Hasse–Witt matrices, unit roots and period integrals

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Abstract
Motivated by the work of Candelas et al. (Calabi–Yau manifolds over finite fields, I. arXiv:hep-th/0012233, 2000) on counting points for quintic family over finite fields, we study the relations among Hasse–Witt matrices, unit-root part of zeta functions and period integrals of Calabi–Yau hypersurfaces in both toric varieties and flag varieties. We prove a conjecture by Vlasenko (Higher Hasse–Witt matrices. Indag Math 29(5):1411–1424, 2018) on unit-root $F$-crystals for toric hypersurfaces following Katz’s local expansion method (1984, 1985) in logarithmic setting. The Frobenius matrices of unit-root $F$-crystals also have close relation with period integrals. The proof gives a way to pass from Katz’s congruence relations in terms of expansion coefficients (1985) to Dwork’s congruence relations (1969) about periods.

1 Introduction
The relations among Hasse–Witt matrices, unit roots of zeta-functions and period integrals were pioneered in Dwork’s work on the variation of zeta-functions of hypersurfaces [1–3]. Some well-known examples are Legendre family

$$y^2 = x(x - 1)(x - \lambda),$$

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see Example 8 in [4]; and the Dwork family

\[ X_0^{n+1} + \cdots + X_n^{n+1} - (n+1)t X_0 \cdots X_n = 0, \]

see 2.3.7.18 and 2.3.8 in [5] and also [6]. For the Dwork family, we define \( \lambda = t^{-n+1}. \) There is a canonical choice of holomorphic \((n-1)\)-forms \( \omega_\lambda \) for these Calabi–Yau families by adjunction formula since they are hypersurfaces in \( \mathbb{P}^n. \) These families have maximal unipotent monodromy on middle dimensional homology at \( \lambda = t^{-(n+1)} = 0. \)

The invariant space under the monodromy is generated by one cycle \( \gamma. \) The period integral \( I_\gamma \) of \( \omega_\lambda \) over the invariant cycle \( \gamma \) near \( \lambda = 0 \) is the unique holomorphic solutions to the corresponding Picard–Fuchs equation. On the other hand, these families are defined over \( \mathbb{Z}. \) We can consider the reductions of these families modulo \( p \) and the Hasse–Witt matrices associated to \( \omega_\lambda. \) According to a theorem of Igusa–Manin–Katz, they are solutions to the Picard–Fuchs equations mod \( p. \) We first state the relations between period integral and Hasse–Witt matrix for the Dwork family, see Yu’s work [6]. The period \( I_\gamma \) is given by hypergeometric series

\[ I_\gamma = F(\lambda) = n F_{n-1} \left( \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}; \lambda \right) = \sum_{r=0}^\infty \frac{\binom{1}{n+1}_r \binom{2}{n+1}_r \cdots \binom{n}{n+1}_r \lambda^r}{(r!)^n}. \]

Let \( p \) be a prime not dividing \( n + 1. \) The Hasse–Witt matrix \( \text{HW}_p \) is given by the truncation of power series \( F(\lambda) \)

\[ \text{HW}_p(\lambda) = (p-1) F(\lambda) = \sum_{r=0}^{p-1} \frac{\binom{1}{n+1}_r \binom{2}{n+1}_r \cdots \binom{n}{n+1}_r \lambda^r}{(r!)^n}. \]

Here \( \binom{k}{F} \) means the truncation of \( F(\lambda) \) with terms of \( \lambda \) of degree less or equal than \( k. \) Let

\[ g(\lambda) = \frac{F(\lambda)}{F(\lambda^p)} \in \mathbb{Z}_p[[\lambda]] \] (1.1)

Then \( g \) is an element in

\[ \lim_{s \to \infty} \frac{\mathbb{Z}_p[\lambda, ( (p-1) F(\lambda) )^{-1} / p^s] \mathbb{Z}_p[\lambda, ( (p-1) F(\lambda) )^{-1}]}{ps^s \mathbb{Z}_p[\lambda, ( (p-1) F(\lambda) )^{-1}] / p^s} \]

and it satisfies Dwork congruences

\[ g(\lambda) \equiv \frac{(p^s-1)(F(\lambda))}{(p^{s-1}-1)(F(\lambda^p))} \mod p^s. \] (1.2)
Especially it is related to Hasse–Witt matrix by

\[ \frac{F(\lambda)}{F(\lambda^p)} \equiv (p-1)F(\lambda) \mod p. \]

Let \( q = p^r \) and \( t \in \mathbb{F}_q \). Assume \( p \nmid n+1, t^{n+1} \neq 0, 1 \) and \( \text{HW}_p(\lambda) \neq 0 \). Consider the zeta-function of Dwork family mod \( p \), the characteristic polynomial of Frobenius action on middle dimensional cohomology has exactly one \( p \)-adic unit root. It is given by

\[ g(\hat{\lambda})g(\hat{\lambda}^p) \ldots g(\hat{\lambda}^{p^{r-1}}) \quad (1.3) \]

with \( \hat{\lambda} \) being the Teichmüller lifting under \( \lambda \to \lambda^p \). Therefore the period integral \( F(\lambda) \) over \( \mathbb{C} \) completely determines the Hasse–Witt matrix of the reduction of the family mod \( p \), and also the unit root part of zeta function when the Hasse–Witt matrix is invertible.

There are many papers studying different families and obtaining analogous results as Dwork family. In [7], Miyatani studied monomial deformation family of certain Calabi–Yau projective hypersurfaces. In [8], Adolphson and Sperber obtained this result for generalized Calabi–Yau hypersurfaces, which are hypersurfaces with Calabi–Yau type middle dimensional cohomology. One corollary of these results is that if two families of Calabi–Yau from above share the same period integral, then they also share a common factor related to the period integral in zeta functions. Similar results appear in examples from mirror symmetry and are called arithmetic mirror symmetry. See Wan’s work [9] on Dwork family and its mirror, results by Doran, Kelly, Salerno, Sperber, Voight and Whitcher [10] on Berglund–Hübsch–Krawitz (BHK) mirror symmetry, and further generalization to certain monomial deformations of Delsarte hypersurfaces by Kloosterman [11].

In this paper, we generalize the relations above to hypersurfaces of Calabi–Yau or general type in \( X \), a complete toric variety or flag variety of dimension \( n \). We now informally summarize the main results in the Calabi–Yau case: in the universal Calabi–Yau hypersurface family in \( X \), there is a canonical degeneration \( Y_{s_0} \) called the large complex structure limit in the mirror symmetry literature, such that there is a unique invariant cycle \( \gamma_0 \in H_{n-1}(Y) \) near \( Y_{s_0} \). There is also a canonical trivialization \( \omega \) of the holomorphic top form for the Calabi–Yau family. The resulting period integral \( \int_{\gamma_0} \omega \) has a power series expansion at \( s_0 \) with integral coefficients. Our first main result Theorem 1.2 says that a certain truncation of this power series equals the Hasse–Witt matrix of the reduction of the family mod \( p \). When the Hasse–Witt matrix is nondegenerate, there is a unique \( p \)-adic unit root of the zeta function. The second main result theorem 1.4 gives an explicit formula of this unit root in terms of the period integral, similar as (1.1), (1.3) by a Dwork congruence relation proved by Vlasenko [12] similar to (1.2).

We first prove the mod \( p \) results. The key algorithm of Hasse–Witt matrix is a generalization of the result on hypersurfaces in \( \mathbb{P}^n \). See Katz’s algorithm 2.7 in [5]. For general hypersurfaces in a Fano variety \( X \), we use the Cartier operator on ambient space to localize the calculation in terms of local expansion similar to [13]. When \( X \)
is toric variety, the algorithm depends on the toric data associated to $X$. The algorithm implies generic invertibility of Hasse–Witt matrices for toric hypersurfaces, generalizing Adolphson and Sperber’s result for $\mathbb{P}^n$ in [14], see remark 2.1 and corollary 2.8. For flag varieties, Bott–Samelson desingularization is used to reduce the calculation to a similar situation as toric varieties. The affine charts on Bott–Samelson varieties also give an explicit algorithm to calculate the power series expansions of period integrals of hypersurfaces in $G/P$. Complete intersections are treated in Sect. 4.

The second part of the paper applies Katz’s local expansion method [13, 15] to prove a conjecture by Vlasenko [12]. The crystalline cohomology of the hypersurface family has an $F$-crystal structure. When the Hasse–Witt matrix is invertible, there exists a unit root part of the $F$-crystal. We consider the $p$-adic approximation of the Frobenius matrix on the unit root part. In particular, the Hasse–Witt matrix is the Frobenius matrix mod $p$. Katz [13] gives a $p$-adic approximation of the Frobenius matrix in terms of the local expansions of top forms on a formal chart along a section of the family. In [12], Vlasenko constructed a sequence of matrices related to a Laurent polynomial $f$ and proved congruence relations similar to Katz’s algorithm in [13]. She conjectured that the $p$-adic limit is the Frobenius matrix for hypersurfaces in $\mathbb{P}^n$ when $f$ is a homogeneous polynomial. According to Corollary 2.4 in Sect. 2, the first matrix $\alpha_1 \mod p$ appeared in [12] is the Hasse–Witt matrix for toric hypersurfaces. So it is natural to generalize Vlasenko’s conjecture to toric hypersurfaces. We give a proof of the conjecture in Sect. 5.

The main contribution of this paper compared with previous results is the cases of general toric hypersurfaces and Calabi–Yau hypersurfaces in flag varieties.

We first recall some notation for period integral and large complex structure limit in Calabi–Yau hypersurfaces in toric varieties and flag varieties.

1.1 Period integral

1. Let $X$ be a smooth semi-Fano variety of dimension $n$ over $\mathbb{C}$. In this paper $X$ is complete toric variety or flag variety $G/P$ with $P$ parabolic subgroup in a complex semi-simple algebraic group $G$.

2. We denote $V^\vee = \mathcal{H}^0(X, \omega_X^{-1})$ to be the space of anticanonical sections.

3. For any nonzero section $s \in V^\vee$, the zero locus $Y_s$ is a Calabi–Yau hypersurface in $X$.

4. Let $B$ be the set of $s \in V^\vee$ such that $Y_s$ is smooth. Then $B$ is Zariski open subset of $V^\vee$ and there is a family of smooth Calabi–Yau varieties $\pi : \mathcal{Y} \to B$ with $Y_s$ as fibers.

5. The section $s$ induces the adjunction isomorphism $\omega_{Y_s} \cong \omega_X \otimes \omega_X^{-1}|_{Y_s}$. The constant function 1 on the right hand side corresponds to a canonical section $\omega_s \in H^0(Y_s, \omega_{Y_s})$. In other words, the section $\omega_s$ is the residue of rational form $\frac{1}{s}$. Putting $\omega_s$ together, we obtain a canonical section of $R^0\pi_*(\omega_{\mathcal{Y}/B})$, denoted by $\omega$. 

\[ \omega \]
1.1.1 Period integral

Consider the local system \( L \) on \( B \) formed by \( H_{n-1}(Y_s, \mathbb{Q}) \), which is the dual of \( R^{n-1} \pi_* \mathbb{Q} \). For any flat section \( \gamma \) of \( L \) on \( U \subset B \) an open subset, the period integral \( I_\gamma \) is defined by \( \int_\gamma \omega \).

1.1.2 Picard–Fuchs system

Let \( D_{V^\vee} \) be the sheaf of linear differential operators generated by \( \text{Der}(V^\vee) \). Then any section \( D \) of \( D_{V^\vee} \) acts on the de-Rham cohomology sheaf \( R^{n-1} \pi_*(\Omega^*(Y/B)) \) via Gauss–Manin connection \( \nabla \). Define the sheaf of Picard–Fuchs system of \( \omega \) to be \( PF(U) = \{ D \in D_{V^\vee}(U) \mid \nabla(D|_{U \cap B}) \omega = 0 \} \). Period integrals are solutions to Picard–Fuchs system. In other words, we have \( DI_\gamma = 0 \) for any \( D \in PF(U) \).

1.1.3 Solution-rank-1 points

Consider the classical solution sheaf

\[
\text{Sol} = \text{Hom}_{D_{V^\vee}}(D_{V^\vee}/PF, \mathcal{O}_{V^\vee}).
\]

The solution rank at a point \( s \in V^\vee \) is defined to be the dimension of the stalk \( \text{Sol}_s \). We will consider the points in \( V^\vee \) having solution rank 1.

1.1.4 Special point \( s_0 \)

There exist special solution-rank-1 points when \( X \) is toric or \( G/P \). The theorem we will state is for those special solution-rank-1 points. We characterize \( s_0 \) up to scaling in terms of its zero locus \( Y_{s_0} \) as follows. If \( X \) is a toric variety, we consider the stratification of \( X \) by the torus action. Then \( Y_{s_0} \) is the union of toric invariant divisors. If \( X = G/P \), we consider the stratification of \( X \) by projected Richardson varieties, see [16]. Then \( Y_{s_0} \) is the union of projected Richardson divisors. Especially, if \( P \) is a Borel subgroup, the divisor \( Y_{s_0} \) is the union of Schubert divisors and opposite Schubert divisors. The corresponding points \( s_0 \) in \( V^\vee \) are proven to have solution rank 1 from GKZ systems and tautological systems by Huang–Lian–Zhu [17]. There is an invariant cycle \( \tilde{\gamma}_0 \in H_n(X - Y_s) \) for \( s \) near \( s_0 \) given by the parallel transport of some real torus \((S^1)^n\) in \( X \setminus Y_{s_0} \), see Sects. 2 and 3. This is homologous to the image of the tube map of an invariant cycle \( \gamma_0 \in H_{n-1}(Y_s) \). The unique solution to GKZ systems in toric cases and tautological systems in the homogeneous cases at \( s_0 \) is realized as the period integral \( I_{\gamma_0} \) over the invariant cycle \( \gamma_0 \). The family of toric Calabi–Yau hypersurfaces has maximal unipotent monodromy near the special point \( s_0 \), which is known as the large complex structure limit. We expect the same result holds for flag varieties.

**Example 1.1** Let \( X = \mathbb{P}^n \) with homogeneous coordinate \([x_0, \ldots, x_n]\). Then \( V \) is identified with space of homogeneous polynomials of degree \( n+1 \). In this case, the special solution-rank-1 point \( s_0 = x_0 \ldots x_n \).
1.2 Hasse–Witt matrix

Next we recall the definition of the Hasse–Witt matrix. Let $k$ be a perfect field of characteristic $p$. Assume $\pi: Y \to S$ is a smooth family of Calabi–Yau variety over $k$ with relative dimension $n - 1$. Let $\omega$ be a trivializing section of $R^0\pi_* (\omega_{Y/S})$. Let $\omega^*$ be the dual section of $R^{n-1}\pi_* (\mathcal{O}_Y)$. The $p$-th power endomorphism of $\mathcal{O}_Y$ induces a $p$-semilinear map $R^{n-1}\pi_* (\mathcal{O}_Y) \to R^{n-1}\pi_* (\mathcal{O}_Y)$ sending $\omega^*$ to $a \omega^*$. Then $HW_p = a$ as a section of $\mathcal{O}_S$ is the Hasse–Witt matrix under the basis $\omega^*$. The choice of $\omega$ for Calabi–Yau hypersurfaces is made by adjunction formula similar to period integral.

1.3 Statement of the theorem

Now we state our main theorem. When $X$ is toric or $G/P$, it has an integral model over $\mathbb{Z}$. Let $s_0 \in H^0(X, K_X^{-1})$ be the special solution-rank-1 point chosen in Sect. 1.1.4. Then $s_0$ can be extended as a basis $s_0 \ldots s_N$ of $H^0(X, K_X^{-1})$. Let $a_0 \ldots a_N$ be the dual basis for $s_0 \ldots s_N$. Suitable choices of $s_0 \ldots s_N$ are still basis considering the $p$-reduction of $X$. See Sects. 2 and 3 for the details of choice of $s_i$ and the integration cycle $\gamma_0$. There exists the following truncation relation between Hasse–Witt matrices of hypersurfaces over $\mathbb{F}_p$ and period integrals. It can also be viewed as a relation between solutions to certain Picard–Fuchs systems over $\mathbb{C}$ and finite fields $\mathbb{F}_p$.

**Theorem 1.2** 1. The Hasse–Witt matrix $HW_p$ defined above is a scalar and it is a polynomial of $a_1$ of degree $p - 1$.

2. The period integral $I_{\gamma_0}$ defined above can be extended as holomorphic function at $s_0$ and has the form $\frac{1}{a_0} P\left(\frac{a_1}{a_0}\right)$, where $P\left(\frac{a_1}{a_0}\right)$ is a Taylor series of $\frac{a_1}{a_0}, \ldots, \frac{a_N}{a_0}$ with integer coefficients.

3. They satisfy the following truncation relation

$$\frac{1}{a_0^{p-1}} HW_p = (p-1) P\left(\frac{a_1}{a_0}\right) \mod p$$

where $(p-1) P\left(\frac{a_1}{a_0}\right)$ is the truncation of $P$ at degree $p - 1$.

**Remark 1.3** The Hasse–Witt matrices count rational point on Calabi–Yau hypersurfaces mod $p$ by Fulton’s fixed point formula [18]. In the case of Calabi–Yau hypersurfaces in toric varieties, the relation between point counting and period integrals has been studied by Candelas, de la Ossa and Rodriguez-Villegas [19].

Next we state the relation between period integrals and unit roots of zeta-function of toric hypersurfaces.

**Theorem 1.4** Let $X$ be a smooth toric variety and $a_0 = 1$. The formal power series $g(a_1) = \frac{P(a_1)}{P(a_1^p)}$ satisfies Dwork congruences

$$g(a_1) \equiv \frac{(p^s-1)(P(a_1))}{(p^s-1)(P((a_1)^p))} \mod p^s$$
and lies in \( \lim_{s \to \infty} \mathbb{Z}_p[a_1, \ldots, a_{1N}, \frac{(p-1)P(a_1)^{-1}}{p^s\mathbb{Z}_p[a_1, \ldots, a_{1N}, (p-1)P(a_1)^{-1}]} \). Let \( \overline{a}_I \in \mathbb{F}_q \), where \( q = p^r \). Assume the hypersurface \( Y \) defined by \( \sum a_Is_I \) is smooth and \( \text{HW}_p(Y) \neq 0 \). Then there exists exactly one \( p \)-adic unit root in the factor of zeta function of \( Y \) corresponding to Frobenius action on \( H_{\text{cris}}^{n-1}(Y) \). It is given by the formula

\[
g(\hat{a}_I)g(\hat{a}_1^p) \ldots g(\hat{a}_r^p)
\]

with \( \hat{a}_I \) being the Teichmüller lifting of \( a_I \) under \( a_I \to a_1^p \).

Similar unit root formulas for general-type toric hypersurfaces and Calabi–Yau hypersurfaces in \( G/P \) are given in Sects. 5 and 6, respectively.

Note added: 1. After our paper was posted, Beukers and Vlasenko posted a paper [20, 21] with overlapping results, which in particular contains another proof of Vlasenko’s conjecture in [12]. We thank them for informing us of their paper; 2. The results of this paper partially inspired a significant recent development in \( p \)-adic string theory in physics [22].

## 2 Local expansions and Hasse–Witt matrices

Now we prove an algorithm of calculating Hasse–Witt matrices for hypersurfaces of \( X \) in terms of local expansions of the sections. The key ingredient is to relate the Hasse–Witt operator of Calabi–Yau or general type families \( Y \) to the Cartier operator on \( X \). Then we apply the algorithm to toric and flag varieties. Especially this recovers the algorithm for \( \mathbb{P}^n \).

We make the following assumptions and fix the list of notation for this section.

1. Let \( k \) be a field of positive characteristic \( p \) and \( X^n \) a smooth projective variety defined over \( k \) and satisfies \( H^n(X, \mathcal{O}) = H_{\text{cris}}^{n-1}(X, \mathcal{O}) = 0 \).
2. Let \( L \) be a base point free line bundle on \( X \) and \( V^\vee = H^0(X, L) \neq 0 \) and \( W^\vee = H^0(X, L \otimes K_X) \neq 0 \). Let \( a_1^\vee, a_2^\vee, \ldots, a_N^\vee \) and \( e_1^\vee, \ldots, e_r^\vee \) be basis of \( V^\vee \) and \( W^\vee \).
3. Consider a smooth family of hypersurfaces \( Y \) over \( S \to V^\vee - \{0\} \). Let \( \mathfrak{X} = X \times S \) and \( i : Y \to \mathfrak{X} \) be the natural embedding. The projections to \( S \) are denoted by \( \pi \).
4. Let \( F_S \) be the absolute Frobenius on \( S \) and \( \mathfrak{X}^{(p)} = \mathfrak{X} \times_{F_S} S \). Then we have absolute Frobenius \( F_X : \mathfrak{X} \to \mathfrak{X} \) and the relative Frobenius \( F_{\mathfrak{X}/S} : \mathfrak{X} \to \mathfrak{X}^{(p)} \). Denote by \( W : \mathfrak{X}^{(p)} \to \mathfrak{X} \) and \( \pi^{(p)} : \mathfrak{X}^{(p)} \to S \) the natural projections. We use the same notation for family \( Y \).

Consider the following diagram. This is a relative version of the diagram used by Katz to obtain an algorithm of Hasse–Witt matrices for hypersurfaces in projective spaces, see [5, Diagram 2.3.7.13].
Proposition 2.2

Let $\emptyset$. Now we can conclude the dual of Hasse–Witt matrix is given by the following

The basis $\text{Definition 2.1}$

Proof

Horizontal maps in the diagram are induced by this map. The map $f^{-1} \cdot \text{W} \rightarrow F_{X/S_\ast} L^{-1}$ is induced by $F\ast W^\ast L^{-1} = F\ast L^{-1}$ composing with multiplication by $f^{-1}$. So the diagram is commutative because both paths from $W^\ast L \rightarrow F_{X/S_\ast} \mathcal{O}_X$ are induced by multiplication by section $f^p$. Then we have a commuting diagram

$$
\begin{align*}
R^{n-1} \pi_\ast^{(p)} (\mathcal{O}_{\mathcal{Y}}) & \rightarrow R^{n} \pi_\ast^{(p)} (W^\ast L^{-1}) \\
\downarrow F & \downarrow F \\
R^{n-1} \pi_\ast^{(p)} (F_{Y/S_\ast} \mathcal{O}_Y) & \rightarrow R^{n} \pi_\ast (L^{-1})
\end{align*}
$$

Since the variety $X$ satisfies $H^n(X, \mathcal{O}) = H^{n-1}(X, \mathcal{O}) = 0$, the two horizontal maps above are isomorphisms. The left vertical map is the Hasse–Witt operator $\text{HW} : F\ast (R^{n-1} \pi_\ast (\mathcal{O}_Y)) \cong R^{n-1} \pi_\ast^{(p)} (\mathcal{O}_{\mathcal{Y}}) \rightarrow R^{n-1} \pi_\ast^{(p)} (F_{Y/S_\ast} \mathcal{O}_Y) \cong R^{n-1} \pi_\ast \mathcal{O}_Y$.

Definition 2.1

The basis $e_1^\gamma \ldots e_r^\gamma$ of $H^0(X, K_X \otimes L)$ induces a basis of $R^0 \pi_\ast^{(\gamma)} (\mathcal{O}_{\mathcal{Y}})$ by residue map and dual basis $e_1 \ldots e_r$ of $R^{n-1} \pi_\ast (\mathcal{O}_Y)$ under Serre duality. The Hasse–Witt matrix $a_{ij}$ is defined by $\text{HW}(F\ast (e_i)) = \sum_j a_{ij} e_j$.

The condition $H^0(X, K_X \times L) \neq 0$ guarantees that the Hasse–Witt matrix is not empty. Now we can conclude the dual of Hasse–Witt matrix is given by the following algorithm in terms of local expansions.

Proposition 2.2

Let $(t_1, \ldots, t_r)$ be local coordinate of $X$ at a point $x$ and $g(t) = \sum a_{i1} t^I$ a formal power series with multi-index $I = (i_1, \ldots, i_n)$. Define the operation $\tau$ by $\tau (g) = \sum_j a_{ij} t^I$ with $I = (p-1, \ldots, p-1) + pJ$. Fix a trivializing section $\xi$ of $L$ on an open neighborhood $U$ of $x$. Then we can view $e_1^\gamma (\xi^{-1})$ as a section of canonical bundle $\omega_X$ on $U$ and we assume it has the form $h_1(t) dt_1 \wedge dt_2 \cdots \wedge dt_n$. Under the same trivialization, we further assume the section $e_1^\gamma f^{-1} (\xi^{-1})$ has the form $g_1(t) dt_1 \wedge dt_2 \cdots \wedge dt_n$. Then the formal power series $\tau (g_1)$ has the form $\tau (g_1) = \sum_j a_{ji} h_j$.

Proof

Let $C_{X/S} : \omega_{X/S} \rightarrow \omega_{X(p)/S}$ be the top Cartier operator. For any coherent sheaf $M$ on $X$, the Grothendieck duality

$$F_{X/S_\ast} \mathcal{H}om(M, \omega_{X/S}) \cong \mathcal{H}om(F_{X/S_\ast} M, \omega_{X(p)/S})$$
is related to $C_{X/S}$ by the natural pairing
\[ F_{X/S} \mathcal{H}om(M, \omega_{X/S}) \otimes F_{X/S} M \to \omega_{X(p)/S} \]
sending $g \otimes m$ to $C_{X/S}(g(m))$. Consider $M = L - p$. Since $F_{X/S} L - p \cong W^* L^{-1}$, we have
\[ F_{X/S} \mathcal{H}om(L - p, \omega_{X/S}) \cong \mathcal{H}om(W^* L^{-1}, \omega_{X(p)/S}). \]
The following morphism is induced by multiplication by $f p^{-1}$
\[ F_{X/S} \mathcal{H}om(L^{-1}, \omega_{X/S}) \to F_{X/S} \mathcal{H}om(L - p, \omega_{X/S}) \cong \mathcal{H}om(W^* L^{-1}, \omega_{X(p)/S}). \]
After applying the functor $R^0 \pi_*(p)$ on both sides, we have a morphism
\[ R^0 \pi_* \mathcal{H}om(L^{-1}, \omega_{X/S}) \to R^0 \pi_* \mathcal{H}om(W^* L^{-1}, \omega_{X(p)/S}). \]
This is the dual of
\[ R^n \pi_*(p) (W^* L^{-1}) \to R^n \pi_*(L^{-1}). \]
On the other hand, the top Cartier operator has the following form in terms of local expansion
\[ C_{X/S} : g(t) dt_1 \wedge dt_2 \cdots \wedge dt_n \mapsto \tau(g) dt_1 \wedge dt_2 \cdots \wedge dt_n. \]
So the conclusion follows.

In the applications of this algorithm, there is either an affine space $\mathbb{A}^n$ or torus $\mathbb{G}_m^n$ as an open subscheme $U$ in $X$. When $U$ is the torus $\mathbb{G}_m^n$ with coordinates $\{(t_1, \ldots, t_n) \mid t_i \neq 0\}$, the argument above still works. So we have the same local expansion formula except that the local expansions $g$ and $h_i$ are Laurent polynomials instead of formal power series. When $X$ is toric variety, the torus chart $\mathbb{G}_m^n$ is given in a natural way. When $X$ is flag variety, the the torus chart $\mathbb{G}_m^n$ naturally appears in the corresponding Bott–Samelson varieties, see Sect. 3. More explicitly, we have the torus version of Proposition 2.2.

**Proposition 2.3** Let $U \cong \mathbb{G}_m^n$ be an open subscheme of $X$ with torus coordinate $U = \{(t_1, \ldots, t_n) \mid t_i \neq 0\}$. Assume $L$ has a trivializing section $\xi$ on $U$. Replacing power series $g, g_i, h_i$ by Laurent polynomials and extend operator $\tau$ by the same formula in Proposition 2.2, the same conclusion holds.

**Proof** The same proof works because the Cartier operator on torus has the same form in terms of torus coordinates
\[ C_{X/S} : g(t) dt_1 \wedge dt_2 \cdots \wedge dt_n \mapsto \tau(g) dt_1 \wedge dt_2 \cdots \wedge dt_n. \]
This can be derived from embedding the torus into affine space $\mathbb{A}^n$ and the Cartier operator being $p^{-1}$-linear.

We first specialize this algorithm to toric hypersurfaces. Let $X$ be a smooth complete toric variety defined by a fan $\sigma$. The 1-dimensional primitive vectors $v_1, \ldots, v_N$ correspond to toric divisors $D_1, \ldots, D_N$. Then $K_X \cong \mathcal{O}_X(-\sum_i D_i)$. Assume $L = \mathcal{O}(\sum a_i D_i)$ with $a_i \geq 1$. Let $\Delta = \{ v \in \mathbb{R}^n \mid \langle v, v_i \rangle \geq -a_i \}$ and $\hat{\Delta}$ the interior of $\Delta$. Then $H^0(X, L)$ has a basis corresponding to $u_i \in \Delta \cap \mathbb{Z}^n$ and $H^0(X, L \otimes K_X)$ has basis $e_i^\vee$ identified with $u_i \in \hat{\Delta} \cap \mathbb{Z}^n$. Let $a_1$ be sections of $\mathcal{O}_S$ and $f = \sum a_1 t^{u_1}$ the Laurent polynomial defining hypersurface family $Y \to S$. Assume $f^{p-1} = \sum A_u t^u$ as Laurent polynomial. Then we can read the Hasse–Witt matrix from the coefficients $A_u$.

**Corollary 2.4** The Hasse–Witt matrix of hypersurface family over $S$ under the basis $e_i^\vee \in \hat{\Delta} \cap \mathbb{Z}^n$ is given by $a_{ij} = A_{pu_j-u_i}$.

**Proof** After certain change of coordinates, we can assume $v_1 \ldots v_n$ form the standard basis of $\mathbb{Z}^n$. Then $(t_1, \ldots, t_n)$ is an affine chart on $X$. We choose a section of $L \otimes K_X$ to be $s_0$ corresponding to origin in $\hat{\Delta} \cap \mathbb{Z}^n$ and a meromorphic section of $K_X \theta = \frac{dt_1 \wedge dt_2 \cdots \wedge dt_n}{t_1 \cdots t_n}$. Let $s = \frac{s_0}{\theta}$ be a meromorphic section of $L$. Then

$$\frac{e_i^\vee}{s} = \frac{t^{u_i}}{t_1 \cdots t_n} dt_1 \wedge dt_2 \cdots \wedge dt_n.$$  

If we view $f$ as a section in $H^0(X, L)$, then

$$\frac{f}{s} = t_1 \cdots t_n \sum a_1 t^{u_1}$$

Hence $\frac{e_i^\vee f^{p-1}}{s^{p}} = g_i dt_1 \wedge dt_2 \cdots \wedge dt_n$ with

$$g_i = \frac{t^{u_i}(\sum a_1 t^{u_1})^{p-1}}{t_1 \cdots t_n} = \sum_u A_u t^{u+u_i-1}.$$

Here $1 = (1, \ldots, 1)$. So

$$\tau(g_i) = \sum_v A_u t^v$$

with $u + u_i - 1 = pv + (p-1)1$. On the other hand, we have

$$\tau(g_i) = \sum_j a_{ji} t^{u_j-1}.$$  

So $a_{ij} = A_{pu_j-u_i}$.  

\(\square\)
Remark 2.5 When $X$ is $\mathbb{P}^n$, Corollary 2.4 gives the same algorithm as Katz [5]. In [12], Vlasenko defines the higher Hasse–Witt matrices for a Laurent polynomial $f$. When $f$ is a homogeneous polynomial of degree $d$, the first matrix $\alpha_1$ in [12] mod $p$ is the Hasse–Witt matrix for the corresponding hypersurface in $\mathbb{P}^n$. The $p$-adic limit of the matrices is conjectured to give the Frobenius matrix of the unit root part of $H^{n-1}_{\text{cris}}(Y)$, which is a dual analogue of matrices defined by Katz [4]. Corollary 2.4 proves that $\alpha_1$ mod $p$ is also the Hasse–Witt matrix for toric hypersurfaces. Hence it is natural to generalize Vlasenko’s conjecture to toric hypersurfaces.

If $X$ is any smooth variety satisfying the assumptions in this section and $L = K_X^{-1}$, then we have a Calabi–Yau family. In this case, the algorithm coincides with the criterion for Frobenius splitting of $X$ respect to $Y_s$. The basis of $H^0(X, L \otimes K_X)$ is chosen to be constant function $1$. The Hasse–Witt matrix is a function on $S$. Under local coordinates $(t_1, \ldots, t_n)$ on $X$, we can choose the trivializing section of $L$ to be $(dt_1 \wedge dt_2 \cdots \wedge dt_n)^{-1}$. The local algorithm in this case is the following.

Corollary 2.6 Let $f = g(t)(dt_1 \wedge dt_2 \cdots \wedge dt_n)^{-1}$. Then the Hasse–Witt matrix $\alpha$ is given by $\tau(g^{p-1})$. More explicitly $\alpha$ is the coefficient of $(t_1 \ldots t_n)^{p-1}$ in local expansion of $g(t)^{p-1}$.

Remark 2.7 For any closed point $s \in S(k)$, the corresponding section $f_s^{p-1} \in H^0(X, \alpha_X^{-1}p)$ determines a Frobenius splitting of $X$ compatible with $Y_s$ if and only if $\alpha(s) \neq 0$. It is also equivalent to $Y_s$ being Frobenius split. Especially, Corollary 2.4 implies the well-known fact that toric variety $X$ is Frobenius split compatibly with torus invariant divisors. See Chapter 1 of [23].

Proof of Theorem 1.2 for toric $X$ Following the previous notation, let $X$ be a smooth complete toric variety and $L = K_X^{-1} = O_X(\sum_i D_i)$. Then the basis of $H^0(X, L)$ is identified with the integral points $u_I$ in the polytope $\Delta = \{v \in \mathbb{R}^n \mid \langle v, v_i \rangle \geq -1\}$. The universal section of $L$ is $f(t) = \sum_i a_I u_I$ with $a_0 = 0, \ldots, 0$. Then $H_W^p$ is the coefficient of constant term in $f^{p-1}$ according to Corollaries 2.4 or 2.6. On the other hand, we consider the period integral

$$I_\gamma = \frac{1}{(2\pi i)^n} \int_{\gamma} \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n f(t)}$$

along the cycle $\gamma: |t_1| = |t_2| = \cdots |t_n| = 1$. We view $I_\gamma$ as a local holomorphic function near the large complex structure limit point defined by $a_0 = 1$ and $a_I = 0, \forall I \neq 0$. If we normalize $f$ by multiplying $a_0^{-1}$, then the period integral $I_\gamma$ is the coefficient of constant term 1 in the Laurent expansion of $f^{-1}$, given by

$$I_\gamma = \frac{1}{a_0} \left(1 + \sum_{k=1}^{\infty} (-1)^k \sum_{k_1 u_{l_1} + \cdots + k_l u_{l_I} = 0, \sum k_j = k, I_j \neq 0} \binom{k}{k_1, k_2, \ldots, k_l} \left(\frac{a_{l_I}}{a_0}\right)^{k_i} \cdots \left(\frac{a_{l_1}}{a_0}\right)^{k_1}\right).$$
The Hasse–Witt matrix

\[
\text{HW}_p = a_0^p \left( 1 + \sum_{k=1}^{p-1} \sum_{k_1 u_{I_1} + \cdots + k_l u_{I_l} = 0, \sum k_j = k, I_j \neq 0} (p-1) (k_1, k_2, \ldots, k_l, p - 1 - k) \right)
\begin{pmatrix}
\left( \frac{a_{I_1}}{a_0} \right)^{k_1} & \cdots & \left( \frac{a_{I_l}}{a_0} \right)^{k_l}
\end{pmatrix}.
\]

Then we apply the congruence relation

\[
\left( \frac{p-1}{k_1, k_2, \ldots, k_l, p - 1 - k} \right) \equiv (-1)^k \left( \frac{k}{k_1, k_2, \ldots, k_l} \right) \mod p
\]
in the two expansions to obtain the conclusion. \(\square\)

### 2.1 Period integrals and Hasse–Witt matrices of general toric hypersurfaces

The same argument in the proof of Theorem 1.2 can also be applied to obtain truncation-type relations between period integrals and Hasse–Witt matrices of general toric hypersurfaces. This also shows that the Hasse–Witt matrices are generically invertible, see Corollary 2.8. The results for hypersurfaces in \(\mathbb{P}^n\) are proved by Adolphson and Sperber in [14]. We follow the notation in Corollary 2.4. Any section \(s\) of \(H^0(X, L \otimes K_X) \otimes \mathcal{O}_S\) determines a section of \(R^0\pi_*(\mathcal{Y}, \omega_{\mathcal{Y}/S})\) via residue map and we can define period integrals in a similar way as Calabi–Yau hypersurfaces. Let \(s_0 = t_1 \cdots t_n (dt_1 \wedge \cdots \wedge dt_n)^{-1} \in H^0(X, K_X^{-1})\) be the large complex structure limit point with zero locus equal to the union of \(D_i\). Let \(e^\vee_i\) be the basis of \(H^0(X, L \otimes K_X)\) corresponding to \(u_i \in \Delta \cap \mathbb{Z}^n\) and denote \(s_i = s_0 \otimes e^\vee_i \in H^0(X, L)\). Let \(f = \sum a_{I} t^{u_{I}}\) be the defining section of \(\mathcal{Y}\). In the Laurent series expression of \(f\), the section \(s_i\) defined above is identified with monomial \(t^{u_i}\). The period integral of \(e^\vee_i\) along the cycle \(\gamma: |t_1| = |t_2| = \cdots |t_n| = 1\) near \(s_j\) is given by

\[
I_{\gamma, i} = \frac{1}{(2\pi \sqrt{-1})^n} \int_{\gamma} \frac{t^{u_i} dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n f(t)}
\]

and it is equal to the coefficient of \(t^{-u_i}\) in the Laurent expansion of \(f^{-1}\) when we normalize \(f\) by multiplying \(a_{j}^{-1}\). On the other hand, the entry \(a_{i j}\) at \(i\)-th row and \(j\)-th column of the Hasse–Witt matrix under the basis \(e^\vee_1 \cdots e^\vee_r\) is given by the coefficient of \(t^{pu_{j} - u_i}\) in the Laurent expansion of \(f^{-1}\). So we have the following

1. The function \(a_{i j}\) on \(S\) are polynomials of \(a_I\) of degree \(p - 1\).
2. The period integral \(I_{\gamma, i}\) is a holomorphic functions at \(s_j\) and has the form \(\frac{1}{a_j} P_i(\frac{a_I}{a_j})\), where \(P_i(\frac{a_I}{a_j})\) is a Taylor series of \(\frac{a_I}{a_j}\) with integer coefficients.
3. They satisfy the following truncation relation
\[
\frac{1}{a_j^{p-1}} a_{ij} \equiv (p^{-1})(P_i \left( \frac{a_j}{a_I} \right)) \mod p,
\]
where \((p^{-1})(P_i \left( \frac{a_j}{a_I} \right))\) is the truncation of \(P\) at degree \(p - 1\).

Since the period integral of
\[
\omega_i = \frac{t^{\mu_i} dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n f(t)}
\]
satisfies the corresponding Gelfand–Kapranov–Zelevinsky hypergeometric differential system, the entries \(a_{ij}\) of Hasse–Witt matrices are mod \(p\) solutions to the same differential system. See [24] for mod \(p\) solutions to general hypergeometric systems. In [14], Adolphson and Sperber also proved the generic invertibility of Hasse–Witt matrices for hypersurfaces in \(\mathbb{P}^n\). Similar idea gives the same result for toric hypersurfaces.

**Corollary 2.8** The Hasse–Witt matrices for generic smooth toric hypersurface are non-degenerate. In other words, the determinant \(\det(a_{ij}) \neq 0\).

**Proof** Consider the determinant of matrix \(B = (B_{ij}) = (p^{-1})(P_i \left( \frac{a_j}{a_I} \right)) = \left( \frac{1}{a_j^{p-1}} a_{ij} \right)\).

The entry \((p^{-1})(P_i \left( \frac{a_j}{a_I} \right))\) has the form
\[
\sum_{k=0}^{p-1} (-1)^k \sum_{u_1 + \cdots + u_k = (k+1)u_j - u_i} \left( \frac{a_{I_1}}{a_1} \right) \cdots \left( \frac{a_{I_k}}{a_k} \right).
\]

The indices \(I_l\) are not necessarily distinct. The constant term in \(B_{ij}\) is \(\delta_{ij}\). Now we prove the constant term in \(\det B\) is 1. Let \(\epsilon\) be a permutation of \(r\)-elements. Assume
\[
\frac{a_{I_1}}{a_1} \cdots \frac{a_{I_1}}{a_1} \frac{a_{I_2}}{a_2} \cdots \frac{a_{I_2}}{a_2} \cdots \frac{a_{I_k}}{a_k} \cdots \frac{a_{I_k}}{a_k} \frac{a_{I_r}}{a_r} \frac{a_{I_r}}{a_r}
\]
is a constant term appearing in the product \(B_{\epsilon(1)} \cdots B_{\epsilon(r)}\). Then all indices \(I_m\) appearing in the numerator correspond to interior integer points \(u_i \in \hat{\Lambda} \cap \mathbb{Z}^n\) and satisfy
\[
u_{I_1} + \cdots + u_{I_k} + u_{\epsilon(i)} = (k_i + 1)u_i.
\]

Consider the vertex \(u_I\) of the convex polytope generated by all \(u_i \in \hat{\Lambda} \cap \mathbb{Z}^n\). Since the convex expression for such \(u_I\) is unique, the indices \(I_1 = \cdots = k_i = \epsilon(l) = l\). Hence other terms in the product does not involve \(u_I\). We can delete the vertices and consider the convex polytope generated by the remaining \(u_i\) and obtain \(\epsilon(i) = i\) inductively. Then the only constant term is 1. \(\square\)
3 Flag varieties

Let $G$ be a simply connected, semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Let $R$ be the root system and $R^{\pm}$ the subsets of positive roots and negative roots. Denote by $B = B^+$ the Borel subgroup and $B^-$ the opposite Borel subgroup with unipotent subgroups $U$ and $U^-$. Let $P$ be a parabolic subgroup of $G$ containing $B$. Now we prove similar proposition for flag variety $X = G/P$ using Corollary 2.6. There is a natural candidate for large complex structure limit in the family of Calabi–Yau hypersurfaces in $G/P$, which is the union of codimension one strata of projections of Richardson varieties, denoted by $Y_0$. See [16, 25–27] for the definition of projections of Richardson varieties and [17] for identification of $Y_0$ as solution-rank-1 point of Picard–Fuchs system. The proof of toric Calabi–Yau families depends on the following fact. There is a torus chart $(\mathbb{G}_m)^n = \text{Spec}(\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$ on $X_{\mathbb{Z}}$ such that the large complex structure limit has the form $s_0 = t_1 \ldots t_n (dt_1 \ldots dt_n)^{-1}$ on $(\mathbb{G}_m)^n$. The top Cartier operator on $(\mathbb{G}_m)^n$ gives Hasse–Witt matrix in terms of local expansions of anticanonical forms $f = g(t)(dt_1 \ldots dt_n)^{-1}$ on $(\mathbb{G}_m)^n$. The same conclusion as Corollary 2.6 still holds under the existence of torus chart $(\mathbb{G}_m)^n$. On the side of period integral, we assume the tube construction of the invariant cycle near $s_0$ is represented by real torus $\gamma = \{ |t_1| = |t_2| = \cdots |t_n| = 1 \}$. Then period integral $I_\gamma = \int_\gamma g(t)^{-1}(dt_1 \ldots dt_n)$ along $\gamma$ has the same local expansion expression of $g(t)^{-1}$ in the proof of Theorem 1.2. Hence the same truncation relation between period integral and Hasse–Witt matrix as Theorem 1.2 still holds. When $X$ is a flag variety, the torus chart $(\mathbb{G}_m)^n$ naturally appears in the Bott–Samelson–Demazure–Hansen type resolution of projections of Richardson varieties. This resolution is also used in the proof of Frobenius splitting for projections of Richardson varieties, see [16].

Now we review Bott–Samelson–Demazure–Hansen varieties as resolutions of singularities of Schubert varieties. See [23, section 2] or [16, 28]. Denote by $\alpha_1 \ldots \alpha_l$ the simple roots of the root system $R$. Let $W$ be the Weyl group and $s_i \in W$ the simple reflection attached to $\alpha_i$. Let $w = s_1 \ldots s_n$ be a reduced expression for $w \in W$ and we denote it by $w = (s_i_1, \ldots, s_i_n)$. Let $P_{ij}$ be the minimal parabolic subgroup corresponding to simple root $\alpha_{ij}$. Then the Bott–Samelson variety $Z_w$ is defined to be $P_{i_1} \times \cdots \times P_{i_l} / B^n$. Here the right action by $B^n$ is defined by $(p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n) = (p_1 b_1 b_1^{-1} p_2 b_2, \ldots, b_n^{-1} p_n b_n)$. The image of $(p_1, \ldots, p_n)$ under the quotient map is denoted by $[p_1, \ldots, p_n]$. The natural map $\psi_w : Z_w \to G/B$ defined by $[p_1, \ldots, p_n] \mapsto p_1 \ldots p_n$ induces a birational map between $Z_w$ and Schubert variety $X_w = \overline{BwB}/B$. Denote by $1$ the identity element in $G$. Let $Z_w^{(j)} = P_{i_1} \times \cdots \times P_{i_l} / B^n$ be a divisor of $Z_w$ via the embedding $[p_1, \ldots, \hat{p}_j, \ldots, p_n] \mapsto [p_1, 1, \ldots, p_n]$. The boundary divisor of $Z_w$ is defined to be $\partial Z_w = Z_w^{(1)} + \cdots + Z_w^{(n)}$. These components have normal crossing intersection at $[1, \ldots, 1]$. Fixing a base field $k$, let $\lambda : B \to k^*$ be a character for $B$ and $k_\lambda$ be the corresponding one-dimensional representation space. Let $L(\lambda) = G \times_B k_\lambda$ be the equivariant line bundle on $G/B$ associated to character $\lambda$ and $L_w(\lambda) = \psi_w^* L(\lambda)$. Then the canonical bundle of $Z_w$ is $\omega_{Z_w} \cong \mathcal{O}_{Z_w}(-\partial Z_w) \otimes L_{\omega_0}(\rho)$ with $\rho$ the sum of fundamental weights.
Next we construct a special section \( s_0 \subset H^0(X, \omega_X^{-1}) \). Since the Picard group of \( G/B \) is generated by opposite Schubert divisors and each opposite Schubert divisor corresponds to a line bundle associated with one fundamental weight (for example, see [23, Exercise 2.2.E(3)]), there is a section \( \sigma \) of \( L(\rho) \) vanishing exactly along all opposite Schubert divisors with multiplicity one. Let \( \tilde{s}_0 \) be the tensor product of \( \psi_{w}^* \sigma \) and the canonical section of \( O_{\omega_{w}}(\partial \omega_{w}) \). Then \( \tilde{s}_0 \) vanishes along \( \partial \omega_{w} \) and preimage of opposite Schubert divisors. Let \( U^-_{ij} \) be the negative unipotent subgroup \( P_i \cap U^+ \). Then \( U^-_{ij} \) is isomorphic to the affine line \( \mathbb{A}^1 \) with coordinate \( t_j \). The natural map \( U^-_{ij} \times \cdots \times U^-_{ij} \subset \mathbb{A}^n \) with coordinate \( (t_1, \ldots, t_n) \). Then \( Z_{\omega_{w}}(j) \) on this affine chart is defined by \( t_j = 0 \). The image of this chart under \( \psi_w \) is inside the opposite Schubert cell \( C_{id} = B^0 \). So \( \tilde{s}_0 \) vanishes with simple zero along coordinate hyperplanes on this chart. After rescaling \( \sigma \), we can take \( \tilde{s}_0 = t_1 \cdots t_n dt_1 \wedge \cdots \wedge dt_n \). For the parabolic group \( P \), let \( W_P \subset W \) be the Weyl group of Levi subgroup of \( P \). Let \( W_P \subset W \) be the set of minimal representatives in cosets \( W/W_P \) and \( w_P \) be the longest element in \( W \). Next we identify \( \tilde{s}_0 \) with a special anticanonical form defined on \( X = G/P \). Given two elements \( v, w \) in \( W \) such that \( v \leq w \) under Bruhat order, the corresponding Richardson variety \( X^v_{w} = X^v \cap X_w \) is defined to be the intersection of Schubert variety \( X_w = BwB/B \) with opposite Schubert variety \( X^v = B^0vB/B \). The image of \( X^v_{w} \) under the natural map \( G/B \to G/P \) is denoted by \( \Pi^v_{w} \). This forms a stratification of \( X \). The codimension one strata form an anticanonical divisor \( \Pi_1 + \cdots + \Pi_s \). See [16, section 3]. (Note that our notation for Schubert variety and opposite Schubert variety is different from [16].) So there is a section \( s_0 \) of anticanonical bundle with simple zeros along \( Y_{s_0} = \Pi_1 \cup \cdots \cup \Pi_s \). The map \( \psi: Z_{w_P} \to G/P \) induces a pull-back map \( \psi^* \) from meromorphic sections of anticanonical bundle on \( G/P \) to those sections on \( Z_{w_P} \).

**Lemma 3.1** The two sections of anticanonical bundles are related \( \tilde{s}_0 = \psi^*(s_0) \) up to a rescaling under the pull-back map \( \psi^* \).

**Proof** We compare the two divisors \( \tilde{s}_0 \) and \( (\psi^*(s_0)) \). Since \( \sigma \) vanishes along opposite Schubert divisors on \( G/B \), then \( \tilde{s}_0 \) vanishes along the preimage of opposite Schubert divisors under \( \psi_{w} \) and \( \partial \omega_{w} \). Let \( C_{w_p} \) be the Schubert cell and \( C_{id} \) the opposite Schubert cell. The restriction of \( \psi_{w_P} : Z_{w_p} = \partial Z_{w_p} \to C_{w_P} \) is an isomorphism. Let \( D_i \) be the irreducible reduced divisors supported on \( C_{w_p} - C_{id} \). Then \( \tilde{s}_0 = \sum_i \psi_{w_P}^{-1}(D_i) + \sum_j Z_{w_p(j)} \). The restriction of projection \( C_{w_P} \to G/P \) is also isomorphism on its image. The divisors \( \Pi_j \) are exactly the complement of the image of \( C_{w_P} \cap C_{id} \). So we have \( (\psi^*(s_0))_{|_{\psi_{w_P}^{-1}(C_{w_P})}} = \sum_i \psi_{w_P}^{-1}(D_i) \). The exceptional locus of \( \psi \) is supported on \( \partial Z_{w_P} \). So \( (\psi^*(s_0)) = \sum_i \psi_{w_P}^{-1}(D_i) + \sum_j n_j Z_{w_p(j)} \) as a meromorphic anticanonical section. Since \( \tilde{s}_0 \) and \( (\psi^*(s_0)) \) are linear equivalent, then \( \sum_j Z_{w_p(j)} \) and \( \sum_j n_j Z_{w_p(j)} \) are linear equivalent. On the other hand, the divisors \( Z_{w_p(1)} \cdots Z_{w_p(n)} \) form a basis for \( \text{Pic}(Z_{w_P}) \), see [23, Exercise 3.1.E(3)]. So \( n_j = 1 \).

\( \square \)
So the Hasse–Witt matrices have similar expansion algorithm as toric case according to the discussion above. On the other hand, the period integral near \( s_0 \) can also be calculated by pulling-back to \( Z_{w,P} \). The cycle \( \gamma : |t_1| = |t_2| = \cdots |t_n| = 1 \) has nontrivial image in \( H_n(X - Y_{s_0}) \) since the integral \( \int_H \frac{1}{s_0} \neq 0 \). This is the unique invariant cycle near \( s_0 \) since dim \( H^n_c(X - Y_0) = 1 \). According to Theorem 1.4 in [17], the period integral \( \int_H \psi_{s_0} \frac{1}{\gamma} = \int_H \psi^*(\frac{1}{\gamma}) \) is the unique holomorphic solution to the Picard–Fuchs system near \( s_0 \). So we proved Theorem 1.2 for flag variety \( X = G/P \). Note that the basis of \( H^{123} \) to the discussion above. On the other hand, the period integral near \( s_0 \) and in [33] 12.4 for general

\[ \text{Remark 3.2} \] The anticanonical form \( s_0 \) appears in [16, 30]. In [30], the form \( s_0 \) is constructed on torus chart of the open Richardson cell \( R_{id} = C^{id} \cap C_w \) and glued together by coordinate transformations. We use the construction in [16] that the complement of \( R_{id} \) is an anticanonical divisor. Lemma 3.1 proves that \( \psi^*(s_0) = t_1 \cdots t_n(d_1 \wedge \cdots \wedge d_n)^{-1} \) on the affine coordinate of \( Z_{w} \), which is the local formula on torus chart appeared in [30]. This gives an explanation of the footnote in section 7 of [30]. The cycle \( \gamma \) appears in [31] 7.1 for complete flag variety \( G/B \), in [32] Theorem 4.2 for Grassmannians and in [33] 12.4 for general \( G/P \).

Now we give an explicit example of the resolution and the anticanonical form \( s_0 \) under the resolution.

\[ \text{Example 3.3} \] Let \( X \) be Grassmannian \( G(2, 4) \). Then \( X = G/P \) with \( G = SL(4) \) and \( P = \begin{pmatrix} *** & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \). The Weyl group is \( S_4 \) and \( W_P = S_2 \times S_2 \). The element

\[ w_p = (13)(24) \] with one choice of reduced expression \( w_p = (23)(34)(12)(23) = s_2 s_3 s_1 s_2 \). So \( Z_{w,P} = P_1 \times P_2 \times P_3 \times P_4/B^4 \) with \( P_1 = \begin{pmatrix} 0 & *** \\ 0 & t_1 & *** \\ 0 & 0 & 0 & * \end{pmatrix} \), \( P_2 = \begin{pmatrix} * & *** \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \), \( P_3 = \begin{pmatrix} t_3 & *** \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \) and \( P_4 = \begin{pmatrix} * & * & * \\ 0 & t_4 & * \\ 0 & 0 & 0 \end{pmatrix} \). The largest Schubert cell is

\[ \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \] \( P/P \) with coordinates \( (a, b, c, d) \). The affine coordinate

\( (t_1, \ldots, t_4) \in \mathbb{A}^4 \) on \( Z_{w,P} \) is

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t_2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
So the map $\psi : Z_{\mu_p} \rightarrow X$ under these local charts is given by

$$a = \frac{1}{t_1t_3}, \quad b = -\frac{t_1 + t_4}{t_1t_2t_3t_4}, \quad c = \frac{1}{t_1}, \quad d = -\frac{1}{t_1t_2} \tag{3.1}$$

Recall the anticanonical section $s_0$ in [17] is given in terms of standard monomials as follows. Let $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$ be the basis of any two plane. The Plücker coordinates $x_{ij}$ are the determinant of $i, j$ columns. The section $s_0 = x_{12}x_{23}x_{34}x_{14}$. In coordinate of Schubert cell, we have $s_0 = -ad(ad - bc)(da \wedge db \wedge dc \wedge dd)^{-1}$. A direct calculation using (3.1) shows that $\psi^*s_0 = t_1t_2t_3t_4(dt_1dt_2dt_3dt_4)^{-1}$. The other sections of $H^0(X, K_X^{-1})$ can also be written as homogeneous polynomials of $x_{ij}$ of degree 4.

**Remark 3.4** The proof for both toric and flag varieties only depends on the following fact. There is a torus chart $(t_1, \ldots, t_n)$ on the complement of $Y_{s_0}$ with $s_0 = t_1 \cdots t_n(d_1 \wedge \cdots \wedge d_t)^{-1}$ on the chart. So Theorem 1.2 can hold for more general ambient spaces $X$ with such an anticanonical form $s_0$. The existence of such $s_0$ also implies the Frobenius splitting of $X$ compatible with $Y_{s_0}$.

## 4 Complete intersections

We further discuss the algorithm for Hasse–Witt matrix for complete intersections.

1. Let $X^n$ be a smooth projective variety defined over field $k$ with positive characteristic $p$. Let $L_1, \ldots, L_s$ be line bundles on $X$ and $E = \bigoplus_i L_i$. Assume the following vanishing conditions $H^i(X, K_X \otimes \wedge^s E) = H^{i-1}(X, K_X \otimes \wedge^{s-i} E) = 0$ for $i = 1, \ldots, s$.

2. Let $V_i = H^0(X, L_i) \neq 0$ and $W = H^0(X, \det E \otimes K_X) \neq 0$. We further assume the zero locus of a generic element of $V^\vee = H^0(X, E)$ is smooth with codimension $s$. Let $e_1^\vee \ldots e_s^\vee$ be a basis of $W^\vee$. Let $f_i$ be the universal section of $L_i$ and $f = (f_1, \ldots, f_s)$ be the universal section of $E$.

3. Let $\mathcal{Y}$ be the family of complete intersections defined by $f$ over the complement of the discriminant locus $\pi : \mathcal{X} \rightarrow \mathbb{X}$ is the embedding of universal family. The projections to $S$ are denoted by $\pi$.

4. Let $F_S$ be the absolute Frobenius on $S$ and $\mathcal{X}^{(p)} = \mathcal{X} \times F_S S$ the fiber product. Then we have absolute Frobenius $\mathcal{F}_S : \mathcal{X} \rightarrow \mathcal{X}$ and the relative Frobenius $F_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X}^{(p)}$. Denote $W : \mathcal{X}^{(p)} \rightarrow \mathcal{X}$ and $\pi^{(p)} : \mathcal{X}^{(p)} \rightarrow S$ to be the projections. The corresponding diagram for family $\mathcal{Y}$ is defined in a similar way.

We repeat the argument in the hypersurfaces using the Koszul resolution

$$0 \rightarrow \wedge^s E^\vee \rightarrow \wedge^{s-1} E^\vee \rightarrow \cdots \rightarrow E^\vee \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y \rightarrow 0.$$
\( \bigoplus_{(j_1, \ldots, j_k)} L^\vee_{j_1} \otimes \cdots \otimes L^\vee_{j_k} \) with ordered tuple \((j_1, \ldots, j_k)\), such that \(j_1, \ldots, j_k\) are distinct and \(f^\vee_{j_1, \ldots, j_k} = \pm f^\vee_{j'_1, \ldots, j'_k}\) if \(j_1, \ldots, j_k\) is a permutation of \(j'_1, \ldots, j'_k\) with signature \(\pm 1\). Then \((f^\vee_{j_1, \ldots, j_k})^p\) is mapped to \((\sum_j f^\vee_{j_1, \ldots, j_k} f_j)^p\). We have similar commutative diagram as (2.1)

\[
\begin{align*}
0 & \longrightarrow W^* \det E^\vee \overset{f}{\longrightarrow} F_X/S_* \det E^\vee \quad \overset{F}{\longrightarrow} \quad i_*O_Y(p) \longrightarrow 0 \\
0 & \longrightarrow F_X/S_* \det E^\vee \overset{f}{\longrightarrow} F_X/S_* \det E^\vee \quad \overset{F}{\longrightarrow} \quad i_*F_Y/S_*O_Y \longrightarrow 0.
\end{align*}
\]

The map \(f^{p-1}: W^* \wedge^k E^\vee \rightarrow F_X/S_* \wedge^k E^\vee\) is induced by multiplication \((f^\vee_{j_1, \ldots, j_k})^p \rightarrow ((f^\vee_{j_1, \ldots, j_k})^p f_{j_1}^{p-1} \ldots f_{j_k}^{p-1})\). So we have commutative diagram

\[
\begin{align*}
R^{n-s} \pi_*^{(p)}(O_Y(p)) & \longrightarrow R^{n-s} \pi_*^{(p)}(W^* \det E^\vee) \\
& \quad \quad \downarrow F \quad \quad \quad \quad \quad \quad \downarrow f^{p-1} \\
R^{n-s} \pi_*^{(p)}(F_Y/S_*O_Y) & \longrightarrow R^{n-s} \pi_*^{(p)}(\det E^\vee)
\end{align*}
\]

with horizontal maps being isomorphisms. The left vertical map is the Hasse–Witt operator HW: \(F_S^* (R^{n-s} \pi_* (O_Y)) \cong R^{n-s} \pi_*^{(p)}(O_Y(p)) \rightarrow R^{n-s} \pi_*^{(p)}(F_Y/S_*O_Y) \cong R^{n-s} \pi_*^{(p)}(\det E^\vee)\). So we have similar definition of Hasse–Witt matrix under basis \(e_1^\vee \ldots e_r^\vee\).

**Definition 4.1** The basis \(e_1^\vee \ldots e_r^\vee\) of \(H^0(X, K_X \otimes \det E)\) induces a basis of \(R^{n-s} \pi_*^{(p)}(\omega_{Y/S})\) by residue map and dual basis \(e_1 \ldots e_r\) of \(R^{n-s} \pi_*^{(p)}(O_Y)\) under Serre duality. The Hasse–Witt matrix \(a_{ij}\) is defined by HW\((F_S^*(e_i)) = \sum_j a_{ij} e_j\).

The same argument in hypersurfaces case gives us the algorithm of computing Hasse–Witt matrix in terms of local expansion of \(f\). Now \(f^{p-1}\) is replaced by \((f_1 \ldots f_s)^{p-1}\). Under the trivialization \(\xi\) of \(E\) under local coordinates \((t_1, \ldots, t_n)\), the section \(e_i^\vee\) has the form \(h_i(t)dt_1 \wedge dt_2 \cdots \wedge dt_n\) and \(\frac{e_i^\vee(f_1 \cdots f_s)^{p-1}}{\xi}\) has the form as \(g_i(t)dt_1 \wedge dt_2 \cdots \wedge dt_n\). Then \(\tau(g_i)\) has the form \(\tau(g_i) = \sum j a_{ij} h_j\).

On the other hand, the period integral has the form \(\int_{\gamma'} \text{Res } \frac{\Omega}{f_1 \cdots f_s} = \int_{\gamma'} \frac{\Omega}{f_1 \cdots f_s}\) where \(\gamma'\) is a cycle in the complement of \(\{f_1 \ldots f_s = 0\}\). So it is the same form as hypersurfaces with \(f\) replaced by \(f_1 \ldots f_s\). The section \(f_1 \ldots f_s\) also defines a subfamily of hypersurfaces in the linear system of \(E\). Both Hasse–Witt matrices and period integrals can be calculated with the same algorithm applied to this subfamily.

So the truncation relation still holds for complete intersections in both toric variety and flag variety. For example, the statement for toric Calabi–Yau complete intersections is as follows. Let \(X\) be a smooth toric variety and \(K_X^{-1} = D_1 + \cdots + D_s\) be a partition of toric divisors. Let \(L_i = O(D_i)\). Let \(f_{ij}\) be a basis of \(H^0(X, L_i)\) consisting of monomials with \(f_{i0}\) the defining section of \(D_i\). The universal section is \(f = (\sum_j b_{ij} f_{ij})_i\). Then the period integral of the unique invariant cycle near \(f_{i0} = (f_{i0})\)
has the form \( \frac{1}{b_{10} \cdots b_{00}} P(b_{1j_1} \cdots b_{js}) \), in which \( P(b_{1j_1} \cdots b_{js}) \) is a Taylor series of \( \frac{b_{1j_1} \cdots b_{js}}{b_{10} \cdots b_{00}} \) with integer coefficients. The degree-(\( p - 1 \)) truncation \((p-1)P(b_{ij})\) multiplied by \((b_{10} \cdots b_{00})^{p-1}\) is a degree-(\( p - 1 \)) polynomial of \( b_{1j_1} \cdots b_{js} \), and gives the Hasse–Witt matrix for the Calabi–Yau complete intersection family.

### 5 Frobenius matrices of toric hypersurfaces

Now we give a proof of the conjecture in [12] for toric hypersurfaces. First we state the conjecture. The notation follows [13]. Let \( k \) be a perfect field of characteristic \( p \). Let \( W = W(k) \) be the ring of Witt vectors of \( k \). Denote by \( \sigma : W \to W \) the absolute Frobenius automorphism of \( W \). For any \( W \)-scheme \( Z \), let \( Z_0 = Z \otimes W k \) be the reduction mod \( p \). Let \( S = \text{Spec}(R) \) be an affine \( W \)-scheme. Denote by \( R_\infty = \varprojlim R/p^s R \) and \( S_\infty = \text{Spf}(R_\infty) \). We fix a Frobenius lifting on \( R \) and denote by \( \sigma \), which is a ring endomorphism \( \sigma : R \to R \) such that \( \sigma(a) = a^p \mod pR \). Let \( X \) be a smooth complete toric variety defined by a fan. The 1-dimensional cones generated by primitive vectors \( v_1, \ldots, v_N \) correspond to toric divisors \( D_i \). Assume \( L = \mathcal{O}(\sum k_i D_i) \) with \( k_i \geq 1 \). Let \( \Delta = \{ v \in \mathbb{R}^n \mid \langle v, u_i \rangle \geq -k_i \} \) and \( \Lambda \) the interior of \( \Delta \). Then \( H^0(X, L) \) has a basis corresponding to \( u_i \in \Delta \cap \mathbb{Z}^n \) and \( H^0(X, L \otimes K_X) \) has basis \( e_i^\vee \) identified with \( u_i \in \Lambda \cap \mathbb{Z}^n \). Let \( f = \sum a_{ij} t^{u_i} \), \( a_{ij} \in R \) be a Laurent series representing a section of \( H^0(X, L) \). Let \( (\alpha_s)_{i,j} \) be a matrix with \( ij \)-th entry equal to the coefficient of \( t^{p^s u_j - u_i} \) in \( (f(t))^{p^s-1} \). The endomorphism \( \sigma \) is also extended entry-wisely to matrices. It is proved in [12] that \( \alpha_s \) satisfies the following congruence relations

**Theorem 5.1** (Theorem 1 in [12])

1. For \( s \geq 1 \),
   \[
   \alpha_s \equiv \alpha_1 \cdot \sigma(\alpha_1) \cdot \cdots \cdot \sigma^{s-1}(\alpha_1) \mod p.
   \]

2. Assume \( \alpha_1 \) is invertible in \( R_\infty \). Then
   \[
   \alpha_{s+1} \cdot \sigma(\alpha_s)^{-1} \equiv \alpha_s \cdot \sigma(\alpha_{s-1})^{-1} \mod p^s.
   \]

3. Under the condition of (2), for any derivation \( D : R \to R \), we have
   \[
   D(\sigma^m(\alpha_{s+1})) \cdot \sigma^m(\alpha_{s+1})^{-1} \equiv D(\sigma^m(\alpha_s)) \cdot \sigma^m(\alpha_s)^{-1} \mod p^{s+m}.
   \]

Suppose that \( f \) defines a smooth hypersurface \( \pi : Y \to S \). We assume \( Y \) satisfies the condition (HLF) in [13], the Hodge cohomology groups \( H^j(Y, \Omega^r_{Y/S}) \) are locally free \( R \)-modules for \( i + j = n - 1 \). We also assume the pair \( (X, Y) \) satisfies (HLF), the Hodge cohomology groups \( H^j(X, \Omega^r_{X/S}(\log Y)) \) are locally free \( R \)-modules for \( i + j = n \).

The comparison theorem gives isomorphism between \( F \)-crystals \( H^n_{cris}(Y_0/S_\infty) \cong H^0_{DR}(Y/S) \otimes_R R_\infty. \) We further assume the family \( Y/S \) satisfies condition HW(\( n-1 \)) in [13], which says for any \( r : R_0 \to K \) with \( K \) perfect field, the Hasse–Witt operator

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$H^{n-1}(Y_r^{(p)}, \mathcal{O}_{Y_r^{(p)}}) \rightarrow H^{n-1}(Y_r, \mathcal{O}_{Y_r})$ is an automorphism. Notice that $\alpha_1 \mod p$ is the Hasse–Witt matrix under the dual basis of $\omega_i = \text{Res}_{t_1^i \cdots t_n^i} \in H^0(Y, \Omega^{n-1}_{Y/S})$ according to Corollary 2.4. The condition $\text{HW}(n-1)$ can be checked by matrix $\alpha_1$. In particular, this condition implies that $\alpha_1 \mod p$ is invertible. The unit-root sub-$F$-crystal $U_0 \subset H^{n-1}_{\text{cris}}(Y_0/S_{\infty})$ and slope $n - 2$ sub-crystal $U_{\leq n-2}$ are defined under this assumption. The quotient $Q_{n-1} = H^{n-1}_{\text{cris}}/U_{\leq n-2}$ is isomorphic to the $p^{n-1}$-twist of the dual $U_0^\vee$ to $U_0$. See section 2 of [13] for the definitions. The projection of $\omega_i$ to $Q_{n-1}$ gives a dual basis of $U_0$. In [12], the Frobenius matrix and connection matrix of $U_0$ are conjectured to be the limits of matrices in Theorem 5.1.

**Conjecture 5.2** [12] The Frobenius matrix is the $p$-adic limit

$$F = \lim_{s \to \infty} \alpha_{s+1} \sigma(\alpha_s)^{-1}.$$ 

The connection matrix is given by

$$\nabla_{D} = \lim_{s \to \infty} D(\alpha_s)(\alpha_s)^{-1}.$$ 

Now we give the proof of this conjecture under an additional assumption on $(X, L)$.

**Theorem 5.3** Let $(X, L)$ be a smooth toric variety with line bundle $L = \mathcal{O}(\sum k_i D_i)$. Let $p_i$ be toric invariant points corresponding to top dimensional cone in the fan decomposition. If a generic section of $L$ does not vanish at some $p_i$, then Conjecture 5.2 is true.

The assumption in the theorem can be checked from toric data, or replaced by the equivalent assumption on the polytope of $|L|$. Let $p_i$ be the intersection of $D_1 \ldots D_n$. Under a transformation of $SL(n, \mathbb{Z})$, we can assume the corresponding cone is generated by standard basis of $\mathbb{R}^n$. Let $f = \sum_i a_i t^{a_i}$ as before. Then under a trivialization of $L$, the universal section $f$ is $t_1^{k_1} \ldots t_n^{k_n}(\sum_i a_i t^{a_i})$. So a generic section $f$ does not vanish at $(t_1, \ldots, t_n) = (0, \ldots, 0)$ means $(-k_1, \ldots, -k_n)$ is a vertex of $\Delta$. The assumption in the theorem is equivalent to that at least one of the vertices of $\Delta$ is the intersection of hyperplanes $\langle v, v_i \rangle = -k_i$, $1 \leq i \leq n$ with $v_1 \ldots v_n$ generating a cone of $X$. Especially, this is satisfied by $X = \mathbb{P}^n$ with $L = \mathcal{O}(d)$, $d \geq n + 1$.

**Proof** The proof follows the ideas in Katz’s proof of Theorem 6.2 in [13]. Consider the $F$-crystal structure on logarithmic crystalline cohomology $H^n_{\text{cris}}(X_0, Y_0) \cong H^n_{DR}(X, Y) \otimes R_{\infty}$. From the long exact sequence

$$\cdots \rightarrow H^n_{DR}(X) \rightarrow H^n_{DR}(X, Y) \rightarrow H^{n-1}_{DR}(Y)(-1) \rightarrow \cdots$$

and $H^n_{DR}(X)$ is concentrated in $H^{k-k}_{\text{cris}}$, the corresponding subcrystal $U_{\leq n-1}$ and quotient $Q_n$ are also defined on $H^n_{\text{cris}}(X_0, Y_0)$ by taking the inverse image of $U_{\leq n-2}$ subcrystal in $H^{n-1}_{\text{cris}}(Y)$ and

$$Q_n(H^n_{\text{cris}}(X_0, Y_0)) \cong Q_{n-1}(H^{n-1}_{\text{cris}}(Y_0))(-1).$$
Here $(-1)$ means the Frobenius action is multiplied by $p$. We also have an isomorphism $H_{cris}^n(X_0, Y_0) = (H^0(X, \Omega^2_X/S(Y)) \otimes R_{\infty}) \bigoplus U_{n-1}$. So we only need to consider the Frobenius matrix acting on projections of log $n$-forms $\omega_i = \frac{t^{ui} dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n f(t)}$ onto $\mathbb{Q}_n$. We can assume the primitive vectors $v_1, \ldots, v_n$ are the standard basis of $\mathbb{R}^n$. The cone generated by $v_1 \ldots v_n$ defines an affine coordinate $(t_1, \ldots, t_n)$ on $X$ which is isomorphic to $\mathbb{A}^n$. First we assume $Y$ is away from $(t_1, \ldots, t_n) = 0$ and consider the formal expansion map at $(t_1, \ldots, t_n) = 0$

$$P : H^m_{DR}(X, Y) \otimes R_{\infty} \rightarrow H^m_{DR}(R_{\infty}[[t_1, \ldots, t_n]]/R_{\infty}).$$

Similar as Katz’s proof of Theorem 6.2 [13], we have the following conjecture

**Conjecture 5.4** $U_{\leq n-1}$ is the kernel of formal expansion map.

Actually a weaker statement that $U_{\leq n-1}$ is contained in the the kernel can imply that $U_{\leq n-1}$ is the kernel, see Remark 5.5. The conjecture might be proved by log version of the theory of de Rham–Witt following Illusie’s proof of Theorem 7.2 in [13]. We will first give the proof of Theorem 5.3 assuming the conjecture and state the method to get around the conjecture at the end. Assume the local expansion of $\frac{1}{f}$ exist in $R[[t_1, \ldots, t_n]][\frac{1}{t_1}, \ldots, \frac{1}{t_n}]$ and has the form

$$\frac{1}{f} = \sum_u A_u t^u.$$

Notice that $f$ may not have an inverse in $R[[t_1, \ldots, t_n]][\frac{1}{t_1}, \ldots, \frac{1}{t_n}]$. We can consider the localization of $R$ by inverting the coefficient $a_{u_0}$ of the vertex $u_0 = (-k_1, \ldots, -k_n)$. Let $a_{u_0} = 1$, then

$$\frac{1}{f} = t^{-u_0} \left( 1 + \sum_{u \neq u_0} a_1 t^{u_1-u_0} \right) = t^{-u_0} \left( 1 + \sum_{u \neq u_0} a_1 t^{u_1-u_0} \right) = \sum_u A_u t^u.$$

So the local expansion of $\omega_i$ has the form

$$\omega_i = \frac{t^{ui} dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n f(t)} = \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n} \cdot \sum_u A_u t^{u+u_i}$$

with all $u + u_i > 0$. Here any multi-index being greater than zero means that each component is greater or equal than zero and at least one component is strictly greater than zero.

Assume the Frobenius action on $\omega_i$ has the form

$$F(\omega_i^{(\sigma)}) \equiv \sum_j f_{ij} \omega_j \mod U_{\leq n-1}$$
and the connection of $\nabla_D$ has the form

$$\nabla(D)(\omega_i) \equiv \sum_j \nabla(D)_{ij} \omega_j \mod U_{\leq n-1}. \tag{5.1}$$

Assume Conjecture 5.4 is true, then

$$F(\omega^{(\sigma)}_i) = \sum_j f_{ij} \omega_j \text{ in } H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty) \tag{5.2}$$

and

$$\nabla(D)(\omega_i) = \sum_j \nabla(D)_{ij} \omega_j \text{ in } H^n_{DR}(R_\infty[[t_1 \ldots t_n]]/R_\infty). \tag{5.3}$$

According to the standard calculation of Frobenius action on $H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty)$, we compare the coefficient of $t^{p^k}v$ for multi-index $v \in \mathbb{Z}^n$ and $v > 0$

$$p^n \sigma(A^i_{u_i}) = \sum_j f_{ij} A^i_{u''_j} \mod p^k \tag{5.4}$$

with multi-indices $u'_i$ and $u''_j$ satisfy $p(u'_i + u_i) = p^k v = u''_j + u_j$. On the other hand, we compare the expansions of $f^{p^s-1} = \sum_u \tilde{A}^{s'}_u t^u$ and $\frac{1}{f}$

$$\sum_u \tilde{A}^{s'}_u t^u = f^{p^s-1} \frac{1}{f} = \left(\sum_u B^{s'}_u t^u\right) \left(\sum_u A^i_{u''} t^u\right). \tag{5.5}$$

So $\tilde{A}^{s'}_u = \sum_{u'+u''=u} B^{s'}_u A^i_{u''}$. We can extend $\sigma$ to a ring homomorphism on the ring of Laurent series with coefficients in $R$ by $\sigma(t^n) = t^{p^n}$. Since $\sigma(f) = f^p + pg$ for some Laurent series $g$, then

$$\sigma(f^{p^s-1}) = \sigma(f)^{p^s-1} = (f^p + pg)^{p^s-1} \equiv f^{p^s} \mod p^s. \tag{5.6}$$

Let $f^{p^s-1} = \sum_u \tilde{A}^{s-1}_u t^u$, then

$$\sum_u \sigma(\tilde{A}^{s-1}_u) t^{pu} = f^{p^s} \sigma\left(\frac{1}{f}\right) = \left(\sum_u B^{s'}_u t^u\right) \left(\sum_u \sigma(A_u) t^{pu}\right) \mod p^s. \tag{5.7}$$

So $\sigma(\tilde{A}^{s-1}_u) \equiv \sum_{u'+pu''=pu} B^{s'}_u \sigma(A_{u''}) \mod p^s$. Let $i$ and $m$ be two integers. Now we compute the action of $\sigma$ on the element of $\alpha_{s-1}$ at $i$-th row and $m$-th column

$$\sigma((\alpha_{s-1})_{im}) = \sigma(\tilde{A}^{s-1}_{p^{s-1}u_m-u_i}) \text{ in terms of } A_u \text{ and } B_u. \tag{5.8}$$

The sum of $B^{s'}_u \sigma(A_{u''})$
with \( u' + pu'' = p(p^{s-1}u_m - u_i) \). The factor \( B_u^s \) is the sum of the terms

\[
\left( \begin{array}{c} p^s \\ k_1, k_2, \ldots, k_l \end{array} \right) a^k_{l_1} \cdots a^k_{l_l}
\]

with \( k_1u_{l_1} + \cdots + k_lu_{l_l} = u' \). Denote by \( v_p \) the \( p \)-adic valuation. Let \( k = \min\{v_p(k_1)\ldots v_p(k_l)\} \) and \( p^kv = p^su_m - u' = p(u'' + u_i) \) in (5.3), then

\[
\sigma(A_{u''}) \equiv \sum_j f_{ij} A_{u''_j} \mod p^k
\]

with \( u' + u'' = p^su_m - u_j \). Since the \( p \)-adic valuation of multinomial has estimate

\[
v_p\left( \begin{array}{c} p^s \\ k_1, k_2, \ldots, k_l \end{array} \right) \geq s - \min\{v_p(k_1)\ldots v_p(k_l)\},
\]

then

\[
\left( \begin{array}{c} p^s \\ k_1, k_2, \ldots, k_l \end{array} \right) a^k_{l_1} \cdots a^k_{l_l} \sigma(A_{u''}) \equiv \sum_j f_{ij} \left( \begin{array}{c} p^s \\ k_1, k_2, \ldots, k_l \end{array} \right) a^k_{l_1} \cdots a^k_{l_l} A_{u''_j} \mod p^s.
\]

So we have

\[
B_u^s \sigma(A_{u''}) \equiv \sum_j f_{ij} B_u^s A_{u''_j} \mod p^s
\]

with \( u' + u'' = p^{s-1}u_m - u_i \) and \( u' + u''_j = p^su_m - u_j \). Summing all such terms implies

\[
p^n \sigma(\alpha_{s-1}) \equiv (f_{ij})\alpha_s \mod p^s
\]

and \( p^n(f_{ij})^{-1} \equiv \alpha_s\sigma(\alpha_{s-1})^{-1} \mod p^{s-n} \). Similar calculation as [13] and congruence relation \( D(B_u^s) \equiv 0 \mod p^s \) imply

\[
D(\alpha_s) \equiv (\nabla(D)_{ij})\alpha_s \mod p^s.
\]

If the coefficient of the vertex \( u_0 \) is zero, we regard \( a_I \) as formal variables and the universal hypersurface family. Then we can prove the result on an open subset of \( S \) and the \( p \)-adic limit formulas hold on the open subset. Since Vlasenko proved the congruence relations in Theorem 5.1 without any constraints on the coefficients, the \( p \)-adic limits always exist. So the limits coincide with Frobenius matrices and connection matrices because they are equal restricted to an open subset of \( S \).
Now we state the proof without assuming Conjecture 5.4. We claim $p^{l(n-1)} F(U_{\leq n-1}) \subset p^l H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty)$. Applying Katz’s argument of extension of scalars in [13, section 7], we only need to prove this when $R$ is the ring of Witt vectors of a perfect field. The Frobenius action on $U_{\leq n-1}$ divides $p^{n-1}$. So there exists $\sigma^{-1}$-linear map $\tilde{F}$ on $U_{\leq n-1}$ such that $\tilde{F} F = F \tilde{F} = p^{n-1}$. On the other hand, the Frobenius action on each element in $H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty)$ has a factor $p^n$. So $p^{n-1} P(U_{\leq n-1}) = P(p^{n-1} U_{\leq n-1}) = P(F \tilde{F} U_{\leq n-1}) \subset p^n H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty)$ and $l$ iterations give $p^{l(n-1)} P(U_{\leq n-1}) \subset p^l H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty)$. Multiplying both (5.1) and (5.2) by $p^{l(n-1)}$, we obtain

$$p^{l(n-1)} F(\omega_i^{(\sigma)}) \equiv p^{l(n-1)} \sum_j f_{ij} \omega_j \mod p^l$$

and

$$p^{l(n-1)} \nabla(D)(\omega_i) \equiv p^{l(n-1)} \sum_j \nabla(D)_{ij} \omega_j \mod p^l$$

Let $s = nl$, similar congruence relation as (5.3) still holds for $k \leq s$

$$p^{n+l(n-1)} \sigma(A_{u_i}) \equiv p^{l(n-1)} \sum_j f_{ij} A_{u_j} \mod p^k$$

with $p(u_i + u_i) = p^k v = u_j + u_j$ and $v > 0$. The same argument shows

$$p^{n+l(n-1)} \sigma(\alpha_{s-1}) \equiv p^{l(n-1)} (f_{ij}) \alpha_s \mod p^s$$

and

$$p^{l(n-1)} D(\alpha_s) \equiv p^{l(n-1)} (\nabla(D)_{ij}) \alpha_s \mod p^s$$

Dividing both sides by $p^{l(n-1)}$ and letting $l \to \infty$, we see that the subsequence $\alpha_s \sigma(\alpha_{s-1})^{-1}$ and $D(\alpha_s)(\alpha_s)^{-1}$ converges to the Frobenius matrix and connection matrix.

Remark 5.5 If $U_{\leq n-1}$ is contained in the the kernel of formal expansion map, then it is exactly the kernel. We only need to show the restriction of expansion map on $H^0(X, \Omega_X^1(Y)) \otimes R_\infty \to H^n_{DR}(R_\infty[[t_1, \ldots, t_n]]/R_\infty)$ is injective. This can be proved by similar argument in the proof of Theorem 5.3 and invertibility of $\alpha_s$.

Remark 5.6 The proof also gives a weaker version of the second and third congruence relations in Vlasenko’s Theorem 5.1. The first congruence relation $\alpha_s \equiv \alpha_1 \cdot \sigma(\alpha_{s-1}) \mod p$ can also be proved geometrically using the argument in Proposition 2.2 and Corollary 2.4. We can consider the $s$-iterated Hasse–Witt operation $H^{n-1}(Y_0^{(p^s)}(\mathcal{O}_{Y_0^{(p^s)}})) \to H^{n-1}(Y_0, \mathcal{O}_{Y_0})$. Using similar commutative diagram 2.1
with the first vertical map $L^{-1} \to L^{-1}$ replaced by the composition $L^{-1} \to L^{-p^s} \to L^{-1}$ with $\xi \mapsto \xi^{p^s} \cdot f^{p^s-1}$, we can see the matrix for $s$-iterated Hasse–Witt operation is given by $\alpha_s \mod p$. Hence $\alpha_s \equiv \alpha_1 \cdot \sigma(\alpha_{s-1}) \mod p$.

### 5.1 Unit root of toric Calabi–Yau hypersurfaces and periods

Now we discuss the relation between unit roots of zeta functions and period integrals for toric Calabi–Yau hypersurfaces. Let $f$ be a Laurent series defining toric Calabi–Yau hypersurfaces. For the sake of simplicity, let $a_0 = 1$ be the constant term (or the coefficient of interior point of $\Delta$). The unique holomorphic period integral at the special solution-1 point or “large complex structure limit” is $I_{\gamma} = \text{constant term of expansion}$.

It can be written as formal power series of $a_I$ with constant term being 1 and denoted by $P(a_I)$. Then

$$\alpha_s \equiv (p^s-1)(P(a_I)) \mod p^s$$

because of the congruence

$$\left(\begin{array}{c} p^s - 1 \\ k_1, k_2, \ldots, k_l, p^s - 1 - k \end{array} \right) \equiv (-1)^k \left(\begin{array}{c} k \\ k_1, k_2, \ldots, k_l \end{array} \right) \mod p^s.$$

Then

$$\alpha_s \sigma(\alpha_{s-1})^{-1} \equiv \frac{(p^s-1)(P(a_I))}{(p^s-1)(P(\sigma(a_I)))} \mod p^s$$

according to Vlasenko’s congruences without any geometric constraints. So the $p$-adic limit $\frac{P(a_I)}{P(\sigma(a_I))}$ exists in $R_\infty$ and equal to the Frobenius matrix. We can fix $\sigma(a_I) = a_I^p$. Then the formal power series

$$g(a_I) = \frac{P(a_I)}{P(a_I^p)}$$

has $p$-adic limit in $\lim_{s \to \infty} \mathbb{Z}_p[a_{I_1} \ldots a_{I_N}, \alpha_1^{-1}] / p^s \mathbb{Z}_p[a_{I_1} \ldots a_{I_N}, \alpha_1^{-1}]$ and it satisfies Dwork congruences

$$g(a_I) \equiv \frac{(p^s-1)(P(a_I))}{(p^s-1)(P(\sigma(a_I)))} \mod p^s.$$
Especially it is related to Hasse–Witt matrix by

\[ \frac{P(a_I)}{P(a_I^p)} \equiv (p-1)(P(a_I)) \pmod{p}. \]

Let \( q = p^r \) and \( a_I \in \mathbb{F}_q \) defining a smooth Calabi–Yau variety \( Y_0 \) over \( \mathbb{F}_q \). Assume the Hasse–Witt matrix \( (p-1)(P(a_I)) \pmod{p} \) is not zero. Then there exist exactly one \( p \)-adic unit root in the factor of zeta function of \( Y_0 \) corresponding to Frobenius action on \( H^{n-1}_{\text{cris}}(Y_0) \). It is given by

\[ g(\hat{a}_I)g(\hat{a}_I^p)\ldots g(\hat{a}_I^{p^{r-1}}) \]

with \( \hat{a}_I \) being the Teichmüller lifting under \( \sigma \). For example, if \( a_I \) has lifting as an integer, then \( \hat{a}_I = \lim_{s\to\infty} a_I^p \).

**Remark 5.7** In [12], the following result about unit roots is proved. When \( S_0 = \text{Spec}(\mathbb{F}_q) \) and \( Y_0 \) is a smooth hypersurface in \( \mathbb{P}^n \), let \( \Phi = F \cdot \sigma(F)\ldots\sigma^{r-1}(F) \). Then the eigenvalues of \( \Phi \) are unit roots of zeta-function of \( Y_0 \). The conjecture proved above implies that the multiplicities of unit roots are also equal. The proof in [12] uses Stienstra’s result on formal groups [34, 35]. See also [6] for the unit root formula for Dwork family using formal groups following Stienstra. It might be possible to give a proof of the conjecture by this approach.

### 6 Frobenius matrices for Calabi–Yau hypersurfaces

Now we discuss the algorithm of Frobenius matrix of Calabi–Yau hypersurfaces in terms of local expansion. Let \( X \) be a smooth Fano variety over \( S \) with ample line bundle \( L \). Let \( f \in H^0(X, L) \) define a smooth hypersurface \( Y \) in \( X \). Let \( \omega_i \) be a basis of \( H^0(X, L \otimes K_X^{-1}) \). This induces a basis of \( H^0(Y, K_Y) \) via adjunction formula. The Hasse–Witt matrix under this basis in terms of local coordinate is given by the algorithm in Sect. 2. Fix a section \( p : S \to X \) on ambient space \( X \) such that \( Y \) is away from \( p \). Let \((t_1, \ldots, t_n)\) be the formal coordinate of \( X \) at \( p \). The proof of Theorem 5.3 depends on \( \omega_i \) having the following form in local expansion. There is a trivialization \( \xi \) of \( L \) along \( p \) such that \( \frac{\omega_i}{\xi} \) has the form \( \frac{t_1^{n_1}dt_1\wedge\cdots\wedge t_n^{n_n}dt_n}{t_1\cdots t_n} \). The matrix \((\alpha_s)_{ij}\) is defined to be the coefficients of \( t^{p^{s-1}}\xi \) in \( \frac{f^{p^{s-1}}}{\xi^{p^{s-1}}} \). This applies to \( L = K_X^{-1} \) with trivialization \( \xi = (dt_1 \wedge \cdots \wedge dt_n)^{-1} \). So we have the following

**Proposition 6.1** Let \( f = g(t)(dt_1 \wedge \cdots \wedge dt_n)^{-1} \) and \( \alpha_s \) is the coefficient of \( (t_1 \ldots t_n)^{p^{s-1}} \) in the local expansion of \( g^{p^{s-1}} \). Then similar congruence relations in Vlasenko’s theorem (Theorem 5.1) still hold

1. For \( s \geq 1 \),

\[ \alpha_s \equiv \alpha_1 \cdot (\alpha_{s-1})^p \pmod{p}. \]
2. Under the condition $HW(n-1)$ and $g(0) \neq 0$, we have

$$\alpha_{n(s+1)} \cdot \sigma(\alpha_{n(s+1)})^{-1} \equiv \alpha_{ns} \cdot \sigma(\alpha_{ns})^{-1} \mod p^{s-n}.$$ 

3. Under the condition of (2), for any derivation $D: R \to R$, we have

$$D(\sigma(\alpha_{n(s+1)})) \cdot \sigma(\alpha_{n(s+1)})^{-1} \equiv D(\sigma(\alpha_{ns})) \cdot \sigma(\alpha_{ns})^{-1} \mod p^{s}.$$ 

The $p$-adic limit $\alpha_{ns} \cdot \sigma(\alpha_{ns})^{-1}$ gives the Frobenius action on the unit-root part $U_0$ of $H_{cris}^{n-1}(Y_0)$ under the basis induced by residue map.

### 6.1 Unit root of Calabi–Yau hypersurfaces in $G/P$

Now we discuss the algorithm for Frobenius matrix of the unit root part of Calabi–Yau hypersurfaces in $X = G/P$. Consider the affine chart $\mathbb{A}^n$ on the Bott–Samelson desingularization of $G/P$ in Sect. 3. This induces a torus chart $\mathbb{G}_m^n$ on $G/P$. We consider the formal polydisc $R\{[t_1, \ldots, t_n]|\frac{1}{t_1}, \ldots, \frac{1}{t_n}\}$ instead of $R\{[t_1, \ldots, t_n]\}$ in the formal expansion map in the proof of Theorem 5.3. The same method gives the algorithm of the unit root part of the Frobenius action.

**Theorem 6.2** Let $f \in H^0(X, K_X^{-1})$ be an anticanonical form defining a smooth Calabi–Yau hypersurface $Y$. Assume $f$ has the form $f = g(t)(dt_1 \wedge \cdots \wedge dt_n)^{-1}$ with $g(t) \in R[t_1, \ldots, t_n][\frac{1}{t_1}, \ldots, \frac{1}{t_n}]$ in the torus chart as above. Assume the hypersurface $Y$ is away from the image of $(t_1, \ldots, t_n) = 0$. Let $a_s$ be the coefficient of $(t_1 \ldots t_n)^{p^s-1}$ in the local expansion of $g^{p^s-1}$. The Hasse–Witt matrix is given by $\alpha_1 \mod p$. The same congruence relations and Frobenius matrix in Proposition 6.1 holds.

**Proof** The function $g(t) = t_1^{-k_1} \ldots t_n^{-k_n} \tilde{g}(t)$ for some $k_i \geq 0$ and $\tilde{g}(t) \in R[t_1, \ldots, t_n]$ does not vanish at $(t_1, \ldots, t_n) = 0$. This is because the torus chart extends to an map on $\mathbb{A}^n$ and $k_i$ are the multiplicity of exceptional divisor. So the index $p^k v$ appearing in the proof of Theorem 5.3 still has positive components. The rest of the proof is the same as toric case.

In Sect. 3, the period integral for the Calabi–Yau hypersurfaces in $G/P$ is reduced to similar algorithm on the torus chart. So the same argument in Sect. 5.1 implies similar relations between periods and the unit roots of zeta-functions of Calabi–Yau hypersurfaces defined on finite fields.

**Remark 6.3** We require the non-vanishing condition in Theorem 5.3 and 6.2 to discuss the local expansion map. But the definition of matrix $\alpha_s$ and congruence relations do not require this condition. So there might be a proof for general cases not depending on local expansions.

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Declarations

Conflict of interest  On behalf of all authors, Chenglong Yu states that there is no conflict of interest.

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