On the Infinitude of Some Special Kinds of Primes

—— Dedicated to the memory of my mother

Shaohua Zhang

School of Mathematics, Shandong University, Jinan, China, 250100
E-mail address: shaohuazhang@mail.sdu.edu.cn

Abstract: The aim of this paper is to try to establish a generic model for the problem that several multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points. More concretely, we introduced briefly the research background—the history and current situation—from Euclid’s second theorem to Green-Tao theorem. We analyzed some equivalent necessary conditions that irreducible univariable polynomials with integral coefficients represent infinitely many primes, found new necessary conditions which perhaps imply that there are only finitely many Fermat primes, obtained an analogy of the Chinese Remainder Theorem, generalized Euler’s function, the prime-counting function and Schinzel-Sierpinski’s Conjecture and so on. Nevertheless, this is only a beginning and it miles to go. We hope that number theorists consider further it.

Keywords: Euclid’s second theorem, Chinese Remainder Theorem, Dirichlet’s theorem, Fermat primes, Schinzel-Sierpinski’s Conjecture, Green-Tao theorem

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1 Research background—the history and current situation—from Euclid’s second theorem to Green-Tao theorem

From ancient to modern times, the study of the infinitude of some special kinds of primes has been one of the most important topics in Number Theory. People usually ask the following questions:

1. Are there infinitely many Fermat primes? Fermat primes are primes of the form $2^{2^x} + 1$. 
2. Are there infinitely many Mersenne primes? Mersenne primes are primes of the form $2^x - 1$, where $x$ is also a prime.

3. Are there infinitely many twin primes?

4. Are there infinitely many primes of the form $x^2 + 1$?

5. Are there infinitely many Sophie Germain primes? A prime $p$ is called a Sophie Germain prime if $2p + 1$ is also prime.

And so on.

Mathematicians throughout history have been fascinated by these problems. However, they are still unanswered. Euclid [1] proved firstly the following result.

**Euclid’s second theorem:** There are infinitely many primes.

Anyone who likes Number Theory must like Euclid’s second theorem. In his book *The book of prime number records* [2], Paulo Ribenboim cited nine and a half proofs of Euclid’s second theorem. In this paper, we listed the references of fifteen distinct proofs again, see [3–17].

Clearly, using Euclid’s method, the ancient Greeks can also prove that there are infinitely many primes of the form $4k - 1$ or $6k - 1$. Using properties of quadratic residues, it is easy to prove that there are infinitely many primes of the form $4k + 1$ or $6k + 1$. Cyclotomic polynomials [18] can be used to prove that there are infinitely many primes of the form $ak + 1$. In 2004, Yoo, Jisang [19] gave another elementary proof of the infinitude of primes of the form $ak + 1$. Especially, in 2005, Robbins Neville [20] gave a simple proof of the infinitude of primes of the form $3k + 1$.

Naturally, a more general problem on primes in arithmetic progressions seems that there should be infinitely many primes of the form $a + bn$, where $a$ and $b$ are integers satisfying $(a, b) = 1$, and either $a \neq 0, b > 0$, or $a = 0, b = 1$. After the time of Euclid, there have been no great improvements on this problem in about 2000 years. Until 1837, using L-series and analytic methods, Dirichlet [21] solved thoroughly it.

**Dirichlet’s theorem:** There are infinitely many primes of the form $a + bn$, where $a$ and $b$ are integers satisfying $(a, b) = 1$, and either $a \neq 0, b > 0$, or $a = 0, b = 1$.

This is a classical and most important theorem which is perceived as a milestone of the study on the infinitude of some special kinds of primes. In the 1890’s, de la Vallée Poussin [22] showed further that the number of
such primes not exceeding a large number $x$ is asymptotic to $x/\varphi(b) \log x$ as $x \to \infty$, where $\varphi(.)$ is Euler’s function.

Clearly, the question of existence of infinitely many primes in arithmetic progressions can be regard as the question of existence of infinitely many prime values of linear polynomials. In 1857, Bouniakowsky [23] considered the case of nonlinear polynomials and stated a conjecture below.

**Bouniakowsky’s conjecture:** If $f(x)$ is an irreducible polynomial with integral coefficients, positive leading coefficient and degree at least 2, and there does not exist any integer $n > 1$ dividing all the values $f(k)$ for every integer $k$, then $f(x)$ is prime for an infinite number of integers $x$.

Concerning the simultaneous values of several linear polynomials, Dickson [24] stated the following conjecture in 1904:

**Dickson’s conjecture:** Let $s \geq 1$, $f_i(x) = a_i + b_i x$ with $a_i$ and $b_i$ integers, $b_i \geq 1$ (for $i = 1, ..., s$). If there does not exist any integer $n > 1$ dividing all the products $\prod_{i=1}^{s} f_i(k)$, for every integer $k$, then there exist infinitely many natural numbers $m$ such that all numbers $f_1(m), ..., f_s(m)$ are primes.

In 1958, by studying the consequences of Bouniakowsky’s conjecture and Dickson’s conjecture, A. Schinzel and W. Sierpinski [25] got the following conjecture:

**Schinzel-Sierpinski conjecture (H hypothesis):** Let $s \geq 1$, and let $f_1(x), ..., f_s(x)$ be irreducible polynomials with integral coefficients and positive leading coefficient. If there does not exist any integer $n > 1$ dividing all the products $\prod_{i=1}^{s} f_i(k)$, for every integer $k$, then there exist infinitely many natural numbers $m$ such that all numbers $f_1(m), ..., f_s(m)$ are primes.

For some details on primes represented by univariate polynomials, see also [26–32]. As for the case of primes represented by polynomials in few variables, it is very complicated and precise conjectures do not seem to have been formulated in the literature for multivariable polynomials still less univariate number-theoretic functions or multivariable number-theoretic functions. However, some notable results on the question of existence of infinitely many prime values of bivariate polynomials have been obtained by using sieve methods.

The problem goes back to Fermat who proved that there are infinitely many primes of the form $x^2 + y^2$. E. Schering [33] and H. Weber [34] proved
that every primitive binary quadratic form (positive if definite) with discriminant different from a perfect square represents infinitely many primes. In 1969, Motohashi, Yoichi [35] proved that there are infinitely many primes of type \( x^2 + y^2 + 1 \). In the early 1970’s, as an improvement of results of Bredihin B. M., Linnik Ju. V. and Motohashi Yoichi [35-37], H. Iwaniec [38, 39] obtained the significant asymptotic formula of the number of primes represented by a primitive quadratic polynomial. In 1997, Fouvry, Etienne and Iwaniec, Henryk [40] proved that there are infinitely many primes of type \( x^2 + y^2 \), where \( x \) is a prime number.

In the above-mentioned sequences of polynomial values in which it has been proved there are infinitely many primes, there are \( \gg x/(\log x)^c \) elements of the sequence up to \( x \), for some fixed \( c > 0 \). Below are two great results on some bivariate polynomials can take on infinitely many prime values.

In 1998, Friedlander, John and Iwaniec, Henryk [41] proved that \( x^2 + y^4 \) takes on a prime value for \( \sim z^{3/4}/\log z \) values \( \leq z \), which implies that there are infinitely many primes of the type \( x^2 + y^4 \). It is a "monumental breakthrough"—reviewed by Andrew Granville.

In 2001, Heath-Brown, D. R. [42] proved that \( x^3 + 2y^3 \) takes on a prime value for \( \sim z^{2/3}/\log z \) values \( \leq z \), which implies that there are infinitely many primes of the type \( x^3 + 2y^3 \). It is "one of the major landmarks of analytic number theory"—reviewed by G. Greaves.

After the work of Friedlander, John and Iwaniec, Henryk in 1998 and Heath-Brown, D. R. in 2001, maybe, the next goal of this line of research is to prove that Landau’s first conjecture [66] is true. Namely, there are infinitely many primes of the form \( x^2 + 1 \). It should be interesting to see.

As for the primes of other forms, such as Wilson primes, Wieferich primes, regular primes, NSW-primes, the primes of form \( \frac{10^r - 1}{9} \) and so on, see many papers or books, for example [43-62].

Finally, we introduce the famous work of Ben Green, Terence Tao and Tamar Ziegler to close this section. In 2004, Ben Green and Terence Tao [63] proved the following brilliant result:

**Green-Tao theorem:** The sequence of prime numbers contains arbitrarily long arithmetic progressions.
Green-Tao theorem is a great support to Dickson’s conjecture and this deep and important result has brought a very significant impact in studying primes. "It is a landmark contribution to additive number theory."—reviewed by Tamar Ziegler. Recently, they further gave important consideration and profound analysis on Dickson’s conjecture [64]. In 2006, Terence Tao and Tamar Ziegler [65] extended Green-Tao theorem to polynomial progressions via the Bergelson-Leibman polynomial Szemerédi theorem.

Based on the aforementioned rich achievements and advancements, and also due to the fact that the universe has been governed by the same laws, we believe that it is possible to establish a generic model for the problem that several multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points. It will be the main aim of this paper. Nevertheless, this is a very intractable task. It seems that the author can not finish well. Our work is only a beginning. We hope that number theorists consider further it.

Next, let’s begin with the simplest case that an irreducible univariable polynomial with integral coefficients represents infinitely many primes.

2 Necessary conditions that an irreducible univariable polynomial with integral coefficients represents infinitely many primes

In this paper, we always restrict that a $k$-variables number-theoretic function $f(x_1, ..., x_k)$ is a map from $\mathbb{N}^k$ to $\mathbb{Z}$. Moreover, we assume that $f(x_1, ..., x_k)$ is a continuous function on $\mathbb{R}^k$, where $\mathbb{R}$ is the set of all real numbers. Specially, an irreducible univariable polynomial $f(x)$ is a map from $\mathbb{N}$ to $\mathbb{Z}$. Of course, a prime number is positive. We do not consider negative primes.

Let $f(x)$ be a univariable polynomial with integral coefficients, we further assume that $f(x)$ is not a constant. Note that pairwise distinct primes are pairwise relatively prime. Thus, we get a natural necessary condition that $f(x)$ represents infinitely many primes.

**Necessary condition A:** There exists an infinite sequence of positive integers $x_1, x_2, ..., x_k, ...$ such that $f(x_1), f(x_2), ..., f(x_k), ...$ are pairwise relatively prime, moreover $f(x_1) > 1, f(x_2) > 1, ..., f(x_k) > 1, ...$.

**Proposition 1:** Necessary condition A implies the following necessary
conditions B, C and D. Moreover, B, C and D are equivalent.

**Necessary condition B:** For any positive integer \( m > 1 \), there exists a positive integer \( x \) such that \( \gcd(f(x), m) = 1 \).

**Necessary condition C:** For any positive integer \( m > 1 \), there exists a positive integer \( x \) such that \( m \) does not divide \( f(x) \). Namely, there does not exist a positive integer \( m > 1 \), such that for any positive integer \( x \), \( m \) divides \( f(x) \).

**Necessary condition D:** For any prime \( p \), there exists a positive integer \( x \) such that \( \gcd(f(x), p) = 1 \). Namely, there does not exist a prime \( p \), such that for any positive integer \( x \), \( p \) divides \( f(x) \).

**Proof of Proposition 1:** Clearly, \( A \Rightarrow B \Rightarrow C \Rightarrow D \). Next, we prove that \( D \Rightarrow B \). For any positive integer \( m > 1 \), we write \( m = \prod_{i=1}^{k} p_i^{e_i} \).

By \( D \), there exists a positive integer \( a_i \) such that \( \gcd(f(a_i), p_i^{e_i}) = 1 \) for \( 1 \leq i \leq k \). By Chinese Remainder Theorem, there exists a positive integer \( x \) such that \( x \equiv a_i \pmod{p_i^{e_i}} \). Note that \( f(x) \) is a polynomial with integral coefficients. (Here, \( f(x) \) need not be irreducible.) Hence, \( f(x) \equiv f(a_i)( \pmod{p_i^{e_i}} ) \) and \( \gcd(f(x), m) = 1 \).

Obviously, if \( f(x) \) represents infinitely many primes, then we must have the following:

**Necessary condition E:** For any positive integer \( m > 1 \), there exists a positive integer \( x \) such that \( \gcd(f(x), m) = 1 \) and \( f(x) > 1 \).

**Necessary condition F:** For any positive integer \( m > 1 \), there exists a positive integer \( x \) such that \( f(x) > 1 \) and \( m \) does not divide \( f(x) \).

**Necessary condition G:** For any prime \( p \), there exists a positive integer \( x \) such that \( \gcd(f(x), p) = 1 \) and \( f(x) > 1 \).

**Proposition 2:** Necessary conditions A and E are equivalent, however, they and F (resp. G) are not always equivalent.

**Proof of Proposition 2:** Let \( f(x) \) be a polynomial with integral coefficients. If the leading coefficient of \( f(x) \) is positive, then A, E, F and G are equivalent by the idea of proof in Proposition 1. Now we consider the case that the leading coefficient of \( f(x) \) is negative. Clearly, in this case, we still have: \( A \Rightarrow E \). Next, we prove that \( E \Rightarrow A \). Since the leading coefficient of \( f(x) \) is negative, hence \( f(x) \) at most represents finitely many
positive integers. Denote the product of these positive integers by $M$. By Necessary conditions $E$, let $m = 2$, there exists a positive integer $x$ such that $f(x) > 1$ and gcd($f(x), 2$) = 1. This implies that $f(x)$ can represent a positive integer greater than 1. Therefore, $M > 1$. By $E$ again, there exists a positive integer $x$ such that $f(x) > 1$ and gcd($f(x), M$) = 1. But $M$ is the product of all positive integers which can be represented by $f(x)$. It is impossible. So, $A$ and $E$ are equivalent. By considering $f(x) = -x^2 + 6$, it is easy to prove that $f(x)$ represents infinitely many positive integers and $f(x) = -x^2 + 6$ does not imply $A$.

Based on Proposition 2, Necessary condition $A$ and $E$ will become our main interest in future study.

**Corollary 1:** Let $f(x)$ be a polynomial with integral coefficients, then $A$ and the following condition are equivalent: the leading coefficient of $f(x)$ is positive, and there does not exist any integer $n > 1$ dividing all the values $f(k)$ for every integer $k$.

**Remark 1:** Some people call Necessary condition $C$ (resp. $D$) Bunyakovsky’s property.

**Remark 2:** Our work in this section shows that there are several equivalent forms of Bouniakowsky’s conjecture. For instance, if $f(x)$ is an irreducible polynomial with integral coefficients, and for any positive integer $m > 1$, there exists a positive integer $a$ such that gcd($f(a), m$) = 1 and $f(a) > 1$, then $f(x)$ represents infinitely many primes.

**Remark 3:** Generalizing our work to the generic cases, one could obtain several equivalent forms of Dickson’s conjecture even Schinzel-Sierpinski’s Conjecture. For example, let $s \geq 1$, and let $f_1(x), ..., f_s(x)$ be irreducible polynomials with integral coefficients, if there exists an infinite sequence of positive integers $x_1, x_2, ..., x_k, ...$ such that $\prod_{i=1}^{s} f_i(x_1), ..., \prod_{i=1}^{s} f_i(x_k), ...$ are pairwise relatively prime, moreover $f_i(x_1) > 1, f_i(x_2) > 1, ..., f_i(x_k) > 1, ...$ for $i = 1, ..., s$, then there exist infinitely many natural numbers $m$ such that all numbers $f_1(m), ..., f_s(m)$ are primes.

We do not know Dickson, Schinzel and Sierpinski whether noticed these equivalent forms. It seems that they focused on Bunyakovsky’s property and believed that if a univariable polynomial $f(x)$ with integral coefficients and the positive leading coefficient has Bunyakovsky’s property, then $f(x)$ represents infinitely many primes. Namely, for any univariable polynomial $f(x)$
with integral coefficients and the positive leading coefficient, Bunyakovsky’s property of \( f(x) \) is the sufficient and necessary condition that \( f(x) \) represents infinitely many primes.

Unfortunately, these conjectures are open for many years. It is time to reconsider them. On one hand, one maybe ask: is Bunyakovsky’s property of \( f(x) \) enough to determine that \( f(x) \) represents infinitely many primes? On the other hand, how to generalize Schinzel-Sierpinski’s Conjecture to the cases of multivariable polynomials with integral coefficients even multivariable number-theoretic functions?

Let \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) be \( s \) multivariable number-theoretic functions from \( \mathbb{N}^k \) to \( \mathbb{Z} \). Assuming that \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points \( (x_1, ..., x_k) \).

Now we generalize Necessary condition A to the generic case as follows.

**Necessary condition H:** There exists an infinite sequence of integral points \( (x_{11}, ..., x_{k1}), ..., (x_{1i}, ..., x_{ki}), ... \) such that \( \prod_{j=1}^{s} f_j(x_{11}, ..., x_{k1}), ..., \prod_{j=1}^{s} f_j(x_{1i}, ..., x_{ki}), ... \) are pairwise relatively prime and \( f_j(x_{11}, ..., x_{ki}) > 1 \) for each \( i \) and \( j \).

Similarly, Necessary condition H and the following necessary condition I are equivalent:

**Necessary condition I:** For any positive integer \( m > 1 \), there exists an integral point \( X = (x_1, ..., x_k) \) such that \( \gcd(\prod_{i=1}^{s} f_i(X), m) = 1 \) and \( f_i(X) > 1 \) for \( 1 \leq i \leq s \).

As we aforementioned, Necessary condition H should be viewed as a natural necessary condition. Based on this observation, we believe that there is always a common necessary condition that any multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points. Surely, at least, it is not weaker than the natural necessary condition and can be called the maximum necessary condition. Once adding appropriate conditions, it perhaps leads to the sufficient condition that multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points. Therefore, it is possible to generalize Schinzel-Sierpinski’s Conjecture.

We also find that using the natural necessary condition H (resp. I) is more convenient than using Bunyakovsky’s property when we treat the multivariable cases, in which we have not the definition of leading coef-
cient even irreducible. Moreover, our work will show that the natural necessary condition perhaps is the maximum necessary condition when $f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k)$ are multivariable polynomials with integral coefficients. For details, see next several sections.

3 Find new necessary conditions

Why do we need to find new necessary conditions? Note that the number-theoretic function $2^{2^x} + 1$ implies the natural necessary condition. Numbers of the form $2^{2^x} + 1$ are called Fermat numbers. Primes of the form $2^{2^x} + 1$ are Fermat primes. Eisenstein proposed as a problem in 1844 the proof that there are infinite number of Fermat primes [2]. Nevertheless, Hardy and Wright [67] conjectured that the number of Fermat primes is finite, although they did not give any reasons and explanations. By factoring Fermat number, many people believe that the conjecture in [67] holds. So far, people do not find a new Fermat primes except for the first four Fermat primes as follows: 5, 17, 257, 65537. If let $x = 0$, then 3 is viewed as a Fermat prime. But we restricted that a number-theoretic function is a map from $N$ to $Z$ in Section 2. Therefore, here, we do not consider 3.

Historically, the problem that the number-theoretic function $f(x) = 2^{2^x} + 1$ represents primes were first studied by Pierre de Fermat, who conjectured that $f(x) = 2^{2^x} + 1$ are prime for all $x \in N \cup \{0\}$. Unfortunately, in 1732, his conjecture was refuted by Leonhard Euler. Euler showed that $f(5) = 4294967297 = 641 \times 6700417$. Euler proved that every prime factor of $f(x)$ must have the form $k \times 2^{x+1} + 1$. For $x = 5$, this means that the only possible factors are of the form $64k + 1$. Euler found the factor 641 when $k = 10$. Lucas refined Euler’s result: Any prime divisor of $f(x)$ is of the form $k \times 2^{x+2} + 1$ whenever $x > 1$.

According to R. P. Brent [92]: "The complete factorization of Fermat numbers $f(6), f(7), ..., $ has been a challenge since Euler’s time. Because the $f(x)$ grow rapidly in size, a method which factors $f(x)$ may be inadequate for $f(x+1)$, No Fermat primes larger than $f(4)$ are known, and a probabilistic argument makes it plausible that only a finite number of $f(x)$ (perhaps only 3, 5, 17, 257, 65537) are prime." As of 2008 it is known that $f(x)$ is composite for $5 \leq x \leq 32$, although complete factorizations of $f(x)$ are known only for $0 \leq x \leq 11$. Below is the list of complete factorizations:

$$f(6) = 18446744073709551617 = 274177 \times 67280421310721 \ [88].$$
\[ f(7) = 59649589127497217 \times p^{22}, \text{ where } p^{22} \text{ is a prime which has } 22 \text{ decimal digits} \ [89]. \]
\[ f(8) = 1238926361552897 \times p^{62} \ [90]. \]
\[ f(9) = 2424833 \times p^{49} \times p^{99} \ [91]. \]
\[ f(10) = 45592577 \times 6487031809 \times p^{40} \times p^{252} \ [92]. \]
\[ f(11) = 319489 \times 974849 \times p^{21} \times p^{22} \times p^{564} \ [92]. \]

Thus, if the conjecture in [67] holds, then the natural necessary condition \( A \) is too weak to make us know more information on the infinitude of some special kinds of primes, and it should be strengthened. Here, we always assume that there is the maximum necessary condition that any multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points.

Another reason, although it seems reluctant, we give it as follows: Considering the infinitude, the technical definition of limit occurred to us. If \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points \((x_1, ..., x_k)\), then there should exist a constant \( c \) such that for every positive integer \( m > c \), there exists an integral point \((x_1, ..., x_k)\) such that \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) are coprime to \( m \) simultaneously. Based on some heuristic observations, for example, by refining Necessary condition \( E \) as follows: for a sufficiently large constant \( c \) and for any positive integer \( m > c \), there exists a positive integer \( x \) such that \( \gcd(f(x), m) = 1 \) and \( m > f(x) > 1 \), we would like to restrict the values of \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) in

\[
Z^*_m = \{ x \in \mathbb{N} | 1 \leq x < m, \gcd(x, m) = 1 \}
\]
in order to know more information that \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) maybe take on infinitely many prime values.

The third reason, let’s look back Necessary condition \( E \) again. Necessary condition \( E \) states that for an integral polynomial \( f(x) \) and for any positive integer \( m > 1 \), there exists a positive integer \( x \) such that \( \gcd(f(x), m) = 1 \) and \( f(x) > 1 \). Therefore, if for such an integral polynomial \( f(x) \), Necessary condition \( E \) satisfies, then there must exist the least positive integer \( n \) such that \( \gcd(f(n), m) = 1 \) and \( f(n) > 1 \). Denote this least positive integer \( n \) by \( S_f(m) \), then for a sufficiently large constant \( c \) and for any positive integer \( m > c \), \( S_f(m) < \left( \frac{m}{L(f)} \right)^{\frac{1}{d}} \), where \( d \) is the degree of \( f(x) \), and, \( L(f) \) is the leading coefficient of \( f(x) \).

For given polynomial \( f(x) \) and every sufficiently large \( m \), Estimating the
upper bound of $S_f(m)$ is an interesting question. For example, let $f(x) = x$, then $S_f(m) < m^{1/2}$ when $m > 30$ by Bonse’s inequalities [68-69]. Moreover, this result can be refined as follows: for any given positive integer $k$, there is a constant $c_k$, when $m > c_k$, $S_f(m) < m^{1/3}$. As another example, generalizing $f(x)$ to the case of number-theoretic functions and defining similarly $S_f(m)$, let $f(x) = 2^x - 1$, then $S_f(m) < \log_2 m$ when $m > 21$ [70]. In fact, when $f(x) = 2^x - 1$, the meaning of $S_f(m)$ is definite. when $\gcd(m, 2) = 1$, $\gcd(m, 2^{x(m)}+1 - 1) = 1$; when $\gcd(m, 2) = 2$, we write $m = 2^t x$ with $\gcd(t, 2) = 1$, then $\gcd(m, 2^{x(t)}+1 - 1) = 1$. When $f(x) = 2^{2^x} + 1$, for any positive integer $m$, we have $\gcd(m, 2^{2^m} + 1) = 1$ because any prime divisor $p$ of $2^{2^m} + 1$ is of the form $k2m+2 + 1$ whenever $m$ is greater than one. So $S_f(m) \leq m$ when $f(x) = 2^{2^x} + 1$.

$S_f(m)$ also can be generalized to the generic case: Let $f_1(x_1, ..., x_k)$, ..., $f_s(x_1, ..., x_k)$ be multivariable number-theoretic functions. If for any positive integer $m$, there exists an integral point $(y_1, ..., y_k)$ such that $f_1(y_1, ..., y_k) > 1$, ..., $f_s(y_1, ..., y_k) > 1$ are all coprime to $m$, then there must exist the shortest integral vector $X = (x_1, ..., x_k)$ such that $f_1(X) > 1$, ..., $f_s(X) > 1$ are all coprime to $m$. Denote this shortest integral vector by $S_{f_1, ..., f_s}(m)$. Then, $S_{f_1, ..., f_s}(m)$ is the generalization of $S_f(m)$.

Estimating the upper bound of $S_f(m)$ also leads to strengthen Necessary condition E as the aforementioned.

Certainly, making the decision of strengthening Necessary condition E should always take a risk. We must verify the sequences of functional values in which it has been proved there are infinitely many primes implies that there exist a constant $c$ such that for every positive integer $m > c$, there exist an integral point such that those corresponding functional values are all in $Z^*_m$.

To begin with, noticed that if $a > 1$ is the smallest integer such that $\gcd(a, m) = 1$, then $a$ is a prime when $m > 2$. Namely, there exists a constant $c = 2$ such that for every positive integer $m > c$, there is a prime in $Z^*_m$. Thus, we proved the case of $s = k = 1$ with $f(x) = x$.

In additionally, note that $\pi_{a, b, x} \sim \frac{x}{\varphi(b) \log x}$ as $x \to \infty$, where $\pi_{a, b, x}$ is the number of prime of the form $a + bx$ with $b > 0, \gcd(a, b) = 1$. This implies that there is a positive constant $c$, when $m > c$, we have $\pi_{a, b, x} > 1 + \log_2 m$. But $m$ has at most $\lceil \log_2 m \rceil$ distinct prime factors. Hence, there is always a prime of the form $a + bx$ in $Z^*_m$ and we proved the case $s = k = 1$ with
\( f(x) = ax + b. \)

Last but not the least, using the similar method, one can show respectively that there is a positive constant \( c \), when \( m > c \), the number of prime of the form \( f(x, y) = x^2 + y^2 + 1 \), \( f(x, y) = x^3 + 2y^3 \), \( f(x, y) = x^2 + y^4 \) and so on \( > 1 + \log_2 m \) in the cases of \( s = 1 \) and \( k = 2 \). It follows immediately the desired consequences. Combining with the above discussions, one could conjecture the following:

**Conjecture 1:** Let \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) be \( s \) multivariable number-theoretic functions. If \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points \((x_1, ..., x_k)\), then there is always a constant \( c \) such that for every positive integer \( m > c \), there exists an integral point \((y_1, ..., y_k)\) such that \( f_1(y_1, ..., y_k) > 1, ..., f_s(y_1, ..., y_k) > 1 \) are all in \( \mathbb{Z}_m^* \).

**Remark 4:** The conjecture is only a necessary condition not a sufficient condition. For example, by Bonse’s inequalities [68-69], one can prove that every positive integer \( m > 30 \), there are positive integers of the form \( x^2 \) in \( \mathbb{Z}_m^* \). But, \( x^2 \) never represents a prime.

Now, we prove that Conjecture 1 implies that there are only finitely many Fermat primes. In fact, if there are infinitely many Fermat primes, then by Conjecture 1, there is always a constant \( c \) such that for every positive integer \( m > c \), there exists a positive integer \( n \) such that \( f(n) = 2^{2^n} + 1 \) in \( \mathbb{Z}_m^* \). Let \( m = \prod_{i=0}^{k-1} f(i) = \prod_{i=0}^{k-1} (2^{2^i} + 1) > c \). Clearly, there is always such a positive integer \( k \) because \( c \) is a constant. Hence, we must have \( n \geq k \) when \( m = \prod_{i=0}^{k-1} (2^{2^i} + 1) \) and \( 2^{2^n} + 1 \) in \( \mathbb{Z}_m^* \). Thus \( f(n) \geq f(k) \). Note that \( f(k) = m + 2 \). Therefore, we have \( f(n) \geq m + 2 \). But, it is impossible since \( 2^{2^n} + 1 \) in \( \mathbb{Z}_m^* \) implies that \( 2^{2^n} + 1 < m \) by the meaning of notation \( \mathbb{Z}_m^* \).

In like manner, this follows immediately Conjecture 1 which implies also that there are only finitely many primes of the form \( n^a + 1 \). Based on the same reason, we maybe foresee that there are only finitely many prime values for several iterative functions. For example, maybe, there are only finitely many prime numbers in the sequence: \( p_1 = 2^2 - 1 = 3, p_2 = 2^3 - 1 = 7, p_3 = 2^7 - 1 = 127, p_4 = 2^{127} - 1, p_5 = 2^{p_4} - 1, \ldots \)

A clear sense is of that such a sequence is so sparse that it can not guarantee that there is always a constant \( c \) such that for every positive integer \( m > c \), there exists a positive integer \( n \) such that \( p_n \) is in \( \mathbb{Z}_m^* \).
Besides, we demand Conjecture 1 to test that the infinitude of some special kinds of primes such as Twins primes, safe primes (co-Sophie-Germain primes), Mersenne primes and so on, which are markedly infinitely many. In [70], we proved that the several number-theoretic functions \( f(x) = x, g(x) = x + 2; f(x) = x, g(x) = 2x + 1; f(x) = x, g(x) = 2^x - 1 \) which perhaps represent simultaneously infinitely many primes imply Conjecture 1. Moreover, by the following quantitative form of Schinzel-Sierpinski’s Conjecture—Bateman-Horn’s conjecture [71], if \( f_1(x), ..., f_s(x) \) are polynomials with integral coefficients, and represent simultaneously infinitely many primes, then Conjecture 1 holds.

**Bateman-Horn’s conjecture:** Let \( s \geq 1 \), and let \( f_1(x), ..., f_s(x) \) be irreducible polynomials with integral coefficients and positive leading coefficient. If there does not exist any integer \( n > 1 \) dividing all the products \( \prod_{i=1}^{s} f_i(k) \), for every integer \( k \), and for every integer \( m > 1 \), the number \( Q(m) \) of integers \( 1 \leq n \leq m \) such that \( f_1(n), ..., f_s(n) \) are all primes is about

\[
C_{f_1, ..., f_s} \frac{1}{\prod_{i=1}^{s} d_i} \sum_{n=2}^{m} \frac{1}{(\log n)^s} \sim C \frac{1}{\prod_{i=1}^{s} d_i} \int_{2}^{m} \frac{dt}{(\log t)^s},
\]

where \( d_i = \deg f_i(x) \), \( C = C_{f_1, ..., f_s} = \prod_p \frac{1 - \psi(p)/p}{1 - 1/p} \) is a very complicated constant and \( \psi(p) \) is the number of solutions \( x, 0 \leq x \leq p - 1 \), of the congruence \( f_1(x) ... f_s(x) \equiv 0 \pmod{p} \).

Roughly speaking, the number \( Q(m) \) is about \( C \frac{1}{\prod_{i=1}^{s} d_i (\log m)^s} \), which of course, implies Conjecture 1 when \( f_1(x), ..., f_s(x) \) are polynomials with integral coefficients. Namely, if polynomials \( f_1(x), ..., f_s(x) \) with integral coefficients represent simultaneously primes for infinitely many integers \( x \), then there is always a constant \( c \) such that for every positive integer \( m > c \), there exists an integers \( y \) such that \( f_1(y) > 1, ..., f_s(y) > 1 \) are all in \( \mathbb{Z}_m^* \).

**Remark 5:** Friedlander John and Granville Andrew [79-81] showed that Bateman-Horn’s asymptotic formula does not always hold and there are infinitely many different polynomials of given degree which take either significantly more or significantly less prime values than expected. However, we believe that Conjecture 1 holds without a proviso when \( f_1(x), ..., f_s(x) \) in Conjecture 1 are irreducible polynomials with integral coefficients and positive leading coefficient. In our another paper Notes on Dickson’s Conjecture, we have proved strictly that Conjecture 1 holds when \( f_1(x), ..., f_s(x) \) in Conjecture 1 are all linear polynomials with integral coefficients and positive leading coefficient, for the details, see [101]. Furthermore, in [101],
we generalize Dickson’s Conjecture to the multivariable case or a system of affine-linear forms on \( N^k \). In [102], we give Dickson’s conjecture on \( Z^n \) and obtain an equivalent form of Green-Tao’s conjecture [64].

Anyway, like the \( \varepsilon - \delta \) definition of limit, Conjecture 1 maybe provides us with another mathematical description for the infinitude of some special kinds of primes.

Conjecture 1 leads to the generalizations of Euler’s function and the prime-counting function, see Section 4. It also yields an analogy of Chinese Remainder Theorem, see Section 5. But, Conjecture 1 is based on the finiteness of Fermat primes which is unproved yet. Everyone is unwilling to see its unreliable basis. So, we only hope that one keeps it in his mind. Conjecture 1 maybe will lead to some correct conjectures.

4 Generalizations of Euler’s function and the prime-counting function

Euler’s totient function \( \varphi(n) \) is a very important number-theoretic function and defined to be the number of positive integers \( x \) less than \( n \) which are relatively prime to \( n \). \( \varphi(n) = \#\{x \in \mathbb{N} | \gcd(x, n) = 1, x < n\} = n \prod_{p | n} (1 - 1/p) \). If we look upon \( x \) as the value of number-theoretic function \( f(x) \), then when \( f(x) = x \), we have \( \varphi(n) = \#\{f(x) \in \mathbb{N}, \gcd(f(x), n) = 1, f(x) < n | x \in \mathbb{N}\} \). Thus, let \( f(x) \) be a number-theoretic function, then one can generalize Euler’s totient function as follows: \( \Phi_f(n) = \#\{f(x) \in \mathbb{Z}_n^* | x \in \mathbb{N}\} \).

More generally, let \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) be \( s \) multivariable number-theoretic functions from \( N^k \) to \( \mathbb{Z} \). one can generalize further \( \Phi_f(n) \) as follows:

\[
\Phi_{f_1, ..., f_s}(n) = \#\{f_1(X) \in \mathbb{Z}_n^*, ..., f_s(X) \in \mathbb{Z}_n^* | X = (x_1, ..., x_k) \in N^k\}.
\]

Now, we generalize another important number-theoretic function — the prime-counting function \( \pi(x) \), which is the number of primes less than or equal to some real number \( x \). Note that pairwise distinct primes are pairwise relatively prime. Consider the number-theoretic function \( f(x) = x \). For any given positive integer \( x > 1 \), consider a special sub-set \( H \) of \( \{1, 2, ..., x\} \) as following: \( \forall a \in H, \text{ we have } a > 1, \text{ and } \forall a \neq b \in H, \text{ we also have } \gcd(a, b) = 1 \). Namely, the elements of \( H \) are pairwise relatively prime.

Denote the set of all such sub-sets of \( \{1, 2, ..., x\} \) by \( M \). Thus, \( M = \{H \subseteq \{1, 2, ..., x\} | \forall a \neq b \in H, \gcd(a, b) = 1, \forall a \in H, a > 1\} \). Clearly,
\[ \pi(x) = \max_{H \subseteq M} \{ \#H \}. \] Namely, \( \pi(x) \) can be viewed as the largest among the cardinality of all sub-sets (in which each element exceeds 1 and pairwise distinct elements are pairwise relatively prime) of \{1, 2, ..., x\}.

Now, let \( f(x) \) be a generic number-theoretic function. Let \( H \) be any sub-set of the image of \( f \). Consider the set \( M = \{ H \subseteq \{1, 2, ..., x\} \mid \forall f(a) \in H, f(a) > 1, \forall f(a) \neq f(b) \in H, \gcd(f(a), f(b)) = 1 \} \). Let \( \Pi_f(x) = \max_{H \subseteq M} \{ \#H \} \). Then, \( \Pi_f(x) \) can be viewed as the generalization of \( \pi(x) \). Denote the number of distinct prime factors of \( x \) by \( \omega(x) \). If we have \( \Pi_f(m) > \omega(m) \), then there is a positive integer \( a \) such that \( f(a) \) is in \( \mathbb{Z}_m^* \), and \( \Phi_f(m) \geq 1 \).

More generally, let \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) be \( s \) multivariable number-theoretic functions, consider the set \( M = \{ H \subseteq \{1, 2, ..., x\} \mid \forall f(X) \in H, f(X) > 1, \forall f_1(X) \neq f_1(Y) \in H, ..., f_s(X) \neq f_s(Y) \in H, \gcd(\prod_{i=1}^{s} f_i(X), \prod_{i=1}^{s} f_i(Y)) = 1 \} \), where integral points \( X, Y \) should be viewed as vectors.

Let \( \Pi_{f_1, ..., f_s}(x) = \max_{H \subseteq M} \{ \#H \} \). Then, \( \Pi_{f_1, ..., f_s}(x) \) can be viewed as the generalization of \( \pi(x) \). Thus, if \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points \( X = (x_1, ..., x_k) \), then, we must have \( \Pi_{f_1, ..., f_s}(x) \to \infty \) as \( x \to \infty \).

Similarly, if \( \Pi_{f_1, ..., f_s}(m) > \omega(m) \) then \( \Phi_{f_1, ..., f_s}(m) \geq 1 \).

Using sieve theory [72-75], one could obtain some asymptotic formulae of \( \Phi_{f_1, ..., f_s}(m) \) and \( \Pi_{f_1, ..., f_s}(m) \). This should become the subject of future publications.

5 An analogy of Chinese Remainder Theorem

Chinese Remainder Theorem [76] states that for given a system of simultaneous linear congruences \( x \equiv a_i \pmod{n_i} \) for \( i = 1, 2, ..., k \) and for which \( n_i \) are pairwise relatively prime positive integers, where \( a_i \) are integers, then this linear system has a unique solution modulo \( n = \prod_{i=1}^{k} n_i \). Particularly, for \( i = 1, 2, ..., k \), if \( \gcd(a_i, n_i) = 1 \), then, this linear system has a unique solution \( x \) in \( \mathbb{Z}_n^* \).

Chinese Remainder Theorem is the greatest theorem in ancient China in my eyes. And it is a very theorem which was named after a unique nation. It is one of the jewels of mathematics and contains in a third-century AD book The Mathematical Classic by Sun Zi by Chinese mathematician Sun Tzu. It reflects a perfect combination of beauty and utility. The famous Fast Fourier
Transform can be even viewed as a special case of its. That is because Chinese Remainder Theorem can be generalized over generic rings and Fourier Transform formula \( f \rightarrow (f(\omega^0), ..., f(\omega^{n-1}))(f' \rightarrow \frac{1}{n}(f'(1), ..., f'((\omega^{-n+1}))) \) is exactly viewed as the isomorphism \( C[x]/x^n - 1 \simeq C[x]/x - \omega \times ... \times C[x]/x - \omega^n \) which is essentially Chinese Remainder Theorem. People said that "it is difficult to image what would happen if there was no Fast Fourier Transform in modern communications". This will enable us to learn better the significance of Chinese Remainder Theorem. In this section, we will give an analogy of its.

Let us look back the proof of Theorem 2 in [70]. In order to prove that there is always a constant \( c \), such that when \( n > c \), there exists \( x \in Z^*_a \) and \( 2x + 1 \in Z^*_a \) with \( x > 1 \), our method is to prove firstly that there exists \( x \in Z^*_a \) and \( 2x + 1 \in Z^*_a \), to prove secondly that there exists \( y \in Z^*_b \) and \( 2y + 1 \in Z^*_b \) with \( \gcd(a, b) = 1 \), to prove lastly that there exists \( z \in Z^*_ab \) and \( 2z + 1 \in Z^*_ab \). This is exactly viewed as Chinese Remainder Theorem which states essentially that if there is an integer in \( Z^*_a \), and there is an integer in \( Z^*_b \) with \( \gcd(a, b) = 1 \), there is an integer in \( Z^*_ab \). We hope certainly that this can be generalized to generic cases as follows.

**An analogy of Chinese Remainder Theorem:** Let \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) be multivariable polynomials with integral coefficients. If \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points, and if \( \gcd(a, b) = 1 \) and there exist integral point \((x_1, ..., x_k)\) and \((y_1, ..., y_k)\) such that \( f_1(x_1, ..., x_k) > 1, ..., f_s(x_1, ..., x_k) > 1 \) are all in \( Z^*_a \) and \( f_1(y_1, ..., y_k) > 1, ..., f_s(y_1, ..., y_k) > 1 \) are all in \( Z^*_b \), then there exists an integral point \((z_1, ..., z_k)\) such that \( f_1(z_1, ..., z_k) > 1, ..., f_s(z_1, ..., z_k) > 1 \) are all in \( Z^*_ab \).

Here, we must explain why the condition that "\( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points" is necessary. That is because if the number of primes is finite, then Chinese Remainder Theorem is false [77], namely, Chinese Remainder Theorem implies Euclid’s second theorem. In fact, \( f(x) = x^3 + 1 \) has not this property because it does not represent infinitely many primes. For example, \( f(1) = 2 \in Z^*_9 \) and \( f(2) = 9 \in Z^*_9 \), but there is not a positive integer \( x \) such that \( f(x) = x^3 + 1 \in Z^*_9 \). Thus, we also obtain another necessary condition that \( f_1(x_1, ..., x_k), ..., f_s(x_1, ..., x_k) \) represent simultaneously primes for infinitely many integral points.

We also find that \( f(n) = 2^n - 1 \) satisfies this necessary condition [70].
However, \( f(n) = 2^{2^n} + 1 \) does not satisfy this necessary condition. For example, \( 5 \in \mathbb{Z}_{51}^* \) and \( 17 \in \mathbb{Z}_{5 \times 257}^* \). But, there is not a Fermat number in \( \mathbb{Z}_{51 \times 5 \times 257}^* \). Does it imply possibly that there are only finitely many Fermat primes again? The answer perhaps is no, at least we have a reason, due to the fact that we only consider the case that \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) are multivariable polynomials with integral coefficients. We do not consider the case of generic number-theoretic functions. In fact, in the case of generic number-theoretic functions, we have not such an analogy of Chinese Remainder Theorem. For instance, let \( f(x) = \begin{cases} 2, & 1 \leq x \leq 2 \\ 3, & 3 \leq x \leq 39 \\ \lfloor x/3 \rfloor, & x \geq 40 \end{cases} \). Clearly, \( f(x) \) represents infinitely many primes. But it has not the similar property of Chinese Remainder Theorem. For instance, \( f(1) = 2 \in \mathbb{Z}_3^* \) and \( f(3) = 3 \in \mathbb{Z}_4^* \), but there is not a positive integer \( x \) such that \( f(x) \in \mathbb{Z}_{12}^* \). (Is there a counterexample of the analogy of Chinese Remainder Theorem when \( f(x_1, \ldots, x_k) \) is a continuous function on \( \mathbb{R}^k \)?) By this example, one maybe ask: is Conjecture 1 true? We do not assert the answer now. But it is possible to lead to generalize Schinzel-Sierpinski’s Conjecture.

**Remark 6:** Conjecture 1 and the analogy of Chinese Remainder Theorem should be equivalent when \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) in Conjecture 1 are multivariable polynomials with integral coefficients.

**Remark 7:** Note that if there are primes in \( \mathbb{Z}_a^* \) and \( \mathbb{Z}_b^* \) respectively, then there is primes in \( \mathbb{Z}_{ab}^* \) when \( \gcd(a, b) = 1 \). Similarly, if multivariable polynomials \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) with integral coefficients represent simultaneously primes for infinitely many integral points, and if \( \gcd(a, b) = 1 \) and there are integral points \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) such that \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) in \( \mathbb{Z}_a^* \) are all primes, and \( f_1(y_1, \ldots, y_k), \ldots, f_s(y_1, \ldots, y_k) \) in \( \mathbb{Z}_b^* \) are all primes, then maybe, there exists an integral point \((z_1, \ldots, z_k)\) such that \( f_1(z_1, \ldots, z_k), \ldots, f_s(z_1, \ldots, z_k) \) in \( \mathbb{Z}_{ab}^* \) are all primes. This is a very interesting problem on the analogy of Chinese Remainder Theorem, and in the simple case, we have: if \( \gcd(m, n) = 1 \), \( \gcd(a, b) = 1 \) with \( b > 1 \) and \( a + bx \in \mathbb{Z}_m^* \) is prime, and \( a + by \in \mathbb{Z}_n^* \) also is prime, then there exists a prime of the form \( a + bz \) in \( \mathbb{Z}_{mn}^* \).

**Remark 8:** When the paper is written here, we feel that it is not difficult to generalize Schinzel-Sierpinski’s Conjecture to the case of multivariable polynomials with integral coefficients. It will be not a pure speculation anymore and become a somewhat reasonable conjecture. For details, see Section 6.
6 Generalizing Schinzel-Sierpinski’s Conjecture to the case of multivariable polynomials

A possible generalization of Schinzel-Sierpinski’s Conjecture is the following:

Let \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) be multivariable polynomials with integral coefficients, if \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) are irreducible over \( \mathbb{Q}[x_1, \ldots, x_k] \), and there is always a constant \( c \) such that for every positive integer \( m > c \), there exists an integral point \((y_1, \ldots, y_k)\) such that \( f_1(y_1, \ldots, y_k) > 1, \ldots, f_s(y_1, \ldots, y_k) > 1 \) are all in \( \mathbb{Z}_m^* \), then \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) represent simultaneously primes for infinitely many integral points \((x_1, \ldots, x_k)\).

However, we do not do this. On one hand, there are many puzzles on the factorization in \( \mathbb{Q}[x_1, \ldots, x_k] \), and maybe, the word ”irreducible” can not explain more. On the other hand, in order to generalize it to the more generic case such as number-theoretic functions, we need a dependable condition to replace the ”irreducible” condition.

For this goal, let us look back on the work of M. Ram Murty [78]: let \( f(x) = \sum_{i=0}^{m} a_i x^i \) be a polynomial of degree \( m \) in \( \mathbb{Z}[x] \) and set \( H = \max_{0 \leq i \leq m-1} |a_i/a_m| \). If \( f(n) \) is prime for some integer \( n \geq H + 2 \), then \( f(x) \) is irreducible in \( \mathbb{Z}[x] \). Based on the work of M. Ram Murty and our aforehand analysis, we give the following conjecture.

**Conjecture 2:** Let \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) be multivariable polynomials with integral coefficients, if there is a positive integer \( c \) such that for every positive integer \( m \geq c \), there exists an integral point \((y_1, \ldots, y_k)\) such that \( f_1(y_1, \ldots, y_k) > 1, \ldots, f_s(y_1, \ldots, y_k) > 1 \) are all in \( \mathbb{Z}_m^* \), and there exists an integral point \((z_1, \ldots, z_k)\) such that \( f_1(z_1, \ldots, z_k) \geq c, \ldots, f_s(z_1, \ldots, z_k) \geq c \) are all primes, then \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) represent simultaneously primes for infinitely many integral points \((x_1, \ldots, x_k)\).

**Remark 9:** Let \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) be multivariable polynomials with integral coefficients from \( N^k \) to \( \mathbb{Z} \), then the following conditions are equivalent:

\((H)\): If there exists an infinite sequence of integral points \((x_{11}, \ldots, x_{k1}), \ldots, (x_{1i}, \ldots, x_{ki})\) such that \( \prod_{j=1}^{s} f_j(x_{11}, \ldots, x_{k1}), \ldots, \prod_{j=1}^{s} f_j(x_{1i}, \ldots, x_{ki}) \) are pairwise relatively prime and \( f_j(x_{11}, \ldots, x_{ki}) > 1 \) for each \( i \) and \( j \).

\((J)\): If there is a positive integer \( c \) such that for every positive integer \( m \geq c \), there exists an integral point \((y_1, y_k)\) such that \( f_1(y_1, \ldots, y_k) > 1 \)
Thus, we deduce an equivalent form of Schinzel-Sierpinski conjecture: Let \( s \geq 1 \), and let \( f_1(x), \ldots, f_s(x) \) be irreducible polynomials with integral coefficients, if there is a positive integer \( c \) such that for every positive integer \( m \geq c \), there exists a positive integer \( a \) such that \( f_1(a) > 1, \ldots, f_s(a) > 1 \) are all in \( \mathbb{Z}_m^* \), then there exist infinitely many natural numbers \( m \) such that all numbers \( f_1(m), \ldots, f_s(m) \) are primes.

Conjecture 2 should view as the sufficient and necessary condition that multivariable polynomials \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) with integral coefficients represent infinitely many primes.

**Remark 10:** Can Conjecture 2 be generalized similarly to the generic case of number-theoretic functions? The author would like to keep it in mind because this problem is unattackable now. For example, let \( f_1 = 3 \cdot 2^{x-1} - 1, f_2 = 3 \cdot 2^x - 1, f_3 = 9 \cdot 2^{2x-1} - 1 \), then, do \( f_1, f_2, f_3 \) represent simultaneously primes for infinitely many \( x \)? Another example, do \( g_1 = 8x + 5, g_2 = x^3 + 2, g_3 = 2^x - 1 \) represent simultaneously primes for infinitely many \( x \)? Particularly, does \( h(x) = 2^x + x \) represent primes for infinitely many \( x \)? And so on.

In the author’s eyes, it perhaps is easy to give a sufficient condition that multivariable number-theoretic functions \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) represent infinitely many primes, but it is difficult to give its sufficient and necessary condition. On this problem, we will try to present a plausible proposal in Section 9.

**Remark 11:** Conjecture 2 leads to the following significative problem:

Let \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) be number-theoretic functions. Assume that there is a positive integer \( c \) such that for every positive integer \( m \geq c \), there exists an integral point \( (y_1, \ldots, y_k) \) such that \( f_1(y_1, \ldots, y_k) > 1, \ldots, f_s(y_1, \ldots, y_k) > 1 \) are all in \( \mathbb{Z}_m^* \), and there exists an integral point \( (z_1, \ldots, z_k) \) such that \( f_1(z_1, \ldots, z_k) \geq c, \ldots, f_s(z_1, \ldots, z_k) \geq c \) are all primes. Since \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) represent simultaneously primes for some integral point \( (x_1, \ldots, x_k) \), hence we can denote the least prime represented simultaneously by \( P_{f_1, \ldots, f_s} \). A significative problem is to estimate the upper bound of \( P_{f_1, \ldots, f_s} \).

Historically, this problem is one of important topics in Number Theory. In the simplest case, denote \( p(l, k) \) the least prime in the arithmetic pro-
gression \( l + kn \) with \((l, k) = 1\), where \( n \) runs through the positive integers, and let \( p(k) \) be the maximum value of \( p(l, k) \) for all \( l \) satisfying \((l, k) = 1\) and \( 1 \leq l \leq k \). Linnik proved that there exist positive \( C \) and \( L \) such that \( p(k) < Ck^L \). Heath-Brown proved that \( p(k) < Ck^{5.5} \) [83]. On the problem of the least prime in an arithmetic progression, Chinese mathematicians and Chengdong Pan, Jingrun Chen, Jianmin Liu and Wei Wang et al. made great contributions, see [93-100]. In the case of irreducible polynomials with degree \( \geq 1 \), McCurley Kevin S., Adleman Leonard M., Odlyzko Andrew M. [27, 82, 84] obtained important results.

Remark 12: Conjecture 2 is the first to mention the existence of primes among the conjectures which conjecture the infinitude of some special kinds of primes. Of course, if one wants to prove that the infinitude, firstly, he must prove the existence. Unfortunately, it is a critical difficulty. In next section, we go on with this problem.

7 The existence of some special kinds of primes

We begin with Euclid in this section. In his beautiful proof of the infinitude of primes, Euclid must know the existence of primes. Of course, the existence of primes is very clear. So he omitted the proof of the existence of primes and supposed that there are only finitely many primes, say \( k \) of them, which denoted by \( p_1, \ldots, p_k \) and constructed directly the number \( 1 + \prod_{i=1}^{k} p_i \) which leads to the contradiction.

As we know, it is very difficult to prove the existence of some special kinds of primes. For example, for every \( k \geq 1 \), we even do not know whether there are always primes \( p \) and \( q \) such that \( p - q = 2k \) or not. Namely, we do not know whether \( f(x) = x \) and \( g(x) = x + 2k \) represent simultaneously primes for some integer \( x \) and every \( k \) so far.

If one does not know whether there are some special kinds of primes or not, can he prove their infinitude? This problem goes back to Euler. By Euler’s identity \( \sum_{n=1}^{\infty} n^{-s} = \prod_{p}(1 - p^{-s})^{-1} \), one could prove that \( \sum_{p} p^{-s} \to \infty \) as \( s \to 1 \) which implies the existence of infinitely many primes. Based on Euler's idea, Dirichlet introduced Characters and proved further \( \sum_{p \equiv a \pmod{b}} p^{-s} \to \infty \) as \( s \to 1 \) which implies that there are infinitely many primes of the form \( a + bx \) when \( \gcd(a, b) = 1 \).

More generally, denote the number of some special kinds of primes not exceeding \( x \) by \( P(x) \). If we can prove \( P(x) \to \infty \) as \( x \to \infty \), then we not
only know the existence of these special kinds of primes, but also know their
infinitude. This is a good method which goes back to Legendre who firstly
conjectured $\pi(x) \approx \frac{x}{\log x - 1.08...}$. Gauss found that a good approximation to
$\pi(x)$ is $li(x) = \int_2^x \frac{dt}{\log t}$. It is easy to prove that $\pi(x) \geq \frac{\log x}{2\log 2}$ which implies
the existence of infinitely many primes again. In 1851, Tchebychev proved
firstly that for all sufficiently large $x$, $0.92\frac{\log x}{\log x} \leq \pi(x) \leq 1.10\frac{\log x}{\log x}$. In 1896,
Hadamard and de la Vallée Poussin proved independently $\pi(x) \sim \frac{x}{\log x}$ (or
equivalently, $\pi(x) \sim li(x)$). This is famous Prime Number Theorem which
implies simply the existence of infinitely many primes.

As we mentioned, by studying the behavior of $P(x)$, one not only can de-
termine the existence and infinitude of some special kinds of primes, but also
know the distribution of these special kinds of primes, this is a quantitative
form and becomes then a main method for studying the infinitude of some
special kinds of primes. However, it also is the most difficult. Next section,
we would like to focus our attention on the natural necessary condition and
try to give a new sufficient condition of the infinitude of some special kinds
of primes. This leads to a new way for determining the existence of these
primes.

8 A sufficient condition that a multivariable number-
theoretic function represents primes for infinitely
many integral points

In this section, we begin with Euclid’s proof of the infinitude of primes.
Euclid’s beautiful proof by contradiction goes as follows: Suppose that there
are only finitely many primes, say $k$ of them, which denoted by $2 = p_1 < \ldots < p_k$. Note that $p_1...p_k + 1 > 1$ and hence it must have a prime factor
which differs from $p_1$, ..., $p_k$ and this leads to a contradiction.

Euclid’s proof is essentially to construct a number $x$ such that $x$ is co-
prime to the product $p_1...p_k$. Note that 2 and 3 are prime. So $|Z_{p_1...p_k}^*| > 1$, by Euler function formula. On the other hand, as we know, if $a$ is the
smallest integer such that $a > 1$ and gcd($a, p_1...p_k$) = 1 then $a$ is prime.
Therefore, there are infinitely many primes since $|Z_{p_1...p_k}^*| > 1$ implies that
there is such an integer $a$ in $Z_{p_1...p_k}^*$. This gives a proof for the infinitude
of primes. Although the proof perhaps is not new, it is enlightened us. This
proof need not construct a new number $x$ such that $x$ is coprime to the
product $p_1...p_k$ but prove directly that there is a number $x > 1$ such that
$x$ is coprime to the product $p_1...p_k$. Hence $x$ has a new prime factor and it leads to a contradiction. By the existence of such a $x$, there must be the least positive integer $x > 1$ which is coprime to the product $p_1...p_k$. Of course, it is prime.

The question of existence of infinitely many primes can be regarded as the question of existence of infinitely many prime values of the polynomial $f(x) = x$. For any positive integer $m > 1$, $f(S_f(m)) = S_f(m)$ always is prime when $f(x) = x$, where $S_f(m)$ is the least positive integer $n$ such that $\gcd(f(n), m) = 1$ and $f(n) > 1$. More generally, let $f(x)$ be a generic number-theoretic function, unfortunately, for any positive integer $m > 1$, $f(S_f(m))$ is not always prime. For example, let $f(x) = 2^x - 1$ and $m = 82677$, $S_f(m) = 11$ and $f(S_f(m)) = 2^{11} - 1 = 23 \times 89$ is not prime. Thus a key fact which states that if $a > 1$ is the smallest integer such that $\gcd(a, m) = 1$ and then $a$ is prime is not true in the generic case. Why is it a key fact that if $a > 1$ is the smallest integer such that $\gcd(a, m) = 1$ and then $a$ is prime. As we know, if the number of primes is finite, then the proposition which states if $a > 1$ is the smallest integer such that $\gcd(a, m) = 1$ and then $a$ is prime is false. Therefore, we want to use this fact. Unfortunately, in the generic case, it is not always true. How to treat with it?

Let’s look back Euclid’s proof again. He considered the product of primes $p_1...p_k$. Similarly, we may consider $p_k!$. In fact, $p_k! + 1$ and $p_1...p_k$ are coprime, which implies the infinitude of primes again. Directly or more expediently, we consider the factorial $n!$ instead of the finite product $p_1...p_k$ of primes. Clearly, so long as $n > p_k$, then it will lead to a contradiction still. Particularly, let $a \in Z_n^*$ be the smallest integer such that $a > 1$ and $\gcd(a, n!) = 1$, then $a$ is prime. This is a key fact. We hope naturally this key fact still is true in the generic case that a number-theoretic function $f(x)$ or $f(x_1, ..., x_k)$ represents infinitely many primes. In the following conjecture 3, we try to give a primary consideration.

Another reason that we would like to consider the factorial is because the factorial can be viewed as a special case of the $\Gamma$ function which is closely related to the distinguished Riemann Hypothesis.

Below is the third reason that we would like to consider the factorial:

We notice that if a number-theoretic function $f(x)$ represents primes for infinitely many natural numbers $x$, then for any positive integer $n$, there is a natural numbers $x$ such that the least prime divisor of $f(x)$ is greater than
Therefore, there must be a least natural numbers \( k \) such that the least prime divisor of \( f(k) \) is greater than \( n \). Namely, \( f(k) (>1) \) is coprime to \( n! \).

We also know that there must be a least natural numbers \( r \) such that \( f(r) (>1) \) is coprime to \( n! \). Of source, \( r = k \). Very naturally, one might believe that \( f(k) = f(r) \) is prime.

The following Proposition 3 further gives some witnesses.

**Proposition 3:** Let \( f(x) \) be a generic number-theoretic function. If there is a constant \( c \) such that for every positive integer \( m > c \), there is a natural number \( y \) such that \( f(y) > 1 \) is in \( Z_m^* \), and if the least number \( f(x) \) which exceeds 1 in \( Z_m^* \) is not prime, then \( f(x) \) represents primes at most for finitely many natural numbers \( x \).

**Proof of Proposition 3:** If \( f(x) \) represents primes for infinitely many natural numbers \( x \), then there is a natural number \( y \) such that \( f(y) > 1 \) is prime. Without loss of generality, assume that \( f(y) \) is the least prime which exceeds \( c \). If \( f(y) > 2 \), then \( 2((f(y) - 1)! > c \). But \( f(y) \) is prime and also is the least natural number which exceeds 1 in \( Z_{2((f(y) - 1))}^* \). By our assumption, this least natural number should be a composite number. This is a contradiction. Therefore, \( f(y) = 2 \) and \( 2 > c \). In this case, note that \( 3 > c \) and we have \( f(y) = 2 \in Z_3^* \). But, 2 is prime and also is the least number which exceeds 1 in \( Z_3^* \). This is a contradiction again. Therefore, Proposition 3 holds.

By the proof of this proposition, we see also that if \( f(x) \) represents primes for infinitely many numbers \( x \), then there are infinitely many numbers \( m \) such that the least number \( f(y) \) which exceeds 1 in \( Z_m^* \) is prime. One could generalize it to the generic case. We also believe naively that if \( f(x) \) represents primes for infinitely many numbers \( x \), then there is a positive integer \( c \) such that for each \( m > c \), if \( f(r) (>1) \) is the least natural numbers of the form \( f(r) \) such that \( f(r) \) is coprime to \( m! \), then \( f(r) \) is prime.

Due to the fact the \( f(x) = ax + b \) with \( \gcd(a, b) = 1 \) represents primes for infinitely many natural numbers \( x \), we now prove that there is a positive integer \( c \) such that for each \( m > c \), and if \( f(r) \) is the least prime of the form \( f(x) = ax + b \) such that \( \gcd(f(r), m!) = 1 \), then \( f(r) < m! \). This is easy to prove. Denote the \( i^{th} \) prime of the form \( f(x) = ax + b \) by \( P_{f,i} \). In [103], we have proved that there is a constant \( C \) depending on \( a \) and \( b \) such that when \( n > C \), \( \prod_{i=1}^{i=n} P_{f,i} \geq P_{f,n+1} \). Let \( k \geq n \) and \( \prod_{i=1}^{i=k} P_{f,i} < m! < \prod_{i=1}^{i=k+1} P_{f,i} \). Clearly, \( \gcd(P_{f,k+1}, m!) = 1 \). So, \( P_{f,k+1} \geq f(r) \). If
integral point \((w, f, l, f)\) are all in \(Z^e\) that point \((x, y, z, w)\) for every positive integer \(m\), the least positive integer such that provided \(n\) an integral point \((x, y, z, w)\), assume that there is a positive integer \(l\), and further generalize it as follows:

Conjecture 3: Let \(f(x_1, \ldots, x_k)\) be a multivariable polynomial with integral coefficients (or a multivariable number-theoretic function), if there is a positive integer \(c\) such that for every positive integer \(m \geq c\), there exists an integral point \((y_1, \ldots, y_k)\) such that \(f(y_1, \ldots, y_k) > 1\) is in \(Z^e_m\), and there exists an integral point \((z_1, \ldots, z_k)\) such that \(f(z_1, \ldots, z_k) \geq c\) is primes, moreover, for any integer \(l\) with \(l \geq r\), if \(f(x_1, \ldots, x_k)\) is the least positive integer such that \(f(x_1, \ldots, x_k) > 1\) is in \(Z^e_l\), then \(f(x_1, \ldots, x_k)\) represents primes, where \(r\) is the least positive integer such that provided \(n \geq r!\), then there exists an integral point \((y_1, \ldots, y_k)\) such that \(f(y_1, \ldots, y_k) > 1\) is in \(Z^e_n\).

As for more generic case of several multivariable number-theoretic functions, it is very complicated. For instance, let \(f_1(x) = x, f_2(x) = x + 180\). It is easy to prove that for each \(n > 5\), there is a least natural numbers \(x\) such that \(f_1(x) = x > 1, f_2(x) = x + 180 > 1\) and \(f_1(x) \times f_2(x) \in Z^e_{n!}\). But when \(n = 6\), we have \(f_1(x) = x = 7\) and \(f_2(x) = 187\) is not prime. We left this question to the readers. However, when \(f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k)\) are multivariable polynomials with integral coefficients, we fix Conjecture 3 and further generalize it as follows:

For \(s > 1\), let \(f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k)\) be multivariable polynomials with integral coefficients, assume that there is a positive integer \(c\) such that for every positive integer \(m \geq c\), there exists an integral point \((y_1, \ldots, y_k)\) such that \(f_1(y_1, \ldots, y_k) > 1, \ldots, f_s(y_1, \ldots, y_k) > 1\) are all in \(Z^e_m\), and there exists an integral point \((z_1, \ldots, z_k)\) such that \(f_1(z_1, \ldots, z_k) \geq c, \ldots, f_s(z_1, \ldots, z_k) \geq c\) are all primes. Then for any integer \(l\) with \(l \geq r\), there exists an integral point \((x_1, \ldots, x_k)\) such that \(f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k)\) are all in \(Z^e_l\) and \(f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k)\) represent simultaneously primes, where \(r\) is the least positive integer such that provided \(n \geq r!\), then there exists an integral point \((w_1, \ldots, w_k)\) such that \(f_1(w_1, \ldots, w_k) > 1, \ldots, f_s(w_1, \ldots, w_k) > 1\) are all in \(Z^e_n\).
9 A sufficient and necessary condition that several multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points

After finishing Section 8, the author felt intensively that there must be a sufficient and necessary condition that several multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points. Therefore, this section was added very recently. The author obtrusively suggested a generalization of Conjecture 2 as follows.

A sufficient and necessary condition: Let \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) be multivariable number-theoretic functions, assume that there is a positive integer \( c \) such that for every positive integer \( m \geq c \), there exists an integral point \( (y_1, \ldots, y_k) \) such that \( f_1(y_1, \ldots, y_k) > 1, \ldots, f_s(y_1, \ldots, y_k) > 1 \) are all in \( Z_m^\ast \), and there exists an integral point \( (z_1, \ldots, z_k) \) such that \( f_1(z_1, \ldots, z_k) \geq c!, \ldots, f_s(z_1, \ldots, z_k) \geq c! \) are all primes. Then the sufficient and necessary condition that \( f_1(x_1, \ldots, x_k), \ldots, f_s(x_1, \ldots, x_k) \) represent simultaneously primes for infinitely many integral points is that for every \( m \), there is an integral point \( (m_1, \ldots, m_k) \) such that \( f_1(m_1, \ldots, m_k), \ldots, f_s(m_1, \ldots, m_k) \) in \( Z_m^\ast \) are all primes.

Due to the fact that we consider the factorial which can be viewed as a special case of the \( \Gamma \) function, we should assume that \( f_1, \ldots, f_s \) are continuous functions on \( R^k \), where \( R \) is the set of all real numbers.

This sufficient and necessary condition implies that there is a positive integer \( c \) such that for every positive integer \( m \geq c \), there exists an integer \( x > 0 \) such that \( 2^{2^x} + 1 \) is in \( Z_m^\ast \). But, if \( 2^{2^n} + 1 \) is the least Fermat number in \( Z_m^\ast \), then, \( 2^{2^n} + 1 \) might is not always prime. For instance, by Stirling’s formula \( n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \) and the factorization of Fermat numbers, we have \( 2^{2^6} + 1 \in Z_{(2^{2^4} + 1)^n}^\ast \), \( 2^{2^7} + 1 \in Z_{(2^{2^5} + 1)^n}^\ast \) and so on, but \( 2^{2^6} + 1 \) and \( 2^{2^7} + 1 \) are the least but not prime respectively. Does it imply that there are only finitely many Fermat primes
again?

Anyway, are these conjectures proposed consistent with each other? Are they reasonable or reliable? The author are waiting for advice of readers. With the development of mathematics, the correct answers will come—we must know, we will know, Hilbert said.

10 Conclusion

I learn from Euclid all the time. This paper is a part of my paper Euclid’s algorithm and the infinitude of some special kinds of primes, in which, his two great number-theoretical achievements—Euclid’s algorithm and his proof for the infinitude of primes, have been studied. Of course, these two significant results are not independent, and, Chinese Remainder Theorem is a bridge between them because Euclid’s algorithm implies Chinese Remainder Theorem which also implies Euclid’s second theorem.

On Euclid’s algorithm, we have done the following work in the paper Euclid’s algorithm and the infinitude of some special kinds of primes: (1) Euclid’s Number-Theoretical Work [77]; (2) Euclid’s Algorithm, Guass’ Elimination and Buchberger’s Algorithm [85]; (3) Euclid’s Algorithm and W Sequences [86]; (4) Euclid’s Algorithm and three public key cryptosystems—RSA Cryptosystem, Elliptic Curve Cryptosystems and Multivariate Public Key Cryptosystems; (5) Euclid’s Algorithm, LLL Algorithm and the number field sieve. On Euclid’s proof for the infinitude of primes, it leads to this paper. Knuth [87] called Euclid’s Algorithm the granddaddy of all algorithms. In the author’s eyes, Euclid’s proof for the infinitude of primes also is the granddaddy of some proofs for the infinitude of some kind special kinds of primes.

In this paper, we try to establish a generic model for the problem of infinitude of some special kinds of primes. More precisely, we try to establish a generic model for the problem that several multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points. We analyzed some equivalent necessary conditions that irreducible univariable polynomials with integral coefficients represent infinitely many primes, found new necessary conditions which perhaps imply that there are only finitely many Fermat primes, generalized Euler’s function, the prime-counting function and Schinzel-Sierpinski’s Conjecture and so on, obtained an analogy of the Chinese Remainder Theorem. Finally, a suf-
ficient and necessary condition that several multivariable number-theoretic functions represent simultaneously primes for infinitely many integral points was proposed. Nevertheless, this is only a beginning and it miles to go. The author would like to cite the comment in Schinzel and Sierpinski’s paper "we do not know what will be the fate of our hypothesis, however, we think that, even if they are refuted, this will not be without profit for Number Theory.” to close this paper. Please let me know any questions, reviews and criticisms at shaohuazhang@mail.sdu.edu.cn. Thank you very much.

11 Acknowledgements

God created the integers. Thank God for the great blessings he has given me to study integers.

I dedicate this paper to her. I regret that I have not proved an excellent number-theoretical theorem for her. What I can do now is to try my best to become a disseminator of some excellent number-theoretical theorems. In [Appendix], I give a list of 100 theorems in Number Theory.

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12 Appendix

100 theorems in Number Theory

"Number Theory is the queen of mathematics"—Guass. For a long time, I want to edit a number-theoretical e-book which includes many excellent theorems and is worthwhile to take my lifetime to learn. Recently, this work is almost completed. I named it "100 theorems in Number Theory". If Number Theory is the queen of mathematics, then these theorems are her pearls. Of course, there are much more than 100 theorems in Number Theory. To follow a principle that Mathematics is essentially simple, and based on the individual opinion and taste, I only pick some theorems which are my favorites and look simple in spite of some proofs are extremely intricate. Namely, the description of these theorems is very easy, although their proofs maybe be extremely difficult. For example, the meaning of Fermat’s last theorem or Green-Tao theorem is very clear, but its proof is difficult to understand. Unfortunately, the proofs of about 51 theorems in this e-book, while intensely enjoyable, do require hard study to grasp. Maybe, this is a basic reading for Number Theory. If one is afraid of meeting the difficulty,
then he always meets the difficulty. Therefore, anyone who loves Number Theory should learn the proofs of approximately 87 theorems. I will try my best to travel for this dream. A Bachelor of Number Theory had better understand the proofs of more or less 60 theorems. A Master of Number Theory had better understand the proofs of more or less 70 theorems. And a Doctor of Number Theory had better understand the proofs of more or less 80 theorems. Below is the list of theorems.

1 The first theorem about Theory of Divisibility (also called division algorithm): Let $a$ and $b$ be integers with $b > 0$. There exist unique integers $q$ and $r$ such that $a = bq + r$ and $0 \leq r < b$.

Remark: This theorem is the basis of Theory of Divisibility. Many number-theoretical texts begin with it. However, Euclid did not do like this. Euclid began his number-theoretical work by introducing his algorithm which states essentially that for two distinct positive integers, replace continually the larger number by the difference of them until both are equal, then the answer is their greatest common divisor. In [77], we showed that Euclid’s algorithm is equivalent with Division algorithm.

2 Euclid’s first theorem[67]: If $p$ is prime, and $p|ab$, then $p|a$ or $p|b$.

3 Euclid’s second theorem[67]: The number of primes is infinite.

4 The fundamental theorem of arithmetic: Every positive integer can be written uniquely (up to order) as the product of prime numbers.

5 The linear congruence theorem: If $a$ and $b$ are any integers and $n$ is a positive integer, then the congruence $ax \equiv b \pmod{n}$ has a solution for $x$ if and only if $b$ is divisible by the greatest common divisor $(a, n)$ of $a$ and $n$. Particularly, when $(a, n) = 1$, the congruence $ax \equiv b \pmod{n}$ has a unique solution modulo $n$.

6 Chinese Remainder Theorem: Given a system of simultaneous linear congruences $x \equiv a_i \pmod{n_i}$ for $i = 1, 2, ..., k$ and for which $n_i$ are pairwise relatively prime positive integers, where $a_i$ are integers, then this linear system has a unique solution modulo $n = \prod_{i=1}^{k} n_i$. Particularly, for $i = 1, 2, ..., k$, if $\gcd(a_i, n_i) = 1$, then, this linear system has a unique solution $x$ in $\mathbb{Z}_n^*$.

7 Fermat’s little theorem: Let $p$ be a prime, if the integer $a$ is not
divisible by \( p \), then \( a^{p-1} \equiv 1 \pmod{p} \). Moreover, \( a^p \equiv a \pmod{p} \) for every integer \( a \).

8 Euler’s theorem: Let \( m \) be a positive integer, and let \( a \) be an integer relatively prime to \( m \). Then \( a^\varphi(m) \equiv 1 \pmod{m} \), where \( \varphi(m) \) is defined to be the number of positive integers less than or equal to \( m \) that are coprime to \( m \).

9 Carmichael’s theorem: If \( (a, m) = 1 \), then \( a^\lambda \equiv 1 \pmod{m} \), where \( \lambda \) is the smallest integer such that \( k^\lambda \equiv 1 \pmod{m} \) for all \( k \) relatively prime to \( m \).

10 Wilson’s theorem: If \( p \) is prime, then \((p-1)! \equiv -1 \pmod{p}\).

11 A theorem of Wolstenholme: If \( p > 3 \) is prime, then the numerator of harmonic number \( 1 + \frac{1}{2} + \ldots + \frac{1}{p-1} \) is divisible by \( p^2 \).

12 Euclid-Euler theorem: \( n \) is even perfect number if and only if \( n = 2^{p-1}(2^p - 1) \), where \( p \) is prime such that \( 2^p - 1 \) is also prime.

13 Lamé’s theorem: Finding the greatest common divisor of integers \( a \) and \( b \) with \( a > b \), Euclid’s algorithm runs in no more than \( 5k \) steps, where \( k \) is the number of (decimal) digits of \( b \).

14 Midy’s theorem: If the period of a repeating decimal for \( \frac{a}{p} \), where \( p \) is prime and \( \frac{a}{p} \) is a reduced fraction, has an even number of digits, then dividing the repeating portion into halves and adding gives a string of 9s. For example, \( \frac{1}{7} = 0.142857 \), \( 142 + 857 = 999 \).

15 Bauer’s theorem: Let \( m > 2 \) be an integer and let \( f(x) \) be an integral polynomial that has at least one real root. Then \( f(x) \) has infinitely many prime divisors that are not congruent to \( 1 \pmod{m} \).

16 Euler’s quadratic residue theorem: Let \( p \) be an odd prime. For every integer \( a \), \( \left( \frac{a}{p} \right) \equiv a^{(p-1)/2} \pmod{p} \), where \( \left( \frac{a}{p} \right) \) is the Legendre symbol.

17 The golden theorem (also called the law of quadratic reciprocity): If \( p \) and \( q \) is distinct odd primes, then \( \left( \frac{q}{p} \right) \left( \frac{p}{q} \right) = (-1)^{(p-1)(q-1)/4} \).

18 Fermat’s theorem on sums of two squares: An odd prime \( p \) is expressible as \( p = x^2 + y^2 \) with \( x \) and \( y \) are integers, if and only if \( p \equiv 1 \pmod{4} \).

19 Lagrange’s four-square theorem: Every positive integer can be
expressed as the sum of four squares of integers.

20 **Fermat polygonal number theorem:** Every positive integer is a sum of at most \( n \) \( n \)-polygonal numbers, where \( n > 2 \) is a positive integer.

21 **A theorem of Carmichael on the \( n \)-th Fibonacci number:** Every Fibonacci number \( f_n \) with \( n \neq 1, 2, 6, 12 \), has at least one characteristic factor which is not a factor of any earlier Fibonacci number.

22 **Lagrange’s continued fraction theorem:** A number is a quadratic surd if and only if its continued fraction expansion is eventually periodic.

23 **Dirichlet’s approximation theorem:** Given any real number \( \theta \) and any positive integer \( n \), there exist integers \( x \) and \( y \) with \( 0 < x \leq n \) such that \( |x\theta - y| < \frac{1}{n} \).

24 **Liouville’s theorem:** For any algebraic number \( x \) with degree \( n > 1 \), there exists \( c > 0 \) such that \( |x - \frac{p}{q}| > \frac{c}{q^n} \) for all rationals \( \frac{p}{q} (q > 0) \).

25 **Van der Waerden’s theorem:** If the set of all positive integers is written as the union of sets of the finite number, then there exists at least a set which contains arbitrarily long arithmetic progressions.

26 **Minkowski’s theorem:** Any convex set in \( \mathbb{R}^n \) which is symmetric with respect to the origin and with volume greater than \( 2^n \) contains a non-zero lattice point.

27 **Bertrand-Chebyshev theorem:** There exists a prime in interval \((n, 2n)\) when \( n > 1 \).

28 **Mills’ theorem:** There exists a real constant \( \theta \) such that \( \lfloor \theta^{3^n} \rfloor \) is prime for all \( n \geq 1 \).

29 **Rosser’s theorem:** Let \( p_n \) be the \( n \)-th prime number, then for \( n > 1 \), \( p_n > n \ln n \).

30 **Beatty’s theorem:** Let \( x \) and \( y \) be positive irrational numbers satisfying \( \frac{1}{x} + \frac{1}{y} = 1 \). Then each positive integer belongs to exactly one of the two sequences \( \{nx\} \) and \( \{ny\} \).

31 **Hurwitz’s Irrational Number Theorem:** For any irrational number \( \theta \), there are infinitely many rational numbers \( \frac{x}{y} \) such that \( |\theta - \frac{x}{y}| < \frac{1}{\sqrt{5}y^2} \).

32 **Blichfeldt’s Theorem:** A bounded set of points \( C \) with area \( A \), can be
translated to a position $C'$ so as to cover a number of lattice points greater than $A$.

33 Ramanujan-Skolem’s theorem: The equation $x^2 + 7 = 2^n$ has solutions in natural numbers $n$ and $x$ just when $n = 3, 4, 5, 7, 15$.

34 Thue’s Theorem: If $f$ is a bivariate form with rational coefficients which is irreducible over the rational numbers and has degree $\geq 3$, and $r$ is a rational number other than 0, then the equation $f(x, y) = r$ has only finitely many solutions in integers $x$ and $y$.

35 Rotkiewicz Theorem: If $n \geq 19$, there exists a Poulet number between $n$ and $n^2$.

Remark: A Poulet number $m$ is a Fermat pseudoprime to base 2, namely, a composite number $m$ satisfying $2^m - 1 \equiv 1 \pmod{m}$.

36 Schnirelmann’s Theorem: There exists a positive integer $C$ such that every sufficiently large integer is the sum of at most $C$ primes.

37 Sierpinski’s Composite Number Theorem: There exist infinitely many positive odd numbers $k$ such that $k \cdot 2^n + 1$ is composite for every integer $n > 0$.

38 Sierpinski’s Prime Sequence Theorem: For any $m$, there exists a number $k$ such that the sequence $\{n^2 + k\}$ contains at least $m$ primes.

39 A theorem of H.Gupta and S.P. Khare: For $2 < k < 1794$, $(\frac{k^2}{k})$ is greater than the product of the first $k$ primes, while for $k \geq 1794$, $(\frac{k^2}{k})$ is less than the product of the first $k$ primes.

40 A theorem of Sylvester and Schur: The product of $k$ consecutive positive integers each exceeding $k$ is divisible by a prime greater than $k$.

41 Theorem of Pillai and Szekeres: For any positive integer $n \geq 17$, there exists a sequence of $n$ consecutive positive integers such that no one of this sequence is relatively prime with all of the others.

42 Erdős-Anning Theorem: An infinite number of points in the plane can have mutual integer distances only if all the points lie on a straight line.

43 A theorem of Erdős and Selfridge: The product of consecutive integers is never a power.
44 **The theorem of Waring-Hilbert:** Every positive integer \( n \) is the sum of at most \( s \) \( k \)-th powers of natural numbers, where \( s = s(k) \) is independent of \( n \).

45 **Mason’s theorem:** Let \( f, g, h \) be three polynomials with no common factors such that \( f + g = h \). Then the number of distinct roots of the three polynomials is either one or greater than their largest degree.

46 **Dirichlet’s theorem:** For any two positive coprime integers \( a \) and \( b \), there are infinitely many primes of the form \( a + bn \).

47 **Chebyshev’s theorem:** For real number \( x \), denote the number of primes less than or equal to \( x \) by \( \pi(x) \). Then there exist positive constants \( A \) and \( B \) such that \( Ax \leq \pi(x) \ln x \leq Bx \).

48 **Prime number theorem:** \( \pi(x) \sim \frac{x}{\ln x} \).

49 **A theorem of G. Robin:** Denote the number of distinct prime factors of \( x \) by \( \omega(x) \), then for every integer \( n \geq 26 \), \( \omega(n) < \frac{\log n}{\log \log n - 1.174} \), with equality when \( n \) is the product of the first 189 primes.

50 **A theorem of M. Agrawal -N. Kayal and N. Saxena:** There is an algorithm determines whether a number is prime or composite within polynomial time.

51 **Brun’s theorem:** The sum of the reciprocals of the twin primes is convergent with a finite value.

52 **Apéry’s theorem:** The number \( \zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots \) is irrational.

53 **A theorem of Motohashi Y.:** There are infinitely many primes of the form \( x^2 + y^2 + 1 \).

54 **A theorem of Fouvry, Etienne and Iwaniec H.:** There are infinitely many primes of type \( x^2 + y^2 \), where \( x \) is a prime number.

55 **A theorem of Friedlander, John and Iwaniec H.:** There are infinitely many primes of type \( x^2 + y^4 \).

56 **A theorem of Heath-Brown:** There are infinitely many primes of form \( x^3 + 2y^3 \).

57 **Linnik’s theorem:** Denote \( p(l, k) \) the least prime in the arithmetic progression \( l + kn \) with \( (l, k) = 1 \), where \( n \) runs through the positive integers,
and let \( p(k) \) be the maximum value of \( p(l, k) \) for all \( l \) satisfying \( (l, k) = 1 \) and \( 1 \leq l \leq k \). Then there exist positive \( C \) and \( L \) such that \( p(k) < Ck^L \).

58 **Heath-Brown’s theorem:** (with the notation above) \( p(k) < Ck^{5.5} \).

59 **Vinogradov’s theorem:** Every sufficiently large odd number can be written as the sum of three primes.

60 **Chen’s theorem:** Every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime.

61 **A theorem of Roth:** For every value \( d \) with \( 0 < d < 1 \), there is a number \( C \) such that every subset \( A \) of \( \{1, 2, 3, ..., N\} \) of cardinality \( dN \) contains a length-3 arithmetic progression, provided \( N > C \).

62 **Szemerédi’s theorem:** Every sequence of integers that has positive upper density contains arbitrarily long arithmetic progressions.

63 **A theorem of J. G. van der Corput:** The primes contain infinitely many arithmetic progressions of length 3.

64 **A theorem of Ben Green:** Any set containing a positive proportion of the primes contains a 3-term arithmetic progression.

65 **A theorem of Balog:** For any \( m > 1 \), there are \( m \) distinct primes \( p_1, ..., p_m \) such that all of the averages \( \frac{p_i + p_j}{2} \) are primes.

66 **Green-Tao theorem:** The primes contain arbitrarily long arithmetic progressions.

67 **A theorem of W. R. Alford - A. Granville and C. Pomerance:** There are infinitely many Carmichael numbers.

68 **Tijdeman’s theorem:** There are at most a finite number of consecutive powers.

69 **Mihailescu’s theorem:** 8 and 9 are the only consecutive powers.

70 **A theorem of Pythagoras’ school:** \( \sqrt{2} \) is irrational.

71 **A theorem of Euler on the irrationality:** The base of the natural logarithm \( e \) is irrational.

72 **A theorem of Lambert:** The ratio \( \pi \) of a circle’s circumference to its diameter is irrational.
73 **Hermite-Lindemann theorem:** $e$ and $\pi$ are all transcendental numbers.

74 **Gelfond-Schneider theorem:** If $\alpha$ and $\beta$ are algebraic numbers (with $\alpha \neq 0, 1$), and if $\beta$ is not a rational number, then any value of $\alpha^\beta$ is a transcendental number.

75 **Six Exponentials Theorem:** Let $(x_1, x_2)$ and $(y_1, y_2, y_3)$ be two sets of complex numbers linearly independent over the rational number field. Then at least one of $e^{x_1 y_1}, e^{x_1 y_2}, e^{x_1 y_3}, e^{x_2 y_1}, e^{x_2 y_2}, e^{x_2 y_3}$ is transcendental.

76 **Baker-Stark theorem:** The only imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with class number 1, where $d$ is a square-free positive integer, are given by $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$.

77 **Thue-Siegel-Roth theorem:** For any given algebraic number $\theta$, and for given $\varepsilon > 0$, the inequality $|\theta - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$ can have only finitely many solutions in coprime integers $p$ and $q$.

78 **A theorem of Yu. V. Nesterenko:** $e$, $\pi$ and $\Gamma\left(\frac{1}{4}\right)$ are algebraically independent.

79 **Matiyasevich’s theorem:** Every recursively enumerable set is Diophantine.

80 **Fifteen Theorem:** If an integral quadratic form with integral matrix represents all positive integers up to 15, then it represents all positive integers.

81 **Erdős-Kac theorem:** If $\omega(n)$ is the number of distinct prime factors of $n$, then for any fixed $a < b$, $\lim_{N \to \infty} \frac{1}{N} \left| \{ n \leq N : a \leq \frac{\omega(n) - \log \log N}{\sqrt{\log \log N}} \leq b \} \right| = \int_a^b \phi(x) dx$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the probability density function of the standard normal distribution.

82 **A theorem of Pomerance and Selfridge:** For any given integer $n$ and $m$ with $n > 0$, there exists a 1-1 correspondence $f : \{1, \ldots, n\} \rightarrow \{m + 1, \ldots, m + n\}$ such that $\gcd(i, f(i)) = 1$ for $1 \leq i \leq n$.

83 **Dirichlet’s unit theorem:** The rank of the group of units in the ring of algebraic integers of a number field $F$ equals to $r_1 + r_2 - 1$, where $r_1$ is the number of real embeddings and $r_2$ the number of conjugate pairs of complex embeddings of $F$. 

42
84 The Fundamental Theorem of Ideal Theory: In the domain of all algebraic integers in an algebraic number field, every nonzero ideal can be represented uniquely (except for order) as a product of powers of distinct prime ideals.

85 Kronecker-weber theorem: Every abelian field is a subfield of a cyclotomic field. Namely, any Galois extension of the field $Q$ of rational numbers whose Galois group is Abelian must be a subextension of a field obtained from $Q$ by adjoining root of unity.

86 Kummer’s theorem: If $p$ is a regular prime which does not divide the class number of the cyclotomic field $Q(\zeta_p)$, then the equation $x^p + y^p = z^p$ is unsolvable in nature number.

87 Hilbert’s basis theorem: Every ideal in the ring $F[x_1, \ldots, x_n]$ is finitely generated, where $F[x_1, \ldots, x_n]$ is a polynomial ring in $n$ variables over a field $F$.

88 Hilbert’s zero theorem: Let $f_1, \ldots, f_m$ and $g$ be polynomials in the ring $F[x_1, \ldots, x_n]$. If each common root of $f_1, \ldots, f_m$ is a root of $g$, then there exists an integer $r$ such that $g^r$ belongs to the ideal generated by $f_1, \ldots, f_m$.

89 Riemann-Roch theorem: Let $X$ be a curve of genus $g$ and $D$ be a divisor on $X$. Then $l(D) - l(K - D) = \text{deg} D + 1 - g$, where $K$ is the canonical divisor on $X$.

90 Hurwitz’s Theorem: Let $f : X \to Y$ be a finite separable morphism of curves, and let $n = \text{deg} f$. Let $R$ be the ramification divisor of $f$. Denote the genus of $X$ and $Y$ by $g(X)$ and $g(Y)$ respectively. Then, $2g(X) - 2 = n(2g(Y) - 2) + \text{deg} R$.

91 Hasse’s theorem: If $H$ is the number of points on the elliptic curve $E$ over a finite field with $q$ elements, then $|H - (q + 1)| < 2\sqrt{q}$.

92 A theorem of Weil: Let the elliptic curve $E$ be defined over a finite field $F_q$ and $m$ a positive integer. Denote the number of points on the elliptic curve $E$ over a finite field $F_{q^m}$ by $N$. Then $N = q^m + 1 - a^m - b^m$, where $a$ and $b$ satisfy $ab = q$ and $a + b = H - (q + 1)$ with the notation above $H$.

93 Rück-Voloch theorem: Let the elliptic curve $E$ be define over a finite field $F_q$. Then the group $E(F_q)$ is isomorphic to a unique direct product of two cyclic groups $Z_m$ and $Z_n$ with $m|n$ and $m|(q - 1)$. 

43
94 A theorem of Mordell: For an elliptic curve $E$ over the rational number field $Q$, the group $E(Q)$ of rational points of $E$ is a finitely-generated abelian group.

95 Mordell-Weil theorem: For an abelian variety $A$ over a number field $K$, the group $A(K)$ of $K$-rational points of $A$ is a finitely-generated abelian group.

96 Faltings’ theorem: Let $C$ be a non-singular algebraic curve over the rational number field of genus $g > 1$. Then the number of rational points on $C$ is finite.

97 Tunnell’s theorem: Let $n$ be a congruent number, if $n$ is odd then $2A_n = B_n$ and if $n$ is even then $2C_n = D_n$, where $A_n = |\{(x, y, z) \in Z^3 : n = 2x^2 + y^2 + 32z^2\}|$, $B_n = |\{(x, y, z) \in Z^3 : n = 2x^2 + y^2 + 8z^2\}|$, $C_n = |\{(x, y, z) \in Z^3 : n = 8x^2 + 2y^2 + 64z^2\}|$, $D_n = |\{(x, y, z) \in Z^3 : n = 8x^2 + 2y^2 + 16z^2\}|$.

98 Mazur’s torsion theorem: The torsion subgroups of the group of rational points on an elliptic curve defined over the rational number field is one of the following fifteen groups: $Z/NZ(1 \leq N \leq 10)$ or $Z/12Z$; $Z/2Z \times Z/2NZ(1 \leq N \leq 4)$.

99 Fermat’s last theorem: If $n > 2$ is a positive integer, then the equation $x^n + y^n = z^n$ is unsolvable in nature number.

100 The Modularity theorem: All rational elliptic curves arise from modular forms.