REGULARIZATION AND MINIMIZATION OF Γ1-STRUCTURES

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Dedicated to Paul Schweitzer.

Abstract. We prove the existence of a minimal (all leaves dense) foliation of codimension one, on every closed manifold of dimension at least 4 whose Euler characteristic is null, in every homotopy class of hyperplanes distributions, in every homotopy class of Haefliger structures, in every differentiability class, under the obvious embedding assumption. The proof uses only elementary means, and reproves Thurston’s existence theorem in all dimensions. A parametric version is also established.

1. Introduction

In the middle of the 1970’s, after works by many authors to endow some particular manifolds with foliations, W. Thurston published fundamental existence theorems for all compact manifolds. We are interested in the codimension one case. Not only his main result [14] builds a foliation $\mathcal{F}$ of codimension one on every closed manifold whose Euler characteristic is null, but it allows to prescribe the homotopy class of the hyperplane distribution tangent to $\mathcal{F}$, and also the homotopy class of $\mathcal{F}$ regarded as a Haefliger structure [2]. Recall that every manifold of dimension at least 3 carries an unnumerable infinity of such classes [12].

It is notorious [15] that the subsequent study of foliations was strained by the widely spread feeling that these works were difficult. Actually, both proofs given in [14] follow the same scheme of construction in three steps. First one makes a foliation with “holes”, that is, parts of the manifold left unfoliated. Second, a substantial construction extends the foliation into them. Third, one adjusts the homotopy class of the underlying Haefliger structure. This third step relies on Mather’s homology equivalence [6] between the classifying spaces $B\text{Diff}^c(R)$ and $\Omega B\Gamma_{1+}^1$. The idea grew that this certainly nonelementary tool was

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unavoidable (and/or the simplicity of $\text{Diff}_+(S^1)$ \cite{3\cite{4}\cite{7}\cite{8}}, for which Thurston calls in some constructions of codimension one foliations in dimension 3), despite the fact that in 1976 Thurston himself did not write so.

“Currently, step 3 seems to involve some nonelementary background about classifying spaces for Haefliger structures and groups of diffeomorphisms.” \cite{14}

Also, in view of the importance of Novikov’s closed leaf theorem for the study of foliations in dimension 3, the question was raised to what extent closed manifolds of dimension at least 4 and whose Euler characteristic is null carry foliations of codimension 1 without compact leaf.

P. Schweitzer proved \cite{9} that every such manifold admits such a foliation of class $C^1$, whose tangent distribution homotopy class and whose concordance class may moreover be prescribed.

The present paper aims both to answer completely this question and to reprove Thurston’s theorem in all dimensions by the only means of elementary differential topology. It doesn’t call for any result concerning diffeomorphisms groups, nor the homotopy type of classifying spaces. The argument is self-contained and constructive. It works in all differentiability classes.

The object under consideration is a Haefliger structure of codimension one — more briefly a $\Gamma_1$-structure — and may be thought as a “foliation with singularities” \cite{2}. It can be defined on a manifold $M$ as a rank-one real vector bundle over $M$, the “microbundle” $\nu\xi$, together with, in its total space, along the zero section $Z(M)$, a germ $\xi$ of codimension-one foliation transverse to every fibre.

The singularities are the points in $M$ where $Z$ is not transverse to $\xi$. If $\xi$ is regular, then the pullback $Z^*\xi$ is a foliation on $M$. One makes no difference between “foliation” and “regular $\Gamma_1$-structure”.

If regular, $\xi$ induces a vector bundle embedding of $\nu\xi$ into $\tau M$, negatively (geometers’ choice!) transverse to the foliation $Z^*\xi$ (see paragraph 2.1 below). Call it the gradient of $\xi$.

A homotopy between two $\Gamma_1$-structures $\xi_0$, $\xi_1$ on the same manifold $M$, is a $\Gamma_1$-structure on the manifold $M \times [0, 1]$ whose restriction to $M \times i$ is $\xi_i$, for $i = 0, 1$. On the other hand, by a homotopy of embeddings of $\nu\xi$ into $\tau M$, one means, as usual, a continuous 1-parameter family of such embeddings.

One works in an arbitrary $C^r$ differentiability class, $0 \leq r \leq \infty$. That is, $M$ is assumed smooth ($C^{\infty}$), and all foliations and $\Gamma_1$-structures
are understood of class $C^{r,\infty}$, i.e. globally $C^r$ and tangentially $C^\infty$. In particular, a continuous tangent hyperplane distribution exists even for $r = 0$. Recall that this is no loss of generality: every $\Gamma_1$-structure of class $C^r$ is easily homotoped to some $\Gamma_1$-structure of class $C^{r,\infty}$.

**THEOREM A.** Let $\xi$ be a $\Gamma_1$-structure on a closed, connected manifold $M$ of dimension $n$, whose microbundle $\nu\xi$ admits an embedding $J$ into $\tau M$. Then:

1. (Thurston [14]) There is a foliation on $M$ homotopic to $\xi$ and whose gradient is homotopic to $J$;
2. If moreover $n \geq 4$, then there is a minimal foliation on $M$ homotopic to $\xi$ and whose gradient is homotopic to $J$.

“Minimal” meaning of course that all leaves are dense. In dimension 3, another proof of (1) was recently given [5], different from that of the present paper, and which also uses only elementary means, and builds a foliation of much more explicit structure than we do here.

**COROLLARY 1.1.** Every closed, connected manifold of dimension at least 4 and whose Euler characteristic is null, admits a smooth minimal foliation of codimension one.

For example, $S^5$ and $S^2 \times S^3$ do. So, in higher dimensions and in the general frame of smooth foliations of codimension 1, there seems to be nothing like Novikov’s closed leaf theorem.

Although it answers a question usually considered open, actually this corollary is not new. In paragraphs 2 to 6 of [14], on an arbitrary closed manifold whose dimension is at least 4 and whose Euler characteristic is null, is manufactured a smooth foliation which is by construction without compact leaf. A very little more care in Thurston’s method makes it minimal. So, this corollary is a “new theorem in Thurston ’76”. On the contrary, the existence of a foliation smooth and minimal, in every homotopy class of $\Gamma_1$-structures, is new.

The proof of theorem A is constructive and essentially local: one performs explicit local modifications of $\xi$ and $J$ until $\xi$ is regular and negatively transverse to $J$. To force these modifications to be homotopies, eventually $\xi$ (resp. $J$) is previously given locally a reflection symmetry, and the modification is then applied twice, respecting this symmetry. This elementary trick spares, at the end of the construction, any need to adjust the $\Gamma_1$-structure class. The final regularization step is much as in [14]: the singular set having been given the form of round singularities [1], each of them is eliminated by a turbulization, at the cost of leaving a “hole”, that is, a part of the manifold left unfoliated. The hole is then filled by Thurston’s construction.
More precisely, the proof consists in the following steps.

**Morsifying the $\Gamma_1$-structure** — In class $C^{r\geq 2}$, it is immediate, by a small generic homotopy, to make $\xi$ Morse-singular, and smooth in a neighborhood of its singularities. The same result is also achieved in the lower differentiability classes, but requires different and less immediate methods.

**Minimizing the $\Gamma_1$-structure** — Then one easily makes $\xi$ minimal by local modifications. Thanks to the symmetry trick, its homotopy class is not changed.

**Morsifying the contact** — A pseudgradient $\nabla \xi$ is fixed for the Morse $\Gamma_1$-structure $\xi$. One may think to $J$ and to $\nabla \xi$ as twisted vector fields, see paragraph 2.1. One defines the negative contact locus $C_-(J, \nabla \xi) \subset M$ as the set of points where $J$ and $\nabla \xi$ are nonpositively colinear. One considers the field $T$ tangential to $\xi$ obtained as the projection of $J$ into $\tau \xi$ parallelly to $\nabla \xi$. Thus for each leaf $L$ of $\xi$, the restriction $T|L$ is a vector field on $L$, singular at every point of $C_-(J, \nabla \xi) \cap L$. After local homotopies of $\xi$ and $J$, all these singularities are given the Morse form. The negative contact locus becomes a union of arcs, each of which $A = [s, s']$ is transverse to $\xi$ except at its origin $s$ and extremity $s'$, which are a pair of singularities of $\xi$ of successive indices $i, i + 1$ (bisingular arc). The union of $A$ with a second arc, also transverse to $\xi$, whose origin is $s'$ and whose extremity is $s$, constitutes a bisingular circle.

**Symmetrizing** — After homotopies of $\xi$ and $J$, these circles are encapsuled by pairs into some domains of $M$ where $\xi$ and $J$ are both invariant by some reflection symmetry, and out of which $\xi$ is negatively transverse to $J$. To get this, the Morse form that the contact has been given is crucial.

**Rounding the singularities** — After a classical method of Asimov, each pair of Morse singularities is easily changed into a pair of round ones.

**Turbulizing the round singularities** — Each round singularity is eliminated by turbulization, at the cost of leaving in the manifold some unfoliated hole. This is immediate.

**Filling the holes** — Each hole is filled by a foliation. Here we follow Thurston. The argument is given in details in an appendix, for the sake of completeness, and also because, as said before, the regularization of Haefliger structures of codimension one has been for long believed to require difficult tools. I have simplified some part of Thurston’s argument, but it remains substantial.

**Generalizations**: 
To fix ideas, in this paper it is assumed that $M$ is closed. It is immediate to generalize theorem A and its proof to $\Gamma_1$-structures on manifolds with boundaries, as follows. One adds to the hypotheses of the theorem that $J$ is tangential to $\partial M$ and that $\xi$ is already negatively transverse to $J$ on $\partial M$; and concludes that after homotopies of $\xi$ and $J$ relative to $\partial M$, the $\Gamma_1$-structure $\xi$ becomes negatively transverse to $J$ on $M$, and moreover minimal if $n \geq 4$.

For example, for any two linear foliations of codimension one $F_0, F_1$ on the 3-torus $T^3$, one easily makes a smooth foliation $F$ on $T^3 \times [0,1]$ such that $F|_i (T^3 \times i) = F_i$ ($i = 0,1$). After the generalization of theorem A, one can moreover make $F$ without interior leaf, which is much less straightforward.

Also, the theorem and its proof adapt to the case of tangential boundaries provided that their first homology group is nonzero, just like in [14] — this generalization is left as an exercise.

Also, the methods given here go not only in every class $C^r$, $0 \leq r \leq \infty$, but more generally for $G$-structures, where $G$ is any pseudogroup of local transformations of the real line $R$ (that is, if one likes better, an open subgroupoid of the groupoid $\Gamma_1^0$ of germs of homeomorphisms of $R$) which verifies the following:

Every nonempty open subset of $R$ has a finitely generated group of homeomorphisms with compact support, noncommutative, belonging to $G$, and which has a locally dense orbit.

For example, theorem A holds as well in the transverse “differentiability” classes $PL$, diadic $PL$, $C^{1+BV}$...

Also, the constructive character of our proof gives at hand a parametric version. To fix ideas, restrict to the smooth class. Given a manifold $M$ and a rank-one vector bundle $\nu$, the space $\Gamma_1(M,\nu)$ of smooth $\Gamma_1$-structures on $M$ whose microbundle is $\nu$, is endowed with the $C^\infty$ topology on germs of hyperplane fields.

**THEOREM B.** Let $M$ be a closed, connected manifold, $\nu$ a rank-one vector bundle over $M$ which admits an embedding $J$ into $\tau M$, and let $\Xi \subset \Gamma_1(M,\nu)$ be a compact subset.

Then there exist an embedding $J'$ of $\nu$ into $\tau M$, homotopic to $J$, and a continuous map $h : \Xi \to \Gamma_1(M \times [0,1],\nu)$

such that for each $\xi \in \Xi$, one has $h(\xi)|(M \times 0) = \xi$, while $h(\xi)|(M \times 1)$ is negatively transverse to $J'$ — and thus in particular, regular.
If moreover \( \dim M \geq 4 \), then one can arrange that every \( h(\xi)|(M \times 1) \) is minimal.

For example, on any closed manifold whose Euler characteristic is null, any compact family of transversely oriented \( \Gamma_1 \)-structures can be continuously homotoped to foliations negatively transverse to a same nonsingular vector field, whose homotopy class can moreover be prescribed.

**Corollary 1.2.** The inclusion of \( \Gamma_1(M, \nu)_{\text{regular}} \) into \( \Gamma_1(M, \nu) \) is \( H_k \)-surjective for every \( k \geq 0 \). If \( \dim M \geq 4 \), then the inclusion of \( \Gamma_1(M, \nu)_{\text{regular and minimal}} \) into \( \Gamma_1(M, \nu) \) is \( H_k \)-surjective for every \( k \geq 0 \).

The question whether \( \Gamma_1(M, \nu) \) is 1-connected relatively to \( \Gamma_1(M, \nu)_{\text{regular}} \), remains open.

It is a pleasure to thank François Laudenbach for generously sharing his interrogations about the actual status of Mather-Thurston’s theorem, and of the simplicity of diffeomorphism groups, in the construction of foliations; and for his listening and observations all along this work.

**2. Proof of Theorem A.**

Let \( \xi \) be an arbitrary \( \Gamma_1 \)-structure of differentiability class \( C^{r, \infty} \), \( 0 \leq r \leq \infty \), on a closed, connected, smooth manifold \( M \) of dimension \( n \), and let \( J \) be a nonsingular smooth twisted vector field (see paragraph 2.1). One will, through successive homotopies of \( \xi \) and \( J \), make \( \xi \) minimal (except in dimension 3) and negatively transverse to \( J \) (thus in particular, regular). The successive \( \Gamma_1 \)-structures and nonsingular twisted vector fields will all be named \( \xi \) and \( J \). One assumes that \( n \geq 3 \), leaving the cases \( n = 1, 2 \) as an exercise.

**2.1. Generalities on \( \Gamma_1 \)-structures.** By definition, a \( \Gamma_1 \)-structure \( \xi \) is the germ, along \( Z(M) \), of an integrable 1-form \( \Omega_\xi \) defined in a neighborhood of \( Z(M) \) in the total space of the fibre bundle \( \nu_\xi \). This form is twisted — by this term we always mean, twisted w.r.t. the bundle \( \nu_\xi \). That is, \( \Omega_\xi \) takes values in \( \nu_\xi \). One can arrange that moreover \( \Omega_\xi \) restricted to each microfibre is the identity. Then \( \Omega_\xi \) is uniquely defined by \( \xi \).

It is convenient to think in terms of twisted vector fields. By a twisted tangent vector one means some \( X \in \nu_\xi^* \otimes \tau M \), that is, at some point \( m \) of \( M \), a linear map from the microfibre \( \nu_m \xi \) to the tangent vector space \( \tau_m M \). Then \( (\Omega_\xi)X \), by which we mean the composition \( (\Omega_\xi) \circ X \), is a linear endomorphism of the line \( \nu_m \xi \), that is, a real number. In
particular, if $X$ is transverse to $\xi$, then this transversality has a well-defined sign.

Of course, any local trivialization of $\nu \xi$ changes $X$ into an ordinary tangent vector.

For example, the embedding $J$ is nothing but a nonsingular twisted vector field.

Also, if $f$ is a function defined on an open subset $U$ of $M$, one calls $f$ a first integral of $\xi$ in $U$ if there is a trivialization $\nu \xi|_U \cong U \times \mathbb{R}$ through which $\Omega \xi = dy - df(x)$ $(x, y) \in U \times \mathbb{R}$.

**Observation 2.1.** Let $f$ be a first integral of $\xi$ in $U$ and let $g$ be a function in $U$ such that $f = g$ on a neighborhood of $\partial U$. Then $\xi$ is homotopic to the $\Gamma_1$-structure on $M$ that coincides with $\xi$ on $M \setminus \text{Supp } (f - g)$ and that admits $g$ as a first integral in $U$.

Indeed in $U \times [0, 1]$ one has the $\Gamma_1$-structure admitting as first integral the function:

$$(x, t) \mapsto (1 - t)f(x) + tg(x)$$

For example, in a $\Gamma_1$-structure one can locally create a pair of Morse singularities in cancellation position, in Smale’s sense. Or one can cancel such a pair. This does not change the homotopy class of $\xi$.

**2.2. Morsifying the $\Gamma_1$-structure.**

**Proposition 2.2.** After some homotopy, $\xi$ is smooth (that is, $C^\infty$) in a neighborhood of its singular set; and every singular point is of Morse type.

Fix a representant for the germ $\xi$, that is, a foliation $\mathcal{X}$ in a neighborhood of the zero section $Z(M)$ in the microbundle, transverse to the fibres, and whose germ along $Z(M)$ is $\xi$.

**Proof of proposition 2.2 in case $r \geq 2$.** — In case $\xi$ is $C^{r \geq 2}$, the proposition is straightforward: for every smooth section $s$ close enough to $Z$, the $\Gamma_1$-structure $s^* \mathcal{X}$ is homotopic to $\xi = Z^* \mathcal{X}$. By Thom’s transversality theorem, for a generic $s$, every singularity $x$ of $s^* \mathcal{X}$ is of Morse type. Then it is easy, pushing $\mathcal{X}$ by a $C^2$-small isotopy with support in a small neighborhood of $s(x)$, to make $\mathcal{X}$ smooth there. Then $s^* \mathcal{X}$ is smooth in a neighborhood of $x$.

The rest of this section is to prove proposition 2.2 in the general case $r \geq 0$. It seems to be not trivial, and needs different methods (which will also be necessary for the parametric version). One will make a continuous section $s$ piecewise linear over a fine triangulation and with
so strong slopes, that it is transverse to $\mathcal{X}$ except over the 0-skeleton; make $\mathcal{X}$ smooth over some neighborhood of this 0-skeleton; change $s$ to some smooth approximation $s_\infty$, still transverse to $\mathcal{X}$ outside this neighborhood; and change $\xi = Z^*\mathcal{X}$ to $s_\infty^*\mathcal{X}$. The smoothing step is far from immediate.

The transversality of $s$, $s_\infty$ to $\mathcal{X}$ far from the vertices will be assured by the transversality of $s^*\mathcal{X}$, $s_\infty^*\mathcal{X}$ to some twisted vector field $X$ linear on simplices.

For $0 \leq k \leq n$, consider the standard $k$-simplex

$$\Delta^k := \{(x_0, \ldots, x_k) \in \mathbb{R}^{k+1}/x_i \geq 0, \sum_{i=0}^k x_i = 1\}$$

For every $\lambda := (\lambda_i)_{0 \leq i \leq k} \in \mathbb{R}^{k+1}$ one has on $\Delta^k$ the usual linear vector field:

$$X_\lambda(x_0, \ldots, x_k) := \left(\sum_{i=0}^k \lambda_i x_i \partial/\partial x_i\right) - \left(\sum_{i=0}^k \lambda_i x_i\right) \sum_{i=0}^k x_i \partial/\partial x_i$$

Trivially, $X_\lambda$ is tangent to every face of $\Delta^k$, and its restriction to each face of positive dimension is linear. Also:

(1) $$\|X_\lambda(x)\| \leq 2\|\lambda\|$$

where $\|X_\lambda(x)\|$ denotes the euclidian norm, while $\|\lambda\|$ is the $\ell^\infty$ norm. Also, the following minoration is easily verified for the derivate of the function $f_\lambda(x) := \sum_i \lambda_i x_i$ by $X_\lambda$. Set:

$$\delta(\lambda) := \inf_{0 \leq i < j \leq k} |\lambda_i - \lambda_j|$$

For every $x \in \Delta^k$, denote $dv(x)$ the distance to the nearest vertex. One has:

(2) $$\delta(\lambda)^2 dv(x)^2/2 \leq \sum_{0 \leq i < j \leq k} x_i x_j (\lambda_i - \lambda_j)^2 = df_\lambda X_\lambda(x)$$

In particular, if the $\lambda_i$'s are two by two distinct, $X_\lambda$ is nonsingular but at the vertices: call it nondegenerate.

Fix a riemannian structure on the rank-one vector bundle $\nu\xi$. Then for every local orientation, this bundle admits a unique local trivialization which is isometric and oriented. Through this trivialization, any smooth local section $s$ of $\nu\xi$ becomes a function and any twisted tangent vector $X$ becomes an ordinary tangent vector. Thus $dsX$ is a well-defined real number, independant on the local orientation.
VOCABULARY 2.3. Given a smooth triangulation $T_r$ on the manifold $M$:

Call a twisted vector field piecewise linear w.r.t. $T_r$ (and nondegenerate) if its restriction to each simplex is linear (and nondegenerate), in the above sense, when viewed in the isometric local trivializations;

Call a section of $\nu \xi$ piecewise linear w.r.t. $T_r$ if it is linear over each simplex, when viewed in the isometric local trivializations.

Also endow $M$ with an auxiliary riemannian metric $g$. Write $\text{InjRad}$ its injectivity radius.

Lemma 2.4. For every positive real number $\text{Slope}$, one has on $M$ a triangulation $T_r$, a twisted vector field $X$ piecewise linear w.r.t. $T_r$, a section $s$ of $\nu \xi$ piecewise linear w.r.t. $T_r$, and a radius $0 < r \leq \text{InjRad}/2$ such that:

1. In $M$ the vertices of $T_r$ are more than $2r$-separated;
2. At every $x \in M$ one has $\|s(x)\| < 1$;
3. If $x$ is at least $r$-distant from the vertices one has:

$$dsX(x) > \text{Slope}\|X(x)\|$$

Proof — Let first $T_r$ be the barycentric subdivision of any triangulation $T_{r_0}$. One can colour the 0-skeleton of $T_r$ with $n+1$ colours such that the extremities of each edge have different colours. Namely, colour each vertex $x$ with the dimension $d(x)$ of the simplex of $T_{r_0}$ whose barycenter is $x$. Define then a section $s$ of $\nu \xi$ over the 0-skeleton of $T_r$ by choosing over every vertex $x$ any of the (one or two) elements of norm $d(x)/2n$.

Obviously, $s$ extends uniquely to a global section over $M$, still denoted $s$, piecewise linear w.r.t. $T_r$; and one has $\|s\| \leq 1/2$.

Now, $s$ also induces a piecewise linear twisted vector field. Namely, for each simplex $\sigma$ of $T_r$, any choice of an orientation of $\nu \xi$ above $\sigma$ defines a unique isometric oriented trivialization of $\nu \xi$ above $\sigma$, which in turn changes the values of $s$ at the vertices of $\sigma$ into real numbers $\lambda_0, \ldots, \lambda_k$, where $k = \dim \sigma$, such that:

$$(3) \quad \|\lambda\| \leq 1/2 \quad \text{and} \quad \delta(\lambda) \geq 1/(2n)$$

Thus one has a linear vector field $X_\lambda$ on $\sigma$. Changing the orientation of $\nu \xi$ over $\sigma$ changes $\lambda$ into $-\lambda$ and thus $X_\lambda$ into $-X_\lambda$. That is, $X_\lambda$ is a twisted vector field on $\sigma$, well-defined by $s$.

The collection of these fields on all the vertices is a global twisted vector field $X$ on $M$, piecewise linear w.r.t. $T_r$.

To find an $r$ verifying both properties (1) and (3) of lemma 2.4, regard $T_r$ as a family of smooth embeddings of the standard $n$-simplex.
\( \Delta^n \) into \( M \). Push the standard euclidian metric of \( \Delta^n \) through these embeddings, and get on \( M \) a piecewise riemannian, length metric \( g(\text{Tr}) \). Set for any two such metrics \( g_0 \), \( g_1 \):

\[
\text{Lip}(g_1/g_0) := \sup_{v \in T_M, v \neq 0} \frac{\|v\|_{g_1}}{\|v\|_{g_0}}
\]

\[
K(g_0, g_1) := \text{Lip}(g_1/g_0) \text{Lip}(g_0/g_1)
\]

Write for short \( K := K(g, g(\text{Tr})) \). Let \( R \) be a positive constant less than \( 2^{-1/2} = \text{diam}(\Delta^n)/2 \). Let \( r := R/\text{Lip}(g(\text{Tr})/g) \). Thus at each vertex \( x \) of \( \text{Tr} \) one has the inclusions of compact balls:

\[
B_{g(\text{Tr})}(x, R/K) \subset B_g(x, r) \subset B_{g(\text{Tr})}(x, R)
\]

By the very choice of \( R \), the \( g(\text{Tr}) \)-balls of radius \( R \) centered at the vertices are two by two disjoint. Thus the \( g \)-balls of radius \( r \) with the same centers are also two by two disjoint. Outside the \( g(\text{Tr}) \)-balls of radius \( R/K \) with the same centers, by equations 3 and 2 one has \( dsX \geq R^2/(8n^2K^2) \). On the other hand, by equations 3 and 1 one has \( \|X(x)\|_g \leq 1 \). Finally:

\[
dsX \geq R^2(8n^2K^2\text{Lip}(g/g(\text{Tr})))^{-1}\|X\|_g
\]

Thus (3) will hold if \( \text{Lip}(g/g(\text{Tr}))K(g, g(\text{Tr}))^2 \) is close enough to 0: it remains to find arbitrarily fine triangulations of \( M \) with some uniform quasiconformality property, and each of which is a barycentric subdivision.

Whitney’s standard subdivisions \([16]\) will provide it.

Namely, start from an arbitrary triangulation \( \text{Tr} \) of \( M \) and consider the sequence of triangulations \( \text{Bar}(\text{Whi}^p(\text{Tr})) \), where \( \text{Bar} \) denotes the barycentric subdivision, while \( \text{Whi} \) denotes the standard subdivision. Recall that the successive standard subdivisions have a uniform self-similarity property. Regarding \( \Delta^n \) as isometrically embedded into \( \mathbb{R}^n \), there is a finite family \( (\delta_i) \) of affine \( n \)-simplices in \( \mathbb{R}^n \), such that for every \( p \geq 0 \), each \( n \)-simplex of the subdivision \( \text{Whi}^p(\Delta^n) \) is the image of some \( \delta_i \) by a homothety-translation of the form \( x \mapsto 2^{-p}x + \text{const} \).

It follows immediately that the sequence of triangulations \( \text{Whi}^p(\text{Tr}) \) verifies

\[
\text{Lip}(g/g(\text{Whi}^p(\text{Tr}))) \to 0
\]

while \( K(g, g(\text{Whi}^p(\text{Tr}))) \) remains bounded. Thus, the sequence of their first barycentric subdivisions has the same properties:

\[
\text{Lip}(g/g(\text{Bar}(\text{Whi}^p(\text{Tr})))) \to 0
\]

while \( K(g, g(\text{Bar}(\text{Whi}^p(\text{Tr})))) \) remains bounded.

\[\bullet\]
Lemma 2.5. The same as lemma 2.4, except that $s$ is smooth instead of being piecewise linear.

This smoothing relies on the fact that piecewise linear fields allow $\epsilon$-invariant partitions of unity, as follows.

Lemma 2.6. Let $X$ be a twisted vector field on $M$, piecewise linear w.r.t. some triangulation $Tr$, and nondegenerate.

Let $N$ be an open cover of $M$ made of a neighborhood $N_\sigma$ for each simplex $\sigma$ of $Tr$. Denote $N_0 := \cup_{\text{dim } \sigma = 0} N_\sigma$ the union of the neighborhoods of the vertices.

Let $\epsilon > 0$.

Then there exists a smooth partition of unity $(\psi_\sigma)$ subordinate to this cover, such that at every point of $M \setminus N_0$, one has w.r.t. any local isometric trivialization of $\nu_\xi$:

$$|d\psi_\sigma X| < \epsilon$$

Proof. Denote $Tr^k$ the $k$-skeleton of $Tr$, for $0 \leq k \leq n$. The dynamics of the field $X$ is trivial since every simplex $\sigma$ of $Tr$ is invariant, and $X|\sigma$ is linear. Fix a local isometric trivialization of $\nu_\xi$, thus $X|\sigma$ becomes an ordinary vector field $X_\lambda$. Let $\text{vmin}(\sigma)$ (resp. $\text{vmax}(\sigma)$) denote the vertex of $\sigma$ where $\lambda$ is minimal (resp. maximal). In the interior $\text{Int}(\sigma) := \sigma \setminus \partial \sigma$, every integral line goes from $\text{vmin}(\sigma)$ to $\text{vmax}(\sigma)$.

By induction on $k$, assume that one already has a family of smooth functions $(\phi_\tau)_{\text{dim } \tau \leq k}$ on $M$, indexed by the simplices of $Tr$ of dimension $\leq k$, such that:

(i) Each $\phi_\tau$ is supported in $N_\tau$ and takes values in $[0,1]$;

(ii) Each derivate $|d\phi_\tau X| < \epsilon$ outside $N_0$;

(iii) $\Phi_k := \sum_{\text{dim } \tau \leq k} \phi_\tau \geq 1$ on a neighborhood $V_k$ of $Tr^k$.

Given a $(k+1)$-simplex $\sigma$ of $Tr$, we seek for a function $\phi_\sigma$ verifying (1) and (2) for $\tau = \sigma$, and such that $\phi_\sigma = 1$ on a neighborhood of $\sigma \setminus (\sigma \cap V_k)$.

Fix a local orientation for $\nu_\xi$ in a neighborhood of $\sigma$, thus turning $X$ into an ordinary vector field. Write $x = (x_i)$ the standard coordinates in $D^{n-1}$ and $y$ the standard coordinate in $D^1$:

$$D^{n-1} := \{x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} / x_1^2 + \cdots + x_{n-1}^2 \leq 1\}$$

$$D^1 = \{y \in \mathbb{R} / |y| \leq 1\}$$

The subdisk $D^k \subset D^{n-1}$ is defined by $x_{k+1} = \cdots = x_n = 0$.

Since $X|\sigma$ is linear, one has a smooth embedding of $D^{n-1} \times D^1$ into $N_\sigma$ such that:

(1) $(D^{n-1} \times D^1) \cap \sigma = D^k \times D^1$.
(2) \( X(x, y) = \partial / \partial y \) for every \( x \in D^k, y \in D^1 \);
(3) \( \sigma \subset (D^{k-1} \times D^1) \cup V_k \);
(4) \( (\partial D^k) \times D^1 \subset V_k \);
(5) \( D^{n-1} \times (\partial D^1) \subset (N_{v_{\text{min}}(\sigma)} \cup N_{v_{\text{max}}(\sigma)}) \).

By (4) the subset \( \text{pr}_1((D^k \times D^1) \setminus V_k) \) is relatively compact in \( \text{Int}(D^k) \).

Let \( \phi : D^k \to [0, 1] \) be a smooth function with compact support in \( \text{Int}(D^k) \) and which is 1 on a neighborhood of this subset. For every \( (x, y) \in D^{n-1} \times D^1 \), let \( \phi(x, y) = \phi(x_1, \ldots, x_k) \).

Inside \( D^{n-1} \times D^1 \), the euclidian metric w.r.t. the standard coordinate system \( (x_1, \ldots, x_{n-1}, y) \) is understood. Let \( A := \sup \| d\phi \| \).

Being piecewise smooth, \( X \) is \( L \)-Lipschitz for some constant \( L \). Fix a radius \( 0 < r < 1 \) such that \( r \leq \epsilon / (2AL) \).

The vector field \( X' := X - \partial / \partial y \) being null on \( D^k \times D^1 \) (property (2) above) and \( L \)-lipschitz on \( D^{n-1} \times D^1 \), in the \( r \)-neighborhood of \( D^k \times D^1 \) one has \( \| X' \| \leq \epsilon / (2A) \) and thus:

\[
|d\phi X| = |d\phi X'| \leq \epsilon / 2
\]

On the other hand, on the interval \((0, r)\) fix a smooth nonnegative function \( u \) with compact support such that \( \int_0^r u(\rho)d\rho = 1 \) and that \( u(\rho) \leq \epsilon / (2L\rho) \) for every \( \rho \). Let :

\[
\psi(x, y) := 1 - \int_0^{||(x_{k+1}, \ldots, x_{n-1})||} u(\rho)d\rho
\]

thus a smooth function on \( D^{n-1} \times D^1 \) with values in \([0, 1] \); equal to 1 in a neighborhood of \( D^k \times D^1 \); null outside the \( r \)-neighborhood of \( D^k \times D^1 \); and one has:

\[
\| d\psi(x, y) \| \leq \epsilon / (2L ||(x_{k+1}, \ldots, x_{n-1})||)
\]

Since moreover \( X' \) is \( L \)-lipschitz:

\[
|d\psi X| = |d\psi X'| \leq \epsilon / 2
\]

Let \( v \) be a smooth nonnegative function on \( D^1 \) with compact support in \( \text{Int} D^1 \), and such that \( v(y) = 1 \) whenever \( (x, y) \not\in N_0 \) (property (5) above). In view of properties (1) and (3) above, the function

\[
\phi_\sigma(x, y) := \phi(x)\psi(x)v(y)
\]

fulfills the demand.

This gives, by induction on \( k \), a family \( (\phi_\sigma) \) satisfying (i), (ii) and (iii) for \( k = n \). Let for every simplex \( \sigma \) of \( Tr \):

\[
\psi_\sigma := \frac{\phi_\sigma}{\Phi_n}
\]
Obviously it is a partition of unity; and one has:

\[ d\psi_\sigma X = \frac{d\phi_\sigma X}{\Phi_n} - \frac{\phi_\sigma \sum_\tau d\phi_\tau X}{\Phi_n^2} \]

whose absolute value is, in view of (i), (ii), and the triangle inequality, at most \((S + 1)\epsilon\) where \(S\) is the number of simplices of \(Tr\). Lemma 2.6 is proved.

\[ \bullet \]

Proof of lemma 2.5 — Let \(Tr\), \(s\), \(X\), \(r\) be as in lemma 2.4. Denote \(S\) the number of simplices of \(Tr\), and \(N_0\) the \(r\)-neighborhood of the vertices of \(Tr\). Let \(\epsilon > 0\) be less than the infimum, over \(M \setminus N_0\), of \(dsX - \text{Slope} \|X\|\).

For each simplex \(\sigma\) of \(Tr\) of positive dimension, the section \(s|\sigma\) extends into a smooth section \(s_\sigma\) of \(\nu \xi\) over some open neighborhood \(N_\sigma\) of \(\sigma\) in \(M\). Moreover, one can arrange that \(\|s_\sigma(x)\| < 1\) and that, at every \(x \in N_\sigma\) not in \(N_0\):

\[ ds_\sigma X(x) > \text{Slope} \|X(x)\| + \epsilon \]

For every vertex \(\sigma\) of \(Tr\), let \(N_\sigma\) be the \(r\)-ball centered at \(\sigma\) and let \(s_\sigma = 0\).

Lemma 2.6 gives an \((\epsilon/S)\)-invariant partition of unity \((\psi_\sigma)\) subordinate to \((N_\sigma)\). Set:

\[ s_\infty := \sum_\sigma \psi_\sigma s_\sigma \]

Clearly \(s_\infty\) is smooth and \(\|s_\infty(x)\| < 1\). At every \(x \in M \setminus N_0\), fix a local isometric trivialization of \(\nu \xi\). Then the sections \(s_\sigma\), \(s_\infty\) are locally turned into functions \(f_\sigma\), \(f_\infty\), and one has:

\[ df_\infty X(x) = \sum_\sigma (d\psi_\sigma X(x)) f_\sigma(x) + \sum_\sigma \psi_\sigma(x) df_\sigma X(x) \]

The first term has absolute value less than \(\epsilon\) (lemma 2.6) while the second is larger than \(\text{Slope} \|X(x)\| + \epsilon\). Thus \(ds_\infty X(x) > \text{Slope} \|X(x)\|\).

\[ \bullet \]

Recall that the rank-1 vector bundle \(\nu \xi\) has been given a riemannian structure. In particular, on its total space \(E \xi\) one has the norm function \(z \mapsto \|z\|\), whose level sets are the leaves of a smooth foliation \(X_0\). Denote:

\[ D \xi := \{z \in E \xi / \|z\| \leq 1\} \]
Endow $E\xi$ with the riemannian structure such that $\mathcal{X}_0$ is orthogonal to the fibres, that each fibre is isometrically embedded into $E\xi$, and that each leaf of $\mathcal{X}_0$ projects locally isometrically onto $(M,g)$.

By a horizontal foliation $\mathcal{F}$ we mean a foliation of the total space $E\xi$ which is transverse to the microfibres. Then $\mathcal{F}$ is the kernel of a unique differential form $\Omega(\mathcal{F})$ of degree 1, twisted (i.e. taking values in $\nu\xi$), whose restriction to each microfibre is the identity.

**Lemma 2.7.** There is a constant $L < \infty$ such that for every horizontal foliation $\mathcal{X}$ and every finite family of disjoint open balls $B_i \subset M$ of radius $\leq \text{InjRad}/2$ there is a horizontal foliation $\mathcal{X}'$ such that:

1. $\mathcal{X}'$ is homotopic to $\mathcal{X}$ as a $\Gamma_1$-structure on $E\xi$;
2. $\mathcal{X}' = \mathcal{X}_0$ over each ball $B_i$;
3. $\|\Omega(\mathcal{X}') - \Omega(\mathcal{X}_0)\|_{D\xi} \leq L\|\Omega(\mathcal{X}) - \Omega(\mathcal{X}_0)\|_{D\xi}$

Here of course $\|\Omega\|_{D\xi}$ denotes the sup norm of the twisted differential 1-form $\Omega$, with respect to the riemannian structures on $\tau E\xi$ and $\nu\xi$, over the compact subset $D\xi \subset E\xi$.

**Proof.** Fix a smooth function $u : [0, 1] \to [0, 1]$ such that $u(t) = 0$ in a neighborhood of 0 and such that $u(t) = t$ in a neighborhood of 1. Let $F_r$ be, for every $r > 0$, the smooth self-mapping of the compact $n$-ball of centre 0 and radius $r$ defined as

$$F_r(x) := u(||x||)||x||^{-1}x$$

Thus $\|DF_r\| \leq \|u'||_{[0,1]}$. Let $F_M$ be the self-mapping of $M$ defined as the identity outside the balls, and, in each ball $B_i$ of radius $r_i$, conjugate of $F_r$ by the exponential map at its centre. So,

$$\|DF_M\| \leq L := \|u'||_{[0,1]}\|D\exp\|\|D(\exp^{-1})\|$$

where $\|D\exp\|$ is the sup norm of the differential of the exponential map on the $\text{InjRad}/2$-neighborhood of the null section, and $\|D(\exp^{-1})\|$ is the sup norm of the differential of its inverse on the set of couples whose distance is at most $\text{InjRad}/2$. Let $F_{E\xi}$ be the self-mapping of $E\xi$ such that in any isometric local trivialization one has:

$$F_{E\xi}(m, v) = (F_M(m), v)$$

Let $\mathcal{X}' := F_{E\xi}^*\mathcal{X}$. Then (1) and (2) are obvious. Since $\Omega(\mathcal{X}_0)$ and $\Omega(\mathcal{X})$ coincide in restriction to each fibre, (3) is also clear.

•

**Proof of proposition 2.2** — Recall that $\xi$ is the germ along $Z(M)$ of a foliation $\mathcal{X}$, defined in an open neighborhood $U$ of $Z(M)$, and transverse to the microfibres. Without loss of generality, $U = E\xi$. 


Indeed, there is a fibre-preserving embedding \( r \) of \( E\xi \) into \( U \), whose germ along \( Z(M) \) is the identity. One only has to change \( \mathcal{X} \) for \( r^*\mathcal{X} \).

Let \( L \) be as in lemma 2.7. Set:

\[
\text{Slope} := L\|\Omega(\mathcal{X}) - \Omega(\mathcal{X}_0)\|_{D\xi}
\]

Apply lemma 2.5 to get a triangulation \( \mathcal{T} \), a radius \( r \) such that the compact balls of radius \( r \) centered at the vertices are two by two disjoint (denote \( N_0 \) their union), a smooth section \( s \) with values in \( D\xi \), and a twisted vector field \( X \) such that at every point of \( M \setminus N_0 \):

\[
dsX > \text{Slope}\|X\|
\]

Notice that \( dsX \) is nothing but \( \Omega(X)_0s_*X \). Apply lemma 2.7 to \( X \) and to these balls, get a horizontal foliation \( \mathcal{X}' \) homotopic to \( \mathcal{X} \) as a \( \Gamma_1 \)-structure on \( E\xi \), coinciding with \( \mathcal{X}_0 \) over \( N_0 \), and such that:

\[
\|\Omega(\mathcal{X}') - \Omega(\mathcal{X}_0)\|_{D\xi} \leq \text{Slope}
\]

Since \( \Omega(\mathcal{X}') - \Omega(\mathcal{X}_0) \) is null on the microfibres, one deduces:

\[
|(\Omega(\mathcal{X}') - \Omega(\mathcal{X}_0))s_*X| \leq \text{Slope}\|X\|
\]

By the triangle inequality, \( \Omega(X')s_*X > 0 \) outside \( N_0 \).

Finally, apply a small generic perturbation to \( s \) over \( N_0 \), to put it in Morse position with respect to \( \mathcal{X}' = \mathcal{X}_0 \). The \( \Gamma_1 \)-structure \( s^*\mathcal{X}' \) then fulfils the demands of proposition 2.2. \( \square \)

2.3. **Reflection symmetry.** In this paper, to make sure that the modifications performed on the \( \Gamma_1 \)-structure don’t change its homotopy class, one will use the following much elementary principle.

Let \( Y \) be a manifold (in practice \( D^1 \) or \( S^1 \)). Endow \( \mathbb{R}^{n-1} \) with the canonical coordinates \( x_1, \ldots, x_{n-1} \).

Write \( D^{n-1} \subset \mathbb{R}^{n-1} \) the compact unit ball; write \( D^{n-2} = D^{n-1} \cap \{x_{n-1} = 0\} \); write \( D_+^{n-1} = D^{n-1} \cap \{x_{n-1} \geq 0\} \); write \( D_-^{n-1} = D^{n-1} \cap \{x_{n-1} \leq 0\} \). In \( D^{n-1} \times Y \), write \( \sigma \) the reflection involution w.r.t. \( D^{n-2} \times Y \):

\[
\sigma : (x_1, \ldots, x_{n-2}, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-2}, -x_{n-1}, y)
\]

**Lemma 2.8.** Let \( \xi_0, \xi_1 \) be two \( \sigma \)-invariant \( \Gamma_1 \)-structures on \( D^{n-1} \times Y \) which coincide in restriction to \( \partial(D^{n-1} \times Y) \). Then \( \xi_0 \) and \( \xi_1 \) are homotopic rel. \( \partial(D^{n-1} \times Y) \).

One puts no constraint on the positions of \( \xi_0, \xi_1 \) with respect to the boundary.
Proof of the lemma — Set:
\[ r(x_1, \ldots, x_{p-1}, x_p, y) := (x_1, \ldots, x_{p-1}, |x_p|, y) \]
Recall that \( \Gamma_1 \)-structures can be pulled back through continuous maps. By hypothesis:
\[ \xi_i = r^*(\xi_i|\mathbb{D}_+^{n-1} \times Y) \]
\( (i = 0, 1) \). On the other hand, \( \mathbb{D}_+^{n-1} \times Y \) retracts by deformation onto its intersection with \( \partial(\mathbb{D}^{n-1} \times Y) \), in restriction to which \( \xi_0 = \xi_1 \). Thus \( \xi_0 \) and \( \xi_1 \) are homotopic rel. \( \partial(\mathbb{D}^{n-1} \times Y) \).

In the same way, to fix the homotopy class of the gradient, one will apply an analogous elementary symmetry principle for vector fields.

**Lemma 2.9.** Let \( V_0, V_1 \) be two \( \sigma \)-invariant nonsingular vector fields on \( \mathbb{D}^{n-1} \times \mathbb{D}^1, n \geq 4 \), which coincide over \( \mathbb{D}^{n-2} \cup \partial(\mathbb{D}^{n-1} \times \mathbb{D}^1) \). Then \( V_0 \) and \( V_1 \) are homotopic rel. \( \partial(\mathbb{D}^{n-1} \times \mathbb{D}^1) \).

One puts no constraint on the positions of \( V_0, V_1 \), with respect to the boundary.

**Proof of the lemma** — Over \( \mathbb{D}^{n-2} \cup \partial(\mathbb{D}^{n-1} \times \mathbb{D}^1) \), one has a \( \sigma \)-invariant homotopy from \( V_0 = V_1 = \partial/\partial y \) over \( \mathbb{D}^{n-1} \cup \partial(\mathbb{D}^{n-1} \times \mathbb{D}^1) \). Then the obstruction \( o_+ \) (resp. \( o_- \)) to homotope \( V_0 \) to \( V_1 \) over \( \mathbb{D}_+^{n-1} \times \mathbb{D}^1 \) (resp. \( \mathbb{D}_-^{n-1} \times \mathbb{D}^1 \)) rel. \( \partial(\mathbb{D}_+^{n-1} \times \mathbb{D}^1) \) (resp. \( \partial(\mathbb{D}_-^{n-1} \times \mathbb{D}^1) \)) belongs to \( \pi_n(\mathbb{S}^{n-1}) \). Since \( V_0 \) and \( V_1 \) are \( \sigma \)-invariant, \( o_+ = o_- \). Since \( n \geq 4 \), one has \( \pi_n(\mathbb{S}^{n-1}) \cong \mathbb{Z}/2\mathbb{Z} \). Thus \( o_+ + o_- = 0 \). That is, \( V_0 \) and \( V_1 \) are homotopic rel. \( \partial(\mathbb{D}^{n-1} \times \mathbb{D}^1) \).

2.4. **Minimizing.** After proposition [2.2] has been applied, \( \xi \) is smooth in a neighborhood of its singularities, each of which is of Morse type. Each singularity may be given two indices \( i, n - i \), depending on the choice of a local transverse orientation.

One will now easily make all leaves of \( \xi \) dense, through a homotopy (and at the price of some new singularities). Here and further down, if some singularity of indices \( 1, n - 1 \) separates its “leaf”, the two components are to be understood as distinct leaves.

First one gets rid of the singularities of index 0 (or \( n \), which is the same) as follows. Close to each of them \( s_0 \), create a pair \( s_1, s_2 \) of singularities of indices 1, 2, in cancellation position. Then \( s_1 \) is in
cancellation position with $s_0$ as well. That is, in a neighborhood of any arc transverse to $\xi$ except at its extremities $s_0$, $s_1$, the $\Gamma_1$-structure $\xi$ admits a first integral (see paragraph 2.1) for which $s_0$, $s_1$ are Morse singularities in cancellation position, in Smale’s sense. Cancel them. By observation 2.1, the homotopy class of $\xi$ is not changed.

Lemma 2.10. Let $\Sigma$ be a closed manifold. Then on $H := D^2 \times \Sigma \times D^1$ there is a Morse $\Gamma_1$-structure $\xi_\Sigma$ such that:

1. In a neighborhood of $\partial H$, the structure $\xi_\Sigma$ coincides with the slice foliation (projection to $D^1$);
2. $\xi_\Sigma$ is homotopic to this slice foliation relatively to $\partial H$;
3. Every leaf of $\xi_\Sigma$ meets $(\partial D^2) \times \Sigma \times D^1$;
4. Every leaf of $\xi_\Sigma$ meeting $(\partial D^2) \times \Sigma \times [-1/3, +1/3]$ is locally dense.

Proof — First when $\Sigma$ is a point. Choose a representation

$$\rho: \pi_1 S_g \to \text{Diff}_c(-1/2, +1/2)$$

where $S_g$ is the compact surface of some genus $g$ with one boundary component; such that $\rho(\partial S_g) = \text{id}$; and such that under $\rho$, the orbit of every point $-1/3 \leq t \leq +1/3$ is dense. To prove theorem A, $g = 1$ is enough; but for its analogue in class $PL$ one has to take $g = 2$. Inside $D^2 \times D^1$, change the projection to $D^1$ into a Morse-singular function $f$ by creating $g$ pairs of singularities of indices 1 and 2 in cancellation position, the singular values being $-2/3$ and $+2/3$. Then, inside $f^{-1}[-1/2, +1/2] \cong S_g \times [-1/2, +1/2]$, change the level sets of $f$ into the suspension of $\rho$. Let $\xi_{\text{pt}}$ be the resulting $\Gamma_1$-structure. Properties 1, 3, 4 are obvious. Doubling $g$ if necessary, it is easy to arrange that $\xi_{\text{pt}}$ is moreover $\sigma$-invariant (see paragraph 2.3). Property 2 follows by lemma 2.8.

For a general $\Sigma$, one stabilizes $\xi_{\text{pt}}$ by $\Sigma$, that is, one endowes $H := D^2 \times \Sigma \times D^1$ with the $\Gamma_1$-structure $\pi^* \xi_{\text{pt}}$ pullback of $\xi_{\text{pt}}$ by the projection $\pi$ onto $D^2 \times D^1$. Obviously, $\pi^* \xi_{\text{pt}}$ is homotopic relatively to $\partial H$ to the slice foliation on $H$.

The singular set of $\pi^* \xi_{\text{pt}}$ consists in $2g$ submanifolds of the form $s \times \Sigma$, where $s$ is a singularity of $f$ of index $i = 1$ or 2. It is easy to change every such singular submanifold into Morse singularities whose indices are between $i$ and $\dim \Sigma + i$: one chooses a small open neighborhood $N^3$ of $s$ in $D^2 \times D^1$ and modifies $\pi^* \xi_{\text{pt}}$ inside $\Sigma \times N^3$, endowed with coordinates $x \in \Sigma$, $z \in N^3$, by changing the local first integral $(x, z) \mapsto f(z)$ into the function $(x, z) \mapsto f(z) + g(x)h(z)$, where $g$ is a Morse function on $\Sigma$, and where $h$ is a smooth function with compact support in $N^3$, positive and constant in a neighborhood of $s$;
and such that $\|Dh\| \leq \|Df\|$ at each point of $N^3$. By observation 2.1, this morsification does not change the homotopy class of the structure relatively to $\partial H$. The result is a Morse $\Gamma_1$-structure $\xi_\Sigma$ which obviously fulfils properties 1 through 4.

Since $\xi$ is Morse-singular and has no singularity of index 0 or $n$, there is in $M$ a finite family of disjoint transverse compact arcs whose union meets all leaves. Each arc $A_k$ has a neighborhood of the form $D^{n-1} \times D^1$, to which the restriction of $\xi$ is the slice foliation. In this $D^{n-1}$ lies a $S^{n-3}$ with trivial normal bundle. Thence a $D^2 \times S^{n-3}$ embedded into $D^{n-1}$. Thus, there is a $H_k \cong (D^2 \times S^{n-3} \times D^1) \subset M$ such that $A_k$ is contained in $D^2 \times S^{n-3} \times [-1/3, +1/3]$ and that $\xi|H_k$ is the slice foliation. Apply lemma 2.10 to $\Sigma := S^{n-3}$, change $\xi$ into $\xi_{S^{n-3}}$ inside every $H_k$, and get a $\Gamma_1$-structure on $M$, homotopic to $\xi$, and every leaf of which is locally dense. Thus every leaf is dense.

2.5. Morsifying the contact. The previous section has left the $\Gamma_1$-structure $\xi$ minimal, and smooth in a neighborhood of its singularities, which are of Morse type (and necessarily of index $\neq 0, n$). In the next sections it will be homotoped to some foliation. In the present section one prepares $J$ and $\xi$ so that the gradient of this foliation will be homotopic to $J$.

To avoid irrelevant technicallities, we give standard forms to $\xi$ and $J$ close to every singularity $s$. Fix once and for all, in a neighborhood of $s$, a smooth local system of coordinates $(x_1, \ldots, x_n)$ in which $\xi$ admits for a first integral the standard quadratic form of rank $n$ and index $i$:

$$Q_i(x_1, \ldots, x_n) := -x_1^2 - x_2^2 - \cdots + x_{i+1}^2 + x_{i+2}^2 + \cdots$$

Call them the standard coordinates at $s$. In the rest of the proof of theorem A, by the words “relative to the singularities” one means: relative to some neighborhood of the singularities.

**Definition 2.11.** The twisted vector field $J$ is standard at the singularities if, in the local standard system of coordinates, $J = \partial/\partial x_1$ or $\partial/\partial x_{i+1}$.

This is immediately obtained by a local homotopy of $J$.

**Definition 2.12.** A pseudogradients for the Morse $\Gamma_1$-structure $\xi$ is a smooth, Morse-singular, twisted vector field $\nabla \xi$ such that:

1. At every point of $M$ regular for $\xi$, the twisted vector $\nabla \xi$ is negatively transverse to $\xi$;
(2) In a neighborhood of every singularity:
\[ \nabla \xi = (-\partial Q_i/\partial x_j)_{1 \leq j \leq n} \]
in the standard coordinates.

One easily builds such a pseudogradient \( \nabla \xi \) by means of a partition of unity.

**Definition 2.13.** The contact locus \( C(J, \nabla \xi) \) (resp. positive contact locus \( C_+(J, \nabla \xi) \)) (resp. negative contact locus \( C_-(J, \nabla \xi) \)) is the set of points of \( M \) where the twisted vector fields \( \nabla \xi \) and \( J \) are colinear (resp. nonnegatively colinear) (resp. nonpositively colinear).

We are naturally interested in the negative one. After a small generic perturbation of \( J \) relative to the singularities, \( J \) is smooth and is transverse to \( -\nabla \xi \) as sections of the unitary twisted tangent bundle, thence \( C_-(J, \nabla \xi) \) is in \( M \) a smooth compact submanifold of dimension one, whose boundary is exactly the set of singularities. This last affirmation is immediate since \( J \) is standard at the singularities.

Also, one easily gets rid of irrelevant technicalities related to the eventual lack of differentiability of \( \xi \):

**Lemma 2.14.** After a small isotopy of \( \xi \), relative to the singularities, \( \xi \) is smooth in a neighborhood of \( C_-(J, \nabla \xi) \).

**Proof.** — Note that \( \nabla \xi \) and \( J \) remain unchanged.

One applies to \( \xi \), successively, two small isotopies, relative to the singularities, and so \( C^1 \)-small that \( \nabla \xi \) remains a pseudogradient for \( \xi \). The first one makes \( \xi \) transverse to \( C_-(J, \nabla \xi) \), but at a finite number of points. This is easy even for \( r = 0 \), since \( C_-(J, \nabla \xi) \) is of dimension 1. It is then also easy, by a second isotopy, to make \( \xi \) smooth in a neighborhood of \( C_-(J, \nabla \xi) \).

Let \( T \) denote the projection of \( J \) into \( \tau \xi \) parallelly to \( \nabla \xi \), thus a twisted vector field on \( M \), undefined at the singularities, smooth in a neighborhood of \( C_-(J, \nabla \xi) \). One has a decomposition:
\[ J = h \nabla \xi + T \]
where \( h \) is a function. The set of zeroes of \( T \) (where \( h < 0 \)) is nothing but the (negative) contact locus minus the singularities. It appears more convenient to think of \( T \) than of \( J \). One will eventually modify \( T \) close to \( C_-(J, \nabla \xi) \), and this will result into a homotopy of \( J \):

**Observation 2.15.** Let \( T' \) be a twisted vector field on \( M \) such that:

1. \( T' \) is tangential to \( \xi \);
(2) $T' = T$ close to the singularities;

(3) $T' = T$ outside a small enough neighborhood of $C_-(J, \nabla \xi)$.

Then $J' := h \nabla \xi + T'$ is homotopic to $J$ relatively to the singularities.

Indeed, if $h < 0$ on the support of $T' - T$, then $(1 - t)J + tJ'$ is nonsingular for every $0 \leq t \leq 1$.

Consider the nature of every point $m \in C_-(J, \nabla \xi)$, not a singularity of $\xi$, as a singularity for the field $T|L_m$, restriction of $T$ to the leaf of $\xi$ through $m$.

Call $m$ nondegenerate if $m$ is a nondegenerate singularity for $T|L_m$. That is, if no eigenvalue of the differential $D_m(T|L_m)$ at $m$ is null.

Then, at $m$ the curve $C_-(J, \nabla \xi)$ is transverse to $\xi$. Moreover, $\xi$ being given a local transverse orientation, we refer to the Poincaré-Hopf index of $T|L_m$ at $m$, that is, the degree of $T$ as a self-mapping of $S^{n-2}$, as the contact degree.

Call $m$ hyperbolic if $m$ is a hyperbolic singularity for the field $T|L_m$. That is, no eigenvalue of the differential $D_m(T|L_m)$ at $m$ is of null real part.

Then, $\xi$ being given a local transverse orientation, we refer to the Morse index of $T|L_m$ at $m$, that is, the number of eigenvalues whose real part is negative, as the contact index.

Call $m$ Morse if $m$ is a Morse singularity for the field $T|L_m$. That is, in a neighborhood of $m$, this field is smoothly conjugate to the gradient field of a nondegenerate quadratic form.

Recalling that $T$ is a twisted vector field, one observes that reversing the transverse orientation of $\xi$ turns the contact degree from $d$ into $(-1)^{n-1}d$, and the contact index from $i$ into $n - 1 - i$. Also, the following is immediate in view of the standard form that $J$ has been given at the singularities.

observation 2.16. Let $s$ be a singularity, of index $i$ with respect to some local transverse orientation of $\xi$. Then every $m \in C_-(J, \nabla \xi)$ close enough to $s$ is Morse. The contact index at $m$ is $i$ if $m$ is above $s$, and $i - 1$ if $m$ is below $s$.

Let $A$ be an embedding of $D^1$ into $M$ transversely to $\xi$ except at its endpoints, which are singularities. The orientation of $A$ induces a local transverse orientation of $\xi$ in a neighborhood, which in turn gives the origin (resp. extremity) singularity a well-defined index $i$ (resp. $i'$). Assume that $i' = i + 1$.

definition 2.17. Then we say that the endpoints have successive indices and that the arc $A$ is bisingular.
Proposition 2.18. After homotopies of $\xi$ and $J$ relative to the singularities, that keep $\xi$ Morse-singular and minimal; and for a appropriate choice of $\nabla\xi$:

1. $\xi$ is still smooth in a neighborhood of $C_-(J, \nabla\xi)$;
2. Every connected component of $C_-(J, \nabla\xi)$ is a bisingular arc;
3. Every negative contact point is Morse.

It may be noticed that none of these prescriptions can in general be fulfilled just by a small perturbation of $J$. Indeed, for a generic $J$:

1. There may be degenerate contact points, where $T$ presents a "saddle-node bifurcation". We shall call them cubic. At such a point $C_-(J, \nabla\xi)$ is tangential to $\xi$ and their contact is quadratic.
2. Moreover the nondegenerate points need not be all hyperbolic. Generically $T$ may have finitely many “Hopf bifurcations”, non-degenerate and nonhyperbolic singular points where the contact index changes by $\pm 2$.
3. Moreover the hyperbolic points need not be Morse: the eigenvalues of $DT$ need not be real.
4. Moreover, $C_-(J, \nabla\xi)$ may have circular components.
5. Moreover, the indices $i$, $i'$ of the endpoints of an arc component need not be successive: if all interior points of the arc are hyperbolic, then the contact index is constant; thus by observation 2.16 $i$ and $i'$ are successive. On the contrary, if the arc contains cubic points or Hopf bifurcations, $i$ and $i'$ need not be successive.

All these phenomena are of course stable by any small perturbation of $J$.

The proof of proposition 2.18 needs some preliminaries.

What is the nature of the degenerate contact points for a generic $J$? As aforesaid, it is the same to think of a generic $T$. In a neighborhood of every contact point, $T$ is just a 1-parameter family of germs of vector fields in $\mathbb{R}^{n-1}$. The generic bifurcations of their singularities are of course well-known: “saddle-node bifurcation”, “fold”. The models are as follows. Endow $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_{n-1}, y$. For $\epsilon = \pm 1$ and for $\lambda = (\lambda_2, \ldots, \lambda_{n-1}) \in (\mathbb{R}^*)^{n-2}$, set:

$$X_y^{\epsilon, \lambda} := (x_1^2 + \epsilon y)\partial / \partial x_1 + \sum_{j=2}^{n-1} \lambda_j x_j \partial / \partial x_j$$

Theorem 2.19. (Shoshitaichvili: [10], [11]). Every generic 1-parameter family $(X_y)$ of smooth vector fields in $\mathbb{R}^{n-1}$, such that $X_0$ has a degenerate singularity at $x_1 = \cdots = x_{n-1} = 0$, is topologically conjugate,
in a neighborhood of \( x_1 = \cdots = x_{n-1} = y = 0 \), to one of the above models.

**Definition 2.20.** Call \( c \in C_-(J, \nabla \xi) \) cubic if \( c \) is degenerate and if in a neighborhood of \( c \), the field \( T \) is topologically conjugated to some \( X^{c, \lambda} \). Call \( c \) perfect if moreover \( T \) is locally smoothly conjugate to \( X^{c, \lambda} \). Given a local transverse orientation of \( \xi \), call \( c \) a death (resp. birth) cubic point if, \( \partial / \partial y \) being assumed positive, one has \( \epsilon = +1 \) (resp. \(-1\)).

**Observation 2.21.** Let \( c \in C_-(J, \nabla \xi) \) be cubic (resp. perfect cubic).

1. At \( c \) the curve \( C_-(J, \nabla \xi) \) has a quadratic tangency with \( \xi \);
2. The contact points close to \( c \) are hyperbolic (resp. Morse); and the contact index varies from \( i \) to \( i' = i \pm 1 \) when crossing \( c \).

Say that \( c \) has indices \((i, i')\).

Also observe an obvious way to create a pair of cubic points.

**Lemma 2.22.** Let \( m \in C_-(J, \nabla \xi) \) be nondegenerate. Give \( \xi \) a local transverse orientation. Let \( 0 \leq i, i' \leq n - 1 \) be such that \((-1)^i\) is the contact degree at \( m \) and that \( i' = i \pm 1 \).

Then, by a homotopy of \( J \) in a small neighborhood of \( m \), one can create on \( C_-(J, \nabla \xi) \) a pair of perfect cubic points: a birth point and a death point, such that the contact index is \( i' \) between them and \( i \) outside them. The other neighboring points remain nondegenerate.

**Proof** — By a first local modification of \( T \), one makes \( m \) a Morse point of contact index \( i \). The creation of the pair is then obvious. As a result, in the local leaf \( L_x \) through any point \( x \) of \( C \) between the cubic points \( c, c' \), the tangential field \( T|L_x \) has three singularities: two of Morse index \( i \) and one, \( x \), of index \( i' \). One has the birth of a pair, in cancellation position, at \( c \), and the death of a pair at \( c' \).  

One also has a tool to change a negative cubic point into a pair of singularities.

**Lemma 2.23.** Let \( c \in C_-(J, \nabla \xi) \) be cubic. Give \( \xi \) a local transverse orientation. Let \( i, i + 1 \) be the neighboring contact indices.

Then, after local homotopies of \( \xi \) and \( J \), the negative contact locus looses a small arc through \( c \), whose endpoints become singularities \( s, s' \) of \( \xi \). If \( c \) is a birth point, the indices of \( s \), \( s' \) are \( i, i + 1 \). If \( c \) is a death point, the indices of \( s \), \( s' \) are \( i + 1, i + 2 \).

This is done in three times.
Lemma 2.24. Let \( c \in C_-(J, \nabla \xi) \) be cubic. Then some homotopy of \( J \) in a neighborhood of \( c \) makes it perfect.

Proof of the lemma — Regard local tangential vector fields as maps into \( \mathbb{R}^{n-1} \). By Shoshitaichvili’s theorem 2.19 there is a neighborhood \( N \) of \( c \) in \( M \), and a \( C^0 \) embedding \( f_0 : N \to \mathbb{R}^n \) such that \( T = X^{\epsilon,L} \circ f_0 \). One arranges that \( N \) is a compact ball on which \( h \neq 0 \), and whose intersection with \( C \) is a single arc. In \( \mathbb{R}^n \) write \( P \) the parabol, set of the zeroes of \( X^{\epsilon,L} \). Obviously there is a continuous homotopy \( (f_t) \) from \( f_0 \) to some smooth embedding \( f_1 : N \to \mathbb{R}^n \), such that for each \( t \) one has \( N \cap C = f_t^{-1}(P) \). One defines a twisted vector field \( T' \) on \( M \) as coinciding with \( T \) in \( M \setminus N \), and in \( N \) with the map :

\[
N \cong (N \times 1) \cup (\partial N \times [0,1]) \to \mathbb{R}^{n-1} : (m,t) \mapsto X^{\epsilon,L}(f_t(m))
\]

So \( T' \) is nonsingular, except on \( C \); and smooth outside \( \partial N \times [0,1] \). Change it into a smooth approximation \( T'' \) inside \( \partial N \times [0,1] \), to make it smooth. Then \( J(T'') := h\nabla \xi + T'' \) is homotopic to \( J \); and \( C \) is also the contact locus of \( \nabla \xi \) with \( J(T'') \); and \( T'' \) is smoothly conjugate to \( X^{\epsilon,L} \) close to \( c \).

Lemma 2.25. Let \( U \) be a nonempty open subset of \( M \), disjoint from \( C_-(J, \nabla \xi) \), and in which \( \xi \) is transversely oriented, and let \( 0 \leq i \leq n-2 \).

(1) Then there is a homotopy of \( \xi \) with support in \( U \) which creates two singularities \( s_i, s_{i+1} \) of indices \( i, i+1 \), and whose effect on \( C_-(J, \nabla \xi) \) is to create a new arc component \( C_-(J, \nabla \xi)_{\text{new}} \) between \( s_i, s_{i+1} \), which presents exactly one degenerate point — a perfect death cubic point. The other points on \( C_-(J, \nabla \xi)_{\text{new}} \) are Morse and their contact indices are \( i, i+1 \).

(2) The same as (1), except that \( C_-(J, \nabla \xi)_{\text{new}} \) presents no degenerate point. The contact index on \( C_-(J, \nabla \xi)_{\text{new}} \) is \( i \).

Proof of lemma 2.25 — (1) This will be verified on a local model where moreover one is reduced to the 2-dimensional case (figure 1, left). After passing to appropriate local coordinates, \( U = \mathbb{R}^n \) endowed with coordinates \( (x_1, \ldots, x_{n-1}, y) \), and in \( U \) the original \( \Gamma_1 \)-structure \( \xi \) admits for a first integral a nonsingular function of the form

\[
f := y + Q(x_1, \ldots, x_{n-1})
\]

where \( Q = \lambda_1 x_1^2 + \cdots + \lambda_{n-1} x_{n-1}^2 \) is a nondegenerate diagonal quadratic form of index \( i \). Moreover in \( U \):

\[
\nabla \xi = (-\partial f/\partial x_1, \ldots, -\partial f/\partial x_{n-1}, -\partial f/\partial y)
\]
Figure 1. Creation of a new component of $C_-(J, \nabla \xi)$ (as dotted lines). The arrows are the descending gradient.

The vector field $J$ being in $U$ nowhere negatively colinear to $\nabla \xi$, one can arrange moreover, after a homotopy with compact support, that in a neighborhood of $D^{n-1} \times D^1 \subset \mathbb{R}^n$, one has $J = \partial / \partial x_1$.

Let $\xi'$ be the $\Gamma_1$-structure on $M$ that coincides with $\xi$ outside $U$ and that admits in $U$ for a first integral the function:

$$g := y - b(\|x\|)b(y) + Q(x_1, \ldots, x_{n-1})$$

where of course $\|x\| := (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$ and where $b$ is a bump function. By observation 2.1, $\xi'$ is homotopic to $\xi$.

Define the pseudogradient $\nabla \xi'$ as equal to $\nabla \xi$ outside $U$, while inside $U$:

$$\nabla \xi' := (-\partial g / \partial x_1, \ldots, -\partial g / \partial x_{n-1}, -\partial g / \partial y)$$

Choose $b$ smooth, nonnegative, even, such that its support is exactly $[-1, +1]$, such that $b = 1$ in a neighborhood of $0$, and such that $b'' = 0$ at exactly two points $\pm y_0$. So $b' = 1$ at exactly two points $y_1, y_2$; and $-1 < y_1 < y_0 < y_2 < 0$. Moreover choose the $\lambda_i$'s large enough with respect to the variations of the bump function, namely:

$$|b'(x)/x| < 2|\lambda_i| \quad (x \in \mathbb{R}, 1 \leq i \leq n-1)$$
Then the following is immediately verified.

The function $g$ has exactly two singularities $(0, \ldots, 0, y_1), (0, \ldots, y_2)$. They are nondegenerate of indices $i, i + 1$. Provided that $b$ has been chosen to coincide with second degree polynomials in neighborhoods of $y_1$ and of $y_2$, the coordinates $x_1, \ldots, x_{n-1}, y$ are standard coordinates for $\xi'$ at these singularities, thus $J$ is also standard there with respect to $\xi'$.

The new contact locus:

$$\frac{\partial g}{\partial x_2} = \cdots = \frac{\partial g}{\partial x_{n-1}} = \frac{\partial g}{\partial y} = 0$$

is contained in the 2-plane $x_2 = \cdots = x_{n-1} = 0$ and its negative component:

$$C_{\text{new}}^- := C_-(J, \nabla \xi') \setminus C_-(J, \nabla \xi)$$

is an arc linking the singularities. The equation of the degenerate contact points:

$$\frac{\partial g}{\partial x_2} = \cdots = \frac{\partial g}{\partial x_{n-1}} = \frac{\partial g}{\partial y} = \frac{\partial^2 g}{\partial y^2} = 0$$

admits a unique solution $c := (x_0, 0, \ldots, y_0)$ of negative contact. One can also arrange, by a appropriate choice of $b$ in a neighborhood of $x_0$ and $y_0$, that $c$ is a perfect cubic point. The details are left as an exercise.

By construction, the nondegenerate new contact points are Morse (another reason for that is that $J$ is also the gradient of a function there.)

(2) Same method as above, but $J$ is homotoped to $-\partial/\partial y$ instead of $\partial/\partial x_1$ (figure 1, right).

Lemmas 2.26. Let $c, c' \in C_-(J, \nabla \xi)$ be perfect cubic. Assume that there is a arc $[c, c']$ embedded in $M$ transversely to $\xi$, such that for the induced local transverse orientation, $c$ and $c'$ have the same contact indices, $c$ is a death point, and $c'$, a birth point. Assume moreover that $[c, c']$ is disjoint from $C_-(J, \nabla \xi)$ except at $c, c'$, and that $b < 0$ on $[c, c']$.

Then some homotopy of $J$ in a neighborhood of $[c, c']$ performs an elementary surgery of index 1 on $C_-(J, \nabla \xi)$ (removing $D^1 \times S^0$ and adding $S^0 \times D^1$) and cancels the two cubic points.

Proof of lemma 2.26 — Let a point $m$ follow $[c, c']$. In the local leaf $L_m$ of $\xi$ through $m$, the tangential vector field $T|L_m$, for $m$ slightly below $c$, presents a pair of Morse singularities of indices $i, i + 1$ in
cancellation position. They cancel at \( m = c \). For \( c < m < c' \), the field \( T|L_m \) is not singular. At point \( m = c' \) one has in \( L_m \) the birth of a pair of singularities of \( T \) of the same indices \( i, i + 1 \). Finally, for \( m \) slightly above \( c' \), the field \( T|L_m \) presents a pair of Morse singularities of indices \( i, i + 1 \) in cancellation position, thus smoothly conjugate to the situation for \( m \) slightly below \( c \). Thanks to the smooth conjugations to the models \( X^{\pm 1,L} \) for both perfect cubic points \( c, c' \), one easily changes \( T \) into \( T' \) which has on \( L_m \) a pair of singularities of indices \( i, i + 1 \) for every \( m \). One changes \( J \) into \( J(T') \), which is homotopic to \( J \) (observation 2.15). Then there is no more death nor birth, that is, the cubic points \( c, c' \) have cancelled. The negative contact set \( C_{-}(J, \nabla \xi) \) has been applied an elementary surgery on index 1.

**Proof of lemma 2.23** — Assume first that \( c \) is a birth point. By lemma 2.24 make it perfect. Slightly below \( c \), change \( \xi \) and \( \nabla \xi \) as in lemma 2.25 to make a new death perfect cubic \( c' \in C_{-}(J, \nabla \xi) \) of the same contact indices \( i, i + 1 \). From \( c' \) to \( c \) there is a short positive transverse arc. Cancel them by lemma 2.26.

The death case follows by reversing the local transverse orientation.

**Proof of proposition 2.18** — As aforesaid, Shositaichvili’s normal form theorem 2.19 for the bifurcations of a singularity of vector fields shows that after some generic perturbation of \( J \), all degenerate negative contact points are cubic. This property will be preserved all along the proof.

First one will make \( C_{-}(J, \nabla \xi) \) a disjoint union of arcs, transverse to \( \xi \) except at their endpoints which will be singularities of successive indices, and each of which will contain an even number of cubic points. Then one will build in \( M \) a 1-manifold \( C'_{-} \) which fulfils the prescriptions and which is cobordant to \( C_{-}(J, \nabla \xi) \). Finally one will apply Thom-Pontryagin’s method to homotope \( J \) into a new field whose negative contact locus with \( \nabla \xi \) will be \( C'_{-} \).

1. One will arrange that each component of \( C_{-}(J, \nabla \xi) \) be an arc whose endpoints have successive indices, and bears an even number of cubic points. The tools for that are lemmas 2.22 and 2.23.

1.1 — To turn every circle component \( S \) of \( C_{-}(J, \nabla \xi) \) into an arc, use lemma 2.22 to create on \( S \), with respect to some local transverse orientation of \( \xi \), a birth cubic point \( c \) and a death cubic point of contact indices, say, 1 and 2. Then apply lemma 2.23 to change \( c \) into
two singularities of indices 1 and 2. The circle component has become an arc component.

After that, \( \nu \xi \) has become orientable in a neighborhood of \( C_-(J, \nabla \xi) \). Fix such an orientation.

1.2 — One now makes every component of \( C_-(J, \nabla \xi) \) contain an even number of cubic points. Let a component \( A \) not be so.

Since, along the arc \( A \), the sign of the contact degree changes at each cubic point, it must be positive on some subarc. There, apply lemma 2.22 to create two cubic points \( c_1 \) (birth), \( c_4 \) (death), the contact index being 0 around the subarc \([c_1, c_4]\), and 1 inside it. Between them, apply 2.22 again to create two cubic points \( c_2 \) (death), \( c_3 \) (birth), the contact index being 2 between \( c_2 \) and \( c_3 \), 1 between \( c_1 \) and \( c_2 \), 1 between \( c_3 \) and \( c_4 \). We can apply lemma 2.23 to either \( c_3 \) or \( c_4 \). In both cases, this creates two singularities of indices 1, 2; and \( A \) is cut into two components. For one of the two cuts, each of the new components bears an even number of cubic points.

1.3 — Once its number of cubic points is even, each component \( A \) of \( C_-(J, \nabla \xi) \) is, in a neighborhood of one of its endpoints \( s \), above \( s \); and in a neighborhood of its other endpoint \( s' \), below \( s' \). Let \( i \) (resp. \( i' \)) be the index of the singularity \( s \) (resp. \( s' \)). Let \( k := i' - i \). One wants to force \( k = 1 \).

Notice that \( k \) is odd. Indeed, along the arc, the contact degree is constant but at the cubic points where it changes; while by observation 2.16 this degree equals \((-1)^i\) close to \( s \), and \((-1)^{i'-1}\) close to \( s' \).

First case: \( k \geq 3 \). It this case, close to \( s \) on \( A \) create, using lemma 2.22 a death point of contact indices \((i, i + 1)\) and a birth point of contact indices \((i + 1, i)\). If \( k \geq 5 \), between them create a second pair of cubic points: a birth point of contact indices \((i + 2, i + 1)\) and a death point of contact indices \((i + 2, i + 1)\). And so on, until on \( A \) starting from \( s \) one has a sequence of \( k - 1 \) cubic points, alternatively death and birth points, and of respective contact indices \((i, i + 1)\), \((i + 1, i + 2)\), \(\ldots\), \((i + k - 2, i + k - 1)\). Change them into \( k - 1 \) pairs of singularities (lemma 2.23). We are left with \( k \) arc components of \( C_-(J, \nabla \xi) \), each of whose has endpoints whose indices are successive, and bears an even number of cubic points.

The second case \( k \leq -1 \) is much alike: close to \( s \) on \( A \) create \( 1 - k \) nested pairs of cubic points such that the \( 1 - k \) first points are alternatively death and birth points, and have respective contact indices \((i, i - 1)\), \((i - 1, i - 2)\), \(\ldots\), \((i + k, i + k - 1)\). Then change them into \( 1 - k \) pairs of singularities (lemma 2.23). We are left with \( 2 - k \) arc components of \( C_-(J, \nabla \xi) \), each of whose has endpoints whose indices are successive, and bears an even number of cubic points.
2. One will build a compact orientable surface $S$ embedded in $M$ such that:

1. The boundary of $S$ is the union of $C_-(J, \nabla \xi)$ with some arcs transverse to $\xi$ and some circles transverse to $\xi$;
2. Over $S$, both vector bundles $\nu \xi$ and $\tau M$ are orientable.

Let $p : \tilde{M} \to M$ be a finite cover of $M$ in which the lifts of both $\nu \xi$ and $\tau M$ are oriented. Let $\tilde{\xi} := p^* \xi$. Let $\tilde{C} \subset \tilde{M}$ be a 1-submanifold that projects one-to-one onto $C_-(J, \nabla \xi)$ . Then each component $\tilde{A}$ of $\tilde{C}$ is an arc which has with $\tilde{\xi}$ an even number of tangency points, alternatively local maxima and local minima. Orient $\tilde{A}$ so that, of these points $c_1 < \cdots < c_{2i}$ , $c_{2i-1}$ is a local maximum and $c_{2i}$ a local minimum ($1 \leq i \leq k$). Write $c_0$ and $c_{2k+1}$ the origin and the extremity of $\tilde{A}$ . For every $c, c' \in \tilde{A}$ , write $[c, c']$ the segment of $\tilde{A}$ between $c$ and $c'$ , oriented from $c$ to $c'$ .

Recall the folkloric “transitivity” properties of codimension-one foliations without closed leaf. They obviously extend without change to Morse-singular $\Gamma_1$-structures:

**Lemma 2.27.** Let $\xi$ be a Morse-singular $\Gamma_1$-structure without closed leaf on a connected manifold $X$ of dimension $\geq 3$ . Let $x, y \in X$ .

1. If $\xi$ is transversely oriented, then $X$ contains an arc from $x$ to $y$ , positively transverse to $\xi$ .
2. If $\xi$ is not transversely orientable, then for any choice of a transverse orientation of $\xi$ at $x$ and of a transverse orientation of $\xi$ at $y$ , there exists an arc from $x$ to $y$ , transverse to $\xi$ , positive at each extremity with respect to the chosen orientation.

Thus, there is in $\tilde{M}$ a positive transverse arc $a_i$ from $c_{2i-1}$ to $c_{2i}$ , $1 \leq i \leq k$ . Let :

$$z_i := a_i \cup [c_{2i}, c_{2i-1}] \quad (1 \leq i \leq k)$$

$$\tilde{A}' := (\bigcup_{i=0}^k [c_{2i}, c_{2i+1}]) \cup (\bigcup_{i=1}^k - a_i)$$

After a small perturbation, the arc $\tilde{A}'$ and the circles $z_1 , \ldots , z_k$ are transverse to $\xi$ . The arcs $\tilde{A}$ , $\tilde{A}'$ and the circles $z_1 , \ldots , z_k$ bound an orientable embedded surface $\tilde{S}(A)$ — of course, the new arc $\tilde{A}'$ and the old one $\tilde{A}$ coincide close to their common endpoints, so $\tilde{S}(A)$ has a kind of spine there, but this will make no problem.

Let $S \subset M$ be the union of the $p(\tilde{S}(A))$’s, where $A$ describes the connected components of $C_-(J, \nabla \xi)$ . One can arrange that $S$ is embedded into $M$ . This is easy, since each $\tilde{S}(A)$ is contained in an arbitrarily small neighborhood of the 1-complex $\tilde{A} \cup \tilde{A}'$. 
Also, in $M$, the projections $p(a_i)$ being transverse to $\xi$, it is easy to make $\xi$ smooth close to them by a small isotopy. So $\xi$ is smooth in a neighborhood of $S$.

Also, since $S$ has the homotopy type of a complex of dimension 1, in a neighborhood one trivializes both orientable vector bundles $\nu_\xi$ and $\tau_M$. So, there, $\nabla_\xi$ and $J$ become not only ordinary vector fields, but even maps into $S^{n-1}$ (except close to the singularities). One can also, to fix ideas, choose the trivialization such that in a neighborhood of $S$ (except close to the singularities), $\nabla_\xi$ is a constant map into $S^{n-1}$.

Define $C'_-$ as the union in $M$ of the arcs $p(A'_i)$ with the circles $p(z_i)$'s: a 1-manifold that coincides with $C_-(J, \nabla_\xi)$ close to the singularities; and transverse to $\xi$ except at the singularities. One has $\partial S = C_-(J, \nabla_\xi) \cup C'_-.$

3. Recall that the Thom-Pontryagin theory associates to $J$ and $\nabla_\xi$ (and in the present situation, outside the singularities), a framing $F$ of the curve $C_-(J, \nabla_\xi)$, that is, a trivialization of the bundle normal to $C_-(J, \nabla_\xi)$ in $M$. Namely, $F$ is mapped by the differential of $J$ to a constant basis of the vector space tangent to $S^{n-1}$ at the point $\nabla_\xi$.

This framing $F$ then extends into a framing $F_S$ of $S \cong ([0, 1/2] \cup (S \times (1/2)))$ in $M \times [0, 1]$. Indeed, $S$ retracts by deformation on $C_-(J, \nabla_\xi)$, and $\tau_S$, $\tau_M|S$ are orientable.

By the Thom-Pontryagin theory, $J$ is homotopic (relatively to the singularities and to the exterior of a small neighborhood of $S$) to some nonsingular twisted vector field $J'$ such that $C_-(J', \nabla_\xi) = C'_-.$

Every connected component of $C_-(J', \nabla_\xi)$ is either a transverse arc whose endpoints are singularities of successive indices, or a transverse circle. One has a decomposition $J' = h'\nabla_\xi + T'$ where $h'$ is a function and where $T'$ is a twisted vector field. The 1-manifold $C_-(J', \nabla_\xi)$ is the set of zeroes of $T'$ where $h' < 0$.

Every negative contact point $m$ of $J'$ with $\nabla_\xi$ is nondegenerate. Indeed, at $m$, the 1-manifold $C_-(J', \nabla_\xi)$ being transverse to $\xi$, the differential of $T'|L_m$ coincides with the frame $F_S(m)$.

One will now change every circle component of $C_-(J', \nabla_\xi)$ into arcs. To this end, by lemma 2.23 one creates on it, by a local homoyopy of $J'$, a pair of cubic points of contact indices, for example, 1 and 2. Those are in turn, by lemma 2.24, changed into four singularities of indices 1, 2, 2, 3, leaving, instead of the circle, two nondegenerate arc components of $C_-(J', \nabla_\xi)$ whose endpoints have successive indices.

Through this proof, up to this point, $\xi$ has been modified by some creations of pairs of singularities in cancellation position. In dimensions $n \geq 4$, these singularities were all of indices between 1 and $n - 1$, and
so obviously $\xi$ has remained minimal. In dimension $n = 3$ it is not the case: during the last step one has created singularities of index 3. One has to get rid of them.

Let $s_3$ be one of them. It is an endpoint of an arc component $A'$ of $C_-(J', \nabla \xi)$, the other endpoint $s_2$ being a singularity of index 2. They are in cancellation position. That is, in a neighborhood of $A'$, the $\Gamma_1$-structure $\xi$ admits a first integral (see paragraph 2.1) for which $s_2$, $s_3$ are Morse singularities in cancellation position, in Smale’s sense. Cancel them. By observation 2.1, the homotopy class of $\xi$ is not changed.

Once $J$ and $\xi$ have been so modified, $C_-(J, \nabla \xi)$ is a disjoint union of bisingular arcs, and along each of these arcs $A$ the contact is nondegenerate. The contact points close to both endpoints of $A$ are Morse of the same contact index $i$ (observation 2.16). Thus a local modification of $T$ along $A$, that does not change its set of zeroes $C_-(J, \nabla \xi)$, turns all of them into Morse contact points of index $i$. Such a modification results into a homotopy of $J$ (observation 2.15). Proposition 2.18 is proved.

2.6. Symmetrizing. One will symmetrize $\xi$ and $J$ in a neighborhood of the singularities. The symmetry domain must contain a transverse circle through each singular leaf.

**DEFINITION 2.28.** A bisingular circle is a piecewise smooth, oriented circle $S \subset M$ such that:

1. $S$ is smooth and transverse to $\xi$, except at two points $s$, $s'$, which are singularities of $\xi$;
2. $\xi$ admits, in a neighborhood of $S$, a transverse orientation compatible with the orientation of $S$ and for which the indices of $s$, $s'$ are successive.

In particular, $S$ contains a unique bisingular arc.

Fix a basepoint $* \in S^1$. Recall that $D^{n-1}$ is the compact unit ball in $\mathbb{R}^{n-1}$, whose standard coordinates are denoted $x_1, \ldots, x_{n-1}$. Let $D^{n-2}$ (resp. $D^{n-1}_+$, $D^{n-1}_-$) be the subset of $D^{n-1}$ defined by $x_{n-1} = 0$ (resp. $x_{n-1} \geq 0$, $x_{n-1} \leq 0$). Recall also the reflection involution:

$$\sigma : D^{n-1} \times S^1 \to D^{n-1} \times S^1 : (x_1, \ldots, x_{n-2}, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-2}, -x_{n-1}, y)$$

**PROPOSITION 2.29.** After homotopies of $\xi$ and $J$, that keep $\xi$ Morse-singular and minimal, there is a finite family of two by two disjoint embeddings:

$$j_k : D^{n-1} \times S^1 \to M \quad (1 \leq k \leq K)$$

such that:
(1) The twisted vector field $J$ is negatively transverse to $\xi$ on $M$ except maybe on the $2K$ open $n$-balls:

$$B^+_k := j_k(\text{Int}(D_+^{n-1}) \times (S^1 \setminus *))$$

$$B^-_k := j_k(\text{Int}(D_-^{n-1}) \times (S^1 \setminus *))$$

(2) For each $k$, both $j^*_k \xi$ and $j^*_k J$ are $\sigma$-invariant;

(3) Each $j^*_k \xi | (D_{n-1}^+ \times S^1)$ has exactly two singularities $s_k, s'_k$, and contains a bisingular circle passing through them, whose bisingular arc is disjoint from $(D_{n-1}^+ \times S^1)$.

One will then denote $D$ the union of the $j_k(D_{n-1}^+ \times S^1)$’s; and $\sigma$ the involution of $D$ whose restriction to each connected component is the reflection symmetry; and $B$ the union of the $B^\pm_k$’s.

Proof — Let $\xi, J, \nabla \xi$ be as in proposition 2.18. In particular $C_-(J, \nabla \xi)$ is a disjoint union of bisingular arcs $A_k$ (definition 2.17) (which have of course nothing to do with the $A_k$’s of the above paragraph 2.4) along each of which $T$, the projection of $J$ tangentially to $\xi$ and parallelly to $\nabla \xi$, is Morse of some tangential index $1 \leq i_k \leq n-2$.

To make $J$ and $\xi$ locally symmetric needs some care. We begin with $\xi$.

**Definition 2.30.** A local first integral $f$ for $\xi$ in a neighborhood of a bisingular arc $A$ is standard w.r.t. $A$ if:

(1) The restriction $f|A$ is an orientation-preserving diffeomorphism from $A$ onto $D^1 := [-1, +1]$;

(2) In a neighborhood of the origin of $A$ one has in the local standard coordinates (definition 2.12):

$$f(x_1, \ldots, x_n) = Q_{i_k}(x_1, \ldots, x_n) - 1$$

(3) In a neighborhood of the extremity of $A$ one has in the local standard coordinates:

$$f(x_1, \ldots, x_n) = Q_{i_k+1}(x_1, \ldots, x_n) + 1$$

By lemma 2.25 (2), on $N := D^{n-1} \times [-2, +2]$ one easily makes a Morse function $\bar{f}$, and a pseudogradient $\nabla \bar{f}$, such that:

1) $\bar{f}(x, y) = y$ and $\nabla \bar{f} = -\partial/\partial y$ in a neighborhood of $\partial N$;

2) $\bar{f}$ has two singularities in $N$, of respective indices $i_k, i_k + 1$, extremities of a bisingular arc $A$;

3) $\bar{f}$ is standard w.r.t. $A$;

4) Some nonsingular vector field $\bar{J}$ on $N$ is homotopic to $-\partial/\partial y$ relatively to $\partial N$, standard at $s, s'$, and its projection $\bar{T}$ tangential to $\bar{f}$ parallelly to $\nabla \bar{f}$ is Morse-singular.
Figure 2. Symmetrization of a bisingular circle \((n = 3, i = 1)\). Full lines are tangential to \(\xi\), dotted lines are transverse and positively oriented. \(bsc, \overline{bsc}\): bisingular circles; \(p\): transverse path; \(\partial P\): boundary of the portal.

Fix \(k\). Obviously \(\xi\) admits a standard first integral \(f\) in a small neighborhood of \(A_k\). There is an embedding \(F\) of \(N\) into this neighborhood such that \(f \circ F = \bar{f}\) on a neighborhood of \(\partial N\). After an appropriate local homotopy of \(\nabla \xi\) and of \(J\), which keeps \(\nabla \xi\) a pseudo-gradient of \(\xi\), and which does not change \(C_-(J, \nabla \xi)\), in \(N_k := F(N)\) one has \(\nabla \bar{\xi} = J = F_\ast(-\partial/\partial y)\). Inside \(N_k\), change \(\xi\) to \(df\) and \(\nabla \xi\) to \(\nabla \bar{f}\) and \(J\) to \(J\). By observation 2.1, the homotopy class of \(\xi\) is not changed. By iv) above, the homotopy class of \(J\) is not changed. The negative contact locus \(C_-(J, \nabla \xi)\) has a new component \(\bar{A}_k := F(A)\), a bisingular arc along which the contact is Morse of the same index \(i_k\) as along \(A_k\) (figure 2).

Let \(\sigma_k\) denote the involution exchanging some small neighborhood of \(\partial A_k\) with some small neighborhood of \(\partial \bar{A}_k\), and that becomes the identity in the local standard coordinates. If these neighborhoods are small enough, then \(\xi, J, \nabla \xi\) and \(T\) are \(\sigma_k\)-invariant. This is the point where the standard form that \(J\) has been given at the singularities is important. Without it, it would be difficult to get such a local conjugation without loosing the results of proposition 2.18.
Every point \( m \) interior to \( A_k \) being in its leaf a Morse singularity of index \( i_k \) for the tangential vector field \( T \), and the same for \( \bar{A}_k \), clearly \( \sigma_k \) extends to an involution, still denoted \( \sigma_k \), exchanging some neighborhood of \( A_k \) with some neighborhood of \( \bar{A}_k \), and such that \( f = f \circ \sigma_k \) (thus \( \sigma_k^* \xi = \xi \)) and \( \sigma_k^* T = T \).

One easily makes for \( \xi \) on \( M \) a second pseudogradient \( \nabla' \xi \) tangential to \( C_-(J, \nabla \xi) \), and \( \sigma_k \)-invariant close to \( A_k \) and \( \bar{A}_k \).

For each \( k \), choose an arc \( I_k \) from \( m_k := A_k \cap \bar{f}^{-1}(0) \) to \( \bar{m}_k := \bar{A}_k \cap \bar{f}^{-1}(0) \), tangential to \( \xi \), and \( \sigma_k \)-invariant close to its endpoints \( m_k , \bar{m}_k \). Changing the intermediate part of \( I_k \) if necessary, arrange that it does not meet the local stable nor unstable manifolds, for the pseudogradient \( \nabla' \xi \), of the four singularities \( \partial A_k , \partial \bar{A}_k \), but of course at \( m_k , \bar{m}_k \). This is possible by the very construction of \( \bar{A}_k \) above: since its endpoints are in cancellation position, their local stable and unstable manifolds cut the intermediate level set \( \bar{f}^{-1}(0) \) into a bouquet of spheres that does not separate \( \bar{f}^{-1}(0) \).

Then push \( I_k \) by the flow of \( \nabla' \xi \), conveniently rescaled, up and down to the singular level sets, and get a portal: a square \( P := D^1 \times D^1 \) topologically embedded in \( M \), and actually smoothly embedded except at its four corners which are the four singularities. One has \( f(x,y) = y \) if \( x \leq 0 \) and \( \bar{f}(x,y) = y \) if \( x \geq 0 \).

Thanks to the transitivity properties of \( \xi \) (lemma 2.27) one also has a simple path \( p : [0,1] \to M \) transverse to \( \xi \) and such that \( p(0) = (0,1) \in P \), \( p'(0) = \partial / \partial y \), \( p(1) = (0,-1) \in P \) and \( p'(1) = \partial / \partial y \). Obviously \( P \cup p([0,1]) \) admits a small neighborhood which is an embedding \( j_k \) of \( D^{n-1} \times S^1 \) into \( M \), and such that the reflection symmetry \( \sigma \) preserves \( j_k^* \xi \), and such that \( \sigma_k \circ j_k \) close to \( j_k^{-1}(A_k) \). Also, we can arrange that \( j_k(D^{n-1} \times S^1) \) contains the loop made of \( A_k \), followed by the half edge \( ([0,1] \times 1) \subset P \), followed by \( p \), followed by the half edge \( ([0,-1] \times (-1)) \subset P \). By perturbation of this loop one gets a bisingular circle containing \( A_k \) and interior to \( j_k(D^{n-1} \times S^1) \). One can arrange that \( j_k(D^{n-1} \times \ast) \) is disjoint from \( A_k \).

It remains to symmetrize \( J \) by homotopy. For short, let \( D \) be the union of the \( j_k(D^{n-1} \times S^1) \)'s. Let \( B \) be the union of the \( B_k^\pm \)'s. Thus \( D \) is equipped with a \( \xi \)-preserving global involution \( \sigma \) which also preserves \( T \) in a compact neighborhood \( Nb \) of \( C_-(J, \nabla \xi) \) contained in \( B \).

For every \( X \subset M \), write \( X_{\text{reg}} \) for \( X \) minus the singularities of \( \xi \). Recall that the nonsingular twisted vector field \( J \) is decomposed over \( M_{\text{reg}} \) as:

\[
J = h \nabla \xi + T
\]
that \( h \) is a continuous function on \( M_{\text{reg}} \); that \( \nabla \xi \) is a pseudogradient of \( \xi \) on \( M \) (definition 2.12); that \( T \) is a twisted vector field on \( M_{\text{reg}} \) tangential to \( \xi \). Thus:

\[
C_-(J, \nabla \xi)_{\text{reg}} = \{ T = 0, h < 0 \}
\]

\[
C_+(J, \nabla \xi)_{\text{reg}} = \{ T = 0, h > 0 \}
\]

Also recall that in a neighborhood of the singularities, \( J \) is standard (definition 2.11).

Thus one easily makes on \( D_{\text{reg}} \) a smooth function \( h' \) such that:

1. \( h' = h \) close to the singularities;
2. \( h' < 0 \) on \( C_-(J, \nabla \xi)_{\text{reg}} \);
3. \( h' > 0 \) on \( C_+(J, \nabla \xi) \cap D_{\text{reg}} \);
4. \( h' > 0 \) on \( D_{\text{reg}} \setminus N b \).
5. Then, changing \( h' \) to \( (h' + h' \circ \sigma)/2 \), one can have moreover \( h' = h' \circ \sigma \).

For every \( t \in [0, 1] \) and at every point in \( D_{\text{reg}} \) let:

\[
J_t := ((1-t)h + th')\nabla \xi + T
\]

At every singularity \( s \), set \( J_t(s) = J(s) \). The twisted vector field \( J_t \) is defined on \( D \). It is nonsingular for every \( t \), since by 2) and 3) at any point in \( D_{\text{reg}} \) one has either \( T \neq 0 \) or \( h, h' > 0 \) or \( h, h' < 0 \).

Then, for every \( t \in [0, 1] \) and at every point in \( D_{\text{reg}} \) let:

\[
J'_t := (1-t/2)J_1 + (t/2)\sigma^*J_1
\]

At every singularity \( s \), set \( J'_t(s) = J(s) \). The twisted vector field \( J'_t \) is defined on \( D \). It is nonsingular for every \( t \), since at every point \( m \in D_{\text{reg}} \):

Either \( h'(m) \neq 0 \) and then, by 5), at \( m \) both \( J_1 = h'\nabla \xi + T \) and \( \sigma^*J_1 \) are transverse to \( \xi \) with the same sign;

or \( h'(m) = 0 \) and then by 2) and 3) one has \( T(m) \neq 0 \), while by 4) and the \( \sigma \)-invariance of \( T \) in \( N b \) one has \( J_1 = T = \sigma^*J_1 \).

The field \( J'_t \) is \( \sigma \)-invariant over \( D \). By 4), it is negatively transverse to \( \xi \) over \( D \setminus B \). Extend it continuously over \( M \), into a twisted vector field still written \( J'_1 \), negatively transverse to \( \xi \) outside \( D \).

Then \( J'_t \) is homotopic to \( J \) over \( M \) through nonsingular twisted vector fields. Indeed, over \( M \setminus \text{Int}(D) \) the fields \( J \) and \( J'_t \) are nowhere nonpositively colinear to \( \nabla \xi \). Likewise, for each \( 0 \leq t \leq 1 \), over \( \partial D \) the fields \( J_t \), \( J'_t \) are nowhere nonpositively colinear to \( \nabla \xi \). Thus this homotopy from \( J \) to \( J'_t \) over \( D \) extends over \( M \).

The rest of the proof of theorem A will rely on the use of round singularities and holes.
2.7. **Round singularities.** The local model for them, which depends on an index $0 \leq i \leq n - 1$, is the function defined for $\theta \in S^1$, $x_s \in \mathbb{R}^{n-i-1}$ and $x_u \in \mathbb{R}^i$ by:

$$f_{i}^{\text{round}}(\theta, x_s, x_u) := \|x_s\|^2 - \|x_u\|^2$$

In a manifold endowed with a $\Gamma_1$-structure $\xi$, by a *round* (or *Bott*) singularity of index $i$ one means a singular circle in a neighborhood of which $\xi$ admits a first integral smoothly conjugate to $f_{i}^{\text{round}}$ (so, it is understood that the stable and unstable fibre bundles are trivial.)

These objects will be used as intermediates for the cancellation of singularities.

Indeed, on the one hand, a pair of Morse singularities of successive indices can eventually be changed into a single round singularity, as follows. Let, in an orientable open subset $U \subset M$, the $\Gamma_1$-structure $\xi$ have a first integral $f$ such that:

1. The function $f$ is Morse and has exactly two singularities $s$, $s'$, whose indices $i$, $i + 1$ are successive;
2. For some descendant pseudogradient vector field in $U$, and for some intermediate value $t$ between $f(s_i)$ and $f(s_{i+1})$, every gradient line going down from $s'$ (resp. up from $s$) meets the level set $f^{-1}(t)$. Thus one has a stable disk $D_{s}^{j-i}(s)$ and an unstable disk $D_{u}^{i+1}(s')$ whose boundaries are two spheres embedded in $f^{-1}(t)$;
3. These spheres meet each other in $f^{-1}(t)$ transversely in exactly two points with opposite signs: Asimov position.

**Lemma 2.31.** (Asimov: [1]) Under these hypotheses, there is a function on $U$ equal to $f$ close to $\partial U$, and whose singular set is a round singularity of index $i$.

On the other hand, the advantage of a round singularity $s^{\text{round}}$ of index $i$ is that it can eventually be *turbulized*, as follows (figure 3). In a neighborhood $N$ of $s^{\text{round}}$ identified with $S^1 \times D^{n-1}$, the $\Gamma_1$-structure $\xi \cong df_{i}^{\text{round}}$ may also be viewed as the pullback $F^*dy$, where $F$ is the mapping into the annulus $S^1 \times D^1$ defined as:

$$F(\theta, x_s, x_u) := (\theta, \|x_s\|^2 - \|x_u\|^2)$$

and where $y$ denotes the coordinate in $D^1$. Choose a diffeomorphism $\phi \in \text{Diff}(\text{Int}D^1)$ such that $\phi(0) \neq 0$, and endow the annulus $S^1 \times D^1$ with the foliation $S\phi$ suspension of $\phi$.

Turbulizing is changing $F^*(dy)$ into $F^*(S\phi)$. 

Figure 3. Turbularization of a round singularity \((n = 3, i = 1)\). Left: before, right: after.

Since \(\phi(0) \neq 0\), in the annulus the circle \(S^1 \times 0\) is transverse to \(S\phi\). Also, \(F\) is a submersion, except on \(s^{\text{round}}\). Ergo, \(F\) is transverse to \(S\phi\), and \(F^*(S\phi)\) is a foliation on \(N\) equal to \(\xi|N\) outside \((f_i^{\text{round}})^{-1}(\text{Supp }\phi)\).

In case the index \(i\) equals 0 or \(n - 1\), one has just changed \(s^{\text{round}}\) into a Reeb component. In contrast, for an intermediate index, no leaf of \(F^*(S\phi)\) is interior to \(N\), and the turbulization is not relative to the boundary \(\partial N \cong S^1 \times S^{n-2}\). Instead, on \(\partial N\) the modification is as follows. Write:

\[
\Sigma^{n-3} := S^{n-i-2} \times S^{i-1}
\]

Thus the trace, on the boundary, of the singular level set \((f_i^{\text{round}})^{-1}(0)\), is \(S^1 \times \Sigma^{n-3}\). It admits in \(\partial N\) a tubular neighborhood \(S^1 \times \Sigma^{n-3} \times D^1\) on which \(\xi\) was originally the slice foliation, i.e. by projection to the \(D^1\) factor. The turbulization has the effect of changing this foliation into the stabilized suspension \(S\phi \times \Sigma^{n-3}\).

Note — In the literature, a turbulization is usually described as changing \(df_i^{\text{round}}\) into the nonsingular integrable 1-form:

\[
df_i^{\text{round}} + (h \circ f_i^{\text{round}})d\theta
\]
which is a particular case of the above construction: the case where the diffeomorphism $\phi$ is the time 1 of the flow of a vector field $h(y)\partial/\partial y$ on the segment $D^1$.

2.8. Holes. By a hole one means a compact manifold with boundary and eventually corners, together with a germ of $\Gamma_1$-structure $\xi$ along the boundary. By filling the hole one means extending this structure into a foliation in the interior.

Many holes will be of the form $H = B \times Y$, where $B$ is a compact $p$-manifold with a connected boundary and where $Y = D^1$ or $S^1$. Moreover $\xi$ will be the slice foliation (i.e. by projection to $Y$) in a neighborhood of $B \times \partial Y$; and transverse to each fibre $x \times Y$, $x \in \partial B$. In other words the transverse boundary $(\partial B) \times Y$ is a foliated $Y$-bundle.

Equivalently, $\xi|((\partial B) \times Y)$ is the suspension of $\pi_1 \partial B$ into $\text{Diff}^+(Y)$. Usually $\partial B = S_1 \times S_2 \times \ldots$ will be a product of spheres of various dimensions $d_i := \dim S_i$, thus $\pi_1 S_i \cong \mathbb{Z}$ or 1. For $d_i = 1$, let $\phi_i \in \text{Diff}^+(Y)$ be the action of the canonical generator of $\pi_1 S_i$. For $d_i \geq 2$, put $\phi_i = \text{id}$.

**NOTATION 2.32.** Write $\mathcal{H}(\phi_1, \phi_2, \ldots)$ the above germ of $\Gamma_1$-structure along $\partial(B \times Y)$.

For example, to every diffeomorphism with compact support $\phi \in \text{Diff}_c(\text{Int} D^1)$ is associated the 3-dimensional canonical hole whose transverse boundary is endowed with the suspension of $\phi$:

$$(D^2 \times D^1, \mathcal{H}(\phi))$$

Recall that by Reeb's stability theorem it is not fillable unless $\phi = \text{id}$.

In order to relax the constraint expressed by Reeb's stability theorem, W. Thurston eventually enlarges any hole $H := B \times D^1$ by a "worm gallery", that is, attaches a handle $W := D^{n-1} \times D^1$ of index 1, the attachment map being an embedding of $D^{n-1} \times y$ into $B \times (-y)$ ($y = \pm 1$). The union of the hole with the handle is made itself a hole by endowing the transverse boundary of the handle with the slice foliation. If $B$ is not connected, one attaches a copy of $W$ to each connected component of $H$. Denote by $H_W$ the resulting enlarged hole.

2.9. Regularizing.

**PROPOSITION 2.33.** Let $S$ be a bisingular circle in a $\Gamma_1$-structure $\xi$. Let $\text{Nb}(S)$ be a neighborhood of $S$ which contains no other singularity.

Then there is a foliation in $\text{Nb}(S)$ which coincides with $\xi$ close to the boundary. Moreover, in case $n \geq 4$, no leaf of this foliation is interior to $\text{Nb}(S)$.
Proof of proposition 2.33. — In Nb(S), restricted if necessary, give ξ the transverse orientation compatible with that of S. Let $s_i$, $s_{i+1}$ be the singularities, of respective indices $i$, $i+1$.

First one will change $s_i$ and $s_{i+1}$ into a couple of round singularities (see section 2.7). Here dimension 3 is special.

In the general case $n \geq 4$: Reversing the local orientation if necessary, one may assume that $i \leq n-3$. On the oriented circle $S$, consider the arc $[s_i, s_{i+1}]$ of origin $s_i$ and extremity $s_{i+1}$. By a modification of ξ in a neighborhood of this arc, one creates beside $S$ a second pair of singularities $\hat{s}_{i+1}$, $\hat{s}_{i+2}$, of respective indices $i+1$, $i+2$, in cancellation position, and such that $\hat{s}_{i+1}$ is close to $s_i$ and slightly above $s_i$, while $\hat{s}_{i+2}$ is close to $s_{i+1}$ and slightly above $s_{i+1}$.

For some local pseudogradient vector field, both a stable disk of $\hat{s}_{i+1}$ and an unstable disk of $\hat{s}_{i+2}$ have their boundaries in some common intermediate leaf $L$, and these boundaries don’t meet each other. After a local modification of the pseudogradient field, these boundaries do meet transversely in $L$ in two points of opposite signs. By lemma 2.31 a local modification of ξ changes $\hat{s}_{i+1}$ and $s_i$ into a round singularity of index $i$.

In the same way, $\hat{s}_{i+2}$ and $s_{i+1}$ are changed into a round singularity of index $i+1$.

In case $n = 3$: Necessarily $i = 1$. On the oriented circle $S$, consider the arc $[s_2, s_1]$ of origin $s_2$ and extremity $s_1$. By a modification of ξ in a neighborhood of this arc, one creates beside $S$ a second pair of singularities $\hat{s}_1$, $\hat{s}_2$, of respective indices 1, 2, in cancellation position, and such that $\hat{s}_1$ is close to $s_2$ and slightly below $s_2$, while $\hat{s}_2$ is close to $s_1$ and slightly above $s_1$. Thus, for some local pseudogradient vector field, both a stable disk of $\hat{s}_1$ and an unstable disk of $s_2$ have their boundaries on some common intermediate leaf $L$, and don’t meet each other. After a local modification of the pseudogradient field, they do meet transversely in $L$ in two points of opposite signs. By lemma 2.31 a local modification of ξ changes $\hat{s}_1$ and $s_2$ into a round singularity of index 1.

In the same way, $\hat{s}_2$ and $s_1$ are also changed into a round singularity of index 1.

Next, in all dimensions, one changes every round singularity $s_{\text{round}}$ into a hole with worm galleries (see section 2.8) as follows (figure 4). Let $i$ be the index of $s_{\text{round}}$, and $S^1 \times D^{n-1}$ be a small tubular neighborhood, as in section 2.7. The intersection of the local singular leaf with the boundary $S^1 \times \partial D^{n-1}$ is a $S^1 \times S^{i-1} \times S^{n-i-2}$ tangential to ξ. It admits
in $M$ a small tubular neighborhood of the form:

$$H := S^1 \times D^1 \times S^{i-1} \times S^{n-i-2} \times D^1$$

the restriction of $\xi$ to which is the slice foliation by projection to the last factor $D^1$. One forgets $\xi$ in the interior of $H$, leaving a hole, and turbulizes $\xi|(M \setminus H)$ around $s^{\text{round}}$ (see section 2.7), using a diffeomorphism $\phi \in \text{Diff}_c(\text{Int}D^1)$ that does not fix the singular value. If $n = 3$, one chooses $\phi$ such that its extension as a diffeomorphism of the circle is tame (definition 4.1). If $n \geq 4$, one takes for $\phi$ a split commutator (definition 4.3). Then the singularity has disappeared, and we are left with the unfoliated hole $H$, the transverse boundary of which is foliated by two suspensions as follows. On the internal boundary

$$S^1 \times (-1) \times S^{i-1} \times S^{n-i-2} \times D^1$$

the factor $S^1$ acts through $\phi$ and the fundamental groups of the spheres $S^{i-1}$, $S^{n-i-2}$ don’t act. The external boundary

$$S^1 \times (+1) \times S^{i-1} \times S^{n-i-2} \times D^1$$

has the slice foliation, i.e. by projection to the last factor $D^1$. 

Figure 4. A round singularity, the hole left unfoliated in order to turbulize it, and the worm’s gallery, for $n = 3$, $i = 1$. 
Close to the bisingular circle $S$, one has a transverse path from the top $y = 1$ of $H$ to its bottom $y = -1$. This is where a bisingular circle is useful. One enlarges the hole $H$ by a worm gallery $W$ that follows this path. In case $i = 1$ or $i = n - 2$, where $H$ is not connected, one adds a worm gallery to each connected component of $H$. One gets an enlarged hole $H_W$.

It remains to fill $H_W$. To establish that it is fillable, consider the hole (abstract in the sense that it is not embedded in $M$):

$$H' := (D^2 \times S^{i-1} \times S^{n-i-2} \times D^1, \mathcal{H}(\text{id}, \text{id}, \text{id}))$$

The transverse boundary of $H'$ and the external transverse boundary of $H$ are isomorphic. One forms their union along those boundaries and gets a new abstract hole:

$$H'' = (D^2 \times S^{i-1} \times S^{n-i-2} \times D^1, \mathcal{H}(\bar{\phi}, \text{id}, \text{id}))$$

where $\bar{\phi}$ is the tame diffeomorphism of the circle extending $\phi$. Thus, proposition 4.2 fills $H''_W$ up. For $n \geq 4$, it is proposition 4.4 that fills $H''_W$ up.

Then, inside the foliated $H''_W$, for $z \in \text{Int}D^2$ close enough to the boundary $\partial D^2$, the trace of the foliation on

$$Z^{n-2} := z \times S^{i-1} \times S^{n-i-2} \times D^1$$

is the slice foliation. In consequence, $H''_W$ minus a small tubular neighborhood of $Z^{n-2}$ is just a filling of $H_W$.

Once this has been done to both round singularities, $\xi$ is nonsingular in $Nb(S)$. For $n \geq 4$, the foliation inside $H''_W$ has no interior leaf (proposition 4.4), and thus it is clear that $Nb(S)$ does not either: proposition 2.33 is established.

2.10. **End of the proof of theorem A.** After $\xi$ has been symmetrized (proposition 2.29), in the symmetry domain $D$ the singularities lie on finitely many bisingular circles $(S_k)$ and their symmetric images $\sigma(S_k)$. By proposition 2.33 (1), one regularizes $\xi$ in some small neighborhood of each $S_k$, and symmetrically close to $\sigma(S_k)$. Let $\xi_{\text{reg}}$ be the resulting foliation on $M$.

By lemma 2.8, $\xi_{\text{reg}}$ is homotopic to $\xi$.

If $n \geq 4$, since $\xi$ was minimal, by proposition 2.33 (2), $\xi_{\text{reg}}$ is also minimal.
It remains to fix the homotopy class of $\nabla \xi_{\text{reg}}$. Dimension 3 is special. In case $n = 3$, we can make $\nabla \xi_{\text{reg}}$ homotopic to $J$ by Wood’s method ([17], see also [5]). This changes $\xi_{\text{reg}}$, introducing new Reeb components, and Wood components. But the class of $\xi_{\text{reg}}$ as a $\Gamma_1$-structure is not changed.

In case $n \geq 4$, one also could apply an easy generalization of Wood’s method, but it would have the inconvenient to introduce compact leaves, thus the resulting foliation would not be minimal. However, in fact, in these dimensions, $\xi_{\text{reg}}$ needs not be changed: its gradient already lies in the homotopy class of $J$.

To prove this, first observe that, $\xi_{\text{reg}}$ and $J$ being both $\sigma$-invariant in $B$, and negatively transverse to $\xi_{\text{reg}}$ outside, clearly $\xi_{\text{reg}}$ admits in $M$ a gradient $\nabla \xi_{\text{reg}}$ which is $\sigma$-invariant in $B$ and coincides with $J$ outside $B$. We cannot apply directly to $J$ and to $\nabla \xi_{\text{reg}}$ the reflection symmetry trick for vector fields (lemma 2.9), since each component of $D$ is homeomorphic to $D^{n-1} \times S^1$ and not to $D^{n-1} \times D^1$. However, we claim that over each $(n-1)$-ball $b_k := j_k(D^{n-1} \times *)$ the vector fields $J$ and $\nabla \xi_{\text{reg}}$ are homotopic rel. $\partial b_k$. Indeed, $\nabla \xi$ is nonsingular there (proposition 2.29 (3)) and $J$ is homotopic to $\nabla \xi$. So it remains to show that $\nabla \xi_{\text{reg}}$ is homotopic to $\nabla \xi$ over $b_k$ rel. $\partial b_k$. We shall now see that this follows from the details of the regularization process. First this process allows to make $\xi_{\text{reg}} = \xi$ except in an arbitrarily small neighborhood of $C_-(J, \nabla \xi)$ (thus far from the $b_k$’s) and except in the worm galleries. Inside each of these galleries $W \cong D^{n-1} \times D^1$ one has a toric gallery $T := D^2 \times T^{n-3} \times D^1$ (see the appendix 4). In the complement $W \setminus T$, both $\xi$ and $\xi_{\text{reg}}$ coincide with the slice foliation, so one can take $\nabla \xi = \nabla \xi_{\text{reg}}$ in this complement.

We can arrange that $p([0,1])$ meets $b_k$ transversely once (see the proof of proposition 2.29), so $T \cap b_k$ is a single slice $D^2 \times T^{n-3} \times *$; and that the point $*$ belongs to $Nb(A)$ (see the proof of lemma 4.5). Then, on $T \cap b_k$, both $\xi$ and $\xi_{\text{reg}}$ are saturated by the $T^{n-3}$ factor. Thus one can choose $\nabla \xi$ and $\nabla \xi_{\text{reg}}$ tangential to the $D^2 \times D^1$ factor at every point. Since $T^{n-3}$ admits a nonsingular vector field, $\nabla \xi$ and $\nabla \xi_{\text{reg}}$ are homotopic rel. $(\partial T) \cap b_k$. The claim is proved.

For each $k$, first homotope $J$ to $\nabla \xi_{\text{reg}}$ over $b_k$, rel. $\partial b_k$. Symmetrically, homotope $J$ to $\nabla \xi_{\text{reg}}$ over $j_k(D^{n-1} \times *)$, rel. $j_k(\partial D^{n-1} \times *)$. Then $J$, $\nabla \xi_{\text{reg}}$ are still $\sigma$-invariant in $D$, and $J = \nabla \xi_{\text{reg}}$ in $M \setminus B$. After a small, $\sigma$-invariant perturbation of $J$ in $D$, one has $J = \nabla \xi_{\text{reg}}$ in a small neighborhood $V_\epsilon(M \setminus B)$. Then lemma 2.9 applies to $J$ and $\nabla \xi_{\text{reg}}$ in every $n$-ball of the form:

$$j_k((D^{n-1} \times S^1) \setminus V_\epsilon((\partial D^{n-1} \times S^1) \cup (D^{n-1} \times *)))$$
thus $\nabla_{\xi_{\text{reg}}}$ is homotopic to $J$ over $M$. Theorem A is proved.

3. Proof of theorem B

A sketch is enough: indeed the above arguments for theorem A are already compatible with parameters.

One is given a rank-one vector bundle $\nu$ over a closed, connected manifold $M$ of dimension $n \geq 3$ (the lower dimensions being left as an exercise), an embedding $J$ of $\nu$ into $\tau M$, and a compact set $\Xi$ of $\Gamma_1$-structures whose bundle is $\nu$.

Morsification — The general proof of proposition 2.2 (review paragraph 2.2) actually simultaneously morsifies every $\xi \in \Xi$. Indeed, in the parametric situation, the compact set $\Xi$ of germs of foliations is represented in the total space $E$ of $\nu$ by a continuous family $(\mathcal{X}_\xi)_{\xi \in \Xi}$ of foliations all defined in a same neighborhood $U$ of $Z(M)$, all transverse to the fibres. One has a fibre-preserving embedding $r$ of $E$ into $U$, whose germ along $Z(M)$ is the identity. Changing each $\mathcal{X}_\xi$ for $r^* \mathcal{X}_\xi$, one can assume that $U = E$. Endow $M$, $\nu$, $E$ with riemannian metrics, just as in paragraph 2.2, let $\mathcal{X}_0$ be as in this paragraph, and let

$$D := \{z \in E : \|z\| \leq 1\}$$

The norms $\|\Omega(\mathcal{X}_\xi)\|_D$ are uniformly bounded. Let $L$ be as in lemma 2.7. Set:

$$\text{Slope} := L \sup_{\xi \in \Xi} \|\Omega(\mathcal{X}_\xi) - \Omega(\mathcal{X}_0)\|_D$$

Just as at the end of paragraph 2.2, apply lemma 2.5 to get a triangulation $\mathcal{T}r$, a radius $r$ such that the compact balls of radius $r$ centered at the vertices are two by two disjoint (denote $N_0$ their union), a smooth section $s$ with values in $D$, and a twisted vector field $X$ such that at every point of $M \setminus N_0$:

$$dsX > \text{Slope}\|X\|$$

Apply lemma 2.7 to each $\mathcal{X}_\xi$ and to these balls. In the proof of this lemma, the construction of $\xi'$ is continuous w.r.t. $\xi$. Thus one gets a continuous family $(\mathcal{X}_\xi')_{\xi \in \Xi}$ of horizontal foliations. Also, one has a homotopy from $\mathcal{X}_\xi$ to $\mathcal{X}_\xi'$ (regarded as $\Gamma_1$-structures on $E$) continuous w.r.t. $\xi$. For every $\xi \in \Xi$, the foliation $\mathcal{X}'$ coincides with $\mathcal{X}_0$ over $N_0$, and verifies:

$$\|\Omega(\mathcal{X}') - \Omega(\mathcal{X}_0)\|_D \leq \text{Slope}$$
By the triangle inequality, $\Omega(X'_s)s, X > 0$ outside $N_0$. After a small generic perturbation of $s$ over $\tilde{N}_0$, it is in Morse position with respect to $\lambda' = \lambda_0$.

Thus one has on $\mathcal{M}$ a continuous family $(\xi' := s^*\lambda'_\xi)_{\xi \in \Xi}$ of Morse-singular $\Gamma_1$-structures, continuously homotopic to $\Xi$. They coincide in a common open neighborhood $N_0 \subset M$ of their singularities, where they are smooth. Moreover on $M \setminus N_0$ they are positively transverse to a same continuous twisted vector field $X$. Thus in $M$ they admit a common pseudogradient $\nabla$ (definition 2.12).

The 1-skeletton $Tr^1$ of the triangulation $Tr$ being tangential to $X$, it is transverse to all the $\xi'$s outside $N_0$. It is not difficult, then, to isotope the $\xi'$s continuously to make them equal in restriction to some neighborhood $N_1$ of $Tr^1$ containing $N_0$.

Review now the minimization process (paragraph 2.4). Every leaf of every $\xi'$ meets $N_1$. This is obvious in view of the fact that, in each $n$-simplex of $Tr$ and outside $N_0$, each $\xi'$ is transverse to the linear vector field $X$, and thus meets $Tr^1$. Thus the minimization process can be performed in $N_1$, where the $\xi'$s coincide, leading to a continuous family of Morse $\Gamma_1$-structures, denoted $(\xi'')_{\xi \in \Xi}$, which are not only minimal in $M$, but already minimal in restriction to $N_1$. In particular, every $\xi''$s is transitive (lemma 2.27) in $N_1$: all the transverse paths necessary to the rest of the proof can be chosen in $N_1$.

Review now the morsification of the contact (paragraph 2.5). Since the complement of $N_1$ retracts by deformation on the $(n-2)$-skeletton of the subdivision dual to $Tr$, one can homotope $J$ to $\nabla$ there. After what, the negative contact locus $C_-(J, \nabla)$ is contained in $N_1$.

So the rest of the proof of theorem A: morsification of the contact, symmetrization and regularization, actually can take place in the open set $N_1$ in restriction to which all the $\xi''$s are equal. Theorem B is proved.

4. APPENDIX: FILLING THE HOLES, AFTER W. THURSTON

This appendix recalls Thurston’s methods to extend the boundary foliation inside the “holes” [13], [14]. Some argument has been simplified.

4.1. FILLING IN DIMENSION 3. The case $n = 3$ is completely different from the higher dimensions, much easier, and well-known. However we recall it for completeness.

Definition 4.1. Call $\phi \in \Diff_+(S^1)$ tame if it belongs to the normal subgroup generated by the rotations.
Of course, this definition is superfluous in the differentiability classes \( C^r \), \( 3 \leq r \leq \infty \), since it is known that \( \text{Diff}_+^r(S^1) \) is simple \([3][4][7]\). However also recall that the simplicity of \( \text{Diff}_+^2(S^1) \) remains open \([8]\). Definition 4.1 allows to spare the use of Mather’s and Herman’s difficult results, and to make the following argument work as well in the lower differentiability classes.

**Proposition 4.2.** \([13]\) Let \( \phi \in \text{Diff}_+(S^1) \) be tame. Then one can fill the hole :

\[
(D^2 \times S^1, \mathcal{H}(\phi))
\]

Recall (section \([2.8]\)) that \( \mathcal{H}(\phi) \) denotes the foliation of \( \partial(D^2 \times S^1) \) by suspension of \( \phi \).

**Proof.** First, in case \( \phi \) is a rotation \( \rho \), the filling is explicit. Regard \( D^2 \) as the compact unit disk in \( \mathbb{C} \). Set :

\[
A := \{ z \in D^2 / |z| \geq 1/2 \}
\]

The suspension of \( \rho \) is the linear foliation on \( \partial D^2 \times S^1 \) by a constant form \( dy - \lambda d\theta \), where \( \lambda \in \mathbb{R} \), \( e^{i\theta} \in \partial D^2 \), \( y \in S^1 \). One extends it to \( A \times S^1 \) as the integrable nonsingular 1-form :

\[
(1 - u(r))dr + u(r)(dy - \lambda d\theta)
\]

where \( u \) is a smooth function on \([1/2, 1]\) which is 0 on a neighborhood of \( 1/2 \) and 1 on a neighborhood of 1. One then fills the remaining central solid torus by a Reeb component, and gets a foliation \( \mathcal{F}(\rho) \) on \( D^2 \times S^1 \) that fills up \( \mathcal{H}(\rho) \).

More generally, if \( \phi = \psi \rho \psi^{-1} \) is conjugate to some rotation \( \rho \), the suspension \( \mathcal{H}(\phi) \) is filled up by the image \( \mathcal{F}(\psi \rho \psi^{-1}) \) of \( \mathcal{F}(\rho) \) through the self-diffeomorphism \( (z, y) \mapsto (z, \psi(y)) \) of \( D^2 \times S^1 \).

Third, given two foliated solid tori \( (D^2 \times S^1, \mathcal{F}(\phi_i)) \) (\( i = 1, 2 \)) whose restrictions to the boundary are suspensions of the diffeomorphisms \( \phi_i \), one attaches them along their boundary parallel \( 1 \times S^1 \) and gets a third foliated solid torus \( (D^2 \times S^1, \mathcal{F}(\phi_1 \phi_2)) \) whose restriction to the boundary is the suspension of \( \phi_1 \phi_2 \).

Hence, for every tame diffeomorphism \( \phi \in \text{Diff}_+(S^1) \), a filling of \( \mathcal{H}(\phi) \). \( \bullet \)

### 4.2. Filling in higher dimensions.

First one specifies the kind of diffeomorphisms of the interval that one will allow as holonomy on the boundary of holes, to be able to fill them.

**Definition 4.3.** Call \( \phi \in \text{Diff}_c(\text{Int}D^1) \) a split commutator if there are two diffeomorphisms \( \alpha, \beta \in \text{Diff}_c(\text{Int}D^1) \) and a point \(-1 < y < +1\)
such that $\phi$ is the commutator $[\alpha, \beta]$ , and $y \not\in \text{Supp } \alpha \cup \text{Supp } \beta$ , but $\text{Supp } \phi$ meets both $[-1, y]$ and $[y, +1]$.

**PROPOSITION 4.4.** [14] Let $n \geq 4$ . Let $\Sigma^{n-3} = T^{n-3}$ or $S^{n-3}$ or $S^p \times S^q$ $(p + q = n - 3)$ be the $(n-3)$-torus, or the $(n-3)$-sphere, or the product of two spheres. Let also $\phi \in \text{Diff}_c(\text{Int}D^1)$ be a split commutator.

Then one can fill the hole enlarged by a worm gallery :

$$(D^2 \times \Sigma^{n-3} \times D^1, H(\phi, \text{id}))_W$$

Moreover the resulting foliation has no leaf interior to $H \cup W$ .

Recall (section 2.8) that $H(\phi, \text{id})$ denotes the stabilization by $\Sigma^{n-3}$ of the germ of foliation along $\partial(D^2 \times D^1)$ that coincides, close to $D^2 \times \pm 1$ , with the projection to $D^1$ , and, on $S^1 \times D^1$ , with the suspension of $\phi$.

The proof occupies the rest of this appendix. Certainly, the simplest way to fill a hole of the kind described in paragraph 2.8 is by the means of a foliated $D^1$-bundle, in the happy case where the representation of $\pi_1 \partial B$ extends to $\pi_1 B$ . The proof of proposition 4.4 will consist in dividing and subdividing the hole into subholes that either fall into this happy case, or can be applied the following lemma.

**LEMMA 4.5.** Assume $n \geq 4$ . Let $\alpha, \beta, \psi_1, \ldots, \psi_{n-3} \in \text{Diff}_c(\text{Int}D^1)$ . Assume that $\psi_1, \ldots, \psi_{n-3}$ commute to each other and that their supports are disjoint from the supports of $\alpha$ and $\beta$ . Assume moreover that they bracket $\alpha$ and $\beta$ , that is, every point in

$$A := \text{Supp } \alpha \cup \text{Supp } \beta$$

is separated from $\partial D^1$ by

$$P := \text{Supp } \psi_1 \cup \ldots \cup \text{Supp } \psi_{n-3}$$

Then the hole :

$$H := (D^2 \times \Sigma^{n-3} \times D^1, H([\alpha, \beta], \psi_1, \ldots, \psi_{n-3}))$$

is fillable without interior leaf.

**Proof of the lemma —** This is the point, in the filling process, where we do not follow strictly Thurston. The interior foliation will indeed be easily built through a appropriate turbulization.

By the bracketting property, $A$ is contained in an interval whose extremities $y_1 < y_2$ both belong to $P$ . Let $f$ be a Morse function on $D^2 \times D^1$ such that $f(z, y) = y$ close to the boundary; which has in the interior of $D^2 \times D^1$ a pair of cancellable singularities $s_1 = (0, t_1)$ of index 1 and $s_2 = (0, t_2)$ of index 2 ; and whose singular values are $f(s_i) = y_i$ $(i = 1, 2)$ . The intermediate level sets are diffeomorphic to
the surface $S$ of genus one with one boundary component. Let $F$ be the stabilization of $f$ by $T^{n-3}$:

$$F : H \to T^{n-3} \times D^1 : (z, \theta_1, \ldots, \theta_{n-3}, y) \mapsto (\theta_1, \ldots, \theta_{n-3}, f(z, y))$$

Also fix a finite union $Nb(A)$ of compact subintervals of $(y_1, y_2)$, which contains $A$ in its interior and is disjoint from $P$.

Endow $T^{n-3} \times D^1$ with the foliation $S(\psi_1, \ldots, \psi_{n-3})$ suspension of $\psi_1, \ldots, \psi_{n-3}$ (foliated $D^1$-bundle). Each singular value $y_i$ $(i = 1, 2)$ belonging to $P$, one can arrange that the hypersurface $T^{n-3} \times y_i$ is transverse to $S(\psi_1, \ldots, \psi_{n-3})$. On the contrary, $Nb(A)$ being disjoint from $P$, one can arrange that $T^{n-3} \times y$ is a leaf of $S(\psi_1, \ldots, \psi_{n-3})$ for every $y \in Nb(A)$.

Observe that the mapping $F$ is transverse to $S(\psi_1, \ldots, \psi_{n-3})$. Indeed $F$ maps, for $i = 1, 2$, the $(n-3)$-torus $0 \times T^{n-3} \times t_i$ diffeomorphically onto $T^{n-3} \times y_i$. So $\mathcal{F} := F^* S(\psi_1, \ldots, \psi_{n-3})$ is a foliation on $H$ and coincides, close to the boundary of the hole, with $\mathcal{H}(\text{id}, \psi_1, \ldots, \psi_{n-3})$. Also, in $f^{-1}(Nb(A)) \times T^{n-3} \cong S \times T^{n-3} \times Nb(A)$, the foliation $\mathcal{F}$ is the slice foliation whose leaves are diffeomorphic to $S \times T^{n-3}$. It just remains to change $\mathcal{F}$ there into the suspension of $\alpha$ and $\beta$ over the surface $S$, stabilized by $T^{n-3}$. Clearly every leaf meets the boundary.

4.3. **Proof of proposition** 4.4 **in the toric case.** W. Thurston solves first the toric case $\Sigma = T^{n-3}$, and then the spheric ones, actually the ones of interest, follow easily.

For $\Sigma = T^{n-3}$ the canvas is as follows. The hole $H_0 = D^2 \times T^{n-3} \times D^1$ will be “rolled up”, that is, its filling will be reduced to the same problem for smaller and smaller subholes $H_k$. These $H_k$’s will also be copies of $D^2 \times T^{n-3} \times D^1$, with more and more holonomy on the boundary, but that point will make no problem. Lemma 4.5 will be used to fill up the successive complements $H_k \setminus H_{k+1}$. Finally $H_{n-3}$ will have become small enough to pass through the worm’s gallery. In other words it will be extended into a round hole $D^2 \times T^{n-3} \times S^1$, the filling of which will also fall to lemma 4.5.

4.3.1. **A 3-dimensional tool.** One first manufactures a 3-manifold $Q^3$, and representations of $\pi_1 Q^3$, that will be a tool to roll up the holes.

Denote $rD^2$ the compact disk in $C$ of center 0 and radius $r$; in particular the segment $[0,1]$ is a radius of the unit disk $D^2$; and fix a basepoint $s \in S^1$. In $D^2 \times S^1$, let $A$ be the annulus $[0,1] \times S^1$; let $i$ be a self-embedding of $D^2 \times S^1$ whose image is $(1/3)D^2 \times S^1$; let $D$
be the disk \((2/3)D^2 \times s\); let \(j\) be a self-embedding of \(D^2 \times S^1\) whose image is a small tubular neighborhood of \(\partial D\), disjoint from the image of \(i\). Let \(Q^3\) be \(D^2 \times S^1\) minus the interiors of the images of \(i\) and \(j\).

Given two commuting diffeomorphisms \(\phi, \psi \in \text{Diff}_c(\text{Int}D^1)\), endow \(\pi_1Q^3\) with the representation:

\[
\rho_{\phi,\psi} : \pi_1Q^3 \to \text{Diff}_c(\text{Int}D^1) : \gamma \mapsto \phi^{A^*}\gamma\psi^{D^*}\gamma
\]

where \(A^*\), \(D^*\) are Poincaré-dual to \(A\) and \(D\) respectively.

The restriction of \(\rho_{\phi,\psi}\) to the three boundary tori of \(Q^3\) is as follows. Let \(p\) be the parallel \(1 \times S^1\) and \(m\) be the meridian \(\partial D^2 \times s\), so \(\rho_{\phi,\psi}(p) = \text{id}\) and \(\rho_{\phi,\psi}(m) = \phi\). One can and does choose \(i\) and \(j\) such that moreover \(\rho_{\phi,\psi}(i(p)) = \psi\), and \(\rho_{\phi,\psi}(i(m)) = \phi\), and \(\rho_{\phi,\psi}(j(p)) = \phi\), and \(\rho_{\phi,\psi}(j(m)) = \psi\).

4.3.2. **Rolling up the holes.** From the above self-embedding \(j\) of \(D^2 \times S^1\), one defines by stabilization a sequence \((J_k)\) of self-embeddings of \(D^2 \times T^{n-3} \times D^1\):

\[
J_k(z, \theta_0, \theta_1, \ldots, \theta_k, \ldots, \theta_{n-4}, y) := (z', \theta_0, \theta_1, \ldots, \theta', \ldots, \theta_{n-4}, y)
\]

where \(j(z, \theta_k) = (z', \theta')\). Let \(J\) be the composition \(J_0J_1\ldots J_{n-4}\). Observe that each \(J_k\) preserves all the coordinates \(\theta_0\), \(\theta_1\), \ldots, except \(\theta_k\), and that the values of \(\theta_k\) on the image of \(J_k\) are close to 0. So the image of \(J\) is contained in a small neighborhood of \(D^2 \times 0 \times \cdots \times 0 \times D^1\). Also, it is disjoint from \(\partial D^2 \times T^{n-3} \times D^1\). So, the image of \(J\) is contained in \(D^{n-1} \times D^1\) for some ball \(D^{n-1} \subset \text{Int}(D^2 \times T^{n-3})\). After an isotopy in \(D^2 \times T^{n-3} \times D^1\), the tangential boundary \(J(D^2 \times T^{n-3} \times \pm 1)\) of the image is contained in the attachments disks of \(W\), that is, the entrance and the exit of the worm’s gallery.

4.3.3. **Choosing the holonomy on the boundaries.** Then we define the diffeomorphisms that will generate the holonomy of the transverse boundaries of the nested holes. To this end, partition \(D^1 = [-1, +1]\) into \(2n - 4\) intervals \([y_k, y_{k+1}]\), where \(2 - n \leq k \leq n - 3\). For each of them choose two noncommuting diffeomorphisms \(\alpha_k, \beta_k \in \text{Diff}_c(\text{Int}D^1)\) with support in the open interval \((y_k, y_{k+1})\), and write \(\gamma_k\) the commutator \([\alpha_k, \beta_k]\). For \(0 \leq k \leq n - 3\), consider \(\phi_k := \gamma_{-k-1}\gamma_k\) and the hole:

\[
H_k := (D^2 \times T^{n-3} \times D^1, \mathcal{H}(\phi_k, \phi_0, \phi_1, \ldots, \phi_{k-1}, \text{id}, \ldots, \text{id}))
\]

Since the diffeomorphism \(\phi\) given in proposition 4.4 is a split commutator, one may choose \(\phi_0\) conjugate to \(\phi\), and one does. So \((H_0)_W\) is the hole that we have to fill up.
Unfortunately lemma $4.5$ does not apply to $H_k$, since precisely the holonomy $\phi_k$ of the compressible factor $\partial D^2$ is the one that is not bracketed by the other ones.

4.3.4. Filling the successive complements. Let us fix $0 \leq k \leq n - 4$, and fill up $H_k \setminus \text{Int} J_k(H_{k+1})$. To this end, since all diffeomorphisms commute two by two, one has a representation

$$\pi_1(Q^3 \times T^{n-4}) \cong \pi_1 Q^3 \times Z^{n-4} \rightarrow \text{Diff}_c(\text{Int} D^1):$$

$$(\gamma, p_1, \ldots, p_k, \ldots, p_{n-4}) \mapsto \rho_{\phi_k, \phi_{k+1}}(\gamma) \phi_0^{p_1} \ldots \phi_{k-1}^{p_{k-1}}$$

(see paragraph $4.3.1$) whose suspension is a foliation $F_k$ on $Q^3 \times T^{n-4} \times D^1$.

The restriction $F_k|{(\partial D^2 \times S^1 \times T^{n-4})}$ is exactly the transverse boundary of $H_k$. The restriction $F_k|{(\partial J(D^2 \times S^1) \times T^{n-4})}$ is nothing but the transverse boundary of $H_{k+1}$. The restriction $F_k|{(\partial i(D^2 \times S^1) \times T^{n-4})}$ is the transverse boundary of the hole:

$$H := (D^2 \times T^{n-3} \times D^1, \mathcal{H}(\phi_k, \phi_0, \phi_1, \ldots, \phi_{k-1}, \phi_{k+1}, \text{id}, \ldots, \text{id}))$$

Which is fillable. Indeed, $\phi_{k+1}$ brackets both diffeomorphisms $\alpha_{-k-1}\alpha_k$ and $\beta_{-k-1}\beta_k$, whose commutator is $\phi_k$. Thus lemma $4.5$ applies to $H$.

So, $F_k$ together with the filling of $H$ fill the hole $H_k \setminus \text{Int}(J_k(H_{k+1}))$.  

4.3.5. Inside the worm’s gallery. After an induction on $k$, one has filled up the complement $H_0 \setminus \text{Int} J(H_{n-3})$. It remains to treat $J(H_{n-3})$.

Recall that the tangential boundary $J(D^2 \times T^{n-3} \times \{\pm 1\})$ is contained in the attachment disks of $W = (D^{n-1} \times D^1, \mathcal{H}((\text{id}))$, that is, the entrance and exit of the worm’s gallery. So there exists also an embedding into the gallery itself,

$$F : D^2 \times T^{n-3} \times D^1 \rightarrow W$$

such that $F(D^2 \times T^{n-3} \times \mp 1) = J(D^2 \times T^{n-3} \times \pm 1)$. One fills up the complement $W \setminus \text{Int} F(D^2 \times T^{n-3} \times D^1)$ with the slice foliation by projection to the last factor $D^1$. So, it remains only to fill the union of $F(D^2 \times T^{n-3} \times D^1)$ with $J(H_{n-3})$, which union is a round hole

$$R := (D^2 \times T^{n-3} \times S^1, \mathcal{H}(\bar{\phi}_{n-3}, \bar{\phi}_0, \ldots, \bar{\phi}_{n-4}))$$

Recall that the diffeomorphisms $\bar{\phi}_0, \ldots, \bar{\phi}_{n-3} \in \text{Diff}_+(S^1)$ are just the $\phi_i$’s extended by the identity into diffeomorphisms of $S^1$. It is convenient to consider that the embedding of $D^1 = [-1, +1]$ into $S^1$ is the standard inclusion into $RP^1 = S^1$.

Although the hole $H_{n-3}$ could not be applied the lemma $4.5$ since the holonomy $\phi_{n-3}$ of the compressible factor $\partial D^2$ is the one that brackets
the other ones, now instead of the interval we have the circle, on which the bracketting relation is symmetric, so the lemma can be applied. Namely, for every \( \phi \in \text{Diff}_c(\text{Int}D^1) \), denote \( \hat{\phi} \) the conjugate of \( \bar{\phi} \) by \( y \mapsto 1/y \). Let \( [-\epsilon, +\epsilon] \subset D^1 \) be a neighborhood of 0 so small that it does not meet \( \text{Supp} \phi_0 \). Then \( R \) splits as the union of two holes, say, short and long.

The short hole:

\[
(D^2 \times T^{n-3} \times [-\epsilon, +\epsilon], \mathcal{H}(\text{id}, \text{id}))
\]

is filled by the slice foliation: projection to the last factor \([−\epsilon, +\epsilon] \). The long hole:

\[
(D^2 \times T^{n-3} \times [-\epsilon^{-1}, \epsilon^{-1}], \mathcal{H}(\hat{\phi}_{n-3}, \hat{\phi}_0, \ldots, \hat{\phi}_{n-4}))
\]

runs all along the worm’s gallery. The fundamental group of the boundary of its base acts through the diffeomorphisms:

\[
\hat{\phi}_0 , \ldots, \hat{\phi}_{n-3} \in \text{Diff}[-\epsilon^{-1}, \epsilon^{-1}]
\]

Now it’s \( \hat{\phi}_0 \) that brackets \( \hat{\alpha}_{n-3} \hat{\alpha}_{2-n} \) and \( \hat{\beta}_{n-3} \hat{\beta}_{2-n} \), and lemma 4.5 does apply to the long hole.

This finishes Thurston’s proof of proposition 4.4 in the toric case.

\[4.4. \text{Proof of proposition 4.4 in the spheric case.} \]

Assume that \( \Sigma^{n-3} \) is a sphere, or the product of two spheres. One wants to fill \( H_W \) where

\[
H := (D^2 \times \Sigma^{n-3} \times D^1, \mathcal{H}(\phi, \text{id}))
\]

One first immediately reduces to the case where \( \Sigma^{n-3} \) has no \( S^0 \) factors — recall that the index \( W \) means a worm gallery added to every connected component. Then, the problem will be reduced to the toric one if one finds an embedding of

\[
H_{\text{toric}} := (D^2 \times T^{n-3} \times D^1, \mathcal{H}(\phi, \text{id}))
\]

into the interior of \( H \) such that the complementary hole \( H \setminus \text{Int}H_{\text{toric}} \) is fillable. At this point, one may reduce again to the case where \( \Sigma^{n-3} \) has no \( S^1 \) factors, that is, is simply connected. Indeed, the case where \( \Sigma^{n-3} \) has a \( S^1 \) factor follows by stabilization by \( S^1 \).

**Lemma 4.6.** The torus \( T^{n-3} \) embeds into the interior of \( D^2 \times \Sigma^{n-3} \) with trivial normal bundle, and such that its projection to \( \Sigma^{n-3} \) is of degree 1.

**Proof of the lemma** — First every torus \( T^p \) has an embedding \( i_p \) into \( \mathbb{R}^{p+1} \) with trivial normal bundle. This is verified e.g. by induction on
if $i_p$ exists then the boundary of a tubular neighborhood of $i_p(T^p)$ in $\mathbb{R}^{n+2}$ is an embedded $T^{p+1}$ with trivial normal bundle.

Next, any point of $\mathbb{R}^{p+1}$ interior to $i_p(T^p)$ is the common center of two hyperspheres between which $i_p(T^p)$ lies, hence actually:

$$i_p(T^p) \subset \text{Int}(D^1 \times S^p)$$

Obviously the projection to $S^n$ is of degree one.

Recall that $\Sigma^{n-3}$ is either a sphere or a product of two spheres.

In case $\Sigma^{n-3} = S^{n-3}$, one has the inclusion $D^1 \subset D^2$ and thus:

$$i_{n-3}(T^{n-3}) \subset D^2 \times S^{n-3}$$

In case $\Sigma^{n-3} = S^p \times S^q$, one makes the cartesian product of the embeddings:

$$i_p \times i_q : T^{n-3} = T^p \times T^q \rightarrow (D^1 \times S^p) \times (D^1 \times S^q) \cong D^2 \times S^p \times S^q$$

Let the torus of dimension $n-3$ be embedded into the interior of $D^2 \times \Sigma^{n-3}$ with trivial normal bundle and such that the projection to $\Sigma^{n-3}$ is of degree one. After excision of a tubular neighborhood of this torus, one gets:

$$B := (D^2 \times \Sigma^{n-3}) \setminus \text{Int}(D^2 \times T^{n-3})$$

Endow the fundamental group $\pi_1 B$ with the representation $\rho(\gamma) := \phi^{\lambda(\gamma)}$ where $\lambda(\gamma)$ is the linking number of the loop $\gamma$ with the $(n-3)$-torus. This linking number is well defined, since $D^2 \times \Sigma^{n-3}$ is simply connected, and since in $D^2 \times \Sigma^{n-3}$ the Poincaré-dual of the torus is null on every absolute 2-cycle.

The linking number restricts to the two connected components of the boundary of $B$ as follows. The restriction to $\partial D^2 \times T^{n-3}$ is by nature Poincaré-dual to the parallel $1 \times T^{n-3}$. Likewise, the restriction of $\lambda$ to $\partial D^2 \times \Sigma^{n-3}$ is Poincaré-dual to the parallel $1 \times \Sigma^{n-3}$, because the projection of the $(n - 3)$-torus to $\Sigma^{n-3}$ is of degree one.

In conclusion, $H \setminus \text{Int}H_{\text{toric}} = B \times D^1$ can be filled by the suspension of the representation $\rho$. This ends Thurston’s proof of proposition 4.4.

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