Time-Varying and Nonlinearly Scaled Consensus of Multiagent Systems: A Generic Attracting Law Approach

Mingxuan Sun and Xing Li

Abstract—This paper presents the design and analysis of the finite/fixed-time scaled consensus for multiagent systems. A study on a generic attracting law, the certain classes of nonlinear systems that admit attractors with finite/fixed-time convergence, is at first given for the consensus purpose. The estimates for the lower and upper bounds on the settling time functions are provided through the two-phase analysis. The given estimates are initial state dependent, but the durations are finite, without regarding the values that the initial states take. According to the generic attracting law, distributed protocols are proposed for multiagent systems with undirected and detail-balanced directed graphs, respectively, where the scaled strategies, including time-varying and nonlinear scales, are adopted. It is shown that the finite/fixed-time consensus for the multiagent system undertaken can still be achieved, even though both time-varying and nonlinear scales are taken among agents. Numerical simulation of two illustrative examples are given to verify effectiveness of the proposed finite-duration consensus protocols.

Index Terms—Scales, finite time convergence, initial conditions, multi-agent systems.

I. INTRODUCTION

In recent years, the problem of distributed cooperative control of multi-agent systems has aroused considerable attention [1], [2], [3], [4], [5]. The broad applications include rendezvous of mobile autonomous robots [6], [7], vehicle formations [1], [3], [8], flocking of mobile sensor networks [9], and so on. Through the mutual cooperation between agents, the large and complex system undertaken could be coped with, and tasks can be accomplished by each single agent. The consensus objective is to apply distributed protocols, that only require interaction information between the local neighborhoods of agents, due to the limitations of communication bandwidth and sensor range, to make the agents reach an agreement on their states. The early study on consensus problems was found in [10], which gave a simple discrete-time model to simulate the emergence of self-organized particle swarm. In [2], a theoretical explanation is provided for the consensus behavior of the Vicsek model, and the convergence analyses for several inspired models are given. In [5], the results in [2] were extended to the case of directed graphs, where interaction information between the local neighborhoods of agents can be exchanged under dynamically changing interaction topologies. Graph Laplacians are important which play a crucial role in stability and convergence analysis of consensus algorithms [3], [4]. It is well known that the algebraic connectivity can measure the convergence rate, and the convergence performance can be improved by increasing the algebraic connectivity. The theoretical framework of the graph Laplacians based consensus of multiagent systems is systematically provided in [11], and references therein. Note that special attentions were paid to the design techniques for the systems on directed graphs [12], [13], [14].

The convergence speed of consensus protocols is usually taken as a type of performance assessment, an indicator of how effectively and efficiently the consensus is achieved. Many conventional schemes pursue an asymptotic or exponential solution which is obtained in an infinite time range. In fact, one would expect that the system consensus occurs in finite time, and maintain it afterwards. Moreover, finite-time convergence is desirable to satisfy special needs, such as better disturbance rejection and robustness against uncertainties. Such a specific requirement is particularly interesting that has caught the attention of researchers. In [15], the normalized and signed versions of the gradient descent flow of a differentiable function were introduced, and it was shown how the proposed nonsmooth gradient flows achieve consensus in finite-time. A general framework for designing finite-time semistable protocols in dynamical networks, was developed in [16], where semistability is the property whereby every system solution converges to a limit point that may depend on the initial condition.

Earlier works were found in [17], [18], which provide an effective way to construct consensus protocols by continuous state feedbacks and bridge the gap between asymptotical consensus and discontinuous finite-time consensus. In the last ten years, increasing research efforts have been dedicated to various finite-time consensus problems, e.g., for systems described by double integrators [19], for systems under the directed and switching topologies [20], and for systems under the time-varying directed topologies with uncertain leader [21]. In [22], a switching consensus protocol was designed to solve the finite-time weighted-average-consensus problem for systems on a fixed directed interaction graph. It should be noted that for the existing finite-time protocol designs, the settling time function depends on the initial state of the agent undertaken. The convergence time cannot be pre-specifiable, as the initial state is not available. Moreover, it takes a long time for the convergence, as the initial state is located far away from the attractor. The finite-time stability was examined in a seminal paper [23], for characterizing that...
there exists an upper-bound on the settling time function of the adopted two-term attracting law (AL), where the term fixed-time was adopted to describe such convergence property. The works reported in [24], [25], [26] showing the early efforts which were made to apply such stability theory to the consensus designs. Furthermore, these problems were solved for second-order systems [27], [28], and higher-order systems [29]. The problem of finite/fixed-time cluster synchronization with pinning control was addressed in [30]. The achievements of consensus are shown to be effective in coping with input delays [28], handling the output feedback protocol design for second-order systems without velocity measurement [31], and addressing robust performance against bounded uncertain disturbances [27], [32], [33], [34]. It was noticed in [35], that for certain ALs, the duration (or the bound of the duration) of the settling time varies with the initial state, and the duration can be exactly calculated for each given initial state. It was also shown that the expression for the duration bound can be obtained, and it was proved to be finite, even as the initial state approaches infinity. In this paper, AL indicates a desired model capable of finite-time tracking, by which the dynamics of the closed-loop system is governed by the desired model. The AL approach is closely related to the pole placement technique taking for example simultaneous coordination of vehicles both in space and on ground, due to the huge difference between the scales of vehicles position and velocity [57]. Related topics are the weighted-average consensus with constant weights [22], and with respect to a monotonic function [38], and bipartite consensus under cooperative and antagonistic interactions [39], [40]. For a complex network composed of two subnets, the problem of scale group consensus was addressed in [41]. In [42], a class of smooth functions was identified, for which one can synthesize distributed algorithms that achieve consensus. In [43], a description of the feasible time-varying formation and an explicit expression of the time-varying reference function were presented. Scaled consensus, allowing both nonlinear and time-varying scales, can be formulated in a unified and general manner for consensus. To the best of our knowledge, however, none of the studies have explored yet, which may be of particular interest as assessed in [22], [38], [39], [40], [42], [43].

In this paper, we investigate the scaled consensus, including both nonlinear and time-varying scales, for multi-agent systems, such that the finite/fixed-time consensus is achieved. Comparing with the existing works, in particular, the main contributions of this paper lie in: i. the scaled consensus with time-varying and nonlinear scales, achieved by all agents in the network; ii. the protocol designs for multiagent systems on undirected and directed graphs, respectively; and iii. a finite/fixed convergent AL that the duration of the settling time function is finite, whatever the values of initial states take. The novelty of our proposed protocol design is its combined use of a generic form of AL, aiming at the improvement of the convergence performance, whereas the double power AL is usually adopted in the related consensus schemes.

The rest of the paper is organized as follows. In Section II we give a problem formulation, with a description about the scaled consensus to be tackled. The convergence results of a generic AL are given in subsection III through the lower and upper bound estimates on the settling time functions. The main results are presented in Section IV. The finite/fixed-time consensus protocols are designed and analyzed, in subsections IV-A and IV-B for systems on undirected and detail-balanced directed graphs, respectively. More related issues are addressed in Section V. The obtained numerical results are presented in Section VI and the conclusion is finally drawn in Section VII.

II. PROBLEM FORMULATION

Let us consider a weighted undirected or directed graph $G = \{V, E, A\}$ ( or $G(A)$ for short), in order to model the interaction topology of the network of $N$ dynamic agents, for which the consensus problem is tackled in this paper. $V = \{1, 2, \cdots, N\}$ is the vertex set, where each vertex represents an agent of the network. $E \subseteq V \times V$ is the set of connected edges. $A[a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency weight matrix, where $a_{ij} > 0$ if and only if $(i, j) \in E$, otherwise, $a_{ij} = 0$, $i, j = \{1, 2, \cdots, N\}$. If all nonzero elements of $A$ are 1, we say $G(A)$ is unweighted. Here assume that no self-loops in the graph, i.e., $(i, i) \notin E$, or $a_{ii} = 0$. For an undirected graph, if there is a connection between two nodes $i$ and $j$, then $a_{ij} = a_{ji} > 0$, and $A$ is symmetric. In contrast, for a directed graph, if there is a connection from nodes $i$ to $j$, then $a_{ij} > 0$, $A$ is not a symmetric matrix. $G(A)$ is referred to as a weighted directed graph, if $i, j \in E \Leftrightarrow a_{ij} > 0$. The degree matrix of $G(A)$ is a diagonal matrix $D = \text{diag}\{d_1, d_2, \cdots, d_N\}$, where the degree $d_i$ of node $i$ is defined as $d_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix of graph $G(A)$ is defined as $L_A = D - A$, which is symmetric. An undirected graph $G(A)$ is connected if there exists a path between $i$ and $j$, $i, j \in \{1, 2, \cdots, N\}$, between any pair of distinct agent $i$ and agent $j$ in $G(A)$. For a directed graph, it is considered to have strong connectivity, if any two different nodes can reach each other.

We consider a group of $N$ agents, which are described by the following first-order differential equation

$$\dot{x}_i = u_i$$

(1)

where $i \in \{1, 2, \cdots, N\}$, $x_i$ is the state of agent $i$ and $u_i$ the control input protocol to be designed. The objective of this paper is to find $u_i$, $i = 1, 2, \cdots, N$, such that the scaled consensus can be achieved for the $N$ agents described by Eq. (1), under undirected and directed interaction topology, respectively.

A multiagent system is said to achieve the finite-duration scaled consensus, with both time-varying and nonlinear scales, if, for any $x_i(0), i \in \{1, 2, \cdots, N\}$, and $\forall i, j \in \{1, 2, \cdots, N\}$,

$$g_i(x_j, t) = g_i(x_i, t), \quad \forall t \geq T$$

(2)

where $T$ is the duration of the settling time, $g_i(x_i, t), i = 1, 2, \cdots, N$, represent the scaling functions, which are assumed to be continuous differentiable, and $g_i(x_i, t) \neq 0$ and $\frac{\partial g_i}{\partial x_i}(x_i, t) \neq 0$. 

\[\text{Figure 1: The network topology.} \]
The scaling can be adopted by a separate manner. The multiple scales takes $s_i(t)g_i(x_i)$ as the scaling functions, which typically satisfies that

1. $s_i(t) \neq 0, i = 1, 2, \ldots, N$, indicating the time-varying scales, and
2. $g_i(x_i) \neq 0, i = 1, 2, \ldots, N$, are the nonlinear scales, satisfying that $\frac{dg_i(x_i)}{dx_i} \neq 0$.

The additional scales adopt the form of $s_i(t) + g_i(x_i)$, with the appropriate requirements. The published scaled consensus designs [50, 51] dealt with constant (multiple) scales, where $s_i$ takes a constant value and $g_i(x_i, t) = x_i$. The nonlinear consensus can be handled by choosing $s_i(t) = 1$ [42]. Similar to [43], we apply the additional scales to address the problem of time-varying consensus. The consensus approach of our paper is a unified one, allowing the above described scaling, and has suitability for broad applications. By the finite-fixed-time consensus we mean that the duration of the settling time of each closed-loop system is finite, whatever value the initial state is. We shall explain this, in Section III, with the definition of the finite-fixed-time stability.

III. A Generic Attracting Law

For the purpose of the scaled consensus, in this section, the convergence performance of certain nonlinear systems are considered, which admit attractors with finite settling time.

Let us begin with the introduction of the concepts about finite-time stability. Consider the scalar nonlinear system $\dot{x} = f(x), f(0) = 0$, and $x(0) = x_0$. Its zero solution, $x = 0$, is said to be globally fixed-time stable, if it is globally finite-time stable and the settling time function, $T(x_0)$, is bounded for arbitrary $x_0$ [23]. For the concepts on asymptotic stability, finite-time stability, and the definition for settling time function, we refer to literature [44]. In this paper, we suggest an attracting law approach, which is applicable for both finite-time and fixed-time stable systems. Note that the settling time function of a finite-time stable system, $T(x_0)$, is continuous if and only if it is continuous at $x = 0$. Hence, as the settling time function is continuous, the settling time is finite for a finite $x_0$. Moreover, for a fixed-time stable system, an upper bound of $T(x_0)$ exists, and the duration of the interval $[0, T(x_0))$ is finite, whatever the value of $x_0$ takes. In this paper, we also wish to give a lower bound of $T(x_0)$, which is helpful for determining the duration, when $x_0$ is not available.

The suggested AL approach is closely related to the pole placement technique for linear systems. The pole placement technique is usually applied for a linear system to realize the model-following purpose, by which the dynamics of the closed-loop system is governed by the desired model capable of exponentially asymptotically tracking. In this paper, AL indicates a desired model ensuring finite-time tracking, and the consensus objective is also to realize the model-following purpose. We can see that the AL approach presented in this paper relies on an extension of the pole placement technique.

For the consensus purpose, we adopt a generic attracting law (GAL), described by the following differential equation

$$\dot{x} = -\rho x - \kappa_1 |x|^{\gamma_1} \text{sgn}(x) - \kappa_2 |x|^{\gamma_2} \text{sgn}(x), \ x(0) = x_0 \quad (3)$$

where $\rho, \kappa_1$ and $\kappa_2$ are positive reals, $\gamma_1 = q/p$, $\gamma_2 = m/n$, and $q, p, m, n$ are odd numbers, satisfying that $q < p$ and $n < m$. The main idea behind the generic use is to accelerate the convergence rate in different phases, based on the properties of the power function. The GAL involves three terms, all in the form of $x^\gamma$, satisfying that for $0 < \gamma' < \gamma''$, $|x|^{\gamma'} < |x|^{\gamma''}$, as $|x| > 1$; and $|x|^\gamma > |x|^{\gamma''}$, as $|x| < 1$. These three terms in GAL indicates all cases of $\gamma$ (i.e., $0 < \gamma < 1$, $\gamma = 1$, and $\gamma > 1$), and in turn specify a generic action for convergence improvement, according to the following useful properties:

P-term (the proportion term, with $\gamma = 1$). The proportional term with $\gamma = 1$ is introduced, in order to speed up the convergence process. With this term, the AL is usually referred to as a fast-AL. When setting $\kappa_1 = \kappa_2 = 0$, the AL reduces to $\dot{x} = -\rho x$, assuring that $x$ converges to zero exponentially, as time increases.

FT-term (the finite-time term, with $0 < \gamma_1 < 1$). Due to the term with $0 < \gamma_1 < 1$, the finite-time convergence of the AL is guaranteed. Obviously, the AL fails to achieve finite-time convergence, as $\kappa_1 = 0$. Note that for $\gamma_1' < \gamma_1''$, $|x|^{\gamma_1'} > |x|^{\gamma_1''}$, as $|x| < 1$. As such, the smaller the $\gamma_1$, the faster the convergence rate.

FD-term (the finite-duration term, with $\gamma_2 > 1$). The term with $\gamma_2 > 1$ is needed to ensure that the duration bound on the settling time function is finite. Without this term, namely, $\kappa_2 = 0$, this AL cannot achieve the bounded-duration convergence rate any more. Note that for $\gamma_2' < \gamma_2''$, $|x|^{\gamma_2'} < |x|^{\gamma_2''}$, as $|x| < 1$. Therefore, the larger the $\gamma_2$, the faster the convergence rate.

For the purpose of comparison, we consider the following AL, a desired model appeared in the related publications,

$$\dot{x} = -\kappa_1 |x|^{\gamma_1} \text{sgn}(x) - \kappa_2 |x|^{\gamma_2} \text{sgn}(x), \ x(0) = x_0 \quad (4)$$

This model has only two terms in the right-hand side of (4), and lacks the proportion term. This model is very popular and the style is unique, in the context of finite-time consensus. We have to make a detailed comparison between this model and the GAL, and clarify the need and the necessity to introduce the GAL.

The results about the mentioned finite/fixed-time convergent ALs are summarized in the following lemmas, which will be used for the consensus designs to be presented.

**Lemma 1:** The origin of (3) is finite-time stable, associated with the property that the duration of the settling time function satisfies, for $|x_0| \geq 1$,

$$\ln \left( \frac{1 + \frac{\rho \kappa_2}{\kappa_1}}{|x_0|^{\gamma_1 - \gamma_2} + \frac{\rho}{\kappa_1}} \right) + \ln \left( 1 + \frac{\rho \kappa_2}{\kappa_1} \right) \leq T(x_0)$$

and for $|x_0| < 1$,

$$\ln \left( 1 + \frac{\rho \kappa_2}{\kappa_1} \right) \leq T(x_0)$$
It is seen from (7) and (8) that
\[
T(x) = \frac{\ln \left( \frac{(1 + \kappa_2 x_0^{1-\gamma_2})}{\kappa_2 (1 - \gamma_1)} \right)}{\kappa_2 (1 - \gamma_1)} + \frac{\ln \left( 1 + \frac{\kappa_2}{\kappa_1} \right)^{x_0^{1-\gamma_1}}}{\kappa_1 (1 - \gamma_1)} \leq T(x_0)
\]
and for \( |x| < 1 \),
\[
\frac{\ln \left( 1 + \frac{\kappa_2}{\kappa_1} \right)^{x_0^{1-\gamma_1}}}{\kappa_2 (1 - \gamma_1)} \leq T(x_0) \leq \frac{1}{\kappa_1 (1 - \gamma_1)} - \frac{1}{\kappa_2 (1 - \gamma_1)} \left( 1 - |x|^{1-\gamma_1} \right)
\]
According to (5)–(6), the finiteness of the duration \(|0, T(x_0)|\) with respect to \( x_0 \) is guaranteed, which can be expressed as, for \( |x| \geq 1 \),
\[
\ln \left( 1 + \frac{\rho + \kappa_2}{\kappa_1} \right) \leq \frac{1}{\rho (1 - \gamma_1)} \left( 1 + \frac{\rho}{\kappa_1} \right) + \frac{1}{\rho (1 - \gamma_1)} \ln \left( 1 + \frac{\rho}{\kappa_1} \right) \leq T(x_0)
\]
for \( |x| < 1 \),
\[
0 \leq T(x_0) \leq \frac{1}{\rho (1 - \gamma_1)} \ln \left( 1 + \frac{\rho}{\kappa_1} \right)
\]
It is seen from (7) and (8) that \( T(x_0) \) is finite for whatever value \( x_0 \) takes, and for \( |x| \geq 1 \),
\[
\ln \left( 1 + \frac{\kappa_2}{\kappa_1} \right)^{x_0^{1-\gamma_1}} \leq \frac{1}{\kappa_1 (1 - \gamma_1)} \leq \frac{1}{\kappa_2 (1 - \gamma_1)} \left( 1 - |x|^{1-\gamma_1} \right)
\]
for \( |x| < 1 \),
\[
0 \leq T(x_0) \leq \frac{1}{\kappa_1 (1 - \gamma_1)} - \frac{1}{\kappa_2 (1 - \gamma_1)} \left( 1 - |x|^{1-\gamma_1} \right)
\]
Whenever \( \rho = 0 \), the system (5) becomes the system (4). Therefore, the convergence rate of the system is improved due to the introduction of the term \(-\rho x\) in (5).

Corollary 1: The finite-time stable system (5) has smaller upper and lower bounds of the settling time duration than those of system (4).

This paper provides the GAL, raised mainly from the requisition for the convergence performance. It should be noted that the mentioned-above GAL is not new, but a well-known technique for fastening upon the rate of convergence of terminal sliding-mode control of dynamic systems, guidance of missiles, etc. However, to the best of our knowledge, previously none of the studies have yet been performed to evaluate the convergence performance of the protocol designs for consensus of multiagent systems, for which the GAL is adopted.

IV. FINITE-TIME SCALED CONSENSUS

On the basis of the GAL, the protocol designs for finite-time scaled consensus and the performance analysis of the multiagent systems undertaken are carried out in this section.

A. Scaled consensus on undirected graphs

Let us denote by \( L_A = [l_{ij}] \in \mathbb{R}^{N \times N} \) the Laplacian of the Gal of \( G(A) \), which is defined as
\[
l_{ij} = \begin{cases} 
\sum_{k=1}^{N} a_{ik}, & i = j \\
-a_{ij}, & i \neq j
\end{cases}
\]
and denote the eigenvalues of \( L_A \) by \( \lambda_1, \ldots, \lambda_N \), satisfying that \( \lambda_1 \leq \cdots \leq \lambda_N \). It always has a zero eigenvalue, i.e., \( \lambda_1 = 0 \), corresponding to the aligned state \( x_N = [1, 1, \ldots, 1]^T \). In addition, as \( G(A) \) is connected, \( \lambda_2 > 0 \) [4], [5]. Hence, for a connected graph, \( L_A \) is positive semi-definite, i.e., all nonzero eigenvalue of \( L_A \) is positive. The Laplacian potential is expressed by
\[
x^T L_A x = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (x_j - x_i)^2
\]
According to the algebraic connectivity,
\[
x^T L_A x \geq \lambda_2 (L_A) x^T x
\]
if \( 1^T x = \sum_{i=1}^{N} x_i = 0 \). Furthermore, for the connected graph \( G(A) \), a slight modification to Lemma 4 in [40] can be given as:
\[
x^T L_A x \geq \lambda_2 (L_A) x^T x
\]
for any \( x \in \mathbb{R}^N \). For our analysis purpose, let us denote \( \chi_g = [g_1, \ldots, g_N]^T \). The following result is in turn established.

Lemma 5: For the connected graph \( G(A) \),
\[
\chi_g^T L_A^2 \chi_g \geq \lambda_2 (L_A) \chi_g^T L_A \chi_g
\]
Proof. Since the undirected \( G(A) \) is connected, \( L_A \) is positive semi-definite and symmetric. There exists a unique positive semi-definite matrix \( M \) such that \( L_A = MM^T = M^2 \) [45]. We note that \( M = M^T \) and \( L_A^2 = M^2 \). Then we obtain \( 1^T x \leq M x = 0 \), which implies \( 1^T x \chi_g = 0 \). It follows from (13) that
\[
\chi_g^T L_A^2 \chi_g = \chi_g^T M^2 \chi_g = (M \chi_g)^T L_A M \chi_g \geq \lambda_2 (L_A) (M \chi_g)^T (M \chi_g) = \lambda_2 (L_A) \chi_g^T L_A \chi_g
\]
Hence, inequality (15) holds.

We shall present a continuous protocol to solve the problem of the scaled consensus for the multi-agent system (11), which interaction topology is modeled by an undirected graph. The proposed control protocol for agent \( i \) is,
\[
u_i = \frac{1}{\kappa_1} \left( \sum_{j=1}^{N} a_{ij} (g_j (x_j, t) - g_i (x_i, t)) \right)^{\gamma_1} + \kappa_2 \left( \sum_{j=1}^{N} a_{ij} (g_j (x_j, t) - g_i (x_i, t)) \right)^{\gamma_2} + \rho \left( \sum_{j=1}^{N} a_{ij} (g_j (x_j, t) - g_i (x_i, t)) - \frac{\partial g_i}{\partial t} \right)
\]
where \( \rho, \kappa_1, \kappa_2 > 0 \), \( \gamma_1 = q/p \), \( \gamma_2 = m/n \), \( m \), \( n \), \( p \), and \( q \) are odd numbers, satisfying that \( m > n \) and \( p > q \), \( \frac{\partial g_i}{\partial x_i} \) and \( \frac{\partial g_i}{\partial t} \) represent the partial derivatives of \( g_i(x_i, t) \) with respect to \( x_i \) and \( t \), respectively, and \( V = \frac{1}{2} \chi^T g L A \chi \) is the Lyapunov function candidate we choose.

**Theorem 1:** Consider the multiagent system (1) with a connected communication topology. Then the protocol (16) achieves the finite/fixed-time consensus, for all initial states.

**Proof.** By the definition of \( L_A \), let us choose

\[
V = \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(g_j(x_j, t) - g_i(x_i, t))^2
\]

Calculating the derivative of \( V \) with respect to time and applying the protocol (16) give rise to

\[
\dot{V} = \sum_{i=1}^{N} \left( \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial t} + \frac{\partial V}{\partial g_i} \frac{\partial g_i}{\partial t} \right)
\]

Due to that \( \frac{q+p}{2p} \in (0, 1) \), \( \frac{m+n}{2m} \in (1, \infty) \),

\[
\dot{V} \leq - \kappa_1 \left( \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(g_j(x_j, t) - g_i(x_i, t)) \right) + \frac{q+p}{2p} - \kappa_2 \left( \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(g_j(x_j, t) - g_i(x_i, t)) \right) \]

To proceed, the following relationship is needed:

\[
\sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij}(g_j(x_j, t) - g_i(x_i, t)) \right)^2 = (L A \chi_0)^T (-L A \chi_0) = \chi_0^T L_A^2 \chi
\]

Substituting (18) into (17), we obtain

\[
\dot{V} \leq - \rho \chi_0^T L_A^2 \chi_0 - \kappa_1 \left( \chi_0^T L_A^2 \chi_0 \right) \frac{q+p}{2p} - \kappa_2 \left( \chi_0^T L_A^2 \chi_0 \right) \frac{m+n}{2m}
\]

It follows by Lemma 3 that

\[
\dot{V} \leq -2\rho \lambda_2(L_A)V - \kappa_1 \left( 2\lambda_2(L_A)V \right) \frac{q+p}{2p} - \kappa_2 \left( 2\lambda_2(L_A)V \right) \frac{m+n}{2m}
\]

Defining \( \Lambda = \sqrt{V} \) leads to, as \( V \neq 0 \),

\[
\dot{\Lambda} = \frac{1}{2V} \dot{V} \leq -2\rho \lambda_2(L_A)V - \kappa_1 \left( 2\lambda_2(L_A)V \right) \frac{q+p}{2p} - \kappa_2 \left( 2\lambda_2(L_A)V \right) \frac{m+n}{2m}
\]

Defining \( \rho' = \rho \lambda_2(L_A), \kappa_1' = \kappa_1 \frac{q+p}{2p}, \lambda_2(L_A) = \frac{2\rho' \lambda_2(L_A)}{q+p}, \) and \( \kappa_2' = \kappa_2 \frac{m+n}{2m} \), we have

\[
\dot{\Lambda} \leq -\rho' \Lambda - \kappa_1' \Lambda^\frac{q+p}{2p} - \kappa_2' \Lambda^\frac{m+n}{2m}
\]
According to (19), and by invoking Lemma 1 the conclusion follows. ■

When setting that \( \rho = 0 \), (16) reduces to the double-power protocol. Namely,

\[
\dot{u}_i = \frac{1}{\partial g_i} \left( \kappa_1 \left( \sum_{j=1}^{N} a_{ij}(g_j(x_j, t)) - g_i(x_i, t) \right) \right) + \kappa_2 \left( \sum_{j=1}^{N} a_{ij}(g_j(x_j, t)) - g_i(x_i, t) \right) - \frac{\partial g_i}{\partial t}
\]

where \( \kappa_1, \kappa_2 > 0 \), \( \gamma_1 = q/p, \gamma_2 = m/n, m, n, p, \) and \( q \) are odd numbers, satisfying that \( m > n, p > q \), and \( V(= \frac{1}{T} \chi L \lambda g) \).

With the two-term protocol, the scaled consensus result can be presented in the following theorem.

**Theorem 2**: System (1) with the connected communication topology achieves the finite/fixed-time consensus for all initial states, under the double-power protocol (20).

**Proof.** It follows when the protocol (20) is applied that

\[
\dot{V} = -\kappa_1 \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij}(g_j(x_j, t)) - g_i(x_i, t) \right)^{\frac{-p}{2p}} - \kappa_2 \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij}(g_j(x_j, t)) - g_i(x_i, t) \right)^{\frac{-n}{2n}}
\]

Noting that (15) holds, we obtain

\[
\dot{V} \leq -\kappa_1 \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_{ij}(g(x_j, t)) - g(x_i, t) \right)^{2} \right)^{\frac{p}{2p}} - \kappa_2 N^{\frac{m-n}{2n}} \left( \chi^T L \chi \right)^{\frac{p}{2p}}
\]

due to that \( \frac{q+p}{2p} \in (0, 1), \frac{m+n}{2n} > 1 \). By Lemma 3 it follows that

\[
\dot{V} \leq -\kappa_1 \left( 2\lambda_2(L_A)V \right)^{\frac{q+p}{2p}} - \kappa_2 N^{\frac{m-n}{2n}} \left( 2\lambda_2(L_A)V \right)^{\frac{m+n}{2n}}
\]

Defining \( \Lambda = \sqrt{V} \) leads to, as \( V \neq 0 \),

\[
\dot{\Lambda} = \frac{1}{2\sqrt{V}} \dot{V} \leq -\kappa_1 \left( 2\lambda_2(L_A)V \right)^{\frac{q+p}{2p}} \frac{1}{2\sqrt{V}} - \kappa_2 N^{\frac{m-n}{2n}} \left( 2\lambda_2(L_A)V \right)^{\frac{m+n}{2n}} \frac{1}{2\sqrt{V}}
\]

Let us define \( \kappa_1' = \kappa_1 2^\frac{2}{m-n} \lambda_2(L_A) \frac{q+p}{2p} \) and \( \kappa_2' = \kappa_2 2^\frac{m+n}{2n} N^\frac{-n-m}{2n} \lambda_2(L_A) \frac{m+n}{2n} \). Then

\[
\dot{\Lambda} \leq -\kappa_1' \Lambda^\frac{\hat{p}}{2} - \kappa_2' \Lambda^\frac{\hat{m}}{2} \quad (21)
\]

Then the conclusion follows from (21), by invoking Lemma 4. ■

The derivations presented in the proofs for Theorems 1-2 ensure that the positive definite function \( \Lambda \) satisfy (3), with the appropriate selection of protocol parameters. In turn, it is seen that Lemma 1 plays an important role in finalizing the analysis and facilitating the proof. Consequently, the settling time finite-duration convergence can be determined, through the chosen protocol parameters, given in (19) in Theorem 1 and (21) in Theorem 2.

**B. Scaled consensus on directed graphs**

Now we address the problem of the nonlinearly-scaled consensus for multi-agent systems on directed graphs. To this end, we introduce the definitions for indegree and outdegree. The indegree and outdegree of node \( i \) of graph \( G \) are defined as: \( d_{in}(i) = \sum_{j=1}^{N} a_{ij} \), and \( d_{out}(i) = \sum_{j=1}^{N} a_{ji} \). Node \( i \) in graph \( G \) is said to be balanced, if \( d_{in}(i) = d_{out}(i) \). Obviously, the indegree of each node in the undirected graph is equal to its outdegree. Hence, every undirected graph is balanced. However, for a directed graph, because the edges between the nodes are directed, \( (i,j) \in E \) does not induce \( (j,i) \in E \), and the adjacency matrix \( A \) is not necessarily symmetrical. The directed graph is balanced, only if the degree of each node is equal to its outdegree.

In addition, the detail balance property is helpful. If a weighted directed graph \( G(A) \) satisfies the detail-balanced condition in weights, there are some real numbers \( p_i > 0, i = \{1, 2, \ldots, N\} \), such that \( p_i a_{ij} = p_j a_{ji} \) for \( \forall i, j \in \{1, 2, \ldots, N\} \). Here, \( p_1, p_2, \ldots, p_N \) are the detailed balance parameters associated to \( G(A) \). The detail balance parameter \( p_1, p_2, \ldots, p_N \) is a positive integer (1 is the only common divisor), and the positive vector \( p = \{p_1, p_2, \ldots, p_N\} \) is not unique. Particularly, if \( p_1 = p_2 = \cdots = p_N \), the calculation can be simplified by letting \( p_i = 1, i = 1, 2, \ldots, N \). Let us denote by \( G(A) = \{V, E, A\} \) the mirror of \( G(A) \), with the same node set as \( G(A) \), and \( \tilde{A} \) the graph Laplacian for \( G(\tilde{A}) \) with adjacency matrix \( \tilde{A} \). The graph Laplacian is defined as \( \tilde{L} = [\tilde{L}_{ij}] \in \mathbb{R}^{N \times N} \), whose elements are given by

\[
\tilde{L}_{ij} = \begin{cases} \sum_{k=1}^{n} p_i p_{ai_k}, & i = j \\ -p_i a_{ij}, & i \neq j \end{cases}
\]

(22)

Note that the adjacency matrix \( \tilde{A} = [\tilde{a}_{ij}] \) is symmetric, and its elements \( \tilde{a}_{ij} = \tilde{a}_{ji} = p_i \tilde{a}_{ij} > 0 \). Since \( G(A) \) is assumed to satisfy the detail-balanced condition in weights, \( L_{\tilde{A}} \) is positive semi-definite and symmetric, and \( 0 = \lambda_1(\tilde{A}) \leq \cdots \leq \lambda_N(\tilde{A}) \), and \( x^T \tilde{L} x \geq \lambda_2(L_{\tilde{A}}) x^T x \) when \( 1^T x = 0, \forall x \in \mathbb{R}^n \). Therefore, we obtain similar property to that described in Lemma 3, i.e., \( \chi^T L_{\tilde{A}} \chi \geq \lambda_2(L_{\tilde{A}}) x^T \chi \).

The theoretical result of Theorem 1 can be directly extended by the discussion on multi-agent systems on directed graphs. Consider system (1) with connected communication topology.
of which the weighted directed graph $G(A)$ is strongly connected and detail-balanced, and the graph Laplacian is defined by (22). The designed control protocol (16) is applicable, as $a_{ij}$ is replaced by $\hat{a}_{ij}$. Then with the revised control protocol for agent $i$, the nonlinearly-scaled finite/fixed-time consensus can be realized.

As for the result of Theorem 2, we modify the double-power protocol (20), by replacing $a_{ij}$ with $\hat{a}_{ij}$. The convergence result can be established with the similar lines to those for Theorem 2. As such, the double-power protocol can be applied to solve the problem of the scaled consensus for the multi-agent system undertaken, which interaction topology is modeled by a detail-balanced directed graph.

V. Discussions

The obtained results of multi-agent consensus on undirected/directed graphs, addressing the scaling issue, are mainly due to the convergence properties of the proposed GAL, presented in Section III. Consequently, the duration of the settling time function of the multi-agent system undertaken, determined by the designed protocol parameters and the network structure, is independent of the initial condition. The convergence rate is improved by our approach, in comparison with the conventional finite-time system approach that one usually adopted. In the published literature, there are many protocol designs, in the context of fixed-time consensus. The performance improvement of such designs can be made by directly applying the GAL-based approach.

Conventionally, one does not use scaling, i.e., $s_i(t) = 1$ and $g_i(x_i) = x_i$. However, we have to use it, when we face the problem of difference scales of agents, e.g., huge difference between the agents’ position and velocity in space and on ground. Researches on simple but useful situations were found in [37], where the scaling functions were taken as $g_i(x_i, t) = x_i/s_i$, and $s_i \in \{1, 2, ..., n\}$. However, a troublesome situation may occur due to the agents’ dynamic structure changes. In order to verify the time-varying and nonlinear scaling being possible, we adopt nonlinear scales, where $g_i(x_i, t)$ is usually nonlinear function of $x_i$ and $t$. The separated scaling functions, $s_i(t)g_i(x_i)$, indicate a direct extension to the existing ones and would be a useful alternative.

One interesting issue is the cooperative and antagonistic interactions in multi-agent systems [40]. It is shown that the state of all agents can be agreed in the case of the same modulus but different symbols. We need to take into account and show that the proposed protocol (16) can be modified to achieve the cooperative and antagonistic behavior in finite duration, which is given as

$$u_i = \frac{1}{\rho \frac{\partial g_i}{\partial x_i}} \left( \rho \sum_{j=1}^{N} a_{ij}(g_j(x_j, t) - \text{sign}(a_{ij})g_i(x_i, t)) + \kappa_1 \left( \sum_{j=1}^{N} a_{ij}(g_j(x_j, t) - \text{sign}(a_{ij})g_i(x_i, t)) \right)^{\gamma_1} \right)$$

Difference exists between the steady situation of the scaled consensus and the behavior of implicit systems and/or algebraic loops. The former is adjustable, through changing the scale, (namely, by designing the $g_j$), while the latter cannot be changed. Many of the published schemes by no means adjust the steady-state of the agents. Our scaled design provides one way for it.

The asymmetry of the Laplacian matrix of the directed graph makes it difficult to choose a suitable Lyapunov function. The detailed balance condition, given in [12] and further specified in [17], [22], paves the way to solve the difficulty. Such a matrix $L_\lambda$ is in fact equivalent to a symmetric Laplace matrix formed in the case of the undirected communication graph.

With strongly connected and detail-balanced topology, the proposed protocols (17), (22) solve the finite time average consensus. By adopting the similar technique, in this paper we deal with the scaled consensus problem. More related treatments, to introduce a Lyapunov function suitable for the analysis on general digraphs, can be found, for instance, [13], [14], which deserve further study for development and application of new scaled consensus techniques.

VI. Numerical Simulation

Two numerical examples are provided in this section to illustrate the effectiveness of the proposed consensus protocols, with different scale settings, for which the finite/fixed-time convergence performance is characterized.

Example 1: Consider a group of six agents whose dynamical behavior is described by (1), and the interaction topology is represented by the undirected graph, shown in Fig. 1 with the following adjacency matrix

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 
\end{bmatrix}$$
The initial states of agents are set as $x_1(0) = -18, x_2(0) = -8, x_3(0) = -5, x_4(0) = 5, x_5(0) = 8,$ and $x_6(0) = 18$. For the numerical simulation, a separate scaling manner of six agents is taken into account, with the following two settings, respectively.

Scale setting C1:

- $g_1(x_1, t) = (0.5\sin(2\pi t) + 1) (5\sin(0.2x_1) + 2x_1)$,
- $g_2(x_2, t) = (-0.5\sin(2\pi t) - 1) (2\sin(0.5x_2) + 2x_2)$,
- $g_3(x_3, t) = (-0.5\sin(2\pi t) - 1) (\sin(x_3) + 2x_3)$,
- $g_4(x_4, t) = (-0.5\sin(2\pi t) - 1) (5\cos(0.2x_4) - 2x_4)$,
- $g_5(x_5, t) = (0.5\sin(2\pi t) + 1) (2\cos(0.5x_5) - 2x_5)$,
- $g_6(x_6, t) = (0.5\sin(2\pi t) + 1) (\cos(x_6) - 2x_6)$.

Note that the given scales in C1 and C2 satisfy $\frac{dg_i(x_i)}{dx_i} \neq 0$.

To illustrate the result of Theorem 1, we apply the protocol (16), with the chosen controller parameters: $\rho = 2, \kappa_1 = 1, \kappa_2 = 1, \gamma_1 = \frac{1}{3}, \gamma_2 = \frac{2}{3}$. According to the definition of $L_A$, the algebraic connectivity of $G(A)$ can be calculated as $\lambda_2(L_A) = 1$. Under the scale setting C1, the numerical results are shown in Figs. 24. The resultant states of each multi-agent are shown in Fig. 2 and Fig. 3 shows the functions $g_i(x_i, t)$, the scaled consensus results by the protocol undertaken. It is seen from Fig. 2 that the states achieve consensus, according to the different scales, which show the the scales’ impact on the consensus results. In Fig. 3, we confirm that the protocol design is efficient for multiagent systems subject to both time-varying and nonlinear scales. By Lemma 1, the lower and upper bounds of the settling time can be calculated as $T_0 = 0.8$, $T_1 = 1.96$, and from Fig. 3, the finite/fixed-time control objective is accomplished. It exhibits that the settling time is actually smaller than the upper bound, and larger than the lower bound of the theoretical estimation. This is due to the given initial states, verifying that the actually settling time heavily depends on the the initial states.

For comparison, the simulation is carried out by applying the protocol (20), and the consensus result is shown in Fig. 5. By Lemma 2, the lower and upper bounds of the settling time are calculated as $T_2 = 1.35$, $T_3 = 4.05$, which gives the reason for the slower convergence rate than that by (16), as shown in Fig. 5.

The simulation results under the scale setting C2 are shown in Figs. 4 and 5. The initial condition of the multiagent undertaken is the same to the above case. The resultant states of each multi-agent are shown in Fig. 4 and the scaled consensus results by the protocol (16), $g_i(x_i, t)$, are given in Fig. 5. It is observed that the designed consensus protocol achieves the scaled consensus with finite/fixed-time convergence, leads to the faster convergence rate than that by applying the protocol (20).

Example 2: Consider a six-agent system with the interaction topology modeled as a weighted directed graph, which adjacency matrix is as follows:

$$A = \begin{bmatrix}
0 & 0.2 & 0 & 0 & 0 & 0 \\
0.4 & 0 & 0.2 & 0 & 0 & 0 \\
1 & 0.5 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0.8 & 0.4 \\
0 & 0 & 0.4 & 0 & 0.4 & 0 \\
0 & 0 & 0.4 & 0.8 & 0 & 0
\end{bmatrix}$$

which is both strongly connected and detail-balanced, as
and (b) by (20) states shown in Fig. 6. By choosing Fig. 6. Communication topology (Example 2)

\[ g(x, t) = x + \frac{1}{1 + 0.1x^2_i}, \quad i = 1, 2, 3, 4, 5, 6 \]

Note that \( \frac{dg_i(x_i)}{dx_i} = 1 + \frac{1-0.1x^2_i}{1+0.1x^2_i} > 0 \).

Under scale settings C3 and C4, we apply the control protocols (16) and (20), respectively. The obtained numerical results are shown in Figs. 7 and Figs. 9 and Figs. 10 respectively. Fig. 7 and Fig. 9 show the resultant state of each agent, which achieve consensus on different scales. In Fig. 7, the converged states are time-varying, mainly due to the time-varying scales, where two groups of states can be observed. One is composed of the agents 1, 5 and 6, the other includes agents 2, 3 and 4. Two groups of states are opposite in sign. In Fig. 9, the converged states keep constant, also because of the time-variant scales, where the states of agents 1 and 3 converge to the same value, the states of agents 4 and 6 converge to the other value, and the converged states of agents 2 and 5 are opposite in sign. It is seen from Figs. 8 and 10 that the scaled consensus described by (20) is realized. Using Lemma 1, the lower and upper bounds of the settling time can be estimated as \( T_0 = 0.87, T_1 = 2.09 \), and Using Lemma 2, \( T_2 = 1.43, T_3 = 4.32 \).

VII. CONCLUSION

For the purpose of consensus of multiagent systems, in this paper, stability results of certain class of nonlinear systems have been presented, which admit attractors with finite/fixed-time convergence. A framework of finite/fixed-time consensus has been provided, according to the finite/fixed-time stability results, and distributed protocols are proposed, which realize the scaled consensus, with time-varying and nonlinear scales. The theoretical analyses of finite/fixed-time convergence have been given for systems with undirected and directed graphs, respectively. The estimates for the upper bounds on the settling time functions are provided, where the given estimates are initial state dependent, but the durations are finite, without regarding the values that the initial states take. It has been shown that the finite/fixed-time scaled consensus for the multiagent system undertaken can be achieved, despite the adopted time-varying and nonlinear scales. The simulation results, showing

shown in Fig. 6. By choosing \( p = [10, 5, 2, 2, 4, 2]^T \), \( p_i a_{ij} = p_j a_{ji} \) for \( \forall i, j \in \{1, 2, 3, \cdots, N\} \). We apply the control protocol (16), with \( a_{ij} \) being replaced by \( \tilde{a}_{ij} \), and the parameter settings: \( \rho = 2, \kappa_1 = 1, \kappa_2 = 1, \gamma_1 = \frac{1}{3}, \gamma_2 = \frac{2}{3} \). The algebraic connectivity of \( G(A) \) can be calculated as \( \lambda_2(L_A) = 0.9383 \). In the simulation, the initial condition are set as \( x_0 = [-12, -5, -3, 12, 5, 3] \).

For scaling the six agents, the following two sets of time-invariant and time-varying scales are taken into account, respectively.

Scale setting C3:

\[ s_1(t) = 0.5\sin(2\pi t) + 1, \quad s_2(t) = -0.5\sin(2\pi t) - 1 \]

\[ s_3(t) = -0.5\sin(2\pi t) + 1, \quad s_4(t) = 0.5\sin(2\pi t) + 1 \]

\[ g_i(x_i) = x_i + \frac{1}{1 + 0.1x^2_i}, \quad i = 1, 2, 3, 4, 5, 6 \]

Scale setting C4:

\[ s_1(t) = 1, \quad s_2(t) = 5, \quad s_3(t) = 1 \]

\[ s_4(t) = -1, \quad s_5(t) = -5, \quad s_6(t) = -1 \]

\[ g_i(x_i) = x_i + \frac{1}{1 + 0.1x^2_i}, \quad i = 1, 2, 3, 4, 5, 6 \]
the desired convergence performance, have been presented to verify effectiveness of the proposed protocols.

REFERENCES

[1] J. R. Lawton, R. W. Beard, and B. Young, “A decentralized approach to formation maneuvers”, *IEEE Transactions on Robotics and Automation*, vol. 19, no. 6, pp. 933-941, 2003.
[2] A. Jadbabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules”, *IEEE Transactions on Automation Control*, vol. 49, no. 9, pp. 988-1001, 2003.
[3] J. A. Fax and R. M. Murray, “Information flow and cooperative control of vehicle formations”, *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 115-120, 2004.
[4] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays”, *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520-1533, 2004.
[5] W. Ren and R. W. Beard, “Consensus seeking in multiagent systems under dynamically changing interaction topologies”, *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655-661, 2005.
[6] J. Cortes, S. Martinez, and F. Bullo, “Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions”, *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1289-1298, 2006.
[7] D. V. Dimarogonas and K. J. Kyriakopoulos, “On the rendezvous problem for multiple nonholonomic agents”, *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 916-922, 2007.
[8] G. Lafferriere, J. S. Caughman, J. J. P. Veerman, and A. Williams, “Decentralized control of vehicle formations”, *Syst and Control Letters*, vol. 54, no. 9, pp. 899-910, 2005.
[9] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, “Flocking in Fixed and Switching Networks”, *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 863-868, 2007.
[10] T. Vicsek, A. Czirók, and E. B. Jacob, “Novel type of phase transition in a system of self-driven particles”, *Physical Review Letters*, vol. 75, no. 6, pp. 1226-1229, 1995.
[11] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems”, *Proceedings of the IEEE*, vol. 95, no. 1, pp. 212-233, 2007.

[12] B. Schurmann, “Stability and adaptation in artificial neural systems”, *Physical Review A*, vol. 40, no. 5, pp. 2681-2688, 1989.

[13] H. Zhang, F. L. Lewis, and Z. Qu, “Lyapunov, adaptive, and optimal design techniques for cooperative systems on directed communication graphs”, *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3026-3041, 2012.

[14] Z. Li, G. Wen, and Z. Duan, “Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs”, *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 1152-1157, 2015.

[15] J. Cortes, “Finite-time convergent gradient flows with applications to network consensus”, *Automatica*, vol. 42, no. 11, pp. 1993-2000, 2006.

[16] G. Hui, W. M. Haddad, and S. P. Bhat, “Finite-time semistability and consensus for nonlinear dynamical networks”, *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1887-1900, 2008.

[17] L. Wang and F. Xiao, “Finite-time consensus problems for networks of dynamic agents”, *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 950-955, 2010.

[18] F. Xiao, L. Wang, J. Chen, and Y. Gao, “Finite-time formation control for multi-agent systems”, *Automatica*, vol. 45, no. 11, pp. 2605-2611, 2009.

[19] X. Wang, S. Li, P. Shi, “Distributed finite-time containment control for double-integrator multiagent systems”, *IEEE Transactions on Cybernetics*, vol. 44, no. 9, pp. 1518-1528, 2014.

[20] C. Y. Li and Z. H. Qu, “Distributed finite-time consensus of nonlinear systems under switching topologies”, *Automatica*, vol. 50, no. 6, pp. 1626-1631, 2014.

[21] X. Su, Y. Wang, X. Yu, et al., “Finite-time control for robust tracking consensus in MASs with an uncertain leader”, *IEEE Transactions on Cybernetics*, vol. 47, no. 5, pp. 1210-1223, 2017.

[22] X. Liu, J. Lam, W. Yu, et al., “Finite-time consensus of multiagent systems with a switching protocol”, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 4, pp. 853-862, 2016.

[23] A. Polyakov, Nonlinear feedback design for fixed-time stabilization of linear control systems, *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106-2110, 2012.

[24] S. Parsegov, A. Polyakov, and P. Shcherbakov, “Fixed-time consensus algorithm for multi-agent systems with integrator dynamics”, *IFAC Proceedings Volumes*, vol. 46, no. 27, pp. 110-115, 2013.

[25] Z. Zuo and L. Tie, “A new class of finite-time nonlinear consensus protocols for multi-agent systems”, *International Journal of Control*, vol. 87, no. 2, pp. 363-370, 2014.

[26] W. Lu, X. Liu, and T. Chen, “A note on finite-time and fixed-time stability”, *Neural Networks*, vol. 81, pp. 11-15, 2016.

[27] J. Fu and J. Wang, “Fixed-time coordinated tracking for second-order multi-agent systems with bounded input uncertainties”, *Systems. Control Letters*, vol. 93, pp. 1-12, 2016.

[28] J. Ni, L. Ling, C. Liu, et al., “Fixed-time leader-following consensus for second-order multiagent systems with input delay”, *IEEE Transactions on Industrial Electronics*, vol. 64, no. 11, pp. 8635-8646, 2017.

[29] Z. Zuo, B. Tian, M. Defoort, et al., “Fixed-time consensus tracking for multiagent systems with high-order integrator dynamics”, *IEEE Transactions on Automatic Control*, vol. 63, No. 2, pp. 563-570, 2018.

[30] X. Liu, T. Chen, “finite-Time and Fixed-Time Cluster Synchronization With or Without Pinning Control”, *IEEE Transactions on Cybernetics*, vol. 48, no. 11, pp. 240-252, 2018.

[31] B. Tian, H. Lu, Z. Zuo, et al., “Fixed-time leader-follower output feedback consensus for second-order multiagent systems”, *IEEE Transactions on Cybernetics*, vol. 49, no. 4, pp. 1545-1550, 2019.

[32] H. Hong, W. Yu, G. Wen, et al., “Distributed robust fixed-time consensus for nonlinear and disturbed multiagent systems”, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 7, pp. 1464-1473, 2017.

[33] X. Liu, D. W. C. Ho, Q. Song, et al., “Finite/Fixed-Time pinning synchronization of complex networks with stochastic disturbances”, *IEEE Transactions on Cybernetics*, vol. 49, no. 6, pp. 2198-2403, 2019.

[34] X. Wei, W. Yu, H. Wang, et al., “An observer based fixed-time consensus control for second-order multi-agent systems with disturbances”, *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 66, No. 2, pp. 247-251, 2019.

[35] M. Sun, “Two-phase attractors for finite-duration consensus of multiagent systems”, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2017, doi:10.1109/TSMC.2017.2785314

[36] S. Roy, “Scaled consensus “, *Automatica*, vol. 51, pp. 259-262, 2015.

[37] D. Meng and Y. Jia, “Robust consensus algorithms for multiscale coordination control of multivehicle systems with disturbances”, *IEEE Transactions on Industrial Electronics*, vol. 63, no. 2, pp. 1107-1119, 2016.

[38] F. Jiang and L. Wang, “Finite-time weighted average consensus with respect to a monotonic function and its application”, *Systems & Control Letters*, vol. 60, pp. 718-726, 2011.

[39] Z. Altafini, “Consensus problems on networks with antagonistic interactions”, *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935-946, 2013.

[40] D. Meng, Y. Jia and J. Du, “Finite-time consensus for multiagent systems with cooperative and antagonistic interactions”, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 4, pp. 762-770, 2017.

[41] J. Yu, Y. Shi, “Scaled group consensus in multiagent systems with first/second-order continuous dynamics”, *IEEE Transactions on Cybernetics*, vol. 48, no. 8, pp. 2259-2271, 2018.

[42] J. Cortes, “Distributed algorithms for reaching consensus on general functions”, *Automatica*, vol. 44, pp. 726-737, 2008.

[43] X. Dong and G. Hu, “Time-varying formation control for general linear multi-agent systems with switching directed topologies”, *Automatica*, vol. 73, pp. 47-55, 2016.

[44] W. M. Haddad and V. Chellabonia, “Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach”, *New Jersey: Princeton University Press*, 2008.

[45] R. A. Horn and C. R. Johnson, “Matrix Analysis”, *Cambridge, U. K.: Cambridge University Press*, 1985.