BANACH SPACES WITH WEAK*-SEQUENTIAL DUAL BALL

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Abstract. A topological space is said to be sequential if every subspace closed under taking limits of convergent sequences is closed. We consider Banach spaces with weak*-sequential dual ball. In particular, we show that if \( X \) is a Banach space with weak*-sequentially compact dual ball and \( Y \subset X \) is a subspace such that \( Y \) and \( X/Y \) have weak*-sequential dual ball, then \( X \) has weak*-sequential dual ball. As an application we obtain that the Johnson-Lindenstrauss space \( JL_2 \) and \( C(K) \) for \( K \) a scattered compact space of countable height are examples of Banach spaces with weak*-sequential dual ball. These results provide a negative solution to a question of A. Plichko, who asked whether the dual ball of a Banach space is weak*-angelic whenever it is weak*-sequential.

1. Introduction

All topological spaces considered in this paper are Hausdorff. The symbol \( w^* \) denotes the weak* topology of the corresponding Banach space. A topological space \( T \) is said to be sequentially compact if every sequence in \( T \) contains a convergent subsequence. Moreover, \( T \) is said to be Fréchet-Urysohn (FU for short) if for every subspace \( F \) of \( T \), every point in the closure of \( F \) is the limit of a sequence in \( F \). Every FU compact space is sequentially compact. A Banach space with weak*-FU dual ball is said to have weak*-angelic dual. Some examples of Banach spaces with weak*-angelic dual are WCG Banach spaces (i.e., Banach spaces generated by a weakly compact set) and, in general, WLD Banach spaces (i.e., Banach spaces whose dual ball with the weak*-topology is Corson). On the other hand, every weak Asplund Banach space and every Banach space without copies of \( \ell_1 \) in the dual have weak*-sequentially compact dual ball [7, Chapter XIII].

In this paper we are going to focus on sequential spaces, which is a generalization of the FU property. If \( T \) is a topological space and \( F \) is a subspace of \( T \), the sequential closure of \( F \) is the set of all limits of sequences in \( F \). \( F \) is said to be sequentially closed if it coincides with its sequential closure. A topological space is said to be sequential if any sequentially closed subspace is closed. Thus, every FU
space is sequential. Another natural generalization of the FU property is countable tightness. A topological space \( T \) is said to have \textit{countable tightness} if for every subspace \( F \) of \( T \), every point in the closure of \( F \) is in the closure of a countable subspace of \( F \). It can be proved that every sequential space has countable tightness. However, whether the converse implication in the class of compact spaces is true is known as the Moore-Mrowka Problem and it is undecidable in ZFC (i.e., in the usual axioms of set-theory)\(^{[2]}\). Therefore, for a compact space \( K \), we have the following implications:

\[
K \text{ is FU} \implies K \text{ is sequential} \implies K \text{ is sequentially compact}
\]

\[
\Downarrow
\]

\[
K \text{ has countable tightness}
\]

In \(^{[20]}\) Question 10] A. Plichko asked whether every Banach space with weak*-sequential dual ball has weak*-angelic dual. In the next section we prove the following theorem, which is applied to prove that the Johnson-Lindenstrauss space \( JL_2 \) provides a negative answer to Plichko’s question:

**Theorem 1.1.** Let \( X \) be a Banach space with weak*-sequentially compact dual ball. Let \( Y \subset X \) be a subspace with weak*-sequential dual ball with sequential order \( \leq \gamma_1 \) and such that \( X/Y \) has weak*-sequential dual ball with sequential order \( \leq \gamma_2 \). Then \( X \) has weak*-sequential dual ball with sequential order \( \leq \gamma_1 + \gamma_2 \).

One of the properties studied by Plichko in \(^{[20]}\) is property \( E \) of Efremov. A Banach space \( X \) is said to have property \( E \) if every point in the weak*-closure of any convex subset \( C \subset B_{X^*} \) is the weak*-limit of a sequence in \( C \). We say that \( X \) has property \( E' \) if every weak*-sequentially closed convex set in the dual ball is weak*-closed. Thus, if \( X \) has weak*-angelic dual, then it has property \( E \); and if \( X \) has weak*-sequential dual ball, then \( X \) has property \( E' \). We also provide a convex version of Theorem 1.1 (see Theorem 2.3).

Other related Banach space properties are the Mazur property and property (C). A Banach space \( X \) has Mazur property if every \( x^{**} \in X^{**} \) which is weak*-sequentially continuous on \( X^* \) is weak*-continuous and, therefore, \( x^{**} \in X \). Notice that if a topological space \( T \) is sequential, then any sequentially continuous function \( f : T \rightarrow \mathbb{R} \) is continuous. Thus, it follows from the Banach-Dieudonné Theorem that every Banach space with weak*-sequential dual ball has the Mazur property. Moreover, property \( E' \) also implies the Mazur property.

A Banach space \( X \) has property (C) of Corson if and only if every point in the closure of \( C \) is in the weak*-closure of a countable subset of \( C \) for every convex set \( C \) in \( B_{X^*} \); (this characterization of property (C) is due to R. Pol \(^{[19]}\)).

Thus, we have the following implications among these Banach space properties:

\[
\text{weak*-angelic dual} \implies \text{weak*-sequential dual ball} \implies \text{weak*-seq. compact dual ball}
\]

\[
\Downarrow
\]

\[
\text{property } E \implies \text{property } E' \implies \text{property (C)}
\]

\[
\Downarrow
\]

Mazur property
Notice that $C([0, \omega_1])$ has weak*-sequentially compact dual ball, but it is not weak*-sequential. Moreover, $\ell_1(\omega_1)$ has the Mazur property \[1\] Section 5], but it does not have property (C).

In [21] p. 352 it is asked whether property (C) implies property $E$. J.T. Moore in an unpublished paper and C. Brech in her PhD thesis [6] provided a negative answer under some additional consistent axioms, but the question is still open in ZFC. Notice that the convex version of Plichko’s question is whether property $E'$ implies property $E$. A negative answer to this question would provide an example of a Banach space with property (C) not having property $E$.

In [10, Lemma 2.5] it is proved that the dual ball of $C(K)$ does not contain a copy of $\omega_1 + 1 = [0, \omega_1]$ when $K$ is a scattered compact space of finite height satisfying some properties. It is also proved in [17] that $C(K)$ has the Mazur property whenever $K$ is a scattered compact space of countable height. We generalize these results by proving that $C(K)$ has weak*-sequential dual ball whenever $K$ is a scattered compact space of countable height (Theorem 3.2).

2. Banach spaces with weak*-sequential dual ball

**Definition 2.1.** Let $T$ be a topological space and $F$ a subspace of $T$. For any $\alpha \leq \omega_1$ we define $S_\alpha(F)$ as the $\alpha$th sequential closure of $F$ by induction on $\alpha$: $S_0(F) = F$, $S_{\alpha+1}(F)$ is the sequential closure of $S_\alpha(F)$ for every $\alpha < \omega_1$ and $S_\alpha(F) = \bigcup_{\beta<\alpha} S_\beta(F)$ if $\alpha$ is a limit ordinal.

Notice that $S_{\omega_1}(F)$ is sequentially closed for every subspace $F$. Thus, a topological space $T$ is sequential if and only if $S_{\omega_1}(F) = F$ for every subspace $F$ of $T$. We say that $T$ has sequential order $\alpha$ if $S_\alpha(F) = F$ for every subspace $F$ of $T$ and for every $\beta < \alpha$ there exists $F$ with $S_\beta(F) \neq F$. Therefore, a topological space $T$ is sequential with sequential order $\leq 1$ if and only if it is FU. We will use the following lemma in the proof of Theorem 3.2.

**Lemma 2.2.** Let $f : K \to L$ be a continuous function, where $K, L$ are topological spaces and $K$ is sequentially compact. Then, $f(S_\alpha(F)) = S_\alpha(f(F))$ for every $F \subset K$ and every ordinal $\alpha$.

**Proof.** The inclusion $f(S_\alpha(F)) \subset S_\alpha(f(F))$ follows from the continuity of $f$.

We prove the other inclusion by induction on $\alpha$. The case $\alpha = 0$ is immediate. Suppose $\alpha = 1$. Take $s \in S_1(f(F))$. Then, there exists a sequence $t_n$ in $F$ such that $f(t_n)$ converges to $s$. Since $K$ is sequentially compact, without loss of generality we may suppose $t_n$ is converging to some point $t$. Then, it follows from the continuity of $f$ that $f(t) = s$. Thus, $s \in f(S_1(F))$.

Now suppose the result is true for every $\beta < \alpha$ and $\alpha \geq 2$. If $\alpha$ is a limit ordinal, then

$$f(S_\alpha(F)) = f(\bigcup_{\beta<\alpha} S_\beta(F)) = \bigcup_{\beta<\alpha} f(S_\beta(F)) = \bigcup_{\beta<\alpha} S_\beta(f(F)) = S_\alpha(f(F)).$$

If $\alpha = \beta + 1$ is a successor ordinal, then

$$f(S_\alpha(F)) = f(S_\beta(S_\beta(F))) = S_\beta(f(S_\beta(F))) = S_\beta(f(F)) = S_{\beta+1}(f(F)).$$

□
Proof of Theorem 1.1. It is enough to prove that if $F \subseteq B_{X^*}$ and $0 \in F^{w^*}$, then $0 \in S_{\gamma_1+\gamma_2}(F)$. Let $R : X^* \rightarrow Y^*$ be the restriction operator. For each finite set $A \subseteq X$ and each $\varepsilon > 0$, define

$$F_{A,\varepsilon} = \{ x^* \in F : |x^*(x)| \leq \varepsilon \text{ for all } x \in A \}.$$  

Since $R$ is weak*-weak* continuous and $0 \in F^{w^*}_{A,\varepsilon}$, we have that

$$0 \in R(F_{A,\varepsilon})^{w^*} = S_{\gamma_1}(R(F_{A,\varepsilon})) = R(S_{\gamma_1}(F_{A,\varepsilon})), $$

where the last equality follows from Lemma 2.2.

Thus, for every finite set $A \subseteq X$ and every $\varepsilon > 0$ we can take $x^*_{A,\varepsilon} \in S_{\gamma_1}(F_{A,\varepsilon})$ such that $R(x^*_{A,\varepsilon}) = 0$.

Therefore, $0 \in G^{w^*}$, where

$$G := \{ x^*_{A,\varepsilon} : A \subseteq X \text{ finite, } \varepsilon > 0 \} \subset Y^* \cap B_{X^*}.$$  

Note that $(Y^* \cap B_{X^*}, w^*)$ is homeomorphic to the dual ball of $(X/Y)^*$ with the weak* topology. Hence

$$0 \in S_{\gamma_2}(G) \subset S_{\gamma_2}(S_{\gamma_1}(F)) = S_{\gamma_1+\gamma_2}(F).$$

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach space, we say that $(y_k)_{k\in\mathbb{N}}$ is a **convex block subsequence** of $(x_n)_{n\in\mathbb{N}}$ if there is a sequence $(I_k)_{k\in\mathbb{N}}$ of subsets of $\mathbb{N}$ with $\max(I_k) < \min(I_{k+1})$ and a sequence $a_n \in [0,1]$ with $\sum_{n\in I_k} a_n = 1$ for every $k \in \mathbb{N}$ such that $y_k = \sum_{n\in I_k} a_n x_n$. A Banach space $X$ is said to have **weak*-convex block compact dual ball** if every bounded sequence in $X^*$ has a weak*-convergent convex block subsequence. Every Banach space containing no isomorphic copies of $\ell_1$ has weak*-convex block compact dual ball [5]. Therefore, every WPG Banach space (i.e. every Banach space with a linearly dense weakly precompact set) also has weak*-convex block compact dual ball.

For any ordinal $\gamma \leq \omega_1$, we say that $X$ has property $\mathcal{E}(\alpha)$ if $S_\alpha(C) = C$ for every convex subset $C$ in $(B_{X^*}, w^*)$. Thus, property $\mathcal{E}$ is property $\mathcal{E}(1)$ and property $\mathcal{E}'$ is property $\mathcal{E}(\omega_1)$. The proof of the following theorem is an immediate adaptation of the proof of Lemma 2.2 and Theorem 1.1.

**Theorem 2.3.** Let $X$ be a Banach space with weak*-convex block compact dual ball. Let $Y \subseteq X$ be a subspace with property $\mathcal{E}(\gamma_1)$ such that $X/Y$ has property $\mathcal{E}(\gamma_2)$. Then $X$ has property $\mathcal{E}(\gamma_1+\gamma_2)$.

**Theorem 2.4.** Let $X$ be a Banach space and $(X_n)_{n\in\mathbb{N}}$ an increasing sequence of subspaces with $X = \bigcup_{n\in\mathbb{N}} X_n$. Suppose that each $X_n$ has weak*-sequential dual ball with sequential order $\alpha_n$. Then $X$ has weak*-sequential dual ball with sequential order $\leq \alpha + 1$, where $\alpha := \sup\{\alpha_n : n \in \mathbb{N}\}$.

**Proof.** Set $R_n : X^* \rightarrow X_n^*$ as the restriction operator for every $n \in \mathbb{N}$. Since the countable product of sequentially compact spaces is sequentially compact and $(B_{X^*}, w^*)$ is homeomorphic to a subspace of $\prod(B_{X_n^*}, w^*)$, it follows that $X$ has weak*-sequentially compact dual ball. In order to prove the theorem, it is enough to prove that if $F \subseteq B_{X^*}$ and $0 \in \overline{F}^{w^*}$, then $0 \in S_{\alpha+1}(F)$. Since $B_{X^*}$ is weak*-sequentially compact, we have that $0 \in \overline{R_n(F)}^{w^*} = S_\alpha(R_n(F)) = R_n(S_\alpha(F))$ for every $n \in \mathbb{N}$, where the last equality follows from Lemma 2.2. Thus, we can take
Let \( x_n^* \in S_\alpha(F) \) such that \( R_\alpha(x_n^*) = 0 \). Now there exists some subsequence of \( x_n^* \) converging to a point \( x^* \in S_{\alpha+1}(F) \). Since \( R_\alpha(x^*) = 0 \) for every \( n \in \mathbb{N} \), we conclude that \( x^* = 0 \).

**Corollary 2.5.** Let \( X \) be a Banach space and \((X_\alpha)_{\alpha<\gamma}\) an increasing sequence of subspaces with \( X = \bigcup_{\alpha<\gamma} X_\alpha \), where \( \gamma \) is a countable limit ordinal. Suppose that each \( X_\alpha \) has weak*-sequential dual ball with sequential order \( \leq \theta_\alpha \). Then \( X \) has weak*-sequential dual ball with sequential order \( \leq \theta + 1 \) where \( \theta := \sup\{\theta_\alpha : \alpha < \gamma\} \).

The next theorem follows from combining Theorem 1.1 and Corollary 2.5:

**Theorem 2.6.** Let \( \gamma \) be a countable ordinal, \( X_\gamma \) a Banach space and \((X_\alpha)_{\alpha\leq\gamma}\) an increasing sequence of subspaces of \( X_\gamma \) such that:

1. \( X_0 \) has weak*-sequential dual ball with sequential order \( \leq \theta \);
2. each quotient \( X_{\alpha+1}/X_\alpha \) has weak*-angelic dual;
3. \( X_\alpha = \bigcup_{\beta<\alpha} X_\beta \) if \( \alpha \) is a limit ordinal;
4. \( X_\gamma \) has weak*-sequentially compact dual ball.

Then each \( X_\alpha \) has weak*-sequential dual ball with sequential order \( \leq \theta + \alpha \) if \( \alpha < \omega \) and sequential order \( \leq \theta + \alpha + 1 \) if \( \alpha \geq \omega \).

**Proof.** It follows from (4) that every \( X_\alpha \) has weak*-sequentially compact dual ball. Thus, the result for \( \alpha < \omega \) follows by applying inductively Theorem 1.1. Suppose \( \alpha \geq \omega \) and \( X_\beta \) has weak*-sequential dual ball with sequential order \( \leq \theta + \beta + 1 \) for every \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, then it follows from (3) and from Corollary 2.5 that \( X_\alpha \) has weak*-sequential dual ball with sequential order

\[
\leq \sup_{\beta<\alpha} (\theta + \beta + 1) + 1 = \theta + \alpha + 1.
\]

If \( \alpha \) is a successor ordinal, then the result is a consequence of Theorem 1.1.

We also have the following convex equivalent version of the previous theorem:

**Theorem 2.7.** Let \( \gamma \) be a countable ordinal, \( X_\gamma \) a Banach space and \((X_\alpha)_{\alpha\leq\gamma}\) an increasing sequence of subspaces of \( X_\gamma \) such that:

1. \( X_0 \) has property \( \mathcal{E}(\theta) \);
2. each quotient \( X_{\alpha+1}/X_\alpha \) has \( \mathcal{E} \);
3. \( X_\alpha = \bigcup_{\beta<\alpha} X_\beta \) if \( \alpha \) is a limit ordinal;
4. \( X_\gamma \) has weak*-convex block compact dual ball.

Then each \( X_\alpha \) has property \( \mathcal{E}(\theta + \alpha) \) if \( \alpha < \omega \) and property \( \mathcal{E}(\theta + \alpha + 1) \) if \( \alpha \geq \omega \).

### 3. Applications

As an application of Theorem 1.1, we obtain that the Johnson-Lindenstrauss space \( JL_2 \) has weak*-sequential dual ball. Let us recall the definition of \( JL_2 \):

Let \( \{N_r : r \in \Gamma\} \) be an uncountable maximal almost disjoint family of infinite subsets of \( \mathbb{N} \). For each \( N_r, \chi_{N_r} \in \ell_\infty \) denotes the characteristic function of \( N_r \). The Johnson-Lindenstrauss space \( JL_2 \) is defined as the completion of \( \text{span}(c_0 \cup \{\chi_{N_r} : r \in \Gamma\}) \subset \ell_\infty \) with respect to the norm

\[
\|x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}}\| = \max \left\{ \|x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}}\|_\infty, \left( \sum_{1 \leq i \leq k} |a_i|^2 \right)^{\frac{1}{2}} \right\}.
\]
where $x \in c_0$ and $\| \cdot \|_\infty$ is the supremum norm in $\ell_\infty$. If we just consider the supremum norm in the definition, then we obtain the space $JL_0$. We refer the reader to [14] for more information about these spaces.

**Theorem 3.1.** The Johnson-Lindenstrauss space $JL_2$ has weak*-sequential dual ball with sequential order 2.

**Proof.** We use the following results proved in [14]:

(i) $JL_2$ has an equivalent Fréchet differentiable norm;
(ii) $JL_2/c_0$ is isometric to $\ell_2(\Gamma)$.

It follows from (i) that $JL_2$ has weak*-sequentially compact dual ball (cf. [12]). It follows from (ii) and Theorem 1.1 that $JL_2$ has weak*-sequential dual ball with sequential order $\leq 2$. Since $JL_2$ does not have weak*-angelic dual (cf. [9, Proposition 5.12]) we have that $JL_2$ has weak*-sequential dual ball with sequential order $2$. \qed

Theorem 3.1 provides an example of a Banach space with weak*-sequential dual ball which does not have weakly*-angelic dual, answering a question of Plichko in [20, Question 10].

For a scattered compact space $K$, we denote by $ht(K)$ the height of $K$, i.e. the minimal ordinal $\gamma$ such that the $\gamma$th Cantor-Bendixson derivative $K^{(\gamma)}$ is discrete. Since every Banach space with weak*-sequential dual ball has the Mazur property, the following theorem improves [17, Theorem 4.1]:

**Theorem 3.2.** Let $K$ be an infinite scattered compact space. If $ht(K) < \omega$, then $C(K)$ has weak*-sequential dual ball with sequential order $\leq ht(K)$. Moreover, if $\omega \leq ht(K) < \omega_1$, then $C(K)$ has weak*-sequential dual ball with sequential order $\leq ht(K) + 1$.

**Proof.** It is well-known that if $K$ is scattered, then $C(K)$ is Asplund and therefore $B_{C(K)^*}$ is weak*-sequentially compact (see, for example, [22]). Denote by $\{K^{(\alpha)} : \alpha \leq \gamma\}$ the Cantor-Bendixson derivatives of $K$, where $\gamma = ht(K)$. For every $\alpha \leq \gamma$, set

$$X_\alpha = \{f \in C(K) : f(t) = 0 \text{ for every } t \in K^{(\alpha)}\}.$$

Since $C(K)$ contains a complemented copy of $c_0$, every finite-codimensional subspace of $C(K)$ is isomorphic to $C(K)$. Therefore, since $X_\gamma$ is a finite-codimensional subspace of $C(K)$, it is isomorphic to $C(K)$. Notice that for every $0 \leq \alpha < \gamma$ we have that $X_{\alpha+1}/X_\alpha$ is isomorphic to $c_0(K^{(\alpha)} \setminus K^{(\alpha+1)})$. Moreover, if $\alpha \leq \gamma$ is a limit ordinal, then $\bigcap_{\beta < \alpha} K^{(\beta)} = K^{(\alpha)}$ and therefore

$$\bigcup_{\beta < \alpha} X_\beta = \{f \in C(K) : \exists \beta < \alpha \text{ with } f(t) = 0 \forall t \in K^{(\beta)}\} = X_\alpha.$$

Now the conclusion follows from Theorem 2.6. \qed

R. Haydon [13] and K. Kunen [18] constructed under CH an FU compact space $K$ such that $B_{C(K)^*}$ does not have countable tightness. Thus, it is not true for a general compact space $K$ that if $K$ is sequential, then $B_{C(K)^*}$ is sequential. We refer the reader to [11] for a discussion on this topic.

It can be easily checked that the space $JL_0$ is isomorphic to a $C(K)$ space where $K$ is a scattered compact space with $ht(K) = 2$ and sequential order 2. Thus, $JL_0$ has weak*-sequential dual ball with sequential order 2.
The known examples in ZFC of sequential compact spaces are all of sequential order \( \leq 2 \). Nevertheless, A.I. Baškirov constructed sequential compact spaces of any sequential order \( \leq \omega_1 \) under the Continuum Hypothesis \( 3 \). A different construction was also given by V. Kannan in \([16]\). Baškirov’s construction is studied in detail in \([1]\) and, as C. Baldovino highlights in \([1, \text{Remark 6.8}]\), these constructions are scattered compact spaces such that the sequential order and the scattering height coincide whenever the sequential order is a successor ordinal.

Moreover, A. Dow constructed under the assumption \( b = c \) a scattered compact space \( K \) of sequential order 4 such that the sequential order and the scattering height coincide \([8]\).

**Corollary 3.3.** Under the Continuum Hypothesis there exist Banach spaces with weak*-sequential dual ball of any sequential order \( < \omega \) and Banach spaces with weak*-sequential dual ball with arbitrarily large countable sequential order.

On the other hand, under \( b = c \), there exist Banach spaces with weak*-sequential dual ball of any sequential order \( \leq 4 \).

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