Appearence of Mother Universe and Singular Vertices in Random Geometries

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We discuss a general mechanism that drives the phase transition in the canonical ensemble in models of random geometries. As an example we consider a solvable model of branched polymers where the transition leading from tree-like to bush-like polymers relies on the occurrence of vertices with a large number of branches. The source of this transition is a combination of the constraint on the total number of branches in the canonical ensemble and a nonlinear one-vertex action. We argue that exactly the same mechanism, which we call constrained mean-field, plays the crucial role in the phase transition in 4d simplicial gravity and, when applied to the effective one-vertex action, explains the occurrence of both the mother universe and singular vertices at the transition point when the system enters the crumpled phase.

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Introduction

Simplicial quantum gravity in four dimensions is known to have two geometrically distinct phases called, in a manner reflecting their basic geometrical features, respectively the elongated and the crumpled phase. For large values of the coupling constant, frequently denoted by $\kappa_2$, that controls the
curvature of the universe and, roughly speaking, corresponds to a regularized version of the inverse Newton constant, the typical universe has a tree–like shape. The branches of this tree consist of so–called “baby universes” that are linked to each other by very narrow bottlenecks resembling the wormholes known from other considerations of Euclidean Quantum Gravity [3]. In this phase, simplicial gravity has been shown to behave like an ensemble of branched polymers with the characteristic values of a Hausdorff dimension $d_H = 2$ and an entropy exponent $\gamma = 1/2$ [4]. For $\kappa_2$ below the critical value [5], simplicial gravity enters the crumpled phase. So–called singular vertices occur on the typical geometry, vertices of a large order that grows linearly with the volume of the ensemble [6,7]. Because of this, almost the whole universe is in the neighbourhood of these singular vertices, which means that the average geodesic distances between simplices depends weakly if at all on the universe’s volume. It is therefore frequently stated that the crumpled phase has a very large or even infinite Hausdorff dimension. The appearance of the singular vertices coincides with a dramatic change in the tree of baby universes; in the elongated phase all the baby universes have the same status and none of them is favoured, whereas here in the crumpled phase one of them becomes a “mother universe” which has a much larger volume than the others and has many baby universes directly linked to it. As we will show, this effect can be easily understood in terms of a constrained mean–field used as an effective model for the tree of baby universes. There is also a clear correlation between the appearance of singular vertices and the typical baby universe structure, as was for example shown in [8] where the correlations between the maximal order of the vertex and the average minbu size were observed at the transition point. It is, for instance, seen that the mother universe always contains the singular vertex. Because of these correlations, we will argue that the occurrence of singular vertices is also caused by the mechanism discussed here.

Branched Polymers and Constrained Mean–field Models

It has been well established by now that the branched polymer model very accurately describes the elongated phase of 4d simplicial gravity [4,9,10]. We will argue that this description in terms of branched polymers can be extended beyond the phase transition into the crumpled phase. First, however, let us define this model and its generalizations, which we call constrained mean–field models.

The branched polymer model describes an ensemble of abstract trees which link different vertices via branches in such a way that there are no closed loops on the corresponding graph. To each vertex we assign a vertex order or branching number $n$, which is defined as the number of branches emerging from this vertex. There is a certain given probability distribution $p(n)$ for the
branching number at each vertex, but the numbers of branches at any two different vertices are independent. If, however, one considers the canonical ensemble of trees with a fixed number of vertices, say \( N \), then the total number of branches on the tree is fixed at \( 2N - 2 \) by the Euler relation. This induces an effective dependence between the vertex orders, results in the multi-point probability

\[
p(n_1, n_2, \ldots, n_N) \sim p(n_1) \cdots p(n_N) \delta_{n_1+\cdots+n_N,2N-2}
\]

(1)

and leads to geometrical correlations on the branched polymers [9,10]. This model, for so-called planar rooted trees and with the power–like probability \( p(n) \sim n^{-\beta} \) for the branching number, has been solved in [12]. At a certain value of \( \beta \) the model was shown to undergo a phase transition between two phases called, respectively, the tree– and bush–like phases. The tree phase, which occurs for \( \beta < \beta_c \), is a genuine branched polymer phase. Above the critical \( \beta \), however, one of the vertices becomes singular and the system enters the bush–like phase, where most branches grow directly out of the singular vertex, or root of the bush. In figure 1 we show the normalized distribution \( \pi(n) \) of vertex orders for different tree sizes \( N \). With increasing \( N \) the distributions approach smoothly the one calculated in the thermodynamic limit. In figure 2 we show distributions for different polymer sizes in the bush phase. One clearly sees the appearance of a singular vertex, or bush root. Moreover, the order of this vertex grows linearly with the polymer size; one could say...
that the bush grows only around its root. Looking at the peak in the probability distribution $\pi(n)$ that corresponds to the root, one can also check that it becomes smaller in inverse proportion to the polymer size, which means that independently of the size there is always the same number of roots per tree (in this case one). The detailed calculations of the vertex order distribution and the full discussion will be presented in the forthcoming publication [13].

The behaviour described here is typical of a large class of models which we call constrained mean-field (or non-interacting constrained) models. We will not give a strict definition of this class, but rather present another, very simple model, which captures all its essential characteristics. The mathematical structure of this model is the same as the branched polymer’s and will neatly explain the behaviour discussed above.

Suppose that we have a lattice model describing a certain local quantity whose states can be enumerated by a discrete positive number $n_i$, where the index $i$ runs over lattice points on which the quantity is defined. We assume that this model does not contain any explicit interactions, i.e. the action is just a simple sum over all the points of the lattice: $S(n) = \sum_{i=1}^{N} s(n_i)$. As it stands, this model is, of course, trivial; but we put in one additional constraint: $\sum_{i=1}^{N} n_i = M$. One realisation of this model would be a system of $M$ balls in $N$ boxes where the one-box action depends only on the number of balls in the box. The partition function of this model is

![Graph showing distribution of branching number in the bush phase of BP.](image)
\[ Z(N, M) = \sum_{n_1, \ldots, n_N} p(n_1) \cdots p(n_N) \delta_{n_1 + \cdots + n_N, M} \] (2)

where \( p(n_i) = e^{-s(n_i)} \).

Without the constraint the one–point effective (or dressed) probability \( \pi(n) \) would just be equal to the bare probability \( p(n) \). Because of the constraint, however, it is given by

\[ \pi(n) = p(n) \frac{Z(N - 1, M - n)}{Z(N, M)} \] (3)

It is worth noting here that the dressed one–point probability \( \pi(n) \) will not change if we add to the action a term linear in \( n_i \) (i.e. \( -\mu n_i \)), as such a term will cancel itself in the numerator and denominator in formula (3). This is actually a very crucial point because it means that it is only the combination of a constraint on the sum of \( n_i \)'s and an action that is non–linear in the \( n_i \)s that makes this model non–trivial.

If we choose \( p(n) \sim n^{-\beta} \) and \( M = \rho N \), we find that, similarly to the branched polymer case, the model has two phases: a “distributed/fluid” phase where every box has, on average, \( M/N \) balls; and a “condensed” phase where one box will contain a number of balls proportional to \( N \), resulting in a peak in the dressed box occupation probability. Obviously, the situation is the same as above. In fact, the branched polymer model can be mapped exactly onto this type of balls–in–boxes model [13]. However, the balls–in–boxes model is more general because we can vary the density \( \rho \) of the balls. It turns out that for fixed \( \beta \) the transition can also be produced by changing the density of the balls [13]. By increasing the density we can bring the system into the condensed phase (see fig. 3).

The Minbu Tree

The physical picture which explains the equivalence of the elongated phase of gravity and branched polymers is that the typical geometry of simplicial gravity is effectively described by the cascade of baby universes weakly interacting through the wormholes. With each elongated configuration of simplicial gravity one can associate an abstract tree in which each baby universe is represented by a vertex and each wormhole by a branch joining those two vertices that correspond to the baby universes which are linked through this wormhole.

The average number \( N \) of vertices on this tree depends on the number \( N_4 \) of simplices on the triangulation. However, only \( N_4 \) is actually fixed in canonical
Fig. 3. The density induced transition in the balls–in–boxes model.

Simulations of simplicial gravity, and $N$ will fluctuate around the average value given by $N_4$. So for the tree of baby universes we do not have a true constraint as we had on the branched polymer, and the formula cannot be expected to be more than a good approximation. We will show that it is indeed very good.

We will, in the following, directly study the underlying tree structure of baby universes. More specifically, we will restrict ourselves to the tree of so–called minbus, which are MInimal Neck Baby Universes [14]. The smallest bottleneck possible on a four–dimensional simplicial manifold consists of five tetrahedra connected to each other in such a way that they form the skeleton of a four–simplex that is not itself a part of the simplicial manifold. Each minbu can have outgrowths on itself, which are then considered its neighbouring minbus. If one confines oneself to spherical topology, one can build a generation tree by pointing from each minbu to its respective ‘parent’. More precisely, one starts from all minbus that have only one wormhole. These form the last generation and are linked by their wormholes to their respective fathers, which make up the next–to–last generation. Repeating this procedure, one recovers the whole tree. For the ensemble of minbu trees one can define geometrical quantities analogous to those in simplicial gravity. The geodesic distance between two vertices on the tree is just the number of links between them. (On the tree there is only one path leading from any one point to any other). The local quantity which plays the role of the curvature is the vertex order. One expects the vertex orders (i. e., the numbers of neighbouring minbus for each baby
universe) to be almost independent from each other, and the main effect of their interaction to come from the constraint. To check this we measured the vertex–vertex correlation function on the minbu trees in the elongated phase ($\kappa_2 > \kappa^c_2$) and compared it to the formula for the correlator for branched polymers: $G(r) \sim re^{-c_2/N}$ ([4]), where $r$ is the geodesic distance as defined above, and $N$ is the number of vertices. The results are presented in figure 4 and one sees that the data fits well the function $G(r)$. Moreover, measuring the size distribution of minbus one can directly find the entropy exponent $\gamma = 1/2$. This means that the ensemble of minbu trees is in a genuine branched polymer phase, and that the constrained mean–field theory works in the whole region of $\kappa_2$ above the critical point. It is now very natural to expect that it will work at the critical point as well; and, as can be seen in figure 5 which shows the distribution of vertex orders for different values of $\kappa_2$, indeed it does. In the elongated phase, the curve falls off very rapidly for higher branching numbers. In the crumpled phase, we see the same separated peak as we did in the branched polymer model, where it corresponds to the bush root. This singular root appears exactly in the critical region, and we found its order to grow linearly with $N$, as expected. The trees change to bushes at this point, and we see that the phase transition is indeed driven by the constrained mean–field mechanism.

Finally, we will show that this transition to the crumpled phase coincides with
Fig. 5. The distribution $\pi(n)$ of branching numbers $n$ on the minbu tree for $\kappa_2 = 1.0$ (solid line) and $\kappa_2 = 1.4$ (dotted line).

the appearance of a mother universe, by which we mean a part of the universe that has a much larger area than all other minbus. It is very intuitive to argue that the minbu which corresponds to the root of the underlying tree structure must have a relatively large size to allow for many outgrowths on it. Indeed, figure 5 shows the clear correlations between the size of a minbu and the order of the corresponding vertex. By the size of the minbu we here mean the size of that part of the minbu obtained by cutting out all direct outgrowths and then 'sewing up' each of the resulting holes with a four–simplex. We also find that the mother universe is not only the largest minbu on the tree, but also contains the singular vertex. In fact, this vertex is the reason why the mother universe has such a large volume, simply because the neighbourhood of the singular vertex contains so many four–simplices. This correlation between the appearance of singular vertices and the change in the minbu structure has already been observed in [8].

**Simplicial gravity**

Comparison of figures 1, 2, 3 with figure 4 and those in [6] clearly shows a great similarity between simplicial 4d gravity, minbu trees, and branched polymers, not only in the elongated phase, but also beyond the transition into the crum-
Minbu branching numbers $n$ as a function of the minbu size $A$ in the crumpled phase ($N_4 = 4000$, $\kappa_2 = 1.0$).

Fig. 6. Minbu branching numbers $n$ as a function of the minbu size $A$ in the crumpled phase ($N_4 = 4000$, $\kappa_2 = 1.0$).

In all these cases the constrained mean–field mechanism plays the crucial role in the transition. In fact, branched polymers are by construction the realization of the mean–field theory. The situation is slightly different for the ensemble of minbu trees, which is an effective theory; here one has to argue that the formula provides a good approximation, and that the corrections to this formula coming from the two– (or multi–) vertex action play a secondary role. However, it should indeed be very plausible that the number of outgrowths from two different minbus are weakly correlated and not in a position to change the main mean–field effect. Very similarly, if one uses the concept of an effective one–vertex action for the orders of vertices in simplicial gravity, one ends up with the constrained mean–field theory as well, due to the constraint on the total sum of vertex orders in the canonical ensemble. In the next two sections, we will discuss numerical results from simulations of 2d and 4d simplicial gravity to support our argument. Finally, as an application of the constrained mean–field scenario, we will use it to explain why singular links are found in the crumpled phase of 4d gravity but singular triangles are not.
In 2d simplicial gravity, the model has been studied with $d$ gaussian fields coupled to the triangulation, with a measure term that directly depends on the vertex order as $n^\beta$. Simulations show that by changing $\beta$ or $d$ one can drive the system from the elongated to the crumpled phase (called collapsed phase in [15]). A typical configuration in this phase looks just like a two-layer pancake. It consists of two discs, each of which is concentrated around a singular vertex of very high order and glued to the other one along its perimeter. In the ideal case, when there is no roughness on the discs, each of them is a set of $n$ triangles, all of which have a common point of order $n$ in the disc’s center. After gluing the discs together, the $n$ points on the perimeter will be of order 4. One recognizes the typical constrained mean-field situation: a constraint on the sum of vertex orders – $\sum o_i = 3N$, where $N$ is the number of triangles – and an effective action, obtained by integrating out the gaussian sector and adding the one-vertex term $\beta \log n$ from the measure force the system to produce singular vertices. When the system grows in size, the orders of the singular points do likewise. Unlike the branched polymer situation, however, the geometrical structure of the two-dimensional manifold forces the appearance of two singular vertices via local geometrical correlations. In fact, it was shown in [16] that by putting a large number of gaussian fields on the vertices of the triangulation, one effectively introduces the measure $n^{-d/2}$.

In four dimensions the main effect will be that of changing the occupation density. As mentioned before, this can trigger the phase transition as well (see figure [3]). By changing the coupling $\kappa_2$, one changes the average vertex order. This value is equal to the average number of four-simplices to which any vertex belongs and can be calculated as $\langle o \rangle = 5N_4/N_0$, where $N_4$ is the number of four-simplices and $N_0$ denotes the number of points. (The factor 5 comes from the fact that each four-simplex has five vertices: $\sum o_i = 5N_4$). By decreasing $\kappa_2$ while keeping $N_4$ constant (i.e., while staying in the canonical ensemble), one lowers $N_0$. In the language of the $N$-boxes, $M$-balls model, we increase the density $\rho = M/N$ by changing the number of boxes. For large $\kappa_2$ in the crumpled phase, $N_0 \approx N_4/4$, and the average vertex order becomes $\langle o \rangle \approx 20$. For $\kappa_2 = 0$ in the crumpled phase, for $N_4 = 32000$ for example, $N_0$ is of order 1000, and hence the average order is $\langle o \rangle \approx 160$, which favours a phase with singular vertices. In fact, for $N_4 = 32000$ the transition already takes place for $\kappa_2 \sim 1.258$, where the average order is $\langle o \rangle \approx 30$; therefore, at $\kappa_2 = 0$ one should expect a very strong domination of singular vertices. Indeed, this turns out to be true. Note that for fixed $N_4$ and $\kappa_2$ the number
$N_0$ is not fixed, so strictly speaking this is not the balls–in–boxes model we have been considering so far. Therefore one should in principle consider $N_0$ as a random variable with a certain smeared distribution. However, the width of this distribution is relatively small, and one expects that its effects can be neglected here.

In terms of the constrained mean–field mechanism, we can now also try to explain why singular links appear in the crumpled phase of 4d simplicial gravity, but singular triangles do not. As we already know, in the crumpled phase the system favours configurations with as few singular vertices as possible, because these can then have the greatest possible order. Consider the neighbourhood of a singular vertex. It has the topology of a 4d ball and a three–dimensional boundary with the topology of a 3d sphere. The number of tetrahedra on the boundary corresponds to the order of the singular vertex in the centre of the ball, and the number of points on the boundary corresponds to the number of links emerging from the sphere. The boundary has its own curvature, since on a three–dimensional sphere one can keep the number of tetrahedra fixed and still lower the number of points. This is exactly what happens when one decreases $\kappa_2$, because this lowers the number of points on the 4d manifold in general, and thus also in the neighbourhood of the singular vertex. When the number of points on the boundary of this neighbourhood becomes small enough, the density–forced transition of the constrained mean–field mechanism is triggered, and another singular vertex appears. Since this point being part of the neighbourhood of the first singular vertex means that there will be a link connecting the two, one has just produced a singular link.

By the same argument, however, one cannot create a singular triangle in this manner: the boundary of a link’s neighbourhood has the topology of a 2d sphere, for which the ratio of triangles and points is fixed by the Euler relation. But if this ratio cannot be changed, it is obviously impossible to trigger a density–forced transition, and there is thus no reason why one of the points on the boundary of the neighbourhood of a singular link should become another singular vertex. Of course, that still leaves us with the possibility that a singular triangle might appear directly, without referring to a singular link already existing on the manifold. But this turns out to be equally unlikely, mainly because the average triangle order is a rather slow–changing function of $\kappa_2$. More specifically, for the average triangle order we have $\langle o \rangle = 10N_4/N_2$, where $N_2$ is the number of triangles on a configuration. Numerical simulations show that for large $N_4$ the ratio $N_2/N_4$ is asymptotically limited by an upper bound of 2.5 in the elongated phase and a lower bound of 2.0 in the crumpled phase; thus, over the whole range of $\kappa_2$, the average triangle order varies only in the very narrow range $\langle o \rangle \in (4, 5)$. Additionally, this range is very close to the minimal triangle order, which is 3. Compare this with the situation for vertices, which have a minimal order of 5 but an average order growing from 20 up to extremely large numbers, and it becomes clear why singular triangles
should be expected to appear much more reluctantly than singular vertices or links.

Note that following the scenario outlined here, there might – or even should – be a range of $\kappa_2$ values below the critical point where singular vertices are already present but singular links are yet missing because the total number of points is still too large. It would certainly be worthwhile to check this.

Summary

To summarize, in this paper we proposed the constrained mean-field scenario as a very simple explanation of the transition mechanism between the crumpled and elongated phases in models of random geometries. Put shortly, one might say that the elongated phase is entropy dominated while the crumpled phase is, in contrast, energy dominated. In the crumpled phase, singular spikes appear in the dressed one–point distributions; depending on context these correspond to either the singular vertices, the mother universe, or the root of the bush. This phase results only from the constraint inherent in a canonical ensemble and is therefore to some extent pathological. For instance, it is the fast increase of the singularity with the system volume, caused only by the constraint, that is responsible for the effectively infinite Hausdorff dimension of geometries in this phase. Since the ensemble of simplicial gravity that is (hopefully) related to the continuum physics is the grand canonical one, this would imply that the crumpled phase has no physical meaning. Since, however, one cannot directly study the unconstrained ensemble on a computer, one should instead try to keep the system in its natural phase, namely the entropy–dominated one. One way of doing this would be to suppress the singular vertices by introducing a cut–off for the maximal vertex order in the distribution. Alternatively, as has already been done in 2d simulations, one could modify the integration measure by adding terms $n^\beta$ (where $n$ is the vertex order), and keep looking for some kind of non–trivial physics in the elongated phase. A third possibility would be to add fermionic degrees of freedom to prevent the vertex–order condensation of the system. Note that the transition to the crumpled phase is rather pathological and (as in 2d) should not be of any physical importance in the context of continuum gravity. In any case, it is known to be highly non–universal: as pointed out in [12] the order of the transition can be easily altered by changing the one–vertex action.

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References

[1] M.E. Agishtein and A.A. Migdal, Nucl.Phys. B385 (1992) 395.
[2] J. Ambjørn and J. Jurkiewicz, Phys. Lett. B278 (1992) 42.
[3] S. Coleman, Nucl.Phys. B310 (1988) 643.
[4] J. Ambjørn and J. Jurkiewicz, Nucl. Phys. B541 (1995) 643.
[5] P. Bialas, Z. Burda, A. Krzywicki and B. Petersson, Nucl. Phys. B472 (1996) 293.
[6] T. Hotta, T. Izubuchi and J. Nishimura, Nucl. Phys. B (Proc. Suppl.), 47 (1995) 609.
[7] S. Catterall, J. Kogut, R. Renken and G. Thorleifsson, Nucl.Phys. B468 (1996) 263.
[8] S. Bilke, Z. Burda, A. Krzywicki and B. Petersson, to appear in Nucl. Phys. B (Proc. Suppl.), Lattice 96.
[9] P. Bialas, Phys. Lett. B373 (1996) 289.
[10] P. Bialas, to appear in Nucl. Phys. B (Proc. Suppl.), Lattice 96.
[11] J. Ambjørn, B. Durhuus, T. Jonsson, Phys. Lett. B244 (1990) 403.
[12] P. Bialas and Z. Burda, Phase Transition in Fluctuating Branched Geometry, hep-lat/9605020 to appear in Phys. Lett. B
[13] P. Bialas and Z. Burda, in preparation.
[14] S. Jain and S.D. Mathur Phys. Lett. B286 (1992) 239.
[15] A. Krzywicki, Nucl. Phys. B (Proc. Suppl) 4 (1988).
[16] J. Ambjørn, B. Durhuus, J. Fröhlich, P. Orland Nucl. Phys. B270 (1986) 457.