Sparse Density Estimation with Measurement Errors

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Abstract

This paper aims to build an estimate of an unknown density of the data with measurement error as a linear combination of functions of a dictionary. Inspired by penalization approach, we propose the weighted Elastic-net penalized minimal $L_2$-distance method for sparse coefficients estimation, where the weights adaptively coming from sharp concentration inequalities. The optimal weighted tuning parameters are obtained by the first-order conditions holding with high-probability. Under local coherence or minimal eigenvalue assumptions, non-asymptotical oracle inequalities are derived. These theoretical results are transposed to obtain the support recovery with high-probability. Then, the issue of calibrating these procedures is studied by some numerical experiments for discrete and continuous distributions, it shows the significant improvement obtained by our procedure when compared with other conventional approaches. Finally, the application is performed for a meteorology data set. It shows that our method has potency and superiority of detecting the shape of multi-mode density compared with other conventional approaches.

Keywords: density estimation, sparsity, measurement error, oracle inequalities, support recovery, multi-mode data.

AMS subject classification: 62G07, 62H12, 62J07

1 Introduction

Over the years, the mixture models have been extensively applied to model unknown distributional shapes. The distributions of real data often shows multi-mode and heterogeneous which involve potential complex variables. The application are frequently used in astronomy, biology, economics, and genomics see McLachlan et al. (2019) and references therein. Due to the flexibility, it also often appear in various distribution-based statistical techniques, such as cluster analysis, discriminant analysis, survival analysis, empirical Bayesian inference. Flexible mixture models can naturally represent how the data are generated as mathematical artifacts. There are theoretical results which show that the mixture can approximate any density in the Euclidean space well. The mixture can be finite (for example, a mixture of several normal distributions). Although the mixture model is inherently attractive to the statistical modelling, it is a well-known difficult to inference, see Balakrishnan et al. (2017). From the computational point of view, the optimization problems of mixture models are non-convex. Existing computational
methods, such as EM and various MCMC algorithms, are capable of making the mixture model fit the data relatively easily. It should be emphasized that the mixture problems are essentially difficult to be unrecognizable, the number of components (says, the order selection) is hard to determine, see Chen and Khalili (1999). There is a large amount of literature on its approximation theory and various methods have been proposed to estimate the components, see DasGupta (2008) and references therein.

Nonparametric and combinatorial density estimation method were studied in Devroye and Lugosi (2001), Biau and Devroye (2005), Meister (2009). These can be used to consistently estimate the amount of the components of the mixture when the components have a known functional form. When the number of candidate components is large, the non-parametric method becomes computationally infeasible. Fortunately, the advance of high-dimensional inference would compensate for this gap and guarantee the correct identification of the mixture components with a probability attending to 1. With the advancement of technology, high-dimensional problems have been being applied to the forefront of statistical researches, and high-dimensional inference method has been applied to the infinite mixture models with a sparse mixture of $p \to \infty$ components, which is an interesting and changeling problem, see Bunea et al. (2010) and Bertin et al. (2011). However, difficult computing challenges have emerged in high-dimensional data, since it needs fast and flexible inference procedures. Calculations are quite cumbersome in high-dimensional problems (especially when $p$ is increasing in $n$ and $p > n$), and the inference of the mixing distribution is still a challenging problem, see Meister (2009). In the error-in-variables density estimation problem, it is hard to find an orthogonal dictionary.

In the real data, we often encounter the situation that the i.i.d. samples $X_i = Z_i + \varepsilon_i$ are contained by some zero-mean measurement error $\{\varepsilon_i\}_{i=1}^n$, see Hall et al. (2008), Meister (2006), Cheng and van Ness (1999). For density estimation of $\{X_i\}_{i=1}^n$, in the non-measurement setting, the SPADES estimator in Bunea et al. (2010) is considered by taking advantage of an orthogonal basis of functions. However, in the measurement-error setting, finding an orthogonal based density function is not easy, see Schennach and Bonhomme (2013). To address the error-variables density estimation, particularly with nonorthogonal base functions, the SPADES method is attractive and makes it easy to deal with. Schennach and Bonhomme (2013) suggests the assumption that the conditional distribution function of $X_i$ given $Z_i$ is known. This condition is somewhat strong since the most conditional distribution is hard to get the explicit formula (except the normal distribution). The strength of SPADES is that the addressing of nonorthogonal base-functions is particularly appealing in work. Our method is an Elastic-net calibration approach which is simpler and more interpretable than the conditional inference procedure proposed by Schennach and Bonhomme (2013). In this paper, we proposed the corrected loss function to debase the measurement error, this is motivated by Nakamura (1990). Due to measurement errors, The main contribution is that we only assume the fluctuation of the based density functions. Note that the solution path of the SPADES estimator may not be a probability density, we have to rescale the estimator such that the obtained estimator is valid density. We derive the honest variable selection consistency based on weighted $\ell_1 + \ell_2$ penalty, while the SPADES only has the equal weights setting which is not plausible in sense of adaptive (data-dependent) penalized estimation. Moreover, we perform the Poisson mixture model to approximate the complex discrete distribution in the simulation part, while existing papers only emphasize on continuous distribution models. Note that the multivariate kernel density estimator can only deal with continuous distribution and it requires a multivariate bandwidths section, while our method is dimensional free (the required tuning parameters are two).

This paper is presented as follows. Section 2 introduces the density estimator with measurement error. We introduce a novel data-dependent weights for Lasso penalty, and the weights are derived by the event of KKT conditions such that the KKT conditions hold with high proba-
bility. In Section 3, we give a condition that can accurately estimate the weight of the mixture, with a probability tending to 1. We show that, in an increasing dimensional mixture model under local coherence assumption, if the tuning parameter is higher than the noise level, the recovery of the mixture component holds with high probability. In Section 4, we also study the performance of our approach on some simulated mixed normal (or Poisson) distributions compared with other conventional methods, which indeed shows the improvement by employing our procedure. Besides, the simulation also demonstrates that our method is better than the traditional EM algorithm even under a low dimensional model. Considering multi-modal density in the meteorology dataset, our proposed estimator has a stronger ability in detecting multiple modes for the underlying distribution, comparing with other methods such as $\ell_1$-penalized estimators or un-weighted Elastic-net estimator. Section 5 is the summary and the proof of theoretical results is given in Section 6.

2 Density Estimation

2.1 Mixture models

Suppose that \( \{Z_i\}_{i=1}^n \) are independent random variables with a common unknown density \( h \in \mathbb{R}^d \). However, the observations are contaminated with measurement errors \( \{\varepsilon_i\}_{i=1}^n \) as latent variables, the observed data are \( X_i = Z_i + \varepsilon_i \). Let \( \{h_j\}_{j=1}^W \) be a series of density functions (such as Gaussian, Poisson), and \( \{h_j\}_{j=1}^W \) are also called basis functions. Assume that the estimator of \( h \) belongs to the linear combination of \( \{h_j\}_{j=1}^W \). The \( Z \sim h_\beta(z) := \sum_{j=1}^W \beta_j h_j(z), \) with \( \sum_{j=1}^W \beta_j = 1 \). (1)

If the base is orthogonal and there are no measurement error, a perfectly natural method is to estimate \( h \) by an orthogonal series of estimators in the form of \( h_\tilde{\beta} \), where \( \tilde{\beta} \) has the coordinates \( \tilde{\beta}_j = \frac{1}{n} \sum_{i=1}^n h_j(X_i) \). However, this estimator depends on the choice of \( W \), and a data-driven selection of \( W \) or threshold needs to be adaptive. Since these models have always been applied to \( W \leq n \), we want to solve more general problems, for instance, the base functions \( \{h_j\}_{j=1}^W \) are not necessarily orthogonal. Here the \( W \) is not necessarily less than \( n \), but the form is the estimator \( h_\beta \) and we can still achieve the best convergence.

Theorem 33.2 in DasGupta (2008) states that any smooth density can be well approximated by the finite mixture is a continuous function. However, Theorem 33.2 in DasGupta (2008) does not make sure that how many components \( W \) are required in the mixture. Thus the hypothesis of the increasing-dimensional \( W \) is reasonable. For discrete distributions, there is also a similar mixture density approximation, see Remark of Theorem 33.2 in DasGupta (2008).

The density of the observed data is surely the linear combination of a series of new based density functions \( \{\tilde{h}_j\}_{j=1}^W \) if we still use the original true parameter \( \beta^* \).

\[
X \sim g(x) := g_{\beta^*}(x) = \sum_{j=1}^W \beta_j^* \tilde{h}_j(x). \tag{2}
\]

Note that the \( \{\tilde{h}_j\}_{j=1}^W \) are not the true mixture density since we wrongly use the contaminated data to fit the unobserved data. The direct estimation of coefficients based on \( \{h_j\}_{j=1}^W \) in equation (1) is imprecise.
2.2 Estimation for sparse density with measurement errors

The aim of this subsection is to construct a sparse estimator for the density \( h(z) := h_{\beta^*}(z) \) as a linear combination of mixture densities.

Recall the definition of the \( L_2(\mathbb{R}^d) \) norm \( \|f\| = (\int_{\mathbb{R}^d} f^2(x)dx)^{\frac{1}{2}} \). For \( f, g \in L_2(\mathbb{R}^d) \), let the inner product be \( \langle f, g \rangle = \int_{\mathbb{R}^d} f(x)g(x)dx \). Note that if the density \( h(z) \) belongs to \( L_2(\mathbb{R}^d) \) and assume that \( \{X_i\}_{i=1}^n \) has the same distribution \( X \), for any \( f \in L_2 \), we have \( \langle f, h \rangle = \int_{\mathbb{R}^d} f(x)h(x)dx = Ef(X) \). If \( h(x) \) is density function for discrete distribution, the integral is changed by summation for defining the inner product \( \langle f, h \rangle := \sum_{k \in \mathbb{Z}^d} f(k)h(k) \).

Let us minimize the \( \|h_\beta - h\|^2 \) on \( \beta \in \mathbb{R}^W \) to obtain the estimate of \( h(z) := h_{\beta^*}(z) \), i.e. minimizing

\[
\|h_\beta - h\|^2 = \|h\|^2 + \|h_\beta\|^2 - 2 \langle h, h_\beta \rangle = \|h\|^2 + \|h_\beta\|^2 - 2Eh_\beta(X) \propto -2Eh_\beta(Z) + \|h_\beta\|^2. \tag{3}
\]

The (3) implies that minimizing the \( \|h_\beta - h\|^2 \) for true observations \( \{Z_i\}_{i=1}^n \) is equivalent to minimizing

\[
-2Eh_\beta(Z) + \|h_\beta\|^2 \approx -\frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \|h_\beta\|^2. \tag{4}
\]

For the convenience analysis of the measurement error, suppose that the base functions \( \{\hat{h}_j(\cdot)\}_{j=1}^W \) in (2) and the i.i.d. observations \( \{X_i\}_{i=1}^n \) have following decomposition:

\[
g_\beta(X_1) = \sum_{j=1}^W \beta_j \hat{h}_j(X_1) \approx \sum_{j=1}^W \beta_j [h_j + e_j](X_1)
\]

Here \( \{e_j(x)\}_{j=1}^W \) are technically assumed to be some orthogonal error functions and the given base function \( \{\hat{h}_j\}_{j=1}^W \) of interest is orthogonal with the error function \( \{e_j(x)\}_{j=1}^W \). Moreover, we assume that the error functions as the perturbation functions have the zero empirical average evaluated at the observed data \( \frac{1}{n} \sum_{i=1}^n e_j(Z_i) \approx Ee_j(Z) = 0 \) for all \( j \). This assumption means that \( \{e_j(x)\}_{j=1}^W \) is an instrumental function for dealing with the misspecified base function \( \{\hat{h}_j\}_{j=1}^W \) in (2). Here, we mimic the idea of instrumental variables in econometrics, and it is supposed that there always exists such functions. For \( g_\beta(X_1) \) we have the following approximation:

\[
-2Eg_\beta(Z) = -2E \sum_{j=1}^W \beta_j [h_j + e_j](Z) = -2Eh_\beta(Z) \approx -\frac{2}{n} \sum_{i=1}^n h_\beta(X_i).
\]

For the observations \( \{X_i\}_{i=1}^n \) with measurement errors \( \{e_i\}_{i=1}^n \), minimizing the \( \|g_\beta - g\|^2 \) is equivalent to minimize \( -2Eg_\beta(X) + \|g_\beta\|^2 \). More specifically, we approximate \( -2Eg_\beta(X) + \|g_\beta\|^2 \) by the argument in below

\[
-2Eg_\beta(Z) + \|g_\beta\|^2 \approx -\frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \sum_{1 \leq i, j \leq W} \beta_i \beta_j \int_{\mathbb{R}^d} [h_i(z) + e_i(z)][h_j(z) + e_j(z)]dz
\]

\[
= -\frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \sum_{1 \leq i, j \leq W} \beta_i \beta_j \int_{\mathbb{R}^d} h_i(z)h_j(z)dz + \sum_{1 \leq i \leq W} \beta_i^2 \int_{\mathbb{R}^d} e_i^2(z)dz
\]

\[
\approx -\frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \|h_\beta\|^2 + c \sum_{1 \leq i \leq W} \beta_i^2.
\]
where the equality stems from the orthogonality assumptions of \( \{e_j(x)\}_{j=1}^W \).

It is plausible to assign more constrains for the candidate set of \( \beta \) in the optimization, for example, the \( \ell_1 \) constrains \( \|\beta\|_1 \leq a \) where \( a \) is the tuning parameter. More adaptively, we prefer to use the weighted \( \ell_1 \) restriction \( \sum_{j=1}^W \omega_j |\beta_j| \leq c \). The weights \( \omega_j \)'s are data-dependent that will be specified later. From the discussion above, now we propose the following Corrected Sparse Density Estimator (CSDE)

\[
\hat{\beta} := \hat{\beta}(\omega_1, \cdots, \omega_W) = \arg \min_{\beta \in \mathbb{R}^W} \left\{ -\frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \|h_\beta\|^2 + 2 \sum_{j=1}^W \omega_j |\beta_j| + c \sum_{j=1}^W \beta_j^2 \right\}
\]  
(5)

where \( c \) is the tuning parameter for \( \ell_2 \)-penalty.

For CSDE, if \( \{h_j\}_{j=1}^W \) orthogonal system, it can be clearly seen that the CSDE estimator is consistent with the soft threshold estimator, and the explicit solution is \( \hat{\beta}_j = \frac{1}{n} \sum_{i=1}^n h_j(X_i) \) and \( x_+ = \max(0, x) \). In this case, we can see that \( \omega_j \) is the threshold of the \( j \)th component of the simple mean estimator \( \hat{\beta} = (\hat{\beta}_1, \cdots, \hat{\beta}_W) \).

From sub-differential of the convex optimization, the corresponding Karush-Kuhn-Tucker conditions (necessary and sufficient first order condition) for minimizer \( (5) \) is

\textbf{Lemma 1.} (KKT conditions in short, see page68 of Buhlmann and van de Geer (2011)) Let \( k \in \{1, 2, \cdots, W\} \) and \( c > 0 \). Then, for CSDE defined in (5), we have

1. \( \hat{\beta}_k \neq 0 \) iff \( \frac{1}{n} \sum_{i=1}^n h_k(X_i) - \sum_{j=1}^W \hat{\beta}_j < h_j, h_k > -c\hat{\beta}_k = w_k \text{sign}(\hat{\beta}_k) \).

2. \( \hat{\beta}_k = 0 \) iff \( \left| \frac{1}{n} \sum_{i=1}^n h_k(X_i) - \sum_{j=1}^W \hat{\beta}_j < h_j, h_k > -c\hat{\beta}_k \right| \leq w_k \).

Since all \( \beta_j^* \) are non-negative, when doing minimization in equation (5), we have to put a non-negative restriction for optimizing (5).

Notwithstanding, various penalties in literature, we prefer to adapt the weight lasso penalty as a convex adaptive \( \ell_1 \) penalization due to computational feasibility and optimal first-order conditions. We require that the larger weights are assigned to the coefficients of unimportant covariates, while the smaller weights are accompanied by important covariates. So the weights represent the importance of the covariates. The larger (smaller) weights shrink to zero more easily (difficultly) than the un-weighted lasso, with appropriate or even optimal weights, it may lead to less bias and more efficient variable selection. The derivation of the weight will be given in Section 2.3.

### 2.3 Data-dependent weights

The weights \( \omega_j \)'s are chosen adequately such that the KKT conditions for stochastic optimization problems have a high probability to be satisfied.

As mentioned before, the weights in (5) rely on the observed data since we calculate the weights which make sure the KKT conditions hold with high probability. The weights lead to weighted Lasso estimates which could have less \( \ell_1 \) estimation error comparing with Lasso estimates, see also the simulation part. Next, the question we need to consider is that what kind of configurations of data-dependent weights can enable the KKT conditions to have a high probability to be satisfied. The fundamental way to get data-dependent weights is to apply a concentration inequality for a weighted sum of independent r.v. Moreover, the weights should be a known function of data without any unknown parameters. There is a criterion that can help to obtain the weight grounded on Bernstein’s concentration inequality in SPADES.
Whereas, the convergence rate of the probability upper bounds of the summation of $n$ independent random variables deviated from its expected value for Bernstein’s concentration inequality is $\exp \left( -\frac{c_1 t^2}{n+c_0 t} \right)$. Contrasting to the Bernstein’s concentration inequality, the McDiarmid’s inequality (also known as the bounded difference inequality which is used for obtaining the desired weights) has a faster convergence rate $\exp \left( -\frac{c_1 t^2}{n} \right)$ in $t$.

**Lemma 2.** Suppose $X_1, \ldots, X_n$ are independent random variables, all values belong to $A$. Let $f : A^n \to \mathbb{R}$ be a function and satisfy the bounded difference conditions

$$\sup_{x_1, \ldots, x_n, x'_n \in A} |f(x_1, \ldots, x_n) - g(x_1, \ldots, x_{s-1}, x'_s, x_{s+1}, \ldots, x_n)| \leq C_s,$$

then for all $t > 0$,

$$P \{|f(X_1, \ldots, X_n) - Ef(X_1, \ldots, X_n)| \geq t\} \leq 2 \exp \left\{-\frac{2t^2}{\sum_{s=1}^n C_s^2} \right\}.$$

We define the KKT conditions of optimization evaluated at $\beta^*$ (it is from the sub-gradient of the optimization function evaluated at $\beta^*$) by the events below:

$$\mathcal{F}_k(\omega_k) := \left\{ \frac{1}{n} \sum_{i=1}^n h_k(X_i) - \sum_{j=1}^W \beta^*_j < h_j, h_k > -c \beta^*_k \leq \omega_k \right\}, k = 1, 2, \ldots, W,$$

where $E h_k(X_i) = \sum_{j=1}^W \beta_j < h_j, h_k >$ (which is free of $X_i$).

Also, we assume that $L_k = \|h_k\|_{\infty} = \max_{1 \leq s \leq n} |h_k(X_i)|$, hence we could check that the following event is verified by bounded difference conditions,

$$\frac{1}{n} \left| \sum_{i=1}^n h_k(X_i) - \sum_{j=1}^W \beta^*_j < h_j, h_k > - \left( \sum_{i \neq s} h_k(X_i) - \sum_{j \neq s} \beta^*_j < h_j, h_k > + h_k(X'_s) - Eh_k(X'_s) \right) \right|$$

$$= \frac{1}{n} \left| h_k(X_s) - h_k(X'_s) + Eh_k(X'_s) - Eh_k(X_s) \right|$$

$$\leq \frac{1}{n} \left( |h_k(X_s) - h_k(X'_s)| + |E h_k(X'_s) - Eh_k(X_s)| \right) \leq \frac{4L_k}{n}.$$

The last inequality above is due to $|h_k(X_i) - Eh_k(X_i)| \leq 2L_k$.

Next, we apply the McDiarmid’s inequality to event $\mathcal{F}_k(\omega_k)$ on the restrict set $\max_{1 \leq j \leq W} |\beta^*_j| \leq B$. Then

$$P(\mathcal{F}_k(\omega_k)) = P \left\{ \left| \frac{1}{n} \sum_{i=1}^n h_k(X_i) - \sum_{j=1}^W \beta^*_j < h_j, h_k > -c \beta^*_k \right| \geq \omega_k \right\}$$

$$\leq P \left\{ \left| \frac{1}{n} \sum_{i=1}^n h_k(X_i) - Eh_k(X_i) \right| + c \beta^*_k \geq \omega_k \right\}$$

$$(0 < \max_{1 \leq j \leq W} |\beta^*_j| \leq B) \leq P \left\{ \left| \frac{1}{n} \sum_{i=1}^n h_k(X_i) - Eh_k(X_i) \right| \geq \omega_k - cB \right\}$$

$$(\text{assume } \omega_k = \omega_k - cB > 0) \leq 2 \exp \left\{ -\frac{2\omega^2_k}{16L_k^2/n} \right\}$$

$$= 2 \exp \left\{ -\frac{n\omega^2_k}{8L_k^2} \right\} =: \delta, \quad 0 < \delta < 1.$$
Solve the last inequality of the above formula,

$$\omega_k := 2\sqrt{2L_k}\sqrt{\frac{1}{n}\log \frac{2W}{\delta}} + cB =: 2\sqrt{2L_k}v(\delta/2) + cB,$$

where \( v = v(\delta) := \sqrt{\frac{1}{n}\log \frac{W}{\delta}} \).

The weight \( \omega_k \) in our paper is different from Bunea et al. (2010) that gives the un-shift version \( (\omega_k^B = 4L_k\sqrt{\frac{1}{n}\log \frac{W}{\delta/2}}) \), due to the Elastic-net penalty. And we define a modified version of event of KKT condition

$$K_k(\omega_k) := \left\{ \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - \sum_{j=1}^{W} \beta_j^* < h_j, h_k > \right\} \leq \tilde{\omega}_k, \quad k = 1, 2, \ldots, W$$

holds with the probability at least \( 1 - 2\exp\left\{ -\frac{n\tilde{\omega}_k^2}{8L_k^2} \right\} \).

### 2.4 We cannot transform the mixture models to linear models!

In this part, we will illustrate that even in the mixture models without measurement error \( [1] \) can’t be partially transformed into the linear model, namely

$$Y = X^T \beta + \varepsilon.$$  

where \( Y \) is the \( n \)-dimensional response variables, \( X \) is the \( W \times n \)-dimensional fixed design matrix, \( \beta \) is a \( W \)-dimensional vector of model parameters, the \( \varepsilon \) is a \( n \times 1 \)-dimensional vector for random error terms with zero mean and finite variance. Consider the least square objective function \( U(\beta) \) for estimating \( \beta \)

$$U(\beta) = (Y - X^T \beta)^T(Y - X^T \beta) = -2Y^T X^T \beta + \beta^T XX^T \beta + Y^T Y.$$  

Minimizing (8) is equivalent to minimizing \( U^*(\beta) \) in the following formula (9)

$$U^*(\beta) = -2Y^T X^T \beta + \beta^T XX^T \beta.$$  

Comparing the objective function (9) with (4), it is easy to obtain

$$Y = \left( \begin{array}{cccc} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{array} \right)^T, \quad \beta = (\beta_1, \beta_2, \cdots, \beta_W)^T, \quad X = \left( \begin{array}{ccc} h_1(X_1) & \cdots & h_1(X_n) \\ \vdots & \ddots & \vdots \\ h_W(X_1) & \cdots & h_W(X_n) \end{array} \right).$$

Substituting \( Y, X, \) and \( \beta \) into a linear regression model, we have

$$\left( \begin{array}{c} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{array} \right)_{n \times 1} = \left( \begin{array}{ccc} h_1(X_1) & \cdots & h_W(X_1) \\ \vdots & \ddots & \vdots \\ h_1(X_n) & \cdots & h_W(X_n) \end{array} \right)_{n \times W} \left( \begin{array}{c} \beta_1 \\ \vdots \\ \beta_W \end{array} \right)_{W \times 1} + \left( \begin{array}{c} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{array} \right)_{n \times 1}.$$  

Then,

$$\varepsilon_i = \frac{1}{n} - \sum_{j=1}^{W} \beta_j h_j(X_i), \quad i = 1, 2, \cdots, n.$$  

(10)
It can be seen from equation (10) that when $n \to \infty$, the value of $\varepsilon_i$ is no longer random since $X$ is a fixed design matrix. Furthermore, if $X$ is a random design, take the expectation on both sides of (10), and we can find that the left side is not equal to the right side, that is,

$$E(\varepsilon_i) = 0 \neq \frac{1}{n} = \frac{1}{n} - \sum_{j=1}^{W} \beta_j Eh_j(X_i).$$

It leads to additional requirement $\sum_{j=1}^{W} \beta_j Eh_j(X_i) = \frac{1}{n} \to 0$ which is meaningless as $n \to \infty$, since all $\beta_j$ and $h_j$ are positive, this is a contradiction to $\sum_{j=1}^{W} \beta_j Eh_j(X_i) > 0$ for all $n$.

Both of the two situations above contradict the definition of the assumed linear regression model and hence we can’t convert the estimation of (1) into the estimation problem of linear models. Thus the existing oracle inequalities are not applicable anymore, we will propose the oracle inequalities in the next section. However, we are able to transform the mixture models to corrected score Dantzig selector such as Belloni et al. (2017). Although Bertin et al. (2011) studies the oracle inequalities for adaptive the Dantzig density estimation, their study does not contain the error-in-variables framework and the support recovery consistency.

3 Sparse Mixture Density Estimation

In this section, we will present the oracle inequalities for estimators $\hat{\beta}$ and $h_\beta$. The core of this section consists of 5 main results, corresponding to the oracle inequalities for estimated density (Theorems 1 and 2, respectively), upper bounds on $\ell_1$-estimation error (Corollaries 1 and 2, respectively), and support consistency (Theorem 3) as the byproduct of Corollary 2.

3.1 Non-asymptotic oracle inequalities

The oracle inequality connects the performance of an obtained estimator with the true parameter which is not available in practice, see Candes (2008) for more discussions. Introduced by Donoho and Johnstone (1994), oracle inequality is a powerful non-asymptotical and analytical tool that seeks to provide the distance from the obtained estimator and a true estimator.

For $\forall \beta \in \mathbb{R}^W$, let

$$I(\beta) = \{ j \in \{1, \cdots, W\} : \beta_j \neq 0 \}$$

be the indices corresponding to the non-zero component of the vector $\beta$, i.e. the support in mathematical language. Sometimes, if there is no ambiguity, we write $I(\beta^*)$ as $I_\ast$. And

$$W(\beta) = \sum_{j=1}^{W} I(\beta_j \neq 0)$$

is the number of its non-zero components, where $I(\cdot)$ represents an indicative function. Let

$$\sigma_j^2 = \text{Var}(h_j(X_1)), \quad 1 \leq j \leq W.$$

Below, we will state the non-asymptotic oracle inequalities for $h_\beta$ (with high probability $1 - \delta(W, n)$ for any integer $W$ and $n$) which measures the $L_2$ distance between $h_\beta$ and $h$. For $\beta \in \mathbb{R}^W$, define the correlation for the two base densities: $h_i$ and $h_j$,

$$\rho_W(i, j) = \frac{\langle h_i, h_j \rangle}{\|h_i\|\|h_j\|}, \quad i, j = 1, \cdots, W.$$
We consider the definition of cumulative local coherence given by Bunea et al. (2010):

$$\rho_s(\beta) = \sum_{i \in I(\beta)} \sum_{j > i} |\rho_W(i, j)|.$$  

Define

$$H(\beta) = \max_{j \in I(\beta)} \frac{\omega_j}{v(\delta/2)||h_j||}, \quad F = \max_{1 \leq j \leq W} \frac{v(\delta/2)||h_j||}{\omega_j} = \max_{1 \leq j \leq W} \frac{||h_j||}{2\sqrt{2L_j}},$$

where $v(\delta) := \sqrt{\frac{1}{n} \log \frac{W}{\delta}}$ and $\bar{\omega}_j := 2\sqrt{2L_j}v(\delta/2)$.

Under the regularity condition for $\rho_s(\beta)$ and the notations above, we present the key result of this paper which lays the foundation for the oracle inequality of the estimated mixed coefficients.

**Theorem 1.** For all $||\beta||_\infty \leq B$, the true base functions $\{h_j\}_{j=1}^W$ satisfies cumulative local coherence assumption

$$12FH(\beta)\rho_s(\beta) \sqrt{W(\beta)} \leq \gamma, \quad (11)$$

and suppose $\max_{1 \leq j \leq W} L_j < \infty$ and $c = \frac{\min_{1 \leq j < W} \bar{\omega}_j}{B}$, then we have the following oracle inequality,

$$\|h_{\hat{\beta}} - h\|^2 + \frac{\alpha_{opt1}(1 - \gamma)}{\alpha_{opt1} - 1} \sum_{j=1}^W \bar{\omega}_j |\hat{\beta}_j - \beta_j| + \frac{\alpha_{opt1}}{\alpha_{opt1} - 1} \sum_{j=1}^W c(\hat{\beta}_j - \beta_j)^2$$

$$\leq \frac{\alpha_{opt1} + 1}{\alpha_{opt1} - 1} \|h_{\hat{\beta}} - h\|^2 + \frac{18\alpha_{opt1}^2}{\alpha_{opt1} - 1} H^2(\beta) v^2(\delta/2) W(\beta)$$

with probability at least $1 - \delta$, where $0 < \gamma \leq 1$ and $\alpha_{opt1} = 1 + \sqrt{1 + \frac{\|h_{\hat{\beta}} - h\|^2}{9H^2(\beta)v^2(\delta/2)W(\beta)}}$.

Let us address the sparse Gram matrix $\psi_W = (< h_i, h_j >)_{1 \leq i, j \leq W}$ with a small number of non-zero element in off-diagonal positions, define $\psi_W(i, j)$ as the element $(i, j)$-th of position $\psi_W$. The condition (11) in Theorem 1 can be transformed to the condition

$$12SH(\beta)\sqrt{W(\beta)} \leq \gamma,$$

where the number $S$ is called the sparse index of matrix $\psi_W$ which is defined as follows:

$$S = |\{(i, j) : i, j \in \{1, \cdots, W\}, i > j \text{ and } \psi_W(i, j) \neq 0\}|,$$

where $|A|$ is the number of elements of set $A$.

Sometimes the assumption (11) does not necessarily imply the positive definiteness of $\psi_W$. Next we give similar oracle inequality that is valid under the hypothesis that the Gram matrix $\psi_W$ is positive definite.

**Theorem 2.** Assume that $\max_{1 \leq j \leq W} L_j < \infty$ and Gram matrix $\psi_W$ are positive definite with minimum eigenvalues greater than or equal to $\lambda_W > 0$. For all $\beta \in \mathbb{R}^W$, we have the following oracle inequality with probability at least $1 - \delta$,

$$\|h_{\hat{\beta}} - h\|^2 + \frac{\alpha_{opt2}}{\alpha_{opt2} - 1} \sum_{j=1}^W \bar{\omega}_j |\hat{\beta}_j - \beta_j| + \frac{\alpha_{opt2}}{\alpha_{opt2} - 1} \sum_{j=1}^W c(\hat{\beta}_j - \beta_j)^2$$

$$\leq \frac{\alpha_{opt2} + 1}{\alpha_{opt2} - 1} \|h_{\hat{\beta}} - h\|^2 + \frac{576\alpha_{opt2}^2}{\alpha_{opt2} - 1} \frac{G^*}{\lambda_W} v^2(\delta/2),$$

where $G^*(\beta) = \sum_{j \in I(\beta)} L_j^2$ and $\alpha_{opt2} = 1 + \sqrt{1 + \frac{\|h_{\hat{\beta}} - h\|^2}{288 \frac{G^*}{\lambda_W} v^2(\delta/2)}}$. 

9
Remark: The argument and result of Theorem 1 in this paper is more refined than the conclusion of Theorem 1 in [Bunea et al. (2010)] for Lasso by letting $\gamma = 1/2$ and $c = 0$. In addition, Theorem 1 and Theorem 2 of this paper respectively give the optimal $\alpha$ value of the density estimation oracle inequalities, namely $\alpha_{opt1}$, $\alpha_{opt2}$. It provides potentially sharper bound for the $\ell_1$-estimation error bound in below.

Next, we will present $\ell_1$-estimation error for the estimator $\hat{\beta}$ by (5), and the weights are defined by (6).

For technical reason, we consider that $\|h_j\| = 1$ for all $j$ in (3), i.e. the based functions are normalized. This normalization mimics the covariates’ standardization procedure when we do some penalized estimation in generalized linear models. For simplicity, we put $L := \max_{1 \leq j \leq W} L_j$.

For any other choice of $v(\delta/2)$ greater than or equal to $\sqrt{1/n \log 2W/\delta}$, the conclusions of Section 3 are valid with high probability. It imposes restriction on the predictive performance of CSDE. As pointed out in [Bunea (2008)], for the $\ell_1$-penalty in the regression, the adjustment sequence $\omega_j$ required for the correct selection is usually larger than the adjustment sequence $\omega_j$ that produces a good prediction. The selection of the mixed density shown below is also true. Specifically, we will take the value $v = v(\delta/2W) = \sqrt{\log(2W^2/\delta)/n}$, let $\alpha_{opt1}, \alpha_{opt2} = 2$, in below we give the corollary of Theorem 1,2.

**Corollary 1.** Given the same conditions as Theorem 1 with $\|h_j\| = 1$ for all $j$, we have the following $\ell_1$-estimation error oracle inequality:

$$
\sum_{j=1}^{W} |\hat{\beta}_j - \beta_j^*| \leq \frac{72\sqrt{2v(\delta/2W)W(\beta^*)} (L + L_{min})^2}{1 - \gamma} L_{min} (12)
$$

with probability at least $1 - \delta/W$.

**Corollary 2.** Given the same conditions as Theorem 2 with $\|h_j\| = 1$ for all $j$, let $\alpha_{opt2} = 2$, we have the following $\ell_1$-estimation error oracle inequality, with probability at least $1 - \delta/W$,

$$
\sum_{j=1}^{W} |\hat{\beta}_j - \beta_j^*| \leq \frac{288\sqrt{2v(\delta/2W)G^*}}{L_{min}\lambda W} L_{j}^2.
$$

where $G^* = \sum_{j \in I^*} L_j^2$.

If the number $W(\beta^*)$ of the mixed indicator elements is much smaller than $\sqrt{n}$, then the inequality (12) guarantees that the estimated $\hat{\beta}$ is close to the true $\beta^*$. For example, the $\ell_1$-estimation error will be presented in the numerical simulation in Section 4. Our results of Corollary 1 and 2 are non-asymptotic which applies to any $W$ and $n$. The oracle inequalities is a guider for us to find an optimal tuning parameter with order $O(\sqrt{\log W/n})$ for sharp estimation error and good prediction performance. This is also an intermediate and crucial result which leads to the main result of correctly identifying the mixture components in Section 3.2. In the following section, we turn to cope with the identification of $I_\ast$. The selection of correct components is derived by the proposed oracle inequalities for the weighted $\ell_1 + \ell_2$ penalty.

### 3.2 Correct support identification of mixture models

In this section, we will study results on support recovery of the CSDE estimator. There are a few versions of support recovery while most of the results are the consistency of $\ell_1$-error and
prediction error. Here we borrow the framework due to Bunea (2008), Zhang and Jia (2017), they give many proof techniques to deal with the correct support identification in linear models by $\ell_1 + \ell_2$ regularization. Let $\hat{I}$ be the set of indicators consisting of non-zero elements of $\hat{\beta}$ in the given (5). In other words, $\hat{I}$ is an estimate of the true variable set $I(\beta^*) := I_*$. We will study that for a given $0 < \varepsilon < 1$,

$$P(\hat{I} = I(\beta^*)) \geq 1 - \varepsilon$$

under some mild conditions.

In order to identify the $I_*$ consistently, we need more assumptions about some special correlation conditions than $\ell_1$-error consistency.

Condition (A):

$$\rho_*(\beta^*) \leq \frac{LL_{\text{min}}\lambda_W}{288G^*}.$$ 

Moreover, we need an additional condition that the minimal signal should be higher than a threshold level and quantified by the order of tuning parameter. We state it as follows:

Condition (B):

$$\min_{j \in I^*} |\beta^*_j| \geq 4\sqrt{2}v(\delta_{2W})L,$$

where $v(\delta_{2W}) := \sqrt{\frac{1}{n}\log \frac{2W^2}{\delta}}$.

When performing simulation, condition (B) is the theoretical guarantee that the smallest magnitude of $\beta_j$ must be greater than a threshold value as a minimal signal condition. It is also called Beta-min condition, see Buhlmann and van de Geer (2011).

**Theorem 3.** Let $0 < \delta < \frac{1}{2}$ be a given value. Assume that both condition (A,B) are true and given the same conditions as Corollary 2, then

$$P(\hat{I} = I_*) \geq 1 - 2\delta(1 + \frac{1}{W}).$$

Under the beta-min condition, the support estimation is very close to the true the support of $\beta_j^*$. The probability of the event $\{\hat{I} = I_*\}$ approximates 1 increasingly when $W$ is growing. The $\hat{\beta}$ recovers the correct support with probability at least $1 - 2\delta(1 + \frac{1}{W})$. The result is non-asymptotic, it is true for any fixed $W$ and $n$.

4 Simulation and Real Data Analysis

Bunea et al. (2010) suppose the spades estimation to deal with the samples for sparse mixture density, and they also derive an algorithm to complement their theoretical result. Their findings successfully handle the high-dimensional adaptive density estimation in some degree. However, their algorithm may be costly and unstable. In this section, we deal with the tuning parameter directly and compare our CSDE method (Elastic-net) with the SPADES method (Lasso) in Bunea et al. (2010) and other similar methods. In all cases here, we fix $n = 100$ for $W = 81, 131, 211, 321$, which is known as the dimension of the unknown parameter $\beta^*$. The performance of each estimator is evaluated by $\ell_1$-estimation error and total variation (TV) distance between the estimator and the true value of $\beta^*$. The total variation (TV) error is defined by:

$$\text{TV}(h_{\beta^*}, h_{\hat{\beta}}) = \int |h_{\beta^*}(x) - h_{\hat{\beta}}(x)| dx.$$
4.1 Tuning parameter selection

In Bunea et al. (2010), the $\lambda_1$ is chosen by coordinate descent method, while the mixture weights are detected by general bisection method (GBM). But in our article, the optimal weight can be computed directly. The $\ell_1$-penalty term $\sum_{j=1}^{W} \omega_j |\beta_j|$ with optimal weights is defined by

$$
\omega_k := 2\sqrt{2}L_k v(\delta/2) + cB,
$$

where $L_j = \|h_j\|_\infty$, which usually can be compute easily for a continuous $h_j$.

For a discrete base density $\{h_j\}_{j=1}^{W}$, it can be estimated as the following approximation by using concentration inequalities from Exercise 4.3.3 of Chow and Teicher (2003):

$$
|\text{med}(X) - E(X)| \leq 2\text{Var}(X),
$$

(13)

$$
\tilde{x} \approx x_{\text{med}} (1 + O(n^{-1})) \approx h^{-1}(L_j) (1 + O(n^{-1})),
$$

(14)

where $\tilde{x}$ and $x_{\text{med}}$ represent the sample mean and sample median respectively in each simulation, then we only need to select the $\lambda_1$ and $c = \lambda_2$, and they can be detected by nesting coordinate descent method. Besides, the precision level is assigned as $\epsilon = 0.001$ in our simulation.

4.2 Multi-modal distributions

First, we examine our method in a multi-modal normal model that is similar to the first model in Bunea et al. (2010). However, our mixture Gaussian has a different variance, which leads the meaningful weight to our estimation. The model is assigned as follow:

$$
h^*_\beta(x) = \sum_{j=1}^{W} \beta^*_j \phi(x|a_j, \sigma_j).
$$

(15)

In practice, the difficulty is to detect the sparse high-dimensional parameter in the sample, which often is not sufficiently large in size. The multi-modal will also increase the complication of the weak signals. We choose $a = 0.5, n = 100$ and:

$$
\beta^* = \left(0_{8}^T, 0.2, 0_{10}^T, 0.1, 0_{5}^T, 0.1, 0_{10}^T, 0.1, 0_{10}^T, 0.1, 0_{10}^T, 0.15, 0_{10}^T, 0.15, 0_{10}^T, 0.1, 0_{W-76}^T\right)^T.
$$

(16)

The variances of the Gaussian distribution are also the signal:

$$
\sigma = \left(1_{20}^T, 0.8_{6}^T, 0.6_{11}^T, 0.4_{11}^T, 0.6_{6}^T, 0.8_{11}^T, 1.2_{W-76}^T\right)^T.
$$

(17)

An acceptable measurement error $e_j(x)$ satisfied $Ee_j(X) = 0$ is chosen as:

$$
h_j(x) + e_j(x) \sim N(Eh_j(X), 1.1 \text{var}(h_j(X))).
$$

(18)

Then we use the sample $x_1, \ldots, x_n$ i.i.d. $\sum_{j=1}^{W} \beta^*_j [h_j(x) + e_j(x)]$, which has the measurement error, to estimate $\beta^*$. We replicate the simulation $N = 100$ times. The results of simulation are presented in Table 1, we can see our method has the more and more excellent performances as the $W$ increases which matches the non-asymptotical results in the previous section. The best performance is far away from the other three method when $W = 321$. It’s worthy to note that the better approximation following the increasing of $W$, matching the and Theorem 3 in our previous section.

We plot the solution path to compare the performance of the four estimators in $\beta_j \in I(\beta)$ for every $W$ in Figure 1 (the result of Enet in $W = 321$ is not be shown due to its exactly poor
Table 1: The mean and standard deviation of the errors in the four estimators of \( \beta^* \) under \( N = 100 \) simulations, with \( n = 100 \). The quasi-optimal \( \lambda_2 \) is \( c = 0.002 \) for Enet, while \( c = 0.027 \) is for the adaptive method.

|         | \( W \) | \( \lambda_1 \) | \( L_1 \) error | \( TV \) error |
|---------|--------|----------------|----------------|--------------|
| Lasso   | 81     | 0.065          | 2.133 (2.467)  | 1.137 (1.115) |
| Enet    |        |                | 2.061 (1.439)  | 1.114 (0.805) |
| AdaLasso|        | 0.053          | 1.922 (2.211)  | 1.258 (1.296) |
| CSDE    |        |                | 2.191 (4.812)  | 1.405 (2.329) |
| Lasso   | 131    | 0.068          | 2.032 (0.985)  | 1.352 (0.712) |
| Enet    |        |                | 2.236 (2.498)  | 1.409 (1.056) |
| AdaLasso|        | 0.056          | 1.880 (2.644)  | 0.972 (1.204) |
| CSDE    |        |                | 1.635 (0.342)  | 0.863 (0.402) |
| Lasso   | 211    | 0.071          | 2.572 (4.187)  | 1.605 (2.702) |
| Enet    |        |                | 2.061 (1.883)  | 1.353 (1.516) |
| AdaLasso|        | 0.058          | 1.764 (1.041)  | 0.832 (0.610) |
| CSDE    |        |                | 1.648 (0.168)  | 0.791 (0.415) |
| Lasso   | 321    | 0.074          | 2.120 (2.842)  | 1.146 (1.115) |
| Enet    |        |                | 10.173 (82.753)| 7.839 (67.887)|
| AdaLasso|        | 0.061          | 2.106 (4.816)  | 0.818 (1.565) |
| CSDE    |        |                | 1.623 (0.085)  | 0.634 (0.199) |

Performance.). These figures also provide strong support for the above analysis. Meanwhile, we plot the probability densities of the several estimators and the true density to complement the visual sensory of the advantage in our method in Figure 2, in which the powerful competency of detecting the multi-mode is shown (whereas other methods only find the most strong signal, ignoring other meaningful but relatively slight signals).

4.3 Mixture of Poisson distributions

In the second set of our simulations, we study the mixture of discrete distribution: mixture Poisson distribution

\[
h_{\beta^*}(x) = \sum_{j=1}^{W} \beta^*_j p(x|\lambda_j = a_j), \tag{19}
\]

where \( a = 0.1 \), and:

\[
\beta^* = \left( 0^T_8, 0.2, 0^T_{10}, 0.1, 0^T_5, 0.1, 0^T_{10}, 0.1, 0^T_5, 0.15, 0^T_{10}, 0.15, 0^T_{10}, 0.1, 0^T_{W-75} \right)^T. \tag{20}
\]

The adjusted weight is calculated by (5), and in discrete distributions, we define \( \langle f, g \rangle = \sum_{k=1}^{\infty} f(k)g(k) \). Meanwhile, the Poisson distribution with measurement error can be treated as a negative binomial distribution. Namely:

\[
p(x|\lambda_i) + e_i(x) \sim NB \left( r, \frac{\lambda_i}{\lambda_i + r} \right), \tag{21}
\]

where \( \lambda_i \) is the mean for \( i \)-th observation and \( r \) is the common dispersion parameter. Here, \( i = 1, 2, \cdots, n = 100 \).
Figure 1: The estimated support of $\beta^*$ by the four types of estimators, and the $W$ is varying. The circles represent the means of the estimators under the four specific approaches, while the half of the vertical lines mean the standard deviations.

For a practical $r$, we choose $r = 6$, which leads an increment $\lambda_i j^2 / r$ in the variance. Similarly, we replicate each simulation to estimate the parameter $N = 100$ times with the sample came from the mixture negative binomial distribution above. The result is shown in Table 2. The result is actually akin with the mixture Gaussian distribution, while the better performance of our method is shown clearly when $W$ is considerable.
Table 2: The mean and standard deviation of the errors in the four estimators of $\beta$ under $N = 100$ simulations. The $\lambda_2$ is chosen as $c = 0.005$ for Enet, while $c = 0.203$ for the adaptive method.

|        | $W$ | $\lambda$ | $L_1$ error | TV error |
|--------|-----|-----------|-------------|----------|
| Lasso  | 81  | 0.048     | 1.796 (0.006) | 0.002 (0.001) |
| Enet   |     |           | 1.796 (0.006) | 0.002 (0.001) |
| AdaLasso |   | 0.138     | 1.811 (0.013) | 0.002 (0.005) |
| CSDE   |     |           | 1.806 (0.008) | 0.003 (0.005) |
| Lasso  | 131 | 0.051     | 1.828 (0.006) | 0.003 (0.001) |
| Enet   |     |           | 1.830 (0.009) | 0.004 (0.002) |
| AdaLasso |   | 0.145     | 1.880 (0.006) | 0.002 (0.005) |
| CSDE   |     |           | 1.834 (0.006) | 0.002 (0.004) |
| Lasso  | 211 | 0.053     | 1.935 (0.010) | 0.005 (0.003) |
| Enet   |     |           | 2.061 (0.014) | 0.007 (0.008) |
| AdaLasso |   | 0.152     | 1.935 (0.008) | 0.005 (0.003) |
| CSDE   |     |           | 1.861 (0.005) | 0.003 (0.002) |
| Lasso  | 321 | 0.055     | 1.927 (0.031) | 0.005 (0.002) |
| Enet   |     |           | 2.123 (0.026) | 0.009 (0.009) |
| AdaLasso |   | 0.158     | 1.938 (0.008) | 0.005 (0.003) |
| CSDE   |     |           | 1.852 (0.002) | 0.002 (0.001) |

4.4 Low dimensional mixture model

Another phenomenon of our method also has more competitive efficacy than some popular methods (such as EM algorithm), even the dimension $W$ is relatively small. To see this, we introduce the following numerical experiments to estimate the weights of the low dimensional mixed Gaussian model: the samples $x_1, \cdots, x_n$ come from the model:

$$h_{\beta^*}(x) = \sum_{j=1}^{W} \beta_j^* \phi(x|\mu_j, \sigma_j).$$

The updating equation for EM algorithm in $t$-th step is:

$$\omega_{ij}^{(t)} = \frac{p_{ij}^{(t)} \phi(x_i; \mu_t, \sigma_t)}{\sum_{s=1}^{W} p_{is}^{(t)} \phi(x_i; \mu_s, \sigma_s)}, \quad \beta_{j}^{(t+1)} = \frac{\sum_{i=1}^{n} \omega_{ij}^{(t)}}{\sum_{i=1}^{n} \sum_{j=1}^{W} \omega_{ij}^{(t)}}.$$

Here we consider two scenarios:

1. $W = 6, \beta = (0.3, 0, 0, 0.3, 0, 0.4)^T, \mu = (0, 10, 20, 30, 40, 50)^T, \sigma = (1, 2, 3, 4, 5, 6)^T$
2. $W = 7, \beta = (0.1, 0, 0, 0.8, 0, 0, 0.1)^T, \mu = (0, 1, 2, 3, 4, 5, 6)^T, \sigma = (0.3, 0.2, 0.2, 0.1, 0.2, 0.2, 0.3)^T$.

For each scenario $n = 50$, and the fitter level (cessation level) in the EM approach and our method is $\epsilon = 10^{-4}$. A well-advised initial value is equal weights.

We replicate the simulation $N = 100$ times, and the optimal tuning parameters stem from CV (so under each simulation they are not the same, albeit they are very close to each other).
Table 3: The eventual simulations result.

| Scenario  | EM          | TV error          | CSDE          | TV error          |
|-----------|-------------|-------------------|---------------|-------------------|
| 1         | 0.2547758(0.1217747) | 0.2045831(0.09807962) | 0.2055284(0.1449865) | 0.1852503(0.104226) |
| 2         | 0.1109372(0.05491049) | 0.1107578(0.05491172) | 0.1090387(0.03715101) | 0.108296(0.03683571) |

4.5 Real data examples

Practically, we consider using our method to estimate some densities in the environmental science field. In the area of meteorology, wind, which is mercurial, has been an advisable object to study for a long time. Take notice of the speed of the wind at one specific location maybe not diverse, so we will use the wind’s azimuth angle with a more sparse density at two locations in China. Concerning many types of research about the estimated density for wind existed, there is a possibility to use our approach to cope with some difficulties in meteorology science.

There have been some very credible meteorological data sets. We would use the ERA5 hourly data in Hersbach et al. [2018] to continue our analysis, regardless of it does be an artificial data computed by the interpolation method. We decide to choose a continental area and a coastal area in China which refers to Beijing Nongzhanguan and Qingdao Coast, respectively. The location of these two area are: [116.3125° E, 116.4375° E] × [39.8125° N, 39.8125° N]. Take notice that the wind in one day may be highly correlated, therefore, using the data at a specific time point of each day in one consecutive period as i.i.d. samples are more reasonable. The sample histogram of 6 am in Beijing Nongzhanguan and at 12 am Qingdao Coast is shown as following. Here we use the data from 2013/01/01 to 2015/12/12.

As we can see, their density does be multi-peak (we used 1095 samples). Now we can use our approach to estimate the multi-mode densities based on a relatively small size of samples, which is only a tiny part of the whole data from 2013/01/01 to 2015/12/12. Because one year has nearly 360 days, we may assume that every day is a latent factor that forms the base density. So the model is designed as

\[ h_{\beta^*}(x) = \sum_{j=1}^{360} \beta_j^* \phi(x|\mu_j, \sigma_j) \]  

(22)

with the mean and variance parameters:

\[ \mu = (1, 2, \ldots, 360)^T, \quad \sigma = t \cdot 1_{360}^T, \]  

(23)

where the \( t \) is seen as the bandwidth (or tuning parameter). With the different sub-samples, the values it computed are different.

Another critical issue is how to choose the tuning parameter \( \lambda_1 \) and \( \lambda_2 \). Then we apply the cross-validation criterion, namely choosing \( \lambda_i \) minimizing the difference of two estimators derived from the separated samples in random dichotomy.

Now start to construct the samples for estimating procedure. Assume that an observatory wants to figure some information about the wind in the 2 areas. However, it doesn’t have intact data due to the limited budget at its inception. The only samples it has are 6 ~ 8 days’ information each month for the 2 areas and these days scatter randomly. This imperfect data does increase the challenge of estimating the trustworthy density, and our premier goal is to compare our method with other previous methods, in which appraising the difference between the complete data sample histogram and the estimated density under each method is for the evaluation. Please notice that the samples are only a tiny part of the whole data, so the \( n \)
Figure 2: The density map of the four estimators’ means and the true dense. The result of Enet in $W = 321$ is not be shown due to its exactly poor performance.
Figure 3: The sample histogram of the azimuth in Beijing Nongzhanguan at 6am and Qingdao Coast at 12am.

is relatively small and \( W = O(n) \) which is the same order of \( n \). The small sample and large dimension setting coincide the non-asymptotical theory provided in the previous section. This real example belongs to the increasing-dimensional issue. In this scenario, the estimating density has been shown in Figure 4.

Evidently, in this practical application, we can see that our method vindicates its more efficient estimating performance and stability from its propinquity of the complete sample histogram, namely the efficacious capacity of detecting the shape of multi-mode density, and the stronger inclination to bear resemblance to each subsample (although some subtle nuances do exist by reason of the different subsample).

5 Summary and Discussions

In this paper, we study the finite sample properties of sparse mixed density estimation models by deriving non-asymptotic oracle inequalities of the weighted \( \ell_1 + \ell_2 \) penalized estimator \( \hat{\beta} \). Our purpose is to estimate the vector \( \beta^* \) by adapting the weighted \( \ell_1 + \ell_2 \) penalty to this unknown sparsity of \( \beta^* \) and then to identify \( I_* \) with high probability. The following estimation problem is also similar to the CSDE.

Aggregate density estimator with measurement errors. Based on the idea of model average, our aim is to aggregate some candidate density models \( h_1, \ldots, h_W \) based on the data \( \{X_i\}_{i=1}^n \) containing measurement errors, It means we need to construct a new aggregated estimator as the convex combination of \( h_1, \ldots, h_W \), which is approximately the best among \( h_1, \ldots, h_W \). The aggregation we consider here in the form of \( h_{\hat{\beta}} \) by appropriately chosen the weight vector \( (\beta_1, \ldots, \beta_W) \in \mathbb{R}^W \).

For the future study, it is interesting and meaningful to do hypothesis testing about the coefficients \( \beta^* \in \mathbb{R}^W \) in sparse mixture models. For a general function \( h : \mathbb{R}^W \rightarrow \mathbb{R}^m \) and a nonempty closed set \( \Omega \in \mathbb{R}^m \), consider

\[
H_0 : h(\beta^*) \in \Omega \quad \text{vs.} \quad H_1 : h(\beta^*) \notin \Omega.
\]
Figure 4: The density map of the four estimators’ approaches for the three random subsamples.

6 Appendix: Proof

For convenience, we first give a preliminary lemma and proof, now define the random variable

\[ M_j = \frac{1}{n} \sum_{i=1}^{n} \{ h_j(X_i) - Eh_j(X_i) \}. \]

Define event \( \mathcal{E} \) by

\[ \mathcal{E} = \bigcap_{j=1}^{W} \{ 2|M_j| \leq \tilde{\omega}_j \}, \]
where \( \tilde{\omega}_k := 2\sqrt{2}L_k \sqrt{\frac{1}{n} \log \frac{W}{\delta^2}} =: 2\sqrt{2}L_k v(\delta/2) \).

Then we have the following lemma which is cornerstone for the proofs in below.

**Lemma 3.** Suppose that \( \max_{1 \leq j \leq W} L_j < \infty \) and \( c = \min_{1 \leq j \leq W} \{ \tilde{\omega}_j \} \), for any \( \beta \in \mathbb{R}^W \), on event \( \mathcal{E} \), we have,

\[
\| h_\beta - h \|^2 + \sum_{j=1}^W \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + \sum_{j=1}^W c(\hat{\beta}_j - \beta_j)^2 \leq \| h_\beta - h \|^2 + 6 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j|.
\]

(24)

**Proof of Lemma 3** According to the definition of \( \hat{\beta} \), for any \( \beta \in \mathbb{R}^W \),

\[
- \frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \| h_\beta \|^2 + 2 \sum_{j=1}^W \omega_j |\hat{\beta}_j| + c \sum_{j=i}^W \hat{\beta}_j^2 \leq - \frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + \| h_\beta \|^2 + 2 \sum_{j=1}^W \omega_j |\hat{\beta}_j| + c \sum_{j=1}^W \hat{\beta}_j^2.
\]

Then

\[
\| h_\beta \|^2 - \| h_\beta \|^2 \leq - \frac{2}{n} \sum_{i=1}^n h_\beta(X_i) - \frac{2}{n} \sum_{i=1}^n h_\beta(X_i) + 2 \sum_{j=1}^W \omega_j |\hat{\beta}_j| - 2 \sum_{j=1}^W \omega_j |\hat{\beta}_j| + c \sum_{j=1}^W \hat{\beta}_j^2 - c \sum_{j=1}^W \hat{\beta}_j^2.
\]

(25)

Note that

\[
\| h_\beta - h \|^2 = \| h_\beta - h_\beta + h_\beta - h \|^2
\]

\[
= \| h_\beta - h_\beta \|^2 + \| h_\beta - h \|^2 + 2 < h_\beta - h, h_\beta - h_\beta >
\]

\[
= \| h_\beta - h \|^2 - 2 < h, h_\beta - h_\beta > + 2 < h_\beta, h_\beta - h_\beta > + \| h_\beta - h_\beta \|^2
\]

\[
= \| h_\beta - h \|^2 - 2 < h, h_\beta - h_\beta > + \| h_\beta \|^2 - \| h_\beta \|^2.
\]

(27)

Combining (26) and (27), we have

\[
\| h_\beta - h \|^2 \leq \| h_\beta - h \|^2 + 2 \sum_{j=1}^W \omega_j |\hat{\beta}_j| - 2 \sum_{j=1}^W \omega_j |\hat{\beta}_j| + c \sum_{j=1}^W \hat{\beta}_j^2 - c \sum_{j=1}^W \hat{\beta}_j^2
\]

\[
- 2 < h, h_\beta - h_\beta > + \frac{2}{n} \sum_{i=1}^n h_\beta(X_i) - \frac{2}{n} \sum_{i=1}^n h_\beta(X_i)
\]

(28)

According to the definition of \( h_\beta(x) \), it gives \( h_\beta(x) = \sum_{j=1}^W \beta_j h_j(x) \) with \( \beta = (\beta_1, \ldots, \beta_n) \). For the 3 terms in (28), we have

\[
-2 < h, h_\beta - h_\beta > + \frac{2}{n} \sum_{i=1}^n h_\beta(X_i) - \frac{2}{n} \sum_{i=1}^n h_\beta(X_i)
\]

\[
= 2 \cdot \frac{1}{n} \sum_{i=1}^n \left( \sum_{j=1}^W \hat{\beta}_j h_j(X_i) - \sum_{j=1}^W \beta_j h_j(X_i) \right) - 2E(h_\beta - h_\beta)(X_i)
\]

\[
= 2 \sum_{j=1}^W \frac{1}{n} \sum_{i=1}^n h_j(X_i)(\hat{\beta}_j - \beta_j) - 2 \sum_{j=1}^W E h_j(X_i)(\hat{\beta}_j - \beta_j)
\]

\[
= 2 \sum_{j=1}^W \left( \frac{1}{n} \sum_{i=1}^n h_j(X_i) - E h_j(X_i) \right)(\hat{\beta}_j - \beta_j).
\]
then,
\[
\|h_\beta - h\|^2 \leq \|h_\beta - h\|^2 + 2 \sum_{j=1}^{W} \left( \frac{1}{n} \sum_{i=1}^{n} h_j(X_i) - Eh_j(X_i) \right) (\hat{\beta}_j - \beta_j) + 2 \sum_{j=1}^{W} \omega_j |\beta_j| - 2 \sum_{j=1}^{W} \omega_j |\hat{\beta}_j| + c \sum_{j=1}^{W} \beta_j^2 - c \sum_{j=1}^{W} \hat{\beta}_j^2.
\]
Conditioning on event $\mathcal{E}$, we have
\[
\|h_\beta - h\|^2 \leq \|h_\beta - h\|^2 + \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + c \sum_{j=1}^{W} (\beta_j - \hat{\beta}_j)^2
\]
Adding $\sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + c \sum_{j=1}^{W} (\beta_j - \hat{\beta}_j)^2$ to both sides of the above inequality, it gives
\[
\|h_\beta - h\|^2 + \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + c \sum_{j=1}^{W} (\beta_j - \hat{\beta}_j)^2
\]
\[
\leq \|h_\beta - h\|^2 + 2 \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + 2 \sum_{j=1}^{W} \omega_j (|\beta_j| - |\hat{\beta}_j|) + c \sum_{j=1}^{W} (\beta_j^2 - \hat{\beta}_j^2) + c \sum_{j=1}^{W} (\beta_j - \hat{\beta}_j)^2.
\]
Note that
\[
c \left[ \sum_{j=1}^{W} (\beta_j^2 - \hat{\beta}_j^2) + \sum_{j=1}^{W} (\beta_j - \hat{\beta}_j)^2 \right] = c \left[ \sum_{j=1}^{W} (\beta_j^2 - \hat{\beta}_j^2 + \beta_j^2 - \hat{\beta}_j^2) \right]
\]
\[
= 2c \sum_{j=1}^{W} \beta_j (\beta_j - \hat{\beta}_j) = 2c \sum_{j \in I(\beta)} \beta_j (\beta_j - \hat{\beta}_j)
\]
\[
\leq 2cB \sum_{j \in I(\beta)} |\beta_j - \hat{\beta}_j| \leq 2 \sum_{j \in I(\beta)} \tilde{\omega}_j |\beta_j - \hat{\beta}_j|,
\]
where the last inequality is due to the assumption $c = \frac{\min_{1 \leq i \leq W} \{|\omega_i|\}}{B} \leq \frac{\tilde{\omega}_j}{B}$.
So we obtain
\[
\|h_\beta - h\|^2 + \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + c \sum_{j=1}^{W} (\beta_j - \hat{\beta}_j)^2
\]
\[
\leq \|h_\beta - h\|^2 + 2 \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + 2 \sum_{j=1}^{W} \omega_j (|\beta_j| - |\hat{\beta}_j|) + 2 \sum_{j \in I(\beta)} \omega_j |\beta_j - \hat{\beta}_j|
\]
\[
\leq \|h_\beta - h\|^2 + 2 \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + 2 \sum_{j=1}^{W} \omega_j (|\beta_j| - |\hat{\beta}_j|) + 2 \sum_{j \in I(\beta)} \omega_j |\beta_j - \hat{\beta}_j|,
\]
where the last inequality stems from $\tilde{\omega}_j \leq \omega_j$ for all $j$.
We know that when $j \in I(\beta)$, $\beta_j \neq 0$, when $j \notin I(\beta)$, $\beta_j = 0$. Considering $|\beta_j| - |\hat{\beta}_j| \leq |\beta_j - \hat{\beta}_j|$ for all $j$, then
\[
2 \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + 2 \sum_{j=1}^{W} \omega_j (|\beta_j| - |\hat{\beta}_j|) \leq 4 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j|.
\]
\[ \|h_\beta - h\|^2 + \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j| + c \sum_{j=1}^{W} (\hat{\beta}_j - \beta_j)^2 \leq \|h_\beta - h\|^2 + 4 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j| + 2 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j| \]

\[ = \|h_\beta - h\|^2 + 6 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j|. \]

Therefore, Lemma 2 is proved.

### 6.1 The Proof of Theorems

According to the construction of \(\tilde{\omega}_j = 2\sqrt{2}L_j \sqrt{\frac{1}{n} \log \frac{2W}{\delta}}\) in the formula (6), the sum of the independent random variables \(\zeta_{ij} = h_i(X_i) - Eh_j(X_i)\) is determined by Hoeffding’s inequality, and \(|\zeta_{ij}| \leq 2L_j\). We obtain

\[ P(\mathcal{E}^C) = P \left( \bigcup_{j=1}^{W} \{ 2|M_j| > \tilde{\omega}_j \} \right) \leq \sum_{j=1}^{W} P(2|M_j| > \tilde{\omega}_j) \]

\[ \leq 2 \sum_{j=1}^{W} \exp \left( - \frac{2n^2 \cdot \tilde{\omega}_j^2 / 4}{4nL_j^2} \right) \]

\[ = 2 \sum_{j=1}^{W} \exp \left( - \log \frac{2W}{\delta} \right) = 2W \cdot \frac{\delta}{2W} = \delta. \]

**Proof of Theorem 1** By using Lemma 3, we need an upper bound on \(\sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j|\).

For easy notation, let

\[ q_j = \hat{\beta}_j - \beta_j, \quad Q(\beta) = \sum_{j \in I(\beta)} |q_j||h_j||, \quad Q = \sum_{j=1}^{W} |q_j||h_j|. \]

According to the definition of \(H(\beta)\), that is, \(H(\beta) = \max_{j \in I(\beta)} \frac{\omega_j}{\sqrt{\delta/2} ||h_j||}\), we have

\[ \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j| \leq \sqrt{\delta/2} H(\beta) Q(\beta). \tag{29} \]

Let \(Q_*(\beta) := \sqrt{\sum_{j \in I(\beta)} q_j^2 ||h_j||^2}\). Using the definition of \(h_\beta(x)\), we obtain

\[ Q_*(\beta) = \sum_{j \in I(\beta)} q_j^2 ||h_j||^2 \]

\[ = \|h_\beta - h_\beta\|^2 - \sum_{i,j \notin I(\beta)} q_iq_j < h_i, h_j > -(2 \sum_{i \notin I(\beta)} \sum_{j \in I(\beta)} q_iq_j < h_i, h_j > + \sum_{i,j \in I(\beta), i \neq j} q_iq_j < h_i, h_j >). \]

As \(i,j \notin I(\beta), \beta_i = \beta_j = 0\), it is easy to see,

\[ \sum_{i,j \notin I(\beta)} q_iq_j < h_i, h_j > q_iq_j \geq 0. \]
Observe that

\[
2 \sum_{i \notin I(\beta)} \sum_{j \in I(\beta)} q_i q_j < h_i, h_j > + \sum_{i,j \in I(\beta), i \neq j} q_i q_j < h_i, h_j > = 2 \sum_{i \notin I(\beta)} \sum_{j \in I(\beta)} q_i q_j < h_i, h_j > + 2 \sum_{i,j \in I(\beta), j > i} q_i q_j < h_i, h_j > = 2 \sum_{i \in I(\beta), j > i} q_i q_j < h_i, h_j > .
\]

By the definitions of \( \rho_W(i,j) \) and \( \rho_*(\beta) \), then

\[
Q^2_*(\beta) \leq \|h_\beta - h_\beta\|^2 + 2 \sum_{i \in I(\beta), j > i} |q_i| |q_j| \|h_i\| \|h_j\| < h_i, h_j > \leq \|h_\beta - h_\beta\|^2 + 2 \rho_*(\beta) \max_{i \in I(\beta), j > i} |q_i| \|h_i\| |q_j| \|h_j\|.
\]

In fact,

\[
\max_{i \in I(\beta)} |q_i| \|h_i\| \leq \sqrt{\sum_{j \in I(\beta)} q_j^2 \|h_j\|^2} = Q_*(\beta), \quad \max_{i \in I(\beta), j > i} |q_j| \|h_j\| \leq \sum_{j=1}^W |q_j| \|h_j\|.
\]

Thus

\[
Q^2_*(\beta) \leq \|h_\beta - h_\beta\|^2 + 2 \rho_*(\beta) Q_*(\beta) \sum_{j=1}^W |q_j| \|h_j\| = \|h_\beta - h_\beta\|^2 + 2 \rho_*(\beta) Q_*(\beta) Q. \tag{30}
\]

By (30), we can get

\[
Q^2_*(\beta) - 2 \rho_*(\beta) Q_*(\beta) Q - \|h_\beta - h_\beta\|^2 \leq 0.
\]

In order to find the upper bound of \( Q_*(\beta) \), apply the properties of the quadratic inequality to the above formula,

\[
Q_*(\beta) \leq \rho_*(\beta) Q + \sqrt{\rho^2_*(\beta) Q^2 + \|h_\beta - h_\beta\|^2} \leq \rho_*(\beta) Q + [\rho_*(\beta) Q + \|h_\beta - h_\beta\|] \leq 2 \rho_*(\beta) Q + \|h_\beta - h_\beta\|. \tag{31}
\]

Note that \( W(\beta) = |I(\beta)| = \sum_{j=1}^W I(\beta_j \neq 0) \), employing Cauchy-Schwarz inequalities, we have

\[
W(\beta) \sum_{j \in I(\beta)} |q_j|^2 \|h_j\|^2 = \sum_{j \in I(\beta)} I(j \in I(\beta)) \sum_{j \in I(\beta)} |q_j|^2 \|h_j\|^2 \geq \left( \sum_{j \in I(\beta)} I(\{j \in I(\beta)\}) \|q_j\| \|h_j\| \right)^2 = Q^2(\beta).
\]

Then

\[
Q^2_*(\beta) = \sum_{j \in I(\beta)} |q_j|^2 \|h_j\|^2 \geq Q^2(\beta) / W(\beta).
\]

Combined with (31), we can get \( Q(\beta) / \sqrt{W(\beta)} \leq Q_*(\beta) \leq 2 \rho_*(\beta) Q + \|h_\beta - h_\beta\| \). Therefore,

\[
Q(\beta) \leq 2 \rho_*(\beta) \sqrt{W(\beta)} Q + \sqrt{W(\beta)} \|h_\beta - h_\beta\|. \tag{32}
\]
By Lemma 3, we have the following inequality established by probability $1 - \delta$ at least.

\[
\|h_\beta - h\|^2 + \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2
\leq \|h_\beta - h\|^2 + 6 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j|
\]

(by (29)) $\leq \|h_\beta - h\|^2 + 6v(\delta/2)H(\beta)Q(\beta)$

(by (32)) $\leq \|h_\beta - h\|^2 + 6v(\delta/2)H(\beta)|2\rho_*(\beta)\sqrt{W(\beta)} \sum_{j=1}^{W} |q_j| \|h_j\| + \sqrt{W(\beta)}\|h_\beta - h_\beta\|$

$= \|h_\beta - h\|^2 + 12v(\delta/2)H(\beta)\rho_*(\beta)\sqrt{W(\beta)} \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j|\|h_j\|\tilde{\omega}_j$

$+ 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h_\beta - h_\beta\|$

$\leq \|h_\beta - h\|^2 + 12FH(\beta)\rho_*(\beta)\sqrt{W(\beta)} \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h_\beta - h_\beta\|$

$\leq \|h_\beta - h\|^2 + \gamma \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h_\beta - h_\beta\|,$

where the second last inequality is from the definition of $F := \max_{1 \leq j \leq W} \frac{v(\delta/2)\|h_j\|}{\omega_j}$, and the last inequality is derived by the assumption $12FH(\beta)\rho_*(\beta)\sqrt{W(\beta)} \leq \gamma,(0 < \gamma \leq 1)$.

Then,

\[
\|h_\beta - h\|^2 + (1 - \gamma) \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2
\leq \|h_\beta - h\|^2 + 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h_\beta - h_\beta\|
\]

$= \|h_\beta - h\|^2 + 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h_\beta - h + h - h_\beta\|
\]

$\leq \|h_\beta - h\|^2 + 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h_\beta - h\| + 6v(\delta/2)H(\beta)\sqrt{W(\beta)}\|h - h_\beta\|.$

Using the elementary inequality $2st \leq s^2/\alpha + \alpha t^2 \ (s, t \in \mathbb{R}, \alpha > 1)$ to the last two terms of the above inequality, it yields

\[
2\{3v(\delta/2)H(\beta)\sqrt{W(\beta)}\}\|h_\beta - h\| \leq \alpha \cdot 9v^2(\delta/2)H^2(\beta)W(\beta) + \|h_\beta - h\|^2/\alpha,
\]

\[
2\{3v(\delta/2)H(\beta)\sqrt{W(\beta)}\}\|h_\beta - h\| \leq \alpha \cdot 9v^2(\delta/2)H^2(\beta)W(\beta) + \|h_\beta - h\|^2/\alpha.
\]

Thus

\[
\|h_\beta - h\|^2 + (1 - \gamma) \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2
\leq \|h_\beta - h\|^2 + 18\alpha v^2(\delta/2)H^2(\beta)W(\beta) + \|h_\beta - h\|^2/\alpha + \|h_\beta - h\|^2/\alpha.
\]
Simplifying, we have

\[ \|h_\beta - h\|^2 + \frac{\alpha(1 - \gamma)}{\alpha - 1} \sum_{j=1}^{W} \tilde{\omega}_j |\hat{\beta}_j - \beta_j| + \frac{\alpha}{\alpha - 1} \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2 \]

\[ \leq \frac{\alpha + 1}{\alpha - 1} \|h_\beta - h\|^2 + \frac{18\alpha^2}{\alpha - 1} H^2(\beta)v^2(\delta/2)W(\beta), \quad \alpha > 1, 0 < \gamma \leq 1. \quad (33) \]

Optimizing \(\alpha\) to obtain the sharp upper bounds for the above oracle inequality

\[ \alpha_{opt} := \arg \min_{\alpha > 1} \left\{ \frac{\alpha + 1}{\alpha - 1} \|h_\beta - h\|^2 + \frac{18\alpha^2}{\alpha - 1} H^2(\beta)v^2(\delta/2)W(\beta) \right\} = 1 + \sqrt{1 + \frac{\|h_\beta - h\|^2}{9H^2(\beta)v^2(\delta/2)W(\beta)}} \]

by the first order condition.

So far, Theorem 1 is proved by substituting \(\alpha_{opt}\) into (33).

**Proof of Theorem 2**

By the minimal eigenvalue assumption for \(\psi_W\), we have

\[ \|h_\beta\|^2 = \| \sum_{j=1}^{W} \beta_j h_j(x) \|^2 = \beta^T \psi_W \beta \geq \lambda_W \|\beta\|^2 \geq \lambda_W \sum_{j \in I(\beta)} \beta_j^2. \quad (34) \]

Using the definition of \(\omega_j\) and assumption \(L_{min} := \min_{1 \leq j \leq W} L_j > 0\), thus

\[ \omega_j = 2L_j \left( \frac{2 \log(2W/\delta)}{n} + \frac{cB}{2L_j} \right) \leq 2L_j \left( \frac{2 \log(2W/\delta)}{n} + \frac{cB}{2L_{min}} \right). \]

Since \(cB = \tilde{\omega}_{min} = 2\sqrt{2}L_{min}v(\delta/2)\) and \(v(\delta/2) = \sqrt{\frac{\log(2W/\delta)}{n}}\), we have

\[ \omega_j \leq 4\sqrt{2}L_jv(\delta/2). \]

Let \(G^*(\beta) = \sum_{j \in I(\beta)} L_j^2\), we can get by the Cauchy-Schwartz inequality

\[ 6 \sum_{j \in I(\beta)} \omega_j |\hat{\beta}_j - \beta_j| \leq 24\sqrt{2}v(\delta/2) \sum_{j \in I(\beta)} L_j |\hat{\beta}_j - \beta_j| \]

\[ \leq 24\sqrt{2}v(\delta/2) \sqrt{\sum_{j \in I(\beta)} L_j^2} \sqrt{\sum_{j \in I(\beta)} (\hat{\beta}_j - \beta_j)^2} \]

\[ \leq 24\sqrt{2}v(\delta/2) \sqrt{\frac{G^*(\beta)}{\lambda_W}} \|h_\beta - h_\beta\|, \quad (35) \]

where the last inequality above is due to

\[ \|h_\beta - h_\beta\|^2 = \sum_{1 \leq i, j \leq W} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) < h_i, h_j \geq \lambda_W \sum_{j \in I(\beta)} (\hat{\beta}_j - \beta_j)^2 \]

from (34).
Let $b(\beta) := 12\sqrt{2v(\delta/2)}\sqrt{\frac{G^*(\beta)}{\lambda_W}}$, Lemma 2 implies

$$
\|h_\beta - h\|^2 + \sum_{j=1}^{W} \tilde{\omega}_j|\hat{\beta}_j - \beta_j| + \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2 \leq \|h_\beta - h\|^2 + 2b(\beta)\|h_\beta - h\|
$$

$$
= \|h_\beta - h\|^2 + 2b(\beta)\|h_\beta - h - h_\beta\|
$$

$$
\leq \|h_\beta - h\|^2 + 2b(\beta)\|h_\beta - h\| + 2b(\beta)\|h_\beta - h\|.
$$

Use inequality $2st \leq s^2/\alpha + \alpha t^2$ ($s, t \in R, \alpha > 1$) for the last two terms on the right side of the above inequality, then

$$
2b(\beta)\|h_\beta - h\| + 2b(\beta)\|h_\beta - h\| \leq \|h_\beta - h\|^2/\alpha + b^2(\beta)\alpha + \|h_\beta - h\|^2/\alpha + b^2(\beta)\alpha
$$

Thus

$$
\|h_\beta - h\|^2 + \sum_{j=1}^{W} \tilde{\omega}_j|\hat{\beta}_j - \beta_j| + \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2 \leq \|h_\beta - h\|^2 + \|h_\beta - h\|^2/\alpha + \|h_\beta - h\|^2/\alpha + 2b^2(\beta)\alpha.
$$

We have

$$
\frac{\alpha - 1}{\alpha} \|h_\beta - h\|^2 + \sum_{j=1}^{W} \tilde{\omega}_j|\hat{\beta}_j - \beta_j| + \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2 \leq \frac{\alpha + 1}{\alpha} \|h_\beta - h\|^2 + 2\alpha b^2(\beta).
$$

Therefore

$$
\|h_\beta - h\|^2 + \frac{\alpha}{\alpha - 1} \sum_{j=1}^{W} \tilde{\omega}_j|\hat{\beta}_j - \beta_j| + \frac{\alpha}{\alpha - 1} \sum_{j=1}^{W} c(\hat{\beta}_j - \beta_j)^2 \leq \frac{\alpha + 1}{\alpha - 1} \|h_\beta - h\|^2 + \frac{2\alpha^2}{\alpha - 1} b^2(\beta)
$$

$$
= \frac{\alpha + 1}{\alpha - 1} \|h_\beta - h\|^2 + \frac{576\alpha^2}{\alpha - 1} G^*(\beta) v^2(\delta/2).
$$

To get the sharp upper bounds for the above oracle inequality, we optimize $\alpha$

$$
\alpha_{opt2} := \arg\min_{\alpha > 1} \left\{ \frac{\alpha + 1}{\alpha - 1} \|h_\beta - h\|^2 + \frac{576\alpha^2}{\alpha - 1} G^*(\beta) v^2(\delta/2) \right\}
$$

$$
= 1 + \sqrt{1 + \frac{\|h_\beta - h\|^2}{288 G^*(\beta) v^2(\delta/2)}},
$$

by the first order condition. Theorem 2 is proved.

**Proof of Corollary 1** Let $L_{min} = \min_{1 \leq j \leq W} L_j$ and $\tilde{\omega}_{min} := \min_{1 \leq j \leq W} \tilde{\omega}_j$. We replace $v(\delta/2)$ in Theorem 1 by the larger value $v(\delta/2W)$. Substitute $\beta = \beta^*$ in Theorem 1, we have

$$
\frac{\alpha_{opt1}(1 - \gamma)}{\alpha_{opt1} - 1} \sum_{j=1}^{W} \tilde{\omega}_j|\hat{\beta}_j - \beta_j^*| \leq \frac{18\alpha_{opt1}^2}{\alpha_{opt1} - 1} H^2(\beta^*) v^2(\delta/2W) W(\beta^*)
$$

by $h = h_{\beta^*}$. Since $\tilde{\omega}_j \geq \tilde{\omega}_{min}$ for all $j$, we get

$$
\sum_{j=1}^{W} |\hat{\beta}_j - \beta_j^*| \leq \frac{18\alpha_{opt1}}{1 - \gamma} \cdot \frac{1}{\tilde{\omega}_{min}} \cdot \max_{j \in I(\beta)} \frac{\omega_j^2}{\|h_j\|^2} \cdot W(\beta^*).
$$
In this case, $\alpha_{\text{opt1}} = 2$ and $\|h_j\| = 1$, thus

$$\|\hat{\beta} - \beta^*\| \leq \frac{36}{1 - \gamma} \cdot \max_{j \in I(\beta)} \frac{\omega_j^2}{\omega_{\min}} \cdot W(\beta^*)$$

$$= \frac{72\sqrt{2}v(\delta/2W)W(\beta^*)}{1 - \gamma} \left(\max_{j \in I(\beta)} (L_j + L_{\min})^2 \right) \leq \frac{72\sqrt{2}v(\delta/2W)W(\beta^*)}{1 - \gamma} \left(\frac{L + L_{\min}}{L_{\min}}\right)^2$$

form $\omega_{\min} = 2\sqrt{2}v(\delta/2W)L_{\min}$, $\omega_j^2 = [2\sqrt{2}v(\delta/2W)]^2 \left[ L_j + \frac{\omega_{\min}}{2\sqrt{2}v(\delta/2W)} \right]^2 = [2\sqrt{2}v(\delta/2W)]^2[L_j + L_{\min}]^2$.

**Proof of Corollary 2** Let $\beta = \beta^*$ in Theorem 2, with $\alpha_{\text{opt2}} = 2$, we replace $v(\delta/2)$ in Theorem 2 by the larger value $v(\delta/2W)$, then

$$\sum_{j=1}^{W} \omega_{\min} |\hat{\beta}_j - \beta_j^*| \leq \sum_{j=1}^{W} \omega_j |\hat{\beta}_j - \beta_j^*| \leq \frac{576\alpha_{\text{opt2}} G^* v^2(\delta/2W)}{\lambda_W}.$$

Thus

$$\sum_{j=1}^{W} |\hat{\beta}_j - \beta_j^*| \leq \frac{576\alpha_{\text{opt2}} G^* v^2(\delta/2W)}{\omega_{\min} \lambda_W} = \frac{576 \cdot 2G^* v^2(\delta/2W)}{2\sqrt{2}v(\delta/2W)L_{\min} \lambda_W} = \frac{288\sqrt{2}G^* v(\delta/2W)}{L_{\min} \lambda_W}.$$

**Proof of Theorem 3** The following lemma is by virtue of KKT conditions. It derives a bound of $P(I_+ \not\subseteq \hat{I})$ which is easily analysed.

**Lemma 4.** *(Proposition 3.3 in [Bunea (2008)])*

$$P(I_+ \not\subseteq \hat{I}) \leq W(\beta^*) \max_{k \in I_+} P(\hat{\beta}_k = 0 \text{ and } \beta_k^* \neq 0).$$

To present the proof of Theorem 3, first we notice,

$$P(\hat{I} \neq I_+) \leq P(I_+ \not\subseteq \hat{I}) + P(\hat{I} \not\subseteq I_+).$$

Next, we control the probability on the right side of the above inequality.

For the control of $P(I_+ \not\subseteq \hat{I})$, by Lemma 4, it remains to find $P(\hat{\beta}_k = 0 \text{ and } \beta_k^* \neq 0)$.

In below, we will use the conclusion of Lemma 2 (KKT condition). Recall that $Eh_k(X_1) = \sum_{j \in I_+} \beta_j^* < h_k, h_j > = \sum_{j=1}^{W} \beta_j^* < h_k, h_j >$. Since we assume that the density of $X_1$ is the mixture density $h_{\beta^*} = \sum_{j \in I_+} \beta_j^* h_j$. So for $k \in I_+$, we have

$$P(\hat{\beta}_k = 0 \text{ and } \beta_k^* \neq 0)$$

$$= P \left(\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - \sum_{j=1}^{W} \hat{\beta}_j < h_j, h_k > \leq 2\sqrt{2}v(\delta/2W)L_k; \beta_k^* \neq 0 \right) \quad (36)$$

$$= P \left(\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) + Eh_k(X_1) - \sum_{j=1}^{W} \hat{\beta}_j < h_j, h_k > \leq 2\sqrt{2}v(\delta/2W)L_k; \beta_k^* \neq 0 \right)$$

$$= P \left(\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) - \sum_{j=1}^{W} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > \leq 2\sqrt{2}v(\delta/2W)L_k; \beta_k^* \neq 0 \right)$$
\[
\begin{align*}
&= P \left( \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) - \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > + \beta_k^* \| h_k \|^2 \leq 2\sqrt{2}v(\delta/2W)L_k \right) \\
&\leq P \left( \| \beta_k^* \| h_k \|^2 - 2\sqrt{2}v(\delta/2W)L_k \leq \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \right) + P \left( \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > \right) \\
&\leq P \left( \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \right) \geq \frac{\| \beta_k^* \| h_k \|^2}{2} - \sqrt{2}v(\delta/2W)L_k \right) \\
&\quad + P \left( \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > \right) \geq \frac{\| \beta_k^* \| h_k \|^2}{2} - \sqrt{2}v(\delta/2W)L_k \right) .
\end{align*}
\]

Similar to Lemma 2, for (37), we use Hoeffding’s inequality. Since \( \| h_k \| = 1 \) for all \( k \). Consider condition (B), \( \min_{k \in \mathcal{L}} |\beta_k^*| \geq 4\sqrt{2}v(\delta/2W)L \) and \( L \geq \max_{1 \leq k \leq W} L_k \), then we have

\[
\begin{align*}
&= P \left( \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \right) \geq \frac{\| \beta_k^* \| h_k \|^2}{2} - \sqrt{2}v(\delta/2W)L_k \right) \\
&\leq P \left( \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \right) \geq \frac{|\beta_k^*|}{2} - \sqrt{2}v(\delta/2W)L \\
&\leq P \left( \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \right) \geq 2\sqrt{2}v(\delta/2W)L - \sqrt{2}v(\delta/2W)L \\
&= P \left( \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \right) \geq \sqrt{2}v(\delta/2W)L \\
&\leq 2 \exp \left\{ -\frac{4n^2v^2(\delta/2W)L^2}{4nL^2} \right\} = 2 \exp \left\{ -nv^2(\delta/2W) \right\} = 2 \exp \left\{ -n \frac{\log(2W^2/\delta)}{n} \right\} = \frac{\delta}{W^2}.
\end{align*}
\]

For the upper bound of (38), using condition (A) and condition (B), considering the definition of \( \rho_*(\beta^*) \) and \( W(\beta^*) \), we have

\[
\begin{align*}
P \left( \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > \right) \geq \frac{\| \beta_k^* \| h_k \|^2}{2} - \sqrt{2}v(\delta/2W)L_k \right) \\
= P \left( \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > \right) \geq \frac{\| \beta_k^* \|}{2} - \sqrt{2}v(\delta/2W)L_k \right) \\
\leq P \left( \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) < h_j, h_k > \right) \geq 2\sqrt{2}v(\delta/2W)L - \sqrt{2}v(\delta/2W)L \\
\leq P \left( \rho_*(\beta^*) \sum_{j \neq k} (\hat{\beta}_j - \beta_j^*) \geq \sqrt{2}v(\delta/2W)L \right) \\
\leq P \left( \sum_{j=1}^{W} |\hat{\beta}_j - \beta_j^*| \geq \frac{\sqrt{2}v(\delta/2W)L}{\rho_*(\beta^*)} \right)
\end{align*}
\]
(by condition (A)) \leq P \left( \sum_{j=1}^{W} |\beta_j - \beta_j'| \geq \frac{288 \sqrt{2G^*} v(\delta/2W)}{L_{\min} \lambda_W} \right) \leq \frac{\delta}{W}.

The last inequality above is by using the $\ell_1$-estimation oracle inequalities in Corollary 2. Therefore, by the definition of $W(\beta^*)$, $W(\beta^*) = |I_\ast| \leq W$, we have

$$p(I_\ast \not\subseteq \hat{I}) \leq W(\beta^*) \max_{k \in I_\ast} P(\hat{\beta}_k = 0) \leq W(\beta^*) \frac{\delta}{W^2} + W(\beta^*) \frac{\delta}{W} \leq W \frac{\delta}{W^2} + W \frac{\delta}{W} = \delta + \delta.$$

For the controlling of $P(\hat{I} \not\subseteq I_\ast)$, let

$$\tilde{\eta} = \arg\min_{\eta \in \mathbb{R}^{W(\beta^*)}} z(\eta),$$

where

$$z(\eta) = -\frac{2}{n} \sum_{i=1}^{n} \sum_{j \in I_\ast} \eta_j h_j(X_i) + \|\sum_{j \in I_\ast} \eta_j h_j\|^2 + \sum_{j \in I_\ast} (4\sqrt{2}v(\delta/2)L_j + 2cB)|\eta_j| + c \sum_{j \in I_\ast} \eta_j^2.$$

Consider the following random event,

$$\bigcap_{k \not\in I_\ast} \left\{ \left| -\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) + \sum_{j \in I_\ast} \tilde{\eta}_j < h_j, h_k \right| \leq 2\sqrt{2}v(\delta/2)L_k \right\} \subseteq \bigcap_{k \not\in I_\ast} \left\{ \left| -\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) + \sum_{j \in I_\ast} \tilde{\eta}_j < h_j, h_k \right| \leq 2\sqrt{2}v(\delta/2W)L \right\} =: \Psi. \quad (41)$$

Let $\tilde{\eta} \in \mathbb{R}^W$ be a vector corresponding to the component of the index set $I_\ast$ having $\tilde{\eta}$ given by equation (40), and the component at other corresponding positions is 0. By Lemma 1, we know that $\tilde{\eta} \in \mathbb{R}^W$ is a solution of \((5)\) on event $\Psi$. It is recalled that $\beta \in \mathbb{R}^W$ is also a solution of \((5)\). Through the definition of indicator set $\hat{I}$, we have $\hat{\beta}_k \neq 0$ for $k \in \hat{I}$. By construction, we obtain $\hat{\eta}_k \neq 0$ for some subset $T \subseteq I_\ast$. The KKT conditions indicate that any two solutions have non-zero components at the same position. Therefore, $\hat{I} = T \subseteq I_\ast$ on event $\Psi$, thus, $P(\hat{I} \not\subseteq I_\ast)$ is a high probability.

Then

$$P(\hat{I} \not\subseteq I_\ast) \leq P(\Psi^c)$$

$$= P \left( \bigcup_{k \not\in I_\ast} \left\{ \left| -\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) + \sum_{j \in I_\ast} \tilde{\eta}_j < h_j, h_k \right| \geq 2\sqrt{2}v(\delta/2W)L \right\} \right) \quad (42)$$

$$\leq \sum_{k \not\in I_\ast} P \left\{ \left| -\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) + \sum_{j \in I_\ast} \tilde{\eta}_j < h_j, h_k \right| \geq 2\sqrt{2}v(\delta/2W)L \right\}$$

$$= \sum_{k \not\in I_\ast} P \left\{ \left| -\frac{1}{n} \sum_{i=1}^{n} h_k(X_i) + Eh_k(X_1) - Eh_k(X_1) + \sum_{j \in I_\ast} \tilde{\eta}_j < h_j, h_k \right| \geq 2\sqrt{2}v(\delta/2W)L \right\}$$

$$= \sum_{k \not\in I_\ast} P \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) - \sum_{j \in I_\ast} (\tilde{\eta}_j - \beta_j^*) < h_j, h_k \right| \geq 2\sqrt{2}v(\delta/2W)L \right\}$$

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In summary, we can get
\[ P \left\{ \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \geq \sqrt{2}v(\delta/2W)L \right\} \]

\[ + \sum_{k \notin I_*} P \left\{ \sum_{j \in I_*} |\tilde{\eta}_j - \beta_j^*| < h_j, h_k \geq \sqrt{2}v(\delta/2W)L \right\} \]  \hspace{1cm} (44)

According to the previously proved (39) formula, we have
\[ \sum_{k \notin I_*} P \left\{ \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \geq \sqrt{2}v(\delta/2W)L \right\} \leq \sum_{k=1}^{W} P \left\{ \frac{1}{n} \sum_{i=1}^{n} h_k(X_i) - Eh_k(X_1) \geq \sqrt{2}v(\delta/2W)L \right\} \leq W \cdot \frac{\delta}{W^2} = \frac{\delta}{W}. \]

For the upper bound of (44), observe Theorem 2, we can use a larger \( v(\delta/2W) \) instead of \( v(\delta/2) \).

Considering the construction of \( \tilde{\eta} \) in (40), we have
\[ P \left( \sum_{j \in I_*} |\tilde{\eta}_j - \beta_j^*| \geq \frac{288 \sqrt{2} G^* v(\delta/2W)}{L_{\min} \lambda W} \right) \leq \frac{\delta}{W}. \]

Similarly, we have
\[ \sum_{k \notin I_*} P \left\{ \sum_{j \in I_*} |\tilde{\eta}_j - \beta_j^*| < h_j, h_k \geq \sqrt{2}v(\delta/2W)L \right\} \leq \sum_{k=1}^{W} P \left\{ \sum_{j \in I_*} |\tilde{\eta}_j - \beta_j^*| \leq \frac{\sqrt{2}v(\delta/2W)L}{\rho_*(\beta^*)} \right\} \]
\[ = \sum_{k=1}^{W} P \left\{ \sum_{j \in I_*} |\tilde{\eta}_j - \beta_j^*| \geq \frac{\sqrt{2}v(\delta/2W)L}{\rho_*(\beta^*)} \right\} \]
(using condition (A)) \[ \leq \sum_{k=1}^{W} P \left\{ \sum_{j \in I_*} |\tilde{\eta}_j - \beta_j^*| \geq \frac{288 \sqrt{2} G^* v(\delta/2W)}{L_{\min} \lambda W} \right\} \leq \sum_{k=1}^{W} \frac{\delta}{W} = \delta. \]

In summary, we can get
\[ P(\hat{I} \neq I_*) \leq P(I_* \not\subseteq \hat{I}) + P(\hat{I} \not\subseteq I_*) \leq \frac{\delta}{W} + \frac{\delta}{W} + \frac{\delta}{W} = 2\delta(1 + \frac{1}{W}). \]

So, Theorem 3 is proved.

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