Analytical Results for Abelian Projection

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Analytic methods for Abelian projection are developed, and a number of results related to string tension measurements are obtained. It is proven that even without gauge fixing, Abelian projection yields string tensions of the underlying non-Abelian theory. Strong arguments are given for similar results in the case where gauge fixing is employed. The subgroup used for projection need only contain the center of the gauge group, and need not be Abelian. While gauge fixing is shown to be in principle unnecessary for the success of Abelian projection, it is computationally advantageous for the same reasons that improved operators, e.g., the use of fat links, are advantageous in Wilson loop measurements.

1. Introduction

Although Abelian projection has a compelling theoretical basis \cite{1,2}, and has notable successes \cite{3} in lattice simulations, there are several fundamental questions which remain unresolved. \cite{4} The most basic issue is the possibility that the successes of abelian projection are not consequences of our insight into the non-abelian dynamics, but simply reflect general field-theoretic principles. A closely related, more practical issue is the choice of the correct or best subgroup to use for projection. As discussed below, gauge invariance is the key physical principle responsible for the success of Abelian projection; furthermore, any subgroup, not necessarily Abelian, will work as long as it contains the center of the gauge group. Details are given in reference \cite{5}.

2. Abelian Projection in Practice

The standard approach to Abelian projection is a three step process. The gauge fields $g_l$ are associated with links of the lattice, and take on values in a compact Lie group $G$. An ensemble of lattice gauge field configurations is generated using standard Monte Carlo methods. Each field configuration in the $G$-ensemble is placed in a particular gauge by site-based gauge transformations $\phi$. The gauge-fixing condition is chosen to preserve gauge invariance for some subgroup $H$ of $G$. A typical gauge-fixing procedure for Abelian projection is to maximize

$$S_{gf} = \lambda \sum_l Tr \left[ g_l M g_l^+ M \right]$$

where $M$ is a traceless, Hermitian matrix that commutes with every element of the subgroup $H$ and the sum is taken over all links. From this ensemble of gauge-fixed field configurations, another ensemble of gauge fields is generated, with the fields taking on values in the subgroup $H$. This is obtained by maximizing

$$S_{proj} [g, h] = \sum_l \left[ \frac{\rho}{2N} Tr \left( g_l^+ h_l + h_l^+ g_l \right) \right]$$

where $h_l \in H$.

For analytical purposes, it is necessary to generalize the projection procedure, so that a single configuration of $g$-fields will be associated with an ensemble of configurations of both $h$-fields and $\phi$-fields. The original fields $g_\mu(x)$ are replaced in the gauge-fixing function and the projection function by $\tilde{g}_\mu(x) = \phi(x) g_\mu(x) \phi^+(x + \hat{\mu})$. For each $g$-field configuration, an ensemble of $\phi$-fields will be generated, weighted by $S_{gf}$. For each $\tilde{g}$-field configuration, an ensemble of $h$-fields will be generated, weighted by $S_{proj}$. The parameters $\lambda$ and $\rho$ control the width of the two distributions, and the original scheme is formally regained in the limit

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\[ \lambda, p \to \infty. \] The distribution of the ensemble of \( g \)-fields is independent of \( \lambda \) and \( p \). The field \( \phi \) is quenched relative to \( g \), and \( h \) is quenched relative to both \( \phi \) and \( g \). This quenching is necessary to preserve the normal lattice definition of gauge-invariant quantities, but introduces complications reminiscent of spin glasses into expressions for expectation values.

### 3. Projection without Gauge Fixing

The simplest and most analytically tractable case is projection without gauge fixing, which is achieved by setting \( \lambda = 0 \). In this case, the integration over the \( \phi \) fields takes any configuration over its entire gauge orbit, thus serving to enforce gauge invariance.

Consider the expectation value of a Wilson loop \( W \) with no self-intersections in a representation \( \beta \) of \( H \), denoted by \( \langle \bar{\chi}^\beta(W) \rangle = \langle \bar{\chi}^\beta(h_1...h_n) \rangle \) where \( \bar{\chi}^\beta \) stands for the appropriate group character. In calculating the expectation value, the weight function for projecting each link can be expanded in the characters of the group \( G \),

\[
\exp \left[ \frac{p}{2N} Tr \left( g^+ h + h^g \right) \right] = \sum_\alpha d_\alpha \chi_\alpha (h^+ g) . \tag{3}
\]

The principal difficulty in the evaluation of the expected value lies in the treatment of factors resulting from the quenching of \( \phi \). To lowest order in the character expansion, systematic application of gauge invariance at all sites along the curve \( W \) collapses the sum into the simple result

\[
\langle \bar{\chi}^\beta(h_1...h_n) \rangle = \sum_\alpha \frac{(\chi_\alpha(p)/c_\alpha(p))^n}{\alpha} \int_H (dh) \bar{\chi}^\beta(h) \chi^\alpha(h^+) \cdot \langle \chi^\alpha(g_1...g_n) \rangle \tag{4}
\]

where the sum is over representations \( \alpha \) of \( G \). This formula has obvious physics content: the Wilson loop as measured in the \( \beta \) representation of \( H \) is given as a sum of Wilson loops in the irreducible representations of \( G \), each weighted by the number of times \( \beta \) appears in \( \alpha \) and by a \( p \)-dependent factor which contributes to the perimeter dependence. This result can be turned into rigorous upper and lower bounds on \( \langle \bar{\chi}^\beta(W) \rangle \) with changes only in the perimeter dependence on the right-hand side of the formula. This leads immediately to area law behavior for \( H \), independent of \( p \):

\[
\hat{\sigma}_\beta = \min_\alpha \sigma_\alpha \tag{5}
\]

where the minimum is taken over all representations \( \alpha \) that have a non-zero contribution. Center symmetry plays an important role here, causing many potential terms to vanish. In the case of \( SU(N) \) projected to \( Z(N) \), a very direct alternative proof has been given recently by Ambjorn and Greensite.

Consider, as an example, the case of \( SU(2) \) projected to \( U(1) \). The string tension is non-zero for the half-integer representations \( j = 1/2, 3/2 \ldots \) due to the \( Z_2 \) center symmetry, but not for the integer representations. A typical result is

\[
\hat{\sigma}_1/2 = \min_{j=1/2, 3/2\ldots} \sigma_j \tag{6}
\]

but \( \hat{\sigma}_1 = 0 \) because of string-breaking in the adjoint representation: \( \sigma_1 = 0 \).

### 4. Projection with Gauge Fixing

Projection combined with gauge fixing (\( \lambda \neq 0 \) is less tractable than the \( \lambda = 0 \) case. However, a strong-coupling expansion in \( \lambda \), which is convergent for sufficiently small \( \lambda \), indicates similar results. The lowest order result can be obtained by expanding the \( \lambda = 0 \) result in \( p \). The form of the corrections is determined by gauge invariance. To order \( \lambda^3 \), the corrections to the area- and perimeter-dependence are determined by

\[
\langle \chi (h_1...h_n) \rangle \approx p^n K_1 \langle \chi (g_1...g_n) \rangle + p^n \lambda^3 K_2 \cdot \sum \langle \chi (g_1 g_2 g_B g_C ... g_n) \rangle \chi^\alpha(g_2 g_B g_C ... g_n) \chi^\beta(h_1...h_n) \tag{7}
\]

where the summation sign indicates that the decoration of \( g_2 \) by the staple \( g_B g_C \) is to be repeated through the entire loop for all directions orthogonal to the loop. The constants \( K_1 \) and \( K_2 \) are power series in \( p \) and \( \lambda \), beginning at order 1. These corrections are shown graphically in
Figure 1. From this expression, we can see that the string tension should still satisfy

$$\tilde{\sigma}_\beta = \min_{\alpha} \sigma_\alpha$$  \hspace{1cm} (8)$$

as in the case of no gauge fixing. The parameters $p$ and $\lambda$ change the perimeter dependence of the Wilson loop expectation value, but not the area dependence. Gauge fixing appears here in a manner quite similar to the use of fat links, and it is likely that fat links would be numerically advantageous when projection is performed without gauge fixing.

5. Conclusions

The success of Abelian projection appears to have its origin in very general considerations. The key principle is local gauge invariance. Ultimately, it is Elitzur's theorem that ensures that observables constructed from the projected field can always be rewritten in terms of gauge-invariant observables of the underlying gauge fields. Abelian dominance is not necessary. Note that at no point in the arguments given above has space-time dimensionality been a consideration, a further indication that Abelian projection does not depend on some particular set of important field configurations.

There is one possible weak point in the gauge-fixed case: the standard gauge fixing algorithm corresponds formally to the limit $\lambda \to \infty$, but the strong-coupling expansion in $\lambda$ has a finite radius of convergence. These arguments will fail for large $\lambda$ if there is, as seems likely, a phase transition along some critical line $\lambda_c(\beta)$, a function of the gauge coupling $\beta$. However, even if there is a phase transition in the gauge-variant sector, it may be that the strong-coupling region and weak-coupling region are in fact connected because the critical line has an end-point. In any event, it remains conceptually difficult to claim that confinement should be understood differently for large $\lambda$ and small $\lambda$, because by construction, the underlying ensemble of non-Abelian gauge fields does not depend on $\lambda$.

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