BV spaces and the perimeters related to Schrödinger operators with inverse-square potentials and applications to the rank-one theorem

Yang Han, Jizheng Huang, Pengtao Li, Yu Liu

Abstract: For $a \geq -\left(\frac{d}{2} - 1\right)^2$ and $2\sigma = d - 2 - ((d - 2)^2 + 4a)^{1/2}$, let

$$
\begin{aligned}
\mathcal{H}_a &= -\Delta + \frac{a}{|x|^2}, \\
\mathcal{H}_\sigma &= 2(-\Delta + \frac{\sigma^2}{|x|^2})
\end{aligned}
$$

be two Schrödinger operators with inverse-square potentials. In this paper, on the domain $Ω \subset \mathbb{R}^d \setminus \{0\}, d \geq 2$, the $\mathcal{H}_a$-BV space $\mathcal{BV}_{\mathcal{H}_a}(Ω)$ and the $\mathcal{H}_\sigma$-BV space $\mathcal{BV}_{\mathcal{H}_\sigma}(Ω)$ related to $\mathcal{H}_a$ and $\mathcal{H}_\sigma$ are introduced, respectively. We investigate a series of basic properties of $\mathcal{BV}_{\mathcal{H}_a}(Ω)$ and $\mathcal{BV}_{\mathcal{H}_\sigma}(Ω)$. Furthermore, we prove that $\mathcal{H}_\sigma$-restricted BV functions can be characterized equivalently via their subgraphs. As applications, we derive the rank-one theorem for $\mathcal{H}_\sigma$-restricted BV functions.

Keywords: rank-one theorem, subgraphs, BV space, Schrödinger operator.

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1 Introduction

In this paper, we will discuss several basic questions of geometric measure theory related to the Schrödinger operator with inverse-square potential:

\[ \mathcal{H}_a = -\Delta + \frac{a}{|x|^2}, \quad a \geq -\left(\frac{d}{2} - 1\right)^2 \]

on the Euclidean space \( \mathbb{R}^d \) with \( d \geq 2 \). More precisely, we interpret \( \mathcal{H}_a \) as the Friedrichs extension of this operator defined initially on \( C_0^\infty(\mathbb{R}^d \setminus \{0\}) \) (cf. [16]). The operator \( \mathcal{H}_a \) often appears in mathematics and physics, and is usually used as scale limit for more complex problems. The references [6, 7, 15, 20, 22] discuss several examples of this situation in physics. These examples range from combustion theory to the Dirac equation with Coulomb potentials, and the study of the disturbance of classical space-time metric such as Schwarzschild and Reissner-Nordström metric. The appearance of \( \mathcal{H}_a \) as a scale limit (from both microscopic and astronomical aspects) is a signal of its unique properties: \( \mathcal{H}_a \) is scale-invariant. In particular, the potential function and Laplace function are equally strong in every length scale. Accordingly, problems involving \( \mathcal{H}_a \) rarely obey simple perturbation theory. This is one of the reasons why we (and many scholars before us) choose this particular operator for further study.

The first aim of this paper is to investigate the class of functions of bounded variation related to \( \mathcal{H}_a \). In the literature, a function of bounded variation, simply a BV-function, is a real-valued function whose total variation is finite. In the multi-variable setting, a function defined on an open subset \( \Omega \subseteq \mathbb{R}^d, d \geq 2, \) is said to have bounded variation provided that its distributional derivative is a vector-valued finite Radon measure over the subset \( \Omega \). Let \( \text{div} \) and \( \nabla \) denote the divergence operator and the gradient operator, respectively, where

\[
\begin{align*}
\nabla u &:= \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_d} \right), \\
\text{div}\varphi &:= \frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} + \cdots + \frac{\partial \varphi_d}{\partial x_d}
\end{align*}
\]

for \( u \in C^1_c(\Omega) \) and \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C^1_c(\Omega, \mathbb{R}^d) \). A function \( u \in L^1(\Omega) \) whose partial derivatives in the sense of distributions are measures with finite total variation \( |Du| \) in \( \Omega \) is called a function of bounded variation, where

\[ |Du|(\Omega) := \sup \left\{ \int_{\Omega} u \ \text{div} \nu \, dx : \nu = (\nu_1, \ldots, \nu_d) \in C_c^\infty(\Omega, \mathbb{R}^d), |\nu(x)| \leq 1, x \in \Omega \right\} < \infty. \]

The class of all such functions will be denoted by \( BV(\Omega) \). The norm of \( BV(\Omega) \) is defined as

\[ \|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega). \]

Importantly, the BV-functions form an algebra of possibly discontinuous functions whose weak first partial derivatives exist and are Radon measures-thanks to this nature, this algebra is frequently used to define generalized solutions of nonlinear problems involving functional analysis, ordinary and partial differential equations, mathematical physics and engineering science.

Essentially, the above definition of the BV-function is corresponding to the Laplace operator \( \Delta \), where \( \mathcal{H}_0 = -\Delta \), since \( \Delta \) can be represented as \( \text{div}(\nabla \cdot \cdot) \). If \( d = 2 \), at this time, \( a \geq -\left(\frac{d}{2} - 1\right)^2 \) implies that \( a \geq 0 \), then \( \mathcal{H}_a \) is a Schrödinger operator with the nonnegative potential and it is positive semi-definite (e.g. cf. [5, Section 3]). If \( d \geq 3 \), the arguments of [16, Section 1.1] imply that
the restriction \( a \geq -(d/2 - 1)^2 \) ensures that the operator \( \mathcal{H}_a \) is positive semi-definite. Moreover, \( \mathcal{H}_a \) can be factorized as

\[
\mathcal{H}_a = -\sum_{i=1}^{d} (A_{-i,a} A_{i,a}),
\]

where

\[
A_{i,a} = \frac{\partial}{\partial x_i} + \sigma \frac{x_i}{|x|^2}, A_{-i,a} = \frac{\partial}{\partial x_i} - \sigma \frac{x_i}{|x|^2}, 1 \leq i \leq d
\]

with

\[
\sigma := \frac{d - 2}{2} - \frac{1}{2} \sqrt{(d - 2)^2 + 4a}.
\]

When \( a = 0 \), \( A_{-i,a} \) and \( A_{i,a} \) are exactly the classical partial derivatives. This fact indicates that the operators \( A_{i,a} \), \( 1 \leq |i| \leq d \), play the same role as the classical partial derivatives \( \frac{\partial}{\partial x_i} \) in \( \mathbb{R}^d \). Based on this observation, we call \( \mathcal{H}_{a} \) the restriction \( a \) of approximation functions, we can deduce that \( H_{a} \) the restriction \( a \) of approximation functions, we can deduce that

\[
\mathcal{H}_a u = -\text{div}_{\mathcal{H}_a} (\nabla_{\mathcal{H}_a} u) = -\Delta u + \frac{a}{|x|^2} u \quad \forall u \in C^2_c(\Omega).
\]

Naturally, we introduce the class of bounded variation functions related to \( \mathcal{H}_a \) denoted by \( BV_{\mathcal{H}_a} (\Omega) \). In Section 2.1, we investigate some basic properties of \( BV_{\mathcal{H}_a} (\Omega) \) including the lower semicontinuity, the structure theorem, the approximation via \( BV \) \( - \text{functions, etc. In Theorem 2.5, we prove that } \mathcal{H}_a \)-BV functions can be approximated by smooth functions. It should be noted that in contrast with Theorem 2 in [13, Section 5.2.2], we need to add a condition (6) in Theorem 2.5, which can be obtained by the Hardy-Sobolev inequality (cf. [8]). Moreover, the \( \mathcal{H}_a \)-perimeter of \( E \subseteq \Omega \) induced by \( BV_{\mathcal{H}_a} (\Omega) \) is introduced in Section 2.2, see (9) below. We obtain the following coarea inequality for \( \mathcal{H}_a \)-BV function: if \( u \in BV_{\mathcal{H}_a} (\Omega) \), then

\[
|\nabla_{\mathcal{H}_a} u|(\Omega) \leq \int_{-\infty}^{+\infty} P_{\mathcal{H}_a}(E_t, \Omega) dt,
\]

where \( E_t = \{ x \in \Omega : u(x) > t \} \) for \( t \in \mathbb{R} \), see Theorem 2.10.

In Section 3, our purpose is to establish the converse of inequality (2). Via choosing a sequence of approximation functions, we can deduce that

\[
\int_{-\infty}^{+\infty} P_{\mathcal{H}_a}(E_t, \Omega) dt \leq \sqrt{2} \int_{\Omega} \left( |\nabla u(x)| + \frac{\sigma}{|x|} |u(x)| \right) dx
\]

for \( u \in BV_{\mathcal{H}_a} (\Omega) \). However, it should be noted that \( |\nabla u(x)| + \frac{\sigma}{|x|} |u(x)| \) can not be dominated by \( |\nabla_{\mathcal{H}_a} u(x)| \). In fact, for example, let \( u(x) = x_1^2, \sigma = -1 \), then via computation, we have

\[
|\nabla u(x)| + \frac{\sigma}{|x|} |u(x)| = 2|x_1| + \frac{x_1^2}{|x|}
\]
and

\[ |\nabla_{\mathcal{H}_\sigma} u(x)| = \left( 4x_1^2 - \frac{3x_1^4}{|x|^2} \right)^{1/2}. \]

Hence, the previous fact holds true. Thus, the converse of (2) does not hold for \( BV_{\mathcal{H}_\sigma}(\Omega) \). With this in mind, we introduce the following subspace of \( BV_{\mathcal{H}_\sigma}(\Omega) \). The classical gradient operator \( \nabla \) and the classical divergence operator \( \text{div} \) in (1) inspire us to introduce the following symmetric gradient operator and symmetric divergence operator related with \( \mathcal{H}_\sigma \) defined as \( \nabla_{\mathcal{H}_\sigma} := \text{div}_{\mathcal{H}_\sigma} (\nabla_{\mathcal{H}_\sigma}) \), 

\[
\left\{\begin{array}{l}
\nabla_{\mathcal{H}_\sigma} u := (A_1, \ldots, A_{d,a} u, A_{1,a} u, \ldots, A_{d,a} u, \ldots), \\
\text{div}_{\mathcal{H}_\sigma} \Phi := A_{1,a} \varphi_1 + \cdots + A_{d,a} \varphi_d + A_{1,a} \varphi_{d+1} + \cdots + A_{d,a} \varphi_{2d}
\end{array}\right.
\]

for \( u \in C^1_c(\Omega) \) and \( \Phi = (\varphi_1, \ldots, \varphi_{2d}) \in C^1_c(\Omega, \mathbb{R}^{2d}) \). The class of bounded variation functions related to \( \mathcal{H}_\sigma \) is denoted by \( BV_{\mathcal{H}_\sigma}(\Omega) \). Based on the definition of \( \nabla_{\mathcal{H}_\sigma} \), in Theorem 3.12, using the equivalence of \( |\nabla \varphi| + \frac{1}{|x|}|\varphi(x)| \) and \( |\nabla_{\mathcal{H}_\sigma} \varphi| \), we prove that (2) can be improved to the following coarea formula:

\[ |\nabla_{\mathcal{H}_\sigma} u|(\Omega) \approx \int_{-\infty}^{+\infty} P_{\mathcal{H}_\sigma}(E_t, \Omega) dt \quad \forall \ u \in BV_{\mathcal{H}_\sigma}(\Omega). \tag{3} \]

By the aid of (3), we deduce that the Sobolev type inequality

\[ \|f\|_{L^{d/(d-1)}(\Omega)} \lesssim |\nabla_{\mathcal{H}_\sigma} f|(\Omega) \quad \forall \ f \in BV_{\mathcal{H}_\sigma}(\Omega) \]

is equivalent to the following isoperimetric inequality

\[ |E|^{1-1/d} \lesssim P_{\mathcal{H}_\sigma}(E, \Omega), \]

where \( E \) is a bounded set with finite \( \mathcal{H}_\sigma \)-perimeter in \( \Omega \).

As an application, we further investigate the rank-one property for \( \mathcal{H}_\sigma \)-variations. The rank-one theorem was first conjectured by L. Ambrosio and E. De Giorgi in [9] (see also [3, 10]). This theorem is of great significance to the application of vector variational problems (lower semicontinuity, relaxation, approximation and integral representation theorem, etc.) and partial differential equations. By introducing new tools and using complex techniques in geometric measure theory, G. Alberti first proved the rank-one theorem in [1]. A simpler proof, based on the area formula and the Reshetnyak continuity theorem, has been given in [2]. Unfortunately, this proof works for particular BV functions only, the monotone ones (gradients of locally bounded convex functions, for instance). Two different proofs of rank-one theorem have been found recently. One of them was proposed by G. De Philippis and F. Rindler, who obtained a new proof from a profound PDEs result [11], and also proved a rank-one property for maps with bounded deformation (BD) firstly. At the same time, A. Massaccesi and D. Vittone in [17] provided another simpler proof of geometric properties by virtue of the properties of subgraphs in Euclidean space. Applying properties related with the horizontal derivatives of a real-valued function with bounded variation and its subgraph, S. Don, A. Massaccesi and D. Vittone obtained a rank-one theorem for the derivatives of vector-valued maps with bounded variation in a class of Carnot groups \( \mathbb{G} \) (see [12]).

One of significant properties of BV functions in \( \mathbb{R}^d \) is that any BV function can be characterized equivalently by its subgraph (cf. [14]). In [17] and [12], such subgraph property is an important tool in the proof of the rank-one theorem. In Section 4, similar to the concept of subgraph in \( \mathbb{R}^d \), we introduce the definition of subgraph of \( \mathcal{H}_\sigma \)-BV function and study the subgraph properties of
\( \widetilde{H}_a \)-BV functions. We point out that in settings of Euclidean spaces \( \mathbb{R}^d \) and Carnot groups \( G \), for \( \varphi \in C^1_c(\Omega, \mathbb{R}^d) \), the integral of the divergence \( \text{div}\varphi \) is zero, i.e.,

\[
\int_{\Omega} \text{div} \varphi(x) \, dx = 0. \quad (4)
\]

However, due to the occurrence of the perturbation term \( \frac{\sigma x_i}{|x|} \) in \( \text{div} \tilde{H}_a \), (4) does not hold for \( \text{div} \tilde{H}_a \).

We introduce the concept of \( \tilde{H}_a \)-restricted variation and perimeter to eliminate the influence of the potential part of the operator \( H_a \) on the subgraph properties. Finally, in Theorem 4.2, we prove that a measurable function \( u \) belongs to \( BV_{\tilde{H}_a}(\Omega) \) if and only if its subgraph is a set of finite \( \tilde{H}_a \)-restricted perimeter in \( \mathbb{R}^d \).

In Section 5, we devote ourselves to a closer analysis of the distributional derivative of a \( \tilde{H}_a \)-restricted BV function \( u \). In analogy with the results obtained in Euclidean space \([4]\) for BV functions, we can write \( \nabla \tilde{H}_a u = \nabla^A_{\tilde{H}_a} u + \nabla^S_{\tilde{H}_a} u \), where \( \nabla^A_{\tilde{H}_a} u \) is absolutely continuous with respect to \( \mathcal{L}^d \) and \( \nabla^S_{\tilde{H}_a} u \) is singular with respect to \( \mathcal{L}^d \). Based on the above decomposition of derivatives, in Section 5, the proof of rank-one theorem related to \( BV_{\tilde{H}_a}(\Omega) \) is given, see Theorem 5.4.

Some notation:

- Throughout this article, we will use \( c \) and \( C \) to denote the positive constants, which are independent of main parameters and may be different at each occurrence. \( U \approx V \) indicates that there is a constant \( c > 0 \) such that \( c^{-1}V \leq U \leq cV \), whose right inequality is also written as \( U \lesssim V \). Similarly, one writes \( V \gtrsim U \) for \( V \geq cU \).

- For convenience, the positive constant \( C \) may change from one line to another and this usually depends on the spatial dimension \( d \), the indices \( p \), and other fixed parameters.

- Let \( \mathbb{N}_0 \) denote the non-negative integers. A \( d \)-dimensional multi-index is a vector \( \alpha \in \mathbb{N}_0^d \), meaning that, \( \alpha = (\alpha_1, \ldots, \alpha_d) \) for \( \alpha_i \in \mathbb{N}_0 \). The derivative of order \( \alpha \) is defined by

\[
\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}
\]

with \( |\alpha| = \sum_{i=1}^d \alpha_i \).

- Let \( \Omega \subset \mathbb{R}^d \) be an open set. Throughout this article, we use \( C(\Omega) \) to denote the space of all continuous functions on \( \Omega \). Let \( k \in \mathbb{N} \cup \{\infty\} \). The symbol \( C^k(\Omega) \) denotes the class of all functions \( f : \Omega \rightarrow \mathbb{R} \) with \( k \) continuous partial derivatives. Denote by \( C^k_c(\Omega) \) the class of all functions \( f \in C^k(\Omega) \) with compact support.

2 \( H_a \)-BV functions

2.1 Basic properties of \( BV_{H_a}(\Omega) \)

In this section, we introduce the \( H_a \)-BV space and investigate its properties. The \( H_a \)-divergence of a vector-valued function

\[
\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C^1_c(\Omega, \mathbb{R}^d)
\]

is defined as

\[
\text{div}_{H_a} \varphi = A_{-1,a} \varphi_1 + \cdots + A_{-d,a} \varphi_d.
\]
By a simple computation, we have
\[ H_a := -\text{div}_{H_a}(\nabla_{H_a} u) = -\left( \nabla - \sigma \frac{x}{|x|^2} \right) \cdot \left( \nabla + \sigma \frac{x}{|x|^2} \right) u \]
\[ = \left( -\Delta - \sigma(d - 2 - \sigma) \frac{1}{|x|^2} \right) u \]
\[ = \left( -\Delta + \frac{a}{|x|^2} \right) u, \]
where
\[ \sigma := \frac{d - 2}{2} - \frac{1}{2} \sqrt{(d - 2)^2 + 4a}. \]
Let \( \Omega \subseteq \mathbb{R}^d \) be an open set. The \( H_a \)-variation of \( f \in L^1(\Omega) \) is defined by
\[ |\nabla_{H_a} f|(\Omega) := \sup_{\varphi \in \mathcal{F}(\Omega)} \left\{ \int_{\Omega} f(x) \text{div}_{H_a} \varphi(x) dx \right\}, \]
where \( \mathcal{F}(\Omega) \) denotes the class of all functions
\[ \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C^1_c(\Omega, \mathbb{R}^d) \]
satisfying
\[ \|\varphi\|_{\infty} = \sup_{x \in \Omega} \left\{ (|\varphi_1(x)|^2 + \cdots + |\varphi_d(x)|^2)^{1/2} \right\} \leq 1. \]
An function \( f \in L^1(\Omega) \) is said to have the \( H_a \)-bounded variation on \( \Omega \) if
\[ |\nabla_{H_a} f|(\Omega) < \infty, \]
and the collection of all such functions is denoted by \( BV_{H_a}(\Omega) \). It follows from Remark 2.3 below that \( BV_{H_a}(\Omega) \) is a Banach space with the norm
\[ \|f\|_{BV_{H_a}(\Omega)} = \|f\|_{L^1(\Omega)} + |\nabla_{H_a} f|(\Omega). \]
For the sake of research, we give the definition of the Sobolev space associated with \( H_a \). [16] has also studied the Sobolev space defined in terms of the operator \((H_a)^{s/2}\) for \( 0 < s < 1 \).

**Definition 2.1.** Suppose \( \Omega \) is an open set in \( \mathbb{R}^d \) for \( d \geq 2 \). Let \( 1 \leq p \leq \infty \). The Sobolev space \( W^{k,p}_{H_a}(\Omega) \) associated with \( H_a \) is defined as the set of all functions \( f \in L^p(\Omega) \) such that
\[ A_{j_1,a} \cdots A_{j_m,a} f \in L^p(\Omega), \quad 1 \leq j_1, \ldots, j_m \leq d \quad \& \quad 1 \leq m \leq k. \]
The norm of \( f \in W^{k,p}_{H_a}(\Omega) \) is given by
\[ \|f\|_{W^{k,p}_{H_a}(\Omega)} := \sum_{1 \leq j_1, \ldots, j_m \leq d, 1 \leq m \leq k} \|A_{j_1,a} \cdots A_{j_m,a} f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}, \]
where \( a \geq -(\frac{d}{2} - 1)^2 \).

In what follows, we will collect some properties of the space \( BV_{H_a}(\Omega) \).

**Lemma 2.2.**
(i) Suppose that \( f \in W^{1,1}_{H_a}(\Omega) \). Then
\[
|\nabla_{H_a}f|(\Omega) = \int_{\Omega} |\nabla_{H_a}f(x)|dx.
\]

(ii) *(Lower semicontinuity)* Suppose that \( f_k \in BV_{H_a}(\Omega), k \in \mathbb{N} \) and \( f_k \to f \) in \( L^1_{loc}(\Omega) \). Then
\[
|\nabla_{H_a}f|(\Omega) \leq \lim inf_{k \to \infty} |\nabla_{H_a}f_k|(\Omega).
\]

*Proof.* (i) For every \( \varphi \in C_c^1(\Omega, \mathbb{R}^d) \) with \( \|\varphi\|_{L^\infty(\Omega)} \leq 1 \), we have
\[
\left| \int_{\Omega} f(x) \text{div}_{H_a} \varphi(x) dx \right| = \left| \int_{\Omega} \nabla_{H_a}f(x) \cdot \varphi(x) dx \right| \leq \int_{\Omega} |\nabla_{H_a}f(x)|dx.
\]
By taking the supremum over \( \varphi \), it is obvious that
\[
|\nabla_{H_a}f|(\Omega) \leq \int_{\Omega} |\nabla_{H_a}f(x)|dx.
\]

Define \( \varphi \in L^\infty(\Omega, \mathbb{R}^d) \) as follows:
\[
\varphi(x) := \begin{cases} 
\frac{\nabla_{H_a}f(x)}{|\nabla_{H_a}f(x)|}, & \text{if } x \in \Omega \text{ and } \nabla_{H_a}f(x) \neq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

It is easy to see that \( \|\varphi\|_{L^\infty(\Omega)} \leq 1 \). Moreover, we can obtain the approximating smooth fields \( \varphi_n := (\varphi_{n,1}, \ldots, \varphi_{n,d}) \) such that \( \varphi_n \to \varphi \) pointwise as \( n \to \infty \), with \( \|\varphi_n\|_{L^\infty(\Omega)} \leq 1 \) for all \( n \in \mathbb{N} \). In fact, for \( m > 0 \), let
\[
\phi_{m,i}(x) := \varphi_i(x) \chi_{B(0,m) \cap \Omega}(x), \quad x \in \Omega,
\]
where \( i = 1, \ldots, d \) and \( \chi_{B(0,m) \cap \Omega}(x) \) is the characteristic function of \( B(0,m) \cap \Omega \). Then \( \phi_{m,i}(x) \to \varphi_i(x) \) as \( m \to \infty \) for any \( x \in \Omega \). For any \( \epsilon > 0 \), there exists a sufficiently large \( m_0 \) such that
\[
|\phi_{m_0,i}(x) - \varphi_i(x)| < \epsilon/2.
\]
Choose \( \{\psi_n\}_{n>0} \subset C_c^\infty(\Omega) \) as an identity approximation and define
\[
\varphi_{n,i}(x) := \psi_n \ast \phi_{m_0,i}(x), \quad x \in \Omega.
\]
Then \( \{\varphi_{n,i}\}_{n \in \mathbb{N}} \subset C_c^\infty(\Omega) \) such that \( \lim_{n \to \infty} \varphi_{n,i}(x) = \phi_{m_0,i}(x) \) for any \( x \in \Omega \). This indicates that for the above \( \epsilon > 0 \), there exists \( N > 0 \) such that for \( n > N \),
\[
|\varphi_{n,i}(x) - \phi_{m_0,i}(x)| < \epsilon/2
\]
and
\[
|\varphi_{n,i}(x) - \varphi_i(x)| \leq |\phi_{m_0,i}(x) - \varphi_i(x)| + |\varphi_{n,i}(x) - \phi_{m_0,i}(x)| < \epsilon,
\]
i.e., \( \varphi_{n,i} \to \varphi_i \) pointwise as \( n \to \infty \), where \( i = 1, \ldots, d \). Then the desired approximating smooth fields \( \varphi_n \) can be obtained. Also, it follows from the facts \( \psi_n \in C_c^\infty(\Omega) \) and \( \|\varphi\|_{L^\infty(\Omega)} \leq 1 \) that
\[
|\varphi_n(x)| = \int_{\Omega} |\psi_n(y)| \cdot |\phi_{m,i}(x-y)|dy \leq 1,
\]
Therefore, (ii) can be proved by the definition of \(|\nabla H_a f|\) with integration by parts derives that for every \(n \geq 1\),

\[
|\nabla H_a f|(\Omega) \geq \int_{\Omega} f(x) \text{div} H_a \varphi_n(x) \, dx \\
= \int_{\Omega} f(x) \left( \left( \frac{\partial}{\partial x_1} - \sigma \frac{x_1}{|x|^2} \right) \varphi_{n,1}(x) + \cdots + \left( \frac{\partial}{\partial x_d} - \sigma \frac{x_d}{|x|^2} \right) \varphi_{n,d}(x) \right) \, dx \\
= - \int_{\Omega} \nabla H_a f(x) \cdot \varphi_n(x) \, dx.
\]

Using the dominated convergence theorem and the definition of \(\varphi\) in (5), we have

\[
|\nabla H_a f|(\Omega) \geq \int_{\Omega} |\nabla H_a f(x)| \, dx
\]

by letting \(n \to \infty\).

(ii) Fix \(\varphi \in C^1_c(\Omega, \mathbb{R}^d)\) with \(\|\varphi\|_{L^\infty(\Omega)} \leq 1\). By the definition of \(|\nabla H_a f_k|\) \((\Omega)\), we have

\[
|\nabla H_a f_k|(\Omega) \geq \int_{\Omega} f_k(x) \text{div} H_a \varphi(x) \, dx.
\]

Since \(\varphi \in C^1(\Omega, \mathbb{R}^d)\) and \(0 \notin \Omega\), there exists a constant \(c > 0\) such that \(|x| > c\) for \(x \in \text{supp} \varphi\), which gives

\[
|\text{div} H_a \varphi(x)| \leq \sum_{i=1}^d \left\{ \left| \frac{\partial \varphi}{\partial x_i}(x) \right| + |\sigma| \frac{|x_i|}{|x|^2} \right\} \leq C.
\]

This indicates that

\[
\left| \int_{\Omega} f_k(x) \text{div} H_a \varphi(x) \, dx - \int_{\Omega} f(x) \text{div} H_a \varphi(x) \, dx \right| \leq C \int_{\Omega} |f_k(x) - f(x)| \, dx,
\]

which, together with the convergence of \(\{f_k\}_{k \in \mathbb{N}}\) to \(f\) in \(L^1_{\text{loc}}(\Omega)\), implies that

\[
\liminf_{k \to \infty} |\nabla H_a f_k|(\Omega) \geq \liminf_{k \to \infty} \int_{\Omega} f_k(x) \text{div} H_a \varphi(x) \, dx = \int_{\Omega} f(x) \text{div} H_a \varphi(x) \, dx.
\]

Therefore, (ii) can be proved by the definition of \(|\nabla H_a f|\) \((\Omega)\) and the arbitrariness of such functions \(\varphi\).

**Remark 2.3.** The space \((\mathcal{BV}_{H_a}(\Omega), \| \cdot \|_{\mathcal{BV}_{H_a}(\Omega)})\) is a Banach space. Firstly, it is easy to check that \(\| \cdot \|_{\mathcal{BV}_{H_a}(\Omega)}\) is a norm. Secondly, let \(\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{BV}_{H_a}(\Omega)\) be a Cauchy sequence. Then \(\{f_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in the Banach space \(L^1(\mathbb{R}^d)\). Finally, using the lower semicontinuity of \(H_a\)-BV functions (cf. Lemma 2.2), we obtain the desired proof.

The following lemma gives the structure theorem for \(H_a\)-BV functions and it can be proved by the Hahn-Banach theorem and the Riesz representation theorem in [4].

**Lemma 2.4.** Let \(u \in \mathcal{BV}_{H_a}(\Omega)\). There exists a unique \(\mathbb{R}^d\)-valued finite Radon measure \(\mu_{H_a}\) such that

\[
\int_{\Omega} u(x) \text{div} H_a \varphi(x) \, dx = \int_{\Omega} \varphi(x) \cdot d\mu_{H_a}(x)
\]

for any \(\varphi \in C^\infty_c(\Omega, \mathbb{R}^d)\) and

\[
|\nabla H_a u|(\Omega) = |\mu_{H_a}|(\Omega),
\]

where \(\mu_{H_a}\) is the total variation of the measure \(\mu_{H_a}\).
Combining the mean value theorem of multivariate functions with an additional condition (6), we can obtain the following approximation result for the $\mathcal{H}_a$-variation. It should be noted that the condition (6) is added due to the singularity of the perturbation term $\sigma \frac{x}{|x|}$ in div$\mathcal{H}_a$ and we explain the relation between the condition (6) and the Hardy-Sobolev inequality.

**Theorem 2.5.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. Assume that $u \in \mathcal{BV}_{\mathcal{H}_a}(\Omega)$ with

$$
\int_{\Omega} |u(y)||y|^{-2}dy < \infty,
$$

then there exists a sequence $\{u_h\}_{h \in \mathbb{N}} \in \mathcal{BV}_{\mathcal{H}_a}(\Omega) \cap C^\infty_c(\Omega)$ such that

$$
\lim_{h \to \infty} \|u_h - u\|_{L^1(\Omega)} = 0
$$

and

$$
\lim_{h \to \infty} \int_{\Omega} |\nabla \mathcal{H}_a u_h(x)|dx = |\nabla \mathcal{H}_a u|(\Omega).
$$

*Proof.* The following proof is similar to that of [13, Section 5.2.2, Theorem 2], but different from its proof we need to use the mean value theorem of multivariate functions and the condition (6).

Via the lower semicontinuity of $\mathcal{H}_a$-$BV$ functions, it suffices to show that for $\varepsilon > 0$, there exists a function $u_\varepsilon \in C^\infty(\Omega)$ such that

$$
\int_{\Omega} |u_\varepsilon(x) - u(x)|dx < \varepsilon
$$

and

$$
|\nabla \mathcal{H}_a u_\varepsilon|(\Omega) \leq |\nabla \mathcal{H}_a u|(\Omega) + \varepsilon.
$$

Fix $\varepsilon > 0$. Given a positive integer $m$, let

$$
\Omega_j := \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) > \frac{1}{m + j} \right\} \cap B(0, m + j), \quad j \in \mathbb{N},
$$

where $\text{dist}(x, \partial \Omega) = \inf\{|x - y| : y \in \partial \Omega\}$. In fact, let $\{\Omega_j\}_{j \in \mathbb{N}}$ be a sequence of subdomains of $\Omega$ such that $\Omega_j \subset \Omega_{j+1} \subset \Omega$, $j \in \mathbb{N}$ and $\bigcup_{j=0}^\infty \Omega_j = \Omega$. Since $|\nabla \mathcal{H}_a u|(:)$ is a measure, we can choose a $m \in \mathbb{N}$ so large that

$$
|\nabla \mathcal{H}_a u|(\Omega \setminus \Omega_0) < \varepsilon.  \tag{7}
$$

Let $U_0 := \Omega_0$ and $U_j := \Omega_{j+1} \setminus \Omega_{j-1}$ for $j \geq 1$. Following the proof of [13, Section 5.2.2, Theorem 2], we conclude that there is a partition of unity subordinate to the covering $\{U_j\}_{j \in \mathbb{N}}$. Thus, there exist functions $\{f_j\}_{j \in \mathbb{N}} \in C^\infty_c(U_j)$ such that $0 \leq f_j \leq 1$, $j \geq 0$ and $\sum_{j=0}^\infty f_j = 1$ on $\Omega_0$.

Given $\varepsilon > 0$ and $u \in L^1(\Omega)$, extended to zero out of $\Omega$, we define the usual regularization

$$
u_\varepsilon(x) := \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} \eta\left(\frac{x - y}{\varepsilon}\right)u(y)dy,
$$

where $\eta \in C^\infty_c(\mathbb{R}^d)$ is a nonnegative radial function satisfying $\int_{\mathbb{R}^d} \eta(x)dx = 1$ and supp$\eta \subset B(0, 1)$. Then for each $j$, there exists $0 < \varepsilon_j < \varepsilon$, such that

$$
\text{supp}((f_j u)_{\varepsilon_j}) \subset U_j,
$$

$$
\int_{\Omega} |(f_j u)_{\varepsilon_j}(x) - f_j u(x)|dx < \varepsilon 2^{-(j+1)},
$$

$$
\int_{\Omega} |(u \nabla f_j)_{\varepsilon_j}(x) - u \nabla f_j(x)|dx < \varepsilon 2^{-(j+1)}. \tag{8}
$$
Define
\[ v_\varepsilon (x) := \sum_{j=0}^{\infty} (uf_j)\varepsilon_j (x). \]

Clearly, \( v_\varepsilon \in C^\infty (\Omega) \) and \( u = \sum_{j=0}^{\infty} uf_j \). Therefore, by a simple computation, we can get
\[ \| v_\varepsilon - u \|_{L^1 (\Omega)} \leq \sum_{j=0}^{\infty} \Omega \int | (f_j u)\varepsilon_j (x) - f_j (x)u(x) | dx < \varepsilon. \]

Consequently, \( v_\varepsilon \to u \) in \( L^1 (\Omega) \) as \( \varepsilon \to 0 \). Now, let \( \varphi \in C^1_c (\Omega, \mathbb{R}^d) \) satisfying \( |\varphi| \leq 1 \). Then
\[
\int_{\Omega} v_\varepsilon (x) \text{div}_{\mathcal{H}_a} \varphi (x) dx
\]
\[ = \int_{\Omega} \left( \sum_{j=0}^{\infty} (uf_j)\varepsilon_j (x) \right) \text{div}_{\mathcal{H}_a} \varphi (x) dx
\]
\[ = \sum_{j=0}^{\infty} \Omega \int (uf_j)\varepsilon_j (x) \left( A_{-1,0} \varphi_1 (x) + A_{-2,0} \varphi_2 (x) + \cdots + A_{-d,0} \varphi_d (x) \right) dx
\]
\[ := I + II,
\]
where
\[
\begin{cases}
I := \sum_{j=0}^{\infty} \Omega \int (uf_j)\varepsilon_j (x) \left( \frac{\partial}{\partial x_1} \varphi_1 (x) + \frac{\partial}{\partial x_2} \varphi_2 (x) + \cdots + \frac{\partial}{\partial x_d} \varphi_d (x) \right) dx;

II := -\sum_{j=0}^{\infty} \Omega \int (uf_j)\varepsilon_j (x) \left( \frac{\sigma x_1}{|x|^2} \varphi_1 (x) + \frac{\sigma x_2}{|x|^2} \varphi_2 (x) + \cdots + \frac{\sigma x_d}{|x|^2} \varphi_d (x) \right) dx.
\end{cases}
\]
As for \( I \), we get
\[
I = \sum_{j=0}^{\infty} \Omega \int (uf_j)\varepsilon_j (x) \text{div} \varphi (x) dx
\]
\[ = \sum_{j=0}^{\infty} \Omega \int (uf_j) (y) \text{div} (\eta_{\varepsilon_j} \ast \varphi (y)) dy
\]
\[ = \sum_{j=0}^{\infty} \Omega \int u (y) \text{div} (f_j (\eta_{\varepsilon_j} \ast \varphi)) (y) dy - \sum_{j=0}^{\infty} \Omega \int u (y) \nabla (f_j (\eta_{\varepsilon_j} \ast \varphi)) (y) dy
\]
\[ = \sum_{j=0}^{\infty} \Omega \int u (y) \text{div} (f_j (\eta_{\varepsilon_j} \ast \varphi)) (y) dy - \sum_{j=0}^{\infty} \Omega \int \varphi (y) \left( \eta_{\varepsilon_j} \ast (u \nabla f_j) (y) - u \nabla f_j (y) \right) dy
\]
\[ := I_1 + I_2,
\]
where in the last equality we have used the fact that \( \sum_{j=0}^{\infty} \nabla f_j (x) = 0 \) on \( \Omega \). Actually, when \( \| \varphi \|_{L^\infty} \leq 1 \), it holds that \( |f_j (\eta_{\varepsilon_j} \ast \varphi) (x)| \leq 1 \), \( j \in \mathbb{N} \), and each point in \( \Omega \) belongs to at most three of the sets \( \{ U_j \}_0^\infty \). Moreover, it follows from (8) that \( |I_2| < \varepsilon \).
As for $II$, we change the order of integration to get

$$ II = -\sum_{j=0}^{\infty} \int_{\Omega} (uf_{j})_{\varepsilon, j}(x) \left\{ \frac{\sigma x_1}{|x|^2} \varphi_1(x) + \frac{\sigma x_2}{|x|^2} \varphi_2(x) + \cdots + \frac{\sigma x_d}{|x|^2} \varphi_d(x) \right\} dx $$

$$ = -\sum_{j=0}^{\infty} \int_{\Omega} \int_{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y)f_{j}(y) \left( \sum_{k=1}^{d} \frac{\sigma y_k}{|y|^2} \varphi_k(x) \right) dy dx $$

$$ - \sum_{j=0}^{\infty} \int_{\Omega} \int_{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y)f_{j}(y) \left( \sum_{k=1}^{d} \left( \frac{\sigma x_k}{|x|^2} - \frac{\sigma y_k}{|y|^2} \right) \varphi_k(x) \right) dy dx $$

$$ = -\sum_{j=0}^{\infty} \int_{\Omega} u(y)f_{j}(y) \left( \sum_{k=1}^{d} \frac{\sigma y_k}{|y|^2} \varphi_k \eta \varepsilon_j(y) \right) dy $$

$$ - \sum_{j=0}^{\infty} \int_{\Omega} \int_{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y)f_{j}(y) \left( \sum_{k=1}^{d} \left( \frac{\sigma x_k}{|x|^2} - \frac{\sigma y_k}{|y|^2} \right) \varphi_k(x) \right) dy dx. $$

Consequently, the above estimate of the term $I_2$ shows that

$$ \left| \int_{\Omega} v_{\varepsilon}(x) \text{div}_{H^a} \varphi(x) dx \right| = |I_1 + I_2 + II| \leq J_1 + J_2 + \varepsilon, $$

where

$$ J_1 := \left| \sum_{j=0}^{\infty} \int_{\Omega} u(y) \text{div}(f_{j}(\eta \varepsilon_j * \varphi)) dy - \sum_{j=0}^{\infty} \int_{\Omega} u(y)f_{j}(y) \left( \sum_{k=1}^{d} \frac{\sigma y_k}{|y|^2} (\varphi_k \eta \varepsilon_j(y)) \right) dy \right| $$

and

$$ J_2 := \left| -\sum_{j=0}^{\infty} \int_{\Omega} \int_{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y)f_{j}(y) \left( \sum_{k=1}^{d} \left( \frac{\sigma x_k}{|x|^2} - \frac{\sigma y_k}{|y|^2} \right) \varphi_k(x) \right) dy dx \right|. $$

Moreover,

$$ J_1 = \left| \sum_{j=0}^{\infty} \int_{\Omega} u(y) \text{div}(f_{j}(\eta \varepsilon_j * \varphi)) dy - \sum_{j=0}^{\infty} \int_{\Omega} u(y)f_{j}(y) \left( \sum_{k=1}^{d} \frac{\sigma y_k}{|y|^2} (\varphi_k \eta \varepsilon_j(y)) \right) dy \right| $$

$$ \leq \left| \int_{\Omega} u(y) \text{div}(f_0(\eta \varepsilon_0 * \varphi)) dy - \int_{\Omega} u(y)f_0(y) \left( \sum_{k=1}^{d} \frac{\sigma y_k}{|y|^2} (\varphi_k \eta \varepsilon_0(y)) \right) dy \right| $$

$$ + \left| \sum_{j=1}^{\infty} \int_{\Omega} u(y) \text{div}(f_{j}(\eta \varepsilon_j * \varphi)) dy - \sum_{j=1}^{\infty} \int_{\Omega} u(y)f_{j}(y) \left( \sum_{k=1}^{d} \frac{\sigma y_k}{|y|^2} (\varphi_k \eta \varepsilon_j(y)) \right) dy \right| $$

$$ \leq |\nabla_{H^a} u|_{\Omega} + \sum_{j=1}^{\infty} |\nabla_{H^a} u|_{(U_j)} $$

$$ \leq |\nabla_{H^a} u|_{\Omega} + |\nabla_{H^a} u|_{\Omega \setminus \Omega_0} $$

$$ \leq |\nabla_{H^a} u|_{\Omega} + 3\varepsilon, $$

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where we have used the fact (7) in the last inequality. Note that \( \psi(x) = x_k |x|^{-2} \), \( \| \varphi \|_{L^\infty} \leq 1 \) and \( \text{supp}(\eta) \subseteq B(0, 1) \). When \( |x - y| < \varepsilon_j < |y|/2 \), by the mean value theorem of multivariate functions, there exists \( \theta \in (0, 1) \) such that

\[
|\psi(x) - \psi(y)| = |y + \theta(x - y)|^{-2} |x - y| \leq C|x - y| |y|^{-2}.
\]

Then, we obtain

\[
J_2 = \left| - \sum_{j=0}^{\infty} \int_{\Omega} \int_{\Omega} \frac{1}{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y) f_j(y) \left\{ \sum_{k=1}^{d} \left( \frac{\sigma x_k}{|x|^2} - \frac{\sigma y_k}{|y|^2} \right) \varphi_k(x) \right\} dy \right| dx
\]

\[
\leq |\sigma| \sum_{j=0}^{\infty} \int_{\Omega} \int_{\Omega} \frac{1}{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) u(y) f_j(y) \left\{ \sum_{k=1}^{d} \left| \frac{x - y}{y + \theta(x - y)} \right|^2 \varphi_k(x) \right\} dy \right| dx
\]

\[
\leq C|\sigma| \sum_{j=0}^{\infty} \int_{\Omega} \int_{\Omega} \frac{1}{\varepsilon_j} \eta \left( \frac{x - y}{\varepsilon_j} \right) |x - y| dy \right| dx |u(y)||f_j(y)||y|^{-2} dy
\]

\[
\leq C \varepsilon_j |\sigma| \int_{\mathbb{R}^d} |\eta(z)| dz \sum_{j=0}^{\infty} \int_{\Omega} |u(y)||f_j(y)||y|^{-2} dy
\]

\[
= C \varepsilon_j |\sigma| \int_{\mathbb{R}^d} |\eta(z)| dz \int_{\Omega} |u(y)| \left\| \sum_{j=0}^{\infty} f_j(y) \right\| |y|^{-2} dy
\]

\[
\leq C \varepsilon_j |\sigma| \int_{\mathbb{R}^d} |\eta(z)| dz
\]

\[
\lesssim |\varepsilon|,
\]

where we have used the condition (6). By taking the supremum over \( \varphi \) and the arbitrariness of \( \varepsilon > 0 \), the theorem can be proved. \( \square \)

**Remark 2.6.** Suppose \( \Omega \) is a bounded domain in \( \mathbb{R}^d \) and the weighted Sobolev space \( W_0^{k,p}(\Omega, |x|^\alpha) \) is the closure of the set \( C_0^\infty(\Omega) \) in the norm

\[
\|u\| := \sum_{|\beta| \leq k} \| \partial^\beta u \|_{L^p(|x|^\alpha)}.
\]

with \( 1 < p < d/k \) and \( -2 \leq \alpha \leq 0 \). It is noted that \( W_0^{k,p}(\Omega, |x|^\alpha) \) is exactly the classical Sobolev space when \( \alpha = 0 \). The condition (6) can be satisfied in one of the following three cases:

1. \( u \in W_0^{1,p}(\Omega, |x|^\alpha) \) for \( 1 < p < -d/\alpha \) with \( -2 \leq \alpha \leq -1 \) and \( d \geq 3 \);
2. \( u \in W_0^{2,2}(\Omega) \) for \( d > 4 \);
3. \( u \) is any bounded function on \( \Omega \) for \( d \geq 3 \).

**Proof.** Firstly, if \( u \in W_0^{1,p}(\Omega) \), then we use the Hardy-Sobolev inequality in [8, p. 259] to obtain

\[
\int_{\Omega} |u(y)||y|^{-2} dy \leq |\Omega|^{1/r} \left( \int_{\Omega} |u(y)|^r |y|^{-2r} dy \right)^{1/r}
\]
\[ \lesssim |\Omega|^{1/r'} \left( \int_{\Omega} |\nabla u|^p \, dy \right)^{1/p} < \infty, \]

where \(1/r - (1 + \alpha)/d = 1/p\) and \(1 < r < d/2\). Secondly, if \(u \in W^{2,2}_0(\Omega)\), then by the Hardy-Sobolev inequality in [19, (1.2)] we obtain

\[
\int_{\Omega} |u(y)||y|^{-2} \, dy \leq |\Omega|^{1/2} \left( \int_{\Omega} |u(y)|^2 |y|^{-4} \, dy \right)^{1/2} \lesssim |\Omega|^{1/2} \left( \int_{\Omega} |\Delta u|^2 \, dy \right)^{1/2} < \infty.
\]

Finally, since \(u\) is bounded on \(\Omega\), then there exists a nonnegative constant \(C\) such that \(|u(x)| \leq C\) for any \(x \in \Omega\). Moreover, there exists a ball \(B(0, r)\) such that \(E \subseteq B(0, r)\), where \(B(0, r)\) denotes a ball with center 0 and radius \(r > 0\). Then, since \(d \geq 3\),

\[
\int_{\Omega} |u(y)||y|^{-2} \, dy \leq C \left( \int_{B(0, r)} |y|^{-2} \, dy \right)^{1/2} < \infty.
\]

All in all, the condition (6) can be satisfied under the above cases.

Moreover, by Lemma 2.2 and Theorem 2.5, we have the following max-min property of the \(H_a\)-variation.

**Theorem 2.7.** Let \(\Omega \subset \mathbb{R}^d\) be an open and bounded domain. Suppose \(u, v \in L^1(\Omega)\) obey the condition (6), then

\[
|\nabla H_a \max\{u, v\}|(\Omega) + |\nabla H_a \min\{u, v\}|(\Omega) \leq |\nabla H_a u|(\Omega) + |\nabla H_a v|(\Omega).
\]

**2.2 Basic properties of \(H_a\)-perimeter**

The \(H_a\)-perimeter of \(E \subseteq \Omega\) can be defined as follows:

\[
P_{H_a}(E, \Omega) = |\nabla H_a 1_E|(\Omega) = \sup_{\varphi \in F(\Omega)} \left\{ \int_E \text{div}_a \varphi(x) \, dx \right\}. \tag{9}
\]

The following conclusion is a direct corollary of Lemma 2.2.

**Corollary 2.8.** (Lower semicontinuity of \(P_{H_a}\)) Suppose \(1_{E_k} \rightarrow 1_E\) in \(L^1_{\text{loc}}(\Omega)\), where \(E\) and \(E_k\), \(k \in \mathbb{N}\), are subsets of \(\Omega\), then

\[
P_{H_a}(E, \Omega) \leq \liminf_{k \to \infty} P_{H_a}(E_k, \Omega).
\]

Moreover, by Theorem 2.7, via choosing \(u = 1_E\) and \(v = 1_F\) for any compact subsets \(E, F\) in \(\Omega\), we immediately obtain the following corollary. Moreover, it follows from [21, Section 1.1 (iii)] that the equality condition of (10) can be similarly obtained.

**Corollary 2.9.** For any compact subsets \(E, F\) in \(\Omega\), we have

\[
P_{H_a}(E \cap F, \Omega) + P_{H_a}(E \cup F, \Omega) \leq P_{H_a}(E, \Omega) + P_{H_a}(F, \Omega). \tag{10}
\]

Especially, if \(P_{H_a}(E \setminus (E \cap F), \Omega) \cdot P_{H_a}(F \setminus (F \cap E), \Omega) = 0\), the equality of (10) holds true.
Proof. Since (10) is valid, we only need to prove its converse inequality holds true under the above condition. Obviously, the condition \( P_{H_a}(E \setminus (E \cap F), \Omega) \cdot P_{H_a}(F \setminus (F \cap E), \Omega) = 0 \) implies that \( P_{H_a}(E \setminus (E \cap F), \Omega) = 0 \) or \( P_{H_a}(F \setminus (F \cap E), \Omega) = 0 \). Suppose \( P_{H_a}(E \setminus (E \cap F), \Omega) = 0 \). Via (10), we have

\[
\begin{align*}
P_{H_a}(E, \Omega) &= P_{H_a}((E \setminus (E \cap F)) \cup (E \cap F), \Omega) \\
&\leq P_{H_a}(E \setminus (E \cap F), \Omega) + P_{H_a}(E \cap F, \Omega) \\
&= P_{H_a}(E \cap F, \Omega).
\end{align*}
\]

Using (9) and \( E \cup F = F \cup (E \setminus (E \cap F)) \), we obtain

\[
\begin{align*}
P_{H_a}(F, \Omega) &= \sup_{\varphi \in F(\Omega)} \left\{ \int_F \text{div}_H \varphi(x) dx \right\} \\
&= \sup_{\varphi \in F(\Omega)} \left\{ \int_{E\setminus F} \text{div}_H \varphi(x) dx - \int_{E \cap (E \setminus F)} \text{div}_H \varphi(x) dx \right\} \\
&\leq \sup_{\varphi \in F(\Omega)} \left\{ \int_{E \setminus (E \cap F)} \text{div}_H \varphi(x) dx \right\} + \sup_{\varphi \in F(\Omega)} \left\{ \int_{E \cap (E \setminus F)} \text{div}_H \varphi(x) dx \right\} \\
&= P_{H_a}(E \cup F, \Omega) + P_{H_a}(E \setminus E \cap F, \Omega) \\
&= P_{H_a}(E \cap F, \Omega).
\end{align*}
\]

Combining (11) with (12) deduces that

\[
P_{H_a}(E, \Omega) + P_{H_a}(F, \Omega) \leq P_{H_a}(E \cup F, \Omega) + P_{H_a}(E \cap F, \Omega),
\]

which derives the desired result. Another case can be similarly proved, we omit the details. \( \square \)

For the \( H_a \)-variation and the \( H_a \)-perimeter, we can prove a coarea inequality for functions in \( BV_{H_a}(\Omega) \).

**Theorem 2.10.** If \( f \in BV_{H_a}(\Omega) \), then

\[
|\nabla_{H_a} f|(\Omega) \leq \int_{-\infty}^{+\infty} P_{H_a}(E_t, \Omega) dt,
\]

where \( E_t = \{ x \in \Omega : f(x) > t \} \) for \( t \in \mathbb{R} \).

**Proof.** Let \( \varphi \in C^1_c(\Omega, \mathbb{R}^d) \) and \( \| \varphi \|_{L^\infty(\Omega)} \leq 1 \). We can easily prove that for \( i = 1, 2, \ldots, d \),

\[
- \int_{\Omega} f(x) \frac{\sigma x_i}{|x|} \varphi_i(x) dx = - \int_{-\infty}^{+\infty} \left( \int_{E_t} \frac{\sigma x_i}{|x|} \varphi_i(x) dx \right) dt
\]

and

\[
\int_{\Omega} f(x) \text{div}_H \varphi(x) dx = \int_{-\infty}^{+\infty} \left( \int_{E_t} \text{div} \varphi(x) dx \right) dt,
\]

where the latter can be seen in the proof of [13, Section 5.5, Theorem 1]. It follows that

\[
\int_{\Omega} f(x) \text{div}_{H_a} \varphi(x) dx = \int_{-\infty}^{+\infty} \left( \int_{E_t} \text{div}_{H_a} \varphi(x) dx \right) dt.
\]
Therefore, we conclude that for all $\varphi$ as above,
\[
\int_{\Omega} f(x) \text{div}_{\mathcal{H}_a} \varphi(x) dx \leq \int_{-\infty}^{+\infty} P_{\mathcal{H}_a}(E_t, \Omega) dt.
\]
Furthermore,
\[
|\nabla_{\mathcal{H}_a} f|(\Omega) \leq \int_{-\infty}^{+\infty} P_{\mathcal{H}_a}(E_t, \Omega) dt.
\]

\[\square\]

3 $\tilde{\mathcal{H}}_\sigma$-BV functions

In order to overcome the deficiency of $\mathcal{BV}_{\mathcal{H}_a}(\Omega)$ in the coarea formula, we turn to study another form of divergence operator and gradient operator related to the operator $\tilde{\mathcal{H}}_\sigma$ which is closely related to the operator $\mathcal{H}_a$. Via a simple computation, we obtain, for $u \in C^2_c(\Omega)$,
\[
\tilde{\mathcal{H}}_\sigma u := -\text{div}_{\tilde{\mathcal{H}}_\sigma} (\nabla \tilde{\mathcal{H}}_\sigma u) = -\left(\nabla - \sigma \frac{x}{|x|^2}, \nabla + \sigma \frac{x}{|x|^2}\right) \cdot \left(\nabla + \sigma \frac{x}{|x|^2}, \nabla - \sigma \frac{x}{|x|^2}\right) u
\]
\[
= -\left(\nabla - \sigma \frac{x}{|x|^2}, \nabla + \sigma \frac{x}{|x|^2}\right) \cdot \left(\nabla + \sigma \frac{x}{|x|^2}\right) u, (\nabla - \sigma \frac{x}{|x|^2}) u
\]
\[
= \left(\Delta - \sigma (d - 2 - \sigma) \frac{1}{|x|^2}\right) u + \left(-\Delta + \sigma ((d - 2 - \sigma) \frac{1}{|x|^2} + 2\sigma \frac{1}{|x|^2})\right) u
\]
\[
= 2\left(-\Delta + \frac{\sigma^2}{|x|^2}\right) u.
\]

The $\tilde{\mathcal{H}}_\sigma$-divergence of a vector valued function
\[
\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_{2d}) \in C^1_c(\Omega, \mathbb{R}^{2d})
\]
is defined as
\[
\text{div}_{\tilde{\mathcal{H}}_\sigma} \Phi := A_{-1,a} \varphi_1 + \cdots + A_{-d,a} \varphi_d + A_{1,a} \varphi_{d+1} + \cdots + A_{d,a} \varphi_{2d}.
\]
For $u \in C^1_c(\Omega)$, the $\tilde{\mathcal{H}}_\sigma$-gradient of $u$ is defined as
\[
\nabla_{\tilde{\mathcal{H}}_\sigma} u := (A_{1,a} u, \ldots, A_{d,a} u, A_{-1,a} u, \ldots, A_{-d,a}) u.
\]

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. The $\tilde{\mathcal{H}}_\sigma$-variation of $f \in L^1(\Omega)$ is defined by
\[
|\nabla_{\tilde{\mathcal{H}}_\sigma} f|(\Omega) = \sup_{\Phi \in \tilde{\mathcal{F}}(\Omega)} \left\{ \int_{\Omega} f(x) \text{div}_{\tilde{\mathcal{H}}_\sigma} \Phi(x) dx \right\},
\]
where $\tilde{\mathcal{F}}(\Omega)$ denotes the class of all functions
\[
\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_{2d}) \in C^1_c(\Omega, \mathbb{R}^{2d})
\]
satisfying
\[
\|\Phi\|_{\infty} = \sup_{x \in \Omega} \left\{ (|\varphi_1(x)|^2 + \ldots + |\varphi_{2d}(x)|^2)^{1/2} \right\} \leq 1.
\]
An function \( f \in L^1(\Omega) \) is said to have the \( \tilde{\mathcal{H}}_\sigma \)-bounded variation on \( \Omega \) if
\[
|\nabla \tilde{\mathcal{H}}_\sigma f|(\Omega) < \infty,
\]
and the collection of all such functions is denoted by \( \mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega) \), which is a Banach space with the norm
\[
\|f\|_{\mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega)} = \|f\|_{L^1(\Omega)} + |\nabla \tilde{\mathcal{H}}_\sigma f|(\Omega).
\]
The \( \tilde{\mathcal{H}}_\sigma \)-perimeter of \( E \subseteq \Omega \) can be defined as follows:
\[
P_{\tilde{\mathcal{H}}_\sigma}(E, \Omega) = \left| \nabla \tilde{\mathcal{H}}_\sigma 1_E \right|(\Omega) = \sup_{\Phi \in \tilde{\mathcal{F}}(\Omega)} \left\{ \int_E \operatorname{div} \tilde{\mathcal{H}}_\sigma \Phi(x) dx \right\}.
\]
It is easy to see that if \( u \) belongs to \( \mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega) \), then \( u \) also belongs to \( \mathcal{BV}_{\mathcal{H}_\alpha}(\Omega) \). In fact, this can be proved by choosing \( \Phi = (\varphi_1, \ldots, \varphi_d, 0, \ldots, 0) \) in the definition of the \( \tilde{\mathcal{H}}_\sigma \)-variation of \( u \).

### 3.1 Basic properties of \( \mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega) \)

In this subsection, using similar methods, we conclude that \( \mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega) \) enjoys several properties as same as those of \( \mathcal{BV}_{\mathcal{H}_\alpha}(\Omega) \). For convenience, we list the following results for \( \mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega) \) and omit the proofs.

**Lemma 3.1.** Let \( u \in \mathcal{BV}_{\tilde{\mathcal{H}}_\sigma}(\Omega) \). There exists a unique \( R^{2d} \)-valued finite Radon measure \( D_{\tilde{\mathcal{H}}_\sigma} u = (D_{A_{j_1,a},u}, \ldots, D_{A_{j_m,a},u}, D_{A_{-1,a},u}, \ldots, D_{A_{-d,a},u}) \) such that
\[
\int \Omega u(x) \operatorname{div} \tilde{\mathcal{H}}_\sigma \Phi(x) dx = \int \Omega \Phi(x) \cdot dD_{\tilde{\mathcal{H}}_\sigma} u
\]
for every \( \Phi \in C_\infty(\Omega, R^{2d}) \) and
\[
|\nabla \tilde{\mathcal{H}}_\sigma u|(\Omega) = |D_{\tilde{\mathcal{H}}_\sigma} u|(\Omega),
\]
where \( |D_{\tilde{\mathcal{H}}_\sigma} u| \) is the total variation of the measure \( D_{\tilde{\mathcal{H}}_\sigma} u \).

Similar to Definition 2.1, we give the definition of the Sobolev space associated with \( \tilde{\mathcal{H}}_\sigma \).

**Definition 3.2.** Suppose \( \Omega \) is an open set in \( R^d \) for \( d \geq 2 \). Let \( 1 \leq p \leq \infty \). The Sobolev space \( W_{\tilde{\mathcal{H}}_\sigma}^{k,p}(\Omega) \) is defined as the set of all functions \( f \in L^p(\Omega) \) such that
\[
A_{j_1,a} \cdots A_{j_m,a} f \in L^p(\Omega), \quad 1 \leq |j_1|, \ldots, |j_m| \leq d, 1 \leq m \leq k.
\]
The norm of \( f \in W_{\tilde{\mathcal{H}}_\sigma}^{k,p}(\Omega) \) is defined as
\[
\|f\|_{W_{\tilde{\mathcal{H}}_\sigma}^{k,p}(\Omega)} := \sum_{1 \leq |j_1|, \ldots, |j_m| \leq d} \|A_{j_1,a} \cdots A_{j_m,a} f\|_{L^p} + \|f\|_{L^p},
\]
where \( a \geq -(\frac{d}{2} - 1)^2 \).

It follows from Definition 3.2 that \( W_{\tilde{\mathcal{H}}_\sigma}^{k,p}(\Omega) \subseteq W_{\mathcal{H}_\alpha}^{k,p}(\Omega) \).

**Lemma 3.3.**
(i) Suppose that \( f \in W^{1,1}_{\tilde{H}_{\sigma}}(\Omega) \). Then
\[
|\nabla_{\tilde{H}_{\sigma}} f| (\Omega) = \int_{\Omega} |\nabla_{\tilde{H}_{\sigma}} f(x)| dx.
\]

(ii) Suppose that \( f_k \in BV_{\tilde{H}_{\sigma}}(\Omega), k \in \mathbb{N} \) and \( f_k \to f \) in \( L^1_{loc}(\Omega) \). Then
\[
|\nabla_{\tilde{H}_{\sigma}} f| (\Omega) \leq \liminf_{k \to \infty} |\nabla_{\tilde{H}_{\sigma}} f_k| (\Omega).
\]

**Theorem 3.4.** Let \( \Omega \subset \mathbb{R}^d \) be an open and bounded domain. Assume that \( u \in BV_{\tilde{H}_{\sigma}}(\Omega) \) satisfying the condition \((6)\), then there exists a sequence \( \{u_h\}_{h \in \mathbb{N}} \in BV_{\tilde{H}_{\sigma}}(\Omega) \cap C^\infty_c(\Omega) \) such that
\[
\lim_{h \to \infty} \|u_h - u\|_{L^1} = 0
\]
and
\[
\lim_{h \to \infty} \int_{\Omega} |\nabla_{\tilde{H}_{\sigma}} u_h(x)| dx = |\nabla_{\tilde{H}_{\sigma}} u| (\Omega).
\]

**Theorem 3.5.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open domain. Suppose \( u, v \in L^1(\Omega) \) and satisfy the condition \((6)\), then
\[
|\nabla_{\tilde{H}_{\sigma}} \max\{u, v\}| (\Omega) + |\nabla_{\tilde{H}_{\sigma}} \min\{u, v\}| (\Omega) \leq |\nabla_{\tilde{H}_{\sigma}} u| (\Omega) + |\nabla_{\tilde{H}_{\sigma}} v| (\Omega).
\]

**Corollary 3.6.** (Lower semicontinuity of \( P_{\tilde{H}_{\sigma}} \)) Suppose \( 1_{E_k} \to 1_E \) in \( L^1_{loc}(\Omega) \), where \( E \) and \( E_k, k \in \mathbb{N} \), are subsets of \( \Omega \), then
\[
P_{\tilde{H}_{\sigma}} (E, \Omega) \leq \liminf_{k \to \infty} P_{\tilde{H}_{\sigma}} (E_k, \Omega).
\]

**Corollary 3.7.** For any compact subsets \( E, F \) in \( \Omega \), we have
\[
P_{\tilde{H}_{\sigma}} (E \cap F, \Omega) + P_{\tilde{H}_{\sigma}} (E \cup F, \Omega) \leq P_{\tilde{H}_{\sigma}} (E, \Omega) + P_{\tilde{H}_{\sigma}} (F, \Omega).
\] Especially, if \( P_{\tilde{H}_{\sigma}} (E \setminus E \cap F, \Omega) \cdot P_{\tilde{H}_{\sigma}} (F \setminus (F \cap E), \Omega) = 0 \), the equality of \((13)\) holds true.

The following lemma gives a scaling relation of the \( \tilde{H}_{\sigma} \)-perimeter in \( \Omega \).

**Lemma 3.8.** For any set \( E \) in \( \Omega \), denote by \( sE \) the set \( \{sx : x \in E\} \). If \( sE \subseteq \Omega \) for \( s > 0 \), then
\[
P_{\tilde{H}_{\sigma}} (sE, \Omega) = s^{d-1} P_{\tilde{H}_{\sigma}} (E, \Omega).
\]

**Proof.** By the definition of the \( \tilde{H}_{\sigma} \)-perimeter, we have
\[
P_{\tilde{H}_{\sigma}} (sE, \Omega) = \sup_{\Phi \in \mathcal{F}(\Omega)} \left\{ \int_{sE} \text{div}_{\tilde{H}_{\sigma}} \Phi(x) dx \right\}
\]
\[
= \sup_{\Phi \in \mathcal{F}(\Omega)} \left\{ \int_E s^{d-1} \left( \sum_{k=1}^{d} \partial_{x_k} x_k \varphi_k (sx) + \sum_{k=1}^{d} \partial_{x_k} \varphi_{d+k} (sx) \right)
\right. \\
\left. - s^{d-1} \left( \sum_{k=1}^{d} x_k |x|^{-2} \varphi_k (sx) - \sum_{k=1}^{d} x_k |x|^{-2} \varphi_{d+k} (sx) \right) \right\} dx.
\]
Then
\[
P_{\tilde{H}_{\sigma}} (sE, \Omega) = s^{d-1} P_{\tilde{H}_{\sigma}} (E, \Omega).
\]
An immediate corollary of Lemma 3.8 is given as follows.

**Corollary 3.9.** Let $B(x, s)$ be the open ball in $\Omega$ centered at $x$ with radius $s$.

$$P_{\tilde{\mathcal{H}}_{\sigma}}(B(x, s), \Omega) = P_{\mathcal{H}_{\sigma}}(B(x, 1), \Omega)s^{d-1}.$$  

**Remark 3.10.** It should be noted that the set $E$ and its complementary set have the same perimeter in the classical case. But unfortunately, for the case of the $\tilde{\mathcal{H}}_{\sigma}$-perimeter, the above fact does not hold. For example, let $\Omega = \mathbb{R}^d \setminus \{0\}$ and $E = B(x, r)$ with $r > 0$. The definition of the $\tilde{\mathcal{H}}_{\sigma}$-perimeter and Corollary 3.9 indicate that

$$P_{\tilde{\mathcal{H}}_{\sigma}}(B(x, r)^c, \Omega) \geq \int_{B(x, r)^c} |y|^{-1} dy = \infty > P_{\tilde{\mathcal{H}}_{\sigma}}(B(x, r), \Omega).$$

### 3.2 Coarea formula and Sobolev’s inequality of $\tilde{\mathcal{H}}_{\sigma}$-BV functions

In what follows, we prove the coarea formula for $\tilde{\mathcal{H}}_{\sigma}$-BV functions. Before proving this result, we give the following lemma.

**Lemma 3.11.** If $f \in C^1(\Omega)$, then

$$|\nabla f(x)| + \frac{\sigma}{|x|}|f(x)| \leq |\nabla \tilde{\mathcal{H}}_{\sigma} f(x)| \leq \sqrt{2}\left(|\nabla f(x)| + \frac{\sigma}{|x|}|f(x)|\right). \quad (14)$$

**Proof.** By the definition of $\nabla \tilde{\mathcal{H}}_{\sigma} f$, we have

$$|\nabla \tilde{\mathcal{H}}_{\sigma} f|^2 = |A_{1,a} f|^2 + \cdots + |A_{d,a} f|^2 + |A_{-1,a} f|^2 + \cdots + |A_{-d,a} f|^2$$

$$= \sum_{i=1}^{d} (|A_{i,a} f|^2 + |A_{-i,a} f|^2)$$

$$= \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(x) + \frac{\sigma x_i}{|x|^2} f(x) \right|^2 + \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(x) - \frac{\sigma x_i}{|x|^2} f(x) \right|^2$$

$$= 2 \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(x) \right|^2 + 2 \sum_{i=1}^{d} \left| \frac{\sigma x_i}{|x|^2} f(x) \right|^2$$

$$= 2 \left| \nabla f(x) \right|^2 + 2 \frac{\sigma^2}{|x|^4} f(x)^2 \left( \sum_{i=1}^{d} |x_i|^2 \right)$$

$$= 2|\nabla f(x)|^2 + 2\frac{\sigma^2}{|x|^2} |f(x)|^2.$$

Then it is easy to see that

$$|\nabla f(x)| + \frac{\sigma}{|x|}|f(x)| \leq \sqrt{2}\left(|\nabla f(x)|^2 + \frac{\sigma^2}{|x|^2} |f(x)|^2\right)^{1/2} = |\nabla \tilde{\mathcal{H}}_{\sigma} f(x)| \leq \sqrt{2}\left(|\nabla f(x)| + \frac{\sigma}{|x|} |f(x)|\right),$$

which derives that (14) is valid. \hfill \Box

**Theorem 3.12.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain. If $f \in BV_{\tilde{\mathcal{H}}_{\sigma}}(\Omega)$ satisfying the condition (6), then

$$|\nabla \tilde{\mathcal{H}}_{\sigma} f|(\Omega) \approx \int_{-\infty}^{+\infty} P_{\tilde{\mathcal{H}}_{\sigma}}(E_t, \Omega) dt, \quad (15)$$

where $E_t = \{x \in \Omega : f(x) > t\}$ for $t \in \mathbb{R}$. 

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Proof. Firstly, suppose 
\[ \Phi = (\varphi_1, \varphi_2, \ldots, \varphi_{2d}) \in C^1_c(\Omega, \mathbb{R}^{2d}). \]

We can easily prove that for \( i = 1, 2, \ldots, d, \)
\[
- \int_\Omega f(x) \frac{\sigma x_i}{|x|^2} \varphi_i(x) dx = - \int_{E_t} \frac{\sigma x_i}{|x|^2} \varphi_i(x) dx dt,
\]
\[
\int_\Omega f(x) \frac{\sigma x_i}{|x|^2} \varphi_{d+i}(x) dx = \int_{-\infty}^{+\infty} \left( \int_{E_t} \frac{\sigma x_i}{|x|^2} \varphi_{d+i}(x) dx \right) dt,
\]
\[
\int_\Omega f(x) \text{div}(\varphi_1(x), \ldots, \varphi_d(x)) dx = \int_{-\infty}^{\infty} \left( \int_{E_t} \text{div}(\varphi_1(x), \ldots, \varphi_d(x)) dx \right) dt,
\]
and
\[
\int_\Omega f(x) \text{div}(\varphi_{d+1}(x), \ldots, \varphi_{2d}(x)) dx = \int_{-\infty}^{\infty} \left( \int_{E_t} \text{div}(\varphi_{d+1}, \ldots, \varphi_{2d}) dx \right) dt,
\]
where the latter can be seen in the proof of [13, Section 5.5, Theorem 1]. It follows that
\[
\int_\Omega f(x) \text{div}_{\tilde{H}_s} \Phi(x) dx = \int_{-\infty}^{\infty} \left( \int_{E_t} \text{div}_{\tilde{H}_s} \Phi(x) dx \right) dt.
\]

Therefore, we conclude that for all \( \Phi \in \tilde{F}(\Omega), \)
\[
\int_\Omega f(x) \text{div}_{\tilde{H}_s} \Phi(x) dx \leq \int_{-\infty}^{+\infty} P_{\tilde{H}_s}(E_t, \Omega) dt.
\]
Furthermore,
\[
|\nabla_{\tilde{H}_s} f|(\Omega) \leq \int_{-\infty}^{+\infty} P_{\tilde{H}_s}(E_t, \Omega) dt.
\]

Secondly, without loss of generality, we only need to verify that
\[
|\nabla_{\tilde{H}_s} f|(\Omega) \geq \int_{-\infty}^{+\infty} P_{\tilde{H}_s}(E_t, \Omega) dt
\]
holds for \( f \in BV_{\tilde{H}_s}(\Omega) \cap C^\infty(\Omega). \) This proof can refer to the idea of [18, Proposition 4.2]. Let
\[
m(t) = \int_{\{x \in \Omega: f(x) \leq t\}} |\nabla f(x)| dx.
\]
It is obvious that
\[
\int_{-\infty}^{\infty} m'(t) dt \leq \int_\Omega |\nabla f(x)| dx.
\]

Define the following function \( g_h \) as
\[
g_h(s) := \begin{cases} 
0, & \text{if } s \leq t, \\
h(s - t), & \text{if } t \leq s \leq t + 1/h, \\
1, & \text{if } s \geq t + 1/h,
\end{cases}
\]
where \( t \in \mathbb{R}. \) Set the sequence \( v_h(x) := g_h(f(x)). \) At this time, \( v_h \to 1_{E_t} \) in \( L^1(\Omega). \) In fact,
\[
\int_\Omega |v_h(x) - 1_{E_t}| dx = \int_{\{x \in \Omega: t < f(x) \leq t + 1/h\}} g_h(f(x)) dx
\]
\[ \left\{ x \in \Omega : t < f(x) \leq t + 1/h \right\} \rightarrow 0. \]

Since \( \{ x \in \Omega : t < f(x) \leq t + 1/h \} \rightarrow \emptyset \) as \( h \rightarrow \infty \), by Lemma 3.11, we obtain

\[
\int_{\Omega} |\nabla \tilde{H}_\sigma v_h(x)| \, dx
\]

\[
= \int_{\{ x \in \Omega : t < f(x) \leq t + 1/h \}} |\nabla \tilde{H}_\sigma (hf(x) - t)| \, dx + \int_{\{ x \in \Omega : f(x) \geq t + 1/h \}} |\nabla \tilde{H}_\sigma 1| \, dx
\]

\[
\leq \sqrt{2} h \int_{\{ x \in \Omega : t < f(x) \leq t + 1/h \}} |\nabla f(x)| \, dx + \sqrt{2} \sigma \int_{\{ x \in \Omega : f(x) \geq t + 1/h \}} \frac{1}{|x|} \, dx + |\sigma| \sqrt{2} \int_{\{ x \in \Omega : f(x) \geq t \}} \frac{|\sigma|}{|x|} \, dx.
\]

Taking the limit \( h \rightarrow \infty \) and using Theorem 3.4, we obtain

\[
|\nabla \tilde{H}_\sigma 1_{E_1}(\Omega) | \leq \limsup_{h \rightarrow \infty} \int_{\Omega} |\nabla \tilde{H}_\sigma v_h(x)| \, dx
\]

\[
= \sqrt{2} m'(t) + \sqrt{2} \int_{\{ x \in \Omega : f(x) \geq t \}} \frac{|\sigma|}{|x|} \, dx.
\] (16)

Integrating (16) reaches

\[
\int_{-\infty}^{+\infty} P_{\tilde{H}_\sigma}(E_t, \Omega) \, dt \leq \sqrt{2} \int_{-\infty}^{+\infty} m'(t) \, dt + \sqrt{2} \int_{-\infty}^{+\infty} \int_{\{ x \in \Omega : f(x) \geq t \}} \frac{|\sigma|}{|x|} \, dx \, dt
\]

\[
\leq \sqrt{2} \int_{\Omega} (|\nabla f(x)| + \frac{|\sigma|}{|x|} |f(x)|) \, dx
\]

\[
\leq \sqrt{2} \int_{\Omega} |\nabla \tilde{H}_\sigma f(x)| \, dx.
\]

Finally, by approximation and using the lower semicontinuity of the \( \tilde{H}_\sigma \)-perimeter, we conclude that (15) holds true for all \( f \in BV_{\tilde{H}_\sigma}(\Omega) \) satisfying the condition (6).

In addition, we can develop the Sobolev’s inequality and the isoperimetric inequality for \( \tilde{H}_\sigma \)-BV functions.

**Theorem 3.13.**

(i) (Sobolev inequality) Let \( \Omega \subset \mathbb{R}^d \) be an open and bounded domain. For all \( f \in BV_{\tilde{H}_\sigma}(\Omega) \) satisfying the condition (6), then

\[
\| f \|_{L^{d/(d-1)}(\Omega)} \leq |\nabla \tilde{H}_\sigma f|(\Omega).
\] (17)

(ii) (Isoperimetric inequality) Let \( E \) be a bounded set of finite \( \tilde{H}_\sigma \)-perimeter in \( \Omega \). Then

\[
|E|^{-1/d} \leq P_{\tilde{H}_\sigma}(E, \Omega).
\] (18)

(iii) The above two statements are equivalent.

**Proof.** (i) Let \( \{ f_k \}_{k \in \mathbb{N}} \subseteq C^\infty_c(\Omega) \cap BV_{\tilde{H}_\sigma}(\Omega) \) be a sequence such that

\[
\left\{ \begin{array}{l}
f_k \rightarrow f \text{ in } L^1(\Omega), \\
\int_{\Omega} |\nabla \tilde{H}_\sigma f_k(x)| \, dx \rightarrow |\nabla \tilde{H}_\sigma f|(\Omega).
\end{array} \right.
\]
Then by Fatou’s lemma and the classical Gagliardo-Nirenberg-Sobolev inequality (see [13]), we have
\[
\|f\|_{L^{d/(d-1)}(\Omega)} \leq \liminf_{k \to \infty} \|f_k\|_{L^{d/(d-1)}(\Omega)} \lesssim \lim_{k \to \infty} \|\nabla f_k\|_{L^1(\Omega)} \lesssim \lim_{k \to \infty} \|\nabla \tilde{\eta}_s f_k\|_{L^1(\Omega)} = |\nabla \tilde{\eta}_s f|_{(\Omega)},
\]
where we have used the relation between $|\nabla f(x)|$ and $|\nabla \tilde{\eta}_s f(x)|$ in Lemma 3.11.

(ii) We can show that (18) is valid via letting $f = 1_E$ in (17).

(iii) Obviously, (i)$\Rightarrow$(ii) has been proved. In what follows, we prove (ii)$\Rightarrow$(i). Assume that $0 \leq f \in C_c^\infty(\Omega)$. By the coarea formula in Theorem 3.12 and (ii), we have
\[
\int_\Omega |\nabla \tilde{\eta}_s f(x)|dx \approx \int_0^\infty |\nabla \tilde{\eta}_s 1_E_t|dx \geq \int_0^\infty |E_t|^{-1/d}dt,
\]
where $E_t = \{x \in \Omega : f(x) > t\}$. Let
\[
f_t = \min\{t, f\} \quad \text{and} \quad \chi(t) = \left(\int_\Omega f_t^{d/(d-1)}(x)dx\right)^{1-1/d} \quad \forall \ t \in \mathbb{R}.
\]
It is easy to see that
\[
\lim_{t \to \infty} \chi(t) = \left(\int_\Omega |f(x)|^{d/(d-1)}dx\right)^{1-1/d}.
\]
In addition, we can check that $\chi(t)$ is nondecreasing on $(0, \infty)$ and for $h > 0$,
\[
0 \leq \chi(t + h) - \chi(t) \leq \left(\int_\Omega |f_{t+h}(x) - f_t(x)|^{d/(d-1)}dx\right)^{1-1/d} \leq h|E_t|^{-1/d}.
\]
Then $\chi(t)$ is locally a Lipschitz function and $\chi'(t) \leq |E_t|^{-1/d}$, a.e. $t \in (0, \infty)$. Hence,
\[
\left(\int_\Omega |f(x)|^{d/(d-1)}dx\right)^{1-1/d} = \int_0^\infty \chi'(t)dt \leq \int_0^\infty |E_t|^{-1/d}dt \lesssim \int_{\mathbb{R}^d} |\nabla \tilde{\eta}_s f(x)|dx.
\]
For all $f \in BV_{\tilde{\eta}_s}(\Omega)$ satisfying the condition (6), we conclude that (17) is valid by approximation and Theorem 3.4.

\[
\square
\]

4 Subgraphs of $\tilde{\eta}_s$-restricted BV functions

The aim of this section is to show that properties of $\tilde{\eta}_s$-restricted BV functions can be described equivalently in terms of their subgraphs.

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. The $\tilde{\eta}_s$-restricted variation of $f \in L^1(\Omega)$ is defined by
\[
|\nabla^R \tilde{\eta}_s f|_{(\Omega)} = \sup_{\Phi \in \bar{F}_R(\Omega)} \left\{ \int_\Omega f(x)\text{div} \tilde{\eta}_s \Phi(x)dx \right\},
\]
where $\bar{F}_R(\Omega)$ denotes the class of all functions
\[
\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_{2d}) \in C_c^1(\Omega, \mathbb{R}^{2d})
\]
satisfying
\[
\|\Phi\|_\infty = \sup_{x \in \Omega} \left\{ (|\varphi_1(x)|^2 + \ldots + |\varphi_{2d}(x)|^2)^{1/2}\right\} \leq 1.
\]
and
\[
\int_{\Omega} \left( \sum_{k=1}^{d} \sigma \frac{x_k}{|x|^2} (\varphi_k(x) - \varphi_{k+d}(x)) \right) dx = 0. \tag{19}
\]

Define a new type BV space as:
\[
\mathcal{BV}^{R}_{\tilde{H}_\sigma}(\Omega) := \{ f \in L^1(\Omega) : |\nabla^{R}_{\tilde{H}_\sigma} f| < \infty \}.
\]

Similarly, it is easy to see that \(\mathcal{BV}^{R}_{\tilde{H}_\sigma}(\Omega) \subseteq \mathcal{BV}_{\tilde{H}_\sigma}(\Omega)\), which also is a Banach space with the norm
\[
\| f \|_{\mathcal{BV}^{R}_{\tilde{H}_\sigma}(\Omega)} = \| f \|_{L^1(\Omega)} + |\nabla^{R}_{\tilde{H}_\sigma} f|\). \tag{20}
\]

The space \(\mathcal{BV}^{R}_{\tilde{H}_\sigma}(\Omega)\) enjoys similar properties as \(\mathcal{BV}_{\tilde{H}_\sigma}(\Omega)\), for example, the lower semicontinuity, the structure theorem, the approximation via \(C^\infty\)-functions, etc.

For \(u \in \mathcal{BV}^{R}_{\tilde{H}_\sigma}(\Omega)\), the subgraph of \(u\) is defined as the measurable subset of \(\Omega \times \mathbb{R}\) given by
\[
S_u := \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\}.
\]

For the sake of simplicity, we introduce the family \(D := (\tilde{A}_{i,a}, \ldots, \tilde{A}_{d+1,a})\) of linearly independent vector fields in \(\mathbb{R}^{2d+1}\) defined for \((x,t) \in \mathbb{R}^d \times \mathbb{R}\) by
\[
\begin{align*}
\tilde{A}_{i,a}(x,t) &= (A_{i,a}(x), 0) \in \mathbb{R}^{d+1} \equiv \mathbb{R}^d \times \mathbb{R} \quad \text{for} \ i = 1, \ldots, d, \\
\tilde{A}_{d+i,a}(x,t) &= (A_{-i,a}(x), 0) \in \mathbb{R}^{d+1} \equiv \mathbb{R}^d \times \mathbb{R} \quad \text{for} \ i = 1, \ldots, d, \\
\tilde{A}_{2d+1,a}(x,t) &= \frac{\partial}{\partial t}.
\end{align*}
\]

Furthermore, we also need to define the so-called \(\tilde{H}_\sigma\)-restricted perimeter in order to achieve our aim.

**Definition 4.1.** Let \(\tilde{\Omega} \subseteq \mathbb{R}^{d+1}\) be an open and bounded domain. The \(\tilde{H}_\sigma\)-restricted perimeter of \(E \subseteq \tilde{\Omega}\) can be defined as
\[
\tilde{P}_{\tilde{H}_\sigma}(E, \tilde{\Omega}) = \sup \left\{ \int_{E} \sum_{i=1}^{d+1} \tilde{A}_{i,a}(x,t) dxdt : \Phi \in \mathcal{F}_{R}(\tilde{\Omega}, \mathbb{R}^{2d+1}) \right\},
\]
where \(\mathcal{F}_{R}(\tilde{\Omega}, \mathbb{R}^{2d+1})\) denotes the class of all functions
\[
\Phi = (\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_{2d+1}) \in C^1_c(\tilde{\Omega}, \mathbb{R}^{2d+1})
\]
satisfying
\[
\| \Phi \|_{\infty} = \sup_{(x,t) \in \tilde{\Omega}} \left\{ \left( |\tilde{\varphi}_1(x,t)|^2 + \cdots + |\tilde{\varphi}_{2d+1}(x,t)|^2 \right)^{1/2} \right\} \leq 1
\]
and
\[
\int_{\tilde{\Omega}} \left( \sum_{k=1}^{d} \sigma \frac{x_k}{|x|^2} (\tilde{\varphi}_k(x,t) - \tilde{\varphi}_{k+d}(x,t)) \right) dxdt = 0. \tag{20}
\]
If $\tilde{\Omega} \subset \mathbb{R}^{d+1}$ is open and $u$ is a function of bounded variation on $\tilde{\Omega}$ with respect to the family $D$, we write the $\mathbb{R}^{2d+1}$-valued distribution in $\tilde{\Omega}$ as

$$D u := (D_{A_{1,a}} u, \ldots, D_{A_{2d+1,a}} u).$$

The following theorem is the natural generalization of some related results about functions of bounded variation on the Euclidean space and Carnot groups (see [14] or [12]). We denote by $\pi : \mathbb{R}^{2d+1} \to \mathbb{R}^{2d}$ the canonical projection obeying $\pi(x,t) = x$ and $\pi_\#$ denotes the associated push-forward of measures.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain and $u \in L^1(\Omega)$ satisfy the condition (6). Then $u$ belongs to $BV^R_{\mathcal{H}_\sigma}(\Omega)$ if and only if its subgraph $S_u$ has finite $\mathcal{H}_\sigma$-restricted perimeter in $\Omega \times \mathbb{R}$, that is,

$$\mathcal{P}_{\mathcal{H}_\sigma}(S_u, \Omega \times \mathbb{R}) < \infty.$$  

Moreover, writing $D'1_{S_u} := (D_{A_{1,a}} 1_{S_u}, \ldots, D_{A_{2d+1,a}} 1_{S_u})$, then we have

1. $\pi_\# D_{A_{1,a}} 1_{S_u} = D_{A_{1,a}} u$ and $\pi_\# D_{A_{2d+1,a}} 1_{S_u} = D_{A_{2d+1,a}} u$; $i = 1, \ldots, d$;
2. $\pi_\# \frac{\partial}{\partial t} 1_{S_u} = -\mathcal{L}^d$, where $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$.  
3. $\pi_\# |D_{A_{1,a}} 1_{S_u}| = |D_{A_{1,a}} u|$ and $\pi_\# |D_{A_{2d+1,a}} 1_{S_u}| = |D_{A_{2d+1,a}} u|$; $i = 1, \ldots, d$;
4. $\pi_\# |\frac{\partial}{\partial t} 1_{S_u}| = \mathcal{L}^d$.

**Proof.** Suppose first that $\mathcal{P}_{\mathcal{H}_\sigma}(S_u, \Omega \times \mathbb{R}) < \infty$. In this case, the measures $D_{A_{1,a}} 1_{S_u}$ can be extended as linear functionals acting on continuous and bounded functions in $\Omega \times \mathbb{R}$ by means of the Lebesgue theorem. We choose a special sequence in $C_c^\infty(\mathbb{R})$, denoted by $\{g_h\}$, such that

$$g_h(t) = \begin{cases} 1, & |t| \leq h, \\ 0, & |t| \geq h + 1, \end{cases}$$

and

$$\int_\mathbb{R} g_h(t) dt = 2h + 1.$$  

Let $\Phi = (\varphi_1, \ldots, \varphi_{2d}) \in C_c^1(\Omega, \mathbb{R}^{2d})$ with $|\Phi| \leq 1$ and satisfy (19). By the dominated convergence theorem, we have

$$\int_{\Omega \times \mathbb{R}} \Phi(x) \cdot d(D'1_{S_u})(x,t) = \lim_{h \to +\infty} \int_{\Omega \times \mathbb{R}} g_h(t) \Phi(x) \cdot d(D'1_{S_u})(x,t)$$

$$= \lim_{h \to +\infty} \int_{\Omega \times \mathbb{R}} 1_{S_u}(x,t) g_h(t) \text{div}_{\mathcal{H}_\sigma} \Phi(x) dx dt$$

$$= \lim_{h \to +\infty} \int_{\Omega} \left( \int_{-\infty}^{u(x)} g_h(t) dt \right) \text{div}_{\mathcal{H}_\sigma} \Phi(x) dx.$$  

Via the definition of $g_h(t)$, for every $z \in \mathbb{R}$ and every $h \in \mathbb{N}$, we get

$$\int_{-\infty}^{z} g_h(t) dt \leq |z| + h + \frac{1}{2}$$

and

$$\lim_{h \to +\infty} \left( \int_{-\infty}^{z} g_h(t) dt - h - \frac{1}{2} \right) = z,$$  

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where we have used the fact
\[ \int_{\Omega} \text{div} \tilde{H}_\sigma \Phi(x) dx = 0. \]

Consequently, using the dominated convergence theorem again, we can deduce that
\[
\int_{\Omega \times \mathbb{R}} \Phi(x) \cdot d(D'1_{S_u})(x,t) = \int_{\Omega \times \mathbb{R}} \Phi(x) \cdot d(D\tilde{H}_u u) = \int_{\Omega} \Phi(x) \cdot d(D\tilde{H}_u u).
\] (21)

In particular, \( u \in BV_{\tilde{H}_\sigma}^{\mathbb{R}}(\Omega) \) and for any open set \( A \subseteq \Omega \), we have
\[
|D\tilde{H}_u u|(A) \leq |D'1_{S_u}|(A \times \mathbb{R}). \] (22)

Before proving the reverse implication, we firstly consider two facts. For any \( \psi \in C^1_c(\Omega) \) one has
\[
\int_{\Omega \times \mathbb{R}} \psi(x) d(D_{\tilde{\Lambda}_{2d+1,1} 1_{S_u}})(x,t) = \int_{\Omega \times \mathbb{R}} \psi(x) d(\frac{\partial}{\partial t} 1_{S_u})(x,t)
\]
\[= \lim_{h \to +\infty} \int_{\Omega \times \mathbb{R}} g_h(t) \psi(x) d(\frac{\partial}{\partial t} 1_{S_u})(x,t)
\]
\[= - \lim_{h \to +\infty} \int_{\Omega \times \mathbb{R}} 1_{S_u}(x,t) g'_h(t) \psi(x) dx dt \]
\[= - \lim_{h \to +\infty} \int_{\Omega} g'_h(t) \int_{-\infty}^{u(x)} \psi(x) dx
\]
\[= - \int_{\Omega} \psi(x) dx. \] (23)

Moreover, for any open set \( A \subseteq \Omega \),
\[
|A| \leq |D_{\tilde{\Lambda}_{2d+1,1} 1_{S_u}}|(A \times \mathbb{R}). \] (24)

Furthermore, if \( \Phi \in C^1_c(\Omega, \mathbb{R}^{2d}) \) satisfies
\[
\int_{\Omega} \left( \sum_{k=1}^{d} \frac{\partial^2 \varphi_k(x) - \varphi_{k+d}(x)}{|x|^2} \right) dx = 0,
\]
by (21) and (23), we can get
\[
\int_{\Omega \times \mathbb{R}} \Phi(x) \cdot d(D1_{S_u})(x,t) = \int_{\Omega} \Phi(x) \cdot d(D\tilde{H}_u u, -\mathcal{L}^d)(x),
\]
which derives for any open set \( A \subseteq \Omega \),
\[
|(D\tilde{H}_u u, -\mathcal{L}^d)|(A) \leq |D1_{S_u}|(A \times \mathbb{R}). \] (25)
Suppose now that \( u \in BV^R_{H_{\sigma}}(\Omega) \) satisfies the condition (6). Let \( A \subset \Omega \) be open. Similarly to Theorem 3.4, we can choose a sequence of smooth functions \( \{u_k\} \) in \( C_c^\infty(A) \cap BV^R_{H_{\sigma}}(A) \) such that \( u_k \to u \) in \( L^1 \) and

\[
\int_A |\nabla^R_{H_{\sigma}} u_k(x)|dx \to |\nabla^R_{H_{\sigma}} u|(A)
\]
as \( k \to \infty \). In the classical case (cf. [14, Theorem 1]), we observe that for any \( \tilde{\Phi} \in C_c^1(A \times \mathbb{R}) \),

\[
\int_{A \times \mathbb{R}} \tilde{\Phi}(x,t) d(\partial_x1_{S_{u_k}}) = \int_A \tilde{\Phi}(x,u_k(x)) \frac{\partial}{\partial x_i} u_k(x)dx, \ i = 1, \ldots, d.
\]

For convenience, we write

\[
\nabla 1_{S_{u_k}} = \left( \frac{\partial}{\partial x_1} 1_{S_{u_k}}, \ldots, \frac{\partial}{\partial x_d} 1_{S_{u_k}} \right)
\]
and

\[
\nabla' 1_{S_{u_k}} = \left( \frac{\partial}{\partial x_1} 1_{S_{u_k}}, \ldots, \frac{\partial}{\partial x_d} 1_{S_{u_k}} \right).
\]

Then for any \( \tilde{\Phi} \in C_c^1(A \times \mathbb{R}) \) with \( |\tilde{\Phi}| \leq 1 \) and satisfying (20), we have

\[
\int_{A \times \mathbb{R}} \tilde{\Phi}(x,t) \cdot d(\nabla' 1_{S_{u_k}})
\]
\[
= \int_{A \times \mathbb{R}} \tilde{\Phi}(x,t) \cdot d\left( \frac{\partial}{\partial x_1} 1_{S_{u_k}}, \ldots, \frac{\partial}{\partial x_d} 1_{S_{u_k}} \right)
\]
\[
= \int_A \tilde{\Phi}(x,u_k(x)) \cdot \left( \frac{\partial}{\partial x_1} u_k(x), \ldots, \frac{\partial}{\partial x_d} u_k(x) \right)dx + \int_A \tilde{\Phi}(x,u_k(x)) \cdot \left( \frac{x_1}{|x|^2} u_k(x), \ldots, \frac{x_d}{|x|^2} u_k(x), -\frac{x_1}{|x|^2} u_k(x), \ldots, -\frac{x_d}{|x|^2} u_k(x) \right)dx
\]
\[
= \int_A u_k(x) \text{div}_{H_{\sigma}} \tilde{\Phi}(x,u_k(x))dx + \int_A \tilde{\Phi}(x,u_k(x)) d(D_{H_{\sigma}} u_k),
\]
where we have used the fact that \( x \to \int_{-\infty}^{u_k(x)} \Phi(x,t)dt \) is in \( C_c^1(A) \). In a similar way,

\[
\int_{A \times \mathbb{R}} \tilde{\Phi}(x,t) d\left( D_{A_{2d+1,1}} 1_{S_{u}} \right) = \int_{A \times \mathbb{R}} \tilde{\Phi}(x,t) d\left( \frac{\partial}{\partial t} 1_{S_{u_k}} \right)
\]
\[
= -\int_A \left( \int_{-\infty}^{u_k(x)} \frac{\partial}{\partial t} \tilde{\Phi}(x,t)dt \right)dx
\]
\[
= -\int_A \tilde{\Phi}(x,u_k(x))dx.
\]

Formulas (26) and (27) imply that for any \( \tilde{\Phi} \in C_c^1(A \times \mathbb{R}, \mathbb{R}^{2d+1}) \) satisfying (20),

\[
\int_{A \times \mathbb{R}} \tilde{\Phi}(x,t) \cdot d(\nabla 1_{S_{u_k}}) = \int_A \tilde{\Phi}(x,u_k(x)) \cdot d(D_{H_{\sigma}} u_k, -L^d)(x).
\]
Since \(1_{S_{u_k}} \to 1_{S_u}\) in \(L^1(A \times \mathbb{R})\), we get
\[
|D1_{S_u}|(A \times \mathbb{R}) = |\nabla 1_{S_u}|(A \times \mathbb{R}) \\
\leq \liminf_{k \to +\infty} |\nabla 1_{S_{u_k}}|(A \times \mathbb{R}) \\
\leq \liminf_{k \to +\infty} |(D\tilde{\mu}_\sigma u_k, -\mathcal{L}^d)|(A) \\
= |(D\tilde{\mu}_\sigma u, -\mathcal{L}^d)|(A) < +\infty,
\]
which indicates that \(\tilde{P}_{\tilde{H}_\sigma}(S_u, \Omega \times \mathbb{R}) < \infty\). Similarly, using the lower semicontinuity, we obtain
\[
|D'1_{S_u}|(A \times \mathbb{R}) \leq |D\tilde{\mu}_\sigma u|(A), \\
|D_{A_{2d+1,a}}1_{S_u}|(A \times \mathbb{R}) \leq |A| < +\infty.
\]

Eventually, statements (i) and (ii) follow from (21) and (23), while statements (iii) and (vi) are consequences of formulas (22), (24), (25), (28) and (29).

Let \(u = (u_1, \ldots, u_m) \in \mathcal{B} \mathcal{H}^R_{\tilde{H}_\sigma}(\Omega, \mathbb{R}^m)\), that is, \(u_i \in \mathcal{B} \mathcal{H}^R_{\tilde{H}_\sigma}(\Omega)\) for \(i = 1, \ldots, m\). By the Lebesgue decomposition theorem for measures, we can decompose its distributional derivatives as
\[
D\tilde{\mu}_\sigma u = D^A_{\tilde{H}_\sigma} u + D^S_{\tilde{H}_\sigma} u,
\]
where \(D^A_{\tilde{H}_\sigma} u\) is absolutely continuous with respect to the Lebesgue measure \(\mathcal{L}^d\) and \(D^S_{\tilde{H}_\sigma} u\) is singular with respect to \(\mathcal{L}^d\). Furthermore, write \(D^A_{\tilde{H}_\sigma} u = Mu\mathcal{L}^d\), where \(Mu \in L_{loc}^1(\Omega, \mathbb{R}^{2d})\) is the approximate differential of \(u\).

In this case, the Radon-Nikodym derivative \(\frac{D^S_{\tilde{H}_\sigma} u}{|D^S_{\tilde{H}_\sigma} u|}\) of \(D^S_{\tilde{H}_\sigma} u\) with respect to its total variation \(|D^S_{\tilde{H}_\sigma} u|\) is a \(|D^S_{\tilde{H}_\sigma} u|\)-measurable map from \(\Omega\) to \(\mathbb{R}^{2d \times m}\). We can define the normal to \(x\) as
\[
\nu_S(x) := \frac{D^S_{\tilde{H}_\sigma} u}{|D^S_{\tilde{H}_\sigma} u|} \in \mathbb{R}^{2d}.
\]

The normal \(\nu_S(x) = ((\nu_S(x))_1, \ldots, (\nu_S(x))_{2d})\) is defined up to sign and it can be canonically identified with a vector at \(x\) by
\[
(\nu_S(x))_i = (\nu_S(x))_1 D_{A_{1,a}}(x) + \ldots + (\nu_S(x))_{2d} D_{A_{d,a}}(x).
\]

We can also consider the polar decomposition \(D\tilde{\mu}_\sigma u = \sigma_u D\tilde{\mu}_\sigma u\), where \(\sigma_u : \Omega \to S^{2d-1}\) is a \(|D\tilde{\mu}_\sigma u|\)-measurable function. If \(u = 1_S\) is the characteristic function of a set \(S \subset \Omega \times \mathbb{R}\) of finite \(\tilde{H}_\sigma\)-restricted perimeter in \(\Omega \times \mathbb{R}\), we write \(D1_S = \nu_S \circ D1_S\) for some Borel function \(\nu_S = ((\nu_S)_1, \ldots, (\nu_S)_{2d+1})\) called inner normal to \(S\).

Via Theorem 4.2, we can deduce the following result.

**Theorem 4.3.** Let \(\Omega \subset \mathbb{R}^d\) be a bounded open domain. If \(u \in \mathcal{B} \mathcal{H}^R_{\tilde{H}_\sigma}(\Omega)\) satisfies the condition (6), define
\[
E := \left\{(x, t) \in \Omega \times \mathbb{R} : (\nu_{S_u})_{2d+1}(x, t) = 0\right\}
\]
and
\[
T := \left\{(x, t) \in \Omega \times \mathbb{R} : (\nu_{S_u})_{2d+1}(x, t) \neq 0\right\}.
\]
Then, the following identities are valid:

\[ v_{S_u}(x,t) = (\sigma_u(x),0) \text{ for } \tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R}) - \text{a.e. } (x,t) \in E; \]  
\[ v_{S_u}(x,t) = \frac{(Mu(x), -1)}{\sqrt{1 + |Mu(x)|^2}} \text{ for } \tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R}) - \text{a.e. } (x,t) \in T; \]

\[ \pi_\#(D1_{S_u}E) = (D^S_{\tilde{\mathcal{H}}_\sigma} u, 0); \]
\[ \pi_\#(D1_{S_u}T) = (D^S_{\tilde{\mathcal{H}}_\sigma} u, -L^d). \]

**Proof.** From the above Theorem 4.2 and similarly to [4, Theorem 2.28], we can decompose the perimeter \( \tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R}) \) into \( |(D_{\tilde{\mathcal{H}}_\sigma} u, -L^d)| \): for every \( x \in \Omega \), there exists a probability measure \( \mu_x \) on \( \mathbb{R} \) such that for every Borel function \( g \in L^1(\Omega \times \mathbb{R}, \tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R})) \),

\[
\int_{\Omega \times \mathbb{R}} g(x,t)d\tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R})(x,t) = \int_{\Omega} \left( \int_{\mathbb{R}} g(x,t)d\mu_x(t) \right) d|\tilde{P}_{\tilde{\mathcal{H}}_\sigma} u, -L^d|(x).
\]

Therefore, for any Borel function \( \Phi : \Omega \to \mathbb{R} \), we can get

\[
\int_{\Omega} \Phi(x)d(D_{\tilde{\mathcal{H}}_\sigma} u, -L^d)(x) = \int_{\Omega} \Phi(x)d\pi_\#(v_{S_u} \tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R}))(x)
\]

\[ = \int_{\Omega \times \mathbb{R}} \Phi(x) v_{S_u}(x,t)d(\tilde{P}_{\tilde{\mathcal{H}}_\sigma}(S_u, \Omega \times \mathbb{R}))(x,t)
\]

\[ = \int_{\Omega} \Phi(x) \left( \int_{\mathbb{R}} v_{S_u}(x,t)d\mu_x(t) \right) d|D_{\tilde{\mathcal{H}}_\sigma} u, -L^d|(x). \]

Since \( D^A_{\tilde{\mathcal{H}}_\sigma} u \) and \( D^S_{\tilde{\mathcal{H}}_\sigma} u \) are mutually singular, we have

\[ |(D_{\tilde{\mathcal{H}}_\sigma} u, -L^d)| = |(D^A_{\tilde{\mathcal{H}}_\sigma} u, -L^d)| + |(D^S_{\tilde{\mathcal{H}}_\sigma} u, 0)| = \sqrt{1 + |Mu|^2}L^d + |D^S_{\tilde{\mathcal{H}}_\sigma} u|,
\]

and (34) gives

\[
\int_{\Omega} \Phi(x)d((Mu,-1)L^d + (\sigma_u,0)|D^S_{\tilde{\mathcal{H}}_\sigma} u|)
\]

\[ = \int_{\Omega} \Phi(x) \left( \int_{\mathbb{R}} v_{S_u}(x,t)d\mu_x(t) \right) d\left( \sqrt{1 + |Mu|^2}L^d + |D^S_{\tilde{\mathcal{H}}_\sigma} u| \right)(x). \]

Let \( I \) denote the subset of \( \Omega \) such that its Lebesgue measure \( |I| = 0 \) and \( |D^S_{\tilde{\mathcal{H}}_\sigma} u|\Omega \setminus I = 0 \). Considering Borel test functions \( \varphi \) such that \( \varphi = 0 \) in \( \Omega \setminus I \), we deduce that for \( |D^S_{\tilde{\mathcal{H}}_\sigma} u|\text{-a.e. } x \in I \) one has

\[ (\sigma_u(x),0) = \int_{\mathbb{R}} v_{S_u}(x,t)d\mu_x(t). \]

Taking the scalar product with \( (\sigma_u(x),0) \) on both sides, we can get

\[ \left( (\sigma_u(x),0), \int_{\mathbb{R}} v_{S_u}(x,t)d\mu_x(t) \right) = 1. \]

Since \( \mu_x(\mathbb{R}) = 1 \) and (for \( |(D_{\tilde{\mathcal{H}}_\sigma} u, -L^d)|\text{-a.e. } x \in \Omega \) \( |v_{S_u}(x,t)| = 1 \) for \( \mu_x\text{-a.e. } t \), we conclude that

\[ v_{S_u}(x,t) = (\sigma_u(x),0) \text{ for } |D^S_{\tilde{\mathcal{H}}_\sigma} u|\text{-a.e. } x \in I \text{ and } \mu_x\text{-a.e. } t \in \mathbb{R}, \]
i.e.,
\[ v_{S_u}(x, t) = (\sigma_u(x), 0) \] for \( \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \)-a.e. \((x, t) \in I \times \mathbb{R}, \) \tag{36}

which implies that \( \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \)-a.e. \((x, t) \in I \times \mathbb{R} \) belongs to \( E \) and that \( \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \)-a.e. \((x, t) \in T \) belongs to \((\Omega \setminus I) \times \mathbb{R}.\)

Using (35) again and letting \( \Phi = 0 \) on \( I, \) we obtain
\[
\int_{\Omega} \Phi(x) \frac{(Mu(x), -1)}{\sqrt{1 + |Mu(x)|^2}} \sqrt{1 + |Mu(x)|^2} dx = \int_{\Omega} \Phi(x) \left( \int_{\mathbb{R}} v_{S_u}(x, t) d\mu_x(t) \right) \sqrt{1 + |Mu(x)|^2} dx.
\]

Then, for a.e. \( x \in \Omega \setminus I, \) we have
\[
\int_{\mathbb{R}} v_{S_u}(x, t) d\mu_x(t) = \frac{(Mu(x), -1)}{\sqrt{1 + |Mu(x)|^2}}
\]

Consequently, for a.e. \( x \in \Omega \setminus I \) and \( \mu_x \text{-a.e.} \ t \in \mathbb{R}, \) we can deduce that
\[
v_{S_u}(x, t) = \frac{(Mu(x), -1)}{\sqrt{1 + |Mu(x)|^2}}
\]

or equivalently, for \( \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \)-a.e. \((x, t) \in (\Omega \setminus I) \times \mathbb{R}, \)
\[
v_{S_u}(x, t) = \frac{(Mu(x), -1)}{\sqrt{1 + |Mu(x)|^2}}
\]

Similarly, it implies that \( \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \)-a.e. \((x, t) \in (\Omega \setminus I) \times \mathbb{R} \) belongs to \( T \) and \( \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \)-a.e. \((x, t) \in E \) belongs to \( I \times \mathbb{R}.\)

Since \( E \) and \( T \) are disjoint, the formulas (30) and (31) can be obtained. Now (32) can be easily deduced due to
\[
\pi_\#(D1_{S_u} \cap E) = \pi_\#(v_{S_u} \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \cap (I \times \mathbb{R}))
\]
\[
= (\sigma_u(x), 0)((D_{\bar{H}_\sigma} u, -L^d) \cap I)
\]
\[
= (D^{S}_{\bar{H}_\sigma} u, 0).
\]

The last formula (33) can be obtained by the formula (36) and similarly,
\[
\pi_\#(D1_{S_u} \cap T) = \pi_\#(v_{S_u} \bar{P}_{\bar{H}_\sigma}(S_u, \Omega \times \mathbb{R}) \cap ((\Omega \setminus I) \times \mathbb{R}))
\]
\[
= \sqrt{1 + |Mu(x)|^2} \left((D_{\bar{H}_\sigma} u, -L^d) \cap (\Omega \setminus I)\right)
\]
\[
= (Mu, -1)L^d.
\]

This completes the proof of this theorem. \(\square\)
5 Rank-one theorem for $\tilde{H}_\sigma$-restricted BV functions

In this section, we prove the rank-one theorem for $\tilde{H}_\sigma$-restricted BV functions in Euclidean spaces by using Theorem 4.2 in Section 4 and Lemma 5.3 below which is a key tool.

Let $\mathcal{H}^{d-1}$ be the standard $(d-1)$-dimensional Hausdorff measure. A set $E \subset \mathbb{R}^d$ is rectifiable if $\mathcal{H}^{d-1}(E) < \infty$ and there exists a (finite or countable) family of $C^1$ hypersurfaces in $\mathbb{R}^d$, denoted by $\{\Sigma_i\}_{i \in \mathbb{N}}$, such that

$$\mathcal{H}^{d-1}(E \setminus \bigcup_{i \in \mathbb{N}} \Sigma_i) = 0.$$  

We define the normal $v_E$ to $E$ as

$$v_E(x) := v_{\Sigma_i}(x) \quad \text{if} \ x \in E \cap \Sigma_i \setminus \bigcup_{j < i} \Sigma_j.$$  

Note that the normal $v_E$ is well-defined (up to sign) $\mathcal{H}^{d-1}$-a.e. on $E$, since the set of points where two $C^1$ hypersurfaces intersect transversally is $\mathcal{H}^{d-1}$-negligible.

Definition 5.1. Let $E$ be of finite $\tilde{P}_{\tilde{H}_\sigma}$-perimeter. The $\tilde{H}_\sigma$-reduced boundary of $E$, denoted by $\partial_{\tilde{H}_\sigma}E$, consists of all points $x \in \mathbb{R}^{d+1}$ for which the following statements hold:

(i) $|D1_E|(B(x, r)) > 0$ for all $r > 0$,  

(ii) if

$$n_r(x, E) := -\frac{D1_E(B(x, r))}{|D1_E|(B(x, r))},$$

then the limit $n(x, E) := \lim_{r \rightarrow 0} n_r(x, E)$ exists with $|n(x, E)| = 1$.

Remark 5.2. Similarly to [23, Section 5.5], we deduce that $|D1_E|(\mathbb{R}^{d+1} - \partial_{\tilde{H}_\sigma}E) = 0$ is true. Consequently, $\partial_{\tilde{H}_\sigma}E$ is $|D1_E|$-measurable and $|D1_E| = |D1_E|\cdot \partial_{\tilde{H}_\sigma}E$. Then by the rectifiability theorem for measures [14, Section 2.1.4, Theorem 2], the rectifiability of $\partial_{\tilde{H}_\sigma}E$ can be obtained. Correspondingly, it follows that $|D1_E| = \mathcal{H}^d \cdot \partial_{\tilde{H}_\sigma}E$ if $E \subset \mathbb{R}^{d+1}$.

Via the lemma in [17, page 3256] or [12, Lemma 3.2], the following result can be derived from the coarea formula in Theorem 3.12.

Lemma 5.3. Let $\Sigma_1, \Sigma_2$ be $C^1$ hypersurfaces in $\mathbb{R}^{d+1}$ with unit normals $v_{\Sigma_1}$ and $v_{\Sigma_2}$. Then the set

$$T := \left\{ p \in \Sigma_1 : \exists q \in \Sigma_2 \cap \pi^{-1}(\pi(p)) \text{ with } (v_{\Sigma_1}(p))_{2d+1} = (v_{\Sigma_2}(q))_{2d+1} = 0 \text{ and } v_{\Sigma_1}(p) \neq \pm v_{\Sigma_2}(q) \right\}$$

is $\mathcal{H}^d$-negligible.

Theorem 5.4. (Rank-one theorem) Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. Assume $u \in BV^R(\Omega, \mathbb{R}^m)$ is a function with bounded $\tilde{H}_\sigma$-variation and satisfies the condition (6). Let $D^S_{\tilde{H}_\sigma}u$ be the singular part of $D_{\tilde{H}_\sigma}u$ with respect to the Lebesgue measure $\mathcal{L}^d$. Then $D^S_{\tilde{H}_\sigma}u$ is a rank-one measure, i.e., the (matrix-valued) function $\frac{D^S_{\tilde{H}_\sigma}u}{|D^S_{\tilde{H}_\sigma}u|}(x)$ has rank one for $D^S_{\tilde{H}_\sigma}u$-a.e. $x \in \Omega$. 

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Proof. We adopt the method of main results in [17] or [12] to give the proof. Let \( u = (u_1, \ldots, u_m) \in BV^R(\Omega, \mathbb{R}^m) \). For any \( i = 1, \ldots, m \), denote by \( D^S_{\mathcal{H}_s} u_i = \sigma_i |D^S_{\mathcal{H}_s} u_i| \) for a \( |D^S_{\mathcal{H}_s} u_i| \)-measurable map \( \sigma_i : \Omega \to S^{2d-1} \). Note that the equality \( \sigma_i = \sigma_{u_i} \) holds \( |D^S_{\mathcal{H}_s} u_i| \)-almost everywhere using the notation in Section 4. Let
\[
S_i := \left\{ (x, t) \in \Omega \times \mathbb{R} : t < u_i(x) \right\}
\]
be the subgraph of \( u_i \). By Theorem 4.2, \( S_i \) has finite \( \mathcal{H}_s \)-perimeter in \( \Omega \times \mathbb{R} \). For convenience, we define the \( \mathcal{H}_s \)-reduced boundary of \( S_i \) as \( \partial_{\mathcal{H}_s} S_i \) and write \( v_i = v_{E_i} \) for the measure-theoretic inner normal to \( S_i \). By Theorem 4.3 and Remark 5.2, we have
\[
|D^S_{\mathcal{H}_s} u_i| = \pi_\#(\mathcal{H}^d \setminus E_i),
\]
where \( E_i := \{ p \in \partial_{\mathcal{H}_s} S_i : (v_i(p))_{2d+1} = 0 \} \) and \( \pi_\# \) denotes push-forward of measures defined in Section 4. The set \( E_i \) is rectifiable and we can assume that it is contained in the union \( \bigcup_{h \in \mathbb{N}} \Sigma^i_h \) of \( C^1 \) hypersurfaces \( \Sigma^i_h \) in \( \mathbb{R}^{d+1} \).

By Theorem 4.3, Remark 5.2 and Lemma 5.3, we apply the well-known properties of rectifiable sets to conclude that the following properties hold for \( \mathcal{H}^d \)-a.e. \( p \in E_1 \cup \ldots \cup E_m \):
\[
v_{\partial_{\mathcal{H}_s} S_i}(p) = (\sigma_i(\pi(p)), 0);
\]
\[
\text{if } p \in \Sigma^i_h, \text{ then } v_i(p) = \pm v_{\Sigma^i_h}(p);
\]
\[
\text{if } p \in \Sigma^i_h \text{ and } q \in E_j \cap \Sigma^j_k \cap \pi^{-1}(\pi(p)), \text{ then } v_{\Sigma^k_h}(p) = \pm v_{\Sigma^j_k}(q).
\]

Via modifying \( E_i \) on an \( \mathcal{H}^d \)-negligible set and \( \sigma_i \) on a \( |D^S_{\mathcal{H}_s} u_i| \)-negligible set, we can assume that the properties (37)-(39) hold everywhere on \( E_i \) and \( \sigma_i = 0 \) on \( \Omega \setminus \pi(E_i) \).

Since \( D^S_{\mathcal{H}_s} u = (\sigma_1 |D^S_{\mathcal{H}_s} u_1|, \ldots, \sigma_m |D^S_{\mathcal{H}_s} u_m|) \) and \( |D^S_{\mathcal{H}_s} u| \) is concentrated on the union \( \pi(E_1) \cup \ldots \cup \pi(E_m) \), so we just need to prove that the matrix-valued function \( (\sigma_1, \ldots, \sigma_m) \) has rank one on the set \( \pi(E_1) \cup \ldots \cup \pi(E_m) \). The proof of the following fact
\[
i, j \in \{1, \ldots, m\}, \ i \neq j, \ x \in \pi(E_i) \implies \sigma_j(x) \in \{0, \sigma_i(x), -\sigma_i(x)\},
\]
derives the desired result.

If \( i, j, x \) are given as above and \( x \not\in \pi(E_j) \), then \( \sigma_j(x) = 0 \). Otherwise, \( x \in \pi(E_i) \cap \pi(E_j) \), i.e., there exist \( p \in E_i \) and \( h \in \mathbb{N} \) such that \( \pi(p) = x \) and \( \sigma_i(x) = \pm v_{\Sigma^i_h}(p) \). Also, there exist \( q \in E_j \) and \( k \in \mathbb{N} \) such that \( \pi(q) = x \) and \( \sigma_j(x) = \pm v_{\Sigma^j_k}(p) \). From (39), we conclude that \( \sigma_j(x) = \pm \sigma_i(x) \). This completes the proof of Theorem 5.4.

\[\Box\]

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