Construction of Anti-Cyclotomic Euler Systems of Abelian Varieties Associated to \( X_1(N) \)

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Abstract. Let \( K \) be an imaginary quadratic field, \( N \) be a positive integer, \( f(z) \) be a newform of level \( \Gamma_1(N) \), and \( A_f \) be the abelian variety associated to \( f \). For each \( \tau \in K \) (\( \Im \tau > 0 \)), we construct a certain point \( P_\tau \) on \( A_f \) defined over an extended ring class field of \( K \) of level \( N \). Our construction generalizes Birch’s construction of the Heegner points to the abelian varieties associated to modular forms of level \( \Gamma_1(N) \) and nontrivial character. Then, we show that \( P_\tau \)'s satisfy the distribution and congruence relations of an Euler system, which implies that it should be possible to apply the Euler system techniques to them to show a relation between the non-torsionness of \( P_\tau \) and the rank of \( A_f(K) \).

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1. INTRODUCTION

In this paper, we present the construction of certain points on the modular curve \( X_1(N) \) (and by extension, on the abelian varieties over \( \mathbb{Q} \) given by irreducible quotients of \( J_1(N) \)). We will argue that our points generalize Birch’s Heegner points (\cite{2}) (which are defined on \( X_0(N) \)) in the sense that

\begin{itemize}
  \item 2010 Mathematics Subject Classification. Primary: 11G, Secondary: 14G05, 14G35.
  \item Key words and phrases. Euler systems, rational points of abelian varieties associated to modular forms.
\end{itemize}
(like Birch’s construction) modular functions and class field theory play an integral role in the construction, and show that they satisfy the conditions (the distribution and congruence relations) of an Euler system.

First, we give a brief description of Birch’s construction of the Heegner points. As is customary, we let \( \mathbb{H} \) denote the upper-half plane, let \( Y_0(N) \) (resp. \( Y_1(N) \)) denote \( \Gamma_0(N) \backslash \mathbb{H} \) (resp. \( \Gamma_1(N) \backslash \mathbb{H} \)), and \( X_0(N) \) (resp. \( X_1(N) \)) denote its compactification by the addition of cusps. Let \( j(\tau) \) (\( \tau \in \mathbb{H} \)) denote the modular elliptic function given by the \( j \)-invariant of \( \Lambda_\tau = (1, \tau) = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau \). It is well-known that \( (j(\tau), j(N\tau)) \) satisfies a certain polynomial equation \( P_N(X, Y) = 0 \), which gives an affine model over \( \mathbb{Q} \) of \( Y_0(N) \). If \( K \) is an imaginary quadratic field, and \( \tau \in K \cap \mathbb{H} \), then \( j(\tau) \) generates a certain ring class field extension of \( K \) by class field theory. Birch noted that for a carefully chosen \( \tau \), \( (j(\tau), j(N\tau)) \) is a point on the affine model \( Y_0(N)/\mathbb{Q} \) over the ring class field (thus can be considered as a point on \( X_0(N)/\mathbb{Q} \)), which he called a Heegner point \( \{2\} \). A Heegner point can be also considered as a point on an elliptic curve \( E \) over \( \mathbb{Q} \) of conductor \( N \), as we will explain shortly.

On the other hand, the authors wanted to find an explicit way to construct a point on \( X_1(N) \) also defined over a certain ring class field extension of \( K \). Noting the role played by \( j(\tau) \) in Birch’s construction, we looked for modular functions that can play a similar role.

In \( [1] \), Baaziz constructed modular functions \( b(\tau), c(\tau) \) of level \( \Gamma_1(N) \) (Section \( [2] \)), which are rational functions of the Weierstrass functions \( \wp(\cdot; \Lambda_\tau), \wp'(\cdot; \Lambda_\tau) \), and generate the function field of \( X_1(N) \). Jeon, Kim, and Lee noted \( [5] \) that \( (b(\tau), c(\tau)) \) gives an affine model (over \( \mathbb{Q} \)) of \( Y_1(N) \). The modular functions \( b(\tau), c(\tau) \) seemed ideal for our purpose.

Our first goal is to define a point analogous to the Heegner points: We define \( P_\tau = (b(\tau), c(\tau)) \in Y_1(N)(\subset X_1(N)) \) for any \( \tau \in K, \Im \tau > 0 \).

Secondly, we find a number field over which \( P_\tau \) is defined. For an order \( \mathcal{O} \) of \( K \), let \( L_{\mathcal{O}, N} \) be the extended ring class field of level \( N \) associated to \( \mathcal{O} \) (see Section \( [2.2] \)). We show that if \( \mathcal{O} \) acts on \( \Lambda_\tau \), then \( P_\tau \in X_1(N)(L_{\mathcal{O}, N}) \) (Corollary \( [2.3] \)).

Thirdly, we show that \( P_\tau \) satisfies the distribution and congruence relations of Euler systems.

Let \( f(z) = \sum_{n=-\infty}^{\infty} a_n(f)q^n \) \((a_1 = 1)\) be a newform of level \( \Gamma_1(N) \) with character \( \epsilon \) (modulo \( N \)), \( A_f \) be the abelian variety given by the quotient of \( J_1(N) \) divided by the ideal of the Hecke algebra generated by \( T_l - a_l(f) \) \((l \nmid N)\), \( U_l - a_l(f) \) \((l \mid N)\), \( \langle l \rangle - \epsilon(l) \) \( ((l, N) = 1) \) where \( l \) runs over all primes, and \( \mu_f : X_1(N) \to J_1(N) \to A_f \) be a modular parametrization (where the map from \( X_1(N) \) to \( J_1(N) \) is given by \( P \mapsto (P - (\infty)) \)). We let \( P_\tau \) also denote \( \mu_f(P_\tau) \in A_f \) by abuse of notation.

Suppose \( K = \mathbb{Q}(\sqrt{D}) \) for some square-free negative integer \( D \). Fix \( \tau_K = \sqrt{D} \) if \( D \not\equiv 1 \pmod{4} \), and \( \frac{\sqrt{D} + 1}{2} \) if \( D \equiv 1 \pmod{4} \). For each positive integer \( c \) prime to \( N \), let
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\[
tau' = \frac{a + \tau_K}{c}
\]

for an integer $a \in \mathbb{Z}$. Then, as mentioned above, $P_{\tau'}$ is defined over $L_{O_c,N}$ where $O_c = \mathbb{Z} + cO_K$.

Suppose $p$ is a prime number prime to $N \cdot \text{disc}(K/\mathbb{Q})$, and let $a_p(f)$ be the $p$-th Fourier coefficient of the $q$-expansion of $f$.

If $(p, c) = 1$, $p \equiv 1 \pmod{N}$, and $p$ is inert over $K/\mathbb{Q}$, then

\[\text{Tr}_{L_{O_{cp},N}/L_{O_c,N}} P_{\tau'}/p = a_p(f)P_{\tau'}.\]

(1) (See Theorem 4.2)

On the other hand, if $p | c$, then

\[\text{Tr}_{L_{O_{cp},N}/L_{O_c,N}} P_{\tau'}/p = a_p(f)P_{\tau'} - \epsilon(p)P_{p\tau'}.\]

(2) (See Theorem 4.4)

Now, suppose $p$ is a prime that is inert over $K/\mathbb{Q}$ and $p \equiv 1 \pmod{N}$, and $(p, c) = 1$. Let $\lambda$ be any prime of $L_{O_c,N}$ lying above $p$, $\lambda'$ be any prime of $L_{O_{cp},N}$ lying above $\lambda$, and $\text{red}_{\lambda}$ and $\text{red}_{\lambda'}$ be reduction maps onto the special fiber of the Néron model (over $\mathbb{Z}_p$) of $A_f$. Then, we have

\[(p + 1) \text{red}_{\lambda'} P_{\tau'}/p = (\text{Frob}_p + p \cdot \epsilon(p) \cdot \text{Frob}_p^{-1}) \text{red}_{\lambda} P_{\tau'}\]
\[= a_p(f) \cdot \text{red}_{\lambda} P_{\tau'}\]

(3) (see Theorem 4.6)

Now, let’s compare them with the conditions of Kolyvagin’s Euler system of the Heegner points ([9] Sections 1, 3). Let $C_\tau$ denote Birch’s Heegner point $(j(\tau), j(N\tau)) \in X_0(N)$. Suppose $f(z)$ is a newform of level $\Gamma_0(N)$ and $A_f$ is the abelian variety associated to $f(z)$ as in the case of $X_1(N)$ (the most prominent case being an elliptic curve over $\mathbb{Q}$). Again, we fix a modular parametrization map $\mu_f : X_0(N) \to A_f$ defined over $\mathbb{Q}$ which satisfies $\mu_f(\infty) = 0$. By abuse of notation, we let $C_\tau$ denote $\mu_f(C_\tau) \in A_f$ as well. By Kolyvagin ([7], [8]), for each $n \in \mathbb{Z}(n > 0)$ we can choose an appropriate $\tau_n \in K \cap \mathbb{H}$ so that $C_{\tau_n}$ is defined over the ring class field $L_{O_n}$ of the order $O_n = \mathbb{Z} + nO_K$, and for a prime $l$ with $(l, N) = 1$, if $l \nmid n$ and $l$ is inert over $K/\mathbb{Q}$, we have the distribution relation

\[\text{Tr}_{L_{O_{nl}/L_{O_n}}} C_{\tau_{nl}} = a_l(f)C_{\tau_n}\]

(4) ([4] Proposition 1). Although, it does not appear in Kolyvagin’s work, we also have that if $l | n$,

\[\text{Tr}_{L_{O_{nl}/L_{O_n}}} C_{\tau_{nl}} = a_l(f)C_{\tau_n} - C_{\tau_n/l}\]

(5)
(see the proof of [13] Proposition 6.1 although the readers should note that Rubin assumes \( a_l(f) = 0 \).

Also, where \( l \) does not divide \( \text{disc}(K/Q) \), \( v \) is any prime of \( K(1) \) above \( l \), and \( w \) is any prime of \( L_{O_1} \) above \( v \), we have the congruence relation

\[
\text{red}_w(C_{\tau l}) = \text{Frob}_l(\text{red}_v(C_{\tau 1})) \tag{6}
\]

(see [7] Proposition 6, and [8] Proposition 1). (1) is clearly analogous to (4) with the extra condition \( p \equiv 1 \pmod{N} \), which we believe will not make much difference in practice. (2) is also clearly analogous to (5). The appearance of \( \varepsilon(p) \) can be easily explained by the fact that modular forms of level \( \Gamma_0(N) \) have a trivial character. (2) should be what Rubin calls the distribution relation in the \( p \)-direction ([17] Remark 2.1.5), and (as Rubin points out) we believe that it can replace the congruence relations in the Euler system techniques. Also it should be noted that (2) indicates a natural connection with Iwasawa Theory.

The main goal of an Euler system is to obtain a sharp bound for the ranks of \( A_f \). For example, Kolyvagin showed that if \( f \) is a newform of level \( \Gamma_0(N) \) and \( N_{K(1)/K}C_1(\in A_f(K)) \) is not torsion, then \( \text{rank} A_f(K) = 1 \). We believe that we can apply the techniques of Euler systems to \( \{ P_{(a+\tau)/c} \} \), and obtain a similar result for \( A_f(K) \) where \( f \) is a newform of level \( \Gamma_1(N) \) (and a more general result in the direction of Iwasawa Theory), and we are hopeful that such a result will be in our subsequent publication.

**Remark 1.1.** There are also Kato’s Euler systems ([6]) defined on \( J(N) \). We note that his Euler systems are “the Euler systems over the cyclotomic fields” whereas our Euler system (as well as the Euler system of the Heegner points) are “the Euler systems over anti-cyclotomic fields.” They are different in the definition, construction, and application.

2. Preliminaries

2.1. Modular functions \( b(\tau) \) and \( c(\tau) \).

Let \( \Gamma = SL_2(\mathbb{Z}) \) be the full modular group, and for any \( N \geq 1 \), \( \Gamma(N) \), \( \Gamma_1(N) \), and \( \Gamma_0(N) \) be the standard congruence groups. Let \( Y_1(N)/\mathbb{Q} \) be the affine curve over \( \mathbb{Q} \) of the moduli schemes of the isomorphism classes of elliptic curves \( E \) with an \( N \)-torsion point. As well-known, \( Y_1(N)_{\mathbb{C}} \) is (isomorphic to) \( (Y_1(N)/\mathbb{Q} \otimes \mathbb{C})^{an} \).

More explicitly, this isomorphism is given by the following: Let \( \Lambda_\tau = (\tau, 1) \) be the lattice in \( \mathbb{C} \) with basis \( \tau \) and 1. Then, the above-mentioned isomorphism (of analytic curves between \( Y_1(N)_{\mathbb{C}} \) and \( (Y_1(N)/\mathbb{Q} \otimes \mathbb{C})^{an} \)) is given by

\[
\tau \mapsto \left( \mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau \right).
\]
The Tate normal form of an elliptic curve with point \( P = (0, 0) \) is as follows:

\[
E = E(b, c) : Y^2 + (1-c)XY - bY = X^3 - bX^2,
\]

and this is nonsingular if and only if \( b \neq 0 \). On the curve \( E(b, c) \) we have the following by the chord-tangent method:

\[
\begin{align*}
P &= (0, 0), \\
2P &= (b, bc), \\
3P &= (c, b-c), \\
4P &= \left( \frac{b(b-c)}{c^2}, -\frac{b^2(b-c-c^2)}{c^3} \right), \\
5P &= \left( -\frac{bc(b-c-c^2)}{(b-c)^2}, \frac{bc^2(b^2-bc-c^3)}{(b-c)^3} \right), \\
6P &= \left( \frac{(b-c)(b^2-bc-c^3)}{(b-c-c^2)^2}, \frac{c(2b^2-3bc-bc^2+c^2)(b-c)^2}{(b-c-c^2)^3} \right).
\end{align*}
\]

In fact, the condition \( NP = O \) in \( E(b, c) \) gives a defining equation for \( X_1(N) \). For example, \( 11P = O \) implies \( 5P = -6P \), so

\[
x_{5P} = x_{-6P} = x_{6P},
\]

where \( x_{nP} \) denotes the \( x \)-coordinate of the \( n \)-multiple \( nP \) of \( P \). Eq. (7) implies that

\[
-\frac{bc(b-c-c^2)}{(b-c)^2} = \frac{(b-c)(b^2-bc-c^3)}{(b-c-c^2)^2}.
\]

Without loss of generality, the cases \( b = c \) and \( b = c + c^2 \) may be excluded. Then Eq. (8) becomes as follows:

\[-b^2c^3 - 6bc^5 + 3b^3c^2 + 9b^2c^4 - 3bc^6 - 3b^4c - 4b^3c^3 + 3b^2c^5 - bc^7 + c^6 + b^5 = 0,
\]

which is one of the equation \( X_1(11) \) called the raw form of \( X_1(11) \). By the coordinate changes \( b = (1-x)xy(1+xy) \) and \( c = (1-x)xy \), we get the following equation:

\[
f(x, y) := y^2 + (x^2+1)y + x = 0.
\]

Now we note that

\[
\left( \mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau \right) = \left( y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \left( \varphi \left( \frac{1}{N}; \Lambda_\tau \right), \varphi' \left( \frac{1}{N}; \Lambda_\tau \right) \right) \right) = \left( y^2 + (1-c(\tau))xy - b(\tau)y = x^3 - b(\tau)x^2, (0, 0) \right),
\]
where $\wp(z; \Lambda_\tau)$ is the Weierstrass elliptic function of the period $\Lambda_\tau$. From \cite{1}, it follows that

$$b(\tau) = -\frac{\wp(\frac{1}{N}; \Lambda_\tau) - \wp(\frac{2}{N}; \Lambda_\tau)}{\wp'(\frac{1}{N}; \Lambda_\tau)^2}, \quad c(\tau) = -\frac{\wp'(\frac{2}{N}; \Lambda_\tau)}{\wp'(\frac{1}{N}; \Lambda_\tau)}$$

are modular functions on $\Gamma_1(N)$ and generate the function field of $X_1(N)$, where the derivative $\wp'$ is with respect to $z$.

2.2. Field of definitions of $b(\tau)$ and $c(\tau)$ for a CM-points $\tau$. Let $F_N$ be the extension of the function field $\mathbb{Q}(j(\tau))$ generated by the Fricke functions indexed by $r \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}^2$ (see \cite{10} Section 4), where $j(\tau)$ is the modular invariant function. By the theory of modular functions, it is known that $F_N$ is the set of all functions in $C(X(N))$ whose Fourier coefficients are in $\mathbb{Q}(\zeta_N)$, $F_1$ is simply $\mathbb{Q}(j(\tau))$, and

$$\text{Gal}(F_N/F_1) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \cong G_N \cdot \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$$

where

$$G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \mid d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

The functions $b(\tau), c(\tau)$ have their Fourier coefficients in $\mathbb{Q}(\zeta_N)$, and they are contained in $F_N$.

**Definition 2.1.**

$$P_\tau = (b(\tau), c(\tau)) \in X_1(N).$$

Let $O$ be an order of conductor $c$ in an imaginary quadratic field $K$. The ring class field of $O$, denoted by $L_O$, is determined via the Existence Theorem of class field theory \cite{3} Theorem 8.6] by the subgroup $P_{K,\mathbb{Z}}(c) \subset I_K(c)$ generated by principal ideals $\alpha O_K \in I_K(c)$ where $\alpha \equiv a \mod cO_K$ for some $a \in \mathbb{Z}$. Here $I_K(c)$ denotes the group of all fractional ideals relatively prime to $c$. This implies that

$$\text{Gal}(L_O/K) \cong I_K(c)/P_{K,\mathbb{Z}}(c) \cong C(O),$$

where $C(O)$ is the class group of $O$. Following \cite{3} we define

$$P_{K,\mathbb{Z},N}(cN) \subset I_K(cN)$$

to be the subgroup generated by the principal ideals $\alpha O_K \in I_K(cN)$ where $\alpha \in O_K$ satisfies

$$\alpha \equiv a \mod cNO_K$$

for some $a \in \mathbb{Z}$ with $a \equiv 1 \mod N$.

It then follows from the Existence Theorem that there exists an extension $L_{O,N}$ called the extended ring class field of level $N$, with Galois group

$$\text{Gal}(L_{O,N}/K) \cong I_K(cN)/P_{K,\mathbb{Z},N}(cN).$$
We note that $L_\mathcal{O,1} = L_\mathcal{O}$ and $L_\mathcal{O,N}$ is a Galois extension of $L_\mathcal{O}$. In particular, if $\mathcal{O} = \mathcal{O}_K$, then $L_{\mathcal{O,N}}$ is equal to the ray class field $K(N)$.

A point $\tau \in K \cap \mathbb{H}$ is a root of $ax^2 + bx + c$ where $a, b, c \in \mathbb{Z}$ are relatively prime with $a > 0$. Then the lattice $L_\tau = [1, \tau]$ is a proper ideal for the order $\mathcal{O} = [1, a\tau]$ (see [4, Theorem 7.7]). As a consequence of Shimura reciprocity we have the following theorem.

**Theorem 2.2.** [4, Theorem 15.16] Fix $\tau \in K \cap \mathbb{H}$ and $\mathcal{O}$ as above and assume that $f(\tau)$ is well-defined for a modular function $f \in \mathcal{F}_N$. Then $f(\tau) \in L_{\mathcal{O,N}}$.

Thus we have the following immediate corollary.

**Corollary 2.3.** Fix $\tau \in K \cap \mathbb{H}$ and $\mathcal{O}$ as above and assume that $b(\tau), c(\tau)$ are defined. Then $b(\tau), c(\tau) \in L_{\mathcal{O,N}}$ and therefore the point $P_\tau$ is defined over $L_{\mathcal{O,N}}$.

**Proof.** This immediately follows from Theorem 2.2 because $b(\tau), c(\tau) \in \mathcal{F}_N$. □

3. **The Main Theorem of Complex Multiplication, and the action of the Galois groups on $P_\tau$**

In this section, we apply Shimura’s theory of complex multiplication to $P_\tau$ to study the action of the Galois groups of extended ring class fields, and in particular, we find the field of definition of $P_\tau$ by other means.

As before, the lattice $(\alpha, \alpha')$ denotes $\mathbb{Z}\alpha + \mathbb{Z}\alpha'$. The following is from [18] Section 5.2 and Section 5.3.

Suppose $\Lambda$ is an arbitrary $\mathbb{Z}$-lattice in $K$. For each rational prime $p$, let $\mathcal{K}_p = K \otimes \mathbb{Q}_p$ and $\Lambda_p = \Lambda \otimes \mathbb{Z}_p$ (so that $\mathbb{A}_K = \prod_p \mathcal{K}_p$). It is worth noting that if $p$ splits completely over $K/\mathbb{Q}$ (so that $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}'$), then $\mathcal{K}_p = \mathcal{K}_p \times \mathcal{K}_p$.

For any $x \in \mathbb{A}_K^*$, we may speak of the $p$-component $x_p$ of $x$ belonging to $\mathcal{K}_p^*$. (In other words, if $p$ is inert, $x_p \in \mathcal{K}_p^*$; if $p$ splits completely, $x_p = (x_p, x_p') \in \mathcal{K}_p^* \times \mathcal{K}_p^*$, and if $p$ is ramified, $x_p \in \mathcal{K}_p^*$ for the unique prime $\mathfrak{p}$ of $\mathcal{O}_K$ above $p$.)

We observe that $x_p\Lambda_p$ is a $\mathbb{Z}_p$-lattice in $\mathcal{K}_p$. It is well-known that there exists a $\mathbb{Z}$-lattice $\Lambda'$ in $K$ such that $\Lambda_p' = x_p\Lambda_p$ for every $p$ ([18] page 116). Then, we define

$$x\Lambda \overset{\text{def}}{=} \Lambda'.$$

The isomorphism $x : K/\Lambda \xrightarrow{\sim} K/x\Lambda$ is given as follows: Since $\mathbb{Q}/\mathbb{Z} = \prod_p \mathbb{Q}_p/\mathbb{Z}_p$ canonically, we have the canonical decomposition $K/\Lambda = \prod_p K_p/\Lambda_p$. There is a well-defined isomorphism given by multiplication $x_p : K_p/\Lambda_p \overset{\times x_p}{\rightarrow}$.
For each prime $p$, we obtain an isomorphism $x : K / \Lambda \to K / x \Lambda$. In other words, $x : K / \Lambda \to K / x \Lambda$ is an isomorphism which makes the following diagram commutative for every prime $p$:

\[
\begin{array}{ccc}
K_p / x_p \Lambda_p & \xrightarrow{x_p} & K_p / x_p \Lambda_p \\
\downarrow & & \downarrow \\
K / \Lambda & \xrightarrow{x} & K / x \Lambda.
\end{array}
\]

The following is by Shimura, et. al.

**Theorem 3.1** (Main Theorem of Complex Multiplication, [18] Chapter 5 Theorem 5.4.) Recall that $K$ is an imaginary quadratic field. Let $\Lambda \subset K$ be a lattice in $K$, $\sigma$ be an automorphism of $\mathbb{C}$ invariant on $K$ (in other words, a $K$-automorphism of $\mathbb{C}$), $s$ be an element of $\mathbb{A}_K^\times$ so that $\sigma|_{K_{ab}} = [s, K]$, and $E$ be an elliptic curve so that there is an analytic isomorphism $\xi : \mathbb{C} / \Lambda \to E$. Then, there is an isomorphism $\xi' : \mathbb{C} / s^{-1} \Lambda \to E^\sigma$ so that the following is commutative:

\[
\begin{array}{ccc}
K / \Lambda & \xrightarrow{\xi} & E_{\text{tors}} \\
\downarrow \sigma & & \downarrow \\
K / s^{-1} \Lambda & \xrightarrow{\xi'} & E^\sigma_{\text{tors}}.
\end{array}
\]

($\xi'$ is uniquely determined by the above property once $\xi$ is fixed.)

Note that the precise definition of $s^{-1} \Lambda$ is given above.

As before, we let $P_\tau = (b(\tau), c(\tau))$ for $\tau \in K \cap \mathbb{H}$, and $\Lambda_\tau = (1, \tau)$.

For any lattice $\Lambda$ in $K$ we have the standard invariants

\[
G_{2n}(\Lambda) = \sum_{\omega \in \Lambda, \omega \neq 0} \omega^{-2n}, \quad g_2(\Lambda) = 60 \cdot G_4(\Lambda), \quad g_3(\Lambda) = 140 \cdot G_6(\Lambda).
\]

Suppose $E_\tau$ is the elliptic curve given by the Weierstrass equation

\[
y^2 = 4x^3 - g_2(\Lambda_\tau)x - g_3(\Lambda_\tau)
\]
so that there is an (analytic) isomorphism

\[
\xi : \mathbb{C} / \Lambda_\tau \to E_\tau \\
z \mapsto (\varphi(z; \Lambda_\tau), \varphi'(z; \Lambda_\tau)).
\]

As in Theorem 3.1, $\sigma$ is any automorphism of $\mathbb{C}$ invariant on $K$, and $s \in \mathbb{A}_K^\times$ satisfies $[s, K] = \sigma|_{K_{ab}}$. By Theorem 3.1, there is an (analytic) isomorphism $\xi' : \mathbb{C} / s^{-1} \Lambda_\tau \to E^\sigma_\tau$ such that the diagram in Theorem 3.1 commutes. As well-known, there is a lattice $\Lambda' = (\omega_1', \omega_2')$ in $K$ such that
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$$g_2(\Lambda_\tau)^\sigma = g_2(\Lambda'), \quad g_3(\Lambda_\tau)^\sigma = g_3(\Lambda'),$$

and

$$\xi'' : \mathbb{C}/\Lambda' \to \mathcal{E}_\tau, \quad z \mapsto (\wp(z; \Lambda'), \wp'(z; \Lambda'))$$

is an analytic isomorphism. Then, the composite map $\mathbb{C}/s^{-1}\Lambda_\tau \xrightarrow{\xi'} \mathcal{E}_\tau \xrightarrow{\xi''^{-1}} \mathbb{C}/\Lambda'$ is an analytic isomorphism, which is given by

$$\mathbb{C}/s^{-1}\Lambda_\tau \xrightarrow{\lambda \cdot} \mathbb{C}/\Lambda'$$

for some $\lambda \in \mathbb{C}^*$ (implying $\Lambda' = \lambda \cdot s^{-1}\Lambda_\tau$). In other words,

$$\xi'(z) = \xi''(\lambda \cdot z)$$

for $z \in \mathbb{C}/s^{-1}\Lambda_\tau$.

Therefore, for any $u \in K/\Lambda_\tau$,

(10) \hspace{1cm} \varphi(u; \Lambda_\tau)^\sigma = \varphi(\lambda \cdot s^{-1}u; \lambda \cdot s^{-1}\Lambda_\tau),

(11) \hspace{1cm} (\wp'(u; \Lambda_\tau))^\sigma = \wp'(\lambda \cdot s^{-1}u; \lambda \cdot s^{-1}\Lambda_\tau).

Suppose $N \cdot \mathcal{O}_K = \prod_{i=1}^k v_i^{n_i}$ ($n_i > 0$) for some primes $v_1, \ldots, v_k$ of $\mathcal{O}_K$.

Suppose an order $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ acts on $\Lambda_\tau$ for some $c \in \mathbb{Z} (c > 0)$ with $(c, N) = 1$.

Suppose $\sigma$ is identity on $L_{\mathcal{O}_c}$. Since $\text{Gal}(L_{\mathcal{O}_c}/K) \cong \mathbb{A}_K^*/K^* \prod_v \mathcal{O}_c^*$, $s \in \mathbb{A}_K^*$ satisfying $[s, K] = \sigma|_{K_{ab}}$ should be indeed $s \in K^* \prod_v \mathcal{O}_c^*$.

Write $s = \mu[\cdots, a_v, \cdots]_v$ where $\mu \in K^*$ and $a_v \in \mathcal{O}_c^*$ for each place $v$ of $K$. By the Chinese remainder theorem, there is $B \in \mathcal{O}_K$ so that $B \equiv (ca_v)^{-1} (\text{mod } v_i^{n_i})$ for every $i = 1, \ldots, k$. Let $C = cB$. Then, $C \in c\mathcal{O}_K \subset \mathcal{O}_c$, and naturally, $C \equiv a_v^{-1} (\text{mod } v_i^{n_i})$.

Then, we have the following formula for the action of $\sigma$ on $P_\tau$.

**Proposition 3.2.** For an $L_{\mathcal{O}_c}$-automorphism $\sigma$ of $\mathbb{C}$, and $C$ defined above, we have

$$b(\tau)^\sigma = -\frac{\left(\wp(C^{-1}_N; \Lambda_\tau) - \wp(C_2^{-1}_N; \Lambda_\tau)\right)^3}{\wp'(C^{-1}_N; \Lambda_\tau)^2},$$

$$c(\tau)^\sigma = -\frac{\wp'(C_2^{-1}_N; \Lambda_\tau)^2}{\wp'(C^{-1}_N; \Lambda_\tau)^2}.$$
Proof. Since we assume \( a_v \in \mathcal{O}^*_v \) for each \( v \), \( a_v^{-1}Z_l(1, \tau) = Z_l(1, \tau) \). Therefore, \([\cdots, a_v, \cdots]^{-1}\Lambda = \Lambda, \) and \( s^{-1}\Lambda = \mu^{-1}\Lambda \).

If \( l \) is a prime, and \( l \nmid N \), then \( \frac{1}{\Lambda} \in Z_l \), thus \( \frac{1}{\Lambda} \equiv 0 \) modulo \( Z_l(1, \tau) \). Since \( a_v \in \mathcal{O}^*_v \) for each \( v \), \( \prod_{v|l} a_v^{-1} \frac{1}{\Lambda} \in \Lambda \), and since \( C \subset \mathcal{O}, C \frac{1}{\Lambda} \in \Lambda \). In other words, \( a_v^{-1} \frac{1}{\Lambda} \equiv C \frac{1}{\Lambda} \) modulo \( \Lambda \).

If \( l|N \) and \( v|l \) for a prime \( l \), \( i.e. \), \( v = v_i \) for some \( i = 1, \cdots, k \), by construction \( C = a_v^{-1} (\text{mod } v_i^{n_i}) \), thus \( C = a_v^{-1} (\text{mod } NO_{K_v}) \). In other words, \( a_v^{-1} \frac{1}{\Lambda} \in \mathcal{O}_{K_v} \). Since \( \mathcal{O}_c \subset \Lambda \), \( a_v^{-1} \frac{1}{\Lambda} \in \Lambda \), \( \text{for each } v_i \). Furthermore, suppose \( \sigma(x) = (\text{mod } \Lambda) = 1 \)

Thus, we have \( b(\tau)^{\sigma} = \frac{\varphi(\lambda^{-1} \frac{1}{\Lambda}; \mu^{-1} \Lambda) - \varphi(\lambda^{-1} \frac{1}{\Lambda}; \mu^{-1} \Lambda)}{\varphi(\lambda^{-1} \frac{1}{\Lambda}; \mu^{-1} \Lambda)} \).

By noting \( \varphi(\lambda^{-1} \frac{1}{\Lambda}; \mu^{-1} \Lambda) = (\lambda^{-1} \frac{1}{\Lambda})^{-2} \varphi(z; \Lambda) \) and \( \varphi(\lambda^{-1} \frac{1}{\Lambda}; \mu^{-1} \Lambda) = (\lambda^{-1} \frac{1}{\Lambda})^{-3} \varphi(z; \Lambda) \), we obtain our claim.

The case for \( c(\tau)^{\sigma} \) is similar. \( \square \)

Furthermore, suppose \( \sigma \) is identity on \( L_{O_c,N} \) (assuming \( (c, N) = 1 \)). Since \( \text{Gal}(L_{O_c,N}/K) \cong \Lambda_K^*/K^* \prod_{v|N} \mathcal{O}^*_v \prod_{i=1}^k 1 + v_i^{n_i} \), we can choose \( s = \mu \cdot [\cdots, a_v, \cdots] \) so that \( a_v \equiv 1 \) (mod \( v_i^{n_i} \)) for \( i = 1, \cdots, k \), thus we can choose \( C \) which satisfies \( C \equiv 1 \) (mod \( N \)). Thus by Proposition 3.2, we have

\( b(\tau)^{\sigma} = b(\tau), \quad c(\tau)^{\sigma} = c(\tau) \).

Therefore, \( P_\tau \) is defined over \( L_{O_c,N} \).

4. Euler systems

4.1. Hecke operators.
Recall that we let $(\alpha, \alpha')$ denote $\mathbb{Z}\alpha + \mathbb{Z}\alpha'$. Also, we will let $\mathbb{Z}_p(\alpha, \alpha')$ denote $\mathbb{Z}_p \cdot \alpha + \mathbb{Z}_p \cdot \alpha'$. We recall that a point on $X_1(N)/\mathbb{Q} \otimes \mathbb{C}$ is given by $(E, P)$ where $E$ is a generalized elliptic curve over $\mathbb{C}$, and $P \in E[N]$. For simplicity, let $(E, P)$ also denote the divisor $(E, P) \in \text{Div}X_1(N)$. First, we note that for a prime $p$ ($p \nmid N$), $T_p$ acts on $\text{Div}X_1(N)$ by

$$T_p : (E, P) \mapsto \sum_C (E/C, P')$$

where $C$ runs over all cyclic subgroups of $E[p]$ of order $p$ (there are $p + 1$ such subgroups), and $P'$ is the image of $P$ under $E \rightarrow E/C$ (see [19] page 5).

In particular, when $(E, P) = (C/\Lambda_\tau, 1/N)$,

$$T_p(E, P) = p - 1 \sum_{j=0}^{p-1} (C/\Lambda_{z+1/p}, 1/N) + C/(1/p, \tau, 1/N).$$

As in Section 3, write

$$N = \prod_{i=1}^k v_i^{n_i}$$

for prime ideals $v_i$ of $\mathcal{O}_K$ and integers $n_i > 0$. We recall from Section 2.2 that when $\mathcal{O}$ is an order of $K$ of conductor $c$ (i.e., $\mathcal{O} = \mathbb{Z} + c\mathcal{O}_K$) with $(c, N) = 1$, $L_{\mathcal{O},N}$ is the extended ring class field of level $N$ which is characterized by

$$\text{Gal}(L_{\mathcal{O},N}/K) \cong I_K(cN)/P_{K,\mathbb{Z},N}(cN) \cong K^*/K^* \prod_{v|N} \mathcal{O}_v^* \prod_{i} (1 + v_i^{n_i}).$$

Let $f(z) = \sum a_n q^n$ be a (standardized) newform of level $\Gamma_1(N)$ and character $\epsilon \pmod{N}$, and $\mu_f : X_1(N) \rightarrow J_1(N) \rightarrow A_f$ be a modular parametrization map where the first map is given by $P \mapsto (P) - (\infty)$ for any $P \in X_1(N)$, and $A_f$ is the quotient of $J_1(N)$ given in the standard way associated to $f$.

**Notation 4.1.** Suppose $P_\tau \in X_1(N)(L)$ for some extension $L$ of $K$. We let $P_\tau$ also denote the image $\mu_f(P_\tau) \in A_f(L)$.

### 4.2. Distribution relations.

Recall that $K = \mathbb{Q}(\sqrt{D})$ where $D$ is a square-free negative integer. Let

$$\tau = \begin{cases} 
\frac{\sqrt{D}}{2} & \text{if } D \not\equiv 1 \pmod{4} \\
\frac{\sqrt{D} + 1}{2} & \text{if } D \equiv 1 \pmod{4}
\end{cases}$$

Suppose $c$ is a positive integer prime to $N$, and let
where $a, c \in \mathbb{Z}$. We note that $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ acts on $(1, \tau')$. As shown in Corollary 2.3 (and also in Section 3), $P_{\tau'} \in L_{\mathcal{O}_c, N}$.

4.2.1. Proof when $p$ is prime to the conductor.

**Theorem 4.2.** Suppose $p$ is a prime number, $(c, pN) = 1$, $p \equiv 1 \pmod{N}$, and $p$ is unramified and inert over $K/\mathbb{Q}$. Then,

$$\text{Tr}_{L_{\mathcal{O}_p, N}/L_{\mathcal{O}_c, N}} P_{\tau'/p} = a_p(f) P_{\tau'}.$$  

($P_{\tau'/p}$ can be replaced by $P_{(\tau' + j)/p}$ for any $j \in \mathbb{Z}$, or $P_{p\tau'}$).

**Proof.** Recall

$$T_p \left( \mathbb{C}/\Lambda_{\tau'}, \frac{1}{N} \right) = \sum_{j=0}^{p-1} \left( \mathbb{C}/\Lambda_{\frac{\tau' + j}{p}}^{p}, \frac{1}{N} \right) + \left( \mathbb{C}/(1, \tau'), \frac{1}{N} \right)$$

$$= \sum_{j=0}^{p-1} \left( \mathbb{C}/\Lambda_{\frac{\tau' + j}{p}}^{p}, \frac{1}{N} \right) + \left( \mathbb{C}/(1, p\tau'), \frac{p}{N} \right)$$

$$= \sum_{j=0}^{p-1} \left( \mathbb{C}/\Lambda_{\frac{\tau' + j}{p}}^{p}, \frac{1}{N} \right) + \left( \mathbb{C}/(1, p\tau'), \frac{1}{N} \right)$$

(the last equality because $p \equiv 1 \pmod{N}$).

For any prime $l(\neq p)$ and integer $j$,

$$\mathbb{Z}_l(1, \frac{\tau' + j}{p}) = \mathbb{Z}_l(1, \tau' + j) = \mathbb{Z}_l(1, \tau') = \mathbb{Z}_l(\frac{1}{p}, \tau').$$

On the other hand,

$$\mathbb{Z}_p(1, \frac{\tau' + j}{p}) = \mathbb{Z}_p(1, \frac{a + cj + \tau}{p})$$

$$= (a + cj + \tau) \cdot \mathbb{Z}_p\left( \frac{1}{a + cj + \tau}, \frac{1}{cp} \right)$$

$$= (a + cj + \tau) \cdot \mathbb{Z}_p\left( \frac{a + cj + \tau}{N_{K/\mathbb{Q}}(a + cj + \tau)}, \frac{1}{p} \right).$$

**Case 1.** $D \not\equiv 1 \pmod{4}$.

First, note

$$\frac{a + cj + \tau}{N_{K/\mathbb{Q}}(a + cj + \tau)} = \frac{a + cj - \tau}{(a + cj)^2 - D}.$$
Since $p$ is unramified and inert over $K/Q$, $(a + cj)^2 - D \not\equiv 0 \pmod{p}$ for any $a + cj$. Thus,

\[ Z_p\left(\frac{a + cj + \tau}{N_{K/Q}(a + cj + \tau)}, \frac{1}{p}\right) = Z_p(a + cj - \tau, \frac{1}{p}) = Z_p(\tau, \frac{1}{p}). \]

**Case 2.** $D \equiv 1 \pmod{4}$. 

\[ \frac{a + cj + \tau}{N_{K/Q}(a + cj + \tau)} = \frac{(a + cj) + (-\tau + 1)}{(a + cj)^2 + (a + cj) + \frac{1 - D}{4}}. \]

For a reason similar to the above, $(a + cj)^2 + (a + cj) + \frac{1 - D}{4} \not\equiv 0 \pmod{p}$ for any $a + cj$. Thus,

\[ Z_p\left(\frac{a + cj + \tau}{N_{K/Q}(a + cj + \tau)}, \frac{1}{p}\right) = Z_p((a + cj) + (-\tau + 1), \frac{1}{p}) = Z_p(\tau, \frac{1}{p}). \]

In either case,

\[ Z_p(1, \frac{\tau' + j}{p}) = (a + cj + \tau) \cdot Z_p\left(\frac{a + cj + \tau}{N_{K/Q}(a + cj + \tau)}, \frac{1}{p}\right) = (a + cj + \tau) \cdot Z_p(\tau, \frac{1}{p}), \]

and since $(c, p) = 1$, 

\[ Z_p(\tau, \frac{1}{p}) = Z_p\left(\frac{a + \tau}{c}, \frac{1}{p}\right) = Z_p(\tau', \frac{1}{p}). \]

For each $j \in \mathbb{Z}$, let 

\[ s_j = (1, 1, \cdots, a + cj + \tau, 1, \cdots) \in A_K^* \]

where $a + cj + \tau \in \mathcal{O}_{K_p}^*$ is the $p$-component. We note $s_j \in \prod_{c \in \mathbb{N}} \mathcal{O}_{c,v}^* \prod_i 1 + v_i^{n_i}$ (because all components of $s_j$ are 1 except for the $p$-component, and $\mathcal{O}_{c,p} = \mathcal{O}_{K_p}$ and $a + cj + \tau \in \mathcal{O}_{K_p}^*$). So far, we have shown 

\[ (1, \frac{\tau' + j}{p}) = s_j \cdot \left(\frac{1}{p}, \tau'\right). \]
Lemma 4.3. (a) \( s_j \frac{1}{N} = \frac{1}{N} \) where \( \frac{1}{N} \) on the left is considered an element of \( K/(\frac{1}{p}, \tau') \), and the one on the right an element of \( K/\Lambda_{\tau'+j} \).

(b) \( \{ a + cj + \tau \}_{j=0,1,\ldots,p-1} \cup \{1\} = \mathcal{O}_{cp,p}^*/\mathcal{O}_{cp,p}^* \cong \mathcal{O}_{K_p}^*/(\mathbb{Z}_p + p\mathcal{O}_{K_p})^* \).

(c) Consequently, \( \{ [s_j, K]_{L\mathcal{O}_{cp,N}} \} \cup \{\text{id}\} = \text{Gal}(L\mathcal{O}_{cp,N}/L\mathcal{O}_{c,N}) \).

Proof. (a) Simply because for every \( v|N \), \( v \)-entry of \( s_j \) is 1, and for every \( l \nmid N \), \( \frac{1}{N} \in \mathbb{Z}_l \).

(b) By brute force, show the elements of the lefthand side are all distinct modulo \( (\mathbb{Z}_p + p\mathcal{O}_{K,p})^* \). Since the righthand side has \( p + 1 \) elements, we obtain our claim.

(c) Because \( \text{Gal}(L\mathcal{O}_{cp,N}/L\mathcal{O}_{c,N}) \cong \mathcal{O}_{cp,p}^*/\mathcal{O}_{cp,p}^* \).

□

Now, choose \( \sigma_j \in \text{Aut}(\mathbb{C}) \) so that \( \sigma_j|_{\mathbb{K}_{ab}} = [s_j, K] \) for \( j = 0, 1, \ldots, p-1 \). Recall

\[ P_{\tau'} = (b(\tau'), c(\tau')) \]

where

\[ b(\tau') = \frac{(\varphi(\frac{1}{N}; \Lambda_{\tau'}) - \varphi(\frac{2}{N}; \Lambda_{\tau'}))^3}{\varphi'(\frac{1}{N}; \Lambda_{\tau'})^2}, \]

\[ c(\tau') = \frac{-\varphi'(\frac{2}{N}; \Lambda_{\tau'})}{\varphi'(\frac{1}{N}; \Lambda_{\tau'})}. \]

As noted earlier,

\[ T_p P_{\tau'} = \sum_{j=0}^{p-1} P_{\frac{\varphi_j+1}{p}} + P_{p\tau'} \]

as divisors on \( X_1(N) \).

Suppose \( E_{p\tau'} : y^2 = 4x^3 - g_2x - g_3 \) with \( g_2 = g_2((\frac{1}{p}, \tau')), g_3 = g_3((\frac{1}{p}, \tau')) \), and \( \varphi \) be the following analytic isomorphism:

\[ \mathbb{C}/(\frac{1}{p}, \tau') \rightarrow E_{p\tau'} \]

\[ z \mapsto (\varphi(z; (\frac{1}{p}, \tau')), \varphi'(z; (\frac{1}{p}, \tau'))). \]
By Theorem 3.1, there is an analytic isomorphism \( \psi : \mathbb{C}/s_j \cdot \left( \frac{1}{p}, \tau' \right) \to E_{pr'}^{\sigma_j^{-1}} \) so that the following diagram is commutative:

\[
\begin{array}{ccc}
K/\left( \frac{1}{p}, \tau' \right) & \xrightarrow{\psi} & (E_{pr'})_{\text{tor}} \\
\times s_j \downarrow & & \downarrow \sigma_j^{-1} \\
K/s_j \cdot \left( \frac{1}{p}, \tau' \right) & \xrightarrow{\psi} & (E_{pr'})_{\text{tor}}^{\sigma_j^{-1}}
\end{array}
\]

Similar to the argument in Section 3, there is some \( x \in \mathbb{C}^* \) so that \( \psi \) is the composite map

\[
\psi : \mathbb{C}/s_j \cdot \left( \frac{1}{p}, \tau' \right) \xrightarrow{x \cdot \psi} \mathbb{C} \cdot s_j \cdot \left( \frac{1}{p}, \tau' \right) \xrightarrow{(\psi, \psi')} E_{pr'}^{\sigma_j^{-1}}.
\]

Recall \( s_j \cdot \left( \frac{1}{p}, \tau' \right) = \Lambda_{p^{-1}} \) (see the discussion before Lemma 4.3), and \( s_j \cdot \frac{1}{N} = \frac{1}{N} \) (Lemma 4.3 (a)). By substituting these into \( b(\cdot) \) and \( c(\cdot) \), we have

\[
b\left( \frac{\tau' + j}{p} \right) = -\frac{(\varphi(s_j \cdot \frac{1}{N} ; x \cdot s_j \cdot \left( \frac{1}{p}, \tau' \right)) - \varphi(s_j \cdot \frac{2}{N} ; x \cdot s_j \cdot \left( \frac{1}{p}, \tau' \right))^3}{\varphi'(s_j \cdot \frac{1}{N} ; s_j \cdot \left( \frac{1}{p}, \tau' \right))^2}
\]

which is equal to

\[
-\frac{(\varphi(x \cdot s_j \cdot \frac{1}{N} ; x \cdot s_j \cdot \left( \frac{1}{p}, \tau' \right)) - \varphi(x \cdot s_j \cdot \frac{2}{N} ; x \cdot s_j \cdot \left( \frac{1}{p}, \tau' \right))^3}{\varphi'(x \cdot s_j \cdot \frac{1}{N} ; x \cdot s_j \cdot \left( \frac{1}{p}, \tau' \right))^2}
\]

because \( \varphi(xz; x\Lambda) = x^{-2}\varphi(z; \Lambda) \), \( \varphi'(xz; x\Lambda) = x^{-3}\varphi'(z; \Lambda) \). By the above commutative diagram, this is equal to

\[
\left[ \frac{(\varphi(\frac{1}{N} ; \left( \frac{1}{p}, \tau' \right)) - \varphi(\frac{2}{N} ; \left( \frac{1}{p}, \tau' \right))^3}{\varphi'(\frac{1}{N} ; \left( \frac{1}{p}, \tau' \right))^2} \right]^{\sigma_j^{-1}} = \left[ \frac{(\varphi(\frac{p}{N} ; \Lambda_{pr'}) - \varphi(\frac{2p}{N} ; \Lambda_{pr'}))^3}{\varphi'(\frac{p}{N} ; \Lambda_{pr'})^2} \right]^{\sigma_j^{-1}}
\]

\[
= b(pr')^{\sigma_j^{-1}}.
\]

(The last equality is because \( p \equiv 1 \pmod{N} \).)

Similarly, \( c\left( \frac{\tau' + j}{p} \right) = c(pr')^{\sigma_j^{-1}} \).

Thus, \( P_{pr'}^{\sigma_j^{-1}} = P_{pr'}^{\sigma_j^{-1}} \), which shows
\[ T_p(P_{\tau'}) = \sum_{j=0}^{p-1} (P_{pp\tau'})^j + (P_{pp\tau'}) = \text{Tr}_{L_{O_{cp}},N}/L_{O_{c},N} (P_{pp\tau'}) \]

(as divisors) by Lemma 4.3 and by the assumption that \( \sigma_j |_{L_{O_{cp}},N} = [s_j, L_{O_{cp},N}/K] \). Since \( T_p \) acts as multiplication by \( a_p(f) \) on the abelian variety \( A_f \), and since \( \text{Gal}(L_{O_{cp},N}/L_{O_{c},N}) \) acts transitively on \( \{ P_{\tau'+j} \} \cup \{ P_{pp\tau'} \} \), we obtain Theorem 4.2.

\[ \square \]

4.2.2. Proof when \( p \) divides the conductor.

**Theorem 4.4.** Suppose \( p \) is a prime number such that \( p | c \) and \( p \nmid N \text{disc}(K/Q) \). Then,

\[ \text{Tr}_{L_{O_{cp},N}/L_{O_{c},N}} P_{\tau'/p} = a_p(f)P_{\tau'} - \epsilon(p)P_{pp\tau'}. \]

\( (P_{\tau'/p} \) can be replaced by \( P_{(\tau'+j)/p} \) for any \( j \in \mathbb{Z} \)) where \( \epsilon \) is the (Nebentypus) character of \( f \).

**Proof.** Similar to Theorem 4.2, we have the following equality of divisors:

\[ T_p \left( \mathbb{C}/\Lambda_{\tau'}, \frac{1}{N} \right) = \sum_{j=0}^{p-1} \left( \mathbb{C}/\Lambda_{\tau'+j/p} \cdot \frac{1}{N} \right) + \left( \mathbb{C}/(1, p\tau'), \frac{p}{N} \right) \]

\[ = \sum_{j=0}^{p-1} \left( \mathbb{C}/\Lambda_{\tau'+j/p} \cdot \frac{1}{N} \right) + \langle p \rangle \left( \mathbb{C}/(1, p\tau'), \frac{1}{N} \right) \]

where \( \langle \cdot \rangle \) is the diamond operator (for the action of \( \langle \cdot \rangle \), see [19] Section 2)

Similar to Theorem 4.2, for any prime \( l \neq p \) and integer \( j \),

\[ Z_l(1, \frac{\tau'+j}{p}) = Z_l(1, \tau') = Z_l(1, \frac{\tau'}{p}). \]

When \( l = p \), we note

\[ Z_p(1, \frac{\tau'+j}{p}) = Z_p(1, \frac{a + cj + \tau}{cp}). \]

Now, we have
We consider \(\frac{\tau' + j}{p}\) to be an element of \(K \otimes \mathbb{Q}_p\) (and in an appropriate context, the \(p\)-component of the adele \(\mathbb{A}_K^+\)). In other words, if \(p\) is inert, it is an element of \(K_p\), and if \(p\) splits completely so that \(p\mathcal{O}_K = pp\), an element of \(K_p \times K_{\overline{p}}\). Similarly, \(Z_p(1, \frac{\tau' + j}{p})\) is considered a lattice inside \(K \otimes \mathbb{Q}_p\).

**Case 1.** \(D \not\equiv 1 \pmod{4}\).

By simple computation,

\[
\frac{x + j}{p} = \frac{[(a + cj) + \tau]}{N_{K/Q}(a + \tau)}
\]

Noting \(a^2 - D \not\equiv 0 \pmod{p}\), it follows that

\[
\frac{\tau' + j}{p} \cdot Z_p(1, \frac{\tau'}{p}) = Z_p(1 + \frac{2ac}{a^2 - D} j - \frac{c^2}{a^2 - D} j\tau', \frac{\tau' + j}{p})
\]

\[
= Z_p(1 + \frac{2ac}{a^2 - D} j - \frac{c^2}{a^2 - D} j\tau' + \frac{c^2 p}{a^2 - D} \cdot (\frac{\tau' + j}{p}, \frac{\tau' + j}{p})
\]

\[
= Z_p(1 + \frac{2ac}{a^2 - D} j + \frac{c^2 p}{a^2 - D} \cdot (\frac{\tau' + j}{p})
\]

\[
= Z_p(1, \frac{\tau' + j}{p})
\]

(the last line because \(p|c\) and \(p \nmid a^2 - D\)).

Noting \(\frac{\tau' + j}{\tau'} = 1 + \frac{ac}{a^2 - D} j - \frac{c}{a^2 - D} j\tau\) as shown above, what we have shown is equivalent to
\[(1 + \frac{ac}{a^2 - D} j - \frac{c}{a^2 - D} j \tau) Z_p(1, \frac{\tau'}{p}) = Z_p(1, \frac{\tau' + j}{p}).\]

**Case 2.** \(D \equiv 1 \mod{4}.

Note

\[N_{K/Q}(a + \tau) = a^2 + a + \frac{1 - D}{4} \not\equiv 0 \mod{p}.

\[(a + cj) + \tau | a + \tau| = a(a + cj) + \frac{1 - D}{4} + a \tau + (a + cj) \tau
= a^2 + acj + \frac{1 - D}{4} + a \tau + (a + cj)(1 - \tau)
= (a^2 + acj + \frac{1 - D}{4} + a + cj) - cj \tau

So, similar to Case 1,

\[\frac{\tau' + j}{\tau'} \cdot Z_p(1, \frac{\tau'}{p}) = Z_p(1 + \frac{2acj + cj}{a^2 + a + \frac{1 - D}{4}} - \frac{c^2 j}{a^2 + a + \frac{1 - D}{4}} \tau', \frac{\tau' + j}{p})
= Z_p(1 + \frac{2acj + cj + c^2 j^2}{a^2 + a + \frac{1 - D}{4}}, \frac{\tau' + j}{p})
= Z_p(1, \frac{\tau' + j}{p})

(the last line because \(\frac{2acj + cj + c^2 j^2}{a^2 + a + \frac{1 - D}{4}} \equiv 0 \mod{p}\)).

Noting \(\frac{\tau' + j}{\tau'} = 1 + \frac{acj + cj}{a^2 + a + \frac{1 - D}{4}} - \frac{c j}{a^2 + a + \frac{1 - D}{4}} \tau\) as shown above, what we have shown is equivalent to
Euler Systems of $X_1(N)$

$$
\left(1 + \frac{acj + cj}{a^2 + a + \frac{1}{4}} - \frac{cj}{a^2 + a + \frac{1}{4}}\right) \cdot Z_p(1, \frac{\tau'}{p}) = Z_p(1, \frac{\tau' + j}{p})
$$

for each $j \in \mathbb{Z}$.

Let $s_j = (1, 1, \cdots, \frac{\tau' + j}{\tau'}, 1, \cdots) \in A^*_{K}$ for $j = 0, 1, \cdots, p - 1$ where $\frac{\tau' + j}{\tau'}$ is the $p$-component. We have shown that in both Case 1 and Case 2,

$$s_j \Lambda_{\tau'} = \Lambda_{\tau' + j}.$$

We note that in both Case 1 and Case 2,

$$\frac{\tau' + j}{\tau'} \in 1 + c(\mathcal{O}_K \otimes \mathbb{Z}_p) \subset (\mathcal{O}_c \otimes \mathbb{Z}_p)^*,$$

which implies $s_j \in \prod_{v \mid N} \mathcal{O}_c^* \prod_i 1 + v_i^{n_i}$.

**Lemma 4.5.** (a) $s_j \frac{1}{N} = \frac{1}{N}$ where $\frac{1}{N}$ on the left is in $K/\Lambda_{\tau'}$, and the one on the right in $K/\Lambda_{\tau' + j}$.

(b) $\left\{\frac{\tau' + j}{\tau'}\right\}_{j=0,1,\ldots,p-1} \cong \left((\mathcal{O}_c) \otimes \mathbb{Z}_p\right)^*/\left((\mathcal{O}_c) \otimes \mathbb{Z}_p\right)^*$.

(c) Consequently, $\left\{[s_j, K]|_{L_{\mathcal{O}_c, N}}\right\}_{j=0,1,\ldots,p-1} = \text{Gal}(L_{\mathcal{O}_c, N}/L_{\mathcal{O}_c, N})$.

**Proof.** Similar to Lemma 4.3 except for (b). As for (b), first suppose $p^n \parallel c$. We show that for any $A, B \in \mathbb{Z}_p$ ($(B, p) = 1$), $1 + Ap^n j + Bp^n j \tau$ for $j = 0, 1, \cdots, p - 1$ are all distinct modulo $\left((\mathcal{O}_c) \otimes \mathbb{Z}_p\right)^*$, and thus $\left\{1 + Ap^n j + Bp^n j \tau\right\}_{j=0,1,\ldots,p-1} = \left((\mathcal{O}_c) \otimes \mathbb{Z}_p\right)^*/\left((\mathcal{O}_c) \otimes \mathbb{Z}_p\right)^*$ by noting that both sides have the same number of elements. \hfill $\square$

Now, choose $\sigma_j \in \text{Aut}(\mathbb{C})$ so that $\sigma_j|_{K_{ab}} = [s_j, K]$ for $j = 0, 1, \cdots, p - 1$.

Similar to Theorem 4.2,

$$b\left(\frac{\tau' + j}{p}\right) = b\left(\frac{\tau'}{p}\right)^{\sigma_j^{-1}}$$

$$c\left(\frac{\tau' + j}{p}\right) = c\left(\frac{\tau'}{p}\right)^{\sigma_j^{-1}}$$

Thus, $P_j^{\sigma_j^{-1}} = P_{\tau' + j}$, which shows

$$T_p(P_{\tau'}) = \sum_{j=0}^{p-1} (P_j^{\sigma_j^{-1}} + (p)(P_{\tau'})) = \text{Tr}_{L_{\mathcal{O}_c, N}/L_{\mathcal{O}_c, N}}(P_j^{\sigma_j^{-1}}) + (p)(P_{\tau'})$$

Since $T_p$ acts as multiplication by $a_p(f)$ on the abelian variety $A_f$, $\langle p \rangle$ as multiplication by $\epsilon(p)$, and $\text{Gal}(L_{\mathcal{O}_{cp},N}/L_{\mathcal{O}_{c},N})$ acts transitively on $\{P_{\tau' + j}\}_{j=0,\ldots,p-1}$, we obtain Theorem 4.4.

4.3. Congruence Relations.

Where $\mathcal{L}$ is a local field of residue characteristic $p$ which is prime to $N$, $\nu$ is the maximal ideal of $\mathcal{O}_{\mathcal{L}}$, $\mathcal{F}$ is $\mathcal{O}_{\mathcal{L}}/\nu$, and $\mathcal{A}$ is the Néron model of $A_f$ over $\mathbb{Z}_p$ (which exists because $(p,N) = 1$), we note that there is the standard reduction map $\text{red}_\nu : A_f(\mathcal{L}) \to A_f/\mathcal{F}$. Also let $J/\mathcal{F}$ denote the fiber of the Néron model of $J_1(N)/\mathbb{Q}$ over $\mathbb{Z}_p$ (which exists because $(p,N) = 1$).

Recall that $\tau = \sqrt{D}$ if $D \not\equiv 1 \pmod{4}$, and $\tau = a + \frac{1}{2}$ if $D \equiv 1 \pmod{4}$, and $\tau' = \frac{a + \tau}{c}$ with $(c,N) = 1$.

Theorem 4.6. Suppose $p$ is a prime that is inert over $K/\mathbb{Q}$, $p \equiv 1 \pmod{N}$, and $(p,c) = 1$. Let $\lambda$ be any prime of $L_{\mathcal{O}_{c},N}$ lying above $p$, and $\lambda'$ be any prime of $L_{\mathcal{O}_{cp},N}$ lying above $\lambda$. Then, we have

$$(p + 1)\text{red}_\lambda P_{\tau'/p} = (\text{Frob}_p + p \cdot \epsilon(p) \cdot \text{Frob}_p^{-1})\text{red}_\lambda P_{\tau'} = a_p(f)\text{red}_\lambda P_{\tau'}.$$ 

Remark 4.7. As we will see in the proof, $P_{\tau'/p}$ can be replaced by $P_{\tau' + j}/p$ for any $j \in \mathbb{Z}$, or $P_{\tau'}/p$.

We state Theorem 4.6 with $\text{Frob}_p + p \cdot \epsilon(p) \cdot \text{Frob}_p^{-1}$ to make it appear similar to Kolyvagin’s congruence relation ((6) in Section 1), but in practice, the statement with $a_p(f)\text{red}_\lambda P_{\tau'}$ might be more suitable.

Proof. As we see in the proof of Theorem 4.2 (see (12)),

$T_p(P_{\tau'}) = \text{Tr}_{L_{\mathcal{O}_{cp},N}/L_{\mathcal{O}_{c},N}}(P_{\tau'})$.

As noted in Theorem 4.2, $P_{\tau'}/p$ can be replaced by $P_{\tau' + j}/p$ for any $j \in \mathbb{Z}$ because they are transitive under the action of $\text{Gal}(L_{\mathcal{O}_{cp},N}/L_{\mathcal{O}_{c},N})$. To be consistent with the notation in the theorem, we replace $P_{\tau'}/p$ with $P_{\tau'}/p$.

Our condition implies that $(p)$ splits completely over $L_{\mathcal{O}_{c},N}/K$, and $\lambda'$ is totally ramified over $L_{\mathcal{O}_{cp},N}/L_{\mathcal{O}_{c},N}$ (therefore $\mathcal{O}_{L_{\mathcal{O}_{cp},N}}/\lambda' \cong \mathcal{O}_{L_{\mathcal{O}_{c},N}}/\lambda \cong \mathcal{O}_{K}/(p)$). Thus, every $\sigma \in \text{Gal}(L_{\mathcal{O}_{cp},N}/L_{\mathcal{O}_{c},N})$ is the identity modulo $\lambda'$. Hence,

$\text{red}_\lambda T_p(P_{\tau'}) = (p + 1)\text{red}_\lambda P_{\tau'/p}$.

On the other hand, by the Eichler-Shimura relation,

$T_p = \text{Frob}_p + \langle p \rangle \text{Ver}_p$.
as action on $J_{/F_p}$. Here Frob$_p$ is the Frobenius and Ver$_p$ is the Verschiebung of the group scheme $J_{/F_p}$ (19 Section 4, (4.1)). We recall that $\langle p \rangle$ acts as multiplication by $\epsilon(p)$ as an action on $A_{/F_p}$.

By recalling that Ver$_p$ is characterized by Frob$_p$ Ver$_p = Ver_p$ Frob$_p = p$, we obtain

$$(p + 1) \text{red}_\chi P_{\tau' / p} = (\text{Frob}_p + p \cdot \epsilon(p) \cdot \text{Frob}_p^{-1}) \text{red}_\lambda P_{\tau'}.$$  

(We may consider Frob$_p$ as an automorphism of finite fields.) Thus, we obtain the first part of our equation.

Then, note the standard result Frob$_p + p \cdot \epsilon(p) \cdot \text{Frob}_p^{-1} = a_p(f)$ on $A_{/F_p}$.

(Or, we can simply argue that $T_p$ acts as multiplication by $a_p$ as an action on $A_{/F_p}$.) Thus, we also obtain the second part of the equation. □

In his original work, Kolyvagin does not make any use of the second distribution relation (5) (see Section 1), which in fact does not seem to appear in his work. This relation is what Rubin calls the distribution relation in the $p$-direction. Instead, he uses congruence relations.

Rubin points out (16, and 17 Remark 2.1.5, and also Section 4.8) that the congruence relations are often unnecessary, or can be derived from other distribution relations if “the Euler system satisfies distribution relations in the $p$-direction” (such as (2) in Section 1, which is Theorem 4.4).

On the other hand, suppose one insists on using a congruence relation. Theorem 4.6 has a more restrictive condition that $p$ is inert over $K/Q$ and $p \equiv 1 \pmod{N}$ (whereas Kolyvagin’s congruence condition applies to all primes prime to the discriminant of $K/Q$). However, a careful reading of 14 Section 4 (which is another work that uses the congruence conditions) indicates that one does not need a congruence relation for every prime, and in fact, it is probably enough that it holds for inert primes. Adding the condition $p \equiv 1$ is probably fine, too.

Either way, we can claim that $\{P_{\tau'}\}$ satisfies enough relations to be an Euler system.

5. Appendix

Our description of the action of the Galois groups on $P_{\tau}$ in Section 3 is not necessarily efficient for practical computation. In this section, we apply Koo, Shin, and Yoon’s work (10 Section 4, 11 Section 2) on the action of the Galois groups of class field extensions on the singular values of modular functions to obtain formulas which may be more readily useful in practice.

Following 10, when $K$ is an imaginary quadratic field of discriminant $d_K$, let

$$\theta = \begin{cases} \sqrt{d_K}/2 & \text{if } d_K \equiv 0 \pmod{4} \\ (-1 + \sqrt{d_K})/2 & \text{if } d_K \equiv 1 \pmod{4}. \end{cases}$$
We let $\mathcal{F}_N$ be the extension of the function field $\mathbb{Q}(j(\tau))$ generated by the Fricke functions indexed by $r \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ (see [10] Section 4). By the theory of modular functions, it is known that $\mathcal{F}_N$ is the set of all functions in $\mathbb{C}(X(N))$ whose Fourier coefficients are in $\mathbb{Q}(\zeta_N)$, $\mathcal{F}_1$ is simply $\mathbb{Q}(j(\tau))$, and $\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$.

where

$$G_N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \mid d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$ 

We note $b(\tau), c(\tau) \in \mathcal{F}_N$.

The action of $\text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ on $\mathcal{F}_N$ can be made explicit through $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \cong G_N \cdot \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ as follows ([11] Section 2): Let $\min(\theta, Q) = x^2 + B\theta x + C\theta \in \mathbb{Z}[x]$, and let

$$W_{N,\theta} = \left\{ \begin{bmatrix} t - B\theta s \\ s \\ C\theta s \\ t \end{bmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : t, s \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

By [10] Proposition 4.1, we can identify $K(\mathcal{N})$ with the field $K(h(\theta) \mid h \in \mathcal{F}_N$ is defined and finite at $\theta$).

Koo, Shin, and Yoon identify $\text{Gal}(K(\mathcal{N})/H)$ with the image of $W_{N,\theta}$ by the following surjection:

**Proposition 5.1** ([10] Proposition 4.2).

$$W_{N,\theta} \to \text{Gal}(K(\mathcal{N})/H) \quad \alpha \mapsto (h(\theta) \mapsto h^\alpha(\theta))$$

If $d_K \leq -7$, then its kernel is $\{\pm I_2\}$. By applying this to $b(\tau)$ and $c(\tau)$, we can compute the action of $\text{Gal}(K(\mathcal{N})/H)$ on $P_{\tau}$. (The computational advantage is that we only need to know $W_{N,\theta}$.)
The action of $\text{Gal}(H/K)$ (and by extension, the action of $\text{Gal}(K(N)/K)$) is given as follows ([10] Proposition 4.3). Let $C(d_K)$ the form class group of discriminant $d_K$, which is the set of primitive positive definite quadratic forms $aX^2 + bXY + cY^2 \in \mathbb{Z}[X,Y]$ under a proper equivalence relation. (See [10] p.351 for an explicit classification of the elements of $C(d_K)$.) We note that $C(d_K) \cong \text{Gal}(H/K)$ ([4] Theorem 7.7).

For a reduced quadratic form $Q = aX^2 + bXY + cY^2$ of discriminant $d_K$, we set

$$\theta_Q = \left( -b + \sqrt{d_K} \right) / 2a,$$

and set $\beta_Q = (\beta_p)_p \in \prod_p \text{GL}_2(\mathbb{Z}_p)$ (where $p$ runs over all primes) for $\beta_p$ given in [10] p.351-352.

**Proposition 5.2** ([10] Proposition 4.3). Assume $d_K \leq -7$ and $N \geq 1$. (Note that by this assumption, $W_{N,\theta}/\{\pm I_2\} \cong \text{Gal}(K(N)/H)$.) Then, we have a bijective map

$$W_{N,\theta}/\{\pm I_2\} \times C(d_K) \to \text{Gal}(K(N)/K)$$

$$(\alpha, Q) \mapsto \left( h(\theta) \mapsto h^{\alpha\beta_Q(\theta_Q)} \right)$$

As [10] notes, there is $\beta \in \text{GL}_2^+(\mathbb{Q}) \cap \text{M}_2(\mathbb{Z})$ with $\beta \equiv \beta_p \pmod{NZ_p}$ for all primes $p$ dividing $N$, and the action of $\beta_Q$ is understood to be that of $\beta$.

The bijection in Proposition 5.2 is not a group isomorphism (but the map in Proposition 5.1 is a group homomorphism), but from the computational perspective, it should not matter.

This bijection gives a computational description of the action of $\text{Gal}(K(N)/K)$ on our functions $b(\tau), c(\tau)$ when $\tau = \theta$. [10] does not explicitly say much when $\tau$ is not $\theta$, but its authors told us that their result can be generalized to any $\tau \in K$. In such a case, the action will be that of $\text{Gal}(L_{O_n,N}/K)$ where $O_n$ is the order of $K$ acting on $\Lambda_\tau$, which is compatible with our work.

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