TATE CYCLES ON SOME QUATERNIONIC SHIMURA VARIETIES MOD $p$

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Abstract. Let $F$ be a totally real field in which $p$ is inert. We continue the study of (generalized) Goren-Oort strata on quaternionic Shimura varieties over finite extensions of $F$. We prove that, when the dimension of the quaternionic Shimura variety is even, the Tate conjecture for the special fiber of the quaternionic Shimura variety holds for the cuspidal $\pi$-isotypical component, as long as the two unramified Satake-parameters at $p$ are not differed by a root of unity.

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1. Introduction

This paper is the third in a series, in which we study the Goren-Oort stratification of quaternionic Shimura varieties and its applications. In the first paper [TX13+a], we gave a global description of each stratum, in terms of a $\mathbb{P}^1$-product bundle over another quaternionic Shimura variety. In the paper [TX13+b], we apply this result to prove the classicality of overconvergent Hilbert modular forms of small slopes. The aim of this paper is to investigate the Goren-Oort strata by viewing them as special cycles on the special fiber of quaternionic Shimura varieties. This eventually leads to a proof of Tate conjecture of the special fiber of the quaternionic Shimura varieties under certain hypotheses. We start by explaining the underlying philosophy using some examples.

1.1. Hilbert modular surface. Let $F$ be a real quadratic field and $p > 2$ be a prime number that is inert in $F/\mathbb{Q}$. Let $\mathbb{A}_F^\infty$ be the ring of finite adèles of $F$, and $K \subset \text{GL}_2(\mathbb{A}_F^\infty)$ be an open compact subgroup hyperspecial at $p$. We consider the Hilbert modular variety $X$ over $\mathbb{Z}(p)$, with $X$ as its special fiber. The main result of Brylinski and Labesse [BL84] says that, up to Frobenius semisimplification, the cuspidal part of the étale cohomology of $X$ is given as follows

$$H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_{\ell})_{\text{cusp}} \cong \bigoplus_{\pi} (\pi^\infty)^K \otimes \rho_{\pi} \otimes (\rho_{\pi} \otimes 2)_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^2)},$$

where the direct sum is taken over all cuspidal automorphic representations whose archimedean components are discrete series of parallel weight 2 and whose $p$-component is unramified, and $\rho_{\pi}$.
is the Galois representation associated to \( \pi \) (in particular, it is unramified at \( p \); so the tensorial induction is just simply a self tensor product.)

We fix an automorphic cuspidal representation \( \pi \) of \( \text{GL}_2(\mathbb{A}_F) \) as above. We put \( H^2_{\text{et}}(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)_{\pi} := (\pi^\infty)^K \otimes_{\pi} \rho_{\pi} \circ (\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)^2)^{-1} \). Let \( \text{Frob}_{p^2} \in \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \) denote the geometric Frobenius element. To simplify our discussion, we assume that

- the two eigenvalues \( \alpha \) and \( \beta \) of \( \rho_{\pi}(\text{Frob}_{p^2}) \) are distinct,
- \( (\pi^\infty)^K \) is one-dimensional, and
- the central character of \( \pi \) is trivial.

Here, the first condition is an essential hypothesis; whereas the last two conditions may be removed. The action of \( \text{Frob}_{p^2} \) on \( H^2_{\text{et}}(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)_{\pi} \) has (generalized) eigenvalues \( \alpha^2, \beta^2 \), and \( \alpha \beta = p^2 \) with multiplicity two. In other words, \( H^2_{\text{et}}(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell(1))_{\pi} \) has a two-dimensional subspace on which the Frobenius acts trivially (or more rigorously speaking, unipotently). According to the prediction of the famous Tate Conjecture, this subspace should be generated by cycle classes of the two archimedean places of \( F \), of the same level \( K \). From this, we get a natural homomorphism

\[
\bigoplus_{i=1}^2 H^0_{\text{et}}(\text{Sh}_{\infty_1, \infty_2}, \mathbb{F}_p, \overline{\mathbb{Q}}_\ell) \xrightarrow{\text{Gysin}} \bigoplus_{i=1}^2 H^0_{\text{et}}(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\text{Gysin}} H^2_{\text{et}}(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell(1))_{\pi}^{\text{Frob}_{p^2} = 1}.
\]

All homomorphisms are equivariant for the prime-to-\( p \) Hecke actions. By Jacquet-Langlands correspondence, \( (\pi^\infty)^K \) appears on each term of the left hand side of (1.1.1) with multiplicity one. By taking \( \pi \)-isotypical parts (or more precisely the \( (\pi^\infty)^K \)-isotypical parts) of (1.1.1), one gets

\[
\bigoplus_{i=1}^2 \overline{\mathbb{Q}}_\ell \xrightarrow{\text{Gysin}} H^2_{\text{et}}(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell(1))_{\pi}^{\text{Frob}_{p^2} = 1}.
\]

Our main result of this paper says that this is an isomorphism. To show this, we consider the natural restriction map:

\[
H^2(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)(1) \xrightarrow{\text{res}} \bigoplus_{i=1}^2 H^2(X_{\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)(1) \cong \bigoplus_{i=1}^2 H^0(\text{Sh}_{\infty_1, \infty_2}, \mathbb{F}_p, \overline{\mathbb{Q}}_\ell).
\]

Its composition with (1.1.1) gives an endomorphism of \( \bigoplus_{i=1}^2 H^0(\text{Sh}_{\infty_1, \infty_2}, \mathbb{F}_p, \overline{\mathbb{Q}}_\ell) \). The key point is to show that, the projection to \( \pi \)-isotypical component of this endomorphism is given by

\[
\begin{pmatrix}
-2p & \alpha + \beta \\
\alpha + \beta & -2p
\end{pmatrix},
\]

whose determinant is equal to \(-\alpha \beta\). Here the entries \( \alpha + \beta \) come from the fact that the morphisms mapping the intersection \( X_1 \cap X_2 \) to \( \text{Sh}_{\infty_1, \infty_2} \) using two \( \mathbb{P}^1 \) parametrizations exactly give the Hecke correspondence \( T_p \) of \( \text{Sh}_{\infty_1, \infty_2} \), and hence we see the evaluation of \( T_p \) at \( \pi \), which is \( \alpha + \beta \). This matrix should be viewed as the \( \pi \)-projection of the intersection matrix of the GO-strata \( X_1 \) and \( X_2 \). From this, we see that when \( \alpha \neq \beta \), these GO-strata gives rise to all Tate cycles in the \( \pi \)-isotypical component of the special fiber of the Hilbert modular surface.
In contrast, if $\alpha = \beta$, $H^2_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(1))_\pi$ is expected to have four dimensional Tate classes. However, the image of the classes of Goren-Oort strata in $H^2_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(1))_\pi$ probably only contributes to only a one-dimensional subspace (see Example 4.6 for the discussion).

1.2. Hilbert modular four-folds. As one tries to generalize the result above to higher dimensional case, one finds quickly that the Goren-Oort stratification does not provide enough interesting cycles. Take a Hilbert modular four-fold $X$ as an example, where the totally real field $F$ defining it has degree 4 and $p$ is inert in $F/\mathbb{Q}$, and the level structure $K \subseteq \text{GL}_2(\mathbb{A}^\infty_F)$ is hyperspecial at $p$. Hence $X$ has an integral model over $\mathbb{Z}_p$ and let $X$ denote its special fiber. Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ with trivial central character whose archimedean components are discrete series of parallel weight 2 and whose $p$-adic component is unramified. We assume again that $(\pi^\infty)^K$ is one-dimensional. Then the $\pi$-isotypical component $H^4_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(2))_\pi$ is isomorphic to $\mu_4^{\otimes 4}$ up to Frobenius semisimplification. Let $\alpha$ and $\beta$ be the (generalized) eigenvalues of $\rho_\pi(\text{Frob}_p)$ so that $\alpha \beta = p^4$. We further assume that $\alpha/\beta \neq \pm 1$. The same computation as before shows that the multiplicity of Frobenius eigenvalue 1 in $H^4_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(2))_\pi$ is $\left(\frac{2}{3}\right) = 6$. As a result, we should expect 6 collections of certain (2-dimensional) strata of $X$ to contribute to this 6-dimensional subspace.

We recall from [TX13+3] the description of the Goren-Oort strata. There are indeed 6 collections of two-dimensional Goren-Oort strata $X_{ij}$ for $\{i, j\} \subseteq \{0, \ldots, 3\}$. Unfortunately, only two of them $X_{02}$ and $X_{13}$ contribute to the correct Tate classes; they are $\mathbb{P}^1$-bundles parametrized by the discrete Shimura varieties for the quaternion algebra $B_{\infty_0, \infty_3}$ over $F$ (which ramifies exactly at all archimedean places). Other strata only contribute the spectrum of one-dimensional representations, since they do not have interesting $H^0$’s.

To get enough strata that contribute to the Tate cycles of $H^4_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(2))_\pi$, we need to look at all codimension one strata $X_0, \ldots, X_3$ first. Each $X_i$ is a $\mathbb{P}^1$-bundle over the Shimura variety for the quaternion algebra $B_{\infty_i, \infty_{i-1}}$. If we consider the Goren-Oort stratification of the Shimura variety for $B_{\infty_0, \infty_3}$ and take the corresponding $\mathbb{P}^1$-bundle, we will get two 2-dimensional subvarieties $X_{11}$ and $X_{12}$ of $X_i$. In fact one of them is equal to a Goren-Oort stratum (either $X_{02}$ or $X_{13}$) and the other one is a completely new collection of subvarieties of $X$, which is a family of $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ parametrized by the discrete Shimura variety for $B_{\infty_0, \infty_3}$. We call them generalized Goren-Oort cycles. In fact each irreducible component is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(-p))$ (see Example 3.9). One can characterize these new subvarieties by looking at the $p^2$-torsion of the universal abelian variety. To sum up, we obtain 4 new two-dimensional subvarieties. Our main theorem says that they, together with the two Goren-Oort strata, span the 6-dimensional Tate classes in $H^4_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(1))_\pi$.

More generally, we prove the following.

**Theorem 1.3** (special case of Theorem 4.4). Let $F$ be a totally real field of even degree $g = [F : \mathbb{Q}]$, and let $p$ be a rational prime inert in $F$. Let $X$ denote the special fiber over $\mathbb{F}_{p^g}$ of the Hilbert modular variety of some level $K \subseteq \text{GL}_2(\mathbb{A}^\infty_F)$ hyperspecial at $p$. Fix an automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$ whose archimedean components are discrete series of parallel weight 2. Suppose that $\pi_p$ is an unramified principal series whose two Satake parameters do not differ by a root of unity.

Then Tate Conjecture holds for the $\pi$-isotypical component $H^3_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(\frac{g}{2}))_\pi$; more precisely, the $\left(\frac{g}{2}\right)$ dimensional subspace of $H^3_{et}(X_{P_p}, \overline{\mathbb{Q}_\ell}(\frac{g}{2}))_\pi$ invariant under the action of $\text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^g})$ is generated by cycle classes of the generalized Goren-Oort cycles of $X$.

We once again point out that the condition on Satake parameters is essential. The contribution from the Goren-Oort strata can be degenerate if the two parameters are the same. See Remark 4.5(2) and Example 4.6.
1.4. Framework for Generalized Goren-Oort cycles. The explicit verification of Tate Conjecture is only a degenerate case of the following study of generalized Goren-Oort cycles. We still start from the Galois representation side. Let $F$ be a totally real field of degree $g$ over $\mathbb{Q}$ and let $p$ be a prime inert in $F/\mathbb{Q}$. We fix a natural number $r \leq g/2$. We consider a regular multiweight $(k, w)$, that is a collection of integers, where $k = (k_1, \ldots, k_g)$ with $k_i \geq 2$ and $k_i \equiv w \mod 2$.

As before, we consider the special fiber $\bar{X}$ over $\mathbb{F}_p$ of the Hilbert modular variety, taking the limit over all tame level structure, but fixing the open compact subgroup at $p$ to be hyperspecial. There is an automorphic local system $L^{(k, w)}$ over $X$. We fix a cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A}_F)$ which are discrete series of weight $(k, w)$ at archimedean places. We focus on the $\pi$-isotypical component of $H^g_{et}(X_{\overline{\mathbb{F}}_p}, L^{(k, w)})[\pi]$, which is (up to Frobenius semisimplification) isomorphic to $\pi^{\otimes g}$ tensored with $\rho_\pi|_{\text{Gal}(\mathbb{F}_p/\mathbb{F}_p)}$, where $\rho_\pi$ is the Galois representation associated to $\pi$. Let $\alpha$ and $\beta$ denote the two eigenvalues of $\rho_\pi(Frob_{\overline{\rho}})$. We assume that $\alpha/\beta$ is not a root of unity.

Instead of looking at the Frobenius eigenvalue $(\alpha/\beta)^{g/2}$, which gives rise to Tate cycles, we consider the Frobenius eigenvalue $\alpha^{g-r} \beta^r$ for all $r \leq \frac{g}{2}$. The associated generalized eigenspace has dimension $(\binom{g}{r})$. Our main result of this paper is the following.

**Theorem 1.5 (Theorem 4.4).** Suppose that $\alpha/\beta$ is not a root of unity. There exist $(\binom{g}{r})$ collection of subvarieties $X_1, \ldots, X_{\binom{g}{r}}$ of $X$ such that

- each $X_i$ is an $r$-times iterated ($\mathbb{P}^1$)-bundle over the special fiber of a Shimura variety\footnote{In fact, we need a slightly funny choice of Deligne homomorphism for these quaternionic Shimura varieties; we refer to the main context of the paper for details.} associated to a quaternion algebra over $F$ which ramifies exactly at $2r$ archimedean places;
- The direct sum of the Gysin maps

$$\bigoplus_{i=1}^{(\binom{g}{r})} H^g_{et}(X_{i, \overline{\mathbb{F}}_p}, L^{(k, w)}|_{X_i})[\pi] \to H^g_{et}(X_{\overline{\mathbb{F}}_p}, L^{(k, w)}(r))[\pi]$$

is an isomorphism on the generalized eigenspace for $Frob_{\overline{\rho}} = \alpha^{g-r} \beta^r / p^{\rho r}$.

In a sense, Theorem 1.3 is a special (and a degenerate) case of this theorem. We in fact prove in Theorem 4.4 a stronger result than stated here for general quaternionic Shimura varieties.

The proof of this theorem is straightforward in that we just need to verify that the left vertical homomorphism (composition of the rest of the diagram) in the following diagram is an isomorphism when restricted to the generalized eigenspace for $Frob_{\overline{\rho}} = \alpha^{g-r} \beta^r / p^{\rho r}$.

$$\bigoplus_{i=1}^{(\binom{g}{r})} H^g_{et}(X_{i, \overline{\mathbb{F}}_p}, L^{(k, w)}|_{X_i})[\pi] \xrightarrow{\text{Gysin}} H^g_{et}(X_{\overline{\mathbb{F}}_p}, L^{(k, w)}(r))[\pi] \xrightarrow{\text{restriction}} \bigoplus_{i=1}^{(\binom{g}{r})} H^g_{et}(X_{i, \overline{\mathbb{F}}_p}, L^{(k, w)}|_{X_i(r)})[\pi],$$

In other words, we need to verify that a $(\binom{g}{r}) \times (\binom{g}{r})$-matrix is invertible. (When $g = 2r$ and $(k, w)$ is of parallel weight 2, this is the intersection matrix of $X_i$'s.) While it is not extremely difficult to compute this matrix and its determinant when both $g$ and $r$ are relatively small, it appears to be quite a non-trivial work if one would like to prove the invertibility of such a matrix for general $g$ and $r$.\footnote{In fact, we need a slightly funny choice of Deligne homomorphism for these quaternionic Shimura varieties; we refer to the main context of the paper for details.}
In this paper, we will show that the entries of the matrix can be computed in a completely combinatorial way; and the matrix then reduces to a version of Gram determinant for meanders, which then follows from some work of mathematical physicists [MS13, GL98]. We confess that this potential link with mathematical physics is probably a mirage. It only suggests strong relation to representation theory. See Remark 3.3.

**Remark 1.6.** We point out a key philosophy suggested by Theorem 1.5 “interesting cycles” on the special fibers of Shimura variety can be predicted by looking at the associated Galois representation. In particular, the multiplicity of certain Frobenius eigenspace is expected to be equal to the number of collections of such interesting cycles.

We also remark that, as pointed out to us by Xinwen Zhu, the name “Goren-Oort” cycles is slightly misleading because the union of the generalized Goren-Oort cycles are supposed to give rise to the *Newton stratification* of \( X \). This viewpoint will be further elaborated in a forthcoming joint work of the authors and David Helm [HTX13].

### 1.7. Generalized Goren-Oort cycles: construction.

The best way (so far) to parametrize the Goren-Oort cycles is to use periodic semi-meanders (mostly for the benefit of later computation of Newton stratification rise to the). For each \( \mathbb{F} \) Goren-Oort cycles is to use periodic semi-meanders (mostly for the benefit of later computation of Newton stratification rise to the).

In particular, the multiplicity of certain Frobenius eigenspace is expected to be equal to the number of special fibers of Shimura variety can be predicted by looking at the associated Galois representation.

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1. **Generalized Goren-Oort cycles: construction.** The best way (so far) to parametrize the Goren-Oort cycles is to use periodic semi-meanders (mostly for the benefit of later computation of the Gysin-restriction matrix). As before, we take \( F \) to be a totally real field of degree \( g \) in which \( p \) is inert.

A **periodic semi-meander** of \( g \) nodes is a graph where \( g \) nodes are aligned equidistant on a section of a vertical cylinder, and are either connected pairwise by non-intersecting curves drawn above the section, or connected by a straight line to \(+\infty\). We use \( r \) to denote the number of arcs. For example, \( \bullet \bullet \bullet \bullet \) and \( 
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\) are both semi-meanders of 6 points with \( r = 2 \) and 3 respectively. An elementary computation shows that there are \( \binom{g}{r} \) semimeanders of \( g \) nodes with \( r \) arcs.

We label the set of \( p \)-adic embeddings \( \mathcal{O}_F \hookrightarrow W(\mathbb{F}_p) = \hat{\mathbb{Z}}^\text{ur}_p \) by \( \tau_1, \ldots, \tau_g \) so that \( \sigma \circ \tau_i = \tau_{i+1} \), where \( \sigma \) is the Frobenius \( W(\mathbb{F}_p) \to W(\mathbb{F}_p) \) and \( \tau_i = \tau_i \mod g \). Let \( X \) denote the special fiber of the Hilbert modular variety. For each \( \mathbb{F}_p \)-point \( x \in X \), we use \( A_x \) to denote the universal abelian variety at \( x \). Let \( \mathcal{D}_x \) denote the covariant Dieudonné module of the \( p \)-divisible group of \( A_x \); it is a \( W(\mathbb{F}_p) \)-module. The \( \mathcal{O}_F \)-action on \( A_x \) induces a natural direct sum decomposition \( \mathcal{D}_x \cong \bigoplus_{i=1}^g \mathcal{D}_{x,i} \), where \( \mathcal{D}_{x,i} \) is the direct summand of \( \mathcal{D}_x \) on which \( \mathcal{O}_F \) acts through \( \tau_i \). It is known that each \( \mathcal{D}_{x,i} \) is free of rank two over \( W(\mathbb{F}_p) \).

The Verschiebung induces a \( \sigma^{-1} \)-semilinear map \( V_i : \mathcal{D}_{x,i+1} \to \mathcal{D}_{x,i} \). (Note that the index might be different from the convention used in other literatures.) The image \( V_i(\mathcal{D}_{x,i+1})/p\mathcal{D}_{x,i} \) is canonically isomorphic to \( \omega_{A'_x,i}^\vee \), the \( \tau_i \)-component of the invariant 1-differentials on \( A'_x \). The latter is a one-dimensional \( \mathbb{F}_p \)-vector space.

We fix \( r \leq g/2 \). To each semi-meander \( a \) considered above, we associate a subvariety \( X_a \) of \( X \) whose \( \mathbb{F}_p \)-points \( x \) are characterized as follows:

- For each curve connecting \( a \) and \( a + d \) for some odd number \( d < g \), we require that

  \[ V_a \circ V_{a+1} \circ \cdots \circ V_{a+d}(\mathcal{D}_{x,a+d+1}) \subseteq p^{(d+1)/2} D_{x,a} \text{ or equivalently } p^{(d+1)/2} | V_a \circ V_{a+1} \circ \cdots \circ V_{a+d}. \]

In fact, the inclusion forces an equality.

For example, the condition for the semimeander \( 
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\) (with \( g = 7 \)) is

\[ p | V_0 V_1, \quad p | V_2 V_3, \quad \text{and } p | V_6 \frac{V_0 V_1}{p} \frac{V_2 V_3}{p} V_4. \]

Using the result from [ITX13+a] it is not difficult to see that \( X_a \) is an \( r \)-times iterated \( \mathbb{P}^1 \)-bundle over the special fiber of quaternionic Shimura variety [3] for the quaternion algebra which ramifies at all archimedean places corresponding to those nodes which are linked to arcs.

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[3] More rigorously, we have a funny choice of Deligne homomorphism for this quaternionic Shimura variety.
The main computation of this paper is to show that the Gysin-restriction matrix agrees with the natural bilinear form on the vector space spanned by the semi-meanders. See Subsection 3.2 for the definition and Theorem 4.3 for the precise statement.

Finally, we make a technical remark: our results involves transferring constructions from the unitary setup to the quaternion case; this is the origin of most notational complications. Moreover, certain descriptions of the GO-cycles have ambiguity; but the ambiguity does not affect the proof of the main theorem.

Structure of the paper. In Section 2, we recall necessary facts about Goren-Oort stratification from [TX13+a]. Some of the proofs are mostly book-keeping but technical; the readers may skip them for the first time reading. In Section 3, we first recall the combinatorics about semi-meanders and then give the definition of the generalized Goren-Oort cycles associated to each periodic semi-meander. In Section 4, we state our main Theorem 4.4 and prove them modulo Theorem 4.3, which says that the Gysin-restriction matrix for Goren-Oort cycles is roughly the same as the Gram matrix of the corresponding periodic semi-meanders. This key theorem is proved in Section 5. The appendix includes a proof of the description of the cohomology of quaternionic Shimura varieties; this is well known to the experts but we keep it there for completeness.

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2. Goren-Oort stratification

We first recall the Goren-Oort stratification of the special fiber of quaternionic Shimura varieties and their descriptions, following [TX13+a]. We tailor our discussion to later application and hence we will focus on certain special cases discussed in loc. cit.

2.1. Notation. For a number field $F$, we use $\text{Gal}_F$ to denote its absolute Galois group, $\mathbb{A}_F$ (resp. $\mathbb{A}_F^\infty$, $\mathbb{A}_F^{\infty,p}$) its ring of adèles (resp. finite adèles, finite adèles away from a rational prime $p$). When $F = \mathbb{Q}$, we suppress the subscript $F$, e.g. by writing $\mathbb{A}^\infty$. Let $\mathcal{O}_F$ denote the idèle of $\mathbb{A}_F^\infty$ which is $p$ at all $p$-adic places and trivial elsewhere. We also normalize the Artin reciprocity map $\text{Art} : \mathbb{A}_F^\times/F^\times \rightarrow \text{Gal}_F^{ab}$ so that a local uniformizer at a finite place $v$ corresponds to a geometric Frobenius element at $v$.

We fix $F$ a totally real field of degree $g > 1$ over $\mathbb{Q}$. Let $\Sigma$ denote the set of places of $F$, and $\Sigma_{\infty}$ the subset of all real places. We fix a prime number $p > 2$ inert in the extension $F/\mathbb{Q}$.

We put $\mathbb{Q}_p = \mathbb{Q}^p$, $\mathbb{Q}_p^\infty$ be its valuation ring, and $k_p$ the residue field. We fix an isomorphism $\iota_p : \mathbb{C} \simeq \mathbb{Q}_p$. Let $\mathcal{O}_p$ denote the unramified extension of $\mathbb{Q}_p$ of degree $g$ in $\mathbb{Q}_p$; let $\mathcal{O}_p^\infty$ be its valuation ring. Post-composition with $\iota_p$ identifies $\Sigma_{\infty} = \text{Hom}(F, \mathbb{R}) = \text{Hom}(\mathbb{Q}_p^\infty, \mathbb{Q}_p^\infty) = \text{Hom}(\mathcal{O}_p, \mathbb{Q}_p^\infty)$. In particular, the absolute Frobenius $\sigma$ acts on $\Sigma_{\infty}$ by sending $\sigma v = v^p$.

Although most of our argument works equally well in the case when $p$ is only assumed to be unramified, we insist to assume that $p$ is inert to largely simplify notation so that the proof of the main result is more accessible. But see Remark 4.5.
2.2. Quaternionic Shimura varieties. Let $S$ be an even set of places of $F$ not containing $p$. Put $S_\infty = S \cap \Sigma_\infty$ and $S'_\infty = \Sigma_\infty - S_\infty$ and $d = \# S'_\infty$. We also fix a subset $T$ of $S'_\infty$. We denote by $B_S$ the quaternion algebra over $F$ ramified exactly at $S$. Let $G_{S,T} = \text{Res}_{F/Q}(B_S^\infty)$ be the associated $\mathbb{Q}$-algebraic group. Here we inserted the subscript $T$ because we will use the following associated Deligne homomorphism

$$h_{S,T}: S(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G_{S,T}(\mathbb{R}) \cong (\mathbb{H}^\times)^{S_\infty - T} \times (\mathbb{H}^\times)^T \times \text{GL}_2(\mathbb{R})^{S_\infty}$$

$$x + y i \longmapsto \left( (1, \ldots, 1), (x^2 + y^2, \ldots, x^2 + y^2), (\left(\begin{array}{cc} x & y \\ -y & x \end{array}\right), \ldots, \left(\begin{array}{cc} x & y \\ -y & x \end{array}\right)) \right).$$

When $T = \emptyset$, the Deligne homomorphism $h_{S,\emptyset}$ is the same as $h_S$ considered in [TX13+a, §3.1]. The $G_{S,T}(\mathbb{R})$-conjugacy class of $h_{S,T}$ is independent of $T$ and is isomorphic to $\mathcal{O}_S = (\mathfrak{h}^\pm)^{S_\infty}$, where $\mathfrak{h}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. Consider the Hodge cocharacter

$$\mu_{S,T} : \mathbb{G}_{m,C} \overset{\mu_{S,T}}{\longrightarrow} \mathcal{O}_{\mathbb{C}} \cong \mathbb{G}_{m,C} \times \mathbb{G}_{m,C} \overset{h_{S,T}}{\longrightarrow} G_{S,T, C}.$$ 

Here, the composition of the natural inclusion $\mathbb{C}^\times = S(\mathbb{R}) \hookrightarrow S(\mathbb{C})$ with the first (resp. second) projection $S(\mathbb{C}) \to C^\times$ is the identity map (resp. the complex conjugation).

The reflex field $F_{S,T}$, i.e. the field of definition of the conjugacy class of $\mu_{S,T}$, is a finite extension of $\mathbb{Q}$ sitting inside $C$ and hence inside $\mathbb{Q}_p$ via $\iota_p$. It is clear that the $p$-adic closure of $F_{S,T}$ in $\overline{\mathbb{Q}}_p$ is contained in $\mathbb{Q}_p$, the unramified extension of $\mathbb{Q}_p$ of degree $g$ in $\overline{\mathbb{Q}}_p$. Instead of working with occasional smaller reflex field, we are content for working with Shimura varieties over $\mathbb{Q}_p$.

We fix an isomorphism $G_{S,T}(\mathbb{Q}_p) \cong \text{GL}_2(F_p)$ and put $K_p = \text{GL}_2(\mathcal{O}_p)$ or occasionally $\text{Iw}_p := (\mathcal{O}_p^\times \mathcal{O}_p)_{\mathcal{O}_p}^\times$ when $\mathcal{O}_S^\times = \emptyset$. We will only consider open compact subgroups $K \subseteq G_{S,T}(\mathbb{A}^\infty)$ of the form $K = K_p K^p$ with $K^p$ an open compact subgroup of $G_{S,T}(\mathbb{A}^\infty)^\circ$. For such a $K$, we have a Shimura variety $Sh_K(G_{S,T})$ defined over $\mathbb{Q}_p$, whose $\mathbb{C}$-points (via $\iota_p$) are given by

$$Sh_K(G_{S,T})(\mathbb{C}) = G_{S,T}(\mathbb{Q})/\mathcal{O}_S \times G_{S,T}(\mathbb{A}^\infty)/K.$$ 

We put $Sh_{K_p}(G_{S,T}) := \lim_{\longleftarrow} K_p Sh_{K_p} K_p(G_{S,T})$. This Shimura variety has dimension $d = \# S'_\infty$. There is a natural morphism of geometric connected components

$$(2.2.1) \quad \pi_0(Sh_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}_p}) \longrightarrow F_{+}^{\times,cl} \backslash \mathbb{A}_{F}^{\infty,\times}/\mathcal{O}_{F_p}^{\times},$$

where $F_+^{\times}$ is the subgroup of totally positive elements of $F^{\times}$, and the superscript cl stands for taking closure in the appropriate topological space. The morphism (2.2.1) is an isomorphism if $S'_\infty \neq \emptyset$ by [De71, Théorème 2.4]. Following the convention in [TX13+a], we shall call the preimage of an element of $x \in F_{+}^{\times,cl} \backslash \mathbb{A}_{F}^{\infty,\times}/\mathcal{O}_{F_p}^{\times}$ under the map (2.2.1) a geometric connected component, denoted by $Sh_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}_p}$, although it is not geometrically connected when $S'_\infty = \emptyset$. The preimage of 1 is called the neutral geometric connected component, which we denoted by $Sh_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}_p}$.

Note that, for different choices of $T$, the Shimura varieties $Sh_K(G_{S,T})$ are isomorphic over $\overline{\mathbb{Q}}_p$ (in fact over $\mathbb{Q}$ if we have not $p$-adically completed the reflex field), but the actions of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ depend on $T$. By Shimura-Deligne’s reciprocity law (cf. [De71] or [TX13+a, Section 2.7]), the
action of $\text{Gal}([\overline{\mathbb{Q}}_p]/\mathbb{Q}_p)$ on $\pi_0(\text{Sh}_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}}_p)$ factors through $\text{Gal}_{f, p} \cong \text{Gal}((\mathbb{Z}^u_\ell)/\mathbb{Z}_p)$, so that the connected components of $\text{Sh}_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}}_p$ are actually defined over $\mathbb{Q}^u_\ell$. More precisely, the action of the geometric Frobenius on $\mathbb{F}_p^\times \setminus \mathbb{A}_F^\times / \mathcal{O}_F^\times$, induced through the morphism \((2.2.1)\), is given by multiplication by the finite idèle \((\mathbb{P}_F)^{2\#T + 1 + \#S_\infty} \in \mathbb{A}_F^\times / \mathbb{Q}_p^\times\). This determines a reciprocity map:
\[
\text{Rec}_p : \text{Gal}_{f, p} \to F^\times \setminus \mathbb{A}_F^\times / \mathcal{O}_F^\times.
\]

Following Deligne’s recipe [De79] of connected Shimura varieties, we put
\[
\mathcal{G}_{S,T,p} := (G_{S,T}(\mathbb{A}^\infty,p)/\mathcal{O}^\times_{F,(p)}) \times \text{Gal}_{f, p},\]
and define $\mathcal{E}_{G_{S,T}}$ as its subgroup consisting of pairs \((x, \sigma)\) such that $\nu(x)$ is equal to $\text{Rec}_p(\sigma)^{-1}$, where $\nu : G_{S,T} \to \text{Res}_{F/\mathbb{Q}}(G_m)$ is the norm map. Here, $\mathcal{O}^\times_{F,(p)}$ denotes the closure $\mathcal{O}^\times_F$ in $G_{S,T}(\mathbb{A}^\infty,p)$.

By Shimura reciprocity law, each geometric connected component $\text{Sh}_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}}_p$ of $\text{Sh}_{K_p}(G_{S,T})_{\mathbb{Z}_p}$ (in the sense above, by taking the preimage of the morphism \((2.2.1)\)) carries an action of the group $\mathcal{E}_{G_{S,T}}$. Conversely, one can recover $\text{Sh}_{K_p}(G_{S,T})$ from $\text{Sh}_{K_p}(G_{S,T})_{\mathbb{Z}_p}$ by first forming the product
\[
\text{Sh}_{K_p}(G_{S,T})_{\mathbb{Z}_p} \times \mathcal{E}_{G_{S,T}} \mathcal{G}_{S,T,p}
\]
and then Galois descending to $\mathbb{Q}_p$. We write $\text{Sh}_{K_p}(G_{S,T})_{\mathbb{Z}_p}$ for the geometric connected component corresponding to $\text{Sh}_{K_p}(G_{S,T})_{\overline{\mathbb{Q}}}_p$.

**Notation 2.3.** Note that $G_{S,T}(\mathbb{A}^\infty)$ depends only on the finite places contained in $S$. In later applications, we will consider only pairs of subsets \((S', T')\) such that $S'$ contains the same finite places as $S$: in that case, we will fix an isomorphism $G_{S',T'}(\mathbb{A}^\infty) \cong G_{S,T}(\mathbb{A}^\infty)$, and denote them uniformly by $G(\mathbb{A}^\infty)$ when no confusions arise. Similarly, we have its subgroup $G(\mathbb{A}^\infty,p) \subseteq G(\mathbb{A}^\infty)$ consisting of elements with $p$-component equal to 1. Thus, we may view $K$ (resp. $K^p$) as an open compact subgroup of $G(\mathbb{A}^\infty)$ (resp. $G(\mathbb{A}^\infty,p)$).

Under this identification, the group $\mathcal{G}_{S,T,p}$ is independent of $S, T$, and we henceforth write $\mathcal{G}_p$ for it. Its subgroup $\mathcal{E}_{G_{S,T}}$ in general depends on the choice of $S$, and $T$. However, the key point is that, if $S'$ and $T'$ is another pair of subsets satisfying similar conditions and $\#S_\infty - 2\#T = \#S'_\infty - 2\#T'$ (which will be the case we consider later in this paper), then the subgroup $\mathcal{E}_{G_{S,T}}$ is the same as $\mathcal{E}_{G_{S',T'}}$.

**Remark 2.4.** Using Proposition 2.10 and Construction 2.15 later, we can access to most of the statements in [TX13+a] which were initially proved for unitary groups and interpreted using connected Shimura varieties. The key point mentioned in Notation 2.3 has the additional benefit that the description of Goren-Oort strata actually descend to quaternionic Shimura varieties because now the subgroup $\mathcal{E}_{G_{S,T}}$ are compatible for different $S$ and $T$’s.

2.5. Automorphic representations. Following [TX13+b] 5.10, we use $\mathcal{M}_{(\ell,w)}$ to denote the set of irreducible automorphic cuspidal representations $\pi$ of $GL_2(\mathbb{A}_F)$ whose archimedean component $\pi_\tau$ for each $\tau \in \Sigma_{\infty}$ is a discrete series of weight $k_\tau - 2$ with central character $x \mapsto x^{w-\tau}$, and whose $p$-component $\pi_p$ is unramified. We write $\pi_{\infty,p}$ to denote the prime-to-$p$ finite part of $\tau$.

We denote by $\rho_\pi : Gal_F \to GL_2(\mathbb{Q}_p)$ the Galois representation attached to $\pi$ normalized so that if $v$ is a finite place of $F$ at which $\pi$ is unramified and $K$ is hyperspecial, the action of a geometric

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6When $S_\infty = \emptyset$ or equivalently when $\text{Sh}_{K_p}(G_{S,T})$ is a zero-dimensional Shimura variety, the action of $\text{Frob}_p$ is given by multiplication by the finite idèle $(\mathbb{P}_F)^{\#T}$ in the center $\text{Res}_{F/\mathbb{Q}} G_m$ of $G_{S,T}$. This gives the canonical model for the discrete Shimura variety in the sense of [TX13+b] 2.8).

7Comparing with [TX13+b] (2.11.3), we dropped the star extension because the center of $G_{S,T}$ is $\text{Res}_{F/\mathbb{Q}} G_m$, which has trivial first cohomology. We also include the Galois part into the definition of $\mathcal{G}$ to simplify notation here.
Frobenius at \( v \) has trace equal to the eigenvalue of usual Hecke operator \( T_v \) on \((\pi^\infty)^K\). Let \( \rho_{p,\pi} \) be the restriction of \( \rho_p \) to \( \text{Gal}_{\mathbb{Q}_p} \) (note that \( \rho_p \) is unramified at \( p \)).

Assume that \( K_p = \mathrm{GL}_2(\mathbb{O}_p) \). The Hecke operators \( T_p \) and \( S_p \) are given by the action on \( \pi_p^{K_p} \) of the double cosets \( K_p(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix})K_p \) and \( K_p^{-1}K_p \), respectively. Then, for \( \pi \in \mathcal{A}_{(\ell,w)} \), the characteristic polynomial of \( \rho_{\pi,p}(\text{Frob}_p) \) is given by

\[
X^2 - T_p(\pi)X + S_p(\pi)p^\ell,
\]

where \( T_p(\pi) \) and \( S_p(\pi) \) denote respectively the eigenvalues of \( T_p \) and \( S_p \) on \( \pi_p^{K_p} \).

We say an automorphic representation \( \pi \in \mathcal{A}_{(k,w)} \) appears in the cohomology of the Shimura variety \( \text{Sh}_K(G_{S,T}) \) if for each finite place \( v \) of \( S \), the local component \( \pi_v \) of \( \pi \) is square integrable so that \( \pi \) is the image of the Jacquet-Langlands correspondence of a unique automorphic representation \( \pi_{B_S} \) of \( G_{S,T}(\hat{\mathbb{A})} = (B_S \otimes \mathbb{Q} \hat{\mathbb{A})^\times} \), and \( (\pi_{B_S}^{\infty})^K \) is nonzero.

**Notation 2.6.** For \( \pi \in \mathcal{A}_{(k,w)} \) and a \( \overline{\mathbb{Q}_l}(G(\mathbb{A}^{\infty,p})) \)-module \( M \), we put

\[
M[\pi] := \text{Hom}_{\overline{\mathbb{Q}_l}(G(\mathbb{A}^{\infty,p}))}((\pi_{B_S}^{\infty,p})^K, M).
\]

By strong multiplicity one for quaternion algebras, \( \pi_{B_S} \) is determined by \( \pi_{B_S}^{\infty,p} \); this justifies the notation for \( M[\pi] \). There is also a finite version: let \( K^\ell \subset G(\mathbb{A}^{\infty,p}) \) be an open compact subgroup so that \( (\pi_{B_S}^{\infty,p})_K^\ell \) is an irreducible module over the prime-to-\( p \) Hecke algebra \( T(K^\ell) := \overline{\mathbb{Q}_l}[K^\ell \otimes (\mathbb{A}^{\infty,p})/K^\ell] \), and \( \pi_{B_S} \) is determined by the \( \mathbb{T}(K^\ell) \)-module \( \pi_{B_S}^{\infty,p} \). Then, for a \( \mathbb{T}(K^\ell) \)-module \( M \), we put

\[
M[\pi] := \text{Hom}_{\mathbb{T}(K^\ell)}((\pi_{B_S}^{\infty,p})_K^\ell, M).
\]

**2.7. An auxiliary CM field.** To use the results in \([\text{TXI3+a}]\), we fix a CM extension \( E/F \) such that

- every place in \( S \) is inert in \( E/F \); and
- the place \( p \) splits as \( qq \) in \( E/F \) if \( \#S^c_\infty \) is even; and it is inert in \( E/F \) if \( \#S^c_\infty \) is odd.

These conditions imply that \( B_S \) splits over \( E \). In later applications, we will need to consider several \( S \) at the same time; but we remark, for all subsets \( S \) involved, the finite places contained in \( S \) are the same, and \( \#S^c_\infty \) will have the same parity.

We shall frequently use the following two finite idéle elements:

1. \( \mathcal{P}_F \) denotes the finite idéle in \( \mathbb{A}^F_\infty \) which is \( p \) at \( p \) and is 1 elsewhere (which we have already introduced in [2.1]);
2. when \( p \) splits into \( qq \) in \( E \), \( q \) denotes the finite idéle in \( \mathbb{A}^\infty_\infty \) which is \( p \) at \( q \), \( p^{-1} \) at \( \bar{q} \), and 1 elsewhere.

Let \( \Sigma_{E,\infty} \) denote the set of complex embeddings of \( E \). We fix a choice of subset \( \tilde{S}_\infty \subseteq \Sigma_{E,\infty} \) such that the natural restriction map \( \Sigma_{E,\infty} \to \Sigma_\infty \) induces an isomorphism \( \tilde{S}_\infty \to \Sigma_\infty \). When \( p \) splits into \( qq \), we use \( \tilde{S}_\infty/q \) (resp. \( \tilde{S}_\infty/q \)) to denote the subset of places in \( \tilde{S}_\infty \) inducing \( q \) (resp. \( \bar{q} \)) through the isomorphism \( \iota_p \). We put \( \Delta_S = \#S_{\infty}/\#S_{\infty}/\bar{q} \). We remark that, all the \( S \)’s we encounter later in this paper will all have the same \( \Delta_S \).

We write \( E_p \) for \( F_p \otimes_F E \); it is the quadratically unramified extension of \( F_p \) if \( p \) is inert and it is \( E_q \times E_{\bar{q}} \) if \( p \) splits. We set \( \mathcal{O}_{E_p} := \mathcal{O}_p \otimes \mathcal{O}_F \mathcal{O}_E \).

We put \( S = (S, \tilde{S}_\infty) \). Put \( T_{E,\tilde{S},T} = T_E = \text{Res}_{E/Q}(\mathcal{G}_m) \), where the subscript \((\tilde{S}, T)\) means that we take the following Deligne homomorphism

\[
h_{E,\tilde{S},T} : S(\mathbb{R}) = \mathbb{C}^\times \longrightarrow T_{E,\tilde{S},T}(\mathbb{R}) = \bigoplus_{\tau \in \Sigma_\infty} (E \otimes_{F,\tau} \mathbb{R}) \cong (\mathbb{C}^\times)^{S_\infty - T} \times (\mathbb{C}^\times)^T \times (\mathbb{C}^\times)^{S_\infty}
\]

\[
z = x + yi \longmapsto \left( (\bar{z}, \ldots, \bar{z}), (z^{-1}, \ldots, z^{-1}), (1, \ldots, 1) \right).
\]
Here, the isomorphism $E \otimes_{F,\tau} \mathbb{R} \simeq \mathbb{C}$ for $\tau \in S_{\infty}$ is given by the chosen embedding $\tau \in \tilde{S}_{\infty}$ lifting $\tau$. One has the system of zero-dimensional Shimura varieties $Sh_{K_E}(T_{E,\tilde{S},T})$ with $\mathbb{C}$-points given by:

$$Sh_{K_E}(T_{E,\tilde{S},T})(\mathbb{C}) = E^{\times,c_1} \setminus T_{E,\tilde{S},T}(\mathbb{A}^\infty)/K_E,$$

for any open compact subgroup $K_E \subseteq T_{E,\tilde{S},T}(\mathbb{A}^\infty) \cong \mathbb{A}_E^\infty \times [p]$. We put $K_{E,p} = \mathcal{O}_{E,p}^{\times} \subseteq T_{E,\tilde{S},T}(\mathbb{Q}_p)$, and write $Sh_{K_{E,p}}(T_{E,\tilde{S},T}) = \lim_{\leftarrow K_E} Sh_{K_{E,p}}(T_{E,\tilde{S},T})$ as the inverse limit over all open compact subgroups $K_E^p \subseteq T_{E,\tilde{S},T}(\mathbb{A}_E^\infty,p)$.

As in Notation 2.3, we identify $\tau_{E,\tilde{S}}$ as the quotient of $\mathcal{O}_{E,\tilde{S}} = \mathcal{O}_{E,\tilde{S}}^{\times} \times_{\mathcal{O}_{p,\tilde{S}}} \mathcal{O}_{p,\tilde{S}}^{\times}$, which is the quotient of $\mathcal{O}_{E,\tilde{S}}$ by taking the spectra of the corresponding rings of integers. Denote by $Sh_{K_E}(T_{E,\tilde{S},T})(\mathbb{F}_p)$ its special fiber. By Shimura’s reciprocity law, the action of the geometric Frobenius $\mathfrak{Frob}_{p^{\mathfrak{F}}}$ of $\mathbb{F}_p^{\mathfrak{F}}$ on $Sh_{K_E}(T_{E,\tilde{S},T})(\mathbb{F}_p)$ is given by:

- when $p$ is inert in $E/F$, multiplication by $(p_{E,p})^{(#S_{\infty} \cdot -#T) - #T} = (p_{E,p})^{#S_{\infty} - 2#T}$ and
- when $p$ splits into $\mathfrak{q}_1 \mathfrak{q}_2$, multiplication by 

$$\frac{2(#S_{\infty} - #T)}{\mathfrak{q}_1 \mathfrak{q}_2} = \frac{2(#S_{\infty} - #T)}{\mathfrak{q}_1 \mathfrak{q}_2} = (p_{E,p})^{#S_{\infty} - 2#T}(\mathfrak{q}_1 \mathfrak{q}_2),$$

where $\mathfrak{q}$ (resp. $\mathfrak{q}_1 \mathfrak{q}_2$) is the finite idèle in $\mathbb{A}_E^\infty$ which is $p$ at the place $\mathfrak{q}$ (resp. $\mathfrak{q}_1 \mathfrak{q}_2$) and is 1 elsewhere, and $\mathfrak{q}_1 \mathfrak{q}_2$ is the idèle defined in [2.7.2] above.

In particular, if $(S', T')$ is another pair above such that $#S'_{\infty} - 2#T' = #S_{\infty} - 2#T$ and $\Delta_{\tilde{S}} = \Delta_{\tilde{S}'}$, if $p$ splits, then there exists an isomorphism of Shimura varieties over $\mathbb{F}_p^{\mathfrak{F}}$:

$$(2.7.1) \quad Sh_{K_{E,p}}(T_{E,\tilde{S},T}) \overset{\sim}{\rightarrow} Sh_{K_{E,p}}(T_{E,\tilde{S}',T})$$

compatible with the Hecke action of $T_{E}(\mathbb{A}^\infty,p)$ on both sides as $K_E^p$ varies.

2.8. A unitary Shimura variety. Let $Z = \text{Res}_{F/Q}(\mathbb{G}_m)$ be the center of $G_{S,T}$. Put $G^\prime_{S} = G_{S,T} \times_Z T_{E,\tilde{S},T}$, which is the quotient of $G_{S,T} \times T_{E,\tilde{S},T}$ by $Z$ embedded anti-diagonally as $z \mapsto (z, z^{-1})$. Consider the product Deligne homomorphism

$$h_{S,T} \times h_{E,\tilde{S},T} : S(\mathbb{R}) = \mathbb{C}^\times \rightarrow (G_{S,T} \times T_{E,\tilde{S},T})(\mathbb{R}),$$

which can be further composed with the quotient map to $G^\prime_{S}$ to get

$$h^\prime_{S} : S(\mathbb{R}) = \mathbb{C}^\times \rightarrow (G_{S,T} \times Z T_{E,\tilde{S},T})(\mathbb{R}) \cong G^\prime_{S}(\mathbb{R}).$$

Note that $h^\prime_{S}$ does not depend on the choice of $T \subseteq S_{\infty}$ (hence the notation), and its conjugacy class is identified with $\mathfrak{h}_{S} = (h_0)^{S_{\infty}}$. Let $K''_p$ denote the (maximal) open compact subgroup $\text{GL}_2(\mathcal{O}_p) \times_{\mathcal{O}_p^{\times}} \mathcal{O}_p^{\times}$ of $G^\prime_{S}(\mathbb{Q}_p)$. We will consider open compact subgroups of the form $K''_p K''_{mp} \subset G^\prime_{S}(\mathbb{A}^\infty,p)$ with $K''_{mp} \subset G^\prime_{S}(\mathbb{A}^\infty,p)$. These data give rise to a Shimura variety $Sh_{K''_p}(G^\prime_{S})$ (defined over $\mathbb{Q}_p$), whose $\mathbb{C}$-points (via $\iota_p$) are given by

$$Sh_{K''_p}(G^\prime_{S})(\mathbb{C}) = G^\prime_{S}(\mathbb{Q}) \setminus (\mathfrak{h}_{S} \times G^\prime_{S}(\mathbb{A}^\infty)) / K''_p.$$

We put $Sh_{K''_p}(G^\prime_{S}) := \lim_{\leftarrow K''_p} Sh_{K''_p}(G^\prime_{S})$; its geometric connected components admit a natural map

$$(2.8.1) \quad \pi_0 \left( Sh_{K''_p}(G^\prime_{S}) \right) \rightarrow \left( E^{\times,c_1} \setminus \mathbb{A}^\infty / \mathcal{O}_p^{\times} \right) \times \left( E^{\times,N_{E/F}=1,c_1} \setminus \mathbb{A}_{E/F}^{\infty,N_{E/F}=1} / \mathcal{O}_E^{N_{E/F}=1} \right).$$

The canonical model of this Shimura variety should be understood in the sense of [TX13+a] Section 2.8.
We write $Sh_{K'}^0(G''_S)_{\overline{\mathbb{Q}}_p}$ for the preimage of some element on the right hand side and $Sh_{K'_p}(G''_S)^\circ_{\overline{\mathbb{Q}}_p}$ for the preimage of $1 \times 1$; they are called a geometric connected component and the neutral geometric connected component of the unitary Shimura variety, respectively.

We can define the group $E''_S$ and $G''_{S,p}$ for the Shimura data $(G''_S, h''_S)$ as in Subsection 2.2 (see e.g. [TX13+a, §2.11] for the recipe). First, we spell out the Shimura reciprocity map

\[(2.8.2) \quad \text{Rec}'_p: \mathcal{G} F_{\overline{\mathbb{Q}}_p} \rightarrow (F^+ \backslash \mathbb{A}^\infty_F \mathcal{O'}_F^\times) \times (E^+_{N_{E/F}=1,\text{cl}} \mathcal{A}'_{E/F=1}/ \mathcal{O'}_{E/F=1}^N_{E/F=1}),\]

where $N_{E/F}$ is the norm from $E$ to $F$. Explicitly,

- when $p$ is inert in $E/F$, $\text{Rec}'_p(\text{Frob}_{\overline{\mathbb{Q}}_p}) = (p_{E/F})^2 \times 1$; and
- when $p$ splits in $E/F$, $\text{Rec}'_p(\text{Frob}_{\overline{\mathbb{Q}}_p}) = (p_{E/F})^2 \times (q)^{2\Delta_S}$.

We put $G''_{S,p} = (G''_S(\mathbb{A}^\infty_F)/\mathcal{O}_{E,(p)}^\infty) \times \mathcal{G} F_{\overline{\mathbb{Q}}_p}$ and define $E''_S$ to be its subgroup of pairs $(x, \sigma)$ such that $\nu''(x)$ is equal to $\text{Rec}'_p(\sigma)^{-1}$, where

$$\nu'': G''_S \rightarrow \text{Res}_{E/F}(\mathbb{G}_m) \times \text{Ker} \left( \text{Res}_{E/F}(\mathbb{G}_m) \rightarrow \text{Res}_{E/F}(\mathbb{G}_m) \right)$$

is the natural morphism from $G''_S$ to its maximal abelian quotient.

If $Sh_{K'_p}(G''_S)^\circ_{\overline{\mathbb{Q}}_p}$ is a geometric connected component of $Sh_{K'_p}(G''_S)_{\overline{\mathbb{Q}}_p}$ in the sense above, then one can recover the entire Shimura variety from this geometric connected component by taking the Galois descent of $Sh_{K'_p}(G''_S)^\circ_{\overline{\mathbb{Q}}_p} \times E''_S, G''_{S,p}$. We use $Sh_{K'_p}(G''_S)^\circ_{\overline{\mathbb{Q}}_p}$ to denote the integral version of $Sh_{K'_p}(G''_S)^\circ_{\overline{\mathbb{Q}}_p}$.

**Remark 2.9.** Similar to Notation 2.3 if $S'$ is another subset of places of $F$ containing the same finite places as $S$ (together with a choice of $S'_\infty$), then $G''_{S'}(\mathbb{A}^\infty)$ is isomorphic to $G''_S(\mathbb{A}^\infty)$. We fix such an isomorphism, and denote them uniformly as $G''(\mathbb{A}^\infty)$. Hence we naturally identify groups $G''_{S,p}$ for different $S$’s.

When $\#S_S = \#S'_{S'_S}$ and $\Delta_S = \Delta_{S'}$ if $p$ splits in $E/F$, the subgroup $E''_{S'} \subset G''_{S',p}$ can also be identified with $E''_{S} \subset G''_{S,p}$. Indeed, in this case the reciprocity map $\text{Rec}'_p$ for $S$ and $S'$ are the same.

**Proposition 2.10.** (1) We have a canonical isomorphism $E_{G_{S,T}} \cong E_{G'^{'}_{S,T}}$. If $\mathbb{Q}^{ur}_p$ denotes the maximal unramified extension of $\mathbb{Q}_p$, the neutral connected component of the Shimura variety $Sh_{K'_p}(G_{S,T})_{\overline{\mathbb{Q}}_p}$ together with the action of $E_{G_{S,T}}$ is isomorphic to $Sh_{K'_p}(G''_S)^\circ_{\overline{\mathbb{Q}}_p}$ together with the action of $E_{G'^{'}_S}$.

(2) The Shimura varieties $Sh_K(G_{S,T})$ (resp. $Sh_{K'}(G''_S)$) admits a canonical integral model over $\mathbb{Z}[p^\infty]$ (resp. over $\mathbb{Z}[p^\infty]$).

**Proof.** For (1), the case when $T = \emptyset$ is treated in [TX13+a]. In general, note that the sequence of morphisms

$$G''_S \leftarrow G_{S,T} \times T_{E,\overline{\mathbb{Q}}_p} \rightarrow G_{S,T}$$

is compatible with the associated Deligne homomorphism, and the conjugacy classes of Deligne homomorphisms into various algebraic groups defined above are canonically identified. Standard facts (e.g. [TX13+a Corollary 2.17]) about Shimura varieties implies (2) and that the series of morphisms of Shimura varieties

$$Sh_{K'_p}(G''_S) \leftarrow Sh_{K'_p}(G'_{S,T}) \times_{\mathbb{Z}[p]} Sh_{K_{E,p}}(T_{E,\overline{\mathbb{Q}}_p}) \rightarrow Sh_{K'_p}(G_{S,T})$$

induces an isomorphism on the neutral connected component; hence by [TX13+a Theorem 3.14] (and [TX13+a Corollary 2.17]), all these Shimura varieties have integral canonical models over $\mathbb{Z}[p^\infty]$, which we still denote by the same notation above. (The canonical integral model for $Sh_{K'_p}(G_{S,T})$

---

9As in the footnote to 2.2.2, we omitted the star product in the definition of this group comparing to [TX13+a (2.11.3)] because the center $Res_{E/F}(\mathbb{G}_m)$ of $G''_{S,p}$ has trivial first cohomology.
descends to \(\mathbb{Z}_{p^a}\). Moreover, different geometric connected components are isomorphic to each other by the Hecke action.

\[\blacksquare\]

**Remark 2.11.** Statement (2) of Proposition 2.10 is of course a consequence of much more general theory of Kisin \([K12]\). However, in this paper, we will need essentially this explicitly relationship between the integral models of \(Sh_K(G_{S,T})\) and those of \(Sh_{K''}(\hat{G}_S'')\).

**Notation 2.12.** To save notations, we still use \(Sh_{K_p}(G_{S,T})\), \(Sh_{K_p}(T_{E,\hat{S},T})\), \(Sh_{K''}(\hat{G}_S'')\)...

to denote the integral model over \(\mathbb{Z}_{p^a}\) or \(\mathbb{Z}_{p^2}\) of the corresponding Shimura variety, and use systematically Roman letters to denote the special fibers of Shimura varieties:

\[Sh_{K_p}(G_{S,T})_{\overline{F}_p} := Sh_{K_p}(G_{S,T})_{\overline{\mathbb{Z}}_{p^a}} \times_{\overline{\mathbb{Z}}_{p^a}} \overline{F}_p, \quad \text{and} \quad Sh_{\tau}(G_{S,T}) := Sh_{\tau}(G_{S,T}) \times_{\overline{\mathbb{Z}}_{p^a}} \overline{F}_{p^a}.\]

for (?) = K or K_p, and \(\ast = \circ\) or \(\bullet\), and

\[Sh_{K_p}(T_{E,\hat{S},T}_{\overline{F}_p})_{\overline{\mathbb{Z}}_{p^2}} := Sh_{K_p}(T_{E,\hat{S},T}) \otimes_{\overline{\mathbb{Z}}_{p^2}} \overline{F}_{p^2}, \quad Sh_{K''}(G_{\hat{S}}'')_{\overline{F}_p} := Sh_{K''}(G_{\hat{S}}') \otimes_{\overline{\mathbb{Z}}_{p^2}} \overline{F}_{p^2}\]

and the same with open compact subgroup \(K_E = K_{p}K_E' \subset T_E(\mathbb{A}^\infty)\) and \(K'' = K''_pK''_E \subset G''(\mathbb{A}^\infty)\). We put \(Sh_{K_p}(G_{\hat{S}}'')_{\overline{F}_p} = Sh_{K_p}(G''_{\hat{S}})_{\overline{\mathbb{Z}}_{p^a}} \times_{\overline{\mathbb{Z}}_{p^a}} \overline{F}_p\) for \(\ast = \circ\) or \(\bullet\).

### 2.13. Automorphic sheaves.

We now study the automorphic sheaves on these Shimura varieties. Fix a prime \(\ell \neq p\), and an isomorphism \(\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell\). By a regular multiweight, we mean a tuple \((k, w) \in \mathbb{Z}_{\Sigma_{\infty}} \times \mathbb{Z}\) such that \(k_r \equiv w \pmod{2}\) and \(k_r \geq 2\) for all \(r \in \Sigma_{\infty}\). Consider the algebraic representation

\[\rho^{(k, w)}(G_{S,T}) = \bigotimes_{r \in \Sigma_{\infty}} (\text{Sym}^{k_r-2} \otimes \det \frac{k_r-w}{2})\]

of \(G_{S,T} \times C \cong \prod_{r \in \Sigma_{\infty}} \text{GL}_2(C)\), where \(\text{std}\) is the standard representation of \(\text{GL}_2(C)\). As explained in \([BM99]\), we have an automorphic \(\overline{\mathbb{Q}}_\ell\)-lisse sheaf \(L^{(k, w)}(G_{S,T})\) on \(Sh_{K_p}(G_{S,T})\) associated to \(\rho^{(k, w)}\). Note that \(L^{(k, w)}_{S,T}\) is pure of weight \((w-2)(\#S_{\infty} + 2\#T)\).

We fix a section \(\hat{\Sigma} \subset \Sigma_{E,\infty}\) of the natural restriction map \(\Sigma_{E,\infty} \to \Sigma_{\infty}\) (which may or may not be related to the choices \(\tilde{S}_{\infty}\)). Let \((k, w)\) be a regular multi-weight. Consider the following one-dimensional representation of \(T_{E,\hat{S},T} \times C \subseteq \prod_{r \in \Sigma_{\infty}} \mathbb{G}_{m, \hat{r}} \times \mathbb{G}_{m, \tilde{r}}:\)

\[\rho_{E,\tilde{S}}^w = \bigotimes_{r \in \Sigma_{\infty}} x^{2-w} \circ \text{pr}_{E,\tilde{r}},\]

where \(\text{pr}_{E,\tilde{r}}\) is the projection to the \(\tilde{r}\)-component and \(x^{2-w}\) is the character of \(C^\times\) given by raising to \((2-w)\)th power. This representation gives rise to a lisse \(\overline{\mathbb{Q}}_\ell\)-étale sheaf \(L_{E,\hat{S},T,\Sigma}^w\) pure of weight \((w-2)(\#S_{\infty} + 2\#T)\) on \(Sh_{K_p}(T_{E,\hat{S},T})\). If \(Sh_{K_E}(T_{E,\hat{S},T})\) is another Shimura variety with \(\#S_{\hat{r}} - 2\#T = \#S_{\hat{r}} - 2\#T\) and \(\gamma : Sh_{K_E}(T_{E,\hat{S},T}) \cong Sh_{K_E}(T_{E,\hat{S},T})\) is the isomorphism \([2.7.1]\), then we have

\[\gamma^*(L_{E,\hat{S},T,\Sigma}^w) \cong L_{E,\hat{S},T,\Sigma}^w.\]

Let \(\alpha_T : G_{S,T} \times T_{E,\hat{S},T} \to \hat{G}_S''\) denote the natural quotient morphism. We have the following diagram.

\[(2.13.2) \quad Sh_{K_p}(G_{S,T}) \xrightarrow{pr_1} Sh_{K_p}(G_{S,T}) \times_{\overline{\mathbb{Z}}_{p^a}} Sh_{K_E,p}(T_{E,\hat{S},T}) \xrightarrow{\alpha_T} Sh_{K''_p}(G''_S) \]

\[\xrightarrow{pr_2} Sh_{K_E,p}(T_{E,\hat{S},T}).\]
By our definition, the tensor product representation $\rho_{E,\Sigma}^{(k,w)} \otimes \rho_{E,\Sigma}^{(k,w)}$ of $G_{S,T} \times T_{E,\Sigma}$ factors through $G_{S,\Sigma}^{(2,1.5.2)}$. This defines a $\mathcal{O}_{\ell}$-lisse sheaf $L_{S,\Sigma}^{(k,w)}$ on $\text{Sh}_{K_p}(G_{S,\Sigma})$ such that we have a canonical isomorphism

$$\alpha_{\tau}(L_{S,\Sigma}^{(k,w)}) \cong \text{pr}_1^*(L_{S,\Sigma}^{(k,w)}) \otimes \text{pr}_2^*(L_{S,\Sigma}^{(k,w)}).$$

Note that the left hand side is independent of $T$.

Put $D = B_S \otimes_F E$. Then our choice of $E/F$ in [2.7] implies that $D \cong M_{2 \times 2}(E)$, which explains the omission of $S$ in our notation. We fix such an isomorphism, and take a maximal order $\mathcal{O}_D \cong M_{2 \times 2}(\mathcal{O}_E)$. Recall that there exists a versal family of abelian varieties $a : A_{S,K_p}^{(2,1.5.1)} \rightarrow \text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.2)})$ on which $E/F$ acts. We now discuss a very important process that allows us to transfer certain correspondences on the unitary Shimura varieties $\text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.2)})$ to the quaternionic Shimura varieties. We point out beforehand that the entire construction involves picking a preimage of some element in the target of the surjective map $\pi'' : G_{S,\Sigma}^{(2,1.5.2)} \rightarrow G_{S,\Sigma}^{(2,1.5.1)}$. Our choice of this preimage, and all choices form a torsor under $\mathcal{O}_D$. Here, “versal” means that the Kodaira-Spencer map for the family $A_{S,K_p}^{(2,1.5.1)}$ is an isomorphism. Using $A_{S,K_p}^{(2,1.5.1)}$, $L_{S,\Sigma}^{(k,w)}$ can be reinterpreted as follows. Put $H_\ell(A_{S,K_p}^{(2,1.5.1)}) = R^1a_*(\mathcal{O}_\ell)$, which is an $\ell$-adic local system on $\text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.2)})$ equipped with an induced action by $M_{2 \times 2}(E)$. For each $\tau \in \Sigma_{E,\infty}$, let $H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau}$ denote the direct summand of $H_\ell(A_{S,K_p}^{(2,1.5.1)})$ on which $E$ acts via $E \xrightarrow{\tau} \mathbb{C} \xrightarrow{i} \mathcal{O}_\ell$. Consider the idempotent $e = (1,0) \in M_{2 \times 2}(E)$. We put $H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau} = e \cdot H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau}$, which is an $\ell$-adic local system on $\text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.2)})$ of rank $2$. We have a canonical decomposition

$$H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau} = \bigoplus_{\tau' \in \Sigma_{E,\infty}} (H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau'} \oplus H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau'}) = \bigoplus_{\tau' \in \Sigma_{E,\infty}} (H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau'}^{\circ,\tau_2} \oplus H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau'}^{\circ,\tau_2^c}),$$

where $\tau^c$ denote the conjugate of $\tau$ under the non-trivial automorphism of $E/F$. Then one has

$$L_{S,\Sigma}^{(k,w)}_{\tau} = \bigotimes_{\tau' \in \Sigma_{E,\infty}} \left( \text{Sym}^{k_\ell - 2} H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau'}^{\circ,\tau_2} \otimes (\wedge^2 H_\ell(A_{S,K_p}^{(2,1.5.1)})_{\tau'}^{\circ,\tau_2}) \right).$$

Remark 2.14. We shall introduce a general construction below to relate the unitary Shimura varieties and the quaternionic Shimura varieties. We point out beforehand that the entire construction is modeled on the following question: By Hilbert 90, we have an exact sequence:

$$1 \rightarrow F^{x,cl} \setminus \mathbb{A}_F^{\infty,\times} / \mathcal{O}_p^{\infty,\times} \rightarrow E^{x,cl} \setminus \mathbb{A}_E^{\infty,\times} / \mathcal{O}_p^{\infty,\times} \twoheadrightarrow E^{x,\mathcal{O}_p^{\infty,\times} / \mathcal{O}_p^{\infty,\times}} \rightarrow E^{x,\mathcal{O}_p^{\infty,\times} / \mathcal{O}_p^{\infty,\times}} = 1,$$

The construction involves picking a preimage of some element in the target of the surjective map above. In general, there is no canonical choice of this preimage, and all choices form a torsor under the group $F^{x,cl} \setminus \mathbb{A}_F^{\infty,\times} / \mathcal{O}_p^{\infty,\times}$. In very special case when the element in the target of the surjective map is trivial, one can have a canonical choice of its preimage, namely the identity element $1$.

Construction 2.15. We now discuss a very important process that allows us to transfer certain correspondences on the unitary Shimura varieties $\text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.2)})$ to the quaternionic Shimura varieties $\text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.1)})$. Throughout this subsection, we assume that we are given two sets of data: $\mathcal{S}', \Sigma$, and $\mathcal{S}', \Sigma'$ as above, and we assume that they satisfy the following conditions:

$$\# \mathcal{S}_\infty - 2 \# \Sigma = \# \mathcal{S}_\infty' - 2 \# \Sigma', \quad \Delta_{\mathcal{S}} = \Delta_{\mathcal{S}'} \quad \text{if } \mathfrak{p} \text{ splits in } E/F,$$

and the finite places contained in $\mathcal{S}$ and those in $\mathcal{S}'$ are the same.

Suppose that we are given a correspondence between two Shimura varieties

$$\text{Sh}_{K_p}(G_{S,\Sigma}^{(2,1.5.2)}) \xleftarrow{\pi''} X \xrightarrow{\pi'''} \text{Sh}_{K_p}(G_{\mathcal{S}',\Sigma}^{(2,1.5.1)}),$$

where the group $G_{S,\mathfrak{p}}^{(2,1.5.2)} \cong G_{\mathcal{S}',\mathfrak{p}}^{(2,1.5.2)}$ acts on all three spaces and the morphisms are equivariant for the actions. We further assume that the fibres of $\pi''$ are geometrically connected.
Step I: We explain how to complete the correspondence above into a diagram

$$
\begin{array}{ccc}
\text{Sh}_{K_p}(G_{S,T}) \times F_p & \xrightarrow{\pi} & \text{Sh}_{K_{E,p}}(T_{E,S,T}) \xrightarrow{\eta^x} \text{Sh}_{K_p}(G_{S,T}) \times F_p \\
\downarrow \alpha\tau & & \downarrow \alpha'' \tau \\
\text{Sh}_{K''_p}(G''_S) & \xleftarrow{\pi''} & X \xrightarrow{\eta''} \text{Sh}_{K''_p}(G''_S),
\end{array}
$$

such that the left square is Cartesian and the top line is equivariant for actions of $G_{S,T,p} \times \bar{A}_{E}^{\infty,x} \cong G_{S,T,p} \times \bar{A}_{E}^{\infty,x}$. For this, we define $Y$ as the Cartesian product on the left square of (2.15.3). So it suffices to lift the morphism $\eta''$ to $\eta^x$. We point out that both $\alpha\tau$ and $\alpha'' \tau$ map every geometric connected component isomorphically to another geometric connected component.

We now separate the discussion (but not in an essential way) depending on whether $S_\infty'$ is empty.

- When $S_\infty' \neq \emptyset$, let $X^0$ denote the preimage $\pi''^{-1}(\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p})$; it is a geometric connected component of $X$. Then it is mapped under $\eta''$ to some geometric connected component $\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p}$. Let $Y^0$ denote the preimage $(\pi^x)^{-1}(\text{Sh}_{K_p}(G_{S,T})_{\bar{F}_p} \times \{1\}$), where 1 denotes the neutral point, namely the image of $1 \in A^{\infty,x}_E$ in $\text{Sh}_{K_{E,p}}(T_{E,S,T})_{\bar{F}_p}$. Then the desired map $\eta^x$ restricted to $Y^0$ must have image in a chosen geometric connected component of $\alpha'^{-1}_\tau(\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p})$, and conversely the map is determined by this choice of geometric connected component, as $\eta^x$ is required to be equivariant for the $G_{S,T,p} \times A^{\infty,x}_E$-action.

   Indeed, once we fix such a choice, say $\text{Sh}_{K_p}(G_{S,T})_{\bar{F}_p} \times \{t\}$ with $t \in E^{\times,cl}\backslash A^{\infty,x}_E / \mathcal{O}_{E_p}^{\times}$ and other choices would differ by multiplication by some element in $F^{\times,cl}\backslash A^{\infty,x}_E / \mathcal{O}_{E_p}^{\times}$ because $\alpha'^{-1}_\tau(\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p})$ is a torsor under this group. Note that $Y$ (resp. $\text{Sh}_{K_p}(G_{S,T})_{\bar{F}_p}$) can be recovered from $Y^0$ (resp. $\text{Sh}_{K_p}(G_{S,T})_{\bar{F}_p} \times \{t\}$) by applying $- \times \varepsilon_{E_S,T}$ to $\varepsilon_{E_S,T}$, by Proposition 2.10 (1), and it embeds into the product $G_{S,T,p} \times E^{\times,cl}\backslash A^{\infty,x}_E / \mathcal{O}_{E_p}^{\times}$ as follows: the morphism from $\varepsilon_{E_S,T}$ to the first factor is the natural embedding and that to the second factor is given by first projecting to the Galois factor and then apply the Shimura reciprocity map as specified in Subsection 2.7, namely, the element Frobenius$^{p^2}$ is taken to $(p_\sigma^{-1})^{\#S_\infty' - 2\#T}$ if $p$ is inert in $E/F$, and to $(p_\sigma^{-1})^{\#S_\infty' - 2\#T}$ if $p$ splits in $E'/F$. One defines thus $\eta^x$ as the morphism obtained by applying $- \times \varepsilon_{E_S,T}$ to $\varepsilon_{E_S,T}$ to the map

$$
Y^0 \xrightarrow{\eta^x \alpha'^{-1}_\tau} \text{Sh}_{K''_p}(G''_S)_{\bar{F}_p} \xrightarrow{\sim} \text{Sh}_{K_p}(G_{S,T})_{\bar{F}_p} \times \{t\},
$$

where the last isomorphism is the inverse of the restriction of $\alpha'^{-1}_\tau$ to $\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p} \times \{t\}$.

- When $S_\infty' = \emptyset$, a slight rewording is needed. Let $X^0$ denote the preimage under $\pi''$ of the $\bar{F}_p$-point $1 \in \text{Sh}_{K''_p}(G''_S)_{\bar{F}_p}$. So it is mapped under $\eta''$ to a point $g'' \in \text{Sh}_{K''_p}(G''_S)_{\bar{F}_p}$. Let $Y^0$ denote the preimage under $\pi^x$ of the $\bar{F}_p$-point $1 \times 1 \in \text{Sh}_{K_p}(G_{S,T})_{\bar{F}_p} \times \text{Sh}_{K_{E,p}}(T_{E_S,T})_{\bar{F}_p}$.

Then the map $\eta^x$ must take $Y^0$ to a point in $\bar{F}_p$-point in $\alpha'^{-1}_\tau(g'')$, and conversely, $\eta^x$ is determined by this choice of such a point by the same argument as above using the fact that $\eta^x$ is equivariant for the $G_{S,T,p} \times A^{\infty,x}_E$-action.

To sum up, one can always define such a lift $\eta^x$, which is canonical unique up to multiplication by an element of $E^{\times,cl}\backslash A^{\infty,x}_E / \mathcal{O}_{E_p}^{\times}$. However, when we know that $\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p}$ is in fact the neutral connected component $\text{Sh}_{K''_p}(G''_S)_{\bar{F}_p}$ in the former case and $g'' = 1$ in the later case, there is a

\[\text{We point out that } E^{\times,cl}\backslash A^{\infty,x}_E / \mathcal{O}_{E_p}^{\times} \text{ is canonically isomorphic to } \mathcal{O}_E^{\times,cl} \backslash A^{\infty,p,x}.\]
canonical choice of such lift, namely, the neutral connected component $\text{Sh}_{K_p}(G_{S',T})_{\mathbb{F}_p} \times \{1\}$ in the former case and $1 \times 1$ in the latter case. So under this additional hypothesis, we can define a canonical map $\eta^\times$.

**Step II:** Once we have constructed the diagram (2.15.3), we want to obtain a correspondence (which is canonical up to an element of $E_{\times,cl} \backslash A_{E,F}^{\infty,\times} / \mathcal{O}_{E,F}^\times$)

(2.15.4) $\text{Sh}_{K_p}(G_{S,T})_{\mathbb{F}_p,\mathfrak{p}^g} \leftarrow Z \overset{\eta}{\rightarrow} \text{Sh}_{K_p}(G_{S',T})_{\mathbb{F}_p,\mathfrak{p}^g}.
$

For this, it suffices to construct (2.15.4) over $\mathbb{F}_p$ which carries equivariant action of $\text{Gal}_{\mathbb{F}_p,\mathfrak{p}^g}$. Starting with the top row of (2.15.3), composing $\eta^\times$ with multiplication by $t^{-1}$ (note that $\text{Sh}_{K_{E,p}}(T_{E,S',T'})$ is in fact a group scheme), we get a correspondence $\eta^\times$

(2.15.5) $\text{Sh}_{K_p}(G_{S,T})_{\mathbb{F}_p} \times \text{Sh}_{K_{E,p}}(T_{E,S',T'})_{\mathbb{F}_p} \overset{\pi^\times_1 \times \eta}{\rightarrow} \text{Sh}_{K_p}(G_{S',T})_{\mathbb{F}_p,\mathfrak{p}^g} \times \text{Sh}_{K_{E,p}}(T_{E,S',T'})_{\mathbb{F}_p},$

which respects the projection to $\text{Sh}_{K_{E,p}}(T_{E,S',T'})_{\mathbb{F}_p} \cong \text{Sh}_{K_{E,p}}(T_{E,S',T'})_{\mathbb{F}_p}$. Taking the fiber of (2.15.5) over 1 of $\text{Sh}_{K_{E,p}}(T_{E,S',T'})_{\mathbb{F}_p}$ gives (2.15.4), but to descend we need to modify the Galois action above (so that the Galois action preserves the fiber over 1) as follows: we change the action of Frobenius on (2.15.5) by further composing with a Hecke action given by $1 \times (p^1_F)^{2#T-#S_{\infty}} \in G(\mathbb{A}_{\infty}) \times A_{E,F,\mathbb{A}^\times}$ if $\mathfrak{p}$ is inert in $E/F$, and $1 \times (p^1_F)^{2#T-#S_{\infty}}(q)^{-1}A_{\mathbb{A}}$ if $\mathfrak{p}$ splits in $E/F$. This way, by usual Galois descent, we get (2.15.4).

**Step III:** We will obtain a sheaf version of the construction above, namely, if in addition, we are given an isomorphism of sheaves

(2.15.6) $\eta^\times : \pi^\times_*(\mathcal{L}_{S,T}^{(k,w)}) \overset{\cong}{\rightarrow} \eta^\times_*(\mathcal{L}_{S',T'}^{(k,w)})$,

then we will construct an isomorphism of sheaves

(2.15.7) $\eta^\times : \pi^\times_*(\mathcal{L}_{S,T}^{(k,w)}) \overset{\cong}{\rightarrow} \eta^\times_*(\mathcal{L}_{S',T'}^{(k,w)})$,

which again depends on the choice of the connected component in Step I and is canonical up to multiplication by an element of $\mathcal{O}_{E,F,\mathbb{A}^\times}$.

First, pulling back (2.15.6) along $\text{pr}_1^\times$ in the commutative diagram (2.15.3), we get

\[ \alpha^\times_\pi_\pi(\eta^\times) : (\pi^\times)^*(\alpha_\pi^\times_\pi(\mathcal{L}_{S,T}^{(k,w)})) \overset{\cong}{\rightarrow} (\eta^\times)^*(\alpha_\pi^\times_\pi(\mathcal{L}_{S',T'}^{(k,w)})). \]

Taking into account of the isomorphism (2.13.3), we have

\[ \alpha^\times_\pi(\eta^\times) : (\pi^\times)^*(\text{pr}_1^\times(\mathcal{L}_{S,T}^{(k,w)}) \otimes \text{pr}_2^\times(\mathcal{L}_{E,S',T,T'}^{w})) \overset{\cong}{\rightarrow} (\eta^\times)^*(\text{pr}_1^\times(\mathcal{L}_{S',T'}^{(k,w)}) \otimes \text{pr}_2^\times(\mathcal{L}_{E,S',T',\Sigma}^{w})). \]

Composing this with the action of $t^{-1}$, we get a morphism

\[ (\pi^\times)^*(\text{pr}_1^\times(\mathcal{L}_{S,T}^{(k,w)}) \otimes \text{pr}_2^\times(\mathcal{L}_{E,S',T,T'}^{w})) \overset{\cong}{\rightarrow} (t^{-1} \circ \eta^\times)^*(\text{pr}_1^\times(\mathcal{L}_{S',T'}^{(k,w)}) \otimes \text{pr}_2^\times(\mathcal{L}_{E,S',T',\Sigma}^{w})). \]

Since we may also identify the sheaves $\mathcal{L}_{E,S',T,T'}^{w}$ with $\mathcal{L}_{E,S',T',\Sigma}^{w}$ using (2.13.1), we may restrict the morphism above to the fiber over the neutral point 1 and get a morphism of sheaves (2.15.7) we want over $\mathbb{F}_p$. (Once again, this morphism is unique up to multiplication with an element of $E_{\times,cl} \backslash A_{E,F}^{\infty,\times} / \mathcal{O}_{E,F}^{\times}$.) To descend it back down to $\mathbb{F}_p$, we modify the action of the Frobenius by composing it with a central Hecke action as in Step II above. This concludes the needed construction.

**Step IV:** Understand the ambiguity appeared in the construction. We call $\eta$ the morphism associated to $\eta''$ with shift $t^{-1}$, where $t \in E_{\times,cl} \backslash A_{E,F}^{\infty,\times} / \mathcal{O}_{E,F}^{\times}$ is the element appeared in Step I. In particular, when $\text{Sh}_{K''}(G^\times_{\mathbb{F}_p}) = \text{Sh}_{K''}(G^\times_{\mathbb{F}_p})_p$ or $t = 1$ in Step I, the canonically defined $\eta$ has

\[ 11 \text{Once again, both } t \text{ and this correspondence depend on the choice of the geometric connected component, and are uniquely defined up to multiplication by an element of } \mathcal{O}_{E,F,\mathbb{A}^\times} / \mathcal{O}_{E,F,\mathbb{A}^\times}. \]
shift 1. An easy computation shows that if a different geometric connected component in Step I was chosen, the element $t^{-1}$ remembering shift would be changed to $(xt)^{-1}$ for some element $x \in F^\times \backslash A_F^\times / \mathcal{O}_p^\times$, and accordingly, the morphism $\eta$ will be multiplied with $x$.

Finally, let us mention where the choice made in Step I is specified later in this paper. In Subsection 3.7 we invoke this construction to define Goren-Oort cycles; this is where the choice will be fixed. Moreover, this choice will retroactively determine the choice we make when applying this construction to define link morphisms in the earlier Subsection 2.21 whenever this subsection is quoted. The shift will allow us to keep track of the choices we made.

**Remark 2.16.** When we have two correspondences as given in Construction 2.15 that includes: subsets of $S_i, T_i$ for $i = 1, 2, 3$ satisfying the condition that $\#S_{i, \infty} - 2 \#T_i$, the subset of $S_i$ of finite places, and $\Delta_{S_i}$ are independent of $i$; and two $G''_{\delta, p}$-equivariant correspondences between Shimura varieties

$$\text{Sh}_{K_p}(G''_{\delta_1}) \xleftarrow{\pi''_1} X_1 \xrightarrow{\eta_1} \text{Sh}_{K_p}(G''_{\delta_2}) \xrightarrow{\pi''_2} X_2 \xrightarrow{\eta_2} \text{Sh}_{K_p}(G''_{\delta_3})$$

with $i = 1, 2$ such that $\pi''_i$ is a fiber bundle with geometric connected fibers. Then we can compose these two correspondences to get a correspondence

$$\text{Sh}_{K_p}(G''_{\delta_1}) \xleftarrow{\pi''_1} X_3 := X_1 \times_{\eta_1''} \text{Sh}_{K_p}(G''_{\delta_2}). \pi''_2 X_2 \xrightarrow{\eta_2''} \text{Sh}_{K_p}(G''_{\delta_3}).$$

Thus we may apply Construction 2.15 to get correspondences on the quaternionic Shimura varieties:

Then the shift of the correspondence $(\pi_3, \eta_3)$ is the product of the shifts of the correspondence $(\pi_1, \eta_1)$ and $(\pi_2, \eta_2)$.

For the rest of this paper, we always assume $K_p = \text{GL}_2(\mathcal{O}_p)$.

2.17. **Hecke operators at $p$.** When $S'_\infty = \emptyset$, namely when the Shimura varieties are discrete ones, we want to relate the Hecke operators at $p$ for the unitary and quaternionic Shimura varieties in a manner similar to above. We assume that $p$ splits in $E/F$ which is the case if we will only encounter later.

Let $Iw_p \subseteq \text{GL}_2(\mathcal{O}_p)$ denote the subgroup consisting of matrices which are upper triangular when modulo $p$. The discussion in this section is designed to cover this case and give a canonical integral model $Sh_{Iw_p}(G_{S, T})$ of the Shimura variety with Iwahoric level structure. We denote by $T_p$ the Hecke correspondence given by diagram:

$$\text{Sh}_{Iw_p}(G_{S, T}) \xrightarrow{\pi_1} \text{Sh}_{K_p}(G_{S, T}) \xrightarrow{\pi_2} \text{Sh}_{K_p}(G_{S, T})$$

where $\pi_1$ is the natural projection, and $\pi_2$ sends the double coset of $x \in G(\mathbb{A}_F)$ to that of $x(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$.

For the unitary Shimura variety, $G''(\mathbb{Q}_p) = \text{GL}_2(F_p) \times_{F_p} (E_q^\times \times E_q^\times)$ and we use $Iw''_p$ to denote the subgroup $Iw_p \times \mathcal{O}_q \times \mathcal{O}_q^\times (\mathcal{O}_q^\times \times \mathcal{O}_q^\times)$. Similarly, we have an integral model $Sh_{Iw''_p}(G''_{\delta})$ of the unitary Shimura variety with this Iwahori level structure. Consider the element $x''_q = (\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}, (1, p)) \in \mathcal{O}_q^\times \times \mathcal{O}_q^\times$. 


The embedding of $F$ into $G'' = G''(Q_p)$, then it gives rise to a Hecke operator $T_q$ corresponding to the double coset $K''_{Q_p} \gamma''_{q} K''_{P}$. Geometrically, it is given by the following diagram

$$(2.17.2)$$

\[
\begin{array}{ccc}
\text{Sh}_{1w''}^\eta(G''_S) & \xrightarrow{\pi''_1} & \text{Sh}_{1w''}^{\eta'}(G''_S) \\
\text{Sh}_{1w''}^{\eta'}(G''_S) & \xrightarrow{\pi''_2} & \text{Sh}_{1w''}^\eta(G''_S).
\end{array}
\]

where $\pi''_1$ is the natural projection, and $\pi''_2$ sends the double coset of $x \in G''(A^\infty)$ to that of $x \gamma''_{q}$. In a language similar to the previous subsection (except that we cannot quote it directly because the morphism $\pi''$ therein would not have geometric connected fibers), we may phrase the relation between the Hecke correspondences $T_p$ and $T_q$ in terms of the following commutative diagram (with $T_q$ vertical on the left and $T_p$ vertical on the right)

\[
\begin{array}{ccc}
\text{Sh}_{K''_{P}}(G''_S) & \xrightarrow{\alpha} & \text{Sh}_{K''_{P}}(G_{S,T}) \times \text{Sh}_{K_{E,P}}(T_{E,\hat{S},T}) \xleftarrow{\text{fiber over 1}} \text{Sh}_{K_{P}}(G_{S,T}) \\
\text{Sh}_{1w''}^\eta(G''_S) & \xrightarrow{\alpha} & \text{Sh}_{1w''}^{\eta'}(G''_S) \times \text{Sh}_{1w''}^{\eta''}(T_{E,\hat{S},T}) \xleftarrow{\text{fiber over 1}} \text{Sh}_{1w''}(G''_S) \\
\text{Sh}_{K''_{P}}(G''_S) & \xrightarrow{\alpha} & \text{Sh}_{K''_{P}}(G_{S,T}) \times \text{Sh}_{K''_{P}}(T_{E,\hat{S},T}) \xleftarrow{\text{fiber over 1}} \text{Sh}_{K''_{P}}(G_{S,T}).
\end{array}
\]

So we may view $T_p$ as the correspondence associated to $T_q$ similar to the previous subsection, with shift $\varpi^{-1} \in E_{\times,cl} \backslash A_{E,\times}^{\infty}/O_{E_p}^{\times}$.

2.18. Links. We recall briefly the notion of links introduced in [TX13+a, §7]. First, put $d$ points aligned equi-distant on a section of a vertical cylinder, one point corresponding to an archimedean embedding of $F$ (also identified with a $p$-adic embedding of $F$ via $\iota_p : C \cong \overline{Q}_p$) so that the Frobenius action is equivalent to shifting the points to the right. For a subset $S$ of places of $F$ as above, we label places in $S_{\infty}$ by a plus sign and places in $S_{\infty}^c$ by a node. We call the entire picture a band corresponding to $S$. We often draw the picture in a 2-dimensional plane, by presenting the points $\gamma_0, \ldots, \gamma_{d-1}$ on a horizontal line. For example, if $F$ has degree 6 over $Q$ and $S_{\infty} = \{\tau_1, \tau_3, \tau_4\}$, then we draw the band as $\bullet + + + \bullet$.

Suppose that $S'$ is another even set of places of $F$ containing exactly the same finite places of $F$ as $S$ and satisfying $\#S_{\infty} = \#S'_{\infty}$. A link $\eta : S \to S'$ is a picture of the following kind. Put the band for $S$ on the top of the band for $S'$ in the same cylinder; draw non-intersecting curves linking each of the nodes from the top band to nodes from the bottom band (and ignore the plus signs). We say a curve is turning to the left (resp. right) if it is so when moving from the top band to the bottom band, in this 2-dimensional picture. The displacement of a curve in $\eta$ is the number of points it travels to the right; it is negative if the curve turns to the left. The total displacement $v(\eta)$ is the sum of displacements of all curves. For example, if $g = 5$, $S_{\infty} = \{\tau_1, \tau_3\}$ and $S'_{\infty} = \{\tau_1, \tau_4\}$, we have a link given by

\[
\eta = \bullet + + + \bullet.
\]

Its total displacement is $v(\eta) = 3 + 3 + 2 = 8$. For another example, the action of Frobenius $\sigma$ twists the band and gives rise to a link $\sigma : S \to \sigma(S)$ for which every curve is turning to the right.
with displacement 1, where $\sigma(S)$ is the set of places containing the same finite places as $S$ but $\sigma(S)_{\infty} = \sigma(S_{\infty})$. Its total displacement is $v(\sigma) = g = \#S_{\infty}$.

For a link $\eta : S \to S'$, we use $\eta^{-1} : S' \to S$ to denote the link obtained by flipping the picture about the equator of the cylinder. For two links $\eta : S \to S'$ and $\eta' : S' \to S''$, we have a natural composition of links $\eta' \circ \eta : S \to S''$ given by putting the picture of $\eta$ on top of the picture of $\eta'$ and joint the nodes corresponding to $S'$. It is obvious that $v(\eta^{-1}) = -v(\eta)$ and $v(\eta' \circ \eta) = v(\eta') + v(\eta)$.

When discussing the relative positions of two nodes of the band associated to $S$, it is convenient to use the following

**Notation 2.19.** For $\tau \in S^c_{\infty}$, let $n_\tau$ be the minimal positive integer such that $\sigma^{-n_\tau} \tau \in S^c_{\infty}$. We put $\tau^- = \sigma^{-n_\tau} \tau$. We use $\tau^+$ to denote the place in $S^c_{\infty}$ such that $(\tau^-)^+ = \tau$.

**Example 2.20.** A link from $S$ to itself can only be an integer power of the fundamental link $\eta_S$, that is to link each $\tau$ to $\tau^+$ by slightly shifted to the right. For example, $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$. The total displacement of a fundamental link is exactly $v(\eta_S) = g = [F : \mathbb{Q}]$.

### 2.21 Link morphisms I.

Let $S$ and $S'$ be two even subsets of places of $F$ consisting of the same finite places and $\#S_{\infty} = \#S'_{\infty}$. Let $\eta : S \to S'$ be a link such that all the curves (if there is any) are turning to the right. We allow the case $S = \Sigma$ (so that there are no curves in the link $\eta$ at all); in that case we say $\eta$ is the trivial link. For each node $\tau \in S^c_{\infty}$, we use $m(\tau)$ to denote the displacement of the curve connected to $\tau$ in $\eta$. Let $S_{\infty}$ and $S'_{\infty}$ be (any) lifts of $S_{\infty}$ and $S'_{\infty}$ as in Subsection 2.7. We have two unitary Shimura varieties $\text{Sh}_{K_{\eta}''}(G''_{S})$ and $\text{Sh}_{K_{\eta}'}(G''_{S'})$ as defined in [2.8]. Since $S$ and $S'$ contain the same finite places so that we can fix an isomorphism $G''_{S}(A_{\infty}) \cong G''_{S'}(A_{\infty})$, and denote them commonly by $G''(A_{\infty})$.

We now recall the definition of the link morphism on $\text{Sh}_{K_{\eta}''}(G''_{S})$ associated to $\eta$ as introduced in [TX13+a] Definition 7.5]. Let $n$ be an integer, which is always taken to be 0 if $p$ is inert in $E$. A link morphism of indentation degree $n$ associated to $\eta$ on $\text{Sh}_{K_{\eta}''}(G''_{S})$ is a pair $(\eta_{(n), z}^{\prime\prime}, \eta_{(n)}^{\prime\prime})$, where

1. $\eta_{(n), z}^{\prime\prime} : \text{Sh}_{K_{\eta}''}(G''_{S}) \to \text{Sh}_{K_{\eta}''}(G''_{S'})$ is a morphism of Shimura varieties that induces a bijection on closed points;
2. $\eta_{(n)}^{\prime\prime} : A''_{S} \to \eta_{(n), z}^{\prime\prime}(A''_{S'})$, is a $p$-quasi-isogeny of abelian varieties compatible with the $O_D$-actions, the polarizations and the level structures such that the following conditions hold:
   a. For each geometric point $x$ of $\text{Sh}_{K_{\eta}''}(G''_{S})$ with image $x' = \eta_{(n), x}^{\prime\prime}(x)$, we use $\tilde{D}(A''_{S}, x')$ to denote the $\tilde{x}$-component of the covariant Dieudonné module of $A''_{S,x}$ for each $\tilde{x} \in \Sigma_{E, \infty}$. If $S_{\infty} \neq \Sigma_{\infty}$, then there exists, for each $\tilde{x} \in S'_{E, \infty}$, some $t_{\tilde{x}} \in \mathbb{Z}$ independent of the point $x$ such that
      $$\tilde{\eta}_{(n), x}^{\prime\prime}(F_{es,A''_{S,x}}(\tilde{D}(A''_{S,x}), \tilde{x})) = p^{t_{\tilde{x}}} \tilde{D}(A''_{S,x},x') \sigma_{m(\tau)}(\tilde{x}),$$
      where $F_{es,A''_{S,x}} : \tilde{D}(A''_{S,x}) \to \tilde{D}(A''_{S,x}, \sigma_{m(\tau)}(\tilde{x})$ is $m(\tau)$-th iteration of the essential Frobenius for $A''_{S,x}$ defined in [TX13+a] §4.2]. If $S_{\infty} = \Sigma_{\infty}$, we require the same condition holds for all $\tilde{x} \in \Sigma_{E, \infty}$ with $m(\tau) = 0$.
   b. When $p$ splits as $qq$ in $E$, the degree of the quasi-isogeny
      $$\eta_{(n), q}^{\prime\prime} : A''_{S[q]\infty} \to \eta_{(n), z}^{\prime\prime}(A''_{S'[q]\infty})$$
      of the $q$-divisible groups is $p^{2n}$.

When the indentation degree $n$ is clear in the context, we write simply $(\eta_{\tilde{x}}(\tilde{x})$ for $(\eta_{(n), z}^{\prime\prime}, \eta_{(n)}^{\prime\prime})$.

For our purpose, the most important property we need is the following
Lemma 2.22 ([TX13+a, Proposition 7.8]). Let \( \eta : S \to S' \) be a link as above. Then there exists at most one link morphism \((\eta'_{(n)\sharp}, \eta''_{(n)})\) with indentation degree \( n \) on \( \text{Sh}_{K''}(G''_{\mathfrak{S}}) \).

Example 2.23. Let \( S \) and \( \mathfrak{S} \) be as in the previous subsection. Let \( \sigma^2 : S \to \sigma^2(S) \) be the second iteration of the Frobenius link on \( S \). Put \( \sigma^2 \mathcal{S} = (\sigma^2(S), \sigma^2(\mathcal{S}_\infty)) \). In [TX13+a, §3.22], we introduced natural morphisms called the twisted (partial) Frobenius

\[
\mathfrak{F}''_{\mathfrak{S}^2} : \text{Sh}_{K''}(G''_{\mathfrak{S}}) \to \text{Sh}_{K''}(G''_{\mathfrak{S}^2\mathfrak{S}})
\]
together with a quasi-isogeny of abelian varieties

\[
\eta''_{\mathfrak{S}^2} : A''_{\mathfrak{S}} \to (\mathfrak{F}''_{\mathfrak{S}^2})^* A''_{\mathfrak{S}^2\mathfrak{S}}.
\]

Such a morphism is characterized by the fact that the morphism \( \eta''_{\mathfrak{S}^2} \) is given by the twice iteration of the relative Frobenius. Then in the language of link morphism introduced above, \((\mathfrak{F}''_{\mathfrak{S}^2}, \eta''_{\mathfrak{S}^2})\) is the link morphism on \( \text{Sh}_{K''}(G''_{\mathfrak{S}}) \) associated to the link \( \eta = \sigma^2 \) of indentation degree 0 if \( p \) is inert in \( E/F \) and of indentation degree \( 2\Delta_{\mathfrak{S}} \) if \( p \) splits in \( E/F \). (See [TX13+a Example 7.7(1)]).

Example 2.24. When \( p \) splits into \( q \bar{q} \) in \( E/F \), consider the Hecke operator \( S_q \) given by multiplication by \( 1 \times q^{-1} \in G''(\mathbb{A}^\infty) = G(\mathbb{A}^\infty) \times \mathbb{A}_E^\times \mathbb{A}_E^\times \) on the unitary Shimura variety. We start with the versal family of abelian varieties \( A''_{\mathfrak{S}} \) on \( \text{Sh}_{K''}(G''_{\mathfrak{S}}) \), and put \( B = A''_{\mathfrak{S}} \otimes_{E,F} q : \mathfrak{q}^{-1} \) equipped with the induced action by \( O_D \). Let \( \phi_q : A''_{\mathfrak{S}} \to B \) denote the natural \( p \)-quasi-isogeny induced by \( O_E \to \mathfrak{q} \bar{q}^{-1} \). We equip \( B \) with the natural prime-to-\( p \) level structure compatible with \( \phi_q \). The polarization \( \lambda_{A''_{\mathfrak{S}}} \) on \( A''_{\mathfrak{S}} \) naturally induces a polarization \( \lambda_B \) on \( B \) such that \( \lambda_{A''_{\mathfrak{S}}} = \phi_q^* \circ \lambda_B \circ \phi_q \).

There is a unique morphism

\[
S_q : \text{Sh}_{K''}(G''_{\mathfrak{S}}) \to \text{Sh}_{K''}(G''_{\mathfrak{S}}),
\]

which, together with \( \phi_q \), gives a link morphism for the trivial link \( \text{id} : S \to S \) of indentation degree \( 2g \).

If we apply Construction 2.15 to the morphism \( S_q \) (with the morphism \( \pi \) there being trivial), we can lift it to an endomorphism of \( \text{Sh}_{K''}(G_{S,T}) \times \text{Sh}_{K,E,p}(T_{\mathfrak{S},T}) \) given by multiplication by \((p_F)^{-1} \times \mathfrak{w}_q^{-2} \in G(\mathbb{A}^\infty) \times \mathbb{A}_E^\times \mathbb{A}_E^\times \). So the endomorphism \( S_p \) given by multiplication by central element \((p_F)^{-1} \) may be viewed as the morphism on the quaternionic Shimura variety obtained by applying Construction 2.15 to the morphism \( \eta'' = S_q \) with shift \( \mathfrak{w}_q^2 \).

2.25. Normalizations of link morphism. Keep the notation of Subsection 2.21 and assume that

- the link morphism \((\eta''_{(n)\sharp}, \eta''_{(n)})\) on \( \text{Sh}_{K''}(G''_{\mathfrak{S}}) \) exists,
- \( \Delta_{\mathfrak{S}} = \Delta_{\mathfrak{S}^2} \) if \( p \) splits in \( E/F \), and
- we are given two subsets \( T \subseteq S_\infty \) and \( T' \subseteq S'_\infty \) such that \#T = \#T'.

Applying Construction 2.15 to the link morphism \((\eta''_{(n)\sharp}, \eta''_{(n)})\) (with \( X = \text{Sh}_{K''}(G''_{\mathfrak{S}}) \) and \( \pi'' \) in (2.15.2) equal to the identity), we get a pair of morphisms

\[
\eta_{(n)\sharp} : \text{Sh}_{K''}(G_{S,T}) \to \text{Sh}_{K''}(G_{S',T'}) \quad \text{and} \quad \eta_{(n)} : \mathcal{L}_{S,T}^{(k,w)} \to \eta_{(n)\sharp} \mathcal{L}_{S',T'}^{(k,w)},
\]

with some shift \( t \in E^\times, \mathfrak{w}_E^\infty / O_{E,p}^\times \) (See Construction 2.15 Step IV). In the sequel, we call \((\eta_{(n)\sharp}, \eta_{(n)})\) (or simply \( \eta_{(n)\sharp} \) for short) the link morphism with indentation degree \( n \) on the quaternionic Shimura variety \( \text{Sh}_{K''}(G_{S,T}) \) with shift \( t \). Note that by Lemma 2.22 and Construction 2.15 for a fixed lifting \( \bar{S} \) of \( S \), indentation degree \( n \) and shift \( t \), there exists at most one link morphism \((\eta_{(n)\sharp}, \eta_{(n)})\) on \( \text{Sh}_{K}(G_{S,T}) \).
The link morphism \((\eta(n), \eta')\) induces a homomorphism of the cohomology groups:

\[
\tilde{\eta}(n) : H_c^*(Sh_{K_p}(G_{S', T})_{\mathbb{F}_p}, L^{(k, w)}_{S', T}) \rightarrow H_c^*(Sh_{K_p}(G_{S, T})_{\mathbb{F}_p}, \eta(n), \eta(L^{(k, w)}_{S', T}))
\]

\[
\frac{\eta(n)^{-1}}{H_c^*(Sh_{K_p}(G_{S, T})_{\mathbb{F}_p}, L^{(k, w)}_{S, T})}
\]

which is equivariant under the Hecke action by \(G(\mathbb{A}^\infty)\)\(^{12}\) and the Galois action by \(\text{Gal}_{\mathbb{F}_p^\infty}\). We fix a square root \(p^{1/2} \in \mathbb{Q}_\ell\) of \(p\). We put

\[
\eta^*_n = \frac{1}{p^\nu(q)/2} \tilde{\eta}(n);
\]

and call it the \textit{normalized link morphism} on the cohomology groups of quaternionic Shimura varieties associated to \(\eta\) with indentation degree \(n\) and shift \(t\). This normalization will be justified in Lemma \ref{lem:2.29}(2). We point out one more time that the (normalized) link morphism depends on the choice of the geometric connected component as in Construction \ref{constr:2.15} Step I, or equivalently the shift \(t\). When the link morphism \(\eta^*_n : Sh_{K_p}(G'_{S}) \rightarrow Sh_{K_p}(G'_{S})\) preserves the neutral connected components, \(t = 1\) is a canonical choice; in that case, \(\eta^*_n\) is canonically defined.

Let \(\eta_1 : S_1 \rightarrow S_2\) and \(\eta_2 : S_2 \rightarrow S_3\) be two links with all curves turning to the right, satisfying the conditions above, i.e. all \(S_i\) have the same set of finite places, \(#S_1, = #S_2, = #S_3,\), \(#T_1 = #T_2 = #T_3\), and \(\Delta_{S_1} = \Delta_{S_2} = \Delta_{S_3}\) if \(p\) splits in \(E/F\). Suppose that there are link morphisms \((\eta^*_{i_{(n_1)}}, \eta^*_{i_{(n_2)}})\) for \(i = 1, 2\) on unitary Shimura varieties with indentation degree \(n_i\).

Then the composed map

\[
\eta^*_{12,(n_12)} : Sh_{K_p}(G'_{S_1}) \xrightarrow{\eta^*_{1(n_1), 12}} Sh_{K_p}(G'_{S_2}) \xrightarrow{\eta^*_{2(n_2), 12}} Sh_{K_p}(G'_{S_3})
\]

together with the composed quasi-isogeny

\[
\eta^*_{12,(n_12)} : A'_{\mathbb{A}_S} \xrightarrow{\eta^*_{1(n_1), 12}} A'_{\mathbb{A}_S} \xrightarrow{\eta^*_{2(n_2), 12}} A'_{\mathbb{A}_S}
\]

gives the (unique) link morphism on the unitary Shimura variety with indentation degree \(n_{12} = n_1 + n_2\) associated to the composed link \(\eta^*_{12,(n_12)} := \eta^*_{2(n_2), 1} \circ \eta^*_{1(n_1), 12}\). From this, we get a link morphism of quaternionic Shimura varieties of indentation degree \(n_{12}\):

\[
\eta_{12,(n_12)} : Sh_{K_p}(G_{S_1, T}) \xrightarrow{\eta_{1(n_1), 12}} Sh_{K_p}(G_{S_2, T_2}) \xrightarrow{\eta_{2(n_2), 12}} Sh_{K_p}(G_{S_3, T_3}),
\]

such that the shift of \(\eta_{12,(n_12)}\) is the product of the shifts of \(\eta_{1(n_1), 12}\) and \(\eta_{2(n_2), 12}\). Moreover, we have \(\eta^*_{12,(n_12)} = \eta^*_{1(n_1), 12} \circ \eta^*_{2(n_2), 12}\) on the cohomology groups of quaternionic Shimura varieties.

\textbf{Proposition 2.26.} Let \(\pi \in \mathcal{A}_{(k, w)}\) be an automorphic representation appearing in the cohomology of the Shimura variety \(Sh_K(G_{S, T})\). Then we have

\[
H_c^*(Sh_K(G_{S, T})_{\mathbb{F}_p}, L^{(k, w)}_{S, T})[\pi] = \begin{cases} \rho_{\pi,d}^\otimes \otimes \det(\rho_{\pi,d})(1)^{\otimes \#T} & \text{if } i = d, \\ 0 & \text{if } i \neq d; \end{cases}
\]

it is equivariant, up to semi-simplicification, for the action of geometric Frobenius \(\text{Frob}_{p^g}\). Explicitly, if \(\alpha_\pi, \beta_\pi\) are the two eigenvalues eigenvalues of \(\rho_{\pi}(\text{Frob}_{p^g})\), then the (generalized) eigenvalues of the action of \(\text{Frob}_{p^g}\) on \((A.3.1)\) are \(p^{2g\#T} \alpha_\pi^{2(i+\#T)} \beta_\pi^{2(d-i+\#T)}\) with multiplicity \(\left(\frac{d}{i}\right)\) for \(0 \leq i \leq d\).

\textbf{Proof.} The first part of the proposition is well known to experts; we defer its proof to the Appendix (see Proposition \ref{prop:A.3}). The explicit description of the action of \(\text{Frob}_{p^g}\) is straightforward. \(\square\)

\textbf{Proposition 2.27.} Assume that \(d = \#S_\infty \neq 0\).

\(^{12}\)Here, \(G(\mathbb{A}^\infty)\) denotes the common finite adelic points of \(G_{S, T}\) and \(G_{S', T}\) according to Notation \ref{not:2.3}.
The link morphism on $\text{Sh}_K(G''_S)$ with indentation degree 0 associated to the link $\eta$, Statement (1) is evident. For (2), we first check easily that the maps given by (a) and (b) are link morphisms with indentation degree 0 associated to the link $\eta$.

Moreover, this link morphism preserves the neutral geometric connected component $\text{Sh}_{K''}(G''_S)^0_F$ and hence induces a canonical link morphism $(\eta^{2d}_S,\eta^{2d},\eta^{2d},\eta^{2d})$ on the quaternionic Shimura variety $\text{Sh}_{K}(G_{S,T})$ with shift 1 for any fixed subset $T \subset S_{\infty}$.

(3) Let

$$(\eta^{2d}_S)^*(0) : H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\overline{F}}, L^{(k,w)}_{S,T}) \to H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\overline{F}}, L^{(k,w)}_{S,T})$$

be the normalized link morphism (2.25.1) induced by $(\eta^{2d}_S,\eta^{2d},\eta^{2d})$. Then we have an equality of operators on cohomology groups:

$$(\eta^{2d}_S)^* = p^{-d} \cdot \text{Frob}_{p^{2g}} \circ S_p^{-d-2\#T},$$

where $S_p$ is the Hecke operator given by central element $\frac{1}{2} \in G(\mathbb{A}_{\infty})$. In particular, for each $\pi \in \mathcal{O}_{(k,w)}$ and each integer $i$ with $0 \leq i \leq d$, $(\eta^{2d}_S)^*$ acts on the (generalized) eigenspace of $\text{Frob}_{p^{2g}}$ on $H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\overline{F}}, L^{(k,w)}_{S,T})[\pi]$ with eigenvalue $p^{-2g\#T} \alpha^2(2i+\#T) / \beta^2(2i+\#T)$ is also a (generalized) eigenspace of $(\eta^{2d}_S)^*$ with eigenvalue $(\alpha / \beta)^2 i^2$. 

**Proof.** Statement (1) is evident. For (2), we first check easily that the maps given by (a) and (b) are link morphisms with indentation degree 0 associated to the link $\eta^{2d}$. This follows easily from Examples 2.23 and 2.24. By the uniqueness of link morphisms (Lemma 2.22), they are the link morphisms we sought for.

We next show that the link morphism in the unitary case preserves the neutral geometric connected component $\text{Sh}_{K''}(G''_S)^0_F$. This is a direct computation using the Shimura reciprocity map (in Subsection 2.8), which we spell out now. Denote by $\Phi^{2g}$ the Frobenius endomorphism of $\text{Sh}_{K''}(G''_S)$ relative to $\overline{F}_{p^{2g}}$. Then $(\eta^{2d}_p)^*$ is nothing but the composition of $\Phi^{2g}$ with the Hecke operator $S_p^{-g}$, where $S_p$ is the Hecke correspondence given by the central element $(\frac{1}{2}, 1) \in G(\mathbb{A}_{\infty}) \times \mathbb{A}_{\infty} \times \mathbb{A}_{\infty} \times \equiv G''(\mathbb{A}_{\infty})$. Recall that the set of geometric connected components of $\text{Sh}_{K''}(G''_S)$ is given by

$$\pi_0(\text{Sh}_{K''}(G''_S)_{/\overline{F}}) \cong \left( F^{e,\text{cl}}_+ \big/ \mathbb{A}_{F}^{\infty,\times} \big/ \mathbb{O}_{F}^{\times} \right) \times \left( E^{*,N_{E/F}=1,\text{cl}}_{\mathbb{A}_{E}}^{\infty,N_{E/F}=1} \big/ \mathbb{O}_{E_{F}}^{N_{E/F}=1} \right).$$

The action of $\Phi^{2g}$ on $\pi_0(\text{Sh}_{K''}(G''_S)_{/\overline{F}})$ coincides with the arithmetic Frobenius $\text{Frob}_{p^{2g}}^{-1} \in \text{Gal}_{p^{2g}}$, which is computed already by 2.8.2. We now list the actions of these operators on the geometric connected components.

| Operator | When $p$ splits | When $p$ is inert |
|----------|-----------------|------------------|
| $\Phi^{2g}$ | $(p_F)^{-2g} \times (q)^{-2d}$ | $(p_F)^{-2g} \times 1$ |
| $S_p$ | $(p_F)^{-2} \times 1$ | $(p_F)^{-2} \times 1$ |
| $S_q$ | $1 \times q^{-2}$ | $N/A$ |

It is now clear that the link morphisms given in (1) and (2) preserve the neutral geometric connected component.

We now turn to the proof of (3). We compute first the canonical lift of our link morphism on $\text{Sh}_K(G_{S,T}) \times \text{Sh}_K(E_{S,T})$ appearing in Construction 2.15 Step I (and the Step II there is trivial as the shift is trivial in our case). It is clearly a composition of the Frobenius endomorphism relative
to $\mathbb{F}_{p^{2g}}$, which we denote by $\Phi^{2g}$, and the action of a Hecke operator given by a central element $x$ in $G(\mathbb{A}_F) \times \mathbb{A}_E^\times$. This central element $x$ is characterized by (and uniquely determined by) the fact that

(a) the resulting link morphism on $\text{Sh}_{K_p}(G_S,T) \times \text{Sh}_{K_E,T}(T_{E,S},T)$ preserves the neutral connected component, and

(b) under the natural projection $G(\mathbb{A}_F) \times \mathbb{A}_E^\times \rightarrow G(\mathbb{A}_F) \times \mathbb{A}_E^\times \mathbb{A}_E^\times \simeq G^d(\mathbb{A}_F)$, $x$ is mapped to the central element $((p_F)^\beta, 1)$ if $p$ is inert in $E/F$ and to $((p_F)^\beta, q^\Delta)$ if $p$ splits in $E/F$.

Taking into account of the action of $(p_F)^\beta$ on the geometric connected component, according to Shimura reciprocity law in Subsections 2.2 and 2.7, is given by

$$S_{\sigma} := \eta \circ \sigma^{2g N} \circ S_{\sigma} \circ \eta : S \rightarrow S$$

is the fundamental link for $S \rightarrow S$ on the $\pi$-component by the scalar $\omega_{\pi}(p^{-1}) = \alpha_{\pi} \beta_{\pi}/p^d$. Now statement (3) follows immediately from the following easy computation:

$$p^{-dg} \times p^{-2gB+1+iR} \beta_{\pi}^{2(d-i+R)} \times \left(\alpha_{\pi} \beta_{\pi}/p^d\right)^{-(2+2R)} = \left(\alpha_{\pi} \beta_{\pi}\right)^{2i-d}.$$



2.28. Link morphisms II. If $\eta : S \rightarrow S'$ is a general link without the assumption that all curves of $\eta$ are turning to the right, there exists an integer $N \geq 0$ such that the composition of the links $\xi := \eta \circ \sigma^{2g N} \circ \eta : S \rightarrow S'$ satisfies the assumption, where $\eta_{\mathcal{G}}$ is the fundamental link for $S \rightarrow S$ of (2.20). Suppose that the link morphism on $\text{Sh}_{K_p}(G_{S'}^d \mathcal{G})$ associated to $\xi$ with indentation degree $n$ exists. Then we put, for each $\pi \in \mathcal{G}(\mathbb{k}, w)$,

$$\eta^*_{(n)} : H^d_{\text{et}}(\text{Sh}_{K_p}(G_{S'}, T_{E,S'}), \mathcal{L}_{G_{S'}, T_{E,S'}}^{(k,w)}) / [\pi] \xrightarrow{((\eta^*_{(2g)})_{(0)})^{-N}} H^d_{\text{et}}(\text{Sh}_{K_p}(G_{S'}, T_{E,S'}), \mathcal{L}_{G_{S'}, T_{E,S'}}^{(k,w)}) / [\pi] \xrightarrow{\xi_{(n)}} H^d_{\text{et}}(\text{Sh}_{K_p}(G_{S'}, T_{E,S'}), \mathcal{L}_{G_{S'}, T_{E,S'}}^{(k,w)}) / [\pi],$$

and call it the normalized link morphism on the cohomology group of quaternionic Shimura varieties associated to $\eta$ with indentation degree $n$. Here the link morphism $(\eta_{G_{S'}}^d)_{(0)}$ is taken to be the canonical one, so that it is invertible by Proposition 2.27. The shift of $\eta^*_{(n)}$ is defined to be the same as that of $\xi^*_{(n)}$ (as $(\eta^*_{G_{S'}}^d)_{(0)}$ has shift 1). By Lemma 2.22 on the uniqueness of link morphism, this definition does not depend on the choice of $N$ (but on the shift of $\xi_{(n)}$) and is compatible with compositions.

Lemma 2.29. (1) For any link $\eta : S \rightarrow S'$, there exist an integer $N > 0$ and another link $\xi : S' \rightarrow S$ turning to the right such that $\xi \circ \eta : S \rightarrow S$ is the same as $\sigma^{2g N} : S \rightarrow S$. 

(2) If \( \eta : S \to S' \) is a link with all curves turning to the right, and the link morphism \( (\eta''_{(n)}, \eta''_{(n)}) \) on \( \text{Sh}_{K_p}(G_\mathbb{S}''_\mathbb{S}) \) with indentation degree \( n \) associated to \( \eta \) exists, then there exists \( N > 0 \) such that the link morphism associated to \( \eta^{-1} \circ (\eta''_{(n)})^N : S' \to S \) of indentation \( -n \) exists.

(3) Let \( \eta : S \to S' \) be the link as in (2), and \( \eta''_{(n)} : \text{Sh}_{K_p}(G_\mathbb{S}) \to \text{Sh}_{K_p}(G_\mathbb{S}') \) be the link morphism with some shift \( t \) obtained by applying Construction 2.15 to \( \eta''_{(n)} \). If \( \eta^{-1} : S' \to S \) denotes the inverse link, then the morphism

\[
(\eta^{-1})_{(-n)}^d : H^d_{\text{et}}(\text{Sh}_{K_p}(G_\mathbb{S}'), \mathcal{L}^{(k,w)}_{G_\mathbb{S}'}) \longrightarrow H^d_{\text{et}}(\text{Sh}_{K_p}(G_\mathbb{S}), \mathcal{L}^{(k,w)}_{G_\mathbb{S}})
\]

with some shift \( t^{-1} \) is the same as the inverse of \( \eta^* \), which has shift \( t \). Moreover, if \( \eta''_{(n), \sharp} \) (or equivalently \( \eta^-_{(n), \sharp} \)) is finite flat of degree \( p^{v(\eta)} \), where \( v(\eta) \) denotes the total displacement of \( \eta \), we have also \( (\eta^{-1})^*_(-n) = p^{-v(\eta)/2} \text{Tr}_{\eta_{(n)}, \sharp} \), where \( \text{Tr}_{\eta_{(n)}, \sharp} \) is the trace map on cohomology induced by the finite flat morphism \( \eta_{(n), \sharp} \).

**Proof.** (1) is obvious. For (2), we may first find \( N \) so that \( \xi := \eta^{-1} \circ (\eta''_{(n)})^N \) has all curves turning to the right. Then we consider the two morphisms

\[
\text{Sh}_{K_p}(G_\mathbb{S}'') \xrightarrow{\eta''_{(n)}} \text{Sh}_{K_p}(G_\mathbb{S}') \xleftarrow{\eta''_{(n), \sharp}} \text{Sh}_{K_p}(G_\mathbb{S}'')
\]

Since the link morphism \( \eta_{(n), \sharp} \) induces a bijection on the closed points, [He12, Proposition 4.8] implies that after possibly enlarging \( N \), the map \( (\eta''_{(n), \sharp})^N \) factors through \( \eta_{(n), \sharp} \), as \( \eta_{(n), \sharp} \circ \xi \). It is easy to see that \( \xi \) gives the required link morphism.

The first part of (3) follows from the uniqueness of link morphism (Lemma 2.22). For the second part of (3), note that the composed morphism

\[
H^d_{\text{et}}(\text{Sh}_{K_p}(G_\mathbb{S}'), \mathcal{L}^{(k,w)}_{G_\mathbb{S}'}) \xrightarrow{p^{v(\eta)/2} \eta^{-1}} H^d_{\text{et}}(\text{Sh}_{K_p}(G_\mathbb{S}), \mathcal{L}^{(k,w)}_{G_\mathbb{S}}) \xrightarrow{\text{Tr}_{\eta_{(n)}, \sharp}} H^d_{\text{et}}(\text{Sh}_{K_p}(G_\mathbb{S}''), \mathcal{L}^{(k,w)}_{G_\mathbb{S}'})
\]

is nothing but the multiplication by \( p^{v(\eta)} \), according to our normalization of \( \eta^* \) (2.25.1). It follows immediately that \( (\eta^{-1})^*_(-n) = p^{-v(\eta)/2} \text{Tr}_{\eta_{(n)}, \sharp} \).

\[ \square \]

2.30. **Goren-Oort divisors.** We recall the definition of Goren-Oort stratification from [TX13+a, Section 4]. We will make essential use of the case of divisors. Let \( \text{Sh}_{K_p}(G_\mathbb{S}) \) be the special fiber a quaternionic Shimura variety of the type as in Subsection 2.2. We fix throughout this paper a choice of lifting \( \mathbb{S}_\infty \) of \( \mathbb{S}_\infty \) and let \( \text{Sh}_{K_p}(G_\mathbb{S}''_\mathbb{S}) \) be the associated unitary Shimura variety.

In [TX13+a, Definition 4.6 and §4.9], we defined, for each \( T \in \mathbb{S}_\infty \), the Goren-Oort divisor \( \text{Sh}_{K_p}(G_\mathbb{S}''_\mathbb{S}_T) \) of \( \text{Sh}_{K_p}(G_\mathbb{S}''_\mathbb{S}) \) at \( T \) as the vanishing locus of the \( T \)-th partial Hasse invariant of the versal family \( \mathbb{A}_\mathbb{S}'' \). Each \( \text{Sh}_{K_p}(G_\mathbb{S}''_\mathbb{S}_T) \) is projective and smooth by [TX13+a, Proposition 4.7]. Transferring these structures to the quaternionic Shimura varieties using Proposition 2.10 one gets a Goren-Oort divisor \( \text{Sh}_{K_p}(G_\mathbb{S})_T \) on \( \text{Sh}_{K_p}(G_\mathbb{S}) \) for each \( T \in \mathbb{S}_\infty \). When \( T = \emptyset \), this is done in [TX13+a, 4.9], and the general case is the same.

For a subset \( J \subset \mathbb{S}_\infty \), we put \( \text{Sh}_{K_p}(G_\mathbb{S}_T) = \cap_{T \in J} \text{Sh}_{K_p}(G_\mathbb{S})_T \) and \( \text{Sh}_{K_p}(G_\mathbb{S}'')_J = \cap_{T \in J} \text{Sh}_{K_p}(G_\mathbb{S}'')_T \). The closed subvarieties \( \text{Sh}_{K_p}(G_\mathbb{S}_T)_J \) (resp. \( \text{Sh}_{K_p}(G_\mathbb{S}'')_J \)) with \( J \) running through the subsets of \( \mathbb{S}_\infty \}) form the Goren-Oort stratification of \( \text{Sh}_{K_p}(G_\mathbb{S})_T \) (resp. \( \text{Sh}_{K_p}(G_\mathbb{S}'')_T \)).

The main results of [TX13+a] give an explicit description of all closed Goren-Oort strata \( \text{Sh}_{K_p}(G_\mathbb{S}'')_J \) (resp. \( \text{Sh}_{K_p}(G_\mathbb{S})_J \)) as an iterated \( \mathbb{P}^1 \)-bundle over another unitary (resp. quaternionic) Shimura variety of the same type. We list results from [TX13+a] that we will make use in this paper. (One more result will be used later in proving Lemma 5.12.)
Proposition 2.31. Let \( \tau \in S^c_\infty \). Assume that \( \tau^- = \sigma^{-\tau} \) is different from \( \tau \) (See [2.19 for the notation). We put \( S_\tau = S \cup \{ r, r^- \} \) and \( T_\tau = T \cup \{ r \} \). Let \( \tilde{S}_{\tau,\infty} \) be the lifting of \( S_{\tau,\infty} \) derived from \( \tilde{S}_\infty \) according to the rule of [TX13+a], and put \( \tilde{S}_\tau = (S_\tau, \tilde{S}_{\tau,\infty}) \); in particular, \( \Delta_{\tilde{S}_\tau} = \Delta_{\tilde{S}} \) when \( p \) is inert in \( E/F \).

(1) There exist a \( \mathbb{P}^1 \)-bundle fibration

\[
\pi''_\tau : \text{Sh}_{K^p}(G''_\delta)_\tau \to \text{Sh}_{K^p}(G''_{\tilde{S}_\tau})_\tau
\]
equivariant for the action of \( G''_{\tilde{S}_\tau} = G''_{\tilde{S}_\tau,p} \) and a \( p \)-quasi-isogeny of abelian schemes on \( \text{Sh}_{K^p}(G''_{\tilde{S}_\tau})_\tau \)

\[
\Phi_{\pi''_\tau} : A''_{\tilde{S}_\tau} \to \pi''_\tau(\text{A''}_{\tilde{S}_\tau}).
\]

By Construction 2.15, this gives rise to a \( \mathbb{P}^1 \)-bundle fibration

\[
\pi_\tau : \text{Sh}_{K^p}(G_{S,T})_\tau \to \text{Sh}_{K^p}(G_{S_\tau,T_\tau})
\]
with some shift \( t_\tau \in E^{x,cl}\A^{\infty,\times}_E/\mathcal{O}^\times_{E_p} \), compatible with the Hecke action of \( G(\mathbb{A}^{\infty,p}) \) as well as an isomorphism of étale sheaves for a regular multi-weight \((k,w)\)

\[
\pi^2_\tau : L^{(k,w)}_{S_{\tau,T}}|_{\text{Sh}_{K^p}(G_{S,T})_\tau} \to \pi^2_\tau(L^{(k,w)}_{S_{\tau,T}}).
\]

The morphisms \( \pi_\tau \) and \( \pi^2_\tau \) are unique up to a central Hecke action by an element of \( F^{x,cl}\A^{\infty,\times}_F/\mathcal{O}_F^\times \).

(2) Let \( \mathcal{O}(1) \) be the tautological quotient line bundle on \( \text{Sh}_{K^p}(G_{S,T})_\tau \) for the \( \mathbb{P}^1 \)-bundle given by \( \pi_\tau \). If \( \tau^- = \sigma^{-\tau} \) is different from \( \tau \), then the normal bundle of the closed immersion \( \text{Sh}_{K^p}(G_{S,T})_\tau \hookrightarrow \text{Sh}_{K^p}(G_{S_\tau,T_\tau})_\tau \) is, up to tensoring a line bundle which is torsion in the Picard group of \( \text{Sh}_{K^p}(G_{S,T})_\tau \), the same as \( \mathcal{O}(-2p^{\tau^r}) = \mathcal{O}(1)^{\otimes(-2p^{\tau^r})} \).

Proof. In statement (1), the existence of \( \pi''_\tau \) is a special case of [TX13+a, Corollary 5.9]. Roughly speaking, this \( \mathbb{P}^1 \)-bundle \( \pi''_\tau \) parametrizes the lines (the Hodge filtration) in the reduced \( \tilde{\tau}^- = \sigma^{-\tau} \)-component of the relative de Rham homology of the versal family \( A''_{\tilde{S}_\tau} \) on \( \text{Sh}_{K^p}(G''_{\tilde{S}_\tau})_\tau \). It is straightforward to check that the condition 2.15.1 is satisfied for the pairs \((\tilde{S},T)\) and \((S_\tau,T_\tau)\). We apply Construction 2.15 to deduce the existence of \((\pi_\tau,\pi^2_\tau)\) from that of \((\pi''_\tau,\Phi_{\pi''_\tau})\).

Statement (2) follows from [TX13+a, Proposition 6.4], when noting that the quaternionic Shimura varieties and the unitary Shimura varieties have the same geometric connected component.

Proposition 2.26(1) implies that we have a morphism

\[
(2.31.1) \quad \pi^*_\tau : H^*_\text{et}(\text{Sh}_K(G_{S_\tau,T_\tau})_\tau, L^{(k,w)}_{S_{\tau,T}}) \to H^*_\text{et}(\text{Sh}_K(G_{S,T})_\tau, L^{(k,w)}_{S_{\tau,T}})
\]
equivariant under \( T(K^p) = \mathcal{O}_F[K^p \setminus G(\mathbb{A}^{\infty,p})/K^p] \). It is canonical up to the action of the central Hecke character, which comes from the ambiguity of choosing the geometric connected component in Construction 2.15.

Theorem 2.32.

(1) If \( \tau_1, \tau_2 \in S^c_\infty \) are two places such that \( \tau_1, \tau_2, \tau_1^-, \tau_2^- \) are distinct, then we have a Cartesian diagram

\[
\begin{array}{ccc}
\text{Sh}_{K^p}(G_{S,T})_{\tau_1,\tau_2} & \xrightarrow{\pi_{\tau_1}} & \text{Sh}_{K^p}(G_{S_1,T_1})_{\tau_2} \\
\downarrow{\pi_{\tau_2}} & & \downarrow{\pi_{\tau_2}} \\
\text{Sh}_{K^p}(G_{S_2,T_2})_{\tau_2} & \xrightarrow{\pi_{\tau_2}} & \text{Sh}_{K^p}(G_{S_2,T_2})_{\tau_2}.
\end{array}
\]
Moreover, the natural morphisms on the cohomology are commutative.

\[
\begin{align*}
H_{et}^*(Sh_{K_p}(G_{S_1},\tau_{r_1})) & \longrightarrow H_{et}^*(Sh_{K_p}(G_{S_1\cup S_2},\tau_{r_1\cup r_2},)F_p, L_{(k,w)^{r_1\cup r_2}}^{(k,w)}) \\
\pi_{r_1}^* & \downarrow \pi_{r_1}^* \downarrow \\
H_{et}^*(Sh_{K_p}(G_{S_2\cup S_3},\tau_{r_2})) & \longrightarrow H_{et}^*(Sh_{K_p}(G_{S_2\cup S_3},\tau_{r_2}, F_p, L_{(k,w)^{r_2}}^{(k,w)})).
\end{align*}
\]

(2) Let \( \tau \in S_\infty^c \) be a place such that \( \tau, \tau^+, \tau^- \) are distinct. Put \( n = n_{r^+} - n_{r^-} \) if \( p \) splits in \( E/F \) and \( n = 0 \) if \( p \) is inert in \( E/F \). Let \( \eta : S_{r^+} = S \cup \{ \tau^+, \tau \} \rightarrow S_\tau = S \cup \{ \tau, \tau^- \} \) be the link given by straight lines except sending \( \tau^- \) to \( \tau^+ \) over \( \tau \):

\[
\begin{array}{ccccccccc}
+ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & + \\
\end{array}
\]

Let \( \eta(n)_{\tau} \) be the morphism defined by the following commutative diagram

\[ (2.32.1) \]

\[
\begin{array}{ccccccccc}
\text{Sh}_{K_p}(G_{S,T})_{\tau^+} & \xleftarrow{\pi_{\tau^+}} & \text{Sh}_{K_p}(G_{S,T})_{(\tau^+, \tau)} & \xrightarrow{\eta(n)_{\tau}} & \text{Sh}_{K_p}(G_{S,T})_{\tau^-} \\
\pi_{\tau^+} & \searrow & \eta(n)_{\tau} & \downarrow & \pi_{\tau^-} \\
\text{Sh}_{K_p}(G_{S^+, \tau_{r^+}}) & & & & \text{Sh}_{K_p}(G_{S^-, \tau_{r^-}})
\end{array}
\]

Then the following statements hold:

(a) The map \( \eta(n)_{\tau} \) is the morphism obtained by applying Construction \[ 2.15 \] to a link morphism on \( Sh_{K_p}''(G_{S^+}''_{\tau^+}) \) with indentation degree \( n \).

(b) If \( t_? \in E^{\times,cl} \bigl( A_{E/F}^\times \bigr) / O_{E/F}^\times \) for \( ? = \tau, \tau^+ \) denotes the shift of the correspondence \( \text{Sh}_{K_p}(G_{S,T})_{\tau^+} \hookrightarrow \text{Sh}_{K_p}(G_{S,T})_{\tau^-} \), then \( \eta(n)_{\tau} \) has shift \( t_{\tau^+} t_{\tau^-}^{-1} \).

(c) The morphism \( \eta(n)_{\tau} \) is finite flat of degree \( p^{\nu(n)} \).

(d) The \( p \)-quasi-isogenies of the universal abelian variety \( Sh_{K_p}(G_{S^+}''_{\tau^+}) \) given by

\[
\pi''_{\tau^+}(A_{G_{S^+}}'')_{|Sh_{K_p}(G_{S^+}''_{\tau^+})} \xrightarrow{\Phi''_{\tau^+}} A_{G_{S^+}}''_{|Sh_{K_p}(G_{S^+}''_{\tau^+})} \xrightarrow{\Phi_{\tau^+}} \pi_{\tau^+}(A_{G_{S^+}}''_{|Sh_{K_p}(G_{S^+}''_{\tau^+})})
\]

induces a link morphism on the sheaves \( \eta_{\tau} : L_{S^+, \tau_{r^+}}^{(k,w)} \longrightarrow \eta_{\tau}(L_{S^-, \tau_{r^-}}^{(k,w)}) \). Then the induced normalized link morphism \( \eta_{\tau}^* \) on the cohomology groups constructed as in \[ 2.25 \] fits into the following commutative diagram:

\[ (2.32.2) \]

\[
\begin{array}{ccccccccc}
H_{et}^*(Sh_{K_p}(G_{S,T})_{\tau^+}, F_p) & \longrightarrow & H_{et}^*(Sh_{K_p}(G_{S,T}(\{\tau^+, \tau\}), F_p) & \longrightarrow & H_{et}^*(Sh_{K_p}(G_{S,T})_{\tau^-}, F_p) \\
\pi_{\tau^+} & \searrow & \eta_{\tau}^* & \downarrow & \pi_{\tau^-} \\
H_{et}^*(Sh_{K_p}(G_{S^+, \tau_{r^+}}), F_p) & \longrightarrow & H_{et}^*(Sh_{K_p}(G_{S^-, \tau_{r^-}}), F_p)
\end{array}
\]

where the upper horizontal arrows are natural restriction maps. Here, for simplification, we have suppressed the sheaves from the notation; for instance, \( H_{et}^*(Sh_{K_p}(G_{S,T})_{\tau^+}, F_p) \) should be read as \( H_{et}^*(Sh_{K_p}(G_{S,T})_{\tau^+}, F_p, L_{S_{\tau}^+, \tau_{r^+}}^{(k,w)} | Sh_{K_p}(G_{S,T})_{\tau^+} ) \).

(3) Assume that \( S_\infty^c = \{ \tau, \tau^- \} \) (and hence \( p \) splits in \( E/F \)). Then \( Sh_{K_p}(G_{S,T}(\{\tau, \tau^-\}) \) is isomorphic to the special fibre of the zero-dimensional Shimura variety \( Sh_{K_p}(G_{S,T}) \) with Iwahoric level at \( p \). Let \( \eta : S_{r^+} \rightarrow S_\tau \) denote the link map (with no curve). Then the link morphism \( \eta_{(\nu_{\tau^-})_{\tau}} : Sh_{K_p}(G_{S^-_{\tau^-}}) \sim Sh_{K_p}(G_{S_\tau}) \) of indentation degree \( 2n_{\tau^-} \) associated to \( \eta \) exists,
and the following diagram

\[
\begin{array}{c}
\text{Sh}_K(G_{S, T})_{\{\tau, \tau^+\}} \\
\downarrow \pi_{\tau} \\
\text{Sh}_K(G_{S, T}) \\
\downarrow \pi_{\tau^-} \\
\text{Sh}_K(G_{S, T - \tau^+}) \cong \eta(\pi_{\tau^-}) \text{Sh}_K(G_{S, T}).
\end{array}
\]

is (the base change to \( \mathbb{F}_p \) of) the Hecke correspondence \( T_p \) on \( \text{Sh}_K(G_{S, T}) \). If \( t_\tau \in E^{x, c1} \backslash A_E^{\infty, x}/C_E^x \) for \( ? = \tau, \tau^+ \) denotes the shift of the correspondence

\[
\text{Sh}_K(G_{S, T}) \xleftarrow{\pi_{\tau^-}} \text{Sh}_K(G_{S, T}) \xrightarrow{\eta(\pi_{\tau^-})} \text{Sh}_K(G_{S, T}).
\]

then \( \eta(\pi_{\tau^-}) \) has shift \( \mathcal{W}_q t_\tau^{-1} t_{\tau^-} \). Moreover, the map induced by the diagram above on cohomology groups

\[
H^0_{\text{et}}(\text{Sh}_K(G_{S, T}, \mathcal{Z}_p)) \xrightarrow{(\eta(\pi_{\tau^-}) \circ \pi_{\tau^-})^*} H^0_{\text{et}}(\text{Sh}_K(G_{S, T})_{\{\tau, \tau^+\}, \mathcal{Z}_p}) \xrightarrow{T_{\pi_{\tau^-}}} H^0_{\text{et}}(\text{Sh}_K(G_{S, T}, \mathcal{Z}_p)).
\]

is the usual Hecke action \( T_p \). Here, as in (2), we have suppressed the sheaves from the notation.

Proof. The analogs of (1), (2)(a) and (2)(c) for unitary Shimura varieties were proved in [TX13+a, Proposition 7.12 and Theorem 7.16]. The statements here follow immediately from Construction 2.15. Statement (2)(b) regarding shifts follows directly from Remark 2.16. Statement (2)(d) is direct by the construction of \( \eta(n)^*_n \) and \( \eta(n)^* \).

For (3), the analogous statement for unitary Shimura varieties \( \text{Sh}_K(G_{S, T}) \) (with \( T_p \) replaced by \( T_\mathcal{W} \)) was proved in [TX13+a, Theorem 7.16(2)]. One deduces (3) using the construction in Subsection 2.17 and computes the shifts by Remark 2.16. \( \square \)

3. Goren-Oort cycles

In this section, we investigate certain generalization of Goren-Oort strata, called Goren-Oort cycles. They are parametrized by certain combinatorial data, called periodic semi-meanders. We will show later that the intersection matrix of generalized Goren-Oort cycles turns out to be the Gram matrix associated to these periodic semi-meanders (which explains our choice of the combinatorics model).

3.1. Periodic semi-meanders. The combinatorics construction that we will use later is related to so-called link representations of periodic Temperley-Lieb algebra, which appear naturally in the study of mathematical physics; see for example [Di90, GL98, MS13]. We will simply state here the main result with minimal input, and refer to [MS13] for a detailed discussion of the mathematical physics background and the proofs.

We slightly modify the definition of periodic semi-meanders to adapt to our situation. Recall that \( F \) is a totally real field of degree \( g \) and \( S, T \) are introduced as in Subsection 2.18 and \( d = \# S_\infty \). We consider the band associated to \( S \) defined as in Subsection 2.18.

A periodic semi-meander for \( S \) is a graph where each node is either connected pairwise by non-intersecting curves (called the arcs) drawn above the band, or connected by a straight line to \( +\infty \). The number of arcs is denoted by \( r \) (so \( r \leq d/2 \)), and the number of straight lines \( d - 2r \) is called the defect of the periodic semi-meander. We use \( \mathcal{B}_S^r \) denote the set of semi-meanders for \( S \) with \( r \) arcs. For example, if \( F \) has degree 7 over \( \mathbb{Q} \), \( r = 2 \), and \( S = \{\infty_1, \infty_4 \} \), we have

\[
\mathcal{B}_S^2 = \{\text{\includegraphics{semi-meander.png}}\}.
\]
When drawing in the plane, points are placed on the horizontal axis at points of coordinates 
\((0, 0), \ldots, (g - 1, 0)\) and the diagram for a periodic semi-meander is taken to be periodic in the 
\(x\)-direction with \(x + g \equiv x\). The curves connecting the points can connect through the imaginary 
boundary line at \(x = -1/2\) and \(x = g - 1/2\). An elementary calculation shows that \(|\mathcal{B}_S^r| = \binom{d}{r}\).

For \(a \in \mathcal{B}_S^r\), we use \(\ell(a)\) to denote the total span of \(a\), that is the sum of the span of all curves 
over the band. For example, the last element above in \(\mathcal{B}_S^2\) has two curves with spans 1 and 5, 
respectively, and hence its total span is 6.

3.2. Gram matrix. We use \(\mathcal{B}_S^r\) to denote the \(\mathbb{Q}_l\)-vector space with basis \(\mathcal{B}_S^r\). We define the Gram 
product to be the following pairing

\[
\langle \cdot | \cdot \rangle_S : \mathcal{B}_S^r \times \mathcal{B}_S^r \to \begin{cases} 
\mathbb{Q}_l(v) & \text{if } r < d/2, \\
\mathbb{Q}_l(T) & \text{if } r = d/2,
\end{cases}
\]

and we extend the Gram product linearly to all of \(\mathcal{B}_S^r\). For \(a, b \in \mathcal{B}_S^r\), we consider the diagram 
\(D(a, b)\) obtained by taking mirror image of \(b\) by a horizontal axis and then connecting the \(d\) nodes 
of \(b\) to that of \(a\) according to their labelings. We put

\[
\langle a | b \rangle_S = \begin{cases} 
0 & \text{if in the diagram } D(a, b), \text{ two straight} \\
(-2)^{m_0}v^{m_v} & \text{lines of } a \text{(or of } b \text{) are connected}, \\
(-2)^{m_0}T^{m_T} & \text{otherwise if } r < d/2, \\
& \text{otherwise if } r = d/2.
\end{cases}
\]

Here, \(m_0\) and \(m_T\) are respectively the numbers of contractible and non-contractible closed loops in 
\(D(a, b)\), and the meaning of \(m_v\) is explained as follows: Assume thus \(r < d/2\) and that no straight 
lines of \(a\) and \(b\) are connected. Let \(\tilde{a}\) denote the periodic semi-meander obtained by removing all 
arcs of \(a\) and turning all nodes connected to an arc into plus signs. Removing all contractible loops 
from \(D(a, b)\) gives rise to a link \(\eta_{\tilde{a}, \tilde{b}}\) from \(\tilde{a}\) to \(\tilde{b}\). Then \(m_v\) is defined to be the total displacement 
v(\(\eta_{\tilde{a}, \tilde{b}}\)) of this link.

Note that only one of \(m_v\) and \(m_T\) can be non-zero by definition.

Example 3.3. The following examples are copied from [MS13].

1. \(a = \ldots\), \(b = \ldots\), \(D(a, b) = \ldots\), the link is

\[
\eta_{\tilde{a}, \tilde{b}} = + + + + + + + + + , \text{ and } \langle a | b \rangle_S = (-2)^{v^{-9}}.
\]

2. \(a = \ldots\), \(b = \ldots\), \(D(a, b) = \ldots\), and \(\langle a | b \rangle_S = 0\).

3. \(a = \ldots\), \(b = \ldots\), and \(D(a, b) = \ldots\), and

\[
\langle a | b \rangle_S = (-2)^{3T^2}.
\]
Remark 3.4. When $S = \emptyset$, the vector space $V^r_S$ is the link representation of so-called periodic Temperley-Lieb algebra $\mathcal{ETLP}_N(T, -2)$ under the notation of [MS13]. (In particular, we specialize the theory to the case when the quantum variable $q = i$.) With respect to the bilinear form we introduced earlier, the representation is $\dagger$-hermitian with respect to the natural involution $\dagger$ on the Temperley-Lieb algebra. Since we will not use the structure of this representation, we simply refer to [MS13, Section 2.3] for further discussion. It seems that the mysterious relationship between this mathematical physics calculation and our Shimura variety calculation probably comes from some common representation theory feature. It might be an intriguing question to ask what quantization could mean for Shimura varieties (or its local analogues) so that the intersection matrix computed in a similar manner as we did later would have a chance to match with the quantized version of the Gram determinant in loc. cit.

The following theorem is essentially the main theorem of [MS13] (which seems to be priorly known to [GL98] using a different argument).

**Theorem 3.5.** Put $t_{d,r} = \sum_{i=1}^{r-1} \binom{d}{i}$. Let $G^r$ denote the Gram matrix $\langle (a|b) \rangle_{a,b \in \mathbb{B}^r}$. Then its determinant is given as follows.

- (When $d$ is even) $\det G^{d/2} = \pm (T^2 - 4)^{t_{d,d/2}}$.
- For $r < d/2$, $\det G^r = \pm (v^g - v^{-g})^{2t_{d,r}}$.

**Proof.** When $S = \emptyset$ (so $d = g$), this is a special case of [MS13, Theorem 4.1]. Indeed, the parameter $\alpha$ in loc. cit. is $T$ in our notation, and since the $\beta$ in loc. cit. is $-2$, the $C_k$ in loc. cit. are equal to $\pm 1$ for all $k$. One easily simplifies their formula to the one stated in this theorem.

The general case requires little modification, but the method of the proof will be important for us later. When $r = \frac{d}{2}$, we just simply ignore all points corresponding to $S_\infty$. So we assume $r < \frac{d}{2}$ from now on. We use $\langle a|b \rangle_d$ to denote the pairing computed by removing all points from $S_\infty$ (and shrink the cylinder accordingly) and hence with displacements computed with respect to only the $d$ nodes. Let $G^r_d$ denote the corresponding matrix. We need to compare $\det G^r_d$ with $\det G^r_S$ by showing that $\det G^r_S$ can be obtained by replacing all $v^d$ in the expression of $\det G^r_d$ by $v^g$.

By the very definition of determinant, $\det G^r_S$ is the sum over all permutations $\sigma$ of the set $\mathbb{B}^r$, of the product of the signature of $\sigma$, and, for every cycle $(a_1 \ldots a_t)$ of the permutation $\sigma$, the product

\[
\langle a_1|a_2 \rangle_S \cdot \langle a_3|a_4 \rangle_S \cdots \langle a_t|a_1 \rangle_S.
\]

The same applies to $\det G^r_d$ except that the product (3.5.1) are taken for the pairing $\langle \cdot|\cdot \rangle_d$. The product (3.5.1), if not zero, is equal to $(-2)^m v^{n_2}$, where $m$ is the sum of total number of contractible loops in the diagrams $D(a_1, a_2), D(a_2, a_3), \ldots, D(a_t, a_1)$, and $n_2$ equals to the total displacement of the composition of the link

\[
\eta_{\delta_1, \delta_2} \circ \cdots \circ \eta_{\delta_2, \delta_3} \circ \eta_{\delta_1, \delta_2},
\]

by the additivity of total displacements as remarked in Subsection 2.18. Note that (3.5.2) is in fact a link from $\delta_1$ to $\delta_2$. So it must be an integer power $n$ of the fundamental link $\eta_{\delta_1}$ defined in Subsection 2.18. In particular, we have $n_2 = n\delta$. Making the same observation for computing $\det G^r_d$, the product (3.5.1) with $\langle \cdot|\cdot \rangle_d$ is instead equal to $(-2)^m v^{n_d}$ with the same $m$ as above, and $n_d$ is the total displacement of (3.5.2) with all points corresponding to $S_\infty$ removed. By the same discussion above, we have $n_d = nd$ with the same $n$ for which (3.5.2) is nth power of the fundamental link.

In conclusion, each term of $\det G^r_S$ can be obtained from the corresponding term of $\det G^r_d$ via replacing $v^d$ by $v^g$. Therefore, $\det G^r_S = \pm (v^g - v^{-g})^{2t_{d,r}}$. □

**Notation 3.6.** Using the illustration of periodic semi-meanders as in (3.1.1), we say an arc $\delta$ lies over another arc $\delta'$ if the contractible closed loop in the picture given by adjoining $\delta$ with
the equator contains $\delta'$ inside. For example, in the list of $\mathcal{M}_2^{\tau}$ in (3.1.1), the last five periodic semi-meanders each has an arc lying over another.

In a periodic semi-meander for $\mathcal{S}$, a basic arc is an arc which does not lie over any other arcs, or equivalently, it is an arc which links some $\tau$ to $\tau^-$ (See (2.19) for the notation). For example, in the list of $\mathcal{M}_2^{\tau}$ in (3.1.1), the five periodic semi-meanders in the first row each has two basic arcs, and the five periodic semi-meanders in the second row each has one basic arc.

It is clear that all periodic semi-meanders have a basic arc except the one with only straight lines. Given a periodic semi-meander $a \in \mathcal{M}_2^{\tau}$ for $\mathcal{S}$ with a basic arc $\delta$ linking two nodes $\tau, \tau^- \in \mathcal{S}_\infty$, we can delete the arc and replace its end-nodes by $+$ to get a periodic semi-meander $a \setminus \delta \in \mathcal{M}_{r-1}^{\tau-1}$ for $\mathcal{S} \cup \{\tau, \tau^-, \tau^+\}$.

3.7. Goren-Oort cycles. We now construct the Goren-Oort cycle $\operatorname{Sh}_{K_p}(G_{S,T})_a$ associated to a periodic semi-meander $a$ for $\mathcal{S}$. The cycle will admit an iterated $\mathbb{P}^1$-bundle morphism $\pi_a : \operatorname{Sh}_{K_p}(G_{S,T})_a \rightarrow \operatorname{Sh}_{K_p}(G_{S_a,T_a})$ for some appropriate subsets $S_a$ and $T_a$ of $\Sigma_\infty$. The correspondence $\operatorname{Sh}_{K_p}(G_{S_a,T_a}) \xleftarrow{\pi_a} \operatorname{Sh}_{K_p}(G_{S,T})_a \rightarrow \operatorname{Sh}_{K_p}(G_{S,T})$ will be constructed using the unitary Shimura varieties; it is not canonical but is characterized by its shift $t_a \in O_{E,(p)}^{\times} \backslash A_E^{\infty,p,\times}$ as explained in Construction 2.15.

We proceed by induction on $\# \mathcal{S}_\infty$. In particular, if $\mathcal{S}_\infty$ is empty, then there is only one Goren-Oort cycle which is the Shimura variety $\operatorname{Sh}_{K_p}(G_{S,T})$ itself. Now suppose the construction is done for Shimura varieties with $\# \mathcal{S}_\infty < d$. We consider a Shimura variety $\operatorname{Sh}_{K_p}(G_{S,T})$ with $\# \mathcal{S}_\infty = d > 1$ and a periodic semi-meander $a$ for $\mathcal{S}$.

If $a$ consists of $g$ straight lines, the corresponding Goren-Oort cycle is just the Shimura variety $\operatorname{Sh}_{K_p}(G_{S,T})$ itself (and the shift is trivial, namely, $1 \in E^{\times,cl} \backslash A_E^{\infty,\times} / O_E^{\infty,p}$). Otherwise, $a$ must contains a basic arc $\alpha$, say the one linking the places $\tau, \tau^- \in \mathcal{S}_\infty$ in order. By Proposition 2.31(1), we have a natural $\mathbb{P}^1$-bundle morphism $\pi_\tau : \operatorname{Sh}_{K_p}(G_{S,T})_\tau \rightarrow \operatorname{Sh}_{K_p}(G_{S,(\tau,\tau^-)\cup\{\tau\}})$. Here we fix for the rest of the paper the choice of a geometric connected component (and hence a shift $t_\tau \in E^{\times,cl} \backslash A_E^{\infty,\times} / O_E^{\infty,p}$) when applying Construction 2.15 to defining the correspondence $\operatorname{Sh}_{K_p}(G_{S,(\tau,\tau^-)\cup\{\tau\}}) \xleftarrow{\pi_\tau} \operatorname{Sh}_{K_p}(G_{S,T})_\tau \rightarrow \operatorname{Sh}_{K_p}(G_{S,T})$. Invoking the construction of Goren-Oort cycles in the case with smaller number of split archimedean places, we have a Goren-Oort cycle $\operatorname{Sh}_{K_p}(G_{S,(\tau,\tau^-)\cup\{\tau\}})_a \setminus \alpha$. We then define the Goren-Oort cycle $\operatorname{Sh}_{K_p}(G_{S,T})_a$ as

$$\operatorname{Sh}_{K_p}(G_{S,T})_a := \pi_\tau^{-1}(\operatorname{Sh}_{K_p}(G_{S,(\tau,\tau^-)\cup\{\tau\}})_a \setminus \alpha).$$

Tracing back the construction, it comes from the study of analogous Goren-Oort cycles on the unitary Shimura varieties; we point out that condition 2.15.1 is always satisfied for the pairs $(\mathbb{S}, T)$ and $(\mathbb{S}_a,T_a)$. If $t_a \setminus \alpha$ denotes the shift for $\operatorname{Sh}_{K_p}(G_{S,(\tau,\tau^-)\cup\{\tau\}})_a \setminus \alpha$, then the shift for $\operatorname{Sh}_{K_p}(G_{S,T})_a$ is just $t_a \setminus \alpha$. By Theorem 2.32(1), we see that the construction does not depend on the choice of basic arc $\alpha$. Finally, we point out that $\mathbb{S}_a$ and $T_a$ are determined as follows: $\mathbb{S}_a$ is obtained by adjoining to $\mathcal{S}$ all end-nodes of the arcs of $a$ to $\mathcal{S}$ and $T_a$ is obtained by adjoining to $T$ all left end-nodes of the arcs of $a$. In particular, the dimension of fibers of $\operatorname{Sh}_{K_p}(G_{S,T})_a$ over $\operatorname{Sh}_{K_p}(G_{S_a,T_a})$ is the same as the codimension of $\operatorname{Sh}_{K_p}(G_{S,T})_a$ in $\operatorname{Sh}_{K_p}(G_{S,T})$, which is $r$.

We fix a multiweight $(k,w)$; recall that $\mathcal{L}^{(k,w)}_{S,T}$ denotes the automorphic $\ell$-adic local system on $\operatorname{Sh}_{K_p}(G_{S,T})$. The same construction above also gives rise to a natural isomorphism $\pi_a^* : \pi_a^* \mathcal{L}^{(k,w)}_{S_a,T_a} \xrightarrow{\sim} \mathcal{L}^{(k,w)}_{S,T} |_{\operatorname{Sh}_{K_p}(G_{S,T})_a}$.

Remark 3.8. It was pointed out to us by X. Zhu that the union of all Goren-Oort cycles associated to periodic semi-meanders with $r$ arcs is exactly the closure of certain Newton strata of the unitary
Shimura variety, transported to the quaternionic side. So maybe the name “Goren-Oort” is slightly misleading, as it usually refers to stratification given by the $p$-torsion subgroup of the universal abelian varieties.

**Example 3.9.** Let $F$ be of degree 6 over $\mathbb{Q}$ and $S = T = \emptyset$. Then $\text{Sh}_K(G_{\emptyset, \emptyset})$ is (the special fiber of) the Hilbert modular variety for $F$. Let $\tau_0, \ldots, \tau_5$ denote the embeddings of $\mathcal{O}_F$ into $\mathbb{Z}_p^{ur}$ so that $\tau_i = \tau_i \pmod{6}$ and $\tau_{i+1} = \sigma \tau_i$. We have a universal abelian variety $A$ over $\text{Sh}_K(G_{\emptyset, \emptyset})$ equipped with an $\mathcal{O}_F$-action.

We consider the periodic semi-meander $a = \bullet \bullet \bullet \bullet \bullet \bullet$. For each $\overline{\mathbb{F}}_p$-point $x \in \text{Sh}_K(G_{\emptyset, \emptyset})$, the Dieudonné module $D_x$ of the universal abelian variety $A_x$ at $x$ decomposes as $D_x = \bigoplus_{i=0}^5 D_{x,i}$, where $\mathcal{O}_F$ acts on the $i$-th factor via $\tau_i$. Let $V_i : D_{x,i+1} \to D_{x,i}$ denote the Verschiebung map for $i \in \mathbb{Z}/5\mathbb{Z}$. Then $x \in \text{Sh}_K(G_{\emptyset, \emptyset})$ if and only if

$$V_1 \circ V_2(D_{x,3}) \subseteq pD_{x,1}, \quad V_4 \circ V_5(D_{x,0}) \subseteq pD_{x,4}, \quad \text{and} \ V_0 \circ V_1 \circ V_2 \circ V_3(D_{x,4}) \subseteq p^2 D_{x,0}.$$ 

In this case, $\text{Sh}_{K_p}(G_{\emptyset, \emptyset})$ is a collection of “iterated $\mathbb{P}^1$-bundles” parametrized by the discrete Shimura variety $\text{Sh}_K(G_{\Sigma_\infty, \{\tau_2, \tau_3, \tau_5\}})$. In fact, one can prove that each geometric connected component is isomorphic to the product of $\mathbb{P}^1$ with the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-p) \oplus \mathcal{O}_{\mathbb{P}^1}(p))$ over $\mathbb{P}^1$.\footnote{More canonically, it is the projective bundle of a rank two bundle $E$ over $\mathbb{P}^1$ which sits inside an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(p) \to E \to \mathcal{O}_{\mathbb{P}^1}(-p) \to 0$; but the extension splits (not canonically though).}

## 4. Cohomology of Generalized Goren-Oort cycles

Let $\text{Sh}_{K_p}(G_{3, T})$ be the special fiber of a quaternionic Shimura variety as in Section 2.2. Using Gysin maps, the cohomology of the Goren-Oort cycles gives rise to part of the cohomology of the big Shimura variety $\text{Sh}_{K_p}(G_{3, T})$.

### 4.1. Generality on étale cohomology.

We recall first some generality on Gysin maps and étale cohomology of iterated $\mathbb{P}^1$-bundles. Let $\ell$ be a fixed prime, $k$ be an algebraically closed field of characteristic different from $\ell$.

Consider a closed immersion $i : Y \hookrightarrow X$ of smooth varieties over $k$ of codimension $r$. The functor of direct image $i_*$ has a right adjoint, denoted by $i^!$. For an $\ell$-adic étale sheaf $\mathcal{F}$ on $X$, $i^! \mathcal{F}$ is the sheaf of sections of $\mathcal{F}$ with support in $Y$. This is a left exact functor, and let $R^{\geq q}i^!$ denote its $q$-th derived functor. Then by the relative cohomological purity [SGA4, XVI, Théorème 3.7], we have $R^{\geq q}i^! \mathcal{O}_X = 0$ for $q \neq 2r$, and a canonical isomorphism $R^{2r}i^! \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_Y(-r)$. Explicitly, the inverse isomorphism $\mathcal{O}_Y \xrightarrow{\sim} R^{2r}i^! \mathcal{O}_Y(r)$ is given by the fundamental class $\text{cl}_Y \in H^{2r}_{et,Y}(X, \overline{\mathcal{O}}_Y(r)) = H^0_{et}(Y, R^{2r}i^! \mathcal{O}_X)$ of $Y$. Now for any $\ell$-adic lisse sheaf $\mathcal{F}$ on $X$, we define the Gysin map as the composite

$$Gysin : H^q_{et}(Y, i^* \mathcal{F}) \xrightarrow{(\text{cl}_Y)} H^{q+2r}_{et,Y}(X, \mathcal{F}) \to H^{q+2r}_{et}(X, \mathcal{F}),$$

where the second map is the natural morphism from cohomology with support in $Y$ to the usual cohomology group.

Let $\pi : X \to Y$ be a $r$-th iterated $\mathbb{P}^1$-bundle of proper and smooth $k$-varieties, i.e. $\pi$ admits a factorization

$$\pi : X_0 := X \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} X_2 \to \cdots \pi_r \to X_r := Y,$$

where each $\pi_i : X_{i-1} \to X_i$ is a $\mathbb{P}^1$-fibration for $1 \leq i \leq r$. Then the trace map

$$\text{Tr}_\pi : R^{2r} \pi_*(\overline{\mathcal{O}}_X(r)) \xrightarrow{\sim} \overline{\mathcal{O}}_Y$$
is an isomorphism. We denote by $\text{cl}_\pi \in H^0(Y, R^{2r} Q\ell(r))$ with $\text{Tr}_\pi(\text{cl}_\pi) = 1$, and call it fundamental class of the fibration $\pi$. For any $Q\ell$-lisse sheaf $F$ on $Y$ and any integer $q \geq 0$, it induces a map

$$\pi_a : H^q_{\text{et}}(X, \pi^* F(r)) \to H^{q-2r}(Y, F \otimes R^{2r} \pi_*(Q\ell(r))),$$

where the first morphism comes from the Leray spectral sequence $E_2^{q,b} = H^q_{\text{et}}(Y, R^b \pi^* F(r)) \Rightarrow H^{q+b}(X, \pi^* F(r))$. Explicitly, $\pi_a$ admits the following description. Put $\pi_{[0,i]} := \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_1$ for $1 \leq i \leq r$. Let $O_{\pi_i}(1)$ be the tautological quotient line bundle of the $\mathbb{P}^1$-bundle $\pi_i$, and $c_1(O_{\pi_i}(1)) \in H^2(X_{i-1}, Q\ell(1))$ be its first Chern class. Put $\xi_i = \pi_{[0,i]}^*(c_1(O_{\pi_i}(1))) \in H^2(X, Q\ell(1))$. By induction on $r$, one deduces easily from [SGA5, Corollaire 2.2.6] a decomposition

$$H^q_{\text{et}}(X, \pi^* F(r)) \cong \bigoplus_{0 \leq j \leq r} \left( \bigoplus_{1 \leq i_1 < \cdots < i_j \leq r} \pi^* H^{q-2j}_{\text{et}}(Y, F(r - j)) \otimes \xi_{i_1} \cup \cdots \cup \xi_{i_j} \right).$$

Then for an element $x = \sum_j \sum_{1 \leq i_1 < \cdots < i_j \leq r} \pi^* (y_{i_1, \ldots, i_j}) \cup \xi_{i_1} \cup \cdots \cup \xi_{i_j}$, one has

$$\pi_a(x) = y_{1, \ldots, r}.$$

In particular, the fundamental class $\text{cl}_\pi$ is the image of $\xi_1 \cup \cdots \cup \xi_r$ in $H^0(X, R^{2r} Q\ell(r))$.

### 4.2. Gysin maps and restrictions

We keep the notation of Section 3.7. The pair of morphisms $(\pi_a, \pi_b)$ induces a sequence of natural homomorphisms, whose composition is denoted by $\text{Gys}_a$,

$$H^{d-2r}_{\text{et}}(\text{Sh}_K(G_{S,T_a})_{\mathbb{A},T_a}, L_{S,T_a}) \xrightarrow{\pi_a^*, \cong} H^{d-2r}_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}) \xrightarrow{\pi_b^*} H^{d-2r}_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}^*) \xrightarrow{\text{Gysin}_{4.1.1}} H^{d}_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}^*(r)).$$

We can also consider the dual picture. We define the morphism $\text{Res}_a$ to be the composition of the following homomorphisms:

$$\text{Res}_a : H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}^*(r)) \xrightarrow{\text{Restriction}} H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}^*) \xrightarrow{\cong} H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}^*) \xrightarrow{\pi_a} H^{d-2r}_{\text{et}}(\text{Sh}_K(G_{S,T_a})_{\mathbb{A},T_a}, L_{S,T_a}^*(r)).$$

It is clear from the construction that both morphisms $\text{Gys}_a$ and $\text{Res}_a$ are equivariant for the action of the tame Hecke algebra $\mathcal{T}(K^p) = \mathbb{Q}_p[K^p \backslash G(\mathbb{A}^{\infty,p})/K^p]$.

The following theorem is the key to prove our main result. We defer its proof to the next section.

**Theorem 4.3.** For $a, b \in \mathbb{B}_c$, we have the following description of the composition

$$H^{d-2r}_{\text{et}}(\text{Sh}_K(G_{S,T_a})_{\mathbb{A},T_a}, L_{S,T_a}^*(r)) \xrightarrow{\text{Gys}_b} H^d_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T}, L_{S,T}^*(r)) \xrightarrow{\text{Res}_a} H^{d-2r}_{\text{et}}(\text{Sh}_K(G_{S,T_a})_{\mathbb{A},T_a}, L_{S,T_a}^*(r)).$$

1. When $(a,b) = 0$, the composition $\text{Res}_a \circ \text{Gys}_b$ factors through the cohomology group $H^{d-2(r+1)}_{\text{et}}(\text{Sh}_K(G_{S,T})_{\mathbb{A},T})$ of some quaternionic Shimura variety of dimension $d - 2(r + 1)$ with $\# T' = \# T + (r + 1)$.
2. When $r < \frac{d}{2}$ and $(a,b) = (-2)^m v_{m,r}$, we can define the induced link $\eta_{a,b} : S_a \to S_b$ as in Subsection 3.2. Then $\text{Res}_a \circ \text{Gys}_b$ is given by

$$(-2)^m \cdot p^{(\ell(a) + \ell(b))/2} \eta_{a,b}(z),$$

where $\eta_{a,b}(z)$ is the morphism on cohomology by (2.25) induced by the link morphism

$$\eta_{a,b}(z) : \text{Sh}_K(G_{S_a,T_a}) \to \text{Sh}_K(G_{S_b,T_b}).$$
of shift $t_at_b^{-1}$ and indentation degree $z$; here $z = \ell(a) - \ell(b)$ if $p$ splits in $E/F$ and $z = 0$ if $p$ is inert.

(3) When $r = \frac{d}{2}$ and $\langle a | b \rangle = (-2)^{m_0}T^{m_T}$, $\text{Res}_{a} \circ \text{Gys}_b$ is given by

$$(-2)^{m_0} \cdot p^{(\ell(a)+\ell(b))/2}(T/p^{g/2})^{m_T} \circ \eta^*_z,$$

where $\eta$ is the trivial link from $S_a$ to $S_b$ and $\eta^*_z$ is the associated link morphism with shift $t_at_b^{-1}$ and indentation degree $z = \ell(a) - \ell(b) - m_Tg$.

We now assume Theorem 4.3 and deduce the main theorem of this paper.

**Theorem 4.4.** Fix an positive integer $r \leq \frac{d}{2}$.

1. For each periodic semi-meander $a \in \mathcal{B}_S$, the Goren-Oort cycle $\text{Sh}_{K_p}(G_{S,T})_a$ of the Shimura variety $\text{Sh}_{K_p}(G_{S,T})$ is a subvariety of codimension $r$, compatible with the change of tame level structure and stable under the action of the tame Hecke correspondences. Moreover, it admits a natural proper smooth morphism

$$\pi_a : \text{Sh}_{K_p}(G_{S,T})_a \rightarrow \text{Sh}_{K_p}(G_{S_a,T_a})$$

to another quaternionic Shimura variety (in characteristic $p$), such that the fibers of $\pi_a$ are $r$-times iterated $\mathbb{P}^1$-bundles. The morphism is equivariant for the tame Hecke correspondences.

2. We fix a cuspidal automorphic representation $\pi \in \mathcal{A}_{(k, w)}$ (see 2.5) so that its associated Galois representation $\rho_\pi$ is unramified at $p$. Let $\alpha_\pi$ and $\beta_\pi$ denote the (generalized) eigenvalues of $\rho_{\pi,p}(\text{Frob}_p)$. Suppose that $\alpha_\pi/\beta_\pi$ is not a $2n$-th root of unity for $n \leq d$ so that $\alpha_\pi^2/\beta_\pi^2$ are distinct from each other for $1 \leq i \leq d$. Then the action of $\text{Frob}_{p^2}$ on the generalized eigenspace of $H^d_{et}(\text{Sh}_{K_p}(G_{S,T})_F, \mathcal{L}_{(k,w)}^{G_{S,T}}(r))[\pi]$ with eigenvalue $\alpha_\pi^2/\beta_\pi^2$ is semi-simple (so that the generalized eigenspace is a genuine eigenspace), and the direct sum of the Gysin morphisms

$$(4.4.1) \bigoplus_{a \in \mathcal{B}_S} H^d_{et}^{G_{S,T}}(\text{Sh}_{K_p}(G_{S,T})_a, \mathcal{L}_{(k,w)}^{G_{S,T}}(r))[\pi] \xrightarrow{\sum_a \text{Gys}_a} H^d_{et}(\text{Sh}_{K_p}(G_{S,T})_F, \mathcal{L}_{(k,w)}^{G_{S,T}}(r))[\pi]$$

induces an isomorphism on $\text{Frob}_{p^{2g}}$-eigenspace with this eigenvalue on both sides.

2′) Keep the notation in (2) but assume that $r = \frac{d}{2}$ (so $d$ is even) and $(k, w) = (2)$. Suppose that $\alpha_\pi/\beta_\pi$ is not an $n$-th root of unity for $n \leq \frac{d}{2}$. Then the $\text{Frob}_{p^2}$-invariant subspace of $H^d_{et}(\text{Sh}_{K_p}(G_{S,T})_F, \mathbb{Q}_\ell(\frac{d}{2}))[\pi]$ is generated by the cycle classes of $\text{Sh}_{K_p}(G_{S,T})_a$ for $a \in \mathcal{B}_S^{d/2}$.

**Proof.** By taking the limit over all tame levels, it suffices to work with level $K_p$. Statement (1) follows from the construction of Goren-Oort cycles in Subsection 3.7 (2′) is clearly a special case of (2). We now focus on the proof of (2). By Proposition 2.26 the morphism (4.4.1) is nothing but

$$(4.4.2) \bigoplus_{a \in \mathcal{B}_S} \rho_{\pi,p}^{\otimes (d-2r)} \otimes (\det \rho_{\pi,p}(1))^\otimes(\#T+r) \longrightarrow \rho_{\pi,p}^{\otimes d} \otimes (\det \rho_{\pi,p}(1))^\otimes\#T(r).$$

Thus the generalized eigenspace for the action of $\text{Frob}_{p^{2g}}$ with eigenvalue

$$(4.4.3) \alpha_\pi^2(\beta_\pi/p^g)^{2(\#T+r)} = \alpha_\pi^2(\beta_\pi/p^g)^{2(\#T+r)} \rho_{\pi,p}^{\otimes d} \otimes (\det \rho_{\pi,p}(1))^\otimes\#T(r).$$

has dimension exactly equal to $\binom{d}{r}$ for both sides of (4.4.2); and the generalized eigenspace on left hand side is a genuine eigenspace (since it is the direct sum of $\binom{d}{r}$-copies of one-dimensional generalized eigenspace). Thus, the proof of (2) and (2′) will be finished if we show that (4.4.2) is injective on the corresponding generalized eigenspace.
We consider the composition of the Gysin morphisms \((4.4.1)\) with the Restriction morphisms:

\[
\bigoplus_{b \in \mathbb{G}_S} H^d_{et}(\text{Sh}_{K_p}(G_{S_{\kappa}}, \tau_b)_{\mathbb{F}_p}, L^{(\ell, w)}_{\mathbb{S}^{\kappa}, \mathbb{T}}) \rightarrow \bigoplus_{a \in \mathbb{G}_S} H^d_{et}(\text{Sh}_{K_p}(G_{S_{\kappa}, \tau_a})_{\mathbb{F}_p}, L^{(\ell, w)}_{\mathbb{S}^{\kappa}, \mathbb{T}}) \quad (4.4.4)
\]

(We switched the first sum from over \(a\) to over \(b\).) Taking the generalized eigenspace for \(\text{Frob}_{b^Q}\) acting on \((4.4.1)\) with the eigenvalue \((4.4.3)\) and using the description Proposition 2.26 we arrive at the following linear map

\[
\bigoplus_{b \in \mathbb{G}_S} \mathbb{Q}_f \rightarrow \bigoplus_{a \in \mathbb{G}_S} \mathbb{Q}_f \quad (4.4.5)
\]
of vector spaces. Choosing a basis, this map is given by a \((d \times d)\)-matrix \(A\) with coefficients in \(\mathbb{Q}_f\).

We explain how this matrix \(A\) is related to the Gram matrix for the periodic semi-meanders. We first normalize \(A\) by multiplying \(A\) on both sides by the same diagonal matrix, whose diagonal component is \(p^{-\ell(a)/2}\) at \(a\); this will kill the auxiliary factor \(p^{(\ell(a)+\ell(b))/2}\) in Theorem 4.3. Let \(B\) denote the product matrix. We will prove that

\[
\det B = \begin{cases} 
\det \mathcal{G}_S^{d/2}|_{r=T_p^d}, & \text{if } r = d/2 \\
\det \mathcal{G}_S^{|r=\eta^{univ}_p}, & \text{if } r < d/2,
\end{cases}
\]

where \(r=T_p^d\) and \(r=\eta^{univ}_p\) are formal substitutions, and \(T_p^d\) and \(\eta^{univ}_p\) are some formal symbols we define later.

We first compare the entries of \(B\) with the entries of \(\mathcal{G}_S^r\) when \(a|b=0\). In this case, by Theorem 4.3(1), \(\text{Res}_a \circ \text{Gys}_S\) factors through

\[
H^{d-2(r+1)}(\text{Sh}_{K_p}(G_{S', \tau})_{\mathbb{F}_p}, L^{(\ell, w)}_{S', \mathbb{T}})\cdot (-1)
\]

for some quaternionic Shimura variety \(\text{Sh}_{K_p}(G_{S', \tau})\) of dimension \(d - 2(r + 1)\). Thanks to the assumption that \(p^2_{\alpha}\beta^2_{\beta} \neq p^2_{\alpha}\beta^2_{\beta}\) are distinct: the cohomology group above does not contain any generalized \(\text{Frob}_{b^Q}\)-eigenspaces with eigenvalue \((4.4.3)\). Thus the \((a,b)\)-entry of \(B\) is zero.

We separate the discussion for \(r < \frac{d}{2}\) and \(r = \frac{d}{2}\). First, suppose that \(r < \frac{d}{2}\). A subtle point of our argument is that we can not directly identify the matrix \(B\) with \(\mathcal{G}_S^r\) entry by entry, because there is no canonical choice of basis on each of the factor of \((4.4.5)\). The proof resembles to that of Theorem 3.5. The determinant of \(B\) is equal to the sum over all permutations \(s\) of the set \(\mathbb{G}_S\), of the product of \(s\), and for every cycle \((a_1, \ldots, a_t)\) of the permutation \(s\), the product

\[
p^{-(\ell(a_1) + \cdots + \ell(a_i))} \cdot (\text{Res}_{a_1} \circ \text{Gys}_{a_2}) \cdot (\text{Res}_{a_2} \circ \text{Gys}_{a_3}) \cdots (\text{Res}_{a_t} \circ \text{Gys}_{a_1}).
\]

Let \(m\) be the sum of total number of contractible loops in the diagrams \(D(a_1, a_2), D(a_2, a_3), \ldots, D(a_1, a_1)\). Then by Theorem 4.3(2), \((4.4.6)\) is of the form \((-2)^m\) times the following composition of link morphisms on cohomology groups:

\[
\eta^{\ast}_{a_1, a_2, z(a_1, a_2)} \circ \eta^{\ast}_{a_2, a_3, z(a_2, a_3)} \circ \cdots \circ \eta^{\ast}_{a_t, a_1, z(a_t, a_1)},
\]
of shift \(\prod_{i=1}^{t-1} (t_{a_i} t_{a_{i+1}}^{-1}) = 1\) and indentation degree

\[
\sum_{i=1}^{t-1} z(a_i, a_{i+1}) + z(a_t, a_1) = \begin{cases} 
\sum_{i=1}^{t-1} (\ell(a_i) - \ell(a_{i+1})) + \ell(a_n) - \ell(a_1) = 0 & \text{if } p \text{ splits in } E/F \\
0 + \cdots + 0 = 0 & \text{if } p \text{ is inert in } E/F.
\end{cases}
\]
So this composition \((4.4.7)\) is the same link morphism associated to some \(n\)-th power of the fundamental link \(\eta_{S_{a_1}}\) for \(S_{a_1}\), with trivial shift and indentation degree 0 (no matter \(p\) splits or not in \(E/F\)).

Note that the link morphism \((\eta^n_{S_{a_1}})^*\) acting on the one-dimensional Frobenius-eigenspace
\[
(4.4.8) \quad (H_{et}^{d-2r}(Sh_{K_p}(G_{S_{a_1}},T_{a_1}), L_{S_{a_1}}))[\pi]\] Frobenius\(2\text{-}s\text{-}\text{eigenspace}

is just the multiplication by a scalar which we denote by \(\lambda_{a_1,n}\). We claim that \(\lambda_{a_1,n}\) does not depend on \(a_1 \in \mathfrak{B}^s\). Indeed, for \(a_1, a' \in \mathfrak{B}^s\) with \(\langle a|a'\rangle \neq 0\), Theorem \(4.3(2)\) gives a link morphism \(\eta_{S_{a_1}S_{a_1}^\prime,(m)} : Sh_{K_p}(G_{S_{a_1}},T_{a_1}) \rightarrow Sh_{K_p}(G_{S_{a_1}^\prime},T_{a_1})\) with some indentation degree \(z\) and some shift, then
\[
(\eta^n_{S_{a_1}})^*(0) = (\eta^n_{S_{a_1}})^* \circ (\eta^n_{S_{a_1}^\prime})^{-1} \circ \eta_{S_{a_1}S_{a_1}^\prime,(m)})^{-1}
\]
provided one of \((\eta^n_{S_{a_1}})^*\) or \((\eta^n_{S_{a_1}^\prime})^*\) exists. When this happens, we must have \(\lambda_{a_1,n} = \lambda_{a_1',n}\). For general \(a\) and \(a'\), we can always find \(a_1 = a, \ldots, a'_l = a_1 \in \mathfrak{B}^s\) such that \(\langle a|a_{l+1}\rangle \neq 0\); so if for some \(n\) the link morphism \((\eta^n_{S_{a_1}})^*\) exists, then it does not depend on \(a\). In the sequel, we put \(\lambda_n = \lambda_{a,n}\) as long as \((\eta^n_{S_{a_1}})^*\) exists for some \(a \in \mathfrak{B}^s\).

We can thus introduce the formal symbol \(\eta_n^{univ}\) such that \((\eta_n^{univ})^n = \lambda_n\) whenever \((\eta^n_{S_{a_1}})^*\) exists for an integer \(n\). Comparing this computation with \(\det \mathfrak{S}^s_B\) in the proof of Theorem \(3.5\) we see that \(\det B\) is obtained by replacing every \(a^g\) in \(\det \mathfrak{S}^s_B\) by \(\eta_n^{univ}\). By Theorem \(3.5\) this means that
\[
\det B = \pm ((\eta_n^{univ} - (\eta_n^{univ})^{-1})^{2d,d/r}.
\]
In particular, \((\eta_n^{univ})^2\) appears in the determinant and hence \((\eta^n_{S_{a_1}})^*\) exists.

Finally, it follows from Proposition \(2.27\) that \((\eta_n^{univ})^{2(d-2r)} = \lambda_{2(d-2r)} = (\alpha_\pi/\beta_\pi)^{d-2r}\). Our assumption implies that \((\alpha_\pi/\beta_\pi)^{d-2r} 
eq 1\); so \((\eta_n^{univ})^2 \neq 1\) and hence \(\det B \neq 0\). This concludes (2).

We now treat the case of \(r = \frac{d}{2}\). Similarly to the discussion above, \(\det B\) is equal to the sum over all permutations \(s\) of the set \(\mathfrak{B}^s\), of the product of the signature of \(s\) and, for every cycle \((a_1 \cdots a_k)\) of the permutation \(s\), the product \((4.4.6)\). In this case \((4.4.6)\) is of the form \((-2)^{m_0}\text{eigentimes}\,(T_p/p^{d/2}m_T^{d/2})\text{times}\,(\det S_{a_1,T_{a_1}})\) to itself with shift
\[
\prod_{i=1}^{T-1}(t_n, t_{a_{i+1}}, t_{-m}^{-m_T}) t_n, t_{a_i}^{-1} t_{-m}^{-m_T, n} = \langle \xi, \xi \rangle^{-m_T}
\]
and indentation degree
\[
\sum_{i=1}^{T} (\ell(a_i) - \ell(a_{i-1}) - m_{T,i}g) = -m_T g
\]
Here \(m_{T,i}\) is the number of non-contractible closed loops in \(D(a_i, a_{i+1})\), and \(m_T = \sum m_{T,i}\). By Example \(2.24\) and the uniqueness of link morphisms (Lemma \(2.22\)), this link morphisms is nothing but the one associated to \(S_{q}^{-m_T/2}\) with shift \(\langle \xi, \xi \rangle^{-m_T}\); this in particular says that \(m_T\) is even. By the second part of Example \(2.24\) we see that this link morphism is exactly \(S_{p}^{-m_T/2}\). Therefore, \((4.4.6)\) is given by
\[
(-2)^{m_0}(T_p/p^{d/2}m_T)(S_p)^{-m_T/2} = (-2)^{m_0}(\alpha_\pi + \beta_\pi)^2/(\alpha_\pi/\beta_\pi)^{m_T/2}.
\]
Comparing this with the computation of \(\det \mathfrak{S}_{a_1}^s\), we see that \(\det B\) is nothing but replacing every \(T_2\) by \(T_p^{n_2} = (\alpha_\pi + \beta_\pi)^2/\alpha_\pi/\beta_\pi\). By Theorem \(3.5\) we see that
\[
\det B = \pm (\alpha_\pi + \beta_\pi)^2/(\alpha_\pi/\beta_\pi - 4)^{1d,d/2} = \pm ((\alpha_\pi - \beta_\pi)/\alpha_\pi/\beta_\pi)^{1d,d/2}.
\]
It is nonzero as long as \(\alpha_\pi \neq \beta_\pi\). This concludes the proof of Theorem \(4.4\).

\textbf{Remark 4.5.} (1) When \(p\) is only assumed to be unramified, the same argument should be able to prove the analogous Theorem 4.4, provided that one can prove the version of Proposition 2.27 for all twisted partial Frobenii.

\footnote{Note that we still need \(\alpha_\pi/\beta_\pi\) to avoid roots of unity to get \((4.4.6)\).}
(2) It would be very interesting to see what happens if \( \alpha_\pi = \beta_\pi \). See Example 4.6 for more discussion.

(3) One should be able to formulate certain version of Tate Conjecture for motives attached to the cohomology of local systems; we however do not further elaborate this viewpoint.

**Example 4.6.** The following example is suggested by a discussion with Zhiwei Yun. Suppose that \( F \) is a real quadratic field, the automorphic form \( \pi \in \mathcal{M}(\mathbb{Q}) \) is defined over \( \mathbb{R} \), and the unramified Satake parameters \( \alpha_\pi = \beta_\pi = \pm p \). Consider the Shimura variety \( X = \text{Sh}_K(G_1(v_1,v_2), \emptyset) \), where \( v_1 \) and \( v_2 \) are two finite prime-to-\( p \) places of \( F \) so that the Shimura variety is compact (for simplicity). We assume that \( \pi \) appears in its cohomology. Then the Goren-Oort cycles contribute only a one-dimensional subspace of the Tate classes of \( H^2_{et}(X_{\mathbb{F}_p}, \mathbb{Q}_\ell(1))[\pi] \), which has dimension 4. Indeed, the intersection matrix \( B \) given above is degenerate. This means that the corresponding intersection matrix is degenerate (having rank 1). Moreover, the intersection matrix on the Neron-Severi group of \( X \) has only one positive signature which corresponds to the class for ample line bundle (which is in the spectrum of the one-dimensional automorphic representations). So degeneration of the intersection matrix means that the contribution from the Goren-Oort cycles is really a one-dimensional subspace of \( H^2_{et}(X_{\mathbb{F}_p}, \mathbb{Q}_\ell(1))[\pi] \).

We also point out that the case \( \alpha_\pi = \beta_\pi = p \) is indeed possible. For example, if \( \pi \) comes from the base change of a usual modular form corresponding to an elliptic curve over \( \mathbb{Q} \) which has good supersingular reduction at \( p \), then the local Satake parameters of \( \pi \) at \( p \) are \( \alpha_\pi = \beta_\pi = p \).

5. Computation of the intersection matrix

The aim of this section is to establish Theorem 4.3 and hence to finish the proof of the main theorems. We keep the notation from the previous section.

**Notation 5.1.** To simplify notation, we suppress the automorphic sheaf \( \mathcal{L}^{(k,w)}_{G,T} \), the level structure \( K_p \), the change of base to \( \mathbb{F}_p \), and the subscript et from the notation, as they are all fixed throughout this section. For example, we write

\[
H^*(\text{Sh}(G_{S,T}))_a(r) = H^*_a(\text{Sh}_{K_p}(G_{S,T})_a,\mathbb{F}_p, \mathcal{L}^{(k,w)}_{G,T}(r) |_{\text{Sh}_{K_p}(G_{S,T})_a}),
\]

where \( \text{“}(r)\text{”} \) denotes the Tate twist. This should not cause any confusion because all the automorphic sheaves are compatible on Goren-Oort cycles.

Before going into the intricate induction, we first handle a few simple but essential cases; the general case will be essentially reduced to these cases.

5.2. The case of \( r = 1 \) and \( a = b \). This is the case where the corresponding periodic semimeanders are given as \( a = b = \bullet \cdots \bullet \) (or their shifts), linking \( \tau \) with \( \tau^- = \sigma^{-n_\tau} \tau \). (Here we did not draw the straight lines on the sides.)

Unwinding the definition, we have the following commutative diagram

\[
\begin{array}{ccc}
H^0(\text{Sh}(G_{S,T})) & \xrightarrow{\text{Res}_a \circ \text{Gys}_a} & H^0(\text{Sh}(G_{S,T})) \\
| & \pi_a^* & | \\
H^0(\text{Sh}(G_{S,T})) & \xrightarrow{\text{Gys}} & H^2(\text{Sh}(G_{S,T}))(1) & \xrightarrow{\text{Restr.}} & H^2(\text{Sh}(G_{S,T}))_a(1)
\end{array}
\]

Recall that \( \text{Sh}(G_{S,T})_a \) is a \( \mathbb{P}^1 \)-bundle over \( \text{Sh}(G_{S,a,T})_a \), hence \( \pi_a^* \) and \( \pi_a \) are both isomorphisms. By the excessive intersection formula [Fu98, §6.3], the composition of the bottom line is given by the cup product with the first Chern class of the normal bundle of the embedding \( \text{Sh}(G_{S,T})_a \hookrightarrow \text{Sh}(G_{S,T}) \), which is isomorphic to \( -2p^{n_\tau} \) times the canonical quotient ample line bundle for the \( \mathbb{P}^1 \)-bundle.
given by $\pi_a^*$, according to Proposition 2.31(2). Therefore, the top line morphism $\text{Res}_a \circ \text{Gys}_a$ is nothing but multiplication by $-2p^n\tau = -2p^\ell(a)$.

5.3. The case of $d = 2$ and $r = 1$ with $a \neq b$. This is the case where the corresponding periodic semi-meanders are given as

\[
a = \cdots \bullet + \cdots + \bullet + \cdots \quad \text{and} \quad b = \cdots \bullet + \cdots + \bullet + \cdots
\]

(or their simultaneous shifts). Let $\tau^-$ denote the node on the left and $\tau$ the one on the right so that $\tau^+ = \tau^-$. Unwinding the definition, the morphism $\text{Res}_b \circ \text{Gys}_a$ is the composition of the following commutative diagram from the upper-left to the lower-right (first rightward and then downward):

\[
\begin{array}{cccccc}
H^0(\text{Sh}(G_{S_b,\tau_b})) & \xrightarrow{\pi_b^*} & H^0(\text{Sh}(G_{S,\tau}^-)) & \xrightarrow{\text{Gysin}} & H^2(\text{Sh}(G_{S,\tau}))(1) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\text{Sh}(G_{S,T}_{\{\tau^-,\tau\}})) & \xrightarrow{\text{Gysin}} & H^2(\text{Sh}(G_{S,T}_{\tau}))(1) & & \downarrow \pi_{a,1} \\
\downarrow \text{Restr.} & & & & \downarrow \text{Restr.} \\
H^0(\text{Sh}(G_{S_a,\tau_a})) & & & & \pi_{a} \downarrow \\
\end{array}
\]

Here, the commutativity of the square follows from the fact that the corresponding morphisms of Shimura varieties form a Cartesian square (as $a$ and $b$ take the vanishing locus of two different partial Hasse-invariants), and $\text{Tr}_{\pi_a}$ is the trace map induced by the finite étale map

\[
\text{Sh}(G_{S,T}_{\{\tau^-,\tau\}}) \leftrightarrow \text{Sh}(G_{S,T}_{\tau}) \xrightarrow{\pi_a} \text{Sh}(G_{S_a,\tau_a}),
\]

and the natural isomorphism between the pullback of the automorphic sheaf on $\text{Sh}(G_{S_a,\tau_a})$ with that on $\text{Sh}(G_{S,T}_{\tau})$.

By Theorem 2.32(3), the diagonal composition from upper-left to lower-right, or equivalently the morphism $\text{Res}_a \circ \text{Gys}_b$, is $T_p \circ (\eta_{(n)})^{-1}$, where $\eta_{(n)}$ is a link morphism associated to the trivial link $\eta : S_b \rightarrow S_a$ with indentation degree $n = 2n_{\tau^-} = -\ell(a) - \ell(b) - g$ and shift $w_q t_q^{-1} t_b$. Thus the inverse $(\eta_{(n)})^{-1} = (\eta_{(-n)})^*$ is the link morphism associated to the link $\eta^{-1}$ with indentation degree $\ell(a) - \ell(b) - g$ and shift $w_q^{-1} t_q t_b^{-1}$. This proves Theorem 4.3(3) for the given case.

5.4. The case of $r = 1$, $d > 2$, and $\langle a, b \rangle = v^{m_v}$. Assume that $m_v > 0$ first. In our situation, the corresponding periodic semi-meanders, up to shifting, are given as

\[
(5.4.1) \quad a = \bullet \cdots \bullet + \cdots + \bullet + \cdots \quad \text{and} \quad b = \bullet \cdots \bullet + \cdots + \bullet
\]

Note that the two arcs in $a$ and $b$ must be adjacent, otherwise, $\langle a, b \rangle = 0$. Let $\tau^-,\tau,\tau^+$ denote the three thick dots in the picture, from left to right; we did not draw other straight lines. So if

$\tau = \sigma^{-n} \tau^+$ and $\tau^- = \sigma^{-n} \tau$, then $m_v = n_{\tau} + n_{\tau^+}$.
Unwinding the definition, the morphism $\text{Res}_a \circ \text{Gys}_{b}$ is the composition of the following commutative diagram from the upper-left to the lower-right:

$$\begin{align*}
H^{d-2}(\text{Sh}(G_{S_b}, T_b)) & \xrightarrow{\pi^*_b} H^{d-2}(\text{Sh}(G_{S_b} T_b)) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S_b} T)) (1) \\
\downarrow \text{Restr.} & & \downarrow \text{Restr.} & \\
H^{d-2}(\text{Sh}(G_{S_b} T)_{\{\tau, \tau^+\}}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S_b} T)_{\tau^+}) (1) \\
\downarrow (\theta^{-1})^*, \cong & & \downarrow \pi_a \dagger & \\
H^{d-2}(\text{Sh}(G_{S_a} T_a)) & & & \\
\end{align*}$$

By Theorem $2.32(2)$, the morphism

$$\theta : \text{Sh}(G_{S_b} T)_{\{\tau, \tau^+\}} \hookrightarrow \text{Sh}(G_{S_b} T)_{\tau^+} \xrightarrow{\pi_b} \text{Sh}(G_{S_b}, T_a)$$

is an isomorphism, and the composition

$$\text{Sh}(G_{S_a} T_a) \xleftarrow{\theta^{-1}} \text{Sh}(G_{S_b} T)_{\{\tau, \tau^+\}} \hookrightarrow \text{Sh}(G_{S_b} T)_{\tau} \xrightarrow{\pi_b} \text{Sh}(G_{S_b}, T_b)$$

is exactly the morphism

$$\eta_{a,b,(z),\sharp} : \text{Sh}(G_{S_a} T_a) \longrightarrow \text{Sh}(G_{S_b}, T_b),$$

associated to the link $\eta_{a,b} : S_a \rightarrow S_b$ given by

$$(5.4.3)$$

with shift $t_a t_b^{-1}$ and indentation degree $z$ equal to $\ell(a) - \ell(b)$ if $p$ splits in $E/F$ and to 0 if $p$ is inert in $E/F$. Therefore, $\text{Res}_a \circ \text{Gys}_{b}$ is exactly $p^{\nu(\eta_{a,b})/2} \eta^*_{a,b,(z)} = p^{m_v/2} \eta^*_{a,b,(z)}$ (note the normalization in $(2.25.2)$), verifying Theorem $4.3(2)$.

We now come to the case where $m_v$ is negative. In this case, the picture of $a$ and $b$ in $(5.4.1)$ are swapped. Then we have a commutative diagram similar to $(5.4.2)$:

$$\begin{align*}
H^{d-2}(\text{Sh}(G_{S_b}, T_b)) & \xrightarrow{\pi^*_b} H^{d-2}(\text{Sh}(G_{S_b} T_b)) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S_b} T)) (1) \\
\downarrow \cong & & \downarrow \text{Restr.} & \\
\cong & & \downarrow \text{Restr.} & \\
H^{d-2}(\text{Sh}(G_{S_b} T)_{\{\tau, \tau^+\}}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S_b} T)_{\tau^+}) (1) \\
\downarrow & & \downarrow \pi_a \dagger & \\
H^{d-2}(\text{Sh}(G_{S_a} T_a)) & & & \\
\end{align*}$$

and the composed diagonal morphism gives $\text{Res}_a \circ \text{Gys}_{b}$. Let $\eta_{b,a} : S_b \rightarrow S_a$ denote the inverse link of $\eta_{a,b}$. Since $a$ and $b$ are obtained by swapping with each other from the previous case, the link morphism $\eta_{b,a,(z),\sharp} : \text{Sh}(G_{S_b}, T_b) \rightarrow \text{Sh}(G_{S_a} T_a)$ with shift $t_b t_a^{-1}$ exists, where $z = \ell(a) - \ell(b)$ if $p$ splits in $E$ and $z = 0$ if $p$ is inert in $E$. Note also that $\eta_{b,a,(z),\sharp}$ is flat of degree $p^{-m_v} = p^{\nu(\eta_{b,a})}$ by Theorem $2.32$. One sees easily that $\text{Res}_a \circ \text{Gys}_{b} = \text{Tr}_{\eta_{b,a,(z),\sharp}}$. By Lemma $2.29(3)$, this is exactly $p^{-m_v/2} (\eta^*_{b,a,(z)})^{-1} = p^{(\ell(a) + \ell(b))/2} \eta^*_{a,b,(z)}$. This proves Theorem $4.3(2)$ in this case.
5.5. **Decomposition of periodic semi-meanders.** Before proceeding to the inductive proof, we discuss certain ways to “decompose” periodic semi-meanders appearing in the induction. Let \( \mathfrak{a} \in \mathfrak{B}_S^b \) be a periodic semi-meander. We call \( \Delta \) a subset of \( r' \) arcs \( (r' \leq r) \) in \( \mathfrak{a} \) saturated, if for each arc \( \delta \) belonging to \( \Delta \), any arc that is contained in the circuit given by adjoining the arc \( \delta \) with the part of the equator between the two end-nodes of \( \delta \), also belongs to \( \Delta \). For example, if \( \mathfrak{a} = \bullet \cdots \bullet \bullet \), the subset \( \Delta = \bullet \cdots \bullet \bullet \bullet \bullet \) is saturated, but \( \bullet \cdots \bullet \bullet \bullet \bullet \bullet \bullet \) is not.

Now fix a saturated \( \Delta \). We use \( \mathfrak{a}^b \) to denote the periodic semi-meander for \( S \) given by all the arcs in \( \Delta \) and then adjoining straight lines to the rest of the nodes. Then \( S_{\mathfrak{a}^b} \) is the union of \( S \) and all nodes connected to an arc in \( \Delta \). We use \( \mathfrak{a}_{\text{res}} = \mathfrak{a} \setminus \Delta \) to denote the periodic semi-meander for \( S_{\mathfrak{a}^b} \) obtained by removing all the arcs in \( \Delta \) including their end-nodes. In the example above, \( \mathfrak{a}^b = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \) and \( \mathfrak{a}_{\text{res}} = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \), where the plus signs indicates points corresponding to \( S_{\mathfrak{a}^b, \infty} \).

By the construction of the Goren-Oort cycles, we have the following commutative diagram, where the middle square is Cartesian.

\[
\begin{array}{ccc}
\text{Sh}(G_{S,T})_{\mathfrak{a}^c} & \xrightarrow{\pi_{\mathfrak{a}}} & \text{Sh}(G_{S,T}) \\
\pi_{\mathfrak{a}^c} & & \pi_{\mathfrak{a}^c} \\
\downarrow & & \downarrow \\
\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}_{\text{res}}} & \xleftarrow{\pi_{\mathfrak{a}_{\text{res}}}} & \text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b}) \\
\downarrow & & \downarrow \\
\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}}} & \xrightarrow{\pi_{\mathfrak{a}}} & \text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}_{\text{res}}} \\
\end{array}
\]

Since the construction of this diagram comes from the unitary Shimura varieties, we point out that, by Remark 2.16, the shift \( t_{\mathfrak{a}} \) is the product of the shift \( t_{\mathfrak{a}^b} \) and the shift \( t_{\mathfrak{a}_{\text{res}}} \) for the correspondence \( \text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}_{\text{res}}} \rightarrow \text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}}) \). From the commutative diagram, we can decompose the morphisms \( \text{Res}_{\mathfrak{a}} \) and \( \text{Gys}_{\mathfrak{a}} \) as compositions as follows:

\[
\text{Gys}_{\mathfrak{a}} : H^{d-2r}(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}_{\text{res}}})) \xrightarrow{\pi_{\mathfrak{a}_{\text{res}}}} H^{d-2r}(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})_{\mathfrak{a}})) \xrightarrow{\text{Gysin}} H^{d-2r}(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})))(r - r')
\]

\[
\pi_{\mathfrak{a}^b} \xrightarrow{\pi_{\mathfrak{a}^b}} H^{d-2r}(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})))(r - r') \xrightarrow{\text{Gysin}} H^d(\text{Sh}(G_{S,T}))(r)
\]

and

\[
\text{Res}_{\mathfrak{a}} : H^d(\text{Sh}(G_{S,T}))(r) \xrightarrow{\text{Restr}} H^d(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b})))(r) \xrightarrow{\pi_{\mathfrak{a}^b, r'}} H^{d-2r'}(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b}))) \xrightarrow{\pi_{\mathfrak{a}_{\text{res}}, r'}} H^{d-2r}(\text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b}))).
\]

Here, to get the decomposition for \( \text{Res}_{\mathfrak{a}} \), we have used the fact that the trace map \( \text{Tr}_{\pi_{\mathfrak{a}}} \) can be factorized as

\[
R^{2r-2r'} \pi_{\mathfrak{a}^b, T_{\mathfrak{a}^b}} \circ \mathfrak{Q}_{\ell}(r) \xrightarrow{\text{Tr}_{\mathfrak{a}^b}} R^{2r-2r'} \pi_{\mathfrak{a}_{\text{res}}}(\mathfrak{Q}_{\ell})(r - r') \xrightarrow{\text{Tr}_{\mathfrak{a}_{\text{res}}}} \mathfrak{Q}_{\ell}.
\]

Summing up everything in short, we obtain thus

\[
\text{Gys}_{\mathfrak{a}} = \text{Gys}_{\mathfrak{a}^b} \circ \text{Gys}_{\mathfrak{a}_{\text{res}}}, \quad \text{Res}_{\mathfrak{a}} = \text{Res}_{\mathfrak{a}_{\text{res}}} \circ \text{Res}_{\mathfrak{a}^b}, \quad \text{and} \quad t_{\mathfrak{a}} = t_{\mathfrak{a}^b} t_{\mathfrak{a}_{\text{res}}}.
\]

We will apply this to appropriate \( \Delta \)'s to reduce the calculation to \( \text{Sh}(G_{S_{\mathfrak{a}^b}, T_{\mathfrak{a}^b}}) \) and reduce the inductive proof essentially to the cases considered above.
5.6. **Decomposition of periodic semi-meanders continued.** We will also encounter the following situation: assume that the set of arcs in a periodic semi-meander **a** is the disjoint union of two *saturated* subset **Δ** and **Δ′**; put $s = \#\Delta$ and $s' = \#\Delta'$ so that $r = s + s'$. We will show that **Δ** and **Δ′** “behave” independently.

We write $\mathbf{a}^b$ (resp. $\mathbf{a}^r$) for the periodic semi-meander for $\mathbb{S}$ given by all arcs in $\mathbf{Δ}$ (resp. $\mathbf{Δ}'$) and then adjoining straight lines to the rest of the nodes. We put $a_{\text{res}}$ (resp. $a'_{\text{res}}$) for the periodic semi-meander for $\mathbb{S}_{\mathbf{a}^b}$ (resp. $\mathbb{S}_{\mathbf{a}^r}$) obtained by removing all arcs in $\mathbf{Δ}$ (resp. $\mathbf{Δ}'$) and replacing all their end-nodes by plus signs.

In this case, in view of the construction of the Goren-Oort cycle $\mathbb{S}(\mathbb{G}_{\mathbf{a}^b}, \mathbb{T})_a$, we could either go through the arcs in $\mathbf{Δ}$ first, or the arcs in $\mathbf{Δ}'$ first. So we have the following commutative Cartesian diagram:

$$
\begin{array}{cccc}
\mathbb{S}(\mathbb{G}_{\mathbf{a}^b}, \mathbb{T}) & \xrightarrow{\pi^b} & \mathbb{S}(\mathbb{G}_{\mathbf{a}^b}, \mathbb{T}_{\mathbf{a}^b}) & \\
\downarrow & & \downarrow & \\
\mathbb{S}(\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r}) & \xrightarrow{\pi^r} & \mathbb{S}(\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r}) & \\
\downarrow & & \downarrow & \\
\mathbb{S}(\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r}) & \xrightarrow{\pi^t_{\text{res}}} & \mathbb{S}(\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r}) & \\
\end{array}
$$

where $\pi_\Delta$ and $\pi_{\Delta'}$ are the morphisms defined by the natural pull-back of upper-right and lower-left Cartesian squares, respectively. If $t_{\mathbf{a}^b}$ (resp. $t_{\mathbf{a}^r}$) denotes the shift of the correspondence from bottom right to top right (resp. bottom left), then Remark 2.16 gives an equality

$$
t_{\mathbf{a}^b} t_{\mathbf{a}^r} = t_a = t_{\mathbf{a}^b} t_{\mathbf{a}^r}
$$

in $E^{\times, c_1} \langle \mathcal{L}^{\infty} \rangle / \mathcal{O}_{\mathbb{C}_q}^\times$.

This implies that both $\pi_\Delta$ and $\pi_{\Delta'}$ are iterated $\mathbb{P}^1$-bundles of relative dimensions $s$ and $s'$ respectively. We use $\pi_\Delta!$ to denote the natural morphism

$$
\pi_\Delta!: H^*_c(\mathbb{S}(\mathbb{G}_{\mathbf{a}^b}, \mathbb{T})_a, \mathbb{F}_p, \mathcal{L}^{(k,w)}_{\mathbb{G}_{\mathbf{a}^b}, \mathbb{T}_{\mathbf{a}^b}}(s)) \xrightarrow{\cong} H^*_c(\mathbb{S}(\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r})_{\text{res}}, \mathbb{F}_p, \mathcal{L}^{(k,w)}_{\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r}}(s))
$$

where the last map is induced by the trace isomorphism $R^2* \pi_\Delta! \mathcal{L}^{(k,w)}_{\mathbb{G}_{\mathbf{a}^b}, \mathbb{T}_{\mathbb{C}}}(s) \cong \mathcal{L}^{(k,w)}_{\mathbb{G}_{\mathbf{a}^b}, \mathbb{T}_{\mathbb{C}}}$. As a consequence of the Cartesian property and Theorem 2.32, we have the following commutative diagram (which is placed into 5.6.1 vertically on the right)

$$
\begin{array}{cccc}
H^{d-2s'}(\mathbb{S}(\mathbb{G}_{\mathbf{a}^r}, \mathbb{T}_{\mathbf{a}^r})_{\text{res}})(s) & \xrightarrow{\pi^t_{\text{res}}!} & H^{d-2s'}(\mathbb{S}(\mathbb{G}_{\mathbb{C}}, \mathbb{T}_{\mathbb{C}}))(s) & \\
\downarrow & & \downarrow & \\
H^{d-2s-2s'}(\mathbb{S}(\mathbb{G}_{\mathbb{C}}, \mathbb{T}_{\mathbb{C}})) & \xrightarrow{\pi^t_{\text{res}}!} & H^{d-2s-2s'}(\mathbb{S}(\mathbb{G}_{\mathbb{C}}, \mathbb{T}_{\mathbb{C}}))(s') & \\
\end{array}
$$

5.7. **Inductive proof of Theorem 4.3.** We now start the proof of Theorem 4.3 by induction on $d = \#\mathbb{S}_{\infty}$ or equivalently the dimension of the Shimura variety $\mathbb{S}(\mathbb{G}, \mathbb{T})$ (and also on $r$ by keeping $d - 2r$ fixed throughout the induction). The base case $d = 0$ and $d = 1$ trivial (as there is no nontrivial periodic semi-meander).

We now assume that Theorem 4.3 holds for all Shimura varieties $\mathbb{S}(\mathbb{K}, \mathbb{T}, \mathbb{S})$ with $\#\mathbb{S}_{\infty} < d$. We now fix $\mathbb{S}, \mathbb{T}$ so that $\#\mathbb{S}_{\infty} = d$. The case of $r = 0$ is clear. We henceforth assume that $r > 0$.

Let $\mathbf{a}, \mathbf{b} \in \mathbb{B}_{\mathbb{S}}$ be as in Theorem 1.3. We fix a basic arc $\delta_\mathbf{b}$ of $\mathbf{b}$, with right end-node $\tau \in \mathbb{S}_{\infty}$ (and left end-node $\tau^{-} \in \mathbb{S}_{\infty}$). As in Subsection 5.5, we use $\mathbf{b}_{\text{res}} \in \mathbb{B}_{\mathbb{S}_{\infty}}(\tau)$, to denote the periodic
semi-meander $b \setminus \delta_b$ obtained by removing $\delta_b$ from $b$ and replacing its end-nodes by plus signs. We will use $\delta_b$ itself to denote the corresponding $b'$, that is, we also view $\delta_b$ as a periodic semi-meander for $S$ with only one arc $\delta_b$ (and $d - 2$ straight lines).

The basic idea is to factor the Gysin map $\text{Gys}_b$ using $\delta_b$, in the sense of Subsection 5.5 and to factor the restriction map $\text{Res}_a$ according to the following list of four cases.

(i) The two nodes $\tau, \tau^-$ are both linked to straight lines in $a$; this forces us to fall into the case (1) of Theorem 4.3.

(ii) There is a (basic) arc $\delta_a$ in $a$ linking $\tau^-$ to $\tau$ from left to right, so that $\delta_a$ and $\delta_b$ form a contractible circle in $D(a, b)$; in other words, one has $\delta_a = \delta_b$ up to deformation the arcs. We shall reduce the proof of Theorem 4.3 to the case for $\tau' = S \cup \{\tau, \tau^-, \tau^+\}, \tau^+ = T \cup \{\tau\}, \delta_a = a')\delta_a$ and $\delta_b \setminus \delta_a$ and it hence follows from the inductive hypothesis. In particular, we shall see that the contractible circle $\delta_a$ and $\delta_b$ contributes a factor of $-2p^t(\delta_b)$. 

(iii) There is an arc $\delta_a$ in $a$ connecting $\tau$ and $\tau^-$ wrapped around the cylinder from right to left; in other words, $\delta_a$ and $\delta_b$ form a non-contractible circle in $D(a, b)$. This can only happen if $r = d/2$. We will show that the composition $\text{Res}_a \circ \text{Gys}_b$ is essentially the $T_p$-operator composed with $\text{Res}_a \setminus \delta_a \circ \text{Gys}_b \setminus \delta_b$ for the Shimura variety with $S' = S \cup \{\tau, \tau^-, \tau^+, \tau^+\}$ and $\tau' = T \cup \{\tau\}$, up to some link morphism which we make explicit later.

(iv) Neither of above happens. Then, in $a$, either $\tau$ is connected by an arc whose other end-node is not $\tau^-$, and/or $\tau^-$ is connected by an arc whose other end-node is not $\tau$. In either case, we will reduce to a case with the two nodes $\tau$ and $\tau^-$ removed, after composing with certain link morphism.

We now treat each of the cases separately.

5.8. Case (i). This is the case when $\tau$ and $\tau^-$ are connected to straight lines in $a$. We have $\langle a \mid b \rangle = 0$; and hence we need to show that Theorem 4.3(1) happens. Let $a^*$ denote the periodic semi-meander for $S$ given by removing the two straight lines of $a$ connected to $\tau$ and $\tau^-$ and reconnecting $\tau$ and $\tau^-$ by a (basic) arc. In particular, $a^* \in \mathcal{B}_S^{-1}$.

By the discussion of Subsection 5.5, we see that the morphisms $\text{Res}_a \circ \text{Gys}_b$ is the composition from top-left to the bottom-right of the following commutative diagram by going first downwards and then rightwards.

$$
\begin{align*}
H^{d-2r}(\text{Sh}(G_{S, \tau_b})) & \xleftarrow{\pi_{\tau_b} \circ \text{Gys}_{\text{res}}} H^{d-2}(\text{Sh}(G_{S, \tau})\delta_b)(r-1) \xrightarrow{\text{Restr.}} H^{d-2}(\text{Sh}(G_{S, \tau})a^*)(r-1) \\
& \xrightarrow{\text{Gysin}} H^d(\text{Sh}(G_{S, \tau})(r) \xrightarrow{\text{Restr.}} H^d(\text{Sh}(G_{S, \tau})a)(r) \xrightarrow{\pi_{a, !}} H^{d-2r}(\text{Sh}(G_{S, \tau_a}))
\end{align*}
$$

Here, the square is commutative because the corresponding morphisms on the Shimura varieties form a Cartesian square. The diagram implies that the $\pi$-component of $\text{Res}_a \circ \text{Gys}_b$ factors through the cohomology group

$$
H^{d-2}(\text{Sh}(G_{S, \tau})a^*)(r-1) \cong H^{d-2(r+1)}(\text{Sh}(G_{S^*, \tau^+}))(r-1)^{\oplus(r+1)}
$$

which is the cohomology of a Goren-Oort cycle for a periodic semimeander with $r + 1$ arcs. This proves Theorem 4.3 in its case (1).

5.9. Case (ii). This is the case when there is a basic arc $\delta_a$ in $a$ linking $\tau^-$ to $\tau$ from left to right and hence $\delta_a$ and $\delta_b$ together form a contractible circle in $D(a, b)$; in other words, one has $\delta_a = \delta_b$. We thus identify $\delta_a$ with $\delta_b$ and write $\delta$ for both, viewed as a periodic semi-meander for $S$ with only
one arc \( \delta \). We write \( a_{\text{res}} = a \setminus \delta \) for the periodic semi-meander for \( S_\delta \) obtained by removing \( \delta \) from \( a \) and turning its end nodes into plus signs.

Using the discussion of Subsection 5.5, the morphism \( \text{Res}_a \circ \text{Gys}_b \) is the composition from the upper-left to the upper-right going all the way around: first downwards to the bottom, then all the way to the right, and finally upwards.

\[
\begin{align*}
H^{d-2r}(\text{Sh}(G_{S_a,T_a})) & \xrightarrow{\text{Gys}_{a_{\text{res}}}} H^{d-2r}(\text{Sh}(G_{S_b,T_b})) \\
H^{d-2}(\text{Sh}(G_{S_a,T_a}))(r - 1) & \xrightarrow{\pi^*_\delta} H^{d-2}(\text{Sh}(G_{S_b,T_b}))(r - 1) \\
H^{d-2}(\text{Sh}(G_{S,T})) & \xrightarrow{\text{Res}_{a_{\text{res}}}} H^d(\text{Sh}(G_{S,T}))(r) \\
H^{d-2}(\text{Sh}(G_{S,T})) & \\
\end{align*}
\]

As in Subsection 5.2, the composition of the bottom line is given by the excessive intersection formula, that is to take the cup product with the first Chern class of the normal bundle of the embedding \( \text{Sh}(G_{S,T})_\delta \hookrightarrow \text{Sh}(G_{S,T}) \), which is \( -2p^{(\delta)} \) times the class of the canonical quotient bundle for the \( \mathbb{P}^1 \)-bundle given by \( \pi_\delta \), according to Proposition 2.31(2). Therefore, the dotted arrow in the middle is simply multiplication by \( -2p^{(\delta)} \). From this, we deduce that

\[
\text{Res}_a \circ \text{Gys}_b = -2p^{(\delta)} \cdot \text{Res}_{a_{\text{res}}} \circ \text{Gys}_{b_{\text{res}}},
\]

where the latter morphism is constructed over the Shimura variety \( \text{Sh}(G_{S_a,T_a}) \) of lower dimension. Note also that \( \ell(a) - \ell(a_{\text{res}}) = \ell(b) - \ell(b_{\text{res}}) = \ell(\delta) \). This concludes the inductive proof in this case.

5.10. Case (iii). This is the case when there is an arc \( \delta_a \) in \( a \) connecting \( \tau \) and \( \tau^- \) wrapped around the cylinder from right to left, and hence \( \delta_a \) and \( \delta_b \) together form a non-contractible circle in \( D(a,b) \). We are forced to have \( d = 2r \) in this case. Moreover, the arc \( \delta_a \) must be lying over all other arcs of \( a \) (if there is any). We now define a list of notations followed by an example.

- let \( \delta_{a\cdot} \) (resp. \( \delta_{b\cdot} \)) denote the periodic semi-meander of two nodes just consisting of \( \delta_a \) (resp. \( \delta_b \)) and its end-nodes;
- let \( a^\ast = a \setminus \delta_a \) denote the periodic semi-meander for \( S_a \) given by removing the arc \( \delta_a \) from \( a \) and replacing the nodes at \( \tau \) and \( \tau^- \) by plus signs;
- let \( a^\circ \) denote the periodic semi-meander for \( S \) given by removing the arc \( \delta_a \) and adjoining two straight lines attached to both \( \tau \) and \( \tau^- \);
- let \( a^\ast \) denote the semi-meander for \( S \) obtained by replacing the arc \( \delta_a \) in \( a \) with \( \delta_b \) instead.

For example, if \( a = \bullet \circ \circ \circ \circ \bullet \) and \( b = \circ \circ \circ \circ \circ \bullet \) and we choose \( \delta_b \) to be the arc of \( b \) linking the first and the last nodes, then \( \delta_a \) is the arc linking the first and the last nodes (but “over” all other arcs). In this case, we have

\[
\delta_{a\cdot} = \bullet + + + + \quad , \quad a^\ast = + \circ \circ \circ \circ + \quad , \quad a^\circ = \downarrow \circ \circ \circ \circ \circ \quad , \quad \text{and} \quad a^\ast = \circ \circ \circ \circ \circ .
\]

By the discussion of Subsection 5.5, we see that the morphism \( \text{Res}_a \circ \text{Gys}_b \) is the composition from top-left to bottom-left of the following diagram, by going first rightward to the end, then
downwards to the bottom, and finally to the left by the long arrow:

$$(5.10.1)$$

$$H^0(\text{Sh}(G_{S_b,T_b}))$$

$$H^{d-2}(\text{Sh}(G_{S,T})(\frac{d}{2} - 1)) \xrightarrow{\pi_{a_b}} H^{d-2}(\text{Sh}(G_{S,T})(\frac{d}{2} - 1)) \xrightarrow{\text{Gysin}} H^d(\text{Sh}(G_{S,T}))$$

$$H^{d-2}(\text{Sh}(G_{S_{\delta_b},T_{a_b}})(\frac{d}{2} - 1)) \xrightarrow{\pi_{a_b}} H^{d-2}(\text{Sh}(G_{S,T})(\frac{d}{2} - 1)) \xrightarrow{\text{Gysin}} H^d(\text{Sh}(G_{S,T}))$$

The top two squares of the above diagram are commutative because the corresponding morphisms on the Shimura varieties form Cartesian squares. The middle rectangle of $(5.10.1)$ is commutative by $(5.6.2)$ (applied with our $\alpha$, $\alpha'$, and $\delta_b$ being the $a$, $\alpha'$, and $\alpha''$ therein, respectively). Now, for the bottom rectangle, we are simply working with the Shimura variety $\text{Sh}(G_{\delta_b,T_b})$ and hence are reduced to the case of $d = 2$. Let $t_{\alpha',a}$ for $? = \emptyset$ or $\ast$ denote the shift of the correspondence from $\text{Sh}(G_{\alpha',T_{a'}})$ to $\text{Sh}(G_{\delta_b,T_{a_b}})$, so that by Subsection $3.7$ we have

$$t_{\alpha',a} = t_{\alpha}t_{a_{\ast b}}^{-1} = t_{\alpha}t_{a_{\ast b}}^{-1}$$

$$t_{\alpha',a'} = t_{\alpha'}t_{b_{\ast a_b}}^{-1}$$

(Note that the way we defined the middle line is to pull back the description of $\text{Sh}(G_{S,T})(\delta_b)$. Using Subsection $3.3$, we see that the dotted downward arrow on the left is nothing but the operator $T_p$ times a link morphism $\eta^*$ associated to the link $\eta : S_a \to S_{a'}$, with indentation degree $-2\ell(\delta_b, \ast)$ and shift

$$t_{\alpha',a}^{-1}t_{a_{\ast b}}^{-1} = t_{\alpha'}t_{b_{\ast a_b}}^{-1}(t_{a_{\ast b}}^{-1})^{-1}.$$

To sum up, the morphism $\text{Res}_{a_b} \circ \text{Gys}_{b_b}$ is the same as the composition of the downward arrows on the left in $(5.10.1)$. In other words, we have

$$\text{Res}_{a_b} \circ \text{Gys}_{b_b} = T_p \circ \eta^* \circ \text{Res}_{a_{\ast b}} \circ \text{Gys}_{b_{\ast a_b}}.$$

Note that we always have the numerical equality

$$-2\ell(\delta_b, \ast) + (\ell(a_{\ast}) - \ell(b_{\ast})) = \ell(a) - \ell(b) - g.$$

This agrees with the combinatorics side and hence concludes the induction of Theorem $4.3$ in case $(iii)$. 

5.11. Case (iv). Recall that $\delta_b$ is a basic arc of $b$ linking $\tau$ with $\tau^-$ from right to left. This is when at least one of $\tau$ and $\tau^-$ is linked to an arc in $a$ that is not connected to the other node. We start with a long list of combinatorics construction, followed by an example.

- Let $\alpha$ be the periodic semi-mearner for $S_{\delta_b}$ given by first adjoining the basic arc $\delta_b$ to $a$ from underneath the band, and then replace the nodes $\tau, \tau^-$ by plus signs to “pull the strings”.  

• Let $a^b$ denote the periodic semi-meander for $S$ that consists of two straight lines at both $\tau$ and $\tau^-$, all arcs in $a$ that do not intersect with these two straight lines, and straight lines at the nodes that are not connected to anything above. Let $r' (< r)$ denote the number of arcs in $a^b$ so that $a^b \in \mathcal{W}^r$.

• Let $a^+_1$ denote the periodic semi-meander for $S_{\delta_b}$ obtained by removing the two straight lines at both $\tau$ and $\tau^-$ from $a^b$ and replacing the nodes at $\tau, \tau^-$ by plus signs.

• We use $a^+_c$ to denote the periodic semi-meander for $S$ given by replacing in $a^b$ the two straight lines connected to $\tau$ and $\tau^-$ by $a^b$.

• We use $\delta_{a^b}$ to denote the periodic semi-meander for $S_{a^b}$ consisting of only one arc $\delta_b$ (and all straight lines of $a^b$).

• We choose an arc $\delta_a$ which has either $\tau$ as its left end node, or $\tau^-$ as its right end node; such an arc $\delta_a$ exists by the assumption of Case (iv). We use $\tau'$ to denote the right endpoint of $\delta_a$; therefore, $\tau'$ is neither one of $\tau$ and $\tau^-$ in the first situation, and it is $\tau' = \tau^-$ in the second situation.

• We use $a^b_{\tau'}$ to denote the periodic semi-meander for $S$ given by deleting from $a^b$ the two straight lines connected to the end-nodes of $\delta_a$ and then adjoining the arc $\delta_a$.

• We use $\delta_{a^b}$ to denote the periodic semi-meander for $S_{a^b}$ consisting of only one arc $\delta_a$ (and all straight lines of $a^b_{\tau'}$).

• We use $\eta_{a^b, a^b_{\tau'}}$ to denote the link from $S_{a^b_{\tau'}}$ to $S_{a^b}$ obtained by “pulling strings” on $D(\delta_b/a^b, \delta_a/a^b)$.

• We use $a^b_{\text{res}}$ to denote the periodic semi-meander for $S_{a^b_{\tau'}}$ given by deleting all arcs in $a$ that has already appeared in $a^b_{\tau'}$, and changing their end-nodes to plus signs.

• We use $a^b_{\text{res}}$ to denote the periodic semi-meander for $S_{a^b}$ nodes given by deleting all arcs in $a^b$ that has already appeared in $a^b_{\tau'}$, and changing their end nodes to plus signs.

• We use $\eta_{a^b, a^b_{\tau'}}$ to denote the link from $S_{a}$ to $S_{a^b_{\tau'}}$ which is the restriction of $\eta_{a^b_{\tau'}, a^b_{\tau'}}$ to $S_{a}$.

• We use $b^b_{\text{res}}$ to denote the semi-meander for $S_{\delta_b}$ obtained by deleting the arc $\delta_b$ and replacing the nodes $\tau, \tau^-$ by plus signs.

For example, if $b$ has a basic arc connecting the node 1 with 2 (starting with node 0 on the left) and if $a = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$, then the arc $\delta_a$ has to be the one connecting nodes 2 and 5 and $\tau'$ is the node 5. We have

\[
\begin{align*}
a^b &= \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet, & a^+ &= + + + + + + + +, & a^\tau &= \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet, \\
a^b_{\tau'} &= \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet, & \delta_{b/a^b} &= + + + + + + + +, & \delta_{a/a^b} &= + + + + + + + +, \\
a^b_{\text{res}} &= + + + + + + + +, & a^b &= + + + + + + + +, & a^b_{\text{res}} &= + + + + + + + +, \\
\eta_{a^b_{\tau'}, a^b_{\tau'}} &= + + + + + + + +, & \eta_{a^b, a^b_{\tau'}} &= + + + + + + + +.
\end{align*}
\]

(In other examples when one of $\tau, \tau'$ is connected to a straight line, the link $\eta_{a^b, a^b_{\tau'}}$ may not be trivial.)

We consider the link morphism associated with $\eta_{a^b_{\tau'}, a^b_{\tau'}}$. We distinguish two cases:

**Case (a).** Suppose that the left node of $\delta_a$ is $\tau$. This is the case of the example above. All the curves in the link $\eta_{a^b_{\tau'}, a^b_{\tau'}}$, except for one denoted by $\xi$, are straight lines, and $\xi$ sends $\tau^-$ to $\tau'$. By Theorem 2.32(2), there exists a link morphism $\eta_{a^b_{\tau'}, a^b_{\tau'}}$ with indentation degree $\ell(\delta_a) - \ell(\delta_b)$ and shift $t_{a^b_{\tau'}} t_{a^b_{\tau'}}^{-1}$ (and also a link morphism on the local system as in Theorem 2.32(2)(d)),
which fits into the following commutative diagram

\[
\begin{array}{ccc}
\text{Sh}(G_{S_{\theta_1}^\circ, T_{\theta_1}^\circ})_{\delta_a} & \xleftarrow{\sim} & \text{Sh}(G_{S_{\theta_1}^\circ, T_{\theta_1}^\circ})_{\{r, r'\}} \\
\pi_{\delta_a} & \searrow & \eta_{\theta_1^s, a_{r'}^s, z} \\
\text{Sh}(G_{S_{\theta_1}^\circ, T_{\theta_1}^\circ}) & \xrightarrow{\sim} & \text{Sh}(G_{S_{\theta_1}^\circ, T_{\theta_1}^\circ}).
\end{array}
\]

By Theorem 2.32(2)(c) \( \eta_{a_{r'}, a_{r'}^s, z} \) is finite flat of degree \( p^y \) with \( y := v(\eta_{a_{r'}, a_{r'}^s, z}) = \ell(\delta_a) + \ell(\delta_b) \).

Consider the \((r - r' - 1)\)-th iterated \( P^1 \)-bundle \( \pi_{a_{r'}^s, a_{r'}^s, z} : \text{Sh}(G_{S_{\theta_1}^\circ, T_{\theta_1}^\circ})_{a_{r'}^s, z} \to \text{Sh}(G_{S_a, T_a}). \)

By applying repeatedly applying [TX13+a, Proposition 7.17] and Construction 2.15, one produces easily a commutative diagram:

\[
(5.11.1) \quad \text{Sh}(G_{S_{\theta_1}^\circ, T_{\theta_1}^\circ})_{a_{r'}^s, z} \xrightarrow{\pi_{a_{r'}^s, a_{r'}^s, z}} X_1 \xrightarrow{\eta_{a_{r'}, a_{r'}^s, z}} X_2 \cdots \xrightarrow{X_{r-r'-2}} X_{r-r'-1} \xrightarrow{\pi_{a_{r'}^s, a_{r'}^s, z}} \text{Sh}(G_{S_a, T_a})
\]

where each \( \pi_i^1 \) and \( \pi_i^0 \) are both \( P^1 \)-fibrations, and the vertical arrows are link morphisms (associated to certain links), the composition of top (resp. bottom) horizontal arrows is \( \pi_{a_{r'}^s, a_{r'}^s, z} \) (resp. \( \pi_{a_{r'}^s, a_{r'}^s, z} \)). Of course, there exist at the same time link morphism \( \eta_{a_{r'}, a_{r'}^s, z} \) and \( \eta_{a_{r'}, a_{r'}^s, z} \) on the étale local systems satisfying a similar commutative diagram. For instance, to construct \( \eta_{a_{r'}, a_{r'}^s, z} : X_1 \to Y_1 \), one chooses a basic arc \( \delta_a \) in \( a_{r'}^s \). Let \( a_{r'}^s \), be the periodic semi-meander obtained by removing \( \delta_a \) from \( a_{r'}^s \) and replacing the nodes of \( c \) by plus signs, and \( a_{r',1}^s \) be the periodic semi-meander obtained by removing from \( a_{r',1}^s \), the straight lines at the nodes of \( c \) and adjoining \( c \). Put \( X_1 := \text{Sh}(G_{S_{a_{r',1}^s}, T_{a_{r',1}^s}})_{a_{r',1}^s} \), and denote by \( \pi_1^1 : \text{Sh}(G_{S_{a_{r',1}^s}, T_{a_{r',1}^s}})_{a_{r',1}^s} \to X_1 \) the \( P^1 \)-fibration given by the arc \( c \). Let \( c^0 \) be the arc \( \eta_{a_{r'}, a_{r'}^s, z} (c) \); it is a basic arc in \( a_{r'}^s \).

We define periodic semi-meanders \( a_{r,1}^s \) and \( a_{r,1}^s \) in the same way as \( a_{r}, a_{r,1}^s \) and \( a_{r,1}^s \) with \( c \) replaced by \( c^0 \). Then we have a \( P^1 \)-fibration

\[
\pi_1^0 : \text{Sh}(G_{S_{a_{r,1}^s}, T_{a_{r,1}^s}})_{a_{r,1}^s} \to Y_1 := \text{Sh}(G_{S_{a_{r,1}^s}, T_{a_{r,1}^s}})_{a_{r,1}^s},
\]

given by \( c^0 \). If \( \eta_{1} : S_{a_{r,1}^s} \to S_{a_{r,1}^s} \) denotes the link induced by \( \eta_{a_{r'}, a_{r'}^s, z} \), then [TX13+a, Proposition 7.17] and Construction 2.15 implies the existence of the link morphism \( \eta_{a_{r}, a_{r,1}^s, z} \) which fits into the left commutative square of (5.11.1). This finishes the construction of \( X_1 \) and \( Y_1 \). The induced link \( \eta_{1} \) has the same property as \( \eta_{a_{r}, a_{r,1}^s, z} \), namely all the curves of \( \eta_{1} \) are straight lines except possible for one. Thus, the rest of (5.11.1) can be constructed inductively in a similar way.

We check easily that the final link morphism \( \eta_{a,a_{r,1}^s, z} \) with shift \((t_{a_{r,1}^s}, t_{a_{r,1}^s}^{-1})t_{a_{r,1}^s}^{-1} \) and indentation degree \( \ell(\delta_a) - \ell(\delta_b) + \ell(a_{r}^s) - \ell(a_{r,1}^s) \) if \( p \) splits in \( E/F \) and degree 0 if \( p \) is inert in \( E/F \). Note also that even through each \( \eta_{a_{r,1}^s, z} \) is not unique (since there are many ways to choose a basic arc of \( a_{r,1}^s \) for instance), the final link morphism \( \eta_{a,a_{r,1}^s, z} \) is uniquely determined by the uniqueness of link morphisms. By [TX13+a, Proposition 7.17(3)] and Construction 2.15, \( \eta_{a,a_{r,1}^s, z} \) is finite flat of degree \( p^y(\eta_{a,a_{r,1}^s, z}) \). We have thus the link normalized
morphisms \( \eta^\ast_{a^\circ, a^\circ} \) and \( \eta^\ast_{a^\circ, a^\circ} \) on the corresponding cohomology groups as defined in\(^{(2.25.1)}\) induced by \( (\eta^\ast_{a^\circ, a^\circ}, \xi^\ast_{a^\circ, a^\circ}) \) and \((\eta^\ast_{a^\circ, a^\circ}, \xi^\ast_{a^\circ, a^\circ})\) respectively.

**Case (b)** Suppose now that the right node of \( \delta_a \) is \( \tau^\ast \). Then the only genuine curve in the link \( \eta_{a^\circ, a^\circ} \) is turning to the left with displacement \( y = \ell(\delta_a) + \ell(\delta_b) \). Let \( \eta_{a^\circ, a^\circ} \) be the inverse link of \( \eta_{a^\circ, a^\circ} \). Applying the discussion in **Case (a)** to \( \eta_{a^\circ, a^\circ} \), one gets a link morphism \( \eta_{a^\circ, a^\circ} : Sh(G_{a^\circ, \tau}) \to Sh(G_{a^\circ, \tau}) \) of indentation degree \( \ell(\delta_b) - \ell(\delta_a) \) and shift \( t_{a^\circ, t_{a^\circ}} \); it in turn induces a link morphism \( \eta_{a^\circ, a^\circ} : Sh(G_{a^\circ, \tau, a^\circ}) \to Sh(G_{a^\circ, \tau, a^\circ}) \) associated to the inverse link of \( \eta_{a^\circ, a^\circ} \) of indentation degree \( \ell(\delta_b) - \ell(\delta_a) + \ell(a^\circ_{a^\circ}) \), and shift \( t_{a^\circ, t_{a^\circ}} \).

By Lemma \( 2.29 \) we get well defined link morphisms on cohomology

\[
\eta^\ast_{a^\circ, a^\circ} = (\eta^\ast_{a^\circ, a^\circ})^{-1} = p^{\nu/2} \eta_{a^\circ, a^\circ}^\ast : H_{et}^{d-2r^\ast-2}(Sh(G_{a^\circ, \tau, a^\circ})) \to H_{et}^{d-2r^\ast-2}(Sh(G_{a^\circ, \tau, a^\circ}))
\]

of indentation degree \( \ell(\delta_a) - \ell(\delta_b) \) associated to the link \( \eta_{a^\circ, a^\circ} \) and

\[
\eta^\ast_{a^\circ, a^\circ} = (\eta^\ast_{a^\circ, a^\circ})^{-1} = p^{\nu/2} \eta_{a^\circ, a^\circ}^\ast : H_{et}^{d-2r^\ast}(Sh(G_{a^\circ, \tau})) \to H_{et}^{d-2r^\ast}(Sh(G_{a^\circ, \tau}))
\]

of indentation degree \( \ell(\delta_b) - \ell(\delta_a) + \ell(a^\circ_{a^\circ}) \) associated to the link \( \eta_{a^\circ, a^\circ} \).

**Lemma 5.12.** Under the above notation, put \( x = \ell(a^\circ_{a^\circ}) - \ell(a^\circ_{a^\circ}) \) and \( y = \ell(\delta_a) + \ell(\delta_b) \). Then in both **Case (a)** and **Case (b)** above, there exists a commutative diagram of cohomology groups:

\[
\begin{array}{ccc}
H_{et}^{d-2r^\ast-2}(Sh(G_{a^\circ, \tau})) & \xrightarrow{Res} & H_{et}^{d-2r^\ast-2}(Sh(G_{a^\circ, \tau})) \\
| p^{\nu/2} \eta^\ast_{a^\circ, a^\circ} & | & | p^{\nu/2} \eta^\ast_{a^\circ, a^\circ} \\
H_{et}^{d-2r^\ast}(Sh(G_{a^\circ, \tau})) & \xrightarrow{Res} & H_{et}^{d-2r^\ast}(Sh(G_{a^\circ, \tau}))
\end{array}
\]

Note that the composite of the top (resp. bottom) two horizontal morphisms above is nothing but \( Res^\ast_{a^\circ, a^\circ} \) (resp. \( Res^\ast_{a^\circ, a^\circ} \)).

**Proof.** The commutativity of the left square is evident. We check commutativity of the right hand side square case by case.

Suppose first we are in **Case (a)**, i.e. \( \delta_a \) has \( \tau \) as its left node. We distinguish three sub-cases:

**Case (a1)** \( \tau^\ast \) is linked to a straight line in \( a \). Then both \( a^\ast_{a^\circ} \) and \( a^\circ_{a^\circ} \) contain no arcs. It follows that \( x = 0 \), and \( \pi_{a^\circ, a^\circ} \) and \( \pi_{a^\circ, a^\circ} \) are isomorphisms. In this case, the commutativity of the right hand side square is trivial.

**Case (a2)** \( \tau^\ast \) is the left node of an arc in \( a \). Then it is easy to see that \( x = y \) and the link \( \eta_{a^\circ, a^\circ} \) contains only straight lines. By **[TX13+a]** Proposition 7.17(3)** and Construction 2.15**, \( \eta_{a^\circ, a^\circ} \) is an isomorphism. Consider the commutative diagram \( (5.11.1) \). Both top and bottom rows are factorizations of \( (r - r^\ast - 1) \)-th iterated \( \mathbb{P}^1 \)-bundles as in \( (1.1.2) \). For each \( 1 \leq i \leq r - r^\ast - 1 \), let \( \xi_i \in H_{et}^2(Sh_{a^\circ, \tau, \tau, a^\circ}) \) (resp. \( \xi_i \in H_{et}^2(Sh_{a^\circ, \tau, \tau, a^\circ}) \)) be the image of the first Chern class of the tautological quotient line bundle of \( \tau_i \) (resp. \( \tau_i^\ast \)) as considered in Subsection 4.1. Note that the only curve in \( \eta_{a^\circ, a^\circ} \) links the left node of an arc in \( a^\circ_{a^\circ} \) to the left node of an arc of \( a^\circ_{a^\circ} \). Then by **[TX13+a]** Proposition 7.17(3)** and Construction 2.15**, there exists a unique integer \( i_0 \) with \( 1 \leq i_0 \leq r - r^\ast - 1 \) such that \( \eta^\ast_{a^\circ, a^\circ} (\xi_i) = p^{\nu/2} \xi^\ast_{i_0} \), and
\(\eta^*_{a, i, a, b} (\xi_i) = \xi'_i\) for all \(i \neq i_0\). Let

\[
z = \sum_{1 \leq j \leq r-r'-1} \left( \sum_{1 \leq i_1 < \cdots < i_j \leq r-r'-1} \pi_{a_i}^* (z_{i_1, \ldots, i_j}) \cup \xi_{i_1} \cup \cdots \cup \xi_{i_j} \right)
\]

be an element of \(H^{d-2r'-2}(Sh(G_{a_i}, \tau_{a_i})_a^c, \odot)\) with \(z_{i_1, \ldots, i_j} \in H^{d-2r'-2-2j}(Sh(G_{a_i}, \tau_{a_i})_a^c, \odot)\). Then one has

\[
\pi_{a_i}^* (p^{y/2} \eta^*_{a_i, a, b} (z)) = \pi_{a_i}^* \eta^*_{a_i, a, b} (\pi_{a_i}^* (z)) = p^y \pi_{a_i}^* (\sum (\eta^*_{a_i, a, b} (\pi_{a_i}^* (z_{i_1, \ldots, i_j})) \cup \xi_{i_1} \cup \cdots \cup \xi_{i_j})
\]

where the forth and fifth equalities used the formula \(4.1.3\). This shows the commutativity of the right square in the Lemma.

**Case (a2)** \(\tau^-\) is the right node of an arc in \(a\). Then \(x = -y\) and \(\eta_{a, a^c}\) contains only straight lines. Hence, \(\eta_{a, a^c, a}\) is an isomorphism as in **Case (a2)**. We want to show

\[
\eta^*_{a, a^c} \circ \pi_{a_i}^* = \pi_{a_i}^* \circ \left( p^{y/2} \eta^*_{a_i, a, b} \right).
\]

The arguments are quite similar as in **Case (a2)**. Let \(\xi_i, \xi'_i\) be as defined in **Case (a2)** for \(1 \leq i \leq r-r'-1\). Then by [TX13+a, Proposition 7.17(3)], we have \(\eta^*_{a_i, a, b} (\xi_i) = \xi'_i\) for all \(1 \leq i \leq r-r'-1\) (this differs from the situation of **Case (a2)** because the only curve in \(\eta^*_{a_i, a, b}\) links the right node of an arc in \(a^c\) to the right node of an arc in \(a^c\)). Then the rest of the computation is the same as in **Case (a2)**.

Consider now the **Case (b)**, i.e. the right node of \(\delta_0\) is \(\tau^-\). We have symmetrically three sub cases:

**Case (b1)** \(\tau\) is linked to a straight line in \(a\). Then as in **Case (a1)**, we have \(x = 0\) and \(\pi_{a_i}^*\) and \(\pi_{a_i}^*\) are both isomorphisms. In this case, the commutativity of the right hand side square is trivial.

**Case (b2)** \(\tau\) is the left node of an arc in \(a\). Then \(x = -y\) and \(\eta_{a, a^c}\) contains only straight lines. Hence, \(\eta^*_{a, a^c, a}\) is an isomorphism as in **Case (a2)**. By [5.11.2] and [5.11.3], the desired commutativity is equivalent to

\[
\text{Tr} \eta^*_{a, a^c, a} \circ \pi_{a_i}^* = \pi_{a_i}^* \circ \text{Tr} \eta^*_{a, a^c, a, b},
\]

which is an easy consequence of the compatibility of trace maps with composition.

**Case (b3)** \(\tau\) is the right node of an arc in \(a\). Then \(x = y\) and \(\eta^*_{a, a^c, a}\) is an isomorphism as in **Case (a2)**. The desired commutativity is equivalent to

\[
\pi_{a_i}^* \circ p^{y/2} (\eta^*_{a_i, a, b})^{-1} = p^y (\eta^*_{a, a^c})^{-1} \circ \pi_{a_i}^* \iff \eta^*_{a, a^c} \circ \pi_{a_i}^* = \pi_{a_i}^* \circ (p^{y/2} \eta^*_{a_i, a, b}).
\]

Thus the situation is exactly the same as **Case (3a)** above (except for switching the roles of \(Sh(G_{a_i}, \tau_{a_i})_a^c, \odot)\) and \(Sh(G_{a_i}, \tau_{a_i})_a^c, \odot)\), and we conclude by the same arguments.

\[\square\]

We continue our inductive proof in **Case (iv)**. Using the discussion in Subsection [5.5], we see that the morphism \(\text{Res}_a \circ \text{Gys}_b\) is the composition of the following diagram from the top-left to the
bottom through first $G_{\text{res}}$ and then all the way to the right and then all the way downwards, and finally through $\pi_{\delta_{a/a'}}!$ and $\text{Res}_{a_{\text{res}}^b}$.

\begin{equation}
H^{d-2r}(\text{Sh}(G_{a,T_a}))
\end{equation}

Here, $x, y$ are the same as in Lemma 5.12 for simplicity, we have omitted the Tate twists from the notation, and each cohomology group $H^a(\ast)$ should be understood as $H^a(\ast)(b)$ with $a - 2b = d - 2r$; for instance, $H^{d-2r'-2}(\text{Sh}(G_{a,\tau_a}^x))$ should be understood as $H^{d-2r'-2}(\text{Sh}(G_{a,\tau_a}^x))(r - r' - 1)$.

We now explain the commutativity of this diagram. The top two squares are commutative because the corresponding morphisms on the Shimura varieties form Cartesian squares. The commutativity of the middle rectangle follows from that of (5.6.2) (applied with our $a_{\tau}$ and $a^b$ therein, respectively). The commutativity of the lower trapezoid follows from applying Subsection 5.4 (applied to the Shimura variety $\text{Sh}(G_{a,\tau_a})$ with our $\delta_{a/\delta}$ and $\delta_{b/\delta}$ being the $a$ and $b$ therein). Finally, the commutativity of the bottom parallelogram is justified by Lemma 5.12.

To sum up, the morphism $\text{Res}_a \circ G_{b}$ is the composition of (5.12.1) from the top-left to the bottom by first going all the way down and through $\eta_{a,a'}^{x,y}$. Thus, we have

\[ \text{Res}_a \circ G_{b} = p^{(x+y)/2} \eta_{a,a'}^{x,y} \circ \text{Res}_{a'} \circ G_{b_{\text{res}}}. \]

We need only to check the claims on indention degrees and shifts for $\text{Res}_a \circ G_{b}$ under the induction hypothesis for $\text{Res}_{a'} \circ G_{b_{\text{res}}}$.

Before this, we point out that our construction contains several instances of decomposing a periodic semi-meander in the sense of Subsection 5.5.

(a) $a$ is decomposed into three steps: $a_+^x$, $\delta_{a/a'}$, and $a_{\text{res}}^y$;

(b) $a^y$ is decomposed into two steps: $a_+^y$ and $a_{\text{res}}^y$;

(c) $b$ is decomposed into two steps $\delta_b$ and $b_{\text{res}}$.

These decomposition gives rise to numerical equalities:

\begin{align}
\ell(a) &= \ell(a_+^x) + \ell(\delta_{a/a'}) + \ell(a_{\text{res}}^y), \quad t_a = t_{a_+^x} t_{\delta_{a/a'}} t_{a_{\text{res}}^y}; \\
\ell(a^y) &= \ell(a_+^y) + \ell(a_{\text{res}}^y), \quad t_{a^y} = t_{a_+^y} t_{a_{\text{res}}^y}; \\
\ell(b) &= \ell(\delta_b) + \ell(b_{\text{res}}), \quad t_b = t_{\delta_b} t_{b_{\text{res}}}. 
\end{align}
The equality for indentation degrees is always trivial when \( p \) is inert; so we assume \( p \) splits. By construction, the indentation degree of the link morphism \( \eta_{\delta_a,\delta_b}^* \) is \( \ell(\delta_a) - \ell(\delta_b) \), and that of the indentation degree of \( \eta_{\delta_a,\delta_b}^* \) is \( \ell(\delta_a) - \ell(\delta_b) + (\ell(a_{\text{res}}^b) - \ell(a_{\text{res}}^a)) \), which is the same as
\[
\ell(a) - \ell(b) - (\ell(a^c) - \ell(b_{\text{res}}))
\]
by \((5.12.2)-(5.12.4)\) (and the easy fact that \( \ell(\delta_a) = \ell(\delta_a/\delta_b^*) \)). This is exactly what we expected for the induction.

So we have
\[
x + y = \ell(a_{\text{res}}^b) - \ell(a_{\text{res}}^a) + \ell(\delta_a) + \ell(\delta_b)
\]
\[
= \ell(a) + \ell(b) - (\ell(a^c) + \ell(b_{\text{res}}))
\]
(by \((5.12.2)-(5.12.4)\)),

which is what we need for the induction.

For the shift, recall that the link morphism \( \eta_{\delta_a,\delta_b}^* \) has shift \((t_{a_{\text{res}}^b} t_{a_{\text{res}}^a})^{-1} t_{b_{\text{res}}^b}^{-1} \). As \( t_{a_{\text{res}}^b} t_{a_{\text{res}}^a}^{-1} = t_{\delta_a/\delta_b^*} t_{\delta_b/\delta_a^*}^{-1} \), it is equal to \( t_a t_b^{-1} (t_{\delta_a} t_{\delta_b}^{-1})^{-1} \) by \((5.12.2)-(5.12.4)\) (and the fact that \( t_{\delta_a} = t_{\delta_b} \) for \( ? = a, b \)). This checks the condition for the induction, and hence concludes the inductive proof of Theorem 4.3.

**Appendix A. Cohomology of quaternionic Shimura varieties**

We include the proof of Proposition 2.26 regarding the cohomology of our “slightly twisted” quaternionic Shimura varieties; it is based on comparing the cohomology with the known case when \( T = 0 \). This is certainly known to the experts, but we could not find the exact version in the literature.

**A.1. Discrete Shimura varieties for \( F^\times \).** Consider a Deligne homomorphism for \( T_{F,T} := \text{Res}_F/Q(\mathbb{G}_m) \) given by
\[
h_T: S(\mathbb{R}) = C^\times \longrightarrow T_F(\mathbb{R}) = (\mathbb{R}^\times)^T \times (\mathbb{R}^\times)^{\Sigma_\infty - T}
\]
\[
z \longmapsto ((|z|^2, \ldots, |z|^2), (1, \ldots, 1))
\]
Under this choice of Deligne homomorphism, we can define a discrete Shimura variety \( \text{Sh}_{K_{T,p}}(T_{F,T}) \) for \( K_{T,p} = O_p^\times \) whose complex points are given by
\[
\text{Sh}_{K_{T,p}}(T_{F,T})(\mathbb{C}) = F^\times,cl \setminus A_{F,\mathbb{C}}^\times / O_p^\times.
\]
It admits a canonical integral model with special fiber \( \text{Sh}_{K_{T,p}}(T_{F,T}) \) over \( F_p \) (in the sense of \([\text{TX13+}, \text{ Section 2.8}]\)), which is determined by the Shimura reciprocity map
\[
\text{Rec}_{T,T,p}: \text{Gal}_{F,p} \longrightarrow F^\times,cl \setminus A_{F,\mathbb{C}}^\times / O_p^\times.
\]
Explicitly, \( \text{Rec}_{T,T,p} \) sends the geometric Frobenius of \( F_p \) to the finite idèle \( (p_F)^T \).

Fix a prime number \( \ell \neq p \). The algebraic representation \( \rho_{T,T}^\ell \) of \( T_{F,T} \times \mathbb{C} \cong \prod_{\tau \in \Sigma_\infty} \mathbb{G}_{m,\tau} \) sending \( x \) to \((x^2, \ldots, x^w)\) gives a lisse \( \overline{q}_\ell \)-étale sheaf \( L_{T,T}^w \) of pure weight \( 2(w - 2)\) on \( \text{Sh}_{K_{T,p}}(T_{F,T}) \).

**A.2. Changing \( T \).** We need to compare the Shimura varieties \( \text{Sh}_{K_p}(G_{S,T}) \) and \( \text{Sh}_{K_p}(G_{S,\emptyset}) \). Note that the natural product morphism
\[
pr: G_{S,\emptyset} \times \text{Res}_F/Q \mathbb{G}_m \to G_{S,T}
\]
is compatible with the Deligne homomorphism \( h_{S,\emptyset} \times h_T \) on the source and \( h_{S,T} \) on the target, i.e. \( pr \circ (h_{S,\emptyset} \times h_T) = h_{S,T} \). This gives a natural morphism of Shimura varieties
\[
(A.2.1) \quad pr_*: \text{Sh}_{K_p}(G_{S,\emptyset}) \times \text{Sh}_{K_{T,p}}(T_{F,T}) \longrightarrow \text{Sh}_{K_p}(G_{S,T}).
\]
induces an isomorphism instead of tensorial induction because \( \rho \) reciprocity map \( \text{Rec} \).

**Proof.** The proposition is known when \( T = \emptyset \) by [BL84, §3.2] (note that we have the tensor product instead of torsional induction because \( \rho_{\pi, p} \) is unramified at \( p \).) For general \( T \), the morphism (A.2.1) induces an isomorphism

\[
H^*_\text{et}(\text{Sh}_K(G_{S,T})_{\overline{F}}, L_{S,T}^{(k,w)})[\pi] = \begin{cases} \rho_{\pi, p} \otimes [\det(\rho_{\pi, p})(1)]^{\#T} & \text{if } i = d, \\
0 & \text{if } i \neq d; \end{cases}
\]

which is, up to semi-simplicification, equivalent for the action of geometric Frobenius \( \text{Frob}_{p^d} \).

This concludes the proof of this Proposition. \( \square \)

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