Black hole tidal charge constrained by strong gravitational lensing

Zs. Horváth1⋆ and L.A. Gergely1

1 Departments of Theoretical and Experimental Physics, University of Szeged, Dóm tér 9, Szeged 6720, Hungary

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Spherically symmetric brane black holes have tidal charge, which modifies both weak and strong lensing characteristics. Even if lensing measurements are in agreement with a Schwarzschild lens, the margin of error of the detecting instrument allows for a certain tidal charge. In this paper we derive the respective constraint on the tidal charge of the supermassive black hole (SMBH) in the center of our galaxy, from the radius of the first relativistic Einstein ring, emerging in strong lensing. We find that even if general relativistic predictions are confirmed by high precision strong lensing measurements, SMBHs could have a much larger tidal charge, than the Sun or neutron stars.

1 Introduction

The Galactic Center is hard to resolve due to source confusion. Absorbing gas and dust renders most of the knowledge about this part of our galaxy to come from observations in radio and infrared (Ghez et al. 2003). It has been estimated (Brown et al. 2005) that cold dark matter remnants of 1000 solar masses (M⊙) and compact star remnants of 1000 M⊙ reside in the inner 0.01 parsec (pc) region. The star population in the inner 0.04 pc, the Sgr A* stellar cluster consists of B stars (Krabbe et al. 1995). In the distance range [0.04 pc, 0.5 pc] Wolf Rayet and OB giant stars are found.

The Galactic Center is dominated by a Supermassive Black Hole (SMBH), with mass 4.31 × 10⁶ M⊙ and distance from the Sun 8330 pc (Gillessen et al. 2012). The orbits of nearby stars depend on the mass and spin of the SMBH. These characteristics were obtained observing of stellar orbits (Will 2008; Merritt et al. 2010).

The first measurements of proper motions of stars within 2400 AU from the center of our Galaxy was published by Eckart and Genzel (1997). A detailed review of 16 years of monitoring stellar orbits around the SMBH using near-infrared (NIR) techniques was given by Gillessen et al. (2009).

With increasing measurement accuracy and the possibility of direct observations of the horizon of the SMBH at the Galactic Center the possibility of testing general relativistic predictions emerges. The measurement accuracy of a specific instrument will allow for certain margin of error for any additional parameter.

The design of the interferometer GRAVITY has reached completion (Gillessen et al. 2010). This will use the four telescopes of the Very Large Telescope as an interferometer in the NIR band. GRAVITY will perform astrometry with 12 microarcsecond (μas) precision in the K band up to 15 magnitudes. The margin of error of its measurements will be Δθ = 12 × 10⁻⁶ as. From the numerous scientific applications of GRAVITY (Gillessen et al. 2010, Section 3.), the observation of relativistic motions near the horizon of Sgr A* will be of uttermost importance.

By such observations of NIR flares the metric near the horizon can be determined (Psaltis 2004). A submm-VLBI array should be able to actually resolve Sgr A* (Falcke, Melia & Agol 2000). As proposed by Will (2008) the no-hair theorem can be tested.

We will constrain the tidal charge, an imprint of the possible 5-dimensional character of gravity in a brane-world scenario (Maartens 2000; Maartens & Koyama 2010). The tidal charge emerges from the Weyl curvature of a 5-dimensional space-time in which our 4-dimensional observable world is embedded. Such a tidal charged black hole solution on a brane was found by Dadhich et al. (2000). In these theories the electric part of the Weyl curvature, non-standard model 5-dimensional fields, asymmetric embedding and a varying brane tension all could act as sources of the effective Einstein equation (Shiromizu, Maeda & Sasaki 2000; Gergely 2003, 2008). Remarkably, the Weyl curvature source term could replace dark matter (Harko & Cheng 2007; Mak & Harko 2004).

Light deflection and weak gravitational lensing by tidal charged brane black holes was investigated by Böhmer, Harko and Lobo (2008); Gergely, Keresztes & Dwornik (2009). One of the results of these investigations was a strict limit imposed on the tidal charge from Solar System measurements. Horváth, Gergely and Hobill (2010) derived analytical results on weak lensing by tidal charge dominated black holes. Bin-Nun (2010b) discussed weak lensing by the Galactic SMBH assumed tidal charged. The angular position of the image and its apparent magnitude were shown to both decrease as function of the tidal charge.

Whisker (2005) discussed strong lensing in two brane-world black hole models, the “U = 0 black hole” and the tidal charged black hole. For the central black hole of our...
galaxy as lens the angular radius where the second and higher relativistic images are confused was given. The angular separation of these relativistic images from the first relativistic image was also presented. Bin-Nun (2010a) has discussed three brane BH solutions, including the tidal charged one. The size and magnifications of the primary and the first two relativistic Einstein rings was given for a source lensed by the SMBH at Sgr A*. Bin-Nun (2011) extends the analysis presented by Bin-Nun (2010b) to the first relativistic images of the stars S2, S6, S14.

In this paper we study strongly relativistic orbits in the Galactic Center assuming that the metric is the one of the tidal charged brane black hole. We focus on 1-loop photon trajectories, as depicted on Fig. 1. We summarize the basic equations concerning the one-loop null geodesics in spherically symmetric, static space-times in Section 2. The formation of the first relativistic Einstein ring, emerging due to strong lensing by the SMBH in the center of our galaxy is discussed in Section 3. Then we investigate the magnitude of the tidal charge, still compatible with the margin of error discussed in Section 3. Then we investigate the magnitude of the tidal charge derived by this method in Section 4.

We present our conclusions which include the constraints on the tidal charge derived by this method in Section 4. We use the geometric units, $G = c = 1$.

## 2 Null geodesics and relativistic Einstein rings in spherically symmetric, static space-times

The general spherically symmetric, static metric is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2d\theta^2 + r^2\sin^2\theta \, d\varphi^2 .$$

In the following we consider metrics with $g_{tt} = r^2, g_{\varphi\varphi} = r^2\sin^2\theta$. Without reducing generality, we can restrict geodesic motions to the plane $\theta = \pi/2$. We also introduce the dimensionless radial coordinate $R = r/M$. The second order radial geodesic equation can be replaced by the first order null condition $0 = g_{tt}(dt/dp)^2 + g_{rr}(dr/dp)^2 + g_{\varphi\varphi}(d\varphi/dp)^2$ where $p$ is a parameter of the null curve). As the metric does not depend on either of the coordinates $t$ or $\varphi$, two constants of motion emerge by integration from the geodesic equations: $L = g_{\varphi\varphi}d\varphi/dp$ the specific angular momentum of the photon with dimension $[\text{length}]^2$; and $E = g_{tt}dt/dp$, proportional to the energy of the photon, the dimension of $E$ being $[\text{length}]$.

Then the $rr$ term can be written in the form $g_{rr}(dr/d\varphi)^2 \times L^2/g_{\varphi\varphi}$, while the $tt$ and $\varphi\varphi$ terms as $E^2/g_{tt}$ and $L^2/g_{\varphi\varphi}$. Hence the null condition becomes

$$0 = \frac{E^2}{g_{tt}} + g_{rr}\left(\frac{dr}{d\varphi}\right)^2 \frac{L^2}{g_{\varphi\varphi}} + \frac{L^2}{g_{\varphi\varphi}} .$$

By reordering the terms, we obtain the equation characterizing the trajectory $r(\varphi)$:

$$\frac{dr}{d\varphi} = \pm \left[\frac{g_{\varphi\varphi}}{g_{rr}} \left(\frac{E^2}{L^2} - \frac{g_{\varphi\varphi}}{g_{tt}} - 1\right)\right]^{1/2} .$$

The sign differentiates between the incoming and outgoing parts of the path.

We will seek a null geodesic curve along which the photons travel from the source $S$ to the observer $O$, while turning around the lens $L$ once (1-loop orbit). The source, the lens and the observer are placed on the same coordinate line $\varphi = 0$ (the optical axis). More specifically $L$ is the origin, $S$ lies at $(\varphi = 0, r = D_{LS})$, while $O$ at $(\varphi = \pi, r = D_L)$. Then the trajectory $r(\varphi)$ of the photon is a monotonically decreasing function from $r = D_{LS}$ to some distance $r = r_{min}$, then increases to $r = D_L$. The distance of minimal approach $r_{min} = r(\varphi_{min})$ is defined by the equation

$$\frac{dr}{d\varphi}(\varphi_{min}) = 0 .$$

For such a one-loop path the total change of the polar angle $\varphi$ from $S$ to $O$ is $\pi + 2\pi$.

The image created by these 1-loop orbits is called the first relativistic Einstein ring. Its angular radius, the first relativistic images was given for a source lensed by the SMBH at Sgr A*. Bin-Nun (2011) extends the analysis of these relativistic images from the first relativistic Einstein ring.

We present our conclusions which include the constraints on the tidal charge derived by this method in Section 4. We use the geometric units, $G = c = 1$. The equality of the right hand sides gives

$$\Theta_E \equiv \arccos \left(\frac{\partial R}{\partial \varphi} \sqrt{g_{RR}}\right)^2 g_{RR} + \left(\frac{\partial \varphi}{\partial \varphi}\right)^2 g_{\varphi\varphi}$$

$$= \arccos \left[\left(\frac{dr}{d\varphi}\right)^2 g_{RR} + g_{\varphi\varphi}\right]^{1/2} .$$

In the last step the parametrization of the curve $R(\varphi)$ was eliminated by multiplying both the numerator and the denominator by $dp/d\varphi$.
izes the Schwarzschild solution of a black hole with mass \( M \). The tidal charged black hole (Dadhich et al. 2000) generalizes the Schwarzschild solution of a black hole with mass \( M \), by allowing for a tidal charge parameter \( q \) in the metric function

\[
g_{tt} = -\frac{1}{g_{rr}} = -1 + \frac{2M}{r} - \frac{q}{r^2}, \tag{6}
\]

of the spherically symmetric, static space-time (1). For \( q < 0 \) there is a horizon at \( M + (M^2 - q)^{1/2} \), while for \( 0 < q < M^2 \) there are two horizons, at \( M \pm (M^2 - q)^{1/2} \). For \( q > M^2 \) the metric describes a naked singularity.

The trajectory (3) becomes

\[
\frac{dR}{d\varphi} = \pm \left( \frac{EM}{L} \right)^2 R^4 - R^2 + 2R - \frac{q}{M^2} \right)^{1/2}, \tag{7}
\]

\( L/ME \) denotes the dimensionless impact parameter. The minimal dimensionless distance \( R_{\text{min}} := r_{\text{min}}/M \) is a root of the polynomial obtained from Eq. (7) by setting the left hand side as 0.

The Einstein angle (5) then becomes

\[
\Theta_E = \arccos \left[ 1 - \left( \frac{L}{ED_L} \right)^2 \left( 1 - \frac{2M}{D_L} + \frac{q}{D_L^2} \right) \right]. \tag{8}
\]

Next we assume a stellar source on the optical axis defined by the SMBH and the GALAXY detector, at a distance \( D_{LS} \) varying in the range \([10 \text{ pc}, 10^5 \text{ pc}]\) and a photon trajectory with one loop about the SMBH. Each value of \( L/ME \) determines a null geodesic curve by Eq. (7), however for most of the values the resulting curve does not reproduce the desired lensing geometry (e.g. the boundary conditions set by \( S \) and \( O \)). Therefore one has to ‘fine-tune’ \( L/ME \) to generate the specific curve which satisfies the conditions preset by the 1-loop orbit. We did this in the following way. After setting the tidal charge \( q \) and the distance \( D_{LS} \), we chose an initial value for the \( L/ME \) (between 1 and 10). Then we evolved the differential equation (7) numerically and calculated the change in the polar angle \( \varphi \) while the photon travels from \( D_{LS}/M \) to \( R_{\text{min}} \) then further to \( D_{LS}/M \). For \( L/ME = 10 \) this angular change was too large (\( \Delta \varphi > 3\pi \)) for any \( q \in [-5M^2, +2M^2] \) and \( D_{LS} \in [10 \text{ pc}, 10^5 \text{ pc}] \). Therefore we reduced its value and repeated the procedure, until it yielded \( \Delta \varphi = 3\pi \) with \( 10^{-12} \) rad accuracy. In this iterative way we obtained the dimensionless impact parameter. Then we inserted this into Eq. (8) to calculate the radius of the first relativistic Einstein ring.

Limits \( q_{\text{min}} \) and \( q_{\text{max}} \) for the tidal charge can be derived from the designed margin of error of the measurements of GRAVITY. The values of \( \Theta_E \) are enlisted for \( D_{LS} = 10 \text{ pc} \) in Table 1. The tidal charge was varied to fit these limits, the Einstein radius changes within \( 2\Delta \Theta \). The limits \( q_{\text{min}} \) and \( q_{\text{max}} \) are defined by

\[
\Theta_E(q_{\text{min}}, D_{LS}) = \Theta_E(0, D_{LS}) + \Delta \Theta, \tag{9}
\]

\[
\Theta_E(q_{\text{max}}, D_{LS}) = \Theta_E(0, D_{LS}) - \Delta \Theta.
\]

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**Table 1** The allowed range of the Einstein ring (column 1.) and the associated range of the tidal charge (columns 5.-6.). Columns 2.-3.: the orbital parameters of the orbits with a polar angle change \( 3\pi \), column 2.: the minimal distance \( R_{\text{min}} \), larger than the horizon in each case, column 3.: the parameter \( L/ME \), column 4.: the normalized radius of the horizon, column 5.: the tidal charge, column 6.: the normalized tidal charge. We chose \( D_{LS} = 10 \text{ pc} \).

| \( \Theta_E \) | \( \pi \text{ rad} \) | \( \frac{L}{ME} \) | \( \frac{r_{\text{min}}}{M} \) | \( q \) | \( 10^{20} \text{ m}^2 \) |
|---|---|---|---|---|---|
| 38 | 4.9 | 7.7 | 3.3 | -1.815 | -4.5 |
| 34 | 4.2 | 6.7 | 2.8 | -1.008 | -2.5 |
| 30 | 3.7 | 6.1 | 2.4 | -0.491 | -1.2 |
| 26 | 3.0 | 5.2 | 2.0 | 0.000 | 0.0 |
| 22 | 2.5 | 4.4 | 1.6 | 0.148 | 0.3 |
| 18 | 1.9 | 3.8 | 1.1 | 0.268 | 0.7 |
| 14 | 1.4 | 3.2 | n. s. | 0.524 | 1.3 |

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Fig. 1 The trajectory of light along which the photons travel from the source \( S \) to the observer \( O \), while turning around the lens \( L \) once, for \( q = 0 \), \( D_L = 8600 \text{ pc} \), \( D_{LS} = 10 \text{ pc} \). On panel (a) the whole geodesics is seen on logarithmic scale. The distorted region in the left and right side of the loop is magnified and shown on a linear scale on panel (b).
Table 2. Bounds on the tidal charge normalized by mass square.

| Object          | Sun    | N.s. Binary | Neutron Star | SMBH |
|-----------------|--------|-------------|--------------|------|
| $|q/\mathcal{M}|_{\text{max}}$ | 0.003   | 2.339       | 227.647      | 4.485|

We have checked that by varying $D_{LS}$ in the domain [10 pc, 10$^5$ pc] the $\Theta_E$ change by less than 1 $\mu$as. With the mass of the SMBH a tidal charge falling in the range $[-1.815, 0.524] \times 10^{20}$ m$^2$ is allowed. This range is consistent with measurements of the first relativistic Einstein ring generated by sources opposite to us with respect to the central SMBH.

4 Concluding Remarks

Bounds on the tidal charge of various astrophysical objects were derived earlier in the literature. For neutron stars a limit of $|q| < 10^7$ m$^2$ was established by Kotrolova et al. (2008) from orbital models of high-frequency quasiperiodic oscillations observed in neutron star binary systems. From the constraint on the brane tension (Germani & Maartens 2001, Eq. (30) ) a weaker limit for negative tidal charges emerges in the following way. The junction condition of the uniform density star and its exterior represented by the tidal charged metric associates a tidal charge to the brane tension limit of neutron stars (Germani & Maartens 2001), column 5.: from forthcoming strong lensing observations on the Galactic SMBH derived in this paper.

We have additionally checked that the second and third relativistic rings (with $\Delta \varphi = 5\pi, 7\pi$, respectively) lead to identical results on the allowed range of $q$, as the first ring does, supporting that the relativistic rings are situated quite close to each other (within 0.03 $\mu$as).

Although the derived constraint on $q$ is much weaker than those from neutron stars or Solar System observations, the dimensionless quantity $q/\mathcal{M}^2$ is not very much different, as shown in Table 2. In fact for this quantity the neutron stars constraints are the weakest, and the Solar System constraints the strongest, while both the neutron star binary systems and the SMBH strong lensing considerations presented in this paper gave comparable constraints.

In summary even if general relativistic predictions on the Galactic SMBH are confirmed by high precision measurements, our investigations show that SMBHs could have a much larger tidal charge, than the Sun or neutron stars.

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