A path-integral representation for the kernel of the evolution operator of general Hamiltonian systems is reviewed. We study the models with bosonic and fermionic degrees of freedom. A general scheme for introducing boundary conditions in the path-integral is given. We calculate the path-integral for the systems with quadratic first class constraints and present an explicit formula for the heat kernel (HK) in this case. These results may be applied to many quantum systems which can be reduced to the Hamiltonian systems with quadratic constraints (confined quarks, Calogero type models, string and p-brain theories, etc.).

1 Boundary conditions for path-integral.
Let us consider a Hamiltonian system in the phase space $M$ with coordinates $Z_A = \{q_a, p_a, \psi_\alpha, \bar{\psi}^\alpha\}$ where $q_a$ and $p_a$ are bosonic coordinates and momenta ($a = 1, \ldots, N$) while $\psi_\alpha$ and $\bar{\psi}^\alpha$ ($\alpha = 1, \ldots, K$) are fermionic degrees of freedom. The action for this system can be written in the form

$$S = \int_0^T dt \left[ \frac{1}{2} Z_A \Omega^{AB} \dot{Z}_B - \mathcal{H}(Z) \right],$$

where $t \in [0, T]$ is an evolution parameter and $\mathcal{H}$ is the Hamiltonian. The variational principle, $\delta S = 0$, gives the equations of motion, $\dot{Z}_A = \Omega^{AB} \partial_B \mathcal{H}$. ($\partial_B \equiv \partial_{\bar{\psi}}^{\alpha}$). Here $\Omega^{AB} = -(-)^{A+B} \Omega_{BA} = \{Z_A, Z_B\}$ is inverse to the matrix

$$\Omega^{AB} = (\partial^A P^B - (-)^{A+B} + AB \partial^B P^A) \Omega^{BA} = -(-)^{A+B} + AB \Omega^{BA}.$$

Note that the symplectic 2-form $\omega$ on the superspace $M$ is defined by a slightly different supermatrix: $\omega = (\partial^A P^B - (-)^{AB} + B \partial^B P^A) dZ_A \wedge dZ_B$. Below we concentrate on the case of a constant supermatrix $\Omega^{AB}$. Then we have $P^A = \frac{1}{2} Z_B \Omega^{BA}$ and the action (1) can be represented in the form

$$S = \int_0^T dt \left[ \frac{1}{2} Z_A \Omega^{AB} \dot{Z}_B - \mathcal{H}(Z) \right].$$

We will investigate the path-integral representation for the HK of the
evolution operator $U$:

$$< Z^f | U(T,0) | Z^i > = \int d\{Z_A\} \exp\{i(S+B)\}$$  \hspace{1cm} (3)

where $d\{Z_A\}$ is a measure over the space of trajectories in $M$. The boundary terms $B$ depending on the initial and final points, $Z^i, Z^f$, are determined by relevant boundary conditions.

**Proposition 1.** Boundary conditions can be specified by the boundary terms in the path-integral (3):

$$B = \frac{1}{2} \left[ Z^i \Omega P_i Z(0) - Z^f \Omega P_f Z(T) \right],$$  \hspace{1cm} (4)

where $Z^f,i_A$ are fixed supervectors in $M$ and matrices $P_{f,i}$ are constant projectors ($P^2 = P$) in $M$ of the rank $(N + K)$. These projectors should satisfy the conditions $P^T \Omega = \Omega (1 - P)$, where $T$ denotes a super-transposition. Substituting the boundary terms (4) into (3) is equivalent to fixing the initial and final states of the system by the conditions:

$$(1 - P_i) Z(0) = Z^i, (1 - P_f) Z(T) = Z^f.$$

**Hint for proof.** Consider the equation $\delta (S + B) = 0$ at the points $t = 0, T$.

**Note.** Different choices of the projectors $P_{f,i}$ lead to different choices of the boundary states in (3). In this way one can write down the HK (3) in coordinate, momentum or holomorphic representations in unified form.

2 Path-integral (3) for quadratic Hamiltonians.

Now, consider systems with general quadratic Hamiltonians

$$\mathcal{H}(Z) = \frac{1}{2} Z_A \left[ \Omega^{AB} A^0_B \right] Z_D,$$  \hspace{1cm} (5)

where the supermatrix $A$ depends on the evolution parameter $t$ and is independent of $Z_A$. The hermiticity of (5) requires that $A^{T} \Omega + \Omega A = 0$, i.e. $A \in \text{osp}(2K|2N)$. With the Hamiltonian (5) the action (2) acquires the form

$$S = \frac{1}{2} \int_0^T dt Z \Omega [\partial_t - A] Z.$$

**Proposition 2.** In the case of the Hamiltonian system with the action (6) and boundary terms (4), the path-integral (3) gives the following representation for the evolution operator $U$:

$$< Z^f | U(T,0) | Z^i > \simeq \text{Ber}^{1/2} \left( \Omega \frac{1}{V^+} \right) \exp \left\{ -i S_{eff}(Z^f, Z^i) \right\},$$  \hspace{1cm} (7)

2
\[ S_{\text{eff}} = \frac{1}{2} \left[ Z^i \Omega \frac{1}{V^-} V^+ Z^i - 2Z^i \Omega \frac{1}{V^+} Z^f + Z^f \Omega V^- \frac{1}{V^-} Z^f \right] \]

where \( V = T \exp \left\{ \int_0^T dt A(t) \right\} \), \( V^{\alpha \beta} = P^\alpha_i V P^\beta_i \) and \( P^-_{f,i} \equiv P_{f,i}, \ P^+_{f,i} \equiv 1 - P_{f,i} \).

**Proof.** Direct computation.

3 Heat kernel for the evolution operator of “Discrete Strings”.

Finally, consider the system with the action (6) and interpret the Hamiltonian (5) as a linear combination of the first class constraints. In this case, the elements of the supermatrix \( A \) play the role of Lagrange multipliers. Note that the fermionic string models can be described by the action of the same type. For this reason we call such systems as “Discrete Strings” (for details see [4], [5], [6], [7]). The Action with boundary terms, \( S' = S + B \) (where \( S \) and \( B \) are defined in (6) and (4)), is invariant under the gauge transformations

\[ Z \rightarrow F(T) Z , \ A \equiv A_M e^M \rightarrow FAF^{-1} + \dot{F} F^{-1} \iff V \rightarrow F(T) VF(0) , \]

where \( F^T \Omega F = \Omega \) (i.e., the gauge group \( G \) is a subgroup of \( Osp(2|2N) \)). Matrices \( e^M \) form a basis in the algebra of \( G \). The boundary terms in \( S' \) fix the gauge group parameters at the final and initial points up to residual gauge transformations: \( [F(T,0), P_{f,i}] = 0 \). Let us denote the corresponding groups by \( G_{f,i} \). Using the freedom of rotating the boundary states \( Z^{f,i} \rightarrow F(T,0) Z^{f,i} \), one can show that the evolution operator \( \tilde{U} \) depends only on the coordinates of the moduli space \( \{V\}/G_{f} \otimes G_i \equiv \overline{V} \), which is an analogue of the Teichmüller space used in string theories.

The standard procedure [1], [6], [8] for quantizing constrained Hamiltonian systems via path-integrals allows to obtain (after fixing a gauge and representing Faddeev-Popov determinant as an integral over ghosts \( \{B, C\} \)) the expression for the propagator of “Discrete Strings”:

\[ K_{f,i} = \int d\{\overline{\nabla}\} < Z^{f} | U(T,0) | Z^{i} > \cdot < B^{f}, C^{f} | \tilde{U}(T,0) | B^{i}, C^{i} > . \quad (8) \]

Here \( d\{\overline{\nabla}\} \) is a measure over the moduli space and the HK for the ghost evolution operator \( \tilde{U} \) is represented by the path-integral:

\[ < B^{f}, C^{f} | \tilde{U}(T,0) | B^{i}, C^{i} > = \int d\{B\} d\{C\} exp\{i(S_{gh} + B_{gh})\} . \quad (9) \]

The ghost action \( S_{gh} \) and boundary terms \( B_{gh} \) have the form

\[ S_{gh} = \int_0^T dt B^N (\delta^M_N \partial_t - \tilde{\Delta}^M_M) C_M , \quad B_{gh} = B^{i} \tilde{P}_i C(0) - B^{f} \tilde{P}_f C(T) , \]

3
where ghosts $B^N$, $C_M$ realize the adjoint representations of gauge group $G$ ($\tilde{A}$ is the adjoint analogue of $A$) and their parity is opposite to the parity of the Lagrange multipliers $A_N$. Constant projectors $\tilde{P}_{f,i}$ are defined by the relations $(\tilde{P}_{f,i})^M_N e^N = P^+_{f,i} e^M P^+_{f,i} + P^-_{f,i} e^M P^-_{f,i}$, which impose some restrictions on the choice of the boundary projectors $\tilde{P}_{f,i}$. As in Proposition 1, the boundary terms $B_{gh}$ fix the boundary states for ghosts: $B(T,0)\tilde{P}_{f,i} = B^f_i$, $(1 - \tilde{P}_{f,i}) C(T,0) = C^f_i$.

Proposition 3. The path-integral (10) for the ghost HK is equal to

$$<B^f, C^f | \tilde{U}(T,0)| B^i, C^i > \approx \tilde{B}\text{er}(\tilde{V}^{+-}) \exp\{i\tilde{S}_{eff}\},$$

$$\tilde{S}_{eff} = \left[ (B^i - B^f \tilde{V}^{+-}) \frac{1}{\tilde{V}^{+-}} - (C^f - \tilde{V}^{++} C^i) - B^f \tilde{V}^{+-} C^i \right],$$

where $\tilde{V} = T \exp\{\int_0^T dt \tilde{A}(t)\}$, $\tilde{V}^{\alpha\beta} = \tilde{P}_{f}^\alpha \tilde{V} \tilde{P}_{i}^\beta$ and $\tilde{P}_{f,i}^\pm \equiv \tilde{P}_{f,i}^- \equiv 1 - \tilde{P}_{f,i}$.

Proof. Direct computation.

Note. We have to stress the following: if the operator $\tilde{V}^{+-}$ has zero eigenvalues, the right-hand side of (10) has to be modified. Namely, we have to remove the zero eigenvalues from the superdeterminant $\tilde{B}\text{er}$ and also multiply the final expression by the product of delta-functions of corresponding zero-modes of the ghost variables $B, C$ (to remove infinities from $\tilde{S}_{eff}$).

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