N + 1)/2 \) is a normalization factor. Anticommutation relation between the Dirac operator and the chirality operator is
\[ \hat{D} \hat{\gamma}_3 + \hat{\gamma}_3 \hat{D} = \frac{2}{\rho_2} \hat{x}_i \text{Ad}(\hat{L}_i) \] (66)
which vanishes in the commutative limit as can be seen from (25). Another type of a Dirac operator which anticommutes with (65) is constructed in [17].

We next investigate the spectrum of the Dirac operator. We consider an eigenstate \( \Psi \) of \( J^2 \) and \( J_3 \), satisfying \( J^2 \Psi_{jm} = j(j+1) \Psi_{jm} \) and \( J_3 \Psi_{jm} = (m+1/2) \Psi_{jm} \), where \( J_i = L_i + \sigma_i/2 \) and \( 1/2 \leq j \leq N + 1/2 \) and \( -l \leq m \leq l \). Acting on \( \Psi_{jm} \), square of the Dirac operator becomes
\[ \rho^2 \hat{D}^2 \Psi_{jm} = \sigma_i [\hat{L}_i, \Psi_{jm}] + [\hat{L}_i, [\hat{L}_i, \Psi_{jm}]] + \Psi_{jm} \\
= ((\text{Ad}(J))^2 - (\text{Ad}(L))^2 - \frac{3}{4}) \Psi_{jm} + (\text{Ad}(L))^2 \hat{\Psi}_{jm} + \hat{\Psi}_{jm} \\
= (j + \frac{1}{2})^2 \hat{\Psi}_{jm}. \] (67)
We have used the relation \( J^2 = L^2 + L \cdot \sigma + 3/4 \). This spectrum is identical with the spectrum in the commutative limit.

### 4 Stability of Classical Solutions

We have argued the noncommutative gauge theories on the fuzzy sphere. This section is devoted to a discussion of stability of the classical solutions. Let us first evaluate classical values of the action for the solutions of (3) and (7).

The action vanishes
\[ S = 0, \] (68)
for commuting matrices (3), and becomes
\[
S = -\frac{1}{24g^2}N(N+1)(N+2)\alpha^4
= -\frac{2}{3g^2}\rho^4 \frac{N+1}{N(N+2)},
\]
(69)
for the fuzzy sphere \(5\) (7). The solution of the fuzzy sphere has lower energy than the solution of commuting matrices, and therefore the fuzzy sphere is more stable than the commuting matrices at the classical level.

We next investigate one-loop effective action\([1, 6]\) around the classical solutions. We decompose the matrices \(A_i\) and \(\psi\) into the classical backgrounds and fluctuations:
\[
A_i = X_i + \tilde{A}_i,
\psi = \chi + \tilde{\psi}.
\]
(70)

We add the following gauge-fixing term\([1]\),
\[
S_{g.f.} = -\frac{1}{g^2}Tr(\frac{1}{2}[X_i, A_i]^2 + [X_i, b][A_i, c]),
\]
(71)
where \(b\) and \(c\) are ghost and anti-ghost, respectively. We expand the action up to the second order of the fluctuations and set \(\chi = 0\):
\[
S = -\frac{1}{g^2}Tr(\frac{1}{2}[X_i, \tilde{A}_j]^2 + [X_i, X_j][\tilde{A}_i, \tilde{A}_j]
- i\alpha\epsilon_{ijk}X_i[\tilde{A}_j, \tilde{A}_k]
- \frac{1}{2}\bar{\psi}\Gamma^i[X_i, \bar{\psi}] - \alpha\bar{\psi}\psi + [X_i, b][X_i, c]).
\]
(72)

Then the one-loop effective action is calculated as
\[
W = -\log \int d\tilde{A}d\bar{\psi}dbdc e^{-S}.
\]
(73)

We first consider the one-loop effective action for the commuting matrices. The contributions from \(\tilde{A}\) and \(\tilde{\psi}\) are calculated as follows
\[
W_B = \frac{1}{2} \sum_{i \neq j} \left( \log(x^{(i)} - x^{(j)})^2 + \log(1 - \frac{4\alpha^2}{(x^{(i)} - x^{(j)})^2}) \right),
\]

\(5\) The classical value of the action depends on the quadratic Casimir. That is, it depends on the representation. We can construct an alternative representation in \(10\). The value of the action is larger than the irreducible representation. Therefore the solutions of reducible representation are unstable\(25\).
\[ W_F = -\frac{1}{2} \sum_{i \neq j} \left( \log(x^{(i)} - x^{(j)})^2 + \log(1 - \frac{4\alpha^2}{(x^{(i)} - x^{(j)})^2}) \right). \] (74)

Contributions from ghosts are included in \( W_B \). We find that the one-loop effective action for commuting matrices vanishes due to the cancellation between bosonic and fermionic contributions as expected from supersymmetry. Therefore there are no net forces between the eigenvalues if we neglect the effect of the fermionic zero modes\[6\]. For the fuzzy sphere, one-loop effective action can be calculated as

\[ W_B = \frac{1}{2} Tr \log(Ad(L))^2, \]
\[ W_F = -\frac{1}{2} Tr \log(Ad(L))^2 - \frac{1}{4} Tr \log(1 + \frac{4 + 3\sigma^i Ad(L_i)}{(Ad(L))^2}). \] (75)

In this case, the one-loop effective action does not vanish because this background preserves no supersymmetry. One-loop quantum corrections to the classical action \( W \) are

\[ W = -\frac{1}{4} Tr \log(1 + \frac{4 + 3\sigma^i Ad(L_i)}{(Ad(L))^2}). \] (76)

Let us evaluate this quantity. This expression can be rewritten using a replacement \( \sigma^i Ad(L_i) \rightarrow j(j + 1) - l(l + 1) - 3/4 = s(2l + 1) - 1/2, \) where \( s = \pm 1/2 \):

\[ W = -\frac{1}{4} \sum_{l} \sum_{s=\pm 1/2} (2l + 1) \log(1 + \frac{3s(2l + 1) + \frac{5}{2}}{l(l + 1)}) \]
\[ = -\frac{1}{2} \sum_{l} (2l + 1) \log \left( \frac{l + 2}{l(l + 1)} \right). \] (77)

In the large \( N \) limit, we have

\[ W \sim 2 \log N. \] (78)

The value of the effective action including both the classical and the one-loop quantum corrections becomes

\[ S_{eff} = -\frac{2}{3g^2\rho^4} \frac{N + 1}{N(N + 2)} + 2 \log N \]
\[ = -\frac{\pi}{6\rho^2 g_Y^2} N(N + 2) + 2 \log N. \] (79)

For large enough \( N \) and \( g_Y = \text{fixed}, \) the fuzzy sphere is more stable than the commuting matrices\[6\]. On the other hand, there is a region of \( S_{eff} > 0 \) if we take \( g_Y \) sufficiently large.

---

\( ^6 \) As studied in papers\[29\], the partition function of \( d = 3 \) supersymmetric reduced models is not convergent against integral over bosonic variables. It implies that eigenvalues are more likely to be separated from each other and the fuzzy sphere solution is unstable nonperturbatively, in contrast to our perturbative analysis. From this point of view, it is more interesting if we can construct nontrivial curved backgrounds from convergent higher dimensional supersymmetric reduced models. We thank Staudacher for informing us of their analysis on reduced model integrals.
so that the second term dominates the first one. In the region, the commuting matrices is more stable than the fuzzy sphere.

5 Supersymmetric Noncommutative Gauge Theory on Fuzzy Sphere

In this section, we consider another action which gives a supersymmetric noncommutative gauge theory on the fuzzy sphere. Supersymmetric gauge theories on a fuzzy sphere is also considered in [22]. The action is defined by

\[ S = \frac{1}{g^2} Tr\left( -\frac{1}{4}[A_i, A_j][A^i, A^j] + \frac{2}{3} i\alpha\epsilon_{ijk} A^i A^j A^k + \frac{1}{2} \bar{\psi} \sigma^i [A_i, \psi] \right). \]  

(80)

It has has the following \( \mathcal{N} = 2 \) supersymmetry:

\[ \delta^{(1)} A_i = i\bar{\epsilon} \sigma_i \psi \]

\[ \delta^{(1)} \psi = \frac{i}{2} ([A_i, A_j] - i\alpha \epsilon_{ijk} A_k) \sigma^{ij} \epsilon, \]  

(81)

and

\[ \delta A^{(2)} = 0 \]

\[ \delta^{(2)} \psi = \xi. \]  

(82)

The transformation for the gaugino in (81) is modified in comparison to the transformation in the usual Yang-Mills reduced models \(^7\). Let us check that these two transformations form the \( \mathcal{N} = 2 \) supersymmetry algebra. Algebra is also modified due to the modification of supersymmetry. We have the following relations:

\( (\delta^{(1)} \epsilon_{1}^{(1)} - \delta^{(1)} \epsilon_{2}^{(1)}) \psi = i[\psi, \lambda] + i\theta_i \frac{\sigma_i}{2} \psi, \)

\( (\delta^{(1)} \epsilon_{1}^{(1)} - \delta^{(1)} \epsilon_{2}^{(1)}) A_i = i [A_i, \lambda] + \epsilon_{ijk} \theta_j A_k, \)  

(83)

where \( \lambda = 2i(\bar{\epsilon}_2 \sigma_j \epsilon_1) A_j \) and \( \theta_j = 2i\alpha(\bar{\epsilon}_2 \sigma_j \epsilon_1). \) The second term in the right hand side is a new term corresponding to \( SO(3) \) rotation. Other commutation relations are calculated as follows,

\( (\delta^{(1)} \epsilon_{1}^{(1)} - \delta^{(2)} \epsilon_{1}^{(2)}) \psi = 0, \)

\( (\delta^{(2)} \epsilon_{1}^{(2)} - \delta^{(1)} \epsilon_{1}^{(1)}) \psi = 0, \)

\( (\delta^{(2)} \epsilon_{2}^{(2)} - \delta^{(1)} \epsilon_{2}^{(1)}) \psi = 0, \)

\( (\delta^{(2)} \epsilon_{2}^{(2)} - \delta^{(1)} \epsilon_{2}^{(1)}) A_i = i [A_i, \lambda] + \epsilon_{ijk} \theta_j A_k, \)  

(83)

\(^7\) In [30], there is a discussion about the supersymmetry transformation on the fuzzy sphere. In the context of string theory, the spherical D2-brane can be a supersymmetric configuration. The fact that our model [31] has the modified supersymmetry [32] which vanishes for the fuzzy sphere is consistent with the comment given in [30]. We thank K.Hashimoto and K.Krasnov for their stimulating comments.
\[ (\delta^{(1)}_\xi \delta^{(2)}_\xi - \delta^{(2)}_\xi \delta^{(1)}_\xi)A_i = -i\bar{\epsilon}\sigma_i\xi, \quad (84) \]

and

\[ (\delta^{(2)}_\xi \delta^{(2)}_\xi - \delta^{(2)}_\xi \delta^{(2)}_\xi)\psi = 0, \]
\[ (\delta^{(2)}_\xi \delta^{(2)}_\xi - \delta^{(2)}_\xi \delta^{(2)}_\xi)A_i = 0. \quad (85) \]

If we take a linear combination of \( \delta^{(1)} \) and \( \delta^{(2)} \) as

\[ \tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}, \]
\[ \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}), \quad (86) \]

we can obtain the following commutation relations up to a gauge symmetry and \( SO(3) \) symmetry,

\[ (\tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\xi - \tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\xi)\psi = 0, \]
\[ (\tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\xi)A_i = -2i\bar{\epsilon}\sigma_i\xi, \]
\[ (\tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\xi)\psi = 0, \]
\[ (\tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\xi)A_i = -2i\bar{\epsilon}\sigma_i\xi, \]
\[ (\tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\xi - \tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\xi)\psi = 0, \]
\[ (\tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\xi - \tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\xi)A_i = 0. \quad (87) \]

We find that these commutation relations indeed show the \( \mathcal{N} = 2 \) supersymmetry algebra. A new feature is the appearance of \( SO(3) \) rotation. This model has the same classical solutions as the previous model, commuting diagonal matrices and the fuzzy sphere. In this model, the fuzzy sphere preserves half of the \( \mathcal{N} = 2 \) supersymmetries since \( (81) \) vanishes for the fuzzy sphere while \( (82) \) does not, and this solution corresponds to a 1/2 BPS background. Looking at the algebra \( (83) \), the remaining supersymmetry on the fuzzy sphere generates \( SO(3) \) rotation instead of a constant shift of \( A_i \). It is natural since translation on a sphere is generated by \( SO(3) \) rotation. On the other hand, commuting matrices break all the supersymmetry.

By expanding the bosonic matrices around the fuzzy sphere solution \( (7) \) as in \( (15) \) and applying the mapping rule which is given in section 2, we can obtain an \( \mathcal{N} = 1 \) supersymmetric \( U(1) \) or \( U(n) \) noncommutative gauge theory on the fuzzy sphere. For simplicity only \( U(1) \) gauge theory is considered in the following. The bosonic part of the action is

\[ S_B = -\frac{\alpha^4 \rho^4}{4g^2} Tr(\hat{F}_{ij}\hat{F}_{ij}) \]
\[ -i\frac{\alpha^4 \rho^4}{2g^2} \epsilon^{ijk} Tr(\frac{1}{\rho}[\hat{L}_i, \hat{a}_j]\hat{a}_k + \frac{1}{3}\delta_i [\hat{a}_j, \hat{a}_k] - \frac{i}{2}\delta_{ijm}\hat{a}^m\hat{a}_k) - \frac{\alpha^2}{6g^2} Tr(\hat{x}_i^2) \]
\[-\frac{\alpha^4 \rho^4}{4g^2}(N + 1) \int \frac{d\Omega}{4\pi}(F_{ij} F_{ij})_* \]
\[-\frac{i}{2g^2} \alpha^4 \rho^3 \epsilon^{ijk}(N + 1) \int \frac{d\Omega}{4\pi} \left( \frac{1}{\rho} (L_i a_j) a_k + \frac{1}{3} a_i [a_j, a_k] - \frac{i}{2\rho} \epsilon_{ijm} a^m a_k \right)_* \]
\[-\frac{\alpha^4}{24g^2} N(N + 1)(N + 2), \quad (88)\]

where $F_{ij}$ is defined in (39).

The fermionic part is

\[ S_F = \frac{\alpha}{2g^2} Tr \bar{\psi} \sigma^i \{ L_i + \rho \hat{a}_i, \psi \} \]
\[ = \frac{\alpha \rho}{2g^2} (N + 1) \int \frac{d\Omega}{4\pi} \left( \frac{1}{\rho} \bar{\psi} \sigma^i L_i \psi + \bar{\psi} \sigma^i [a_i, \psi] \right)_*. \quad (89)\]

This supersymmetric noncommutative gauge theory has the following $\mathcal{N} = 1$ supersymmetry:

\[ \delta a_i = \frac{i}{\alpha \rho} \bar{\psi} \gamma_i \psi \]
\[ \delta \psi = \frac{i \epsilon^2 \rho^2}{2} F_{ij} \sigma^{ij} \psi. \quad (90)\]

Applying this transformation twice generates translation on the sphere, that is the rotation.

Let us consider a commutative limit. The action in the commutative limit is obtained by the same procedure as in section 2. In terms of the gauge field $b_a$ and the scalar field $\phi$, the bosonic part becomes

\[ S_B = -\frac{1}{2g_{YM}^2 \rho^2} \int d^2 x \sqrt{g} (F_{ab} F^{ab} + \frac{2i}{\sqrt{g}} \epsilon^{ab} F_{ab} \phi + (D_a \phi)(D^a \phi) - 2 \rho^2). \quad (91)\]

This is the same as (43). The fermionic part becomes

\[ S_F = \frac{1}{2g_{YM}^2 \rho^2} \int d\Omega (\bar{\psi} \gamma^a D_a \psi + \bar{\psi} \gamma_3 [\phi, \psi]). \quad (92)\]

We have rescaled as $\psi \rightarrow \alpha^{3/2} \psi$. The difference between the action obtained in this section and the action obtained in section 2 is that the former is supersymmetric while the latter is not. The $\mathcal{N} = 1$ supersymmetry transformation (44) is rewritten in terms of the gauge field and the scalar field as

\[ \delta \phi = i \bar{\psi} \gamma_3 \psi \]
\[ \delta b^a = i \bar{\psi} \gamma^a \psi \quad (93)\]

and

\[ \delta \psi = \frac{i}{2} (F_{ab} \gamma^{ab} - 2 \rho \gamma^3 \phi + \rho (D_a \phi)[\gamma^3, \gamma^a]) \epsilon' \quad (94)\]

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where \( b^a \equiv g^{ab} b_b \), \( \gamma^3 = x_i \sigma_i / \rho \), \( \gamma^a = K^a \sigma_i \) and \( \epsilon' = \alpha^{1/2} \epsilon \).

We next consider stability of the two classical solutions against quantum corrections in the same manner as discussed in section 4. One-loop corrections to the commuting matrices are calculated as

\[
W_B = \frac{1}{2} \sum_{i \neq j} ( \log(x^{(i)} - x^{(j)})^2 + \log(1 - \frac{4\alpha^2}{(x^{(i)} - x^{(j)})^2}) ) ,
\]
\[
W_F = -\frac{1}{2} \sum_{i \neq j} \log(x^{(i)} - x^{(j)})^2 .
\]

Then, the value of the effective action including both classical and one-loop quantum effects is given by

\[
S_{CM}^{\text{eff}} = \frac{1}{2} \sum_{i \neq j} \log(1 - \frac{4\alpha^2}{(x^{(i)} - x^{(j)})^2}) .
\]

From this expression, we can read off that the eigenvalues tend to collapse into a single ball whose size is of order \( \alpha \). This shows that there exist a minimum length scale set by \( \alpha \) in this theory. Since there are \( N(N+1)/2 \) pairs of eigenvalues, \( S_{CM}^{\text{eff}} \) is negative and of order \( N^2 \).

For the fuzzy sphere, one-loop correction can be calculated as

\[
W_B = \frac{1}{2} Tr \log(Ad(L))^2 ,
\]
\[
W_F = -\frac{1}{2} Tr \log(Ad(L))^2 ,
\]

and the effective action for the fuzzy sphere does not receive quantum corrections perturbatively as expected. Therefore the value of the effective action is

\[
S_{FS}^{\text{eff}} = -\frac{2}{3g^2 \rho^4} \frac{N + 1}{N(N + 2)} = -\frac{\pi}{6\rho^2 g^2_y M} N(N + 2) .
\]

In this case, since both of (96) and (98) are negative valued and of the same order \( N^2 \), it is difficult to conclude perturbative stability of the fuzzy sphere at this level.

6 Summary

In this paper, we have studied noncommutative gauge theories on the fuzzy sphere. We considered two different types of supersymmetric three-dimensional matrix model actions with a Chern-Simons term. These models have a classical solution which represents a fuzzy sphere. By expanding the models around this solution, we have obtained noncommutative
gauge theories on the fuzzy sphere. The gauge field acquires a mass due to the Chern-Simons term.

We have discussed two large $N$ limits. One corresponds to a commutative limit. By taking this limit, we obtained a gauge theory on a commutative sphere. Another limit corresponds to a decompactifying limit. A noncommutative gauge theory on the fuzzy sphere becomes a noncommutative gauge theory on a noncommutative plane.

Two types of three-dimensional supersymmetric matrix model actions we have considered in this paper have different supersymmetry properties. The first type contains a Majorana mass term in order to preserve the original supersymmetry in the flat space. The second one does not have any other term than the Yang Mills and the Chern Simons terms but it is invariant under a modified supersymmetry. The fuzzy sphere preserves supersymmetry in the second type while it does not in the first one. Therefore the second one will be more natural from the fuzzy sphere point of view. On the other hand, as for the commuting matrices, the situation is opposite. A solution with commuting matrices preserves the supersymmetry in the first type and eigenvalues can move freely at least perturbatively. In the second type, eigenvalues collapse into a small ball whose size is of the noncommutative scale.

We have also investigated the stability of the fuzzy sphere against quantum fluctuations. By calculating the one-loop effective action, we showed that in the first model the fuzzy sphere is stable for fixed $g_{YM}$ in the large $N$ limit. However, if we take $g_{YM}$ sufficiently large with the large $N$ limit, the fuzzy sphere becomes unstable and decays into a set of diagonal eigenvalues. On the other hand, the fuzzy sphere is stable against quantum corrections because of the nature of a BPS state in the second model.

Finally let us comment on ambiguities of operator orderings and corresponding freedom for mapping from matrices to functions. Matrices on the fuzzy sphere are expanded by a noncommutative spherical harmonics as discussed in section 2. The ordering of the noncommutative coordinates in the noncommutative spherical harmonics corresponds to the Weyl type ordering. Therefore a product of the noncommutative spherical harmonics becomes Moyal product in the decompactifying limit. On the other hand, a stereographic projection from a fuzzy sphere to a noncommutative complex plane as discussed in the appendix B enables us to construct a normal ordered type basis. After mapping from matrices to functions, a product of functions are written by the so called Berezin product. Since such noncommutative products are known for general Kähler manifolds, it would be an interesting problem to obtain noncommutative gauge theories on more general curved backgrounds from matrix models.
A Star product on fuzzy sphere

In this section we summarize the noncommutative products on fuzzy sphere. We mainly follow [14]. We have expanded functions and matrices as (16) and (19):

\[ a(\Omega) = \sum_{l=0}^{l=N} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega), \]  
\[ \hat{a} = \sum_{l=0}^{l=N} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}. \]  

Normalization is fixed as in (18) and (21). Combining (A.1) and (A.2), we can obtain a map between the field and the function:

\[ a(\Omega) = \frac{1}{N+1} \sum_{lm} Tr(\hat{Y}_{lm}^\dagger \hat{a}) Y_{lm}(\Omega). \]  

Let us consider the product of the two spherical harmonics, \( \hat{Y}_{lm} \) and \( \hat{Y}_{l'm'} \). We have required that maximum value of \( l \) is \( N \). This product is expanded by the spherical harmonics and it contains \( \hat{Y}_{l+l'} \). We assume that \( N \) is large such that \( l + l' \) does not exceed \( N \). We define the noncommutative product on the fuzzy sphere as

\[ a \star b(\Omega) \equiv \frac{1}{N+1} \sum_{lm} Tr(\hat{a} \hat{b} \hat{Y}_{lm}^\dagger) Y_{lm}(\Omega) \]
\[ = \frac{1}{N+1} \sum_{lm} \sum_{l'm'} \sum_{l''m''} \int d\Omega' d\Omega'' Y_{l'm'}^*(\Omega') Y_{l''m''}^*(\Omega'') a(\Omega') b(\Omega'') \]
\[ Tr(\hat{Y}_{l'm'} \hat{Y}_{l''m''} \hat{Y}_{lm}^\dagger) Y_{lm}(\Omega). \]  

Components of the spherical harmonics can be given using Wigner Eckart theorem:

\[ (\hat{Y}_{lm})_{ss'} = \langle N/2, s | \hat{Y}_{lm} | N/2, s' \rangle = (-1)^{N-s} \frac{N}{2-s} \frac{N}{2} \frac{N}{2} \sqrt{(2l+1)(N+1)} \]  

where \( s, s' = -N/2, -N/2 + 1, \ldots, 0, \ldots, N/2 \). Then we can calculate the trace of three spherical harmonics:

\[ \frac{1}{N+1} Tr(\hat{Y}_{l_{m_1}m_1} \hat{Y}_{l_{m_2}m_2} \hat{Y}_{l_{m_3}m_3}) \]
\[ = (-1)^{N+l_1+l_2+l_3} l_1 l_2 l_3 m_1 m_2 m_3 \frac{N}{2} \frac{N}{2} \frac{N}{2} \sqrt{(2l_1+1)(2l_2+1)(2l_3+1)(N+1)}. \]
where $\cdots$ and $\{\cdots\}$ are Wigner’s $3j$-symbol and $6j$-symbol respectively. $\{\cdots\}$ behaves as \( N^{-3/2} \) for $N \to \infty$ [14]. $\cdots$ becomes zero only when $m_1 + m_2 + m_3 = 0$. By substituting this quantity into (A.4), we get the explicit expression of the noncommutative product on the fuzzy sphere.

Commutator of two matrices is

$$[\hat{a}, \hat{b}] = \sum_{l_1 m_1} \sum_{l_2 m_2} a_{l_1 m_1} b_{l_2 m_2} [\hat{Y}_{l_1 m_1}, \hat{Y}_{l_2 m_2}]$$

$$= \sum_{l_1 m_1} \sum_{l_2 m_2} \sum_{l_3 m_3} a_{l_1 m_1} b_{l_2 m_2} f_{l_1 m_1 l_2 m_2}^{l_3 m_3} \hat{Y}_{l_3 m_3}. \quad (A.7)$$

\( f_{l_1 m_1 l_2 m_2}^{l_3 m_3} \) is zero when \( l_1 + l_2 + l_3 = \text{even} \) and is given as follows when \( l_1 + l_2 + l_3 = \text{odd} \):

$$f_{l_1 m_1 l_2 m_2}^{l_3 m_3} = 2 \sqrt{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)N + 1} (-1)^{N-1} \left\{ l_1 \ l_2 \ l_3 \right\}_{m_1 \ m_2 \ m_3}. \quad (A.8)$$

This quantity behaves as \( N^{-1} \) when $N \to \infty$.

**B Wick type star product**

Here we explain the star product on the fuzzy sphere which corresponds to a normal ordered operator product [31, 32, 33] and derive mapping rules from matrix models to field theories with the normal ordered products à la Berezin [31].

We first rescale $\hat{x}_i$ as $\hat{y}_i = \hat{x}_i / \rho$ where $\hat{x}_i$’s are defined by (7). Then

$$[\hat{y}_i, \hat{y}_j] = i \beta \epsilon_{ijk} \hat{y}_k, \quad (B.1)$$

where $\beta = \alpha / \rho$. Thus \( \hat{y}_i^2 = 1 \) and $\beta^2 = 4 / N(N+2)$. We define the stereographic projection of $\hat{y}_i$ as

$$\hat{z} = \frac{1}{\sqrt{2}} \hat{y} - \hat{x}, \quad \hat{z}^\dagger = \frac{1}{\sqrt{2}} \hat{y}^\dagger + \hat{x}, \quad (B.2)$$

where $\hat{y}_\pm = \hat{y}_1 \pm i \hat{y}_2 / \sqrt{2}, \hat{x} = 2(1 - \hat{y}_3)^{-1} \). Note that the complex coordinate projected from the sphere of radius $\rho$ is $\hat{w} = \rho \hat{z}$. By the above definition, we obtain the commutation relation

$$[\hat{z}, \hat{z}^\dagger] = \beta \hat{x} \left( 1 + |\hat{z}|^2 - \frac{1}{2} \hat{x} (1 + \frac{\beta}{2} |\hat{z}|^2) \right), \quad (B.3)$$

\( \beta \) See [34, 35] about Weyl ordered star product on Kähler manifold.
and by using $y_i^2 = 1$ we can solve $\hat{\chi}$ in terms of $\hat{z}$ and $\hat{z}^\dagger$:

$$
\beta \hat{\chi} = 2 + \beta \hat{\xi} - 1 - \sqrt{4\hat{\xi} - 1 + \beta^2 \hat{\xi}^{-2}},
$$

(B.4)

where $|\hat{z}|^2 = \hat{z}\hat{z}^\dagger$ and $\hat{\xi} = 1 + \beta|\hat{z}|^2$. For sufficiently large $N$, $\hat{\chi} \sim 1 + |\hat{z}|^2$ and the commutation relation is simplified as

$$
[\hat{z}, \hat{z}^\dagger] \sim \frac{1}{N}(1 + |\hat{z}|^2)^2.
$$

(B.5)

As discussed in [32, 33, 36], the operator $\hat{z}$ can be written in terms of an annihilation operator of a usual harmonic oscillator as

$$
\hat{z} = f(\hat{n} + 1)\hat{a},
$$

(B.6)

where

$$
[\hat{a}, \hat{a}^\dagger] = 1, \quad \hat{n} = \hat{a}^\dagger\hat{a},
$$

(B.7)

and

$$
f(\hat{n}) = \frac{\sqrt{N - \hat{n} + 1}}{\sqrt{N/2(N + 1) + N/2 - \hat{n}}} \sim \frac{1}{\sqrt{N + 1 - \hat{n}}}.
$$

(B.8)

Eq.(B.8) is derived from a correspondence[33]

$$
|N/2; m\rangle \leftrightarrow |n\rangle,
$$

(B.9)

where $|N/2; m\rangle$ and $|n\rangle$ are eigenstates of $\hat{L}_3$ and $\hat{n}$ respectively. Thus the eigenvalue of $\hat{L}_3$ is given by $m = n - N/2$ where $0 \leq n \leq N$. To construct a normal ordered star product, we define the following coherent state $|z\rangle$.

$$
|z\rangle = M(|z|^2)^{-1/2} \sum_{n=0}^{N} \frac{z^n}{[\gamma(n)]!} |n\rangle,
$$

(B.10)

where $M(|z|^2)$ is a normalization factor, $\gamma(n) \equiv \sqrt{n}f(n)$ and

$$
[\gamma(n)]! = \begin{cases} \gamma(n)\gamma(n-1)\cdots\gamma(1) & (n \neq 0) \\ 1 & (n = 0) \end{cases}.
$$

Eq.(B.10) satisfies the condition of the coherent state only for sufficiently large $N$ and when we can neglect the state of $|N + 1\rangle$. The normalization factor is determined from $\langle z|z\rangle = 1$ as

$$
M(|z|^2) = \sum_{n=0}^{N} \frac{|z|^{2n}}{[\gamma(n)]^2!} \sim \sum_{n=0}^{N} [N + 1 - n]! \frac{|z|^{2n}}{n!} = (1 + |z|^2)^N.
$$

(B.11)

Also the completeness condition

$$
1 = \sum_{n} |n\rangle\langle n| = (N + 1) \int d\mu(z, \bar{z}) |z\rangle\langle z|
$$

(B.12)
determines the measure \(d\mu\) as

\[
d\mu(z, \bar{z}) \sim \frac{idz \wedge d\bar{z}}{2\pi(1 + |z|^2)^2}.
\] (B.13)

(See [33] for details.)

Normal ordered operators can be expanded in terms of \(|n\rangle\langle m|\)

\[
\hat{Z}_{nm} = |n\rangle\langle m| = \left(\hat{a}^\dagger\right)^n |0\rangle\langle 0| \frac{\hat{a}^m}{\sqrt{m!}} = \frac{(\hat{z}^\dagger)^n}{[\gamma(n)]!} |0\rangle\langle 0| \frac{z^m}{[\gamma(m)]!},
\] (B.14)

where \(|0\rangle\langle 0| = \sum_k c_k (\hat{z}^\dagger)^k \hat{z}^k\) and \(c_k\) is determined by \(M(x)^{-1} = \sum_k c_k x^k\). The corresponding orthonormalized basis of functions are defined by

\[
Z_{nm}(z, \bar{z}) = \langle z|\hat{Z}_{nm}|z \rangle = \frac{z^n z^m}{[\gamma(n)]! [\gamma(m)]! M(|z|^2)}.
\] (B.15)

Thus any operators

\[
\hat{a} = \sum_{nm} \hat{a}_{nm} \hat{Z}_{nm}
\] (B.16)

are mapped to corresponding functions as

\[
\hat{a} \to a(z, \bar{z}) = \langle z|\hat{a}|z \rangle = \sum_{nm} \hat{a}_{nm} Z_{nm}(z, \bar{z}).
\] (B.17)

The trace of an operator is mapped to an integral over the projected plane

\[
\frac{1}{N+1} Tr \to \int d\mu(z, \bar{z}).
\] (B.18)

The analytic continuation of this function is defined by

\[
a(z, \bar{\eta}) = \frac{\langle \eta|\hat{a}|z \rangle}{\langle \eta|z \rangle},
\] (B.19)

where \(\langle \eta|z \rangle = M(|\eta|^2)^{-\frac{1}{2}} M(|z|^2)^{-\frac{1}{2}} M(\bar{\eta}z)\). Using this definition, a product of matrices is mapped to the star product as

\[
a \star b(z, \bar{\xi}) = \langle z|\hat{a}\hat{b}|z \rangle = (N+1) \int d\mu(\eta, \bar{\eta}) \langle \eta|z \rangle^2 \frac{\langle \eta|\hat{a}|\eta \rangle}{\langle \eta|\eta \rangle} \frac{\langle \eta|\hat{b}|z \rangle}{\langle \eta|z \rangle}
\]

\[
= (N+1) \int d\mu(\eta, \bar{\eta}) a(\eta, \bar{\xi}) \frac{M(\bar{\eta}z) M(|\eta|^2) M(|z|^2)}{M(|z|^2)} b(z, \bar{\eta})
\]

\[
\sim \frac{1}{\lambda + 1} \int \frac{id\eta \wedge d\bar{\eta}}{2\pi(1 + |\eta|^2)^2} a(\eta, \bar{\xi}) \left[\frac{(1 + z\bar{\eta})(1 + \eta\bar{z})}{(1 + |\eta|^2)(1 + |z|^2)}\right]^\frac{1}{2} b(z, \bar{\eta}),
\] (B.20)

where \(\lambda = 1/N\). This star product is nothing but the Berezin product on the sphere[31].

From the definition of \(\hat{Z}_{nm}\), the corresponding function satisfies the following simple relations

\[
Z_{nm} \star Z_{rs} = \delta_{mr} Z_{ns},
\] (B.21)
and
\[ \int d\mu(z, \bar{z}) \bar{Z}_{nm} \ast Z_{n'm'} = \delta_{nn'} \delta_{mm'}. \] (B.22)

Next we write down functions corresponding to the operators \( \hat{y} \)'s. \( \hat{y}_\pm, \hat{y}_3 \) are rewritten by \( \hat{z}, \hat{z}^\dagger \) and \( \hat{n} \) as
\[
\begin{align*}
\hat{y}_3 &= \beta(\hat{n} - N/2), \\
\sqrt{2}\hat{y}_+ &= (1 - \hat{y}_3) \hat{z}^\dagger, \\
\sqrt{2}\hat{y}_- &= \hat{z}(1 - \hat{y}_3). 
\end{align*}
\] (B.23)

Thus, the corresponding functions can be expanded by \( Z_{nm} \) as
\[
\begin{align*}
y_3 &= \beta \sum_{n=0}^{N} (n - N/2)Z_{nn}, \\
y_+ &= (1 - y_3) \ast \bar{z} = \sum_{n=0}^{N-1} \hat{y}_n Z_{n+1 n}, \\
y_- &= z \ast (1 - y_3) = \sum_{n=0}^{N-1} \hat{y}_n Z_{n+1 n}, 
\end{align*}
\] (B.24)

where
\[
\hat{y}_n = (1 - \beta(n + 1 - N/2)) \gamma(n + 1) = \beta \sqrt{n + 1} \sqrt{N - n}. 
\] (B.25)

Also the functions corresponding to the operators \( \hat{z}, \hat{z}^\dagger \) can be expanded as
\[
\begin{align*}
z &= \sum_{n=0}^{N-1} \gamma(n + 1)Z_{n+1 n}, \\
\bar{z} &= \sum_{n=0}^{N-1} \gamma(n + 1)Z_{n+1 n}. 
\end{align*}
\] (B.26)

Now we will derive differential operators corresponding to the adjoint operators \( Ad(\hat{y}_i) \) which are necessary to rewrite the matrix model action in terms of field theory on the projected plane. We first define differential operators
\[
\begin{align*}
k_z &= \frac{1}{N} M^{-1} \partial_z M \sim \frac{\bar{z}}{1 + |z|^2} + \frac{1}{N} \partial_z, \\
k_{\bar{z}} &= \frac{1}{N} M^{-1} \partial_{\bar{z}} M \sim \frac{z}{1 + |z|^2} + \frac{1}{N} \partial_{\bar{z}}. 
\end{align*}
\] (B.27)

Note that the derivatives above act not only on \( M \) but on functions on the right of \( M \). We can easily obtain the following relations
\[
\begin{align*}
z k_z Z_{nm} &= m' Z_{nm}, \quad \bar{z} k_{\bar{z}} Z_{nm} = n' Z_{nm}, \\
k_z Z_{nm} &= \frac{m'}{\gamma(m)} Z_{n-1 m}, \quad k_{\bar{z}} Z_{nm} = \frac{n'}{\gamma(n)} Z_{n-1 m}, \\
z Z_{nm} &= \gamma(m + 1) Z_{n+1 m}, \quad \bar{z} Z_{nm} = \gamma(n + 1) Z_{n+1 m}, 
\end{align*}
\]

\( \hat{n} \) can be written in terms of \( \hat{z}, \hat{z}^\dagger \). However it is not necessary here.
where \( n' = n/N \), \( m' = m/N \). By using these relations, we have

\[
[\hat{y}_3, \hat{Z}_{nm}] \rightarrow \beta(n - m)Z_{nm} = \beta N(\bar{z}k_\bar{z} - zk_z)Z_{nm} \sim \beta(\bar{z}\partial_\bar{z} - z\partial_z)Z_{nm}
\]

and hence the adjoint operator in matrix models \( Ad(\hat{y}_3) \) becomes the following differential operator after mapping matrix models to field theories:

\[
Ad(\hat{y}_3) = [\hat{y}_3, \cdot] \rightarrow \beta(\bar{z}\partial_\bar{z} - z\partial_z). \tag{B.28}
\]

We can also obtain similar rules for adjoint operators:

\[
Ad(\sqrt{2}\hat{y}_+^\prime) \rightarrow -\beta(\partial_\bar{z} + \bar{z}^2\partial_z), \tag{B.29}
\]

\[
Ad(\sqrt{2}\hat{y}_-^\prime) \rightarrow \beta(\partial_\bar{z} + z^2\partial_z), \tag{B.30}
\]

\[
Ad(\hat{y}_i^\prime)^2 \rightarrow -\beta^2(1 + |z|^2)^2\partial_\bar{z}\partial_z. \tag{B.31}
\]

Using mapping rules \( (B.17),(B.18),(B.20) \) and \( (B.29) \sim (B.31) \), the matrix model actions can be written in terms of field theories on the projected plane.

In the commutative limit, the action is simplified by writing the vector fields in terms of projected ones as \( (39) \). Killing vectors on the projected plane are given

\[
K_+^z = -\frac{i}{\sqrt{2}}z, \quad K_+^\bar{z} = -\frac{i}{\sqrt{2}}\bar{z},
\]

\[
K_-^z = \frac{i}{\sqrt{2}}\bar{z}^2, \quad K_-^\bar{z} = \frac{i}{\sqrt{2}},
\]

\[
K_3^z = -iz, \quad K_3^\bar{z} = iz,
\]

which can be seen from \( (B.29) \sim (B.31) \). Thus the field strength of \( U(1) \) gauge field in the commutative limit can be written in terms of \( z, \bar{z} \) as

\[
F_{+3} = -\frac{i}{\rho^2}K_+^aK_3^b(\partial_a b_b - \partial_b b_a) = -\frac{i}{2\rho^2}(1 - |z|^2)(1 + |z|^2)F_{z\bar{z}},
\]

\[
F_{+3} = -\frac{i}{\rho^2}K_3^aK_3^b(\partial_a b_b - \partial_b b_a) = -\frac{i}{\sqrt{2}\rho^2}\bar{z}(1 + |z|^2)F_{\bar{z}z},
\]

\[
F_{-3} = -\frac{i}{\rho^2}K_-^aK_3^b(\partial_a b_b - \partial_b b_a) = \frac{i}{\sqrt{2}\rho^2}z(1 + |z|^2)F_{z\bar{z}}, \tag{B.32}
\]

where

\[
F_{z\bar{z}} = \partial_\bar{z}b \bar{z} - \partial_z b_z. \tag{B.33}
\]

We have set the scalar field \( \phi = 0 \) for simplicity. Also we get

\[
Tr F^2 \rightarrow 2(N + 1) \int d\mu(z, \bar{z})(F_{12}^2 + F_{23}^2 + F_{31}^2)
\]

\[
= 2(N + 1) \int d\mu(z, \bar{z})(-F_{+3}^2 + 2F_{+3}F_{-3})
\]

\[
= \frac{2(N + 1)}{\rho^4} \int d\mu(z, \bar{z})(1 + |z|^2)^2 F_{z\bar{z}}^2
\]

\[
= \frac{16(N + 1)}{\rho^4} \int d\mu(z, \bar{z})F_{ab}F^{ab}, \tag{B.34}
\]

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where
\[ g^{zz} = g^{\bar{z}\bar{z}} = 0, \quad g^{\bar{z}z} = g^{z\bar{z}} = \frac{(1 + |z|^2)^2}{4}. \] (B.35)

This is the well-known result.

Finally we consider the flat limit (B.35). Here, it should not be discussed around the north pole but the south pole because of the definition (B.2). So \( \hat{y}_3 \) can be approximated as
\[ \hat{y}_3 \sim -\beta N/2 \sim -1, \] (B.36)
and a rescaled coordinate corresponding to \( x' \) is
\[ w' = i\sqrt{\frac{2}{N}} w = i\sqrt{\frac{2}{N}} \rho z = i\rho' z. \] (B.37)

Thus we can rewrite the star product (B.20) in the flat case:
\[ M(|z|^2) \sim (1 + |z|^2)^N \]
\[ = (1 - \frac{|w'|^2}{\rho^2})^N \]
\[ = \left( (1 - \frac{|w'|^2}{\rho^2}) - \frac{N}{\rho^2} |w'|^2 \right) \rightarrow e^{-\frac{N}{\rho^2} |w'|^2}, \] (B.38)

\[ d\mu(z, \bar{z}) \sim \frac{idz \wedge d\bar{z}}{2\pi(1 + |z|^2)^2} \]
\[ = \frac{(1 - \frac{|w'|^2}{\rho^2})^{-2} - idw' \wedge d\bar{w}'}{2\pi \rho^2} \rightarrow \frac{idw' \wedge d\bar{w}'}{2\pi \rho^2}, \] (B.39)

\[ a \star b(w', \bar{w}') = \frac{1}{\nu} \int \frac{id\eta' \wedge d\bar{\eta}'}{2\pi} a(\eta', \bar{w}') e^{\frac{i}{2}(w'\eta' + \bar{w}'\bar{\eta}') \frac{1}{\rho^2} (w'\bar{\eta}' - \bar{w}'\eta')} b(w', \bar{\eta'}), \] (B.40)

where \( \nu = -\rho^2/(N + 1) \sim -\alpha^2/2 \). Thus the star product becomes the Berezin product on the flat plane.

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