A CONDENSED PROOF OF THE DIFFERENTIAL
GROTHENDIECK–RIEMANN–ROCH THEOREM

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(Communicated by Varghese Mathai)

Dedicated to my father, Kar-Ming Ho

Abstract. We give a direct proof that the Freed–Lott differential analytic
index is well defined and a condensed proof of the differential Grothendieck–
Riemann–Roch theorem. As a byproduct we also obtain a direct proof that
the \( \mathbb{R}/\mathbb{Z} \) analytic index is well defined and a condensed proof of the \( \mathbb{R}/\mathbb{Z} \)
Grothendieck–Riemann–Roch theorem.

1. Introduction

Differential \( K \)-theory, the differential extension of topological \( K \)-theory, has been
studied intensively in the last decade. Basically, a differential \( K \)-theory class con-
sists of an equivalence class \([E, h, \nabla, \phi]\) of a Hermitian bundle with connection and
a differential form, with the connection and form related nontrivially.

The mathematical motivation for differential \( K \)-theory can be traced to Cheeger–
Simons differential characters \([8]\), the unique differential extension of ordinary co-
homology \([15]\), and to work of Karoubi \([13]\). It is thus natural to look for differential
extensions of generalized cohomology theories such as topological \( K \)-theory. Various
definitions of differential \( K \)-theory have been given, notably by Bunke–Schick
\([5]\), Freed–Lott \([10]\), Hopkins–Singer \([12]\) and Simons–Sullivan \([16]\). By work of \([6]\),
these models of differential \( K \)-theory are all isomorphic. For a detailed survey of
differential \( K \)-theory, see \([7]\).

The Atiyah-Singer family index theorem can be formulated as the equality of
the analytic and topological pushforward maps
\[
\text{ind}^{\text{an}} = \text{ind}^{\text{top}} : K(X) \to K(B).
\]

Applying the Chern character, we get the Grothendieck–Riemann–Roch theorem,
the commutativity of
\[
\begin{array}{ccc}
K(X) & \xrightarrow{\text{ch}} & H^{\text{even}}(X; \mathbb{Q}) \\
\xrightarrow{\text{ind}^{\text{an}}} & & \xrightarrow{f_{X/B}} \text{Todd}(X/B) \cup (-) \\
K(B) & \xrightarrow{\text{ch}} & H^{\text{even}}(B; \mathbb{Q})
\end{array}
\]

Analogous theorems hold in differential \( K \)-theory. Bunke–Schick proved the
differential Grothendieck–Riemann–Roch theorem (dGRR) \([5]\) Theorem 6.19]; i.e.,
for a proper submersion $\pi : X \to B$ of even relative dimension, the following diagram is commutative:

$$
\begin{array}{ccc}
\hat{K}_{BS}(X) & \xrightarrow{\hat{ch}_{BS}} & \hat{H}^{even}(X; \mathbb{R}/\mathbb{Q}) \\
\downarrow \text{ind}_{BS}^{an} & & \downarrow \int_{X/B} \text{Todd}(\nabla^{V}X)^{\ast}(\cdot) \\
\hat{K}_{BS}(B) & \xrightarrow{\hat{ch}_{BS}} & \hat{H}^{even}(B; \mathbb{R}/\mathbb{Q})
\end{array}
$$

Here $\hat{H}(X; \mathbb{R}/\mathbb{Q})$ is the ring of differential characters, $\hat{ch}_{BS}$ is the Bunke–Schick differential Chern character [5, §6.2], $\text{ind}_{BS}^{an}$ is the Bunke–Schick differential analytic index [5, §3] and $\int_{X/B} \text{Todd}(\nabla^{V}X)^{\ast}(\cdot)$ is a modified pushforward of differential characters [5, §6.4]. The notation is explained more fully in later sections. On the other hand, Freed–Lott proved the differential family index theorem [10, Theorem 7.32]

$$
\text{ind}_{FL}^{an} = \text{ind}_{FL}^{top} : \hat{K}_{FL}(X) \to \hat{K}_{FL}(B),
$$

where $\text{ind}_{FL}^{an}$ and $\text{ind}_{FL}^{top}$ are the Freed–Lott differential analytic index [10, Definition 3.11] and the differential topological index [10, Definition 5.33]. Applying the differential Chern character $\hat{ch}_{FL}$ yields the dGRR [10, Corollary 8.23]. Since $\text{ind}_{BS}^{an} = \text{ind}_{FL}^{an}$ [5, Corollary 5.5], the two dGRR theorems are essentially the same.

Both proofs of the dGRR are involved and yield much more information than the dGRR alone. In particular, the fact that $\text{ind}_{FL}^{an}$ is well defined follows a posteriori from the differential family index theorem. The main results of this paper are a direct proof that $\text{ind}_{FL}^{an}$ is well defined and a condensed proof of dGRR. Note that without the direct proof that $\text{ind}_{FL}^{an}$ is well defined we cannot compute $\hat{ch}_{FL}(\text{ind}_{FL}^{an}(\mathcal{E}))$ without using the differential family index theorem.

We first prove these theorems in the special case where the family of kernels of the Dirac operators has constant dimension, i.e., $\text{ker}(D^{E}) \to B$ is a superbundle. The proof of $\text{ind}_{FL}^{an}$ being well defined makes use of the variational formula of the Bismut-Cheeger eta form, and the proof of the dGRR relies on a result of Bismut [3, Theorem 1.15], which allows us to shorten the existing proofs at the expense of using this theorem. The general case follows from a standard perturbation argument as in [10, §7]. It is stated in [3, p. 23] that Bismut’s theorem extends to the general case.

J. Lott proved the equality [14, Corollary 3]

$$
\text{ind}_{L}^{an} = \text{ind}_{L}^{top} : K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z}) \to K_{L}^{-1}(B; \mathbb{R}/\mathbb{Z})
$$

of an analytic and topological index in his geometric model of $K_{L}(X; \mathbb{R}/\mathbb{Z})$. This index theorem and the corresponding GRR theorem are consequences of the Freed-Lott differential family index theorem. Thus we also obtain a direct proof that the analytic index $\text{ind}_{L}^{an}$ is well defined, and a condensed proof of the corresponding GRR theorem. Indeed, Bismut already stated in [3, p. 17] (without proof) that [3, Theorem 1.15] implies this GRR theorem. For proofs of these theorems without using differential $K$-theory, see [11].
The next two sections contain the necessary background material. Section 2 reviews Cheeger–Simons differential characters, their multiplication and some properties of pushforward. Section 3 reviews Freed–Lott differential $K$-theory, the construction of the Freed–Lott differential analytic index and the Freed–Lott differential Chern character. The main results of the paper are proved in Section 4.

2. Cheeger–Simons differential characters

2.1. Definition of differential characters. We recall Cheeger–Simons differential characters [8] with coefficients in $\mathbb{R}/\mathbb{Q}$. Let $X$ be a manifold. The ring of differential characters of degree $k \geq 1$ is

$$\hat{H}^k(X; \mathbb{R}/\mathbb{Q}) = \{ f \in \text{Hom}(Z_{k-1}(X), \mathbb{R}/\mathbb{Q}) | \exists \omega_f \in \Omega^k(X) \text{ such that } f \circ \partial = \omega_f \},$$

where $\cdot : \Omega^k(X) \to C^k(X; \mathbb{R}/\mathbb{Q})$ is an injective homomorphism defined by $\omega(c_k) := \int_{c_k} \omega \mod \mathbb{Q}$. It is easy to show that $\omega_f$ is a closed $k$-form with periods in $\mathbb{Q}$ and is uniquely determined by $f \in \hat{H}^k(X; \mathbb{R}/\mathbb{Q})$. In the following hexagon, the diagonal sequences are exact, and every triangle and square commutes [8, Theorem 1.1]:

$$\begin{array}{ccccccccc}
0 & \to & H^{k-1}(X; \mathbb{R}/\mathbb{Q}) & \xrightarrow{\alpha} & H^k(X; \mathbb{R}/\mathbb{Q}) & \xrightarrow{\partial_1} & 0 \\
& \downarrow{i_1} & \downarrow{r} & \downarrow{\delta_2} & \downarrow{\delta_1} & \downarrow{s} & \downarrow{d} & \downarrow{\Omega^k(X)} & \downarrow{0} \\
H^{k-1}(X; \mathbb{R}) & \xrightarrow{\beta} & \hat{H}^k(X; \mathbb{R}/\mathbb{Q}) & \xrightarrow{r} & H^k(X; \mathbb{R}) & & & & \Omega^k_{\mathbb{Q}}(X) & \xrightarrow{0} \\
& \downarrow{\Omega^{k-1}(X)} & \downarrow{\Omega^k_{\mathbb{Q}}(X)} & \downarrow{\delta_2} & \downarrow{\Omega^k(\mathbb{Q})} & \downarrow{0} & \downarrow{0} & \downarrow{0} & \downarrow{0} \\
0 & \to & \Omega^{k-1}(X) & \xrightarrow{\beta} & \Omega^k_{\mathbb{Q}}(X) & \xrightarrow{r} & 0 & \xrightarrow{d} & \Omega^k(\mathbb{Q}) & \xrightarrow{0} \\
\end{array}$$

The maps are defined as follows: $r$ is induced by $\mathbb{Q} \hookrightarrow \mathbb{R}$,

$$i_1([z]) = z|Z_{k-1}(X), \ i_2(\omega) = \overline{\omega}|Z_{k-1}(X), \ \delta_1(f) = \omega_f \text{ and } \delta_2(f) = [c],$$

where $[c] \in H^k(X; \mathbb{Q})$ is the unique cohomology class satisfying $r[c] = [\omega_f]$, and $\Omega^k_{\mathbb{Q}}(X)$ consists of closed forms with periods in $\mathbb{Q}$. (We will not use the other maps.) The character diagram uniquely characterizes differential extension of ordinary cohomology [15].

Invariant polynomials for $U(n)$ have associated characteristic classes and differential characters. In particular, for a Hermitian vector bundle $E \to X$ with a metric $h$ and a unitary connection $\nabla$, the differential Chern character is the unique natural differential character [8, Theorem 2.2]

$$\hat{\text{ch}}(E, h, \nabla) \in \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$$

such that

$$\delta_1(\hat{\text{ch}}(E, h, \nabla)) = \text{ch}(\nabla) \text{ and } \delta_2(\hat{\text{ch}}(E, h, \nabla)) = \text{ch}(E).$$

We will write $\hat{\text{ch}}(E, h, \nabla)$ as $\hat{\text{ch}}(E, \nabla)$ in the sequel.
2.2. Multiplication of differential characters. In [8] the multiplication of differential characters is defined. Let $E : \Omega^{k_1}(X) \times \Omega^{k_2}(X) \to C^{k_1+k_2-1}(X; \mathbb{R})$ be a natural chain homotopy between the wedge product $\wedge$ and the cup product $\cup$, i.e., for $\omega_1 \in \Omega^{k_1}(X)$, we have
\[ \delta E(\omega_1, \omega_2) + E(d\omega_1, \omega_2) + (-1)^{k_1}E(\omega_1, d\omega_2) = \omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2 \]
as cochains. Note that any two choices of $E$ are naturally chain homotopic. For $f \in \tilde{H}^{k_1}(X; \mathbb{R}/\mathbb{Q})$ and $g \in \tilde{H}^{k_2}(X; \mathbb{R}/\mathbb{Q})$, define $f \ast g \in \tilde{H}^{k_1+k_2}(X; \mathbb{R}/\mathbb{Q})$ by
\[ f \ast g = \left( T_f \cup \omega_g + (-1)^{k_1}\omega_f \cup T_g + T_f \cup \delta T_g + E(\omega_f, \omega_g) \right) \cdot \pi_{k_1+k_2-1}(X), \]
where $T_f, T_g \in C^{k_1-1}(X, \mathbb{R})$ are lifts of $f$ and $g$.

**Proposition 1** ([8] Theorem 1.11). Let $f \in \tilde{H}^{k_1}(X; \mathbb{R}/\mathbb{Q})$ and $g \in \tilde{H}^{k_2}(X; \mathbb{R}/\mathbb{Q})$. Then:

1. $f \ast g$ is independent of the choice of the lifts $T_f$ and $T_g$,
2. $f \ast (g \ast h) = (f \ast g) \ast h$ and $f \ast g = (-1)^{k_1k_2}g \ast f$,
3. $\omega_{f \ast g} = \omega_f \wedge \omega_g$ and $c_{f \ast g} = c_f \cup c_g$, i.e., $\delta_1$ and $\delta_2$ are ring homomorphisms,
4. if $\phi : N \to M$ is a smooth map, then $\phi^*(f \ast g) = \phi^*(f) \ast \phi^*(g)$,
5. if $\theta \in \Omega^*(X)$, then $i_2(\theta) \ast f = i_2(\theta \wedge \omega_f)$,
6. if $[c] \in H^k(X; \mathbb{R}/\mathbb{Q})$, then $f \ast i_1([c]) = (-1)^{k_1}i_1([c_f] \cup [c])$.

2.3. Pushforward of differential characters. The pushforward of differential characters is defined in [12] §3.4. We only consider proper submersions $\pi : X \to B$ with closed fibers of relative dimension $n$, where the definition [10] §8.3 is straightforward: for $k \geq n$,
\[ \int_{X/B} : \tilde{H}^k(X; \mathbb{R}/\mathbb{Q}) \to \tilde{H}^{k-n}(B; \mathbb{R}/\mathbb{Q}), \quad \left( \int_{X/B} f \right)(z) = f(\pi^{-1}(z)). \]

Let $\int_{X/B}$ denote the pushforward of both forms and cohomology classes.

**Proposition 2** ([12] §3.4). Let $f \in \tilde{H}^k(X; \mathbb{R}/\mathbb{Q})$, $[c] \in H^{k-1}(X; \mathbb{R}/\mathbb{Q})$ and $\theta \in \Omega^{k-1}(X)/\Omega^{k-1}(X)$. Then:

1. $\delta_1(\int_{X/B} f) = \int_{X/B} \omega_f$,
2. $\delta_2(\int_{X/B} f) = \int_{X/B} [c_f]$,
3. $\int_{X/B} i_1([c]) = i_1(\int_{X/B} [c])$.
4. $\int_{X/B} i_2(\theta) = i_2(\int_{X/B} \theta)$. 
3. Freed–Lott differential \( K \)-theory

3.1. Definition of Freed–Lott differential \( K \)-theory. In this subsection we review Freed–Lott differential \( K \)-theory \[10\].

The Freed–Lott differential \( K \)-group \( \tilde{K}_{FL}(X) \) is the abelian group generated by quadruples \( E = (E, h, \nabla, \phi) \), where \( (E, h, \nabla) \rightarrow X \) is a complex vector bundle with a hermitian metric \( h \) and a unitary connection \( \nabla \), and \( \phi \in \frac{\Omega^\text{odd}(X)}{\text{Im}(d)} \). The only relation is \( \mathcal{E}_1 = \mathcal{E}_2 \) if and only if there exists a generator \( (F, h^F, \nabla^F, \phi^F) \) of \( \tilde{K}_{FL}(X) \) such that \( E_1 \oplus F \cong E_2 \oplus F \) and \( \phi_1 - \phi_2 = CS(\nabla^E_2 \oplus \nabla^F, \nabla^E_1 \oplus \nabla^F) \).

In the following hexagon, the diagonal sequences are exact, and every triangle and square commutes \[10\]:

\[
\begin{array}{c}
\omega \\
\downarrow \delta \\
\alpha \\
\downarrow i \\
0 \\
\end{array}
\begin{array}{c}
K^{-1}(X;\mathbb{R}/\mathbb{Z}) \\
\downarrow -B \\
K(X;\mathbb{Z}) \\
\downarrow \text{ch}_R \\
0 \\
\end{array}
\begin{array}{c}
H^\text{odd}(X;\mathbb{R}) \\
\downarrow \beta \\
\tilde{K}_{FL}(X) \\
\downarrow j \\
\Omega^\text{odd}(X) \\
\downarrow d \\
\Omega^\text{even}(X;\mathbb{R}) \\
\downarrow \text{dr} \\
0 \\
\end{array}
\begin{array}{c}
\Omega^\text{odd}(X;\mathbb{R}) \\
\downarrow \delta \\
\Omega^\text{even}(X;\mathbb{R}) \\
\downarrow \text{ch}_{\text{FL}} \\
0 \\
\end{array}
\begin{array}{c}
\Omega^\text{even}(X;\mathbb{R}/\mathbb{Q}) \\
\downarrow \text{dr} \\
\Omega^\text{even}(X;\mathbb{R}/\mathbb{Q}) \\
\downarrow \text{dr} \\
0 \\
\end{array}
\]

where \( \text{ch}_R := r \circ \text{ch} : K(X) \rightarrow H^\text{even}(X;\mathbb{R}) \), and

\[
\Omega^\bullet_{\text{BU}}(X) := \{ \omega \in \Omega^\bullet_{d=0}(X) | [\omega] \in \text{Im}(\text{ch}^\bullet : K^\bullet(\mathbb{R}/\mathbb{Z}) \rightarrow H^\bullet(X;\mathbb{Q})) \},
\]

where \( \bullet \in \{ \text{even, odd} \} \). The maps are defined as follows:

- \( \delta(\mathcal{E}) = [E] \), \( \text{ch}_{\tilde{K}_{FL}}(\mathcal{E}) = \text{ch}(\nabla) + d\phi \), \( j(\phi) = (0, 0, d, \phi) \),
- \( i \) is the natural inclusion map,
- \( d \) is the de Rham map, and
- the sequence of maps \( (\alpha, \beta, \text{ch}_R) \) can be regarded as the Bockstein sequence in \( K \)-theory as we may identify \( H^\bullet(X;\mathbb{R}) \) with \( K^\bullet(X;\mathbb{R}) \) via the Chern character.

The Freed–Lott differential Chern character \( \text{ch}_{\text{FL}} : \tilde{K}_{FL}(X) \rightarrow \tilde{H}^\text{even}(X;\mathbb{R}/\mathbb{Q}) \) is defined by

\[
\text{ch}_{\text{FL}}(\mathcal{E}) = \text{ch}(E, \nabla) + i_2(\phi),
\]

where \( \text{ch}(E, \nabla) \) is given in (3) and \( i_2 \) is in (2).

3.2. Freed–Lott differential analytic index. The Freed–Lott differential analytic index of a generator \( \mathcal{E} = (E, h, \nabla^E, \phi) \in \tilde{K}_{FL}(X) \) is roughly given by the geometric construction of the analytic index of \( (E, h, \nabla) \) with a modified pushforward of the form \( \phi \).

In more detail, let \( \pi : X \rightarrow B \) be a proper submersion of even relative dimension \( n \), and let \( T^V X \rightarrow X \) be the vertical tangent bundle, which is assumed to have a metric \( g^{TV} \). A given horizontal distribution \( T^H X \rightarrow X \) and a Riemannian metric \( g^{TB} \) on \( B \) determine a metric on \( TX \rightarrow X \) by \( g^{TX} := g^{TV} \oplus \pi^* g^{TB} \).
If $\nabla^{TX}$ is the corresponding Levi-Civita connection, then $\nabla^{TV} X := P \circ \nabla^{TX} \circ P$ is a connection on $TV X \to X$, where $P : TX \to TV X$ is the orthogonal projection. $TV X \to X$ is assumed to have a spin$^c$ structure. Denote by $SV X \to X$ the spinor$^c$ bundle associated to the characteristic Hermitian line bundle $L^V X \to X$ with a unitary connection $\nabla^{L^V} X$. Define a connection $\widehat{\nabla}^{TV} X$ on $SV X \to X$ by

$$\widehat{\nabla}^{TV} X := \nabla^{TV} X \otimes \nabla^{L^V},$$

where $\nabla^{TV} X$ also denotes the lift of $\nabla^{TV} X$ to the local spinor bundle. The Todd form $\text{Todd}(\widehat{\nabla}^{TV} X)$ of $SV X \to X$ is defined by

$$\text{Todd}(\widehat{\nabla}^{TV} X) := \widehat{A}(\nabla^{TV} X) \wedge e^{\frac{1}{2}c_1(\nabla^{L^V} X)}.$$  

For $k \geq n$, the modified pushforward of forms $\pi_* : \Omega^k(X) \to \Omega^{k-n}(B)$ [10, (3.2)] is defined by

$$\pi_*(\phi) = \int_{X/B} \text{Todd}(\widehat{\nabla}^{TV} X) \wedge \phi.$$  

It induces a map, still denoted by $\pi_* : \Omega^{\text{odd}}(X) / \text{Im}(d) \to \Omega^{\text{odd}}(B) / \text{Im}(d)$.

We briefly recall the definition of the Bismut–Cheeger eta form $\tilde{\eta}(E) \in \Omega^{\text{odd}}(B) / \text{Im}(d)$ associated to $E \in \tilde{K}_{FL}(X)$. With the above setup, consider the infinite-rank superbundle $\pi_* E \to B$, where the fibers at each $b \in B$ is given by

$$(\pi_* E)_b := \Gamma(X_b, (SV X \otimes \mathcal{E})|_{X_b}).$$

Recall that $\pi_* E \to B$ admits an induced Hermitian metric and a connection $\nabla^{\pi_* E}$ compatible with the metric [1] §9.2, Proposition 9.13. For each $b \in B$, the canonically constructed Dirac operator

$$D_b^E : \Gamma(X_b, (SV X \otimes \mathcal{E})|_{X_b}) \to \Gamma(X_b, (SV X \otimes \mathcal{E})|_{X_b})$$

gives a family of Dirac operators, denoted by $D_b^E : \Gamma(X, SV X \otimes \mathcal{E}) \to \Gamma(X, SV X \otimes \mathcal{E})$. Assume the family of kernels $\ker(D_b^E)$ has locally constant dimension; i.e., $\ker(D_b^E) \to B$ is a finite-rank Hermitian superbundle. Let $P : \pi_* E \to \ker(D_b^E)$ be the orthogonal projection, $h^{\ker(D_b^E)}$ be the Hermitian metric on $\ker(D_b^E) \to B$ induced by $P$, and $\nabla^{\ker(D_b^E)} := P \circ \nabla^{\pi_* E} \circ P$ be the connection on $\ker(D_b^E) \to B$ compatible to $h^{\ker(D_b^E)}$.

The (scaled) Bismut-superconnection $A_t^E : \Omega(B, \pi_* E) \to \Omega(B, \pi_* E)$ [2, Definition 3.2] (see also [1] Proposition 10.15 and [9] (1.4)) is defined by

$$A_t^E := \sqrt{t}D^E + \nabla^{\pi_* E} - \frac{c(T)}{4\sqrt{t}},$$

where $c(T)$ is the Clifford multiplication by the curvature 2-form of the fiber bundle. The Bismut–Cheeger eta form $\tilde{\eta}(E)$ [4, (2.26)] (see also [9] and [11] Theorem 10.32) is defined by

$$\tilde{\eta}(E) := \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \text{Str} \left( \frac{dA_t^E}{dt} e^{-A_t^E} \right) dt.$$  

It satisfies

$$d\tilde{\eta}(E) = \int_{X/B} \text{Todd}(\widehat{\nabla}^{TV} X) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^{\ker(D_b^E)}).$$

The Freed–Lott differential analytic index $\text{ind}^{an}_{FL} : \tilde{K}_{FL}(X) \to \tilde{K}_{FL}(B)$ is

$$\text{ind}^{an}_{FL}(E) = (\ker(D^E), h^{\ker(D^E)}, \nabla^{\ker(D^E)}, \pi_*(\phi) + \tilde{\eta}(E)).$$
4. Main results

In this section we give a direct proof that \( \text{ind}^{an}_{\text{FL}} \) is well defined and give a condensed proof of the dGRR. We first recall a theorem of Bismut [3]. In the setup of §3.2, with the fibers spin and \( \ker(D^E) \to B \) assumed to form a superbundle, we have

\[
\hat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\eta(E)) = \int_{X/B} \hat{A}(T^V X, \nabla^{T^V X}) \ast \hat{\text{ch}}(E, \nabla^E) \tag{5}
\]

[3, Theorem 1.15]. If the fibers are only spin\(^c\), \( \eta(E) \) has the obvious modification

\[
\hat{\text{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\eta(E)) = \int_{X/B} \hat{\text{Todd}}(T^V X, \nabla^{T^V X}) \ast \hat{\text{ch}}(E, \nabla^E), \tag{6}
\]

for \( \hat{\text{Todd}}(T^V X, \nabla^{T^V X}) \in \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \) the differential character associated to the Todd form and the Todd class as in [3], and similarly for \( \hat{A}(T^V X, \nabla^{T^V X}) \). We will write \( \hat{\text{Todd}}(T^V X, \nabla^{T^V X}) \) as \( \hat{\text{Todd}}(\nabla^{T^V X}) \) in the sequel. Note that \( \eta(E) \) and \( \eta(F) \) extend to the general case where \( \ker(D^E) \to B \) does not form a bundle [3, p. 23].

4.1. Freed–Lott differential analytic index. In this subsection we prove that \( \text{ind}^{an}_{\text{FL}} \) is well defined.

**Proposition 3.** Let \( \pi : X \to B \) be a proper submersion with closed spin\(^c\) fibers of even relative dimension. If \( \mathcal{E} = \mathcal{F} \in \hat{K}_{\text{FL}}(X) \), then

\[
\text{ind}^{an}_{\text{FL}}(\mathcal{E}) = \text{ind}^{an}_{\text{FL}}(\mathcal{F}).
\]

**Proof.** Let \( f := \text{ind}^{an}_{\text{FL}}(\mathcal{E}) - \text{ind}^{an}_{\text{FL}}(\mathcal{F}) \). Since there exists a generator \( G \) in \( \hat{K}_{\text{FL}}(X) \) such that \( E \oplus G \cong F \oplus G \) and \( \phi^E - \phi^F = \text{CS}(\nabla^F \oplus \nabla^G, \nabla^E \oplus \nabla^G) \) up to an exact form, it follows that \( \ker(D^E) \oplus \ker(D^G) \cong \ker(D^F) \oplus \ker(D^G) \) and therefore \( \delta(f) = [\ker(D^E) - \ker(D^F)] = 0 \) in \( K(B) \). By [4] there exists a unique \( \omega \in \frac{\text{O}^{\text{odd}}(B)}{\text{O}_{\text{BU}}^{\text{odd}}(B)} \) such that \( j(\omega) = f \). Since

\[
\left( \ker(D^E), h^{\ker(D^E)}, \nabla^{\ker(D^E)}, \eta(E) + \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \phi^E \right) = \left( \ker(D^F), h^{\ker(D^F)}, \nabla^{\ker(D^F)}, \eta(F) + \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \phi^F \right) + j(\omega),
\]

it follows that, up to an exact form,

\[
\text{CS}(\nabla^{\ker(D^E)} \oplus \nabla^{\ker(D^F)}), \nabla^{\ker(D^E)} \oplus \nabla^{\ker(D^F)}) = \text{CS}(\nabla^{\ker(D^E)} \oplus \nabla^{\ker(D^F)}), \nabla^{\ker(D^E)} \oplus \nabla^{\ker(D^F)} \oplus d)
\]

\[
= \eta(F) + \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \phi^F + \omega - \eta(E) - \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \phi^E
\]

\[
= \omega + \eta(F) - \eta(E) + \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \text{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G),
\]

is well defined and give a con-
and hence
\[\omega = \text{CS}(\nabla_{\ker(D^E)} \oplus \nabla_{\ker(D^G)}, \nabla_{\ker(D^F)} \oplus \nabla_{\ker(D^G)}) + \tilde{\eta}(\mathcal{E}) - \tilde{\eta}(\mathcal{F}) + \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \wedge \text{CS}(\nabla^F \oplus \nabla^G, \nabla^E \oplus \nabla^G)\]
in \(\frac{\Omega^\text{odd}(B)}{\Omega^\text{BU}(B)}\). We prove that \(\omega \in \Omega^\text{odd(}B\text{)}.\) Since the variational formula of the Bismut-Cheeger eta form is given by
\[\tilde{\eta}(\mathcal{F}) - \tilde{\eta}(\mathcal{E}) = \text{CS}(\nabla_{\ker(D^E)} \oplus \nabla_{\ker(D^G)}, \nabla_{\ker(D^F)} \oplus \nabla_{\ker(D^G)}) + \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \wedge \text{CS}(\nabla^F \oplus \nabla^G, \nabla^E \oplus \nabla^G) \mod \Omega^\text{odd exact}(B),\]
where \(\Omega^\text{odd exact}(B)\) denotes the ring of exact odd forms on \(B\), it follows that \(\omega \in \Omega^\text{odd exact}(B) \subseteq \Omega^\text{odd BU}(B)\). Thus \(j(\omega) = 0\).

4.2. Differential Grothendieck–Riemann–Roch theorem. In this subsection we give a condensed proof of the dGRR.

**Theorem 1.** Let \(\pi : X \to B\) be a proper submersion with closed spin\(^c\) fibers of even relative dimension. Then the following diagram is commutative:
\[
\begin{array}{ccc}
\hat{K}_{\text{FL}}(X) & \xrightarrow{\partial_{\text{FL}}} & \hat{H}^\text{even}(X; \mathbb{R}/\mathbb{Q}) \\
\text{ind}_{\text{FL}}^\text{an} \downarrow & & \downarrow \text{int}_{\text{FL}}^{\text{an}}(\hat{\nabla}^T_V X)^{\ast(\cdot)} \\
\hat{K}_{\text{FL}}(B) & \xrightarrow{\partial_{\text{FL}}} & \hat{H}^\text{even}(B; \mathbb{R}/\mathbb{Q});
\end{array}
\]
i.e., for \(\mathcal{E} = (E, h, \nabla, \phi) \in \hat{K}_{\text{FL}}(X)\), we have
\[\hat{\text{ch}}_{\text{FL}}(\text{ind}_{\text{FL}}^\text{an}(\mathcal{E})) = \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \ast \hat{\text{ch}}_{\text{FL}}(\mathcal{E}).\]

**Proof.** Observe that
\[
(7) \quad f := \hat{\text{ch}}_{\text{FL}}(\text{ind}_{\text{FL}}^\text{an}(\mathcal{E})) - \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \ast \hat{\text{ch}}_{\text{FL}}(\mathcal{E})
\]
does not depend on the choice of \(\phi\) in \(\mathcal{E} = (E, h, \nabla, \phi) \in \hat{K}_{\text{FL}}(X)\). To see this, note that
\[
(8) \quad \hat{\text{ch}}_{\text{FL}}(\text{ind}_{\text{FL}}^\text{an}(\mathcal{E})) = \hat{\text{ch}}(\text{ind}^\text{an}(E, h, \nabla^E)) + i_2\left( \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \wedge \phi \right) + i_2(\tilde{\eta})
\]
and
\[
(9) \quad \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \ast \hat{\text{ch}}_{\text{FL}}(\mathcal{E})
\]
\[\quad = \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \ast \hat{\text{ch}}(E, \nabla^E) + \int_{X/B} \text{Todd}(\hat{\nabla}^T_V X) \ast i_2(\phi).
\]

\(^1\)The author would like to thank J.-M. Bismut and S. Goette for pointing out this formula.
Using Proposition 1(2) and Proposition 2(4), (5), we get

\[
i_2 \left( \int_{X/B} \text{Todd}(\hat{\nabla}^T V_X) \wedge \phi \right) = \int_{X/B} i_2(\text{Todd}(\hat{\nabla}^T V_X) \wedge \phi) \\
= \int_{X/B} i_2(\phi \wedge \text{Todd}(\hat{\nabla}^T V_X)) \\
= \int_{X/B} i_2(\phi) \ast \text{Todd}(\hat{\nabla}^T V_X) \\
= \int_{X/B} \text{Todd}(\hat{\nabla}^T V_X) \ast i_2(\phi).
\]

(10)

It follows from (8), (9) and (10) that

\[
f = \hat{\text{ch}}(\text{ind}^\text{an}(E, h, \nabla_E)) + i_2(\tilde{\eta}(E)) - \int_{X/B} \text{Todd}(\hat{\nabla}^T V_X) \ast \hat{\text{ch}}(E, \nabla_E).
\]

(11)

Thus proving (7) is zero is equivalent to proving (11) is zero, which follows from (5).

\[\square\]

**Acknowledgements**

The author would like to thank several people. First of all he would like to thank Steven Rosenberg for suggesting this problem and for many stimulating discussions. Second he would like to thank Bai-Ling Wang for his comments on the Bismut-Cheeger eta form, and Ulrich Bunke for kindly pointing out an error in Proposition 3 in a previous version of this paper. Third, he would like to thank Jean-Michel Bismut and Sebastian Goette for providing him with many valuable insights about the variational formula of the Bismut-Cheeger eta form used in the proof of Proposition 3. Last but not least he would like to thank the referee for helpful comments.

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