The non-Debye, i.e., non-exponential, behavior characterizes a large plethora of dielectric relaxation phenomena. Attempts to find their theoretical explanation are dominated either by considerations rooted in the stochastic processes methodology or by the so-called fractional dynamics based on equations involving fractional derivatives which mimic the non-local time evolution and as such may be interpreted as describing memory effects. Using the recent results coming from the stochastic approach we link memory functions with the Laplace (characteristic) exponents of infinitely divisible probability distributions and show how to relate the latter with experimentally measurable spectral functions characterizing relaxation in the frequency domain. This enables us to incorporate phenomenological knowledge into the evolution laws. To illustrate our approach we consider the standard Havriliak-Negami and Jurlewicz-Weron-Stanislavsky models for which we derive well-defined evolution equations. Merging stochastic and fractional dynamics approaches sheds also new light on the analysis of relaxation phenomena which description needs going beyond using the single evolution pattern. We determine sufficient conditions under which such description is consistent with general requirements of our approach.

I. INTRODUCTION

The Debye relaxation (abbreviated by a subscript $D$) describes the response of an ideal non–interacting population of dipoles subjected to an alternating external electric field. The simplest example of such a phenomenon is the step–like outer field which switching off results in the time decay of induced polarization described by the relaxation function $n(t)$. Its time behaviour is governed by the differential equation $\dot{n}_D(t) = -\tau^{-1} n_D(t)$ where $\dot{n}_D(t) = dn_D(t)/dt$ and $\tau$ denotes a material constant called the relaxation time. Solution to this elementary equation required to satisfy the initial condition $n(0) = 1$ reads $n_D(t) = \exp(-t/\tau)$, i.e., the usual exponential law. If the system response is not ideal and mutual interaction of dipoles influences the relaxation process then its description based on the Debye pattern does not work any longer and to get agreement with experimental data one has to give up the paradigm of exponential decay. Among efforts focused on generalizations of the time evolution equation the overriding example is to replace the constant relaxation time $\tau^{-1}$ by a time dependent transition rate function $r(t, \tau)$ \cite{4, 5, 8, 13}. This leads to the equation

$$\dot{n}(t) = -r(t, \tau) n(t)$$

whose solution obeying the initial condition $n(0) = 1$ reads $n(t) = \exp\left(-\int_0^t r(\xi, \tau) d\xi\right)$. The transition rate $r(t, \tau) = -\dot{n}(t)/n(t)$ remains constant only for the Debye relaxation and it is a serious challenge how to determine it for other processes. The basic experimental source of information is the broadband dielectric spectroscopy \cite{17}. It provides us with phenomenological knowledge on dispersive and absorptive properties of the medium in the frequency range of several decades of magnitude, next encoded in the complex valued, frequency dependent, dielectric permittivity $\varepsilon(\omega)$ \cite{2}. However, if one utilizes these data to find the time dependence of relaxation functions (it may be calculated explicitly using analytically known expressions which fit the data \cite{4, 5, 8, 13}) then immediately encounters the singularity of $r(t, \tau)$ at $t = 0$. The widely used Kohlrausch-Williams-Watts (KWW) model of the stretched exponential time decay $n_{KWW}(t) = \exp\left(-\left(t/\tau\right)^\alpha\right)$ leads to $r_{KWW}(t, \tau) = \alpha \tau(t/\tau)^{\alpha-1}$ which for physically typical values $\alpha \in (0,1)$ is obviously singular at $t = 0$. The presence of such a singularity means that $r(t, \tau)$ cannot be experimentally handled, or credibly extrapolated, for short times. Measurements in a neighbourhood close to $t = 0$ become uncontrolled and unfaithful as may give completely different results for $n_{KWW}(t)$ even if repeated in the same experimental conditions. Similar situation is met for another phenomenological models of non-Debye relaxations within which calculations of $r(t, \tau)$ usually lead to singular ratios of functions belonging to the Mittag-Leffler family. Thus, to get the time evolution equations reliable for non-Debye relaxation one does need another approach - an alternative, still belonging to the field of macroscopic physics, is to go beyond investigations of the time evolution in terms of simple ordinary differential equations. Among popular and widely accep-
ted proposals to solve the problem one finds methods of fractional dynamics based on using equations which involve fractional derivatives. This provides us with many valuable results but has also a serious disadvantage: rules and/or laws of fractional dynamics leaves open the question which type of fractional derivative should be chosen as the most suitable for the problem under consideration and how to relate it to phenomenological observations. The choice is frequently done ad hoc and many a time justified only by the statement that the proposed model is solvable. This is neither physically nor mathematically satisfactory and has motivated us to develop a novel scheme which starting point is the integral version of Eq. (1)

\[ n(t) = 1 - \int_0^t r(\xi, \tau) n(\xi) \, d\xi, \]

made to be non-local in time by introducing the time smearing \( r(\xi, \tau) \to r(t - \xi, \tau) \)

\[ n(t) = 1 - \int_0^t r(t - \xi, \tau) n(\xi) \, d\xi \tag{2} \]

which reflects memory effects modifying the time evolution law. We shall show that merging this with the results obtained in the framework of the stochastic processes based approach to relaxation phenomena (for comprehensive literature see [28–32]) we are able to built into proposed scheme the phenomenological knowledge of relaxation functions coming from the broadband dielectric spectroscopy.

Physicists who investigate relaxation processes using extensively the probabilistic and statistical methods customary are used to treat Eq. (2) as a kinetic equation related to some stochastic process governed by an infinitely divisible distribution and to write it down in the form of the master equation

\[ n(t) = 1 - B(\tau, p) \int_0^t M(t - \xi) n(\xi) \, d\xi, \tag{3} \]

separating out the memory function \( M(t) \) and the transition rate constant \( B(\tau, p) \equiv B \) which, besides of the relaxation time, may involve also other parameters \( p \). In the Laplace domain the solution of Eq. (3) is

\[ \hat{n}(s) = \{s [1 + B \hat{M}(s)]\}^{-1}. \tag{4} \]

Thus it is the knowledge of the Laplace transformed memory function \( \hat{M}(s) \) being supposed to play a role of quantity essential for further considerations. Indeed, it does play - the formalism developed within the stochastic approach to relaxation processes allows one to connect \( \hat{M}(s) \) with the Laplace exponent \( \hat{\Psi}(s) \) determined by the probability distribution governing the stochastic process [28, 31]. The expected relation takes the form

\[ \hat{M}(s) = [\hat{\Psi}(s)]^{-1}, \tag{5} \]

and links dynamical and stochastic aspects of the relaxation phenomena according to

\[ \hat{n}(s) = \{s [1 + B/\hat{\Psi}(s)]\}^{-1}. \tag{6} \]

The relations Eq. (4) and consequently Eq. (5) are unique and open the way to use methods worked out in the theory of positive definite functions ([1, 34, 35], for a brief introduction to their properties see Appendix A). From the Lévy–Khintchine formula [34, 35] we learn that for \( s \in \mathbb{R}_+, \hat{\Psi}(s) \) is a complete Bernstein function (CBF). Properties of CBFs imply also that \( s/\hat{\Psi}(s) = \hat{\Phi}(s) \) is a CBF as well and that there exists a completely monotone function (CMF) associated with CBF - this comes out directly from the fact that the algebraic reciprocal of any CBF is CMF [37, 38]. Collecting these facts together we conclude that when describe the relaxation we deal with two couples of functions: the first of them, related to Eq. (3), consists of the memory function \( M(t) \) and the Laplace exponent \( \hat{\Psi}(s) \), whereas the second one involves the Laplace exponent \( \hat{\Phi}(s) \) and a new memory function \( k(t) \) which, through the Laplace transform, corresponds to \( \hat{\Phi}(s) \) according to

\[ \hat{k}(s) = [\hat{\Phi}(s)]^{-1} = \hat{\Psi}(s)/s. \tag{7} \]

It is natural to ask the question - what is the equation related to the function \( k(t) \)? From the just quoted properties of CBFs we know that for \( s \) restricted to \( \mathbb{R}_+, \hat{k}(s) \) is CMF. So its time domain analogue \( k(t) \) may be interpreted as the memory function which defines the well-posted Cauchy problem

\[ \int_0^t k(t - \xi) \nu(\xi) \, d\xi = -B \nu(t) \tag{8} \]

with the initial condition \( \nu(0) = 1 \). Eqs. (4) and (5) stem from the same stochastic process and as such should be read out as mutually alternative descriptions of the same physical reality. Thus we can identify the relaxation functions \( n(t) \) and \( \nu(t) \). To trace this step better note that the Laplace exponents \( \hat{\Psi}(s) \) and \( \hat{\Phi}(s) \) are joined by \( \hat{\Phi}(s) \hat{\Psi}(s) = s \). Rewriting this relation for the memory kernels in the Laplace domain we obtain

\[ \hat{k}(s) \hat{M}(s) = s^{-1}, \tag{9} \]

which, by the usual convolution property, in the time domain equals to

\[ \int_0^t k(\xi) M(t - \xi) \, d\xi = \int_0^t k(t - \xi) M(\xi) \, d\xi = 1 \tag{10} \]

i.e., gives the condition which says that the memory kernels \( k(t) \) and \( M(t) \) constitute the Sonine pair [11, 12], the property long–time noticed and applied in the theory of integral equations. Comparing Eqs. (4) and/or (5) with Eq. (1) we see that they describe the time smearing \( r(t, \tau) n(t) \) and/or \( \hat{n}(t) \), respectively. As just
remarked from the physical point of view both smearings should give the same results, i.e., starting from Eq. [3] we should be able to obtain Eq. [4] and vice versa. Simultaneously, we should recover essential objects describing the relaxation, namely the memory, relaxation and spectral functions if the Laplace exponent of its underlying stochastic process is known. In the view of that the crucial result comes to play - the Laplace exponent is uniquely connected to the spectral function which real and imaginary parts are measured in dielectric spectroscopy and provide us with basic empirical information on the relaxation process being studied.

Forthcoming part of the paper is devoted to the explanation how just sketched construction works and explained how just sketched construction works and its special cases: the Cole-Cole (CC) and the Cole-Davidson (CD) models. An example of "atypical" relaxation is given by the Jonscher-Weron-Stanislavsky (JWS) relaxation patterns. More precisely, the article is devoted to deriving explicit forms of Eqs. [3] and [4] adjusted to the HN and JWS models. In Sec. II we recall some basic facts concerning the HN model and identify relaxation functions relevant for these models. The time domain memory functions are calculated in Sec. III giving results used in Sec. IV to write down explicitly Eqs. [3] and [4] as well as their solutions. In Sec. V we propose an extension of our scheme to encounter processes which description goes beyond using single relaxation pattern - these we shall call "multichannel" processes. We show that our approach remains self-consistent for a process governed by a sum of the Cole-Cole patterns parametrized by \( \alpha \) and \( b \) satisfying \( \alpha + b = 1 \). Such a conclusion is expected in view of results obtained recently within the stochastic approach [3]. The paper is resumed in Conclusions and completed by Appendices added to clarify some not widely known mathematical aspects of the exposition.

II. NON-DEBYE RELAXATION MODELS - A BRIEF RECOLLECTION

The non-Debye relaxation processes can be divided into "typical" and "atypical" phenomena. The words "typical" and "atypical" are referred to Jonscher's universal relaxation law (URL) [14] which gives the asymptotics of dielectric permittivity \( \hat{\varepsilon}(i\omega) \) as the function of frequency \( \omega \):

\[
\hat{\varepsilon}(i\omega) \propto (i\omega \tau)^{\alpha - 1}, \quad \omega \tau \ll 1
\]

\[
\Delta \hat{\varepsilon}(i\omega) = \varepsilon_0 - \hat{\varepsilon}(i\omega) \propto (i\omega \tau)^b, \quad \omega \tau \gg 1,
\tag{9}
\]

where the static permittivity \( \varepsilon_0 \) is the limit of \( \hat{\varepsilon}(i\omega) \) for \( \omega \to 0 \). The parameters \( 1 - \alpha \) and \( b \) belong to the range \((0, 1)\). "Typical" relaxation models mean that \( b \geq a - 1 \). Their encompass the HN model and its special cases: the Cole-Cole (CC) and the Cole-Davidson (CD) models. An example of "atypical" relaxation is given by the JWS model for which \( b < a - 1 \) which also includes the CC model.

The handful objects investigated in the relaxation physics are spectral functions defined as the ratios of the frequency dependent permittivities \( \phi(i\omega) = [\hat{\varepsilon}(i\omega) - \varepsilon_\infty]/[\varepsilon_0 - \varepsilon_\infty] \). Spectral functions, although being of purely empirical origin, are representable by simple expressions which analytic form comes from fitting the data. If transformed to the time domain the spectral functions bear the name of the response functions \( \phi(\cdot; t) \) introduced either by taking the time derivative of the relaxation function with the sign minus, namely \( \phi(\cdot; t) = -\phi(\cdot; t) \) or as the inverse Laplace transform \( L^{-1}[\phi(\cdot; s)] \) with \( \cdot \) denoting all parameters used to specify the model.

For the HN relaxation we have

\[
\hat{\phi}_{HN}(\alpha, \beta; i\omega \tau) = \left[1 + (i\omega \tau)^{\alpha - 1}\right]^{-\beta} \quad \text{and} \quad \phi_{HN}(\alpha, \beta; t) = \tau^{-1}(t/\tau)^{\alpha\beta - 1} E_{\alpha,\beta}^\beta[-(t/\tau)^\alpha]
\]

with \( \alpha, \beta \in (0, 1) \) and \( E_{\alpha,\beta}^\gamma(x) \) being the three parameter Mittag-Leffler function, see Appendix B. For \( \beta = 1 \) and \( \alpha \in (0, 1) \) it reduces to the CC relaxation whereas for \( \beta \in (0, 1) \) and \( \alpha = 1 \) it becomes the CD model. If \( \alpha = \beta = 1 \) then we have the Debye relaxation. For the general form of the HN model the exponents \( a \) and \( b \) appearing in Jonscher's URL are equal to \( 1 - a = \alpha \beta \) and \( b = \alpha \beta \).

In the case of JWS model we have

\[
\hat{\phi}_{JWS}(i\omega) = 1 - \left[1 + (i\omega \tau)^{-\alpha}\right]^{-\beta} \quad \text{and} \quad \phi_{JWS}(t) = \delta(t) - t^{-1} E_{\alpha,0\beta}^\beta[-(t/\tau)^\alpha],
\]

where \( \delta(t) \) stands for the Dirac \( \delta \)-function, \( \alpha \in (0, 1) \), and \( \alpha \beta \leq 1 \). The exponents in Eq. [9] are equal \( 1 - \alpha = \alpha \beta \) and \( b = \alpha \beta \). The JWS model becomes the CC relaxation for \( \alpha \in (0, 1) \) and \( \beta = 1 \) and the Debye relaxation for \( \alpha = 1 \) and \( \beta = 1 \) analogously as it takes place in the HN model.

The relaxation function \( n(t) \) is connected to the spectral function \( \hat{\phi}(i\omega) \) by the the inverse Laplace transform

\[
n(t) = L^{-1}[1 - \hat{\phi}(i\omega)]/(i\omega; t).
\tag{10}
\]

It implies that the relaxation function \( n_{HN}(t) \) relevant for the HN model [4] reads

\[
n_{HN}(t) = 1 - (t/\tau)^{\alpha\beta} E_{\alpha,1+\alpha\beta}^{\beta}[-(t/\tau)^\alpha],
\]

while for the JWS model it is equal to [4]

\[
n_{JWS}(t) = E_{\alpha,1}^{\beta}[-(t/\tau)^\alpha].
\]

Because of the future use of the CC relaxation function we quote it explicitly

\[
n_{CC}(t) = E_{\alpha}^{\beta}[-(t/\tau)^\alpha].
\]

For \( \beta = 1 \) it comes immediately from \( n_{JWS}(t) \) while for \( n_{HN}(t) \) emerges after some calculation effort presented in [8].
Recall that the transition rate \( r(t, \tau) = -\dot{n}(t)/n(t) = \phi(t)/n(t) \). The use of asymptotics of \( \phi(t) \) and \( n(t) \) for \( t \rightarrow 0 \) (given in Eqs. (3.29), (3.30)) as well as in \( \text{ibid.}, \) Eqs. (3.45), (3.46)) enables one to write

\[
\hat{r}_{\text{HN}}(t, \tau) \sim \frac{\alpha \beta}{\tau} \frac{(t/\tau)^{\alpha \beta - 1}}{\Gamma(1 + \alpha \beta) - (t/\tau)^{\alpha \beta}},
\]

and

\[
\hat{r}_{\text{JWS}}(t, \tau) \sim \frac{\alpha \beta}{\tau} \frac{(t/\tau)^{\alpha - 1}}{\Gamma(1 + \alpha) - \beta (t/\tau)^\alpha},
\]

where \( \alpha, \beta \in (0, 1) \). As it has been mentioned in the Introduction for any \( \alpha, \beta \in (0, 1) \) we arrive for \( t = 0 \) at a power-like singularity which vanishes for \( \alpha = \beta = 1 \), i.e., for the Debye relaxation.

### III. MEMORY FUNCTIONS \( M(t) \) AND \( k(t) \)

The combination of Eqs. (3) and (10) taken in the Laplace domain results in \( \hat{n}(s) = s^{-1} - B \hat{M}(s) \hat{n}(s) \) or \( s \hat{n}(s) = 1 - \hat{\phi}(s) \) and leads to

\[
\hat{M}(s) = B^{-1} \hat{\phi}(s)/[s \hat{n}(s)],
\]

which implies

\[
\hat{M}(s) = B^{-1} \{[\hat{\phi}(s)]^{-1} - 1\}^{-1}.
\]

The use of Eqs. (13) and (14) enables us to write

\[
\hat{k}(s) = B \hat{n}(s)/\hat{\phi}(s),
\]

and next

\[
\hat{k}(s) = B s^{-1} \{[\hat{\phi}(s)]^{-1} - 1\}.
\]

The memory functions, as seen from the fact that the Laplace transform \( \hat{r}(s, \tau) = L[\phi(t)/n(t); s] \) differs from \( \hat{M}(s) \) and \( \hat{k}(s) \), are not related in any straightforward way to the transition rate \( r(t, \tau) \). More informative is the connection between the memory functions \( \hat{M}(s) \) and \( \hat{k}(s) \) and the spectral function \( \hat{\phi}(s) \). Namely, for large \( [\hat{\phi}(s)]^{-1} \) (i.e., small \( [\hat{\phi}(s)] \)) we can write

\[
\hat{M}(s) \propto \hat{\phi}(s) \quad \text{and} \quad \hat{k}(s) \propto [s \hat{\phi}(s)]^{-1}.
\]

which implies that \( \hat{M}(s) \) has the same asymptotics as \( \hat{\phi}(s) \), whereas \( \hat{k}(s) \) behaves as \( [s \hat{\phi}(s)]^{-1} \). Thus the asymptotics of \( \hat{M}(s) \) and \( \hat{k}(s) \) may be guessed from the asymptotics of the spectral function \( \hat{\phi}(s) \) at \( \tau |s| \ll 1 \) known from the Jonscher URL.

In the next two subsections we shall focus our interest on finding the memory functions \( \hat{M}(t) \) and \( \hat{k}(t) \) for the HN and JWS models.

### Memory functions for the HN model

From (13) we get the Laplace transformed memory function \( \hat{M}_{\text{HN}}(\alpha, \beta; s) \) for the HN relaxation

\[
\hat{M}_{\text{HN}}(\alpha, \beta; s) = B^{-1} \{[1 + (\tau s)^\alpha]^{\beta} - 1\}^{-1}.
\]

Its Laplace inversion is known \[13, 22\]

\[
\hat{M}_{\text{HN}}(\alpha, \beta; t) = (Bt)^{-1} \sum_{r \geq 1} \left( \frac{t}{\tau} \right)^{\alpha \beta} E_{\alpha, \beta, r}(-t/(\tau)^\alpha).
\]

Using Eq. (16) and expressing \( E_{\alpha, \beta, r}(-t/(\tau)^\alpha) \) by the formula Eq. (15) we get the integral representation

\[
\hat{M}_{\text{HN}}(\alpha, \beta; t) = B^{-1} \int_0^\infty e^{-u \tau - \alpha} \left( \frac{u}{\tau^\alpha} \right)^\beta g_\alpha(u, t) \frac{du}{u},
\]

where the shorthand

\[
g_\alpha(u, t) = u^{-1/\alpha} g_\alpha(tu^{-1/\alpha})
\]

has been used to denote the one-sided Lévy stable distribution \( g_\alpha(x) \) with \( \alpha \in (0, 1) \) and \( x > 0 \) \[22, 21\]. (This class of distributions appears also in the KWW relaxation \[7, 22\]). The special cases of \( \hat{M}_{\text{HN}}(\alpha, \beta; t) \) for \( \alpha = 1 \) and/or \( \beta = 1 \), i.e., the D, CD and CC relaxations, respectively, can be derived either from the series \[13\] or from the just given integral form of \( \hat{M}_{\text{HN}}(\alpha, \beta; t) \). Because the CC relaxation will be used in the sequel we signify the memory function \( M_{\text{CC}}(\alpha; t) \), which coincides with \( \hat{M}_{\text{HN}}(\alpha, 1; t) \)

\[
M_{\text{CC}}(\alpha; t) = B^{-1} \frac{\tau_{-\alpha}^{-1}}{\Gamma(\alpha)}, \quad \alpha \in (0, 1).
\]

The memory function \( k_{\text{HN}}(\alpha, \beta; t) \) coupled to \( M_{\text{HN}}(\alpha, \beta; t) \) is obtained in the Laplace space from Eq. (13) with the spectral function \( \hat{\phi}_{\text{HN}}(\alpha, \beta; s) \) being used. That gives

\[
\hat{k}_{\text{HN}}(\alpha, \beta; s) = B s^{-1} \{[\tau_{-\alpha}^{\alpha} (\tau_{-\alpha}^{\beta} + s^\alpha)^\beta - 1\}
\]

and in the time domain becomes

\[
k_{\text{HN}}(\alpha, \beta; t) = B \{[\tau/t]^{\alpha \beta} E_{\alpha, 1-\alpha, \beta}^{-1}(-t/(\tau)^\alpha) - 1\}. \quad (19)
\]

To earn the CC relaxation we take \( k_{\text{HN}}(\alpha, 1; t) \). In this case we apply the definition of the Mittag-Leffler polynomials – displayed in Eq. \[15\] – according to which \( E_{1-\alpha}^{-1}(x) = \Gamma(1 - \alpha)^{-1} - x \). That allows one to express \( k_{\text{HN}}(\alpha, 1; t) \) as

\[
k_{\text{CC}}(\alpha; t) = B \tau_{-\alpha, \beta}^{\alpha} \frac{\Gamma(1 - \alpha)^{-1}, \quad \alpha \in (0, 1),}.
\]

Note that in the Laplace domain memory functions \( M_{\text{CC}}(\alpha; s) \) and \( k_{\text{CC}}(\alpha; s) \), \( \alpha \in (0, 1) \), take particularly simple forms

\[
M_{\text{CC}}(\alpha; s) = B^{-1} (\tau s)^\alpha; \quad k_{\text{CC}}(\alpha; s) = B \tau_{-\alpha}^{-\alpha} s^{-1+\alpha}.
\]

(21)
Memory functions for the JWS model

For the JWS model we get

\[ \hat{M}_{\text{JWS}}(\alpha, \beta; s) = B^{-1} \left[ s^{-\alpha} (\tau^{-\alpha} + s^{\alpha})^{\beta} - 1 \right], \]

which for \( \beta = 1 \) is tantamount to \( \hat{M}_{\text{CC}}(\alpha; s) \) whose Laplace transform inversion gives \( k_{\text{CC}}(\alpha; t) \). The inverse Laplace transform of \( \hat{M}_{\text{JWS}}(\alpha, \beta; s) \) allows one to determine the memory function \( M_{\text{JWS}}(\alpha, \beta; t) \) in an unique way. It is

\[ M_{\text{JWS}}(\alpha, \beta; t) = B^{-1} \left\{ t^{-1} E_{\alpha,0}^{\beta} \left[ - (t/\tau)^{\alpha} \right] - \delta(t) \right\}. \]

The memory function \( \hat{k}_{\text{JWS}}(\alpha, \beta; s) \) received from Eq. \( \text{[14]} \) gives

\[ \hat{k}_{\text{JWS}}(\alpha, \beta; s) = B \left\{ s \left[ 1 + (\tau s)^{-\alpha} \right] - s \right\}^{-1}, \quad (22) \]

which for \( \beta = 1 \) implies \( \hat{k}_{\text{CC}}(\alpha; s) = \hat{M}_{\text{CC}}(\alpha; s) \).

To compute \( k_{\text{JWS}}(\alpha, \beta; t) \) we represent \( \hat{k}_{\text{JWS}}(\alpha, \beta; s) \) as \( (Bx/s)/(1-x) \) where \( x = [1 + (\tau s)^{-\alpha}]^{-\beta} \). Assuming that \( |x| < 1 \) we can express \( \hat{k}_{\text{JWS}}(\alpha, \beta; s) \) as

\[ \hat{k}_{\text{JWS}}(\alpha, \beta; s) = B \sum_{r \geq 0} \frac{s^{-1}}{\left[ 1 + (\tau s)^{-\alpha} \right]^{\beta(r+1)}} \]

whose Laplace transform reads

\[ k_{\text{JWS}}(\alpha, \beta; t) = B \sum_{r \geq 0} E_{\alpha,1}^{\beta(r+1)} \left[ - (t/\tau)^{\alpha} \right]. \]

Applying the integral form of \( E_{\alpha,1}^{\beta} \left[ - (t/\tau)^{\alpha} \right] \) given by Eq. \( \text{[13]} \) and interchange the order of integration and summation, we represent \( k_{\text{JWS}}(\alpha, \beta; t) \) as follows

\[ k_{\text{JWS}}(\alpha, \beta; t) = B \int_0^{\infty} e^{-\tau^{-\alpha} u} u^{\beta-1} \]

\[ \times L^{-1} \left\{ E_{\beta,1}(u^{\beta} s^{\alpha}) s^{\alpha-1} e^{-u s^{\alpha}} ; t \right\} \partial u. \quad (23) \]

By virtue of \( \text{[10]} \), Eq. \( (14) \) we write \( E_{\beta,1}(u^{\beta} s^{\alpha}) = -u^{-\beta} s^{-\alpha} E_{\beta,0}(u^{-\beta} s^{-\alpha}) \), so Eq. \( \text{[23]} \) can be expressed as

\[ k_{\text{JWS}}(\alpha, \beta; t) = -B \int_0^{\infty} e^{-\tau^{-\alpha} u} u^{\beta-1} \]

\[ \times L^{-1} \left\{ E_{\beta,0}(u^{-\beta} s^{-\alpha}) s^{\alpha-1} e^{-u s^{\alpha}} ; t \right\} \partial u. \]

Now, using the series form of the two parameter Mittag-Leffler function and once again the integral Eq. \( \text{[14]} \) we get

\[ k_{\text{JWS}}(\alpha, \beta; t) = -B \sum_{r \geq 0} E_{\alpha,1}^{-\beta(r+1)} \left[ - (t/\tau)^{\alpha} \right]. \quad (24) \]

In the Laplace domain it reads

\[ \hat{k}_{\text{JWS}}(\alpha, \beta; s) = -B \sum_{r \geq 0} \frac{s^{-1}}{\left[ 1 + (s^{\alpha})^{-\beta} \right]^{-\beta r}}. \quad (25) \]

which can be calculated for \( \| 1 + (s^{\alpha})^{-\beta} \| < 1 \). So we obtain Eq. \( \text{[22]} \). In this way we apply the procedure presented in \( \text{[8, 10]} \). Accordingly, the right hand side expression in Eq. \( \text{[22]} \) can be expressed in the series form in two different ways which gives the solution under assumption \( \| 1 + (\tau s)^{-\alpha} \| > 1 \) or \( \| 1 + (\tau s)^{-\alpha} \| < 1 \).

Above, we have shown that in both these regions the solutions are equivalent. The CC relaxation model is also obtained from Eq. \( \text{[24]} \) by the appropriate setting \( \beta = -\alpha \). That enables one to write Eq. \( \text{[23]} \) in the form:

\[ k_{\text{JWS}}(\alpha, 1; t) = B \int_0^{\infty} e^{-\tau^{-\alpha} u} u^{\beta-1} \left\{ 1 + (u \tau s)^{-\alpha} \right\} \partial u, \]

which boils down to Eq. \( \text{[20]} \).

IV. THE TIME EVOLUTION EQUATIONS

Two types of the time smearing

To show that two types of the time smearing applied to Eq. \( \text{[11]} \) lead, under the condition \( \text{[5]} \), to the same result we begin with taking the time derivative of Eq. \( \text{[14]} \), viz.

\[ \dot{n}(t) = -B \frac{d}{dt} \int_0^t M(t - \xi)n(\xi) \partial \xi \]

\[ = -B \frac{d}{dt} \int_0^t M(\eta)n(t - \eta) \partial \eta. \]

In turn, thanks to the standard procedure involving Leibniz rule, replacement of the derivative \( \frac{d}{dt} \rightarrow -\frac{d}{d\eta} \) and integration by parts, we transform the latter into

\[ -\dot{n}(t) = B M(t) + B \int_0^t M(\eta)\dot{n}(t - \eta) \partial \eta \]

\[ = B M(t) + B \int_0^t M(t - \xi)\dot{n}(\xi) \partial \xi, \quad (26) \]

which is in fact the integral equation for the response function \( \phi(t) \). Multiplying Eq. \( \text{[20]} \) by \( k(T - t) \) and integrating it with respect to \( t \) on \([0, T]\) we arrive at the equation which right-hand side involves the time smeared \( \dot{n}(t) \). Consequently,

\[ \int_0^T k(T - t)\dot{n}(t) \partial t = -B \int_0^T k(T - t)M(t) \partial t \]

\[ - B \int_0^T \left[ \int_0^t k(T - t)M(t - \xi)\dot{n}(\xi) \partial \xi \right] \partial t. \quad (27) \]

The integration domain of the double integral is of right triangle form so, setting \( T - t = u \) we can rewrite the
second line of Eq. 27 into

\[ -B \int_0^T \left[ \int_0^t (T-t) M(t-\xi) \dot{n}(\xi) d\xi \right] dt = -B \int_0^T \left[ \int_0^{T-\xi} k(u) M(T-\xi - u) du \right] \dot{n}(\xi) d\xi. \]

If the memory kernels \( M(t) \) and \( k(t) \) satisfy Eq. 25 then the RHS above equals \(-Bn(t) + B \) and we get Eq. 26. Thus, we can claim that the condition 24 makes Eq. 25 equivalent to Eq. 26. We can also conclude that the smearing of \( r(t, \tau)n(t) \) can be exchanged by the smearing of the first time derivative \( \dot{n}(t) \) as it has been shown in 10.

The Cole-Cole model: Riemann-Liouville integrals and Caputo derivatives

The simplest example which illustrates the meaning and practical usefulness of the relation between Eqs. 23 and 24 is given by the CC model. Eq. 25 with \( M_{CC}(\alpha; t) \) of Eq. 17 reads

\[ n_{CC}(t) = 1 - \tau^{-1}(I^\alpha n_{CC})(t), \]  

(28)

where

\[ (I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0 \]

is the Riemann–Liouville fractional integral equivalently defined as the negative fractional derivative in the Riemann–Liouville sense, namely \( I_0^-\alpha = D_0^\alpha \). Taking the Riemann-Liouville fractional derivative \( D^\alpha f(x) = (\frac{d}{dx})^{\alpha} f(x) \) of both sides of Eq. 24 we get

\[ (D^\alpha n_{CC})(t) = D^\alpha 1 - \tau^{-\alpha}(D^\alpha I^\alpha n_{CC})(t), \]  

(29)

which is nothing else than Eqs. (2.141) and (2.106) of [23] for \( \alpha \in (0,1) \). Standard relation \( D^\alpha C = C x^{-\alpha}/\Gamma(1-\alpha) \) valid for any constant \( C \) and the semigroup property \( (D^\alpha D^-\alpha f)(x) = f(x) \) allow one to rewrite Eq. 29 as

\[ (D^\alpha n_{CC})(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = -\tau^{-\alpha}n_{CC}(t). \]  

(30)

Two terms in the left hand side of Eq. 30, with \( \alpha \in (0,1) \), reduce to the fractional derivative in the Caputo sense \( (C^\alpha D^{-\alpha} n_{CC})(t) \) which coincides with \( (I^{1-\alpha} D^-\alpha n_{CC})(t) \), also with \( \alpha \in (0,1) \). Hence, the integral equation 28 becomes the integro–differential equation

\[ C^\alpha D^\alpha n_{CC}(t) = -\tau^{-\alpha}n_{CC}(t), \]  

(31)

which we recognize as Eq. 27 with the memory function \( k_{CC}(\alpha; t) \) given by Eq. 24. Thus our approach reproduces classical forms of the evolution equations governing the CC model: Eq. 28 is equivalent to [4, Eq. (3.7.43)], Eq. 30 is analogous to [4, Eq. (3.7.49)], and Eq. 31 is the same as [4, Eq. (3.10)].

The Havriliak-Negami and Jurlewicz-Weron-Stanislavsky models: more complicated pseudo-operators enter the game

As remarked in the first subsection of Sec. 11 it is natural to treat Eqs. 28 and 29 as equivalent. Thus, to get information on the process, it is enough to find and solve one of these equations. Knowing the memory functions \( k_{HN}(\alpha, \beta; t) \) and \( M_{JWS}(\alpha, \beta; t) \) we can get relevant forms of Eqs. 28 and 29 without tedious calculations involving operator and power series methods [4, 13, 28]. If we focus our interest on the HN relaxation then we take Eq. 28 with memory function \( k_{HN}(\alpha, \beta; t) \). It immediately yields the equation proposed in [4, Eq. (3.40)]:

\[ C(D^\alpha + \tau^{-\alpha})^\beta n(t) = -\tau^{-\alpha\beta}, \]  

(32)

where the pseudo–operator on the left hand side belongs to the class of Prabhakar–like integral operators and for the considered case it is defined as [4, Eq. (B.23)]

\[ C(D^\alpha + \tau^{-\alpha})^\beta n(t) = \int_0^t (t-\xi)^{-\alpha\beta} E_{\alpha,1-\alpha\beta} \left[ \frac{(-t-\xi)^\alpha}{\tau^{\alpha\beta}} \right] \dot{n}(\xi) d\xi. \]  

(33)

Note that we are keeping the notation of [4, Appendix B] and it has to be pointed out here that the pseudo–operator 25 must be distinguished from the pseudo–operator \( (C^\alpha D^\alpha + \tau^{-\alpha})^\beta \) involving the Caputo derivative (also considered in the paper [4]). This is evidently seen from fact that using Eq. (B6) it can be concluded \( C(D^\alpha + \tau^{-\alpha})^\beta n(t) \). Hence, the integrand in the formula Eq. (34) can be rewritten due to the substitution

\[ t^{-\alpha} E_{\alpha,0}^{-\beta}(-\tau^{-\alpha} t^\alpha) = D^{1-\alpha\beta} [t^{-\alpha\beta} E_{\alpha,1-\alpha\beta}^{-\beta}(-\tau^{-\alpha} t^\alpha)]. \]  

(34)

Note that this expression is quoted in Eq. 13. Thereafter, we take the Riemann-Liouville fractional derivative \( D^\beta \) of both sides of the resulting equation and make use of the semigroup property \( (D^\mu D^\nu f)(x) = (D^{\mu+\nu} f)(x) \) [23, Eq. (2.127)] taken for \( \mu = \alpha\beta \) and \( \nu = 1-\alpha\beta \). Thus, we obtain

\[ \int_0^t \frac{d}{dt} \left\{ (t-\xi)^{-\alpha\beta} E_{\alpha,1-\alpha\beta}^{-\beta}(-\tau^{-\alpha} (t-\xi)^\alpha) \right\} n(\xi) d\xi = D^\alpha n(t). \]  

(35)

Interchange (in a legitimate way) of the order of integration and derivation enables us to represent the left-hand
side of the equation above as the action of the pseudoooperator \( (D^\alpha + \tau^{-\alpha})^\beta \), *viz.*

\[
\frac{d}{dt} \int_0^t (t - \xi)^{-\alpha\beta} E_{\alpha,1-\alpha\beta} \left[ \frac{(t - \xi)^\alpha}{\tau^\alpha} \right] n(\xi) \, d\xi
= (D^\alpha + \tau^{-\alpha})^\beta n(t).
\]

Using this Eq. \( \text{35} \) takes the form of kinetic equation

\[
(D^\alpha + \tau^{-\alpha})^\beta n(t) = \frac{t^{-\alpha\beta}}{\Gamma(1-\alpha\beta)}, \quad (36)
\]

which should be completed with a suitable initial condition. Eq. \( \text{39} \), in the context of the JWS model, first appeared in \( \text{29} \), Eq. (4.3) being justified by under–and overshooting subordination technique applied for the anomalous diffusion \( \text{27} \). Eq. \( \text{39} \) was also obtained using the integral transform methods in \( \text{4} \).

V. TOWARDS MULTICHANNEL DESCRIPTION OF RELAXATION PHENOMENA

Hitherto, we have considered models describing the relaxation phenomena assumed to be underpinned by a single stochastic process \( U \) characterized by the Laplace exponent \( \Psi(s) \). The complete Bernstein character of \( \Psi(s) \) allows to introduce a pair of coupled memory functions related to this process, namely \( M(t) \) and \( k(t) \), whose Laplace transforms are given as \( M(s) = [\Psi(s)]^{-1} \) and \( k(s) = \Psi(s)/s \) and consequently satisfy a constraint \( M(s)k(s) = s^{-1} \). The latter leads to the conclusion that integral or integro-differential equations Eqs. \( \text{38} \) and \( \text{19} \) which even if formulated in a two-fold way emerge from the same stochastic process \( U \) and describe the same physical reality.

In what follows we are going to pay attention on an extension of the presented approach which sheds light on relaxation processes assumed to be described by effective spectral functions composed of more elementary ones, say \( \phi_j(s), j = 1, 2 \) being e.g. any of the standard non–Debye patterns. Each such component, if treated separately, leads to the complete monotone memory function and is related to its own Laplace exponent \( \Psi_j(s) \).

In the framework of the linear response approach we may consider a simple model for which the effective spectral function is represented in the form of a linear combination \( \hat{\phi}(s) = C_1 \phi_1(s) + C_2 \phi_2(s) \), with positive coefficients \( C_1 \) and \( C_2 \) next put equal to 1 to simplify the exposition. Obviously \( \hat{\phi}(s) \) is CMF and may be written as

\[
\hat{\phi}(s) = [1 + \Psi_1(s)]^{-1} + [1 + \Psi_2(s)]^{-1}.
\]

The following question immediately arises - is it possible to relate \( \hat{\phi}(s) \) with a single process arisen as a cumulative effect of processes \( U_1 \) and \( U_2 \), say? To answer such a question we should determine the new Laplace exponent \( \hat{\Psi}(s) \) which would satisfy \( \hat{\phi}(s) = [1 + \hat{\Psi}(s)]^{-1} \) and be a CBF. We have

\[
\hat{\phi}(s) = \hat{\phi}_1(s) + \hat{\phi}_2(s) = \frac{1}{1 + \Psi_1(s)} + \frac{1}{1 + \Psi_2(s)} = \frac{1 + 1}{1 + \Psi_1(s) + \Psi_2(s)} = \frac{1}{1 + \Psi(s)}.
\]

From that we conclude \( \hat{\Psi}(s) = [1 - \hat{\phi}(s)]/[1 - [1 - \hat{\phi}(s)] \}, \)
which after substituting Eq. \( \text{37} \) mutatis mutandis takes the series expansion form

\[
\Psi(s) = \sum_{r \geq 0} \left( \frac{1}{1 + \Psi_1(s) + \Psi_2(s)} \right)^{r+1}.
\]

We point out that this series expansion is well–defined, being for \( x = \Phi_1(s), y = \Phi_2(s) \) the estimate

\[
\frac{xy - 1}{(1 + x)(1 + y)} < 1, \quad x, y > 0
\]

valid. This implies the geometric series expansion \( \text{38} \). On the other hand \( \hat{\Psi}(s) \) is CBF when it is the series of CBFs and/or the first term in the series is completely monotone. For any CBF \( \Psi_1(s) \) and \( \Psi_2(s) \) the numerator of this expression is CMF as a reciprocal of CBF.

To guarantee complete monotonicity of the whole expression it will be sufficient to show that its denominator is a CBF which comes from the property that any fraction CMF/CBF is CBF. Obviously \( 1 + \Psi_1(s) + \Psi_2(s) \) is the complete Bernstein function which helps us to analyse analogous property of \( \Psi_1(s)\Psi_2(s)/[1 + \Psi_1(s) + \Psi_2(s)] \). Using \( \text{34} \) Proposition 7.1 our CBF \( \hat{\Psi}(s) \) will be CBF.

Recalling the analytic forms of \( \hat{\Psi}(s) \) relevant for the standard non-Debye relaxation patterns

| \( M(s) \) | \( \Psi(s) \) |
| --- | --- |
| CC | \( B^{-1} \{ (\tau s)^\alpha \} \}^{-1} \) | \( B(\tau s)^\alpha \) |
| HN | \( B^{-1} \{ [1 + (\tau s)^\alpha]^\beta - 1 \}^{-1} \) | \( B s \{ [1 + (\tau s)^\alpha]^\beta - 1 \} \) |
| JWS | \( B^{-1}[1 + (\tau s)^{-\alpha}]^\beta - 1 \) | \( B s \{ [1 + (\tau s)^{-\alpha}]^\beta - 1 \} \) |

we see that this condition may be satisfied only for \( \Psi_1(s) = s^\alpha, \Psi_2(s) = s^\beta \), which fulfill the additional constraint \( \alpha + \beta = 1 \). Therefore, we conclude that our simple “two channel” model, if based on the sum of Cole-Cole patterns, share properties of standard “one channel” models and may be proposed as a prospective challenger for description phenomena which exhibit more than one characteristic time or go beyond the Jonscher URL. This confirms theoretical results of \( \text{33} \) and also justifies approach proposed in \( \text{3} \) to interpret experimental data obtained in studies of magnetic relaxation and magneto-calorimetric effect.
VI. CONCLUSIONS

We have considered the HN and JWS relaxation patterns starting from the basic assumption that relaxation models are governed by integral or integro-differential equations with kernels which mimic memory effects and in such a way influence the time behaviour of relaxing systems. Equations been studied are of the Volterra type and involve the time non-localities, expressed either through the integral operator with the time smeared integral kernel denoted by \( M \) or contain the generalized (fractional) differential operator involving the time non-local kernel \( k \). If the kernels \( M \) and \( k \) satisfy the Sonine condition then both equations yield equivalent results for the relaxation function \( n(t) \) has been looked for.

Simultaneously, such derived equations stem from the analysis of the stochastic processes which govern the relaxation and mathematically are described by infinite divisible distributions. It implies that the memory functions \( M \) and \( k \) responsible for behavior of the physical system are uniquely connected with the Laplace exponent which is mathematical characterization of the underlying process. Our approach merges dynamical and statistical/stochastic aspects of the relaxation phenomena through the positive-valued functions which properties guarantee that both physical as well as mathematical requirements of the theory are satisfied.

The simplest illustration relevant for our construction are equations relevant for the Cole-Cole relaxation pattern. They contain either the Riemann–Liouville fractional integral [28] or the fractional derivatives in the Caputo sense [31] and are solvable by the Mittag-Leffler function. Physically the first type of these equations describes the smearing of \( r_{CC}(t, \tau)n_{CC}(t) \) and another one characterizes the smearing of \( n_{CC}(t) \).

Analogical, however much complicated, situation we meet for the HN and JWS models. Nevertheless pseudo–operators traditionally used in their description may be treated as connected with the same types of the time-smearing: the pseudo–operator \( C(D^\alpha + \tau^{-\alpha})^\beta n(t) \) reflects the time-smearing \( \dot{n}(t) \) whereas \( (D^\alpha + \tau^{-\alpha})^\beta n(t) \) is related to the time-smeread product of \( n(t) \) and the transition rate \( r(t, \tau) \). We are convinced that our approach provides us with mathematically well-defined integral or integro-differential counterparts of pseudo–differential operators and will be helpful to clarify this duality.

Resumming the paper - the presented approach enables us to explain, in the framework of one scheme, the origin of the time evolution equations used to describe the relaxation phenomena. This flows out from two sources: first, the relaxation phenomena are rooted in stochastic processes which general (mathematical) properties govern their physical properties and second, evolution equations describing dynamics of the relaxation phenomena involve memory effects encoded by functions directly related to objects which characterize underlying stochastic processes. Moreover, our scheme opens possibilities to develop and verify methods leading to the consistent description of multichannel processes.

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Appendix A: Classes of positive functions: completely monotone, Bernstein and completely Bernstein

The completely monotone functions (CMFs) \( F(x) \) are non-negative functions on \( \mathbb{R}_+ \) whose all derivatives exist and alternate, i.e.,

\[
(-1)^n F^{(n)}(x) \geq 0, \quad n = 0, 1, \ldots,
\]

where \( F^{(n)}(x) = \frac{d^n F(x)}{dx^n} \). According to the Bernstein theorem [34]: \( x \in [0, \infty) \rightarrow F(x) \in \text{CMF} \) iff

\[
F(x) = \int_0^\infty \exp(-xt)g(t)dt \quad \text{(A1)}
\]

where \( g(t) \geq 0 \) for \( t \in [0, \infty) \). Eq. (A1) uniquely connects the CMF with the non-negative integrable function. It should be also remembered that the product and convex sum of two CMFs is a CMF.

The Bernstein functions (BF) are non-negative functions on \( \mathbb{R}_+ \) having CMF derivative [34]: \( h(s) > 0 \) is a BF if

\[
(-1)^{n-1}h^{(n)}(s) \geq 0, \quad n = 1, 2, \ldots.
\]

Moreover, \( h(s) \) is a BF iff it admits the representation

\[
h(s) = a + bs + \int_0^\infty (1 - e^{-\xi}) d\mu(\xi),
\]

where \( a, b \geq 0 \) and \( \mu \) (the Lévy measure) is a positive measure on \( (0, \infty) \) satisfying \( \int_{\mathbb{R}_+} \xi/(\xi + 1) d\mu(\xi) < \infty \), see [1, Theorem 2.6] or [34, Theorem 3.2]. The following properties of the BF deserve to be mentioned:

(BF1) the convex sum of two BFs is another BF;

(BF2) the composition of BFs is also a BF, i.e \( h_1(h_2(s)) \) is a BF for all \( h_1 \) and \( h_2 \) being BFs;

(BF3) the composition of a CMF and a BF is another CMF, i.e. \( F(h(s)) \) is a CMF.
The complete Bernstein functions (CBF) \( \text{CBF} \) are defined: \( c(s) \) is a CBF, \( s > 0 \), if \( c(s)/s \) is the Laplace transform of a CMF restricted to the positive semiaxis or, equivalently, the Stieltjes transform of a positive function restricted to this domain - this comes from the fact all Stieltjes functions are completely monotone. CBFs is a subclass of BFs so the properties (BF1)-(BF3) can be also rewritten for CBFs. Thus,

\( \text{(CBF1)} \) the convex sum of two CBFs is another CBF;

\( \text{(CBF2)} \) the composition of CBFs is a CBF;

\( \text{(CBF3)} \) the composition of a CMF and a CBF is another CMF;

are complemented by

\( \text{(CBF4)} \) if \( c(s) \) is a CBF then \( s/c(s) \) is a CBF;

\( \text{(CBF5)} \) if \( \alpha, \beta \in (0, 1) \) such that \( \alpha + \beta \leq 1 \) then \( c_1^\alpha \cdot c_2^\beta \) is a CBF for all \( c_1 \) and \( c_2 \) being CBFs.

For our purposes the most important is the property CBF4.

### Appendix B: Three parameter Mittag-Leffler function

The three parameter Mittag-Leffler function is defined through the power series

\[
E_{\alpha,\nu}^\mu(x) = \sum_{r \geq 0} \frac{\mu \cdot z^r}{r! \Gamma(\nu + \alpha r)},
\]

where \( \Re(\alpha), \Re(\nu), \Re(\mu) > 0 \) and \( z \in \mathbb{C} \), \( \mu \) denotes the Pochhammer symbol (raising factorial) equal to \( \Gamma(\mu + r) / \Gamma(\mu) = \mu(\mu + 1) \ldots (\mu + r - 1) \in \mathbb{N}_0 \). The Pochhammer symbol for \( \mu = 1 \) is equal to \( r ! \) and \( \text{B1} \) depends on two parameters \( \alpha \) and \( \nu \) only. This case is named the two parameter Mittag-Leffler (Wiman) function and it is quoted as \( E_{\alpha,\nu}(z) = E_{1,\nu}^\mu(z) \). For \( \mu = \nu = 1 \) Eq. \( \text{B1} \) reduces to the one parameter (standard) Mittag-Leffler function \( E_{\nu}(z) = E_{1,1}^1(z) \) which generalizes the exponential function. For \( \alpha, \beta, \gamma > 0, \beta - \alpha \gamma > 0 \) and \( z \in \mathbb{R} \), the three, two and one parameter Mittag-Leffler functions are completely monotone. They find numerous applications in the relaxation theory coming from the frequently met shape of their Laplace transform

\[
\mathcal{L}[t^{\nu-1} E_{\alpha,\nu}^\mu(\lambda t^\alpha); s] = s^{-\nu} (1 - \lambda s^{-\alpha})^{-\mu}
\]

for \( \Re(\nu), \Re(s) > 0, |s| > |\lambda|^{1/\Re(\alpha)} \). The fractional derivative in Riemann–Liouville sense of \( t^{\nu-1} E_{\alpha,\nu}^\mu(\lambda t^\alpha) \) yields Eq. (5.1.34) of \[7\] namely

\[
\{D^\beta [t^{\nu-1} E_{\alpha,\nu}^\mu(\lambda t^\alpha)]\}(x) = x^{\nu-\beta-1} E_{\alpha,\nu-\beta}(\lambda x^\beta).
\]

The general form of the integral representation of \( E_{\alpha,\nu}^\mu(\lambda t^\alpha) \) is given by \[10\] Eq. (15). In the paper we need only two cases, namely

\[
E_{\alpha,1}^\nu(-\lambda t^\alpha) = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-\lambda u} u^{\gamma-1} \mathcal{L}^{-1}[s^{\gamma-1} e^{-us}; t] \, du
\]

and derived in \[8\] Eq. (11) or \[10\] Eq. (17)

\[
E_{\alpha,\alpha\delta}^\nu(-\lambda t^\alpha) = t^{1-\alpha \mu} \int_0^\infty e^{-\lambda u} u^{\mu-1} g_{\alpha}(u, t),
\]

where \( g_{\alpha}(u, t) = \mathcal{L}^{-1}[e^{-us}; t] \). Moreover, \( g_{\alpha}(u, t) = u^{-1/\alpha} g_{1}(1, tu^{-1/\alpha}) = u^{-1/\alpha} g_{1}(t, u^{-1/\alpha}) \) and \( g_{\alpha}(\sigma) \) with \( \sigma > 0 \) and \( \alpha \in (0, 1) \) is the one-sided Lévy stable distribution \[20\], \[21\]. From the definition \( \text{B1} \) we can also derive the Mittag-Leffler polynomials \[10\] Eq. (22)

\[
E_{\alpha,\nu}^n(x) = \sum_{r=0}^n \binom{n}{r} \frac{(-x)^r}{\Gamma(\nu + r)}, \quad \alpha, \nu > 0,
\]

which also satisfied properties \[12\] and \[13\].

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[1] Ch. Berg, Stieltjes-Pick-Bernstein-Schoenberg and their connection to completely monotonicity, in "Positive Define Functions: From Schoenberg to Space-Time challenges", edited by J. Mateu and E. Porcu, (Dep. Math. of Univ. Jaume I, Castellon, Spain, 2008).

[2] C. J. F. Böttcher and P. Bordewijk, Theory of Electric Polarization, Elsevier, Amsterdam 1978.

[3] M. Fitta, R. Pelka, P. Konieczny, M. Balanda, Multifunctional molecular magnets: magnetocaloric effect in octacyanometallates, Crystals 9 (2019) 9.

[4] R. Garrappa, F. Mainardi, and G. Maione, Models of dielectric relaxation on completely monotone functions, Frac. Calc. Appl. Anal. 19(5) (2016) 1105–1160; corrected version available in arXiv: 1611.04028.

[5] W. G. Glöckle and T. F. Nonnenmacher, Fox function representation of non-Debye relaxation processes, J. Stat. Phys. 71 (1993) 741–757.

[6] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, "Mittag-Leffler Functions, Related Topics and Applications" (Springer, Berlin, 2014).

[7] K. Görka, K. A. Penson, G. Dattoli, and G. H. E. Duchamp, Operator solutions for fractional Fokker–Planck equations, Phys. Rev. E 85 (2012) 031138.

[8] K. Görka, A. Horzela, L. Bratek, K. A. Penson, and G. Dattoli, The Haussdorff-Neumann relaxation and its relatives: the response, relaxation and probability density functions, J. Phys. A: Math. Theor. 51 (2018) 135202.

[9] K. Görka, A. Horzela, and T. K. Pogany, A note on the article "Anomalous relaxation model based on the fractional derivative with a Prabhakar-like kernel" [Z. Angew. Math. Phys. (2019) 70: 42]; Z. Angew. Math. Phys. 70 (2019) 141.
[10] K. Görsk and A. Horzela, The Volterra type equations related to the non-Debye relaxation, Commun. Nonlinear Sci. Numer. Simulat. 85 (2020) 105246.

[11] K. Görsk, A. Horzela, E. K. Lenzi, G. Pagnini, and T. Sandev, Generalized Cattaneo (telegrapher’s) equation in modeling anomalous diffusion phenomena, Phys. Rev. E 102 (2020) 022128.

[12] A. Hanyga, A comment on a controversial issue: a generalized fractional derivative cannot have a regular kernel, Fract. Calc. Appl. Anal. 23 (2020) 211.

[13] R. Hilfer, H-function representations for stretched exponential relaxation and non-Debye susceptibilities in glassy systems, Phys. Rev. E 65 (2002) 061510.

[14] A. K. Jonscher, The universal dielectric response and its physical significance, IEEE Transactions on Electrical Insulation 27 (1992) 407.

[15] A. A. Khamzin, R. R. Nigmatullin, and I. I. Popov, Justification of the empirical laws of the anomalous dielectric relaxation in the framework of the memory function formalism, Fract. Calc. Appl. Anal. 17 (2014) 247.

[16] A. N. Kochubei, General fractional calculus, evolution equations, and renewal processes, Integr. Equations. Oper. Theory 71 (2011) 583–600.

[17] F. Kremer and A. Schönhals, Broadband Dielectric Spectroscopy, Berlin Heidelberg Springer Verlag 2003.

[18] F. Mainardi, A note on the equivalence of fractional equations to differential equations with varying coefficients, Mathematics 6 (2018), Article ID=8, 5pp.

[19] R. R. Nigmatullin and Ya. E. Ryabov, Cole-Davidson dielectric relaxation as a self-similar relaxation process, Phys. Stat. Sol. 19 (1948) 1115.

[20] K. A. Penson and K. Görsk, Exact and explicit probability densities for one-sided Lévy stable distributions, Phys. Rev. Lett. 105 (2010) 210604.

[21] H. Pollard, The representation of $e^{-x^a}$ as a Laplace integral, Bull. Amer. Math. Soc. 52 (1946) 908.

[22] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_a(-x)$, Bull. Amer. Math. Soc. 54 (1948) 1115.

[23] I. Podlubny, ”Fractional Differential Equations”, (Academic Press, San Diego, 1999).

[24] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971) 7–15.

[25] C. F. A. E. Rosa and E. Capelas de Oliveira, Relaxation equations: fractional models, J. Phys. Math. 5 (2015) 1000146.

[26] T. Sandev, Ž. Tomovski, J. L. Dubbeldam, and A. V. Chechkin, Generalized diffusion-wave equation with memory kernel, J. Phys. A: Math. Theor. 52 (2019) 015201.

[27] A. Stanislavsky and K. Weron, Anomalous diffusion with under- and overshooting subordination: A competition between the very large jumps in physical and operational times, Phys. Rev. E 82 (2010) 051120.

[28] A. Stanislavsky, K. Weron, and A. Weron, Anomalous diffusion approach to non-exponential relaxation in complex physical systems, Commun. Nonlinear Sci. Numer. Simulat. 24 (2015) 117–126.

[29] A. Stanislavsky and K. Weron, Atypical Case of the Dielectric Relaxation Responses and its Fractional Kinetic Equation, Frac. Calc. Appl. Anal. 19(1) (2016) 212–228.

[30] A. Stanislavsky and K. Weron, Stochastic tools hidden behind the empirical dielectric relaxation laws, Rep. Prog. Phys. 80 (2017) 036001.

[31] A. Stanislavsky and A. Weron, Control of the transient subdiffusion exponent at short and long times, Phys. Rev. Research 1 (2019) 023006.

[32] A. Stanislavsky and K. Weron, Fractional-calculus tools applied to study the nonexponential relaxation in dielectrics in V. E. Tarasov (ed.) ”Handbook of Fractional Calculus with Applications. Volume 5. Applications in Physics, Part B” (De Gruyter, Berlin, 2019), 53–70.

[33] A. Stanislavsky and A. Weron, Accelerating and retarding anomalous diffusion: A Bernstein function approach, Phys. Rev. E 101 (2020) 052119.

[34] R. L. Schilling, R. Song, and Z. Vondraček, ”Bernstein Functions: Theory and Applications” (De Gruyter, Berlin, 2010).

[35] R. L. Schilling, An introduction to Lévy and Feller processes, in From Lévy–type processes to parabolic SPDEs. Adv. Courses Math. (CRM Barcelona, Birkhäuser – Springer, Cham, 2016), 1–126.

[36] In mathematical literature the Laplace exponent is better known as the Lévy, or characteristic exponent of some stochastic process $U$.

[37] The role of Bernstein, complete Bernstein and completely monotone functions as elucidating problems of anomalous diffusion, in particular as providing tools which allow to judge probabilistic interpretation of solutions, has been noticed quite recently and is the subject of still growing interest.

[38] If $\Psi(s)$ is CBF for $s \in \mathbb{R}_+$ then Eq. (4) guarantees that the memory function $M(t)$ is CMF.

[39] The inverse Laplace transform of $f(t)$ is given by the Bromwich integral $f(t) = L^{-1}[\hat{f}(s); t] = \int_0^\infty \exp(st)\hat{f}(s) \, ds/(2\pi i)$. The (direct) Laplace transform is equal to $\hat{f}(s) = L[f(t); s] = \int_0^\infty \exp(-ts)f(t) \, dt$. 