A Practical Approach for Exponentiation of QED Corrections in Arbitrary Processes

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It is a well-known fact that, among the electroweak corrections, QED radiation gives the largest contribution and the needed precision requires a re-summation of the large logarithms which show up in perturbation theory. For annihilation processes, $e^+ e^- \rightarrow \bar{f}f$, initial state radiation is a definable, gauge-invariant, concept and one has general tools to deal with it; the structure function approach and also the parton-shower method. However, when one tries to apply the algorithm to four-fermion processes that include non-annihilation channels a problem is faced: is it still possible to include QED corrections by making use of the standard tools? A systematization is attempted of several, recently proposed, algorithms. In particular, it is shown that starting from the exponentiation of soft photons one can still derive a description of QED radiation in terms of structure functions, i.e. the kernel for the hard scattering is convoluted with generalized structure functions where each of them is no longer function of one scale. Each external, charged, fermion leg brings a factor $x^{\alpha A - 1}$ where $\alpha$ is the fine-structure constant, $0 \leq x \leq 1$ and $A$ is a function which depends on the momenta of the charged particles.

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1 Introduction

In QED, and more generally in the standard model (SM), the most important terms in radiative corrections to various processes at high energy or at large momentum transfer are those that contain large logarithms of the type $\ln \left( \frac{Q^2}{m^2} \right)$, where $Q^2$ stands for a large kinematical variable. The quantity $m$ is the mass of a light, charged particle, which emits photons, for example, the leptons or the light quarks. The origin of these large logarithms is connected to the presence of collinear divergences in the perturbative expansion. For QED in particular they arise whenever a photon is radiated in the direction of an incoming (outgoing) light fermion and we compute an exclusive observable. Collinear divergences are present both in virtual and in real corrections.

As far as virtual corrections are concerned, they appear as a consequence of a mass singular renormalization procedure. Suppose we have a Green function $F(q,...)$, where $q$ is some external momentum of a loop diagram. Then the renormalization procedure is mass singular if

$$F^{\text{ren}}(q,...) = F(q^2,...) - F(0,...),$$

that is, the subtraction has been performed at $q^2 = 0$.

The collinear divergences enter $F^{\text{ren}}$ through $F(0,...)$, in the limit $-q^2 \gg m^2$, where

$$F^{\text{ren}} \sim \ln n \frac{-q^2}{m^2}, \quad n = 1, 2.$$  

When we consider, instead, the real corrections, terms like $\Delta^{-1} = (p - k)^2 + m^2$ will arise as a consequence of a fermion of momentum $p$ and mass $m$ emitting a photon of momentum $k$. The singularity will arise if we set $m = 0$. With $p^2 = k^2 = 0$ and $\vec{k} \parallel \vec{p}$ the denominator in $\Delta$ goes to zero. If we now integrate over $k$, then the expression for the diagram in which $\Delta$ appears becomes singular. Strictly speaking, collinear divergences are only present in theories where massless quanta couple to each other, which is not the case for QED, or for the SM, where the theory becomes almost massless, at high energies; that is, almost mass singular, due to the presence of low mass fermions. In those cases where we need a very high accuracy of theoretical predictions, the presence of large logarithms calls for a re-summation of the perturbation series, in spite of the low value of the coupling constant. The method for this summation was actually developed within QCD and is based on the factorization theorems, which allow us to split the contributions of large and small distances. As a result of this re-summation procedure one starts with the cross-section for the so-called hard process; that is, the process with large kinematic variables, which is subsequently convoluted with the structure function of the initial (final) particles $[1]$.

For a given scale $Q$, the hard part of the cross-section is determined by distances of the order $1/Q$ and is expressed in terms of the coupling constant at these distances, $\alpha(Q^2)$. This hard cross-section contains no large logarithms and is evaluated in perturbation theory. The whole contribution of large distances is inserted into the structure functions, which obey an equation of the renormalization-group type.
The structure functions describe the radiative corrections to a given process in the so-called leading logarithmic approximation (LLA):

\[ \frac{\alpha}{\pi} L = \frac{\alpha}{\pi} \ln \frac{Q^2}{m^2}, \]

where \( Q \) represents some large scale inherent to the problem under consideration. Therefore, in LLA we re-sum all terms of the form \((\alpha/\pi L)^n\). Sometimes it is possible to go beyond the LLA to guarantee the correct evaluation of the next-to-leading terms, \(\alpha/\pi (\alpha/\pi L)^n\). The latter, usually included in the so-called \(K\)-factor, are not universal but require instead a comparison with the explicit calculation of the cross-section up to the two-loop level.

However, for a general process we are not always in this favorable situation. Very often, therefore, one can find the statement that the choice of the appropriate scale in the structure functions is mandatory. This is a jargon for 'implementing the correct exponentiation factor in multi-photon emission'.

The outline of the paper will be as follows: in Section 2 we introduce the problem of a general treatment of QED radiation for arbitrary processes which do not allow a gauge invariant definition of initial state radiation. In Section 3.1 – 3.3 we review the classic Yennie, Frautschi and Suura exponentiation algorithm (hereafter YFS), completely reviewed in terms of dimensional regularization for both virtual and real infrared terms. In particular we give the derivation of the YFS virtual form-factor in Section 4. In Section 5 we shown that the YFS real form-factor can be exactly evaluated in \(n\)-dimensions without having to split a soft part from the hard one. In Section 6 we introduce an approximation to the exact result which is much simpler to handle in practical computations. The extension from two emitters to an arbitrary number is discussed in Section 7. As a consequence of our approximation the result will be naturally expressed in terms of generalized structure functions with an overall exponent which is discussed in Section 8. Additional refinements are introduced and discussed in Section 9. Some general considerations are introduced in Section 10 that introduce the problems that one encounters in going beyond the lowest order result. Conclusions are shown in Section 11. Finally, the most relevant integrals used in the paper are explicitly shown in appendix.

2 Background of the problem

The great success of high precision LEP physics is intimately linked to the possibility of splitting initial state QED radiation from the rest in a meaningful way and of using accurate determinations of the structure functions for the incoming \(e^+e^-\) beams, the so-called \(s\)-channel structure functions.

For LEP 2 physics, although the required theoretical precision is not comparable to what we need around the \(Z\) resonance, we are not in the same fortunate situation. First of all, initial versus final state radiation is not longer a meaningful concept. Secondly, a large fraction of processes are not dominated by annihilation and, therefore, the standard methods of using
s-channel structure functions fail to reproduce the correct result. Nevertheless, the language of structure functions is a useful one and it is desirable to include at least the bulk of large radiative corrections. Hence, structure functions are still applied for these processes, but some large uncertainty remains, connected with what is usually referred to as the problem of selecting the right scale. In a word, the choice of the energy scale is not a trivial issue.

For processes where some exact perturbative calculation exists the scale inherent to structure functions can be determined by matching the two languages but this is not possible in general. The keyword in all these cases is, according to commonly accepted jargon, to ‘select a suitable scale without knowing the exact one (two) loop calculation’. Needless to say this attempt is utopistic, although some ingenuous strategy has been devised in recent times. In particular we refer to some interesting work that can be found in ref. [4] and in ref. [5]

More or less, all these attempts amount to start with some sort of naive soft + virtual photon approximation, not the one commonly employed in the Yennie, Frautschi and Suura formalism [6], but rather something as in the following expression:

$$\frac{d\sigma_{\text{soft}}(s)}{d\Omega} = \frac{d\sigma_0(s)}{d\Omega} \exp \left[ -\frac{\alpha}{\pi} \ln \left( \frac{E}{\Delta E} \right) \sum_{i,j} Q_i Q_j \epsilon_i \epsilon_j \ln \left( \frac{1 + \beta_{ij}}{1 - \beta_{ij}} \right) \right]$$

$$\beta_{ij}^2 = 1 + \left( \frac{m_i m_j}{p_i \cdot p_j} \right)^2,$$  \hspace{1cm} (4)

where $m_i (p_i)$ are the mass(momentum) of $i$-th charged particle, $\Delta E$ is the maximum energy of the soft photon (the boundary between soft- and hard-photons), $E$ is the beam energy, and $Q_i$ the electric charge in unit of the $e^+$ charge. The factor $\epsilon_j$ is $-1$ for the initial particles and $+1$ for the final particles. The indices $i, j$ run over all the charged particles in the initial and final states.

We will say that any formulas as in Eq.(4) includes all double-logarithms. This terminology is not ordinarily accepted, so we stress that it means including, prior to integration, all terms proportional to

$$\ln(\Delta E/E) \ln(Q^2/m_f^2),$$

(5)

where $Q^2$ denotes, generically, some large scale present in the problem. It must be stressed that Eq.(4) misses to incorporate virtual corrections and that the single-logarithmic part is omitted. The adopted strategy is to implement a structure function approach [1] on the Born kernel for the process with the opinion that one is able to make an educate guess about the scale in these structure functions, something as in

$$\sigma_{\text{tot}}(s) = \int dx_- dy_- dx_+ dx_+ dx_{+u} dx_{+d} D_{e^-}(x_-, Q^2_{f-}) D_{e^-}(y_-, Q^2_{f-}) \times D_{e^+}(x_+, Q^2_{f+}) D_{u}(x_u, Q^2_{f_u}) D_{d}(x_d, Q^2_{f_d}) \sigma_0(s),$$

(6)

where all scales, $Q^2_{f-}$ etc are guessed. Note that, for definiteness, we have taken a specific example.

Another problem arises in adapting structure functions to $t$-channel processes. A non-accelerated charged particle cannot radiate. Consider now an incoming electron of momentum $p_in$, an outgoing
electron of momentum $p_{\text{out}}$ and the corresponding radiation of any number of photons from both legs, before and after the hard scattering, with total momentum $K$. Furthermore, let $Q = p_{\text{in}} - p_{\text{out}}$ and consider processes dominated by the region where $Q^2 \sim 0$. The adopted strategy would be to fold the hard scattering cross-section with structure functions at virtualness $Q^2$.

$$\beta \ (1 - x)^{\beta/2-1}, \quad \beta = \frac{2 \alpha}{\pi} \left( \ln \frac{-Q^2}{m_e^2} - 1 \right),$$

where $x$ is the probability of finding the electron within an electron with longitudinal momentum fraction $x$; however, the limit $Q^2 \to 0$ (no radiation) will not be correctly reproduced. Of course, one can switch off the radiation for $Q^2 \leq Q^2_{\text{min}}$ with $Q^2_{\text{min}}$ chosen ad hoc, but this represents an artificial solution. Another solution is to replace the logarithm of the structure function with a power law behavior for small values of $Q^2$ but a continuous solution valid in any regime and not only for $Q^2 \gg m_e^2$ or $Q^2 \ll m_e^2$ is desirable.

Furthermore, all approaches that are missing virtual photonic corrections – at least the universal, process independent, YFS form-factor – simulate their effect by imposing an effective lower energy cutoff on the photon energy, $E_c$, and require $\ln(E_{\text{max}}/E_c) = \mathcal{O}(1)$, where $E_{\text{max}}$ is the maximum photon energy.

The situation is slightly better for those $t$-channel processes where a perturbative calculation exists, see for instance the two-photon process $e^+e^- \to e^+e^- \mu^+\mu^-$. Here the comparison between structure functions and the $\mathcal{O}(\alpha)$ calculation [7] allows us to select a scale $t$ once a $K$-factor is included, box diagrams are neglected (but luckily enough their contribution is strongly suppressed) and one stays away from the forward scattering region where large deviations are expected and seen.

Our approach is aimed to systematize this basic idea and, therefore, it must be clearly stated that it does not represent an attempt to move beyond the correct treatment of those terms that may lead to double-logarithmic enhancement. Simply, we start from the fundamental process of exponentiation of Yennie, Frautschi and Suura [6] and try to understand its correct interplay with the language of structure functions. Complete virtual corrections are not included, therefore only the universal YFS factor is included, collinear single-logarithms are again missing since, basically, only soft photons enter into the scheme. However, the issue of ‘selecting the scale’ turns out to be much less arbitrary than in any previous approach.

In particular, there is a situation where the exclusion of hard photons represents a bad approximation to the exact result. Consider two final state emitters, then owing to the Kinoshita, Lee and Nauenberg [8] theorem (hereafter KLN) the corresponding corrections are always small and free of large logarithms. They can become more sizeable only for tight cuts on the invariant mass of the emitting pair. The typical example is in the corrections to $e^+e^- \to \mu^+\mu^-$ where the $\mathcal{O}(\alpha)$ inclusive corrections are represented by the well-known factor $3/4(\alpha/\pi)$. This result, however, requires adding the matrix element for single hard bremsstrahlung and the complete $\mathcal{O}(\alpha)$ virtual corrections. For a general process with many fermions in the final state the knowledge of this matrix element is usually missing and, unless tight cuts are imposed on the invariant masses of the final state pairs, we cannot reproduce the correct KLN limit.
In principle hard photons can be included for arbitrary processes in the so-called collinear approximation, namely hard photons are allowed only within a small cone surrounding each charged external fermion. In this case, however, we are not allowed to integrate over the whole phase space of the photon since, in this case, there is no gauge invariant leading – sub-leading splitting of collinear radiation and the latter, in any case, will not be suppressed. Even more, the complete $\mathcal{O}(\alpha)$ virtual corrections are needed as well and this can only be derived on a process-by-process basis.

YFS exponentiation has been widely used in the past [8] and we have no pretension to be adding any substantial improvement. The only goal of this paper is to clarify the extraction of structure functions from the YFS program, without having to introduce a soft-hard separation in the YFS form factor, and extending their validity beyond the asymptotic region where all the invariants of the process are much larger than all fermion masses. Therefore, the main difference is that we adopt a slightly different variant of the YFS-approach where no separation is made between the soft and the hard region; rather we re-formulate in modern language an old proposal by Chahine [11].

3 QED corrections

The material in this section is well known [8] and we go through it with the main motivation of establishing notations and conventions. As an example consider the process

$$e^+(p_1) + e^-(p_2) \rightarrow f(p_4) + f(p_3), \quad (8)$$

3.1 Soft photon exponentiation

The cross section for the emission of an extra soft photon of momentum $k$ is

$$d\sigma \sim \frac{\alpha}{4\pi^2} d\sigma_0 \int \frac{d^3k}{k_0} \left( \sum_{i=1,4} \frac{\theta_i p_i}{p_i \cdot k} \right)^2, \quad (9)$$

where $d\sigma_0$ is the non-radiative differential cross-section. The variables $\theta_i$ are defined as follows:

|            | $\theta_i$  |
|------------|-------------|
| in - part. | $-Q_e$      |
| in - antipart. | $+Q_e$    |
| out - part. | $+Q_f$     |
| out - antipart. | $-Q_f$   |

and they satisfy conservation of charge, i.e. $\sum_i \theta_i = 0$. We define the usual eikonal factor,

$$j_\mu(k) = \sum_i \theta_i \frac{p_{i\mu}}{p_i \cdot k}, \quad (11)$$

5
which satisfies current conservation,

$$j \cdot k = \sum_i \theta_i = 0.$$  \hfill (12)

Next we consider the generalization of process Eq.(8), $e^+e^-$ annihilation into several fermion-antifermion pairs and an arbitrary number of photons:

$$e^+(p_+) e^- (p_-) \to \bar{f}_1 + f'_1 + \ldots + \bar{f}_l + f'_l + n \gamma.$$  \hfill (13)

In our approach intermediate vector boson are unstable particles and never appear in the final state. When the emitted photons are soft the corresponding amplitude is approximated, by standard methods, and reads as follows:

$$M^\text{soft}_n = (i e)^n M_0 \prod_{a=1}^n \epsilon(k_a) \cdot j(k_a),$$

$$j_\mu(k_a) = \sum_{i=1,2l} \theta_i \frac{p_{i\mu}}{p_i \cdot k_a},$$  \hfill (14)

where $\epsilon$ is the photon polarization vector, $\epsilon(k) \cdot k = 0$, and $M_0$ is the non-radiative amplitude for the process. Note that, in soft approximation, only photons emitted by external charged fermions are relevant. This fact is connected with gauge invariance as it can be illustrated by considering $e^+e^- \to ZZ\gamma$. There are two diagrams in Born approximation, direct and crossed conversions. If we consider only emission from the external $e^\pm$ lines, the Ward identity $k \cdot M^\text{ext} = 0$ is violated. However $k \cdot M^\text{soft}_\text{ext} = 0$, so that $M^\text{soft}_\text{ext}$ is gauge invariant and the full identity is restore by including $M^\text{int}$, i.e. emission from the internal electron line which, however, is neither infrared nor collinear divergent.

The cross-section for Eq.(14), for an infinite number of emitted soft photons, is

$$\sigma = \sum_{n=0,\infty} \frac{1}{n!} \int dPS_{\text{non-rad}} dPS_{\text{rad}} \sum_{\text{spins}} |M_n|^2 (2\pi)^{-4} \delta^4 \left(p_+ + p_- - \sum_{i=1}^{2l} q_i - \sum_{j=1}^n k_j\right),$$  \hfill (15)

$$dPS_{\text{non-rad}} = (2\pi)^{6l} \prod_{i=1}^{2l} d^4 q_i \delta^+(q_i^2 + m_i^2),$$

$$dPS_{\text{rad}} = (2\pi)^{3n} \prod_{j=1}^n d^4 k_j \delta^+(k_j^2),$$  \hfill (16)

where we have split the total phase-space into radiation and non-radiation phase spaces. We easily derive

$$\sigma \sim \sum_{n=0}^{\infty} \frac{1}{n!} \int dPS_{\text{non-rad}} \sum_{\text{spins}} |M_0|^2 (2\pi)^{3n} \prod_{j=1}^n d^4 k_j \delta^+(k_j^2)$$

$$\times |e j^\mu(k_j)|^2 (2\pi)^{-4} \delta^4 \left(p_+ + p_- - \sum_{i=1}^{2l} q_i - \sum_{j=1}^n k_j\right).$$  \hfill (17)
According to the classical treatment one writes
\[
\delta^4 \left( K - \sum_{j=1}^{n} k_j \right) = \int \frac{d^4x}{(2\pi)^4} \exp \left\{ i K \cdot x - i \sum_{j} k_j \cdot x \right\},
\]
\[
K = p_+ + p_- - \sum_{i=1}^{2l} q_i.
\] (18)

This result can be cast into the following form:
\[
\sigma \sim \int \prod_{i=1}^{2l} d^4 q_i \delta^+ \left( q_i^2 + m_i^2 \right) (2\pi)^{6l-4} \sum_{\text{spins}} |M_0|^2 E \left( p_+ + p_- - \sum_i q_i \right),
\] (19)

where the spectral function for the photon has been introduced:
\[
E(K) = \int \frac{d^4x}{(2\pi)^4} \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} \exp \left( i K \cdot x \right),
\]
\[
E_n(x) = \int \prod_{j=1}^{n} d^4 k_j \delta^+ (k_j^2) |e j^\mu(k_j) |^2 \exp \left( -i k_j \cdot x \right).
\] (20)

The photons can be re-summed to all orders giving
\[
E(K) = \int \frac{d^4x}{(2\pi)^4} \exp \left( i K \cdot x \right) \frac{1}{n!} [F(x)]^n,
\]
\[
F(x) = \frac{1}{(2\pi)^3} \int d^4 k \exp \left( -i k \cdot x \right) \delta^+ (k^2) |e j^\mu(k) |^2.
\] (21)

The Dirac delta-function, expressing four-momentum conservation, is therefore replaced inside Eq.(20) by the photon spectral function. The latter is defined through a Fourier-transform,
\[
E(K) = \frac{1}{(2\pi)^4} \int d^4 x \exp \left( i K \cdot x \right) E(x),
\]
\[
E(x) = \exp \left\{ \frac{\alpha}{2\pi^2} \int d^4 k \exp \left( -i k \cdot x \right) \delta^+ (k^2) |j^\mu(k) |^2 \right\}.
\] (22)

\(E(K)\) is the spectral function describing radiation and \(\alpha = e^2/4\pi\) is the fine-structure constant. An important property, following from charge conservation is
\[
\sum_{i,j} \theta_i \theta_j \frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k} = -\sum_{i<j} \theta_i \theta_j \left| \frac{p_i^\mu}{p_i \cdot k} - \frac{p_j^\mu}{p_j \cdot k} \right|^2,
\] (23)

and, therefore, we derive
\[
E(x) = \exp \left\{ -\frac{\alpha}{2\pi^2} \sum_{i<j} \theta_i \theta_j \int d^4 k \exp \left( -i k \cdot x \right) \delta^+ (k^2) \left| \frac{p_i^\mu}{p_i \cdot k} - \frac{p_j^\mu}{p_j \cdot k} \right|^2 \right\}.
\] (24)
Note that there is no delta-function expressing four-momentum conservation inside Eq.(20), not for the full process nor for the soft limit. Therefore, we are not allowed to use relations connected to momentum conservation. Later in this paper we will adopt a coplanar approximation for the exact spectral function, the difference between the two to be treated perturbatively, which will introduce again conservation but, this time, expressed in terms of the process where one incorporates photons emitted along the directions of the charged fermions.

3.2 Re-organization of the perturbative expansion

The exponentiation of infrared divergences is rigorous only in the limit where all photon momenta go to zero. The procedure adopted here will be to define an approximation to the exact cross section where the matrix element for the process are replaced by their soft limit. However, we do not introduce a cutoff separating hard from soft in the exponentiation, therefore the approximated result receives contributions from photon momenta which are large as well as soft. Perturbation theory is then reorganized by evaluating the difference between the exact and the approximated contributions. The main goal of the present investigation will be to establish a link between our approximation and the structure function language. Note that, by virtue of Eq.(12), our approximation satisfies \( U(1) \) gauge invariance.

Let \( M_n \) denote the complete amplitude for the production of a final state \( \{q\} \) with the emission of \( n \) photons,

\[
M_n (p_+,p_-; \{q\}; k_1, \ldots, k_n), \quad \rho_n = \sum_{\text{spin}} \left| M_n \right|^2.
\]  

(25)

The entire procedure amounts to construct a perturbative expansion which starts with an approximation that embodies the desired features of re-summation. We introduce, as usual, the factor

\[
J(k) = |e \varepsilon(k) \cdot j(k)|^2.
\]  

(26)

Then the complete amplitude squared can be written as

\[
\rho_n = \beta_0 \prod_{l=1}^n J(k_l) + \sum_{i=1}^n \prod_{l \neq i}^n J(k_l) \beta_1(k_i) + \ldots + \beta_n (k_1, \ldots, k_n).
\]  

(27)

A solution for the infrared-finite residuals \( \beta \) is

\[
\beta_0 = \rho_0, \quad \beta_1(k) = \rho_1(k) - \rho_0 J(k), \quad \text{etc.}
\]  

(28)

giving a perturbative expansion for the cross-section that starts with

\[
\sigma = \frac{1}{(2\pi)^4} \int d^4x \exp (i K \cdot x) E(x) \int dPS_q \left[ \beta_0 \\
+ \frac{1}{(2\pi)^4} \int d^4k \delta^+(k^2) \exp (-ik \cdot x) \beta_1(k) + \ldots \right].
\]  

(29)
Here $dPS_q$ is the phase-space for the final state fermions. At this point we can ask about the validity of a result that includes only $\beta_0$. The singular behavior of the amplitude for any radiative process is better examined in terms of the so-called dipole formalism. For simplicity let us consider the case of only two emitters, the generalization to an arbitrary number being straightforward, both outgoing. An approximation to the true amplitude squared that incorporates the correct singular behavior in the massless fermion limit is represented by

$$\left| M \right|^2 \sim \left| M_{\text{non-rad}} \right|^2 g_{ij},$$

$$g_{ij} = \frac{z}{y p_i \cdot p_j} \left[ \frac{2}{1 - z(1 - y)} - 1 - z \right],$$

$$y = \frac{p_i \cdot k}{p_i \cdot p_j + p_i \cdot k + p_j \cdot k}, \quad z = \frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k}. \quad (30)$$

It is easily seen that the infrared limit corresponds to $k \to 0$ or $y \to 0$, $z \to 1$ while the collinear limit to $y \to 0$, independently of $z$. Therefore the exponentiation procedure correctly accounts for photons that are infrared or infrared & collinear but not for photons that are hard & collinear. In the coplanar approximation, which receives contributions from photon momenta which are large as well as soft, we include all photons, soft or collinear that originates from

$$\frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k}, \quad (31)$$

that, however, does not reproduce the correct results of Eq.(30). In particular $\beta_1$ is essential to correctly reproduce the KLN result.

### 3.3 Inclusion of virtual corrections

Let $M_0$ be the Born amplitude for the process where $n$ photons are radiated. Let $M_1$ be the same amplitude with one loop corrections included. Therefore we obtain

$$M = \exp (\alpha B) \left[ M_0 + (M_1 - \alpha B M_0) + \ldots \right],$$

$$B = -\frac{i}{8 \pi^3} \sum_{i<j} B_{ij} \theta_i \theta_j. \quad (32)$$

$B_{ij}$ is the universal, i.e. process independent, YFS virtual factor. Virtual corrections are operatively included through the following procedure:

1) $\sigma \to \left| \exp (\alpha B) \right|^2 \sigma,$

2) $\beta_0 \to \sum_{\text{spins}} \left[ \left| M_0 \right|^2 + 2 \text{Re} M_0^* (M_1 - \alpha B M_0) \right] = \beta_{00} + \beta_{01}, \quad (33)$

3) $\beta_1 \to \beta_1,$
where the $M_0$ in $\beta_0$ denotes the one-loop corrected amplitude with no real photons and $\beta_1$ gives the Born amplitude with one emitted photon. The cross-section becomes

\[
\sigma = \int dPS_q \mid \exp(\alpha B) \mid^2 E (p_+ + p_- - \sum q) (\beta_{00} + \beta_{01}) + \frac{1}{(2\pi)^4} \int dPS_q d^4k \delta^+(k^2) d^4K \mid \exp(\alpha B) \mid^2 E(K) \frac{1}{(2\pi)^7} \int d^4x \exp \{ i (p_+ + p_- - \sum q - k - K) \cdot x \} \beta_{10}(k) + \ldots
\]

(34)

4 Evaluation of $B_{ij}$ in dimensional regularization

Consider an arbitrary process containing charged incoming and outgoing fermions, each with charge $Q_i$ and momentum $p_i$. The YFS $B_{ij}$ function in Eq.(32) describes infrared divergent virtual photons associated with the external charged lines, therefore independent of the internal details of the process. Let us consider the definition of $B_{ij}$ in dimensional regularization, i.e. for $n \neq 4$:

\[
B_{ij} = \mu^{4-n} \int \frac{d^4k}{k^2} B_{ij} = \mu^{4-n} \int \frac{d^4k}{k^2} \left[ \frac{(2 \epsilon_i p_i - k)\mu}{k^2 - 2 \epsilon_i \epsilon_j k \cdot k} + \frac{(2 \epsilon_j p_j + k)\mu}{k^2 + 2 \epsilon_j \epsilon_j k \cdot k} \right]^2,
\]

(35)

where $\epsilon_i = \pm 1$. The connection between $\epsilon_i$ and $\theta_i$ is as follows: each particle has a charge $Q_i = 0, -1, 2/3, -1/3$, then $\theta_i = \epsilon_i Q_i$, or $\epsilon_i = \pm 1$ for incoming(outgoing) fermion $i$ (outgoing(incoming) anti-fermion $i$). Using

\[
k^2 - 2 \epsilon_i p_i \cdot k = (k - \epsilon_i p_i)^2 + m_i^2 \equiv (2),
\]

\[
k^2 + 2 \epsilon_j p_j \cdot k = (k + \epsilon_j p_j)^2 + m_j^2 \equiv (3),
\]

(36)

we derive the following decomposition:

\[
B_{ij} = \frac{1}{(2)^2} \left[ 4 \left( p_i^2 - \epsilon_i \epsilon_j p_i \cdot k \right) + k^2 \right] + \frac{1}{(3)^2} \left[ 4 \left( p_j^2 + \epsilon_j \epsilon_j p_j \cdot k \right) + k^2 \right]
\]

\[
+ \frac{2}{(2)(3)} \left[ 4 \epsilon_i \epsilon_j p_i \cdot p_j + 2 \left( \epsilon_i p_i - \epsilon_j p_j \right) \cdot k - k^2 \right],
\]

(37)

which allows us to derive $B_{ij}$ in terms of standard one-loop scalar functions [12]. We have three terms

\[
B_{ij} = \sum_{l=1}^{3} B'_{ij},
\]

(38)

where the first one can be written as

\[
B_{ij}^1 = i \pi^2 \left\{ B_0 \left( p_i^2, 0, m_i \right) - 2 \frac{\partial}{\partial m_i^2} \left[ 2 p_i^2 B_0 \left( p_i^2, 0, m_i \right) + p_i^2 B_1 \left( p_i^2, 0, m_i \right) \right] \right\} \mid_{p_i^2 = -m_i^2}.
\]

(39)
The various terms are computed as follows [13]:

\[
B_0 \left( -m^2, 0, m \right) = \frac{1}{\bar{\varepsilon}} - \ln \frac{m^2}{\mu^2} + 2, \\
\frac{\partial}{\partial m^2} B_0 \left( p^2; 0, m \right) \mid_{p^2 = -m^2} = -\frac{1}{2m^2} \left( \frac{1}{\bar{\varepsilon}} + \ln \frac{m^2}{\mu^2} \right), \\
\frac{\partial}{\partial m^2} B_1 \left( p^2; 0, m \right) \mid_{p^2 = -m^2} = \frac{1}{m^2},
\]

(40)
giving the first term,

\[
B_{ij}^1 = i\pi^2 \left[ -3 \left( \frac{1}{\bar{\varepsilon}} + \ln \frac{m^2}{\mu^2} \right) + 4 \right].
\]

(41)

Similarly we derive

\[
B_{ij}^2 = i\pi^2 \left[ -3 \left( \frac{1}{\bar{\varepsilon}} + \ln \frac{m^2}{\mu^2} \right) + 4 \right].
\]

(42)

Although not strictly necessary, we have carefully distinguished ultraviolet (1/\varepsilon) poles from infrared (1/\varepsilon') ones,

\[
\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln \pi, \quad \frac{1}{\varepsilon'} = \frac{2}{\varepsilon'} + \gamma + \ln \pi,
\]

(43)

with regulators that satisfy \( \bar{\varepsilon} + \varepsilon = 0 \). The last term is the sum of three contributions,

\[
B_{ij}^3 = i\pi^2 \left\{ 8 \varepsilon_i p_i \cdot p_j C_0 - 4 P_{-ij} \cdot \left[ C_{11} \varepsilon_i p_i - C_{12} P_{+ij} \right] - 2 B_0 \left( P_{+ij}^2; m_i, m_j \right) \right\},
\]

(44)

where the \( C \)-functions have arguments

\[
p_1 = -\varepsilon_i p_i, \quad p_2 = P_{+ij}, \quad m_1 = 0, m_2 = m_i, m_3 = m_j,
\]

(45)

and where we have introduced

\[
P_{\pm ij} = \varepsilon_i p_i \pm \varepsilon_j p_j.
\]

(46)

The function \( C_0 \) has a well-known representation,

\[
C_0 = \frac{1}{2} \left( F_1 \frac{1}{\bar{\varepsilon}} + F_2 \right),
\]

(47)

where \( F_{1,2} \) are given in terms of variables

\[
y_{ij} = \frac{1}{P_{+ij}} \left[ P_{+ij}^2 + m_j^2 - m_i^2 \pm \lambda^{1/2} \left( -P_{+ij}^2, m_i^2, m_j^2 \right) \right].
\]

(48)

Through the paper

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2 \left( xy + xz + yz \right),
\]

(49)
represents the Källen’s λ-function. The following result holds:

\[
F_1 = \frac{1}{P_{+ij}^2} \left( \ln \left( 1 - \frac{1}{y_1} \right) - \ln \left( 1 - \frac{1}{y_2} \right) \right),
\]

\[
F_2 = F_1 \ln \frac{P_{+ij}^2 - i \varepsilon}{\mu^2} + \frac{1}{P_{+ij}^2} \left[ \ln \left( \frac{y_1^{ij}}{y_2^{ij}} \right) - \ln \left( y_1^{ij} \right) \right],
\]

\[
f(x, y) = \frac{1}{2} \ln \left( 1 - \frac{1}{y} \right) \ln \left[ (y-1)(x-y)^2 \right] - \text{Li}_2 \left( \frac{1-y}{x-y} \right) + \text{Li}_2 \left( \frac{-y}{x-y} \right).
\]

\text{Li}_2(z) \text{ is the standard di-logarithm. The higher rank three-point functions can be reduced to scalar ones as follows:}

\[- C_{11} \epsilon_i p_i^\mu + C_{12} P_{+ij}^\mu = \left[ \tilde{C}_{11} + C_0 \right] \epsilon_i p_i^\mu + \tilde{C}_{12} \epsilon_j p_j^\mu. \tag{51} \]

In the above equation the \( \tilde{C} \)-functions have arguments

\[ p_1 = \epsilon_i p_i, \quad p_2 = \epsilon_j p_j, \quad m_1 = m_i, \quad m_2 = 0, \quad m_3 = m_j. \tag{52} \]

Following standard reduction techniques we obtain

\[
\tilde{C}_{11} = -\frac{1}{d} \left( m_j^2 R_1 + \epsilon_i \epsilon_j p_i \cdot p_j R_2 \right),
\]

\[
\tilde{C}_{12} = -\frac{1}{d} \left( m_i^2 R_2 + \epsilon_i \epsilon_j p_i \cdot p_j R_1 \right),
\]

\[
R_1 = \frac{1}{2} \left[ B_0 \left( P_{+ij}^2; m_i, m_j \right) - B_0 \left( -m_j^2; 0, m_j \right) + f_1 C_0 \right],
\]

\[
R_2 = \frac{1}{2} \left[ -B_0 \left( P_{+ij}^2; m_i, m_j \right) + B_0 \left( -m_i^2; 0, m_i \right) + f_2 C_0 \right],
\]

\[
f_1 = 2 m_i^2, \quad f_2 = -2 \epsilon_i \epsilon_j p_i \cdot p_j, \quad d = m_i^2 m_j^2 - (p_i \cdot p_j)^2, \tag{53} \]

giving the final result for \( B^3 \),

\[
B_{ij}^3 = 2 i \pi^2 \left[ 4 \epsilon_i \epsilon_j p_i \cdot p_j C_0 + B_0 \left( P_{+ij}^2; m_i, m_j \right) - \sum_{l=i}^j B_0 \left( -m_l^2; 0, m_l \right) \right]. \tag{54} \]

If we use the explicit expressions for the scalar integrals we obtain

\[
C_0 = \frac{1}{2} \left( \frac{1}{\varepsilon} + \ln \frac{P_{+ij}^2 - i \varepsilon}{\mu^2} \right) F_1 + \frac{1}{2} F_2^{\text{rest}},
\]

\[
B_0 \left( -m^2; 0, m \right) = -\frac{1}{\varepsilon} - \ln \frac{m^2}{\mu^2} + 2, \quad B_0 \left( P_{+ij}^2; m_i, m_j \right) = -\frac{1}{\varepsilon} - \ln \frac{m_i m_j}{\mu^2} + F_3. \tag{55} \]
where $F_3^{\text{rest}}$ is the part of $F_2$ not proportional to $F_1$, see Eq.(50). Furthermore, $F_3$ is the finite part of the $B_0$-function, given by

$$F_3(p^2; m_i, m_j) = \frac{m_i^2 - m_j^2}{2p^2} \ln \frac{m_i^2}{m_j^2} + 2 - \frac{\Lambda}{p^2} \ln \frac{\Lambda^2 - i\varepsilon + m_i^2 + m_j^2 - \Lambda}{2m_im_j},$$  

with $\Lambda^2 = \lambda(-p^2, m_i^2, m_j^2)$. Using these results we find

$$B_{ij} = 2i\pi^2 \left\{2 \left(-1 + \epsilon_i \epsilon_j p_i \cdot p_j F_1 \right) \frac{1}{\varepsilon} - \sum_{l=ij} \ln \frac{m_i^2}{\mu^2} 
+ \epsilon_i \epsilon_j p_i \cdot p_j \left[F_1 \ln \frac{P^{2+ij}_i - i\varepsilon}{\mu^2} + F_2^{\text{rest}} \right] + F_3 \right\},$$

showing that the virtual infrared pole originates from a $C_0$ function. $F_1$ and $F_2^{\text{rest}}$ are defined in Eq.(50), $F_3$ in Eq.(56). Let $Q_{ij} = P_{+ij}$, we will consider two limiting cases.

4.1 The case of large invariant

If $m_i = m_j = m$ the results simplify into

$$F_1 = \frac{2}{P^{2+ij}_i \beta_{ij}} \ln \frac{\beta_{ij} + 1}{\beta_{ij} - 1},
F_2^{\text{rest}} = \frac{1}{P^{2+ij}_i \beta_{ij}} \left[\ln \frac{\beta_{ij} + 1}{\beta_{ij} - 1} \ln \frac{m_i^2 \beta_{ij}^2}{P^{2+ij}_i} - 2 \sum_{l=\pm1} \text{Li}_2 \left(\frac{\beta_{ij} + l}{2 \beta_{ij}}\right)\right],$$

with $\beta_{ij}^2 = 1 + 4m^2/P^{2+ij}_i$. Furthermore

$$F_3 = 2 - \beta_{ij} \ln \frac{\beta_{ij} + 1}{\beta_{ij} - 1}.$$  

In the limit $|Q_{ij}| \gg m^2$, where $Q_{ij} = P_{+ij}$, we obtain

$$F_1 \sim \frac{2}{Q_{ij}^2} \ln \frac{Q_{ij}^2 - i\varepsilon}{m^2},$$

$$F_2^{\text{rest}} = -\frac{1}{Q_{ij}^2} \left[\ln^2 \frac{Q_{ij}^2 - i\varepsilon}{m^2} + \frac{\pi^2}{3}\right],$$

$$F_3 \sim -\ln \frac{Q_{ij}^2 - i\varepsilon}{m^2} + 2.$$
4.2 The case of small invariant

Consider now the opposite limit where \(|Q_{ij}^2| \ll m_i^2, m_j^2\). We easily derive

\[
F_1 \sim \frac{1}{m_j^2 - m_i^2} \ln \frac{m_j^2}{m_i^2}, \quad F_2 \sim \frac{1}{2} \frac{1}{m_j^2 - m_i^2} \left[\ln^2 \frac{m_j^2}{\mu^2} - \ln^2 \frac{m_i^2}{\mu^2}\right].
\]  

(61)

Note that, for \(m_i = m_j = m\) this further simplifies into

\[
F_1 \sim \frac{1}{m^2}, \quad F_2 \sim \frac{1}{2} \frac{1}{m^2} \ln \frac{m^2}{\mu^2}.
\]  

(62)

Furthermore the two-point function is

\[
B_0 (0; m_i, m_j) = -\frac{1}{\varepsilon} - 1 - \frac{1}{m_j^2 - m_i^2} \left[\ln^2 \frac{m_j^2}{\mu^2} - m_i^2 \ln \frac{m_i^2}{\mu^2}\right].
\]  

(63)

which, for equal masses, gives

\[
B_0 (0; m, m) = -\frac{1}{\varepsilon} - \ln \frac{m^2}{\mu^2}.
\]  

(64)

Collecting the various terms and introducing \(r = m_j^2/m_i^2\) we obtain

\[
B_{ij} = i \pi^2 \left\{\left[\frac{r+1}{r-1} \ln r - 2\right] \frac{1}{\varepsilon} + \frac{r+1}{r-1} \left[\ln^2 \frac{m_j^2}{\mu^2} - \ln^2 \frac{m_i^2}{\mu^2}\right]
\right.
\]

\[
+ \frac{3-r}{r-1} \ln \frac{m_i^2}{\mu^2} + \frac{1-3r}{r-1} \ln \frac{m_j^2}{\mu^2} + 2\right\},
\]  

(65)

which, for equal masses gives

\[
B_{ij} \left(Q^2, m_i = m_j = m\right) \to 0, \quad \text{for} \quad Q^2 \to 0.
\]  

(66)

5 The two-particle radiation factor

The photon spectral function is defined in Eq. (22). Remarkably enough the exponent in \(E(x)\) can be computed exactly \([1]\). We will show it by computing \(E(x)\) for the case of only two emitters. This underlying ingredient will be referred to as the two-particle radiation factor, \(R_{ij}\). In dimensional regularization it reads as follows:

\[
R_{ij} = \mu^{4-n} \int d^n k \exp (-i k \cdot x) \frac{\delta^+(k^2)}{p_i \cdot k p_j \cdot k},
\]  

(67)

where \(\mu\) is the arbitrary unit of mass and

\[
p_i^2 = -m_i^2, \quad q_{ij}^2 = (p_i + p_j)^2.
\]  

(68)
Furthermore $\delta^+ \selects$ positive energies,
\begin{equation}
\delta^+ (k^2) = \theta(k_0) \delta(k^2). \tag{69}
\end{equation}

In order to evaluate $\mathcal{R}_{ij}$ we introduce a parameter $\rho_{ij} \defined$ by the following relations \cite{14}:
\begin{equation}
p = \rho_{ij} p_i, \quad q = p_j, \quad (p - q)^2 = 0. \tag{70}
\end{equation}
Hence we have a solution
\begin{equation}
\rho_{ij}^\pm = \frac{1}{2m_i^2} \left[ -q_{ij}^2 - m_i^2 - m_j^2 \pm \lambda^{1/2} ( -q_{ij}^2, m_i^2, m_j^2 ) \right]. \tag{71}
\end{equation}
If $q_{ij}^2 < 0$ we select $\rho^+$, while $\rho^-$ is chosen when $q_{ij}^2 > 0$. Next we introduce a Feynman parameter $u$ and define
\begin{equation}
P = q + ( p - q ) \ u. \tag{72}
\end{equation}
With our choice for $\rho$ it follows that $P^2 < 0$ and we will compute $\mathcal{R}$ in the frame where
\begin{equation}
P = 0, \quad P_0 = M. \tag{73}
\end{equation}

Therefore we obtain
\begin{equation}
\mathcal{R} = \rho \int_0^1 du \, J,
J = \frac{\pi^{n/2 - 1} \mu^{4-n}}{M^2 \Gamma(n/2 - 1)} \int_0^\infty dk \, k^{n-5} \int_{-1}^{+1} dy \, (1 - y^2)^{n/2 - 2} \exp{\{ -i \, k \, (y r - x_0) \}}, \tag{74}
\end{equation}
where $r = |x|$. If we write
\begin{equation}
\mathcal{R} = \rho \frac{\pi^{n/2 - 1} \mu^{4-n}}{\Gamma(n/2 - 1)} \int_0^1 du \, K, \tag{75}
\end{equation}
then the function $H$ becomes
\begin{align*}
H &= \int_0^\infty dk \, k^{n-5} \int_{-1}^{+1} dy \, \exp\{ -i \, k \, (y r - x_0) \} \, (1 - y^2)^{n/2 - 2} \\
&= \left( \frac{2}{r} \right)^{n/2 - 3/2} \pi^{1/2} \Gamma(n/2 - 1) \int_0^\infty dk \, \exp\{ i \, k x_0 \} \, k^{n/2 - 7/2} \, J_{n/2 - 3/2}(kr), \tag{76}
\end{align*}
where $J_\nu$ is a Bessel-function. As a consequence we arrive at the following result:
\begin{align*}
\mathcal{R} &= 2^{n/2 - 3/2} \, \rho \, \pi^{n/2 - 1/2} \, (\mu r)^{1-n} \int_0^1 du \, \frac{K}{M^2}, \\
K &= K_r + i \, \epsilon(x_0) \, K_i = \int_0^\infty dk \, \exp\{ i \, x_0 \, k \} \, k^{n/2 - 7/2} \, J_{n/2 - 3/2}(k). \tag{77}
\end{align*}
Real and imaginary parts of $\mathcal{R}$ will be computed separately.
5.1 The real part

We start by computing the real part of $K$, Eq. (77),

$$K_r = \int_0^\infty dk \cos(\xi k) \, k^{n/2-7/2} \, J_{n/2-3/2}(k), \quad \xi = \frac{x_0}{r}, \quad 0 < \xi < \infty. \quad (78)$$

This function will be considered separately in the two regions $0 < \xi < 1$ and $1 < \xi < \infty$. For the former we find

$$0 < \xi < 1, \quad K_r = 2^{n/2-7/2} \frac{\Gamma(n/2 - 2)}{\Gamma(3/2)} \, {}_2F_1 \left( \frac{n - 4}{2}, \frac{1}{2}; \frac{1}{2}; \xi^2 \right), \quad (79)$$

where ${}_2F_1$ is the standard hypergeometric function \[10\]. Let $n = 4 + \varepsilon'$, with $\varepsilon' \geq 0$, then we can use the Laurent’s expansion of the $\Gamma$-function

$$\Gamma \left( \frac{\varepsilon'}{2} \right) = \frac{2}{\varepsilon'} - \gamma + \mathcal{O}(\varepsilon'), \quad (80)$$

and one of the transformation properties of the hypergeometric function to arrive at the following form:

$$_2F_1 \left( \frac{1 - \varepsilon'}{2}, \frac{1}{2}; \frac{1}{2}; \xi^2 \right) = \left( 1 - \xi^2 \right)^{1-\varepsilon'/2} \, {}_2F_1 \left( \frac{1 - \varepsilon'}{2}, 1; \frac{1}{2}; \xi^2 \right). \quad (81)$$

Since we are in the region $0 \leq \xi^2 \leq 1$ the following relation holds:

$$_2F_1 \left( \frac{1 - \varepsilon'}{2}, 1; \frac{1}{2}; \xi^2 \right) = \frac{\Gamma(1/2)}{\Gamma(1/2 - \varepsilon'/2)} \sum_{l=0}^{\infty} \frac{\Gamma(l + 1/2 - \varepsilon'/2) \Gamma(l + 1)}{\Gamma(l + 1/2)} \frac{\xi^{2l}}{l!}. \quad (82)$$

The function $\Gamma(\varepsilon'/2)$ shows an infrared pole and, therefore, we must expand the hypergeometric function in powers of $\varepsilon'$. The whole procedure is cumbersome and, essentially, requires to obtain derivative of $\Gamma$-function with respect to the parameters. Using the following expansion for the Euler gamma-function

$$\Gamma(a + \lambda \varepsilon') = \Gamma(a) \left[ 1 + \lambda \psi(a) \, \varepsilon' + \mathcal{O}(\varepsilon'^2) \right], \quad (83)$$

where it appears the Euler $\psi$-function, we obtain

$$\Gamma(\varepsilon'/2) = \Gamma(1/2) \sum_{l=0}^{\infty} \frac{\Gamma(2 + l)}{\Gamma(1/2 - \varepsilon'/2)} \frac{\xi^{2l}}{l!}. \quad (84)$$

As a consequence we are to consider the following expansion

$$\Gamma(\varepsilon'/2) = \sum_{l=0}^{\infty} \frac{\Gamma(2 + l)}{l!} \left[ \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} + l \right) \right] \xi^{2l} + \mathcal{O}(\varepsilon'^2). \quad (85)$$

As a consequence we are to consider the following expansion

$$\, {}_2F_1 \left( \frac{1 - \varepsilon'}{2}, 1; \frac{1}{2}; \xi^2 \right) = \left( 1 - \xi^2 \right)^{-1} + \frac{\varepsilon'}{2} E_1(\xi^2) + \mathcal{O}(\varepsilon'^2),$$

$$E_1(\xi^2) = \sum_{l=0}^{\infty} \left[ \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} + l \right) \right] \xi^{2l}.$$
The series in $E_1$ can be re-summed as follows. First we write

$$\psi \left( l + \frac{1}{2} \right) - \psi \left( \frac{1}{2} \right) = \int_0^1 dx \, x^{-1/2} \frac{1 - x^l}{1 - x}, \tag{86}$$

and successively we obtain

$$\sum_{l=0}^{\infty} \left[ \psi \left( l + \frac{1}{2} \right) - \psi \left( \frac{1}{2} \right) \right] z^l = \sum_{l=0}^{\infty} \int_0^1 dx \, x^{-1/2} \frac{1 - x^l}{1 - x} z^l, \tag{87}$$

$$= \frac{z}{1 - z} \int_0^1 dx \, x^{-1/2} (1 - zx)^{-1}$$

The re-summation gives a rather simple result,

$$E_1(\xi) = -\frac{\xi}{1 - \xi^2} \ln \frac{1 + \xi}{1 - \xi}, \tag{88}$$

For $K_r$ in this region we find

$$K_r = (2 \pi)^{-1/2} \left[ \frac{2}{\epsilon' \gamma} - \gamma + \ln 2 - \ln \left( 1 - \xi^2 \right) - \xi \ln \frac{1 + \xi}{1 - \xi} \right]. \tag{89}$$

The result for $R_r$ in the region $0 \leq \xi \leq 1$ is

$$R_r = \pi \rho \int_0^1 \frac{du}{M^2} \left[ \frac{2}{\epsilon' \gamma} - \gamma + \ln \pi + 2 \ln 2 - \ln \mu^2 x^2 - \xi \ln \frac{1 + \xi}{1 - \xi} \right]. \tag{90}$$

Now we turn to the complementary region $1 \leq \xi \leq \infty$, where

$$K_r = 2^{3/2 - n/2} \xi^{4-n} \cos \left( \frac{n - 4}{2} \pi \right) \frac{\Gamma(n-4)}{\Gamma(n-1/2)} \, {}_2F_1 \left( \frac{n}{2} - 2, \frac{n-3}{2}; \frac{n-1}{2}; \xi^{-2} \right), \tag{91}$$

and where we will use

$${}_2F_1 \left( \frac{\epsilon'}{2}, 1 + \frac{\epsilon'}{2}; \frac{3 + \epsilon'}{2}; \xi^{-2} \right) = \left( 1 - \frac{1}{\xi^2} \right)^{1-\epsilon'/2} \, {}_2F_1 \left( \frac{3}{2}, 1; \frac{3 + \epsilon'}{2}; \xi^{-2} \right). \tag{92}$$

Since we are in the region where $0 \leq \xi^{-2} \leq 1$ the following result holds:

$${}_2F_1 \left( \frac{3}{2}, 1; \frac{3 + \epsilon'}{2}; \xi^{-2} \right) = \left( 1 - \frac{1}{\xi^2} \right)^{-1} - \frac{\epsilon'}{2} E_2(\xi) + O(\epsilon''), \tag{93}$$

$$E_2(\xi) = \sum_{l=0}^{\infty} \left[ \psi \left( l + \frac{3}{2} \right) - \psi \left( \frac{3}{2} \right) \right] \xi^{-2l}.$$

The series for $E_2$ can be re-summed by using the following identity:

$$\psi \left( \frac{3}{2} + l \right) - \psi \left( \frac{3}{2} \right) = \int_0^1 dx \, x^{1/2} \frac{1 - x^l}{1 - x}, \tag{94}$$
and, consequently, we get
\[
\sum_{l=0}^{\infty} \left[ \psi\left(l + \frac{3}{2}\right) - \psi\left(\frac{3}{2}\right) \right] z^l = \sum_{l=0}^{\infty} \int_0^1 dx \frac{1}{x^{1/2}} \frac{1-z^l}{1-x} \frac{z^l}{1-z^l} = \frac{z}{1-z} \int_0^1 dx \frac{x^{1/2}}{1-zx}. \tag{95}
\]

We obtain the following result for \(E_2\):
\[
E_2(\xi) = \left(1 - \frac{1}{\xi^2}\right)^{-1} \left[\xi \ln \frac{\xi + 1}{\xi - 1} - 2\right], \tag{96}
\]
and
\[
K_r = \left(\frac{2}{\pi}\right)^{1/2} \left[\frac{1}{\varepsilon} - \frac{1}{2} \gamma + \frac{1}{2} \ln 2 - \frac{1}{2} \ln \left(1 - \frac{1}{\xi^2}\right) - \frac{1}{2} \xi \ln \frac{\xi + 1}{\xi - 1}\right]. \tag{97}
\]

As far as the real part is concerned we have
\[
0 \leq \xi \leq 1 \quad R_r = \pi \rho \int_0^1 \frac{du}{M^2} \left[\frac{2}{\varepsilon^2} - \gamma + \ln \pi + 2 \ln 2 - \ln \mu^2 x^2 - \xi \ln \frac{1 + \xi}{1 - \xi}\right],
\]
\[
1 \leq \xi \leq \infty \quad R_r = \pi \rho \int_0^1 \frac{du}{M^2} \left[\frac{2}{\varepsilon^2} - \gamma + \ln \pi + 2 \ln 2 - \ln(-\mu^2 x^2) - \xi \ln \frac{\xi + 1}{\xi - 1}\right]. \tag{98}
\]

5.2 The imaginary part

The imaginary part of the two-particle radiator is given in terms of hypergeometric functions,
\[
K_i = \int_0^\infty dk \sin (\xi k) \frac{k^{n/2-7/2}}{n/2-3/2} J_{n/2-3/2}(k). \tag{99}
\]
This can again be written in terms of hypergeometric functions,
\[
0 \leq \xi \leq 1 \quad K_i = 2^{n/2-5/2} \xi \Gamma\left(\frac{n-3}{2}\right) \frac{1}{2} F_1\left(\frac{n-3}{2}, 0, \frac{3}{2}; \xi^2\right),
\]
\[
1 \leq \xi \leq \infty \quad K_i = 2^{3/2-n/2} \xi^{4-n} \frac{\Gamma(n-4)}{\Gamma(n/2-1/2)} \sin \left(\frac{n-4}{2}\right) \Gamma\left(\frac{n-4}{2}\right) \left(\frac{n-3}{2}; \xi^{-2}\right) \times \frac{1}{2} F_1\left(\frac{n-3}{2}, \frac{n-4}{2}, \frac{n-1}{2}; \xi^{-2}\right). \tag{100}
\]

We use the following expansion,
\[
\sin \left(\frac{n-4}{2}\right) \Gamma\left(\frac{n-4}{2}\right) = \frac{\pi}{2} \Gamma(n-3) + O(n-4), \tag{101}
\]
18
to show that, as expected, the imaginary part has no infrared poles and, therefore, we may set \( n = 4 \). The result is

\[
0 \leq \xi \leq 1 \quad K_i = 2^{-1/2} \pi^{1/2} \xi \, _2F_1\left(\frac{1}{2}, 0; \frac{3}{2}; \xi^2\right),
\]

\[
1 \leq \xi \leq \infty \quad K_i = 2^{-1/2} \pi^{1/2} \, _2F_1\left(\frac{1}{2}, 0; \frac{3}{2}; \xi^{-2}\right).
\] (102)

It is straightforward to derive that

\[
_2F_1\left(\frac{1}{2}, 0; \frac{3}{2}; \xi^2\right) = 1.
\] (103)

The final result for the imaginary part reads as follows:

\[
0 \leq \xi \leq 1 \quad R_i = \rho \pi^2 \int_0^1 \frac{du}{M^2} \xi,
\]

\[
1 \leq \xi \leq \infty \quad R_i = \rho \pi^2 \int_0^1 \frac{du}{M^2}.
\] (104)

In order to cast the final result into a more compact form we introduce an infinitesimal quantity \( \delta \) such that

\[
x_0 \to x_0 + i \delta, \quad \delta \to 0+,
\] (105)

It follows that \( x^2 \to r^2 - x_0^2 - i x_0 \delta \) and

\[
\ln(x^2) \to \ln \left(x^2 - i x_0 \delta\right) = \left\{ \begin{array}{ll}
\ln(x^2) & \text{for } x^2 > 0 \\
\ln(-x^2) - i \pi \epsilon(x_0) & \text{for } x^2 < 0
\end{array} \right.
\]

5.3 The complete result

Collecting the results for \( R = R_r + i \epsilon(x_0) R_i \), we obtain for \( 0 \leq \xi \leq 1 \)

\[
R_r = \rho \pi \int_0^1 \frac{du}{M^2} \left[ \frac{1}{\xi} + 2 \ln(2 - \gamma) - \ln(2 - \gamma) - \ln \mu^2 x^2 - \xi \ln \frac{1 + \xi}{1 - \xi} \right],
\]

\[
R_i = \rho \pi^2 \int_0^1 \frac{du}{M^2} \xi,
\] (106)

and for \( 1 \leq \xi \leq \infty \),

\[
R_r = \rho \pi \int_0^1 \frac{du}{M^2} \left[ \frac{1}{\xi} + 2 \ln(2 - \gamma) - \ln(-\mu^2 x^2) - \xi \ln \frac{\xi + 1}{\xi - 1} \right],
\]

\[
R_i = \rho \pi^2 \int_0^1 \frac{du}{M^2}.
\] (107)

Here we have introduced

\[
\frac{2}{\varepsilon'} = \frac{1}{\varepsilon} - \gamma - \ln \pi.
\] (108)
The total result is

\[ \mathcal{R} = \mathcal{R}_r + i \epsilon(x_0) \mathcal{R}_i. \]  

(109)

For \( 1 \leq \xi \leq \infty \) it follows \( x^2 \leq 0 \) and, therefore

\[ \ln(-x^2) - i \pi \epsilon(x_0) \to \ln x^2, \]  

(110)

while, for \( 0 \leq \xi \leq 1 \) we can replace

\[ \ln \frac{1 + \xi}{1 - \xi} + i \pi \epsilon(x_0) \to \ln \frac{\xi + 1}{\xi - 1}. \]  

(111)

Therefore the function \( \mathcal{R} \) is defined on the whole \( \xi \)-axis by the following expression:

\[ \mathcal{R} = \rho \pi \int_0^1 \frac{du}{M^2} \left[ \frac{1}{\xi} + 2 \left( \ln 2 - \gamma \right) - \ln \mu^2 x^2 - \xi \ln \frac{\xi + 1}{\xi - 1} \right] \]

\[ = \rho \pi \int_0^1 \frac{du}{M^2} \left[ \Delta_{\text{IR}} - \ln \mu^2 x^2 - \xi \ln \frac{\xi + 1}{\xi - 1} \right], \]

\[ \Delta_{\text{IR}} = \frac{1}{\xi} + 2 \left( \ln 2 - \gamma \right), \]

\[ \xi = \frac{x_0}{r}, \quad x_0 = x_0 + i \delta, \quad \delta \to 0+. \]  

(112)

To get the final form of our result we must express all quantities in covariant form. Therefore we have

\[ M^2 = -P^2, \quad x_0 = -\frac{P \cdot x}{M}, \quad \xi^2 = \frac{(P \cdot x)^2}{(P \cdot x)^2 + P^2 x^2}. \]  

(113)

### 6 Radiation from two legs

Consider the case when the photon can be emitted by two external charged fermions only. The corresponding radiator is

\[ \mathcal{R}(p_i, p_j) = -\frac{\theta_i \theta_j}{2 \pi^2} \left[ p_i^2 \mathcal{R}_{ii} + p_j^2 \mathcal{R}_{jj} - 2 p_i \cdot p_j \mathcal{R}_{ij} \right], \]  

(114)

and we need the photon spectral function,

\[ E_{ij}(x) \equiv E(p_i, p_j; x) = \frac{1}{(2 \pi)^4} \int d^4 K \exp\{ -i K \cdot x + \alpha \mathcal{R}(p_i, p_j) \}. \]  

(115)

We rewrite the radiator as

\[ \mathcal{R}_{ij} = \rho_{ij} \pi \left( \Delta_{\text{IR}} - \ln \mu^2 x^2 \right) \int_0^1 \frac{du}{M^2} \rho_{ij} \pi r_{ij}, \]

\[ = \rho_{ij} \pi \left( \Delta_{\text{IR}} - \ln \mu^2 x^2 \right) \frac{1}{q^2 - p^2} \ln \frac{p^2}{q^2} + \rho_{ij} \pi r_{ij}, \]

\[ = -\rho_{ij} \pi \ln \left( e^{-\Delta_{\text{IR}}} \mu^2 x^2 \right) \frac{1}{q^2 - p^2} \ln \frac{p^2}{q^2} + \rho_{ij} \pi r_{ij}. \]  

(116)
In principle there is no problem in evaluating the finite part of the radiator explicitly, with a result that contains several di-logarithms. However, we have to exponentiate it as in Eq. (115) and, successively, we must compute the Fourier transform of the result: we will not be able to proceed any further with the complete expression. Using Eq. (113) we get

\[ P^2 = q^2 + (p^2 - q^2) \ u, \quad P \cdot x = q \cdot x + (p - q) \cdot x \ u, \quad (117) \]

and \( \xi \) becomes

\[ \xi^2 = \frac{(a + b \ u)^2}{A \ u^2 + 2 \ B \ u + C}, \quad a = q \cdot x, \quad b = (p - q) \cdot x, \quad C = (q \cdot x)^2, \quad B = q \cdot x (p - q) \cdot x + \frac{1}{2} \ (p^2 - q^2) \ x^2, \quad A = b^2. \quad (118) \]

Therefore, one integral is immediate

\[ \int_0^1 \frac{du \ M^2}{M^2} = \frac{1}{p^2 - q^2} \ln \frac{p^2}{q^2}, \quad (119) \]

while the remaining one starts with

\[ \int_0^1 \frac{\xi \ M^2}{M^2} \ln \frac{\xi + 1}{\xi - 1} = - \int_0^1 \frac{a + bu}{(c + du) \ U} \ln \frac{a + bu + U}{a + bu - U}, \quad (120) \]

where \( U^2 = A \ u^2 + 2 \ B \ u + C \) and, moreover, \( c = q^2, d = p^2 - q^2 \). If needed the last integral can be computed through the substitution

\[ t = \frac{u}{U - \sqrt{C}}. \quad (121) \]

### 6.1 Coplanar approximation

The Fourier transform of Eq. (113) cannot be computed in closed form, therefore we change strategy and introduce an approximated formulas which is much simpler to handle in practical computations [11]. This approximation is the coplanar one, where the effective photon momentum is constrained to lie in the plane formed by \( p_i \) and \( p_j \), so that the spectral function turns out to be proportional to \( \delta^2(K_\perp) \), where \( K_\perp \) is the transverse component of \( K \). This coplanar approximation reads

\[ R^c(p_i, p_j) = - \frac{\theta_i \theta_j}{\pi} \left\{ \ln \left( - e^{-\Delta_{iR}^2} \frac{\mu^2 \ p_i \cdot x \ p_j \cdot x}{s_{ij}} \right) \left[ 1 + \rho_{ij} \frac{p_i \cdot p_j}{p_j^2 - \rho_{ij}^2 p_i^2} \ln \frac{\rho_{ij}^2 p_i^2}{p_j^2} \right] + \frac{1}{2} + \frac{\pi^2}{6} \right\}, \quad (122) \]

where we have used the fact that for \( i = j \) \( \rho_{ii} = 1, M^2 = m_i^2 \). Several new quantities have been introduced,

\[ \Delta_{iR}^2 = \frac{1}{\varepsilon} - 2 \gamma + \frac{3}{2}, \quad s_{ij} = \left[ 1 + \frac{Q_{ij}^2}{m_i m_j} \right]^{1/2} m_i m_j, \quad Q_{ij}^2 = (\varepsilon_i p_i + \varepsilon_j p_j)^2. \quad (123) \]
Note that $s_{ij}$ satisfies the following asymptotic behavior.

\[
s_{ij} \sim \left[ |Q_{ij}^2| m_i m_j \right]^{1/2}, \quad |Q_{ij}^2| \gg m_i^2, m_j^2, \]
\[
s_{ij} \sim m_i m_j, \quad |Q_{ij}^2| \ll m_i^2, m_j^2.
\]

The same result is rewritten as

\[
R_c(p_i, p_j) = -A_{ij} \ln \left( -e^{-\Delta_{16}^2 \frac{\mu^2 p_i \cdot x p_j}{s_{ij}}} \right) + \delta_{ij},
\]

with a function $A_{ij}$ defined by

\[
A_{ij} = \frac{\theta_i \theta_j}{\pi} \left[ 1 - \rho_{ij} \frac{p_i \cdot p_j}{m_j^2 - \rho_{ij}^2 m_i^2} \ln \left( \frac{\rho_{ij}^2 m_i^2}{m_j^2} \right) \right], \quad \delta_{ij} = -\frac{\theta_i \theta_j}{\pi} \left( \frac{1}{2} + \frac{\pi^2}{6} \right).
\]

There is some element of ambiguity in the definition of the coplanar factor of Eq.(122), which disappears when the difference (exact - coplanar) is properly included. It remains, however, when the result is expressed solely in terms of the coplanar approximation. This is connected to the fact that collinear logarithms, e.g. $\ln(Q^2/m^2)$, are not fully accounted in the exponentiation and only double-logarithms, of the form $\ln(\Delta E/E) \ln(Q^2/m^2)$, are properly included.

In other words, what we can do is as follows:

1. to exponentiate according to the YFS recipe and we do that, in principle, by including the full eikonal factor, no soft limit;
2. to compensate with respect to the complete answer by including infrared safe residuals, $\beta_1$ or more, see Eq.(28);
3. to translate the bulk of exponentiation into structure functions (see Eq.(136) below) including the rest into a remainder (see Eq.(131) below) which could be handled numerically.

From this point of view it really does not matter which approximation we start with, the only problem being that we, in general, do not control $\beta_1$ and neglect remainders. This is why the coplanar approximation is aimed to be as accurate as possible.

The main characteristics that an approximation to the exact spectral function has to satisfy are: a) the possibility of extracting the typical form of the solution of the evolution equations for fermion(anti-fermion) distributions in the soft limit and b) the correct exponentiation of the leading logarithms. Furthermore, it should respect the correct scaling behavior and return no radiation for $p_i = p_j$. The particular choice made in Eq.(123) will be commented in Section 8.

If we now write

\[
A_{ij} = \theta_i \theta_j A_{ij},
\]

\[
\text{(127)}
\]
it is easily seen that $A_{ij}$ is non-positive $\forall i, j$. Indeed we can immediately derive that

$$A_{ij} \propto -\mu^2 \int d\Omega_k A_{ij}^\mu A_{\mu ij},$$

$$A_{\mu ij} = \left( \frac{p_{ij}}{p_i \cdot k} - \frac{p_{ji}}{p_j \cdot k} \right), \quad (128)$$

where the integration is over the angular variables of the photon. From $A_{ij} \cdot k = 0$ and $k^2 = 0$ it follows that $A_{ij} \cdot A_{ij} \geq 0$. In coplanar approximation we have

$$E^c (p_i, p_j; K) = \frac{1}{(2\pi)^4} \int d^4 x \exp \{ i K \cdot x + \alpha \mathcal{R}^c (p_i, p_j) \}. \quad (129)$$

The total radiator will be the sum of its coplanar approximation and a remainder,

$$\mathcal{R} (p_i, p_j) = \mathcal{R}^c (p_i, p_j) + \mathcal{R}^{rest} (p_i, p_j). \quad (130)$$

In this way we obtain the spectral function $E(K)$ as

$$E_{ij}(K) = E (p_i, p_j; K) = \frac{1}{(2\pi)^4} \int d^4 x \exp \{ i K \cdot x + \alpha \mathcal{R}^c (p_i, p_j) + \alpha \mathcal{R}^{rest} (p_i, p_j) \}$$

$$= \int d^4 K' \Phi (K - K') \frac{1}{(2\pi)^4} \int d^4 x \exp \{ i K' \cdot x + \alpha \mathcal{R}^c \},$$

$$\Phi (K) = \frac{1}{(2\pi)^4} \int d^4 y \exp \{ i K \cdot y + \alpha \mathcal{R}^{rest} \}. \quad (131)$$

The exact spectral function is now written as the convolution of some flux $\Phi$ with a kernel integral that we may cast in an appropriate form

$$\mathcal{H}(K) = \frac{1}{(2\pi)^4} \int d^4 x \exp \{ i K \cdot x + \alpha \mathcal{R}^c \}$$

$$= \frac{1}{(2\pi)^4} \int d^4 x \exp (i K \cdot x + \alpha \delta_{ij}) \left[ -e^{\Delta R} \mu^2 \frac{p_i \cdot x p_j \cdot x}{s_{ij}} \right]^{-\alpha A_{ij}}. \quad (132)$$

The flux-function can be expanded in powers of $\alpha$, giving

$$\Phi (K) = \delta^4 (K) + \frac{\alpha}{(2\pi)^4} \int d^4 y \exp (i K \cdot y) \mathcal{R}^{rest} + \mathcal{O} \left( \alpha^2 \right). \quad (133)$$

For the kernel $\mathcal{H}$ we use the fact that $x_0$ is defined with a small imaginary part, or

$$i p \cdot x \rightarrow i p \cdot x + \delta. \quad (134)$$

Furthermore, the relation

$$(i p \cdot x)^{-s} = \frac{1}{\Gamma (s)} \int_0^{\infty} d\sigma \sigma^{s-1} \exp (-i p \cdot x \sigma), \quad (135)$$

23
is used to obtain
\[ \mathcal{H}(K) = \frac{1}{(2\pi)^4} \int d^4x \exp(iK \cdot x + \alpha \delta_{ij}) \left[ -e^{-\Delta_{\text{IR}}^2} \frac{\mu^2 p_i \cdot x p_j \cdot x}{s_{ij}} \right]^{\alpha A_{ij}} \]

\[ = \left[ e^{-\Delta_{\text{IR}}^2} \frac{\mu^2}{s_{ij}} \right]^{-\alpha A_{ij}} \frac{e^{\alpha \delta_{ij}}}{\Gamma^2(\alpha A_{ij})} \int_0^\infty d\sigma d\sigma' (\sigma \sigma')^{\alpha A_{ij}-1} \delta^4(\sigma p_i + \sigma' p_j - K). \]  

Eq. (136) exhibits the typical form of a structure function, i.e.

\[ \frac{\alpha A_{ij}}{\Gamma(\alpha A_{ij} + 1)} \sigma^{\alpha A_{ij}-1}. \]  

(137)

Note that the quantity \( \Delta_{\text{IR}}^c \) inside Eq. (136) exhibits the infrared pole. Eq. (135) is valid for \( \text{Re}\sigma > 0 \) and, therefore, Eq. (136) is valid only if \( \text{Re} A_{ij} > 0 \), which is not always the case due to the \( \theta_i \theta_j \) factor. We will examine in the next section the generalization to positive exponents.

The key relation, therefore, is Eq. (135) which expresses the fact \( i p \cdot x \) through the Mellin transform of an exponential that, in turn, re-establish energy-momentum conservation explicitly. For alternative uses of Mellin type representations see the first of ref. [8].

To summarize our findings the coplanar approximation to the exact photon spectral function is given by

\[ E^c(p_i, p_j; K) = F_{\text{IR}} \int d^4K' \int_0^\infty d\sigma_i d\sigma_j \Phi(K') \left[ \frac{\alpha A_{ij}}{\Gamma(\alpha A_{ij} + 1)} \right]^2 (\sigma_i \sigma_j)^{\alpha A_{ij}-1} \times \delta^4(\sigma_i p_i + \sigma_j p_j - K - K'), \]

\[ F_{\text{IR}} = \left[ e^{-\Delta_{\text{IR}}^2} \frac{\mu^2}{s_{ij}} \right]^{-\alpha A_{ij}} e^{\alpha \delta_{ij}}. \]  

(138)

There is an important property that any spectral function must satisfy, corresponding to the fact that a non-accelerated charged particle cannot radiate. From Eq. (11) and from Eq. (128) it follows that both the exact and the coplanar spectral function satisfy

\[ E^{\text{ex},c}(p, p; K) \propto \delta^4(K), \]  

(139)

since both \( j^\mu \) and \( A_i^\mu \) are zero.

A convenient inclusion of the flux-function \( \Phi \) towards the exact spectral function is achievable with the help of Eq. (138): let us assume that both \( p_i \) and \( p_j \) denote outgoing momenta, then according to a well-know strategy which is adopted in the dipole formalism [13], we define transformed momenta \( \tilde{p}_i, \tilde{p}_j \). Let us introduce the following quantities:

\[ \lambda_{ij} = \lambda \left( -K^2, \sigma_i^2 m_i^2, \sigma_j^2 m_j^2 \right), \quad \lambda_j = \lambda \left( -(\sigma_j p_j + K')^2, -K^2, \sigma_j^2 m_j^2 \right), \]  

(140)

then the transformed momenta are

\[ \tilde{p}_{i\mu} = \sigma_i p_{i\mu} + K'_\mu + (\sigma_j p_{j\mu} - \tilde{p}_{j\mu}), \]

\[ \tilde{p}_{j\mu} = \left( \frac{\lambda_{ij}}{\lambda_j} \right)^{1/2} \sigma_j \left( p_{j\mu} - \frac{K \cdot p_j}{K^2} K_\mu \right) + \frac{1}{2} \frac{K^2 - \sigma_j^2 m_j^2 + \sigma_i^2 m_i^2}{K^2} K_\mu. \]  

(141)
By construction they satisfy
\[ \sigma_i p_i + \sigma_j p_j + K' = \tilde{p}_i + \tilde{p}_j = K \] (142)
and
\[ \tilde{p}_i^2 = -\sigma_i^2 m_i^2, \quad \tilde{p}_j^2 = -\sigma_j^2 m_j^2. \] (143)
Moreover, if the phase space element is written as
\[ d\phi (p_i, p_j; K'; K) = \prod_{l,j=1}^{\infty} d^4 p_l \delta^4 (p_l^2 + m_l^2) \delta^4 (\sigma_i p_i + \sigma_j p_j - K - K'), \] (144)
after the transformation we get the following decomposition,
\[ d\phi (p_i, p_j; K'; K) = d\phi (\tilde{p}_i, \tilde{p}_j; 0; K) \otimes [dK'], \] (145)
which implies
\[ E (p_i, p_j; K) = F_{IR} \int_0^\infty d\sigma_i d\sigma_j \left[ \frac{\alpha A_{ij}}{\Gamma (\alpha A_{ij} + 1)} \right]^2 (\sigma_i \sigma_j)^{\alpha A_{ij} - 1} \Phi_{int} \delta^4 (\tilde{p_i} + \tilde{p_j} - K), \]
\[ \Phi_{int} = \int [dK'] \Phi (K'). \] (146)
The explicit form of the integration measure \([dK']\) depends on the nature of \(p_i, p_j\), i.e. incoming and/or outgoing and the successive integration represents the difficult part of the procedure. We will not investigate it any further.

Note that the coplanar approximation satisfies the scaling property
\[ E^c (p_i, p_j; \lambda K) = \lambda^{2 \alpha A_{ij} - 4} E^c (p_i, p_j; K), \] (147)
and it is symmetric, i.e. \(E^c (p_i, p_j; K) = E^c (p_j, p_i; k)\).

An additional comment concerns the correct interpretation of the parameters \(\sigma, \sigma'\). In the standard approach the structure function represents the probability of finding an electron (or any fermion) within an electron (or any fermion) with a longitudinal momentum fraction \(\sigma\). In the standard approach, therefore, the emitted photons, typically representing initial state radiation in \(e^+e^-\)-annihilation are strictly collinear. There is a subtle point here [16]: consider \(e^+e^-\) annihilation processes, dominated or not by \(s\)-channel diagrams. Let \(p\) be the four-momentum of the incoming electron in the laboratory system,
\[ p = \frac{1}{2} \sqrt{s} (0, 0, \beta, 1), \quad \beta^2 = 1 - 4 \frac{m_e^2}{s}. \] (148)
The electron, before interacting, emits soft and collinear photons. Let \(k = k_1 + k_2 + \ldots\) be the total four-momentum of the radiated photons. Thus
\[ k = \frac{1}{2} \sqrt{s} (1 - x) (0, 0, 1, 1), \] (149)
so that $k^2 = 0$, as requested by collinear, massless, photons. Usually one can work with the massless approximation for the electron taking part in the hard scattering, thus an on-shell (massless) electron can emit a bunch of massless, collinear, photons and remain on its (massless) mass shell. But the electron mass cannot be neglected in particular cases and, after radiation, the electron finds itself in a virtual state having four-momentum

$$\hat{p} = p - k = \frac{1}{2} \sqrt{s} \ (0, 0, \beta - 1 + x, x),$$

(150)

with $x$ being the fraction of energy remaining after radiation. As a consequence, the electron is put off its mass shell,

$$\hat{p}^2 = -m_e^2 + \frac{1}{2} (1 - \beta)(1 - x) s \sim -x m_e^2 \text{ for } m_e \to 0.$$  

(151)

When considering the whole process we introduce $p_{\pm}$ for the incoming $e^\pm$ in the laboratory system. Once radiation has been emitted the momenta will be denoted by $\hat{p}_{\pm}$ with

$$\hat{p}_{\pm} = \frac{1}{2} \sqrt{s} \ (0, 0, \mp(\beta - 1 + x_{\pm}), x_{\pm}).$$

(152)

The total four-momentum becomes

$$\hat{P} = \hat{p}_+ + \hat{p}_- = \frac{1}{2} \sqrt{s} \ (0, 0, x_- - x_+, x_- + x_+),$$

(153)

with a corresponding invariant mass $\hat{P}^2 = -x_+ x_- s = \hat{s}$.

In our approach things are different since $K$ inside Eq.(146) is, by no means, restricted by the condition $K^2 = 0$. In other words $K$, instead of being collinear, is coplanar and we have no problem in dealing with relations like $\hat{p} = (1 - x) p$ and $p^2 = -m_e^2$ simultaneously. Needless to say coplanar includes collinear.

### 7 Extension to $n$ emitters

The result of the previous sector should now be generalized to an arbitrary number of emitters. With $n$ emitters we have $n(n - 1)/2$ pairs and $n(n - 1)$ $\sigma$-variables. The corresponding spectral function becomes

$$E(K) = \frac{1}{(2\pi)^4} \int d^4 x \ exp \ (i \ K \cdot x) \ \prod_{l+1}^N \ \exp \ \{ \alpha \ R^c_l + \alpha \ R^\text{rest}_l \},$$

(154)

where $N = n(n - 1)/2$ and $l$ runs over all pairings of external, charged, fermion lines. It follows

$$E(K) = \int d^4 K' \ \Phi(K') \ \left\{ \frac{1}{(2\pi)^4} \int d^4 x \ \exp \ [i \ (K - K') \cdot x + \alpha \ \sum_l R^c_l] \right\},$$

$$\Phi(K) = \frac{1}{(2\pi)^4} \int d^4 y \ \exp \ (i \ K \cdot y) \ \prod_{l=1}^N \ \exp \ \{ \alpha \ R^\text{rest}_l \}.$$  

(155)
Consider now
\[
\mathcal{H}_N(K - K') = \frac{1}{(2\pi)^4} \int d^4x \exp\left[ i (K - K') \cdot x + \alpha \sum_{l=1}^N R^c_l \right]
\]
\[
= \frac{1}{(2\pi)^4} \int d^4K_N \frac{1}{(2\pi)^4} \int d^4x \exp\left\{ i K_N \cdot x + \alpha R^c_N \right\}
\]
\[
\times \int d^4y \exp\left\{ i (K - K' - K_N) \cdot y + \alpha \sum_{l=1}^{N-1} R^c_l \right\}.
\]
(156)

This result can be transformed into
\[
\mathcal{H}_N(K - K') = \frac{1}{(2\pi)^4} \int d^4K_N \int d^4y \exp\left\{ i (K - K' - K_N) \cdot y + \alpha \sum_{l=1}^{N-1} R^c_l \right\}
\]
\[
\times C_N \int_0^\infty d\sigma_N d\sigma'_N (\sigma_N\sigma'_N)^{\alpha A_N - 1} \delta^4(\sigma_Np_{i_N} + \sigma'_Np_{j_N} - K_N).
\]
(157)

with
\[
C_N = \frac{1}{\Gamma^2(\alpha A_N)}.
\]
(158)

This equation can be written as
\[
\mathcal{H}_N(K - K') = C_N \int_0^\infty d\sigma_N d\sigma'_N (\sigma_N\sigma'_N)^{\alpha A_N - 1} \mathcal{H}_{N-1}(K - K' - \sigma_Np_{i_N} - \sigma'_Np_{j_N}),
\]
\[
\mathcal{H}_0(K) = \delta^4(K).
\]
(159)

The process can be iterated until we reach the final result
\[
\mathcal{H}_N(K - K') = \prod_{l=1}^N C_l \int_0^\infty d\sigma_l d\sigma'_l (\sigma_l\sigma'_l)^{\alpha A_l - 1} \delta^4\left(K - K' - \sum_{l=1}^N (\sigma_l p_{i_l} + \sigma'_l p_{j_l})\right).
\]
(160)

Define
\[
x_k = \sum_l (\sigma_l + \sigma'_l) |_{i_l,j_l \in \{k\}}, \quad 1 \leq k \leq n,
\]
(161)

where \(n\) is the number of emitters, \(N\) the corresponding number of pairs and \(\{k\}\) is the set of pairs that have one \(k\)-line. Then
\[
\sum_{l=1}^N (\sigma_l p_{i_l} + \sigma'_l p_{j_l}) = \sum_{k=1}^n x_k p_k.
\]
(162)

We recall that Eq.\(135\) is valid only for negative exponents, i.e. for positive \(A_{ij}\). Since some of the \(A_{ij}\) may be negative, depending on the product \(\theta_i\theta_j\), we are forced to consider an alternative derivation. Starting from
\[
E(K) = \int d^4K' \Phi(K') \mathcal{H}(K - K'),
\]
\[
\mathcal{H}(K) = \frac{1}{(2\pi)^4} \int d^4x \exp\left\{ i K \cdot x + \alpha \sum_l R^c_l \right\},
\]
(163)
we derive

$$
\mathcal{H}(K) = \frac{1}{(2\pi)^4} \prod_{l=1}^{N} \left[ e^{-\Delta_{IR} \frac{\mu^2}{s_{ij}}} \right]^{-\alpha A_l} \int d^4x \exp (i K \cdot x) \prod_{i=1}^{n} (i p_i \cdot x)^{-\alpha \sum_{l \in \{i\}} A_l}. \tag{164}
$$

As before $l_i$ denotes the set of all pairs that include the $i$-line. If the sum

$$
A_i = \sum_{l \in \{i\}} A_l \tag{165}
$$

is positive the derivation follows as before. In the case of $A_i$ negative we have to consider another integral representation,

$$
(i p_i \cdot x)^{-\alpha A_i} = \frac{1}{\Gamma(\alpha A_i)} \int_0^\infty d\sigma \sigma^{\alpha A_i - 1} \left[ \exp (-i \sigma p_i \cdot x) - 1 \right], \tag{166}
$$

which is valid for

$$
-1 < \text{Re}(\alpha A_i) < 0. \tag{167}
$$

By standard arguments it follows

$$
\mathcal{H}(K) = \prod_{l=1}^{N} \left[ e^{-\Delta_{IR} \frac{\mu^2}{s_{ij}}} \right]^{-\alpha A_l} e^{\alpha \delta_{ij}} \prod_{i=1}^{n} \int_0^\infty d\sigma_i \sigma^{\alpha \sum_{l \in \{i\}} A_l - 1}
\times \left[ 1 - \theta \left( - \sum_{l \in \{i\}} A_l \right) \mathcal{P}(\sigma_i) \right] \delta^4 \left( \sum_{i=1}^{n} \sigma_i p_i - K \right), \tag{168}
$$

where we have introduced a projector $\mathcal{P}$

$$
\mathcal{P}(\sigma_i) \delta^4 \left( \sum_{j=1}^{n} \sigma_j p_j - K \right) = \delta^4 \left( \sum_{j \neq i} \sigma_j p_j - K \right). \tag{169}
$$

### 7.1 The case $2 \rightarrow 2$

For a $2 \rightarrow 2$ process we have $n = 4$ external particles and $N = 6$ pairs of emitters. Let us assume that all exponent are positive and consider

$$
\mathcal{H}_6 = \prod_{l=1}^{6} C_l \int_0^\infty d\sigma_l d\sigma_l' \left( \sigma_l \sigma_l' \right)^{\alpha A_l - 1} \delta^4 \left( K - K' - \sum_{l=1}^{N} (\sigma_l p_i + \sigma_l' p_j) \right). \tag{170}
$$

With the following identification

| $i$ - line | $j$ - line | $l$ - pair |
|-----------|-----------|------------|
| 1         | 2         | 1          |
| 1         | 3         | 2          |
| 1         | 4         | 3          |
| 2         | 3         | 4          |
| 2         | 4         | 5          |
| 3         | 4         | 6          |

(171)
we obtain
\[ \sum_{l=1}^{6} (\sigma_l p_l + \sigma'_l p'_{l,i}) = (\sigma_1 + \sigma_2 + \sigma_3) p_1 + (\sigma'_1 + \sigma_4 + \sigma_5) p_2 + (\sigma'_2 + \sigma'_4 + \sigma_6) p_3 + (\sigma'_3 + \sigma'_5 + \sigma'_6) p_4. \] (172)

Therefore the object to compute is
\[ \mathcal{R}_6 = \int_{-\infty}^{\infty} \prod_{i=1}^{4} dx_i \int_{0}^{+\infty} \prod_{i'=1}^{6} d\sigma_i d\sigma'_i C_i (\sigma_i \sigma'_i)^{\alpha A_i-1} \delta^4 \left( K - K' - \sum_{i=1}^{4} x_i p_i \right) \]
\[ \times \delta (x_1 - \sigma_1 - \sigma_2 - \sigma_3) \delta (x_2 - \sigma'_1 - \sigma_4 - \sigma_5) \]
\[ \times \delta (x_3 - \sigma'_2 - \sigma'_4 - \sigma_6) \delta (x_4 - \sigma'_3 - \sigma'_5 - \sigma'_6). \] (173)

With \( \mathcal{R}_6 \) expressed as the integral of some \( \Sigma_6 \),
\[ \mathcal{R}_6 = \int_{-\infty}^{\infty} \prod_{i=1}^{4} dx_i \delta^4 \left( K - K' - \sum_{i=1}^{4} x_i p_i \right) \Sigma_6, \] (174)
we perform the various \( \sigma \) integrations, starting with the trivial ones,
\[ \Sigma_6 = \int_{0}^{+\infty} \prod_{l=2}^{6} d\sigma_l \prod_{i'=1}^{6} (x_1 - \sigma_2 - \sigma_3)^{\alpha A_1-1} (x_2 - \sigma_4 - \sigma_5)^{\alpha A_1-1} \]
\[ \times \sigma_2^{\alpha A_2-1} (x_3 - \sigma'_4 - \sigma_6)^{\alpha A_2-1} \sigma_3^{\alpha A_3-1} (x_4 - \sigma'_5 - \sigma'_6)^{\alpha A_3-1} \]
\[ \times \sigma_4^{\alpha A_4-1} (\sigma'_4)^{\alpha A_4-1} \sigma_5^{\alpha A_5-1} (\sigma'_5)^{\alpha A_5-1} \sigma_6^{\alpha A_6-1} (\sigma'_6)^{\alpha A_6-1}. \] (175)

After integration a set of consistency conditions will emerge,
\[ x_1 \geq \sigma_2 + \sigma_3, \quad x_2 \geq \sigma_4 + \sigma_5, \]
\[ x_3 \geq \sigma'_4 + \sigma_6, \quad x_4 \geq \sigma'_5 + \sigma'_6. \] (176)

The next step is to perform the \( \sigma_2 \)-integration,
\[ \int_{0}^{\infty} d\sigma_2 (\sigma_2)^{\alpha A_2-1} (x_1 - \sigma_3 - \sigma_2)^{\alpha A_1-1} = B (\alpha A_1, \alpha A_2) \left( x_1 - \sigma_3 \right)^{\alpha (A_1 + A_2)-1}, \] (177)
which requires \( x_1 \geq \sigma_3 \) and where \( B \) is the Euler’s beta-function. Next we integrate over \( \sigma_3 \),
\[ B (\alpha A_1, \alpha A_2) \int_{0}^{\infty} d\sigma_3 (\sigma_3)^{\alpha A_3-1} (x_1 - \sigma_3)^{\alpha (A_1 + A_2)-1} = \frac{\prod_{i=1,3} \Gamma (\alpha A_i)}{\Gamma (\alpha \sum_{i=1,3} A_i)} \left( x_1 \right)^{\alpha \sum_{i=1,3} A_i-1}. \] (178)

Also the \( \sigma_4, \sigma_5 \) integrations give
\[ \int_{0}^{\infty} d\sigma_4 (\sigma_4)^{\alpha A_4-1} (x_2 - \sigma_5 - \sigma_4)^{\alpha A_1-1} = B (\alpha A_1, \alpha A_4) \left( x_2 - \sigma_5 \right)^{\alpha (A_1 + A_4)-1}, \] (179)
\[ B(\alpha A_1, \alpha A_4) \int_0^\infty d\sigma_5 (\sigma_5)^{\alpha A_5 - 1} (x_2 - \sigma_5)^{\alpha (A_1 + A_4) - 1} \]

\[ = \frac{\Gamma(\alpha A_1) \Gamma(\alpha A_4) \Gamma(\alpha A_5)}{\Gamma(\alpha (A_1 + A_4 + A_5))} (x_2)^{\alpha (A_1 + A_4 + A_5) - 1}, \] 

(180)

Similar results hold for the \(\sigma_6, \sigma_4'\) and \(\sigma_5', \sigma_6'\) integration, with a total result

\[ \Sigma_6 = \prod_{i=1}^{6} \Gamma^2(\alpha A_i) \prod_{i=1}^{4} (x_i)^{\alpha A_{(i)} - 1}, \]

\[ A_{(1)} = A_1 + A_2 + A_3, \quad A_{(2)} = A_1 + A_4 + A_5, \]

\[ A_{(3)} = A_2 + A_4 + A_6, \quad A_{(4)} = A_3 + A_5 + A_6, \] 

(181)

which can be easily generalized to

\[ \Sigma_N = \prod_{i=1}^{N} \Gamma^2(\alpha A_i) \left[ \prod_{i=1}^{n} \Gamma \left( \alpha \sum_{k \in \{k_i\}} A_k \right) \right]^{-1} \prod_{i=1}^{n} (x_i)^{\alpha \sum_{k \in \{k_i\}} A_k - 1}, \] 

(182)

where, for \(i\) fixed, the index \(k \in \{k_i\}\) runs over all pairs that contain the line \(i\) and \(\forall i, x_i \geq 0\).

8 Combining real and virtual corrections

The emission of a real photon from the \(ij\) pair is described by a structure function language with an exponent \(\alpha A_{ij} - 1\), with

\[ A_{ij} = \frac{\theta_i \theta_j}{\pi} \left[ 1 - \rho_{ij} \frac{p_i \cdot p_j}{m_i^2 - \rho_{ij}^2 m_j^2} \ln \frac{\rho_{ij}^2 m_i^2}{m_j^2} \right]. \] 

(183)

For virtual photons the overall, universal, factor

\[ B = -\frac{i}{8 \pi^2} \sum_{i<j} \theta_i \theta_j B_{ij}, \] 

(184)

is exponentiated and the relevant quantity is \(\exp(2 \alpha \Re B)\). For real radiation we have another overall exponentiation where the infrared-divergent object, for each pair, is given in Eq.(136),

\[ \left[ e^{-\Delta^c_{\text{IR}} \frac{\mu^2}{s_{ij}}} \right]^{-\alpha A_l} e^{\alpha \delta_{ij}} = \exp \left\{ \alpha A_{ij} \left[ \Delta^c_{\text{IR}} - \ln \frac{\mu^2}{s_{ij}} \right] + \alpha \delta_{ij} \right\}, \] 

(185)

where \(l = \{ij\}\) is the emitting pair that we are considering. Here

\[ \Delta^c_{\text{IR}} = \frac{1}{\varepsilon} - 2 \gamma + \frac{3}{2}, \quad \delta_{ij} = -\frac{\theta_i \theta_j}{\pi} \left( \frac{1}{2} + \frac{\pi^2}{6} \right) \] 

(186)
Structure function language means that the overall, real + virtual, exponent is multiplied by
\[ \prod_i \beta_i \frac{\beta_i^{-1}}{\Gamma(\beta_i + 1)} \]  
(187)
where \( i \) runs over external charged lines. Moreover
\[ \beta_i = \alpha \sum_{k \in k_i} A_k, \]  
(188)
where the sum is limited to those pairs containing \( i \).

8.1 IR finite exponent

Once virtual and real exponentiation are combined we have cancellation of the infrared pole and some global remainder that reads as follows:

\[
\frac{\alpha}{\pi} \theta_i \theta_j F_{ij} = \alpha \left\{ A_{ij} \left[ \Delta_{\text{IR}} + \ln \frac{s_{ij}}{\mu^2} \right] - \frac{\theta_i \theta_j}{\pi} \left( \frac{1}{2} + \frac{\pi^2}{6} \right) \right\} - \frac{i \alpha}{4 \pi^2} \theta_i \theta_j \text{Re} B_{ij}
\]
\[
= \frac{\alpha \theta_i \theta_j}{\pi} \left\{ \left[ 1 - \rho_{ij} \frac{p_i \cdot p_j}{m_j^2 - \rho^2_{ij} m_i^2} \ln \frac{\rho^2_{ij} m_i^2}{m_j^2} \right] \left( \Delta_{\text{IR}}^c + \ln \frac{s_{ij}}{\mu^2} \right) \right. 
- \left. \frac{1}{2} - \frac{\pi^2}{6} + \text{Re} B_{ij}^\text{IR} \frac{1}{\varepsilon} + \text{Re} B_{ij}^\text{fin} \right\},
\]  
(189)
where \( 1/\varepsilon \) shows the virtual infrared pole. The residue and the finite part of the virtual corrections are

\[
B_{ij}^\text{IR} = -1 + \epsilon_i \epsilon_j p_i \cdot p_j F_1^{ij},
\]
\[
B_{ij}^\text{fin} = - \ln \frac{m_i m_j}{\mu^2} + \epsilon_i \epsilon_j p_i \cdot p_j \left[ F_1 \ln \frac{Q^2 - i\epsilon}{\mu^2} + F_2^{\text{rest}} \right] + \frac{1}{2} F_3.
\]  
(190)

Here \( F_1 \) is expressed as

\[
Q = \epsilon_i p_i + \epsilon_j p_j,
\]
\[
y_{1,2} = \frac{1}{2 Q^2} \left[ Q^2 + m_j^2 - m_i^2 \pm \lambda^{1/2} \left( -Q^2, m_i^2, m_j^2 \right) \right],
\]
\[
F_1^{ij} = \frac{1}{Q^2 (y_1 - y_2)} \left[ \ln \left( 1 - \frac{1}{y_2} \right) - \ln \left( 1 - \frac{1}{y_1} \right) \right].
\]  
(191)

Furthermore introduce a shorthand notation for the Källen’s function,
\[ \lambda \left( -Q^2, m_i^2, m_j^2 \right) = \Lambda^2. \]  
(192)
We see from Eq.(70) that $\rho_{ij}$ is also a solution of the equation

\[(\rho_{ij} p_i - p_j)^2 = 0.\]  

(193)

If $\epsilon_i \epsilon_j = +1$ we find

$$\rho_{ij} = \frac{1}{2m_i^2} \left[ -Q^2 - m_i^2 - m_j^2 + \Lambda \right].$$ 

(194)

If instead $\epsilon_i \epsilon_j = -1$ we have

$$\rho_{ij} = \frac{1}{2m_i^2} \left[ Q^2 + m_i^2 + m_j^2 + \Lambda \right].$$ 

(195)

In both cases we derive a noticeable relation,

$$\frac{\rho}{m_j^2 - \rho^2 m_i^2} = \frac{1}{\Lambda}.$$ 

(196)

Moreover, it is straightforward to show that

$$\frac{1}{Q^2 (y_1 - y_2)} = \frac{1}{\Lambda}.$$ 

(197)

Consider now the quantity $Y$ defined as

$$Y = \frac{y_1 (y_2 - 1)}{y_2 (y_1 - 1)}.$$ 

(198)

It follows that $Y$ can be expressed as

$$Y = \frac{(Q^2 + m_i^2 + m_j^2 + \Lambda)^2}{4 m_i^2 m_j^2} = \frac{4 m_i^2 m_j^2}{(Q^2 + m_i^2 + m_j^2 - \Lambda)^2}.$$ 

(199)

Similarly we obtain

$$\frac{\rho_{ij}^2 m_i^2}{m_j^2} = \left\{ \begin{array}{ll} \frac{(Q^2 + m_i^2 + m_j^2 - \Lambda)^2}{4 m_i^2 m_j^2} & \text{if } \epsilon_i \epsilon_j = +1 \\ \frac{(Q^2 + m_i^2 + m_j^2 + \Lambda)^2}{4 m_i^2 m_j^2} & \text{if } \epsilon_i \epsilon_j = -1 \end{array} \right.$$ 

In other words, an important result can be derived, namely

$$\ln \frac{\rho_{ij}^2 m_i^2}{m_j^2} = -\epsilon_i \epsilon_j \ln Y, \quad Y = \frac{y_1 (y_2 - 1)}{y_2 (y_1 - 1)}.$$ 

(200)

The IR-finite exponent is therefore $\alpha/\pi \theta_i \theta_j F_{ij}$, with

$$F_{ij} = 1 - 2 \gamma - \frac{\pi^2}{6} + \ln \frac{s_{ij}}{m_i m_j} + \frac{1}{2} \text{Re} F_3 + \epsilon_i \epsilon_j \frac{p_i \cdot p_j}{\Lambda} \left\{ \left[ \ln \frac{|Q^2|}{s_{ij}} \right] - \frac{3}{2} + 2 \gamma \right\} \ln Y + \text{Re} f_{2,\text{rest}}.$$ 

(201)
and with

\[ L_{ij} = \ln \left| \frac{Q^2}{m_i m_j} \right|, \quad F_{2}^{\text{rest}} = \frac{1}{\Lambda} f_{2}^{\text{rest}}. \]  

(202)

### 8.2 Asymptotic limits and general considerations

**a) the case** \( Q^2 \gg m^2 \)

It is important to show the asymptotic behavior of this exponent in the region \( |Q^2| \gg m_i^2, m_j^2 \). We easily obtain that

\[ \Lambda \sim -Q^2, \quad y_1 \sim -\frac{m_i^2}{Q^2}, \quad y_2 \sim 1 - \frac{m_j^2}{Q^2}, \]  

(203)

giving the asymptotic limit of \( Y \) as

\[ Y \sim \frac{m_i^2 m_j^2}{(Q^2)^2}. \]  

(204)

Using the asymptotic behavior of Eq.(60) and also Eq.(124) we derive

\[ F_{ij} \sim 2 \gamma (L_{ij} - 1) - \frac{3}{2} L_{ij} + 2 - \frac{2}{3}, \]  

(205)

which shows, among other things, a cancellation of the \( \ln^2 \) terms.

We can easily check that the exponent \( \beta_{ij} = \alpha A_{ij} - 1 \) has the usual asymptotic behavior. With

\[ s = -(p_i + p_j)^2, \quad m_i = m_j = m, \quad \theta_i \theta_j = -1, \]  

(206)

in the limit \( s \gg m^2 \) we get

\[ \alpha A_{ij} \sim \frac{\alpha}{\pi} \left( \ln \frac{s}{m^2} - 1 \right). \]  

(207)

The result of Eq.(203) explains the normalization in the definition of the coplanar factor. The infrared finite overall exponent has, in the asymptotic region \( |Q^2| \gg m^2 \), the correct behavior to reproduce the exponentiation commonly employed to describe initial state radiation in \( e^+e^- \)-annihilation, at least up to terms \( O(\alpha^2) \) and without hard photons,

\[ G(x) = \frac{\beta}{\Gamma(\beta+1)} x^{\beta-1} \exp \left\{ -\beta \gamma + \delta^{V+S} \right\}, \]

\[ \delta^{V+S} = \frac{\alpha}{\pi} \left( \frac{3}{2} \ln \frac{s}{m_e^2} - 2 + \frac{\pi^2}{3} \right), \quad \beta = \frac{2 \alpha}{\pi} \left( \ln \frac{s}{m_e^2} - 1 \right). \]  

(208)

In the above result \( G \) is the so-called radiator function which is connected to structure functions \( D \) by the relation

\[ G(x) = \int_1^x dz \; D(z) \; D\left(\frac{x}{z}\right). \]  

(209)
Note that Eq. (208) is sometimes written as

\[ G(x) = \frac{\beta}{\Gamma(\beta + 1)} x^{\beta - 1} \exp\left\{-\beta \gamma + \delta_{\text{YFS}}\right\} (1 + \delta_S), \]

\[ \delta_{\text{YFS}} = \frac{\alpha}{\pi} \left( \frac{1}{2} \ln \frac{s}{m_e^2} - 1 + \frac{\pi^2}{3} \right), \]

\[ \delta_S = \frac{1}{2} \beta + \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 \ln^2 \frac{s}{m_e^2}, \]

(210)
a form which follows from the evaluation of the YFS soft form-factor

\[ \beta \ln \varepsilon + 2 \delta_{\text{YFS}}, \]

ε being the parameter introduced to limit the multiplicity of very soft photons.

Moreover, if we neglect constant terms, our exponent of Eq. (201) reproduces the leading behavior of the Gribov-Lipatov solution \[ \epsilon \] of the evolution equation for the electron structure function,

\[ D(x) = \frac{x^{\eta/2 - 1}}{\Gamma(\eta/2)} \exp\left\{\frac{\eta}{4} \left( \frac{3}{2} - 2 \gamma \right) \right\}, \quad \eta = -6 \ln \left( 1 - \frac{\alpha}{3 \pi} \ln \frac{s}{m_e^2} \right), \]

(212)
which is valid in the soft limit. The factor \( \delta_{ij} \) inserted in Eq. (122) has the purpose of reproducing the constant terms of Eq. (208), therefore increasing the accuracy of the approximation.

In conclusion the coplanar approximation, in the limit \( x_i \ll 1, \forall i \) coincides precisely with the exact expression resulting from the soft-photon re-summation, as given for \( e^+e^- \rightarrow \gamma^* \) in the classic YFS treatment. We want to stress that, for a general process, there are ambiguities in the choice of overall exponent in Eq. (201). Indeed the asymptotic factor

\[ -\frac{3}{2} L_{ij} + 2 - \frac{\pi^2}{3}, \]

(213)
which is sub-leading, is tailored to reproduce the asymptotic form of the soft + virtual one loop corrections to the vertex \( e^+e^-\gamma \), i.e. \( 2 \Re F_{\text{dirac}} + \delta_{\text{soft}} \). Single logarithms and constant terms in the overall exponent can never be exact, unless the full virtual + hard part of the spectrum is included. Therefore, the accuracy of the result is always limited to

\[ \beta x^{\beta - 1} \left[ 1 + O(\alpha L_{\text{coll}}) \right]. \]

(214)

As mentioned in Section 2 the missing parts of the hard photon spectrum and of virtual corrections violate the well-known KLN result that the inclusive corrections are always small and free of large logarithms for a pair of final state emitters. Therefore, the accuracy of our result is, in this case, controlled only if tight cuts are imposed on the invariant masses of the final state pairs. To give a concrete example the cross-section for \( e^+e^- \rightarrow f \bar{f} \) that includes exact \( O(\alpha) \) final state radiation
\[
\sigma_c(s) = \frac{\alpha}{4\pi} Q^2 \sigma^0(s) \left\{ -2(1-z)^2 + 4 \left( z + \frac{z^2}{2} + 2 \ln(1-z) \right) \ln \frac{s}{m_f^2} \right. \\
+ z \left( 1 + \frac{z}{2} \right) \ln z + 2\zeta(2) - 2\text{Li}_2(1-z) - 2 \ln (1-z) + \frac{5}{4} - 3z - \frac{z^2}{4} \right\}, \tag{215}
\]

where \( z = M^2 \left( f \bar{f} \right) / s \). The difference between the factor

\[
\ln(1-z) \left[ \ln \frac{s}{m_f^2} - 1 \right], \tag{216}
\]

which is then exponentiated and the full result of Eq.(216) is due to hard, virtual and real, photons and is responsible for the correct limit \( z \to 0 \),

\[
\sigma_c = \sigma^0(s) \left( 1 + \frac{3\alpha}{4\pi} Q^2 \right), \tag{217}
\]

b) the case \( Q^2 \ll m^2 \)

Finally we consider the case of one incoming/one outgoing electron. For

\[
t = -(p_i - p_i)^2, \quad -t \ll m_e^2, \tag{218}
\]

we obtain

\[
\rho_{ij} \sim 1 + \left( -\frac{t}{m_e^2} \right)^{1/2}, \quad \alpha A_{ij} \sim \frac{\alpha}{\pi} \left( -\frac{t}{m_e^2} \right)^{1/2}, \tag{219}
\]

showing a power law behavior in the exponent, to be compared with the logarithmic one for \(-t \gg m_e^2\).

When \( Q^2 \ll m^2_i, m^2_j \), we derive

\[
\Lambda \sim \left( m^2_i - m^2_j \right), \quad \rho_{ij} \sim \frac{m^2_i + m^2_j}{2 m^2_i}. \tag{220}
\]

Therefore the real emission is controlled by a coefficient

\[
\mathcal{A}_{ij} = 1 - \rho_{ij} \frac{p_i \cdot p_j}{m^2_j - \rho_{ij}^2 m^2_i} \ln \frac{\rho_{ij}^2 m^2_i}{m^2_j} \sim 1 + 2 \frac{(1+r)^2}{4 - r^2(1+r)^2} \ln \frac{r(1+r)}{2}, \tag{221}
\]

with \( r = m^2_i/m^2_j \), which, for equal masses, reproduces the correct limit

\[
\mathcal{A}_{ij} \left( Q^2, m_i = m_j = m \right) \to 0, \quad \text{for} \quad Q^2 \to 0, \tag{222}
\]

where a non-accelerated charge does not radiate. From these result we see that the normalization of the coplanar factor in Eq.(122) has been chosen to avoid appearance of spurious logarithms,
ln\(Q^2/m^2\), for \(Q^2 \ll m^2\). One may wonder whether the limit of very small momentum-transfer is relevant for any physical situation. Consider the process \(e^+(p_+) + e^-(p_-) \to e^-(q_-) + X(q_x)\) and define

\[
Q = p_- - q_-, \quad y = \frac{p_+ \cdot Q}{p_+ \cdot p_-}.
\]

In the region of forward \(e^-\) scattering we have

\[
Q^2 \geq m_e^2 \frac{y^2}{1 - y},
\]

where \(y\) is bounded by

\[
\frac{M_0^2}{s} \leq y \leq 1 - \frac{m_e}{\sqrt{s}}, \quad s = -(p_+ + p_-)^2,
\]

and \(M_0\) is the minimum invariant mass of the \(X\)-system. Whenever this mass is very low with respect to \(s\) we may reach the regime \(Q^2 \ll m_e^2\).

As a final consideration, note that we are treating all charged fermionic lines on the same footing since only this combination of radiation is a meaningful gauge-invariant concept. For a general process, therefore, the interference between different legs is a fundamental part of the QED corrections and not an additional minor effect. Interference must be meaningfully definable, in particular when one exponentiates, see a discussion in ref. [19]. In certain situations interference is responsible for changing the scale in the leading logarithms but it should be present, as a matter of principle, also for those situations, as in forward scattering, where the typical scale is not large with respect to fermion masses. Strictly speaking the exponentiation of ‘soft’ initial-final interference is not accurate enough and, therefore, insufficient to describe annihilation processes around resonances. We will not dwell upon this subject any longer and refer to [20].

### 9 Further refinements

There are several reasons to increase the value of our approximation and, for the sake of simplicity, we start our considerations by examining the case of a \(2 \to 2\) process. The result may be cast into the following form,

\[
\sigma \propto \int dPS_2 \int d^4K \Phi(K') \int \prod_{i=1}^4 dx_i \\
\times \delta^4(p_+ + p_- - q_+ - q_- - K' - \sum_i x_ip_i) \prod_i \frac{\beta_i}{\Gamma(1 + \beta_i)} (x_i)^{\beta_i - 1} \beta_0,
\]

where \(\beta_0\) is the Born matrix element, \(\beta_i = \alpha A_i\) and

\[
p_1 = p_+, \quad p_2 = p_-, \quad p_3 = q_-, \quad p_4 = q_+.
\]
Furthermore $dPS_2$ is the two-body phase-space. If we neglect terms of $\mathcal{O}(\alpha)$ in the flux-function, i.e. $\Phi(K') = \delta^4(K')$, the argument of the delta-function in Eq. (226) becomes

$$(1 - x_1) \ p_+ + (1 - x_2) \ p_- - (1 + x_3) \ q_- - (1 + x_4) \ q_-.$$  \hfill (228)$$

Next we introduce scaled momenta,

$$\hat{p}_+ = (1 - x_1) \ p_+, \quad \hat{p}_- = (1 - x_2) \ p_-,$$
$$\hat{q}_+ = (1 + x_4) \ q_+, \quad \hat{q}_- = (1 + x_3) \ q_-,$$  \hfill (229)$$

and derive four-momentum conservation in terms of the radiative process which incorporates the photons emitted along the directions of the charged fermions. There seems to be a clash between kinematics and matrix element; the original procedure requires a reorganization of the perturbative expansion which starts with the matrix element in soft approximation while the delta-function expressing conservation is kept exact, transformed with other ingredients into the photon spectral function which is again approximated to introduce conservation at the level of scaled momenta, as it would appear in the structure function language. In any intermediate step we are not authorized to use energy-momentum conservation since there is no delta-function to use.

The lowest order factor in the perturbative expansion of the squared matrix element, $\beta_0$, is however constructed with non-scaled momenta. Typically, we will have

$$\beta_0 \propto q_- \cdot p_- \ q_+ \cdot p_+ + q_- \cdot p_+ \ q_+ \cdot p_-.$$  \hfill (230)$$

Let us change variables according to

$$x_+ = 1 - x_1, \quad x_- = 1 - x_2,$$
$$\frac{1}{y_+} = 1 + x_4, \quad \frac{1}{y_-} = 1 + x_3,$$  \hfill (231)$$

so that the set of momenta satisfying conservation is

$$\hat{p}_+ = x_+ \ p_+, \quad \hat{p}_- = x_- \ p_-,$$
$$\hat{q}_+ = \frac{q_+}{y_+}, \quad \hat{q}_- = \frac{q_-}{y_-},$$
$$\prod_{i=1}^4 dx_i = \frac{1}{y_+^2 y_-^2} \ dx_+ dx_- dy_+ dy_-.$$  \hfill (232)$$

As a consequence the Born matrix element can be cast into the following form:

$$\beta_0 \propto \frac{y_+ y_-}{x_+ x_-} \left[ \hat{p}_- \cdot \hat{q}_- \hat{p}_+ \cdot \hat{q}_+ + \hat{p}_- \cdot \hat{q}_- \hat{p}_+ \cdot \hat{q}_+ \right].$$  \hfill (233)$$

Furthermore, $\beta_0$ will contain an overall factor $s^3$ from the $s$-channel propagator and from the root of the Källen function. Given the relation between $s$ and $\hat{s}$,

$$s = -(p_+ + p_-)^2 = \frac{\hat{s}}{x_+ x_-},$$  \hfill (234)$$

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we obtain
\[ \beta_0 = \frac{\pi}{2 s} \int d\hat{t} \hat{\beta}_0^{\text{inv}}, \]  
(235)
and, after transforming variables,
\[ \prod_{i=1}^{4} \int dx_i \hat{\beta}_0^{\text{inv}} = \int dx_+ dx_- dy_+ dy_- \frac{x_+^2 x_-^2 \hat{\hat{t}}^2 + \hat{u}^2}{s^3}. \]  
(236)

In deriving this result we have used scaled invariants, defined by
\[ \hat{s} = - (\hat{p}_+ + \hat{p}_-)^2 = - (\hat{q}_+ + \hat{q}_-)^2, \]
\[ \hat{t} = - (\hat{p}_+ - \hat{q}_+)^2 = - (\hat{p}_- - \hat{q}_-)^2, \]
\[ \hat{u} = - (\hat{p}_+ - \hat{q}_-)^2 = - (\hat{q}_+ - \hat{p}_-)^2. \]  
(237)

From phase space considerations and from positivity for all \( x_i \) it follows that
\[ 0 \leq x_\pm, y_\pm \leq 1. \]  
(238)

As a next step, we will show that it is possible to introduce, in a way that is consistent with the perturbative approach, a new lowest order result: it contains \( \hat{\beta}_0^{\text{inv}} \) instead of \( \beta_0^{\text{inv}} \),
\[ \hat{\beta}_0^{\text{inv}} = \frac{\hat{t}^2 + \hat{u}^2}{s^3}, \]  
(239)
the difference between the two formulations being of order \( \alpha \). This difference is known and computable so that perturbation theory indeed starts with a radiative kernel and the re-summation of all photons emitted along the directions of the external, charged, fermions. Let us consider this difference in more detail. First of all it is zero for \( x_+ = \ldots y_- = 1 \). Next consider the following function:
\[ F(z, \beta) = \int_1^z dx \mathcal{D}_\beta(x) f(x), \]  
(240)
where \( \beta = \alpha A - 1 \) and where we have also introduced the distribution
\[ \mathcal{D}_\beta = \beta (1 - x)^{\beta - 1}. \]  
(241)

Adding and subtracting a term we derive
\[ F(z, \beta) = f(1) (1 - z)^\beta + \beta \int_z^1 dx \frac{f(x) - f(1)}{1 - x} (1 - x)^\beta. \]  
(242)
From this result the distribution can be computed. For instance, to second order in \( \beta \), we have
\[ \mathcal{D}_\beta = \delta(x - 1) + \beta \left\{ \ln (1 - z) + \left[ \frac{1}{1 - x} \right]_+ \right\} + \beta^2 \left\{ \frac{1}{2} \ln^2 (1 - z) + \left[ \frac{\ln(1 - x)}{1 - x} \right]_+ \right\} + \mathcal{O} (\beta^3). \]  
(243)
The ‘+’-distribution is defined, as usual, by its action on a generic test function $g(x)$:

$$
\int_z^1 dx \, g(x) f_+(x) = \int_z^1 dx \, [g(x) - g(1)] f(x).
$$

(244)

Consider now a simple example, where $f(x) = x^2 \hat{f}(x)$; it follows that

$$
F(z, \beta) = \int_z^1 \beta (1 - x)^{\beta - 1} x^2 \hat{f}(x) = \int_z^1 dx \beta (1 - x)^{\beta - 1} \hat{f}(x)
\times \left\{ 1 + \beta \left[ \frac{1}{\hat{f}(x)} \int_z^1 dy (1 + y) \hat{f}(y) + (1 - x^2) \ln (1 - x) \right] + O(\beta^2) \right\}.
$$

(245)

Therefore, the perturbative expansion is controlled by the parameter $\beta = \alpha A - 1$ and we may compute the kernel cross-section with scaled momenta and fold it with the appropriate factors of $\beta(1 - x)^{\beta - 1}$. The difference with a kernel cross-section computed with non-scaled momenta and the successive application of the correct four-momentum conservation appears only at the next order in $\beta$ and can be re-adjusted order-by-order in perturbation theory.

There are two reasons why one should rescale momenta in the kernel cross-section. In any process with a resonance in the annihilation channel this procedure includes the possibility of a correct description of the radiative return directly in lowest order. There is more, this procedure is sometimes requested by a correct treatment of gauge invariance. Consider, for instance, the process $e(p) + P(P) \rightarrow e(p') + X$. We write

$$
d\sigma = \frac{1}{2P \cdot p} \frac{e^2 W_{\mu\nu} T_{\mu\nu}'}{(q^2)^2} \frac{d^4 p'}{(2\pi)^3} \delta^+(p'^2 + m_e^2) \delta^4(p - q - p').
$$

(246)

The factor $\delta^4(p - q - p')$ is successively promoted to become the spectral function $E(p - q - p')$ which, in coplanar approximation, generates

$$
E^c(p - q - p') \rightarrow \delta^4((1 - \sigma)p - (1 + \sigma')p' - q),
$$

(247)

so that the kinematics is specified by

$$
q = \hat{p} - \hat{p}', \quad \hat{p} = xp = (1 - \sigma) p, \quad \hat{p}' = \frac{p'}{y} = (1 + \sigma') p',
$$

(248)

while the leptonic tensor in soft approximation is extracted as

$$
T_{\mu\nu} = \frac{1}{2} q^2 \delta_{\mu\nu} + p_{\mu} p'_{\nu} + p_{\nu} p'_{\mu} = \frac{1}{2} q^2 \delta_{\mu\nu} + \frac{y}{x} [\hat{p}_{\mu} \hat{p}'_{\nu} + \hat{p}_{\nu} \hat{p}'_{\mu}]
\times \hat{T}_{\mu\nu} + \left( \frac{y}{x} - 1 \right) [\hat{p}_{\mu} \hat{p}'_{\nu} + \hat{p}_{\nu} \hat{p}'_{\mu}],
$$

(249)

and gauge invariance is respected only by $\hat{T}$, namely $q_{\mu} \hat{T}^{\mu\nu} = q_{\nu} \hat{T}^{\mu\nu} = 0$. 

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One should remember that, at least in principle, some of the exponents $\beta_i$ inside Eq. (226) could be negative. In this case we should write

$$\int \prod_i x_i^{\beta_i - 1} \left[ 1 - \theta (-\beta_i) \mathcal{P}(x_i) \right] \hat{\beta}_0,$$

which, by setting some of the $x_i$ to zero in the integrand, introduces the opportune subtraction on the original result. After changing variables, we obtain

$$\int_0^1 dx_+ dx_- dy_+ dy_- (1 - x_+)^{\beta_1 - 1} (1 - x_-)^{\beta_2 - 1} \left( \frac{1}{y_-} - 1 \right)^{\beta_3 - 1} \left( \frac{1}{y_+} - 1 \right)^{\beta_4 - 1}$$

$$\times \frac{x_+^2 x_-^2}{y_+ y_-} \left[ 1 - \theta (-\beta_1) \mathcal{P}(x_+ - 1) \right] \ldots \left[ 1 - \theta (-\beta_4) \mathcal{P}(y_+ - 1) \right] \hat{\beta}_{\text{inv}}^0.$$

(251)

Each subtraction sets one parameter to one and corresponds, figuratively speaking, to disallow radiation from the corresponding leg.

Furthermore, one should also take into proper account the kinematical constraints on the process. We must require that the invariant mass of the outgoing fermion – anti-fermion pair be at least $2m_f$ which implies

$$x_+ x_- y_+ y_- \geq 4 \frac{m_f^2}{s}, \quad s = -(p_+ + p_-)^2,$$

(252)

and $x_\pm, y_\pm$ cannot be zero for non-zero fermion masses.

10 A strategy for computing $\beta_1$

The real improvement upon the present implementation of QED radiative corrections in generic $2 \rightarrow n$ processes requires to go beyond $\beta_0$ in Eq. (28). Therefore, to go beyond the present approximation one has to compute $\beta_1$ in Eq. (28) or, at least to include the collinear singularity of the hard photon. Consider once more the process

$$e^+ e^- \rightarrow nf + \gamma,$$

(253)

and let

$$M = M_\mu e^\mu(k), \quad M_0 \equiv M \left( e^+ e^- \rightarrow nf \right).$$

(254)

Let $i$ be an external fermion, for instance outgoing. Then

$$M_\mu = i e^\nu(p_i) \left( T^i_\mu + R^i_\mu \right), \quad T^i_\mu = -i \frac{Q_i}{2 p_i \cdot k} \gamma_\mu \left( \not{p_i} + \not{k} \right) T^i(p_i + k),$$

(255)
where, for simplicity we have assumed massless fermions. $T$ represents the contribution where the photon is emitted by the $p_i$-line with residual amplitude $T^i(p_i + k)$ and $R$ represents the rest. Gauge invariance requires $k \cdot M = 0$ or

$$\overline{\psi}(p_i) k \cdot R^i = i Q_i \overline{\psi}(p_i) T^i(p_i + k).$$

(256)

Consider vectors $Q, n$ and $k_\perp$, with $Q^2 = n^2 = 0$ and $k_\perp \cdot Q = k_\perp \cdot n = 0$ and introduce

$$k_\mu = z Q_\mu + k_\perp \mu - \frac{k_\perp^2}{2 z} \frac{n_\mu}{Q \cdot n},$$

$$p_\mu = (1 - z) Q_\mu - k_\perp \mu - \frac{k_\perp^2}{2 (1 - z)} \frac{n_\mu}{Q \cdot n},$$

(257)

giving

$$p_i^2 = k^2 = 0, \quad 2 p_i \cdot k = -\frac{k_\perp^2}{z(1 - z)}.$$

(258)

Using the relation

$$p_i + k = Q + \mathcal{O}\left(k_\perp^2\right),$$

(259)

we derive

$$\sum_{\text{spins}} | M \cdot \epsilon |^2 = 2 Q_i^2 \epsilon^2 (1 - z) \frac{1 + (1 - z)^2}{k_\perp^2} \sum_{\text{spins}} | M_0 (p_i \to p_i + k) |^2 + \mathcal{O}(1).$$

(260)

The above result shows factorization of the collinear divergence. This representation solves many problems, for the leading $k_\perp$ behavior we must consider only external fermions and we do not need to have a precise knowledge of the residuals $R^i$. Therefore we do not care about including internal $W$-bosons emitting photons.

However, the procedure is not gauge invariant, gauge violation occurring at $\mathcal{O}(k_\perp^2)$, sub-leading w.r.t. leading $\ln k_\perp$ corrections. There are two possibilities. Either photons are allowed only within a cone (with half-opening $\delta$) surrounding each charged external fermion and we identify a leading, gauge-invariant, $\ln(E \delta/m)$ behavior with sub-leading gauge non-invariant contributions heavily suppressed or we integrate over the whole phase space of the photon. For the latter we may still identify and compute collinear logarithms without having to compute the exact $\mathcal{O}(\alpha)$ matrix element but the scale in the logarithm becomes arbitrary.

Therefore, a rigorous result cannot do without the exact, $\mathcal{O}(\alpha)$ matrix element and any approximation is not free from ambiguities.

### 11 Conclusions

One of the main ingredients in all calculations aimed to a very accurate control of high-energy-physics observables is represented by the re-summation of large QED corrections. This
procedure is usually performed by introducing structure functions. The scale that controls the large logarithms to be re-summed, as well as the $K$-factor which one introduces to increase the accuracy of the calculation are based on some algorithm where one starts from the evolution equation for the structure function itself and seek for a solution which factorizes the re-summed Gribov-Lipatov \cite{17} term and which, through an iterative method, matches with the finite, second order, result of Barbieri, Mignaco and Remiddi \cite{21}. The procedure is to some extent based on the soft limit.

The are are two classes of problems when one wants to generalize this algorithm to more complex processes, with many fermions in the final state. First of all the choice of the scale describing the evolution of the structure function is ambiguous and it is not even clear that one can have a realistic description with just one scale in situations where the dominant contribution to the process is far from the asymptotic regime. Secondly, an exact fixed order calculation is generally missing for the process and not only the scale in the dominant logarithms is ambiguous but also one has no control over the sub-leading logarithms.

From this point of view, all claims that are based on general arguments as factorization theorems or renormalization group equation are usually void. The only safe and rigorous argument that one can apply is the exact soft-photon re-summation, as given in the classic YFS treatment.

The YFS algorithm has been further developed and modelled for its use in MonteCarlo programming and we have no pretension to be adding any substantial improvement, as clearly stated in the introduction. At the same time we cannot offer any claim pointing to the complete solution of the problem of a very precise implementation of QED radiation for processes more complex than $e^+e^-$-annihilation into fermion-antifermion pairs. This simple statement should not be confused with a failure of the method. In Section 3.2 we have repeated the classical argument that perturbation theory and exponentiation can me made consistent with residuals that are infrared finite. The accuracy at stake is confined in those ingredients that are missing just because of some technical inadequacy in controlling the full content of higher orders.

What we have done is an attempt to systematize all arguments and speculations that have appeared in recent times in the literature. We started with the well-known YFS re-summation procedure, expressed in the modern language of dimensional regularization for infrared divergences. Based on this result we have adopted a slightly different approach where we, nevertheless, pursue the simple picture in which the whole effect of soft-photon emission is described by a real-photon spectral weight function. However, in our approach we avoid the introduction of a cutoff. In this respect we follow an old proposal by Chahine \cite{11} by introducing an approximation to the exact spectral function which retains the important properties and incorporates the expected peaking of the emitted photons along the direction of charged particles. Also in this case we have completely reformulated the algorithm in modern language.

The result of our investigation allows to write a corrected cross-section where the kernel for the hard scattering is convoluted with generalized structure functions where each of them is no longer function of one scale. Each external, charged, fermion leg brings a factor $x^{\alpha A - 1}$ where $\alpha$ is the fine-structure constant and $A$ is a function which depends on the momenta of the charged particles.
A preliminar account of these results has been given in [22].

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13 Appendix A

The evaluation of the two-particle radiation function is based on Fourier cosine(sine) transforms of Bessel functions [23],

\[
g(y) = \int_0^\infty dx \, \cos(xy) \, x^{2\mu-1} \, J_{2\nu}(\alpha x),
\]

\[
0 \leq y \leq \alpha, \ g(y) = \frac{2^{2\mu-1} \alpha^{-2\mu} \Gamma(\mu+\nu)}{\Gamma(1+\nu-\mu)} \_{2}F_{1}\left(\nu+\mu,\mu-\nu;\frac{1}{2};\frac{y^2}{\alpha^2}\right),
\]

\[
\alpha \leq y \leq \infty, \ g(y) = \frac{(a/2)^{2\nu} y^{-2\nu-2\mu} \Gamma(2\nu+2\mu) \cos(\nu \pi + \mu \pi)}{\Gamma(2\nu+1)} \times \_2F_1\left(\nu+\mu,\nu+\mu+\frac{1}{2};2\nu+1;\frac{\alpha^2}{y^2}\right), \tag{261}
\]

which is valid for

\[- \operatorname{Re} \nu < \operatorname{Re} \mu < \frac{3}{4}. \tag{262}\]

\[
g(y) = \int_0^\infty dx \, \sin(xy) \, x^{2\mu-1} \, J_{2\nu}(\alpha x),
\]

\[
0 \leq y \leq \alpha, \ g(y) = 4^\mu \alpha^{-2\mu-1} y \frac{\Gamma(\frac{1}{2}+\nu+\mu)}{\Gamma(\frac{1}{2}+\nu-\mu)} \_{2}F_{1}\left(\frac{1}{2}+\nu+\mu,\frac{1}{2}+\mu-\nu;\frac{3}{2};\frac{y^2}{\alpha^2}\right),
\]

\[
\alpha \leq y \leq \infty, \ g(y) = \left(\frac{a}{2}\right)^{2\nu} y^{-2\nu-2\mu} \frac{\Gamma(2\nu+2\mu)}{\Gamma(2\nu+1)} \sin(\nu \pi + \mu \pi) \times \_2F_1\left(\frac{1}{2}+\nu+\mu,\nu+\mu;2\nu+1;\frac{\alpha^2}{y^2}\right), \tag{263}
\]

which is valid for

\[- \operatorname{Re} \nu - \frac{1}{2} < \operatorname{Re} \mu < \frac{3}{4}, \quad \alpha > 0. \tag{264}\]

Another useful integral is [23]

\[
\int_{-1}^{+1} dx \, \exp(i \, zx) \left(1 - x^2\right)^{-1/2} = 2^\nu \pi^{1/2} \Gamma\left(\nu + \frac{1}{2}\right) z^{-\nu} J_\nu(z), \tag{265}
\]

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which is valid for
\[ \text{Re} \nu > - \frac{1}{2}. \] (266)

Further, the following integral representation holds \[23\]
\[ \psi(s + \alpha) - \psi(s + \beta) = \int_0^1 dx \frac{x^{s-1} x^\beta - x^\alpha}{1 - x}, \quad \text{Re} s > -\text{Re} \alpha, -\text{Re} \beta. \] (267)

We have used two different integral representations for the \( \Gamma \)-function \[23\],
\[ z^{-s} \Gamma(s) = \int_0^\infty dx \frac{x^{s-1} e^{-zx}}{1 - x}, \quad \text{Re} z > 0, \quad \text{Re} s > 0, \] (268)
and a second one due to Cauchy and Saalschütz \[24\]
\[ z^{-s} \Gamma(s) = \int_0^\infty dx \frac{x^{s-1} \left[ e^{-zx} - \sum_{m=0}^{n} \frac{(-zx)^m}{m!} \right]}{1 - x}, \] (269)
which is valid for
\[ \text{Re} z > 0, \quad -(n + 1) < \text{Re} s < -n. \] (270)
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