On asymptotic continuity of functions of quantum states

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A useful kind of continuity of quantum states functions in asymptotic regime is so-called asymptotic continuity. In this paper we provide general tools for checking if a function possesses this property. First we prove equivalence of asymptotic continuity with so-called robustness under admixture. This allows us to show that relative entropy distance from a convex set including maximally mixed state is asymptotically continuous. Subsequently, we consider arrowing - a way of building a new function out of a given one. The procedure originates from constructions of intrinsic information and entanglement of formation. We show that arrowing preserves asymptotic continuity for a class of functions (so-called subextensive ones). The result is illustrated by means of several examples.

I. INTRODUCTION

One of basic issues of Quantum Information Theory is to evaluate operational quantities such as capacities of quantum (usual of teleportation) channel [1, 2], compression rates [3] or localisable information rates [4, 5]. The quantities are usually defined in spirit of Shannon – in asymptotic regime of many uses of channel or many copies of state. Apart from such operational quantities one also considers mathematical functions, that are expected to reflect somehow those features of states or channels. To this end, one chooses functions, that satisfy some requirements. For example, most of entanglement measures are mathematical functions, that do not increase under local operations and classical communication [3, 8]. Other examples are correlation measures (see e.g. [6, 7, 8]). Such functions turn out to be very useful, as they often provide upper or lower bounds for operational quantities. in asymptotic regime, the functions are especially useful, if they are asymptotically continuous. The prototype for asymptotic continuity is Fannes inequality [3] for von Neumann entropy $S(\rho) = -\text{Tr} \log \rho$, which says that for any states $\rho$ and $\sigma$ with $||\rho - \sigma||_1 \leq 1/2$ we have

$$|S(\rho) - S(\sigma)| \leq ||\rho - \sigma||_1 \log d + \eta(||\rho - \sigma||_1)$$

where $\eta(x) = -x \log x$, $d$ is dimension of Hilbert space. The important feature of this stronger form of continuity, is that the right-hand-side scales logarithmically with dimension of Hilbert space. This kind of inequality, was first applied in quantum information theory in [14, 15] to provide lower bound for compression rates of mixed signal states (interestingly, the question of achievability of the bound is in general still open). Subsequently, it was applied to entanglement theory [16, 17], which lead, in particular, to methods of providing bounds for distillable entanglement and entanglement cost [17, 18]. Asymptotic continuity has become an important tool in proving irreversibility of pure states transformations (see [18] and references therein).

In [19, 20] two measures of entanglement have been proven to satisfy Fannes-like inequality (i.e. to be asymptotically continuous) – entanglement of formation $E_F$ [3] and relative entropy of entanglement [21]. In [22] asymptotic continuity of conditional entropy $S(A|B) = S(B|A) - S(AB) = S(\rho_B) - S(\rho_{AB})$ have been proven, where the right-hand-side depend only on dimension of system $A$. This allowed to prove asymptotic continuity of third measure of entanglement – squashed entanglement [22]. The importance of asymptotic continuity was made even more transparent in [24] where it was shown, that a convex and so called subextensive function, if not asymptotically continuous, it behaves in a quite weird way: namely, after removing one qubit, it can change at arbitrarily large amount. Clearly, it is very important to know whether a function is asymptotically continuous or not. Yet it is usually rather a difficult task. The aim of this paper is to provide general tools for checking asymptotic continuity. First, we show that the latter is equivalent to so called "robustness under admixtures", i.e. a function is asymptotically continuous, if it does not change too much under admixing any state with a small weight. Using it, we prove that relative entropy distance from any convex set including maximally mixed state is asymptotically continuous, extending therefore result of [20] where it was proven for compact and convex sets.

Next, we consider a procedure, called arrowing, of building new functions out of given functions. The procedure originates both from classical privacy theory [25, 26] – where the prototype was so-called intrinsic information – as well as from entanglement theory, since it includes as a special case the other procedure called convex roof [27], the prototype of which was entanglement of formation [3]. Since arrowing is commonly used in different contexts (see quite recent application [28]), it is important to be able to check the properties of arrowed versions of different functions. We provide here a quite general result, showing that for subextensive functions such procedure preserve asymptotic continuity, i.e. if an original function is asymptotically continuous, so is its "arrowed" version. We then apply it to show, that some tripartite entanglement measure [18, 29] as well as so called mixed convex roof of quantum mutual information introduced in [25] are asymptotically continuous.
II. BASIC DEFINITIONS

In this section we will introduce some definitions which we will use throughout this paper. **Set of states.** A positive operator \( \rho \in \mathcal{S} \) with \( \text{Tr}\rho = 1 \), acting on Hilbert space \( \mathcal{H} \) we will call state. Set of all states will be denoted by \( \mathcal{S}(\mathcal{H}) \). (We will deal with finite dimensional Hilbert spaces). A state is called pure, if it is of the form \( |\psi\rangle\langle\psi| \) where \( \psi \in \mathcal{H} \). Otherwise it is called mixed state.

**Von Neumann entropy** \( S(\rho) \) for a state \( \rho \) is given by formula:

\[
S(\rho) = -\text{Tr}\rho \log \rho
\]

(2)

We use base 2 logarithm in this paper.

**Relative entropy** for states \( \rho \) and \( \sigma \) is defined as:

\[
S(\rho|\sigma) = \text{Tr}\rho \log \rho - \text{Tr}\rho \log \sigma
\]

(3)

**Trace norm** of an operator \( A \) is given by:

\[
||A||_1 = \text{Tr}\sqrt{AA^\dagger}
\]

(4)

where \( A^\dagger \) stands for Hermitian conjugation.

**Measurement.** We will consider measurements with finite number of outcomes, represented by finite sets of operators \( \mathcal{M} = \{A_i\} \) satisfying \( \sum_i A_i A_i^\dagger = I \). Slightly abusing terminology, we will call the measurements POVMs (Positive Operator Valued Measure).

**Subextensivity** A function \( f : \mathcal{S}(\mathcal{H}) \to \mathbb{R} \) is subextensive if

\[
\forall_{\rho} \exists M \quad f(\rho) \leq M \log d
\]

(5)

where \( M \) is constant, \( d = \dim \mathcal{H} \).

**Definition 1** Let \( f \) be a real-valued function \( f : \mathcal{S}(\mathcal{C}^d) \to \mathbb{R} \) and \( \rho_1, \rho_2 \) are states acting on Hilbert space \( \mathcal{C}^d \) and \( \varepsilon = ||\rho_1 - \rho_2||_1 \). Then a function is asymptotically continuous if fulfills the following condition

\[
\forall_{\rho_1,\rho_2} |f(\rho_1) - f(\rho_2)| \leq K_1\varepsilon \log d + O(\varepsilon)
\]

(6)

where \( K_1 \) is constant and \( O(\varepsilon) \) is any function, which satisfies the condition that \( O(\varepsilon) \) converges to 0 when \( \varepsilon \) converges to 0 and depends only on \( \varepsilon \). (In particular, it does not depend on dimension).

**Definition 2** Let \( f \) be a real-valued function \( f : \mathcal{S}(\mathcal{C}^d) \to \mathbb{R} \) and \( \rho_1, \rho_2 \) are states acting on Hilbert space \( \mathcal{C}^d \). Then a function is robust under admixtures if

\[
\forall_{\rho_1,\rho_2} \forall_{\delta>0} |f((1-\delta)\rho_1 + \delta\rho_2) - f(\rho_1)| \leq K_2\delta \log d + O(\delta)
\]

(7)

where \( K_2 \) is constant and \( O(\delta) \) is any function, which satisfies the condition that \( O(\delta) \) converges to 0 when \( \delta \) converges to 0 and depends only on \( \delta \). (In particular, it does not depend on dimension).

**Remark.** Notice that usually for asymptotic continuity or robustness under admixtures we will not require fulfilling conditions (6) and (7) for whole range of \( \varepsilon \) or \( \delta \). We will rather restrict to some limited subset of positive real value of \( \varepsilon \) or \( \delta \) (limited by 1 or \( \frac{1}{2} \), for example.)

III. ASYMPTOTIC CONTINUITY AND ROBUSTNESS UNDER SMALL ADMIXTURES.

In this section we prove equivalence between asymptotic continuity and robustness under admixtures of function. This is an extension of result of [24], where it is proved that if a function \( f \), under admixtures does not change more than a constant, and subextensive then is also asymptotically continuous.

**Proposition 1** Let \( f \) be a function \( f : \mathcal{S}(\mathcal{C}^d) \to \mathbb{R} \) then the function is asymptotically continuous if only if is robust under admixtures.
Remark. This proposition can be also proved when we do not require "Lipschitz type" continuities, but rather "Cauchy type" ones. (See appendix)

Proof.

\(\Rightarrow\)

We assume that function is asymptotically continuous. This implies

\[
|f((1-\delta)\varrho_1 + \delta \varrho_2) - f(\varrho_1)| \leq K_1|\varrho_1 - ((1-\delta)\varrho_1 + \delta \varrho_2)||_1 \log d + O(||\varrho_1 - ((1-\delta)\varrho_1 + \delta \varrho_2)||_1) = \tag{8}
\]

\[
= K_1|\delta \varrho_1 - \delta \varrho_2||_1 \log d + O(||\delta \varrho_1 - \delta \varrho_2||_1) \leq 2K_1 \delta \log d + O(\delta) \tag{9}
\]

Lets take \(K_2 = 2K_1\). Then

\[
|f((1-\delta)\varrho_1 + \delta \varrho_2) - f(\varrho_1)| \leq K_2 \delta \log d + O(\delta) \tag{10}
\]

\(\Rightarrow\)

We will base on result of Refs. \([22]\) (see also \([30]\)), which can be viewed as a sort of generalized Tales theorem

\[
\forall \varrho_1, \varrho_2 \exists \sigma, \gamma_1, \gamma_2 \quad \sigma = (1-\varepsilon)\varrho_1 + \varepsilon \gamma_1 = (1-\varepsilon)\varrho_2 + \varepsilon \gamma_2 \tag{11}
\]

where \(\varrho_1, \varrho_2, \sigma, \gamma_1, \gamma_2\) are states acting on Hilbert space and \(\varepsilon = ||\varrho_1 - \varrho_2||_1\). Using it we obtain:

\[
|f(\varrho_2) - f(\varrho_1)| \leq |f(\varrho_2) - f(\sigma)| + |f(\sigma) - f(\varrho_1)| \tag{12}
\]

\[
|f((1-\varepsilon)\varrho_2 + \varepsilon \gamma_2) - f(\varrho_2)| + |f((1-\varepsilon)\varrho_1 + \varepsilon \gamma_1) - f(\varrho_1)| \leq 2K_2 \varepsilon \log d + 2O(\varepsilon) \tag{13}
\]

so that we can take \(K_1 = 2K_2\). Then

\[
|f(\varrho_2) - f(\varrho_1)| \leq K_1 \varepsilon \log d + O(\varepsilon) \tag{14}
\]

This ends the proof.

A. Application: asymptotic continuity of relative entropy distance from convex set of states.

In \([20]\) it was shown that so called relative entropy distance from convex, compact set including maximally state \(\varrho\) is asymptotically continuous. The proof was quit involved. Here, basing on Proposition 1 we present a more general result, where we do not require compactness of the set. Moreover our proof is straighter.

Relative entropy of distance \(E^D_R\) is defined as follows

\[
E^D_R(\varrho) = \inf_{\sigma \in D} S(\varrho|\sigma) \tag{15}
\]

where \(D\) is a convex set of state including maximally mixed state, \(\varrho \in \mathcal{C}^d\).

We start with lemma:

Lemma 1 Relative entropy of distance \(E^D_R\) fulfills the following condition

\[
|E^D_R((1-\varepsilon)\varrho + \varepsilon \sigma) - E^D_R(\varrho)| \leq 2\varepsilon \log d + H(\varepsilon) \tag{16}
\]

where \(H(\varepsilon) = -\varepsilon \log \varepsilon - (1-\varepsilon) \log(1-\varepsilon)\)

Proof.

First we show the that \(E^D_R\) satisfies the following inequality

\[
\sum_k p_k E^D_R(\varrho_k) - E^D_R(\sum_k p_k \varrho_k) \leq S(\sum_k p_k \varrho_k) - \sum_k p_k S(\varrho_k) \tag{17}
\]

This fact was shown for relative entropy distance from separable states in \([31]\), but it is also true for relative entropy distance from any convex set of states. Here we repeat this proof for \(E^D_R\) defined in \([15]\). Notice that for \(\varrho = \sum_k p_k \varrho_k\)

\[
S(\varrho|\sigma) = S(\sum_k p_k \varrho_k|\sigma) = \text{Tr}(\sum_k p_k \varrho_k \log(\sum_k p_k \varrho_k) - \sum_k p_k \varrho_k \log \sigma) = \tag{18}
\]

\[
= \text{Tr}(\sum_k p_k (\varrho_k \log \varrho_k - \varrho_k \log \sigma + \varrho_k \log \varrho - \varrho_k \log \varrho_k)) = \tag{19}
\]

\[
= \sum_k p_k S(\varrho_k|\sigma) + \sum_k p_k S(\varrho_k) - S(\varrho) \tag{20}
\]
Let \( \sigma \in \mathcal{D} \) be a state such that \( E_R^\sigma = S(\rho) - \delta \). Then we can rewrite
\[
E_R(\rho) = \sum_k p_k S(\rho_k|\sigma) + \sum_k p_k S(\rho_k) - S(\rho) - \delta \geq \sum_k p_k E_R(\rho_k) + \sum_k p_k S(\rho_k) - S(\rho) - \delta
\]  
\tag{21}
\]
Since by definition of \( E_R^\sigma \) \( \delta \) can be arbitrarily small, we obtain
\[
\sum_k p_k E_R(\rho_k) - E_R(\sum_k p_k \rho_k) \leq S(\sum_k p_k \rho_k) - \sum_k p_k S(\rho_k)
\]  
\tag{22}
We use also fact that \( 32 \)
\[
S(\sum_k p_k \rho_k) \leq \sum_k p_k S(\rho_k) + H(\{p_k\})
\]  
\tag{23}
and that relative entropy distance is convex function, what is implied by convexity of quantum relative entropy in two arguments. Notice also that \( E_R \) is bounded by \( \log d \), because \( \mathcal{D} \) includes maximally mixed state (so \( E_R \leq S(\rho|_{\mathcal{D}}^\perp) = \log d - S(\rho) \leq \log d \)). Then we have
\[
|E_R((1 - \varepsilon) \rho + \varepsilon \sigma) - E_R(\rho)| = |E_R((1 - \varepsilon) \rho + \varepsilon \sigma) - (1 - \varepsilon) E_R(\rho) - \varepsilon E_R(\sigma) - \varepsilon E_R(\rho) + \varepsilon E_R(\sigma)| \leq \log d - S(\rho) \leq \log d. \]
\tag{24}
\[
|E_R((1 - \varepsilon) \rho + \varepsilon \sigma) - (1 - \varepsilon) E_R(\rho) - \varepsilon E_R(\sigma)| + \varepsilon |E_R(\rho)| + \varepsilon |E_R(\sigma)| \leq S((1 - \varepsilon) \rho + \varepsilon \sigma) - (1 - \varepsilon) S(\rho) - \varepsilon S(\sigma) + \varepsilon \log d + \varepsilon \log d \leq H(\varepsilon) + 2\varepsilon \log d \tag{27}
\]
This ends the proof.

**Remark.** Note that the main feature of \( E_R^\sigma \) responsible for robustness under admixtures, are the following:
1) \( E_R^\sigma \) satisfy inequality :
\[
|E_R(\sum_k p_k \rho_k) - \sum_k p_k E_R(\rho_k)| \leq H(\{p_k\}) \tag{28}
\]
2) \( E_R^\sigma \) is bounded by \( \log d \).

**Lemma 2** Relative entropy of distance \( E_R \) is asymptotic continuous i.e.
\[
|E_R(\rho) - E_R(\sigma)| \leq 4\varepsilon \log d + 2H(\varepsilon) \tag{29}
\]
where \( H(\varepsilon) = -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon) \) and \( \varepsilon = ||\rho - \sigma||_1 \).

**Proof.**
\( E_R \) is robust under admixtures so under Proposition \( 1 \) is also asymptotically continuous.

**IV. ASYMMETRIC CONTINUITY OF FUNCTIONS BUILT BY ”ARROWING”**

In this section we consider "arrowing" — a construction that from given function \( f \) creates a new function denoted by \( f_\downarrow \). The definition is motivated by intrinsic information and its generalizations \( 32, 34, 35 \). The new function \( f_\downarrow \) is defined on enlarged system as follows

**Definition 3** For any function \( f : S(\mathcal{H}_X) \to R \) acting on states of system \( X \), we define function \( f_\downarrow : S(\mathcal{H}_X \otimes \mathcal{H}_E) \to R \) as follows
\[
f_\downarrow(\rho_{XE}) = \inf_{\{A_i\}} \sum_i p_i f(\rho_X^i)
\]  
\tag{30}
\]
where infimum is taken over all finite POVM’s \( \{A_i\} \) performed on system \( E \) and
\[
p_i = \text{Tr}(I_X \otimes A_i) \rho_{XE}, \quad \rho_X^i = \frac{1}{p_i} \text{Tr}_E(I_X \otimes A_i, \rho_{XE} I_X \otimes A_i^\dagger) \tag{31}
\]
i.e. \( p_i \) is probability of outcome \( i \), and \( \rho_X^i \) is the state of system \( X \) given outcome \( i \) was obtained.
Remark. We can define modified version of previous function as follows:

**Definition 4** For any function \( f : \mathcal{S}(\mathcal{H}_X) \rightarrow \mathbb{R} \) acting on states of system \( X \), we define function \( f_\uparrow : \mathcal{S}(\mathcal{H}_X \otimes \mathcal{H}_E) \rightarrow \mathbb{R} \) as follows

\[
f_\uparrow(\rho_{XE}) = \sup A_i \sum p_i f(\rho_X^i) \tag{32}
\]

where supremum is taken over all finite POVM’s \( \{A_i\} \) performed on system \( E \) and

\[
p_i = \text{Tr}(I_X \otimes A_i) \rho_{XE}, \quad \rho_X^i = \frac{1}{p_i} \text{Tr}_E(I_X \otimes A_i \rho_{XE} I_X \otimes A_i^\dagger) \tag{33}
\]

i.e. \( p_i \) is probability of outcome \( i \), and \( \rho_X^i \) is the state of system \( X \) given outcome \( i \) was obtained.

All features of \( f_\downarrow \) presenting in this paper are also valid for function \( f_\uparrow \).

We have the following lemma, which is proven in Sec. IX:

**Lemma 3** The infimum in the definition of \( f_\downarrow \) is achievable.

We will show in this section, that that asymptotic continuity and subextensivity of function \( f \) implies asymptotic continuity of \( f_\downarrow \). Thus in a sense, arrowing preserves asymptotic continuity. Let us stress that all the involved systems are finite-dimensional.

We will need the following definition:

**Definition 5** Given a function \( f \) defined on states of a system \( X \), we define its conditional version \( F \) for a quantum-classical state of a system \( XE \)

\[
\rho_{XE}^{qc} = \sum p_i \rho_X^i \otimes |i\rangle_E \langle i| \tag{34}
\]

as follows

\[
F(\rho_{XE}^{qc}) = \sum p_i f(\rho_X^i) \tag{35}
\]

If the quantum classical state was obtained from state \( \rho_{XE} \) by a POVM \( \mathcal{M} \) performed on system \( E \) we will also use notation \( F(\rho_{XE}, \mathcal{M}) \equiv F(\rho_{XE}^{qc}) \).

Let us now present the main result of this section.

**Proposition 2** Let \( f \) be a function defined on states of system \( X \), which is subextensive and asymptotic continuous. Then function \( f_\downarrow \) is also asymptotically continuous. Moreover, the constant in asymptotic continuity condition depends only on dimension of system \( X \).

**Proof.**

Let \( \rho_{XE} \) and \( \sigma_{XE} \) be states and \( \varepsilon = \|\rho_{XE} - \sigma_{XE}\|_1 \). Let \( \mathcal{M}_\rho = \{A_k^\rho\} \) and \( \mathcal{M}_\sigma = \{A_k^\sigma\} \) be the optimal measurements for \( \rho \) and \( \sigma \) respectively (i.e. the ones achieving infimum in definition of \( f_\downarrow \)) where \( \sum_k A_k^\rho A_k^\rho\dagger = I_E, \sum_k A_k^\sigma A_k^\sigma\dagger = I_E \).

For measurement \( \mathcal{M}_\sigma \), let \( p_k \) and \( q_k \) be probabilities of outcomes if a state was \( \rho \) and \( \sigma \) respectively. The resulting states on system \( X \), given the outcome was \( k \), we will denote by \( \rho_k \) and \( \sigma_k \) respectively. Due to asymptotic continuity (see sec. III) we assume that

\[
|f(\rho_{XE}) - f(\sigma_{XE})| \leq K\varepsilon \log d_X + O(\varepsilon) \tag{36}
\]

and due to subextensivity

\[
|f(\rho)| \leq M \log d_X \tag{37}
\]
for any state \( \rho \) on system \( X \), where \( d_X = \dim \mathcal{H}_X \) and \( M \) and \( K \) are constants. Then we have the following estimate

\[
f_4(\rho_{XE}) - f_4(\sigma_{XE}) = F(\rho_{XE}, \mathcal{M}_\phi) - F(\sigma_{XE}, \mathcal{M}_\sigma) \leq F(\rho_{XE}, \mathcal{M}_\sigma) - F(\sigma_{XE}, \mathcal{M}_\sigma) = \tag{38}
\]

\[
= \sum_k p_k f(\rho^k_X) - \sum_k q_k f(\sigma^k_X) \leq \left| \sum_k p_k f(\rho^k_X) - \sum_k q_k f(\sigma^k_X) \right| = \tag{39}
\]

\[
\geq \left| \sum_k p_k f(\rho^k_X) - p_k f(\sigma^k_X) + p_k f(\sigma^k_X) - q_k f(\sigma^k_X) \right| \leq \tag{40}
\]

\[
\leq \sum_k \left( p_k |f(\rho^k_X) - f(\sigma^k_X)| + |p_k - q_k||f(\sigma^k_X)| \right) \leq \tag{41}
\]

\[
\leq \sum_k p_k \varepsilon_k K \log d_X + \varepsilon M \log d_X + O(\varepsilon) \leq K_1 \varepsilon \log d_X + O(\varepsilon) \tag{42}
\]

where \( \varepsilon_k = ||\rho^k_X - \sigma^k_X||_1 \) and \( K_1 = 2K + M \). The last two steps of the above estimate are implied by asymptotic continuity, subextensivity of the function \( f \) and the following facts (see 22):

\[
\sum_k |p_k - q_k| \leq \varepsilon \tag{43}
\]

and

\[
\sum_k p_k \varepsilon_k \leq 2\varepsilon \tag{44}
\]

The inequality \( 42 \) we get via the following estimate

\[
\sum_k |p_k - q_k| = \left\| \sum_k p_k |k\rangle \langle k| - \sum_k q_k |k\rangle \langle k| \right\|_1 \leq \left\| \sum_k p_k \rho^k_X \otimes |k\rangle \langle k| - \sum_k q_k \sigma^k_X \otimes |k\rangle \langle k| \right\|_1 = \tag{45}
\]

\[
|||I_X \otimes \Lambda_\sigma) \rho_{XE} - (I_X \otimes \Lambda_\sigma) \sigma_{XE}||_1 \leq ||\rho_{XE} - \sigma_{XE}||_1 = \varepsilon. \tag{46}
\]

where \( \Lambda_\sigma \) is a completely positive map induced by POVM \( \mathcal{M}_\sigma \) as follows

\[
\Lambda_\sigma(\cdot) = \sum_k \text{Tr}[A_k^\sigma(\cdot) A_k^{\sigma \dagger}] |k\rangle \langle k| \tag{47}
\]

We have used here the fact that trace norm does not increase under completely positive trace preserving maps 30. The inequality \( 44 \) is proven as follows

\[
\varepsilon = ||\rho_{XE} - \sigma_{XE}||_1 \geq \sum_k |p_k \rho^k_X \otimes |k\rangle \langle k| - q_k \sigma^k_X \otimes |k\rangle \langle k| ||_1 = \sum_k ||p_k \rho^k_X - q_k \sigma^k_X||_1 \tag{48}
\]

\[
\geq \sum_k \left( ||p_k \rho^k_X - \rho^k_X||_1 - ||p_k \sigma^k_X - q_k \sigma^k_X||_1 \right) \tag{49}
\]

\[
= \sum_k p_k ||\rho^k_X - \sigma^k_X||_1 - \sum_k |p_k - q_k| \geq \sum_k p_k \varepsilon_k - \varepsilon \tag{50}
\]

Analogously we can show that

\[
f_4(\sigma_{XE}) - f_4(\rho_{XE}) = F(\sigma_{XE}, \mathcal{M}_\sigma) - F(\rho_{XE}, \mathcal{M}_\sigma) \leq F(\sigma_{XE}, \mathcal{M}_\sigma) - F(\rho_{XE}, \mathcal{M}_\sigma) \leq K_1 \varepsilon \log d_X + O(\varepsilon) \tag{51}
\]

Thus we obtain

\[
|f_4(\rho_{XE}) - f_4(\sigma_{XE})| \leq K_1 \varepsilon \log d_X + O(\varepsilon) \tag{52}
\]

This ends the proof.

**Remark.** In the proof we have used the fact that the infimum in definition of \( f_4 \) is achievable. However it is not essential: the proof that does not use it is very similar to the above one.

Finally, consider modification of the function \( f_4 \), where we do not optimize over all POVM’s, but only over complete POVM’s, for which the operators \( A_k \) are of rank one.
Definition 6 For any function $f : S(\mathcal{H}_X) \to R$ acting on states of system $X$, we define function $f_{\text{cpl}}^\downarrow : S(\mathcal{H}_X \otimes \mathcal{H}_E) \to R$ as follows

$$f_{\downarrow}^\text{cpl}(\rho_{XE}) = \inf_{\{A_i\}} \sum p_i f(\rho_{iX}^i)$$

where infimum is taken over all finite POVM’s $\{A_i\}$ with elements $A_i$ being of rank one. The notation is the same as in Def. 3

Again, the infimum in the above definition can be achieved, see Sec. IX. We then obtain

Proposition 3 Let $f$ be a function defined on states of system $X$, which is subextensive and asymptotic continuous. Then function $f_{\downarrow}^\text{cpl}$ is also asymptotically continuous. Moreover, the constant in asymptotic continuity condition depends only on dimension of system $X$.

The proof is analogous to the proof of Prop. 2

V. APPLICATIONS

A. Measure of classical correlation $C_{\rightarrow}$

This proposition implies asymptotic continuity of measure of classical correlation $C_{\rightarrow}$ defined as follows [9]:

$$C_{\rightarrow}(\rho_{AB}) = \max_{B_i^i|B_i} S(\rho_A) - \sum p_i S(\rho_A^i)$$

where $B_i^i|B_i$ is a POVM performed on subsystem $B$, $\rho_A^i = tr_B(I \otimes B_i \rho_{AB} I \otimes B_i^i)$ is remaining state of $A$ after obtaining the outcome $i$ on $B$, and $p_i = tr_{AB}(I \otimes B_i \rho_{AB} I \otimes B_i^i)$. Notice that we can rewrite $C_{\rightarrow}$:

$$C_{\rightarrow}(q_{AB}) = \max_{B_i^i|B_i} \sum p_i (S(\sum \rho_A^i) - S(\rho_A^i))$$

So $C_{\rightarrow}$ is a kind of function build by "arrowing", where $f : S(\mathcal{H}_A) \to R$ acting on states of system $A$ if of the form:

$$f(\rho_A^i) = S(\sum p_i \rho_A^i) - S(\rho_A^i)$$

Function $f$ is asymptotically continous, beacuse entropy von Neumann $S$ possess this feature. So whereby of Proposition [2]quantity $C_{\rightarrow}$ is also asymptotically continous.

B. Intrinsic conditional information

Consider the following function called intrinsic conditional information: $I(X;Y \downarrow E)$ between $X$ and $Y$ given $E$ defined as

$$I(X;Y \downarrow E) = \inf_{P_{E|\bar{E}}} \inf \sum p(\bar{e}) I(X;Y|\bar{E} = \bar{e})$$

where $P_{E|\bar{E}}$ is a classical channel, $I(X;Y|\bar{E} = \bar{e})$ is mutual information between $X$ and $Y$ given $\bar{E} = \bar{e}$ and $p(\bar{e})$ is probability that we have outcome $\bar{e}$ on subsystem $\bar{E}$. The quantity $I(X;Y|\bar{E} = \bar{e}) = \sum p(\bar{e}) I(X;Y|\bar{E} = \bar{e})$ is called conditional information. It is known [37] that infimum in definition of intrinsic conditional information is achievable. It is enough to take minimum over $P_{E|\bar{E}}$ with the system $\bar{E}$ of size of $E$.

One easily finds, that that intrinsic information is a particular case of "arrowing". Indeed, for a given classical channel $P_{E|\bar{E}}$ with conditional probabilities $\{p_{e|\bar{e}}\}$ we consider POVM given by Kraus operator $A_{\bar{e}} = \sum_{e} \sqrt{p_{e|\bar{e}}} |\bar{e}\rangle \langle e|$. Now, if we embedded in natural way our distribution into set of quantum states, then we see that our definition [3] reproduces the above quantity.

If we notice that the mutual information itself is asymptotically continuous (it is sum of entropies, each of them being asymptotically continuous due to Fannes inequality [1]), then we will see that the asymptotic continuity of intrinsic conditional information follows from our theorem.
VI. CONVEX ROOF FUNCTIONS

Here we present asymptotic continuity of functions constructed from other asymptotically continuous function $f$ by means of convex roof $\hat{f}$. We will distinguish between pure and mixed convex roof. The pure convex roof is a generalization of definition of entanglement of formation $E_F$ given in \cite{3}. It was proposed and investigated in Ref. \cite{27} and called there just convex roof.

A. Pure convex roof

**Definition 7** For a function $f$ defined on pure states its pure convex roof $\hat{f}$ is a function defined on all states, given by

$$\hat{f}(\varrho) = \inf_{\{p_k, \psi_k\}} \sum_k p_k f(\psi_k)$$

where infimum is taken over all finite pure ensembles $\{p_k, \psi_k\}$, satisfying $\varrho = \sum p_k |\psi_k\rangle\langle\psi_k|$.  

It is useful to represent convex roof in a different way (cf. \cite{19}), to make explicit, that operation of pure convex roof is actually arrowing. Indeed, for any state $\varrho$ acting on Hilbert space $\mathcal{H}_X$ of dimension $d_X$ we can construct its purification i.e. pure state $\varphi_{\varrho}$ acting on Hilbert space $\mathcal{H}_X \otimes \mathcal{H}_E$ (with $\dim \mathcal{H}_E = \dim \mathcal{H}_X$) such that

$$\text{Tr}_{\mathcal{H}_E} \varphi_{\varrho} = \varrho$$

Moreover for any pure decomposition of $\varrho$, given by $\{p_k, \psi_k\}$ there exists a complete POVM on $\mathcal{H}_{\text{anc}}$ which gives such ensemble on system $X$, and vice versa: any POVM gives rise to some pure decomposition.

Then we can rewrite $\hat{f}$ as infimum over measurements $\mathcal{M}$

$$\hat{f}(\varrho) = \inf_{\sum p_k |\psi_k\rangle\langle\psi_k|=\varrho} \sum_k p_k f(\psi_k)$$

Consequently, we have

$$\hat{f}(\varrho_X) = f_{\text{cpl}}(\varphi_{\varrho}^{X_E})$$

where the equality holds for arbitrarily fixed purification $\varphi_{\varrho}^{X_E}$ of the state $\varrho_X$. Having rewritten pure convex roof in terms of arrowed function, we can easily prove its asymptotic continuity, by use of the proposition 2.

**Proposition 4** Let $f$ be a function, which is subextensive and asymptotically continuous. Then its convex roof $\hat{f}$ is also asymptotically continuous.

**Proof.**

We will use following inequalities \cite{35}:

$$1 - F(\varrho, \sigma) \leq \frac{1}{2} ||\varrho - \sigma||_1 \leq \sqrt{1 - F(\varrho, \sigma)}$$

where $F(\varrho, \sigma) = \sqrt{\varrho \sigma} \sqrt{\varrho}$ is fidelity \cite{39, 40}. The fidelity can be also expressed as follows

$$F(\varrho, \sigma) = \sup ||\langle \psi_\varrho | \psi_\sigma \rangle||$$

where supremum is taken over all $\psi_\varrho$ and $\psi_\sigma$, which are purifications of states $\varrho$ and $\sigma$. The supremum is achievable.

Consider now arbitrary states $\varrho$ and $\sigma$ let $\varepsilon = ||\varrho - \sigma||_1$. We want to estimate $\hat{f}(\varrho) - \hat{f}(\sigma)$. Since the representation \cite{61} does not depend on the choice of purification, we take such purifications $\psi_\varrho$ and $\psi_\sigma$, that

$$F(\varrho, \sigma) = F(\psi_\varrho, \psi_\sigma)$$

Then we have

$$||\langle \psi_\varrho | \psi_\varrho \rangle - |\psi_\sigma\rangle\langle\psi_\sigma| ||_1 \leq 2 \sqrt{1 - F(\psi_\varrho, \psi_\sigma)} = 2 \sqrt{1 - F(\varrho, \sigma)} \leq 2 \sqrt{||\varrho - \sigma||_1/2} = \sqrt{2 \varepsilon}$$

Since we assume that $f$ is asymptotically continuous and subextensive, we can use Prop. 2 to get

$$|\hat{f}(\varrho) - \hat{f}(\sigma)| = |f_{\text{cpl}}(\psi_\varrho) - f_{\text{cpl}}(\psi_\sigma)| \leq K \sqrt{2 \varepsilon} \log d_X + O(\sqrt{2 \varepsilon})$$

This ends the proof. 

**Remark.** Notice that however we have here $\sqrt{2 \varepsilon}$ instead of $\varepsilon$, but we think that it does not change essence of condition referring asymptotic continuity.
VII. MIXED CONVEX ROOF

Analogously to pure convex roof we can define mixed convex roof.

**Definition 8** Let $f$ be a function and $\rho$ be a state then we can define function mixed convex roof $\hat{f}$ as follows

$$\hat{f}(\rho) = \inf_{\{p_k, \rho_k\}} \sum_k p_k f(\rho_k)$$

where infimum is taken over all ensembles $\{p_k, \rho_k\}$, where $\rho = \sum p_k \rho_k$.

Similarly as in the case of pure convex roof we can show that

$$\hat{f}(\rho_X) = f_\downarrow(\psi_{XE}^\rho)$$

where, again, $\psi_{XE}^\rho$ is arbitrarily fixed purification of $\rho_X$.

Therefore, with analogous proof as that of Prop. 4, we obtain

**Proposition 5** Let $f$ be subextensive and asymptotically continuous function then function mixed convex roof $\hat{f}$ is also asymptotically continuous.

VIII. APPLICATIONS

A. Pure convex roof of measure of entanglement for tripartite pure states

Consider the quantity $E$ which is equal to sum of measure of entanglement for bipartite state applied for subsystem of tripartite state:

$$E(\rho_{ABC}) = E_R(\rho_{AB}) + S(\rho_C)$$

where $S$ is von Neumann entropy and $\rho_{AB} = Tr_C \rho_{ABC}$, $\rho_C = Tr_{AB} \rho_{ABC}$ and $E_R$ is relative entropy distance from set of separable states. Now, we can consider pure convex roof of function $E$:

$$\hat{E}(\rho_{ABC}) = \inf_{\rho_{ABC} = \sum p_k |\psi_k\rangle \langle \psi_k|_{ABC}} \sum_k p_k E(|\psi_k^k_{ABC}\rangle)$$

Note that $E$ is subextensive and asymptotically continuous, because relative entropy distance and entropy possess these feature. Thus the Proposition 4 implies that convex roof of this function $\hat{E}$ is also asymptotically continuous.

B. Entanglement of formation

Proposition 4 implies asymptotic continuity of entanglement of formation $E_F$ (which was first shown in [41]) defined as

$$E_F(\rho_{AB}) = \inf_{\rho_{AB} = \sum p_k |\psi_k\rangle \langle \psi_k|_{AB}} \sum_k p_k S_A(|\psi_k\rangle)$$

where $S_A$ is a von Neumann entropy of subsystem $A$ of state. In original definition infimum is taken over all pure ensembles, but notice that in this case infimum over all ensembles reduce to infimum over pure ensembles. Thus

$$E_F(\rho_{AB}) = \inf_{\rho_{AB} = \sum p_k \rho_k} \sum_k p_k S_A(\rho_k)$$

This is implied by concavity of von Neumann entropy:

$$\sum_k p_k S_A(\rho_k) = \sum_k p_k S_A(\sum q_i^k |\varphi_i^k\rangle \langle \varphi_i^k|) \geq \sum_k p_k \sum_{i,k} q_i^k S_A(|\varphi_i^k\rangle) = \sum_{i,k} p_k q_i^k S_A(|\varphi_i^k\rangle)$$

So for every mixed ensemble we can find a pure ensemble which gives no greater value of function $E_F$ than a mixed ensemble.
C. Pure and mixed convex roof of mutual information.

Now, we show example of function for which pure and mixed convex ro of are not equal to each other. Consider the following functions:

\[
\hat{I}_M(\rho_{AB}) = \inf_{\rho_{AB} = \sum_k p_k |\psi_k\rangle\langle\psi_k|} \sum_k p_k I_M(|\psi_k\rangle)
\]

(74)

\[
\hat{I}_M(\rho_{AB}) = \inf_{\rho_{AB} = \sum_k p_k \rho_k} \sum_k p_k I_M(\rho_k)
\]

(75)

where \( I_M \) is mutual information \( I_M = S_A(\rho_{AB}) + S_B(\rho_{AB}) - S(\rho_{AB}) \). In our terminology, the functions are pure and convex roof of quantum mutual information. The second one was introduced in [25]. Notice that for a pure convex roof we have

\[
\hat{I}_M(\rho_{AB}) = 2 \inf_{\rho_{AB} = \sum_k p_k |\psi_k\rangle\langle\psi_k|} \sum_k p_k S_A(|\psi_k\rangle) = 2E_F(\rho_{AB})
\]

(76)

Let \( \rho_{as} \) be antysymmetric state state:

\[
\rho_{as} = \frac{1}{d^2 - d}(I - V)
\]

(77)

where \( V \) is a unitary flip operator \( V \) acting on Hilbert space \( \mathcal{C}^d \otimes \mathcal{C}^d \) system defined by \( V \phi \otimes \varphi = \varphi \otimes \phi \). We know that \[12\]

\[ E_F(\rho_{as}) = 1 \]

(78)

So \( \hat{I}_M(\rho_{as}) = 2 \). Then we have the following inequality

\[
\hat{I}_M(\rho_{as}) \leq I_M(\rho_{as}) = 2 \log d - S(\rho_{as}) = \log \frac{2d}{d - 1}
\]

(79)

So for \( d \geq 3 \) we have that \( \hat{I}_M(\rho_{as}) \neq \hat{I}_M(\rho_{as}) \).

IX. ACHIEVING INFIMUM IN DEFINITION OF ARROWING

We prove that in definition of arrowing the infimum is achievable, so that it can be replaced by minimum. First we prove the following lemma.

**Lemma 4** Let \( \{p_i\} \) be a probability distribution then any convex combination \( \sum_i p_i x_i, \) where \( x_i = (\rho_i, f(\rho_i)) \), equal to \( \sum_p (\rho_i, f(\rho_i)) \) can be written as a convex combination \( \sum q_i (\rho_i, f(\rho_i)) \) consisting of \( n + 1 \) (or less) ingredients, where \( n \) is a dimension of space on which is acting \( x_i \). So

\[
\sum_i p_i \rho_i = \sum_{i=1}^{n+1} q_i \rho_i \quad \text{and} \quad \sum_i p_i f(\rho_i) = \sum_{i=1}^{n+1} q_i f(\rho_i)
\]

(80)

**Proof.** Let \( \tilde{f} = \sum_i p_i f(\rho_i) \) where \( \tilde{g} = \sum_i p_i \rho_i \) is a state acting on Hilbert space \( \mathcal{H} \). Let \( x_i = (\rho_i, f(\rho_i)) \) be a point from a convex set \( \mathcal{S} = \text{co}(\rho_i, f(\rho_i)) \). Then

\[
(\tilde{g}, \tilde{f}) = (\sum_i p_i \rho_i, \sum_i p_i f(\rho_i)) = \sum_i p_i (\rho_i, f(\rho_i)) \in \mathcal{S}
\]

(81)

Using Caratheodory ‘s Theorem we have that there exists such set of probability distribution consisting of \( n + 1 \) or less elements that

\[
(\rho, \tilde{f}) = \sum_i q_i (\rho_i, f(\rho_i))
\]

(82)
Proposition 6 Let $f$ be a function, then the following conditions are equivalent:

1) $\forall \varepsilon > 0 \exists \delta > 0 \quad \forall \rho, \sigma \ |\rho - \sigma|_1 \leq \delta \implies |f(\rho) - f(\sigma)| \leq K_1 \varepsilon \log d + O(\varepsilon)$  

2) $\forall \varepsilon > 0 \exists \delta > 0 \quad \forall \rho, \sigma \ |f((1 - \delta)\rho + \delta\sigma) - f(\rho)| \leq K_2 \varepsilon \log d + O(\varepsilon)$

$K_1, K_2$ are constants and $O(x)$ is any function that satisfies (i) $O(x)$ converges to 0 when $x$ converges to 0 and (ii) $O(x)$ depends only on $x$ (so in our particular case, it will not depend on dimension).

Proof.

$1 \Rightarrow 2^*$ Let $\varepsilon > 0$ be fix then there exists such $\delta > 0$ that for any states $\rho$ and $\sigma$, the following conditions is fulfilled

$$||\rho - \sigma||_1 \leq \delta \implies |f(\rho) - f(\sigma)| \leq K_1 \varepsilon \log d + O(\varepsilon)$$

Notice that there exists such $\delta_1 = \frac{\delta}{4}$

$$||\rho - ((1 - \delta_1)\rho + \delta_1\sigma)||_1 = \delta_1 ||\rho - \sigma||_1 \leq 2\delta_1 = \delta$$
this implies that
\[ |f((1 - \delta_1)\varrho + \delta_1\sigma) - f(\varrho)| \leq K_1\varepsilon \log d + O(\varepsilon) = K_2\varepsilon \log d + O(\varepsilon) \] \hspace{1cm} (93)

"2 \Rightarrow 1"

Let \( \varepsilon > 0 \) then there exists such \( \delta > 0 \) that
\[ \forall_{\varrho,\sigma} \quad |f((1 - \delta)\varrho + \delta\sigma) - f(\varrho)| \leq K_2\varepsilon \log d + O(\varepsilon) \] \hspace{1cm} (94)

Let \( \varrho_1, \varrho_2 \) be state that
\[ ||\varrho_1 - \varrho_2||_1 = \delta \] \hspace{1cm} (95)

Analogously to proof of Theorem \[ \square \]

\[ \exists_{\sigma,\gamma_1,\gamma_2} \quad \sigma = (1 - \delta_1)\varrho_1 + \delta_1\gamma_1 = (1 - \delta_1)\varrho_2 + \delta_1\gamma_2 \] \hspace{1cm} (96)

\[ |f(\varrho_2) - f(\varrho_1)| \leq |f(\varrho_2) - f(\sigma)| + |f(\sigma) - f(\varrho_1)| = \\
|f(\varrho_2) - f((1 - \delta_1)\varrho_2 + \delta_1\gamma_2)| + |f((1 - \delta_1)\varrho_1 + \delta_1\gamma_1) - f(\varrho_1)| \leq 2K_2\log d + 2O(\varepsilon) = K_1\log d + O(\varepsilon) \] \hspace{1cm} (98)

This ends the proof. \[ \square \]
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