Integrable Supersymmetric Fluid Mechanics from Superstrings

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Abstract

Following the construction of a model for the planar supersymmetric Chaplygin gas, supersymmetric fluid mechanics in (1+1)-dimensions is obtained from the light-cone parametrized Nambu-Goto superstring in (2+1)-dimensions. The lineal model is completely integrable and can be formulated neatly using Riemann coordinates. Infinite towers of conserved charges and supercharges are exhibited. They form irreducible representations of a dynamical (hidden) SO(2,1) symmetry group.

1 Introduction

The Galileo invariant equations governing isentropic fluids in one spatial dimension (continuity and Euler equations for the density $\rho$ and velocity $v$) are completely integrable for polytropic gases (pressure $\propto \rho^n$) [1], and are accompanied by the usual hallmarks of complete integrability: Lax pairs, infinite number of constants of motion, etc. [2] Especially interesting is the Chaplygin gas ($n = -1$) because this model possesses the further hidden symmetry of (2+1)-dimensional Poincaré invariance, which is a consequence of the fact that this fluid model in (1+1)-dimensional spacetime devolves
from the Nambu-Goto model for a string moving on the plane, after the parameterization invariance of the latter is fixed [3].

In this Letter we enlarge the lineal Chaplygin gas to include anti-commuting Grassmann variables, so that the extended model supports a supersymmetry. This is achieved by considering a superstring moving on a plane and again fixing the parameterization invariance. The construction is analogous to what has already been done in one higher dimension: the Nambu-Goto action for a supermembrane in (3+1)-dimensions gives rise, in a specific parameterization, to a supersymmetric planar Chaplygin gas [4]. Lineal and planar supersymmetric fluid models appear to be the only possible examples of the supersymmetric Nambu-Goto/fluid connection. For a higher dimensional generalization, the reduction program would begin with a $p$-brane in $D = p + 2$ spacetime, giving rise to a fluid in $D = p + 1$ spacetime. While there are no constraints on $p$ in the purely bosonic case, supersymmetric extensions are greatly constrained: the brane-scan for “fundamental” super $p$-branes (i.e. with only scalar supermultiplets in the worldvolume) contains only the above two cases cases [3], $p = 2$ in $D = 4$ and $p = 1$ in $D = 3$. As we demonstrate, the supersymmetric extension enjoys the same integrability properties as the purely bosonic, lineal Chaplygin gas, as a consequence of the complete integrability for the dynamics of the superstring on the plane.

2 Superstring Formulation

We begin with the Nambu-Goto superstring in $D = 3$:

$$I = -\int d\tau d\sigma \left\{ \sqrt{g} - i\epsilon^{ij}\partial_i X^\mu \bar{\psi}_\mu \gamma_j \psi \right\},$$  \hspace{1cm} (2.1)

where

$$g = -\det\{\Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu}\},$$  \hspace{1cm} (2.2)

$$\Pi^\mu_i = \partial_i X^\mu - i\bar{\psi}\gamma^\mu \partial_i \psi.$$  \hspace{1cm} (2.3)

In these expressions $\mu, \nu$ are spacetime indices running over 0, 1, 2 and $i, j$ are worldsheet indices denoting $\tau$ and $\sigma$. We now go to the light-cone gauge where we define $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^2)$. $X^+$ is identified with the timelike parameter $\tau$, $X^-$ is renamed $\theta$, and the remaining transverse component $X^1$.
is renamed \(x\). We can choose a two-dimensional Majorana representation for the \(\gamma\)-matrices:

\[
\gamma^0 = \sigma^2, \quad \gamma^1 = -i\sigma^3, \quad \gamma^2 = i\sigma^1,
\]

such that \(\psi\) is a real, two-component spinor. A remaining fermionic gauge choice sets

\[
\gamma^+ \psi = 0,
\]

where \(\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^2)\). Thus \(\psi\) is further reduced to a real, one-component Grassmann field. Finally we define the complex conjugation of a product of Grassmann fields \((\psi_1 \psi_2)^* = \psi_1^* \psi_2^*\) so as to eliminate \(i\) from Grassmann bilinears in our final expression. The light-cone gauge-fixed Lagrange density becomes:

\[
\mathcal{L} = -\sqrt{g\Gamma} + \sqrt{2}\psi \partial_\sigma \psi, \tag{2.4}
\]

where

\[
\begin{align*}
g &= (\partial_\sigma x)^2, \tag{2.5} \\
\Gamma &= 2\partial_\tau \theta - (\partial_\tau x)^2 - 2\sqrt{2}\psi \partial_\tau \psi + \frac{u^2}{g}, \tag{2.6} \\
u &= \partial_\sigma \theta - \partial_\tau x \partial_\sigma x - \sqrt{2}\psi \partial_\tau \psi. \tag{2.7}
\end{align*}
\]

In the above equations, \(\partial_\sigma\) and \(\partial_\tau\) denote partial derivatives with respect to the spacelike and timelike worldsheet coordinates. The canonical momenta

\[
\begin{align*}
p &= \frac{\partial \mathcal{L}}{\partial (\partial_\tau x)} = \sqrt{\frac{g}{\Gamma}}(\partial_\tau x + \frac{u}{g} \partial_\sigma x), \tag{2.8} \\
\Pi &= \frac{\partial \mathcal{L}}{\partial (\partial_\tau \theta)} = -\sqrt{\frac{g}{\Gamma}}, \tag{2.9}
\end{align*}
\]

satisfy the constraint equation

\[
p \partial_\sigma x + \Pi \partial_\sigma \theta - \sqrt{2}\Pi \psi \partial_\sigma \psi = 0 \tag{2.10}
\]

and can be used to recast \(\mathcal{L}\) into the form

\[
\mathcal{L} = p \partial_\sigma x + \Pi \partial_\sigma \theta + \frac{1}{2\Pi}(p^2 + g) + \sqrt{2}\psi \partial_\sigma \psi - \sqrt{2}\Pi \psi \partial_\tau \psi + u(p \partial_\sigma x + \Pi \partial_\sigma \theta - \sqrt{2}\Pi \psi \partial_\sigma \psi), \tag{2.11}
\]

where \(u\) is now a Lagrange multiplier enforcing the constraint. We use the remaining parameterization freedom to fix \(u = 0\) and \(\Pi = -1\) and perform
a hodographic transformation, interchanging independent with dependent variables [4]. The partial derivatives transform by the chain rule:

\[
\begin{align*}
\partial_\sigma &= (\partial_\sigma x) \partial_x = \sqrt{g} \partial_x , \\
\partial_\tau &= \partial_t + (\partial_\tau x) \partial_x = \partial_t + v \partial_x ,
\end{align*}
\]

and the measure transforms with a factor of \(1/\sqrt{g}\). Finally, after renaming \(\sqrt{g}\) as \(\sqrt{2\lambda/\rho}\), we obtain the Lagrangian for the Chaplygin “super” gas in (1+1)-dimensions (below and in what follows the overdot denotes derivative with respect to time \(t\)):

\[
L = \frac{1}{\sqrt{2\lambda}} \int \mathrm{d}x \left\{ -\rho (\dot{\theta} - \frac{1}{2} \psi \dot{\psi}) - \frac{1}{2} \rho v^2 - \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \partial_x \psi \right\},
\]

where according to (2.8) and (2.10) (at \(u = 0\) and \(\Pi = -1\))

\[
v = p = \partial_x \theta - \frac{1}{2} \psi \partial_x \psi .
\]

We have used \(\rho\) and \(v\) in anticipation of their role as the fluid density and velocity, and we demonstrate below that they indeed satisfy appropriate equations of motion. For convenience we have also rescaled \(\psi\) everywhere by a factor of \(2^{-3/4}\). The Lagrangian (2.14) agrees with the limiting case of the planar fluid in [4]. We note that as for the planar case, a more straightforward derivation leads to the fluid Lagrangian of (2.14) with \(\rho\) integrated out. Specifically, if the parameterization freedom is used directly to equate the spacelike and timelike coordinates \(\sigma\) and \(\tau\) with \(x\) and \(t\), we obtain

\[
L' = -\int \mathrm{d}x \left( \sqrt{2\dot{\theta} - \psi \dot{\psi}} + v^2 - \frac{1}{2} \psi \partial_x \psi \right),
\]

where \(v\) is defined as in (2.15). This form of the Lagrangian can be obtained from (2.14) after \(\rho\) is eliminated using the equations of motion for \(\theta\) and \(\psi\), shown below.

3 The Supersymmetric Chaplygin Gas
3.1 Equations of Motion

The following equations of motion are obtained by variation of the Lagrangian (2.14):

\[ \dot{\rho} + \partial_x (\rho v) = 0, \]  
\[ \dot{\psi} + \left( v + \frac{\sqrt{2\lambda}}{\rho} \right) \partial_x \psi = 0, \]  
\[ \dot{\theta} + v \partial_x \theta = \frac{1}{2} v^2 + \frac{\lambda}{\rho^2} - \frac{\sqrt{2\lambda}}{2\rho} \psi \partial_x \psi, \]  
\[ \dot{v} + v \partial_x v = \partial_x \left( \frac{\lambda}{\rho^2} \right). \]

Naturally, there are only three independent equations of motion as (3.4) is obtained from (3.2), (3.3) and (2.15). Equations (3.1) and (3.4) are seen to be just the continuity and Euler equations for the Chaplygin gas. Note that these do not see the Grassmann variables directly.

We now pass to the Riemann coordinates, which for this system are (velocity ± sound speed \( \sqrt{2\lambda/\rho} \)):

\[ R_{\pm} = \left( v \pm \frac{\sqrt{2\lambda}}{\rho} \right). \]

In terms of the Riemann coordinates, the equations of motion obtain the form

\[ \dot{R}_{\pm} = -R_{\pm} \partial_x R_{\pm}, \]  
\[ \dot{\psi} = -R_+ \partial_x \psi, \]  
\[ \dot{\theta} = -\frac{1}{2} R_+ R_- - \frac{1}{2} R_+ \psi \partial_x \psi. \]

The equations in (3.6) contain the continuity and Euler equations and are known to be integrable [2]. It is readily verified that equation (3.7) for \( \psi \) is solved by any function of \( R_- \),

\[ \psi = \Psi(R_-), \]

and hence the fluid model is completely integrable. That this is the case should come as no surprise considering that we began with an integrable world-sheet theory.
At this point it may seem curiously asymmetric that equation (3.7) for the Grassmann field should contain the $R_+$ Riemann coordinate and not the $R_-$ companion coordinate. In fact, the reverse would have been the case if the sign of the $\sqrt{2\lambda}$ term in (2.14) had been opposite. The entire model is consistent with this substitution, which is just the choice of identifying $\sqrt{g}$ with plus or minus the sound speed $\sqrt{2\lambda}/\rho$.

The energy-momentum tensor is constructed from (2.14), and its components are

\begin{align*}
T^{00} &= \mathcal{H} = \frac{1}{2} \rho v^2 + \frac{\lambda}{\rho} - \frac{\sqrt{2\lambda}}{2} \psi \partial_x \psi, \\
T^{01} &= \mathcal{P} = \rho v, \\
T^{10} &= \frac{\rho v}{2} R_+ R_- - \frac{\sqrt{2\lambda}}{2} R_+ \psi \partial_x \psi, \\
T^{11} &= \rho R_+ R_-.
\end{align*}

The expected conserved quantities of the system, the generators of the Galileo group, are verified to be time-independent using the equations of motion. We have

\begin{align*}
N &= \int dx \rho, \\
P &= \int dx \rho v, \\
H &= \int dx \left( \frac{1}{2} \rho v^2 + \frac{\lambda}{\rho} - \frac{\sqrt{2\lambda}}{2} \psi \partial_x \psi \right), \\
B &= \int dx \rho (x - vt) = \int dx x - tP,
\end{align*}

Although some generators look purely bosonic, there are still Grassmann fields hidden in $v$ according to its definition (2.15).

In going to Riemann coordinates, we can observe a ladder of conserved charges of the form (3.18)

\begin{align*}
I_n^\pm &= \int dx \rho R_n^\pm.
\end{align*}

The first few values of $n$ above give

\begin{align*}
I_0^+ &= N \\
I_1^+ &= P \pm \sqrt{2\lambda} \Omega \\
I_2^+ &= 2H.
\end{align*}
where $\Omega$ is used to denote the length of space $\int dx$. (Note that $I^{-}_{x}$, would correspond to the Hamiltonian of the theory with $\sqrt{2\lambda}$ replaced by its negative).

Ref. [4] identified two different supersymmetry generators, which correspond in one space dimension to the time independent quantities:

\begin{equation}
\tilde{Q} = \int dx \rho \psi,
\end{equation}

\begin{equation}
Q = \int dx \rho \left( v - \frac{\sqrt{2\lambda}}{\rho} \right) \psi.
\end{equation}

These are again but special cases ($n = 0$ and $n = 1$) of a ladder of conserved supercharges described by

\begin{equation}
Q_{n} = \int dx \rho R_{n}^{-} \psi.
\end{equation}

We see that the supercharges evaluated on the solution (3.9) reproduce the form of the bosonic charges (3.18).

Let us observe that there exist further bosonic and fermionic conserved charges. For example, one may verify that the bosonic charges

\begin{equation}
\int dx \rho R_{n}^{m} \left( \frac{\partial_{x} R_{\pm}}{\rho} \right)^{m},
\end{equation}

\begin{equation}
\int dx \rho R_{n}^{m} \left( \frac{\psi \partial_{x} \psi}{\rho} \right)
\end{equation}

are conserved, as are the fermionic charges

\begin{equation}
\int dx \rho R_{n}^{m} \left( \frac{\partial_{x} \psi}{\rho} \right).
\end{equation}

Conserved expressions involving higher derivatives may also be constructed. The conservation of these quantities is easily understood when the string worldsheet variables are used. Then the above are written as $\int d\sigma R_{n}^{m} \left( \partial_{\sigma} R_{\pm} \right)^{m}$, $\int d\sigma R_{n}^{m} \left( \psi \partial_{\sigma} \psi \right)$, and $\int d\sigma R_{n}^{m} \left( \partial_{\sigma} \psi \right)$, respectively. Furthermore when $R_{\pm}$ are evaluated on solutions, they become functions of $\tau \pm \sigma$ [8], so that integration over $\sigma$ extinguishes the $\tau$ dependence, leaving constant quantities.
3.2 Canonical Structure

The equations of motion (3.1-3.3) can also be obtained by Poisson bracketing with the Hamiltonian (3.10) if the following canonical brackets are postulated:

\[
\{ \theta(x), \rho(y) \} = \delta(x - y), \tag{3.28}
\]

\[
\{ \theta(x), \psi(y) \} = \frac{\psi}{2\rho} \delta(x - y), \tag{3.29}
\]

\[
\{ \psi(x), \psi(y) \} = -\frac{1}{\rho} \delta(x - y), \tag{3.30}
\]

where the last bracket, containing Grassmann arguments on both sides is understood to be the anti-bracket. With these one verifies that the conserved charges in (3.14)-(3.17) generate the appropriate Galileo symmetry transformations on the dynamical variables \( \rho, \theta, \) and \( \psi \). Correspondingly the supercharges (3.22),(3.23) generate the super transformations

\[
\tilde{\delta}\rho = 0 \quad \delta\rho = -\eta \partial_x (\rho \psi) \tag{3.31}
\]

\[
\tilde{\delta}\theta = -\frac{1}{2} \eta \psi \quad \delta\theta = -\frac{1}{2} \eta R_+ \psi \tag{3.32}
\]

\[
\tilde{\delta}\psi = -\eta \quad \delta\psi = -\eta \psi \partial_x \psi - \eta R_- \tag{3.33}
\]

which leave the Lagrangian (2.14) invariant. The algebra of the bosonic generators reproduces the algebra of the (extended) Galileo group, the extension residing in the bracket \( \{ B, P \} = -N \). The algebra of the supercharges is

\[
\{ \bar{\eta}Q, \eta Q \} = 2\bar{\eta}\eta H \tag{3.34}
\]

\[
\{ \bar{\eta}Q, \eta \bar{Q} \} = \bar{\eta}\eta N \tag{3.35}
\]

\[
\{ \bar{\eta}Q, \eta Q \} = \bar{\eta}\eta (P - \sqrt{2}\lambda \Omega) \tag{3.36}
\]

\[
\{ B, Q \} = \bar{Q}. \tag{3.37}
\]

4 Further Symmetries of the Fluid Model

As mentioned above, since the fluid model descends from the superstring, it should possess an enhanced symmetry beyond the Galileo symmetry in
(1+1)-dimensions. In fact, the following conserved charges effecting time rescaling and space-time mixing \[4\] are also verified:

\[
D = \int dx (t \mathcal{H} - \rho \theta), \quad (4.1)
\]

\[
G = \int dx (x \mathcal{H} - \theta \mathcal{P}), \quad (4.2)
\]

\(G\) is sometimes referred to as the “anti-boost” because of its transformations on extended space-time \[9\]. The Galileo generators supplemented by \(D\) and \(G\) together satisfy the Lie algebra of the (2+1)-dimensional Poincaré group, with \(N\), \(P\), and \(H\) corresponding to the three translations and with \(B\), \(D\) and \(G\) forming the (2+1)-dimensional Lorentz group \(SO(2, 1)\):

\[
\{B, D\} = B, \quad \{G, B\} = D, \quad \{D, G\} = G, \quad (4.3)
\]

with Casimir

\[
C = B \circ G + G \circ B + D \circ D. \quad (4.4)
\]

Adjoining the supercharges results in the super-Poincaré algebra of (2+1)-dimensions. The Lorentz charges do not belong to the infinite towers of constants of motion mentioned earlier. Rather, they act as raising and lowering operators. One verifies for the \(Q_n\) and \(I^+_n\): \[9\]

\[
\{B, I^+_n\} = -n I^+_n, \quad \{D, I^+_n\} = (n - 1) I^+_n, \quad \{G, I^+_n\} = \left(\frac{n}{2} - 1\right) I^+_n+1, \quad (4.5)
\]

\[
\{B, Q_n\} = -n Q_{n-1}, \quad \{D, Q_n\} = (n - \frac{1}{2}) Q_n, \quad \{G, Q_n\} = \left(\frac{n}{2} - \frac{1}{2}\right) Q_{n+1},
\]

The brackets with the \(I^-_n\) do not close, but the \(I^-_n\) can be modified by the addition of another tower of constant quantities, namely those of \(5,2\)\):

\[
\tilde{I}^-_n = I^-_n - \sqrt{2\lambda} n(n - 1) \int dx R^{-2}_n \psi \partial_x \psi. \quad (4.6)
\]

\[\text{Note that the } \{B, I^+_1\} \text{ bracket coincides with } \{B, 2H\}, \text{ which should equal } -2P \text{ according to the Galileo algebra. But the above result, viz. } -2I^+_1, \text{ gives } -2(P + \sqrt{2\lambda} \Omega). \text{ This central addition arises from a term of the form}
\]

\[
\int dx dy \sqrt{2\lambda} x \frac{\partial}{\partial x} \delta(x - y),
\]

whose value is ambiguous, depending on the order of integration.
The modified constants obey the same algebra as $I_n^+$

\[
\{B, \tilde{I}_n^+\} = -n\tilde{I}_{n-1}^-, \quad \{D, \tilde{I}_n^-\} = (n-1)\tilde{I}_n^-, \quad \{G, \tilde{I}_n^-\} = (\frac{n}{2} - 1)\tilde{I}_{n+1}^-.
\]

(4.7)

Evidently $I_n^+, \tilde{I}_n^-, \text{and } Q_n$ provide irreducible, infinite dimensional representations for $SO(2,1)$, with the Casimir, in adjoint action, taking the form $l(l+1)$, and $l = 1$ for $I_n^+, \tilde{I}_n^-$, and $l = 1/2$ for $Q_n$.

Finally we inquire about the algebra of the towers of extended charges $I_n^+, \tilde{I}_n^-, \text{and } Q_n$. While some (bosonic) brackets vanish, others provide new constants of motion like those in (3.25)-(3.27) and their generalizations with more derivatives. Thus it appears that one is dealing with an open (super) algebra.

5 Conclusions

We have presented an integrable, supersymmetric fluid model with additional, “dynamical” symmetry tracing back to its origin in the superstring. Besides the planar case in [4], this is the only other dimensionality for a supersymmetric Chaplygin gas that can be obtained by going to the light-cone gauge in a super $p$-brane.

It remains an open question what other fluid interactions can be obtained from the rich factory of branes. For example, string theory $D$-branes have gauge fields living on them. Such gauge fields would presumably remain in passing to a fluid model and may thus provide a model of magnetohydrodynamics from $D$-branes. It might also be worthwhile to explore whether the fluid models with Grassmann variables are suited to describing the physical properties of fluids with spin degrees of freedom.

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