Lyra black holes

*F. Rahaman*, A. Ghosh* and M. Kalam**

Abstract

Long ago, since 1951, Lyra proposed a modification of Riemannian geometry. Based on the Lyra's modification on Riemannian geometry, Sen and Dunn constructed a field equation which is analogous to Einstein's field equations. Furthermore, Sen and Dunn gave series type solution to the static vacuum field equations. Retaining only a few terms, we have shown that their solutions correspond to black holes (we call, Lyra black holes). Some interesting properties of the Lyra black holes are studied.

1. INTRODUCTION

To unify gravity with other fundamental forces remains an elusive goal for theoretical physicists. Einstein developed general theory of relativity, in which gravitation is described in terms of Riemannian geometry. Soon after his discovery, it has been realized that only Riemannian geometry could not help to unify gravitation and electromagnetism in a single space time geometry. For that reason, several modifications of Riemannian geometry have been proposed time to time. At first, Weyl [1] persuaded a modification of Riemannian geometry to unify gravity with electromagnetic field. But this theory was not accepted as it was based on the non-integrability condition of a vector under parallel transport. Three decades later, in 1951, Lyra [2], proposed a modification of Riemannian geometry by introducing a gauge or scale function which removes the non-integrability condition of a vector under parallel transport. This modification of Riemannian geometry is known as Lyra's geometry.

---

0 Pacs Nos : 04.20.Gz; 04.50.+h
Key words and phrases: Lyra geometry, Black holes

*Dept. of Mathematics, Jadavpur University, Kolkata-700 032, India
E-Mail:farook_rahaman@yahoo.com

** Dept. of Phys., Netaji Nagar College for Women, Regent Estate, Kolkata-700092, India
Sen and Dunn [3] proposed a new scalar tensor theory of gravitation and constructed the field equations analogous to Einstein’s field equations based on Lyra’s geometry as

\[ R^{ik} - \frac{1}{2} g^{ik} R - \frac{3}{2} (x^0)^2 x^{0,i} x^{0,k} + \frac{3}{4} (x^0)^2 g^{ik} x^{0,j} x^0_j = -8\pi G (x^0)^{-2} T^{ik} \]  

(1)

Here the scalar field is characterized by the function \( x^0 = x^0(x^i) \), where \( x^i \) are coordinates in the four dimensional Lyra manifold and other symbols have their usual meaning as in Riemannian geometry.

Furthermore, Sen and Dunn [3] gave a series type solutions to the static vacuum field equations. Jeavons et al [4] have pointed out that the original field equation proposed by Sen and Dunn may still prove to be heuristically useful even though they are not derivable from the usual variational principle. Several authors have applied this alternative theory to study cosmological models [5], topological defects [6] and various other applications. Recently, Casana et al [7] have studied Dirac field, Scalar and vector Massive fields and Massless DKP field in Lyra geometry. In this article, we shall focus the solutions obtained by Sen and Dunn [3] and try to improve the status of the solutions. Retaining only a few terms in their solutions, we shall show that their solutions correspond to black holes. We shall also study some interesting properties of the Lyra black holes subsequently.

Since our target is to provide a proper understanding about how standard Schwarzschild solution gets modified through the introduction of gauge function. Implications are obtained through the study of geodesic equation in such spacetimes and the test particles and light rays follow geodesics of the geometry. In the vacuum case, equation (1) is algebraically identical to the conformally mapped Einstein equations [4]. Following the arguments of Bhadra et al [8] and Kar et al [9], one could exploit the Einstein frame results without going to all detailed calculations starting from the beginning.

The layout of the paper is as follows: In section 2, the reader is reminded about the vacuum solution obtained by Sen and Dunn. In section 3, some properties of the solutions is described. In section 4, we study the motion of test particles whereas in section 5, we study the gravitational lensing due to Lyra black hole. The paper ends with a short discussion.
2. LYRA BLACK HOLES:

The matter free region surrounding a massive spherically symmetric body has the static spherically symmetric metric structure as

\[ ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \] (2)

Sen and Dunn has given a series solution to the field equation (1) for the metric (2) as

\[ e^\nu = D + C\phi(r) \] (3)

\[ e^\lambda = \frac{Ar^4(\phi')^2}{D + C\phi(r)} \] (4)

\[ \phi = \Sigma_{r=0}^{\infty}a_n r^{-n} \] (5)

D,C,A are arbitrary constants.

The coefficients \( a_n \) are given by \( a_0 \), arbitrary, \( Aa_1^2 = D + Ca_0 \), \( a_2 = 0 \), \( a_3 \) arbitrary, \( a_n \), \( n > 3 \), are determined by the following recurrence relation

\[
\begin{align*}
a_{n-1}[(D + Ca_0)(n - 1)(n - 4)] - Aa_1 \Sigma_{k=3}^{n-1}(k - 1)(n - k + 1)a_{k-1}a_{n-k+1} \\
- A \Sigma_{l=3}^{n-1}[(l - 1)a_{l-1}] \Sigma_{k=2}^{n-l+2}(k - 1)(n - l - k + 3)a_{k-1}a_{n-l-k+3} \\
- \Sigma_{l=2}^{n-1}a_{n-l}a_{l-1}(n - l)(2l - n - 1) = 0
\end{align*}
\]

Also

\[ x^0 = k.exp\int[-(\frac{8}{3r^2} + \frac{4}{3r^2} \phi^0 \phi')^\frac{1}{2}]dr \] (6)

where \( k \) is a constant.
Retaining only a few terms, we write equations (3), (4) and (7) as

\[ e^\nu = C \left( b_0 + \frac{a_1}{r} + \frac{a_3}{r^3} + \frac{a_4}{r^4} \right) \]  

(7)

\[ e^\lambda = \frac{C}{e^\nu} \left( a_1^2 + \frac{6a_1a_3}{r^2} + \frac{8a_1a_4}{r^4} + \frac{9a_2^2}{r^6} \right) \]  

(8)

\[
(x^0)^{-1} \frac{dx_0}{dr} = 2C_0 [r^{-2} + \frac{a_4}{a_3r^3} - \frac{a_1a_2^2 + 3a_3^3}{2a_1a_3^2r^4}]
\]  

(9)

where \( b_0C = D + Da_0 \) and \( C_0^2 = \frac{2a_3}{a_1} \).

If we impose the usual boundary conditions at infinity i.e. \( e^\nu \) and \( e^\lambda \) tend to 1 as \( r \to \infty \), then one gets,

\[ D + Ca_0 = 1 \]  

(10)

and

\[ D + Ca_0 = Aa_1^2 \]  

(11)

These imply

\[Cb_0 = 1 \]  

(12)

and

\[ a_1^2 = \frac{1}{A} \]  

(13)

i.e.

\[ a_1 = \pm \frac{1}{\sqrt{A}} \]  

(14)
For vanishing scale function \( i.e. x_0 = 0 \), these solutions reduce exactly to the Schwarzschild solution. Thus when \( a_3 = 0 \ i.e. a_n = 0 \) for \( n > 1 \), one gets

\[
e^{\nu} = b_0 C \pm \frac{C}{\sqrt{A}r} = 1 \pm \frac{C}{\sqrt{A}r} \quad (15)
\]

\[
e^{\lambda} = \frac{Aa_3^2}{b_0 C + \frac{a_1 C}{r}} = \frac{1}{1 \pm \frac{C}{\sqrt{A}r}} \quad (16)
\]

Since equations (15) and (16) represent Schwarzschild black hole solution, one should take negative sign and \( \frac{C}{\sqrt{A}} = 2M' = M \) (say), ( \( M' \), mass of the black hole ).

Thus one can write the solution (3) and (4) as

\[
e^{\nu} = 1 - \frac{M}{r} + \frac{M\sqrt{A}a_3}{r^3} + \frac{M^2\sqrt{A}a_3}{r^4} \quad (17)
\]

\[
e^{\lambda} = \frac{(1 - \frac{6\sqrt{A}a_3}{r^2} - \frac{8M\sqrt{A}a_3}{r^3} + \frac{9a_3^2A}{r^4})}{1 - \frac{M}{r} + \frac{M\sqrt{A}a_3}{r^3} + \frac{M^2\sqrt{A}a_3}{r^4}} \quad (18)
\]

These solutions represent black holes and we call it, Lyra black holes.
3. Properties of the Lyra black holes solutions:

At the horizon, \( e^\nu = 0 \) i.e.

\[
r^4 - Mr^3 + pMr + pM^2 = 0 \tag{19}
\]

\[ p = \sqrt{Aa_3} \]

Since, here two variation of signs, by Descartes Rule of Sign, it has either two positive roots or no positive root.

![Figure 1: \( e^\nu \) has two zeros ( positive ), for \( p = 1 \), \( M = 10 \)](image)

The general solutions of equation (19) is (see appendix-1 for details calculations)

\[
r = r_{\pm} \tag{20}
\]

\( r = r_{\pm} \) correspond to two horizons.

\[ r_+ \text{ outer horizon and } r_- \text{ inner horizon} \]

The kretschmann scalar

\[
K = \frac{[(\Delta^2)']^2}{\sigma^4} - (\Delta^2)'(\Delta^2)'(\sigma^2)'\frac{(\sigma^2)'}{4\sigma^8} + \frac{4}{r^2}(\frac{[(\Delta^2)']^2}{\sigma^4} - (\Delta^2)(\Delta^2)'(\sigma^2)'\frac{(\sigma^2)'}{2\sigma^8} + (\Delta^4)\frac{[(\sigma^2)']^2}{2\sigma^8}) + \frac{4}{r^4}[1 - \frac{\Delta^2}{\sigma^2}]^2 \tag{21}
\]

\[ e^\nu = \Delta^2 = 1 - \frac{M}{r} + \frac{M\sqrt{Aa_3}}{r^3} + \frac{M^2\sqrt{Aa_3}}{r^4} \text{ and } e^\lambda = \frac{\sigma^2}{\Delta^2}, \]

where \( \sigma^2 = 1 - \frac{6\sqrt{Aa_3}}{r^2} - \frac{8M\sqrt{Aa_3}}{r^4} + \frac{9a_3^2A}{r^6} \]

is finite at \( r_\pm \) and is divergent at \( r = 0 \), indicating that \( r_+ \) and \( r_- \) are regular horizons and the singularity locates at \( r = 0 \).
Now we consider the case when the equation (19) has only one positive root. For

\[ 72p^3 + 63p^2 M + \sqrt{144p^2 + 204pM^2} (6p^2 + pM^2) = 9pM^4 \]  

(22)

there is only one positive root of \( e^\nu = 0 \) at \( r = r_0 \), where

\[ r_0 = \frac{12p + \sqrt{144p^2 + 204pM^2}}{6M} \]  

(23)

(see appendix-2 for details calculations)

Two horizons \( r_+ \) and \( r_- \) match to form a regular event horizon while \( r = 0 \) is still a singularity in this case.

Now we calculate the entropy \( S \) and Hawking temperature \( T_H \) of the Lyra black holes:

\[ S = \pi r_{\text{horizon}}^2 \]  

(24)

\[ T_H = \frac{1}{\sqrt{-g_{tt}g_{rr}}} \frac{d}{dr} (-g_{tt}) \bigg|_{r=r_{\text{horizon}}} = \frac{\left(-\frac{M}{r_{\text{horizon}}^2} + \frac{3pM}{r_{\text{horizon}}^3} + \frac{4pM^2}{r_{\text{horizon}}^4}\right)}{\sqrt{1 - \frac{6p}{r_{\text{horizon}}^2} - \frac{8pM}{r_{\text{horizon}}^3} + \frac{9p^2}{r_{\text{horizon}}^4}}} \]  

(25)

where \( r_{\text{horizon}} \) has been given in equation (20) or in (23).

4. Motion of test particle:

Let us consider a test particle having mass \( m_0 \) moving in the gravitational field of the Lyra black hole described by the metric ansatz (2).

So the Hamilton-Jacobi [ H-J ] equation for the test particle is [10]

\[ g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} + m_0^2 = 0 \]  

(26)

where \( g_{ik} \) are the classical background field (2) and \( S \) is the standard Hamilton’s characteristic function.
For the metric (2) the explicit form of H-J equation (26) is [10]

$$e^{-\nu}(\partial S/\partial t)^2 - e^{-\lambda}(\partial S/\partial r)^2 - \frac{1}{r^2}(\partial S/\partial \theta)^2 - \frac{1}{r^2 \sin^2(\partial S/\partial \varphi)} + m_0^2 = 0$$  \hspace{1cm} (27)$$

where $e^\nu$ and $e^\lambda$ are given in equations (17) and (18).

In order to solve this partial differential equation, let us choose the $H - J$ function $S$ as [11]

$$S = -E.t + S_1(r) + S_2(\theta) + J.\varphi$$  \hspace{1cm} (28)$$

where $E$ is identified as the energy of the particle and $J$ is the momentum of the particle. The radial velocity of the particle is

( for detailed calculations, see ref.[11])

$$\frac{dr}{dt} = \frac{e^\nu}{E\sqrt{e^\lambda}} \sqrt{\frac{E^2}{e^\nu} + m_0^2 - \frac{p_0^2}{r^2}}$$  \hspace{1cm} (29)$$

where $p_0$ is the separation constant.

The turning points of the trajectory are given by $\frac{dr}{dt} = 0$ and as a consequence the potential curve are

$$\frac{E}{m_0} = \sqrt{e^\nu\left(\frac{p_0^2}{m_0^2r^2} - 1\right)} \equiv V(r)$$  \hspace{1cm} (30)$$

In a stationary system $E$ i.e. $V(r)$ must have an extremal value. Hence the value of $r$ for which energy attains it extremal value is given by the equation

$$\frac{dV}{dr} = 0$$  \hspace{1cm} (31)$$

Hence we get the following equation as

$$M^5 - \frac{2p^2r^4}{m_0^2} + \left(\frac{p_0^2M}{m_0^2} + \frac{2p^2M}{m_0^2} - 3Mp\right)r^3 - 4pM^2r^2 - \left(\frac{3p_0^2Mp}{m_0^2} + \frac{2p^3M}{m_0^2}\right)r - \left(\frac{2p^3M^2}{m_0^2} + \frac{4p_0^2M^2}{m_0^2}\right) = 0$$  \hspace{1cm} (32)$$

This is an algebraic equation of degree five with negative last term , so this equation must have at least one positive root. So the bound orbits are possible for the test particle i.e. particle can be trapped by Lyra black hole. In other words, Lyra black hole always exerts attractive gravitational force on the surrounding matter.
5. Geodesic and Lensing in Lyra background:

We shall next consider the geodesic equations for the spherically symmetric metric (2), which are given by[12]

\[ \dot{r}^2 \equiv \left( \frac{dr}{d\tau} \right)^2 = e^{-\lambda}[e^{-\nu}E^2 - J^2/r^2 - L] \] (33)

\[ \dot{\phi} \equiv \frac{d\phi}{d\tau} = \frac{J}{r^2} \] (34)

\[ \dot{t} \equiv \frac{dt}{d\tau} = Ee^{-\nu} \] (35)

where the motion is considered in the \( \theta = \frac{\pi}{2} \) plane and constants \( E \) and \( J \) are identified as the energy per unit mass and angular momentum respectively about an axis perpendicular to the invariant plane \( \theta = \frac{\pi}{2} \). Here \( \tau \) is the affine parameter and \( L \) is the Lagrangian having values 0 and 1 respectively for null and time like particles. The metric coefficients \( e^\nu \) and \( e^\lambda \) given in equations (17) and (18) yield the equation for radial geodesic ( \( J = 0 \)):

\[ \dot{r}^2 \equiv \left( \frac{dr}{dt} \right)^2 = \frac{(1 - \frac{M}{r} + \frac{Mp}{r^2} + \frac{pM^2}{r^3})^2}{1 - \frac{6p}{r^2} - \frac{8Mp}{r^3} + \frac{9p^2}{r^4}} \] (36)

This gives,

\[ \pm t = r + M \ln r + \frac{4pM}{r^2} + \frac{3p - \frac{M^2}{2}}{r^2} + \frac{5pM^2}{2r^3} + \frac{3pM(M^2 - P)}{r^4} \] (37)

[ retaining terms upto \( \frac{1}{r^4} \) ]

Also, one can find the \( \tau - r \) relationship as

\[ \pm E\tau = r + \frac{3p}{r^2} + \frac{8pM}{r^2} - \frac{3Mp^2}{r^4} \] (38)

[ retaining terms upto \( \frac{1}{r^4} \) ]

Here the features are in sharp contrast with what happens in a Schwarzschild space time.
Figure 2: $t-r$ relation, for $p = 1, M = 10$

Figure 3: $\tau - r$ relation, for $p = 1, M = 10, E = 1$
Lensing (Bending of light rays):

For photons, the trajectory equations (33) and (34) yield

\[
\left(\frac{dU}{d\phi}\right)^2 = e^{-\lambda-\nu}a^2 - \frac{U^2}{e^\lambda}
\]  

(39)

where \( U = \frac{1}{r} \) and \( a^2 = \frac{E^2}{M^2} \).

Equation (39) gives,

\[
\int \frac{dU}{[a + (3ap - \frac{1}{2a})U^2 + (4apM + \frac{M}{2a})U^3]} = \phi
\]  

(40)

[ neglecting \( U^4 \) and higher terms ]

This integral can not be solved analytically. But, if we assume \( 3ap = \frac{1}{2a} \), then one can easily solve to yield

\[
\phi = \frac{1}{4aMp + \frac{M}{2a}} \left[ \frac{1}{6A^2} \ln \frac{(U + A)^2}{U^2 - AU + A^2} + \frac{1}{A\sqrt{3}} \arctan \frac{2U - A}{A\sqrt{3}} \right]
\]  

(41)

where \( A^3 = \frac{a}{4aMp + \frac{M}{2a}} \).

For, \( U = 0 \), one gets,

\[
\phi = \frac{1}{4aMp + \frac{M}{2a}} \left[ \frac{5\pi}{6A\sqrt{3}} \right]
\]  

(42)

and bending comes out as

\[
\Delta \phi = \pi - 2\phi = \pi \left[ 1 - \frac{\sqrt{50}(7Mp)^{\frac{1}{3}}}{42M\sqrt{p}} \right]
\]  

(43)
5. DISCUSSIONS:

Rather discovering a new black hole solution, we have highlighted vacuum spherically symmetric solution within the framework of Lyra geometry obtained by Sen and Dunn and have shown that these solutions correspond to black holes, which have been overlooked in the previous study. Similar to Schwarzschild black hole, this solution is characterized by mass but nature of the solutions are sharply contrast to the Schwarzschild solutions. The Lyra black hole has two horizons but under certain condition (see equation (22)), two horizons coincide. We have shown that Lyra black hole always exerts attractive gravitational force on the surrounding matter. We have subsequently studied geodesics and gravitational lensing in these space times. Thus in this work, we focus new black hole solutions within the framework of Lyra geometry. We have studied some phenomenological consequences of this black hole so that where there is any hope of testing this in astrophysical observations. It would be interesting to study the thermodynamic and stability properties of this black hole. Work in this direction is in progress and could be noted else where.

Acknowledgements

F.R is thankful to Jadavpur University and DST, Government of India for providing financial support under Potential Excellence and Young Scientist scheme. MK has been partially supported by UGC, Government of India under Minor Research Project scheme. We are thankful to the anonymous referee for his several valuable comments and constructive suggestions.
References

[1] H Weyl, Sitzber.Preuss.Akad. Wiss. 465 (1918). Reprinted (English version) in L O’Raifeartaigh, The Dawning of Gauge Theory, Princeton Series in Physics, Princeton (1997).

[2] Lyra, G, Math Z 54,52 (1951).

[3] Sen D. K and Dunn K. A, J. Math. Phys 12, 578 (1971).

[4] J S Jeavons et al, J. Math. Phys. 16, 320 (1975)

[5] Bharma K. S, Aust. J. Phys. 27, 541 (1974); Karadi T.M and Borikar S.M, Gen Rel. Grav. 1, 431 (1978); A Beesham, Astrophys.Space Sci. 127, 189 (1986); Matyjasek J, Astrophys.Space Sci. 207,313 (1993); T. Singh and G.P. Singh, J. Math. Phys. 32, 2456 (1991); G.P. Singh and K. Desikan, Pramana 49, 205 (1997);
F. Rahaman, J.K. Bera, Int.J.Mod.Phys.D10,729(2001);
F.Rahaman et al, Astrophys.Space Sci.288,483(2003);
F.Rahaman et al, Astrophys.Space Sci.294,219(2005).

[6] F.Rahaman, Int.J.Mod.Phys.D9,775(2000);
F.Rahaman, Int.J.Mod.Phys.D10, 579(2001);
F.Rahaman, Astrophys.Space Sci.283,151(2003);
F.Rahaman et al, Int.J.Mod.Phys.D10,735(2001);
F. Rahaman, P. Ghosh, Fizika B13,719 (2004);
F. Rahaman, R. Mukherji, Astrophys.Space Sci.288, 493 (2003)

[7] R Casana, C A M de Melo and B Pimentel, gr-qc/0509096; gr-qc/0509117; hep-th/0501085

[8] A Bhadra et al , Class.Quan.Grav.23, 6101 (2006)

[9] S Kar et al , Phys.Rev.D 67, 044005(2003)

[10] Landau L and Lifschitz E, Classical theory of fields, Pergamon Press, Oxford (1975).

[11] S. Chakraborty, Gen. Rel. Grav. 28, 1115(1996);
    S. Chakraborty, F. Rahaman, Pramana 51,689(1998)

[12] Weinberg S, Gravitation and Cosmology (2005), John Wiley and Sons (Asia) Pvt. Ltd, Singapore.
Appendix-1:

At the horizon, $e^\nu = 0$ i.e.

$$r^4 - Mr^3 + pMr + pM^2 = 0$$

One can write the above equation as

$$(r^2 - ar + b)^2 - (lr + m)^2 \equiv r^4 - Mr^3 + pMr + pM^2$$

The solutions are given by

$$r = \frac{a \mp l \pm \sqrt{(a \mp l)^2 - 4(b \pm m)}}{2}$$

where $a, l, m, b$ are given by (comparing with the original equation)

$$a = \frac{M^2}{2}, \quad l = \frac{-bM - pM}{2\sqrt{b^2 - pM^2}}, \quad m^2 = b^2 - pM^2$$

and $b$ satisfies the following equation

$$b^3 - \frac{5}{4}pM^2b - \frac{1}{8}(pM^4 + p^2M^2) = 0$$

Using Cardan method, one can easily solve the above equation to yield 'b' and consequently $a, m$ and $l$ would be determined.

Finally, one could find $r = r_{\text{horizon}}$ at the points

$$\frac{M^4}{4} \left[1 \pm \frac{(p + [A + B]^\frac{1}{3} + [A - B]^\frac{1}{3})}{\sqrt{([A + B]^\frac{1}{3} + [A - B]^\frac{1}{3})^2 - pM^2}}\right]$$

$$\pm \frac{M^2}{4} \left[1 \pm \frac{(p + [A + B]^\frac{1}{3} + [A - B]^\frac{1}{3})}{\sqrt{([A + B]^\frac{1}{3} + [A - B]^\frac{1}{3})^2 - pM^2}}\right] - 4[A + B]^\frac{1}{3} + [A - B]^\frac{1}{3} \pm \sqrt{([A + B]^\frac{1}{3} + [A - B]^\frac{1}{3})^2 - pM^2}$$

where $A = \frac{1}{16}(pM^4 + p^2M^4)$ and $B = \frac{1}{16} \sqrt{\frac{1}{16}(pM^4 + p^2M^4)^2 - \frac{125}{256}p^3M^6}$

Here $r_\pm$ correspond to two positive roots (two horizons).
Appendix-2:

Conditions for double roots of an equation \( f(r) = 0 \) are \( f(r_0) = 0 \) and \( f'(r_0) = 0 \).

Now \( f(r_0) = 0 \) and \( f'(r_0) = 0 \) imply

\[
r^4 - Mr^3 + pMr + pM^2 = 0
\]

\[
4r^3 - 3Mr^2 + pM = 0
\]

From the above equations, one gets

\[
-r^3 + 3pr + 4pM = 0
\]

The last two equations give

\[
3Mr^2 - 12pr - 17pM = 0
\]

Solving the above equation to yield

\[
r = \frac{12 \pm \sqrt{144p^2 + 204pM^2}}{6M}
\]

Putting the value of 'r' in \( f'(r) = 0 \), we get the required condition as (taking positive sign)

\[
72p^3 + 63p^2M^2 - 9pM^4 + \sqrt{144p^2 + 204pM^2}(6p^2 + pM^2) = 0
\]