Modified teleparallel gravity with higher-derivative torsion terms

Giovanni Otalora\textsuperscript{1,2,3,}\textsuperscript{*} and Emmanuel N. Saridakis\textsuperscript{3,4,5,}\textsuperscript{†}

\textsuperscript{1}Departamento de Matemática, ICE, Universidade Federal de Juiz de Fora, Minas Gerais, Brazil
\textsuperscript{2}Instituto de Física Teórica, UNESP-Universidade Estadual Paulista Caixa Postal 70532-2, 01156-970, São Paulo, Brazil
\textsuperscript{3}Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile
\textsuperscript{4}Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece
\textsuperscript{5}CASPER, Physics Department, Baylor University, Waco, TX 76798-7310, USA

We construct $F(T, (\nabla T)^2, \Box T)$ gravitational modifications, which are novel classes of modified theories arising from higher-derivative torsional terms in the action, and are different than their curvature analogue. Applying them in a cosmological framework we obtain an effective dark energy sector that comprises of the novel torsional contributions. We perform a detailed dynamical analysis for two specific examples, extracting the stable late-time solutions and calculating the corresponding observables. We show that the thermal history of the universe can be reproduced, and it can result in a dark-energy dominated, accelerating universe, where the dark-energy equation-of-state parameter lies in the quintessence regime, or may exhibit the phantom-divide crossing during the cosmological evolution. Finally, the scale factor behaves asymptotically either as a power-law or as an exponential, in agreement with observations.

\textbf{PACS numbers:} 04.50.Kd, 98.80.-k, 95.36.+x

\section{I. INTRODUCTION}

The early and late time accelerated expansions of the universe are probably the most surprising findings in modern cosmology and establish a serious challenge to our current knowledge of physics. There are two main ways that one could follow in order to describe them. The first is to maintain general relativity (as the gravitational theory and modify the content of the universe by introducing new, exotic components, such as the inflaton field(s) \cite{1} or the dark energy sector \cite{2,3}. The second is to modify the gravitational theory itself, constructing a theory with additional degrees of freedom that can drive acceleration, but which still possesses general relativity as a particular limit \cite{4}.

Most works in modified gravity start from the standard gravitational formulation, which is based on curvature, and modify the Einstein-Hilbert action, with the simplest extended model being the $F(R)$ one \cite{5}. Also, other modifications to gravity can arise from a Planck-scale deformed dispersion relation and effective spacetime metric that depends of the energy, momentum or spin of the probe particle \cite{6,7}. Nevertheless, one can equally well build modified gravitational theories starting from the torsional gravitational formulation, and in particular from the Teleparallel Equivalent of General Relativity (TEGR) \cite{8,9,10}. Since in this theory the gravitational Lagrangian is the torsion scalar $T$, the simplest extended scenario is to extend it to $F(T)$ theory \cite{11,12,13} (see \cite{14} for a review). Note that although at the level of equations TEGR is completely equivalent with general relativity, $F(T)$ is a different class of modified gravity than $F(R)$ gravity, and therefore its cosmological implications bring novel features, either at late-times \cite{15,16} or at the inflationary epoch \cite{17}.

Nevertheless, in curvature-based modified gravity one can construct more complicated extensions of the Einstein-Hilbert action by introducing higher-order terms, such as $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $(\nabla R)^2$, the Gauss-Bonnet combination, $R\Box R$, $R\Box^{\lambda} R$, etc, and moreover couple these terms to an additional scalar field and its derivatives \cite{18}, since such terms could be justified due to quantum corrections or through a fundamental gravitational theory (for instance such terms appear in the string effective Lagrangian or in Kaluza-Klein theories, when the mechanism of dimensional reduction is used \cite{19}, \cite{20}), or in quantum-gravity-like effective actions at scales closed to the Planck one \cite{21}. In principle, one could follow the same direction in torsional gravity, i.e construct gravitational modifications using higher-order torsional terms. For instance, one could construct the teleparallel equivalent of the Gauss-Bonnet combination and insert arbitrary functions of it in the Lagrangian \cite{22}, or extend the procedure to the teleparallel equivalent of Lovelock gravity \cite{23}.

In this work, and inspired by the corresponding curvature based modification \cite{24}, we are interested in constructing novel torsional gravitational modifications using higher-derivative, $(\nabla T)^2$ and $\Box T$ terms, i.e theories that are characterized by the Lagrangian $F(T, (\nabla T)^2, \Box T)$, and investigate their cosmological implications. The plan of the work is as follows: In Section II we construct $F(T, (\nabla T)^2, \Box T)$ gravity and we apply it in a cosmological framework, extracting the cosmological equations and calculating various observables. In Section III we analyze in detail two specific models, performing a dynamical analysis in order to reveal the global features of the corresponding cosmological behavior. Finally, in Section IV we summarize the obtained results.
II. THE MODEL

In this section we construct modified teleparallel gravitational theories with higher derivative torsion contributions, extracting the general field equations, and we apply them in a cosmological framework.

A. \( F(T, (\nabla T)^2, \Box T) \) gravity

In teleparallel gravity the dynamical field is the vierbein \( e^A_{\mu} \), which forms an orthonormal base for the tangent space at each point of a manifold. It is related to the metric through

\[
g_{\mu\nu} = \eta_{AB} e^A_{\mu} e^B_{\nu}, \tag{1} \]

where greek indice span the coordinate space and latin indices span the tangent space. Additionally, one introduces the Weitzenböck connection \[ \Gamma^A_{\mu\nu} \equiv e^A_{\lambda} \partial_\mu e^\lambda_{\nu} \]

and the gravitational field is described by the torsion tensor

\[
T^\rho_{\mu\nu} \equiv e^A_{\lambda} \left( \partial_\mu e^A_{\nu} - \partial_\nu e^A_{\mu} \right). \tag{2} \]

Hence, the Lagrangian of the theory is the torsion scalar \( T \), constructed by contractions of the torsion tensor as

\[
T = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\rho\mu\nu} + T^{\rho\mu\nu} T_{\rho\mu\nu}. \tag{3} \]

In the simplest torsional modified gravity, and inspired by similar procedures in curvature gravity, one extends the Lagrangian to an arbitrary function \( F(T) \), resulting to \( F(T) \) gravity, \[17, 18, 20] However, one could be inspired by the higher-derivative curvature modifications, and construct torsional modified gravity using higher derivative torsional terms, like \((\nabla T)^2\) and \(\Box T\). Hence, in this work we consider theories of the form

\[
S = \frac{1}{2} \int d^4x e F(T, (\nabla T)^2, \Box T) + S_m(e^A_{\mu}, \Psi_m), \tag{4} \]

where \((\nabla T)^2 = \eta^{AB} e^A_{\mu} e^B_{\nu} \nabla_\mu T \nabla_\nu T\) and \(\Box T = \eta^{AB} e^A_{\mu} e^B_{\nu} \nabla_\mu \nabla_\nu T = g^{\mu\nu} \nabla_\mu \nabla_\nu T\), and where \( e = \det(e^A_{\mu}) = \sqrt{-g} \) (for simplicity we have set the light speed \( c = 1 \) and the gravitational constant \( \kappa^2 = 8\pi G \) = 1). Note that in the above total action we have also considered a general matter action comprised of general fields \( \Psi_m \), allowing also for an arbitrary coupling with the vierbein. Finally, we mention that for simplicity in the present work we follow the usual, “pure-tetrad”, approach to torsional modified gravity, while the extension to the inclusion of a general spin connection is straightforward, following \[29].

Using for simplicity the notation \( X_1 = (\nabla T)^2 \) and \( X_2 = \Box T \), as well as \( F_T = \partial F/\partial T \), \( F_{X_\alpha} = \partial F/\partial X_\alpha \), with \( a = 1, 2 \), variation of action \[1] with respect to the vierbein leads to the following field equations:

\[
\frac{1}{e} \partial_\mu \left( e F_T e^A_{\mu} S^A_{\tau} \rho \right) - F_T e^A_{\mu} S^A_{\nu\rho} T^{\mu\nu}_{\tau\rho} + \frac{1}{4} e^A F
\]

\[
+ \frac{1}{4} \sum_{a=1}^{2} \left\{ F_{X_a} \frac{\partial X_a}{\partial e^A_{\rho}} - \frac{1}{e} \left[ \partial_\mu \left( e F_{X_a} \frac{\partial X_a}{\partial e^A_{\mu}} - \partial_\mu \partial_\nu \left( e F_X \frac{\partial X_a}{\partial e^A_{\mu}} \right) \right) \right] \right\}
\]

\[
- \frac{1}{4e} \partial_\lambda \partial_\mu e_\nu \left( e F_{X_2} \frac{\partial X_2}{\partial e^A_{\nu}} \right) \equiv -\frac{1}{2} e^A_{\tau} T^{(m)}_{\tau}. \tag{5} \]

where we have defined the “superpotential” \( S^A_{\mu\rho} = \frac{1}{2} \left( K^\mu_{\nu\rho} + \delta^\mu_{\nu} T^\rho_{\theta\phi} - \delta^\rho_{\mu} T^\theta_{\phi\nu} \right) \), with \( K^\mu_{\nu\rho} = -\frac{1}{2} \left( T^{\mu\rho}_{\nu\rho} - T^{\nu\rho}_{\mu\rho} - T^{\rho\mu}_{\nu\rho} \right) \) the contortion tensor. Note that in the right hand side of \[5\] we have defined the matter energy momentum tensor as

\[
e^A_{\tau} T^{(m)}_{\tau} \equiv -\frac{1}{2} \frac{\partial S^A_m}{\partial X_a}. \tag{6} \]

Finally, since the covariant derivative of the matter energy-momentum tensor in every theory where matter is minimally coupled to gravity is zero, as long as the matter Lagrangian is diffeomorphism invariant \[30\], we deduce that if in action \[4\] we make the usual consideration that the matter action does not have an arbitrary coupling with gravity (i.e with the vierbein) but only a minimal coupling, then its covariant derivative is indeed zero. This can be verified explicitly too, by taking in this case the covariant derivative of \[5\].

Equations \[5\] contain higher-order derivatives as expected. However, this is not necessarily an indication of Ostrogradsky instabilities \[31, 32\] since the above modified gravity has not been formulated in the Einstein frame. Thus, the higher-order derivatives may be just an indication of extra degrees of freedom, as it is the case in many gravitational modifications, like \( f(R) \) gravity \[33\]. One could try to transform the model in the Einstein frame, however in torsional modified gravities such transformations do not exist, or at least they are not known yet \[19, 33\]. Hence, the only safe method to examine whether the present constructions have any ghost or Laplacian instabilities, or extract the sub-classes that are free of such instabilities, is through a robust Hamiltonian analysis. Such a necessary investigation lies beyond the scope of the present work, which is a first study on the subject, and hence it is left for a separate project.

B. Cosmological equations

In order to proceed to the cosmological applications of the above theory, we consider a flat Friedmann-Robertson-Walker (FRW) background space-time with
metric $ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j$, which arises from the vierbein

$$e^A_{\mu} = \text{diag}(1, a(t), a(t), a(t)), \quad (7)$$

where $a(t)$ is the scale factor. In such a geometry, and assuming as usual that the matter action includes only a minimal coupling to gravity (i.e with the vierbein), the field equations \([5]\) give rise to the two Friedmann equations as

$$F_T H^2 + (24H^2 F_{X_1} + F_{X_2}) \left(3\dot{H} + \dot{H}^2\right) H + 12H \dot{F}_{X_1} T + 24H^2 \ddot{F}_{X_1} + \left(4H + 3H^2\right) \ddot{F}_{X_2} + 24HF_{X_1} \left(4H + 3H^2\right) \ddot{H} = -\frac{\rho_m}{2}, \quad (8)$$

where

$$\dot{F}_T = F_{TT} \dot{T} + \sum_{a=1}^{2} F_{TX_a} \dot{X}_a, \quad (10)$$

$$\dot{F}_{X_a} = F_{X_a T} \dot{T} + \sum_{b=1}^{2} F_{X_a X_b} \dot{X}_b, \quad (11)$$

$$\ddot{F}_{X_a} = \left[F_{X_a T T} \ddot{T} + \sum_{b=1}^{2} F_{X_a X_b} \ddot{X}_b \right] \dot{T} + \sum_{b=1}^{2} \left[F_{X_a T X_b} \ddot{T} + \sum_{c=1}^{2} F_{X_a X_b X_c} \ddot{X}_c \right] \dot{X}_b \quad (12)$$

$$\ddot{F}_{X_a} = \frac{\partial \ddot{F}_{X_a}}{\partial T} \ddot{T} + \sum_{b=1}^{2} \frac{\partial \ddot{F}_{X_a}}{\partial X_b} \ddot{X}_b, \quad (13)$$

with $F_{TT} = \partial^2 F/\partial T^2$, $F_{T T T} = \partial^3 F/\partial T^3$, $F_{X_a X_b} = \partial^2 F/\partial X_a \partial X_b$ and $F_{X_a X_b X_c} = \partial^3 F/\partial X_a \partial X_b \partial X_c$. Additionally, note that in the FRW geometry \([7]\), the torsion scalar \([9]\), as well as the functions $X_1$ and $X_2$, become

$$T = -6H^2, \quad (14)$$

$$X_1 = 144H^2 \dot{H}^2, \quad (15)$$

$$X_2 = -12 \left(\dot{H} \left(\dot{H} + 3H^2\right) + H \ddot{H}\right). \quad (16)$$

Finally, note that in the Friedmann equations \([5],[9]\) we have assumed that the matter energy-momentum tensor corresponds to a perfect fluid with energy density $\rho_m$ and pressure $p_m$.

The Friedmann equations \([5],[9]\) can be re-written in the standard form

$$3H^2 = \rho_{DE} + \rho_m, \quad (17)$$

$$\dot{H} = \rho_m + p_m + \rho_{DE} + \rho_{DE}, \quad (18)$$

where the energy density and pressure of the effective dark energy sector are respectively defined as

$$\rho_{DE} = \frac{\dot{F}^2}{2} - 6H^2 F_T^2 + 3H^2$$

$$-6F_{X_1} H^2 - 6H \left(F_{X_2} + 24H^2 F_{X_1}\right) \left(3\dot{H} + \dot{H}^2\right)$$

$$-144H^3 \dot{H} F_{X_1} - 6H \left(3H^2 - \dot{H}\right) F_{X_2} - 6H^2 \ddot{F}_{X_2}, (19)$$

$$p_{DE} = -3H^2 - 2\dot{H} + 2 \left(F_T \dot{H} + H \ddot{F}_T\right)$$

$$+ 24H \left[2H \dot{H} + 3 \left(\dot{H} + H^2\right) \ddot{H}\right] F_{X_1}$$

$$+ 12H \dot{H} \ddot{F}_{X_2} + 24H^2 \ddot{F}_{X_1} + \left(\dot{H} + 3H^2\right) \ddot{F}_{X_2}$$

$$+ 24H^2 F_{X_1} \ddot{H} + H \dddot{F}_{X_2} + 24F_{X_1} H^2 \left(12H^2 + \dot{H}\right)$$

$$+ 24HF_{X_1} \left(4H + 3H^2\right) \ddot{H}\right). \quad (20)$$

As we mentioned earlier, since we have considered a matter sector minimally coupled to gravity and diffeomorphism invariant, the covariant derivative of the matter energy-momentum tensor is zero, which in the case of perfect fluid in FRW geometry leads to

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (21)$$

Hence, in this case, the Friedmann equations \([17],[18]\) imply

$$\ddot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0. \quad (22)$$

In the following we will consider the matter fluid to be of barotropic nature, namely $p_m = (\gamma - 1)\rho_m$, with $w_m = \gamma - 1$ its equation-of-state parameter. Similarly, we can define the effective dark energy equation-of-state parameter as

$$w_{DE} = \frac{p_{DE}}{\rho_{DE}}. \quad (23)$$

Lastly, concerning cosmological investigations it proves convenient to introduce the standard density parameters $\Omega_m = \frac{\rho_m}{3H^2}$ and $\Omega_{DE} = \frac{\rho_{DE}}{3H^2}$, as well as the total equation-of-state parameter as

$$w_{tot} = \frac{p_{DE} + p_m}{\rho_{DE} + \rho_m}, \quad (24)$$

which is immediately related to the deceleration parameter $q$ through

$$q = \frac{1}{2} (1 + 3w_{tot}), \quad (25)$$
and hence acceleration occurs when \( q < 0 \).

In summary, in the cosmological scenario of modified gravity with higher-order torsional derivatives, one obtains an effective dark energy sector that comprises of these novel torsional terms. As we observe from the specific expressions, although TEGR coincides completely with general relativity at the level of equations, the corresponding modified scenario is different from its curvature analogue. This is a common feature of all torsional modified gravities, namely that they do not coincide with their curvature analogues, despite the fact that their starting theories are equivalent. Hence, since the present scenario is a novel class of gravitational modification, it is both interesting and necessary to investigate its cosmological applications. This is performed in the next section.

III. COSMOLOGICAL DYNAMICS

In the previous section we presented a torsional modified gravity based on the use of higher derivative terms, and we applied it in a cosmological framework. As we saw, in such a scenario we have obtained an effective dark energy sector which arises from the novel, higher-derivative torsional terms. In this section we are interested in investigating in detail the cosmological dynamics, using the powerful method of dynamical-system analysis [34], which allows to by-pass the complexities of the equations and reveal the global behavior of the system.

In order to perform the phase-space analysis of the cosmological scenario at hand, we have to introduce suitable dimensionless auxiliary variables that will bring the system of cosmological equations into its autonomous form [34]. For a system of order \( l \) we introduce the following dimensionless variables [34]:

\[
Z_1 = H, \quad Z_2 = \frac{\dot{H}}{H} t, \ldots, Z_{l+1} = \frac{H}{H} + \frac{1}{T},
\]

with \( l = 1, \ldots, n \), the number of overdots. Then, the field equations can be rewritten in the form of an autonomous system [34]

\[
\frac{dZ_1}{dN} = Z_1 Z_2, \\
\frac{dZ_2}{dN} = Z_3 - 2Z_2^2, \\
\vdots \\
\frac{dZ_{l+1}}{dN} = Z_{l+2} - (l + 1)Z_2Z_{l+1},
\]

where we have introduced the e-folds number \( N = \log a \). The system is truncated at the variable \( Z_{l+2} \). Thus, the dimensionless variable \( Z_{l+2} = Z_{l+2}(Z_1, Z_2, \ldots, Z_{l+1}) \) is calculated from the field equations.

The critical points \( (Z_1^*, Z_2^*, \ldots, Z_{l+1}^*) \) of the above dynamical system can be extracted by imposing the conditions \( \frac{dZ_1}{dN} = \frac{dZ_2}{dN} = \ldots = \frac{dZ_{l+1}}{dN} = 0 \). Observing [26] and [27] we can easily deduce that a de Sitter critical point is realized if

\[
Z_1^* > 0, \quad Z_2^* = Z_3^* = \ldots = Z_{l+2}^* = 0,
\]

since in this case we immediately obtain \( a(t) \sim e^{Ht} \), with \( H = Z_1^* > 0 \) and \( l = 1, \ldots, n \). Similarly, a power-law form for the scale factor is realized if

\[
Z_1^* = 0, \quad Z_2^* = l(l + 1)Z_2^{l(l+1)},
\]

for \( l = 1, \ldots, n \), in which case asymptotically we have \( a(t) \sim t^p \) with \( p = -1/Z_2^* \) (note that \( Z_2^* \) for \( l = 1, \ldots, n \) can be non-zero although both the numerators and denominators in their definitions [26] tend to zero).

Finally, perturbing the system linearly around these critical points, and expressing the perturbations equations in terms of a perturbation matrix, allows one to determine the type and stability of each critical point by examining the eigenvalues of this matrix [34].

In the following we apply this procedure to two specific \( F(T, (\nabla T)^2, \Box T) \) models.

A. Model I: \( F(T, (\nabla T)^2, \Box T) = T + \alpha_1 (\nabla T)^2 + \alpha_2 e^{\frac{\alpha_1}{T^2}} \)

Let us start our analysis by a simple scenario, in which the action does not depend on \( X_2 \equiv \nabla T \) but only on \( X_1 \equiv (\nabla T)^2 \), i.e a scenario of the form

\[
F(T, X_1, X_2) = T + \alpha_1 X_1 + \alpha_2 e^{\frac{\alpha_1}{T^2}} = T + \alpha_1 (\nabla T)^2 + \alpha_2 e^{\frac{\alpha_1}{T^2}},
\]

where \( \alpha_1, \alpha_2 \) and \( \delta \) are constants. As described above, we introduce the following three dimensionless variables

\[
Z_1 = H, \quad Z_2 = \frac{\dot{H}}{H} t, \quad Z_3 = \frac{H}{H} + \frac{1}{T},
\]

In terms of these variables, the torsion scalar from [14] and the function \( X_1 \) from [15] become

\[
T = -6Z_2^2, \quad X_1 = 144Z_1^6 Z_2^2.
\]

Hence, the system of the two Friedmann equations [9] is written in its autonomous form as

\[
\frac{dZ_1}{dN} = Z_1 Z_2, \\
\frac{dZ_2}{dN} = Z_3 - 2Z_2^2, \\
\frac{dZ_3}{dN} = Z_4 - 3Z_2 Z_3,
\]

where the function \( Z_4(Z_1, Z_2, Z_3) \) is calculated from Eqs. [17] and [18] and is given in [22]. Additionally, in
terms of the auxiliary variables, and using (19), (20) and (23), (25), we can express the observables as

$$\Omega_{DE} = \frac{\alpha_2}{48Z_1^4} e^{\frac{Z_2^2}{2}} \left[ 18Z_1^2(4Z_2^2 - Z_3 - 3Z_2^3)\delta + 4Z_2^2(3Z_2^2 - Z_3)\delta^2 - 81Z_1^4 \right] + \frac{2\alpha_3}{3} (3Z_2^2 - 2Z_3 - 6Z_2),$$

$$w_{DE} = \gamma - 1 + \left[ 162Z_1^6(2Z_2 + 3\gamma) \right] \times \left\{ \alpha_2 e^{\frac{Z_2^2}{2}} \left[ 18Z_1^2(Z_3 - 4Z_2^2 + 3Z_2^3)\delta + 4Z_2^2(Z_3 - 3Z_2^3)\delta^2 + 81Z_1^4 \right] + 324\alpha_3 Z_1^6(2Z_3 - 3Z_2^2 + 6Z_2) \right\}^{-1},$$

and

$$q = -1 - Z_2.$$  

The scenario of Model I admits four physical critical points (i.e. real and corresponding to $0 \leq \Omega_{DE} \leq 1$), which are displayed in Table I along with their existence conditions. In the same Table we include the asymptotic behavior of the scale factor $a(t)$ along with the conditions for expansion and acceleration, as well as the corresponding values of the dark energy density parameter $\Omega_{DE}$, of the dark energy equation-of-state parameter $w_{DE}$, and of the deceleration parameter $q$. We have defined $Z_{1\pm} = \frac{1}{\alpha_1}(\pm\sqrt{\alpha_1(\alpha_1 - \frac{\delta}{3})})$.

Table I. The physical critical points of the system (33) of Model I: $F(T, (\nabla T)^2, \Box T) = T + \frac{\alpha_1(\nabla T)^2}{T^2} + \alpha_2 e^{\frac{\delta(\nabla T)^2}{T}}$, their existence and stability conditions, the asymptotic behavior of the scale factor $a(t)$ along with the conditions for expansion and acceleration, and the corresponding values of the dark energy density parameter $\Omega_{DE}$, of the dark energy equation-of-state parameter $w_{DE}$, and of the deceleration parameter $q$. We have defined $Z_{1\pm} = \frac{1}{\alpha_1}(\pm\sqrt{\alpha_1(\alpha_1 - \frac{\delta}{3})})$.

| Name | $\{Z_1, Z_2, Z_3\}$ | Existence | $a(t)$ | $\Omega_{DE}$ | $w_{DE}$ | $q$ | Expansion Acceleration | Stability |
|------|---------------------|-----------|-------|-------------|---------|-----|---------------------|----------|
| $P_1$ | $\left\{ 0, -\frac{\sqrt{2}}{2}, \frac{9\sqrt{2}}{2} \right\}$ | Always | $t^{\frac{2}{\alpha_1}}$ | $\frac{3(4-\gamma)\alpha_1}{2}$ | $\gamma - 1$ | $-1 + \frac{3\gamma}{2}$ | Always | $\gamma < \frac{2}{3}$ | Stable |
| $P_2-$ | $\{0, Z_2^2, 2Z_2^2\}$ | $\alpha_1 < 0$ or $\alpha_1 \geq \frac{1}{6}$ | $t^{-\frac{Z_2^2}{2}}$ | 1 | $-1 - \frac{2Z_2^2}{3}$ | $-1 - Z_2^2$ | Always | $\alpha_1 > \frac{1}{6}$ | No | Saddle |
| $P_2+$ | $\{0, Z_2^2, 2Z_2^2\}$ | $\alpha_1 > 0$ or $\alpha_1 \geq \frac{1}{6}$ | $t^{-\frac{Z_2^2}{2}}$ | 1 | $-1 - \frac{2Z_2^2}{3}$ | $-1 - Z_2^2$ | Always | $\alpha_1 > \frac{1}{6}$ | Stable |
| $P_3$ | $\left\{ \sqrt{\frac{\alpha_1}{6}}, 0, 0 \right\}$ | $\alpha_2 < 0$ | $e^{\sqrt{\frac{\alpha_1}{6}}}t$ | 1 | $-1$ | $-1$ | Always | Always | Stable |

The projection of the phase-space evolution on the $Z_1-Z_2$ plane, for Model I: $F(T, (\nabla T)^2, \Box T) = T + \frac{\alpha_1(\nabla T)^2}{T^2} + \alpha_2 e^{\frac{\delta(\nabla T)^2}{T}}$, for $\gamma = 1$, $\alpha_1 = 1.2$ and $\alpha_2 = 0$, in units where $8\pi G = 1$. The universe is attracted by the quintessence-like stable point $P_{2+}$, marked by the red bullet.
region of the parameter value $\alpha_1$, but it can never be accelerating. However, it is a saddle point and therefore it cannot represent the late-time universe.

Point $P_{2+}$ corresponds to an expanding, dark-energy dominated universe, which can be accelerating for a large region of the model parameter $\alpha_1$, with the scale factor having a power-law form. Its corresponding dark energy equation-of-state parameter lies always in the quintessence regime ($w_{DE} \geq -1$), and it can acquire values very close to the observed ones for large model parameter $\alpha_1$ (for instance $w_{DE} \approx -0.98$ for $\alpha_1 = 10$ in units where $8\pi G = 1$). This fixed point can be stable for a large region of the model parameters.

Point $P_3$ corresponds to an expanding, de Sitter solution, with $\Omega_{DE} = 1$ and equation of state $w_{DE} = w_{tot} = -1$. This fixed point is always accelerating and it is an attractor for a large region of the model parameters.

In summary, $P_{2+}$ and $P_3$ are the most important solutions in the scenario at hand, since they are both stable and possess observables in agreement with observations.

In order to present the above behavior in a more transparent way, we evolve the cosmological equations numerically and in Fig. 2 we depict the corresponding phase-space behavior projected on the $Z_1 - Z_2$ plane, for given values of the model parameters $\alpha_1$, $\alpha_2$ and $\delta$. As we can see, in this specific example the universe results in the dark-energy dominated, accelerating, quintessence-like stable point $P_{2+}$.

Apart from the correct late-time behavior, one should examine whether at early and intermediate times one can reconstruct the standard thermal history of the universe too. Hence, we numerically integrate the Friedmann equations (17, 18), including for completeness the radiation sector, and in the upper graph of Fig. 2 we depict the evolution of the various density parameters, namely $\Omega_{DE0}$ (solid line), $\Omega_{m0}$ (dashed line), and $\Omega_{r}$ (dotted line). In the lower graph we present the evolution of the dark-energy (dotted line) and total (solid line) equation-of-state parameters. The universe is attracted by the de Sitter fixed point $P_3$. For the numerics we have imposed $\Omega_{DE0} \approx 0.72$, $\Omega_{m0} \approx 0.28$, $w_{DE0} \approx -0.94$ and $w_{r0} \approx -0.68$ at present ($z = 0$), in agreement with observations.

**B. Model II:**

$$F(T, (\nabla T)^2, \Box T) = T + \frac{\beta_1 (\nabla T)^2}{T} + \frac{\beta_2 (\Box T)^2}{T^3} + \beta_3 e^{\frac{\sigma X_2}{T^2}}$$

Let us now consider a scenario in which the action does not depend on $X_1 \equiv (\nabla T)^2$ but only on $X_2 \equiv \Box T$, i.e. a scenario of the form

$$F(T, X_1, X_2) = T + \frac{\beta_1 X_2}{T} + \frac{\beta_2 X_2^2}{T^3} + \beta_3 e^{\frac{\sigma X_2}{T^2}}$$

$$T + \frac{\beta_1 \Box T}{T} + \frac{\beta_2 (\Box T)^2}{T^3} + \beta_3 e^{\frac{\sigma X_2}{T^2}}, \quad \text{(37)}$$

with $\beta_1$, $\beta_2$, $\beta_3$ and $\sigma$ the model parameters. We introduce the following five dimensionless variables

$$Z_1 = H, \quad Z_2 = \frac{\dot{H}}{H^2}, \quad Z_3 = \frac{\ddot{H}}{H^3},$$

$$Z_4 = \frac{\dddot{H}}{H^5}, \quad Z_5 = \frac{\ddot{H}}{H^4}. \quad \text{(38)}$$

In terms of these variables, the torsion scalar from (14) and the function $X_2$ from (16) become

$$T = -6Z_1^2,$$

$$X_2 = -12Z_1^2 [Z_3 + Z_2 (3 + Z_2)]. \quad \text{(39)}$$
Table II. The physical critical points of the system (B1) of Model II: $F(T, (\nabla T)^2, (\nabla T^2)^2) = T + \frac{\beta_1 T^3 + \beta_2 (T^2)^2}{2} + \beta_3 e^{\frac{\beta_3 T}{2}}$ for $\beta_2 = \frac{7\beta_3}{24}$, their existence and stability conditions, the asymptotic behavior of the scale factor $a(t)$ along with the conditions for expansion and acceleration, and the corresponding values of the dark energy density parameter $\Omega_{DE}$ of the dark energy equation-of-state parameter $w_{DE}$, and of the deceleration parameter $q$. We have defined $A_\pm = \frac{1}{14} \sqrt{625 - 28\beta_3 (\frac{12}{\beta_3} + 24)}$, $B_\pm = \frac{-17\pm\sqrt{625 - 28\beta_3 (\frac{12}{\beta_3} + 24)}}{14}$, and $s(\gamma) = \frac{3\gamma (2+3\gamma)(7\gamma - 16)(21\gamma - 34)}{544}$.

Hence, the system of the two Friedmann equations (8, 9) is written in its autonomous form as

$$\frac{dZ_1}{dN} = Z_1 Z_2,$$
$$\frac{dZ_2}{dN} = Z_1 - 2Z_2^2,$$
$$\frac{dZ_3}{dN} = Z_1 - 3Z_2 Z_3,$$
$$\frac{dZ_4}{dN} = Z_1 - 4Z_2 Z_4,$$
$$\frac{dZ_5}{dN} = Z_1 - 5Z_2 Z_5. \quad (40)$$

The function $Z_6(Z_1, ..., Z_5)$ is calculated from Eqs. (17) and (18), and is given in (B2). Moreover, in terms of the auxiliary variables and using (19) (20) and (23) (25), we express the observables $\Omega_{DE}$, $w_{DE}$ and $q$ respectively as

$$\Omega_{DE} = \frac{\beta_3}{34992Z_1^6} e^{\frac{Z_1}{(2Z_1 + 2Z_2 + 3Z_3 + 3Z_4 + 3Z_5)^{3/2}}} \left\{ -5832Z_1^6 e^{\frac{18Z_1}{(2Z_1 + 2Z_2 + 3Z_3 + 3Z_4 + 3Z_5)^{3/2}}}, \frac{324Z_1^6}{18Z_1^2} \left\{ Z_4 + 3(1 - Z_2) Z_3 - 6Z_2^2 - 12Z_3^2 \right\}^2 \left\{ 2Z_5 - 2Z_2^2 - 6Z_3^2 - 2Z_4 - 3Z_5 \right\} 3Z_3 - 3Z_2^2 + 33Z_2 \right\}^2 \right\},$$

$$w_{DE} = \gamma - 1 - 11664Z_1^6 \left\{ 2Z_2 + 3\gamma \right\}$$

\[\times \left\{ \left\{ Z_1^3 + 3(1 - Z_2) Z_3 - 6Z_2^2 - 12Z_3^2 \right\}^2 \sigma^2 \left\{ 2Z_5 - 2Z_2^2 - 6Z_3^2 - 2Z_4 - 3Z_5 \right\} 3Z_3 - 3Z_2^2 + 33Z_2 \right\}^2 \right\},$$

$$q = -1 - Z_2. \quad (43)$$

The system (40) cannot be analytically handled in the case of general $\beta_1$ and $\beta_2$. Hence, in order to analytically extract its critical points we need to assume a relation between these two coupling parameters. Without loss of generality, and in order to simplify the expressions, we consider $\beta_2 = \frac{7\beta_3}{34}$. In this case, the scenario of Model
II admits six physical critical points (i.e. real and corresponding to $0 \leq \Omega_{DE} \leq 1$), which are displayed in Table I along with their existence conditions. In the same Table we include the asymptotic behavior of the scale factor $a(t)$ along with the conditions for expansion and acceleration, as well as the corresponding values of the dark energy density parameter $\Omega_{DE}$ calculated from (11), of the dark energy equation-of-state parameter $w_{DE}$ from (12), and of the deceleration parameter $q$ from (43). As we can see from the coordinates of the critical points $Q_1$, $Q_{2\pm}$ and $Q_{3\pm}$ in Table I, they satisfy the constraint (29), and thus they correspond to power-law solutions. On the other hand, the critical point $Q_4$ satisfies the constraint (28), and thus it corresponds to a de Sitter solution. Finally, we include the stability conditions, arising from the investigation of Appendix II.

Point $Q_1$ corresponds to an expanding universe in which the dark-energy density parameter lies in the interval $0 < \Omega_{DE} < 1$, and therefore it can alleviate the coincidence problem, however for usual dust matter it cannot lead to acceleration. It is a saddle one, and hence it cannot attract the universe at late times, nevertheless it could be the state of the universe for large intermediate-time intervals, describing the matter era.

Points $Q_{2\pm}$ and $Q_{3\pm}$ correspond to a dark-energy dominated ($\Omega_{DE} = 1$) expanding universe, which however is always decelerating, and thus not favored by observations. Both points are saddle, and therefore they cannot be the stable late-time solutions of the universe. Additionally, point $Q_{2+}$ corresponds to a contracting universe, and since it is saddle it cannot attract the universe at late times.

Point $Q_{3+}$ corresponds to a dark-energy dominated universe, which can be expanding and accelerating for a large region of the model parameter $\beta_1$, with the scale factor having a power-law form. Its corresponding dark energy equation-of-state parameter lies always in the quintessence regime, and it can acquire values very close to the observed ones for large model parameter $\beta_1$ (for instance $w_{DE} \approx -0.98$ for $\beta_1 = 10$ in units where $8\pi G = 1$). This point can be stable for a large region of the model parameters, in particular for the same range that it is accelerating.

Point $Q_4$ corresponds to a de Sitter solution with $\Omega_{DE} = 1$ and equation of state $w_{DE} = w_{tot} = -1$, it is always accelerating, and it can be stable for a large region of the model parameters.

In summary, $Q_{3+}$ and $Q_4$ are the most important solutions in the scenario at hand, since they are both stable and possess observables in agreement with observations.

In order to present the above behavior more transparently, we evolve the cosmological system numerically and in Fig. 3 we present the corresponding phase-space behavior projected on the $Z_1 - Z_2$ plane. As we observe, in this specific example the universe results in the dark-energy dominated, accelerating, quintessence-like stable point $Q_{3+}$.

Although the present Model II can correctly describe the universe at late times, it cannot provide a very satisfactory behavior at intermediate times, since the exponential term in the Lagrangian (37) cannot remain small for sufficiently large times in order for the standard matter epoch to be reproduced, but still increase at late times in order to drive acceleration (the exponent $(\Gamma T/T^3)$ of the exponential oscillates around zero). Hence, the obtained matter era has smaller duration than the observed one. One may bypass this problem by adding higher order terms in the exponent, such as the quadratic quotient $(\Gamma T/T^3)^2$, which can regularize the oscillations and thus keep the effective dark energy sector very small during a sufficient time in order to reproduce successfully the matter era. However, such a construction would result to a more complicated model, with more complicated phase-space behavior, whose detailed investigation, although necessary and interesting, lies beyond the scope of the present work.

IV. DISCUSSIONS AND FINAL REMARKS

In the present work we constructed classes of modified gravitational theories using higher-derivative torsional terms. In particular, since we know that in curvature gravity one may construct theories with higher-order derivatives in the extended action, which could be justified as arising from higher-loop corrections in the high-curvature regime, one could in principle follow the same direction in torsion formulation of gravity and consider similar corrections to the simple action of Teleparallel Equivalent of general relativity (TEGR). As it is a common feature of all torsional modified gravities, althoughTEGR coincides completely with general relativity at the
level of equations, the corresponding modified scenario with higher-derivative torsion terms, such as \((\nabla T)^2\) and \(\Box T\), is different from its curvature analogue, i.e. it is a novel class of gravitational modification. Hence, it is both interesting and necessary to investigate its cosmological applications.

Extracting the general cosmological equations, we saw that we obtained an effective dark energy sector that comprises of the novel torsional contributions. Similarly to the curvature analogue models, for suitable constructions these novel terms can have a significant contribution, although they arise from higher-order derivatives. We then used the powerful method of dynamical system analysis in order to bypass the complexities of the equations and obtain information about the global, asymptotic behavior of the universe. In particular, we extracted the stable critical points of the scenario, which can thus be the late-time state of the universe, calculating moreover the corresponding observables, such as the various density parameters and the deceleration and dark-energy equation-of-state parameters, as well as the asymptotic form of the scale factor. We examined two specific scenarios, one based on \((\nabla T)^2\) and one based on \(\Box T\) terms.

In the first Model we found that for a wide range of the model parameters the universe can result in a dark-energy dominated, accelerating universe, where the dark-energy equation-of-state parameter lies in the quintessence regime. Additionally, apart from the correct late-time behavior, the model can describe the thermal history of the universe, i.e. the successive sequence of radiation, matter and dark energy epochs, which is a necessary requirement for any realistic scenario. Moreover, during the evolution the dark-energy equation-of-state parameter may exhibit the phantom-divide crossing, which is an additional advantage. Similarly, in the second Model, although the global behavior is richer, we also found that the universe will be led to a dark-energy dominated, accelerating, quintessence-like state, for a wide region of the parameter space. In both Models the scale factor behaves asymptotically either as a power law or as an exponential law, while for large parameter regions the exact value of the dark-energy equation-of-state parameter can be in great agreement with observations.

We mention that the above behavior has been obtained without the use of an explicit cosmological constant, and it is a pure results of the novel higher-derivative torsion terms.

In summary, as we can see, modified gravity with higher-derivative torsion terms can be very efficient in describing the evolution of the universe at the background level. However, before considering it as a successful candidate for the description of nature it is necessary to perform a detailed investigation of its perturbations, since perturbative instabilities may always arise (for instance this is the case in the initial version of Hořava-Lifshitz gravity [36], in the initial version of de Rham-Gabadadze-Tolley massive gravity [37], etc). Although such a detailed and complete analysis of the cosmological perturbations is necessary, its various complications and lengthy calculations make it more convenient to be examined in a separate project [38]. Nevertheless, for the moment we would like to mention that in the case of simple \(f(T)\) gravity, the perturbations of which were examined in detail [39], one may obtain instabilities, but there are many classes of \(f(T)\) ansatzes and/or parameter-space regimes, where the perturbations are well-behaved. This feature is a good indication that we could expect to find a similar behavior in modified gravity with higher torsion derivatives too, although we need to indeed verify this under a thorough perturbation analysis.

ACKNOWLEDGMENTS

The authors would like to thank A. A. Deriglazov, W. G. Ramírez, M. Krššák and F. S. N. Lobo for useful comments. G.O. would like to thank CAPES (Programm PNPD) for financial support. This work was partially supported by the JSPS KAKENHI Grant Number JP 25800136 and the research-funds given by Fukushima University (K. Bamba). This article is also based upon work from COST action CA15117 (CANTATA), supported by COST (European Cooperation in Science and Technology).

Appendix A: Stability analysis of Model I:

\[
F(T, (\nabla T)^2, \Box T) = T + \frac{\alpha_1 (\nabla T)^2}{\Box T} + \alpha_2 e^{\frac{2(\nabla T)^2}{\Box T}}
\]

In this Appendix we investigate the stability of Model I: \(F(T, (\nabla T)^2, \Box T) = T + \frac{\alpha_1 (\nabla T)^2}{\Box T} + \alpha_2 e^{\frac{2(\nabla T)^2}{\Box T}}\) of subsection IIIA. The autonomous form of the system is given in [33], namely

\[
\begin{align*}
\frac{dZ_1}{dN} &= Z_1 Z_2, \\
\frac{dZ_2}{dN} &= Z_3 - 2Z_2^2, \\
\frac{dZ_3}{dN} &= Z_4 - 3Z_2 Z_3,
\end{align*}
\]
where the function $Z_4(Z_1, Z_2, Z_3)$ is calculated from Eqs. [17] and [18] and it reads

\[
Z_4 = -\left\{ \begin{array}{l}
154 Z_1^2 Z_2 \left\{ Z_2^2 + [(\gamma + 1)Z_2 - 10 Z_2^2] Z_3 \\
+ 16 Z_2^4 - 3 (1 + \gamma) Z_2^3 \right\} \delta^2 \\
- 486 \delta Z_1^2 \left[ (4Z_2 - \gamma - 1) Z_3 - 7 Z_2^3 + 4 (1 + \gamma) Z_2^2 - 3 \gamma Z_2 \right] \\
+ 8 Z_2^3 (Z_3 - 3 Z_2^2) 2 \delta^3 + 2187 \gamma Z_2^2 \\
- 1458 Z_2^8 \alpha_1 \left[ 2 (4Z_2 - 3 (1 + \gamma)) Z_3 - 6 Z_2^3 + 9 \gamma Z_2^2 - 18 \gamma Z_2 \right] \\
+ 2187 Z_2^8 (2 Z_2 + 3 \gamma) \right\} \\
\times \left\{ 9 Z_1^2 \left[ \alpha_2 \delta (2 Z_2^2 \delta + 9 Z_2^2) e^{2 \gamma Z_2} + 324 Z_2^6 \alpha_1 \right] \right\}^{-1}.
\] (A2)

The autonomous system [A1] admits four physical critical points (i.e. real and corresponding to $0 \leq \Omega_{DE} \leq 1$), which are displayed in Table I along with their existence conditions.

In order to examine the stability of these critical points, we perform linear perturbations around them as $Z_i = Z_i^* + \delta Z_i$, and thus we extract the perturbation equations as $U' = M \cdot U$, where $U$ is the column vector of the perturbations $\delta Z_i$, and $M$ is the $3 \times 3$ matrix that contains the coefficients of the perturbation equations. The non-zero components of $M$ read

\[
M_{11} = Z_2, \\
M_{12} = Z_1, \\
M_{22} = -4 Z_2, \\
M_{23} = 1, \\
M_{31} = \frac{\partial Z_1}{\partial Z_1}, \\
M_{32} = \frac{\partial Z_2}{\partial Z_2} - 3 Z_3, \\
M_{33} = \frac{\partial Z_3}{\partial Z_3} - 3 Z_2.
\] (A3)

Hence, as usual, the eigenvalues of $M$ determine the type and stability of the specific critical point. In particular, if the eigenvalues have negative real parts then the critical point is stable, if they have positive real parts then the critical point is unstable, and if they have real parts of different sign then the critical point is a saddle one.

For the fixed point $P_1$ the three eigenvalues $\mu_i$ write as

\[
\mu_1 = -\frac{3 \gamma}{2},
\]

\[
\mu_{2,3} = \frac{1}{4 \alpha_1} \left\{ -3 (2 - \gamma) \alpha_1 \\
\pm \left[ 9 (2 - \gamma)^2 \alpha_1^2 - 12 \alpha_1 (2 - 3 (4 - \gamma) \gamma \alpha_1) \right] \right\}. \] (A4)

From the corresponding value of $\Omega_{DE} = \frac{3 (4 - \gamma) \gamma}{2 (1 + \gamma) \gamma}$ depicted in Table I we deduce that the physical condition $0 < \Omega_{DE} < 1$ requires $0 < \alpha_1 < \frac{2}{3 (4 - \gamma) \gamma}$. Hence, for this region $P_1$ is always stable. In particular, for $0 < \alpha_1 < \frac{2}{3 (4 - \gamma) \gamma}$ it is a stable node, whereas for $0 < \alpha_1 < \frac{2}{3 (4 - \gamma) \gamma}$ it is a stable spiral.

For the fixed point $P_2$, the eigenvalues write as

\[
\mu_1 = Z_{2-}^*,
\]

\[
\mu_{2,3} = \frac{1}{2} \left\{ -3 (1 + \gamma + Z_{2-}^*) \pm \left[ 9 (1 + \gamma + Z_{2-}^*)^2 - 2 Z_{2-}^* \left( 12 + 3 \gamma + 4 Z_{2-}^* \right) + \frac{9 \gamma}{\alpha_1 Z_{2-}^*} \right]^{1/2} \right\}. \] (A5)

with

\[
Z_{2-}^* = 3 \left[ -\alpha_1 - \sqrt{\alpha_1 (\alpha_1 - \frac{1}{6})} \right]/\alpha_1.
\]

Hence, this point is stable if

\[
\frac{9 \gamma}{\alpha_1 Z_{2-}^*} - 2 Z_{2-}^* \left( 12 + 3 \gamma + 4 Z_{2-}^* \right) < 0,
\]

\[-1 - \gamma < Z_{2-}^* < 0. \] (A6)

Thus, in the physical range $0 \leq \gamma < 2$, $P_2$ is always saddle.

For the fixed point $P_{2+}$ the eigenvalues write as

\[
\mu_1 = Z_{2+}^*,
\]

\[
\mu_{2,3} = \frac{1}{2} \left\{ -3 (1 + \gamma + Z_{2+}^*) \pm \left[ 9 (1 + \gamma + Z_{2+}^*)^2 - 2 Z_{2+}^* \left( 12 + 3 \gamma + 4 Z_{2+}^* \right) + \frac{9 \gamma}{\alpha_1 Z_{2+}^*} \right]^{1/2} \right\}. \] (A7)

with

\[
Z_{2+}^* = 3 \left[ -\alpha_1 + \sqrt{\alpha_1 (\alpha_1 - \frac{1}{6})} \right]/\alpha_1.
\]

Therefore, this solution is an attractor if it satisfies the conditions

\[
\frac{9 \gamma}{Z_{2+}^* \alpha_1} - 2 Z_{2+}^* \left( 12 + 3 \gamma + 4 Z_{2+}^* \right) < 0,
\]

\[-1 - \gamma < Z_{2+}^* < 0. \] (A8)

Thus, in the physical range $0 \leq \gamma < 2$, $P_{2+}$ is stable for $\alpha_1 > \frac{2}{3 (4 - \gamma) \gamma}$.

For the fixed point $P_3$ the eigenvalues are given by

\[
\mu_1 = -3 \gamma,
\]

\[
\mu_{2,3} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{3 \alpha_2}{2 (\delta + \alpha_1 \alpha_2)}}. \] (A9)

This critical point is an attractor if it satisfies the condition $\frac{3 \alpha_2}{\delta + \alpha_1 \alpha_2} > 0$. More specifically, it is a stable node for $0 < \frac{\alpha_2}{\delta + \alpha_1 \alpha_2} < \frac{3}{2}$, while it is a stable spiral for $\frac{\alpha_2}{\delta + \alpha_1 \alpha_2} > \frac{3}{2}$. 

Appendix B: Stability analysis of Model II:

In this Appendix we investigate the stability of Model II: $F(T, (\nabla T)^2, \Delta T) = T + \frac{\beta_1 (\nabla T)^2}{2} + \frac{\beta_2 (\nabla T)^4}{4} + \beta_3 e^{\frac{\Delta T}{T}}$.

The autonomous system (B1) admits six physical critical points (i.e. real and corresponding to $0 \leq \Omega_{DE} \leq 1$), which are displayed in Table I along with their existence conditions (without loss of generality, and in order to simplify the expressions, we consider $\beta_2 = \frac{\gamma}{2}$). Similarly to Appendix A we perform linear perturbations around these critical points as $Z_i = Z_i^* + \delta Z_i$ and in this case the non-zero components of the $5 \times 5$ perturbation matrix read as

$$
\begin{align*}
M_{11} &= Z_2, & M_{12} &= Z_1, \\
M_{22} &= -4Z_2, & M_{23} &= 1, \\
M_{32} &= -3Z_3, & M_{33} &= -3Z_2, \\
M_{44} &= 1, & M_{42} &= -4Z_4, \\
M_{44} &= -4Z_2, & M_{45} &= 1, \\
M_{51} &= \frac{\partial Z_0}{\partial Z_1}, & M_{52} &= \frac{\partial Z_6}{\partial Z_2} - 5Z_5, \\
M_{53} &= \frac{\partial Z_6}{\partial Z_3}, & M_{54} &= \frac{\partial Z_6}{\partial Z_4}, \\
M_{55} &= \frac{\partial Z_6}{\partial Z_5} - 5Z_2.
\end{align*}
$$

where the function $Z_6(Z_1, ..., Z_5)$ is calculated from Eqs. (17) and (18) and it reads as

$$
Z_6 = \left\{ 324Z_1^4 \left\{ \beta_3 \sigma^2 \left( \frac{|Z_4 + Z_3 (Z_3 + 3)|^2}{4Z_2} - 432Z_1^6 \right) \right\} \right\}
\times \left\{ (6Z_1^3 + 12Z_2 - Z_4 + 3Z_2 Z_3 - 3Z_3)^{\gamma^2 - 18Z_1^2 (Z_4 - 3Z_2 Z_3 + 3Z_3 - 6Z_2^2 - 12Z_2)} \right\}
\times \left\{ 3Z_5 + (3\gamma - 41Z_2 + 12) Z_4 - 9Z_2^2 + [51Z_2^2 - 3(3\gamma + 56)Z_2 + 9\gamma + 9] Z_3 + 228Z_2^4 - 6(3\gamma - 61) Z_2^2 - 36(\gamma + 1) Z_2^2 \right\} \sigma^3
\times \left\{ 32 (4Z_2 + 39 - 39Z_2^2 - 570Z_2^2 + 484Z_2 - 97 + 78Z_2 + 6Z_4 + 72Z_5^2 - 283Z_2^3 - 609Z_2^2 + 383Z_2^3 + 18 - 99\gamma Z_2 - 114Z_2 \right\} \sigma^2
\times \left\{ 17496Z_1^6 \left( 4Z_4 - 48Z_2 Z_3 + 11\gamma Z_3 + 12Z_3 + 84Z_3^2 - 43Z_2^2 - 48Z_2^2 + 33\gamma Z_2 \right) \sigma + 31492Z_1^6 \right\} \beta_3 \ e^{\frac{|Z_4 + Z_3 (Z_3 + 3)|^2}{4Z_2}}.
\right.

Concerning point $Q_1$, the corresponding eigenvalues are given by

$$
\begin{align*}
\mu_1 &= \frac{3\gamma}{2}, \\
\mu_{2,3} &= \frac{3}{4} (2 - \gamma) - \sqrt{C_+} \pm \sqrt{C_-}, \\
\mu_{4,5} &= \frac{3}{4} (2 - \gamma) + \sqrt{C_+} \pm \sqrt{C_-},
\end{align*}
$$

where we have defined the functions

$$
C_\pm = \frac{3}{224 \beta_1} \left[ 485 \beta_1 \pm \sqrt{7\beta_1 (8704 - 25233\beta_1)} \right].
$$

Although we cannot examine the sign of the real parts of the above eigenvalues analytically, numerically one can see that point $Q_1$ is always a saddle one.

In the case of fixed points $Q_{2, \pm}$ and $Q_{3, \pm}$ the first eigenvalue is obviously $\mu_1 = Z_{2, \pm}^2$. However, the other two eigenvalues are complicated and hence we do not give
The only point that can be stable is $Q_{3+}$, in the region $\beta_1 > 0.3$, while all the others points are saddle in their respective ranges of existence.

Finally, for the fixed point $Q_4$ the eigenvalues write as

\[
\mu_1 = -3\gamma, \quad (B8)
\]
\[
\mu_{2,3} = \frac{1}{2} \left[ -3 - \sqrt{D_+} \pm \sqrt{D_-} \right], \quad (B9)
\]
\[
\mu_{4,5} = \frac{1}{2} \left[ -3 + \sqrt{D_+} \pm \sqrt{D_-} \right], \quad (B10)
\]

where we have defined

\[
D_\pm = \frac{-144\beta_3 (3\sigma + \beta_1\beta_3) \pm 18E}{12\sigma^2 + 7\beta_1\beta_3^2} + \frac{9}{2}, \quad (B11)
\]
\[
E = \frac{1}{4} \left( 4\sigma^2 + \frac{7}{3} \beta_1 \beta_3^2 \right)^{1/2} \times \left[ 36\sigma^2 - 576\beta_3 \sigma + (128 - 171\beta_1) \beta_3^2 \right]^{1/2}. \quad (B12)
\]

Numerically, we find that fixed point $Q_4$ is always an attractor (stable node or stable spiral).

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