Higgs Bundles in Geometry and Arithmetic

Kang Zuo

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A Higgs bundle on a complex manifold $X$ is a pair $(E, \theta)$ consisting of a holomorphic vector bundle $E$ on $X$ and an $\text{End}(E)$-valued 1-form $\theta$ satisfying the condition $\theta \wedge \theta = 0$. The form $\theta$ is called the Higgs field. A major development in the field was made by N. Hitchin [Hit87] and C. Simpson [Sim88] in the so-called Hitchin–Simpson correspondence, a powerful tool in complex algebraic and analytic geometry. They followed the physicists in solving the Yang–Mills–Higgs partial differential equations on semistable Higgs bundles. Generalizations of this approach to more abstract settings in algebraic geometry have been highly successful [LSYZ19, EG17, Lan15, Ara19, KYZ20b, KYZ20a, YZ20].

In the project proposed here we introduce the notions of deformation Higgs bundle and Riemann–Finsler metric on the moduli space of polarized varieties. We also use the Higgs–de Rham flow in the $p$-adic setting. These are the key novelties in our program. These tools enable us to attack the following problems:

- **The Shafarevich conjecture on the finiteness of isomorphism classes of families of higher dimensional varieties.**
- **A folklore conjecture on the bigness of the fundamental group of moduli spaces of smooth projective varieties with semi-ample canonical line bundles.**

These conjectures are major current roadblocks preventing the progress in the Shafarevich program.

In the $p$-adic nonabelian Hodge Theory, we develop and explore further a theory of Higgs bundles on varieties over $p$-adic fields. Two directions of applications are

- **The Faltings’ $p$-adic Simpson correspondence.**
- **The construction of motivic local systems over $p$-adic curves in connection with Drinfeld’s work on the Langlands program.**

Below, we explain the above fundamental notions in our approach and the problems we will consider.

I. Higgs Bundles on Moduli spaces of Manifolds and the Shafarevich Program.

The Shafarevich conjecture over function fields of one variable, proved by Parshin and Arakelov [Ara71], states that: a. If a smooth algebraic curve $U$ parametrizes a non-isotrivial family of smooth projective curves, then $U$ is a hyperbolic curve, i.e. $U$ carries a metric of constant negative curvature. b. There are only finitely many non-isotrivial families parametrized by $U$ up to isomorphism.

In algebraic geometry, moduli spaces are spaces whose points parametrize solutions to a specific geometric problem. One of the most important moduli spaces is the moduli space $M_g$ parameterizing algebraic manifolds with semi-ample canonical line bundle and with a fixed Hilbert polynomial $h$. A family $f: V \to U$ of $n$-dimensional complex manifolds gives rise to a classifying map $\phi: U \to M_h$ and the family is called of maximal variation if $\dim U = \dim \phi(U)$. Kodaira and Spencer studied the variation of the complex structure of the family $f$ by introducing the notion of the Kodaira–Spencer map $\tau^{n,0}: T_U \to R^1 f_*T_{V/U}$. This map can be extended to a log-smooth compactification $f: X \to Y$, i.e., to the logarithmic Kodaira-Spencer map $\tau^{n,0}: T_Y(\log S) \to R^1 f_*T_{X/Y}(\log \Delta)$. Griffiths [Grif88a, Grif88b, Grif70] introduced the notion of polarized variation of Hodge structures on $U$ by examining how the Hodge structures on the Betti cohomology of the fibers of $f$ vary. Simpson [Sim88] introduced a basic notion in nonabelian Hodge theory, the so-called system of Hodge bundles $(E, \theta)$, to be the associated graded Higgs bundle of a polarized variation of Hodge structures. These theories are of fundamental importance in the study of the global geometry of families. The works of Viehweg–Zuo [VZ01, VZ02, VZ03, VZ05], combined the above-mentioned theories by extending the logarithmic Kodaira-Spencer map to the higher direct images of wedge products $T^q_{X/Y}(\log \Delta) := \wedge^q T_{X/Y}(\log \Delta)$ of the relative log tangent sheaf $\tau^{n-q,q}: T_Y(\log S) \otimes R^q f_*T^q_{X/Y}(\log \Delta) \to R^{q+1} f_*T^{q+1}_{X/Y}(\log \Delta)$ and constructed the logarithmic (graded) deformation Higgs bundle $(F, \tau) := (\bigoplus_q R^q f_*T^q_{X/Y}(\log \Delta), \bigoplus_q \tau^{n-q,q})$ for the family $f: X \to Y$. Moreover, Viehweg–Zuo constructed a comparison map $\rho: (F, \tau) \to (E, \theta) \otimes A^{-1}$, where $(E, \theta)$ is the Higgs bundle associated to the variation of Hodge structures on the middle cohomology of a new family $g: Z \to Y$ built from a cyclic covering of $X$ by taking roots out of sections of the relative pluricanonical linear system twisted by an anti-ample line bundle $A^{-1}$ on $Y$. Via the maximal non-zero iteration $\tau^n: S^n T_Y(\log S) \to \ker(\theta^{n-m-1,m+1}) \otimes A^{-1}$ of the Kodaira–Spencer map of $(F, \tau)$ (also known as the
Griffiths–Yukawa coupling), where $\ker(\theta^{p,q})$ is the kernel of the Higgs field $\theta^{p,q}$ which is semi-negative [Zuo00], one obtains a big subsheaf $A \subset S^m \Omega^1_{\mathbb{C}}(\log S)$. In the analytic setting, one defines a complex Finsler pseudometric $ds^2_{CF}$ on $T_X(-\log S)$ by taking the $m$-th root out of the product of the Hodge metric and the Fubini-Study metric on $\ker(\theta^{m-m-1,m+1}) \otimes \Omega^{m-1}$ via the map $\tau^m$. One shows that $ds^2_{CF}$ has the holomorphic sectional curvature bounded above by a negative constant. The negative holomorphic sectional curvature derived as above plays the same role as the negative holomorphic sectional curvature associated to horizontal period maps in Hodge theory. It implies Brody and Kobayashi hyperbolities of $U$ in [VZ03, PTW19] and [Den18a]. Also, the big Picard theorem was recently proven in [DLSZ19].

I.1. Finiteness of Families and Geometric Characterization of Non-Rigid Families. The Finiteness problem proposed by the Shafarevich program for the isomorphism classes of families of higher dimensional varieties may be reduced to two subproblems: the boundedness of families and the rigidity of families. The boundedness of families over a fixed base has been proven by various people, see for example [Fal83a, VZ01, KL10]. The rigidity problem is crucial for solving the conjecture on the finiteness and is subtle. Recall that a family $f : V \to U$ is called non-rigid if the family can be non-trivially extended over $U \times \Delta$, where $\Delta$ is a disc. One can construct non-rigid families of higher dimensional varieties by simply taking the product of two families with lower-dimensional fibers. Under certain additional assumptions we expect the rigidity holds true. As an initial step, a recent work of Javanpeykar–Sun–Zuo [JSZ20] shows that the $N$-pointed Shafarevich conjecture is true: let $C$ be a smooth projective curve of genus $g$ and $S$ a divisor on $C$ with given points $\{c_1, \ldots, c_N\} \subset C \setminus S$. Then there are only finitely many isomorphism classes of families $f : X \to C$ of $n$-folds with semi-ample canonical line bundles and bad reduction at $S$ and with the fixed isomorphism classes of the fibers over $\{c_1, \ldots, c_N\}$ for $N \geq 2\left( 2g - 2 + \#(S) \right)$. Currently, Javanpeykar-Lu-Sun-Zuo are making new progress on this type of problems and we expect that the one-pointed Shafarevich conjecture holds true. The proofs crucially rely on the deformation Higgs bundle and the complex Finsler metric whose curvature is bounded above by a negative constant. At the final step we study the rigidity problem in the original Shafarevich program. Given a non-rigid family $f : X \to C$ over a fixed base curve $C$ and with a fixed degeneration locus $S_C$, the boundedness of families over a fixed base with a fixed degeneration locus implies that $f$ extends to a non-trivial family over a product space $f : X \to C \times T$, where $T$ is a projective curve and with the degeneration locus $S_C \times T \cup C \times S_T$.

Problem 1. a. Show that the deformation Higgs bundle for the family $f : X \to C \times T$ with the projections $p : C \times T \to C$ and $q : C \times T \to T$ decompose in the following form $(F, \tau) = p^*(F, \tau)_C \otimes q^*(F, \tau)_T \otimes (F, \tau)^\prime$, where $(F, \tau)_C$ and $(F, \tau)_T$ are graded logarithmic Higgs bundles over $(C, S_C)$ and $(T, S_T)$. b. Show that a family $f : X \to C \times T$ of canonically polarized manifolds over a product base with maximal variation decomposes as the product of two families $f_C : X_C \to C$ and $f_T : X_T \to T$ with lower dimensional fibers after taking a ramified cover of $X$ (up to blow-ups and blow-downs). The statement in Problem 1.b. should hold in general only for fibers of general type as Faltings has found examples of non-rigid families of abelian varieties whose generic fibers are simple abelian varieties. The investigation of non-rigid families of hypersurfaces in projective spaces will be the first non-trivial test for Problem 1. In [VZ05], Viehweg–Zuo have constructed non-rigid families of hypersurfaces in $\mathbb{P}^n$. For simplicity, here we consider hypersurfaces in $\mathbb{P}^3$ of degree $d \geq 4$. Any set of $d$ (pairwise) distinct points in $\mathbb{P}^1$ corresponds to a homogenous polynomial in two variables of degree $d$ which has this set as its set of roots. For any two such homogeneous polynomials $f_d(x_1, x_2)$ and $g_d(y_1, y_2)$, the equation $f_d(x_1, x_2) + g_d(y_1, y_2) = 0$ defines a smooth hypersurface $X_{f_d,g_d}$ in $\mathbb{P}^3$. By varying the roots of $f_d$ and $g_d$ (and thus varying $f_d$ and $g_d$), one obtains a family $f : X \to \Sigma_d \times \Sigma_d$ in $\mathbb{P}^3$, where $\Sigma_d$ is the moduli space of $d$ distinct points in $\mathbb{P}^1$. Viehweg-Zuo have observed that the family $f$ is, up to ramified covers and blow-ups and blow-downs, the product of the family $C_{f_d} \to \Sigma_d$ defined by the equation $x_3^d = f_d(x_1, x_2)$ and the family $C_{g_d} \to \Sigma_d$ defined by $y_3^d = g_d(y_1, y_2)$. In fact we have enough evidence to be convinced that

Conjecture 2. Any smooth fiber of a non-rigid family of hypersurfaces in $\mathbb{P}^3$ of degree $d$ is defined by an equation of the form $f_d(x_1, x_2) + g_d(y_1, y_2) = 0$ up to a projective transformation.

Viehweg-Zuo have found a close relationship between the relative graded Jacobian ring and the system of Hodge bundles by showing that the graded pieces of Jacobian ring of a hypersurface $X \subset \mathbb{P}^n$ coincide with the eigenspaces of the $\mathbb{Z}/2\mathbb{Z}$-action on the middle-dimensional Hodge cohomology of the cyclic cover of $\mathbb{P}^n$ ramified over $X$ as modules together with the Kodaira-Spencer multiplication. Using Deligne’s result on tensor decomposition of a polarized variation of Hodge structures over a product base one shows that a decomposition of the base of a family $f : X \to C \times T$ of hypersurfaces in the projective space $\mathbb{P}^n$ leads to a tensor product decomposition of the relative Jacobian ring as modules together with the Kodaira-Spencer multiplication. Moreover, Problem 1.a. has a positive answer in this case. It is well-known that the Jacobian ring of a smooth hypersurface determines the isomorphism type of the hypersurface itself. We hope
that the above decomposition of the relative Jacobian ring (as a module) as well as the decomposition of the deformation Higgs bundle will lead to some significant geometric consequences on the fibers, as in Conjecture 2. Motivated by Problem 1.b. we pose the following problem which gives a criterion for rigidity.

**Problem 3.** Show that any family \( f: V \to U \) of maximal variation and containing a fiber with the generically ample cotangent sheaf is rigid.

### I.2. Bigness of Fundamental Groups of Moduli Spaces of Projective Manifolds.

Milnor [Mil68] introduced a growth function \( \ell \) associated to a finitely generated group \( G \) as follows: for each positive integer \( s \) let \( \ell(s) \) be the number of distinct group elements which can be expressed as words of length \( \leq s \) with a fixed choice of generators and their inverses. Milnor proved that \( \pi_1(M) \) of a compact Riemannian manifold \( M \) with all Riemannian sectional curvatures less than zero has exponential growth, i.e., \( \ell(s) \geq a^s \) for some \( a > 1 \). The proof relies on G"unther's volume comparison theorem on the exponential growth of the volume of the geodesic ball on the universal cover \( \tilde{M} \). We propose the following problem.

**Problem 4.** Let \( U \) be a base space parameterizing polarized manifolds with semi-ample line bundle and with maximal variation. Prove that \( \pi_1(U) \) grows at least exponentially.

Similar to the approach of proving the complex hyperbolicity on \( U \) we would like to construct a Riemann-Finsler metric on \( U \) via the maximal non-zero iteration of Kodaira–Spencer map, whose curvature has certain negativity properties. Wu–Xin generalized Milnor’s theorem to the Finsler setting by showing that the G"unther’s volume comparison theorem holds for Riemann-Finsler metric with negative flag curvature (an analogue of the Riemannian sectional curvature in Finsler geometry) [BCS00, WX07]. In our situation the complex Finsler metric (the sum of several such metrics from different cyclic covers) naturally induces a non-degenerated Riemann-Finsler metric \( ds_{\text{EF}}^2 \) on \( U \). We are aware that in general the Riemannian curvature decreasing principle does not hold true for real submanifolds. Very recently, together with S. Lu and R.R. Sun, we observed that the fact that pluriharmonicity of the composition of the horizontal period map with the projection to the symmetric space of non-compact type implies decreasing Riemannian curvature still holds true in a weak form and we expect that this weak form of the negative sectional curvature property can show G"unther’s volume comparison inequality and get the solution of Problem 4 by Milnor’s original argument.

### II. Higgs Bundles in \( p \)-adic Nonabelian Hodge Theory and Motivic Local Systems.

In the \( p \)-adic world, Fontaine and Faltings introduced the notion of Fontaine–Faltings module \( MF^\nabla(X) := \{(V, Fil^s, \nabla, \Phi)\} \) over a smooth proper scheme \( X \) over a \( p \)-adic number ring \( O_{K_p} \) (more precisely on the \( p \)-adic completion \( \hat{X} \)). A Fontaine–Faltings module consists of a filtered de Rham bundle endowed with a relative Frobenius \( \Phi \). The Fontaine-Laffaille-Faltings \( p \)-adic Riemann-Hilbert correspondence produces a functor \( \mathbb{D} \) sending a Fontaine–Faltings module to a crystalline representation of \( \pi_1^\text{et}(X_K) \) ([FL82], [Fal89]). Faltings [Fal05] has proposed a \( p \)-adic Simpson correspondence to describe the so-called generalized representations of the geometric fundamental group \( \pi_1^\text{et}(X_{\overline{\mathbb{Q}}_p}) \) in terms of Higgs bundles. Faltings asked whether semi-stable Higgs bundles \( (E, \theta) \) with trivial Chern classes \( c_i(E) = 0 \) correspond precisely to genuine representations. Scholze [Sch13] has made exciting progress in \( p \)-adic Hodge theory. He introduced the proétale site \( X_{\text{proet}} \) of \( X \) as a refinement of the usual étale topology on \( X \); using this topology, he defined the Rham representations as being associated to an \( \mathcal{O}_{\mathbb{B}_{dR}} \)-module equipped with a Hodge filtration. By taking the category \( \mathcal{H}IG \) of semi-stable graded Higgs bundles over \( \mathcal{X} \), Lan–Sheng–Zuo [LSZ19] introduced the notions Higgs–de Rham flow and periodic Higgs bundle, a \( p \)-adic analogue of Yang–Mills–Higgs equation over the complex numbers. They established a \( p \)-adic Simpson correspondence between the category of stable periodic Higgs bundles \( \mathcal{H}IG^{\text{per}} \subset \mathcal{H}IG \) and the category \( \mathcal{R}EP^{\text{cryt}} \) of crystalline representations of \( \pi_1^\text{et}(X_K) \) when \( X \) has good reduction. The construction relies on the fundamental work of Ogus–Vologodsky [OV07] of Cartier transform in characteristic \( p \), and the work of Simpson on the existence of graded semi-stable Hodge filtration [Sim10]. The notion of Higgs–de Rham flow has already applications in both algebraic and arithmetic geometry, see recent works by [LSYZ19, EG17, Lan15, Ara19, KYZ20b, KYZ20a].

#### II.1. \( p \)-adic Simpson Correspondence and Semi-Stable Higgs Bundles.

For simplicity we just work with a smooth projective curve \( X \) over \( W(k) \). Given a Higgs bundle \( (E, \theta) \) over \( X \) with nilpotent Higgs field of degree \( < p \). By Faltings’ Simpson correspondence for integral generalized representation of \( \pi_1^\text{proet} \). We believe that the heart part for proving Faltings conjecture lies in the characteristic \( p \) case, and the stability of Higgs bundle plays the crucial role.

#### II.1.1. Faltings’ Conjecture over Characteristic \( p \) and Preperiodic Higgs de Rham Flow.

Given a semi-stable Higgs bundle \( (E, \theta)_0 \) on \( X_0 = X \pmod{p} \) of degree zero and with a nilpotent Higgs field of degree \( < p \), we obtain a preperiodic Higgs de Rham flow by running semi-stable Higgs de Rham flow
over $X_k$ with the initial $(\bar{E}, \theta)_0$

where the Hodge filtrations in the semi-stable de Rham bundles are grading semi-stable. The Lan-Sheng-Zuo’s functor $\alpha$ over $k$ is defined by sending the direct sum of the periodic part of the flow with the $f$-periodic isomorphism $\Psi$ to the direct sum of their inverse Cartier transform endowed with the Hodge filtration $(\bigoplus_{i=0}^{f-1}(\bar{E}, \theta)_{i+1}), \Psi) \xrightarrow{\alpha} (\bigoplus_{i=0}^{f-1}(\bar{V}, \nabla, \bar{F}t^i)_{i+1}, C^{-1}(\Psi))$, which is indeed a $p$-torsion Fontaine-Faltings module, endowed with a natural $\mathbb{F}_{p^f}$-action, and therefore corresponds to a $\mathbb{F}_{p^f}$-crystalline representation $\mathbb{L}^{\text{cry}}_{(E, \theta)_0}$. This correspondence restricted to the geometric $\pi_1$ coincides with Faltings’ Simpson correspondence. For a morphism $\sigma : \mathcal{Y} \to \mathcal{X}$ and a small Higgs bundle $(E, \theta)/\mathcal{X}$, Faltings introduced the notion of the twisted pull-back $\sigma^*(E, \theta)$ of $(E, \theta)$ in [Fal05], which is compatible with the pull-back of generalized representations under Faltings’ Simpson correspondence.

**Conjecture 5.** Given a preperiodic Higgs-de Rham flow as above, there exists a morphism $\sigma : \mathcal{Y} \to \mathcal{X}$ such that $\sigma^*(E, \theta)_0 \simeq \sigma^*(E, \theta)_e$.

**Remark 6.** Conjecture 5 implies that a semi-stable nilpotent Higgs bundle over $X_k$ of degree zero corresponds to an $\mathbb{F}_{p^f}/(\pi_{p^{e-1}(p-1)})$-geometric local system.

**II.1.2. Lifting of Higgs Bundles and Local Systems at Truncated Level.** Given a Higgs bundle $(E, \theta)$ over $\mathcal{X}_{O_{C,p}}$ such that $(E, \theta)$ over $X_{O_{C,p}/O_{C,p}}$ corresponds to a genuine geometric local system $\mathbb{L}$, by taking a Galois base change $\sigma : \mathcal{Y}_{O_{C,p}} \to \mathcal{X}_{O_{C,p}}$, étale on the generic fiber killing $\mathbb{L}$, we may assume Faltings’ twisted pull-back $\sigma^*(E, \theta)$ on $Y_{O_{C,p}/O_{C,p}}$ is isomorphic to the trivial Higgs bundle $(\mathcal{O}^*, 0)$. One can measure the difference between $\sigma^*(E, \theta)$ and $(\mathcal{O}^*, 0)$ over $Y_{O_{C,p}/O_{C,p}}$ by a class $\eta_{\text{Hig}}$ in $H^1_{\text{Hig}}(Y_{O_{C,p}/O_{C,p}}, \text{End}(\mathcal{O}^*, 0))$ as the torsor space for lifting Higgs bundles from modulo $p$ to modulo $p^2$ over the trivial Higgs bundle.

**Problem 7.** Work out the Faltings’ $p$-adic Simpson correspondence between the torsor space for lifting Higgs bundles and the torsor space for lifting geometric local systems and show that $(E, \theta)$ corresponds to a genuine geometric local system $\mathbb{L}$.

Given a generic punctured hyperbolic curve $(C, S)/\mathbb{F}_q$, in [LSYZ19] we showed that there exists a so-called canonically lifted $(C, S)/W(\mathbb{F}_q)$ regarding that the uniformizing Higgs bundle $\theta : L \xrightarrow{\sim} L^{-1} \otimes \Omega^1_c(\log S)$ corresponds to a crystalline local system on $(C \setminus S)/W(\mathbb{F}_q)[1/p]$, where $L$ is the logarithmic Theta characteristic of $(C, S)$. This theorem shall be considered as the Higgs bundle incarnation of Mochizuki’s $p$-adic Teichmüller theory [Moc96]. The techniques developed in attacking problems in II. 1.1. and II.1.2 shall allow us to work out the far-reaching generalization for $p$-adic Teichmüller theory for arbitrary $p$-adic punctured hyperbolic curves.

**Problem 8.** (p-adic Teichmüller theory) Show that the uniformizing Higgs bundle over a $p$-adic punctured hyperbolic curve $(C, S)/K_p$ corresponds to a geometric $\mathbb{C}_p$-local system $\rho^{\text{uni}}$, which is invariant under the natural $\text{Gal}(\bar{K}_p/K_p)$-action. Furthermore, if $(C, S) \neq (C', S')$ then $\rho^{\text{uni}} \neq \rho'^{\text{uni}}$.

**II.1.3. Galois-Action on Geometric $\pi_1$ and a $p$-adic Analogue of $\mathbb{C}^*$-action on Higgs Fields.** The action of Galois group $\text{Gal}(\bar{K}_p/K_p)$ on $X_{\bar{K}_p}$ induces a natural action on the category of generalized representations. In [YZ20] by carefully checking the construction of Faltings’ Simpson correspondence one finds that the corresponding action on the category of Higgs bundles can be basically described as the Galois action on the usual category of Higgs bundles with the extra action on Higgs fields twisted by a 1-cocycle induced by the element $\xi$ in the Fontaine periods ring $B^{+}_{IR}$. Consequently, we show that a generalized representation corresponding to a graded Higgs bundle over $K_p$ is $\text{Gal}(\bar{K}_p/K_p)$-invariant and conversely, a Higgs bundle corresponding to a Galois-invariant generalized representation is precisely graded and defined over a finite extension of $K_p$ for rank $E \leq 2$, and with a nilpotent Higgs field in the general case. Together with the expected results from II.1.1. and II.1.2. we shall get a better understanding of the Galois action on the geometric $\pi_1$ and, in particular, Grothendieck’s anabelian geometry.
II.2. Rank-2 $p$-adic Motivic Local Systems on a Punctured Curve and Langlands Correspondence over Function Fields. Let $(C,S)$ be a smooth projective curve over complex numbers together with a set of $n$ punctures $S$. Simpson shows that the category of abelian schemes over $C$ of $GL_2$ type with bad reductions over $S$ is equivalent to the category of rank-2 motivic Higgs bundles on $C$ with logarithmic-parabolic structure on $S$. We plan to study rank-2 motivic Higgs bundles using $p$-adic nonabelian Hodge theory and Langlands correspondence over function fields in char. $p$. Let $(C,S)_{W(k)}$ denote a smooth projective curve with a set of $n$ punctures $S$ over $W(k)$, and $\mathcal{HIG}$ denotes the moduli space of rank-2 graded stable Higgs bundles over $(C,S)$ with log-parabolic structure over $S$. Clearly a motivic Higgs bundle on an arithmetic scheme must be periodic over all unramified places. Conversely, we ask:

Problem 9. Study those rank-2 log periodic Higgs bundles over $(C,S)/W(F_q)$, which are motivic.

In [SYZ17] one considers rank-2 stable logarithmic Higgs bundles over the projective line $(P^1,S)$ with $n$ punctures and with the parabolic structure attached to one puncture in $S$ of weight $\{1/2,1/2\}$. The moduli space $\mathcal{HIG}$ contains irreducible components of dimensions $n-3,n-5,n-7,\ldots$. Sun–Yang–Zuo constructed infinitely many $GL_2(Z_p)$-crystalline local systems over the generic fibre $(P^1 \setminus S)_{\hat{Q}_p}$ by running Higgs–de Rham flow on $\mathcal{HIG}$ and showing that the set of periodic Higgs bundles over $Z_p$ is Zariski dense in the components of $\mathcal{HIG}$ of maximal dimension. That should be also true for a general punctured hyperbolic curve and follow from the deformation theory developed in [KYZ20c]. We raise:

Problem 10. Given a punctured hyperbolic curve $(C,S)/W(F_q)$, show that the set of periodic Higgs bundles over $W(F_{p^n})$, $n \in \mathbb{N}$ is Zariski dense in $\mathcal{HIG}$. Are they motivic?

Problem 10 is a $p$-adic analogue of a very recent program of Esnault-Kerz [EK20]. They conjecture that motivic points are Zariski dense in the character variety of $\ell$-adic local systems.

For $S = \{0,1,\infty,\lambda\}$, the moduli space $\mathcal{HIG}$ parameterizes rank-2 graded stable Higgs bundles over $(P^1,S)/W(k)$ of the form $(O \oplus O(-1), O \xrightarrow{\theta \in \mathbb{P}^1} (−1) \oplus \Omega^1_{P^1}(\log S))$ and $\theta$ has a single zero ($\theta_0 \in P^1$). One makes an identification $\mathcal{HIG} = P^1$ by sending $(E,\theta) \in \mathcal{HIG}$ to $\theta_0 \in P^1$. We take the elliptic curve $(E_\lambda,y^2 = x(x-1)(x-\lambda), 0 := \infty)$ as the double cover of $P^1$ ramified over $S$. Recently Krishnamoorthy, Yang and Zuo are making new progress. We show that Problem 10 has a positive answer for $(P^1,\{0,1,\infty,\lambda\})$ for $p < 50$ and $E_\lambda$ is a supersingular elliptic curve.

Conjecture 11. ([SYZ17]) The set of periodic Higgs bundles over $(P^1,\{0,1,\infty,\lambda\})/W(F_q)$, $n \in \mathbb{N}$ carries a ”group law” from the elliptic curve $E_\lambda$, i.e. a Higgs bundle $(E,\theta) \in \mathcal{HIG}(W(F_{p^n}))$ is $f$-periodic if and only if the preimage of the zero of the Higgs field $\pi^{-1}(\theta_0) \in E_\lambda$ is a $(p^f \pm 1)$-torsion point.

Langlands Correspondence over Function Fields of Characteristic $p$. Given a punctured curve $(C,S)_{W(F_q)}$, and picking up a field isomorphism $\sigma: \hat{Q}_p \to \hat{Q}_q$, by Deligne’s conjecture on $p$-to-$l$ companions solved by Abe [Abe13] and via overconvergent $F$-isocrystals one obtains an inclusion from the set of (modulo $p$) stable logarithmic periodic Higgs bundles $\mathcal{HIG}^{perf} \hookrightarrow \mathcal{R}E\mathcal{P}^{\ell-adic}$, the set of irreducible $GL_2(\hat{Q}_l)$-local systems over $(C \setminus S)_{\hat{Q}_l}$. By Drinfeld [Dri83] the $\ell$-adic local system corresponding to $(E,\theta)$ together with its full companions will be realized by an abelian scheme over $(C \setminus S)_{\hat{Q}_l}$ of $GL_2$-type. One studies the existence of grading stable Hodge filtrations attached to realizations of $F$-crystals over $(C,S)_{W(F_q)}$, applies the logarithmic Grothendieck-Messing deformation theorem and shows that this abelian scheme lifts over $W(F_{p^n})$ for the case $(P^1,\{0,1,\infty,\lambda\})$. In general, we pose

Problem 12. Find conditions for abelian schemes of $GL_2$-type over $(C,S)_{\hat{Q}_l}$ arising from Langlands correspondence for rank-2 $\ell$-adic local systems to be liftable over $W(F_q)$.

$p$-adic Nonabelian Hodge Theory and Dynamical Systems over $\mathbb{F}_p$. Conjecture 11 is motivated by Sun–Yang–Zuo’s work [SYZ17]. The Higgs–de Rham flow on $(P^1,\{0,1,\infty,\lambda\})$ induces a self map $\psi_\lambda$ on the moduli space $\mathcal{HIG}(\simeq \mathbb{P}_{\hat{Q}_l})$ of rank-2 logarithmic Higgs bundles with the parabolic structure of the weight $(1/2,1/2)$ attached to a puncture. As is explained in II.1.1, a point Higgs bundle $z \in \mathcal{HIG}$ corresponds to a crystalline local system on $\mathbb{P}^1 \setminus \{0,1,\infty,\lambda\}$ iff $z$ is a periodic point of $\psi_\lambda$.

Conjecture 13. The self-map $\psi_\lambda$ coincides with the multiplication by $p$ map on the elliptic curve $\pi: E_\lambda \to \mathbb{P}^1$ as the double cover ramified on $S$ via $\pi$
We confirmed this conjecture via explicit computation for cases that the characteristic $p \leq 50$, see [SYZ17].

Note that self-maps on $\mathbb{P}^1$ descended from $n$-multiplication by maps on an elliptic curve $\pi : E \to \mathbb{P}^1$ ($n \in \mathbb{Z}$) are called Lattès maps ([Mil06]). If we vary $\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, we obtain a family of self-maps $\psi_{\lambda}$. In the setting of complex dynamics, there is a rigidity theorem for stable families of self-maps on $\mathbb{P}^1$ due to McMullen [Mc87]. It states that any such family is either of Lattès type (i.e. constructed from $n$-multiplication map on $E_\lambda$ by varying $\lambda$) or trivial (all its members are conjugate by Möbius transformations). We hope that the techniques in complex dynamics will inspire us to investigate the self-map $\psi_{\lambda}$ induced from $p$-adic nonabelian Hodge theory by deforming the parameter $\lambda$.

**Problem 14.** Study self maps arising from $p$-adic Higgs-de Rham flows in connection with the rigidity theorem of McMullen on dynamical systems on the complex projective line.

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3. Bibliography.

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