Waveform relaxation for low frequency coupled field/circuit differential-algebraic models of index 2

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Abstract

Motivated by the task to design quench protection systems for superconducting magnets in particle accelerators we address a coupled field/circuit simulation based on a magneto-quasistatic field modeling. We investigate how a waveform relaxation of Gauss-Seidel type performs for a coupled simulation when circuit solving packages are used that describe the circuit by the modified nodal analysis. We present sufficient convergence criteria for the coupled simulation of FEM discretised field models and circuit models formed by a differential-algebraic equation (DAE) system of index 2. In particular, we demonstrate by a simple benchmark system the drastic influence of the circuit topology on the convergence behavior of the coupled simulation.

1 Introduction

Lumped circuit models, such as modified nodal analysis (MNA), are well-established in electrical engineering. However, they neglect the spatial dimension and therefore distributed phenomena like the skin effect. For certain devices, this may lead to inaccuracies of unacceptable magnitude in the simulation, e.g. for electric machines \cite{14} or the quench protection system of superconducting magnets in particle accelerators \cite{1}. These cases call for field/circuit coupling \cite{16}, \cite{3}. To solve such coupled systems, it is often advisable to use waveform relaxation (WR) \cite{7}, since this iterative method allows for dedicated step sizes and suitable solvers for the different subsystems, and even for the use of proprietary blackbox solvers. The coupled field/circuit model considered here is a DAE in the time domain after space discretisation of the field system. It is well-known that WR can suffer from instabilities for DAEs unless an additional contraction criterion is satisfied \cite{7}, \cite{11}. This work presents coupled field/circuit models, which are DAEs of index 2 \cite{5}, for the case where WR is convergent and the case where it diverges. Furthermore, generalizing a convergence criterion of \cite{11}, a topological and easy-to-check criterion is provided. Finally, we present numerical simulations verifying the topological convergence criterion.

2 Field/Circuit Model

To describe the electromagnetic (EM) field part, we consider a magnetoquasistatic approximation of Maxwell’s equations in a reduced magnetic vector potential formulation \cite{4}. This leads to the curl-curl eddy current partial differential equation (PDE). The circuit side is formulated with the MNA \cite{6}.

For the numerical simulation of the coupled system, the method of lines is used with a finite element
(FE) discretisation. Altogether, this leads to a time-dependent coupled system of DAE initial value problems (IVPs), described by

\[
\begin{align*}
M\dot{a} + K(a)a - X_{im} &= 0, \\
X^\top \dot{a} &= v_c, \\
E(x)\dot{x} + f(t,x) &= P_{im}, \\
P^\top x - v_c &= 0.
\end{align*}
\]

The first Equation (1) represents the space-discrete field model based on the matrices

\[
(M)_{ij} = \int_{\Omega} \sigma \omega_i \cdot \omega_j \, dV, \quad (K(a))_{ij} = \int_{\Omega} v(a) \nabla \cdot \omega_i \cdot \nabla \omega_j \, dV,
\]

which follow from the Ritz-Galerkin approach using a finite set of Nédélec basis functions \(\omega\). The strong monotonicity of the underlying nonlinear material law, i.e. the BH-curve \[13\].

Assumption 2(a) follows naturally if a Ritz-Galerkin formulation (3) is chosen. The second Assumption 2(b) will be guaranteed by appropriate boundary and gauging conditions. Thirdly, the full column rank Assumption 2c) follows from the fact that the columns are discretisations of different coils that are located in spatially disjoint subdomains. Finally, the monotonicity Assumption 2(d) follows from the fact that the columns are discretisations of different coils that are located in spatially disjoint subdomains.

The assumptions are in agreement with previous formulation in the literature, e.g. \[2, 15\]. The first Assumption 2(a) reflects the global passivity of the respective elements \[8\]. Considering well-known relations between incidence matrices and circuit topology, Assumption 2(b) excludes the electrically forbidden configurations of loops of voltage sources and cutsets of current sources \[5\].

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\]

which follow from the Ritz-Galerkin approach using a finite set of Nédélec basis functions \(\omega\). The strong monotonicity of the underlying nonlinear material law, i.e. the BH-curve \[13\]. In general, the field model is a multiport element such that the circuit coupling is established via multiple currents and voltages, i.e., vector-valued \(i_m\) and \(v_c\). However, for simplicity of notation we assume a two-terminal device in the following.

The circuit Equation (2) can be expanded into

\[
E(x) = \begin{pmatrix}
\mathcal{L}_C(e) & 0 & 0 \\
0 & -L(i_L) & 0 \\
0 & 0 & 0
\end{pmatrix}
\quad f(t,x) = \begin{pmatrix}
g_R(e) + A_L i_L + A_V i_V + q_L(t) \\
A_L e \\
A_V e - q_V(t)
\end{pmatrix},
\quad P = \begin{pmatrix}
A_m \\
0
\end{pmatrix}
\]

using the definitions \(\mathcal{L}_C(e) := A_C C(A_C^\top e) A_C^\top\), \(g_R(e) := A_R g_R e\) and \(x = (e, i_L, i_V)\) where \(A_*\) are the usual incidence matrices and \(L(\cdot), C(\cdot)\) are state-dependent square matrices describing inductances and capacitances. The function \(g(\cdot)\) describes the voltage-current relation of resistive elements. Finally, \(x\) collects all node potentials \(e\), currents through branches with voltage sources \(i_V\) and inductors \(i_L\). The circuit system shall fulfill the following properties:

Assumption 3 It holds (a) \(g, C, L\) are Lipschitz continuous, \(g\) is strongly monotone and \(C, L\) are uniformly positive definite, (b) \(q_L\) and \(q_V\) are continuously differentiable, (c) \(A_V\) has full column rank and \((A_C, A_V, A_R, A_L)\) has full row rank.

Assumption 3 reflects the global passivity of the respective elements \[8\]. Considering well-known relations between incidence matrices and circuit topology, Assumption 3(b) excludes the electrically forbidden configurations of loops of voltage sources and cutsets of current sources \[5\].
3 Waveform Relaxation and Convergence

We consider the Gauß-Seidel WR method. Applied to the coupled system \(1\)–\(2\), this yields the scheme

\[
\begin{align*}
\dot{a}^k + K(a^k)a^k - X_{m}^k &= 0, \\
E(x^k)x^k + f(t,x^k) &= P_i_{m}, \\
X^T\dot{a}^k &= v_c^{-1}, \\
P^T\dot{a}^k &= v_i^0.
\end{align*}
\]

(6) (7)

The superscript \(k\) denotes the iteration index. A common choice for the initial guess \(v_i^0\) is constant extrapolation of the initial value.

We shall proceed as follows:

1. Lemmata \(4\) and \(6\) provide a DAE-decoupling of the EM field DAE \(1\) and the MNA DAE \(2\), respectively.
2. Definition \(5\) introduces the concept of parallel CVR paths. Assuming their existence and exploiting the previous decoupling Lemmata, Lemma \(7\) yields a DAE-decoupling of the coupled WR iteration \(6\)–\(7\). Notably, it reveals the structure of its inherent ODE, given by \(\dot{\phi}\) in Equation (11).
3. The convergence Theorem \(8\) is a simple consequence of the previous Lemmata; it shows that the existence of parallel CVR paths guarantees convergence of the WR scheme \(6\)–\(7\).

For visual reasons, we shall write column vectors as \((a,b,c)\).

**Lemma 4** Let Assumption \(2\) hold. Then, for a given source term \(v_c\), there exists a coordinate transformation \((w,u) = T^{-1}a\) and a system of the form

\[
\begin{align*}
\dot{u} + A_1u &= A_2v_c, \\
w &= Bu, \\
i_m &= G_1u + G_2v_c
\end{align*}
\]

(8)

such that \((a,i_m)\) solves Equation \(1\) if and only if \((u,w,i_m)\) solves Equation \(8\).

**Proof:** For better readability and shortness we present the proof only for the slightly more restrictive case where \(X^TM = 0\), which is usually satisfied.

We equivalently transform the field DAE with new coordinates \(T\alpha = a:\)

\[
\begin{align*}
T^MT\alpha + T^TK(T\alpha)T\alpha - T^TX_{m} &= 0, \\
X^T\alpha &= v_c
\end{align*}
\]

(9)

The transformation matrix \(T := (T_{ker} X T_{\perp})\) is constructed such that the columns of \(T_{ker}\) and \(T_{\perp}\) form a basis of \(\ker M \cap \ker X^T\) and \((\ker M)^\perp\), respectively. It is nonsingular indeed, since its construction and Assumption \(2\) combined with \(X^TM = 0\) guarantee that \(\text{im}X \perp \text{im}T_{ker}\) and \(\text{im}T_{\perp} \perp \text{im}(T_{ker})\).

With \(\alpha = (w,u)\) and \(u = (u_1,u_2)\), the transformed DAE \(9\) has the detailed form

\[
\begin{align*}
T_{ker}^TK(T\alpha)T_{ker}w + T_{ker}^TK(T\alpha)(X T_{\perp})u &= 0, \\
X^TK(T\alpha)T\alpha - X^TX_{m} &= 0, \\
T_{\perp}^MT_{\perp}u_2 + T_{\perp}^TK(T\alpha)T\alpha &= 0, \\
X^T\alpha &= v_c.
\end{align*}
\]

The underlined matrices are nonsingular due to Assumption \(2\) and Equation \(8\) is obtained by inversion and insertion.

**Definition 5** A CVR path in a circuit is a path which consists of only capacitances, voltages sources and resistances. An element has a parallel CVR path, if its incident nodes are connected by a CVR path.

**Lemma 6** Let Assumption \(3\) hold. Then, for a given source term \(i_m\), there exists a coordinate transformation \((y,z_1,z_2) = T^{-1}x\) and a system of the form

\[
\begin{align*}
\dot{y} &= f_0(t,y,z_1,z_2,u), \\
z_1 &= g_1(t,y,z_2,z_2,u), \\
z_2 &= g_2(t) + QPi_m, \\
v_c &= P^T(T(y,z_1,z_2))
\end{align*}
\]

(10a) (10b)

with \(f_0,g_1,g_2\) uniformly globally Lipschitz continuous \(\forall t\) and \(g_2 \in C^1\) such that

1. \((x,v_c)\) solves Equation \(3\) if and only if \((y,z_1,z_2,v_c)\) solves Equation \(10\).
Let Assumptions 2 and 3 hold. If each EM field element has a parallel CVR path, then there exists a coordinate transformation \((r, s) = T^{-1}(a, x)\) and a system of the form

\[
\dot{s}^k = \phi(t, s^k, s^{k-1}), \quad \dot{r}^k = \varphi(t, s^k)
\]

with \(\phi\) uniformly globally Lipschitz continuous \(\forall t\) and \(\varphi\) continuous such that \((a^k, \dot{a}^k, s^k, v^k)\) solves Equations (6)-7 if and only if \((\dot{s}^k, \dot{r}^k)\) solves Equation 11.

**Proof:** We apply Lemmas 4-6 to the iterated subsystems 4-7. This yields an equivalent system

\[
\begin{align*}
\dot{u}^k &= -A_1 u^k + A_2 v^{k-1}, \quad w^k = Bu^k, \quad \dot{r}_m = G_1 u^k + G_2 v^{k-1}, \\
\dot{y}^k &= f_0(t, \dot{y}^k, \dot{z}^k, z^k, u^k), \quad \dot{z}_1^k = g_1(t, \dot{y}^k, \dot{z}_2^k, z^k), \quad \dot{z}_2^k = g_2(t), \quad \dot{v}^k = P^T T(\dot{y}^k, \dot{z}_1^k, \dot{z}_2^k).
\end{align*}
\]

Since each field element has a parallel CVR path, \(\dot{z}^k = g(t)\) does not depend on \(u^k\) anymore.

We insert \(\dot{v}^k = P^T \dot{y}^k = P^T T(\dot{y}^k, \dot{z}_1^k, \dot{z}_2^k)\) and \(\dot{z}_1^k = g_1(t, \dot{y}^k, \dot{z}_2^k, z^k)\) and \(\dot{z}_2^k = g_2(t)\) therein to obtain, with \(\dot{g}_1(t, \dot{y}^k, \dot{z}_2^k, u^{k-1}) = g_1(t, \dot{y}^k, \dot{z}_2^k, z^k), \dot{g}_2(t, u^{k-1})\),

\[
\dot{u}^k = \Phi_2(t, u^k, \dot{y}^k, u^{k-1}, \dot{v}^k) := -A_1 u^k + A_2 P^T T(\dot{y}^k, \dot{z}_1^k, \dot{z}_2^k). \quad (\text{11})
\]

Insertion of \(\dot{z}_1^k, \dot{z}_2^k, \dot{v}^k\) into \(f_0\) yields

\[
\dot{y}^k = \Phi_1(t, u^k, \dot{y}^k) := f_0(t, \dot{y}^k, \dot{g}_1(t, \dot{y}^k, \dot{z}_2^k, z^k), \dot{g}_2(t, u^k), \dot{g}_2(t), u^k).
\]

Hence, defining \(\dot{s}^k := (u^k, \dot{y}^k)\) and \(\dot{\varphi} := (\Phi_1, \Phi_2)\), the sequence \((\dot{s}^k, \dot{r}^k)\) is given implicitly by an ODE recursion of the form \(\dot{s}^k = \Phi(t, \dot{s}^k, \dot{r}^k)\).

The algebraic constraint of Equation 11 is obtained with \(r^k = (w^k, \dot{z}_1^k, \dot{z}_2^k, \dot{v}^k)\) and \(s^k = (u^k, \dot{y}^k)\) and \(\varphi(t, s^k) = (Bu, Gu, g_1(t, \dot{y}^k, \dot{z}_2^k, z^k), \dot{g}_2(t), u^k, \dot{g}_2(t)).\)

Clearly, \((\dot{s}^k, \dot{r}^k)\) solves Equation 11 if and only if \(\dot{a}^k := (u^k, \dot{w}^k, \dot{\dot{a}}^k, \dot{\dot{z}}_1^k, \dot{\dot{z}}_2^k, \dot{v}^k)\)] solves Equations 12-14, and \(\dot{a}^k\) solves Equations 12-14 if and only if \((u^k, \dot{a}^k, x^k, v^k)\) solves Equations 6-7.

We deduce the main result of this work:

**Theorem 8** If each EM field element of the coupled system 1-2 has a parallel CVR path, then the WR scheme 6-7 is uniformly convergent to the exact solution of 1-2.

**Proof:** The ODE part of Equation 11 is a WR scheme for ODEs with Lipschitz continuous vector field \(\Phi\). It is well-known that such schemes are unconditionally convergent on bounded time intervals [7]. The convergence of \(\dot{s}^k\) clearly implies the convergence of \((\dot{s}^k, \dot{r}^k)\) defined by 11. Due to the equivalence provided by Lemma 6-7, it follows that the original scheme 6-7 is convergent.

**Remark 9** The convergence result holds for arbitrary continuous initial guesses \(x^0\) and for bounded intervals of arbitrary size, see e.g. [7] [12].
4 Numerical Examples

To illustrate the convergence behaviour of the WR scheme according to the derived criteria, we consider the toy example circuits in Figures 2a and 2b. Both are described with MNA [2] and the (arbitrary) parameters $R = 1\, \Omega$, $L = 5\, H$, $C = 1\, F$, $i_s(t) = \sin(2t) + 5\sin(20t)$ and $v_s(t) = \sin(t) + \sin(20t)$ are set. The eddy current Equation (1) is solved on the single phase isolation transformer shown in Figure 1. For simplicity, a zero current is imposed on the secondary coil (dark orange) and only the primary coil is coupled to the circuit.

The WR algorithm is applied on the simulation time window $T = [0\, 0.8] \, s$ and the internal time integration is performed with the implicit Euler scheme with time step size $\delta t = 10^{-2} \, s$.

The theoretical result is illustrated by the successful simulation, see Figure 3a, of the model shown in Figure 2a which satisfies the convergence criterion of Theorem 8. However, numerical simulations of the model shown in Figure 2b show that WR can diverge indeed if the criterion is not satisfied.

5 Conclusions

In this work, we have presented a space-discretised coupled field/circuit model, which is a DAE of index 2, and a simulation of this model by means of WR. Furthermore, we have provided an easy-to-check topological convergence criterion for a class of coupled DAE/DAE systems of index 2.

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Figure 3: Monolithic (“mon”) and WR solution for $k = 1, 2$ iterations.

References

[1] L. Bortot et. al. STEAM: A hierarchical co-simulation framework for superconducting accelerator magnet circuits. 28(3), 2018.
[2] I. Cortes Garcia, H. De Gersem, and S. Schöps. A structural analysis of field/circuit coupled problems based on a generalised circuit element. 83(1):373–394, 2020.
[3] I. Cortes Garcia et. al. Optimized field/circuit coupling for the simulation of quenches in superconducting magnets. 2(1):97–104, 2017.
[4] C. R. I. Emson and C. W. Trowbridge. Transient 3d eddy currents using modified magnetic vector potentials and magnetic scalar potentials. 24(1):86–89, 1988.
[5] D: Estévez Schwarz and C. Tischendorf. Structural analysis of electric circuits and consequences for MNA. 28(2):131–162, 2000.
[6] C.-W. Ho, A. E. Ruehli, and P. A. Brennan. The modified nodal approach to network analysis. 22(6):504–509, 1975.
[7] E. Lelarasmee, A. E. Ruehli, and A. L. Sangiovanni-Vincentelli. The waveform relaxation method for time-domain analysis of large scale integrated circuits. 1(3):131–145, 1982.
[8] M. Matthes. Numerical Analysis of Nonlinear Partial Differential-Algebraic Equations: A Coupled and an Abstract Systems Approach. Logos Verlag Berlin GmbH, 2012.
[9] D. Meeker. Finite Element Method Magnetics, version 4.2 (25feb2018 build) edition, 2018. User’s Manual.
[10] P. Monk. Finite Element Methods for Maxwell’s Equations. Oxford University Press, 2003.
[11] J. Pade and C. Tischendorf. Waveform relaxation: a convergence criterion for differential-algebraic equations. 81(4): 1327-1342, 2019.
[12] J. Pade. Convergence criteria for waveform relaxation on differential-algebraic systems: a topological approach for circuits. Ph.D. Thesis, HU Berlin. In Preparation, 2020
[13] C. Pechstein. Multigrid-Newton-Methods For Nonlinear-Magnetostatic Problems, Master’s Thesis, University of Linz, 2004.
[14] S. J. Salon. Finite Element Analysis of Electrical Machines. Kluwer, 1995.
[15] S. Schöps. Multiscale Modeling and Multirate Time-Integration of Field/Circuit Coupled Problems. VDI Verlag. Fortschritt-Berichte VDI, Reihe 21.
[16] S. Schöps, H. De Gersem, and A. Bartel. A cosimulation framework for multirate time-integration of field/circuit coupled problems. 46(8):3233–3236, 2010.