Solutions of DC OPF are Never AC Feasible

Kyri Baker

Abstract—In this paper, we analyze the relationship between generation dispatch solutions produced by the DC optimal power flow (DC OPF) problem and generation solutions produced by the AC optimal power flow (AC OPF) problem. While there has been much previous work in analyzing the approximation error of the DC assumption, the AC feasibility of the DC OPF problem has not been fully explored, although difficulty achieving AC feasibility is known in practice. Here, we consider the set of feasible active power generation values in a standard DC OPF problem and the set of feasible active power generation values in a standard AC OPF problem given a set of network loads. Under some very light assumptions, we show that the intersection of these sets is the empty set; i.e., that no solution to the DC OPF problem will satisfy the AC power flow constraints. This has implications for continuing to use the standard DC approximation in current grid operation as well as in research.

I. INTRODUCTION

The DC optimal power flow (DCOPF) approximation of the AC optimal power flow (ACOPF) problem is widely used in the current power system due to its convexity and computational benefits that allow it to be solved on fast timescales. However, it is well-known that the DC OPF can provide, in some cases, a poor approximation of actual AC power flows and locational marginal prices (LMPs), resulting in physically unrealizable system states. A report from the Federal Energy Regulatory Commission (FERC) states that the differences in objective function value and approximation error between DC OPF and AC OPF. Outside of analyzing the economic dispatch optimization problem neglects modeling power flows and transmission losses throughout the network altogether, simply optimizing for lowest-cost active power dispatch:

\[
\text{min}_{\mathbf{p}_g} \quad \sum_{j \in \mathcal{G}} a_j p_{g,j}^2 + b_j p_{g,j} + c_j \quad \text{(1a)}
\]

s.t. : \( \sum_{j \in \mathcal{G}} p_{g,j} - \sum_{i \in \mathcal{N}} p_{i,j} = 0 \quad \text{(1b)} \)

\[
p_{g,j} \leq p_{g,j} \leq \bar{p}_{g,j}, \quad \forall j \in \mathcal{G} \quad \text{(1c)}
\]

K. Baker is an Assistant Professor in the Department of Civil, Environmental, and Architectural Engineering at the University of Colorado Boulder, Boulder, CO, USA. Email: kyri.baker@colorado.edu

that the DC OPF is not “typically” AC infeasible; in fact, it is always AC infeasible. This fact must be formally shown to illustrate the challenges with using the DC approximation to represent physically realizable solutions.

In this paper, we show that under some light assumptions, the set of active power generation set points within the feasible region of the DC OPF problem and the set of active power generation set points within the feasible region of the AC OPF problem have an empty intersection. While this has been observed in practice, we demonstrate mathematically why the DC OPF is never AC feasible. Similarly, we also show that the solution to the economic dispatch (ED) problem is also never AC feasible. This has implications for future power system operation, considering many system operators still use DC OPF, and may suggest a move towards more physically representative approximations of the AC OPF problem or towards using line loss models within DC OPF [8].

II. OPTIMAL DISPATCH AND POWER FLOW

We first briefly summarize three common ways of optimizing generation dispatch in transmission networks. Define coefficients \( a_j, b_j, \) and \( c_j \) as the operational costs associated with generator \( j \). Let set \( \mathcal{G} \) be the set of all generators in the network, \( \mathcal{N} \) be the set of all buses in the network, \( \mathcal{L} \) be the set of all lines (branches) in the network, \( \Omega_i \) be the set of buses connected to bus \( i \), and \( \mathcal{G}_j \) be the set of generators connected to bus \( j \). Define \( p_{i,j} (q_{i,j}) \) as the total active (reactive) power consumption at bus \( j \), \( p_{g,j} (q_{g,j}) \) as the active (reactive) power output of generator \( j \), and \( p_{g,j} (q_{g,j}) \) and \( \bar{p}_{g,j} (\bar{q}_{g,j}) \) are lower and upper limits on active (reactive) power generation, respectively. Let \( \mathbf{p}_g \) be a vector comprising the active power generation \( p_{g,j} \) at each generator \( j \in \mathcal{G} \) and \( \mathbf{q}_l \) be a vector comprising the active power consumption at each bus \( i \in \mathcal{N} \). The complex voltage at bus \( i \) has magnitude \( |v_i| \) and phase angle \( \theta_i \), and the difference in phase angle between buses \( i \) and \( m \) is written as \( \theta_{im} \). Constant parameters \( G_{im} \) and \( B_{im} \) are the conductance and susceptance of line \( im \), respectively.

A. Economic Dispatch

The economic dispatch optimization problem neglects modeling power flows and transmission losses throughout the network altogether, simply optimizing for lowest-cost active power dispatch:

\[
\text{min}_{\mathbf{p}_g} \quad \sum_{j \in \mathcal{G}} a_j p_{g,j}^2 + b_j p_{g,j} + c_j \quad \text{(1a)}
\]

s.t. : \( \sum_{j \in \mathcal{G}} p_{g,j} - \sum_{i \in \mathcal{N}} p_{i,j} = 0 \quad \text{(1b)} \)

\[
p_{g,j} \leq p_{g,j} \leq \bar{p}_{g,j}, \quad \forall j \in \mathcal{G} \quad \text{(1c)}
\]
In the ED formulation, (1b) ensures that the total generated active power equals the total consumed active power in the network. Constraint (1c) enforces upper and lower limits on active power generation at each generator.

**B. AC Optimal Power Flow**

The AC Optimal Power Flow (AC OPF) model is typically considered the ground truth for estimating physical power flows throughout the network.

\[
\begin{align*}
\min_{p_i} & \sum_{j \in G} a_j p_{g,j}^2 + b_j p_{g,j} + c_j \\
\text{s.t.:} & \\
& |v_i| \sum_{m \in \Omega_i} |v_m| (G_{im} \cos(\theta_{im}) + B_{im} \sin(\theta_{im})) = p_{l,i} - \sum_{j \in \Omega_i} p_{g,j}, \quad \forall i \in \mathcal{N} \quad (2b) \\
& |v_i| \sum_{m \in \Omega_i} |v_m| (G_{im} \sin(\theta_{im}) - B_{im} \cos(\theta_{im})) = q_{l,i} - \sum_{j \in \Omega_i} q_{g,j}, \quad \forall i \in \mathcal{N} \quad (2c) \\
& p_{g,j} \leq p_{g,j} \leq \bar{p}_{g,j}, \quad \forall j \in \mathcal{G} \quad (2d) \\
& q_{g,j} \leq q_{g,j} \leq \bar{q}_{g,j}, \quad \forall j \in \mathcal{G} \quad (2e) \\
& |v| \leq |v_i| \leq \bar{v}, \quad \forall i \in \mathcal{N}. \quad (2f)
\end{align*}
\]

where \( v \) is a 2n-dimensional vector comprising the unknown voltage magnitudes and angles.

**C. DC Optimal Power Flow**

The constraints within the DC Optimal Power Flow problem (DC OPF) are linear approximations of the actual nonlinear AC power flows. The DC approximation is derived from multiple physical assumptions and observations. First, in transmission networks, the line resistance \( R_{im} \) is typically significantly less than the line reactance \( X_{im} \); thus, the conductance \( G_{im} \) can be approximated to zero and the susceptance \( B_{im} \) can be approximated to \( \frac{1}{X_{im}} \). Second, the phase angle difference between any two buses is typically small and usually does not exceed 30°. From this, we can use the small angle approximation to approximate \( \sin(\theta_{im}) \approx \theta_{im} \) and \( \cos(\theta_{im}) \approx 1 \). Third, transmission level voltage magnitudes are typically very close to 1.0 p.u. during normal operation.

Lastly, from these initial assumptions, and confirmed by the fact that reactive power is a localized phenomenon that cannot travel long distances, we see that the magnitude of the reactive power flow on the lines (denote this as \( Q_{im} \) for line \( im \)) is significantly less than the magnitude of the active power flow on the lines (denote this as \( P_{im} \) for line \( im \)). Using these assumptions when studying the AC power flow equations, equations (2b) and (2c) simplify and we are left with the DC OPF problem

\[
\begin{align*}
\min_{p_i} & \sum_{j \in \mathcal{G}} a_j p_{g,j}^2 + b_j p_{g,j} + c_j \\
\text{s.t.:} & \\
& p_{l,i} - \sum_{j \in \mathcal{G}} p_{g,j} = \sum_{m \in \Omega_i} B_{im} \theta_{im}, \quad \forall i \in \mathcal{N} \\
& - F_{im} \leq B_{im} \theta_{im} \leq F_{im}, \quad \forall i \in \mathcal{N} \quad (3c)
\end{align*}
\]

where \( F_{im} \) represents the limit on the magnitude of the line flows on line \( im \). Note that physically, transmission line flows are limited by the amount of current that can safely flow through the line. We can write the current flow limit on line \( im \), \( |I_{im}| \), in terms of the complex power \( S_{im} \), real power \( P_{im} \), and reactive power \( Q_{im} \) flowing from bus \( i \) to bus \( m \) and the complex voltage at bus \( i \):

\[
|I_{im}| = \left| \left( \frac{S_{im}}{v_i} \right)^* \right| = \left( \frac{(P_{im}^2 + Q_{im}^2)}{|v_i|^2} \right),
\]

and by using the DC approximations stated above, namely that \( P_{im} >> Q_{im} \) and \( |v_i| = 1.0 \) p.u., we can write the current flow limit in terms of power

\[
|I_{im}| \approx \frac{P_{im}}{|v_i|} \approx P_{im},
\]

which justifies (3c). More information on the derivation of the DC approximation, including a full explanation of assumptions, can be found in [8].

**III. AC Feasibility**

Towards determining the AC feasibility of the economic dispatch and DC OPF problems, define the set of feasible active power generation vectors \( p_g \) for a given set of loads \( p_l \) satisfying (1) as \( \mathcal{Y}_{ED} \subset \mathbb{R}^{|\mathcal{G}|} \), the set of feasible \( p_g \) satisfying (3) as \( \mathcal{Y}_{DC} \subset \mathbb{R}^{|\mathcal{G}|} \), and the set of feasible \( p_g \) satisfying (2) as \( \mathcal{Y}_{AC} \subset \mathbb{R}^{|\mathcal{G}|} \).

**Definition** AC feasibility. A vector of active power generation values \( p_g \) is called AC feasible if \( p_g \in \mathcal{Y}_{AC} \).

Next, we state two assumptions that are typically assumed when solving AC OPF problems.

**Assumption 1.** Assume that the loads in the network are modeled as constant (P, Q) loads [9].

**Assumption 2.** Assume that power is flowing on at least one line in the network; i.e., for all \( p_g \in \mathcal{Y}_{AC} \), \( \sum_{j \in \mathcal{G}} p_{g,j} > \sum_{i=1}^{\mathcal{N}} p_{l,i} \) due to line losses.

**Lemma 1. Solutions to (1) are never AC feasible.**

**Proof.** Define \( p_i := |v_i| \sum_{m \in \Omega_i} |v_m| (G_{im} \cos(\theta_{im}) + B_{im} \sin(\theta_{im})) \). Then, notice that \( \sum_{i \in \mathcal{N}} p_i = \sum_{i \in \mathcal{N}} p_{l,i} - \sum_{j \in \mathcal{G}} p_{g,j} \). If a solution satisfies the constraints in (1), we have \( \sum_{j \in \mathcal{G}} p_{g,j} \geq \sum_{i \in \mathcal{N}} p_{l,i} \). Thus, \( \sum_{i \in \mathcal{N}} p_i = 0 \). However, from Assumption 2, we must have \( \sum_{j \in \mathcal{G}} p_{g,j} > \sum_{i \in \mathcal{N}} p_{l,i} \). Thus \( \sum_{i \in \mathcal{N}} p_i \neq 0 \), meaning we have a contradiction and there exists no \( p_g \) that is an element of both \( \mathcal{Y}_{ED} \) and \( \mathcal{Y}_{AC} \); thus \( \mathcal{Y}_{ED} \cap \mathcal{Y}_{AC} = \emptyset \). □
Theorem 1. Solutions to \( \mathcal{Y}_{AC} \) are never AC feasible.

Proof. Consider constraint \((3b)\). Summing the left-hand side and right-hand side over all \( i \in \mathcal{N} \) yields

\[
\sum_{i \in \mathcal{N}} p_{li,i} - \sum_{j \in \mathcal{G}} p_{g,j} = \sum_{i \in \mathcal{N}} \sum_{m \in \Omega_i} B_{im} \theta_{im}. \tag{4}
\]

Since \( B_{im} = B_{mi} \) and \( \theta_{im} = -\theta_{mi} \), we are left with

\[
\sum_{i \in \mathcal{N}} p_{li,i} - \sum_{j \in \mathcal{G}} p_{g,j} = 0. \tag{5}
\]

By Lemma 1, we thus have \( \mathcal{Y}_{DC} \cap \mathcal{Y}_{AC} = \emptyset \).

A. Notes

Consider relaxing the constraint \( \sum_{i \in \mathcal{N}} p_{li,i} = \sum_{j \in \mathcal{G}} p_{g,j} \) to \( \sum_{i \in \mathcal{N}} p_{li,i} \leq \sum_{j \in \mathcal{G}} p_{g,j} \) in \((1)\). Due to the nonnegativity of the generator parameters \( a_i \) and \( b_i \) and \( p_g \) and thus the convexity of the objective function, the optimal solution to the ED problem will be reached at \( \sum_{i \in \mathcal{N}} p_{li,i} = \sum_{j \in \mathcal{G}} p_{g,j} \); i.e., where the power balance constraint is binding. We now have the result \( \mathcal{Y}_{ED} \cap \mathcal{Y}_{AC} \neq \emptyset \); however, we also have the optimal solution \( p_g^* \notin \mathcal{Y}_{AC} \), so in practice, the obtained optimal solution to \((1)\) will not satisfy \((4)\). This indicates that even if there exists a non-empty intersection between feasible sets, the obtained solution may not satisfy \((4)\). This difference is the root of the cause for DC OPF solutions never being AC feasible.

IV. CONCLUDING STATEMENTS

In this note, we demonstrated how a solution to DC OPF and ED, under light assumptions, will not satisfy the AC power flow equations. While this has been observed in practice, and the employment of iterative “AC feasibility” techniques are currently being used by system operators to modify the original DC OPF problem, it had not been shown that a solution satisfying both the the standard DC and AC OPF does not exist. This indicates that the standard DC OPF may not be a representative model for grid operators to continue using. Inclusion of renewable energy, energy storage, controllable loads, etc. could also potentially change these results.

REFERENCES

[1] “Recent ISO software enhancements and future software and modeling plans,” www.ferc.gov/industries/electric/indust-act/tto/tto-isosoft-2011.pdf.
[2] K. Dvijotham and D. K. Molzahn, “Error bounds on the DC power flow approximation: a convex relaxation approach,” in 2016 IEEE 55th Conference on Decision and Control (CDC), Dec 2016, pp. 2411–2418.
[3] Z. Yang, H. Zhong, Q. Xia, and C. Kang, “Solving OPF using linear approximations: fundamental analysis and numerical demonstration,” IET Gen., Trans., and Dist., vol. 11, no. 17, pp. 4115–4125, 2017.
[4] T. J. Overbye, X. Cheng, and Y. Sun, “A comparison of the AC and DC power flow models for LMP calculations,” in 37th Annual Hawaii International Conference on System Sciences (HICSS’04), 2004.
[5] S. H. Low, “Convex relaxation of optimal power flow part I: Formulations and equivalence,” IEEE Transactions on Control of Network Systems, vol. 1, no. 1, pp. 15–27, March 2014.
[6] D. Shchetinin, T. T. De Rubira, and G. Hug, “On the construction of linear approximations of line flow constraints for AC optimal power flow,” IEEE Trans. on Power Systems, vol. 34, no. 2, pp. 1182–1192, March 2019.
[7] S. Mhanna, G. Verbić, and A. C. Chapman, “Tight LP approximations for the optimal power flow problem,” in 2016 Power Systems Computation Conference (PSCC), June 2016, pp. 1–7.
[8] R. Eldridge, R. P. O’Neill, and A. Castillo, “Marginal loss calculations for the DCOPF,” FERC Technical Report on Loss Estimation, Jan. 2017.
[9] W. F. Tinney and C. E. Hart, “Power flow solution by Newton’s method,” IEEE Transactions on Power Apparatus and Systems, vol. PAS-86, no. 11, pp. 1449–1460, Nov 1967.
[10] M. Cain, R. P. O’Neill, and A. Castillo, “History of optimal power flow and formulations,” FERC Technical Report, last modified Aug. 2013.
[11] D. Zelenke, S. Chatziioannou, and D. Molzahn, “Inexact convex relaxations for AC optimal power flow: Towards AC feasibility,” arXiv preprint arXiv:1902.04815, Oct. 2019.