\( \mathcal{E}' \) AS AN ALGEBRA BY MULTIPLICATIVE
CONVOLUTION

Dietmar Vogt

In the present paper we continue the investigations in [6]. For that purpose we define the \textit{multiplicative convolution} in the space \( \mathcal{E}'(\mathbb{R}^d) \) of distributions of compact support on \( \mathbb{R}^d \) by \( (T \star S)g = T_x S_y f(xy) \). Equipped with this multiplication \( \mathcal{E}'(\mathbb{R}^d) \) is a commutative algebra. Each \( T \in \mathcal{E}'(\mathbb{R}^d) \) defines a convolution operator \( N_T : S \mapsto T \star S \) in \( \mathcal{E}'(\mathbb{R}^d) \). For open \( \Omega, \Omega' \subset \mathbb{R}^d \) we show that \( N_T \mathcal{E}'(\Omega') \subset \mathcal{E}'(\Omega) \) if and only if \( \text{supp } T \cdot \Omega' \subset \Omega \), in particular, \( N_T \) defines an operator in \( L(\mathcal{E}'(\Omega)) \) if and only if \( \text{supp } T \subset V(\Omega) \) where \( V(\Omega) \) denotes the set of dilations acting on \( \Omega \). In this case \( N_T = M_T^* \) where \( M_T \in L(\mathcal{E}(\Omega)) \) is the Hadamard operator on \( C^\infty(\mathbb{R}^d) \) defined by \( M_T f(y) = T_x f(xy) \) that is \( N_T \) is the dual operator of \( M_T \).

This allows us an elegant access to the theory of Hadamard operators on \( \mathcal{E}'(\Omega) \) for open sets \( \Omega \subset \mathbb{R}^d \), that is, of operators which allow all monomials as eigenvectors. These have been studied and characterized in [6]. The algebra \( M(\Omega) \) of such operators is a closed subalgebra of \( L(C^\infty(\Omega)) \). It is isomorphic to the algebra \( (\mathcal{E}'(V(\Omega)), \star) \) and we determine the topology induced from \( L_b(C^\infty(\Omega)) \) on \( \mathcal{E}'(V(\Omega)) \). We use this isomorphism to show that the algebra \( M(\Omega) \) is isomorphic to an algebra of holomorphic functions around zero where the multiplication is the classical Hadamard multiplication, that is, multiplication of the Taylor coefficients. Hadamard-type operators assigned to distributions with support \( \{(1, \ldots, 1)\} \) are called Euler operators and we study global solvability for such operators on open subsets of \( \mathbb{R}^d \).

Hadamard operators on the space \( \mathcal{A}(\Omega) \) of real analytic functions have been studied and characterized in Domański-Langenbruch [1], [2], [3]. Analogous problems as in the present paper for real analytic functions are studied in Domański-Langenbruch-Vogt [4]. While the problems are analogous, the results, the methods and the difficulties to overcome are, in part, quite different.

\textit{2010 Mathematics Subject Classification.} Primary: 47L80, 47B38. Secondary: 46H35, 46E10.

\textit{Key words and phrases:} Operators on smooth functions, distributions with compact support, algebra of operators, monomials as eigenvectors.
The present paper was originally written as a draft for the author’s use to serve as the mathematical basis for lectures the author delivered in 2014 (Istanbul Analysis Seminar, Nordwestdeutsches Funktionalanalysis-Kolloquium in Paderborn, DMV-PTM Mathematical Meeting in Poznań). It has maintained much of this character, for which the author apologizes. In particular, for proper references, citations and much of the notation, and also for the background of the problem we refer to [6].

1 Multiplicative convolution on $\mathcal{E}'$

On $\mathcal{E}'(\mathbb{R}^d)$ we define the multiplicative convolution:

**Definition 1** For $T, S \in \mathcal{E}'(\mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}^d)$ we set $(T \ast S)f = T_x(S_y f(xy)).$

We freely use this notation, in different context, also for other combinations of $T, S$ and $f$, whenever the right hand side makes sense. As multiplication on $\mathcal{E}'(\mathbb{R}^d)$ it has the following basic properties:

**Proposition 1.1** 1. $(\mathcal{E}'(\mathbb{R}^d), \ast)$ is a commutative algebra.
2. $\delta_1$ is the unit element.
3. The multiplication is hypocontinuous.
4. $\text{supp} (T \ast S) \subset \text{supp} T \cdot \text{supp} S.$

**Proof:** 1. To show commutativity we remark that for any $\alpha \in \mathbb{N}_0^d$ we have

$$(T \ast S)x^\alpha = T x^\alpha \cdot S x^\alpha$$

Therefore $T \ast S$ and $S \ast T$ coincide on the polynomials which are dense in $C^\infty(\mathbb{R}^d)$.

2. is trivial. To show 3. we have, with suitable $K$ and $p$ for $T$, resp. $L$ and $q$ for $S$, and a constant $C > 0$ depending only on $L, p, q$ the following estimates:

$$| (T \ast S)f | = | T_x(S_y f(xy)) |$$
$$\leq \| T \|^*_K, p \sup_{x \in K, |\alpha| \leq p} | S_y(y^\alpha f^{(\alpha)}(xy)) |$$
$$\leq \| T \|^*_K, p \| S \|_{L^q_y} \sup_{x \in K, |\alpha| \leq p} \sup_{y \in L, |\beta| \leq q} | \partial^\beta_y(y^\alpha(f^{(\alpha)}(yx)) |$$
$$\leq C \| T \|^*_K, p \| S \|^*_{L^q_y} \sup_{z \in KL, |\gamma| \leq pq} | f^{(\gamma)}(z) |.$$

This shows 3. and implies 4. $\square$

Let $\Omega'$ and $\Omega$ be open subsets of $\mathbb{R}^d$. From Proposition 1.1 we obtain immediately
Corollary 1.2 If $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\text{supp} \: T \cdot \Omega' \subset \Omega$, then $N_T : S \mapsto T \ast S$ defines a map in $L(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega))$.

Proof: We have only to remark that $\mathcal{E}'(\Omega')$ is bornological and that, by Proposition 1.1.11, the map $N_T$ sends equicontinuous (=bounded) sets into equicontinuous (=bounded) sets.

We want now to describe the maps $N \in L(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega))$ for which there is $T \in \mathcal{E}'(\mathbb{R}^d)$ with $N = N_T$. We denote the set of these maps by $N(\Omega', \Omega)$.

Lemma 1.3 For $N \in L(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega))$ the following are equivalent:
1. $N(S_1) \ast S_2 = S_1 \ast N(S_2)$ for all $S_1, S_2 \in \mathcal{E}'(\Omega')$
2. There is $T \in \mathcal{E}'(\mathbb{R}^d)$ such that $N(S) = T \ast S$ for all $S \in \mathcal{E}'(\Omega')$.

Proof: To show 1. $\Rightarrow$ 2. we choose $\eta \in \Omega' \cap \mathbb{R}^d$. Then for all $S \in \mathcal{E}'(\Omega')$ we have $N(S) \ast \delta_\eta = S \ast N(\delta_\eta)$ and therefore $N(S) = T \ast S$ with $T = N(\delta_\eta) \ast \delta_1/\eta$. The reverse direction is obvious.

For two sets $X, Y \subset \mathbb{R}^d$ we set $V(X, Y) := \{ \eta \in \mathbb{R}^d : \eta X \subset Y \}$, $V(X) = V(X, X)$ and $Y_x = \{ \eta \in \mathbb{R}^d : \eta x \in Y \}$.

Lemma 1.4 If $X$ is open and $Y$ is closed, then $V(X, Y)$ is closed and $V(X, Y) = \bigcap_{x \in X \cap \mathbb{R}^d} \frac{1}{x} Y$. If, in particular, $Y$ is compact, then also $V(X, Y)$.

Proof: Because $X \cap \mathbb{R}^d$ is dense in $X$ we have

$$V(X, Y) = \{ \eta \in \mathbb{R}^d : \eta (X \cap \mathbb{R}^d) \subset Y \} = \bigcap_{x \in X \cap \mathbb{R}^d} \frac{1}{x} Y.$$

Since $\frac{1}{x} Y$ is closed for all $x \in \mathbb{R}^d$ the set $V(X, Y)$ is closed, same argument in the case of compact $Y$. \hfill \Box

Theorem 1.5 For $T \in \mathcal{E}'(\mathbb{R}^d)$ the following are equivalent:
1. $N_T \in L(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega))$. 
2. $N_T \in L(C'(\Omega'), \mathcal{E}'(\Omega))$. 
3. $\text{supp} \: T \subset V(\Omega', \Omega)$.

Proof: Since 3. $\Rightarrow$ 1. and 1. $\Rightarrow$ 2. are obvious we have only to show 2. $\Rightarrow$ 3.

If $N_T \in L(C'(\Omega'), \mathcal{E}'(\Omega))$ we have for every $\eta \in \Omega'$ that $\text{supp} \: (T \ast \delta_\eta) \subset \Omega$. We fix an open $\omega' \subset \subset \Omega'$. Because of the continuity of $N_T$ there is a compact subset $L \subset \Omega$ such that $\text{supp} \: (T \ast \delta_\eta) \subset L$ for all $\eta \in \omega'$. For $\eta \in \mathbb{R}^d$ we have $\text{supp} \: (T \ast \delta_\eta) = \eta \text{supp} \: T$. So we have shown that $\text{supp} \: T \subset \bigcap_{\eta \in X \cap \mathbb{R}^d} \frac{1}{\eta} L = V(\omega', L) \subset V(\omega', \Omega)$. The equality comes from Lemma 1.4 Since $V(\Omega', \Omega) = \bigcap_{\omega' \subset \subset \Omega'} V(\omega', \Omega)$ the proof is complete. \hfill \Box
2 Transposed description

Let \( N_T \in N(\Omega', \Omega) \), then the transposed operator \( N_T^* \in L(C^\infty(\Omega), C^\infty(\Omega')) \) is given by

\[
(N_T^*f)(x) = \langle \delta_x, N_T^*f \rangle = \langle T \star \delta_x, f \rangle = T_y f(xy) =: (M_T f)(x),
\]

that is \( N_T^* = M_T \). We extend this to complete characterization:

**Theorem 2.1** For \( M \in L(C^\infty(\Omega), C^\infty(\Omega')) \) the following are equivalent:
1. There is \( T \in \mathcal{E}'(V(\Omega', \Omega)) \) such that \( M = N_T^* \).
2. There is \( T \in \mathcal{E}'(V(\Omega', \Omega)) \) such that \( M = M_T \).
3. \( M \) admits all monomials as eigenvectors.

**Proof:** The equivalence of 1. and 2. is obvious. Likewise 2. \( \Rightarrow \) 3. is clear.
We show 3. \( \Rightarrow \) 1. We set \( N := M^* \in L(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega)) \) and have to show that \( N(S_1) \star S_2 = S_1 \star N(S_2) \) for all \( S_1, S_2 \in \mathcal{E}'(\Omega') \). Due to the density in \( C^\infty(\Omega') \) of the polynomials we have to show the equality only on the set of all monomials.

For \( f(x) = x^\alpha \) we obtain

\[
(N(S_1) \star S_2)f = N(S_1)_x(S_2)_y(xy)^\alpha = S_1(Mx^\alpha)S_2(x^\alpha) = m_\alpha S_1(x^\alpha)S_2(x^\alpha)
\]

where \( Mx^\alpha = m_\alpha x^\alpha \). For \( (S_1 \star N(S_2))f \) we obtain the same, which shows the result. \( \square \)

The operators described in Theorem 2.1 are called \textit{operators of Hadamard type}, or also \textit{multipliers} and they have been subject of many investigations where the equivalence of 2. and 3. was one of the main results.

**Definition 2** For open \( \Omega \in \mathbb{R}^d \) we set \( M(\Omega) := \{M \in L(C^\infty(\Omega)) : M \text{ Hadamard type} \} \).

We state two basic facts:

1. \( M(\Omega) \) is a closed subalgebra of \( L_b(C^\infty(\Omega)) \).
2. \( T \mapsto M_T \) an algebra isomorphism \( \mathcal{E}'(V(\Omega)), \mathcal{E}'(\Omega)) \to M(\Omega) \).

The first follows from \( Mx^\alpha \subset \text{span}\{x^\alpha\} \) for all \( \alpha \), the second is a simple calculation.
3 Topologies

In this section we will study the Problem: Which topology does the algebraic isomorphism \( T \mapsto N_T \) from \( \mathcal{E}'(V(\Omega)) \) to \( N_b(\Omega', \Omega) \) induce on \( \mathcal{E}'(V(\Omega)) \)? Here \( \ast_b \) denotes the topology of uniform convergence on bounded sets.

The solution will be given in several steps. First we need some notation:

For \( \Omega' \subset \subset \Omega \) open we set

\[
U = U_{\Omega'} = V(\overline{\Omega'}, \Omega) = \{ \eta \in \mathbb{R}^d : \eta \Omega' \subset \subset \Omega \}
\]

and

\[
NC_b(\Omega', \Omega) = \{ N_T : T \in \mathcal{E}'(U) \} \cap L_b(C'(\overline{\Omega'}), \mathcal{E}'(\Omega)).
\]

It is easily seen that \( U_{\Omega'} \) is an open neighborhood of \( V(\Omega) \).

**Proposition 3.1** \( \mathcal{E}'(U) \cong N_b(\Omega', \Omega) = NC_b(\Omega', \Omega) \) are topologically isomorphic. The first isomorphism is given by \( T \mapsto N_T \).

**Proof:** From Proposition 1.1 and its proof it is clear that \( T \mapsto N_T \) sends every bounded set of \( \mathcal{E}'(U) \) into a bounded set of \( L_b(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega)) \). Since \( \mathcal{E}'(U) \) is bornological the map is continuous. Clearly the identical map \( N_b(\Omega', \Omega) \rightarrow NC_b(\Omega', \Omega) \) is continuous.

Let \( B \subset L_b(C'(\overline{\Omega'}), \mathcal{E}'(\Omega)) \) be bounded, then \( \tilde{B} = \{ \varphi(\delta_\eta) : \eta \in \overline{\Omega'}, \varphi \in B \} \) is a bounded set in \( \mathcal{E}'(\Omega) \). Therefore there is a compact set \( K \subset \Omega \) such that \( \tilde{B} \) is bounded in \( \mathcal{E}'(K) \).

We fix now \( \eta \in \overline{\Omega'} \cap \mathbb{R}^d \) and obtain, by use of the proof of Lemma 1.3,

\[
\{ T : N_T \in B \} = \{ \delta_1/\eta \ast N_T(\delta_\eta) : N_T \in B \} \subset \{ \delta_1/\eta \ast S : S \in \tilde{B} \}.
\]

We set \( L = \bigcap_{\eta \in \overline{\Omega'}, \eta \neq 0} \frac{1}{\eta} K = V(\overline{\Omega'}, K) \) (see Lemma 1.4). Then \( L \) is a compact subset of \( U \) and \( \{ T : N_T \in B \} \) is a bounded subset of \( \mathcal{E}'(L) \), since it is a bounded subset of \( \mathcal{E}'(\mathbb{R}^d) \) and the support of all of its elements is contained in \( L \). So it is a bounded subset of \( \mathcal{E}'(U) \).

We have shown that the map \( \mathcal{N} : \mathcal{E}'(U) \rightarrow NC_b(\Omega', \Omega) \) is continuous and surjective. Moreover \( \mathcal{N}^{-1}(B) \) is bounded for every bounded \( B \subset L_b(C'(\overline{\Omega'}), \mathcal{E}'(\Omega)) \) By Baernstein’s Lemma (see [3]) \( T \mapsto N_T \) is a topological imbedding of \( \mathcal{E}'(U) \) into \( L_b(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega)) \), which proves the result. \( \square \)

Let now \( \omega_1 \subset \subset \omega_2 \subset \subset \ldots \) be an exhaustion of \( \Omega \). We put \( U_n := U_{\omega_n} \). Then \( U_1 \supset U_2 \supset \ldots \) is a decreasing sequence of open neighborhoods of \( V(\Omega) \) with \( \bigcap_n U_n = V(\Omega) \).

\[ \text{Page 5} \]
Definition 3 $E_t'(V(\Omega)) := \lim_{n} \text{proj}_n E'(U_n)$.

Clearly the $t$-topology does not depend on the choice of the exhaustion. $E'(U_n)$ (DF)-space hence $E_t'(V(\Omega))$ is (PDF)-space.

We set $N_b(\Omega) := N_b(\Omega, \Omega)$ and $NC_b(\Omega) = NC_b(\Omega, \Omega)$.

Theorem 3.2 $E_t'(V(\Omega)) \cong N_b(\Omega) = NC_b(\Omega)$ topologically, the first isomorphism is established by $T \mapsto M_T$.

Proof: This follows immediately from Proposition 3.1 by forming projective limits. \(\square\)

4 Examples

$\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < y < 2\} \Rightarrow V(\Omega) = \{(x, 1) : x \in \mathbb{R}\}$ with $r_n \searrow 1 : \omega_n = \{(x, y) \in \mathbb{R}^2 : |x| < n, r_n < y < 2/r_n\} \Rightarrow U_n = \{(x, y) : 1/r_n < y < r_n\}$

$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x, 1 < y < 2\} \Rightarrow V(\Omega) = \{(x, 1) : x > 0\}$ with $r_n \searrow 1 : \omega_n = \{(x, y) \in \mathbb{R}^2 : 1/n < |x| < n, r_n < y < 2/r_n\} \Rightarrow U_n = \{(x, y) : 0 < x, 1/r_n < y < r_n\}$

In both cases $V(\Omega)$ is closed in some $U_n$. If this is the case,

- then it is closed in all $U_m$ for $m \geq n$
- $E_t'(V(\Omega)) \cong E'(V(\Omega))$ equipped with the canonical (DF)-topology inherited from $E'(U_n)$.
- $V(\Omega)$ is locally compact, $\sigma$-compact and $E_t'(V(\Omega)) = \text{ind}_{K \subset V(\Omega)} E'(K)$.

We have shown:

Theorem 4.1 If $V(\Omega)$ is closed in some $U_n$, then $V(\Omega)$ is locally compact, $\sigma$-compact and $M(\Omega) \cong E'(V(\Omega))$ with its standard (DF)-topology.

Example 4.2 If $\overline{V(\Omega)} \setminus V(\Omega)$ is finite then the theorem applies.

Lemma 4.3 $y \in \overline{V(\Omega)} \setminus V(\Omega)$ then there is $j$ with $y_j = 0$.  

6
Proof: If \( y \in V(\Omega) \) and \( \min_j |y_j| \geq 2\varepsilon > 0 \) then \( d(yx, \partial\Omega) \geq \varepsilon d(x, \partial\Omega) \) for all \( x \in \Omega \), since that holds for an approximating sequence in \( V(\Omega) \). Hence \( yx \in \Omega \). Therefore for \( y \in V(\Omega) \setminus V(\Omega) \) there is \( j \) such that \( y_j = 0 \). \( \square \)

**Theorem 4.4** If \( d = 1 \) then \( V(\Omega) \) is locally compact, \( \sigma \)-compact and \( M(\Omega) \cong \mathcal{E}'(V(\Omega)) \) with its standard \((DF)\)-topology.

Proof: By Lemma 4.3 we have \( V(\Omega) \setminus V(\Omega) \subset \{0\} \). By Theorem 4.1 the result follows. \( \square \)

5 Relation to classical convolution

We set \( Q_+ = \{ x \in \mathbb{R}^d : x_j > 0 \text{ for } j = 1, \ldots, d \} \) and restrict our attention to open sets \( \Omega \subset Q_+ \). The map \( \text{Log}(x) = (\log x_1, \ldots, \log x_d) \) defines a diffeomorphism from \( Q_+ \) onto \( \mathbb{R}^d \) whose inverse is \( \text{Exp}(x) = (\exp x_1, \ldots, \exp x_d) \).

The map \( C_{\text{Log}} : \varphi \to \varphi \circ \text{Log} \) is an isomorphism from \( C^\infty(\mathbb{R}^d) \) onto \( C^\infty(Q_+) \) whose inverse is \( C_{\text{Exp}} \), defined analogously. Then the transposed maps \( C_{\text{Log}}^* : \mathcal{E}'(Q_+) \to \mathcal{E}'(\mathbb{R}^d) \) and \( C_{\text{Exp}}^* : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(Q_+) \) are isomorphisms.

The crucial observation relating the two kinds of convolution is:

**Lemma 5.1** \( T \ast S = C_{\text{Exp}}^* (C_{\text{Log}}^* T \ast (C_{\text{Log}}^* S)) \) for all \( T, S \in \mathcal{E}'(Q_+) \).

A subset \( X \subset Q_+ \) will be called multiplicatively convex (mconv) if with \( x, y \in X \) also \( x^t y^{1-t} \in X \) for all \( 0 < t < 1 \). For a subset \( X \subset Q_+ \) we define the multiplicatively convex hull \( \text{mconv}X \) as the smallest mconv set, which contains \( X \). We have \( \text{mconv}X = \text{Exp}(\text{conv}(\text{Log}X)) \) where \( \text{conv} \) denotes the convex hull. Explicitly we have

\[
\text{mconv}X = \left\{ \prod_j x_j^{\lambda_j} : x_j \in X, 0 < \lambda_j, \sum_j \lambda_j = 1 \right\}
\]

where the sums and products are finite and the powers are taken coordinatewise.

We obtain the multiplicative analogue of the Theorem of Lions.

**Theorem 5.2** \( \text{mconv}(\text{supp}(T \ast S)) = \text{mconv}(\text{supp} T) \cdot \text{mconv}(\text{supp} S) \) for all \( T, S \in \mathcal{E}'(Q_+) \).

For the following we might exploit Theorem 5.2 but we prefer to use the direct transfer of properties of classical convolution operators.
First we consider the case of \( \text{supp} \ T = \{ 1 \} \). The operators \( M_T \), resp. \( N_T \), are then called \textit{Euler operators} . In this case \( T \) has the form \( T = \sum_{|\alpha| \leq p} c_\alpha \theta_\alpha^{(T)} \). Since \( M_{\delta_1^{(T)}} = (-1)^{|\alpha|} x^\alpha \partial^\alpha \) the Euler operator \( M_T \) has the form \( M_T = \sum_{|\alpha| \leq p} b_\alpha x^\alpha \partial^\alpha \).

With \( \theta_j = x_j \partial_j \) and different coefficients it can also be written as \( M_T = \sum_{|\alpha| \leq p} c_\alpha \theta_\alpha^{(T)} \).

Let now \( \Omega \subset Q_+ \) be open. If we set, keeping the notation, \( P(z) = \sum_{|\alpha| \leq p} c_\alpha z^\alpha \), then \( P(\partial) C^\infty(\log \Omega) = C^\infty(\log \Omega) \) if, and only if, \( \log \Omega \) is \( \partial \)-convex. This means that for every compact \( K \subset \log \Omega \) there is a compact \( L \subset \log \Omega \) with the following property: if \( S \in \mathcal{E}'(\log \Omega) \) and \( P(-\partial) S \in \mathcal{E}'(K) \) then \( S \in \mathcal{E}'(L) \).

Now, it is obvious that this condition simply carries over to \( P(\partial) \) and \( \Omega \), and we obtain

**Theorem 5.3** \( P(\partial) C^\infty(\Omega) = C^\infty(\Omega) \) if, and only if, \( \Omega \) is \( P(\partial) \)-convex, that is, for every compact \( K \subset \Omega \) there is a compact \( L \subset \Omega \) with the following property: if \( S \in \mathcal{E}'(\Omega) \) and \( P(\partial^*) S \in \mathcal{E}'(K) \) then \( S \in \mathcal{E}'(L) \).

Here \( \theta_j^* \) denotes the transpose of \( \theta_j \), that is, \( \theta_j^* T = -\partial_j (x_j T) \). By Theorem 5.2 this implies:

**Corollary 5.4** If \( \Omega \) is mconvex then all Euler operators are surjective in \( C^\infty(\Omega) \).

Staying with mconvex \( \Omega \subset Q_+ \) we study a more general situation. We fix \( T \in \mathcal{E}'(Q_+) \) and set \( \Omega' = V(\text{supp} \ T, \Omega) = V(\text{mconv supp} \ T, \Omega) \). It is easy to see that \( \Omega' \) is open and mconvex. Clearly \( \text{supp} \ T \subset V(\Omega', \Omega) \), that is, \( N_T \in L(\mathcal{E}'(\Omega'), \mathcal{E}'(\Omega)) \). Let \( K \subset \Omega \) be compact and mconvex. Assume moreover that \( S \in \mathcal{E}'(Q_+) \) and \( \text{supp} \ (T \ast S) \subset K \), then also \( \text{mconv supp} \ T \ast \text{mconv supp} \ S = \text{mconv supp} \ (T \ast S) \subset K \). This implies that \( \text{supp} \ S \subset L := V(\text{mconv supp} \ T, K) \). Since \( \text{supp} \ T \subset \mathbb{R}^d \) we have \( L \subset \frac{1}{\eta} K \) for any \( \eta \in \text{supp} \ T \). Therefore \( L \) is compact. We have shown:

**Lemma 5.5** If \( \Omega \subset Q_+ \) is open and mconvex, \( T \in \mathcal{E}'(Q_+) \) and \( \Omega' = V(\text{supp} \ T, \Omega) \), then for every mconvex compact set \( K \subset \Omega \) there is a compact set \( L \subset \Omega' \) such that the following holds: if \( S \in \mathcal{E}'(Q_+) \) and \( \text{supp} \ (T \ast S) \subset K \) the \( \text{supp} \ S \subset L \).

To get solvability conditions for our multiplicative convolution equations we need the equivalent of an elementary solution. We set for \( T \in \mathcal{E}'(Q_+) \)

\[ \tilde{T}(z) := T_x(x^{-iz}), \quad z \in \mathbb{C}^d \]

and remark that \( \tilde{T} = \mathcal{F}^*_{\log T}, \) \( \mathcal{F} \) denoting the Fourier-transform.

**Definition 4** An entire function \( J \) is said to be slowly decreasing if there exist positive numbers \( a, b, c \) such that for each point \( x \in \mathbb{R}^d \) we can find a point \( y \in \mathbb{R}^d \)
with
(1) $|x - y| \leq a \log(1 + |x|)$,
(2) $|J(y)| \geq b/(1 + |y|^c)$.

This notation is due to Ehrenpreis and we obtain the following multiplicative analogue to the Theorem of Ehrenpreis on completely inversible operators:

**Theorem 5.6** For $T \in \mathcal{E}'(Q_+)$ the following are equivalent
1. $M_T$ is surjective in $C^\infty(Q_+)$.
2. There exists $E \in \mathcal{D}'(Q_+)$ such that $T \ast E = \delta_1$.
3. $\tilde{T}$ is slowly decreasing.

**Proof:** 1. is equivalent to the surjectivity of $f \mapsto (C^*_\log T) \ast f$ in $C^\infty(\mathbb{R}^d)$. By the Theorem of Ehrenpreis this is equivalent to $C^*_\log \tilde{T}$ being slowly decreasing and this is equivalent to 3.

3. means that $C^*_\log \tilde{T}$ is slowly decreasing and this is, by the Theorem of Ehrenpreis, equivalent to the existence of a distribution $W \in \mathcal{D}'(\mathbb{R}^d)$ with $(C^*_\log T) \ast W = \delta$. Given 3. we set $E = C^*_\log W$ and obtain

$$T \ast E = C^*_\log ((C^*_\log T) \ast (C^*_\log E)) = C^*_\log ((C^*_\log T) \ast W) = C^*_\log \delta = \delta_1.$$  

Given 2. we set $W = C^*_\log E$ and proceed like before. \hfill \Box

Putting things together we obtain

**Theorem 5.7** If $\Omega \subset Q_+$ is m-convex and open, $T \in \mathcal{E}'(Q_+)$, $\tilde{T}$ slowly decreasing and $\Omega' = V(\text{supp} T, \Omega)$, then $M_T : C^\infty(\Omega') \to C^\infty(\Omega)$ is surjective.

**Proof:** First we remark that, by Theorem 5.6 $M_T \mathcal{E}(\Omega')$ is dense in $\mathcal{E}(\Omega)$. Due to the Surjectivity Criterion we have to show that for any bounded set $B \subset \mathcal{E}'(\Omega)$ the set $N_T^{-1}B$ is bounded in $\mathcal{E}'(\Omega)$. There is compact, m-convex $K \subset \Omega$ such that $B \subset \mathcal{E}'(K)$ and bounded there. By Lemma 5.5 there is compact $L \subset \Omega'$ such that $N_T^{-1} \mathcal{E}'(K) \subset \mathcal{E}'(L)$. Then $\ast$-convolution with $E$ (notation of Theorem 5.6) yields a continuous linear map $R(N_T) \cap \mathcal{E}'(K) \to \mathcal{E}'(L)$ inverting $N_T$. So $N_T^{-1}B = E \ast (R(N_T) \cap B)$ is bounded in $\mathcal{E}'(L) \subset \mathcal{E}'(\Omega')$. \hfill \Box

An interesting fact is the following: for fixed $z \in \mathbb{C}^d$ we have $M_T(\xi \to \xi^z)[x] = T_y(xy)^z = T_z(\xi^z) x^z$. So all functions $x^z$ are eigenfunctions of the operator $M_T$ with eigenvalue $\tilde{T}(iz)$. In particular we have $m_\alpha = \tilde{T}(i\alpha)$ where $m_\alpha, \alpha \in \mathbb{N}_0^d$ is the multiplier sequence for the Hadamard operator $M_T$.

This all can be done for each of the $2^d$ quadrants, but not for the whole euclidean space, as the following simple example shows. Take the operator $x \frac{\partial}{\partial z}$. It is easily seen to be surjective on $C^\infty(Q_{+/-})$, but it is not surjective on $C^\infty(\mathbb{R})$ since $xf'(x)$ is $0$ in $0$ for any $f$. 

9
6 Laurent representation theorem

For \( T \in \mathcal{E}'(\mathbb{R}^d) \) and \( z \in \mathbb{C}^d, \, z_j \neq 0 \) for all \( j \) we set \( \mathcal{C}(z) = \prod_{j=1}^{d} \frac{1}{z_j} \). For any subset \( B \subset \mathbb{R}^d \) we define

\[
\mathcal{W}(B) = \{ z \in \mathbb{C}^d : \xi_j \neq z_j \text{ for all } \xi \in B \text{ and } j = 1, \ldots, d \}.
\]

For \( z \notin \mathcal{W}(\text{supp } T) \) we define \( \mathcal{C}_T(z) = T_\xi(\mathcal{C}(\xi - z)) = (T \ast \mathcal{C})(z) \).

Since \( \mathcal{C} \) defines a distribution on \( \mathbb{C}^d \) also \( \mathcal{C}_T = T \ast \mathcal{C} \) is a distribution on \( \mathbb{C}^d \) which is a holomorphic function on \( \mathcal{W}(\text{supp } T) \).

On \( \mathcal{D}'(\mathbb{C}^d) \) we set \( \bar{\partial}^1 := \pi^{-d} \frac{\partial}{\partial \bar{z}_1} \cdots \frac{\partial}{\partial \bar{z}_d} \), then \( \mathcal{C} \) is a fundamental solution for \( \bar{\partial}^1 \) and therefore \( \bar{\partial}^1 \mathcal{C}_T = T \).

If \( \mathcal{W}(\text{supp } T) \subset \{ \xi \in \mathbb{R}^d : |\xi|_\infty \leq R \} \) then \( (\mathbb{C} \setminus [-R, +R]^d) \subset \mathcal{W}(\text{supp } T) \) and \( \mathcal{C}_T \) extends to a holomorphic function on \( (\hat{\mathbb{C}} \setminus [-R, +R]^d) \), \( \hat{\mathbb{C}} \) denoting the Riemann sphere. \( \mathcal{C}_T(z) = 0 \) outside \( \mathbb{C}^d \). \( \mathcal{C}_T \) is the unique solution of \( \bar{\partial}^1 u = T \) with this property.

For \( \min_j |z_j| > R \) the function \( \mathcal{C}_T(z) \) is defined and holomorphic and it has the expansion

\[
\mathcal{C}_T(z) = \frac{1}{z_1 \cdots z_d} \sum_{\alpha \in \mathbb{N}_0^d} T_\xi(\xi^\alpha) \frac{1}{z^\alpha} = \sum_{\alpha \in \mathbb{N}_0^d} m_\alpha \frac{1}{z^{\alpha + 1}}.
\]

We have proved the following

**Proposition 6.1** Let \( B \subset \mathbb{R}^d \) be compact and closed with respect to multiplication. The algebra \( (\mathcal{E}'(B), \ast) \) is algebra-isomorphic to the algebra of all distributions on \( \mathbb{C}^d \) which are holomorphic on \( (\mathbb{C} \setminus [-R, +R]^d) \) for some \( R > 0 \), regular with value 0 in all infinite points of \( \mathbb{C}^d \) and zero solutions for \( \bar{\partial}^1 \) outside \( B \), equipped with Hadamard multiplication of the coefficients of their Laurent expansion around \( (\infty, \ldots, \infty) \).

We will now give a description in terms properties of the functions on \( (\mathbb{C} \setminus \mathbb{R})^d \). All of the following is well known. We indicate the proofs for the convenience of the reader.

**Theorem 6.2** 1. If \( f = \mathcal{C}_T \) on \( (\mathbb{C} \setminus \mathbb{R})^d \) for \( T \in \mathcal{E}'(\mathbb{R}^d) \) the there is \( p \in \mathbb{N}_0 \) such that

\[
\sup_{z \in (\mathbb{C} \setminus \mathbb{R})^d} |f(z)| |y_1 \cdots y_d|^p < +\infty
\]

where \( y = x + iy \).
2. If \( f \in H((\mathbb{C} \setminus [-R, +R])^d) \) for some \( R > 0 \) is regular with value 0 in all infinite points of \( \mathbb{C}^d \) and fulfills (1), then there is \( T \in \mathcal{E}'(\mathbb{R}^d) \) such that \( f = \mathcal{C}_T \).

3. If \( f = \mathcal{C}_T \) the for \( x \in \mathbb{R}^d \) we have:

\[
\sigma(f) \text{ is compact and } \sigma(f) \subset V(\Omega).
\]

Here \( \sigma(f) \) denotes the hyperfunction support of \( f \).

We recall the definition:

Definition 5 \( x \not\in \sigma(f) \) if in a complex neighborhood of \( x \) we have \( f = f_1 + \cdots + f_d \) where \( f_j \) extends in the \( j \)-th variable holomorphically across \( \mathbb{R} \).
7 Hadamard representation theorem

In a next step we want to change the equivalence into one with Hadamard multiplication of power series. For that we observe that for \( z \in \mathbb{C}^d \cap \mathcal{W}(\text{supp} \, T) \) we have

\[
C_T(z) = \frac{1}{z_1 \cdots z_d} \mathcal{E}_T \left( \prod_j \frac{1}{1 - \xi_j \cdot 1/z_j} \right)
\]

and therefore for \( z \in \mathbb{C}^d \) such that \( 1/z \in \mathcal{W}(\text{supp} \, T) \)

\[
T_\xi \left( \prod_j \frac{1}{1 - \xi_j z_j} \right) = \frac{1}{z_1 \cdots z_d} C_T(1/z).
\]

We set

\[
C_T(z) = T_\xi \left( \prod_j \frac{1}{1 - \xi_j z_j} \right)
\]

on

\[ W(\text{supp} \, T) := \{ z \in \mathbb{C}^d : x_j z_j \neq 1 \text{ for all } x \in \text{supp} \, T \text{ and } j = 1, \ldots, d \}. \]

We have \((\mathbb{C} \setminus \mathbb{R} \cup [-R,+R])^d \subset W(\text{supp} \, T)\) for some \( R > 0 \) and for \( |z|_\infty < R \)

\[
C_T(z) = \sum_\alpha m_\alpha x^\alpha.
\]

This establishes an algebra isomorphism of \( M(\Omega) \cong \mathcal{E}'(V(\Omega)) \) to an algebra of holomorphic functions with Hadamard multiplication, whose properties we get from the previous case by the formula

\[
C_T(z) = \frac{1}{z_1 \cdots z_d} C_T(1/z).
\]

References

[1] P. Domański, M. Langenbruch, Representation of multipliers on spaces of real analytic functions, Analysis 32 (2012), 137–162.

[2] P. Domański, M. Langenbruch, Algebra of multipliers on the space of real analytic functions of one variable, Studia Math. 212 (2012), 155–171.

[3] P. Domański, M. Langenbruch, Hadamard multipliers on spaces of real analytic functions, Adv. Math. 240 (2013), 575–612.
[4] P. Domaniński, M. Langenbruch, D. Vogt, Hadamard type operators on spaces of real analytic functions in several variables, to appear in *J. Funct. Anal.* (2015).

[5] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.

[6] D. Vogt, Operators of Hadamard type on spaces of smooth functions, *Math. Nachr.* **288** (2015), 353–361.

Bergische Universität Wuppertal, FB Math.-Nat., Gauß-Str. 20, D-42119 Wuppertal, Germany
e-mail: dvogt@math.uni-wuppertal.de