A Sierpiński triangle geometric algorithm for generating stronger structures

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Abstract. In the twenty-first century, in the era of 3D printers and advanced technologies for the production of complex parts, it becomes possible to create objects that have a fractal structure. This gives relevance to the attempts of applying the knowledge of fractal geometry. This paper presents effectiveness confirmation and feasibility studies of using a fractal algorithm for making stronger structures. The algorithm is based on the Sierpiński fractal. The essence of the algorithm is to increase the buckling strength of the rods by fixing their midpoints. This principle makes it possible to create complex flat and three-dimensional structures, as well as to solve some engineering problems. Analytical strength calculations were run in Wolfram Mathematica, whilst SolidWorks, a CAD and CAE program, were used to compute the strength.

1. Introduction
Fractal sets and geometric fractals in particular are a fairly popular topic of modern research. Benoit Mandelbrot laid the foundations of the fractal theory in the second half of the 20th century. In addition to being of interest for pure science, fractals have useful applications, e.g., in electronics, where they help design antennas with special properties [1]. Geometric fractals, or rather structures that utilize fractal patterns, are used in architecture. Paper [2] investigates the applicability of the Takagi prefractal in roofing. This study presents the results of testing the strength of the proposed prefractal models. The strength properties of structures generated by applying fractal patterns are detailed in [3, 4]. It demonstrates a significant increase in strength per unit mass. However, running more iterations of the hierarchy diminishes the effect of using the proposed design technique. Improving the strength of structures by using novel geometric solutions is a promising area of research, the results of which may help implement innovative engineering solutions and reduce the consumption of expensive materials.

2. Formulation of the problem
Researchers today are tasked with inventing lighter-weight but stronger products [5]. Many approaches are known, falling mostly within either of two categories. One category revolves around using advanced materials and their composites [6], modifying coatings and innovations in processing and machining [5]. The other one is based on optimizing the material structure [7] or the structural geometry [8], which calls for geometric solutions. Optimizing the geometry of a structure requires strength estimates as a proof of effectiveness. Computer simulations based on the finite element method constitute the most relevant applicable testing method as of today. Studies presented in [9, 10] make active use of this approach. If the
elements making up the structure have relatively simple geometry, the finite element method should be complemented with analytical calculations [11], which require using mathematical software for more reliable results.

This paper applies a geometric approach to solving the problem of creating stringer yet lightweight structures; the approach produces results that are applicable to a variety of materials. Fractal geometry is one specifically understudied method [3, 4]. Therefore, the goal hereof is to develop and test the effectiveness of a geometric algorithm for generating structures that can withstand large loads while saving materials. Running more prefractal iterations should make the algorithm proportionately more effective, which must be tested by running analytical calculations based on the mechanics of materials coupled with SolidWorks simulations. Analytical calculations are to be run in Wolfram Mathematica. Hereinafter, fractal algorithm effectiveness is defined as the improvement in the strength of a structure against a similar non-fractal structure of the same mass subjected to the same load.

3. Theoretical foundations

Structural parts have to withstand tension, compression, bending, torsion, etc. Critical load magnitude is determined by the maximum value of stress in the part material. Only normal stresses occur in central tension and compression, whereby its magnitude depends on the cross-sectional area and on the applied load. Bending and torsion have moments, the magnitude of which depends on the distance to the point of force application; if the levers are large, these moments can cause enormous stresses. For this reason, structure designers seek to make such structures, the parts of which will be subject to tension and compression rather than bending, see, e.g., suspension bridges. Thus, for the purposes of this research, the bearing of tensile and compressive loads is of importance.

Consider a cylindrical rod of constant length and mass, which is subject to compression. Geometric methods cannot increase its cross-sectional area without creating a critical section elsewhere in the rod, i.e., cannot make it stronger. Rod destruction will occur at a lower load due to buckling, if the cross-sectional area to the rod length section is sufficiently small. At stresses within the proportional limit of the material, the Euler formula can be used to find the critical (buckling) load:

$$P_b = \pi E J (KL)^2,$$

where $E$ is Young's modulus of the rod material, $J$ is the minimum area moment of inertia of the cross-section, $L$ is the unsupported length of the rod, and $K$ is rod effective length factor.

In case of a round rod, this formula applies:

$$P_{fa} = m^2 \pi E \left[ \frac{4 \rho L^2}{(KL)^2} \right]^2$$

If the material has constant properties ($p = \text{const}, E = \text{const}$) and length ($L = \text{const}$ and $K = \text{const}$), the critical load and the load determined by the factor of safety are:

$$P_b = k_1 m,$$

$$P_b = k_1 m^2,$$

where $k_1$ and $k_2$ are constants.

Figure 1 shows the corresponding curves.
Figure 1. Total strength as a combination of the critical load and the load determined by the factor of safety.

LK can be reduced by making the rod shorter or by adding pin joint in the middle. LK will be halved in this case, while $P_b$ will quadruple. As the mass decreases, two more joints can be added in the middle of each of the remaining rod parts, again quadrupling $P_{fa}$. This only applies to rods operating in the low-mass domain (to the left of point K, Figure 1). The constraint $P_b < P_{fa}$ can be kept by repeating this step. This algorithm is iterative and can be implemented using a fractal composed of rods and based on the Sierpiński triangle (Figure 2.). Hereinafter, SRT stands for Sierpiński rod triangle.

Figure 2. The first three iterations of constructing a Sierpinski rod triangle.

The study concerns the plane case, i.e., rods can only buckle, and the nodes can only move within the SRT plane. All the segments in the following diagrams are circular rods whose circular section does not change lengthwise.
If the triangle base vertices are pivoted, and point force is applied to the third vertex towards the triangle center, then the side rods AF and AD will bear the bulk of the load. The remaining rods will provide support to prevent the buckling of AD and AF. If the ends of the rod AD are fixed, and force is applied to the vertex F, FD will bear the bulk of the load. Refer to AD, AF, and DF as the base rods, and to the rest as the supporting frame. This term will be used to refer to a set of rods in a fractal structure, the main function of which is to prevent the buckling of those rods that bear the bulk of the target load.

For the purposes hereof, it is necessary to confirm that at certain values of the whole-structure mass, raising the mass of the supporting frame further, which is a result of running more iterations, will not reduce the overall efficiency, that is, the increase in buckling resistance will outstrip the increase in mass.

The maximum specific strength of the SRT is the product of two functions $f_1$ and $f_2$. The specific strength is defined herein as the ratio of the critical load that the nth iteration prefractal can withstand to the critical load of the triangle composed of the base rods $P_0$ at the optimal mass.

$$f_1 = P_0 \times 4^n,$$

where $R_0$ is the cross-sectional radius of a base rod, and $n$ is the iteration of the prefractal. Formula (2) is valid only for the mass of a prefractal, the base rods of which bear a load that corresponds to the point K, Figure 1.

$$f_2 = \frac{m_0}{m_n},$$

(2)

where $m_0$ is the triangle mass, and $m_n$ is the mass of the prefractal of the nth iteration. It is possible to calculate the mass if the radii of the rods are known. Then the maximum specific strength of the SRT can be found as:

$$f_0 = f_1 \times f_2 = 4^n P_0 \times \left[ 3\rho L \pi R_0^2 + \rho L \pi R_1^2 \sum_{i=1}^{n} \left( \frac{3}{2} \right)^i \right]^{-1},$$

(3)

where $\rho$ is the density of the material. To find $f_0$, find the $R_0$ to $R_1$ ratio first.

4. Results and discussion

Ratios of the rod radii for the first-iteration SRT are found below for two assumptions:

1) The base rods are compressed but do not buckle. Then the supporting frame must withstand part of the load transmitted by the base rods when compressed (Figure 4 (a)).
2) The base rods buckle but are not compressed. Then the supporting frame must withstand the load resulting from the buckling of the base rods (Figure 4 (b)). The figure shows the buckling diagram at maximum load on the rods of the supporting frame.

\[ \Delta L_{bdy} = \Delta L_{bey} + v_E \]  

By replacing the force equations with Euler’s critical loads (1), all the coefficients pertaining to the material are canceled out, deriving the following equation:

\[
R_{1}^{(1)} = R_0 \left\{ \frac{3R_0^4}{2L_0^2} + \left( \frac{24L_0^2 R_0^6 + 9R_0^8}{2L_0^2} \right)^{1/2} \right\}^{1/4} \times \left\{ R_0^4 + \frac{3R_0^4}{2L_0^2} + \left( \frac{24L_0^2 R_0^6 + 9R_0^8}{2L_0^2} \right)^{1/2} \right\}^{1/4} \cos 60
\]

Simplified:

\[ R_{1}^{(1)} = \left[ \frac{3R_0^4}{2L_0^2} + \left( \frac{24L_0^2 R_0^6 + 9R_0^8}{2L_0^2} \right)^{1/2} \right]^{1/2} \]

The difference between the equations (5) and (6) stems from the difference in load between the rod sections AB and BD, which contributes insignificantly.

The second approximation returns:

\[ R_{1}^{(2)} = \frac{R_0^2}{L_0} \left( \pi^2 - 6 \right)^{1/2} \]

All the variables pertaining to the material are canceled out as well.

If the rod radius to length ratio is small, the value of (7) is negligible compared to the value of (6), so only the latter is used.

When applying the first approximation to solve problems similar to those in Figure 4 but for the second-iteration SRT, one needs to solve a system of equations for deformations (4) and for strength. In this case, the material variables cannot be canceled out, meaning that a general equation similar to
cannot be obtained. For this reason, radius ratios used hereinafter for base rods and supporting rods were found by computer simulation.

SolidWorks simulations of the SRT model prove the analytical conclusions for the first-iteration SRT. The tested model has the following specifications: triangle height of 100 mm, i.e., the side is about 115.47 mm. The material is ABS (the plan is to further test 3D-printed mockups).

Figure 5 shows curves plotted on the computer simulation output; it shows the optimal radii: \( R_0 \approx 1.6460, R_1 \approx 0.3050. \)

![Image](image.png)

**Figure 5.** Critical load as a function of \( R_1 \) in buckling testing of a 3-gram structure.

This point \( (R_1 = 0.3050) \) is the maximum of the curve showing the stage, in which the base rods and the supporting frame rods buckle simultaneously (Figure 6 (b)).
Figure 6. SRT model simulation results, first iteration: (a) insufficient, (b) optimal, and (c) excessive mass of the supporting frame.

Calculations by (3) return $R_0 = 1.6459$ and $R_1 = 0.3062$. The error of $R_1$ here is less than 0.3%, and the error of strength is less than 0.03%. Note that the errors are rather conditional, since they are below the SolidWorks critical load error for this specific problem.

Figure 7 shows maximum loads that the structure can withstand as a function of its mass. The curves of loads by factor of safety coincide for the first-iteration prefractal and for the base triangle, since their masses differ only slightly. The difference between simulation results and analytical calculations at large masses is attributable to the fact that the rods overlap to a greater extent at the nodes of the structure, and only simulations adjust for it.

Figure 7. Critical loads as functions of mass.
Maximum load curves for the second and third-iteration prefractals are simulation-based. A curve similar to the one in Figure 5 was plotted to find the optimal $R_0$ to $R_1$ ratio at each point with a certain mass. Figure 8 shows the specific strength curves.

![Figure 8. Specific strength of SRT prefractals.](image)

5. Discussion
The points highlighted in Figure 8 correspond to the point K in Figure 1 and split the curve into a hyperbolic section and a linear section. Fig. 8 shows that the fractal algorithm boosts the efficiency of the structure for certain loads. It clearly demonstrates that the proposed algorithm multiplies the specific strength, at least in the first few iterations. Unlike in [3], the effect of complicating the supporting framework increases iteration to iteration. However, the described dependencies may change when running more iterations, e.g., due to the increasing effect of the second assumption, equation (7).

In practice, SRTs have very few applications. The idea behind this paper is to show that structures could be made stronger (more efficient) for certain loads by applying the geometric principle fractal structuring. The principle consists in connecting the midpoints of the rods of a structure to each other.
iteratively in order to reduce the rod effective length factor across the structure. The algorithm is quite simple to code, which might potentially enable automatic modeling of lightweight yet strong fractal structures. The study of fractal structures generated by mathematical programs using analytical expressions, as in the study [12], is promising. Figure 9 shows a solution to the problem of increasing the buckling resistance of a single rod compressed by the force $P$.

![Figure 9. Example of using the fractal algorithm to improve the buckling resistance of a rod.](image)

6. Summary and conclusion

Note that in practice, the supporting frame of a fractal structure can have an additional mechanical function, for example, the SRT and similar structures will be much stronger than a simple triangle when exposed to a force applied perpendicularly to the midpoint of one of its sides. Besides, the supporting frame will be able to prevent the destruction of the product when exposed complex loads it was not designed for.

Further research may additionally focus on the ability of fractals to withstand shock loads by distributing the impact energy evenly among the elements. Similar structures have been proven shock-resistant in [13].

Planar fractals can be generalized to three-dimensional space [14]. The algorithm underlying SRT generation is no exception. In this case, a rod-shaped Sierpiński tetrahedron will be obtained (Figure 10).

![Figure 10. Three-dimensional SRT generalization.](image)

Further research will be devoted to the detailed study of spatial structures, as well as to the solid fractal structures similar to the lattices described in [15].
References

[1] Anguera J et al. 2020 Fractal antennas: An historical perspective Fractal and Fractional 4

[2] Rian I M 2019 FracShell: From Fractal Surface to a Lattice Shell Structure Digital Wood Design pp 1459-79

[3] Rayneau-Kirkhope D et al. 2012 Hierarchical space frames for high mechanical efficiency: Fabrication and mechanical testing Mechanics Research Communications 46 pp 41–46

[4] Rayneau-Kirkhope D, Mao Y and Farr R 2012 Ultralight fractal structures from hollow tubes Physical review letters 109 № 20 pp 204–301

[5] Lohr C et al. 2019 Polymer-Steel-Sandwich-Structures: Influence of Process Parameters on the Composite Strength Key Engineering Materials. Trans Tech Publications Ltd 809 pp 266–273

[6] Zhu G et al. 2020 Comparative study on metal/CFRP hybrid structures under static and dynamic loading International J. of Impact Engineering 141 pp 103–109

[7] Zhang X et al. 2020 3D printing boehmite gel foams into lightweight porous ceramics with hierarchical pore structure J. of the European Ceramic Society 40 №. 3 pp 930–934

[8] Berger J B, Wadley H N G and McMeeking R M 2017 Mechanical metamaterials at the theoretical limit of isotropic elastic stiffness Nature 543 №. 7646 pp 533–537

[9] Kader M A et al. 2020 Novel design of closed-cell foam structures for property enhancement Additive Manufacturing 31 p40

[10] Jiang H et al. 2020 Bioinspired multilayered cellular composites with enhanced energy absorption and shape recovery Additive Manufacturing 36 pp 101–430

[11] Yin S et al. 2019 Effects of architecture level on mechanical properties of hierarchical lattice materials International Journal of Mechanical Sciences 157 pp 282–292

[12] Beglov I A 2020 Generation of the surfaces via quasi-rotation of higher order J. of Phys.: Conf. Ser. 1546 012032

[13] Madke R R and Chowdhury R 2020 Anti-impact behavior of auxetic sandwich structure with braided face sheets and 3D re-entrant cores Composite Structures 236

[14] Zhikharev L A 2015 Generalization to the three-dimensional space of the Pythagorean and Koch fractals. Part 1 Geometry and graphics 3 №. 3 pp 24–37

[15] Schaedler T A et al. 2011 Ultralight metallic microlattices Science 334 №. 6058 pp 962–965

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