Approximating of functions from Holder classes $H^\alpha[0, 1]$ by Haar wavelets

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Abstract. In the present work, a new direct computational method for solving definite integrals based on Haar wavelets is introduced. The definite integral of the functions from Holder classes is replaced with the approximation of the function by Haar wavelets and the calculation of definite integrals is reduced to the problem of solving algebraic equation formed by the Fourier coefficients in terms of Haar wavelets. Based on the properties of the Haar wavelets it is shown that the such approximations much better approximate the value of the integrals for the functions from Holder classes. The Error analysis of the approximation method are worked out in the classes of Holder to show the efficiency of the new method and connection of the module of difference with smoothness of the function is established. Finally, some numerical examples of the implementation the method for the functions from Holder classes are presented.

1. Introduction

Numerical integration has several applications in mathematics, physics and engineering. There are many works devoted to the approximations in terms of quadrature rule for numerical integration. Quadrature rule is based on polynomial interpolation. Interpolating polynomials are used to find weights corresponding to nodes. The quadrature rule may cause to deal with the high degree polynomial interpolation if the number of node points are large. The large number of node points are needed to get high accuracy in numerical integration. In order to overcome difficulties in numerical integration Haar wavelets is introduced. The motivation again comes from approximation theory, the function might be difficult to work with, some integrals of the functions cannot be found exactly, some require special functions which themselves are a challenge to compute, and others are so complex that finding the antiderivative is impossible.

A lot of work has been done in this area in the context of power series and Fourier series. Wavelets theory will provide a new method for decompose, analyze and synthesize a function or signals. Wavelet expansions are an attempt to improve on Fourier series and other classical expansions. There are various types of wavelets with different properties, such as Haar wavelets, Daubechies’ orthonormal of compact support and Chebyshev wavelets [1, 2, 3]. According to wavelets research, Haar wavelets which is mathematically the simplest orthonormal wavelets with compact support. Application of the
Haar wavelets methods had been widely used in solving different numerical approximation problems, for example solving linear and nonlinear integral equations in [4, 5]. In this present paper we have consider this method as suggested in [6] but the error analysis, we have extended to a wider class of function meaning that, the wavelets approximate function is estimated in Holder space comparing they approach in using differential class function.

The organization of this paper is as follows. In section 2 and 3, Haar wavelets and numerical integration using Haar wavelets has been described. In section 4 and 5, Holder space and error analysis that satisfied the Holder space is given and in section 6 numerical results are reported. Further discussion are drawn in Section 7.

2. Haar wavelets
In this paper we deal with the problems of approximation of the integrals of the functions from Holder classes $H^\alpha[0, 1]$ by Haar wavelets. The wavelets, in general, are very important tools for analyzing the signals in engineering sciences and consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. Here we give definition of the Haar wavelets.

We start from introducing the mother Haar wavelet. Let introduce Haar mother wavelet:

$$H(x) = \begin{cases} 
1 & \text{for } x \in [0, \frac{1}{2}) \\
-1 & \text{for } x \in [\frac{1}{2}, 1) \\
0 & \text{elsewhere},
\end{cases}$$

Next we define

$$h_n(x) = 2^j H(2^j x - k), n = 2^j + k, j = 0, 1, 2, ..., k = 0, 1, ..., 2^j - 1.$$ 

All functions $\{h_n(x)\}, n = 1, 2, 3, ...$ are defined on subintervals of $[0, 1)$ and are derived from $H(x)$ by considering two operations that is dilation and translation. The integer $j$ indicates the level of the wavelet and $k$ is the translation parameter. The Haar wavelets form the orthogonal system:

$$\int_0^1 h_n(x) h_\ell(x) \, dx = \begin{cases} 
1 & n = \ell = 2^j + k \\
0 & n \neq \ell.
\end{cases}$$

Let define scaling Haar wavelet

$$h_0(x) = \begin{cases} 
1 & \text{for } x \in [0, 1) \\
0 & \text{elsewhere}.
\end{cases}$$

It can be shown that the sequence $\{h_n(x)\}_{n=0}^\infty$ is a complete orthonormal base in $L^2(-\infty, +\infty)$.

3. Numerical integration using Haar wavelets
We consider the integral

$$\int_0^1 f(x) \, dx.$$ 

Any function $f \in L^2[0, 1]$ can be expanded as

$$f(x) = \sum_{n=0}^\infty \langle f, h_n \rangle h_n(x),$$

where the coefficients are defined as follows

$$\langle f, h_n \rangle = \int_0^1 f(x) h_n(x) \, dx, n = 0, 1, 2, 3, ....$$
For the square integrable function in the interval $[0, 1]$ the approximation by the first $2M - 1$ term of its expansion has a form

$$f(x) \approx \sum_{n=0}^{2M-1} a_n h_n(x),$$  \hspace{1cm} (1)

where $M = 2^{j_1}$ and $J_1$ is the maximum level of Haar wavelets.

Since

$$\int_0^1 h_n(x) \, dx = 0, \quad n = 1, 2, 3, \ldots,$$

and

$$\int_0^1 h_0(x) \, dx = 1,$$

we obtain

$$\int_0^1 f(x) \, dx \approx \sum_{n=0}^{2M-1} a_n \int_0^1 h_n(x) \, dx \approx a_0.$$ \hspace{1cm} (2)

In equation (2) it is sufficiently clear that, Haar approximation involves only one coefficient in the evaluation of the definite integral. To calculate the Haar coefficient $a_0$ we consider the nodal points

$$x_k = \frac{k + 0.5}{2M}, \quad k = 0, 1, \ldots, 2M - 1.$$

The discretized form of (1) can be written as

$$f(x_k) \approx \sum_{n=0}^{2M-1} a_n h_n(x_k), \quad k = 0, 1, \ldots, 2M - 1.$$ \hspace{1cm} (3)

For further advantage of Haar wavelet approximation is that we do not need to solve the above system which is computationally hectic for large values of $M$. The summation of the left and right side by $k$ from 0 to $2M - 1$ gives us an easy formula to calculate the value of the coefficient $a_0$:

$$\sum_{k=0}^{2M-1} f(x_k) = a_0 \sum_{k=0}^{2M-1} h_0(x_k) + a_1 \sum_{k=0}^{2M-1} h_1(x_k) + \ldots + a_{2M-1} \sum_{k=0}^{2M-1} h_{2M-1}(x_k).$$

From definition we will have that all summations $\sum_{k=0}^{2M-1} h_l(x_k) = 0, \quad l = 1, 2, \ldots, 2M - 1.$

We proved that the solution of the system (3) for $a_0$ has a form as follows:

$$a_0 = \frac{1}{2M} \sum_{k=0}^{2M-1} f(x_k).$$

Applying the ideas above we establish the following formula for numerical integral equation

$$\int_0^1 f(x) \, dx \approx \frac{1}{2M} \sum_{k=0}^{2M-1} f(x_k) = \frac{1}{2M} \sum_{k=0}^{2M-1} f\left(\frac{k + 0.5}{2M}\right).$$
4. Holder space $H^\alpha[0, 1]$

The non-differentiable functions are important for application matters. The importance of studying continuous but nowhere differentiable functions was emphasized a long time ago. It is possible for a continuous function to be sufficiently irregular so that its graph is a fractal. This observation points out to a connection between the lack of differentiability of such a function and the dimension of its graph. Irregular functions arise naturally in various branches of physics. It is well known that the graphs of projections of Brownian paths are nowhere differentiable and have dimension 3/2. A generalization of Brownian motion called fractional Brownian motion gives rise to graphs having dimension between 1 and 2. Typical Feynmann paths, like the Brownian paths are continuous but nowhere differentiable. Also, passive scalars advected by a turbulent fluid can have is scalar surfaces which are highly irregular, in the limit of the diffusion constant going to zero. Attractors of some dynamical systems have been shown to be continuous but nowhere differentiable. All these irregular functions are characterized at every point by a local Holder exponent typically lying between 0 and 1.

Let proceed with the definition of the Holder classes of order $\alpha \in (0, 1)$. The set of all continuous functions on $[0, 1]$, which satisfies the inequality:

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \forall x, y \in [0, 1]$$

is called a Holder space of order $\alpha$ and denoted by $H^\alpha[0, 1]$. The Holder spaces are nested as the follows:

$$H^\alpha \subset H^\beta, \quad \alpha < \beta$$

Holder spaces are medium spaces between $C[0, 1]$ and $C^1[0, 1]$ such that:

$$C^1[0, 1] \subset H^\alpha[0, 1] \subset C[0, 1], \quad 0 < \alpha < 1$$

5. Error analysis

In comparison with error estimation to the previous errors [7], we have extended the error estimation to a wider class of functions from Holder space that will produce better result.

**Theorem.** Let $u(t) \in H^\alpha[0, 1], \; 0 < \alpha < 1$, then

$$\|u - u_N\|_{L_2[0, 1]} \leq \frac{L^2}{4(4^\beta - 1)N^{2\alpha}}.$$  

**Proof:** Let $u(t)$ is defined on $[0, 1]$ and belong to the Holder class $H^\alpha[0, 1]$, such that

$$H^\alpha([0, 1]) = \{ f : \forall t_1, t_2 \in [0, 1] : |u(t_1) - u(t_2)| \leq L|t_1 - t_2|^\alpha, 0 < \alpha < 1 \}.$$

Suppose $u_k(t)$ is the following approximation of $u(t)$,

$$u_N(t) = \sum_{n=0}^{N-1} u_n h_n(t)$$

where, $N = 2^{\beta+1}, \; \beta = 0, 1, 2, \ldots$, then

$$u(t) - u_N(t) = \sum_{n=N}^{\infty} u_n h_n(t) = \sum_{n=2^{\beta+1}}^{\infty} u_n h_n(t)$$
By taking the error in $L^2[0, 1]$ norm, thus

$$
\|u - u_N\|^2 = \int_0^1 (u(t) - u_N(t))^2 \, dt = \sum_{n=2^{\beta+1}}^{\infty} \sum_{m=2^{\beta+1}}^{\infty} u_n u_m \int_0^1 h_n(t) h_m(t) \, dt = \sum_{n=2^{\beta+1}}^{\infty} u_n^2
$$

where

$$
u_n = \langle u(t), h_n(t) \rangle = \int_0^1 u(t) h_n(t) \, dt.
$$

From the condition that if $u \in H^\alpha[0, 1]$ then $|u(t + h) - u(t)| \leq L |h|^\alpha$, $L > 0$, $h \in [0, 1]$

$$
|u_n| \leq 2^\frac{\alpha}{2} \int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} |u(t) - u(t + 2^{-j-1})| \, dt \\
\leq 2^\frac{\alpha}{2} L 2^\alpha(-j-1) \int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} \, dt \\
= L 2^\alpha(-j-1) \frac{1}{2^{-j-1}}.
$$

Hence, we have arrived to the final estimation for the coefficients

$$
u_n^2 \leq L^2 2^{-2\alpha(j+1)-j-2}.
$$

Therefore,

$$
\|u - u_N\|^2 = \sum_{n=2^{\beta+1}}^{\infty} u_n^2 \leq \sum_{n=2^{\beta+1}}^{\infty} L^2 2^{-2\alpha(j+1)-j-2} = \frac{L^2}{4^{\alpha+1}} \sum_{j=\beta+1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-2\alpha j - j} = \frac{L^2}{4^{\alpha+1}} \sum_{j=\beta+1}^{\infty} 2^{-2\alpha j},
$$

$$
= \frac{L^2}{4^{(\alpha+1)} N^{2\alpha}}.
$$

Finally

$$
\|u(t) - u_n(t)\|^2 \leq \frac{L^2}{4^{(4\alpha - 1)} N^{2\alpha}}
$$

This completes proof of the Theorem.
6. Numerical examples
In this section we will show how the obtained results can be applied to approximate the integrals function from Holder classes $H^\alpha[0, 1]$.

Example 1.
Let calculate the following integral using Haar wavelet approximation.

$$\int_0^1 x^{\frac{1}{p}} (1 - x)^{\frac{1}{p}} \, dx \simeq \frac{1}{2M} \sum_{k=0}^{2M-1} \left( \frac{k + 0.5}{2M} \right)^{\frac{1}{p}} \left( 1 - \frac{k + 0.5}{2M} \right)^{\frac{1}{p}}.$$

First we show that $f(x) = x^{\frac{1}{p}} (1 - x)^{\frac{1}{p}} \in H^\frac{1}{p}[0, 1]$. For any $p > 1$ we prove that

$$\left| x^{\frac{1}{p}} (1 - x)^{\frac{1}{p}} - y^{\frac{1}{p}} (1 - y)^{\frac{1}{p}} \right| \leq c|x - y|^{\frac{1}{p}}.$$

Since $a^p + b^p \leq (a + b)^p$ for all $p > 1$ and $a, b > 0$, then let $a = x^{\frac{1}{p}} - y^{\frac{1}{p}}$ and $b = y^{\frac{1}{p}}$.

Now, one easily show that

$$\left( x^{\frac{1}{p}} - y^{\frac{1}{p}} \right)^p + \left( y^{\frac{1}{p}} \right)^p \leq \left( x^{\frac{1}{p}} - y^{\frac{1}{p}} + y^{\frac{1}{p}} \right)^p,$$

$$\left( x^{\frac{1}{p}} - y^{\frac{1}{p}} \right) \leq (x - y)^{\frac{1}{p}}.$$

Therefore,

$$|f(x) - f(y)| = \left| x^{\frac{1}{p}} (1 - x)^{\frac{1}{p}} - y^{\frac{1}{p}} (1 - y)^{\frac{1}{p}} \right|,$$

$$= \left| \left( x^{\frac{1}{p}} - y^{\frac{1}{p}} \right) (1 - x)^{\frac{1}{p}} + y^{\frac{1}{p}} (1 - x)^{\frac{1}{p}} - (1 - y)^{\frac{1}{p}} \right|,$$

$$\leq |x - y|^{\frac{1}{p}} + |x - y|^{\frac{1}{p}} = 2|x - y|^{\frac{1}{p}}.$$

Example 2. As the second example consider the following integral:

$$\int_0^1 \sin^{\frac{1}{2}}(x) \, dx \simeq \frac{1}{2M} \sum_{k=0}^{2M-1} \sin^{\frac{1}{2}} \left( \frac{k + 0.5}{2M} \right).$$

Let $f(x) = \sin^\alpha(x)$ and proofed that $f(x) \in H^\alpha[0, 1]$.

$$|f(x) - f(y)| = |\sin^\alpha(x) - \sin^\alpha(y)|,$$

$$\leq |\sin(x) - \sin(y)|^{\alpha} = |2 \sin \left( \frac{y-x}{2} \right) \cos \left( \frac{x+y}{2} \right)|,$$

$$\leq 2^\alpha |\frac{y-x}{2}|^\alpha \cdot 1 = |x - y|^\alpha.$$

Example 3.

$$\int_0^1 x^{\frac{1}{p}} (1 - x^{\frac{1}{p}})^{\frac{1}{p}} \, dx \simeq \frac{1}{2M} \sum_{k=0}^{2M-1} \left( \frac{k + 0.5}{2M} \right)^{\frac{1}{p}} \left( 1 - \frac{k + 0.5}{2M} \right)^{\frac{1}{p}}.$$

Let $f(x) = x^{\frac{1}{p}} (1 - x^{\frac{1}{p}})^{\frac{1}{p}}$ and proofed that $f(x) \in H^\frac{1}{p}[0, 1]$.

$$|f(x) - f(y)| = \left| x^{\frac{1}{p}} (1 - x^{\frac{1}{p}})^{\frac{1}{p}} - y^{\frac{1}{p}} (1 - y^{\frac{1}{p}})^{\frac{1}{p}} \right|,$$

$$= \left| x^{\frac{1}{p}} \left( (1 - x^{\frac{1}{p}})^{\frac{1}{p}} - (1 - y^{\frac{1}{p}})^{\frac{1}{p}} \right) + (1 - y^{\frac{1}{p}})^{\frac{1}{p}} \left( x^{\frac{1}{p}} - y^{\frac{1}{p}} \right) \right|.$$
Since \( \left| \left( 1 - \frac{1}{x^a} \right)^{\frac{1}{b}} - \left( 1 - \frac{1}{y^a} \right)^{\frac{1}{b}} \right| \leq \left| \frac{1}{x^a} - \frac{1}{y^a} \right|^{\frac{1}{b}} \) and \( \left| \left( 1 - \frac{1}{y^a} \right)^{\frac{1}{b}} \right| \leq 1, \) then
\[
|f(x) - f(y)| \leq \left| \frac{1}{x^a} - \frac{1}{y^a} \right|^{\frac{1}{b}} + \left| \frac{1}{x^p} - \frac{1}{y^p} \right|,
\]
\[
\leq |x - y|^{\frac{1}{ab}} + |x - y|^{\frac{1}{p}}.
\]
In particular, if \( ab = p. \) then
\[
|f(x) - f(y)| \leq 2|x - y|^{\frac{1}{p}}.
\]

**Table 1.** Absolute error for different value of J of Example 1, Example 2 and Example 3.

| Haar | Example 1 | Example 2 | Example 3 |
|------|------------|-----------|------------|
| J=4  | 6.698E-04  | 3.243855E-04 | 4.156066E-04 |
| J=5  | 2.737E-04  | 1.159275E-04 | 1.435076E-04 |
| J=6  | 8.400E-05  | 4.12975E-05  | 4.95815E-05  |
| J=7  | 2.971E-05  | 1.46795E-05  | 1.71562E-05  |

7. Discussion
Babolian and Shahsavaran [7] has prove the error analysis for Haar wavelets. The only drawback is its error analysis is limited to functions belonging to \( C^1(R) \). Here we have extended the error analysis of [7] to a wider class of functions, which is called the Holder class \( H^s(R) \). We are able to classify some of \( H^s(R) \) functions and also give numerical approximation by Haar wavelets in Example 1-3.

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