An optimal piecewise cubic nonconforming finite element scheme for the planar biharmonic equation on general triangulations

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Abstract This paper presents a nonconforming finite element scheme for the planar biharmonic equation, which applies piecewise cubic polynomials ($P_3$) and possesses $O(h^2)$ convergence rate for smooth solutions in the energy norm on general shape-regular triangulations. Both Dirichlet and Navier type boundary value problems are studied. The basis for the scheme is a piecewise cubic polynomial space, which can approximate the $H^4$ functions with $O(h^2)$ accuracy in the broken $H^2$ norm. Besides, a discrete strengthened Miranda-Talenti estimate ($\nabla_2 \cdot \nabla_2$) = ($\nabla \cdot \nabla$), which is usually not true for nonconforming finite element spaces, is proved. The finite element space does not correspond to a finite element defined with Ciarlet’s triple; however, it admits a set of locally supported basis functions and can thus be implemented by the usual routine. The notion of the finite element Stokes complex plays an important role in the analysis as well as the construction of the basis functions.

Keywords biharmonic equation, optimal cubic finite element scheme, general triangulation, discretized Stokes complex, discrete strengthened Miranda-Talenti estimate

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1 Introduction

In order to obtain a simpler interior structure, in the study of the numerical analysis of partial differential equations, lower-degree polynomials are often expected to be used with respect to the same convergence rate. Finite element schemes with polynomials of degrees not higher than $k$ for $H^m$ problem that possess convergence rates of $O(h^{k+1-m})$ in the energy norm for smooth solutions are called \textbf{optimal}. According to [34], this illustrates both the highest accuracy with respect to certain degree of polynomials and the smallest shape function space with respect to certain convergence rate, and is a critical characteristic for the finite element methodology. Motivated by the fundamental problem, in this paper, we are concerned with whether and how the optimal finite element scheme can be constructed for the biharmonic equation with piecewise cubic polynomials on general triangulations.
1.1 A brief review of relevant works

Various works on optimal schemes can be found focusing mainly on low-order problems. For the lowest-differentiation-order ($H^1$) elliptic problems, the standard Lagrangian elements can yield the optimal approximation on the simplicial grids of an arbitrary dimension. Furthermore, the optimal nonconforming element spaces of $k$-th degrees have also been constructed (see, e.g., [12,13,20] for the cases where $k = 1$, $k = 2$ and $k = 3$, respectively, and [5] for general $k$). For higher-differentiation-order ($H^m$, $m > 1$) elliptic problems, minimal-degree approximations have been studied with the lowest accuracy-order. Specifically, when the subdivision comprises simplexes, a systematic family of nonconforming finite elements has been proposed by Wang and Xu [48] for $H^m$ elliptic partial differential equations in $\mathbb{R}^n$ for any $n \geq m$ with polynomials with degree $m$. Besides, the constructions of finite element functions that do not depend on cell-by-cell definitions can be found in [29,41,62], wherein minimal-degree finite element spaces are defined on general quadrilateral grids for $H^1$ and $H^2$ problems. In contrast, the construction of higher-accuracy-order optimal schemes is complicated for higher-differentiation-order problems, even the planar biharmonic problem, a simple fundamental model problem.

Conforming finite elements for the biharmonic equation requires the $C^1$ continuity assumption. It is well known that with polynomials of degrees $k \geq 5$, spaces of $C^1$ continuous piecewise polynomials can be constructed with local basis. Moreover, these spaces perform optimal approximations of $H^2$ functions with sufficient smoothness [2,16,36,57,58]. With polynomials of degrees $2 \leq k \leq 4$, spaces of $C^1$ continuous piecewise polynomials can be shown to provide the optimal approximation when the triangulation is of some special structures, such as the Powell-Sabin and Powell-Sabin-Heindl triangulations [26,42,43], the criss-cross triangulations [59], the Hsieh-Clough-Tocher triangulation [10] and the Sander-Veubeke triangulation [18,45]. The conditions on the grids can be relaxed, but they are generally required on at least some part of the triangulation [9,39,40]. On general triangulations, as is shown in [17], the optimal approximation cannot be obtained with $C^1$ continuous piecewise polynomials of degrees $k < 5$. It is illustrated in [1] that not all the basis functions can be determined locally on general grids. We would particularly recall a counterexample that, as studied in [4,14,15], the $C^1-P_3$ scheme is only $O(h)$-order convergent in the energy norm for smooth solutions on a triangulation obtained by subdividing a rectangular domain with three groups of parallel lines (see Figure 1), which is even though one of the simplest and most regular triangulations.

In contrast, a nonconforming finite element methodology, namely, the Morley element [37], which uses piecewise quadratic polynomials with a convergence rate of $O(h)$, was shown to perform optimally for $k = 2$. The element has been generalized to the Morley-Wang-Xu family [48] for any order problem in any dimension. However, to the best of our knowledge, optimal piecewise cubic or quartic finite element schemes (either conforming or nonconforming) for a planar biharmonic equation have not been discovered. We remark that several $O(h^2)$ ordered finite element methods are designed with piecewise cubic polynomials enriched with higher-degree bubbles (see, e.g., [24,30,31,49]), though we are concerned with methods with exactly piecewise cubic polynomials. For a biharmonic problem in higher dimensions and other problems with higher orders, greater difficulties can be expected.

![Figure 1](image-url) On the triangulations of this profile, the optimal finite element scheme for the biharmonic equation with piecewise cubic polynomials, conforming or nonconforming, is not yet known.
1.2 Main results in the present paper

In this paper, a space $B^3_h$ is constructed with piecewise cubic polynomials, whose subspaces $B^3_{h^x}$ and $B^3_{h^y}$ are proved to provide the optimal approximation of $H^2 \cap H^1_0$ and $H^2_0$, respectively. Finite element schemes that apply the two subspaces to the biharmonic equation with Navier and Dirichlet boundary conditions, respectively, are both nonconforming, but the consistency errors are both of $O(h^2)$-order for solutions in $H^4(\Omega)$. The schemes can also resolve sharply the regularity of the exact solutions when they are in $H^s(\Omega)$ with $2 < s \leq 3$ only. The finite element schemes are optimal on any shape regular triangulations and on both convex and nonconvex polygonal domains.

Furthermore, it is proved that $(\nabla_h^2 w_h, \nabla_h^2 v_h) = (\Delta_h w_h, \Delta_h v_h)$ for any two functions $w_h, v_h \in B^3_h$. We call this the discrete strengthened Miranda-Talenti estimate. The Miranda-Talenti estimate relates the $L^2$ norms of the Hessian and the Laplacian of an $H^2$ function on convex domains, which reads (see [22,35])

$$\| \nabla^2 v \|_{0,\Omega} \leq \| \Delta v \|_{0,\Omega}, \quad v \in H^2(\Omega) \cap H^1_0(\Omega), \quad \Omega \text{ being convex.}$$  \hspace{1cm} (1.1)

This estimate can play important roles in applications (see, e.g., [38,47]). For polygonal domains, a strengthened Miranda-Talenti estimate is proved in [23] that

$$\| \nabla^2 v \|_{0,\Omega} = \| \Delta v \|_{0,\Omega}, \quad v \in H^2(\Omega) \cap H^1_0(\Omega),$$  \hspace{1cm} (1.2)

and the domain $\Omega$ is not necessarily convex. This estimate cannot generally be inherited by nonconforming $H^2$ finite elements, and $(\Delta_h, \nabla_h)$ may lead to the degenerate bilinear form on, e.g., the Morley element space even imposed with homogeneous boundary conditions. In this paper, we show a discrete analogue of (1.2) for the space $B^3_h$. This property makes the finite element spaces potentially usable for, e.g., the discretization of the bi-Laplacian operator $\Delta \Delta$ with the varying coefficient $A$. The two approaches of implementing the schemes are suggested. One is to figure out their local basis functions: the finite element scheme does not correspond to a finite element in Ciarlet’s triple; but the other is to decompose the finite element scheme to three decoupled subproblems, which are either a Poisson system or a Stokes system, to solve sequentially. Note that the optimal solvers for a discrete Poisson system and a Stokes problem have been very well developed, and the latter approach suggests a method to implement and to solve the finite element problem in a friendly way.

1.3 Main technical ingredient of the present paper

For the nonconforming finite element space $B^3_h$, to control the consistency error, sufficient restrictions on the interfacial continuity have to be imposed across the edges of the cells. However, the constraints on the continuity are overdetermined in comparison to local shape functions; hence, the global finite element space does not correspond to a local finite element defined with Ciarlet’s triple. The functions can be viewed as nonconforming splines. Consequently, several challenges arise in both theoretical analysis and practical implementation, even on counting the dimension of the space. To overcome these challenges, in this paper, indirect methods are adopted; the construction and utilization of discretized Stokes complexes constitute the bulk of the task in the construction of the space and schemes. This indirect approach is viewed as the main ingredient of the paper.

Discretized Stokes complexes are finite element analogs of the 2D Stokes complexes (or the de Rham complex with the enhanced regularity), which read corresponding to the boundary condition

$$0 \xrightarrow{\text{inclusion}} H^2_0 \xrightarrow{\nabla} (H^1_0)^2 \xrightarrow{\text{rot}} L^2_0 \xrightarrow{\text{f.}} 0$$  \hspace{1cm} (1.3)

and

$$0 \xrightarrow{\text{inclusion}} H^2 \cap H^1_0 \xrightarrow{\nabla} (H^1_0)^2 \cap H_0(\text{rot}) \xrightarrow{\text{rot}} L^2_0 \xrightarrow{\text{f.}} 0.$$  \hspace{1cm} (1.4)

In the complex, the combination of the successive two operators vanishes, and the kernel of the latter one is exactly the range of the former one. The finite element complexes have been widely used (see [3]), and in this paper, the important role they play is four-folded:
(1) It is used for the error analysis. We construct two discretized Stokes complexes that start with finite element spaces $B^3_{h,0}$ and $B^3_{h,t}$, respectively, and connect the error analysis to the discretization error of the auxiliary finite element discretization of the Stokes problem. This connection works for the approximation of $B^3_{h,0}$ ($B^3_{h,t}$) in the energy norm for smooth functions (see Subsections 3.1 and 3.2) and for the entire discretization error with solutions without high smoothness (see Subsection 3.5); see more in Remark 3.12. For both highly and lowly smooth solutions, the error estimation in the energy norm holds on both convex and non-convex domains.

(2) The discrete strengthened Miranda-Talenti estimate is, again, proved by the aid of the discretized Stokes complex.

(3) Furthermore, though the finite element space does not correspond to a finite element defined in Ciarlet’s triple, the finite element spaces do admit a set of basis functions, each of which is supported in a patch of a vertex or a patch of an edge. Once again, the discretized Stokes complexes play crucial roles in proving the existence of the locally supported basis functions.

(4) Finally, we remark, beyond bringing ease in constructing and analyzing the schemes, the discretized Stokes complex is also helpful to the implementation and numerical solution of the systems by the aid of the discretized Poisson and discretized Stokes systems; please also refer to [19,21,27,44,53,54,63] for relevant discussions.

The rest of this paper is organized as follows. Section 2 presents some finite element spaces and finite element complexes. Section 3 presents two optimal nonconforming finite element schemes, including the construction and theoretical analysis for the two kinds of boundary value problems, respectively. Two approaches of implementation are given in Section 4. Finally, in Section 5, some conclusions and further discussions are given.

2 Finite element spaces and finite element complexes

2.1 Preliminaries

In what follows, we use $\Omega$ to define a simply connected polygonal domain, and $\Gamma_j$, $j = 1 : J_S$ for the segments of the boundary $\Gamma = \partial \Omega$. We use $\nabla$, curl, div, rot and $\nabla^2$ to define the gradient operator, curl operator, divergence operator, rot operator, and Hessian operator, respectively. As usual, we use $H^2(\Omega)$, $H^2_0(\Omega)$, $H^1(\Omega)$, $H^1_0(\Omega)$, $H(\text{rot},\Omega)$, $H_0(\text{rot},\Omega)$ and $L^2(\Omega)$ to define certain Sobolev spaces and specifically, define

$$L^2_0(\Omega) := \left\{ w \in L^2(\Omega) : \int_{\Omega} w dx = 0 \right\}, \quad H^1_0(\Omega) := (H^1_0(\Omega))^2$$

and

$$H^1_0(\Omega) = (H^1(\Omega))^2 \cap H_0(\text{rot},\Omega).$$

Furthermore, we denote by “$\cdot$” vector-valued quantities, while $\bar{v}^1$ and $\bar{v}^2$ denote the two components of the function $v$. We use $\langle \cdot, \cdot \rangle$ to represent $L^2$ inner product, and $\langle \cdot, \cdot \rangle$ to denote the duality between a space and its dual. Without ambiguity, we use the same notation $\langle \cdot, \cdot \rangle$ for different dualities, and it can occasionally be treated as $L^2$ inner product for certain functions. We use the subscript “$\cdot_h$” to define the dependence on triangulation. In particular, an operator with the subscript “$\cdot_h$” indicates that the operation is performed cell by cell. Finally, $\Xi^2$ denotes the equality up to a constant. The hidden constants depend on the domain, and when triangulation is involved, they also depend on the shape regularity of the triangulation, but they do not depend on $h$ or any other mesh parameter.

Let $T_h$ be a shape-regular triangular subdivision of $\Omega$ with the mesh size $h$ such that $\bar{\Omega} = \bigcup_{T \in T_h} \bar{T}$, and every boundary vertex is connected to at least one interior vertex. Denote by $E_h$, $E^1_h$, $E^2_h$, $\mathcal{X}_h$, $\mathcal{X}^1_h$, $\mathcal{X}^2_h$, $\mathcal{X}^{e}_{h}$ and $\mathcal{X}^{b}_{h}$ the set of edges, interior edges, boundary edges, vertices, interior vertices, boundary vertices and corners, respectively. For any edge $e \in E_h$, denote by $n_e$ and $t_e$ the unit normal and tangential vectors of $e$, respectively, and denote by $\jump{\cdot}_e$ the jump of a given function across $e$; if particularly $e \in E^b_h$, 


correction
\[ \left[ \right]_e \text{ stands for the evaluation of the function on } e. \text{ The subscript } \cdot_e \text{ can be dropped when there is no ambiguity.} \]

Let \( A_{h}^{b,k} := \{ a \in A_h^b, a \text{ is connected to } A_h^b \text{ by } e \in E_h^i \} \) and \( A_{h}^{i,-1} := A_h^i \setminus A_{h}^{b,k} \); furthermore, with \( A_{h}^{i,-(k-1)} \neq \emptyset \) let \( A_{h}^{i,-(k-1)} := \{ a \in A_h^{i,-(k-1)}, a \text{ is connected to } A_h^{i,-(k-1)} \text{ by } e \in E_h^i \} \) and \( A_{h}^{i,-k} := A_{h}^{i,-(k-1)} \setminus A_{h}^{b,k} \).

The smallest \( k \) such that \( A_{h}^{i,-(k-1)} = A_{h}^{b,k} \) is called the number of levels of the triangulation.

For a triangle \( T \), we use \( P_k(T) \) to define the set of polynomials on \( K \) of degrees not higher than \( k \). In a similar manner, \( P_k(e) \) is defined on an edge \( e \). We define \( P_k(T) = P_k(T)^2 \) and similarly \( P_k(e) \) defined. We use \( a_i, i = 1, 2, 3 \) for the vertices of \( T \) in an anticlockwise order, \( e_i, i = 1, 2, 3 \) for the edges opposite to \( a_i \), respectively, and \( \lambda_i, i = 1, 2, 3 \) the barycentric coordinates.

Also, we denote basic finite element spaces by

- \( L_k^2 := \{ v \in H^1(\Omega) : w|_{T} \in P_k(T), \forall T \in T_h \} \), \( L_k^2 := L_k^2 \cap H_0^1(\Omega) \), \( k \geq 1 \);
- \( P_k^0 := \{ v \in L^2(\Omega) : w|_{T} \in P_k(T) \}, P_k^0 := P_k^0 \cap L_0^2(\Omega), k \geq 0 \);
- \( S_k^h := P_k^0 \cap H^1(\Omega), S_k^h := (S_k^h)^2, S_k := S_k^h \cap \text{rot}(\Omega) \) and \( S_k^h := S_k^h \cap H_0^1(\Omega), k \geq 1 \);
- \( G_k^0 := \{ v \in P_k^0 : \int_{T} p_k v \, d\Omega = 0, \forall p_k \in P_{k-1}(e), \forall e \in E_h^i \}, G_k^0 := \{ v \in G_k^0 : \int_{T} p_k v \, d\Omega = 0, \forall p_k \in P_{k-1}(e), \forall e \in E_h^i \}, G_k^2 : (G_k^2)^2, G_k^2 := \{ v \in G_k^2 : \int_{T} p_k v \, d\Omega = 0, \forall p_k \in P_{k-1}(e) \} \) and \( G_k^2 := (G_k^2)^2, k \geq 1 \).

Namely, \( S_k^h (S_k^h) \) consists of continuous functions, and \( G_k^0 (G_k^0) \) consists of \((k-1)\)-th-order moment-continuous functions. Particularly, the space \( G_k^2 (G_k^2) \) corresponds to the famous Fortin-Soulie element (see [20]). The following stability result is well known.

**Lemma 2.1** ([See [20]]). There exists a generic constant \( C \) depending on the domain and the regularity of the triangulation such that

\[
\sup_{\forall v_h \in \mathcal{G}_h^2 \| \nabla v_h \|_{0,\Omega} = 1} (\text{div} v_h, q_h) \geq C ||q_h||_{0,\Omega} \quad \forall q_h \in P_k^0, \tag{2.1}
\]

**Remark 2.2.** By the symmetry between the two components of \( H^1(\Omega) \), Lemma 2.1 remains true when “\( \text{div} v_h \)” is replaced by “\( \text{rot} v_h \)”.

Let

\[
B_{h}^{2} := \{ \phi_h : \phi_h \big|_{T} \in \text{span}\{ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2/3, \forall T \in T_h \} \}
\]

and evidently the first-order moments of \( \phi_h \) vanish along any edge of \( T_h \). We call it the moment bubble. Let \( B_{h}^{2} = (B_{h}^{2})^2 \). Then \( G_2^{0} = S_2^{h} \oplus B_2^{h} \); however, \( G_2^{0} = S_2^{h} + B_2^{h} \) is not a direct sum (see [20]). The decomposition can be generalized to even \( k \) (see [5]). We can indeed figure out the structure of a bigger space.

**Lemma 2.3.** It holds that \( \mathcal{G}_h^{2} = S_2^{h} \oplus B_2^{h} \).

**Proof.** It is evident that \( S_2^{h} \oplus B_2^{h} \) is direct, and \( \mathcal{G}_h^{2} \subset S_2^{h} \oplus B_2^{h} \). We turn to the other direction.

Given \( w_h \in \mathcal{G}_h^{2} \), for any corner \( c \) of \( \Omega \) and any cell \( T_1 \) with \( c \) being its vertex (see Figure 2), there is a unique \( b_{T_1} \), a vector moment bubble on \( T_1 \) such that \( (w_h|_{T_1} - b_{T_1})(c) = 0 \). Now, set

\[
w_h^1 = w_h - \sum_{T : \text{connected to corner}} b_T.
\]
where we can easily divide for any $p$ for Lemma 2.4. Evidently, $\tilde{h} \in B_2$, such that $\tilde{h} \in B_2$. Finally, we obtain a $w^2_h \in G_{ht}$ such that $w_h - w^2_h \in B^2_{ht}$, $w^2_h(c) = 0$ on any corner $c$, and $w^2_h \cdot t_e = 0$ on any boundary edge $e$.

Note that $w^2_h(c) = 0$ at any corner $c$, and thus we can divide $w^2_h$ in the space $G_{ht}$ as

$$w_h = \sum_{j=1}^{J_S} w^{2,j}_h \cdot n_j + w^{2,0}_h,$$

where $w^{2,j}_h = 0$ on $\Gamma \setminus \Gamma_j$ and $n_j$ is the normal vector of $\Gamma_j$ for $1 \leq j \leq J_S$, and $w^{2,0}_h \in G^2_{ht}$. Furthermore, we can easily divide $w^{2,j}_h = w^{3,j}_h + b^j_h$, where $w^{3,j}_h \in S^2_h$ and $w^{3,j}_h |_{\Gamma \setminus \Gamma_j} = 0$, and $b^j_h \in B^2_{ht}$, $1 \leq j \leq J_S$.

Summing all the above, we obtain

$$w_h = \sum_{j=1}^{J_S} w^{3,j}_h \cdot n_j + w^{2,0}_h + \left[ \sum_{j=1}^{J_S} b^j_h \cdot n_j + w_h - w^2_h \right] =: w^3_h + w^{2,0}_h + \text{Res.}$$

As $w^3_h \in S^2_{ht}$, $w^{2,0}_h \in S^2_{ht} + B^2_{ht}$ and $\text{Res} \in B^2_{ht}$, this shows $G^2_{ht} \subset S^2_{ht} + B^2_{ht}$. The proof is completed. 

**Lemma 2.4.** For any $w_h, \psi_h \in G^2_{ht}$, it holds that

$$(\nabla_h w_h, \nabla_h \psi_h) = (\text{div}_h w_h, \text{div}_h \psi_h) + (\text{rot}_h w_h, \text{rot}_h \psi_h).$$

**Proof.** Firstly, (2.2) holds for any $w_h, \psi_h \in S^2_{ht} \subset H^1_0(\Omega)$. Secondly, (2.2) holds for any $w_h \in G^2_{ht}$ and $\psi_h \in B_{ht}$; actually, for any $K \in T_h$,

$$\int_K \nabla w_h : \nabla \psi_h = -\int_K \Delta w_h \cdot \psi_h + \int_{\partial K} \delta_n \cdot w_h \psi_h$$
\[- \int_K \Delta w_h \cdot \psi_h = - \int_K (\nabla \text{div} + \text{curlrot}) w_h \cdot \psi_h \]
\[- \int_K (\nabla \text{div} + \text{curlrot}) w_h \cdot \psi_h = \int_K (\text{div} \psi_h \cdot w_h + \int_{\partial K} (\text{div} \psi_h) \cdot n + \text{rot} \psi_h \cdot t) \]
\[= \int_K \text{div} w_h \text{div} \psi_h + \int_K \text{rot} w_h \text{rot} \psi_h; \]

here, we have used the fact that \( \partial_n \psi_h \), \( \text{div} \psi_h \) and \( \text{rot} \psi_h \) are all the linear polynomials along the edges of \( K \) and that the first-order moments of \( \psi_h \) vanish along the edges of \( K \).

Thus we only have to check if \( \dim(\nabla \psi_h) \cdot \psi_h \) is established for every pair of the parts, and the proof is completed.

Remark 2.5. It is known that (2.2) holds for \( H^1 \) functions but in general not for nonconforming finite element functions (such as the Crouzeix-Raviart element functions). This lemma reveals that the nonconforming space \( G^2_{h \ell} \) is in some sense like a conforming one.

2.2 An auxiliary finite element Stokes complex

Given a grid \( T_h \), define
- \( A^3_{h 0} := \{ w_h \in L^2(\Omega) : w_h|_T \in P_3(T); w_h(a) \text{ is continuous at } a \in \mathcal{X}_h \} \);
- \( A^3_{h 0} := \{ w_h \in A^3_{h 0} : w_h(a) = 0 \text{ at } a \in \mathcal{X}_h \} \);
- \( G^2_{h 0} := \{ \psi \in (P^2_h)^2 : \int_T \psi \cdot e = 0, \forall e \in \mathcal{E}_h \} \); and
- \( G^2_{h 0} := \{ \psi \in G^2_{h 0} : \int_e \psi \cdot e = 0, \forall e \in \mathcal{E}_h \} \).

Lemma 2.6. A finite element exact complex is given by
\[
\begin{align*}
0 & \xrightarrow{\text{inclusion}} A^3_{h 0} \xrightarrow{\nabla} G^2_{h 0} \xrightarrow{\text{rot}} P^1_{h 0} \xrightarrow{\int} 0.
\end{align*}
\] (2.3)

Proof. We adopt the standard counting technique.

Firstly, by Lemma 2.1,
\[
P^1_{h 0} = \text{rot}_h G^2_{h 0} \subset \text{rot}_h G^2_{h 0} \subset P^1_{h 0}.
\]

Secondly,
\[
\nabla_h A^3_{h 0} \subset \{ \psi_h \in G^2_{h 0} : \text{rot}_h \psi_h = 0 \}.
\]

Thus we only have to check if \( \dim(\nabla_h A^2_{h 0}) + \dim(P^1_{h 0}) = \dim(G^2_{h 0}) \). Observing that \( \dim(A^3_{h 0}) = \#(\mathcal{X}_h) + 7\#(\mathcal{T}_h) \), \( \dim(G^2_{h 0}) = \#(\mathcal{E}_h) + 9\#(\mathcal{T}_h) \) and \( \dim(P^1_{h 0}) = 3\#(\mathcal{T}_h) - 1 \), we obtain the identity by the Euler formula. The proof is completed. \( \square \)
2.3 Finite element spaces for $H^2$ and discretized Stokes complexes

Define

\[ B_h^3 := \{ v \in \mathbb{P}_h^3 : v \text{ is continuous at } a \in X_h; \int_e [v] = 0 \text{ and } \int_e p_e [\partial_n v] = 0, \forall p_e \in P_1(e), \forall e \in E_h \} , \]

\[ B_{h0}^3 := \{ v \in B_h^3 : v(a) = 0, a \in X_h^b; \int_e v = 0, \forall v \in E_h^b \} , \]

and

\[ B_{h0}^3 := \{ v \in B_h^3 : \int_e p_e [\partial_n v] = 0, \forall p_e \in P_1(e), \forall e \in E_h^b \} . \]

According to the boundary conditions on $B_{h0}^3$, we can recognize them as for $H^2$ problems.

**Remark 2.7.** Note that given $v_h \in B_h^3$, on every cell $T$, $v_h |_{T} \in P_3(T)$ but is related to 12 restrictions. We cannot expect $B_h^3$ to correspond to a finite element defined with Ciarlet’s triple.

**Lemma 2.8.** It holds that $B_{h0}^3 = \{ w_h \in A_{h0}^3 : \nabla_h w_h \in G_{h0}^2 \}$ and $B_{h0}^3 = \{ w_h \in A_{h0}^3 : \nabla_h w_h \in G_{h0}^2 \}$.

**Proof.** By an elementary calculus, the continuity restriction of $B_h^3$ implies that $\int_e p_e [\partial_t v_h] = 0$ for any $p_e \in P_1(e)$, any $e \in E_h^b$ and any $v_h \in B_h^3$. Also, $\int_e p_e [\partial_t v_h] = 0$ for any $p_e \in P_1(e)$ and any $e \in E_h^b$.

Then by the definitions of $B_{h0}^3$ and $A_{h0}^3$,

\[ B_{h0}^3 \subset \{ w_h \in A_{h0}^3 : \nabla_h w_h \in G_{h0}^2 \} . \]

On the other hand, given $w_h \in A_{h0}^3$ such that $\nabla_h w_h \in G_{h0}^2$, it holds that $\int_e [\partial_n w_h] p_e = \int_e [\partial_t w_h] p_e = 0$ for any $e \in E_h$ and $p_e \in P_1(e)$. This implies $w_h \in B_{h0}^3$. Namely, $B_{h0}^3 = \{ w_h \in A_{h0}^3 : \nabla_h w_h \in G_{h0}^2 \}$.

Similarly, $B_{h0}^3 = \{ w_h \in A_{h0}^3 : \nabla_h w_h \in G_{h0}^2 \}$ can be proved, and all the proof is completed.

**Theorem 2.9** (Discrete strengthened Miranda-Talenti estimate). It holds for $w_h, v_h \in B_{h0}^3$ that

\[ (\nabla_h w_h, \nabla_h v_h) = (\Delta_h w_h, \Delta_h v_h). \] (2.4)

**Proof.** By Lemma 2.4, as $\nabla_h B_{h0}^3 \subset G_{h0}^2$,

\[ (\nabla_h w_h, \nabla_h v_h) = (\text{div} \nabla_h w_h, \text{div} \nabla_h v_h) + (\text{rot} \nabla_h w_h, \text{rot} \nabla_h v_h) = (\Delta_h w_h, \Delta_h v_h), \quad \forall w_h, v_h \in B_{h0}^3 . \]

The proof is completed. \( \square \)

**Remark 2.10.** The lemma reveals that the functions in $B_{h0}^3$ play closely to the $H^2$ conforming functions.

**Theorem 2.11.** Two discretized Stokes complexes are given by

\[ 0 \xrightarrow{\text{inclusion}} B_{h0}^3 \xrightarrow{\nabla_h} G_{h0}^2 \xrightarrow{\text{rot}_h} \frac{\mathbb{P}_1}{h_0} \xrightarrow{\int} 0 \]

and

\[ 0 \xrightarrow{\text{inclusion}} B_{h0}^3 \xrightarrow{\nabla_h} G_{h0}^2 \xrightarrow{\text{rot}_h} \frac{\mathbb{P}_1}{h_0} \xrightarrow{\int} 0 . \] (2.6)

**Proof.** By Lemmas 2.1, given $p_h \in \mathbb{P}_1$, there exists $\sigma_h \in G_{h0}^2$ such that $\text{rot}_h \sigma_h = p_h$. Furthermore, given $\tau_h \in G_{h0}^2$ such that $\text{rot}_h \tau_h = 0$, by Lemma 2.6, there exists $w_h \in A_{h0}^3$ such that $\tau_h = \nabla_h w_h$. Furthermore, by Lemma 2.8, $w_h \in B_{h0}^3$. Therefore, (2.5) is proved. Similarly, (2.6) can be proved. \( \square \)

**Remark 2.12.** A key feature for the proof of Theorem 2.11 is to construct a bigger finite element complex to cover (see, e.g., (2.5)); this is accomplished by Lemma 2.6 where a finite element complex is constructed, where the same piecewise polynomial space with the lower regularity is used corresponding to (2.5). A dual way can be to use a bigger piecewise polynomial space with the same regularity. A different proof of (2.5) can be found along this line in [60].
3 Optimal nonconforming finite element schemes for biharmonic equations

We consider the biharmonic equation with \( f \in L^2(\Omega) \):

\[
\text{Dirichlet type } \begin{cases}
  \Delta^2 u = f & \text{in } \Omega, \\
  u = \partial_n u = 0 & \text{on } \partial\Omega
\end{cases}
\quad (3.1)
\]

and

\[
\text{Navier type } \begin{cases}
  \Delta^2 z = f & \text{in } \Omega, \\
  z = \Delta z = 0 & \text{on } \partial\Omega.
\end{cases}
\quad (3.2)
\]

The variational problems are, respectively,

- to find \( u \in H^2_0(\Omega) \) such that
  \[
  (\nabla^2 u, \nabla^2 v) = (f, v), \quad \forall v \in H^2_0(\Omega),
  \]

- to find \( z \in H^2(\Omega) \cap H^1_0(\Omega) \) such that
  \[
  (\nabla^2 z, \nabla^2 v) = (f, v), \quad \forall v \in H^2 \cap H^1_0(\Omega).
  \]

In this section, we consider the nonconforming finite element discretization for them:

- find \( u_h \in B^3_{h0} \) such that
  \[
  a_h(u_h, v_h) := (\nabla^2_h u_h, \nabla^2_h v_h) = (f, v_h), \quad \forall v_h \in B^3_{h0};
  \]

- find \( z_h \in B^3_{ht} \) such that
  \[
  a_h(z_h, v_h) = (f, v_h), \quad \forall v_h \in B^3_{ht}.
  \]

By the weak continuity of \( B^3_{ht} \), \( |\cdot|_{2,h} \) (namely, \( \|\nabla^2_h \cdot\|_{0,\Omega} \) is a norm on \( B^3_{ht} \), and (3.5) and (3.6) are well posed.

The main result of this section is contained in the theorem below.

**Theorem 3.1.** Let \( u, u_h, z \) and \( z_h \) be solutions of (3.3), (3.5), (3.4) and (3.6), respectively. Then with a generic constant \( C \) depending on \( \Omega \) and the regularity of the grid only, it holds for \( u, z \in H^m(\Omega), m = 3, 4 \) that

\[
\|\nabla^2_h (u - u_h)\|_{0,\Omega} \leq C(h^{m-2}|u|_{m,\Omega} + h^2\|f\|_{0,\Omega})
\]

and

\[
\|\nabla^2_h (z - z_h)\|_{0,\Omega} \leq C(h^{m-2}|z|_{m,\Omega} + h^2\|f\|_{0,\Omega}).
\]

Moreover, when \( \Omega \) is convex,

\[
\|\nabla_h (u - u_h)\|_{0,\Omega} \leq C(h^{m-1}|u|_{m,\Omega} + h^3\|f\|_{0,\Omega})
\]

and

\[
\|\nabla_h (z - z_h)\|_{0,\Omega} \leq C(h^{m-2+\delta}|z|_{m,\Omega} + h^3\|f\|_{0,\Omega}), \quad 1/2 < \delta \leq 1.
\]

When \( \Omega \) is specifically a rectangle, \( \delta = 1 \).

We postpone the proof of Theorem 3.1 after some technical lemmas.

### 3.1 Approximation property of \( B^3_{h0} \) for smooth functions

First of all, we define an interpolator to \( B^3_{h0} \). Given \( w \in H^3(\Omega) \cap H^2_0(\Omega) \), set \( \varphi := \nabla w \), and then

\[
\varphi \in H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad \text{rot } \varphi = 0.
\]

Indeed, \((\varphi, p \equiv 0)\) solves the incompressible Stokes equation...
Then, by Theorem 2.11, there exists a unique interpolation operator \( I_{h0}^B : H^3(\Omega) \cap H^3_0(\Omega) \rightarrow B_{h0}^3 \) by
\[
I_{h0}^B w := w_h.
\]

**Lemma 3.2.** There exists a constant \( C \) such that for any \( w \in H^3_0(\Omega) \cap H^m(\Omega), m = 3, 4, \) it holds for \( k = 2 \) that
\[
|w - I_{h0}^B w|_{k,h}^2 \leq C \sum_{T \in T_h} h_T^{m-2k} |w|_{m,T}^2.
\] (3.14)

If \( \Omega \) is convex, then (3.14) holds for \( k = 1, 2 \).

**Proof.** Set \( \varphi = \nabla w \) and \( \varphi_h = \nabla h I_{h0}^B w \). Then \( \varphi \) and \( \varphi_h \) are, respectively the solutions (velocity component) of the problems (3.11) and (3.12). By the standard theory for the Stokes problem (see [7, 20]),
\[
|\varphi - \varphi_h|_{k-1,h}^2 \leq C \sum_{T \in T_h} h_T^{2m-2k} |\varphi_t|_{m-1,T}^2
\]
with \( k = 2 \) on general polygonal domains and \( k = 1, 2 \) on convex domains, and \( m = 3, 4 \). Therefore,
\[
|w - I_{h0}^B w|_{k,h}^2 = |\varphi - \varphi_h|_{k-1,h}^2 \leq C \sum_{T \in T_h} h_T^{2m-2k} |\varphi_t|_{m-1,T}^2 = C \sum_{T \in T_h} h_T^{2m-2k} |w|_{m,T}^2.
\]
The proof is completed. \( \square \)

### 3.2 Approximation of \( B_{ht}^3 \) for smooth functions

Again, we firstly define an interpolator to \( B_{ht}^3 \). Given \( w \in H^3(\Omega) \cap H^3_0(\Omega) \) such that \( \Delta w|_\Gamma = 0 \), set \( \varphi := \nabla w \). Then \( \varphi \in H^3(\Omega) \cap H_0(\text{rot}, \Omega), \text{rot} \varphi = 0 \) and \( (\text{div} \varphi)|_\Gamma = 0 \). Indeed, \((\varphi, p \equiv 0)\) solves the incompressible Stokes equation
\[
\begin{aligned}
(\nabla \varphi, \nabla \psi) + (\text{rot} \varphi, p) &= (-\Delta \varphi, \psi), \quad \forall \psi \in H^1(\Omega) \cap H_0(\text{rot}, \Omega), \\
(\text{rot} \varphi, q) &= 0, \quad \forall q \in L^2_0(\Omega).
\end{aligned}
\] (3.15)

Here, we have used the boundary condition \( \Delta w|_\Gamma = 0 \). Now, choose \((\varphi_h, p_h) \in G_{ht}^2 \times P_{ht}^1\) such that
\[
\begin{aligned}
(\nabla h \varphi_h, \nabla h \psi h) + (\text{rot} h \varphi_h, p_h) &= (-\Delta \varphi, \psi h), \quad \forall \psi_h \in G_{ht}^2, \\
(\text{rot} h \varphi_h, q_h) &= 0, \quad \forall q_h \in P_{ht}^1.
\end{aligned}
\] (3.16)

Then by Theorem 2.11, there exists a unique \( w_h \in B_{ht}^3 \) such that \( \nabla h w_h = \varphi_h \). By this way, we define an interpolation operator \( I_{ht}^B \) to \( B_{ht}^3 \) for \( w \in H^3(\Omega) \cap H^3_0(\Omega) \) and \( \Delta w|_\Gamma = 0 \) by
\[
I_{ht}^B w := w_h.
\] (3.17)
There exists a constant $C$ such that for any $w \in H_0^1(\Omega) \cap H^m(\Omega)$ such that $\Delta w |_{\Gamma} = 0$, $m = 3, 4$, it holds that
\[
|w - \Pi_h^2 w|^2_{\Omega} \leq C \sum_{T \in T_h} h_T^2 |w|^2_{m,T}. \tag{3.18}
\]
If $\Omega$ is convex, then
\[
|w - \Pi_h^2 w|^2_{\Omega} \leq C \sum_{T \in T_h} h_T^{2m-4+\kappa} |w|^2_{m,T} \quad \text{with } 1 < \kappa \leq 2. \tag{3.19}
\]
If specifically $\Omega$ is rectangle, $\kappa = 2$.

**Proof.** By definition, the interpolation error of $\Pi_h^2 w$ is the discretization error of (3.16), and (3.18) and (3.19) can be obtained by standard technique (with $\Omega$ either convex or nonconvex). We only have to note that the regularity of the auxiliary Stokes problem on the convex domains can be affected under the boundary condition of this kind. Specifically, please refer to [6] for the full regularity of (3.2) and thus of the auxiliary Stokes problem (3.15) on rectangles. This explains why $\kappa$ can be 2 on rectangular domains.

### 3.3 Convergence analysis of the nonconforming schemes

For suitable $\varphi$ and $\psi$, define the bilinear forms
\[
R_h^1(\varphi, \psi) := (\nabla^2 \varphi, \nabla^2 \psi) + (\nabla \Delta \varphi, \nabla \psi), \tag{3.20}
\]
\[
R_h^2(\varphi, \psi) := (\nabla \Delta \varphi, \nabla \psi) + (\Delta^2 \varphi, \psi) \tag{3.21}
\]
and
\[
R_h(\varphi, \psi) := R_h^1(\varphi, \psi) - R_h^2(\varphi, \psi). \tag{3.22}
\]

**Lemma 3.4.** There exists a constant $C$ such that it holds for any $\varphi \in H_0^2(\Omega) \cap H^k(\Omega)$ with $\Delta^2 \varphi \in L^2(\Omega), w_h \in B_{h0}^1 + H_0^2(\Omega)$, and $k = 3, 4$ that
\[
R_h^1(\varphi, w_h) \leq C h^{k-2} |\varphi|_{k,\Omega} \|\nabla^2 w_h\|_{0,\Omega}, \tag{3.23}
\]
\[
R_h^2(\varphi, w_h) \leq C (h^{k-2} |\varphi|_{k,\Omega} + h^2 \|\Delta^2 \varphi\|_{0,\Omega}) \|\nabla^2 w_h\|_{0,\Omega}. \tag{3.24}
\]

**Proof.** Given $e \in \mathcal{E}_h$, by the definition of $B_{h0}^1$, $\int_e p_e [\partial_{\text{r}} w_h]_e = 0, p_e \in P_1(e)$; for the tangential direction, $\int_e p_e [\partial_{\text{t}} w_h]_e = (p_e(L_e)[w_h]_e(L_e) - p_e(R_e)[w_h]_e(R_e)) - \int_e \partial_{\text{t}} p_e [w_h]_e = 0$. Hence,
\[
\int_e p_e [\nabla w_h]_e = 0, \quad \forall p_e \in P_1(e), \quad e \in \mathcal{E}_h. \tag{3.25}
\]
Therefore, (3.23) follows by standard techniques.

Now, define $\Pi_h^2$ the nodal interpolation to $L^2_{h0}$ such that
\[
(\Pi_h^2 w)(a) = w(a), \quad \forall a \in \mathcal{X}_h \quad \text{and} \quad \int_e (\Pi_h^2 w) = \int_e w, \quad \forall e \in \mathcal{E}_h. \tag{3.26}
\]
It is easy to verify that the operator is well defined. Moreover,
\[
\int_T e \cdot \nabla (w - \Pi_h^2 w) = 0, \quad \forall e \in \mathbb{R}^2 \quad \text{and} \quad T \in T_h, \quad \text{provided } w \in H_0^2(\Omega) + B^3_{h0}. \tag{3.27}
\]
By Green’s formula,
\[
(\Delta^2 \varphi, \Pi_h^2 w) = - (\nabla \Delta \varphi, \nabla \Pi_h^2 w_h). \tag{3.28}
\]
Therefore,
\[
R_h^2(\varphi, w_h) = (\nabla \Delta \varphi, \nabla (w_h - \Pi_h^2 w_h)) + (\Delta^2 \varphi, w_h - \Pi_h^2 w_h) =: I_1 + I_2. \tag{3.29}
\]
By (3.27),
\[ I_1 = \inf_{\varphi \in (P_h^{1})^2} (\nabla \Delta \varphi - e_0, \nabla_h (\Pi_h^2 w_h - w_h)) \leq C h^{k-2} \| \varphi \|_{k, \Omega} \| \nabla_h^2 w_h \|_{0, \Omega}. \]

Furthermore,
\[ I_2 \leq C h^2 \| \Delta^2 u \|_{0, \Omega} \| \nabla_h^2 w_h \|_{0, \Omega}. \]

Summing all the above proves (3.24) and completes the proof.

Similarly, we have the lemma below.

**Lemma 3.5.** There exists a constant $C$ such that it holds for any $\varphi \in H^1_0(\Omega) \cap H^k(\Omega)$ with $\Delta^2 \varphi \in L^2(\Omega)$ and $(\Delta \varphi)|_{\Gamma} = 0$, $w_h \in B^{k+2}_h \cap H^k(\Omega)$ and $k = 3, 4$ that
\[ R^1_h(\varphi, w_h) \leq C h^{k-2} \| \varphi \|_{k, \Omega} \| \nabla_h^2 w_h \|_{0, \Omega}, \tag{3.29} \]
\[ R^2_h(\varphi, w_h) \leq C (h^{k-2} \| \varphi \|_{k, \Omega} + h^2 \| \Delta^2 \varphi \|_{0, \Omega}) \| \nabla_h^2 w_h \|_{0, \Omega}. \tag{3.30} \]

**Proof of Theorem 3.1.** The proof follows a similar approach to the one in [46] with some technical modifications. By the Strang lemma,
\[ \| \nabla^2_h (u - u_h) \|_{0, \Omega} \leq \inf_{v_h \in B_{\alpha, h}} \| \nabla^2_h (u - v_h) \|_{0, \Omega} + \sup_{v_h \in B_{\alpha, h}(\Omega)} \frac{\langle \nabla^2 u, \nabla^2_h v_h \rangle - \langle f, v_h \rangle}{\| \nabla^2_h v_h \|_{0, \Omega}}. \]

The approximation error estimate follows by Lemma 3.2. By Lemma 3.4,
\[ \| \nabla_h^2 (u - u_h) \|_{0, \Omega} \leq C (h^{m-1} \| u \|_{m, \Omega} + h^3 \| f \|_{0, \Omega}) \| \nabla_h^2 v_h \|_{0, \Omega}, \]
which completes the proof of (3.7).

Now, we turn our attention to the proof of (3.9) for convex $\Omega$. Define $u^\Pi_h = \Pi_{0, h}^B u$. Then by Lemma 3.2,
\[ \| \nabla_h^2 (u - u^\Pi_h) \|_{0, \Omega} \leq C h^{k-2} \| u \|_{k, \Omega}, \quad j = 1, 2. \]
Denote by $\Pi_h^1$ the nodal interpolation onto $L_{1, h}^1$. Then $\Pi_h^1(u^\Pi_h - u_h) \in H^1_0(\Omega)$. Set $\varphi \in H^1(\Omega) \cap H^2(\Omega)$ such that
\[ \langle \nabla^2 \varphi, \nabla^2 v \rangle = \langle \nabla \Pi^1_h(u^\Pi_h - u_h), \nabla v \rangle, \quad \forall v \in H^2_0(\Omega). \]

Then when $\Omega$ is convex, $\| \varphi \|_{3, \Omega} \leq \| \Pi^1_h(u^\Pi_h - u_h) \|_{1, \Omega}$. By Green’s formula,
\[ \| \nabla \Pi^1_h(u^\Pi_h - u_h) \|_{2, \Omega} = \| \nabla \Delta \varphi, \nabla \Pi^1_h(u^\Pi_h - u_h) \|_{2, \Omega} = \| \nabla \Delta \varphi, \nabla \Pi^1_h(u^\Pi_h - u_h) \|_{2, \Omega} = \| \nabla \Delta \varphi \cdot \nabla (Id - \Pi^1_h)(u^\Pi_h - u_h) - (\nabla \Delta \varphi \cdot \nabla (u^\Pi_h - u_h)) \|_{2, \Omega} = I_1 + I_2 + I_3. \]

Furthermore, set $\varphi^\Pi_h = \Pi_{0, h}^B \varphi$ and
\[ I_3 = \| \nabla^2 \varphi, \nabla^2_h (u - u_h) + R^1_h(\varphi, u - u_h) \|_{2, \Omega} \]
\[ = \langle \nabla^2 \varphi, \nabla^2_h (u - u_h) + R^1_h(\varphi, u - u_h) \|_{2, \Omega} = \langle \nabla^2 \varphi, \nabla^2_h (u - u_h) + R^1_h(\varphi, u - u_h) \|_{2, \Omega} = \| \nabla \Delta \varphi \cdot \nabla (u^\Pi_h - u_h) \|_{2, \Omega} \]
\[ \| \nabla \Pi^1_h(u^\Pi_h - u_h) \|_{2, \Omega} \leq C \| \varphi \|_{3, \Omega} (h^{m-1} \| u \|_{m, \Omega} + h^3 \| \Delta^2 u \|_{0, \Omega}) \]
and
\[ \| \nabla \Pi^1_h(u^\Pi_h - u_h) \|_{2, \Omega} \leq C (h^{m-1} \| u \|_{m, \Omega} + h^3 \| \Delta^2 u \|_{0, \Omega}). \]

Finally,
\[ \| \nabla_h (u - u_h) \|_{0, \Omega} \leq \| \nabla_h (u - u^\Pi_h) \|_{0, \Omega} + \| \nabla_h (u^\Pi_h - u_h) \|_{0, \Omega} \]
\[ \leq \| \nabla_h (u - u^\Pi_h) \|_{0, \Omega} + \| \nabla_h [(u^\Pi_h - u_h) - \Pi^1_h(u^\Pi_h - u_h)] \|_{0, \Omega} + \| \nabla \Pi^1_h(u^\Pi_h - u_h) \|_{0, \Omega} \]
\[ \leq C (h^{m-1} \| u \|_{m, \Omega} + h^3 \| \Delta^2 u \|_{0, \Omega}). \]

The proofs of (3.8) and (3.10) are basically the same. The convergence rate for the $H^1$ norm of the error is slightly lost due to the lost of the regularity of the model problem (3.1) on general convex polygons.
3.4 A variant formulation for the bi-Laplacian equation with the varying coefficient

The bi-Laplacian equation $\Delta(\Delta u) = f$, where $A$ is a smooth non-constant coefficient with positive lower and upper bounds, is frequently dealt with in applications. The equation arises in, e.g., the Helmholtz transmission eigenvalue problem in acoustics (see, e.g., [11, 32, 50]). The variational problem is to find $u \in H^3_0(\Omega)$ such that

$$ (A \Delta u, \Delta v) = (f, v), \quad \forall \ v \in H^3_0(\Omega). \quad (3.31) $$

Correspondingly, we consider the nonconforming finite element discretization: find $u_h \in B^3_{h_0}$ such that

$$ \bar{a}_h(u_h, v_h) := (A \Delta_h u_h, \Delta_h v_h) = (f, v_h), \quad \forall \ v_h \in B^3_{h_0}. \quad (3.32) $$

Lemma 3.6. The finite element problem (3.32) admits a unique solution.

Proof. By Theorem 2.9, the bilinear form $\bar{a}_h(\cdot, \cdot)$ is coercive on $B^3_{h_0}$ with respect to the norm $\cdot_{2,h}$. The well-posedness of (3.32) follows by the Lax-Milgrem lemma. The proof is completed. \hfill \Box

Theorem 3.7. Let $u$ and $u_h$ be solutions of (3.31) and (3.32), respectively. Then with a generic constant $C$ depending on $A$, $\Omega$ and the regularity of the grid only, it holds for $u \in H^m(\Omega)$, $m = 3, 4$ that

$$ \|\nabla^2_h (u - u_h)\|_{0,\Omega} \leq C(h^{m-2}|u|_{m,\Omega} + h^2\|f\|_{0,\Omega}). \quad (3.33) $$

Moreover, when $\Omega$ is convex,

$$ \|\nabla_h (u - u_h)\|_{0,\Omega} \leq C(h^{m-1}|u|_{m,\Omega} + h^3\|f\|_{0,\Omega}). \quad (3.34) $$

Remark 3.8. For the bi-Laplacian equation with the non-constant coefficient $A$, the finite element scheme of the formulation (3.32) is a natural alternative. When the formulation (3.32) is used on, e.g., the Morley element, the scheme is not well posed without extra stabilisations. Higher regularity of $B^3_{h_0}$ here makes it fit for the formulation (3.32).

Remark 3.9. Similarly, by Theorem 2.9, a bilinear form induced by $(A \Delta_h \cdot, \cdot)$ can be used for $\Delta(A \Delta u) = f$ with the Navier type boundary condition.

3.5 Discussion on the case where the solution does not possess the full regularity

In this subsection, we study the convergence analysis of the scheme when the exact solution does not possess the full regularity, i.e., when $u \in H^s(\Omega)$ for some $s < 3$ only. Similar to the discussion in [33], the a priori error estimate for solutions with the low regularity can be useful for the definition of nonlinear approximation classes and for the analysis of the quasi-optimality of adaptive finite element methods for the biharmonic equation. Again, we begin with an estimate for an auxiliary Stokes problem.

Lemma 3.10 (See [33, Theorem 3.1, Remark 3.4 and Theorem 3.6]). Given $g \in L^2(\Omega)$, let $(u, p) \in H^1_0(\Omega) \times L^2_0(\Omega)$ be such that

$$ \begin{cases} 
(\nabla u, \nabla v) + (p, \text{rot} v) = (g, v), & \forall \ v \in H^1_0(\Omega), \\
(\text{rot} u, q) = 0, & \forall \ q \in L^2_0(\Omega)
\end{cases} \quad (3.35) $$

and $(u_h, p_h) \in C^2_{h_0} \times P^1_{h_0}$ be such that

$$ \begin{cases} 
(\nabla_h u_h, \nabla_h v_h) + (p_h, \text{rot} h v_h) = (g, v_h), & \forall \ v_h \in C^2_{h_0}, \\
(\text{rot} h u_h, q_h) = 0, & \forall \ q_h \in P^1_{h_0}
\end{cases} \quad (3.36) $$
Suppose furthermore \((u, p) \in H^{1+s}(\Omega) \times H^s(\Omega)\) with \(0 < s \leq 1\). Then

\[
\|u - u_h\|_{1,h} + \|p - p_h\|_{0,\Omega} \leq Ch^s(\|u\|_{1+s,\Omega} + \|p\|_{s,\Omega} + \left( \sum_{T \in \mathcal{T}_h} h_T^2 \inf_{\psi \in P_1(T)} \|g - \psi\|_{0,T}^2 \right)^{1/2}).
\]

**Theorem 3.11.** Given \(f \in L^2(\Omega)\), let \(u\) and \(u_h\) be the solutions of \((3.3)\) and \((3.5)\), respectively. If \(u \in H^{2+s}(\Omega)\) for some \(0 < s < 1\), then

\[
\|\nabla_h^2 (u - u_h)\|_{0,\Omega} \leq Ch^s(\|u\|_{2+s,\Omega} + \|f\|_{0,\Omega}).
\]

**Proof.** Following \((3.26)\), let \(\Pi_h^2\) be the nodal interpolation to \(L^2_{\Omega_0}\), and let \(\Pi_h^{RT1}\) be the nodal interpolation to the first degree (incomplete quadratic) rotated Raviart-Thomas element space for \(H_0(\text{rot}, \Omega)\). Then the commutating diagram \(\nabla \Pi_h^2 w = \Pi_h^{RT1} \nabla w\) holds for reasonable \(w\) (see \([7]\)). Define \(r \in H_0^1(\Omega)\) such that \((\nabla r, \nabla s) = (f, s), \forall s \in H_0^1(\Omega)\). Let \((\varphi, p) \in H_0^1(\Omega) \times L^2_0(\Omega)\) be such that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\nabla \varphi, \nabla \psi) + (p, \text{rot} \psi) = (\nabla r, \psi), \quad \forall \psi \in H_0^1(\Omega),
(q, \text{rot} \varphi) = 0, \quad \forall q \in L^2_0(\Omega).
\end{array} \right.
\end{aligned}
\]

Then \(\varphi = \nabla u\); this fact is well known, and can be found in, e.g., \([19, \text{Theorem 2.1}]\). Furthermore, \(\|\varphi\|_{1+s,\Omega} \leq \|u\|_{2+s,\Omega}\) and \(\|p\|_{s,\Omega} \leq C(\|u\|_{2+s,\Omega} + \|f\|_{0,\Omega})\).

Now, let \(u_h^{(1)} \in B_{h0}^0\) be such that

\[
(\nabla_h^2 u_h^{(1)}, \nabla_h^2 v_h) = (f, \Pi_h^2 v_h), \quad \forall v_h \in B_{h0}^3,
\]

let \((\varphi_h^{(1)}, p_h^{(1)}) \in C_{h0}^2 \times P_{h0}^1\) be such that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\nabla_h \varphi_h^{(1)}, \nabla_h \psi_h) + (p_h^{(1)}, \text{rot}_h \psi_h) = (\nabla r_h, \Pi_h^{RT1} \psi_h), \quad \forall \psi_h \in C_{h0}^2,
(q_h, \text{rot}_h \varphi_h^{(1)}) = 0, \quad \forall q_h \in P_{h0}^1.
\end{array} \right.
\end{aligned}
\]

and let \((\varphi_h^{(2)}, p_h^{(2)}) \in C_{h0}^2 \times P_{h0}^1\) be such that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\nabla_h \varphi_h^{(2)}, \nabla_h \psi_h) + (p_h^{(2)}, \text{rot}_h \psi_h) = (\nabla r_h, \psi_h), \quad \forall \psi_h \in C_{h0}^2,
(q_h, \text{rot}_h \varphi_h^{(2)}) = 0, \quad \forall q_h \in P_{h0}^1.
\end{array} \right.
\end{aligned}
\]

As \((f, \Pi_h^2 v_h) = (\nabla r_h, \nabla \Pi_h^2 v_h) = (\nabla r_h, \Pi_h^{RM1} \nabla v_h)\), by Theorem 2.11, we obtain \(\varphi_h^{(1)} = \nabla_h u_h^{(1)}\). This is relevant to and slightly different from the decomposition as Lemma 4.12 below. Furthermore,

\[
\|\nabla_h (u_h - u_h^{(1)})\|_{0,\Omega} \leq \sup_{0 \neq v_h \in B_{h0}^3} \frac{(f, v_h - \Pi_h^2 v_h)}{\|\nabla_h v_h\|_{0,\Omega}} \leq C h^2 \|f\|_{0,\Omega},
\]

\[
\|\nabla_h (\varphi_h^{(1)} - \varphi_h^{(2)})\|_{0,\Omega} \leq \sup_{0 \neq v_h \in C_{h0}^2, \text{rot}_h \psi_h = 0} \frac{(\nabla r_h, \psi_h - \Pi_h^{RT1} \psi_h)}{\|\nabla_h \psi_h\|_{0,\Omega}} \leq C h \|f\|_{0,\Omega},
\]

and by Lemma 3.10,

\[
\|\varphi_h^{(2)} - \varphi\|_{1,h} \leq Ch^s(\|\varphi\|_{1+s,\Omega} + \|p\|_{s,\Omega} + \|\nabla r\|_{0,\Omega}) \leq Ch^s(\|u\|_{2+s,\Omega} + \|f\|_{0,\Omega}).
\]
Summing all the above, we have
\[ \|u - u_h\|_{2,h} \leq C(\|\nabla u - \varphi_h^{(2)}\|_{1,h} + \|\nabla_h (\varphi_h^{(1)} - \varphi_h^{(2)})\|_{0,\Omega} + \|\nabla_h^2 (u_h - u_h^{(1)})\|_{0,\Omega}) \leq Ch^4(\|u\|_{2+s,\Omega} + \|f\|_{0,\Omega}). \]

The proof is completed.

**Remark 3.12.** We note that the case of Theorem 3.11 happens generally for non-convex domains. However, in the proof, we do not need any regularity assumption associated with convex or non-convex domains. A similar procedure can be adopted for problems with the Navier type boundary condition. The approach utilized here is a completely indirect one without analysis on the approximation of \(B_h^3\), and it can also be adopted to the cases where the exact solutions possess the full regularity with the auxiliary Stokes problem (3.38) replaced by (3.11) and (3.15), respectively. We present both of the two approaches in this paper for a comparison.

## 4 On the implementation of the schemes

In this section, we present two approaches to implement the schemes. One is to figure out the locally supported basis functions of \(B_h^3\) and \(B_h^2\), and the other is to decompose the finite element system to three sub-problems to be solved sequentially. The former approach makes the scheme fit for the general finite element programming procedure, and the latter approach, as the sub-problems are Poisson systems and a Stokes system, makes the finite element problems easier to implement and to solve.

### 4.1 Locally supported basis functions of the finite element spaces

#### 4.1.1 Structure of the weakly rot-free space

Let \(T\) be a triangle with \(a_i\) and \(e_i\), \(i = 1, 2, 3\), being its vertices and edges (see Figure 3). Define
\[ P_2(\text{rot}, w; T) := \left\{ p \in P_2(T) : \int_T \text{rot} p = 0 \right\}. \]

Then \(\dim(P_2(\text{rot}, w; T)) = 11\).

Define
- \(\eta^{a_i}_i\) such that \(\eta^{a_i}_i(a_j) = (\delta_{ij}, 0)^T\), and \(\int_{e_k} \eta^{a_i}_i = 0\), \(i, j, k = 1, 2, 3\);
- \(\eta^{b_i}_i\) such that \(\eta^{b_i}_i(a_j) = (0, \delta_{ij})^T\), and \(\int_{e_k} \eta^{b_i}_i = 0\), \(i, j, k = 1, 2, 3\);
- \(\eta_{a_i}\), such that \(\eta_{a_i}(a_j) = 0\), and \(\int_{e_k} \eta_{a_i} \cdot t_{ek} = 1\) and \(\int_{e_j} \eta_{a_i} \cdot n_{ej} = 0\), where \(t_{ek}\), \(a_i\) is the unit tangential vector along \(e_k\) starting from \(a_i\) and \(n_{ej}\) is the outward normal of \(e_j\), \(i, j, k = 1, 2, 3\), \(i \neq k\);
- \(\eta_{c_i}\), such that \(\eta_{c_i}(a_j) = 0\), \(\int_{e_k} \eta_{c_i} \cdot t_{ek} = 0\), where \(t_{ek}\) is the anti-clockwise tangential vector along \(e_k\), and \(\int_{e_k} \eta_{c_i} \cdot n_{ek} = \delta_{ik}\), \(i, j, k = 1, 2, 3\).

**Lemma 4.1.** All \(\eta^{a_i}_i\), \(\eta^{b_i}_i\), and \(\eta_{c_i}\) for \(i = 1, 2, 3\) and any two of \(\eta_{a_i}\) among \(i = 1, 2, 3\) form a basis of \(P_2(\text{rot}, w; T)\).

**Proof.** Evidently, any two of \(\eta_{a_i}\) among \(i = 1, 2, 3\) are linearly independent and \(\eta_{a_1} + \eta_{a_2} + \eta_{a_3} = 0\). Secondly, all \(\eta^{a_i}_i\), \(\eta^{b_i}_i\), and \(\eta_{c_i}\) for \(i = 1, 2, 3\) are linearly independent. Furthermore, all \(\eta^{a_i}_i\), \(\eta^{b_i}_i\), and \(\eta_{c_i}\) for \(i = 1, 2, 3\) and any two of \(\eta_{a_i}\) among \(i = 1, 2, 3\) are linearly independent.

Analogically, define \(S_2^1(\text{rot}, w; 0) := \{ v \in S_2^1 : (\text{rot}_h v, q) = 0, \forall q \in P^0_0 \}\); \(S_2^1(\text{rot}, w; 0) := \{ v \in S_2^1 : (\text{rot}_h v, q) = 0, \forall q \in P^0_0 \}\); and \(S_2^1(\text{rot}, w; 0) := \{ v \in S_2^1 : (\text{rot}_h v, q) = 0, \forall q \in P^0_0 \}\).
Meanwhile, for \( a \in X_h \), denote by \( P_a \) the union of triangles of which \( a \) is a vertex, namely the patch associated with \( a \); for \( e \in E_h \), denote by \( P_e \) the patch associated with \( e \). With respect to \( a \in X_h \), let the functions in \( S^2_h \) be (see Figure 4)

- \( \varphi^x_a \) such that \( \varphi^x_a(a) = (1, 0)^\top \), \( \varphi^x_a(a') = 0 \) on \( a \neq a' \in X_h \), and \( \int_e \varphi^x_a = 0 \) on \( e' \in E_h \);
- \( \varphi^y_a \) such that \( \varphi^y_a(a) = (0, 1)^\top \), \( \varphi^y_a(a') = 0 \) on \( a \neq a' \in X_h \), and \( \int_e \varphi^y_a = 0 \) on \( e' \in E'_h \);
- \( \varphi_{P_a} \) such that \( \varphi_{P_a}(a') = 0 \) on \( a' \in X_h \), \( \int_e \varphi_{P_a} = 0 \) on \( e \in E_h \) and \( a \notin e \), and \( \int_e \varphi_{P_a} \cdot t_e \cdot n_{e,P_a} = 1 \) and \( \int_e \varphi_{P_a} \cdot n_{e,P_a} = 0 \) on \( e \subset P_a \) and \( a \in e \), where \( t_e \cdot P_a \) is the unit tangential vector along \( e \) starting from \( a \) and \( n_{e,P_a} \) is the anticlockwise normal direction of \( e \) with respect to \( P_a \);
- \( \varphi_e \) such that \( \int_e \varphi_e \cdot t_e = 0 \), \( \int_e \varphi_e \cdot n_e = 1 \), and \( \varphi_e \) vanishes on \( \Omega \setminus \tilde{P}_e \).

**Lemma 4.2.** The set \( \{ \varphi^x_a, \varphi^y_a, \varphi_{P_a}, \varphi_e \}_{a \in X_h, e \in E_h} \) forms a basis of \( S^2_{h0}(\text{rot}, w_0) \); namely,

\[
S^2_{h0}(\text{rot}, w_0) = \text{span}\{ \varphi^x_a \}_{a \in X_h} \oplus \text{span}\{ \varphi^y_a \}_{a \in X_h} \oplus \text{span}\{ \varphi_e \}_{e \in E_h} \oplus \text{span}\{ \varphi_{P_a} \}_{a \in X_h}.
\]

**Proof.** By direct calculation, the functions \( \varphi^x_a, \varphi^y_a, \varphi_{P_a} \) and \( \varphi_e \) all belong to \( S^2_{h0}(\text{rot}, w_0) \). By their definitions, the functions \( \{ \varphi^x_a, \varphi^y_a, \varphi_{P_a}, \varphi_e \}_{a \in X_h, e \in E_h} \) are linearly independent, and the summation \( \text{span}\{ \varphi_{P_a} \}_{a \in X_h} + \text{span}\{ \varphi^x_a, \varphi^y_a, \varphi_e \}_{a \in X_h, e \in E_h} \) is direct. Since

\[
\dim(S^2_{h0}(\text{rot}, w_0)) = \dim(S^2_{h0}) - \dim(P_{h0}) = \#((N)^0_h) + 3\#(X_h^h),
\]

it remains for us to show that \( \{ \varphi_e \}_{e \in E^h} \) are linearly independent.

Assume that there exist \( \{ \alpha_a \}_{a \in X} \subset \mathbb{R} \) with \( \alpha_a = 0 \) for \( a \in X_h \) such that \( \psi = \sum_{a \in X_h} \alpha_a \varphi_{P_a} \equiv 0 \).

By the definition of \( \varphi_{P_a} \), for any \( e \in E^h \), \( \int_{f_e} \psi \cdot n_e = |\alpha_{a_e^l} - \alpha_{a_e^r}| \), where \( a_e^l \) and \( a_e^r \) are the two ends of \( e \); thus, \( \alpha_{a_e^l} = \alpha_{a_e^r} \) for every \( e \in E^h \). Since \( \alpha_a = 0 \) for \( a \in X_h \), \( \alpha_a = 0 \) for \( a \in X_h^{b+1} \) (see Figure 6); recursively, we obtain \( \alpha_a = 0 \) for \( a \in X_h^{b+j} \) level by level, and finally show \( \alpha_a = 0 \) for all \( a \in X_h \).

The proof is completed by noting that the two sides of (4.1) have the same dimension.
Figure 4 Illustration of a vertex patch $P_a$ (solid line); (a) $a$ is an interior vertex; (b) $a$ is a boundary vertex.

Figure 5 Illustration of an edge patch $P_e$ (solid line); (a) $e$ is an interior edge; (b) $e$ is a boundary edge.

Figure 6 Recursively diminishing from outside to inside. The ■’s denote boundary vertices; the •’s denote vertices of $X_{h+1}^b$, and they are proved vanishing according to ■’s; the ▲’s denote vertices of $X_{h+2}^b$, and they are proved vanishing according to •’s. The procedure is repeated, recursively.

Remark 4.3. The supports of $\{\tilde{\varphi}_x^a, \tilde{\varphi}_y^a, \tilde{\varphi}_P_a\}_{a \in X_h^i}$ are each a vertex patch shown in Figure 4(a), and the supports of $\{\tilde{\varphi}_e\}_{e \in E_h^i}$ are each an edge patch shown in Figure 5(a).
Lemma 4.6. It holds that
\[ S_{ht}^2(\text{rot}, w0) = S_{ht0}^2(\text{rot}, w0) \oplus \text{span}\{\varphi^b_a : a \in X^b_h \setminus X^c_h\} \oplus \text{span}\{\varphi_c : c \in E^b_h\}. \]

Proof. By Lemma 4.1, \( \varphi^b_a \) with \( a \in X^b_h \setminus X^c_h \) are linearly independent, and the right-hand side is a direct sum included on the left-hand side. On the other hand,
\[
\dim(S_{ht}^2(\text{rot}, w0)) = \dim(S_{ht0}^2(\text{rot}, w0)) = \dim(S_{ht0}^2(\text{rot}, w0)) + \dim(\text{span}\{\varphi^b_a : a \in X^b_h \setminus X^c_h\}) + \dim(\text{span}\{\varphi_c : c \in E^b_h\}) = \dim(\text{right-hand side}).
\]
This proves the assertion. \( \square \)

Remark 4.5. The supports of \( \{\varphi^b_a : a \in X^b_h \setminus X^c_h\} \) are each a vertex patch shown in Figure 4(b), and the supports of \( \{\varphi_c : c \in E^b_h\} \) are each an edge patch shown in Figure 5(b).

4.1.2 Structure of the piecewise rot-free space

Let \( P_2(\text{rot}, 0; T) = \{p \in P_2(T) : \text{rot}p = 0\} \) and \( \dim(P_2(\text{rot}, 0; T)) = 9 \). Denote by \( \phi_T \) the bubble function \((\lambda^2 + \Delta^2) - 2/3\), and define a mapping \( F_T : P_2(\text{rot}, w0; T) \rightarrow P_2(\text{rot}, 0; T) \) by
\[
F_T\eta = \eta + \phi, \quad \phi \in \text{span}\{(\phi_T, 0)^\top, (0, \phi_T)^\top\} \quad \text{such that rot}(F_T\eta) = 0.
\]

Lemma 4.6. \( F_T \) is well defined.

Proof. Direct calculation leads to
\[
\int_T \text{rot}(\mu\phi_T, \nu\phi_T)^\top = 0, \quad \int_T \text{rot}(\mu\phi_T, \nu\phi_T)^\top x = \nu/2|T|, \quad \int_T \text{rot}(\mu\phi_T, \nu\phi_T)^\top y = -\mu/2|T|.
\]
Thus for any \( \eta \in P_2(\text{rot}, w0; T) \), we can always find \( \phi \in \text{span}\{(\phi_T, 0), (0, \phi_T)\} \) such that
\[
\int_T \text{rot}(\eta + \phi) = 0, \quad \int_T \text{rot}(\eta + \phi)x = 0 \quad \text{and} \quad \int_T \text{rot}(\eta + \phi)y = 0.
\]
Namely, \( \text{rot}(\eta + \phi) = 0 \). This confirms the well-posedness of \( F_T \). \( \square \)

It can be verified that \( F_T(\eta_{a_1}^x + \eta_{a_2}^y + \eta_{a_3}^z) = 0 \) and \( F_T(\eta_{a_1}^y + \eta_{a_2}^x + \eta_{a_3}^x) = 0 \).

Lemma 4.7. Any two \( F_T\eta_{a_i}^x \) among \( i = 1, 2, 3 \), any two \( F_T\eta_{a_i}^y \) among \( i = 1, 2, 3 \), any two \( F_T\eta_{a_i}^z \) among \( i = 1, 2, 3 \), and all \( F_T\eta_{e_i} \) for \( i = 1, 2, 3 \) form a basis of \( P_2(\text{rot}, 0) \).

Let \( G^2_h(\text{rot}, 0) := \{v \in G^2_h : \text{rot}_hv = 0\} \), \( G^2_{ht0}(\text{rot}, 0) := \{v \in G^2_{ht0} : \text{rot}_{ht0}v = 0\} \) and \( G^2_h(\text{rot}, 0) := \{v \in G^2_{ht0} : \text{rot}_{ht0}v = 0\} \). Define an operator \( F_h : S_{ht}^2(\text{rot}, w0) \rightarrow G^2_h(\text{rot}, 0) \) by
\[
F_h\varphi_h = \varphi_h + \phi_h, \quad \phi_h \in B^2_{ht0} \quad \text{such that rot}_h(F_h\varphi_h) = 0.
\]

Similarly, \( F_h \) is well defined. Indeed, \( (F_h\varphi_h)|_T = F_T(\varphi_h|_T) \), and \( F_h\varphi_h \) has the same support as \( \varphi_h \).
Lemma 4.9. \( F_h \) is a bijection between \( S^{2}_{h0}(\text{rot},w_0) := \{ v \in S^{2}_{h0} : (\text{rot} v, q) = 0, \forall q \in \mathbb{P}^{0}_{h0} \} \) and \( G^{2}_{h0}(\text{rot},0) := \{ v \in G^{2}_{h0} : \text{rot} h v = 0 \} \), and a bijection between \( S^{2}_{h0}(\text{rot},w_0) := \{ v \in S^{2}_{h0} : (\text{rot} v, q) = 0, \forall q \in \mathbb{P}^{0}_{h0} \} \) and \( G^{2}_{h0}(\text{rot},0) := \{ v \in G^{2}_{h0} : \text{rot} h v = 0 \} \).

Proof. Since \( S^{2}_{h0} \cap B_{h0} = \{0\} \), \( F_h \) is an injection on \( S^{2}_{h0}(\text{rot},w_0) \).

Given \( \gamma_h \in G^{2}_{h0}(\text{rot},0) \), decompose it to \( \gamma_h = \gamma^1_h + \gamma^2_h \) such that \( \gamma^1_h \in S^{2}_{h0} \) and \( \gamma^2_h \in B_{h0} \). As \( \text{rot} (\gamma_h|T) = 0 \) and \( \int_{T} \text{rot}(\phi_T,0) = \int_{T} \text{rot}(0,\phi_T) = 0 \) on every cell \( T \), \( \int_{T} \text{rot}(\gamma^1_h|T) = 0 \). Namely, \( \gamma^1_h \in S^{2}_{h0}(\text{rot},w_0) \).

By this way, \( F_h \) is a bijection between \( S^{2}_{h0}(\text{rot},w_0) \) and \( G^{2}_{h0}(\text{rot},0) \).

Similarly, we can prove \( F_h S^{2}_{h0}(\text{rot},w_0) = G^{2}_{h0}(\text{rot},0) \), and the proof is completed.

By Lemmas 4.2, 4.7 and 4.8, we can prove the lemmas below.

Lemma 4.9. The set \( \{ F_h \varphi_a, F_h \varphi_a^y, F_h \varphi_a^w, F_h \varphi_a, F_h \varphi_e \}_{a \in X^1_h, e \in E^1_h} \) forms a basis of \( G^{2}_{h0}(\text{rot},0) \).

Lemma 4.10. It holds that
\[
G^{2}_{h0}(\text{rot},0) = G^{2}_{h0}(\text{rot},0) \oplus \text{span}\{ F_h \varphi_a^b : a \in X^2_h \setminus X^1_h \} \oplus \text{span}\{ F_h \varphi_e^c : e \in E^2_h \}.
\]

4.1.3 Locally supported basis functions of \( B^{3}_{h0} \) and \( B^{3}_{ht} \)

By the exact sequences (2.6) and (2.5), we got to know that the piecewise gradient \( \nabla_h \) is a bijection between \( B^{3}_{ht} \) and \( G^{2}_{h0}(\text{rot},0) \), and between \( B^{3}_{ht} \) and \( G^{2}_{h0}(\text{rot},0) \). In the sequel, we denote by \( (\nabla_h)^{-1} \) the inverse operator of \( \nabla_h \) from \( G^{2}_{h0}(\text{rot},0) \) to \( B^{3}_{h0} \). \( (\nabla_h)^{-1} \) is a bijection from \( G^{2}_{h0}(\text{rot},0) \) onto \( B^{3}_{h0} \).

Now, given \( \psi_h \in G^{2}_{h0}(\text{rot},0) \), we can construct the exact sequence (2.6) on the support of \( \psi_h \), and find a unique \( w_h \in B^{3}_{ht} \) supported on the support of \( \psi_h \) such that \( \nabla_h w_h = \psi_h \) and thus \( w_h = (\nabla_h)^{-1} \psi_h \).

Namely, \( (\nabla_h)^{-1} \) preserves the locality of the support of functions in \( G^{2}_{h0}(\text{rot},0) \).

Now we are going to show that \( B^{3}_{ht} \) admits a set of basis functions with vertex-patch-based supports.

Theorem 4.11. The space \( B^{3}_{ht} \) admits a set of basis functions, and each is supported in a patch of some vertex.

Proof. By Lemma 4.9, the set
\[
\{ (\nabla_h)^{-1} F_h \varphi_a^y, (\nabla_h)^{-1} F_h \varphi_a, (\nabla_h)^{-1} F_h \varphi_a, (\nabla_h)^{-1} F_h \varphi_e \}_{a \in X^1_h, e \in E^1_h} \quad (4.4)
\]
forms a basis of \( B^{3}_{h0} \). By the locality preservation of \( (\nabla_h)^{-1} \), (4.4) is a basis each supported in the patch of a vertex, including edge patches. Furthermore,
\[
B^{3}_{ht} = B^{3}_{h0} \oplus \text{span}\{ (\nabla_h)^{-1} F_h \varphi_a^b : a \in X^2_h \setminus X^1_h \} \oplus \text{span}\{ (\nabla_h)^{-1} F_h \varphi_e^c : e \in E^2_h : c \in E^2_h \},
\]
and a locally supported basis of \( B^{3}_{ht} \) follows. The proof is completed.

All these basis functions can be obtained by straightforward calculation, as \( F_h \) preserves the locality of the supports and can be done cell by cell and \( (\nabla_h)^{-1} \) preserves the locality of the supports and can be done patch by patch. It is worth noticing that the piecewise kernel of \( \nabla_h \) consists of constants; the solutions of \( (\nabla_h)^{-1} \) on cells are matched together by the continuity of the evaluation on vertices of \( B^{3}_{ht} \).

Though the space \( B^{3}_{h0} \) does not correspond to a finite element defined by Ciarlet’s triple, these locally supported basis functions play the same role as that by the usual nodal basis functions. Substituting these functions into the common routine generates finite element codes in a standard way.
4.2 Implementation by the decomposition

In this subsection, alternatively, we suggest a decomposition procedure, and the schemes (3.5) and (3.6) can be implemented without the explicit construction of the basis functions.

**Lemma 4.12.** Let $u_\lambda$ be obtained by the following procedure:

1. Find $r_h \in A_{h0}^3$ such that

   \[ (\nabla_h r_h, \nabla_h s_h) = (f, s_h), \quad \forall s_h \in A_{h0}^3. \]  \hfill (4.6)

2. With $r_h$ obtained, find $(\varphi_h, p_h) \in G_{h0}^2 \times P_{h0}^1$ such that

   \[
   \begin{cases}
   (\nabla_h \varphi_h, \nabla_h \psi_h) + (p_h, \text{rot}_h \psi_h) = (\nabla_h r_h, \psi_h), & \forall \psi_h \in G_{h0}^2, \\
   (q_h, \text{rot}_h \varphi_h) = 0, & \forall q_h \in P_{h0}^1.
   \end{cases}
   \]  \hfill (4.7)

3. With $\varphi_h$ obtained, find $u_\lambda \in A_{h0}^3$ such that

   \[ (\nabla_h u_h, \nabla_h v_h) = (\varphi_h, \nabla_h v_h), \quad \forall v_h \in A_{h0}^3. \]  \hfill (4.8)

Let $u_h$ be the solution of (3.5). Then $u_\lambda = u_h$.

**Proof.** Firstly, it is evident that all the three subproblems admit a unique solution. Let $r_h$, $\varphi_h$ and $u_\lambda$ be, respectively the solutions of (4.6)–(4.8). Since $\text{rot}_h \varphi_h = 0$, by Theorem 2.11, $\varphi_h \in \nabla_h B_{h0}^3 \subset \nabla_h A_{h0}^3$; furthermore, $\nabla_h u_\lambda = \varphi_h$. Now given any $s_h \in B_{h0}^3$, by (4.7),

\[ (\nabla_h u_h, \nabla_h s_h) = (\nabla_h \varphi_h, \nabla_h (\nabla_h s_h)) = (\nabla_h r_h, \nabla_h s_h) = (f, s_h); \]

namely, $u_\lambda$ solves (3.5). By the uniqueness of the solution of (3.5), the proof is completed. \hfill \Box

**Remark 4.13.** For the implementation, we note and use the fact that $G_{h0}^2 = S_{h0}^2 \oplus B_{h0}^2$.

Similarly we have the lemma below.

**Lemma 4.14.** Let $z_\lambda$ be obtained by the following procedure:

1. Find $r_h \in A_{h0}^3$ such that

   \[ (\nabla_h r_h, \nabla_h s_h) = (f, s_h), \quad \forall s_h \in A_{h0}^3. \]  \hfill (4.9)

2. With $r_h$ obtained, find $(\varphi_h, p_h) \in G_{ht}^2 \times P_{h0}^1$ such that

   \[
   \begin{cases}
   (\nabla_h \varphi_h, \nabla_h \psi_h) + (p_h, \text{rot}_h \psi_h) = (\nabla_h r_h, \psi_h), & \forall \psi_h \in G_{ht}^2, \\
   (q_h, \text{rot}_h \varphi_h) = 0, & \forall q_h \in P_{h0}^1.
   \end{cases}
   \]  \hfill (4.7)

3. With $\varphi_h$ obtained, find $z_\lambda \in A_{h0}^3$ such that

   \[ (\nabla_h z_h, \nabla_h v_h) = (\varphi_h, \nabla_h v_h), \quad \forall v_h \in A_{h0}^3. \]

Let $z_h$ be the solution of (3.6). Then $z_\lambda = z_h$.  

It is worthy of noting that, the scheme (4.6) is not a convergent one for the Poisson equation, but it is well posed based on the continuity of $A_{h0}^3$ on vertices. With the formulations presented in Lemmas 4.12 and 4.14, the spaces used for Poisson equations and Stokes problems only are easy to formulate; the system only needs solving two Poisson systems and one Stokes system one by one, each of which can be solved with various optimal solvers in a friendly way.

**Remark 4.15.** The decompositions as in Lemmas 4.12 and 4.14 can also be established for (3.32) and the one in Remark 3.9, respectively.

5 Conclusion and discussion

In this paper, based on theoretical analysis by an indirect approach, a constructive answer is given to the question whether an optimal scheme can be designed for the biharmonic equation with piecewise cubic polynomials on general triangulations; particularly, the schemes work optimally on triangulations shown in Figure 1. Beside the theoretical values, the scheme can find its application onto practical problems. The author’s unpublished preprint [60] was the space $B_h^3$ reported for the first time and was the optimal accuracy for sufficiently smooth solutions for the biharmonic equation proved. The discrete strengthened Miranda-Talenti estimate on $B_h^3$ was firstly reported in the author’s another unpublished preprint [61]. Since then, consequent works have been devoted onto the application of $B_h^3$ for practical problems. For example, $B_h^3$-based high-accuracy schemes are published for the transmission eigenvalue problems in [51] and [52]; we particularly refer there for many numerical experiments on relevant fourth-order problems. In [51] it is also shown how the guideline in Subsection 4.1 can be followed step by step to construct local basis functions. The practical usage of the scheme can be thus illustrated.

This paper relies on construction and utilization of discretized Stokes complexes based on the $G_{h0}^2-P_{h0}^1$ pair. The space $G_h^k$ with $k = 3$ corresponds to the Crouzeix-Falk pair studied in [12]. In that paper, Crouzeix and Falk proved that the pair $G_{h0}^3-P_{h0}^2$ is stable “for most reasonable meshes”. Moreover, they presented a conjecture that the pair is stable “for any triangulation of a convex polygon satisfying the minimal angle condition and containing an interior vertex”. Recently, some triangulations where $G_{h0}^3-P_{h0}^2$ is stable or at least $\text{div}G_{h0}^2 = P_{h0}^2$ are introduced in [25]. This hints the possibility to generalize the concept for optimal quartic element schemes (see [60] for details). The connections among the finite elements for the biharmonic equation, the Stokes problem and the Poisson equations may be generalized to a higher dimension and even higher-order problems; this could be of interest in the future.
in a dual way, such as the mixed element method, local discontinuous Galerkin (DG) method, hybridized DG method, central DG (CDG) method, weak Galerkin method, virtual element method and so forth. We remark that the literature on the related works in this context is vast, but we would not discuss them in this paper. Moreover, based on the space $B^3_0(B^2_0)$, DG schemes can be designed. One may be able to construct, for example, a weakly over-penalized interior penalty (IP) method (like [8]) or an interior penalty discontinuous Galerkin (IPDG) method with optimal convergence rate robust with respect to the penalization parameter [61] with piecewise cubic polynomials. More discussions may be carried out by the aid of the framework of [28].

The finite element functions given can be viewed as nonconforming spline functions. Beside the indirect approach mainly adopted here, a direct interpolator may be constructed with respect to the locally supported basis, and may find applications for kinds of elliptic perturbation problems. This can be studied in the future. We also refer to [55] for relevant discussions. The approach of nonconforming spline functions has shown useful in constructing low-degree-polynomial schemes with relatively high accuracy; please refer to [41, 55, 56, 62] for some examples for $H^1$ and $H^2$ problems. The approach is expected to be extended to cases of higher-order problems in relatively low dimensions, particularly cases not yet covered in the Morley-Wang-Xu family, and will be discussed in the future.

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