Planar 3-SAT with a Clause/Variable Cycle

Alexander Pilz

Institute of Software Technology, Graz University of Technology.
apilz@ist.tugraz.at

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Abstract

In the Planar 3-SAT problem, we are given a 3-SAT formula together with its incidence graph, which is planar, and are asked whether this formula is satisfiable. Since Lichtenstein’s proof that this problem is NP-complete, it has been used as a starting point for a large number of reductions. In the course of this research, different restrictions on the incidence graph of the formula have been devised, for which the problem also remains hard.

In this paper, we investigate the restriction in which we require that the incidence graph is augmented by the edges of a Hamiltonian cycle that first passes through all variables and then through all clauses, in a way that the resulting graph is still planar. We show that the problem of deciding satisfiability of a 3-SAT formula remains NP-complete even if the incidence graph is restricted in that way and the Hamiltonian cycle is given. This complements previous results demanding cycles only through either the variables or clauses.

The problem remains hard for monotone formulas and instances with exactly three distinct variables per clause. In the course of this investigation, we show that monotone instances of Planar 3-SAT with three distinct variables per clause are always satisfiable, thus settling the question by Darmann, Döcker, and Dorn on the complexity of this problem variant in a surprising way.

1 Introduction

Let $\phi$ be a Boolean formula in conjunctive normal form (CNF) and let $G_\phi$ be a graph whose vertices are the variables and the clauses of $\phi$ such that (1) every edge of $G_\phi$ is between a variable and a clause and (2) there is an edge between a variable $v$ and a clause $c$ if and only if $v$ occurs in $c$ (negated or unnegated). We call $\phi$ a CNF formula and $G_\phi$ is called the incidence graph of $\phi$. A CNF formula is a 3-SAT formula if every clause contains at most three variables. (We will also discuss the case where every clause contains exactly three distinct variables.) The Planar 3-SAT problem asks whether a given 3-SAT formula $\phi$ is satisfiable, given that $G_\phi$ is a planar graph. This problem has been shown to be NP-complete by Lichtenstein [21]. (In contrast to the general version, a PTAS is known for maximizing the number of satisfied clauses for the planar version of the 3-SAT problem [16].) See Figure 1 for drawings of an incidence graph.

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Reducing from Planar 3-SAT is a standard technique to show NP-hardness of problems in computational geometry. In these reductions, the vertices and edges of $G_\phi$ are replaced by gadgets (consisting of geometric objects) that influence each other. However, it is often useful to have further restrictions on $G_\phi$ or on how $G_\phi$ can be embedded.

Lichtenstein’s reduction already contains such a restriction: the problem remains NP-complete even if the graph remains planar after adding a cycle whose vertices are exactly the variables of the formula [21], and this cycle is part of the input. We call it a variable cycle. This fact allows for placing the variable vertices along a line, connected to “three-legged” clauses above and below that line (stated more explicitly in [18]; see Figure 2). De Berg and Khosravi [7] showed that it is also possible to have all literals in the clauses above the line to be positive, and all literals in clauses below the line negative. In Lichtenstein’s reduction, one may as well add a clause cycle whose vertices are exactly the clauses of the formula while keeping the graph planar [20]. We call the Planar 3-SAT variants in which we require a variable cycle and a clause cycle Var-Linked Planar 3-SAT and Clause-Linked Planar 3-SAT, respectively.\(^1\) Even though the incidence graphs constructed by Lichtenstein can be augmented with both a clause cycle [20] and a variable cycle, one cannot adapt Lichtenstein’s construction to always obtain both cycles without any crossings.\(^2\)

Our research is motivated by the following problem that attempts to combine these restrictions. See Figure 1 (right) for an accompanying illustration.

**Definition 1 (Linked Planar 3-SAT).** Let $G_\phi = (C \cup V, E)$ be the incidence graph of a 3-SAT formula $\phi$, where $C$ is the set of clauses and $V$ is the set of variables of $\phi$. Further, let $\kappa$ be a Hamiltonian cycle of $C \cup V$ that first visits all elements of $C$ and then all elements of $V$. Suppose that the union of $G_\phi$ and $\kappa$ is a planar graph. The Linked Planar 3-SAT problem asks, given $\phi$, $G_\phi$, and $\kappa$, whether $\phi$ is satisfiable.

\(^1\)While Lichtenstein’s definition of planar 3-SAT [21] already requires the cycle through the variables, this property is often considered an explicit restriction. We thus follow the terminology of Fellows et al. [11] to emphasize when the variable cycle is needed.

\(^2\)If not stated otherwise, we will implicitly require the incidence graphs augmented by the additional edges to be planar throughout this paper.
Related problems in which all variables or all clauses can be drawn incident to the unbounded face are known to be in P, due to results by Knuth [17], and Kratochvíl and Krívánek [19], respectively. In particular, this is the case when there is a variable cycle and a path connecting all clauses or vice versa. One way of tackling the Linked Planar 3-SAT problem could be to show that \( G_\phi \) has bounded treewidth. For such instances, the satisfiability of \( \phi \) can be decided in polynomial time [12]. (This generalizes the above-mentioned results, as every \( k \)-outerplanar graph has treewidth at most \( 3k - 1 \) [4]3; see also Demaine’s lecture notes [10].) However, with the right perspective on the Linked Planar 3-SAT problem, it will be easy to observe that there are formulas whose incidence graph has a grid minor with a linear number of vertices (and thus such graphs have unbounded treewidth). It is the same perspective through which we will show NP-completeness of the problem in Section 2, using a reduction from Planar 3-SAT. We note that requiring an arbitrary Hamiltonian cycle is not a restriction: As the incidence graph is bipartite, it is known that its page number is two [8]; hence, we can always add a Hamiltonian cycle through the variables and clauses in a planar way (possibly re-using edges of the incidence graph).

\[ (v_1 \lor v_3 \lor \neg v_4) \]
\[ (\neg v_1 \lor \neg v_2 \lor \neg v_3) \]
\[ (\neg v_3 \lor \neg v_3 \lor v_4) \]
\[ (v_2 \lor v_3 \lor v_4) \]

Figure 2: A “three-legged” Planar 3-SAT instance with variables on a line similar to [18, p. 425]. (There, two-variable clauses are transformed to three-variable clauses that contain one literal twice, a construction that is not necessary but possible for the initial graph in our reduction.)

1.1 Motivation

Restrictions on the problem to reduce from can make NP-hardness reductions simpler. For reductions from Planar 3-SAT, it is common to actually reduce from Var-Linked Planar 3-SAT, using the variable cycle, in particular the “three-legged” embedding of [18]. Also, the clause cycle has been used [20, 11]. For an exhaustive survey on the numerous variants of Planar 3-SAT, see the thesis of Tippenhauer [26].

One motivation for considering Linked Planar 3-SAT is the framework for showing NP-hardness of platform games by Aloupis et al. [2]. In this class of reductions from 3-SAT to such games, a player’s character starts at a specified position and traverses all variable gadgets, making a decision on their truth value. Clauses connected to the satisfied literal can be “unlocked” by visiting these clauses. Finally the player’s character has to traverse

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3An outerplanar graph is 1-outerplanar, and a \( k \)-outerplanar graph is \( (k - 1) \)-outerplanar after removing the vertices incident to the unbounded face.
all clause gadgets to reach the finish (called the “check path”). The framework then requires a game-specific implementation of the gadgets for start, finish, variables, clauses, and crossovers. Reducing from \textsc{Linked Planar 3-SAT} removes this dependency on crossover gadgets. (In particular, Theorem 9 can be used to show that the traversal through the literals can be done without crossings.) See [2, Section 2.1] for a more detailed description.

1.2 Results

We first prove that \textsc{Linked Planar 3-SAT} is NP-complete. In the following sections, we refine the construction to show that restricted variants of the problem remain hard as well. In particular, we do this for formulas without negated and unnegated variables in the same clause (\textsc{Monotone Planar 3-SAT}), and formulas for which the edges to negated variables are all on the same side of $\kappa$ (recall that $\kappa$ is the Hamiltonian cycle that first visits all variables and then all clauses). Also, we may require that all clauses contain exactly three distinct variables. A cycle $\kappa$ in instances of \textsc{Positive Planar 1-in-3-SAT} (which requires exactly one true literal in each clause) also keeps the problem hard.

Finally, we discuss settings in which the planarity constraint is fulfilled only by satisfiable formulas. In particular, we show that planar CNF formulas with at least four variables per clause are always satisfiable. The same holds for instances of \textsc{Monotone Planar 3-SAT} with exactly three variables per clause. This solves an open problem by Darmann, Döcker, and Dorn [5, 6], who show that the corresponding problem with at most three variables per clause remains NP-complete with bounds on the variable occurrences, which refines the result of de Berg and Khosravi [7].

2 NP-hardness of Linked Planar 3-SAT

We now show that \textsc{Linked Planar 3-SAT} is NP-hard by reducing \textsc{Planar 3-SAT} to it. We are thus given a 3-SAT formula $\tilde{\phi}$ with variable set $\tilde{V}$, clause set $\tilde{C}$, and incidence graph $G_{\tilde{\phi}}$. Our goal is to construct another formula $\phi$ that is an instance of \textsc{Linked Planar 3-SAT} with incidence graph $G_{\phi}$ and Hamiltonian cycle $\kappa$ such that $\phi$ is satisfiable if and only if $\tilde{\phi}$ is. The construction of $G_{\phi}$ will be given by an embedding.

We start by producing a suitable embedding of the initial graph $G_{\tilde{\phi}}$. We require an embedding $\Gamma$ of $G_{\tilde{\phi}}$ on the integer grid, with edges drawn as $x$-monotone curves. (Our construction can easily be modified for non-$x$-monotone edges, but this assumption facilitates the presentation.) We further require that the size of the grid is polynomial in the size of $\tilde{\phi}$. It is well-known that a planar graph with $n$ vertices can be embedded with straight-line edges on an $O(n) \times O(n)$ grid in $O(n)$ time [9, 25]. We can take such an embedding and perturb the clause vertices s.t. each variable has even $x$-coordinate, and the $x$-coordinate of each clause is odd. This can be done without introducing crossings by choosing the grid sufficiently large and scaling $\Gamma$; a blow-up by a factor polynomial in $n$ is sufficient. More specifically, we scale $\Gamma$ by a polynomial multiple of 2 and increase the $x$-coordinate of each clause vertex by 1. (It will become apparent that our reduction merely uses the combinatorial structure of the embedding; however, it seems easier to describe the construction with a fixed straight-line embedding of $G_{\tilde{\phi}}$ on the integer grid.)

\footnote{We can consider the smallest horizontal distance between a vertex and a non-incident edge. This distance $v$ is rational with numerator and denominator quadratic in the largest coordinate. By multiplying the $x$-coordinates of the vertices by $2/v$, rounding, and again multiplying by 2, we get the desired embedding, similar to [1, Lemma 6.1].}
Figure 3: Left: Drawings of the paths $\kappa_C$ (solid) and $\kappa_V$ (dotted), containing the clauses and variables of $\phi$, respectively. They intersect a rectangle $R$ (gray) in vertical segments. The incidence graph is drawn inside $R$. Right: A similar construction can be used to show that all the incidence graphs we obtain can be augmented by either a clause cycle or a variable cycle (and not just two paths obtained from $\kappa$).

With a suitable drawing of $G_{\tilde{\phi}}$ at hand, we start the drawing of $G_\phi$ with the curve that will contain the cycle $\kappa$ (we add its vertices later). It can be partitioned into two paths, one that will contain the elements of the variable set $V(\kappa_V)$ and one for the elements of the clause set $(\kappa_C)$. In our drawing shown in Figure 3 (left), we obtain a rectangular region $R$, whose intersection with $\kappa$ consists of vertical line segments of unit distance, in alternation belonging to $\kappa_V$ and $\kappa_C$. We call them the clause segments and variable segments, respectively. We assume that the segments are placed on the integer grid with unit distance, with the variable segments having even $x$-coordinates and the clause segments having odd $x$-coordinates. In other words, $R$ represents a grid in which the columns are traversed in alternation by $\kappa_V$ and $\kappa_C$.

Place $\Gamma$ inside $R$. By the construction of $\gamma$, all its variable vertices have even $x$-coordinates and are thus on variable segments, and its clause vertices are on clause segments. We obtain $G_\phi$ by replacing the edges of $\Gamma$ by gadgets, consisting of subgraphs of $G_{\tilde{\phi}}$. In the construction, we will make use of “cyclic implications”, effectively copying the value of a variable; the graph $G_\phi$ will contain many pairs of variables $x$ and $x'$ with a clause $c_x = (\neg x \lor x')$ and a clause $c'_x = (x \lor \neg x')$. Clearly, $x = x'$ in any satisfying assignment. We depict the negation in such a clause $c_x$ by an arrow from $x$ to $c_x$, and from $c_x$ to $x'$ (where the arrows are also edges of the incidence graph). In general, we use the convention that an arrow from a variable to a clause denotes that the variable occurs negated in that clause, while an arrow from the clause to the variable means that the variable occurs unnegated.

We replace each edge $e$ of $\Gamma$ by a sequence of so-called connector gadgets. A connector gadget consists of two variables $x$ and $x'$, and two clauses $c_x = (\neg x \lor x')$ and $c'_x = (\neg x' \lor x)$, implying $x = x'$ in any satisfying truth assignment. The variable vertices are placed on the intersections of $e$ with two consecutive variable segments in $R$, and the clause vertices are placed on the clause segment between them (also close to the crossing of $e$ and the clause segment), effectively subdividing $\kappa$. An edge in $\Gamma$ connecting a variable $\tilde{v} \in \tilde{V}$ to a clause...
\[ \tilde{c} \in \tilde{C} \] that crosses \( \kappa \) can be replaced by a sequence of connector gadgets in a sufficiently small neighborhood of the edge, as shown in Figure 4. Thus, in the resulting drawing, we have subdivided \( \kappa \) to remove crossings with \( e \), and the resulting formula is satisfiable if and only if the initial formula is satisfiable.

Replacing the edges of \( \Gamma \) by the connector gadgets results in a drawing of \( G_\phi \). As all edges are \( x \)-monotone, all crossings with \( \kappa \) are replaced. Thus this drawing is planar and contains a number of vertices that is polynomial in \( |C| \), as the number of crossings of an edge in \( \Gamma \) with clause and variable segments (and thus the number of vertices needed to replace the edge) is bounded by the grid size. Also, all “new” clauses contain only two variables (we will see a modification of the reduction without this property). As none of the edges in the resulting embedding of \( G_\phi \) crosses the initially drawn cycle for \( \kappa \), \( G_\phi \) can be augmented by \( \kappa \) along that cycle maintaining planarity. Finally, observe that \( \phi \) is satisfiable if and only if \( \tilde{\phi} \) is: the two variables of a connector gadget must have the same value, and the clauses not part of the connector gadget are the clauses of \( \tilde{C} \) in which we replaced variables by others that have to be equal. We thus obtain our main result.

**Theorem 2.** Linked Planar 3-SAT is \( \text{NP-complete} \).

Let us illustrate the reduction explicitly starting with a VAR-LINKED PLANAR 3-SAT instance. Figure 5 shows an example of the reduction starting with the “three-legged” drawing shown in Figure 2. We can multiply the coordinates of the vertices (and of the bends) by four; this gives enough freedom to embed the vertices accordingly. Observe that the vertices and bends of such an embedding have coordinates of absolute value at most \( 4 \cdot 3 |\tilde{C}| \) (and in this representation, observe that we only need to blow the grid up by a factor of 2). We replace each horizontal bar representing a variable by a sequence of connector gadgets as shown in Figure 5. Note that all variables in the gadget have the same value due to the cyclic implication, and all its vertices have distance at most one from the original segment. The original vertex representing a clause, say, \( \tilde{c} = (v_1 \lor v_3 \lor \neg v_4) \) is replaced by a clause \( c = (v'_1 \lor v'_3 \lor \neg v'_4) \). Observe that due to the cyclic implication, the variables \( v_1, v_2, \ldots \) need to have the same values as their surrogates \( v'_1, v'_2, \ldots \). Since all coordinates of \( \Gamma \) are multiples of four, the gadgets can be drawn inside the “thickened” legs of the initial drawing, and thus the new drawing remains crossing-free.

Observe that our construction also does not change the number of satisfiable assignments, i.e., the reduction is \textit{parsimonious}. Since counting the number of satisfiable assignments to a planar 3-SAT formula is \#P-complete [15], this also holds for LINKED PLANAR 3-SAT.
Figure 5: A construction based on the three-legged drawing in Figure 2. A clause $\tilde{c} = (v_1 \lor v_3 \lor \neg v_4)$ of $\phi$ in $\Gamma$ is replaced by the clause $c = (v'_1 \lor v'_3 \lor \neg v'_4)$. The truth value of the variables on the $x$-axis is “transported” to the new clause by the according cyclic implications.

3 Further variants

We use the main idea of drawing the cycle $\kappa$ as shown in Figure 3 to obtain similar reductions for variants of the planar satisfiability problem. If the initial problem is known to be hard, we merely need to find an according connector gadget to replace the crossings of edges of $G_{\tilde{\phi}}$ and $\kappa$.

3.1 Positive Planar 1-in-3-SAT

It is easy to transform our reduction to be an instance of 1-IN-3-SAT, where exactly one variable is true: we get the “main” clauses directly from an initial 1-IN-3-SAT instance, and the connector gadgets are the same. We therefore impose two more requirements on the instance. First, all clauses should have three elements, and second, all literals should be positive.

Definition 3 (Positive Planar 1-in-3-SAT [24]). Given a formula $\phi$ in which each clause contains exactly three distinct unnegated literals, and an embedding of the incidence graph $G_{\phi}$, the Positive Planar 1-in-3-SAT problem asks whether there exists a satisfying assignment of $\phi$ such that exactly one variable in each clause is true.

Mulzer and Rote [24] show that deciding satisfiability of Positive Planar 1-in-3-SAT instances is hard, even if the incidence graph can be augmented by a variable cycle.

Theorem 4. The Positive Planar 1-in-3-SAT problem remains NP-complete even for problem instances that are also instances of the Linked Planar 3-SAT problem.

Proof. The reduction works again by blowing up the grid embedding of the initial Positive Planar 1-in-3-SAT incidence graph by a polynomial factor $f$ and increase the $x$-coordinate of each clause by 1. We choose $f$ to be a multiple of 8, as (i) all variable vertices should have even $x$-coordinates (and are thus placed on the appropriate part of $\kappa$), (ii) the number of clause segments between a variable and a clause (before increasing its $x$-coordinate by 1) is a multiple of two (because of the gadget width described below), and (iii) there are at least three variable segments between each clause vertex of the initial instance and each of its
variables, even after increasing the $x$-coordinate of each clause vertex by 1 (to have sufficient space for “detours” described below). As before, $f$ is chosen such that moving the clauses does not produce any crossings in the straight-line embedding. Then, we use two clauses sharing two variables to produce a connector gadget, as shown in Figure 6. The two clauses $(x, a, b)$ and $(x', a, b)$ ensure that $x = x'$.

The construction works analogous to the simpler connector gadget of the previous section, except for the following caveat. The connector gadgets have width 4, and this is the reason for requiring condition (ii): We placed each clause of the initial formula at positions with $x$-coordinate $8k + 1$, for some integer $k$. The sketch on the right of Figure 6 shows how to connect the clause to the connector gadgets. (Note the “detour” one chain of connector gadgets may have to take in the vicinity of the initial clause to connect to it if all three edges of the clause emanate to the right in the initial drawing; this does not interfere with other gadgets as $f$ is a multiple of 8.)

3.2 Exactly three distinct variables per clause

If we require the formula to have exactly three distinct variables in each clause, the reduction can also be modified accordingly. Mansfield [22] showed how to extend Lichtenstein’s construction to obtain a planar 3-SAT formula with exactly three different variables per clause by constructing such a formula with planar incidence graph and a variable that is false in every satisfying assignment.

**Theorem 5.** The LINKED PLANAR 3-SAT problem remains NP-complete even if each clause contains exactly three different variables.

**Proof.** We reduce from the non-linked version of the problem by drawing an incidence graph on our grid and replace edge parts by gadgets that transport the truth settings of a variable. To this end, we use a modified connector gadget shown in Figure 7, which assure that two variables $x$ and $x'$ always have the same value: When removing $u$ from the shown formula, we get $x \Rightarrow a$ and $a \Rightarrow x'$, as well as $x \Rightarrow b$ and $b \Rightarrow x'$. Since $u$ occurs unnegated in the clauses containing $a$, and it occurs negated in the clauses containing $b$, one of the two implication pairs will make sure that $x \Rightarrow x'$. More formally, we have $(\neg x \lor a \lor u) \land (\neg a \lor u \lor x')$, which entails $(\neg x \lor a \lor u \lor x')$, and $(\neg x \lor b \lor \neg u) \land (\neg b \lor \neg u \lor x')$, which entails $(\neg x \lor \neg u \lor x')$. These two clauses have $(\neg x \lor x')$ as a resolvent. Analogously, the clauses containing $u'$ imply $x' \Rightarrow x$.

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5We thank an anonymous referee for simplifying the previously-used gadget.

6This restriction has been re-discovered, e.g., in [15].
We note that in Mansfield’s reduction it is no longer shown that there exists a variable cycle, so we cannot rely on a “three-legged” embedding but rather use a straight-line one. As in the previous reduction, the connector gadgets have width 4. We thus again use a suitable blow-up factor and place each clause of the initial formula at positions with $x$-coordinate $8k + 1$, for some integer $k$.

Note that the bi-implication implemented by the connector gadget has sixteen different truth assignments, independent of whether $x$ is true or false. As the Planar 3-SAT problem is known to be #P-complete even if each clause contains exactly three distinct variables [15], we can add connector gadgets to transform any such instance into a Linked Planar 3-SAT instance. For each connector gadget, the number of solutions is multiplied by 16.

**Theorem 6.** The Linked Planar 3-SAT problem remains #P-complete even if each clause contains exactly three different variables.

### 3.3 Monotonicity restrictions

We can add the following list of restrictions to the setting for which the problem remains hard. Both can be shown by reducing from Monotone Planar 3-SAT.

A clause is monotone if it contains either only negated or only unnegated literals; a formula is monotone if all its clauses are monotone.

**Definition 7 (Monotone Planar 3-SAT).** Let $G_\phi = (C \cup V, E)$ be the incidence graph of a 3-SAT formula $\phi$, where $C$ is the set of clauses and $V$ is the set of variables of $\phi$, and each clause of $C$ is monotone. The Monotone Planar 3-SAT problem asks whether $\phi$ is satisfiable.

This problem has been shown to be NP-complete by de Berg and Khosravi [7]; they actually show that the problem remains hard even if there is a variable cycle separating the clauses with the negated variables from the ones with the unnegated variables. That is, the incidence graph can be drawn in a rectilinear way, with the variables on the $x$-axis and exactly the clauses with the negated occurrences below the $x$-axis. We can use their result to show hardness of the according problem in our setting.

**Theorem 8.** The Linked Planar 3-SAT problem remains NP-complete even if all clauses are monotone.
Figure 8: A variant of the connector gadget in which all clauses are monotone. We have $x \neq x' = x$ in the middle. The gadget can be further split to have each variable vertex of degree at most three (right).

**Proof.** We reduce from MONOTONE PLANAR 3-SAT. The reduction is the same as for general LINKED PLANAR 3-SAT, but with a different connector gadget. We use two sub-gadgets that contain two clauses, one all positive one all negative, that ensure that two variables are not equal. Two such gadgets in sequence form a new connector gadget. See Figure 8. We can deal with the position of the clause being off by one to the end of each connector gadget in the same way as in Figure 7 (right).

Lichtenstein already showed that PLANAR 3-SAT remains NP-complete even if we require that the variable cycle partitions the edges of every variable vertex into those leading to a negated occurrence and those leading to an unnegated one [21, Lemma 1]. (As he mentions, this also implies that one could split a variable vertex into two literal vertices while preserving planarity.) Note that there is a subtle difference to the restriction of de Berg and Khosravi [7] to MONOTONE PLANAR 3-SAT, as the side of the variable cycle to which the edges to the negated occurrences emanate is not fixed globally. The following set of related restrictions could be particularly interesting for further reductions.

**Theorem 9.** The LINKED PLANAR 3-SAT problem remains NP-complete even if, for each clause, the edges corresponding to positive occurrences emanate to the interior of the cycle $\kappa$, and the ones to negated occurrences to the exterior. In addition, each variable occurs in at most three clauses.

**Proof.** We reduce from the MONOTONE PLANAR 3-SAT variant of de Berg and Khosravi [7], using a “three-legged” embedding. The construction (see Figure 9) uses cycles consisting of the variables $x_1, \ldots, x_k$ and $\overline{x_1}, \ldots, \overline{x_k}$ and clauses $(\neg x_i \vee x_{i+1})$ and $(\neg \overline{x_i} \vee \overline{x_{i+1}})$ for all valid indices, as well as the clauses $(x_1 \vee \overline{x_k})$ and $(\neg \overline{x_1} \vee \overline{x_k})$. The latter thus entails $(\neg x_1 \vee \overline{x_k})$. We therefore have $x_i = x_j$, $\overline{x_i} = \overline{x_j}$, and $x_i \neq \overline{x_j}$ for $i, j \leq k$. The variable vertices are placed from left to right with increasing indices. Thus, the variables on the upper part of the cycle have the opposite value of those on the lower part. For a clause with positive literals in the MONOTONE PLANAR 3-SAT instance, we connect the variables as in Figure 9.

We therefore have variable gadgets that consist of said cycles; see the three bottom-most cycles in Figure 9. These variable gadgets are connected to the clause $c$ (which represents the clause of the initial formula) either directly (for the middle variable gadget), or via another cycle (for the left and the right variable gadget). For these connector gadgets, we again have that the variables on the upper part have the opposite value as those of the lower part; in the way a variable gadget and a connector gadget are connected, a variable on the upper part of the variable gadget has the same value as one on the lower part of the connector gadget. The clause $c$ has two edges to the left (which therefore correspond to negated occurrences) and one to the right (an unnegated occurrence). In the setting shown in Figure 9, we consider a variable set to true if the variables on the lower part of the variable gadget are true. A clause $\overline{c} = (x \vee y \vee z)$ is thus transformed to $c = (\neg \overline{x_1} \vee \neg \overline{y_1} \vee z_k)$. We can have variable
Figure 9: All clauses have negative literals to the left and positive literals to the right. The variable state is transported by cycles where the variables on top have the negative value of the variables at the bottom. A clause \( \tilde{c} = (x \lor y \lor z) \) is transformed to \( c = (\neg x_i \lor \neg y_j \lor z_k) \). The “three-legged” embedding of the clause is indicated by the gray contour.

gadgets of arbitrary width simply by having larger cycles, and connect the clauses according to the “three-legged” embedding. The resulting construction is thus crossing-free, and the formula is satisfiable if and only if the initial one is.

Observe that the connector gadgets in Figure 9 use only clauses with two variables. The reduction therefore also works for Planar 2-SAT instances. While Planar 2-SAT can be solved in polynomial time, it is known to be \#P-complete [28]. As our gadgets are parsimonious (i.e., do not change the number of solutions), they can be applied to show \#P-completeness.

**Corollary 10.** The Linked Planar 2-SAT problem is \#P-complete. It remains \#P-complete even if for each clause, the edges corresponding to positive occurrences emanate to the interior of the cycle \( \kappa \), and the ones to negative literals to the exterior.

Note that making each variable occur in at most three clauses requires that there are clauses with at most two (different) literals, as every CNF formula with exactly three literals per clause and at most three occurrences per variable is satisfiable [27].

Darmann, Döcker, and Dorn [5, 6] showed how to reduce the number of times a variable occurs. For the variant of Monotone Planar 3-SAT that requires exactly three different variables per clause, the complexity was previously unknown [5, 6]. Surprisingly, it turns out that such instances are always satisfiable, as discussed in the next section.

4 Remark: different cycles through clauses and variables

Observe that, for all our constructions, \( G_\phi \) still allows for adding a variable cycle \( H \), as well as a clause cycle \( H' \), as shown in Figure 3 (right). But these cycles will, in general, cross mutually. Also, they will cross the cycle \( \kappa \). Recall that if \( H \) did not cross \( H' \) or \( \kappa_C \), the problem would be solvable in polynomial time [10].

Using the gadgets and the two cycles shown in Figure 3 (right), we observe that Planar 3-SAT remains NP-complete even if these cycles exist for all variants mentioned. While the variable and clause cycles have been identified in [21] and [20], respectively, it seems to have been unknown for the variants using exactly three variables per clause (even though Mansfield’s construction [22] can be embedded to obtain the cycles). Also, it seems that clause
cycles have not been considered for Monotone Planar 3-SAT and Planar Positive 1-in-3-SAT.

5 Properties forcing satisfiability

Planarity is a rather drastic combinatorial restriction on the structure of a graph. While NP-completeness of 3-SAT is preserved in the planar setting, further properties may lead not only to polynomial-time algorithms (as for Planar NAE-SAT [23]), but also to instances that are always satisfiable.

Theorem 11. Every instance of Planar SAT in which each clause has three negated or three unnegated occurrences of three distinct variables is satisfiable. A satisfying assignment can be found in quadratic time.

Proof. Consider any plane drawing of the incidence graph of the formula. For each clause vertex, we can add a 3-cycle consisting of three of its variable vertices that are either all negated or all unnegated, as shown in Figure 10. (For clauses with three variables this is similar to a Y-Δ transform, a common operation to replace a vertex of degree 3 by a 3-cycle: connect two variables by a curve in a neighborhood of the two edges connecting them to the clause.) Observe that, after removing the clause vertices, the resulting graph is still planar. It is thus 4-colorable [3] and we may consider any 4-coloring of the variables. Set the variables of the vertices with colors 1 and 2 to true, and the others to false. A 3-cycle contains three different colors, and thus a 3-cycle through tree (monotone) variable vertices has at least one variable set to true and one variable set to false.

Corollary 12. Every instance of Monotone Planar SAT with at least three distinct variables per clause is satisfiable.

Corollary 13. Every instance of Planar SAT with at least five distinct variables per clause is satisfiable.

For at least four variables per clause, we can give a similar result, closing the gap between Corollary 13 and Mansfield’s result [22], using less heavy machinery than the Four-Color theorem. The proof makes use of Hall’s theorem [13], inspired by a technique by Tovey [27]: each subset of \( k \) clauses has at least \( k \) variables occurring in it.

Lemma 14. Let \( G = (B \cup C, E) \) be a bipartite planar graph with parts \( B \) and \( C \) such that all vertices in \( C \) have degree at least four. Then there exists a matching covering every vertex of \( C \).
\textbf{Proof.} Consider any plane embedding of $G$ and let $F$ be its set of faces. Thus, by Euler’s formula, we have $|B| + |C| - |E| + |F| = 1 + k$, where $k \geq 1$ is the number of connected components of $G$. Every edge has two incidences with faces, and since $G$ is bipartite, every face has at least four incidences with edges. We get

$$2|E| = \sum_{f \in F} |f| \geq 4|F|$$

(1)

(where $|f|$ is the number of sides of the face $f$). Combining Euler’s formula and (1) to $|E| \geq 2(1 + k - |B| - |C| + |E|)$ gives the bound

$$2|B| + 2|C| - 2 - 2k \geq |E| .$$

(2)

Further, $|E| \geq 4|C|$, as $G$ is bipartite and every element of $C$ is incident to at least four edges. Combining this with (2) results in

$$|C| \leq |B| - 1 - k .$$

(3)

Finally, observe that every subset of $C$ plus its neighbors in $B$ also induce a graph in which the analogue of (3) holds. As, for every subset of $C$, the set of adjacent elements in $B$ has at least the same cardinality, there is a matching on $G$ that covers $C$ by Hall’s theorem [13]. \hfill \Box

\textbf{Theorem 15.} Each CNF formula with planar incidence graph of $n$ vertices and at least four distinct variables per clause has a satisfying assignment, which can be found in $O(n^{1.5})$ time.

\textbf{Proof.} Let $\phi$ be a planar SAT instance with vertex set $V$ and clause set $C$, and let $G_\phi = (V \cup C, E)$ be the associated incidence graph. By Lemma 14, there is a matching on $G_\phi$ covering $C$, which assigns a distinct variable to each clause (i.e., a system of distinct representatives for the clauses). If we set the according literal to true, we get an assignment satisfying $\phi$. The matching can be found using the algorithm by Hopcroft and Karp [14]. \hfill \Box

Satisfiability of such formulas also follows from Theorem 11: A clause with four variables that does not have three negated or three unnegated occurrences is of the form $(a \lor b \lor \neg c \lor \neg d)$, for some variables $a, b, c, d$. Such a clause can be replaced in an equisatisfiable formula by two clauses using one additional variable $u$: $(a \lor b \lor u) \land (\neg u \lor \neg c \lor \neg d)$. Replacing all such clauses in the original formula therefore results in a new equisatisfiable formula that has either three negated or three unnegated occurrences in each clause. Further, if the original formula had a planar incidence graph, there is also one for the new formula. (Alternatively, we can think of replacing such clauses with a construction involving two 3-cycles in the proof of Theorem 11.)

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