A compactification of the universal moduli space of principal $G$-bundles

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Abstract. We construct a compactification of the universal moduli space of semistable principal $G$-bundles over $\overline{M}_g$, the fibers of which over singular curves are the moduli spaces of $\delta$-semistable singular principal $G$-bundles

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§1 INTRODUCTION

Let \( X \) be a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field, \( \mathbb{C} \), of characteristic 0 and \( G \) a reductive algebraic group. In his seminal work [15, 16], A. Ramanathan proved the existence of a projective moduli space, \( M_X^{(s)}(G) \), of (semi)stable principal \( G \)-bundles on \( X \) with fixed topological type. When \( G = \text{GL}_n \), this moduli space happen to be isomorphic to the classical moduli space of (semi)stable vector bundles constructed by D. Mumford and C. S. Seshadri (see [12, 22]).

An alternative construction of this moduli space was given later by A. Schmitt. Given a faithful representation \( \rho : G \hookrightarrow \text{SL}(V) \) of dimension \( r \), a singular principal \( G \)-bundle is a pair \((\mathcal{E}, \tau)\) consisting on a vector bundle \( \mathcal{E} \) of rank \( r \) and degree 0 together with a non-trivial morphism of algebras \( \tau : S^*(V \otimes \mathcal{E})^G \to \mathcal{O}_X \). Giving \( \tau \) is the same as giving a morphism \( X \to \text{Hom}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee) \). The singular principal \( G \)-bundle \((\mathcal{E}, \tau)\) is said to be honest if \( \tau \) takes values in the subscheme of local isomorphisms \( \text{Isom}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee)/G \). The heart of Schmitt’s work consists of the following observation,

\[
\left\{ \text{isomorphism classes of principal } G \text{-bundles on } X \right\} \cong \left\{ \text{isomorphism classes of pairs } (\mathcal{E}, \tau) : \begin{array}{l}
\text{where } \mathcal{E} \text{ is a locally free sheaf of rank } r \text{ with trivial determinant and } \\
\tau : X \to \text{Isom}_{\mathcal{O}_X}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee)/G
\end{array} \right\}.
\]  (1)

In [19], it is proved the existence of a projective moduli space, \( \text{SPB}_X(\rho)^{\delta(s)}_P \), of \( \delta \)-semistable singular principal \( G \)-bundles with Hilbert polynomial \( P \) on a smooth projective curve, for \( \delta \in \mathbb{Q}_{>0} \) and \( \rho : G \hookrightarrow \text{SL}(V) \) given. Furthermore, it is proved that for \( \delta \gg 0 \), every \( \delta \)-semistable singular principal \( G \)-bundle is honest and that \( \text{SPB}_X(\rho)^{\delta(s)}_P \) happens to be isomorphic to \( M_X^{(s)}(G) \).

At this point, a natural problem is to consider a degeneration of \( X \) along a discrete valuation ring \( A \), whose special fiber is a stable curve of genus \( g \) and to describe what the limit of a semistable principal \( G \)-bundle on \( X \) looks like when we approach the special fiber. When \( G = \text{GL}_n \) the solution is well known and is a torsion free sheaf [2]. In case of a more general reductive group \( G \), A. Schmitt’s approach seems to be more suitable to handle this problem (see also [5] for a different alternative). In fact, it is shown in [11, Theorem 6.4] that given \( \delta \in \mathbb{Q} \geq 0 \) large enough, there exists a limit and is a \( \delta \)-semistable singular principal \( G \)-bundle.

In this work, the goal is to prove the existence of a compactification of the moduli problem defined by pairs \((X, P)\), where \( X \) is a smooth projective curve of genus \( g \) and \( P \) is a principal \( G \)-bundle. The approach we will follow is A.
Schmitt’s approach so, thanks to Equation (11), we will be able to substitute $P$ by its corresponding singular principal $G$-bundle.

1.1. — The strategy and known results

The steps we follow for the construction of a compactification of the universal moduli space of principal $G$-bundles are summarized below, and are similar to those presented in the case of $G = GL_n$ (see [14]).

1.1.1. — The fiber-wise problem Let $X$ be a stable curve of genus $g$. The moduli space $\text{SPB}_X(\rho)_P^{\delta(s)}$ of $\delta$-semistable singular principal $G$-bundles with Hilbert polynomial $P$ is constructed in several steps.

1. It is shown that there are natural numbers, $a$ and $b$, large enough such that to every singular principal $G$-bundle $\tau : S^*(V \otimes \mathcal{F})^G \to \mathcal{F}_X$ with Hilbert polynomial $P$ we can assign a swamp $\phi_\tau : (\mathcal{V}(\mathcal{F})^a)^b \to \mathcal{F}_X$ with Hilbert polynomial $P$, and this mapping, $\tau \mapsto \phi_\tau$, is injective. Then a singular principal $G$-bundle $(\mathcal{F}, \tau)$ is said to be $\delta$-semistable if the corresponding swamp $(\mathcal{F}, \phi_\tau)$ is $\delta$-semistable. Thus, the construction of the moduli space of singular principal $G$-bundles is basically reduced to the construction of the moduli space of swamps.

2. It is shown that there is a natural number $N$ large enough such that for every $\delta$-semistable swamp, $(\mathcal{F}, \phi)$, and for every $k \geq N$, $\mathcal{F}(k)$ is generated by global sections and $h^1(X, \mathcal{F}(k)) = 0$. This implies that every $\delta$-semistable swamp $(\mathcal{F}, \phi)$ defines a point, $[(q, \phi)]$, in the projective scheme

$$\text{Quot}^P_{C^P(k) \otimes O_X(-k)/X/\mathbb{C}} \times \mathbb{P}((((C^P(k))^a)^b)^\lor \otimes H^0(X, O_X(ak)))$$

where $q : C^P(k) \otimes O_X(-k) \to \mathcal{F}$ and $C^P(k) \simeq H^0(X, \mathcal{F}(k))$ is a fixed isomorphism.

3. It is shown that, given $k \geq N$, there is a natural number $L$ large enough such that for every $l \geq L$, the closed immersion

$$i_{k,l} : \text{Quot}^P_{C^P(k) \otimes O_X(-k)/X/\mathbb{C}} \to \mathbb{P}((C^P(k) \otimes O_X(l-k)))$$

defined by the Grothendieck embedding of the Quot scheme composed with Plücker embedding satisfies the following property: a point $(i_{k,l} \times \text{id})([(\mathcal{F}, \phi)])$ in the projective scheme

$$\mathbb{P}((C^P(k) \otimes H^0(X, O_X(l-k)))) \times \mathbb{P}(((V^a)^b)^\lor \otimes H^0(X, O_X(ak)))$$

is GIT semistable with respect to the action of $SL_{P(k)}$ and respect to certain polarization $\mathcal{L}$ if and only if $(\mathcal{F}, \phi)$ is $\delta$-semistable and $C^P(k) \simeq H^0(X, \mathcal{F}(k))$.

4. The construction of both moduli spaces, $\text{SPB}_X(\rho)_P^{\delta(s)}$ and $T_X^{\delta(s)}$, is as follows. It is shown that there exist closed subschemes $Z_{band}(X) \subset Z_{sw}(X)$
of the projective scheme given in Equation (2) parametrizing (rigidified) singular principal $G$-bundles and swamps respectively. The action of $\text{SL}_{P(k)}$ on this scheme induces an action of both schemes $Z_{\text{bund},g}(X)$ and $Z_{\text{sw},g}(X)$. From Step 3, we conclude that $Z^\text{ss}_{\text{sw},g}(X)/\text{SL}_{P(k)}$ exists and is projective, which in turn implies that $Z^\text{ss}_{\text{bund},g}(X)/\text{SL}_{P(k)}$ exists and is projective as well. Again, from Step 3, we conclude that $Z^\text{ss}_{\text{sw},g}(X)/\text{SL}_{P(k)}$ and $Z^\text{ss}_{\text{bund},g}(X)/\text{SL}_{P(k)}$ are coarse moduli spaces for $\delta$-semistable swamps and $\delta$-semistable singular principal $G$-bundles.

1.1.2.—The relative problem

Fix integers $g \geq 2$, $d = 10(2g - 2)$ and $M = d - g$. Consider the Hilbert functor

$$\text{Hilb}_{d,g}(S) = \left\{ \begin{array}{l} Z \rightarrow S \times \mathbb{P}^M \\ \varphi \downarrow S \\ Z_s \text{ being a} \\ \text{projective curve of genus} \\ g \text{ and degree } d \forall s \in S \end{array} \right\}.$$ 

Let us denote by $H_{g,d,M}$ the representative of $\text{Hilb}_{d,g}$ and $H_g \subset H_{g,d,M}$ the locus of non-degenerate, 10-canonical stable curves of genus $g$. The scheme $H_g$ is a locally closed subscheme and it is a nonsingular, irreducible quasi-projective variety.

Fix an isomorphism for each stable curve, $X$, of genus $g$, $\mathbb{C}^{M+1} \simeq H^0(X, \omega_X^{10})$ and let $h(s) = ds - g + 1$ be a polynomial in $s$ of degree 1. Recall that there exists an integer $s_0$ such that $\forall s \geq s_0$, $i'_s : H_{d,g,N} \rightarrow \text{Grass}(h(s), H^0(P^M, O_{P^M}(s)))$ is a closed immersion. Moreover, there exists $s_1$ such that the GIT linearized problem $i'_s$ satisfies: (1) $H_g$ belongs to the semistable locus, (2) $H_g$ is closed in the semistable locus. Finally, the action of $\text{SL}_{M+1}$ on $\mathbb{P}^M$ induces an action on $H_g$, and Gieseker shows that $\overline{M}_g = H_g/\text{SL}_{M+1}$. The scheme $H_g$ is endowed with a universal family $U_g \rightarrow H_g$ called the universal curve.

1. It is shown that all the numbers appearing in the fiber-wise problem (that need to be large enough) do not depend on the base curve. That is, there are $a, b, N, L$ that works for every stable curve of genus $g$.

2. Relative parameter spaces $Z_{\text{bund},g} \rightarrow H_g$ and $Z_{\text{sw},g} \rightarrow H_g$ for families of $\delta$-semistable singular principal $G$-bundles and families of $\delta$-semistable swamps on the fibers of $U_g \rightarrow H_g$ are constructed. Again $Z_{\text{bund},g} \subset Z_{\text{sw},g}$.

3. $Z_{\text{bund},g}$ and $Z_{\text{sw},g}$ are embedded in the projective scheme $\mathbb{P}(H_1) \times \mathbb{P}(H_2) \times \mathbb{P}(H_3)$, where

$$H_1 = (((V)^{\otimes a})^{\otimes b})^\vee \otimes H^0(\mathbb{P}^M, O_{\mathbb{P}^M}(ak))$$

$$H_2 = \bigwedge_{P(l)} (H^0(\mathbb{P}^M, O_{\mathbb{P}^M}(s)))$$

$$H_3 = \bigwedge_{P(k)} (\mathbb{C}^{P(k)} \otimes H^0(\mathbb{P}^M, O_{\mathbb{P}^M}(l - k)))$$

(3)

4. The natural action of $\text{SL}_{P(k)} \times \text{SL}_{M+1}$ on $\mathbb{P}(H_1) \times \mathbb{P}(H_2) \times \mathbb{P}(H_3)$ induces an action on $Z_{\text{bund},g}$ and $Z_{\text{sw},g}$. Then it is shown that $Z^\text{ss}_{\text{sw},g}/(\text{SL}_{P(k)} \times \text{SL}_{M+1})$ exists and is projective, which in turn implies that $Z^\text{ss}_{\text{bund},g}/(\text{SL}_{P(k)} \times \text{SL}_{M+1})$ exists and is projective as well. Finally, Step 1 implies that $Z^\text{ss}_{\text{bund},g}/(\text{SL}_{P(k)} \times \text{SL}_{M+1})$
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$\text{SL}_{M+1}$) coarsely represents the moduli functor for pairs, and that there is $\delta \in \mathbb{Q}_{>0}$ large enough such that the fibers over nonsingular curves are precisely the moduli spaces of semistable principal $G$-bundles over them (quoted out by the group of automorphisms of the base curve).

1.1.3.—Known results  Regarding the fiber-wise problem, it is worth to note that the moduli space of $\delta$-semistable swamps have been constructed in [9] over any projective scheme of pure dimension one, even in the relative case, while the existence of a projective moduli space of $\delta$-semistable singular principal $G$-bundles over any stable curve (and also in the relative case) has been proved in [11]. However, the problem of the uniform behavior of the numerical parameters remains unsolved, and it is essential for the construction of the universal compactification. Regarding the relative problem, an important fact is that the actions of the groups $\text{SL}_n$ and $\text{SL}_{M+1}$ commute with each other, which makes the relative GIT problem easier to handle (see [14] for the case of vector bundles). This fact allows us to make use of some technical results proved in [14] to show the existence of a projective moduli space for $\delta$-semistable swamps over $\overline{M}_g$.

1.2. — Outline of the paper

This paper is organized as follows. In Section 2 we deal with Step 1 of Paragraph 1.1.2, and the main result is Theorem 2.6. Applying it to the key steps of the construction of the moduli space of $\delta$-semistable swamps, we show the uniform behavior along $\overline{M}_g$ of the numerical parameters involved in such construction. We end up with the existence of a projective moduli space of $\delta$-semistable swamps of given Hilbert polynomial over a stable curve of genus $g$. This result has been proved in [9], but it has been included for the sake of clarity of the exposition.

In Section 3 we prove the existence of a coarse projective moduli space for the moduli functor defined by pairs $(X, (\mathcal{F}, \phi))$, $X$ being a stable curve of genus $g$ and $(\mathcal{F}, \phi)$ a $\delta$-semistable swamp of given Hilbert polynomial. The forgetful map defines a morphism between this moduli space and $\overline{M}_g$.

Finally, in Section 4 we show the existence of a coarse projective moduli space for the moduli functor defined by pairs $(X, (\mathcal{F}, \tau))$, $X$ being a stable curve of genus $g$ and $(\mathcal{F}, \tau)$ a $\delta$-semistable singular principal $G$-bundle of given rank and degree 0. As in the case of swamps there is a morphism to $\overline{M}_g$. This, together with the results given in [11, 17, 20, 21] and Section 2 implies that the fibers of the above morphism over nonsingular curves are precisely the classical moduli spaces of principal $G$-bundles constructed by A. Ramanathan, and they form an open subset of the constructed moduli space.

The base field will be taken to be an algebraically closed field of characteristic 0. The word curve will mean a $\mathbb{C}$-scheme of finite type and pure dimension one.

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Our goal is to prove Theorem 2.6. The construction of the moduli space $T_{X, P, a, b, \mathcal{F}}^{\delta-(s)s}$ for a single stable curve $X$ of genus $g$ does not possess extra difficulties (see [9] for the construction on a projective scheme of pure dimension 1). However, we
will go over the key steps of such construction in order to exhibit how Theorem 2.6 shows the uniformity of the parameters \( N, L \) mentioned in Paragraph 1.1.2. This, together with [11, Theorem 5.4], eventually shows the main property of the universal moduli space that will be constructed.

Although the aim is to construct moduli spaces over stable curves, the results of this section hold true for Cohen-Macaulay curves, so the results are stated for this more general case.

### 2.1. — Vector bundles on the projective line

Let \( \mathcal{E} \) be a vector bundle of rank \( r \) on the projective line \( \mathbb{P}^1_k \). By \( \mathfrak{G} \), we know that there are integers \( n_1 \geq \ldots \geq n_r \) such that

\[
\mathcal{E} \simeq \bigoplus_{i=1}^{r} \mathcal{O}(n_i)
\]

The tuple \((n_1, \ldots, n_r)\) is defined as the type of \( \mathcal{E} \) and it is denoted by \( \tau(\mathcal{E}) \). We will denote by \( \tau_{\min}(\mathcal{E}) \) (respectively, \( \tau_{\max}(\mathcal{E}) \)) the minimum (respectively, maximum) integer of the type \( \tau(\mathcal{E}) \) of \( \mathcal{E} \), that is, \( \tau_{\min}(\mathcal{E}) = n_r \) (respectively, \( \tau_{\max}(\mathcal{E}) = n_1 \)).

Let \( r, m \in \mathbb{N} \) and \( d \in \mathbb{Z} \) be three integers. Observe that there are finitely many isomorphism classes of locally free sheaves on \( \mathbb{P}^1 \) of rank \( r \), degree \( d \) and such that \( h^0(\mathbb{P}^1, \mathcal{E}) = m \). Such isomorphism classes are determined by the tuples of integers \( n_1 \geq \ldots \geq n_r \) verifying the equations

\[
\sum_{i=1}^{r} \delta(n_i) = m, \quad \text{where } \delta(n_i) = \begin{cases} n_i + 1 & \text{if } n_i \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
\sum_{i=1}^{r} n_i = d
\]

Let us denote by \( N(r, d, m) \) the set of tuples of integers \( n_1 \geq \ldots \geq n_r \) satisfying Equation (4) and by \( S_-(r, d, m) \) (resp. \( S_+(r, d, m) \)) the minimum (resp. maximum) among the integers \( i \in \mathbb{Z} \) that appear as the smaller (resp. largest) integer in a tuple of \( N(r, d, m) \). Therefore, any integer \( n \in \mathbb{Z} \) that appears in a tuple \((n_1, \ldots, n_r) \in N(r, d, m) \) satisfies that \( S_+(r, d, m) \geq n \geq S_-(r, d, m) \).

**Lemma 2.1.** Let \( \mathcal{E} \) be a locally free sheaf of rank \( r \), degree \( d \) and \( h^0(\mathbb{P}^1, \mathcal{E}) = m \). Then, for every \( n \geq -S_-(r, d, m) \), \( \mathcal{E}(n) \) is generated by its global sections and \( h^1(\mathbb{P}^1, \mathcal{E}(n)) = 0 \).

**Proof.** Let \( \tau(\mathcal{E}) = (n_1, \ldots, n_r) \) be the type of \( \mathcal{E} \), that is, \( \mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}(n_i) \). Then, \( \mathcal{E}(n) \) is generated by global sections if and only if \( n + n_i \geq 0 \) for all \( i \), that is, if and only if \( n \geq -n_i \) for all \( i \). However, \( n_i \geq S_-(r, d, m) \) for every \( i \) by definition, so taking \( n \geq -S_-(r, d, m) \) we get the desired result. On the other hand \( h^1(\mathbb{P}^1, \mathcal{E}(n)) = \sum_{i=1}^{r} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 - n_i - n)) \), and \( h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 - n_i - n)) = 0 \) for every \( n \geq -S_-(r, d, m) \), so we are done. \( \square \)

### 2.2. — A uniform boundedness result on Cohen-Macaulay curves

Let \( X \) be a Cohen-Macaulay projective and connected curve of genus \( g \) together with a very ample invertible sheaf, \( \mathcal{O}_X(1) \), of degree \( h \). Given a coherent sheaf,
\( \mathcal{F} \), on \( X \) its \textit{degree} and its \textit{slope} are defined as

\[
\deg(\mathcal{F}) := \chi(\mathcal{F}) - r\chi(\mathcal{O}_X), \\
\mu'(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\alpha},
\]

respectively. Nevertheless, we will also use the quantity

\[
\mu(\mathcal{F}) := \frac{\chi(\mathcal{F})}{\alpha} = \frac{r(1-g) + \deg(\mathcal{F})}{\alpha},
\]

as the slope of \( \mathcal{F} \). As always, \( \alpha \) is the multiplicity of \( \mathcal{F} \) (the degree one coefficient of its Hilbert polynomial) and \( r = \alpha/h \) is its rank.

\textbf{Remark 2.2.} Observe that the degree of a coherent sheaf \( \mathcal{F} \) is determined by its Hilbert polynomial, the degree of the very ample line bundle we have fixed, \( \mathcal{O}_X(1) \), and the genus \( g \) of the curve, since

\[
\deg(\mathcal{F}) = \chi(\mathcal{F}) = P_\mathcal{F}(0) - \frac{P'_\mathcal{F}(n)}{h}(1-g)
\]

\( P'_\mathcal{F}(n) \) being the derivative of the polynomial \( P_\mathcal{F}(n) \).

A coherent sheaf on \( X \) is of pure dimension 1 if \( \text{dim}(\text{Supp}(\mathcal{F})) = 1 \) for every \( \mathcal{G} \subseteq \mathcal{F} \). Recall that a coherent sheaf of pure dimension 1, \( \mathcal{F} \), is semistable if for any subsheaf \( \mathcal{F}' \subset \mathcal{F} \),

\[
\mu'(\mathcal{F}') \leq \mu'(\mathcal{F}).
\]

Note that this is the same as saying that \( \mu(\mathcal{F}') \leq \mu(\mathcal{F}) \). Recall also that for any coherent sheaf of pure dimension 1, \( \mathcal{F} \), there is a unique filtration (Harder-Narasimhan filtration)

\[
0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}
\]

such that the quotients \( \mathcal{F}_i/\mathcal{F}_{i-1} \) are semistable sheaves with decreasing slopes. Note that this is true independently on the definition of the slope, \( \mu \) or \( \mu' \), that we use. As usual, we will use the following notation,

\[
\mu_{\text{max}}(\mathcal{F}) = \max\{\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) | i = 1, \ldots, k\} = \mu(\mathcal{F}_1), \\
\mu_{\text{min}}(\mathcal{F}) = \min\{\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) | i = 1, \ldots, k\} = \mu(\mathcal{F}/\mathcal{F}_{k-1}).
\]

Recall that for any subsheaf \( \mathcal{G} \subseteq \mathcal{F} \), \( \mu(\mathcal{G}) \leq \mu_{\text{max}}(\mathcal{F}) \).

Let \( \mathcal{L} \) be a line bundle of degree \( d \). By [13, Proposition 6] there exists a finite surjective morphism \( f : X \to \mathbb{P}^1 \) such that \( f^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_X(1) \) and [10, Theorem 23.1] implies that \( f \) is a flat, so \( \mathcal{E} := f_*\mathcal{L} \) is locally free. Since \( f \) is finite and \( f^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_X(1) \), we know that \( P_{\mathcal{E}}(n) = P_{\mathcal{E}}(n) \). Assuming \( n \gg 0 \), we get

\[
P_{\mathcal{E}}(n) = \text{rk}(\mathcal{E}) \cdot n + \text{rk}(\mathcal{E}) + \deg(\mathcal{E}), \\
P_{\mathcal{E}}(n) = h \cdot n + (1 - g) + d.
\]

hence, \( \text{rk}(\mathcal{E}) = h \) and \( \deg(\mathcal{E}) = 1 - g - h + d \). Therefore, \( \mathcal{E} \) being locally free implies that there are integers \( a_1(f) \geq \ldots \geq a_h(f) \) such that

\[
\mathcal{E} := f_*\mathcal{L} = \bigoplus_{i=1}^h \mathcal{O}_{\mathbb{P}^1}(a_i(f)), \\
1 - g - h + d = \sum_{i=1}^h a_i(f), \ a_1(f) \geq \ldots \geq a_h(f) \in \mathbb{Z}
\]
Definition 2.3. Let $X$, $\mathcal{O}_X(1)$, $f$, $\mathcal{L}$ be as above. We define the $f$-type of $\mathcal{L}$ as the tuple $(a_1(f), \cdots, a_h(f))$, and it is denoted by $\tau_f(\mathcal{L})$.

Lemma 2.4. Let $g, h, m \in \mathbb{N}$ and $d \in \mathbb{Z}$. There are integers $S_-, S_+$ depending only on $g, h, m, d$ such that for any Cohen-Macaulay projective and connected curve of genus $g$ with a very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, any line bundle $\mathcal{L}$ on $X$ of degree $d$ and $h^0(X, \mathcal{L}) = m$, and any finite morphism $f : X \to \mathbb{P}^1$ such that $f^* \mathcal{O}(1) \simeq \mathcal{O}_X(1)$, the following holds true:

$$S_+ \geq \tau_{f,\text{max}}(\mathcal{E}), \quad \tau_{f,\text{min}}(\mathcal{E}) \geq S_-, \quad \text{where } \mathcal{E} := f_* (\mathcal{L}) .$$

Furthermore, for every $n \geq -S_-$, $\mathcal{L}(n)$ is generated by its global sections and $h^1(X, \mathcal{L}(n)) = 0$.

Proof. By Equation (6), $\text{rk}(\mathcal{E}) = h$ and $\text{deg}(\mathcal{E}) = 1 - g - h + d$. On the other hand $h^0(\mathbb{P}^1, \mathcal{E}) = h^0(X, \mathcal{L}) = m$. Then, $S_- := S_-(h, 1 - g - h + d, m)$ and $S_+ := S_+(h, 1 - g - h + d, m)$ satisfy the inequality given in Equation (7). Besides, if $n \geq -S_-$, $\mathcal{E}(n)$ is generated by global sections and $h^1(\mathbb{P}^1, \mathcal{E}(n)) = 0$ by Lemma 2.4. This implies that $h^1(X, \mathcal{L}(n)) = 0$, since $h^1(X, \mathcal{L}(n)) = h^1(\mathbb{P}^1, \mathcal{E}(n))$, and that $\mathcal{L}(n)$ is generated by its global sections, since the adjunction map, $f^* f_* \to \text{id}$, is surjective for finite morphisms.

Proposition 2.5. Let $g, r, h \in \mathbb{N}$ and $n \in \mathbb{Z}$. There exists a constant $C \in \mathbb{Z}$, depending only on $g, h, n$, such that for every Cohen-Macaulay projective and connected curve $X$ of genus $g$ with a very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, we have $\mu_{\text{max}}(\mathcal{E}) \leq C$, where $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_X(n_i)$ with $n_i \leq n$.

Proof. Let $f : X \to \mathbb{P}^1$ be a finite morphism such that $f^* \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_X(1)$. Then, $f_* \mathcal{E} = \oplus_{i=1}^r \mathcal{O}_X(n_i)$, where $\mathcal{E} = f_* \mathcal{O}_X = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i(f))$. By Equation (6), we know that $\text{rk}(\mathcal{E}) = h$ and $\text{deg}(\mathcal{E}) = 1 - g - h$. Let $\mathcal{F} \subset \mathcal{E}$ be a subsheaf of multiplicity $\alpha$. Obviously, $f_* \mathcal{F}$ is locally free, so there are integers $t_1, \ldots, t_{\alpha}$ such that $f_* \mathcal{F} = \oplus_{i=1}^{\alpha} \mathcal{O}_X(t_i)$. Let $n'$ be the maximum among $n_1, \ldots, n_\alpha$. Since $f_* \mathcal{F} \subset f_* \mathcal{E}$, we deduce that $t_i \leq \tau_{\text{max}}(\mathcal{E}) + n' \leq \tau_{\text{max}}(\mathcal{E}) + n$, which implies that $\text{deg}(f_* \mathcal{F}) \leq \alpha (\tau_{\text{max}}(\mathcal{E}) + n)$. On the other hand, $\text{deg}(\mathcal{F}) = \alpha + \text{deg}(f_* \mathcal{F}) - \alpha (1 - g)$, so $\text{deg}(\mathcal{F}) \leq \alpha (1 + \tau_{\text{max}}(\mathcal{E}) + n - \frac{\alpha}{h} (1 - g)$ and, therefore, $\mu(\mathcal{F}) \leq 1 + \tau_{\text{max}}(\mathcal{E}) + n - \frac{1}{h} (1 - g)$. Finally, from Lemma 2.4, there is a constant $S_+$, depending only on $g$ and $h$, such that $\tau_{\text{max}}(\mathcal{E}) \leq S_+$. Therefore, $\mu(\mathcal{F}) \leq 1 + S_+ + n - \frac{1}{h} (1 - g) =: C$.

Let $k \in \mathbb{Z}$ be an integer. A coherent sheaf, $\mathcal{F}$, on $X$ is $k$-regular if

$$H^1(X, \mathcal{F}(k - i)) = 0, \quad \text{for all } i > 0.$$ 

Clearly, if $\mathcal{F}$ is $k$-regular and $k' > k$ then $\mathcal{F}$ is also $k'$-regular. From Serre’s vanishing theorem, it follows that there is always an integer $k$ such that $\mathcal{F}$ is $k$-regular. The regularity of $\mathcal{F}$ is defined by

$$\text{reg}(\mathcal{F}) = \inf\{k \in \mathbb{Z} : \mathcal{F} \text{ is } k\text{-regular}\}.$$ 

Given a projective scheme $X$, and a family of equivalence classes of coherent sheaves whose Hilbert polynomials belong to a finite set in $\mathbb{Z}[n]$, we can decide whether the family is bounded or not by looking at the regularity of the members of the family (see [8, Lemma 1.7.6]).
Theorem 2.6. Let \( g, h, C \in \mathbb{N} \) and \( P \) a finite set of polynomials of degree one with integral coefficients. There is a natural number \( N_0 \) depending only on \( P, C, g, h \) such that for every Cohen-Macaulay projective and connected curve of genus \( g \geq 2 \) with a very ample line bundle \( \mathcal{O}_X(1) \) of degree \( h \) and every family \( E \) of equivalence classes of coherent sheaves of pure dimension 1, \( \mathcal{F} \), with Hilbert polynomial in \( P \) that satisfy \( \mu_{\text{max}}(\mathcal{F}) \leq C \), we have \( h^1(X, \mathcal{F}(k)) = 0 \) for all \( k \geq N_0 \) and \( \mathcal{F}(k) \) is generated by its global sections. Therefore, \( \operatorname{reg}(\mathcal{F}) \leq N_0 \) and the family is bounded.

Proof. Since \( X \) is Cohen-Macaulay, by Serre duality theorem, we have that
\[
h^1(X, \mathcal{F}(k)) = \dim(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}(k), \omega_X)), \quad \forall k \in \mathbb{Z}.
\]
Suppose that \( h^1(X, \mathcal{F}(k)) \neq 0 \) and \( k \geq 0 \). Then, there is a nonzero morphism \( f' : \mathcal{F} \to \omega_X(-k) \).

Define \( \mathcal{K} := \ker(f') \) and \( \mathcal{N} := \operatorname{im}(f') \) and consider the exact sequence
\[
0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{N} \to 0
\]
Then, we have
\[
deg(\mathcal{N}) = \deg(\mathcal{F}) - \deg(\mathcal{K}) = \deg(\mathcal{F}) - \alpha_{\mathcal{F}} \mu(\mathcal{K}) \geq \deg(\mathcal{F}) - \alpha_{\mathcal{F}} C =
\]
\[
P_{\mathcal{F}}(0) - \frac{P'_{\mathcal{F}}(n)}{h}(1 - g) - \alpha_{\mathcal{F}} C \geq
\]
\[
\geq B_0 := \min\{P_{\mathcal{F}}(0) - \frac{P'_{\mathcal{F}}(n)}{h}(1 - g) - i \cdot C \mid i \in [1, \alpha_{\text{max}}], P_{\mathcal{F}}(n) \in \mathcal{P}\}
\]
\( \alpha_{\text{max}} \) being the maximum among the degree one coefficients of the polynomials in \( \mathcal{P} \). Note that \( B_0 \) is a constant which depends only on \( \mathcal{P}, h \) and the genus \( g \). The injective morphism \( \mathcal{N} \to \omega_X(-k) \) induces an injective morphism \( \mathcal{N}(k) \to \omega_X \).

Then, we have
\[
deg(\mathcal{N}(k)) = \alpha_{\mathcal{N}} \cdot k + \deg(\mathcal{N}) \geq k + B_0,
\]
and therefore,
\[
2g - 2 = \deg(\omega_X) = \deg(\mathcal{N}(k)) + \deg(\omega_X/\mathcal{N}(k)) \geq
\]
\[
\geq k + B_0 + \deg(\omega_X/\mathcal{N}(k)). \tag{8}
\]
Let us find a bound of \( \deg(\omega_X/\mathcal{N}(k)) \). Denote \( \mathcal{J} := \omega_X/\mathcal{N}(k) \). Since \( f \) is finite, we have a surjection \( f_*(\omega_X) \to f_* \mathcal{J} \).

Denote by \( \mathcal{J} \) the torsion subsheaf of \( f_* \mathcal{J} \) and by \( \mathcal{U} = f_* \mathcal{J}/\mathcal{J} \) the locally free subsheaf. We have \( f_* \mathcal{J} = \mathcal{U} \oplus \mathcal{J} \), so \( \deg(f_* \mathcal{J}) \geq \deg(\mathcal{U}) \), and it is enough to give a bound for \( \mathcal{U} \). Observe that we have a surjective morphism \( f_*(\omega_X) \to \mathcal{U} \), and that
\[
f_*(\omega_X) = \bigoplus_{i=1}^h \mathcal{O}(a_i)
\]
\[\mathcal{U} = \bigoplus_{i=1}^T \mathcal{O}(b_j)\]
From the above surjection, we deduce that for every $j = 1, \ldots, T$ there exists an index $i = 1, \ldots, h$ such that $b_j \geq a_i$. Therefore,

$$\deg(\mathcal{U}) = \sum_{i=1}^{T} b_i \geq \sum_{i=1}^{T} a_{i_k} \geq T \cdot A$$

where $A := \min\{a_i\}$. Now, by Proposition 2.4, there exists an integer $S = S(g,h,m)$ depending only on $g,h,m := h^0(X, \Omega_X) = g$ such that $A \geq S$. Therefore, $\deg(\mathcal{U}) \geq T \cdot S \geq T_0 \cdot S$ where $T_0 := \min\{0, h \cdot S\}$. Finally, from Equation \[5\], we get

$$2g - 2 \geq k + B_0 + T_0 \cdot S.$$ (9)

Let $N'_0$ be the smaller integer such that $N'_0 + B_0 + T_0 \cdot S > 2g - 2$. Then $h^1(X, \mathcal{F}(k)) = 0$ for all $k \geq N'_0$ and $N'_0$ only depends on $P,C,g,h$.

For the last part, let $x \in X$ be a closed point and define $\mathcal{I} := \mathcal{I}(-x) = \mathcal{I} \otimes \mathcal{O}_X(-x)$. Since $\mathcal{I}$ is a pure dimension 1 coherent sheaf, $\mathcal{I}$ is of pure dimension 1 as well, so $\mathcal{I} \subset \mathcal{F}$ and $\mu_{\text{max}}(\mathcal{I}) \leq C$. Observe also that

$$P_{\mathcal{I}}(n) = P_{\mathcal{F}}(n) - d(x),$$

where $d(x) := \dim \mathcal{F}_x/m_x \mathcal{F}_x$.

The function $d(x)$ is clearly bounded, that is, there are positive constants $B_{\text{inf}}$ and $B_{\text{sup}}$ such that $B_{\text{inf}} \leq d(x) \leq B_{\text{sup}}$ for every $x \in X$. Moreover, $B_{\text{inf}}$ and $B_{\text{sup}}$ depend only on the Hilbert polynomial $P(n)$. We can argue now as above and we arrive at the equation

$$2g - 2 \geq k + B_0 - B_{\text{sup}} + T_0 \cdot S.$$ (8)

Let $N_0(\geq N'_0)$ be the smaller integer such that $N_0 + B_0 - B_{\text{sup}} + T_0 \cdot S > 2g - 2$. Then $h^1(X, \mathcal{F}(-x)(k)) = h^1(X, \mathcal{F}(k)) = 0$ for all $k \geq N_0$, so $\mathcal{F}(k)$ is generated by its global sections and $h^1(X, \mathcal{F}(k)) = 0$ for every $k \geq N_0$ and $N_0$ only depends on $P,C,g,h$. \hfill \Box

**Corollary 2.7.** Let $g,h,C,C' \in \mathbb{N}$ and $P$ a finite set of polynomials of degree one with integral coefficients. There is a natural number $N_1$ depending only on $g,h,C,C',P$ such that for every Cohen-Macaulay projective and connected curve of genus $g \geq 2$ with a very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, every family $E$ of equivalence classes of coherent sheaves of pure dimension one, $\mathcal{F}$, with Hilbert polynomial in $P$ that satisfy $\mu_{\text{max}}(\mathcal{F}) \leq C$ and every family $E'$ of equivalence classes of subsheaves $\mathcal{F}' \subset \mathcal{F}$, with $\mathcal{F} \in E$, such that $|\deg(\mathcal{F}')| \leq C'$, we have $h^1(X, \mathcal{F}'(k)) = 0$ for all $k \geq N_1$ and $\mathcal{F}'(k)$ is generated by its global sections. Therefore, $\text{reg}(\mathcal{F}') \leq N_1$ and the family is bounded.

**Proof.** Let $\mathcal{F} \in E$ and $\mathcal{F}' \subset \mathcal{F}$ a coherent sheaf in $E'$. Obviously $\mu_{\text{max}}(\mathcal{F}') \leq C$. Since $|\deg(\mathcal{F}')| \leq C'$, there are only finitely many possible polynomials in the set of Hilbert polynomials of the members of $E'$. Then, applying Theorem 2.6, we conclude. \hfill \Box

### 2.3. Uniform boundedness of $\delta$-semistable swamps

Let $X$ be a Cohen-Macaulay projective and connected curve of genus $g$. We will generalize the definition of $\delta$-semistability for swamps by simply substituting
the ranks by the multiplicities of the coherent sheaves. A direct application of Theorem 2.6 will allow us to prove Theorem 2.11 which in turn will imply that the families of \( \delta \)-semistable swamps of fixed type are uniformly bounded along the moduli space of stable curves of genus \( g \), \( \overline{M}_g \).

**Definition 2.8.** Let \( a, b \in \mathbb{N} \) and \( \mathcal{D} \) an invertible sheaf. A swamp over \( X \) of type \( (a, b, \mathcal{D}) \) is a pair \((\mathcal{F}, \phi)\) where \( \mathcal{F} \) is a coherent \( \mathcal{O}_X \)-module and \( \phi \) is a non-zero morphism of \( \mathcal{O}_X \)-modules, \( \phi : (\mathcal{F} \otimes a) \otimes b \rightarrow \mathcal{D} \).

**Definition 2.9.** Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module on \( X \). A weighted filtration, \((\mathcal{F}_*, m)\), of \( \mathcal{F} \) is a filtration

\[
\mathcal{F}_* \equiv (0) \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_t \subset \mathcal{F}_{t+1} = \mathcal{F},
\]

equipped with positive numbers \( m_1, \ldots, m_t \in \mathbb{Q}_{>0} \). We adopt the following convention: the one-step filtration is always equipped with \( m = 1 \). A filtration is called saturated if the quotients \( \mathcal{F}/\mathcal{F}_i \) are torsion free sheaves.

Let \( \phi : (\mathcal{F} \otimes a) \otimes b \rightarrow \mathcal{D} \) be a swamp on \( X \) and let \((\mathcal{F}_*, m)\) be a weighted filtration. For each \( \mathcal{F}_i \) denote by \( \alpha_i \) its multiplicity and just by \( \alpha \) the multiplicity of \( \mathcal{F} \). Define the vector

\[
\Gamma = \sum_{1}^{t} m_i \Gamma^{(\alpha_i)},
\]

where \( \Gamma^{(l)} = (l - \alpha, \ldots, l - \alpha, l, \ldots, l) \). Let us denote by \( J \) the set

\[
J = \{ \text{ multi-indices } I = (i_1, \ldots, i_n) | I_j \in \{1, \ldots, t + 1\} \}. \tag{10}
\]

Define

\[
\mu(\mathcal{F}_*, m, \phi) = \min_{I \in J} \{ \Gamma_{\alpha_1} + \ldots + \Gamma_{\alpha_n} | \phi (\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \otimes b \neq 0 \}. \tag{10}
\]

**Definition 2.10.** Let \( \delta \in \mathbb{Q}_{>0} \) be a positive rational number. A swamp \((\mathcal{F}, \phi)\) is \( \delta \)-(semi)stable if for each weighted filtration \((\mathcal{F}_*, m)\) the following holds

\[
\sum_{1}^{t} m_i (\alpha P_{\mathcal{F}_i} - \alpha_i P) + \delta \mu(\mathcal{F}_*, m, \phi) (\leq 0). \tag{11}
\]

**Theorem 2.11.** Let \( a, g, h \in \mathbb{N} \) and let \( P(n) \in \mathbb{Z}[n] \) be a polynomial of degree one and \( \delta \in \mathbb{Q}_{>0} \). There exists a natural number \( N_2 \in \mathbb{N} \) depending only on \( P(n), a, \delta, g, h \) such that for every Cohen-Macaulay projective and connected curve of genus \( g \), \( X \), with very ample line bundle \( \mathcal{O}_X(1) \) of degree \( h \) and for every coherent sheaf of pure dimension one \( \mathcal{F} \) with Hilbert polynomial \( P(n) \) appearing in a \( \delta \)-(semi)stable swamp of type \( (a, -,-) \), \( \mathcal{F}(k) \) is generated by its global sections and \( h^1(X, \mathcal{F}(k)) = 0 \) for every \( k \geq N_2 \). In particular, the set \( E \) of equivalence classes of coherent sheaves of pure dimension one with Hilbert polynomial \( P(n) \) appearing in \( \delta \)-semistable swamps of type \( (a, -,-) \) is bounded.

**Remark 2.12.** The notation \( (a, -,-) \) means that we allow swamps \( \phi : (\mathcal{F} \otimes a)^b \rightarrow \mathcal{D} \), whoever \( b \in \mathbb{N} \) and \( \mathcal{D} \in \text{Pic}(X) \) are.
Proof. Let $X$ be a Cohen-Macaulay projective and connected curve of genus $g$ $b \in \mathbb{N}$ a natural number, $\mathcal{D}$ an invertible sheaf and $(\mathcal{F}, \phi)$ a $\delta$-semistable swamp of type $(a, b, \mathcal{D})$ with Hilbert polynomial $P(n)$. Let $\mathcal{F}_1 \subset \mathcal{F}$ be a subsheaf and consider the one-step flag $0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{F}$. The computation of the semistability condition given in Equation (11) leads to

$$\mu(\mathcal{F}_1) \leq C := \mu(\mathcal{F}) + a(\alpha - 1)\delta$$

$C$ being a constant depending only on $P, h, a, \delta, g$. Now, by Theorem 2.6, there exists a natural number $N_2 \in \mathbb{N}$ depending only on $P(n), C, g, h$ (thus, on $P(n), a, \delta, g$) such that $h^1(X, \mathcal{F}(k)) = 0$ and $\mathcal{F}(k)$ is generated by its global sections for every $k \geq N_2$. Finally, by (8, Lemma 1.7.6), the set $E$ of equivalence classes of coherent sheaves of pure dimension one with Hilbert polynomial $P(n)$ appearing in $\delta$-semistable swamps of type $(a, -, -)$ is bounded.

2.4. — Comparison of $\delta$-semistability and GIT semistability

Let $g \in \mathbb{N}, \delta \in \mathbb{Q}_{>0}, N_2 \in \mathbb{N}$ as in Theorem 2.11 and let $k \geq N_2$. For any stable curve of genus $g \geq 2$, $X$, we can consider the Quot scheme Quot_{P \in X} O_{X}(-k)/X/\mathbb{C}$, where the polarization that is being fixed is the one given by $\omega_X^{\otimes 10}$. Due to the Grothendieck embedding composed with the Plücker embedding, we can see this Quot scheme as a closed subscheme of certain projective space $\mathbb{P}^M$, where $M$ depends only on $g, k, P$ and on a new natural number $l \in \mathbb{N}$. This last parameter has to be with the Plücker map, and is taken large enough for this map to be a closed immersion. The lower bound for $l$ to satisfy this property does not depend on the base curve. Therefore, Theorem 2.11 implies that for every $X$ as before, and every $\delta$-semistable swamp, $(\mathcal{F}, \phi)$, the coherent sheaf $\mathcal{F}$ defines a point in $\mathbb{P}^M$. The same will be proved for the morphisms $\phi$, showing that every $\delta$-semistable swamp defines a point in some projective space that does not depend on the base curve. An important issue that remains unsolved is to show that $\delta$-semistability coincides with the GIT semistability in this projective space whatever the base curve we are working on. This will be controlled making the natural number $l$ large enough, and the problem will be solved by giving a lower bound independent on the base curve.

We will go over the key results that lead to Theorem 2.23 (see [4] for the case of nonsingular curves) to show how to find a lower bound for $l$ independent of the base curve.

2.4.1. — $\delta$-semistability and sectional semistability

Let $X$ be a Cohen-Macaulay projective and connected curve of genus $g$, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and suppose we have a filtration, $\mathcal{F}_i$, of $\mathcal{F}$. We will denote by $\alpha^i$ the multiplicity of $\mathcal{F}/\mathcal{F}_i$ and by $\alpha_i$ the multiplicity of $\mathcal{F}_i$ (thus, $\alpha(\mathcal{F}) = \alpha_i + \alpha^i$). Let now $P(n) \in \mathbb{Z}[n]$ be a polynomial, $\alpha, d$ rational numbers such that $P(n) = \alpha n + \frac{\alpha}{h}(1 - g) + d$, and $k$ a natural number. Then, we define:

1. $S^*$ is the set of $\delta$-semistable swamps $(\mathcal{F}, \phi)$ with $\mathcal{F}$ a coherent sheaf of pure dimension one with Hilbert polynomial $P$. 

2. $S'_m$ is the set of swamps $(\mathcal{F}, \phi)$ with $\mathcal{F}$ a coherent sheaf of pure dimension one with Hilbert polynomial $P$, and such that
\[ \sum_{i=1}^{t} m_i(\alpha^h(X, \mathcal{F}_i(k)) - \alpha_i P(k)) + \delta \mu(\mathcal{F}_*, m, \phi) \leq 0 \]
for every weighted filtration $(\mathcal{F}_*, m)$.

3. $S''_m$ is the set of swamps $(\mathcal{F}, \phi)$ with $\mathcal{F}$ a coherent sheaf of pure dimension one with Hilbert polynomial $P$, and such that
\[ \sum_{i=1}^{t} m_i(\alpha^i P(m) - \alpha^h(X, \mathcal{F}_i(P))) + \delta \mu(\mathcal{F}_*, m, \phi) \leq 0, \]
for every weighted filtration $(\mathcal{F}_*, m)$.

4. $S_N = (\cup_{k \geq N} S''_k) \cup S^*$, $N \in \mathbb{N}$.

**Lemma 2.13.** Let $a, g, h \in \mathbb{N}$ and let $P(n) \in \mathbb{Z}[n]$ be a polynomial of degree one and $\delta \in \mathbb{Q}_{>0}$. There exists natural numbers $N_3, C_0 \in \mathbb{N}$ depending only on $P(n), a, \delta, g, h$ such that for every Cohen-Macaulay projective and connected curve of genus $g$, $X$, with very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, if $(\mathcal{F}, \phi) \in S_N$ is a swamp of type $(a, -,-)$ then, for all saturated weighted filtrations $(\mathcal{F}_*, m)$ and for all $C \geq C_0$, the following holds for all $i$:
\[ \deg(\mathcal{F}_i) - \alpha_i \mu_s \leq C, \text{ where } \mu_s := \frac{d - a \delta}{\alpha} \text{ and } d := \deg(\mathcal{F}) \]
and either 1) $C \leq \deg(\mathcal{F}_i) - \alpha_i \mu_s$, or
2.a) $h^0(X, \mathcal{F}_i(k)) < \alpha_i (P(k) - a \delta)$, if $(\mathcal{F}, \phi) \in S^*$ and $k \geq N_3$
2.b) $\alpha^i (P - a \delta) < \alpha (P_{\mathcal{F}i} - a \delta)$ if $(\mathcal{F}, \phi) \in \cup_{k \geq N_1} S''_k$.

**Proof.** The proof is the same one as in [4 Lemma 2.6]. An easy calculation permits to given an explicit expression for $N_3$ and $C_0$:
\[ C_0 = \max\{a \delta, \alpha^2 + B - r(1 - g) + d\} + 1, \]
\[ N_3 = \max\{0, B' + \frac{C_0}{\alpha} - \mu_s\} + 1, \]
where $[-]$ denotes de integral part, $B$ is the constant given in [24 Corollary 1.7], which depends only on $P$ and $h$, and $B' := B + \frac{(1 - g)}{h}$. \hfill $\Box$

**Lemma 2.14.** Let $a, r, g, h \in \mathbb{N}$, $P(n) \in \mathbb{Z}[n]$ a polynomial of degree one and $\delta \in \mathbb{Q}_{>0}$. There exists a natural number $N_4 \in \mathbb{N}$ depending only on $P(n), a, \delta, g, h$ such that for every Cohen-Macaulay projective and connected curve of genus $g$, $X$, with very ample line bundle $\mathcal{O}_X(1)$ of degree $h$ the following holds true: for every swamp $(\mathcal{F}, \phi) \in S_N$ on $X$ of type $(a, -,-)$ and for every subsheaf $\mathcal{F}' \subset \mathcal{F}$ with $\mathcal{F}' \in S_0$, both, $\mathcal{F}(k)$ and $\mathcal{F}'(k)$, are generated by global sections and $h^1(X, \mathcal{F}(k)) = h^1(X, \mathcal{F}'(k)) = 0$ for every $k \geq N_4$. 

\[ \frac{1}{13} \]
Proof. It is easy to show that if a swamp $(\mathcal{F}, \phi)$ belongs to $S_N$ then
\[
\mu_{\text{max}}(\mathcal{F}) \leq \mu_s + C + \frac{1 - \varrho}{h}, \quad \text{where } C \geq C_0
\]
This implies that, for every swamp $(\mathcal{F}, \phi) \in S_N$ and every subsheaf $\mathcal{F}' \subset \mathcal{F}$, the inequality $\mu_{\text{max}}(\mathcal{F}') \leq \mu_s + C + \frac{1 - \varrho}{h}$ holds true. On the other hand, every swamp $(\mathcal{F}, \phi) \in S_N$ has Hilbert polynomial $P(n)$ and the set of Hilbert polynomials of subsheaves $\mathcal{F}' \subset \mathcal{F}$ is finite. Moreover, the coefficients of these Hilbert polynomials have lower and upper bounds that depend only on $P(n), a, \delta, g, h$. Therefore, we conclude by applying Theorem 2.16.

Proposition 2.15. Let $X$ be a projective scheme of dimension less or equal than two, $\mathcal{O}_X(1)$ a very ample line bundle and let $j : X \hookrightarrow \mathbb{P}^n$ be a closed immersion such that $j^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_X(1)$. If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves of $\mathcal{O}_X$-modules that are $k_1$- and $k_2$-regular respectively, then $\mathcal{F} \otimes \mathcal{G}$ is $(k_1 + k_2)$-regular.

Proof. Since $j : X \hookrightarrow \mathbb{P}^n$ is a finite morphism, we know that $H^i(X, \mathcal{F}(n)) = H^i(\mathbb{P}^n, j_*((\mathcal{F}(n))(n))$ and $H^i(X, \mathcal{G}(n)) = H^i(\mathbb{P}^n, j_*((\mathcal{G}(n)))$. This implies that $j_*\mathcal{F}$ and $j_*\mathcal{G}$ are $k_1$- and $k_2$-regular respectively, and obviously $(j_*\mathcal{F})_y = (j_*\mathcal{G})_y = 0$ for every $y \in \mathbb{P}^n \setminus X$. Therefore, by [23, Proposition 1.5], $j_*\mathcal{F} \otimes j_*\mathcal{G}$ is $(k_1 + k_2)$-regular, that is,
\[
H^i(\mathbb{P}^n, (j_*\mathcal{F} \otimes j_*\mathcal{G})(k_1 + k_2 - i)) = 0
\]
for every $i \geq 0$. Since $j$ is a closed immersion, we deduce that $j_*\mathcal{F} \otimes j_*\mathcal{G} = j_*((\mathcal{F} \otimes \mathcal{G})$ and, therefore,
\[
H^i(X, (\mathcal{F} \otimes \mathcal{G})(k_1 + k_2 - i)) = H^i(\mathbb{P}^n, j_*((\mathcal{F} \otimes \mathcal{G})(k_1 + k_2 - i)) =
\]
\[
= H^i(\mathbb{P}^n, j_*((\mathcal{F} \otimes \mathcal{G}))(k_1 + k_2 - i)) =
\]
\[
= H^i(\mathbb{P}^n, (j_*\mathcal{F} \otimes j_*\mathcal{G})(k_1 + k_2 - i)) = 0
\]

Then we have

Theorem 2.16. Let $a, r, g, h \in \mathbb{N}$, $P(n) \in \mathbb{Z}[n]$ a polynomial of degree one and $\delta \in \mathbb{Q}_{>0}$. There exists a natural number $N_5 \in \mathbb{N}$ depending only on $P(n), a, \delta, g, h$ such that for every Cohen-Macaulay projective and connected curve of genus $g$, $X$, with very ample line bundle $\mathcal{O}_X(1)$ of degree $h$ the following properties of swamps, $(\mathcal{F}, \phi)$, of type $(a, r, -)$ with $\mathcal{F}$ a coherent sheaf of pure dimension one and with Hilbert polynomial $P(n)$, are equivalent:

1) $(\mathcal{F}, \phi)$ is $\delta$-(semi)stable.
2) $\forall (\mathcal{F}, m)$ we have $\sum_i m_i(\alpha_1 P(k) - \alpha_1 P'(k)) + \delta \mu(\mathcal{F}, m, \phi) \leq 0$.
3) $\forall (\mathcal{F}, m)$ we have $\sum_i m_i(\alpha_1 P(k) - \alpha_1 P'(k)) + \delta \mu(\mathcal{F}, m, \phi) \leq 0$.

Furthermore, for any swamp satisfying these conditions, we have $h^1(X, \mathcal{F}(k)) = 0$.

Proof. By Proposition 2.15 and Lemma 2.14, $S_N$ and $S_0$ are bounded (for all $N$), and we can find $N_5 > N_4, N_3, N_2, N_1, N_0$, depending only on the numerical input data, such that sheaves $\mathcal{F}$ in $S$ and $S_0$ are $N_5$-regular and $F_1 \otimes \ldots \otimes F_a$ is $a \cdot N_5$-regular for all $\mathcal{F}_1, \ldots, \mathcal{F}_a$ is $S_0$. The rest of the proof is as in [3, Theorem 2.5].
Corollary 2.17. Let $a, g, h \in \mathbb{N}$, $P(n) \in \mathbb{Z}[n]$ a polynomial of degree one and $\delta \in \mathbb{Q}_{>0}$. Let $X$ be Cohen-Macaulay projective and connected curve of genus $g$, $X$, with very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, $(\mathcal{F}, \phi)$ a $\delta$-semistable swamp of type $(a, -,-)$, $k \geq N_5$, and assume that there is a weighted filtration $(\mathcal{F}, m)$ such that

$$\sum_{i=1}^{t} m_i(ah^0(X, \mathcal{F}_i(k)) - \alpha_i P(k)) + \delta \mu(\mathcal{F}, m, \phi) = 0. \quad (13)$$

Then $\mathcal{F}_i \in S_0$ and $h^0(X, \mathcal{F}_i(k)) = P_{\mathcal{F}_i}(k)$ for all $i$.

Proof. If $\mathcal{F}_i \in S_0$, then $P_{\mathcal{F}_i}(k) = h^0(X, \mathcal{F}_i(k))$. If $\mathcal{F}_i$ do not belongs to $S_0$, then the second alternative of Lemma 2.13 holds, so

$$ah^0(X, \mathcal{F}_i(k)) < \alpha_i(P(k) - a\delta). \quad (14)$$

Let $T' \subset T = \{1, \ldots, t\}$ be the subset of those $i$ for which $\mathcal{F}_i \in S_0$. Let $(\mathcal{F}', m)$ the corresponding subfiltration. Then

$$\sum_{i=1}^{t} m_i(ah^0(X, \mathcal{F}_i(k)) - \alpha_i P(k)) + \delta \mu(\mathcal{F}, m, \phi) \leq$$

$$\leq \left(\sum_{i \in T'} m_i(ah^0(X, \mathcal{F}_i(k)) - \alpha_i P(k)) + \delta \mu(\mathcal{F}', m', \phi) \right)$$

$$\leq \left(\sum_{i \in T'} m_i(ah^0(X, \mathcal{F}_i(k)) - \alpha_i P(k)) + \delta \mu(\mathcal{F}', m', \phi) \right) + \left(\sum_{i \in T - T'} m_i(a \alpha_i P(k)) - \alpha_i P(k)\right) - a \alpha_i (\mathcal{F}, \phi, m') \leq 0. \quad (15)$$

Equation (13) implies that the inequalities in (15) become equalities, so $T = T'$, $\mathcal{F}_i \in S_0$ for all $i$, and we are done. The last part follows from Lemma 2.13. \qed

2.4.2.—The parameter space Let $X$ be a Cohen-Macaulay projective and connected curve of genus $g$, $D \in \mathbb{N}$ a natural number, $e \in \mathbb{Z}$ an integer and let $\mathcal{D}$ be an invertible sheaf on $X$ of degree $e$ with $h^0(X, \mathcal{D}) = D$. Let us fix a polynomial $P$ of degree one with integral coefficients and $a, b \in \mathbb{N}$. Given $k \in \mathbb{N}$, let $\mathcal{H} \subset \text{Quot}_{X}(P(k) \otimes \mathcal{O}_X(-k)/X/\mathbb{C})$ be the subscheme of the quotient scheme parametrizing quotients $V \otimes \mathcal{O}_X(-k) \to \mathcal{F}$ of pure dimension one. By Theorem 2.16 and Lemma 2.4, there is a natural number $N \in \mathbb{Z}$, depending only on the numerical input data, such that for every $k \geq N$ and for each $\delta$-(semi)stable swamp, $(\mathcal{F}, \phi)$, of type $(a, b, \mathcal{D})$, we have that $\mathcal{F}(k)$ is generated by its global sections, $h^1(X, \mathcal{F}(k)) = 0$, and such that $\mathcal{D}(k)$ is generated by global sections and $h^1(X, \mathcal{D}(k)) = 0$ as well.

Fix such a natural number $k > N$ and let $n = P(k)$. For $l$ large enough, there is a projective embedding (Grothendieck embedding composed with the Plücker embedding),

$$\mathcal{H} \hookrightarrow \mathbb{P}(\bigwedge^{P(l)}(\mathbb{C}^n \otimes H^0(X, \mathcal{O}_X(l-k)))).$$
Consider the vector space $Z_1 = \left((V^\otimes a)^\oplus b\right)^V \otimes H^0(X, \mathcal{D}(ak))$. The functor of points of the projective space $\mathbb{P}(Z_1)$ is given by

$$\mathbb{P}(Z_1)^*(T) = \begin{cases} \text{equivalence classes of invertible quotients} \\ \left(\left((V^\otimes a)^\oplus b\right)^V \otimes H^0(X, \mathcal{D}(ak))^{\vee}\right) \otimes \mathcal{O}_T \to \mathcal{L} \end{cases}$$

over $T$. (16)

For any scheme $T$ we define a family of $\delta$-(semi)sable swamps with Hilbert polynomial $P$ and uniform multi-rank $r$ parametrized by $T$ as a tuple $(\mathcal{F}_T, \phi_T, \mathcal{N})$, where $\mathcal{F}_T$ is a family of coherent sheaves of pure dimension one with uniform multi-rank $r$ on $X \times T$ and Hilbert polynomial $P$ flat over $T$, $\mathcal{N}$ is an invertible sheaf on $T$ and $\phi_T$ is a morphism

$$\phi_T : \left((\mathcal{F}_T^\otimes a)^\oplus b\right)^\vee \to \pi_X^* \mathcal{D} \otimes \pi_T^* \mathcal{N}$$

such that for each point $t \in T$ the pair $(\mathcal{F}_{T,t}, \phi_{T,t})$ is $\delta$-(semi)sable.

Consider the moduli problem defined by the functor

$$\text{Swamps}^{\delta-(semi)sable}_{P,\mathcal{D},a,b}(T) = \begin{cases} \text{isomorphism classes of} \\ \delta-(semi)sable \text{ swamps of pure dimension one} \\ (\mathcal{F}_T, \phi_T, \mathcal{N}) \text{ of uniform multi-rank $r$} \\ \text{and with Hilbert polynomial $P$} \end{cases}.$$ (18)

The strategy to give a coarse solution for the above moduli problem consists of, first rigidify the problem and give a fine solution and, finally, quot out that solution by the automorphisms of the rigidifying datum. The rigidifying datum will consists in giving an isomorphism

$$g_T : V \otimes \mathcal{O}_T \simeq \pi_T^* \mathcal{F}_T(k).$$

Thus, we consider the functor (for $k$ fixed as in the introduction)

$$\text{rigSwamps}^{k}_{P,\mathcal{D},a,b}(T) = \begin{cases} \text{isomorphism classes of tuples} (\mathcal{F}_T, \phi_T, \mathcal{N}, g_T) \\ \text{where} (\mathcal{F}_T, \phi_T, \mathcal{N}) \text{ is a swamp with} \\ \text{Hilbert polynomial $P$ and $g_T$ is a morphism} \\ g_T : V \otimes \mathcal{O}_T \to \pi_T^* \mathcal{F}_T(k) \text{ such that} \\ \text{the induced morphism} V \otimes \mathcal{O}_{X \times T} \to \mathcal{F}_T(k) \text{ is surjective} \end{cases}.$$ (19)

where two tuples, $(\mathcal{F}_T, \phi_T, \mathcal{N}, g_T)$ and $(\mathcal{F}_T', \phi_T', \mathcal{N}', g_T')$, are isomorphic if there exists an isomorphism $(f, \alpha)$ between $(\mathcal{F}_T, \phi_T, \mathcal{N})$ and $(\mathcal{F}_T', \phi_T', \mathcal{N}')$ such that $\pi_T^*(f(k)) \circ g_T = g_T$.

**Theorem 2.18.** The functor $\text{rigSwamps}^{k}_{P,\mathcal{D},a,b}$ is represented by a closed sub-scheme $Z_{k,\mathcal{D}}(X)$ of $\mathcal{H} \times \mathbb{P}(Z_1)$.

**Proof.** This is proved as in the case of irreducible projective curves (see [1, 4]) \(\square\)

2.4.3. $\delta$-semistability and GIT semistability Let $Z_{k,\mathcal{D}}(X) \subset Z_{k,\mathcal{D}}'(X)$ be the closure of the locus representing $\delta$-semistable swamps. Consider the projections

$$p_H : Z_{k,\mathcal{D}}(X) \to \mathcal{H}$$

$$p_P : Z_{k,\mathcal{D}}(X) \to \mathbb{P}(Z_1)$$
and define a polarization on $Z_{k, q}(X)$ by
\begin{equation}
\mathcal{O}_{Z_{k, q}(X)}(n_1, n_2) := p_H^* \mathcal{O}_H(n_1) \otimes p_J^* \mathcal{O}_J(Z_1)(n_2),
\end{equation}
n_1 and n_2 being positive integers such that
\begin{equation}
\frac{n_1}{n_2} = \frac{P(l) - \dim(V)}{\dim(V) - s\delta}.
\end{equation}

The natural action of $SL(V)$ on $H \times P(Z_1)$ preserves the projective scheme $Z_{k, q}(X)$ and the linearizations on $\mathcal{O}_H(1)$ and $\mathcal{O}_J(Z_1)(1)$ induces a linearization on $\mathcal{O}_{Z_{k, q}(X)}(n_1, n_2)$.

**Lemma 2.19.** There is a positive integer $A$, depending only on the numerical input data $P, a$, such that it is enough to check the $\delta$-semistability condition (17) for weighted filtrations with $m_i < A$.

**Proof.** Note that a swamp is $\delta$-(semi)stable if and only if Equation (11) holds for every integral weighted filtration, i.e., filtrations with integral weights. Now, the result follows from [14] Lemma 1.4 changing ranks by multiplicities. Observe that the upper bound $A$ does not depend either on $b$ nor on the sheaf $\mathcal{O}$. □

Let $F$ be a finite dimensional vector space, $X$ a Cohen-Macaulay projective and connected curve of genus $g$, $\mathcal{O}_X(1)$ a very ample line bundle of degree $h$ and let $k \in \mathbb{Z}$ be an integer. Given a quotient $q : F \otimes \mathcal{O}_X(k) \to \mathcal{F}$ and a vector subspace $F' \subset F$, we will denote by $\mathcal{F}_{F'}$ the subsheaf $q(F' \otimes \mathcal{O}_X(k))$.

**Lemma 2.20.** Consider the above situation with fixed $F$ and $k$, and assume that $h^1(X, \mathcal{O}_X(l_0)) = 0$ for certain $l_0 \in \mathbb{N}$. Then, for every $l > -k + l_0$ the following holds true: for every quotient sheaf $q : F \otimes \mathcal{O}_X(k) \to \mathcal{F}$ and for every vector subspace $F' \subset F$, $h^1(X, \mathcal{F}_{F'}(l)) = 0$ and $H^0(q(l))(F' \otimes W) = H^0(X, \mathcal{F}_{F'}(l))$, where $W = H^0(X, \mathcal{O}_X(l + k))$.

**Proof.** Consider the exact sequence
\begin{equation*}
\begin{aligned}
0 & \longrightarrow \mathcal{K}_{F'}(l) \longrightarrow F' \otimes \mathcal{O}_X(l + k) \longrightarrow \mathcal{F}_{F'}(l) \longrightarrow 0
\end{aligned}
\end{equation*}

Taking global sections we arrive at a surjection
\begin{equation*}
F' \otimes H^1(X, \mathcal{O}_X(l + k)) \longrightarrow H^1(X, \mathcal{F}_{F'}(l)) \longrightarrow 0
\end{equation*}

Since $h^1(X, \mathcal{O}_X(l_0)) = 0$, clearly $h^1(X, \mathcal{O}_X(p)) = 0$ for every $p \geq l_0$. Therefore, if $l > -k + l_0$ we get $h^1(X, \mathcal{F}_{F'}(l)) = 0$. On the other hand, combining Proposition 2.19 and Theorem 2.6 we deduce that there is a constant $l_1 \in \mathbb{N}$, depending only on the numerical input data such that for every $l \geq l_1$, $h^1(X, \mathcal{F}_{F'}(l)) = 0$ and $\mathcal{K}_{F'}(l)$ is generated by its global sections. Thus, for every $l \geq \max\{l_0, l_1\}$ we have $H^0(q(l))(F' \otimes W) = H^0(X, \mathcal{F}_{F'}(l))$. □

**Proposition 2.21.** Let $g, h, D, a, b \in \mathbb{N}$, $P(n) \in \mathbb{Z}[n]$ a polynomial of degree one and let $k \geq N$ a natural number. There is a natural number $L_0 \in \mathbb{N}$ depending only on the numerical input data such that for every $l \geq L_0$ the following holds true: for every Cohen-Macaulay projective and connected curve of genus $g$ with
a very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, the point $(q, \phi) \in Z_{k, \mathcal{O}}(X)$ is GIT-(semi)stable with respect to $\mathcal{O}_{Z(k, \mathcal{O})(n_1, n_2)}$ if and only if for every weighted filtration $(F_\bullet, m)$ of $F := \mathbb{C}^{P(k)}$,

$$n_1 \left( \sum_{i=1}^{l} m_i ((\dim(F_i) P(l) - \dim(F) P_{\mathcal{F}_i}(l))) + n_2 \delta \mu(\phi, F_\bullet, m) \right) \leq 0. \quad (22)$$

Furthermore, there is an integer $A_2$ depending only on the numerical input data such that it is enough to consider weighted filtrations with $m_i \leq A_2$.

**Proof.** It follows as in [4, Proposition 3.4] and applying Lemma 2.19. The last part follows by applying the same argument given in Lemma 2.19. \hfill \Box

**Proposition 2.22.** Let $g, h, D, a, b, l_0 \in \mathbb{N}$, $P(n) \in \mathbb{Z}[n]$ a polynomial of degree one and let $k \geq N$ be a natural number. There is a natural number $L_1 \in \mathbb{N}$ depending only on the numerical input data such that for every $l \geq L_0$ the following holds true: for every Cohen-Macaulay projective and connected curve of genus $g$ with a very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, such that $h^1(X, \mathcal{O}_X(l_0)) = 0$, a point $(q, \phi)$ is GIT-(semi)stable if and only if for all weighted filtrations $(\mathcal{F}_\bullet, m)$ of $\mathcal{F}$,

$$\sum_{i=1}^{l} m_i((\dim(F) F_{\mathcal{F}_i} - \epsilon_i(F_\bullet) \delta)(P - a \delta) - (P_{\mathcal{F}_i} - \epsilon_i(F_\bullet) \delta)(\dim(F) - a \delta)) \leq 0,$$

where $F := \mathbb{C}^{P(k)}$. Furthermore, if $(q, [\phi])$ is GIT-semistable, then the map $f_q : F \to H^0(X, \mathcal{F}(k))$ induced by the quotient $q$ is injective.

**Proof.** Using the polarization given in Equation (21), the inequality of Proposition 2.21 becomes

$$\sum_{i=1}^{l} m_i((\dim(F_i) - \epsilon(F_\bullet) \delta)(P(l) - a \delta) - (P_{\mathcal{F}_i}(l) - \epsilon_i(F_\bullet) \delta)(\dim(F) - a \delta)) \leq 0,$$

for every $l \geq L_0$, so it is, in fact, an inequality of polynomials. Now, the proposition follows as in [4, Proposition 3.5]. \hfill \Box

**Theorem 2.23.** Let $g, h, D, a, b \in \mathbb{N}$, $P(n) \in \mathbb{Z}[n]$ a polynomial of degree one and let $k \geq N$ be a natural number. There is a natural number $L \in \mathbb{N}$ depending only on the numerical input data such that for every $l \geq L$ the following holds true: for every Cohen-Macaulay projective and connected curve of genus $g$ with a very ample line bundle $\mathcal{O}_X(1)$ of degree $h$, the point $(q, \phi) \in Z_{k, \mathcal{O}}(X)$ is GIT-(semi)stable with respect to $\mathcal{O}_{Z(k, \mathcal{O})(n_1, n_2)}$ if and only if the corresponding swamp $(\mathcal{F}, \phi)$ is $\delta$-(semi)stable and the linear map $f_q : F \to H^0(X, \mathcal{F}(k))$ is an isomorphism, where $F := \mathbb{C}^{P(k)}$.

**Proof.** 1) We will see that if $(q, [\phi])$ is GIT-(semi)stable then $(\mathcal{F}, \phi)$ is $\delta$-(semi)stable and $q$ induces the isomorphism. From Equation (22) and the polarization defined in Equation (21), we deduce that

$$\sum_{i=1}^{l} m_i((\dim(F_{\mathcal{F}_i}) - \epsilon_i(F_\bullet) \delta)\alpha - \alpha_i(\dim(F) - a \delta)) \leq 0,$$
or, equivalently

\[ \sum_{i=1}^{t} m_i (\dim(F_{\mathcal{F}_i}) \alpha_i - \alpha_i \dim(F)) + \delta \mu(\mathcal{F}, m, \phi) \leq 0. \]  

(23)

Since \( \dim(F) = P(k) \) and \( P(k) \leq h^0(\mathcal{F}_i(k)) + h^0(\mathcal{F}^i(k)) \), inequality (23) becomes

\[ (\sum_{i=1}^{t} m_i (\alpha_i P(k) - \alpha h^0(X, \mathcal{F}^i(k))) + \delta \mu(\mathcal{F}, m, \phi) \leq 0. \]  

(24)

Applying [9, Lemma 2.3] and [25, Lemma 1.17], and by a similar argument as given in [4, Theorem 3.5], we deduce that \( \mathcal{F} \) has pure dimension one. Now, by Theorem 2.16, \((\mathcal{F}, \phi)\) is \( \delta \)-semistable. Finally, the quotient \( q \) induces a linear map

\[ f_q : F \rightarrow H^0(X, \mathcal{F}(k)). \]

Let \( F' \subseteq F \) be its kernel. Obviously, \( \mathcal{F}_{F'} = 0 \) and \( \mu(\phi, F' \subseteq F) = \text{adim}(F') \). Proposition 2.21 give us

\[ n_1 \dim(F')P(l) + n_2 \dim(F') \leq 0 \]

hence \( F' = 0 \), so \( f_q \) is injective. Since \((\mathcal{F}, \phi)\) is \( \delta \)-semistable, \( \dim(F) = h^0(X, \mathcal{F}(m)) \) so \( f_q \) is, in fact, an isomorphism.

2) Assume \((\mathcal{F}, \phi)\) is \( \delta \)-(semi)stable and that \( q \) induces an isomorphism \( f_q : F \cong H^0(X, \mathcal{F}(k)) \). Since \( f_q \) is an isomorphism then \( F_{\mathcal{F}'} = H^0(X, \mathcal{F}'(k)) \) for any subsheaf \( \mathcal{F}' \subseteq \mathcal{F} \). Thus, by Theorem 2.16, we have

\[ \sum_{i=1}^{t} m_i (\text{adim}(F_{\mathcal{F}_i}) - \alpha_i P(k)) + \delta \mu(\mathcal{F}, m, \phi) \leq 0 \]  

(25)

for all weighted filtrations. Observe that the left-hand side of Equation (25) is precisely the leading coefficient of the polynomial

\[ \sum_{i=1}^{t} m_i ((\dim(F_{\mathcal{F}_i}) - \epsilon_i(\mathcal{F}) \delta)(P - a) - (P_{\mathcal{F}_i} - \epsilon_i(\mathcal{F}) \delta)(\dim V - a \delta)). \]

We deduce that if we have a strict inequality in Equation (26) then,

\[ \sum_{i=1}^{t} m_i ((\dim(F_{\mathcal{F}_i}) - \epsilon_i(\mathcal{F}) \delta)(P - s) - (P_{\mathcal{F}_i} - \epsilon_i(\mathcal{F}) \delta)(\dim F - s \delta)) < 0. \]

If \((\mathcal{F}, \phi)\) is strictly \( \delta \)-semistable, by Theorem 2.16 there is a filtration \((\mathcal{F}, m)\) giving an equality in \(25\)

\[ \sum_{i=1}^{t} m_i (\text{adim}(F_{\mathcal{F}_i}) - \alpha_i P(k)) + \delta \mu(\mathcal{F}, m, \phi) = 0. \]  

(26)

Note that

\[ \sum_{i=1}^{t} m_i ((\dim(F_{\mathcal{F}_i}) - \epsilon_i \delta)(P - a \delta) - (P_{\mathcal{F}_i} - \epsilon_i \delta)(\dim F - a \delta)) = \]

\[ = \sum_{i=1}^{t} m_i ((\dim(F_{\mathcal{F}_i})P - \dim(F)P_{\mathcal{F}_i}) + \delta(P_{\mathcal{F}_i} a - \epsilon_i P) - \]

\[ - \delta(\dim(F_{\mathcal{F}_i}) a - \epsilon_i \dim(F))). \]
The degree one coefficient of this polynomial is given by
\[
\sum_{i=1}^{t} m_i((\dim(F_{F_i})\alpha - \dim(F)\alpha_i) + \delta(\alpha_i a - \epsilon_i a)) = \\
\sum_{i=1}^{t} m_i(\alpha \dim(F_{F_i}) - \alpha_i P(k)) + \delta \mu(\mathcal{F}_\bullet, m, \phi) = 0,
\]
which is equal to 0 because of Equation (26). Using the equalities \(P(n) = (\dim(F) - \alpha k) + \alpha n, P_{F_i}(n) = (\dim(F_{F_i}) - \alpha_i k) + \alpha_i n\) (this last equality follows from Corollary 2.17) and again Equation (26), it follows that the constant coefficient of this polynomial is also 0. This implies that if \((\mathcal{F}, \phi)\) is \(\delta\)-(semi)stable and \(f_1\) is an isomorphism then
\[
\sum_{i=1}^{t} m_i((\dim(F_{F_i}) - \epsilon_i(\mathcal{F}_\bullet)\delta)(P - s\delta) - (P_{F_i} - \epsilon_i(\mathcal{F}_\bullet)\delta)(\dim(F - s\delta))(\leq)0.
\]
Now, the result follows from Proposition 2.22. \(\square\)

**Theorem 2.24.** Fix a polynomial \(P\), natural numbers \(a\) and \(b\), a rational number \(\delta \in \mathbb{Q}_{>0}\) and a locally free sheaf \(\mathcal{D}\) on \(X\). There is a projective scheme \(\mathcal{T}_{X,P,a,b,\mathcal{D}}^{\delta,ss}\) and an open subscheme \(\mathcal{T}_{X,P,a,b,\mathcal{D}}^{\delta,ss} \subset \mathcal{T}_{X,P,a,b,\mathcal{D}}^{\delta,ss}\) together with a natural transformation
\[
\alpha^{(s)s}_{\mathcal{D},P,a,b}: \text{Swamps}_{P,\mathcal{D},a,b}^{\delta,ss} \rightarrow h_{\mathcal{T}_{X,P,a,b,\mathcal{D}}^{\delta,ss}}
\]
with the following properties:
1) For every scheme \(\mathcal{M}\) and every natural transformation
\[
\alpha': \text{Swamps}_{P,\mathcal{D},a,b}^{\delta,ss} \rightarrow h_{\mathcal{M}},
\]
there exists a unique morphism \(\varphi: \mathcal{T}_{X,P,a,b,\mathcal{D}}^{\delta,ss} \rightarrow \mathcal{M}\) with \(\alpha' = h(\varphi) \circ \alpha^{(s)s}_{\mathcal{D},P,a,b}\).
2) The scheme \(\mathcal{T}_{X,P,a,b,\mathcal{D}}^{\delta,ss}\) is a coarse moduli space for the functor \(\text{Swamps}_{P,\mathcal{D},a,b}^{\delta,ss}\)

**Proof.** This follows as in [4, Theorem 1.8]. \(\square\)

§3

**THE UNIVERSAL MODULI SPACE OF SWAMPS**

Our goal is to prove the existence of a coarse projective moduli space, \(\mathcal{T}_{g,P,a,b}^{\delta,ss}\), for the moduli functor
\[
\text{Swamps}_{g,P,a,b}^{\delta,ss}(T) = \begin{cases} 
\text{isomorphism classes of pairs } (X_T, (\mathcal{F}_T, \phi_T, \mathcal{N})) \\
\text{where } X_T \text{ is a semistable curve of genus } g \\
\text{over } T \text{ and } (\mathcal{F}_T, \phi_T, \mathcal{N}) \text{ is a } \delta\text{-}(semi)stable \\
torsion free swamp of uniform multi-rank } r \\
\text{over } T \text{ and with Hilbert polynomial } P
\end{cases}.
\]
satisfying that there is a natural map \(\Theta_{sw}: \mathcal{T}_{g,P,a,b}^{\delta,ss} \rightarrow \overline{M}_g\) such that for any stable curve \([X] \in \overline{M}_g, \Theta_{sw}^{-1}([X]) = \mathcal{T}_{X,P,a,b}^{\delta,ss}/\text{Aut}(X)\).
3.1. — Gieseker construction of $\overline{M}_g$

Fix integers $g \geq 2$, $d = 10(2g - 2)$ and $M = d - g$. Consider the Hilbert functor

$$\operatorname{Hilb}_{d,g}(T) = \left\{ Z \hookrightarrow T \times \mathbb{P}^M \bigg| T \text{ projective curve of genus } g \text{ and degree } d \forall t \in T \right\}.$$ 

This is just the Quot scheme $\text{Quot}_{\mathcal{O}_{\mathbb{P}^M}/\mathbb{P}^M}(T)$, where $h(s) = ds - g + 1$, so there exists a natural number $s_0$ such that for every $s \geq s_0$

$$i'_s : H_{d,g,M} \hookrightarrow \text{Grass}(h(s), H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(s))^\vee),$$

$H_{d,g,M}$ being the representative of $\text{Hilb}_{d,g}$. Given a stable curve $X$ of genus $g$, $\omega_X^{10}$ is a very ample line bundle with $\dim(H^0(X, \omega_X^{10})) = M + 1$. Thus, once an isomorphism $\mathbb{C}^{M+1} \simeq H^0(X, \omega_X^{10})$ is fixed, $\omega_X^{10}$ embeds $X$ in the projective space $\mathbb{P}^M$, the image being a projective curve of genus $g$ and degree $d$. Therefore, $X \in \text{Hilb}_{d,g}(\mathbb{C})$.

Let us denote by $H_g \subset H_{d,g,M}$ the locus of non-degenerate, 10-canonical stable curves of genus $g$. The scheme $H_g$ is a locally closed subscheme and it is a nonsingular, irreducible quasi-projective variety. The action of $\text{SL}_{M+1}$ on $\mathbb{P}^M$, the image being a projective curve of genus $g$ and degree $d$. Therefore, $X \in \text{Hilb}_{d,g}(\mathbb{C})$.

1. $H_g$ belongs to the semistable locus,
2. $H_g$ is closed in the semistable locus,

from what follows that $\overline{M}_g = H_g/\text{SL}_{M+1}$ exists and is projective. The scheme $H_g$ is endowed with a universal family

$$U_g \xleftarrow{\psi} H_g \times \mathbb{P}^M \xrightarrow{pr_2} \mathbb{P}^M \xrightarrow{pr_1} H_g$$ (27)

called the universal curve of genus $g$. We will denote by $\nu : U_g \to \mathbb{P}^M$ the second projection.

The projective scheme $H_g$ has the following important feature. For any closed point $h \in H_g$, $\psi$ induces a closed immersion $\psi_h : X_h \to \mathbb{P}^M$, $X_h$ being the fiber of $\mu$ over $h \in H_g$, which satisfies that $\psi_h^* \mathcal{O}_{\mathbb{P}^M}(1) \simeq \omega_{X_h}^{10}$ (see [3, Proposition 2.0.0]). Therefore, $\nu^* \mathcal{O}_{\mathbb{P}^M}(1)|_{X_h} \simeq \omega_{X_h}^{10}$ for every $h \in H_g$.

3.2. — Projective embedding of the relative Quot scheme

Consider the relatively very ample line bundle $\nu^* \mathcal{O}_{\mathbb{P}^M}(1) = : \mathcal{O}_{U_g}(1)$ on the universal curve. Define

$$\mathcal{Q}_g^P(\mu, k, P) := \text{Quot}_{\mathbb{P}^N \otimes \mathcal{O}_{U_g}(-k)/U_g/H_g}^P \subset \text{Quot}_{\mathbb{P}^N \otimes \mathcal{O}_{U_g}(-k)/U_g/H_g}^P,$$

where $k \in \mathbb{N}$ and $n = P(k)$, the open and closed subscheme of quotients with uniform multi-rank $r$. By construction, there is a canonical projective morphism

$$\mathcal{Q}_g^P(\mu, k, P) \xrightarrow{\pi} H_g$$ (28)

21
and the fibered product

\[ \mathbb{Q}_g^r(\mu, k, P) \times_{H_g} U_g \xrightarrow{\theta} \mathbb{Q}_g^r(\mu, k, P) \]

is equipped with a universal quotient

\[ q_{U_g} : \mathbb{C}^n \otimes \phi^* \mathcal{O}_{U_g}(-k) \rightarrow \mathcal{E} \rightarrow 0 \quad (29) \]

flat over \( \mathbb{Q}_g(\mu, k, f) \) (see [7]). From Equations (27) and (28) we find a closed immersion

\[ \mathbb{Q}_g^r(\mu, k, P) \times_{H_g} U_g \xrightarrow{id \times \psi} \mathbb{Q}_g^r(\mu, k, P) \times_{H_g} (H_g \times \mathbb{P}^M) \simeq \mathbb{Q}_g^r(\mu, k, P) \times \mathbb{P}^M \]

We can push the universal quotient (29) forward to \( \mathbb{Q}_g^r(\mu, k, P) \times \mathbb{P}^M \) and compose it with the natural surjection \( \mathbb{C}^n \otimes \phi^* \mathcal{O}_{U_g}(-k) \rightarrow \mathbb{C}^n \otimes (id \times \psi)_* \phi^* \mathcal{O}_{U_g}(-k) \). This, in turn, induces an exact sequence,

\[ 0 \rightarrow \mathcal{K} \hookrightarrow \mathbb{C}^n \otimes \phi^* \mathcal{O}_{U_g}(-k) \rightarrow \mathcal{E} \rightarrow 0, \]

all the sheaves being flat over \( \mathbb{Q}_g^r(\mu, k, P) \).

We deduce that there exists an integer \( l_1 \) such that for all \( l > l_1 \) and each \( \xi \in \mathbb{Q}_g^r(\mu, k, P) \)

1. \( h^1(\mathbb{P}^M, \mathcal{K} \otimes \mathcal{O}_{\mathbb{P}^M}(l)) = 0 \)
2. \( h^0(\mathbb{P}^M, \mathcal{K} \otimes \mathcal{O}_{\mathbb{P}^M}(l)) = P(l) \)
3. \( h^1(\mathbb{P}^M, \mathcal{K} \otimes \mathcal{O}_{\mathbb{P}^M}(l)) = 0 \)

Then for \( l > l_1 \) we have a quotient \( (l > k) \),

\[ q_\xi : \mathbb{C}^n \otimes H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(l - k)) \rightarrow H^0(\mathbb{P}^M, \mathcal{K}_\xi(l)) \rightarrow 0. \]

We conclude that for all \( l > l_1(l > k) \) there is a well-defined morphism

\[ i_l : \mathbb{Q}_g^r(\mu, k, P) \rightarrow \text{Grass}(P(l), \mathbb{C}^n \otimes H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(l - k))). \]

\[ \xi \mapsto q_\xi \]

There is, in fact, an integer \( l_2 \) such that for all \( l > l_2(l > k) \) there is a closed immersion

\[ \pi \times i_l : \mathbb{Q}_g^r(\mu, k, P) \hookrightarrow H_g \times \text{Grass}(P(l), \mathbb{C}^n \otimes H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(l - k))), \quad (30) \]

and composing with the Plücker embedding we get the desired projective embedding,

\[ \pi \times i_l : \mathbb{Q}_g^r(\mu, k, P) \hookrightarrow H_g \times \mathbb{P}(H_3), \quad (31) \]

with \( H_3 := \bigwedge^{P(l)}(\mathbb{C}^n \otimes H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(l - k))). \)
3.3. — Projective embedding of swamps data

The results proved so far show that there exists $k \in \mathbb{N}$ large enough such that any pair $(X,(\mathcal{F},\phi))$, given by a stable curve of genus $g$ (with polarization $\mathcal{O}_X(1) := \omega_X^{\otimes 10}$) and a $\delta$-semistable swamp with Hilbert polynomial $P$, defines a point in the relatively projective $H_\mu$-scheme

$$Q_\mu^g(\mu,k,P) \times \mathbb{P}((((\mathbb{C}^n)^{\otimes a})^{\otimes b})^\vee \otimes \mu_*\mathcal{O}_U(ak)) \rightarrow H_g$$

The natural number $k \in \mathbb{N}$ is fixed as before. Since $h^1(X,\omega_X^{\otimes 10}) = 0$ and $h^0(X,\omega_X^{\otimes 10}) = 10(2g-2) - g + 1$ for every stable curve, we have that $\mu_*\mathcal{O}_U(ak)$ is locally free, and that $R^1\mu_*\mathcal{O}_U(ak) = 0$. Consider now the projective bundle

$$\mathbb{P}((((\mathbb{C}^n)^{\otimes a})^{\otimes b})^\vee \otimes \mu_*\mathcal{O}_U(ak))$$

Since $\mu_*\mathcal{O}_U(ak) = pr_1^*\psi_*(\psi^*pr_2^*\mathcal{O}_M(ak))$, the natural surjection

$$pr_2^*\mathcal{O}_M(ak) \rightarrow \psi_*\psi^*pr_2^*\mathcal{O}_M(ak),$$

induces a diagram

$$\begin{array}{ccc}
pr_1^*pr_2^*\mathcal{O}_M(ak) & \xrightarrow{v_k} & pr_1^*\psi_*\psi^*pr_2^*\mathcal{O}_M(ak) \\
\mathcal{O}_H \otimes \subset H^0(\mathbb{P}^M, \mathcal{O}_M(ak)) & \cong & \mu_*\mathcal{O}_U(ak)
\end{array}$$

and therefore a morphism $\mathcal{O}_H \otimes \subset H^0(\mathbb{P}^M, \mathcal{O}_M(ak)) \rightarrow \mu_*\mathcal{O}_U(ak)$, which will be denoted by $v_k$. By Serre’s theorem, there is a natural number $N' \in \mathbb{N}$, that may be taken grater than $N$ (see Paragraph 2.4.2), such that if $k \geq N'$ then $v_k$ is surjective and we have a closed immersion

$$\begin{array}{ccc}
\mathbb{P}((((\mathbb{C}^n)^{\otimes a})^{\otimes b})^\vee \otimes \mu_*\mathcal{O}_U(ak)) & \xrightarrow{\text{closed}} & H_g \times \mathbb{P}(H_1) \\
\downarrow \pi & & \downarrow \pi \\
H_g & & H_g
\end{array}$$

where $H_1 = ((((\mathbb{C}^n)^{\otimes a})^{\otimes b})^\vee \otimes H^0(\mathbb{P}^M, \mathcal{O}_M(ak)))$.

3.4. — Parameter space for swamps and its linearization

Consider the fibered product

$$Y := Q_\mu(\mu,n,P) \times_{H_g} \mathbb{P}((((\mathbb{C}^n)^{\otimes a})^{\otimes b})^\vee \otimes \mu_*\mathcal{O}_U(ak))$$

Giving $\pi_2$ is the same as giving a quotient invertible sheaf

$$(((\mathbb{C}^n)^{\otimes a})^{\otimes b})^\vee \otimes w^*\mu_*\mathcal{O}(ak) \rightarrow \mathcal{L}$$

(33)
on $Y$, which is the same as giving a nonzero morphism
\[
\phi_Y: \left(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right) \otimes O_Y\right) \to w^*\mu_* O_{U_g}(ak) \otimes L,
\]
(34)
while giving $\pi_1$ is the same as giving a quotient sheaf
\[
q_Y: \mathbb{C}^\omega \otimes O_{Y \times H_g U_g} \to \mathcal{E}(k)
\]
(35)
on $Y \times H_g U_g$. Now, we can pull (35) back to $Y \times H_g U_g$, and we get
\[
\phi_Y^\prime: \left(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right) \otimes O_{Y \times H_g U_g}\right) \to \pi_Y^\prime w^*\mu_* O_{U_g}(ak) \otimes \pi_Y^\prime L.
\]
(36)
From Equation (35) and Equation (36) we can form the following diagram
\[
\begin{array}{c}
0 \rightarrow \mathcal{E} \rightarrow \left(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right) \otimes O_{Y \times H_g U_g}\right) \rightarrow (\mathcal{E}(k)^{\otimes a})^{\otimes b} \rightarrow 0 \\
\downarrow \phi_Y^\prime \downarrow \downarrow \downarrow \downarrow \downarrow \\
\pi_Y^\prime w^*\mu_* O_{U_g}(ak) \otimes \pi_Y^\prime L \rightarrow \pi_Y^\prime w^*\mu_* O_{U_g}(ak) \otimes \pi_Y^\prime L \\
\downarrow \downarrow \downarrow \downarrow \\
\pi_Y^\prime \mu_* O_{U_g}(ak) \otimes \pi_Y^\prime L
\end{array}
\]
Since $\pi_Y^\prime \mu_* O_{U_g}(ak) \otimes \pi_Y^\prime L$ is flat over $Y$, there exists a closed subscheme
\[
\mathcal{Z} \subset \mathcal{Q}_g^\prime(\mu, k, P) \times H_g P(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right) \otimes \mu_* O_{U_g}(ak))
\]
characterized by the fact that $\overline{\phi_Y^\prime}|_\mathcal{Z} = 0$. Restricting the diagram to $\mathcal{Z}$ we show that $\overline{\phi_Y^\prime}$ lifts to $(\mathcal{E}(k)^{\otimes a})^{\otimes b}$, that is, it factorises through a morphism
\[
\phi_\mathcal{Z}: \left(\mathcal{E}(k)^{\otimes a}\right)^{\otimes b} \rightarrow \pi_{U_g}^\prime O_{U_g}(ak) \otimes \pi_Y^\prime L.
\]
(37)
Then, the closed subscheme $\mathcal{Z} \subset Y$ carries a universal family of swamps over $H_g$
\[
\begin{align*}
q_\mathcal{Z}: \mathbb{C}^\omega \otimes O_{\mathcal{Z} \times H_g U_g}(-k) & \rightarrow \mathcal{E}|_\mathcal{Z}, \\
\phi_\mathcal{Z}: \left(\mathcal{E}(k)^{\otimes a}\right)^{\otimes b} & \rightarrow \pi_{U_g}^\prime O_{U_g}(ak) \otimes \pi_Y^\prime L.
\end{align*}
\]
(38)
The group $\text{SL}_{M+1}$, acting on $H_g$, induces naturally actions on $\mathcal{Q}_g^\prime(\mu, k, P)$ and $\mathbb{P}(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right)^{\otimes b} \otimes \mu_* O_{U_g}(ak))$, and the group $\text{SL}_n$, acting on $\mathbb{C}^n$, induces actions on $\mathcal{Q}_g^\prime(\mu, k, P)$ and $\mathbb{P}(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right)^{\otimes b} \otimes \mu_* O_{U_g}(ak))$ as well. Therefore, the group $\text{SL}_{M+1} \times \text{SL}_n$ is acting on $\mathcal{Q}_g^\prime(\mu, k, P) \times \mathbb{P}(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right)^{\otimes b} \otimes \mu_* O_{U_g}(ak))$ and the actions commute with each other.
Combining Section 3.2 and Section 3.3 we find a projective embedding
\[
\begin{align*}
\mathcal{Z}, \\
\mathcal{Q}_g^\prime(\mu, k, P) \times H_g P(\left(\left(\mathbb{C}^\omega\right)^{\otimes b}\right)^{\otimes b} \otimes \mu_* O_{U_g}(ak)) \\
H_g \times \text{Grass}(\mathbb{P}(l), \mathbb{C}^\omega \otimes H^0(\mathbb{P}^M, O_{\mathbb{P}^M}(l-k))) \times \mathbb{P}(H^1)
\end{align*}
\]
Finally we can conclude that for large $k, l, s$ we have a closed immersion

$$Z \hookrightarrow \text{Grass}(h(s), H^0(P^n, \mathcal{O}_{P}(s))) \times \text{Grass}(P(l), \mathcal{O}(\mathcal{O}(l - k))) \times \mathcal{P}(H_1)$$

Considering the Plücker embedding, we finally get the closed immersion

$$j_{s,t,k} : Z \hookrightarrow P(H_1) \times P(H_2) \times P(H_3),$$

where

$$H_1 = (((C^n)^{\times a})^{\times b})^\vee \otimes H^0(P^n, \mathcal{O}_{P}(ak)))$$

$$H_2 = \bigwedge P(l) H^0(P^n, \mathcal{O}_{P}(s)))$$

$$H_3 = \bigwedge (\mathcal{C}^{\otimes H^0(P^n, \mathcal{O}_{P}(l - k)))},$$

and which is $SL_{M+1} \times SL_n$-equivariant. For each $i = 1, 2, 3$, denote by $\pi_i$ the projection onto the $i$th factor, $\pi_i : P(H_1) \times P(H_2) \times P(H_3) \to P(H_i)$. Let $\pi_1^* \mathcal{O}(\alpha) \otimes \pi_3^* \mathcal{O}(\gamma)$ be a polarization on $P(H_1) \times P(H_3)$ and recall that

$$H^0(P(H_1), \mathcal{O}_{P(H_1)}(\alpha)) = S^\alpha H_1$$

$$H^0(P(H_3), \mathcal{O}_{P(H_3)}(\gamma)) = S^\gamma H_3$$

We have a canonical surjection

$$(S^\alpha H_1 \otimes S^\gamma H_3) \otimes \mathcal{O}_{P(H_1) \times P(H_3)} \twoheadrightarrow \pi_1^* \mathcal{O}_{P(H_1)}(\alpha) \otimes \pi_3^* \mathcal{O}_{P(H_3)}(\gamma)$$

and then a canonical morphism, $s : P(H_1) \times P(H_3) \to P(S^\alpha H_1 \otimes S^\gamma H_3)$, which is in fact a closed immersion (the $(\alpha, \gamma)$ Segre embedding). Define $J := S^\alpha H_1 \otimes S^\gamma H_3$. Clearly $s^* \mathcal{O}_{P(J)}(1) = \pi_1^* \mathcal{O}_{P(H_1)}(\alpha) \otimes \pi_3^* \mathcal{O}_{P(H_3)}(\gamma)$. Consider now the composition:

$$j_{s,t,k} : Z \hookrightarrow P(H_1) \times P(H_2) \times P(H_3) \overset{\pi_2 \times s}{\twoheadrightarrow} P(H_2) \times P(J).$$

For the polarization $\mathcal{O}(\beta, 1)$ on $P(H_2) \times P(J)$, with $\beta \in \mathbb{N}$, we have

$$(\pi_2 \times s)^* \mathcal{O}(\beta, 1) = \pi_1^* \mathcal{O}_{P(H_1)}(\alpha) \otimes \pi_3^* \mathcal{O}_{P(H_3)}(\beta) \otimes \pi_3^* \mathcal{O}_{P(H_3)}(\gamma).$$

From now onwards, $\alpha$ and $\gamma$ are assumed to satisfy

$$\frac{\alpha}{\gamma} = \frac{P(l) - P(k)}{P(k) - a\delta}$$

as in the fiber-wise problem.
3.5. — Construction of the universal moduli space

Let $\xi : \text{SL}_{M+1} \to \text{SL}(H_2)$ and $\omega : \text{SL}_{M+1} \to \text{SL}(J)$ be two rational representations of $\text{SL}_{M+1}$. Denote by

$$\rho_{H_2} : \mathbb{P}(H_2) \times \mathbb{P}(J) \to \mathbb{P}(H_2),$$

the projections onto $\mathbb{P}(H_2)$. The following two results will be applied to $\mathbb{P}(H_2) \times \mathbb{P}(J)$.

**Proposition 3.1.** [14] Proposition 7.1.1 There exists $\beta_0 = \beta_0(\xi, \omega)$ such that $\forall \beta > \beta_0,$

$$\rho_{H_2}^{-1}(\mathbb{P}(H_2)^s) \subset (\mathbb{P}(H_2) \times \mathbb{P}(J))_{[\beta, 1]}^{s'}. \tag{42}$$

**Proposition 3.2.** [14] Proposition 7.1.2 There exists $\beta_1 = \beta_1(\xi, \omega)$ such that $\forall \beta > \beta_1,$

$$(\mathbb{P}(H_2) \times \mathbb{P}(J))_{[\beta, 1]}^{ss} \subset \rho_{H_2}^{-1}(\mathbb{P}(H_2)^{ss}). \tag{43}$$

The superscripts $\{s', ss\}, \{s'', ss''\}, \{s, ss\}$ will denote stability (semistability) with respect $\text{SL}_{M+1}, \text{SL}_n$ and $\text{SL}_{M+1} \times \text{SL}_n$ respectively, while the subscripts $[\cdot, \cdot]$ (or $[\cdot, \cdot, \cdot]$) denotes the polarization the semistability condition is being analyzed respect with. Define $Z_{\alpha, \beta, \gamma}^{ss} := Z \cap (\mathbb{P}(V) \times \mathbb{P}(W) \times \mathbb{P}(H))_{\alpha, \beta, \gamma}^{ss}$.

**Proposition 3.3.** The quotient $T_{\alpha, \beta, \gamma}^{ss} : Z_{\alpha, \beta, \gamma}^{ss} / (\text{SL}_{M+1} \times \text{SL}_n)$ exists and is projective.

**Proof.** By above propositions we can find $\beta > \max\{\beta_0, \beta_1\}$ such that

$$\rho_{H_2}^{-1}(\mathbb{P}(H_2)^s') \subset (\mathbb{P}(H_2) \times \mathbb{P}(J))_{[\beta, 1]}^{s'}, \tag{44}$$

$$\rho_{H_2}^{-1}(\mathbb{P}(H_2)^{ss'}) \subset (\mathbb{P}(H_2) \times \mathbb{P}(J))_{[\beta, 1]}^{ss'}. \tag{45}$$

If we apply $(id \times s)^{-1}$ to Equation (42) and Equation (43) we get

$$(id \times s)^{-1}\rho_{H_2}^{-1}(\mathbb{P}(H_2)^s') \subset (id \times s)^{-1}(\mathbb{P}(H_2) \times \mathbb{P}(J))_{[\beta, 1]}^{s'} \tag{46}$$

From the fact that $H_g \subset \mathbb{P}(H_2)^s'$ and Equation (44), we find

$$H_g \times \mathbb{P}(H_1) \times \mathbb{P}(H_3) \subset (\mathbb{P}(H_1) \times \mathbb{P}(H_2) \times \mathbb{P}(H_3))_{[\alpha, \beta, \gamma]}^{s'}. \tag{47}$$

Therefore we have

$$Z \subset (\mathbb{P}(H_1) \times \mathbb{P}(H_2) \times \mathbb{P}(H_3))_{[\alpha, \beta, \gamma]}^{s'}. \tag{48}$$
Since $H_g$ is closed in $\mathbf{P}(H_2)^{ss'}$ we know that
\[
H_g \times \mathbf{P}(H_1) \times \mathbf{P}(H_3) \subset \rho_{H_2}^{-1}(\mathbf{P}(H_2)^{ss'})
\] (47)
Now, we have the following diagram
\[
\begin{array}{ccc}
Z'(\rho) & \rightarrow & (\mathbf{P}(H_1) \times \mathbf{P}(H_2) \times \mathbf{P}(H_3))^{ss'}_{[\alpha,\beta,\gamma]} \\
\downarrow & & \downarrow \\
H_g \times \mathbf{P}(H_1) \times \mathbf{P}(H_3) & \subset & \rho_{H_2}^{-1}(\mathbf{P}(H_2)^{ss'})
\end{array}
\]
Since $Z$ is projective over $H_g$, $\varepsilon$ is closed, so
\[
Z \subset (\mathbf{P}(H_1) \times \mathbf{P}(H_2) \times \mathbf{P}(H_3))^{ss'}_{[\alpha,\beta,\gamma]} \subset \rho_{H_2}^{-1}(\mathbf{P}(H_2)^{ss'})
\]
is closed in $\rho_{H_2}(\mathbf{P}(H_2)^{ss'})$. Since $\zeta$ is open and both spaces are open we conclude
that $Z$ is closed in $(\mathbf{P}(H_1) \times \mathbf{P}(H_2) \times \mathbf{P}(H_3))^{ss'}_{[\alpha,\beta,\gamma]}$. Since
\[
(\mathbf{P}(H_1) \times \mathbf{P}(H_2) \times \mathbf{P}(H_3))^{ss'}_{[\alpha,\beta,\gamma]} \subset (\mathbf{P}(H_1) \times \mathbf{P}(H_2) \times \mathbf{P}(H_3))^{ss'}_{[\alpha,\beta,\gamma]},
\]
and
\[
Z^{ss}_{[\alpha,\beta,\gamma]} = Z \cap (\mathbf{P}(V) \times \mathbf{P}(W) \times \mathbf{P}(H))^{ss}_{[\alpha,\beta,\gamma]},
\]
we deduce that $Z^{ss}_{[\alpha,\beta,\gamma]}$ is closed in $(\mathbf{P}(V) \times \mathbf{P}(W) \times \mathbf{P}(H))^{ss}_{[\alpha,\beta,\gamma]}$. Therefore,
\[
T_{P,g,a,b}^{[s]} := Z^{ss}_{[\alpha,\beta,\gamma]} / (\text{SL}_{M+1} \times \text{SL}_n)
\] (48)
exists and is projective.

**Proposition 3.4.** The points of $Z^{ss}_{[k,\alpha,\beta]}$ correspond exactly with pairs $(X, (q, \phi))$ where $X$ is a 10-canonical stable curve of genus $g$, $q : \mathbb{C}^n \otimes \mathcal{O}_X(-k) \rightarrow \mathcal{F}$ is a quotient of uniform multi-rank $r$ such that $H^0(q(k)) : \mathbb{C}^n \rightarrow H^0(X, \mathcal{F}(k))$ is an isomorphism, $(\mathcal{F}, \phi)$ being a $\delta$-(semi)stable swamp of type $(a, b, \mathcal{O}_X)$ and with Hilbert polynomial $P(n)$.

**Proof.** If we prove that there are equalities
\[
Z^{ss}_{[k,\alpha,\beta]} = Z^{ss'}_{[k,\alpha,\beta]},
\]
then the result will follow from Theorem 2.23 and the construction of $Z^{ss}_{[\alpha,\beta,\gamma]}$. To prove the equalities we use the Hilbert-Mumford criterium. Consider the three representations
\[
\begin{align*}
\xi^3 : & \text{SL}_{M+1} \times \text{SL}_n \rightarrow \text{SL}_{M+1} \rightarrow \text{SL}(S^3(H_2)) \\
\omega_1 : & \text{SL}_{M+1} \times \text{SL}_n \rightarrow \text{SL}(S^\alpha(H_1)) \\
\omega_2 : & \text{SL}_{M+1} \times \text{SL}_n \rightarrow \text{SL}(S^\gamma(H_3))
\end{align*}
\]
Show that we obviously have the inclusion $Z^{ss}_{[\alpha,\beta,\gamma]} \subset Z^{ss'}_{[\alpha,\beta,\gamma]}$. The representations $\omega_1$ and $\omega_2$ give us a representation
\[
\omega = \omega_1 \otimes \omega_2 : \text{SL}_{M+1} \times \text{SL}_n \rightarrow \text{SL}(J)
\]
Let $\xi \in \mathcal{Z}_{\{\alpha, \beta, \gamma\}}^{s_{\xi}}$ and define $(\xi_1, \xi_2) := j_{s, t, k}(\xi)$, $j_{s, t, k}$ being the closed immersion given in Equation (40). Let $\lambda : \mathcal{G}_m \to \text{SL}_{M+1} \times \text{SL}_n$ be a nontrivial 1-PS given by

$$
\begin{align*}
\lambda_1 : \mathcal{G}_m &\to \text{SL}_{M+1} \\
\lambda_2 : \mathcal{G}_m &\to \text{SL}_n
\end{align*}
$$

Now the result follows by a similar argument given in [14, Proposition 2.8.1].

**Theorem 3.5.** The projective scheme $\mathcal{T}_{\mathcal{P}, g, a, b}^{\delta, (s)s}$ is a coarse moduli space for the moduli functor $\textbf{Swamps}_{\mathcal{P}, g, a, b}^{\delta, (s)s}$.

**Proof.** Let $T$ be a scheme and $\eta \in \textbf{Swamps}_{\mathcal{P}, g, a, b}^{\delta, (s)s}(T)$, which consists on

$$
\eta = \begin{cases} 
\pi : X_T \to T \text{ a flat family of stable curves of genus } g \\
\text{with relative polarization } \mathcal{O}_{X_T}(1) := \omega_{X_T/T}^{10} \\
\mathcal{F}_T \text{ a flat family of torsion free sheaves with Hilbert polynomial } P(n) \\
\text{and uniform multi-rank } r, \text{ and } \mathcal{N} \text{ an invertible sheaf on } T \\
\phi : (\mathcal{F}_T^{\delta})^{\oplus b} \to \pi^* \mathcal{N} \text{ a } \delta\text{-}(semi)stable swamp }
\end{cases}
$$

Let $k \geq N'$ be the natural number fixed for the construction of $\mathcal{T}_{\mathcal{P}, g, a, b}^{\delta, (s)s}$ (see Section 3.3). Then $R^1\pi_*\mathcal{F}_T(k) = 0$ and $\pi_*\mathcal{F}_T(k)$ is locally free of rank $n = P(k)$. On the other hand, $R^1\pi_*\mathcal{O}_{X_T}(1) = 0$ and $\pi_*\mathcal{O}_{X_T}(1)$ is locally free of rank $N + 1 = 10(2g - 2) - g + 1$. Let $\{U_i\}$ be an open cover of $T$ such that $(\pi_*\mathcal{F}_T(k))|_{U_i} \simeq \mathbb{C}^n \otimes \mathcal{O}_{U_i}$ and $(\pi_*\mathcal{O}_{X_T}(1))|_{U_i} \simeq \mathbb{C}^{N+1} \otimes \mathcal{O}_{U_i}$. This trivializations induce surjections

$$
\begin{align*}
q_{2,W_i} : \mathbb{C}^{N+1} \otimes \mathcal{O}_{X_T}|_{W_i} &\to \mathcal{O}_{X_T}(1)|_{W_i}, \\
q_{1,W_i} : \mathbb{C}^n \otimes \mathcal{O}_{X_T}(-k)|_{W_i} &\to \mathcal{F}_T|_{W_i},
\end{align*}
$$

(50) (51)

where $W_i = \pi^{-1}(U_i)$. Composing $\phi(ak)|_{W_i}$ with $(q_{1,W_i}(k)^{\oplus a})^{\oplus b}$ and taking the pushforward by $\pi$, we get a morphism

$$
\phi_i : ((\mathbb{C}^n)^{\oplus a})^{\oplus b} \otimes \mathcal{O}_T|_{U_i} \to (\mathcal{N}|_{U_i}) \otimes \pi_*\mathcal{O}_{X_T}(ak)|_{U_i}
$$

The first surjection, Equation (50), embeds $q_{2,W_i} : W_i \hookrightarrow U_i \times \mathbb{P}^M$, while the second surjection, Equation (51), defines a map $q_{1,W_i} : U_i \to \mathbb{Q}_g^\vee(\mu, k, P)$. Finally, the morphism $\phi_i$ defines a map $\psi_i : U_i \to \mathbb{P}((((\mathbb{C}^n)^{\oplus a})^{\oplus b})^{\vee} \otimes \mu_*\mathcal{O}_{U_g}(ak))$ as well. Therefore, $q_{1,W_i}$ and $\psi_i$ define a map:

$$
\psi_i : U_i \to \mathbb{Q}_g^\vee(\mu, k, P) \times \mathbb{P}((((\mathbb{C}^n)^{\oplus a})^{\oplus b})^{\vee} \otimes \mu_*\mathcal{O}_{U_g}(ak))
$$

Consider two open subsets $U_i, U_j$, and let $U_{ij} = U_i \cap U_j$ be the intersection. Then, $\phi_i$ and $\phi_j$ define maps

$$
\phi_i : U_{ij} \to \mathbb{Q}_g^\vee(\mu, k, P) \times \mathbb{P}((((\mathbb{C}^n)^{\oplus a})^{\oplus b})^{\vee} \otimes \mu_*\mathcal{O}_{U_g}(ak))
$$

$$
\phi_j : U_{ij} \to \mathbb{Q}_g^\vee(\mu, k, P) \times \mathbb{P}((((\mathbb{C}^n)^{\oplus a})^{\oplus b})^{\vee} \otimes \mu_*\mathcal{O}_{U_g}(ak))
$$

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which differ by a $U_{ij}$-valued point of the group scheme $\text{SL}_n \times \text{SL}_{M+1}$ and that take values in $\mathcal{Z}_{\alpha, \beta, \gamma}^{ss}$ because $\eta$ is $\delta$-semistable. Therefore, there is a well-defined morphism $T \to T_{P,g,a,b}^{\delta-(s)s}$. This shows the existence of a natural transformation

$$\Psi : \text{Swamps}_{P,g,a,b}^{\delta-(s)s} \to \text{Hom}(-, T_{P,g,a,b}^{\delta-(s)s}).$$

Let $\mathcal{M}$ be a scheme and suppose that there exists a natural transformation $\Psi' : \text{Swamps}_{P,g,a,b}^{\delta-(s)s} \to \text{Hom}(-, \mathcal{M})$

There is a canonical member $\phi_{\text{univ}} \in \text{Swamps}_{P,g,a,b}^{\delta-(s)s} \mathcal{Z}_{\alpha, \beta, \gamma}^{ss}$ corresponding to the universal family (see Equation (38)). The morphism

$$\Psi' (\phi_{\text{univ}}) : \mathcal{Z}_{\alpha, \beta, \gamma}^{ss} \to \mathcal{M}$$

is $\text{SL}_n \times \text{SL}_{M+1}$-invariant, so it descends to a morphism $\Psi' (\phi_{\text{univ}}) : T_{P,g,a,b}^{\delta-(s)s} \to \mathcal{M}$ which defines a map $\Phi' : \text{Hom}(-, T_{P,g,a,b}^{\delta-(s)s}) \to \text{Hom}(-, \mathcal{M})$. Clearly, the triangle

$$\text{Swamps}_{P,g,a,b}^{\delta-(s)s} \xrightarrow{\Psi} \text{Hom}(-, T_{P,g,a,b}^{\delta-(s)s}) \xrightarrow{\Psi'} \text{Hom}(-, \mathcal{M})$$

commutes. Finally, the equality $\text{Swamps}_{P,g,a,b}^{\delta-(s)s}(C) = \text{Hom}(\text{Spec} C, T_{P,g,a,b}^{\delta-(s)s})$ follows from Proposition 3.4, so $T_{P,g,a,b}^{\delta-(s)s}$ is a coarse moduli space

\section{A Compactification of the Universal Moduli Space of Principal Bundles}

Let $G$ be a reductive algebraic group and $\rho : G \hookrightarrow \text{SL}(V)$ be a finite and faithful representation of dimension $r$, $\delta \in \mathbb{Q}_{>0}$. Given a natural number $g \geq 2$, we will show that there is a projective coarse moduli space, $\text{SPB}(\rho)^{\delta-(s)s}_{P,g}$, for the moduli functor

$$\text{SPB}(\rho)^{\delta-(s)s}_{P,g} (T) = \left\{ \begin{array}{cl} \text{isomorphism classes of pairs } (X_T, (\mathcal{F}_T, \tau_T)) & \text{over } T \text{ and } (\mathcal{F}_T, \tau_T) \text{ is a } \delta-(\text{semi})\text{stable singular principal } G\text{-bundle of uniform multi-rank } r \\
\text{over } T \text{ with Hilbert polynomial } P & \end{array} \right\},$$

together with a morphism $\Theta_{ab} : \text{SPB}(\rho)^{\delta-(s)s}_{P,g} \to \overline{\text{M}}_g$. We will assume that each curve brings with it as polarization the very ample invertible sheaf given by $\omega_X^\otimes 10$, which has degree $h = 10(2g - 2)$. In addition, we will show that if $P(n) = (rh)n + r(1 - g)$ and $\delta$ is very large, then $\Theta_{ab}^{-1}([X]) = \text{M}_X^{(s)s}(G)/\text{Aut}(X)$ for every smooth projective curve of genus $g$. 

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4.1. — The fiber-wise problem

Let $X$ be a stable curve over $\mathbb{C}$ of genus $g$. A singular principal $G$-bundle over $X$ is a pair $(\mathcal{F}, \tau)$ where $\mathcal{F}$ is a torsion free sheaf of rank $r$ and $\tau$ is a morphism of $\mathcal{O}_C$-algebras, $\tau : S^s(V \otimes \mathcal{F})^G \to \mathcal{O}_C$, which is not just the projection onto the zero degree component.

Consider a singular principal $G$-bundle on $X$, $\tau : S^s(V \otimes \mathcal{F})^G \to \mathcal{O}_X$. We can fix $s \in \mathbb{N}$ such that $S^s(V \otimes \mathcal{F})^G$ is generated by the submodule $\bigoplus_{i=0}^s S^i(V \otimes \mathcal{F})^G$. Let $d \in \mathbb{N}^s$ be such that $\sum id_i = s!$. Then we have:

$$\bigotimes_{i=1}^s (V \otimes \mathcal{F})^{\otimes id_i} \to \bigotimes_{i=1}^s S^d_i (S^i(V \otimes \mathcal{F})) \to \bigotimes_{i=1}^s S^d_i (S^i(V \otimes \mathcal{F}))^G \to \mathcal{O}_X \quad (52)$$

Adding up these morphisms as $d \in \mathbb{N}$ varies we find a swamp (see [17])

$$\text{Swamp}(\tau) := \phi_\tau : (V \otimes \mathcal{F})^{\otimes s!} \to \mathcal{O}_X. \quad (53)$$

Define $a := s!$ and $b = N$. Then, by [11], there is an $s \in \mathbb{N}$ large enough such that the map

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\
\text{of singular principal} \\
G\text{-bundles} \end{array} \right\} \to \left\{ \begin{array}{c} \text{isomorphism classes} \\
\text{of swamps} \\
\text{of type } (a, b, \mathcal{O}_X) \end{array} \right\} \quad (54)$$

is injective and depends only on the numerical input data, and not on the base curve.

**Definition 4.1.** Let $\delta \in \mathbb{Q}_{>0}$. A singular principal $G$-bundle is said to be $\delta$-semi(stable) if its associated swamp is $\delta$-semi(stable).

Therefore, there is a natural transformation from the functor

$$\text{SPB}(\rho)_{P}^{\delta(s)s} (S) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\
\text{families of } \delta\text{-}(semi)stable singular} \\
G\text{-bundles on } X \text{ parametrized} \\
\text{by } S \text{ of uniform multi-rank } r \\
\text{and with Hilbert polynomial } P \end{array} \right\}.$$

\text{to the functor}

$$\text{Swamps}_{P, \mathcal{L}_X, a, b}^{\delta(s)s}(T) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\
\text{ } \delta\text{-}(semi)stable torsion free swamps} \\
\text{of } (V \otimes \mathcal{F}_T, \phi_T, N) \text{ of uniform multi-rank } r \\
\text{and with Hilbert polynomial } P \end{array} \right\}.$$

This allows proving, following standard arguments (see [11]), the next theorem.

**Theorem 4.2.** [11 Theorem 6.3] There is a projective scheme $\text{SPB}(\rho)_{P}^{\delta(s)s}$ and an open subscheme $\text{SPB}(\rho)_{P}^{\delta(s)s} \subset \text{SPB}(\rho)_{P}^{\delta(s)s}$ together with a natural transformation

$$\alpha^{(s)s} : \text{SPB}(\rho)_{P}^{\delta(s)s} \to h_{\text{SPB}(\rho)_{P}^{\delta(s)s}}$$

with the following properties:

1) For every scheme $\mathcal{N}$ and every natural transformation $\alpha' : \text{SPB}(\rho)_{P}^{\delta(s)s} \to h_{\mathcal{N}}$, there exists a unique morphism $\varphi : \text{SPB}(\rho)_{P}^{\delta(s)s} \to \mathcal{N}$ with $\alpha' = h(\varphi) \circ \alpha^{(s)s}$.

2) The scheme $\text{SPB}_{P}(\rho)^{\delta} = \text{SPB}^{\delta(s)s}$ is a coarse projective moduli space for the functor $\text{SPB}_{P}(\rho)^{\delta(s)s}$. 

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4.2. — The relative problem

Consider the Quot scheme \( Q_g^r(\mu, k, P) = \text{Quot}^r_{C^0 \otimes \mathcal{O}_{U_g}(-k)/U_g/H_g} \), that was considered in the case of swamps, and the affine \( H_g \)-scheme defined by

\[
H(V, s, k) := \bigoplus_{i=1}^s \text{Hom}_{H_g}(S^i(V \otimes \mathbb{C}^n) \otimes \mathcal{O}_{H_g}, \mu_s \mathcal{O}_{U_g}(ik)) \to H_g.
\]

By [11] and the results of Section 2, we know that for \( s, k \) large enough, every \( \delta \)-semistable singular principal \( G \)-bundle over a semistable curve of genus \( g \) determines a point in \( Q_g^r(\mu, k, P) \times_{H_g} H(V, s, k) \). Denote by \( \pi \) and \( \overline{\pi} \) the projections

\[
(Q_g^r(\mu, k, P) \times_{H_g} H(V, s, k)) \times_{H_g} U_g \to U_g,
\]

\[
(Q_g^r(\mu, k, P) \times_{H_g} H(V, s, k)) \times_{H_g} U_g \to H_g,
\]

respectively. The goal now is to put a scheme structure on the locus given by the points \(([q], [k]) \in Q_g^r(\mu, k, P) \times_{H_g} H(V, s, k)\) that comes from a morphism of algebras \( S^* (V \otimes \mathcal{F})^G \to \mathcal{O}_Y \). For the sake of clarity, let us denote \( \Xi_g := (Q_g^r(\mu, k, P) \times_{H_g} H(V, s, k)) \times_{H_g} U_g \). On \( \Xi_g \), there are universal morphisms

\[
\varphi^i : S^i(V \otimes W) \otimes \mathcal{O}_{\Xi_g} \to \overline{\pi}^* \overline{\pi}_* (\mathcal{O}_{U_g}(ik))
\]

that composed with the evaluation maps \( \overline{\pi}^* \overline{\pi}_* (\mathcal{O}_{U_g}(ik)) \to \mathcal{O}_{U_g}(ik) \) lead to

\[
S^i(V \otimes W) \otimes \mathcal{O}_{\Xi_g} \to \mathcal{O}_{\Xi_g}(ik) = \mathcal{O}_{\Xi_g}(ik)
\]

Summing up these morphisms over all \( i \) we find

\[
\varphi_{\Xi_g} : \mathcal{Y}_{\Xi_g} := \bigoplus_{i=1}^s S^i(V \otimes W \otimes \mathcal{O}_{\Xi_g}(-k)) \to \mathcal{O}_{\Xi_g}.
\]

Now \( \varphi \) gives a morphism \( \tau_{\Xi_g} : S^* (\mathcal{Y}_{\Xi_g}) \to \mathcal{O}_{\Xi_g} \). Consider now the universal quotient, \( q_{U_g} : C^n \otimes \phi^* \mathcal{O}_{U_g}(-k) \to \mathcal{F} \), on \( Q_g^r(\mu, k, P) \times_{H_g} U_g \). Pulling it back to \( \Xi_g \) we get a quotient \( q_{\Xi_g} : C^n \otimes \mathcal{O}_{\Xi_g}(-k) \to \mathcal{F} \), and therefore a chain of surjections

\[
S^* (V \otimes C^n \otimes \mathcal{O}_{\Xi_g}(-k)) \xrightarrow{S^* (1 \otimes q_{\Xi_g})} S^* (V \otimes \mathcal{F})
\]

Let us denote by \( \beta \) the composition of these morphisms and consider the diagram

\[
0 \to \text{Ker}(\beta) \xrightarrow{\tau_{\Xi_g}} S^* (\mathcal{Y}_{\Xi_g}) \xrightarrow{\beta} S^* (V \otimes \mathcal{F})^G \to 0
\]

As in the fiber-wise problem, there exists a closed subscheme \( D_g \subset Q_g^r(\mu, k, P) \times_{H_g} H(V, s, k) \) over which the morphism \( \tau_{\Xi_g} : S^* (\mathcal{Y}_{\Xi_g}) \to \mathcal{O}_{\Xi_g} \) lifts to a morphism of
algebras $S^*(V \otimes \mathcal{F}|_{D_g})^G \to \mathcal{O}_{D_g}$. There are two groups acting on $D_g$. The action of the group $\text{SL}_{M+1}$ on $H_g$ lifts to an action on $D_g$, while the group $\text{GL}_n$ is acting on both, $Q^s_\mu(\mu, k, P)$ and $H(V, s, k)$. The group $\text{GL}_n$ leaves invariant $D_g$, so $\text{GL}_n$ is acting on $D_g$ as well. Again, as in the fiber-wise problem, we can study the quotient by studying separately the actions of $\mathbb{C}^*$ and $\text{SL}_n$ (see [17, Section 4.2]).

Let $Z$ be the parameter space for swamps $\phi : ((V \otimes \mathcal{F})^{\otimes n})^{\oplus b} \to \mathcal{O}_X$ (see Subsection 3.4). The injective map defined in Equation (54) shows that there is a well-defined morphism of $H_g$-schemes

$\text{Swamp} : D_g \to Z$

which is $\mathbb{C}^*$-invariant and $\text{SL}_n$-equivariant, injective and proper. Thus, it induces a morphism

$\overline{\text{Swamp}} : D_g := D_g//\mathbb{C}^* \to Z$

which is $\text{SL}_n$-equivariant, injective and proper. Finally, by Definition 4.1 and Theorem 3.3 we conclude that $\text{SPB}(\rho)^{\delta-(s)s}_{P,g}$ exists and is projective.

**Theorem 4.3.** The projective scheme $\text{SPB}(\rho)^{\delta-(s)s}_{P,g}$ is a coarse moduli space for the moduli functor $\text{SPB}(\rho)^{\delta-(s)s}_{P,g}$.

**Proof.** Follows from standard arguments as those given in [17, Proposition 4.1.1, Proposition 4.2.1, Theorem 4.2.2.] and Theorem 3.5. \qed

### 4.3. — The fibers over the nonsingular locus $M_g$

Assume now that $P(n) = (rh)n + r(1 - g)$. This is the Hilbert polynomial of coherent sheaves of uniform multi-rank $r$ and degree 0. We have constructed a projective scheme $\text{SPB}(\rho)^{\delta-(s)s}_{P,g}$ together with a map $\Theta_{sw} : \text{SPB}(\rho)^{\delta-(s)s}_{P,g} \to M_g$ satisfying that for any stable curve $[X] \in M_g$,

$$\Theta_{sw}^{-1}(X) = \text{SPB}_X(\rho)^{\delta-(s)s}_{P}/\text{Aut}(X).$$

By [18, Theorem 3.3.1], there exists $\delta_\infty \in \mathbb{Q}_{>0}$ large enough, depending only on $a, b, P$, such that the associated swamp of any $\delta$-semistable singular principal $G$-bundle on a smooth projective curve of genus $g$, $(\mathcal{E}, \tau) \in \text{SPB}_X(\rho)^{\delta-(s)s}_{P}$, is generically semistable, and by [20, Corollary 4.1.2] and [21, Remark 2.3.4.4], this means that $(\mathcal{E}, \tau)$ is honest and $\mathcal{E}$ is a semistable vector bundle. Finally, form [17, Paragraph 5.1], we deduce that $\text{SPB}_X(\rho)^{\delta-(s)s}_{P} = M_X(G)$ for any $\delta > \delta_\infty$ and any smooth projective curve of genus $g$. Therefore, if we assume $\delta > \delta_\infty$, we have

$$\Theta_{sw}^{-1}(X) = M_X(G)/\text{Aut}(X)$$

for every smooth projective curve of genus $g$, thus, $\text{SPB}(\rho)^{\delta-(s)s}_{P,g}$ is a compactification of the moduli problem defined by pairs $(X, P)$ where $X$ is a smooth projective curve of genus $g$ and $P$ is a principal $G$-bundle.
4.4. — Open problems and future work

Observe that two questions remain open. First of all, a good compactification, in the sense of Pandharipande, requires for $\Theta_{\text{ns}}^{-1}(M_g)$ to be a dense open subset of $\text{SPB}(\rho)^{\delta(s)\text{ss}}_{P,g}$. Since it is obviously open, that remains to be solved is the part regarding the density. Note also that this is equivalent to the following statement: Let $G$ be a reductive group, $\rho: G \hookrightarrow \text{SL}(V)$ a faithful representation of dimension $r$, $\delta \in \mathbb{Q}_{>0}$ larger than $\delta_{\infty}$ and $g \geq 2$ a natural number. Let $X$ be a stable curve of genus $g$ and $(\mathcal{F}, \tau)$ a $\delta$-semistable singular principal $G$-bundle of degree 0 and rank $r$. Then, there is a discrete valuation ring $(\mathcal{O}, m, \mathbb{C})$, a flat family of stable curves of genus $g$, $X_T \rightarrow T := \text{Spec}(\mathcal{O})$, the generic fiber $X_{T,\mu}$ is smooth and $X_{T,\mu} \simeq X$, and a flat family of $\delta$-semistable singular principal $G$-bundles $(\mathcal{F}_T, \tau_T)$ such that $(\mathcal{F}_{T,\mu}, \tau_{T,\mu}) \simeq (\mathcal{F}, \tau)$. By [14, Lemma 9.2.3], the extension of the sheaf $\mathcal{F}$ is ensured, so the problem we have to deal with is the extension of the morphism of algebras $\tau: S^*(V \otimes \mathcal{F})^G \rightarrow \mathcal{O}_X$. This problem has been addressed, in part, by X. Sun in the series of papers [26, 27] for the case $G = \text{SL}_n$.

On the other hand, recall that the fibers over smooth curves are given in terms of semistable principal $G$-bundles, while the fibers over singular curves are given in terms of $\delta$-semistable singular principal $G$-bundles. The last are parameter-dependent objects while the former are not. Accordingly, the second problem consists on giving a purely geometric and parameter-independent description of these objects (see [19] for the case of irreducible curves). We have proved that for $\delta$ large enough every $\delta$-semistable singular principal $G$-bundle is semistable (in some sense) and honest, that is, it defines a principal $G$-bundle in a dense open subset of the base curve. Furthermore, there is a large $\delta$ that works for every stable curve of a given genus $g \geq 2$ as in the smooth case. This result will appear in a forthcoming paper.

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