Variational bound on energy dissipation in turbulent shear flow

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We present numerical solutions to the extended Doering–Constantin variational principle for upper bounds on the energy dissipation rate in plane Couette flow, bridging the entire range from low to asymptotically high Reynolds numbers. Our variational bound exhibits structure, namely a pronounced minimum at intermediate Reynolds numbers, and recovers the Busse bound in the asymptotic regime. The most notable feature is a bifurcation of the minimizing wavenumbers, giving rise to simple scaling of the optimized variational parameters, and of the upper bound, with the Reynolds number.

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Rigorous results are rare in the theory of turbulence. There are no analytical solutions to the Navier–Stokes equations for fully developed turbulence, and numerical simulations can only cope with flows at comparatively low Reynolds numbers, due to the enormous number of degrees of freedom involved. The recent formulation by Doering and Constantin of a novel variational principle for computing rigorous upper bounds on quantities characterizing turbulent flows [1] has therefore met with considerable interest. When resorting to a variational principle, one does no longer aim at solving the equations of motion exactly, but rather uses these equations to derive inequalities that bound the relevant quantities, such as the rate of energy dissipation or other transport properties. The hope is to obtain variational inequalities that are, on the one hand, still technically manageable, but capture essential features of the full dynamical fluid system on the other.

This spirit has been very much alive already some 25 years ago in the Howard–Busse theory [4–6]. The renewed excitement about such methods stems from an obvious question: Does the new principle provide new physical insight?

In this Letter, we answer this question for energy dissipation in plane Couette flow. We have devised a numerical scheme that allows us to exhaust the Doering–Constantin variational principle, with the extension introduced in Ref. [7], for the entire range from low to asymptotically high Reynolds numbers. Intriguingly, we find that the solution to this intricate problem is organized by a simple feature: the minimizing wavenumbers bifurcate at a certain intermediate Reynolds number, thereby determining the asymptotic scaling of the optimal upper bound on the energy dissipation rate. As a consequence of this bifurcation, this upper bound exhibits some structure at about those Reynolds numbers where typical laboratory shear flows become turbulent.

We start from the equations of motion

\begin{align}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \nu \Delta \mathbf{u}, \\
\nabla \cdot \mathbf{u} &= 0
\end{align}

for an incompressible fluid confined between two infinitely extended, rigid plates; \(\nu\) is the kinematic viscosity and \(p\) the kinematic pressure. The lower plate, coinciding with the plane \(z = 0\) of a Cartesian coordinate system, is at rest, whereas the upper one at \(z = h\) is sheared with constant velocity \(U\) in positive \(x\)-direction. This yields the no-slip boundary conditions

\[ \mathbf{u}(x, y, 0, t) = 0, \quad \mathbf{u}(x, y, h, t) = U\hat{x} \]

(\(\hat{x}\) is the unit vector in \(x\)-direction); periodic boundary conditions are imposed in \(x\)- and \(y\)-direction. The time-averaged rate of energy per mass dissipated in the periodicity volume \(\Omega\) is given by

\[ \varepsilon_T = \frac{1}{T} \int_0^T dt \left\{ \frac{\nu}{\Omega} \int d^3x \left[ \sum_{i,j=x,y,z} (\partial_j u_i)^2 \right] \right\}. \]

Our aim is to compute a rigorous upper bound on the long-time limit of the non-dimensionalized dissipation rate

\[ c_e(Re) = \lim_{T \to \infty} \frac{\varepsilon_T}{U^3 h^{-1}}, \]

where \(Re = Uh/\nu\) is the Reynolds number.

Let us recall [7] that below the energy stability limit \(Re_{ES} \approx 82.65\) we have the exact identity \(c_e(Re) = Re^{-1}\). Moreover, the laminar flow gives rise to the lower bound \(c_e(Re) = Re^{-1}\) for all \(Re\). In order to calculate an upper bound on \(c_e\) for \(Re \geq Re_{ES}\), we employ the background flow approach pioneered by Doering and Constantin. Decomposing the actual velocity field \(\mathbf{u}\) into a stationary, divergence-free “background flow” \(\mathbf{U}\) (not to be confused with the physical mean flow), which has to carry the physical boundary conditions, and the deviations \(\mathbf{v}\) from that flow,

\[ \mathbf{u}(x, t) = \mathbf{U}(x) + \mathbf{v}(x, t), \]

inserting this decomposition into Eq. (4), and utilizing the equations of motion (1)–(2) when eliminating the deviations \(\mathbf{v}\), one derives an inequality that bounds \(c_e\) in...
terms of $U$. We confine ourselves to background flows of the form
\[ U(x) \equiv U\phi(\zeta) \hat{x}, \]
where $\zeta \equiv z/h$, and introduce a dimensionless balance parameter $a > 1$, which weights the contribution to $\varepsilon_T$ that stems from the cross-term containing both background flow and deviations. This parameter constitutes the same homogeneous boundary conditions as the $\zeta \equiv z/h$ parameter, which weights the contribution to $\varepsilon_T$ that stems from the cross-term containing both background flow and deviations.

The essential point now is that the candidate profiles are restricted by a spectral constraint: for each admissible $\phi$, all eigenvalues $\lambda$ of the linear eigenvalue problem
\[ \lambda V = -2h^2 \Delta V + R \phi' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} V + \nabla P, \]
\[ 0 = \nabla \cdot V, \]
have to be positive; the eigenvectors $V(x)$ have to satisfy the same homogeneous boundary conditions as the deviations $v(x,t)$. The number $R$ introduced here is the rescaled Reynolds number,
\[ R \equiv \frac{a}{a-1} Re. \]

Computationally, the main task is to calculate, for every single candidate profile $\phi$, that rescaled Reynolds number $R_c(\phi)$ for which the lowest eigenvalue of (10) passes through zero. When $a$ is adjusted such that the r.h.s. of the inequality (8) is minimized for fixed $\phi$ and $Re$, subject to the condition (11), this $\phi$ produces via (8) an upper bound on $c_e$ for all $Re \leq R_c(\phi)$. That is, $R_c(\phi)$ marks the highest $Re$ up to which $\phi$ is admissible to the variational principle (5). Exploiting the periodic boundary conditions, each eigenvalue $\lambda$ is labeled by two wavenumbers, $k_x$ and $k_y$. We first keep these wavenumbers fixed and compute that number $R = R_0(\phi)(k_x, k_y)$ for which the lowest eigenvalue $\lambda_{k_x, k_y}$ becomes zero, and then determine $R_c(\phi)$ by minimizing over all $k_x$ and $k_y$. The details of this tedious procedure will be given in Ref. (10).

We use variational profiles of the type displayed in Fig. 1. boundary layer segments of width $\delta$, modeled by polynomials of order $n$, are connected by a linear piece with slope $p$, such that the profiles are $n-1$ times continuously differentiable at the matching points $z/h = \delta$ and $z/h = 1 - \delta$. Thus, we have the three variational parameters $\delta, n,$ and $p$. These profiles have not been chosen ad libitum, but have been shown to correctly capture the asymptotics of the dissipation bound in a closely related model problem (13).

The optimized profiles resulting from the variational principle are depicted in Fig. 2. Considering the response of the optimized profile parameters to the increase of $Re$, we can clearly distinguish five different regimes. (i) Up to $Re = Re_{ES}$ we have the laminar regime with $\phi(\zeta) = \zeta$. (ii) For $Re_{ES} \leq Re \leq Re_1 \approx 160$ the laminar profile is deformed. The parameters $\delta$ and $n$ remain at their values 0.5 and 3, respectively, while the slope $p$ decreases from 1 to almost 0. (iii) In the following regime, $Re_1 \leq Re \leq Re_2 \approx 670$, boundary layers develop. Here $n$ still remains fixed, $p$ increases again, while $\delta$ decreases to its minimal value 0.14. (iv) Then, for $Re_2 \leq Re \leq Re_3 \approx 1845$, $n$ increases dramatically from 3 to 34, thus steepening the profile in the immediate vicinity of each boundary, thereby effectively generating a new internal boundary layer within each boundary segment. As a consequence, $\delta$ increases back to 0.5, so that the boundary segments finally join together. (v) For $Re \geq Re_3$ the variational parameters obey simple scaling laws: $\delta = 0.5$ remains fixed, the profile slope at the boundary becomes
\[ \phi'(0) \sim n \propto Re, \]
while the slope at the midpoint is given by
\[ \phi'(\frac{1}{2}) = p \propto Re^{-1}. \]

The size of the internal boundary layers, i.e., the range close to the boundaries where the optimized profiles are steep, is measured by $1/\phi'(0)$ and hence proportional to $Re^{-1}$. It needs to be emphasized, however, that the optimal variational profiles do not necessarily resemble the physically realized mean flow profiles.

A key for understanding the above scaling is provided by the behavior of the minimizing wavenumbers. Remarkably, the minimizing $k_y$ is zero for all $Re$, reflecting the fact that there is no distinguished length scale in shear direction. Even more important, the minimizing $k_y$ bifurcates at
\[ Re_B \approx 460, \]
as shown in Fig. 3. This means that for $Re > Re_B$ the two lowest eigenvalues of Eqs. (10) are degenerate and pass through zero simultaneously; it is a key property of the variational principle that both corresponding minima of $R_0(\phi)(0, k_y)$ adopt identical values. The lower $k_y$-branch ($k_{y,1}$) approaches a constant value for $Re \to \infty$, namely, the very same $\delta$ that also determines the energy stability limit, whereas the upper branch ($k_{y,2}$) scales proportionally to $Re$. Both findings are intimately connected to the power-law behavior of the profile expressed by Eqs. (12) and (13). For Reynolds numbers above $Re_B$ two different length scales, $k_{y,1}^{-1}$ and $k_{y,2}^{-1}$, appear on the
stage, which can be identified \[10\] as the extension of the profile’s flat part in the interior and the effective widths of the (internal) boundary layers. The regime (iii) can thus be considered as a cross-over regime from deformed profiles with no definite length scales to profiles with well-developed boundary layers.

The optimal variational bound on the energy dissipation rate \( c_\varepsilon \) that results from these ingredients is depicted in Fig. 4. Immediately above \( Re_{ES} \) this upper bound \( \overline{\varepsilon} \) increases, exhibits a maximum, and then decreases roughly proportional to \( Re^{-1/4} \) in regime (iii). The occurrence of the bifurcation changes this behavior: the bound passes through a distinct minimum at those \( Re \) where the internal boundary layers start to develop, and finally ascends to its asymptotic value

\[
\lim_{Re \to \infty} \overline{\varepsilon}(Re) = 0.01087(1) \tag{15}
\]

This asymptotic bound has to be compared to the bound calculated by Busse in the framework of his Optimum Theory \[11\], which refers to the limit \( Re \to \infty \) and reads

\[
\lim_{Re \to \infty} c_\varepsilon(Re) \lesssim 0.010 \tag{16}
\]

The prediction that the bound furnished by the variational principle \[8\] should coincide in the limit \( Re \to \infty \) with the Busse bound \[11\] has been made by Kerswell \[12\], who could bring the variational principle into a form to which Busse’s so-called multi-\( \alpha \) solutions can be applied. Thus, Kerswell’s conclusion had to rely on all assumptions inherent in the Optimum Theory, whereas we have actually constructed a rigorous solution to the variational principle \[8\]. In this way, we could not only confirm the correctness of Busse’s asymptotic result, but provide a rigorous bound on \( c_\varepsilon \) for all \( Re \). Our bound shows non-trivial structure: a pronounced minimum followed by a \( Re \)-range between 1000 and 1800 in which the bound’s curvature changes its sign. This occurs in regime (iv), i.e., at those \( Re \) where one observes the onset of turbulence in typical laboratory shear flows.

In order to illustrate the importance of the proper implementation of the spectral constraint, we compare in Fig. 4 the bound obtained in this work to the bound derived previously by Doering and Constantin \[14\] with the help of elementary functional estimates. These estimates are convenient to evaluate, but over-satisfy the spectral constraint and result in the structureless bound \( c_\varepsilon(Re) \leq 1/(8\sqrt{2}) \approx 0.088 \) for \( Re \geq 8\sqrt{2} \). The improvements due to both the introduction of the balance parameter \( a \) and the evaluation of the less restrictive actual spectral constraint amounts to almost an order of magnitude, namely to a factor of 8 in the asymptotic regime. All upper bounds in Fig. 4 become asymptotically independent of the Reynolds number, which means that they are in accordance with the scaling implied by classical turbulence theories.

To conclude, we have found a scheme for solving the extended Doering–Constantin variational principle \[14\] for upper bounds on energy dissipation in plane Couette flow. The bound established here is rigorous for all Reynolds numbers, recovers the Busse bound in the asymptotic regime, and represents in the important regime of intermediate \( Re \) the best upper bound that has been calculated so far. Moreover, we have pinned down the feature that organizes the solutions to the variational principle, namely the bifurcation of the minimizing wave-numbers illustrated in Fig. 3. Based on these results, it is now possible to obtain similar bounds of comparable quality also for other flows of interest.

However, comparing the bound with the experimental data indicated in Fig. 3 it is clear that this is not the end of the story. The variational bound lies an order of magnitude above the dissipation rates measured by Reichardt \[14\] and Lathrop et al. \[15\], and these data do not seem to approach a constant value, but still decrease with \( Re \). If one could improve the variational principle such that this decrease is captured, and if it did persist asymptotically, one could make definite statements about possible intermittency corrections to classical scaling \[16\].

To proceed along these lines is possibly one of the greatest challenges in the rigorous theory of turbulence.

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FIG. 1. Variational test profiles $\phi(\zeta)$ versus $\zeta = z/h$.

FIG. 2. Metamorphosis of the optimized variational profiles with the increase of the Reynolds number. We have depicted the most important $Re$-range on a logarithmic scale, beginning with the energy stability limit $Re_{ES}$ and ending in the scaling regime $Re > Re_3$.

FIG. 3. Minimizing wavenumber(s) $k_y$ belonging to the upper bound on $c_\varepsilon$ displayed in Fig. 4. The wavenumbers of the upper branch correspond to the inverse boundary layer thickness of the profiles, while those of the lower branch reflect the inverse extension of the flat interior profile part.

FIG. 4. Bounds on $c_\varepsilon$ for the plane Couette flow. Points denote the upper bound $c_\varepsilon(Re)$ derived from the variational principle (8); the solid line on the left is the lower bound $c_\varepsilon(Re) = Re^{-1}$. The asymptotic value of the upper bound, $\lim_{Re \to \infty} c_\varepsilon(Re) = 0.01087(1)$, lies slightly above, but within the uncertainty span of Busse’s asymptotic result (16).

FIG. 5. A synopsis of bounds on $c_\varepsilon$ and experimental data. — Slanted straight line on the left: lower bound $c_\varepsilon(Re) = Re^{-1}$. — Topmost horizontal line: upper bound obtained by Doering and Constantin in Refs. [1,13] with the help of an over-restrictive profile constraint and piecewise linear profiles; $c_\varepsilon(Re) \approx 0.088$ for $Re > 11.32$. — Heavy dots: upper bound obtained in this work from the variational principle supplemented by the actual spectral constraint (10), cf. Fig. 4; $c_\varepsilon(Re) \to 0.01087(1)$. — Joining dashed line: asymptotic upper bound derived by Busse in Refs. [5,6]; $c_\varepsilon(Re) \to 0.010(1)$. The shaded area denotes the estimated uncertainty of this bound. — Triangles: experimental dissipation rates for the plane Couette flow measured by Reichardt [14]. — Circles: experimental dissipation rates for the Taylor–Couette system with small gap as measured by Lathrop, Fineberg and Swinney [15].