Sklyanin algebras revisited

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Abstract

It is proved that the Sklyanin algebra on 4 generators and 6 relations modulo an ideal is isomorphic to a dense sub-algebra of the noncommutative torus. As a corollary one gets a functor between the categories of elliptic curves and noncommutative tori.

Key words and phrases: elliptic curve; Sklyanin algebra; noncommutative torus

MSC: 14H52 (elliptic curves); 16R10 (associative algebras); 46L85 (noncommutative topology)

A. The Sklyanin algebra is an algebra of the noncommutative polynomials over the field \( \mathbb{C} \) having \( n \geq 3 \) generators and \( \frac{n(n-1)}{2} \) quadratic relations; such an algebra is known to be a coordinate ring of an elliptic curve \( \mathcal{E}_\tau \cong \{ \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) : \Im(\tau) > 0 \} \) and a generic point \( \eta \in \mathcal{E}_\tau \). To give an idea, we shall denote by

\[
\theta(z) = \sum_{j \in \mathbb{Z}} (-1)^j e^{2\pi i (jz + \frac{j(j-1)}{2}\tau)}, \quad z \in \mathbb{C},
\]

(1)
a theta function on the lattice \( \mathbb{Z} + \mathbb{Z}\tau \); for an integer \( n \geq 1 \) and \( r \in \mathbb{Z}/n\mathbb{Z} \) one considers a product

\[
\theta_r(z) = \prod_{j=1}^n \theta \left( z + \frac{r}{n} + \frac{j-1}{n} \right) e^{2\pi i \left( rz + \frac{r(r-1)}{2n} + \frac{z}{n} \right)}.
\]

(2)

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A Sklyanin algebra is the (graded) polynomial algebra \( \mathbb{C}\langle x_1, \ldots, x_n \rangle \) on \( n \geq 3 \) non-commuting variables \( x_i \) subject to \( \frac{n(n-1)}{2} \) quadratic relations

\[
\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{\theta_r(\eta)\theta_{j-i-r}(-\eta)} x_{j-r}x_{i+r} = 0,
\]

where \( \eta \in \mathcal{E}_\tau \), \( i, j \in \mathbb{Z}/n\mathbb{Z} \) and \( i \neq j \); such an algebra will be denoted by \( Q_n(\mathcal{E}_\tau, \eta) \). The fundamental result of Sklyanin says that for each \( \tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\} \) and a generic set of points \( \eta \in \mathcal{E}_\tau \) the algebra \( Q_n(\mathcal{E}_\tau, \eta) \) satisfies the Serre Duality:

\[
\text{QGr} \ (Q_n(\mathcal{E}_\tau, \eta)) \cong \text{Qcoh} \ (\mathcal{E}_\tau),
\]

where \( \text{QGr} \) is a category of the quotient graded modules over the algebra \( Q_n(\mathcal{E}_\tau, \eta) \) modulo torsion and \( \text{Qcoh} \) a category of the quasi-coherent sheaves on the elliptic curve \( \mathcal{E}_\tau \); they say therefore that the Sklyanin algebra \( Q_n(\mathcal{E}_\tau, \eta) \) is a noncommutative coordinate ring of the elliptic curve \( \mathcal{E}_\tau \) and a point \( \eta \in \mathcal{E}_\tau \). Clearly, for an \( \eta = \text{Const} \) formula (4) implies equivalence of the categories of elliptic curves \( \mathcal{E}_\tau \) and the Sklyanin algebras \( Q_n(\mathcal{E}_\tau) \).

B. Let \( S^1 \) be a unit circle in the complex plane \( \mathbb{C} \); denote by \( L^2(S^1) \) the Hilbert space of the square integrable complex valued functions on \( S^1 \). Fix a real number \( \theta \in [0, 1) \); for every \( f(e^{2\pi it}) \in L^2(S^1) \) we shall consider two bounded linear operators \( U \) and \( V \) which act by the formulas:

\[
\begin{align*}
Uf(e^{2\pi it}) &= f(e^{2\pi i(t-\theta)}) \\
Vf(e^{2\pi it}) &= e^{2\pi it}f(e^{2\pi it}).
\end{align*}
\]

It follows immediately from the definition that

\[
\begin{align*}
VV &= e^{2\pi i\theta}UV, \\
UU^* &= U^*U = E, \\
VV^* &= V^*V = E,
\end{align*}
\]

where \( U^* \) and \( V^* \) are the adjoint operators of \( U \) and \( V \), respectively, and \( E \) is the identity operator. The noncommutative torus (an irrational rotation algebra) is the \( C^* \)-algebra of bounded linear operators on \( L^2(S^1) \) generated by \( U \) and \( V \); we shall denote such an algebra by \( \mathcal{A}_\theta \); \( \mathbb{I} \), §12.3. The algebras \( \mathcal{A}_\theta \) are fundamental in dynamics, analysis and noncommutative geometry \( \mathbb{I} \). A \( * \)-algebra of noncommutative polynomials \( \mathbb{C}\langle U, V, U^*, V^* \rangle \) modulo an ideal...
generated by relations (6) will be denoted by \( A_0^0 \); the completion of \( A_0^0 \) in the operator norm on the Hilbert space \( L^2(S^1) \) is isomorphic to the \( C^* \)-algebra \( A_\theta \) [11], pp. 204-205.

C. A covariant functor \( F \) from the category of elliptic curves to a category of the noncommutative tori was constructed in [4]; the functor is not injective and \( \ker(F) \cong (0, \infty) \). The construction of \( F \) was based on geometry of measured foliations and used the so-called Hubbard-Masur Theorem instead of the Serre Duality; see item D for a discussion. Since the Sklyanin algebras are also functorial with respect to elliptic curves, it seems reasonable to wonder if the algebras \( Q_n(E) \) and \( A_\theta \) are related. Thus we have the following

**Main problem.** *To find a relationship (if any) between the Sklyanin algebras and noncommutative tori.*

The aim of present note is to show that indeed such a relationship exists. We shall understand by \( Q_n(E_{\tau}) \) a Sklyanin algebra \( Q_n(E, \frac{1}{n}) \), i.e. \( \eta \in E_{\tau} \) is the point of order \( n \). Put \( n = 4 \) and let \( Q_4(E_{\tau}) \) be the Sklyanin algebra on generators \( u, v, u^* \) and \( v^* \); here *-symbol is a formal notation not related to an involution. For a real \( \mu \in (0, \infty) \) let \( I_\mu \subset Q_4(E) \) be a (two-sided) ideal generated by the polynomials \( uu^* = vv^* = \mu^{-1}e \), where \( e \) is the identity of the algebra \( Q_4(E_{\tau}) \). We shall write \( \cong_c \) for an isomorphism of algebras over the field \( \mathbb{C} \); our main result can be formulated as follows.

**Theorem 1** For every \( \tau \in \mathbb{H} \) there exist \( \theta \in [0, 1) \) and \( \mu \in (0, \infty) \) such that

\[
Q_4(E_{\tau}) / I_\mu \cong_c A_\theta^0. \tag{7}
\]

**Proof of theorem 1**. We shall split the proof in a series of lemmas starting with the following elementary

**Lemma 1** The system of equations (6) implies the following system of six quadratic equations:

\[
\begin{align*}
vu &= e^{2\pi i \theta} uv, \\
uu^* &= u^* u, \\
vv^* &= v^* v, \\
uv^* &= e^{2\pi i \theta} v^* u, \\
u^* v &= e^{2\pi i \theta} vu^*, \\
v^* u^* &= e^{2\pi i \theta} u^* v^*. 
\end{align*}
\tag{8}
\]

\(^1\)The result was proved for the so-called Effros-Shen algebras; however it is valid for the noncommutative tori as well since the latter embed into the Effros-Shen algebras, see details in [4].
Proof. Indeed, the three first equations of (8) follow immediately from equations (6). We shall proceed stepwise for the rest of (8).

(i) Let us prove that equations (6) imply \( uv^* = e^{2\pi i \theta} u^* v \). It follows from \( UU^* = E \) and \( VV^* = E \) that \( UU^* VV^* = E \). Since \( UU^* = U^* U \) we can bring the last equation to the form \( U^* U VV^* = E \) and multiply the both sides by the constant \( e^{2\pi i \theta} \); thus one gets the equation \( U^* (e^{2\pi i \theta} UV) V^* = e^{2\pi i \theta} \). But \( e^{2\pi i \theta} UV = VU \) and our main equation takes the form \( U^* VUV^* = e^{2\pi i \theta} \).

We can multiply on the left the both sides of the equation by the element \( U \) and thus get the equation \( UU^* VUV^* = e^{2\pi i \theta} U \); since \( UU^* = E \) one arrives at the equation \( VUU^* = e^{2\pi i \theta} U \).

Again one can multiply on the left both sides by the element \( V \) and thus get the equation \( V^* VUV^* = e^{2\pi i \theta} V^* U \); since \( V^* V = E \) one gets the required identity \( UV^* = e^{2\pi i \theta} V^* U \).

(ii) Let us prove that equations (6) imply \( u^* v = e^{2\pi i \theta} u^* v \). As in the case (i), it follows from the equations \( UU^* = E \) and \( VV^* = E \) that \( VV^* UU^* = E \). Since \( VV^* = V^* V \) we can bring the last equation to the form \( V^* VU^* = E \) and multiply the both sides by the constant \( e^{-2\pi i \theta} \); thus one gets the equation \( V^* (e^{-2\pi i \theta} VU) U^* = e^{-2\pi i \theta} \). But \( e^{-2\pi i \theta} VU = UV \) and our main equation takes the form \( V^* UVU^* = e^{-2\pi i \theta} \).

We can multiply on the left the both sides of the equation by the element \( V \) and thus get the equation \( V^* UVU^* = e^{-2\pi i \theta} V \); since \( VV^* = E \) one arrives at the equation \( UUU^* = e^{-2\pi i \theta} V \).

Again one can multiply on the left both sides by the element \( U^* \) and thus get the equation \( U^* UVU^* = e^{-2\pi i \theta} U^* V \); since \( U^* U = E \) one gets the equation \( VU^* = e^{-2\pi i \theta} U^* V \). Multiplying both sides by constant \( e^{2\pi i \theta} \) we obtain the required identity \( U^* V = e^{2\pi i \theta} VU^* \).

(iii) Let us prove that equations (6) imply \( v^* u^* = e^{2\pi i \theta} u^* v^* \). Indeed, it was proved in (i) that \( UV^* = e^{2\pi i \theta} V^* U \); we shall multiply on the right this equation by the equation \( U^* U = E \). Thus one arrives at the equation \( UV^* U^* U = e^{2\pi i \theta} V^* U \).

Notice that in the last equation one can cancel \( U \) on the right thus bringing it to the simpler form \( UV^* U^* = e^{2\pi i \theta} V^* \).

We shall multiply on the left both sides of the above equation by the element \( U^* \); one gets therefore \( U^* UV^* U^* = e^{2\pi i \theta} U^* V^* \). But \( U^* U = E \) and the left hand side simplifies giving the required identity \( V^* U^* = e^{2\pi i \theta} U^* V^* \).
Remark 1 The converse of lemma 1 is not true an obstacle being the equations $uu^* = e$ and $vv^* = e$.

Proof. Every equation following from (8) must be homogeneous, i.e. invariant of the substitution $u' = ku, v' = kv, (u^*)' = ku^*$ and $(v^*)' = kv^*$ for a $k \in \mathbb{C} - \{0\};$ but equations $uu^* = e$ and $vv^* = e$ are not homogeneous. □

Lemma 2 For every $\tau \in \mathbb{H}$ there exist $\theta \in [0, 1)$ and $\mu \in (0, \infty)$ such that

$$Q_4(\mathcal{E}_\tau) \cong \mathbb{C} \cdot \mathcal{U}_{\theta, \mu},$$

where $\mathcal{U}_{\theta, \mu}$ is the polynomial algebra $\mathbb{C}\langle u, v, u^*, v^* \rangle$ modulo an ideal generated by equations (8) and the scaled identity $u^* = \mu e$.

Proof. Recall that the algebra of skew polynomials (over $\mathbb{C}$) is an algebra with the generators $\{x_1, \ldots, x_n\}$ and relations $x_i x_j = q_{ij} x_j x_i \quad 1 \leq i, j \leq n$, where $q_{ij} \in \mathbb{C} - \{0\}$. Since $\eta \in \mathcal{E}_\tau$ is a point of order $n$, the Sklyanin algebra $Q_n(\mathcal{E}_\tau, \eta)$ is isomorphic to an algebra of skew polynomials on $n$ generators, see [2], Remark 1 and [3], §1; we shall use the following explicit formulas defining this isomorphism. Let $\{y_1, \ldots, y_r; t_1, \ldots, t_r\}$ be generators of a skew polynomial algebra, where $t_i = e^{2\pi i z_i}$; then

$$x_j = \sum_{k=1}^r \frac{\theta_j(z_k)}{\theta(z_1 - z_k) \cdots \theta(z_k - z_k) \cdots \theta(z_r - z_k)} y_k,$$

(10)

where $\{x_1, \ldots, x_{2r}\}$ are generators of the Sklyanin algebra $Q_{2r}(\mathcal{E}_\tau, \eta)$ and the symbol $\theta(z_k - z_k)$ means that the term $\theta(z_k - z_k)$ is omitted, see Feigin & Odesskii [3], §2.

Put $n = 4$; in this case $\eta = \frac{1}{4}$ and relations (3) in the algebra $Q_4(\mathcal{E}_\tau)$ take the form:

\begin{equation}
\begin{cases}
x_1 x_2 - x_2 x_1 - \tau (x_3 x_4 + x_4 x_3) = 0, \\
x_1 x_2 + x_2 x_1 - (x_3 x_4 - x_4 x_3) = 0, \\
x_1 x_3 - x_3 x_1 - \frac{1}{3}(x_4 x_2 + x_2 x_4) = 0, \\
x_1 x_3 + x_3 x_1 - (x_4 x_2 - x_2 x_4) = 0, \\
x_1 x_4 - x_4 x_1 + \frac{4r+1}{r+1}(x_2 x_3 + x_3 x_2) = 0, \\
x_1 x_4 + x_4 x_1 - (x_2 x_3 - x_3 x_2) = 0.
\end{cases}
\end{equation}

(11)

see [9], p.260 and [10], Example 8.5. By definition of the Sklyanin algebra $Q_4(\mathcal{E}_\tau)$ point $\eta = \frac{1}{4}$ is a point of order 4; therefore $Q_4(\mathcal{E}_\tau)$ is isomorphic to
an algebra of skew polynomials. Let us show that such an algebra is $U_{\theta,\mu}$. Indeed, $U_{\theta,\mu}$ is an algebra of skew polynomials with the structure constants

\[
\begin{align*}
q_{13} &= q_{24} = 1, \\
q_{12} &= q_{14} = e^{2\pi i \theta}, \\
q_{23} &= q_{34} = e^{-2\pi i \theta}.
\end{align*}
\]

and therefore one can use formulas (10) to construct generators $x_j$ of the Sklyanin algebra $Q_4(E_{\tau})$. Notice that $U_{\theta,\mu}$ is a two-parameter family of non-isomorphic skew polynomial algebras and therefore the map $U_{\theta,\mu} \to Q_4(E_{\tau})$ is surjective. Thus for every $\tau \in \mathbb{H}$ one can find $\theta \in [0, 1)$ and $\mu \in (0, \infty)$ such that $Q_4(E_{\tau}) \cong \mathbb{C} U_{\theta,\mu}$. □

**Lemma 3** $U_{\theta,\mu} / I_{\mu} \cong \mathbb{C} A^0_{\theta}$.

**Proof.** The algebra $A^0_{\theta}$ is an algebra of polynomials in four non-commuting variables $u, v, u^*$ and $v^*$ over $\mathbb{C}$ subject to the relations

\[
\begin{align*}
vv^* &= e^{2\pi i \theta} u^* u = e, \\
vu &= e^{2\pi i \theta} u v, \\
uu^* &= u^* u = e, \\
v^* v &= v^* v = e,
\end{align*}
\]

where $e$ is the identity, see [11], p.204. By lemma 1 equations (13) imply equations (8); thus $A^0_{\theta}$ is isomorphic to the algebra $U_{\theta,\mu}$ modulo an ideal generated by the scaled identity $e' = \mu e$. To calculate this ideal, notice that $e = \mu^{-1} e' = uu^* = vv^*$ and therefore it coincides with the ideal $I_{\mu}$. □

Theorem 1 follows from lemmas 2 and 3. □

D. Let $\mathfrak{E}$ be a category of elliptic curves over the complex numbers whose arrows are isomorphisms of the curves; let $\mathfrak{A}$ be a category of noncommutative tori whose arrows are stable isomorphisms of the tori. An application of theorem 1 is as follows.

**Corollary 1** There exists a (covariant) non-injective functor $F : \mathfrak{E} \to \mathfrak{A}$ such that $\text{Ker}(F) \cong (0, \infty)$.

\[\text{2The tori $A_{\theta}$ and $A_{\theta'}$ are said to be stably isomorphic if $A_{\theta} \otimes K \cong A_{\theta'} \otimes K$, where $K$ is the $C^*$-algebra of compact operators; such an isomorphism happens if and only if } \theta' = \frac{ad + bc}{cd^2 + d}, \text{ where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1.\]
Proof. Let us notice that the algebras $A^0_\theta$ and $A_\theta$ define each other; therefore the existence of functor $F$ follows from the Serre Duality (4) and theorem 1. The kernel of $F$ consists of those elliptic curves $E_\tau$ for which $\theta = \text{Const}$; according to formula (7) such curves are parameterized by the reals $\mu \in (0, \infty)$. □

The corollary was originally proved using geometry of measured foliations on the two-dimensional torus [4]; such a foliation is given by its slope $\theta$ and a (constant) transverse measure $\mu \in (0, \infty)$. The Hubbard-Masur Theorem says that every elliptic curve is defined by a measured foliation on the two-torus; thus such a theorem replaces the Serre Duality. It is important to notice that the method of measured foliations extends to the algebraic curves of any genus $g$ [5] while Sklyanin’s theory is restricted to the case $g = 1$; the author hopes such a general theory will appear in the future.

Elsewhere a quotient of the Sklyanin algebra on four generators is proved isomorphic to a coordinate ring of the noncommutative three-dimensional sphere [1].

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