Frequency depending permittivity of Coulomb system with the Bose–Einstein condensate

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Abstract. The low-frequency second-order singularity is found in the permittivity of isotropic and homogeneous model system of charged particles consisting of electrons and boson nuclei. This singularity is caused by the existence of a Bose–Einstein condensate for nuclei. The result obtained for the model under consideration leads to the existence of the “nuclei superconductivity”. Since the system permittivity is obtained for arbitrary strong Coulomb interaction, we suppose that the model can describe also atomic systems with interatomic interaction. In this connection the possible applicability of this model system to weakly interacted Bose systems and to superfluid He II, usually considering as the system of neutral “atoms”, is discussed. In parallel, we discuss the unsolved and principal question about bosonization to clarify which kind of objects should be considered as real bosons, in particular, in He II. In the model under consideration in two-component Coulomb system only nuclei can be considered as bosons, assuming that nuclear forces can provide bosonization of protons and neutrons. The result obtained can be experimentally verified in weakly non-ideal Bose gases and superfluid He II. This is especially actual due to recently discovered electrical activity in superfluid helium.

1. Introduction
Experimental detection of a Bose–Einstein condensate (BEC) in rarefied gases of alkali elements [1–3] not only opened a new field of research in the ultralow temperature region [4, 5], but also was an indirect confirmation of the possibility of using the BEC concept in the microscopic theory of superfluid He II [6, 7]. At the same time the existence of Bose condensate in a bulk He II is not experimentally verified.

At the interpretation of the BEC phenomena, it is a priori considered that the corresponding fluids consist of electrically neutral “atoms” as initial microparticles. In particular, the BEC appearance in rarefied gases is associated with confinement of alkali element atoms in magnetic traps due to diamagnetism of these atoms. Magnetic traps are filled with rarefied gas which is cooled by a laser with a frequency lower than the absorption frequency of corresponding atoms (see [8] for more details). In this case, the interaction of laser radiation with atoms is described in terms of atomic polarizability within the dipole approximation (see [9] for more details).
Note, that the use of the polarizability means that the atom consists of a nucleus and electrons localized near it.

In turn, atoms of He II have zero dipole moment. Therefore, the experimental results [10–12], showing electrical activity of He II appeared rather unexpected. Till now, it is unclear whether the observed absorption of the microwave radiation corresponds to excitation of some characteristic modes related to superfluid component or to interaction with the normal component of the fluid. To date, there are a few theoretical models to describe this effect (see [13–17] and references therein). No less interesting is the observation of the superradiant Rayleigh scattering from a BEC, detected in [18–20], which differs from Rayleigh scattering of electromagnetic field in normal fluids (see [21] and references therein).

We would like to pay attention to that the consistent theoretical study of electromagnetic phenomena in medium implies the consideration of substance as a system of charged electrons and nuclei interacting with each other according to the Coulomb law (Coulomb system—CS, see [22–25], and references therein). This approach successfully describes not only fully-ionized plasmas but also low temperature systems. Usually, the properties of matter are considered in the frame the chemical model, where the system is a mix of atoms, quasifree electrons and ions with interaction between all of them via some approximate interaction potentials. The advantage of consideration on the basis of Coulomb system approach, especially for such simple elements as hydrogen and helium, is caused by knowledge of the explicit form of the interaction—Coulomb potential. In some cases, on this basis the exact results for arbitrary strong interaction can be found. The consideration in the paper is based only on the Penrose–Onsager concept for the BEC and commutation relations for the operators of creation and annihilation of particles. Therefore, there is no necessity to use approximations on the intensity of Coulomb interparticle interaction to find the general structure and characteristic peculiarities of frequency dependence of the dielectric permittivity. Naturally, to find the concrete structure of the functions defining the obtained dielectric permittivity, the further assumptions about the values of the Coulomb interaction parameters are necessary.

Here we confine by the description of two-component CS in the non-relativistic approximation. In this case, we consider electrons and nuclei as point-like particles, not taking into account the structure of nuclei. Herewith, atom is a composite particle that is the bound state of the nucleus and the corresponding number of electrons, providing electrical neutrality of the atom. The consideration of nucleus as the composite particle consisting of nucleons (composite boson for even nucleons in the nuclei), is beyond the scope of this paper. This reasonable simplification can be connected, as we suppose, with non-electromagnetic nature of nuclear forces (the detail analysis of this statement cannot be justified completely, because the notion of elementary bosons is not precisely defined at the present day for composite particles and the types of interparticle forces inside. A detailed analysis of this statement is beyond the scope of the paper. Nevertheless, this problem is shortly discussed in the section Conclusion).

The standard model of superfluid He assumes that helium atoms are the point particles. This model can be successfully used to study thermodynamic properties of various substances in the insulating state, although the explicit form of the interaction potential between atoms in such a model remains unknown function. However, the use of this model is incorrect to study electromagnetic properties of substance.

Some theoretical aspects of electrical activity in He II have been considered in [14, 15, 26] on the basis of CS approach. In this paper we study the linear electromagnetic properties of the Coulomb system of electrons and Bose nuclei, assuming the opportunity for BEC of nuclei. It is naturally to apply our results to He II, taking into account that it is the unique bulk superfluid substance and the experiments on electric activity were performed in He II. In this connection the difference between the CS and the term “plasma” should be emphasized. Plasma is a specific CS in which the average density of electrons in delocalized states (scattering states)
is comparable to the average density of all electrons in a material under consideration [27]. In turn, the neutral fluid is a CS in which the average density of delocalized electrons is extremely low. Hence, such a system in many cases can be described as a system of initial atoms (see [28, 29] and references therein). Within the adiabatic approximation for a subsystem of nuclei, the initial atom can be considered as a nucleus with electronic states localized near it. In this case, the size of the initial atom, which is defined by the distribution of the inhomogeneous electron density near the nucleus of the corresponding atom, should be much smaller than the average distance between initial atoms [29]. However, the use of the adiabatic approximation leads to ambiguity at determining the pair interaction potential between initial atoms.

Additionally to this ambiguity, there is also a problem of many-particle interactions of atoms (see, e.g., [30]) whose effect cannot be reliably estimated.

At the same time, as shown in [31], the form of the pair interaction potential of atoms has of fundamental importance for describing the quantum neutral fluid with a BEC. This also leads to the necessity to develop a corresponding theory, based on the Coulomb model of matter to define the effective interaction potential of atoms in the fluid considering as a CS (see, e.g., [32]). An alternative version widely used in the plasma theory is based on the equivalent consideration of electrons and nuclei in the CS without the adiabatic approximation for the subsystem of nuclei. To describe the CS with the BEC, this approach is in the early development stage [26], although the study of the so-called charged Bose gas which is a model single-component system of charged bosons in a compensating background has a long history [33, 34]. However, for description of neutral fluids, in which nuclei can be considered as charged bosons, the one-component charged Bose gas model cannot be used. The formation of atoms and requires a strong interaction between electrons and nuclei in the two-component CS. Note, that the use of CS approach in principle provides a universal way to describe substance in state of isolator and conductor (see [31–35] and references therein). In fact, the division of substances on isolators and conductors caused by essential difference of the conductivity values is conditional. The application of the functional approach in quantum-field theory methods allows, in principle, take into account the effects of the strong interaction between electrons and nuclei to describe atomic-like states [22].

As a result, we come to the conclusion that the use of the CS model corresponds to the consistent consideration of matter, including the presence of the BEC. At the same time, we assume that the electrons and nuclei in the CS are elementary particles within the used non-relativistic approximation. It means that the Coulomb interaction of charged particles cannot affect statistics of electrons and nuclei. From this point of view, electrons as fermions cannot form the single-particle electronic BEC, but affect formation of the BEC for nuclei. The main results of the paper can be summarized as the following. The frequency dependent dielectric permittivity for such system below the Bose condensation temperature for nuclei is found and the existence of the second order singularity of dielectric permittivity in the limit of low frequency is shown. The obtained current is similar to the London’s equation for the superconducting electron current, but for the system under consideration the appropriate current is determined by the Bose condensed nuclei. In this sense we introduce the notion “superconductivity of nuclei”.

The problem of the BEC for so-called composite particles, in particular, initial atoms, is not discussed in the present paper. In section 2, we consider the definition of the BEC for boson nuclei in the CS. On this basis, in section 3, we study the features of the frequency-dependent permittivity of the CS in the presence of a BEC for nuclei. The nuclei of 4He can be considered as true bosons, since we eliminate from the description the nuclear forces and, therefore, assume that these nuclei are the point particles, as we mentioned above.
2. Bose–Einstein condensate for nuclei

When considering the non-relativistic CS, we suppose that nuclei (subscript \( c \)) are the point-like elementary bosons, which results in the formation of a BEC for nuclei at low temperatures.

According to the general definition proposed by Penrose and Onsager [36], the existence of a BEC is associated with the anomalous spatial behavior of the equilibrium one-particle density matrix, which was called the off-diagonal long-range order (ODLRO) [37]. For the homogeneous and isotropic CS, in which the one-particle density matrix of nuclei has the form \( \gamma_c(\mathbf{r}, \mathbf{r}') = \gamma_c(|\mathbf{r} - \mathbf{r}'|) \), this statement is written as

\[
\lim_{|\mathbf{r} - \mathbf{r}'| \to \infty} \gamma_c(\mathbf{r}, \mathbf{r}') = n_c^{\text{BEC}} \neq 0, \quad \gamma_c(\mathbf{r}, \mathbf{r}') = \langle \hat{\Psi}_c^+(\mathbf{r}) \hat{\Psi}_c(\mathbf{r}') \rangle,
\]

where \( n_c^{\text{BEC}} \) is the density of the number of nuclei in a BEC, \( \hat{\Psi}_c^+(\mathbf{r}) \) and \( \hat{\Psi}_c(\mathbf{r}) \) are the field operators of creation and annihilation for nuclei, respectively, and angle brackets mean averaging over the Gibbs distribution. In the normal CS, \( n_c^{\text{BEC}} = 0 \), i.e., BEC is absent. It is clear that the existence of a BEC in the CS when using definition (1) can be caused only by boson nuclei. Note, that the CS for the case of electron superconductivity is characterized by the existence of the ODLRO for the two-particle electron density matrix [38].

As is well known, for the thermodynamic equilibrium in the CS is necessary to satisfy the quasi-neutrality condition (see [38] for more details), which is written as

\[
\sum_{a=e,c} z_a e n_a = 0, \quad (2)
\]

where \( n_a = \langle \mathcal{N}_a \rangle / V \) is the average density for the number of particles of type \( a \) with charge \( z_a e \) and mass \( m_a \) in the volume \( V \), \( \mathcal{N}_a = \int d\mathbf{r} \hat{\Psi}_a^+(\mathbf{r}) \hat{\Psi}_a(\mathbf{r}) \) is the operator of the total number of particles of type \( a \); subscript \( e \) corresponds to electrons.

It should be considered that averaging over the Gibbs distribution in the statistical theory corresponds to the thermodynamic equilibrium state only after transition to the thermodynamic limit \( \langle \mathcal{N}_a \rangle \to \infty, V \to \infty, n_a = \langle \mathcal{N}_a \rangle / V = \text{const} \). This means that in calculating the average values, a system should be initially considered in a very large (macroscopic), but finite volume \( V \), and then the transition to the thermodynamic limit should be performed [39].

To pass to the thermodynamic limit in equation (1), the field operators \( \hat{\Psi}_c^+(\mathbf{r}) \) and \( \hat{\Psi}_c(\mathbf{r}) \) are written as

\[
\hat{\Psi}_c^+(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_s \sum_p \hat{c}^+_{p,s} \exp(-i\mathbf{p}\mathbf{r}), \quad \hat{\Psi}_c(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_s \sum_p \hat{c}_{p,s} \exp(i\mathbf{p}\mathbf{r}), \quad (3)
\]

where \( \hat{c}^+_{p,s} \) and \( \hat{c}_{p,s} \) are the creation and annihilation operators, respectively, for nuclei with momentum \( \hbar \mathbf{p} \) and spin projection \( s \). Taking into account the possible application to He II, we consider below the case of nuclei having zero spin. Note, that in the non-relativistic approach the interaction is non-dependable from the spin of bosons. However, particle spin affects on their statistics (for electrons—Fermi–Dirac statistics; for Bose nuclei—Bose–Einstein statistics). Taking into account relation (3), we can represent the one-particle density matrix for nuclei in the homogeneous and isotropic CS as the Fourier series

\[
\gamma_c(|\mathbf{r} - \mathbf{r}'|) = \frac{f_c^{(V)}(\mathbf{p} = 0)}{V} + \frac{1}{V} \sum_{\mathbf{p} \neq 0} f_c^{(V)}(\mathbf{p}) \exp(i\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')), \quad (4)
\]

where \( f_c^{(V)}(\mathbf{p}) \) is the average occupation number of nuclei with momentum \( \hbar \mathbf{p} \) (or the single-particle distribution function over momenta), the subscript \( (V) \) means that a given function corresponds to a system in a very large (macroscopic), but finite volume \( V \). Taking into account equations (1) and (4), the average density of the number of nuclei in a BEC is written as

\[
n_c^{\text{BEC}} = \frac{f_c^{(V)}(\mathbf{p} = 0)}{V}. \quad (5)
\]
Hence, the average occupation number of nuclei with zero momentum \( \langle \hat{N}_0 \rangle = f_c(\mathbf{p} = 0) = \langle c_0^+ c_0 \rangle = n_c^{\text{BEC}} \) is a macroscopic quantity which is exactly the definition of BEC. As a result, after transition to the thermodynamic limit, the distribution function over momenta for nuclei in the presence of BEC has the form

\[
f_c(\mathbf{p}) = \langle \hat{N}_0 \rangle \delta_{\mathbf{p},0} + f_c^{\text{over}}(\mathbf{p}) [1 - \delta_{\mathbf{p},0}],
\]

where \( f_c^{\text{over}}(\mathbf{p}) \) is the single particle distribution function for nuclei in the “overcondensate” state at \( \mathbf{p} \neq 0 \). In this case, the average density of the number of nuclei \( n_c = \langle \hat{N}_c \rangle / V \) is given by

\[
n_c = \lim_{V \to \infty} \frac{1}{V} \sum_{\mathbf{p}} f_c(\mathbf{p}) = n_c^{\text{BEC}} + n_c^{\text{over}},
\]

\[
n_c^{\text{over}} = \lim_{V \to \infty} \frac{1}{V} \sum_{\mathbf{p} \neq 0} f_c(\mathbf{p}) = \int \frac{d^3 p}{(2\pi)^3} f_c^{\text{over}}(\mathbf{p}),
\]

where \( n_c^{\text{over}} \) is the density of the number of nuclei in overcondensate states.

Thus, for calculating the average values corresponding to the thermodynamic equilibrium in the CS in the presence of BEC, it is first necessary to consider the initial system in a very large, but finite volume, and then, after separating singular terms corresponding to the macroscopic number of nuclei in a BEC, to perform the transition to the thermodynamic limit. It is easy to verify that a similar statement takes place when considering the inhomogeneous system with a BEC [40].

3. CS permittivity in the presence of BEC for nuclei

As is known [41], electromagnetic properties of the homogeneous and isotropic CS under the influence of a weak electromagnetic field are completely defined by the permittivity tensor

\[
\varepsilon_{\alpha\beta}(q, \omega) = (\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}) \varepsilon^{tr}(q, \omega) + \frac{q_\alpha q_\beta}{q^2} \varepsilon^l(q, \omega),
\]

where \( \varepsilon^{tr}(q, \omega) \) and \( \varepsilon^l(q, \omega) \) are, respectively, the transverse and longitudinal permittivity accounting for the spatial and frequency dispersion. In the case under consideration we restrict ourselves by the long-wavelength limit \( q \to 0 \) of the dielectric permittivity

\[
\lim_{q \to 0} \varepsilon^{tr}(q, \omega) = \lim_{q \to 0} \varepsilon^l(q, \omega) = \varepsilon(\omega), \quad \lim_{q \to 0} \varepsilon_{\alpha\beta}(q, \omega) = \varepsilon(\omega) \delta_{\alpha\beta}.
\]

It is shown below that in this limit the interesting physical consequences related to the low frequency singularity in the dielectric permittivity appear.

Within the linear response theory, the function \( \varepsilon(\omega) \) is defined by the relations (see [42] for more details)

\[
\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \varphi(\omega) = \int_0^\infty dt \exp(i\omega t) f_\varphi(t),
\]

\[
f_\varphi(t) = -\frac{i}{3\hbar V} \{ [\hat{I}^\beta(t), \hat{I}^\beta(0)] \}, \quad \lim_{t \to \infty} f_\varphi(t) = 0.
\]

Here and below, the commutator \([\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \langle \hat{A} \rangle \) means the averaging with the Gibbs partition function for the system

\[
\langle \hat{A} \rangle = \frac{\text{Tr} \exp\{- (\hat{H} - \sum_a \mu_a \hat{N}_a)/T\} \hat{A}}{\text{Tr} \exp\{- (\hat{H} - \sum_a \mu_a \hat{N}_a)/T\}}.
\]

Here \( \hat{H} \) is the Hamiltonian of the CS at the temperature \( T \) (in energy units), \( \mu_a \) is the chemical potential for particles of type \( a \), \( \omega_p \) is the plasma frequency defined by the total number of
electrons and nuclei in the CS,
\[ \omega_p^2 = \sum_a \omega_a^2, \quad \omega_a^2 = \frac{4\pi z_a^2 e^2 n_a}{m_a}, \]  
(13)
where \( \omega_a \) is the plasma frequency for particles of type \( a \), \( \hat{I}_a^\beta = \sum_a z_a e \hat{n}_a^\beta \) is the operator of the total electric current, \( \hat{I}_a^\beta \) is the operator of the total flux of the number of particles of type \( a \),
\[ \hat{I}_a^\beta = -\frac{i\hbar}{2m_a} \int \! dr \{ \hat{\Psi}_a^+(r) \nabla_{\beta} \hat{\Psi}_a(r) - \nabla_{\beta} \hat{\Psi}_a^+(r) \hat{\Psi}_a(r) \}, \]  
(14)
and \( \hat{H} \) is the exact Hamiltonian of the CS in the non-relativistic approximation. Hereafter, summation over repeated indices is meant. Relations (11) and (12) should be understood in the thermodynamic limit. In this case, the time correlation function \( f_{\phi}(t) \) is such that the time integral (12) defining \( \phi(\omega) \) converges at any real frequencies \( \omega \) [43]. From equations (14) and (15), it immediately follows that
\[ \frac{d\hat{I}_a^\beta}{dt} = \frac{1}{m_a} \sum_{b \neq a} \int \! dr_1 \! dr_2 \nabla_{r_1,\beta} U_{ab}(r_1 - r_2) \hat{n}_a(r_1) \hat{n}_b(r_2), \]  
(16)
where \( U_{ab}(r) \) is the Coulomb interaction potential for charged particles of types \( a \) and \( b \): \( \hat{n}_a(r) = \hat{\Psi}_a^+(r) \hat{\Psi}_a(r) \). Let us first consider some particular cases of the general equations (11) and (12). For the simplest case of charged one-component Bose gas without BEC which is a particular case of the one-component plasma (OCP) model, the permittivity \( \varepsilon_{\text{OCP}}(\omega) \), according to equations (11)–(16) can be written explicitly as
\[ \varepsilon_{\text{OCP}}(\omega) = 1 - \frac{\omega_c^2}{\omega^2}, \]  
(17)
This relation follows straightforward from the general formulae (11) and (12) and the equality \( \hat{I}(t) = \hat{I}(0) \) for the OCP.

Further, we study the frequency dispersion of the permittivity \( \varepsilon(\omega) \) in the static \( \omega \to 0 \) for the two-component normal CS which is an adequate model of the real matter consisting of electrons and nuclei of one chemical element (without assumption about the presence of BEC of nuclei, which is considered below). For the normal CS, the function \( \varepsilon(\omega) \) in the static limit \( \omega \to 0 \) has the singularity
\[ \varepsilon(\omega) \big|_{\omega \to 0} \to \frac{4\pi i \sigma_{\text{st}}}{\omega}, \]  
(18)
where \( \sigma_{\text{st}} = \lim_{\omega \to 0} \sigma(\omega) \) is the static conductivity which is nonzero at a nonzero temperature for all known materials. The classification of materials by the static conductivity (electrical conductivity) on “conductors” (high conductivity), “dielectrics” (low conductivity), and “semiconductors” (intermediate conductivity strongly depending on external conditions) is conditional. Therefore, the value of \( \sigma_{\text{st}} \) for all materials is finite, although it is lower for dielectrics than for conductors by many orders of magnitude. Relation (18) is a consequence of the known general formula relating the dynamic conductivity \( \sigma(\omega) \) to the permittivity \( \varepsilon(\omega) \) (see, e.g., [41])
\[ \varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega}, \]  
(19)
In this case, the static permittivity \( \varepsilon_{\text{st}} = \lim_{\omega \to 0} \text{Re} \varepsilon(\omega) \) for normal system has no singularities on \( \omega \) (see, e.g., [44, 45]).
To establish the correspondence between relation (11) based on the linear response theory and limit relation (18), we use the operator equation

\[ \frac{d\hat{P}_a^\beta}{dt} = \hat{I}_a^\beta, \quad \hat{P}_a^\beta = \int d\mathbf{r} \mathbf{r}_\beta n_a(\mathbf{r}). \] (20)

Then, twice integrating in parts in definition (11) for function \( \varphi(\omega) \) (see, e.g., [43]) and taking into account equation (20), we find for the permittivity \( \varepsilon(\omega) \)

\[ \varepsilon(\omega) = \varepsilon_p(\omega) + \frac{4\pi i f_p(\infty)}{\omega}, \quad \varepsilon_p(\omega) = 1 + 4\pi \alpha(\omega) - \frac{1}{\omega^2}(\omega_p^2 - \Omega_p^2), \] (21)

\[ \alpha(\omega) = \int_0^\infty dt \exp(i\omega t)f_\alpha(t), \quad f_\alpha(t) = f_p(t) - f_p(\infty), \] (22)

\[ f_p(t) = \frac{i}{3\hbar V}\langle[\hat{P}^\beta(t), \hat{P}^\beta(0)]\rangle, \quad f_p(\infty) = \lim_{t \to \infty} f_p(t), \] (23)

\[ \Omega_p^2 = \frac{4\pi i}{3\hbar V}\langle[\hat{P}^\beta(0), \hat{P}^\beta(0)]\rangle. \] (24)

Definitions (23) and (24) should be understood in the thermodynamic limit. In this case, the transition to the limit \( t \to \infty \) for the time correlation function \( f_p(t) \) when calculating \( f_p(\infty) \) should be performed after passing to the thermodynamic limit (see [46] for more details). Let us pay attention that relations (21)–(24) are valid under the condition

\[ \lim_{t \to \infty} \frac{1}{V}\langle[\hat{I}^\beta(t), \hat{P}^\beta(0)]\rangle = 0, \quad \lim_{t \to \infty} \frac{1}{V}\langle[\hat{P}^\beta(t), \hat{I}^\beta(0)]\rangle = 0, \] (25)

which provide the fulfillment of the limit relation (18) [43].

According to the above justification, to consider possibility of the BEC existence in the CS, for calculation of \( \Omega_p^2 \) (24), it is necessary (taking into account equations (3) and (4)), to perform the transition from the coordinate representation of the operators \( \hat{I}_a^\beta \) and \( \hat{P}_a^\beta \) in definitions (14) and (20) to the momentum representation (see, e.g., [47]). Assuming the nuclei BEC existence the transition to the momentum representation permits to take into account the presence of the BEC by the simple calculation, using equation (6)

\[ \hat{I}_a^\beta = \sum_s \sum_{p \neq 0} \frac{\hbar p_a^\beta}{m_a} \hat{a}_p^+ \hat{a}_{ps}, \quad \hat{P}_a^\beta = i \sum_s \sum_p \hat{a}_p^+ \nabla_p \hat{a}_{p} \] (26)

Using the commutation relations for the creation and annihilation operators, from equation (26) we find

\[ \langle[\hat{I}_a^\beta(0), \hat{P}_a^\beta(0)]\rangle = -3i \hbar \sum_s \sum_{p \neq 0} f_a^{(V)}(p, s), \quad f_a^{(V)}(p, s) = \langle \hat{a}_p^+ \hat{a}_{ps} \rangle^{(V)}. \] (27)

Substituting equation (27) into (24) and taking into account equations (7) and (8), we find

\[ \Omega_p^2 = \omega_p^2 - \omega_{BEC}^2, \quad \omega_{BEC}^2 = \frac{4\pi z e^2 n_{BEC}}{m_e}. \] (28)

Thus, according to equations (21)–(24),

\[ \varepsilon(\omega) = 1 + 4\pi \alpha(\omega) - \frac{\omega_{BEC}^2}{\omega^2} + \frac{4\pi i f_p(\infty)}{\omega}. \] (29)

In general, this result shows the presence of the dissipative part in the dielectric permittivity which for both normal and superconductive CS has a first-order singularity in the limit of low frequencies \( \omega \to 0 \). However, the appearance of the second-order singularity for the real part in the dielectric permittivity for low frequency \( \omega \to 0 \) leads to a result similar to the London’s model of the “electronic” superconductor, but for the considered CS with BEC of nuclei. It leads
to the appearance of non-damped superconductive current in the system under consideration. This part of the current exists even in the case when the external field is switched off. As is known, the part of the current connected with the second order peculiarity on $\omega$ in the real part of the dielectric permittivity is non-dissipative [48] For the normal CS, $\omega_{\text{BEC}}^2 = 0$; hence, a comparison of equations (18) and (29) shows that

$$\sigma_{\text{st}} = f_p(\infty).$$

(30)

According to equation (22) the value $\alpha(\omega)$ has no peculiarity at low $\omega$ (see [39] for more details). We should emphasize that the low-frequency singularity $-\omega_{\text{BEC}}^2/\omega^2$ of dielectric permittivity for the one-component model of the Coulomb system in Bose condensed state has been found in RPA approximation [34]. The result (29) and (30) is valid for two-component system and arbitrary strong Coulomb interaction. This circumstance allows, in principle, to compare the model under consideration with the real systems.

In the specific limiting state of “true insulator” for which $\sigma_{\text{st}} = 0$, the permittivity $\varepsilon^{\text{TI}}(\omega)$ takes the form [49, 50]

$$\varepsilon^{\text{TI}}(\omega) = 1 + 4\pi\alpha^{\text{TI}}(\omega),$$

(31)

so that the quantity $\alpha(\omega)$ has the meaning of polarization and has no singularities in the static limit $\omega \to 0$ (see equation (22)).

Let us now consider the quantity from (29) in the presence of BEC for nuclei $\omega_{\text{BEC}}^2 \neq 0$. In this case, a higher-order singularity appears in the permittivity $\varepsilon(\omega)$ at $\omega \to 0$ in addition to singularity (18). In other words, according to equation (29), in the presence of BEC for nuclei, we have

$$\varepsilon(\omega) \mid_{\omega \to 0} \to -\frac{\omega_{\text{BEC}}^2}{\omega^2}. $$

(32)

It follows from relation (32), taking into account the Fresnel formulas (see, e.g., [51]) as well as from the analogy with the reflection of external electromagnetic waves of low frequency from “electronic” superconductor, that an electromagnetic wave with a low frequency $\omega \to 0$, incident on the interface the CS with a BEC for nuclei, will be completely reflected from the surface. It is valid since, according to equations (29), (30), (32), in the case $\omega \to 0$ the inequality $\text{Im}\varepsilon(\omega) \ll |\text{Re}\varepsilon(\omega)|$ between the imaginary and real parts of dielectric permittivity is fulfilled.

To calculate the reflection coefficient and the depth of field penetration, the dispersion equation for the transversal electromagnetic waves and the unknown functions in (29) should be accounted. To find the reflection in the semi-bounded problem the dielectric permittivity for the homogeneous case is usually used (see, e.g., [41] and references therein). In other words, a weak electromagnetic field (this means the opportunity to neglect the nonlinear response of the system on external field [41]) will not penetrate into the CS in the presence of the BEC for nuclei when $\omega \to 0$ and dissipation is negligible in the sense of the above mentioned inequality for the imaginary and real parts of dielectric permittivity.

Furthermore, if we proceed from the definition of the dynamic conductivity $\sigma(\omega)$ as a proportionality factor between the electric current density $J(k, \omega)$ and the electric field strength $E(k, \omega)$ in the weak inhomogeneity limit $k \to 0$ (see [41] for more details),

$$J(k, \omega) = \sigma(\omega)E(k, \omega),$$

(33)

taking into account equations (19), (21), (28), (32), we find in the limit $\omega \to 0$ that

$$J(k, \omega) = \frac{2e^2 n_{\text{BEC}}}{m_c\omega}E(k, \omega).$$

(34)

If we now consider the relation between $E(k, \omega)$ and the vector potential of the electromagnetic field $A(k, \omega)$ for the limit $q \to 0$: $E(k, \omega) = i\omega A(k, \omega)/c$, where $c$ is the speed of light, we find
in the limit $\omega \to 0$

$$J(k, \omega) = -\frac{z^2 e^2 n_{\text{BEC}}}{m_c} A(k, \omega). \quad (35)$$

As is easily seen, relation (35) is similar to the London’s equation for the superconducting electron current [52]. According to equations (19), (29), (33)–(35), in the general case of arbitrary frequencies $\omega$, the electric current density $J(k, \omega)$ in a weak electromagnetic field is defined by the vector potential $A(k, \omega)$ in the form

$$J(k, \omega) = J^{\text{BEC}}(k, \omega) + J^{\text{over}}(k, \omega), \quad (36)$$

$$J^{\text{BEC}}(k, \omega) = -\frac{z^2 e^2 n_{\text{BEC}}}{m_c} A(k, \omega), \quad (37)$$

$$J^{\text{over}}(k, \omega) = \frac{i\omega \sigma_{\text{st}} + \omega^2 \alpha(\omega)}{c} A(k, \omega). \quad (38)$$

Representation (36) is similar to the separation of the electric current density in the weakly inhomogeneous electromagnetic field for the superconducting and normal components in the “electronic superconductivity” theory (see, e.g., [53]). We note, that such classification is conditional, since the current density $J^{\text{over}}(k, \omega)$, as well as the quantities $\sigma_{\text{st}}$ and $\alpha(\omega)$, also depend on the existence of BEC in the CS. It follows from relations (36)–(38) that the superconductor behavior in a finite-frequency $\omega$ electromagnetic field will be changed significantly in comparison with the static case ($\omega \to 0$) (35), since in this limit dissipation is completely absent.

4. Conclusion

To the end we return shortly to the main obtained results. The frequency dependent dielectric permittivity for the CS with the BEC of nuclei is firstly investigated. We found the existence of the second order singularity of dielectric permittivity at low frequency $\omega \to 0$ for the temperatures below the BEC transition of nuclei. The obtained result is similar to the London’s model for “electronic” superconductor, but for the CS with the BEC of nuclei. It is the reason to introduce the notion “superconductivity of nuclei”. We also show that there is analogue of the Meissner effect for the system with BEC of nuclei: the external electromagnetic wave of a low frequency $\omega \to 0$ cannot penetrate in a sample.

We come to the conclusion that the “superconductivity of nuclei” occurs in the homogeneous and isotropic CS in the presence of BEC for nuclei. As follows from the above consideration, the “superconductivity of nuclei” is not directly related to the static conductivity $\sigma_{\text{st}}$ (30). The last one is defined by the interaction of electrons with nuclei in the “overcondensate” states. Thus, the transition to the state of the “superconductivity of nuclei” is possible in principle for conductors, as well as for semiconductors and dielectrics. In this connection, we note that the insulator–superconductor transition is exist when considering the electronic superconductivity [54–57].

These statements are the direct consequences of the fact that the derivation of relations (32), (35) is based only on the definition of the BEC concept via ODLRO for the one-particle density matrix (by use the relations (1), (4) and (6) below some critical temperature of the second order transition in the state with the BEC of nuclei) without use of the perturbation theory on the interparticle interaction in the CS (in contrast to the electronic superconductivity theory).

In this sense relation (29) as well as the subsequent relations (36)–(38) are explicit (similarly to the the explicit relation for CS in equilibrium state, to compare see, e.g., [58], where we do not use the perturbation theory). However, it is obvious that specific calculations of the values of $n_{\text{BEC}}$, $\sigma_{\text{st}}$, and $\alpha(\omega)$ requires the use of these or those approximate methods. At the same time, the conclusion about superconductivity of nuclei is valid for arbitrary interaction parameters if only the system remains isotropic and homogeneous.
The sole essential assumption of the present consideration is related with the choice of a true boson in helium. In the accepted theory the role of the true boson plays the composed quasiparticle—helium atom. Since the Pauli theorem [59] about relation between spin and statistics is valid only for elementary particles, the question arises for what kind of the composed particles and interparticle interaction one can use the Bose–Einstein statistics (see, e.g., [60–62])

In this work we study the hypothesis that it is possible for nuclei, where the nuclear forces provide the bound state. We suppose that the statistics cannot depend from the intensity of Coulomb interaction between electrons and nuclei and creation of the atomic bound state. This hypothesis can explain the recent experiments in He II [10–12] and due to closeness of the atomic and nuclear masses of 4He has no contradiction with the theory of superfluidity, related in the case under consideration to Bose-condensation of He II nuclei.

At the same time, the problem of composite bosons is still open and needs deep investigations (see, e.g., [62–64] and references therein). Depending on correct choice of the true composite bosons the results of this work can be considered as a description of real physical systems or only as explicitly solvable model.

In the case of realization of the first opportunity, the BEC currently obtained in rarefied gases is strongly inhomogeneous and seems inappropriate for the “superconductivity of nuclei” straightforward observation. The superfluid He II is an appropriate object for experimental confirmation of the above results. It should also be emphasized that an experimental validation of the existence of the “superconductivity of nuclei” is of fundamental importance for solving the problem of the relation between the superfluidity phenomenon and the existence of BEC.

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