Students’ structure sense ability in solving quadratic equation problems

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Abstract. Structure sense is the ability of students to see any relation between symbols, to manipulate, and to use algebraic structures. This is considered an important ability for students as algebra is the core topic within high school mathematics. An important skills need to be acquired by students in algebra is solving equations, such as solving quadratic equations. To address this, this research aims to understand students’ structure sense ability by investigating strategies used in solving quadratic equations of the form of a square of difference of two variables. To reach these aims, a qualitative descriptive method was conducted in the form of four quadratic equation tasks done by twelve Senior High School Students from one of senior high schools in Bandung. In this study, each student was required to work on each task using two different strategies if possible. The result from the students’ answers informed that the majority used structure sense strategy. However, there were mistakes conducted by fifty percent and more students completing both the procedural and structure sense strategy. In conclusion, we found out that students’ structure sense ability is good despite students experienced some difficulties, whether they used the procedural strategy or involved their structure sense.

1. Introduction

Algebra is the core topic in high school mathematics learning as it is the opening “gate” to the university mathematics [1]. Nevertheless, according to Novotna and Hoch [2], many high school students have weaknesses in algebraic manipulation. This is because they considered algebra as abstract symbols, their way of understanding the symbols without any algebraic connection because of a misconception formed at the beginning of learning [3].

Kieran and Sfard [4] proposed that this term is associated with students’ inabilities to manipulate symbols as they prefer to solve it based on standard procedure. They do not see any characteristics from the symbols, including in solving the problem by applying prior algebraic structure [4]. The skills to interpret an algebraic expression, manipulate, and solve it using a familiar structure accurately are defined as structure sense ([2], [5], [6], [7]). Presenting structure sense when solving algebraic problems shows relational than instrumental understanding [8].

The term structure sense is first used by Linchevski and Livneh [9] as an ability in which students use the equivalent algebraic structure of expression both creatively and flexibly. Hoch and Dreyfus [10] described students’ skills involving structure sense as the ability to see at first the algebraic expression and choose the right manipulation technique before applying the algebraic transformation automatically. Hoch and Dreyfus [5] developed structure sense to examine its definition to explained students’
difficulties in factoring an algebraic expression. Meanwhile, Hoch and Dreyfus [7] used the structure sense’ definition that has been developed to see students’ ability to recognize an algebraic structure.

As mentioned before, several international studies on structure sense have been done to investigate student proficiency in algebra, meanwhile, in Indonesia, the study in this and related topic is limited ([11], [12], [13]). This leads us to investigate how Indonesian students’ ability on dealing with algebraic problems. While in previous studies, they examined students’ structure sense by only factoring the algebraic expression ([10], [5], [7]), we decided to develop the study by solving quadratic equations. It means that we will examine students’ structure sense by factoring and their final solution. Generally, Indonesian students have been taught several procedures that can be used to solve a quadratic equation in their school. The procedures are factoring, completing the squares, and using the quadratic formulas [14].

Therefore, to investigate the students’ ability in structure sense, the result of students’ answers will be distinguished into two strategies, namely structure sense strategy and procedural strategy. The student displays the procedural strategy if s/he directly uses a common quadratic equation procedure without considering the efficiency when using the procedure (e.g. [14]). As an example, when solving the equation \((2x - 3)^2 - 7(2x - 3) + 12 = 0\), the student does not substitute \((2x - 3)\) to make a more simple equation, which can be solved efficiently using factoring strategies but transform the equation into \(4x^2 - 26x + 42 = 0\) and use the quadratic formula to find the solution [11].

The student is said to use structure sense strategy if s/he can: (1) recognize a familiar structure in its simplest form; (2) deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form; (3) choose appropriate manipulations to make best use of a structure ([2], [5], [6], [7]). As an example, when solving the equation \((x - 2)^4 - (x + 2)^4 = 0\) using structure sense strategy, the student can see \((x - 2)^2\) and \((x + 2)^2\) as single entities to get simpler equation and use it to manipulate quadratic equation into \(((x - 2)^2 - (x + 2)^2)((x - 2)^2 + (x + 2)^2) = 0\) [11].

2. Methods
A qualitative descriptive method was used to investigate the structure sense ability from twelve senior high school students in one of senior schools in Bandung (from twelfth-grade classes, advanced, intermediate, and low in mathematics proficiency). To do so, first, we developed four quadratic equation tasks in the form of \(a^2 - 2ab + b^2 = k \iff (a - b)^2 = k\) based on structure sense characteristics in high school (see Table 1). Second, we examined the task to students and asked them to solve each task with two different solution strategies if possible. The duration of the test was 80 minutes. Finally, we analyzed their strategies based on structure sense characteristics ([2], [5], [6], [7]).

| No | Task | Structure Sense Theory |
|----|------|------------------------|
| 1  | \(x^2 - 2xy + y^2 = 16\)  
Find the value of \(x - y\) | Recognize a familiar structure in its simplest form |
| 2  | \(x^4 + y^4 - 2x^2y^2 - 81 = 0\) 
Find the value of \(x^2 - y^2\) | Deal with a compound term as a single entity |
| 3  | \((2x + 1)^2 - 2(2x + 1)(x + 2) + (x + 2)^2 = 36\) 
Find the value of \(x\) | Choose appropriate manipulations to make best use of a structure |
| 4  | \(x^2 + 2x + 1 - 2(x + 1)(2x + 3) + (2x + 3)^2 = 9\) 
Find the value of \(x\) | Choose appropriate manipulations to make best use of a structure |
3. Result and Discussion

Table 2 shows the number of students’ answers and strategies. For example, viewed from the final answer, six students find the solution correct and six others incorrect in task number 1. From twelve students who did the task number 1, eleven students used structure sense strategy while a student used procedural. We found that there was a decrease in the number of students working with structure sense but there was an exception for the last number. Although the number of students with the ability in structure sense is high, based on table 2, the number of students who did the final answer incorrectly is high as well, 6 students for task 1, 3, 4, and 9 students for task 2. These mistakes were done by students completing both the procedural and structure sense strategy. In addition, there was a student in number 2 and 4 could not solve the equations.

Table 2. Students’ answers and strategies solving quadratic equation problems in the form of \( a^2 - 2ab + b^2 = k \) \( \iff (a - b)^2 = k \)

| Task | Correct | Incorrect | Empty | Strategy |
|------|---------|-----------|-------|----------|
|      | 1       | 2         | 3     | 4        |
| 1    | 6       | 6         | 0     | 11       |
| 2    | 2       | 9         | 1     | 9        |
| 3    | 6       | 6         | 0     | 5        |
| 4    | 5       | 6         | 1     | 7        |

For Task 1, a representative of a student with structure sense could recognize the structure \( a^2 - 2ab + b^2 = k \) in equation \( x^2 - 2xy + y^2 = 16 \) (see Figure 1 left part). Therefore, s/he simplified it into \( (x - y)^2 = 16 \) and determined the value of \( x - y \) in which has two solutions, 4 or \(-4\). Meanwhile, a representative of a student who lacks of structure sense tried to find the solution with a quadratic formula (see Figure 1 right part). Additionally, s/he seemed to have a difficulty on determining the value of \( b \) and \( c \) to input them into the formula, it is supposed to be \( b = -2y \) and \( c = y^2 - 16 \). Hence, s/he could not finished it due to the wrong interpretation. However, five out of eleven students with structure sense did mistakes when determining the value of \( x - y \). They considered the value of \( x - y \) was only 4 and predicted the answer by replacing the variable \( x \) and \( y \) with any numbers (variable \( x \) with 4 and \( y \) with \(-4\)) so brought it to the inaccurate or even incorrect final solution. We consider these students still have structure sense ability on the first characteristic by Novotna and Hoch [2] despite their mistakes when finding the final answer.

Figure 1. Representative examples of student solution strategies on Task 1

For Task 2, students with structure sense were able to see \( x^2 \) and \( y^2 \) as single entities, recognized the structure \( a^2 - 2ab + b^2 = k \) in the equation \( x^4 + y^4 - 2x^2y^2 = 8 \) then simplified it into \( (x^2 - y^2)^2 = 81 \) (see Figure 2 left part). Students found the value of \( x^2 - y^2 \) accurately, 9 or \(-9\). Meanwhile in Figure 2 right part is a representative of a student who did not see \( x^2 \) and \( y^2 \) as single
entities thus s/he did not recognize the structure in the equation. Moreover, the student seemed to do a mistake by applying the wrong concept so s/he was unable to find the requested solution. We infer the student who solved the equation like this lack of structure sense. However, even nine students possess structure sense ability on second characteristic [2], the mistake on the concept of quadratic value performed by seven of them. They assumed the value of \( x^2 - y^2 \) only 9.

![Figure 2. Representative examples of student solution strategies on Task 2](image)

For Task 3, Figure 3, left part, shows an example of a student with structure sense used variable \( a \) to replace \((2x + 1)\) and \( b \) to \((x + 2)\) then brought the equation to the structure \( a^2 - 2ab + b^2 = 36 \). These steps indicate that the student not only saw the compound term \((2x + 1)\) and \((x + 2)\) as single entities but also chose an appropriate manipulation to give them ease on recognizing the structure \( a^2 - 2ab + b^2 = k \). This strategy suits the last structure sense characteristic [2]. In the last step, s/he found the final solution by changing variable \( a \) and \( b \) to their real expressions. While in Figure 3, right part, gives an example of a student strategy without structure sense. Instead of doing manipulation on \((2x + 1)\) and \((x + 2)\) to reveal a simpler equation, s/he chose to do inefficient algebraic transformation into the equation \( x^2 - 2x - 35 = 0 \) then solved it with factoring the equation. Also, two of seven students with this strategy did an error calculation so they got the incorrect final solution.

Moreover, there were four students who also recognized the existence of structure \( a^2 - 2ab + b^2 = k \) in the equation. However, they were mistaken to simplify the expression \((2x + 1)^2 - 2(2x + 1)(x + 2) + (x + 2)^2\) into the same expression as \((a - b)^2\). Instead of writing \((2x + 1)\) and \((x + 2)\), they wrote \((2x + 1)^2\) and \((x + 2)^2\) as the representative of \( a \) and \( b \), resulting in them to write the incorrect answer. In this case, it is defined as partial structure sense which means mistakes that students do when transforming an algebraic expression into a particular structure even if they notice the existence of a structure in an algebraic expression [5].

![Figure 3. Representative examples of student solution strategies on Task 3](image)
For Task 4, the representative of a student with structure sense solved it as follows: \( x^2 + 2x + 1 - 2(x + 1)(2x + 3) + (2x + 3)^2 = 9 \) \( \iff \) \( (x + 1)^2 - 2(x + 1)(2x + 3) + (2x + 3)^2 = 9 \), and so \( x = -5 \) or \( x = 1 \) (see Figure 4 left part). The student’s accuracy in choosing the manipulation technique as s/he chose to factor the expression \( (x^2 + 2x + 1) \) into \( (x + 1)^2 \) and used variables \( a \) and \( b \) to replace \( (x + 1) \) and \( (2x + 3) \) make them easier to recognize the structure \( a^2 - 2ab + b^2 = k \) in the equation. The student easily found the final solution by changing back variable \( a \) and \( b \) to their real expressions. By choosing appropriate manipulation, they suit the last structure sense characteristic [2]. However, the same mistake as they did in Task 1 and Task 2, the concept of quadratic value, performed by six of the seven students did with structure sense strategy, thus they assumed the value of \( x \) is only \(-5\).

Figure 4, right part, gives an example that from the first step, the student did not realize the existence of \( (x + 1)^2 \) as another form of \( x^2 + 2x + 1 \). Instead of factoring \( x^2 + 2x + 1 \) into \( (x + 1)^2 \), s/he focused to extend the expression \(-2(x + 1)(2x + 3)\) and \((2x + 3)^2\) so it did not reveal a repetition in \((x + 1)\) and \((2x + 3)\). This gives us information that the student was not able to choose which expression to manipulate and what manipulation technique is appropriate to use on the structure. Therefore, the student does not involve a structure sense ability to solve the equation and this makes her/him continue with algebraic operations until it comes to factoring procedures to find the final solution.

**Figure 4. Representative examples of student solution strategies on Task 4**

4. Conclusion
Our finding on students’ strategy in solving four tasks about quadratic equation in the form of \( a^2 - 2ab + b^2 = k \) \( \iff \) \( (a - b)^2 = k \) shows that most students use structure sense strategy, although there are few of them that belong to partial structure sense. Furthermore, in accordance with the complexity of the task is increasingly higher, the ability of students’ structure sense tend to be reduced then start to switch to the procedural strategy. Even though the number of students with structure sense ability is relatively high, the fact that many of them acquire an incorrect/less precise final solution. However, students who lack structure sense ability in this study have an inability to view compound term as a single entity in the equation and inability to choose appropriate manipulation so that students are unaware there is a specific structure that can be transformed into a simpler equation. It can be concluded that having structure sense ability for solving quadratic equations is more complex than factoring algebraic expression. It is revealed that for some students who have this ability, they still did mistakes in finding the right final solution.

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