A SCHOEN THEOREM FOR MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. In this paper we prove that a complete minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane, must be a model surface. Moreover, we give a geometric description of minimal ends of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$.

1. Introduction

In the early eighties, R. Schoen [20] proved a beautiful theorem about minimal surfaces in Euclidean space. Namely, a complete and connected minimal surface immersed in $\mathbb{R}^3$ with two embedded ends of finite total curvature is a catenoid.

In his article, R. Schoen described the structure of finite total curvature ends minimally embedded in $\mathbb{R}^3$, relying on the results of A. Huber [9] and R. Osserman [14] about the Weierstrass representation of such ends.

At the beginning of this century, the discovery of a generalized Hopf differential by U. Abresch and H. Rosenberg [1] stimulated the study of minimal surfaces in three-dimensional homogeneous manifolds. Many new embedded and complete minimal surfaces have been found in $\mathbb{H}^2 \times \mathbb{R}$. In particular J. Pyo [15] and F. Morabito and M. Rodriguez [12] have constructed, independently, a family of minimal embedded annuli with finite total curvature. Each end of such annuli is asymptotic to a vertical geodesic plane. We call each of such surfaces a two ends model surface.

In this article we prove the following theorem.

Main Theorem. A complete and connected minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane, is a two ends model surface.

Following the same spirit of Schoen’s work, we describe the full geometry of minimal ends of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. The study of such ends was first developed by the first author and H. Rosenberg in [8].

The paper is organized as follows.

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In Section 2, we study the geometry of minimal ends of finite total curvature. The main geometric property is that horizontal sections of finite total curvature ends converge towards a horizontal geodesic. In Section 3, we prove the Main Theorem. In the Appendix, we study the geometry of curves with bounded curvature in the hyperbolic plane.

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2. Minimal ends with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$

In this section we give the geometrical structure of a finite total curvature end. We rely on the complex analysis involved in the theory of minimal surfaces [7], [8], [18] and on the theory of harmonic maps developed by Z. Han, L. Tan, A. Treiberg and T. Wan [6] and Y. N. Minsky in [11].

Let $M$ be a Riemann surface and let $X = (F, h) : M \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal and minimal immersion. The map $F : M \to \mathbb{H}^2$ is harmonic and $h$ is a harmonic function on $M$. Let $z$ be a local conformal coordinate on $M$ and let $\sigma$ be the conformal factor of the hyperbolic metric on $\mathbb{H}^2$. We set

$$Q(F) := (\sigma \circ F)^2 F_z \overline{F}_z dz^2 = \phi(z) dz^2,$$

then $Q(F)$ is a quadratic holomorphic differential globally defined on $M$, known as the quadratic Hopf differential associated to $F$. Moreover we have $(h_z)^2 (dz)^2 = -Q(F)$ (see [18] Proposition 1), so that $Q(F)$ has two square roots globally defined on $M$. We denote by $\sqrt{\phi} dz$ the square root of $Q(F)$, so that

$$h = -2 \text{Re} \int i \sqrt{\phi} dz = 2 \text{Im} \int \sqrt{\phi} dz.$$

The metric induced on $M$ by the immersion $X$ is

$$ds^2 = (\sigma \circ F)^2 \left| F_z + |F_z|^2 \right| |dz|^2.$$

From a result by A. Huber [9, Theorem 15], we deduce that a minimal end $E$ of finite total curvature is parabolic, so that it can be parametrized by $U := \{ z \in \mathbb{C} \mid |z| > 1 \}$. Let $X = (F, h) : U \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal and complete parametrization of the end, $E = X(U)$.

As it is shown in [8], $\phi = (\sigma \circ F)^2 F_z \overline{F}_z$ has a finite number of zeroes in $U$, then, up to restrict $U$, we can assume that $\phi$ has no zeros on $U$. Moreover, it is proved in [8], that $\phi$ extends meromorphically to the puncture $z = \infty$. Therefore, $\phi$ has the following form

$$\phi(z) = \left( \sum_{k \geq 1} \frac{a_k}{z^k} + P(z) \right)^2,$$  (1)
where \( P \) is a polynomial function. If we choose \( \sqrt{\phi} = \sum_{k \geq 1} \frac{a_{-k}}{z^k} + P(z) \), then
\[
h = 2 \text{Im} \int \left( \sum_{k \geq 1} \frac{a_{-k}}{z^k} + P(z) \right) dz.
\]
Since the height function is well defined on \( U \), the real part of \( a_{-1} \) is zero. Let \( \beta \in \mathbb{R} \) such that \( a_{-1} = i\beta \).

**Lemma 2.1.** The polynomial function \( P \) is not identically zero.

**Proof.** Assume by contradiction that \( P \equiv 0 \). If \( a_{-1} = 0 \) we obtain that
\[
\int_U |\phi(z)| dA < \infty,
\]
and it is shown in \[8\] that the minimal end \( E \) would have finite area. From \[4\] (Theorem 3 and Remark 4) we deduce that for any \( p \in E \) and for any real number \( \mu < d_E(p, \partial E) \), we have \( \text{Area}(B(p, \mu)) \geq \pi\mu^2 \), where \( B(p, \mu) \) is the geodesic disk in \( E \) centered at \( p \), with radius \( \mu \). Considering a suitable diverging sequence of points \( (p_n) \) in \( E \), we deduce that \( E \) has infinite area. This gives a contradiction.

Assume now that \( a_{-1} \neq 0 \). Since \( a_{-1} = i\beta \), we obtain (up to an additive constant)
\[
h = 2 \text{Im} \int \left( \sum_{k \geq 1} \frac{a_{-k}}{z^k} \right) dz = 2\beta \log|z| + o(1),
\]
where \( o(1) \) is a function depending of \( z \) and \( o(1) \to 0 \) when \( |z| \to \infty \).

For \( R > 1 \), let \( A_R = \{ R \leq |z| \leq 2R \} \). Thus, \( X(A_R) \) is a compact and minimal annulus immersed in \( \mathbb{H}^2 \times \mathbb{R} \), whose boundary has two connected components. For \( R \) large enough, the vertical distance between those two boundary components is larger than \( 2\pi \), while the family of the catenoids stays in a slab of height smaller than \( \pi \) \[13, Proposition 5.1\]. Therefore, we can compare \( X(A_R) \) with the catenoids and obtain a contradiction by the maximum principle. This concludes the proof. \( \square \)

**Definition 2.1.** Let \( m \geq 0 \) be the degree of \( P \). We will say that \( E \) is an end of degree \( m \).

Let \( E \) be an end of degree \( m \). Up to a change of variable, we can assume that the coefficient of the leading term of \( P \) is one. Then, for suitable complex number \( a_0, \ldots, a_{m-1} \), one has
\[
P(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0 \quad \text{and} \quad \sqrt{\phi} = z^m (1 + o(1)). \tag{2}
\]
For any \( R > 1 \), we set \( U_R := \{ z \in \mathbb{C} \mid |z| > R \} \), \( S_R := \{ z \in \mathbb{C} \mid |z| = R \} = \partial U_R \) and \( E_R := X(U_R) \).

We set
\[
W(z) := \int \sqrt{\phi(z)} dz = \int \left( \sum_{k \geq 1} \frac{a_{-k}}{z^k} + a_0 + \cdots + z^m \right) dz,
\]
so that \( h(z) = 2 \text{Im} W(z) \). If \( \beta = 0 \), the function \( W \) is well defined on \( U \). If \( \beta \neq 0 \), the function \( W \) is only locally defined and has a real period equal to \(-2\pi\beta\). We denote by \( \theta \in \mathbb{R} \) a determination of the argument of \( z \in U \), therefore

\[
\frac{1}{2} h(z) = \text{Im} W(z) = \beta \log|z| + \left( \frac{|z|^{m+1}}{m+1} \right) \left( \sin(m+1)\theta + o(1) \right)
\]

and, locally

\[
\text{Re} W(z) = -\beta \theta + \left( \frac{|z|^{m+1}}{m+1} \right) \left( \cos(m+1)\theta + o(1) \right).
\]

**The image of \( W \) and the level sets of \( \text{Im} W \)**

**Definition 2.2.** (1) For any \( R \geq 1 \), a *semi-complete curve* in \( U_R \) is the image of a map \( c : [0, +\infty[ \to U_R \) such that \( |c(t)| = \frac{1}{t} \to +\infty \).

(2) Let \( c : [0, +\infty[ \to U_R \) be a semi-complete curve and let \( \theta_0 \) be a real number. We say that the image of \( c \) has the ray \( \{re^{i\theta_0}, r > 0\} \) as asymptotic direction, if \( \theta(t) \to \theta_0 \), where \( \theta(t) \) is the determination of the argument of \( c(t) \) in \([\theta_0 - \pi, \theta_0 + \pi]\).

From formula (3) above, by a continuity argument we deduce the following facts.

**Lemma 2.2.** (1) There exists \( R_0 > 1 \) so that, for \( k = 0, \ldots, 2m+1 \) and for any \( R \geq R_0 \), the function \( \text{Im} W \) is strictly monotone along the pairwise disjoint arcs

\[
A_k(R) := \{z \in S_R, \quad \frac{k\pi}{m+1} - \frac{\pi}{10(m+1)} < \arg(z) < \frac{k\pi}{m+1} + \frac{\pi}{10(m+1)}\}.
\]

(2) For any fixed \( C \in \mathbb{R} \) one has

- If \((z_n)\) is a sequence of complex numbers such that \(|z_n| \to \infty \) and \( \text{Im} W(z_n) \equiv C \), then \( \sin \left((m+1) \arg z_n\right) \to 0 \).
- There exists \( r(C) > R_0 \) such that, for any \( R \geq r(C) \), there are exactly \( 2m+2 \) points \( \text{Re} e^{i\theta_k} \), \( k = 0, \ldots, 2m+1 \), on the circle \( S_R \) verifying \( \text{Im} W(\text{Re} e^{i\theta_k}) = C \) and \( \text{Re} e^{i\theta_k} \in A_k(R) \). Moreover, we have \( \theta_k \to \frac{k\pi}{m+1} \).
- For any \( R \geq r(C) \), the set \( U_R \cap \{\text{Im} W(z) = C\} \) is composed of \( 2m+2 \) semi-complete curves \( H_k(C, R) \), \( k = 0, \ldots, 2m+1 \). Moreover \( H_k(C, R) \) has the ray \( \{re^{i\frac{k\pi}{m+1}}, r > 0\} \) as asymptotic direction.

Let \( k = -1, \ldots, 2m+1 \). We take \( C = 0 \) in Lemma 2.2 and define \( H_k(R) := H_k(0, R) \), \( R_1 := r(0) > R_0 \). Moreover set \( \alpha_k := \frac{k\pi}{m+1} \). Then, we deduce the following result.

**Corollary 2.1.** For any \( R \geq R_1 \), the level set \( U_R \cap \{\text{Im} W(z) = 0\} \) is composed of the \( 2m+2 \) semi-complete curves \( H_k(R) \), \( k = 0, \ldots, 2m+1 \), having the following properties (see Figure 1(a)).
Each curve $H_k(R)$ has a unique boundary point that belongs to the open arc $A_k(R)$.

Each curve $H_k(R)$ has the ray $\{re^{i\frac{k\pi}{m+1}}, r > 0\}$ as asymptotic direction.

Each curve $H_k(R)$ is contained in the truncated sector $\Delta_k(R)$ defined as follows

$$\Delta_k(R) := \{|z| > R \text{ and } \alpha_k - \frac{\pi}{10(m+1)} < \arg(z) < \alpha_k + \frac{\pi}{10(m+1)}\}$$

Let us state some consequences of the properties of the harmonic function $\text{Im} W$.

Let $C_0 > 0$ be a real number such that $C_0 > \max\{|\text{Im} W(z)|, z \in S_{R_1}\}$. Let $R_2$ be a real number satisfying $R_2 > r(C_0), r(-C_0), R_1$, where $r(C_0), r(-C_0)$ and $R_1 = r(0)$ are as in Lemma 2.2. Note that the set $U_{R_3} \cap \{\text{Im} W(z) = C_0\}$ is composed of $m + 1$ proper and complete curves without boundary $L_0^+, \ldots, L_m^+$ (see Figure 1(b)).

For each $j = 0, \ldots, m$, the level curve $L_j^+$ is contained in the domain of $C$ which does not contain 0 and which is bounded by $H_{2j+1}(R_1), H_{2j+2}(R_1)$ and an arc of $S_{R_1}$ contained in the arc $\{z \in S_{R_1} \mid \alpha_{2j} - \frac{\pi}{10(m+1)} < \arg(z) < \alpha_{2j+1} + \frac{\pi}{10(m+1)}\}$.

In the same way, the set $U_{R_3} \cap \{\text{Im} W(z) = -C_0\}$ is composed of $m + 1$ proper and complete curves without boundary $L_1^-, \ldots, L_m^-$. Each level curve $L_j^-$ is contained in the domain of $C$ which does not contain 0 and which is bounded by $H_{2j+1}(R_1), H_{2j+2}(R_1)$ and an arc of $S_{R_1}$ contained in the arc $\{z \in S_{R_1} \mid \alpha_{2j+1} - \frac{\pi}{10(m+1)} < \arg(z) < \alpha_{2j+2} + \frac{\pi}{10(m+1)}\}$, we set $H_{2m+2}(R_1) := H_0(R_1)$.

For each level curve $L_j^\pm$, we denote by $L_j^\pm$, the connected component of $C \setminus L_j^\pm$ which does not contain the circle $S_{R_1}$.
For each $k = 0, \ldots, 2m + 1$ we define the open set $\Omega_k$ setting:

$$\Omega_k := \begin{cases} 
\mathcal{L}_{k-1}^- \cup \mathcal{L}_k^+ \cup \Delta_k(R_2) & \text{if } k \text{ is even,} \\
\mathcal{L}_{k-1}^- \cup \mathcal{L}_{k-1}^- \cup \Delta_k(R_2) & \text{if } k \text{ is odd,}
\end{cases} \quad (5)$$

where we set $\mathcal{L}^-_m := \mathcal{L}^-_0$ (see Figure 2).

By construction, we have that each $\Omega_k$ is a simply connected domain and, setting $\mathcal{U} := \cup_{k=0}^{2m+1} \Omega_k$, we have $U_{R_2} \subset \mathcal{U} \subset U_{R_1}$. Moreover, we have

$$\Omega_k \subset \{|z| > R_1 \text{ and } \alpha_{k-1} - \frac{\pi}{10(m+1)} < \arg(z) < \alpha_{k+1} + \frac{\pi}{10(m+1)}\},$$

(recall that $\alpha_{-1} := -\pi/(m+1)$ and $\alpha_{2m+2} := 2\pi$).

We summarize the above construction as follows.

**Lemma 2.3.** Let $C$ be a real number, then the following facts hold.

1. If $C > C_0$, then
   
   - the level set $\{\text{Im} W(z) = C\} \cap \mathcal{U}$ is composed of $m + 1$ proper and complete curves without boundary $L_0(C), \ldots, L_m(C)$ satisfying $L_j(C) \subset \mathcal{L}^+_j$ and, therefore, $L_j(C) \subset \Omega_{2j} \cap \Omega_{2j+1}$, $j = 0, \ldots, m$.
   - the level set $\{\text{Im} W(z) = -C\} \cap \mathcal{U}$ is composed of $m + 1$ proper and complete curves without boundary $L_0(-C), \ldots, L_m(-C)$ satisfying $L_j(-C) \subset \mathcal{L}^-_j$ and, therefore, $L_j(-C) \subset \Omega_{2j+1} \cap \Omega_{2j+2}$, $j = 0, \ldots, m$, where $\Omega_{2m+2} := \Omega_0$. 

![Figure 2. The domains $\Omega_k$ for $m = 2$](image-url)
(2) If $-C_0 \leq C \leq C_0$ then the level set $\{ \text{Im} W(z) = C \} \cap U$ is composed of $2m + 2$ proper curves $H_k(C) \subset \Omega_k$, $k = 0, \ldots, 2m + 1$, satisfying the same properties as the level curves $H_k(R_2)$ in Corollary 2.1 with $R = R_2$.

**Proposition 2.1.** For $k = 0, \ldots, 2m + 1$ and for any $z \in \Omega_k$ we choose the determination of $\text{arg } z$ in the interval $[\alpha_{k-1} - \frac{\pi}{10(m+1)}, \alpha_{k+1} + \frac{\pi}{10(m+1)}]$. Then, the restriction of $W$ to $\Omega_k$ is a well defined complex function, denoted by $W_k$. Furthermore, $W_k : \Omega_k \to \mathbb{C}$ is one-to-one and defines a conformal diffeomorphism from $\Omega_k$ onto a simply connected domain $\tilde{\Omega}_k := W_k(\Omega_k)$ in the $w$ complex plane.

**Proof.** Since $\Omega_k$ is a simply connected domain which does not contain the origin, the function $W$ is well defined on $\Omega_k$.

Let $z_1, z_2 \in \Omega_k$ be such that $W_k(z_1) = W_k(z_2)$. We deduce from Lemma 2.3 that for any $C \in \mathbb{R}$, the level set $\{ \text{Im} W_k(z) = C \}$ has a unique connected component in $\Omega_k$. Therefore, $z_1$ and $z_2$ belong to the same level curve $L \subset \Omega_k$. Since $W_k'(z) = \sqrt{\phi(z)}$ and $\phi$ does not vanish on $U$, we deduce that the function $\text{Re } W$ is strictly monotonous on $L$. We conclude that $z_1 = z_2$ as desired. \qed

In the $w$-complex plane the domains $\tilde{\Omega}_k$, $k = 0, \ldots, 2m + 1$, defined in Proposition 2.1 have a nice structure, that will be crucial in the following.

**Corollary 2.2.** Let $k$ be an even number, $k = 2j$. Then, $\tilde{\Omega}_k$ is the complementary of a horizontal half-strip. The non horizontal component of $\partial \tilde{\Omega}_k$ is a compact arc that is the image by $W_k$ of the boundary arc of $\Omega_k$ in $A_k(R_2)$ joining $L^+_j$ and $L^-_{j-1}$. Thus, $\text{Im } W$ is strictly monotonous along such non horizontal component and $\text{Re } w$ is bounded from above by a real number $a_k$ for any $w \in \partial \tilde{\Omega}_k$ (see Figure 3(a)).

If $k$ is an odd number, then $\tilde{\Omega}_k$ has a similar description, except that on the half-strip the real part of $w$ is now bounded from below, i.e. for some real number $b_k$ we have $\text{Re } w > b_k$ for any $w \in \partial \tilde{\Omega}_k$ (see Figure 3(b)).

We get a proof of Corollary 2.2 by invoking Lemma 2.3.

**Figure 3.** The domains $\tilde{\Omega}_k$
By the equalities in (2), we can take $R_2$ in (5) large enough so that
\[
\frac{1}{2} |z|^m < \sqrt{|\phi(z)|} < 2 |z|^m
\]
when $|z| \geq R_2$. With this choice, we can prove the following result.

**Lemma 2.4.** There is a real constant $c_1 > 0$ such that, for any $z$ satisfying $|z| > 2R_2$, there exists $k \in \{0, \ldots, 2m + 1\}$ such that
\[
z \in \Omega_k \quad \text{and} \quad d_\phi(z, \partial \Omega_k) > c_1 |z|,
\]
where $d_\phi$ stands for the distance on $\Omega_k$ with respect to the $\phi$-metric given by $|\phi(z)| |dz|^2$.

**Proof.** First assume that $m \geq 1$. Let $z \in U$ such that $|z| \geq 2R_2$. Recall that $\alpha_k = \frac{k\pi}{m+1}$ for $k = -1, \ldots, 2m + 2$. There exists a unique $k \in \{0, \ldots, 2m + 1\}$ such that either $(\alpha_k + \alpha_{k+1})/2 \leq \arg z \leq \alpha_{k+1}$ or $\alpha_k \leq \arg z \leq (\alpha_k + \alpha_{k+1})/2$. Without loss of generality, we can assume that the latter occurs. Therefore $z \in \hat{\Omega}_k$.

For any $k = 0, \ldots, 2m + 1$, we define the following rays:
\[
D_k := \{\rho e^{i(\alpha_{k+1} - \pi/10(m+1))}, \quad \rho \geq R_2\},
\]
\[
D'_k := \{\rho e^{i(\alpha_k + \pi/10(m+1))}, \quad \rho \geq R_2\}.
\]

By assumption, $z$ belongs to the subdomain $\hat{\Omega}_k$ of $\Omega_k$ bounded by $D_k$, $D'_k$ and the arc $\Gamma(R_2)$ of $S_{R_2}$ corresponding to $\alpha_{k-1} + \pi/10(m+1) \leq \theta \leq \alpha_{k+1} - \pi/10(m+1)$ (see Figure 4).

Then, we have
\[
d_\phi(z, \partial \Omega_k) \geq \min \{d_\phi(z, D_k), \ d_\phi(z, D'_k), \ d_\phi(z, \Gamma(R_2))\}.
\]
Let $\gamma : [0, 1] \to \Omega_k$ be any smooth arc satisfying $\gamma(0) = z$, $\gamma(1) \in D_k$ and $|\gamma(t)| > R_2$ for any $t \in [0, 1]$. Denoting by $L_\phi(\gamma)$ the length of $\gamma$ for the $\phi$-metric and using (8), we have:

$$L_\phi(\gamma) = \int_0^1 \sqrt{|\phi(\gamma(t))|} |\gamma'(t)| \, dt \geq \frac{1}{2} \int_0^1 |\gamma(t)|^m |\gamma'(t)| \, dt \geq \frac{R_2^m}{2} \int_0^1 |\gamma'(t)| \, dt,$$

so that $L_\phi(\gamma) \geq (1/2) R_2^m L(\gamma)$, where $L(\gamma)$ is the Euclidean length of $\gamma$. Since $4\pi/10(m+1) \leq \alpha_k - \pi/10(m+1) - \arg z \leq 9\pi/10(m+1) < \pi/2$ we get

$$L(\gamma) > d(z, D_k) > \sin \left( \frac{4\pi}{10(m+1)} \right) |z|$$

where $d(z, D_k)$ stands for the Euclidean distance between $z$ and $D_k$. From the last inequality, we deduce

$$d_\phi(z, D_k) \geq \frac{R_2^m}{2} \sin \left( \frac{4\pi}{10(m+1)} \right) |z|. \quad (7)$$

Let now $\gamma : [0, 1] \to \Omega_k$ be any smooth arc satisfying $\gamma(0) = z$, $\gamma(1) \in D_k$ and $|\gamma(t)| > R_2$ for any $t \in [0, 1]$. In the same way, we can show that

$$L_\phi(\gamma) \geq \frac{R_2^m}{2} L(\gamma) \geq \frac{R_2^m}{2} d(z, D_k').$$

Since $9\pi/10(m+1) \leq \arg z - \alpha_{k-1} - \pi/10(m+1) \leq 14\pi/10(m+1)$ we obtain

$$d_\phi(z, D_k') \geq \min \left\{ \sin \left( \frac{9\pi}{10(m+1)} \right), \sin \left( \frac{14\pi}{10(m+1)} \right) \right\} \frac{R_2^m}{2} |z|. \quad (8)$$

Finally, let $\gamma : [0, 1] \to \Omega_k$ be any smooth arc satisfying $\gamma(0) = z$, $|\gamma(1)| = R_2$ and $|\gamma(t)| > R_2$ for any $t \in [0, 1]$. As before we have

$$L_\phi(\gamma) \geq \frac{R_2^m}{2} L(\gamma) \geq \frac{R_2^m}{2} (|z| - R_2).$$

Since $|z| > 2R_2$ we get

$$d_\phi(z, S_{R_2}) > \frac{R_2^m}{4} |z|. \quad (9)$$

Now, using estimates (7), (8) and (9), we are done in the case $m \geq 1$.

Now we consider the case $m = 0$. Then, there are only two domains: $\Omega_0, \Omega_1$, and we have $\alpha_0 = 0$, $\alpha_1 = \pi$ and $\alpha_2 = 2\pi$.

Let $z \in U$ such that $|z| \geq 2R_2$. For some $k \in \{0, 1\}$, we have either $(\alpha_k + \alpha_{k+1})/2 \leq \arg z \leq \alpha_{k+1}$ or $\alpha_k \leq \arg z \leq (\alpha_k + \alpha_{k+1})/2$. Without loss of generality, we can assume that the former occurs and that $k = 0$, that is: $\pi/2 \leq \arg z \leq \pi$ and, therefore, $z \in \Omega_1$.

We set

$$D := \{ \rho e^{i\pi/10}, \rho \geq R_2 \}, \quad D' := \{ \rho e^{-i\pi/10}, \rho \geq R_2 \}.$$
We have $d(z, D) \geq |z|/2$ and $d(z, D') \geq d(z, D)$. Moreover, it can be shown in the same way as in the case $m \geq 1$, that $d_\phi(z, S_{R_2}) > |z|/4$. We obtain that $d_\phi(z, \partial \Omega_k) > |z|/4$, which concludes the proof. \hfill \square

\textbf{Remark 2.1.} For $k = 0, \ldots, 2m + 1$, the map $W_k : \Omega_k \to \tilde{\Omega}_k$ is a conformal diffeomorphism. Since $W_k'(z) = \sqrt{\phi(z)}$, $W_k$ is an isometry when $\Omega_k$ is equipped with the $\phi$-metric $|\phi(z)| |dz|^2$ and $\tilde{\Omega}_k$ is equipped with the Euclidean metric $|dw|^2$.

We denote by $\tilde{\Omega}_k : \Omega_k \to \tilde{\Omega}_k$ the inverse function of $W_k$.

The image of the level sets of $\text{Im} W$ by the harmonic map $F$

Let $N := (n_1, n_2, n_3)$ be the unit normal vector field along the end $E$ such that $(X_x, X_y, N)$ has the positive orientation. We get from \cite{[18]} Proposition 4] that $n_3 = |F_x| - |F_y|$. We define a function (possibly with poles) $\omega$ on $U_R$ setting \cite{[7]} Formula 14]

\begin{equation}
 n_3 = \tanh \omega. \tag{10}
\end{equation}

For $k = 0, \ldots, 2m + 1$, we denote the restriction of $\omega$ to $\Omega_k$ by $\omega_k$. The function $\tilde{\omega}_k : \tilde{\Omega}_k \to \mathbb{R}$ is defined by setting $\tilde{\omega}_k(w) := (\omega_k \circ Z_k)(w)$ for any $w \in \tilde{\Omega}_k$.

The induced metric $ds^2$ on $U$ reads as

\begin{equation}
 ds^2 = 4 \cosh^2(\omega) |\phi| |dz|^2, \tag{11}
\end{equation}

see \cite{[7]} Equation 14].

\textbf{Remark 2.2.} Since $\phi$ has no zero on $U$, the function $\omega$ has no pole and the tangent plane of $E$ is never horizontal. This means that the end $E$ is transversal to any slice $\mathbb{H}^2 \times \{t\}$. Thus, the intersection of $E$ with any slice is composed of analytic curves.

Let us denote by $\Delta_x$ (resp. $\Delta_w$) the laplacian on $\Omega$ (resp. $\tilde{\Omega}$) with respect to the Euclidean metric $|dz|^2$ (resp. $|dw|^2$). Since $\Delta_x \omega_k = 2 \sinh(2\omega_k) |\phi(z)|$ (see \cite{[7]} Equation 13]), we deduce

\begin{equation}
 \Delta_w \tilde{\omega}_k = 2 \sinh(2\tilde{\omega}_k). \tag{12}
\end{equation}

For any $w \in \tilde{\Omega}_k$ we denote by $d_k(w)$ the Euclidean distance between $w$ and the boundary of $\tilde{\Omega}_k$.

The following estimate \cite{[13]} can be found in \cite{[8]} (see also \cite{[11]} Lemma 3.3]).

\textbf{Proposition 2.2.} There exists a constant $K_0 > 0$ such that for $k = 0, \ldots, 2m + 1$ and for any $w \in \Omega_k$ with $d_k(w) > 1$, we have

\begin{equation}
 |\tilde{\omega}_k(w)| \leq \frac{K_0}{\cosh d_k(w)} < 2K_0 e^{-d_k(w)}. \tag{13}
\end{equation}

Consequently, the tangent planes to the end become vertical at infinity.
The last assertion is a consequence of the estimate (13) and Lemma 2.4.

We recall that the energy density of the harmonic function $F$ with respect to the metric $|\phi(z)||dz|^2$ on $U$ and the hyperbolic metric on $\mathbb{H}^2$ is the real function defined on $U$ by

$$e(z) := \frac{(\sigma \circ F)^2(z)}{|\phi(z)|} \left(|F_z|^2 + |F_{\overline{\eta}}|^2\right)$$

Then one has

$$e(z) = \frac{|F_z|}{|F_{\overline{\eta}}|} + \frac{|F_{\overline{\eta}}|}{|F_z|} = 2 \cosh 2\omega.$$ 

where the first equality follows from the definition of $\phi$ and the second equality follows from the definition of $\omega$. Observe that $e^{2\omega} = \frac{|F_z|}{|F_{\overline{\eta}}|}$.

For $k = 0, \ldots, 2m + 1$ we denote by $\tilde{F}_k$ the harmonic map $\tilde{F}_k := F \circ Z_k : \tilde{\Omega}_k \to \mathbb{H}^2$.

Recall that the relation between the coordinate $z$ in $\Omega$ and the coordinate $w$ in $\tilde{\Omega}$ is $w = W(z)$ and $\frac{dw}{dz} = 1/\sqrt{\phi \circ Z}$. The energy density $\tilde{e}$ of $\tilde{F}$ with respect to the Euclidean metric $|dw|^2$ on $\tilde{\Omega}$ and the hyperbolic metric on $\mathbb{H}^2$ is defined on $\tilde{\Omega}$ by

$$\tilde{e}(w) := (\sigma \circ \tilde{F})^2(w)\left(|\tilde{F}_w|^2 + |\tilde{F}_{\overline{\eta}}|^2\right)$$

As before, we have

$$\tilde{e}(w) = \frac{|\tilde{F}_w|}{|\tilde{F}_{\overline{\eta}}|} + \frac{|\tilde{F}_{\overline{\eta}}|}{|\tilde{F}_w|} = 2 \cosh 2\tilde{\omega}.$$ 

Thus $\tilde{e}(w) = e(z)$, if $z = Z(w)$.

**Definition 2.3.** Let $\kappa : U \to \mathbb{R}$ be defined as follows: for any $z_0 \in U$, $\kappa(z_0)$ is the geodesic curvature in $\mathbb{H}^2$ (with respect to the normal orientation induced by the unit normal vector field $N$ on $E$) of the connected component of $F(\{\text{Im} W = \text{Im} W(z_0)\})$ passing through the point $F(z_0)$.

For any $k = 0, \ldots, 2m + 1$, let $\tilde{\kappa} : \tilde{\Omega}_k \to \mathbb{R}$ be defined by setting $\tilde{\kappa}(w_0) = \kappa(z_0)$, where $w_0 = W(z_0)$.

As a consequence of Remark 2.2, we have that the function $\kappa$ is analytic.

**Lemma 2.5.** Fix a number $k \in \{0, \ldots, 2m + 1\}$ and consider the simply connected domain $\Omega_k$ defined in (5). Then, setting $w = u + iv$ on $\tilde{\Omega}_k$, the pullback by the harmonic map $\tilde{F}_k : \tilde{\Omega}_k \to \mathbb{H}^2$ of the hyperbolic metric $\sigma^2(\xi)|d\xi|^2$ is given by

$$\tilde{F}_k^*(\sigma^2(\xi)|d\xi|^2) = 4 \cosh^2 \tilde{\omega}_k \, du^2 + 4 \sinh^2 \tilde{\omega}_k \, dv^2.$$ 

Moreover, for any horizontal coordinate curve $\tilde{\gamma} := \{v = \text{const}\}$ in $\tilde{\Omega}_k$, the absolute value of the geodesic curvature $\tilde{\kappa}$ of the curve $\tilde{F}_k(\tilde{\gamma})$ in $\mathbb{H}^2$ is given by

$$|\tilde{\kappa}(w)| = \frac{1}{2 \cosh \tilde{\omega}_k} \left|\frac{\partial \tilde{\omega}_k}{\partial v}\right|(w).$$

(15)
for any $w \in \tilde{\gamma}$.

Proof. A straightforward computation shows that
\[
F_k^*(\sigma^2(\xi)d\xi d\bar{\xi}) = \phi(z)dz^2 + \bar{\phi}(z)d\bar{z}^2 + \epsilon(z)|\phi(z)|dzd\bar{z}.
\]

Since $dw = \sqrt{\phi(z)}dz$ and $d\bar{w} = \sqrt{\phi(z)}d\bar{z}$, in the coordinate $w = u + iv$, we have
\[
F_k^*(\sigma^2(\xi)d\xi d\bar{\xi}) = (\bar{e} + 2)du^2 + (\bar{e} - 2)dv^2 = 4 \cosh^2 \tilde{\omega}_k \, du^2 + 4 \sinh^2 \tilde{\omega}_k \, dv^2.
\]
Then equality (14) is proved.

Now, let $w_0 \in \tilde{\gamma}$ and assume $\tilde{\omega}_k(w_0) \neq 0$. Then, by (14), the pullback by $F_k$ of the hyperbolic metric is a regular metric in a neighborhood of $w_0$ in $\tilde{\Omega}_k$. Consequently, the geodesic curvature of $F_k(\tilde{\gamma})$ at $w_0$ is given by
\[
\tilde{\kappa}(w_0) = -\frac{1}{2} \frac{1}{4 \cosh^2 \tilde{\omega}_k} \frac{1}{2|\sinh \tilde{\omega}_k|} \frac{\partial}{\partial v} (4 \cosh^2 \tilde{\omega}_k)(w_0) = -\frac{1}{2} \frac{1}{\cosh \tilde{\omega}_k} \frac{\sinh \tilde{\omega}_k}{|\sinh \tilde{\omega}_k|} \frac{\partial \tilde{\omega}_k}{\partial v}(w_0),
\]
(see [10, Formula (42.8)]). Therefore, the proof is finished in the case $\tilde{\omega}_k(w_0) \neq 0$.

Assume now that $\tilde{\omega}_k(w_0) = 0$. If $\tilde{\omega}_k$ vanishes identically in a neighborhood of $w_0$, then the tangent plane of the minimal end $E$ is always vertical in a open neighborhood of $X(Z_k(w_0))$. This means that such neighborhood is contained in a vertical cylinder in $\mathbb{H}^2 \times \mathbb{R}$. Since $E$ is minimal, the vertical cylinder is a part of a vertical geodesic plane and, by analyticity, the whole end $E$ is contained in the geodesic plane. Consequently the curve $F_k(\tilde{\gamma})$ is a part of a geodesic of $\mathbb{H}^2$ and formula (15) is trivially satisfied. If $\tilde{\omega}_k$ is not identically zero in a neighborhood of $w_0$, then there exists a sequence $(w_n)_{n \in \mathbb{N}}$ in $\tilde{\Omega}_k$ converging to $w_0$ such that $\tilde{\omega}_k(w_n) \neq 0$ for any $n > 0$. Since formula (15) holds at any point $w_n$ and $|\kappa|$ is a continuous function, then (15) holds also at $w_0$.

The following Proposition is crucial in order to understand the geometry of the horizontal sections.

**Proposition 2.3.** Let $z_0 \in U$ and let $\kappa(z_0)$ be the geodesic curvature of the level curve $F(\{\text{Im} \, W(z) = \text{Im} \, W(z_0)\}) \subset \mathbb{H}^2$. We set $R_3 = \max\{2R_2, 2/c_1\}$, where $c_1 > 0$ is the constant given by Lemma 2.4. Then, there exists a constant $c_2 > 0$ such that, for any $z_0 \in U_{R_3}$, we have
\[
|\kappa(z_0)| < c_2 e^{-c_1|z_0|}.
\]

Proof. Let $z_0 \in U$ be any point such that $|z_0| > R_3$. It follows from Lemma 2.4 that there exists $k \in \{0, \ldots, 2m + 1\}$ such that
\[
z_0 \in \Omega_k \quad \text{and} \quad d_{\phi}(z_0, \partial \Omega_k) > c_1|z_0|.
\]
Setting $w_0 := W(z_0) \in \tilde{\Omega}_k$, we get $d_k(w_0) := d_k(w_0, \partial \tilde{\Omega}_k) = d_{\phi}(z_0, \partial \Omega_k)$, where $d_k$ denotes the Euclidean distance in $\tilde{\Omega}_k$. Therefore, we obtain $d_k(w_0) > 2$, since $c_1|z_0| > c_1R_3 > 2$. 

□
Let $\tilde{D}$ be the unit disk in the $w$-complex plane, centered at $w_0$, thus $\tilde{D} \subset \tilde{\Omega}_k$. For any $w \in \tilde{D}$, we denote by $d(w)$ the Euclidean distance between $w$ and $\partial \tilde{D}$. Recall that the function $\tilde{\omega}_k$ satisfies Equation [12] on $\tilde{\Omega}_k$. We restrict $\tilde{\omega}_k$ to $\tilde{D}$ and we apply the interior $a$-priori gradient estimate for the Poisson Equation [5, Theorem 3.9], then

$$\sup_{\tilde{D}} (d(w) |\nabla \tilde{\omega}_k|) < K_1 \left( \sup_{\tilde{D}} |\tilde{\omega}_k| + 2 \sup_{\tilde{D}} d^2(w) |\sinh 2\tilde{\omega}_k| \right),$$

for some constant $K_1 > 0$, where $\nabla$ means the Euclidean gradient.

Since $d(w_0) = 1$ and $d(w) \leq 1$ for any $w \in \tilde{D}$, we get

$$|\nabla \tilde{\omega}_k|(w_0) < K_1 \left( \sup_{\tilde{D}} |\tilde{\omega}_k| + 2 \sup_{\tilde{D}} |\sinh 2\tilde{\omega}_k| \right)$$

for any $w \in \tilde{D}$. Using the inequality $\cosh(t - 1) > \frac{e^t}{10}$ for any $t \in \mathbb{R}$ we obtain

$$|\nabla \tilde{\omega}_k|(w_0) < K_1 \left(10K_0 e^{-d_k(w_0)} + 2 \sinh(20K_0 e^{-d_k(w_0)})\right).$$

The function $x \mapsto \frac{\sinh x}{x}$ is strictly increasing for $x > 0$. As $d_k(w_0) > 2$, then we obtain that $\sinh(20K_0 e^{-d_k(w_0)}) < e^2 \sinh(20K_0 e^{-2}) e^{-d_k(w_0)}$. This proves that there exists a constant $\delta > 0$ such that

$$|\nabla \tilde{\omega}_k|(w_0) < \delta e^{-d_k(w_0)}, \quad (16)$$

for any $w_0 \in \tilde{\Omega}_k$ such that $d_k(w_0) > 2$. From formula (16) and from the previous computations, setting $c_2 := \delta/2$, we conclude that

$$|\tilde{\kappa}(w_0)| < c_2 e^{-d_k(w_0)}.$$  

As $\kappa(z_0) = \tilde{\kappa}(w_0)$ and $d_k(w_0) := d_\phi(z_0, \partial \Omega_k) > c_1 |z_0|$ by Lemma 2.3, this completes the proof. $\square$

In view of Lemma 2.3 let us sum up some notations previously established. For any $C \in \mathbb{R}$ and for $k = 0, \ldots, 2m + 1$, $H_k(C, R) \subset U \subset U$, denotes the semi-complete level curve of the function $\text{Im} W$ whose asymptotic direction is $\{re^{i\alpha_k}, \, r > 0\}$, where $\alpha_k = k\pi/(m + 1)$. That is

$$\text{Im} W(z) = C \quad \text{for any} \quad z \in H_k(C, R) \quad \text{and} \quad \arg z \xrightarrow{|z| \to \infty} \alpha_k, \quad z \in H_k(C, R).$$

For any $C > C_0$, we denote by $L_j(C)$ (resp. $L_j(-C)$), $j = 0, \ldots, m$, the proper and complete level curves given by $\{\text{Im} W = C\}$ (resp. $\{\text{Im} W = -C\}$). We have, for any $R$, $H_{2j}(C, R) \cup H_{2j+1}(C, R) \subset L_j(C)$ and $H_{2j+1}(-C, R) \cup H_{2j+2}(-C, R) \subset L_j(-C)$, $j = 0, \ldots, m$, where $H_{2m+2}(-C, R) := H_0(-C, R)$. 


The notion of convergence in the $C^1$ topology in the next Theorem is given in the statement of Definition \[1.1\].

**Theorem 2.1.** (1) For any $C \in \mathbb{R}$, let $r(C)$ be defined as in Lemma \[2.2\]. Then, for any $C \in \mathbb{R}$, for any $k \in \{0, \ldots, 2m+1\}$ and $R > r(C)$, the level curve $F(H_k(C, R)) \subset \mathcal{U} \subset \mathbb{H}^2$ is a proper semi-complete curve which has no limit point in $\mathbb{H}^2$ and with a unique asymptotic point in $\partial_\infty \mathbb{H}^2$.

(2) For any $C_1, C_2 \in \mathbb{R}$ and for any $k \in \{0, \ldots, 2m+1\}$, the level curves $F(H_k(C_1, R)), F(H_k(C_2, R)) \subset \mathbb{H}^2$ are asymptotic. That is, there exist a sequence $(p_n)$ on $H_k(C_1, R)$ and a sequence $(q_n)$ on $H_k(C_2, R)$ such that $|p_n|, |q_n| \to \infty$, and $d_{\mathbb{H}^2}(F(p_n), F(q_n)) \to 0$. Consequently, $F(H_k(C_1, R))$ and $F(H_k(C_2, R))$ have the same asymptotic point $\tilde{\theta}_k \in \partial_\infty \mathbb{H}^2$.

(3) For $k = 0, \ldots, 2m+1$, the asymptotic points $\theta_k$ and $\theta_{k+1}$ are distinct, ($\theta_{2m+2} := \theta_0$).

(4) Let $j \in \{0, \ldots, m\}$.

- When $C \to +\infty$, then the proper and complete level curves $F(L_j(C)) \subset \mathbb{H}^2$ converge for the $C^1$ topology to the geodesic in $\mathbb{H}^2$ with asymptotic boundary $\{\theta_{2j}, \theta_{2j+1}\}$.
- When $C \to +\infty$, then the proper and complete level curves $F(L_j(-C)) \subset \mathbb{H}^2$ converge for the $C^1$ topology to the geodesic in $\mathbb{H}^2$ with asymptotic boundary $\{\theta_{2j+1}, \theta_{2j+2}\}$, ($\theta_{2m+2} := \theta_0$).

**Proof.** Assertion (1) is a straightforward consequence of the curvature estimates given in Proposition \[2.3\] together with Proposition \[1.1\].

Let us prove Assertion (2). Let $k \in \{0, \ldots, 2m+1\}$. By Lemma \[2.3\] for any $R > r(C_i)$, we have $H_k(C_i, R) \subset \Omega_k$, $i = 1, 2$. From formula (1) we deduce that $\text{Re} W(z) \xrightarrow{|z| \to \infty} +\infty$, $z \in H_k(C_i, R)$ (resp. $-\infty$) if $k$ is an even (resp. odd) number, $i = 1, 2$.

Assume now that $k$ is even (the argument is analogous in the other case). Then, by the geometry of the sets $W(\Omega_k)$ (see Corollary \[2.2\]), setting $\tilde{p}_n := n + iC_1$ and $\tilde{q}_n := n + iC_2$, for any $n \in \mathbb{N}$ large enough, we have $\tilde{p}_n \in W(H_k(C_1)) \subset \Omega_k$ and $\tilde{q}_n \in W(H_k(C_2)) \subset \Omega_k$. Moreover, setting $\tilde{\gamma}_n := \{(1-t)\tilde{p}_n + t\tilde{q}_n, 0 \leq t \leq 1\}$, we have $\tilde{\gamma}_n \subset \tilde{\Omega}_k$.

Let us set $p_n = Z_k(\tilde{p}_n)$, $q_n = Z_k(\tilde{q}_n)$ and $\gamma_n = Z_k(\tilde{\gamma}_n)$ where $Z_k : \tilde{\Omega}_k \to \Omega_k$ is the inverse function of $W$ restricted to $\Omega_k$ as defined in Remark \[2.1\]. Thus, we have:

- $\partial \gamma_n = \{p_n, q_n\}$, $p_n \in H_k(C_1, R)$ and $q_n \in H_k(C_2, R)$.
- $\text{Re} W(z) = n$, for any $z \in \gamma_n$.

The distance between $F_k(p_n)$ and $F_k(q_n)$ in $\mathbb{H}^2$ is smaller than the length of $F_k(\gamma_n)$ which, by construction, is equal to the length of $\tilde{F}_k(\tilde{\gamma}_n)$.

We need to prove the following Claim.
Claim. Let $\gamma \subset \gamma_n$ be an open arc along which the restriction of $\omega$ to $\Omega_k$ vanishes. Then, $F_k(\gamma) \subset \mathbb{H}^2$ is reduced to a single point.

Indeed, let $c : [0,1[ \to \gamma \subset \gamma_n$ be a smooth parametrization of $\gamma$. Since $\text{Re} W(c(t)) \equiv n$ for $0 < t < 1$, setting $w = W(z)$ and differentiating with respect to $t$, we get $\frac{dw}{dz}, \frac{dc}{dt} + \left( \frac{dw}{dz} \right) \left( \frac{dc}{dt} \right) = 0$. Moreover, as $\omega_k(c(t)) \equiv 0$ we have $|F_\gamma| = |F_\gamma|$. We deduce from formula (3) that $F_k \left( \frac{dc}{dt} \right) = (F_k \circ c)(t) \equiv 0$, which proves the Claim.

Consequently, the length of $F_k(\gamma_n)$ in $\mathbb{H}^2$ is equal to the length of $\tilde{\gamma}_n$ with respect to the pseudo-metric (14) on $\tilde{\Omega}_k$, denoted by $L_k(\tilde{\gamma}_n)$.

Corollary 2.2 yields that for any $w \in \tilde{\gamma}_n$ and for $n$ large enough we have

$$d_k(w) := d_k(w, \partial \tilde{\Omega}_k) \geq n - a_k,$$

where, as usual, $d_k$ is the Euclidean metric on $\tilde{\Omega}_k$. We have

$$d_{\mathbb{H}^2}(F_k(p_n), F_k(q_n)) = d_{\mathbb{H}^2}(\tilde{F}_k(p_n), F_k(q_n)) \leq L_{\mathbb{H}^2}(\tilde{F}_k(\gamma_n)) = L_k(\tilde{\gamma}_n)$$

where $d_{\mathbb{H}^2}$ (resp. $L_{\mathbb{H}^2}$) is the distance (resp. the length) in the hyperbolic metric.

As $u = \text{Re} W \equiv n$ along $\tilde{\gamma}_n$, we obtain

$$L_k(\tilde{\gamma}_n) = 2|C_1 - C_2| \int_0^1 \left| \sinh \tilde{\omega}_k(\tilde{\gamma}_n(t)) \right| dt \leq 2 \sinh \left( \frac{K_0}{\cosh(n - a_k)} \right)|C_1 - C_2|,$$

where the inequality comes from formula (13). Hence, we have $d_{\mathbb{H}^2}(F_k(p_n), F_k(q_n)) \to 0$ when $n \to +\infty$. We conclude that $F(H_k(C_1, R))$ and $F(H_k(C_2, R))$ have the same asymptotic point $\theta_k \in \partial_\infty \mathbb{H}^2$ and this completes the proof of Assertion (2).

Let us prove Assertion (3). Assume, for instance, that $\theta_0 = \theta_1$. Then, for any $C > C_0$, there exists a complete level curve $L_0(C) \subset \Omega_0$ such that $F(L_0(C)) \subset \mathbb{H}^2$ is a proper and complete curve with $\partial_\infty F(L_0(C)) = \{ \theta_0 \}$. We deduce from Proposition 2.3 and formula (3) that for $C$ large enough, the absolute value of the geodesic curvature of $F(L_0(C)) \subset \mathbb{H}^2$ is smaller than $1/4$. Let $\Gamma \subset \mathbb{H}^2$ be any complete geodesic such that $\theta_0 \in \partial_\infty \Gamma$. Then, we obtain a contradiction with the maximum principle, comparing $F(L_0(C))$ with the family of complete curves $\gamma_p$, $p \in \Gamma$, orthogonal to $\Gamma$ at $p$, with constant curvature $1/2$, and such that $\theta_0$ belongs to the asymptotic boundary of the mean convex component of $\mathbb{H}^2 \setminus \gamma_p$. This completes the proof of Assertion (3).

Assertion (4) is a straightforward consequence of Assertion (3), Propositions 2.3 and 4.2.}

Remark 2.3. (1) We deduce from Theorem 2.1 that the asymptotic boundary of $F(U)$ is composed of exactly $2m + 2$ points, counting with multiplicity. In particular, if $m = 0$ then $\partial_\infty F(U)$ has exactly two distinct points.

(2) Observe that the $2m + 2$ asymptotic points $\theta_0, \ldots, \theta_{2m+1}$ of $F(U)$ need not to be distinct. They even not need to be well ordered as we can see in some examples found by J. Pyo and M. Rodriguez [16].
We can construct artificial examples for which the asymptotic points are not distinct: just consider the covering maps \( \psi_n : U \to U \), \( n \geq 2 \), defined by \( \psi_n(z) = z^n \), and the minimal ends \( X_n := X \circ \psi_n : U \to \mathbb{H}^2 \times \mathbb{R} \).

Let us state two results that have an independent interest in this theory.

**Theorem 2.2.** Let \( X := (F, h) : U \to E \subset \mathbb{H}^2 \times \mathbb{R} \) be a complete minimal end with finite total curvature. Then, \( E \) is properly immersed.

**Proof.** Let \( m \in \mathbb{N}^* \) be the degree of the end \( E \) (see Definition 2.1). Let \( (p_n) \) be a sequence in \( U \) such that \( |p_n| \to +\infty \). We want to show that \( (X(p_n)) \) is not a bounded sequence in \( \mathbb{H}^2 \times \mathbb{R} \). Up to choosing a subsequence, we can assume that there is \( k \in \{0, \ldots, 2m + 1\} \) such that, for any \( n \), we have \( p_n \in \Omega_k \). Recall that \( h(z) = 2\text{Im} W(z) \) for any \( z \) in \( U \).

If \( h(p_n) \to \infty \) we are done.

Assume that the sequence \( (h(p_n)) \) of real number is bounded. Thus, up to considering a subsequences, there exists a real number \( C_1 \) such that \( h(p_n) \to C_1 \). We set \( S(C_1) := \{z \in \Omega_k \mid h(z) = C_1\} \). Thus, \( S(C_1) \) is either a complete curve or a semi-complete curve. As in the proof of Theorem 2.1, we can construct a sequence \( (q_n) \) in \( \Omega_k \) such that

\[
\forall n \in \mathbb{N}, \quad q_n \in S(C_1) \quad \text{and} \quad d_{\mathbb{H}^2}(F(p_n), F(q_n)) \to 0.
\]

Since \( F(S(C_1)) \) has no limit point in \( \mathbb{H}^2 \) and has an asymptotic point \( p_\infty \in \partial_\infty \mathbb{H}^2 \) (see (1) in Theorem 2.1), we get that \( F(q_n) \to p_\infty \) and consequently \( F(p_n) \to p_\infty \).

Therefore, the sequence \( X(p_n) \) is not bounded, which concludes the proof. \( \square \)

**Theorem 2.3.** Let \( E \subset \mathbb{H}^2 \times \mathbb{R} \) be a minimal, complete end with finite total curvature. Then, there exists a constant \( c_3 = c_3(E) \) such that, for any \( p \in E \), we have

\[
|K_E(p)| \leq c_3 e^{-d_E(p, \partial E)},
\]

where \( K_E \) denotes the intrinsic Gauss curvature and \( d_E(\cdot, \partial E) \) stands for the intrinsic distance on \( E \).

**Remark 2.4.** A straightforward consequence of Theorem 2.3 is the following: for any complete and connected minimal surface \( \Sigma \subset \mathbb{H}^2 \times \mathbb{R} \) with finite total curvature, and for any \( p_0 \in \Sigma \), there exists a constant \( c_4 = c_4(p_0, \Sigma) \) such that for any \( p \in \Sigma \) we have

\[
|K_\Sigma(p)| \leq c_4 e^{-d_\Sigma(p, p_0)}
\]

**Proof.** Let \( m \in \mathbb{N} \) be the degree of the end \( E \). We consider the open sets \( \Omega_k \subset U \), \( k = 0, \ldots, 2m + 1 \), as defined in (5) and the real number \( R_3 > 1 \) given in Proposition 2.3. In this proof we will use the notation previously established for the function \( W = \text{Im} \int \sqrt{\phi(z)} \, dz \), \( \tilde{\Omega}_k = W(\Omega_k) \), and \( Z_k : \tilde{\Omega}_k \to \Omega_k \).

Let \( p \in E \) and let \( z \in U \) such that \( X(z) = p \). Assume first that \( |z| > R_3 \). Therefore, by Lemma 2.4, there exists \( k \in \{0, \ldots, 2m + 1\} \) such that \( z \in \Omega_k \) and \( d_\phi(z, \partial \Omega_k) > c_1 |z| \).
We set \( w = W(z) \), so that \( w \in \tilde{\Omega}_k \). We deduce from formula (11) that the metric \( \tilde{d}^2 \) induced on \( \tilde{\Omega}_k \) by the minimal immersion \( \tilde{X} := X \circ Z_k : \tilde{\Omega}_k \to \mathbb{H}^2 \times \mathbb{R} \) is given by

\[
\tilde{d}^2 = 4 \cosh^2 \tilde{\omega}_k(w) |dw|^2.
\]

Therefore, we obtain

\[
K_E(p) = K_{\tilde{d}^2}(w) = -\frac{\tanh \tilde{\omega}_k}{4 \cosh^2 \tilde{\omega}_k} \Delta \tilde{\omega}_k - \frac{1}{4 \cosh^4 \tilde{\omega}_k} |\nabla \tilde{\omega}_k|^2.
\]

It follows from the proof of Proposition 2.3 that \( d_k(w) > 2 \). Now, by a straightforward computation, using formulas (12), (13) and (16) (where \( \delta = 2c_2 \)), we obtain

\[
|K_{\tilde{d}^2}(w)| < c_3 e^{-2d_k(w)}
\]

for some constant \( c_3 > 0 \) which does not depend on \( w \). Observe that

\[
d_E(p, \partial E) = d_E(\tilde{X}(w), \partial E) \geq d_E(\tilde{X}(w), \tilde{X}(\partial \tilde{\Omega}_k)) = d_{\tilde{d}^2}(w, \partial \tilde{\Omega}_k).
\]

From the comparison of the metric \( d_{\tilde{d}^2} \) with the Euclidean metric, we infer

\[
d_{\tilde{d}^2}(w, \partial \tilde{\Omega}_k) \geq 2d_k(w).
\]

Formula (17) follows by the previous inequalities for \( |z| > R_3 \), i.e. outside a compact subset of the end \( E \). Finally, it suffices to observe that the continuous function \( p \mapsto |K_{E}(p)| e^{d_E(p, \partial E)} \) is bounded on any compact subset of \( E \).

\[
\square
\]

3. Complete minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) with finite total curvature

The aim of this section is to prove the Main Theorem stated in the Introduction. The proof make essential use of the geometric properties of the horizontal sections of a finite total curvature end, that were established in Section 2.

In the following, \( \mathbb{H}^2 \times \{0\} \) will be identify with \( \mathbb{H}^2 \).

**Definition 3.1.** Let \( X := (F, h) : U \to \mathbb{H}^2 \times \mathbb{R} \) be a conformal and complete minimal annular end, where \( U := \{|z| > 1\} \), and let \( \gamma \subset \mathbb{H}^2 \) be a geodesic. We say that the end \( X(U) \subset \mathbb{H}^2 \times \mathbb{R} \) is asymptotic to the vertical geodesic plane \( \gamma \times \mathbb{R} \) if, for any real number \( C \) with \( |C| \) large enough, \( X^{-1}(\{t = C\}) \) is a complete curve of \( U \) and if, for any \( \varepsilon > 0 \), there exists a compact subset \( K \subset U \) such that the distance between any point of \( X(U \setminus K) \) and \( \gamma \times \mathbb{R} \) is smaller than \( \varepsilon \).

We deduce from Lemma 2.1 and Theorem 2.1 that an annular minimal end with finite total curvature is asymptotic to a vertical geodesic plane if, and only if, \( m = 0 \). That is, if \( P \) is a nonzero constant polynomial function.

**Definition 3.2.** Let \( \gamma \subset \mathbb{H}^2 \) be a geodesic. We say that a nonempty set \( S \subset \mathbb{H}^2 \times \mathbb{R} \) is a horizontal graph with respect to the geodesic \( \gamma \), if for any equidistant line \( \tilde{\gamma} \) of \( \gamma \) and for any \( t \in \mathbb{R} \), the curve \( \tilde{\gamma} \times \{t\} \) intersects \( S \) at most at one point.

**Remark 3.1.** We notice that a different notion of horizontal graph appears in [17], in order to treat different kinds of problems about minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \).
Proposition 3.1. Let $\gamma_1$ and $\gamma_2$ be two distinct geodesics in $\mathbb{H}^2$ with a common asymptotic point. Then, there is no complete, connected, immersed minimal surface with finite total curvature and two ends, one being asymptotic to $\gamma_1 \times \mathbb{R}$ and the other asymptotic to $\gamma_2 \times \mathbb{R}$.

Proof. (see Figure 5). We set $\partial_\infty \gamma_1 = \{a_\infty, p_\infty\}$ and $\partial_\infty \gamma_2 = \{b_\infty, p_\infty\}$, so that $p_\infty$ is the common asymptotic point of $\gamma_1$ and $\gamma_2$. We denote by $\gamma_\perp$ the geodesic such that $\gamma_1$ is the reflection of $\gamma_2$ across $\gamma_\perp$, we then have $p_\infty \in \partial_\infty \gamma_\perp$. We denote by $\gamma_0$ the geodesic such that $\partial_\infty \gamma_0 = \{a_\infty, b_\infty\}$. Observe that $\gamma_0$ meets $\gamma_\perp$ orthogonally at some point $p_0 \in \gamma_0 \cap \gamma_\perp$. For any $s > 0$ we denote by $p_s$ the point in the half geodesic $[p_0, p_\infty] \subset \gamma_\perp$ such that $d_{\mathbb{H}^2}(p_0, p_s) = s$.

For any $s > 0$ let $\gamma_s$ be the geodesic orthogonal to $\gamma_\perp$ at $p_s$. We set $P_s := \gamma_s \times \mathbb{R}$.

Assume by contradiction that there exists a complete and connected minimal surface $\Sigma$ with finite total curvature and two ends, one asymptotic to $\gamma_1 \times \mathbb{R}$ and the other asymptotic to $\gamma_2 \times \mathbb{R}$. By a result from A. Huber [9, Theorems 13 and 15], such surface is parametrized by a Riemann surface $M$ conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at two points $z_1, z_2$, $M \simeq \overline{M} \setminus \{z_1, z_2\}$. We denote by $X = (F, h) : M \to \Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ the minimal and conformal immersion. Thus, $F : M \to \mathbb{H}^2$ is a harmonic map and $h : M \to \mathbb{R}$ is a harmonic function.

We want to show that $\Sigma$ is a horizontal graph with respect to $\gamma_\perp$, afterwards we will derive a contradiction to conclude that such surface does not exist.

For any $s > 0$, we denote by $P_s^-$ the component of $((\mathbb{H}^2 \times \mathbb{R}) \setminus P_s$ containing $\{p_0\} \times \mathbb{R}$ and we denote by $P_s^+$ the other component. Thus $\{p_\infty\} \times \mathbb{R} \subset \partial_\infty P_s^+$. For any $s > 0$ we set $\Sigma_s^-' := \Sigma \cap P_s^-$, $\Sigma_s^+ := \Sigma \cap P_s^+$ and we denote by $\Sigma_s^{-*}$ the reflection of $\Sigma_s^-$ across the vertical geodesic plane $P_s$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Figure 5.}
\end{figure}
For any $\rho > 0$, we denote by $L^1_\rho$ (resp. $L^2_\rho$) the equidistant line of $\gamma_\perp$ with distance $\rho$, intersecting $\gamma_1$ (resp. $\gamma_2$). We denote by $C_\rho$ the domain of $\mathbb{H}^2 \times \mathbb{R}$ bounded by $(L^1_\rho \cup L^2_\rho) \times \mathbb{R}$ and we set $Q_\rho := (\mathbb{H}^2 \times \mathbb{R}) \setminus C_\rho$. Thus, $\{a_\infty\} \times \mathbb{R}, \{b_\infty\} \times \mathbb{R} \subset \partial_\infty Q_\rho$ for any $\rho > 0$. Since each end of $\Sigma$ is asymptotic to one of the vertical planes $\gamma_i \times \mathbb{R}$, $i = 1, 2$, for any $s > 0$ there exists $\rho_s > 0$, large enough, such that $\Sigma \cap Q_{\rho_s} \subset P_s^-$. Therefore, as a consequence of Propositions 2.3 and 4.3, we have that if $s$ is small enough, then for any $t \in \mathbb{R}$, the level set $\Sigma \cap (\mathbb{H}^2 \times \{t\})$ is a horizontal graph with respect to the geodesic $\gamma_\perp$. Consequently, there exists $s_1 > 0$ such that $\Sigma_{s_1}^- \in \Sigma_{s_1}^+$ is a horizontal graph with respect to $\gamma_\perp$ and $\Sigma_{s_1}^- \cap \Sigma_{s_1}^+ = \emptyset$ for any $0 < s < s_1$.

We set $$ I := \{\sigma \geq 0 \mid \Sigma^-_{s} \cap \Sigma^+_{s} = \emptyset, \text{ for any } 0 < s \leq \sigma\}. $$

In order to ensure that $\Sigma$ is a horizontal graph, we must show that $I = [0, +\infty[$. The set $I$ is nonempty because $[0, s_1] \subset I$.

We set $s_2 = \sup I$. If $s_2 = +\infty$ we are done. Assume that $s_2 \neq +\infty$. By a continuity argument we have $\Sigma^-_{s_2} \cap \Sigma^+_{s_2} = \emptyset$, so that $s_2 \in I$.

Recall that one end of $\Sigma$ is asymptotic to $\gamma_1 \times \mathbb{R}$ and the other end is asymptotic to $\gamma_2 \times \mathbb{R}$. Moreover, from Lemma 2.3, formula 3 and Proposition 2.3, it follows that, for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for any $t > t_0$, the intersection $\Sigma \cap (\mathbb{H}^2 \times \{t\})$ is composed of two complete curves, $c_1^t$ and $c_2^t$ verifying

$$ \partial_{-\infty} c_1^t = \{a_\infty, p_\infty\}, \quad \partial_{-\infty} c_2^t = \{b_\infty, p_\infty\} \quad \text{and} \quad \sup_{c_1^t} |\kappa(q)| < \varepsilon, \quad i = 1, 2, $$

where $\kappa$ denotes the geodesic curvature. From Proposition 4.2, we deduce that $c_i^t$ is $C^1$-close to $\gamma_i$, $i = 1, 2$, if $\varepsilon$ is small enough. Analogously $\Sigma \cap (\mathbb{H}^2 \times \{-t\})$ is composed of two complete curves, $c_1^{-t}$ and $c_2^{-t}$, $C^1$-close to $\gamma_1$ and $\gamma_2$ respectively.

**Claim 1.** There exist $t_0 > 0$ and $\eta_1 > 0$ such that $\Sigma^-_{s_2 + \eta_1} \cap (\mathbb{H}^2 \times \{t \mid t > t_0\})$ is a horizontal graph with respect to the geodesic $\gamma_\perp$ and $\Sigma^-_{s_2 + \eta_1} \cap (\mathbb{H}^2 \times \{t \mid t > t_0\}) = \emptyset$.

We set $p_1^t = c_1^t \cap P_{s_2}$. Observe that the (nonoriented) angle between $c_1^t$ and the equidistant line to $\gamma_\perp$ passing through $p_1^t$ is close to the angle between the same equidistant line and $\gamma_1$ at the common point. Hence, there is $\alpha \in (0, \pi/2)$ such that, if $t_0$ is large enough, this angle is larger than $\alpha$ for any $t > t_0$, and the same is true for the analogous angle defined for $P_{s_2} \cap c_2^t, P_{s_2} \cap c_1^t$ and $P_{s_2} \cap c_2^t$. Therefore, there exists $\eta_1 > 0$ such that $\Sigma^-_{s_2 + \eta_1} \cap (\mathbb{H}^2 \times \{t \mid t > t_0\})$ is a horizontal graph with respect to $\gamma_\perp$ and $\Sigma^-_{s_2 + \eta_1} \cap (\mathbb{H}^2 \times \{t \mid t > t_0\}) = \emptyset$. Then the claim is proved.

**Claim 2.** There exists $\eta_2 > 0$ such that $\Sigma^-_{s_2 + \eta_2} \cap (\mathbb{H}^2 \times \{t \mid t \leq t_0\})$ is a horizontal graph with respect to the geodesic $\gamma_\perp$ and $\Sigma^{+}_{s_2 + \eta_2} \cap (\mathbb{H}^2 \times \{t \mid t \leq t_0\}) = \emptyset$.

Observe first that, at any point of $\Sigma \cap P_{s_2}$, the equidistant line to $\gamma_\perp$ passing through this point is not tangent to $\Sigma$. Indeed, suppose that at some point $p \in \Sigma \cap P_{s_2}$ the
equidistant line to $\gamma_\perp$ passing through $p$ is tangent to $\Sigma$. Thus $\Sigma$ is orthogonal to $P_{s_2}^-$ at $p$ and, therefore, $\Sigma_{s_2}^-$ and $\Sigma_{s_2}^+$ are tangent at the point $p$ of their common boundary. Since $\Sigma_{s_2}^- \cap \Sigma_{s_2}^+ = \emptyset$, the boundary maximum principle would imply that $\Sigma_{s_2}^- = \Sigma_{s_2}^+$. This gives a contradiction, since the asymptotic boundary of $\Sigma$ is not symmetric with respect to any vertical geodesic plane $P_s$.

Therefore, since $(\Sigma \cap P_{s_2}) \cap \{|t| \leq t_0\}$ is compact, there is $\beta \in (0, \pi/2)$ such that the nonoriented angle between the equidistant lines to $\gamma_\perp$ and $\Sigma$ at any point of $(\Sigma \cap P_{s_2}) \cap \{|t| \leq t_0\}$ is larger than $\beta$. By a compactness argument again, there is $\eta_2 > 0$ such that $\Sigma_{s_2+\eta_2}^- \cap (\mathbb{H}^2 \times \{|t| \leq t_0\})$ is a horizontal graph and $(\Sigma_{s_2+\eta_2}^- \cap \Sigma_{s_2+\eta_2}^+) \cap (\mathbb{H}^2 \times \{|t| \leq t_0\}) = \emptyset$. This proves the claim.

We set $\eta = \min\{\eta_1, \eta_2\}$. From Claims 1 and 2, we get that $s_2 + \eta \in I$. This gives a contradiction with the maximality of $s_2$. Therefore, $I = [0, +\infty]$ and $\Sigma$ is a horizontal graph with respect to $\gamma_\perp$.

Now we can conclude the proof.

Let $Q(F)$ be the quadratic Hopf differential associated to $F$. We know that $Q(F)$ is holomorphic on $M$ and has a pole at the ends $z_1, z_2 \in M$. Since each end is asymptotic to a vertical geodesic plane, the degree of each end is $m = 0$. Therefore, each end is a pole of order 4 of $Q(F)$. According to the Riemann relation for $Q(F)$, we have that

$$\text{Pole}(Q(F)) - \text{Zero}(Q(F)) = 2\chi(M),$$

thus $\text{Zero}(Q(F)) = \text{Pole}(Q(F)) - 2\chi(M) = 8 - 2\chi(M) \geq 4$. Consequently, there exists $z_0 \in M$ which is a zero of $Q(F)$. From formula (11) we deduce that $z_0$ is a pole of $\omega$ and then, the tangent plane of $\Sigma$ at $X(z_0)$ is horizontal (see formula (10)).

Let $s' > 0$ such that $X(z_0) \in P_{s'}$. We get a contradiction by the boundary maximum principle since, on one hand $\Sigma_{s'}^+ \cap \Sigma_{s'}^- = \emptyset$ (since $\Sigma$ is a horizontal graph with respect to $\gamma_\perp$) but on the other hand $\Sigma_{s'}^+$ and $\Sigma_{s'}^-$ are tangent at their common boundary point $X(z_0)$.

**Proposition 3.2.** Let $\gamma_1$ and $\gamma_2$ be two distinct geodesics in $\mathbb{H}^2$ intersecting at some point. Then, there is no complete, connected, immersed minimal surface with finite total curvature and two ends, one being asymptotic to $\gamma_1 \times \mathbb{R}$ and the other asymptotic to $\gamma_2 \times \mathbb{R}$.

**Proof.** We set $\{w\} := \gamma_1 \cap \gamma_2$. We denote by $\alpha$ and $\beta$ the two geodesics passing through $w$ such that the reflection of $\gamma_1$ across $\alpha$ is $\gamma_2$ and the reflection of $\gamma_1$ across $\beta$ is $\gamma_2$. Thus, $\alpha$ intersects $\beta$ orthogonally at $w$ (see Figure 6).

We choose an orientation on $\alpha$ and $\beta$. For any $s \in \mathbb{R}$, we denote by $p_s$ (resp. $q_s$) the point of $\alpha$ (resp. $\beta$) whose signed distance to $w$ is $s$, observe that $q_0 = p_0 = w$. Furthermore, for any $s \in \mathbb{R}$, we denote by $P_s$ (resp. $Q_s$) the vertical geodesic plane passing through $p_s$ (resp. $q_s$) and orthogonal to the geodesic $\alpha$ (resp. $\beta$), note that $P_0 = \beta \times \mathbb{R}$ and $Q_0 = \alpha \times \mathbb{R}$.

We set $P_0^+ := \cup_{s>0} P_s$, $P_0^- := \cup_{s<0} P_s$, $Q_0^+ := \cup_{s>0} Q_s$ and $Q_0^- := \cup_{s<0} Q_s$. 

Assume that there exists a complete minimal surface $\Sigma$ with finite total curvature and two ends, one being asymptotic to $\gamma_1 \times \mathbb{R}$ and the other being asymptotic to $\gamma_2 \times \mathbb{R}$. Using the Alexandrov reflection principle with respect to the vertical planes $P_s$, $s \in \mathbb{R}$, we can show, as in the proof of Proposition 3.1, that $\Sigma \cap P_0^+$ is a horizontal graph with respect to the geodesic $\alpha$ and so is $\Sigma \cap P_0^-$. In the same way, we can show that $\Sigma \cap Q_0^+$ and $\Sigma \cap Q_0^-$ are both horizontal graphs with respect to the geodesic $\beta$.

Now we proceed as in the proof of [20, Theorem 3, Case 1]. We set $L := P_0 \cap Q_0 = \{w\} \times \mathbb{R}$. Since $\Sigma \cap P_0^+$ and $\Sigma \cap P_0^-$ are horizontal graphs with respect to $\alpha$, the self intersection set $S$ of $\Sigma$ is contained in $P_0$. By the same argument, we have $S \subset Q_0$, so that $S \subset L$. Since an end of $\Sigma$ is asymptotic to $\gamma_1 \times \mathbb{R}$ and the other end is asymptotic to $\gamma_2 \times \mathbb{R}$, we have $S \neq \emptyset$. By the analyticity of the sets $\Sigma \cap P_0$ and $\Sigma \cap Q_0$, we get that $S = L$. Moreover, since $\Sigma \cap P_0^+$ and $\Sigma \cap P_0^-$ are horizontal graphs with respect to $\alpha$, we deduce that $\Sigma \cap Q_0 = L$. Analogously, $\Sigma \cap P_0 = L$. Therefore, $\Sigma \setminus L$ consists of four connected components $\Sigma_i$, $i = 1, \cdots, 4$ with:

$$
\Sigma_1 \subset P_0^+ \cap Q_0^+, \quad \Sigma_2 \subset P_0^+ \cap Q_0^-, \quad \Sigma_3 \subset P_0^- \cap Q_0^- \quad \text{and} \quad \Sigma_4 \subset P_0^- \cap Q_0^+.
$$

Denoting by $\sigma$ the rotation about the vertical geodesic $L$ with angle $\pi$, the reflection principle shows that $\sigma(\Sigma_1) = \Sigma_3$, so that $\Sigma' := \Sigma_1 \cup \Sigma_3$ is a smooth and complete minimal surface embedded in $\mathbb{H}^2 \times \mathbb{R}$. For the same reason, $\Sigma'' := \Sigma_2 \cup \Sigma_4$ is also a smooth and complete minimal surface embedded in $\mathbb{H}^2 \times \mathbb{R}$. Therefore, we get that $\Sigma = \Sigma' \cup \Sigma''$ and consequently, the minimal surface is not connected. We arrive to a contradiction, which concludes the proof.

Now we can restate the Main Theorem, announced in the Introduction, in the following way.

**Theorem 3.1.** Let $\Sigma$ be a complete, connected minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and two ends. Assume that each end is asymptotic to a vertical geodesic plane $\gamma_i \times \mathbb{R}$, where each $\gamma_i$, $i = 1, 2$, is a geodesic.
Then, we have $\gamma_1 \cap \gamma_2 = \emptyset$, $\partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset$. Furthermore, $\Sigma$ is a properly embedded annulus and is a two ends model surface.

In order to prove Theorem 3.1 we fix some notations and prove some lemmas.

**Notations.** Let $\gamma_1, \gamma_2 \subset \mathbb{H}^2$ be two geodesics satisfying
\[
\gamma_1 \cap \gamma_2 = \emptyset \quad \text{and} \quad \partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset. \tag{18}
\]
We denote by $\gamma_0 \subset \mathbb{H}^2$ the geodesic orthogonal to both $\gamma_1$ and $\gamma_2$. We set $p_1 = \gamma_1 \cap \gamma_0$ and $p_2 = \gamma_2 \cap \gamma_0$. We call $p_0$ the middle point of the segment of $\gamma_0$ between $p_1$ and $p_2$. We denote by $\Gamma \subset \mathbb{H}^2$ the geodesic passing through $p_0$ and orthogonal to $\gamma_0$ (see Figure 7).

![Figure 7](image)

In the following lemmas the surface $\Sigma$ satisfies the hypothesis of Theorem 3.1.

**Lemma 3.1.** Suppose that $\gamma_1$ and $\gamma_2$ are geodesics satisfying the properties in (18). Then, the surface $\Sigma$ is symmetric with respect to the vertical geodesic plane $\gamma_0 \times \mathbb{R}$ and each component of $\Sigma \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to $\Gamma$.

**Proof.** We choose an orientation on $\Gamma$. For any $s \in \mathbb{R}$, we denote by $q_s$ the unique point in $\Gamma$ whose signed distance to $p_0$ is $s$, thus $q_0 = p_0$. For any $s \in \mathbb{R}$, we denote by $\widetilde{\gamma}_s \subset \mathbb{H}^2$ the geodesic orthogonal to $\Gamma$ and passing through $q_s$. Observe that $\widetilde{\gamma}_0 = \gamma_0$.

For any $s \in \mathbb{R}$, we set $Q_s := \widetilde{\gamma}_s \times \mathbb{R}$. Moreover, for any $s \neq 0$, we denote by $Q^+_s$ the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus Q_s$ containing $\{p_0\} \times \mathbb{R}$ and by $Q^-_s$ the other component. For any $s \neq 0$ we set $\Sigma^- := \Sigma \cap Q^-_s$, $\Sigma^+_s := \Sigma \cap Q^+_s$ and we denote by $\Sigma^-_s*$ the reflection of $\Sigma^-_s$ across the vertical geodesic plane $Q^-_s$.

As in the proof of Proposition 3.1 (Claim 1 and Claim 2), we can show that for any $s \neq 0$, $\Sigma^-_s$ is a horizontal graph with respect to $\Gamma$ and that $\Sigma^-_s* \cap \Sigma^+_s = \emptyset$.

Then, passing to the limit for $s \to 0$ from both sides, we conclude that $\Sigma$ is symmetric with respect to $\gamma_0 \times \mathbb{R}$ and that each component of $\Sigma \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to $\Gamma$. \qed
Remark 3.2. It follows from the proof of Lemma 3.1 that the tangent plane at any point of \( \Sigma \setminus (\gamma_0 \times \mathbb{R}) \) is never horizontal.

Lemma 3.2. Suppose that \( \gamma_1 \) and \( \gamma_2 \) are geodesics satisfying the properties in \([18]\). Then, the surface \( \Sigma \) is symmetric with respect to the vertical geodesic plane \( \Gamma \times \mathbb{R} \) and each component of \( \Sigma \setminus (\Gamma \times \mathbb{R}) \) is a horizontal graph with respect to \( \gamma_0 \).

Proof. Let \( d > 0 \) be the distance between \( p_0 \) and \( p_1 \), thus \( d = d_{\mathbb{H}^2}(p_0, p_1) = d_{\mathbb{H}^2}(p_0, p_2) \).

For any \( s \in [0, d] \) we denote by \( \tilde{p}_s \in \gamma_0 \) the unique point between \( p_0 \) and \( p_1 \), whose distance to \( p_0 \) is \( s \). Thus \( p_0 = \tilde{p}_0 = p_1 \).

We denote by \( \Gamma_s \subset \mathbb{H}^2 \) the geodesic orthogonal to \( \gamma_0 \) and passing through \( \tilde{p}_s \), thus \( \Gamma_d = \gamma_1 \). We set \( P_s := \Gamma_s \times \mathbb{R} \). For any \( s \in [0, d[ \) we denote by \( P_s^- \) the connected component of \( \mathbb{H}^2 \times \mathbb{R} \setminus P_s \) containing \( \{p_1\} \times \mathbb{R} \) and by \( P_s^+ \) the other component. \( P_s \) is a horizontal graph with respect to \( \gamma_0 \). Consequently, the intersection of a finite number of Jordan curves whose interiors are mutually disjoint.

We set \( \Sigma_s^- := \Sigma \cap P_s^- \) and \( \Sigma_s^+ := \Sigma \cap P_s^+ \).\( \Sigma_s^- \) and \( \Sigma_s^+ \) denote the reflection of \( \Sigma_s^- \) across \( P_s \). We give to the geodesic \( \gamma_0 \) the orientation going from \( p_1 \) to \( p_2 \).

We will say that \( \Sigma_s^- \leq \Sigma_s^+ \) if \( \Sigma_s^- \) remains under \( \Sigma_s^+ \) with respect to the orientation of \( \gamma_0 \). Using Propositions \([23, 24]\) and \([13]\), it can be shown that there exists \( \varepsilon > 0 \) such that for any \( s \in [d - \varepsilon, d[ \), \( \Sigma_s^- \) is a horizontal graph with respect to \( \gamma_0 \). Therefore, for any \( s \in [d - \varepsilon/2, d[ \), we have \( \Sigma_s^- \leq \Sigma_s^+ \).

We set
\[
I = \{ s \in [0, d] \mid \Sigma_r^- \leq \Sigma_r^+ \text{ for any } r \in [d - s, d[ \}.
\]

We have \( I \neq \emptyset \), since \( \varepsilon/2 \in I \). We set \( s_0 := \sup I \) and want to prove that \( s_0 = d \).

Assume that \( s_0 \neq d \). By continuity we get that \( \Sigma_{s_0}^- \leq \Sigma_{s_0}^+ \). On the other hand we have \( \Sigma_{s_0}^- \neq \Sigma_{s_0}^+ \), since the asymptotic boundaries of those two parts are not equal. Observe that \( \partial \Sigma_{s_0}^- = \Sigma \cap P_{s_0} \) is compact and the boundary maximum principle shows that \( \Sigma \) is never orthogonal to \( P_{s_0} \) along their intersection. We deduce that there exists \( \varepsilon_1 > 0 \) such that \( \Sigma_{d-s_0-\varepsilon_1}^- \) is a horizontal graph with respect to \( \gamma_0 \) and \( \Sigma_r^- \leq \Sigma_r^+ \) for any \( r \in [d - s_0 - \varepsilon_1, d[ \), which gives a contradiction with the maximality of \( s_0 \). We deduce that \( s_0 = d \) and then \( \Sigma_0^- \leq \Sigma_0^+ \).

Using the same arguments coming from the other side, that is from \( p_2 \) to \( p_0 \), we can show that \( \Sigma_0^+ \geq \Sigma_0^- \). We conclude that \( \Sigma_0^+ = \Sigma_0^- \), that is \( \Sigma \) is symmetric with respect to \( P_0 = \Gamma \times \mathbb{R} \), as desired.

Lemma 3.3. Suppose that \( \gamma_1 \) and \( \gamma_2 \) are geodesics satisfying the properties in \([18]\). Then, the surface \( \Sigma \) is symmetric with respect to some slice \( \mathbb{H}^2 \times \{t_0\} \) and each component of \( \Sigma \setminus (\mathbb{H}^2 \times \{t_0\}) \) is a vertical graph.

Proof. From Lemma 3.1 and 3.2 we know that each component of \( \Sigma \setminus (\Gamma \times \mathbb{R}) \) is a horizontal graph with respect to \( \gamma_0 \) and each component of \( \Sigma \setminus (\gamma_0 \times \mathbb{R}) \) is a horizontal graph with respect to \( \Gamma \). Consequently, the intersection \( \mathcal{C} := \Sigma \cap (\Gamma \times \mathbb{R}) \) is composed of a finite number of Jordan curves whose interiors are mutually disjoint.
For $i = 1, 2$, we call $\Sigma_i$ the component of $\Sigma \setminus (\Gamma \times \mathbb{R})$ which is asymptotic to the
vertical plane $\gamma_i \times \mathbb{R}$. For any $t \in \mathbb{R}$, we set $\Pi_t := \mathbb{R}^2 \times \{t\}$. Let $t_1 \in \mathbb{R}$ be such that
$\partial \Sigma_1 \cap \Pi_{t_1} = \emptyset$ for any $t > t_1$ and $\partial \Sigma_1 \cap \Pi_{t_1} \neq \emptyset$. Such $t_1$ exists since $\partial \Sigma_1 = C$ is compact.
For any $t \in \mathbb{R}$ we set: $\Pi_t^+ := \mathbb{R}^2 \times \{s \mid s > t\}$, $\Pi_t^- := \mathbb{R}^2 \times \{s \mid s < t\}$, $\Sigma_{i,t}^+ = \Sigma_1 \cap \Pi_{t}^+$, $\Sigma_{i,t}^- = \Sigma_1 \cap \Pi_{t}^-$ and we denote by $\Sigma_{i,t}^{+\ast}$ the reflection of $\Sigma_{i,t}^+$ across $\Pi_t$. Moreover, $\Sigma_{i,t}^{+\ast} \geq \Sigma_{i,t}^-$ means that $\Sigma_{i,t}^{+\ast}$ stays above $\Sigma_{i,t}^-$. 

Claim 1. For any $t > t_1$ we have $\Sigma_{i,t}^{+\ast} \geq \Sigma_{i,t}^-$. Consequently, $\Sigma_{i,t_1}^+$ is a vertical graph.
Indeed, for $t > t_1$ we know from Lemma 3.2 that the intersection $\Sigma_1 \cap \Pi_t$ is a complete
curve, that is a horizontal graph with respect to $\gamma_0$ and whose asymptotic boundary
is $\partial_{\infty} \gamma_1 \times \{t\}$. For any $s \in \mathbb{R}$ we denote by $T_s$ the horizontal translation along $\gamma_0$ of
directed length $s$ in the direction going from $p_2$ to $p_1$.
Suppose that $\Sigma_{i,t}^{+\ast}$ does not remain above $\Sigma_{i,t}^-$. Then, for $\varepsilon > 0$ small enough, the
translated $T_{\varepsilon}(\Sigma_{i,t}^{+\ast})$ does not remain above $\Sigma_{i,t}^-$. Observe that $\partial_{\infty} T_{\varepsilon}(\Sigma_{i,t}^{+\ast}) \cap \partial_{\infty} \Sigma_{i,t}^- = \emptyset$.
Moreover, $\partial T_{\varepsilon}(\Sigma_{i,t}^{+\ast}) \cap \Sigma_{i,t}^- = \emptyset$, for any $s > 0$, since $\Sigma_1$ is a horizontal graph with respect to $\gamma_0$. We deduce that, for $\varepsilon > 0$ small enough, the part of $\Sigma_{i,t}^-$ which remains above
$T_{s}(\Sigma_{i,t}^{+\ast})$ has compact closure. Therefore, there exists $s_1 > 0$ such that $T_{s_1}(\Sigma_{i,t}^{+\ast}) \cap \Sigma_{i,t}^- = \emptyset$
for any $s > s_1$ and $T_{s_1}(\Sigma_{i,t}^{+\ast}) \cap \Sigma_{i,t}^- \neq \emptyset$. This means that $T_{s_1}(\Sigma_{i,t}^{+\ast})$ and $\Sigma_{i,t}^-$ are tangent at
some point and one surface remains in one side of the other, which gives a contradiction
with the maximum principle and proves the claim.

Claim 2. For any $\varepsilon > 0$ small enough, we have $\Sigma_{i,t_1-\varepsilon}^{+\ast} \geq \Sigma_{i,t_1-\varepsilon}^-$. 
For any $s \in \mathbb{R}$, let $T_s$ be the horizontal translation defined as in Claim 1. Since $\Sigma_1$ is a horizontal graph with respect to $\gamma_0$, we have $T_s(\Sigma) \cap \Sigma_1 = \emptyset$ for any $s > 0$. Furthermore, since the whole surface $\Sigma$ is symmetric with respect to $\Gamma \times \mathbb{R}$ we have that $T_s(\Sigma) \cap \Sigma_1 = \emptyset$ for any $s \neq 0$. Let $\mathcal{D} \subset \Gamma \times \mathbb{R}$ be the bounded domain with boundary $\mathcal{C}$. Since $\Sigma$ is connected, we have $T_s(\mathcal{D}) \cap \Sigma = \emptyset$ for any $s \neq 0$.
Let $\varepsilon > 0$ such that $\mathcal{C}_{t_1-\varepsilon}^{+\ast} \geq \mathcal{C}_{t_1-\varepsilon}^-$, where $\mathcal{C}_{t_1-\varepsilon}^{+\ast}$ etc. are obviously defined. Then, using
an argument analogous to that of Claim 1, considering translations $T_s$, it can be shown
that $\Sigma_{i,t_1-\varepsilon}^{+\ast} \geq \Sigma_{i,t_1-\varepsilon}^-$. This proves Claim 2.

Now we can conclude the proof. Since $\Sigma$ is composed of a finite number of Jordan curves, there exists a component, say $\mathcal{C}$, and a real number $t_0 < t_1$ satisfying $\Sigma_{i,t}^{+\ast} \geq \Sigma_{i,t}^-$ for any $t > t_0$, such that at least
one of the following properties occurs:

1. $\mathcal{C}_{t_0}^{+\ast} \geq \mathcal{C}_{t_0}^-$ and $\mathcal{C}_{t_0}^{+\ast}$ is tangent to $\mathcal{C}_{t_0}^-$ at some interior point.
2. $\mathcal{C}_{t_0}^{+\ast} \geq \mathcal{C}_{t_0}^-$ and $\mathcal{C}_{t_0}^{+\ast}$ and $\mathcal{C}_{t_0}^-$ are tangent along their common boundary.

Recall that $\Sigma$ is orthogonal to $\Gamma \times \mathbb{R}$ along $\mathcal{C}$, since $\Sigma = \Sigma \cap (\Gamma \times \mathbb{R})$ and $\Sigma$ is symmetric
with respect to $\Gamma \times \mathbb{R}$.
Thus, in the first case, applying the boundary maximum principle to the surfaces $\Sigma_{i,t_0}^{+\ast}$ and $\Sigma_{i,t_0}^-$, we conclude that $\Sigma_{i,t_0}^{+\ast} = \Sigma_{i,t_0}^-$ and then $\Sigma_{t_0}^{+\ast} = \Sigma_{t_0}^-$. 
In the second case we apply the boundary maximum principle to the surfaces $\Sigma_{t_0}^{+*}$ and $\Sigma_{t_0}^{-}$ in order to infer $\Sigma_{t_0}^{+*} = \Sigma_{t_0}^{-}$.

Consequently, the surface $\Sigma$ is symmetric with respect to the horizontal plane $\Pi_{t_0} = \mathbb{H}^2 \times \{t_0\}$, as desired. 

\begin{remark} \ \ It follows from the proof of Lemma 3.3 that the tangent plane at any point of $\Sigma \setminus (\mathbb{H}^2 \times \{t_0\})$ is never vertical. \end{remark}

\begin{proof} of Theorem 3.1. \ \ The maximum principle shows that $\gamma_1 \neq \gamma_2$, since $\Sigma$ is not a vertical plane. We know from Proposition 3.1 that $\partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset$ and from Proposition 3.2 that $\gamma_1 \cap \gamma_2 = \emptyset$. Thus, the geodesics $\gamma_1$ and $\gamma_2$ satisfy the properties \begin{equation} \tag{18} \end{equation}.

Therefore, we deduce from Lemmas 3.1, 3.2 and 3.3 that $\Sigma$ is symmetric with respect to the vertical planes $\gamma_0 \times \mathbb{R}$ and $\Gamma \times \mathbb{R}$ and also with respect to the slice $\Pi_0 := \mathbb{H}^2 \times \{0\}$ (up to a vertical translation).

We call $S_0$ the reflection across the slice $\Pi_0$, $S_\Gamma$ the reflection across the vertical plane $\Gamma \times \mathbb{R}$ and $S_{\gamma_0}$ the reflection across the vertical plane $\gamma_0 \times \mathbb{R}$.

For any real number $s \neq 0$, we denote by $\Gamma_s$ the equidistant line to $\Gamma$ with distance equal to $|s|$, which intersects $\gamma_0$ between $p_0$ and $p_1$ (resp. $p_0$ and $p_2$) if $s > 0$ (resp. $s < 0$). We set $\Gamma_0 = \Gamma$. For any $s \in \mathbb{R}$, we set $P_s := \Gamma_s \times \mathbb{R}$.

We define $\Sigma^+ := \Sigma \cap (\mathbb{H}^2 \times \{0\}, +\infty)$]. Since $\Sigma^+$ is a vertical graph, the tangent plane is never vertical along $\Sigma^+$. Consequently, $\Sigma^+$ intersects any $P_s$ tranversally. Since each component of $\Sigma^+ \setminus (\gamma_0 \times \mathbb{R})$ is a horizontal graph with respect to $\Gamma$ and since $\Sigma$ is symmetric with respect to $\Pi_0$ and $\gamma_0 \times \mathbb{R}$, we deduce that for any $s \in \mathbb{R}$ the intersection $\Sigma \cap P_s$ consists in a Jordan curve. Therefore, $\Sigma$ is homeomorphic to an annulus. Since $\Sigma$ has finite total curvature, we get that $\Sigma$ is conformally parametrized by $C^* := C \setminus \{0\}$.

Let $X : C^* \to \Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a conformal parametrization of $\Sigma$. Thus, the symmetry $S_\Gamma$ corresponds to a anticonformal diffeomorphism $s_\Gamma : C \cup \{\infty\} \to C \cup \{\infty\}$ satisfying $s_\Gamma(0) = \infty$ and $s_\Gamma(\infty) = 0$. Since the set of fixed points of $S_\Gamma$ in $\Sigma$ is a Jordan curve, the set of fixed points of $s_\Gamma$ is a circle $c_\Gamma$. Up to a conformal change of coordinates, we can assume that $c_\Gamma \subset C$ is the unit circle centered at the origin. Thus, we get $s_\Gamma(z) = 1/z$ for any $z \in C^*$.

Then, we denote by $s_{\gamma_0} : C \cup \{\infty\} \to C \cup \{\infty\}$ the anticonformal diffeomorphism corresponding to $S_{\gamma_0}$. The set of fixed points of $s_{\gamma_0}$ in $C^*$ is a straight line $L_\gamma$ passing to and punctured at the origin. Up to a rotation we can assume that $L_\gamma = \{\text{Re} \ z = 0\}$. Thus, we have $s_{\gamma_0}(z) = -\overline{z}$ for any $z \in C$.

At last, let us call $s_0 : C \cup \{\infty\} \to C \cup \{\infty\}$ the anticonformal diffeomorphism corresponding to $S_0$. The set of fixed points of $s_0$ in $C^*$ is a straight line $L$ passing and punctured at the origin. Since we have $(S_0 \circ S_{\gamma_0}) \circ (S_0 \circ S_{\gamma_0}) = \text{Id}$ on $\Sigma$, we must have $(s_0 \circ s_{\gamma_0}) \circ (s_0 \circ s_{\gamma_0}) = \text{Id}$ on $C^*$. Thus, $L$ must be orthogonal to $L_\gamma$ and we get $L = \{\text{Im} \ z = 0\}$. Then, we have $s_0(z) = \overline{z}$ for any $z \in C$. 

We can assume that $\Sigma^*$ of $\Sigma_0^*$ conformal parametrization of $\Sigma_0^*$ recover the whole surface $\Sigma$ by applying the symmetries $S^*_0$, $S^*_1$ and $S^*_2$ to the closure of $\Sigma_0^*$. We can assume that $\Sigma_0^*$ is parametrized by the subset
\[ U_0 := \{ z \in \mathbb{C} \mid |z| > 1, \text{ Re } z < 0, \text{ Im } z > 0 \}. \]

Since $\Sigma_0^*$ is simply connected, we can consider its conjugate $\Sigma_0^{**}$ which is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ uniquely defined up to an ambient isometry. From now on, for any object $x$ relative to $\Sigma_0^*$ we denote by $x^*$ the corresponding object relative to the conjugate surface $\Sigma_0^{**}$. Thus, $X^* := (F^*, h^*) : U_0 \to \Sigma_0^{**} \subset \mathbb{H}^2 \times \mathbb{R}$ is a conformal parametrization of $\Sigma_0^{**}$. Observe that the boundary of $\Sigma_0^*$ is composed of three arcs:

1. A semi-complete curve $b_1 \subset (\gamma_0 \times \mathbb{R})$ with boundary point $q$.
2. A compact arc $b_2 \subset (\Gamma \times \mathbb{R})$ with boundary $q$ and $\tilde{q}$.
3. A semi-complete curve $b_3 \subset \Pi_0$ with boundary point $\tilde{q}$.

In order to visualize the following discussion we consider the model of the unit disk for $\mathbb{H}^2$ (see Figure 8). Up to an isometry we can assume that $q^* = 0 \in \mathbb{H}^2$.

Since $b_1$ is contained in a vertical plane, its conjugate $b_1^*$ must be a horizontal half-geodesic issue from 0 (see [3], end of Section 4.1). We can assume that $\partial_{\infty} b_1^* = \{ i \}$. Moreover, since $b_2$ is contained in a vertical plane, its conjugate $b_2^*$ must be a compact geodesic arc orthogonal to $b_1^*$ with endpoints $\tilde{q}^*$ and 0. Finally, since $b_3$ is contained in a horizontal plane, its conjugate $b_3^*$ must be a vertical half-geodesic issue from $\tilde{q}^*$. We can assume that $b_3^* = \tilde{q}^* \times \{ t > 0 \}$.

We denote by $C \subset \mathbb{H}^2$ the geodesic passing through $\tilde{q}^*$, having $i$ in its asymptotic boundary and we denote by $C_0$ the segment of $C$ with boundary points $\tilde{q}^*$ and $i$. We call $\mathcal{D}$ the domain of $\mathbb{H}^2$ bounded by $b_1^*$, $b_2^*$ and $C_0$ such that $\partial_{\infty} \mathcal{D} = \{ i \}$.

In order to prove that $\Sigma$ is a two ends model surface, it is enough to prove that $\Sigma_0^*$ is a vertical graph over $\mathcal{D}$, with infinite data on $C_0$ and zero data on the two sides $b_1^*$ and $b_2^*$ (see [15]).

We call $\sigma$ the reflection in $\mathbb{H}^2 \times \mathbb{R}$ across the vertical geodesic containing $b_3^*$.

Since $\Sigma_0^*$ is parametrized by $U_0$, the interior of $\Sigma_0^* \cup \sigma(\Sigma_0^*)$ is parametrized by
\[ \tilde{U}_0 := \{ z \in \mathbb{C} \mid |z| > 1, \text{ Re } z < 0 \}. \]

We recall that $\phi^* = -\phi$, $h^* = -2 \text{ Re } W$, since $h = 2 \text{ Im } W$ and $h^*$ is the harmonic conjugate of $h$. Thus $W^* = -iW$. We may suppose that $h^* > 0$ on $\tilde{U}_0$. We are able to study the behavior of $F^*$ and $h^*$ on $\tilde{U}_0$ in the same way as we did for $F$ and $h$ in Section 2. By Lemma 2.3 and Theorem 2.1 we infer that $m = 0$ for the end of $\Sigma$ parametrized by $\{|z| > 1\}$. We deduce that for each $\mu > 0$ the level set $\{ h^* = \mu \}$ is
composed of complete curves, each one having a part with asymptotic direction the ray \( \{ ri \mid r > 0 \} \) and the other part with asymptotic direction the ray \( \{ -ri \mid r > 0 \} \).

Each one of those level curves intersects \( L^{-} := \{ \text{Im} z = 0 \} \cap \tilde{U}_{0} \).

Since \( \Sigma_{0} \cup S_{0}(\Sigma_{0}) \) is a horizontal graph with respect to \( \Gamma \), we get that the tangent plane to it is never horizontal. Therefore, \( \phi \neq 0 \) on \( \tilde{U}_{0} \). Since \( h \equiv 0 \) on \( L^{-} \), we deduce that \( h^{*} \) is strictly monotonous on \( L^{-} \). Therefore, for any \( \mu > 0 \), the level set \( \{ h^{*} = \mu \} \) is composed of a unique complete curve \( L_{\mu} \). Since \( h \) is strictly monotonous along any curve \( L_{\mu} \), we deduce that \( W^{*} \) is one-to-one on \( \tilde{U}_{0} \). The results established in Theorem 2.1 yields that the projections of the curves \( L_{\mu} \) on \( \mathbb{H}^{2} \) converge to the geodesic \( C \) when \( \mu \to +\infty \).

Let us call \( B_{i} \) the geodesic containing \( b_{i}^{*} \), \( i = 1, 2 \).

**Claim 1.** We have \( \Sigma_{0}^{*} \subset D \times \mathbb{R} \).

By construction, \( B_{1} \) and \( B_{2} \) meet orthogonally at the origin and \( B_{2} \) is the geodesic with asymptotic points \((-1,0), (1,0)\). For \( s > 0 \), we call \( Q_{s} \subset \mathbb{H}^{2} \times \mathbb{R} \) the vertical plane orthogonal to \( B_{2} \) at \((s,0)\) and we call \( Q_{s}^{-} \) the component of \((\mathbb{H}^{2} \times \mathbb{R}) \setminus Q_{s} \) which does not contain \( D \times \mathbb{R} \).

Observe that \( \partial_{\infty} \Sigma_{0}^{*} = \{ i \} \times \mathbb{R} \). Moreover, if \( (z_{n}) \) is a sequence in \( U_{0} \) such that \( h^{*}(z_{n}) \to +\infty \), then we have \( d_{\mathbb{H}^{2}}(F^{*}(z_{n}), C) \to 0 \). Consequently, for any \( s > 0 \), the intersection \( \Sigma_{0}^{*} \cap Q_{s}^{-} \) is either empty or have compact closure. Assume the latter is true. In this case, we find \( s \) such that \( \Sigma_{0}^{*} \cap Q_{s}^{-} = \emptyset \). Then, we start to decrease \( s \). By the maximum principle, we can decrease \( s \) till \( 0 \) and obtain that \( \Sigma_{0}^{*} \cap Q_{s}^{-} = \emptyset \) for any \( s > 0 \). Therefore, \( \Sigma_{0}^{*} \) remains in the component of \((\mathbb{H}^{2} \times \mathbb{R}) \setminus (B_{1} \times \mathbb{R})\) containing \( D \times \mathbb{R} \).
By the same reasoning as above we can prove that $\Sigma_0^*$ remains in the component of $(\mathbb{R}^2 \times \mathbb{R}) \setminus (B_2 \times \mathbb{R})$ containing $D \times \mathbb{R}$ and also $\Sigma_0^*$ remains in the component of $(\mathbb{R}^2 \times \mathbb{R}) \setminus (C \times \mathbb{R})$ containing $D \times \mathbb{R}$. We conclude that $\Sigma_0^* \subset D \times \mathbb{R}$.

**Claim 2.** We have $F^*(U_0) \subset D$. Furthermore, the map $F^* : U_0 \to D$ is proper.

Let $Pr : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the projection on the first component. Since $F^* = Pr \circ X^*$ we deduce from Claim 1 that $F^*(U_0) \subset D$.

We must prove that for any compact set $K \subset D$, $(F^*)^{-1}(K)$ is a compact subset of $U_0$. In order to prove it, it is enough that for any sequence $(z_n)$ in $(F^*)^{-1}(K)$, there is a subsequence of $(z_n)$ converging in $(F^*)^{-1}(K)$.

Since $K$ is far from the geodesic $C$, the height function is bounded on $K$. Therefore, there exists a constant $\mu > 0$ such that $(F^*)^{-1}(K)$ remains in the subset of $U_0$ bounded by the level set $L_\mu$ and the half-axis $\{iy \mid y > 0\}$. Let $(z_n)$ be a sequence in $(F^*)^{-1}(K)$. Suppose that $(z_n)$ is not bounded. Therefore, there exists a subsequence $(z_{\varphi(n)})$ of $(z_n)$ such that $|z_{\varphi(n)}| \to +\infty$. As in the proof of Theorem 2.1 (Assertion 2), we can show that there exists a sequence $(iy_n)$ such that $y_n \to +\infty$ and $d_{\mathbb{R}^2}(F^*(z_{\varphi(n)}), F^*(iy_n)) \to 0$. But this is a contradiction since we have, by construction, $F^*(z_{\varphi(n)}) \in K$ and $F^*(iy_n) \to i \in \partial_{\infty}\mathbb{R}^2$.

Thus $(z_n)$ is a bounded sequence of $U_0$. Therefore, since $U_0 \subset C$, we can extract a subsequence $(z_{\varphi(n)})$ converging to some point $z \in \overline{U}_0$. We want to show that $z \in (F^*)^{-1}(K)$.

Observe that $F^*$ maps the boundary of $U_0$ onto the boundary of $D$, and moreover $d_{\mathbb{R}^2}(K, \partial D) > 0$, since $K$ is a compact subset of $D$. Therefore, we deduce that $z \in U_0$.

Since $F^*$ is continuous and $K$ is compact, we obtain $F^*(z) \in K$, from which we get $z \in (F^*)^{-1}(K)$. Therefore $F^* : U_0 \to D$ is a proper map, as desired.

**Claim 3.** We have $F^*(U_0) = D$ and $\Sigma_0^*$ is a vertical graph over $D$.

We know from Claim 2 that $F^*(U_0) \subset D$. Therefore, it suffices to prove that $F^*(U_0)$ is a closed and open subset of $D$.

It is known that $n_3 = \pm n_3$ (see [1] Remark 9]). On the other hand, since $\Sigma_0$ is a vertical graph, we have $n_3 \neq 0$ along $\Sigma_0$. We deduce that the tangent plane is never vertical along $\Sigma_0^*$ and that the map $F^* : U_0 \to D \subset \mathbb{R}^2$ is a local smooth diffeomorphism. Then, $F^*$ is an open map. As $U_0$ is open, we get that $F^*(U_0)$ is an open subset of $D$.

Now we prove that $F^*(U_0)$ is also a closed subset of $D$.

Let $(q_n)$ be a sequence in $F^*(U_0)$ converging to some point $q \in D$. We want to prove that $q \in F^*(U_0)$.

We set $K := \{q\} \cup \{q_n, n \in \mathbb{N}\}$, then $K$ is a compact subset of $D$. From Claim 2 we deduce that $(F^*)^{-1}(K)$ is a compact subset of $U_0$. For any $n \in \mathbb{N}$ there exists $z_n \in (F^*)^{-1}(K)$ such that $F^*(z_n) = q_n$. Therefore, we can extract a subsequence $(z_{\varphi(n)})$ which converges to some $z \in (F^*)^{-1}(K) \subset U_0$. Since $F^*$ is continuous we obtain $F^*(z_{\varphi(n)}) \to F^*(z)$, that is $q_{\varphi(n)} \to F^*(z)$. We deduce that $q = F^*(z)$ and then $q \in F^*(U_0)$, therefore $F^*(U_0)$ is a closed subset of $D$. Consequently we get $F^*(U_0) = D$. 


Hence, the map $F^*: U_0 \to D$ is a local smooth diffeomorphism. Moreover $F^*$ is proper and surjective. We deduce that it is a covering map. Since $D$ is connected and simply connected and $U_0$ is connected, we deduce that $F^*$ is a global diffeomorphism from $U_0$ onto $D$, that is $\Sigma_0^*$ is a vertical graph over $D$, as desired. \hfill \Box

4. Appendix. Basic geometry in $\mathbb{H}^2$

In this section, we establish some background material about $C^2$-curves in $\mathbb{H}^2$ whose absolute value of geodesic curvature is strictly smaller than one. We observe that the condition on the curvature implies that such curve is embedded, see for example [19 Proposition 2.6.32].

**Proposition 4.1.** Let $c : [0, +\infty[ \to \mathbb{H}^2$ be a regular $C^2$-curve with infinite length. Let $\kappa(t)$ be the geodesic curvature of $c$ at the point $c(t)$ and assume $|\kappa(c(t))| < k < 1$, for any $t \geqslant 0$. Then, the curve $C := c([0, +\infty[)$ has no limit point in $\mathbb{H}^2$, and the asymptotic boundary of $C$ consists in only one point $\{p_\infty\} = \partial_\infty C$.

**Proof.** If $k = 0$ then $C$ is a part of a geodesic and the assertions are obvious. Therefore we assume that $0 < k < 1$.

**Claim 1.** $C$ has no limit point in $\mathbb{H}^2$.

Indeed, assume by contradiction that there exists $p \in \mathbb{H}^2$ and a sequence of positive numbers $(t_n)$ such that $t_n \to +\infty$ and $p_n := c(t_n) \to p$.

Assume first that there exists a point $q \in C$, $q = c(t_0)$ for some $t_0 > 0$, such that $C$ is orthogonal at $q$ to the geodesic passing through $q$ and $p$. Let $H_q \subset \mathbb{H}^2$ be the horocycle through $q$, tangent to the curve $C$, such that $p$ belongs to the convex component of $\mathbb{H}^2 \setminus H_q$. Noticing that $|\kappa(c(t))| < 1$, and the absolute value of the curvature of the horocycles is 1. Thus, the maximum principle for curves, see [19 Theorem 2.6.27], ensures that $C_0 := c([t_0, +\infty[)$ belongs to the nonconvex component of $\mathbb{H}^2 \setminus H_q$ and then, $p$ cannot not be in the closure of $C$.

Hence we infer that the function $t \mapsto d_{\mathbb{H}^2}(c(t), p)$ is strictly decreasing.

For $t > 0$ we denote by $\alpha(t) \in [0, \pi]$, the nonoriented angle at $c(t)$ between the tangent vector $c'(t)$ and the geodesic segment $[c(t), p]$. Since the function $t \mapsto d_{\mathbb{H}^2}(c(t), p)$ is strictly decreasing we have that $\alpha(t) \in [0, \pi/2]$ for any $t > 0$.

Actually, we have $\alpha(t) \to 0$ as $t \to +\infty$. Indeed, assume by contradiction that there exist a sequence $(t_n)$, and a real number $\alpha_0 \in (0, \pi/2)$, such that $t_n \to +\infty$ and $\alpha(t_n) \to \alpha_0$. For any $n \in \mathbb{N}$, we denote by $\gamma_n$ the geodesic of $\mathbb{H}^2$ through $c(t_n)$ tangent to $C$. Let us denote by $H_n \subset \mathbb{H}^2$, $n \in \mathbb{N}$, the horocycle through $c(t_n)$ tangent to the curve $C$ and contained in the same component of $\mathbb{H}^2 \setminus \gamma_n$ as $p$. Therefore, for $n$ large enough, the point $c(t_n)$ is very close to $p$ and the angle $\alpha(t)$ is very close to $\alpha_0$. This would imply, for $n$ large enough, that $p$ belongs to the convex component of $\mathbb{H}^2 \setminus H_n$ and this would give again a contradiction with the maximum principle for curves. Therefore we get that $\alpha(t) \to 0$ as $t \to +\infty$. 


To conclude the argument, we choose for $\mathbb{H}^2$ the model of the unit disk equipped with the metric $g_\mathbb{D} = \lambda^2(z) |dz|^2$, where $\lambda(z) = 2/(1-|z|^2)$. We can assume that $p = 0$ and that $C$ is parametrized by arclength.

In polar coordinates we have $c(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t))$ where $r(t) = |c(t)| > 0$ and $\theta(t) \in \mathbb{R}$. We set $\partial_r := (\cos \theta, \sin \theta)$. Since $\alpha(t) \to 0$, we have

$$\frac{\langle c'(t); \partial_r \rangle_{\mathbb{D}}}{|\partial_r|_{\mathbb{D}}} \to -1,$$

where the scalar product and the norm are considered with respect to the metric $g_\mathbb{D}$.

From which we get that $\lambda(c(t)) r'(t) \to -1$. Using that $c(t) \to p = 0$ for $t \to \infty$, we obtain $\lambda(c(t)) \to 2$ for $t \to \infty$, therefore $r'(t) \to -1/2$ and then $r(t) \to -\infty$ as $t \to +\infty$. This is a contradiction and this concludes the proof of the claim.

Since $C$ is embedded and has no limit point in $\mathbb{H}^2$, we deduce that its asymptotic boundary is not empty. Let $p_\infty \in \partial_\infty C$ be an asymptotic point of $C$.

**Claim 2.** $p_\infty$ is the unique asymptotic point of $C$.

Let $\Gamma \subset \mathbb{H}^2$ be a complete curve with constant curvature $k$ such that $p_\infty \notin \partial_\infty \Gamma$ and $p_\infty$ belongs to the asymptotic boundary of the convex component of $\mathbb{H}^2 \setminus \Gamma$.

Let $\gamma \subset \mathbb{H}^2$ be a geodesic intersecting $\Gamma$ such that $p_\infty \in \partial_\infty \gamma$. For any $s \in \mathbb{R}$, let $\Gamma_s$ be the translated copy of $\Gamma$ along $\gamma$ at distance $|s|$, towards $p_\infty$ if $s > 0$, and in the opposite direction otherwise. We denote by $\Gamma_s^+$ the convex component of $\mathbb{H}^2 \setminus \Gamma_s$. Thus, we have $\Gamma_0 = \Gamma$ and $p_\infty \in \partial_\infty \Gamma_s^+$ for any $s \geq 0$. Observe that $\cap_{s \geq 0} \partial_\infty \Gamma_s^+ = \{p_\infty\}$.

If we assume that for any $s > 0$, there exists $t_s > 0$ such that $c([t_s, +\infty]) \subset \overline{\Gamma_s^+}$, then we deduce that $p_\infty$ is the unique asymptotic point of $C$, as desired. Therefore, we are left with the proof of such assumption.

Suppose by contradiction that there exists $r > 0$ such that for any $t > 0$ the curve $c([t, +\infty])$ is not entirely contained in $\overline{\Gamma_r^+}$. Therefore, there is an arc $C_1 \subset C$ such that $\partial C_1 \subset \Gamma_r$ and $C_1 \cap \Gamma_r^+ = \emptyset$, that is $C_1$ stays outside $\Gamma_r^+$. Note that $\overline{C_1}$ is a compact arc with boundary on $\Gamma_r$. Considering the curves $\Gamma_s$, for $s$ going from $r$ to $-\infty$, we get a real number $\rho < r$ such that $C_1 \subset \overline{\Gamma_{\rho}^+}$ and $C_1$ and $\Gamma_\rho$ are tangent at some interior point of $C_1$. This gives a contradiction by the maximum principle, keeping in mind the hypothesis about the curvature of $C$ and $\Gamma_\rho$ and the fact that $C_1$ belongs to the closure of the convex component of $\mathbb{H}^2 \setminus \Gamma_\rho$. This concludes the proof. $\square$

**Definition 4.1.** (1) Let $(p_n)$ be a sequence in $\mathbb{H}^2$ converging to some point $p \in \mathbb{H}^2$. Let $v \in T_p \mathbb{H}^2$ and $v_n \in T_{p_n} \mathbb{H}^2$ be non zero vectors. Assuming that $p_n \neq p$, we denote by $c_n$ the geodesic passing through $p$ and $p_n$, and by $T_n$ the translation along $c_n$ such that $T_n(p) = p_n$. If $p_n = p$, we set $T_n = Id$. Let $\alpha_n \in [0, \pi]$ be the non-oriented angle between $v_n$ and $T_n(v)$.

We say that the sequence $(v_n)$ converges to $v$ (denoted shortly by $v_n \to v$) if $\alpha_n \to 0$ and $|v_n|_{\mathbb{H}^2} \to |v|_{\mathbb{H}^2}$. 

(2) Let $\gamma$ be a geodesic in $\mathbb{H}^2$ and $(\gamma_n)$ be a sequence of complete and regular $C^1$-curves in $\mathbb{H}^2$. We say that the sequence $(\gamma_n)$ converges to $\gamma$ in the $C^1$ topology if:

(i) For any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for any $n > n_0$ the curve $\gamma_n$ stays in the region of $\mathbb{H}^2$ bounded by the two equidistant lines of $\gamma$ with distance $\varepsilon$ from $\gamma$.

(ii) Let $p \in \gamma$, and let $(p_n), p_n \in \gamma_n$, be any sequence converging to $p$. Let $v_n \in T_{p_n} \gamma_n$ be a unit tangent vector of $\gamma_n$ at $p_n$. If the sequence $v_n$ converges to a unit vector $v \in T_p \mathbb{H}^2$, then $v \in T_p \gamma$, that is $v$ is tangent to $\gamma$ at $p$.

**Proposition 4.2.** Let $\gamma \subset \mathbb{H}^2$ be a geodesic. Let $(\gamma_n)$ be a sequence of complete and regular $C^2$-curves such that:

- $\partial_{\infty} \gamma_n = \partial_{\infty} \gamma$ for any $n \in \mathbb{N}$.
- $\sup_{q \in \gamma_n} \{|\kappa_{\gamma_n}(q)|\} \xrightarrow{n \to \infty} 0$, where $\kappa_{\gamma_n}(q)$ is the geodesic curvature of $\gamma_n$ at point $q \in \gamma_n$.

Then, the sequence $(\gamma_n)$ converges to $\gamma$ in the $C^1$ topology.

**Proof.** Let us prove (i) of Definition 4.1.

Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have

$$\tanh \varepsilon > \sup_{q \in \gamma_n} \{|\kappa_{\gamma_n}(q)|\}.$$  

Denote by $L_1^\varepsilon$ and $L_2^\varepsilon$ the two equidistant lines of $\gamma$ with distance $\varepsilon$ from $\gamma$. It suffices to show that, for any $n > n_0$, the curve $\gamma_n$ belongs to the convex component of both $\mathbb{H}^2 \setminus L_1^\varepsilon$ and $\mathbb{H}^2 \setminus L_2^\varepsilon$. We prove that fact for $L_2 := L_1^\varepsilon$. The proof for $L_2^\varepsilon$ is analogous.

Let $\gamma'$ be any geodesic of $\mathbb{H}^2$ different from $\gamma$, intersecting $\gamma$. Let $p_\infty \in \partial_{\infty} \mathbb{H}^2$ be the asymptotic point of $\gamma'$ which stays in the asymptotic boundary of the nonconvex component of $\mathbb{H}^2 \setminus L_2$. Consider the translations along $\gamma'$, towards $p_\infty$. Assume by contradiction that for any $n_0$ there exists $n > n_0$ such that $\gamma_n$ does not belong to the convex component of $\mathbb{H}^2 \setminus L_2$. There exists a translated copy $L'_2$ of $L_2$ such that:

- $L'_2$ intersects the curve $\gamma_n$ at one point $q_n$.
- $L'_2$ and $\gamma_n$ are tangent at $q_n$.
- $\gamma_n$ belongs to the closure of the convex component of $\mathbb{H}^2 \setminus L'_2$.

Since the geodesic curvature of $L'_2$ is $\tanh \varepsilon$ (with respect to the normal direction pointing towards the convex component of $\mathbb{H}^2 \setminus L'_2$) and since $\tanh \varepsilon > \sup_{q \in \gamma_n} \{|\kappa_{\gamma_n}(q)|\}$, we obtain a contradiction with the maximum principle. This completes the proof of (i).

Now, we prove (ii) of Definition 4.1.

By contradiction, assume that the unit vector $v \in T_p \mathbb{H}^2$ is not tangent to $\gamma$.

Let $\varepsilon > 0$ and let $L \subset \mathbb{H}^2$ be one of the two complete curves passing through $p$, tangent to $v$ whose absolute value of the geodesic curvature is $\tanh \varepsilon$. Since $v$ is not tangent to $\gamma$, if $\varepsilon$ is small enough, then the curve $L$ separates the two points of the asymptotic boundary of $\gamma$, say $p_\infty$ and $q_\infty$. 


As in the Definition 4.1, we denote by $c_n$ the geodesic passing through $p$ and $p_n$. Let $T_n$ be the hyperbolic translation along $c_n$ such that $T_n(p) = p_n$. Let $R_n$ be the rotation in $\mathbb{H}^2$ around $p_n$ such that $R_n(T_n(v)) = v_n$. Therefore, $L_n := (R_n \circ T_n)(L)$ is a complete curve through $p_n$, tangent to $\gamma_n$ at $p_n$, with constant (absolute value) curvature equal to $\tanh \epsilon$. If $n$ is large enough then the curve $L_n$ separates $p_\infty$ and $q_\infty$, since this is true for $L$.

Observe that, if $n$ is large enough, we have $\sup_{q \in \gamma_n} \{\kappa_{\gamma_n}(q)\} < \tanh \epsilon$. Consequently, using the maximum principle for curves in the same way as before, we can show that $\gamma_n$ entirely belongs to the closure of one of the two components of $\mathbb{H}^2 \setminus L_n$. But this gives a contradiction with the assumption that $\partial_\infty \gamma_n = \partial_\infty \gamma = \{p_\infty, q_\infty\}$. We conclude that $v$ is tangent to $\gamma$, as desired.

\begin{remark}
We can extend Definition 4.1 to any dimensional hyperbolic space $\mathbb{H}^n$, $n \geq 2$. Moreover, we can prove in the same way as in Proposition 4.2 that if $\Pi \subset \mathbb{H}^n$ is a geodesic hyperplane and if $(\Pi_n)$ is a sequence of complete and regular $C^2$-hypersurfaces of $\mathbb{H}^n$ such that $\partial_\infty \Pi_n = \partial_\infty \Pi$ for any $n$ and $\sup_{q \in \Pi_n} \{H_n(q)\} \to 0$, where $H_n(q)$ denotes the mean curvature of $\Pi_n$ at $q$, then the sequence $(\Pi_n)$ converges $C^1$ to $\Pi$.
\end{remark}

\begin{proposition}
Let $\gamma_1 \subset \mathbb{H}^2$ be a geodesic and let $p_\infty \in \partial_\infty \mathbb{H}^2$ such that $p_\infty \not\in \partial_\infty \gamma_1$. For any $\rho > 0$, let $L_\rho$ be the equidistant line to $\gamma_1$ whose distance to $\gamma_1$ is $\rho$, such that $p_\infty$ belongs to the asymptotic boundary of the nonconvex component of $\mathbb{H}^2 \setminus L_\rho$.

Let $0 < k < 1$ and let $c : [0, \infty[ \to \mathbb{H}^2$ be a regular $C^2$-curve such that $\partial_\infty c([0, \infty[) = \{p_\infty\}$ and such that $|\kappa(c(t))| < k$ for any $t \geq 0$. Set $\rho_0 = \max\{d_{\partial\mathbb{H}^2}(c(0), \gamma_1), \tanh^{-1}(k)\}$.

Then, for any $\rho > \rho_0$, one has the following facts.

1. The equidistant line $L_\rho$ cuts the curve $c([0, \infty[)$ at a unique point. Therefore, there exists $t_0 > 0$ such that the curve $c([t_0, \infty[)$ is a horizontal graph with respect to $\gamma_1$.

2. The equidistant line $L_\rho$ is transversal to $c([0, \infty[)$.
\end{proposition}

\begin{proof}
Let $\rho > \rho_0$ and let $C = c([0, \infty[)$. Since $\partial_\infty C = p_\infty$, the equidistant line $L_\rho$ must intersect the curve $C$ at least at one point. Assume by contradiction that $L_\rho$ cut $C$ at two or more points. By construction, $c(0)$ belongs to the convex component of $\mathbb{H}^2 \setminus L_\rho$. Let $p_1 \in L_\rho \cap C$ be the first intersection point from $c(0)$. The boundary maximum principle for curves shows that the curves $L_\rho$ and $C$ are not tangent at $p_1$. Let $p_2 \in L_\rho \cap C$ be the first intersection point after $p_1$. Thus, the whole arc of $C$ between $p_1$ and $p_2$ belongs to the nonconvex component of $\mathbb{H}^2 \setminus L_\rho$. Now we obtain a contradiction with the maximum principle in the following way.

Let $p_\infty' \in \partial_\infty \mathbb{H}^2$ be a point in the asymptotic boundary of the convex component of $\mathbb{H}^2 \setminus L_\rho$. Let $\gamma \subset \mathbb{H}^2$ be the geodesic such that $\partial_\infty \gamma = \{p_\infty, p_\infty'\}$. Considering the hyperbolic translations along $\gamma$ towards $p_\infty$, we obtain a translated copy $L_\rho'$ of $L_\rho$ such that:

- $L_\rho'$ intersects the arc of $C$ between $p_1$ and $p_2$ at one point $q$.



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\begin{itemize}
  \item $L'_\rho$ and the arc of $C$ between $p_1$ and $p_2$ are tangent at $q$.
  \item The arc of $C$ between $p_1$ and $p_2$ belongs to the closure of the convex component of $\mathbb{H}^2 \setminus L'_\rho$.
\end{itemize}

Since the geodesic curvature of $L'_\rho$ is $\tanh \rho$ (with respect to the normal direction pointing towards the convex component of $\mathbb{H}^2 \setminus L'_\rho$) and since $\tanh \rho > k > \sup_{q \in C} \{|\kappa(C)|\}$, we obtain a contradiction with the maximum principle.

So Assertion (1) is proved. Now we prove Assertion (2).

Suppose, by contradiction, that for some $\rho > \rho_0$, the equidistant line $L_\rho$ is tangent to the curve $C$ at some point $p_1$. We set $C_1 = c([\rho_0, \infty])$. Then, the first part of the proof shows that the curve $C_1 \setminus \{p_1\}$ remains in the nonconvex component of $\mathbb{H}^2 \setminus L_\rho$.

Therefore, for $\varepsilon > 0$ small enough, the equidistant line $L_{\rho+\varepsilon}$ intersects the curve $C$ at least at two different points near $p_1$, giving a contradiction with assertion (1). □

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