Positroids, Plabic Graphs, & Scattering Amplitudes in \textsc{Mathematica}

Jacob L. Bourjaily

\textit{Department of Physics, Harvard University, Cambridge, MA 02138}

\textbf{Abstract:} The many intricate connections between scattering amplitudes, on-shell diagrams, and the positroid stratification of the Grassmannian has recently been described in detail. In order to facilitate the exploration of this rich correspondence, we have prepared a public \textsc{Mathematica} package called “\texttt{positroids}” which includes an array of useful tools including those for the construction of canonical coordinates for positroid configurations, the drawing of representative on-shell (plabic) graphs, and the evaluation of on-shell differential forms. This note documents the functions made available by the \texttt{positroids} package; the package’s source code together with a \textsc{Mathematica} notebook containing many detailed examples of its functionality are included with this note’s submission files on the \texttt{arXiv}.
1. Introduction

The recent work of [1] presents a comprehensive summary of the extensive correspondence between “on-shell diagrams”, [2–6], the Grassmannian contour integral described in [7–11], scattering amplitudes in planar, maximally supersymmetric (“$\mathcal{N} = 4$”) Yang-Mills (SYM), [12–18], and the combinatorics and geometry of what is known as the positroid stratification of the Grassmannian, [19,20]. At the heart of this story is the fact that scattering amplitudes can be represented (to all loop orders) in terms of on-shell diagrams, and that (reduced) on-shell diagrams can be fully characterized combinatorially by permutations—associated with left-right paths; moreover, these same permutations label the configurations of the positroid stratification of the Grassmannian $G(k,n)$ of $k$-planes in $n$ dimensions. These strata are naturally endowed with positive coordinates $\alpha_i$ and a canonical volume-form, [21, 22], which when expressed in terms of positive coordinates, is simply: $d\log(\alpha_1)\wedge\cdots\wedge d\log(\alpha_d)$. Because on-shell diagrams can be directly represented (and computed) as integrals of this invariant volume-form over their corresponding positroid configurations, this makes the evaluation of on-shell diagrams exceedingly simple.
We can illustrate this rich correspondence with the following example:

\[
C(\alpha) \equiv \begin{pmatrix}
1 & \alpha_8 (\alpha_5 + \alpha_1 \alpha_8) & \alpha_2 \alpha_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_{10} (\alpha_{10} \alpha_{13} + \alpha_4) & \alpha_{47} & 0 \\
-\alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 1 & \alpha_6 (\alpha_3 + \alpha_2 \alpha_6)
\end{pmatrix} \in G(4,9)
\]

Starting with the on-shell diagram on the left, we find that it would be labeled (via left-right paths) by the permutation denoted \(\sigma \equiv \{7, 5, 6, 10, 8, 9, 13, 11, 12\}\); this permutation also labels the Grassmannian configuration drawn (projectively\(^{1}\)) on the right—a configuration represented by the matrix \(C(\alpha)\) above, parameterized by the positive coordinates \(\alpha_i\); in terms of \(C(\alpha)\), the corresponding on-shell ‘function’ \(f_\sigma\) associated with the diagram is determined by the integral at the bottom.

We will not review these ideas here, but instead refer the interested reader to reference [1] for a thorough introduction and summary together with a more comprehensive list of references to the existing literature.

In order to help facilitate further investigation along these lines, however, we have prepared a MATHEMATICA package called “positroids”—which is documented in this note. The positroids package makes available many of the essential tools required to investigate the myriad connections between on-shell physics, scattering amplitudes, and the combinatorics and geometry of the positroid stratification of the Grassmannian.

\(^{1}\)Here, each dot represents a column of the matrix \(C(\alpha)\) viewed projectively as a point in \(\mathbb{P}^3\).
2. The Mathematica Package \texttt{positroids}

2.1 Obtaining the positroids Package and Demonstration Notebook

From the abstract page for this paper on the arXiv, follow the link “other formats” (below the option for “PDF”) and download the “source” for the submission. The source-file will contain\(^2\) the \texttt{positroids} package’s main source-code (\texttt{positroids.m}), together with a demonstration notebook (\texttt{positroids_package_demo.nb}) which describes with detailed examples many of the functions defined by the package.

2.2 Using the positroids Package

Upon obtaining the package, users should open and evaluate the Mathematica notebook \texttt{positroids_package_demo.nb}; this notebook will copy \texttt{positroids.m} to the user’s Application directory for Mathematica; this will allow \texttt{positroids} to be started in any future notebook via the command:

\begin{verbatim}
In[1]:= <<positroids.m
Out[1]:=
\end{verbatim}

(If the file “\texttt{positroids.m}” has not yet been copied to the user’s Application directory, then the package can be initialized by saving a notebook to the directory where \texttt{positroids.m} is located, and evaluating “\texttt{SetDirectory[NotebookDirectory[]]}” prior to the command “\texttt{<<positroids.m}”.)

\( ^2\)Occasionally, the “source” file downloaded from the arXiv is saved without any extension; this can be ameliorated by manually appending “.tar.gz” to the name of the downloaded file.
3. Functions Defined by positroids

3.1 Operations on Permutations Labeling Positroids

- **boundary[permutation]**: returns a list of permutation labels for positroid cells in the boundary, $\partial$, of the cell labeled by *permutation*. For example, the co-dimension one boundaries of the generic configuration in $G_+(3,6)$ are:

  ```plaintext
  In[1] := boundary[{4,5,6,7,8,9}]
  Out[1] := {{5,4,6,7,8,9},{4,6,5,7,8,9},{4,5,7,6,8,9},
  {4,5,6,8,7,9},{4,5,6,7,9,8},{3,5,6,7,8,10}}
  ```

- **checkOperator[permutation][operator]**: returns:
  - 0 if the minor (operator) vanishes for the cell labeled by *permutation*;
  - 1 if minor (operator) is non-vanishing for the cell labeled by *permutation*.

  ```plaintext
  In[1] := {checkOperator[{3,5,6,7,8,10}][{1,2,3}],
  checkOperator[{3,5,6,7,8,10}][{2,3,4}]}
  Out[1] := {0,1}
  ```

- **cyclicize[permutation]**: returns a sorted list of non-repeating permutations in the same cyclic class as *permutation*.

  ```plaintext
  In[1] := cyclicize[{6,5,8,7,10,9,12,11}]
  Out[1] := {{4,7,6,9,8,11,10,13},{6,5,8,7,10,9,12,11}}
  ```

- **cyclicRep[permutation]**: returns the lexicographically-first permutation in the cyclic-class of *permutation*.

- **decorate[permutation]**: takes an ‘ordinary’ permutation $\sigma$ of $n$ integers, and returns a *decorated* affine permutation $\widehat{\sigma}$, adding $n$ to the image of any $a$ such that $\sigma(a) < a$. For example, applying decorate to the ordinary permutation associated with the on-shell diagram given in the introduction (section 1) gives:

  ```plaintext
  In[1] := decorate[{7,5,6,1,8,9,4,2,3}]
  Out[1] := {7,5,6,10,8,9,13,11,12}
  ```

- **dimension[permutation]**: gives the dimension of the positroid stratum labeled by *permutation*; for example,

  ```plaintext
  In[1] := dimension[{3,5,7,6,8,14,10,12,11,13,16,21}]
  Out[1] := 20
  ```
• **dualGrassmannian[permutation__]:** if permutation labels a \((2n-4)\)-dimensional cell in the ‘momentum-space’ Grassmannian \(\hat{C} \in G(k+2, n)\), then dualGrassmannian returns the permutation label of the corresponding, \(4k\)-dimensional cell in the ‘momentum-twistor’ Grassmannian \(C \in G(k, n)\), and vice-versa; e.g.,

```mathematica
In[1]:= dualGrassmannian[{6,5,8,7,10,9,12,11}]
dualGrassmannian@%
Out[1]:={2,5,4,7,6,9,8,11}
{6,5,8,7,10,9,12,11}
```

• **eulerCharacteristicTable[n_,k__]:** returns a table indicating the numbers of \(d\)-dimensional cells in the positroid stratification of \(G_+(k,n)\). (This data is generated using the combinatorial results of [20].)

```mathematica
In[1]:= eulerCharacteristicTable[10,5]
Out[1]:=

| dim |      # |      # | dim |
|-----|-------|-------|-----|
| 24  | 1     | 10    | 25  |
| 22  | 55    | 220   | 23  |
| 20  | 715   | 2002  | 21  |
| 18  | 4985  | 11240 | 19  |
| 16  | 23210 | 44220 | 17  |
| 14  | 78087 | 128100| 15  |
| 12  | 195315| 276450| 13  |
| 10  | 362175| 437112| 11  |
| 8   | 482670| 482940| 9   |
| 6   | 432060| 339360| 7   |
| 4   | 228102| 126420| 5   |
| 2   | 54600 | 16800 | 3   |
| 0   | 3150  | 252   | 1   |

|       |       |
|-------|-------|
| 1865126 | 1865125 |

Euler Characteristic: 1865126 – 1865125 = 1
```

• **intersectionNumber[permutation__,m__4]:** returns the number of isolated solutions to \(m\)-dimensional kinematical constraints, \(\Gamma^m(C)\), where \(C\) is the positroid labeled by permutation. Supposing that \(C\) is an \((m k)\)-dimensional cell in the momentum-twistor Grassmannian\(^3\), intersectionNumber counts the number of

\(^3\)intersectionNumber also tests \((2n-4)\)-dimensional cells in the momentum-space Grassmannian.
isolated points $C^* \in C \cap Z^\perp$, where $Z$ is a generic configuration in $G(m,n)$.

\[ \text{In[1]} := \{ \text{intersectionNumber}[\{6,5,8,7,10,9,12,11\}], \text{intersectionNumber}[\{15,14,8,7,21,20,19,13,12,26, \linebreak 25,24,18,17,31,30,29,23,22,36\}] \} \]
\[ \text{Out[1]} := \{2,34\} \]

- \textbf{inverseBoundary}[$\text{permutation}$]: returns a list of permutation labels for positroid cells in the inverse-boundary, $\partial^{-1}$, of the cell labeled by $\text{permutation}$.

- \textbf{legalPermQ}[$\text{permutation}$]: tests whether the list $\text{permutation}$ does in fact denote a (possibly decorated) permutation on $n$ integers.

- \textbf{necklace}[$\text{permutation}$]: if $\text{permutation}$ labels a positroid configuration $C \in G(k,n)$, then \textbf{necklace} returns a list of $n$, $k$-tuples $A^{(a)} \equiv (A_1^{(a)}, \ldots, A_k^{(a)})$ denoting the lexicographically-minimal non-vanishing minors starting from each column $a$. Recall from [1,19] how the necklace encodes the list of all ranks of consecutive chains of columns of $C$; for example, the necklace for the configuration in $G(4,8)$ labeled by the permutation \{3,7,6,10,9,8,13,12\} would be given by,

\[
\begin{align*}
A^{(8)} &= (8\ 9\ 10\ 13) \\
A^{(7)} &= (7\ 8\ 9\ 10) \\
A^{(6)} &= (6\ 7\ 9\ 10) \\
A^{(5)} &= (5\ 6\ 7\ 10) \\
A^{(4)} &= (4\ 5\ 6\ 7) \\
A^{(3)} &= (3\ 4\ 5\ 7) \\
A^{(2)} &= (2\ 3\ 4\ 5) \\
A^{(1)} &= (1\ 2\ 4\ 5)
\end{align*}
\] (3.1)

This data is generated by the function \textbf{necklace} according to:

\[ \text{In[1]} := \text{necklace}[\{3,7,6,10,9,8,13,12\}] \]
\[ \text{Out[1]} := \{\{1,2,4,5\},\{2,3,4,5\},\{3,4,5,7\},\{4,5,6,7\}, \linebreak \{5,6,7,2\},\{6,7,1,2\},\{7,8,1,2\},\{8,1,2,5\}\} \]
• **necklaceR**[**permutation**__]: if **permutation** labels a positroid configuration \( C \in G(k,n) \), then **necklace** returns a list of \( n \), \( k \)-tuples \( \hat{A}^{(a)} \equiv (\hat{A}_1^{(a)}, \ldots, \hat{A}_k^{(a)}) \) denoting the lexicographically-maximal non-vanishing minors starting from each column \( a \). Like the more familiar ‘Grassmannian necklace’ generated by the function **necklace**, this data similarly encodes the ranks of all consecutive chains of columns of \( C \); to see this, consider the ‘reverse’ necklace for the configuration in \( G(4,8) \) labeled by the permutation \( \{3,7,6,10,9,8,13,12\} \),

\[
\hat{A}^{(8)} = (4\ 5\ 6\ 7), \\
\hat{A}^{(7)} = (2\ 4\ 5\ 6), \\
\hat{A}^{(6)} = (2\ 3\ 4\ 5), \\
\hat{A}^{(5)} = (7\ 2\ 3\ 4), \\
\hat{A}^{(4)} = (7\ 8\ 2\ 3), \\
\hat{A}^{(3)} = (7\ 8\ 1\ 2), \\
\hat{A}^{(2)} = (4\ 7\ 8\ 1), \\
\hat{A}^{(1)} = (4\ 5\ 7\ 8)
\]

This data is generated by the function **necklaceR** according to:

\[\text{In}[1]:= \text{necklaceR}[\{3,7,6,10,9,8,13,12\}]\]
\[\text{Out}[1]:= \{\{4,5,7,8\},\{1,4,7,8\},\{1,2,7,8\},\{2,3,7,8\},\{2,3,4,7\},\{2,3,4,5\},\{2,4,5,6\},\{4,5,6,7\}\}\]

• **nonSingularQ**[**permutation**, **twistorDimension**__4]: tests whether or not a configuration labeled by **permutation** has non-vanishing support for generic kinematical data—**nonSingularQ** returns True iff \( \text{intersectionNumber}[\text{permutation}] > 0 \).

• **parityConjugate**[**permutation**__]: if **permutation** labels a configuration \( C \in G(k,n) \), then **parityConjugate** returns the permutation labeling the geometrically dual configuration \( C^\perp \in G(n-k,n) \). For example,

\[\text{In}[1]:= \text{parityConjugate}[\{3,5,6,7,10\}]\]
\[\text{parityConjugate}@%\]
\[\text{Out}[1]:= \{4,5,7,6,8,9\}\]
\[\{3,5,6,7,8,10\}\]

# 7
permToGeometry[permutation_, removeVanishingQ_: True]: returns a (formatted) table of planes of various ranks spanned by consecutive chains of column-vectors of the configuration labeled by permutation; distinguished planes are highlighted in blue, and (for visual clarity) the option removeVanishingQ (True by default) causes any vanishing columns of the configuration to be suppressed.

For example, consider the positroid configuration labeled by the permutation \{3, 7, 6, 10, 9, 8, 13, 12\},

\[
\begin{array}{c|c}
\text{consec. chains of columns} & \text{span} \\
(1) (2) (3) (4) (5) (6) (7) (8) & \mathbb{P}^0 \\
(123) (34) (45) (56) (678) (81) & \mathbb{P}^1 \\
(56781) (81234) (3456) & \mathbb{P}^2 \\
\end{array}
\]

For this, the function permToGeometry would produce:

\[
\text{In}[1]:= \text{permToGeometry}[\{3,7,6,10,9,8,13,12\}] \\
\text{Out}[1]:= \left(\begin{array}{c}
(1) (2) (3) (4) (5) (6) (7) (8) \\
(1 2 3) (3 4) (4 5) (5 6) (6 7 8) (8 1) \\
(3 4 5 6) (5 6 7 8 1) (8 1 2 3 4)
\end{array}\right)
\]

(Here, all the ‘distinguished’ maximal planes (\(a+1, \ldots, \sigma(a)-1\) have been highlighted in blue—as the rest of the table follows form knowledge of these planes.)

permutationK[permutation_]: returns the “\(k\)” associated with permutation. More explicitly, if \(C \equiv (c_1, \ldots, c_n)\) is the positroid configuration labeled by permutation, then permutationK returns rank\{\(c_1, \ldots, c_n\}\}; as such, \(C \in G(k, n)\). E.g.,

\[
\text{In}[1]:= \text{permutationK}[\{3,7,6,10,9,8,13,12\}] \\
\text{Out}[1]:= 4
\]

preferredGauge[permutation_]: returns the lexicographically first non-vanishing minor of the configuration \(C \in G(k, n)\); equivalently, if \(C \equiv (c_1, \ldots, c_n)\) is the configuration labeled by permutation, then preferredGauge returns the lexicographically-minimal set of column-labels \((a_1, \ldots, a_k)\) such that rank\{\(c_{a_1}, \ldots, c_{a_k}\}\} = k.
• \texttt{randomCell[n,k,d,exclusionsQ=True]}: gives the permutation label of a randomly-generated, \(d\)-dimensional positroid cell \(C \in G(k,n)\). If the optional argument \texttt{exclusionsQ} is \texttt{True} (its default value), then \texttt{randomCell} tries to find a configuration for which all columns are non-vanishing (when possible).

• \texttt{rotate[rotation_][permutation_]}: returns the permutation labeling a positroid cell whose columns are rotated (positively) relative to that of \texttt{permutation} by \texttt{rotation}; more specifically, given a positroid cell \(C = (c_1, \ldots, c_n) \in G(k,n)\) labeled by \texttt{permutation}, \texttt{rotate} returns the label of the configuration \(C' = (c'_1, \ldots, c'_n)\) where \(a' = a + \text{rotation}\).

• \texttt{storeBoundaries}: a global variable which by default is set to \texttt{False}; when \texttt{True}, boundary information is stored in memory so that boundary computations need not be repeated.
3.2 Positroid Coordinates and Matrix Representatives

- \texttt{bridgeToMinors[permutation]}: using the BCFW-bridge construction of coordinates for the configuration \( C \) labeled by \texttt{permutation}, \texttt{bridgeToMinors} expresses each BCFW-bridge coordinate \( \alpha_i \) directly in terms of the minors of \( C \). For example, the (canonical) BCFW-bridge decomposition of the permutation \( \{3, 7, 6, 10, 9, 8, 13, 12\} \) generates the matrix representative:

\[
\begin{pmatrix}
1 & \alpha_9 & 0 - \alpha_5 - \alpha_5 \alpha_6 & 0 & 0 & 0 \\
0 & 1 & \alpha_8 & \alpha_7 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_3 + \alpha_6 & \alpha_3 \alpha_4 & 0 - \alpha_1 \\
0 & 0 & 0 & 0 & 1 & \alpha_4 & \alpha_2 & 0
\end{pmatrix}
\]

It is a highly non-trivial fact that such bridge coordinates \( \alpha_i \) can be expressed as ratios of \textit{monomials} involving the minors of \( C \). The explicit correspondence is given by the function \texttt{bridgeToMinors}, as in the following example:

\[
\begin{array}{c}
\text{In[1]:=} \quad \text{bridgeToMinors[\{3,7,6,10,9,8,13,12\}]//nice} \\
\text{Out[1]:=} \quad \alpha_1 \rightarrow (1258) \quad \alpha_2 \rightarrow (1247) \quad \alpha_3 \rightarrow (1245)(1278)(4567) \\
\alpha_4 \rightarrow (1245)(1267)(4578) \quad \alpha_5 \rightarrow (1278)(1456) \quad \alpha_6 \rightarrow (1246)(4578) \\
\alpha_7 \rightarrow (1478) \quad \alpha_8 \rightarrow (1245) \quad \alpha_9 \rightarrow (2456)
\end{array}
\]

- \texttt{matrixCharts[permutation]}: gives a list of (distinct) matrix-representatives of the positroid configuration \( C \) labeled by \texttt{permutation}, using each of the (cyclically-distinct) columns as the ‘minimal’ column used in the construction of BCFW-bridge coordinates for \( C \).

- \texttt{matrixToPerm[matrix]}: returns the permutation which labels the positroid cell represented by \texttt{matrix}. If \texttt{matrix} is given in terms of unspecified variables, then \texttt{matrixToPerm} assumes that all such take on \textit{generic} values.

- \texttt{permToMatrix[permutation,transpositionScheme_0]}: returns a matrix-representative of the positroid cell labeled by \texttt{permutation} given in terms of BCFW-bridge coordinates obtained using one of several possible bridge-decomposition schema—with the default scheme being “0”, that corresponding to the canonical or ‘lexicographic’ scheme described in reference [1]. The possible decomposition schema are those described for \texttt{transpositionChain}.
• `positiveQ[matrix_]`: returns `True` if the *all* ordered minors of the matrix are strictly non-negative; if `matrix` is parameterized by unspecified variables, then `positiveQ` returns `True` if `matrix` is positive (in the previous sense) when all its unfixed variables are given random, *positive* values.

• `transpositionChain[permutation_, scheme_]`: returns a list `{transpositionList, permutationList, seedGauge}` containing the complete BCFW-bridge decomposition of `permutation` into ‘adjacent’ transpositions according to the scheme denoted `scheme`. (This data is obviously redundant, but such redundancy proves somewhat useful to have at hand.) The possible schema include:

| scheme             | Description                                                                 |
|--------------------|-----------------------------------------------------------------------------|
| 0 (default)        | the canonical or ‘lexicographic’ decomposition scheme                         |
| ‘cyclic’           | a scheme which attempts to decompose `permutation` into a sequence of bridges in a way which preserves cyclic symmetry (if any exists) |
| 1                  | a scheme which correctly orients all (2n−4)-dimensional cells in the momentum-space Grassmannian |
| −1                 | a scheme which correctly orients all 4k-dimensional cells in the momentum-twistor Grassmannian |

To illustrate the different transposition schemes, the *lexicographic* scheme (“`scheme=0`”) would decompose the permutation `{4, 7, 6, 9, 11, 10, 13}`,

"0" (Lexicographic) Bridge Decomposition Scheme

| \( \tau \) | 1 2 3 4 5 6 7 8  | BCFW shift |
|------------|-----------------|------------|
| (12)       | 4 7 6 9 8 11 10 13 | \( c_2 \mapsto c_2 + \alpha_{12} c_1 \) |
| (23)       | 7 4 6 9 8 11 10 13 | \( c_3 \mapsto c_3 + \alpha_{11} c_2 \) |
| (34)       | 7 6 4 9 8 11 10 13 | \( c_4 \mapsto c_4 + \alpha_{10} c_3 \) |
| (23)       | 7 9 6 4 8 11 10 13 | \( c_5 \mapsto c_5 + \alpha_9 c_2 \) |
| (12)       | 9 7 6 4 8 11 10 13 | \( c_6 \mapsto c_6 + \alpha_8 c_1 \) |
| (35)       | 9 7 8 4 6 11 10 13 | \( c_7 \mapsto c_7 + \alpha_7 c_3 \) |
| (23)       | 9 8 7 4 6 11 10 13 | \( c_8 \mapsto c_8 + \alpha_6 c_2 \) |
| (56)       | 9 8 7 4 11 6 10 13 | \( c_9 \mapsto c_9 + \alpha_5 c_5 \) |
| (35)       | 9 8 11 4 7 6 10 13 | \( c_{10} \mapsto c_{10} + \alpha_4 c_3 \) |
| (57)       | 9 8 11 4 10 6 7 13 | \( c_{11} \mapsto c_{11} + \alpha_3 c_5 \) |
| (25)       | 9 10 11 4 8 6 7 13 | \( c_{12} \mapsto c_{12} - \alpha_2 c_2 \) |
| (58)       | 9 10 11 4 13 6 7 8 | \( c_{13} \mapsto c_{13} + \alpha_1 c_5 \) |
This decomposition would give rise to the following representative of the cell:

\[
\begin{pmatrix}
1 (\alpha_8 + \alpha_{12}) & (\alpha_9 + \alpha_{11})\alpha_8 & \alpha_8\alpha_9\alpha_10 & 0 & 0 & 0 & 0 \\
0 & 1 & (\alpha_6 + \alpha_9 + \alpha_{11}) (\alpha_6 + \alpha_9)\alpha_{10} & (\alpha_6\alpha_7 - \alpha_2 - \alpha_2\alpha_3) - \alpha_2\alpha_5 - \alpha_2\alpha_3 & 0 \\
0 & 0 & 1 & \alpha_{10} & (\alpha_4 + \alpha_7) & \alpha_4\alpha_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \alpha_5 & \alpha_3 - \alpha_1 & 0
\end{pmatrix}
\]

However, the “cyclic” scheme would decompose the permutation according to:

“cyclic” Bridge Decomposition Scheme

| \(\tau\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| (12) | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 |
| (34) | 7 | 4 | 6 | 9 | 8 | 11 | 10 | 13 |
| (56) | 7 | 4 | 9 | 6 | 8 | 11 | 10 | 13 |
| (78) | 7 | 4 | 9 | 6 | 11 | 8 | 10 | 13 |
| (23) | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| (34) | 7 | 9 | 4 | 6 | 11 | 8 | 13 | 10 |
| (67) | 7 | 9 | 6 | 4 | 11 | 8 | 13 | 10 |
| (78) | 7 | 9 | 6 | 4 | 11 | 13 | 8 | 10 |
| (12) | 7 | 9 | 6 | 4 | 11 | 13 | 10 | 8 |
| (35) | 0 | 7 | 6 | 4 | 11 | 13 | 10 | 8 |
| (56) | 0 | 7 | 11 | 4 | 6 | 13 | 10 | 8 |
| (72) | 0 | 7 | 11 | 4 | 13 | 6 | 15 | 8 |

BCFW shift

\(c_2 \leftrightarrow c_2 + \alpha_{12}c_1\)
\(c_4 \leftrightarrow c_4 + \alpha_{11}c_3\)
\(c_6 \leftrightarrow c_6 + \alpha_{10}c_5\)
\(c_8 \leftrightarrow c_8 + \alpha_9c_7\)
\(c_3 \leftrightarrow c_3 + \alpha_8c_2\)
\(c_4 \leftrightarrow c_4 + \alpha_7c_3\)
\(c_7 \leftrightarrow c_7 + \alpha_6c_6\)
\(c_8 \leftrightarrow c_8 + \alpha_5c_7\)
\(c_2 \leftrightarrow c_2 + \alpha_4c_1\)
\(c_5 \leftrightarrow c_5 + \alpha_3c_3\)
\(c_6 \leftrightarrow c_6 + \alpha_2c_5\)
\(c_2 \leftrightarrow c_2 - \alpha_1c_7\)

which would result in the following matrix-representative:

\[
\begin{pmatrix}
1 (\alpha_4 + \alpha_{12}) & \alpha_4\alpha_8 & \alpha_4\alpha_8\alpha_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & (\alpha_7 + \alpha_{11}) & 0 & -\alpha_2 - \alpha_2\alpha_6 - \alpha_2\alpha_6\alpha_9 \\
0 & 0 & 0 & 0 & 1 (\alpha_3 + \alpha_{10}) & \alpha_3\alpha_6 & \alpha_3\alpha_6\alpha_9 \\
0 & \alpha_1 & \alpha_1\alpha_8 & \alpha_1\alpha_8\alpha_{11} & 0 & 0 & 1 (\alpha_5 + \alpha_9)
\end{pmatrix}
\]
3.3 Drawing On-Shell (Plabic) Graphs and Left-Right Paths

- \texttt{plabicGraph[\texttt{permutation\_optionsList\_defaultOptions}]:} draws a reduced plabic graph whose left-right path would be given by \texttt{permutation}. (Here, ‘plabic’ is used somewhat loosely, as the default behavior of \texttt{plabicGraph} is to draw graphs with monovalent and bivalent vertices when the inclusion of such is warranted.)

There are many possible options for \texttt{plabicGraph}; the principle of these are:

| option      | value          | description                                                                 |
|-------------|----------------|----------------------------------------------------------------------------|
| \texttt{chainOption} | ⋆ 0            | uses the lexicographic bridge decomposition scheme                          |
|             | ″cyclic″      | uses the bridge decomposition scheme ″cyclic″                               |
|             | ±1            | uses the bridge decomposition scheme set by ±1                             |
| \texttt{rotation} | ⋆ 0            | constructs the graph using whatever bridge decomposition scheme is specified, but where particle ‘1’ is considered minimal |
|             | a             | same as above, but with cyclic ordering beginning with particle (1−a)       |
| \texttt{LRpaths} | ⋆ {}          | does not show any left-right paths                                          |
|             | \{a,...,b\}  | draws the left-right paths which start at legs a,...,b                     |
|             | ″All″         | draws all left-right paths                                                 |
| \texttt{directed} | ⋆ False       | results in an undirected graph                                             |
|             | True          | results in a directed graph whose perfect orientation follows from the BCFW bridge-decomposition scheme used |
| \texttt{edgeLabels} | ⋆ False       | does not label any edges of the graph                                       |
|             | True          | labels the BCFW-bridge edges by the bridge coordinates associated with each |
| \texttt{faceLabels} | ⋆ False       | does not label any faces of the graph                                       |
|             | True          | labels each face of the graph with a label “f_i”                          |
|             | ″A″           | labels the faces of the graph with A-variable labels                       |
|             | ″X″           | labels the faces of the graph with X-variable labels                       |
| \texttt{showRemovable} | ⋆ True       | shows monovalent vertices attached to legs which are self-identified under \texttt{permutation} |
|             | False         | removes boundary legs which are self-identified under \texttt{permutation} |
| \texttt{orientation} | ⋆ 1           | draws the external legs with clockwise ordering                           |
|             | -1            | draws external legs with counterclockwise ordering                         |
| \texttt{bipartiteQ} | ⋆ False       | allows for trees of same-colored vertices                                  |
|             | True          | collapses all same-colored trees giving a bipartite graph                  |
Here, each default option has been marked with a “⋆”. In addition to these options, there are many which control stylistic and aesthetic details of the graphs generated by `plabicGraph`; the principle among these include:

| option         | value | description                                                                                     |
|----------------|-------|-----------------------------------------------------------------------------------------------|
| angle          | ⋆ 0   | draws the graph with particle 1 toward the middle-left                                           |
|                | θ     | rotates the graph by θ radians in the clockwise direction                                        |
| extLabels      | ⋆ “Auto” | labels the external legs {1,...,n}                                                                |
|                | {a,...,b} | labels the external legs {a,...,b}                                                                |
| labelSpacing   | ⋆ 0.65 | places the external particle labels at a distance of 0.65 from the boundary                      |
| font           | ⋆ “Times” | uses the font “Times” for all labels                                                              |
| fontSize       | ⋆ 36  | sets the “FontSize” for external labels to be 36pt                                              |
| imageSize      | ⋆ 300 | sets the “ImageSize” of the Graphics output to be 300                                              |
| radius         | ⋆ 4   | sets the graph’s boundary as a circle with radius 4 units                                         |
| vertexSize     | ⋆ 0.325 | draws all vertices with size of 0.325 units                                                       |
| lineThickness  | ⋆ 3.5 | draws edges with AbsoluteThickness[3.5]                                                           |
| LRpathThickness| ⋆ 4   | draws left-right paths with AbsoluteThickness[4]                                                  |
| LRpathDistance | ⋆ 0.275 | draws left-right paths with a distance of 0.275 units from the graph’s edges                     |
| LRArrowHeadSize| ⋆ 0.0655 | sets the size of left-right path Arrowheads to be 0.0655                                        |
| edgeArrowSize  | ⋆ 0.05 | sets the size of directed-edge Arrowheads to be 0.05                                             |
| outerCircle    | ⋆ True| draws a disc at the boundary, enclosing the graph                                                |

Wherever sufficiently obvious, we have left as implicit the possible alternative settings allowed these options.

For a complete list of options for `plabicGraph`—together with the default value for each—consult the global variable “defaultPlabicGraphOptions”; users should find most detailed features of the output to be pliable through simple experimentation with the options.

- `plabicGraphData[permutation__,transpositionScheme__0]`: given the permutation label for a positroid configuration a some `transpositionScheme`, `plabicGraphData` returns the pair `{edgeList,faceList}`, where `edgeList` lists each (directed) edge `i→j` (carrying a weight `w`) as a triple `{i,j,w}`; and `faceList` lists the vertices along each face of the graph, listing them with clockwise ordering.
- `resetGraphDefaults`: a function which automatically resets all the default graph options for the function `plabicGraph`. (By default, `plabicGraph` remembers any options explicitly set until the length of permutations being drawn is changed.)
3.4 Physical Operations, Kinematics, and Scattering Amplitudes

- \textbf{bcfwPartitions}[n_,k_]: returns a list of pairs \{(n_L, k_L), (n_R, k_R)\} which are bridged-together to compute the \(n\)-point \(\text{N}^{(k-2)}\text{MHV}\) tree-amplitude \(A_n^{(k)}\) according to

\[
A_n^{(k)} = \sum_{(n_L, k_L), (n_R, k_R)} A_{n_L}^{(k_L)} \otimes_{\text{BCFW}} A_{n_R}^{(k_R)}.
\] (3.6)

- \textbf{bcfwTermNames}[n_,k_]: returns a formatted list of terms appearing in the BCFW tree-amplitude where MHV and \(\text{MHV}\) amplitudes have been collected together.

\begin{verbatim}
In[1]:= bcfwTermNames[6,3]
bcfwTermNames[14,7][[9598]]
Out[1]:= \{A_5^{(3)} \otimes A_3^{(1)}, A_4^{(2)} \otimes A_4^{(2)}, A_3^{(2)} \otimes A_5^{(2)}\}
\end{verbatim}

- \textbf{generalTreeContour}[a_:0,b_:0,bridgeChoice_:0][n_,k_]: returns a list of permutation labels for the positroid cells which together give the \(n\)-particle \(\text{N}^{(k-2)}\text{MHV}\) tree-amplitude obtained using the white-to-black BCFW-bridge attached to legs \((1 \ n)\) for \textit{bridgeChoice} ‘0’ (the default) or \((n \ 1)\) for \textit{bridgeChoice} ‘1’, and for which the lower-point amplitudes appearing in the recursion have been recursed (using the same bridge-choice) attached to legs \((1 + a, n_L + a)\) and \((1 + b, n_R + b)\) of the left and right amplitudes, respectively.

\begin{verbatim}
In[1]:= identitySigns[{{3,2,4,5,6,7},{2,4,3,5,6,7},{2,3,5,4,6,7},{2,3,4,6,5,7},{2,3,4,5,7,6},{1,3,4,5,6,8}}]
Out[1]:= \{1,-1,1,-1,1,-1\}
\end{verbatim}

- \textbf{nRatioContour}[n_,k_]: produces the same output as:

\begin{verbatim}
permToResidue/@dualGrassmannian/@treeContour[n,k].
\end{verbatim}

- \textbf{nTreeContour}[n_,k_]: produces the same output as:

\begin{verbatim}
permToResidue/@treeContour[n,k].
\end{verbatim}
• **permToResidue[permutation_]**: for a permutation labeling either a \((2n-4)\)-dimensional cell \(C\) in the momentum-space Grassmannian or a \(4k\)-dimensional cell in the momentum-twistor Grassmannian, `permToResidue` uses the globally defined momentum twistors \(Zs\) and the corresponding (or alternatively-defined) global, spinor variables \(Ls\) and \(Lbs\) (\(\lambda\) and \(\tilde{\lambda}\), respectively), to find the isolated point(s) \(C^*\in C\) which solve the kinematical constraints and returns a pair (or list of pairs if more than one solution exists) \(\{\mu^*, C^*\}\) where \(\mu^*\) is the positroid measure evaluated at the point \(C^*\) which solves the kinematical constraints.

The on-shell function corresponding to a positroid cell labeled by \(\sigma\), when evaluated at whatever kinematical data is given, would be given by,

\[
f_{\sigma} = \mu^* \times \delta^{k \times 4}(C^* \cdot \tilde{\eta}), \quad \text{or} \quad f_{\sigma} = \mu^* \times \delta^{k \times 4}(C^* \cdot \eta),
\]

if \(\sigma\) labels a cell in the momentum-space Grassmannian or the momentum-twistor Grassmannian, respectively.

• **positiveZs[nParticles_]**: it is sometimes convenient to evaluate analytic expressions involving spinor-helicity variables or momentum-twistors using explicit kinematical data; under such circumstances, there are some conveniences afforded by using “well-chosen” kinematical data.

Reasons for preferring one choice over another include: having all Lorentz invariants be integer-valued and relatively small; having all dual-conformal cross ratios positive (so as to avoid branch-ambiguities when evaluating the polylogarithms that arise in scattering amplitudes at loop-level); and possibly to have all Lorentz-invariants be distinct (either to help reconstruct an analytic expression or to avoid ‘accidental’ cancelations). Of these, the following momentum-twistors meet the first two desires spectacularly:

\[
Zs \equiv \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & \binom{n}{0} \\
2 & 3 & 4 & 5 & \cdots & \binom{n+1}{1} \\
3 & 6 & 10 & 15 & \cdots & \binom{n+2}{2} \\
4 & 10 & 20 & 35 & \cdots & \binom{n+3}{3}
\end{pmatrix}.
\]

The function `positiveZs[16]` is evaluated when the `positroids` package is first loaded, allowing amplitudes involving as many as 16 particles to be evaluated without specific initialization.

• **randomTreeContour[n_, k_]**: returns the list of permutation labels for cells occurring in the BCFW tree-amplitude formula, where the lower-point amplitudes have been recursed using randomly-chosen legs.
\* \textbf{setupUsingSpinors}[\textit{lambdaList}, \textit{lambdaBarList}]: sets up the global variables \( L_s \) and \( L_{bs} \) for \( \lambda \) and \( \bar{\lambda} \), respectively, and defines the global \((n \times 4)\) matrix \( Z_s \) for momentum twistorss for use in numerical evaluation.

\* \textbf{setupUsingTwistors}[\textit{twistorList}]: sets up the global \((n \times 4)\) matrix \( Z_s \) encoding the momentum-twistor kinematical data, and defines the auxiliary variables \( L_s \) and \( L_{bs} \) for \( \lambda \) and \( \bar{\lambda} \), respectively.

\* \textbf{superComponent}[\textit{component}][\textit{superFunction}]: for the purposes of the \textit{positroids} package, a \textit{superFunction} must be given by a pair \( \{f,C\}\)—an ordinary function \( f(1, \ldots, n) \) of the kinematical variables times a fermionic \( \delta \)-function of the form,

\[
\delta^{k \times 4}(C \cdot \eta) \equiv \prod_{I=1}^{4} \left( \bigoplus_{a_1 < \cdots < a_k} (a_1 \cdots a_k) \eta^I_{a_1} \cdots \eta^I_{a_k} \right), \tag{3.9}
\]

where \( C \) is an \((n \times k)\)-matrix of ordinary functions, and for each \( a = 1, \ldots, n, \eta_a \) is a fermionic (anti-commuting) variable. To be clear, we consider each particle as a Grassmann coherent state of the form,

\[
|a\rangle = |a\rangle_I + \frac{1}{2!} \eta^I_a |a\rangle_{IJ} + \frac{1}{3!} \eta^I_a \eta^J_a \eta^K_a |a\rangle_{IJK} + \frac{1}{4!} \eta^I_a \eta^J_a \eta^K_a \eta^L_a |a\rangle_{IJKL}.
\]

If we use \( r_a \) to denote the \( R \)-charge of the \( a \)-th particle according to,

| field | helicity | \( R \)-charge \( (r_a) \) | short-hand for \( r_a \) |
|-------|----------|------------------|-----------------|
| \( |a\rangle_I \) | +1 | \{I\} | \( p/2(\Leftrightarrow \{4\}) \) |
| \( |a\rangle_{IJ} \) | +\frac{1}{2} | \{I, J\} | — |
| \( |a\rangle_{IJK} \) | 0 | \{I, J, K\} | \( m/2(\Leftrightarrow \{1, 2, 3\}) \) |
| \( |a\rangle_{IJKL} \) | -1 | \{1, 2, 3, 4\} | \( m \) |

then \textbf{superComponent}[\( r_1, \ldots, r_n \)][\textit{superFunction}] returns the \textit{component} function of \textit{superFunction} proportional to,

\[
\prod_{a=1}^{n} \prod_{I \in r_a} \eta^I_a. \tag{3.10}
\]

—that is, the component-function involving the states:

\[
|1\rangle_{r_1} \cdots |n\rangle_{r_n}. \tag{3.11}
\]

For example, the \( \{ -, +, -, +, +, +, +, + \} \) component of the 8-particle \( N^2 \)MHV amplitude \( A^{(4)}_8 \) proportional to,

\[
(\eta^1_1 \eta^2_2 \eta^3_3 \eta^4_4)(\eta^1_5 \eta^2_5 \eta^3_5 \eta^4_5)(\eta^1_7 \eta^2_7 \eta^3_7 \eta^4_7), \tag{3.12}
\]

would be extracted be obtained by evaluating, (compare with e.g. \[23,24\]):

\[
\text{In}[1]:= \text{Total[superComponent[m,p,m,p,m,p,m]//} \text{permToResidue/@treeContour[8,4]]}
\]

\[
\text{Out}[1]:= \frac{908416}{39375}
\]
• \texttt{termsInBCFW}[n, k, \ell=0]: gives the number of (non-vanishing) terms generated by the BCFW-recursion for the $\ell$-loop, $n$-point $N^{(k-2)}$MHV amplitude. ($\ell$ must be either 0 or 1 as the number of terms is scheme-dependent beyond 1-loop).

\begin{verbatim}
In[1]:= termsInBCFW[6,3]
out[1]:= 3

In[2]:= termsInBCFW[6,3,1]
out[2]:= 16
\end{verbatim}

• \texttt{treeContour}[n, k]: returns the list of permutation labels for positroid cells which together give the $n$-particle $N^{(k-2)}$MHV tree-amplitude, using the \textit{default} recursion scheme; \texttt{treeContour}[n, k] is equivalent to \texttt{generalTreeContour}[0,0,0][n,k].

\begin{verbatim}
In[1]:= treeContour[6,3]
out[1]:= {{4,5,6,8,7,9},{3,5,6,7,8,10},{4,6,5,7,8,9}}
\end{verbatim}
3.5 Aesthetic and General Purpose Functions

- **explicitly expression_positiveQ_true**: picks random (integers) for all variables occurring in an expression; if positiveQ is True (its default value), then the random assignments are taken to be positive. (If expression includes angle-brackets ⟨⋯⟩, explicitify will evaluate these assuming random kinematical data.)

- **exportToPDF[fileName_][figure_]**: saves a PDF version of figure to the file fileName using outlined fonts (and with other minor processing).

- **mod2[objectList_]**: returns the elements of objectList which occur an odd number of times in objectList. This can be useful, for example, if one wants to explicitly verify that \( \partial^2 = 0 \mod 2 \):

  ```math
  In[1]:= mod2[Join@@(boundary/@boundary[randomCell[8,4,12]])]
  Out[1]:={}
  ```

- **nice[expression_]**: formats expression to display ‘nicely’ by making replacements such as \( ab[x\cdots y] \mapsto ⟨x\cdots y⟩ \), \( α[1] \mapsto α_1 \), etc., and by writing any level-zero matrices in MatrixForm.

- **niceTime[timeInSeconds_]**: converts a time measured in seconds timeInSeconds, to human-readable form. For example,

  ```math
  In[1]:= niceTime[299792458]
niceTime[3.1415926535]
  Out[1]:= 9 years, 182 days
            3 seconds, 141 ms
  ```

- **random[objectList_]**: returns a random element from (the first level of) objectList.

- **randomSubset[subsetLength_][objectList_]**: returns a randomly-chosen subset of length subsetLength from among the list objectList.

- **timed[expression_]**: evaluates expression and prints a message regarding the time required for evaluation.

---

4The file is saved to the same directory as Export[]: either NotebookDirectory[] or Directory[] (if the former does not exist).
Acknowledgements

This work emerged directly out of research in collaboration with Nima Arkani-Hamed, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, and Jaroslav Trnka, and was greatly encouraged by early discussions with Pierre Deligne, Bob MacPherson, and Mark Goresky. We are especially grateful to Jaroslav Trnka, Nima Arkani-Hamed, and Freddy Cachazo for their helpful comments and suggestions regarding the package’s documentation and the examples described in the demonstration notebook. We are indebted to Thomas Lam and David Speyer for their invaluable assistance in the development of a combinatorial test for kinematical support. This work was supported in part by the Harvard Society of Fellows, a grant from the Harvard Milton Fund, Department of Energy contract DE-FG02-91ER40654, and by the generous hospitality of the Institute for Advanced Study.
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