Properties of a New Class of Lattice Dirac Operators

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A new class of lattice Dirac operators $D$ have been recently proposed on the basis of the generalized Ginsparg-Wilson relation, $\gamma_5(D\gamma_5) + (\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2}$, where $k$ is a non-negative integer. We discuss the index theorem and locality properties for this general class of lattice Dirac operators.

1. Introduction

We have recently witnessed a remarkable progress in the treatment of lattice fermions. A basic algebraic relation which lattice Dirac operators should satisfy was clearly stated by Ginsparg and Wilson[1], and an explicit solution to the algebra was given[2]. This solution exhibits quite interesting chiral properties[3][4][5], including locality properties[6][7]. In the mean time, a new class of lattice Dirac operators $D$ have been proposed on the basis of the algebraic relation[8]

$$\gamma_5 D + D\gamma_5 = 2aD(a\gamma_5 D)^{2k}\gamma_5 D$$

(1.1)

where $k$ stands for a non-negative integer, and $k = 0$ corresponds to the ordinary Ginsparg-Wilson relation[1]. In the following sections we discuss the properties of the operators satisfying this general Ginsparg-Wilson relation.

2. A new class of lattice Dirac operators

We first define

$$H_{(2k+1)}$$

$$= \frac{1}{2}\gamma_5 \left( 1 + D_W^{(2k+1)\dagger} \frac{1}{\sqrt{D_W^{(2k+1)\dagger} D_W^{(2k+1)}}} \right)$$

(2.2)

where $D_W^{(2k+1)\dagger} = i(C^\mu)^{2k+1} + (B)^{2k+1} - \left( \frac{m_0}{a} \right)^{2k+1}$

(2.3)

where $C^\mu$ is the covariant difference operator and $B$ is the covariant Wilson term. The solution $D$ of the general Ginsparg-Wilson relation(1.3) is then given by

$$D = \frac{1}{a}\gamma_5 (H_{(2k+1)})^{1/(2k+1)}$$

(2.4)

where $k = 0$ corresponds to the overlap Dirac operator[2]. When the parameter $m_0$ is chosen as $0 < m_0 < 2r$ in these operators, the free propagators have a single massless pole and are free from species doublers. We next see that the case of $k \geq 1$ is better than $k = 0$ on the scaling property for $a \to 0$. For the operators(2.4), in the near continuum configurations we see the behaviours as follows,

$$D \approx i\slashed{D} + a^{2k+1}(\gamma_5 i\slashed{D})^{2k+2}$$

(2.5)

where $\slashed{D} = \gamma_\mu (\partial_\mu + igA_\mu)$. The first terms in these expressions stand for the leading terms in chiral symmetric terms, and the second terms stand for the leading terms in chiral symmetry breaking terms. This shows that one can improve the chiral symmetry for larger $k$. As another manifestation of this property, the spectrum of the operators with $k \geq 1$ is closer to that of the continuum operator in the sense that the small eigenvalues of $D$ accumulate along the imaginary axis (which is a result of taking a $2k + 1$-th root)[4], compared to the overlap Dirac operator for which the eigenvalues of $D$ draw a perfect circle in the complex eigenvalue plane.
3. Index theorem

We first examine the lattice chiral symmetry\[^5\]. The fermion action which has the generalized operators\[^2,3\] is invariant under the global transformation:

\[
\begin{align*}
\psi &\rightarrow \psi' = \psi + i e \Gamma_5 \psi, \\
\bar{\psi} &\rightarrow \bar{\psi}' = \bar{\psi} + i e \bar{\psi} \gamma_5 \Gamma_5 \gamma_5,
\end{align*}
\]

where \(\Gamma_5 = \gamma_5 - (a \gamma_5 D)^{2k+1}\). But the fermion measure produces the Jacobian factor:

\[
D \bar{\psi}' D \psi' = J D \bar{\psi} D \psi,
\]

\[
J = \exp(-2 i e T r \Gamma_5).
\]

In this expression \(T r \Gamma_5\) is regarded as the chiral anomaly on the lattice. We next analyze the index relation for the generalized Dirac operators. We consider the eigenvalues \(\lambda_n\) and the eigenmodes \(\phi_n\) of the hermitian operator \(\gamma_5 D\). The zero modes are defined by

\[
(\gamma_5 D) \phi_0 = 0.
\]

Using \(\Gamma_5(\gamma_5 D) + (\gamma_5 D) \Gamma_5 = 0\) from the general algebra, we obtain

\[
(\gamma_5 D)(\gamma_5 \phi_0) = 0.
\]

Therefore we can assign the chiralities for zero modes:

\[
\gamma_5 \phi_0^{(\pm)} = \pm \phi_0^{(\pm)}
\]

The other modes \((\lambda_n \neq 0)\) are defined by

\[
(\gamma_5 D) \phi_n = \lambda_n \phi_n,
\]

\[
(\gamma_5 D) \Gamma_5 \phi_n = - \lambda_n \Gamma_5 \phi_n.
\]

And then we obtain \((\phi_n, \Gamma_5 \phi_n) = 0\) for \(\lambda_n \neq 0\). From the above analysis \(T r \Gamma_5\) is written as follows \((n = 0\) stands for \(\lambda_n = 0)\):

\[
T r \Gamma_5 = \sum_n (\phi_n, \Gamma_5 \phi_n) = \sum_{n=0} \phi_n, \Gamma_5 \phi_n + \sum_{n \neq 0} \phi_n, \Gamma_5 \phi_n
\]

\[
= \sum_{n=0} \phi_n, \Gamma_5 \phi_n = \sum_{n=0} \phi_n, \gamma_5 \Gamma_5 \phi_n
\]

\[
= \sum_{n=0} \phi_n, (\gamma_5 - (a \gamma_5 D)^{2k+1}) \phi_n
\]

\[
= \sum_{n=0} \phi_n, \gamma_5 \phi_n
\]

\[
n_+ - n_- = \text{index}
\]

This is the index theorem on the lattice. Next we evaluate \(T r \Gamma_5\) for \(a \rightarrow 0\). The local version of the trace is

\[
\begin{align*}
tr \Gamma_5(x) &= tr[\gamma_5 - (\gamma_5 a D)^{2k+1}] \\
&= -\frac{1}{2} tr \left[ D_W^{(2k+1)} + \frac{1}{\sqrt{(D_W^{(2k+1)})^t D_W^{(2k+1)}}} \right]
\end{align*}
\]

In this expression \(D_W^{(2k+1)}\) includes the lattice spacing \(a\). Then we expand the above expression in \(a\) and take \(a \rightarrow 0\). After the straightforward calculations, we obtain

\[
tr \Gamma_5(x) = I_{2k+1}(r, m_0) g^2 tr e^{i \mu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}.
\]

We can show that \(I_{2k+1}(r, m_0)\) is independent of \(k, r\) and \(m_0\) and that \(I_{2k+1}(r, m_0) = 1/32\pi^2\). Therefore we recover the index theorem in the continuum theory:

\[
n_+ - n_- = \int d^4 x g^2 32\pi^2 tr e^{i \mu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}.
\]

in the naive continuum limit for any operator in \[^2,3\].

4. Locality property

In this section we discuss the locality property\[^10\]. We first see the locality of the free operator. The generalized Dirac operators in the free fermion case are written as

\[
H(p) = \gamma_5 \left( \frac{1}{2} \right) \left( \frac{1}{\sqrt{F_k}} \right) \left( \frac{1}{\sqrt{\tilde{M}_k}} \right)
\times \left\{ \left( \sqrt{F_k + \tilde{M}_k} \right) \left( \frac{1}{\sqrt{F_k}} \right) - \left( \sqrt{F_k - \tilde{M}_k} \right) \left( \frac{1}{\sqrt{\tilde{M}_k}} \right) \right\},
\]

(4.12)

where

\[
\tilde{M}_k = \sum_\mu (1 - c_\mu) 2^{k+1} - 1
\]

\[
F_k = (s^2)^{2k+1} + \tilde{M}_k^2
\]

\[
c_\mu = \cos a p_\mu, \quad s = \sum_\mu \gamma_\mu \sin a p_\mu,
\]

\[
s^2 = \sum_\mu (\sin a p_\mu)^2.
\]
Here \( H = a \gamma_5 D \) and we set \( r = 1 \) and \( m_0 = 1 \). Now we want to know if \( H \) is exponentially local. For this purpose we note the next statement: If \( H(p) \) is differentiable for infinite times with respect to \( p \), \( H(x,y) \) is exponentially local [12].

Considering the one-dimensional case for simplify, this proof goes as follows,

\[
\frac{\partial}{\partial p^l} H(p) = \frac{\partial}{\partial p^l} \int dx \, e^{ipx} H(x) = \int dx \, (ix)^l H(x) e^{ipx} < \infty
\]

This last inequality for any \( l \) implies

\[ \|H(x)\| < C e^{-\theta x}, \quad \theta > 0. \]

Now all we need to do is to know if \( H(p) \) is infinite times differentiable. From the expression of \( H(p)(4.12) \), the infinite times differentiability means that the following terms,

\[
\left( \frac{1}{\sqrt{F_k}} \right)^{(2k+1)} \left( \sqrt{F_k + \tilde{M}_k} \right)^{(2k+1)}
\]

\[
\left( \sqrt{F_k - \tilde{M}_k} \right)^{(2k+1)}
\]

are infinite times differentiable. Noting that there is a mass gap in \( F_k \), we can prove this [10]. Therefore the free operator \( H \) is local. We also presented a crude estimate of the localization length, examining the singularity of the Dirac operators. This suggests that the operators with gauge fields are local for sufficiently weak background gauge fields. On the other hand, it is difficult to see the locality of the interacting operator \( H \) directly. But it may be true that if \( H^{(2k+1)} = H^{(2k+1)} \) is local, \( H \) is also local. In the one-dimensional integral, one has

\[
\int_{-\infty}^{\infty} dy \, \text{exp}[-|x-y|/L] \text{exp}[-|y-z|/L]
\]

\[ = (L+|x-z|) \text{exp}[-|x-z|/L]. \quad (4.14) \]

This shows that a multiplication of two operators, which decay exponentially, produces an operator which decays with the same exponential factor up to a polynomial prefactor. A generalization of this relation suggests that a suitable \((2k+1)\)-th root of an exponentially decaying operator gives rise to an operator with an identical localization length for any finite \( k \). This holds in the case of the free fermion operator. Therefore we examined the locality of interacting \( H^{(2k+1)} \), as a direct extension of the analysis in the overlap Dirac operator [11]. For \( H^{(2k+1)}(2.2) \) we showed that \( \|D_{W}^{(2k+1)} D_{W}^{(2k+1)} \| > 0 \) for all finite \( k \) when \( \|1 - U_{\mu\nu}\| \) is very small. This means \( H^{(2k+1)} \) is exponentially local. As for the localization domain for \( k = 1 \), for example, we see

\[ \|1 - U_{\mu\nu}\| < \frac{1}{2 \times 10^5} \rightarrow \|D_{W}^{(3)} D_{W}^{(3)}\| > 0 \]

Since this analysis gives a conservative estimate, we expect that the actual locality domain of gauge field strength could be much larger. But the operators spreads over more lattice points for larger \( k \).

5. Summary

We discussed the properties of the Dirac operators satisfying the algebraically generalized Ginsparg-Wilson relation. Our analysis indicates an infinite number of lattice Dirac operators which have good properties. For the future work it is important to see the locality of interacting operator more precisely.

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