On Base Station Localization for State Estimation over Lossy Networks

Ufuk Topcu, Kenneth Hsu, and Kameshwar Poolla

September 24, 2008

Abstract

We consider a state estimation problem where observations are made by multiple sensors. These observations are communicated over a lossy wireless network to a central base station that computes estimates via a Kalman filter. The goal is to determine the optimal location of the base station under a certain class of packet loss probability models. It is shown in the two sensor case that the base station is optimally located at one of the sensor locations. Empirical evidence suggests that the result holds in some generality.

1 Introduction

The recent confluence of low-cost sensing, communication, and computation technologies has led to much interest in the development of wireless sensor networks for estimation and control. These wireless sensor networks provide for an economical means of extracting greater performance and efficiency in a variety of applications, such as manufacturing and chemical plant processes [?], indoor climate control (HVAC) [?], environment monitoring [?], electrical power distribution [?], and automatic traffic flow [?]. There are many advantages that wireless sensor networks hold over their wired counterparts. For example, the fact that communication is performed wirelessly eliminates any physical connection between the various nodes in the network. This enables wireless sensor networks to be implemented without any major infrastructure overhauls. This also allows for the design of networks that can gracefully accept dynamic changes to its...
structure, such as the addition or subtraction of nodes, or in the case of mobile sensor networks, changing communication topologies.

While the physical disconnection within the network allows for many practical and performance related advantages, it also introduces many fundamental issues (e.g. loss/delay of information, limits on communication bandwidth, power constraints [?]) that are of lesser concern in wired configurations. Issues such as packet loss/delay are crucial in estimation and control problems where information flow is assumed to be uninterrupted, and much work has been done to examine performance in the presence of these complications [?, ?, ?].

In this paper, we consider the problem where (noisy) observations of a dynamic process are made by multiple sensors at fixed locations. These sensors communicate over a lossy wireless network to a base station that computes the optimal state estimate by processing the data via a Kalman filter. Since the physical distance between the sensors and the base station affects the packet loss probability (which in turn affects the quality of the estimates), our goal is to determine the optimal position of the base station. A similar problem of locating control logic has been studied in [?].

There has been much recent work on decentralized estimation problems where no single node is burdened with the majority of the computational tasks [?, ?]. However, implementation of these ideas require “smart” sensors that have local computational abilities. While these approaches have been shown to possess various performance advantages, they are not always realizable. In order to examine more fundamental issues regarding the implementation of wireless sensor networks, we restrict our attention to the case where stationary sensors merely sense and communicate, and possess no computational power. We leave the more general problem for future work.

The remainder of the paper is organized as follows. In Section 2 we give a brief overview of Kalman filtering. Section 3 contains our problem formulation and main results. Some discussion regarding more general cases can be found in Section 4.

2 Preliminaries

2.1 Kalman Filtering

We begin by giving a brief overview of the canonical state estimation problem. Consider a process whose dynamics are modeled by the (possibly time-varying) state space equation

\[ x_{k+1} = Ax_k + w_k, \]

where \( x_k \in \mathbb{R}^n \) is the state vector at time \( k \), and \( w \) is a sequence of independent Gaussian random vectors with zero mean and covariance matrix \( Q \). The initial condition \( x_0 \) is
unknown and modeled as a Gaussian random variable with zero mean and covariance matrix $P_0$.

We monitor this process via measurements of the form

$$y_k = Cx_k + v_k,$$

where $C$ is possibly time-varying, and $v$ is a sequence of independent Gaussian random vectors with zero mean and covariance matrix $R$. The sequences $w$ and $v$ are assumed to be uncorrelated. The minimum variance estimate of the sequence $x$ is then given recursively by the Kalman filter

$$K_k = AP_kC^*(CP_kC^* + R)^{-1}$$

$$\hat{x}_{k+1} = A\hat{x}_k + K_k(y_k - C\hat{x}_k)$$

$$P_{k+1} = AP_kA^* + Q - K_kCP_kA^*.$$ (2c)

Here, $M^*$ denotes the complex conjugate transpose of a matrix $M$.

Due to the increasing availability of high quality, low-cost sensors, we are often confronted with the situation where we have access to measurements from multiple sensors. That is, suppose that we monitor the process via $N$ sensors, whose measurements are represented as

$$y_{1k} = C_1x_k + v_{1k}$$

$$\vdots$$

$$y_{Nk} = C_Nx_k + v_{Nk}.$$ (2d)

Here, $v^1, \ldots, v^2$ are sequences of independent Gaussian random vectors with zero mean and covariance matrices $R_1, \ldots, R_N$, respectively. The sequences $v^1, \ldots, v^N$ and $w$ are assumed to be uncorrelated. The resulting state estimates are again given by (2), where we now define

$$C = \text{blkcol}(C_j), \ j = 1, \ldots, N$$

$$R = \text{blkdiag}(R_j), \ j = 1, \ldots, N.$$ (2e)

Here, blkcol denotes the vertical stacking of its arguments, and blkdiag denotes a block diagonal matrix of its arguments.

### 2.2 Kalman Filtering with Packet Losses

We now discuss the minimum variance state estimation problem in the presence of packet losses. Let $\Omega$ be the power set of $\{1, \ldots, N\}$ and let each subset $\omega \in \Omega$ be given the standard simple order. Since the Kalman filtering equations (2) yield the minimum
variance state estimate for linear time-varying systems, we obtain the following optimal filter for state estimation with packet losses

\[
K_k = AP_kF_{\omega_k}^*(F_{\omega_k}P_kF_{\omega_k}^* + G_{\omega_k})^{-1}
\]
\[
\hat{x}_{k+1} = A\hat{x}_k + K_k(y_k - F_{\omega_k}\hat{x}_k)
\]
\[
P_{k+1} = AP_kA^* + Q - K_kF_{\omega_k}P_kA^*
\]

for \(\omega_k \in \Omega\setminus\phi\),

and

\[
\hat{x}_{k+1} = A\hat{x}_k + Q
\]
\[
P_{k+1} = AP_kA^* + Q
\]

for \(\omega_k = \phi\),

where

\[
F_{\omega_k} = \text{blkcol}(C_j), \quad j \in \omega_k
\]
\[
G_{\omega_k} = \text{blkdiag}(R_j), \quad j \in \omega_k.
\]

For a similar treatment, see [?, ?].

3 Base Station Placement

3.1 Problem Formulation

We will model the packet arrival process for each sensor as a sequence of independent binomial random variables. More specifically, for \(i = 1, \ldots, N\), let \(\lambda_i\) be the probability that the observation from sensor \(i\) is received. Since the sensor locations are fixed, a given base station location gives rise to distances \(d_1, \ldots, d_N\). We will assume that the probability of packet arrival decreases as the distance \(d_i\) between a sensor and the base station increases.

Note that the sequences \((F_\omega)_k\) and \((G_\omega)_k\) both depend on whether packets have been received or lost, and hence the sequence \((P_k)_k\) is itself random. We will then aim to minimize (with respect to the Loewner ordering [?], i.e., \(X \succeq Y\) if \(X - Y\) is positive semi-definite) the expected estimation error covariance. That is, we consider the problem

\[
\min \mathbb{E}[P_{k+1}|P_k],
\]

where the minimization is performed over the set of possible base station locations. If we define for each nonempty \(\omega \in \Omega\)

\[
\alpha_\omega = \prod_{j \in \omega} \lambda_j \prod_{j \in \{1, \ldots, N\}\setminus \omega} (1 - \lambda_j)
\]
\[
K_k = AP_kF_\omega^*(F_\omega P_k F_\omega^* + G_\omega)^{-1},
\]
and $\alpha_\omega = 0$ for $\omega = \phi$, we can write
\[ E[P_{k+1}|P_k] = AP_kA^* + Q - \sum_{\omega \in \Omega} \alpha_\omega K_\omega F_\omega P_k A^*. \]

As stated, the above optimization problem is a difficult nonlinear programming problem that can be attacked by the appropriate solvers. Since it is not feasible for the location of the base station to change over time, we will insist that the solution to the above problem minimizes $E[P_{k+1}|P_k]$ for any $P_k$. Consequently, we shall hereafter restrict our attention to an illuminating instance of the above problem.

### 3.2 The 2 Sensor Case

In this section, we consider the case where measurements are obtained from 2 homogeneous sensors, $s_1$ and $s_2$. Without loss of generality, these sensors are located at $\xi_1 = 0$ and $\xi_2 = 1$ on the real line. The goal is to find the best location $d \in [0, 1]$ for the base station. Note that optimal solution must lie in the interval $[0, 1]$. The situation $d = 0$ (or $d = 1$) corresponds to the base station being physically wired to one of the sensors, and wireless communication is performed only by the other sensor.

Although there may be many factors that influence the packet loss probability, we shall consider only the effects due to the physical distance between the communicating nodes. Since the sensors have identical broadcasting capabilities, it is then natural to model the probability of packet loss in the following manner. Let $f : [0, 1] \rightarrow [0, 1]$ be convex and decreasing, with $f(0) = 1$. When the base station is located at some $d \in [0, 1]$, the probability that a packet is received from sensor $s_1$ is $f(d)$, and the probability that a packet is received from sensor $s_2$ is $f(1 - d)$. The constraint that $f(0) = 1$ captures the implicit assumption that no packet is lost when a sensor is affixed to the base station and communication is not performed wirelessly.

#### 3.2.1 Vector-valued Measurements, Same Covariances

We will first restrict our attention to the case where
\[
\begin{align*}
C_1 &= C_2 = C \in \mathbb{R}^{m \times n} \\
R_1 &= R_2 = R \in \mathbb{R}^{m \times m}.
\end{align*}
\]

If we model the probability of packet loss as described above, we can examine the estimation error covariance as a function of $d$. For ease of notation, let us define
\[
\begin{align*}
M_1 &= AP_k C^* (CP_k C^* + R)^{-1} CP_k A^* \\
M_2 &= AP_k C^* (CP_k C^* + R/2)^{-1} CP_k A^* \\
S &= AP_k A^* + Q.
\end{align*}
\]
The estimation error covariance $P_{k+1}$ then assumes values as follows:

| $P_{k+1}$ | Probability |
|-----------|-------------|
| $S - M_2$ | $f(d)f(1-d)$ |
| $S - M_1$ | $f(d) - 2f(d)f(1-d) + f(1-d)$ |
| $S$ | $(1 - f(d))(1 - f(1-d))$ |

Note that in the case where packets from both sensors are lost, the resulting covariance corresponds to that of propagating the previous state estimate. The expected error covariance can then be computed as

$$E[P_{k+1}|P_k] = S - f(d)f(1-d)M_1 - (f(d) - 2f(d)f(1-d) + f(1-d))M_2.$$ 

We have the following result.

**Proposition 1** Let $C_1 = C_2 = C \in \mathbb{R}^{m \times n}$ and $R_1 = R_2 = R \in \mathbb{R}^{m \times m}$. Let $f : [0,1] \rightarrow [0,1]$ be twice differentiable, convex, and decreasing, with $f(0) = 1$. Then, $E[P_{k+1}|P_k]$ is matrix concave in $d$ on $[0,1]$.

**Proof** Define

$$J_{k+1}(d) = E[P_{k+1}|P_k] = AP_kA^* + Q - f(d)f(1-d)M_2 - (f(d) - 2f(d)f(1-d) + f(1-d))M_1.$$ 

Taking the second derivative yields

$$J''_{k+1}(d) = f''(d)(f(1-d) - 1)M_1 + f''(1-d)(f(d) - 1)M_1 - 2f'(d)f'(1-d)(2M_1 - M_2) + (f''(d)f(1-d) + f(d)f''(1-d))(M_1 - M_2).$$ 

Since $f$ is convex and decreasing with $f(0) = 1$, we have $f'(x) \leq 0$ and $f''(x) \geq 0$ for $x \in [0,1]$. Consequently, $J''(d) \leq 0$, and hence $E[P_{k+1}|P_k]$ is matrix concave on $[0,1]$ (see [?], p.110). □

**Corollary 1** Let $C_1 = C_2 = C \in \mathbb{R}^{m \times n}$ and $R_1 = R_2 = R \in \mathbb{R}^{m \times m}$. Let $f : [0,1] \rightarrow [0,1]$ be twice differentiable, convex, and decreasing, with $f(0) = 1$. Then, the base station is optimally located at one of the sensor positions.
3.2.2 Scalar-valued Measurements, Same Covariances

A similar result may be obtained in the case where $C_1 = C_2 = C \in \mathbb{R}^{1 \times n}$ and the noise covariances $R_1$ and $R_2$ are not necessarily equal. We will need the following preliminary result.

**Lemma 1** Let $C_1 = C_2 = C \in \mathbb{R}^{1 \times n}$ and define

$$
T = CP_kC^* \\
M_0 = AP[C]^*[T + R_1 \ T \ T + R_2]^{-1}[C]PA^* \\
M_1 = APC^*(T + R_1)^{-1}CPA^* \\
M_2 = APC^*(T + R_2)^{-1}CPA^*.
$$

Then, $M_0 - M_1 - M_2 \geq 0$.

**Proof** Let us define

$$
\begin{bmatrix}
\alpha & \beta \\
\beta & \gamma
\end{bmatrix} = \begin{bmatrix}
T + R_1 & T \\
T & T + R_2
\end{bmatrix}, \text{ and } U = \begin{bmatrix}
C \\
C
\end{bmatrix} P_kA^*.
$$

Then, $M_0 - M_1 - M_2$ can be written as

$$
M_0 - M_1 - M_2 = U^* \left( \begin{bmatrix}
\alpha & \beta \\
\beta & \gamma
\end{bmatrix}^{-1} - \begin{bmatrix}
\alpha & 0 \\
0 & \gamma
\end{bmatrix}^{-1} \right) U
$$

$$
= U^* \begin{bmatrix}
\frac{1}{\alpha - \beta^2 \gamma^{-1}} & \frac{\beta \gamma^{-1}}{\alpha - \beta^2 \gamma^{-1}} \\
\frac{\beta \gamma^{-1}}{\alpha - \beta^2 \gamma^{-1}} & \frac{1}{\alpha - \beta^2 \gamma^{-1}}
\end{bmatrix} U
$$

$$
= U^* \begin{bmatrix}
\frac{1}{\alpha - \beta^2 \gamma^{-1}} & \frac{\beta \gamma^{-1}}{\alpha - \beta^2 \gamma^{-1}} \\
\frac{\beta \gamma^{-1}}{\alpha - \beta^2 \gamma^{-1}} & \frac{1}{\alpha - \beta^2 \gamma^{-1}}
\end{bmatrix} \begin{bmatrix}
1 & -\frac{\beta \gamma^{-1}}{\alpha - \beta^2 \gamma^{-1}} \\
0 & 0
\end{bmatrix} U
$$

$$
- U^* \begin{bmatrix}
-\alpha^{-1} & 0 \\
0 & 0
\end{bmatrix} U
$$

$$
= AP_kC^* \left( \frac{(1 - \gamma^{-1} \beta)^2}{\alpha - \beta^2 \gamma^{-1}} - \alpha^{-1} \right) CP_kA^*.
$$

where the second equality follows from the matrix inversion lemma [?]. Now, let

$$
H = \frac{(1 - \gamma^{-1} \beta)^2}{\alpha - \beta^2 \gamma^{-1}} - \alpha^{-1}
$$

$$
= \frac{(1 - \gamma^{-1} \beta)^2 - \alpha^{-1}(\alpha - \beta^2 \gamma^{-1})}{\alpha - \beta^2 \gamma^{-1}}.
$$
The desired result follows from noticing that the denominator is positive and the numerator satisfies

\[
(1 - \gamma^{-1}\beta)^2 - \alpha^{-1}(\alpha - \beta^2\gamma^{-1}) \\
= -\beta\gamma^{-1} \left( 2 - \frac{CP_kC^*}{CP_kC^* + R_2} - \frac{CP_kC^*}{CP_kC^* + R_1} \right) \\
\leq 0.
\]

\[\square\]

**Proposition 2** Let \(C_1 = C_2 = C \in \mathbb{R}^{1 \times n}\) and suppose that \(R_1\) and \(R_2\) are not necessarily equal. Let \(f : [0, 1] \to [0, 1]\) be twice differentiable, convex, and decreasing, with \(f(0) = 1\). Then, \(E[P_{k+1}|P_k]\) is matrix concave in \(d\) on \([0, 1]\).

**Proof** Define

\[
T = CP_kC^* \\
M_0 = AP\left[C^* \begin{bmatrix} C \\ T + R_1 \\ T + R_2 \end{bmatrix}^{-1} C \right]PA^* \\
M_1 = APC^*(T + R_1)^{-1} CPA^* \\
M_2 = APC^*(T + R_2)^{-1} CPA^*.
\]

We can then write

\[
J_{k+1}(d) = E[P_{k+1}|P_k] = AP_kA^* + Q \\
- f(d)f(1 - d)M_0 - f(d)(1 - f(1 - d))M_1 \\
- f(1 - d)(1 - f(d))M_2
\]

Taking the second derivative yields

\[
J''_{k+1}(d) = f''(d)(f(1 - d) - 1)M_1 \\
+ f''(1 - d)(f(d) - 1)M_2 \\
- 2f'(d)f'(1 - d)(M_1 + M_2 - M_0) \\
+ f''(d)f(1 - d)(M_2 - M_0) \\
+ f(d)f''(1 - d)(M_1 - M_0).
\]

Since \(f\) is convex and decreasing with \(f(0) = 1\), we have \(f'(x) \leq 0\) and \(f''(x) \geq 0\) for \(x \in [0, 1]\). From Lemma 1 we have \(M_1 + M_2 - M_0 \leq 0\). As a result, \(J''(d) \leq 0\), so \(E[P_{k+1}|P_k]\) is matrix concave on \([0, 1]\) (see [?], p.110).

\[\square\]
Corollary 2 Let $C_1 = C_2 = C \in \mathbb{R}^{1 \times n}$ and suppose that $R_1$ and $R_2$ are not necessarily equal. Let $f : [0, 1] \to [0, 1]$ be twice differentiable, convex, and decreasing, with $f(0) = 1$. Then, the base station is optimally located at one of the sensor positions. If $R_1 \succeq R_2$, then the base station is optimally located at the second sensor.

3.2.3 An Example

We now offer an illustrative example to complement our main results.

Suppose that there are two sensors, located respectively at the points 0 and 1 on the real line. Let us model our packet arrival probability function as

$$f(d) = e^{-d}.$$

Note that $f$ is convex and decreasing on $[0, 1]$ with $f(0) = 1$. The process being monitored is described by the state space equations in (1), where all the eigenvalues of $A \in \mathbb{R}^{2 \times 2}$ lie in the unit circle and $C \in \mathbb{R}^{1 \times 2}$. The trace of the estimation error covariance matrices is plotted in Figure 1 for the case where the base station is situated at the point 0 and the case where it is situated at the point 0.5.

![Figure 1: Covariances for $d = 0$ (dashed) and $d = 0.5$ (solid).](image)

4 General Cases

We now offer some commentary on extending our main results.

Although the previous results were confined to the case $C_1 = C_2$, simulations suggest that they also hold in more generality. Consider the case where $N$ sensors located
colinearly are available to make observations. In general, we may have different values for $C_1, \ldots, C_N$ and different values for $R_1, \ldots, R_N$. In this case, empirical evidence supports the extension of our main result in the sense that the expected error covariance is piecewise concave over the intervals defined by the location of the sensors. Figure 2 illustrates this situation for the case $N = 3$.

![Figure 2: Piecewise concavity for multiple colinear sensors.](image)

In the case where the convex hull of the sensor locations has a nonempty interior in $\mathbb{R}^2$, it can be shown that the optimal base station location may lie at a point other than the sensor locations. Consider 3 sensors, located at the vertices of an equilateral triangle with side length 0.5. For $P_k = I$, $f(d) = e^{-d}$, $C_1 = C_2 = C_3 = 0.6$, and unit noise covariances, the expected error covariance (as a function of base station location) has a local minimum at the centroid, as illustrated in Figure 3.

![Figure 3: Local minimum in the interior of the convex hull.](image)
5 Conclusion

We considered the problem of positioning a central computational node within a wireless sensor network for the purpose of state estimation. In the presence of potential packet losses, the optimal estimate is given by a time-varying Kalman filter. We then strived to determine the optimal location of the base station so as to minimize the expected conditional estimation error covariance. It was shown for the case of a 2-sensor configuration that under a certain class of packet loss probability models, the base station is optimally located at one of the sensor positions.