ON WIGNER’S THEOREM

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For Mike Freedman, on the occasion of his 60th birthday

Abstract. Wigner’s theorem asserts that any symmetry of a quantum system is unitary or antiunitary. In this short note we give two proofs based on the geometry of the Fubini-Study metric.

The space of pure states of a quantum mechanical system is the projective space $\mathbb{P} \mathcal{H}$ of lines in a separable complex Hilbert space $(\mathcal{H}, \langle -, - \rangle)$, which may be finite or infinite dimensional. It carries a symmetric function $p: \mathbb{P} \mathcal{H} \times \mathbb{P} \mathcal{H} \to [0, 1]$ whose value $p(L_1, L_2)$ on states $L_1, L_2 \in \mathbb{P} \mathcal{H}$ is the transition probability: if $\psi_i \in L_i$ is a unit norm vector in the line $L_i$, then

$$p(L_1, L_2) = |\langle \psi_1, \psi_2 \rangle|^2.$$

Let $\text{Aut}_{\text{qtm}}(\mathbb{P} \mathcal{H})$ denote the group of symmetries of $(\mathbb{P} \mathcal{H}, p)$, the group of quantum symmetries. A fundamental theorem of Wigner$^1$ [Wi, §20A, §26], [Ba], [We, §2A] expresses $\text{Aut}_{\text{qtm}}(\mathbb{P} \mathcal{H})$ as a quotient of linear and antilinear symmetries of $\mathcal{H}$. This note began with the rediscovery of a formula which relates the quantum geometry of $(\mathbb{P} \mathcal{H}, p)$ to a more familiar structure in differential geometry: the Fubini-Study Kähler metric on $\mathbb{P} \mathcal{H}$. It leads to two proofs of Wigner’s theorem, Theorem 8 of this note, based on the differential geometry of projective space.

The proofs here use more geometry than the elementary proofs [Ba], [We, §2A]. We take this opportunity to draw attention to Wigner’s theorem and to the connection between quantum mechanics and projective geometry. It is a fitting link for a small tribute to Mike Freedman, whose dual careers in topology and condensed matter physics continue to inspire.

Let $d: \mathbb{P} \mathcal{H} \times \mathbb{P} \mathcal{H} \to \mathbb{R}^\geq 0$ be the distance function associated to the Fubini-Study metric.

Theorem 1. The functions $p$ and $d$ are related by

$$(2) \quad \cos(d) = 2p - 1.$$

As a gateway into the literature on ‘geometric quantum mechanics’, where (2) can be found,$^2$ see [BH] and the references therein.

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$^1$As I learned in [Bo, p. 74], this theorem was first asserted in a 1928 joint paper [VNW, p. 207] of von Neumann and Wigner, though with only a brief justification. A more complete account appeared in Wigner’s book (in the original German) in 1931.

$^2$Notice that (2) is equivalent to $p = \cos^2(d/2)$. 
Corollary 3. $\text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ is the group of isometries of $\mathbb{P}\mathcal{H}$ with the Fubini-Study distance function.

Remark 4. If $\mathcal{H}$ is infinite dimensional, then $\mathbb{P}\mathcal{H}$ is an infinite dimensional smooth manifold modeled on a Hilbert space. Basic notions of calculus and differential geometry carry over to Hilbert manifolds [L]. The Myers-Steenrod theorem asserts that a distance-preserving map between two Riemannian manifolds is smooth and preserves the Riemannian metric. That theorem is also true on Riemannian manifolds modeled on Hilbert manifolds [GJR]. So in the sequel we use that a distance-preserving map $\mathbb{P}\mathcal{H} \rightarrow \mathbb{P}\mathcal{H}$ is smooth and is an isometry in the sense of Riemannian geometry.

The tangent space to $\mathbb{P}\mathcal{H}$ at a line $L \subset \mathcal{H}$ is canonically $T_L \mathbb{P}\mathcal{H} \cong \text{Hom}_\mathbb{C}(L, L^\perp)$, where $L^\perp \subset \mathcal{H}$ is the orthogonal complement to $L$, a closed subspace and therefore itself a Hilbert space. If $f_1, f_2 \colon L \rightarrow L^\perp$, then the Fubini-Study hermitian metric is defined by

$$\langle f_1, f_2 \rangle = \text{Tr}(f_1^* f_2).$$

The adjoint $f_1^*$ is computed using the inner products on $L$ and $L^\perp$. The composition $f_1^* f_2$ is an endomorphism of $L$, hence multiplication by a complex number which we identify as the trace of the endomorphism. If $\ell \in L$ has unit norm, then the map

$$\text{Hom}_\mathbb{C}(L, L^\perp) \rightarrow L^\perp$$

$$f \mapsto f(\ell)$$

is a linear isometry for the induced metric on $L^\perp \subset \mathcal{H}$. The underlying Riemannian metric is the real part of the hermitian metric (5); it only depends on the real part of the inner product on $\mathcal{H}$.

Proof of Theorem 1. Equation (2) is obvious on the diagonal in $\mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H}$, as well as if $\text{dim} \mathcal{H} = 1$. Henceforth we rule out both possibilities. Fix $L_1 \neq L_2 \in \mathbb{P}\mathcal{H}$ and let $V$ be the 2-dimensional space $L_1 + L_2 \subset \mathcal{H}$. The unitary automorphism of $\mathcal{H} = V \oplus V^\perp$ which is +1 on $V$ and −1 on $V^\perp$ induces an isometry of $\mathbb{P}\mathcal{H}$ which has $\mathbb{P}V$ as a component of its fixed point set. It follows that $\mathbb{P}V$ is totally geodesic. Therefore, to compute $d(L_1, L_2)$ we are reduced to the case of the complex projective line with its Fubini-Study metric: the round 2-sphere.

Let $e_1 \in L_1$ have unit norm and choose $e_2 \in V$ to fill out a unitary basis $\{e_1, e_2\}$. Then $\lambda e_1 + e_2 \in L_2$ for a unique $\lambda \in \mathbb{C}$. If $\lambda = 0$ then it is easy to check that $d = \pi$ and $p = 0$, consistent with (2), so we now assume $\lambda \neq 0$. Identify $\mathbb{P}V \setminus \{\mathbb{C} \cdot e_2\}$ with $\mathbb{C} \cdot (e_1 + \mu e_2) \leftrightarrow \mu$. Use stereographic projection from the north pole $(1,0)$ in Euclidean 3-space $\mathbb{R} \times \mathbb{C}$ to identify $\{0\} \times \mathbb{C} \approx S^2 \setminus \{(1,0)\}$, where $S^2 \subset \mathbb{R} \times \mathbb{C}$ is the unit sphere. Under these identifications we have

$$L_1 \leftrightarrow (-1,0)$$

$$L_2 \leftrightarrow \left(-\frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}, \frac{2|\lambda|^2}{|\lambda|^2 + 1}, \frac{1}{\lambda}\right)$$

from which $\cos(d) = (|\lambda|^2 - 1)/(|\lambda|^2 + 1)$ can be computed as the inner product of vectors in the 3-dimensional vector space $\mathbb{R} \oplus \mathbb{C}$. Since $p = |\lambda|^2/(|\lambda|^2 + 1)$, equation (2) is satisfied. \hfill \square

3The proof depends on the existence of geodesic convex neighborhoods, proved in [L, §VIII.5]. For the Fubini-Study metric on $\mathbb{P}\mathcal{H}$ such neighborhoods may easily be constructed explicitly. I thank Karl-Hermann Neeb for his inquiry about the Myers-Steenrod theorem in infinite dimensions.
A real linear map $S : \mathcal{H} \to \mathcal{H}$ is antiunitary if it is conjugate linear and
\[
\langle S \psi_1, S \psi_2 \rangle = \overline{\langle \psi_1, \psi_2 \rangle} \quad \text{for all } \psi_1, \psi_2 \in \mathcal{H}.
\]

Let $G(\mathcal{H})$ denote the group consisting of all unitary and antiunitary operators on $\mathcal{H}$. In the norm topology it is a Banach Lie group $[M]$ with two contractible components; the same is true in the compact-open topology $[FM, \text{Appendix D}]$. The identity component is the group $U(\mathcal{H})$ of unitary transformations. Any $S \in G(\mathcal{H})$ maps complex lines to complex lines, so induces a diffeomorphism of $\mathbb{P}\mathcal{H}$, and since $S$ preserves the real part of $\langle -, - \rangle$ the induced diffeomorphism is an isometry. The unit norm scalars $T \subset G(\mathcal{H})$ act trivially on $\mathbb{P}\mathcal{H}$, so there is an exact\footnote{We assume $\dim \mathcal{H} > 1$.} sequence of Lie groups
\[
1 \longrightarrow T \longrightarrow G(\mathcal{H}) \longrightarrow \text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H}).
\]

Note that $T$ is not central since antiunitary maps conjugate scalars.

**Theorem 8** (Wigner $[Wi]$). The homomorphism $G(\mathcal{H}) \to \text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ is surjective: every quantum symmetry of $\mathbb{P}\mathcal{H}$ lifts to a unitary or antiunitary operator on $\mathcal{H}$.

By Corollary 3 the same is true for isometries of the Fubini-Study metric, and indeed we prove Wigner’s Theorem by computing the group of isometries.

**Remark 9.** If $\rho : G \to \text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ is any group of quantum symmetries, then the surjectivity of $G(\mathcal{H}) \to \text{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ implies the extension (7) pulls back to a twisted central extension of $G$. The twist is the homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$ which tells whether a symmetry lifts to be unitary or antiunitary. The isomorphism class of this twisted central extension is then an invariant of $\rho$. This is the starting point for joint work with Greg Moore $[FM]$ about quantum symmetry classes and topological phases in condensed matter physics.

**Example 10.** $\mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1$ with the Fubini-Study metric is the round 2-sphere of unit radius. Its isometry group is the group $O(3)$ of orthogonal transformations of $SO(3)$. The identity component $SO(3)$ is the image of the group $U(2)$ of unitary transformations of $\mathbb{C}^2$. The other component of $O(3)$ consists of orientation-reversing orthogonal transformations, such as reflections, and they lift to antiunitary symmetries of $\mathbb{C}^2$. In this case the group $G(\mathcal{H})$ is also known as $\text{Pin}^c(3)$; see $[ABS]$.

We present two proofs of Theorem 8. The first is based on the following standard fact in Riemannian geometry.

**Lemma 11.** Let $M$ be a Riemannian manifold, $p \in M$, and $\phi : M \to M$ an isometry with $\phi(p) = p$. Suppose $B_r \subset T_p M$ is the open ball of radius $r$ centered at the origin and assume the Riemannian exponential map $\exp_p$ maps $B_r$ diffeomorphically into $M$. Then in exponential coordinates $\phi \big|_{B_r}$ equals the restriction of the linear isometry $d\phi_p$ to $B_r$.

**Proof.** If $\xi \in B_r$, then $\exp_p(\xi) = \gamma_\xi(1)$, where $\gamma_\xi : [0, 1] \to M$ is the unique geodesic which satisfies $\gamma_\xi(0) = p$, $\dot{\gamma}_\xi(0) = \xi$. Since $\phi$ maps geodesics to geodesics, $\phi \circ \exp_p = \exp_p \circ d\phi_p$ on $B_r$, as desired. \qed
If $\rho: [0,r') \to [0,r)$ is a diffeomorphism for some $r' > 0$, then

$$\xi \mapsto \exp_p(\rho(|\xi|)\xi)$$

maps $B_r'$ diffeomorphically into $M$, and $\phi$ in this coordinate system is also linear.

First Proof of Theorem 8. Let $\phi: \mathbb{P}^\mathcal{H} \to \mathbb{P}^\mathcal{H}$ be an isometry. Composing with an isometry in $G(\mathcal{H})$ we may assume $\phi(L) = L$ for some $L \in \mathbb{P}^\mathcal{H}$. The tangent space $T_L\mathbb{P}^\mathcal{H}$ is canonically $\text{Hom}_C(L, L^\perp)$, and also $f \in \text{Hom}_C(L, L^\perp)$ determines $\Gamma_f \in \mathcal{H}$ by $\Gamma_f \subset \mathcal{H} = L \oplus L^\perp$ is the graph of $f$. We claim $f \mapsto \Gamma_f$ has the form (12) for some $\rho: [0, \infty) \to [0, \pi)$. It suffices to show that for any $f \in \text{Hom}_C(L, L^\perp)$ of unit norm, the map $t \mapsto \Gamma_{tf}$ traces out a (reparametrized) geodesic in a parametrization independent of $f$. As in the proof of Theorem 1 this reduces to $\text{dim} \mathcal{H} = 2$ and so to an obvious statement about the round 2-sphere. It follows from Lemma 11 that $\phi$ is a real isometry $S \in \text{End}_R(\text{Hom}_C(L, L^\perp))$. It remains to prove that $S$ is complex linear or antilinear; then we extend $S$ by the identity on $L$ to obtain a unitary or antiunitary operator on $\mathcal{H} = L \oplus L^\perp$.

If $\text{dim} \mathcal{H} = 2$ then Theorem 8 can be verified (see Example 10), so assume $\text{dim} \mathcal{H} > 2$. Identify $\text{Hom}_C(L, L^\perp) \approx L^\perp$ as in (6). Since $S \in \text{End}_R(\text{Hom}_C(L^\perp))$ maps complex lines in $L^\perp$ to complex lines, there is a function $\alpha: L^\perp \setminus \{0\} \to \mathbb{C}$ such that $S(i\xi) = \alpha(\xi)S(\xi)$ for all nonzero $\xi \in L^\perp$. Fix $\xi \neq 0$ and choose $\eta \in L^\perp$ which is linearly independent. Then

$$S(i(\xi + \eta)) = \alpha(\xi + \eta)[S(\xi) + S(\eta)] = \alpha(\xi)S(\xi) + \alpha(\eta)S(\eta)$$

from which $\alpha(\xi) = \alpha(\eta)$. Applied to $i\xi, \eta$ we learn $\alpha(\xi) = \alpha(i\xi)$. On the other hand,

$$-S(\xi) = S(-\xi) = \alpha(i\xi)S(i\xi) = \alpha(i\xi)\alpha(\xi)S(\xi),$$

whence $\alpha(\xi)^2 = -1$. By continuity either $\alpha \equiv i$ or $\alpha \equiv -i$, which proves that $S$ is linear or $S$ is antilinear. \hfill \square

The second proof leans on complex geometry.

**Lemma 13.** An isometry $\phi: \mathbb{P}^\mathcal{H} \to \mathbb{P}^\mathcal{H}$ is either holomorphic or antiholomorphic.

**Proof.** Let $I: T\mathbb{P}^\mathcal{H} \to T\mathbb{P}^\mathcal{H}$ be the (almost) complex structure. Then $I$ is parallel with respect to the Levi-Civita covariant derivative, since $\mathbb{P}^\mathcal{H}$ is Kähler, and so therefore is $\phi^*I$. We claim any parallel almost complex structure $J$ equals $\pm I$; the lemma follows immediately.

If $J$ is parallel, then it commutes with the Riemann curvature tensor $R$. Compute at $L \in \mathbb{P}^\mathcal{H}$ and identify $T_L\mathbb{P}^\mathcal{H} \approx L^\perp$, as in (6). Then if $\xi, \eta \in L^\perp$ and $(\xi, \eta) = 0$, since $\mathbb{P}(L \oplus \mathbb{C}\cdot \xi \oplus \mathbb{C}\cdot \eta) \subset \mathbb{P}^\mathcal{H}$ is totally geodesic and has constant holomorphic sectional curvature one [KN, §IX.7], we compute

$$R(\xi, I\xi)\xi = -|\xi|^2 I\xi,$$

$$R(\xi, I\xi)\eta = -\frac{1}{2} |\xi|^2 I\eta.$$

It follows that $J$ preserves every complex line $K = \mathbb{C}\cdot \xi \subset L^\perp$ and commutes with $I$ on $K$. Therefore, $J = \pm I$ on $K$. By continuity, the sign is independent of $K$ and $L$. \hfill \square
**Second Proof of Theorem 8.** First, recall that if $U$ is finite dimensional, then every holomorphic symmetry of $\mathbb{P}U$ is linear. The proof is as follows. Let $\mathcal{L} \to \mathbb{P}U$ be the canonical holomorphic line bundle whose fiber at $L \in \mathbb{P}U$ is $L$. A holomorphic line bundle on $\mathbb{P}U$ is determined by its Chern class, so $\phi^* \mathcal{L} \cong \mathcal{L}$. Fix an isomorphism; it is unique up to scale. There is an induced linear map on the space $H^0(\mathbb{P}U; \mathcal{L}^*) \cong U^*$ of global holomorphic sections:

$$
\phi^* : H^0(\mathbb{P}U; \mathcal{L}^*) \longrightarrow H^0(\mathbb{P}U; \phi^* \mathcal{L}^*) \cong H^0(\mathbb{P}U; \mathcal{L}^*).
$$

The transpose $\hat{\phi}$ of (14) is the desired linear lift of $\phi$.

Let $\phi : \mathbb{P}H \to \mathbb{P}H$ be an isometry. After composition with an element of $G(\mathcal{H})$ we may, by Lemma 13, assume $\phi$ is holomorphic and fixes some $L \in \mathbb{P}H$. Let $U \subset \mathcal{H}$ be a finite dimensional subspace containing $L$. Then the pullback of $\mathcal{L}_\mathcal{H} \to \mathbb{P}H$ to $\phi^* \mathcal{L}_\mathcal{H} |_{\mathbb{P}U} \to \mathbb{P}U$ has degree one, so is isomorphic to $\mathcal{L}_U \to \mathbb{P}U$, and there is a unique isomorphism which is the identity on the fiber over $L$. A functional $\alpha \in H^* \mathcal{H}$ restricts to a holomorphic section of $\phi^* \mathcal{L}_\mathcal{H} |_{\mathbb{P}U} \to \mathbb{P}U$, so by composition with the isomorphism $\phi^* \mathcal{L}_\mathcal{H} |_{\mathbb{P}U} \cong \mathcal{L}_U$ to an element of $U^*$. The resulting map $\hat{\alpha}^* \to U^*$ is linear, and its transpose $\hat{\phi} : U \to \mathcal{H}$ is the identity on $L$. Let $U$ run over all finite dimensional subspaces of $\mathcal{H}$ to define $\hat{\phi} : \mathcal{H} \to \mathcal{H}$. The uniqueness of the isomorphism $\phi^* \mathcal{L}_\mathcal{H} |_{\mathbb{P}U} \cong \mathcal{L}_U$ implies that $\hat{\phi}$ is well-defined and a linear lift of $\phi$. It is unitary since $\phi$ is an isometry. 

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