DUAL MIXED VOLUMES AND ISOSYSTOLIC INEQUALITIES

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Abstract. The theory of dual mixed volumes is extended to star bodies in cotangent bundles and is used to prove several isosystolic inequalities for Hamiltonian systems and Finsler metrics.

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1. INTRODUCTION

A celebrated theorem of Pu (see [25]) states that the volume of any Riemannian metric on the projective plane is bounded below by \(2/\pi\) times the square of the length of the shortest non-contractible geodesic. Equality holds if and only if the metric is of constant curvature. In the same paper, Pu investigates analogues of this result in other homogeneous spaces under the condition that the metrics involved be conformal to an invariant metric. In [6], M. Berger considered infinitesimal deformations of metrics in compact symmetric spaces of rank one and proved, among other results, that if \(g_t\) is a one-parameter family of Riemannian metrics on \(\mathbb{R}^n\) such that \(g_0\) is the standard invariant metric of constant curvature, then there exists a second family of Riemannian metrics, \(h_t\), that agrees to first order with \(g_t\) at \(t = 0\) and satisfies the isosystolic inequality

\[
\frac{\text{sys}_1^v(\mathbb{R}^n, h_t)}{\text{vol}(\mathbb{R}^n, h_t)} \leq \frac{\text{sys}_1^v(\mathbb{R}^n, g_0)}{\text{vol}(\mathbb{R}^n, g_0)}.
\]

Here \(\text{sys}_1(\mathbb{R}^n, g)\), the 1-systole of \((\mathbb{R}^n, g)\), denotes the length of the shortest non-contractible geodesic for the metric \(g\).

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This paper extends many of Pu’s sharp isosystolic inequalities and Berger’s infinitesimal isosystolic inequality \cite{11} to Finsler metrics and Hamiltonian systems. In mechanical terms these inequalities provide lower bounds for the Liouville volume enclosed by an energy surface in terms of the action of periodic solutions of the system on that energy level.

The first part of the paper extends the theory of dual mixed volumes to star bodies in cotangent bundles. This theory is used to define relative invariants for pairs of Finsler metrics on a compact \( n \)-dimensional manifold \( M \), \( \tilde{W}_k(M; L, L_0) \), \( 1 \leq k \leq n - 1 \), and to prove the inequality

\[
\tilde{W}_k(M; L, L_0)^n \leq \frac{\text{vol}(M, L)^{n-k}}{\text{vol}(M, L_0)^{n-k}}.
\]

Equality holds if and only if \( L \) is a constant multiple of \( L_0 \).

The second part of the paper considers three different sets of hypotheses under which the inequality

\[
\tilde{W}_k(M; L, L_0)^n \geq \frac{\text{sys}_k(M, L)}{\text{sys}_k(M, L_0)}
\]

holds. To recall, for \( k > 1 \), the \( k \)-systole of a Riemannian or Finsler manifold \((M, L)\), \( \text{sys}_k(M, L) \), is the infimum of the areas of all \( k \)-dimensional submanifolds that are not homologous to zero. The notion of area and volume on a Finsler manifold we shall use is that of Holmes and Thompson \cite{15, 4}, which from the Hamiltonian viewpoint is more natural than the Hausdorff measure.

Under these hypotheses, inequalities (2) and (3) yield the isosystolic inequality

\[
\frac{\text{sys}_k(M, L)^n}{\text{sys}_k(M, L_0)^n} \leq \tilde{W}_k(M; L, L_0)^n \leq \frac{\text{vol}(M, L)^{n-k}}{\text{vol}(M, L_0)^{n-k}}.
\]

In the first set of hypotheses \( M \) is a homogeneous space, \( L_0 \) is an invariant Finsler metric, \( L \) is conformal to \( L_0 \), and the result is a fairly straightforward generalization of the results in \cite{25}. The second and third sets of hypotheses are more subtle.

**Theorem 1.1.** Let \( L_0 \) be a Finsler metric on \( \mathbb{R}P^n \) such that its geodesic flow is symplectically conjugate to the geodesic flow of a metric of constant curvature. If \( L \) is conformal to \( L_0 \), then

\[
\frac{\text{sys}_1(M, L)^n}{\text{sys}_1(M, L_0)^n} \leq \tilde{W}_{n-1}(M; L, L_0)^n \leq \frac{\text{vol}(M, L)}{\text{vol}(M, L_0)}.
\]

**Theorem 1.2.** Let \( L_0 \) be a Finsler metric on \( \mathbb{R}P^n \) such that its geodesic flow is periodic. If the geodesic flow of a Finsler metric \( L \) on \( \mathbb{R}P^n \) commutes with that of \( L_0 \), then

\[
\frac{\text{sys}_1(M, L)^n}{\text{sys}_1(M, L_0)^n} \leq \tilde{W}_{n-1}(M; L, L_0)^n \leq \frac{\text{vol}(M, L)}{\text{vol}(M, L_0)}.
\]

It is not clear if there is any difference between the hypotheses on \( L_0 \) in the previous theorems. The geodesic flow of a Finsler metric on \( \mathbb{R}P^2 \) is periodic if and only if it is symplectically conjugate to the geodesic flow of a metric of constant curvature. In higher dimensions this is also the case for all known examples.

Theorem \cite{12, 12} together with the technique of averaging of Hamiltonian systems (see \cite{24} and \cite{11}) yields the following generalization of Berger’s result.
Theorem 1.3. If $L_t$ is a smooth path of Finsler metrics on $\mathbb{RP}^n$ such that the geodesic flow of $L_0$ is periodic, then there exists another smooth path of Finsler metrics, $K_t$, that agrees to first order with $L_t$ at $t = 0$ and satisfies the isosystolic inequality
\[
\frac{\text{sys}_n^*(\mathbb{RP}^n, K_t)}{\text{vol}(\mathbb{RP}^n, K_t)} \leq \frac{\text{sys}_n^*(\mathbb{RP}^n, L_0)}{\text{vol}(\mathbb{RP}^n, L_0)}.
\]

Unfortunately, the Finsler extension of Pu’s theorem for Riemannian metrics on the projective plane lies beyond the reach of these techniques. This extension is, however, an easy consequence of the main result of S. Ivanov in \cite{Ivanov} and would follow immediately from Theorem 1.1 if the following Finsler generalization of the uniformization theorem were true:

**Uniformization conjecture.** If $L$ is a reversible Finsler metric on $\mathbb{RP}^2$, then there exists a smooth positive function $\rho$ such that the geodesic flow of the metric $\rho L$ is periodic.

It is important to remark that the extensive work on coarse isosystolic inequalities and systolic freedom by Babenko, Gromov, Katz, and Suciu (see, for example, \cite{Babenko, Gromov, Katz, Suciu}) applies unchanged to Finsler metrics.

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2. Dual mixed volumes on cotangent bundles

The theory of dual mixed volumes was introduced by E. Lutwak in \cite{Lutwak1, Lutwak2} as a version of the Brunn-Minkowski theory in which averages of areas of projections of convex bodies, *quermassintegrals*, are replaced by averages of areas of central sections of star-shaped bodies. The theory plays a key role in the solution of the Busemann-Petty problem and other problems in geometric tomography (\cite{Busemann, Petty}). It also has applications to integral geometry (e.g., \cite{Lutwak3}) and the theory of valuations (\cite{Valuations1, Valuations2}).

In this section we present a straightforward extension of the dual theory to star bodies in cotangent bundles. In simple terms, a star body in a cotangent bundle is a choice of a star-shaped body in each cotangent space that varies continuously with the base point. The following is the precise definition:

**Definition 2.1.** Let $M$ be a compact manifold whose boundary need not be empty. A *star Hamiltonian* is a continuous function $H : T^*M \to [0, \infty)$ that is

- positive outside the zero section;
- positively homogeneous of degree one (i.e., $H(tp) = tH(p)$ whenever $t > 0$);
- proper.

We shall say a star Hamiltonian is *smooth* if it is smooth outside the zero section. A subset $A \subset T^*M$ is said to be a *star body* if
\[
A = D^*(M, H) := \{p \in T^*M : H(p) \leq 1\}
\]
for some star Hamiltonian.
Examples of star bodies are the unit codisc bundles of Riemannian or Finsler metrics and, at least in this paper, the most interesting applications will deal with these.

A very useful tool in the study of star bodies is the radial function: if \( A \) is a star body with Hamiltonian \( H \) and \( U \) is a "model" star body with Hamiltonian \( H_0 \) and boundary \( \partial U \), the radial function, \( \rho_A \), of \( A \) (with respect to \( U \)) is the restriction of \( 1/H \) to \( \partial U \). It’s easy to see that any continuous, positive function on \( \partial U \) is the radial function of some star body, and that the map \( p \mapsto \rho_A(p) \) is a homeomorphism between the boundary of \( U \) and the boundary of \( A \).

In the classical theory of dual mixed volumes, the model star-shaped body is the unit sphere. However, most of the basic results are independent of the choice of model body, and in great part the flexibility of the extension of the theory to cotangent bundles is due to this. Most of the results in this paper are obtained by comparing star bodies to different, more symmetric bodies.

An elementary property of star bodies is that they form a lattice: finite unions and intersections of star bodies are star bodies. Indeed, the radial function for the union of two star bodies is the maximum of their radial functions, and the radial function of the intersection of two star bodies is the minimum of their radial functions. More importantly, we can dilate and add star bodies:

**Definition 2.2.** Let \( A \) and \( B \) be star bodies with radial functions \( \rho_A \) and \( \rho_B \). If \( \lambda \) is a positive real number, we denote by \( \lambda A \) the star body with radial function \( \lambda \rho_A \) and by \( A + B \), the radial sum of \( A \) and \( B \), the star body with radial function \( \rho_A + \rho_B \).

An intrinsic, more geometric description of the radial sum goes as follows: if \( p \) and \( p' \) are two covectors in \( T^*_x M \), we define their radial sum, \( p + p' \), as \( p + p' \) if they belong to the same one-dimensional subspace and as zero otherwise. The radial sum, \( A + B \), of two star bodies \( A \) and \( B \) in \( T^* M \) is the union over all points \( x \) in \( M \) of the sets

\[
(A + B)_x := \{ p + p' : p \in A \cap T^*_x M, p' \in B \cap T^*_x M \}.
\]

Radial functions are also useful in describing a topology on the set of star bodies.

**Definition 2.3.** Let us fix a model body and define the distance between two star bodies \( A \) and \( B \) with radial functions \( \rho_A \) and \( \rho_B \) as the maximum of \( |\rho_A - \rho_B| \). The topology induced by this metric is independent of the choice of model body used to define the radial functions, and will be called the radial Hausdorff topology.

The theory of dual mixed volumes is the study of the interaction between the radial sum and the volume of star bodies. An advantage of extending the classical theory to cotangent bundles is that we may use the natural symplectic volume on these spaces:

Let \( M \) be a smooth \( n \)-dimensional manifold and let \( \pi : T^* M \to M \) be its cotangent bundle. The canonical 1-form \( \alpha \) on \( T^* M \) is the form whose value at a tangent vector \( v_p \in T_p T^* M \) equals \( p(\pi_*(v_p)) \). In local canonical coordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\), \( \alpha \) takes the form

\[
\alpha = \sum_{i=1}^n p_i dq_i.
\]
The symplectic form on $T^*M$ is defined as the 2-form $\omega := d\alpha$ and the (Liouville) volume form is $\omega^n/n!$.

Notice that when $(M, g)$ is an $n$-dimensional Riemannian manifold the volume of its unit codisc bundle equals the Riemannian volume of $(M, g)$ times the volume of the Euclidean unit ball of dimension $n$.

The volume of a star body can be easily described in terms of its radial function:

**Proposition 2.1.** Let $A$ be a star body in the cotangent bundle of an $n$-dimensional manifold $M$ and let $\rho_A$ be its radial function. If we set $\Omega := \alpha \wedge (d\alpha)^{n-1}/n!$, the volume of $A$, $V(A)$, is given by the integral

$$\int_{\partial U} \rho_A^n \Omega.$$

**Proof.** We give a proof in the case where both the model body $U$ and the star body $A$ have smooth boundaries. The general result follows by a standard approximation argument.

By Stokes formula we have that

$$V(A) := \int_A \omega^n/n! = \int_{\partial A} \Omega.$$

The map $\delta : \partial U \to \partial A$ defined by $\delta(p) = \rho(p)p$ is a diffeomorphism and since $\delta^*\alpha = \rho \alpha$, we have that $\delta^*\Omega equals \rho^n\Omega$. We conclude that

$$V(A) = \int_{\partial A} \Omega = \int_{\delta(\partial U)} \Omega$$

$$= \int_{\partial U} \delta^*\Omega = \int_{\partial U} \rho^n\Omega. \qed$$

**Definition 2.4.** Let $A_1, \ldots, A_n$ be $n$ star bodies in the cotangent of a compact $n$-dimensional manifold $M$ and let $\rho_1, \ldots, \rho_n$ be their radial functions. The dual mixed volume of $A_1, \ldots, A_n$, denoted by $\tilde{V}(A_1, \ldots, A_n)$, is defined as the integral

$$\tilde{V}(A_1, \ldots, A_n) := \int_{\partial U} \rho_1 \cdots \rho_n \Omega.$$

The next proposition shows that dual mixed volumes do not depend on the choice of model body that is used in the definition of the radial functions.

**Proposition 2.2.** Let $A_1, \ldots, A_k$ be star bodies in the cotangent bundle of an $n$-dimensional compact manifold $M$. The volume of $\lambda_1 A_1 + \cdots + \lambda_k A_k$ is an $n$-th-degree polynomial in the $\lambda_i$,

$$V(\lambda_1 A_1 + \cdots + \lambda_k A_k) = \sum \lambda_{i_1} \cdots \lambda_{i_n} \tilde{V}(A_{i_1}, \ldots, A_{i_n}),$$

where the sum is taken over all positive integers less than $k$. 


Proof.

\[ V(\lambda_1 A_1 + \cdots + \lambda_k A_k) = \int_{\partial U} (\lambda_1 \rho_1 + \cdots + \lambda_k \rho_k)^n \Omega \]

\[ = \int_{\partial U} \left( \sum \lambda_i \cdots \lambda_n \rho_{i_1} \cdots \rho_{i_n} \right) \Omega \]

\[ = \sum \lambda_i \cdots \lambda_n \tilde{V}(A_{i_1}, \ldots, A_{i_n}). \]

\[ \square \]

Some of the basic properties of dual mixed volumes are:

- Continuity with respect to the radial Hausdorff topology.
- Positivity: \( \tilde{V}(A_1, \ldots, A_n) > 0. \)
- Homogeneity: \( \tilde{V}(\lambda_1 A_1, \ldots, \lambda_n A_n) = \lambda_1 \cdots \lambda_n \tilde{V}(A_1, \ldots, A_n), \lambda_i > 0. \)
- Strict monotonicity: if \( A_i \subset B_i \) for all \( i \), then

\[ \tilde{V}(A_1, \ldots, A_n) \leq \tilde{V}(B_1, \ldots, B_n). \]

Equality holds if and only if \( A_i = B_i \) for all \( i \).
- \( \tilde{V}(A, A, A) = V(A). \)

Some useful abbreviations are

\[ \tilde{V}_k(A, B) := \tilde{V}(A, \underbrace{A, \ldots, A}_{n-k}, B, \ldots, B) \quad \text{and} \quad \tilde{W}_k(A) := \tilde{V}_k(A, U)/V(U). \]

In particular, notice that \( \tilde{W}_k(A) \) is just the average of the \( (n-k) \)-th power of the radial function of \( A \):

\[ \tilde{W}_k(A) = \frac{1}{V(U)} \int_{\partial U} \rho_i^{n-k} \Omega. \]

One of the most basic results about dual mixed volumes is the following inequality:

**Theorem 2.1** (Main inequality). If \( A_1, \ldots, A_n \) are star bodies in the cotangent bundle of an \( n \)-dimensional compact manifold,

\[ \tilde{V}(A_1, \ldots, A_n)^n \leq V(A_1) \cdots V(A_n). \]

Moreover, the equality if and only if all the star bodies are dilations of each other.

**Proof.** In terms of the radial functions \( \rho_1, \ldots, \rho_n \) of the star bodies \( A_1, \ldots, A_n \), we must show that

\[ \left( \int_{\partial U} \rho_1 \cdots \rho_n \Omega \right)^n \leq \int_{\partial U} \rho_1^n \Omega \cdots \int_{\partial U} \rho_n^n \Omega. \]

To do this set \( \nu_i := \rho_i^n, 1 \leq i \leq n, \) and consider the quantity

\[ \frac{n}{\left( \int \nu_1 \cdots \nu_n \Omega \right)^{1/n}} \left( \int \frac{\nu_1}{\Omega} \right)^{1/n} \cdots \left( \int \frac{\nu_n}{\Omega} \right)^{1/n}, \]

where we have suppressed the region of integration, \( \partial U \), to simplify the notation. By the arithmetic-geometric mean inequality, this quantity is less than

\[ \int \frac{1}{n} \left( \frac{\nu_1}{\int \nu_1 \Omega} + \cdots + \frac{\nu_n}{\int \nu_n \Omega} \right) \Omega = 1. \]
Equality holds if and only if all radial functions are multiples of each other. The result follows. □

**Corollary 2.1** (Dual Minkowski inequalities). If $A$ and $B$ are star bodies in the cotangent bundle of an $n$-dimensional compact manifold,

$$\tilde{V}_1(A, B)^n \leq V(A)^{n-1}V(B) \quad \text{and} \quad \tilde{V}_{n-1}(A, B)^n \leq V(A)V(B)^{n-1}.$$

**Theorem 2.2** (Dual Brunn-Minkowski inequality). If $A$ and $B$ are star bodies in the cotangent bundle of an $n$-dimensional compact manifold,

$$V(A \dot{+} B)^{1/n} \leq V(A)^{1/n} + V(B)^{1/n}.$$  

Equality holds if and only if the star bodies are dilations of each other.

**Proof.** Let $A_1, A_2,$ and $B$ be star bodies. Using the additivity of dual mixed volumes and the first of the dual Minkowski inequalities, we have that

$$\tilde{V}_1(B, A_1 \dot{+} A_2) = \tilde{V}_1(B, A_1) + \tilde{V}_1(B, A_2) \leq V(B)^{(n-1)/n} \left( V(A_1)^{1/n} + V(A_2)^{1/n} \right).$$

In the particular case when $B = A_1 \dot{+} A_2$, we also have that left hand side of the previous inequality, $\tilde{V}_1(B, A_1 \dot{+} A_2)$, is equal to $V(A_1 \dot{+} A_2)$ and the dual Brunn-Minkowski inequality follows immediately. □

3. Invariance of dual mixed volumes

A basic remark in the classical theory is that dual mixed volumes are invariant under the special linear group. This is because special linear transformations preserve both radial sums and volumes. However, something that seems to have escaped notice until now is that the symmetry group is actually much larger: any volume-preserving transformation of $\mathbb{R}^n$ that is positively homogeneous of degree one preserves dual mixed volumes. In this section we extend this remark to our generalized setting.

**Definition 3.1.** A diffeomorphism $\phi : T^*M \setminus 0 \to T^*M \setminus 0$ is said to be a special homogeneous transformation if it is positively homogeneous of degree one (i.e., $\phi(tp) = t\phi(p)$, $t > 0$) and preserves the form $\Omega$. Equivalently, a special homogeneous transformation is a volume-preserving transformation that is positively homogeneous of degree one.

**Theorem 3.1.** Let $M$ be a compact manifold and let $\phi : T^*M \setminus 0 \to T^*M \setminus 0$ be a special homogeneous transformation. If $A$ and $B$ are star bodies in the cotangent bundle of $M$, then

$$V(A \dot{+} B) = V(\phi(A) \dot{+} \phi(B)).$$

The proof of this theorem depends on the following trivial lemma:

**Lemma 3.1.** Let $\phi : T^*M \setminus 0 \to T^*M \setminus 0$ be a diffeomorphism that is positively homogeneous of order one. If $\rho_A$ is the radial function of a star body $A$ with respect to the model body $U$, then $\rho_A \circ \phi^{-1}$ is the radial function of $\phi(A)$ with respect to $\phi(U)$. 
Proof of theorem 3.1. Using the lemma and the formula for the volume of the radial sum $\phi(A) + \phi(B)$ in terms of its radial function with respect to $\phi(U)$, we have that

$$V(\phi(A) + \phi(B)) = \int_{\phi(\partial U)} (\rho_A \circ \phi^{-1} + \rho_B \circ \phi^{-1})^n \Omega.$$ 

Since $\phi^{-1*}\Omega = \Omega$, we may write this integral as

$$\int_{\phi(\partial U)} \phi^{-1*}[(\rho_A + \rho_B)^n\Omega] = \int_{\partial U} (\rho_A + \rho_B)^n \Omega = V(A + B).$$

□

Theorem 3.2. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be star bodies in the cotangent bundle of a compact $n$-dimensional manifold $M$. If there exists a special homogeneous transformation $\phi: T^*M \setminus 0 \to T^*M \setminus 0$ such that $\phi(A_i) \subset B_i$, for all $i$, then

$$\tilde{V}(A_1, \ldots, A_n) \leq \tilde{V}(B_1, \ldots, B_n).$$

Proof. Because of the basic monotonicity property of dual mixed volumes, we only need to show that

$$\tilde{V}(\phi(A_1), \ldots, \phi(A_n)) = \tilde{V}(A_1, \ldots, A_n).$$

To do this, notice that for any positive numbers $\lambda_1, \ldots, \lambda_n,$

$$\sum \lambda_{i_1} \cdots \lambda_{i_n} \tilde{V}(A_{i_1}, \ldots, A_{i_n}) = V(\lambda_1 A_1 + \cdots + \lambda_n A_n) = V(\phi(\lambda_1 A_1 + \cdots + \lambda_n A_n)) = V(\lambda_1 \phi(A_1) + \cdots + \lambda_n \phi(A_n)) = \sum \lambda_{i_1} \cdots \lambda_{i_n} \tilde{V}(\phi(A_{i_1}), \ldots, \phi(A_{i_n})).$$

This immediately implies the invariance of dual mixed volumes under special homogeneous transformations.

Let us remark that $\tilde{V}(A_1, \ldots, A_n)$ is a non trivial invariant of $A_1, \ldots, A_n$ in the sense that it is not a function of their symplectic volumes. In fact, assume that the $A_i$’s are not dilations of each other and that all have unit volume. If $B$ is another star body with unit volume, the main inequality tells us that

$$\tilde{V}(A_1, \ldots, A_n) < 1 = \tilde{V}(B, \ldots, B),$$

and so the dual mixed volume cannot be a function of the symplectic volume.

An important class of special homogeneous transformations are diffeomorphisms from $T^*M \setminus 0$ to itself that preserve the canonical form $\alpha$. These transformations are known as homogeneous canonical transformations.

While the results in this section show that dual mixed volumes are invariant under homogeneous canonical transformations, their invariance under the larger special group of homogeneous transformations makes them uninteresting as symplectic invariants.
4. Finsler metrics and optical Hamiltonians

In this section, we will quickly review the basic concepts in Finsler geometry that will be used throughout the rest of the paper.

Roughly speaking, a Finsler metric on a manifold $M$ is a smooth choice of norm on each tangent space of $M$. However, in this paper we shall sometimes work with a slightly more general geometric structure where the norm of a vector $v$ is not necessarily equal to the norm of $-v$. Moreover, the unit sphere of this “non-symmetric” norm on each tangent space $T_x M$ must be quadratically convex: its principal curvatures are positive for any auxiliary Euclidean structure on $T_x M$.

Definition 4.1. A Finsler metric on a manifold $M$ is a function $L : TM \to [0, \infty)$ that is

- positive and smooth outside the zero section;
- positively homogeneous of degree one;
- for each $x \in M$ the unit tangent sphere in $T_x M$ is quadratically convex.

The metric $L$ is said to be reversible if for any tangent vector $v$, we have that $L(v) = L(-v)$.

Among the examples of reversible Finsler metrics we find Riemannian metrics, submanifolds of Minkowski spaces (i.e., normed spaces whose unit spheres are quadratically convex), and the Hilbert geometries. Some examples of non-reversible Finsler metrics are the Katok examples (see [17] and [33]) of Finsler spheres with only two closed geodesics, Bryant’s examples of Finsler metrics on the 2-sphere with constant curvature ([7, 8]), and the image of any Riemannian metric by a small homogeneous canonical transformation.

In many cases, we will prefer to work with the duals of Finsler metrics:

Definition 4.2. An optical Hamiltonian on (the cotangent bundle of) a manifold $M$ is a function $H : T^* M \to [0, \infty)$ that is

- positive and smooth outside the zero section;
- positively homogeneous of degree one;
- for every $x \in M$ the unit cotangent sphere in $T^*_x M$ is quadratically convex.

The Hamiltonian $H$ is said to be reversible if for any covector $p$, we have that $H(p) = H(-p)$.

The duality between Finsler metrics and optical Hamiltonians is given by the Legendre transform. This geometric transformation is best explained on a single vector space:

Let $V$ be a finite-dimensional real vector space and let $S \subset V$ be a quadratically convex hypersurface enclosing the origin. If $v$ is a point in $S$, there is a unique covector $\xi \in V^*$ such that the hyperplane $\xi = 1$ is tangent to $S$ at $v$ and the half-space $\xi \leq 1$ contains $S$. The map that sends $v$ to $\xi$ is called the Legendre transform and will be denoted by

$$\mathcal{L} : S \to V^* .$$

The image of $S$ under $\mathcal{L}$, the dual of $S$, is denoted by $S^*$ and is also a quadratically convex hypersurface that encloses the origin. If we now use $S^* \subset V^*$ to define the Legendre transform

$$\mathcal{L}^* : S^* \to V ,$$
it is easy to see that \((S^*)^* = S\) and that
\[
\mathcal{L}(\mathcal{L}^*(\xi)) = \xi \quad \text{and} \quad \mathcal{L}^*(\mathcal{L}(v)) = v .
\]

If \(L\) is a Finsler metric on a manifold \(M\), we can perform the above construction on each tangent space \(T_xM\), \(x \in M\), and define a diffeomorphism between \(TM \setminus 0\) and \(T^*M \setminus 0\) which we shall still call the Legendre transform and denote by \(\mathcal{L}\). With this notation, the function \(H := L \circ \mathcal{L}^{-1}\) is an optical Hamiltonian. Conversely, if \(H\) is an optical Hamiltonian and \(\mathcal{L}^*\) is its Legendre transform, \(L := H \circ \mathcal{L}^{-1}\) is a Finsler metric. Using the standard terminology from classical mechanics, we shall say that \(L\) is the Lagrangian of \(H\), and \(H\) is the Hamiltonian of \(L\).

Given a Finsler metric \(L\) on a manifold \(M\), we define the length of a smooth curve \(\gamma : [a, b] \to M\) by
\[
\text{length}(\gamma) := \int_a^b L(\dot{\gamma}(t)) dt .
\]
When \(L\) is reversible, this defines a length structure and a metric on \(M\). In general the length of a curve is only invariant under reparameterizations that preserve the orientation.

The \textit{geodesics} of a Finsler manifold \((M, L)\) are those curves that satisfy the Euler-Lagrange equations for the Lagrangian \(L\). In the reversible case geodesics locally minimize length, and in the non reversible case they locally minimize oriented length.

If \(H\) is an optical Hamiltonian, we denote the sublevel set \(H \leq 1\) by \(D^*(M, H)\) and its boundary, the level surface \(H = 1\), by \(S^*(M, H)\). On this surface we use the canonical 1-form \(\alpha\) to define the \textit{Reeb vector field} \(X_H\) by the equations
\[
d\alpha(X_H, \cdot) = 0 \quad \text{and} \quad \alpha(X_H) = 1 .
\]

The integral curves of the Reeb vector field are usually called the \textit{characteristics} of \(S^*(M, H)\). A basic fact that we will use in the next two sections is that if \(H\) is an optical Hamiltonian and \(\gamma\) is a characteristic, then the projection of \(\gamma\) to \(M\) is a geodesic for the associated Finsler metric, \(L\). Moreover, the image of \(\gamma\) under the Legendre transform of \(H\) is the velocity curve of this geodesic. Conversely if \(c\) is a geodesic, then the image of \(\dot{c}\) under the Legendre transform of \(L\) is a characteristic of \(S^*(M, H)\). The length of \(c\) can be computed “upstairs” as the \textit{action} of \(\gamma\):
\[
\text{action}(\gamma) := \int_\gamma \alpha .
\]
Notice that if a homogeneous canonical transformation \(\phi : T^*M \setminus 0 \to T^*M \setminus 0\) preserves \(S^*(M, H)\), then it sends characteristics to characteristics and preserves their action.

One of the advantages of the Hamiltonian viewpoint in Finsler geometry, is that it suggests a natural definition of volume:

**Definition 4.3.** Let \(L\) be a Finsler metric on an \(n\)-dimensional manifold \(M\) and let \(H\) be its Hamiltonian. The \textit{Holmes-Thompson volume} of \((M, L)\) is defined as the symplectic volume of the set
\[
D^*(M, H) := \{ p \in T^*M : H(p) \leq 1 \} .
\]
divided by the volume of the Euclidean unit ball of dimension $n$. The $k$-area of a $k$-dimensional submanifold of $M$ is defined as the Holmes-Thompson volume of the submanifold with its induced Finsler metric.

This definition, with its connections to symplectic geometry, convex geometry ([15, 28]), the Fourier transform ([29, 3]), and integral geometry ([27, 2, 26]), has marked advantages over the Hausdorff measure. Nevertheless, there is a simple relationship between these two important notions of volume:

**Theorem 4.1** (Durán, [11]). Let $(M, L)$ be a Finsler manifold with finite Hausdorff measure. The Holmes-Thompson volume of $(M, L)$ is less than or equal to its Hausdorff measure. Equality holds if and only if the metric is Riemannian.

Note that an optical Hamiltonian on a compact manifold is a star Hamiltonian, and that $D^*(M, H)$ is a star body. This allows us to use the theory of dual mixed volumes to define relative invariants of Finsler metrics:

**Definition 4.4.** Let $L_1$ and $L_2$ be two Finsler metrics on a compact manifold $M$ and let $H_1$ and $H_2$ be their respective Hamiltonians. We define the $k$-th dual mixed volume of $L_1$ and $L_2$ as the quantity

$$\tilde{V}_k(M; L_1, L_2) := \tilde{V}_k(D^*(M, H_1), D^*(M, H_2)).$$

Whenever we consider the first metric as a model metric against which other metrics are to be compared (for example, an invariant metric on a homogeneous space) we shall denote it by $L_0$ and define

$$\tilde{W}_k(M, L) := \frac{1}{V(D^*(M, H_0))} \tilde{V}_k(M; L, L_0).$$

Note that the main inequality and the definition of the Holmes-Thompson volume easily imply the following result:

**Proposition 4.1.** If $L$ and $L_0$ are Finsler metrics on an $n$-dimensional compact manifold $M$, then

$$\tilde{W}_k(M, L)^n \leq \frac{\text{vol}(M, L)^n}{\text{vol}(M, L_0)^n}.$$

5. **Finslerian extensions of Pu’s theorem**

In this section, we apply the theory of dual mixed volumes to extend the works of Loewner and Pu on isosystolic inequalities to the Finsler setting.

The $k$-th systole of a Finsler manifold $(M, L)$, denoted by $\text{sys}_k(M, L)$, is defined the infimum of the volumes of all $k$-dimensional submanifolds not homologous to zero. When $k = 1$, it is usual to change the definition to the infimum of the lengths of all non-contractible curves on $M$.

For star Hamiltonians on cotangent bundles of manifolds that are not simply connected, we define the 1-systole as the infimum of the actions of all closed characteristics of $S^*(M, H)$ that project to non-contractible curves on $M$. A theorem of Cieliebak (Theorem 2 in Part II of [9]) guarantees the existence of such characteristics for smooth star Hamiltonians on a multiply-connected compact manifold.

As a rule, sharp isosystolic inequalities are only known for metrics in certain conformal classes. The methods in the next two sections allow us to go somewhat
further, but in the present section all the results will be connected with notions of conformality. The first of these notions is a straight-forward generalization of conformality for Riemannian metrics.

**Definition 5.1.** Two Finsler metrics $L_1$ and $L_2$ on a manifold $M$ are said to be conformal if there exists a diffeomorphism $\varphi : M \to M$ and a smooth positive function $\rho$ on $M$ such that $\rho L_2 = \varphi^* L_1$.

Since 1-systoles and volumes are invariant under homogeneous canonical transformations that are isotopic to the identity, any sharp isosystolic inequality we prove for a class of Finsler metrics will hold for all star Hamiltonians obtained from the Hamiltonians of these metrics by composition with some homogeneous canonical transformation that is isotopic to the identity. A somewhat more subtle notion of conformality that involves homogeneous canonical transformations is as follows:

**Definition 5.2.** Two star Hamiltonians $H_1$ and $H_2$ on the cotangent of a manifold $M$ are said to be s-conformal if there exists a homogeneous canonical transformation $\phi : T^* M \setminus 0 \to T^* M \setminus 0$ and a smooth positive function $\rho$ on $M$ such that $\rho H_2 = H_1 \circ \phi$.

Note the slight abuse of notation in identifying the function $\rho$ with its pull-back to $T^* M$.

The definition of s-conformal Hamiltonians is not useful in Riemannian geometry since even small homogeneous canonical transformations do not generally send Riemannian metrics to Riemannian metrics. However, in the Finsler and Hamiltonian setting, this notion allows us to recognize that hidden symmetries can play a role in the proof of sharp isosystolic inequalities.

The two main results of this section are:

**Theorem 5.1.** Let $M$ be a compact homogeneous space and let $L_0$ be an invariant Finsler metric on $M$. If $L$ is a Finsler metric conformal to $L_0$, then

$$\frac{\text{sys}^n_k(M, L)}{\text{vol}^n(M, L)} \leq \frac{\text{sys}^n_k(M, L_0)}{\text{vol}^n(M, L_0)}.$$  

Moreover, equality holds if and only if $L$ is isometric to a multiple of $L_0$.

**Theorem 5.2.** Let $L_0$ be the standard Riemannian metric of curvature one on $\mathbb{R}P^n$ and let $H_0$ be its Hamiltonian. If $L$ is a Finsler metric on $\mathbb{R}P^n$ whose Hamiltonian $H$ is s-conformal to $H_0$, then

$$\frac{\text{sys}^n_k(\mathbb{R}P^n, L)}{\text{vol}(\mathbb{R}P^n, L)} \leq \frac{\text{sys}^n_k(\mathbb{R}P^n, L_0)}{\text{vol}(\mathbb{R}P^n, L_0)}.$$  

Moreover, equality holds if and only if $H = H_0 \circ \phi$ for some homogeneous canonical transformation $\phi$.

The idea of the proof theorem 5.1 is very simple: by proposition 4.1, we know that

$$\tilde{W}_{n-k}(M, L)^n \leq \frac{\text{vol}(M, L)^k}{\text{vol}(M, L_0)^k}.$$  

Since $k$-systoles and volumes are invariant under isometries, we may assume that $L = \rho L_0$ with $\rho$ a smooth positive function on $M$. The proof reduces to proving
the inequality
\[ W_{n-k}(M, \rho L_0) \geq \frac{\text{sys}_k(M, \rho L_0)}{\text{sys}_k(M, L_0)}. \]

This hinges on interpreting \( W_{n-k}(M, L_0) \) as different averages of the radial function. For this we shall need two trivial lemmas:

**Lemma 5.1.** Let \( L_0 \) be a Finsler metric on an \( n \)-dimensional manifold \( M \). If \( \rho \) is a smooth positive function on \( M \), the quantity \( W_{n-k}(M, \rho L_0) \) is the average of the \( k \)-th power of \( \rho \) over the manifold \( M \):
\[
W_{n-k}(M, \rho L_0) = \frac{1}{\text{vol}(M, L_0)} \int_M \rho^k \, dV^0,
\]
where \( dV^0 \) is the density for the Holmes-Thompson volume on \((M, L_0)\).

**Lemma 5.2.** Let \( G \) be a compact Lie group and let \( \mu \) be the Haar measure on \( G \) normalized so that the measure of \( G \) equals one. If \( Q \) is a compact manifold on which \( G \) acts transitively and \( \nu \) is an invariant measure on \( Q \), then for any function \( f \in L^1(Q, \nu) \), we have that
\[
\int_G f(g \cdot x) \, d\mu = \frac{1}{\nu(Q)} \int_Q f(y) \, d\nu.
\]
In other words, the average of the pullback of \( f \) to \( G \) equals the average of \( f \) on \( Q \).

Putting both lemmas together, we have the following proposition:

**Proposition 5.1.** Let \( M \) be homogeneous space under the left-action of a compact Lie group \( G \) and let \( \mu \) be the Haar measure on \( G \) normalized so that the measure of \( G \) equals one. If \( L_0 \) is an invariant Finsler metric on \( M \) and \( \rho \) is a smooth positive function, then
\[
W_{n-k}(M, \rho L_0) = \int_G \rho^k (g \cdot x) \, d\mu
\]
for any point \( x \in M \).

**Proof.** Applying lemma 5.2 with \( Q := M \), \( f := \rho^k \), and \( \nu := dV^0 \), the volume density of the Holmes-Thompson volume of \((M, L_0)\), we obtain that
\[
\int_G \rho^k (g \cdot x) \, d\mu = \frac{1}{\text{vol}(M, L_0)} \int_M \rho^k \, dV^0.
\]
By lemma 5.1 this quantity equals \( W_{n-k}(M, \rho L_0) \). \( \square \)

**Proposition 5.2.** Let \( M \) be a compact homogeneous space and let the model metric \( L_0 \) be an invariant Finsler metric on \( M \). If \( \rho \) is a smooth positive function on \( M \), then
\[
W_{n-k}(M, \rho L_0) \geq \frac{\text{sys}_k(M, \rho L_0)}{\text{sys}_k(M, L_0)}.
\]

**Proof.** By proposition 5.1 we have that
\[
W_{n-k}(M, \rho L_0) = \int_G \rho^k (g \cdot x) \, d\mu
\]
for any point \( x \in M \).
If \( N_i \subset M \) is a sequence of \( k \)-dimensional submanifolds not homologous to zero whose \( k \)-dimensional volumes decrease to sys\(_k\)(\( M, \rho L_0 \)),

\[
\bar{W}_{n-k}(M, \rho L_0) \text{ vol}_k(N_i, L_0) = \int_N \left( \int_G \rho^k \, d\mu \right) dV_k^0
= \int_G \left( \int_{g(N_i)} \rho^k \, dV_k^0 \right) d\mu.
\]

Since \( \rho^k \, dV_k^0 \) is the \( k \)-area density of the Finsler metric \( \rho L_0 \) and \( g(N_i) \) is not homologous to zero, the last integral is greater than or equal than the \( k \)-systole of \( (M, \rho L_0) \). Therefore, for all \( i \) we have that

\[
\bar{W}_{n-k}(M, \rho L_0) \text{ vol}_k(N_i, L_0) \geq \text{sys}_k(M, \rho L_0).
\]

Taking the limit as \( i \) tends to infinity yields the desired inequality.

The proof of theorem 5.2 is somewhat more subtle because we need to average with respect to hidden symmetries. Like in the case of theorem 5.1, everything boils down to proving the following result:

**Proposition 5.3.** Let \( L_0 \) be the standard Riemannian metric of curvature one on \( \mathbb{R}P^n \) and let \( H_0 \) be its Hamiltonian. If \( L \) is a Finsler metric on \( \mathbb{R}P^n \) whose Hamiltonian \( H \) is \( s \)-conformal to \( H_0 \), then

\[
\bar{W}_{n-1}(M, L) \geq \frac{\text{sys}_1(M, L)}{\text{sys}_1(M, L_0)}.
\]

**Proof.** Since \( H \) is \( s \)-conformal to \( H_0 \), there exists a smooth positive function \( \rho \) on \( M \) and a homogeneous canonical transformation \( \phi \) such that \( \rho H = H_0 \circ \phi \). Notice the slight abuse notation in denoting \( \rho \) and its pullback to the cotangent bundle by the same symbol.

If we lift the standard action of \( SO(n+1) \) on \( \mathbb{R}P^n \) to \( T^*\mathbb{R}P^n \) and conjugate by \( \phi \) we obtain a left-action of

\[
SO(n+1) \times (T^*\mathbb{R}P^n \setminus 0) \longrightarrow (T^*\mathbb{R}P^n \setminus 0)
\]

by homogeneous canonical transformations that preserve the Hamiltonian \( H_0 \circ \phi =: K_0 \). Notice that the geodesic flow of \( K_0 \) is symplectically conjugate to that of \( H_0 \) and, therefore, all geodesics are closed of length \( \pi \).

By lemma 5.2 we have that

\[
\bar{W}_{n-1}(\mathbb{R}P^n, L) = \int_{SO(n+1)} \rho(g \cdot p) \, d\mu.
\]

Mimicking the proof of proposition 5.2, we let \( \gamma \) be a closed characteristic of \( S^*(\mathbb{R}P^n, K_0) \) and write

\[
\bar{W}_{n-1}(\mathbb{R}P^n, L) \text{ sys}_1(\mathbb{R}P^n, L_0) = \int_\gamma \left( \int_{SO(n+1)} \rho \, d\mu \right) \alpha
= \int_{SO(n+1)} \left( \int_{g(\gamma)} \rho \alpha \right) \, d\mu.
\]

We would like to argue that the integral of \( \rho \alpha \) along \( g(\gamma) \) is the length of some non-contractible curve in the metric \( L \), and conclude that it must be greater than \( \text{sys}_1(M, L) \). In order to do this we must show that the curve \( g(\gamma) \) is the image under
the Legendre transform of $L$ of the velocity curve of some non-contractible curve in $\mathbb{R}P^n$. Since, as a rule, homogeneous canonical transformations mix momentum and position, this is not entirely obvious.

We start by noticing that since $g$ preserves both $K_0$ and the canonical 1-form $\alpha$, $g(\gamma)$ is also a characteristic of $S^*(\mathbb{R}P^n, K_0)$, and, hence, its image under the Legendre transform of $K_0$ is the velocity curve of some curve in $c$ in $\mathbb{R}P^n$. Since $SO(n+1)$ is connected, the curve $c$ cannot be contractible. Now, we use that $\rho H = K_0$ and that $\rho$ is constant on the fibers to conclude that the Legendre transform of $H$ sends the curve $g(\gamma)$ to the velocity curve of a suitable reparameterization of $c$.

We conclude that

$$
\tilde{W}_{n-1}(\mathbb{R}P^n, L) \sys_1(\mathbb{R}P^n, L_0) = \int_{SO(n+1)} \left( \int g(\gamma) \rho^k \alpha \right) d\mu
\geq \sys_1(\mathbb{R}P^n, L).
$$

In the Riemannian case, isosystolic inequalities on the two-dimensional torus and the projective plane are substantially strengthened by the fact that any Riemannian metric on these surfaces is conformal to a homogeneous metric. This is utterly false in the Finsler case, nevertheless it seems that any reversible Finsler metric on $\mathbb{R}P^2$ is s-conformal to the standard Riemannian metric of curvature one:

**Uniformization conjecture.** If $L$ is a reversible Finsler metric on the projective plane, $\mathbb{R}P^2$, then there exists a smooth positive function of $\mathbb{R}P^2$ such that the Finsler metric $\rho L$ has periodic geodesic flow.

Using the results of this section, the uniformization conjecture would lead to an alternate proof of Ivanov’s generalization of Pu’s theorem.

**Theorem 5.3** (Ivanov, [16]). Any reversible Finsler metric $L$ on the projective plane satisfies the inequality

$$\frac{2}{\pi} \sys_1(\mathbb{R}P^2, L) \leq \vol(\mathbb{R}P^2, L).$$

Equality holds if and only if the geodesic flow of $L$ is periodic.

6. Isosystolic inequalities for commuting Hamiltonians

In this section, we move away from notions of conformality and establish isosystolic inequalities under a different set of hypotheses. Namely, we shall consider star Hamiltonians with periodic flow and star Hamiltonians commuting with them.

**Theorem 6.1.** If $H_0$ is a smooth star Hamiltonian on $T^*\mathbb{R}P^n$ with periodic flow, all simple characteristics on $H_0 = 1$ project to closed non-contractible curves in $\mathbb{R}P^n$ and

$$\frac{\sys_1(\mathbb{R}P^n, H_0)}{\vol(\mathbb{R}P^n, H_0)} = \frac{2\pi^n}{(n+1)\varepsilon_{n+1}},$$

where $\varepsilon_k$ is the volume of the Euclidean unit ball of dimension $k$.

**Proof.** The periodicity of the flow implies that all simple characteristics are homotopic. Since by Theorem 2, Part II, of [14], some closed characteristic on $H_0 = 1$
projects to a non-contractible curve in $\mathbb{R}P^n$, all characteristics project to non-contractible curves.

The proof of equality relies on the results and techniques of Weinstein ([30]) and Yang ([31]). They proved that the volume of a Riemannian metric on the $n$-sphere whose geodesic flow is periodic with period $\ell$ is equal to the volume of the Euclidean $n$-sphere of radius $\ell/2\pi$. Their techniques are symplectic and topological, and carry over without modification to smooth star Hamiltonians on the cotangent bundle of the sphere or real projective space. □

**Theorem 6.2.** If $H_0$ is a smooth star Hamiltonian on $T^*\mathbb{R}P^n$ with periodic flow and $H$ is a star Hamiltonian that is constant along the orbits of $X_{H_0}$, then

$$\frac{\text{sys}_n(\mathbb{R}P^n, H)}{\text{vol}(\mathbb{R}P^n, H)} \leq \frac{\text{sys}_n(\mathbb{R}P^n, H_0)}{\text{vol}(\mathbb{R}P^n, H_0)}.$$ 

Moreover, equality holds if and only if $H$ is a fixed multiple of $H_0$.

Since for any star body $A \subset T^*M$

$$\tilde{W}_{n-1}(A) = \frac{1}{V(A)} \int_{\partial U} \rho \alpha \geq \min \rho_\alpha,$$

the proof of the theorem above reduces to that of the following inequality:

**Proposition 6.1.** Let $H$ and $H_0$ be as in theorem 6.2. If $\rho$ denotes the radial function of the star body $H \leq 1$, then

$$\min \rho \geq \frac{\text{sys}_1(\mathbb{R}P^n, H)}{\text{sys}_1(\mathbb{R}P^n, H_0)}.$$

The gist of the proof is to show that if $\lambda := \min \rho$ is attained at a point $p_0 \in S^*(M, H_0)$, then the integral curve of $X_H$ on $S^*(M, H)$ with initial condition $p := \rho(p_0)p_0$ is closed, projects to a non-contractible curve, and its action equals $\lambda \text{sys}_1(\mathbb{R}P^n, H_0)$.

**Lemma 6.1.** Consider the map $T: S^*(\mathbb{R}P^n, H_0) \to S^*(\mathbb{R}P^n, H)$ defined by $p \mapsto (\rho(p)p)$. If $p$ is a critical point of $\rho$, then $T_*(X_{H_0}(p))$ is a positive multiple of $X_H(\rho(p)p)$

**Proof.** Since

$$T^*(d\alpha|T_*(X_{H_0})) = (d\rho \wedge \alpha)|X_{H_0},$$

we have that when $d\rho(p) = 0$, $d\alpha|T_*(X_{H_0})$ must be zero. The kernel of $d\alpha$ on $S^*(\mathbb{R}P^n, H)$ is one dimensional and, hence, $T_*(X_{H_0}(p))$ is a multiple of $X_H(\rho(p)p)$. To see that it is a positive multiple we remark that

$$\alpha(T_*(X_{H_0})) = (T^*\alpha)(X_{H_0}) = p\alpha(X_{H_0}) = \rho.$$

which is always positive. □

**Proof of proposition 6.1.** We now proceed to build a closed characteristic on $S^*(\mathbb{R}P^n, H_0)$ that is closed, projects to a non-contractible curve in $\mathbb{R}P^n$, and such that its action equals $\lambda \text{sys}(\mathbb{R}P^n, H_0)$.

Let $p_0$ be a point in $S^*(\mathbb{R}P^n, H_0)$ where the minimum of $\rho$ is attained, and let $\sigma(t)$ be a closed characteristic with $\sigma(0) = p_0$. Notice that since $\rho$ is constant along characteristics, $\rho(\sigma(t))$ is constant and all the points $\sigma(t)$ are critical points of $\rho.$
This, together with the fact that $T^\ast(X_{H_0}(p))$ is a positive multiple of $X_H(p(p)p)$ whenever $p$ is a critical point, implies that

$$\gamma(t) := T(\sigma(t)) = \lambda\sigma(t)$$

is a characteristic of $S^\ast(\mathbb{R}P^n, H)$. Moreover, $\gamma$ projects down to the same curve as $\sigma$. By Theorem 6.1, this implies that $\gamma$ projects to a non-contractible curve on $\mathbb{R}P^n$.

It is easy to compute the action of $\gamma$:

$$\int_\gamma \alpha = \int_{T(\sigma)} \alpha = \int_\sigma T^\ast\alpha = \int_\sigma \lambda\alpha = \lambda\sys(\mathbb{R}P^n, H_0).$$

$$\square$$

Identifying $SO(3)$ with $\mathbb{R}P^3$ and recalling (see [1]) that left-invariant Hamiltonians poisson-commute with the Hamiltonian of the bi-invariant metric on $SO(3)$, we have the following corollary:

**Corollary 6.1.** If $L$ a Finsler metric on $SO(3)$ that is conformal to a left-invariant Finsler metric, then

$$\frac{\sys(\mathbb{R}P^n, L)^3}{\vol(\mathbb{R}P^n, L)} \leq \pi.$$ 

Equality holds if and only if $L$ is bi-invariant.

## 7. Hamiltonian averaging and isosystolic inequalities

The main result of this section is a Hamiltonian generalization of Berger’s infinitesimal isosystolic inequality for Riemannian metrics on real projective spaces ([6]).

**Theorem 7.1.** If $H_t$ is a smooth path of smooth star Hamiltonians on $T^\ast\mathbb{R}P^n$ such that the flow of $H_0$ is periodic, then there exists another smooth path of smooth star Hamiltonians, $K_t$, that agrees to first order with $H_t$ at $t = 0$ and satisfies the isosystolic inequality

$$\frac{\sys^n(\mathbb{R}P^n, K_t)}{\vol(\mathbb{R}P^n, K_t)} \leq \frac{\sys^n(\mathbb{R}P^n, H_0)}{\vol(\mathbb{R}P^n, H_0)}.$$ 

For the proof we will need the following mild adaptation of a classical (and easy) result in the theory of normal forms of Hamiltonian systems (see lemma 3.3 in [10]):

**Lemma 7.1.** Let $H_0$ be a star Hamiltonian on $T^\ast\mathbb{R}P^n$ whose flow is periodic. Any Hamiltonian $H$ on $T^\ast\mathbb{R}P^n \setminus 0$ that is homogeneous of degree one can be written in a unique way as $E + \{H_0, F\}$, where $E$ and $F$ are also homogeneous Hamiltonians of degree one and $\{H_0, E\} = 0$.

**Proof of theorem 7.1.** By theorem 6.2, we know that all star Hamiltonians of the form $H_0 + tE$ with $\{H_0, E\} = 0$ satisfy the isosystolic inequality. Moreover, If $F$ is a Hamiltonian that is homogeneous of degree one, and $\phi_t$ is its flow, then for sufficiently small $t$, the function $H_0 + tE \circ \phi_t =: K_t$ is a star Hamiltonian and satisfies the isosystolic inequality.
Writing \( H_t = H_0 + tH_1 + O(t^2) \), we see that it is possible to find \( E \) and \( F \) such that \( K_t \) and \( H_t \) agree up to order one at \( t = 0 \). Indeed

\[
\frac{dK_t}{dt}(0) = E + \{H_0, F\},
\]

and, according to lemma \( \square \) we may always find \( E \) and \( F \) such that \( E + \{H_0, F\} = H_1 \).

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