Reduction of Rota’s basis conjecture to a problem on three bases

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Abstract

It is shown that Rota’s basis conjecture follows from a similar conjecture that involves just three bases instead of \( n \) bases.

Key words: common independent sets, non-base-orderable matroid, odd wheel

1 Introduction

In 1989, Rota formulated the following conjecture, which remains open.

**Conjecture 1 (Rota’s basis conjecture)** Let \( M \) be a matroid of rank \( n \) on \( n^2 \) elements that is a disjoint union of \( n \) bases \( B_1, B_2, \ldots, B_n \). Then there exists an \( n \times n \) grid \( G \) containing each element of \( M \) exactly once, such that for every \( i \), the elements of \( B_i \) appear in the \( i \)th row of \( G \), and such that every column of \( G \) is a basis of \( M \).

Partial results towards this conjecture may be found in [1,2,3,4,5,6,7,8,12,14,15]. Now consider the following conjecture.

**Conjecture 2** Let \( M \) be a matroid of rank \( n \) on \( 3n \) elements that is a disjoint union of 3 bases. Let \( I_1, I_2, \ldots, I_n \) be disjoint independent sets of \( M \), with \( 0 \leq |I_i| \leq 3 \) for all \( i \). Then there exists an \( n \times 3 \) grid \( G \) containing each element of \( M \) exactly once, such that for every \( i \), the elements of \( I_i \) appear in the \( i \)th row of \( G \), and such that every column of \( G \) is a basis of \( M \).

The main purpose of the present note is to make the following observation.

**Theorem 3** Conjecture 2 implies Conjecture 1.

Our proof is inspired by the proof of Theorem 4 in [10].
**PROOF.** Since Conjecture 1 is known if \( n \leq 2 \), we may assume that \( n \geq 3 \). Let \( M \) be given as in the hypothesis of Conjecture 1. Define a **transversal** to be a subset \( \tau \subseteq M \) that contains exactly one element from each \( B_i \). Define a **double partition** of \( M \) to be a pair \((\beta, \tau)\) where \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) is a partition of \( M \) into \( n \) pairwise disjoint bases \( \beta_i \) and \( \tau = (\tau_1, \tau_2, \ldots, \tau_n) \) is a partition of \( M \) into \( n \) pairwise disjoint transversals. Given a double partition \((\beta, \tau)\), define
\[
\mu(\beta, \tau) = \sum_{i \neq j} |\beta_i \cap \tau_j|.
\]
Observe that if \( \mu(\beta, \tau) = 0 \) then necessarily \( \beta_i = \tau_i \) for all \( i \), and then Rota’s basis conjecture follows—just let the \((i, j)\) entry of \( G \) be \( B_i \cap \tau_j \).

So let \((\beta, \tau)\) be an arbitrary double partition with \( \mu(\beta, \tau) > 0 \). We show how to construct a double partition \((\beta', \tau')\) with \( \mu(\beta', \tau') < \mu(\beta, \tau) \); the proof is then complete, by infinite descent, since by hypothesis there exists at least one double partition. Since \( \mu(\beta, \tau) > 0 \), there exist \( \beta_i \) and \( \tau_j \) with \( i \neq j \) such that \( \beta_i \cap \tau_j \neq \emptyset \). Since \( n \geq 3 \), there also exists \( k \) such that \( i, j, \) and \( k \) are all distinct. It will simplify notation to assume that \( i = 1, j = 2, \) and \( k = 3 \); no generality is lost, and it will be convenient to be able to reuse the index variables \( i \) and \( j \) below. Let \( S = \beta_1 \cup \beta_2 \cup \beta_3 \), let \( T = \tau_1 \cup \tau_2 \cup \tau_3 \), and let \( M' = M \setminus S \) (i.e., \( M \) restricted to the ground set \( S \)).

For each \( i \), let \( I_i = B_i \cap T \setminus S \). Then \( I_i \) is an independent subset of the matroid \( M' \), and \(|I_i| \leq |B_i \cap T| \leq 3 \). The \( I_i \) are pairwise disjoint because the \( B_i \) are pairwise disjoint. Therefore we may apply Conjecture 2 to obtain an \( n \times 3 \) grid \( G' \) whose columns \( \beta'_1, \beta'_2, \) and \( \beta'_3 \) are disjoint bases of \( M' \) (and therefore are bases of \( M \)) and whose \( i \)th row contains the elements of \( I_i \).

To construct the desired double partition \((\beta', \tau')\), let \( \beta' = \beta \) except with \( \beta_1, \beta_2, \) and \( \beta_3 \) replaced with \( \beta'_1, \beta'_2, \) and \( \beta'_3 \) respectively. Similarly, let \( \tau' = \tau \) except with \( \tau_1, \tau_2, \) and \( \tau_3 \) replaced with \( \tau'_1, \tau'_2, \) and \( \tau'_3 \), which are defined as follows. Let \( G'' \) be any \( n \times 3 \) grid whose columns \( \tau'_1, \tau'_2, \) and \( \tau'_3 \) replace \( \tau_1, \tau_2, \) and \( \tau_3 \), respectively. Its \( (i, j) \)th entry agrees with that of \( G' \) whenever that entry is in \( I_i \). Clearly \( G'' \) exists (though it may not be unique). Let \( \tau'_j \) be the \( j \)th column of \( G'' \), for \( j = 1, 2, 3 \).

It is easily verified that what we have done is to regroup the elements of \( M' \) into three new bases and to regroup the elements of \( T \) into three new transversals in such a way that the contribution to \( \mu(\beta', \tau') \) from intersections of the new bases with the new transversals is reduced to zero, and such that the total of the other contributions to \( \mu \) is unchanged. Thus the overall value of \( \mu \) is reduced, as required. \( \square \)

Careful inspection of the above proof shows that it is easily adapted to prove a stronger statement than Theorem 3. Let \( C(k) \) denote the statement obtained by replacing ‘3’ with ‘\( k \)’ throughout Conjecture 2. Then the above argument,
mutatis mutandis, yields the following result.

**Theorem 4** For any \( \ell \geq k \geq 2 \), \( C(k) \) implies \( C(\ell) \).

In particular, proving \( C(k) \) for any fixed \( k \) would prove Rota’s basis conjecture (in fact a stronger statement, namely \( C(n) \)) for all \( n \) greater than or equal to that fixed \( k \).

It is therefore natural to ask why we have formulated Conjecture 2 as \( C(3) \) rather than as \( C(2) \). The reason is that \( C(2) \) is false. The simplest counterexample is a well-known stumbling block that is partly responsible for the fact that there is no known general “matroid union intersection theorem,” i.e., a criterion for determining the minimum number of common independent sets that a set with two matroid structures on it can be partitioned into. Namely, take \( M(K_4) \), the graphic matroid of the complete graph on four vertices, and let the \( I_i \) be the three pairs of non-incident edges of \( K_4 \). Another counterexample arises from a matroid that Oxley [11] calls \( J \). Representing \( J \) by vectors in Euclidean 4-space, we can for example let

\[
\begin{align*}
I_1 &= \{(-2, 3, 0, 1), (0, 0, 1, 1)\} \\
I_2 &= \{(0, 2, 0, 1), (1, 0, 3, 1)\} \\
I_3 &= \{(1, 0, 0, 1), (0, 1, 2, 1)\} \\
I_4 &= \{(0, 1, 0, 1), (4, 0, 0, 1)\}
\end{align*}
\]

It may be possible to construct other examples from non-base-orderable matroids such as those in [9].

Despite these counterexamples to \( C(2) \), we believe that Conjecture 2 is plausible. Using a database of matroids with nine elements kindly supplied by Gordon Royle [13], we have computationally verified Conjecture 2 for the case \( n = 3 \).

In an earlier version of this paper, the formulation of Conjecture 2 did not require the \( I_i \) to be independent. A counterexample to that version of the conjecture was found by Colin McDiarmid. Take the complete graph on the vertex set \( \{1, 2, 3, 4\} \), and create an extra copy of the three edges incident to vertex 4. Call the edges 12, 13, 14, 23, 24, 34, 14’, 24’, 34’, and let \( I_1 = \{14, 14', 23\} \), \( I_2 = \{24, 24', 13\} \), and \( I_3 = \{34, 34', 12\} \). More generally, as pointed out by an anonymous referee, if \( k \) is odd, then a wheel with \( k - 1 \) copies of each of its \( k \) spokes yields a counterexample to \( C(k) \) if the \( I_i \) are not required to be independent.

In closing, we speculate that Conjecture 2 might be provable using the following strategy. First, develop a modified version of \( C(2) \) that says that the conclusion holds provided certain “obstructions” (such as \( M(K_4) \) and \( J \)) are absent. Then use Rado’s theorem (12.2.2 of [11]), or a suitable strengthening
of it, to construct a first column of $G$ in such a way that the remaining $2n$ elements are obstruction-free. Applying the modified version of $C(2)$ would then yield the desired result. The analysis of obstructions should hopefully be tractable since there are only 3 columns to consider.

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