Scalar products of the open XYZ chain with non-diagonal boundary terms

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Abstract

With the help of the F-basis provided by the Drinfeld twist or factorizing F-matrix of the eight-vertex solid-on-solid (SOS) model, we obtain the determinant representations of the scalar products of Bethe states for the open XYZ chain with non-diagonal boundary terms. By taking the on shell limit, we obtain the determinant representations (or Gaudin formula) of the norms of the Bethe states.

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1 Introduction

One of the most challenging and important problems in theory of quantum integrable models, having obtained the spectrum and eigenstates of the corresponding Hamiltonians, is to construct exact and manageable expressions of correlation functions (or scalar products of Bethe states) \[1,2\]. This problem is also fundamental to enlarge the range of applications of these models in the realm of condensed matter physics, quantum information theory. There are two approaches in the literature for computing the correlation functions of a quantum integrable model. One is the vertex operator method (see e.g. \[3,4,5,6,7,8\]) which works only on an infinite lattice, and another one is based on the detailed analysis of the structure of the Bethe states \[9,10\]. As for the second approach which usually works for models with finite size, it is well known that in the framework of quantum inverse scattering method (QISM) \[2\] Bethe states are obtained by applying pseudo-particle creation operators to reference state (pseudo-vacuum). However, the apparently simple action of creation operators is plagued with non-local effects arising from polarization clouds or compensating exchange terms on the level of local operators. This makes the direct calculation of correlation functions of models with finite size challenging.

Progress has recently been made on the second approach with the help of the Drinfeld twists or factorizing F-matrices \[11\]. Working in the F-basis provided by the F-matrices, the authors in \[12,13\] managed to calculate the form factors and correlation functions of the XXX and XXZ chains with periodic boundary condition (or closed chains) analytically and expressed them in determinant forms. Then the determinant representation of the scalar products and correlation functions of the supersymmetric t-J model \[14\] and its q-deformed model \[15\] with periodic boundary condition was obtained within the corresponding F-basis given in \[16\].

It was noticed \[17,18\] that the F-matrices of the closed XXX and XXZ chains also make the pseudo-particle creation operators of the open XXX and XXZ chains with diagonal boundary terms polarization free. This is mainly due to the fact that the closed chain and the corresponding open chain with diagonal boundary terms share the same reference state \[19\]. However, the story for the open XXZ chain with non-diagonal boundary terms is quite different \[20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35\]. Firstly, the reference state (all spin up state) of the closed chain is no longer a reference state of the open chain with
non-diagonal boundary terms [21, 22, 26]. Secondly, at least two reference states (and thus two sets of Bethe states) are needed [36] for the open XXZ chain with non-diagonal boundary terms in order to obtain its complete spectrum [37, 38]. As a consequence, the F-matrix found in [12] is no longer the desirable F-matrix for the open XXZ chain with non-diagonal boundary terms. With the help of the F-matrices of SOS models given in [39, 40], the domain wall (DW) partition function of the six-vertex model with a non-diagonal reflecting end [41, 42] (or the DW partition function of the trigonometric SOS model with reflecting end [43]), the explicit and completely symmetric expressions [40] of the two sets of Bethe states and the determinant representation [44] of scalar products of these Bethe states for the open XXZ chain with non-diagonal boundary terms have been obtained.

It is well known that among solvable models elliptic ones stand out as a particularly important class due to the fact that most trigonometric and rational models can be obtained from them by certain limits. In this paper, we focus on the most fundamental elliptic model—the XYZ spin chain [45] whose trigonometric/rational limit gives the XXZ/XXX chain. Basing the work [39] we have succeeded in obtaining the DW partition function of the eight-vertex model with a non-diagonal reflecting end [46] (or the DW partition function of the eight-vertex SOS model with reflecting end [47]) and the explicit and completely symmetric expressions of the two sets of Bethe states of the open XYZ chain with non-diagonal boundary terms [48]. In this paper, we shall investigate the determinant representations of scalar products of the Bethe states of the open XYZ chain with non-diagonal boundary terms specified by the non-diagonal K-matrices (2.14) and (2.16) given by [49, 50].

The paper is organized as follows. In section 2, we briefly describe the open XYZ chain with non-diagonal boundary terms, and introduce the pseudo-particle creation operators and the two sets of Bethe states of the model. In section 3, we introduce the face picture of the model and express the scalar products in terms of the operators in the face picture. In section 4, we use the F-matrix of the eight-vertex SOS model to construct the completely symmetric and polarization free representations of the pseudo-particle creation/annihilation operators in the F-basis. In the face picture, with the help of the F-basis, we obtain the determinant representations of the scalar products of Bethe states and the determinant representations (or Gaudin formula) of the norms of the Bethe states in section 5. In section 6, we summarize our results and give some discussions.
2 The inhomogeneous spin-$\frac{1}{2}$ XYZ open chain

Let us fix $\tau$ such that $\text{Im}(\tau) > 0$ and a generic complex number $\eta$. Introduce the following elliptic functions

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (u, \tau) = \sum_{n=-\infty}^{\infty} \exp \left\{ i\pi \left[ (n+a)^2 \tau + 2(n+a)(u+b) \right] \right\},$$

$$\theta^{(j)}(u) = \theta \left[ \begin{array}{c} \frac{1}{2} - \frac{j}{2} \\ \frac{1}{2} \end{array} \right] (u, 2\tau), \quad j = 1, 2; \quad \sigma(u) = \theta \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (u, \tau).$$

The $\sigma$-function satisfies the so-called Riemann identity:

$$\sigma(u + x)\sigma(u - x)\sigma(v + y)\sigma(v - y) - \sigma(u + y)\sigma(u - y)\sigma(v + x)\sigma(v - x) = \sigma(u + v)\sigma(u - v)\sigma(x + y)\sigma(x - y),$$

which will be useful in the following. Moreover, for any $\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 \in \mathbb{Z}_2$, we can introduce a function $\sigma_{\alpha}(u)$ as follow

$$\sigma_{\alpha}(u) = \theta \left[ \begin{array}{c} \frac{1}{2} + \frac{\alpha_1}{2} \\ \frac{1}{2} + \frac{\alpha_2}{2} \end{array} \right] (u, \tau), \quad \alpha_1, \alpha_2 \in \mathbb{Z}_2.$$  

The above definition implies the identification $\sigma_{(0,0)}(u) = \sigma(u)$.

Let $V$ be a two-dimensional vector space $\mathbb{C}^2$ and $\{\epsilon_i | i = 1, 2\}$ be the orthonormal basis of $V$ such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. The well-known eight-vertex model R-matrix $\overline{R}(u) \in \text{End}(V \otimes V)$ is given by

$$\overline{R}(u) = \begin{pmatrix} a(u) & d(u) \\ b(u) & c(u) \\ c(u) & b(u) \\ d(u) & a(u) \end{pmatrix},$$

The non-vanishing matrix elements are [45]

$$a(u) = \frac{\theta^{(1)}(u) \theta^{(0)}(u + \eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(0)}(\eta) \sigma(u + \eta)}, \quad b(u) = \frac{\theta^{(0)}(u) \theta^{(1)}(u + \eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(0)}(\eta) \sigma(u + \eta)};$$

$$c(u) = \frac{\theta^{(1)}(u) \theta^{(1)}(u + \eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(1)}(\eta) \sigma(u + \eta)}; \quad d(u) = \frac{\theta^{(0)}(u) \theta^{(0)}(u + \eta) \sigma(\eta)}{\theta^{(1)}(0) \theta^{(1)}(\eta) \sigma(u + \eta)}.$$  

Our $\sigma$-function is the $\vartheta$-function $\vartheta_1(u)$ [51]. It has the following relation with the Weierstrassian $\sigma$-function $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2} \sigma(u)$ with $\eta_1 = \pi^2 (\frac{1}{b} - 4 \sum_{n=1}^{\infty} \frac{n^{2\eta_1}}{1 - q^{2n}})$ and $q = e^{i\tau}$.  

4
Here \( u \) is the spectral parameter and \( \eta \) is the so-called crossing parameter. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE)

\[
R_{1,2}(u_1 - u_2)R_{1,3}(u_1 - u_3)R_{2,3}(u_2 - u_3) = R_{2,3}(u_2 - u_3)R_{1,3}(u_1 - u_3)R_{1,2}(u_1 - u_2).
\] (2.7)

Throughout we adopt the standard notation: for any matrix \( A \in \text{End}(V) \), \( A_j \) (or \( A^j \)) is an embedding operator in the tensor space \( V \otimes V \otimes \cdots \), which acts as \( A \) on the \( j \)-th space and as identity on the other factor spaces; \( R_{i,j}(u) \) is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the \( i \)-th and \( j \)-th ones.

One introduces the “row-to-row” (or one-row) monodromy matrix \( T(u) \), which is a \( 2 \times 2 \) matrix with elements being operators acting on \( V \otimes N \), where \( N = 2M \) (\( M \) being a positive integer),

\[
T_0(u) = R_{0,N}(u - z_N)R_{0,N-1}(u - z_{N-1}) \cdots R_{0,1}(u - z_1).
\] (2.8)

Here \( \{z_j|j = 1, \cdots, N\} \) are arbitrary free complex parameters which are usually called inhomogeneous parameters.

Integrable open chain can be constructed as follows [19]. Let us introduce a pair of K-matrices \( K^{-}(u) \) and \( K^{+}(u) \). The former satisfies the reflection equation (RE)

\[
R_{1,2}(u_1 - u_2)K^{-}(u_1)R_{2,1}(u_1 + u_2)K^{-}(u_2) = K^{-}(u_2)R_{1,2}(u_1 + u_2)K^{-}(u_1)R_{2,1}(u_1 - u_2),
\] (2.9)

and the latter satisfies the dual RE

\[
R_{1,2}(u_2 - u_1)K^{+}(u_1)R_{2,1}(-u_1 - u_2 - 2\eta)K^{+}(u_2) = K^{+}(u_2)R_{1,2}(-u_1 - u_2 - 2\eta)K^{+}(u_1)R_{2,1}(u_2 - u_1).
\] (2.10)

For open spin-chains, instead of the standard “row-to-row” monodromy matrix \( T(u) \) (2.8), one needs to consider the “double-row” monodromy matrix \( \mathbb{T}(u) \)

\[
\mathbb{T}(u) = T(u)K^{-}(u)\hat{T}(u), \quad \hat{T}(u) = T^{-1}(-u).
\] (2.11)

Then the double-row transfer matrix of the XYZ chain with open boundary (or the open XYZ chain) is given by

\[
\tau(u) = tr(K^{+}(u)\mathbb{T}(u)).
\] (2.12)
The QYBE and (dual) REs lead to that the transfer matrices with different spectral parameters commute with each other \[19\]: \[[\tau(u), \tau(v)]=0\]. This ensures the integrability of the open XYZ chain.

In this paper, we consider the K-matrix \(K^{-}(u)\) which is a generic solution \[49, 50\] to the RE \[2.9\] associated with the R-matrix \[2.5\]

\[
K^{-}(u) = k_{0}^{-}(u) + k_{x}^{-}(u) \sigma^{x} + k_{y}^{-} \sigma^{y} + k_{z}^{-}(u) \sigma^{z}, \tag{2.13}
\]

where \(\sigma^{x}, \sigma^{y}, \sigma^{z}\) are the Pauli matrices and the coefficient functions are

\[
k_{0}^{-}(u) = \frac{\sigma(2u) \sigma(\lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{2} + \xi)}{2 \sigma(u) \sigma(-u + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi + u) \sigma(\lambda_{2} + \xi + u)},
\]

\[
k_{x}^{-}(u) = \frac{\sigma(2u) \sigma(\lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{1} + \lambda_{2} + \xi)}{2 \sigma(\lambda_{1}) \sigma(-u + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi + u) \sigma(\lambda_{2} + \xi + u)},
\]

\[
k_{y}^{-}(u) = \frac{i \sigma(2u) \sigma(\lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{1} + \lambda_{2} + \xi)}{2 \sigma(\lambda_{1}) \sigma(-u + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi + u) \sigma(\lambda_{2} + \xi + u)},
\]

\[
k_{z}^{-}(u) = \frac{\sigma(2u) \sigma(\lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{1} + \lambda_{2} + \xi)}{2 \sigma(\lambda_{1}) \sigma(-u + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi + u) \sigma(\lambda_{2} + \xi + u)}.
\tag{2.14}
\]

At the same time, we introduce the corresponding dual K-matrix \(K^{+}(u)\) which is a generic solution to the dual reflection equation \[2.10\] with a particular choice of the free boundary parameters according to those of \(K^{-}(u)\) \[2.13\] \& \[2.14\]:

\[
K^{+}(u) = k_{0}^{+}(u) + k_{x}^{+}(u) \sigma^{x} + k_{y}^{+} \sigma^{y} + k_{z}^{+}(u) \sigma^{z}, \tag{2.15}
\]

with the coefficient functions

\[
k_{0}^{+}(u) = \frac{\sigma(-2u + 2\eta) \sigma(\lambda_{1} + \lambda_{2} + \eta - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{2} + \xi)}{2 \sigma(-u - \eta) \sigma(u + \eta + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi - u - \eta) \sigma(\lambda_{2} + \xi - u - \eta)},
\]

\[
k_{x}^{+}(u) = \frac{\sigma(-2u + 2\eta) \sigma(\lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{2} + \xi)}{2 \sigma(\lambda_{1}) \sigma(-u - \eta) \sigma(u + \eta + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi - u - \eta) \sigma(\lambda_{2} + \xi - u - \eta)},
\]

\[
k_{y}^{+}(u) = \frac{i \sigma(-2u + 2\eta) \sigma(\lambda_{1} + \lambda_{2} + \eta - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{2} + \xi)}{2 \sigma(\lambda_{1}) \sigma(-u - \eta) \sigma(u + \eta + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi - u - \eta) \sigma(\lambda_{2} + \xi - u - \eta)},
\]

\[
k_{z}^{+}(u) = \frac{\sigma(-2u + 2\eta) \sigma(\lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi) \sigma(\lambda_{2} + \xi)}{2 \sigma(\lambda_{1}) \sigma(-u - \eta) \sigma(u + \eta + \lambda_{1} + \lambda_{2} - \frac{1}{2}) \sigma(\lambda_{1} + \xi - u - \eta) \sigma(\lambda_{2} + \xi - u - \eta)}.
\tag{2.16}
\]
The K-matrices $K^\pm (u)$ depend on four free boundary parameters \{\(\lambda_1, \lambda_2, \xi, \bar{\xi}\)\}. It is very convenient to introduce a vector $\lambda \in V$ associated with the boundary parameters \{\(\lambda_i\)\},
\[
\lambda = \sum_{k=1}^{2} \lambda_k \epsilon_k. \tag{2.17}
\]

### 2.1 Vertex-face correspondence

Let us briefly review the face-type R-matrix associated with the six-vertex model.

Set
\[
\hat{i} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{2} \sum_{k=1}^{2} \epsilon_k, \quad i = 1, 2, \quad \text{then} \quad \sum_{i=1}^{2} \hat{i} = 0. \tag{2.18}
\]

Let $\mathfrak{h}$ be the Cartan subalgebra of $A_1$ and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalizable $\mathfrak{h}$-module is a complex finite dimensional vector space $W$ with a weight decomposition $W = \bigoplus_{\mu \in \mathfrak{h}^*} W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x v = \mu(x) v$, ($x \in \mathfrak{h}$, $v \in W[\mu]$). For example, the non-zero weight spaces of the fundamental representation $V_{\Lambda_1} = \mathbb{C}^2 = V$ are
\[
W[\hat{i}] = \mathbb{C} \epsilon_i, \quad i = 1, 2. \tag{2.19}
\]

For a generic $m \in V$, define
\[
m_i = \langle m, \epsilon_i \rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j \rangle, \quad i, j = 1, 2. \tag{2.20}
\]

Let $R(u, m) \in \text{End}(V \otimes V)$ be the R-matrix of the eight-vertex SOS model \[45\] given by
\[
R(u; m) = \sum_{i=1}^{2} R(u; m)^{ii}_{ii} E_{ii} \otimes E_{ii} + \sum_{i \neq j} \left\{ R(u; m)^{ij}_{ij} E_{ii} \otimes E_{jj} + R(u; m)^{ji}_{ij} E_{ji} \otimes E_{ij} \right\}, \tag{2.21}
\]
where $E_{ij}$ is the matrix with elements $(E_{ij})^l_k = \delta_{jk} \delta_{il}$. The coefficient functions are
\[
R(u; m)^{ii}_{ii} = 1, \quad R(u; m)^{ij}_{ij} = \frac{\sigma(u)\sigma(m_{ij} - \eta)}{\sigma(u + \eta)\sigma(m_{ij})}, \quad i \neq j, \tag{2.22}
\]
\[
R(u; m)^{ji}_{ij} = \frac{\sigma(\eta)\sigma(u + m_{ij})}{\sigma(u + \eta)\sigma(m_{ij})}, \quad i \neq j, \tag{2.23}
\]
and $m_{ij}$ is defined in (2.20). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation (or the star-triangle relation) \[45\]
\[
R_{1,2}(u_1 - u_2; m - \eta h^{(3)}) R_{1,3}(u_1 - u_3; m) R_{2,3}(u_2 - u_3; m - \eta h^{(1)}) = R_{2,3}(u_2 - u_3; m) R_{1,3}(u_1 - u_3; m - \eta h^{(2)}) R_{1,2}(u_1 - u_2; m). \tag{2.24}
\]
Here we have adopted

\[ R_{1,2}(u, m - \eta h^{(3)}) v_1 \otimes v_2 \otimes v_3 = (R(u, m - \eta \mu) \otimes \text{id}) v_1 \otimes v_2 \otimes v_3, \quad \text{if } v_3 \in W[\mu]. \quad (2.25) \]

Moreover, one may check that the R-matrix satisfies the weight conservation condition,

\[ [h^{(1)} + h^{(2)}, R_{1,2}(u; m)] = 0, \quad (2.26) \]

the unitary condition,

\[ R_{1,2}(u; m) R_{2,1}(-u; m) = \text{id} \otimes \text{id}, \quad (2.27) \]

and the crossing relation

\[ R(u; m)_{ij}^{kl} = \varepsilon_i \varepsilon_j \frac{\sigma(u)\sigma((m - \eta \hat{\eta})_{21})}{\sigma(u + \eta)\sigma(m_{21})} R(-u - \eta; m - \eta \hat{\eta})_{ki}^{jk}, \quad (2.28) \]

where

\[ \varepsilon_1 = 1, \quad \varepsilon_2 = -1, \quad \text{and } \hat{1} = 2, \quad \hat{2} = 1. \quad (2.29) \]

Let us introduce two intertwiners which are 2-component column vectors \( \phi_{m,m - \eta}(u) \) labelled by \( \hat{1}, \hat{2} \). The \( k \)-th element of \( \phi_{m,m - \eta}(u) \) is given by

\[ \phi_{m,m - \eta}^{(k)}(u) = \theta^{(k)}(u + 2m), \quad (2.30) \]

where the functions \( \theta^{(j)}(u) \) are given in (2.2). Explicitly,

\[ \phi_{m,m - \eta_{\hat{1}}}(u) = \begin{pmatrix} \theta^{(1)}(u + 2m_1) \\ \theta^{(2)}(u + 2m_1) \end{pmatrix}, \quad \phi_{m,m - \eta_{\hat{2}}}(u) = \begin{pmatrix} \theta^{(1)}(u + 2m_2) \\ \theta^{(2)}(u + 2m_2) \end{pmatrix}. \quad (2.31) \]

One can prove the following identity \[52\]

\[ \det \begin{vmatrix} \theta^{(1)}(u + 2m_1) & \theta^{(1)}(u + 2m_2) \\ \theta^{(2)}(u + 2m_1) & \theta^{(2)}(u + 2m_2) \end{vmatrix} = C(\tau) \sigma(u + m_1 + m_2 - \frac{1}{2}) \sigma(m_{12}), \]

where \( C(\tau) \) is non-vanishing constant which only depends on \( \tau \). This implies that the two intertwiner vectors \( \phi_{m,m - \eta}(u) \) are linearly independent for a generic \( m \in V \).

Using the intertwiner vectors, one can derive the following face-vertex correspondence relation \[45\]

\[ \overline{R}_{1,2}(u_1 - u_2) \phi_{m,m - \eta_{\hat{1}}}(u_1) \phi_{m,m - \eta_{\hat{1}}}(u_2) = \sum_{k,l} R(u_1 - u_2; m)_{ij}^{kl} \phi_{m,m - \eta_{\hat{1}}}(u_1) \phi_{m,m - \eta_{\hat{1}}}(u_2). \quad (2.32) \]
Then the QYBE (2.7) of the vertex-type R-matrix \( \bar{R}(u) \) is equivalent to the dynamical Yang-Baxter equation (2.24) of the SOS R-matrix \( R(u, m) \). For a generic \( m \), we can introduce other types of intertwiners \( \bar{\phi}, \tilde{\phi} \) which are both row vectors and satisfy the following conditions,

\[
\bar{\phi}_{m,m-\eta \bar{\mu}}(u) \phi_{m,m-\eta \hat{\nu}}(u) = \delta_{\mu \nu}, \quad \tilde{\phi}_{m+\eta \hat{\mu},m}(u) \phi_{m+\eta \hat{\nu},m}(u) = \delta_{\mu \nu},
\]

from which one can derive the relations,

\[
\sum_{\mu=1}^{2} \phi_{m,m-\eta \bar{\mu}}(u) \bar{\phi}_{m,m-\eta \bar{\mu}}(u) = \text{id}, \quad (2.34)
\]

\[
\sum_{\mu=1}^{2} \phi_{m+\eta \hat{\mu},m}(u) \tilde{\phi}_{m+\eta \hat{\mu},m}(u) = \text{id}. \quad (2.35)
\]

With the help of (2.32)-(2.35), we obtain,

\[
\tilde{\phi}_{m+\eta \hat{k},m}(u_1) \bar{R}_{1,2}(u_1 - u_2) \phi_{m+\eta \hat{l},m}(u_2) = \sum_{i,j} R(u_1 - u_2; m)_{ij}^{kl} \bar{\phi}_{m+\eta(i+j),m+\eta \hat{j}}(u_1) \phi_{m+\eta(k+i),m+\eta \hat{k}}(u_2), \quad (2.36)
\]

\[
\bar{\phi}_{m+\eta \hat{k},m}(u_1) \tilde{R}_{1,2}(u_1 - u_2) \phi_{m+\eta \hat{l},m}(u_2) = \sum_{i,j} R(u_1 - u_2; m)_{ij}^{kl} \bar{\phi}_{m+\eta(i+j),m+\eta \hat{j}}(u_1) \phi_{m+\eta(k+i),m+\eta \hat{k}}(u_2), \quad (2.37)
\]

\[
\bar{\phi}_{m,m-\eta \hat{l}}(u_2) \bar{R}_{1,2}(u_1 - u_2) \phi_{m,m-\eta \hat{l}}(u_1) = \sum_{i,j} R(u_1 - u_2; m)_{ij}^{kl} \bar{\phi}_{m,m-\eta(k+i),m-\eta \hat{i}}(u_1) \phi_{m,m-\eta(i+j),m-\eta \hat{j}}(u_2), \quad (2.38)
\]

\[
\tilde{\phi}_{m,m-\eta \hat{l}}(u_2) \tilde{R}_{1,2}(u_1 - u_2) \phi_{m,m-\eta \hat{l}}(u_1) = \sum_{i,j} R(u_1 - u_2; m)_{ij}^{kl} \tilde{\phi}_{m,m-\eta(k+i),m-\eta \hat{i}}(u_1) \phi_{m,m-\eta(i+j),m-\eta \hat{j}}(u_2). \quad (2.39)
\]

In addition to the Riemann identity (2.3), the \( \sigma \)-function enjoys the following properties:

\[
\sigma(2u) = \frac{2 \sigma(u) \sigma_{(0,1)}(u) \sigma_{(1,0)}(u) \sigma_{(1,1)}(u)}{\sigma_{(0,1)}(0) \sigma_{(1,0)}(0) \sigma_{(1,1)}(0)}, \quad (2.40)
\]

\[
\sigma(u + 1) = -\sigma(u), \quad \sigma(u + \tau) = e^{-2i\pi(u+\frac{1}{2}+\frac{\tau}{2})} \sigma(u), \quad (2.41)
\]

where the functions \( \sigma_a(u) \) are given by (2.4). Using the above identities and the method in [52], after tedious calculations, we can show that the K-matrices \( K^\pm(u) \) given by (2.13) and (2.15) can be expressed in terms of the intertwiners and diagonal matrices \( K(\lambda|u) \) and
\( \tilde{K}(\lambda|u) \) as follows

\[
K^-(u)_t^{i} = \sum_{i,j} \phi_{\lambda-\eta(i-j)}^{(s)}(u) \tilde{K}(\lambda|u)_t^{i} \left( \tilde{\phi}_{\lambda, \lambda-\eta}^{(t)}(-u) \right),
\]

\[
K^+(u)_t^{i} = \sum_{i,j} \phi_{\lambda-\eta(j-i)}^{(s)}(-u) \tilde{K}(\lambda|u)_t^{i} \left( \tilde{\phi}_{\lambda-\eta(j-i), \lambda-\eta(j-i)}^{(t)} \right).
\]

(2.42)

(2.43)

Here the two diagonal matrices \( K(\lambda|u) \) and \( \tilde{K}(\lambda|u) \) are given by

\[
K(\lambda|u) \equiv \text{Diag}(k(\lambda|u)_1, k(\lambda|u)_2) = \text{Diag} \left( \frac{\sigma(\lambda_1 + \xi - u)}{\sigma(\lambda_1 + \xi + u)}, \frac{\sigma(\lambda_2 + \xi - u)}{\sigma(\lambda_2 + \xi + u)} \right),
\]

\[
\tilde{K}(\lambda|u) \equiv \text{Diag}(\tilde{k}(\lambda|u)_1, \tilde{k}(\lambda|u)_2)
\]

\[
= \text{Diag} \left( \frac{\sigma(\lambda_{12} - \eta) \sigma(\lambda_1 + \tilde{\xi} + u + \eta)}{\sigma(\lambda_{12}) \sigma(\lambda_1 + \xi - u - \eta)}, \frac{\sigma(\lambda_{12} + \eta) \sigma(\lambda_2 + \tilde{\xi} + u + \eta)}{\sigma(\lambda_{12}) \sigma(\lambda_2 + \xi - u - \eta)} \right).
\]

(2.44)

(2.45)

Although the vertex type K-matrices \( K^+(u) \) given by (2.13) and (2.15) are generally non-diagonal, after the face-vertex transformations (2.42) and (2.43), the face type counterparts \( K(\lambda|u) \) and \( \tilde{K}(\lambda|u) \) become simultaneously diagonal. This fact enabled the authors in [26, 36] to diagonalize the transfer matrices \( \tau(u) \) (2.12) by applying the generalized algebraic Bethe ansatz method developed in [22].

### 2.2 Two sets of eigenstates

In order to construct the Bethe states of the open XYZ model with non-diagonal boundary terms specified by the K-matrices (2.14) and (2.16), we need to introduce the new double-row monodromy matrices \( T^+(m|u) \) [22, 40, 53]:

\[
T^-(m|u)_i^{0} = \tilde{\phi}_{m-\eta(j-i),m-\eta}^{0}(u) T_0(u) \phi_{m,m-\eta}^{0}(-u),
\]

\[
T^+(m|u)_i^{j} = \prod_{k \neq j} \frac{\sigma(m_{jk})}{\sigma(m_{jk} - \eta)} \phi_{m-\eta(j-i),m-\eta(j-i)}^{t_0}(u) \left( T^+(u) \right)^{t_0} \phi_{m,m-\eta}^{t_0}(-u),
\]

(2.46)

(2.47)

where \( t_0 \) denotes transposition in the 0-th space (i.e. auxiliary space) and \( T^+(u) \) is given by

\[
\left( T^+(u) \right)^{t_0} = T^{t_0}(u) \left( K^+(u) \right)^{t_0} \tilde{T}^{t_0}(u).
\]

(2.48)

These double-row monodromy matrices, in the face picture, can be expressed in terms of the face type R-matrix \( R(u;m) \) (2.21) and K-matrices \( K(\lambda|u) \) (2.44) and \( \tilde{K}(\lambda|u) \) (2.45) (for the details see section 3).
So far only two sets of Bethe states (i.e., eigenstates) of the transfer matrix for the models with non-diagonal boundary terms have been found \[54\, 56\, 53\]. These two sets of states are

\[
|\{v_i^{(1)}\}^{(I)}\rangle = T^+(\lambda + 2\eta \hat{1})|v_i^{(1)}\rangle |\ldots T^+(\lambda + 2M\eta \hat{1})|v_M^{(1)}\rangle|\Omega^{(I)}(\lambda)\rangle,
\]

\[
|\{v_i^{(2)}\}^{(II)}\rangle = T^-(\lambda - 2\eta \hat{2})|v_i^{(2)}\rangle |\ldots T^-(\lambda - 2M\eta \hat{2})|v_M^{(2)}\rangle|\Omega^{(II)}(\lambda)\rangle,
\]

where the vector \(\lambda\) is related to the boundary parameters \(2.17\). The associated reference states \(|\Omega^{(I)}(\lambda)\rangle\) and \(|\Omega^{(II)}(\lambda)\rangle\) are

\[
|\Omega^{(I)}(\lambda)\rangle = \phi_1^{1,N}\phi^{2,\lambda + (N-1)\eta\lambda}(z_1)\phi^{2,\lambda + (N-2)\eta\lambda}(z_2)\ldots \phi_N^{N,\lambda}(z_N),
\]

\[
|\Omega^{(II)}(\lambda)\rangle = \phi_1^{1,N}\phi^{2,\lambda - \eta\lambda}(z_1)\phi^{2,\lambda - 2\eta\lambda}(z_2)\ldots \phi_N^{N,\lambda}(z_N).
\]

It is remarked that \(\phi^k = \text{id} \otimes \text{id} \cdots \otimes \hat{\phi} \otimes \text{id} \cdots\).

If the parameters \(\{v_k^{(1)}\}\) satisfy the first set of Bethe ansatz equations given by

\[
\frac{\sigma(\lambda_2 + \xi + v^{(1)}_a \lambda(\lambda_2 + \xi + v^{(1)}_a)\sigma(\lambda_1 + \xi - v^{(1)}_a)}{\sigma(\lambda_2 + \xi + v^{(1)}_a + \eta)\sigma(\lambda_2 + \xi - v^{(1)}_a - \eta)}
\]

\[
= \prod_{k \neq a}^{M} \frac{\sigma(v^{(1)}_a + v^{(1)}_k + 2\eta)\sigma(v^{(1)}_a - v^{(1)}_k - \eta)}{\sigma(v^{(1)}_a + v^{(1)}_k)\sigma(v^{(1)}_a - v^{(1)}_k)}
\]

\[
\times \prod_{k=1}^{2M} \frac{\sigma(v^{(1)}_a + z_k)\sigma(v^{(1)}_a - z_k)}{\sigma(v^{(1)}_a + z_k + \eta)\sigma(v^{(1)}_a - z_k + \eta)}, \quad \alpha = 1, \ldots, M, \quad (2.53)
\]

the Bethe state \(|v_1^{(1)}, \ldots, v_M^{(1)}\rangle\) becomes the eigenstate of the transfer matrix with eigenvalue \(\Lambda^{(1)}(u)\) given by \[53\]

\[
\Lambda^{(1)}(u) = \frac{\sigma(\lambda_2 + \xi - u)\sigma(\lambda_1 + \xi + u)\sigma(\lambda_1 + \xi - u)\sigma(2u + 2\eta)}{\sigma(\lambda_2 + \xi - u - \eta)\sigma(\lambda_1 + \xi - u - \eta)\sigma(\lambda_1 + \xi + u)\sigma(2u + \eta)}
\]

\[
\times \prod_{k=1}^{M} \frac{\sigma(u + v^{(1)}_k)\sigma(u - v^{(1)}_k - \eta)}{\sigma(u + v^{(1)}_k + \eta)\sigma(u - v^{(1)}_k)}
\]

\[
+ \frac{\sigma(\lambda_2 + \xi + u + \eta)\sigma(\lambda_1 + \xi + u + \eta)\sigma(\lambda_2 + \xi - u - \eta)\sigma(2u)}{\sigma(\lambda_2 + \xi - u - \eta)\sigma(\lambda_1 + \xi + u)\sigma(\lambda_2 + \xi + u)\sigma(2u + \eta)}
\]

\[
\times \prod_{k=1}^{M} \frac{\sigma(u + v^{(1)}_k + 2\eta)\sigma(u - v^{(1)}_k + \eta)}{\sigma(u + v^{(1)}_k + \eta)\sigma(u - v^{(1)}_k)}
\]

\[
\times \prod_{k=1}^{2M} \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + \eta)\sigma(u - z_k + \eta)}.
\]

(2.54)
If the parameters \( \{v_{k}^{(2)}\} \) satisfy the second Bethe Ansatz equations

\[
\begin{align*}
\frac{\sigma(\lambda_1 + \xi + v_{\alpha}^{(2)})\sigma(\lambda_1 + \xi - v_{\alpha}^{(2)})\sigma(\lambda_2 + \xi + v_{\alpha}^{(2)})\sigma(\lambda_2 + \xi - v_{\alpha}^{(2)})}{\sigma(\lambda_1 + \xi + v_{\alpha}^{(2)})\sigma(\lambda_1 + \xi - v_{\alpha}^{(2)})\sigma(\lambda_2 + \xi + v_{\alpha}^{(2)})\sigma(\lambda_2 + \xi - v_{\alpha}^{(2)})}
&= \prod_{k \neq \alpha} \frac{\sigma(v_{\alpha}^{(2)} + v_{k}^{(2)} + 2\eta)\sigma(v_{\alpha}^{(2)} - v_{k}^{(2)} + \eta)}{\sigma(v_{\alpha}^{(2)} + v_{k}^{(2)})\sigma(v_{\alpha}^{(2)} - v_{k}^{(2)} - \eta)} \\
&\times \prod_{k=1}^{2M} \frac{\sigma(\eta + z_{k})\sigma(\xi + z_{k})}{\sigma(\eta + \xi + z_{k})\sigma(\xi - z_{k} + \eta)}, \quad \alpha = 1, \cdots, M,
\end{align*}
\]

(2.55)

the Bethe states \(|v_{1}^{(2)}, \cdots, v_{M}^{(2)}\rangle^{(II)}\) yield the second set of the eigenstates of the transfer matrix with the eigenvalues \([54, 30]\).

\[
\Lambda^{(2)}(u) = \frac{\sigma(2u + 2\eta)\sigma(\lambda_1 + \xi - u)\sigma(\lambda_2 + \xi + u)\sigma(\lambda_2 + \xi - u)}{\sigma(2u + \eta)\sigma(\lambda_1 + \xi - u - \eta)\sigma(\lambda_2 + \xi - u - \eta)\sigma(\lambda_2 + \xi + u)} \\
\times \prod_{k=1}^{M} \frac{\sigma(u + v_{k}^{(2)} + 2\eta)\sigma(u - v_{k}^{(2)} + \eta)}{\sigma(u + v_{k}^{(2)} + \eta)\sigma(u - v_{k}^{(2)} - \eta)} \\
+ \frac{\sigma(2u)\sigma(\lambda_1 + \xi + u + \eta)\sigma(\lambda_2 + \xi + u + \eta)\sigma(\lambda_1 + \xi - u - \eta)}{\sigma(2u + \eta)\sigma(\lambda_1 + \xi - u - \eta)\sigma(\lambda_2 + \xi + u)\sigma(\lambda_1 + \xi + u)} \\
\times \prod_{k=1}^{M} \frac{\sigma(u + v_{k}^{(2)} + 2\eta)\sigma(u - v_{k}^{(2)} + \eta)}{\sigma(u + v_{k}^{(2)} + \eta)\sigma(u - v_{k}^{(2)} - \eta)} \\
\times \prod_{k=1}^{2M} \frac{\sigma(u + z_{k})\sigma(u - z_{k})}{\sigma(u + \xi + z_{k})\sigma(u - z_{k} + \eta)}.
\]

(2.56)

### 3 Scalar products

It was shown that in order to compute correlation functions of the closed chain \([2]\) and the open chain with diagonal boundary terms \([17, 18]\), one suffices to calculate the scalar products of an on-shell Bethe state and a general state (an off-shell Bethe state). The aim of this paper is to give the explicit expressions of the following scalar products of the open XYZ chain with non-diagonal boundary terms:

\[
S^{(I)}(\{u_{\alpha}\}; \{v_{i}^{(1)}\}) = \langle \{u_{\alpha}\} | \{v_{i}^{(1)}\} \rangle^{(I)}, \quad S^{(II)}(\{u_{\alpha}\}; \{v_{i}^{(1)}\}) = \langle \{u_{\alpha}\} | \{v_{i}^{(1)}\} \rangle^{(II)},
\]

(3.1)

\[
S^{(I)}(\{u_{\alpha}\}; \{v_{i}^{(2)}\}) = \langle \{u_{\alpha}\} | \{v_{i}^{(2)}\} \rangle^{(I)}, \quad S^{(II)}(\{u_{\alpha}\}; \{v_{i}^{(2)}\}) = \langle \{u_{\alpha}\} | \{v_{i}^{(2)}\} \rangle^{(II)},
\]

(3.2)

where the dual states \(\langle \{u_{\alpha}\} | \) and \(\langle \{u_{\alpha}\} \rangle^{(II)}\) are given by

\[
\langle \{u_{\alpha}\} | = \langle \Omega^{(I)}(\lambda) | T^{-}(\lambda - 2(M - 1)\eta|u_{M})^{2} \cdots T^{-}(\lambda|u_{1})^{2}, \quad (3.3)
\]

\[
\langle \{u_{\alpha}\} \rangle^{(II)} = \langle \Omega^{(II)}(\lambda) | T^{+}(\lambda + 2(M - 1)\eta|u_{M})^{2} \cdots T^{+}(\lambda|u_{1})^{2}, \quad (3.4)
\]
and \( \langle \Omega^{(I)}(\lambda) \rangle, \langle \Omega^{(II)}(\lambda) \rangle \) are

\[
\langle \Omega^{(I)}(\lambda) \rangle = \tilde{\phi}^1_{\lambda, \lambda - \eta}(z_1) \cdots \tilde{\phi}^N_{\lambda, \lambda - 2M\eta}(z_N), \\
\langle \Omega^{(II)}(\lambda) \rangle = \tilde{\phi}^1_{\lambda + 2M\eta, \lambda}(z_1) \cdots \tilde{\phi}^N_{\lambda + \eta, \lambda}(z_N).
\]

Some remarks are in order. The parameters \( \{u_\alpha \} \) in (3.3)-(3.4) are free parameters, namely, they do not need to satisfy the Bethe ansatz equations. Moreover, even if these parameters are required satisfy the associated Bethe ansatz equations (2.53) and (2.55) respectively, the corresponding dual states (3.3) and (3.4) are not the eigenstates of the model (in contrast with those of the open XXZ chain with diagonal boundary terms [18]). Such a phenomena was already found for the open XXZ chain with non-diagonal boundary terms [44].

The K-matrices \( K^\pm(u) \) given by (2.13) and (2.15) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (2.42) and (2.43), the face type counterparts \( K(\lambda|u) \) and \( \tilde{K}(\lambda|u) \) given by (2.44) and (2.45) simultaneously become diagonal. This fact suggests that it would be much simpler if one performs all calculations in the face picture.

### 3.1 Face picture

Let us introduce the face type one-row monodromy matrix (c.f (2.8))

\[
T_F(l|u) \equiv T^F_{0,1\ldots N}(l|u)
\]

\[
= R_{0,N}(u - z_N; l - \eta \sum_{i=1}^{N-1} h^{(i)}) \cdots R_{0,2}(u - z_2; l - \eta h^{(1)}) R_{0,1}(u - z_1; l),
\]

\[
= \begin{pmatrix}
T_F(l|u)_1 & T_F(l|u)_2 \\
T_F(l|u)_1^2 & T_F(l|u)_2^2
\end{pmatrix}
\]

(3.7)

where \( l \) is a generic vector in \( V \). The monodromy matrix satisfies the face type quadratic exchange relation [55, 56]. Applying \( T_F(l|u)_{ij} \) to an arbitrary vector \( |i_1, \ldots, i_N\rangle \) in the \( N \)-tensor product space \( V^\otimes N \) given by

\[
|i_1, \ldots, i_N\rangle = \epsilon^1_{i_1} \cdots \epsilon^N_{i_N},
\]

we have

\[
T_F(l|u)_{ij}^i |i_1, \ldots, i_N\rangle \equiv T_F(m|l|u)_{ij}^i |i_1, \ldots, i_N\rangle
\]
in terms of the one-row monodromy matrix operator

The above double-row monodromy matrix operators $T_m$ where $m = l - \eta \sum_{k=1}^{N} i_k$. With the help of the crossing relation (2.28), the face-vertex correspondence relation (2.32) and the relations (2.33), following the method developed in [22, 40, 48], we find that the scalar products (3.1)-(3.2) can be expressed in terms of the face-type double-row monodromy operators as follows:

$$S^{I,II}({\{u_0}\};{v^{(2)}_1}) = \langle 1, \ldots, 1 | T_F^-(\lambda - 2(M - 1)\eta \hat{1}, \lambda | u_M) \hat{1} \ldots T_F^-(\lambda, \lambda | u_1) \hat{1} \times T_F^-(\lambda + 2\eta \hat{1}, \lambda | v^{(2)}_1) \hat{1} \ldots T_F^-(\lambda + 2M\eta \hat{1}, \lambda | v^{(2)}_M) \hat{2}, \ldots, 2 \rangle,$$

$$S^{II,\hat{I}}({\{u_0\};{v^{(1)}_1}}) = \langle 2, \ldots, 2 | T_F^+(\lambda, \lambda + 2(M - 1)\eta \hat{2} | u_M) \hat{1} \ldots T_F^+(\lambda, \lambda | u_1) \hat{1} \times T_F^+(\lambda, \lambda - 2\eta \hat{1} | v^{(1)}_1) \hat{1} \ldots T_F^+(\lambda, \lambda - 2M\eta \hat{1} | v^{(1)}_M) \hat{1}, \ldots, 1 \rangle,$$

$$S^{\hat{I},I}({\{u_0\};{v^{(1)}_1}}) = \langle 1, \ldots, 1 | T_F^-(\lambda - 2(M - 1)\eta \hat{1}, \lambda | u_M) \hat{2} \ldots T_F^-(\lambda, \lambda | u_1) \hat{2} \times T_F^+(\lambda, \lambda + 2\eta \hat{1} | v^{(1)}_1) \hat{1} \ldots T_F^+(\lambda, \lambda + 2M\eta \hat{1} | v^{(1)}_M) \hat{1}, \ldots, 1 \rangle,$$

$$S^{II,\hat{II}}({\{u_0\};{v^{(2)}_1}}) = \langle 2, \ldots, 2 | T_F^+(\lambda, \lambda + 2(M - 1)\eta \hat{2} | u_M) \hat{1} \ldots T_F^+(\lambda, \lambda | u_1) \hat{1} \times T_F^-(\lambda - 2\eta \hat{2}, \lambda | v^{(2)}_1) \hat{1} \ldots T_F^-(\lambda - 2M\eta \hat{2}, \lambda | v^{(2)}_M) \hat{2}, \ldots, 2 \rangle.$$

The above double-row monodromy matrix operators $T_F^-(m, \lambda | u)^2$ and $T_F^+(\lambda, m | u)^1$ are given in terms of the one-row monodromy matrix operator $T_F(m; l | u)^j$ [48]

$$T_F^-(m, \lambda | u)^2 = \frac{\sigma(m_2)}{\sigma(\lambda_2)} \prod_{k=1}^{N} \frac{\sigma(u + z_k)}{\sigma(u + z_k + \eta)} \times \left\{ \frac{\sigma(\lambda_1 + \xi - u)}{\sigma(\lambda_1 + \xi + u)} T_F(m, \lambda | u)^2 T_F(m + \eta \hat{2}, \lambda + \eta \hat{2} | u - \eta)^2 - \frac{\sigma(\lambda_2 + \xi - u)}{\sigma(\lambda_2 + \xi + u)} T_F(m + 2\eta \hat{2}, \lambda | u)^2 T_F(m + \eta \hat{1}, \lambda + \eta \hat{1} | u - \eta)^2 \right\},$$

$$T_F^+(\lambda, m | u)^1 = \prod_{k=1}^{N} \frac{\sigma(u + z_k)}{\sigma(u + z_k + \eta)} \times \left\{ \frac{\sigma(\lambda_1 - \eta) \sigma(\lambda_1 + \xi + u + \eta)}{\sigma(m_1 - \eta) \sigma(\lambda_1 + \xi - u - \eta)} T_F(\lambda + 2\eta \hat{2}, m + 2\eta \hat{2} | u)^2 T_F(\lambda + \eta \hat{2}, m + \eta \hat{2} | u - \eta)^2 - \frac{\sigma(\lambda_2 - \eta) \sigma(\lambda_2 + \xi + u + \eta)}{\sigma(m_2 + \eta) \sigma(\lambda_2 + \xi - u - \eta)} T_F(\lambda, m + 2\eta \hat{2} | u)^2 T_F(\lambda + \eta \hat{2}, m + \eta \hat{2} | u - \eta)^2 \right\}.$$
In the next section we use the Drinfeld twist (or factorizing F-matrix) of the eight-vertex SOS model proposed in [39] to construct the polarization free forms of the two sets of pseudo-particle creation/annihilation operators \( T_F^\pm \) given by (3.14) and (3.15), which allow us to construct the explicit expressions of the scalar products (3.10)-(3.13).

4 F-basis

In this section, after briefly reviewing the result [39] about the Drinfeld twist [11] (factorizing F-matrix) of the eight-vertex SOS model, we obtain the explicit expression of the double rows monodromy operator \( T_\pm(F) \) given by (3.14) and (3.15) in the F-basis provided by the F-matrix.

4.1 Factorizing Drinfeld twist \( F \)

Let \( S_N \) be the permutation group over indices 1, \ldots, \( N \) and \( \{ s_i | i = 1, \ldots, N - 1 \} \) be the set of elementary permutations in \( S_N \). For each elementary permutation \( s_i \), we introduce the associated operator \( R_{s_1 \ldots N}^s \) on the quantum space

\[
R_{s_1 \ldots N}^s(l) \equiv R^s_i(l) = R_{i,i+1}(z_i - z_{i+1})|l\rangle - \eta \sum_{k=1}^{i-1} h^{(k)}(|l\rangle),
\]

where \( l \) is a generic vector in \( V \). For any \( s, s' \in S_N \), operator \( R_{s_1 \ldots N}^{ss'} \) associated with \( ss' \) satisfies the following composition law [12, 39, 16, 15]:

\[
R_{s_1 \ldots N}^{ss'}(l) = R_{s(1 \ldots N)}^{s'}(l) R_{s_1 \ldots N}^s(l).
\]

Let \( s \) be decomposed in a minimal way in terms of elementary permutations,

\[
s = s_{\beta_1} \ldots s_{\beta_p},
\]

where \( \beta_i = 1, \ldots, N - 1 \) and the positive integer \( p \) is the length of \( s \). The composition law (4.2) enables one to obtain operator \( R_{s_1 \ldots N}^s \) associated with each \( s \in S_N \). The dynamical quantum Yang-Baxter equation (2.24), the weight conservation condition (2.26) and the unitary condition (2.27) guarantee the uniqueness of \( R_{s_1 \ldots N}^s \). Moreover, one may check that \( R_{s_1 \ldots N}^s \) satisfies the following exchange relation with the face type one-row monodromy matrix (3.7)

\[
R_{s_1 \ldots N}^s(l) T_{0,s(1 \ldots N)}^F(l|u) = T_{0,s(1 \ldots N)}^F(l|u) R_{s_1 \ldots N}^s(l - \eta h^{(0)}), \quad \forall s \in S_N.
\]
Now, we construct the face-type Drinfeld twist \( F_{1\ldots N}(l) \equiv F_{1\ldots N}(l; z_1, \ldots, z_N) \) on the \( N \)-fold tensor product space \( V^{\otimes N} \), which satisfies the following three properties \([39, 16, 15]\):

I. lower – triangularity;

II. non – degeneracy;

III. factorizing property : \( R^s_{1\ldots N}(l) = F_{s(1\ldots N)}^{-1}(l) F_{1\ldots N}(l) \), \( \forall s \in \mathcal{S}_N \). \( (4.7) \)

Substituting \((4.7)\) into the exchange relation \((4.4)\), we have

\[
F_{s(1\ldots N)}^{-1}(l) F_{1\ldots N}(l) T_{0,1\ldots N}^F(l|u) = T_{0,s(1\ldots N)}^F(l|u) F_{s(1\ldots N)}^{-1}(l - \eta h^{(0)}) F_{1\ldots N}(l - \eta h^{(0)}).
\] \( (4.8) \)

Equivalently,

\[
F_{1\ldots N}(l) T_{0,1\ldots N}^F(l|u) F_{1\ldots N}^{-1}(l - \eta h^{(0)}) = F_{s(1\ldots N)}(l) T_{0,s(1\ldots N)}^F(l|u) F_{s(1\ldots N)}^{-1}(l - \eta h^{(0)}).
\] \( (4.9) \)

Let us introduce the twisted monodromy matrix \( \tilde{T}_{0,1\ldots N}^F(l|u) \) by

\[
\tilde{T}_{0,1\ldots N}^F(l|u) = F_{1\ldots N}(l) T_{0,1\ldots N}^F(l|u) F_{1\ldots N}^{-1}(l - \eta h^{(0)}) = \left( \begin{array}{c} \tilde{T}_F(l|u)_1^1 \ \\ \tilde{T}_F(l|u)_2^1 \end{array} \right) \left( \begin{array}{c} \tilde{T}_F(l|u)_1^2 \ \\ \tilde{T}_F(l|u)_2^2 \end{array} \right).
\] \( (4.10) \)

Then \((4.9)\) implies that the twisted monodromy matrix is symmetric under \( \mathcal{S}_N \), namely,

\[
\tilde{T}_{0,1\ldots N}^F(l|u) = \tilde{T}_{\tilde{0},s(1\ldots N)}^F(l|u), \quad \forall s \in \mathcal{S}_N.
\] \( (4.11) \)

Define the F-matrix:

\[
F_{1\ldots N}(l) = \sum_{s \in \mathcal{S}_N} \sum_{\{\alpha_i\} = 1}^{2^N} \prod_{i=1}^{N} P_{\alpha(i)}^{s(i)} R^s_{1\ldots N}(l),
\] \( (4.12) \)

where \( P^{i}_{\alpha} \) is the embedding of the project operator \( P_{\alpha} \) in the \( i \)th space with matric elements \( (P_{\alpha})_{kl} = \delta_{kl} \delta_{k\alpha} \). The sum \( \sum^{\ast} \) in \((4.12)\) is over all non-decreasing sequences of the labels \( \alpha_{s(i)} \):

\[
\alpha_{s(i+1)} \geq \alpha_{s(i)} \quad \text{if} \quad s(i+1) > s(i),
\] \( (4.13) \)

\[
\alpha_{s(i+1)} > \alpha_{s(i)} \quad \text{if} \quad s(i+1) < s(i).
\] \( (4.14) \)

From \((4.14)\), \( F_{1\ldots N}(l) \) obviously is a lower-triangular matrix. Moreover, the F-matrix is non-degenerate because all its diagonal elements are non-zero. It was shown in \([39]\) that the F-matrix also satisfies the factorizing property \((4.7)\).

\( ^2 \)In this paper, we adopt the convention: \( F_{s(1\ldots N)}(l) \equiv F_{s(1\ldots N)}(l; z_{s(1)}, \ldots, z_{s(N)}) \).
4.2 Completely symmetric representations

In the F-basis provided by the F-matrix \((4.12)\), the twisted operators \(\tilde{T}_F(l|u)^i_2\) defined by \((4.10)\) become polarization free \([39]\). Here we present the results relevant for our purpose

\[
\tilde{T}_F(l|u)^2_2 = \frac{\sigma(l_{21} - \eta)}{\sigma(l_{21} - \eta + \eta(H, \epsilon_1))} \otimes i \left( \frac{\sigma(u - z_i)}{\sigma(u - z_i + \eta)} \right) \quad \text{,} \tag{4.15}
\]

\[
\tilde{T}_F(l|u)^2_1 = \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u - z_i + l_{12})}{\sigma(u - z_i + \eta)} E_{12}^i \otimes j \neq i \left( \frac{\sigma(u - z_i)\sigma(z_i - z_j + \eta)}{\sigma(u - z_j + \eta)\sigma(z_i - z_j)} \right) \quad \text{,} \tag{4.16}
\]

\[
\tilde{T}_F(l|u)^2_2 = \frac{\sigma(l_{21} - \eta)}{\sigma(l_{21} + \eta)} \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u - z_i + l_{21} + \eta + \eta(H, \epsilon_1 - \epsilon_2))}{\sigma(u - z_i + \eta)} \sigma(l_{21} + \eta + \eta(H, \epsilon_1 - \epsilon_2)) \times E_{21}^i \otimes j \neq i \left( \frac{\sigma(u - z_i + \eta)}{\sigma(z_i - z_j)} \right) \quad \text{,} \tag{4.17}
\]

where \(H = \sum_{k=1}^{N} h^{(k)}\). Applying the above operators to the arbitrary state \(|i_1, \ldots, i_N\rangle\) given by \((3.8)\), we have

\[
\tilde{T}_F(m, l|u)^2_2 = \frac{\sigma(l_{21} - \eta)}{\sigma(l_{21} - m_1 - \eta)} \otimes i \left( \frac{\sigma(u - z_i)}{\sigma(u - z_i + \eta)} \right) \quad \text{,} \tag{4.18}
\]

\[
\tilde{T}_F(m, l|u)^2_1 = \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u - z_i + l_{12})}{\sigma(u - z_i + \eta)} E_{12}^i \otimes j \neq i \left( \frac{\sigma(u - z_i)\sigma(z_i - z_j + \eta)}{\sigma(u - z_j + \eta)\sigma(z_i - z_j)} \right) \quad \text{,} \tag{4.19}
\]

\[
\tilde{T}_F(m, l|u)^2_2 = \frac{\sigma(l_{21} - \eta)}{\sigma(m_{21} - 2\eta)} \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u - z_i + m_{21} - \eta)}{\sigma(u - z_i + \eta)} \sigma(m_{21} - \eta) \times E_{21}^i \otimes j \neq i \left( \frac{\sigma(u - z_i)}{\sigma(u - z_j + \eta)} \right) \quad \text{.} \tag{4.20}
\]

With the help of the Riemann identity \((2.3)\), we find that the two pseudo-particle creation operators \((3.14)\) and \((3.15)\) in the F-basis simultaneously have the following completely symmetric polarization free forms:

\[
\tilde{T}_F^{-1}(m, \chi|u)^2_1 = \frac{\sigma(m_{12})}{\sigma(m_1 - \lambda_2)} \prod_{k=1}^{N} \frac{\sigma(u + z_k)}{\sigma(u + z_k + \eta)}
\]
\[
\sum_{i=1}^{N} \frac{\sigma(\lambda_1 + \xi - z_i)\sigma(\lambda_2 + \xi + z_i)\sigma(2u)\sigma(\eta)}{\sigma(\lambda_1 + \xi + u)\sigma(\lambda_2 + \xi + u)\sigma(u - z_i + \eta)\sigma(u + z_i)} \times E_{12}^{i} \otimes_{j \neq i} \left( \frac{\sigma(u - z_i)\sigma(u + z_i + \eta)\sigma(z_i - z_j)}{\sigma(u - z_j + \eta)\sigma(u + z_j)\sigma(z_i - z_j)} \right)_{(j)},
\]
(4.21)

\[
\tilde{T}_{F}^{+}(\lambda, m|u)_{1/2} = \frac{\sigma(m_{21} + \eta)}{\sigma(m_{2} - \lambda_{1})} \prod_{k=1}^{N} \frac{\sigma(u + z_{k})}{\sigma(u + z_{k} + \eta)} \times \sum_{i=1}^{N} \frac{\sigma(\lambda_2 + \xi - z_i)\sigma(\lambda_1 + \xi + z_i)\sigma(2u + 2\eta)\sigma(\eta)}{\sigma(\lambda_1 + \xi - u - \eta)\sigma(\lambda_2 + \xi - u - \eta)\sigma(u + z_i)\sigma(u - z_i + \eta)} \times E_{21}^{i} \otimes_{j \neq i} \left( \frac{\sigma(u - z_i)\sigma(u + z_i + \eta)}{\sigma(u - z_j + \eta)\sigma(u + z_j)} \right)_{(j)}.
\]
(4.22)

The very polarization free form (4.21) of \(\tilde{T}_{F}^{+}(m, \lambda|u)_{1/2}\) enabled the authors in [16] to succeed in obtaining a single determinant representation of the domain wall partition function of the eight-vertex model with a non-diagonal reflecting end.

5 Determinant representations of the scalar products

Due to the fact that the states \(|1, \ldots, 1\rangle, |2, \ldots, 2\rangle\) and their dual states \langle 1, \ldots, 1|, \langle 2, \ldots, 2|\) are invariant under the action of the F-matrix \(F_{1 \ldots N}(l)\) (4.12), the calculation of the scalar products (3.10)-(3.13) can be performed in the F-basis. Namely,

\[
S^{I,II}(\{u_{a}\}; \{v_{i}^{(2)}\}) = \langle 1, \ldots, 1|\tilde{T}_{F}^{+}(\lambda - 2(M - 1)\eta \hat{1}, \lambda|u_{M})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda|u_{1})_{1/2} \\
\times \tilde{T}_{F}^{+}(\lambda + 2\eta \hat{1}, \lambda|v_{1}^{(2)})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda + 2M\eta \hat{1}, \lambda|v_{M})_{1/2}|2, \ldots, 2\rangle,
\]
(5.1)

\[
S^{II,II}(\{u_{a}\}; \{v_{i}^{(1)}\}) = \langle 2, \ldots, 2|\tilde{T}_{F}^{+}(\lambda, \lambda + 2(M - 1)\eta \hat{1}, \lambda|u_{M})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda|u_{1})_{1/2} \\
\times \tilde{T}_{F}^{+}(\lambda, \lambda - 2\eta \hat{2}|v_{1}^{(1)})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda - 2M\eta \hat{2}|v_{M})_{1/2}|1, \ldots, 1\rangle,
\]
(5.2)

\[
S^{I,I}(\{u_{a}\}; \{v_{i}^{(1)}\}) = \langle 1, \ldots, 1|\tilde{T}_{F}^{+}(\lambda - 2(M - 1)\eta \hat{1}, \lambda|u_{M})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda|u_{1})_{1/2} \\
\times \tilde{T}_{F}^{+}(\lambda, \lambda + 2\eta \hat{1}|v_{1}^{(1)})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda + 2M\eta \hat{1}|v_{M})_{1/2}|1, \ldots, 1\rangle,
\]
(5.3)

\[
S^{II,I}(\{u_{a}\}; \{v_{i}^{(2)}\}) = \langle 2, \ldots, 2|\tilde{T}_{F}^{+}(\lambda, \lambda + 2(M - 1)\eta \hat{1}, \lambda|u_{M})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda|u_{1})_{1/2} \\
\times \tilde{T}_{F}^{+}(\lambda, \lambda - 2\eta \hat{2}, \lambda|v_{1}^{(2)})_{1/2} \ldots \tilde{T}_{F}^{+}(\lambda, \lambda - 2M\eta \hat{2}, \lambda|v_{M})_{1/2}|2, \ldots, 2\rangle.
\]
(5.4)

In the above equations, we have used the identity: \(\hat{1} = -\hat{2}\). Thanks to the polarization free representations (4.21) and (4.22) of the pseudo-particle creation/annihilation operators, we can obtain the determinant representations of the scalar products.
5.1 The scalar products $S^{I,II}$ and $S^{II,I}$

It was shown [16] that the scalar product $S^{I,II}(\{u_\alpha\}; \{v^{(2)}_i\})$ (resp. $S^{II,I}(\{u_\alpha\}; \{v^{(1)}_i\})$) can be expressed in terms of some determinant no matter the parameters $\{v^{(2)}_i\}$ (resp.$\{v^{(1)}_i\}$) satisfy the associated Bethe ansatz equations or not. In this subsection we do not require these parameters being the roots of the Bethe ansatz equations. Let us introduce two functions

$$\mathcal{Z}_N^{(I)}(\{\tilde{u}_j\}) \equiv S^{I,II}(\{u_\alpha\}; \{v_i\}) = \langle 1, \ldots, 1 | \tilde{T}_F^- (\lambda-2(M-1)\eta\bar{1}, \lambda|\tilde{u}_N)^2 \ldots \tilde{T}_F^- (\lambda+2M\eta\bar{1}, \lambda|\tilde{u}_1)^2 | 2, \ldots, 2 \rangle, (5.5)$$

$$\mathcal{Z}_N^{(II)}(\{\tilde{u}_j\}) \equiv S^{II,I}(\{u_\alpha\}; \{v_i\}) = \langle 2, \ldots, 2 | \tilde{T}_F^+ (\lambda, \lambda+2(M-1)\eta\bar{2}|\tilde{u}_N)^2 \ldots \tilde{T}_F^+ (\lambda, \lambda-2M\eta\bar{2}|\tilde{u}_1)^2 | 1, \ldots, 1 \rangle, (5.6)$$

where $N$ free parameters $\{\tilde{u}_j|J=1, \ldots N\}$ are given by

$$\tilde{u}_i = u_i \text{ for } i = 1, \ldots, M, \quad \text{and} \quad \tilde{u}_{M+i} = v_i \text{ for } i = 1, \ldots, M. \quad (5.7)$$

Note that these functions $\mathcal{Z}_N^{(I)}(\{\tilde{u}_j\})$ and $\mathcal{Z}_N^{(II)}(\{\tilde{u}_j\})$ correspond to the partition functions of the eight-vertex model with domain wall boundary conditions and one reflecting end [16] specified by the non-diagonal K-matrices (2.13) and (2.15) respectively [46].

The polarization free representations (4.21) and (4.22) of the pseudo-particle creation/annihilation operators allowed ones [46] to express the above functions in terms of the determinants representations of some $N \times N$ matrices as follows:

$$\mathcal{Z}_N^{(I)}(\{\tilde{u}_j\}) = \prod_{k=1}^M \frac{\sigma(\lambda_{12} + 2k\eta)\sigma(\lambda_{12} - 2k\eta + \eta)}{\sigma(\lambda_{12} + k\eta)\sigma(\lambda_{12} - k\eta + \eta)} \prod_{i=1}^N \frac{\sigma(\tilde{u}_i + z_i)}{\sigma(\tilde{u}_i + z_i + \eta)} \prod_{\alpha<\beta} \frac{\sigma(\bar{u}_\alpha - \bar{u}_\beta)\sigma(\bar{u}_\alpha + \bar{u}_\beta + \eta \bar{1})\prod_{k<1} \sigma(z_k - z_1)\sigma(z_1 + z_k)}{\sigma(z_k - z_k)\sigma(z_1 + z_k)}, \quad (5.8)$$

$$\mathcal{Z}_N^{(II)}(\{\tilde{u}_j\}) = \prod_{k=1}^M \frac{\sigma(\lambda_{21} + \eta - 2k\eta)\sigma(\lambda_{21} - \eta + 2k\eta)}{\sigma(\lambda_{21} - k\eta)\sigma(\lambda_{21} + k\eta)} \prod_{i=1}^N \frac{\sigma(\tilde{u}_i + z_i)}{\sigma(\tilde{u}_i + z_i + \eta)} \prod_{\bar{\alpha}>\bar{\beta}} \frac{\sigma(\bar{u}_\alpha - \bar{u}_\beta)\sigma(\bar{u}_\alpha + \bar{u}_\beta + \eta \bar{2})\prod_{k<1} \sigma(z_k - z_1)\sigma(z_1 + z_k)}{\sigma(z_k - z_k)\sigma(z_1 + z_k)}, \quad (5.9)$$

where the $N \times N$ matrices $\mathcal{N}^{(I)}(\{\bar{u}_\alpha\}; \{z_i\})$ and $\mathcal{N}^{(II)}(\{\bar{u}_\alpha\}; \{z_i\})$ are given by

$$\mathcal{N}^{(I)}(\{\bar{u}_\alpha\}; \{z_i\})_{\alpha,j} = \frac{\sigma(\eta)\sigma(\lambda_1 + \xi - z_j)}{\sigma(\bar{u}_\alpha - z_j)\sigma(\bar{u}_\alpha + z_j + \eta)\sigma(\lambda_1 + \xi + \bar{u}_\alpha)} \frac{\sigma(\lambda_2 + \xi + z_j)}{\sigma(\lambda_2 + \xi + \bar{u}_\alpha)\sigma(\bar{u}_\alpha - z_j + \eta)\sigma(\bar{u}_\alpha + z_j)}, \quad (5.10)$$
\[ \mathcal{N}^{(II)}(\{\bar{u}_\alpha\}; \{z_i\})_{\alpha,j} = \frac{\sigma(\eta)\sigma(\lambda_2 + \bar{\xi} - z_j)}{\sigma(u_\alpha - z_j)\sigma(u_\alpha + z_j + \eta)}\frac{\sigma(\lambda_2 + \bar{\xi} - z_j)}{\sigma(\lambda_1 + \bar{\xi} + z_j)\sigma(2u_\alpha + 2\eta)} \times \frac{\sigma(\lambda_1 + \bar{\xi} - u_\alpha - \eta)}{\sigma(u_\alpha - z_j + \eta)\sigma(\bar{u}_\alpha + z_j)}. \] (5.11)

The above single determinant representations are crucial to construct the determinant representations of the remaining scalar products \(S^{I,I}\) and \(S^{II,II}\) in the next subsection.

5.2 The scalar products \(S^{I,I}\) and \(S^{II,II}\)

Let us introduce two sets of functions \(\{H_j^{(I)}(u; \{z_i\}, \{v_i\})|j = 1, \ldots, M\}\) and \(\{H_j^{(II)}(u; \{z_i\}, \{v_i\})|j = 1, \ldots, M\}\)

\[ H_j^{(I)}(u; \{z_i\}, \{v_i\}) = F_1(u) \prod_{l=1}^{N} \frac{\sigma(u + z_i)}{\sigma(u + z_i + \eta)} \frac{\prod_{k \neq j} \sigma(u + v_k + 2\eta)\sigma(u - v_k + \eta)}{\prod_{k \neq j} \sigma(v_k + u + 2\eta)\sigma(v_k - u - \eta)}, \] (5.12)

\[ H_j^{(II)}(u; \{z_i\}, \{v_i\}) = F_3(u) \prod_{l=1}^{N} \frac{\sigma(u - z_i)}{\sigma(u - z_i + \eta)} \frac{\prod_{k \neq j} \sigma(v_k + u + 2\eta)\sigma(v_k - u + \eta)}{\sigma(v_k + u + \eta)\sigma(u - v_j)\sigma(2u + \eta)}, \] (5.13)

where the coefficients \(\{F_i(u)|i = 1, 2, 3, 4\}\) are

\[ F_1(u) = \sigma(\lambda_2 + \bar{\xi} + u + \eta)\sigma(\lambda_2 + \bar{\xi} - u - \eta)\sigma(\lambda_1 + \bar{\xi} - u - \eta)\sigma(\lambda_1 + \xi + u + \eta), \] (5.14)

\[ F_2(u) = \sigma(\lambda_2 + \bar{\xi} - u)\sigma(\lambda_2 + \bar{\xi} + u)\sigma(\lambda_1 + \bar{\xi} + u)\sigma(\lambda_1 + \xi - u), \] (5.15)

\[ F_3(u) = \sigma(\lambda_2 + \bar{\xi} - u - \eta)\sigma(\lambda_2 + \bar{\xi} - u + \eta)\sigma(\lambda_1 + \bar{\xi} + u + \eta)\sigma(\lambda_1 + \xi - u - \eta), \] (5.16)

\[ F_4(u) = \sigma(\lambda_2 + \bar{\xi} + u)\sigma(\lambda_2 + \bar{\xi} - u)\sigma(\lambda_1 + \bar{\xi} - u)\sigma(\lambda_1 + \xi + u). \] (5.17)

Let us consider the scalar product \(S^{I,I}(\{u_\alpha\}; \{v_i^{(1)}\})\) defined by (5.2). The expression (5.3) of \(S^{I,I}(\{u_\alpha\}; \{v_i^{(1)}\})\) under the F-basis and the polarization free representations (4.21) and (4.22) of the pseudo-particle creation/annihilation operators allow us to compute the scalar product following the similar procedure as that in [3] for the bulk case as follows.

In front of each operators \(\hat{T}_F^-\) in (5.3), we insert a sum over the complete set of spin states \(|j_1, \ldots, j_i\rangle\rangle\), where \(|j_1, \ldots, j_i\rangle\rangle\) is the state with \(i\) spins being \(\epsilon_2\) in the sites \(j_1, \ldots, j_i\) and \(2M - i\) spins being \(\epsilon_1\) in the other sites. We are thus led to consider some intermediate
functions of the form
\[ G^{(i)}(u_1, \ldots, u_i | j_{i+1}, \ldots, j_M; \{v_1^{(1)}\}) = \ll j_{i+1}, \ldots, j_M, \tilde{T}_F^{-}(\lambda-2(i-1)\eta \hat{1}, \lambda | u_1) \frac{1}{2} \ldots \tilde{T}_F^{-}(\lambda, \lambda | u_1)^2 \times \tilde{T}_F^{+}(\lambda, \lambda+2\eta \hat{1} | v_1^{(1)}) \frac{1}{2} \ldots \tilde{T}_F^{+}(\lambda, \lambda+2M\eta \hat{1} | v_M^{(1)})^2 | 1, \ldots, 1), \]
\[ i = 0, 1, \ldots, M, \quad (5.18) \]

which satisfy the following recursive relation:
\[ G^{(i)}(u_1, \ldots, u_i | j_{i+1}, \ldots, j_M; \{v_1^{(1)}\}) = \sum_{j \neq j_{i+1}, \ldots, j_M} \ll j_{i+1}, \ldots, j_M, \tilde{T}_F^{-}(\lambda-2(i-1)\eta \hat{1}, \lambda | u_1) \frac{1}{2} \ldots \tilde{T}_F^{-}(\lambda, \lambda | u_1)^2 | j, j_{i+1}, \ldots, j_M, \gg \]
\[ \times G^{(i-1)}(u_1, \ldots, u_{i-1} | j, j_{i+1}, \ldots, j_M; \{v_1^{(1)}\}), \quad i = 1, \ldots, M. \quad (5.19) \]

Note that the last of these functions \( \{G^{(i)} | i = 0, \ldots, M\} \) is precisely the scalar product \( S^{I,J}(\{u_\alpha\}; \{v_1^{(1)}\}) \), namely,
\[ G^{(M)}(u_1, \ldots, u_M; \{v_1^{(1)}\}) = S^{I,J}(\{u_\alpha\}; \{v_1^{(1)}\}), \quad (5.20) \]

whereas the first one,
\[ G^{(0)}(j_1, \ldots, j_M; \{v_1^{(1)}\}) = \ll j_1, \ldots, j_M, \tilde{T}_F^{+}(\lambda, \lambda+2M\eta \hat{1} | v_M^{(1)})^2 | 1, \ldots, 1), \]

is closely related to the partition function computed in [46]. Solving the recursive relations (5.19), we find that if the parameters \( \{v_k^{(1)}\} \) satisfy the first set of Bethe ansatz equations (2.53) the scalar product \( S^{I,J}(\{u_\alpha\}; \{v_1^{(1)}\}) \) has the following determinant representation
\[ S^{I,J}(\{u_\alpha\}; \{v_1^{(1)}\}) = \prod_{k=1}^{M} \left( \frac{\sigma(\lambda_{12}+2\eta-2k\eta)\sigma(\lambda_{12}-\eta+2k\eta)}{\sigma(\lambda_{12}-(k-1)\eta)\sigma(\lambda_{12}+k\eta)} \prod_{i=1}^{N} \frac{\sigma(u_k-z_i)\sigma(v_k^{(1)}-z+i)}{\sigma(u_k-z_i+\eta)\sigma(v_k^{(1)}-z+i+\eta)} \right) \]
\[ \times \frac{\text{det} \mathcal{N}^{(I)}(\{u_\alpha\}; \{v_1^{(1)}\})}{\prod_{\alpha<\beta} \sigma(u_\alpha-u_\beta)\sigma(u_\alpha+u_\beta+\eta)\prod_{k=1}^{M} \sigma(v_k^{(1)}-v_1^{(1)}+v_1^{(1)}+\eta)}, \quad (5.21) \]

where the \( M \times M \) matrix \( \mathcal{N}^{(I)}(\{u_\alpha\}; \{v_1^{(1)}\}) \) is given by
\[ \mathcal{N}^{(I)}(\{u_\alpha\}; \{v_1^{(1)}\})_{\alpha,j} = \frac{\sigma(\eta)\sigma(2u_\alpha)\sigma(2v_1^{(1)}+2\eta)H^{(I)}_j(\{u_\alpha\}; \{v_1^{(1)}\})}{\sigma(\lambda_{1}+\xi+u_\alpha)\sigma(\lambda_{2}+\xi+u_\alpha)\sigma(\lambda_{2}+\xi-v_j^{(1)}-\eta)\sigma(\lambda_{1}+\xi-v_j^{(1)}-\eta)}, \quad (5.22) \]
Using the similar method as above, we have that the scalar product $S^{II,II}(\{u_\alpha\}; \{v_i^{(2)}\})$ has the following determinant representation provided that the parameters $\{v_k^{(2)}\}$ satisfy the second set of Bethe ansatz equations \([2.49]\)

\[
S^{II,II}(\{u_\alpha\}; \{v_i^{(2)}\}) = \prod_{k=1}^{M} \left\{ \frac{\sigma(\lambda_1 + \eta + 2k\eta)\sigma(\lambda_2 + \eta + 2k\eta)}{\sigma(\lambda_1 + \eta + (k-1)\eta)\sigma(\lambda_2 + \eta + k\eta)} \prod_{l=1}^{N} \frac{\sigma(u_k + z_l)\sigma(v_k^{(2)} + z_l)}{\sigma(u_k + z_l + \eta)\sigma(v_k^{(2)} + z_l + \eta)} \right\} \\
\times \frac{\det \mathcal{N}^{(II)}(\{u_\alpha\}; \{v_i^{(2)}\})}{\prod_{\alpha<\beta} \sigma(u_\alpha - u_\beta)\sigma(u_\alpha + u_\beta + \eta)\prod_{k>l} \sigma(v_k^{(2)} - v_l^{(2)})\sigma(v_k^{(2)} + v_l^{(2)} + \eta)},
\]

(5.23)

where the $M \times M$ matrix $\mathcal{N}^{(II)}(\{u_\alpha\}; \{v_i^{(2)}\})$ is given by

\[
\mathcal{N}^{(II)}(\{u_\alpha\}; \{v_i^{(2)}\})_{\alpha,j} = \frac{\sigma(\eta)\sigma(2u_\alpha + \eta)\sigma(2v_j^{(2)})H_j^{(II)}(u_\alpha; \{z_l\}; \{v_i^{(2)}\})}{\sigma(\lambda_2 + \xi - u_\alpha - \eta)\sigma(\lambda_1 + \xi - u_\alpha - \eta)\sigma(\lambda_2 + \xi + v_j^{(2)})\sigma(\lambda_1 + \xi + v_j^{(2)})}.
\]

(5.24)

Now we are in position to compute the norms of the Bethe states which can be obtained by taking the limit $u_\alpha \to v_\alpha^{(i)}$, $\alpha = 1, \ldots, M$. The norm of the first set of Bethe state \([2.49]\) is

\[
\mathcal{N}^{I,I}(\{v_\alpha^{(1)}\}) = \lim_{u_\alpha \to v_\alpha^{(i)}} S^{I,I}(\{u_\alpha\}; \{v_i^{(1)}\}) \\
= \prod_{k=1}^{M} \left\{ \frac{\sigma(\lambda_1 + 2\eta - 2k\eta)\sigma(\lambda_2 + \eta + 2k\eta)}{\sigma(\lambda_1 - (k-1)\eta)\sigma(\lambda_2 + k\eta)} \prod_{l=1}^{N} \frac{\sigma^2(v_k^{(1)} - z_l)}{\sigma^2(v_k^{(1)} - z_l + \eta)} \right\} \\
\times \prod_{\alpha<\beta} \frac{\sigma(v_\alpha^{(1)} + v_\beta^{(1)})\sigma(v_\alpha^{(1)} - v_\beta^{(1)} - \eta)}{\sigma(v_\alpha^{(1)} - v_\beta^{(1)})\sigma(v_\alpha^{(1)} + v_\beta^{(1)} + \eta)} \det \Phi^{(I)}(\{v_\alpha^{(1)}\}),
\]

(5.25)

where the matrix elements of $M \times M$ matrix $\Phi^{(I)}(\{v_\alpha\})$ are given by

\[
\Phi^{(I)}_{\alpha,j}(\{v_\alpha\}) = \frac{\sigma(\eta)\sigma(\lambda_2 + \xi - v_\alpha)\sigma(\lambda_1 + \xi + v_\alpha)\sigma(\lambda_1 + \xi - v_\alpha)}{\sigma(\lambda_1 + \xi + v_\alpha)\sigma(\lambda_2 + \xi - v_j - \eta)\sigma(\lambda_1 + \xi - v_j - \eta)} \\
\times \frac{\sigma(2v_\alpha)\sigma(2v_j + 2\eta)}{\sigma(2v_\alpha + \eta)\sigma(2v_j + \eta)} \prod_{l=1}^{N} \frac{\sigma(v_\alpha - z_l + \eta)}{\sigma(v_\alpha - z_l)} \\
\times \frac{\partial}{\partial v_\alpha} \ln \left\{ \frac{\sigma(\lambda_2 + \xi + v_j + \eta)\sigma(\lambda_2 + \xi - v_j - \eta)\sigma(\lambda_1 + \xi + v_j + \eta)\sigma(\lambda_1 + \xi + v_j - \eta)}{\sigma(\lambda_2 + \xi - v_j)\sigma(\lambda_2 + \xi + v_j)\sigma(\lambda_1 + \xi + v_j)\sigma(\lambda_1 + \xi - v_j)} \right\} \prod_{l=1}^{N} \frac{\sigma(v_j + z_l)\sigma(v_j - z_l)}{\sigma(v_j + z_l + \eta)\sigma(v_j - z_l + \eta)} \prod_{k \neq j} \frac{\sigma(v_j + v_k + 2\eta)\sigma(v_j - v_k + \eta)}{\sigma(v_j + v_k)\sigma(v_j - v_k - \eta)}.
\]

(5.26)
the norm of the second set of Bethe state \( [2,50] \) is given by

\[
N^{II,II}(\{v_{α}^{(2)}\}) = \lim_{u_{α} \rightarrow v_{α}^{(2)}} S^{II,II}(\{u_{α}\};\{v_{i}^{(2)}\})
\]

\[
= \prod_{k=1}^{M} \frac{σ(λ_{12}+kη)σ(λ_{21}−η+2kη)}{σ(λ_{12}+kη)σ(λ_{21}+(k−1)η)} \prod_{l=1}^{N} \frac{σ^{2}(v_{k}^{(2)}+zi)}{σ^{2}(v_{k}^{(2)}+zi+η)}
\]

\[
\times \prod_{α≠β} \frac{σ(v_{α}^{(2)}+v_{β}^{(2)}+2η)σ(v_{α}^{(2)}−v_{β}^{(2)}−η)}{σ(v_{α}^{(2)}−v_{β}^{(2)})σ(v_{α}^{(2)}+v_{β}^{(2)}+η)} \detΦ^{II}(\{v_{α}^{(2)}\}),
\]

where the matrix elements of \( M × M \) matrix \( Φ^{II}(\{v_{α}\}) \) are given by

\[
Φ^{II}_{α,β}(\{v_{α}\}) = \frac{σ(η)σ(λ_{2}+λ_{α}+η)σ(λ_{1}+λ_{β}+η)}{σ(λ_{1}−λ_{α}−η)σ(λ_{2}+λ_{β}+η)} \frac{σ(2v_{α}+2η)}{σ(2v_{α}−2η)} \prod_{l=1}^{N} \frac{σ(v_{α}−zi)}{σ(v_{α}−zi+η)}
\]

\[
\times \frac{∂}{∂v_{α}} \ln \left\{ \frac{σ(λ_{2}−v_{β}+η)σ(λ_{2}+v_{β}+η)σ(λ_{1}+v_{β}+η)}{σ(λ_{2}+v_{β}+η)σ(λ_{2}−v_{β}+η)σ(λ_{1}+v_{β}−η)} \prod_{l=1}^{N} \frac{σ(v_{j}+zi)}{σ(v_{j}−zi+η)} \prod_{k≠j} \frac{σ(v_{j}+v_{k}+2η)}{σ(v_{j}−v_{k}−η)} \right\}.
\]

Moreover, one may check that if the parameters \( \{u_{α}\} \) satisfy the Bethe ansatz equations (i.e. on shell) but different from \( \{v_{α}^{(1)}\} \) the corresponding scalar products \( S^{II,II}(\{u_{α}\};\{v_{i}^{(1)}\}) \) or \( S^{II,II}(\{u_{α}\};\{v_{i}^{(2)}\}) \) vanishes, namely, the corresponding Bethe states are orthogonal.

6 Conclusions

We have studied scalar products between an on-shell Bethe state and a general state (or an off-shell Bethe state) of the open XYZ chain with non-diagonal boundary terms, where the non-diagonal K-matrices \( K^{±}(u) \) are given by \( (2.14) \) and \( (2.16) \). In our calculation the factorizing F-matrix \( (4.12) \) of the eight-vertex SOS model, which leads to the polarization free representations \( (4.21) \) and \( (4.22) \) of the associated pseudo-particle creation/annihilation operators, has played an important role. It is found that the scalar products can be expressed in terms of the determinants \( (5.28) \), \( (5.39) \), \( (5.21) \) and \( (5.23) \). By taking the on shell limit, we obtain the determinant representations (or Gaudin formula) \( (5.25)−(5.26) \) and \( (5.27)−(5.28) \) of the norms of the Bethe states. However it should be emphasized that, in contrast with
those of the open XXZ chain with diagonal boundary terms, the dual states (3.3) and (3.4) are generally no longer the eigenstates of the open chain with non-diagonal boundary terms even if the parameters satisfy the associated Bethe ansatz equations.

Now let us consider the various trigonometric limits of our results. The definition of the elliptic function (2.1) implies the following asymptotic behaviors

\[
\lim_{\tau \to +i\infty} \sigma(u) = -2e^{i\pi \tau} \sin \pi u + \cdots,
\]

\[
\lim_{\tau \to +i\infty} \theta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (u, \tau) = 1 + \cdots.
\]

Let consider the first trigonometric limit, i.e. redefining the parameters as follows

\[
u \to \frac{u}{\pi}, \quad \lambda_i \to \frac{\lambda_i - \frac{1}{4} + \frac{3\tau}{4}}{\pi}, \quad i = 1, 2, \tag{6.1}
\]

\[
\xi \to \frac{\xi + \frac{1}{4} - \frac{3\tau}{4}}{\pi}, \quad \bar{\xi} \to \frac{\bar{\xi} + \frac{1}{4} - \frac{3\tau}{4}}{\pi}. \tag{6.2}
\]

Then taking the limit \(\tau \to +i\infty\), the resulting K-matrices \(K^\pm(u)\) given by (2.14) and (2.16) become the non-diagonal trigonometric K-matrices considered in [44], and our results (5.8), (5.9), (5.21), (5.23), (5.25)-(5.26) and (5.27)-(5.28) recover those of [44] for the open XXZ chain with non-diagonal boundary terms. On the other hand, if we redefine the parameters as follows (c.f. (6.1)-(6.2))

\[
\xi \to \frac{\xi - \lambda_2 + \frac{i}{2} \ln 2}{\pi}, \quad \bar{\xi} \to \frac{\bar{\xi} - \lambda_2 + \frac{i}{2} \ln 2}{\pi}, \tag{6.3}
\]

\[
u \to \frac{u}{\pi}, \tag{6.4}
\]

and then take the limit: \(\tau \to +i\infty\) and \(\lambda_1 \to +i\infty\), the resulting K-matrices \(K^\pm(u)\) given by (2.14) and (2.16) become the diagonal trigonometric K-matrices and our result recovers that of [17, 18] for the open XXZ chain with diagonal boundary terms.

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