Polymer Bose–Einstein Condensates

E. Castellanos†
Departamento de Física, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, A. P. 14-740, 07000 México D. F., México.

G. Chacón-Acosta†
Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa, Artificios 40, México D. F. 01120, México

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In this work we analyze a non–interacting one dimensional polymer Bose–Einstein condensate in an harmonic trap within the semiclassical approximation. We use an effective Hamiltonian coming from the polymer quantization that arises in loop quantum gravity. We calculate the number of particles in order to obtain the critical temperature. The Bose–Einstein functions are replaced by series, whose high order terms are related to powers of the polymer length. It is shown that the condensation temperature presents a shift respect to the standard case, for small values of the polymer scale. In typical experimental conditions, it is possible to establish a bound for \( \lambda^2 \) up to \( \lesssim 10^{-16} \text{m}^2 \). To improve this bound we should decrease the frequency of the trap and also decrease the number of particles.

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I. INTRODUCTION

Several quantum gravity models suggest that the relation between energy and momentum of microscopic particles must be modified as a consequence of the quantum structure of space–time [1–9]. A deformed dispersion relation, emerges as an ideal tool in the search for possible effects related to the quantum structure of space–time. Nevertheless, the most difficult aspect in the search of experimental evidence relevant for the quantum gravity problem is the smallness of the possible effects [3, 4]. Unfortunately, due to this fact, the possible bounds for the deformation parameters, open a wide range of possible magnitudes, which implies a significant challenge.

Among the models that introduce such deformations, polymer quantum systems are simple quantum mechanical models quantized in a similar way as in loop quantum gravity [10, 11]. These systems provide scenarios where some characteristics of the full theory, specially the discrete nature of space, can be explored in a simpler context [12–16]. Indeed, in polymer quantum mechanics the momentum operator can not be defined, hence a regularized operator is proposed by the introduction of the so-called polymer length scale [10, 11]. Particularly, from this model an effective Hamiltonian which contains some trace of the granularity of space can be drawn [17].

The use of Bose–Einstein condensates as a tool in the search of quantum gravity effects, for instance, in the context of Lorentz violations, or to provide phenomenological constrains on Planck scale physics, has produced several interesting research [5–7, 12–22] (and references therein). In these works, the possible effects arising from Planck scale physics by looking at some modifications in the thermodynamic properties of Bose-Einstein condensates was explored. Since its observation with the help of magnetic traps [27–30], the phenomenon of Bose–Einstein condensation, from the experimental and theoretical point of view, has spurred an enormous amount of publications [30–51, 58, 59] (and references therein). In particular the condensates have been studied in different spatial dimensions [34–52]. Although, a one dimensional condensate can never be reached, it is possible to obtain a quasi one dimensional condensate, by using extremely anisotropic traps [34, 52–55]. Additionally, the condensation process of a bosonic gas in one dimension, trapped in a box or in an harmonic oscillator potential, has a very particular behavior. In this situation, apparently the condensation is not possible in the thermodynamic limit if one assumes that the ground state energy is zero, or equivalently, if the associated chemical potential is zero at the condensation temperature. However, taken into account the minimum energy associated to the system, the condensation process is possible at finite temperature [34, 35, 49, 59].

The aim of this work is to analyze the properties of a polymer one dimensional condensate trapped in a harmonic oscillator potential, within the semiclassical approximation, in order to obtain representative bounds for the polymer length scale. We will show also that for this one dimensional system the functional form of the relevant thermodynamic quantities, particularly, for the number of particles, can be expressed by series of the so–called Bose–Einstein functions. This series converges when the polymer scale is small enough. Finally, by the introduction of the ground state energy associated to this system, we calculate the shift in the usual condensation temperature caused by the polymer scale \( \lambda^2 \). These facts

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*Electronic address: ecastellanos@fis.cinvestav.mx
†Electronic address: gchacon@correo.cua.uam.mx
allow us to bound the polymer scale by using one-dimensional finite size systems.

II. POLYMER QUANTIZATION

Let us now present some of the main results of polymer quantization \[10, 11\]. This quantization arises from loop quantum gravity \[56\]. In the loop or polymer representation the corresponding Hilbert space \(\mathcal{H}_{\text{poly}}\) is spanned by the basis states \(\{|x_j\}\), whose coefficients have a suitable fall-off \[10\], with the following inner product

\[
\langle x_i | x_j \rangle = \delta_{i,j},
\]

where \(\delta_{i,j}\) is the Kronecker delta. The polymer Hilbert space can be represented as \(\mathcal{H}_{\text{poly}} = L^2(\mathbb{R}_d, d\mu_d)\), where \(d\mu_d\) is the corresponding Haar measure, being \(\mathbb{R}_d\) the real line endowed with the discrete topology, that is, the dual of the Bohr compactification of the real line \(\mathbb{R}\). The basic operators in this quantization are the position and translation. The position operator \(\hat{x}\) acts as usual by multiplication

\[
\hat{x}|x_j\rangle = x_j|x_j\rangle,
\]

while the translation operator \(\hat{V}(\lambda)\) moves to a position of arbitrary distance \(\lambda\)

\[
\hat{V}(\lambda)|x_j\rangle = |x_j - \lambda\rangle.
\]

In the Schrödinger quantization, the operator \(\hat{V}(\lambda)\) is weakly continuous in \(\lambda\) and the momentum operator is not well defined. Due to this fact, in the present work we restrict ourselves to the one-dimensional case.

III. CONDENSATION TEMPERATURE

Let us start with a one-dimensional effective polymer Hamiltonian \[17\], given by

\[
H = \frac{\hbar^2}{2m\lambda^2} \sin^2 \left( \frac{\lambda p_x}{\hbar} \right) + U(x)
\]

where \(U(x) = m\omega_t^2x^2/2\) is the one dimensional harmonic oscillator potential that model the trap, and \(\lambda\) is the so-called polymer length scale. Therefore, the semiclassical energy spectrum associated to \(n\) can be expressed as follows

\[
\epsilon_p = \frac{\hbar^2}{2m\lambda^2} \sin^2 \left( \frac{\lambda p_x}{\hbar} \right) + \frac{m\omega_t^2x^2}{2}.
\]

The associated one-dimensional spatial density can be written as \[34, 59\]

\[
n(x) = \frac{1}{2\pi\hbar} \int \frac{dp_x}{e^{\beta(\epsilon_p - \mu)} - 1},
\]

where, as usual, \(\beta = 1/\kappa T\), being \(\kappa\) the Boltzmann constant, and \(\mu\) is the corresponding chemical potential. Consequently, the total number of particles of the system is

\[
N = \int n(x)dx.
\]

Let us calculate the spatial density \(n(x)\) for the polymer case. To do so we need to calculate the integral \[6\] substituting the energy \[5\]

\[
n(x) = \frac{1}{2\pi\hbar} \int Z^{-1}(x) \exp \left( \frac{\beta \hbar^2}{2m\lambda^2} \sin^2 \left( \frac{\lambda p_x}{\hbar} \right) \right) - 1,
\]

where \(Z(x) = \exp \left[ \beta \left( \mu - \frac{m\omega_t^2x^2}{2} \right) \right] \). The integrand of \[8\] can be replaced by a geometric series

\[
n(x) = \frac{1}{2\pi\hbar} \int \sum_{j=1}^{\infty} \left( Z(x) e^{-\frac{\beta \hbar^2}{2m\lambda^2} \sin^2 \left( \frac{\lambda p_x}{\hbar} \right)} \right)^j dp_x.
\]
The last integral can be recognized as a modified Bessel function of first kind \[60\], if we introduce a regulator for the integral that corresponds to the polymer length \(\lambda\) \[15\], with the result

\[
n(x) = \frac{1}{2\sqrt{\pi}} \frac{\hbar}{\lambda} \sum_{j=1}^{\infty} Z^{j}(x) e^{-\frac{j^2\hbar^2}{4m\lambda^2}} I_0 \left( \frac{j^2\hbar^2}{4m\lambda^2} \right). \tag{11}\]

We obtain the total number of particles by integrating \[11\] over all space. The only \(x\)-dependent function is \(Z(x)\), and then the corresponding integration is straightforward

\[
N = \frac{1}{2} \sum_{j=1}^{\infty} \frac{\hbar}{\lambda} Z^{j}(x) e^{-\frac{j^2\hbar^2}{4m\lambda^2}} I_0 \left( \frac{j^2\hbar^2}{4m\lambda^2} \right). \tag{12}\]

When \(\lambda \ll 1\), we can use the asymptotic expansions of the modified Bessel functions \[61, 62\]

\[
I_0(u) \approx e^u \sqrt{2\pi u} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2u)^k k!} \Gamma \left( k + \frac{1}{2} \right). \tag{13}\]

Consequently, the total number of particles is given approximately by

\[
N = \frac{\kappa T}{\hbar \omega_x} \sum_{k=0}^{\infty} g_{k+1}(x) \left( -\lambda^2 \right)^k \left( \frac{2m\kappa T}{\hbar^2} \right)^k \frac{\Gamma \left( k + \frac{1}{2} \right)}{k! \Gamma \left( \frac{1}{2} - k \right)}. \tag{14}\]

where \(g_{k+1}(x)\) are the so-called Bose–Einstein functions, being \(z = e^{\beta\mu}\) the fugacity \[58\]. Series \[14\] is not convergent for arbitrary values of the polymer length. However, the approximation \[13\] is valid only for \(\lambda \ll 1\). We notice that the leading term turns to be \(g_1(x)\), the standard result \[34, 58\]. The second term can be considered as a correction of order \(\lambda^2\), and is proportional to \(g_2(x)\)

\[
N = \frac{\kappa T}{\hbar \omega_x} \left[ g_1(x) + \lambda^2 \frac{2m\kappa T}{\hbar^2} g_2(x) + O(\lambda^4) \right]. \tag{15}\]

It is well known that a one-dimensional condensate cannot exist in the thermodynamic limit, due to the divergent behavior of the Bose–Einstein function \(g_1(x)\). Nevertheless, if we take into account the ground state energy of the system, we are able to obtain a well defined condensation temperature \[34, 48, 58\]. The ground state energy associated with our system was already calculated in \[10, 13, 10\] and is given by

\[
\epsilon_0 = \hbar \omega_x \frac{d^2}{\lambda^2} \left( 1 + \frac{\lambda^4}{8d^4} a_0 \left( \frac{4d^4}{\lambda^4} \right) \right), \tag{16}\]

where \(d^2 = \hbar/ma^2\), is the characteristic length of the one-dimensional oscillator, and \(a_0(x)\) is the Mathieu characteristic function \[64\]. However, in the limit \(\lambda \ll 1\) one can regain the usual ground state energy plus corrections caused by the polymer scale

\[
\epsilon_0 = \frac{\hbar \omega_x}{2} - \frac{\lambda^2}{32} m \omega_x^2 + O(\lambda^4). \tag{17}\]

At the condensation temperature, the value of chemical potential \(\mu\) takes the minimum energy associated to the system. Substituting \[17\] into \[15\] and by using the properties of Bose–Einstein functions when \(\epsilon_0/\kappa T < 1\), \[58\] we can reexpress \[15\] at the condensation temperature as follows

\[
N = \frac{\kappa T_c}{\hbar \omega_x} \left[ g_1(z_c) + \lambda^2 \frac{m\kappa T_c}{2\hbar^2} g_2(z_c) + O(\lambda^4) \right], \tag{18}\]

where \(z_c = e^{-\epsilon_0/\kappa T_c}\).

Setting \(\lambda = 0\) we recover the usual expression for the number of particles \[34, 55\]

\[
N = \frac{\kappa T_0}{\hbar \omega_x} \ln \left( \frac{2\kappa T_0}{\hbar \omega_x} \right), \tag{19}\]

where \(T_0\) is the usual condensation temperature in one dimension without polymer modifications. By using an iterating procedure \[49\], the condensation temperature turns to be

\[
\kappa T_0 \approx h \omega_x N \ln \frac{2N}{N}. \tag{20}\]

Bose–Einstein condensates in one or two dimensions have been extensively studied \[34, 53, 54, 55\] (and references therein). The one dimensional condensation is seemingly not possible in the thermodynamic limit, in other words, the condensation temperature tends to zero when the number of particles tends to infinity \[34, 33, 54, 53, 50\]. Nevertheless, if one takes into account the associated ground energy, the condensation is possible at finite temperature. Finite size effects are needed in one dimensional systems, in order to make the condensation possible.

In order to calculate the condensation temperature \(T_c\), from \[15\], we introduce the following well known expressions for the Bose–Einstein functions \[58\]

\[
g_1(-e^{\mu c}/\kappa T_c) \approx \ln(\kappa T_c/\epsilon_0), \tag{21}\]

\[
g_2(-e^{\mu c}/\kappa T_c) \approx \zeta(2) - \frac{\kappa T_c}{\epsilon_0} \left[ 1 + \ln \left( \frac{\kappa T_c}{\epsilon_0} \right) \right]. \tag{22}\]

After some algebraic manipulation, we are able to express the number of particles as follows

\[
N = \frac{\kappa T_c}{\hbar \omega_x} \ln \left( \frac{2\kappa T_c}{\hbar \omega_x} \right) \left( 1 - \frac{\lambda^2}{2d^2} \right) - \frac{7}{8} \lambda_c^2 \frac{\kappa T_c}{\hbar \omega_x} \left( 1 - \zeta(2) \frac{8}{7} \frac{\kappa T_c}{\hbar \omega_x} \right) + O(\lambda^4), \tag{23}\]

where \(\lambda_c^2 = 2\pi l^2/m\kappa T_c\) is the standard one dimensional thermal wave length. Notice that the leading term is identical to \[19\], with \(T_c\) instead of \(T_0\). From \[23\], we can notice that there are two kinds of corrections to the number of particles, associated with the polymer length. One is due to the ratio \(\lambda^2/d^2\), that is related to the effective size of the ground state of the harmonic trap. The
second kind of correction is due to $\lambda^2/\Lambda^2$, this term can be interpreted as a pseudo–interaction within the system [22].

To calculate the shift caused by the polymer scale $\lambda^2$, let us expand [23] at first order in $\lambda^2 = 0$ and $T_c = T_0$, recalling that $T_0$ is the usual condensation temperature given by equation (20). The resulting shift $\Delta T_c$ is

$$\frac{T_c - T_0}{T_0} = \frac{\Delta T_c}{T_0} \approx -\frac{\lambda^2}{2d^2} \left[ \ln 2N - \frac{7}{8} + \frac{2N}{\ln 2N} \right].$$

(24)

Notice that the possibility to obtain a measurable correction associated to the polymer scale $\lambda^2 (\delta T_c^2)$ requires that, if $\Delta(T_c)$ is the experimental error, then $\Delta(T_c) \lesssim |\delta T_c^2|$, which in our case this entails

$$\Delta(T_c) \lesssim \left| \frac{\lambda^2}{2d^2} \left[ \ln 2N - \frac{7}{8} + \frac{2N}{\ln 2N} \right] \right|. \quad \text{(25)}$$

In order to obtain representative bounds associated with the polymer scale, let us appeal to the current high precision experiments for $\frac{49}{19}K$ in a 3–Dim condensate [31]. Although, from the experimental point of view there is no a real one dimensional condensate, it is possible to construct a quasi one dimensional condensate just by using extremely anisotropic traps [34]. Then, in principle, the use of this experimental accuracy is justified to bound the polymer scale $\lambda^2$. The shift in the condensation temperature with respect to the ideal result, caused by the interactions among the constituents of the gas is about $5 \times 10^{-2}$ with a $1\%$ of error [31]. For instance, under typical conditions the number of particles is about $N \sim 10^6 - 10^5$ in three dimensions [34]; for one dimensional systems, the number of particles can be estimated up to $N \sim 10^4$ [24, 54, 55]. Using typical frequencies of order 100 Hz, allows us to bound the polymer scale up to $\lambda^2 \lesssim 10^{-16} m^2$, which is notable. Indeed, as far as we know, this is the first bound associated to $\lambda$, from low energy, earth based experiments.

Notice that, if we increase the number of particles for a fixed trap frequency of order $10^3$Hz, the bound associated to the polymer scale decreases. On the other hand, for a fixed number of particles of order $\sim 10^4$, again the bound decreases, when increasing $\omega$. These facts suggest, that finite size systems are required in order to obtain representative bounds for $\lambda$.

IV. DISCUSSION

Quantum gravity phenomenology suggests that the quantum structure of space-time could have some effect on the dynamics of particle motion. These effects can be modeled, for instance, as deformations of dispersion relations. Particularly, the quantization that arises in loop quantum gravity applied to the motion of a quantum particle in one dimension (known as polymer quantization), gives a regularized Hamiltonian together with the introduction of a length parameter known as polymer scale. There are many proposals in the experimental search of these effects, however, is of special interest to study the Bose–Einstein condensates as they experimentally offer high accuracy measurements, which allows to establish bounds to the polymer length scale.

The main goal of this paper was to constrain the polymer length parameter by studying an non–interacting one dimensional polymer Bose–Einstein condensate in an harmonic trap within the semiclassical approximation. We have proved that the corresponding expression (14) generalize the so-called Bose–Einstein functions that appears in the standard case, to a series in terms of powers of the polymer length scale. Using typical experimental values for the number of particles in the condensate, the mass, and the frequency of the harmonic trap, we were able to establish a bound for the polymer length up to $\lambda^2 \lesssim 10^{-16} m^2$. To improve this bound, finite size systems in one dimension are required.

Notice that if we consider the ratio $(\lambda/d)^2$ as an effective dimensionless parameter then in principle, we are able to compare with the deformation parameters suggested in other approaches [23–25] by using condensates. We can see that the order of magnitude are quite similar and opens the possibility to relate different scenarios as was recently suggested [57].

It is possible to consider more general traps like power law potentials, the 3D case, and interactions among the constituents of the gas, to improve further the bound on the polymer scale, and will be presented elsewhere [20].

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[1] G. Amelino-Camelia, Int. J. Mod. Phys. D9, 1663 (2003).

[2] G. Amelino-Camelia, *Quantum-Gravity Phenomenology,*
[3] V. A. Kostelecký, R. Lehnert, Phys. Rev. D 63, 065008, (2001).
[4] G. Amelino-Camelia, Lect. Notes Phys. 541, 1 (2000); Quantum Gravity Phenomenology, [arXiv:0806.0339]
[5] G. Amelino-Camelia, C. Laemmerzahl, F. Mercati, and G. M. Tino, Constraining the Energy–Momentum Dispersion Relation with Planck–Scale Sensitivity Using Cold Atoms, Phys. Rev. Lett. 103, 171302 (2009)
[6] F. Mercati, D. Mazon, G. Amelino-Camelia, J.M. Carmona, J.L. Cortes, J. Indurain, C. Lammerzahl, and G.M. Tino, Class. Quant. Grav. 27, 215003 (2010).
[7] G. Amelino-Camelia, [gr-qc/0808029] Nature 398, 216 (1999).
[8] L. Smolin, Three roads to quantum gravity (Basic Books, 2002).
[9] J. Alfaro, H.A. Morales-Tecotl, L.F. Urrutia, Phys. Rev. D66, 124006 (2002).
[10] A. Ashtekar, S. Fairhurst and J. L. Willis, Class. Quantum Grav. 20 1031 (2003).
[11] G. M. Tino, Condensate Fluctuations of a Trapped, ideal Bose Gas, Phys. Rev. Lett. 87, 130402, (2001).
[12] G. Chacón-Acosta, E. Manrique, L. Dagdug and H. Morales-Técotl, in progress.
[13] G. Chacón-Acosta, in progress.
[14] G. Chacón-Acosta, L. Dagdug and H. Morales-Técotl, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA) 7, 110 (2011).
[15] G. Chacón-Acosta, E. Manrique, L. Dagdug and H. Morales-Técotl, Class. Quantum Grav. 24, 2603 (2007).
[16] G. Chacón-Acosta, E. Manrique, L. Dagdug and H. Morales-Técotl, AIP Conf. Proc. 1396, 99 (2011).
[17] D. Colladay and P. McDonald, Statistical Mechanics and Lorentz Violation, Phys. Rev. D70 (2004), 125007.
[18] D. Colladay and P. McDonald, Bose–Einstein condensates as a probe for Lorentz violation, Phys. Rev. D 73 (2006), 105006.
[19] A. Camacho, White Dwarfs as Test Objects of Lorentz Violations, Class. Quantum Grav. 23 (2006), pp. 7355-7368.
[20] E. Castellanos, A. Camacho, Critical Points in a Relativistic Bosonic Gas Induced by the Quantum Structure of Spacetime, Gen. Rel. Grav. 41, 2677-2685, (2009).
[21] E. Castellanos, A. Camacho, Stability of Bose–Einstein Condensates in a Lorentz Violating Scenario, Modern Physics Letters A, Vol. 25, No. 6, 459–469, (2010).
[22] E. Castellanos, C. Laemmerzahl, Modified Bosonic Gas Trapped in a Generic 3-dim Power Law Potential, [arXiv:1202.3806] (2011).
[23] E. Castellanos, C. Laemmerzahl, Ideal–Modified Bosonic Gas Trapped in a Generic 3-dim Power Law Potential, Modern Physics Letters A Vol. 27, No. 31 (2012).
[24] E. Castellanos, Planck Scale Physics and Bogoliubov in a Bose–Einstein Condensate, [arXiv:1212.2700] (2012).
[25] E. Castellanos, G. Chacon–Acosta, in progress.
[26] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[27] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).
[28] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. 78, 985 (1997).
[29] M. R. Anderson et al, Science 269, 198–201, (1995).
[30] R. P. Smith, R. L. D. Campbell, N. Tammuz, and Z. Hadzibabic, Phys. Rev. Lett. 105, 250403 (2011).
[31] V. Bagnato, D. E. Pritchard, D. Kleppner, Bose-Einstein Condensation with a Finite number of Particles in a Particle Number in a Trapped Bose-Einstein Condensate, Phys. Rev. A 56, 455 (1997).
[32] L. Salasnich, Critical Temperature of an Interacting Bose Gas in a Generic Power-Law Potential Int. J. Mod. Phys. B 16, 2185 (2002).
[33] O. Zobay, Mean–field analysis of Bose–Einstein condensation in general power-law potentials J. Phys. B 37, 2593 (2004).
[34] A. Jaouadi, M. Telmini, and E. Charron, Bose–Einstein Condensation with a Finite number of Particles in a Bose–Einstein Condensate Fluctuations of a Trapped, ideal Bose Gas Phys. Rev. A 54, 5048 (1996).
[35] S. Grossmann and M. Holthaus, On Bose–Einstein condensation in harmonic traps, Phys. Lett. A 208 (1995).
[36] S. Grossmann and M. Holthaus, Microcanonical fluctuations of a Bose systems ground state occupation number, Phys. Rev. E 54, (1996).
[37] S. Grossmann and M. Holthaus, Fluctuations of the Particle Number in a Trapped Bose–Einstein Condensate, Phys. Rev. Lett. Vol 79 (1997).
[38] M. R. Andrews et al, Phys. Rev. Lett. 79, 553–556 (1997).
[39] M. R. Andrews et al, Phys. Rev. Lett. 80, 2967–556 (1998).
[40] V. I. Yukalov, Principal Problems in Bose–Einstein Condensation of Dilute Gases Laser Phys. Lett. 1, 435-461 (2004).
[41] V. I. Yukalov, Modified semiclassical approximation for trapped Bose gases Phys. Rev. A 72, 033608 (2005).
[42] V.I. Yukalov, Basics of Bose-Einstein Condensation, [arXiv:1105.4992v1] (2011).
[43] Vitaly V. Kocharovsky, Vladimir V. Kocharovsky, M. Holthaus, C. H. Raymond Ooi, A. Svidzinsky, W. Ketterle, and M. O. Scully, Fluctuations in Ideal and Interacting Bose–Einstein Condensates: From the laser phase transition analogy to squeezed states and Bogoliubov quasiparticles, [cond-mat.stat-mech] (2006).
[44] E. Karabulut, M. Koyuncu, M. Tomak, Physica A 389, 1371, (2010).
[45] Z. Yan, Phys. Rev. A 59, 4657 (1999).
[46] A. Gorlitz, et. al., Phys. Rev. Lett. 87, 130402, (2001).
[55] F. Schreck, et. al. Phys. Rev. Lett. 87, 080403, (2001).
[56] J. M. Velhinho, Class. Quantum Grav. 24, 3745 (2007).
[57] B. Majumder, S. Sen, Phys. Lett. B, 717, 291 (2012)
[58] R. K. Pathria, Statistical Mechanics (Butterworth, 1996).
[59] C. J. Pethick and H. Smith, Bose–Einstein Condensation in Diluted Gases (Cambridge Univ. Press, 2006).
[60] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions. Dover (1968).
[61] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7th Ed. Academic Press (2007).