Mean field exact solutions showing charge density wave crossover at low fillings in the fractional quantum Hall regime

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A general analytical framework for the determination of the mean field states at arbitrary rational filling factors for the 2DEG in FQHE regime is given. Its use allows to obtain analytic expressions for the solutions at filling factors of the form \( \nu = 1/q \) for arbitrary odd \( q \). The analysis can be performed for two general classes of states characterized by \( \gamma = 1 \) or \( \gamma = 1/2 \) particles per unit cell. Instead of the periodic peaks of the Wigner solid solution, the new states show electron densities forming percolating ridges that may favor an energy decrease through correlated ring of exchange contributions. Therefore, we estimate that they can realize mean field versions of the so called Hall Crystal (HC) states. The obtained analytic HC solution shows the same crystalline symmetry that the corresponding WC state in its class \( \gamma = 1 \), but a qualitatively different charge density distribution. The energy dependence of the corresponding HC and WC states on the filling factor is also evaluated here for the class \( \gamma = 1/2 \). The results show a crossover between HC state and the Wigner crystal, close to filling 1/7. Therefore, transitions may occur from one to the other as the electron density is varied. This result is consistent with recent experimental findings.

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I. INTRODUCTION

The quantum Hall effect has been the source of continued interest for already a quarter of a century. As the samples have become cleaner and the temperature made lower unexpected structures have been seen, suggesting a richer physics than originally thought. Although the system was believed to simply transit from a liquid state to a Wigner Crystal as the electron density is decreased, recent observations show that new structures develop that may signal transitions between states of different symmetry. Numerical calculations also find a variety of states among which a system may transit, notably at Landau levels higher than the first.

The formation of charge density wave states in the thermodynamic limit may be studied using the Hartree-Fock approximation. The formalism allows for the consideration of states of various symmetries, unidirectional waves, triangular or square lattices, etc. The criterion normally used to choose among them is to select the state of lowest energy. Yet none of such approximate states is the true ground state, so the effect of correlations is a consideration to have in mind. In particular, depending on the distribution and relative phases of the occupied single particle states the correlations may be enhanced or suppressed. Since experiments show cases in which two different phases appear it is worthwhile to explore the various symmetries allowing for self-consistent solutions.

In this work we generalize the formalism given in reference for the particular \( \nu=1/3 \) filling, to perform an analytic treatment of the Hartree-Fock problem in the lowest Landau level (LLL) at arbitrary filling. We also will discuss in more detail particular solutions at filling factors \( \nu = 1/q \) for arbitrary odd \( q \). For each value of \( q \) two classes of states are considered and indexed by the number of particles per unit cell \( \gamma = 1 \) and \( \gamma = 1/2 \). For each one of these two groups, two types of mean field states are associated. One in which the electrons build up well localized gaussian-like peaks: the Wigner Crystal (WC),and another in which the charge density forms hexagonal ridges increasingly resembling a beehive as the electron density is decreased: the Hall crystal(HC) state. Therefore, the low density limit of this new state differs qualitatively from the one in the Wigner Crystal states. One can then expect that correlations arising from ring exchange will affect more this state, since percolating paths are open throughout the structure. For the particular \( \gamma = 1/2 \) class, we find a crossover between the WC and HC state, showing that while the Wigner Crystal (WC) in the class has lower energy at fillings below 1/7, the new HC state however, has lower energy at larger fillings. Thus, even within mean field theory both states are competing as precursors to the ground state.

Before ending this section, let us comment on the physical relevance of the parameter \( \gamma \), the number of electron per unit cell. Its importance is related in one sense, with the fact that it determines the symmetry of the mean filed solutions. The value \( \gamma = 1 \) is associated to the Yoshioka-Lee(YL) WC state which in the Hartree-Fock framework shows the lowest energy pere particle. Moreover, within the same class \( \gamma = 1 \) there is a HC state which shows a bee
where \( A \) is fractional or integer. Denoting by \( T \) the constant. The operator \( n \) being a unit vector normal to the plane containing the electrons. Assuming triangular symmetry, the lattice formed by the electrons is then invariant under translations in the set of vectors

\[
R = n_1 a_1 + n_2 a_2, \quad n_1, n_2 = 0, \pm 1, \pm 2, \ldots;
\]

\[
a_1 = a (1, 0),
\]

\[
a_2 = a \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad a = \sqrt{\frac{4\pi\phi}{\sqrt{3} r_o}}.
\]

Appendix A. The Fourier components of the periodic charge density verify the simple relation

\[
\phi \nu = \gamma. \quad (1)
\]

The Fock operator in the LLL may then be written in the form

\[
H_{HF} = \sum_Q v(Q) \exp(-\frac{r_o^2 Q^2}{4}) T_{r_2 n \times Q}, \quad (2)
\]

where

\[
v(Q) = 2\pi r_o^2 \rho(Q) \exp\left(\frac{r_o^2 Q^2}{4}\right) \left( 1 - \delta_{Q,0} \right) \exp(-\frac{r_o^2 Q^2}{4}) - \sqrt{\frac{\pi}{2}} I_0(\frac{r_o^2 Q^2}{4}) \epsilon_o r_o. \quad (3)
\]

Here \( r_o = \sqrt{\frac{\hbar c}{eB}} \) is the magnetic length, \( I_0(u) \) a modified Bessel function and \( \epsilon_o \) the background dielectric constant. The operator \( T_a \) displaces the function it acts upon in \( -a \) adding a magnetic phase factor, as defined in Appendix A. The Fourier components of the periodic charge density \( \rho(x) \) are defined as usual,

\[
\rho(Q) = \frac{1}{A_{cell}} \int dx \, \rho(x) \exp(iQ \cdot x), \quad (4)
\]

where \( A_{cell} \) is the unit cell area

\[
A_{cell} = n a_1 \times a_2 = 2\pi r_o^2 \phi, \quad (5)
\]

\( n \) being a unit vector normal to the plane containing the electrons. Assuming triangular symmetry, the lattice formed by the electrons is then invariant under translations in the set of vectors

\[
R = n_1 a_1 + n_2 a_2, \quad n_1, n_2 = 0, \pm 1, \pm 2, \ldots;
\]

\[
a_1 = a (1, 0),
\]

\[
a_2 = a \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad a = \sqrt{\frac{4\pi\phi}{\sqrt{3} r_o}}.
\]
The vectors $Q$ in Eq. (4) span all points in the reciprocal lattice and are given by

$$Q = Q_1 s_1 + Q_2 s_2$$

$$Q_1, Q_2 = 0, \pm 1, \pm 2, \ldots$$

$$s_1 = -\frac{1}{\phi^2_o} n \times a_2,$$

$$s_2 = \frac{1}{\phi^2_o} n \times a_1,$$

$$s_i a_j = 2\pi \delta_{ij}. \tag{9}$$

The Fourier components of the density obey the sum rule

$$\sum_{Q} |2\pi r_o^2 \rho(Q)|^2 \exp\left(\frac{r_o^2 Q^2}{2}\right) = \nu(1 - \nu). \tag{10}$$

where the term $Q = 0$ is omitted from the sum. This relation states that within the mean field approximation the liquid state of uniform density ($\rho(Q) = 0$ all finite $Q$) in the LLL is only possible at filling one. At fractional fillings the right hand side is finite and so must be at least one finite wave-vector Fourier component of the charge density.

III. BLOCK DIAGONALIZATION OF THE FOCK OPERATOR

The mean field hamiltonian (2) describes an electron in a periodic potential and a perpendicular magnetic field, a case for which many results are known. An important property is that the single particle spectrum in the LLL is arranged in non-overlapping bands, each with the same number of states. The number of bands equals the numerator of the flux per cell, the latter assumed a rational. For filling $\nu = p/q$, with $p, q$ prime to each other, Eq.(1) yields a flux $\phi = q/p$ per plaquette. Assuming $\gamma$ to be a rational this flux is then also a rational number.

We consider in what follows two simple cases that illustrate how different values of the flux $\phi$ are to be treated. One is when this number is an integer, and another when it is half an integer. For simplicity we set $\gamma = 1, 1/2$ and $\nu = 1/q$, so that $\phi = q, q/2$, respectively. Our results cover the more general case $\gamma = p, p/2$ and $\nu = p/q$, giving rise to the same values of the flux we include in the following discussion. Other cases may be treated using similar methods to the ones described below.

A. Integer flux quanta per unit cell

This case is important since it corresponds to the standard WC, in which each unit cell captures a full electron charge. The flux traversing a plaquette is $q$ so that the single electron spectrum will have $q$ bands, one of which is completely filled and the others empty. Since these bands do not overlap the WC state has thus a gap for all values of $q$, whether even or odd. Because an essential feature of experiment at not too low filling fractions is the different behavior at even and odd values of such quantity, the WC state is not a good candidate for being the mean field precursor to the true ground state.

Owing to definition (9) the magnetic translations entering the Fock operator have the form

$$T_{r}^n \times Q = T_{-\frac{q}{q} n} Q_1 s_1 + T_{-\frac{q}{q} n} Q_2 s_2. \tag{11}$$

Since the flux piercing the unit cell is an integral number $q$ of flux quanta, the set of translation operators $T_R$ for all the $R$ defined in (6) commute among themselves, allowing to find common eigenfunctions to all of them. This is not the case for the translations (11) since the original unit cell is partitioned in smaller sectors if $q$ is greater than 1. The basis we shall construct defines a set of $q$-dimensional subspaces, which are closed under the action of translations (11) for all values of $Q$.

A first step in finding the basis is to define a set of eigenfunctions $\chi_k(x)$ of a translation in the vector $-a_1/q$ for each value of the momentum $p = \hbar k$. Expressed as linear combinations of the functions $\varphi_k(x)$ defined in Appendix A, we write them in the form

$$\chi_k(x) = \sum_{s = -\frac{q-1}{2}}^{\frac{q-1}{2}} c_s(k) T_{-\frac{s}{q} a_1} \varphi_k(x), \tag{12}$$
where, for definiteness, we have assumed $q$ to be odd. These functions must obey the condition

$$T_{-\frac{q}{2}a_1} \chi_k(x) = \lambda \chi_k(x).$$

One finds for the eigenvalues $\lambda$ and coefficients $c_s$ the set of $q$ solutions

$$\lambda^{(r)}(k) = \exp(i \frac{k a_1}{q} + \frac{2\pi r}{q})$$

$$c_s^{(r)}(k) = \frac{1}{\sqrt{q}} \exp(-i \frac{s k a_1}{q} - i \frac{2\pi r s}{q})$$

$$r = -\frac{q-1}{2}, \ldots, \frac{q-1}{2}.$$

Substituting in (12) yields the $q$ eigenfunctions

$$\chi_{k}^{(r)}(x) = \frac{1}{\sqrt{q}} \sum_{s=-\frac{q+1}{2}}^{\frac{q-1}{2}} \exp(-i \frac{s k a_1}{q} - i \frac{2\pi r s}{q}) T_{-\frac{q}{2}a_1} \varphi_k(x),$$

$$k \equiv k + n s_1 + m s_2, \quad n, m = 0, \pm 1, \pm 2, \ldots$$

The last relation expresses the fact that the states in the new basis are equivalent upon a shift of $k$ in any linear combination with integer coefficients, of the unit cell vectors of the reciprocal lattice corresponding to the periodicity of the density. The equivalence follows from the following properties: (a) the functions $\varphi_k(x)$ are eigenfunctions of any translation $T_R$ for lattice vectors $R$ given by (6), (b) the operator $T_R$ commutes with all translations entering in the definition of $\chi_{k}^{(r)}(x)$, and (c) relation (A8) in Appendix A, stating the equivalence between magnetic translations acting on $\varphi_k(x)$ and a shift in the momentum labelling these functions.

Let us now inspect the effect of a magnetic translation in $a_2/q$ on the new functions. If such a transformation leaves the $q$-plets invariant, then the matrix reduction of the Hartree-Fock Hamiltonian will follow. One has,

$$T_{\frac{1}{2}a_2} \chi_{k}^{(r)}(x) = \frac{1}{\sqrt{q}} \sum_{s=-\frac{q+1}{2}}^{\frac{q-1}{2}} \exp(-i \frac{s k a_2}{q} - i \frac{2\pi r s}{q}) T_{\frac{1}{2}a_2} T_{-\frac{q}{2}a_1} \varphi_k(x).$$

After using (A4) for changing the order of the two operators within the sum, it follows that

$$T_{\frac{1}{2}a_2} \chi_{k}^{(r)}(x) = \exp(-i \frac{k a_2}{q}) \chi_{k}^{(|r-1|)}(x),$$

where the square bracket defines the number in the set $\{-(q-1)/2, \ldots, (q-1)/2\}$ which is equivalent, modulo $q$, to the integer in the argument. Thus, a magnetic translation in $\frac{1}{2}a_2$ just turns one function in the $q$-plet into another. Besides the properties already discussed, the basis can be checked to obey

$$T_{a_1} \chi_{k}^{(r)}(x) = \exp(-i \frac{k a_1}{q}) \chi_{k}^{(r)}(x),$$

$$T_{a_2} \chi_{k}^{(r)}(x) = \exp(-i \frac{k a_2}{q}) \chi_{k}^{(r)}(x),$$

$$P \chi_{k}^{(r)}(x) = \chi_{-k}^{(-r)}(x),$$

where the parity transformation $P$ is defined as usual, $P \chi_{k}^{(r)}(x) = \chi_{k}^{(r)}(-x)$.

From the above considerations it follows that the $q$-dimensional subspace spanned by the functions $\chi_{k}^{(r)}(x)$ at fixed values of $k$ is left invariant by the action of the operators (11) for arbitrary values of $Q_1$ and $Q_2$. Since the Fock hamiltonian involves just a sum of such translations, it leaves invariant these $q$-dimensional subspaces, as well. It is of interest to note that this is a purely kinematic result which does not depend on the form of the interaction potential.

In the new basis, the $q^2$ matrix elements of the hamiltonian (2) can be readily found to have the convenient form

$$h_k^{(r',r)} = \langle \chi_{k}^{(r')}|H_{HF}|\chi_{k}^{(r)}\rangle$$

$$= \sum_Q v(Q) \exp(-\frac{q^2 Q^2}{4}) \exp\left(-i k \cdot n \times Q + i \frac{q}{2} \Pi Q^2 (2r + Q_1)\right) \delta_{r',|r-Q_1|}. $$
The problem has thus been reduced to the self-consistent diagonalization of a \( d \)-dimensional matrix for each value of the wave vector \( k \). For a sample of surface \( S \) the degeneracy \( D = BS/\phi_0 \) of the Landau level of the non interacting problem is then split into \( d \) bands that span their range as \( k \) covers the Brillouin zone, each holding exactly \( D/q \) single-particle states.

### B. The \( \gamma = 1/2 \) class

We next turn our attention to the case \( \nu = 1/q, \gamma = 1/2 \) for which, following Eq. (11), \( \phi = q/2 \). This case is particularly interesting because if \( q \) is even, say \( q = 2r \), then \( \phi = r \) and the single particle spectrum has just \( r = q/2 \) bands. Since \( \nu = (1/2)/r \) only the lowest of these energy bands has occupied states, yet it is only half filled. The Fermi energy is at the center of the band and the state is metallic. By contrast, if \( q \) is odd there are \( q \) bands in the spectrum, one of which is completely filled and the others empty, leaving the Fermi level in a gap. Even and odd filling fraction denominators thus show qualitatively different behavior, a metal or an insulator, as experiment demands. This remarkable property makes this state a reasonable candidate to be the mean field precursor to the true ground state of the system.

The magnetic translations defining the Fock operator have now the form,

\[
T_{\frac{q}{2} n \times Q} = T_{-\frac{q}{2} Q_2 a_1 + \frac{q}{2} Q_1 a_2}.
\]

(22)

We define a doublet invariant under magnetic translations in the vectors \( a_1 \) and \( a_2 \),

\[
\varphi^\sigma_k(x) = \frac{1}{\sqrt{2}}(\varphi_k(x) + \sigma exp(-ia_2.k)T_{a_2}\varphi_k(x)), \quad \sigma = \pm 1.
\]

(23)

These functions obey

\[
T_{a_1}\varphi^\sigma_k(x) = exp(-ia_1.k)\varphi^{-\sigma}_k(x),
\]

\[
T_{a_2}\varphi^\sigma_k(x) = \sigma exp(-ia_2.k)\varphi^\sigma_k(x),
\]

(24)

being turned into each other by translations along the axes, save for a phase factor. In analogy with the previous subsection we define next a set of eigenfunctions of the translations \( T_{\frac{q}{2} a_1} \), with the form

\[
\chi^{(r,\sigma)}_k(x) = \frac{1}{\sqrt{q}} \sum_{s=-\frac{q-1}{2}}^{\frac{q-1}{2}} exp(i \frac{2k.a_1 s}{q} - i \frac{2\pi r s}{q})T_{\frac{q}{2} a_1} \varphi^\sigma_k(x),
\]

(25)

\[
k \equiv k + n s_1/2 + m s_2/2, \quad n, m = 0, \pm 1, \pm 2, ... \]

Note that the last line indicates for this case that the equivalence of states is now under shifts in half the reciprocal lattice unit cell vectors. This is related to fact that magnetic translations in vectors (6) are non commuting, so that they have no common eigenfunctions. However, translations in twice the spatial unit cell vectors, are commuting operations. Similarly as in the previous subsection, the equivalence follows after considering that, (a) the functions \( \varphi^\sigma_k(x) \) are eigenfunctions of any operator \( T_{2R} \) for lattice vectors \( R \) given by (6), (b) \( T_{2R} \) commutes with all the translations entering in the definition of \( \chi^{(r,\sigma)}_k(x) \) through the original functions \( \varphi^\sigma_k(x) \) and (c) relation (A8) in Appendix A, expressing the equivalence between magnetic translation on \( \varphi^\sigma_k(x) \), with a shift in the momentum labelling these functions.

Aside from relations (24) these functions satisfy the following transformations

\[
T_{\frac{q}{2} a_1} \chi^{(r,\sigma)}_k(x) = exp(i \frac{2k.a_1}{q} + i \frac{2\pi r}{q}) \chi^{(r,\sigma)}_k(x),
\]

(26)

\[
T_{\frac{q}{2} a_2} \chi^{(r,\sigma)}_k(x) = exp(-i \frac{2k.a_2}{q}) \chi^{(r-2,\sigma)}_k(x).
\]

(27)

Consider now fixed values of the quantum numbers \( (k, \sigma) \). Relations (26) and (27) directly show that all translations included in the Fock operator leave invariant the \( d \)-dimensional subspace spanned by the set \( \chi^{(r,\sigma)}_k(x) \), all \( r \).

Using the commutation properties and relations (26) and (27) the matrix elements of the hamiltonian (2) can now be written as
\begin{equation}
\Psi^{(b,\sigma)}(x) = \sum_{r=-\frac{1}{2}}^{1} g^{b}_{r}(k) \chi^{(r,\sigma)}_{k}(x),
\end{equation}

\begin{equation}
\sum_{r=-\frac{1}{2}}^{\frac{1}{2}} g^{\ast b}_{r}(k) g^{b}_{r}(k) = 1.
\end{equation}

Again the mean field problem has been reduced to the diagonalization of a $q$-dimensional matrix, but now this must be done for each value of the wave vector $k$ and the index $\sigma$. An interesting outcome is that the matrix representing the Hamiltonian is identical for the two values of $\sigma$, so that its $q$ eigenvalues are twice degenerate. The obtained block diagonalization of the problem directly indicates a procedure for obtaining approximate self-consistent solutions. One starts assuming an initial density function showing the periodicity in the lattice $\mathbf{R}$ and performs its Fourier series in terms of the momenta in the reciprocal. This furnishes the initial Fourier component of the density $\rho(\mathbf{q})$ lattice cell. Then, defining a 2D uniformly spaced partition of the sets of vectors $\mathbf{q}$ for which the HF states are inequivalent, the set of Hamiltonian matrices associated to these momenta values are diagonalized. Note that these matrices are fully defined after specifying the initial Fourier component of the density. Then, the information in this diagonalization process furnishes the values of the eigenfunctions coefficients. Next, the density to be employed in a new step can be calculated from the formula

\begin{equation}
\rho(x) = \sum_{k} \left| \sum_{\sigma=\pm 1} \sum_{r=-(q-1)/2}^{(q-1)/2} g^{0}_{r}(k) \chi^{(r,\sigma)}_{k}(x) \right|^{2}.
\end{equation}

The coefficients $g^{0}_{r}(k)$ are the lowest energy eigenvectors of the reduced Hamiltonian block matrix for each $k$ value. Since these matrix blocks do not depend on $\sigma$ the eigenvectors are $g^{0}_{r}(k)$, $r = (q-1)/2, ..., -(q-1)/2$ are also $\sigma$ independent. That is, the same is to be used for both values of $\sigma$ in the $\gamma = 1/2$ case. Further, this new density value is again Fourier expanded in the reciprocal lattice vectors and the Fourier coefficients employed to define new Hamiltonian block matrices to be diagonalized. Then, the described process should be developed iteratively up to the arrival to a convergence. To a situation in which the density at any new steps closely approaches the one the previous step.

\section{Analytic Form of the HC States at $\nu = 1/q$ for the Two $\gamma = 1, \frac{1}{2}$ Classes}

As discussed at the end of the last section, further progress will normally require a numerical routine that diagonalizes self-consistently either (21) or (28), where the solutions obtained at the end generate the same potential $\rho(\mathbf{Q})$ that gave rise to them. There is a special situations, however, for which further analytic progress is possible. Because these states qualitatively differs from the one in the usual WC, in a way suggesting the connection with the states that gave rise to them. There is a special situations, however, for which further analytic progress is possible. Because these states qualitatively differs from the one in the usual WC, in a way suggesting the connection with the states that gave rise to them.
where \( b = 0, 1, \ldots, q - 1 \) is the band index, and the label \( \sigma = \pm 1 \) is to be omitted if \( \gamma = 1 \). Because these functions are in the LLL, in our sign convention \( \epsilon = -|\epsilon| \) they must be of the form\(^{19}\)

\[
\Psi(z, z^*) = F(z^*) \exp\left(-\frac{zz^*}{4r_0^2}\right),
\]

where \( z = x + iy, z^* = x - iy, \) and \( F(z^*) \) is an analytic function of its argument. With no loss of generality we choose one of the special zeroes to be at the origin. As one approaches this point one expects that asymptotically

\[
F(z^*) \sim (z^*)^s, \quad s = \gamma(q - 1).
\]

Thus the function \( \Psi \) itself and its first \( s - 1 \) derivatives must vanish at the origin, giving in all \( s \) independent equations to be satisfied. These, together with the normalization condition \( (31) \), total \( s + 1 \) equations, sufficient to determine the \( q \) coefficients \( \{ g_r^0(k), r = \frac{-q-1}{2}, \ldots, \frac{q-1}{2} \} \) of the occupied orbitals. Specifically, for \( \gamma = 1 \) one has \( s + 1 = q \), while for \( \gamma = 1/2 \) one has \( 2s + 1 = q \), the factor of two arising from the double valued index \( \sigma \). Thus the full structure of the single particle wave functions are determined after solving the considered set of equations for the coefficients.

Let us examine in more detail the two states for the class \( \gamma = 1 \) and the simpler case \( q = 3 \). The WC has been reported in the literature for this case\(^9\), but the HC has not. Solving for the coefficients \( g_r^0(k), g_r^2(k), g_r^1(k) \) defining the filled orbitals and substituting them in the equation \( \rho(x) = \sum_k |g_r^0(k)\chi_k(x)|^2 \)

\[
(32)
\]

where the first sum runs over all momenta \( k \) in the Brillouin zone, the particle density may be obtained. The result is shown in Fig. 1 (b), where the density for the Wigner Crystal solution at the same filling, Fig 1 (a), has been included for comparison. Notice the presence of sharp hexagonal channels surrounding low density regions where the density vanishes as the fourth power of the distance. The fact that the density percolates the structure much like the wax in a bee hive marks the essential difference between the HC and WC states. In the latter the charge density is made up essentially of gaussian functions centered at lattice points as seen in Fig. 1(a). As the filling fraction decreases these gaussian peaks become sharper and the limit of a classical point-electron WC is approached.

Further, insertion of the calculated density in Eq. \( 41 \) and use of this result in Eq.(3) allows finding the associated eigenvalues \( \epsilon^r, r = 0, 1, 2 \) by diagonalizing the 3x3 matrix \( (21) \). Three bands are obtained, that span their range as \( k \) covers the Brillouin zone.

The bands dispersion relations are illustrated in Fig. 2. They are quite narrow, with the lowest (the filled one) well separated from the rest by a sizable gap. The same pattern was found for larger values of \( q \). The LLL having been split into \( q \) separate bands appears to yield the spectrum associated with a value of the magnetic field reduced by a factor \( 1/q \). This feature is reminiscent of the composite fermion theory, which interprets the FQHE as the integer QHE of composite fermions in a field reduced by the same factor.\(^{20,21}\)

The energy per particle of the HC solution we have just discussed is found through

\[
\epsilon = \frac{1}{N} \sum_k \frac{\epsilon^r(k)}{2} = -0.362 \frac{e^2}{\varepsilon_0 r_0}, \quad (33)
\]
FIG. 2: Dispersion relation for the three bands into which the lowest Landau level is split by the action of the Coulomb interaction, for the same case as in Fig. 1. Energy units are $\frac{e^2}{\varepsilon_0 a}$.

The value obtained is 7% above the WC solution associated to the $\gamma=1$ class, for which case $\epsilon = -0.388\frac{e^2}{\varepsilon_0 a}$. The usual minimal energy criterion thus points to the WC as a better candidate for the ground state and the HC an exited state, a conclusion that persists for larger values of $q$. Still, how these only approximate mean field solutions are affected by correlations is largely unknown save for second order perturbation corrections.

FIG. 3: Energy per particle for different values of $q = 1/\nu$ for the Wigner Crystal (dashed line) and the Hall Crystal (broken line) both for the class $\gamma = 1/2$. Units are as in Fig. 2. Lines are drawn to guide the eye.

A more interesting case corresponds to the class $\gamma = 1/2$ for $q=3$. A similar analytic procedure as employed above gives the energy for both the HC states. However, it should be noticed that this special solution was already discussed in reference [13]. Concretely, the set of three equations

$$\sum_{r=-1}^1 g_r^b(k) x_k^{(r,\sigma)}(0) = 0, \sigma = \pm 1$$ (34)

$$\sum_{r=-1}^1 g_r^b(k) g_r^b(k) = 1,$$ (35)

for the coefficients of the functions defining the HC states through its zero should now be solved. Note that in this case $\gamma = 1/2$ the three equations correspond to the vanishing of the functions at the origin for the two values of $\sigma=\pm 1$ plus the normalization condition.

Finally, the dependence of the energy per particle on the values of $q$ for the class $\gamma = 1/2$ was evaluated, for the WC as well as for the HC states. The results are shown in Fig. 3. Notice that the curves intersect at about filling
1/7, suggesting that phase transitions may occur between the HC and the WC states associated to the class having half particle per unit cell ($\gamma = 1/2$), as the magnetic field is varied. It should be stressed that, although the mean field results for the energy in the class of states $\gamma = 1/2$, depicted in Fig. 3 are higher than for the ones in the class $\gamma = 1$ (-0.388 $e^2/r_0$ at $\nu = 1/3$ for the WC state in this class), these states could yet enter to compete in having lower energies after the inclusion of the correlations. Moreover, the fact that recent experiments detect phase transitions in varying the filling factors as well as the temperature, could evidence these states as alternative excited configurations which could become the ground ones after the temperature and the filling factor are varied. These possibilities are consistent with the experimental data reported in Ref.\textsuperscript{2}. The evaluations of the energy per particle, for the WC and HC states of class $\gamma = 1/2$, were done by numerically solving the HF problem. The numerical solutions were obtained by using the same programs employed in Ref.\textsuperscript{10}, which implements an algorithm being equivalent to the one sketched at the end of Section 3. In fact the analytical results presented in Sections 2 and 3, can be interpreted as obtained in order to perform in an analytic way a great deal of the numerical steps in the programs employed for obtaining the results in Ref.\textsuperscript{10}.

V. SUMMARY AND CONCLUSIONS

We have shown that analytical solutions to the Hartree-Fock problem for a two dimensional electron gas at filling factors $\nu = p/q$ may be obtained for charge density wave states whose unit cell include $\gamma$ electrons, with this number fractional or integral. Our results follow from symmetry considerations and the construction of a special set of common eigenfunctions to all magnetic translations in a lattice vector. The spectrum of single particle states is shown to be organized in multiplets of dimension $q$ in such a basis, which are left invariant by the action of the Fock operator. The matrix elements representing the Fock operator in each multiplet are explicitly determined.

We treat in detail the case $\nu = 1/q$ for the two classes holding one ($\gamma = 1$) or a half electron ($\gamma = 1$) per plaquette. In particular for each of such classes, we analytically determine a special state whose low density limit is not a Wigner Crystal. We call this state a Hall Crystal. It is characterized by occupied orbitals all of which have a zero of order $\gamma(q-1)$ at a point of high symmetry in each cell. The charge density is made of percolating hexagonal ridges expected to provide easy paths for correlated ring exchange that would lower the energy.\textsuperscript{21} For the particular case, $\gamma = 1/2$, the mean field energy was evaluated here as a function of the filling factor and shows a cross-over from a regime in which the Wigner Crystal like state has lower energy, to one in which the Hall Crystal one has lower energy. In contrast, the same evaluations for the class $\gamma = 1$ down to fillings as low as 1/9, indicates that the Wigner Crystal like state has the lowest energy within our mean field approximation. The correlation energy affecting in a different way both states may alter this ordering, however.

In concluding let us expose some perspectives for the extension of the work. Let us recall states in the lowest Landau level obeying periodic boundary conditions must vanish at a number of points equaling the degeneracy of the non-interacting system.\textsuperscript{22} For $\nu = 1/q$ this number is $qN_e$, where $N_e$ is the total number of electrons in the sample. But, for example, the Hall Crystal states at $\gamma = 1$ attaches $(q-1)N_e$ of them to fixed high symmetry periodic points in the electron lattice, leaving $N_e$ zeroes whose spatial location depends on the quantum number $k$ labelling each occupied state. Also, the associated determinant will vanish as the first power as two particles approach each other. However, the Laughlin state does so as a power $q$. This feature decreases the direct Coulomb energy in the Laughlin states by keeping particles as far as possible from each other, exhibiting built-in correlations which mean field solutions lack. However, the just described property opens the interesting idea of constructing new correlated states optimizing short as well as long range correlations starting form the HC states discussed in this paper. For this purpose, we suggest here to substitute the common factor of radial functions of the $N_e$ particles which appear in the HC state determinant, by Jastrow factors having the same number of zeroes that factor. In this way, the so constructed wave-functions will show $q$ zeros when any two particle joins. But moreover, it can be expected that they will even more optimize the energy, since they also incorporate the long range crystalline correlations present in the Hall Crystal states. We hope the presented results and ideas for the extension of the work will stimulate further research directed to check whether the corrections to the mean field approach early introduced by one of the authors (F.C) could furnish a general theory for the FQHE.

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APPENDIX A: EIGENFUNCTIONS OF MAGNETIC TRANSLATIONS

It is well known that Bloch-like states may be constructed by placing a seed function at each lattice point and attaching to it an appropriate phase factor. In the presence of a magnetic field this procedure may be implemented using the zero angular momentum eigenfunction

$$\phi(x) = \frac{1}{\sqrt{2\pi r_o}} \exp\left(-\frac{x^2}{4r_o^2}\right)$$

and forming the sum\(^{11,12,22}\)

$$\varphi_k(x) = \frac{1}{N_k} \sum_\ell (-1)^{\ell_1 \ell_2} \exp(i \mathbf{k} \cdot \mathbf{\ell}) \, T_\ell \, \phi(x),$$

$$N_k = \sqrt{N_{\phi_0}} \sum_\ell (-1)^{\ell_1 \ell_2} \exp(i \mathbf{k} \cdot \mathbf{\ell} - \frac{\ell^2}{4r_o^2}).$$

The summation indices \(\ell_1, \ell_2\) run over all integers, defining a planar lattice \(L\) through \(\ell = \ell_1 \mathbf{b}_1 + \ell_2 \mathbf{b}_2\) with the unit cell intercepting one flux quantum, so that \(\mathbf{b}_1 \times \mathbf{b}_2 = 2\pi r_o^2\). Displacements of the seed function are effected by magnetic translation operator \(T_\mathbf{a}\) in the vector \(\mathbf{a}\), whose action on any function \(f\) is defined by\(^{26}\)

$$T_\mathbf{a} \, f(x) = \exp\left(\frac{ie}{\hbar c} \mathbf{A}(\mathbf{a}) \cdot \mathbf{x}\right) \, f(x - \mathbf{a}).$$

Here the vector potential is assumed in the axial gauge \(\mathbf{A}(x) = B(-x_2, x_1, 0)/2\) and the electron charge \(e\) is taken with its negative sign \((e = -|e|)\). These translations in general do not commute,

$$T_{\mathbf{a}_1} T_{\mathbf{a}_2} = \exp(\frac{2ie}{\hbar c} \mathbf{A}(\mathbf{a}_1) \cdot \mathbf{a}_2) T_{\mathbf{a}_2} T_{\mathbf{a}_1},$$

$$= \exp(\frac{ie}{\hbar c} \mathbf{A}(\mathbf{a}_1) \cdot \mathbf{a}_2) T_{\mathbf{a}_1 + \mathbf{a}_2}.$$

In the special case of displacements in any vector belonging to \(L\), however, since the flux trapped by any parallelogram bounded by lattice vectors is an integral number of flux quanta, all translations commute.

One can easily check that the functions \(\varphi_k\) are eigenstates of translations in any lattice vector, satisfying the eigenvalue equation

$$T_\mathbf{\ell} \, \varphi_k(x) = \lambda_k(\mathbf{\ell}) \, \varphi_k(x),$$

$$\lambda_k(\mathbf{\ell}) = (-1)^{\ell_1 \ell_2} \exp(-i \mathbf{k} \cdot \mathbf{\ell}).$$

Arranged in a Slater determinant these functions are exact solutions of the Hartree-Fock problem\(^{12,23,24,25}\). This strong property arises from the fact that the HF single particle hamiltonian commutes with all translations leaving \(L\) invariant\(^{12}\) The functions \([A2]\) are common eigenfunctions of the commuting magnetic translations. Moreover, the set of eigenvalues \([A6]\) uniquely determines them. Therefore, the HF hamiltonian associated with the Slater determinant can not change those eigenvalues and the \(\varphi_k\) should be eigenfunctions.

An important property of the basis functions \([A2]\) is that an arbitrary translation is equivalent to a shift in the momentum label, modulo a phase factor\(^{11}\). Operating twice with the translation operator involving an arbitrary vector \(\mathbf{a}\) and a vector in the lattice \(\ell\), and using Eqs. \([A4]\) and \([A5]\) one readily gets,

$$T_\mathbf{a} T_\mathbf{\ell} \, \varphi_k(x) = \lambda_k(\mathbf{\ell}) T_\mathbf{a} \varphi_k(x),$$

$$= \exp\left(\frac{2ie}{\hbar c} \mathbf{A}(\mathbf{a}) \cdot \mathbf{\ell}\right) T_\mathbf{\ell} T_\mathbf{a} \, \varphi_k(x),$$

which can also be written as

$$T_\mathbf{\ell} T_\mathbf{a} \, \varphi_k(x) = \lambda_{k+\frac{|e|}{2\pi} \mathbf{A}(\mathbf{a}) \cdot \mathbf{\ell}} T_\mathbf{a} \varphi_k(x).$$

Then, taking into account that the set of eigenvalues defines uniquely the wave-functions modulo a phase, it follows that

$$T_\mathbf{a} \, \varphi_k(x) = F_k(\mathbf{a}) \, \varphi_{k+\frac{|e|}{2\pi} \mathbf{A}(\mathbf{a})}(x),$$

\([A8]\)
where

\[ F_k(a) = \frac{\varphi_k(0)}{\varphi_k + \frac{c}{2\pi} A(a)} \]  \hspace{1cm} (A9)