The asymptotic complexity of partial sorting

*How to learn large posets by pairwise comparisons*

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Abstract

The expected number of pairwise comparisons needed to learn a partial order on $n$ elements is shown to be at least $n^2/4 - o(n^2)$, and an algorithm is given that needs only $n^2/4 + o(n^2)$ comparisons on average. In addition, the optimal strategy for learning a poset with four elements is presented.

**Key words:** algorithm, average-case, oracle, poset, sorting.

1 Introduction

Traditional sorting is about learning a linear order. Its complexity is often measured by the number of pairwise comparisons a sorting algorithm needs on average, which is known to be $\Theta(n \log n)$. It is a straightforward generalization to ask for algorithms which learn a partial order by pairwise comparisons, a task that could be termed *partial sorting*. Let us designate the set of all strict partial orders on $n = \{0, 1, \ldots, n - 1\}$ by $\mathcal{P}(n)$. This set has $2n^2/4 + o(n^2)$ many elements (cf. [2]), and each pairwise comparison of elements of $n$ has at most three possible results. A trivial lower bound for the expected number of comparisons needed to learn some $P \in \mathcal{P}$ is therefore $\log_3 |\mathcal{O}(n)| = \frac{\log_2 |\mathcal{O}(n)|}{\log_2 3} + o(n^2)$, since in a rooted tree with $\ell$ leaves in which each node has at most $r$ children, the average leaf-root-distance is at least $\log_r \ell$.

In this paper, a lower bound of $n^2/4 - o(n^2)$ is proved, which is larger than the above by a factor of $\log_2 3 \approx 1.58$. In other words, any learning algorithm for large posets must expect to compare at least about half of all pairs. Moreover, it will be shown that there are indeed algorithms whose expected running time is just $n^2/4 + o(n^2)$. Both results use the fact that for (very) large $n$, almost all posets have a specific three-leveled shape.

To underline the asymptotic nature of the results presented below, Figure 1 shows as a contrast the optimal poset learning strategy for $n = 4$ which has been determined by a recursive computer search. Each node is a possible state (up to (dual) isomorphisms), and the node’s diagram shows all relations (like in a Hasse diagram) and all incomparabilities (represented by dotted lines) known in that state. The edges show which states can arise from which others, where loops indicate dualization. Those states in which there is only one possible type of comparison are framed with thinner lines, so the other
nodes already determine the actual strategy. Its average running time is 5.461 comparisons, compared to 6 pairs and a trivial lower bound of $\log_3 |P(4)| = \log_3 219 \approx 4.905$ comparisons. The optimal strategy for $n = 5$ takes 8.744 comparisons on average, while $\binom{5}{2} = 10$ and $\log_3 |P(5)| = \log_3 4231 \approx 7.601$.

2 The lower bound

Given $P \in \mathcal{P}(n)$ and $a, b \in n$, the pairwise comparison $\{a, b\}$ determines $P|_{\{a, b\}}$, that is, provides the information whether $a P b$, $b P a$, or neither. Let us define the covering and anti-covering relations of $P$ by

$$P^\lor := P \setminus P^2 \quad \text{and} \quad P^\land := \{(x, y) \in n \times n : x \neq y, Px \subseteq Py, \text{and } yP \subseteq xP\} \setminus P,$$

where $Py = \{x : x P y\}$ and $xP = \{y : x P y\}$. We consider algorithms which learn a partial order $P \in \mathcal{P}(n)$ given a number $n \geq 1$ and an oracle for $P$, which is just a subroutine that performs a pairwise comparison in $P$. The algorithms can learn $P$ only through oracle calls, each of which is assumed to take constant time. For any such algorithm $\varphi$, let $c_\varphi(P)$ be the number of pairwise comparisons the algorithm needs until it knows $P$. Then $e_\varphi(n) := \sum_{P \in \mathcal{P}(n)} c_\varphi(P)/|\mathcal{P}(n)|$ is the expected number of pairwise comparisons for that algorithm. Finally, let $Q(P) := \{\{a, b\} : a P^\lor b \text{ or } a P^\land b\}$.

**Lemma 1** $c_\varphi(P) \geq |Q(P)|$ for all $\varphi$ and $P$.

**Proof.** Assume that $\varphi$ claims to know $P$ but has not compared the pair $\{a, b\} \in Q(P)$. If $a P^\lor b$, put $P' := P \setminus \{(a, b)\}$, while if $a P^\land b$, put $P' := P \cup \{(a, b)\}$. Then $P'$ is a partial order that would erroneously be recognized as $P$ by $\varphi$. \qed

For $R \in \mathcal{P}(4)$, for example, the average cardinality of $Q(R)$ is about 4.849 which is smaller than the trivial lower bound of 4.905. But for $R \in \mathcal{P}(5)$ it is about 7.958 which improves the trivial lower bound of 7.601.

For the rest of this section, assume that $n$ is a multiple of 4. Let $L(n)$ be the set of all ordered partitions $(A, B, C)$ of $n$ with $|A| = |C| = n/4$ and $|B| = n/2$. Put $T(n) := \bigcup_{(A,B,C) \in L(n)} T_{ABC}(n)$, where $T_{ABC}(n)$ is the set of all $P \in \mathcal{P}(n)$ which fulfill (i) $x = y$ or $Px \not\subseteq Py$ or $yP \not\subseteq xP$ for all $(x, y) \in A^2 \cup B^2 \cup C^2$, and (ii) $aP \cap B \cap C \neq \emptyset$ and $A \cap Pb \neq \emptyset$ if $aP \cap B \neq \emptyset$ and $A \cap Pb \neq \emptyset$ for all $(a, b, c) \in A \times B \times C$. In particular, these posets consist of a lower level $A$ of $n/4$ minimal elements, an antichain $B$ of size $n/2$ building the middle level, and an upper level $C$ of $n/4$ maximal elements, and no $C$-element covers an $A$-element. Moreover, (i) and (ii) imply that $Q(P) = Q_{ABC} := (A \times B) \cup (B \times C)$.

**Lemma 2** $\frac{|T(4m)|}{|\mathcal{P}(4m)|} = 1 - o(1/m)$.

**Proof.** Let $n = 4m$. Improving upon the original asymptotics of Kleitman and Rothschild \[\text{[3]}, \text{Brightwell, Prömel, and Steger}\text{[1]}\] showed that for some $K > 1$, $|\mathcal{P}(n)| =
\(|S(n)| (1 + O(K^{-n}))\), where \(S(n) := \bigcup_{(A,B,C) \in L(n)} S_{ABC}(n)\) and \(S_{ABC}(n)\) is the set of all \(P \in \mathcal{P}(n)\) with \(P^\land \subseteq Q_{ABC}\) and \(|P_x| > 1\) for all \(x \in B \cup C\). On the other hand, it is easy to see that \(T(n) \subseteq S(n)\) and \(|T(n)|/|S(n)| = 1 - o(1/n)\). Hence
\[
1 - |T(n)|/|\mathcal{P}(n)| = 1 - \frac{1-o(1/n)}{1+O(K^{-n})} = o(1/n).
\]

Because \(P \in T(n)\) implies \(|Q(P)| = n^2/4\), it follows that \(e_\varphi(n)\) has a lower bound of \(n^2/4 - o(n^2)\). Table 2 compares \(n^2/4\) with \(\log_3 |\mathcal{P}(n)|\) for some small values of \(n\) (based on numbers from [2]).

3 A simple algorithm

Consider the algorithm \(\varphi_3\) listed in Figure 2 which learns a partial order on \(N\). If \(N\) is a multiple of 4, the strategy of \(\varphi_3\) first assumes that \(P\) is a member of \(T(n)\). If the assumption is true, \(\varphi_3\) will determine the corresponding level partition \((A,B,C)\) in \(o(n^2)\) expected time so that it can afterwards compare exactly the \(n^2/4\) pairs in \(Q(P) = Q_{ABC}\). In the asymptotically unlikely case that \(P \notin T(n)\) it will detect that fact and perform a comparison of all pairs.

Although this is obviously not the best possible strategy, the amount of time \(\varphi_3\) “wastes” becomes negligible for \(N \to \infty\).

**Theorem 1** \(\varphi_3\) is an asymptotically optimal poset learning algorithm in the sense that \(e_\varphi_3(N) = N^2/4 + o(N^2)\).

**Proof.** Let \(P \in \mathcal{P}(N)\). Because of lines 20–21, \(\varphi_3\) learns \(P\) completely.

Let \(U(n) := \bigcup_{(A,B,C) \in L(n)} U_{ABC}(n) \supseteq T(n)\), where \(U_{ABC}(n) \supseteq T_{ABC}(n)\) is the set of all \(P \in \mathcal{P}(n)\) with \(P^\land \subseteq Q_{ABC}\). We may assume that \(P|_n \in U_{A_0B_0C_0}(n)\) for some \((A_0,B_0,C_0) \in L(n)\), since by Lemma 3 \(P|_n \in U(n)\) is true with a probability converging to 1 as \(N \to \infty\). Note that \(\alpha_n := 1 - |T(n)|/|U(n)| = o(1/n)\) is an upper bound for the probability that at some point in \(\varphi_3\), either \(A \not\subseteq A_0, B \not\subseteq B_0,\) or \(C \not\subseteq C_0\).

Conditional to \(P|_n \in U_{A_0B_0C_0}(n)\), the event \(x \parallel y\) has probability \(1/2\) independently for all \((x, y) \in Q_{A_0B_0C_0}\). Hence one can estimate the expected number of pairwise comparisons in iteration \(k\) of the main loop as follows.

(i) Assume that \(k \in B_0\). For \(j := 1\) and \(2 \leq i \leq r\), the disjunction in lines 9–12 is violated with probability at most \(1/2 + \alpha_n\). Hence iteration \(k\) takes an expected number of at most
\[
2 \left[ \sum_{i=2}^{r} i \left( \frac{1}{2} + \alpha_n \right)^{i-2} + (n-1) \left( \frac{1}{2} + \alpha_n \right)^{-2} \right] < 2 \left( \frac{1}{2} - \alpha_n \right)^{-2} + n \left( \frac{1}{2} + \alpha_n \right)^{-1}
\]
pairwise comparisons in this case.

(ii) Assume that, on the other hand, \(k \in A_0 \cup C_0\). For \(1 \leq i \leq m\), the probability that both the conditions of lines 15–16 are violated is at most \(1/2 + \alpha_n\) so that in this case iteration \(k\) takes an expected number of at most
\[
2 \left[ \sum_{i=1}^{m} i \left( \frac{1}{2} + \alpha_n \right)^{i-1} + (n-1) \left( \frac{1}{2} + \alpha_n \right)^{m} \right] < 2 \left( \frac{1}{2} - \alpha_n \right)^{-2} + n \left( \frac{1}{2} + \alpha_n \right)^{m}
\]
over all iterations.
We have seen that, given \( r \) and \( m \), iteration \( k \) takes in both cases an expected number of at most \( (16 + 4n(\frac{1}{2} + \alpha_n)^a)\beta_n \) pairwise comparisons, where \( a := \min\{r, m\} \) and \( \beta_n \to 1 \). At the beginning of iteration \( k \), for \( 0 \leq a \leq \frac{k}{2} \), the probability that \( r = a \) and \( m = k - a \) is at most

\[
\left( \frac{n/2}{a} \right) \left( \frac{n/2}{k-a} \right) + \alpha_n = \left( \frac{k}{a} \right) \left( \frac{n-k}{n/2-a} \right) + \alpha_n < 2^{n-k} \left( \frac{k}{a} \right) + \alpha_n,
\]

and so is the probability that \( m = a \) and \( r = k - a \). In contrast, the probability that \( r + m \neq k \) is at most \( \alpha_n \). In all, iteration \( k \) takes an expected number of at most

\[
2 \sum_{a=0}^{[k/2]} \left( \frac{2^{n-k}}{\binom{n/2}{a}} \right) + \alpha_n \leq O(n) \frac{2^{n-k}}{\binom{n/2}{a}} \sum_{a=0}^{k} \left( \frac{k}{a} \right) \left( \frac{1}{2} + \alpha_n \right)^a 1^{k-a} + o(n)
\]

\[
= O(n) \frac{2^{n-k}}{\binom{n/2}{a}} \left( \frac{3}{4} + \frac{\alpha_n}{4} \right)^k + o(n) = O(n^{3/2})\left( \frac{3}{4} + \frac{\alpha_n}{4} \right)^k
\]

pairwise comparisons, so that the total expected number of comparisons in lines 1–18 is \( O(n^{3/2}) \). If \( P \in T(N) \) then \( P \) is uniquely determined in line 20, hence the expected number of comparisons in lines 19–21 is \( N^2/4 + o(N^2) \), proving the theorem.

\[\square\]

References

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[2] Heitzig, J. and Reinhold, J. (2000) The number of unlabeled orders on fourteen elements, Order 17, 333–341.

[3] Kleitman, D. J. and Rothschild, B. L. (1975) Asymptotic enumeration of partial orders on a finite set, Trans. Amer. Math. Soc. 205, 205–220.
Figure 1: Optimal learning strategy for 4-element posets
Table 1: Comparison of lower bounds for $e_\varphi(n)$ for small $n$

| $n$  | 4   | 8   | 12  | 13  | 14  | 15  | 16  |
|------|-----|-----|-----|-----|-----|-----|-----|
| $n^2/4$ | 4   | 16  | 36  | 42.25 | 49  | 56.25 | 64  |
| $\log_3 |\mathcal{P}(n)|$ | 4.91 | 18.10 | 36.93 | 42.41 | 48.19 | 54.26 | 58.52 |

Figure 2: The asymptotically optimal algorithm $\varphi_3$

input: oracle $\mathcal{C}$ for pairwise comparisons in a partial order $P$ on $N$

output: $P$

1. put $A = B = C = \emptyset$
2. find largest $n \leq N$ with $4|n$

main loop:
3. for $k$ from 0 to $n - 1$ do
4. put $r = |A \cup C|$ and $m = |B|$
5. assume $A \cup C = \{x_1, \ldots, x_r\}$, $B = \{y_1, \ldots, y_m\}$, and $n \setminus \{k\} = \{x_1, \ldots, x_{n-1}\} = \{y_1, \ldots, y_{m-1}\}$

inner loop:
6. for $i$ from 1 to $n - 1$ do
7. call $\mathcal{C}(k, x_i)$
8. if, for some $j < i$, either
9. $x_i P k P x_j$, or
10. $x_j P k P x_i$, or
11. $(x_i P k, x_j \in A)$, but not $x_j P k$), or
12. $(k P x_i, x_j \in C)$, but not $k P x_j$)
13. then add $k$ to $B$ and continue in main loop
14. call $\mathcal{C}(k, y_i)$
15. if $i \leq m$ and $k P y_i$ then add $k$ to $A$ and continue in main loop
16. if $i \leq m$ and $y_i P k$ then add $k$ to $C$ and continue in main loop
end (of inner loop)
17. if $k$ is maximal ($\iff k P y_i$ for no $i$) then add $k$ to $C$
18. if $k$ is minimal ($\iff y_i P k$ for no $i$) then add $k$ to $A$
end (of main loop)

for all $(x, y) \in Q_{ABC} \cup (n \times (N \setminus n))$ call $\mathcal{C}(x, y)$
19. if the calls so far did not determine $P$ uniquely then for all remaining pairs $(x, y)$ call $\tilde{\mathcal{C}}(x, y)$
20. compute and print $P$. 

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