Local rings of embedding codepth 3.
A classification algorithm

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17 February 2014

Abstract

Let $I$ be an ideal of a regular local ring $Q$ with residue field $k$. The length of the minimal free resolution of $R = Q/I$ is called the codepth of $R$. If it is at most 3, then the resolution carries a structure of a differential graded algebra, and the induced algebra structure on $\text{Tor}_Q^*(R, k)$ provides for a classification of such local rings.

We describe the Macaulay 2 package CodepthThree that implements an algorithm for classifying a local ring as above by computation of a few cohomological invariants.

1 Introduction and notation

Let $R$ be a commutative noetherian local ring with residue field $k$. Assume that $R$ has the form $Q/I$ where $Q$ is a regular local ring with maximal ideal $n$ and $I \subseteq n^2$. The embedding dimension of $R$ (and of $Q$) is denoted $e$. Let

$$F = 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be a minimal free resolution of $R$ over $Q$. Set $d = \text{depth} R$; the length $c$ of the resolution $F$ is by the Auslander–Buchsbaum formula

$$c = \text{proj.dim}_Q R = \text{depth} Q - \text{depth}_Q R = e - d,$$

and one refers to this invariant as the codepth of $R$. In the following we assume that $c$ is at most 3. By a theorem of Buchsbaum and Eisenbud [3, 3.4.3] the resolution $F$ carries a differential graded algebra structure, which induces a unique graded-commutative algebra structure on $A = \text{Tor}_Q^*(R, k)$. The possible structures were identified by Weyman [5] and by Avramov, Kustin, and Miller [2]. According to the multiplicative structure on $A$, the ring $R$

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*Part of this work was done while the authors visited MSRI during the Commutative Algebra program in spring 2013. LWC was partly supported by NSA grant H98230-11-0214.
belongs to exactly one of the classes designated $B$, $C(c)$, $G(r)$, $H(p, q)$, $S$, and $T$. Here the parameters $p$, $q$, and $r$ are given by

$$p = \text{rank}_k(A_1 \cdot A_1), \quad q = \text{rank}_k(A_1 \cdot A_2), \quad \text{and} \quad r = \text{rank}_k(\delta: A_2 \to \text{Hom}_k(A_1, A_3)),$$

where $\delta$ is the canonical map. See [1, 2, 5] for further background and details.

When, in the following, we talk about classification of a local ring $R$, we mean the classification according to the multiplicative structure on $A$. To describe the classification algorithm, we need a few more invariants of $R$. Set

$$l = \text{rank}_Q F_1 - 1 \quad \text{and} \quad n = \text{rank}_Q F_c;$$

the latter invariant is called the type of $R$. The Cohen–Macaulay defect of $R$ is $h = \dim R - d$. The Betti numbers $\beta_i$ and the Bass numbers $\mu_i$ record ranks of cohomology groups,

$$\beta_i = \beta^R_i(k) = \text{rank}_k \text{Ext}^i_R(k, k) \quad \text{and} \quad \mu_i = \mu_i(R) = \text{rank}_k \text{Ext}^i_R(k, R).$$

The generating functions $\sum_{i=0}^{\infty} \beta_i t^i$ and $\sum_{i=0}^{\infty} \mu_i t^i$ are called the Poincaré series and the Bass series of $R$.

2 The algorithm

For a local ring of codepth $c \leq 3$, the class together with the invariants $e$, $c$, $l$, and $n$ completely determine the Poincaré series and the Bass series of $R$; see [1]. Conversely, one can determine the class of $R$ based on $e$, $c$, $l$, $n$, and a few Betti and Bass numbers; in the following we describe how.

Lemma 1. For a local ring $R$ of codepth 3 the invariants $p$, $q$, and $r$ are determined by $e$, $l$, $n$, $\beta_2$, $\beta_3$, $\beta_4$, and $\mu_{e-2}$ through the formulas

$$p = n + le + \beta_2 - \beta_3 + \binom{e-1}{3}, \quad q = (n - p)e + l\beta_2 + \beta_3 - \beta_4 + \binom{e-1}{4}, \quad \text{and} \quad r = l + n - \mu_{e-2}.$$

Proof. The Poincaré series of $R$ has by [1, 2.1] the form

$$\sum_{i=0}^{\infty} \beta_i t^i = \frac{(1 + t)^{e-1}}{1 - t - lt^2 - (n - p)t^3 + qt^4 + \cdots},$$

and expansion of the rational function yields the expressions for $p$ and $q$.

One has $d = e - 3$ and the Bass series of $R$ has, also by [1, 2.1], the form

$$\sum_{i=0}^{\infty} \mu_i t^i = t^d n + (l - r)t + \cdots.$$ 

expansion of the rational function now yields the expression for $r$. \qed
Proposition 2. A local ring $R$ of codepth 3 can be classified based on the invariants $e$, $h$, $l$, $n$, $\beta_2, \beta_3, \beta_4$, $\mu_{e-2}$, and $\mu_{e-1}$.

Proof. First recall that one has $h = 0$ and $n = 1$ if and only if $R$ is Gorenstein; see [3, 3.2.10]. In this case $R$ is in class $C(3)$ if $l = 2$ and otherwise in class $G(l + 1)$.

Assume now that $R$ is not Gorenstein. The invariants $p$, $q$, and $r$ can be computed from the formulas in Lemma 1. It remains to determine the class, which can be done by case analysis. Recall from [1, 1.3 and 3.1] that one has

| Class | $p$ | $q$ | $r$ |
|-------|-----|-----|-----|
| $T$   | 3   | 0   | 0   |
| $B$   | 1   | 1   | 2   |
| $G(r)$ [$r \geq 2$] | 0   | 1   | $r$ |
| $H(p,q)$ | $p$ | $q$ | $q$ |

In case $q \geq 2$ the ring $R$ is in class $H(p,q)$; for $q = 0,1$ the case analysis shifts to $p$.

In case $p = 0$ the distinction between the classes $G(r)$ and $H(0,q)$ is made by comparing $q$ and $r$; they are equal if and only if $R$ is in class $H(0,q)$.

In case $p = 1$ the distinction between the classes $B$ and $H(1,q)$ is made by comparing $q$ and $r$; they are equal if and only if $R$ is in class $H(1,q)$.

In case $p = 3$ the distinction between the classes $T$ and $H(3,q)$ is drawn by the invariant $\mu_{e-1}$. Recall the relation $d = e - 3$; expansion of the expressions from [1, 2.1] yields $\mu_{e-1} = \mu_{e-2} + ln - 2$ if $R$ is in $T$ and $\mu_{e-1} = \mu_{e-2} + ln - 3$ if $R$ is in $H(3,q)$.

In all other cases, i.e. $p = 2$ or $p \geq 4$, the ring $R$ is in class $H(p,q)$. □

Remark 3. One can also classify a local ring $R$ of codepth 3 based on the invariants $e$, $h$, $l$, $n$, $\beta_2, \ldots, \beta_5$, and $\mu_{e-2}$. In the case $p = 3$ one then discriminates between the classes by looking at $\beta_5$, which is $\beta_4 + \ell \beta_3 + (n - 3) \beta_2 + \tau$ with $\tau = 0$ if $R$ is in class $H(3,q)$ and $\tau = 1$ if $R$ is in class $T$. However, it is not possible to classify $R$ based on Betti numbers alone. Indeed, rings in the classes $B$ and $H(1,1)$ have identical Poincaré series and so do rings in the classes $G(r)$ and $H(0,1)$.

Remark 4. A local ring $R$ of codepth $c \leq 2$ can be classified based on the invariants $c$, $h$, and $n$. Indeed, if $c \leq 1$ then $R$ is a hypersurface; i.e. it belongs to class $C(c)$. If $c = 2$ then $R$ belongs to class $C(2)$ if and only if it is Gorenstein ($h = 0$ and $n = 1$); otherwise it belongs to class $S$.

Algorithm 5. From Remark 4 and the proof of Proposition 2 one gets the following algorithm that takes as input invariants of a local ring of codepth $c \leq 3$ and outputs its class.

INPUT: $c$, $e$, $h$, $l$, $n$, $\beta_2, \beta_3, \beta_4$, $\mu_{e-2}$, $\mu_{e-1}$

- In case $c \leq 1$ set $Class = C(c)$
- In case $c = 2$
  - If ($h = 0$ and $n = 1$) then set $Class = C(2)$
  - Else set $Class = S$
In case $c = 3$
  
  - if $(h = 0$ and $n = 1$ ) then set $r = l + 1$
    - if $r = 3$ then set $\text{Class} = C(3)$
    - else set $\text{Class} = G(r)$
  - else compute $p$ and $q$
    - if $(q \geq 2$ or $p = 2$ or $p \geq 4$ ) then set $\text{Class} = H(p, q)$
    - else compute $r$
      - In case $p = 0$
        - if $q = r$ then set $\text{Class} = H(0, q)$
        - else set $\text{Class} = G(r)$
      - In case $p = 1$
        - if $q = r$ then set $\text{Class} = H(1, q)$
        - else set $\text{Class} = B$
      - In case $p = 3$
        - if $\mu_{e-1} = \mu_{e-2} + ln - 2$ then set $\text{Class} = T$
        - else set $\text{Class} = H(3, q)$

**OUTPUT:** $\text{Class}$

**Remark 6.** Given a local ring $R = Q/I$ the invariants $e$ and $h$ can be computed from $R$, and $c$, $l$, and $n$ can be determined by computing a minimal free resolution of $R$ over $Q$. The Betti numbers $\beta_2, \beta_3, \beta_4$ one can get by computing the first five steps of a minimal free resolution $F$ of $k$ over $R$. Recall the relation $d = e - c$; the Bass numbers $\mu_{e-2}$ and $\mu_{e-1}$ one can get by computing the cohomology in degrees $d+1$ and $d+2$ of the dual complex $F^* = \text{Hom}_R(F, R)$. For large values of $d$, this may not be feasible, but one can reduce $R$ modulo a regular sequence $x = x_1, \ldots, x_d$ and obtain the Bass numbers as $\mu_{d+i}(R) = \mu_i(R/(x))$; cf. [3, 3.1.16].

### 3 The implementation

The *Macaulay 2* package *CodepthThree* implements Algorithm 5. The function *torAlgClass* takes as input a quotient $Q/I$ of a polynomial algebra, where $I$ is contained in the irrelevant maximal ideal $\mathfrak{N}$ of $Q$. It returns the class of the local ring $R$ obtained by localization of $Q/I$ at $\mathfrak{N}$. For example, the local ring obtained by localizing the quotient

$$Q[x, y, z]/(xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4)$$

is in class $G(2)$; see [4]. Here is how it looks when one calls the function *torAlgClass*.

Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "CodepthThree";
i2 : Q = QQ[x,y,z];
i3 : I = ideal (x*y^2,x*y*z,y*z^2,x^4-y^3*z,x*z^3-y^4);
o3 : Ideal of Q
i4 : torAlgClass (Q/I)
o4 = G(2)

Underlying \texttt{torAlgClass} is the workhorse function \texttt{torAlgData} which returns a hash table with the following data:

| Key     | Value                                                                 |
|---------|-----------------------------------------------------------------------|
| "c"     | codepth of $R$                                                        |
| "e"     | embedding dimension of $R$                                             |
| "h"     | Cohen–Macaulay defect of $R$                                           |
| "m"     | minimal number of generators of defining ideal of $R$                 |
| "n"     | type of $R$                                                            |
| "Class" | (non-parametrized) class of $R$                                        |
| "p"     | rank of $A_1 \cdot A_1$                                              |
| "q"     | rank of $A_1 \cdot A_2$                                              |
| "r"     | rank of $\delta : A_2 \to \text{Hom}_k(A_1, A_3)$                    |
| "PoincareSeries" | Poincaré series of $R$                                      |
| "BassSeries"   | Bass series of $R$                                                  |

In the example from above one gets:

i5 : torAlgData(Q/I)

\[
\begin{align*}
2 & + 2T - T - T + T \\
1 - T - 4T - 2T + T
\end{align*}
\]

c => 3
Class => G
e => 3
h => 1
m => 5
n => 2
p => 0

\[
\begin{align*}
2 \\
2 & - (1 + T)
\end{align*}
\]

PoincareSeries => ----------------------

To facilitate extraction of data from the hash table, the package offers two functions \texttt{torAlgDataList} and \texttt{torAlgDataPrint} that take as input a quotient ring and a list of keys. In the example from above one gets:
As discussed in Remark 6, the computation of Bass numbers may require a reduction modulo a regular sequence. In our implementation such a reduction is attempted if the embedding dimension of the local ring $R$ is more than 3. The procedure involves random choices of ring elements, and hence it may fail. By default, up to 625 attempts are made, and with the function `setAttemptsAtGenericReduction`, one can change the number of attempts. If none of the attempts are successful, then an error message is displayed:

```plaintext
i11 : torAlgClass R
stdio:11:1:(3): error: Failed to compute Bass numbers. You may raise the number of attempts to compute Bass numbers via a generic reduction with the function `setAttemptsAtGenericReduction` and try again.

i12 : setAttemptsAtGenericReduction(R,25)

i13 : torAlgClass R
o13 = G(2)
```

Notice that the maximal number of attempts is $n^2$ where $n$ is the value set with the function `setAttemptsAtGenericReduction`. 

```
2
(1 + T)
```

```
\{3, G, 0, 1, 2, \ldots\} \\
\frac{2}{1 - T - 4T - 2T + T}
```

```
io6 : List
```
Notes. Given $Q/I$ our implementation of Algorithm 5 in $torAlgData$ proceeds as follows.

1. Check if a value is set for $attemptsAtBassNumbers$; if not use the default value 25.

2. Initialize the invariants of $R$ (the localization of $Q/I$ at the irrelevant maximal ideal) that are to be returned; see the table in Section 3.

3. Handle the special case where the defining ideal $I$ or $Q/I$ is 0. In all other cases compute the invariants $c, e, h, m (= l + 1)$, and $n$.

4. If possible, classify $R$ based on $c, e, h, m$, and $n$. At this point the implementation deviates slightly from Algorithm 5, as it uses that all rings with $c = 3$ and $h = 2$ are of class $H(0, 0)$; see [1, 3.5].

5. For rings not classified in step 3 or 4 one has $c = 3$; cf. Remark 4. Compute the Betti numbers $\beta_2, \beta_3$, and $\beta_4$, and with the formula from Lemma 1 compute $p$ and $q$. If possible classify $R$ based on these two invariants.

6. For rings not classified in steps 3–5, compute the Bass numbers $\mu_{e-2}$ and $\mu_{e-1}$. If $d = e - 3$ is positive, then the Bass numbers are computed via a reduction module a regular sequence of length $d$ as discussed in above. Now compute $r$ with the formula from Lemma 1 and classify $R$.

7. The class of $R$ together with the invariants $c, l = m - 1$, and $n$ determine its Bass and Poincaré series; cf. [1, 2.1].

If $I$ is homogeneous, then various invariants of $R$ can be determined directly from the graded ring $Q/I$. If $I$ is not homogeneous, and $R$ hence not graded, then functions from the package $LocalRings$ are used.

References

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