Counter-propagating solitons in microresonators

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Solitons occur in many physical systems when a nonlinearity compensates wave dispersion. Their recently demonstrated formation in microresonators has opened a new research direction for nonlinear optical physics. Soliton mode locking also endows frequency microcombs with the enhanced stability necessary for miniaturization of spectroscopy and frequency metrology systems. These microresonator solitons orbit around a closed waveguide path and produce a repetitive output pulse stream at a rate set by the roundtrip time. Here, counter-propagating solitons that simultaneously orbit in an opposing sense (clockwise/counter-clockwise) are studied. Despite sharing the same spatial mode family, their roundtrip times can be precisely and independently controlled. Furthermore, a state is possible in which both the relative optical phase and relative repetition rates of the distinct soliton streams are locked. This state allows a single resonator to produce dual-soliton frequency-comb streams with different repetition rates, but with a high relative coherence that is useful in both spectroscopy and laser ranging systems.

The recent demonstration of optical solitons in microresonators has opened a new chapter in nonlinear optical phenomena. These dissipative solitons use the Kerr nonlinearity to balance wave dispersion and to compensate cavity loss. The resulting dissipative Kerr solitons (DKSs) exhibit Raman-related phenomena and optical Cherenkov radiation, and can form ordered arrays called soliton crystals. Soliton mode locking also creates a new and very stable frequency microcomb with distinct advantages over earlier microcombs. Broadening of these combs by dispersive-wave generation within the microresonator enables offset frequency measurement for comb self-referencing. Also, the soliton repetition rate has excellent phase noise stability, while its spectral envelope is stable and reproducible, so the resulting microcombs are suitable for dual-comb spectroscopy and laser ranging systems.

Dissipative Kerr solitons orbit in closed optical paths within optical whispering-gallery microresonators. These paths support both clockwise (CW) and counter-clockwise (CCW) whispering gallery modes, which, in principle, could allow two soliton frequency combs to coexist in the same spatial mode family. In this Letter, counter-propagating (CP) solitons are generated by counter-pumping on a single microcavity resonance (Fig. 1a). These CP solitons have several properties. First, because the pump wave of each DKS comb is also a tooth of the corresponding frequency comb (that is, the pump is phase-coherent with the soliton), the tuning of two CP pumps also causes an offset in the optical frequency of the two soliton pulse streams. Moreover, the two pumps can be derived from a single seed laser by radiofrequency-rate tuning of optical modulators (Fig. 1b). The optical power of solitons in one direction is used to servo-lock the pump laser to a certain frequency detuning relative to the cavity mode, where it would eliminate the need for independent and mutually locked frequency combs.

To produce solitons the output from a continuous-wave fibre laser is amplified and split using a directional coupler to pump the CW and CCW modes of a microcavity resonance using a fibre taper coupler (experimental set-up in Fig. 1b). Two AOMs are used to control the pump power and frequencies in each pumping direction. The AOM frequency control allows precise tuning of the soliton repetition rates, as detailed in the following, while the

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amplitude of the AOM transmission is used during triggering of the solitons\(^2,3\). The residual transmitted pump power is filtered by a fibre Bragg grating filter (FBG). The CP solitons are stabilized indefinitely using the active capture technique, which servo-controls the pump frequency using the soliton power\(^1,5\). It is found that application of this locking technique to only one of the soliton pulse streams automatically locks the other pulse stream. The system can be controllably triggered and locked with a single or multiple solitons in each propagation direction.

Figure 1c presents measured optical spectra and autocorrelation traces (insets) for typical CW and CCW soliton streams corresponding to one soliton circulating in each direction. The CP solitons are typically several hundreds of femtoseconds in duration. The microresonator, a high-Q silica wedge design\(^4\) (3 mm diameter), provides a soliton roundtrip time of 46 ps (Fig. 1c, inset). The microresonator features anomalous dispersion at the pumping wavelength near a soliton roundtrip time of 46 ps (Fig. 1c, inset). The microresonator is engineered to produce minimal avoided mode crossings over the optical band of the solitons\(^2\). A zoom-in of the spectrum in Fig. 2a is provided in the upper panel of Fig. 2b. The zoom-in shows two strong central spectral peaks that differ by 60 kHz. These peaks give the fundamental repetition rates associated with the CW and CCW soliton streams. The weaker, non-central beats appearing in Fig. 2a and in the upper panel of Fig. 2b are inter-soliton beat frequencies between comb teeth belonging to different soliton combs. These beat frequencies are equally separated by the difference in the repetition rates (60 kHz). As an aside, the maxima at the extreme wings of the spectrum in Fig. 2a are caused by the mode-crossing distortion in the comb spectra seen in Fig. 1c near 1.542 nm. An interferogram showing the electrical time trace of the co-detected dual-soliton pulse streams is shown in Fig. 2c. This time trace can be understood as a stroboscopic interference of the respective soliton pulses on the detector. The strobing occurs at the rate difference (\(\Delta f = f_{\text{cw}} - f_{\text{ccw}}\)) of the two soliton streams, giving the repetitive signal a period of 16.5 \(\mu\)s. By varying the pump detuning \(\Delta v\) it is possible to observe tuning of the repetition rate difference \(\Delta f\) as shown in Fig. 2d. A theoretical fit, discussed in the Methods, is provided as a red line in the figure.

Near \(\Delta v = 0\) in Fig. 2d, locking to the same repetition rate (degenerate locking) is observed over a range of \(\Delta v\) around 150 kHz. The associated electrical zoom-in spectrum under this locked condition is shown in the lower panel of Fig. 2b. Importantly, nearly all of the weaker peaks that appear in Fig. 2a disappear as a result of this degenerate locking. This can be understood to result from the high relative temporal stability of the two pulse streams. In particular, under the locking condition, inter-soliton pulse mixing on the photodetector, which is guaranteed under conditions of unequal repetition rates, now requires strict spatial–temporal alignment of the two pulse streams at the detector. Consistent with this physical picture, the interferogram trace is observed to show no periodic strobing behaviour (Supplementary Fig. 3). This locking behaviour is believed to occur through backscattering of pump light, which can induce four-wave mixing on the soliton comb lines and subsequently induce locking. A study of this
mechanism is included in the Supplementary Section II. As an aside, the weak sidebands in the lower panel of Fig. 2b are believed to result from pump light from one direction interfering on the detector with comb lines of the soliton from the opposing direction.

In addition to degenerate locking, the soliton pulse streams are also observed to lock when their repetition rates are different. Figure 3a illustrates the principle of this locking mechanism, showing hypothetical soliton spectra for CW and CCW directions. A zoom-in of the higher-frequency portion of the spectra is shown, where the respective soliton spectral lines are superimposed next to shaded areas representing the cavity resonances. The mode index \( \mu = 0 \), which is by convention the optical pump, is also indicated. As required for DKS generation, this pump frequency and the other soliton comb teeth are red-detuned in frequency relative to their respective cavity resonances. At \( \mu = 0 \), the two pump lines are separated by the pump frequency difference, \( \Delta \nu \). Under conditions where these pump frequencies are well separated so that degenerate rate locking does not occur (Fig. 2d), the soliton with the more strongly red-detuned pump will feature a slightly lower repetition rate on account of the self-Raman effect (discussed above and in the Methods). Accordingly, the CW and CCW comb lines will shift in frequency so as to become more closely spaced as \( \mu \) decreases. For a certain negative value of \( \mu \), the CW and CCW comb lines will achieve closest spectral separation. In the illustration, this occurs at comb tooth \( \mu = \pi \) where CW and CCW comb lines have frequency separation \( \delta = \Delta \nu + r \Delta f \). Backscattering within the resonator will couple power between these nearly resonant lines. This power coupling is shown in the Methods to induce locking.
This result shows that pulse rates have a relative stability completely determined by the radiofrequency signal used to set the pump frequency offset, $\Delta v$. As a result, the beat signal between the CCW and CW solitons exhibits very high stability when the system is locked in this way. The above relation also shows that the locked CP solitons play the role of a frequency divisor of the pump frequency difference into the pulse-rate difference frequency, $\Delta f$. The phase noise of the rate difference is therefore $r^2$ lower than the phase noise of the relative pump signal,

$$S_{\Delta f} = \frac{1}{r^2} S_{\Delta v}$$

(2)

where $S_{\Delta f}$ and $S_{\Delta v}$ are the phase noise spectral density functions of the inter-soliton fundamental beat signal and the pump difference signal, respectively.

Figure 3b illustrates the effect of the locking condition on the electrical spectrum produced by photodetection of combined CCW and CW soliton streams. Under unlocked conditions, the electrical spectrum will feature two distinct spectra with spacing $\Delta f$. However, under locked conditions the difference in the frequency of the comb teeth at $\mu = 0$ (that is, optical pumps) is an integer multiple of the difference in the repetition rates. As result, the two electrical spectra merge to form a single spectrum.

Figure 3c shows a typical measured RF spectrum in the unlocked state. It is obtained by Fourier transforming the photodetected interferogram recorded over 1 s. Here, the beat between the CCW and CW comb teeth corresponding to the pumps is indicated and is very stable. However, all other spectral lines are noisier. This noise results from fluctuations of the absolute pump frequencies, which in turn induce fluctuations in the CW and CCW soliton repetition rates.

The resulting frequency noise is multiplied with each comb tooth index relative to the pump comb tooth. Note that the high stability of the pump line in Fig. 3c confirms the relative optical coherence of the solitons as a result of pumping from a common laser source.

In contrast to the unlocked case, the RF spectrum in the locked state (Fig. 3d) is a set of narrow, spectral lines with a 50 dB signal-to-noise ratio (SNR) at the 1 Hz resolution bandwidth (RBW). In this measurement, $\Delta v$ is set to be 1.5 MHz, which is 60 times $\Delta f = 25$ kHz. Locking is also observed at other $\Delta v$, including several of the points in Fig. 2d and in Fig. 2a. To illustrate the stability improvement that results from locking, Fig. 3e plots the average frequency spacing between neighbouring RF comb lines in Fig. 3c and Fig. 3d versus beat note number (relative to the pump). Locking results in a collapse to a sub-hertz stability. To test the frequency division model noted above (equations (1) and (2)) three inter-soliton beatnotes are shown in Fig. 3f. The fundamental beatnote (difference in repetition rates) has the narrowest linewidth, while beatnotes at increasing multiples of 25 kHz are wider. A more quantitative illustration of frequency division on noise is provided in Fig. 3g, where the phase noise of intersoliton beats is plotted versus the beatnote frequency at phase noise spectra offsets of 1 Hz and 10 Hz. The quadratic scaling predicted in equation (2) is apparent in the plots (solid lines). It is noted that the inferred linewidth for the lowest-order beatnote is 40 $\mu$Hz (assuming that it is limited by white frequency noise).

It is important to note that stability improvement in both the interferogram and its Fourier transform (Fig. 3d) are directly transferrable to achieving improved sensitivity of dual-comb spectroscopy and dual-comb LIDAR systems. For example, the spectrum shown in Fig. 3d would carry the absorption spectra information in a dual-comb spectroscopy measurement. Typically, distinct locked frequency combs would be used to generate this spectrum. In the present case, the two combs are provided by a single resonator and locking occurs through the CP soliton interaction. The resulting improvement in interferogram stability compared to previous microcomb dual-comb spectroscopy results is demonstrated in the Supplementary Section III. Soliton locking is possible over a set of repetition rates subject to the constraint imposed by the coupling of respective $r$th comb teeth as detailed here.

In summary, CP solitons have been demonstrated in a high-Q optical microresonator. Both the relative repetition rates and the relative spectral location for the clockwise and counter-clockwise directions are independently tuned by tuning of the corresponding optical pumping frequencies. Two distinctly different locking phenomena have also been observed. In the first, the repetition rates lock to the same value. The pumping frequencies are different when this locking occurs, so the two soliton comb spectra are offset slightly in the optical frequency, but have identical comb line spacings. The interferogram of the two pulse trains has no baseband time dependence when this locking occurs. In the second form of locking, the pumps are typically tuned apart to larger difference frequencies and the solitons are observed to lock at different repetition rates with a difference that divides into the pump-frequency difference. The origin of this locking is associated with optical locking of two comb teeth, one from each soliton. Because the two pumps are derived from the same laser, this additional comb tooth locking effectively results in the two comb spectra being locked at two different positions in their spectra. The resulting high level of mutual soliton coherence is observable in the baseband inter-soliton beat spectra, which feature very narrow spectral lines spaced by the difference in the locked soliton repetition rates. In effect, this second form of locking creates two frequency combs in the same device, with distinct repetition rates and optical frequencies, but that are optically locked. This is potentially useful in dual-comb spectroscopy and dual-comb LIDAR applications, where it would obviate the need for two separate frequency combs and the associated inter-comb locking hardware. Finally, note that, although single clockwise and counter-clockwise solitons have been generated here, it is also possible to create states containing multiple solitons.

Note added in proof: During the proofreading stage, we became aware of another work on a Fabry–Pérot dissipative Kerr soliton (DKS) source, whereby the Fabry–Pérot microcavity geometry also features counter-propagating solitons.

Methods

Methods and any associated references are available in the online version of the paper.
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Author contributions
Experiments were conceived by Q.-F.Y., X.Y., K.Y.Y. and K.V. Analysis of results was conducted by Q.-F.Y., X.Y., K.Y.Y. and K.V. Q.-F.Y. and X.Y. performed measurements. K.Y.Y. fabricated devices. All authors participated in writing the manuscript.

Additional information
Supplementary information is available in the online version of the paper. Reprints and permissions information is available online at www.nature.com/reprints. Publisher’s note: Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations. Correspondence and requests for materials should be addressed to K.V.

Competing financial interests
The authors declare no competing financial interests.
Methods
Repetition rate and offset control of CP solitons. The Raman soliton self-frequency shift (SSFS) $D_0$ is dependent on the pump-cavity detuning $\delta \omega$ according to\[15\]

$$\Omega_0 = -\frac{32D_0^2r_A^2}{75\kappa D_2} \delta \omega^2$$

(3)

where $r_A$ is the Raman shock time, $\kappa$ is the cavity decay rate, and $D_2$ is the free-spectral-range (second-order dispersion) at mode $\mu = 0$ (the pumping mode). The soliton repetition rate $f$ is coupled to the SSFS as

$$2n f = D_1 + \frac{\Omega_0 D_2}{15n}$$

(4)

Therefore the difference in the repetition rates (interferogram rate) between CP solitons with cavity–pump detuning $\delta \omega_{\text{cav}}$ and $\delta \omega_{\text{cav}}$ is given by

$$f_{\text{cav}} - f_{\text{cav}} = \frac{-16D_2\tau_A (\delta \omega_{\text{cav}} - \delta \omega_{\text{cav}})}{15n}$$

(5)

The second form of this equation uses $\Delta \omega = \omega_{\text{cav}} - \omega_{\text{cav}} = 2\pi n \nu$ and is applied for the theoretical plot in Fig. 2d. A plot of equation (5) versus $\Delta \nu$ for various detunings $\delta \omega_{\text{cav}}$ is provided in the Supplementary Fig. 1 and explains how comb offset and rate difference can be controlled independently.

Locking of CP solitons. Here, the locking of CP solitons at different repetition rates is modelled. The dispersive Kerr soliton comb is governed by the Luganino–Lefever equation augmented by the Raman term$^{15,20}$. The presence of scattering centres can induce coupling between the CP solitons as follows:

$$\frac{\partial A(t, \omega)}{\partial t} = -\frac{\kappa}{2} A + \frac{D_2}{2} \frac{\partial^2 A}{\partial \omega^2} + F_A + \chi |A|^2 A + \chi r_A D_2 \frac{\partial |A|^2}{\partial \phi} + \int_0^\infty \Gamma(0) B(t-\delta \omega_{\text{cav}}) e^{i \omega t} d\delta \omega$$

(6)

$$\frac{\partial B(t, \omega)}{\partial t} = -\frac{\kappa}{2} B + \frac{D_2}{2} \frac{\partial^2 B}{\partial \omega^2} + F_B + \chi |B|^2 B + \chi r_A D_2 \frac{\partial |B|^2}{\partial \phi} + \int_0^\infty \Gamma(0) A(t-\delta \omega_{\text{cav}}) e^{i \omega t} d\delta \omega$$

(7)

where $A$ and $B$ denote the slowly varying field envelopes of the CW and CCW solitons, respectively, $\phi$ is the angular coordinate in the rotational frame, $A$ is the normalized Kerr nonlinear coefficient$^{15,20}$, $F_A$ and $F_B$ denote the normalized continuous-wave pump term for field $A$ and $B$, respectively, and $\Gamma(0)$ represents the backscattering coefficient in the laboratory frame.

Considering the spectral misalignment of CP comb lines presented in Fig. 3a, it is assumed that only the rth comb line will induce inter-soliton coupling. Accordingly, the equation of motion for the soliton field amplitude $A$, equation (6), is reduced to the following:

$$\frac{\partial A(t, \omega)}{\partial t} = -\frac{\kappa}{2} A + r_A D_2 \frac{\partial |A|^2}{\partial \phi} + \int_0^\infty \Gamma(0) B(t-\delta \omega_{\text{cav}}) e^{i \omega t} d\delta \omega$$

(8)

where the expansion $B(t, \omega_{\text{cav}}) e^{i \omega_\text{cav} t} = \sum_\nu b_\nu e^{i \omega_\nu t}$ is used to extract the rth comb line from soliton field $B$. A similar equation of motion to equation (8) holds for amplitude $B$ (with corresponding expansion $A(t, \omega_{\text{cav}}) = \sum_\nu a_\nu e^{i \omega_\nu t}$). Coupling coefficient $G$ is given by $G = \int \gamma(t) \exp(-2i\omega t) d\omega$

The soliton field amplitude in the presence of the soliton self-frequency shift can be expressed as$^{15,27}$

$$A = B_i \text{sech}(|\phi - \phi_{\text{cav}}|/D_1 \tau_A) e^{i \psi_A (|\beta_{\text{cav}}| - \phi_{\text{cav}})}$$

(9)

where $B_i$ and $\tau_A$ are the pulse amplitude and duration, respectively, $\mu_A$ is the mode number of the soliton spectral maximum ($\mu = 0$ is the mode number of the pump mode). This mode number is related to the soliton self-frequency shift by $\Omega_0 = \mu_D_2$. $\phi_{\text{cav}}$ is a constant phase determined by the pump$^{15,20}, \psi_{\text{cav}}$ is the peak position of the CW soliton, which is coupled to $\mu_A$ by$^{15}$

$$\frac{\partial \phi_{\text{cav}}}{\partial t} = \mu_AD_2$$

(10)

The soliton energy $E_A$ and spectral maximum mode number $\mu_A$ are given by

$$E_A = \sum_\nu |a_\nu|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |A|^2 d\phi = B_i^2 \tau_A D_1/\pi$$

(11)

and

$$\mu_A = \sum_\nu |b_\nu|^2 = i \int_{-\infty}^{\infty} \left( A \frac{\partial A}{\partial \phi} - A^\dagger \frac{\partial A^\dagger}{\partial \phi} \right) d\phi$$

(12)

Taking the time derivative of equation (12) and substituting $\delta \Lambda/\delta t$ using equation (8), the equation of motion for $\phi_{\text{cav}}$ is obtained as

$$\frac{\partial \phi_{\text{cav}}}{\partial t} = -\kappa \text{sec} \frac{G^2 r_D D_2}{2nE_A} \int_{-\infty}^{\infty} \left( \frac{\partial |A|^2}{\partial \phi} \right)^2 d\phi$$

(13)

$$-\frac{1}{2nE_A} \int_{-\infty}^{\infty} \left( G \mu e^{-i \omega t} \frac{\partial A}{\partial \phi} - irG A^\dagger e^{i \omega t} \right) d\phi$$

The second term on the right-hand side corresponds to the steady-state Raman-induced centre shift$^{14,15}$ and is denoted by $R_A$. The third term is the soliton spectral shift caused by coupling to the opposing CP soliton through its comb tooth $b_t$. By using $A = \sum_\nu a_\nu e^{i \omega_\nu t}$, equation (13) yields

$$\frac{\partial \phi_{\text{cav}}}{\partial t} = -\kappa \text{sec} \frac{G^2 r_D D_2}{2nE_A} \left( a_\nu b_\nu^* - c.c. \right)$$

(14)

$$= -\kappa \text{sec} \frac{G^2 r_D D_2}{2nE_A} \left| a_b b_t \right| $$

where $\theta = (\psi_{A} - \psi_{B} - \psi_{C})$, with the phases $\psi_{A}$ and $\psi_{B}$ of the comb lines $a_i$ and $b_i$ given by the following expression:

$$\psi_{A} = \psi_{B} - rB_D$$

(15)

$$\psi_{B} = \psi_{B} - rB_D + \Delta \omega t$$

(16)

Also, $\psi_{C}$ is the phase of the backscatter coefficient $G$. The time dependence of $\psi_{C}$ can be derived from equation (10) as

$$\frac{\partial \psi_{C}}{\partial t} = -rB_D$$

(17)

Similarly, the derivative of the phase of $b_t$ is given by

$$\frac{\partial \psi_{B}}{\partial t} = -rB_D + \Delta \omega$$

(18)

Therefore the time derivative of the phase term $\theta = (\psi_{A} - \psi_{B} - \psi_{C})$ is given by

$$\frac{\partial \theta}{\partial t} = \Delta \omega + rB_D - \mu_A$$

(19)

Similar to equation (14), a parallel equation exists for soliton $B$ and is given by

$$\frac{\partial \psi_{B}}{\partial t} = -rB_D - 2\nu |a_b b_t |$$

(20)

Taking a time derivative of equation (19) and using equation (14) and equation (20) gives the following equation of motion for the relative phase $\theta$:

$$\frac{\partial ^2 \theta}{\partial t^2} = -2rB_D \left( 1 - \frac{E_A}{E_B} \right) |a_b b_t | \sin \theta + 2\nu \delta t$$

(21)

where $2\nu \delta t = \Delta \omega + rB_D - \mu_A$ is the frequency difference between the rth comb lines induced by the shifted pumps and Raman SSFS when the CP solitons have no interaction. The above equation is similar to the Alder equation of injection locking$^{28}$, only with an additional second-order time-derivative term. Setting the time derivatives of $\theta$ equal to zero gives the locking bandwidth $\omega_L$ of $\delta$ as

$$\omega_L = 4\nu |b_t|^2 \delta \omega = \frac{4r^2 B_D}{k} \left( \frac{1}{E_A} + \frac{1}{E_B} \right) |a_b b_t |$$

(22)
It is noted that for the parameters in this experiment, this solution is stable. Moreover, equation (19) gives $\delta = 0$, so that the pump frequency difference $\Delta \nu$ is divided by the repetition rate difference as follows:

$$\Delta f = -\frac{\Delta \nu}{r}$$ (23)

which is equation (1) in the main text.

**Parameters.** In the measurement, the loss rate is $\kappa/2\pi = 1.5$ MHz, and $D_2/2\pi = 16$ kHz and $r = -60$. For a soliton with $\tau_s = 150$ fs, the mode number of the Raman SSFS is $\mu_R \sim -20$ and the ratio $|a|^2/E^2 = D_1/\tau_s \text{sech}^2\left[\pi(r - \mu_R)/D_1\tau_s/2\right]/8 = 7 \times 10^{-4}$. As the CP solitons have similar powers, the locking bandwidth is estimated as $\omega_L \sim |G|/4$. In this case a backscattering rate of 4 kHz can provide a 1 kHz locking bandwidth.

**Data availability.** The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

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