Vector-valued \( q \)-variational inequalities for averaging operators and the Hilbert transform

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Abstract. Recently, the authors have established \( L^p \)-boundedness of vector-valued \( q \)-variational inequalities for averaging operators which take values in the Banach space satisfying the martingale cotype \( q \) property in Hong and Ma (Math Z 286(1–2):89–120, 2017). In this paper, we prove that the martingale cotype \( q \) property is also necessary for the vector-valued \( q \)-variational inequalities, which was a question left open in the previous paper. Moreover, we also prove that the UMD property and the martingale cotype \( q \) property can be characterized in terms of vector valued \( q \)-variational inequalities for the Hilbert transform.

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1. Introduction. It is well known that in the setting of vector-valued harmonic analysis, many results are related to the properties of the geometry of Banach spaces. In particular, the so-called UMD, and convexity properties of Banach spaces have become the most fundamental tools in the study of vector-valued harmonic analysis. The \( L^p \)-boundedness of vector-valued singular integral operators is an important problem in this area of research. Burkholder [4] first proved that whenever a Banach space \( X \) has the UMD property, the Hilbert transform is bounded on \( L^p(\mathbb{R}^d; X) \) for any \( p \in (1, \infty) \). Later, Bourgain [2] pointed out that the UMD property is indeed necessary for the boundedness of the Hilbert transform. Hytönen [12] proved that the two-sided Littlewood–Paley–Stein \( g \)-function estimate is equivalent to the UMD property of a Banach space \( X \). Moreover, the martingale cotype \( q \) plays an important role in vector-valued analysis. The martingale cotype \( q \) was introduced by Pisier [25] to characterize the properties of uniformly convex Banach spaces.
In [27], Xu provided a characterization of the martingale cotype $q$ property via the vector-valued Littlewood–Paley theory associated to the Poisson kernel on the unit circle. Later on, Martínez et al. [24] further characterized this property through general Littlewood–Paley–Stein theory. We refer the reader to [15] for an extensive study on vector-valued harmonic analysis.

On the other hand, variational inequalities have received a lot of attention in the past decades for many reasons, one of which is that we can measure the speed of convergence for the family of operators under consideration. In 1976, Lépingle [21] applied the regularity of Brownian motion to obtain the first scalar-valued variational inequality for martingales. Ten years later, motivated by the development of Banach space geometry, Pisier and Xu [26] provided another proof of Lépingle’s result based on the stopping time argument. Almost at the same time Bourgain [3] used Lépingle’s results to establish variational inequalities for the ergodic averages of a dynamical system. Bourgain’s work and Pisier-Xu’s method have inspired a new research direction in ergodic theory and harmonic analysis. In [16,17], Jones et al. proved a variational inequalities for the ergodic averages. Campbell et al. in [5] established the variational inequalities for the Hilbert transform and averaging operators, and truncated singular integrals with the homogeneous type kernel [6]. Recently, Ding et al. [8] obtained the variational inequalities for truncated singular integrals with rough type kernels. We recommend [7,9,10,13,14,18–20,22,23] and references therein for related work.

Among these works, some weighted norm and Banach lattice-valued variational inequalities in harmonic analysis have also been established. It is worthwhile to mention that the first and third author [11] obtained the vector-valued variational inequalities for the ergodic averages and averaging operators whenever the underlying Banach spaces are of martingale cotype $q$, while Hytönen et al. [14] proved a variational inequality for the vector-valued Walsh-Fourier series when the underlying Banach spaces belong to a subclass of UMD spaces satisfying tile-type $q$. In these two papers, the finite cotype (or type) is also shown to be necessary for the variational inequalities to hold. However, regarding the sharp necessary conditions for the variational inequalities, it is still unknown to the best of the authors’ knowledge.

To further demonstrate our motivation of the present paper, let us start with some notations. Let $X$ be a Banach space and $a_t(x) : (0, \infty) \times \mathbb{R}^d \to X$ a vector-valued function. For $q \in [1, \infty)$, the seminorm $V_q$ is defined by

$$V_q(a_t(x) : t \in Z) = \sup_{0 < t_0 < \cdots < t_J} \left( \sum_{j=0}^{J} \left\| a_{t_{j+1}}(x) - a_{t_j}(x) \right\|^q \right)^{1/q},$$

where $Z$ is a subset of $(0, \infty)$ and the supremum runs over all finite increasing sequences in $Z$. Let $\{\Sigma_n\}_{n \geq 0}$ be an increasing sequence of $\sigma$-algebras on a probability space $(\Omega, \Sigma, \mathbb{P})$. The associated conditional expectations are denoted by $\{E_n\}_{n \geq 0}$. We say that the Banach space $X$ is of martingale cotype $q$ with $2 \leq q < \infty$ if there exists an absolute constant $C_q$ such that for all $f \in L^q(\Omega; X)$,
\[
\sum_{n \geq 0} \| \mathbb{E}_n f - \mathbb{E}_{n-1} f \|_{L^q(\Omega; X)}^q \leq C_q \| f \|_{L^q(\Omega; X)}^q,
\]
with the convention \( \mathbb{E}_{-1} f = 0 \).

In [26], Pisier and Xu proved the following vector-valued variational inequalities for martingales.

**Theorem A.** Let \( X \) be a Banach space.

(i) If \( X \) is of martingale cotype \( q_0 \) with \( 2 \leq q_0 < \infty \), then for every \( 1 < p < \infty \) and \( q > q_0 \), there exists a positive constant \( C_{p,q} \) such that

\[
\| V_q(\mathbb{E}_k f : k \in \mathbb{N}) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \; \forall f \in L^p(\Omega; X). \tag{1.1}
\]

(ii) Suppose inequality (1.1) holds for some \( p \in (1, \infty) \) and \( q \in [2, \infty) \). Then \( X \) is of martingale cotype \( q \).

Note that the above conclusion (ii) is quite easy in the martingale setting. Indeed, using a standard extrapolation argument, we can assume that inequality (1.1) holds for \( p = q \). Then the conclusion follows trivially from the definitions. The situation changes dramatically in harmonic analysis.

Let \( f : \mathbb{R} \to X \) be locally integrable. Let \( B_t \) be an open ball centered at the origin and its radius equal to \( t \) \((t > 0)\). Then the central averaging operators are defined as follows

\[
A_t f(x) = \frac{1}{|B_t|} \int_{B_t} f(x - y)dy = \frac{1}{|B_t|} \int_{\mathbb{R}^d} f(y) \mathbf{1}_{B_t}(x - y)dy, \; x \in \mathbb{R},
\]
where \( |\cdot| \) denotes the Lebesgue measure.

As previously mentioned, the authors obtained in [11, Theorem 2.1] the \( L^p \)-boundedness of vector-valued variational inequalities for averaging operators when the underlying Banach space satisfies the martingale cotype property. However, the reverse characterization that is similar to (ii) of Theorem A was not obtained in [11]. In this paper, we prove the following theorem which includes this reverse characterization.

**Theorem 1.1.** Let \( X \) be a Banach space.

(i) If \( X \) is of martingale cotype \( q_0 \) with \( 2 \leq q_0 < \infty \), then for any \( 1 < p < \infty \) and \( q_0 < q < \infty \), there exists a constant \( C_{p,q} \) such that

\[
\| V_q(A_t f : t > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \; \forall f \in L^p(\mathbb{R}; X). \tag{1.2}
\]

(ii) Let \( 2 \leq q < \infty \), and for some \( p \in (1, \infty) \), there exists a positive constant \( C_{p,q} \) such that

\[
\| V_q(A_t f : t > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \; \forall f \in L^p(\mathbb{R}; X). \tag{1.2}
\]

Then \( X \) is of martingale cotype \( q \).

Conclusion (i) has been obtained in [11]. To show conclusion (ii), we use the transference techniques several times and essentially follow Xu’s argument in his remarkable paper [27] based on Bourgain’s transference method. In fact, many modifications in the proof of statement (ii) are surely necessary in the present setting, see Section 2 for details.
Before stating our second result, we introduce some more notations. Recall that a Banach space $X$ is said to have the UMD property if for all scalars $r_n = \pm 1$, $n = 1, \ldots, N$, there exists a positive constant $C_{p,X}$ independent of $N$ such that for all $1 < p < \infty$ and all $X$-valued $L^p$-martingale differences \( \{df_n\}_{n \geq 1} \),
\[
\left\| \sum_{n=1}^{N} r_n df_n \right\|_{L^p(\Omega;X)} \leq C_{p,X} \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(\Omega;X)}.
\]

Let $S(\mathbb{R})$ be the set of Schwartz functions on $\mathbb{R}$. Define $f \in S(\mathbb{R}) \otimes X \subseteq L^p(\mathbb{R};X)$ by setting $f(x) = \sum_{k=1}^{n} \varphi_k(x)b_k$, where $\varphi_1, \ldots, \varphi_n \in S(\mathbb{R})$ and $b_1, \ldots, b_n \in X$. For $f \in S(\mathbb{R}) \otimes X$, the truncated Hilbert transform is defined by
\[
H_\epsilon f(x) = \frac{1}{\pi} \int_{|y| > \epsilon} \frac{f(x-y)}{y} dy.
\]

The second result of this paper is on vector-valued variational inequalities for the Hilbert transform.

**Theorem 1.2.** Let $X$ be a Banach space.

(i) Let $2 \leq q_0 < \infty$, and $X$ be of martingale cotype $q_0$ satisfying the UMD property. Then for any $1 < p < \infty$ and $q_0 < q < \infty$, there exists a constant $C_{p,q}$ such that
\[
\|V_q(H_\epsilon f : \epsilon > 0)\|_{L^p} \leq C_{p,q} \|f\|_{L^p}, \forall f \in L^p(\mathbb{R};X).
\]

(ii) Let $2 \leq q < \infty$. If for some $p \in (1, \infty)$, there exists a positive constant $C_{p,q}$ such that
\[
\|V_q(H_\epsilon f : \epsilon > 0)\|_{L^p} \leq C_{p,q} \|f\|_{L^p}, \forall f \in L^p(\mathbb{R};X),
\]
then $X$ satisfies the UMD property and is of martingale cotype $q$.

With Theorem 1.1 at hand, the proof of Theorem 1.2 is relatively easy but some of the arguments seem new. We use again the transference techniques several times for the proof of both (i) and (ii), see Section 3 for details.

Throughout this paper, by $C$ we always denote a positive constant that may vary from line to line.

2. The proof of Theorem 1.1. In this section, we closely follow Xu’s argument [27] to prove Theorem 1.1. The main ingredient is Bourgain’s transference method. We begin by introducing some notations.

Let $P_t(\theta) = \sum_n e^{-t|n|}e^{2\pi in\theta}$ be the Poisson kernel of the unit disc. For $f \in L^p([0,1];X)$, denote the Poisson integral
\[
P_t f(\xi) = P_t * f(\xi) = \int_0^1 f(e^{2\pi i\theta}) \left( \sum_n e^{-t|n|}e^{2\pi in(\xi-\theta)} \right) d\theta.
\]
Let \( m \) be a positive integer and \( 1 \leq k \leq m \). Let \( a_k(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_{k-1}}) \) be a trigonometric polynomial defined on \([0, 1]^{k-1}\) with value in the Banach space \( X \),
\[
a_k(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_{k-1}}) = \sum_{|m_1| \leq N_1} \cdots \sum_{|m_{k-1}| \leq N_{k-1}} x_{m_1, \ldots, m_{k-1}} e^{2\pi i m_1 \theta_1} \cdots e^{2\pi i m_{k-1} \theta_{k-1}},
\]
with the convention \( a_k = x_0 \) when \( k = 1 \). Let \( b_k \) be a trigonometric polynomial with complex coefficients such that \( \hat{b}(0) = 0 \),
\[
b_k(e^{2\pi i \theta_k}) = \sum_{1 \leq |j| \leq M_k} y_j e^{2\pi i j \theta_k}.
\]
Let \( \{n_k\}_{1 \leq k \leq m} \) be a sequence of positive integers, and
\[
a_{k,(n)}(\theta) = a_k(e^{2\pi i (\theta_1+n_1)} \cdots e^{2\pi i (\theta_{k-1}+n_{k-1})}) = b_k(e^{2\pi i (\theta_k+n_k)}),
\]
\[
f(\theta) = \sum_{k=1}^{m} a_{k,(n)}(\theta) b_{k,(n)}(\theta) = \sum_{k=1}^{m} f_{k,(n)}(\theta).
\]
Note that in the following, we fix \( \theta_1, \ldots, \theta_m \). Then all these functions are considered as trigonometric polynomials in \( e^{i\theta} \). The next lemma is a discrete version of Xu’s [27, Lemma 3.5].

**Lemma 2.1.** Let \( a_{k,(n)}, b_{k,(n)}, \) and \( f_{k,(n)} \) be defined as above. For any \( \epsilon > 0 \), we can choose an increasing sequence \( \{n_k\}_{1 \leq k \leq m} \) of positive integers and a decreasing sequence \( \{l_k\}_{0 \leq k \leq m} \subset [0, +\infty] \) such that for all \( \theta \in [0, 1] \),
\[
\|\mathbb{P}_t f_{k,(n)}(\theta)\| < \frac{\epsilon}{2^k}, \quad \forall \ t \geq l_k - 1, \tag{2.1}
\]
and for any \( 0 \leq t \leq l_k \),
\[
\sum_{j=1}^{k} \|\mathbb{P}_{l_k} f_{j,(n)}(\theta) - \mathbb{P}_t f_{j,(n)}(\theta)\| < \epsilon. \tag{2.2}
\]

**Proof.** First we set \( l_0 = +\infty \) and \( n_1 = 1 \). Suppose we have already defined \( +\infty = l_0 > l_1 > \cdots > l_{k-1} \) and \( 1 = n_1 < \cdots < n_k \) with the required properties. Now, we define \( l_k \) and \( n_{k+1} \) as the following.

Since \( f \) is a trigonometric polynomial, \( \mathbb{P}_t * f \) uniformly converges to \( f \) as \( t \to 0^+ \) on \( \theta \in [0, 1] \). Hence we can choose an \( l_k \) sufficiently close to 0 such that (2.2) holds for any \( 0 < t < l_k \). Next, we choose \( n_{k+1} \) such that (2.1) holds at the step \( k+1 \).

According to the definition of \( f_{k,(n)} \), we know that there is a finite integer \( N \) such that \( |m_1 n_1 + \cdots + m_k n_k| < N \) for all \( |m_1| \leq N_1, \ldots, |m_k| \leq N_k \). On the other hand, since \( \hat{b}_{k+1}(0) = 0 \), if \( |n_{k+1}| \) is sufficiently big (precisely, \( n_{k+1} \in \mathbb{Z} \setminus [-N, N] \)), then \( |m_1 n_1 + \cdots + m_k n_k + j n_{k+1}| \) is positive for all \( |m_1| < N_1, \ldots, |m_k| \leq N_k \) and \( 1 \leq |j| \leq M \). Hence by the definition of the Poisson integral, we can choose \( n_{k+1} \) such that (2.1) holds at step \( k+1 \).

Finally, let \( l_m = 0 \), and then we have set up the required sequences \( \{n_k\}_{1 \leq k \leq m} \) and \( \{l_k\}_{0 \leq k \leq m} \).

The following lemma is cited from [11].
Lemma 2.2. Let $X$ be a Banach space. If inequality (1.2) holds for some $p$, then for every $1 < p < \infty$, we have
\[
\|V_q(\mathbb{P}_t f : t > 0)\|_{L^p} \leq C_{p,q} \|f\|_{L^p}, \quad \forall f \in L^p([0,1];X). \tag{2.3}
\]

By the method of the proof of [27, Theorem 3.1], we now prove Theorem 1.1.

Proof of Theorem 1.1. As mentioned earlier, the first and third author have proved part (i) in [11, Theorem 2.1]. Hence, it remains to prove part (ii).

Let $\Omega = \{-1,1\}^\mathbb{N}$ be the dyadic group equipped with its normalized Haar measure $P$, and $m$ be a positive integer. According to [25], we know that to prove that the Banach space $X$ is of martingale cotype $q$, it suffices to prove that there exists a constant $C$ such that for all finite Walsh-Paley martingales $M = \{M_k\}_{1 \leq k \leq m}$, $1 \leq k \leq m$, set
\[
\sum_{k=1}^{m} \mathbb{E}\|M_k - M_{k-1}\|^q \leq C_q \sup_k \mathbb{E}\|M_k\|^q, \tag{2.4}
\]
where $M_{-1} = 0$.

Fix such a martingale $M = \{M_k\}_{1 \leq k \leq m}$. Let $\{\varepsilon_k\}_{1 \leq k \leq m}$ be the coordinate sequence of $\Omega$. By the representation of Walsh-Paley martingales (cf. [15]), we know that there exist functions $\phi_k : \{-1,1\}^{k-1} \rightarrow X$ such that
\[
M_k - M_{k-1} = \phi_k(\varepsilon_1, \ldots, \varepsilon_{k-1})\varepsilon_k, 1 \leq k \leq m,
\]
where $\varepsilon_k \in \{-1,1\}^k$ and $\phi_1$ is a constant belonging to $X$.

Following [1], we use the transference method from $\Omega$ to the group $[0,1]^\mathbb{N}$. For $1 \leq k \leq m$, set
\[
a_k(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_{k-1}}) = \phi_k(\text{sgn}(\cos 2\pi \theta_1), \ldots, \text{sgn}(\cos 2\pi \theta_{k-1})),
\]
\[
b_k(e^{2\pi i \theta_k}) = \text{sgn}(\cos 2\pi \theta_k).
\]

Since $a_k(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_{k-1}}) \in L^q([0,1]^{k-1};X)$ and $b_k(e^{2\pi i \theta_k}) \in L^1([0,1])$, by Cesàro summability, we know that for any $\varepsilon > 0$, there are two trigonometric polynomials $a'_k$ and $b'_k$ with values respectively in $X$ and $\mathbb{C}$ such that $\|a'_k - a_k\|_q < \varepsilon$ and $\|b'_k - b_k\|_q < \varepsilon$. Note that $\int_0^1 \text{sgn}(\cos 2\pi \theta)d\theta = 0$. Then $b'_k(0) = 0$. By this consideration, we assume that $a_k$ and $b_k$ are trigonometric polynomials with values respectively in $X$ and $\mathbb{C}$ and $b'_k(0) = 0$ such that $\|f_{k,(n)} - (M_k - M_{k-1})\|_q < \varepsilon$.

In the following proof, we keep all notations introduced in Lemma 2.1. By Lemma 2.1, there exist sequences $\{n_i\}_{1 \leq i \leq m}$ and $\{l_i\}_{0 \leq i \leq m}$ such that (2.1) and (2.2) hold.

Firstly, for $i = 1$, we have
\[
\left\| \sum_{k=1}^{m} (\mathbb{P}_{l_1} f_{k,(n)} - \mathbb{P}_{l_0} f_{k,(n)}) - f_{1,(n)} \right\| \leq \left\| \mathbb{P}_{l_1} f_{1,(n)} - f_{1,(n)} \right\| + \left\| \sum_{k=2}^{m} \mathbb{P}_{l_1} f_{k,(n)} \right\|.
\]
Combining (2.2) with (2.1), we obtain the following estimate
\[
\left\| \mathbb{P}_{l_1} f_{1,(n)} - f_{1,(n)} \right\| < \varepsilon, \quad \left\| \sum_{k=2}^{m} \mathbb{P}_{l_1} f_{k,(n)} \right\| \leq \sum_{k=2}^{m} \frac{\varepsilon}{2^k} < \varepsilon.
\]
Hence for $i = 1$, we get
\[ \| \sum_{k=1}^{m} (\mathbb{P}_{l_1} f_{k,(n)} - \mathbb{P}_{l_0} f_{k,(n)}) - f_{i,(n)} \| \leq \epsilon. \]

For $1 < i \leq m$, we first have the following inequality
\[ \| \sum_{k=1}^{m} (\mathbb{P}_{l_i} f_{k,(n)} - \mathbb{P}_{l_{i-1}} f_{k,(n)}) - f_{i,(n)} \|
\leq \| \sum_{k=1}^{i-1} (\mathbb{P}_{l_i} f_{k,(n)} - \mathbb{P}_{l_{i-1}} f_{k,(n)}) \| + \| \sum_{k=i}^{m} (\mathbb{P}_{l_i} f_{k,(n)} - \mathbb{P}_{l_{i-1}} f_{k,(n)}) - f_{i,(n)} \|
= I + II. \]

By (2.2), the first term $I$ is controlled by
\[ I < \sum_{k=1}^{i-1} \| \mathbb{P}_{l_i} f_{k,(n)} - \mathbb{P}_{l_{i-1}} f_{k,(n)} \| < \epsilon. \]

To handle part $II$, one can observe the following inequality
\[ II \leq \| \sum_{k=i+1}^{m} \mathbb{P}_{l_i} f_{k,(n)} \| + \| \mathbb{P}_{l_i} f_{i,(n)} - f_{i,(n)} \| + \| \sum_{k=i}^{m} \mathbb{P}_{l_{i-1}} f_{k,(n)} \|
= III + IV + V. \]

For $III$ and $V$, by (2.1), we obtain
\[ III \leq \sum_{k=i}^{m} \| \mathbb{P}_{l_i} f_{k,(n)} \| \leq \sum_{k=i+1}^{m} \frac{\epsilon}{2^k} < \epsilon, \text{ and } V \leq \sum_{k=i}^{m} \| \mathbb{P}_{l_{i-1}} f_{k,(n)} \| \leq \sum_{k=i}^{m} \frac{\epsilon}{2^k} < \epsilon. \]

For $IV$, using (2.2) again, we have
\[ IV = \| \mathbb{P}_{l_i} f_{i,(n)} - f_{i,(n)} \| < \epsilon. \]

Therefore, we have $\| \sum_{k=1}^{m} (\mathbb{P}_{l_i} f_{k,(n)} - \mathbb{P}_{l_{i-1}} f_{k,(n)}) - f_{i,(n)} \| \leq 3\epsilon$. By this estimate, we see that
\[
\int_{[0,1]^m} \int_{[0,1]} \| f_{i,(n)}(\theta) \|^q d\theta d\theta_1 \cdots d\theta_m
\leq C_q \int_{[0,1]^m} \int_{[0,1]} \left\| \sum_{k=1}^{m} (\mathbb{P}_{l_i} f_{k,(n)}(\theta) - \mathbb{P}_{l_{i-1}} f_{k,(n)}(\theta)) \right\|^q d\theta d\theta_1 \cdots d\theta_m
+ C_q \int_{[0,1]^m} \int_{[0,1]} \left\| \sum_{k=1}^{m} (\mathbb{P}_{l_i} f_{k,(n)}(\theta) - \mathbb{P}_{l_{i-1}} f_{k,(n)}(\theta)) - f_{i,(n)}(\theta) \right\|^q d\theta d\theta_1 \cdots d\theta_m
\leq C_q \int_{[0,1]^m} \int_{[0,1]} \left\| \sum_{k=1}^{m} (\mathbb{P}_{l_i} f_{k,(n)}(\theta) - \mathbb{P}_{l_{i-1}} f_{k,(n)}(\theta)) \right\|^q d\theta d\theta_1 \cdots d\theta_m + C_q \epsilon^q.\]
Hence we have
\[
\sum_{i=1}^{m} \int_{[0,1]^m} \int_{[0,1]} \left\| f_{i,(n)}(\theta) \right\|^q d\theta d\theta_1 \cdots d\theta_m
\]
\[
\leq C_q \sum_{i=1}^{m} \int_{[0,1]^m} \int_{[0,1]} \left\| \sum_{k=1}^{m} (P_{i,1} f_{k,(n)}(\theta) - P_{i-1,1} f_{k,(n)}(\theta)) \right\|^q d\theta d\theta_1 \cdots d\theta_m
\]
\[
+ C_q m \epsilon^q.
\]

On the other hand, by our assumption that inequality (1.2) holds, and Lemma 2.2, we have
\[
\int_{[0,1]^m} \int_{[0,1]} \sum_{i=1}^{m} \left\| \sum_{k=1}^{m} (P_{i,1} f_{k,(n)}(\theta) - P_{i-1,1} f_{k,(n)}(\theta)) \right\|^q d\theta d\theta_1 \cdots d\theta_m
\]
\[
\leq C_q \int_{[0,1]^m} \int_{[0,1]} \left\| \sum_{k=1}^{m} f_{k,(n)}(\theta) \right\|^q d\theta d\theta_1 \cdots d\theta_m.
\]

Combining (2.5) with (2.6), we have
\[
\sum_{i=1}^{m} \int_{[0,1]^m} \int_{[0,1]} \left\| f_{i,(n)}(\theta) \right\|^q d\theta d\theta_1 \cdots d\theta_m
\]
\[
\leq C_q \int_{[0,1]^m} \int_{[0,1]} \left\| \sum_{k=1}^{m} f_{k,(n)}(\theta) \right\|^q d\theta d\theta_1 \cdots d\theta_m + C_q m \epsilon^q.
\]

By the Fubini theorem and the change of variables
\[
(\theta_1, \ldots, \theta_m) \mapsto (\theta_1 - n_1 \theta, \ldots, \theta_m - n_m \theta),
\]
we obtain
\[
\sum_{k=1}^{m} \mathbb{E} \| M_k - M_{k-1} \|^q \leq C_q \mathbb{E} \| M_m \|^q + C_q m \epsilon^q.
\]

Taking \( \epsilon \to 0 \), we deduce the desired inequality (2.4). Thus, we proved that \( X \) is of martingale cotype \( q \). This completes the proof of part (ii). \( \square \)

3. The proof of Theorem 1.2. In this section, we prove Theorem 1.2. We begin by introducing some notations.

Let \( P(x) = \frac{1}{\pi} \frac{1}{1+x^2} \) be the Poisson function on \( \mathbb{R} \). For \( \epsilon > 0 \), recall the Poisson kernel \( P_\epsilon(x) = \frac{1}{\pi} P(\epsilon^{-1} x) \) and the conjugate Poisson kernel \( Q_\epsilon(x) = \frac{1}{\pi} \frac{x}{x^2 + \epsilon^2} \). Then we denote the convolutions \( P_\epsilon * f \) and \( Q_\epsilon * f \) by \( P_\epsilon f \) and \( Q_\epsilon f \), respectively.

**Lemma 3.1.** Let \( X \) be a Banach space and \( 2 \leq q < \infty \). If there exist some \( p \in (1, \infty) \) and a positive constant \( C_{p,q} \) such that
\[
\| V_q(H_\epsilon f : \epsilon > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \forall f \in L^p(\mathbb{R}; X),
\]
(3.1)
then $X$ has the UMD property and we have the following inequality
\[ \| V_q(P_{\epsilon} f : \epsilon > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \forall f \in L^p(\mathbb{R}; X). \] (3.2)

**Proof.** Note that variational inequalities imply the pointwise convergence for the family of operators under consideration. Then by condition (3.1), we have
\[ \| H_{\epsilon,1/\epsilon} f(x) \| \to \| H f(x) \|, \text{for a.e. } x \in \mathbb{R}, \text{ as } \epsilon \to 0, \]
where $H_{\epsilon,1/\epsilon} f(x) = \frac{1}{\epsilon} \int_{\epsilon < |y| < 1/\epsilon} \frac{f(x-y)}{y} dy$. Furthermore,
\[ \| H f(x) \|_{L^p(\mathbb{R}; X)} = \left\| \lim_{\epsilon \to 0} \| H_{\epsilon,1/\epsilon} f(x) \| X \right\|_{L^p(\mathbb{R})} \leq \| V_q(H_{\epsilon} f(x) : \epsilon > 0) \|_{L^p(\mathbb{R}; X)} \leq C_{p,q} \| f \|_{L^p(\mathbb{R}; X)}. \]

Hence, by the $L^p$-boundedness of the vector-valued Hilbert transform, we know that the Banach space $X$ has the UMD property (cf. [15, Theorem 5.1.1]).

It remains to prove inequality (3.2). First, by using the formula $Q_{\epsilon} f(x) = \int_{-\infty}^{\infty} H_{\epsilon y} f(x) \frac{2y}{(1+y^2)^2} dy$, we have
\[ V_q(Q_{\epsilon} f(x) : \epsilon > 0) \leq C V_q(H_{\epsilon} f(x) : \epsilon > 0). \]
Then we can obtain inequality (3.2) by the following two estimates,
\[ \| V_q(Q_{\epsilon} f : \epsilon > 0) \|_{L^p} \leq C_{p} \| V_q(H_{\epsilon} f : \epsilon > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \]

and
\[ \| V_q(P_{\epsilon} f : \epsilon > 0) \|_{L^p} = \| V_q(Q_{\epsilon} (H f) : \epsilon > 0) \|_{L^p} \leq C_{p,q} \| H f \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, \]
where in the last inequality we have used that the Banach space $X$ has the UMD property.

**Proof of Theorem 1.2.** For part (i), the proof just follows the scalar case as in [5], the key observation is that the operator $H_{\epsilon} f - Q_{\epsilon} f$ can be divided into four averaging operators. And note that the UMD property is used for proving the $L^p$-boundedness of the operator $V_q(Q_{\epsilon} f : \epsilon > 0)$. We omit the details.

For part (ii), using Lemma 3.1, we obtain that the Banach space $X$ has the UMD property. Moreover, the first and third author [11, Lemma 3.4] have applied the transfer principle of [9] to deduce the inequality $\| V_q(P_{\epsilon} f : \epsilon > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}$ from the inequality (3.2). Indeed, the inequality $\| V_q(P_{\epsilon} f : \epsilon > 0) \|_{L^p} \leq C_{p,q} \| f \|_{L^p}$ implies that the Banach space $X$ is of martingale cotype $q$, see Section 2 for more details. This completes the proof of part (ii) of Theorem 1.2.

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