Research Article

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Lions-type theorem of the $p$-Laplacian and applications

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Abstract: In this article, our aim is to establish a generalized version of Lions-type theorem for the $p$-Laplacian. As an application of this theorem, we consider the existence of ground state solution for the quasilinear elliptic equation with the critical growth.

Keywords: Lions theorem; quasilinear elliptic equation; singular potential; critical exponent; ground state solution

MSC: 35A15; 35R11; 35J92

1 Introduction

Consider the quasilinear elliptic equation:

$$-\Delta_p u + V(|x|)|u|^{p-2}u = f(u), \quad x \in \mathbb{R}^N,$$

(Ω)

where $N \geq 3$, $p \in (1, \sqrt{N})$, $V(|x|) = \frac{A}{|x|^p}$, $a \in (0, p)$, $A > 0$ is a real constant, and $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian. Here, the nonlinearity $f$ is given by

$$(F_1) \quad f(u) = |u|^{p_a-2}u + |u|^{q-2}u + \lambda |u|^{p^*-2}u,$$

where $\lambda > 0$, $p_a = p + \frac{p^* a}{p(N-a)-ap^*}$, $p^* = \frac{Np}{N-p}$ and $q \in \left(\frac{p(N+a-p)}{N-p}, p^*\right)$.

Remark 1.1. The generalized version of the Berestycki-Lions conditions for the nonlinearity $f$ is given as follows:

(1) $f \in C(\mathbb{R}, \mathbb{R})$, there exists $C > 0$ such that $|sf(s)| \leq C(|s|^{p_a} + |s|^{p^*})$ for $s \in \mathbb{R}$;

(2) $\lim_{s \to 0} \frac{L(s)}{|s|^{p_a}} = 1$ and $\lim_{s \to 0} \frac{f(s)}{|s|^p} = 1$, where $F(s) := \int_0^s f(t)dt$; and

(3) there exists $s_0 \in \mathbb{R} \setminus \{0\}$ such that $F(s_0) \neq 0$.

In view of (F2), clearly, the nonlinearity $f$ satisfies the generalized version of the Berestycki-Lions conditions (F2)-(F4). This is the “almost optimal” choice of the nonlinearity $f$.

As is well known, for $p = 2$, equation (Ω) is a model for describing the stationary state of reaction-diffusion equations in population dynamics [7]. It also arises in several other scientific fields such as plasma physics, condensed matter physics and cosmology [6]. The existence of solution of equation (Ω) has been studied extensively by modern variational methods under various hypotheses on the singular potential $V$ and the nonlinearity $f$. Let us briefly recall some related results. For $p = 2$ and $V(|x|) = \frac{1}{|x|^p}$, the existence and nonexistence of solutions to equation (Ω) have been studied in [3, 4, 15, 17, 20, 23]. For $p \neq 2$, the nonexistence results of equation (Ω) were presented in [1, 5, 8, 9, 14, 16, 18] and the references therein.

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For \( p \in (1, N) \) and \( V(x) = \frac{1}{|x|^p} \), Ghoussoub-Yuan [9] investigated the equation:

\[
-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u = |u|^{p-2}u, \ x \in \mathbb{R}^N, \tag{1.1}
\]

where \( N \geq 3 \), \( p \in (1, N) \) and \( \mu \in \left( 0, \left( \frac{N-p}{p} \right)^p \right) \), and established the existence of solutions to equation (1.1) by using the variational methods. Abdellaoui-Peral [1] considered the equation:

\[
-\Delta_p u - \frac{\mu + k(x)}{|x|^p}|u|^{p-2}u = |u|^{p-2}u, \ x \in \mathbb{R}^N, \tag{1.2}
\]

and discussed the existence and nonexistence of solutions to equation (1.2) under different assumptions on \( k(x) \) by applying the concentration compactness principle and Pohožaev-type identity. Filippucci-Pucci-Robert [8] considered the problem:

\[
-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u = |u|^{p-2}u + \frac{|u|^{p^*}}{|x|^p}, \ x \in \mathbb{R}^N, \tag{1.3}
\]

where \( s \in (0, N) \) and \( p_s^* = \frac{p(N-s)}{N-p} \) is the Hardy-Sobolev critical exponent, and obtained the existence results of equation (1.3) by the choice of a suitable energy level for the mountain pass theorem and analysis of concentration.

Su-Wang-Willem [18] dealt with a generalized version with the singular potential:

\[
-\Delta_p u + V(|x|)|u|^{p-2}u = Q(x)f(u), \ x \in \mathbb{R}^N, \tag{\text{\textcopyright}}
\]

where \( 1 < p < N \), and \( V \) and \( Q \) satisfy

\( (V_1) \) \( V \in C(0, \infty), V > 0 \) and there exist real numbers \( a \) and \( a_0 \) such that

\[
\liminf_{r \to \infty} \frac{V(r)}{r^a} > 0 \quad \text{and} \quad \liminf_{r \to 0} \frac{V(r)}{r^{a_0}} > 0.
\]

\( (Q_1) \) \( Q \in C(0, \infty), Q > 0 \) and there exist real numbers \( b \) and \( b_0 \) such that

\[
\limsup_{r \to \infty} \frac{Q(r)}{r^b} < \infty \quad \text{and} \quad \limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < \infty.
\]

They attained the radial inequalities with respect to the parameters \( a, a_0, b, b_0 \), then established main results on continuous and compact embeddings and the existence of solution to equation (\text{\textcopyright}). Badiale-Guida-Rolando [5] generalized the embedding results under different conditions on \( V \) and \( Q \), and explored the existence of solution to equation (\text{\textcopyright}) with the sub-critical and super-critical growth.

If \( a = a_0 = -\alpha \) and \( b = b_0 = 0 \) in conditions \( (V_1) \) and \( (Q_1) \), equation (\text{\textcopyright}) reduces to (\text{\textcopyright}). Let us introduce the result on continuous and compact embeddings described in [18].

**Proposition 1.1.** Suppose that \( N \geq 3 \) and \( p \in (1, N) \). Then we have

\[
\begin{align*}
W^{1,p}_{rad}(\mathbb{R}^N, a) & \hookrightarrow L^r(\mathbb{R}^N), \quad r \in [p_a^*, p^*], \ a \in (0, p), \\
W^{1,p}_{rad}(\mathbb{R}^N, a) & \hookrightarrow L^r(\mathbb{R}^N), \quad r \in [p^*, p_a^*], \ a \in (p, \frac{N-1}{p-1}), \\
W^{1,p}_{rad}(\mathbb{R}^N, a) & \hookrightarrow L^r(\mathbb{R}^N), \quad r \in [p^*, \infty), \ a \in \left[ \frac{N+1}{p-1}, p \right].
\end{align*}
\]

Furthermore, the embeddings are compact if \( r \neq p_a^* \) and \( r \neq p^* \), where \( p_a^* = \frac{p(N-1)+pa}{p(N-1)-ap-1} = p + \frac{p^*a}{p(N-1)-ap-1}, \ p^* = \frac{Np}{N-2p} \) and \( W^{1,p}_{rad}(\mathbb{R}^N, a) : = D^{1,p}_{rad}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, a) \) is the set of radial functions in \( W^{1,p}(\mathbb{R}^N, a) \).

It is very natural to ask whether there exists a solution to equation (\text{\textcopyright}) with the embedding top index \( p^* \) and bottom index \( p_a^* \). To the best of our knowledge, it seems that so far there is no affirmative answer in the literature.
From Proposition 1.1, the embeddings
\[ W^{1,p}_{rad}([\mathbb{R}^N], \alpha) \rightarrow L^p(\mathbb{R}^N) \quad \text{and} \quad W^{1,p}_{rad}([\mathbb{R}^N], \alpha) \rightarrow L^{p^*_a}(\mathbb{R}^N) \]
are not compact. As a result, it is difficult to prove that the Palais-Smale (minimizing) sequence is strongly convergent if we seek solutions of equation (Ω) with the critical exponent.

Lions [10] considered the noncompact embedding problem by the concentration-compactness principle: *Only vanishing, dichotomy or tightness are possible.* If one can exclude vanishing and dichotomy, then tightness occurs. It is not difficult to rule out vanishing. But sometimes it is hard to exclude the dichotomy. Therefore, it becomes interesting to ask under what conditions dichotomy cannot occur? In [11, pp. 232], Lions gave a useful answer.

**Proposition 1.2.** (Lions Theorem) Let \( N \geq 3 \) and \( p \in (1, N) \). Let \( \{u_n\} \subset W^{1,p}(\mathbb{R}^N) \) be any bounded sequence satisfying
\[ \text{(Condition A)} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^r \, dx > 0 \quad \text{for} \quad r \in (p, p^*). \]
Then there exists \( \{y_n\} \subset \mathbb{R}^N \) such that the sequence \( \bar{u}_n(x) = u_n(x + y_n) \) converges weakly and a.e. to \( \bar{u} \neq 0 \) in \( L^1_{loc}(\mathbb{R}^N) \).

Following [10, 11], we can derive a similar result as follows immediately.

**Proposition 1.3.** (Lions-type theorem 1) Let \( N \geq 3 \) and \( p \in (1, N) \). Let \( \{u_n\} \subset W^{1,p}_{rad}(\mathbb{R}^N), \alpha \) be any bounded sequence satisfying
\[ \text{(Condition B)} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^r \, dx > 0, \quad \text{where} \]
\[
\begin{aligned}
    r \in (p^*_a, p^*) &\quad \alpha \in (0, p), \\
    r \in (p^*, p^*_a) &\quad \alpha \in (p, N-1/p), \\
    r \in (p^*, \infty) &\quad \alpha \in [N-1/p, \infty). \\
\end{aligned}
\]
Then the sequence \( \{u_n\} \) converges weakly and a.e. to \( u \neq 0 \) in \( L^1_{loc}(\mathbb{R}^N) \).

From Conditions A and B, we see that Propositions 1.2 and 1.3 provide a technical tool to the cases: (a) the nonlinearity \( f \) neither has the embedding top index nor embedding bottom index, and (b) the nonlinearity \( f \) has either the embedding top index or embedding bottom index. However, Propositions 1.2 and 1.3 become invalid when \( f \) contains both embedding top and bottom indices. Hence, we shall establish a more generalized result for \( \alpha \in (0, p) \) as follows.

**Theorem 1.1.** Suppose that \( N \geq 3, p \in (1, N) \) and \( \alpha \in (0, p) \). Let \( \{u_n\} \subset W^{1,p}_{rad}(\mathbb{R}^N), \alpha \) be any bounded sequence satisfying
\[ \text{(Condition C)} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^r \, dx > 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^\alpha \, dx > 0. \]
Then the sequence \( \{u_n\} \) converges weakly and a.e. to \( u \neq 0 \) in \( L^1_{loc}(\mathbb{R}^N) \).

Before presenting the existence of ground state solution to equation (Ω), let us state the regularity properties of any nonnegative weak solutions of equation (Ω) with \( \alpha \in (0, p) \).

**Theorem 1.2.** Suppose that \( N \geq 3, p \in (1, N), \alpha \in (0, p) \) and condition \( (F_2) \) holds. If \( U \) is a nonnegative weak solution of equation (Ω), then the following statements are true.
(i) \( U \in L^r(\mathbb{R}^N) \) for \( r \in [p^*_a, \infty] \).
(ii) \( U \) is a positive solution.
(iii) \( U \) satisfies the Pohožaev-type identity:
\[
\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla U|^p \, dx + \frac{N-a}{p} \int_{\mathbb{R}^N} \frac{A}{|x|^a} |U|^p \, dx = N \int_{\mathbb{R}^N} F(U) \, dx.
\]
As an application of Theorems 1.1 and 1.2, when the nonlinearity \( f \) satisfies condition \((F_1)\), i.e. equation \((\Omega)\) takes the form
\[
-Au + \frac{A}{|x|^a} |u|^{p-2}u = |u|^{q-2}u + |u|^{q-2}u + \lambda |u|^{p_\ast-2}u, \quad x \in \mathbb{R}^N, \tag{S_{p_\ast}}
\]
we have

**Theorem 1.3.** Assume that \( N \geq 3, p \in (1, \sqrt{N}) \), \( \alpha \in (0, p) \), \( q \in \left(\frac{p(N-p+\alpha)}{N-p}, p^*\right) \) and condition \((F_1)\) holds. Suppose that
\[
\left(\frac{a}{p} S_{p_\ast}^\alpha \right)^{p_\ast} \geq \lambda > 0,
\]
where \( \lambda \) is a constant, \( S_a \) and \( S \) are the best constants of the following inequalities [2, 19]:
\[
S_a \left( \int_{\mathbb{R}^N} |u|^{p_\ast} \, dx \right)^{\frac{p}{p_\ast}} \leq \|u\|_{W^{1,p}(\mathbb{R}^N), \alpha}^p, \quad u \in W^{1,p}(\mathbb{R}^N, \alpha), \tag{1.4}
\]
and
\[
S \left( \int_{\mathbb{R}^N} |u|^{p_\ast} \, dx \right)^{\frac{p}{p_\ast}} \leq \|u\|_{D^{1,p}(\mathbb{R}^N)}^p, \quad u \in D^{1,p}(\mathbb{R}^N). \tag{1.5}
\]
Then equation \((S_{p_\ast})\) has a positive ground state solution \( u \in W^{1,p}(\mathbb{R}^N, \alpha) \).

The rest of this paper is organized as follows. In Section 2, we briefly introduce some useful notations and inequalities. In Sections 3-5, we prove Theorems 1.1-1.3, respectively.

## 2 Preliminaries

For \( N \geq 3 \) and \( p \in (1, N) \), let
\[
D^{1,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \left| \int_{\mathbb{R}^N} |\nabla u|^p \, dx < \infty \right. \right\}
\]
with the semi-norm
\[
\|u\|_{D^{1,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla u|^p \, dx.
\]

Let
\[
W^{1,p}(\mathbb{R}^N, \alpha) := \left\{ u \in D^{1,p}(\mathbb{R}^N) \left| \frac{A}{|x|^\alpha} |u|^p \, dx < \infty \right. \right\} = D^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, \alpha)
\]
with the norm
\[
\|u\|_{W^{1,p}(\mathbb{R}^N, \alpha)} := \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |u|^p \, dx.
\]

The following inequalities will play a crucial role in the proof of Theorem 1.3:
\[
S_a \left( \int_{\mathbb{R}^N} |u|^{p_\ast} \, dx \right)^{\frac{p}{p_\ast}} \leq \|u\|_{W^{1,p}(\mathbb{R}^N, \alpha)}^p, \quad u \in W^{1,p}(\mathbb{R}^N, \alpha), \tag{2.1}
\]
and
\[
S \left( \int_{\mathbb{R}^N} |u|^{p_\ast} \, dx \right)^{\frac{p}{p_\ast}} \leq \|u\|_{D^{1,p}(\mathbb{R}^N)}^p, \quad u \in D^{1,p}(\mathbb{R}^N). \tag{2.2}
\]

A measurable function \( u : \mathbb{R}^N \to \mathbb{R} \) belongs to the Morrey space with the norm \( \|u\|_{\mathcal{M}_{q, \omega}(\mathbb{R}^N)} \), where \( q \in [1, \infty) \) and \( \omega \in (0, N] \), if and only if
\[
\|u\|_{\mathcal{M}_{q, \omega}(\mathbb{R}^N)} := \sup_{R>0, x \in \mathbb{R}^N} R^{\omega-N} \int_{B(x, R)} |u(y)|^q \, dy < \infty.
\]
\textbf{Lemma 2.1.} ([12], Refined Sobolev inequality with the Morrey norm) For $N \geq 3$ and $p \in (1, N)$, there exists $C > 0$ such that for $i$ and $\theta$ satisfying $\frac{p}{p^*} \leq i < 1$ and $1 \leq \theta < \theta^*$, we have

$$
\left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}} \leq C \|u\|^i_{W^{1, p}(\mathbb{R}^N)} \|u\|^{1-i}_{L^{\max(p, \frac{pN}{N-\theta})}(\mathbb{R}^N)},
$$

for $u \in D^{1, p}(\mathbb{R}^N)$. It follows from Lemma 2.1 and $W^{1, p}_{rad}(\mathbb{R}^N, \alpha) \subset D^{1, p}_{rad}(\mathbb{R}^N) \subset D^{1, p}(\mathbb{R}^N)$ that

$$
\left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}} \leq C \|u\|^{i}_{W^{1, p}(\mathbb{R}^N, \alpha)} \|u\|^{1-i}_{L^{\max(p, \frac{pN}{N-\theta})}(\mathbb{R}^N)}, \quad u \in W^{1, p}_{rad}(\mathbb{R}^N, \alpha). \tag{2.3}
$$

To prove the generalized version of Lions-type theorem, we need the following technical lemma.

\textbf{Lemma 2.2 ([18]).} Let $N \geq 3$, $p \in (1, N)$ and $\alpha \in (0, p)$. Then the inequality

$$
\sup_{|x| > 0} \left| \frac{u(x)}{|x|^{p-1}} \right| \leq \frac{C}{|x|^{p-1}} \|u\|_{W^{1, p}_{rad}(\mathbb{R}^N, \alpha)}
$$

holds for $u \in W^{1, p}_{rad}(\mathbb{R}^N, \alpha)$.

Throughout this article, we will use the symbol $C$ to denote a generic constant, possibly varying from line to line. However, special occurrences will be denoted by $C_1$, $\tilde{C}$ or the like.

\section{Proof of Theorem 1.1}

In this section, by applying the refined Sobolev inequality with the Morrey norm and Lemma 2.2, we prove a generalized version of Lions-type theorem.

\textbf{Proof of Theorem 1.1.} We separate the proof into four steps.

\textbf{Step 1.} Note that $\{u_n\}$ is a bounded sequence in $W^{1, p}_{rad}(\mathbb{R}^N, \alpha)$. Then, up to a subsequence, we assume

$$
u_n \rightharpoonup u \text{ in } W^{1, p}_{rad}(\mathbb{R}^N, \alpha), \quad u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N, \quad u_n \rightarrow u \text{ in } L^p_{loc}(\mathbb{R}^N).
$$

According to (2.3) and \textbf{Condition C}, there exists a positive constant $C$ such that for any $n$ there holds

$$
\|u_n\|_{\mathcal{M}(p, \nu, p)(\mathbb{R}^N)} \geq C > 0.
$$

On the other hand, from [12, pp 809] we note that $\{u_n\}$ is bounded in $W^{1, p}_{rad}(\mathbb{R}^N, \alpha)$ and

$$
W^{1, p}_{rad}(\mathbb{R}^N, \alpha) \rightarrow D^{1, p}_{rad}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \rightarrow \mathcal{M}(p, N, \nu, p)(\mathbb{R}^N).
$$

Then we have

$$
\|u_n\|_{\mathcal{M}(p, \nu, p)(\mathbb{R}^N)} \leq C
$$

for some $C > 0$ independent of $n$. Hence, there exists a positive constant $C_0$ such that for any $n$ there holds

$$
C_0 \leq \|u_n\|_{\mathcal{M}(p, \nu, p)(\mathbb{R}^N)} \leq C_0^{-1}.
$$

From this inequality, we deduce that for any $n \in \mathbb{N}$ there exist $\sigma_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$
\sigma_n^p \int_{B(x_n, \sigma_n)} |u_n(y)|^p \, dy \geq \|u_n\|_{\mathcal{M}(p, \nu, p)(\mathbb{R}^N)}^p - \frac{\sigma_n^p}{2n} \geq C_1 > 0. \tag{3.1}
$$

\textbf{Step 2.} We show $\lim_{n \to \infty} \sigma_n = \bar{\sigma} > 0$, where $\bar{\sigma} \in (0, \infty)$.
It suffices to show that \( \lim_{n \to \infty} \sigma_n \neq \infty \). Otherwise, we suppose \( \lim_{n \to \infty} \sigma_n = \infty \). In view of the boundedness of \( \{u_n\} \) and **Condition C**, we get

\[
0 < \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dy \leq C.
\]

Since \( N \geq 3, p \in (1, N) \) and \( \alpha < p \), we have

\[
-p + N \left( 1 - \frac{p}{p_\alpha} \right) < 0 \Leftrightarrow p_\alpha < p^*.
\]

(3.2)

It follows from (3.1)-(3.2) that

\[
0 < C_1 \leq \sigma_n^p \int_{B(x_n, \sigma_n)} |u_n(y)|^p \, dy
\]

\[
\leq \sigma_n^p \left( \frac{\omega_{N-1} \sigma_n^N}{p_\alpha} \right)^{\frac{\alpha - p}{p_\alpha}} \left( \int_{B(x_n, \sigma_n)} |u_n(y)|^{p_\alpha} \, dy \right)^{\frac{p}{p_\alpha}}
\]

\[
= \sigma_n^{-p} \left( \frac{\omega_{N-1} N}{p_\alpha} \right)^{\frac{\alpha - p}{p_\alpha}} \left( \int_{B(x_n, \sigma_n)} |u_n(y)|^{p_\alpha} \, dy \right)^{\frac{p}{p_\alpha}}
\]

\[
\leq C \sigma_n^{-p} \left( \frac{\omega_{N-1}}{N} \right)^{\frac{\alpha - p}{p_\alpha}} \left( \int_{\mathbb{R}^N} |u_n(y)|^{p_\alpha} \, dy \right)^{\frac{p}{p_\alpha}}
\]

\[
\Rightarrow 0, \text{ as } n \to \infty,
\]

where \( \omega_{N-1} \) is the volume of the unit sphere in \( \mathbb{R}^N \). This is a contradiction.

By the Bolzano-Weierstrass theorem, up to a subsequence, still denoted by \( \{\sigma_n\} \), there exists \( \bar{\sigma} \in [0, \infty) \) such that

\[
\lim_{n \to \infty} \sigma_n = \bar{\sigma}.
\]

We now show \( \sigma_n = \bar{\sigma} \neq 0 \). Suppose on the contrary that \( \lim_{n \to \infty} \sigma_n = \bar{\sigma} = 0 \). By using the uniform boundedness of \( \{u_n\} \) in \( W^{1,p}_{rad}(\mathbb{R}^N, \sigma) \), we have

\[
C \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} |u_n|^p \, dy \right)^{\frac{1}{p}} \leq \lim_{n \to \infty} \|u_n\|^p_{W^{1,p}_{rad}(\mathbb{R}^N, \sigma)} \leq \tilde{C}.
\]

It follows from Hölder’s and Sobolev’s inequalities that

\[
\int_{B(0, \sigma_n)} |u_n|^{p_\alpha} \, dy \leq \left( \int_{B(0, \sigma_n)} \right)^{\frac{p}{p_\alpha}} \left( \int_{B(0, \sigma_n)} |u_n|^p \, dy \right)^{\frac{p_\alpha}{p}}
\]

\[
\leq \sigma_n^{-\frac{p}{p_\alpha}} \left( \int_{B(0, \sigma_n)} \right)^{\frac{p}{p_\alpha}} \left( \int_{\mathbb{R}^N} |\nabla u_n|^p \, dy \right)^{\frac{p_\alpha - p}{p}} \int_{B(0, \sigma_n)} |\nabla u_n|^p \, dy
\]

\[
\leq CS^{-\frac{p}{p_\alpha}} \left( \int_{B(0, \sigma_n)} \right)^{\frac{p}{p_\alpha}} \int_{B(0, \sigma_n)} |\nabla u_n|^p \, dy.
\]

Similarly, for each \( z \in \mathbb{R}^N \) we have

\[
\int_{B(z, \sigma_n)} |u_n|^{p_\alpha} \, dy \leq CS^{-\frac{p}{p_\alpha}} \left( \int_{B(z, \sigma_n)} \right)^{\frac{p}{p_\alpha}} \int_{B(z, \sigma_n)} |\nabla u_n|^p \, dy.
\]
Covering $\mathbb{R}^N$ by balls of radius $\sigma_n$, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N + 1$ balls, we find
\[
\int_{\mathbb{R}^N} |u_n|^p \, dy \leq C(N + 1) S^{-\frac{\omega_{N-1}}{p}} \left( \int_{B(0, \sigma_n)} \frac{x^p}{r^p} \, dy \right) \int_{\mathbb{R}^N} |\nabla u_n|^p \, dy
\]
\[
\leq C(N + 1) S^{-\frac{\omega_{N-1}}{p}} \left( \int_{B(0, \sigma_n)} \frac{x^p}{r^p} \, dy \right) \int_{\mathbb{R}^N} |\nabla u_n|^p \, dy
\]
\[
= C(N + 1) S^{-\frac{\omega_{N-1}}{p}} \left( \frac{\omega_{N-1}}{N} \right)^{1-\frac{\omega_{N-1}}{p}} \sigma_n^{N(1-\frac{\omega_{N-1}}{p})},
\]
where $\omega_{N-1}$ is the volume of the unit sphere in $\mathbb{R}^N$.

Taking $n \to \infty$ and applying $\lim_{n \to \infty} \sigma_n = 0$ leads to
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dy \leq C(N + 1) S^{-\frac{\omega_{N-1}}{p}} \left( \frac{\omega_{N-1}}{N} \right)^{1-\frac{\omega_{N-1}}{p}} \sigma_n^{N(1-\frac{\omega_{N-1}}{p})} = 0,
\]
which yields a contradiction to $\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dy > 0$ given in **Condition C**.

Using $\lim_{n \to \infty} \sigma_n = \bar{\sigma} \neq 0$, up to a subsequence, we have $\sigma_n \in (\frac{\bar{\sigma}}{2}, 2\bar{\sigma})$ and
\[
\frac{2p}{\bar{\sigma}} \int_{B(x_n, \sigma_n)} |u_n(y)|^p \, dy \geq C_1 > 0,
\]
which gives
\[
\int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geq \frac{C_1 \bar{\sigma}^p}{2p} > 0. \tag{3.3}
\]

**Step 3.** We show that $\{x_n\}$ is a bounded sequence.

By way of contradiction, we can assume $|x_n| \to \infty$ as $n \to \infty$. According to Lemma 2.2, we have
\[
\sup_{|x| = 0} |u_n(x)| \leq \frac{C}{|x|^{\frac{p(N-1)-q(p-1)}{p}}} \left\| u_n \right\|_{W^{1,p}(\mathbb{R}^N, \alpha)} \leq \frac{C}{|x|^{\frac{p(N-1)-q(p-1)}{p}}}.
\]

For any $\left(\frac{C \bar{\sigma}^p}{2p}||B(0,2\bar{\sigma})||\right)^{\frac{1}{2}} > \varepsilon > 0$, there exists an $M > 0$ such that for any $n > M$ there holds
\[
|u_n(x)| \leq \frac{C}{\left| |x_n| - \sigma_n \right|^{\frac{p(N-1)-q(p-1)}{p}}} \leq \varepsilon, \quad x \in B^c(0, \left| |x_n| - \sigma_n \right|).
\]

From $B(x_n, \sigma_n) \subset B^c(0, \left| |x_n| - \sigma_n \right|)$, it follows that
\[
\int_{B(x_n, \sigma_n)} |u_n(y)|^p \, dy \leq \varepsilon^p \int_{B(x_n, \sigma_n)} \, dy = \varepsilon^p |B(x_n, \sigma_n)| = \varepsilon^p |B(0, \sigma_n)| \leq \varepsilon^p |B(0, 2\bar{\sigma})| < \frac{C_1 \bar{\sigma}^p}{2p}.
\]

**Step 4.** Note that $\{x_n\}$ is bounded. There exists $0 < \hat{C} < \infty$ such that $0 \leq |x_n| < \hat{C}$. In view of $\lim_{n \to \infty} \sigma_n = \bar{\sigma} \neq 0$, up to a subsequence, we have
\[
B(x_n, \sigma_n) \subset B(0, |x_n| + \sigma_n) \subset B(0, \hat{C} + 2\bar{\sigma}).
\]

From (3.1), we deduce that
\[
0 < C_1 \bar{\sigma}^p = C_1 \lim_{n \to \infty} \sigma_n^p
\]
\[
\leq \lim_{n \to \infty} \left( \sigma_n^p \lim_{n \to \infty} \left( \frac{\sigma_n^p}{\left| |x_n| - \sigma_n \right|^{\frac{p(N-1)-q(p-1)}{p}}} \right) \right)
\]
\[
= \lim_{n \to \infty} \int_{B(x_n, \sigma_n)} |u_n|^p \, dy
\]
\[
\leq \lim_{n \to \infty} \int_{B(0, \hat{C}+2\bar{\sigma})} |u_n|^p \, dy.
\]
Applying the embedding $W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \to L^{1,p}_{rad}(\mathbb{R}^N) \to L^{p}_{loc}(\mathbb{R}^N)$, we obtain $u \neq 0$. \qed
4 Proof of Theorem 1.2

In this section, we prove that any nonnegative weak solutions of equation (Ω) with \( \alpha \in (0, p) \) have additional regularity properties.

**Lemma 4.1.** Assume that all the conditions described in Theorem 1.2 hold. For each \( L > 2 \), define

\[
U_L(x) = \begin{cases} 
U(x) & \text{if } U(x) \leq L, \\
L & \text{if } U(x) > L.
\end{cases}
\]

Set \( \bar{U}_L = U_L^{p(|\mu - 1|)} \), where \( \mu > 1 \). Let \( \bar{C}_t \) be the best embedding constant from \( W^{1,p}_{\text{rad}}(\mathbb{R}^N, \alpha) \) to \( L'(\mathbb{R}^N) \) for \( t \in [p^*, p] \). Then we have

\[
\left( \int_{\mathbb{R}^N} |U_U^{p(|\mu - 1|)}|^p \right)^{\frac{1}{p}} \leq \frac{C_\mu}{\bar{C}_t} \left[ \int_{\mathbb{R}^N} U^{p_2} - p |U_U^{p(|\mu - 1|)}|^p \right] + \int_{\mathbb{R}^N} U^{p - p_1} |U_U^{p(|\mu - 1|)}|^p \right]. \tag{4.1}
\]

**Proof.** Let \( U \) be a nonnegative ground state solution of equation (Ω). We show that \( \bar{U}_L \in W^{1,p}_{\text{rad}}(\mathbb{R}^N, \alpha) \). By a straightforward calculation, we get

\[
\int_{\mathbb{R}^N} |\nabla(UU_L^{p(|\mu - 1|)})|^p dx = \int_{\mathbb{R}^N} |U_L^{p(|\mu - 1|)}|U + p(\mu - 1)UU_L^{p(|\mu - 1|) - 1} \nabla U_L|^p dx \leq 2^p \int_{\mathbb{R}^N} |U_L^{p(|\mu - 1|)}|\nabla U|^p dx + (2p)^p(\mu - 1)^p \int_{\mathbb{R}^N} |UU_L^{p(|\mu - 1|) - 1}|\nabla U_L|^p dx = 2^p \int_{\mathbb{R}^N} |U_L^{p(|\mu - 1|)}|\nabla U|^p dx + (2p)^p(\mu - 1)^p \int_{\{x: U(x) \leq L\}} |U_L^{p(|\mu - 1|)}|\nabla U|^p dx \leq 2^p L^{p(|\mu - 1|)}(1 + (\mu - 1)^p) \int_{\mathbb{R}^N} |\nabla U|^p dx,
\]

and

\[
\int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |UU_L^{p(|\mu - 1|)}|^p dx \leq L^{p(|\mu - 1|)} \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |U|^p dx.
\]

This implies that \( \bar{U}_L \in W^{1,p}_{\text{rad}}(\mathbb{R}^N, \alpha) \).

Note that \( U \) is a nonnegative ground state solution of equation (Ω). Then

\[
\int_{\mathbb{R}^N} |\nabla U|^p - 2 \nabla U \nabla \varphi dx + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |U|^p - 2 U \varphi dx = \int_{\mathbb{R}^N} f(U) \varphi dx.
\]

Substituting \( \bar{U}_L \) into the above equation, we get

\[
\int_{\mathbb{R}^N} |\nabla \bar{U}_L|^p - 2 \nabla \bar{U}_L \nabla \bar{U}_L dx + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |UU_L^{p(|\mu - 1|)}|^p dx = \int_{\mathbb{R}^N} f(U) \bar{U}_L dx. \tag{4.2}
\]

Since

\[
\int_{\mathbb{R}^N} UUU_L^{p(|\mu - 1|) - 1} |\nabla U|^p - 2 U \nabla U \nabla U_L dx = \int_{\{x: U(x) \leq L\}} UU_L^{p(|\mu - 1|)} |\nabla U|^p dx \geq 0,
\]

it follows that

\[
\int_{\mathbb{R}^N} |\nabla \bar{U}_L|^p - 2 \nabla \bar{U}_L \nabla \bar{U}_L dx
\]

\[
= \int_{\mathbb{R}^N} UU_L^{p(|\mu - 1|)} |\nabla U|^p dx + p(\mu - 1) \int_{\mathbb{R}^N} UUU_L^{p(|\mu - 1|) - 1} |\nabla U|^p - 2 U \nabla U \nabla U_L dx
\]

\[
\geq \int_{\mathbb{R}^N} UU_L^{p(|\mu - 1|)} |\nabla U|^p dx. \tag{4.3}
\]
Note that
\[ \int_{\mathbb{R}^n} |\nabla (UU_L^{\mu-1})|^p \, dx \leq [2^p + (2p)^p (\mu - 1)^p] \int_{\mathbb{R}^n} U_L^{\mu-1} |\nabla U|^p \, dx \]
\[ \leq (2p)^p \mu^p \int_{\mathbb{R}^n} U_L^{\mu-1} |\nabla U|^p \, dx. \]  \tag{4.4} 

It follows from (4.2)-(4.4) and Proposition 1.1 that
\[ \hat{c}_t \left( \int_{\mathbb{R}^n} |UU_L^{\mu-1}|^t \, dx \right)^{\frac{p}{p-1}} \leq (2p)^p \mu^p \left( \int_{\mathbb{R}^n} |\nabla U|^2 |\nabla U_L| \, dx \right) + \int_{\mathbb{R}^n} \frac{A}{|x|^d} |UU_L^{\mu-1}|^p \, dx \]
\[ \leq C \mu^p \left[ \int_{\mathbb{R}^n} U_L^{p^*-p} |UU_L^{\mu-1}|^p \, dx + \int_{\mathbb{R}^n} U_L^{p^*-p} |UU_L^{\mu-1}|^p \, dx \right]. \]

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We divide the proof into three parts.

**Part (i).** We first show the $L^\infty$ estimate of $U$ by the following four steps.

**Step 1.** We claim that
\[ \left( 1 + \int_{\mathbb{R}^n} |U|^p \, dx \right)^{\frac{p}{p-1}} \leq \infty, \]
where $\mu_1 := 1 + \frac{p^*-p^*}{p^*}$.

Taking $\bar{a} \in \mathbb{R}^+$ and using Hölder’s inequality, we can derive
\[ \int_{\mathbb{R}^n} U^{p^*-p} |UU_L^{\mu-1}|^p \, dx \]
\[ \leq \bar{a}^{p^*-p} \int_{\{x \in U(x) \leq 2\}} U^{p^*-p} |UU_L^{\mu-1}|^p \, dx + \int_{\{x \in U(x) \geq 2\}} U^{p^*-p} |UU_L^{\mu-1}|^p \, dx \]
\[ \leq \bar{a}^{p^*-p} \int_{\mathbb{R}^n} |U|^{p^*-p} \, dx + \left( \int_{\{x \in U(x) \geq 2\}} U^{p^*} \, dx \right)^{\frac{p^*}{p}} \left( \int_{\mathbb{R}^n} |UU_L^{\mu-1}|^p \, dx \right)^{\frac{p}{p^*}}. \]

We choose $\bar{a}$ such that
\[ \left( \int_{\{x \in U(x) \geq 2\}} U^{p^*} \, dx \right)^{\frac{p^*}{p}} \leq \frac{C \mu^p}{2}. \]

Substituting the above two inequalities into (4.1) with the choice of $t = p^*$, we get
\[ \left( \int_{\mathbb{R}^n} |UU_L^{\mu-1}|^p \, dx \right)^{\frac{p}{p^*}} \leq \frac{2C \mu^p}{C_{p^*}} \left[ \int_{\mathbb{R}^n} U^{p_1^*-p} |UU_L^{\mu-1}|^p \, dx + \bar{a}^{p^*-p^*} \int_{\mathbb{R}^n} |U|^{p^*-p} U_L^{p_1^*-p} \, dx \right]. \]

Taking the limit as $L \to \infty$ in the above inequality leads to
\[ \left( \int_{\mathbb{R}^n} |U|^p \, dx \right)^{\frac{p}{p^*}} \leq \frac{2C \mu^p}{C_{p^*}} \int_{\mathbb{R}^n} |U|^{p^*_1} \, dx. \]

Let $p_1^* + \mu - p \in (p_1^*, p^*)$ and choose $\mu \in \left( 1, 1 + \frac{p^*_1 - p_1^*}{p^*} \right]$. Then
\[ U \in L^{p^*_1} (\mathbb{R}^n). \]

Hence, we have
\[ U \in L^{p^*_1} (\mathbb{R}^n), \quad \bar{p}_1 \in [p_1^*, p^*] \cup \left( p^*, \frac{p^* - p_1^*}{p^*} \left( 1 + \frac{p^*_1 - p_1^*}{p^*} \right) \right). \]
We now choose \( \mu_1 := 1 + \frac{p^* - p}{p} \). Then we obtain \( p^* \mu_1 \in \left[ p^*, p^* \left( 1 + \frac{p^* - p}{p} \right) \right] \) and
\[
\left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p^* - 1}} < \infty.
\]

**Step 2.** We show that
\[
\left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p^* - 1}} \leq (C \mu_2)^{\frac{1}{p^* - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p^* - 1}},
\]
where \( \mu_2 - 1 = \left( \frac{p^*}{p} \right) (\mu_1 - 1) \).

Let \( \mu_2 := 1 + \frac{p^*}{p} (\mu_1 - 1) \). Then \( p^* + p \mu_2 - p = p^* \mu_1 \) and \( \mu_2 - 1 = \left( \frac{p^*}{p} \right) (\mu_1 - 1) \). Let \( \mu \in [\mu_1, \mu_2] \). We find
\[
p^*_a < p^*_a + p \mu - p < p^* + p \mu - p \leq p^* \mu_1.
\]

From Lemma 4.1, we get
\[
\left( \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p - 1}} \leq C \mu_2 \left[ \int_{\mathbb{R}^N} |U|^{p^* \mu_1 + p - p} \, dx + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_1 - p} \, dx \right] < \infty.
\]

For each \( \mu \in [\mu_1, \mu_2] \), we have \( U \in L^{p^*}(\mathbb{R}^N) \) and
\[
U \in L^{\frac{p^*}{p}}(\mathbb{R}^N), \quad \hat{p}_2 \in [p^*_a, p^* \mu_1] \cup [p^* \mu_1, p^* \mu_2].
\]

Set \( \mu = \mu_2 \). Then
\[
\left( \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p - 1}} \leq C \mu_2 \left[ \int_{\mathbb{R}^N} |U|^{p^* \mu_1 + p - p} \, dx + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_1 - p} \, dx \right] < \infty.
\]

Let \( a_2 = \frac{p^*(p^* - p)}{p(\mu_2 - 1)} \) and \( b_2 = p^*_a + p \mu_2 - p - a_2 \). Then \( \frac{p^* b_2}{b_2 - a_2} = p^* + p \mu_2 - p \). It follows from Young’s inequality that
\[
\int_{\mathbb{R}^N} |U|^{p^* \mu_1 + p - p} \, dx = \int_{\mathbb{R}^N} |U|^{a_2} \, dx \leq C \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_1 - p} \, dx \right).
\]

Thus, we deduce
\[
\left( \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p - 1}} \leq C \mu_2 \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_1 - p} \, dx \right).
\]

For every \( x_1, x_2 > 0 \), we know that
\[
(x_1 + x_2)^{\frac{p}{2}} \leq x_1^{\frac{p}{2}} + x_2^{\frac{p}{2}}.
\]

We then obtain
\[
\left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p - 1}} \leq 1 + \left( \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p - 1}} \leq C \mu_2 \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_1 - p} \, dx \right).
\]

That is,
\[
\left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p^* - 1}} \leq (C \mu_2)^{\frac{1}{p^* - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{1}{p^* - 1}} \leq (C \mu_2)^{\frac{1}{p^* - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* \mu_1} \, dx \right)^{\frac{p}{p^* - 1}}.
\]
Step 3. We show that
\[
(1 + \int_{\mathbb{R}^N} |U|^{p'}^\mu_3 \, dx)^{\frac{p}{p' \mu_3 - 1}} \leq (C \mu_3)^{\frac{p}{p' \mu_3 - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p}^\mu \, dx \right)^{\frac{p}{p' \mu_3 - 1}},
\]
where \( \mu_3 - 1 = (\frac{p}{p'}) (\mu_2 - 1) \).

Let \( \mu_3 := 1 + \frac{p}{p'} (\mu_2 - 1) \). Then \( p^* + p \mu_3 - p = p^* \mu_2 \) and \( \mu_3 - 1 = (\frac{p}{p'}) (\mu_2 - 1) \). Let \( \mu \in [\mu_2, \mu_3] \). Then
\[
p^*_\mu < p^*_\mu + p \mu - p < p^* + p \mu - p \leq p^* \mu_2.
\]
So we get
\[
\left( \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx \right)^{\frac{p}{p^*}} \leq \frac{C \mu_3}{p^*} \left[ \int_{\mathbb{R}^N} |U|^{p^* + p \mu_3 - p} \, dx + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_3 - p} \, dx \right] < \infty.
\]
For each \( \mu \in [\mu_2, \mu_3] \), we have \( U \in L^{\tilde{p}}(\mathbb{R}^N) \) and
\[
U \in L^{\tilde{p}_1}(\mathbb{R}^N), \quad \tilde{p}_1 \in [p^*_\mu, p^*_\mu_3] \cap [p^* \mu_2, p^* \mu_3].
\]
Set \( \mu = \mu_3 \). Then
\[
\left( \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx \right)^{\frac{p}{p^*}} \leq \frac{C \mu_3}{p^*} \left[ \int_{\mathbb{R}^N} |U|^{p^* + p \mu_3 - p} \, dx + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_3 - p} \, dx \right] < \infty.
\]
Let \( a_3 = \frac{p^* (p^*_\mu)}{p^* (p^*_\mu - 1)} \) and \( b_3 = p^*_\mu + p \mu_3 - p - a_3 \). Then \( \frac{p^* a_3}{p^* - a_3} = p^* + p \mu_3 - p \). It follows from Young’s inequality that
\[
\int_{\mathbb{R}^N} |U|^{p^* + p \mu_3 - p} \, dx \leq C \left( 1 + \int_{\mathbb{R}^N} |U|^{p^* + p \mu_3 - p} \, dx \right).
\]
Hence, we obtain
\[
(1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx)^{\frac{p}{p^* \mu_3 - 1}} \leq (C \mu_3)^{\frac{p}{p^* \mu_3 - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx \right)^{\frac{p}{p^* \mu_3 - 1}}.
\]
Step 4. Iterating the above process and recalling that
\[
p^* + p \mu_{i+1} - p = p^* \mu_i, \quad i \geq 1 \text{ and } i \in \mathbb{N},
\]
we have
\[
\mu_{i+1} - 1 = \left( \frac{p^*}{p} \right)^i (\mu_1 - 1)
\]
and
\[
\left( 1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu_{i+1} \, dx \right)^{\frac{p}{p \mu_{i+1} - 1}} \leq (C \mu_{i+1})^{\frac{p}{p \mu_{i+1} - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx \right)^{\frac{1}{\mu_{i+1} - 1}}
\]
\[
= (C \mu_{i+1})^{\frac{p}{p \mu_{i+1} - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx \right)^{\frac{p}{p \mu_{i+1} - 1}}.
\]
For \( m \in \mathbb{N} \), we further get
\[
\prod_{i=1}^{m} \left( C \mu_{i+1} \right)^{\frac{p}{p \mu_{i+1} - 1}} \left( 1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu \, dx \right)^{\frac{p}{p \mu_{i+1} - 1}}
\]
\[
\geq \left( 1 + \int_{\mathbb{R}^N} |U|^{p^*}^\mu_{m+1} \, dx \right)^{\frac{p}{p \mu_{m+1} - 1}}
\]
\[
\geq \left( \int_{\mathbb{R}^N} |U|^{p^*}^\mu_{m+1} \, dx \right)^{\frac{p}{p \mu_{m+1} - 1}}.
\]
If \( \int_{\mathbb{R}^N} |U|^p \mu_{m+1} \, dx \leq 1 \), then

\[
\prod_{i=1}^{m}(C \mu_{i+1})^{\frac{p}{n_{i+1}-1}} \left( 1 + \int_{\mathbb{R}^N} |U|^p \mu_1 \, dx \right)^{\frac{p}{n_{i+1}-1}} \geq \left( \int_{\mathbb{R}^N} |U|^p \mu_{m+1} \, dx \right)^{\frac{p}{n_{m+1}-1}} = \|U\|_{L^p(\mathbb{R}^N)}^{\frac{p}{n_{m+1}-1}}.
\]

That is,

\[
\|U\|_{L^p(\mathbb{R}^N)} \leq \prod_{i=1}^{m}(C \mu_{i+1})^{\frac{p}{n_{i+1}-1}} \left( 1 + \int_{\mathbb{R}^N} |U|^p \mu_1 \, dx \right)^{\frac{p}{n_{i+1}-1}}.
\]

(4.5)

If \( \int_{\mathbb{R}^N} |U|^p \mu_{m+1} \, dx > 1 \), then

\[
\prod_{i=1}^{m}(C \mu_{i+1})^{\frac{p}{n_{i+1}-1}} \left( 1 + \int_{\mathbb{R}^N} |U|^p \mu_1 \, dx \right)^{\frac{p}{n_{i+1}-1}} \geq \left( \int_{\mathbb{R}^N} |U|^p \mu_{m+1} \, dx \right)^{\frac{p}{n_{m+1}-1}} = \|U\|_{L^p(\mathbb{R}^N)}^{\frac{p}{n_{m+1}-1}}.
\]

That is,

\[
\|U\|_{L^p(\mathbb{R}^N)} \leq \prod_{i=1}^{m}(C \mu_{i+1})^{\frac{p}{n_{i+1}-1}} \left( 1 + \int_{\mathbb{R}^N} |U|^p \mu_1 \, dx \right)^{\frac{p}{n_{i+1}-1}}.
\]

(4.6)

Note that

\[
\lim_{m \to \infty} \prod_{i=1}^{m}(C \mu_{i+1})^{\frac{p}{n_{i+1}-1}} = \lim_{m \to \infty} e^{p \sum_{i=1}^{m} \left( \frac{\ln C}{n_{i+1}} + \frac{\ln \mu_{i+1}}{n_{i+1}} \right)}.
\]

For the series \( \sum_{i=1}^{\infty} \frac{\ln C}{\mu_{i+1}-1} \), using the root test, we get

\[
\lim_{i \to \infty} \sqrt[i]{\frac{\ln C}{\mu_{i+1}-1}} = \lim_{i \to \infty} \sqrt[i]{\frac{\ln C}{\left( \frac{p}{p^*} \right)^i (\mu_1 - 1)}} = \frac{p}{p^*} < 1,
\]

which indicates \( \sum_{i=1}^{\infty} \frac{\ln C}{\mu_{i+1}-1} < \infty \).

For the series \( \sum_{i=1}^{\infty} \frac{\ln \mu_{i+1}}{\mu_{i+1}-1} \), by using the ratio test, we find

\[
\lim_{i \to \infty} \frac{\ln \mu_{i+2}}{\ln \mu_{i+1}} \cdot \frac{\mu_{i+1}-1}{\ln \mu_{i+1}} = \frac{p}{p^*} \lim_{i \to \infty} \frac{\ln \left( 1 + \frac{p}{p^*} (\mu_{i+1} - 1) \right)}{\ln \mu_{i+1}}
\]

\[
< \frac{p}{p^*} \lim_{i \to \infty} \frac{\ln \left( \frac{p}{p^*} (\mu_{i+1} - 1) \right)}{\ln \mu_{i+1}}
\]

\[
= \frac{p}{p^*} \lim_{i \to \infty} \left( \frac{\ln \frac{p}{p^*}}{\ln \mu_{i+1}} + \frac{\ln \mu_{i+1}}{\ln \mu_{i+1}} \right)
\]

\[
< 1,
\]

which implies \( \sum_{i=1}^{\infty} \frac{\ln \mu_{i+1}}{\mu_{i+1}-1} < \infty \). Hence, we have \( \prod_{i=1}^{\infty} (C \mu_{i+1})^{\frac{p}{n_{i+1}-1}} < \infty \).

Letting \( m \to \infty \) in (4.5) and (4.6), we obtain

\[
\|U\|_{L^p(\mathbb{R}^N)} < \infty.
\]
Lemma 5.1. We rewrite equation (Ω) as $-\Delta_p U = g(x, U)$, where
\[
g(x, U) = -\frac{A}{|x|^a} U + f(U).
\]
For all $\Omega \subset \subset \mathbb{R}^N \setminus \{0\}$, there exists $C(\Omega) > 0$ such that $|g(x, U)| \leq C(\Omega)(1 + |U|^{p-1})$ for $x \in \Omega$. It follows from Theorem 1.2 (i) and [21, Theorem 1 and Proposition 1] that $U \in C^1(\mathbb{R}^N \setminus \{0\})$. Finally, the strict positivity follows form the strong maximum principle [22].

Part (ii). From Proposition 1.1, we know that $J_1$ follows form the strong maximum principle [22]. Consequently, the proof is completed.

Part (iii). Applying Theorem 1.2 (ii) and following [8, Claim 5.3], we can derive the Pohožaev-type identity
\[
\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla U|^p dx + \frac{N-a}{p} \int_{\mathbb{R}^N} \frac{A}{|x|^a} |U|^p dx = N \int_{\mathbb{R}^N} F(U) dx.
\]
Consequently, the proof is completed.

5 Proof of Theorem 1.3

As we see, equation $(S_{p,a})$ is variational and its solutions are the critical points of the functional defined in $W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ by
\[
J(u) := \frac{1}{p} |u|^p_{W^{1,p}_{rad}(\mathbb{R}^N, \alpha)} - \frac{A}{p_a} \int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx.
\]
From Proposition 1.1, we know that $J \in C^1(W^{1,p}_{rad}(\mathbb{R}^N, \alpha), \mathbb{R})$. It is easy to see that if $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ is a critical point of $J$, i.e.
\[
0 = \langle J'(u), \phi \rangle = \int_{\mathbb{R}^N} (|u|^{p-2}u\nabla \phi + \frac{A}{|x|^a} |u|^{p-2}u) dx
\]
\[
- \frac{A}{p_a} \int_{\mathbb{R}^N} |u|^p dx - \int_{\mathbb{R}^N} |u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx
\]
for all $\phi \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$, then $u$ is a weak solution of equation $(S_{p,a})$.
Define
\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) > 0,
\]
where
\[
\Gamma = \left\{ \gamma \in C\left([0,1], W^{1,p}_{rad}(\mathbb{R}^N, \alpha)\right) \mid \gamma(0) = 0, \gamma(1) < 0 \right\}.
\]
Define
\[
N = \{ u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \mid \langle J'(u), u \rangle = 0, u \neq 0 \},
\]
and let
\[
\tilde{c} = \inf_{u \in N} J(u).
\]
It is easy to check that $c = \tilde{c} = \hat{c} := \inf_{u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)} \sup_{t \geq 0} J(tu) > 0$.

Lemma 5.1. Assume that all the conditions described in Theorem 1.3 hold. Then
\[
0 < c < c^* := \min\left\{ \frac{\alpha}{Np} \left( \frac{1}{\lambda} \right)^{\frac{\alpha}{p-\alpha}} S_{a,\alpha}^{\frac{\lambda^p}{p}}, \frac{\alpha}{N} \lambda S_{\alpha}^{\frac{\lambda^p}{p-\alpha}} \right\}.
\]

Proof. If $\left( \frac{\alpha}{p} S_{a,\alpha}^{\frac{\lambda^p}{p-\alpha}} S_{\alpha}^{\frac{\lambda^p}{p-\alpha}} \right)^{\frac{p}{p-\alpha}} \geq \lambda > 0$, then
\[
\frac{\alpha}{Np} \left( \frac{1}{\lambda} \right)^{\frac{\alpha}{p-\alpha}} S_{a,\alpha}^{\frac{\lambda^p}{p-\alpha}} \geq \frac{1}{N} S_{\alpha}^{\frac{\lambda^p}{p-\alpha}}.
\]
We choose
\[
\omega_\sigma(x) = \frac{Ca^{n/p}}{(\sigma^{p/\alpha} + |x|^{p/\alpha})^{N-p}}
\]
and
\[
||w_\sigma||_{D^{1,p}(\mathbb{R}^N)}^p = ||w_1||_{D^{1,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |w_1|^p dx = \int_{\mathbb{R}^N} |w_\sigma|^p dx = S^{-p}.
\]
A straightforward calculation gives
\[
\int_{\mathbb{R}^N} \frac{|w_\sigma|^p}{|x|^a} dx \leq C \int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha}} \frac{\sigma^{N-p}}{(\sigma^{p/\alpha} + |x|^{p/\alpha})^{N-p}} dx = \int_{0}^{\infty} \frac{1}{\rho^{a}} \frac{1}{(\sigma^{p/\alpha} + \rho^{p/\alpha})^{N-p}} \rho^{N-1-a} d\rho.
\]
\[
\leq C \int_{0}^{\infty} \frac{\rho^{N-1-a}}{(\sigma^{p/\alpha} + \rho^{p/\alpha})^{N-p}} d\rho + C \int_{1}^{\infty} \frac{\rho^{N-1-a}}{(\sigma^{p/\alpha} + \rho^{p/\alpha})^{N-p}} d\rho.
\]
It is not difficult to see that
\[
\int_{0}^{1} \frac{\rho^{N-1-a}}{(\sigma^{p/\alpha} + \rho^{p/\alpha})^{N-p}} d\rho \leq \int_{0}^{1} \frac{\rho^{N-1-a}}{(\sigma^{p/\alpha})^{N-p}} d\rho < \infty.
\]
In view of \(p \in (1, \sqrt{N})\), we have \(N - 1 - a - \frac{p(N-p)}{p-1} < -1\) and
\[
\int_{1}^{\infty} \frac{\rho^{N-1-a}}{(\sigma^{p/\alpha})^{N-p}} d\rho < \infty.
\]
This implies that \(w_\sigma \in W_{rad}^{1,p}(\mathbb{R}^N, \alpha)\). So we have
\[
\int_{\mathbb{R}^N} |w_\sigma|^p dx = \sigma^{p-\frac{p(N-p)}{p-1}} \int_{\mathbb{R}^N} |w_1|^p dx,
\]
\[
\int_{\mathbb{R}^N} |w_\sigma|^q dx = \sigma^{q-\frac{q(N-q)}{q-1}} \int_{\mathbb{R}^N} |w_1|^q dx,
\]
\[
\int_{\mathbb{R}^N} \frac{1}{|x|^a} |w_\sigma|^p dx = \sigma^{p-a} \int_{\mathbb{R}^N} \frac{1}{|x|^a} |w_1|^p dx,
\]
and \(\lim_{t \to \infty} J(tw_\sigma) = -\infty\).
Let \(t_0 > t_0 > 0\) satisfy
\[
\sup_{t \geq 0} J(tw_\sigma) = J(t_0w_\sigma) \quad \text{and} \quad J(t_0w_\sigma) < 0.
\]
Taking \(\gamma(t) = t_0w_\sigma\), we get
\[
c \leq \max_{t \in [0,1]} J(\gamma(t)) = J(t_0w_\sigma).
\]
Note that
\[
0 = \frac{d}{dt} \bigg|_{t=t_0} J(tw_\sigma)
\]
\[
= \left(t_0^{-1} - t_0^{-1}\right) ||w_1||_{D^{1,p}(\mathbb{R}^N)}^p + \sigma^{p-a} t_0^{-1} \int_{\mathbb{R}^N} \frac{A}{|x|^a} |w_1|^p dx
\]
\[
- \lambda \sigma^{p-\frac{p(N-p)}{p-1}} t_0^{-1} \int_{\mathbb{R}^N} |w_1|^p dx - \sigma^{q-\frac{q(N-q)}{q-1}} t_0^{-1} \int_{\mathbb{R}^N} |w_1|^q dx.
\]
Let \( t_0 := \limsup t_n \). We claim that \( t_0 < \infty \). Otherwise, we assume that \( t_0 = \infty \). Taking an upper limit as \( \sigma \to 0 \) in (5.1), we get
\[
\limsup_{\sigma \to 0} \left( t_0^{p-1} \| w_1 \|_{D^{1,p}(\mathbb{R}^N)}^p + \sigma^{p-a} t_0^{p-1} \int_{\mathbb{R}^N} \frac{A}{|x|^a} |w_1|^p \, dx \right) \\
= \limsup_{\sigma \to 0} \left( t_0^{p-1} \| w_1 \|_{D^{1,p}(\mathbb{R}^N)}^p + \lambda \sigma^{p-\frac{p(N+a)}{p-1}} t_0^{p-1} \int_{\mathbb{R}^N} \xi_1^p \, dx \right) \\
+ \sigma^{q-\frac{p(N+a)}{p-q}} t_0^{q-1} \int_{\mathbb{R}^N} |w_1|^q \, dx \right).
\]
(5.2)

It follows from \( p < p^* \) and \( \limsup t_n = \infty \) that
\[
\limsup_{\sigma \to 0} t_0^{p-1} \left( \| w_1 \|_{D^{1,p}(\mathbb{R}^N)}^p + \sigma^{p-a} \int_{\mathbb{R}^N} \frac{A}{|x|^a} |w_1|^p \, dx \right) \\
< \limsup_{\sigma \to 0} t_0^{p-1} \| w_1 \|_{D^{1,p}(\mathbb{R}^N)} \\
\leq \limsup_{\sigma \to 0} \left( t_0^{p-1} \| w_1 \|_{D^{1,p}(\mathbb{R}^N)} + \lambda \sigma^{p-\frac{p(N+a)}{p-1}} t_0^{p-1} \int_{\mathbb{R}^N} \xi_1^p \, dx \right) \\
+ \sigma^{q-\frac{p(N+a)}{p-q}} t_0^{q-1} \int_{\mathbb{R}^N} |w_1|^q \, dx \right).
\]

This contradicts (5.2). That is, \( t_0 < \infty \).

Passing to an upper limit as \( \sigma \to 0 \) in (5.1) leads to
\[
0 = \left( t_0^{p-1} - t_0^{p-1} \right) \| w_1 \|_{D^{1,p}(\mathbb{R}^N)},
\]
which implies
\[
t_0^{p-1} - t_0^{p-1} = 0 \quad \text{and} \quad t_0 = 1.
\]

Let \( \{ \sigma_n \} \) be a sequence such that \( \sigma \to 0 \) as \( n \to \infty \). Then, up to a subsequence, still defined by \( \{ \sigma_n \} \), we have
\[
t_0 = \frac{t_0}{2} < t_{\sigma_n}.
\]
(5.3)

Hence, we can choose \( \delta > 0 \) small enough such that
\[
t_0 < \frac{t_0}{2} < t_0 \neq t_0.
\]

Set
\[
g(t) = \frac{t^p}{p} - \frac{t^{p^*}}{p^*} \quad \text{and} \quad g'(t) = t^{p-1} - t^{p-1}.
\]

Then, we have \( g'(t_0) = 0, g'(t) < 0 \) for \( t > t_0 \), and \( g'(t) > 0 \) for \( t < t_0 \). Hence, \( g(t) \) attains its maximum at \( t_0 \).

That is,
\[
g(t) < g(t_0) = \frac{1}{N}
\]
(5.4)

for any \( t \neq t_0 \).

It follows from \( q > \frac{p(N+a-p)}{N} \) and (5.3)-(5.4) that
\[
J(t_0 w_0) = \left( t_0^p - t_0^{p^*} \right) \| w_1 \|_{D^{1,p}(\mathbb{R}^N)} + \delta^{p-a} t_0 \int_{\mathbb{R}^N} A \frac{1}{|x|^a} |w_1|^p \, dx - \frac{\delta^{q-\frac{p(N+a)}{p-1}}}{q} t_0^q \int_{\mathbb{R}^N} |w_1|^q \, dx
\]
\[
\leq \left( \frac{t_0^p}{p} - \frac{t_0^{p^*}}{p^*} \right) \| w_1 \|_{D^{1,p}(\mathbb{R}^N)}
\]
\[
= \left( \frac{t_0^p}{p} - \frac{t_0^{p^*}}{p^*} \right) S^{\frac{p^*}{p}}
\]
\[
< \frac{1}{N} S^{\frac{p^*}{p}}
\]
for sufficiently small \( \delta \). Consequently, we arrive at the desired result.
\[
\square
\]
5.1 Perturbation Equation

Applying Theorems 1.1 and 1.2, it is easy to prove the existence of positive ground state solution of the following equation (with small $\varepsilon > 0$, see [18]):

$$-\Delta_p u + \frac{A}{|x|^{q-2}} u = |u|^{p^*-2} u + |u|^{q-2} u + \lambda |u|^p x + \varepsilon^2 u, \quad x \in \mathbb{R}^N. \quad (S_{p^*+\varepsilon})$$

Set the energy functional of equation $(S_{p^*+\varepsilon})$ as follows:

$$J_{\varepsilon}(u) := \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + \frac{A}{|x|^a} |u|^p)dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q dx - \frac{1}{p^*-\varepsilon} \int_{\mathbb{R}^N} |u|^{p^*-\varepsilon} dx - \frac{\lambda}{p^*} \int_{\mathbb{R}^N} |u|^p dx + \varepsilon \int_{\mathbb{R}^N} |u|^p x + \varepsilon^2 dx.$$

Let $v_{\varepsilon}$ be a positive ground state solution of equation $(S_{p^*+\varepsilon})$. For all $\varphi \in W_{rad}^{1,p}(\mathbb{R}^N)$, it follows that

$$0 = (J'_{\varepsilon}(v_{\varepsilon}), \varphi) = \int_{\mathbb{R}^N} (|\nabla v_{\varepsilon}|^p + \frac{A}{|x|^a} |v_{\varepsilon}|^p)\nabla v_{\varepsilon} \nabla \varphi - \frac{1}{q} \int_{\mathbb{R}^N} |v_{\varepsilon}|^q \varphi dx - \frac{1}{p^*-\varepsilon} \int_{\mathbb{R}^N} |v_{\varepsilon}|^{p^*-\varepsilon} \varphi dx,$$

$$0 = P_{\varepsilon}(v_{\varepsilon}) := \frac{1}{p^*} \int_{\mathbb{R}^N} |\nabla v_{\varepsilon}|^p dx + \frac{N-a}{pN} \int_{\mathbb{R}^N} A |v_{\varepsilon}|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} |v_{\varepsilon}|^q dx - \frac{1}{p^*-\varepsilon} \int_{\mathbb{R}^N} |v_{\varepsilon}|^{p^*-\varepsilon} dx,$$

and

$$c_{\varepsilon} = J_{\varepsilon}(v_{\varepsilon}) - \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v_{\varepsilon}|^p + \frac{A}{|x|^a} |v_{\varepsilon}|^p)dx - \frac{1}{q} \int_{\mathbb{R}^N} |v_{\varepsilon}|^q dx - \frac{1}{p^*-\varepsilon} \int_{\mathbb{R}^N} |v_{\varepsilon}|^{p^*-\varepsilon} dx.$$

We then have the following lemma for equation $(S_{p^*+\varepsilon})$.

**Lemma 5.2.** Assume that all the conditions described in Theorem 1.3 hold. Then the following statements are true.

(i) For each $u \in W_{rad}^{1,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $\tau_{\varepsilon} > 0$ such that $P_{\varepsilon}(u_{\tau_{\varepsilon}}) = 0$ for $\varepsilon \in (0, \varepsilon_0)$, where

$$u_{\tau}(x) = \begin{cases} u(\frac{x}{\tau}), & \tau > 0, \\ 0, & \tau = 0. \end{cases}$$

Moreover, we have $J_{\varepsilon}(u_{\tau_{\varepsilon}}) = \max_{\tau \geq 0} J_{\varepsilon}(u_{\tau}).$

(ii) $c_{\varepsilon} = c_{\varepsilon}^p$ for $\varepsilon \in (0, \varepsilon_0)$, where

$$c_{\varepsilon} := \inf \{ J_{\varepsilon}(u) | u \in W_{rad}^{1,p}(\mathbb{R}^N), \text{ and } J'_{\varepsilon}(u) = 0 \},$$

and

$$c_{\varepsilon}^p := \inf \{ J_{\varepsilon}(u) | u \in W_{rad}^{1,p}(\mathbb{R}^N), \text{ and } P_{\varepsilon}(u) = 0 \}.$$

(iii) $\limsup_{\varepsilon \to 0} c_{\varepsilon} \leq c.$

(iv) $c_{\varepsilon} \geq 0$ for $\varepsilon \in [0, \varepsilon_0]$.

(v) Let $\varepsilon_n \to 0^+$ and $\{v_{\varepsilon_n}\} \subset W_{rad}^{1,p}(\mathbb{R}^N)$ satisfy

$$J_{\varepsilon_n}(v_{\varepsilon_n}) = c_{\varepsilon_n}, \quad P_{\varepsilon_n}(v_{\varepsilon_n}) = 0, \quad J'_{\varepsilon_n}(v_{\varepsilon_n}) = 0.$$

Then, $\{v_{\varepsilon_n}\}$ is bounded in $W_{rad}^{1,p}(\mathbb{R}^N)$ and $\lim inf_{n \to \infty} c_{\varepsilon_n} > 0.$
Proof. (i) For each \( u \in W^{1,p}_{rad}(\mathbb{R}^N, a) \setminus \{0\} \), we set

\[
\varphi_\varepsilon(t) = J_\varepsilon(u_t) = \frac{t^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^{p} \, dx + \frac{t^{N-a}}{p} \int_{\mathbb{R}^N} \frac{A|u|^{p}}{|x|^{a}} \, dx - \frac{t^{N}}{q} \int_{\mathbb{R}^N} |u|^{q} \, dx
\]

A direct calculation gives

\[
\varphi'_\varepsilon(t) = \frac{N - p}{p} t^{N-p-1} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{N - a}{p} t^{N-a-1} \int_{\mathbb{R}^N} \frac{A|u|^p}{|x|^a} \, dx - \frac{N}{q} t^{N-1} \int_{\mathbb{R}^N} |u|^q \, dx
\]

In view of \( N \geq 3 \), \( p \in (1, \sqrt{N}) \) and \( a \in (0, p) \), we find that \( \varphi'_\varepsilon(t) > 0 \) for small \( t > 0 \) and \( \lim_{t \to 0} \varphi'_\varepsilon(t) < 0 \). Then there exists \( t_\varepsilon > 0 \) such that \( \varphi'_\varepsilon(t_\varepsilon) = 0 \) and \( J_\varepsilon(u_{t_\varepsilon}) = \max J_\varepsilon(u_t) \). Moreover, \( P_\varepsilon(u_{t_\varepsilon}) = \frac{1}{\varepsilon^p} \lim_{t \to 0} J'_\varepsilon(t) = 0 \).

(ii). On one hand, Theorem 1.2 implies that \( c_\varepsilon \geq c_\varepsilon^p \) for \( \varepsilon \in [0, \varepsilon_0] \). On the other hand, we have

\[
c_\varepsilon = c_\varepsilon^{mp} := \inf_{\gamma \in C} \sup_{t \in [0,1]} J_\varepsilon(\gamma(t)) > 0,
\]

where

\[
\Gamma = \left\{ \gamma \in C \left( [0,1], W^{1,p}_{rad}(\mathbb{R}^N, a) \right) \mid \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0 \right\}.
\]

It is easy to see that there exists \( \tau_1 \) large enough such that \( J_\varepsilon(u_{t_\varepsilon}) < 0 \). Hence, we can choose \( \gamma(t) = u_{t_\varepsilon} \).

Using Lemma 5.2 (i), we have \( c_\varepsilon^{mp} \leq \max_{t \geq 0} J_\varepsilon(u_t) = J_\varepsilon(u_{t_\varepsilon}) \). Since \( u \) is arbitrary, we obtain \( c_\varepsilon^{mp} \leq c_\varepsilon^p \) and \( c_\varepsilon = c_\varepsilon^p \) for \( \varepsilon \in (0, \varepsilon_0] \).

(iii). For any \( \delta \in (0, \frac{1}{2}) \), there exists \( u \in W^{1,p}_{rad}(\mathbb{R}^N, a) \setminus \{0\} \) with \( P(u) = 0 \) such that \( J(u) < c + \delta \). In view of \( P(u) = 0 \), we get

\[
\frac{N - p}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{N - a}{p} \int_{\mathbb{R}^N} \frac{A|u|^p}{|x|^a} \, dx = \frac{N}{q} \int_{\mathbb{R}^N} |u|^q \, dx + \frac{\Lambda N}{p} \int_{\mathbb{R}^N} |u|^{p^*_a} \, dx + \frac{N}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} \, dx > 0.
\]

Then there exists \( \tilde{\tau} > 0 \) large enough such that

\[
J(u_{\tilde{\tau}}) = \frac{\tilde{\tau}^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{\tilde{\tau}^{N-a}}{p} \int_{\mathbb{R}^N} \frac{A|u|^p}{|x|^a} \, dx - \frac{\tilde{\tau}^N}{q} \int_{\mathbb{R}^N} |u|^q \, dx
\]

\[
\leq - 1.
\]

We now show the continuity of \( \frac{\tilde{\tau}^N}{p^*_a} \int_{\mathbb{R}^N} |u|^{p^*_a} \, dx \) and \( \frac{\tilde{\tau}^{N-p}}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} \, dx \) on \( (\tau, \varepsilon) \in [0, \tilde{\tau}] \times (0, \varepsilon_0) \).

Firstly, it is easy to check the continuity of \( \frac{\tilde{\tau}^N}{p^*_a} \int_{\mathbb{R}^N} |u|^{p^*_a} \, dx \) and \( \frac{\tilde{\tau}^{N-p}}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} \, dx \) on \( (\tau, \varepsilon) \in [0, \tilde{\tau}] \times (0, \varepsilon_0) \).

Secondly, let \( 0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0 \). Then \( p_a^* + \varepsilon_1 < p_a^* + \varepsilon_2 < p^* \). It follows from Hölder’s and Young’s inequalities that

\[
\frac{\tilde{\tau}^N}{p^*_a} \int_{\mathbb{R}^N} |u|^{p_a^*} \, dx \leq \frac{p^* - p_a^*}{p^* - p_a^* - \varepsilon_1} \int_{\mathbb{R}^N} |u|^{p_a^* + \varepsilon_1} \, dx + \frac{\varepsilon_2 - \varepsilon_1}{p^* - p_a^* - \varepsilon_1} \int_{\mathbb{R}^N} |u|^{p^*} \, dx,
\]

which gives

\[
\frac{\tilde{\tau}^{N-p}}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} \, dx - \int_{\mathbb{R}^N} |u|^{p_a^* + \varepsilon_1} \, dx \leq \frac{\varepsilon_1 - \varepsilon_2}{p^* - p_a^* - \varepsilon_1} \int_{\mathbb{R}^N} |u|^{p^*} \, dx + \frac{\varepsilon_2 - \varepsilon_1}{p^* - p_a^* - \varepsilon_1} \int_{\mathbb{R}^N} |u|^{p^*} \, dx.
\]

That is,

\[
\left| \int_{\mathbb{R}^N} |u|^{p_a^* + \varepsilon_1} \, dx - \int_{\mathbb{R}^N} |u|^{p_a^* + \varepsilon_1} \, dx \right| \leq \frac{\varepsilon_1 - \varepsilon_2}{p^* - p_a^* - \varepsilon_1} \int_{\mathbb{R}^N} |u|^{p^*} \, dx - \int_{\mathbb{R}^N} |u|^{p^*} \, dx.
\]
From (5.5), it is not difficult to see the continuity of \( \int_{\mathbb{R}^N} |u|^p \, dx \) on \( \varepsilon \in (0, \varepsilon_0) \). Similarly, we can prove the continuity of \( \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \) on \( \varepsilon \in (0, \varepsilon_0) \) too.

Thirdly, let \( f_1(\tau, \varepsilon) = \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \), \( f_2(\tau, \varepsilon) = \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \) and \( g_1(\varepsilon) = \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \) and \( g_2(\varepsilon) = \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \). Then \( f_1(\tau, \varepsilon) \cdot g_1(\varepsilon) \) and \( f_2(\tau, \varepsilon) \cdot g_2(\varepsilon) \) are continuous on \((\tau, \varepsilon) \in [0, \bar{\tau}] \times (0, \varepsilon_0)\).

Finally, by using the continuity of \( \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \) and \( \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx \) on \((\tau, \varepsilon) \in [0, \bar{\tau}] \times (0, \varepsilon_0)\), there exists \( \bar{\varepsilon} > 0 \) such that for all \( \varepsilon \in (0, \bar{\varepsilon}) \) and \( \tau \in [0, \bar{\tau}] \) there holds

\[
|J_\varepsilon(u_\tau) - J(u)| \\
= \tau^N \left| \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx - \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx + \frac{1}{p - \varepsilon} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx - \frac{1}{p \tau} \int_{\mathbb{R}^N} |u|^p \, dx \right| \\
\leq \tau^N \left| \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx - \frac{\lambda}{p\alpha} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx + \tau^N \frac{1}{p - \varepsilon} \int_{\mathbb{R}^N} |u|^{p - \varepsilon} \, dx - \frac{1}{p \tau} \int_{\mathbb{R}^N} |u|^p \, dx \right| \\
< \delta,
\]

which implies

\[
J_\varepsilon(u_\tau) \leq -\frac{1}{2} \varepsilon \in (0, \bar{\varepsilon}).
\]

Note that \( J_\varepsilon(u_{\bar{\tau}}) > 0 \) for \( \tau \) small enough. Then there exists \( \bar{\varepsilon} \in (0, \bar{\varepsilon}) \) such that \( \frac{d^2}{d\varepsilon^2} J_\varepsilon(u_{\tau}) \big|_{\varepsilon = \bar{\tau}} = 0 \). By Lemma 5.2 (i), we know \( J(u_{\bar{\tau}}) \leq J(u) \). Thus, for any \( \varepsilon \in (0, \bar{\varepsilon}) \) there holds

\[
c_\varepsilon \leq J_\varepsilon(u_\tau) \leq \delta \leq J(u) + \delta < c + 2\delta.
\]

Hence, \( \limsup_{\varepsilon \to 0} c_\varepsilon \leq c \).

(iii) By a direct calculation, we have

\[
c_\varepsilon = J_\varepsilon(v_\varepsilon) - P_\varepsilon(v_\varepsilon) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx + \frac{\alpha}{pN} \int_{\mathbb{R}^N} A|v_\varepsilon|^p \, dx \geq 0.
\]

(v) By virtue of Lemma 5.2 (iii), we have

\[
c + 1 \geq c_\varepsilon = J_\varepsilon(v_\varepsilon) - P_\varepsilon(v_\varepsilon) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx + \frac{\alpha}{pN} \int_{\mathbb{R}^N} A|v_\varepsilon|^p \, dx \geq C||v_\varepsilon||_{W^{1,p}_{rad}(\mathbb{R}^N, a)}^p.
\]

Namely, \( \{v_\varepsilon\} \) is bounded in \( W^{1,p}_{rad}(\mathbb{R}^N, a) \).

It follows from (2.1)-(2.2) that

\[
0 = P_{\varepsilon_n}(v_{\varepsilon_n}) \geq C||v_{\varepsilon_n}||_{W^{1,p}_{rad}(\mathbb{R}^N, a)}^p - C||v_{\varepsilon_n}||_{W^{1,\sigma}_{rad}(\mathbb{R}^N, a)}^q - C||v_{\varepsilon_n}||_{W^{1,\sigma}_{rad}(\mathbb{R}^N, a)}^q - C||v_{\varepsilon_n}||_{W^{1,\sigma}_{rad}(\mathbb{R}^N, a)}^q,
\]

which implies that there exists \( C > 0 \) independent of \( n \) such that

\[
||v_{\varepsilon_n}||_{W^{1,p}_{rad}(\mathbb{R}^N, a)} \geq C.
\]

Hence, we obtain \( \liminf_{n \to \infty} c_{\varepsilon_n} > 0 \). \( \square \)

### 5.2 Ground State Solution

In this subsection, by using the perturbation method [13] and Pohožaev-type identity [1], we present the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We separate our proof into two steps.

**Step 1.** We take \( \varepsilon \to 0 \) in equation (S\(_{p^*+\varepsilon}\)). For each small \( \varepsilon_n \), there exists a positive ground state solution \( v_{\varepsilon_n} \). Using Lemma 5.2 (iii), we have

\[
c + 1 \geq c_{\varepsilon_n} = J_{\varepsilon_n}(v_{\varepsilon_n}) - \frac{1}{p\alpha} (J_{\varepsilon_n}(v_{\varepsilon_n}), v_{\varepsilon_n}) \geq \left( \frac{1}{p} - \frac{1}{p\alpha} \right) ||v_{\varepsilon_n}||_{W^{1,p}_{rad}(\mathbb{R}^N, a)}^p.
\]
This implies that \( \{v_{\varepsilon_n}\} \) is bounded in \( W^{1,p}_r(\mathbb{R}^N, \alpha) \). Then, up to a subsequence, we assume that
\[
v_{\varepsilon_n} \rightharpoonup v \quad \text{in} \quad W^{1,p}_r(\mathbb{R}^N, \alpha), \quad v_{\varepsilon_n} \to v \quad \text{a.e. in} \quad \mathbb{R}^N, \quad v_{\varepsilon_n} \to v \quad \text{in} \quad L'(\mathbb{R}^N), \quad r \in (p_*, p').
\]

For any \( \varphi \in W^{1,p}_r(\mathbb{R}^N, \alpha) \), as \( n \to \infty \), we claim that
\[
\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p - \varepsilon_n - 2} v_{\varepsilon_n} \varphi \, dx = \int_{\mathbb{R}^N} |v|^{p - 2} v \varphi \, dx + o(1) \tag{5.6}
\]
and
\[
\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p - \varepsilon_n - 2} v_{\varepsilon_n} \varphi \, dx = \int_{\mathbb{R}^N} |v|^{p - 2} v \varphi \, dx + o(1). \tag{5.7}
\]

Here, we only show (5.6), because the proof of (5.7) can be processed in a similar manner. For any \( \varepsilon > 0 \), there exists a sufficiently large \( R > 0 \) such that
\[
\int_{|x| > R} |v_{\varepsilon_n}|^{p - \varepsilon_n - 2} v_{\varepsilon_n} \varphi \, dx - \int_{|x| > R} |v|^{p - 2} v \varphi \, dx \\
\leq \int_{|x| > R} |v_{\varepsilon_n}|^{p - \varepsilon_n - 2} |\varphi| \, dx + \int_{|x| > R} |v|^{p - 2} |\varphi| \, dx \\
\leq \left( \int_{|x| > R} |v_{\varepsilon_n}|^{p - \varepsilon_n} \right)^{1 - \frac{1}{p - \varepsilon_n}} \left( \int_{|x| > R} |\varphi|^{p - \varepsilon_n} \right)^{\frac{1}{p - \varepsilon_n}} \\
+ \left( \int_{|x| > R} |v|^{p} \right)^{1 - \frac{1}{p}} \left( \int_{|x| > R} |\varphi|^{p'} \right)^{\frac{1}{p'}} < \frac{\varepsilon}{2}.
\]

On the other hand, note that \( \{v_{\varepsilon_n}\} \) is bounded in \( W^{1,p}_r(\mathbb{R}^N, \alpha) \). There exists \( C > 0 \) such that
\[
\left( \int_{|x| \leq R} |v_{\varepsilon_n}|^{p - \varepsilon_n} \right)^{1 - \frac{1}{p - \varepsilon_n}} < C.
\]

In view of \( E \subset \mathbb{R}^N \) and small \( \varepsilon_n > 0 \), it follows from Holder’s and Young’s inequalities that
\[
\int_E |\varphi|^{p - \varepsilon_n} \, dx \\
\leq \left( \int_E |\varphi|^{p_\varepsilon} \, dx \right)^{\frac{\varepsilon}{p - \varepsilon_n}} \left( \int_E |\varphi|^{p'} \, dx \right)^{\frac{p - p_\varepsilon}{p - p_\varepsilon}} \\
\leq \frac{\varepsilon}{p - p_\varepsilon} \int_E |\varphi|^{p'} \, dx + \frac{p_* - p_\varepsilon - \varepsilon_n}{p_* - p_\varepsilon} \int_E |\varphi|^{p} \, dx \\
\leq \int_E |\varphi|^{p} \, dx + \int_E |\varphi|^{p'} \, dx.
\]

For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that when \( E \subset \{ x \in \mathbb{R}^N : ||x|| \leq R \} \) with \( |E| < \delta \) there holds
\[
\int_E |v_{\varepsilon_n}|^{p - \varepsilon_n - 2} v_{\varepsilon_n} \varphi \, dx \\
\leq \left( \int_E |v_{\varepsilon_n}|^{p - \varepsilon_n} \right)^{1 - \frac{1}{p - \varepsilon_n}} \left( \int_E |\varphi|^{p - \varepsilon_n} \, dx \right)^{\frac{1}{p - \varepsilon_n}} \\
\leq \left( \int_E |v_{\varepsilon_n}|^{p - \varepsilon_n} \right)^{1 - \frac{1}{p - \varepsilon_n}} \left( \int_E |\varphi|^{p} \, dx + \int_E |\varphi|^{p'} \, dx \right)^{\frac{1}{p - \varepsilon_n}} < C \varepsilon,
\]

where the last inequality is true due to the absolute continuity of \( \int_E |\varphi|^{p} \, dx \) and \( \int_E |\varphi|^{p'} \, dx \).
Making use of the fact $|v_{e_n}|^{p'-e_n-2}v_{e_n}\varphi \to |v|^{p'-2}v\varphi$ a.e. in $\mathbb{R}^N$, by Vitali’s convergence Theorem, we have

$$
\int_{|x| \leq R} |v_{e_n}|^{p'-e_n-2}v_{e_n}\varphi dx = \int_{|x| < R} |v|^{p'-2}v\varphi dx + \frac{c}{2}.
$$

Then

$$
\int_{\mathbb{R}^N} |v_{e_n}|^{p'-e_n-2}v_{e_n}\varphi dx - \int_{|x| < R} |v|^{p'-2}v\varphi dx \\
\leq \int_{|x| > R} |v_{e_n}|^{p'-e_n-2}v_{e_n}\varphi dx - \int_{|x| > R} |v|^{p'-2}v\varphi dx \\
+ \int_{|x| \leq R} |v_{e_n}|^{p'-e_n-2}v_{e_n}\varphi dx - \int_{|x| \leq R} |v|^{p'-2}v\varphi dx \\
\leq \epsilon.
$$

Hence, we arrive at (5.6).

It follows from (5.6) and (5.7) that

$$
0 = \langle f_{e_n}(v_{e_n}), \varphi \rangle
= \int_{\mathbb{R}^N} (\nabla v_{e_n})^{p'-2} \nabla v_{e_n} \nabla \varphi + \frac{A}{|x|^a} |v_{e_n}|^{p'-2} v_{e_n} \varphi dx - \int_{\mathbb{R}^N} |v_{e_n}|^{q'-2} v_{e_n} \varphi dx \\
- \int_{\mathbb{R}^N} |v_{e_n}|^{p'-e_n-2} v_{e_n} \varphi dx - \lambda \int_{\mathbb{R}^N} |v_{e_n}|^{p'+e_n-2} v_{e_n} \varphi dx \\
= \int_{\mathbb{R}^N} (\nabla v)^{p'-2} \nabla \varphi + \frac{A}{|x|^a} |v|^{p'-2} v\varphi dx - \int_{\mathbb{R}^N} |v|^{q'-2} v\varphi dx \\
- \int_{\mathbb{R}^N} |v|^{p'-2} v\varphi dx - \lambda \int_{\mathbb{R}^N} |v|^{p'+2} v\varphi dx \\
= \langle f(v), \varphi \rangle.
$$

This indicates that $v$ is a weak solution of equation (Q).

**Step 2.** We claim that $v \not\equiv 0$.

In view of $\langle f_{e_n}(v_{e_n}), v_{e_n} \rangle = 0$, we get

$$
\int_{\mathbb{R}^N} |\nabla v_{e_n}|^p dx + \int_{\mathbb{R}^N} \frac{A|v_{e_n}|^p}{|x|^a} dx = \int_{\mathbb{R}^N} |v_{e_n}|^q dx + \int_{\mathbb{R}^N} |v_{e_n}|^{p'-e_n} dx + \lambda \int_{\mathbb{R}^N} |v_{e_n}|^{p'+e_n} dx. \tag{5.8}
$$

It follows from Holder’s and Young’s inequalities that

$$
\int_{\mathbb{R}^N} |v_{e_n}|^{p'+e_n} dx \leq \left( \int_{\mathbb{R}^N} |v_{e_n}|^p dx \right)^{\frac{p'+e_n}{p-\rho_n}} \left( \int_{\mathbb{R}^N} |v_{e_n}|^\rho dx \right)^{\frac{p-\rho_n}{p'-\rho_n}} \\
\leq \frac{p'-p^*_a - \epsilon_n}{p'-p^*_a} \int_{\mathbb{R}^N} |v_{e_n}|^p dx + \frac{\epsilon_n}{p'-p^*_a} \int_{\mathbb{R}^N} |v_{e_n}|^\rho dx, \tag{5.9}
$$

and

$$
\int_{\mathbb{R}^N} |v_{e_n}|^{p'-e_n} dx \leq \left( \int_{\mathbb{R}^N} |v_{e_n}|^p dx \right)^{\frac{e_n}{p-\rho_n}} \left( \int_{\mathbb{R}^N} |v_{e_n}|^\rho dx \right)^{\frac{p-\rho_n}{p'-\rho_n}} \\
\leq \frac{\epsilon_n}{p'-p^*_a} \int_{\mathbb{R}^N} |v_{e_n}|^p dx + \frac{p^*-p^*_a - \epsilon_n}{p'-p^*_a} \int_{\mathbb{R}^N} |v_{e_n}|^\rho dx. \tag{5.10}
$$

Substituting (5.9) and (5.10) into (5.8) leads to

$$
\int_{\mathbb{R}^N} |\nabla v_{e_n}|^p dx + \int_{\mathbb{R}^N} \frac{A|v_{e_n}|^p}{|x|^a} dx \\
\leq \left( 1 + \frac{\epsilon_n(\lambda-1)}{p'-p^*_a} \right) \int_{\mathbb{R}^N} |v_{e_n}|^p dx + \left( \lambda + \frac{\epsilon_n(1-\lambda)}{p'-p^*_a} \right) \int_{\mathbb{R}^N} |v_{e_n}|^\rho dx + \int_{\mathbb{R}^N} |v_{e_n}|^q dx. \tag{5.11}
$$

It suffices to show that there exists $C > 0$ such that $C \leq \|v_{e_n}\|_{W^{1,p}(\mathbb{R}^N, \alpha)}$. Otherwise, we assume that $\|v_{e_n}\|_{W^{1,p}(\mathbb{R}^N, \alpha)} \to 0$. Then it yields $c_{e_n} \to 0$, which contradicts $c_{e_n} > 0$, see Lemma 5.2 (v).
It follows from (5.11) and $0 < C \leq \|v_{e_n}\|_{W^{2,p}(\mathbb{R}^N,d)}^p$ that only one of the following statements is true.

\begin{align*}
\text{(i) } & \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx \to 0 \text{ and } C \leq \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx; \text{ or } \\
\text{(ii) } & C \leq \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx \text{ and } \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx \to 0; \text{ or } \\
\text{(iii) } & C \leq \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx \text{ and } C \leq \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx.
\end{align*} \hfill (5.12)

We first exclude (i) in (5.12). Suppose on the contrary that (5.12) (i) holds. It follows from (5.12) (i) and (5.11) that

\begin{align*}
\int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx & \leq \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx + \int_{\mathbb{R}^N} A|v_{e_n}|^p\,dx \\
& \leq \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx \\
& \lesssim \frac{C}{p^{\frac{p}{p^*}}} \left( \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx \right)^{\frac{p^*}{p}},
\end{align*}

which gives

\begin{align*}
S^{\frac{p^*}{p}} \lesssim \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx.
\end{align*}

In view of Lemma 5.2, we get

\begin{align*}
c \geq c_{e_n} \\
= f_{e_n}(v_{e_n}) - \frac{1}{N} P_{e_n}(v_{e_n}) \\
= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_{e_n}|^2\,dx + \frac{\alpha}{pN} \int_{\mathbb{R}^N} A|v_{e_n}|^p\,dx \\
\geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx \\
\geq \frac{1}{N} S^{\frac{p^*}{p}}.
\end{align*}

This yields a contradiction with the fact of $c < \min \left\{ \frac{1}{N} S^{\frac{p^*}{p}}, \frac{\alpha}{pN} \left( \frac{1}{\lambda} \right)^{\frac{p^*}{p}} S^{\frac{p^*}{p}} \right\}$, see Lemma 5.1. Hence, (5.12) (i) can not occur.

We now exclude (ii) in (5.12). Suppose on the contrary that (5.12) (ii) holds. It follows form (5.12) (ii) and (5.11) that

\begin{align*}
\int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx + \int_{\mathbb{R}^N} A|v_{e_n}|^p\,dx \lesssim \lambda \int_{\mathbb{R}^N} |v_{e_n}|^p\,dx \\
\lesssim \lambda S^{\frac{p^*}{p}} \left( \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx + \int_{\mathbb{R}^N} A|v_{e_n}|^p\,dx \right)^{\frac{p^*}{p}},
\end{align*}

which gives

\begin{align*}
\left( \frac{1}{\lambda} \right)^{\frac{p^*}{p}} S^{\frac{p^*}{p}} \lesssim \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p\,dx + \int_{\mathbb{R}^N} A|v_{e_n}|^p\,dx.
\end{align*}
Using Lemma 5.2 yields

\[
c \geq c_{e_n} = J_{e_n}(v_{e_n}) - P_{e_n}(v_{e_n}) \\
\geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p \, dx + \frac{a}{pN} \int_{\mathbb{R}^N} \frac{A|v_{e_n}|^p}{|x|^\alpha} \, dx \\
\geq \frac{a}{pN} \left( \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p \, dx + \int_{\mathbb{R}^N} \frac{A|v_{e_n}|^p}{|x|^\alpha} \, dx \right) \\
\geq \frac{a}{pN} \left( \frac{1}{\lambda} \right)^{\frac{p}{p-\alpha}} S_a^{\frac{p}{p-\alpha}}.
\]

This contradicts the fact of \( c < \min \left\{ \frac{1}{N} \hat{\alpha}_p, \frac{a}{\lambda} \left( \frac{1}{\lambda} \right)^{\frac{p}{p-\alpha}} S_a^{\frac{p}{p-\alpha}} \right\} \), see Lemma 5.1. Hence, (5.12) (ii) can not occur either.

We now draw a conclusion that (5.12) (iii) is true. By virtue of Theorem 1.1, we have \( v \not\equiv 0 \). In view of Theorem 1.2, \( P(\nu) = 0 \) and the weakly lower semi-continuity of the norm, we obtain

\[
c \leq J(\nu) = J(\nu) - P(\nu) \\
= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \nu|^p \, dx + \frac{a}{pN} \int_{\mathbb{R}^N} \frac{A|\nu|^p}{|x|^\alpha} \, dx \\
\leq \liminf_{N \to \infty} \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_{e_n}|^p \, dx + \frac{a}{N} \int_{\mathbb{R}^N} \frac{A|v_{e_n}|^p}{|x|^\alpha} \, dx \\
= J_{e_n}(v_{e_n}) - \frac{1}{N} P_{e_n}(v_{e_n}) \\
= c_{e_n} \leq c.
\]

Consequently, \( v \) is a positive ground state solution. \( \square \)

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References

[1] B. Abdellaoui and I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the \( p \)-Laplacian with a critical potential, Ann. Mat. Pura Appl. 182 (2003), 247-270.
[2] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geometry 11 (1976), no. 4, 573-598.
[3] M. Badiale and S. Rolando, A note on nonlinear elliptic problems with singular potentials, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 17 (2006), 1-13.
[4] M. Badiale, V. Benci and S. Rolando, A nonlinear elliptic equation with singular potential and applications to nonlinear field equations, J. Eur. Math. Soc. 9 (2007), 355-381.
[5] M. Badiale, M. Guida and S. Rolando, Compactness and existence results for the \( p \)-Laplace equations, J. Math. Anal. Appl. 451 (2017), 345-370.
[6] V. Benci and D. Fortunato, Variational Methods in Nonlinear Field Equations. Solitary Waves, Hylomorphic Solitons and Vortices, Springer Monogr. Math., Springer, Cham, 2014.
[7] P. C. Fife, Asymptotic states for equations of reaction and diffusion, Bull. Amer. Math. Soc. 84 (1978), 693-728.
[8] R. Filippucci, P. Pucci and F. Robert, On a \( p \)-Laplace equation with multiple critical nonlinearities, J. Math. Pures Appl. 91 (2009), no. 2, 156-177.
[9] N. Ghoussoub and C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Hardy and Sobolev exponents, Trans. Amer. Math. Soc. 352 (2000), no. 12, 5703-5743.
[10] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 2, 109–145.
[11] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 4, 223–283.
[12] G. Palatucci and A. Pisante, *Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 799-829.

[13] N. S. Papageorgiou and Y. P. Zhang, *Constant sign and nodal solutions for superlinear (p,q)-equations with indefinite potential and a concave boundary term*, Adv. Nonlinear Anal. **10** (2021), 76-101.

[14] P. Pucci and R. Servadei, *Nonexistence for p-Laplace equations with singular weights*, Commun. Pure Appl. Anal. **9** (2010), no. 5, 1421-1438.

[15] S. Rolando, *Multiple nonradial solutions for a nonlinear elliptic problem with singular and decaying radial potential*, Adv. Nonlinear Anal. **8** (2019), 885-901.

[16] Y. Su, H. Chen, S. Liu and G. Che, *Ground state solution of p-Laplacian equation with finite many critical nonlinearities*, Complex Variables and Elliptic Equations, **66** (2021), no. 2, 283-311.

[17] J. Su, Z. Wang and M. Willem, *Nonlinear Schrödinger equations with unbounded and decaying potentials*, Commun. Contemp. Math. **9** (2007), 571-583.

[18] J. Su, Z. Wang and M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations **238** (2007), 201-219.

[19] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353-372.

[20] S. Terracini, *On positive entire solutions to a class of equations with a singular coefficient and critical exponent*, Adv. Differential Equations **1** (1996), 241-264.

[21] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126-150.

[22] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191-202.

[23] C. Vincent and S. Phatak, *Accurate momentum–space method for scattering by nuclear and Coulomb potentials*, Phys. Rev. **10** (1974), 391-394.