Maxima $Q$-index of graphs with forbidden odd cycles

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Abstract

Let $q(G)$ be the largest eigenvalue of the signless Laplacian of $G$. Let $S_{n,k}$ be the graph obtained by joining each vertex of a complete graph of order $k$ to each vertex of an independent set of order $n-k$. The main result of this paper is the following theorem:

Let $k \geq 3$, $n \geq 110k^2$, and $G$ be a graph of order $n$. If $G$ has no $C_{2k+1}$, then $q(G) < q(S_{n,k})$, unless $G = S_{n,k}$.

This result proves the odd case of the conjecture in [M.A.A. de Freitas, V. Nikiforov, and L. Patuzzi, Maxima of the $Q$-index: forbidden 4-cycle and 5-cycle, submitted, preprint available at arXiv:1308.1652].

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1 Introduction

Given a graph $G$, the $Q$-index of $G$ is the largest eigenvalue $q(G)$ of its signless Laplacian $Q(G)$. Recall the following general problem in extremal graph theory:

How large $q(G)$ can be if $G$ is a graph of order $n$, with no subgraph isomorphic to some forbidden graph $F$?

This problem has been solved for several classes of forbidden subgraphs; in particular, in [7] it has been solved for forbidden cycles $C_4$ and $C_5$. For longer cycles, a general conjecture has been stated in [7].

Let $S_{n,k}$ be the graph obtained by joining each vertex of a complete graph of order $k$ to each vertex of an independent set of order $n-k$; in other words, $S_{n,k} = K_k \vee K_{n-k}$. Also, let $S_{n,k}^+$ be the graph obtained by adding an edge to $S_{n,k}$.

Conjecture 1 Let $k \geq 2$ and let $G$ be a graph of sufficiently large order $n$. If $G$ has no $C_{2k+1}$, then $q(G) < q(S_{n,k})$, unless $G = S_{n,k}$. If $G$ has no $C_{2k}, then q(G) < q(S_{n,k}^+)$, unless $G = S_{n,k}^+$.

In [9], Conjecture 1 was solved asymptotically by the following results.

Theorem 2 If $k \geq 2$, $q(G) \geq n+2k-2$, then $G$ contains cycle of length $l$ whenever, $3 \leq l \leq 2k+2$.

By using some techniques provied in [9] and some careful analysis we will give the complete solution of the odd case of Conjecture 1.

Theorem 3 Let $k \geq 3$, $n \geq 110k^2$, and let $G$ be a graph of order $n$. If $G$ has no $C_{2k+1}$, then $q(G) < q(S_{n,k})$, unless $G = S_{n,k}$.

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2 Notation and supporting results

For graph notation and concepts undefined here, we refer the reader to [2]. For introductory material on the signless Laplacian see the survey of Cvetković [3] and its references. In particular, let $G$ be a graph, and $X$ and $Y$ be disjoint sets of vertices of $G$. We write:
- $V(G)$ for the set of vertices of $G$, $E(G)$ for the set of edges of $G$;
- $\nu(G)$ for the number of vertices of $G$, $e(G)$ for the number of edges of $G$;
- $G[X]$ for the graph induced by $X$, and $e(X)$ for $e(G[X])$;
- $G_w$ for the graph induced by $V(G) \setminus \{w\}$;
- $e(X,Y)$ for the number of edges joining vertices in $X$ to vertices in $Y$;
- $\Gamma_G(u)$ (or simply $\Gamma(u)$) for the set of neighbors of a vertex $u$, and $d_G(u)$ (or simply $d(u)$) for $|\Gamma(u)|$.

We write $P_k$, $C_k$, and $K_k$ for the path, cycle, and complete graph of order $k$.

Here we state several known results, all of which are used in the following proofs. We start with a classical theorem of Erdős and Gallai [6].

**Lemma 4** Let $k \geq 1$. If $G$ is a graph of order $n$, with no $P_{k+2}$, then $e(G) \leq kn/2$, with equality holding if and only if $G$ is a union of disjoint copies of $K_{k+1}$.

The following structural extension of Lemma 1 has been established in [8].

**Lemma 5** Let $k \geq 1$ and let the vertices of a graph $G$ be partitioned into two sets $A$ and $B$. If

$$2e(A) + e(A,B) > (2k-2)|A| + k|B|,$$

then there exists a path of order $2k$ or $2k+1$ with both endvertices in $A$.

Let $c(G)$ denote the circumference, i.e., the size of a longest cycle of $G$. The following result is one case of Dirac theorem (see [5]).

**Lemma 6** Let $G$ be a graph with $\delta(G) \geq 2$. Then $c(G) \geq \delta(G) + 1$ holds.

To state the next result set $L_{t,k} := K_1 \lor tK_k$, i.e., $L_{t,k}$ consists of $t$ complete graphs of order $k+1$, all sharing a single common vertex. In [11], Ali and Staton gave the following stability result.

**Lemma 7** Let $k \geq 1$, $n \geq 2k+1$, $G$ be a graph of order $n$, and $\delta(G) \geq k$. If $G$ is connected, then $P_{2k+2} \subseteq G$, unless $G \subseteq S_{n,k}$, or $n = tk + 1$ and $G = L_{t,k}$.

For the proof we also need the following two upper bounds on $q(G)$. Lemma 8 can be traced back to Merris [10].

**Lemma 8** For every graph $G$,

$$q(G) \leq \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\}.$$

If $G$ is connected, equality holds if and only if $G$ is regular or semiregular.

Finally, it is worth also to mention the following result, due to Das [4].

**Lemma 9** If $G$ is a graph with $n$ vertices and $m$ edges, then

$$q(G) \leq \frac{2m}{n-1} + n - 2.$$
In [7], it was pointed out that when $k \geq 2$ and $n > 5k^2$,\
\[ q(S_{n,k}) > n + 2k - 2 - \frac{2k(k-1)}{n+2k-3} > n + 2k - 3. \]
Then for a graph $G$ with $q(G) \geq q(S_{n,k})$, we have\
\[ n + 2k - 2 - \frac{2k(k-1)}{n+2k-3} < q(S_{n,k}) \leq q(G) \leq \frac{2e(G)}{n-1} + n - 2, \]
which implies\
\[ 2e(G) > 2k(n-1) - 2k(k-1) + \frac{4k(k-1)^2}{n+2k-3}, \]
and then\
\[ 2e(G) \geq 2k(n-1) - 2(k^2 - k) + 2, \]
i.e.,\
\[ e(G) \geq kn - k^2 + 1. \quad (1) \]
Given a graph $G$ and a vertex $u \in V(G)$, note that (see [7])\
\[ \sum_{v \in \Gamma(u)} d(v) = 2e(\Gamma(u)) + (\Gamma(u), V(G) \setminus \Gamma(u)). \]
We first determine a crucial property of $G$.

**Lemma 10** Let $k \geq 2$, $n > 5k^2$, and let $G$ be a graph of order $n$. If $G$ has no $C_{2k+1}$ and $q(G) \geq q(S_{n,k})$, then $\Delta(G) = n - 1$.

**Proof** For short, set $q = q(G)$ and $V = V(G)$. Let $w$ be a vertex for which\
\[ d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \]
is maximal. We shall show that if $d(w) \neq n - 1$, then\
\[ q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) < q(S_{n,k}). \]
Note first that if $d(w) \leq 2k - 2$, then\
\[ d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \leq d(w) + \Delta(G) \leq 2k - 2 + n - 1 = n + 2k - 3 < q(S_{n,k}). \]
So hereafter we shall assume that $d(w) \geq 2k - 1$.
Set $A = \Gamma(w)$, $B = V(G) \setminus (\Gamma(w) \cup \{w\})$. Obviously, $|A| = d(w)$ and $|B| = n - d(w) - 1$. The assumption $C_{2k+1} \not\subseteq G$ implies that the graph $G[V \setminus \{w\}]$ contains no path $P_{2k}$ with both endvertices in $A$. Therefore, Lemma [5] implies that\
\[ d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) = d(w) + 1 + \frac{2e(A) + e(A, B)}{d(w)} \]
\[ \leq d(w) + 1 + \frac{(2k-2)d(w) + k(n - d(w) - 1)}{d(w)} \]
\[ = d(w) + \frac{k(n-1)}{d(w)} + k - 1. \]
The function \( x + k(n - 1)/x \) is convex for \( x > 0 \), so the maximum of the

\[
d(w) + \frac{k(n - 1)}{d(w)}
\]

is attained for the minimum or maximum admissible values for \( d(w) \). When \( k \geq 2 \), \( n > 5k^2 \), if taking \( d(w) = 2k - 1 \), or \( d(w) = n - 2 \), we easily find that

\[
d(w) + \frac{k(n - 1)}{d(w)} + k - 1 = 2k - 1 + \frac{k(n - 1)}{2k - 1} + k - 1 \leq n + 2k - 3 < q(S_{n,k}),
\]

or

\[
d(w) + \frac{k(n - 1)}{d(w)} + k - 1 = n - 2 + \frac{k(n - 1)}{n - 2} + k - 1 \leq n + 2k - 2 - \frac{2k(k - 1)}{n + 2k - 3} < q(S_{n,k}).
\]

So we obtain \( d(w) = n - 1 \). \( \square \)

**Lemma 11** Let \( k \geq 3 \), \( n \geq 110k^2 \), \( 1 \leq t \leq k^2 + k - 3 \), and let \( G \) be a graph of order \( n \). If \( G_w = G_1 \cup G_2 \), where \( G_1 \) and \( G_2 \) are disjoint, \( P_{2k} \not\subseteq G_1 \), and \( v(G_2) = t \), then \( q(G) < q(S_{n,k}) \).

**Proof** Assume for a contradiction that \( q(G) \geq q(S_{n,k}) \). We may suppose that \( w \) is a dominating vertex of \( G \), and \( G_2 \) is isomorphic to \( K_t \), otherwise, we may add some edges to \( G \), while \( q(G) \) will not decrease. Denote by \( G_0 \) the the graph obtained from \( G \) by removing \( G_2 \). In view of \( P_{2k} \not\subseteq G_1 \), then by Lemma 4 we have

\[
e(G_1) \leq (k - 1)(n - t - 1),
\]

and then

\[
e(G_0) = e(G_1) + n - t - 1 \leq k(n - t - 1).
\]

Lemma 4 implies that

\[
q(G_0) \leq \frac{2k(n - t - 1)}{n - t - 1} + n - t - 2 = n + 2k - t - 2.
\]

Let \((x_1, \ldots, x_n)^T\) be a unit eigenvector to \( q(G) \). By symmetry the entries corresponding to vertices of \( G_2 \) have the same value \( x \). From the eigenequations for \( Q(G) \) we see that

\[
(q(G) - n + 1)x_w = \sum_{i \in V(G) \setminus \{w\}} x_i \leq \sqrt{(n - 1)(1 - x_w^2)},
\]

and noting that

\[
q(G) \geq q(S_{n,k}) > n + 2k - 3,
\]

so

\[
x_w^2 \leq \frac{n - 1}{(q(G) - n + 1)^2 + n - 1} \leq \frac{n - 1}{n - 1 + 4(k - 1)^2} < 1 - \frac{4(k - 1)^2}{n + 4k^2}.
\]

(2)

Also, noting that \( k^2 + k - 3 \), we have

\[
x = \frac{x_w}{q(G) - 2t + 1} \leq \frac{x_w}{n + 2k - 2t - 2} \leq \frac{x_w}{n - 2k^2 + 4}.
\]

(3)
When $t \geq 1$, $n \geq 110k^2$, by using (2) and (3) we find that
\[
q(G) = \sum_{ij \in E(G)} (x_i + x_j)^2 = \sum_{ij \in E(G_0)} (x_i + x_j)^2 + t(x + x_w)^2 + 2t(t - 1)x^2
\leq q(G_0)(1 - tx^2) + t(x + x_w)^2 + 2t(t - 1)x^2
< n + 2k - t - 2 + t \left(1 + \frac{1}{(n - 2k^2 + 4)^2} + \frac{2}{n - 2k^2 + 4} + \frac{2(t - 1)}{(n - 2k^2 + 4)^2}\right)x_w^2
< n + 2k - t - 2 + t \left(1 + \frac{3}{n - 2k^2 + 4}\right) \left(1 - \frac{4(k - 1)^2}{n + 4k^2}\right)
< n + 2k - 2 - \frac{4(k - 1)^2}{n + 4k^2} + \frac{3}{n - 2k^2 + 4}
< n + 2k - 2 - \frac{2k(k - 1)}{n - 2k^2 + 4}
< q(S_{n,k}).
\]

Therefore $q(G) < q(S_{n,k})$, and this contradiction completes the proof. \(\square\)

We will call vertex $v$ a center vertex of graph $S_{n,k}$, if $d(v) = n - 1$ holds.

**Lemma 12** If $G_w = \bigcup_{i=1}^k S_{n_i,k-1}, k \geq 3$, and $t \geq 2$ hold, then we have $q(G) < q(S_{n,k})$.

**Proof** We may suppose $d_G(w) = n - 1$, otherwise we may add some edges to $G$, and $q(G)$ will not decrease. We first consider $t = 2$, that is to say,
\[
G_w = S_{n_1,k-1} \cup S_{n_2,k-1}.
\]

Let $u_1, \cdots, u_{k-1}$ be all the center vertices of $S_{n_1,k-1}$, and $v_1, \cdots, v_{k-1}$ be all the center vertices of $S_{n_2,k-1}$. Let $x = (x_1, \ldots, x_n)^T$ be a unit eigenvector to $q(G)$. Then by symmetry we have $x_{u_1} = \cdots = x_{u_{k-1}}$, and $x_{v_1} = x_{v_2} = \cdots = x_{v_{k-1}}$. Without loss of generality we assume that $x_{u_1} \geq x_{v_1}$.

Now combine the components $S_{n_1,k-1}$ and $S_{n_2,k-1}$ into $S_{n_1+n_2,k-1}$, and let $u_1, \cdots, u_{k-1}$ be the center vertices of $S_{n_1+n_2,k-1}$. Denote by $G'$ be the graph obtained from $G$ by the above perturbation. Set
\[
W = V(S_{n_2,k-1}) \setminus \{v_1, v_2, \cdots, v_{k-1}\}.
\]

Then
\[
q(G') - q(G) \geq x^TQ(G')x - x^TQ(G)x
= \sum_{i \in W} (k - 1) \left[(x_{u_1} + x_i)^2 - (x_{v_1} + x_i)^2\right] + (k - 1)^2(x_{u_1} + x_{v_1})^2 - 2(k - 1)(k - 2)x_{v_1}^2
> 0.
\]

Noting that $G' = S_{n,k}$, then we have
\[
q(G) < q(G') = q(S_{n,k}).
\]

When $t \geq 3$, we may prove the lemma by using induction on $t$ and applying the above perturbation to $G$ repeatedly. □

**Lemma 13** Let $G$ be a graph of order $n$ with $e(G) \geq (k - 1)n - (k^2 - k - 1).$ If $P_{2k} \not\subseteq G$, then there exists an induced subgraph $H \subseteq G$ with $\delta(H) \geq k - 1$ and $\nu(H) \geq n - (k^2 - k - 1)$. □
**Proof** Define a sequence of graphs, \( G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r \) using the following procedure.
\[
G_0 := G; \\
i := 0; \\
\textbf{while } \delta(G_i) \leq k-1 \textbf{ do begin} \\
\quad \text{select a vertex } v \in V(G_i) \text{ with } d(v) = \delta(G_i); \\
\quad G_{i+1} := G_i - v; \\
\quad i := i + 1; \\
\textbf{end.}
\]
Note that the while loop must exit before \( i = k^2 - k \). Indeed, by \( P_{2k} \not\subseteq G_i \) Lemma \( \text{[4]} \) implies that
\[
e(G_i) \leq (k-1)(n-i).
\]
On the other hand,
\[
e(G_i) \geq e(G) - i(k-2) \geq (k-1)n - (k^2 - k - 1) - i(k - 2).
\]
Then from \( \text{[4]} \) and \( \text{[5]} \), we have \( i \leq k^2 - k - 1 \). Letting \( H = G_r \), where \( r \) is the last value of the variable \( i \), the proof is completed. \( \square \)

3 Proof of Theorem \( \text{[3]} \)

**Proof of Theorem \( \text{[3]} \)** Assume for a contradiction that \( q(G) \geq q(S_{n,k}) \). By virtue of Lemma \( \text{[1]} \) we suppose \( w \) is a dominating vertex of \( G \). Then from \( \text{[1]} \), we have
\[
e(G_w) \geq (k-1)(n-1) - (k^2 - k - 1).
\]
By taking \( G_w \) as the graph \( G \) in Lemma \( \text{[1]} \) we may obtain an induced subgraphs \( H \) of \( G_w \) such that \( \delta(H) \geq k-1 \) and \( \nu(H) \geq (n-1) - (k^2 - k - 1) \). Write
\[
H = \bigcup_{i=1}^{t} H_i, \text{ and } \nu(H_i) = h_i, t \geq 1.
\]
By virtue of Dirac theorem (see Lemma \( \text{[3]} \), \( \delta(H_i) \geq k-1 \geq 2 \) implies that \( C_l \subseteq H_i, l \geq k \). Then a component of \( G_w \) contains at most one graphs of \( \{ H_1, H_2, \cdots, H_t \} \) as an induced subgraph, otherwise \( P_{2k+1} \not\subseteq G_w \) and then \( C_{2k+1} \subseteq G \). Now for each \( 1 \leq i \leq t \), let \( F_i \) be the component of \( G_w \), which contains \( H_i \) as an induced subgraph. And set
\[
G_w = \bigcup_{i=1}^{t} F_i \cup F_0.
\]
Obviously, \( P_{2k} \not\subseteq F_i \) for any \( 0 \leq i \leq t \).

We claim that \( h_i \geq 2k - 1 \) for each \( 1 \leq i \leq t \). Otherwise the order of the component \( F_i \) satisfies
\[
k \leq h_i \leq \nu(F_i) \leq 2k - 2 + k^2 - k - 1 = k^2 + k - 3.
\]
Then by virtue of Lemma \( \text{[1]} \) we obtain the contradiction \( q(G) < q(S_{n,k}) \). Similarly we calim that \( F_0 = \emptyset \), otherwise \( 1 \leq \nu(F_0) \leq k^2 - k - 1 \), by Lemma \( \text{[1]} \) we also obtain a contradiction.

Since \( P_{2k} \not\subseteq H_i \), from Lemma \( \text{[7]} \) we deduce two cases for the structure of any \( H_i \).
\( \text{(a)} \) \( H_i \subseteq S_{h_i,k-1} \), and then we have
\[
e(H_i) \leq (k-1)h_i - \frac{k(k-1)}{2}.
\]
\( \text{(b)} \) \( H_i = L_{h_i,k-1} \), and then we have
\[
e(H_i) = \frac{k(h_i - 1)}{2} < (k-1)h_i - \frac{k(k-1)}{2}.
\]

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We claim that $H_i \neq L_{h_i,k-1}$. Assume for a contradiction that $H_i = L_{h_i,k-1}$, then 
\[ e(H) \leq (k-1)(\nu(H) - h_i) - \frac{k(k-1)}{2} + \frac{k(h_i-1)}{2}. \]

On the other hand, from the procedure of Lemma 13, we know that 
\[ e(H) \geq e(G_w) - (n-1 - \nu(H))(k-2) \geq (k-1)(n-1) - (k^2 - k - 1) - (n-1 - \nu(H))(k-2). \]

Therefore 
\[ (k-1)(n-1) - (k^2 - k - 1) - (n-1 - \nu(H))(k-2) \leq (k-1)(\nu(H) - h_i) - \frac{k(k-1)}{2} + \frac{k(h_i-1)}{2}, \]

which implies that $h_i < k + 1$, and this is a contradiction to $h_i \geq 2k - 1$. So $H_i$ is a subgraph of $S_{h_i,k-1}$.

Assume now that $I$ is the independent set of $H_i$ of order $h_i - (k-1)$, and set $J = V(H_i) \setminus I$. Clearly, $\delta(H_i) \geq k - 1$ implies that every vertex of $I$ is joined to every vertex in $J$; hence, for any two vertices in $I$ there exists a path of order $2k - 1$ with them as endvertices. If $u$ is a vertex in $V(F_1) \setminus V(H_i)$ and $\Gamma(u) \cap V(H_i) \neq \emptyset$, then we have $\Gamma_{F_i}(u) \subseteq J$, since $P_{2k} \not\subseteq F_i$. Furthermore, for any vertex $v \in V(F_1) \setminus V(H_i)$ we have $\Gamma(v) \cap V(H_i) \neq \emptyset$, and $\Gamma_{F_i}(v) \subseteq J$. Therefore $(V(F_1) \setminus V(H_i)) \cup I$ is an independent set of $F_i$, and then $F_i$ is a subgraph of $S_{P_{2k},k-1}$. Thus 
\[ G_w = \bigcup_{i=1}^t F_i, \]

where $F_i$ is a subgraph of $S_{P_{2k},k-1}$. Note that $q(G)$ will not decrease when adding some edges to $G$. If $t \geq 2$, then by Lemma 12 we deduce the contradiction $q(G) < q(S_{n,k})$. If $t = 1$, we have $q(G) \leq q(S_{n,k})$ with equality holding if and only if $G_w = S_{n-1,k-1}$ and then $G = S_{n,k}$. \hfill $\square$

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