Inflation in Multidimensional Quantum Cosmology

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Abstract

We extend to multidimensional cosmology Vilenkin’s prescription of tunnelling from nothing for the quantum origin of the observable Universe. Our model consists of a $D+4$-dimensional spacetime of topology $\mathcal{R} \times S^3 \times S^D$, with a scalar field (“chaotic inflaton”) for the matter component. Einstein gravity and Casimir compactification are assumed. The resulting minisuperspace is 3-dimensional. Patchwise we find an approximate analytic solution of the Wheeler–DeWitt equation through which we discuss the tunnelling picture and the probability of nucleation of the classical Universe with compactifying extra dimensions. Our conclusion is that the most likely initial conditions, although they do not lead to the compactification of the internal space, still yield (power-law) inflation for the outer space. The scenario is physically acceptable because the inner space growth is limited to $\sim 10^{11}$ in 100 e-foldings.

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of inflation, starting from the Planck scale.

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I. INTRODUCTION

A multidimensional spacetime is often required by fundamental physics [1]. From the cosmological point of view, it is interesting to exploit the phenomenology related to the extra space–like dimensions (the “internal space”). Since such extra dimensions have not left any trace in the observable Universe from the nucleosynthesis onward, the question is what has kept them small and (almost) static, i.e., how to have a stable compactification. A static solution has been provided using an effective potential with quantum effects [2] (Casimir compactification scheme) or a condensed antisymmetric tensor [3] (monopole compactification scheme). Later on, the effective action for quantum effects and its low temperature limit have been evaluated [4] for more general, non static, backgrounds, of cosmological interest. In a closed cosmological model characterized by two scale factors, $a(t)$ associated to the ordinary 3–dimensional space and $b(t)$ to the extra dimensions, the general formula for the effective potential assumes a simple expression in the flat–space limit \[ a(t) \gg b(t). \] This is the Casimir potential commonly used in literature, which we will make use of in this paper.

The scale factor $b(t)$ can be treated as a scalar field, the dilaton, in a 4–dimensional spacetime, under the action of the Casimir potential. It has been shown [5] that for the dilaton field to be an ordinary field, i.e., for it to have a canonical kinetic term, an appropriate and unique field redefinition is required. This point is often neglected in the current literature, but at the price of an intrinsical instability [6].

Once the dynamics of the internal space is reduced to that of an ordinary scalar field in four dimensions, one may ask whether the dilaton can drive inflation [1] in external space. The answer has been worked out in the Casimir [7] and in the monopole [8] model. In both cases the theory includes a cosmological constant. The resulting potentials, qualitatively very similar, have a local maximum that, for specific, fine tuned, initial conditions, makes possible some sort of inflation. But, as Halliwell [8] has shown explicitly for the monopole case, and as we will discuss later for the Casimir case, the top of the potential is not flat.
enough for the duration of the inflationary period to be satisfactory. Thus we are forced to introduce another scalar field, the *inflaton*, with its own appropriate potential. It has been shown [9], indeed, that in this way, it is possible to have an inflationary stage, without, in the meantime, destroying compactification, provided suitable initial conditions are given. Let us go through this specific model in some detail since it will be adopted also in the present paper.

The model [9] consists of:

- an *N*-dimensional universe of topology $\mathcal{R} \times S^3 \times S^D$, with $N = D + 4$;
- quantum corrections of the Casimir type, *i.e.*, one–loop vacuum fluctuations of matter fields in a compact space;
- a scalar field $\phi$ driving inflation;
- a multidimensional cosmological constant tuned to ensure that no effective 4–dimensional cosmological constant appears.

After dimensional reduction [10] and field redefinition [5], the internal space can be treated as an ordinary scalar field $\sigma$ in Einstein theory. The whole dynamics—standard 4–dimensional model with two scalar fields, $\sigma$ (dilaton) and $\phi$ (inflaton)—is then derived from the action:

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G_N} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\sigma, \phi) \right].
\] (1.1)

In (1.1), $R$ is the 4–dimensional scalar curvature and

\[
U(\sigma, \phi) = V_0(\sigma) + e^{-D\sigma/\sigma_0} V_1(\phi)
\] (1.2)

is the combined potential for the two scalar fields. The $\sigma$ field is related to the scalar factor of the internal space $b$ through
\[
\sigma = \sigma_0 \ln \frac{b}{b_0}, \quad \text{where} \quad \sigma_0 \equiv \sqrt{\frac{D(D + 2)}{16\pi G_N}},
\]

and \(b_0\) is the expected present size of the radius of the internal space, corresponding to the static ground state \(\sigma = 0\). Such a state exists if one assumes a potential \(V_0(\sigma)\) of the Casimir type:

\[
V_0(\sigma) = \frac{(D - 1)\sigma_0^2}{(D + 4)b_0^2} \left[ \frac{2}{D + 2} e^{-2(D+2)\sigma/\sigma_0} + e^{-D\sigma/\sigma_0} \right] - \frac{D + 4}{D + 2} e^{-(D+2)\sigma/\sigma_0}.
\]

The \(\phi\) field can drive the inflation of external space if we assume an appropriate potential \(V_1(\phi)\). For example, in the chaotic inflation scenario, \(\phi\) is a “classical” field with the U–shaped potential:

\[
V_1(\phi) = \lambda \phi^4.
\]

Then the (total) potential (1.2) is shown in Fig. 2. For sufficiently small initial values of \(|\phi|\), there exists a channel of stability where the classical universe can undergo an inflationary stage in external space while remaining compactified in internal space (“slow rolling” of the inflaton along the potential in the \(\phi\)–direction). The \(\phi\) field ends its evolution with damped oscillations around its ground state \(\phi = 0\), leading the universe to the present Friedmannian stage, if one assumes \(b_0 < 10^5 \sqrt{Dl_{Pl}}\).

In this paper, we look for inflationary solutions, with particular regard to the scenario described above [9], in the framework of quantum cosmology [11]. In fact, through the wave function of the Universe \(\Psi\) we can evaluate the probability of different initial conditions (although this evaluation is a problem in itself, as we will discuss with some more detail in Section IIIB). To find a specific \(\Psi\) we need solving the Wheeler–DeWitt equation with a given law of boundary conditions. Two prescriptions are commonly used in literature: the “no–boundary” conditions proposed by Hartle and Hawking [12], and Vilenkin’s “tunnelling from nothing” [13,14]. The former have been more widely used, but the latter usually give a larger measure of classical solutions with sufficient inflation [11,15].
Therefore (see also [16]) we will consider only Vilenkin conditions in this paper. They can be stated as follows:

- take only the outgoing modes of the wave function at the singular boundary of the superspace;

- impose a finiteness condition on the wave function.

In quantum cosmology the superspace is the configuration space of the universe (3–geometries, local configurations of matter fields): the classical spacetime corresponds to the region of the superspace in which the wave function oscillates with large phase values. The non–singular boundary of the superspace is that part of the boundary that includes 3–geometries given through a slicing of a regular 4–geometry: the rest of the boundary is called “singular”.

In many one– or two–dimensional minisuperspace models [11] Vilenkin conditions describe a classical universe that nucleates via a tunnelling from the non–singular boundary of the minisuperspace (the nothing) through the superpotential barrier. Although such a picture does not always hold, the Vilenkin wave function still selects inflationary initial conditions for a wide class of models, in 4–dimensional cosmology. In particular, it has success [14] in the chaotic inflation scenario (Einstein gravity plus chaotic inflaton). In this paper we extend Vilenkin’s idea to multidimensional cosmology, introducing\(^1\) in this chaotic scenario the dynamics of the dilaton.

In Section II, we derive the Wheeler–DeWitt equation for our model. The resulting minisuperspace is 3–dimensional, which makes the analysis very difficult. We evaluate the

\(^1\)Quantum cosmology with Hartle–Hawking boundary conditions for a model Einstein gravity plus dilaton was studied by Okada and Yoshimura [7]. They also very briefly comment that the Vilenkin boundary conditions predict that the nucleation of a classical universe like our one would be exponentially suppressed. As we mentioned above, in such a model, the inflationary scenario is not satisfying already at the classical level. Thus it is necessary to introduce the inflaton \(\phi\).
wave function, in the semiclassical limit, in all its interesting subregions. In Section III we comment about the tunnelling picture and the meaning of nothing in multidimensional cosmology, we discuss the inflationary solutions and the quantum stability of the internal space, and we present our conclusions.

II. THE WAVE FUNCTION

A. The Wheeler–DeWitt equation

We assume that the external space has the metric of a 3–sphere of radius \(a\), and the internal space that of a \(D\)–sphere of radius \(b\). Let us express all the space–time coordinates in units of \(\sqrt{2G_N/3\pi}\), and let us introduce the dimensionless fields

\[
\sigma^{\text{New}} = \sqrt{\frac{4\pi G_N}{3}} \sigma, \quad \phi^{\text{New}} = \sqrt{\frac{4\pi G_N}{3}} \phi; \tag{2.1}
\]

at the same time we drop the superscript \(\text{New}\). In our model the metric of the world “4–dimensional space–time + \(\sigma, \phi\) fields” is

\[
ds^2 = \frac{2G_N}{3\pi} \left( N^2 dt^2 - a^2(t)d\Omega_3^2 \right), \tag{2.2}
\]

After a simple integration, the action \(\text{(1.1)}\) becomes

\[
S = \int dt \frac{N}{2} \left[ -\frac{a\dot{a}^2}{N^2} + \frac{a^3}{N^2} \left( \dot{\sigma}^2 + \dot{\phi}^2 \right) + a - a^3 U(\sigma, \phi) \right], \tag{2.3}
\]

where a dot denotes time derivatives. The spatial degrees of freedom of the inflaton have also been frozen in the minisuperspace scheme. The potential in \(\text{(2.3)}\) is now

\[
U(\sigma, \phi) = V_D(\sigma) + e^{-D\sigma/\sigma_D} V(\phi)
\]

\[
V_D(\sigma) = K \left[ \frac{2}{D+2} e^{-2(D+2)\sigma/\sigma_D} + e^{-D\sigma/\sigma_D} \right. \left. \frac{D+4}{D+2} e^{-(D+2)\sigma/\sigma_D} \right],
\]

\[
V(\phi) = \frac{\lambda \phi^4}{4},
\]

\[
K = \frac{2(D-1)\sigma_D^2}{(D+4)b_0^2}, \quad \sigma_D = \sqrt{\frac{D(D+2)}{12}}. \tag{2.4}
\]
The Wheeler–DeWitt equation in the minisuperspace of coordinates \( a, \sigma, \phi \), is then

\[
\frac{1}{a} \frac{\partial^2 \Psi}{\partial a^2} - \frac{1}{a^3} \left( \partial^2_{\sigma} + \partial^2_{\phi} \right) - w(a, \sigma, \phi) \Psi(a, \sigma, \phi) = 0 ,
\]

where

\[
w(a, \sigma, \phi) = a \left[ 1 - a^2 U(\sigma, \phi) \right] ,
\]

is the superpotential of the universe.

Since we want to find the most probable initial conditions for the classical motion of the universe, we consider the semiclassical wave function (i.e., the lowest order in the WKB expansion [11]). In (2.5), we assumed a simple factor ordering in the superhamiltonian: in the semiclassical limit the arbitrariness of the choice of the factor ordering does not affect the solution. The equation

\[
a = a_*(\sigma, \phi) = \frac{1}{\sqrt{U(\sigma, \phi)}}
\]

defines a surface of constant superpotential \( w = 0 \) in the minisuperspace. Eq. (2.6) describes a superpotential barrier in the \( a \)-direction: Eq. (2.7) separates the region \( 0 < a < a_*(\sigma, \phi) \), below the barrier, from the region \( a > a_*(\sigma, \phi) \), beyond the barrier. The formal analogy between the Wheeler–DeWitt equation and a “zero energy” Schrödinger equation (at least when the kinetic contributions of the matter fields are negligible) and this structure of the superpotential suggest that the universe can nucleate at the (presumed) classical/quantum boundary \( a = a_*(\sigma, \phi) \) through a quantum tunnelling process from the configuration \( a = 0 \).

To investigate this possibility, we have to evaluate the Vilenkin wave function. We will look for the approximate analytic solution of the Wheeler–DeWitt equations in the relevant regions of the minisuperspace. In this section we will proceed as follows. First we show that a behaviour of “nothing state” for the wave function is present under the barrier. Then we split the minisuperspace in two regions: i) the region of small \( \phi \), i.e., where the term with \( V(\phi) \) in the potential is negligible; ii) the region of large \( \phi \) where, on the contrary, the term with \( V(\phi) \) is dominant. We evaluate the Vilenkin wave function in these regions in the
WKB limit. Finally, we confirm our results using the method of the constant $w$ surfaces, developed by Halliwell [8].

**B. The solution of nothing**

We now look for the wave function in the region where $a$ is small, i.e., under the barrier of superpotential:

$$a^2 \ll a_{\star}^2(\sigma, \phi).$$

(2.8)

The Wheeler–DeWitt equation (2.5) reduces locally to

$$\left(a^2 \partial_{a}^2 - \partial_{\sigma}^2 - \partial_{\phi}^2 - a^4\right)\Psi = 0.$$

(2.9)

Then Eq. (2.9) is reducible, through the substitution $\Psi = \Theta(a, \sigma)\Gamma(\phi)$, to two decoupled equations in $\Theta$ and $\Gamma$, parametrically depending on the separation constant $E$, whose value can be determined by a matching of the solutions found in nearby regions. In the following we adopt the heuristic method of Halliwell [8], who neglects the separation constant with respect to the superpotential in the regions of the minisuperspace where the latter is large in modulus. This is equivalent to assume [18] that the wave function is asymptotically $\phi$–independent, where the superpotential also has this property. Indeed, for $U(\sigma, \phi) \ll 1$, values of $a$ such that $a^2 \gg 1$ also belong to region (2.8). For them, the superpotential in Eq. (2.9) is, in modulus, much greater than 1. The separation constant can be neglected and we can assume that, at the WKB lowest order, the wave function is proportional to the solution $\Theta(a)$ of the equation

$$\left(d_a^2 - a^2\right)\Theta(a) = 0.$$

(2.10)

Introducing the auxiliary variable $\Gamma(a) = \Theta(a)/a^{1/2}$ and the transformation $v = a^2/2$, Eq. (2.10) reduces to the modified Bessel equation

$$v^2 d_v^2 \Gamma + v d_v \Gamma - \left(v^2 + \frac{1}{16}\right) \Gamma = 0.$$

(2.11)
whose independent solutions are the well known modified Bessel functions of order 1/4, \( I_{1/4}(v) \) and \( K_{1/4}(v) \). Going back to the old variables, we find the growing solution \( a^{1/2}I_{1/4}(a^2/2) \) and the decreasing solution \( a^{1/2}K_{1/4}(a^2/2) \) in the \( a \)-direction. To select one of them, we impose a matching condition with the solution (2.17), discussed in the following subsection, that holds for small |\( \phi \)|. Considering only the dominant exponential factors for \( a \gg 1 \), the matching in the intersection of region (2.13) with region (2.8), gives

\[
\Psi = a^{1/2}K_{1/4}(\frac{a^2}{2}) ,
\]

solution that satisfies locally the Vilenkin boundary condition. Eq. (2.12) is the well known solution of nothing [13]: it has been found by Vilenkin in the limit of small \( a \) in the 4–dimensional model with topology \( \mathcal{R} \times S^3 \) and inflaton, without dilaton. This wave function is monotonic, does not depend on the matter fields, and it has a peak, around \( a = 0 \), whose width is of the order of the Planck length.

Eq. (2.12) is the wanted solution of nothing. Nevertheless, because of solution (2.26), the picture of tunnelling from nothing through the superpotential barrier cannot be straightforwardly extended to our multidimensional model. We will discuss in detail this problem in Section III A.

C. Nucleation of the classical Universe?

1. The minisuperspace region of small |\( \phi \)|

In this case the condition

\[
V(\phi) \ll e^{D\sigma/\sigma_D}V_D(\sigma)
\]

holds. The Wheeler–DeWitt equation can be rewritten as

\[
\left\{ a^2\partial_a^2 - \partial_\sigma^2 - \partial_\phi^2 - a^4 \left[ 1 - a^2V_D(\sigma) \right] \right\} \Psi(a, \sigma, \phi) = 0 .
\]

Then Eq. (2.14) is reducible, through the substitution \( \Psi = \Theta(a, \sigma)\Gamma(\phi) \), to two decoupled equations in \( \Theta \) and \( \Gamma \), parametrically depending on the separation constant \( E \). Following
again the heuristic method of Halliwell \[8\], of neglecting the separation constant with respect to the superpotential in the regions of the minisuperspace where the latter is large in modulus, to the WKB lowest order, $\Psi$ is then proportional to the solution of equation

$$\left\{ a^2 \partial_a^2 - \partial_\sigma^2 - a^4 \left[ 1 - a^2 V_D(\sigma) \right] \right\} \Theta(a, \sigma) = 0 .$$

(2.15)

The solution of Eq.(2.15) with Vilenkin boundary conditions is well known \[14\] in the region of the minisuperspace where the potential $V_D(\sigma)$ is "sufficiently flat," \textit{i.e.}, where

$$\left| \frac{dV_D(\sigma)}{d\sigma} \right| \ll \max \left\{ V_D(\sigma), \frac{1}{a^2} \right\} .$$

(2.16)

Such a solution, to the lowest order of the WKB expansion, is

$$\Theta(a, \sigma) = \begin{cases} \exp \left\{ - \left[ 1 - (1 - V_D(\sigma)a^2)^{3/2} \right] / [3V_D(\sigma)] \right\} & (a^2 V_D(\sigma) < 1) \\ \exp \left\{ - \left[ 1 + i (V_D(\sigma)a^2 - 1)^{3/2} \right] / [3V_D(\sigma)] \right\} & (a^2 V_D(\sigma) > 1) \end{cases}$$

(2.17)

In our model, nevertheless, unlike in Ref. \[14\], the potential $V_D(\sigma)$ has a strongly asymmetric form for $\sigma > 0$ and $\sigma < 0$. Then condition (2.16) and solutions (2.17) do not hold in an important region of the minisuperspace (Fig. 3): the region $a^2 V_D(\sigma) > 1$, $\sigma < 0$. For $\sigma \to -\infty$, in particular, the barrier of superpotential becomes narrow with respect to the configuration $a = 0$: thus, this region is particularly interesting for tunnelling conditions. To find out the wave function here, we adopt the following procedure. Under the transformation $\alpha = \log a$, the Wheeler–DeWitt equation (2.5) becomes

$$\left\{ \partial_\alpha^2 - \partial_\sigma^2 - \partial_\phi^2 - e^{4\alpha} \left[ 1 - e^{2\alpha} U(\sigma, \phi) \right] \right\} \Psi(\alpha, \sigma, \phi) = 0 .$$

(2.18)

In the semiclassical limit, we can omit the first derivative: this is equivalent to a particular choice \[8\] of the factor ordering in the Wheeler–DeWitt equation, to which, in the semiclassical limit, the wave function is insensitive. The Wheeler–DeWitt equation is now

$$\begin{cases} \left[ \partial_\alpha^2 - \partial_\sigma^2 - \partial_\phi^2 - W(\alpha, \sigma, \phi) \right] \Psi(\alpha, \sigma, \phi) = 0 \\ W(\alpha, \sigma, \phi) = e^{4\alpha} \left[ 1 - e^{2\alpha} U(\sigma, \phi) \right] \end{cases} .$$

(2.19)
Repeating the preceding discussion, see Eq. (2.15), we find that $\Psi$ is proportional to the solution $\Theta$ of the equation

$$\left\{ \partial_\alpha^2 - \partial_\sigma^2 - e^{4\alpha} \left[ 1 - e^{2\alpha} V_D(\sigma) \right] \right\} \Theta(\alpha, \sigma) = 0 .$$  \hfill (2.20)

For

$$a^2 V_D(\sigma) \gg 1 , \quad \sigma \ll -\sigma_D/D ,$$  \hfill (2.21)

Eq. (2.20) becomes

$$\left[ \partial_\alpha^2 - \partial_\sigma^2 + \frac{2K}{D+2} \exp \left( 6\alpha - 2 \frac{D+2}{\sigma_D} \sigma \right) \right] \Theta(\alpha, \sigma) = 0 .$$  \hfill (2.22)

Under the rotation

$$\bar{\alpha} = \frac{1}{g_D} \left( - \frac{D+2}{3\sigma_D} \alpha + \sigma \right)$$

$$\bar{\sigma} = \frac{1}{g_D} \left( \alpha - \frac{D+2}{3\sigma_D} \sigma \right)$$

$$g_D = \left\{ \left[ (D+2)/3\sigma_D \right]^2 - 1 \right\}^{1/2}$$  \hfill (2.23)

(for $D$ positive integer, $g_D$ is always a positive number smaller than 1), Eq. (2.22) becomes separable:

$$\left[ \partial_{\bar{\alpha}}^2 - \partial_{\bar{\sigma}}^2 + \frac{2K}{D+2} \exp (6\bar{\sigma}_D) \right] \Theta(\bar{\alpha}, \bar{\sigma}) = 0 .$$  \hfill (2.24)

Again the contribution of the separation constant is negligible where the superpotential becomes sufficiently large. This is the region where $\bar{\sigma}$ is large (i.e., condition (2.21) holds true). Proceeding as before $\Theta$ turns out to be asymptotically proportional to the solution $\Omega(\bar{\sigma})$ of the equation

$$\left[ -d_{\bar{\sigma}}^2 + \frac{2K}{D+2} \exp (6\bar{\sigma}_D) \right] \Omega(\bar{\sigma}) = 0 .$$  \hfill (2.25)

Eq. (2.25) admits monotonic independent solutions that can be written (in the original variables $a, \sigma$) as
\[ \Omega = \exp \left\{ \pm \sqrt{\frac{2K}{D + 2a^3g_D}} e^{-\frac{(D+2)\sigma}{\sigma_D}} \right\}. \] (2.26)

Vilenkin’s regularity condition \(|\Psi| < \infty\) selects the decreasing mode of \(\Omega\) in the region \(a \to \infty\) and/or \(\sigma \to -\infty\) of the singular boundary.

Note that, with an analogous method, it is possible to derive the solution of Eq. (2.15) in the region

\[ a^2V_D(\sigma) \gg 1, \quad \sigma \gg \sigma_D/D, \] (2.27)

where again Eqs. (2.16), (2.17) do not hold. Here the Wheeler–DeWitt equation becomes

\[ \left[ \partial^2_{\tilde{\alpha}} - \partial^2_{\tilde{\sigma}} + K \exp \left( 6\tilde{\alpha} - \frac{D\sigma}{\sigma_D} \right) \right] \Theta(\tilde{\alpha}, \tilde{\sigma}) = 0, \] (2.28)

from which, under the rotation

\[ \tilde{\alpha} = \frac{1}{g_D} \left( \alpha - \frac{D\sigma}{6\sigma_D} \right), \]
\[ \tilde{\sigma} = \frac{1}{g_D} \left( - \frac{D\alpha}{6\sigma_D} + \sigma \right), \]
\[ g_D = \left[ 1 - \left( \frac{D}{6\sigma_D} \right)^2 \right]^{1/2} \] (2.29)

(for \(D\) positive integer, \(g_D\) is always real, positive and smaller than 1) we get

\[ \left[ \partial^2_{\tilde{\alpha}} - \partial^2_{\tilde{\sigma}} + K e^{(6g_D\tilde{\alpha})} \right] \Theta(\tilde{\alpha}, \tilde{\sigma}) = 0. \] (2.30)

For large \(\tilde{\alpha}\), \(i.e.,\) in the region (2.27), Eq. (2.30) admits independent solutions that, in the original variables, can be written as

\[ \Theta = \exp \left( \pm i \frac{\sqrt{K}}{3g_D} a^3 e^{-D\sigma/(2\sigma_D)} \right). \] (2.31)

Of the two oscillating modes (2.31), Vilenkin’s boundary conditions select only the outgoing one (\(i.e.,\) the second in (2.31)) at the singular boundary of the minisuperspace. Note that in the region (2.27), the amplitude of the oscillations of the wave function changes slowly: contrary to what happens in the strip defined by \(a^2V_D > 1\) and \(\sigma \sim \sigma_{\text{top}}\) [see (2.16) and (2.17); \(\sigma_{\text{top}}\) is the value of \(\sigma\) for which \(V_D(\sigma)\) has a local maximum], here the wave function is not strongly peaked.
2. The minisuperspace region of large $|\phi|$

In this case

$$V(\phi) \gg e^{D\alpha/\sigma_D}V_D(\sigma). \quad (2.32)$$

For large $a$,

$$a^2 \gg a^2_*(\sigma, \phi), \quad (2.33)$$

Eq. (2.19) reduces to

$$\left[ \partial^2_{\bar{\alpha}} - \partial^2_{\bar{\sigma}} - \partial^2_{\bar{\phi}} + \exp \left( 6\alpha - \frac{D\sigma}{\sigma_D} \right) V(\phi) \right] \Psi = 0. \quad (2.34)$$

Through the rotation (2.29), Eq. (2.34) can be written

$$\left[ \partial^2_{\bar{\alpha}} - \partial^2_{\bar{\sigma}} - \partial^2_{\bar{\phi}} + e^{(6gD\bar{\alpha})} V(\phi) \right] \Psi = 0. \quad (2.35)$$

Once again, since the superpotential of Eq. (2.35) does not depend on $\bar{\sigma}$, we can disregard the $\bar{\sigma}$–dependence of the wave function where the superpotential is large in modulus, i.e., in the region (2.32) and (2.33). The wave function is thus proportional to the solution $\Sigma(\bar{\alpha}, \phi)$ of equation

$$\left[ \partial^2_{\bar{\alpha}} + e^{(6gD\bar{\alpha})} V(\phi) \right] \Sigma = 0. \quad (2.36)$$

Under the transformation

$$\bar{\alpha} = \frac{1}{gD} \log(gDq),$$

$$\phi = \frac{1}{gD} x, \quad (2.37)$$

Eq. (2.36) becomes

$$\left[ \partial^2_q + \frac{1}{q} \partial_q - \frac{1}{q^2} \partial^2_x + q^4 V(x) \right] \Sigma = 0. \quad (2.38)$$

For $q^2V(x) \gg 1$ (and with a factor ordering $p = 1$ in the notation of Ref. [14], Eq. (2.38) is formally identical to a Wheeler–DeWitt equation for which the Vilenkin wave function is already known. Under the further limit
to the lowest WKB order,

$$\Sigma(q, x) = \exp \left[-\frac{1}{3V(x)}\right] \exp \left[-i\frac{q^3}{3}V_{\frac{1}{3}}(x)\right]. \quad (2.40)$$

Going back to the original variables, the wave function can be written

$$\Psi = \exp \left[-\frac{1}{3g_D^4V(\phi)}\right] \exp \left[-i\frac{a^3}{3g_D}e^{-D\sigma/3\sigma}g_D^2\right], \quad (2.41)$$

in the region of intersection of (2.32), (2.33) and

$$a^2 \gg a^2(\sigma, \phi)e^{-2D\sigma/3\sigma}g_D^2, \quad (2.42)$$

Note that in the absence of the dilaton ($\sigma = 0$) and of extra dimensions ($D = 0$, i.e., $g_D = 1$) solution (2.41) reduces just to the Vilenkin solution for the model $\mathcal{R} \times S^3$ with inflaton $[14]$.

D. The method of null surfaces

It is possible to confirm qualitatively the structure of the wave function found in the different regions of the minisuperspace, using the method of the surfaces of constant superpotential $[8]$. In our model the null surfaces of constant superpotential have equation

$$a^2 = \frac{4}{6U \pm \sqrt{(\partial_\sigma U)^2 + (\partial_\phi U)^2}}, \quad (2.43)$$

while the surfaces of null superpotential have equation

$$a^2 = \frac{1}{U}, \quad (2.44)$$

and $a = 0$. The surfaces of equation (2.43) separate the regions of the minisuperspace where the surfaces of constant superpotential are of opposite kind (time– or space–like). The surfaces of equation (2.44) separate the regions of the minisuperspace where the surfaces of constant superpotential are of opposite sign. Sufficiently far from the surfaces of null
superpotential, the local comparison between the sign and the kind of the surfaces of constant superpotential allows us to understand qualitatively the behaviour, whether monotonic or oscillating, of the wave function (Table 1). Eqs. (2.43) and (2.44) confirm the structure of the wave function evaluated explicitly in this section.

In particular, in the region of small $|\phi|$, (2.43) reduces to

$$a^2 = \frac{4}{6V_D(\sigma) \pm d_\sigma V_D(\sigma)} ,$$

(2.45)

and (2.44) reduces to

$$a^2 = \frac{1}{V_D(\sigma)} .$$

(2.46)

Eqs. (2.45) and (2.46) formally coincide with the expressions found by Halliwell [8] for a model with $D = 2$ extra dimensions, with monopole potential $V_D(\sigma)$ and without inflaton $\phi$. Through them (Figs.3 and 4), we find out that in the main part of the region $a > a_*(\sigma, \phi)$, $\sigma > 0$, the wave function is oscillating (not only in the strip $\sigma \sim \sigma_{top}$ and for $\sigma \gg \sigma_D/D$, in the region $a > a_*(\sigma, \phi)$, where it has been explicitly derived); it is monotonic in the remaining part (not only for $a < a_*(\sigma, \phi)$ or $a > a_*(\sigma, \phi)$, $\sigma \ll -\sigma_D/D$).

In the region of large $|\phi|$ (2.32), provided $|V^{-1}dV/d\phi| \ll 1$, Eq. (2.43) becomes

$$a^2 = \frac{2e^{D\sigma/\sigma_D}}{3V(\phi) [1 \pm D/6\sigma_D]} ,$$

(2.47)

and (2.44) reduces to

$$a^2 = \frac{e^{D\sigma/\sigma_D}}{V(\phi)} .$$

(2.48)

Since for any $D$ integer and positive $0.3 < D/6\sigma_D < 0.6$, such a result confirms (Fig.5) the oscillating behaviour (2.41), beyond the superpotential barrier ($a^2 \gg a^2_*(\sigma, \phi)$), and the monotonic behaviour (2.12) below ($a^2 \ll a^2_*(\sigma, \phi)$). Thus in such a region of the minisuperspace, the superpotential barrier defines, roughly speaking, the classically allowed and the classically forbidden regions. For $\sigma$ negative and large, outside the region of large $|\phi|$, (2.32), this is no longer true.
III. DISCUSSION

A. Tunnelling picture in multidimensional cosmology

The picture of tunnelling from nothing through a superpotential barrier has been criticized [17] or rephrased [15,18]. We point out some new features thereof that were not included in its original formulation.

First, the very definition of nothing is not clear when treating multidimensional space-times. In 4-dimensional cosmology, nothing is the non-singular boundary of the superspace, \( i.e. \), that part of the boundary of the superspace that includes 3-geometries given through a slicing of a regular 4-geometry [13,14]. In the equivalent 4-dimensional model, the extra dimensions play the role of a matter scalar field \( \sigma \): the non-singular boundary of the minisuperspace is then the configuration \( a = 0, |\sigma| < \infty, |\phi| < \infty \). We will call it external nothing since the internal space is assumed to be non-zero (\( a = 0, 0 < b < \infty, |\phi| < \infty \)).

It is also possible to figure out a total nothing defined as \( a = b = 0, |\phi| < \infty \). However, only external nothing is acceptable when Casimir or monopole schemes are used. In fact, vacuum fluctuations in the former scheme give a finite contribution only for \( b \neq 0 \); analogously the antisymmetric tensor field introduced in the latter one is regular on the internal \( D \)-sphere only for \( b \neq 0 \). In both cases, the configurations with \( b = 0 \) (\( i.e. \), total nothing \( a = b = 0, |\phi| < \infty \) and internal nothing \( 0 < a < \infty, b = 0, |\phi| < \infty \)) do not belong to the non-singular boundary of the minisuperspace, but to the singular one.\(^2\)

External nothing is the only nothing configuration classically stable because of the superpotential barrier (at least when the kinetic energy of the matter fields is negligible) and over which the wave

\(^2\)It is worth reminding that when one assumes the Hartle–Hawking boundary conditions [12] for the cosmic wave function in the path integral approach, something analogous happens [8]. Summing over compact Euclidean 4-geometries and over matter fields configurations that are regular over them, initial conditions \( a = 0, b \neq 0 \) of the paths are assumed.
function is peaked. In our model it is the best candidate for the tunnelling picture, but now a problem arises with the tunnelling itself.

In quantum cosmology the tunnelling picture can meet troubles due to the hyperbolic nature of the Wheeler–DeWitt equation. Already in Ref. [17], it has been pointed out that in models such as non minimally coupled scalar field and Bianchi type–IX, the superpotential barrier can disappear, leaving the configuration of nothing exposed to the Lorentzian \((w < 0)\) region of the minisuperspace. This fact raises the question of whether one could apply the Vilenkin boundary conditions at all. We have got a similar problem, but in our case the barrier never disappears. In the quantum mechanics analog of the tunnelling, one would expect a nucleation to be more likely where the barrier becomes thinner. In our model, the barrier becomes narrow with respect to the external nothing for \(U \to \infty\), \(i.e.,\) for both the two configurations \(\sigma \to -\infty, |\phi| \leq \infty\) and \(\sigma < \infty, |\phi| \to \infty\). Nevertheless beyond the barrier \((i.e.,\) for \(a > a_*(\sigma, \phi)\)), \(\Psi\) is monotonic for small \(|\phi|, \sigma \ll -\sigma_D/D\) (while it oscillates for large \(|\phi|\)). This proves that the nucleation of the semiclassical universe in one of the two configurations \((\sigma \to -\infty, |\phi| \leq \infty)\) is not possible, while in the standard tunnelling picture they should be equally probable. In this sense the tunnelling through the superpotential barrier picture does not hold. We will argue, following Ref. [18] (although, differently from here, there is an exposed nothing in the Bianchi type–IX considered there), that this is not a real problem. The tunnelling picture in quantum cosmology was born only as a formal analogy between the Vilenkin cosmic wave function and the tunnelling wave function of the usual quantum mechanics. This analogy is limited to a finite class of models and to particular regions of the superspace. Indeed the superpotential barrier does not always separate classically allowed regions \((i.e.,\) where \(\Psi\) oscillates) from forbidden ones \((i.e.,\) where \(\Psi\) is monotonic) in the space of configurations, contrary to the usual potential barrier in quantum mechanics. This happens because the kinetic form of the gravitational superhamiltonian \(H\) has a hyperbolic structure, while in a usual quantum system it is elliptic. So, to changes in sign of the superpotential \((i.e.,\) regions of the minisuperspace above and below the barrier) does not necessarily correspond, in the classical constraint \(H = 0,\) the
fact that real “velocities” become imaginary, or viceversa. The tunnelling analogy can hold only when the kinetic form has a definite sign with respect to the superpotential. In our model, this does not happen in one of the two configurations where the superpotential barrier becomes narrow, because here the kinetic energy of matter fields is not negligible. That is to say that
\[
\left(\hat{p}_x^2 + \hat{p}_\phi^2\right) \Psi = -\frac{1}{a^3} \left(\partial_x^2 + \partial_\phi^2\right) \Psi,
\]
gives a relevant contribution in the Wheeler–DeWitt equation \(\hat{H} \Psi = 0\) (\(\hat{p}\) operator of the canonical momenta). Nevertheless, Vilenkin boundary conditions have their physical meaning \([13–15]\) that does not depend on the tunnelling analogy. Selecting only outgoing modes for the wave function at the singular boundary, they fix a “time direction” in minisuperspace: the direction of this probability flux toward the boundary. In the following of the paper we will refer to this causal meaning of Vilenkin boundary conditions. We remark that the causal conditions of quantum cosmology still select an inflationary scenario in 4–dimensional models when the tunnelling analogy does not hold anymore: for example, in anisotropic models \([18]\).

As a last comment, we note that since dimensional reduction \([10]\) is not valid anymore for small \(a\) (\(i.e.,\) where the flat–space limit \(a \gg b\) fails), the wave function might not have the form of the “nothing solution” \((2.12)\) in the region \(a^2 \ll a^2(\sigma, \phi)\) of minisuperspace. Nevertheless, as we have just shown, Vilenkin boundary conditions can be applied also when the tunnelling picture fails, and the semiclassical wave function is peaked (see Section \([11,13]\) on asymptotic classical solutions in the region \(a \gg b\), where dimensional reduction holds.

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3More precisely, at that part of the singular boundary close to where \(\Psi\) oscillates.
B. Interpretation of the wave function. The classical limit.

The interpretation of the wave function of the universe is still matter of debate. In this paper we will adopt a minimal interpretative criterion. Let us consider the semiclassical wave function: we will assume that only the strong peaks of $|\Psi|^2$ select classical correlations among the dynamical variables of the universe $^{[15]}$. Weak variations of $|\Psi|^2$ do not select any correlation. Such a criterion only needs the lowest WKB order and does not require either the normalization or even the normalizability $^4$ of the wave function (for this reason in Section $^{[1]}$ we omitted everywhere the pre–exponential WKB factors and the normalization constants: note that the Vilenkin wave function derived above is not normalizable). In literature other criteria to interpret the wave function can be found. Since they introduce measures that are factorized to $|\Psi|^2$ in the semiclassical limit, it is reasonable to expect results qualitatively similar $^{[11]}$ in presence of strong peaks.

The structure of the wave function found in the previous section is the following. For small $a$ (i.e., $a^2 \ll a^2_\star(\sigma, \phi)$), under the superpotential barrier, $\Psi$ has the asymptotic behaviour (2.12) of Vilenkin nothing state. It is peaked on $a = 0$, monotonically decreasing for increasing $a$, independently of the matter fields $\sigma, \phi$. For large $a$ (i.e., $a^2 \gg a^2_\star(\sigma, \phi)$), beyond the superpotential barrier, we recognize two behaviours.

a) For small $|\phi|$ (i.e., in the region (2.13)), $\Psi$ is weakly dependent on $\phi$. This is due to the weak local dependence on $\phi$ of $U(\sigma, \phi)$. $\Psi$ is monotonic for $\sigma < 0$, and oscillating for $\sigma > 0$. The amplitude of the oscillations has a peak around $\sigma = \sigma_{top}$, where $V_D(\sigma)$ has a local maximum. Here

$$|\Psi|^2 = \exp \left[-\frac{2}{3V_D(\sigma)}\right].$$

b) For large $|\phi|$ (i.e., in the intersection of the regions (2.32), (2.33), (2.42)), $\Psi$ oscillates.

$^4$An alternative way to solve the problem of the non normalizability of the wave function is the introduction of conditional probabilities: see Refs. $^{[11, 12]}$. 

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The amplitude of the oscillations increases quickly for increasing $|\phi|$ (converging to a finite limit, at the lowest WKB order). Here

$$|\Psi|^2 = \exp\left[-\frac{2}{3g_D^4 V(\phi)}\right].$$

(3.2)

So the amplitude of the oscillations of $\Psi$, but not their phase, depends weakly on $\sigma$.

To the lowest WKB order, the peaks in the oscillating structure of $\Psi$ corresponding to (3.1) and (3.2) are $a$–independent. They depend only on the potential respectively of the dilaton and of the inflaton. (Note that $1 > g_D^4 > 0.4$ when $0 < D < \infty$). The potential of the inflaton is unbounded in the region of large $|\phi|$. The potential of the dilaton is, on the contrary, limited around $\sigma = \sigma_{\text{top}}$. However $V_D(\sigma_{\text{top}}) \gg 1$ for values of the equilibrium radius smaller than the Planck scale, $b_0 \ll D^{1/2}$; in such a case the two peaks of $\Psi$ could be of comparable amplitude. The quantitative comparison between the probabilities of nucleation of the universe in the regions corresponding to the two peaks is impossible without introducing conditional probabilities [11]. Nevertheless neither one of the two peaks (3.1) and (3.2) corresponds to a universe that is compatible with that observable today. Let’s analyze the classical evolution of a typical universe selected by them.

In the semiclassical limit the cosmic wave function is strongly peaked on the trajectories of superspace that satisfy the first integral

$$p = \nabla F,$$

(3.3)

of the classical equations, where $F$ is the phase of the wave function, $p$ are classical canonical momenta and $\nabla$ is the gradient on the superspace [11].

From (2.41) and (3.3), we find, in the region of large $|\phi|$, the first integral

$$\frac{\dot{a}}{a} = \frac{1}{g_D} \sqrt{V(\phi)} e^{-D\sigma/2\sigma_D},$$

$$\dot{\sigma} = \frac{D}{6\sigma_D g_D} \sqrt{V(\phi)} e^{-D\sigma/2\sigma_D},$$

$$\dot{\phi} = -\frac{d\phi V(\phi)}{6g_D \sqrt{V(\phi)}} e^{-D\sigma/2\sigma_D}.$$
Here $|\Psi|^2$ selects just large $|\phi|$, while it is weakly depending on $\sigma$: all the values of $\sigma$ in this region are almost equally probable. Eqs. (3.4) are verified by an ensemble of classical solutions of “velocity” $\dot{\phi}/\dot{\phi} < 0, \dot{\sigma} > 0$: thus the typical universe selected by the causal conditions in this region rolls down along the profile of the potential $U$ toward the configuration $\sigma = +\infty$ (i.e., $b = \infty$). In particular for $|\phi| \gg 1$, we have $\dot{\sigma}^2 \gg \dot{\phi}^2$, and in the region $|\phi| \gg 1$, $\exp D\sigma/\sigma > V(\phi)$, necessarily $\dot{\phi}^2$ gets asymptotically to zero. This happens because the potential of matter fields is sufficiently flat locally only in the $\phi$-direction (i.e., $|\partial_\phi U/U| \simeq |d_\phi V/V| \ll 1$). The typical classical universe follows then a trajectory almost parallel to the $\sigma$ axis, that can be written, sufficiently far from $\sigma_{in}$, as

$$a \propto t^p, \quad p = \frac{D + 2}{D},$$

$$\sigma \simeq \frac{2\sigma_D}{D} \ln \left[ \frac{D^2}{12g_D\sigma_D^2} \sqrt{V(\phi_{in})} t \right],$$

$$\dot{\phi} \simeq \dot{\phi}_{in}, \quad (3.5)$$

(everywhere a subfixed $in$ indicates an initial value). This is a power–law inflation [19], though rather weak for large $D$. The classical universe corresponding to the peak of the semiclassical function in the region of large $|\phi|$, undergoes an eternal inflation with an unstable internal space (note that $a/b \propto t$).

The peak (3.1) on the strip $\sigma \simeq \sigma_{\text{top}}$ of the region of small $|\phi|$ selects instead the first integral

$$\frac{\dot{a}}{a} = \sqrt{V_D(\sigma_{\text{top}})} \left[ 1 - \frac{\mu^2}{4V_D(\sigma_{\text{top}})} (\sigma - \sigma_{\text{top}})^2 \right],$$

$$\dot{\sigma} = \frac{\mu^2}{6\sqrt{V_D(\sigma_{\text{top}})}} (\sigma - \sigma_{\text{top}}) \left[ 1 + \frac{\mu^2}{4V_D(\sigma_{\text{top}})} (\sigma - \sigma_{\text{top}})^2 \right],$$

$$\dot{\phi} = 0, \quad (3.6)$$

where $\mu^2 = -d_\phi^2 V_D|_{\sigma = \sigma_{\text{top}}}$, which corresponds to two different scenarios, still driven by the dilaton. For initial fluctuation of $\sigma$ to the right of $\sigma_{\text{top}}$ (i.e., $\sigma_{in} > \sigma_{\text{top}}$), the universe rolls down along the profile of $U$ again, towards the configuration $\sigma = +\infty$: it undergoes an eternal inflation with an unstable internal space. For initial fluctuation of $\sigma$ to the left of
$\sigma_{\text{top}}$ (i.e., $\sigma < \sigma_{\text{top}}$), the universe starts with an exponential inflation ($a \propto \exp(\sqrt{V_D(\sigma_{\text{top}})}t)$, followed\(^5\) by a Friedmannian stage, i.e., damped oscillations of $\sigma$ around its ground state $\sigma = 0 \ (b = b_0)$: the internal space is stable. Nevertheless, this peak does not select a good inflationary solution since the flat region of $V_D(\sigma)$ around $\sigma_{\text{top}}$ is not sufficiently wide for quantum fluctuations to be negligible during the evolution of the dilaton throughout the region. In fact, (3.6) describes an exponential inflation that lasts as long as

$$t < t_* \equiv \frac{3}{\mu^2} \sqrt{\frac{V_D(\sigma_{\text{top}})}{\mu^2}} \log \frac{4V_D(\sigma_{\text{top}})}{\mu^2 \delta_{\text{in}}^2},$$

(3.7)

where $\delta_{\text{in}}^2 \equiv (\sigma_{\text{top}} - \sigma_{\text{in}})^2$. To solve the horizon and flatness problems, it is commonly required that the exponential phase lasted at least 65 e–foldings. Then

$$t_* \geq \frac{65}{\sqrt{V_D(\sigma_{\text{top}})}}.$$

(3.8)

From (3.7) and (3.8) we can see that the classical initial condition for the dilaton must be very close to $\sigma_{\text{top}}$ to obtain sufficient inflation:

$$\delta_{\text{in}}^2 \leq \left[ \frac{\mu^2}{4V_D(\sigma_{\text{top}})} \right]^{-1} \exp \left[ -\frac{65\mu^2}{3V_D(\sigma_{\text{top}})} \right],$$

(3.9)

where $\mu^2/V_D(\sigma_{\text{top}}) \simeq 12$ for large $D$.

Around $\sigma_{\text{top}}$, the square of the amplitude of the wave function (3.1) is locally approximated by a Gaussian

$$|\Psi|^2 \propto \exp \left[ -\frac{(\sigma - \sigma_{\text{top}})^2}{2\delta \sigma^2} \right],$$

(3.10)

where we have introduced the mean square fluctuation

$$\delta \sigma^2 \equiv \frac{3}{2\mu^2} V_D^2(\sigma_{\text{top}}).$$

(3.11)

\(^5\)The approximations introduced in the calculation of $\Psi$, as in Eq.(3.6), hold only around $\sigma_{\text{top}}$. To study the following evolution we must impose, to the equation of motions, the initial conditions select by $\Psi$ around $\sigma_{\text{top}}$. 

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From (3.9) and (3.11) it follows that
\[
\frac{\delta \sigma^2}{\delta_{in}^2} > \frac{D}{6}e^{182} \gg 1
\] (3.12)
for values of the internal radius consistent with observational constraints, \(b_0 < 10^{17}\) Planck units. Thus quantum fluctuations of the dilaton span a much larger range than the one of the classical initial conditions that give a good inflation. This is just due to the fact that \(V_D(\sigma)\) is not sufficiently flat around \(\sigma_{top}\), i.e., \(\mu^2/V_D(\sigma_{top}) \gg 1\).

Halliwell [11] gets an analogous result for a model with \(D = 2\), no inflaton and a monopole compactification scheme, using Hartle–Hawking boundary conditions. Indeed this is why an inflaton must be introduced: the dilaton alone does not lead to realistic inflationary scenarios. The initial conditions of slow–rolling of the inflaton in the \(\phi\)–direction, along the profile of \(U\), inside the stability channel are not selected by any of the amplitude peaks of the oscillating wave function.

### C. The semiclassical cutoff

According to Vilenkin [11,14], the introduction of a cutoff is required for the semiclassical approach (minisuperspace scheme, Einstein gravity, etc.) to be valid: in fact the unbounded growth of the potential of matter fields could take their energy density above the Planck scale.

As an example, for the 4–dimensional model with topology \(R \times S^3\) and a chaotic inflaton \(\phi\), for the measure of the classical solutions in the minisuperspace
\[
dP = e^{-2/3V(\phi)}d\phi ,
\] (3.13)
Vilenkin introduces a cutoff at the Planckian boundary \(V(\phi) = 1\). The reason is that the measure (3.13) grows monotonically with |\(\phi\)|, converging to a finite limit. Although for \(V(\phi) > 1\) the semiclassical approximation breaks down, the growing behaviour of (3.13) for \(V(\phi) < 1\) suggests that the universe nucleates most likely in the region \(V(\phi) > 1\). In this
region quantum corrections to the Einstein action (terms of higher order in $R$) could become relevant and reverse the growth of the measure at high energies \[20\].

We apply an analogous prescription to the potential of matter fields $U(\sigma, \phi)$ in our model, in the region of the minisuperspace where $\Psi$ oscillates (the semiclassical cutoff $U = 1$ takes away the whole region of the $\sigma - \phi$ plane where $\sigma$ is negative, but large in modulus: however here $\Psi$ is monotonic). For small $\sigma$ (i.e., $|\sigma| \ll \sigma_D/D$), the cutoff takes away the subregion $V(\phi) > 1$ of the region of large $|\phi|$. In the region of small $|\phi|$ the peak of $\Psi$ around $\sigma = \sigma_{\text{top}}$ is below the cutoff only for $b_0 \gg \sqrt{D}$. For $\sigma$ sufficiently large (i.e., $\sigma \gg \sigma_D/D$), the semiclassical cutoff takes away only the section $V(\phi) > \exp(D\sigma/\sigma_D)$ of the region of large $|\phi|$, while the region

$$\exp(D\sigma/\sigma_D) > V(\phi) \gg K \quad , \quad \sigma \gg \sigma_D/D$$

(3.14)
is still below the cutoff, for any value of $b_0$.

As a result the strong peak (3.2) of $\Psi$ still selects the large values of $|\phi|$, for the classical universe in the region (3.14), for any $b_0$, even after the cutoff $U = 1$ has been introduced. This happens because $|\Psi|^2$ depends locally only on $V(\phi)$, not on $U(\sigma, \phi)$, unlike in the region of small $|\phi|$, where $|\Psi|^2$ depends on $U(\sigma, \phi) \simeq V_D$. It is easy to understand why, generalizing the results in Ref. \[14\] to an $n$–dimensional minisuperspace. In a problem of superpotential barrier like (2.5) and (2.6), the outgoing mode of the Vilenkin wave function depends on the potential of the matter fields through $|\Psi|^2 = \exp(-2/3U)$, only where all the kinetic contributions of the matter fields are negligible, i.e., only where their potential $U$ satisfies the conditions of sufficient flatness:

$$\left| \frac{\partial U}{\partial \phi_j} \right| \ll \max \left\{ U, 1/a^2 \right\} ,$$

(3.15)
for all $j$’s (i.e. in each matter field direction). This condition is satisfied in the strip $\sigma \simeq \sigma_{\text{top}}$ of the region of small $|\phi|$, but not in the region of large $|\phi|$, where, for $|\phi| \gg 1$, the potential is sufficiently flat only in the $\phi$ direction.

Thus in the strip $\sigma \simeq \sigma_{\text{top}}$ of the region of small $|\phi|$, the semiclassical approximation which we used to find the peak of the wave function (3.1), does not hold anymore for
\( U \simeq V_D(\sigma) > 1, \text{ i.e., for } b_0 < \sqrt{D}. \) These are the values for which this peak becomes comparable to the peak in large \( |\phi| \). Since \( V_D(\sigma) \) acts like an effective cosmological constant for the classical solutions selected by Eqs. (3.1) and (3.6) we expect that quantum corrections to the Einstein action decrease the probability of local nucleation at high energies \[20\]. The peak of \( \Psi \) at large \( |\phi| \) could then remain dominant also for \( b_0 < \sqrt{D} \). We remind that the peak at large \( |\phi| \) corresponds to inflationary solutions.

### D. Conclusions

When Vilenkin boundary conditions are applied to solving the Wheeler-DeWitt equation, quantum cosmology predicts a nucleation from nothing of a classical universe in inflationary evolution \[14\]. For this reason we have extended the analysis to an \((N = 4 + D)\)-dimensional model with topology \( \mathcal{R} \times S^3 \times S^D \) in the equivalent “Casimir \( \sigma\)-dilaton plus chaotic \( \phi\)-inflaton” scheme of Einstein gravity.

In this case, the minisuperspace is 3-dimensional and the tunnelling analogy is no longer valid. Nevertheless, Vilenkin’s boundary conditions can be generalized in terms that are independent of the tunnelling picture and in this form they can be applied to our problem too. We have obtained a patchwork of approximate analytic solutions of the Wheeler-DeWitt equation. The patchwork covers all the relevant regions of the minisuperspace so that we have a complete picture of the behaviour of the Universe in this model. Once the solution is known, information are extracted on the basis of the following criterion: where the wave-function is oscillatory rather than exponential, there the Universe starts its classical evolution.

A first consideration is in order. In 2-dimensional minisuperspace models, this criterion leads to the conclusion that the classical evolution starts at the top (i.e. the highest point allowed by the semiclassical cut-off) of the potential of the inflaton field. Our calculations show that this is still true in our 3-dimensional minisuperspace. However the same criterion cannot be extended to the potential of the dilaton \( \sigma \). In fact, for \( \sigma \to -\infty \), and \( \phi \simeq \text{const} \)
\(U(\sigma, \phi) \to \infty\), but the wave function results to be exponential.

Our main conclusion is that Vilenkin wave function predicts inflation, in multidimensional cosmology too. In fact, at the classical birth of the Universe, we find that the law of growth of the ordinary space is \(a \propto t^p\), with \(p = (D + 2)/D > 1\) (power-law inflation). The inflationary phase is driven by the initially large and almost constant value of the inflaton \(\phi\). On the contrary, no preferred initial value of the dilaton \(\sigma\) is selected by the wave function, and \(\dot{\sigma} \gg \dot{\phi}\). This means that only a very small subset of the initial conditions leads to an evolution with \(\sigma\) in the channel of the potential that gives stability to the internal space. In the large majority of the cases, during inflation the internal space expands. However, the growth of \(\sigma\) is only logarithmic, i.e. it is slow enough to make the value of the radius of the internal space at the end of inflation compatible with the known physics. In fact, defining \(N \equiv \ln(a_{\text{fin}}/a_{\text{in}})\), the number of e-folds of inflation, one finds from (3.16) that

\[
\frac{b_{\text{fin}}}{b_{\text{in}}} = e^{2N/pD} = e^{2N/(D+2)},
\]

where \(b = L \exp \sigma\) is the radius of the internal space (\(L\) is a constant). Present accelerators energies, of the order of 100 GeV, exclude the existence of an internal space at the scale of \(10^{17}\) Planck units and, in fact, for \(N = 60\) and \(D = 6\), we obtain \(b_{\text{fin}}/b_{\text{in}} \approx 10^{6.5} \ll 10^{17}\). It is easily checked that the maximum allowed value for \(b_{\text{fin}}/b_{\text{in}}\) is reached after \(N = 157\) e-folds, i.e. no fine-tuning of \(N\) is required.\(^6\)

Of course the problem of the smooth connection of the inflationary phase to the ordinary radiation dominated, Friedmann-Robertson-Walker phase (graceful-exit problem) is not solved in this simplified model. We remark that, in addition, the mechanism for the graceful exit must also provide the stop of the growth of the internal space.

\(^6\) We assumed a chaotic potential (1.3) for the inflaton. In Ref. [3], a model with a \(W\)-shaped potential, such that assumed in the inflationary scenarios with a phase transition, was also studied. Since for large \(|\phi|\) the \(W\)-shaped potential reduces to (1.3), we expect analogous conclusions about the quantum instability of the internal space. We are investigating this case too.
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Figure captions

**Fig. 1** – The total potential for the matter fields ($\phi$-inflaton and $\sigma$-dilaton).

**Fig. 2** – The minisuperspace section of the small $|\phi| = \text{const}$, $\sigma < 0$ in which the total potential is $U \simeq V_D(\sigma)$. The bold line is the boundary $W = 0$ of the superpotential barrier. The dotted line is the null-curve of constant superpotential (i.e., where the supermetric is zero). From a comparison with Table 1, one can get the general behaviour, oscillating or monotonic, of the wave function, far away from the $W = 0$ line.

**Fig. 3** – As in Fig. 2, but now $\sigma > 0$.

**Fig. 4** – The minisuperspace section of “large” $|\phi|$, $\sigma = \text{const}$, in which $U(\sigma, \phi) \simeq e^{-D\sigma/\sigma_D} V(\phi)$, but $|\partial_\sigma U| \gg |\partial_\phi U|$. The bold and the dotted lines have the same meaning that in Fig. 2.
TABLES

| $w = \text{constant}$ | Spacelike           | Timelike    |
|-----------------------|---------------------|-------------|
| $w < 0$               | oscillatory         | monotonic   |
| $w > 0$               | monotonic           | oscillatory |

TABLE I. Surfaces of constant superpotential and the behaviour of the wavefunction, far from the surfaces $W = 0$
This figure "fig1-1.png" is available in "png" format from:

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This figure "fig2-2.png" is available in "png" format from:

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