A POTENTIAL THEORETIC CHARACTERIZATION OF COMPACTNESS OF THE $\bar{\partial}$-NEUMANN PROBLEM

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ABSTRACT. We give a potential theoretic characterization for compactness of the $\bar{\partial}$-Neumann problem on smooth bounded pseudoconvex domains in $\mathbb{C}^n$.

Let $\Omega$ be a domain in $\mathbb{C}^n$ with $C^\infty$-smooth boundary. The domain $\Omega$ is said to be pseudoconvex if the Levi form of $\Omega$, the restriction of the complex Hessian of a defining function onto complex tangent space, is positive semi-definite on the boundary, $b\Omega$, of $\Omega$. On bounded pseudoconvex domains, Hörmander [Hör65] showed that the $\bar{\partial}$-Neumann operator on $\Omega$, the solution operator for $\Box = \bar{\partial}\partial + \partial\bar{\partial}$ and $\bar{\partial}$ is the Hilbert space adjoint of $\bar{\partial}$). We refer the reader to [CS01, Str10] for more information about the $\bar{\partial}$-Neumann problem.

Compactness of the $\bar{\partial}$-Neumann operator is important to study as it is weaker than global regularity [KN65] and it interacts with the boundary geometry. For example, even though the general case is still open, in some cases it is known that existence of an analytic disc in the boundary is an obstruction for compactness of the $\bar{\partial}$-Neumann problem (see, for example, [FS98, FS01, Şah06, ŞS06, Str10]). Recently, Straube and Munasinghe [MS07] (see also [Mun06]) studied compactness using geometric conditions involving short time flows on the boundary (this was done in $C^2$ earlier by Straube [Str04]). Çelik and Straube [CS09] (see also [Cel08]) explored compactness in relation to the so called “compactness multipliers”.

Compactness of the $\bar{\partial}$-Neumann problem has been studied using some potential theoretic conditions by Catlin [Cat84] using property $(P)$ and later by McNeal [McN02] using property $(\tilde{P})$. In this paper we would like to give a new potential theoretic characterization for compactness of the $\bar{\partial}$-Neumann problem. We refer the reader to [Str10, Proposition 4.2] for other equivalent conditions. We would like to note that a similar characterization has been done by Haslinger in [Has].

Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^n, K \subset b\Omega$, and $U$ be an open neighborhood of $K$. We denote the $L^2$ norm and Sobolev $-1$ norm of a function $f \in L^2(\Omega)$ by $\|f\|$ and $\|f\|_{-1}$, respectively. Let $I = \{i_1, i_2, \ldots, i_p\} \subset \mathbb{N}$ such that $j_1 < i_2 < \cdots < i_p$. Then we use the notation $dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}$ and
\begin{align*}
dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}. \quad & \text{Define} \\
C^\infty_{0, (p, q)}(U) = \left\{ \sum_{|I| = p, |J| = q} f_{IJ} dz_I \wedge d\bar{z}_J : f_{IJ} \in C^\infty(U) \right\} \\
\text{for } 0 \leq q \leq n. \quad & \text{Define } \lambda_{(p, q)}(U) \text{ as follows: for } 0 \leq p \leq n \text{ and } 1 \leq q \leq n - 1 \text{ let us define} \\
\lambda_{(p, q)}(U) = \inf \left\{ \frac{\| \bar{\partial} f \|^2 + \| \bar{\partial}^* f \|^2}{\| f \|^2} : f \in \text{Dom}(\bar{\partial}^* \cap C^\infty_{0, (p, q)}(U), f \neq 0 \right\} \\
\text{and} \\
\lambda_{(p, 0)}(U) = \inf \left\{ \frac{\| \bar{\partial} f \|^2}{\| f \|^2} : f \in (\text{Ker} \bar{\partial})^\perp \cap C^\infty_{0, (p, 0)}(U), f \neq 0 \right\} \\
\text{where } (\text{Ker} \bar{\partial})^\perp \text{ is the orthogonal complement of } (\text{Ker} \bar{\partial}) \text{ in } L^2_{p, 0}(\Omega) \text{ (square integrable } (p, 0)\text{-forms on } \Omega). \text{ Notice that } \lambda_{(p, q)}(U) \leq \lambda_{(p, q)}(V) \text{ if } V \subset U. \text{ In this paper a finite type is meant in the sense of D’Angelo } [D’A82] \text{ and infinite type point means a point that is not finite type.}
\end{align*}

**Theorem 1.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$ and $0 \leq p \leq n, 0 \leq q \leq n - 1$, be given. Then the following are equivalent:

(i) the $\bar{\partial}$-Neumann operator $N_{(p, q)}$ of $\Omega$ is compact on $L^2_{p, 0}(\Omega)$,

(ii) for any compact set $K \subset b\Omega$ and $M > 0$ there exists an open neighborhood $U$ of $K$ such that $\lambda_{(p, q)}(U) > M$,

(iii) for any $M > 0$ there exists an open neighborhood $U$ of the set of infinite type points in $b\Omega$ such that $\lambda_{(p, q)}(U) > M$.

**Remark 1.** The definition of $\lambda_{(p, q)}$ is closely connected to the so-called compactness estimates (see [1] in the proof of Theorem [1] as well as Morey-Kohn-Hörmander formula (see, for example, [CS01] Proposition 4.3.1) or [Str10] Proposition 2.4) and property (P) of Catlin.

One can show that the Morey-Kohn-Hörmander formula implies that for a smooth bounded pseudoconvex $\Omega \subset \mathbb{C}^n$, a non-positive function $b \in C^2(\overline{\Omega})$, and $u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap C^1_{(p, q)}(\overline{\Omega})$ we have

\begin{align*}
\sum_{j, k}^{n} \sum_{j_{k}}^{\prime} \int_{\Omega} e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{j, j_{k}} u_{j_{k} j} dV \leq \| \bar{\partial} u \|^2 + \| \bar{\partial}^* u \|^2.
\end{align*}

where $u = \sum_{j, k}^{\prime} \sum_{k=1}^{n} u_{j, j_{k}} dz^j \wedge d\bar{z}_k \wedge d\bar{z}_k$ and the prime indicates that the sum is taken over strictly increasing $(p, q - 1)$-tuples $(j, k)$. If the domain $\Omega$ satisfies property (P) then one can choose $b$ to be bounded from below by $-1$ and with arbitrarily large complex Hessian on the boundary of $\Omega$. Then on a small neighborhood on the boundary the Hessian is still large. Hence $\lambda_{(p, q)}(U)$ will be arbitrarily large for a sufficiently small neighborhood $U$ of $b\Omega$. 


We would like to give a simple example below to show that one can use this characterization to show that, in some cases, compactness of the $\overline{\partial}$-Neumann problem excludes analytic disks from the boundary. We do not claim any originality in this example as it is a special case of Catlin's result [FS01, Proposition 1].

**Example 1.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ such that $\Omega \subset \{z \in \mathbb{C}^2 : \text{Im}(z_2) < 0\}$ and $\{z \in \mathbb{C}^2 : \text{Im}(z_2) = 0, |z_1|^2 + |z_2|^2 < 1\} \subset \partial \Omega$.

**Claim:** The $\overline{\partial}$-Neumann operator on $\Omega$ is not compact.

**Proof of the Claim:** There exist positive numbers $a_1 < a_2$ such that

$$D_1 \times W_1 \subset \Omega \cap \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\} \subset D_2 \times W_2$$

where $D_1 = \{z \in \mathbb{C} : |z| < 2/3\}$, $D_2 = \{z \in \mathbb{C} : |z| < 2\}$, and

$$W_1 = \{z = re^{i\theta} \in \mathbb{C} : 0 < r < a_1, -2\pi/3 < \theta < -\pi/3\},$$

$$W_2 = \{z = re^{i\theta} \in \mathbb{C} : 0 < r < a_2, -4\pi/3 < \theta < \pi/3\}.$$

Let $\phi(z_1, z_2) = f(z_1)g_j(z_2)d\overline{z}_1$ where $f \in C_0^\infty(D_1)$ and $f \not\equiv 0$. Later on we will choose $g_j \in C_0^\infty(\{z \in \mathbb{C} : |z| < j^{-2}\})$ so that $\phi_j \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*)$. There exists $a_3 > 0$ such that $D_1 \times W \subset \Omega$, where $W = \{z \in \mathbb{C} : \text{Im}(z) < 0, |z| < a_3\}$, and $\phi_j(z_1, z_2) = 0$ for $z \in \overline{\Omega} \setminus D_1 \times W$ and $j^{-2} < a_3$. Then for $j^{-2} < a_3$ we have

$$\frac{||\overline{\partial}\phi_j||^2 + ||\overline{\partial}^*\phi_j||^2}{||\phi_j||^2} \leq \frac{||\overline{\partial}g_j(z_2)||^2}{||g_j(z_2)||^2} \frac{||f||^2}{||f||^2} + \frac{||\overline{\partial}f(z_1)||^2}{||f||^2} \frac{||g(z_2)||^2}{||g||^2} \frac{||\overline{\partial}\phi_j||^2}{||\phi_j||^2}$$

Let us choose real valued non-negative functions $\chi_j \in C_0^\infty(\mathbb{R}^2, j^{-2})$ such that $\chi_j(-t) = \chi_j(t)$ and $\chi(t) = 1$ for $|t| \leq \frac{1}{4j}$. Since $z^{-2}$ is not integrable on $W_1 \cap B(0, \varepsilon)$ for any $\varepsilon > 0$, we can choose a positive real number $\alpha_j$ so that

$$\int_{W_1 \cap B(0, 1/j)} \frac{||\chi_j'||^2}{|z_2 - i\alpha_j|^2} dV(z_2) \leq \int_{W_1 \cap B(0, 1/j)} \frac{||\chi_j^2||^2}{|z_2 - i\alpha_j|^2} dV(z_2).$$

Now we define $g_j(z_2) = \chi_j(|z_2|^2) \tau_j(z_2)(z_2 - i\alpha_j)^{-1}$ where $\tau_j \in C^\infty(\mathbb{R})$ such that $\tau_j(z) \equiv 1$ for $\text{Im}(z) \leq 0$ and $\tau_j(z) \equiv 0$ for $\text{Im}(z) \geq \alpha_j/2$. Then we have $\phi_j \in C_0^\infty(U_j) \cap \text{Dom}(\overline{\partial}^*)$ where $U_j = \{z \in \mathbb{C} : |z| < 2^{-1} + j^{-1}\} \times \{z \in \mathbb{C} : |z| < j^{-2}\}$ and

$$||\frac{\partial g_j}{\partial z_2}||_{L^2(W_2 \cap B(0, 1/j))} \leq ||g_j||_{L^2(W_1 \cap B(0, 1/j))}.$$
Hence, we constructed a sequence of $(0,1)$-forms $\{\phi_j\}$ such that $\phi_j \in C^\infty_{0,(0,1)}(U_j) \cap \text{Dom}(\bar{\tau}^\dagger)$ where

$$K = \{ z \in \mathbb{C}^2 : |z_1| \leq 1/2, z_2 = 0 \} = \bigcap_{j=1}^\infty U_j \subset b\Omega$$

and $\frac{\|\bar{\tau}\phi_j\|^2 + \|\bar{\tau}^* \phi_j\|^2}{\|\phi_j\|^2}$ stays bounded as $j \to \infty$. Hence, by Theorem [2] the $\bar{\tau}$-Neumann operator on $\Omega$ is not compact.

**Proof of Theorem [2]**

Proof of Theorem [2] We will show the equivalences for $0 \leq p \leq n$ and $1 \leq q \leq n - 1$. The proof can be mimicked for the case $q = 0$ using the following: compactness of $N_0$ is equivalent to the following compactness estimate: for all $\varepsilon > 0$ there exists $D_\varepsilon > 0$ such that

$$\|g\|^2 \leq \varepsilon \|\bar{\tau}g\|^2 + D_\varepsilon \|g\|^2_{L^\infty}$$

for $g \in (\text{Ker} \bar{\tau})^\perp \cap \text{Dom}(\bar{\tau})$.

First let us prove that $(i)$ implies $(ii)$. Assume that the $\bar{\tau}$-Neumann operator of $\Omega$ is compact, and there exist $K \subset b\Omega$ and $M > 0$ such that $\lambda(p,q)(U) < M$ for all open neighborhoods $U$ of $K$. We may assume that there exist sequences of open neighborhoods $\{U_k\}$ of $K$ and nonzero $(p,q)$-forms $\{f_k\}$ such that

1. $U_{k+1} \Subset U_k$, $K \subset \bigcap_{k=1}^\infty U_k \subset b\Omega$, $f_k \in \text{Dom}(\bar{\tau}^\dagger) \cap C^\infty_{0,(p,q)}(U_k)$,
2. $\|f_k\|^2 = 1$, and $\|\bar{\tau}f_k\|^2 + \|\bar{\tau}^* f_k\|^2 < M$ for $k = 1, 2, 3, \ldots$

Since $K \subset \bigcap_{k=1}^\infty U_k \subset b\Omega$ (hence $K$ has measure 0 in $\mathbb{C}^n$) and $f_k \in C^\infty_{0,(p,q)}(U_k)$, by passing to a subsequence if necessary, we may assume that $\|f_k - f_l\|^2 \geq 1/2$. Compactness of the $\bar{\tau}$-Neumann operator is equivalent to the following so called compactness estimate (see [Str10] Proposition 4.2 or [FS01] Lemma 1)): for all $\varepsilon > 0$ there exists $D_\varepsilon > 0$ such that

$$\|g\|^2 \leq \varepsilon (\|\bar{\tau}g\|^2 + \|\bar{\tau}^* g\|^2) + D_\varepsilon \|g\|^2_{L^\infty}$$

for $g \in (\text{Ker} \bar{\tau})^\perp \cap \text{Dom}(\bar{\tau})$.

Choose $\varepsilon = \frac{1}{16M}$. Since $\text{Dom}(\bar{\tau}^\dagger) \cap C^\infty_{0,(p,q)}(U_k) \subset \text{Dom}(\bar{\tau}^\dagger) \cap \text{Dom}(\bar{\tau})$ using $(1)$ and ii. above we get

$$\|f_k - f_l\|^2_{L^\infty} \geq \frac{1}{4D_\varepsilon} > 0 \text{ for } k \neq l$$

The imbedding from $L^2(\Omega)$ to $W^{-1}(\Omega)$ is compact and $\{f_k\}$ is a bounded sequence in $L^2_{(p,q)}(D)$. Hence $\{f_k\}$ has a convergent subsequence in $W_{(p,q)}^{-1}(\Omega)$. This contradicts with $(2)$.

$(ii)$ obviously implies $(iii)$ so we will skip this part.

Next let us prove that $(iii)$ implies $(i)$. Let $K$ be the set of infinite type points in $b\Omega$ and $u \in \text{Dom}(\bar{\tau}^\dagger) \cap C^\infty_{(p,q)}(\Omega)$. Assume that $\lambda(p,q)(U_k) > k$ where $\{U_k\}$ is a sequence of open neighborhoods of $K$ such that $U_{k+1} \Subset U_k$ and $K \subset \bigcap_{k=1}^\infty U_k \subset b\Omega$. Let $\varphi_k \in C^\infty_0(U_k)$ such that $0 \leq \varphi_k \leq 1$ and $\varphi_k \equiv 1$ in a neighborhood of $K$. Define $\psi_k = 1 - \varphi_k$. Notice that $\psi_k$ is supported away from $K$. In following
estimates, $C_k$ and $C_{k,\varepsilon}$ are general constants meaning that the constants depend on the subscripts only but they might change at each step. Away from $K$ we have subelliptic estimates as $b\Omega \setminus K$ is the set of finite type points (see [Cat87]). Hence, there exists $s > 0$ for all $\varepsilon > 0$ there exists $D_\varepsilon > 0$ such that

$$
\|\psi_k u\|_s^2 \leq \varepsilon\|\psi_k u\|_s^2 + D_\varepsilon\|\psi_k u\|_s^{2-1}
$$

(3)

$$
\leq \varepsilon C_k(\|\overline{\partial}(\psi_k u)\|_2^2 + \|\overline{\partial}^* (\psi_k u)\|_2^2) + C_{k,\varepsilon}\|u\|_s^{2-1}
$$

$$
\leq \varepsilon C_k(\|\overline{\partial}u\|_2^2 + \|\overline{\partial}^* u\|_2^2 + \|u\|_2^2) + C_{k,\varepsilon}\|u\|_s^{2-1}
$$

The first inequality follows because $L^2$ imbedds compactly into $W^s$ for $s > 0$. We used the compactness estimate for the second inequality. If we use $\lambda_{(p,q)}(U_k) > k$ we get:

$$
\|\varphi_k u\|_s^2 \leq \frac{1}{k}(\|\overline{\partial}(\varphi_k u)\|_2^2 + \|\overline{\partial}^* (\varphi_k u)\|_2^2)
$$

(4)

$$
\leq \frac{1}{k}(\|\overline{\partial}u\|_2^2 + \|\overline{\partial}^* u\|_2^2) + D_k\|\varphi_k u\|_s^2
$$

where $\varphi_k \equiv 0$ in a neighborhood of $K, D_k > 0$, and $\varphi_k \equiv 1$ in a neighborhood of the support of $\varphi_k$. Calculations that are similar to ones in (3) show that

$$
\|\varphi_k u\|_s^2 \leq \varepsilon \hat{C}_k(\|\overline{\partial}u\|_2^2 + \|\overline{\partial}^* u\|_2^2 + \|u\|_2^2) + \hat{C}_{k,\varepsilon}\|u\|_s^{2-1}
$$

(5)

By choosing $\varepsilon, \varepsilon' > 0$ small enough and combining (3) and (5) we get the following estimate: for all $k = 1, 2, 3, \cdots$ there exists $M_k > 0$ such that

$$
\|u\|_s^2 \leq \frac{2}{k}(\|\overline{\partial}u\|_2^2 + \|\overline{\partial}^* u\|_2^2 + \|u\|_2^2) + M_k\|u\|_s^{2-1} \quad \text{for } u \in \text{Dom}(\overline{\partial}^*) \cap C_\infty^{(p,q)}(\overline{\Omega})
$$

(6)

We note that $\text{Dom}(\overline{\partial}^*) \cap C_\infty^{(p,q)}(\overline{\Omega})$ is dense in $\text{Dom}(\overline{\partial}^*) \cap \text{Dom}(\overline{\partial})$. Therefore, the above estimate (6) holds on $\text{Dom}(\overline{\partial}^*) \cap \text{Dom}(\overline{\partial})$. That is, the $\overline{\partial}$-Neumann operator of $\Omega$ is compact on $(p,q)$-forms for $0 \leq p \leq n$ and $1 \leq q \leq n - 1$.

\[\square\]

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