Abstract

We establish sharp $W^{2,p}$ regularity estimates for viscosity solutions of fully nonlinear elliptic equations under minimal, asymptotic assumptions on the governing operator $F$. By means of geometric tangential methods, we show that if the recession of the operator $F$ – formally given by $F^*(M) := \infty^{-1} F(\infty M)$ – is convex, then any viscosity solution to the original equation $F(D^2 u) = f(x)$ is locally of class $W^{2,p}$, provided $f \in L^p$, $p > d$, with appropriate universal estimates. Our result extends to operators with variable coefficients and in this setting they are new even under convexity of the frozen coefficient operator, $M \mapsto F(x_0, M)$, as oscillation is measured only at the recession level. The methods further yield BMO regularity of the hessian, provided the source lies in that space. As a final application, we establish the density of $W^{2,p}$ solutions within the class of all continuous viscosity solutions, for generic fully nonlinear operators $F$. This result gives an alternative tool for treating common issues often faced in the theory of viscosity solutions.

Keywords: Fully nonlinear elliptic equations; regularity theory; apriori $W^{2,p}$ estimates.

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1 Introduction

In this article, we investigate interior $W^{2,p}$-regularity estimates for viscosity solutions of second order fully nonlinear elliptic equations

$$F(D^2 u) = f(x).$$

The key novelty of the present work is that $F : S(d) \to \mathbb{R}$ is assumed to be convex (or concave) only at the ends of $S(d)$, that is, only when $\|D^2 u\| \approx \infty$. Under such a mild condition, we establish local $W^{2,p}$ estimates for solutions to (1) in terms of the $L^p$ norm of $f$. Important consequences for the general theory of fully nonlinear PDEs are then derived.

Regularity theory for viscosity solutions of second order fully nonlinear elliptic equations has been a central topic in the field of analysis of PDEs since the trailblazing...
works of N. Krylov and M. Safonov [18, 19] on Harnack inequality for non-divergence form (linear) elliptic equations, back at the beginning of the 1980’s. In turn, solutions to homogeneous equations, $F(D^2 u) = 0$, are of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. The next major chapter in the regularity theory of such equations comes a few years later when L. Evans [9] and N. Krylov [14, 15] proved separately that solutions to convex (or concave) homogeneous equations are locally of class $C^{2,\alpha}$. Still under the (natural) assumption that $F$ is convex, L. Caffarelli, in his seminal paper [3], established the foundations of the $W^{2,p}$ theory for fully nonlinear elliptic equations.

Whether $W^{2,p}$ regularity estimates were available for any uniformly elliptic fully nonlinear equation challenged the community for over thirty years. The problem was settled in the negative by N. Nadirashvili and S. Vlăduț, see [21, 22, 23]. We also mention here the examples from [7] of linear elliptic operators with piecewise constant coefficients whose solutions fail to be in $W^{2,p}$.

Sharp hessian integrability theory for Equation (1) is indeed an intricate mathematical puzzle. In turn, special hidden structures hold the key to the validity of $W^{2,p}$ a priori estimates. Often such hidden structures are not perceived by classical methods and techniques used in the study of PDEs, and alternative approaches must be considered. In this present work, we tackle this issue by means of the so-called geometric tangential analysis. This comprises a series of techniques relating a given problem to an auxiliary one through a genuinely geometric structure. The core of the geometric tangential analysis is to build a path that touches the original problem of interest and connects it with an auxiliary, model-problem. Among the diverse manners we have to build such a path, we bring up in this work the notion of recession function: given a fully nonlinear elliptic operator $F$ defined on $S(d)$, we denote by $F^*$ the limiting operator

$$F^*(M) := \lim_{\mu \to 0} \mu F(\mu^{-1} M), \quad (2)$$

for $M \in S(d)$. Such a limiting operator appears naturally in the study of free boundary problems ruled by fully nonlinear equations, where hessian blow-up is expected through the phase transition. Hence, the free boundary condition, that is, the equation satisfied along the free boundary is prescribed by the $F^*$ rather than $F$, see for instance [24, 1]. In a related reasoning, improved $C^{1,\alpha}$ estimates based on limiting profiles of the operator have been recently announced in [26].

Hereafter, the operator $F^*$ will be called the recession function associated with $F$. This nomenclature is borrowed from the realm of convex analysis [25], and can be found in applications of its techniques to numerous problems, ranging from Economics (e.g. production theory, see [10]) to Physics (e.g. continuum mechanics, see [2]). In this tradition, a convex set $A \subset \mathbb{R}^d$ is said to recedes in the direction of $y \in \mathbb{R}^d$, for $y \neq 0$, if

$$x + \mu y \in A \quad (3)$$

for every $\mu \geq 0$ and $x \in A$. The set of all vectors $y \in \mathbb{R}^d$ for which (3) is satisfied is called the recession cone of $A$, denoted $\text{rec}(A)$. Consider further a convex function $f : \mathbb{R}^d \to \mathbb{R}$ and denote its epigraph by $\text{epi}(f)$. A straightforward reasoning yields that $\text{rec}(\text{epi}(f))$ is on its turn the epigraph of a function, say $f^*$. Hence, $f^*$ is said to be the
recession function associated with $f$. It can be shown that, under a few conditions, we have

$$f^*(y) = \lim_{\mu \to 0} \mu \left( f\left(x + \mu^{-1}y\right) - f(x) \right);$$

(4)

see [25]. By taking $x \equiv 0$ and assuming that $f(0) = 0$, we recover (2) through the limit in (4).

Insofar as the heuristics of the geometric tangential methods are concerned, by making appropriate assumptions on $F^*$, we expect to import regularity from the ends of $S(d)$ back to $F$. The tangential path in this case is parametrized by $\mu > 0$. Our main result is stated in the following theorem:

**Theorem 1.1 (A priori $W^{2,p}$ estimate).** Let $F : S(d) \to \mathbb{R}$ be uniformly elliptic, $p > d$, and $u$ be a viscosity solution of

$$F(D^2 u) = f(x), \quad \text{in } B_1.$$

Assume $F^*$ has a priori $C^{1,1}$ estimates. Then $u \in W^{2,p}(B_{1/2})$ and there exists $C > 0$ so that

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right).$$

(5)

Theorem 1.1 accommodates a fairly general class of fully nonlinear operators $F$, as we shall exemplify when time comes. In turn, a large class of problems can be treated by the methods developed in this article. As a striking application of the results proven in this paper, we will verify that the set of $W^{2,p}$ viscosity solutions is dense in the set of all continuous viscosity solutions of a given class of fully nonlinear equations, in a sense that will be made precise later. This fact enables one to bypass, in many cases, the formalism of the viscosity solution language, when addressing a priori regularity estimates, or any target property that is closed under uniform convergence.

The remainder of this paper is organized as follows. In section 2 we present the set-up under which we shall work in this paper. We also discuss the main ideas and insights concerning the proof of Theorem 1.1. In section 3 we present few examples for which our results can be directly applied to. In section 4 we revisit some tools and elements required in the study of hessian integrability estimates for solutions to non-divergence form equations. In section 5 we discuss the idea of linking the regularity theory of the original operator $F$ to the (better) one of the recession operator $F^*$. This is done by an appropriate path along the set of all $(\lambda, \Lambda)$-elliptic equations. In section 6 we deliver the proof of Theorem 1.1 while in section 7 we discuss generalization to variable coefficient equations, $F(x, D^2 u) = f(x)$. For the latter, we only measure the oscillation of the coefficients at the recession level, which may be strictly less than the oscillation of the coefficients of the original operator. Hence, Theorem 1.1 gives a new information even in the classical setting where $M \mapsto F(\lambda_0, M)$ is assumed to be convex. In section 8 we address the borderline case $p = \infty$, that is, we provide BMO interior estimates for $D^2 u$ in terms of BMO norm of the source $f$. Finally, in section 9 we show our sharp integrability estimates can be applied to establish the density of $W^{2,p}$ solutions within the set of all $C^0$-viscosity solutions.
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2 Main assumptions and outline of the proof

In this section, we detail the main assumptions and set-up under which we shall work on the present paper. The space of all real, $d \times d$ symmetric matrices is denoted by $S(d)$. This is a $\frac{d(d+1)}{2}$-dimensional space. It is a classical result that all matrices $M \in S(d)$ are diagonalizable. For $M \in S(d)$, we define

$$\|M\| := \sum_{i=1}^{d} |e_i|,$$

where $e_1, e_2, \ldots, e_d$ are the eigenvalues of $M$. We recall that any two norms in $S(d)$ are equivalent. Hessians of $d$-dimensional $C^2$ functions, $D^2u$, lie in $S(d)$ and in this article we are interested in second order nonlinear partial differential equations of the form $F(D^2u) = f(x)$, where $F : S(d) \to \mathbb{R}$. Here is the main assumption upon the operator $F$, supposed to hold throughout the entire paper:

A 1. The operator $F$ is uniformly elliptic, with ellipticity constants $\lambda, \Lambda$. That is, for $M \in S(d)$, we have

$$\lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|,$$

for every $N \geq 0$. We further assume, with no loss of generality, that $F(0) = 0$.

An operator satisfying A1 is also referred to as $(\lambda, \Lambda)$-elliptic. It is sometimes convenient to express ellipticity in terms of the extremal Pucci operators:

$$\mathcal{P}^-_{\lambda, \Lambda}(M) := \Lambda \cdot \text{Trace}(M^-) + \lambda \cdot \text{Trace}(M^+),$$

$$\mathcal{P}^+_{\lambda, \Lambda}(M) := \Lambda \cdot \text{Trace}(M^+) + \lambda \cdot \text{Trace}(M^-).$$

Assumption A1 is equivalent to

$$\mathcal{P}^-_{\lambda, \Lambda}(M - N) \leq F(M) - F(N) \leq \mathcal{P}^+_{\lambda, \Lambda}(M - N),$$

for all $M, N \in S(d)$. Under such a monotonicity assumption on $F$, the notion of viscosity solutions provides an appropriate definition of weak solutions for Equation (1).

Definition 2.1. A continuous function $u \in C^0(B_1)$ is said to be a viscosity subsolution to (1) in $B_1$ if whenever one touches the graph of $u$ from above by a smooth function $\varphi$ at $x_0 \in B_1$ (i.e. $\varphi - u$ has a local minimum at $x_0$), there holds

$$F(D^2\varphi(x_0)) \geq f(x_0).$$
Similarly, \( u \) is a viscosity supersolution to (1) if whenever one touches the graph of \( u \) from below by a smooth function \( \phi \) at \( y_0 \in B_1 \), there holds

\[
F(D^2\phi(y_0)) \leq f(y_0).
\]

We say \( u \) is a viscosity solution to (1) if it is a subsolution and a supersolution.

The key goal of this present work is to show that sharp hessian integrability estimates for viscosity solutions to (1) is controlled by the behavior of the operator \( F \) at the ends of its space of definition. This brings us to the notion of recession function, which we formally define in the sequel.

**Definition 2.2** (Recession function). The recession function \( F^*(M) \) associated with the fully nonlinear operator \( F \) is given by

\[
F^*(M) := \lim_{\mu \to 0} F_\mu(M) := \lim_{\mu \to 0} \mu F(\mu^{-1}M).
\]

Notice that \( F, F_\mu := \mu F(\mu^{-1}M) \) and \( F^* \) have the same ellipticity constants. It is also relevant to observe that the recession function is always homogeneous of degree one. The primary assumption we make in this article as to derive sharp \( W^{2,p} \) estimates for viscosity solutions of (1) is that the recession operator \( F^* \) has a rich enough \textit{a priori} regularity theory.

**A 2.** We assume that the recession function \( F^* \) associated with the operator \( F \) exists and has \textit{a priori} \( C^{1,1}_{\text{loc}} \) estimates. That means \( F^*(D^2h) = 0 \) in \( B_1 \) in the viscosity sense, implies \( h \in C^{1,1}(B_{1/2}) \) and

\[
\|h\|_{C^{1,1}(B_{1/2})} \leq C_* \|h\|_{L^\infty(B_1)},
\]

for a constant \( C_* \geq 1 \).

In particular, if \( F^* \) happens to be concave (or convex), \( A^2 \) is immediately satisfied due to Evans-Krylov Theorem. This is the case when \( F \) is suitably modified outside a ball of \( S(d) \). For instance, if \( F \) is supposed to be concave (or convex) in \( S(d) \setminus B_R \), for some \( R \gg 0 \). This constitutes an important class of examples and we write it down for future references.

**Example 1.** Let \( F,G : S(d) \to \mathbb{R} \) be uniformly elliptic operators with, say, \( G \) homogeneous of degree one and convex. Assume \( F(M) = G(M) \) for all \( \|M\| \geq R \) for some \( R \gg 1 \). Then \( F^*(M) = G(M) \).

We further comment that existence and uniqueness of the recession function is not \textit{per se} required. By ellipticity, \( \{F_\mu\}_{\mu > 0} \) is locally pre-compact in \( S(d) \), hence, up to a subsequence, \( F_\mu \) always converge to a recession function \( F^* \). Assumption \( A^2 \) should be understood as a condition on any recession function of \( F \).

Per the classical constraints on the notion of viscosity solutions, we will require that the source function \( f \) is continuous in \( B_1 \).
A 3. We assume that \( f \in C^0(B_1) \).

Upon such a constraint, and in accordance to Caffarelli’s theory, our \( W^{2,p} \)-regularity estimate is understood as an \textit{a priori} estimate. By using weaker notions of viscosity solutions, it is possible to work under integrability assumption on \( f \), c.f. [5]. This is an immediate consequence of the stability properties of \( L^p \) viscosity solutions. Besides, we expect that one can relax the integrability condition on \( f \) in the sense of Escauriaza, see [8].

2.1 Outline of the proof

Through the rest of this section, we will discuss the insights and main strategies for proving Theorem 1.1. Intuitively, \( F^* \) “governs” Equation (1) in the region where the hessian of \( u \) blows-up. Hence, \textit{a priori} \( C^{1,1} \) estimates available for \( F^* \) set a competing inequality which, in turn, should yield a better measure decay on

\[
\Theta_K := \{ x \in B_{1/2} : D^2 u(x) > K \},
\]

for \( K \gg 1 \) sufficiently large. Of course, there are obvious difficulties in carrying out the above mentioned reasoning. For instance, in principle there is no information on the set where the hessian will be large, and in general \( \Theta_K \) are very irregular sets. We should also caution the readers to the existence of a \( C^{1,1} \setminus C^2 \) viscosity solution to a fully nonlinear elliptic equation, [21]. Hence, it is not possible to establish continuity of the hessian out from the regularity theory of its recession function, even if \( F^* = \Delta \).

Alternatively, we will approach the problem with the aid of geometric tangential methods, where \( F^* \), along with its regularity theory, is regarded as a \textit{target} profile. If we denote by \( F_\mu := \mu F(\mu^{-1} M) \), the map \( \mu \mapsto F_\mu \) provides a tangential path within the manifold consisting of \((\lambda, \Lambda)\)-elliptic equations linking the regularity theory of \( F_1 = F \) with the (better) one available for \( F^* \).

Within the heuristics of geometric tangential methods, \textit{universal} regularity theory should be understood as a relatively open property, in the sense that if \( F^* \) has \textit{a priori} \( C^{1,1}_{\text{loc}} \) estimates then near \( F^* \) operators should provide a “slightly weaker” smoothness effect. Hence, owing to \( C^{1,1} \approx W^{2,\infty} \) estimates for \( F^* \), one should be able to reach \( W^{2,p} \) regularity, \( p < \infty \), for operators “near” \( F^* \). Hence, one should expect that for \( 0 < \mu \leq \mu_0 \ll 1 \), the operator \( F_\mu \) belongs to such a neighborhood and then one may transfer such an estimate back to the original \( F = F_1 \).

In section [3] we recur to compactness methods to prove an appropriate approximation lemma, which justifies rigorously the above discussion. Carrying out this analysis, with additional results and techniques from [3], we can deliver a proof of the following result:

\textbf{Proposition 2.1.} Let \( u \) be a bounded viscosity solution of (1) and assume that (13) hold. Suppose further that

\[
\|u\|_{L^\infty(B_1)} \leq 1 \text{ and } \|f\|_{L^p(B_1)} \leq \epsilon,
\]
For $\mu = \mu(p)$ sufficiently small, but still positive, $F_\mu$ enters within a neighborhood of $F^*$ for which $W^{2,p}$ estimates are available. Such an improved estimate is transported down to the original operator $F = F_1$ through the tangential path.

for a small $\epsilon > 0$. Then, $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C,$$

where $C > 0$ depends on the dimension $d$, $F$ and a priori $C^{1,1}$ estimates for $F^*$.

Routine arguments combined with the conclusion of Proposition 2.1 yield a proof of Theorem 1.1.

Remark 2.1. The $W^{2,p}$ estimate from Theorem 1.1 depends not only on universal constants, but actually on a priori $C^{1,1}$ estimates for $F^*$ and on the modulus of convergence $F_\mu \rightarrow F^*$. Namely, in Lemma 5.1 to be later presented, we can define $\omega \colon (0, 1) \rightarrow \mathbb{R}_+$ as

$$\omega(\epsilon) := \sup \left\{ \mu_0 > 0 : \|\mu F(\mu^{-1}M) - F^*(M)\| \leq \epsilon (1 + \|M\|), \forall M \in S(d), \forall \mu \leq \mu_0 \right\}.$$  

The constant $C > 0$ appearing in $W^{2,p}$ estimate (5) depends on dimension, $\lambda$, $\Lambda$, $C^*$ and $\omega$.

3 Further examples

Problems in differential geometry have been a profitable source of examples of fully nonlinear partial differential equations. Let $S \subset \mathbb{R}^{d+1}$ be a hypersurface and let $A$ de-
note its second fundamental form. Notable examples of fully nonlinear elliptic equations arising in differential geometry are of the form

\[ F(A) = F(\kappa_1, \ldots, \kappa_d) = \phi, \]

where the eigenvalues \( \kappa_1, \ldots, \kappa_d \) of \( A \) are the principal curvatures of \( S \), and \( \phi \) is a given function. When \( \phi > 0 \) and \( F \) is symmetric, \( S \) is said to be a Weingarten surface. Specializing \( F \) to be an \( r \)-th elementary symmetric function \( \sigma_r \) and setting \( \phi \equiv 1 \) recovers important geometric quantities. For example, up to a constant, \( r = 1 \) gives the mean curvature, \( r = 2 \) yields the scalar curvature, and \( r = d \) gives the Gauss-Kronecker curvature of \( S \). See [27]

Another remarkable class of fully nonlinear equations appears in the study of certain absolutely area minimizing submanifolds of \( \mathbb{C}^d \approx \mathbb{R}^{2d} \). These are referred as special Lagrangian manifolds and arise, for example, in calibrated geometry. These objects retain an intrinsic connection with fully nonlinear elliptic PDEs, through what is called the special Lagrangian equation

\[ \mathcal{L}(M) := \sum_{i=1}^d \arctan \lambda_i = \Theta. \quad (6) \]

If \( M = (x, \nabla u(x)) \) is a minimal surface in \( \mathbb{R}^{2d} \), then, it is a special Lagrangian manifold and intrinsically geometric information about \( M \) may depend on the regularity of the hessian of \( u \). It is known that the level set of (6) are convex if and only if \( |\Theta| \geq (d - 2)\pi/2 \), see [31]. When \( d = 3 \) and \( |\Theta| < \pi/2 \), given any \( \delta > 0 \), the authors in [30] build up solutions to (6) that are not in \( C^{1,\delta} \).

While the special Lagrangian equation is not uniformly elliptic, it behaves like so for \( C^{1,1} \) solutions. Hence, it is natural to inquire what is the smoothing effect of \( \mathcal{L} \) when added small increment diffusions, say \( \mathcal{L}_\epsilon := \mathcal{L} + \epsilon \Delta \).

**Example 2** (Perturbation of the special Lagrangian equation). Let \( 0 < \alpha_1, \alpha_2, \ldots, \alpha_d < +\infty \) and consider

\[ F(M) := \sum_{i=1}^d (\alpha_i \lambda_i + \arctan \lambda_i). \]

This is a uniformly elliptic operator, and one easily computes

\[ F^*(M) = \alpha_i \lambda_i; \]

i.e., the recession function of the perturbed special Lagrangian operator is, up to a change of variables, the laplacian operator. From Theorem [17] such operator has a priori \( W^{2,p} \) estimates. In view of the counterexamples from [30], these estimates cannot be uniform with respect to \( |(\alpha_1, \cdots, \alpha_d)| \). Nonetheless, it follows that any (possibly singular) solution of (6) can be approximated by \( W^{2,p} \) solutions of small perturbations of the original operator.

Regularity estimates based on the analysis of recession functions turn out to be an efficient tool for analyzing perturbation of geometric equations. In the sequel, we analyze a few further examples for which our main theorem can be applied.
Example 3. Let $q \in 2\mathbb{N} + 1$ be an odd number. Consider the eigenvalue “$q$-momentum” operator

$$F_q(M) = F_q(\lambda_1, \cdots, \lambda_d) = \sum_{i=1}^{d} \left( 1 + \lambda_i^q \right)^{1/q} - d,$$

where $\lambda_1, \cdots, \lambda_d$ are the eigenvalues of the matrix $M \in S(d)$. Notice that $F_q$ is neither concave nor convex. However, one easily computes

$$\mu F_q \left( \mu^{-1} M \right) = \sum_{i=1}^{d} \left( \mu^q + \lambda_i^q \right)^{1/q} - \mu d.$$

Thus,

$$F_q^*(M) = \lim_{\mu \to 0} \mu F_q \left( \mu^{-1} M \right) = \sum_{i=1}^{d} \lambda_i,$$

i.e., $F_q^*$ is the laplacian operator, for any $q \in 2\mathbb{N} + 1$. One easily observes that the recession convergence above is uniform in $q$; not ellipticity though. The limiting operator obtained as $q \to \infty$,

$$F_{\infty}(M) := \begin{cases} 0 & \text{for } \|M\| < 1 \\ \text{Trace}(M) & \text{for } \|M\| \geq 1, \end{cases}$$

prescribes no equation within the unit ball $B_1$. It would be interesting, in future research, to investigate whether $W^{2,p}$-regularity estimate provided by Theorem 1.1 is uniform in $q$.

Another illustrative example we bring up here concerns equations with trigonometric oscillatory dependence:

Example 4. Let $0 < \alpha_1, \alpha_2, \cdots, \alpha_d < +\infty$ and consider

$$F(M) := \sum_{i=1}^{d} \left( (1 + \alpha_i) \lambda_i + \sin \lambda_i \right).$$

This is a uniformly elliptic operator, with ellipticity constants $\lambda = \inf \alpha_i$ and $\Lambda = \sup \alpha_i + 1$. We compute

$$F^*(M) = (1 + \alpha_i) \lambda_i,$$

i.e., modulus a change of coordinates, $F^*$ is the laplacian operator.

4 Some geometric-measure tools

In this section, we gather few tools and elements involved in the proof of Theorem 1.1. Throughout, we revisit the by now well-established a priori $W^{2,\delta}$ estimates for solutions to (1). Although these are well-known facts, they are presented here for completeness. Most of the proofs are omitted in what follows; the reader is referred to [3] and [4, Chapter 7], where the complete arguments are presented.
Proposition 4.1 (Proposition 7.4, [4]). Assume that \( u \in S(f) \) in \( B_1 \) and \( A_7 \) and \( A_3 \) hold. Then, for some universal number \( \delta > 0 \), we have \( u \in W^{2,\delta}(B_{1/2}) \) and
\[
\|u\|_{W^{2,\delta}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right),
\]
where \( C > 0 \) is also a universal constant.

Proposition 4.1 is paramount in the study of a decay rate for the measure of a certain family of subsets of \( B_1 \). We proceed by presenting a definition.

Definition 4.1 (Chapter 7, pp. 62, [4]). Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain and \( u \in C(\Omega) \). For \( M > 0 \) and \( H \subset \Omega \), we define
\[
G_M(u, H) = \{ x_0 \in H \mid \exists \text{P concave paraboloid touching } u \text{ from below at } x_0 \}.
\]
Analogously, we can define \( \overline{G}_M : \)
\[
\overline{G}_M(u, H) = \{ x_0 \in H \mid \exists \text{P convex paraboloid touching } u \text{ from above at } x_0 \}.
\]
In addition, we have
\[
A_M(u, H) = H \setminus G_M(u, H)
\]
and
\[
\overline{A}_M(u, H) = H \setminus \overline{G}_M(u, H).
\]
Finally,
\[
G_M(u, H) = G_M(u, H) \cap \overline{G}_M(u, H),
\]
and
\[
A_M(u, H) = A_M(u, H) \cup \overline{A}_M(u, H).
\]

To prove that \( D^2u \) belongs to \( L^p(B_{1/2}) \), we have to study the summability of
\[
\sum_{k=1}^{\infty} M^{pk} \|A_M(B_{1/2})\|.
\]
This is due to the following well-known result on measure theory:

Lemma 4.1. Let \( g \) be a nonnegative and measurable function on \( \Omega \) and \( \mu_g \) be its distribution function, i.e.,
\[
\mu_g(t) = \|x \in \Omega | g(x) > t\|, \quad t > 0.
\]
Let \( \eta > 0 \) and \( M > 1 \) be constants. Then, for \( 0 < p < \infty \),
\[
g \in L^p(\Omega) \iff \sum_{k=1}^{\infty} M^{pk} \mu_g(\eta M^k) := S < \infty
\]
and
\[
C^{-1}S \leq \|g\|_{L^p(\Omega)}^p \leq C (|\Omega| + S),
\]
where \( C > 0 \) is a constant depending only on \( \eta, M \) and \( p \).
Proof. For the proof, we refer the reader to [4, Lemma 7.3].

Lemma 4.2. Assume that $\Omega$ is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Let $u$ be a continuous function in $\Omega$ with $\|u\|_{L^\infty(\Omega)} \leq 1$, $u \in \mathcal{N}(f)$ in $B_{6\sqrt{d}}$. Assume further that $\|f\|_{L^p(B_{6\sqrt{d}})} \leq \delta_0$. Then,
\[ |\mathcal{G}_M(u, \Omega) \cap Q_1| \geq 1 - \sigma, \]
where $0 < \sigma < 1$, $\delta_0 > 0$ and $M > 1$ are universal constants.

The proof of Lemma 4.2 follows closely the one put forward in [4, Lemma 7.5] and is omitted here.

Lemma 4.3. Let $\Omega$ be a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Assume that $u$ is a continuous functions in $\Omega$ such that $u \in \mathcal{N}(f)$ in $B_{6\sqrt{d}}$. Assume further that $\|f\|_{B_{6\sqrt{d}}} \leq \delta_0$ and $\mathcal{G}_{1}(u, \Omega) \cap Q_1 \neq \emptyset$. Then
\[ |\mathcal{G}_{M}(u, \Omega) \cap Q_1| \geq 1 - \sigma, \]
where $0 < \sigma < 1$, $\delta_0$ and $M > 1$ are universal constants.

As before, we omit the proof of Lemma 4.3 and refer the reader to [4, Lemma 7.6]. Next, we recall an elementary result derived from the Calderón-Zygmund cube decomposition:

Lemma 4.4. Let $A \subset B \subset Q_1$ be measurable sets and assume that $0 < \delta < 1$ is such that

1. $|A| \leq \delta$;
2. if $Q$ is a dyadic cube such that $|A \cap Q| > \delta |Q|$, then $\hat{Q} \subset B$, where $\hat{Q}$ is the predecessor of $Q$.

Then, we have
\[ |A| \leq \delta |B|. \]

Lemma 4.5. Under the hypotheses of Lemma 4.2 extend $f$ by zero outside $B_{6\sqrt{d}}$ and let
\[ A = A_{M+1}(u, \Omega) \cap Q_1, \]
\[ B = \left( A_{M}(u, \Omega) \cap Q_1 \right) \cap \left\{ x \in Q_1 | m(f^n)(x) \geq (c_1M^\sigma)^n \right\}. \]

Then,
\[ |A| \leq \sigma |B|, \]
where $0 < \sigma < 1$, $\delta_0$, $M > 1$ and $c_1 > 0$ are universal constants.

The proof of Lemma 4.5 relies on standard arguments, together with an auxiliary function $\tilde{u}$ determined by
\[ \tilde{u}(y) = \frac{2^{2i}}{M^\sigma} u \left( x_0 + \frac{1}{2^i} y \right). \] (7)

Later in the present paper, we refer to (7).

In the next lemma, we recall a result on the decay rate of the measure of certain sets.
Lemma 4.6. Under the hypotheses of Lemma 4.2

\[ |A_t(u, \Omega) \cap Q_1| \leq c_2 t^{-\mu}, \quad \forall \ t > 0, \tag{8} \]

where \( c_2 \mu > 0 \) are universal constants. If, in addition, \( u \in S(f) \) in \( B_{6\sqrt{d}} \), then

\[ |A_t(u, \Omega) \cap Q_1| \leq c_2 t^{-\mu}, \quad \forall \ t > 0. \tag{9} \]

We conclude this section by combining the above building-block lemmas, yielding a proof of Proposition 4.1.

Proof of Proposition 4.1. We recur to a standard covering argument and assume that \( u \in S(f) \) in \( B_{6\sqrt{d}} \), with \( \|u\|_{L^\infty(B_{6\sqrt{d}})} \leq 1 \) and \( \|f\|_{L^d(B_{6\sqrt{d}})} \leq \delta_0 \). Set \( \delta = \mu/2 \). Because of \( (9) \), we have

\[ \sum_{k \geq 1} M^{\frac{\mu}{2}} |A_{M^k}(B_{6\sqrt{d}}) \cap Q_1| \leq C, \]

for a universal constant \( C > 0 \), where we have used \( \Omega = B_{6\sqrt{d}} \). Furthermore,

\[ A_{M^k}(u, B_{1/2}) \subset A_{M^k}(u, B_{6\sqrt{d}}) \cap Q_1; \]

therefore,

\[ \sum_{k \geq 1} M^{\frac{\mu}{2}} |A_{M^k}(B_{1/2})| \leq C. \]

Define \( \Theta \) by

\[ \Theta(x) := \inf \{ M | x \in G_M(B_{1/2}) \}. \]

Hence,

\[ \mu_0(t) \leq |A_t(B_{1/2})|. \tag{10} \]

The inequality in \( (10) \) and the Lemma 4.1 then lead to

\[ \|D^2 u\|_{L^{\frac{d}{d+1}(2)}} \leq C \|\Theta\|_{L^{\frac{d}{d+1}(2)}} \leq C. \]

\[ \square \]

5 A new geometric tangential path

Here we find a linking path from the regularity theory of the original operator \( F \) to the one of the recession operator \( F^* \). From the heuristics of the geometric tangential methods, this fact allows us to import the estimates available for \( F^* \) back to the original operator through this chosen path. As expected, this pass-through mechanism is not without a cost; if the auxiliary problem has a given property, the original one inherits a slightly weaker variation of it. However, as we will see in section 6, this is just enough to obtain sharp \( W^{2,p} \) estimates for \( (1) \).
Lemma 5.1 (Local uniform convergence). Let $F$ be a uniformly elliptic operator and assume $F^*$ exists. Then, for every $\epsilon > 0$ there is $\mu_0 > 0$ such that, for every $\mu < \mu_0$ we have
\[
|\mu F(\mu^{-1} M) - F^*(M)| \leq \epsilon (1 + \|M\|),
\]
for every $M \in S(d)$.

Proof. The proof of Lemma 5.1 is elementary, but we carry it out as a courtesy to the readers. Since all $F_{\mu}$ are uniformly elliptic operators, by the Arzelà-Ascoli Theorem, up to a subsequence, $F_{\mu}$ converges uniformly in every compact sets of $S(d)$ to $F^*$.

Hence, given $\epsilon > 0$ there exists a $\delta > 0$ so that
\[
|\mu F(\mu^{-1} M) - F^*(M)| \leq \epsilon,
\]
for all matrices $M$ such that $\|M\| \leq 1$ and all $\mu < \delta$. Now let $M$ be a matrix with $\|M\| > 1$. For any $\mu < \delta$, we look at
\[
\mu_1 = \|M\|^{-1} \mu < \mu < \delta
\]
and from above,
\[
|\mu_1 F\left(\mu_1^{-1} M \frac{M}{\|M\|}\right) - F^*(\frac{M}{\|M\|})| \leq \epsilon.
\]
Since $\mu^{-1} \frac{M}{\|M\|} = \mu^{-1} M$, and using the fact that $F^*$ is homogeneous of degree one, we conclude the proof of the lemma. \hfill \Box

Next Lemma is instrumental in ensuring that solutions of (11) can be approximated by $F^*$-harmonic functions, in certain compact functional spaces.

Lemma 5.2 (Approximation Lemma). Let $u \in C(B_1)$ be a viscosity solution of
\[
F_{\mu}(D^2 u) = f(x) \quad \text{in } B_1
\]
with $\|u\|_{L^\infty(B_1)} \leq 1$, and assume that $A^{[1]}$ are satisfied. Given $\delta > 0$, there exists $\epsilon = \epsilon(\delta, d, F)$ such that, if
\[
\|f\|_{L^p(B_1)} \leq \epsilon \quad \text{and} \quad \mu < \epsilon,
\]
there exists $h \in C^{1,1}(B_{3/4})$, solution to
\[
F^*(D^2 h) = 0 \quad \text{in } B_{3/4},
\]
and
\[
\|u - h\|_{L^\infty(B_{1/2})} \leq \delta.
\]

Proof. We prove the lemma by the way of contradiction. Suppose its statement is false. Then, there exists $\delta_0 > 0$ and a sequence of functions $(u_j)_{j \in \mathbb{Z}}$ and $(f_j)_{j \in \mathbb{Z}}$ satisfying
\[
F_{\mu_j}(D^2 u_j) = f_j(x) \quad \text{in } B_1,
\]
with $\|u_j\|_{L^\infty(B_1)} \leq 1$ and $\|f_j\|_{L^p(B_1)} \leq \epsilon(\delta_0, d, F)$.
where $\mu_j$, $\|f_j\|_{L^p(B_1)} = o(1)$ and such that

$$\|u_j - h\|_{L^p(B_{1/2})} > \delta_0,$$

(13)

for every $h$ solution of (12).

On the other hand, because of standard results in elliptic regularity theory, we have that $(u_j)_{j\in\mathbb{N}} \subset C^{0,\alpha}(B_1)$. It means that, through a subsequence if necessary,

$$u_j \to u_{\infty} \quad \text{in the } C^{0,\alpha}_{\text{loc}}(B_1) \text{ topology.}$$

Furthermore, $F_{\mu_j}$ converges uniformly in compact sets of $S(d)$ to $F^*$ and $f_j \to 0$. By classical stability results in the theory of viscosity solutions,

$$F^*(D^2 u_{\infty}) = 0 \quad \text{in } B_{3/4}.$$

Setting $h \equiv u_{\infty}$ in (13) one is led to a contradiction for $j$ sufficiently large, and the proof is concluded. □

**Remark 5.1.** In the proof of Lemma 5.2 we have used a compactness argument based on the fact that $(u_j)_{j\in\mathbb{N}} \subset C^{0,\alpha}(B_1)$ for some $\alpha \in (0,1)$. Though this inclusion suffices for the purposes of the lemma, one can, in fact, establish better regularity results; see for example [26].

**Remark 5.2.** We notice that

$$u - h \in S\left(\frac{\Lambda}{h}, \Lambda, f(x) - F(D^2 h)\right);$$

it follows from the fact that $h \in C^{1,1}$; see [4, Proposition 2.13].

### 6 Proof of Theorem 1.1

In the sequel, we pursue a finer geometric-measure analysis of the sets $G_M$. This builds upon the approximation lemma to prove Proposition 2.1, yielding $W^{2,p}$-regularity for solutions of (1). Apart from the geometric tangential insight, the analysis carried in this section follows closely the one developed by Caffarelli in [3], see also [4, Chapter 7].

**Lemma 6.1.** Let $u \in C(B_1)$ be a viscosity solution of $F(D^2 u) = f(x)$ in $B_{3/4}$ such that $\|u\|_{L^\infty(B_{1/4})} \leq 1$ and $-|x|^2 \leq u(x) \leq |x|^2$ in $B_1 \setminus B_{3/4}$. Assume further that the assumptions of Lemma 5.2 hold true. Then,

$$|G_M(u, B_1) \cap Q_1| \geq 1 - \rho,$$

where $M > 0$ depends only on $d$ and the choice of $\epsilon$ in Lemma 5.2 will be determined by $\rho$.  

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Proof. Consider the function \( h \) from Lemma 5.2 restricted to \( B_{1/2} \). From standard results in elliptic regularity theory we have \( h \in C^2(B_{1/2}) \) and \( \| h \|_{C^1(B_{1/2})} \leq c(d) c_e \). Extend \( h \) outside \( B_{1/2} \) continuously, such that \( h = u \) in \( B_1 \setminus B_{3/4} \) and \( \| u - h \|_{L^\infty(B_{1/2})} \leq 2 \); therefore,

\[
-2 - |x|^2 \leq h(x) \leq |x|^2 + 2
\]

in \( B_1 \setminus B_{1/2} \). Hence, there exists \( N > 0 \) so that

\[
Q_1 \subset G_N(h, B_1).
\]  

\[ (14) \]

Set

\[
\omega \doteq \rho_0 (u - h),
\]

where \( \rho_0 > 0 \) is a constant to be determined later, depending only on the hypotheses of Lemma 5.2. Hence, \( \omega \) satisfies the assumptions of Lemma 4.2. By applying Lemma 4.6 one obtains

\[
|A_t(\omega, B_1) \cap Q_1| \leq c_2 t^{-\mu},
\]

for every \( t > 0 \). Therefore,

\[
|A_s(u - h, B_1) \cap Q_1| \leq C \rho^d c_2 s^{-\mu},
\]

for some constant \( C > 0 \). It follows that

\[
|G_N(u - h, B_1) \cap Q_1| \geq 1 - C \rho^d \geq 1 - \rho_0,
\]

by choosing appropriate constants in Lemma 5.2. Finally, because of (14), we get

\[
|G_N(u, B_1) \cap Q_1| \geq 1 - \rho_0.
\]

\[ \square \]

Lemma 6.2. Let \( u \in C(\Omega) \) be a viscosity solution of \( F(D^2 u) = f(x) \) in \( B_8 \sqrt{d} \). Assume further that the assumptions of Lemma 5.2 hold true. Then,

\[
G_1(u, \Omega) \cap Q_3 \neq \emptyset \implies |G_M(u, \Omega) \cap Q_1| \geq 1 - \rho,
\]

where \( M \) and \( \rho \) are as in Lemma 6.7.

Proof. Take \( x_1 \in G_1(u, \Omega) \cap Q_3 \); then, there exists an affine function \( L_1 \) so that

\[
-\frac{|x - x_1|^2}{2} \leq u(x) - L_1(x) \leq \frac{|x - x_1|^2}{2}
\]

in \( \Omega \). Define

\[
v(x) \doteq \frac{u(x) - L_1(x)}{c(d)}
\]

in \( \Omega \). Define
and take $c(d)$ large enough so that $\|v\|_{L^\infty(B_{\sqrt{d}})} \leq 1$ and

$$-|x|^2 \leq v(x) \leq |x|^2$$

in $\Omega \setminus B_{\sqrt{d}}$. By noticing that $v$ solves

$$\frac{1}{c(d)} F(c(d)D^2v) = \frac{f(x)}{c(d)}$$

in $B_{\sqrt{d}}$ and recurring to Lemma 6.1, one obtains $|G_M(v, \Omega) \cap Q_1| \geq 1 - \rho$. Therefore,

$$|G(c(d)M(u, \Omega) \cap Q_1)| \geq 1 - \rho.$$  \square

**Lemma 6.3.** Let $0 < \epsilon_0 < 1$ and $u$ be a viscosity solution of $F(\mu D^2u) = f(x)$ in $B_{\sqrt{d}}$. Assume that

$$\|u\|_{L^\infty(B_{\sqrt{d}})} \leq \epsilon \quad \text{and} \quad \|f\|_{L^d(B_{\sqrt{d}})} \leq \epsilon,$$

where $\mu$, $\epsilon < \epsilon_0$. Extend $f$ by zero outside $B_{\sqrt{d}}$ and set

$$A \doteq A_{\epsilon^{d+1}}(u, B_{\sqrt{d}}) \cap Q_1,$$

$$B \doteq (A_{\epsilon^d}(u, B_{\sqrt{d}}) \cap Q_1) \cup \{x \in Q_1 | m(f^d(x)) \geq (c_3M^d)^d\}.$$

Then,

$$|A| \leq \epsilon_0 |B|,$$

where $M > 1$ depends only on $d$ and $c_\epsilon$ and $c_3$, $\epsilon > 0$ depend only on $d$, $c_\epsilon$, $\epsilon_0$, $\lambda$ and $\Lambda$.

**Proof.** The proof is based on the Lemma 4.4. Notice that $|u| \leq 1 \leq |x|^2$ in $B_{\sqrt{d}} \setminus B_{\sqrt{d}}$. Hence, by setting $\Omega \equiv B_{\sqrt{d}}$ in Lemma 6.1 one obtains

$$|G_{M^{d+1}}(u, B_{\sqrt{d}}^d) \cap Q_1| \geq 1 - \rho.$$ 

To conclude the proof, we have to check that if $Q$ is a dyadic cube of $Q_1$ such that

$$|A_{\epsilon^{d+1}}(u, B_{\sqrt{d}}^d) \cap Q_1| \geq |A \cap Q_1| \geq \rho |Q_1|,$$

then $\bar{Q} \subset B$. Suppose not, i.e., there exists $x_1 \in \bar{Q}$ so that

$$x_1 \in \bar{Q} \cap G_{M^d}(u, B_{\sqrt{d}}),$$

and

$$m(f^d)(x_1) \leq (c_3M^d)^d.$$

Before we proceed, we notice that if $Q$ is a dyadic cube of $Q_1$, we have $Q = Q_{1/2}(x_0)$ for some $i \geq 0$ and $x_0 \in Q_i$. Define $\tilde{u}$ by (7) and let $\tilde{\Omega}$ be the image of $\Omega$ by

$$x = x_0 + \frac{1}{2^i}y,$$
where we take $\Omega$ to be $B_{8\sqrt{d}}$. It remains to check that $\tilde{u}$ satisfies the hypotheses of Lemma 6.2. Because $B_{8\sqrt{d}/2}(x_0) \subset B_{8\sqrt{d}}$, one has $B_{8\sqrt{d}/2} \subset \tilde{\Omega}$. Hence, $\tilde{u}$ is a viscosity solution of

$$G(D^2\tilde{u}) = \tilde{f},$$

where

$$G(D^2w) = \frac{1}{M^d} F(M^d D^2 w),$$

and

$$\tilde{f}(y) = \frac{1}{M^d} f \left(x_0 + \frac{1}{2^i} y \right).$$

Notice that $|x_1 - x_0| \leq 3/2^{i+1}$ yields $B_{8\sqrt{d}/2}(x_0) \subset Q_{19\sqrt{d}/2}(x_1)$. Hence, (17) implies

$$\|\tilde{f}\|_{L^d(B_{8\sqrt{d}})} \leq \frac{2^{2i}}{M^d} \int_{Q_{19\sqrt{d}/2}(x_1)} |f(x)|^d \, dx \leq C_d c_3^d,$$

by taking $c_3$ small enough, it follows that

$$\|\tilde{f}\|_{L^d(B_{8\sqrt{d}})} \leq \rho.$$

Finally, because of (16), we also have $G_1(\tilde{u}, \tilde{\Omega}) \cap Q_3 \neq \emptyset$. Therefore,

$$|G_M(\tilde{u}, \tilde{\Omega}) \cap Q_1| \geq (1 - \rho)|Q_1|,$$

that is,

$$|G_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap \bar{Q}| \geq (1 - \rho)|\bar{Q}|,$$

which leads to a contradiction in face of (15), finishing the proof.

Next we present the proof of Proposition 2.1.

**Proof of Proposition 2.1.** Let $M$ be as in Lemma 6.3 and take $\rho$ such that

$$\rho M^p = \frac{1}{2}.$$

For $k \geq 0$, set

$$\alpha_k = |A_M(u, B_{8\sqrt{d}}) \cap Q_1|, \quad \beta_k = \left| \left\{ x \in Q_1 : m(f^d)(x) \geq (c_3 M^d)^d \right\} \right|.$$

Lemma 6.3 implies that $\alpha_{k+1} \leq \rho(\alpha_k + \beta_k)$, which leads to

$$\alpha_k \leq \rho^k + \sum_{i=0}^{k-1} \rho^{k-i} \beta_i. \quad (19)$$

Clearly, $f^d \in L^{p/d}$; hence, $m(f^d) \in L^{p/d}$ and

$$\|m(f^d)\|_{L^{p/d}} \leq C \|f^d\|_{L^{p/d}} \leq C.$$
Therefore, by Lemma 4.1 we have
\[ \sum_{k \geq 0} M^k \beta_k \leq C. \] (20)

On the other hand, we have
\[ \mu_0(t) \leq \left| A_t(B_{1/2}) \right| \leq \left| A_t(B_{1/2}) \cap Q_1 \right|. \]

By recurring once again to Lemma 4.1, it suffices to verify that
\[ \sum_{k \geq 1} M^k \alpha_k \leq C, \]
which follows from (19) and (20), as follows:
\[
\sum_{k \geq 1} M^{2^k} \alpha_k \leq \sum_{k \geq 1} \left( \rho M^p \right)^k + \sum_{k \geq 1} \sum_{i=0}^{k-1} \rho^{k-i} M^{p(k-i)} M^i \beta_i
\]
\[
= \sum_{k \geq 1} 2^{-k} \left( \sum_{i \geq 0} M^i \beta_i \right) \left( \sum_{j \geq 1} 2^{-j} \right) \leq C.
\]
\[ \square \]

7 Estimates for variable coefficient equations

In this section, we comment on a generalization of Theorem 1.1 for fully nonlinear elliptic operators with variable coefficients, \( F : B_1 \times S(d) \to \mathbb{R} \). Herein we assume assumption A1 holds uniformly in \( x \in B_1 \), that is, for any \( x \in B_1 \) and any \( M \in S(d) \), there holds
\[ \lambda \| N \| \leq F(x, M + N) - F(x, M) \leq \Lambda \| N \|, \]
for every \( N \geq 0 \). We also assume \( F(x, 0) = 0 \). Following Caffarelli [3], we measure the oscillation of the coefficients of an elliptic operator \( F : B_1 \times S(d) \to \mathbb{R} \) around a point \( x_0 \in B_1 \) by
\[
\beta_F(x, x_0) := \sup_{M \in S(d) \setminus \{0\}} \frac{|F(x, M) - F(x_0, M)|}{\|M\|}.
\]

Now, for any \( \mu > 0 \) and any \( (x, M) \in B_1 \times S(d) \), we define \( F_\mu(x, M) := \mu F(x, \mu^{-1} M) \) and the recession function
\[ F^*(x, M) := \lim_{\mu \to 0} F_\mu(x, M), \]
provided it exists for all pair \( (x, M) \in B_1 \times S(d) \). While it is plain to check that
\[ \beta_F(x, x_0) = \beta_{F^*}(x, x_0), \]
the oscillation of the recession function can indeed be smaller than the original one. This is the case, for instance, when the operator satisfies \( F(x, M) = G(M) \), for all \( M \in S(d) \setminus B_{R} \) (see for instance construction from section 9). Indeed, the recession function \( F^*(x, M) \) is simply \( G^*(M) \); a constant coefficient operator.
Theorem 7.1. Let $F: B_1 \times \mathcal{S}(d) \to \mathbb{R}$ be uniformly elliptic, $p > d$, and $u$ be a viscosity solution of
\[ F(x, D^2 u) = f(x), \quad \text{in } B_1. \]
Assume $F^*(x, M)$ is uniformly convex with respect to $M$ and for some constants $\alpha > 0$ and $C > 0$, there holds
\[ \int_{B_r} \beta F^*(x, x_0)^d x \leq Cr^d \alpha, \quad (21) \]
for $r \leq r_0 \ll 1$ and all $x_0 \in B_1$. Then $u \in W^{2, p}(B_1/2)$ and there exists $C > 0$ so that
\[ \|u\|_{W^{2, p}(B_1/2)} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right). \]

Proof. The proof of Theorem 7.1 follows closely the reasoning from section 6. The key observation is that the version of Lemma 5.2 for operators with variable coefficients can be proven by similar analysis. Indeed, if $u \in C(B_1)$ is a viscosity solution of $F_{\mu}(x, D^2 u) = f(x)$ in $B_1$, with $\|u\|_{L^\infty(B_1)} \leq 1$, then given $\delta > 0$, there exists $\epsilon > 0$ such that, if $\|f\|_{L^p(B_1)} \leq \epsilon$ and $\mu < \epsilon$, we can find a function $h \in C^{2, \beta}(B_3/4)$, satisfying
\[ F^*(x, D^2 h) = 0 \quad \text{in } B_3/4, \quad (22) \]
and
\[ \|u - h\|_{L^\infty(B_1/2)} \leq \delta. \]
The same contradiction argument employed in the proof of Lemma 5.2 assures the existence of a solution $h$ that is $\delta$-close to $u$. That $h$ is of class $C^{2, \beta}(B_3/4)$ follows from hypothesis (21) combined with Caffarelli’s Schauder regularity theory. Owing to the existence of $h$, as well as its universal regularity theory beyond $C^{1, \alpha}$, one can continue the analysis of section 6 with minor modifications. For example, as regards the proofs of Lemmas 6.1 and 6.2, these remain the same. Lemma 6.3 must accommodate condition (21) as an assumption. The geometric-measure arguments from section 4 conclude the proof. □

As a final remark, condition (21) refers to a sort of Hölder continuity of the coefficients in $L^d$ sense. If necessary, one can relax such a hypothesis to weaker continuity assumption on $\beta F^*$, namely Dini continuity suffices. We do not want to enter in this issue here other than mentioning that indeed Theorem 7.1 yields novel information even for convex equations, as the oscillation of the coefficients of the recession function can be strictly less then the original operator.

8 A priori BMO type estimates

In this section, we discuss the borderline case $p = \infty$. It is well established that boundedness of the source function $f$ does not imply, in general, that the hessian is bounded,
even if $F$ is the laplacian operator. That is, sharp $W^{2,p}$-regularity estimates fail in the limit case $p = \infty$. Recall a function $g$ is said to belong to the $p$-BMO space if

$$
\sup_{\rho > 0} \frac{1}{\rho^d} \int_{B_{\rho}} |g(x) - \langle g \rangle_{\rho}|^p \, dx \leq C,
$$

for a constant $C > 0$ independent of $\rho$. Hereafter in the paper

$$
\langle g \rangle_{\rho} := \int_{B_{\rho}} g(x) \, dx.
$$

The ultimate goal of this section is to show that solutions of (1) have hessians in $p$-BMO($B_{1/2}$), for every $p > d$, provided $f$ is in $p$-BMO($B_1$) and $F = F^*$ outside a large ball $B_K \subset S(d)$. That is, in this section we work under the extra assumption:

**A 4.** There exists a constant $L \gg 1$, such that $F^* = F^*$ for all $M \in S(d)$, with $\|M\| \geq L$.

The recession function $F^*$ has a priori $C^{2,\alpha}$ interior estimates.

Assumptions A4 and A2 are typical of some geometric PDEs, where convexity of $c$-level sets are verified for $c \gg 1$. This is also the case of special approximating operators, as the ones we will build up in section 9. The main result of this section is then:

**Theorem 8.1.** Let $u$ be a viscosity solution of (1), assume A1-A4 are satisfied, and $f \in p$-BMO($B_1$), for some $p > d$. Then, $D^2u \in p$-BMO($B_{1/2}$) and

$$
\|D^2u\|_{p\text{-BMO}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{p\text{-BMO}(B_1)} \right).
$$

It is worth commenting that Theorem 8.1 closely relates to, and generalizes to some extent, developments reported in [6] (c.f. [6, Theorem A]). The proof of Theorem 8.1 will be divided into two steps. Initially we show the existence of good approximating paraboloids, $P_\mu$, and in the sequel we apply Theorem 1.1 for a normalized sequence of functions related to the problem. The analysis here is similar to the one put forward in [26] and so we just comment on the necessary modifications. Hereafter constants that depend only on the ellipticity, dimension and $C^{2,\alpha}$ interior estimates for $F^*$ will be called universal.

**Proposition 8.1.** Under the assumptions of Theorem 8.1, there exist two positive universal constants, $\mu_0$ and $r$, such that if $u$ is a solution of $F_\mu(D^2u) = f$, with $\mu + \|f\|_{L^r} \leq \mu_0$ and $\|u\|_{L^r} \leq 1$, then, there exists a paraboloid $P$, with universal controlled norm $\|P\| \leq C$ satisfying

$$
\sup_{B_r} |u - P| \leq r^2.
$$

**Proof.** The proof is based on compactness arguments, similar to the one carried out in [28] and in [26]. We omit the details. \qed

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**Proof of Theorem 8.1**

**Step 1** (Existence of approximating quadratic polynomials). Let \( u \) be a solution of \( F(D^2 u) = f \). An appropriate dilatation of \( u \), \( v(x) := \delta_1 u(\delta_2 x) \) verifies \( \|v\|_\infty \leq 1 \) and solves

\[
F_\mu(D^2 v) = \tilde{f},
\]

with \( \|\tilde{f}\|_{p-BMO(B_1)} + \mu \leq \mu_0 \). The choices for \( \delta_1 \) and \( \delta_2 \) depend only on \( \|u\|_\infty, \|f\|_{p-BMO(B_1)} \) and universal data. We will prove the \( p \)-BMO estimate for \( v \), which clearly gives the corresponding one for \( u \). The first step is to show, by finite induction process, the existence of quadratic polynomials

\[
P_k(x) := a_k + b_k \cdot X + \frac{1}{2} x^t M_k x,
\]

verifying

\[
F^*(M_k) = \langle \tilde{f} \rangle_1, \quad \text{(24)}
\]

\[
\sup_{B_k} |v - P_k| \leq r^{2k}, \quad \text{(25)}
\]

\[
|a_k - a_{k-1}| + r^{k-1} |b_k - b_{k-1}| + r^{2(k-1)} |M_k - M_{k-1}| \leq C r^{2(k-1)}. \quad \text{(26)}
\]

The constant \( r \) appearing in (25) and (26) is the one from Proposition 8.1. We set \( P_0 = P_{-1} = 0 \), and the first step \( k = 0 \) is immediately satisfied. Suppose \( k = 0, 1, \cdots, n \) have been checked, define the new function \( v_m : B_1 \to \mathbb{R} \) by

\[
v_m(X) := \frac{(v - P_m)(r^m X)}{r^{2m}}.
\]

From induction hypothesis, \( \|v_m\|_\infty \leq 1 \) and

\[
\mu F \left( \mu^{-1} \left( D^2 v_m + M_i \right) \right) = \tilde{f}(r^i x).
\]

From uniform convergence, the operators

\[
F_m(M) := F(M + M_m) \quad \text{and} \quad F^*_m(M) := F^*(M + M_m),
\]

are uniformly close. Also, since \( F^*(M_m) = \langle f \rangle_1 \), from the smallness condition on \( \|\tilde{f}\|_{p-BMO} \), the equation

\[
F^*_m(D^2 \xi) = \langle f \rangle_1
\]

satisfies the same a priori \( C^{2,\alpha} \)-regularity estimate as the original \( F^* \) and it is under the assumption of Proposition 8.1. Hence, we can find a quadratic polynomial \( \tilde{P} \) such that

\[
\|v - \tilde{P}\|_{L^{\infty}(B_1)} \leq r^2. \quad \text{(27)}
\]
Step 2 (BMO estimate). Define \( P_{i+1}(X) \) := \( P_i(X) + r^{2i} \hat{P}(r^{-i}X) \) and rescale (27) back to obtain the \( m + 1 \) step of induction. To conclude, for \( \rho > 0 \), choose \( m \) such that \( 0 < r^{m+1} < \rho \leq r^m \) and we apply Theorem 1.1 to \( v_m \) as to obtain

\[
\frac{1}{p^m} \int_{B_r} |D^2 v(y) - M_m|^p dy \leq \frac{1}{r} \cdot \frac{1}{p^m} \int_{B_{r^m}} |D^2 v(y) - M_m|^p dy
\]

\[
= \int_{B_{r^m}} |D^2 v_m(x)|^p dx
\]

\[
\leq C,
\]

and the proof of Theorem 8.1 is complete. \( \square \)

We conclude this section with a remark about the relation of inclusion involving the spaces \( p - \text{BMO} \), for different values of \( 1 < p < \infty \). Indeed, for \( 1 < p, q < \infty \), one always has

\[
p - \text{BMO}(B_1) = q - \text{BMO}(B_1) \quad \text{and} \quad \|u\|_{p^{-\text{BMO}(B_1)}} \propto \|u\|_{q^{-\text{BMO}(B_1)}}.
\]

(28)

This fact is a consequence of the John-Nirenberg inequality: assume \( u \in \text{BMO}(B_1) \) and let \( \rho > 0 \) be so that \( Q_\rho \subset B_1 \); then, there exist constants \( C > 0 \) and \( \alpha > 0 \), depending only on the dimension, for which

\[
\left| \left\{ x \in Q_\rho : |u(x) - \langle u \rangle_\rho| > \lambda \right\} \right| \leq C e^{-\alpha \|u\|_{\text{BMO}(B_1)}} |Q_\rho|.
\]

See [13]. This inequality yields

\[
\int_{Q_\rho} |u(x) - \langle u \rangle_\rho|^p dx \leq p \int_0^{\infty} \lambda^{p-1} C e^{-\alpha \|u\|_{\text{BMO}(B_1)}} d\lambda \left| Q_\rho \right|,
\]

which in turn, combined with

\[
p \int_0^{\infty} \lambda^{p-1} C e^{-\alpha \|u\|_{\text{BMO}(B_1)}} d\lambda \leq C \|u\|_{p^{-\text{BMO}(B_1)}}^p,
\]

establishes (28). Hence the condition \( p > d \) in Theorem 8.1 can be removed and the following a priori estimate:

\[
\|D^2 u\|_{q^-\text{BMO}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{p^-\text{BMO}(B_1)} \right),
\]

holds for any \( 1 < p, q < +\infty \), c.f. Theorem 8.1 and [13 Lemma 3].

9 \( W^{2,p} \)-density in the class of viscosity solutions

In this section, we show that \( W^{2,p} \) solutions are dense in the set of \( C^0 \)-viscosity solutions. We start by recalling the formalism of the class of all solutions of all \((\lambda, \Lambda)\)-uniform elliptic equations.
Definition 9.1. Let \( f \) be a function in \( B_1 \) and \( 0 < \lambda \leq \Lambda \) be positive numbers. We denote by \( S(\lambda, \Lambda, f) \) the set of all continuous functions \( u \) in \( B_1 \) such that \( \mathcal{P}^\lambda_\Lambda(D^2u) \geq f(x) \) in the viscosity sense. Analogously, \( \bar{S}(\lambda, \Lambda, f) \) denotes the set of all continuous functions \( u \) in \( B_1 \) such that \( \mathcal{P}^\lambda_\Lambda(D^2u) \leq f(x) \) in the viscosity sense. The class of all continuous viscosity solutions among all \((\lambda, \Lambda)\)-elliptic equations is defined as

\[
S(\lambda, \Lambda, f) := S(\lambda, \Lambda, f) \cap \bar{S}(\lambda, \Lambda, f).
\]

Heuristically, a function \( u \) belongs to \( S(\lambda, \Lambda, f) \) if it is a continuous viscosity solution of a variable coefficient equation

\[
F(x, D^2u) = f(x),
\]

where \( F \) is \((\lambda, \Lambda)\)-uniform elliptic. A cornerstone result in the theory of fully nonlinear elliptic equations is that functions in \( S(\lambda, \Lambda, f) \) are universally Hölder continuous. \[4\]

It is known that Hölder continuous estimates are the best available for solutions to measurable coefficient equations.

Our next result shows that any continuous viscosity solution can be approximated by a \( W^{2,p} \)-viscosity solution.

Theorem 9.1. Let \( f \in L^p(B_1) \), \( F : B_1 \times S(d) \to \mathbb{R} \) a \((\lambda, \Lambda)\)-elliptic operator and \( u \) continuous viscosity solution of \( F(x, D^2u) = f(x) \) in \( B_1 \). Given \( \delta > 0 \), there exists a sequence of functions \( \{u_j\}_{j \geq 1} \subset W^{2,p}_\text{loc}(B_1) \cap S(\lambda - \delta, \Lambda + \delta, f) \) that converges locally uniformly to \( u \).

Proof. We construct a sequence of operators \( F_j : B_1 \times S(d) \to \mathbb{R} \) as follows: given \( \delta > 0 \), consider the convex (extremal) operator

\[
L_\delta(M) := (\Lambda + \delta) \sum_{e_i > 0} e_i + (\lambda - \delta) \sum_{e_i < 0} e_i,
\]

where \( e_i \) are the eigenvalues of the matrix \( M \in S(d) \). In the sequel we set

\[
F_j(x, M) := \max\{F(x, M), L_\delta(M) - C_j\},
\]

where \( C_j \) is a divergent sequence of positive numbers to be determined a posteriori. From \((\lambda, \Lambda)\)-ellipticity of \( F \), we verify that

\[
F(x, M) \geq \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i \\
\geq \lambda \sum_{e_i > 0} e_i - \Lambda \|M\| \\
= L_\delta(M) - (\Lambda + \delta - \lambda) \sum_{e_i > 0} e_i - (\lambda - \delta) \sum_{e_i < 0} e_i - \Lambda \|M\| \\
\geq L_\delta(M) - (2\Lambda - \lambda + \delta) \|M\| \\
\geq L_\delta(M) - C_j,
\]

provided \( \|M\| \leq j \) and \( C_j := j(2\Lambda - \lambda + \delta) \). This shows that

\[
F_j = F \text{ in } B_j \subset S(d).
\]

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We now compute the recession function of $F^j$. For that, we look at the tangential path $F^j(x, M) := \mu F^j(x, M^{-1}) = \max\{F_\mu(x, M), L_0(M) - \mu C_j\}$. Since $F_\mu$ is $(\lambda, \Lambda)$-elliptic for all $\mu$, we can estimate

$$F_\mu(x, M) \leq \Lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i = L_0(M) - \delta \sum_{e_i > 0} e_i + \delta \sum_{e_i < 0} e_i \leq L_0(M) - \delta \|M\| \leq L_0(M) - \mu C_j,$$

provided $\|M\| \geq \frac{\mu C_j}{\delta}$. In particular, we conclude that $F^j(x, M) = L_0(M) - C_j$ outside the ball of radius $\sim C_j$ and that $(F^j)^* = L_0 -$ a convex operator, with a priori $C^{2,\alpha}$ estimates due to Evans and Krylov Theorem.

Fig. 2: This figure illustrates the construction of the operator $F^j$: the graph of an arbitrary $(\lambda, \Lambda)$-elliptic operator $F$ lies inside the $(\lambda, \Lambda)$-cone of ellipticity. By tilting a little bit the opening of the cone and placing its vertex at a very negative axis point, the boundary of the new cone, which contains the graph of $L_0 - C_j$, stays below the original $(\lambda, \Lambda)$-cone within $B_j$, and above in the complement of the ball of radius $\sim C_j$.

Thus, from Theorem 7.1 for each $j \geq 1$ fixed, the constructed operator $F^j$ has a priori $W^{2,p}$ interior estimates. That is, there exist constants $\kappa_j \geq 1$ such that, for any
viscosity solution \( v \) of
\[
F^j(x, D^2 v) = g(x),
\]
where holds
\[
\|v\|_{W^{2,p}(B_1/2)} \leq \kappa_j (\|v\|_{L^\infty(B_1)} + \|g\|_{L^p(B_1)}).
\]
Finally, we construct \( u_j \) to be the viscosity solution of the Dirichlet problem
\[
\begin{aligned}
F^j(x, D^2 u_j) &= f(x) \quad \text{in } B_1 \\
u_j &= u \quad \text{on } \partial B_1.
\end{aligned}
\]
From the discussion above, each \( u_j \) is locally in \( W^{2,p} \). Also, since \( F^j = F \) within \( B_j \), it follows by stability of viscosity solutions and uniqueness of the Dirichlet problem that, up to a subsequence, \( u_j \to u \) locally in the \( C^{0,\alpha} \)-topology. The proof of the Theorem is complete.

We notice that for constant coefficient equations, \( F(D^2 u) = f(x) \), Theorem 9.1 yields a sequence of \( W^{2,p} \) approximating functions that converge in the \( C^{1,\alpha}_{\text{loc}} \)-topology. One should compare Theorem 9.1 with the result from [12] and also from [16, 17].

This corpus of results has an important consequence for the theory of fully nonlinear elliptic PDEs. In fact, an effective tool in the study of viscosity solutions of homogeneous fully nonlinear equations is the mechanism devised by R. Jensen [11], known as inf-sup convolutions, or upper/lower \( \epsilon \)-envelope. The regularizing effects of this procedure gives semi-concave sub-solutions or semi-convex super-solutions. It now follows from Theorem 9.1 that, when aiming to establish a property closed under uniform limits, one can assume, with no loss, that solutions of non-homogeneous equations, \( F(x, D^2 u) = f(x) \) are locally of class \( W^{2,p} \); setting up a more comfortable starting point for further developments of the theory.

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