Solutions of Kapustin-Witten equations for $ADE$-type groups

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Abstract

Kapustin-Witten (KW) equations are encountered in the localization of the topological $\mathcal{N} = 4$ SYM theory. Mikhaylov has constructed model solutions of KW equations for the boundary 't Hooft operators on a half space. Direct proof of the solutions boils down to check a boundary condition. There are two computational difficulties in explicitly constructing the solutions to higher rank Lie algebra. The first one is related to the commutation of generators of Lie algebra. We derived an identity which effectively reduces this computational difficulty. The second one involves the number of ways from the highest weights to other weights in the fundamental representation. For $ADE$-type gauge groups, we found an amazing formula which can be used to rewrite the solutions of KW equations. This new formula of solutions bypass above two computational difficulties.
1 Introduction

The maximally supersymmetric Yang-Mills theory in four dimensions can be twisted in
three ways to obtain topological field theories. One of the twists appears to be relevant for
the geometric Langlands program called the GL twist [2]. It can be applied to the descrip-
tion of the Khovanov homology of knots [4, 7]. The Chern-Simons theory is effectively
induced on the boundary of a four-dimensional manifold. The supersymmetry conditions
lead to the generalized Bogomolny equations [2] which is called Kapustin-Witten (KW)
equations now.

As described in [4], on a half space $V$ of the form $V = \mathbb{R}^3 \times \mathbb{R}_+$, the KW equations are

$$F - \phi \wedge \phi + * d_A \phi = 0 = d_A * \phi,$$

where $d_A$ is the covariant exterior derivative associated with a connection $A$, and $\phi$ is
one-form valued in the adjoint of the gauge group $G$. Different reductions of the KW
equations lead to other well know equations e.g., Nahm’s equations, Bogomolny equa-
tions or Hitchin equations. Through electric-magnetic duality, the natural Chern-Simons
observables correspond to the boundary ’t Hooft or surface ope rators in four dimensional
gauge theory. These operators are defined by prescribing the singular behavior of the
fields similarly as the supersymmetry boundary conditions in the model.

These model solutions with ’t Hooft operator as boundary conditions were first dis-
cussed in [4] for $SU(2)$ gauge group. For higher rank groups, solutions were constructed
for special values of the magnetic weight in [8]. For any simple compact gauge group, after
reducing to a Toda system, [1] V.Mikhaylov conjectured a formula of the model solutions
for the boundary ’t Hooft operator with general magnetic weight. Model solutions for
the $SU(n)$ groups were also obtained in [1] for the boundary surface operator. For other
related work on these equations, see[5][6].

Proof of the conjecture of the solutions requires to check a boundary condition. This
has been completed for $SU(n)$ group in [1]. In order to check the boundary condition, we
need construct the solutions explicitly. Unfortunately, there are two ’NP’-like computa-
tional difficulties with the increasing rank of Lie algebra. One difficulty is related to the
communication of generators of Lie algebra. Another difficulty involves the number of
paths from the highest weight to an arbitrarily weight in the fundament representation.
The purpose of this paper is to resolve these computational difficulties. In section 2, we
review the construction of the time independent solutions with boundary ’t Hooft opera-
tors. The KW equations can be reduced to a Toda system. The formula of the solutions
was conjectured in a simple way by matching boundary conditions of the half space in [1].
In section 3, we illustrate how to construct the solutions precisely through an example.
Then we derive an identity using the characteristics of Lie algebra. This identity effec-
tively reduces the computational difficulty of the commutation of operators. We give an
example to show the second difficulty related to ways from highest weight to a certain
weight in the fundament representation. In section 4, for Lie algebras of ADE type, we
find an amazing formula which can be used to construct solutions of KW equations. We
have checked this formula by using all the solutions constructed in [1]. In the Appendix
A, we present another observation in the study of the KW equations. In the Appendix
B, other ’t Hooft operator solutions checked by different methods are collected.
2 The KW equations and boundary conditions

We take \( V \) to be the half space \( x^3 \geq 0 \) in a Euclidean space with coordinates \( x^0, \ldots, x^3 \). The three spatial coordinates are denoted by \( x^1 + ix^2 = z, x^3 = y \), and \( |z| = r \). The boundary ’t Hooft operator lies along the line \( z = 0, y = 0 \). In [1], Mikhaylov reduced the Kapustin-Witten equations to a Toda systems and conjectured a formula of the solutions in a simple way. We review this formula following [1] closely to which we refer the reader for more details.

2.1 Reduction of the KW equations

As explained in [4], for time-independent solutions one can take \( A_0 = \phi_3 = 0 \). Introducing the three operators

\[
\begin{align*}
D_1 &= 2\partial z + A_1 + iA_2, \\
D_2 &= \partial y + A_3 - i\phi_0, \\
D_3 &= \phi_1 - i\phi_2,
\end{align*}
\]

then the KW equations (1.1) can be reformulated as

\[
\begin{align*}
[D_i, D_j] &= 0, \quad i,j = 1..3, \\
\sum_{i=1}^{3} [D_i, D_i^+] &= 0.
\end{align*}
\]

The Eq.(2.2) have a larger symmetry than Eq.(1.1). They are invariant under element belongs to a complexified gauge group \( G_C \). While the remaining equation (2.3) can be interpreted as a moment map constraint for this complexified gauge group [4]. Concretely, the Eq. (2.3) is

\[
4F_{z\bar{z}} - 2iD_3\phi_0 + [\varphi, \varphi^\dagger] = 0,
\]

where \( \varphi = \phi_1 - i\phi_2 \).

For the solution of Eq.(2.2), one can take a gauge in which \( A_1 = A_2 = A_3 = 0 \) and \( \phi_0 = 0 \). Then the remaining equations imply that \( \varphi \) is holomorphic and independent of \( y \). Assuming \( \varphi_0(z) \) is a solution of Eq.(2.2), one can apply a gauge transformation \( g \in G_C \) to \( \varphi_0(z) \) and substitute the resulting solution into the moment map equation (2.3), then

\[
4\partial_z (\partial_z h^{-1}) + \partial_y (\partial_y h^{-1}) + [\varphi_0^\dagger(z), h\varphi_0(z)h^{-1}] = 0,
\]

where \( h = g^\dagger g \). Let \( \mathfrak{h} \subset \mathfrak{g}_C \) be a real Cartan subalgebra of the split real form of \( \mathfrak{g}_C \). Choosing \( g = \exp(\Psi) \) for \( \Psi \in \mathfrak{h} \), then Eq.(2.5) be reduced to

\[
\Delta_{\mathfrak{h}}\Psi + \frac{1}{2}[\varphi_0^\dagger(z), e^{2\Psi}\varphi_0(z)e^{-2\Psi}] = 0.
\]

In the Chevalley basis, for a simple roots \( \alpha_i \), denote the corresponding raising and lowering operators by \( E_i^\pm \), and the corresponding coroots by \( H_i \). Then the commutation relations of these operators are

\[
[H_i, H_j] = 0, \quad [H_i, E_j^\pm] = \pm A_{ji}E_j^\pm, \quad [E_i^+, E_j^-] = \delta_{ji}H_j.
\]

The ’t Hooft operators are corresponding to the elements of the cocharacter lattice \( \Gamma_{\mathfrak{ch}}^\vee \in \mathfrak{h} \) which is the lattice of homomorphisms \( \text{Hom}(\mathbb{C}^*, G_C) \). Let \( g(z) = z^\omega, \omega = \sum k_i H_i \in \Gamma_{\mathfrak{ch}}^\vee \).
be such a homomorphism corresponding to ’t Hooft operator. Using Weyl equivalence, one can transform $\omega$ to the positive Weyl chamber such that

$$ r_i = \alpha_i(\omega) \geq 0. \quad (2.8) $$

Since the lattice $\Gamma^\vee_{ch}$ lies inside the dual root lattice $\Gamma^*_r$, the numbers $r_i$ are integer.

Let $\varphi_1 = \sum_i E_i^+$ be a representative of the principal nilpotent orbit in the algebra. We take the solution of the holomorphic equations (2.2) to be of the form

$$ \varphi_0(z) = g(z)\varphi_1 g^{-1}(z). \quad (2.9) $$

Using the commutation relations (2.7), the above formula become

$$ \varphi_0(z) = \sum_i z^{r_i} E_i^+. \quad (2.10) $$

For this solution, with making a real gauge transformation $g = \exp(\Psi)$, $\Psi \in \mathfrak{h}$, the fields become

$$ A_a = -i\epsilon_{ab}\partial_b \Psi, \quad a, b = 1..2, \\
\phi_0 = -i\partial_y \Psi, \quad A_3 = 0, \\
\varphi = e^\Psi \varphi_0 e^{-\Psi}. \quad (2.11) $$

Taking a change of variables $\Psi = \frac{1}{2} \sum_{i,j} A_{ij}^{-1} H_i \psi_j$, the Eq. (2.6) gives the following system of equations,

$$ \sum_j A_{ij}^{-1} \Delta_3 \psi_j - r^2 \psi_s = 0. \quad (2.12) $$

A convenient parameterization

$$ \psi_i = q_i - 2m_i \log r, \quad m_i = r_i + 1, \quad (2.13) $$

which brings the Eq. (2.12) to the scale invariant form. For the scale invariant solutions, $q_i$ depend only on the ratio $y/r$. Setting $y/r = \sinh \sigma$, then the equations (2.12), leads to the Toda form,

$$ \ddot{q}_i - \sum_j A_{ij} e^{q_j} = 0, \quad (2.14) $$

where the dots denote derivatives with respect to $\sigma$.

**Boundary conditions:**

The boundary condition on the plane $y = 0$ away from the defect is determined by prescribing the singular behaviour of the fields [11 9]. In the model solution, the gauge field is $A_0 = A_1 = A_2 = A_3 = 0$, the normal component of one form is $\phi_3 = 0$, and the tangent components of the one-form have a singularity

$$ \phi_0 = \frac{t_3}{y}, \quad \varphi = \frac{t_1 - it_2}{y}, \quad (2.15) $$
where $t_i \in \mathfrak{g}_c$ are the images of a principle embedding of the $\mathfrak{su}(2)$ subalgebra. A convenient representative of this conjugacy class is given by

$$
t_3 = \frac{i}{2} \sum_i B_i H_i ,
$$

$$
t_1 - it_2 = \sum_i \sqrt{B_i} E_i^+, \quad B_i = 2 \sum_j A_{ij}^{-1} . \tag{2.16}
$$

If $\delta^\vee \in \mathfrak{h}$ is the dual of the Weyl vector, then $t_3 = i \delta^\vee$. Let $\Delta_s$ be the set of weights of the fundamental representation $\rho_s$ and $\Lambda_s$ be the highest weight. Then weight $w \in \Delta_s$ of level $n(w)$ can be represented as

$$
w = \Lambda_s - \sum_{i=1}^{n(w)} \alpha_{j_i}, \quad \alpha_i \in \Delta . \tag{2.17}
$$

The lowest weight can be formulated as $\Lambda_i = \Lambda - \sum_j n_j \alpha_j$ which relates to the height $B_i$ as follow

$$
B_i = \sum_j n_j . \tag{2.18}
$$

Following [4], the Toda system Eq.(2.14) have a simple solution,

$$
q_i = -2 \log \sinh \sigma + \log B_i . \tag{2.19}
$$

The corresponding fields are,

$$
A_a = i e^{\frac{3}{2} x_i} \omega, \quad \phi_0 = \frac{i}{2y} \sum_i B_i H_i, \quad \varphi = \frac{1}{y} \sum (z/\bar{z})^{1/2} B_i^{1/2} E_i^+. \tag{2.20}
$$

A gauge transformation $\hat{g} = (\bar{z}/z)^{\omega/2}$ brings it to the form of Eq.(2.15)

$$
A_a = 0, \quad \phi_0 = \frac{i}{2y} \sum_i B_i H_i, \quad \varphi = \frac{1}{y} \sum B_i^{1/2} E_i^+. \tag{2.21}
$$

So, the boundary condition require the functions $q_i$ approach at $\sigma \to 0$ the model solution

$$
\sigma \to 0 : \quad q_j = -2 \log \sigma + \log B_j + \ldots. \tag{2.22}
$$

In parametrization $\chi_i = \sum_j A_{ij}^{-1} q_j$, these boundary condition can be formulated as

$$
\sigma \to 0 : \quad e^{-\chi_i} = \sigma^{B_i} \prod_k B_k^{-A_{ik}^{-1}} + \ldots \to 0 . \tag{2.23}
$$

For $\sigma \to \infty$, the fields must be non-singular along line $r = 0$,

$$
\sigma \to \infty : \quad q_i = -2 m_i \sigma + \log(4 C_j) + O(e^{-\sigma}) , \quad m_i = r_i + 1 ,
$$

where constants $C_j$ fixed from the boundary condition at $\sigma = 0$. The last term $O(e^{-\sigma})$ is expected from the general properties of the open Toda systems Eq.(2.14). In terms of variables $\chi_i$ the boundary condition is

$$
\sigma \to \infty : \quad \chi_i = -2 \lambda_i \sigma + \eta_i + O(e^{-\sigma}) , \tag{2.24}
$$

where $\eta_i$ are functions of constants $C_j$, and $\lambda_i = \sum_j A_{ij}^{-1} m_j$. 


2.2 The solution

Setting $\chi = \sum_i \chi_i H_i$ and $\hat{\omega} = \sum_i \lambda_i H_i$, in terms of the notations of the previous subsection, one have

$$\hat{\omega} = \omega + \delta^\vee.$$  

(2.25)

Since $r_i = \alpha_i(\omega)$, $\alpha_i(\delta^\vee) = 1$, we have

$$m_i \equiv r_i + 1 = \alpha_i(\hat{\omega}).$$  

(2.26)

Let $\Lambda_s$, $s = 1, \ldots, \text{rank}(g)$, be the highest weights of the fundamental representations $\rho_s$ of the Lie algebra $g_c$. We will denote $|\Lambda_s\rangle$ as the highest weight vector of unit norm in the representation $\rho_s$.

In [1], Milkhaylov first constructed a solution starting from “initial values” (2.23) at $\tau \rightarrow \infty$ with constants $C_j$. Then he fixed these constants by matching the second boundary condition (2.23). Solution of the open Toda system (2.14) at time $\tau$ is related to solution at different times $\tau$ through

$$\exp(-\chi_\tau(\sigma)) = \exp(-\chi_\tau(\tau)) \exp \left[ (\tau - \sigma) \hat{\omega} + \sum_j e^{\gamma_j(\tau)/2} (E_j^+ + E_j^-) \right] |\Lambda_s\rangle.$$  

(2.27)

The following formula can be used to calculate the above limit explicitly

$$e^{A+B} = \sum_m \int_0^1 dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 e^{(1-t_m)A} B e^{(t_m-t_{m-1})A} B \cdots Be^{t_1 A}. $$  

(2.28)

Choosing operators $A = \tau \sum_j e^{-m_j \tau} \sqrt{-4C_j} E_j^-, B = 2(-\tau + \sigma) \hat{\omega}$, Eq. (2.28) leads to

$$\exp(A+B) |\Lambda\rangle = \sum_{m=0}^{\infty} \sum_{k=0}^{m} e^{A_k} \frac{1}{\prod_{j \neq k} (A_k - A_j)} B \cdots B |\Lambda\rangle.$$  

Substituting operators $A$ and $B$, this result can be written in the following compact form

$$e^{-\chi_\tau(\sigma)} = e^{-\eta_s} \sum_{u \in \Delta_s} \left[ \exp(2\sigma w(\hat{\omega})) (v_u(\hat{\omega})|v_u(\hat{\omega})\rangle (-1)^{n(w)} \prod_{l=1}^{n(w)} C_{2l} \right],$$  

(2.29)

where vector $|v_u(\hat{\omega})\rangle$ is

$$|v_u(\hat{\omega})\rangle = \sum_s \frac{n(w)}{w(\hat{\omega}) - w_u(\hat{\omega})} E_j^- \cdots E_{j_1}^- |\Lambda\rangle.$$  

(2.30)

The notation $s$ enumerate ways from the highest weight $\Lambda$ a certain weight $w$ which corresponds to a sequence $\Lambda = w_1, w_2, \ldots, w_{n(w)}, w_{n(w)+1} = w$. 

5
Constants $C_i$ are fixed by matching boundary condition Eq. (2.24)

$$
\sum_{w \in \Delta_s} \left[ \langle v_w(\hat{\omega}) | v_w(\hat{\omega}) \rangle (-1)^{n(w)} \prod_{l=1}^{n(w)} C_{j_l} \right] = 0 .
$$

In [1], Mikhaylov conjectured that this solution is unique

$$
C_i = \prod_{\beta_j \in \Delta_+} (\beta_j(\hat{\omega}))^{2(\alpha_i,\beta_j)/(\beta_j,\beta_j)},
$$

(2.31)

where $\Delta_+$ is the set of positive roots.

With this assumption, and substituting explicit expression for the constants $\eta_i$ in terms of $C_j$, Eq. (2.29) becomes

$$
e^{-\chi_s(\sigma)} = 2^{-B_s} \sum_{w \in \Delta_s} \left[ \exp \left( 2 \sigma w(\hat{\omega}) \right) \langle v_w(\hat{\omega}) | v_w(\hat{\omega}) \rangle (-1)^{n(w)} \prod_{\beta_a \in \Delta_+} (\beta_a(\hat{\omega}))^{-2(w,\beta_a)/(\beta_a,\beta_a)} \right]
$$

(2.32)

In the case of $A_n$ algebra, the above formula has be proved in [1]. The first step is to simplify it to be a Weyl invariant form. Since the fundamental representations of $A_n$ are minuscule, the Weyl invariant expression can be further simplified by Weyl transformations from the highest weight term. The final rewritten formula is simple enough to check the boundary condition (2.23) directly.

### 3 Checking the boundary condition

We summarize results of the solution in the last section. The ’t Hooft operator is labeled by cocharacter $\omega \in \Gamma_{\delta}^{\vee}$, and $\hat{\omega} = \omega + \delta^{\vee}$. $\Delta$ is the set of simple roots $\alpha_i$ and $E_\alpha$ are the raising generators corresponding to the simple roots. Then the explicit fields on the solution are

$$
\varphi = \frac{1}{r} \sum_{\alpha \in \Delta} \exp \left[ \alpha(i\omega \theta + \frac{1}{2} \chi(\sigma)) \right] E_\alpha ,
$$

$$
\phi_0 = -\frac{i}{2\rho} \partial_{\sigma} \chi(\sigma) ,
$$

$$
A = -i \left( \hat{\omega} + \frac{1}{2} \sqrt{y^2 + r^2} \partial_{\sigma} \chi(\sigma) \right) d\theta .
$$

where $\chi(\sigma) = \sum \chi_i(\sigma) H_i$. The functions $\chi_i(\sigma)$ are conjectured in Eq. (2.32). In order to prove this conjecture, we need to check following boundary condition Eq. (2.23)

$$
\sigma \to 0 : e^{-\chi_s(\sigma)} = 0 .
$$

(3.1)

In the first subsection, we checked the boundary condition Eq. (2.32) for the fundamnet representation $\rho_1$ of $A_2$ algebra. We generalize some formulas of computation to general Lie algebra $g$ at the same time. In the second subsection, we derived an identity which
effectively simplifies the calculation of the communication of generators of Lie algebra. First, we introduce following notations for conveniences

\[
E_w = \exp(2\sigma w(\hat{\omega}))(1)^{n(w)} \\
W_w = \langle v_w(\hat{\omega})|v_w(\hat{\omega}) \rangle \\
F_w = \prod_{\beta_a \in \Delta_+} (\beta_a(\hat{\omega}))^{-2\langle w, \beta_a \rangle/\langle \beta_a, \beta_a \rangle},
\]

which lead to

\[
e^{-\chi_s(\sigma)} = 2^{-B_s} \sum_{w \in \Delta_s} [E_w \cdot W_w \cdot F_w].
\]

In section 4, we will find that there is a close relationship between term \(W_w\) and term \(F_w\) for Lie algebras of ADE type. This relationship can be used to rewrite the solutions in another form.

### 3.1 Example: fundament representation \(\rho_1\) of \(A_2\).

The highest weight is \(\Lambda_1 = [1, 0]\). And there are three weights \([1, 0], [-1, 1], [0, -1]\) in the fundament representation \(\rho_1\)

\[
[1, 0] \xrightarrow{\alpha_1} [-1, 1] \xrightarrow{\alpha_2} [0, -1].
\]

According Eq.(2.18), the height \(B_1\) is 2.

For the simple root \(\alpha_i\) and the fundamental weight \(\omega_i\), we have the following identities

\[
\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}, \quad \alpha_i = \sum_j A_{ij} \omega_j.
\]

The Cartan matrix of \(A_2\) is

\[
A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix},
\]

which leads to

\[
\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2\omega_1 - 2\omega_2 \\ -\omega_1 + 2\omega_2 \end{pmatrix}.
\]

The positive roots of \(A_2\) are \(\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}\) and the lengths of two simple roots are \(|\alpha_2|^2 = 1, \ |\alpha_1|^2 = 2\). For a general weight \(w = \lambda_1\omega_1 + \lambda_2\omega_2\), using Eq.(2.26) we have

\[
w(\hat{\omega}) = (\lambda_1\omega_1 + \lambda_2\omega_2)(\hat{\omega}) = (\lambda_1, \lambda_2)A_j^{-1} \begin{pmatrix} \alpha_1(\hat{\omega}) \\ \alpha_2(\hat{\omega}) \end{pmatrix} = (\lambda_1, \lambda_2) \left( \frac{2m_1 + m_2}{3m_1 + 2m_2} \right).
\]

Now, we calculate the factor \(F_w = \prod_{\beta_a \in \Delta_+} (\beta_a(\hat{\omega}))^{-2\langle w, \beta_a \rangle/\langle \beta_a, \beta_a \rangle}\) in Eq.(2.32). For the first two positive roots \(\alpha_1, \alpha_2\) in \(\Delta_+\), we have

\[
\beta_a = \alpha_1 : \quad \langle \alpha_1(\hat{\omega}) \rangle^{-\langle \omega, \alpha_1^\vee \rangle} = m_1^{-\lambda_1} \\
\beta_a = \alpha_2 : \quad \langle \alpha_2(\hat{\omega}) \rangle^{-\langle \omega, \alpha_2^\vee \rangle} = m_2^{-\lambda_2}
\]

(3.5)

For the third positive root \(\beta_a = \alpha_1 + \alpha_2\), we have

\[
\langle w, \alpha_1 + \alpha_2 \rangle = \lambda_1 + \lambda_2, \quad \langle \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 \rangle = 2.
\]
Then, we get
\[ \beta_3 = \alpha_1 + \alpha_2 : \quad ((\alpha_1 + \alpha_2)(\hat{\omega}))^{-2(w, \alpha_1 + \alpha_2)/(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)} = (m_1 + m_2)^{-(\lambda_1 + \lambda_2)}. \quad (3.6) \]

Combining Eq. (3.5) and Eq. (3.6), for a general weight \( w = \lambda_1 \omega_1 + \lambda_2 \omega_2 \), we have
\[ F_w = \frac{1}{m_1^{\lambda_1} m_2^{\lambda_2} (m_1 + m_2)^{\lambda_1 + \lambda_2}} \]

It is straightforward to generalize the above result to a general semisimple Lie algebra \( \mathfrak{g} \) with weights \( w = \sum_{i=1}^{\text{rank}(\mathfrak{g})} \lambda_i \omega_i \) in a fundament representation. The inner product of positive root \( \beta_a = \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i \alpha_i \) is
\[ \langle \beta_a, \beta_a \rangle = \sum_{i=1}^{\text{rank}(\mathfrak{g})} \sum_{j=1}^{\text{rank}(\mathfrak{g})} a_i a_j \langle \alpha_i, \alpha_j \rangle \frac{|\alpha_j|^2}{2} = \sum_{i=1}^{\text{rank}(\mathfrak{g})} \sum_{j=1}^{\text{rank}(\mathfrak{g})} a_i a_j A_{ij} \frac{|\alpha_j|^2}{2} \]

Another two factors in \( F_w \) are
\[ \langle w, \beta_a \rangle = \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i \langle w, \alpha_i \rangle \frac{|\alpha_i|^2}{2} = \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i \lambda_i \frac{|\alpha_i|^2}{2}, \quad \beta_a(\hat{\omega}) = \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i \alpha_i(\hat{\omega}) = \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i m_i \]

Substituting the above results into Eq. (3.2), we have
\[ F_w = \prod_{\beta_a \in \Delta_+} (\beta_a(\hat{\omega}))^{-2\langle w, \beta_a \rangle/\langle \beta_a, \beta_a \rangle} = \prod_{\beta_a \in \Delta_+} \left( \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i m_i \right)^{-2\sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i \lambda_i \frac{|\alpha_i|^2}{2}} \]

This compact form only involves basic data of Lie algebra \( \mathfrak{g} \) and simple algebraic calculation. It is convenient for computer program to work on.

Next, we calculate terms \( E_w \) and \( W_w \) in Eq. (2.32). Firstly, we introduce some basic identities. For the highest weight \( \Lambda \), we have
\[ E_i^+ |\Lambda\rangle = 0, \quad H_i |\Lambda\rangle = \lambda_i |\Lambda\rangle. \]

And we will use the following communication relationship frequently
\[ \langle \Lambda | E_i^+ E_i^- | \Lambda \rangle = \langle \Lambda | [E_i^+, E_i^-] + E_i^- E_i^+ | \Lambda \rangle = \langle \Lambda | H_i + E_i^- E_i^+ | \Lambda \rangle = \lambda_i. \]

We calculate the corresponding terms \( E_w \cdot W_w \cdot F_w \) for the three in the fundament representation \( \rho_1 \) as following

- \([1, 0] \): the level is \( n([1, 0]) = 0 \). We have
  \[ W_{[1, 0]} = \langle \Lambda | \Lambda \rangle = 1, \]
  and
  \[ E_{[1, 0]} = \exp[2\sigma([1, 0])(\hat{\omega})](-1)^0 = \exp[\frac{2}{3} \sigma(2m_1 + m_2)]. \]
According to Eq. (3.12), we get
\[ F_{[1,0]} = \frac{1}{(m_1)(m_2 + m_1)}. \] (3.8)

Combining the above three factors, we have
\[ E_{[1,0]} \cdot W_{[1,0]} \cdot F_{[1,0]} = \exp\left[\frac{2}{3} \sigma(2m_1 + m_2)\right] \frac{1}{m_1(m_2 + m_1)}. \] (3.9)

- $[-1, 1]$: the level is $n([-1, 1]) = 1$. We have
\[ E_{[-1,1]} = \exp[2\sigma([-1, 1])(\hat{\omega})](-1)^1 = -\exp[\frac{2}{3} \sigma(-m_1 + m_2)]. \]
According to Eq. (3.2), we get
\[ F_{[-1,1]} = \frac{m_1}{m_2}. \] (3.10)

The vector for the second weight is
\[ |v_{[-1,1]}(\hat{\omega})\rangle = \frac{1}{([1,0])(\hat{\omega}) - ([{-1,1}])^{\hat{\omega}}} E_{\omega^{-1}} |\Lambda\rangle = \frac{1}{m_1} E_{\omega^{-1}} |\Lambda\rangle \]
and the inner product of this vector is
\[ W_{[-1,1]} = \langle v_{[-1,1]}(\hat{\omega})|v_{[-1,1]}(\hat{\omega})\rangle = \langle \Lambda|E_{\omega^{-1}}E_{\omega^{-1}}^\dagger |\Lambda\rangle = \frac{1}{m_1^2} \langle \Lambda|H_{\omega^{-1}}|\Lambda\rangle = \frac{1}{m_1^2}. \]
Combining the above three factors for the second weight, we have
\[ E_{[-1,1]} \cdot W_{[-1,1]} \cdot F_{[-1,1]} = -\exp[\frac{2}{3} \sigma(-m_1 + m_2)] \frac{1}{m_1 m_2}. \] (3.11)

- $[0, -1]$: the level is $n([0, -1]) = 2$. We have
\[ E_{[0,-1]} = \exp[-\frac{2}{3} \sigma(m_1 + 2m_2)](-1)^2. \]
According to Eq. (3.2), we get
\[ F_{[0,-1]} = m_2(m_1 + m_2). \] (3.12)

The vector for the third weight is
\[ |v_{[0,-1]}(\hat{\omega})\rangle = \frac{1}{w(\hat{\omega}) - w_2(\hat{\omega})}, \frac{1}{w(\hat{\omega}) - w_1(\hat{\omega})} E_{\omega^{-1}} E_{\omega^{-1}} |\Lambda\rangle = \frac{1}{-m_2}, \frac{1}{-m_1 - m_2} E_{\omega^{-1}} E_{\omega^{-1}} |\Lambda\rangle, \]
and the conjugate vector is
\[ \langle v_{[0,-1]}(\hat{\omega})\rangle = \langle \Lambda|E_{\omega^{-1}}E_{\omega^{-1}}^\dagger \frac{1}{m_2(m_1 + m_2)}. \]
The inner product is
\[
W_{[0,-1]} = \langle v_{[0,-1]}(\hat{\omega})|v_{[0,-1]}(\hat{\omega})\rangle = \frac{1}{(m_2(m_1 + m_2))^2} \langle \Lambda|E_{\omega^{-1}}E_{\omega^{-1}}^\dagger |\Lambda\rangle
\]
\[ = \frac{1}{(m_2(m_1 + m_2))^2}. \]
Combining the above results for the third weight, we have
\[ E_{[0,-1]} \cdot W_{[0,-1]} \cdot F_{[0,-1]} = \exp[\frac{2}{3} \sigma(m_1 + 2m_2)] \frac{1}{m_2(m_1 + m_2)}. \] (3.13)
To summarize, substituting all the Eq. (3.9), Eq. (3.11) and Eq. (3.13) to the formula (3.2), we have
\[
e^{-\chi_1(\sigma)} = 2^{-2}(E_{[1,0]} \cdot W_{[1,0]} \cdot F_{[1,0]} + E_{[-1,1]} \cdot W_{[-1,1]} \cdot F_{[-1,1]} + E_{[0,-1]} \cdot W_{[0,-1]} \cdot F_{[0,-1]})
\]
\[
\frac{1}{4} \left( \frac{\exp[\frac{2}{3}\sigma(2m_1 + m_2)]}{m_1(m_2 + m_1)} - \frac{\exp[\frac{2}{3}\sigma(-m_1 + m_2)]}{m_1 m_2} + \frac{\exp[\frac{2}{3}\sigma(m_1 + 2m_2)]}{m_2(m_1 + m_2)} \right)
\]
which have been given in \[1\]. It is easy to check that above formula satisfy the boundary condition Eq. (3.1)
\[
\sigma \to 0 : e^{-\chi_1(\sigma)} = 0.
\]

From above derivations, we find that the calculation of the communication of operators in \(W_w\) will be a boring job in checking the boundary condition Eq. (3.2). In a similar situation, it is unrealistic for a personal computer in finite time to work out the inner product of a vector with more than ten Virasoro operators \(L_n\) acting on the highest weight vector. With increasing the rank of Lie algebra \(g\), the calculation work increases rapidly if realizing the communication of operators directly.

In this example there is only one way reaching arbitrarily weight in the fundamental representation \(\rho_1\). With increasing the rank of Lie algebra \(g\), the number of weights as well as the number of ways reaching arbitrarily weight increase rapidly in the fundamental representation \(\rho_1\). This will become clear in an example in the next section.

### 3.2 One useful identity

In this subsection we derive one identity which can be used to reduce the computation of the communication of generators of Lie algebra in \(W_w\). For the highest weight \(\Lambda = \Sigma a \lambda_a \omega_a\) in a representation, according to communication relations (2.7), we have the following basic identity,
\[
E_a^+ E_j^+ \cdots E_j^{[n]} |\Lambda\rangle = (\delta_{a,jn} H_a + E_j^+ E_j^+ \cdots E_j^+ |\Lambda\rangle)
\]
\[
= \sum_{i=1}^{n} E_j^+ E_j^{[n-i]} \cdots E_j^+ |\Lambda\rangle
\]
\[
= \sum_{i=1}^{n} \delta_{a,i} (\lambda_a - \sum_{i=1}^{n-1} A_{i,a}) E_j^+ E_j^{[n-i]} \cdots E_j^+ |\Lambda\rangle
\]
where the hat means omitting the corresponding term. In a special situation,
\[
E_i^+(E_i^-)^n |\Lambda\rangle = (H_i + E_i^- E_i^+)(E_i^-)^{n-1} |\Lambda\rangle
\]
\[
= \sum_{i=1}^{n-1} (E_i^-)^l H_i (E_i^-)^{n-1-l} |\Lambda\rangle
\]
\[
= \sum_{i=1}^{n-1} (E_i^-)^l (\lambda_i - (n - 1 - l) A_{ii}) (E_i^-)^{n-1-l} |\Lambda\rangle
\]
\[
= n(\lambda_i - (n - 1)) (E_i^-)^{n-1} |\Lambda\rangle
\]
The following identity is one of the main results we get in this paper.

**Proposition 1** For the highest weight \(\Lambda = \Sigma a \lambda_a \omega_a\) in a representation, we have
\[
E_i^+(E_i^-)^n \prod_{i=1}^{m} E_j^+ |\Lambda\rangle = n(\lambda_i - (n - 1) - \sum_{b=1}^{m} A_{j,b}) (E_i^-)^{n-1} \prod_{i=1}^{m} E_j^- |\Lambda\rangle + (E_i^-)^n E_i^+ \prod_{i=1}^{m} E_j^- |\Lambda\rangle
\]
Proof: According to Eq. (3.15), we have

\[
L.H.S = \sum_{a=0}^{n-1} (E_i^-)^a H_i (E_i^-)^{n-1-a} \prod_{i=1}^{m} E_{j_i^-}|\Lambda\rangle + (E_i^-)^n E_i^+ \prod_{i=1}^{m} E_{j_i^-}|\Lambda\rangle
\]

\[
= \sum_{a=0}^{n-1} (E_i^-)^a (\lambda_i - (n - 1 - a)A_{ii} - \sum_{b=1}^{m} A_{j_b,i}(E_i^-)^{n-1-a} \prod_{i=1}^{m} E_{j_b^-})|\Lambda\rangle + (E_i^-)^n E_i^+ \prod_{i=1}^{m} E_{j_b^-}|\Lambda\rangle
\]

\[
= n(\lambda_i - (n - 1) - \sum_{b=1}^{m} A_{j_b,i}(E_i^-)^{n-1} \prod_{i=1}^{m} E_{j_b^-})|\Lambda\rangle + (E_i^-)^n E_i^+ \prod_{i=1}^{m} E_{j_b^-}|\Lambda\rangle.
\]

When \( n = 0 \), the formula reduce to Eq. (3.15). When \( m = 0 \), we recover Eq. (3.14). An important fact found is that the following factor

\[
(\lambda_i - (n - 1) - \sum_{b=1}^{m} A_{j_b,i})
\]

is zero from time to time. When this factor is zero, the first term on the right hand side of the formula in the Proposition can be omitted. This potential zero term greatly decrease the computation work of communication of operators in \( W_w \).

We give one example to illustrate the vanishing property of the factor (3.16). First, we introduce following fact which is helpful in practical computation.

**Proposition 2** For the highest weight \( \Lambda_i = [0, \cdots, 1, \cdots, 0] \) of the fundament representation \( \rho_i \) and a state

\[
|\nu_w(\bar{\omega})\rangle = f(E_i^-)E_j^-|\Lambda_i\rangle, \ i \neq j
\]

where \( f \) is a polynomial function of the generators of Lie algebra \( g \), we have

\[
\langle g(E_i^+)|\nu_w(\bar{\omega})\rangle \equiv 0
\]

with arbitrary states \( \langle g(E_i^+)\rangle \).

**Proof:** First, we communicate all these operators in \( f(E_i^-) \) sequentially to the left side of all \( E_i^+ \), then these operators \( E_i^- \) annihilate the lowest weight state \( \langle \Lambda \rangle \). Because of the following identity

\[
H_k E_{k_n}^- \cdots E_{k_1}^-|\Lambda \rangle = c_k E_{k_n}^- \cdots E_{k_1}^-|\Lambda \rangle,
\]

these operators \( H_k \) appearing after the communicating \( [E_k^+, E_k^-] \) can be seen as undetermined constants \( c_k \). Then there are only operators \( E_i^+ \) and \( E_i^- \) left. If no operator \( E_i^+ \) is left to acting on \( E_j^-|\Lambda_i\rangle \), the operator \( E_j^- \) will communicate all the operators \( E_i^+ \) and annihilate the state \( \langle \Lambda \rangle \) which leads to the conclusion. If there is at least one \( E_j^+ \) left, we have

\[
\langle g(E_i^+)|\nu_w(\bar{\omega})\rangle = \langle \cdots E_j^+ E_j^-|\Lambda_i\rangle = \langle \cdots (H_j + E_j^- E_j^+)|\Lambda_i\rangle = 0,
\]

where \( H_j|\Lambda_i\rangle = 0 \) because \( i \neq j \). And \( E_j^+ \) annihilate highest weight state \( |\Lambda_i\rangle \).

**Example:** As shown by Fig. (1), there are four paths reaching weight \([-3, 1]\) from the highest weight \( \Lambda = [0, 1] \) in the fundament representation \( \rho_2 \) of \( G_2 \). The path that will be handed is
The Cartan matrix of property reduces much computation work. For the second term, we have, We calculate the following inner product which is part of $W_{[-3,1]}$. Using proposition 1, we have

$$\begin{align*}
W'_{[-3,1]} &= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ (E_1^+)^2 | (E_1^-)^2 E_2^- E_1^- E_2^- (E_1^-)^3 E_2^- | \Lambda \rangle \\
&= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ \{(-2 - 2(2A_{21} + 4A_{11})) E_1^- E_2^- E_1^- E_2^- (E_1^-)^3 E_2^- \} | \Lambda \rangle \\
&= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ \{(-2A_{21} - 3A_{11}) E_2^- (E_1^-)^3 E_2^- + (E_1^-)^2 E_2^- E_1^- E_2^- (-3 \cdot 2 - 3A_{21}) (E_1^-)^2 E_2^- \} | \Lambda \rangle \\
&= 3 \langle 4W'_{[-3,1]} + 2W'_{[-3,1]_2} + 4W'_{[-3,1]_3} \rangle
\end{align*}$$

In the last formula, we see that the first two terms within the braces are omitted because of the zero factor. The third term is

$$\begin{align*}
W'_{[-3,1]} &= 3 \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ (-2A_{21} + 3A_{11}) E_1^- E_2^- E_1^- E_2^- (E_1^-)^2 E_2^- \\
&= 3 \langle 4W'_{[-3,1]} + 2W'_{[-3,1]_2} + 4W'_{[-3,1]_3} \rangle
\end{align*}$$

where we denote the three none zero terms as $W'_{[-3,1]_1}, W'_{[-3,1]_2}, W'_{[-3,1]_3}$ respectively. For the first one, we have

$$\begin{align*}
W'_{[-3,1]_1} &= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ | E_1^- E_2^- E_1^- E_2^- (E_1^-)^2 E_2^- | \Lambda \rangle \\
&= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ \{(-3A_{12} - 2A_{22} + \lambda_2) E_1^- E_2^- (E_1^-)^3 E_2^- \} | \Lambda \rangle \\
&= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ (E_1^-)^2 E_2^- \{(-2A_{12} + 2A_{22} + \lambda_2) (E_1^-)^2 E_2^- \} | \Lambda \rangle \\
&= 3 \langle \Lambda | E_2^+ (E_1^+)^3 | (E_1^-)^3 E_2^- | \Lambda \rangle \\
&= 3 \cdot 36
\end{align*}$$

As expected, the factor $(\lambda_i - (n-1) - \sum_{b=1}^m A_{b,i})$ becomes zero frequently. This vanishing property reduces much computation work. For the second term, we have,

$$\begin{align*}
W'_{[-3,1]} &= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ E_2^+ (E_1^-)^2 E_2^- (E_1^-)^2 E_2^- | \Lambda \rangle \\
&= \langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ E_1^+ \{(-2 - 2(2A_{12} + A_{22}) + 2\lambda_2) E_2^- (E_1^-)^2 | \Lambda \rangle \\
&= 0
\end{align*}$$
For the third one, we have
\[ W'_{[-3,1]_3} = \langle \Lambda | E_1^+ (E_1^+)^3 E_2^+ E_1^+ \{(E_1^-)^2 (-2A_{12} - 2A_{22} + \lambda_2) E_1^- E_2^- E_2^+ \right) \\
+ (E_1^-)^2 E_1^+ (-A_{12} - A_{22} + \lambda_2) E_1^- E_2^- | \Lambda \rangle \\
= -\langle \Lambda | E_2^+ (E_1^+)^3 E_2^+ \{(3 \cdot 2 - 3(2A_{21} + A_{11})}(E_1^-)^2 E_1^- E_2^- \right) \\
+ (E_1^-)^3 E_2^- (-A_{12} - A_{22} + \lambda_2) E_2^- | \Lambda \rangle \\
= -\langle \Lambda | E_2^+ (E_1^+)^3 \{6(E_1^-)^2 (-A_{12} - A_{22} + \lambda_2) E_2^- E_2^+ + 6 (E_1^-)^3 E_2^- E_1^- \right) \\
+ 3(E_1^-)^3 (-2 + 2\lambda_2) E_2^- | \Lambda \rangle \\
= 0 \]

Combining all above results, the inner product is
\[ W'_{[-3,1]} = 3(4W'_{[-3,1]_1} + 2W'_{[-3,1]_2} + 4W'_{[-3,1]_3}) \]
\[ = 3(4 \cdot 3 \cdot 36 + 2 \cdot 0 + 4 \cdot 0) \]
\[ = 36 \cdot 36 \]

As we see, in the process of above computation, the term \( (\lambda_i - (n - 1) - \sum_{b=1}^{\infty} A_{j,b,i}) \) is to be zero frequently. We find it is a common phenomenon after performing many examples. By virtue of this vanishing factor, the computational efficiency is improved remarkably and the computation of the factor \( W_w \) in \( e^{-\chi_2(\sigma)} \) is simplified.

Unfortunately, there is another computation difficulty as pointed out in the end of section 3.1. We need to consider all the ways \( s \) reaching to \( w \) from the highest to define the vector \( |v_w(\tilde{\omega})\rangle \). As shown in Fig.2, each branch node increase the number of paths. There are ten paths reaching to weight \([0,0,0,0]\) from the highest weight to define the vector \( |v_{[0,0,0,0]}(\tilde{\omega})\rangle \). And twenty-five vectors \( |v_w(\tilde{\omega})\rangle \) need to be considered to compute \( e^{-\chi_2} \). With the rank of \( g \) rising, the number of paths reaching a weight from the highest weight as well as the number of weights are increasing rapidly. Note that we record the inner product of vector \( |v_{[-3,1]}(\tilde{\omega})\rangle \) in one page. But for vector \( |v_{[0,-1]}(\tilde{\omega})\rangle \), we need more than twenty pages to record the whole calculation process of the inner product. For these weights in the \( G_2 \) theory, we can compute the factors \( W_w \) by hand, but it is unrealistic to compute factors \( W_w \) by hand for Lie algebra of higher rank. In fact, it is even difficult for personal computer to work out \( e^{-\chi_2} \) in \( D_4 \) theory. However, in the next section, we find another construction of the solutions of \( KW \) equations for semisimple Lie algebras of \( ADE \) type. This new formula of solutions don’t involve factor \( W_w \). Thus it bypass computational difficulties contained in this factor \( W_w \).

### 4 Reconstruction of the solutions

In this section, we find an amazing formula that can be used to reformulate solutions of \( KW \) equations for Lie algebras of \( ADE \) type. The new formula of solutions can not only avoid computing the communication of operators but also avoid the difficulty related to the number of paths \( s \) in the definition of vector \( |v_w(\tilde{\omega})\rangle \) at same time. We will give an example and more results in Appendix to support our proposal.
Figure 2: Weights in the fundamental representation $\rho_s$ of $D_4$. Each branch node increases the number of paths s. There are ten paths reaching to weight $[0, 0, 0, 0]$ from the highest weight to define the vector $|v_{[0,0,0,0]}(\omega)\rangle$. Twenty-five vectors $|v_w(\omega)\rangle$ need to be considered to compute $e^{-\chi^2}$.

4.1 An amazing identity.

For a weight $w = \Lambda_s - \sum_{l=1}^{n(w)} a_{j_l}$, $a_{j_l} \in \Delta$ in the fundamental representation $\rho_s$, the corresponding vector $|v_w(\omega)\rangle$ Eq.(2.30) is

$$|v_w(\omega)\rangle = \sum_s \prod_{a=1}^{n(w)} \frac{1}{w(\omega) - w_a(\omega)} E_{j_n(w)}^{-} \cdots E_{j_1}^{-} |\Lambda_s\rangle.$$

Let’s consider term

$$\langle v_w(\omega) | v_w(\omega) \rangle \prod_{\beta_a \in \Delta_+} (\beta_a(\omega))^{-2\langle w, \beta_a \rangle / \langle \beta_a, \beta_a \rangle}, \quad (4.1)$$

we have following conjecture which can be used to construct solutions of $\text{KW}$ equations.

**Conjecture 1**

For a weight $w \in \Delta_s$ in the fundamental representation $\rho_s$. Denoting Eq.(3.7) as

$$F_w = \prod_{\beta_a \in \Delta_+} (\beta_a(\omega))^{-2\langle w, \beta_a \rangle / \langle \beta_a, \beta_a \rangle} = \prod_{\beta_a \in \Delta_+} \left( \sum_{i=1}^{\text{rank}(g)} a_i m_i \sum_{j=1}^{\text{rank}(g)} a_{i,j} A_{ij} \right)^{-\frac{1}{2}} = A_w / B_w, \quad (4.2)$$

where $A_w$ and $B_w$ have no common factor. For the simple-laced Lie algebras ($A, D, E_6, E_7, E_8$), if $w \neq [0, 0, \cdots, 0]$, we conjecture that

$$W_w \cdot F_w = \langle v_w(\omega) | v_w(\omega) \rangle \prod_{\beta_a \in \Delta_+} (\beta_a(\omega))^{-2\langle w, \beta_a \rangle / \langle \beta_a, \beta_a \rangle} \propto \frac{1}{A_w \cdot B_w}, \quad (4.3)$$

For $G_2$ gauge group with triple line in Dynkin diagram, we have found a similar result. For a weight $w \in \Delta_s$, $w \neq [0, 0, \cdots, 0]$ in the fundamental representation $\rho_s$ of $G_2$, we have

$$\langle v_w(\omega) | v_w(\omega) \rangle \prod_{\beta_a \in \Delta_+} (\beta_a(\omega))^{-2\langle w, \beta_a \rangle / \langle \beta_a, \beta_a \rangle} \propto \frac{1}{A_w \cdot B_w}, \quad (4.3)$$

where $A_w$ and $B_w$ are defined as in Eq.$(4.2)$. The solutions of $\text{KW}$ equation for $G_2$ gauge group have been completely determined in appendix of $[1]$ (Eq.(3.17) is part of $W_{[-3,1]}$ for the fundamental representation $\rho_s$ of $G_2$).

In the following, we reanalyze the example in section 3.1 to illustrate the identity $(4.2)$.

---

1 we present another conjecture related to the communication of Lie operators in the fundamental representation in the appendix.
**Example:** fundamental representation $\rho_1$ with highest weight $[1, 0]$ of $A_2$ theory.

- $[1, 0]$: from Eq.(3.8), we have
  \[ F_{[1, 0]} = \frac{1}{(m_1)(m_2 + m_1)} \]
  which implies $A_w = 1$ and $B_w = (m_1)(m_2 + m_1)$. According to Eq.(4.3), we have
  \[ W_{[1, 0]} \cdot F_{[1, 0]} = \frac{1}{A_w \cdot B_w} = \frac{1}{m_1(m_2 + m_1)}. \]

- $[-1, 1]$: from Eq.(3.10), we have
  \[ F_{[-1, 1]} = \frac{m_1}{m_2}. \]
  which implies $A_w = m_1$ and $B_w = m_2$. According to Eq.(4.3), we have
  \[ W_{[-1, 1]} \cdot F_{[-1, 1]} = \frac{1}{A_w \cdot B_w} = \frac{1}{m_1m_2}. \]

- $[0, -1]$: from Eq.(3.12), we have
  \[ F_{[0, -1]} = m_2(m_1 + m_2). \]
  which implies $A_w = m_2(m_1 + m_2)$ and $B_w = 1$. According to Eq.(4.3), we have
  \[ W_{[0, -1]} \cdot F_{[0, -1]} = \frac{1}{A_w \cdot B_w} = \frac{1}{m_2(m_1 + m_2)}. \]

These results of terms $W_{[1, 0]} \cdot F_{[1, 0]}$, $W_{[-1, 1]} \cdot F_{[-1, 1]}$ and $W_{[0, -1]} \cdot F_{[0, -1]}$ are all consistent with the results discussed in section 3.1.

For weight $w = [0, \cdots, 0]$, since $F_w = 1$ in Eq.(3.2), one could speculate that $A_w = B_w$ which means $W_w \cdot F_w = \frac{1}{w}$. However, this naive guess is not collect. A counterexample $W_{[0, 1, 0, 0]} \cdot F_{[0, 1, 0, 0]}$ in the fundamental representation $\rho_2$ of $D_4$ is given in Appendix B.

We can reformulate $e^{-\chi_s(\sigma)}$ using the boundary condition Eq.(2.23) when the weight $[0, \cdots, 0]$ is in the weight space $\Delta_s$ of the fundamental representation $\rho_s$. According to the boundary condition Eq.(2.23), we have
\[
\sum_w W_w \cdot F_w|_{\sigma=0} = 0.
\]
This implies
\[
W_{[0, \cdots, 0]} \cdot F_{[0, \cdots, 0]} = -\sum_{w'} W_{w'} \cdot F_{w'} = -\sum_{w'} \frac{1}{A_{w'} \cdot B_{w'}}, \tag{4.4}
\]
where $w'$ denotes the exclusion of $[0, \cdots, 0]$. Thus, we get another method to construct solutions of $KW$ equation as follow.
Proposition 3  For the simple-laced Lie group \((A, D, E_6, E_7, E_8)\), when \([0, 0, \cdots, 0]\) is not in the weight space \(\Delta_s\) of the fundament representation \(\rho_s\), then

\[
e^{-\chi_s(\sigma)} = 2^{-B_s} \sum_{w \in \Delta_s} [E_w \cdot W_w \cdot F_w] = 2^{-B_s} \sum_{w \in \Delta_s} \left[ E_w \cdot \frac{1}{A_w \cdot B_w} \right],
\]

where \(A_w\) and \(B_w\) are defined in Eq.\((4.2)\). When \([0, 0, \cdots, 0] \in \Delta_s\), using Eq.\((4.4)\), we have

\[
e^{-\chi_s(\sigma)} = 2^{-B_s} \left( \sum_{w \in \Delta_s'} \left[ E_w \cdot \frac{1}{A_w \cdot B_w} \right] + W_{[0, \cdots, 0]} \cdot F_{[0, \cdots, 0]} \right) = 2^{-B_s} \sum_{w \in \Delta_s'} \frac{1}{A_w \cdot B_w} [E_w - 1]
\]

where \(\Delta_s'\) denotes the exclusion of weight \([0, \cdots, 0]\) in \(\Delta_s\).

This new formula of solutions of \(KW\) equations only involve in terms \(A_w, B_w,\) and \(E_w\) which can be calculated by simple algebraic relationship. In the Appendix B, we compute the solutions of \(KW\) equations using above formula for several algebras. These solutions are all consistent with the results computed by the Mikhaylov’s conjecture.

5 Summary and open problems

In [1], Mikhaylov have conjectured solutions of \(KW\) equations for a boundary ’t Hooft operator. In order to prove the formula of solutions, we need to check a boundary condition. In this paper, we discuss the construction of these solutions further. There are two difficulties to construct the solutions Eq.\((2.32)\) explicitly. One is related to the communication of generators of Lie algebra in \(W_w\). With the rank of Lie algebra \(g\) rising, the calculation task is increases rapidly if communicating operators directly. We derived an identity (Proposition 1) which effectively simplifies the calculation. The computational efficiency is improved remarkably, since the following factor \((3.16)\)

\[
(\lambda_i - (n - 1) - \sum_{b=1}^{m} A_{js,b,i})
\]

is vanishing from time to time in the computation process. The other difficulty involves the number of paths \(s\) reaching a weight from the highest weight in the fundamental representation. With the rank of \(g\) increasing, the number of weight \(w\) in the fundament representation and paths \(s\) increase rapidly. We found a formula to represent factors \(W_w \cdot F_w\) by the co-prime numerator \(A_w\) and denominator \(B_w\) of \(F_w\). This formula can be used to construct the solutions of \(KW\) equations for gauge groups of \(ADE\) type. The solutions Eq.\((4.5),(4.6)\) don’t involve \(W_w\), so they bypass these two computational difficulties.

Clearly more work is needed. The proof of the formula of solutions \((2.32)\) for a general gauge group \(G\) is still an open problem. The Conjecture 1 also need to be proved. This conjecture imply that maybe we can prove the formula \((2.32)\) of solutions for gauge groups of \(D_n, E_n\) type by Mikhaylov’s proof in the \(A_n\) case. It is also interesting to construct solutions of \(KW\) equations for the boundary surface operator of arbitrary gauge group \(G\) on half space. Instead of one side boundary, we can consider a two-sided problem on \(\mathbb{R}^3 \times I\), where \(I\) is a compact interval with ’t Hooft operator or surface operator in the boundary [4].
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A Another conjecture

Conjecture 2 we present another observation as follow,

- If the weight space $\Delta_s$ of the fundamental representation $\rho_s$ don’t contain the weight $[0,0,\cdots,0]$, then for a weight $w \in \Delta_s$, we have
  
  $$\langle \Lambda_s | E_{i_1}^+ (w) \cdots E_{i_n}^+ (w) | E_{j_1}^- (w) \cdots E_{j_n}^- (w) | \Lambda_s \rangle = 1$$

  where $E_{j_n}^- (w) \cdots E_{j_1}^- (w)$ and $E_{i_n}^+ (w) \cdots E_{i_1}^+ (w)$ need’t to be the same sequences from the highest weight $\Lambda_s$ to weight $w$.

Example: for the fundament representation $\rho_2$ for $A_3$,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Weights in the fundament representation $\rho_2$ of $A_3$}
\end{figure}

We have the following two unit norms

$$\langle \Lambda_2 | E_2^+ E_1^+ E_3^+ | E_3^- E_1^- E_2^- | \Lambda_2 \rangle = 1, \quad \langle \Lambda_2 | E_2^+ E_3^+ E_1^+ | E_2^- E_1^- E_3^- | \Lambda_2 \rangle = 1$$

which correspond to two sequences from the highest weight $[0,1,0]$ to $w$ as shown in Fig.3. We have following inner product where the vector and the conjugate vector are corresponding to two different paths reaching weight $[-1,1,-1]$ from the highest weight,

$$\langle \Lambda_2 | E_2^+ E_1^+ E_3^+ | E_1^- E_2^- E_3^- | \Lambda_2 \rangle = 1, \quad \langle \Lambda_2 | E_2^+ E_3^+ E_1^+ | E_3^- E_1^- E_2^- | \Lambda_2 \rangle = 1$$

- If the weight space $\Delta_s$ of the fundamental representation $\rho_s$ contains zero weight $[0,0,\cdots,0]$, then for a weight $w \in \Delta_s, n(w) < n([0,0,\cdots,0])$, we have
  
  $$\langle \Lambda_s | E_{i_1}^+ (w) \cdots E_{i_n}^+ (w) | E_{j_1}^- (w) \cdots E_{j_n}^- (w) | \Lambda_s \rangle = 1$$

  where $E_{j_n}^- (w) \cdots E_{j_1}^- (w)$ and $E_{i_n}^+ (w) \cdots E_{i_1}^+ (w)$ don’t need to be the same sequences from the highest weight $\Delta_s$ to weight $w$. 

17
\[ B \quad \text{Summary of some relevant results} \]

In Appendix of [11], Mikhaylov collect solutions of KW equations for the algebras \( A_1, A_2, A_3, B_2 \) and \( G_2 \). We collect more explicit formulas for the \( 't \) Hooft operator solutions for other algebras. We present the completely solutions for \( A_4 \) and collect \( e^{-\chi_1(\sigma)} \) and \( e^{-\chi_2(\sigma)} \) for \( D_4 \). We check these results by Proposition [3]. In fact, \( e^{-\chi_2(\sigma)} \) for \( D_4 \) can’t be worked out by computer using Mikhaylov’s initial formula. Different methods lead to completely consistency results. We also present some results of \( F_4 \) algebra. We point out that the result \( e^{-\chi_1(\sigma)} \) for \( E_6 \) not presented here need more then five pages to record.

\[ A_4, \ [0, 0, 0, 0] \]

\[
\exp(-\chi_1) = \frac{1}{64} \left( -e^{1/4(m_1+2m_2+3m_3+3m_4)} + e^{1/6(m_1+2m_2+2m_3+3m_4)} - m_1 m_2 m_3 m_4 (m_1 + m_2 + m_3 + m_4) \right)
\]

\[
[0, 1, 0, 0] :
\exp(-\chi_2) = \frac{1}{64} \left( m_1 \left( m_1 + m_2 \right) \left( m_2 + m_3 \right) \left( m_1 + m_2 + m_3 + m_4 \right) \right)
\]

\[
[0, 0, 1, 0] :
\exp(-\chi_3) = \frac{1}{64} \left( m_1 \left( m_1 + m_2 \right) \left( m_2 + m_3 \right) \left( m_1 + m_2 + m_3 + m_4 \right) \right)
\]

\[
[0, 0, 0, 1] :
\exp(-\chi_4) = \frac{1}{64} \left( m_1 \left( m_1 + m_2 \right) \left( m_2 + m_3 \right) \left( m_1 + m_2 + m_3 + m_4 \right) \right)
\]
\[ F_4, [1, 0, 0, 0] : \\
\exp(-\chi_1) = 2^{-\left[0,1,0,0\right]}(EWF'(\sigma) + WF_{[0,0,0,0]}) \\
= \begin{bmatrix}
\frac{1}{3 \cosh(2m_2 m_1 m_4)} \\
\frac{m_1 m_2^2 m_3 m_4^2}{(m_1 + m_2)^2 (m_1 + m_3)^2} \\
\frac{m_1 m_2^2 m_3 m_4^2}{(m_1 + m_2)^2 (m_1 + m_3)^2} \\
\frac{12 (m_1^2 + 3m_1 m_2 + 3m_2^2)}{m_1^2 m_2^2 (m_1 + m_2)^2 (m_1 + m_3)^2 (m_2 + m_3)^2 (m_1 + 2m_2 + 3m_3)^2}
\end{bmatrix}
\]

We split the contribution \( \sum_{w \in \Delta_4} [E_w \cdot W_w \cdot F_w] \) in \([2.32]\) into two parts. One part is the term \( WF_{[0,0,0,0]} \) stands for the term \( E_w \cdot W_w \cdot F_w \) with weight \( w = [0, \ldots, 0] \). And the other part is the sum of other terms denoted as \( EWF'(\sigma) \). From the above expression of \( \exp(-\chi_1) \), it is easy to find

\[ WF_{[0,0,0,0]} = \frac{12 (m_1^2 + 3m_1 m_2 + 3m_2^2)}{m_1 m_2^2 m_3 m_4^2 (m_1 + m_2)^2 (m_1 + m_3)^2 (m_2 + m_3)^2 (m_1 + 2m_2 + 3m_3)^2}. \]

\[ D_4, [1, 0, 0, 0] : \\
\exp(-\chi_2) = 2^{-\left[0,1,0,0\right]}(EWF'(\sigma) + WF_{[0,0,0,0]}) \\
= \begin{bmatrix}
\frac{1}{3 \cosh(2m_1 m_4)} \\
\frac{m_1 m_2 m_3 m_4^2}{(m_1 + m_2)^2 (m_1 + m_3)^2 (m_2 + m_3)^2 (m_1 + 2m_2 + 3m_3)^2}
\end{bmatrix}
\]

\[ [0, 1, 0, 0] : \\
\exp(-\chi_3) = 2^{-\left[0,1,0,0\right]}(EWF'(\sigma) + WF_{[0,0,0,0]}) \\
= \begin{bmatrix}
\frac{1}{3 \cosh(2m_1 m_4)} \\
\frac{m_1 m_2 m_3 m_4^2}{(m_1 + m_2)^2 (m_1 + m_3)^2 (m_2 + m_3)^2 (m_1 + 2m_2 + 3m_3)^2}
\end{bmatrix}
\]

\[ -WF_{[0,0,0,0]} \]
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