A geometric preferential attachment model with fitness

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Abstract

We study a random graph $G_n$, which combines aspects of geometric random graphs and preferential attachment. The resulting random graphs have power-law degree sequences with finite mean and possibly infinite variance. In particular, the power-law exponent can be any value larger than 2.

The vertices of $G_n$ are $n$ sequentially generated vertices chosen at random in the unit sphere in $\mathbb{R}^3$. A newly added vertex has $m$ edges attached to it and the endpoints of these edges are connected to old vertices or to the added vertex itself. The vertices are chosen with probability proportional to their current degree plus some initial attractiveness and multiplied by a function, depending on the geometry.

Keywords: Complex networks; Random geometric graphs; Scale free graphs; Preferential attachment; Power-law distributions; Ad hoc networks

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1 Introduction

Preferential attachment models are proposed by Barabási and Albert [1] as models for large-scale networks like the Internet, electrical networks, telephone networks, and even complex biological networks. These networks grow in time, because, for example, new routers, transform houses, switchboards or proteins are added to the network. The behavior can be modeled by means of a random graph process. A random graph process is a stochastic process that describes a random graph evolving with time. At each time step, the random graph is updated using a given rule of growth, which will be specified later.

In literature a number of different rules of growth are explored. For example, each time step we add or remove edges/vertices [2], or, more advanced, copy parts of the graph [9]. Furthermore, there is freedom in the choice how to connect endpoints of newly added edges. Mostly, one randomly chooses the endpoint over the vertices, or proportional to the degree. Another possibility is to assign to each vertex a fitness. In [3, 4] additive fitness is explored where one chooses proportional to the degree plus some (random) value. In [5, 10] multiplicative fitness is explored where each vertex has a random fitness and one chooses a vertex proportional to the degree times the fitness. In this paper we use a constant additive fitness and a variant of multiplicative fitness, depending on the distance between vertices.

Many large networks of interest have power-law degree sequences, by which we mean that the number of vertices with degree $k$ falls off as $k^{-\tau}$ for some exponent $\tau > 1$. The parameter $\tau$ is called the power-law exponent. Depending on the value of $\tau$ we classify the following three categories: the infinite mean case, the finite mean and infinite variance case, and the finite variance case, which corresponds to $\tau \in (1, 2)$, $\tau \in (2, 3)$ and $\tau > 3$, respectively.

These categories are of interest, because the behavior of the typical distance is determined by the power-law exponent $\tau$. Results in the literature show that if $\tau \in (1, 2)$ the typical distance is bounded by some constant, if $\tau \in (2, 3)$ the typical distance is concentrated around $\log \log n$ and if $\tau > 3$ it is concentrated around $\log n$, where $n$ is the number of vertices of the graph, see [13, 14, 15, 16, 10].

A large number of graph models have been introduced to describe complex networks, but often the underlying geometry is ignored. In general it is difficult to get rigorous results for properties like the degree distribution, typical distances or diameter, even if one disregards the geometry. However, in wireless ad-hoc
networks the geometry is of great importance, since in these networks nodes are spread over some surface and nodes can only communicate with neighbors within a certain range, depending on the geometry.

In this paper we will rely on the geometric preferential attachment (GPA) model introduced in [6] and extended in [7] by the same authors. The GPA model is a variant of the well known Barabási-Albert (BA) model. In the BA model new vertices are added to the graph one at a time. Each new vertex is connected to an old vertex and the newly added vertex. For instance, let the multiplicative constant be 1 if the vertices are at distance at most \( r_n \), and otherwise zero. The latter attachment rule essentially describes the construction of a simplified wireless ad-hoc network.

### 1.1 Definition of the model

In this section we will introduce the Geometric Preferential Attachment model with fitness (GPAF). The GPAF model is described by a random graph process \( \{G_{\sigma}\}_{\sigma=0}^{n} \), which we will study for large values of \( n \). For \( 0 \leq \sigma \leq n \), each vertex of the graph \( G_{\sigma} = (V_{\sigma}, E_{\sigma}) \) is positioned on the sphere \( S \subset \mathbb{R}^3 \). The radius of the sphere \( S \) is taken equal to \( 1/(2\sqrt{\pi}) \), so that, conveniently, \( \text{Area}(S) = 1 \). The vertices of the graph \( G_{\sigma} \) are given by \( V_{\sigma} = \{1, 2, \ldots, \sigma\} \) and \( E_{\sigma} \) is the set of edges. The position of vertex \( v \in V_{\sigma} \) in the graph \( G_{\sigma} \) is given by \( x_{v} \in S \) and the degree at time \( \sigma \) is given by \( d_{\sigma}(v) \).

In total we need 4 parameters to describe the GPAF model. The first parameter of the model is \( m = m(n) > 0 \), which is the number of edges added in every time step. The second parameter is \( \alpha \geq 0 \), which is a measure of the bias toward self-loops. The third one is \( \delta > -m \), which is the initial attractiveness of a vertex. And, finally, the fourth parameter is a function \( F_{\sigma} : [0, \pi] \rightarrow \mathbb{R}_{+} \), where the value \( F_{\sigma}(u) \) is an indicator of the attraction between two vertices at distance \( u \).

Before we give the model definition, we first will explain the use of the parameter \( \alpha \). Assume that the graph \( G_{\sigma} \) is given, consisting of the vertices \( V_{\sigma} \). We construct the graph \( G_{\sigma+1} \) by choosing vertex \( x_{\sigma+1} \) uniformly at random in \( S \) and add it to \( G_{\sigma} \) with \( m \) directed edges emanating from the vertex \( x_{\sigma+1} \). Let, for \( \sigma = 1, 2, \ldots, n-1 \),

\[
T_{\sigma,n}(u) = \sum_{v=1}^{\sigma} (d_{\sigma}(v) + \delta) F_{\sigma}(|x_{v} - u|),
\]

where \( |x_{v} - u| \in [0, \pi] \) is the angular distance from \( u \) to \( u_{0} \) along a circle with radius \( 1/(\sqrt{2}) \) over the surface \( S \). Furthermore, let the endpoints of the \( m \) emanating edges be given by the vertices \( v^{(1)}_{\sigma+1}, \ldots, v^{(m)}_{\sigma+1} \).

Intuitively, we would like to choose the endpoints at random (with replacement) from \( V_{\sigma} \), such that \( v \in V_{\sigma} \) is chosen with probability

\[
\mathbb{P}_{\sigma}(v^{(i)}_{\sigma+1} = v) = \frac{(d_{\sigma}(v) + \delta) F_{\sigma}(|x_{v} - x_{\sigma+1}|)}{T_{\sigma,n}(x_{\sigma+1})},
\]

where \( \mathbb{P}_{\sigma}(\cdot) = \mathbb{P}(\cdot | G_{\sigma}, x_{\sigma+1}) \). However, the above given rule of growth is not well-defined. To see this, consider the simplified model for wireless ad-hoc networks, i.e., \( F_{\sigma}(x) = 1(x \leq r_n) \). Then, for any \( \sigma \), there is a positive probability that there are no vertices within reach of the newly added vertex \( x_{\sigma+1} \) and therefore \( T_{\sigma,n}(x_{\sigma+1}) = 0 \).

Introducing self-loops solves this problem and for this the additional parameter \( \alpha \) is introduced. We will follow the solution given by the authors of [7] for the GPA model:

#### Rules of growth for \( \alpha > 0 \):

- **Initial Rule** \((\sigma = 0)\): To initialize the process, we start with \( G_0 \) being the empty graph.

- **Growth Rule** (at time \( \sigma + 1 \)): We choose vertex \( x_{\sigma+1} \) uniformly at random in \( S \) and add it to \( G_{\sigma} \) with \( m \) directed edges emanating from the vertex \( x_{\sigma+1} \). Let the endpoints of the \( m \) emanating edges be given by the vertices \( v_{\sigma+1}^{(1)}, \ldots, v_{\sigma+1}^{(m)} \). We choose the endpoints at random (with replacement) from \( V_{\sigma} \), such that \( v \in V_{\sigma} \) is chosen with probability

\[
\mathbb{P}_{\sigma}(v^{(i)}_{\sigma+1} = v) = \frac{(d_{\sigma}(v) + \delta) F_{\sigma}(|x_{v} - x_{\sigma+1}|)}{\max\{T_{\sigma,n}(x_{\sigma+1}), \alpha \mathbb{E}[T_{\sigma,n}(x_{\sigma+1})]/2\}},
\]
\[ P_\sigma(v^{(i)}_{\sigma+1} = \sigma + 1) = 1 - \frac{T_{\sigma,n}(x_{\sigma+1})}{\max\{T_{\sigma,n}(x_{\sigma+1}), \alpha E[T_{\sigma,n}(x_{\sigma+1})]/2\}}, \quad (1.3) \]

for \( i \in \{1, 2, \ldots, m\} \).

The above given random graph model is well defined, since the denominator is always strictly positive. Indeed, the following lemma calculates the value of \( E[T_{\sigma,n}(x_{\sigma+1})] \) which is strictly positive.

**Lemma 1.1** For any fixed point \( v \in S \),
\[
\int_S F_n(|v - u|) \, du = I_n, \quad (1.4)
\]
where
\[
I_n = \frac{1}{2} \int_{x=0}^{\pi} F_n(x) \sin x \, dx.
\]
As a consequence, if \( U \) is a randomly chosen point from \( S \), then
\[
E[T_{\sigma,n}(U)] = I_n(2m + \delta)\sigma. \quad (1.5)
\]

**Proof.** First note that \( I_n \) does not depend on \( v \) due to rotation invariance. Thus, without loss of generality, we can assume that \( v \) is at the north pole of the sphere. Using spherical coordinates, we find \( du = r_0^2 \sin \theta d\theta d\varphi \), where \( r_0 = 1/(2\sqrt{\pi}) \), and \( |v - u| = \theta \), so that:
\[
\int_S F_n(|v - u|) \, du = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} r_0^2 F_n(\theta) \sin \theta \, d\theta d\varphi = 2\pi r_0^2 \int_{\theta=0}^{\pi} F_n(\theta) \sin \theta \, d\theta = I_n.
\]

For the second claim we calculate the expected value of \( T_{\sigma,n}(U) \), (1.1), conditional on the graph \( G_\sigma \):
\[
E[T_{\sigma,n}(U) | G_\sigma] = \sum_{v=1}^{\sigma} (d_\sigma(v) + \delta) E[F_n(|x_v - U|) | G_\sigma]
\]
\[
= I_n \sum_{v=1}^{\sigma} (d_\sigma(v) + \delta) = I_n(2m + \delta)\sigma,
\]
where we apply (1.4) on \( E[F_n(|x_v - U|) | G_\sigma] \), since \( v \leq \sigma \) and, therefore,
\[
E[F_n(|x_v - U|) | G_\sigma] = \int_S F_n(|x_v - u|) \, du = I_n. \quad (1.6)
\]
Hence, \( E[T_{\sigma,n}(U)] = E[E[T_{\sigma,n}(U) | G_\sigma]] = I_n(2m + \delta)\sigma. \quad \square \)

We use the abbreviations, for \( u \in S \),
\[
M_{\sigma,n}(u) = \max\{T_{\sigma,n}(u), \alpha \Theta I_n \sigma\} \quad \text{and} \quad A_{\sigma,n}(u) = F_n(|u - x_{\sigma+1}|), \quad (1.7)
\]
where
\[
\Theta = \Theta(\delta, m) = (2m + \delta)/2. \quad (1.8)
\]
As a consequence, we can rewrite the attachment rules as
\[
P_\sigma(v^{(i)}_{\sigma+1} = v) = \frac{(d_\sigma(v) + \delta)A_{\sigma,n}(x_v)}{M_{\sigma,n}(x_{\sigma+1})} \quad \text{and} \quad P_\sigma(v^{(i)}_{\sigma+1} = \sigma + 1) = 1 - \frac{T_{\sigma,n}(x_{\sigma+1})}{M_{\sigma,n}(x_{\sigma+1})}, \quad (1.9)
\]
for \( i \in \{1, 2, \ldots, m\} \).

**Remark 1.2** In the above description we add directed edges to the graph and therefore we construct a directed graph. For questions about the connectedness and diameter of the graph we ignore the direction of the edges, but we need the direction of the edges in the proofs of the main results.
Remark 1.3 In this paper we will illustrate the theorems using the canonical functions
\begin{align}
F_n^{(0)}(u) &\equiv 1, & F_n^{(1)}(u) &= 1(|u| \leq r_n) & \text{and} & F_n^{(2)}(u) &= \frac{1}{\max\{n^{-\psi}, u\}^\beta},
\end{align}
where $r_n \geq n^{\varsigma - 1/2}$, $\varsigma < 1/2$, $\psi < 1/2$ and $\beta \in (0, 2) \cup (2, \infty)$. The canonical function $F_n^{(0)}$ implies that the vertices are chosen proportional to the degree, and, furthermore, the geometry is ignored, the model is then equivalent to the PARID model, see [3] or section \S 1.2. The function $F_n^{(1)}$ implies that a new vertex can only connect to vertices at distance at most $r_n$. Finally, canonical function $F_n^{(2)}$ implies that vertices are chosen proportional to the degree, and, in contrast to $F_n^{(0)}$, will prefer vertices close to the new vertex, since $F_n^{(2)}$ is non-increasing as a function in $u$.

\subsection*{1.2 Heuristics and main results}

Using the results of [7], which is a special case of our model when $\delta = 0$, together with the results of the PARID model, introduced in [3], we will predict how the power-law exponent of the degree sequence will behave.

Consider the PARID random graph process $\{G_{\sigma}^n\}_{\sigma \geq 0}$ as introduced in [3] with constant weights equal to $m$. For this special case, we give a brief description of the model.

The construction of the PARID graph $G_{\sigma}^n = (V_{\sigma}^n, E_{\sigma}^n)$ depends on the graph $G_{\sigma-1}^n$. The rule of growth is as follow: add a vertex to the graph $G_{\sigma-1}^n$ and from this vertex emanates $m$ edges. The endpoints of these $m$ edges are chosen independently (with replacement) from the vertices of $G_{\sigma-1}^n$. The probability that vertex $v \in V_{\sigma-1}^n$ is chosen is proportional to the degree of vertex $v$ plus $\delta$, more specifically:

\begin{equation}
P(\text{choose vertex } v \mid G_{\sigma}^n) = \frac{d_\sigma(v) + \delta}{(2m + \delta)\sigma}.
\end{equation}

If $\alpha \leq 2$, $\delta > -m$ and $F_n = F_n^{(0)}$, then the GPAF model coincides with the PARID model where the weight of each vertex is set to $m$. Note that, for the chosen parameters,

$$A_{\sigma,n}(x_v) = F_n^{(0)}(|x_v - x_{\sigma+1}|) = 1, \quad T_n^{(0)} = \frac{1}{2} \int_{x=0}^\pi \sin(x) \, dx = 1,$$

and
\begin{equation}
\alpha \Theta T_n^{(0)} \sigma \leq 2 \Theta \sigma = (2m + \delta)\sigma = \sum_{v \in V_\sigma} (d_\sigma(v) + \delta) = T_{\sigma,n}^{(0)}(x_{\sigma+1}),
\end{equation}

thus $M_{\sigma,n}^{(0)}(x_{\sigma+1}) = T_{\sigma,n}^{(0)}(x_{\sigma+1}) = (2m + \delta)\sigma$. Therefore, the equations (1.9) turns into (1.11), since

\begin{equation}
P_\sigma(v_{\sigma+1}^{(1)} = v) = P_\sigma(\text{choose vertex } v) = \frac{(d_\sigma(v) + \delta)A_{\sigma,n}(x_v)}{M_{\sigma,n}^{(0)}(x_{\sigma+1})} = \frac{d_\sigma(v) + \delta}{(2m + \delta)\sigma}. \quad \text{and}
\end{equation}

Furthermore, note that for these parameters there are no self-loops, since

\begin{equation}
P_\sigma(v_{\sigma+1}^{(1)} = \sigma + 1) = 1 - \frac{T_{\sigma,n}^{(0)}(x_{\sigma+1})}{M_{\sigma,n}^{(0)}(x_{\sigma+1})} = 1 - \frac{M_{\sigma,n}^{(0)}(x_{\sigma+1})}{M_{\sigma,n}^{(0)}(x_{\sigma+1})} = 0.
\end{equation}

For the PARID model, we know that the power-law exponent is $3 + \delta/m$, thus we expect that the power-law exponent in our model is $3 + \delta/m$ if $\alpha \leq 2$ and $F_n = F_n^{(0)}$. For $\alpha > 2$, $\delta = 0$ and $F_n$ satisfying some mild condition, see (1.12), we know from [7] that the power-law exponent is $1 + \alpha$, which is independent of $F_n$.

We will show in this paper that the power-law exponent is given by $1 + \alpha(1 + \delta/2m)$, which generalizes the two mentioned papers [6, 7]. More precisely, let $N_k(\sigma)$ denote the number of vertices of degree $k$ in $G_{\sigma}$ and let $\bar{N}_k(\sigma)$ be its expectation. We will show that:

\textbf{Theorem 1.4 (Behavior of the degree sequence)} Suppose that $\alpha > 2$, $\delta > -m = m(n)$ and in addition that for $n \to \infty$,

\begin{equation}
\int_{x=0}^\pi F_n(x)^2 \sin x \, dx = O\left(\frac{n^\theta I_n^2}{\alpha}\right), \quad \text{where } \theta < 1 \text{ is a constant.}
\end{equation}

Then there exists a constant $\gamma_1 > 0$ such that for all $k = k(n) \geq m$,

\begin{equation}
\bar{N}_k(n) = n e^{\phi_k(m, \alpha, \delta)} \left(\frac{m}{k}\right)^{1+\alpha(1+\delta/2m)} + O(n^{1-\gamma_1}), \quad k \geq \log n.
\end{equation}
where \( \phi_k(m, \alpha, \delta) \) tends to a constant \( \phi_{\infty}(m, \alpha, \delta) \) as \( k \to \infty \).

Furthermore, for each \( \epsilon > 0 \) and \( n \) sufficiently large, the random variables \( N_k(n) \) satisfy the following concentration inequality
\[
P \left( \left| N_k(n) - \bar{N}_k(n) \right| \geq I_n^2 n^{\max(1/2, 2/\alpha) + \epsilon} \right) \leq e^{-n^\epsilon}.
\] (1.14)

**Remark 1.5** Note that the power-law exponent in (1.13) does not depend on the choice of the function \( F_n \). We will see in the proof, that \( F_n \) manifest itself only in the error terms.

All the given canonical functions in Remark 1.3 do satisfy the condition given by (1.12): it should be evident that for \( F_n^{(0)} \), the constants \( I_n \) and \( \theta \) are given by \( I_n^{(0)} = 1 \) and \( \theta^{(0)} = 0 \), respectively. Furthermore, in [7] it is shown that one can take \( I_n^{(1)} \sim r_n^2/4 \), \( I_n^{(2)} = O(1) \) if \( \beta \in (0, 2) \), and \( I_n^{(2)} \sim 2(\beta-2) \), if \( \beta > 2 \), and hence we can take \( \theta^{(1)} = 0 \), \( \theta^{(2)} = 0 \) and \( \theta^{(2)} = 2\psi \), respectively.

Before we consider the connectivity and diameter of \( G_n \), we place some additional restrictions on the function \( F_n \). These restrictions are necessary to end up with a graph which is with high probability connected.

Keep in mind the function \( F_n^{(1)}(u) = 1 \{ |u| \leq r_n \} \), then it should be clear that \( r_n \) should not decrease too fast, otherwise we end up with a disconnected graph.

Let \( \rho_n = \rho(\mu, n) \) be such that
\[
\mu I_n = \frac{1}{2} \int_{x=0}^{\rho_n} F_n(x) \sin x \, dx,
\]
for some \( \mu \in (0, 1] \).

We will call \( F_n \) smooth (for some value of \( \mu \)) if

1. \( F_n \) is monotone non-increasing;
2. \( n \rho_n^2 \geq L \log n \), for some sufficiently large \( L \);
3. \( \rho_n^2 F_n(2\rho_n) \geq c_3 I_n \), for some constant \( c_3 \) which is bounded from below.

Before stating the theorem, we will give an intuitive meaning of \( \rho_n \). To that end, consider the function \( F_n^{(1)}(u) = 1 \{ |u| \leq r_n \} \) and use the fact that if \( r_n = O(n^{-1/2-\epsilon}) \) for some \( \epsilon > 0 \), then the limiting graph is not connected, see [11]. It should be intuitively clear that in the limit, each newly added vertex \( u_n \) should connect to at least one other vertex. Thus, there should be at least one vertex within distance \( r_n \) of \( x_n \). At time \( n \) there are \( n - 1 \) vertices and the probability that at least one of these vertices is at distance at most \( r_n \) of vertex \( x_n \), denoted by \( p_c(n, r_n) \), is at most \( C(n-1)r_n^2 \), for some constant \( C \). On the other hand, we see that if \( r_n = O(n^{-1/2-\epsilon}) \) then \( p_c(n, r_n) = O(n^{-2\epsilon}) \) tends zero for large \( n \), and, as a consequence, in the limit the graph is not connected. If, as is our assumption, \( r_n > n^{\epsilon-1/2} \), then \( p_c(n, r_n) \to \infty \).

Interpret \( \rho_n \) for general \( F_n \) as the radius. The condition that \( p_c(n, r_n) \to \infty \) is replaced by \( n \rho_n^2 > L \log n \). Then, intuitively, condition S2 implies that for general \( F_n \), the value \( p_c(n, \rho_n) \), does not tend to zero and implies that the limiting graph is connected. The conditions S1 and S3 are technicalities, combined they ensure that the ‘area’ due to the radius \( \rho_n \) is sufficiently large: condition S1 states that \( F_n \) is monotone non-increasing and combined with S3 one can show that the ‘area’ within radius \( 2\rho_n \) is \( (2\rho_n)^2 F_n(2\rho_n) \), which is at least \( 4c_3 I_n \).

**Theorem 1.6** If \( \alpha \geq 2 \) and \( F_n \) is smooth, \( m \geq K \log n \), and \( K \) is sufficiently large constant, then with high probability

- \( G_n \) is connected;
- \( G_n \) has diameter \( O(\log n/\rho_n) \).

**Remark 1.7** All the canonical functions are smooth. It should be evident that one can take for \( F_n^{(0)} \): \( \mu^{(0)} = 1 \), \( \rho^{(0)} = 1 \) and \( c_3^{(0)} = 1 \) for \( F_n^{(0)} \). For \( F_n^{(1)}(u) \) one can take for example \( \mu^{(1)} \sim 1/4 \) and \( \rho^{(1)} = r_n/2 \) and \( c_3^{(1)} \sim 1 \). Finally, \( F_n^{(2)} \) is also smooth, we refer to [7] for the precise values of \( \rho^{(2)} \), \( \mu^{(2)} \) and \( c_3^{(2)} \).

We end with a sharper result on the diameter, however we, also, need stronger restrictions on the function \( F_n \). We will call \( F_n \) tame if there exists strictly positive constants \( C_1 \) and \( C_2 \) such that

- \( (T1) \) \( F_n(x) \geq C_1 \) for \( 0 \leq x \leq \pi \);
- \( (T2) \) \( I_n \leq C_2 \).
Theorem 1.8 If $\alpha \geq 2$, $\delta > -m$ and $F_n$ is tame and $m \geq K \log n$, and $K$ sufficiently large, then with high probability

- $G_n$ is connected;
- $G_n$ has diameter $O(\log_{n} n)$.

Remark 1.9 It should be evident that the function $F_n^{(\alpha)}$ is tame, since one can take $C_1 = C_2 = 1$. If $\beta \in (0,2)$ then we also have that $F_n^{(\alpha)}$ is tame, since

$$F_n^{(\alpha)}(x) \geq \pi^{-\beta}, \text{ for } 0 \leq x \leq \pi, \quad \text{and} \quad I_n^{(\alpha)} = \frac{1}{2} \int_{\pi}^{\pi} x^{-\beta} \sin x \, dx \leq \frac{\pi^{2-\beta}}{2(2-\beta)}.$$  

Remark 1.10 If we consider the configuration model (CM), see §1.3, [15, 16] or the Poissonian random graph (PRG), see [12, 10], then the typical distance depends on the power-law exponent. If the power-law exponent is larger than 3, then the typical distance is of $O(\log n)$, where $n$ is the number of vertices in the graph, and if the power-law exponent is between 2 and 3 then the typical distance is of $O(\log \log n)$. On before hand, it is not clear if this holds for the GPAF model. Theorem 1.8, only states an upper bound on the diameter, independent of $\delta$.

If $F_n = F_n^{(\alpha)} \equiv 1$ and $\delta \in (-m,0)$ then the authors of [14] show that the diameter in the graph $G_n$ fluctuates around $\log \log n$. If $F_n(u) = F_n^{(\alpha)}(u) = 1\{\{u\} \leq r_n\}$, then, intuitively, the diameter depends only on $r_n$, since $r_n$ determines the maximal length of an edge, and we conjecture that the diameter is at least of order $\log n$.

1.3 Related work

In this section we consider random graph models, which are related to the Geometric Preferential Attachment model with fitness (GPAF).

As mentioned earlier the model is related to the Albert-Barabási (BA) model. In the BA-model the power-law exponent $\tau$ is limited to the value 3, which was proven by Bollobás and Riordan.

Cooper and Frieze introduced in [2] a very general model preferential attachment model. In this model it is both possible to introduce new vertices at each time step or to introduce new edges between old vertices. Due to the weights with which edges of the new vertices are attached to old vertices and the adding of edges between old vertices, the power-law exponent $\tau$ can obtain any value $\tau > 2$.

In [4] the authors overcome the restriction $\tau \geq 3$ in a different way, by choosing the endpoint of an edge proportional to the in-degree of a vertex plus some initial attraction $A > 0$. This is identical by choosing the endpoint of an edge proportional to the degree of a vertex plus some amount $\delta = A - m > -m$, as done in the PARID model (cf. [3]). The power-law exponent in [4] is given by $\tau = 3 + \delta/m$. Note that for $\delta = 0$ we obtain the BA model. The authors of [3] show more rigorously some of the results in [4].

Both in [2] and in [3] it is allowed to add a random number of edges $W$, with the introduction of a new vertex. In case the mean of $W$ is finite the power-law exponent is given by $\tau = 3 + \delta/E[W]$. Hence, if $P(W = m) = 1$ for some integer $m \geq 1$ then we see that $\tau = 2 + \delta/m \geq 2$, since we can choose for $\delta$ any value in $(-m,0)$.

In [6, 7] the authors add geometry to the BA model, which corresponds to the GPAF model, introduced above, with $\delta = 0$. Due to a technical difficulty the model has an additional parameter, called $\alpha > 2$. As a consequence of this restriction they only obtain power-law exponents greater than 3, since the power-law exponent is given by $\tau = \alpha + 2$.

By combining the GPA and PARID model, we obtain the GPAF model, introduced in this paper. Due to the additional parameter $\delta$, it is in this model possible to obtain any power-law exponent $\tau$ bigger than 2.

1.4 Overview of the paper

The remainder of this paper is divided into three sections. In §2 we will derive a recurrence relation for the expected number of vertices of a given degree. In §3 we will present a coupling between the graph process and an urn scheme, which will be used in §4 to show that the number of vertices with a given degree is concentrated around its mean.
2 Recurrence relation for the expected degree sequence

In this section we will establish a recurrence relation for \( \tilde{N}_k(\sigma) = \mathbb{E}[N_k(\sigma)] \), the expected number of vertices with degree \( k \) at time \( \sigma \), which is claim (1.13) of Theorem 1.4. From this recurrence relation, we will show that

\[
\tilde{N}_k(\sigma) \sim \sigma p_k,
\]

where \( p_k \sim k^{-(1+\alpha(1+\delta/2m))} \), as \( k \to \infty \). The proof of claim (1.13) depends on a lemma, which is crucial for the proof. This lemma states that for sufficiently large \( n \) the value \( M_{\sigma,n}(x_{\sigma+1}) \) is equal to \( \alpha \Theta_{I_n} \sigma \), with high probability. This is a consequence of the fact that \( T_{\sigma,n}(x_{\sigma+1}) \) is concentrated around its mean \( \mathbb{E}[T_{\sigma,n}(x_{\sigma+1})] = 2\Theta_{I_n} \sigma < \alpha \Theta_{I_n} \sigma \), see (1.5) and (1.8), which is the content of the next lemma.

Lemma 2.1 If \( \alpha > 2, \delta > -m, \sigma = 0, 1, 2, \ldots, n \), and \( U \) is chosen randomly from \( S \) then

\[
\mathbb{P} \left( |T_{\sigma,n}(U) - \mathbb{E}[T_{\sigma,n}(U)]| > \Theta_{I_n} \left( \sigma^{2/\alpha} + \sigma^{1/2} \log \sigma \right) \log n \right) = O \left( n^{-2} \right).
\]

The proof of this lemma is deferred to §4.1.

We will allow that \( m \) depends on \( n \), thus \( m = m(n) \), as already pointed out previously. In establishing the recurrence relation for \( \tilde{N}_k(\sigma) \), we will rely on the derivation for \( \delta = 0 \) in [7, Section 3.1].

At each time, we add a new vertex from which \( m \) edges are emanating, and for each of these \( m \) emanating edges we need to choose a vertex-endpoint. The first possibility for a vertex to have degree \( k \) at time \( \sigma + 1 \) is that the degree at time \( \sigma \) was equal to \( k \) and that none of the \( m \) endpoints, emanating from \( x_{\sigma+1} \), attaches to the vertex. Furthermore, ignoring for the moment the effect of selecting the same vertex twice or more, the vertex could also have degree \( k - 1 \) at time \( \sigma \) and having one endpoint attached to it at time \( \sigma + 1 \). Finally, it is also possible that the newly added vertex \( x_{\sigma+1} \) has degree \( k \). The total number of vertex-endpoints with degree \( k \) is distributed as \( \text{Bin}(m, p_k(\sigma)) \), where

\[
p_k(\sigma) = \sum_{v \in D_k(\sigma)} \frac{(k+\delta)A_{\sigma,n}(x_v)}{M_{\sigma,n}(x_{\sigma+1})},
\]

and \( D_k(\sigma) \subset V_\sigma \) is the set of vertices with degree \( k \) in the graph \( G_\sigma \). Similarly, the number of vertex-endpoints with degree \( k - 1 \) is distributed as \( \text{Bin}(m, p_{k-1}(\sigma)) \). If the newly added vertex \( x_{\sigma+1} \) ends up with degree \( k \), then this vertex has \( k - m \) self-loops. The number of self-loops, \( d_{\sigma+1}(\sigma+1) - m \), is distributed as \( \text{Bin}(m, p) \), where

\[
p = 1 - T_{\sigma,n}(x_{\sigma+1})/M_{\sigma,n}(x_{\sigma+1}).
\]

For \( k \geq m \), this leads to,

\[
\mathbb{E}[N_k(\sigma + 1)|G_\sigma, x_{\sigma+1}] = N_k(\sigma) - mp_k(\sigma) + mp_{k-1}(\sigma) + \mathbb{E}[1_{d_{\sigma+1}(\sigma+1) = k} | G_\sigma, x_{\sigma+1}] + O(m p_k(G_\sigma, x_{\sigma+1})) + O(m \mathbb{E}[\eta_k(G_\sigma, x_{\sigma+1})]),
\]

where \( \eta_k(G_\sigma, x_{\sigma+1}) \) denotes the probability, conditionally on \( G_\sigma \), that the same vertex-endpoint is chosen at least twice and at most \( k \) times.

Taking expectations on both sides of (2.3), we obtain

\[
\tilde{N}_k(\sigma + 1) = \tilde{N}_k(\sigma) - m \mathbb{E} \left[ \sum_{v \in D_k(\sigma)} \frac{(k+\delta)A_{\sigma,n}(x_v)}{M_{\sigma,n}(x_{\sigma+1})} \right] + \mathbb{E} \left[ \sum_{v \in D_{k-1}(\sigma)} \frac{(k-1+\delta)A_{\sigma,n}(x_v)}{M_{\sigma,n}(x_{\sigma+1})} \right] + \mathbb{P} (d_{\sigma+1}(\sigma+1) - m = k - m) + O(m \mathbb{E}[\eta_k(G_\sigma, x_{\sigma+1})]).
\]

Let

\[
B_\sigma = \{|T_{\sigma,n}(x_{\sigma+1}) - 2\Theta_{I_n} \sigma| < C_1 \Theta_{I_n} \sigma^\gamma \log n \},
\]

where \( \max\{2/\alpha, \theta\} < \gamma < 1 \) and \( C_1 \) is some sufficiently large constant. If

\[
\sigma \geq t_0 = t_0(n) = (\log n)^{2/(1-\gamma)},
\]

then
then \( B_n \) implies that for sufficiently large \( n \),

\[
T_{\sigma,n}(x_{\sigma+1}) \leq 2\Theta I_n \sigma + C_1 \Theta I_n \sigma^\gamma \log n = 2\Theta I_n \sigma (1 + O(\log^{-1} n)) \leq \alpha \Theta I_n \sigma,
\]

since \( \alpha > 2 \), and, hence, with high probability

\[
M_{\sigma,n}(x_{\sigma+1}) = \max\{T_{\sigma,n}(x_{\sigma+1}), \alpha \Theta I_n \sigma\} = \alpha \Theta I_n \sigma.
\]

Next, we consider each term on the right hand side of (2.4) separately, for \( \sigma = 1, 2, \ldots, n \). For the first two terms on the right hand side of (2.4) we will use that \( p_k(\sigma) \) is a probability and that \( \mathbb{P}(B_n^c) = O(n^{-2}) \), for \( \sigma > t_0 \), see Lemma 2.1, which yields

\[
\mathbb{E}[p_k(\sigma)] = \mathbb{E}[p_k(\sigma) | B_n] \mathbb{P}(B_n) + \mathbb{E}[p_k(\sigma) | B_n^c] \mathbb{P}(B_n^c) = \mathbb{E}[p_k(\sigma) | B_n] + O(n^{-2}).
\]  

(2.6)

Also, using that \( N_k(\sigma) \leq \sigma \),

\[
\mathbb{E}[N_k(\sigma) | B_n] = \frac{\mathbb{E}[N_k(\sigma)] - \mathbb{E}[N_k(\sigma) | B_n^c] \mathbb{P}(B_n^c)}{\mathbb{P}(B_n)} = \tilde{N}_k(\sigma) + O(\sigma n^{-2}).
\]  

(2.7)

For \( \sigma \) sufficiently large, using (1.4), (1.7), (2.1) and (2.7),

\[
\mathbb{E}[p_k(\sigma) | B_n] = \frac{k + \delta}{\alpha \Theta I_n \sigma} \mathbb{E}\left[ \sum_{v \in D_k(\sigma)} A_{\sigma,n}(x_v) \bigg| B_n \right] = \frac{k + \delta}{\alpha \Theta I_n \sigma} \mathbb{E}\left[ \sum_{v \in D_k(\sigma)} \int_S F_n(|x_v - u|) \, du \bigg| B_n \right] = \frac{(k + \delta)\mathbb{E}[N_k(\sigma) | B_n]}{\alpha \Theta I_n \sigma} = \frac{(k + \delta)\tilde{N}_k(\sigma)}{\alpha \Theta I_n \sigma} + O(kn^{-2}).
\]  

(2.8)

Combining (2.6) and (2.8), we obtain

\[
\mathbb{E}[p_k(\sigma)] = \mathbb{E}\left[ \sum_{v \in D_k(\sigma)} \frac{(k + \delta)A_{\sigma,n}(x_v)}{M_{\sigma,n}(x_{\sigma+1})} \right] = \frac{(k + \delta)\tilde{N}_k(\sigma)}{\alpha \Theta I_n \sigma} + O(kn^{-2}),
\]  

(2.9)

for \( \sigma \geq t_0 = t_0(n) = (\log n)^{2/(1-\gamma)} \). The above statement remains true when we replace \( k \) by \( k - 1 \).

For the third term on the right hand of (2.4), one can show that for \( \sigma \geq t_0 \), using that \( d_{\sigma+1}(x_{\sigma+1}) - m \) has a binomial distribution, see (2.2), thus

\[
\mathbb{P}(d_{\sigma+1}(x_{\sigma+1}) - m = k - m | B_n) = \binom{m}{k-m} \mathbb{E}[p^{k-m}(1-p)^{2m-k} | B_n] = \binom{m}{k-m} \left( 1 - \frac{\gamma}{\alpha} \right)^{k-m} \left( \frac{2}{\alpha} \right)^{2k-m} (1 + O(\sigma^{-1} \log n)),
\]

where we refer to [7, §3.1] for the derivation of the above result. It follows that

\[
\mathbb{P}(d_{\sigma+1}(x_{\sigma+1}) = k) = \mathbb{P}(d_{\sigma+1}(x_{\sigma+1}) - m = k - m | B_n) \mathbb{P}(B_n) + O(\mathbb{P}(B_n^c)) = \binom{m}{k-m} \left( 1 - \frac{\gamma}{\alpha} \right)^{k-m} \left( \frac{2}{\alpha} \right)^{2k-m} + O(\sigma^{-1} \log n),
\]  

(2.10)

where we refer to [7] for the derivation of the error term \( O(\sigma^{-1} \log n) \).

For the fourth and final term on the right hand side of (2.4), we use

\[
\eta_k(G_{\sigma}, x_{\sigma+1}) = O\left( \min\left\{ \sum_{i=m}^{k} \sum_{v \in D_i(\sigma)} \frac{(i + |\delta|)^2 A_{\sigma,n}(x_v)^2}{M_{\sigma,n}(x_{\sigma+1})^2}, 1 \right\} \right),
\]

which generalizes Equation (5) in [7]. Using similar arguments that led to (2.9), one can show for

\[
sigma > t_1 = t_1(n) = n^{(\gamma+\theta)/2\gamma} \quad \text{and} \quad k \leq k_0 = k_0(n) = n^{(\gamma-\theta)/4},
\]  

(2.11)
that

\[ \mathbb{E}[m \eta_{k}(G_{\sigma}, x_{\sigma+1})] = O \left( \frac{k^{2} n^{\delta}}{m} \right) = O \left( \sigma^{\gamma-1} \right). \] (2.12)

Substituting (2.9), (2.10) and (2.12) in (2.4), we end up with the following recurrence relation:

\[
\tilde{N}_{k}(\sigma + 1) = \tilde{N}_{k}(\sigma) - \frac{m}{\alpha \Theta} (k + \delta) \tilde{N}_{k}(\sigma)/\sigma + \frac{m}{\alpha \Theta} (k - 1 + \delta) \tilde{N}_{k-1}(\sigma)/\sigma \\
+ 1_{\{m \leq k \leq 2m\}} \left( \frac{m}{k - m} \right) (1 - 2\alpha^{-1})^{k - m} (2\alpha^{-1})^{2k - m} + O \left( \sigma^{\gamma-1} \log(n) \right),
\] (2.13)

for \( k \geq m \) and \( \tilde{N}_{m-1}(\sigma) = 0 \) for all \( \sigma \geq 0 \). The above recurrence relation depends on \( \sigma \) and \( k \). Consider the limiting case, i.e., \( \sigma \rightarrow \infty \), and assume that for each \( k \) the limit

\[ \tilde{N}_{k}(\sigma)/\sigma \rightarrow p_k \] (2.14)

exists. If this is indeed the case, then in the limit the recurrence relation (2.13) yields:

\[
p_k = \frac{m}{\alpha \Theta} (k - 1 + \delta) p_{k-1} - \frac{m}{\alpha \Theta} (k + \delta) p_k \\
+ 1_{\{m \leq k \leq 2m\}} \left( \frac{m}{k - m} \right) (1 - 2\alpha^{-1})^{k - m} (2\alpha^{-1})^{2k - m},
\]

where \( k \geq m \) and \( m_{k-1} = 0 \). By induction, we then obtain, for \( k > 2m \),

\[
p_k = \frac{m}{\alpha \Theta} (k - 1 + \delta) p_{k-1} = \frac{k - 1 + \delta}{k + \delta + \frac{m}{\alpha \Theta}} p_{k-1} = \frac{\Gamma(m + 1 + \delta + \frac{m}{\alpha \Theta}) \Gamma(k + \delta)}{\Gamma(m + \delta) \Gamma(k + 1 + \delta + \frac{m}{\alpha \Theta})} p_{2m}.
\]

Using that \( \Gamma(t + a)/\Gamma(t) \sim t^{a} \) for \( a \in [0, 1) \) and \( t \) large, we can rewrite the above equation as follows:

\[
p_k = \phi_k(m, \alpha, \delta) \left( \frac{m}{k} \right)^{1 + \frac{m}{\alpha \Theta}} = \phi_k(m, \alpha, \delta) \left( \frac{m}{k} \right)^{1 + \alpha(1 + \delta/2m)},
\]

where \( \phi_k(m, \alpha, \delta) = O(1) \) and tends to the limit \( \phi_{\infty}(m, \alpha, \delta) \) depending only on \( m, \alpha \) and \( \delta \) as \( k \rightarrow \infty \). Finally, following the proof in [7, from equation (15) up to the end of the proof], which shows that there exists a constant \( M \) independent from \( n \), such that

\[ |\tilde{N}_{k}(\sigma) - p_k| \leq M(n^{1-(\gamma-\theta)/4} + \sigma^{\gamma} \log n), \] (2.15)

for all \( 0 \leq \sigma \leq n \) and \( m \leq k \leq k_0(n) \). Thus, the assumption (2.14) is satisfied. By picking \( \gamma_1 > 0 \) sufficiently small, we can replace the right hand of (2.15) by \( n^{1-\gamma_1} \), and one obtains the claim (1.13).

3 Coupling

In this section we make preparations for the proofs of Lemma 2.1 and the concentration result in Theorem 1.4, see (1.14). In this section we take \( \tau \in \{1, \ldots, n\} \) fixed and we consider the graph process up to time \( \tau - 1 \) resulting in the graph \( G_{\tau-1} \). At time \( \tau \) we apply the Growth Rule, see §1.1, twice on \( G_{\tau-1} \), independently of each other, which results in the graphs \( G_{\tau} \) and \( \tilde{G}_{\tau} \). The idea is to compare the graphs \( G_{\tau} \) and \( \tilde{G}_{\tau} \) over time by considering \( G_{\sigma} \) and \( \tilde{G}_{\sigma} \) for \( \tau \leq \sigma \leq n \). To this end, we will introduce two urn processes. The urns consist of weighted and numbered balls. Instead of choosing a vertex-endpoint \( v \in V_{\sigma+1} \) at time \( \sigma+1 \) by (1.2) and (1.3), we will draw (with replacement) a ball proportional to its weight and then the vertex-endpoint is given by the number on the ball.

The coupling between the urns will be introduced in four steps. The first step is to introduce for any \( \sigma \geq \tau \) two urns. Secondly, we will introduce a probabilistic coupling between the two urn processes. Thirdly, we will describe the coupling between the graph processes \( G_{\sigma} \), \( G_{\sigma} \) and the two urn processes. Finally, we consider the vertex-ends \( v^{(i)}_{\sigma} \) and \( \tilde{v}^{(i)}_{\sigma} \), for \( i = 1, 2, \ldots, m \), in the graphs \( G_{\sigma} \) and \( \tilde{G}_{\sigma} \), respectively, and we will calculate the probability that \( v^{(i)}_{\sigma} \neq \tilde{v}^{(i)}_{\sigma} \).
3.1 The two urns

In this section we describe the contents of the urns corresponding to the graphs $G_\sigma$ and $\hat{G}_\sigma$, for $\sigma = \tau, \tau + 1, \ldots, n$, and we give an alternative way of choosing the vertex-endpoints using the urns.

Fix two graph processes $\{G_s\}$ and $\{\hat{G}_s\}$ such that the graphs up to time $\tau - 1$ are identical, i.e., $G_s = \hat{G}_s$ for $s = 0, 1, 2, \ldots, \tau - 1$, and that $x_s = \hat{x}_s$, for $s = \tau + 1, \tau + 2, \ldots, n$. Thus, the points $x_\tau$ and $\hat{x}_\tau$ will differ from each other, and, as a consequence, also, the edge sets $E_\sigma$ and $\hat{E}_\sigma$, for $s = \tau, \tau + 1, \ldots, \sigma$, will be different. Finally, we assume, without loss of generality, that $T_{\sigma,n}(x_{\sigma+1}) \leq \hat{T}_{\sigma,n}(x_{\sigma+1})$.

Next, we will describe the contents of the urns $U_\sigma$ and $\hat{U}_\sigma$ given the graphs $G_\sigma$ and $\hat{G}_\sigma$, and the newly added vertex $x_{\sigma+1}$. We will use the following abbreviations:

$$T_{\sigma,n} = T_{\sigma,n}(x_{\sigma+1})$$
$$M_{\sigma,n} = M_{\sigma,n}(x_{\sigma+1}).$$

(3.1)

Furthermore, if $e$ is an edge, then we denote by $\text{TO}(e)$ the endpoint of the edge. Thus, if edge $e$ is added at time $t$, emanating from the vertex $t$, points to a vertex $s \in V_t$ then $\text{TO}(e) = s$.

**Contents of the urns:**

- For each edge $e \in E_\sigma$, such that $\text{TO}(e) \neq \tau$, there is a white ball in $U_\sigma$ of weight $A_{\sigma,n}(x_{\text{TO}(e)})$ and numbered $\text{TO}(e)$. Similarly, for each edge in $e \in E_\sigma$, such that $\text{TO}(e) \neq \tau$, there is a white ball in $\hat{U}_\sigma$ of weight $A_{\sigma,n}(x_{\text{TO}(e)}) = A_{\sigma,n}(x_{\text{TO}(e)})$ and numbered $\text{TO}(e)$. Observe that $x_{\text{TO}(e)} = x_{\text{TO}(e)}$ since $\text{TO}(e) \neq \tau$.
- For each vertex $v \in V_\sigma \setminus \{\tau\}$ there is a red ball in each of the urns $U_\sigma$ and $\hat{U}_\sigma$ of weight $(m + \delta)A_{\sigma,n}(x_v)$ and numbered $v$.
- For the vertex $\tau$ there is in $U_\sigma$ a purple ball of weight $(d_\sigma(\tau) + \delta)A_{\sigma,n}(x_\tau)$ and number $\tau$, and in $\hat{U}_\sigma$ there is a red ball of weight $(\hat{d}_\sigma(\tau) + \delta)A_{\sigma,n}(\hat{x}_\tau)$ and numbered $\tau$.
- For the vertex $\sigma + 1$ each of the urns $U_\sigma$ and $\hat{U}_\sigma$ contain a green ball of weight $(\alpha \Theta I_\sigma \sigma - T_{\sigma,n})^+$, where $(\cdot)^+ = \max\{0, \cdot\}$, and numbered $\sigma + 1$. Furthermore, we add only to $U_\sigma$ a blue ball of weight $((\alpha \Theta I_\sigma \sigma - T_{\sigma,n})^+ - (\alpha \Theta I_\sigma \sigma - \hat{T}_{\sigma,n})^+)^+$ and numbered $\sigma + 1$.

**Remark 3.1** The total weight of the white and red balls in $U_\sigma$ are given by

$$\sum_{e \in E_\sigma} A_{\sigma,n}(x_{\text{TO}(e)}) 1_{\{\text{TO}(e) \neq \tau\}}$$
$$\sum_{v \in V_\sigma \setminus \{\tau\}} (m + \delta)A_{\sigma,n}(x_v),$$

respectively, and the weight of the purple ball in $U_\sigma$ can be rewritten as

$$(d_\sigma(\tau) + \delta)A_{\sigma,n}(x_\tau) = \sum_{e \in E_\sigma} A_{\sigma,n}(x_{\text{TO}(e)}) 1_{\{\text{TO}(e) = \tau\}} + (m + \delta)A_{\sigma,n}(x_\tau).$$

Therefore, the total weight of the white, red and purple balls in $U_\sigma$ is equal to:

$$\sum_{e \in E_\sigma} A_{\sigma,n}(x_{\text{TO}(e)}) + \sum_{v \in V_\sigma} (m + \delta)A_{\sigma,n}(x_v) = \sum_{v \in V_\sigma} (d_\sigma(v) + \delta)A_{\sigma,n}(x_v) = T_{\sigma,n}.$$

Furthermore, from (1.7), and some easy calculation, the total weight of all the balls in $U_\sigma$ is $M_{\sigma,n}$. Similarly, the total weight of the white, red and orange balls in the urn $\hat{U}_\sigma$ is $\hat{T}_{\sigma,n}$ and the total weight of all the balls in $\hat{U}_\sigma$ is, precisely $\hat{M}_{\sigma,n}$.

The weight of a ball depends on the time $\sigma$, the color of the ball and the number on the ball. Let $b$ be a ball in $U_\sigma$ or $\hat{U}_\sigma$, then we define the weight function $w_\sigma$ as

$$w_\sigma(b) = \begin{cases} A_{\sigma,n}(x_{\xi(b)}) & \text{if } b \text{ is white}, \\ (m + \delta)A_{\sigma,n}(x_{\xi(b)}) & \text{if } b \text{ is red}, \\ (d_\sigma(x_\tau) + \delta)A_{\sigma,n}(x_\tau) & \text{if } b \text{ is purple}, \\ (\hat{d}_\sigma(\hat{x}_\tau) + \delta)A_{\sigma,n}(\hat{x}_\tau) & \text{if } b \text{ is orange}, \\ (\alpha \Theta I_\sigma \sigma - T_{\sigma,n})^+ & \text{if } b \text{ is green}, \\ ((\alpha \Theta I_\sigma \sigma - T_{\sigma,n})^+ - (\alpha \Theta I_\sigma \sigma - \hat{T}_{\sigma,n})^+)^+ & \text{if } b \text{ is blue}, \end{cases}$$

(3.2)
where \( \xi(b) \) is the number on the ball. Observe that the number and the color together determine the weight of a ball.

We identify a set \( B \subset U_\sigma \) or \( B \subset \hat{U}_\sigma \) of distinct balls by the set of pairs \((c,k)\), where \( c \) denotes the color and \( k \) the number of the ball. For any set \( B \) of distinct balls, define

\[
\|B\|_\sigma = \sum_{b \in B} w_\sigma(b).
\]

We will draw the balls \( \{\hat{b}_\sigma^{(i)}\}_{i=1}^m \) with replacement from the urn \( U_\sigma \) proportional to the weight. Let \( \{\hat{b}_\sigma^{(i)}\}_{i=1}^m \) be the sequence of balls drawn from \( \hat{U}_\sigma \), then it is easy to show that

\[
P(\xi(\hat{b}_{\sigma+1}^{(i)}) = v \mid U_\sigma) = P(v_{\sigma+1}^{(i)} = v \mid G_\sigma, x_{\sigma+1})
\]

and

\[
P(\xi(\hat{b}_{\sigma+1}^{(i)}) = v \mid \hat{U}_\sigma) = P(v_{\sigma+1}^{(i)} = v \mid \hat{G}_\sigma, \hat{x}_{\sigma+1}),
\]

for \( v \in V_{\sigma+1} \). As an example we will show (3.3) for \( v \in V_{\sigma+1} \setminus \{b, \sigma + 1\} \). Observe that in this case the left hand side of (3.3) corresponds to the probability on the event that we draw the red ball numbered \( v \) or one of the \( d_\sigma(v) - m \) white balls, thus

\[
P(\xi(\hat{b}_{\sigma+1}^{(i)}) = v \mid U_\sigma) = \frac{(m + \delta)A_{\sigma,n}(x_v) + (d_\sigma(v) - m)A_{\sigma,n}(x_v)}{\|U_\sigma\|_\sigma}
\]

\[
= \frac{(d_\sigma(v) + \delta)A_{\sigma,n}(x_v)}{\|U_\sigma\|_\sigma} = P(v_{\sigma+1}^{(i)} = v \mid G_\sigma, x_{\sigma+1}),
\]

by (1.9), since \( \|U_\sigma\|_\sigma = M_{\sigma,n} \) (see Remark 3.1).

### 3.2 The joint distribution of drawing balls

In this section we describe how we simultaneously draw the balls from the urns \( U_\sigma \) and \( \hat{U}_\sigma \). As before, we will assume that \( T_{\sigma,n} \leq \hat{T}_{\sigma,n} \), or, equivalently, \( \|U_\sigma\|_\sigma \leq \|\hat{U}_\sigma\|_\sigma \), see Remark 3.1. In the last part of this section we calculate the probability of the event \( \{\hat{b}_\sigma^{(i)} \neq \hat{b}_\hat{\sigma}^{(i)}\} \), for \( i = 1, 2, \ldots, m \), and \( \tau \leq \sigma \leq n \), i.e., the event that the two balls \( \hat{b}_\sigma^{(i)} \) and \( \hat{b}_\hat{\sigma}^{(i)} \) in the \( i \)th draw do not agree on number or color, which we call a **mismatch**.

Define the following sets

\[
\begin{align*}
R_\sigma &= U_\sigma \setminus \hat{U}_\sigma, & C_\sigma &= U_\sigma \cap \hat{U}_\sigma \\
L_\sigma &= \hat{U}_\sigma \setminus U_\sigma,
\end{align*}
\]

where, as before, we compare the balls by color and number.

**Remark 3.2** By construction, \( L_\sigma \) only contains white and orange balls, \( C_\sigma \) contains only white, red and green balls, and \( R_\sigma \) contains only white, purple and blue balls. Furthermore, concerning the weights, we have the following relations

\[
\|C_\sigma\|_\sigma + \|R_\sigma\|_\sigma = \|U_\sigma\|_\sigma \quad \text{and} \quad \|C_\sigma\|_\sigma + \|L_\sigma\|_\sigma = \|\hat{U}_\sigma\|_\sigma.
\]

Next, we give the joint distribution of drawing balls from the urns \( U_\sigma \) and \( \hat{U}_\sigma \).

**The joint distribution:** Draw, with replacement, \( m \) balls \( \hat{b}_\sigma^{(i)}, \ldots, \hat{b}_\sigma^{(m)} \) from \( U_\sigma \). For convenience we write \( \hat{b}_\sigma^{(i)} = \hat{b}_{\sigma+1}^{(i)} \) for \( i = 1, \ldots, m \). For each \( i \), we define \( \hat{b}_\sigma^{(i)} = \hat{b}_{\sigma+1}^{(i)} \) by

- If \( \hat{b}_\sigma^{(i)} \in C_\sigma \) then, with probability

  \[
  \begin{align*}
  P(U_\sigma) &= \frac{w_\sigma(b)}{\|U_\sigma\|_\sigma} \quad \text{and} \quad P(\hat{U}_\sigma) = \frac{w_\sigma(b)}{\|\hat{U}_\sigma\|_\sigma}.
  \end{align*}
  \]

  

- If \( \hat{b}_\sigma^{(i)} \in R_\sigma \), then we choose \( \hat{b}_\sigma^{(i)} \) from \( L_\sigma \), i.e., we choose \( b \in L_\sigma \) with probability \( w(b) / \|L_\sigma\|_\sigma \); observe that the quotient in (3.5) is bounded by 1, because, as remarked earlier, \( \|U_\sigma\|_\sigma \leq \|\hat{U}_\sigma\|_\sigma \).

**The marginal distributions:** Denote by \( \hat{P}(\cdot) \) the joint probability measure under the above introduced coupling. Furthermore, let \( \hat{P}_\sigma(\cdot) = \hat{P}(\cdot \mid U_\sigma, \hat{U}_\sigma) \). We will show that under the coupling

\[
\hat{P}_\sigma(b^{(i)} = b) = \frac{w_\sigma(b)}{\|U_\sigma\|_\sigma} = P(b^{(i)} = b \mid U_\sigma)
\]
and
\[ \tilde{\mathbb{P}}_\sigma(\hat{b}^{(i)} = b) = \frac{w_\sigma(b)}{\|U_\sigma\|_\sigma} = \mathbb{P}(\hat{b}^{(i)} = b | \hat{U}_\sigma), \]  
(3.7)

for \( b \in U_\sigma \) and \( b \in \hat{U}_\sigma \), respectively. The claim (3.6) is true by construction. For the claim (3.7), if \( b \in \hat{U}_\sigma \), then this implies that \( b \in C_{\sigma} \) or \( b \in L_\sigma \), but not in both. Firstly, assume \( b \in C_{\sigma} \), then
\[ \tilde{\mathbb{P}}_\sigma(\hat{b}^{(i)} = b) = \tilde{\mathbb{P}}_\sigma(\hat{b}^{(i)} = b | b^{(i)} = b) \tilde{\mathbb{P}}_\sigma(b^{(i)} = b) = \frac{\|U_\sigma\|_\sigma w_\sigma(b)}{\|U_\sigma\|_\sigma} = \frac{w_\sigma(b)}{\|U_\sigma\|_\sigma}. \]

Secondly, if \( b \in L_\sigma \), then
\[ \tilde{\mathbb{P}}_\sigma(\hat{b}^{(i)} = b) = \tilde{\mathbb{P}}_\sigma(\hat{b}^{(i)} = b | b^{(i)} \in C_{\sigma}) \tilde{\mathbb{P}}_\sigma(b^{(i)} \in C_{\sigma}) \]
\begin{align*}
&+ \tilde{\mathbb{P}}_\sigma(\hat{b}^{(i)} = b | b^{(i)} \in R_{\sigma}) \tilde{\mathbb{P}}_\sigma(b^{(i)} \in R_{\sigma}) \\
&= \frac{w_\sigma(b)}{\|L_\sigma\|_\sigma} \left( 1 - \frac{\|U_\sigma\|_\sigma}{\|U_\sigma\|_\sigma} \right) \frac{\|C_{\sigma}\|_\sigma + \frac{w_\sigma(b)}{\|U_\sigma\|_\sigma}}{\|C_{\sigma}\|_\sigma + \frac{\|R_{\sigma}\|_\sigma}{\|U_\sigma\|_\sigma}} - \frac{\|C_{\sigma}\|_\sigma}{\|U_\sigma\|_\sigma} \\
&= \frac{w_\sigma(b)}{\|L_\sigma\|_\sigma} \frac{\|C_{\sigma}\|_\sigma}{\|U_\sigma\|_\sigma} \frac{\|U_\sigma\|_\sigma}{\|U_\sigma\|_\sigma} \\
&= \frac{w_\sigma(b)}{\|U_\sigma\|_\sigma},
\end{align*}
where we used in the last step the relations given by (3.4). Hence, also, the claim (3.7) is true.

### 3.3 The joint growth rule between coupled graphs

Fix \( \tau \in \{1,2,\ldots,n\} \), as before, and consider the graph process \( \{G_s\}_{s=0}^{\tau-1} \). Let \( \{\hat{G}_s\}_{s=0}^{\tau-1} \) be an identical copy of \( \{G_s\}_{s=0}^{\tau-1} \), and choose at time \( \tau \) the position \( x_\tau \) and \( \hat{x}_\tau \) in \( G_\tau \) and \( \hat{G}_\tau \), respectively, at random in \( S \), independently of each other. Using the urns, we will describe the growth of the graphs \( G_\sigma \) and \( \hat{G}_\sigma \) over time.

At time \( \tau \) we apply the Growth Rule, independently, on the graphs \( G_{\tau-1} \) and \( \hat{G}_{\tau-1} \). Then at time \( \sigma+1 \), for \( \sigma \geq \tau \), let \( x_{\sigma+1} \) randomly chosen from \( S \) and set \( \hat{x}_{\sigma+1} = x_{\sigma+1} \). Let \( U_\sigma \) and \( \hat{U}_\sigma \) the urns correspond to \( (G_\sigma, x_{\sigma+1}) \) and \( (\hat{G}_\sigma, \hat{x}_{\sigma+1}) \), respectively. Note that this is precisely the setting as described in §3.1 and, as a consequence, we can use the results of §3.2. Draw with replacement \( m \) balls, \( \{b^{(i)}_{\sigma+1}\}_{i=1}^{m} \), from \( U_\sigma \), then the vertex-endpoints of \( x_{\sigma+1} \) are given \( \{\xi(b^{(i)}_{\sigma+1})\}_{i=1}^{m} \). We, also, draw with replacement \( m \) balls, \( \{\hat{b}^{(i)}_{\sigma+1}\}_{i=1}^{m} \), from \( \hat{U}_\sigma \), and construct \( G_{\sigma+1} \) in the same way.

### 3.4 The probability on a mismatch

The event of a mismatch of vertex-endpoints in the graphs \( G_\sigma \) and \( \hat{G}_\sigma \), \( \sigma \geq \tau \), can be expressed in terms of drawing balls from the urns \( U_\sigma \) and \( \hat{U}_\sigma \), since
\[ \{\hat{v}_{\sigma+1}^{(i)} \neq \hat{v}_{\sigma+1}^{(i)}\} = \{\xi(b_{\sigma+1}^{(i)}) \neq \xi(\hat{b}_{\sigma+1}^{(i)})\} \subset \{b_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}\}. \]

(3.8)

Thus, we will concentrate on the probability of a mismatch between the drawn balls from the urns. Without loss of generality, we assumed that \( \|U_\sigma\|_\sigma \leq \|\hat{U}_\sigma\|_\sigma \) or, equivalently, \( T_{\sigma,n} \leq \hat{T}_{\sigma,n} \). Using the joint distribution of the urns, see Section 3.2, and (3.4), we obtain
\begin{align*}
\tilde{\mathbb{P}}_\sigma(\hat{b}_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}) &= 1 - \sum_{b \in C_{\sigma}} \tilde{\mathbb{P}}_\sigma(\hat{v}_{\sigma+1}^{(i)} = b | b_{\sigma+1}^{(i)} = b) \tilde{\mathbb{P}}_\sigma(b_{\sigma+1}^{(i)} = b) \\
&= 1 - \sum_{b \in C_{\sigma}} \frac{\|U_\sigma\|_\sigma}{\|U_\sigma\|_\sigma} \frac{w(b)}{\|U_\sigma\|_\sigma} = 1 - \frac{\|C_{\sigma}\|_\sigma}{\|U_\sigma\|_\sigma} \frac{\|U_\sigma\|_\sigma}{\|U_\sigma\|_\sigma} = \frac{\|L_\sigma\|_\sigma}{\|U_\sigma\|_\sigma}. \tag{3.9}
\end{align*}

By (1.7) and Remark 3.1, we can bound the denominator on the right hand side of (3.9) from below by
\[ \|\hat{U}_\sigma\|_\sigma \geq \|U_\sigma\|_\sigma = M_{\sigma,n} \geq \alpha \Theta I_\sigma. \]

Next, we consider the numerator on the right hand side of (3.9). The set \( L_\sigma \) only contains white balls and the orange ball, see Remark 3.2. Therefore, compare (3.2), the total weight of \( L_\sigma \) can be written as
\[ \|L_\sigma\|_\sigma = \sum_{h \in E_\sigma} A_{\sigma,n}(x_h) + (\hat{d}_\sigma(\tau) + \delta) A_{\sigma,n}(\hat{x}_\tau), \]

12
Thus, the probability on a mismatch between balls is bounded from above by 1.6 and 1.8, can be proved almost immediately using the proofs in §7, but this is not true for Theorem 1.4. In this section we will prove the main results, i.e. Theorem 1.4, 1.6 and 1.8. The diameter results, Theorem 4 Proof of the main results

where

(\Theta/m) \leq 2 \frac{E[\Delta^\sigma(U)]}{\alpha \Theta I_n \sigma} \leq 2 I_n \frac{E[\Delta^\sigma(U)]}{\alpha \Theta I_n \sigma}

(4.3)

Lemma 4.1 Under the conditions of Theorem 1.4, let \sigma \geq 1 and U randomly chosen in S, then for some constant C > 0,

E[\Delta^\sigma(U)] \leq C m I_n \left( \frac{\sigma}{\sigma - 1} \right)^{\frac{1}{\sigma - 1}} \log \sigma,

and, as a consequence,

E[\Delta^\sigma(U)] \leq C m I_n \left( \frac{\sigma}{\sigma - 1} \right)^{\frac{1}{\sigma - 1}},

since (\Theta/m)^{-1} < 2.
The proof of the above lemma is deferred to §4.3.

**Remark 4.2** For the proof of the main result, we need that the number of mismatches is of $o(\sigma)$, which implies that the exponent in (4.4) should be smaller than 1, i.e., $m/\alpha \Theta < 1$. For $\alpha > 2$ and $\delta > -m$ this is indeed the case:

$$\frac{m}{\Theta} \leq \frac{2m}{m + (m + \delta)} < \frac{2m}{m} \leq 2,$$

thus $m/\alpha \Theta < 1$.

If $\delta = 0$, which is precisely the model introduced in [7], then the condition simplifies to $1/\alpha < 1$, which is a weaker condition than the condition used in [7]: $2/\alpha < 1$. Nevertheless, we cannot get rid of the condition $\alpha > 2$, because we need that the event $B_\sigma$ occurs with high probability, see (2.5).

### 4.1 Proof of Lemma 2.1

In this section we will prove Lemma 2.1 using the Azuma-Hoeffding inequality, which provides exponential bounds for the tails of a special class of martingales:

**Lemma 4.3** Let $\{X_\tau\}_{\tau \geq 0}$ be a martingale process with the property that, with probability 1, there exists a sequence of positive constants $\{e_\tau\}_{\tau \geq 1}$ such that

$$|X_\tau - X_{\tau - 1}| \leq e_\tau,$$

for all $\tau \geq 1$. Then, for every $\lambda > 0$,

$$\mathbb{P}(|X_0 - X_0| \geq \lambda) \leq 2 \exp \left\{ -\frac{\lambda^2}{4 \sum_{\tau=1}^{\infty} e_\tau^2} \right\}.$$

For a proof of this lemma, we refer to [8].

We will apply Lemma 4.3 by taking a Doob-type martingale $X_\tau = \mathbb{E}[T_{\sigma,n}(U) | G_\tau]$, where $U$ is chosen at random in $S$. By convention, we let $G_0$ be the empty graph, then

$$X_0 = \mathbb{E}[T_{\sigma,n}(U)] \quad \text{and} \quad X_\sigma = T_{\sigma,n}(U).$$

At each time step $s$ we add a new vertex and $m$ edges, see the **Growth Rule** in Section 1.1, call this an action. We call an action $A$ acceptable if the action can be applied with positive probability. Furthermore, denote by $\mathcal{A}(G)$ the set of all acceptable actions that can be applied on the graph $G$.

Clearly,

$$\left| \mathbb{E}[T_{\sigma,n}(U)|G_{\tau - 1}] - \mathbb{E}[T_{\sigma,n}(U)|G_\tau] \right| \leq \sup_{G_{\tau - 1}} \sup_{A \in \mathcal{A}(G_{\tau - 1})} \left| \mathbb{E}[T_{\sigma,n}(U)|G_{\tau - 1}(A)] - \mathbb{E}[T_{\sigma,n}(U)|G_{\tau - 1}(\hat{A})] \right|,$$

where the first supremum is taken over all possible graphs $G_{\tau - 1}$. Next, fix the graph $G_{\tau - 1}$ and let $G_{\tau} = G_{\tau - 1}(A)$ be the graph by applying the action $A$ on the graph $G_{\tau - 1}$. Similarly, define $\hat{G}_{\tau} = G_{\tau - 1}(\hat{A})$. Thus, one can rewrite the right hand side of (4.5) as

$$e_\tau = \sup_{G_{\tau - 1}} \sup_{A \in \mathcal{A}(G_{\tau - 1})} \left| \mathbb{E}[T_{\sigma,n}(U)|G_{\tau}] - \mathbb{E}[\hat{T}_{\sigma,n}(U)|\hat{G}_{\tau}] \right|.$$

Using the triangle inequality, the above implies, under the coupling,

$$e_\tau \leq \sup_{G_{\tau - 1}} \sup_{A \in \mathcal{A}(G_{\tau - 1})} \mathbb{E}\left[ |T_{\sigma,n}(U) - \hat{T}_{\sigma,n}(U)| \right].$$

(4.6)

We claim that, independently of $G_{\tau - 1}, A$ and $\hat{A}$, and, for $\sigma \geq \tau$,

$$\mathbb{E}\left[ |T_{\sigma,n}(U) - \hat{T}_{\sigma,n}(U)| \right] \leq \tilde{C} m I_0(m/\tau)^{2/\alpha},$$

(4.7)
where the proof of this claim is deferred to the end of this section. Thus, using (4.6) and (4.7),
\[
\sum_{\tau=1}^{\sigma} e^2 \leq \tilde{C}^2 m^2 T_n^2 \sigma^{4/\alpha} \sum_{\tau=1}^{\sigma} \tau^{-4/\alpha} = O \left( m^2 T_n^2 (\sigma^{4/\alpha} + \sigma \log \sigma) \right).
\]

To show the above, let \( \beta = 4/\alpha \), then \( \beta \in (0, 2) \). If \( \beta \in (1, 2) \), then \( \sum_{\tau=1}^{\sigma} \tau^{-\beta} < \infty \), and, if \( \beta \in (0, 1] \), then
\[
\sigma^\beta \sum_{\tau=1}^{\sigma} \tau^{-\beta} \leq \sigma^\beta + \sigma \int_{1}^{\sigma} x^{-1} \, dx = \sigma^\beta + \sigma \log \sigma.
\]

From Lemma 4.3 we then obtain for some constant \( C_1 \),
\[
P \left( |T_{\sigma,n}(U) - \mathbb{E}[T_{\sigma,n}(U)]| \geq C_1 m I_n (b^{2/\alpha} + \sigma^{1/2} \log \sigma)(\log n)^{1/2} \right) \leq 2e^{-2 \log n}.
\]

By taking \( n \) sufficiently large, we can replace \( C_1 (\log n)^{1/2} \) by \( \log n \), which is, precisely, the statement of Lemma 2.1, given the claim (4.7).

**Proof of claim (4.7):** Denote by \( d_\sigma(v) \) the in-degree of the vertex \( v \) at time \( \sigma \), and observe that
\[
d_\sigma(v) = d_\sigma(v) + m, \quad (4.8)
\]
since each vertex has by construction \( m \) edges pointing outward to other vertices or itself. Furthermore, we denote by \( y^{(i)}_s \) the position of the \( i \)th vertex-endpoint at time \( s \), thus
\[
y^{(i)}_s = x^{(i)}_{v^{(i)}_s}. \quad (4.9)
\]

By construction of \( G_\tau \) and \( \hat{G}_\tau \), we can apply the coupling introduced in §3.3. Rewrite \( T_{\sigma,n}(U) \), see (1.1), using (4.8), (4.9), and the coupling, as
\[
T_{\sigma,n}(U) = \sum_{v=1}^{\sigma} d_\sigma(v) F_n(|x_v - U|) + \delta \sum_{v=1}^{\sigma} F_n(|x_v - U|)
\]
\[
= \sum_{v=1}^{\sigma} d_\sigma^- (v) F_n(|x_v - U|) + (m + \delta) \sum_{v=1}^{\sigma} F_n(|x_v - U|)
\]
\[
= \sum_{s=1}^{\sigma} \sum_{i=1}^{m} F_n(|y^{(i)}_s - U|) + (m + \delta) \sum_{v=1}^{\sigma} F_n(|x_v - U|)
\]
\[
= \sum_{s=1}^{\sigma} \sum_{i=1}^{m} F_n(|x_{\xi^{(i)}_s} - U|) + (m + \delta) \sum_{v=1}^{\sigma} F_n(|x_v - U|).
\]

Up to and including time \( \tau - 1 \) both graphs are identical, thus the absolute difference
\[
\left| T_{\sigma,n}(U) - \hat{T}_{\sigma,n}(U) \right|
\]
equals to
\[
\left| \sum_{s=1}^{\sigma} \sum_{i=1}^{m} \left( F_n(|x_{\xi^{(i)}_s} - U|) - F_n(|x_{\xi^{(i)}_s} - U|) \right) + (m + \delta) \left( F_n(|x_\tau - U|) - F_n(|\hat{x}_\tau - U|) \right) \right|.
\]

Using the triangle inequality and (4.2), we obtain
\[
\left| T_{\sigma,n}(U) - \hat{T}_{\sigma,n}(U) \right| \leq \Delta_\sigma(U) + (m + \delta) \left( F_n(|x_\tau - U|) + F_n(|\hat{x}_\tau - U|) \right).
\]

Taking expectations on both sides of the above display, and using (1.6) and (4.3), yields
\[
\hat{E} \left[ |T_{\sigma,n}(U) - \hat{T}_{\sigma,n}(U)| \right] = 2I_n (\hat{E} \Delta_\sigma) + (m + \delta). \quad (4.10)
\]

Applying Lemma 4.1 on (4.10) finally results in
\[
\hat{E} \left[ |T_{\sigma,n}(U) - \hat{T}_{\sigma,n}(U)| \right] \leq \tilde{C} m I_n (\sigma/\tau)^{2/\alpha},
\]
for some constant \( \tilde{C} \). This is precisely the claim (4.7).

\[ \square \]
4.2 Proof of the main results

In this section we show the main results. The proof of Theorem 1.4 is almost similar to the proof of Lemma 2.1. The diameter results, i.e., Theorem 1.6 and 1.8 will be proved by using the proofs in [7].

Proof of Theorem 1.4: The first part of Theorem 1.4, i.e., claim (1.13), has been proved in Section 2. For the second part, i.e., claim (1.14), we now give a proof, which is similar to the proof of Lemma 2.1. Therefore, we follow the proof of the previous Section 4.1, where we now choose $X = \mathbb{E}[N_k(n) | G_\tau]$ instead of $X = \mathbb{E}[T_{\sigma,n}(U) | G_\tau]$. Similar to (4.5), we have that

$$\left| \mathbb{E}[N_k(n) | G_{\tau-1}] - \mathbb{E}[N_k(n) | G_\tau] \right|$$

$$\leq \sup_{A \in \mathcal{A}(G_{\tau-1})} \sup_{\tilde{A} \in \mathcal{A}(G_{\tau-1})} \left| \mathbb{E}[N_k(n) | G_{\tau-1}(A)] - \mathbb{E}[N_k(n) | G_{\tau-1}(\tilde{A})] \right| .$$

Using the coupling, we can bound the right hand side in the above display by twice the number of mismatches, since each mismatch can influence at most two edges. Thus,

$$\left| \mathbb{E}[N_k(n) | G_{\tau-1}] - \mathbb{E}[N_k(n) | G_\tau] \right| \leq 2\mathbb{E}[\Delta^*_c] .$$

Therefore, we can take $c_\tau = 2\mathbb{E}[\Delta^*_c]$ and we, again, can apply Lemma 4.3, as done in the previous section, which proves claim (1.14) and hence Theorem 1.4.

Proof of Theorem 1.6: The proof is almost identical to the proof of [7, Theorem 2]. To apply this proof for general $\delta > -m$, we only need to replace the constant $c_3$ in [7] by $c_3^*$, where $c_3^* = c_3/2$ and $c_3$ is the constant of condition S3, see Section 1.2. This will be explained in more detail now.

Pick $\mu$ and $\rho_n = \rho_n(\mu, F_n)$ such that $F_n$ is smooth for $\mu$, see conditions S1, S2 and S3, see Section 1.2. Fix $u \in S$ and denote by $A_{\rho_n}$ the spherical cap with center $u$ and radius $\rho_n$, then there exists positive constants $c_1$ and $c_2$, independent of $\rho_n$, such that

$$A_{\rho_n} = \int_{\{w \in S : \|w - u\| \leq \rho_n\}} dw \in [c_1 \rho_n^2, c_2 \rho_n^2], \quad (4.11)$$

which is shown in [7]. Furthermore, in [7] the authors consider the graph at certain time steps $t_s$, where $s$ is a positive integer, such that the area of the spherical cap is given by

$$s/2 \leq A_{\rho_n, t_s} \leq 3s/2. \quad (4.12)$$

In the proof of [7, Theorem 2], the essential step is the statement that the probability that $v_{t_s}$ chooses vertex $v \in V_{t_s}$, assuming that $|x_{t_s} - x_v| \leq 2\rho_n$, is at least $\frac{2c_1c_3}{\Theta s}$, i.e.,

$$\mathbb{P}(v_{t_s} = v | G_{t_s-1}) \geq \frac{2c_1c_3}{\Theta s} .$$

In our model this is still true, when we replace $c_3$ by $c_3^* = c_3/2$, since, using the assumptions S1, S2 and S3, (4.11) and (4.12),

$$\mathbb{P}(v_{t_s} = v | G_{t_s-1}) \geq \frac{(d_{t_s}(v) + \delta)F_n(|x_{t_s} - x_v|)}{\alpha \Theta I_{t_s}} \geq \frac{(m + \delta)F_n(2\rho_n)}{\alpha \Theta I_{t_s}}$$

$$\geq \frac{2(m + \delta)A_{\rho_n}F_n(2\rho_n)}{\alpha \Theta I_{t_s}} \geq \frac{2(m + \delta)c_3^2 \rho_n^2 F_n(2\rho_n)}{\alpha \Theta I_{t_s}}$$

$$\geq \frac{2(m + \delta)c_1 c_3}{\alpha \Theta} \geq \frac{(m + \delta)2c_1 c_3}{\Theta} \geq \frac{2c_1 c_3}{\alpha \Theta} \geq \frac{2c_1 c_3}{\Theta} \geq \frac{2c_1 c_3^*}{\alpha \Theta} ,$$

where we used that $(m + \delta)/\Theta = 1 + \delta/(2m + \delta) > 1/2$ for $-m < \delta \leq 0$ and $(m + \delta)/\Theta \geq 1 > 1/2$ for $\delta > 0$. If we replace the constant $c_3$ by $c_3^*$ in the proof of Theorem 2, then the proof of [7] holds without further modifications.

Proof of Theorem 1.8: For $\delta = 0$ the proof is given by the proof of Theorem 3 in [7]. The constant $\lambda = C_1/C_2$ in the proof of [7, Theorem 3] should be replaced by $\lambda = (C_1 + \delta)/2C_2$, then the proof holds verbatim.
4.3 Bounding the expected number of mismatches

In this section we will prove Lemma 4.1. In the proof of the lemma, we rely on two claims, which will be stated now. The first claim bounds for any vertex and all time steps the expected degree:

\[
\hat{E}[d_v(\sigma) + \delta] \leq mC \left( \frac{\sigma}{\tau} \right)^a,
\]

(4.13)

where \(C\) is some constant and

\[
a = m/\alpha \Theta.
\]

(4.14)

The second claim is a technical one, which bounds the expectation of

\[
Q_\sigma = \sum_{h \in E_\sigma} A_{\sigma,n}(x_h) 1\{\tau_{\sigma,n} \leq \hat{\tau}_{\sigma,n} \} + \sum_{h \in E_\sigma} A_{\sigma,n}(\hat{x}_h) 1\{\tau_{\sigma,n} > \hat{\tau}_{\sigma,n} \}
\]

(4.15)

from above. More precisely, for any \(\sigma \geq \tau\),

\[
\hat{E}[Q_\sigma] \leq I_n \left( \hat{E}[\Delta_\sigma] + \hat{E}[d_\sigma(\tau) + \delta] + \hat{E}\left[\hat{d}_\sigma(\tau) + \delta\right] + 2\Theta \right).
\]

(4.16)

Next, we will assume that the claims (4.13) and (4.16) do hold and we will show that Lemma 4.1 follows from these two claims. After the proof of Lemma 4.1, we will prove both claims separately.

**Proof of Lemma 4.1:** Let \(\tau < \sigma \leq t\), then the number of mismatches is recursively defined as

\[
\Delta_{\sigma+1} = \Delta_{\sigma} + \sum_{i=1}^{m} 1\{\hat{b}_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}\},
\]

and, therefore,

\[
\hat{E}[\Delta_{\sigma+1}] = \hat{E}[\Delta_{\sigma}] + m \hat{P}(\hat{b}_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}),
\]

(4.17)

since we draw the balls with replacement.

Combining (3.11) and (3.12), yields

\[
\hat{P}\left(\hat{b}_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}\right) \leq \frac{Q_\sigma + (d_\sigma(\tau) + \delta)A_{\sigma,n}(x_\tau) + (\hat{d}_\sigma(\tau) + \delta)A_{\sigma,n}(\hat{x}_\tau)}{\alpha \Theta I_n \sigma}.
\]

(4.18)

Observe from (1.4) that

\[
\hat{E}[d_\sigma(\tau)A_{\sigma,n}(x_\tau) \mid d_\sigma(\tau), x_\tau] = d_\sigma(\tau)\hat{E}[F(|x_\tau - x_{\sigma+1}|) \mid d_\sigma(\tau), x_\tau]
\]

\[
= d_\sigma(\tau) \int S F(|x_\tau - u|) du = d_\sigma(\tau)I_n,
\]

and, hence

\[
\hat{E}[(d_\sigma(\tau) + \delta)A_{\sigma,n}(x_\tau)] = \hat{E}[d_\sigma(\tau) + \delta] I_n.
\]

Similarly, \(\hat{E}\left[\hat{d}_\sigma(\tau) + \delta\right] A_{\sigma,n}(\hat{x}_\tau) = \hat{E}[-d_\sigma(\tau) + \delta] I_n\). Thus, taking expectations on both sides of (4.18), and using (4.16), results in

\[
\hat{P}(\hat{b}_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}) \leq \frac{\hat{E}[Q_\sigma] + (\hat{E}[d_\sigma(\tau) + \delta] I_n) + (\hat{E}[\hat{d}_\sigma(\tau) + \delta] I_n)}{\alpha \Theta I_n \sigma}.
\]

(4.19)

Substituting (4.13) and (4.16) in (4.19), yields

\[
\hat{P}(\hat{b}_{\sigma+1}^{(i)} \neq \hat{b}_{\sigma+1}^{(i)}) \leq \frac{\hat{E}[\Delta_\sigma] + (2\Theta + 4mC(\sigma/\tau)^a)}{\alpha \Theta \sigma}.
\]

for some sufficiently large constant \(C > 0\). Therefore, we can bound the right hand side of equation (4.17) by,

\[
\hat{E}[\Delta_{\sigma+1}] \leq \hat{E}[\Delta_\sigma] \left(1 + \frac{2\Theta + 4mC(\sigma/\tau)^a}{\alpha \Theta \sigma}\right) = \hat{E}[\Delta_\sigma] \left(1 + \frac{\sigma}{\sigma} + 1 + 4C(\sigma/\tau)^a\right).
\]

17
where we used that \( a = m/\alpha \Theta ~ (4.14), 1/\alpha \leq 1/2 \) and \( m/\Theta \leq 2 \). Finally, by taking the constant \( C \) larger, we can replace the above inequality by
\[
\mathbb{E}[\Delta_{\sigma+1}] \leq \mathbb{E}[\Delta_\sigma] \left(1 + \frac{a}{\sigma}\right) + C\tau^{-n}\sigma^{a-1}.
\]

We will now prove an upper bound for \( \mathbb{E}[\Delta_{\sigma+1}] \). To this end, we consider the solution of the recurrence relation \( q(\sigma + 1) = q(\sigma)(1 + a/\sigma) + b(\sigma) \), for \( \sigma > \tau \), with initial condition \( q(\tau) = c \). The solution is given by
\[
q(\sigma) = \frac{\Gamma(\sigma + a)}{\Gamma(\sigma)} \sum_{s=\tau}^{\sigma-1} b(s) \Gamma(s + 1) \Gamma(s + 1 + a) + c \frac{\Gamma(\sigma + a)}{\Gamma(\sigma)} \Gamma(\tau + a).
\]
The above solution implies, that if one takes \( b(s) = C\tau^{-n}s^{a-1} \leq C\tau^{-n}\Gamma(s+1) \), then
\[
\mathbb{E}[\Delta_\sigma] \leq q(\sigma) = \frac{\Gamma(\sigma + a)}{\Gamma(\sigma)} \sum_{s=\tau}^{\sigma-1} b(s) \Gamma(s + 1) \Gamma(s + 1 + a) \leq C\tau^{-n}\frac{\Gamma(\sigma + a)}{\Gamma(\sigma)} \sum_{s=\tau}^{\sigma} \Gamma(s + a).
\]
The summation can be bounded from above by \( \log \sigma \), therefore, for some constant \( C \),
\[
\mathbb{E}[\Delta_\sigma] \leq C \left(\frac{\sigma}{\tau}\right)^a \log \sigma.
\]
This proves (4.4) and hence Lemma 4.1, given the claims (4.13) and (4.16). \( \square \)

**Proof of (4.13):** Note that, see (1.7) and (1.9),
\[
\mathbb{E}_\sigma[d_{\sigma+1}(v) + \delta] = (d_\sigma(v) + \delta) + \mathbb{E}_\sigma[d_{\sigma+1}(v) - d_\sigma(v)]
\]
\[
= (d_\sigma(v) + \delta) \left(1 + m \frac{A_{\sigma,n}(x_v)}{M_{\sigma,n}(x_{\sigma+1})}\right) \leq (d_\sigma(v) + \delta) \left(1 + m \frac{A_{\sigma,n}(x_v)}{\alpha \Theta I_n \sigma}\right).
\]

Therefore, by taking expectations on both sides in the above display, and using (1.6), the value of \( \mathbb{E}[d_{\sigma+1}(v) + \delta] \) is bound from above by
\[
\mathbb{E} \left[ (d_\sigma(v) + \delta) \left(1 + m \frac{E[A_{\sigma,n}(x_v) | G_\sigma]}{\alpha \Theta I_n \sigma} \right) \right] = \left(1 + \frac{m}{\alpha \Theta \sigma}\right) \mathbb{E}[d_\sigma(v) + \delta] \cdot \mathbb{E} \left[ (d_\sigma(v) + \delta) \right].
\]

Thus, by induction, and using \( a = m/\alpha \Theta \),
\[
\mathbb{E}[d_{\sigma+1}(v) + \delta] = \left(\frac{\sigma + a}{\sigma}\right) \mathbb{E}[d_\sigma(v) + \delta] \geq \frac{\Gamma(\sigma + a + 1)}{\Gamma(\sigma + 1)} \mathbb{E}[d_\sigma(v) + \delta].
\]

Finally, note that \( d_\sigma(v) \leq 2m \) and that \( \delta \) is a constant, which implies the claim (4.13). \( \square \)

**Proof of (4.16):** Observe that for \( i = 1, \ldots, m \) and \( s \geq \tau \), see (4.9),
\[
1 \{v^{(i)}(t) \neq \hat{v}_s^{(i)}\} + 1 \{v^{(i)}(t) = \hat{v}_s^{(i)}\} = 1 \{v^{(i)}(t) \neq \hat{v}_s^{(i)}\} = 1 \{x^{(i)}(t) \neq \hat{v}_s^{(i)}\},
\]
(4.20)
since, \( x_i = \hat{v}_s \) if \( i \neq \tau \) and \( x_\tau \neq \hat{v}_s \). The above statement is stronger than (3.8). Using the definition of \( \mathcal{E}_\sigma \), see (3.10), and (4.20), we have that
\[
\sum_{h \in \mathcal{E}_\sigma} A_{\sigma,n}(x_h) \leq \sum_{s=\tau}^\sigma \sum_{i=1}^m A_{\sigma,n}(y_s^{(i)}) 1 \{y_s^{(i)} \neq \hat{v}_s^{(i)}\},
\]
and we can bound \( \sum_{h \in \mathcal{E}_\sigma} A_{\sigma,n}(x_h) \) in a similar way. Hence, we can bound (4.15) from above by
\[
Q_\sigma \leq \sum_{s=\tau}^\sigma \sum_{i=1}^m 1 \{y_s^{(i)} \neq \hat{v}_s^{(i)}\} \left(A_{\sigma,n}(y_s^{(i)}) 1 \{T_{s,n} \leq \hat{T}_{s,n}\} + A_{\sigma,n}(\hat{y}_s^{(i)}) 1 \{T_{s,n} > \hat{T}_{s,n}\}\right)
\]
\[
= \sum_{s=\tau}^\sigma \sum_{i=1}^m 1 \{y_s^{(i)} \neq \hat{v}_s^{(i)}\} A_{\sigma,n}(\hat{y}_s^{(i)})
\]
\[
+ 1 \{T_{s,n} \leq \hat{T}_{s,n}\} \sum_{s=\tau}^\sigma \sum_{i=1}^m 1 \{y_s^{(i)} \neq \hat{v}_s^{(i)}\} \left(A_{\sigma,n}(y_s^{(i)}) - A_{\sigma,n}(\hat{y}_s^{(i)})\right).
\]
(4.21)
Next, we will show that the rightmost double sum of (4.21) can be bounded by
\[(m + \delta)(A_{\sigma,n}(\hat{x}_\tau) - A_{\sigma,n}(x_\tau)).\]

For this, we rewrite \(T_{\sigma,n}\) as
\[
T_{\sigma,n} = \sum_{s=1}^{\sigma} \left( d_s(\sigma) + \delta \right) A_{\sigma,n}(x_s) = \sum_{s=1}^{\sigma} d_s(\sigma) A_{\sigma,n}(x_s) + (m + \delta) \sum_{s=1}^{\sigma} A_{\sigma,n}(x_s).
\]

Note that
\[
\sum_{s=1}^{\sigma} d_s(\sigma) A_{\sigma,n}(x_s) = \sum_{s=1}^{\sigma} \left( \sum_{i=s}^{m} \sum_{i=1}^{t} 1\{v_t(i) = s\} \right) A_{\sigma,n}(x_s)
\]
\[
= \sum_{i=1}^{\sigma} \left( \sum_{t=i}^{m} \sum_{s=1}^{t} 1\{v_t(i) = s\} A_{\sigma,n}(x_s) \right) = \sum_{i=1}^{\sigma} \sum_{s=1}^{t} A_{\sigma,n}(y_t(i)).
\]

Therefore,
\[
T_{\sigma,n} = \sum_{s=1}^{\sigma} \sum_{i=1}^{m} A_{\sigma,n}(y_s(i)) + (m + \delta) \sum_{s=1}^{\sigma} A_{\sigma,n}(x_s)
\]
and a similar result hold for \(\hat{T}_{\sigma,n}\). The difference of these two expressions equals:
\[
T_{\sigma,n} - \hat{T}_{\sigma,n} = \sum_{s=1}^{\sigma} \sum_{i=1}^{m} \left( A_{\sigma,n}(y_s(i)) - A_{\sigma,n}(\hat{y}_s(i)) \right) + (m + \delta)(A_{\sigma,n}(x_\tau) - A_{\sigma,n}(\hat{x}_\tau)),
\]
or
\[
\sum_{s=1}^{\sigma} \sum_{i=1}^{m} \left( A_{\sigma,n}(y_s(i)) - A_{\sigma,n}(\hat{y}_s(i)) \right) = T_{\sigma,n} - \hat{T}_{\sigma,n} - (m + \delta)(A_{\sigma,n}(x_\tau) - A_{\sigma,n}(\hat{x}_\tau)).
\]

Hence,
\[
1\{T_{\sigma,n} \leq \hat{T}_{\sigma,n}\} \sum_{s=1}^{\sigma} \sum_{i=1}^{m} \left( A_{\sigma,n}(y_s(i)) + A_{\sigma,n}(\hat{y}_s(i)) \right) \leq (m + \delta)(A_{\sigma,n}(x_\tau) - A_{\sigma,n}(\hat{x}_\tau)). \tag{4.22}
\]

Substituting (4.22) in (4.21) yields,
\[
Q_\sigma \leq \sum_{s=\tau}^{\sigma} \sum_{i=1}^{m} 1\{y_s(i) \neq \hat{y}_s(i)\} A_{\sigma,n}(\hat{y}_s(i)) + (m + \delta)(A_{\sigma,n}(x_\tau) + A_{\sigma,n}(\hat{x}_\tau)).
\]

Taking the conditional expectation with respect to the graphs \(G_\sigma\) and \(\hat{G}_\sigma\) results in
\[
\hat{E}\left[ Q_\sigma \mid G_\sigma, \hat{G}_\sigma \right] = \sum_{s=\tau}^{\sigma} \sum_{i=1}^{m} \left( \hat{E}\left[ A_{\sigma,n}(y_s(i)) \mid G_\sigma, \hat{G}_\sigma \right] + (m + \delta) \left( \hat{E}\left[ A_{\sigma,n}(x_\tau) \mid G_\sigma, \hat{G}_\sigma \right] + \hat{E}\left[ A_{\sigma,n}(\hat{x}_\tau) \mid G_\sigma, \hat{G}_\sigma \right] \right) \right). \tag{4.23}
\]

For any fixed value \(x \in S\) and using (1.4), we have that
\[
\hat{E}\left[ A_{\sigma,n}(x) \mid G_\sigma, \hat{G}_\sigma \right] = \hat{E}\left[ F(|x - x_{\sigma+1}|) \mid G_\sigma, \hat{G}_\sigma \right] = I_n,
\]
which, in turns, yields that (4.23) can be rewritten as
\[
\hat{E}\left[ Q_\sigma \mid G_\sigma, \hat{G}_\sigma \right] = I_n \left( \sum_{s=\tau}^{\sigma} \sum_{i=1}^{m} 1\{y_s(i) \neq \hat{y}_s(i)\} + 2(m + \delta) \right).
\]

Note that (4.20) implies
\[
\sum_{s=\tau}^{\sigma} \sum_{i=1}^{m} 1\{y_s(i) \neq \hat{y}_s(i)\} = \sum_{s=\tau}^{\sigma} \sum_{i=1}^{m} 1\{v_t(i) \neq \hat{v}_t(i)\} + \sum_{s=\tau}^{\sigma} \sum_{i=1}^{m} 1\{v_t(i) = \hat{v}_t(i) = \tau\} \leq \Delta_\sigma + d_\sigma(\tau).
\]
Thus, 

\[
\mathbb{E}[Q_\sigma | G_\sigma, \hat{G}_\sigma] \leq I_n \left( \Delta_\sigma + d_\sigma(\tau) + 2(m + \delta) \right) = I_n \left( \Delta_\sigma + (d_\sigma(\tau) + \delta) + (2m + \delta) \right).
\]

For convenience, we will use the following weaker statement:

\[
\mathbb{E}[Q_\sigma | G_\sigma, \hat{G}_\sigma] \leq I_n(\Delta_\sigma + (d_\sigma(\tau) + \delta) + (\hat{d}_\sigma(\tau) + \delta) + 2\Theta),
\]

where we replaced \((2m + \delta)\) by \(2\Theta\), see (1.8). Finally, by taking the expectation on both sides in the above display, we obtain the claim (4.16).

\[\square\]

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