Efficient simulation for incompressible turbulent flow using lattice Boltzmann model

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Abstract

Lattice Boltzmann Model is suitable for parallel computation. But it is known that a solution using LBGK model becomes unstable in the case using coarse mesh in high Reynolds number region. In order to overcome this shortcoming, Entropic Lattice Boltzmann model and Quasi-equilibrium Lattice Boltzmann model have been developed. However, comparison between the results obtained by these three methods for turbulence flow has not been done yet. In this study, we applied LBGK, ELBM and QELBM to 2-dimensional, incompressible, homogeneous, isotropic turbulence and compare the stability, accuracy and computational effort. As the result, enhancement of stability is confirmed by using ELBM and QELBM. Especially, it was found in the case of simulation using coarse grid, ELBM is more stable, and in the case of simulation at high Ma region, QELBM is more stable.

Keywords: Lattice Boltzmann model, Homogeneous decaying turbulence, incompressible flow, High Reynolds number.

1. Introduction

Lattice Boltzmann Model (LBM) has been developed as the method which can solve macro-scale fluid dynamics through meso-scale approach by calculating translation and collision of particle. The simple algorithm of LBM, which can solve the Navier-Stokes equation for incompressible flow without solving the Poisson equation of pressure and which only require the variables on the nearest neighbor grid point, is suitable for parallel computation. But it is known that a solution using the lattice Bhatnagar-Gross-Krook model (LBGK), which widely used as a single-time relaxation model of LBM, becomes unstable in the case of the calculation using coarse mesh in high Reynolds number flow region. In order to overcome this shortcoming, Entropic Lattice Boltzmann model (ELBM) \cite{1-9} and Quasi-equilibrium Lattice Boltzmann model (QELBM) \cite{10-12} have been developed. ELBM can enhance stability by satisfying the second principle of the thermodynamic by imposing the monotonicity and the minimality of the H-function, whereas QELBM adjusts the bulk viscosity by using two kinds of relaxation times to get the stable solution. The application of LBM to turbulence flow has been reported \cite{8}, however, comparison between LBGK, ELBM and QELBM has not been done yet. In this study, in order to check the suitability of the method for turbulence flow, LBGK, ELBM and QELBM are applied to 2-dimensional, incompressible, homogeneous, isotropic decaying turbulence and investigate the enhancement of stability, accuracy and computational effort.

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2. Numerical methods

2.1. Entropic lattice Boltzmann model

ELBM is different mainly on two points from standard LBGK model. First, the equilibrium distribution function is derived from minimization of $H$ function under the conserving of mass and momentum. The discrete $H$ function is given as follow

$$H(f) = \sum_{i=0}^{q-1} f_i \ln \left( \frac{f_i}{w_i} \right)$$

where, $f_i$ is the distribution function in $i$ direction, and $q$ is the number of direction of speed, in this model $q=9$. By calculating minimization of Eq.(1), local velocity equilibrium distribution function in $i$ direction $f_i^{eq}$ is obtained as product form as follow

$$f_i^{eq} = \rho \prod_{j=1}^{d} \left( 2 - \sqrt{1 + 3u_j^2} \right) \left( \frac{2u_j + \sqrt{1 + 3u_j^2}}{1 - u_j} \right)^{\gamma_q}$$

where, $\rho$ is the fluid density, $d$ is the number of spatial dimension, $u_j$ is the component of macroscopic velocity in $j$ direction. In the second point, the relaxation time of ELBM is locally adjusted in such a way that the monotonicity of the $H$-function is satisfied through the relaxation parameter $\alpha$. The parameter $\alpha$ is determined by solving following equation by which the monotonicity of the $H$-function is guaranteed.

$$H(f) = H(f + \alpha A)$$

where, $A$ represents the local non-equilibrium value of distribution function, $f^{eq}$ is $f$. Once $\alpha$ is given by solving Eq. (3), the distribution function at new time step can be obtained by following time developed lattice BGK equation

$$f_i(x+c_j,t+1) = f_i(x,t) + \frac{\alpha}{2\tau} [f_i^{eq}(x,t) - f_i(x,t)]$$

where, $\tau$ is the relaxation time used in LBGK and determined by following equation,

$$\nu = \frac{1}{6} (2\tau - 1)$$

where, $\nu$ is kinematic viscosity of fluid. As shown in right hand side of Eq.(4), when $\alpha = 2$, ELBM is equivalent to LBGK. In this study, in order to solve the nonlinear equation Eq. (5) with respect to $\alpha$, Newton-Raphson method is used.

2.2. Quasi-equilibrium lattice Boltzmann model

QELBM has been developed by Asinaly et al [10]. In QELBM, in order to enhance stability two kinds of relaxation processes are used for collision term as shown in Eq. (6).

$$\partial_t f + u \cdot \nabla f = \frac{1}{\tau_f} (f - f_c) - \frac{1}{\tau_s} (f_c - f_M)$$

where $f$ is velocity distribution function and $u$ is macroscopic velocity vector. The first term of right hand side in Eq.(6) is relaxation process from $f$ to a constraint equilibrium distribution function $f_c$ by relaxation time $\tau_f$ and the second term is one from $f_c$ to an equilibrium distribution function $f_M$ by relaxation time $\tau_s$. $f_M$ is determined by minimizing $H$-function under the condition so that conservation of mass and momentum are satisfied, whereas the $f_c$ is determined by minimizing $H$-function under the condition so that conservation of diagonal component of stress tensor is satisfied in addition to mass and momentum conservations. By using quasi-equilibrium distribution function $f_{QE}$ defined as follows

$$f_{QE} = \frac{\tau_f}{\tau_s} f_M + \left( 1 - \frac{\tau_f}{\tau_s} \right) f_c$$
Eq.(6) can be represented in a similar manner with LBGK model as follows.

\[ \partial_t f + u \cdot \nabla f = -\frac{1}{\tau_f} \left( f - f_{eq} \right) \]  

(8)

The relaxation times \( \tau_f \) and \( \tau_s \) have the relationship with kinematic viscosity \( \nu \) and bulk viscosity \( \xi \) through the sound speed \( c_s \), respectively, as follows,

\[ \nu = \tau_f c_s^2, \quad \xi = \tau_s c_s^2 \]  

(9)

Since \( \xi \) is free tunable parameter in the case of incompressible flow limit, QELBM can enhance stability by adjusting \( \xi \). As shown in Eq.(2)-Eq.(4), in the case that the kinematic viscosity is equal to bulk viscosity, QELBM is equivalent to LBGK model.

2.3. Computational conditions

In this study, the two-dimensional isotropic homogeneous decaying turbulence in a square computational region with a side length of \( L \) is calculated. This problem might be old-fashioned, however, useful for comparisons of basic feature of turbulence flow. The initial condition is given so that the following equation is satisfied,

\[ E(k) = \frac{1}{2} \sum |\vec{\omega}(k_1,k_2)|^2, \quad k^2 = k_1^2 + k_2^2 \]  

(10)

where \( \vec{\omega} \) is vorticity in wave space and \( E(k) \) is energy spectra defined as follows,

\[ E(k) = \frac{2}{3} k \exp \left( -\frac{2}{3} k \right) \]  

(11)

As boundary condition, the periodic boundary condition is imposed in \( x \) and \( y \) directions. the uniform grid with the interval of unity and D2Q9 velocity model is used as computational grid and velocity model, respectively. The four kinds of computational grid \( 64 \times 64, 128 \times 128, 256 \times 256, 1024 \times 1024 \) are used in order to check the effect of grid resolution. Two kinds of Mach number, \( \text{Ma} = 0.07, 0.10 \) and \( 0.14 \), based on sound of speed \( c_s \) and characteristic velocity \( U \) (=0.04, 0.06 and 0.08) is used. To simulate high Reynolds number turbulence, initial integral scale Reynolds number is set to \( R_L = 26457 \) which is expressed as \( R_L = \Omega \sqrt{\nu \eta} \), \( \Omega \) and \( \eta \) denote the total energy and the enstrophy dissipation rate, which are defined as

\[ \Omega(t) = \int_0^\infty E(k)dk \]  

(12)

\[ \eta(t) = 2\nu \int_0^\infty k^4 E(k)dk \]  

(13)

3. Results and discussions

3.1. Stability

Table 1 shows the minimum number of grid points required for obtaining stable solution for various \( \text{Ma} \) numbers. It is found by the table 1 that stability of ELBM is better than that of LBGK and QELBM at \( \text{Ma} = 0.07 \), whereas at \( \text{Ma}=0.14 \), stability of QELBM is better than that of the other methods. These differences of stability range might be caused by the difference of mechanisms for enhancing stability, that is, in order to enhance stability ELBM uses numerical viscosity, which is effective for instability due to the lack of grid resolutions, whereas, QELBM tunes the bulk viscosity, which is effective for suppressing the compressible wave occurring at high \( \text{Ma} \) number region.
Table 1 Minimum number of grid points required for obtaining stable solution

| Ma    | LBGK | ELBM | QELBM |
|-------|------|------|-------|
| 0.07  | 256×256 | 64×64 | 256×256 |
| 0.10  | N.A.  | 64×64 | 256×256 |
| 0.14  | N.A.  | N.A.  | 256×256 |

3.2. Flow pattern and Energy spectra

Fig.1 and Fig.2 show the vorticity distributions and energy spectra \( k^3E(k) \) obtained by LBGK, ELBM and QELBM in the case using the grid of 1024×1024. We can see that vorticity distributions and \( k^3E(k) \) obtained by the three methods agree well with each other. Therefore it is considered that the three methods are equivalent in the case of resolved simulation. Fig.3 shows vorticity distribution and \( k^3E(k) \) obtained by ELBM in the case using the coarse grid 128×128 at which LBGK and QELBM cannot get stable solution. It was found that despite the coarse grid, vortices and \( k^3E(k) \) is reasonably captured within the grid resolution. ELBM result using 64×64 grid, which is not shown in figure, cannot capture the vortex structure accurately due to bad grid resolution. Fig.4 shows vorticity distribution and \( k^3E(k) \) obtained by QELBM at Ma=0.14. It is found that by using QELBM, vorticity distributions and \( k^3E(k) \) can be correctly predicted even in high Ma number region.

![Fig. 1. Comparison of vorticity distribution between LBGK, ELBM and QELBM (*=1.0, 1024×1024, Ma=0.07).](image)

![Fig.2. Comparison of energy spectra between LBGK, ELBM and QELBM (*=1.0, 1024×1024, Ma=0.07).](image)
3.3. Computational effort

The CPU time of three methods are compared. The comparisons were carried out by the single core use calculation. As shown in this table 2, LBGK is the fastest in the case of the same grid resolution and the same Ma number because of the additional calculations for collision term of ELBM and QELB, but when we compare the CPU time under the condition of grid resolution and Ma number where stable and reasonable solution can be obtained, CPU time of ELBM and QELBM becomes shorter than that of LBGK model as shown in table 3. Especially, CPU time of ELBM is reduced until about 30% of LBGK model. Whereas CPU time of QELBM is not dramatically shorter than that of LBGK model, however, in the case of 3-D calculation, since QELBM can use smaller velocity model, D3Q13 model, which cannot be used single relaxation time model such as ELBM and LBGK, therefore it is expected that the advantage of QELBM will further expands[11-12].

Table 2 Comparison of CPU time at the same grid resolution and Ma number ($t^*=1.0$)

|                | CPU time ($\times 10^3$) | Ratio |
|----------------|--------------------------|-------|
| LBGK(1024×1024, Ma=0.7) | 1.94                      | 1.0   |
| ELBM(1024×1024, Ma=0.7)  | 7.83                      | 4.0   |
| QELBM(1024×1024, Ma=0.7) | 3.69                      | 1.9   |
Table 3 Comparison of CPU time at the condition that stable and reasonable solution can be obtained ($t^* = 1.0$)

| Model                      | CPU time | Ratio |
|----------------------------|----------|-------|
| LBGK($256 \times 256$, $Ma = 0.7$) | 30.3     | 1.0   |
| ELBM($128 \times 128$, $Ma = 0.10$) | 10.2     | 0.34  |
| QELBM($256 \times 256$, $Ma = 0.14$) | 28.8     | 0.95  |

4. Conclusions

In this study, we applied LBGK, ELBM, and QELBM to 2-dimensional, incompressible, homogeneous, isotropic decaying turbulence and compare the stability, accuracy, and computational effort. Enhancement of stability for LBGK model is found by using ELBM and QELBM. Especially, in the simulation using coarse grid such as $128 \times 128$, ELBM is the most stable method and in the simulation at high $Ma$ region such as $Ma = 0.14$, QELBM is the most stable one. By comparing flow pattern and energy spectra, it was found that QELBM simulation at high $Ma$ number and ELBM one using low grid resolution can get comparable results with those of LBGK model at low $Ma$ number using fine grid. For computational effort, LBGK is the fastest in the case of the same grid resolution and the same $Ma$ number, but when we consider the condition of grid resolution and $Ma$ number where stable and reasonable solution can be obtained, ELBM and QELBM is more efficient than that of LBGK model.

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