ADDITIONAL SYMMETRIES AND STRING EQUATION OF THE CKP HIERARCHY

JINGSONG HE†‡, KELEI TIAN†, ANGELA FOERSTER‡ AND WEN-XIU MA§

† Department of Mathematics, USTC, Hefei, 230026 Anhui, P. R. China
‡ Instituto de Física da UFRGS, Av. Bento Gonçalves 9500, Porto Alegre, RS - Brazil
§ Department of Mathematics, University of South Florida Tampa, FL 33620-5700, USA

ABSTRACT. Based on the Orlov and Shulman’s M operator, the additional symmetries and the string equation of the CKP hierarchy are established, and then the higher order constraints on \( L^n \) are obtained. In addition, the generating function and some properties are also given. In particular, the additional symmetry flows form a new infinite dimensional algebra \( W_{1+∞}^C \), which is a subalgebra of \( W_{1+∞} \).

Keywords: CKP hierarchy, additional symmetries, string equation
Mathematics Subject Classification(2000): 17B80, 37K05, 37K10
PACS(2003): 02.30.Ik

1. Introduction

Since its introduction in 1980, the Kadomtsev-Petviashvili(KP) hierarchy [1, 2] is one of the most important research topics in the area of classical integrable systems. In particular, the study of its symmetries plays a central role in the development of this theory. In this context, additional symmetries, which correspond to a special kind of symmetries depending explicitly on the independent variables \( t_n \) of the KP hierarchy, have been analyzed using two different approaches. In the first one, the explicit form of the additional symmetry flows (action on the wave function, or equivalently on the Lax operator \( L \)) of the KP hierarchy was given by Orlov and Shulman(OS) [6] through a novel operator \( M \), which can be used to form a centerless \( W_{1+∞} \) algebra. Actually, these results go back to some previous works on the \( t \) (time variable) and \( x \) (space variable) dependent symmetries of the KP equation, which forms an infinite dimensional Lie algebra, founded by Oevel and Fuchssteiner [3, 4], and Chen, Lee and Lin [5]. In the second approach, there exists the Sato Bäcklund symmetry defined by the vertex operator \( X(λ, µ) \) acting on the \( τ \) function of the KP hierarchy, with \( X(λ, µ) \) serving as a generating function of the \( W_{1+∞} \) algebra [1]. It is quite surprising that no direct connection was realized between these two types of symmetries for a long period of time. But, in fact, the \( W_{1+∞} \)-algebra of additional symmetry flow defined by OS can be lifted to its central extension by acting on the \( τ \) function of the KP hierarchy [7–9]. In this process, the Adler-Shiota-van Moerbeke(ASvM) formula plays a crucial role. Almost at the same time, Dickey presented a very elegant and compact proof of the ASvM formula [10]. He also found the action of the additional symmetries on the Grassmannian and gave a straightforward derivation of the action of the additional symmetries on the \( τ \)-functions [11].

It is well known that there are two kinds of sub-hierarchies of KP, a BKP hierarchy [1] and a CKP hierarchy [12]. For the BKP hierarchy, its Virasoro constraints and the ASvM formula have been constructed by Johan van de Leur [13,14] using an algebraic formalism. Very recently,
an alternative proof of the ASvM formula of the BKP hierarchy was given by Tu [15] by means of Dickey’s method [10]. So, it would be natural to ask if some corresponding results related to the additional symmetries in the CKP hierarchy also applies. The question concerning the similarities and differences between the BKP and CKP hierarchies is also very relevant in this scenario. Obviously, in contrast to the BKP hierarchy, we can not find an ASvM formula for CKP because this hierarchy does not possess a sole \( \tau \)-function. However this fact does not destroy the existence of additional symmetries and the string equation for the CKP hierarchy, as we shall deliver. The main purpose of this article is to construct the additional symmetries and string equations for the CKP hierarchy providing an answer to all these relevant questions.

The organization of this paper is as follows. We recall some basic facts for the CKP hierarchy in section 2. The OS’s \( M \) operator, additional symmetry and string equation are discussed in sections 3, 4 and 5, respectively. Section 6 is devoted to conclusions and discussions.

2. CKP Hierarchy

Let \( L \) be the pseudo-differential operator,

\[
L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + u_3 \partial^{-3} + \cdots,
\]

and then the KP hierarchy is defined by the set of partial differential equations \( u_i \) with respect to independent variables \( t_j \)

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \cdots.
\]

Here \( B_n = (L^n)_+ = \sum_{k=0}^n a_k \partial^k \) denotes the non-negative powers of \( \partial \) in \( L^n \), \( \partial = \partial/\partial x \), \( u_i = u_i(x = t_1, t_2, t_3, \cdots) \). The other notation \( L^n = L^n - L^n_+ \) will be needed by the sequent text. \( L \) is called the Lax operator and eq.\((2.2)\) is called the Lax equation of the KP hierarchy. In order to define the CKP hierarchy, we need a formal adjoint operation * for an arbitrary pseudo-differential operator \( P = \sum_{i} \xi_i \partial^i \), \( P^* = \sum_{i} (-1)^i \partial^i \xi_i \). For example, \( \partial^* = -\partial \), \( (\partial^{-1})^* = -\partial^{-1} \), and \( (AB)^* = B^*A^* \) for two operators. The CKP hierarchy [12] is a reduction of the KP hierarchy by the constraint

\[
L^* = -L,
\]

which compresses all even flows of the KP hierarchy, i.e. the Lax equation of the CKP hierarchy has only odd flows ,

\[
\frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1}, L], \quad n = 0, 1, 2, \cdots.
\]

Thus \( u_i = u_i(t_1, t_3, t_5, \cdots) \) for the CKP hierarchy.

The Lax equation of the CKP hierarchy can be given by the consistent conditions of the following set of linear partial differential equations

\[
Lw(t, \lambda) = \lambda w(t, \lambda), \quad \frac{\partial w(t, \lambda)}{\partial t_{2n+1}} = B_{2n+1}w(t, \lambda), \quad t = (t_1, t_3, t_5, \cdots).
\]

Here \( w(t, \lambda) \) is identified as a wave function. Let \( \phi \) be the wave operator(or Sato operator) of the CKP hierarchy \( \phi = 1 + \sum_{i=1}^\infty w_i \partial^{-i} \), then the Lax operator and the wave function admit the following representation

\[
L = \phi \partial \phi^{-1}, \quad w(t, \lambda) = \phi(t)e^{\xi(t, \lambda)} = \hat{w}e^{\xi(t, \lambda)},
\]
in which \(\xi(t, \lambda) = \lambda t_1 + \lambda^2 t_3 + \cdots + \lambda^{2n+1} t_{2n+1} + \cdots\), \(\hat{w} = 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \frac{w_3}{\lambda^3} + \cdots\). It is easy to show that the Lax equation is equivalent to Sato equation

\[
\frac{\partial \phi}{\partial t_{2n+1}} = -L_{2n+1} \phi, \tag{2.7}
\]

and the constraint on \(L\) in eq.(2.3) is transformed to the constraint on the wave operator

\[
\phi^* = \phi^{-1}. \tag{2.8}
\]

The eq.(2.8) is a crucial condition to construct the additional symmetries of the CKP hierarchy, which will affect the action of the additional symmetry on \(\phi\). It leads to a distinct definition of the additional symmetry in comparison to the cases of the KP and BKP hierarchies.

3. Orlov-Schulman’s M operator and its adjoint

The Orlov and Shulman’s \(M\) operator [6] is also applicable to construct the additional symmetries of the CKP hierarchy when all of the even independent variables \(t_{2n}\) are frozen. Thus define

\[
M = \phi \Gamma \phi^{-1}, \quad \Gamma = \sum_{i=0}^{\infty} (2i + 1)t_{2i+1} \partial^{2i} = t_1 + 3t_3 \partial^2 + 5t_5 \partial^4 + \cdots. \tag{3.1}
\]

A direct calculation shows that the operator \(M\) satisfies

\[
[L, M] = 1, \quad \partial_{t_{2n+1}} M = [B_{2n+1}, M], \quad M w(t, z) = (\partial_z w(t, z)). \tag{3.2}
\]

Further,

\[
\frac{\partial M^m}{\partial t_{2n+1}} = [B_{2n+1}, M^m], \quad \frac{\partial M^m L^l}{\partial t_{2n+1}} = [B_{2n+1}, M^m L^l] \tag{3.3}
\]

with the help of \(\frac{\partial L^l}{\partial t_{2n+1}} = [B_{2n+1}, L^l]\). Moreover, on the space of wave functions \(w(t, z)\), \([L, M] = 1\) and \([z, \partial_z] = -1\) induce an anti-isomorphic between \((L, M)\) and \((z, \partial_z)\), and

\[
M^m L^l w(t, z) = z^l (\partial_z^m w(t, z)), \quad L^l M^m w(t, z) = \partial_z^m (z^l w(t, z)), m, l \in \mathbb{Z}, m \geq 0. \tag{3.4}
\]

On the other hand, we need some information of the adjoint wave function \(w^*(\text{which is the wave function of the adjoint system } L^*)\) and \(M^*\)(the formal adjoint of \(M\)). For the CKP hierarchy, we have

\[
w^*(t, z) = (\phi^*)^{-1} e^{-\xi(t, z)} = \phi e^{\xi(t, -z)} = w(t, -z), \tag{3.5}
\]

and

\[
M^* = (\phi \Gamma \phi^{-1})^* = (\phi^{-1})^* \Gamma^* \phi^* = M \tag{3.6}
\]

by using eq.\((2.8)\), \(\Gamma^* = \Gamma\) and \(\xi(t, -z) = -\xi(t, z)\). Furthermore, \(L^*\) and \(M^*\) satisfy

\[
[L^*, M^*] = [-L, M] = -1, \tag{3.7}
\]

and

\[
L^* w^*(t, z) = z w^*(t, z), \quad \partial_{t_{2n+1}} w^*(t, z) = -B^*_{2n+1} w^*(t, z), \quad M^* w^*(t, z) = -\partial_z w^*(t, z). \tag{3.8}
\]
4. Additional Symmetries

We are now in a position to define the additional flows, and then to prove that they are symmetries, which are called additional symmetries of the CKP hierarchy. Similar to the case of the BKP [15], we introduce additional independent variables $t_{m,l}^*$ and define the action of the additional flows on the wave operator as

$$\frac{\partial \phi}{\partial t_{m,l}^*} = -(A_{m,l})_\phi \phi,$$

(4.1)

where $A_{m,l} = A_{m,l}(L, M)$ are monomials in $L$ and $M$ and their explicit forms are undetermined.

**Proposition 4.1.** The additional flows act on $L$ and $M$ as

$$\frac{\partial L}{\partial t_{m,l}^*} = -[(A_{m,l})_\phi, L], \quad \frac{\partial M}{\partial t_{m,l}^*} = -[(A_{m,l})_\phi, M]$$

(4.2)

**Proof** By performing the derivative on $L$ (2.6) and using eq.(4.1), we get

$$(\partial_{t_{m,l}^*} L) = (\partial_{t_{m,l}^*} \phi) \frac{\partial \phi^{-1}}{\partial \phi} + \phi \frac{\partial}{\partial \phi} (A_{m,l})_\phi \phi^{-1} = -(A_{m,l})_\phi L + L(A_{m,l})_\phi = -[(A_{m,l})_\phi, L].$$

For the action on $M$ given in eq.(3.1), there exists similar derivation as ($\partial_{t_{m,l}^*} L$), i.e.

$$(\partial_{t_{m,l}^*} M) = (\partial_{t_{m,l}^*} \phi) \frac{\partial \phi^{-1}}{\partial \phi} + \phi \frac{\partial}{\partial \phi} (A_{m,l})_\phi \phi^{-1} = -(A_{m,l})_\phi M + M(A_{m,l})_\phi = -[(A_{m,l})_\phi, M].$$

Here the fact that $\Gamma$ does not depend on the additional flows variables $t_{m,l}^*$ has been used. □

**Corollary 4.1.**

$$\frac{\partial L^n}{\partial t_{m,l}^*} = -[(A_{m,l})_\phi, L^n], \quad \frac{\partial M^m}{\partial t_{m,l}^*} = -[(A_{m,l})_\phi, M^m], \quad \frac{\partial A_{n,k}}{\partial t_{m,l}^*} = -[(A_{m,l})_\phi, A_{n,k}]$$

(4.3)

$$\frac{\partial A_{n,k}}{\partial t_{2n+1}} = [B_{2n+1}, A_{n,k}]$$

(4.4)

**Proof** We present here only the proof of the first equation. The others can be proved in a similar way. The derivative of $L^n$ with respect to $t_{m,l}^*$ leads to

$$\frac{\partial L^n}{\partial t_{m,l}^*} = \frac{\partial L}{\partial t_{m,l}^*} L^{n-1} + \frac{\partial L}{\partial t_{m,l}^*} L^{n-2} + \cdots + \frac{\partial L}{\partial t_{m,l}^*} L^{n-k} + \frac{\partial L}{\partial t_{m,l}^*} L^{n-k} = \sum_{k=1}^{n} L^{k-1} \frac{\partial L}{\partial t_{m,l}^*} L^{n-k}$$

and then taking $\frac{\partial L}{\partial t_{m,l}^*} = -[(A_{m,l})_\phi, L]$ into the above formula, which is followed by

$$\frac{\partial L^n}{\partial t_{m,l}^*} = -\sum_{k=1}^{n} L^{k-1}[(A_{m,l})_\phi, L] L^{n-k} = -[(A_{m,l})_\phi, L^n].$$

□
Proposition 4.2. The additional flows \( \frac{\partial}{\partial t_{m,l}} \) commute with the CKP hierarchy flows \( \frac{\partial}{\partial t_{2n+1}} \), i.e.

\[
[\partial_{t_{m,l}}, \partial_{t_{2n+1}}] = 0
\]

Thus they are symmetries of the CKP hierarchy. Here \( \partial_{t_{m,l}} \), \( \partial_{t_{2n+1}} \).

Proof According to the definition,

\[
[\partial_{t_{m,l}}, \partial_{t_{2n+1}}] = \frac{\partial}{\partial t_{m,l}} \left( \partial_{t_{2n+1}} \phi \right) - \partial_{t_{2n+1}} \left( \partial_{t_{m,l}} \phi \right),
\]

and using the action of the additional flows and the CKP flows on \( \phi \), then

\[
[\partial_{t_{m,l}}, \partial_{t_{2n+1}}] \phi = -\partial_{t_{m,l}}(L_{2n+1} \phi) + \partial_{t_{2n+1}}((A_{m,l})_+ \phi)
\]

By the corollary 4.1 and eq. (2.7), it equals

\[
[\partial_{t_{m,l}}, \partial_{t_{2n+1}}] \phi = \left( (A_{m,l})_- L_{2n+1} \right)_- \phi + \left( (L_{2n+1}) - (A_{m,l})_+ \right) \phi
\]

In the second equality of the above derivation, \( [L_{2n+1}, (A_{m,l})_-] = [L_{2n+1}, (A_{m,l})_-] = 0 \). The last equality holds because \((P_-)_- = P_- \) for arbitrary pseudo-differential operator \( P \).

Remark: The generators of the additional symmetries \( A_{m,l} \) are also different with the counterparts of the BKP hierarchy, which has different constraints on \( \phi \) [15,16].

Proposition 4.3. For the CKP hierarchy, it is sufficient to ask

\[
A_{m,l}^* = -A_{m,l}.
\]

Thus we can let

\[
A_{m,l} = M^m L^l - (-1)^l L^l M^m
\]

Proof The action of the additional flows \( \partial_{t_{m,l}} \) on the adjoint wave operator \( \phi^* \) can be obtained by two different ways. The first is to do a formal adjoint operation on eq. (3.1),

\[
\partial_{t_{m,l}} \phi^* = -\phi^* (A_{m,l})_-.
\]

The second is to do a derivative with respect to \( t_{m,l} \) on \( \phi^* \) and use the constraint relation \( \phi^* = \phi^{-1} \),

\[
\partial_{t_{m,l}} \phi^* = \partial_{t_{m,l}} \phi^{-1} = -\phi^{-1}(\partial_{t_{m,l}} \phi) \phi^{-1} = \phi^{-1} (A_{m,l})_- = \phi^* (A_{m,l})_-.
\]

By comparing eq. (4.8) and eq. (4.9), we have

\[
(A_{m,l})_- = - (A_{m,l})_-,
\]

and thus it is sufficient to let \( A_{m,l} = -A_{m,l}^* \). Moreover, \((M^m L^l)^* = (L^l)^*(M^m)^* = (-1)^l L^l M^m\),

so \( A_{m,l} = M^m L^l - (-1)^l L^l M^m \) satisfies the requirement of eq. (4.10).
Proposition 4.4. Acting on the space of the wave operator \( \phi, \partial_{m,l}^* \) forms new centerless \( W_{1+\infty}^C \)-subalgebra of centerless \( W_{1+\infty} \).

Proof By using \( A_{n,k}^* = -A_{n,k} \), it is easy to compute

\[
([A_{m,l}, A_{n,k}])^* = A_{n,k}^* A_{m,l}^* - A_{m,l}^* A_{n,k}^* = A_{n,k} A_{m,l} - A_{m,l} A_{n,k} = -[A_{m,l}, A_{n,k}],
\]

which shows that they form a close set under the commutator operation. Therefore it can be expanded by other generator,

\[
[A_{m,l}, A_{n,k}] = \sum_{p,q} C_{nk,ml}^{pq} A_{p,q},
\]

and so

\[
[A_{m,l}, A_{n,k}] = \sum_{p,q} C_{nk,ml}^{pq} (A_{p,q})_-
\]

This fact will be used in the computation of commutator of \( \partial_{m,l}^* \). Now we start to do this. By using eq. (4.1),

\[
[\partial_{m,l}^*, \partial_{n,k}^*] \phi = \partial_{m,l}^* (\partial_{n,k}^* \phi) - \partial_{n,k}^* (\partial_{m,l}^* \phi) = -\partial_{m,l}^* ((A_{n,k})_\phi) + \partial_{n,k}^* ((A_{m,l})_\phi)
\]

\[
= -(\partial_{m,l}^* A_{n,k})_\phi - (A_{n,k})_\phi (\partial_{m,l}^* \phi) + (\partial_{n,k}^* A_{m,l})_\phi + (A_{m,l})_\phi (\partial_{n,k}^* \phi).
\]

On account of eq. (4.1) again and eq. (4.3),

\[
[\partial_{m,l}^*, \partial_{n,k}^*] \phi = \sum_{p,q} C_{nk,ml}^{pq} (A_{p,q})_\phi = \sum_{p,q} C_{nk,ml}^{pq} (A_{p,q})_\phi,
\]

which is equivalent to

\[
[\partial_{m,l}^*, \partial_{n,k}^*] = \sum_{p,q} C_{nk,ml}^{pq} \partial_{p,q}^*.
\]

To have a better understanding of the additional symmetry flows, we provide several typical examples.

Corollary 4.2. From eq (4.7) we get \( A_{m,1} = 2LM^m - m M^{m-1}, m \geq 1, m \in Z \), thus the corresponding flows on \( L \) are

\[
\frac{\partial L}{\partial t_{m,1}^*} = [-2(LM^m)_- + m(M^{m-1})_-, L].
\]

Let \( m = 1 \),

\[
\frac{\partial L}{\partial t_{1,1}^*} = [-2(LM)_-, L] = 2L + 2[(LM)_+, L]
\]

inverses

\[
\frac{\partial u_i}{\partial t_{1,1}^*} = 2\left( \frac{\partial u_i}{\partial x} + (i+1)u_i + \sum_{j=1}^{\infty} (2j+1) t_{2j+1} \frac{\partial u_i}{\partial t_{2j+1}} \right), i = 1, 2, 3, \ldots
\]
Proof Eqs. (4.14) and (4.15) are obtained by definition. For the eq. (4.16), first of all, we should find $LM$, which is expressed by

$$LM = \phi \partial \Gamma \phi^{-1} = \phi \partial x \phi^{-1} + \sum_{i=1}^{\infty} (2i + 1)t_{2i+1}\phi \partial^{2i+1}\phi^{-1}$$

(4.17)

with the help of eq. (2.6) and eq. (3.1). Furthermore, by $\partial x = x\partial + 1$ and $\partial^{-i}x = x\partial^{-i} - i\partial^{-i-1}$,

$$\phi \partial x \phi^{-1} = (x\partial + 1 + w_1x)\phi^{-1} + \left(\sum_{i=2}^{\infty} (w_i x \partial^{-(i-1)} - iw_i \partial^{-i})\right) \phi^{-1},$$

and then

$$(\phi \partial x \phi^{-1})_+ = x\partial + 1.$$  

(4.18)

Here the $\phi^{-1} = 1 - w_1 \partial^{-1} + \cdots$ is used. Taking eq. (4.18) into $LM$, we have

$$(LM)_+ = x\partial + 1 + \sum_{i=1}^{\infty} (2i + 1)t_{2i+1}L^{2i+1}_+.$$  

(4.19)

Taking eq. (4.19) back into eq. (4.15),

$$\partial^*_{i+1} L = 2L + 2[x\partial, L] + 2\sum_{i=1}^{\infty} (2i + 1)t_{2i+1}L^{2i+1}_+,$$

which shows

$$\partial u_i \over \partial t^{*}_{i+1} = 2 \left( x \frac{\partial u_i}{\partial x} + (i + 1)u_i \right) + 2 \sum_{j=1}^{\infty} (2j + 1)t_{2j+1} \frac{\partial u_i}{\partial t^{2j+1}},$$

(4.20)

on account of

$$[x\partial, L] = -\partial + \sum_{i=1}^{\infty} \left( x \frac{\partial u_i}{\partial x} + iu_i \right) \partial^{-i}.$$  

(4.22)

This is the end of the proof.

□

Corollary 4.3. From eq (4.14) we get $A_{1,l} = -lL^{l-1}$ when $l$ is even, thus the corresponding flows on $L$ are

$$\partial^*_{i+l} L = l[(L^{l-1})_-, L] = \begin{cases} 0, & \text{for } l = -2, -4, -6, \cdots; \\ -l(\partial^{-l-1} L), & \text{for } l = 2, 4, 6, \cdots. \end{cases}$$

(4.23)

Motivated by the results on the KP hierarchy [9, 10] and the BKP hierarchy [14, 15], we can also define one generating function of additional symmetries

$$Y^C (\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}(A_{m,m+l})_-$

$$= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} ((M^m L^{l+m}) - (-1)^l+m(L^{l+m} M^m))_-, \quad (4.24)$$
which can be expressed by a simple form in the sequent proposition. To this end, we need three well known and useful technique lemmas.

**Lemma 4.1.** ([2] §6.2.5) For two pseudo-differential operators $P$ and $Q$, the identity

$$\text{res}_z[(Pe^{xz})(Qe^{-xz})] = \text{res}_z[PQ^\ast]$$

is true.

**Lemma 4.2.** Let $P$ be a pseudo-differential operators $P = \sum p_i \partial^i$, then

$$P = \sum \partial^i \tilde{p}_i, \text{ and } P_\ast = \sum \partial^{-i} \text{res}_\delta (\partial^{-1} P).$$  \tag{4.26}

**Lemma 4.3.** ([2] §6.3.2(ii)) If $f(z) = \sum_{-\infty}^\infty a_i z^{-i}$, then

$$\text{res}_z[\zeta^{-1}(1 - z/\zeta)^{-1} + z^{-1}(1 - \zeta/z)^{-1}]f(z) = f(\zeta).$$  \tag{4.27}

(Here $(1 - z/\zeta)^{-1}$ is understood as a series in $\zeta^{-1}$ while $(1 - \zeta/z)^{-1}$ is a series in $z^{-1}$.)

**Proposition 4.5.**

$$Y^C(\lambda, \mu) = w(t, \mu) \partial^{-1} w(t, -\lambda) + w(t, -\lambda) \partial^{-1} w(t, \mu)$$  \tag{4.28}

Proof. Starting from eq.(2.6) and eq.(4.1), and using the first two lemmas at the first and second step, we get

$$(M^m L^{l+m})_\ast = \sum_{i=1}^\infty \partial^{-i} \text{res}_\delta [\partial^{-1} \phi \Gamma^m \partial^{l+m} \phi^{-1}]$$

$$= \sum_{i=1}^\infty \partial^{-i} \text{res}_\delta [(\partial^{-1} \phi \Gamma^m \partial^{l+m} e^{\xi(t,z)}) ((\phi^{-1})^* e^{-\xi(t,z)})]$$

$$= \sum_{i=1}^\infty \partial^{-i} \text{res}_\delta [(z^{l+m} \partial^{-1} \phi \Gamma^m e^{\xi(t,z)}) (w^*(t, z))]$$

$$= \sum_{i=1}^\infty \partial^{-i} \text{res}_\delta [(z^{l+m} \partial^{-1} \phi (\partial_t^m e^{\xi(t,z)}) (w^*(t, z))]$$

$$= \sum_{i=1}^\infty \partial^{-i} \text{res}_\delta [(z^{l+m} \partial^{i-1} (\partial_t^m e^{\xi(t,z)}) (w^*(t, z)))]$$

$$= \sum_{i=1}^\infty \partial^{-i} \text{res}_\delta [(z^{l+m} (\partial_t^m w(t, z)) (i-1)) (w^*(t, z))]$$

$$= \text{res}_\delta [z^{l+m} (\partial_t^m w(t, z)) \partial^{-1} w^*(t, z)].$$

Above, the final step is due to the identity $f \partial^{-1} = \partial^{-1} f = \partial^{-1} f_x \partial^{-1} = \sum \partial^{-i} f^{(i-1)}$ inversely.

Here $f_x = (\partial_x f)$ and $f^{(i-1)} = \frac{\partial^{i-1} f}{\partial x^{i-1}}$, $f$ is a $C^\infty$ function in $x$. Similarly,

$$(L^{l+m} M^m)_\ast = \text{res}_\delta [(\partial_t^m z^{l+m} w(t, z)) \partial^{-1} w^*(t, z)].$$
Taking them back into $Y^C$, which becomes
\[
Y^C(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \text{res}_z \left[ z^{l+m}(\partial_z w(t, z)) \partial^{-1} w^*(t, z) \right] \\
+ \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^{\infty} (-\lambda)^{-l-m-1} \text{res}_z \left[ (\partial_z z^{l+m} w(t, z)) \partial^{-1} w^*(t, z) \right] \\
= \text{res}_z \left[ \sum_{n=-\infty}^{+\infty} \frac{z^n}{\lambda^{n+1}} w(t, z + \mu - \lambda) \partial^{-1} w^*(t, z) \right] \\
+ \text{res}_z \left[ \sum_{n=-\infty}^{+\infty} \frac{(z + \mu - \lambda)^n}{(-\lambda)^{n+1}} w(t, z + \mu - \lambda) \partial^{-1} w^*(t, z) \right] \\
= w(t, \mu) \partial^{-1} w(t, -\lambda) + w(t, -\lambda) \partial^{-1} w(t, \mu).
\]

Here the final step is due to eq. (4.27).

Moreover, proposition 4.5 can also provide an explanation (from the view of symmetry) for the form of the Lax operator of constrained the CKP hierarchy [17]. To this end, let
\[
\phi_1(t) = \int d\mu \rho_1(\mu) w(t, \mu), \quad \phi_2(t) = \int d\lambda \rho_2(\lambda) w(t, -\lambda),
\]
and $\rho_i$ are some suitable weighting function, then they satisfy
\[
(\partial_{t_{2n+1}} \phi_i) = (B_{2n+1} \phi_i).
\]

Therefore, there exists symmetry reduction of the CKP hierarchy,
\[
L^k = L^k_+ + \phi_1 \partial^{-1} \phi_2 + \phi_2 \partial^{-1} \phi_1,
\]
which is the form of the Lax operator of the constrained CKP hierarchy [17].

5. **String Equation**

In corollary 4.3 we did not mention the case when $l$ is odd. This case is more interesting because it is related to so called ”string equation” [18] and thus deserves to be discussed separately with more details. In this section, from now on, we set $l = 2k$ and $k = 1, 2, 3, \ldots$

\[
A_{1, -(l-1)} = ML^{-(l-1)} + L^{-(l-1)} M = 2ML^{-(l-1)} - (l - 1)L^{-l},
\]
\[
[A_{1, -(l-1)}, L^l] = -2l.
\]

Therefore, we get a special action of the additional flows on $L^l$
\[
\partial_{t_{1, -(l-1)}} L^l = [-A_{1, -(l-1)}]_-, L^l] = [(A_{1, -(l-1)})_+, L^l] + [-A_{1, -(l-1)}, L^l] \\
= 2[(ML^{-(l-1)})_+, L^l] + 2l.
\]

We can get the following proposition on the string equation.

**Proposition 5.1.** If $L^l$ is a differential operator and it is independent of the additional variables $t^*_{1, -(l-1)}$, then
\[
[L^l, \frac{1}{2}(ML^{-(l-1)})_+] = [L^{2k}, \frac{1}{2k} ML^{-(2k-1)} - \frac{1}{2} \frac{2k - 1}{2k} L^{-2k}] = 1
\]
is a string equation $[P, Q] = 1$ (P and Q are two differential operators) of the CKP hierarchy.
Corollary 5.1. The string equations can be also expressed by CKP flows as

\[ -\frac{1}{2k} \sum_{n \geq k+1} (2n-1)t_{2n-1} \partial_{t_{2n-2k-1}} L^{2k} = 1 \]  

(5.5)

Proof By a direct calculation, the left hand side of eq.(5.4) becomes

\[ \frac{1}{2k} \sum_{n \geq k+1} (2n-1)t_{2n-1} \partial_{t_{2n-2k-1}} L^{2k} = -\frac{1}{2k} \sum_{n \geq k+1} (2n-1)t_{2n-1} \partial_{t_{2n-2k-1}} L^{2k}. \]

This completes the proof.

Corollary 5.2. If \( L^l \) and \( M \) satisfy the string equations eq.(5.4), then

\[ (ML^{-(l-1)})_+ = \frac{l-1}{2} L^{-l}, (L^{-(l-1)} M)_- = -\frac{l-1}{2} L^{-l}. \]  

(5.6)

Here \( l = 2k \) as before.

The \( Q = \frac{1}{l} (ML^{-(l-1)})_+ \) is of infinite order. Let assume that \( t_i (i \geq q+l) \) are frozen, i.e. there exist only nonzero finite independent variables \( t_i (i \leq q+l) \), then \( Q \) has order \( q \).

The string equations in eq.(5.4) impose some restrictions on the Lax operator of the CKP hierarchy. In general, we can proceed further to get more higher order constraints on \( L \) similar to what Adler and Moerbeke [19] have done for the KP hierarchy. To this purpose, we start with a special generator,

\[ A_{j,p+l+j} = M^j L^{p+l+j} + L^{p+l+j} M^j \]  

(5.7)

getting by the definition of generator of the additional symmetry eq.(4.7), with \( j = 1, 3, 5, \ldots \), \( p = -1, 0, 1, 2, 3, \ldots \). The fact \( pl + j \) is an odd number is used to get the form of the \( A_{j,p+l+j} \).

Lemma 5.1. With same the conditions as proposition 5.2, then

\[ (M^j L^{p+l+j})_- = \begin{cases} \prod_{r=0}^{j-1} \left( \frac{l-1}{2} - r \right) \cdot L^{-l}, & p = -1, \\ 0, & p = 0, 1, 2, 3, \ldots \end{cases} \]  

(5.8)
Proof The corollary 5.2 shows that the lemma holds for \( j = 1 \) and \( p = -1 \). Firstly, according to the induction method, we assume that the lemma is verified for a given \( j \) and \( p = -1 \), i.e.,

\[
((M^j L^{-l+j})_\text{\text{-}} = \prod_{r=0}^{j-1} \left( \frac{l-1}{2} - r \right) \cdot L^{-l},
\]

and then prove that the lemma is also true for this \( j \), but \( p \geq 0 \). So let \( p \geq 0 \), note that \( L^{(p+1)l} \) is a differential operator, then

\[
(M^j L^{pl+j})_\text{\text{-}} = ((M^j L^{j-l})_\text{\text{-}} L^{(p+1)l})_\text{\text{-}}.
\]

Taking the assumption into it,

\[
(M^j L^{pl+j})_\text{\text{-}} = \left( \prod_{r=0}^{j-1} \left( \frac{l-1}{2} - r \right) \cdot L^{-l} L^{(p+1)l} \right)_\text{\text{-}} = 0
\]

by using the fact that \( L^{pl} \) is a differential operator. Secondly, assume again that the lemma is true a given \( j \) and \( p = -1 \), and then prove that the lemma holds for the \( j+1 \) and \( p = -1 \). To this end, by using identity \([M, L^j] = -jL^{-1}\), we compute

\[
(M^{j+1} L^{-l+j+1})_\text{\text{-}} = (M^j ML^j L^{-l+1})_\text{\text{-}} = (M^j L^j ML^{-l+1})_\text{\text{-}} - j (M^j L^{j-l})_\text{\text{-}}.
\]

By using formula \( M^j L^j = (M^j L^{j+pl})_{pl=0} = 0 \) we have just proven previously and the assumption, then

\[
(M^{j+1} L^{-l+j+1})_\text{\text{-}} = (M^j L^j (ML^{-l+1})_\text{\text{-}} - j \prod_{r=0}^{j-1} \left( \frac{l-1}{2} - r \right) \cdot L^{-l}
\]

\[
= \frac{l-1}{2} (M^j L^{j-l})_\text{\text{-}} - j \prod_{r=0}^{j-1} \left( \frac{l-1}{2} - r \right) \cdot L^{-l}
\]

\[
= \prod_{r=0}^{j} \left( \frac{l-1}{2} - r \right) \cdot L^{-l},
\]

as we expected. This completes the proof of the lemma.

Lemma 5.2. With same the conditions as proposition 5.2, then

\[
(L^{pl+j} M^j) = \begin{cases} 
\prod_{r=0}^{j-1} \left( \frac{l-1}{2} - r \right) \cdot L^{-l}, & p = -1, \\
0, & p = 0, 1, 2, 3, \cdots
\end{cases}
\]

Proof The proof of this lemma is similar to the proof above.

Proposition 5.2. Let \( l = 2k \) as before, \( j = 1, 3, 5, \cdots, p = -1, 0, 1, 2, 3, \cdots \). If \( L^l \) satisfies the string equation eq. (5.4) and \( L^l \) is a differential operator, then

\[
\partial_{t^*_j, pl+j} L^l = 0
\]

Proof According to the action of additional symmetry flows on the Lax operator, i.e., eq. (5.12), and the explicit form of \( A_{j, pl+j} \) in eq. (5.7),

\[
\partial_{t^*_j, pl+j} L^l = - \left( (M^j L^{pl+j} + L^{pl+j} M^j)_\text{\text{-}}, L^l \right).
\]

Taking the above two lemmas into it, the proposition is proved.
The equations in (5.10) are called Virasoro (for j=1) and higher Virasoro constraints. As Dickey [11] has done for the KP hierarchy, one can get the explicit form of the Virasoro generators and its action on the \( \tau \) function from eq. (5.10). However, for the CKP hierarchy this procedure can not be performed due to the non-existence of a sole \( \tau \)-function.

6. Conclusions and Discussions

To summarize, we have constructed the additional symmetries in eq. (4.1) together with proposition 4.3 of the CKP hierarchy, and further showed that they form a new infinite algebra \( W^C_\infty \) in proposition 4.4. We have presented a simple expression in proposition 4.5 for the generating function of the symmetries. We have also derived the string equation in proposition 5.1 and the associated higher order constraints on \( L \) in proposition 5.2. Particularly, the string equations were also given by the actions of the CKP flows on \( L^j \) in corollary 5.1. Our results show that CKP hierarchy has, indeed, different properties related to additional symmetry comparing with the BKP hierarchy.

Acknowledgments This work is supported by the NSF of China under Grant No. 10301030 and No. 10671187, and SRFDP of China. Support of the joint post-doc fellowship of TWAS(Italy) and CNPq(Brazil) at UFRGS is gratefully acknowledged. J. H. thanks Professors Li Yishen, Cheng Yi(USTC,China) and F. Calogero(University of Rome "La Sapienza", Italy) for long-term encouragements and supports.
ADDITIONAL SYMMETRIES AND STRING EQUATION

REFERENCES

[1] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, in Nonlinear Integrable Systems- Classical and Quantum Theory, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983) p. 39-119.

[2] L. A. Dickey, Soliton Equations and Hamiltonian Systems (2nd Edition)(World Scientific, Singapore, 2003).

[3] W. Oevel, B. Fuchssteiner, Explicit fromulas for symmetries and conservation laws of the Kadomtsev-Petviashvili equation, Phys. Lett. A 88, 323-327(1982).

[4] B. Fuchssteiner, Mastersymmetries, Hilger order time-dependent symmetries and conserved densities of nonlinear evolution equation, Prog. Theor. Phys. 70, 1508-1522(1983).

[5] H. H. Chen, Y. C. Lee and Jeng-Eng Lin, On a new hierarchy of symmetry for the Kadomtsev-Petviashvili equation, Physica D 9,439-445(1983).

[6] A. Yu. Orlov, E. I. Schulman, Additional symmetries of integrable equations and conformal algebra reprensetaion, Lett. Math. Phys. 12, 171-179(1986).

[7] P. Van Moerbeke, Integrable foundation of string theory, in Lectures on integrable systems, P.163-267(World Scientific, Singapore, 1994)

[8] M. Adler, T. Shiota, P. van Moerbeke, A Lax representation for the Vertex operator and the central extension, Comm. Math. Phys. 171, 547-588(1995).

[9] M. Adler, T. Shiota, P. van Moerbeke, From the $w_{\infty}$-algebra to its central extension: a $\tau$-function approach, Phys. Lett. A 194, 33-43(1994).

[10] L. A. Dickey, On additional symmetries of the KP hierarchy and Sato’s Backlund transformation, Comm. Math. Phys. 167, 257-293(1995).

[11] L. A. Dickey, Additional symmetries of KP, Grassmannian, and the string equation, Mod. Phys. Lett. A8, 1259-1272(1993).

[12] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, KP hierarchy of Orthogonal symplectic type–transformation groups for soliton equations VI, J. Phys. Soc. Japan.50, 3813-3818(1981).

[13] Johan van de Leur, The $n$-th reduced BKP hierarchy, the string equation and $BW_{1+\infty}$-constraints, Acta Applicandae Mathematicae 44, 185-206(1996)(see also arXiv:hep-th/9411067).

[14] Johan van de Leur, The Adler-Shiota-van Moerbeke formula for the BKP hierarchy, J. Math. Phys. 36, 4940-4951(1995)

[15] M. H. Tu, On the BKP hierarchy: Additional symmetries, Fay identity and Adler-Shiota- van Moerbeke formula(preprint, arXiv:nlin.SI/0611053).

[16] K. Takasaki, Quasi-classical limit of BKP hierarchy and $W$-infinity symmetries, Lett. Math. Phys. 28, 177-185(1993).

[17] I. Loris, On reduced CKP equations, Inverse Problems. 15, 1099-1109(1999).

[18] M. Douglas, Strings in less than one dimension and the generalized KdV hierarchies Phys. Lett. B238, 176-180(1990).

[19] M. Adler, P. van Moerbeke, A matrix solution to two-dimensional $W_p$-gravity, Comm. Math. Phys. 147, 25-56(1992).