Loop Equations and Virasoro Constraints in Matrix Models

Yu. Makeenko*

Institute for Theoretical and Experimental Physics
SU-117259 Moscow, USSR

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Abstract

In the first part of the talk, I review the applications of loop equations to the matrix models and to 2-dimensional quantum gravity which is defined as their continuum limit. The results concerning multi-loop correlators for low genera and the Virasoro invariance are discussed.

The second part is devoted to the Kontsevich matrix model which is equivalent to 2-dimensional topological gravity. I review the Schwinger–Dyson equations for the Kontsevich model as well as their explicit solution in genus zero. The relation between the Kontsevich model and the continuum limit of the hermitean one-matrix model is discussed.

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1 Introduction

The relevance of matrix models to the problem of genus expansion of Feynman graphs goes back to the original work by ’t Hooft [Hoo74]. An explicit solution for the simplest case of the hermitean one-matrix model had been first obtained by Brézin, Itzykson, Parisi and Zuber [BIPZ78] in genus zero and then extended in [Bes79, IZ80, BIPZ80] to next few genera.

The modern interest in matrix models is associated with the context of statistical theories on random lattices and discretized random surfaces [Kaz85, Dav85, ADF85, KKM85] as well as with the conformal field theory approach to 2D quantum gravity [Pol87, KPZ88, Dav88, DK89]. A connection between continuum limits of the matrix model and minimal conformal models had been conjectured by Kazakov [Kaz89] on the basis of genus-zero results.

The whole genus expansion of 2D quantum gravity has been constructed in [BK90, DS90, GM90b] taking the ‘double scaling limit’ of the (hermitean) one-matrix model. Moreover, the specific heat turns out to obey a (non-perturbative) equation of the Korteweg–de Vries type so that a relation between the continuum limit of the matrix models and integrable theories emerges [GM90a, BDSS90].

While these results were obtained using orthogonal polynomial technique, one more method — that of loop (or Schwinger–Dyson) equations — is custom in studies of matrix models. Loop equations had been proposed originally for Yang-Mills theory both on a lattice [Foe79, Egu79, Wei79] and in the continuum [MM79, Pol80] (for a review, see [Mig83]) and then were applied [PR80, Fri81, Wad81] to matrix models. A modern approach to loop equation which is based on its interpretation as a Laplace equation on the loop space can be found in [Mak88, Mak89, HM89]. The recent applications of loop equations to 2D quantum gravity have been initiated by Kazakov [Kaz89]. The role of loops in 2D quantum gravity is played by boundaries of a 2-dimensional surface.

Ref. [Kaz89] deals with genus zero. The whole set of loop equations for 2D quantum gravity was first obtained by David [Dav90] taking the ‘double scaling limit’ of the corresponding equations for the hermitean matrix model. As was shown in [Mig83, Dav90, AM90], these equations can be unambiguously solved order by order of genus expansion. However, this solution is non-perturbatively unstable [Dav90] as it should be for 2D euclidean quantum gravity.

One of the most interesting results which are obtained with the aid of loop equations is the fact that the partition function of 2D quantum gravity in an external background is the $\tau$-function of KdV hierarchy which is subject to additional Virasoro constraints [FKN91, DVV91]. This proves a conjecture of Douglas [Dou90]. The existence of Virasoro algebra was extended to the case of the matrix model at finite $N$ in [AJM90, GMM*91, IM91, MM90] while the relation to the continuum Virasoro algebra of [FKN91, DVV91] had been studied by M$^4$ [MMMM91].

The second application of loop equations concerns the relation between 2D quantum and topological [LPW88, MS89] gravities. As Witten [Wit90] had conjectured, these two gravities are equivalent. This conjecture has been verified in genus zero and genus one
for 2D quantum gravity coincide with the recursion relations between correlators in 2D topological gravity which were obtained by Verlindes [VV91].

From the mathematical point of view, a solution of 2D topological gravity is equivalent [Wit90] to calculations of intersection indices on the moduli space, $\mathcal{M}_{g,s}$, of curves of genus $g$ with $s$ punctures. Interesting results for this problem have been obtained recently by Kontsevich [Kon91] who has represented the partition function of 2D topological gravity as that of a (hermitean) matrix model in an external field. It is worth noting that the Kontsevich matrix model is associated with the continuum theory. Therefore, it should be directly related to the ‘double scaling limit’ of the standard one-matrix model [BK90, DS90, GM90b].

The Kontsevich model can be studied by the method of loop equations. As has been shown recently by Semenoff and the author [MS91a], the Schwinger-Dyson equations for the hermitean one-matrix model in an external field, which is equivalent to the Kontsevich model, can be represented as a set of Virasoro constraints imposed on the partition function. The large-$N$ solution of these equations, which is known from the work of Kazakov and Kostov [KK89], solves the Kontsevich model in genus zero [MS91a] showing explicitly the equivalence of 2D topological and quantum gravities to this order.

The fact that the partition function of the Kontsevich model obeys the same set of Virasoro constraints [FKN91, DVV91] as the continuum limit of the hermitean one-matrix model has been proven recently by Witten [Wit91] using diagrammatic expansion and by (A.M.)$^3$ [MMM91] using the Schwinger–Dyson equations. This demonstrates an equivalence of 2D topological and quantum gravities to any order of genus expansion.

In the first part of the talk, I review some works [AM90, AJM90, Mak90, MMMM91] on applications of loop equations both to $N \times N$ matrix models at finite $N$ and to 2D quantum gravity which is defined as their continuum limit. The results for multi-loop correlators in low genera and the Virasoro invariance both at finite $N$ and in the continuum are discussed. The second part is devoted to the Kontsevich matrix model which is equivalent to 2-dimensional topological gravity. The Schwinger–Dyson equations for the Kontsevich model as well as their explicit solution in genus zero [MS91a] is reviewed. Some original results concerning the Kontsevich matrix model are reported.
2 Matrix Models and 2D Quantum Gravity

2.1 Loop equation for hermitean matrix model

The hermitean matrix model is defined by the partition function

\[ Z_N^H = \int \mathcal{D} M \exp - \operatorname{tr} V(M) \]  

where \( M \) is the \( N \times N \) hermitean matrix. \( V \) stands for a generic potential

\[ V(p) = \sum_{k=0}^{\infty} t_k p^k. \]  

The coupling \( t_k \) plays here the role of a source for the operator \( \operatorname{tr} M_k \) while \( V(p) \) is a source for the Laplace image of the Wilson loop \( \operatorname{tr} [1/(p - M)] : \)

\[ \operatorname{tr} V(M) = \int_{-i\infty+0}^{+i\infty+0} \frac{dp}{2\pi i} V(p) \operatorname{tr} \frac{1}{p - M}. \]  

The correlators \( \langle \operatorname{tr} M^{k_1} \ldots \operatorname{tr} M^{k_m} \rangle_c \), where the average is defined with the same measure as in (2.1), can be obtained differentiating \( \log Z_N^H \) w.r.t. \( t_{k_1}, \ldots, t_{k_m} \) while loop correlators can be obtained by applying

\[ \frac{\delta}{\delta V(p)} = - \sum_{k=0}^{\infty} p^{-k-1} \frac{\partial}{\partial t_k} \]  

so that the \( m \)-loop correlator reads

\[ W^H(p_1, \ldots, p_m) \equiv \langle \frac{1}{p_1 - M} \ldots \frac{1}{p_m - M} \rangle_c = \frac{\delta}{\delta V(p_1)} \ldots \frac{\delta}{\delta V(p_m)} \log Z_N^H. \]  

To calculate the actual values for the given model, say for the matrix model with cubic potential, one should, after differentiations, put \( t_k \)'s equal their actual values, say \( t_k = 0 \) for \( k > 3 \) in the case of cubic potential.

The loop equation can be derived using the invariance of the integral under an (infinitesimal) shift of \( M \) and reads

\[ \int_{C_1} \frac{d\omega}{2\pi i (p - \omega)} W^H(\omega) = (W^H(p))^2 + \frac{\delta}{\delta V(p)} W^H(p). \]  

The contour \( C_1 \) encircles singularities of \( W^H(\omega) \) so that the integration is a projector picking up negative powers of \( p \). Eq.(2.6) is supplemented with the asymptotic condition

\[ p W^H(p) \rightarrow N \text{ as } p \rightarrow \infty \]  

which is a consequence of the definition (2.5).

Notice that one obtains the single (functional) equation for \( W^H(p) \). This is due to the fact that \( \operatorname{tr} V(M) \) contains a complete set of operators. Such an approach is advocated in [DVV91, Mak90]. The set of equations for multi-loop correlators (2.5), which is considered in [Dav90, FKN91], can be obtained from Eq.(2.6) by \( m - 1 \)-fold application of \( \delta/\delta V(p_i) \).

The system of the standard Schwinger-Dyson equations for the connected correlators \( \langle \operatorname{tr} M^{k_1} \ldots \operatorname{tr} M^{k_m} \rangle_c \) can be then obtained by expanding in powers of \( p_1^{-1}, \ldots, p_m^{-1} \).
2.2 Solution in $1/N$

Eq. (2.6) can be solved order by order of the expansion in $1/N^2$ (the genus expansion). The second term on the r.h.s. represents the connected correlator of two Wilson loops and is, in our normalization, of order 1 as $N \to \infty$ while two other terms are of order $N^2$ since $W^H(p)$ and $V(p)$ are of order $N$. Therefore one can omit it as $N \to \infty$ (which corresponds to genus zero = the planar limit).

The simplest (one-cut) solution of Eq. (2.6) as $N \to \infty$ reads \cite{Mig83}

$$2W^H(0) = V'(p) - M(p) \sqrt{(\omega - x)(\omega - y)}$$

(2.8)

where

$$M(p) = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(p) - V'(\omega)}{(p - \omega)\sqrt{(\omega - x)(\omega - y)}}$$

(2.9)

is a polynomial of degree $K - 2$ if $V(p)$ is that of degree $K$. The ends of the cut, $x$ and $y$, are determined from the asymptotics (2.7):

$$0 = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - x)(\omega - y)}; \quad 2N = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - x)(\omega - y)} \equiv W(x, y).$$

(2.10)

For the even potential ($V(-p) = V(p)$), the first of these equations yields $y = -x = \sqrt{z}$ which simplifies formulas. This case is called the reduced hermitean matrix model.

The multi-loop correlators in the planar (genus zero) limit can be obtained by varying according to the r.h.s. of Eq. (2.5). The 2-loop correlator reads \cite{AJM90}

$$W^H(0)(p, q) = \frac{1}{4(p - q)^2} \left\{ \frac{2pq - (p + q)(x + y) + 2xy}{\sqrt{(p - x)(p - y)}\sqrt{(q - x)(q - y)}} \right\}$$

(2.11)

while an expression for the 3-loop correlator is given in \cite{AJM90}. Note that the 2-loop correlator (2.11) depends on the potential, $V$, only via $x$ and $y$ but not explicitly. This is not the case for all other multi-loop correlators.

To calculate $1/N^2$ correction to (2.8) one needs $W^H(0)(p, p)$ which enters the r.h.s. of Eq. (2.6). Eq. (2.11) yields

$$W^H(0)(p, p) = \frac{(x - y)^2}{16(p - x)^2(p - y)^2}.$$  

(2.12)

and one can now obtain $W^H(1)(p)$ by an iteration of Eq. (2.6). The result

$$W^H(1)(p) = \frac{1}{\sqrt{(p - x)(p - y)}} \int_{C_1} \frac{d\omega}{2\pi i} \frac{1}{(\omega - p)M(\omega)} \frac{(x - y)^2}{16(\omega - x)^2(\omega - y)^2}$$

(2.13)

is unambiguous \cite{Mig83, Dav90} provided that one requires analyticity of $W^H(1)(p)$ at zeros of $M(p)$. This procedure of iterative solution can be pursued order by order of $1/N$. \cite{Mig83}
2.3 Continuum loop equation

The continuum limit of the reduced hermitean matrix model is reached as $N \rightarrow \infty$ while $K - 1$ conditions $W^{(n)}(z_c) = 0$ ($W(z) \equiv W(-\sqrt{z}, \sqrt{z})$) with $n = 1, \ldots, K - 1$ are imposed on the couplings, $t_k$, in addition to \eqref{2.10} at $K$th multi-critical point. 2D quantum gravity corresponds to $K = 2$. The 'double scaling limit' can be obtained if one expands around the critical point:

$$p \rightarrow \sqrt{z_c} + \frac{a \pi}{2 \sqrt{z_c}}; \quad z \rightarrow \sqrt{z_c} - \frac{a \sqrt{\Lambda}}{2 \sqrt{z_c}},$$

\hspace{1cm} \quad \eqref{2.14}

so that $\pi$ and $\Lambda$ play the role of continuum momentum and cosmological constant, respectively. The dimensionful cutoff $a$ should depend on $N$ such that the string coupling constant $G = N^{-2} a^{-2 K - 1}$ would remain finite as $N \rightarrow \infty$ \cite{BK90, DS90, GM90b}.

To obtain the continuum limit of loop correlators \eqref{2.5}, it is convenient to introduce the even parts

$$W^{\text{even}}(p_1, \ldots, p_m) \equiv \frac{\delta}{\delta V^{\text{even}}(p_1)} \cdots \frac{\delta}{\delta V^{\text{even}}(p_m)} \log Z^{\text{reduced}}_N$$

\hspace{1cm} \quad \eqref{2.15}

where $Z^{\text{reduced}}_N$ and $V^{\text{even}}(p)$ means, respectively, \eqref{2.1} and \eqref{2.2} with $t_{2k+1} = 0$:

$$V^{\text{even}}(p) = \sum_{k=0}^{\infty} t_{2k} p^{2k}, \quad \frac{\delta}{\delta V^{\text{even}}(p)} = - \sum_{k=0}^{\infty} p^{-2k-1} \frac{\partial}{\partial t_{2k}}.$$ 

\hspace{1cm} \quad \eqref{2.16}

$W^{\text{even}}(p_1, \ldots, p_m)$ differs from $W^{\text{H}}(p_1, \ldots, p_m)$ by correlators of products of traces of odd powers of $M$. Near the critical point, one gets

$$W^{\text{H}}(p_1, \ldots, p_m) \rightarrow 2^{m-1} W^{\text{even}}(p_1, \ldots, p_m).$$ 

\hspace{1cm} \quad \eqref{2.17}

This formula can be proven analyzing loop equations or by a direct inspection of multi-loop correlators \cite{AJM90}.

The continuum loop correlators can be obtained by the multiplicative renormalization \cite{Dav90, AM90, FKN91}

$$W^{\text{H}}_{2N}(p_1, \ldots, p_m) \rightarrow 2^m a^{-m} G^{\frac{1}{2} m - 1} W_{\text{cont}}(\pi_1, \ldots, \pi_m) \quad \text{for } m \geq 3$$

\hspace{1cm} \quad \eqref{2.18}

while additional subtractions of genus zero terms are needed for $m = 1$ and $m = 2$:

$$W^{\text{H}}_{2N}(p) - \frac{1}{2} V'(p) \rightarrow \frac{1}{a \sqrt{G}} (2W_{\text{cont}}(\pi) - J'(\pi)) ,$$

\hspace{1cm} \quad \eqref{2.19}

$$W^{\text{H}}_{2N}(p_1, p_2) \rightarrow 4 a^{-2} W_{\text{cont}}(\pi_1, \pi_2) + \frac{1}{a^2 (\sqrt{\pi_1} + \sqrt{\pi_2})^2 \sqrt{\pi_1 \pi_2}}.$$ 

\hspace{1cm} \quad \eqref{2.20}

For latter convenience, $W^{\text{H}}_{2N}(p_1, \ldots, p_m)$ on the l.h.s.'s of these formulas is the multi-loop correlator for the $2N \times 2N$ reduced hermitean matrix model.

$J(\pi)$ on the r.h.s. of Eq.\eqref{2.19} plays the role of a source for the continuum Wilson loop:
\[
J(\pi) = \sum_{n=0}^{\infty} T_n \pi^{n+\frac{1}{2}}, \quad \frac{\delta}{\delta J(\pi)} = -\sum_{n=0}^{\infty} \pi^{-n-3/2} \frac{\partial}{\partial T_n}, \tag{2.22}
\]

with \( T_k \) being sources for operators with definite scale dimension. Therefore, Eqs. (2.18), (2.20) can be derived from Eq. (2.19) by varying w.r.t. \( J(\pi) \).

The continuum loop equation can be obtained from (2.6) substituting (2.19), (2.20):

\[
\int_{C_1} \frac{d\Omega}{2\pi i} \frac{J'(\Omega)}{(\pi - \Omega)} W_{\text{cont}}(\Omega) = (W_{\text{cont}}(\pi))^2 + G \frac{\delta W_{\text{cont}}(\pi)}{\delta J(\pi)} + \frac{G}{16\pi^2} + \frac{T_0^2}{16\pi}. \tag{2.23}
\]

This equation describes what is called the ‘general massive model’. It corresponds to arbitrary \( J(\pi) \) and interpolates between different multi-critical points. For \( K^{th} \) multi-critical point, one puts, after varying w.r.t. \( J(\pi) \), \( T_n = 0 \) except for \( n = 0 \) and \( n = K \).

### 2.4 Genus expansion

Eq. (2.23) can be solved order by order in \( G \) (genus expansion) analogously to that of Sect. 2.2. If \( J(\pi) \) is a polynomial \( (T_n = 0 \text{ for } n > K) \), \( K - 1 \) lower coefficients of the asymptotic expansion of \( W_{\text{cont}}(\pi) \) are not fixed while solving in \( 1/\pi \) and should be determined by requiring the one-cut analytic structure in \( \pi \). The continuum analog of (2.8) reads

\[
2W^{(0)}_{\text{cont}}(\pi) = \int_{C_1} \frac{d\Omega}{2\pi i} \frac{J'(\Omega)}{(\pi - \Omega)} \frac{\sqrt{\pi + u}}{\sqrt{\Omega + u}} \tag{2.24}
\]

where \( u \) versus \( \{T\} \) is determined from the asymptotic behavior.

This asymptotic relation can be obtained comparing \( 1/\pi \) terms in Eq. (2.23). Denoting the derivative w.r.t. \( x = -T_0/4 \) by \( D \), it is convenient to represent this relation as

\[
x = \int_{C_1} \frac{d\Omega}{2\pi i} J'(\Omega) D W_{\text{cont}}(\Omega). \tag{2.25}
\]

For the ansatz (2.24), one gets

\[
DW^{(0)}_{\text{cont}}(\pi) = \frac{1}{\sqrt{\pi + u}} - \frac{1}{\sqrt{\pi}} \tag{2.26}
\]

Eq. (2.26) can be extended to any genera using the representation \([GM90a, BDSS90]\)

\[
DW_{\text{cont}}(\pi) = 2\langle x | (\pi + u(x) - \frac{1}{4} GD^2)^{-1} | x \rangle - \frac{1}{\sqrt{\pi}} = 2 \sum_{n=1}^{\infty} R_n[u] \equiv 2R(\pi) \tag{2.27}
\]

where the diagonal resolvent of Sturm-Liouville operator is expressed via the Gelfand-Dikii differential polynomials \([GD75]\)

\[
R_n[u] = 2^{-n-1} \left( \frac{G}{8} D^2 - u - D^{-1}uD \right)^n \cdot 1. \tag{2.28}
\]

Substituting the r.h.s. of Eq. (2.27) into Eq. (2.25), one obtains the string equation \([GM90a, BDSS90]\)
The fact that the ansatz (2.27) does satisfy Eq.(2.23) is shown in [DVV91]. To this aid, one applies the operator
\[
\Delta = -\frac{G}{16}D^4 + (u + \pi)D^2 + \frac{1}{2}(Du)D,
\] (2.30)
which annihilates \(W_{\text{cont}}(\pi)\) given by Eq.(2.27), to Eq.(2.23). The result vanishes provided \(u\) satisfies Eq.(2.29) and
\[-2\frac{\delta}{\delta J(\pi)}u = D^2W_{\text{cont}}(\pi)
\] (2.31)
whose expansion in \(1/\pi\) reproduces the KdV hierarchy \(\partial u/\partial T_n = DR_{n+1}[u]\).

Comparing (2.21) and (2.31), one concludes that
\[Z_{\text{cont}} = \exp\left\{-\frac{2}{G} \int_0^x dy(x-y)u(y) + \Phi(T_1, T_2, \ldots)\right\}
\] (2.32)
where the perturbative solution of Eq.(2.29), which satisfies \(u(0) = 0\), is chosen and an integration ‘constant’ \(\Phi(T_1, T_2, \ldots)\) depends on all \(T\)’s except for \(T_0\). At the given multi-critical point, this \(\Phi\) is unessential so that the r.h.s. of Eq.(2.32) coincides with the continuum partition function which was obtained in [BK90, DS90, GM90b] using the method of orthogonal polynomials.

The general procedure of solving Eq.(2.23) order by order in \(G\) can be now formulated as follows. One should first solve Eq.(2.29) to find \(u\) versus \(x\) and \(\{T_n\}\) (this is perturbatively unambiguous). Then Eq.(2.27) determines \(D W_{\text{cont}}(\pi)\) while \(W_{\text{cont}}(\pi)\) itself can be obtained by integrating
\[W_{\text{cont}}(\pi) = 2\int^x dx R(\pi)
\] (2.33)
where the integration ‘constant’ can be expressed via \(\Phi\) entering Eq.(2.32). This constant becomes unessential for \(K\)th multi-critical point when \(T_n = 0\) except for \(n = 0\) and \(n = K\) so that \(u\) depends only on the cosmological constant \(\Lambda\):
\[\Lambda^{K/2} = \frac{x4^{K+1}(K!)^2}{(2K + 1)(2K)!T_K}.
\] (2.34)

2.5 Multi-loop correlators in 2D quantum gravity

A formula which is similar to Eq.(2.33) exists for the multi-loop correlators:
\[W_{\text{cont}}(\pi_1, \ldots, \pi_m) = 2\int^x dx_1 \frac{\delta}{\delta J(\pi_1)} \cdots \frac{\delta}{\delta J(\pi_{m-1})} R(\pi_m),
\] (2.35)
where the integration ‘constant’ depends again on \(T_1, T_2, \ldots\).

Since \(R(\pi)\) depends on \(T\)’s only implicitly via \(u\), the following chain rule can be used for calculations
\[\delta = 2\int^{+\infty} dR(u) \frac{\delta}{\delta J(\pi)}
\] (2.36)
with $\delta/\delta u(x)$ being the standard variational derivative. The expansion of Eq.\((2.36)\) in $1/\pi$ reproduces the standard (commuting) KdV flows. An advantage of Eq.\((2.36)\) is that it allows to obtain results without solving the string equation \((2.29)\). Therefore, to calculate the multi-loop correlator for a given multi-critical point, one can substitutes the solution of Eq.\((2.29)\) only for this multi-critical point and should not solve it for arbitrary $T$’s.

An alternative way of calculating correlators in 2D quantum gravity is to take the continuum limit \((2.14)\) of formulas of Sect.2.2 with the aid of the renormalization \((2.18)–(2.20)\). For the case of pure gravity (the $K = 2$ critical point), the explicit form of $W_{cont}(\pi)$ is known for genus zero \([Dav90]\) and genus one \([AM90]\):

$$W_{cont}^{(0+1)}(\pi) = -\frac{5}{4}T_2 \left[ (\pi - 1)\sqrt{\pi + \sqrt{\Lambda}} \right] - \frac{G}{90T_2} \frac{(\pi + \frac{5}{2}\sqrt{\Lambda})}{\Lambda(\pi + \sqrt{\Lambda})^{5/2}}$$ \((2.37)\)

where $[\ldots]_-$ means subtraction of the term which diverges as $\pi \to \infty$ as well as that of order $O(\pi^{-1}2/3)$. The multi-loop correlators are known \([AM90, AJM90]\) for genus zero:

$$W_{cont}^{(0)}(\pi_1, \pi_2) = \frac{\left( \sqrt{\pi_1} + \sqrt{\Lambda} - \sqrt{\pi_2} + \sqrt{\Lambda} \right)^2}{4(\pi_1 - \pi_2)^2\sqrt{\pi_1}\sqrt{\pi_2} + \sqrt{\Lambda}} + \frac{1}{4(\sqrt{\pi_1} + \sqrt{\pi_2})^2\sqrt{\pi_1}\pi_2}$$ \((2.38)\)

As is mentioned above, the additional subtraction is needed only for $m \leq 2$.

Analogous formulas can be obtained for higher multi-critical points. For $K = 3$, Eq.\((2.38)\) remain unchanged while the analog of Eq.\((2.37)\) reads \([AM90]\):

$$W_{cont}^{(0+1)}(\pi) = -\frac{7}{4}T_3 \left[ (\pi^2 - 1)\pi + \frac{3}{2} \right] \sqrt{\pi + \sqrt{\Lambda}} - \frac{4G}{315T_3} \frac{(\pi + \frac{7}{2}\sqrt{\Lambda})}{\Lambda^{3/2} + \sqrt{\Lambda}}$$ \((2.40)\)

The above expressions for multi-loop correlators agree with those obtained recently \([GL91, MMS91, MSS91, MS91a]\) for the Liouville theory.

### 2.6 Complex matrix model

The complex matrix model is defined by the partition function

$$Z_N^C = \int \mathcal{D}M\mathcal{D}M^\dagger \exp - \text{tr} \, V^{\text{even}}(MM^\dagger)$$ \((2.41)\)

where the integral goes over $N \times N$ complex matrices and $V^{\text{even}}$ is given by \((2.16)\). As is seen from this formula, the complex matrix model resembles the reduced hermitean one. However, it differs by combinatorics as well as by the fact that averages of traces of odd powers of $M$ do not appear. The model \((2.41)\) has been studied in \([Mor91, AMP91]\) using...
The variables $t_{2k}$ in (2.41) play the role of sources for the operators $\text{tr}(MM^\dagger)^k$ while $V^{\text{even}}$ is a source for the Wilson loop $\text{tr}[p/(p^2 - MM^\dagger)]$ (comp. (2.3)):

$$\text{tr} V^{\text{even}}(MM^\dagger) = \int_{-\infty}^{+i\infty} \frac{dp}{2\pi i} V^{\text{even}}(p) \text{tr} \frac{p}{(p^2 - MM^\dagger)}.$$ (2.42)

The analog of (2.5) reads

$$W^C(p_1, \ldots, p_m) = \delta V^{\text{even}}(p_1) \cdots \delta V^{\text{even}}(p_m) \log Z_N^C$$ (2.43)

which leads to the following loop equation for the complex matrix model [Mak90]

$$\int_{C_1} \frac{d\omega}{4\pi i} \frac{V^{\text{even}}(\omega)}{(p - \omega)} W^C(\omega) = (W^C(p))^2 + \frac{\delta}{\delta V^{\text{even}}(p)} W^C(p).$$ (2.44)

This equation should be supplemented with the asymptotic condition same as (2.7).

Comparison of Eq. (2.44) and loop equation for the reduced hermitian model yields for genus zero:

$$W^C_N(p_1, \ldots, p_m) = \frac{1}{2} W^{\text{even}}_{2N}(p_1, \ldots, p_m),$$ (2.45)

where the correlators $W^{\text{even}}$ for the reduced hermitian model are defined in Sect.2.3. Notice that the correlator for $N \times N$ complex matrix model enters the l.h.s. while that for $2N \times 2N$ reduced hermitian one enters the r.h.s.. This guarantees the asymptotic condition (2.7). The coefficient $1/2$ in Eq. (2.45) leads to the following relation between the partition functions for genus zero:

$$Z_N^C \propto \sqrt{Z_{2N}^{\text{reduced}}}.$$ (2.46)

Due to the relation (2.45), $4W^C(p)$ to the leading order in $1/N$ is given by the r.h.s. of Eq. (2.4) with $V$ replaced by $V^{\text{even}}$ and $y = -x = \sqrt{z}$. The multi-loop correlator to leading order in $1/N$ can be then calculated by varying according to (2.43). The analog of (2.11), (2.12) reads

$$W^{C(0)}(p, q) = \frac{1}{4(p^2 - q^2)^2} \left\{ \frac{2p^2q^2 - zp^2 - zq^2}{\sqrt{p^2 - z}\sqrt{q^2 - z}} - 2pq \right\}, \quad W^{C(0)}(p, p) = \frac{z^2}{16p^2(p^2 - z)^2}.$$ (2.47)

Moreover, an explicit expression for arbitrary multi-loop correlators exists [AJM90] for the complex matrix model even far from the critical point (comp. (2.39)):

$$W^{C(0)}(p_1, \ldots, p_m) = \left( \frac{1}{W'(z)} \frac{\partial}{\partial z} \right)^{m-3} \frac{1}{2zW'(z)} \prod_{i=1}^{m} \frac{z}{2(p_i^2 - z)^{3/2}} \text{ for } m \geq 3.$$ (2.48)

As is proven in [Mor91, AMP91, Mak90, AJM90], the complex and hermitian matrix models belong to the same universality class in the ‘double scaling limit’. This implies, in particular, that the continuum limit of all multi-loop correlators coincide with those of Sects.2.4,2.5. They do not coincide, generally speaking, for higher genera far from the critical points. Eqs. (2.45), (2.46) remain valid, however, to arbitrary order of genus expansion near the critical points. Using (2.17), one concludes that $W^C_N(p_1, \ldots, p_m)$ has
2.7 Loop equations as Virasoro constraints

The loop equation (2.6) can be represented as a set of Virasoro constraints imposed on the partition function. Eq. (2.6) can be rewritten, using the definitions (2.2) and (2.4), as

\[\frac{1}{Z_N} \sum_{n=-1}^{\infty} \frac{1}{p^{n+2}} L_n^H Z_N^H = 0 \]  

(2.49)

where the operators

\[ L_n^H = \sum_{k=0}^{\infty} k t_k^2 \frac{\partial}{\partial t_{k+n}} + \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial t_k \partial t_{n-k}} \]  

(2.50)

satisfy [AJM90, GMM*91] Virasoro algebra

\[ [L_n^H, L_m^H] = (n - m)L_{n+m}^H. \]  

(2.51)

Therefore, Eq. (2.6) is represented as the Virasoro constraints

\[ L_n^H Z_N^H = 0 \text{ for } n \geq -1. \]  

(2.52)

These constraints manifest the invariance of the integral on the r.h.s. of (2.4) under the shift of integration variable \( \delta M = \epsilon \cdot M^{n+1} \) with \( n \geq -1 \) [AJM90, MM90].

It is impossible, however, to make in (2.50), (2.52) the reduction to even times. For even \( n \), this reduction can be done for the first term on the r.h.s. of (2.50) but not for the second one. Therefore, there exist no Virasoro constraints imposed on \( Z_N^H \text{ at finite } N \).

A set of Virasoro operators built up from the even times, \( t_{2k} \), arises for the complex matrix model. The loop equation (2.44) can be represented as Virasoro constraints

\[ L_n^C Z_N^C = 0 \text{ for } n \geq 0; \]  

(2.53)

\[ L_n^C = \sum_{k=0}^{\infty} k t_{2k} \frac{\partial}{\partial t_{2k+n}} + \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial t_{2k} \partial t_{2(n-k)}}. \]  

(2.54)

The Virasoro invariance is now related [AJM90, MM90] to the change \( \delta M = \epsilon(MM^\dagger)^n M \) with \( n \geq 0 \).

Analogously, the continuum loop equation (2.23) can be represented as Virasoro constraints which are imposed on \( Z_{\text{cont}} \) defined by (2.21). Using (2.22), one proves that Eq. (2.23) is equivalent to the continuum Virasoro constraints [FKN91, DVV91]

\[ \mathcal{L}_n^\text{cont} Z_{\text{cont}} = 0 \text{ for } n \geq -1; \]  

(2.55)

\[ \mathcal{L}_n^\text{cont} = \sum_{k=0}^{\infty} (k + 1/2)T_k \frac{\partial}{\partial T_{k+n}} + G \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial T_k \partial T_{n-k}} + \delta_{0,n} \frac{\partial}{\partial T_n} + \frac{\delta_{-1,n} T_n^2}{16} + \frac{\delta_{-1,n} T_n^2}{16}. \]  

(2.56)

The relation between the continuum Virasoro constraints (2.55), (2.56) and those at finite \( N \) can be studied [MMM99] without referring to loop equations. Introducing

\[ T_n = \sum_{k=0}^{\infty} \sqrt{G} a^{n+1/2} k t_{2k} \Gamma(k+1/2) - 4N \sqrt{G} a n_{0,0}, \]  

(2.57)
or, vice versa,
\[ kt_{2k} - 2N \delta_{k,0} = \sum_{n \geq k} (-)^{k-n} a^{-n-1/2} T_n \Gamma(n + 3/2) \sqrt{G(n - k)! \Gamma(k + 1/2)}, \]  
(2.58)

and rescaling the partition function
\[ Z^C_N \rightarrow \tilde{Z}^C_N = e^{-\frac{1}{4} \sum_{A_{mn}} T_m T_n Z^C_N}, \]  
(2.59)

\[ A_{mn} = (-)^{n+m} \frac{\Gamma(n + 3/2) \Gamma(m + 3/2)}{2\pi(n + m + 1)(n + m + 2)n!m!} G^{-1} a^{-m-n-1}, \]  
(2.60)

one gets from (2.52), (2.53)
\[ \tilde{L}^C_n \tilde{Z}^C_N = 0 \quad \text{for} \quad n = -1; \quad \tilde{L}^C_n \tilde{Z}^C_N = (-)^n \frac{1}{16a^n} \tilde{Z}^C_N \quad \text{for} \quad n \geq 0, \]  
(2.61)

\[ \tilde{L}^C_n = \sum_{k=0}^{\infty} (k + 1/2) T_k \frac{\partial}{\partial T_{k+n}} + G \sum_{0 \leq k \leq n-1} \frac{\partial^2}{\partial T_k \partial T_{n-k-1}} + \frac{\delta_{n,0}}{16} + \frac{\delta_{n,-1} T_0^2}{16G}. \]  
(2.62)

The variables \( \{\tilde{T}\} \) are related to \( \{T\} \) by
\[ T_n = \tilde{T}_n + a \frac{n}{n + 1/2} \tilde{T}_{n-1} - 4N \sqrt{G} a \delta_{n,0} \]  
(2.63)

so that the difference disappears as \( a \rightarrow 0 \).

Eqs. (2.57), (2.58) are the standard transition [Kaz89] to operators with definite scale dimensions in the continuum. The rescaling (2.59) makes \( \tilde{Z}_N^C \) finite as \( a \rightarrow 0 \) so that \( \tilde{Z}_N^C \rightarrow Z_{cont} \). While the operators \( \tilde{L}^C_n \) tend to \( L_{cont}^n \) defined by (2.56) as \( a \rightarrow 0 \), the \( a \rightarrow 0 \) limit is not permutable with differentiating \( \tilde{Z}_N^C \) w.r.t. \( T_n \). This is why \( \tilde{L}^C_n \tilde{Z}^C_N \) are nonvanishing (even singular for \( n \geq 1 \)) as \( a \rightarrow 0 \). These terms do not appear [MMMM91], however, when \( \tilde{L}^C_n \) ’s act on
\[ e^{-\frac{1}{4} \sum_{A_{mn}} T_m T_n \sqrt{Z_{2N}^{\text{reduced}}}} \rightarrow Z_{cont} \]  
(2.64)

(comp. (2.46), (2.59)). Thus, the l.h.s. of Eq. (2.64) defines the proper continuum partition function which is annihilated by the continuum Virasoro operators (2.50).
3 Kontsevich Model and 2D Topological Gravity

3.1 2D topological gravity as Kontsevich model

The starting point in demonstrating an equivalence between 2D topological gravity and the Kontsevich model is the Witten’s geometric formulation [Wit90] of 2D topological gravity. In this formulation, one calculates the correlation functions of $s$ operators $\sigma_{n_1}(x_1), \ldots, \sigma_{n_s}(x_s)$ with definite (non-negative integer) scale dimensions $n_i$, living on a 2-dimensional Riemann surface $\Sigma$ of genus $g$. Those are expressed [Wit90] via the intersection indices

$$\left\langle \sigma_{n_1}(x_1) \cdots \sigma_{n_s}(x_s) \right\rangle_g = \int \prod_i c_1(L_{(i)})^{n_i} N(n_i)$$

where $c_1(L_{(i)})$ is the first Chern class of the line bundle (which is the cotangent space to a curve at $x_i$) over the moduli space, $\mathcal{M}_{g,s}$, of curves of genus $g$ with $s$ punctures and the integral goes over $\bar{\mathcal{M}}_{g,s}$. The normalization factor $\prod_i N(n_i)$, which is related to the normalization of the operators $\sigma$, is to be fixed below. The r.h.s. of Eq.(3.1) is non-vanishing only if

$$\sum_i n_i = 3g - 3 + s,$$

i.e. the (complex) dimension of $\mathcal{M}_{g,s}$. Notice the crucial property of correlators in topological theories — those depend only on the dimensions $n_1, \ldots, n_s$ and genus $g$ but not on the metric on $\Sigma$ and, therefore, not on positions of the punctures $x_1, \ldots, x_s$.

It is convenient to introduce the set of couplings $t_n$ which play the role of sources for the operators $\sigma_n$. The genus $g$ contribution to the free energy then reads

$$F_g[t] = \left\langle \exp \left( \sum_n t_n \int \sigma_n \right) \right\rangle_g$$

while the correlator on the l.h.s. of Eq.(3.1) can be obtained by differentiating $F_g[t]$ w.r.t. $t_{n_1}, \ldots, t_{n_s}$ since the correlators do not depend on $x_1, \ldots, x_s$. The total free energy can be obtained from the genus expansion

$$F[t; \lambda] = \sum_g \lambda^{2g-2} F_g[t]$$

with $\lambda^2$ being the string coupling constant. Note, that due to the relation (3.2), the $\lambda$-dependence of $F$ can be absorbed by the rescaling of $t_n$:

$$F[\ldots, t_{n_1}, \ldots; \lambda] = F[\ldots, \lambda^{\frac{3}{2}(n-1)}t_{n_1}, \ldots; 1].$$

The Kontsevich approach [Kon91] to evaluate $F[t; \lambda]$, given by Eq.(3.4), is based on a combinatorial calculation of the intersection indices on $\mathcal{M}_{g,s}$. Let us represent Eq.(3.3) as

$$F_g[t] = \sum_{s \geq 0} \sum_{n_1, \ldots, n_s} \frac{1}{s!} F_{g,s}^{(n_1, \ldots, n_s)}$$

We use in this part of the talk the Witten’s normalization [Wit90] of 2D topological gravity ($N(n) = n!$).
where
\[ F_{g,s}^{(n_1,\ldots,n_s)} = \langle \sigma_{n_1}(x_1) \cdots \sigma_{n_s}(x_s) \rangle_g t_{n_1} \cdots t_{n_s}. \]  
(3.7)

The last quantity can be interpreted as a contribution from a band graph (or a fat graph in Penner’s terminology [Pen88]) of genus \( g \) with \( s \) loops and three bands linked at each vertex. These graphs were introduced in quantum field theory by ’t Hooft [Hoo74]. The original Riemann surface with \( s \) punctures can be obtained from this band graph by shrinking the boundaries of bands (forming loops) into the punctures.

As is proven by Kontsevich [Kon91],
\[ \sum_{n_1,\ldots,n_s} \langle \sigma_{n_1}(x_1) \cdots \sigma_{n_s}(x_s) \rangle_g \prod_i \frac{(2n_i - 1)!!}{N(n_i)} \text{tr} \Lambda^{-2n_i-1} = \sum_\text{graphs}_{g,s} \sum_{\text{#(vert.)}} \prod_{\text{links}_{i,j}} \frac{2^{-(\text{#(vert.)})}}{\text{#(aut.)}} \frac{2}{\Lambda_i + \Lambda_j} \]  
(3.8)

where \( \Lambda_i \) are eigenvalues of a \( N \times N \) hermitean matrix \( \Lambda \) and the sum goes over the connected band graphs with \( \text{#(vert.)} \) vertices. The product goes over the links of the graph. Each link carries two indices \( i, j \) which are continuous along the loops while each of \( s \) traces on the l.h.s. corresponds to the summation over the index along one of \( s \) loops. The combinatorial factor \( \text{#(aut.)} \) in the denominator is due to a symmetry of the graph.

Substituting Eq.(3.8) into Eqs.(3.7), (3.6), identifying
\[ t_n = \frac{\lambda(2n - 1)!!}{N(n)} \text{tr} (\Lambda^{-2n-1}) \]  
(3.9)

and making use of Eq.(3.5), one represents the r.h.s. of Eq.(3.4) in the form of the logarithm of the partition function
\[ Z_{\text{Konts}}[\Lambda; \lambda] \equiv \int D\!X e^{\text{tr}(\frac{2i}{\Lambda} X^3 - \frac{1}{2} \Lambda X^2)} \]  
(3.10)

where the integral goes over the hermitean \( N \times N \) matrix \( X \). The original normalization of [Kon91] corresponds to \( \lambda = -1 \).

There is, however, a subtlety in the identification (3.9). The point is that, for an \( N \times N \) matrix \( \Lambda^{-1} \), \( \text{tr} (\Lambda^{-k}) \) are independent only for \( 1 \leq k \leq N \) while, say \( \text{tr} (\Lambda^{-N-1}) \) is reducible. All \( \text{tr} (\Lambda^{-2n-1}) \) become independent, as it should be for the sources in 2D topological gravity, as \( N \to \infty \). Therefore
\[ \log Z_{\text{Konts}}[\Lambda; \lambda] \to F[t; \lambda]. \]  
(3.11)

only as \( N \to \infty \). The equality (3.11) is valid in a sense of an asymptotic expansion at large \( \Lambda \) with each term being finite providing \( \Lambda \) is positively defined.

Let us explain the Kontsevich results from the viewpoint of the standard analysis of the matrix model (3.10). \( Z_{\text{Konts}}[\Lambda; \lambda] \) admits the perturbative expansion in \( \lambda \) that starts from the term \( O(\lambda) \). This term corresponds to the contribution of three puncture operators in genus zero [Wit90]:
where $\lambda^2$ in the denominator emerges because of Eq. (3.9). The contribution of a generic graph with #(vert.) vertices, #(link) links and $s$ loops is proportional to

$$\lambda^{\#(\text{vert.})/2}N^s = (\lambda N)^s\lambda^{2g-2}$$

in an agreement with Eq. (3.8).

Notice that while $N \to \infty$, all terms of the perturbative expansion in $\lambda$ contribute to $F[t; \lambda]$ in contrast to the standard large-$N$ expansion by ‘t Hooft [Hoo74] when an expansion in $1/N^2$ emerges so that $N = \infty$ corresponds to planar graphs only. The ‘t Hooft case can be reproduced if $\lambda \sim N^{-1}$. Then $F \sim N^2$ while

$$W_{\text{Konts}}[\Lambda; 1/N] \equiv \frac{1}{N^2} \log Z_{\text{Konts}}[\Lambda; 1/N] \to \frac{1}{N^2} F[t; 1/N]$$

is finite. Therefore, $W_{\text{Konts}}[\Lambda; 0] = F_0[t]$ can be obtained in the ‘t Hooft planar limit. This fact has been utilized in [MS91a] to solve the Kontsevich model in genus zero.

### 3.2 The Schwinger-Dyson equations

The Kontsevich model can be studied using the custom methods of solving matrix models. Since $\Lambda$ in Eq. (3.10) is a matrix, the standard orthogonal polynomial technique can not be applied. For this reason, the method of Schwinger-Dyson equations has been applied to this problem [MS91a, MMM91].

To derive the Schwinger–Dyson equations, it is convenient to make a linear shift of the integration variable in the numerator on the r.h.s. of Eq. (3.10). Modulo an unessential constant, one gets

$$Z_{\text{Konts}}[\Lambda; \lambda] = \prod_i \sqrt{\Lambda_i} \prod_{i>j} (\Lambda_i + \Lambda_j) e^{-\frac{\Lambda^3}{3}} Z \left[ \frac{\Lambda^2}{(2\lambda)^{3/2}} \right]$$

where

$$Z[M] = \int D\!X \ e^{tr(-\frac{X^3}{3}+MX)}$$

and the Gaussian integral in the denominator has been calculated.

$Z[M]$ which is defined by Eq. (3.16) is the standard partition function of the hermitean one-matrix model in an $N \times N$ matrix external field $M$. This external field problem, which is analogous to the corresponding problem [BN81, BG80, BRT81] for the unitary matrix model, has been studied recently in [GN91, MS91a]. While the representation (3.10) is convenient for constructing the perturbation theory expansion, the partition function $Z[M]$ is convenient for deriving Schwinger-Dyson equations.

The partition function (3.16) depends on $N$ invariants, $m_i$, — the eigenvalues of $M$. Let us perform the integral over angular variables in the standard way [Z81, Meh81] to express $Z[M]$ as the integral over $x_i$ — the eigenvalues of $X$. Modulo an irrelevant multiplicative constant, the result reads
where $\Delta[m] = \prod_{i<j} (m_i - m_j)$ is the Vandermonde determinant.

The Schwinger–Dyson equations result from the following change of variables in (3.17): $x_i \rightarrow x_i + \epsilon_n x_i^{n+1}$ for $i = 1, \ldots, N$ and $n \geq -1$ in full analogy to the matrix model without external field [AJM90, LM91, MM90]. Noticing that $x_i$ in the integrand can be replaced by $\frac{\partial}{\partial m_i}$ when applied to $\Delta[m]Z[m]$, the set of Schwinger–Dyson equations can be written in the form [MS91a]

$$L_n \Delta[m]Z[m] = 0 \quad \text{for} \quad n \geq -1$$

with

$$L_n = \sum_i \left\{ - \left( \frac{\partial}{\partial m_i} \right)^{n+3} + \left( \frac{\partial}{\partial m_i} \right)^{n+1} m_i + \frac{1}{2} \sum_{k=0}^{n} \sum_{j \neq i} (\frac{\partial}{\partial m_i})^k (\frac{\partial}{\partial m_j})^{n-k} \right\}. \quad (3.19)$$

It is easy to verify by a direct calculation that these operators obey Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m}. \quad (3.20)$$

The Virasoro generators (3.19) annihilate the totally antisymmetric function $\Delta[m]Z[m]$. One can easily construct the generators $L_n$ which annihilate $Z[m]$ itself. Let us introduce for this purpose the ‘long’ derivatives

$$\nabla_i \equiv \Delta^{-1}[m] \frac{\partial}{\partial m_i} \Delta[m] = \frac{\partial}{\partial m_i} + \sum_{j \neq i} \frac{1}{m_i - m_j} \Delta[m]$$

which commute one with each other. The Virasoro constraints (3.18), (3.19) now take the form [MS91a]

$$L_n Z[m] = 0 \quad \text{for} \quad n \geq -1$$

and

$$L_n = \sum_i \left\{ -(\nabla_i)^{n+3} + (\nabla_i)^{n+1} m_i + \frac{1}{2} \sum_{k=0}^{n} \sum_{j \neq i} (\nabla_i)^k (\nabla_j)^{n-k} \right\}. \quad (3.23)$$

Due to the commutativity of $\nabla$’s, the order in the last term in unessential. As follows from the definition (3.21), the generators (3.23) obey the Virasoro algebra commutation relations, same as (3.20).

The Virasoro constraints (3.22), (3.23) turn out to be equivalent to the following equation

$$\left\{ (\partial_i)^2 + \sum_{j \neq i} \frac{1}{m_i - m_j} (\partial_i - \partial_j) - \frac{m_i}{N} \right\} Z[m] = 0 \quad (3.24)$$

which is called in [MS91a] the ‘master equation’. As is shown in [GN91, MMM91], Eq.(3.24) results from shifting $X$ in Eq.(3.16) by an arbitrary (hermitean) matrix while Eqs. (3.18), (3.19) (or (3.22), (3.23)) result from the shift $X \rightarrow X + \epsilon_n X^{n+1}$.

3.3 The genus-zero solution
Brézin and Gross [BG80] with the aid of the Riemann–Hilbert method. The corresponding solution had been first found by Kazakov and Kostov [KK89] and is discussed in [GN91, MS91a].

Substituting this solution into Eq.(3.15) which expresses the partition function of the Kontsevich model via that for the hermitean matrix in an external field, one gets in genus zero

$$F_0 = \frac{1}{N} \sum_i \left\{ \frac{1}{3} (\Lambda_i^2 - 2u)^2 + u\sqrt{\Lambda_i^2 - 2u} + \frac{u^3}{6} - \frac{\Lambda_i^3}{3} \right\} + \frac{1}{2N} \sum_j \left[ \log (\Lambda_i + \Lambda_j) - \log \left( \sqrt{\Lambda_i^2 - 2u} + \sqrt{\Lambda_j^2 - 2u}\right) \right]$$

(3.25)

where $u[\Lambda]$ is determined by

$$u = \frac{1}{N} \sum_i \frac{1}{\sqrt{(\Lambda_i^2 - 2u)}}.$$  

(3.26)

The solution (3.25) is similar to the strong coupling solution of [BG80] while Eq.(3.26) has emerged to guarantee correct analytic properties. It is important that the r.h.s. of Eq.(3.25) is stationary w.r.t. $u$ due to Eq.(3.26). For a constant field $\Lambda_i = (6g)^{-\frac{2}{3}}$, this solution recovers the results of Brézin et al. [BIPZ78] for the case of a cubic interaction.

Eq.(3.26) can be rewritten in the form of the string equation of a ‘general massive model’ [BDSS90] in genus zero. To this aim, let us expand the r.h.s. of Eq.(3.26) in $u$ and substitute

$$t_n = \frac{1}{N} \frac{(2n - 1)!! n!}{n!} \sum_i \Lambda_i^{-2n-1}.$$  

(3.27)

This equation is nothing but Eq.(3.9) with $N(n) = n!$ which fixes the normalization [Wit90] of 2D topological gravity and $\lambda = \frac{1}{N}$ as is prescribed by Eq.(3.14). We rewrite Eq.(3.26) finally as

$$u = \sum_{n=0}^{\infty} t_n u^n.$$  

(3.28)

The precise form of the genus-zero string equation can be obtained by the well-known shift [DW90]: $t_1 \rightarrow t_1 + 1$.

### 3.4 Relation to 2D topological and quantum gravities

It is instructive to compare the solution (3.25) of the Kontsevich model with known results for the partition functions of 2D topological and quantum gravities in genus zero. To obtain the perturbative expansion of $F_0[t]$, one solves Eq.(3.28) by iterations in $u$:

$$u = t_0 + t_0 t_1 + t_0 t_1^2 + t_0^2 t_2 + \ldots$$  

(3.29)

with $t_n \sim t_0^{2n+1}$, and substitutes the result into the r.h.s. of Eq.(3.25) which is expanded
While the complicated structure of the perturbative expansion of $F_0[t]$ represents the variety of planar band graphs (taken with appropriate combinatorial coefficients), great simplifications occur for derivatives of $F_0[t]$. Let us define $D$ by

$$D = 2 \sum_i \frac{\partial}{\partial (\Lambda_i^2)}.$$  \hspace{1cm} (3.30)

Then, by a direct differentiation of Eq.(3.25), one gets

$$D F_0 = \frac{t_0^2}{2} + \frac{u^2}{2} - \sum_{k=0}^{\infty} t_k \frac{u^{k+1}}{(k+1)}.$$  \hspace{1cm} (3.31)

This expression is again stationary w.r.t. $u$ due to Eq.(3.28) so that one more application of $D$ yields

$$D^2 F_0 = u - t_0(1 + t_1).$$  \hspace{1cm} (3.32)

To compare Eq.(3.25) with the known solution of 2D topological gravity in genus zero, let us notice that $D$ defined by (3.30) can be rewritten using Eq.(3.27) as

$$D = -\sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_{n-1}}.$$  \hspace{1cm} (3.33)

This is exactly the operator entering the puncture equation [DW90] which reads in genus zero:

$$\frac{\partial F_0}{\partial t_0} = \frac{t_0^2}{2} - D F_0.$$  \hspace{1cm} (3.34)

Therefore, one gets from Eq.(3.31)

$$\frac{\partial F_0}{\partial t_0} = \sum_{k=0}^{\infty} t_k \frac{u^{k+1}}{(k+1)} - \frac{u^2}{2},$$  \hspace{1cm} (3.35)

which is a true formula that gives in particular

$$\frac{\partial^2 F_0}{\partial t_0^2} = u.$$  \hspace{1cm} (3.36)

Since $t_0$ is the cosmological constant, one sees from this formula $u$ to be the string susceptibility.

Using Eq.(3.36), one can immediately calculate the critical index $\gamma_{string}$. For $K^{th}$ multi-critical point, when one puts all $t_n = 0$ except for $n = 0$ and $n = K$, Eq.(3.28) yields $u \propto x^{\frac{1}{K}}$, $\gamma_{string} = -\frac{1}{K}$ in full analogy to [Kaz89]. Notice, however, that the solution (3.25) is associated with the continuum interpolating model while in the standard case one ‘interpolates’ by a matrix model whose couplings should be turned to critical values in order to reach the continuum limit.

Finally, let us mention that the genus-zero solution (3.25) can be rewritten exactly in the form of that for the hermitean one-matrix model in the continuum limit. Let us first note that Eq.(3.33) can be viewed as an integrated version of Eq.(3.30):
where $u(t_0)$ is a solution of Eq. (3.28) which is considered as a function of $t_0$ at fixed values of $t_n$ for $n \geq 1$. The integration constant is fixed by the fact that $u(0) = 0$ which is nothing but the condition that chooses the perturbative solution of the string equation.

One more integration of Eq. (3.37) yields

$$F_0 = \int_0^{t_0} dx(t_0 - x)u(x) + \Phi(t_1, t_2, \ldots).$$

(3.38)

which coincides with the representation (2.32) of the free energy for 2D quantum gravity. As has been proven recently [Wit91, MMM91], the Kontsevich model obeys the same set of Virasoro constraints (2.55), (2.56) as 2D quantum gravity. This demonstrates an equivalence of 2D topological and quantum gravities to arbitrary genus.

4 Concluding remarks

Loop equations turned out to be a useful tool in studies of matrix models as well as of their continuum limit associated with 2D quantum gravity with matter. The point is that loop equations are literally the Virasoro constraint imposed on the partition function. In the continuum limit, this Virasoro symmetry represents the underlying conformal invariance.

The appearance of new symmetries of loop equations (as well as the very idea of the ‘double scaling limit’ [BK90, DS90, GM90]) is very interesting from the viewpoint of multi-dimensional loop equations (see [Mig83]). A step along this line has been done in [FKN91, DVV91, Goe91] where the $W$-algebras were associated with the continuum limit of multi-matrix models. It would be interesting to find an analog of this symmetry for multi-matrix models at finite $N$.

The fact that the Kontsevich matrix model is a solution of the continuum Virasoro constraints (and, therefore, of the continuum loop equations) throws light on the origin of Virasoro constraints. It is non-trivial that this matrix model appears as a solution of the continuum loop equation. This seems to be analogous to the known property [Mig83] of multi-dimensional loop equations which possess solutions differing from the original Yang–Mills path integral.
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