SINGULAR RIEMANNIAN FLOWS AND CHARACTERISTIC NUMBERS

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Abstract. Let $M$ be an even-dimensional, oriented closed manifold. We show that the restriction of a singular Riemannian flow on $M$ to a small tubular neighborhood of each connected component of its singular stratum is foliated-diffeomorphic to an isometric flow on the same neighborhood. We then prove a formula that computes characteristic numbers of $M$ as the sum of residues associated to the infinitesimal foliation at the components of the singular stratum of the flow.

1. Introduction

In [3], P. Baum and J. Cheeger give a formula for the characteristic numbers of a manifold endowed with a singular isometric flow in terms of integrals of characteristic forms over the singular stratum of the flow. In our paper, we start with a singular Riemannian flow, not assumed to be an isometric flow. Our first result describes the structure of singular Riemannian flows near the singular stratum of the flow. Using this, we obtain a formula similar to [3] for the characteristic numbers of the manifold.

The following definitions and facts about Riemannian foliations can be found in [12]. Let $(M,\mathcal{F})$ be a singular foliation of a smooth, connected, compact manifold $M$ that is smooth in the sense of Sussman and Stefan ([16], [17]). This means that for each leaf $L \in \mathcal{F}$, each $x \in L$, and each $v \in T_xL$, there exists a smooth vector field $V$ on $M$ such that $V(x) = v$ and $V(y) \in T_y\mathcal{F}$ for all $y \in M$. If, in addition, there is a Riemannian metric $g$ on $M$ such that every geodesic that is perpendicular at one point to a leaf is perpendicular to every leaf it meets, we say that the triple $(M,\mathcal{F},g)$ is a singular Riemannian foliation.

If all the leaves of $\mathcal{F}$ have the same dimension, $\mathcal{F}$ is called regular. In this case, the condition on $g$ given above is equivalent to $g$ being a bundle-like metric.

Let the stratum $\Sigma_r \subseteq M$ denote the union of leaves of dimension $r$. Then the restriction of $\mathcal{F}$ and $g$ to each $\Sigma_r$ is a Riemannian foliation with bundle-like metric. The stratum corresponding to leaves of the smallest dimension is a compact submanifold, called the minimal stratum. The stratum corresponding to leaves of maximal dimension is open and dense in $M$ and is called the regular stratum. The closures of the leaves of a singular Riemannian foliation are submanifolds, and the restriction of $\mathcal{F}$ to one of these leaf closures is a [transversally locally homogeneous] regular Riemannian foliation.

A singular Riemannian flow is a singular Riemannian foliation such that the maximal dimension of each leaf is one.

We say that a smooth vector field $X$ on a smooth manifold $M$ is a transverse Killing vector field if there exists a Riemannian metric on $M$ such that the singular flow generated by $X$ is a singular Riemannian flow. If the zero set $\Sigma$ of $X$ is nondegenerate, meaning the
normal Hessian of \( X \) is invertible at \( \Sigma \), we say that \( X \) is a **nondegenerate transverse Killing vector field**. One can always construct a nondegenerate transverse Killing vector field corresponding to any oriented singular Riemannian flow. We remark that in other sources the term “transverse Killing” implies a choice of metric on the normal bundle to the foliation, but we do not specify this metric in our definition.

We start by establishing the structure of an oriented singular Riemannian flow \((M, \mathcal{F}, g)\) in the tubular neighborhood of a component of the singular stratum \( \Sigma := \Sigma_0 \) in Theorem 2.7 and Corollary 2.9. These theorems resemble slice theorems (such as in [13], [1], [11]), but the new results in this paper are stronger for flows in that they apply to the entire tubular neighborhood of a singular stratum rather than to the neighborhood of a singular leaf. We show that there exists a new metric \( g' \) on \( M \) for which \((M, \mathcal{F}, g')\) is a singular Riemannian flow on \( M \) that restricts to an isometric flow on the tubular neighborhood. Note that every vector field that generates an isometric flow for some metric on \( M \) is automatically a nondegenerate transverse Killing vector field and thus generates a singular Riemannian flow. It is easy to construct transverse Killing fields that are not global Killing vector fields for any metric; equivalently, there are singular Riemannian flows that are not foliated-diffeomorphic to singular isometric flows. See Examples 4.5 and 4.6. In addition, a nonorientable flow is considered in Example 5.6. In Section 3, we establish some technical results which help to localize computations to the tubular neighborhood of \( \Sigma \).

The main result of the paper is Theorem 4.2. In this theorem, we provide the formula that computes characteristic numbers of an even-dimensional, oriented closed manifold as the sum of residues at the components of the zero set of a nondegenerate transverse Killing vector field that generates a singular Riemannian flow. We prove that the Lie derivative of the field induces an isometric flow on the normal bundle of each component of the singular stratum, and the residue at this component is defined in terms of the invariants of this action. In the case when the singular Riemannian flow is not orientable, the argument is easily handled by Theorem 5.2. These theorems specialize to the results in [3] in the case when the singular Riemannian flow is in fact a global isometric flow for some metric. One simple consequence, Corollary 5.3, is the formula for the Euler characteristic,

\[
\chi(M) = \sum_j \chi(\Sigma_j),
\]

where \( \Sigma_j \) are the components of the singular stratum of a possibly nonorientable Riemannian flow \((M, \mathcal{F}, g)\). This formula was previously known when \( \Sigma_j \) are the zero sets of a Killing vector field; see [9]. See also Corollary 5.3 for a new formula for the signature of a manifold endowed with a singular Riemannian flow whose singular stratum is a finite set of points.

We now briefly discuss the history of this problem. In the celebrated paper [4], R. Bott showed how to compute the Pontryagin and other characteristic numbers from isolated singular points of holomorphic vector fields or of infinitessimal isometries. In [5], he generalized his result in the holomorphic case to allow vector fields whose zero sets are submanifolds. In [2], M. Atiyah and I. Singer used the \( G \)-signature theorem, a special case of the index theorem, to give the formula for the characteristic numbers of a singular isometric flow in terms of integrals of characteristic forms over the singular stratum of the flow. In [3], P. Baum and J. Cheeger use purely differential-geometric and Stokes’ theorem techniques to derive the same result. One consequence of all these results is that if there exists a nonvanishing Killing vector field on a closed Riemannian manifold, then all of its characteristic numbers
vanish. In [6], Y. Carrière showed that any Riemannian manifold with a nonsingular Riemannian flow has Gromov minimal volume zero, so as a consequence all of the characteristic numbers of that manifold are zero, consistent with Theorem 4.2. In [10], X. Mei considered a singular Riemannian foliation and a variant of curvature coming from the curvature of the normal bundle to the foliation. The author gave a formula for the residue of a characteristic polynomial of this type of curvature at a connected component of the singular stratum.

These are not the same as the residues used to compute the characteristic numbers of the manifold, which are computed in this paper.

2. Structure of the foliation near the singular stratum

Let \((M,F,g)\) be an oriented singular Riemannian flow on a smooth, connected, compact manifold \(M\), and let \(\Sigma\) be the singular stratum of \(F\); note that each singular leaf is a point in \(\Sigma\). In this section, we show that the metric can be modified to another metric such that the flow is still a singular Riemannian flow and now its restriction to a tubular neighborhood of \(\Sigma\) is an isometric flow.

2.1. The flow on a normal disk. By [12, Proposition 6.3], the singular stratum \(\Sigma\) of the flow \(F\) is an embedded submanifold of \(M\), which may have more than one connected component. Let \(\exp^\perp : N\Sigma \to M\) be the normal exponential map for \(N\Sigma \subseteq TM|\Sigma\). For any submanifold \(V \subseteq M\), let \(\text{Tub}_\varepsilon(V)\) denote the set of points of distance \(< \varepsilon\) from \(V\) in \(M\). We also let \(N(V,\varepsilon) = (\exp^\perp)^{-1}(\text{Tub}_\varepsilon(V)) \subseteq NV\). Fix \(x \in \Sigma\). Consider the normal disk \(D_\varepsilon(x) = \exp^\perp(N_x\Sigma) \cap \text{Tub}_\varepsilon(\Sigma)\). We will show in this section that \(F\) restricts to a Riemannian flow on \(D_\varepsilon(x)\).

Let \(B_\varepsilon(x) \subseteq M\) denote the ball of radius \(\varepsilon\) around \(x\), and so \(N(x,\varepsilon)\) is the subset of \(T_xM\) consisting of vectors of length \(< \varepsilon\). There exists \(\varepsilon > 0\) such that the exponential map \(\exp : N(x,\varepsilon) \to B_\varepsilon(x)\) is a diffeomorphism. For \(0 < \lambda \leq 1\) the homothetic transformation \(h_\lambda : B_\varepsilon(x) \to B_{\lambda\varepsilon}(x)\) is defined by \(h_\lambda(v) = \lambda v\). By the distance-preserving property, the foliation restricts to each sphere of radius \(r\) centered at \(x\) for \(0 < r < \varepsilon\); let \(\mathcal{F}|_{B_\varepsilon} = \mathcal{F}|_{B_{\lambda\varepsilon}}\).

**Lemma 2.1.** (Special case of the homothetic transformation lemma. [12, Section 6.2])

For sufficiently small \(\varepsilon > 0\), the homothetic transformation \(h_\lambda\) maps \(\mathcal{F}|_{B_\varepsilon}\) to \(\mathcal{F}|_{B_{\lambda\varepsilon}}\).

Let \(S(\Sigma,r) \xrightarrow{p_r} \Sigma\) denote the fiber bundle of spheres of radius \(r > 0\) in \(N\Sigma\). Let \(S_r\Sigma = \exp^\perp(S(\Sigma,r))\). For sufficiently small \(r\), \(S_r\Sigma\) is an embedded submanifold such that the restriction \((S_r\Sigma, \mathcal{F}|_{S_r\Sigma}, g|_{S_r\Sigma})\) is a nonsingular Riemannian flow ([12, Section 6.2]). Let \(\pi_r = p_r \circ (\exp^\perp)^{-1} : S_r\Sigma \to \Sigma\) be the sphere bundle projection.

**Lemma 2.2.** For sufficiently small \(r\) and each \(x \in \Sigma\), the singular Riemannian foliation \(\mathcal{F}\) restricts to the submanifold \(\pi_r^{-1}(x)\), and \(\mathcal{F}|_{\pi_r^{-1}(x)}\) is a nonsingular Riemannian flow.

**Proof.** For \(x \in \Sigma\), let \(V_r(x)\) be the set of points of distance \(r\) from \(x\) in \(M\), where \(r\) is sufficiently small. Then the Riemannian flow \(\left(M \setminus \Sigma, \mathcal{F}|_{M \setminus \Sigma}\right)\) restricts to a Riemannian flow on \(V_r(x)\), since the geodesics through \(x\) must remain orthogonal to every point of each fixed leaf of \(\left(M \setminus \Sigma, \mathcal{F}|_{M \setminus \Sigma}\right)\). Also, by [12, Sections 6.2 through 6.4], \(\mathcal{F}\) restricts to a nonsingular Riemannian flow on \(S_r\Sigma\). Thus, the Riemannian flow must also restrict to \(V_r(x) \cap S_r\Sigma\), which is \(\pi_r^{-1}(x)\), because of the following argument. Clearly, \(\pi_r^{-1}(x) \subseteq V_r(x)\),
since all of its points are a distance \( r \) from \( x \), and likewise \( \pi_r^{-1}(x) \subseteq S_r \Sigma \) by construction. Since \( \pi_r^{-1}(x) \) and \( V_r(x) \cap S_r \Sigma \) are smooth spheres and of the same dimension for small \( r \), we have \( V_r(x) \cap S_r \Sigma = \pi_r^{-1}(x) \).

\[\square\]

2.2. \textbf{Infinitesimal flow on the normal bundle to the singular stratum.} We will replace the metric on \( \text{Tub}_\varepsilon(\Sigma) \) with a linearized metric, as follows. Let \( g^\Sigma \) be the original metric restricted to \( T \Sigma \), and let \( \tilde{g}^\Sigma \) be the pullback of \( g^\Sigma \) to the horizontal space \( \mathcal{H} \subset T(N(\Sigma, \varepsilon)) \subset T(N \Sigma) \) given by the normal Levi-Civita connection. Let \( g^\perp \) be the metric on \( N_y \Sigma \) for each \( y \in \Sigma \), and let \( g^\perp \) be the corresponding translation-invariant metric on the fibers of \( N \Sigma \to \Sigma \). We define the metric on the total space of \( N(\Sigma, \varepsilon) \) as \( g^\Sigma \oplus g^\perp \). We may then transplant this metric to \( \text{Tub}_\varepsilon(U) \) via \((\exp^\perp)_*\). We call the resulting metric the \textbf{linearized metric} \( g_L \) on \( \text{Tub}_\varepsilon(U) \). By the results of [12] Section 6.4, applied to \( \Sigma \), \((\text{Tub}_\varepsilon(U), \mathcal{F}|_{\text{Tub}_\varepsilon(U)}, g_L)\) is a singular Riemannian foliation, as is \((N(\Sigma, \varepsilon), (\exp^\perp)^{-1}(\mathcal{F}|_{\text{Tub}_\varepsilon(U)}), \tilde{g}^\Sigma \oplus \tilde{g}^\perp)\). We remark that since the foliation is 1-dimensional away from \( \Sigma \), the exponential map takes the leaves of the infinitesimal foliation onto the leaves of \( \mathcal{F} \). Further, the foliation is invariant under homothetic transformations with respect to the stratum \( \Sigma \).

**Lemma 2.3.** For each \( y \in \Sigma \), the restriction of \( \mathcal{F} \) to \( \exp^\perp(N_y \Sigma \cap N(\Sigma, \varepsilon)) \) is an isometric flow in the linearized metric. Similarly, the restriction of \((\exp^\perp)^{-1}(\mathcal{F}|_{\text{Tub}_\varepsilon(U)})\) to \( N_y \Sigma \cap N(\Sigma, \varepsilon) \) is also an isometric flow.

**Proof.** For fixed \( y \in \Sigma \), by Lemma 2.2 the restriction of \( \mathcal{F} \) to \( D = \exp^\perp(N_y \Sigma \cap N(\Sigma, \varepsilon)) \) is a Riemannian flow (for the original metric) on a metric ball of radius \( \varepsilon \) and centered at \( y \), so that the spheres at each radius \( r \) with \( 0 < r < \varepsilon \) are foliated by \( \mathcal{F} \), and such that the homothetic transformation \( r \mapsto r' \) preserves the foliation ([12] Section 6.2). The foliation induces a foliation of the unit sphere \( S_y D \) in the tangent space \( T_y D \), where \( v, w \) are on the same leaf if and only if \( \exp_y(tv) \) and \( \exp_y(tw) \) are on the same leaf for a fixed \( t \in (0, \varepsilon) \) [and hence for all \( t \in (0, \varepsilon) \) by the homothetic transformation property]. We claim that the induced foliation \( \mathcal{F}_S \) on \( S_y D \) with the Euclidean metric is a Riemannian flow. If not, then there exist two nearby leaves of a foliation chart of \( \mathcal{F}_S \) that are not locally equidistant in the Euclidean metric on \( S_y D \). Then it follows that the corresponding leaves of \( \mathcal{F} \) close to the origin \( y \) are also not locally equidistant on a foliation chart of the metric sphere of radius \( t \) for small \( t \). This is a contradiction. Then, since \( \mathcal{F}_S \) is a Riemannian flow on the round sphere \( S_y D \), by [3], the foliation \( \mathcal{F}_S \) is a foliation arising from an isometric flow. \[\square\]

2.3. \textbf{The structure of the tubular neighborhood.} The restriction of the normal exponential map \( \exp^\perp : N(\Sigma, \varepsilon) \to \text{Tub}_\varepsilon(\Sigma) \) is a diffeomorphism. We have seen in the previous section and in Lemma 2.3 that the oriented singular Riemannian flow is conjugate through \( \exp^\perp \) to a singular Riemannian foliation on \( N(\Sigma, \varepsilon) \) such that the restriction of the flow to each fiber \( N_y \Sigma \cap N(\Sigma, \varepsilon) \) is a linear isometric flow whose only fixed point is the origin. Thus the normal bundle has even rank.

More specifically (see [3]), each such flow on a single fiber \( N_y \Sigma \) of the normal bundle is conjugate to a linear isometric flow on \( \mathbb{C}^k \) with no fixed points other than the origin, where \( 2k \) is dimension of \( N_y \Sigma \). Thus, by the representation theory of isometric actions, there exists
an isomorphism \( N_y \Sigma \cong \mathbb{C}^k \) such that the isometric flow \( \gamma : \mathbb{R} \to U(k, \mathbb{C}) \) has the form

\[
\gamma(t) = \begin{pmatrix}
\exp(i\alpha_1 t) & 0 & 0 & 0 \\
0 & \exp(i\alpha_2 t) & \vdots & \vdots \\
\vdots & 0 & \ddots & 0 \\
0 & \cdots & 0 & \exp(i\alpha_k t)
\end{pmatrix}
\]  

(2.1)

acting on \( z \in \mathbb{C}^k \), where \( \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k \) are nonzero real constants. We select the constants so that \( \alpha_1^2 + \ldots + \alpha_k^2 = 1 \). A family of such flows, in other words the flow on the total space \( N \Sigma \) dependent on the base point \( y \in \Sigma \), may move the planes \( P_i = \{ z : z_j = \text{constant for } j \neq i \} \) but we claim must only multiply the \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) by a scalar. Because of the normalization, the scalar must be \( \pm 1 \), so the foliation may only change its orientation. We now prove this claim in the following lemma, proposition, and corollary.

**Lemma 2.4.** Let \( \gamma : \mathbb{R} \to U(k, \mathbb{C}) \) be a family as above, which depends on the choice of \((\alpha_1, \ldots, \alpha_k) \in S^{k-1}\). Let \( C(z) = C(z_1, \ldots, z_k) \) denote the real dimension of the closure of the set \( \{ \gamma(t) z : t \in \mathbb{R} \} \) in \( S^{2k-1} \subset \mathbb{C}^k \). Then the maximum value of \( C(z) \) for \( z \) in any open set is \( \dim_\mathbb{Q} \{ \alpha_1, \ldots, \alpha_k \} = k - \dim_\mathbb{Q} \{ \beta \in \mathbb{Q}^k : \beta \cdot \alpha = 0 \} \).

**Proof.** This is well–known and generalizes the Kronecker foliation. \( \Box \)

**Proposition 2.5.** Let \( I \) be any closed interval of positive length, and let \( \alpha : I \to S^{k-1} \), with \( \alpha(s) = (\alpha_1(s), \ldots, \alpha_k(s)) \), be a continuous, nonconstant path. Define \( D_\alpha : I \to \mathbb{Z} \) by

\[
D_\alpha(s) = \dim_\mathbb{Q} \{ \beta \in \mathbb{Q}^k : \beta \cdot \alpha(s) = 0 \} \quad \text{for } s \in I.
\]

Then \( D_\alpha \) is not constant.

**Proof.** We induct on \( k \geq 1 \). If \( k = 1 \), the statement is true vacuously. Now, suppose that for some \( i \geq 1 \), the statement is true for \( 1 \leq k \leq i \). Now let \( k = i + 1 \), so that \( \alpha : I \to S^i \) is a continuous, nonconstant path. Let \( [a_0, b_0] = I \), and we enumerate \( \mathbb{Q}^{i+1} \setminus \{0\} = \{ \beta_1, \beta_2, \ldots \} \).

We construct a nested sequence of intervals \([a_n, b_n]_{n \geq 1}\) such that \( 0 \leq a_n < b_n \leq 1 \), \( \alpha|_{[a_n, b_n]} \) is not constant, and \( \beta_j \cdot \alpha(s) \neq 0 \) for all \( s \in [a_n, b_n] \) and all \( 1 \leq j \leq n \). To construct \([a_n, b_n]\) from \([a_{n-1}, b_{n-1}]\), let \( U_n = \{ s \in [a_{n-1}, b_{n-1}] : \beta_j \cdot \alpha(s) \neq 0 \} \). The set \( U_n \) is open in \([a_{n-1}, b_{n-1}]\). If \( U_n = [a_{n-1}, b_{n-1}] \), we let \([a_n, b_n] = [a_{n-1}, b_{n-1}]\) and continue. If \( U_n = \emptyset \), then \( \alpha|_{[a_{n-1}, b_{n-1}]} \) is a nonconstant map to the lower-dimensional sphere \( S^j = S^j \cap \{ x \in \mathbb{R}^{i+1} : \beta_n \cdot x = 0 \} \), and by the induction hypothesis \( D_\alpha \) is not constant (because it is \( 1 + D_\alpha \) where \( D_\alpha \) is the corresponding dimension on \( S^j \)), so the statement is true for \( k = i + 1 \), and we stop. Otherwise, \( U_n \) is a proper, nonempty open subset of \([a_{n-1}, b_{n-1}]\). Then \( \beta_n \cdot \alpha(s) \) is nonconstant (since it is nonzero on \( U_n \) and 0 on the complement) on every connected open subset of \( U_n \), making \( \alpha \) nonconstant on any connected open subset of \( U_n \), i.e. on an open interval \( I_n \subseteq U_n \).

We then let \([a_n, b_n]\) be a closed interval within \( I_n \), sufficiently large so that \( \alpha \) is not constant on \([a_n, b_n]\). If this process of constructing \([a_n, b_n]\) terminates, we are done, as mentioned before; otherwise we choose an \( x_0 \in \bigcap_{n \geq 0} [a_n, b_n] \). Since \( \beta_j \cdot \alpha(x) \neq 0 \) for all \( j \), \( D_\alpha(x) = 0 \).

Thus, if \( D_\alpha(s) \neq 0 \) for any \( s \in I \), we have proved the case \( k = i + 1 \). Otherwise, we have that \( D_\alpha(s) = 0 \) for all \( s \in I \). However, this is not possible, because of the following. Since \( \alpha \) is not constant and continuous, there exist \( s_1, s_2 \in I \) such that \( \alpha(s_1) \) and \( \alpha(s_2) \) are not on the same line. But then, there exists \( \beta \in \mathbb{Q}^{i+1} \setminus \{0\} \) contained in the intersection of the open half spaces \( \beta \cdot \alpha(s_1) < 0 \) and \( 0 < \beta \cdot \alpha(s_2) \). By the intermediate value theorem, there

\footnote{The idea of proof was suggested to us by G. Gilbert.}
Corollary 2.6. (Rigidity of isometric flows on spheres) For some connected open set \( U \) of a manifold \( \Sigma \), suppose that \( \mathbb{C}^k \times U \) contains an oriented singular Riemannian flow with associated bundle-like metric such that the flow restricts to a singular isometric flow on each \( \mathbb{C}^k \times \{s\} \) of the form \((z_1, z_2, \ldots, z_k) \mapsto (\exp(i\alpha_1(s)t)z_1, \exp(i\alpha_2(s)t)z_2, \ldots, \exp(i\alpha_k(s)t)z_k)\)
for some \( \alpha(s) = (\alpha_1(s), \ldots, \alpha_k(s)) \in S^{k-1} \). Then \( \alpha \) is constant on \( U \).

Proof. By the previous two propositions, if \( \alpha \) is not constant on \( U \), then there is a one-parameter family \( \alpha(s) \) with \( s \in U \) such that the top dimensions of the leaf closures of the flow restricted to spheres will change discontinuously. In such a situation, it is not possible that the distance between leaves is locally constant, so that the singular flow cannot be Riemannian. \( \square \)

As a result, the restriction of our foliation to a ball in \( N\Sigma \) over each point of a connected component of \( \Sigma \) must be the set of orbits of the group action
\[
(z_1, z_2, \ldots, z_k) \mapsto (\exp(i\alpha_1(s)t)z_1, \exp(i\alpha_2(s)t)z_2, \ldots, \exp(i\alpha_k(s)t)z_k),
\]
where the subspaces corresponding to the span of all \( z_j \) for equal \( \alpha_j \) may vary with \( y \in \Sigma \). In order to determine the allowable variations in these subspaces of the normal spaces \( N_y\Sigma \), we first must understand all orientation-preserving isometries of \( \mathbb{C}^k \) fixing the origin \( y \) and preserving the foliation. In particular, the isometry \( L = L_y \) at \( y \) must preserve the tangent bundle of the foliation. This means that the vector field is determined by the matrix multiplication:
\[
V_x = Mx = \begin{pmatrix}
0 & -\alpha_1 & 0 & 0 & 0 & \cdots & 0 \\
\alpha_1 & 0 & 0 & 0 & \vdots & \vdots & \\
0 & 0 & -\alpha_2 & 0 & \vdots & \vdots & \\
0 & 0 & \alpha_2 & 0 & 0 & \vdots & \\
0 & \vdots & \vdots & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 & 0 & -\alpha_k \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \alpha_k & 0
\end{pmatrix}
\begin{pmatrix}
R_1 & 0 & 0 & 0 \\
0 & R_2 & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & R_k
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\vdots \\
x_k
\end{pmatrix}
\]
must be preserved up to a positive scalar multiple. This means that the push forward of the isometry maps \( V \) to a positive scalar multiple of itself. That is, we require that
\[
LMx = \lambda M(Lx)
\]
for some \( \lambda > 0 \), for all \( x \). Since \( M \) and \( L \) are invertible, taking determinants shows that \( \lambda = 1 \).

We renumber the \( \{\alpha_j\} \) to distinct \( \beta_j \) with multiplicities \( \mu_j \). A simple linear algebra argument yields the following. Using the standard inclusion of \( GL_k(\mathbb{C}) \) in \( GL_{2k}(\mathbb{R}) \), we have
\[
\{ L \in GL_{2k}(\mathbb{R}) : LM = ML \} = \left\{ L = \begin{pmatrix}
B_1 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & B_p
\end{pmatrix} \in GL_k(\mathbb{C}) : B_j \in GL_{\mu_j}(\mathbb{C}) \right\}.
\]
This implies that the tubular neighborhood of each connected component of the singular stratum of the Riemannian flow is diffeomorphic to a neighborhood of the zero section in a direct sum of complex vector bundles \(E_1, \ldots, E_p\) over the singular stratum. The flow is multiplication by \(\exp(i\alpha_j t)\) on each \(E_j\), where \((\alpha_1, \ldots, \alpha_p)\) is a locally constant vector. Considering the torus action by \((z_1, \ldots, z_p) \in T^p\) on the vector bundle by
\[
(z_1, \ldots, z_p) (v_1, \ldots, v_p) \mapsto (z_1 v_1, \ldots, z_p v_p),
\]
the flow is a subgroup of this torus action. We average the metric over the torus flow to get a new metric \(g_1\), so that the Riemannian flow is an isometric flow near each connected component of the singular stratum for this new metric. We summarize this discussion in the following theorem.

**Theorem 2.7.** The restriction of the oriented singular Riemannian flow \((M, \mathcal{F})\) to a tubular neighborhood of a connected component of the singular stratum is foliated-diffeomorphic to an isometric flow on the same tubular neighborhood.

**Lemma 2.8.** Let \((U, \mathcal{F})\) be a smooth, oriented flow on a smooth manifold \(U\), and suppose that there exist two bundle-like metrics \(g, \tilde{g}\) for the flow. Let \(\psi : U \to [0,1]\) be a basic function. Then there exists a smooth bundle-like metric \(g_\psi\) for \((U, \mathcal{F})\) such that \(g_\psi = g\) on \(\psi^{-1}(0)\) and \(g_\psi = \tilde{g}\) on \(\psi^{-1}(1)\).

**Proof.** Near any point we may choose a foliation chart with adapted coordinates \((x, y) \in \mathbb{R} \times \mathbb{R}^q\). A metric on this chart is a bundle-like metric if and only if it has the form
\[
\theta \otimes \theta + \sum_{\alpha, \beta=1}^q h_{\alpha \beta} \, dy^\alpha \otimes dy^\beta,
\]
where \(\theta\) is a one-form such that \(\theta(\partial_x) > 0\) and \(h_{\alpha \beta}\) is a positive definite symmetric matrix of basic functions; see \cite{14} Section IV, Proposition 4.2. In fact, \(\theta\) is always dual to a unit vector field tangent to the flow and \(T\mathcal{F}^\perp = \ker \theta\); thus, up to a sign \(\theta\) is defined globally. We express \(g\) in this form as
\[
g = \theta \otimes \theta + \sum_{\alpha, \beta=1}^q h_{\alpha \beta}(y) \, dy^\alpha \otimes dy^\beta = g_T + g_N.
\]
Similarly, we have
\[
\tilde{g} = \tilde{\theta} \otimes \tilde{\theta} + \sum_{\alpha, \beta=1}^q \tilde{h}_{\alpha \beta}(y) \, dy^\alpha \otimes dy^\beta = \tilde{g}_T + \tilde{g}_N.
\]
Let \(\Pi : TU \to T\mathcal{F}\) be the orthogonal projection defined by the first metric \(g\). Since \(\tilde{\theta}(\partial_x) > 0\), also \((\Pi^* \tilde{\theta})(\partial_x) > 0\). We define a new bundle-like metric \(\overline{g}\) by
\[
\overline{g} = (\Pi^* \tilde{\theta}) \otimes (\Pi^* \tilde{\theta}) + \sum_{\alpha, \beta=1}^q \tilde{h}_{\alpha \beta}(y) \, dy^\alpha \otimes dy^\beta = \overline{g}_T + \overline{g}_N.
\]
Note that the bundles \(T\mathcal{F}\) and \(T\mathcal{F}^\perp\) agree for both \(g\) and \(\overline{g}\), since \(\ker \theta = \ker (\Pi^* \tilde{\theta})\). Since \(\tilde{\theta}\) and \(\Pi^* \tilde{\theta}\) are globally defined up to a sign, it is clear that \(\overline{g}\) and \(\overline{g}\) are homotopic through a global homotopy transforming \(\tilde{\theta}\) to \(\Pi^* \tilde{\theta}\). Next, we may homotope \(\overline{g}\) to \(g\) by separately homotoping \(\overline{g}_T\) to \(g_T\) on \(T\mathcal{F}\) and \(\overline{g}_N\) and \(g_N\) on \(T\mathcal{F}^\perp\). With a simple concatenation and
adjustments to the parameter to make the concatenation smooth, a smooth homotopy connecting \( g \) to \( \tilde{g} \) can be formed, and this homotopy is a homotopy of bundle-like metrics. If we replace the homotopy parameter with the single basic function \( \psi \), it is easy to check that the resulting metric \( g_\psi \) is bundle-like for \((U, F)\).

\[ \square \]

**Corollary 2.9.** Given a nondegenerate transverse Killing field \( X \) on a compact, smooth manifold \( M \), there exists a metric on \( M \) for which the restriction of \( X \) to a tubular neighborhood of the singular stratum \( \Sigma \) is a Killing vector field, and such that each component of \( \Sigma \) is a totally geodesic submanifold.

**Proof.** Use the Lemma to patch the original metric \( g \) with the tubular neighborhood metric \( \tilde{g} \) in the Theorem above using a basic cutoff function \( \rho \) that is 1 in a neighborhood of \( \Sigma \) and is 0 outside a larger neighborhood; the new metric \( g_\rho \) does the job.

\[ \square \]

3. Modification of the metric and localization

It is enough to consider the case where the dimension of our manifold \( M \) is \( 2m \), since the characteristic numbers vanish on odd-dimensional manifolds.

3.1. **Estimate on the complement of a tubular neighborhood of the singular stratum.** Our manifold \( M \) is endowed with an oriented singular Riemannian flow \( F \) whose tangent bundle is given by the span of the vector field \( X \). As in Corollary 2.9, we choose a metric \( g \) on \( M \) and an \( \varepsilon > 0 \) such that the restriction of \( X \) to the tubular neighborhood \( T_\varepsilon \Sigma \) is a Killing vector field and that \( F \) is a singular Riemannian flow globally with \( g \) bundle-like. We now modify this metric to a new metric \( g_t \) as follows. For \( t > 0 \), let \( g_t \) denote the metric on \( M \) defined by

\[
g_t(X,Y) = \begin{cases} g(X,Y) & \text{if } X \in (T_F)^\perp \\ \rho(t,r) g(X,X) & \text{if } X \in T_F. \end{cases}
\]

Here, \( \rho(t,r) \) is a function on \( M \) defined as

\[
f(r) = \begin{cases} 0 & \text{if } r \leq \frac{\varepsilon}{3} \\ 1 & \text{if } r \geq \frac{2\varepsilon}{3} \\ \text{smooth, increasing} & \text{if } \frac{\varepsilon}{3} < r < \frac{2\varepsilon}{3}. \end{cases}
\]

**Lemma 3.1.** For every \( t > 0 \), the singular foliation \((M, F, g_t)\) is Riemannian and restricts to an isometric flow on \( T_\varepsilon \Sigma \).

**Proof.** First we consider the restriction of \( F \) to the tube \( T_\varepsilon \Sigma \), where \( X \) is a Killing field for \( g \). Note that this is still the tube of radius \( \varepsilon \) in the new metric \( g_t \). The flow of \( X \) preserves the tangent space \( T_F \) and normal space \( N_F \) and the metric on \( N_F \). Since \( X\rho = 0 \), the Leibniz rule implies that the Lie derivative \( \mathcal{L}_X g_t^{\text{tan}} \) of the tangential metric is still zero. Then it follows that \( \mathcal{L}_X g_t = 0 \). Next, outside of \( T_\varepsilon \Sigma \), the tangential metric is multiplied by the scalar \( t^2 \), so the leaves of the foliation remain equidistant, so that the metric \( g_t \) is still bundle-like outside of \( T_\varepsilon \Sigma \).

\[ \square \]

**Proposition 3.2.** Let \( g_t \) be the family of Riemannian metrics on the manifold \( M \). Let \( \omega_t \) be a characteristic form, an ad (SO(2m))-invariant polynomial in the curvature \((M, g_t)\). Then

\[
\int_{M \setminus T_\varepsilon \Sigma} \omega_t \to 0
\]

as \( t \to 0 \).
Proof. Similar to the Carrierè argument in [6, Lemmas 1.1 - 1.2], the sectional curvatures of \((M, g_t)\) on \(M \setminus T_\varepsilon \Sigma\) remain bounded as \(t \to 0\). The \(2m\)-degree part of \(\omega_t\) is a polynomial with coefficients independent of \(t\) in the sectional curvatures of \((M, g_t)\) times the volume form of \(g_t\). Since the volume form of \(g_t\) is \(t\) times the volume form of \(g\), the result follows. □

3.2. Estimate on the boundary of the tubular neighborhood. We restrict our attention to \(T_\varepsilon \Sigma\), on which \(X\) is a Killing field for the metric \(g_t\).

We let

\[
\omega_t = \phi(K_t, ..., K_t),
\]

where \(\phi\) is a polynomial that is homogeneous of degree \(m\) and \(K_t\) is the Riemannian curvature two-form. Following [4, Lemma 2], on \(T_\varepsilon \Sigma \setminus \Sigma\), there exists a \(2m-1\) form \(\eta_t\) defined as

\[
\eta_t = \alpha \left\{ \phi^1_t + \phi^2_t d\alpha + \ldots + \phi^m_t (d\alpha)^{m-1} \right\}.
\]

Here, \(\alpha\) is the one-form defined as

\[
\alpha(Y) = \frac{\langle X, Y \rangle}{\langle X, X \rangle} = \frac{\langle X, Y \rangle_t}{\langle X, X \rangle_t},
\]

and for \(1 \leq j \leq m\),

\[
\phi^j_t = \binom{m}{j} \phi(L_t, ..., L_t, K_t, ..., K_t),
\]

where \(L_t\) is the endomorphism of \(TM\) defined by

\[
L_t = \mathcal{L}_X - \nabla^g_t X.
\]

From [4, Lemma 2],

\[
i(X)(\omega_t - d\eta_t) = 0,
\]

which then implies

\[
\omega_t - d\eta_t = 0 \quad (3.1)
\]

since \(i(X)\) injective on top-degree forms on \(T_\varepsilon \Sigma \setminus \Sigma\).

Proposition 3.3. With \(\eta_t\) and \(\varepsilon\) as above,

\[
\int_{\partial T_\varepsilon \Sigma} \eta_t \to 0
\]

as \(t \to 0\).

Proof. We have \(\dim \partial T_\varepsilon \Sigma = 2m - 1\). Let \(i : \partial T_\varepsilon \Sigma \to M\) be the inclusion, so that the form \(i^*\eta_t\) is well-defined, and

\[
\int_{\partial T_\varepsilon \Sigma} \eta_t := \int_{\partial T_\varepsilon \Sigma} i^*\eta_t.
\]

Observe that \(i^*\eta_t\) is a polynomial in sectional curvatures and Christoffel symbols times the volume form of \(\partial T_\varepsilon \Sigma\). From Carrière’s proof in [6, Lemmas 1.1 - 1.2], all the sectional curvatures and Christoffel symbols remain bounded as \(t \to 0\), and the volume form of \(i^*g_t\) is by construction \(t\) times the volume form \(i^*g\), since \(X\) is tangent to \(\partial T_\varepsilon \Sigma\). The result follows. □
4. Computation of the characteristic numbers

Let \( M \) be a compact, oriented, Riemannian manifold of dimension \( 2m \), endowed with a singular Riemannian flow \( \mathcal{F} \) corresponding to the given transverse Killing vector field \( X \). Let \( FM \xrightarrow{\pi} M \) denote the oriented orthonormal frame bundle of \( M \), and let \( K \) be the \( \mathfrak{o}(2m) \)-valued curvature two-form on \( FM \). Let \( \phi \) be an ad (SO \((2m)) \)-invariant symmetric form of degree \( m \) on \( \mathfrak{o}(2m) \). Then the function \( \phi(K) := \phi(K, \ldots, K) \) is the pullback of the closed form on \( M \), which by abuse of notation we also denote \( \phi(K) \). The number \( \int_M \phi(K) \) is the characteristic number associated to \( \phi \):

\[
C_\phi := \int_M \phi(K),
\]

which is independent of the metric (and curvature) on \( M \). In particular, we will use \( K_t \), the Riemannian curvature two-form with respect to the metric \( g_t \) defined in Section 3.1. Let \( \omega_t = \phi(K_t, \ldots, K_t) \).

As in the previous sections, we fix an appropriate \( \varepsilon > 0 \) and let \( T_\varepsilon \Sigma \) denote the tubular neighborhood of the singular stratum \( \Sigma \) of \( \mathcal{F} \), on which \( X \) generates a \( g_t \)-isometric flow. Then, using Stoke’s Theorem and \( \omega_t = d\eta_t \) in \( T_\varepsilon \Sigma \) from (3.1),

\[
C_\phi = \int_{M \setminus T_\varepsilon \Sigma} \omega_t + \int_{T_\varepsilon \Sigma} \omega_t
= \int_{M \setminus T_\varepsilon \Sigma} \omega_t + \lim_{\delta \to 0} \int_{T_\varepsilon \Sigma - \delta \Sigma} \omega_t
= \int_{M \setminus T_\varepsilon \Sigma} \omega_t + \int_{\partial T_\varepsilon \Sigma} \eta_t - \lim_{\delta \to 0} \int_{\partial T_\delta \Sigma} \eta_t
\]

As \( t \to 0 \), by Proposition 3.2 Proposition 3.3, we obtain the following.

**Lemma 4.1.** With the notation above, the characteristic number associated to \( \phi \) satisfies

\[
C_\phi = -\lim_{\delta \to 0} \int_{\partial T_\delta \Sigma} \eta_t = -\lim_{\delta \to 0} \int_{\partial T_\delta \Sigma} \eta_0,
\]

where for small \( \delta \), the restriction of \( \eta_t \) to \( \partial T_\delta \Sigma \) does not depend on \( t \) and corresponds to the fixed metric \( g \).

We now introduce the notation of the main theorem, much of which is similar to that in [3, Section 1]. Given any invertible linear transformation \( A \in \mathfrak{o}(2s) \), there exists an orthonormal basis \( \{e_1, \ldots, e_{2s}\} \) for \( \mathbb{R}^{2s} \) such that \( Ae_{2j-1} = \lambda_j e_{2j} \) and \( Ae_{2j} = -\lambda_j e_{2j-1} \) and \( \lambda_j \geq 0 \) for each \( j \). The numbers \( \lambda_j \) are called **skeigen-values**. It is well-known that if \( \psi \) is an ad (SO \((2s)) \)-invariant symmetric complex-valued polynomial on \( \mathfrak{o}(2s) \), there exists a unique polynomial \( \hat{\psi} : \mathbb{R}^{s+1} \to \mathbb{C} \) such that

\[
\psi(A) = \hat{\psi}(\lambda_1, \ldots, \lambda_s).
\]

for any such transformation \( A \). The Pfaffian \( \chi(A) \) of \( A \) is a particular example; \( \chi(A) = \hat{\chi}(\lambda_1, \ldots, \lambda_s) = \pm \lambda_1 \ldots \lambda_s \), where the positive sign is chosen exactly when \( e_1, \ldots, e_{2m} \) is a positively oriented basis of \( \mathbb{R}^{2s} \).

As we have seen in the proof of Lemma 2.3 and Corollary 2.9, the given nondegenerate transverse Killing field \( X \) with singular set \( \Sigma \), its linearization restricts to each \( N_x \Sigma \) to be a
Killing field. The restriction of its Lie derivative to \( N_x \Sigma \) is a nonsingular skew-symmetric automorphism \( P_x (\mathcal{L}_X|_{\Gamma(N_x \Sigma)}) \), where \( P_x : T_x M \to N_x \Sigma \) is the orthogonal projection. Further we multiply the endomorphism by a positive scalar \( c_x \) so that the resulting skeigen-values \( \{ \alpha_j \} \) satisfy \( \sum \alpha_j^2 = 1 \) and each \( \alpha_j \) is nonzero. Let \( \Lambda'_X = c_x P_x (\mathcal{L}_X|_{\Gamma(N_x \Sigma)}) \). We extend \( \Lambda'_X \) by zero on \( T \Sigma \) to define the endomorphism \( \Lambda_X : TM|_{\Sigma} \to TM|_{\Sigma} \). By the results of Section 2.3, and in particular Corollary 2.6, the skigen-values of \( \Lambda_{\nu} \) \( \{ \nu \} \) the skigen-values of \( \Lambda_X \) are constant on each connected component of \( \Sigma \). Let \( \mu_0, \mu_1, ..., \mu_{\tau} \) be the distinct skigen-values of \( \Lambda_X \). Furthermore, \( TM|_{\Sigma} \) is the direct sum of skigen-bundles \( TM|_{\Sigma} = E_0 \oplus E_1 \oplus ... \oplus E_{\tau}, \) where \( E_0 = T\Sigma \) and

\[
(E_j)_x = -\mu_j^2 \text{ eigenspace of } (\Lambda_X)_x^2
\]

For each \( j \geq 1 \), \( E_{\lambda_j} \) can be endowed with the complex structure \( \frac{1}{\mu_j} (\Lambda_X)^2 \big|_{E_{\nu_j}} \) with induced orientation. We orient \( E_0 = T\Sigma \) so that the orientation agrees with the induced orientation from \( TM \). We set the real fiber dimension of \( E_j \) to be \( 2m_j \), so that \( \sum_{j=0}^\tau m_j = m = \frac{1}{2} \dim M \).

We now introduce forms \( a_j \); in the case where \( E_0, ..., E_{\tau} \) are direct sums of line bundles, they are the first Chern forms (or, classes if considered as elements of \( H^* (\Sigma) \)) of the line bundle components. In general, let \( a_1, ..., a_m \) be such that

1. The \( i^{\text{th}} \) Pontryagin class of \( E_0 \) is the \( i^{\text{th}} \) symmetric function of \( a_1^2, ..., a_{m_0}^2 \), and its Euler class is \( a_1 ... a_{m_0} \).
2. For \( i = 1, ..., \tau \), the \( k^{\text{th}} \) Chern class of \( E_i \) the \( k^{\text{th}} \) elementary symmetric function of those \( a_j^2 \) such that \( m_0 + ... + m_{i-1} + 1 \leq j \leq m_0 + ... + m_i \).

Let \( \lambda_1, ..., \lambda_m \) be the list of real numbers \( 0, ..., 0, \mu_1, ..., \mu_1, ..., \mu_\tau, ..., \mu_\tau \), so that they are the skigen-values of \( \Lambda_X \). We define

\[
\psi (\Lambda_X) := \hat{\psi} (\lambda_1 + a_1, ..., \lambda_m + a_m).
\]

One specific example we will use is

\[
\chi (\Lambda_X') = (\lambda_{m_0+1} + a_{m_0+1}) ... (\lambda_m + a_m).
\]

There is a technical change we need to make in the case \( \Sigma \) is a point, in which case \( TM = E_1 \oplus ... \oplus E_{\tau} \), and it may be the case that the orientation induced from the complex structures on the \( E_j \) does not produce the given orientation of \( TM \). In this case, we instead let

\[
\psi (\Lambda_X) := \hat{\psi} (-\lambda_1 - a_1, \lambda_2 + a_2, ..., \lambda_m + a_m).
\]

**Theorem 4.2.** Let \( (M, g) \) be a compact, oriented Riemannian manifold of dimension \( 2m \) that is endowed with an oriented singular Riemannian foliation \( \mathcal{F} \). Let \( X \) be a nondegenerate transverse Killing vector field on \( M \) whose span is \( T\mathcal{F} \). Let \( \phi \) be an \( \text{ad} (\text{SO}(2m)) \)-invariant symmetric form of degree \( m \) on \( \mathfrak{o}(2m) \). Then the characteristic number \( \phi (M) \) defined by \( \phi \) satisfies

\[
\phi (M) = \sum_j \frac{\phi (\Lambda_X)}{\chi (\Lambda_X')^{\nu_j}} [\Sigma_j],
\]

where \( \Sigma_j \) are the connected components of the singular stratum \( \Sigma \) of \( \mathcal{F} \).
Proof. The vector field $X$ globally generates a singular Riemannian flow. For $p$ near $\Sigma$, we replace $X$ with $\tilde{X} = \frac{d}{dt} \exp^1 \left( \exp \left( t\Lambda_X^* \right) \left( \exp^1(p) \right)^{-1} \right)\bigg|_{t=0}$ where $\exp^1 : N\Sigma \to M$ is the normal exponential map and $\exp : \mathfrak{so}(2k) \to SO(2k)$ is the Lie group exponential with $2k = 2m - \dim \Sigma$. The flow of this vector field is the same as the flow of $X$, and in the metric $g_1$ from Theorem $2.7$, $\tilde{X}$ is an isometric flow near $\Sigma$. Lemma $4.1$ shows that we need only calculate each

$$-\lim_{\delta \to 0} \int_{\partial T_{p}\Sigma} \eta_1.$$ 

We refer to [3] proof of Theorem C] for the calculation of the residue, where the calculation is local and only uses the fact $\tilde{X}$ is Killing in the small tubular neighborhood. Since the final formula of the limit is the same for both $X$ and $\tilde{X}$, the result follows. □

Remark 4.3. For the special case where $\Sigma_j$ is an isolated fixed point $p$,

$$\frac{\phi (\Lambda_X)}{\chi (\Lambda_X^*)} \mid_{\Sigma_j} = \frac{\phi (\Lambda_X)}{\chi (\Lambda_X^*)} (p) = \frac{\hat{\psi} (\lambda_1 + a_1,\lambda_2 + a_2,\ldots,\lambda_m + a_m)}{(\lambda_1 + a_1)(\lambda_2 + a_2)\ldots(\lambda_m + a_m)} (p).$$

Remark 4.4. The theorem above can easily be adapted to the case where the characteristic numbers come from the curvature of a more general foliated vector bundle over $M$. In this case, $X$ acts canonically on such a bundle.

Example 4.5. The following singular foliation is from [15], Section 3.4. Consider the foliation on $S^4$ defined as follows. Let $v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be an eigenvector of a symmetric matrix $B \in SL(2,\mathbb{Z})$ with positive irrational eigenvalues. We consider $S^4$ to be a suspension of $S^3 \subseteq \mathbb{C}^2$, and we foliate each $S^3$ by the curves $t \mapsto (\exp(it\alpha) z_1, \exp(it\beta) z_2)$. This nonsingular isometric flow on $S^3$ extends to an isometric flow of $S^4$, with two fixed points at the poles. Note that each generic leaf closure of the flow is a two-dimensional torus. A tubular neighborhood of such a torus is isometric to a solid torus of the form $D^2 \times T^2$, where $D^2$ is a two-dimensional disk, and where the boundary of this tube is a (rectangular) 3-torus $S^1 \times T^2$. Choose two tubes $\text{Tube}_1$ and $\text{Tube}_2$ like this inside $S^4$ that are isometric and disjoint. We glue the two boundary components of $S^4 \setminus \{\text{Tube}_1 \cup \text{Tube}_2\}$ via the $3 \times 3$ matrix $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, which is a foliated diffeomorphism between the boundary components. This is equivalent to attaching a handle. The result is the manifold

$$M = \left\{ S^4 \setminus \{\text{Tube}_1 \cup \text{Tube}_2\} \right\} / \sim,$$

where the equivalence relation $\sim$ is given by the gluing map described above. For a small interval $I$, we use the product metric on $(\partial \text{Tube}_1) \times I \cong (\partial \text{Tube}_2) \times I$, and using a basic partition of unity (a partition of unity that is constant on the leaves) we patch this to the original metric on $S^4 \setminus \{\text{Tube}_1 \cup \text{Tube}_2\}$ using Lemma $2.3$. The original foliation induces a singular Riemannian flow on $M$ with this metric. It was shown in [15], Section 3.4 that this flow is not isometric. In fact, we certainly could attach more handles as desired. In any case, we could now compute for example the Euler characteristic of this manifold using
Theorem 4.2. The residue at each pole is 1, so that
\[ \chi(M) = 1 + 1 = 2. \]
The same result could be obtained from the Hopf index theorem.

Example 4.6. Consider the manifold \( \mathbb{C}P^m \), with homogeneous coordinates \([z_0, \ldots, z_m]\). Consider the isometric flow parametrized by the curves \( t \mapsto [z_0, \exp(it\alpha_1)z_1, \ldots, \exp(it\alpha_m)z_m] \), where \((\alpha_1, \ldots, \alpha_m)\) is an eigenvector of a specific matrix \( A \in SL(m, \mathbb{Z}) \), where \( \{\alpha_1, \ldots, \alpha_m\} \) is linearly independent over \( \mathbb{Q} \). This isometric flow has \( m + 1 \) fixed points \([1, 0, \ldots, 0],[0, 1, 0, \ldots, 0], \ldots, [0, 0, \ldots, 0, 1] \). Similar to the last example, we note that generic leaf closures of the flow are \( m \)-dimensional tori. A tubular neighborhood of such a torus is isometric to a tube of the form \( D^m \times T^m \), where \( D^m \) is a \( m \)-dimensional disk, and where the boundary of this tube is of the form \( S^{m-1} \times T^m \). Choose two tubes Tube_1 and Tube_2 like this inside \( \mathbb{C}P^m \) that are isometric and disjoint. We glue the two boundary components of \( \mathbb{C}P^m \setminus \{\text{Tube}_1 \cup \text{Tube}_2\} \) via the map \( \text{id} \times A \), which is a foliated diffeomorphism between the boundary components. The result is the manifold
\[ M = \{\mathbb{C}P^m \setminus \{\text{Tube}_1 \cup \text{Tube}_2\}\} / \sim, \]
where the equivalence relation \( \sim \) is given by the gluing map described above. For a small interval \( I \), we use the product metric on \((\partial \text{Tube}_1) \times I \cong (\partial \text{Tube}_2) \times I\), and using a basic partition of unity we patch this to the original metric on \( \mathbb{C}P^m \setminus \{\text{Tube}_1 \cup \text{Tube}_2\} \) using Lemma 2.8. The original foliation induces a singular Riemannian flow on \( M \) with this metric. Similar to what is shown in [15, Section 3.4], we can see that this flow is not isometric. However, we may apply Theorem 4.2 to compute the signature of \( M \). On a small neighborhood of each singular point \([0, \ldots, z_j = 1, 0, \ldots, 0]\), the foliation has the form of the flow
\[ (z_0, z_1, \ldots, z_j, \ldots, z_m) \mapsto (\exp(-it\alpha_j)z_0, \exp(it(\alpha_1 - \alpha_j))z_1, \ldots, \hat{z}_j, \ldots, \exp(it(\alpha_m - \alpha_j))z_m), \]
letting \( \alpha_0 = 0 \). Then the residue calculation for the signature gives (see Corollary 4.3 below),
\[ \sigma(M) = \sum_{j=0}^{m} \prod_{i=0}^{m} \text{sgn}(\alpha_i - \alpha_j) = \begin{cases} 1 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases} \]
We see that the surgery did not alter the signature.

5. The non-orientable case.

Even if the closed manifold \( M \) is orientable, it is possible that there exists a singular Riemannian foliation on it that is not orientable; see Example 5.6. In this case, it is easy to modify the argument of the paper to compute characteristic numbers of the manifold using a calculation at the singular stratum of the foliation.

Lemma 5.1. Let \((M, \mathcal{F}, g)\) be a non-orientable, one-dimensional singular Riemannian foliation of an oriented, Riemannian manifold \( M \). Then there exists an oriented singular Riemannian flow \((\tilde{M}, \tilde{\mathcal{F}}, \pi^*g)\) on a double cover \( \pi : \tilde{M} \rightarrow M \) such that the regular leaves of \( \tilde{\mathcal{F}} \) are double covers of the regular leaves of \( \mathcal{F} \).

Proof. Every foliation chart \( U \subseteq M \) contains a dense subset saturated by one-dimensional plaques of \( \mathcal{F} \). Thus, \( U \) can be endowed with one of two possible leafwise orientations. We
construct the foliation \((\tilde{M}, \tilde{\mathcal{F}})\) by making the foliation charts from sets of the form \(U \times o(U)\), where \(U\) is a foliation chart and \(o(U)\) is a choice of leafwise orientation of \(U\). The plaques are the subsets of the form \(P \times o(U)\), where \(P\) is a plaque of \(\mathcal{F}\) in \(U\). The properties follow easily from the definition, and the orientations \(o(U)\) produce the leafwise orientation of \(\tilde{\mathcal{F}}\).

The fact that characteristic numbers are multiplicative over finite covers yields the following.

**Theorem 5.2.** Let \((M, \mathcal{F}, g)\) be a non-orientable, one-dimensional singular Riemannian foliation of a compact, oriented, Riemannian manifold \(M\) of dimension \(2m\). Let \((\tilde{M}, \tilde{\mathcal{F}}, \pi^* g)\) be the corresponding double cover as in Lemma 5.1. Let \(X\) be a nondegenerate transverse Killing vector field on \(\tilde{M}\) whose span is \(T\tilde{\mathcal{F}}\). Let \(\phi\) be an \(\text{ad}(\text{SO}(2m))\)-invariant symmetric form of degree \(m\) on \(o(2m)\). Then the characteristic number \(\phi(M)\) defined by \(\phi\) satisfies

\[
\phi(M) = \frac{1}{2} \sum_j \phi(\Lambda_X) \chi(\Lambda_{X^p}) \left[\tilde{\Sigma}_j\right],
\]

where \(\tilde{\Sigma}_j\) are the connected components of the singular stratum \(\tilde{\Sigma}\) of \(\tilde{\mathcal{F}}\).

**Corollary 5.3.** Let \((M, \mathcal{F}, g)\) be a possibly nonorientable one-dimensional singular Riemannian foliation of a compact, oriented, Riemannian manifold \(M\) of dimension \(4\ell\) whose singular stratum consists of isolated points \(p_1, ..., p_n\). For each \(j\), by Theorem 2.7 there exists a Killing vector field \(X_j\) such that the restriction of \(\mathcal{F}\) to a ball centered at \(p_j\) is the flow of \(X_j\). We define the *index* \(\varepsilon_j\) of the singular point \(p_j\) as follows. The isometric flow corresponding to \(X_j\) has the local form (2.1) on \(C^{2\ell}\) with constants \(\alpha_1, ..., \alpha_{2\ell}\). We define the sign

\[
\varepsilon_j = \pm \prod_{i=1}^{2\ell} \text{sgn}(\alpha_i),
\]

where the \(\pm\) is chosen depending on whether the orientation of \(C^{2\ell}\) agrees with the orientation of \(M\) at \(p_j\) or not. It is easy to check that this number is independent of the choice of coordinates and of the choice of \(X_j\). Even if the orientation of \(X_j\) is reversed, the product \(\prod_{i=1}^{2\ell} \text{sgn}(\alpha_i)\) is invariant.

**Corollary 5.5.** Let \((M, \mathcal{F}, g)\) be a possibly nonorientable one-dimensional singular Riemannian foliation of a compact, oriented, Riemannian manifold \(M\) of dimension \(4\ell\) whose singular stratum consists of isolated points. Then the signature satisfies \(\sigma(M) = \sum_j \varepsilon_j\), where the sum is over the singular points and \(\varepsilon_j\) is the index of the \(j^{th}\) singular point.
This result is new for the case where $(M,\mathcal{F},g)$ is not an isometric flow.

Below is an example where an orientable 4-manifold is endowed with a one-dimensional, non-orientable singular Riemannian foliation.

**Example 5.6.** Consider a nonorientable disk bundle over a Klein bottle. More specifically, let $K = \mathbb{R}^2 / \langle \tau, \sigma \rangle$, where $\tau(x,y) = (x,y+1)$, $\sigma(x,y) = (x+1,1-y)$. Consider $D \times \mathbb{R}^2$, where $D$ is the unit disk in $\mathbb{C}$. We extend $K$ to be a disk bundle $B$ over $K$ by letting $B = D \times \mathbb{R}^2 / \langle \tau, \sigma \rangle$, where $\tau(z,x,y) = (z,x,y+1)$ and $\sigma(z,x,y) = (z,x+1,1-y)$. Then $B$ is an orientable 4-manifold with boundary $\partial B = \partial D \times \mathbb{R}^2 / \langle \tau, \sigma \rangle$. We foliate $B$ by circles of the form $C_{r,x,y} = \{(z,x,y) / \langle \tau, \sigma \rangle : |z| = r\}$ with $0 \leq r \leq 1$. Now we glue two copies of $B$ with opposite orientation together to form and orientable 4-manifold $M = B \sqcup_{\partial B} B'$, which can be endowed with a metric such that the circles $C_{r,x,y}$ form a singular Riemannian foliation that is not orientable. Since the double cover $\tilde{M}$ is diffeomorphic to $T^2 \times S^2$, all characteristic numbers of $M$ are zero.

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