On a generalization of $P_3(n)$

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We present a formula for $P_3(n)$ the number of partitions of a positive number $n$ into 3 $s$-gonal numbers, by using representations of posets over $\mathbb{N}$.

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1. Introduction

In [10] Lehmer denoted $P_k(n)$ the number of partitions of a natural number $n$ into $k$ integral squares $\geq 0$, and solved almost completely the equation $P_k(n) = 1$. Lehmer claimed that the general problem of finding a formula for $P_k(n)$ was a problem of great complexity. The case $k = 3$ was studied by Grosswald, A. Calloway, and J. Calloway in [4], and Grosswald solved (essentially) the problem, giving the number of partitions of an arbitrary integer $n$ into $k$ squares (taking into account that, he didn’t distinguish between partitions that contains zeros and those that do not) [5].

The main goal of this paper is to give a formula for $P_3(n)$, where the $s$-th polygonal number of order or rank $r$, $p^s_r$, is given by the formula (often, 0 is included as a polygonal number [3,7]),

$$p^s_r = \frac{1}{2}[(s-2)r^2 - (s-4)r].$$

We must note that, for the particular case $s = 3$ Hirschhorn and Sellers proved the identity $P_3^3(27n + 12) = 3P_3^3(3n + 1)$, via generating functions manipulations and some combinatorial arguments [9]. Instead of generating functions, we shall give the formula for $P_3^3(n)$ (note that $P_3(n) = P_3^3(n)$) by using representations of posets over the set of natural numbers $\mathbb{N}$, which have been used in [11] to give criteria for natural numbers which are expressible as sums of three polygonal numbers of positive rank.

2. Preliminaries

2.1. Posets

An ordered set (or partially ordered set or poset) is an ordered pair of the form $(\mathcal{P}, \leq)$ of a set $\mathcal{P}$ and a binary relation $\leq$ contained in $\mathcal{P} \times \mathcal{P}$, called the order (or the partial order) on $\mathcal{P}$, such that $\leq$ is reflexive, antisymmetric and transitive [1]. The elements of $\mathcal{P}$ are called the points of the ordered set.
Let $\mathcal{P}$ be an ordered set and let $x, y \in \mathcal{P}$ we say $x$ is covered by $y$ if $x < y$ and $x \leq z < y$ implies $z = x$.

Let $\mathcal{P}$ be a finite ordered set. We can represent $\mathcal{P}$ by a configuration of circles (representing the elements of $\mathcal{P}$) and interconnecting lines (indicating the covering relation). The construction goes as follows.

1. To each point $x \in \mathcal{P}$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$.
2. For each covering pair $x < y$ in $\mathcal{P}$, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.
3. Carry out (1) and (2) in such a way that
   a) if $x < y$, then $p(x)$ is lower than $p(y)$,
   b) the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

A configuration satisfying (1)-(3) is called a Hasse diagram or diagram of $\mathcal{P}$. In the other direction, a diagram may be used to define a finite ordered set; an example is given below, for the ordered set $\mathcal{P} = \{a, b, c, d, e, f\}$, in which $a < b < c < d < e$, and $f < c$.

![Fig. 1](image_url)

We have only defined diagrams for finite ordered sets. It is not possible to represent the whole of an infinite ordered set by a diagram, but if its structure is sufficiently regular it can often be suggested diagrammatically.

An ordered set $C$ is called a chain (or a totally ordered set or a linearly ordered set) if and only if for all $p, q \in C$ we have $p \leq q$ or $q \leq p$ (i.e., $p$ and $q$ are comparable).

Let $\mathcal{P}$ be a poset and $S \subset \mathcal{P}$. Then $a \in S$ is a maximal element of $S$ if $a \leq x$ and $x \in S$ imply $a = x$. We denote the set of maximal elements of $S$ by Max $S$. If $\mathcal{P}$ (with the order inherited from $\mathcal{P}$) has a top element, $\top$ (i.e., $s \leq \top$ for all $s \in S$), then Max $S = \{\top\}$; in this case $\top$ is called the greatest (or maximum) element of $S$, and we write $\top = \text{max } S$.

Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are (disjoint) ordered sets. The disjoint union $\mathcal{P}_1 + \mathcal{P}_2$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the ordered set formed by defining $x \leq y$ in $\mathcal{P}_1 + \mathcal{P}_2$ if and only if either
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$x, y \in \mathcal{P}_1$ and $x \leq y$ in $\mathcal{P}_1$ or $x, y \in \mathcal{P}_2$ and $x \leq y$ in $\mathcal{P}_2$. A diagram for $\mathcal{P}_1 + \mathcal{P}_2$ is formed by placing side by side diagrams for $\mathcal{P}_1$ and $\mathcal{P}_2$.

2.2. Partitions and Representations of posets over $\mathbb{N}$

As usual in this paper $\mathbb{N}$ denotes the set of natural numbers, while $\mathbb{N} \setminus \{0\}$ is the set of positive integers.

We denote $t_k = p_k^3 = \frac{k(k-1)}{2}$ the $k$-th triangular number $k \in \mathbb{Z}$, and $s_k = p_k^4 = k^2$ is the $k$-th square number.

A partition of a positive integer $n$ is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^r \lambda_i = n$. The $\lambda_i$ are called the parts of the partition [6]. A composition is a partition in which the order of the summands is considered.

Often the partition $\lambda_1, \lambda_2, \ldots, \lambda_r$ will be denoted by $\lambda$ and we sometimes write $\lambda = (1^j 2^i 3^f \ldots)$ where exactly $f_i$ of the $\lambda_j$ are equal to $i$. Note that $\sum_{i=1}^r f_i i = n$.

The partition function $p(n)$ is the number of partitions of $n$. Clearly $p(n) = 0$ when $n$ is negative and $p(0) = 1$, where the empty sequence forms the only partition of zero.

Let $(\mathbb{N}, \leq)$ be the set of natural numbers endowed with its natural order and $(\mathcal{P}, \leq')$ a poset. A representation of $\mathcal{P}$ over $\mathbb{N}$ [11] is a system of the form

$$\Lambda = (\Lambda_0 ; (n_x, \lambda_x) \mid x \in \mathcal{P}),$$

where $\Lambda_0 \subseteq \mathbb{N}, \Lambda_0 \neq \emptyset$, $n_x \in \mathbb{N}, \lambda_x$ is a partition with parts in the set $\Lambda_0$, and $|\lambda_x|$ is the size of the partition $\lambda_x$, in particular if $n_x = 0$ then we consider $\lambda_x = 0$. Further

$$x \leq' y \Rightarrow n_x \leq n_y, \quad |\lambda_x| \leq |\lambda_y|, \quad \max \{\lambda_x\} \leq \max \{\lambda_y\}.$$  \hfill (2)

2.3. The associated graph

A Graph is a pair $G = (V, E)$ of sets satisfying $E \subseteq V^2$, thus the elements of $E$ are 2-elements subsets of $V$, such that $V \cap E = \emptyset$. The elements of $V$ are the vertices of the graph $G$, the elements of $E$ are its edges. A graph with vertex set $V$ is said to be a graph on $V$. The vertex set of a graph is referred to as $V(G)$, its edge set as $E(G)$. We write $v \in G$ to a vertex $v \in V(G)$ and $e \in G$, for an edge $e \in E(G)$, an edge $\{x, y\}$ is usually written as $xy$ or $yx$.

A vertex $v$ is incident with an edge $e$; if $v \in e$, then $e$ is an edge at $v$. The two vertices incident with an edge are its endvertices or ends, and an edge joins its ends. A path is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\}, \quad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the $x_i$ are all distinct. The vertices $x_0$ and $x_k$ are linked by $P$ and are called its ends, the vertices $x_1, \ldots, x_{k-1}$ are the inner vertices of $P$. The number of edges of a path is its length, and the path of length $k$ is denoted by $P^k$. We often refer to a path
by the natural sequence of its vertices writing $P = x_0 x_1 \ldots x_k = x_0 \| x_k$, and calling $P$ a path from $x_0$ to $x_k$ [2].

Given a representation $\Lambda$ for an poset $(P, \leq)$ in [11,12] it was defined its associated graph, $\Gamma_\Lambda$ which has as set of vertices the points of $P$, and containing all information about partitions of the numbers $n_x$. That is $\Gamma_\Lambda$ is represented in such a way that to each vertex of the graph it is attached, either a number $n_x$ given by the representation or one part of a partition of some $n_y$ representing some $y \in P$ such that $x \leq y$.

As an example, we consider [11] an infinite sum of infinite chains pairwise incomparable $R$ in such a way that $R = \sum_{i=0}^{\infty} C_i$, where $C_j$ is a chain such that $C_j = v_{0j} < v_{1j} < v_{2j} < \ldots$. It is defined a representation over $\mathbb{N}$ for $R$, by fixing a number $n \geq 3$ and assigning to each $v_{ij}$ the pair $(n_{ij}, \lambda_{ij}) = (3 + (n - 2)i + (n - 1)j, (3 + (n - 2)i + (n - 1)j)\lambda)$, we note $R_n$ this representation, and write $v_{ij} \in R_n$ whenever it is assigned the number $n_{ij} = 3 + (n - 2)i + (n - 1)j$ to the point $v_{ij} \in R$ in this representation. Fig. 2 below suggests the Hasse diagram for this poset with its associated graph $\Gamma_{p^n_k}$ which attaches to each vertex $v_{ij}$ the number $3 + (n - 2)i + (n - 1)j$, $i, j \geq 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{hasse_diagram.png}
\caption{Hasse diagram for a poset with its associated graph $\Gamma_{p^n_k}$}
\end{figure}

$v_{t \cdot (i-1)} \in l.b.p,$ (left boundary path), $t_{-1} = 0.$
We say that a path $P$ and the inner vertices of $P$ do not lie on any non-trivial path of $\Gamma_{p_k}^n$. The representations $\mathcal{R}_n$ defined above induce an equivalence relation $\sim$ on $\mathcal{R}$ in such a way that $v_{ij} \sim v_{kl}$ if and only if $n_{ij} = n_{kl}$. We denote $[v_{ij}]$ the class of the point $v_{ij} \in \mathcal{R}$. Hence the points of

$$\mathcal{R} = \bigcup_{k \geq 1, m \geq 0} [v_{(2k-1)(7k-3)+m(2(k+1)-1)}] \cup \bigcup_{t \geq 0, s \geq 0} [v_{(2+7t+3s)(2(i+s)+3)}] \subset \mathcal{R}$$

do not lie on any non-trivial path of $\Gamma_{p_k}^n$, if $\mathcal{R}$ is represented in such a way that $(n_{ij}, \lambda_{ij}) = (2i + j, (2i + j)^2)$, $i, j \geq 0$.

We say that a path $P \in \Gamma_{p_k}^n$ is admissible if and only if either $P = P_{i0}$, or $P = P_{i0}P_{ij}$, where $P_{i0}$, $P_{ij}$ are paths such that

$$P_{i0} = v_{t_0} \parallel v_{t_0(i+1)}$, \quad P_{ij} = v_{t_j(i+1)} \parallel v_{(t_j+1)(i+j+2)},$$

and the inner vertices of $P_{i0}$ have the form $v_{th(h+1)}$, $-1 < h < i$, while the inner vertices for $P_{ij}$ have the form $v_{(t_i+1)(i+i+2)}$, $-1 < l < j$. Or $P = P_{i0}P_{ij}P_{ijk}$, where $P_{ijk}$ is a path such that

$$P_{ijk} = v_{(t_i+t_j)(i+j+2)} \parallel v_{(t_i+t_j+t_k)(i+j+k+3)}$, \quad -1 < k < j.$$

If for $s \geq 3$ fixed, we consider the representation $\mathcal{R}_s$ then the numbers $n_{t_i(i+1)}$ representing the vertices $v_{t_i(i+1)} \in \mathcal{R}_s$ of the left boundary path, l.b.p are expressible in the form $n_{t_i(i+1)} = p_i^1 + p_i^2 + p_i^s$, $s$ is a fixed index, $i \geq -1$. Furthermore if $i_0 \geq 0$ is a fixed number, and $v_{t_{i_0}(i_0+1)} \in \text{l.b.p}$ then $n_{t_{i_0}(i_0+1)} \in \text{l.b.p}$, can be expressible in the form $n_{t_{i_0}(i_0+1)} = p_{i_0} + p_{i_0} + p_{i_0} + p_{i_0}$, and a number $n_{t_{i_0}+t_{i_0}+t_{i_0}}(i_0+j+k+3)$ has the form $n_{t_{i_0}+t_{i_0}+t_{i_0}}(i_0+j+k+3) = p_{i_0} + p_{i_0} + p_{i_0} + p_{i_0}$. In [11] has been used this facts to state the following theorem:

**Theorem 1.** A number $m \in \mathbb{N} \setminus \{0\}$ is the sum of three $n$-gonal numbers of positive rank if and only if $m$ represents a vertex $v_{ij} \in \mathcal{R}_n$ in a non-trivial component of $\Gamma_{p_k}^n$.

For example, if $\mathcal{R}_3$ is a representation such that $(n_{ij}, \lambda_{ij}) = (i+j, (i+j)^3)$, $i, j \geq 0$, and $S_i$ is the family of subsets of $\mathcal{R}$ such that for $i \geq 0$, $S_i = \{v_{(t_i+1)(i+2)} | 0 \leq j \leq t_i+1\}$ then $V(\Gamma_{p_k}^n) = \bigcup_i (S_i \cup [v_{i0}] \cup [v_{00}])$, because every natural number is expressible as a sum of three or fewer triangular numbers.

**3. The main result**

If $v_{i0} \in \Gamma_{p_k}^n$ belongs to an non-trivial component, and $v_{i0} \in \mathcal{R}_s$ then there exists an admissible path $P = v_{i0} \parallel v_{i0}$, which has associated a family of compositions (see 2.2) in such a way that if $t > 0$ then

$$n_{t_{i0}(i0+1)} = p_{i0} + p_{i0} + p_{i0}$$. \quad if $P = P_{i0}$,

$$n_{t_{i0}(i0+1)} = p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0}$$. \quad if $P = P_{i0}P_{i0}$,

$$n_{t_{i0}(i0+1)} = p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0}$$. \quad and

$$n_{t_{i0}(i0+1)} = p_{i0} + p_{i0} + p_{i0} + p_{i0} + p_{i0}$$. \quad if $P = P_{i0}P_{i0}P_{i0}$.\quad (3)
Thus we say that two admissible paths $P, Q$ are equivalent if and only if they have associated the same partitions. If $P = v_{00} \parallel v_{ij}$ is an admissible path then we note $[P]$ the class of $P$. Therefore $[P]$ can be determined by fixing the end $v_{ij}$, we note $v_{ij}(P)$ whenever a vertex $v_{ij}$ has been fixed due to this condition (i.e., $v_{ij}(P) = [P]$), $w_{ij}(Q) = \emptyset$ for $Q \in [P]$, $w_{ij} \in [v_{ij}]$, and $w_{ij} \neq v_{ij}$). For example $v_{ij(\epsilon + 1)}(P, a) = [P, a]$ only contains the admissible path $P_{1, a}.

If $v_{ij}(P)$ is the set of admissible paths of the form $v_{00} \parallel u_{rs}$ where $u_{rs} \in [v_{ij}]$ then we note

$$A(v_{ij}) = \{ w_{ij} \in [v_{ij}] \mid w_{ij}(P) = [P], P \in v_{ij}(P) \},$$

therefore $v_{ij}(P) = \bigcup_{w_{ij} \in A(v_{ij})} w_{ij}(P)$.

If $A(v_{ij}) \neq \emptyset$ and $w_{ij} \in A(v_{ij})$ then $\delta_{g}(w_{ij})$ denotes the number of classes of admissible paths at $w_{ij}$. If $A(v_{ij}) = \emptyset$ then $\delta_{g}(v_{ij}) = 0$.

The next theorem is a consequence of theorem 1 and the equations 2.

Theorem 2. Let $P_{s}(n)$ denote the number of partitions of $n \in \mathbb{N}$ into three $s$-gonal numbers, $s \geq 3$, then

$$P_{s}(n) = \begin{cases} 
\sum_{w_{ij} \in A(v_{ij})} \delta_{g}(w_{ij}), & \text{if } A(v_{ij}) \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases}$$

where $v_{ij} \in \mathcal{R}_{s}$ and $n_{ij} = n$.

For example in $\mathcal{R}_{4}$, $[v_{(10,5)}] \cap A(v_{(10,5)}) = \{ v_{(10,5)}, v_{(7,7)} \}$, $\delta_{g}(v_{(10,5)}) = 1$, $\delta_{g}(v_{(7,7)}) = 1$.

If $P = v_{(0,0)} \parallel v_{(0,1)} \parallel v_{(6,5)} \parallel v_{(7,7)}$, $Q = v_{(0,0)} \parallel v_{(10,5)}$ then

$$v_{(7,7)}(P) = \{ v_{(0,0)}, v_{(0,1)}, v_{(6,5)}, v_{(7,7)} \},$$

$$v_{(10,5)}(Q) = \{ v_{(0,0)} \parallel v_{(10,5)} \}. $$

Therefore $P_{4}(38) = 2$.

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