**ORIGINAL ARTICLE**

**Invariant hypercomplex structures and algebraic curves**

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**Abstract**
We show that $U(k)$-invariant hypercomplex structures on (open subsets) of regular semisimple adjoint orbits in $\mathfrak{gl}(k, \mathbb{C})$ correspond to algebraic curves $C$ of genus $(k - 1)^2$, equipped with a flat projection $\pi : C \to \mathbb{P}^1$ of degree $k$, and an antiholomorphic involution $\sigma : C \to C$ covering the antipodal map on $\mathbb{P}^1$.

**KEYWORDS**
adjoint orbits, algebraic curves, Hilbert schemes of morphisms, hypercomplex structures

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53C26, 14H40, 17B08

Invariant hyperkähler metrics on adjoint orbits of a reductive complex Lie group $G^\mathbb{C}$ have been constructed by Kronheimer [10, 11] in the case of nilpotent and semisimple orbits, and by Biquard [5] and Kovalev [9] for arbitrary orbits. In the case of regular orbits, the metrics constructed by these authors are parametrized by triples $(\tau_1, \tau_2, \tau_3)$ of elements of a Cartan subalgebra. For a generic $\zeta \in \mathbb{P}^1$, this hyperkähler manifold $M(\tau_1, \tau_2, \tau_3)$ is isomorphic, as a complex symplectic manifold, to the regular adjoint $G^\mathbb{C}$-orbit of an $l^\prime \in \mathfrak{g}^\mathbb{C}$, the semisimple part of that equals $(\tau_2 + i\tau_3) + 2i\tau_1\zeta + (\tau_2 - i\tau_3)\zeta^2$. D'Amorim Santa-Cruz [6] constructed a much larger class of $G$-invariant pseudo-hyperkähler metrics on (open subsets of) regular adjoint orbits, parametrized by an arbitrary real spectral curve $S$, that is, a real section of $(\mathfrak{g}^\mathbb{C} \otimes \mathcal{O}_{\mathbb{P}^1}(2))/G^\mathbb{C}$. This manifold $M(S)$ has several connected components; in particular there is always a component where the metric is positive-definite, and a component where the metric is negative-definite. A rather surprising (at least to the authors) result in [2] is that already for $G^\mathbb{C} = SL(3, \mathbb{C})$ there exist components on that the metric is indefinite.

Twistor theory implies almost immediately that any $G$-invariant and locally $G^\mathbb{C}$-homogeneous pseudo-hyperkähler manifold can be obtained from the D'Amorim Santa-Cruz construction. The main purpose of this article is to show that this is no longer the case for hypercomplex manifolds. Let $Q$ denote the two sphere of complex structures defining the hypercomplex structure of a hypercomplex manifold. We prove:

**Theorem A.** Let $(M, Q)$ be a connected hypercomplex manifold with a free triholomorphic action of $PU(k)$ and such that:

1. for any complex structure in $Q$, the local action of $PGL(k, \mathbb{C})$ is transitive with the infinitesimal stabilizer isomorphic to the centralizer of a regular element;
2. for a generic complex structure in $Q$, the stabilizer in (1) is reductive (i.e., a Cartan subalgebra).

Then there exists a connected reduced locally complete intersection (lci) algebraic curve $C$ of (arithmetic) genus $g = (k - 1)^2$, equipped with a flat projection $\pi : C \to \mathbb{P}^1$ of degree $k$ and an antiholomorphic involution $\sigma$ covering the antipodal map on $\mathbb{P}^1$, such that $M/PU(k)$ is canonically isomorphic to an open subset of $(\text{Jac}^{g-1}(C) \backslash (\Theta \cup \Delta))^\sigma$, where $\Theta$ is the theta divisor and $\Delta$ is the divisor of invertible sheaves $L$ of degree $g - 1$ such that the shifted Petri map

$$H^0(C, L(1)) \otimes H^0(C, L^*(1) \otimes \mathcal{O}_C) \to H^0(C, K_C(2)),$$

where $L(i) = L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(i)$, is not an isomorphism. The action of $\sigma$ on $\text{Jac}^{g-1}(C)$ is given by $L \mapsto (\sigma^* L)^* \otimes K_C$. 

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Conversely, given \( \pi : C \to \mathbb{P}^1 \) as above, there exists a canonical \( \text{PU}(k) \)-invariant hypercomplex structure, satisfying (1) and (2), on a principal \( \text{PU}(k) \)-bundle over an open subset of \( (\text{Jac}^{g-1}(C) \setminus (\Theta \cup \Delta))^{\sigma} \).

The (real) dimension of the moduli space of such \((C, \pi, \sigma)\) is \(2k^2 - 2k\). The curves in the D’Amorim Santa-Cruz construction are embeddable in \( T\mathbb{P}^1 \), and therefore satisfy \( h^0(C, \pi^*\mathcal{O}_{\mathbb{P}^1}(2)) \geq 4\). However, a generic curve of genus \((k - 1)^2\) with a flat projection of degree \( k \) to \( \mathbb{P}^1 \) has \( h^0(C, \mathcal{O}(2)) = 3 \) if \( k \geq 3 \). Therefore, we can conclude that for \( k > 2 \) a generic \( U(k) \)-invariant locally \( GL(k, \mathbb{C}) \)-homogeneous hypercomplex manifold is not pseudo-hyperkähler. In fact, these hypercomplex manifolds cannot even admit a hypercomplex moment map: as soon as they do, they must be pseudo-hyperkähler.

We also remark that a canonical hypercomplex structure exists on a principal bundle over any connected component \((\text{Jac}^{g-1}(C) \setminus (\Theta \cup \Delta))^{\sigma} \). The structure group (and hence the symmetry group of the hypercomplex structure) will, however, usually vary between different components of \((\text{Jac}^{g-1}(C) \setminus \Theta)^{\sigma} \) (among different real forms of \( \text{PGL}(k, \mathbb{C}) \)).

## 1 | BACKGROUND MATERIAL

### 1.1 | Hypercomplex structures and generalizations

A hypercomplex structure on a smooth manifold \( M \) is given by a subbundle of \( \text{End}(TM) \) that is isomorphic to quaternions and the almost complex structures corresponding to unit quaternions are all integrable. In particular, a hypercomplex structure gives a decomposition \( T\mathbb{C}M \cong E \otimes \mathbb{C}^2 \) of the complexified tangent bundle into the trivial rank 2 bundle and a quaternionic bundle \( E \). Moreover, the tensor product of the quaternionic structure on \( E \) and the standard quaternionic structure on \( \mathbb{C}^2 \) \((z, w) \mapsto (-\bar{w}, \bar{z})\) is the complex conjugation on \( T\mathbb{C}M \). A (pseudo)-hyperkähler manifold is a hypercomplex manifold equipped with a compatible (pseudo)-Riemannian metric. Both hypercomplex and hyperkähler manifold arise via twistor theory as spaces of \( \sigma \)-invariant (known as real) sections of a holomorphic submersion \( \pi : Z \to \mathbb{P}^1 \) with normal bundle \( \bigoplus \mathcal{O}(1) \), where \( \sigma : Z \to Z \) is an anti-holomorphic involution covering the antipodal map on \( \mathbb{P}^1 \).

We are going to consider more general structures. For one, it is useful to consider complex analogues of these notions. Thus, a \( \mathbb{C} \)-hypercomplex manifold is a complex manifold \( M \) such that its holomorphic tangent bundle decomposes as \( E \otimes \mathbb{C}^2 \) and, for each nonzero \( v \in \mathbb{C}^2 \), the distribution \( E \otimes v \) is integrable. This is the geometry on the space of all sections of the twistor fibration with normal bundle \( \bigoplus \mathcal{O}(1) \). Second, we want to relax the hypercomplex condition itself, so that we can describe the geometry on a larger part of the Kodaira moduli space of sections of \( \pi : Z \to \mathbb{P}^1 \), where the normal bundle jumps. As argued in \([3, 4]\), the relevant geometry is that of 2-Kronecker structures.

A 2-Kronecker structure (of rank \( 2n \)) on a complex manifold \( M^{4n} \) consists of a holomorphic vector bundle \( E \) of rank \( 2n \) on \( M \) and a bundle map \( \alpha : E \otimes \mathbb{C}^2 \to TM \) that is injective on \( \alpha|_{E_m \otimes v} \) for every \( m \in M \) and \( v \in \mathbb{C}^2 \).

A quaternionic 2-Kronecker structure on a real manifold \( M^{4n} \) consists of a quaternionic bundle \( E \) (of complex) rank \( 2n \) and a bundle map \( \alpha : E \otimes \mathbb{C}^2 \to T\mathbb{C}M \) that is injective on \( \alpha|_{E_m \otimes v} \) for every \( m \in M \) and \( v \in \mathbb{C}^2 \), and intertwines the tensor product of quaternionic structures of \( E \) and \( \mathbb{C}^2 \) with complex conjugation on \( T\mathbb{C}M \).

A 2-Kronecker structure (quaternionic or not) is integrable, if \( T\mathbb{C}M = \alpha(E \otimes v) \) is an involutive subbundle for every \( v \in \mathbb{C}^2 \setminus \{0\} \).

Thus, hypercomplex (resp. \( \mathbb{C} \)-hypercomplex) structures are integrable quaternionic 2-Kronecker structures (resp. integrable 2-Kronecker structures) such that \( \alpha \) is an isomorphism.

### 1.2 | Pseudo-hyperkähler metrics of D’Amorim Santa-Cruz

Let \( G \) be a compact Lie group and \( G^\mathbb{C} \) its complexification. We denote by \( \pi \) the natural projection

\[
\pi : \mathfrak{g}^\mathbb{C} \otimes \mathcal{O}(2) \to \left( \mathfrak{g}^\mathbb{C} \otimes \mathcal{O}(2) \right)/G^\mathbb{C} \cong \left( \mathfrak{h}^\mathbb{C} \otimes \mathcal{O}(2) \right)/W \cong \bigoplus_{i=1}^{r} \mathcal{O}(2d_i),
\]

where \( r = \text{rank } G \) and the \( d_i \) are the degrees of \( \text{Ad } G^\mathbb{C} \)-invariant polynomials forming a basis of \( \mathbb{C}[\mathfrak{g}^\mathbb{C}]^{G^\mathbb{C}} \). We denote by \( \tilde{\mu} \) the composition of \( \mu \) with the map induced by \( \pi \) on global sections. A regular adjoint \( G^\mathbb{C} \)-orbit corresponds to a point in \( \mathfrak{h}^\mathbb{C}/W \). Therefore, if a section \( A(\zeta) \) of \( \mathfrak{g}^\mathbb{C} \otimes \mathcal{O}(2) \) is regular for every \( \zeta \), then its \( G^\mathbb{C} \)-orbit is identified with a section of \( \left( \mathfrak{h} \otimes \mathcal{O}(2) \right)/W \cong \bigoplus_{i=1}^{r} \mathcal{O}(2d_i) \). We shall call any such section \( S \) a spectral curve.
**Definition 1.1.** A section $A(\xi) = A_0 + A_1 \xi + A_2 \xi^2$ of $\mathfrak{g}^C \otimes \mathcal{O}(2)$ is called regular if $A(\xi)$ is regular for every $\xi$. It is called strongly regular if it is regular and the centralizers of $A(\xi)$ span, as $\xi$ varies in $\mathbb{P}^1$, the whole $\mathfrak{g}^C$.

**Remark 1.2.** For $G = U(k)$ strong regularity means that the coefficients of the powers $A(\xi)^i$, $i = 1, \ldots, k - 1$ are linearly independent in the space of complex $(k \times k)$-matrices.

Let $X_S$ be the submanifold of $\pi^{-1}(S)$ consisting of regular elements of $\mathfrak{g}^C \otimes \mathcal{O}(2)$. Strongly regular sections of $X_S$ which are also real in the sense that $A(\xi) = (T_2 + iT_3) + 2iT_1 \xi + (T_2 - iT_3) \xi^2$, where $T_i \in \mathfrak{g}$, are called twistor lines by D’Amorim Santa-Cruz. They form a manifold $M(S)$, shown by D’Amorim Santa-Cruz [6] to be pseudo-hyperkähler. Indeed, he shows that $M(S)$ has a real bilinear form $g_0$, compatible with hypercomplex structure, such that corresponding holomorphic 2-forms on fibers of $X_S$ (e.g., $g_0(I_2 \cdot, \cdot) + ig_0(I_3 \cdot, \cdot)$ for $I_1$) coincide with the Kostant–Kirillov–Souriau symplectic forms on adjoint orbits. This means that the fundamental forms $g_0(I_i \cdot, \cdot)$, $i = 1, 2, 3$, are nondegenerate and, consequently $g_0$ is a pseudo-hyperkähler metric.

**Remark 1.3.** If we consider all strongly regular sections, not just the real ones, we obtain a complex manifold equipped with a $\mathbb{C}$-hyperkähler structure. The $\mathbb{C}$-hypercomplex structure of this manifold extends to a 2-Kronecker structure on the manifold of all regular sections. Indeed, it follows from the arguments of D’Amorim Santa-Cruz that $H^1(A, N(-1)) = 0$ for any regular section, where $N$ is the normal bundle of $A$ in $X_S$. Therefore, the degrees of rank one direct summands of $N$ can be only 0, 1, 2.

## 2 | HYPERCOMPLEX STRUCTURES FROM ALGEBRAIC CURVES

Let $C$ be a connected and reduced lci algebraic curve of genus $g$ and $L$ a globally generated and non-special (i.e., $h^1(L) = 0$) invertible sheaf of degree $d = g + k - 1$, $k \geq 2$. Then $h^0(L) = k$, and for any basis $z = (s_1, \ldots, s_k)$ of $H^0(C, L)$, we obtain a morphism $\phi_L = \phi_{L, z} : S \rightarrow \mathbb{P}^{k-1}$ defined by

$$x \mapsto [f_1(x), \ldots, f_k(x)] \in \mathbb{P}^{k-1},$$

where $s_i(x) = f_i(x)e$, $i = 1, \ldots, k$, for some local holomorphic section $e$ of $L$. Clearly changing the basis $z$ by an overall multiplicative scalar does not change $\phi_{L, z}$.

Conversely, if $\phi : C \rightarrow \mathbb{P}^{k-1}$ is a holomorphic map such that the Hilbert polynomial $p(m) = \chi(\phi^*\mathcal{O}_{\mathbb{P}^{k-1}}(m))$ is equal to $dm - g + 1$, and $\phi(C)$ is not contained in a hyperplane, then $L = \phi^*\mathcal{O}_{\mathbb{P}^{k-1}}(1)$ has degree $d = g + k - 1$ and $\phi = \phi_{L, z}$ for the basis $\phi^* z$, where $(z_i)$ is the standard basis of $H^0(\mathbb{P}^{k-1}, \mathcal{O}_{\mathbb{P}^{k-1}}(1))$.

We denote the moduli space of invertible sheaves of degree $l$ by $\text{Jac}^l(C)$, and set:

$$\text{Jac}^{g+k-1}_0(C) = \{L \in \text{Jac}^{g+k-1}(C); \text{L is globally generated}, \ h^1(L) = 0\}.$$

Moreover, $F^{g+k-1}_0(C)$ will denote the principal $\text{PGL}(k, \mathbb{C})$-bundle over $\text{Jac}^{g+k-1}_0(C)$, the fibers of which consist of frames of $H^0(C, L)$ modulo scalars. We have a bijection

$$F^{g+k-1}_0(C) \xrightarrow{\sim} \text{Mor}^p_0(C, \mathbb{P}^{k-1}), \quad (2.1)$$

where $\text{Mor}^p_0(C, \mathbb{P}^{k-1})$ is the Hilbert scheme of nondegenerate (i.e., such that the image of $C$ is not contained in a hyperplane) morphisms with Hilbert polynomial $p(m) = dm - g + 1$.

**Proposition 2.1.** The bijection (2.1) is a biholomorphism of complex manifolds.

**Proof.** Since $\text{Jac}^{g+k-1}(C)$ is smooth, so is $F^{g+k-1}_0(C)$. Let $\phi = \phi_{L, z}$ be an element of $\text{Mor}^p_0(C, \mathbb{P}^{k-1})$. The pullback of the Euler sequence on $\mathbb{P}^{k-1}$ is

$$0 \rightarrow \mathcal{O}_C \rightarrow H^0(C, L)^* \otimes L \rightarrow \phi^* T_{\mathbb{P}^{k-1}} \rightarrow 0. \quad (2.2)$$
Since $h^1(L) = 0$, $h^1(\phi^* T_{p_{k-1}}) = 0$, and hence $\text{Mor}_0^P(C, p_{k-1})$ is smooth at $\phi$ (see, e.g., [7]). Moreover, the tangent space at $\phi$ is canonically identified with $H^0(C, \phi^* T_{p_{k-1}})$, and the long exact sequence of (2.2) yields

$$0 \to \mathbb{C} \longrightarrow H^0(C, L)^* \otimes H^0(C, L) \longrightarrow H^0(C, \phi^* T_{p_{k-1}}) \longrightarrow H^1(C, \mathcal{O}_C) \to 0.$$  (2.3)

The subspace $H^0(C, L)^* \otimes H^0(C, L)/C$ of $H^0(C, \phi^* T_{p_{k-1}})$ is precisely the subspace generated by fundamental vector fields (corresponding to a change of frame), while $H^1(C, \mathcal{O}_C)$ corresponds to variations of the invertible sheaf $L$. This describes the differential of (2.1) and shows that it is an isomorphism at every point of $F_{g+k-1}^0(C)$.

We now assume that the arithmetic genus of $C$ is equal to $(k - 1)^2$, and that $C$ is equipped with a fixed flat projection $\pi : C \to \mathbb{P}^1$ of degree $k$. If $C$ is smooth, then the number of branch points is $2k^2 - 2k$, and so the dimension of the moduli space parametrizing such pairs $(C, \pi)$ is $2k^2 - 2k$ (more precisely this is the dimension of the Hurwitz scheme parametrizing simple branch coverings of $\mathbb{P}^1$ of degree $k$ with $2k^2 - 2k$ branch points).

Denote by $\mathcal{O}_C(1)$ the line bundle $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$. This is an invertible sheaf of degree $k$, and the map $L \mapsto L(1)$ is an isomorphism $\text{Jac}^{g-1}(C) \to \text{Jac}^{g+k-1}(C)$. We denote by $\text{Mor}_{g+k-1}^P(C)$ the subset consisting of $L$ such that $L(-1) \not\in \Theta_C$. We have the corresponding open subsets $F_{g+k-1}^*(C)$ of $F_{g+k-1}^0(C)$ and $\text{Mor}_{g+k-1}^P(C, p_{k-1})$ of $\text{Mor}_{g+k-1}^P(C, p_{k-1})$.

We now define a natural $\text{PGL}(k, \mathbb{C})$-invariant integrable 2-Kronecker structure on $M = F_{g+k-1}^*(C) \cong \text{Mor}_{g+k-1}^P(C, p_{k-1})$. The bundle $E$ is

$$E|_{\phi} = H^0(C, \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-1)),$$

and the morphism $\alpha : E \otimes \mathbb{C}^2 \to TM$ is given by multiplication by elements of $\pi^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2$. If $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ vanishes at $\zeta$, then we have an exact sequence

$$0 \to \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-1) \otimes \mathcal{O}_C(\zeta) \to \phi^* T_{p_{k-1}} \to \phi^* T_{p_{k-1}}|_{\pi^{-1}(\zeta)} \to 0,$$

and $\alpha|_{E \otimes \mathbb{C}^2}$ is the induced map on global sections. Hence, $\alpha$ defines a 2-Kronecker structure (of rank $1/2 \dim M$). Moreover, the condition $h^1(L(-1)) = 0$ and (2.2) imply that $h^1(C, \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-1)) = 0$, and hence the the image of $E \otimes \mathbb{C}$ are precisely those sections of $\phi^* T_{p_{k-1}}$ that vanish on $\pi^{-1}(\zeta)$. This is the tangent space to the subvariety of $\text{Mor}_{g+k-1}^P(C, p_{k-1})$ consisting of morphisms that restrict to a fixed morphism over $\pi^{-1}(\zeta)$ (see, e.g., [7]). The 2-Kronecker structure is therefore integrable.

**Proposition 2.2.** The above 2-Kronecker structure is $\mathbb{C}$-hypercomplex on the subset of $F_{g+k-1}^*(C)$ corresponding to $L \in \text{Jac}_{g+k-1}^*(C)$ such that the natural multiplication map

$$H^0(C, L) \otimes H^0(C, L^* \otimes K_C(2)) \longrightarrow H^0(C, K_C(2))$$

is an isomorphism.

**Proof.** The Euler sequence on $\mathbb{P}^1$ gives

$$0 \to \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-2) \longrightarrow \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-1) \otimes \mathbb{C}^2 \longrightarrow \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C \to 0,$$

and hence $\alpha$ is an isomorphism if and only if $h^1(C, \phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-2)) = 0$ (since $\chi(\phi^* T_{p_{k-1}} \otimes \mathcal{O}_C(-2)) = 0$ from (2.2)). After tensoring (2.2) by $\mathcal{O}_C(-2)$ and taking the long exact sequence, this is equivalent to the natural map

$m : H^1(C, \mathcal{O}_C(-2)) \to H^0(C, L^* \otimes H^1(C, L(-2)) \simeq \text{Hom}(H^0(C, L), H^1(C, L(-2)))$

being an isomorphism. Observe that this map is simply $m(\phi)(s) = \phi s$. Using Serre duality, the dual map is the multiplication

$$H^0(C, L) \otimes H^0(C, K_C L^*(2)) \to H^0(C, K_C(2)).$$

□
Example 2.3. The 2-Kronecker structure is certainly degenerate if \( L(-1) \) is a theta characteristic. Indeed, in this case \( K_C L^*(2) \cong L \) and the multiplication map \( H^0(C, L) \otimes H^0(C, L) \to H^0(C, K_C(2)) \) vanishes on the skew-symmetric part of the tensor product.

2.1 | Reality conditions

Suppose now that \( C \) is equipped with an antiholomorphic involution \( \sigma \) covering the antipodal map on \( \mathbb{P}^1 \). We consider the following induced involution on \( \text{Jac}^{g+k-1}(C) \):

\[
\sigma : L \mapsto (\sigma^* L)^* \otimes K_C(2),
\]

and \( L \) will be called real if \( \sigma(L) \cong L \), that is, \( \sigma^* L \cong L^* \otimes K_C(2) \).

If \( L \) is a real, then the pullback \( s \to \sigma^* s \) induces an antiholomorphic isomorphism

\[
H^0(C, L) \to H^0(C, K_C L^*(2)). \tag{2.4}
\]

In order to obtain a hypercomplex structure, we need to extend this real structure to \( F_{g+k-1}^C \). If we tensor the pullback to \( C \) of the Euler sequence on \( \mathbb{P}^1 \) with \( K_C \), we obtain

\[
0 \to K_C \to K_C(1) \otimes \mathbb{C}^2 \to K_C(2) \to 0, \tag{2.5}
\]

and hence a natural surjective homomorphism

\[
\gamma : H^0(C, K_C(2)) \to H^1(C, K_C) \cong \mathbb{C}. \tag{2.6}
\]

The sequence (2.5) is compatible with the real structure and therefore we can define, for a real invertible sheaf \( L \in \text{Jac}^{g+k-1}(C) \), a hermitian form on \( H^0(C, L) \) by

\[
\langle s, t \rangle = \gamma(s \sigma^* t). \tag{2.7}
\]

Lemma 2.4. If \( L(-1) \notin \Theta_C \), then \( \langle , \rangle \) is nondegenerate.

Proof. Let \( s \in H^0(C, L) \). Let \( \zeta_0 \) be a point such that \( \pi^{-1}(\zeta_0) \) consists of distinct points and \( s \) does not vanish at any of them. We can find a section \( t \) of \( L \) that does not vanish at exactly one point of \( \pi^{-1}(-1/\bar{\zeta_0}) \). Then \( u = st \sigma^* t \) does not vanish at exactly one point of \( \pi^{-1}(\zeta_0) \). If \( \gamma(u) = 0 \), then (2.5) implies that \( u = v_1 + (\zeta - \zeta_0)v_2 \) for some \( v_1, v_2 \in H^0(C, K_C(1)) \). Therefore, \( u \) does not vanish at exactly one point \( p \) of \( \pi^{-1}(\zeta_0) \). But this means that \( v_1 \in H^0(C,K_C(p)) \setminus H^0(C,K_C) = \emptyset \). This contradiction implies that \( \gamma(st) \neq 0 \). \( \square \)

We thus obtain a quaternionic 2-Kronecker structure on the subset \( V \) of \( F_{g+k-1}^C \) where the sheaves are real and the fibers consist of unitary frames. Since we are interested in \( PU(k) \)-invariant hypercomplex structure, we denote by \( \tilde{M}(C, \pi) \) the subset of \( V \) where (2.7) is either positive- or negative-definite. Its open subset, where the condition of Proposition 2.2 is satisfied will be denoted by \( M(C, \pi) \).

Thus \( M(C, \pi) \) is a \( U(k) \)-invariant hypercomplex manifold (resp. \( U(k) \)-invariant quaternionic 2-Kronecker manifold).

Remark 2.5. \( M(C, \pi) \) must be nonempty at least for \( (C, \pi) \) in a neighborhood of \( \mathcal{M}_0 \) inside the real locus of the Hurwitz scheme \( \mathcal{H}_{k,(k-1)^2} \), where \( \mathcal{M}_0 \) consists of real \( (C, \pi) \) such that \( C \) is embedded in \( TP^1 \) and \( \pi \) is the restriction of \( TP^1 \to P^1 \).

2.2 | Relation to D’Amorim Santa-Cruz manifolds

D’Amorim Santa-Cruz’s manifolds \( M(S) \) (cf. Section 1.2) for \( G = U(k) \) are included in the above construction and correspond to pairs \( (C, \pi) \) such that \( C = S \) is embedded in \( TP^1 \) (with the original projection \( \pi : C \to P^1 \) equal to the one arising
from the embedding). Indeed, if \( C \) is embedded in \( TP^1 \), then it follows from results of Beauville \([1]\) that \( \text{Jac}_{g+k-1}(C) \) corresponds to \( GL(k, C) \)-conjugacy classes of quadratic \( \mathfrak{gl}(k, C) \)-valued polynomials. The \( C \)-hypercomplex structures on the space of strongly regular sections and on the manifold defined in Proposition 2.2 are easily seen to be the same, as are their restrictions to the real parts. The hermitian form \( \langle , \rangle \) on \( H^0(C, L) \) is then that of Hitchin \([8, \S 6]\).

Proposition 2.2 provides now an algebro-geometric characterization of strong regularity. A quadratic polynomial defines a strongly regular section if and only if the corresponding invertible sheaf satisfies the condition of Proposition 2.2.

### 2.3 Complex structures

We return to the hypercomplex manifold \( M(C, \pi) \) defined above. It consists of \( \sigma \)-invariant nondegenerate morphisms \( \pi: C \to \mathbb{P}^{k-1} \) that correspond to invertible sheaves \( L \) of degree \( g + k - 1 \) such that

1. \( L(-1) \notin \Theta_C \);
2. the multiplication map \( H^0(C, L) \otimes H^0(C, K_C(L^*(-2))) \to H^0(C, K_C^2) \) is an isomorphism;
3. the hermitian form \( \langle , \rangle \) is definite.

For every \( \zeta \in \mathbb{P}^1 \) the fiber \( C_\zeta = \pi^{-1}(\zeta) \) of \( \pi: C \to \mathbb{P}^1 \) can be identified with a fat point \( \sum k_i p_i \) in \( \mathbb{C} \), that is a zero-dimensional scheme \( \bigcup_{i=1}^s \text{Spec} \mathbb{C}[t]/((t-t_i)^{k_i}) \), with \( t_i \) distinct. The restriction of \( \phi \) to \( C_\zeta \) must be nondegenerate, since the image of \( \phi \) being contained in a hyperplane is equivalent to a section of \( L \) vanishing on \( \pi^{-1}(\zeta) \), which contradicts (1) above. Thus, for any complex structure \( I_\zeta \in Q \) of \( M(C, \pi) \) corresponding to \( \zeta \in \mathbb{P}^1 \), we have a local equivariant biholomorphism from \( \sum k_i p_i \) to \( \mathbb{P}^{k-1} \). This latter manifold is easily described:

**Lemma 2.6.** The (smooth) space of nondegenerate morphisms from \( \sum k_i p_i \) to \( \mathbb{P}^{k-1} \) is equivariantly isomorphic to \( GL(k, C)/Z(J) \), where \( Z(J) \) is the centralizer of the Jordan normal form matrix \( J = J_{t_1, k_1} \oplus \cdots \oplus J_{t_s, k_s} \) with distinct \( t_i \in \mathbb{C} \).

**Proof.** The image of each multiple point \( k_ip_i \) can span at most a \( (k_i - 1) \)-hyperplane in \( \mathbb{P}^{k-1} \). Therefore, if \( \phi: \sum k_i p_i \to \mathbb{P}^{k-1} \) is nondegenerate, then there exists a direct sum decomposition \( \mathbb{C}^k = \bigoplus_{i=1}^s V_i \), \( \dim V_i = k_i \), such that \( \phi|_{k_ip_i} \) is a non-degenerate morphism to \( \mathbb{P}(V_i) \). Using the action of \( GL(k, C) \), we can assume that each \( V_i \) is spanned by the corresponding set of coordinate vectors. Using now the action of \( GL(V_i, C) \), we can move \( \phi|_{k_ip_i} \) to \( t \to [1, t, \ldots, t^{k_i-1}] \) in \( PV_i \). Thus, the space of nondegenerate morphisms from \( \sum k_i p_i \) to \( \mathbb{P}^{k-1} \) is acted upon transitively by \( GL(k, C) \) and the stabilizer of

\[
\begin{bmatrix}
1 & t & \ldots & t^{k_1-1} \\
1 & t & \ldots & t^{k_2-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & t & \ldots & t^{k_s-1}
\end{bmatrix}
\]

consists of linear transformations \( (g_1, \ldots, g_s) \in \bigoplus_{i=1}^s GL(k_i, C) \) such that

\[
g_i(1, t, \ldots, t^{k_i-1})^T = \alpha_i(t)(1, t, \ldots, t^{k_i-1})^T, \quad i = 1, \ldots, s,
\]

for a polynomial \( \alpha_i(t) \) of degree \( k_i - 1 \). The result follows. \( \square \)

Therefore, \( (M(C, \pi), I_\zeta) \) is locally equivariantly biholomorphic to \( GL(k, C)/Z(J) \). This latter manifold is biholomorphic to the adjoint orbit of \( J \), but the biholomorphism is not canonical. It becomes canonical only in the case of D’Amorim Santa-Cruz manifolds, where there is also a complex symplectic form and the corresponding moment map embedding \( (M(C, \pi), I_\zeta) \) into \( \mathfrak{gl}(k, C) \).

### 3 PROOF OF THEOREM A

We have already shown how to construct a hypercomplex structure with required properties from an algebraic curve. Conversely, let \( M \) be a hypercomplex manifold with properties stated in Theorem A. We view \( M \) as a \( U(k) \)-invariant manifold (with the center acting trivially) and, for any \( m \in M \) and for every complex structure \( I_\zeta \in Q, \zeta \in \mathbb{P}^1 \), we denote
by \( \mathfrak{z}_m, \zeta \) the infinitesimal stabilizer of \( m \) in \( \mathfrak{gl}(k, \mathbb{C}) \). Since the antipodal map on \( \mathbb{P}^1 \) corresponds to \( I \mapsto -I \) on \( Q \), we have the following reality condition:

\[
\mathfrak{z}_m, -1/\overline{\zeta} = (\mathfrak{z}_m, \zeta)^*, \quad \forall \zeta \in \mathbb{P}^1.
\] (3.1)

**Lemma 3.1.** The family \( \mathfrak{z}_m, \zeta \in \mathbb{P}^1 \), is a holomorphic vector bundle of rank \( k \) and degree \( k - k^2 \).

**Proof.** \( \mathfrak{z}_m, \zeta \) is the kernel of the map

\[
\mathfrak{gl}(k, \mathbb{C}) = u(k) \oplus iu(k) \longrightarrow T_m M, \quad \rho_1 + i\rho_2 \longmapsto X_{\rho_1} + I_\zeta X_{\rho_2},
\]

where \( X_\zeta \) is the fundamental vector field corresponding to \( \zeta \in u(k) \). This map is \( \mathbb{C} \)-linear with respect to \( I_\zeta \). As \( \zeta \) varies, the complex spaces \( (T_m M, I_\zeta) \) form the vector bundle \( \mathfrak{K} \oplus \mathbb{C}^{n} \) on \( \mathbb{P}^1 \), where \( n = k^2 - k = \dim \mathbb{C}M \). Thus, the family in the statement is the kernel of \( \mathfrak{gl}(k, \mathbb{C}) \rightarrow \mathfrak{K} \oplus \mathbb{C}^{n} \rightarrow 0 \). Since the stalks of \( \mathfrak{K} \) all have dimension \( k \), \( \mathfrak{K} \) is locally trivial. The rank and the degree follow from the exact sequence defining \( \mathfrak{K} \). □

The centralizer of a regular element in \( \mathfrak{gl}(k, \mathbb{C}) \) is a commutative subalgebra of \( \text{Mat}_{k,k}(\mathbb{C}) \) (with respect to matrix multiplication). Thus the vector bundle \( \mathfrak{z}_m \) defined in the above lemma is a sheaf of \( k \)-dimensional commutative algebras, locally free as a sheaf of \( \mathfrak{K} \) modules. Therefore, its \( \text{Spec} \) is an algebraic curve \( C_m \) with a flat projection \( \pi_m : C_m \rightarrow \mathbb{P}^1 \) of degree \( k \). The vector bundle \( \mathfrak{z}_m \) is then \( \pi_m^* \mathfrak{K} \). Since the algebras \( \mathfrak{z}_m, \zeta \) are isomorphic for all \( m \), so are the curves \( C_m \) and the projections \( \pi_m : C_m \rightarrow \mathbb{P}^1 \). We denote by \( \pi : C \rightarrow \mathbb{P}^1 \) the abstract curve and its projection, isomorphic to any \( \pi_m : C_m \rightarrow \mathbb{P}^1 \). The reality conditions imply that \( C \) is equipped with an antiholomorphic involution covering the antipodal map.

**Lemma 3.2.** \( C \) is connected, reduced, lci, and of genus \( (k - 1)^2 \).

**Proof.** Connectedness of \( C \) is equivalent to the dimension of the trivial summand of \( \mathfrak{z}_m \approx \pi_m^* \mathfrak{K} \) being equal to 1. This trivial summand corresponds to \( \rho_1 + i\rho_2 \in \mathfrak{gl}(k, \mathbb{C}) \) such that \( X_{\rho_1} + I_\zeta X_{\rho_2} = 0 \) for every \( \zeta \). This is equivalent to \( X_{\rho_1} = X_{\rho_2} = 0 \). Since we assumed that \( PU(k) \) acts freely, the trivial summand of \( \mathfrak{z}_m \) is the center of \( \mathfrak{gl}(k, \mathbb{C}) \), hence it is one-dimensional, and \( C \) is connected. Choose now a local section \( c \) of \( \mathfrak{z}_m \), which generates \( \mathfrak{z}_m, \zeta \) as algebra. We can then embed \( C \) locally into \( \mathbb{C}^2 \) as

\[
\{(\zeta, \eta) \in \mathbb{C}^2; \det(\eta - c(\zeta)) = 0\}.
\]

This shows that \( C \) is lci. It is also reduced, since \( \mathfrak{z}_m, \zeta \) is a Cartan subalgebra for generic \( \zeta \), and hence \( c(\zeta) \) has generically \( k \) distinct eigenvalues. Finally, the genus of \( C \) is \( (k - 1)^2 \), since \( 1 - g = \chi(\mathfrak{K}) = \chi(\mathfrak{z}_m) \). □

Consider now the vector bundle \( E \approx \mathcal{O}_{\mathbb{P}^1}(-1)^\oplus k \). Since \( \mathfrak{z}_m, \zeta \subset \mathfrak{gl}(k, \mathbb{C}) \), \( E \) has the structure of a \( \mathfrak{z}_m \)-module, that is, a \( \pi_\zeta \mathcal{O}_C \)-module. Since \( \mathfrak{z}_m, \zeta \) are centralizers of regular elements, \( E \) is locally isomorphic to \( \pi_\zeta \mathcal{O}_C \), and hence \( E \approx \pi_m^* L_m \) for an invertible sheaf \( L_m \) on \( C \). Moreover, \( h^0(L_m) = h^1(L_m) = 0 \), and so \( L_m \in \text{Jac}^{2-1}(C) \setminus \Theta \).

We need to show that the sheaves \( L_m \) are \( \sigma \)-invariant in the sense of Theorem A. Suppose that \( \zeta \) is a locally free sheaf of centralizers of regular elements of \( \mathfrak{gl}(k, \mathbb{C}) \), such that its \( \text{Spec} \) is isomorphic to \( C \). Let \( L \) be the corresponding invertible sheaf arising from the \( \zeta \)-module structure on \( E \approx \mathcal{O}_{\mathbb{P}^1}(-1)^\oplus k \). Then:

**Lemma 3.3.** The sheaf corresponding to \( \mathfrak{z}^T \) is \( L^* \otimes K_C \).

**Proof.** The \( \mathfrak{z}^T \)-module structure on \( E \) is isomorphic to \( \text{Hom}_{\mathbb{P}^1}(\pi_* L, \mathcal{O}(-2)) \) (where \( \pi_* L \) is, by definition, \( E \) with its \( \mathfrak{z} \)-module structure). Since

\[
\pi_* \text{Hom}_C(L, K_C) \approx \text{Hom}_{\mathbb{P}^1}(\pi_* L, K_{\mathbb{P}^1}),
\]

the claim follows. □

This lemma and (3.1) show that \( L_m(1) \) is indeed real for \( m \in M \) (in the sense of Section 2.1). Moreover, the proof also implies that the antiholomorphic isomorphism (2.4) is simply the conjugation on \( H^0(\mathbb{P}^1, E(1)) \approx C^k \), and the hermitian
form \((2.7)\) is the standard hermitian form on \(\mathbb{C}^k\). We thus obtain a natural smooth map \(\Psi : M \to \bar{M}(C, \pi)\), where the latter manifold was defined at the end of Section 2.1. We claim that \(\Psi\) is injective:

**Lemma 3.4.** \(L_m \simeq L_{m'}\) if and only if \(m\) and \(m'\) belong to the same \(U(k)\)-orbit.

**Proof.** \(L_m\) and \(L_{m'}\) are isomorphic if and only if the two \(\pi_* O_C\) module structures on \(E\) are isomorphic. Given the reality condition \((3.1)\) and the fact that \(\text{Aut}(E) \simeq GL(k, \mathbb{C})\), we conclude that \(L_m \simeq L_{m'}\) if and only if there exists \(g \in U(k)\) such that \(g z_m \xi^{-1} = z_{m'} \xi\) for all \(\xi \in \mathbb{P}^1\). Choose \(\xi\) so that the stabilizer \(z_{m'} \xi\) is a Cartan subalgebra. Owing to assumption (1) in the theorem, there exists an element \(h\) of \(GL(k, \mathbb{C})\) (for the complex structure \(I_{\xi}\)) such that \(h \cdot m = m'\). This means, however, that \(h\) normalizes the Cartan subalgebra \(z_{m'} \xi\), and so \(h = w t\), where \(t \in \exp z_{m'} \xi\) and \(w \in S_k \subset U(k)\). Therefore, \(m\) and \(m'\) belong to the same \(U(k)\)-orbit. \(\square\)

We can extend \(\Psi\) to a neighborhood of \(M\) in \(M^C\), and it will remain injective there. Since this extension is a holomorphic bijection onto its image, it is a biholomorphism, and hence \(\Psi\) is a diffeomorphism onto its image, which must be an open subset of \(\bar{M}(C, \pi)\). The hypercomplex structure of \(M\) is also the one constructed in the previous section, and Proposition 2.2 implies that the sheaves \(L_m\) do not belong to the divisor \(\Delta^\pi\). The proof is complete.

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