Rate Region Frontiers for $n$-user Interference Channel with Interference as Noise

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Abstract—This paper presents the achievable rate region frontiers for the $n$-user interference channel when there is no cooperation at the transmit nor at the receive side. The receiver is assumed to treat the interference as additive thermal noise and does not employ multiuser detection. In this case, the rate region frontier for the $n$-user interference channel is found to be the union of $n$ hyper-surface frontiers of dimension $n-1$, where each is characterized by having one of the transmitters transmitting at full power. The paper also finds the conditions determining the convexity or concavity of the frontiers for the case of two-user interference channel, and discusses when a time sharing approach should be employed with specific results pertaining to the two-user symmetric channel.

I. INTRODUCTION

The capacity region of a two-user communication channel has been an open problem for about 30 years [1], [2]. Information-theoretic bounds through achievable rate regions have been proposed, most famously with the Han-Kobayashi region [3]. The capacity of the Gaussian interference channel under strong interference has been found in [4]. Recent results on the two-user interference channel to within one bit of capacity have been shown in [5]. The aforementioned referenced literature focused on the two-user interference channel from an information-theoretic point of view. This work presents the frontiers for the achievable rate regions for $n$-user interference channel when the interference is treated as additive noise and no multiuser detection is employed. Examples where we encounter the need to define such rate regions are found in multicell communications, in addition to mesh and sensor networks where the preference is in employing low-complexity transceivers.

The system setup is presented in section II. Section III discusses the achievable rate frontiers for the two-user interference channel. The $n$-user generalization is treated in section IV. Section V focuses on characterizing the two-user interference channel in terms of convexity or concavity and when a time sharing approach should be used, with specific results to the symmetric channel.

II. SYSTEM SETUP

The $n$-user interference channel is presented in Fig. 1 with $n$ transmitters and $n$ receivers. The $i^{th}$ transmitter transmits its signal $x_i$ to the intended $i^{th}$ receiver with a power $P_i$. The receivers have independent additive white Gaussian noise with a power noise variance of $\sigma_n^2$. Each transmitter is assumed to have a maximum power constraint of $P_{\text{max}}$. No cooperation is assumed between the nodes at the transmit side or at the receive side. The transmitters have a single antenna each, and they communicate over frequency flat channels. $g_{i,j}$ denotes the channel power gain received at the $i^{th}$ receiver from the $j^{th}$ transmitter, and there are no constraints over the values or distributions of $g_{i,j}$. Therefore $g_{i,i}$ is the channel gain of the $i^{th}$ desired signal, where as $g_{i,j}, j \neq i$ represent the interfering channel gains. $\mathbf{P}$ is the transmit power vector of length $n$, where the $i^{th}$ element $P_i$ denotes the transmit power of the $i^{th}$ transmitter. Treating the interference as additive noise throughout this paper and with no multiuser detection employed, $C_i$ denotes the maximum reliable rate of communication between the $i^{th}$ transmitter and the $i^{th}$ receiver. Therefore the achievable rate for the $i^{th}$ transmit-receive pair is written as:

$$C_i(\mathbf{P}) = \log_2 \left(1 + \frac{g_{i,i}P_i}{\sigma_n^2 + \sum_{j \neq i} g_{i,j}P_j} \right). \quad (1)$$

The objective of this work is to find the achievable rate region for the $n$ transmit-receive pairs. Section III finds the achievable rate region frontier for the two-user channel, and section IV generalizes the frontier for the $n$-user case.

III. ACHIEVABLE RATE REGION FRONTIER FOR TWO-USER INTERFERENCE CHANNEL

This section studies the two-user interference channel. In this case, (1) can be expressed in function of $P_1$ and $P_2$ as $C_i(P_1, P_2), i = 1, 2$. For notational brevity, the channel power...
The two-user interference channel is depicted in Fig. 2. $C_1$ and $C_2$ can therefore be written as:

$$C_1(P_1, P_2) = \log_2 \left(1 + \frac{aP_1}{1 + bP_2}\right),$$ (2)

$$C_2(P_1, P_2) = \log_2 \left(1 + \frac{cP_2}{1 + dP_1}\right).$$ (3)

Our objective is to find a frontier of the achievable rate region of $\{C_1, C_2\}$ through the power control of $P_1$ and $P_2$, where each transmitter is subject to a maximum power constraint of $P_{\text{max}}$.

Fig. 3 illustrates an example of rate region for a two-user interference channel, with $C_1$ and $C_2$ as the x-axis and the y-axis, respectively. From (2), $C_1$ in monotonically increasing in $P_1$ and monotonically decreasing in $P_2$, thus the point $C_1(P_{\max}, 0)$, alternatively annotated as point C on the x-axis in the figure, represents the maximum value $C_1$ can obtain. Similarly for the y-axis, the maximum value that $C_2$ achieves is $C_2(0, P_{\max})$, annotated as the point A on the y-axis. The point B has the following coordinates of $C_1(P_{\max}, P_{\max})$ and $C_2(P_{\max}, P_{\max})$.

A. Rate Region Frontier Formulation

The rate region frontier can be found by setting $C_1$ to a value $R$, then $R$ is swept over the full range of $C_1$, i.e. from 0 to $C_1(P_{\max}, 0)$, while finding the maximum $C_2$ value that can be achieved for each $R$. Hence, for a constant rate $C_1 = R$,

$$C_1(P_1, P_2) = R = \log_2 \left(1 + \frac{aP_1}{1 + bP_2}\right).$$ (4)

Therefore the relation between $P_1$ and $P_2$ is obtained as follows:

$$P_1 = \frac{1}{a}(1 + bP_2)(2^R - 1).$$ (5)

From (5), $C_2(P_1, P_2)$ can now be written in function of one parameter as $C_2(P_2)$, specifically:

$$C_2(P_2) = \log_2 \left(1 + \frac{cP_2}{1 + \frac{d}{a}(1 + bP_2)(2^R - 1)}\right).$$ (6)

It is important to analyze the behavior of $C_2(P_2)$ in terms of $P_2$. This is presented in the following lemma:

Lemma III.1 Setting $C_1$ at a constant rate, $C_1(P_1, P_2) = R$, $C_2(P_2)$ is a monotonically increasing function in $P_2$.

Proof: The proof is provided in the Appendix.

Remark: A direct implication of the monotonicity of the relation in (6) is that if $C_2$ is equal to a constant $C_2^*$ for the rate of $C_1 = R$, then: i) there is a unique $P_2^*$ that achieves $C_2^*$, ii) when $P_2^*$ is determined, then $P_1 = P_1^*$ is uniquely defined from (5), iii) from $P_1^*$ and $P_2^*$, $C_1$ is uniquely defined as $C_1 = R = C_1^*$ from (4). Thus $P_1^*$ and $P_2^*$ uniquely define a point in the rate region with coordinates $C_1^*$ and $C_2^*$.

In other words, any point in the rate region is achieved solely by a unique power tuple. This leads to what we denote by potential lines $\Phi$ in the rate region which are formed by holding one of the power parameters constant to a certain value, and sweeping the other parameter over its full power range. For instance, the potential line $\Phi(; P_2)$ is formed by sweeping $P_1$ from 0 to $P_{\max}$ and holding $P_2$ at $P_{\max}$. Based on the uniqueness property just discussed, these potentials lines along the $P_1$ dimension are therefore non-touching, i.e. $\Phi(;P_2^*)$ and $\Phi(;P_2')$ do not intersect if $P_2^* \neq P_2'$.

The rate region frontier then simplifies to finding the maximum value $C_2(P_2)$ could achieve for any value of $C_1(P_1, P_2) = R$. Effectively, this is formulated as:

$$\arg \max_{P_2} \quad C_2(P_2)$$ subject to \quad $C_1(P_1, P_2) = R$ \quad $P_i \leq P_{\max}$ \quad $i = 1, 2$. (7)

B. $C_2$ frontier for $0 \leq R \leq C_1(P_{\max}, P_{\max})$

As (4) is monotonically increasing in $P_1$ and monotonically decreasing in $P_2$, $R$ can only exceed the value $C_1(P_{\max}, P_{\max})$ when $P_2$ is less than $P_{\max}$. Thus $P_2 = P_{\max}$ is attainable only when $0 \leq R \leq C_1(P_{\max}, P_{\max})$, and $P_2$ needs to be less than $P_{\max}$ otherwise. From the proof provided in Lemma III.1 where (6) is proved to be monotonically increasing in $P_2$, and for the following range of $R$: $0 \leq R \leq C_1(P_{\max}, P_{\max})$, the solution to (7) is:

$$\arg \max_{P_2} C_2(P_2) = P_{\max}.$$ (8)
Therefore in this range of $R$ or equivalently of $C_1$, using (3) and (5), $C_2$ is expressed in function of $C_1$ as follows:

$$C_2(C_1) = \log_2 \left(1 + \frac{cP_{\text{max}}}{1 + \frac{b}{c}(1 + dP_{\text{max}})(2^{C_1} - 1)}\right).$$

C. $C_2$ frontier for $C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)$

Using symmetry of the previous result, for a constant rate $C_2 = R$, there is a linear relation between $P_1$ and $P_2$. And thus $C_1(P_1, P_2)$ can be written in function of one parameter $P_1$ as follows:

$$C_1(P_1) = \log_2 \left(1 + \frac{aP_1}{1 + \frac{b}{c}(1 + dP_1)(2^{C_1} - 1)}\right).$$

And by symmetry of the result in Lemma III.1, $C_1(P_1)$ is monotonically increasing in $P_1$. Thus by symmetry, for the following range of $R$:

$$0 \leq \tilde{R} \leq C_2(P_{\text{max}}, P_{\text{max}}),$$

we have:

$$\arg \max_{P_1} C_1(P_1) = P_{\text{max}}.$$  

Therefore for this range of $\tilde{R}$, $P_1 = P_{\text{max}}$ is attainable and maximizes $C_1(P_1)$. Correspondingly, $C_1$ spans the following range:

$$C_1(P_{\text{max}}, P_{\text{max}}) \leq C_1 \leq C_1(P_{\text{max}}, 0).$$  

So for the range of $\tilde{R}$ in (9) and the range of $C_1$ in (10), $P_1 = P_{\text{max}}$ describes the frontier. Therefore the values of $C_1$ at the frontier are:

$$C_1(P_{\text{max}}, P_2) = \log_2 \left(1 + \frac{aP_{\text{max}}}{1 + bP_2}\right).$$

Hence for $C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)$, the value of $P_2$ that will achieve the frontier follows as:

$$P_2 = \frac{1}{b} \left(\frac{aP_{\text{max}}}{2^{C_1} - 1} - 1\right).$$  

So effectively the value found in (11) is the answer for (7) for the range of $C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)$.

D. Achievable Rate Region Frontier

This subsection consolidates the two results to fully describe the rate region frontier. For a value of $c_1$ that sweeps the full range of $C_1$, we have:

- for $0 \leq c_1 \leq C_1(P_{\text{max}}, P_{\text{max}})$
  $$\arg \max_{P_2} C_2(P_2) = P_{\text{max}}$$

  and the frontier, denoted by $\mathcal{F}_1 = \Phi(\cdot; P_{\text{max}})$, is expressed as:

$$C_2(c_1) = \log_2 \left(1 + \frac{cP_{\text{max}}}{1 + \frac{d}{a}(1 + bP_{\text{max}})(2^{C_1} - 1)}\right).$$  

(12)

- for $C_1(P_{\text{max}}, P_{\text{max}}) \leq c_1 \leq C_1(P_{\text{max}}, 0)$
  $$\arg \max_{P_2} C_2(P_2) = \frac{1}{b} \left(\frac{cP_{\text{max}}}{2^{C_1} - 1} - \frac{d}{a}\right)$$

  and the frontier, denoted by $\mathcal{F}_2 = \Phi(\cdot; P_{\text{max}})$, is expressed as:

$$C_2(c_1) = \log_2 \left(1 + \frac{cP_{\text{max}}}{1 + \frac{d}{a}(1 + bP_{\text{max}})(2^{C_1} - 1)}\right).$$

Fig. 4. A 3-user interference channel rate region
by sweeping the full range of $P_1$ and $P_3$, and holding $P_2$ at $P_{\text{max}}$.

When $P_3 = 0$, the same setup and results that were described in section III applies. Specifically, for the rate range of $0 \leq C_1 \leq C_1(P_{\text{max}}, P_{\text{max}}, 0)$ and $0 \leq C_2 \leq C_2(0, P_{\text{max}}, 0)$ and $C_3 = 0$, the frontier can be described as $\Phi(:, P_{\text{max}}, 0)$, which is the line from point A to point B in Fig. 4. As $P_3$ increases, we want to describe the subsequent effect and how it is traced in the rate region.

Revisiting the equation in (1), a constant $P_3$ has the effect of just an additive noise term in $C_1(P)$ and $C_2(P)$. Hence, all the previous results in section III are applicable for any value of $P_3$ in describing the frontier for $C_1$ and $C_2$: as the effect of $P_3$ can be lumped in the noise term. Thus for the range of $0 \leq C_1 \leq C_1(P_{\text{max}}, P_{\text{max}}, P_3)$ and $0 \leq C_2 \leq C_2(0, P_{\text{max}}, P_3)$, where $P_3$ is constant, the frontier line on $C_1$ and $C_2$ is $\Phi(:, P_{\text{max}}, P_3)$, i.e., characterized by having $P_2 = P_{\text{max}}$. Consequently, the potential lines (or surfaces) concept in the 3-user case carries through.

Next, the frontier on $C_3$ needs to be described. For each value of $P_3$, $\Phi(:, P_{\text{max}}, P_3)$ traces one of the highlighted curves in Fig. 4. For these collection of lines to form a frontier we want to prove that at each increasing value of $P_3$ these potential lines monotonically increase in the $C_3$ dimension. This is obvious from the $C_3$ and $P_3$ relation in (1). The maximum value of $C_3$ that can be achieved in this case is when $P_3 = P_{\text{max}}$, i.e. $C_3(P_{\text{max}}, P_{\text{max}}, P_{\text{max}})$. Therefore the highlighted frontier surface in Fig. 4 is the potential surface $\Phi(:, P_{\text{max}}, :)$.

By symmetry of interchanging $P_1$, $P_2$ and $P_3$, we find that the $3$-user rate region frontier is determined through the union of three surfaces: $\Phi(P_{\text{max}}, :)$, $\Phi(:, P_{\text{max}}, :)$ and $\Phi(:, :, P_{\text{max}})$. And $F$ is expressed as:

$$ F = \text{Convex Hull}\{F_i \cup F_2 \cup F_3\}, $$

where $F_i$ is a potential surface $\Phi(\cdot)$ with $P_{\text{max}}$ in the $i$th power position. (Note that the intersection of potential surfaces is a potential line, as two of the dimensional inputs become equal, i.e. $\Phi(P_{\text{max}}, P_{\text{max}}, :) \in F_1$ and $\Phi(P_{\text{max}}, P_{\text{max}}, :) \in F_2$.)

B. $n$-user generalization

The case for $n$-user generalization can be done by induction. For the $n^{th}$ added dimension to the existing $n-1$ dimensions problem, the additional power effect of $P_n$ can be lumped in the additive noise term of the existing expressions, and thus the results for $C_1, \ldots, C_{n-1}$ hold and carry through. The frontier on $C_n$ is monotonically increasing in $P_n$, and can be maximized with $P_n = P_{\text{max}}$ for the appropriate range in $C_1, \ldots, C_{n-1}$. Invoking symmetry we can generalize over all the rate ranges, therefore arriving to the following theorem.

**Theorem IV.1** The achievable rate region frontier of the $n$-user channel is:

$$ F = \text{Convex Hull}\{\cup_{i=1}^{n} F_i\}, $$

where $F_i$ is a hyper-surface of $n-1$ dimensions characterized by holding the $i^{th}$ transmitter at full power $P_{\text{max}}$.

Using the notation introduced in this section, $F_i$ is effectively $\Phi(:, \ldots, P_{\text{max}}, \ldots, :)$ with $P_{\text{max}}$ at the $i^{th}$ power position.

V. CHARACTERISTICS OF THE ACHIEVABLE RATE REGION FOR TWO-USER INTERFERENCE CHANNEL

Treating the two-user interference channel in more details, this section studies the behavior of the rate region frontiers $F_1$ and $F_2$ in terms of convexity and concavity. In addition, it discusses when a time-sharing approach should be employed, with specific results pertaining to the symmetric channel.

A. Convexity or Concavity of the Frontiers

The frontier $F_2$ in (12) depends on $P_1$ through the following relation of $c_1$ and $P_1$:

$$ P_1 = \frac{1}{a} (1 + b P_{\text{max}}(2c_1^a - 1)). $$

Therefore the second derivative of $F_2$ with respect to $c_1$ leads to the following expression:

$$ \frac{\partial^2 F_2}{\partial c_1^2} = (\theta + adP_1)^2 - (a - \theta)(a - \theta + acP_{\text{max}}), $$

where $\theta = \frac{dP_{\text{max}}}{P_{\text{max}}}$. Therefore if the frontier line is concave (i.e. $\frac{\partial^2 F_2}{\partial c_1^2} \leq 0$) then the enclosed region is convex, i.e. the straight line connecting any two points in the rate region is entirely enclosed in the rate region. Let $\Re(\cdot)$ be the real operation, and defining the quantity $Q_1$ as:

$$ Q_1 = \Re\left(\sqrt{(\theta - \theta)(a - \theta + acP_{\text{max}}) - \theta}\right), $$

then it suffices to study the convexity or concavity by examining the sign of $Q_1$, where $Q_1$ is derived such as:

$$ \text{sign}\left(\frac{\partial^2 F_2}{\partial c_1^2}\right) = \text{sign}(P_{\text{max}} - Q_1). $$
Thus the convexity or concavity of the frontier line $F_2$ is governed by:

- $Q_1 \leq 0$: the frontier line $F_2$ is convex, and the region bounded by $F_2$ is concave. As $P_1 - Q_1 \geq 0$ for all the range of $P_1$.
- $Q_1 \geq P_{\text{max}}$: the frontier line $F_2$ is concave, and the region bounded by $F_2$ is convex. As $P_1 - Q_1 \leq 0$ for all the range of $P_1$.
- $0 < Q_1 < P_{\text{max}}$, the frontier line exhibits a non-stationary inflection point when $P_1 = Q_1$, and $F_2$ is neither convex nor concave between the point extremities of $\Phi(0, P_{\text{max}})$ and $\Phi(P_{\text{max}}, P_{\text{max}})$. In this case:
  - for $0 < P_1 \leq Q_1$: the line $\Phi(0 : Q_1, P_{\text{max}})$ is concave, i.e. the frontier segment between point $\Phi(0, P_{\text{max}})$ (the point A in Fig. 5) and point $\Phi(Q_1, P_{\text{max}})$ (the point E in Fig. 5) is concave.
  - for $Q_1 \leq P_1 < P_{\text{max}}$: the line $\Phi(Q_1 : P_{\text{max}}, P_{\text{max}})$ is convex, i.e. the frontier segment between point $\Phi(Q_1, P_{\text{max}})$ (the point E in Fig. 5) and point $\Phi(P_{\text{max}}, P_{\text{max}})$ (the point B in Fig. 5) is convex.

By symmetry, the frontier line $F_1$ exhibits the following behavior: it is convex when $Q_2 \leq 0$, and it is concave when $Q_2 \geq P_{\text{max}}$, and it exhibits a non-stationary inflection point when $P_2 = Q_2$ – specifically it is convex for $0 < P_2 \leq Q_2$ and concave for $Q_2 \leq P_2 < P_{\text{max}}$. Hereby $Q_2$ is defined as:

$$Q_2 = \frac{\mathbb{R} \sqrt{(c - \beta)(c - \beta + acP_{\text{max}})) - \beta}}{cb},$$

with $\beta = (b + bdP_{\text{max}})$.

When describing the full rate region frontier through $F_1 \cup F_2$, the rate region is convex if both $F_1$ and $F_2$ are concave, and the rate region is concave otherwise. Therefore, whenever the frontier (or segment thereof) is concave, it will describe the convex hull of the rate region instead of a time sharing solution. Fig. 5 illustrates an example where the frontier $F_1$ is concave, and the frontier $F_2$ exhibiting a non-stationary inflection point E. In this case the convex hull rate region is found by operating along the concave frontier $F_1$, and time-sharing between point B and point E, and operating along the concave segment of $F_2$ between point E and point A.

### B. Optimality of Time Sharing

This subsection investigates the optimality of time sharing between operating points in the rate region. For instance, whenever the rate region frontier segment is convex (equivalently, the enclosed rate region is concave) then operating with time sharing between the extremities of the curve is optimal than operating along the log-defined frontier. Analyzing the $F_2$ frontier, and referring to the Fig. 3 it follows:

- $Q_1 \leq 0$: (i.e. $F_2$ frontier is convex) it is optimal to apply time-sharing through the following options:
  - between point A and point B,
  - between point A and point $\Phi(P_{\text{max}}, Q_2)$ if $F_1$ exhibits a non-stationary inflection point,
  - between point A and a point on the concave segment of $F_1$.

These depend on how the parameters $a$, $b$, $c$, $d$, and $P_{\text{max}}$ would lead to a convex hull region. This can be done by evaluating and comparing each of the candidate solution aforementioned.

- $Q_1 \geq P_{\text{max}}$: (i.e. $F_2$ frontier is concave) the potential line $\Phi(0 : P_{\text{max}})$ is optimal, and no time sharing is to be employed.

### 1) Time sharing between Points A, B, and C in Fig. 3

Discounting the case when $F_1$ or $F_2$ exhibit non-stationary inflection point for simplicity, and focusing on the case of $Q_1 \leq 0$ and $Q_2 \leq 0$, it is important to know when time sharing between point A and point C is better than time sharing through the intermediate point B. This is done by comparing the straight line connecting points A and C, and the coordinates of B. It follows that operating with time sharing between the points (or system states) A and C (i.e. one transmitter only transmitting at a certain point) is optimal when:

$$\frac{(1 + cP_{\text{max}})(1 + dP_{\text{max}})}{1 + cP_{\text{max}} + dP_{\text{max}}} \geq \left(1 + aP_{\text{max}} + bP_{\text{max}}\right)^{\gamma}$$

with $\gamma = \log_2(1 + cP_{\text{max}})/\log_2(1 + aP_{\text{max}})$.

### 2) Symmetric two-user interference channel: For the symmetric two-user interference channel, $a = c$, and $b = 0$. In this case, (19) simplifies and leads to the following theorem:

**Theorem V.1** Time-sharing operation with one transmitter active at full power at a time is optimal when

$$b \geq \frac{\sqrt{1 + aP_{\text{max}}}}{P_{\text{max}}}.$$
Remark: For high SNR (i.e. $aP_{\text{max}} \gg 1$), (20) reduces to $bP_{\text{max}} \geq \sqrt{aP_{\text{max}}}$, which interestingly coincides with recent results in [5, eq.(3)].

This indicates that when the cross interfering power gain $b$ exceeds the right hand side value in (20), denoted by $b^*$, it is optimal to operate with one transmitter at a time. Fig. 6 uses $b = 2$, which is larger than $b^* = \sqrt{2}$ obtained from (20) for $a = 1$ and $P_{\text{max}} = 1$. By contrast in Fig. 3, the value of $b = 1$ is adopted and the behavior exhibited is different as expected.

In addition, we subsequently prove that the expression in (20) is a sufficient condition for both frontiers $F_1$ and $F_2$ to be convex, i.e. $Q_1$ and $Q_2$ are always $\geq 0$.

Proof: For the symmetric case, $Q_1 = Q_2 = Q_{\text{sym}}$ can be written as

$$Q_{\text{sym}} = \frac{\Re((\sqrt{a-\theta})(a-\theta+a^2P_{\text{max}})) - \theta}{ab},$$

where $\theta = b + b^2 P_{\text{max}}$. $Q_{\text{sym}}$ can also be written in this form:

$$Q_{\text{sym}} = \frac{\Re(\sqrt{T_1T_2}) - \theta}{ab},$$

where $T_1 = a - \theta = a - b - b^2 P_{\text{max}}$, and $T_2 = a - \theta + a^2 P_{\text{max}}$. From the expression in (20), $a$ can be alternatively upper-bounded as $a \leq (b^2 P_{\text{max}} - 1)/P_{\text{max}}$. Therefore, $T_1$ is upper-bounded as:

$$T_1 \leq -1/P_{\text{max}} - b. \quad (21)$$

From (21), $T_1$ is always negative. $T_2$ however can be positive or negative. Each case is evaluated as follows:

- $T_2 \geq 0$: then $\Re(\sqrt{T_1T_2}) = 0$, and as $\theta$ is always positive, then $Q_{\text{sym}} \leq 0$.
- $T_2 \leq 0$: $\Re(\sqrt{T_1T_2})$ is $\geq 0$. In this case, the numerator of $Q_{\text{sym}}$ can be written as:

$$\text{num}(Q_{\text{sym}}) = \sqrt{(\theta - a)(\theta - a - a^2P_{\text{max}})} - \theta.$$

Given the fact that $(\theta - a - a^2P_{\text{max}}) \leq (\theta - a)$, then num$(Q_{\text{sym}})$ can be upper-bounded as:

$$\text{num}(Q_{\text{sym}}) \leq \sqrt{(\theta - a)^2} - \theta \leq \leq a \leq 0.$$

Hence the frontiers $F_1$ and $F_2$ are convex. Therefore, when $b$ satisfies the equation in (20), the frontiers lines are always convex, and time-sharing with only one transmitter active at a time is optimal.

VI. CONCLUSIONS

The achievable rate region frontiers for the $n$-user interference channel were presented when there is no cooperation at the transmit side nor at the receive side. The receivers do not employ multiuser detection, and the interference is considered as additive noise. Results were first found for the two-user interference channel. The 3-user interference channel was treated next to show the effect of adding the additional dimension, and subsequently the $n$-user interference channel generalization results followed. The $n$-user rate region is found to be the convex hull of the union of $n$ hyper-surfaces each of dimension $n - 1$. Each hyper-surface frontier $F_i$ is defined by having the $i^{th}$ transmitter transmitting at its full power $P_{\text{max}}$.

The two-user interference channel was further studied regarding the convexity or the concavity of the frontiers. Conditions when the frontier is convex or concave or exhibiting a non-stationary inflection point were also obtained. Whenever the log-defined frontier line is convex then a time sharing solution is optimal. For the symmetric two-user case, the condition was found to indicate when time-sharing between the points $\Phi(P_{\text{max}},0)$ and $\Phi(0,P_{\text{max}})$ (i.e. one transmitter solely transmitting at full power at a certain time) is to be used, rather than time-sharing through the point $\Phi(P_{\text{max}},P_{\text{max}})$ (i.e. both transmitters transmitting at full power). That condition was also proven to be sufficient to ensure that both frontiers ($F_1$ and $F_2$) will in fact always be convex.

APPENDIX

Proof that the equation (6), $C_2(P_2)$ is monotonically increasing in $P_2$.

Proof: Effectively (6) is in the form of $f(1 + g(x))$. As $f(\cdot)$ is monotonically increasing in its argument, it suffices to prove that $g(x)$ is monotonically increasing in $x$. Therefore define $g(P_2)$ as:

$$g(P_2) = \frac{acP_2}{a + d(1 + bP_2)(2R - 1)},$$

$$\frac{\partial g(P_2)}{\partial P_2} = \frac{ac}{a + d(1 + bP_2)(2R - 1)} - \frac{acP_2b(2R - 1)}{(a + d(1 + bP_2)(2R - 1))^2}$$

$$= \frac{a^2c + acd(1 + bP_2)(2R - 1) - acP_2db(2R - 1)}{(a + d(1 + bP_2)(2R - 1))^2}$$

$$= \frac{a^2c + acd(2R - 1) + acdbP_2(2R - 1) - acdbP_2(2R - 1)}{(a + d(1 + bP_2)(2R - 1))^2}$$

$$= \frac{ac(a + d(2R - 1))}{(a + d(1 + bP_2)(2R - 1))^2}. \quad (22)$$

The numerator in (22) is $\neq 0$ if $a \neq 0$ and $c \neq 0 (a = 0$ or $c = 0$ are the trivial cases where the rate region is either a line or the point zero). As $R \geq 0$, then $(2R - 1) \geq 0$. Thus $\partial g(P_2)/\partial P_2$ is always $> 0$ for non-trivial cases of $a$ and $c$. Thus $g(P_2)$ is monotonically increasing in $P_2$, and equivalently $C_2(P_2)$ is monotonically increasing in $P_2$. ■

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