Direct and inverse source problems for two-term time-fractional diffusion equation with Hilfer derivative

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Direct and inverse source problems for two-term time-fractional diffusion equation with Hilfer derivative
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Abstract

In this paper, we investigate direct and inverse source problems for the diffusion equation with two-term generalized fractional derivative (Hilfer derivative) in a rectangular domain. Using spectral expansion method, we derive two-term fractional differential equation together with appropriate initial condition (Cauchy problem). Based on solution of that Cauchy problem, we represent solution of formulated problems as a combination of sinus and multinomial Mittag-Leffler function of two variables. Imposing certain conditions to the given data, we prove uniform convergence of certain infinite series.

MSC 2010: 35R11, 26A33

Keywords: Hilfer derivative, multi-term fractional diffusion equation, inverse source problem, multivariate Mittag-Leffler function.

1 Inverse source problem.

Consider the following diffusion equation with Hilfer derivatives

$$D^{\alpha_1,\beta_1}_{0t}u(t, x) + \mu D^{\alpha_2,\beta_2}_{0t}u(t, x) - u_{xx}(t, x) = g(x)$$

in a rectangular domain \( \Omega = \{(t, x) : 0 < t < T, 0 < x < 1\} \). Here \( \alpha_i, \beta_i (i = 1, 2) \), \( \mu, T \) are given real numbers such that \( T > 0, 0 < \alpha_2 < \alpha_1 < 1, 0 \leq \beta_i \leq 1 \),

$$D^{\alpha,\beta}_{0t} f(t) = \left( I^{(\alpha-\beta)(n-\alpha)}_{0t} \left( I^{(1-\beta)(n-\alpha)}_{0t} f \right) \right)(t), \ t > 0$$

is a Hilfer fractional derivative of the order \( \alpha \) and type \( \beta \) with \( n-1 < \alpha \leq n \in \mathbb{N} \) (see [1]),

$$I^{\alpha}_{0t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} f(z)dz, \ t > 0$$

is the Riemann-Liouville fractional integral of order \( \alpha > 0 \) such that \( I^{\alpha}_{0t} f(t) = f(t) \) (see, for example [2]).

We note that in case of \( \beta = 0 \), Hilfer derivative (2) coincides with the Riemann-Liouville derivative and in case of \( \beta = 1 \) with the Caputo derivative [1].

Problem 1 is to find a pair of functions \( \{u(t, x), g(x)\} \) from the class of functions

$$W_1 = \{u(t, x) : u \in C_{-1}[0, T], I^{(1-\beta_i)(1-\alpha_i)}_{0t} u \in C_{-1}[0, T], u \in C[0, 1], u_{xx} \in C(0, 1); g(x) \in C[0, 1] \},$$

which satisfies equation (1) in \( \Omega \) together with boundary conditions

$$u(t, 0) = u(t, 1) = 0, \ 0 \leq t \leq T$$

$$u(t, 0) = u(t, 1) = 0, \ 0 \leq t \leq T$$
and initial condition
\[
\lim_{t \to +0} I_{0t}^{(1-\beta_1)(1-\alpha_1)} u(t, x) = 0, \quad 0 \leq x \leq 1, \tag{5}
\]
and also overdetermining condition
\[
u(T, x) = \varphi(x), \quad 0 \leq x \leq 1. \tag{6}
\]
Here \(\varphi(x)\) is a given function such that \(\varphi(0) = \varphi(1) = 0\).

We remind the definition of the space \(C^m_\alpha\).

**Definition.** A function \(f(x), x > 0\), is said to be in the space \(C^m_\alpha\), \(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\), \(\alpha \in \mathbb{R}\) if there exists a real number \(p, p > \alpha\), such that
\[
f^m(x) = x^p f_1(x) \quad \text{with a function } f_1 \in C[0, \infty), \tag{9}
\]
where \(C^0_\alpha = C_\alpha\).

Hilfer derivative has appeared in the theoretical modeling of a broadband dielectric relaxation spectroscopy for glasses [3]. Some properties and applications of this derivative were studied in [4-7]. We will use in this work result of the work [8], where explicit solution of the modified Cauchy problem for multi-term fractional differential equation with Hilfer derivatives found by operational method.

Separation of variables lead us to the following spectral problem
\[
X''(x) - \lambda X(x) = 0, \quad X(0) = X(1) = 0, \tag{1}
\]
whose eigenvalues are \(\lambda_k = (k\pi)^2, k \in \mathbb{Z}\) and corresponding eigenfunctions \(\{X_k = \sin k\pi x\}\). Since this system forms complete orthogonal basis, we can expand solution of the problem 1 by the following series:
\[
u(t, x) = \sum_{k=1}^{\infty} U_k(t) \sin k\pi x, \quad 0 \leq t \leq T \tag{7}
\]
\[
g(x) = \sum_{k=1}^{\infty} g_k \sin k\pi x, \tag{8}
\]
where
\[
U_k(t) = 2 \int_0^1 u(t, x) \sin k\pi x dx, \quad k = 1, 2, \ldots \tag{9}
\]
\[
g_k = 2 \int_0^1 g(x) \sin k\pi x dx, \quad k = 1, 2, \ldots \tag{10}
\]
Substituting (7)-(8) into (1), we get
\[
D_{0t}^{\alpha_1, \beta_1} U_k(t) + \mu D_{0t}^{\alpha_2, \beta_2} U_k(t) + (k\pi)^2 U_k(t) = g_k. \tag{11}
\]
Initial condition (5) gives us
\[
\lim_{t \to +0} I_{0t}^{(1-\beta_1)(1-\alpha_1)} U_k(t) = 0. \tag{12}
\]
According to [8], solution of (11) together with (12) has a form
\[
U_k(t) = g_k \int_0^t z^{\alpha_1-1} E_{(\alpha_1-\alpha_2, \alpha_1), \alpha_1} \left( -\mu z^{\alpha_1-\alpha_2}, -(k\pi)^2 z^{\alpha_1} \right) dz, \tag{13}
\]
where
\[
E_{(\alpha_1-\alpha_2, \alpha_1), \alpha_1} \left( z^{\alpha_1-\alpha_2}, -(k\pi)^2 z^{\alpha_1} \right) = \frac{1}{\Gamma(\alpha_1)} \int_0^z \frac{t^{\alpha_1-1}}{(k\pi)^2 + t^{\alpha_1}} dt.
\]
where

$$E_{(\alpha-\beta,\alpha)}(x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{n!}{n!(n-i)!} \Gamma(\rho + \alpha n - \beta i) x^i y^{n-i}$$

(14)

is a particular case (in two variables) of multi-variate Mittag-Leffler function [9] with $\alpha, \beta, \rho > 0$.

Applying the following formula (see, for example [10])

$$\int_0^t z^\alpha E_{(\alpha-\beta,\alpha)}(m_1 z^\alpha, m_2 z^\alpha) \, dz = t^\alpha E_{(\alpha-\beta,\alpha)}(m_1 t^\alpha, m_2 t^\alpha)$$

from (13) we obtain

$$U_k(t) = g_k \cdot t^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu t^{\alpha_1-\alpha_2}, -(k\pi)^2 T^{\alpha_1}).$$

(15)

In order to define $g_k$ we use over-determined condition (6), which will take a form

$$U_k(T) = \varphi_k,$$

(16)

where

$$\varphi_k = 2 \int_0^1 \varphi(x) \sin k\pi x dx, \quad k = 1, 2, ...$$

From (15)-(16) we deduce

$$g_k = T^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu T^{\alpha_1-\alpha_2}, -(k\pi)^2 T^{\alpha_1})$$

with

$$T^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu T^{\alpha_1-\alpha_2}, -(k\pi)^2 T^{\alpha_1}) \neq 0.$$

(17)

Finally, $U_k(t)$ will have a form

$$U_k(t) = \frac{T^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu t^{\alpha_1-\alpha_2}, -(k\pi)^2 T^{\alpha_1})}{T^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu T^{\alpha_1-\alpha_2}, -(k\pi)^2 T^{\alpha_1})} \varphi_k.$$

(18)

Now, let us estimate $U_k(t)$. Since

$$\frac{t^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu t^{\alpha_1-\alpha_2}, -(k\pi)^2 t^{\alpha_1})}{T^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\alpha_1),\alpha_1+1}(-\mu T^{\alpha_1-\alpha_2}, -(k\pi)^2 T^{\alpha_1})} \leq C_1 (C_1 = const > 0),$$

we have

$$|U_k(t)| \leq \frac{C_1}{(k\pi)^2} \left| \varphi_k^{(2)} \right|,$$

(19)

where

$$\varphi_k^{(2)} = -2 \int_0^1 \varphi''(x) \sin k\pi x dx.$$

We need more "strong estimate" for $U_k(t)$ in order to provide convergence of infinite series corresponding for $u_{xx}(t,x)$. Precisely,

$$|U_k(t)| \leq \frac{C_2}{(k\pi)^3} \left| \varphi_k^{(3)} \right|,$$

(20)
where $C_2$ is a positive constant and

$$\varphi_k^{(3)} = -2 \int_0^1 \varphi'''(x) \cos k\pi x \, dx.$$ 

We have to impose more condition to the given function $\varphi(x)$ in order to guarantee uniform convergence of the following series

$$u_{xx}(t, x) = \sum_{k=1}^{\infty} U_k(t)(k\pi)^2 \sin k\pi x. \quad (21)$$

In fact, considering (20), we have

$$|u_{xx}(t, x)| \leq \sum_{k=1}^{\infty} |U_k(t)| (k\pi)^2 \leq \sum_{k=1}^{\infty} \frac{C_2}{k\pi} \left| \varphi_k^{(3)} \right|.$$ 

If we use $2ab \leq a^2 + b^2$, we have

$$|u_{xx}(t, x)| \leq \sum_{k=1}^{\infty} \left( \frac{C_2^2}{4(k\pi)^2} + \left| \varphi_k^{(3)} \right|^2 \right).$$

Due to $\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} = 1/6$ and $\sum_{k=1}^{\infty} |\varphi|^2 \leq \|\varphi\|_{L_2(0,1)}$, we assume that

$$\varphi(x) \in C^2[0,1], \varphi'''(x) \in L_2(0,1), \varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = 0, \quad (22)$$

then by Weierstrass M-test theorem, we can conclude that series (21) uniformly converges.

Proof of the uniform convergence of series corresponding to the functions $u(t, x), D_{0t}^{\alpha_1}u(t, x) (i = 1, 2)$ and $g(x)$ can be done by similar way considering (2), (3), (17), (19), but with less conditions to the given function $\varphi(x)$.

The uniqueness of the solution of the problem 1, can be obtained based on the completeness of the system $\{\sin k\pi x, k = 1, 2, \ldots \}$ in $L_2$. In fact, if we consider corresponding homogeneous problem, i.e. $\varphi(x) = 0$, from (18) we will get $U_k(t) \equiv 0$, which implies

$$\int_0^1 u(t, x) \sin k\pi x \, dx = 0, \quad 0 \leq t \leq T.$$ 

Due to the completeness of the system $\{\sin k\pi x, k = 1, 2, \ldots \}$ in $L_2$, we will get $u(t, x) \equiv 0$ in $\Omega$.

We proved the following theorem:

**Theorem 1.** If conditions (17) and (22) are valid, then the problem 1 is uniquely solvable and solution is represented by

$$u(t, x) = \sum_{k=1}^{\infty} \frac{t^{\alpha_1-1}E_{\alpha_1-\alpha_2, \alpha_1}(\alpha_1-\alpha_2, t^{\alpha_1}(-(k\pi)^2)^{\alpha_1} \varphi_k \sin k\pi x,}{T^{\alpha_1-1}}$$

$$g(x) = \frac{1}{\sum_{k=1}^{\infty} T^{\alpha_1-1}E_{\alpha_1-\alpha_2, \alpha_1}(\alpha_1-\alpha_2, t^{\alpha_1}(-(k\pi)^2)^{\alpha_1} \varphi_k \sin k\pi x.}$$
2 Direct problem.

Now let us consider the following direct problem.

**Problem 2.** To find a solution of the equation

\[ D^{\alpha_1, \beta_1}_{0t} u(t, x) + \mu D^{\alpha_2, \beta_2}_{0t} u(t, x) - u_{xx}(t, x) = \tilde{g}(t, x) \]  \hspace{1cm} (23)

from the class of functions

\[ W_2 = \{ u(t, x) : u \in C_{-1}[0, T], I^{(1-\beta_1)(1-\alpha_1)}_{0t} u \in C^1_1[0, T], \]
\[ u \in C[0, 1], u_{xx} \in C(0, 1) \}, \]

satisfying conditions (4) and (5).

Here \( \tilde{g}(t, x) \) is a given function.

Similarly as in the case of problem 1, we search solution in the form of

\[ u(t, x) = \sum_{k=1}^{\infty} \bar{U}_k(t) \sin k \pi x, \quad 0 \leq t \leq T. \]  \hspace{1cm} (24)

Substituting (24) into (23) we get

\[ D^{\alpha_1, \beta_1}_{0t} U_k(t) + \mu D^{\alpha_2, \beta_2}_{0t} U_k(t) + (k \pi)^2 U_k(t) = \bar{g}_k(t), \]  \hspace{1cm} (25)

where

\[ \bar{g}_k(t) = 2 \int_0^1 \tilde{g}(t, x) \sin k \pi x dx. \]

Solution of (25) satisfying initial condition

\[ \lim_{t \to +0} I^{(1-\beta_1)(1-\alpha_1)}_{0t} \bar{U}_k(t) = 0 \]

has a form

\[ U_k(t) = \int_0^t z^{\alpha_1 - 1} E_{(\alpha_1 - \alpha_2, \alpha_1)}(-\mu z^{\alpha_1 - \alpha_2}, -(k \pi)^2 z^{\alpha_1}) \bar{g}_k(t - z) dz. \]  \hspace{1cm} (26)

For the estimation of \( \bar{U}_k(t) \) we use two different estimation of the function (14). Precisely, first of them is

\[ |E_{(\alpha-\beta, \alpha)}(x, y)| \leq \frac{C_3}{1 + |x|} \]  \hspace{1cm} (27)

with \( C_3 = \text{const} > 0 \), which is proved in [9]. Another one is

\[ |E_{(\alpha-\beta, \alpha)}(x, y)| \leq \frac{C_4}{1 + |x + y|}, \]  \hspace{1cm} (28)

with \( C_4 = \text{constant} > 0 \), which has the following additional condition to the fractional orders (see [10], lemma 1.3)

\[ \Gamma(\rho + n(\alpha - \beta) + k\beta) > \Gamma(\rho + n(\alpha - \beta)), \quad n, k \in \mathbb{N}, \quad n \geq k. \]  \hspace{1cm} (29)
If we use estimation (27), we get

\[ |\bar{U}_k(t)| \leq \frac{C_5}{(k\pi)^2} \left| \bar{g}_k^{(2)}(t) \right|, \]

\[ \bar{g}_k^{(2)}(t) = -2 \int_0^1 \frac{\partial^2 \bar{g}(t, x)}{\partial x^2} \sin k\pi x dx. \]

As we mentioned in previous case, we need another estimation for the \( \bar{U}_k(t) \) in order to guarantee uniform convergence of infinite series corresponding to the function \( u_{xx}(t, x) \), namely

\[ |\bar{U}_k(t)| \leq \frac{C_6}{(k\pi)^3} \left| \bar{g}_k^{(3)}(t) \right|, \] (30)

\[ \bar{g}_k^{(3)}(t) = -2 \int_0^1 \frac{\partial^3 \bar{g}(t, x)}{\partial x^3} \cos k\pi x dx. \]

We impose the following conditions to the \( \bar{g}(t, x) \):

\[ \frac{\partial^2 \bar{g}(t, x)}{\partial x^2} \in C[0, 1], \quad \frac{\partial^3 \bar{g}(t, x)}{\partial x^3} \in L_2(0, 1), \]

\[ \bar{g}(t, 0) = \bar{g}(t, 1) = 0, \quad \left. \frac{\partial^2 \bar{g}(t, x)}{\partial x^2} \right|_{x=0} = \left. \frac{\partial^2 \bar{g}(t, x)}{\partial x^2} \right|_{x=1} = 0 \] (31)

in order to get

\[ |u_{xx}(t, x)| \leq C_7 + \left\| \bar{g}_k^{(3)}(t) \right\|, \] (32)

where \( C_7 \) is a positive constant.

Now, if we use estimation (28), we obtain

\[ |\bar{U}_k(t)| \leq \frac{C_8}{(k\pi)^3} \left| \bar{g}_k^{(1)}(t) \right|, \]

\[ \bar{g}_k^{(1)}(t) = 2 \int_0^1 \frac{\partial \bar{g}(t, x)}{\partial x} \cos k\pi x dx. \]

Consequently, in order to provide the uniform convergence of series

\[ u_{xx}(t, x) = \sum_{k=1}^{\infty} \bar{U}_k(t)(k\pi)^2 \sin k\pi x \] (33)

we impose the following condition to the given function \( \bar{g}(t, x) \):

\[ \frac{\partial \bar{g}(t, x)}{\partial x} \in C[0, 1], \quad \frac{\partial^2 \bar{g}(t, x)}{\partial x^2} \in L_2(0, 1), \quad \bar{g}(t, 0) = \bar{g}(t, 1) = 0, \] (34)

which yields

\[ |u_{xx}(t, x)| \leq C_9 + \left\| \bar{g}_k^{(2)}(t) \right\|. \]

Using Weierstrass M-test theorem, one can easily prove the uniform convergence of (33). The uniqueness of the solution for the problem 2 can be proved similarly to the proof of the problem 1.
Hence, we proved the following theorems:

**Theorem 2.** If condition (31) is valid, then problem 2 is uniquely solvable and solution is represented by

\[
    u(t, x) = \sum_{k=1}^{\infty} \sin k\pi x \int_0^t \frac{z^{\alpha_1-1}}{E(\alpha_1-\alpha_2, \alpha_1)} \left(-\mu z^{\alpha_1-\alpha_2}, -(k\pi)^2 z^{\alpha_1}\right) \hat{g}_k(t-z) \, dz. \tag{35}
\]

**Theorem 3.** If condition (29) and (34) are valid, then problem 2 is uniquely solvable and solution is represented by (35).

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