HOMOGENEOUS MULTIGRID FOR EMBEDDED DISCONTINUOUS GALERKIN METHODS

PEIPEI LU, ANDREAS RUPP, AND GUIDO KANSCHAT

Abstract. We introduce a homogeneous multigrid method in the sense that it uses the same embedded discontinuous Galerkin (EDG) discretization scheme for Poisson’s equation on all levels. In particular, we use the injection operator developed in [LRK20] for HDG and prove optimal convergence of the method under the assumption of elliptic regularity. Numerical experiments underline our analytical findings.

Keywords. Multigrid method, embedded discontinuous Galerkin, Poisson equation.

1. Introduction

As described in [CGSS09], the embedded discontinuous Galerkin (EDG) method can be obtained from the hybridizable discontinuous Galerkin (HDG) methods by replacing the space for the hybrid unknown by an overall continuous space. Thus, the stiffness matrix is significantly smaller, and its size and sparsity structure coincide with those of the stiffness matrix of the statically condensed continuous Galerkin method. Additionally, the condition number of the resulting EDG system is smaller than the one of the HDG system. However, [CGSS09] underlines that the computational advantage has to be balanced against the fact that the approximate solutions of the primary and flux unknowns both lose a full order of convergence.

EDG schemes and their variants have gained some popularity over the last decade. They have, for example, been successfully applied to advection–diffusion [FS17], Stokes [RW20], Euler and Navier–Stokes equations [PNC11, NPC15], distributed optimal control for elliptic problems [ZZS18], Dirichlet boundary control for advection–diffusion [CFSZ19], and compared to stabilized, residual-based finite elements [Kam16]. However, to the best of our knowledge, no multigrid method
is available for EDG schemes. Thus, we propose the first (homogeneous) multigrid method for the embedded discontinuous Galerkin method.

Homogeneous multigrid methods use the same discretization scheme on all levels. Such methods are important, since they have the same mathematical properties on all levels. They are also advantageous from a computational point of view, since their data structures and execution patterns are more regular.

Our considerations are based on the analysis techniques for multigrid methods applied to HDG discretizations. The first of these methods has been introduced in [CDGT13, Tan09], while similar results have been obtained for hybrid Raviart–Thomas (RT) schemes in [GT09]. However, all these schemes fall back to linear finite elements and therefore cannot be called “homogeneous”. The first homogeneous multigrid method for hybrid discontinuous Galerkin schemes has finally been introduced in [LRK20].

Thus, the structure of the analysis conducted in this manuscript is similar to the one in [LRK20] and uses the same notation and some results from [LRK20], but the proof technique demonstrated in the following is significantly different.

The remainder of this paper is structured as follows: In Section 2, we briefly review the EDG method for the considered elliptic PDE. Furthermore, an overview over the used function spaces, scalar products, and operators is given. Section 3 is devoted to a brief explanation of the multigrid and states the assumptions for its main convergence result. Sections 4 and 5 verify the assumptions of the main convergence result, while Section 6 underlines its validity by numerical experiments. Short conclusions wrap up the paper.

2. Model equation and discretization

We consider the Dirichlet boundary value problem for Poisson’s equation

\[ -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \] (2.1)

defined on a polygonally bounded domain \( \Omega \subset \mathbb{R}^d \). The flux vector is \( \mathbf{q} = -\nabla u \). In the analysis, we will assume elliptic regularity, namely \( u \in H^2(\Omega) \) if \( f \in L^2(\Omega) \), such that there is a constant \( c > 0 \) for which

\[ |u|_{H^2(\Omega)} \leq c \|f\|_{L^2(\Omega)} \] (2.2)

holds. The domain \( \Omega \) is discretized by a hierarchical sequence of triangulations \( T_\ell \) for \( \ell = 1, \ldots, L \). We assume that each simplicial mesh is topologically regular such that each facet of a cell is either a facet of another cell or on the boundary. The sequence is shape regular in the usual sense. The sequence is constructed recursively from a coarse mesh \( T_0 \) by refinement, such that each cell of the mesh \( T_{\ell-1} \) on level \( \ell - 1 \) is the union of cells of \( T_\ell \). The meshes are assumed quasi-uniform.
such that the typical diameter of a cell of the mesh on level $\ell$ is $h_\ell$. Finally, we assume that refinement from one level to the next is bounded in the sense that there is a constant $c_{\text{ref}} > 0$ with

$$h_\ell \geq c_{\text{ref}} h_{\ell-1}. \quad (2.3)$$

By $F_\ell$ we denote the set of faces of $T_\ell$. The subset of faces on the boundary is

$$F^D_\ell := \{ F \in F_\ell : F \subset \partial \Omega \}. \quad (2.4)$$

Moreover, we define $F^T_\ell := \{ F \in F_\ell : F \subset \partial T \}$ as the set of faces of a cell $T \in T_\ell$. We identify the set $F_\ell$ as a set of faces with the union of these faces as a subset of $\Omega$, such that the notion of function spaces $C^k(F_\ell)$ and $L^2(F_\ell)$ are meaningful. The latter is equipped with the inner product

$$\langle \langle \lambda, \mu \rangle \rangle_\ell = \sum_{T \in T_\ell} \int_{\partial T} \lambda \mu \, d\sigma, \quad (2.5)$$

and its induced norm $\| \mu \|^2_\ell = \langle \langle \mu, \mu \rangle \rangle_\ell$. Note that interior faces appear twice in this definition such that expressions like $\langle \langle u, \mu \rangle \rangle_\ell$ with possibly discontinuous $u|_T \in H^1(T)$ for all $T \in T_\ell$ and $\mu \in L^2(F_\ell)$ are defined without further ado. Additionally, we define an inner product commensurate with the $L^2$-inner product in the bulk domain, namely

$$\langle \lambda, \mu \rangle_\ell = \sum_{T \in T_\ell} \frac{|T|}{|\partial T|} \int_{\partial T} \lambda \mu \, d\sigma \cong \sum_{F \in F_\ell} h_F \int_F \lambda \mu \, d\sigma. \quad (2.6)$$

Its induced norm is $\| \mu \|^2_\ell = \langle \langle \mu, \mu \rangle \rangle_\ell$.

Let $p \geq 1$ and $P_p$ be the space of multivariate polynomials of degree up to $p$. EDG method can be obtained from corresponding HDG methods by replacing the HDG skeletal space by the EDG skeletal space

$$\tilde{M}_\ell := \left\{ \lambda \in C^0(F_\ell) \middle| \begin{array}{l} \lambda|_F \in P_p \quad \forall F \in F_\ell, \; F \not\subset \partial \Omega \\ \lambda|_F = 0 \quad \forall F \in F_\ell, \; F \subset \partial \Omega \end{array} \right\}. \quad (2.7)$$

The EDG method involves a local solver on each mesh cell $T \in T_\ell$ which can be understood as an approximate Dirichlet to Neumann map on each mesh cell. It is written in mixed form, producing cellwise approximate diffusion solutions $u_T \in V_T$ and $q_T \in W_T$, respectively, by solving for given boundary values $\lambda$

$$\int_T q_T \cdot p_T \, dx - \int_T u_T \nabla \cdot p_T \, dx = - \int_{\partial T} \lambda p_T \cdot \nu \, d\sigma$$

$$- \int_T q_T \cdot \nabla v_T \, dx + \int_{\partial T} (q_T \cdot \nu + \tau_\ell v_T) v_T \, d\sigma = \tau_\ell \int_{\partial T} \lambda v_T \, d\sigma \quad (2.8a)$$

for all $v_T \in V_T$, and all $p_T \in W_T$. Here, $\nu$ is the outward unit normal with respect to $T$ and $\tau_\ell > 0$ is the penalty coefficient of the method.
We choose $V_T = \mathcal{P}_p$. Then, choosing $W_T = \mathcal{P}_p^d$ yields an analogue of the so called hybridizable local discontinuous Galerkin (LDG-H) scheme, i.e., the embedded local discontinuous Galerkin (LDG-E) scheme.

Our current analysis is in fact limited to this case and other choices require a modification of Lemma A.1 and Lemma A.2.

While the local solvers are implemented cell by cell, it is helpful for the analysis to combine them by concatenation. To this end, we introduce the spaces

$$V_\ell := \{ v \in L^2(\Omega) \mid v|_T \in V_T, \ \forall T \in \mathcal{T}_\ell \},$$

$$W_\ell := \{ q \in L^2(\Omega; \mathbb{R}^d) \mid q|_T \in W_T, \ \forall T \in \mathcal{T}_\ell \}. \tag{2.9}$$

Hence, the local solvers define a mapping

$$\tilde{M}_\ell \rightarrow V_\ell \times W_\ell$$

$$\lambda \mapsto (U_\ell \lambda, Q_\ell \lambda), \tag{2.10}$$

where for each cell $T \in \mathcal{T}_\ell$ holds $U_\ell \lambda = u_T$ and $Q_\ell \lambda = q_T$. In the same way, we define operators $U_\ell f$ and $Q_\ell f$ for $f \in L^2(\Omega)$, where now the local solutions are defined by the system

$$\int_T q_T \cdot p_T \, dx - \int_T u_T \nabla \cdot p_T \, dx = 0 \tag{2.11a}$$

$$- \int_T q_T \cdot \nabla v_T \, dx + \int_{\partial T} (q_T \cdot \nu + \tau_\ell u_T) v_T \, d\sigma = \int_T f v_T \, dx. \tag{2.11b}$$

Once $\lambda$ has been computed, the EDG approximation to the solution of the Poisson problem and its gradient on mesh $\mathcal{T}_\ell$ will be computed as

$$u_\ell = U_\ell \lambda + U_\ell f$$
$$q_\ell = Q_\ell \lambda + Q_\ell f \tag{2.12}$$

The global coupling condition is derived through a discontinuous Galerkin version of mass balance and reads: Find $\lambda \in \tilde{M}_\ell$, such that for all $\mu \in \tilde{M}_\ell$

$$\sum_{T \in \mathcal{T}_\ell} \sum_{F \notin F_T} \int_F (q_\ell \cdot \nu + \tau_\ell (u_\ell - \lambda)) \, d\sigma = 0. \tag{2.13}$$

EDG can be formulated in a condensed version as finding $\lambda \in \tilde{M}_\ell$ such that

$$a_\ell(\lambda, \mu) = b(\mu) \quad \forall \mu \in \tilde{M}_\ell \tag{2.14a}$$

with the bilinear form $a_\ell(\cdot, \cdot)$ and linear form $b_\ell(\cdot)$ defined by

$$a_\ell(\lambda, \mu) = \langle Q_\ell \lambda, Q_\ell \mu \rangle_\Omega + \langle \tau_\ell (U_\ell \lambda - \lambda), (U_\ell \mu - \mu) \rangle_{\ell}, \tag{2.14b}$$

$$b_\ell(\mu) = \langle U_\ell \mu, f \rangle_\Omega. \tag{2.14c}$$

where $(\cdot, \cdot)_\Omega$ is the standard inner product in $L^2(\Omega)$ and $L^2(\Omega; \mathbb{R}^d)$, respectively. Note that this bilinear form is defined in the same way.
as the one for the HDG method. And since $\tilde{M}_\ell$ is a subspace of the corresponding space of the HDG method, it is symmetric and positive definite [CGL09]. Hence, it is a scalar product, and induces a norm denoted by $\| \cdot \|_{a_\ell}$.

We associate an operator $A_\ell : \tilde{M}_\ell \to \tilde{M}_\ell$ with the bilinear form $a_\ell(\cdot, \cdot)$ by the relation

$$\langle A_\ell \lambda, \mu \rangle_\ell = a_\ell(\lambda, \mu) \quad \forall \mu \in \tilde{M}_\ell.$$  \hfill (2.15)

Additionally, we introduce $V^c_\ell := \{ u \in H^1_0(\Omega) : u|_T \in P_1(T) \forall T \in T_\ell \}$, \hfill (2.16)

and the $L^2$ projections

$$\Pi^0_\ell : H^2(\Omega) \cap H^1_0(\Omega) \to \tilde{M}_\ell, \quad \langle \Pi^0_\ell u, \mu \rangle_\ell = \langle \gamma_\ell u, \mu \rangle_\ell \quad \forall \mu \in \tilde{M}_\ell,$$  \hfill (2.17)

$$\Pi^c_\ell : H^1_0(\Omega) \to V^c_\ell, \quad (\Pi^c_\ell u, w)_0 = (u, w)_0 \quad \forall w \in V^c_\ell,$$  \hfill (2.18)

$$\Pi^d_\ell : H^1(\Omega) \to V_\ell, \quad (\Pi^d_\ell u, w)_0 = (u, w)_0 \quad \forall w \in V_\ell,$$  \hfill (2.19)

with trace operator $\gamma_\ell$ to be used in our analysis. Obviously, we have

$$\| u - \Pi^c_\ell u \|_\ell + \| u - \Pi^0_\ell u \|_\ell \lesssim h^2_\ell |u|_2, \quad \text{(trace approx.)}$$  \hfill (2.20)

$$\| u - \Pi^c_\ell u \|_0 + \| u - \Pi^0_\ell u \|_0 \lesssim h_\ell |u|_1, \quad \text{(L}^2 \text{ approx.)}$$  \hfill (2.21)

$$|\Pi^c_\ell u|_1 \lesssim |u|_1. \quad \text{(H}^1 \text{ stab.)}$$  \hfill (2.22)

Here and in the following, $\lesssim$ has the meaning of smaller than or equal to up to a constant only dependent on the regularity constant of the mesh family and $c_{\text{ref}}$.

3. Multigrid method and main convergence result

We consider a standard (symmetric) V-cycle multigrid method for (2.14) (cf. [BPX91]). Since we deal with noninherited forms, we use [DGTZ07] as a base for our convergence analysis. Section 3.2 recites the multigrid method and Section 3.3 states the main convergence result. First, we recall an estimate for eigenvalues and condition numbers of the matrices $A_\ell$.

Lemma 3.1. Suppose that $T_\ell$ is quasiuniform. Then, there are positive constants $C_1$ and $C_2$ independent of $\ell$ such that

$$C_1 \| \lambda \|^2_\ell \leq a_\ell(\lambda, \lambda) \leq C_2 \beta_\ell h_\ell^{-2} \| \lambda \|_{a_\ell}^2, \quad \forall \lambda \in \tilde{M}_\ell,$$  \hfill (3.1)

where $\beta_\ell := 1 + (\tau_\ell h_\ell)^2$.

Proof. This is a Corollary of [CDGT13, Theo. 3.2] exploiting the fact that the skeletal space of EDG is a subset of the skeletal space for HDG. \hfill $\square$
This implies that for the stiffness matrix, we can bound the condition number $\kappa_\ell$ by
\[ \kappa_\ell \lesssim \beta_\ell h_\ell^{-2} \quad (3.2) \]
which implies that for all choices of $\tau_\ell$ satisfying $\tau_\ell \lesssim h_\ell^{-1}$ the condition number grows at most like $h_\ell^{-2}$.

### 3.1. The injection operator $I_\ell$

The difficulty of devising an “injection operator” $I_\ell: \tilde{M}_{\ell-1} \rightarrow \tilde{M}_\ell$ originates from the fact that the finer mesh has edges which are not refinements of the edges of the coarse mesh. In order to assign reasonable values to these edges, we construct the injection operator similar to the HDG injection operator of [LRK20] in three steps. First, we introduce the continuous finite element space
\[ V_\ell^c := \{ u \in H_0^1(\Omega) \mid u|_T \in P_p(T) \quad \forall T \in \mathcal{T}_\ell \}, \quad (3.3) \]
and define the shape function basis on each mesh cell $T$ by a Lagrange interpolation condition with respect to support points $x$. Afterwards, the continuous extension operator
\[ U_\ell^c : \tilde{M}_\ell \rightarrow V_\ell^c, \quad (3.4) \]
can be defined using those interpolation conditions
\[ [U_\ell^c \lambda](x) = \begin{cases} \lambda(x) & \text{if } x \text{ is located on a face,} \\ [U_\ell \lambda](x) & \text{if } x \text{ is in the interior of a cell.} \end{cases} \quad (3.5) \]
Note that due to the continuity of the EDG method in vertices (and edges in three dimensions), no special handling of degrees of freedom there is needed and they are covered by the first line of the definition of $U_\ell^c \lambda$.

Since $V_{\ell-1}^c \subset V_\ell^c$, there is a natural embedding
\[ I_\ell^c : V_{\ell-1}^c \rightarrow V_\ell^c \]
\[ u \mapsto u. \quad (3.6) \]

On $V_\ell^c$ the trace on edges is well defined, such that we can write
\[ \gamma_\ell : V_\ell^c \rightarrow \tilde{M}_\ell \]
\[ u \mapsto \gamma_\ell u. \quad (3.7) \]

Using these three operators, we define the injection operator $I_\ell$ as their concatenation, namely
\[ I_\ell : \tilde{M}_{\ell-1} \rightarrow \tilde{M}_\ell \]
\[ \lambda \mapsto \gamma_\ell I_\ell^c U_\ell^c \lambda. \quad (3.8) \]

Since the EDG approximation space $\tilde{M}_\ell$ is a subspace of the HDG approximation space, the following Lemma is straightforward from [LRK20, Lem. 2.1].
Lemma 3.2 (Boundedness). When $\tau_\ell = \frac{c}{h_\ell}$, the injection operator $I_\ell$ is bounded in the sense that

$$a_\ell(I_\ell \lambda, I_\ell \lambda) \lesssim a_{\ell-1}(\lambda, \lambda) \quad \forall \lambda \in \hat{M}_{\ell-1}. \tag{3.9}$$

After the injection operator $I_\ell$ has been defined, we introduce two operators from $\tilde{M}_\ell$ to $\tilde{M}_{\ell-1}$, which replace the $L^2$-projection and the Ritz projection of conforming methods, respectively. They are $\Pi_{\ell-1}$ and $P_{\ell-1}$ defined by the conditions

$$\Pi_{\ell-1} : \tilde{M}_\ell \to \tilde{M}_{\ell-1}, \quad \langle \Pi_{\ell-1} \lambda, \mu \rangle_{\ell-1} = \langle \lambda, I_\ell \mu \rangle_\ell \quad \forall \mu \in \tilde{M}_{\ell-1}. \tag{3.10}$$

$$P_{\ell-1} : \tilde{M}_\ell \to \tilde{M}_{\ell-1}, \quad a_{\ell-1}(P_{\ell-1} \lambda, \mu) = a_\ell(\lambda, I_\ell \mu) \quad \forall \mu \in \tilde{M}_{\ell-1}. \tag{3.11}$$

The operator $\Pi_{\ell-1}$ is used in the implementation, while $P_{\ell-1}$ is key to the analysis.

3.2. Multigrid algorithm. Assume that we have an injection operator $I_\ell : \tilde{M}_{\ell-1} \to \tilde{M}_\ell$ for grid transfer. Actually, this has been defined in section 3.1. Assume further a smoother denoted by $R_\ell : \tilde{M}_{\ell-1} \to \tilde{M}_{\ell-1}$. In this manuscript we consider point smoothers in terms of Jacobi or Gauss-Seidel iterations, respectively. Denote by $R_\ell^i$ the adjoint operator of $R_\ell$ with respect to $\langle \cdot, \cdot \rangle_\ell$ and define $R_\ell^i$ by

$$R_\ell^i = \begin{cases} R_\ell & \text{if } i \text{ is odd}, \\ R_\ell^i & \text{if } i \text{ is even}. \end{cases} \tag{3.13}$$

Let $m \geq 1$ denote the number of smoothing steps. We recursively define the multigrid operator of the refinement level $\ell$

$$B_\ell : \tilde{M}_\ell \to \tilde{M}_\ell. \tag{3.14}$$

First, $B_0 = A_0^{-1}$. For $\ell > 0$ and for $\mu \in \tilde{M}_\ell$ define $B_\ell \mu$ as follows: let $x^0 = 0 \in \tilde{M}_\ell$.

1. Define $x^i \in \tilde{M}_\ell$ for $i = 1, \ldots, m$ by

$$x^i = x^{i-1} + R_\ell^i(\mu - A_\ell x^{i-1}). \tag{3.15}$$

2. Set $y^0 = x^m + I_\ell q$, where $q \in \tilde{M}_{\ell-1}$ is defined as

$$q = B_{\ell-1} \Pi_{\ell-1}(\mu - A_\ell x^m). \tag{3.16}$$

3. Define $y^i \in \tilde{M}_\ell$ for $i = 1, \ldots, m$ as

$$y^i = y^{i-1} + R_\ell^{i+m}(\mu - A_\ell y^{i-1}). \tag{3.17}$$

4. Let $B_\ell \mu = y^m$. 
3.3. **Main convergence result.** The analysis of the multigrid method is based on the framework introduced in [DGTZ07]. There, convergence is traced back to three assumptions. Let \( \lambda^j_\ell \) be the largest eigenvalue of \( A_\ell \), and

\[
K_\ell := (1 - (1 - R_\ell A_\ell)(1 - R_\ell^\dagger A_\ell)) A_\ell^{-1}. \tag{3.18}
\]

Then, there exists constants \( C_1, C_2, C_3 > 0 \) independent of the mesh level \( \ell \), such that there holds

- **Regularity approximation assumption:**
  \[
  |a_\ell(\lambda - I_\ell P_{\ell-1}\lambda, \lambda)| \leq C_1 \frac{\|A_\ell \lambda\|_\ell^2}{\lambda^A} \quad \forall \lambda \in \tilde{M}_\ell. \quad (A1)
  \]
- **Stability of the “Ritz quasi-projection” \( P_{\ell-1} \) and injection \( I_\ell :\)
  \[
  \|\lambda - I_\ell P_{\ell-1}\lambda\|_{a_\ell} \leq C_2 \|\lambda\|_{a_\ell} \quad \forall \lambda \in \tilde{M}_\ell. \quad (A2)
  \]
- **Smoothing hypothesis:**
  \[
  \|\lambda\|_{\ell}^2 \leq C_3 \langle K_\ell \lambda, \lambda \rangle_\ell. \quad (A3)
  \]

Theorem 3.1 in [DGTZ07] reads

**Theorem 3.3.** Assume that (A1), (A2), and (A3) hold. Then for all \( \ell \geq 0 \),

\[
|a_\ell(\lambda - B_\ell A_\ell \lambda, \lambda)| \leq \delta a_\ell(\lambda, \lambda), \tag{3.19}
\]

where

\[
\delta = \frac{C_1 C_2}{m - C_1 C_3} \quad \text{with} \quad m > 2C_1 C_3. \tag{3.20}
\]

Thus, in order to prove uniform convergence of the multigrid method, we will now set out to verify these assumptions.

4. **Proof of (A1)**

To show (A1), we follow the lines of [DGTZ07, Sect. 4] and verify the assumption of

**Theorem 4.1 (Sufficient conditions for (A1)).** If \( \tau_\ell h_\ell \lesssim 1 \) and

\[
\|\lambda - I_\ell P_{\ell-1}\lambda\|_\ell \lesssim h_\ell^2 \|A_\ell \lambda\|_\ell \tag{B1}
\]

holds for all \( \lambda \in \tilde{M}_\ell \) and for all \( \ell \), (A1) is satisfied.

**Proof.**

\[
|a_\ell(\lambda - I_\ell P_{\ell-1}\lambda, \lambda)| \overset{2.15}{=} |\langle \lambda - I_\ell P_{\ell-1}\lambda, A_\ell \lambda \rangle_\ell| \leq \|\lambda - I_\ell P_{\ell-1}\lambda\|_\ell \|A_\ell \lambda\|_\ell \overset{(B1)}{\lesssim} h_\ell^2 \|A_\ell \lambda\|_\ell^2 \overset{\text{Lem. 3.1}}{\lesssim} \frac{\|A_\ell \lambda\|_\ell^2}{\lambda^A}. \tag{4.1}
\]

\[\square\]
In order to prove (B1), we construct some auxiliary quantities. Let
\[ V_{p+3}^c = \{ u \in H_0^c(\Omega) \mid u|_T \in P_{p+3}(T) \ \forall T \in T_l \}. \] (4.2)

For all \( \lambda \in \hat{M}_l \) define \( S_\ell \lambda \in V_{p+3}^c \) satisfying
\[ (S_\ell \lambda, v)_T = (U_\ell \lambda, v)_T \quad \forall v \in P_p(T), \] (4.3a)
\[ (S_\ell \lambda, \eta)_F = (\lambda, \eta)_F \quad \forall \eta \in P_{p+1}(F), F \subset \partial T, \] (4.3b)
\[ S_\ell \lambda(a) = \lambda(a) \quad a \text{ is a vertex of } T. \] (4.3c)

Actually, this is also the definition of the \( H^1 \)-conforming finite element in [GR86, Lem. A.3]. By the standard scaling argument and using Lemma A.2 when \( \tau_\ell h_\ell \lesssim 1 \), we have
\[ \| S_\ell \lambda \|_0 \simeq \| \lambda \|_\ell \quad \forall \lambda \in \hat{M}_l. \] (4.4)

That is, \( (S_\ell, S_\ell) \)_0 is an inner product on \( \hat{M}_l \). Thus, for all \( \lambda \in \hat{M}_l \), there is a \( \phi_\lambda \in \tilde{M}_l \) such that
\[ (S_\ell \phi_\lambda, S_\ell \mu)_0 = a_\ell(\lambda, \mu) = (A_\ell \lambda, \mu)_\ell \quad \forall \mu \in \tilde{M}_l. \] (4.5)

We denote \( f_\lambda = S_\ell \phi_\lambda \), define \( \tilde{u} \) as solution of
\[ - \Delta \tilde{u} = f_\lambda \quad \text{in } \Omega, \quad \tilde{u} = 0 \quad \text{on } \partial \Omega \] (4.6)

and let \( \tilde{\lambda}_\ell \in \tilde{M}_l \) be the EDG approximation of (4.6), i.e.,
\[ a_\ell(\tilde{\lambda}_\ell, \mu) = (f_\lambda, U_\ell \mu)_0 \quad \forall \mu \in \tilde{M}_l. \] (4.7)

From (4.4) and (4.5), we have for \( \mu = \phi_\lambda \) that
\[ \| f_\lambda \|_0^2 = \| S_\ell \phi_\lambda \|_0^2 = (A_\ell \lambda, \phi_\lambda)_\ell \lesssim \| A_\ell \lambda \|_\ell \| \phi_\lambda \|_\ell \lesssim \| A_\ell \lambda \|_\ell \| S_\ell \phi_\lambda \|_0 \] (4.8)

which implies that
\[ \| f_\lambda \|_0 \lesssim \| A_\ell \lambda \|_\ell. \] (4.9)

Lemma 4.2. If \( w \in \nabla_{\ell} \), then
\[ S_{\ell \gamma} w = U_{\ell \gamma} w = w. \] (4.10)

Lemma 4.3. Assuming that \( \tau_\ell h_\ell \lesssim 1 \), we have for all \( \lambda \in \hat{M}_l \)
\[ |S_{\ell \lambda}| \lesssim \| Q_{\ell \lambda} \|_0. \] (4.11)

Proof. First, the definition of \( U_{\ell} \) and Lemma A.1 imply that
\[ \| U_{\ell} \lambda - U_{\ell} \lambda \|_0 \lesssim \| U_{\ell} \lambda - U_{\ell} \lambda \|_\ell = \| \lambda - U_{\ell} \lambda \|_\ell \lesssim h_\ell \| Q_{\ell \lambda} \|_0. \] (4.12)

Additionally, using the inverse inequality, [CLX14, Lem. 3.3] (stating that \( \| Q_{\ell \lambda} + \nabla U_{\ell} \lambda \|_0 \lesssim h_\ell^{-1} \| U_{\ell} \lambda - \lambda \|_\ell \)), and the aforementioned inequality
\[ \| U_{\ell} \lambda \|_1 \lesssim h_\ell^{-1} \| U_{\ell} \lambda \|_{1,T_\ell} + h_\ell^{-1} \| U_{\ell} \lambda - U_{\ell} \lambda \|_0 \] (4.13)
\[ \lesssim \| Q_{\ell \lambda} \|_0 + h_\ell^{-1} \| U_{\ell} \lambda - \lambda \|_\ell \lesssim \| Q_{\ell \lambda} \|_0. \]
Here, \( | \cdot |_{1,T} \) denotes the broken (i.e. elementwise) \( H^1 \)-seminorm of an elementwise \( H^1 \) function. Thus,

\[
\| S_\ell \lambda - \Pi_\ell U_\ell^c \lambda \|_0 \overset{\text{Lem. 4.2}}{=} \| S_\ell \lambda - S_\ell \gamma_\ell \Pi_\ell U_\ell^c \lambda \|_0 \overset{\text{(4.4)}}{=} \| \lambda - \gamma_\ell \Pi_\ell U_\ell^c \lambda \|_\ell \\
= \| \gamma_\ell U_\ell^c \lambda - \gamma_\ell \Pi_\ell U_\ell^c \lambda \|_\ell \lesssim \| U_\ell^c \lambda - \Pi_\ell U_\ell^c \lambda \|_0
\]

where the second inequality holds since, \( \langle \cdot, \cdot \rangle_\ell \) is commensurate with the \( L^2 \) inner product in the bulk domain.

\[
|S_\ell \lambda|_1 \leq |S_\ell \lambda - \Pi_\ell U_\ell^c \lambda|_1 + |\Pi_\ell U_\ell^c \lambda|_1 \overset{\text{inv. ineq.}}{\lesssim} h_\ell^{-1} \| S_\ell \lambda - \Pi_\ell U_\ell^c \lambda \|_0 + |U_\ell^c \lambda|_1 \overset{\text{(4.14)}}{\lesssim} |U_\ell^c \lambda|_1 \overset{\text{(4.13)}}{\lesssim} \| Q_\ell \lambda \|_0.
\]

\( \square \)

**Lemma 4.4.** When \( \tau_\ell = \frac{c}{h_\ell} \), let for \( \lambda \in \tilde{M}_\ell \tilde{\lambda}_\ell \in \tilde{M}_\ell \) be the solution of (4.7). We have

\[
\| \lambda - \tilde{\lambda}_\ell \|_{a_\ell} \leq h_\ell \| A_\ell \lambda \|_\ell, \\
\| \lambda - \tilde{\lambda}_\ell \|_\ell \leq h_\ell^2 \| A_\ell \lambda \|_\ell.
\]

To prove this result, we need the following EDG convergence result

**Lemma 4.5.** When \( \tau_\ell = \frac{c}{h_\ell} \), let for \( \tilde{\lambda}_\ell \in \tilde{M}_\ell \) be the solution of (4.7). We have

\[
\| \tilde{\lambda}_\ell - \Pi_\ell^0 \tilde{u} \|_\ell \lesssim h_\ell^2 |\tilde{u}|_2
\]

**Proof.** Lemma 3.1 in [CGSS09] states that

\[
\| \tilde{q} - \tilde{q}_\ell \|_0^2 \lesssim \| \tilde{q} - \Pi_\ell^{RT} \tilde{q} \|_0^2 + \| I_\ell^{int} \tilde{u} - \tilde{u} \|_2^2 + \frac{1}{\tau_\ell} \| \Pi_\ell^{RT} \tilde{q} - \Pi_\ell^0 \tilde{q} \|_2^2
\]

where \( \tilde{q} = -\nabla \tilde{u} \), \( \Pi_\ell^{RT} \) is similar to the standard Raviart–Thomas projection, but has fewer constraints, cf. [CGSS09, (3.2)]. That is, for all \( T \in \mathcal{T}_\ell \), the projection \( \Pi_\ell^{RT} \) suffices

\[
(\Pi_\ell^{RT} \tilde{q}, \upsilon)_T = (\tilde{q}, \upsilon)_T \quad \forall \upsilon \in \mathcal{P}_d(T), \\
(\Pi_\ell^{RT} \tilde{q}, \cdot, \eta)_F = (\tilde{q}, \cdot, \eta)_F \quad \forall \eta \in \mathcal{P}_d(F),
\]

for all \( F \subset \partial T \), but one. \( I_\ell^{int} \) is the continuous interpolant obeying the Dirichlet constraints. Plugging this into [CGSS09, Theo. 2.3], we obtain

\[
\| \tilde{\lambda}_\ell - \Pi_\ell^0 \tilde{u} \|_\ell \leq h_\ell \| \tilde{q} - \tilde{q}_\ell \|_0 \lesssim h_\ell^2 |\tilde{u}|_2.
\]

\( \square \)
Proof of Lemma 4.4. Since
\[ a_\ell(\lambda, \mu) = (f_\lambda, S_\ell\mu)_0 \quad \text{and} \quad a_\ell(\tilde{\lambda}_\ell, \mu) = (f_\lambda, \mathcal{U}_\ell\mu)_0, \]
we have
\[ a_\ell(\lambda - \tilde{\lambda}_\ell, \mu) = (f_\lambda, S_\ell\mu - \mathcal{U}_\ell\mu)_0. \]
This yields for all \( \mu \in \tilde{M}_\ell \)
\[ \|S_\ell\mu - \mathcal{U}_\ell\mu\|_0 = \|S_\ell\mu - \Pi^d_\ell S_\ell\mu\|_{L^2} \leq h_\ell\|S_\ell\mu\|_1 \overset{\text{Lem. 4.3}}{\lesssim} h_\ell\|Q_\ell\mu\|_0 \leq h_\ell\|\mu\|_{a_\ell}. \]
Setting \( \mu = \lambda - \tilde{\lambda}_\ell \) in (4.22), we get
\[ \|\lambda - \tilde{\lambda}_\ell\|_{a_\ell} \overset{(4.9)}{\lesssim} h_\ell\|f_\lambda\|_0 \lesssim h_\ell\|\lambda - \tilde{\lambda}_\ell\|_{a_\ell}. \]
This also implies that
\[ \|\lambda - \tilde{\lambda}_\ell\|_{a_\ell} \lesssim h_\ell\|f_\lambda\|_0 \overset{(4.9)}{\lesssim} h_\ell\|A_\ell\lambda\|_{\ell}. \]
This is the first inequality.
In the following, we will utilize the duality argument to prove the lemma’s second inequality: Suppose
\[- \Delta \psi = S_\ell(\lambda - \tilde{\lambda}_\ell) \quad \text{in} \quad \Omega \quad \text{and} \quad \psi = 0 \text{ on } \partial \Omega \]
and that \( \tilde{\rho}_\ell \) is the EDG approximation of \( \psi \) on the skeleton, which means
\[ a_\ell(\tilde{\rho}_\ell, \mu) = (S_\ell(\lambda - \tilde{\lambda}_\ell), \mathcal{U}_\ell\mu) \quad \forall \mu \in \tilde{M}_\ell. \]
Moreover, let \( \rho_\ell \) be the solution of
\[ a_\ell(\rho_\ell, \mu) = (S_\ell(\lambda - \tilde{\lambda}_\ell), S_\ell\mu) \quad \forall \mu \in \tilde{M}_\ell. \]
Similar to the estimation of \( \|\lambda - \tilde{\lambda}_\ell\|_{a_\ell} \), we have
\[ \|\rho_\ell - \tilde{\rho}_\ell\|_{a_\ell} \lesssim h_\ell\|S_\ell(\lambda - \tilde{\lambda}_\ell)\|_0. \]
Taking \( \mu = \lambda - \tilde{\lambda}_\ell \) in (4.28), we receive
\[ \|S_\ell(\lambda - \tilde{\lambda}_\ell)\|_0^2 = a_\ell(\rho_\ell, \lambda - \tilde{\lambda}_\ell) \overset{(4.22)}{=} a_\ell(\lambda - \tilde{\lambda}_\ell, \rho_\ell - \gamma_\ell \Pi_\ell \psi) \overset{\text{Lem. 4.2}}{=} a_\ell(\rho_\ell - \tilde{\rho}_\ell, \lambda - \tilde{\lambda}_\ell) + a_\ell(\tilde{\rho}_\ell - \gamma_\ell \Pi_\ell \psi, \lambda - \tilde{\lambda}_\ell) \leq \|\lambda - \tilde{\lambda}_\ell\|_{a_\ell} (\|\rho_\ell - \tilde{\rho}_\ell\|_{a_\ell} + \|\tilde{\rho}_\ell - \gamma_\ell \Pi_\ell \psi\|_{a_\ell}). \]
This can be further estimated noting that
\[ \|\tilde{\rho}_\ell - \gamma_\ell \Pi_\ell \psi\|_{a_\ell} \overset{\text{Lem. 3.1}}{\lesssim} h_\ell^{-1} \|\tilde{\rho}_\ell - \gamma_\ell \Pi_\ell \psi\|_{\ell} \overset{\text{Lem. 4.5}}{\lesssim} h_\ell^{-1} \left( h_\ell^2 \|\psi\|_2 + \|\Pi_\ell^2 \psi - \psi\|_{\ell} + \|\psi - \Pi_\ell \psi\|_{\ell} \right) \]

\[ \overset{\text{trace & } L^2 \text{ approx}}{\lesssim} h_\ell^{-1} \left( h_\ell^2 \|\psi\|_2 \right) \lesssim h_\ell\|S_\ell(\lambda - \tilde{\lambda}_\ell)\|_0. \]
Using this inequality combined with (4.29) and (4.30), we have
\[ \|S_\ell(\lambda - \tilde{\lambda}_\ell)\|_0^2 \lesssim h_\ell \|S_\ell(\lambda - \tilde{\lambda}_\ell)\|_0 \|\lambda - \tilde{\lambda}_\ell\|_{a_\ell}. \] (4.32)

By the lemma's first inequality and (4.4)
\[ \|\lambda - \tilde{\lambda}_\ell\|_\ell \simeq \|S_\ell(\lambda - \tilde{\lambda}_\ell)\|_0 \lesssim h_\ell \|\lambda - \tilde{\lambda}_\ell\|_{a_\ell} \lesssim h_\ell^2 \|A_\ell \lambda\|_\ell. \] (4.33)

**Lemma 4.6.** Provided \( \tau_\ell h_\ell \lesssim 1 \), the injection operator \( I_\ell: \tilde{M}_{\ell-1} \to \tilde{M}_\ell \) satisfies
\[ I_\ell \gamma_{\ell-1} w = \gamma_\ell w \quad \text{if } w \in V_{\ell-1}^c, \] (4.34)
\[ \|I_\ell \mu\|_\ell \lesssim \|\mu\|_{\ell-1} \quad \forall \mu \in \tilde{M}_{\ell-1}. \] (4.35)

**Proof.** One easily verifies that for all \( w \in \overline{V}_{\ell-1}^c \)
\[ U_{\ell-1}^c \gamma_{\ell-1} w = w. \] (4.36)
This implies the equality. By the standard scaling argument and Lemma A.2 we can also get the inequality via
\[ \|I_\ell \mu\|_\ell = \|\gamma_\ell U_{\ell-1}^c \mu\|_\ell \lesssim \|U_{\ell-1}^c \mu\|_0 \lesssim \|\mu\|_{\ell-1}. \] (4.37)

**Lemma 4.7.** Suppose that \( w \in H^2(\Omega) \) is the solution of
\[ -\Delta w = g \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega, \] (4.38)
where \( g \in L^2(\Omega) \), and \( w_\ell, \tilde{w}_{\ell-1} \) are its EDG approximates with respect to \( T_\ell \) and \( T_{\ell-1} \), respectively. If \( \tau_\ell = \frac{h_\ell}{h_{\ell-1}} \), we obtain
\[ \|w_\ell - I_\ell \tilde{w}_{\ell-1}\|_\ell \lesssim h_\ell^2 \|g\|_0. \] (4.39)

**Proof.** Defining \( \tilde{w}_{\ell-1} \in \overline{V}_{\ell-1}^c \) as the CG solution of \( w \) and \( \gamma_{\ell-1} \tilde{w}_{\ell-1} \) as its trace on \( \Sigma_{\ell-1} \), we obtain
\[ \|w_\ell - I_\ell \tilde{w}_{\ell-1}\|_\ell \] (4.40)
\[ \leq \|w_\ell - \Pi_\ell^2 w\|_\ell + \|\Pi_\ell^2 w - I_\ell \gamma_{\ell-1} \tilde{w}_{\ell-1}\|_\ell + \|I_\ell \gamma_{\ell-1} \tilde{w}_{\ell-1} - I_\ell w_{\ell-1}\|_\ell. \]
Then, by the approximation properties of EDG (cf. Lemma 4.5), CG, inverse inequality, \( \|v\|_\ell \lesssim \|v\|_0 \) for \( v \in V^c_\ell \), and Lemma 4.6, we obtain
\[ \|w_\ell - \Pi_\ell^2 w\|_\ell \lesssim h_\ell^2 \|g\|_0, \] (4.41)
\[ \|\Pi_\ell^2 w - I_\ell \gamma_{\ell-1} \tilde{w}_{\ell-1}\|_\ell = \|\Pi_\ell^2 w - \gamma_{\ell-1} \tilde{w}_{\ell-1}\|_\ell \lesssim h_\ell^2 \|g\|_0, \] (4.42)
\[ \|I_\ell \gamma_{\ell-1} \tilde{w}_{\ell-1} - I_\ell w_{\ell-1}\|_\ell \lesssim \|\gamma_{\ell-1} \tilde{w}_{\ell-1} - w_{\ell-1}\|_\ell \] (4.43)
\[ \leq \|\gamma_{\ell-1} \tilde{w}_{\ell-1} - \Pi_{\ell-1}^2 w\|_\ell + \|\Pi_{\ell-1}^2 w - w_{\ell-1}\|_\ell \lesssim h_\ell^2 \|g\|_0. \]
This gives the result. \( \square \)

**Lemma 4.8.** (B1) holds if \( \tau_\ell \simeq h_\ell^{-1} \).
Proof. Let $\tilde{\lambda}_{\ell-1} \in \tilde{M}_{\ell-1}$ be the solution of

$$a_{\ell-1}(\tilde{\lambda}_{\ell-1}, \mu) = (f_\lambda, U_{\ell-1} \mu) \quad \forall \mu \in \tilde{M}_{\ell-1},$$

(4.44)
i.e., $\tilde{\lambda}_{\ell-1}$ is the EDG solution of (4.6) on $\tilde{M}_{\ell-1}$. By Lemma 4.7 and (4.9), we have

$$\|\tilde{\lambda}_\ell - I_{\ell} \tilde{\lambda}_{\ell-1}\|_\ell \lesssim h_\ell^2 \|f_\lambda\|_0 \lesssim h_\ell^2 \|A_\ell \lambda\|_\ell.$$ (4.45)

Denoting $e_{\ell-1} = \tilde{\lambda}_{\ell-1} - P_{\ell-1} \lambda$, we can conclude via

$$a_{\ell-1}(P_{\ell-1} \lambda, \mu) = a_\ell(\lambda, I_{\ell} \mu) = (f_\lambda, S_{\ell} I_{\ell} \mu) \quad \forall \mu \in \tilde{M}_{\ell-1}$$ (4.46)

that

$$a_{\ell-1}(e_{\ell-1}, \mu) = (f_\lambda, U_{\ell-1} \mu - S_{\ell} I_{\ell} \mu)_0.$$ (4.47)

Noting that for $\mu = \gamma_{\ell-1} w$ with $w \in V_{\ell-1}$ by Lemmas 4.2 & 4.6

$$U_{\ell-1} \mu = w = S_{\ell} I_{\ell} \mu,$$ (4.48)

which means $a_{\ell-1}(e_{\ell-1}, \mu) = 0$. Similar to the estimation in Lemma 4.4 we can use the duality argument to receive

$$\|e_{\ell-1}\|_{\ell-1} \lesssim h_{\ell-1} \|e_{\ell-1}\|_{a_{\ell-1}}.$$ (4.49)

Thus, we can deduce that for all $\mu \in \tilde{M}_{\ell-1}$

$$\|S_{\ell} I_{\ell} \mu - U_{\ell-1} \mu\|_0 \leq \|S_{\ell} I_{\ell} \mu - U_{\ell} I_{\ell} \mu\|_0 + \|U_{\ell} I_{\ell} \mu - U_{\ell-1} \mu\|_0$$

$$\lesssim h_\ell \|I_{\ell} \mu\|_{a_\ell} + h_\ell \|\mu\|_{a_{\ell-1}} \lesssim h_\ell \|\mu\|_{a_{\ell-1}}.$$ (4.50)

Here, the second inequality is obtained using (4.23) and Lemma A.3 and the last inequality is Lemma 3.2.

Taking $\mu = e_{\ell-1}$ in (4.47) and using (4.50), we have

$$\|e_{\ell-1}\|^2_{a_{\ell-1}} \lesssim \|f_\lambda\|_0 h_\ell \|e_{\ell-1}\|_{a_{\ell-1}},$$ (4.51)

that is

$$\|e_{\ell-1}\|_{a_{\ell-1}} \lesssim h_\ell \|f_\lambda\|_0 \lesssim h_\ell \|A_\ell \lambda\|_\ell.$$ (4.52)

Using (4.49), this results in

$$\|e_{\ell-1}\|_{\ell-1} \lesssim h_\ell^2 \|A_\ell \lambda\|_\ell$$ (4.53)

and by triangle inequality (first inequality), Lemma 4.4 & 4.6, (4.45) (second inequality), and (4.53) (last inequality)

$$\|\lambda - I_{\ell} P_{\ell-1} \lambda\|_\ell \leq \|\lambda - \tilde{\lambda}_\ell\|_\ell + \|\tilde{\lambda}_\ell - I_{\ell} \tilde{\lambda}_{\ell-1}\|_\ell + \|I_{\ell} \tilde{\lambda}_{\ell-1} - I_{\ell} P_{\ell-1} \lambda\|_\ell$$

$$\lesssim h_\ell^2 \|A_\ell \lambda\|_\ell + \|e_{\ell-1}\|_{\ell-1} \lesssim h_\ell^2 \|A_\ell \lambda\|_\ell.$$ (4.54)

$\square$
5. Proof of (A2) and (A3)

The proof of (A2) is a simple consequence of Lemma 3.2 with $P_{\ell-1}\lambda$ instead of $\lambda$ and the following lemma which can be obtained similar to [LRK20, Lem. 4.2]:

**Lemma 5.1.** The “Ritz quasi-projection” $P_{\ell-1} : \tilde{M}_\ell \rightarrow \tilde{M}_{\ell-1}$ is stable in the sense that for all $\lambda \in \tilde{M}_\ell$, we have

$$\|P_{\ell-1}\lambda\|_{a_{\ell-1}} \lesssim \|\lambda\|_{a_\ell}. \quad (5.1)$$

Thus, we can deduce that

$$a_\ell(\lambda - I_{\ell-1} P_\ell \lambda, \lambda - I_{\ell-1} P_\ell \lambda) \leq a_\ell(\lambda, \lambda) - 2a_\ell(I_{\ell-1} P_\ell \lambda, I_{\ell-1} P_\ell \lambda) + C a_{\ell-1}(P_{\ell-1}\lambda, P_{\ell-1}\lambda), \quad (5.2)$$

For the proof of (A3), we heavily rely on [BP92] (where (A3) is denoted (2.11)). Theorems 3.1 and 3.2 of [BP92] ensure that (A3) holds if the subspaces satisfy a “limited interaction property” which holds, because each degree of freedom (DoF) only “communicates” with other DoFs which are located on the same face as the DoF or on the other faces of the two adjacent elements.

6. Numerical Experiments

To evaluate the multigrid method for EDG numerically, we consider the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \quad (6.1a)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (6.1b)$$

where $f$ is chosen as one on the unit square $\Omega = (0,1)^2$. The implementation is based on the FFW toolbox from [BGG*] and employs the
Gauss–Seidel smoother. It uses a Lagrange basis and the Euclidean inner product in the coefficient space instead of the inner product $\langle \cdot,\cdot \rangle_\ell$. These two inner products are equivalent up to a factor of $h_\ell^2$. The numerical experiments are conducted on a successively refined mesh sequence of which the initial mesh is depicted in Figure 1. The iteration process for approximating $\mathbf{x}$ in $A\mathbf{x} = b$ representing the discrete version of (6.1) is stopped if
\[
\frac{\|b - A\mathbf{x}_{\text{iter}}\|_2}{\|b\|_2} < 10^{-6},
\]
and the initial value $\mathbf{x}$ on mesh level $\ell$ is the solution on level $\ell - 1$, describing a nested iteration. The numbers of iteration steps are shown in Table 1 and appear to be independent of the mesh level, as predicted by our analysis. Additionally, the numbers are fairly small, such that we can conclude that we actually have an efficient method. Finally, we see that the choice of $\tau \in \{\frac{1}{h}, 1\}$ does not significantly influence the number of iterations. We did experiments for polynomial degrees up to three, and observed that the iteration counts remain well bounded; nevertheless, we expect rising counts for higher degrees, as we use a point smoother.

Additionally, we tested the correctness of our implementation by employing a right hand side leading to the solution $u = \sin(2\pi x) \sin(2\pi y)$. The estimated orders of convergence (EOC) of the primary unknown $u$ computed as
\[
\text{EOC} = \log \left( \frac{\|u - u_{\ell-1}\|_{L^2(\Omega)}}{\|u - u_{\ell}\|_{L^2(\Omega)}} \right) / \log(2),
\]
and the secondary unknown $q$ of the HDG method are reported in Table 2. Iteration counts are almost identical to those in Table 1, such

| smoother | one step | two steps |
|----------|----------|-----------|
| mesh level | 1 2 3 4 5 6 | 1 2 3 4 5 6 |
| $\tau = \frac{1}{h}$ | 6 7 7 6 6 6 | 4 5 5 5 4 4 |
| $\tau = 1$ | 6 7 7 6 6 6 | 4 5 5 5 4 4 |
| $\tau = \frac{1}{h}$ | 7 7 7 7 7 7 | 5 4 4 4 4 4 |
| $\tau = 1$ | 7 7 7 7 7 7 | 5 4 4 4 4 4 |
| $\tau = \frac{1}{h}$ | 9 9 9 9 9 9 | 6 6 6 5 5 5 |
| $\tau = 1$ | 9 9 9 9 9 9 | 6 6 6 5 5 5 |

Table 1. Numbers of iterations with one and two smoothing steps for $f \equiv 1$. The polynomial degree of the EDG method is $p$. 
Table 2. Estimated orders of convergence (EOC) for primary unknown \( u \) and secondary unknown \( q \) when the polynomial degree of the EDG method is \( p \) and \( u = \sin(2\pi x)\sin(2\pi y) \).

that we do not report them here. As opposed to the HDG method, we see that the choice \( \tau = \frac{1}{h} \) is not suboptimal as compared to \( \tau = 1 \).

7. Conclusions

In the previous pages, we proposed a homogeneous multigrid method for EDG. We proved analytically that this method converges independently of the mesh size. Numerical examples have shown that the condition numbers are not only independent of the mesh size but also reasonably small. As a consequence, we have been enabled to efficiently solve linear systems of equations arising from EDG discretizations of arbitrary order.

Appendix A. Used results

Here, we summarize the results from other sources that we used in the proofs of our propositions.

\textbf{Lemma A.1.} Let \( \mu \) be any function in \( \tilde{M}_\ell \). The following statement holds:

\[ \| \sqrt{\tau} (U_\ell \mu - \mu) \|_\ell \lesssim \sqrt{h_\ell \tau_\ell} \| \mathcal{Q}_\ell \mu \|_0. \]  

(A.1)

Thus, if \( \tau_\ell h_\ell \lesssim 1 \),

\[ \| U_\ell \mu - \mu \|_\ell \lesssim h_\ell \| \mathcal{Q}_\ell \mu \|_0. \]  

(A.2)

\textit{Proof.} The first inequality is \cite[Lemma 3.4 (iv)]{CDGT13} whose right hand side is estimated using \cite[Lemma 3.4 (v)]{CDGT13}. The second inequality follows after multiplication with \( h_\ell \) and exploiting the definitions of \( \| \cdot \|_\ell \) and \( \| \cdot \|_\ell \).

\textbf{Lemma A.2.} If \( \tau_\ell h_\ell \lesssim 1 \), the local solution operators obeys

\[ \| U_\ell \mu \|_0 \lesssim \| \mu \|_\ell \]  

(A.3)
**Proof.** This is Theorem 3.1 in [CDGT13], where we use that the constant becomes independent of $h_ℓ$ if $τ_ℓh_ℓ ≲ 1$. □

**Lemma A.3** (Lemma 4.3 in [LRK20]). The DG reconstructions of the injection operator admits the estimate

$$\|U_{ℓ−1}μ − U_{ℓ}I_{ℓ}μ\|_0 ≲ h_ℓ\|μ\|_{a_{ℓ−1}}, \quad ∀μ ∈ M_{ℓ−1}. \quad (A.4)$$

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