On algebraic geometric and computer algebra aspects of mirror symmetry

N.M. Glazunov

Glushkov Institute of Cybernetics NAS,
03187 Glushkov prospect 40, Kiev-187, Ukraine,
fax: (38 044) 266-7418,
e-mail: glanm@yahoo.com

Abstract. We survey some algebraic geometric aspects of mirror symmetry and duality in string theory. Some applications of computer algebra to algebraic geometry and string theory are shortly reviewed.

1 Introduction

This paper aims to be accessible to those with no previous experience in algebraic geometry; only some basic familiarity with linear and polynomial algebra, group theory, topology and category theory will be assumed. Symmetry principles always played an important role in mathematics and physics. Development of theoretical physics in direction of string theory enlarged the context of symmetry considerations and included in it the notion of duality. String theory has following ingredients: (i) base space (open or closed string) $\Sigma$; (ii) target space $M$; (iii) fields: $X \rightarrow \Sigma \rightarrow M$; (iv) action $S = \int \mathcal{L}(X, \varphi)$. Let $G$ be a group such that $G \supset SU(3) \times SU(2) \times U(1)$. Recall that if $\mathcal{L}(G\Phi) = \mathcal{L}(\Phi)$ then $\mathcal{L}$ is $G$–invariant, or $G$–symmetry. In string theory one of the beautiful symmetries is the radius symmetry $R \rightarrow 1/R$ of circle, known as $T$–duality \cite{2,3} and \cite{4} and references there in. Authors of papers \cite{5,6} conjectured that a similar duality might exist in the context of string propagation on Calabi-Yau (CY) manifolds, where the role of the complex deformation on one manifold get exchanged with the Kähler deformation on the dual manifold. A pair of manifolds satisfying this symmetry is called mirror pair, and this duality is called mirror symmetry. From the point of view of physicist which did the remarkable discovery mirror symmetry is a type of duality that means that we may take two types of string theory and compactify them in two different ways and achieve ”isomorphic” physics \cite{7}. Or in the case of a pair of Calabi-Yau threefolds $(X, Y)$ P. Aspinwall are said \cite{8} that $X$ and $Y$ to be a mirror pair if and only if the type IIA string compactified on $X$ is ”isomorphic to” the $E_8 \times E_8$ heterotic string compactified on $Y$. If in the case $X$ is Calabi-Yau threefold then $Y$ is the product of a $K3$ surface and elliptic curve and the following data specifies the theory \cite{8}.
1. A Ricci-flat metric on $Y$.
2. A $B$–field $\in H^2(K3 \times E, \mathbb{R}/\mathbb{Z})$.
3. A vector bundle $V \rightarrow (K3 \times E)$ with a connection satisfying the Yang-Mills equations.
4. A dilation + axion, $\Phi \in C$.

C. Vafa defines the notion of mirror of a Calabi-Yau manifold with a stable bundle. Lagrangian and special Lagrangian submanifolds appear in this situation. Mathematicians also work hard upon the problems of mirror symmetry, although it is difficult in some cases to attribute to a researcher the identifier ”mathematician” or ”physicist”. V. Batyrev gives construction of mirror pairs using Gorenstein toric Fano varieties and Calabi-Yau hypersurfaces in these varieties [3]. M. Kontsevich in his talk at the ICM’94 gave a conjecture interpretation of mirror symmetry as a ”shadow” of an equivalence between two triangulated categories associated with $A_\infty$– categories [11]. His conjecture was proved in the case of elliptic curves by A. Polishchuk and E. Zaslow [11]. The aim of the paper is to provide a short and gentle survey of some algebraic aspects of mirror symmetry and duality with examples - without proofs, but with (a very restricted) guides to the literature.

2 Algebraic geometric preliminaries

This section gives a basic introduction to algebraic geometric aspects of mirror symmetry beginning with a description of how Calabi-Yau manifolds arise from ringed spaces. A more detailed treatment of algebraic geometric material of this section may be found in [12, 13], the terminology and notation of which will be followed here.

Let $X$ be a topological space and $\text{Cov}(X)$ an open covering of $X$. It is well known [13] that $\text{Cov}(X)$ forms the category. Let $\text{Cat}$ be a category (of sets, abelian groups, rings, modules). The presheaf is a contravariant functor $\mathcal{F}$ from $\text{Cov}(X)$ to $\text{Cat}$. If for example $\text{Cat}$ is the category $\text{Ab}$ of abelian groups then $\mathcal{F} : \text{Cov}(X) \to \text{Ab}$ is the presheaf of abelian groups. Elements $f \in \mathcal{F}(U)$ is called sections of the presheaf $\mathcal{F}$.

If $i : U \subset V$ then we shall denote by $\rho^V_U$ the morphism $\mathcal{F}(i) : \mathcal{F}(V) \to \mathcal{F}(U)$. Functor $\mathcal{F}$ is the contravariant and we apply morphisms from the left to the right. Hence for any open sets $U \subset V \subset W$

$$\rho^W_U = \rho^V_U \rho^W_V.$$

Let $U \subset X$ be any open subset of $X$ and $\bigcup U_\alpha = U$ its open covering. A presheaf $\mathcal{F}$ on a topological space $X$ is called the sheaf if the following conditions are satisfied:

1) if $\rho^U_{U_\alpha} s_1 = \rho^U_{U_\alpha} s_2$ for $s_1, s_2 \in \mathcal{F}(U)$ and for any $U_\alpha$, then $s_1 = s_2$.

2) if $s_\alpha \in \mathcal{F}(U_\alpha)$ are such that $\rho^U_{U_\alpha \cap U_\beta} s_\alpha = \rho^U_{U_\alpha \cap U_\beta} s_\beta$, then there exists $s \in \mathcal{F}(U)$ such that $s_\alpha = \rho^U_{U_\alpha} s$ for all $U_\alpha$.

The ringed space is the pair $(X, \mathcal{O})$, where $X$ is a topological space and $\mathcal{O}$ is a sheaf of rings on $X$. A morphism of ringed spaces

$$\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is the class of maps $(\varphi, \psi_U)$, where $\varphi$ is the continuous map $\varphi : X \to Y$, and $\psi_U$ is a homomorphism $\psi_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ for any open set $U \subset Y$ such that the diagram

$$\begin{diagram}
\mathcal{O}_X(\varphi^{-1}(V)) \arrow{e}{\rho^V_U} \arrow{s}{\psi_U} \mathcal{O}_X(\varphi^{-1}(U)) \\
\arrow{sw}{\psi_V}
\end{diagram}$$
is commutative for any $U$ and $V$, $U \subseteq V \subseteq Y$. For a ringed space $(X, \mathcal{O}_X)$ and an open $U \subseteq X$ the restriction of the sheaf $\mathcal{O}_X$ on $U$ defines the ringed space $(U, \mathcal{O}_{X|U})$.

Let $\mathcal{O}_X$ be the sheaf of smooth (differentiable) functions on a topological space $X$. Then any smooth (differentiable) manifold $X$ with the sheaf $\mathcal{O}_X$ is a ringed space $(X, \mathcal{O}_X)$. Respectively let $X$ be a Hausdorff topological space and $\mathcal{O}_X$ a sheaf on $X$. Let it satisfies conditions: (i) $\mathcal{O}_X$ is the sheaf of algebras over $\mathbb{C}$; (ii) $\mathcal{O}_X$ is a subsheaf of the sheaf of continuous complex valued functions. Let $W$ be a domain in $\mathbb{C}^n$ and $\mathcal{O}_{an}$ the sheaf of analytical functions on $W$. The ringed space $(X, \mathcal{O}_X)$ is called the complex analytical manifold if for any point $x \in X$ there exists a neighbourhood $U \ni x$ such that $(U, \mathcal{O}_{X|U}) \simeq (W, \mathcal{O}_{an})$ (here $\simeq$ denotes the isomorphism of ringed spaces).

The fibre space is the object $(E, p, B)$, where $p$ is the continuous surjective (= on) mapping of a topological space $E$ onto a space $B$ (in our consideration $B = M$ is a complex analytical manifold or algebraic variety), and $p^{-1}(b)$ is called the fibre above $b \in B$. Both the notation $p : E \to B$ and $(E, p, B)$ are used to denote a fibration, a fibre space, a fibre bundle or a bundle.

Vector bundle is fibre space each fibre $p^{-1}(b)$ of which is endowed with the structure of a (finite dimension) vector space $V$ over skew-field $K$ such that the following local triviality condition is satisfied: each point $b \in B$ has an open neighborhood $U$ and a $V$-isomorphism of fibre bundles $\phi : p^{-1}(U) \to U \times V$ such that $\phi|_{p^{-1}(b)} : p^{-1}(b) \to b \times V$ is an isomorphism of vector spaces for each $b \in B$.

$dim V$ is said to be the dimension of the vector bundle.

Let $X$ be a complex analytical manifold, $U$ an element of open covering of $X$, $z_1, z_2, \cdots z_n$ coordinates on $U$. Let $\varphi$ be the hermitian positively defined form on $X$, $\varphi(z, \bar{z}) = \sum c_{\alpha, \beta} dz_\alpha \cdot d\bar{z}_\beta$ the hermitian metric associated to $\varphi$. An Hermitian bundle over complex analytical manifold $X$ consists of a vector bundle over $X$ and a choice of $C^\infty$ Hermitian metric on the vector bundle over complex manifold $X(C)$, which is invariant under antiholomorphic involution of $X(C)$.

Let $\varphi(z, \bar{z})$ be a hermitian metric on the tangent bundle on $X$ and

$$\omega = (i/2) \sum c_{\alpha, \beta} dz_\alpha \cdot d\bar{z}_\beta$$

the two dimensional differential form associated to $\varphi(z, \bar{z})$.

The hermitian metric $\varphi$ on a complex analytical manifold $X$ is called kählerian if the differential form $\omega$ associated to $\varphi$ is closed. The Kähler manifold is the complex analytical manifold with a kählerian metric. We shall use in contrast to [1] some another definition of Calabi-Yau (CY) manifold. The definition is based on the theorem of Yau who proved Calabi’s conjecture that a complex Kähler manifold of vanishing first Chern class admits a Ricci-flat metric.

**Definition 1** A complex Kähler manifold is called Calabi-Yau (CY) manifold if it has vanishing first Chern class.

Examples of the CY-manifolds include, in particular, elliptic curves $E$, $K3$–surfaces and their products $E \times K3$. Let $(X, \omega, \Omega)$ be a complex manifold (real dimension $= 2n$) with

$$\omega^n/n! = (-1)^{n(n-1)/2} (i/2)^n \cdot \Omega \cdot \overline{\Omega}.$$ 

It is said that a $n$–dimensional submanifold $L \subseteq X$ is special Lagrangian (s-lag) $\iff$...
Example 1 Let $X$ be an elliptic curve $E$. Then $\omega = c(i/2)dz \wedge d\bar{z}$, $\Omega = cdz$. S-lag $L \subset E$ are straight lines with slope determined by $\arg c$.

Let $(U, \mathcal{O}_X|_U) \simeq \text{Spec}A$ for a commutative ring $A$. In the case the neighbourhood $U \ni x$ is called the affine neighbourhood of the point $x$.

The scheme $S$ is the ringed space $(X, \mathcal{O}_X)$ with the condition: for any point $x \in X$ there is an affine neighbourhood $V \ni x$ such that $(V, \mathcal{O}_X|_V) \simeq \text{Spec}A$.

2.1 Blow-ups

Blowing up is a well known method of constructing complex manifolds $M$. There are points on the manifolds that are not divisors on $M$. Blow up is the construction that transforms points of complex manifolds to divisors. For instance in the case of two dimensional complex manifolds (complex surface) $N$ it consists of replacing a point $p \in N$ by a projective line $\mathbb{P}(1)$ considered as the set of limit directions at $p$.

Example 2 Let $\pi : M_2 \to \mathbb{C}^2$ be the blow-up of $\mathbb{C}^2$ at the point $0 \in \mathbb{C}^2$. Then $M_2$ is a two dimensional complex manifold that defined by two local charts. In coordinates $\mathbb{C}^2 = (z_1, z_2)$, $\mathbb{P}(1) = \{[0, l_1] \}$ the manifold $M_2$ is defined in $\mathbb{P}(1) \times \mathbb{C}^2$ by quadratic equations $z_i l_j = z_j l_i$. Thus $M_2$ is a line bundle over Riemann sphere $\mathbb{P}(1)$. $\pi^{-1}(0) = \mathbb{P}(1)$ is called the divisor of the blow up (the exceptional divisor).

Recently a large class of CY orbifolds in weighted projective spaces have been proposed. C. Vafa have predicted and S. Roan [17] have computed the Euler number of all the resolved CY hypersurfaces in a weighted projective space $\mathbb{WCP}(4)$.  

2.2 Vector Bundles over Projective Algebraic Curves

Let $X$ be a projective algebraic curve over algebraically closed field $k$ and $g$ the genus of $X$. Let $\mathcal{VB}(X)$ be the category of vector bundles over $X$. Grothendieck have shown that for a rational curve every vector bundle is a direct sum of line bundles. Atiyah have classified vector bundles over elliptic curves. The main result is [18]:

Theorem 1 Let $X$ be an elliptic curve, $A$ a fixed base point on $X$. We may regard $X$ as an abelian variety with $A$ as the zero element. Let $\mathcal{E}(r, d)$ denote the the set of equivalence classes of indecomposable vector bundles over $X$ of dimension $r$ and degree $d$. Then each $\mathcal{E}(r, d)$ may be identified with $X$ in such a way that $\det : \mathcal{E}(r, d) \to \mathcal{E}(1, d)$ corresponds to $H : X \to X$, where $H(x) = hx = x + x + \cdots + x$ ($h$ times), and $h = (r, d)$ is the highest common factor of $r$ and $d$.

Curve $X$ is called a configuration if its normalization is a union of projective lines and all singular points of $X$ are simple nodes [19]. For each configuration $X$ can assign a non-oriented graph $\Delta(X)$, whose vertices are irreducible components of $X$, edges are its singular and an edge is incident to a vertex if the corresponding component contains the
**Theorem 2**

1. \( \mathcal{VB}(X) \) contains finitely many indecomposable objects up to shift and isomorphism if and only if \( X \) is a configuration and the graph \( \Delta(X) \) is a simple chain (possibly one point if \( X = \mathbb{P}^1 \)).
2. \( \mathcal{VB}(X) \) is tame, i.e. there exist at most one-parameter families of indecomposable vector bundles over \( X \), if and only if either \( X \) is a smooth elliptic curve or it is a configuration and the graph \( \Delta(X) \) is a simple cycle (possibly, one loop if \( X \) is a rational curve with only one simple node).
3. Otherwise \( \mathcal{VB}(X) \) is wild, i.e. for each finitely generated \( k \)-algebra \( \Lambda \) there exists a full embedding of the category of finite dimensional \( \Lambda \)-modules into \( \mathcal{VB}(X) \).

Let \( X \) be an algebraic curve. How to normalize it? There are several methods, algorithms and implementations for this purpose. A new algorithm and implementation is presented in [20].

### 3 Complexes, homotopy categories, cohomologies and quasiisomorphisms

Here we recall the relevant properties of complexes, derived categories, cohomologies and quasiisomorphisms referring to [13, 22] for details and indication of proofs.

The **cochain complex**

\[
(K^\bullet, d) = \{K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \xrightarrow{d} \cdots\}
\]

is the sequence of abelian groups and differentials \( d : K^p \rightarrow K^{p+1} \) with the condition \( d \circ d = 0 \). For a category \( \text{Cat} \) we denote by \( \text{ObCat} \) it’s objects and by \( \text{MorCat} \) it’s morphisms.

Let \( A \) be an abelian category, \( Kom(A) \) the category of complexes over \( A \). Furthermore, there are various full subcategories of \( Kom(A) \) whose respective objects are the complexes which are bounded below, bounded above, bounded in both sides. Now recall (by [22]) the notion of homotopy morphism.

**Lemma-definition 1**

(i) Let \( K^\bullet, L^\bullet \) be two complexes over abelian category \( A \), \( k = k^i, k^i : K^i \rightarrow L^{i-1} \) a sequence of morphisms between elements of the complexes.

Then the maps

\[
h^i = k^{i+1}d_K^i + d_L^{i-1}k^i : K^i \rightarrow L^i
\]

form the morphism of complexes

\[
h = kd + dk : K^\bullet \rightarrow L^\bullet.
\]

The morphism \( h \) is called homotopic to zero \( (h \sim 0) \).

(ii) morphisms \( f, g : K^\bullet \rightarrow L^\bullet \) is called homotopic, if \( f - g = kd + dk \sim 0 \) \( (f \sim g) \), \( k \) is called homotopy.

(iii) If \( f \sim g \), then \( H^\bullet(f) = H^\bullet(g) \), where the map \( H^\bullet \) is induced on cohomologies of complexes.

The **homotopy category** \( K(A) \) is defined by the following way:
Mor K(A) = Mor Kom(A) by the module of homotopy equivalence.

Let X be a topological space and $K^\bullet, L^\bullet$ be complexes of sheaves over X. The quasiisomorphism is the map

$$f : K^\bullet \rightarrow L^\bullet$$

which induces the isomorphism of cohomological sheaves

$$f_* : \mathcal{H}^q(K^\bullet) \rightarrow \mathcal{H}^q(L^\bullet), \ q \geq 0.$$

4 Connections

Consider the connection in the context of algebraic geometry. Let $S/k$ be the smooth scheme over field $k$, $U$ an element of open covering of $S$, $\mathcal{O}_S$ the structure sheaf on $S$, $\Gamma(U, \mathcal{O}_S)$ the sections of $\mathcal{O}_S$ on $U$. Let $\Omega^1_{S/k}$ be the sheaf of germs of 1−dimension differentials, $\mathcal{F}$ be a coherent sheaf. The connection on the sheaf $\mathcal{F}$ is the sheaf homomorphism

$$\nabla : \mathcal{F} \rightarrow \Omega^1_{S/k} \otimes \mathcal{F},$$

such that, if $f \in \Gamma(U, \mathcal{O}_S)$, $g \in \Gamma(U, \mathcal{F})$ then

$$\nabla(fg) = f \nabla(g) + df \otimes g.$$ 

There is the dual definition. Let $\mathcal{F}$ be the locally free sheaf, $\Theta^1_{S/k}$ the dual to sheaf $\Omega^1_{S/k}$, $\partial \in \Gamma(U, \Theta^1_{S/k})$. The connection is the homomorphism

$$\rho : \Theta^1_{S/k} \rightarrow \text{End}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}).$$

$$\rho(\partial)(fg) = \partial(f)g + f \rho(\partial).$$

4.1 Integration of connections

Let $\Omega^i_{S/k}$ be the sheaf of germs of $i$−dimensional differential forms on $S$. Particularly $\Omega^1_{S/k}$ is the cotangent bundle over $S$. Let $\omega \in \Omega^i_{S/k}, \ f \in \Gamma$ and

$$\nabla_i(\omega \otimes f) = d\omega \otimes f + (-1)^i \omega \wedge \nabla(f).$$

Hence, $\nabla_i$ define the sequence of homomorphisms of sheaves

$$\mathcal{F} \xrightarrow{\nabla} \Omega^1_{S/k} \otimes \mathcal{F} \xrightarrow{\nabla} \Omega^2_{S/k} \otimes \mathcal{F} \rightarrow \ldots$$

The sequence is the complex if $\nabla \circ \nabla_1 = 0$. In this case the connection $\nabla$ is integrable.

Example 3 Let $\mathcal{F} = \mathcal{O}_S$ be the structural sheaf on $S$. Then

$$\nabla : \mathcal{O}_S \rightarrow \Omega^1_{S/k} \otimes \mathcal{O}_S \simeq \Omega^1_{S/k}.$$ 

Hence $\nabla(f) = df$, $\rho : \Theta_{S/k} \rightarrow \mathcal{O}_S$. This connection $\nabla$ is integrable because it defines the de Rham complex

$$\Omega^\bullet \otimes \mathcal{O}_S \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \ldots$$
Example 4 Let $\mathcal{LC}$ be a locally constant sheaf on $S/k$ such that $\mathcal{LC} \simeq k^n$ (local coefficients) as sheaves. Let $\mathcal{F} = \mathcal{LC} \otimes \mathcal{O}_S$, $v \in \Gamma(U, \mathcal{LC})$, $f \in \Gamma(U, \mathcal{F})$. Then there is a canonical connection $\nabla(v \otimes f) = df \otimes v$:

$$\nabla : \mathcal{F} \to \Omega^1_S \otimes \mathcal{F}.$$ 

For a connection $\nabla : \mathcal{F} \to \Omega^1_S \otimes \mathcal{F}$ a section $s \in \Gamma(U, \mathcal{F})$ is horizontal if $\nabla(s) = 0$.

Let now $S$ be a complex manifolds, $\text{ShC}(S/k)$ the category of sheaves with a connection and $\text{LC}(S)$ the category of local coefficients over $S$. Let $(\mathcal{F}, \nabla) \in \text{ShC}(S/k)$ be a sheaf with connection. We can define the functor

$$F^n : (\mathcal{F}, \nabla) \mapsto \{\text{the sheaf of germs of horizontal sections of } \mathcal{F}\}.$$ 

and it’s inverse $F^n^{-1}$.

**Proposition 1** The functors $F^n$ and $F^n^{-1}$ give the equivalence of category $\text{ShC}(S/k)$ and $\text{LC}(S)$.

5 Moduli spaces in string theory

Mirror symmetry connects with geometrical deformations of complex and Kähler structures on CY-manifolds. So we have to know moduli spaces of complex and Kähler structures on CY-manifolds.

5.1 Moduli spaces

The theory of moduli spaces [15, 16] has, in recent years, become the meeting ground of several different branches of mathematics and physics - algebraic geometry, instantons, differential geometry, string theory and arithmetics. Here we recall some underlying algebraic structures of the relation. In previous section we have reminded the situation with vector bundles on projective algebraic curves $X$. On $X$ any first Chern class $c_1 \in H^2(X, \mathbb{Z})$ can be realized as $c_1$ of vector bundle of prescribed rank (dimension) $r$. How to classify vector bundles over algebraic varieties of dimension more than 1? This is one of important problems of algebraic geometry and the problem has closed connections with gauge theory in physics and differential geometry. Mumford [15] and others have formulated the problem about the determination of which cohomology classes on a projective variety can be realized as Chern classes of vector bundles? Moduli spaces are appeared in the problem. What is moduli? Classically Riemann claimed that $3g - 3$ (complex) parameters could be for Riemann surface of genus $g$ which would determine its conformal structure (for elliptic curves, when $g = 1$, it is needs one parameter). From algebraic point of view we have the following problem: given some kind of variety, classify the set of all varieties having something in common with the given one (same numerical invariants of some kind, belonging to a common algebraic family). For instance, for an elliptic curve the invariant is the modular invariant of the elliptic curve.

Let $\mathbf{B}$ be a class of objects. Let $\mathbf{S}$ be a scheme. A family of objects parametrized by the
$X_s : s \in S, X_s \in B$

equipped with an additional structure compatible with the structure of the base $S$. Parameter varieties is a class of moduli spaces. These varieties is very convenient tool for computer algebra investigation of objects that parametrized by the parameter varieties. We have used the approach for investigation of rational points of hyperelliptic curves over prime finite fields [24].

**Example 5** Let $\omega_1, \omega_2 \in \mathbb{C}$, $\text{Im}(\omega_1/\omega_2) > 0$, $\Lambda = n\omega_1 + m\omega_2$, $n, m \in \mathbb{Z}$ be a lattice. Let $H$ be the upper half plane. Then $H/\Lambda = E$ be the elliptic curve. Let

$$y^2 = x^3 + ax + b = (x - e_1)(x - e_2)(x - e_3), \quad 4a^3 + 27b^2 \neq 0,$$

be the equation of $E$. Then the differential of first kind on $E$ is defined by formula

$$\omega = dx/y = dx/(x^3 + ax + b)^{1/2}.$$

**Periods of $E$**:

$$\pi_1 = 2 \int_{e_2}^{e_2} \omega, \quad \pi_2 = 2 \int_{e_3}^{e_3} \omega.$$

The space of moduli of elliptic curves over $\mathbb{C}$ is $\mathbb{A}^1(\mathbb{C})$. Its completion is $\mathbb{C}P(1)$.

For $K3$–surfaces the situation is more complicated but in some case is analogous [21]:

**Theorem 3** The moduli space of complex structure on market $K3$–surface (including orbifold points) is given by the space of possible periods.

Some computational aspects of periods and moduli spaces are considered in author’s note [25].

### 6 Some categorical constructions

Every compact symplectic manifold $Y, \omega$ with vanishing first Chern class, one can associate a $A_\infty$–category whose objects are essentially the Lagrangian submanifolds of $Y$, and whose morphisms are determined by the intersections of pairs of submanifolds. This category is called Fukaya’s category and is denoted by $\mathcal{F}(Y)$ [10]. Let $(X, Y)$ be a mirror pair. Let $M$ be any element of the mirror pair. The bounded derived category $D^b(M)$ of coherent sheaves on $M$ is obtained from the category of bounded complexes of coherent sheaves on $M$ [22]. In the case of elliptic curves A. Poleshchuk and E. Zaslov have proved [11]:

**Theorem 4** The categories $D^b(E_q)$ and $\mathcal{F}^0(\overline{E}^t)$ are equivalent.

Recently A. Kapustin and D. Orlov have suggested that Kontsevich’s conjecture must be modified: coherent sheaves must be replaced with modules over Azumaya algebras, and the Fukaya category must be ”twisted” by closed 2-form [23].
7 Computer algebra aspects

Computer algebra applications to classical algebraic geometry are well known. Most of them are based on the method of Gröbner bases [26, 27]. They include the decomposition of algebraic varieties, rational parametrization of curves and surfaces [28, 29], inversion of birational maps [30], the normalization of affine rings [24]. Some computer algebra results presented on CAAP-2001 can be used for computations in algebraic geometry and string theory. These are results on computation of toric ideals [31] and on computation of cohomology [32]. Talks of V. Gerdt include also result on computation in Yang-Mills mechanics [33]. Some recent papers include description of efficient algorithms computing the homology of commutative differential graded algebras [34] and computing the complete Hopf algebra structure of the 1-homology of purely quadratic algebra [35].

For the future research let us mention that it might be as well to have a tool, namely computer algebra for computation with (i) various moduli spaces; (ii) deformations (deformation of complex structure and deformation of Kähler structure); (iii) $A_\infty$-categories; (iv) geometric Fourier transform.

Conclusions

In the paper we tried to give an algebraic geometric framework for some aspects of mirror symmetry. This framework includes rather restricted context of mirror symmetry and string theory. But it is based on a simple and unified mathematical base. Some applications of computer algebra to algebraic geometry and string theory are shortly reviewed.

Acknowledgements

I would like to thanks the organizers of the CAAP’2001 (Dubna), SymmNMPh’2001 (Kiev) and SAGP’99 (Mirror Symmetry in String Theory, CIRM, Luminy) for providing a very pleasent environment during the conferences.

References

[1] B. Greene, String theory on Calabi-Yau manifolds, Preprint, 1997, hep-th/9702153.

[2] L. Dixon, Lecture at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology. Trieste, 1987.

[3] W. Lerche, C. Vafa, and N. Warner, Chiral rings in N=2 superconformal theories, Nucl. Phys., 1989, B324, pp. 421-430.

[4] A. Giveon, M. Porrati,E. Rabinovici, Target Space Duality in String Theory, Phys.Rept., 1994, N 244, pp. 77-202.

[5] P. Candelas, M. Lynker, R. Schimmrank. Nucl. Phys., 1990, B330, p. 49.

[6] B. Greene, R. Plesser, Nucl. Phys., 1990, B338, p. 15.
[7] C. Vafa, Geometric Physics, Proceedings of ICM’98, Berlin, Documenta Math. J. DMV, 2000, pp. 430-439.

[8] P. Aspinwall, String theory and duality, ibid, pp. 229-238.

[9] V. Batyrev, Mirror symmetry and toric geometry, ibid, pp. 239-248.

[10] M. Kontsevich, Homological algebra of mirror symmetry, Proceedings of ICM’94, Zurich, Birkhauser, 1995, pp. 120-139.

[11] A. Polishuk, E. Zaslow, Categorical mirror symmetry: the elliptic curves, Adv. Theor. Math. Phys., 1998, N 2, pp. 443-470.

[12] I.R. Shafarevitch. Foundations of Algebraic Geometry. Moscow: Nauka, 1988, v.1, v.2 (in Russian).

[13] Ph. Griffiths, J. Harris. Principles of algebraic geometry. A Wiley Int. Pub. N.Y., 1978.

[14] R. Godement. Topologie algebrique et theorie des faisceaux. Hermann, Paris.

[15] D. Mamford, Towards an enumerative geometry of the moduli space of curves. Arithmetic and Geometry Vol. II, Progress in Math., 1983.

[16] J. Harris, J. Morrison, Moduli of curves, Berlin-N.Y, Springer, 1998.

[17] S. Roan, Algebraic geometry and physics, Preprint, 2000, math-ph/0011038

[18] M. Atiyah, Vector bundles over elliptic curves. Proc. London Math. Soc., 1957, Vol. vii, N 27, pp. 414-452.

[19] Ju. Drozd, G. Greuel, On vector bundles over projective curves, ”Representation Theory and Computer Algebra”, Proc. of Int. Conference, (March 18-23, 1997, Kyiv) Kyiv, Institute of Mathematics, 1997, p. 41.

[20] W. Decker, T. de Jong, G. Greuel, G. Pfister, The normalization: A new algorithm, implementation and comparisons. ”Computational methods for representations of groups and algebras”. Proceedings of the Euroconference, (April 1-5, 1997, Essen) Basel, Birkhäuser, 1999, pp. 177-185

[21] W. Barth, C. Peters, and van der Ven., Compact complex surfaces, New York, Springer, 1984.

[22] S.I. Gelfand, Yu. I. Manin, Methods of homological algebra, Moscow, Nauka, 1988.

[23] A. Kapustin, D. Orlov, Remarks on A-branes, Mirror Symmetry, and the Fukaya category, Preprint, 2001, hep-th/0109098

[24] N.M. Glazunov, On moduli spaces, equidistribution, bounds and rational points of algebraic curves. Ukrainian Math. Journ., 2001, N 9, pp. 1174-1183.
[25] N.M. Glazunov, Computation of Moduli, Periods and Modular Symbols. IMACS-ACA’98. Session ”Algebra and Geometry”. Prague, 1998. [http://math.unm.edu/ACA/1998/sessions/geom/glazunov/index.html](http://math.unm.edu/ACA/1998/sessions/geom/glazunov/index.html)

[26] Buchberger B. An algorithm for finding a bases. PhD thesis, Universitst Innsbruck, Institut fur Mathematik, 1965.

[27] Buchberger B. Groebner bases: an algorithmic method in polynomial ideal theory. Recent Trends in Multidimensional Systems Theory, chapter 6. D. Riedel Publ. Comp, 1985.

[28] J. Sendra, F. Winkler. Symbolic parametrization of curves. J. Symb. Comput., 1991, 12, pp. 607-632.

[29] J. Schicho. Rational parametrization of surfaces. J. Symb. Comput., 1998, 26, pp. 1-29.

[30] J. Schicho. Inversion of Birational Maps with Gröbner Bases. Lond. Math. Soc. Lect. Note Ser. Cambridge: Cambridge Univ. Press. 1998, 251, pp. 495-503.

[31] V.P. Gerdt, Yu.A. Blinkov. Janet trees in computing of toric ideals. Computer algebra and its applications to physics (CAAP-2001), Dubna, 2001, p. 19.

[32] V.V. Kornyak. Extraction of ”minimal” cochain subcomplexes. ibid, p. 10.

[33] V.P. Gerdt, A.M. Khvedelidze, D.M. Mladenov. On light-cone version of SU(2) Yang-Mills mechanics. ibid, p. 12.

[34] V. Álvarez, J. Armario, P. Real, B. Silva. Homological perturbation theory and computability of Hochschild and cyclic homologies of CDGAS. Int. Conf. on Secondary calculus and cohomological physics, Moscow, 1997, pp. 1-15.

[35] M. Jimenez, P. Real. ”Coalgebra” structures on 1-homological models for commutative differential graded algebras. preprint, 2000, 15 p.