CONSISTENCY AROUND A CUBOCTAHEDRON

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ABSTRACT. In this paper, we describe new results arising from a search for lattice equations that are consistently placed on cuboctahedra. These results extend the well-known ABS equations that are consistent on cubes. Our search was motivated by $\tau$-functions related to discrete Painlevé equations.

1. INTRODUCTION

It is a natural question to ask when discrete versions of integrable systems are themselves integrable. This question has led to very active searches for partial difference equations (PΔEs), which have properties closely related to those of integrable PDEs, such as the Korteweg-de Vries equation. Recent results have focused on PΔEs given by polynomial equations placed in a consistent way on faces of cubes, which turn out to have the desired integrability properties. In this paper, we consider the extension of this search to cuboctahedra.

We were led to make this generalization because of PΔEs that arise naturally in our study of discrete Painlevé equations. (Details of the latter study will be given in another paper.) We start with a brief overview of the geometric setting to which we were led. Consider the net graphs (i.e., polygons that can be folded along edges to become faces of a polyhedron) in each of the following two cases. A cuboctahedron is a polyhedron with 8 triangular faces and 6 square faces. Figure 1.1 shows a net graph of a cuboctahedron. In contrast, Figure 1.2 shows a net graph of a cube.

Figure 1.1. A net graph of a cuboctahedron.
Figure 1.2. A net graph of a cube.

One of the most interesting discoveries of recent times is the recognition of PΔEs consistent around cubes as integrable systems. (§1.3 provides more detail.) The study of consistency associates a PΔE with a quadrilateral face. Label its vertices by \((x, y, z, w)\) and associate a polynomial \(Q(x, y, z, w)\) to that face. Now assign a function \(u(i, j)\) to each vertex, in such a way that its values on neighbouring vertices (joined by an edge) are obtained by evaluating it under the iteration \(i \mapsto i \pm 1\) and/or \(j \mapsto j \pm 1\). Then the associated PΔE is given by the equation

\[
Q(u(i, j), u(i + 1, j), u(i, j + 1), u(i + 1, j + 1)) = 0.
\]

Although not all faces of a cuboctahedron are quadrilateral (see Figure 1.1), it turns out we can consider PΔEs on a series of quadrilaterals, some of which are faces of a cuboctahedron and others of which lie in the interior of the same cuboctahedron. We search for equations that define vertex values consistently in such a configuration. (See §2.2 for further details.) A major motivation of our paper is to find PΔEs consistent around cuboctahedra to deduce whether they are integrable.

In our previous work, we found a way to obtain PΔEs satisfying consistency properties by focusing on \(\tau\) functions of discrete Painlevé equations [14, 15]. Starting from discrete Painlevé equations, which occur in the lower half of Sakai’s classification [23], we obtained certain ABS-type equations [12–15, 17].

However, in recent studies of discrete Painlevé equations occurring in the top half of Sakai’s diagram, we obtained lattice equations that are quite different. Below we assume \(l, m \in \mathbb{Z}\) define points on a lattice. From equations that are of \(A_2^{1+}\)-type in the Sakai’s classification (see Table 1.1) we obtained the PΔE

\[
\frac{\hat{U}}{U} = \frac{(\alpha_{lm} - c_1)U - (\alpha_{lm} - (-1)^{l+m}c_2)\hat{U}}{(\alpha_{lm} + c_1)\hat{U} - (\alpha_{lm} + (-1)^{l+m}c_2)U},
\]

(1.1)

where

\[
\alpha_{lm} = \alpha_1(l + m + 1) - \alpha_2(l - m).
\]

(1.2)

Here, \(c_1, c_2, \{\alpha_1(k)\}_{k \in \mathbb{Z}}\) and \(\{\alpha_2(k)\}_{k \in \mathbb{Z}}\) are complex parameters, \(U\) denotes the dependent variable, \(U = U_{l,m}\), and bar and hat denote its forward iteration in \(l\)- and \(m\)-directions respectively. Furthermore, we discovered that this lattice equation is consistent, not around a cube, but around a cuboctahedron. See Remark 2.15 for the precise statement of this property, which we will refer to as the CACO property.

1.1. Main Results. Our main result is expressed by Theorem 2.13, which gives a classification of quad-equations around a cuboctahedron. These quad-equations lead to PΔEs that are consistent on a 3D lattice of quadrilaterals given by overlapping cuboctahedra as described in §§2.1 and 2.2. For conciseness, we state the theorem and all necessary lemmas in this paper without the details of the proofs. Comprehensive proofs will be given in a companion paper [10].
1.2. **Notation and Definitions.** In this section, we define notation and useful nomenclature that will be used throughout the paper.

**Definition 1.1.** Let \( Q = Q(x, y, z, w) \) be a multivariable polynomial over \( \mathbb{C} \).

(a) The polynomial \( Q \) is said to be a multi-affine polynomial, if it has the following form:

\[
Q = A_1xyzw + A_2xyz + A_3xyz + A_4xyz + \cdots + A_{16},
\]

where \( A_j, j = 1, \ldots, 16 \), are complex parameters.

(b) The multi-affine polynomial \( Q \) is said to be irreducible over \( \mathbb{C} \), if the equation \( Q = 0 \) can be solved for each argument, and the solution is a rational function of the other three arguments.

(c) The equation \( Q = 0 \) is said to be a quad-equation, if \( Q \) is an irreducible multi-affine polynomial.

(d) If \( Q'(x, y, z, w) \) is rational in each argument, and can be written as \( Q' = Q/R \), where \( R \) is a polynomial and \( Q \) is an irreducible multi-affine polynomial, then \( Q' = 0 \) is also called a quad-equation.

In what follows, for simplicity, we sometimes use the term “quad-equation \( Q \)” instead of “quad-equation \( Q = 0 \)”. Moreover, when we use this term, it will mean that \( Q \) is given as an irreducible multi-affine polynomial.

**Remark 1.2.** Notice that the condition of irreducibility implies non-vanishing of certain coefficients occurring in a quad-equation \( Q \). We do not restate such conditions separately in the body of the paper.

**Definition 1.3.** We remark that quad-equations given by polynomials \( P \) and \( Q \) are equivalent when \( P \) is a constant multiple of \( Q \). In this case, we write

\[
P \equiv Q.
\]

Moreover, given two sets of quad-equations \( \{ P_1, \ldots, P_n \} \) and \( \{ Q_1, \ldots, Q_n \} \) such that there exists a permutation \( \sigma \in S_n \) with

\[
P_1 \equiv Q_{\sigma(1)}, \ldots, P_n \equiv Q_{\sigma(n)},
\]

we write

\[
\{ P_1, \ldots, P_n \} = \{ Q_1, \ldots, Q_n \}.
\]

1.3. **Background.** Integrable systems are widely applicable models of science, occurring in fluid dynamics, particle physics and optics. The prototypical example is the famous Korteweg-de Vries (KdV) equation whose solitary wave-like solutions interact elastically like particles, leading to the invention of the term soliton. It is then natural to ask what discrete versions of such equations are also integrable. This question turns out to be related to consistency conditions for polynomials associated to faces of cubes as we explain below.

Reductions of integrable PDEs lead to Painlevé equations, which first arose in the search for new transcendental functions in the early 1900’s [6, 7, 21]. Again a natural question is to ask whether discrete versions exist with analogous properties. This question led to the discovery of second-order difference equations called the discrete Painlevé equations [8, 16, 22]).

It is now well-known that discrete Painlevé equations have initial value spaces with geometric structures that can be identified with affine Weyl groups [23]. Sakai showed that there are 22 types of initial value spaces as shown in Table 1.1.
Table 1.1. Types of spaces of initial values.

| Discrete type          | Type of space of initial values                                      |
|-----------------------|---------------------------------------------------------------------|
| Elliptic              | $A^{(1)}_0$                                                          |
| Multiplicative        | $A^{(1)*}_0, A^{(1)}_1, A^{(1)}_2, \ldots, A^{(1)}_8, A^{(1)}_{x'}$ |
| Additive              | $A^{(1)**}_0, A^{(1)}_*^1, A^{(1)}_*^2, D^{(1)}_4, \ldots, D^{(1)}_8, E^{(1)}_{6}, E^{(1)}_{7}, E^{(1)}_{8}$ |

Integrable discrete systems were discovered [19, 20, 22] from mappings that turn out to be consistent on multi-dimensional cubes. These are quad-equations in the sense of Definition 1.1. In [1–5], Adler-Bobenko-Suris et al. classified quad-equations satisfying the following properties. Consider a cube as shown in Figure 1.3. We assume that each face is associated with a quad-equation, and label these by $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$ as indicated in Figure 1.3, where the arguments of each polynomial are given in terms of the vertices as follows:

\[
\begin{align*}
\mathcal{A}(x_0, x_1, x_2, x_{12}) &= 0, \\
\tilde{\mathcal{A}}(x_3, x_{31}, x_{23}, x_{123}) &= 0, \\
\mathcal{B}(x_0, x_2, x_3, x_{23}) &= 0, \\
\tilde{\mathcal{B}}(x_1, x_{12}, x_{31}, x_{123}) &= 0, \\
\mathcal{C}(x_0, x_3, x_1, x_{31}) &= 0, \\
\tilde{\mathcal{C}}(x_2, x_{23}, x_{12}, x_{123}) &= 0.
\end{align*}
\]  

(1.7a, 1.7b, 1.7c)

Note that the six equations (1.7) provide 3 different ways to evaluate $x_{123}$ in terms of the other vertices.

**Definition 1.4** (CAC and tetrahedron properties).

(a) When the 3 results for $x_{123}$ given by equations (1.7) are equal, this system of quad-equations is said to be 3D consistent or consistent around a cube (CAC).

(b) When the result for $x_{123}$ turns out to depend only on $x_1, x_2, x_3,$ and $x_0$ depends only on $x_{12}, x_{23},$ and $x_{31},$ the system (1.7) is said to have the tetrahedron property.

![Figure 1.3. A cube with variables $x_0, \ldots, x_{123}$ associated with vertices and equations $\mathcal{A}, B, C, \tilde{A}, \tilde{B}, \tilde{C}$ associated with faces.](image)

A system of equations (1.7) with the CAC property can be interpreted as a PΔE on a cubic lattice $\mathbb{Z}^3$, by identifying iterates of a variable $U_{l,m,n}$ with the values at each vertex. On a 2-dimensional plane in this lattice we write $U_{l,m} = U_{l,m}$ and denote

\[ U = U_{l,m}, \quad \overline{U} = U_{l+1,m}, \quad \tilde{U} = U_{l,m+1}. \]  

(1.8)

Moreover, we associate parameters $\alpha = \alpha_l, \beta = \beta_m$ with each direction $l, m \in \mathbb{Z}$. We refer to such PΔEs as ABS equations.

It turns out that the CAC equations classified by Adler-Bobenko-Suris et al. in the cubic lattice $\mathbb{Z}^3$, contain many well known integrable PΔEs, including:
• discrete Schwarzian KdV equation [18, 19]

\[
\frac{(U - \overrightarrow{U})(\overrightarrow{U} - \overleftarrow{U})}{(U - \overleftarrow{U})(U - \overrightarrow{U})} = \alpha \frac{\overrightarrow{U} - \beta \overleftarrow{U}}{\alpha U - \beta U};
\]

(1.9)

• lattice modified KdV equation [1, 18, 20]

\[
\frac{\overrightarrow{U}}{U} = \frac{\alpha U - \beta \overleftarrow{U}}{\alpha U - \beta U}.
\]

(1.10)

• lattice potential KdV equation [9, 18]

\[
(U - \overrightarrow{U})(U - \overleftarrow{U}) = \alpha - \beta.
\]

(1.11)

1.4. Outline of the paper. In this paper, we list systems of quad-equations arising from consistency conditions on a cuboctahedron and its interior octahedron. Definitions and a statement of main results (see Theorem 2.13) are provided in §2. These rely on parametric conditions that are listed in detail in Appendix A.

2. Classification of quad-equations on a cuboctahedron

In this section, we consider quad-equations that are placed on quadrilaterals that appear as faces of a cuboctahedron together with three additional quadrilaterals that appear in its interior. These additional quadrilaterals are interior to octahedra that occur naturally inside a cuboctahedron (see Figure 2.1). We list the results of our classification for systems of equations that are self-consistently defined on such a configuration of 9 quadrilaterals. We name this property the CACO property.

Before we start the analysis of consistency, we explain our setting. Consider a cuboctahedron, as drawn in Figure 2.1. We fix one interior octahedron by the choice of a pair of parallel triangles, which are labelled by \(u_1, u_3, u_5\) and \(u_2, u_4, u_6\) respectively. Notice that we could have chosen a different pair of parallel triangles to define the octahedron, but this is equivalent to the pair we chose by rotation of the cuboctahedron.

In this paper, we do not consider the case when more than one such interior octahedron is included in the analysis. Our analysis suggests that the inclusion of additional octahedra leads to overdetermined equations.

The planes that pass through the vertices \(\{u_1, u_2, u_1, u_3\}, \{u_2, u_6, u_5, u_3\}\) and \(\{u_6, u_4, u_3, u_1\}\) give 3 quadrilaterals that lie in the interior of the cuboctahedron. These add to the 6 quadrilaterals that occur as faces of the cuboctahedron. This collection of 9 quadrilaterals is labelled \(\{Q_i, i = 1, \ldots, 9\}\), with \(Q_7, Q_8, Q_9\) being those that are internal to the octahedron. (See Figure 2.3.)

Throughout the paper, we denote the standard basis for \(\mathbb{R}^3\) by \(\{\epsilon_1, \epsilon_2, \epsilon_3\}\).

2.1. Consistency around an octahedron property. In this section, we give a self-contained definition of a consistency around an octahedron.

Consider the octahedron centered around the origin whose six vertices are given by

\[
V = \{\pm(\epsilon_1 + \epsilon_3), \pm(\epsilon_2 + \epsilon_3), \pm(\epsilon_1 + \epsilon_2)\}.
\]

(2.1)

We assign the variables \(u(I)\) to the vertices \(I \in V\) and impose the following relations:

\[
Q_1 (u_{4}, u_2, u_1, u_3) = 0, \quad Q_2 (u_2, u_6, u_5, u_3) = 0, \quad Q_3 (u_6, u_4, u_3, u_1) = 0,
\]

(2.2)

where \(Q_i, i = 1, 2, 3,\) are quad-equations and

\[
u_1 = u(\epsilon_2 + \epsilon_3), \quad u_2 = u(-\epsilon_1 - \epsilon_3), \quad u_3 = u(\epsilon_1 + \epsilon_2), \quad u_4 = u(-\epsilon_2 - \epsilon_3),
\]

\[
u_5 = u(\epsilon_1 + \epsilon_3), \quad u_6 = u(-\epsilon_1 - \epsilon_2).
\]

(2.3)
A cuboctahedron labelled with vertices $v_1, \ldots, v_6, u_1, \ldots, u_6$.

An octahedron labelled with vertices $u_1, \ldots, u_6$.

Figure 2.1. A cuboctahedron and an interior octahedron.

The planes that pass through the vertices $\{u_4, u_2, u_1, u_5\}$, $\{u_2, u_6, u_5, u_3\}$ and $\{u_6, u_4, u_3, u_1\}$ give 3 quadrilaterals that lie in the interior of the octahedron (see Figure 2.2). The quadrilaterals $Q_i$, $i = 1, 2, 3$, are assigned to the quadrilaterals.

Figure 2.2. The octahedron around the origin.

Definition 2.1. The octahedron with quad-equations $\{Q_1, Q_2, Q_3\}$ is said to have a consistency around an octahedron (CAO) property, if each quad-equation can be obtained from the other two equations. An octahedron is said to be a CAO octahedron, if it has the CAO property.

2.2. Consistency around a cuboctahedron property. In this section, we give the definition of a consistency around a cuboctahedron. We consider the cuboctahedron centered around the origin whose twelve vertices are given by

$$V = \left\{ \pm \epsilon_i \pm \epsilon_j \mid i, j \in \mathbb{Z}, 1 \leq i < j \leq 3 \right\}. \quad (2.4)$$

We assign the variables $u(l)$ to the vertices $l \in V$ and impose the following relations:

- $Q_1(u_5, u_1, v_5, v_4) = 0$, $Q_2(v_2, v_1, u_2, u_4) = 0$, $Q_3(u_3, u_5, v_3, v_2) = 0$, \quad (2.5a)
- $Q_4(v_6, v_5, u_6, u_2) = 0$, $Q_5(u_1, u_3, v_1, v_6) = 0$, $Q_6(v_4, v_3, u_4, u_6) = 0$, \quad (2.5b)
- $Q_7(u_4, u_2, u_1, u_5) = 0$, $Q_8(u_2, u_6, u_5, u_3) = 0$, $Q_9(u_6, u_4, u_3, u_1) = 0$. \quad (2.5c)
where $Q_i, i = 1, \ldots, 9$, are quad-equations and

\[
\begin{align*}
  u_1 &= u(e_2 + e_3), \quad u_2 = u(-e_1 - e_3), \quad u_3 = u(e_1 + e_2), \quad u_4 = u(-e_2 - e_3), \\
  u_5 &= u(e_1 + e_3), \quad u_6 = u(-e_1 - e_2), \quad v_1 = u(e_2 - e_3), \quad v_2 = u(-e_1 - e_3), \\
  v_3 &= u(e_1 - e_2), \quad v_4 = u(-e_2 + e_3), \quad v_5 = u(-e_1 + e_3), \quad v_6 = u(-e_1 + e_2). 
\end{align*}
\] (2.6)

Note that quad-equations $Q_i, i = 1, \ldots, 6$, are assigned to the faces of the cuboctahedron. Moreover, $u_i, i = 1, \ldots, 6$, collectively form the vertices of an octahedron and quad-equations $Q_i, i = 7, 8, 9$, are assigned to the quadrilaterals that appear as sections passing through four vertices of the octahedron. (See Figure 2.1.)

**Definition 2.2** (CACO property). The cuboctahedron with quad-equations $\{Q_1, \ldots, Q_9\}$ is said to have a *consistency around a cuboctahedron (CACO)* property, if the following properties hold.

(i): The octahedron with quad-equations $\{Q_1, Q_8, Q_9\}$ has the CAO property.

(ii): Assume that $u_1, \ldots, u_6$ are given so as to satisfy $Q_i = 0, i = 7, 8, 9$, and, in addition, $v_k$ is given, for some $k \in \{1, \ldots, 6\}$. Then, quad-equations $Q_i, i = 1 \ldots, 6$, determine the variables $v_j, j \in \{1, \ldots, 6\}\setminus\{k\}$, uniquely.

A cuboctahedron is said to be a *CACO cuboctahedron*, if it has the CACO property.

**Definition 2.3** (Square property). The CACO cuboctahedron with quad-equations $\{Q_1, \ldots, Q_9\}$ is said to have a *square property*, if there exist polynomials $K_i = K_i(x, y, z, w), i = 1, 2, 3$, where

\[
\begin{align*}
  \deg_x K_i = \deg_y K_i = 1, \quad 1 \le \deg_z K_i, \quad \deg_w K_i,
\end{align*}
\] (2.7)

satisfying

\[
\begin{align*}
  K_1(v_1, u_1, u_4, v_4) = 0, \quad K_2(v_2, u_2, u_5, v_5) = 0, \quad K_3(v_3, u_3, u_6, v_6) = 0. \quad (2.8)
\end{align*}
\]

Then, each equation $K_i = 0$ is called a *square equation*.

To explain Definition 2.2 in plainer language, we use an orthogonal projection of a cuboctahedron to two dimensions, shown in Figure 2.3. Note that the vertices labelled $u_1, \ldots, u_6$ form a hexagram, while the convex hull of the projection which connects the vertices $v_1, \ldots, v_6$ form a hexagon. Below, we refer to the collection of all such vertices in terms of $u(p)$, by using the notation of Equation (2.6).

**Remark 2.4.** Assume that all vertices in the inner hexagram and a vertex in the outer hexagon of an orthogonal projection of a cuboctahedron are given. (Refer to Figure 2.3.)

Label the given vertex in the outer hexagon by $u(p)$, for some $p$. Then, there are two ways of obtaining the value of $u(-p)$. One is obtained by using quad-equations that occur in the walk from $u(p)$ to $u(-p)$ clockwise around the outer hexagon, while the other is by using quad-equations in the anti-clockwise walk around the hexagon. The CACO property ensures that $u(-p)$ is determined uniquely regardless of the direction.

For example, if $p = e_2 - e_3$, then $u(p) = v_1$, while $u(-p) = v_4$. Walking in a clockwise direction means using quad-equations $Q_i, i = 2, 3, 6$, while the anti-clockwise direction means using quad-equations $Q_i, i = 5, 4, 1$. 

We now explain how to associate quad-equations with PDEs in three-dimensional space. This requires us to consider overlapping cuboctahedra that lead to two-dimensional tessellations consisting of quadrilaterals. For each given cuboctahedron, there are twelve overlapping cuboctahedra. One such pair of overlapped cuboctahedra is shown in Figure 2.4.

The two cuboctahedra in Figure 2.4 are coloured yellow and blue to distinguish them. Note that one of the vertices of the yellow cuboctahedron forms the centre of the blue one, and vice versa, one of the vertices of the blue octahedron forms the centre of the yellow one. Note also that there are four shared vertices, which are coloured green in the figure.

Notice that there are pairs of quadrilateral faces that form 2D planes in any sequence of overlapped cuboctahedra. For example, in Figure 2.4, the bottom faces form such a plane and so do the top faces. To fix notation, we take each cuboctahedron to be a copy of the one drawn in Figure 2.1a. So, for example, the bottom face of each cuboctahedron corresponds to the quadrilateral $Q_2$ and the top face corresponds to $Q_1$.

The twelve overlapping cuboctahedra around a given one provide six directions of tiling by quadrilaterals. For later convenience, we label directions by $\epsilon_i \pm \epsilon_j$, $1 \leq i < j \leq 3$, where $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ form the standard basis of $\mathbb{R}^3$. Vertices labelled in this way form the following set

$$\Omega = \left\{ \sum_{i=1}^{3} l_i \epsilon_i \right\} | l_i \in \mathbb{Z}, l_1 + l_2 + l_3 \in 2\mathbb{Z} \right\}. \quad (2.9)$$

Such vertices are interpreted as being iterated on each successive cuboctahedron. We are then led to 6 partial difference equations that occur on these iterated cuboctahedra:

$$P_1 \left( u_{13}, u_{23}, u_{13}, u_{23} \right) = 0, \quad P_2 \left( u_{13}, u_{12}, u_{12}, u_{13} \right) = 0, \quad (2.10a)$$
where $u = u(I)$ and $I \in \Omega$. Here, $P_i$, $i = 1, \ldots, 6$, are quad-equations, and subscripts $\tilde{i}$ and $\tilde{j}$ mean $I \rightarrow I + \epsilon_i$ and $I \rightarrow I - \epsilon_j$, respectively.

Conversely, given $I \in \Omega$, we obtain the cuboctahedron centered around $I$. We refer to its quad-equations as before by $\{Q_I(I), \ldots, Q_9(I)\}$. Moreover, the overlapped region gives an octahedron centred around $I + \epsilon_3$, and we label its quad-equations by $\{\tilde{Q}_I(I), \tilde{Q}_2(I), \tilde{Q}_3(I)\}$.

Each such quad-equation is identified with the 6 partial difference equations given in Equations (2.10) in the following way. Firstly, for $Q_1, \ldots, Q_9$, we use

$$
P_3 \left( u \frac{\partial^3}{\partial u^3}, u \frac{\partial^3}{\partial u^3}, u \frac{\partial^3}{\partial u^3} \right) = 0, \quad P_4 \left( u \frac{\partial^4}{\partial u^4}, u \frac{\partial^4}{\partial u^4}, u \frac{\partial^4}{\partial u^4} \right) = 0,
$$

(2.10b)

$$
P_5 \left( u \frac{\partial^5}{\partial u^5}, u \frac{\partial^5}{\partial u^5}, u \frac{\partial^5}{\partial u^5} \right) = 0, \quad P_6 \left( u \frac{\partial^6}{\partial u^6}, u \frac{\partial^6}{\partial u^6}, u \frac{\partial^6}{\partial u^6} \right) = 0,
$$

(2.10c)

where $u = u(I)$ and $I \in \Omega$. Here, $P_i$, $i = 1, \ldots, 6$, are quad-equations, and subscripts $\tilde{i}$ and $\tilde{j}$ mean $I \rightarrow I + \epsilon_i$ and $I \rightarrow I - \epsilon_j$, respectively.

Figure 2.4. The two overlapping cuboctahedra, one with yellow and one with blue. The green vertices and edges correspond to where the two cuboctahedra overlap.

We are now in a position to define the CACO property for PDEs.

**Definition 2.5** (CACO property for PDEs). We transfer the definition of CACO and square properties to the system of PDEs (2.10) in the obvious way, as follows.

(a) The system of PDEs (2.10) is said to have the CACO property, if the following properties hold.

(i): The cuboctahedra with quad-equations $\{Q_1(I), \ldots, Q_9(I)\}$ have the CACO property.

(ii): The octahedra with quad-equations $\{\tilde{Q}_1(I), \tilde{Q}_2(I), \tilde{Q}_3(I)\}$ have the CAO property.
Moreover, the system \((2.10)\) is said to have the square property, if the cuboctahedra with quad-equations \([Q_1(I), \ldots, Q_0(I)]\) have the square property.

(c) Each PAE in \((2.10)\) is said to have the CACO property, if the system \((2.10)\) has the CACO property.

**Remark 2.6.** Property (ii) in Definition 2.5 means that the octahedra arising from the overlaps between cuboctahedra have the CAO property. Note that these octahedra are different from the interior octahedra arising inside each individual cuboctahedron.

2.3. Classification of quad-equations on a cuboctahedron. In this section, we classify the quad-equations on a cuboctahedron by using the conditions (CO1)–(CO3) defined below. Note that the following description is given only on one individual cuboctahedron.

Consider the system of quad equations on a cuboctahedron as described by Equations (2.5). We assume that the cuboctahedron with quad-equations \([Q_1, \ldots, Q_0]\) satisfies the following properties.

**Condition 2.7.** The quad-equations \([Q_1, \ldots, Q_0]\) satisfy:

(CO1) It has the CACO property.
(CO2) It has the square property.
(CO3) There exist irreducible multi-affine polynomials \(P_i\), \(i = 1, 2, 3\), such that the system of equations is given by

\[
\begin{align*}
Q_1 &= P_1(u_5, u_1, v_3, A_1^{(j)}, \ldots, A_1^{(j)}), \quad Q_2 &= P_1(v_1, u_2, u_4; A_1^{(j)}, \ldots, A_1^{(j)}), \\
Q_3 &= P_2(u_3, v_3, A_2^{(k)}, A_2^{(k)}), \quad Q_4 &= P_2(v_6, u_6, u_2; A_2^{(k)}, A_2^{(k)}), \\
Q_5 &= P_3(u_1, v_1, v_5, A_3^{(j)}, A_3^{(j)}), \quad Q_6 &= P_3(v_4, u_4, u_6; A_3^{(j)}, A_3^{(j)}), \\
Q_7 &= P_4(u_4, u_2, u_1; B_1^{(j)}, B_1^{(j)}), \quad Q_8 &= P_4(u_2, u_6, u_5; B_1^{(j)}, B_1^{(j)}), \\
Q_9 &= P_5(u_6, u_3, u_4; B_3^{(j)}, B_3^{(j)}), \quad Q_0 &= P_5(v_6, u_3, u_5; B_3^{(j)}, B_3^{(j)}),
\end{align*}
\]

where \(k_1, k_2, k_3 \in \mathbb{Z}_{20}\), and \(A_1^{(j)}\) and \(B_1^{(j)}\) are complex parameters. If \(A_1^{(j)} = 0\), then \(B_1^{(j)} = 0\), and vice versa.

Clearly, if the cuboctahedron undergoes a reflection, analogous equations should hold. This leads to the consideration of a symmetry group, generated by the reflections on the vertices of the cuboctahedron. This motivates the definition of the group:

\[
G^{(CO)} = \langle s_{12}, s_{23}, s_{13}, \iota \rangle,
\]

where \(s_{12}, s_{23}, s_{13}\) and \(\iota\) are transformations defined by the following actions:

\[
\begin{align*}
s_{12} : & \quad u_1 \leftrightarrow u_5, \quad u_2 \leftrightarrow u_4, \quad v_1 \leftrightarrow v_2, \quad v_3 \leftrightarrow v_6, \quad v_4 \leftrightarrow v_5, \\
s_{23} : & \quad u_2 \leftrightarrow u_6, \quad u_3 \leftrightarrow u_5, \quad v_1 \leftrightarrow v_4, \quad v_2 \leftrightarrow v_3, \quad v_5 \leftrightarrow v_6, \\
s_{13} : & \quad u_1 \leftrightarrow u_3, \quad u_4 \leftrightarrow u_6, \quad v_1 \leftrightarrow v_6, \quad v_2 \leftrightarrow v_5, \quad v_3 \leftrightarrow v_4, \\
\iota : & \quad u_1 \leftrightarrow u_4, \quad u_2 \leftrightarrow u_5, \quad u_3 \leftrightarrow u_6, \quad v_1 \leftrightarrow v_4, \quad v_2 \leftrightarrow v_5, \quad v_3 \leftrightarrow v_6,
\end{align*}
\]

or, equivalently, the following actions on the standard basis:

\[
\begin{align*}
s_{12} : & \quad e_1 \leftrightarrow e_2, \quad s_{23} : e_2 \leftrightarrow e_3, \quad s_{13} : e_1 \leftrightarrow e_3, \quad \iota : \quad [e_1, e_2, e_3] \leftrightarrow [-e_1, -e_2, -e_3].
\end{align*}
\]

The transformation \(s_{12}\) can be visualized as the reflection across a slice of the cuboctahedron that divides it into two equal pieces. The slice in this case is taken through the vertices \([u_3, u_6]\) and the center of the edge connecting the vertices \([u_1, u_5]\). Similarly, the transformations \(s_{23}\) and \(s_{13}\) are reflections across two other ways of slicing the cuboctahedron. The slice corresponding to \(s_{23}\) goes through the vertices \([u_1, u_4]\) and the center of the edge connecting the vertices \([u_2, u_5]\), while the one for \(s_{13}\) goes through the vertices \([u_2, u_3]\) and the center of the edge connecting the vertices \([u_1, u_3]\). Moreover, the transformation \(\iota\)
corresponds to the reflection of vertices about the centre of the cuboctahedron. (See Figure 2.1.)

We now collect the transformations under which quad-equations on a cuboctahedron remain invariant. Using the Möbius transformations and the transformation group $G^{(CO)}$, we define the equivalence relation $\sim_{co}$ as follows.

**Definition 2.8.** The cuboctahedron with quad-equations $\{Q_1, \ldots, Q_6\}$ is said to be equivalent under $\sim_{co}$ to the cuboctahedron with quad-equations $\{Q'_1, \ldots, Q'_6\}$, i.e.

$$\{Q_1, \ldots, Q_6\} \sim_{co} \{Q'_1, \ldots, Q'_6\},$$

if the following properties hold. Let a Möbius transformation of the variables $u_i$ and $v_j$, where $i, j = 1, \ldots, 6$, be denoted by $r$ and elements of $G^{(CO)}$ be denoted $\sigma$. Then we have

$$\{Q_1, \ldots, Q_6\} = \{r \sigma Q'_1, \ldots, r \sigma Q'_6\}. \tag{2.18}$$

Our purpose in this section is to obtain the quotient set $\{Q_1, \ldots, Q_6\}/ \sim_{co}$. Using the square property, we obtain the following lemma.

**Lemma 2.9.** Let $N3$ and $N1$ be polynomials given by

$$N3(x, y, z, w; A_1, A_2, A_3, A_4) = A_1 xy + A_2 zw + A_3 xw + A_4 yz, \tag{2.19}$$

$$N1(x, y, z, w; A_1, A_2, A_3, A_4) = A_1 (x + z)(y + w) + A_2 (x + z) + A_3 (y + w) + A_4. \tag{2.20}$$

**Under the equivalence relation $\sim_{co}$, the square property leads to three canonical classes of polynomials $P_i$, $i = 1, 2, 3$, in Condition 2.7, which are given by**

$$P_1 = N3(x, y, z, w; A_1, A_2, A_3, A_4),$$

$$P_2 = N3(x, y, z, w; A_1, A_2, A_3, A_4),$$

$$P_3 = N3(x, y, z, w; A_1, A_2, A_3, A_4).$$

$$P_1 = N3(x, y, z, w; A_1, A_2, A_3, A_4),$$

$$P_2 = N3(x, y, z, w; A_1, A_2, A_3, A_4),$$

$$P_3 = N3(x, y, z, w; A_1, A_2, A_3, A_4).$$

$$P_1 = N1(x, y, d_1 z, d_2 w; A_1, A_2, A_3, A_4),$$

$$P_2 = N1(x, y, d_2 z, d_3 w; A_1, A_2, A_3, A_4),$$

$$P_3 = N1(x, y, d_3 z, d_1 w; A_1, A_2, A_3, A_4).$$

where $d_{i j}, j = 1, 2, 3, \text{ are complex parameters satisfying}$

$$d_1 d_2 d_3 = -1. \tag{2.24}$$

These canonical classes lead to corresponding quad-equations $\{Q_1, \ldots, Q_6\}$ given by (2.13) with $k_1 = k_2 = k_3 = 4$ and $|P_1, P_2, P_3|$ being one of (2.21)–(2.23). The resulting three types of quad-equations are expressed by Lemmas 2.10, 2.11, and 2.12 respectively.

Each type leads to lengthy expressions satisfied by the parameters in the quad-equations. Due to the length of these expressions, the details are collected in Appendix A, to enable concise references to the conditions on parameters in the statements of the following lemmas.

**Lemma 2.10 (Type I).** Consider the set of polynomials $\{P_1, P_2, P_3\}$ defined by (2.21). Then, the cuboctahedron with quad-equations (2.13) has the CACO property and the square property, if and only if the parameters satisfy

$$B_1^{(1)} = B_4^{(1)} - B_3^{(1)}, \quad B_2^{(2)} = B_2^{(1)} B_3^{(3)} - B_3^{(3)}, \quad B_2^{(3)} = B_4^{(1)} - B_2^{(1)} B_3^{(3)}, \quad B_4^{(1)} = B_2^{(1)} - B_3^{(3)} B_3^{(3)} \tag{1.II},$$

and one of the conditions (I-1) and (I-2) (or equivalent conditions under the equivalence relation $\sim_{co}$).
Lemma 2.11 (Type II). Consider the set of polynomials \( \{P_1, P_2, P_3\} \) defined by (2.22). Then, the cuboctahedron with quad-equations (2.13) has the CACO property and the square property, if and only if the parameters satisfy

\[
C^{(13)} = C^{(22)} = C^{(31)} = C^{(44)} = 0, \quad \text{(II.a)}
\]
\[
C^{(12)}, C^{(14)}, C^{(21)}, C^{(23)}, C^{(32)}, C^{(34)}, C^{(41)}, C^{(43)} \neq 0, \quad \text{(II.b)}
\]
\[
A_1^{(0)}, A_2^{(0)}, A_3^{(0)}, A_4^{(0)}, B_1^{(0)}, B_2^{(0)}, B_3^{(0)}, B_4^{(0)} \neq 0 \quad (i = 1, 3), \quad \text{(II.c)}
\]

along with condition (1.11) and one of the conditions (II-1.2) (or equivalent conditions under the equivalence relation \( \sim_{e_0} \)). Here, the parameters \( C^{(i)} \) are given by (A.1)–(A.24).

Lemma 2.12 (Type III). Consider the set of polynomials \( \{P_1, P_2, P_3\} \) defined by (2.23). Then, the cuboctahedron with quad-equations (2.13) has the CACO property and the square property, if and only if the parameters satisfy

\[
B_1^{(2)} = B_3^{(1)} - B_4^{(1)} B_2^{(3)}, \quad B_2^{(2)} = B_3^{(1)} B_1^{(3)} - B_4^{(1)} B_2^{(3)}, \quad \text{(III)}
\]
\[
B_3^{(2)} = B_4^{(1)} B_3^{(3)} - B_2^{(1)} B_4^{(3)}, \quad B_4^{(2)} = B_4^{(1)} B_3^{(3)} - B_2^{(1)} B_4^{(3)},
\]

and one of the conditions (Type III-1-1)–(Type III-3-16) in Appendix A.3 (or equivalent conditions under the equivalence relation \( \sim_{e_0} \)).

The results of Lemmas 2.10–2.12 lead to the following theorem.

**Theorem 2.13.** Under the equivalence relation \( \sim_{e_0} \), any cuboctahedron with quad-equations \( \{Q_1, \ldots, Q_5\} \) satisfying the properties (CO1)–(CO3) is equivalent to the cuboctahedron with quad-equations (2.13), where the set of polynomials \( \{P_1, P_2, P_3\} \) is given by one of (2.21)–(2.23), and the conditions of the parameters are listed in Lemmas 2.10–2.12.

2.4. Example: a system of P\( \Delta \)Es which has the CACO property. In this section, we provide an example of a system of quad-equations with the CACO property. The example is given by Equation (1.1) which occurs as an equation on a quadrilateral arising in a system of P\( \Delta \)Es, which has the CACO property.

Let us consider the following system of P\( \Delta \)Es:

\[
P_1 \left( u_{17}, u_{27}, u_{12}, u_{23} \right) = N^3 \left( u_{17}, u_{27}, u_{12}, u_{23}; a_2, a_3, a_4 \right) = 0, \quad (2.25a)
\]
\[
P_2 \left( u_{17}, u_{27}, u_{12}, u_{23} \right) = N^3 \left( u_{17}, u_{27}, u_{12}, u_{23}; a_6, a_7, a_8 \right) = 0, \quad (2.25b)
\]
\[
P_3 \left( u_{17}, u_{27}, u_{12}, u_{23} \right) = N^3 \left( u_{17}, u_{27}, u_{12}, u_{23}; a_{10}, a_{11}, a_{12} \right) = 0, \quad (2.25c)
\]
\[
P_4 \left( u_{23}, u_{13}, u_{17}, u_{27} \right) = N^3 \left( u_{23}, u_{13}, u_{17}, u_{27}; a_1, a_2, a_3 \right) = 0, \quad (2.25d)
\]
\[
P_5 \left( u_{13}, u_{12}, u_{17}, u_{27} \right) = N^3 \left( u_{13}, u_{12}, u_{17}, u_{27}; a_5, a_6, a_7 \right) = 0, \quad (2.25e)
\]
\[
P_6 \left( u_{12}, u_{23}, u_{17}, u_{27} \right) = N^3 \left( u_{12}, u_{23}, u_{17}, u_{27}; a_9, a_{10}, a_{11}, a_{12} \right) = 0, \quad (2.25f)
\]

where \( u = u(l), \ l = \sum_{i=1}^{3} l_{i} \in \Omega \) (recall that the set of vertices \( \Omega \) on a cuboctahedron is defined by (2.9)) and

\[
a_1 = a_{12} + (-1)^{i+j} \delta_2 - (-1)^{j+i} \delta_3, \quad a_2 = a_{12} - (-1)^{j+i} \delta_2 + (-1)^{j+i} \delta_3, \quad (2.26a)
\]
\[
a_3 = a_{31} - c - (-1)^{j+i} \delta_1, \quad a_4 = a_{31} + c - (-1)^{j+i} \delta_1, \quad (2.26b)
\]
\[
a_5 = a_{23} - (-1)^{i+j} \delta_3 - (-1)^{j+i} \delta_1, \quad a_6 = a_{23} - (-1)^{j+i} \delta_3 + (-1)^{j+i} \delta_1, \quad (2.26c)
\]
\[
a_7 = a_{32} - c + (-1)^{j+i} \delta_2, \quad a_8 = a_{32} + c + (-1)^{j+i} \delta_2, \quad (2.26d)
\]
\[
a_9 = a_{31} + (-1)^{i+j} \delta_1 - (-1)^{j+i} \delta_2, \quad a_{10} = a_{31} - (-1)^{i+j} \delta_1 + (-1)^{j+i} \delta_2, \quad (2.26e)
\]
\[
a_{11} = a_{31} - c + (-1)^{j+i} \delta_3, \quad a_{12} = a_{31} + c + (-1)^{j+i} \delta_3, \quad (2.26f)
\]

where

\[
a_{ij} = a_i(l_i) - a_j(l_j), \quad i, j \in \{1, 2, 3\}. \quad (2.27)
\]
Here, \([\alpha_1(l), \alpha_2(l), \alpha_3(l)]\),\(l \in \mathbb{Z}\), \(c\), and \(\delta_j, j = 1, 2, 3\), are complex parameters.

We can easily verify that the system (2.25) has the CACO property (see Definition 2.5).

Remark 2.14. The system (2.25) has the square property (see Definition 2.5). Define the polynomial \(K = K(x, y, z, w; A_1, A_2, A_3, A_4, A_5, A_6)\) by

\[
K = (A_1^2(x-z)(y-w) + A_2^2(x-y)(w-z))(y-z)
+ (A_3 - A_4 - A_5 + A_6)((x-z)(y-w) + A_2(x-y)(w-z))(y + z)
+ (A_3 - A_6)(A_4 - A_5)(y-z)(yz - xy - xz + wx)
+ ((A_3 - A_4)y - (A_4 - A_6)z)((A_3 - A_5)z - (A_2 - A_6)y)(x-w)
+ (A_4 - A_5)^2(x-w)yz.
\]

Remark 2.15. The system of equations (2.25) is classified in Type 1 and satisfies the condition (1-1).
APPENDIX A. CONDITIONS IN LEMMAS 2.10–2.12

This appendix collects and presents all the explicit conditions needed for the statements of Lemmas 2.10–2.12. (Note that any terms appearing in the denominators of the equations in this appendix are assumed to be non-zero.)

A.1. For Lemma 2.10. Lemma 2.10 describes system of quad-equations of Type I, which has two sub-cases called I-1 and I-2. The first case is given by

\[ \begin{align*}
(A_1^{(1)} A_2^{(1)} B_1^{(1)} + A_2^{(1)} A_3^{(1)} B_1^{(1)} + A_3^{(1)} A_4^{(1)} B_1^{(1)} B_4^{(1)}) &= -(A_1^{(2)} A_2^{(2)} B_1^{(2)} + A_2^{(2)} A_3^{(2)} B_1^{(2)} + A_3^{(2)} A_4^{(2)} B_1^{(2)} B_4^{(2)}), \\
((A_1^{(1)} A_2^{(1)} B_1^{(1)} + A_2^{(1)} A_3^{(1)} B_1^{(1)} + A_3^{(1)} A_4^{(1)} B_1^{(1)} B_4^{(1)}) (A_2^{(2)} A_3^{(2)} B_1^{(2)} + A_3^{(2)} A_4^{(2)} B_1^{(2)} + A_4^{(2)} A_1^{(2)} B_1^{(2)} B_4^{(2)})) &= -(A_1^{(1)} A_2^{(1)} B_1^{(1)} + A_2^{(1)} A_3^{(1)} B_1^{(1)} + A_3^{(1)} A_4^{(1)} B_1^{(1)} B_4^{(1)}), \\
((A_1^{(2)} A_2^{(2)} B_1^{(2)} + A_2^{(2)} A_3^{(2)} B_1^{(2)} + A_3^{(2)} A_4^{(2)} B_1^{(2)} B_4^{(2)}) (A_2^{(1)} A_3^{(1)} B_1^{(1)} + A_3^{(1)} A_4^{(1)} B_1^{(1)} + A_4^{(1)} A_1^{(1)} B_1^{(1)} B_4^{(1)})) &= -(A_1^{(2)} A_2^{(2)} B_1^{(2)} + A_2^{(2)} A_3^{(2)} B_1^{(2)} + A_3^{(2)} A_4^{(2)} B_1^{(2)} B_4^{(2)}).
\end{align*} \]

(I-1)

and the images of these equations under the symmetry 1 \(\leftrightarrow\) 2 and 3 \(\leftrightarrow\) 4 applied to the subscripts.

A.2. For Lemma 2.11.

There are two sub-cases in Type II, which correspond to the choice of sign in the equations below. (Type II-1: all positive signs, Type II-2: all negative signs):

\[ \begin{align*}
C^{(32)} &= \pm C^{(12)}, & C^{(33)} &= \pm C^{(15)}, & C^{(34)} &= \pm C^{(14)}, & C^{(35)} &= \pm C^{(11)}, \\
C^{(36)} &= \pm C^{(16)}, & C^{(41)} &= \pm C^{(21)}, & C^{(42)} &= \pm C^{(26)}, & C^{(43)} &= \pm C^{(23)}, \\
C^{(44)} &= \pm C^{(24)}, & C^{(45)} &= \pm C^{(25)}, & C^{(46)} &= \pm C^{(24)}.
\end{align*} \]

Here, the parameters \(c^{(1)}(1), \ldots, c^{(1)}(4), \ldots, c^{(4)}(1), \ldots, c^{(4)}(6)\) are given by

\[ \begin{align*}
C^{(11)} &= A_1^{(1)} A_2^{(1)} A_3^{(1)} A_4^{(1)} B_1^{(1)}, \\
C^{(12)} &= A_1^{(1)} A_2^{(1)} A_3^{(1)} A_4^{(1)} B_1^{(1)}, \\
C^{(13)} &= A_1^{(1)} A_2^{(1)} A_3^{(1)} A_4^{(1)} B_1^{(1)}, \\
C^{(14)} &= A_1^{(1)} A_2^{(1)} A_3^{(1)} A_4^{(1)} B_1^{(1)}, \\
C^{(15)} &= A_1^{(1)} A_2^{(1)} A_3^{(1)} A_4^{(1)} B_1^{(1)}, \\
C^{(16)} &= A_1^{(1)} A_2^{(1)} A_3^{(1)} A_4^{(1)} B_1^{(1)}.
\end{align*} \]

A.3. For Lemma 2.12.
\[ C^{(3)} = A_1^{(1)} B_2^{(4)} + A_2^{(3)} B_3^{(3)} + A_3^{(3)} A_4^{(3)} + A_4^{(3)} B_3^{(4)}, \quad \text{(A.15)} \]
\[ C^{(4)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.16)} \]
\[ C^{(5)} = A_3^{(3)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.17)} \]
\[ C^{(6)} = A_3^{(3)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.18)} \]

\[ C^{(7)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.19)} \]
\[ C^{(8)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.20)} \]
\[ C^{(9)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.21)} \]
\[ C^{(10)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.22)} \]
\[ C^{(11)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.23)} \]
\[ C^{(12)} = A_1^{(1)} B_2^{(4)}(A_2^{(3)} B_3^{(3)} A_4^{(3)} + A_2^{(3)} B_3^{(3)} A_3^{(3)} + A_2^{(3)} B_3^{(3)} A_2^{(3)} + A_2^{(3)} B_3^{(3)} A_1^{(3)}), \quad \text{(A.24)} \]

where
\[ \tilde{B}_1 = B_1^{(1)} B_2^{(3)} + B_1^{(2)} B_3^{(3)}, \quad \tilde{B}_2 = B_1^{(1)} B_3^{(1)} + B_1^{(2)} B_3^{(3)}, \quad \tilde{B}_3 = B_1^{(1)} B_3^{(3)} + B_4^{(1)} B_4^{(3)}. \quad \text{(A.25)} \]
\[ \tilde{B}_4 = B_1^{(2)} B_3^{(1)} + B_5^{(2)} B_5^{(3)}. \quad \text{(A.26)} \]

A.3. For Lemma 2.12.

Type III contains 32 subcases, which are listed below.

(Type III-1-1): the condition
\[ A_1^{(1)} = B_1^{(1)} = A_1^{(2)} = B_1^{(2)} = A_1^{(3)} = B_1^{(3)} = 0, \quad \text{(III-1)} \]

and
\[ \begin{align*}
A_3^{(3)} &= \frac{A_2^{(2)} A_3^{(3)} A_4^{(3)}}{A_1^{(1)} A_2^{(2)} A_3^{(3)}}, \\
B_3^{(3)} &= -\frac{A_2^{(3)} B_3^{(1)} d_3 (A_4^{(1)} B_4^{(1)} + A_4^{(1)} B_4^{(1)} d_3^2 d_3)}, \\
B_4^{(3)} &= \frac{A_2^{(3)} B_3^{(1)} d_3 (A_4^{(1)} A_2^{(3)} A_3^{(3)} (1 - d_3) + A_4^{(1)} A_2^{(3)} (1 - d_3))}{2 A_2^{(3)} A_3^{(3)} (1 + d_3 - 2 A_4^{(1)} d_3)}, \\
B_4^{(5)} &= \frac{2 A_2^{(3)} A_3^{(3)} (1 + d_3 - 2 A_4^{(1)} d_3)}{2 A_2^{(3)} B_3^{(1)} d_3}. \\
\end{align*} \quad \text{(III-1-1)} \]

(Type III-1-2): the condition (III-1) and
\[ \begin{align*}
A_3^{(3)} &= -\frac{A_2^{(2)} A_3^{(3)} A_4^{(3)}}{A_1^{(1)} A_2^{(2)} A_3^{(3)}}, \\
A_4^{(3)} &= \frac{A_2^{(3)} A_3^{(3)} B_3^{(1)} d_3 (1 + d_3) + A_2^{(3)} A_4^{(1)} (1 - d_3 d_3)}{A_2^{(3)} A_3^{(3)} (1 + d_3)}, \\
A_4^{(3)} &= \frac{A_2^{(3)} A_3^{(3)} (1 + d_3) + A_2^{(3)} A_4^{(1)} (1 - d_3 d_3)}{A_2^{(3)} A_3^{(3)} (1 + d_3)}. \\
\end{align*} \quad \text{(III-1-2)} \]

(Type III-1-3): the condition (III-1) and
\[ A_3^{(3)} = -\frac{A_2^{(2)} A_3^{(3)} A_4^{(3)}}{A_1^{(2)} A_3^{(3)} A_4^{(3)}}, \quad A_4^{(3)} = \frac{A_2^{(3)} A_3^{(3)} B_3^{(1)} d_3}{A_2^{(1)} A_3^{(3)}}, \quad d_2 = -1. \quad \text{(III-1-3)} \]

(Type III-1-4): the condition (III-1) and
\[ A_3^{(3)} = -\frac{A_2^{(2)} A_3^{(3)} A_4^{(3)}}{A_1^{(2)} A_3^{(3)} A_4^{(3)}}, \quad d_2 = d_3 = -1. \quad \text{(III-1-4)} \]

(Type III-2-1): the condition
\[ A_1^{(1)} = B_1^{(1)} = 0, \quad A_1^{(2)}, B_1^{(2)}, A_3^{(3)}, B_3^{(3)} \neq 0, \quad \text{(III-2)} \]
and
\[
\begin{align*}
B_2^{(1)} &= \frac{A_1^{(1)} B_3^{(1)}}{A_1^{(1)}}, & B_3^{(1)} &= -\frac{A_2^{(2)} B_4^{(3)} - A_3^{(2)} B_3^{(3)}}{A_2^{(2)}}, \\
A_4^{(1)} &= \frac{A_1^{(1)} A_1^{(3)}(A_2^{(2)} A_3^{(2)} - A_2^{(1)} A_4^{(1)})}{A_3^{(1)} A_4^{(2)} A_1^{(3)}}, \\
B_4^{(1)} &= -\frac{(A_4^{(1)} A_1^{(3)} + A_2^{(1)} A_3^{(3)}) B_3^{(1)}}{A_2^{(1)} A_1^{(3)}}, & d_2 &= d_3 = -1.
\end{align*}
\] (Type III-2-2): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= \frac{A_1^{(1)} B_3^{(1)}}{A_1^{(1)}}, & B_3^{(1)} &= -\frac{(A_3^{(2)} A_1^{(3)} + A_1^{(2)} A_3^{(2)}) B_1^{(3)}}{A_1^{(2)} A_3^{(2)}}, \\
B_4^{(1)} &= -\frac{(A_4^{(1)} A_1^{(3)} + A_2^{(1)} A_3^{(3)}) B_3^{(1)}}{A_2^{(1)} A_1^{(3)}}, & d_2 &= d_3 = -1.
\end{align*}
\] (Type III-2-3): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= \frac{A_1^{(1)} B_3^{(1)}}{A_2^{(1)}}, & A_2^{(2)} &= \frac{A_2^{(2)} B_4^{(1)}(1 - d_2) + A_3^{(2)} A_4^{(1)} A_1^{(3)}}{B_2^{(2)} d_2^2}, \\
A_3^{(2)} &= \frac{A_2^{(2)} A_1^{(3)}}{A_3^{(2)}}, & A_3^{(3)} &= \frac{A_2^{(2)} A_1^{(3)}(A_2^{(2)} B_4^{(1)}(1 - d_2) + A_3^{(2)} B_3^{(1)} d_2^2)}{A_3^{(2)} A_2^{(1)} A_1^{(3)}}, \\
B_3^{(1)} &= -\frac{A_3^{(2)} B_4^{(1)}(1 - d_2)}{A_3^{(2)} d_2^2}, & B_4^{(1)} &= -\frac{A_3^{(2)} B_3^{(1)}(1 - d_2)}{A_3^{(2)} d_2^2}, & d_3 &= -\frac{1}{d_2^2}.
\end{align*}
\] (Type III-2-4): the condition (III-2) and
\[
\begin{align*}
A_3^{(3)} &= -\frac{(A_3^{(2)} B_4^{(1)}(1 - d_2) + A_4^{(2)} B_1^{(1)} d_2^2)}{A_3^{(2)} B_2^{(1)} d_2^2}, & A_3^{(1)} &= \frac{A_3^{(2)} B_1^{(1)}}{B_3^{(1)}}, \\
A_4^{(3)} &= \frac{A_3^{(2)} B_4^{(1)}}{A_3^{(2)}}, & A_4^{(4)} &= \frac{A_3^{(2)} A_1^{(3)} B_4^{(1)}(1 - d_2) + A_3^{(2)} A_4^{(3)} B_2^{(1)} d_2^2}{A_3^{(2)} A_1^{(3)} B_2^{(1)} d_2^2}, \\
A_3^{(2)} &= \frac{A_3^{(2)} A_1^{(3)}}{A_1^{(2)}}, & B_3^{(3)} &= -\frac{A_3^{(2)} B_3^{(1)}(1 - d_2)}{A_3^{(2)} d_2^2}, & d_3 &= -\frac{1}{d_2^2}.
\end{align*}
\] (Type III-2-5): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= -\frac{A_1^{(1)} B_3^{(1)}}{A_2^{(1)}}, & A_2^{(2)} &= \frac{A_2^{(2)} A_3^{(1)}(1 - d_2)}{A_3^{(1)} d_2^2(1 + d_2)}, \\
A_4^{(3)} &= \frac{2 A_4^{(1)} A_3^{(2)}(1 - d_2) + 2 A_2^{(1)} A_3^{(2)}(1 + d_2^2)}{A_2^{(1)} d_2^2(1 + d_2^2)}, & A_4^{(4)} &= \frac{A_3^{(2)} A_1^{(3)} A_3^{(3)}}{A_2^{(1)} A_1^{(3)} d_2^2(1 + d_2^2)} - \frac{A_3^{(2)} B_4^{(1)}(1 - d_2)}{A_3^{(2)} d_2^2(1 + d_2^2)}, \\
A_3^{(3)} &= \frac{A_1^{(1)} A_3^{(1)}(1 - d_2)}{A_3^{(1)} d_2^2(1 + d_2)}, & B_3^{(3)} &= \frac{A_3^{(2)} B_3^{(1)}(1 + d_2^2)}{A_3^{(2)} d_2^2(1 + d_2)}, & d_3 &= \frac{1}{d_2^2}.
\end{align*}
\] (III-2-5)
(Type III-2-6): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= -\frac{A_2^{(4)}}{A_2^{(1)}}, \\
A_4^{(1)} &= \frac{1 - d_2 A_2^{(4)} A_3^{(3)} B_4^{(1)} + 2 A_2^{(1)} A_3^{(3)} B_3^{(1)} d_2^2}{A_2^{(4)}(1 - d_2)} - A_1^{(3)} B_3^{(1)} (1 + d_2), \\
A_2^{(2)} &= A_1^{(3)} B_3^{(1)} (1 - d_2) + A_3^{(3)} B_3^{(1)} (1 - d_2) d_2^2 + A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 + d_2^2), \\
A_4^{(2)} &= \frac{A_2^{(1)} A_3^{(3)} B_4^{(1)} (1 + d_2)}{A_2^{(3)} A_3^{(3)} (1 + d_2)} + \frac{A_3^{(3)} B_4^{(1)} (1 + d_2^2)}{A_2^{(3)} A_3^{(3)} (1 + d_2^2)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 + d_2^2), \\
A_2^{(3)} &= \frac{2 A_2^{(1)} A_3^{(3)} (1 - d_2)}{A_2^{(3)} A_3^{(3)} (1 - d_2)} + \frac{B_3^{(1)} d_2^2 (1 + d_2)}{A_2^{(3)} A_3^{(3)} (1 + d_2^2)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 + d_2^2), \\
A_4^{(3)} &= \frac{A_2^{(1)} A_3^{(3)} (1 - d_2)}{A_2^{(3)} A_3^{(3)} (1 - d_2)} + \frac{B_3^{(1)} d_2^2 (1 + d_2)}{A_2^{(3)} A_3^{(3)} (1 + d_2^2)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 + d_2^2), \\
d_3 &= \frac{1}{d_2^2}.
\end{align*}
\]  

(Type III-2-7): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= \frac{A_2^{(1)} B_3^{(1)}}{A_2^{(1)}}, \\
A_4^{(1)} &= \frac{A_3^{(3)} A_3^{(3)} A_3^{(3)}}{A_2^{(4)} A_3^{(3)} B_3^{(1)}}, \\
A_2^{(2)} &= \frac{A_2^{(1)} A_3^{(3)} (1 - d_2)}{A_2^{(3)} A_3^{(3)} (1 - d_2)} + A_3^{(3)} A_3^{(3)} A_3^{(3)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 - d_2), \\
A_4^{(2)} &= \frac{A_2^{(1)} A_3^{(3)} (1 - d_2)}{A_2^{(3)} A_3^{(3)} (1 - d_2)} + A_3^{(3)} A_3^{(3)} A_3^{(3)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 - d_2), \\
d_2 &= d_3 = 1.
\end{align*}
\]  

(Type III-2-8): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= \frac{A_2^{(1)} B_3^{(1)}}{A_2^{(1)}}, \\
A_4^{(1)} &= \frac{A_2^{(1)} B_4^{(1)}}{A_2^{(1)}}, \\
A_2^{(2)} &= \frac{A_2^{(1)} B_3^{(1)} (1 - d_2)}{A_2^{(1)} A_3^{(3)} (1 - d_2)} - A_3^{(3)} A_3^{(3)} A_3^{(3)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 - d_2), \\
A_4^{(2)} &= \frac{A_2^{(1)} B_3^{(1)} (1 - d_2)}{A_2^{(1)} A_3^{(3)} (1 - d_2)} - A_3^{(3)} A_3^{(3)} A_3^{(3)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 - d_2), \\
d_2 &= d_3 = 1.
\end{align*}
\]  

(Type III-2-9): the condition (III-2) and
\[
\begin{align*}
B_2^{(1)} &= -\frac{A_2^{(1)} B_3^{(1)}}{A_2^{(1)}}, \\
A_4^{(1)} &= \frac{1 - d_2 A_2^{(1)} A_3^{(3)} B_4^{(1)} + 2 A_2^{(1)} A_3^{(3)} B_3^{(1)} d_2^2}{A_2^{(1)} (1 - d_2)} - A_1^{(3)} B_3^{(1)} (1 + d_2), \\
A_2^{(2)} &= A_2^{(1)} B_3^{(1)} (1 + d_2^2) + A_1^{(3)} A_3^{(3)} (1 - d_2), \\
A_4^{(2)} &= \frac{A_2^{(1)} B_3^{(1)} (1 + d_2^2)}{A_2^{(1)} A_3^{(3)} (1 + d_2)} + \frac{A_1^{(3)} A_3^{(3)} (1 + d_2)}{A_2^{(1)} A_3^{(3)} (1 + d_2)}, \\
A_2^{(3)} &= \frac{A_2^{(1)} A_3^{(3)} (1 - d_2)}{A_2^{(1)} A_3^{(3)} (1 - d_2)} + \frac{B_3^{(1)} d_2^2 (1 + d_2)}{A_2^{(1)} A_3^{(3)} (1 + d_2^2)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 + d_2^2), \\
A_4^{(3)} &= \frac{A_2^{(1)} A_3^{(3)} (1 - d_2)}{A_2^{(1)} A_3^{(3)} (1 - d_2)} + \frac{B_3^{(1)} d_2^2 (1 + d_2)}{A_2^{(1)} A_3^{(3)} (1 + d_2^2)} - A_1^{(3)} A_3^{(3)} B_4^{(1)} (1 + d_2^2), \\
d_3 &= \frac{1}{d_2^2}.
\end{align*}
\]  

(Type III-2-10): the condition (III-2) and
\[
\begin{align*}
A_2^{(1)} &= \frac{A_2^{(1)} A_3^{(3)} B_3^{(1)} + A_3^{(3)} B_3^{(1)}}{A_2^{(1)} B_3^{(1)}}, \\
A_4^{(1)} &= A_3^{(1)} A_2^{(2)} (A_2^{(3)} A_3^{(3)} - A_3^{(3)} A_3^{(3)}) - A_2^{(2)} (A_2^{(3)} B_3^{(1)} + A_3^{(3)} B_3^{(1)}) \\
&\quad + A_2^{(1)} A_3^{(3)} - A_3^{(1)} B_3^{(1)} (B_3^{(1)} B_3^{(1)} - B_1^{(3)} B_3^{(1)}), \\
A_2^{(2)} &= \frac{A_2^{(1)} A_3^{(3)} B_3^{(1)} B_3^{(1)} (1 + d_2)}{A_2^{(3)} A_3^{(3)} B_3^{(1)}} \\
&\quad + A_2^{(1)} A_3^{(3)} - A_3^{(1)} B_3^{(1)} (B_3^{(1)} B_3^{(1)} - B_1^{(3)} B_3^{(1)}) \\
A_4^{(2)} &= \frac{B_2^{(3)} A_2^{(1)} B_3^{(1)} - A_3^{(3)} B_1^{(3)} B_3^{(1)}}{A_2^{(1)} A_3^{(3)} A_2^{(3)} A_3^{(3)} A_2^{(3)} A_3^{(3)} - A_2^{(3)} A_3^{(3)} A_2^{(3)} A_3^{(3)} A_2^{(3)} A_3^{(3)}}, \\
d_2 &= d_3 = 1.
\end{align*}
\]
(Type III-2-11): the condition (III-2) and
\[
\begin{align*}
A_3^{(2)} &= B_3^{(2)} = 0, \quad A_4^{(1)} = A_3^{(1)} A_2^{(2)}, \\
A_4^{(2)} &= \frac{A_1^{(1)} A_2^{(2)} (A_3^{(2)} B_3^{(3)} - A_2^{(3)} B_4^{(3)})}{A_2^{(1)} A_1^{(2)} B_1^{(3)}} + \frac{A_1^{(2)} B_4^{(2)} B_4^{(3)} - B_4^{(1)} B_3^{(3)}}{B_3^{(3)} B_1^{(3)}}, \\
A_2^{(3)} &= \frac{A_1^{(2)} B_3^{(3)}}{B_1^{(3)}}, \quad B_3^{(3)} = \frac{(A_2^{(1)} A_3^{(2)} B_3^{(3)} + A_2^{(2)} A_1^{(2)} B_4^{(3)}) B_1^{(3)}}{A_1^{(2)} B_2^{(1)} + A_3^{(2)} B_3^{(3)}}, \\
d_2 &= -1, \quad d_3 = 1.
\end{align*}
\] (III-2-11)

(Type III-2-12): the condition (III-2) and
\[
\begin{align*}
A_3^{(3)} &= B_3^{(3)} = 0, \quad A_4^{(1)} = A_3^{(1)} A_3^{(3)}, \quad A_2^{(3)} &= \frac{A_2^{(2)} B_3^{(3)}}{B_3^{(3)}}, \\
A_2^{(3)} &= \frac{A_1^{(1)} A_3^{(2)} B_3^{(3)} - A_2^{(2)} B_4^{(3)}}{A_3^{(1)} A_1^{(2)} B_1^{(3)}}, \quad A_3^{(1)} B_4^{(3)} (A_2^{(1)} B_1^{(3)} + A_1^{(1)} B_3^{(1)}) - A_3^{(1)} B_3^{(1)} B_1^{(3)} = A_1^{(2)} B_1^{(3)} B_1^{(3)}, \\
B_4^{(1)} &= \frac{A_2^{(2)} B_3^{(3)}}{A_1^{(2)}}, \quad d_2 = d_3 = 1.
\end{align*}
\] (III-2-12)

(Type III-3-1): the condition
\[
A_1^{(i)} = B_1^{(i)} = 1, \quad i = 1, 2, 3, 
\] (III-3)
and
\[
\begin{align*}
a_4^{(1)} &= b_4^{(1)} = -\frac{(A_2^{(1)} + A_2^{(2)} + B_2^{(1)})(A_2^{(1)} + A_3^{(3)} + B_3^{(1)})}{1 + A_2^{(2)} + A_3^{(3)} + B_3^{(3)}}, \\
b_4^{(2)} &= -(A_2^{(1)} + A_2^{(2)} + B_2^{(1)})(A_2^{(1)} + A_3^{(3)} + B_3^{(3)}), \\
b_4^{(3)} &= b_4^{(3)} = \frac{(1 + A_2^{(1)} + A_3^{(3)} + B_3^{(3)})(A_2^{(2)} + A_2^{(3)} + B_2^{(3)})}{A_2^{(2)} + A_2^{(3)} + B_2^{(3)}}, \\
B_3^{(1)} &= 1 + B_3^{(2)}, \quad d_2 = d_3 = -1.
\end{align*}
\] (III-3-1)

Here, parameters $a_4^{(i)}$ and $b_4^{(i)}$, $i = 1, 2, 3$, are given by
\[
\begin{align*}
a_4^{(i)} &= A_4^{(i)} - A_2^{(i)} A_3^{(i)}, \quad b_4^{(i)} &= B_4^{(i)} - B_2^{(i)} B_3^{(i)}.
\end{align*}
\] (A.27)

Note that under the condition (III-3) and setting (A.27), the condition (III) can be rewritten as
\[
B_3^{(1)} = 1 + B_3^{(2)}, \quad B_2^{(2)} = B_3^{(3)} - b_4^{(3)}, \quad B_3^{(2)} = B_2^{(1)} + b_4^{(1)}, \quad B_4^{(2)} = B_4^{(1)} b_4^{(2)}. 
\] (A.28)

(Type III-3-2): the condition (III-3) and the following condition with (A.27):
\[
\begin{align*}
d_4^{(1)} &= \frac{(A_2^{(2)} - B_2^{(1)})(1 + d_2)}{1 + d_2}, \quad d_4^{(2)} = -\frac{(A_2^{(2)} - B_2^{(1)})(A_2^{(2)} - B_2^{(3)})(1 + d_2)}{(1 - d_2)^2}, \\
d_4^{(3)} &= \frac{(A_2^{(2)} - B_2^{(3)})(1 + d_2)}{(1 - d_2)^2}, \quad b_4^{(1)} = -\frac{(A_2^{(2)} - B_2^{(3)})(1 + d_2)}{(1 - d_2)^2}, \\
b_4^{(2)} &= \frac{A_3^{(1)} - B_3^{(3)}(1 + d_2)}{(1 - d_2)^2 d_2}, \quad A_3^{(1)} = -\frac{B_2^{(1)} (1 + d_2^2) - 2 A_2^{(2)} d_2}{(1 - d_2) d_2^2}, \\
b_4^{(3)} &= 1 + B_2^{(3)}, \quad A_2^{(3)} = -\frac{B_3^{(1)} (1 + d_2^2) - 2 A_2^{(2)} d_2}{(1 - d_2)^2 d_2}, \\
B_3^{(1)} &= 1 + B_3^{(2)}, \quad A_2^{(3)} = -\frac{B_3^{(1)} (1 + d_2^2) - 2 A_2^{(2)} d_2}{(1 - d_2)^2 d_2}, \\
A_3^{(3)} &= \frac{1 - 2 d_2 (1 + B_3^{(2)} d_2)}{2 (1 - d_2)}, \quad A_2^{(3)} = -\frac{1 + 2 d_2^2 B_3^{(3)}}{2 (1 - d_2)} + d_2, \\
d_3 &= d_2, \quad d_2 = d_2^2 + 2 d_2^4 = 0.
\end{align*}
\] (III-3-2)
(Type III-3-3): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\left\{ \begin{array}{l}
  d_4^{(1)} &= \frac{A_2^{(2)} - B_2^{(1)}(1 - d_2)}{d_2^2 - d_2^2(1 + d_2)}, \\
  d_4^{(2)} &= \left( \frac{A_2^{(2)}B_2^{(3)}}{2} + \frac{A_2^{(2)}B_2^{(1)}}{2} \right) - (A_2^{(2)} + B_2^{(1)}(A_3^{(2)} + B_3^{(2)})), \\
  b_4^{(1)} &= -\left( \frac{A_2^{(2)}}{A_2^{(2)} + B_2^{(1)}} \right), \\
  b_4^{(2)} &= -\left( \frac{B_2^{(3)}(1 + d_2)}{A_2^{(2)} + B_2^{(1)}(A_3^{(2)} + B_3^{(2)})} \right), \\
  A_3^{(3)} &= -\left( \frac{d_2^4(1 + d_2)}{d_2^2(1 - d_2)} \right), \\
  B_2^{(3)} &= -1 - \left( \frac{1}{2d_2^2} + \frac{1}{d_2^2} + \frac{A_2^{(1)}(1 + d_2)}{d_2^2(1 + d_2)} \right), \\
  d_3 &= -d_2, \quad 1 - d_2^2 + 2d_2^4 = 0.
\end{array} \right.
\end{align*}
\]

(III-3-3)

(Type III-3-4): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\left\{ \begin{array}{l}
  d_4^{(1)} &= \frac{(1 + d_2^2d_3^2)(A_2^{(2)}(1 + d_2) - B_2^{(1)}(1 + d_2))}{d_2^2(1 - d_2)^2(1 + d_2)}, \\
  d_4^{(2)} &= \left( \frac{B_2^{(3)}(1 + d_2) - A_2^{(2)}(1 + d_3)}{A_2^{(2)}(1 + d_2) - B_2^{(1)}(1 + d_3)} \right), \\
  d_4^{(3)} &= \frac{(1 + d_2^2d_3^2)(B_2^{(3)}(1 + d_2) - A_2^{(3)}(1 + d_1))}{d_2^2d_3^2(1 - d_2^2)(1 + d_2)}, \\
  b_4^{(1)} &= -\left( \frac{d_2^2d_3^2(1 - d_2^2)(1 + d_3)}{d_2^2d_3^2(1 + d_2)} \right), \\
  b_4^{(2)} &= -\left( \frac{d_2^2d_3^2(1 - d_2^2)(1 + d_3)}{d_2^2d_3^2(1 + d_2) - A_2^{(3)}(1 + d_3)} \right), \\
  A_3^{(3)} &= -\left( \frac{A_2^{(2)}(1 + d_3)}{d_2^2d_3(1 - d_3)} \right), \\
  B_3^{(3)} &= -\left( \frac{1 + d_2^2d_3^2}{d_2^2d_3^2(1 + d_2)} \right), \\
  A_3^{(3)} &= -\left( \frac{1 - d_2^3d_3^2}{d_2^2d_3^2(1 + d_2)} \right), \\
  B_2^{(3)} &= -\left( \frac{1 - d_2^3d_3^2}{d_2^2d_3^2(1 + d_2)} \right), \\
  A_3^{(3)} &= -\left( \frac{1 + d_2^2d_3^2 + A_2^{(1)}d_2d_3(1 + d_3)}{d_2d_3(1 + d_2)} \right), \\
  1 + d_2^2d_3^2 + d_2^4d_3^2 + d_2^2d_3^4 &= 0.
\end{array} \right.
\end{align*}
\]
(Type III-3-5): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\begin{cases}
  a_4^{(1)} = \frac{(2 + i \sqrt{2})A_3^{(1)} - 2A_2^{(2)}}{3}, & a_4^{(2)} = \frac{(2 + i \sqrt{2})A_3^{(1)} - 6A_2^{(2)}}{12}, \\
  b_4^{(1)} = \frac{(1 + i \sqrt{2})A_3^{(1)}}{3} + i \sqrt{2}A_2^{(2)}, & b_4^{(3)} = \frac{i d_4}{\sqrt{2}}, \\
  B_2^{(1)} = \frac{(2 + i \sqrt{2})A_3^{(1)}}{6}, & B_3^{(1)} = 1 - i \sqrt{2} + (2 - i \sqrt{2})A_2^{(1)}, \\
  A_2^{(3)} = \frac{(2 - i \sqrt{2})(2A_2^{(2)} - d_4^{(3)})}{2}, & A_3^{(3)} = \frac{2 + i \sqrt{2}}{2} + A_2^{(1)}, \\
  B_2^{(3)} = -i \sqrt{2} + (2 - i \sqrt{2})A_2^{(1)}, & B_3^{(3)} = A_3^{(2)} - \frac{d_4^{(3)}}{2}, & d_2 = d_3 = \frac{i}{\sqrt{2}}.
\end{cases}
\end{align*}
\]

where \(i = \sqrt{-1}.\)

(Type III-3-6): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\begin{cases}
  a_4^{(1)} = \frac{(2 - i \sqrt{2})A_3^{(1)} - 2A_2^{(2)}}{3}, & a_4^{(2)} = \frac{(2 - i \sqrt{2})A_3^{(1)} - 6A_2^{(2)}}{12}, \\
  b_4^{(1)} = \frac{(1 + i \sqrt{2})A_3^{(1)}}{3} - i \sqrt{2}A_2^{(2)}, & b_4^{(3)} = \frac{i d_4}{\sqrt{2}}, \\
  B_2^{(1)} = \frac{(2 - i \sqrt{2})A_3^{(1)}}{6}, & B_3^{(1)} = 1 + i \sqrt{2} + (2 + i \sqrt{2})A_2^{(1)}, \\
  A_2^{(3)} = \frac{(2 + i \sqrt{2})(2A_2^{(2)} - d_4^{(3)})}{2}, & A_3^{(3)} = \frac{2 + i \sqrt{2}}{2} + A_2^{(1)}, \\
  B_2^{(3)} = i \sqrt{2} + (2 + i \sqrt{2})A_2^{(1)}, & B_3^{(3)} = A_3^{(2)} - \frac{d_4^{(3)}}{2}, & d_2 = d_3 = -\frac{i}{\sqrt{2}}.
\end{cases}
\end{align*}
\]

(Type III-3-7): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\begin{cases}
  a_4^{(1)} = \frac{-(2 - i \sqrt{2})A_3^{(1)} + 2A_2^{(2)}}{3}, & a_4^{(2)} = \frac{-(2 - i \sqrt{2})A_3^{(1)} + 2A_2^{(2)}}{12}, \\
  A_2^{(3)} = \frac{(2 - i \sqrt{2})(2A_2^{(2)} - 3d_4^{(2)})}{3}, & A_3^{(3)} = \frac{2i + \sqrt{2}}{3}(3i - (2i + \sqrt{2})A_2^{(1)}), \\
  B_2^{(1)} = \frac{4iA_2^{(2)} - (\sqrt{2} - 3A_2^{(1)})}{3\sqrt{2}}, & B_3^{(1)} = 1 + i \sqrt{2} - \frac{(2 - i \sqrt{2})A_2^{(1)}}{3}, \\
  b_4^{(1)} = \frac{(1 + i \sqrt{2})A_3^{(1)} + i \sqrt{2}A_2^{(2)}}{3}, & b_4^{(3)} = -i \sqrt{2} - \frac{(2 - i \sqrt{2})A_2^{(1)}}{3}, \\
  B_2^{(3)} = \frac{(1 - 2i \sqrt{2})A_2^{(1)}}{3} - \frac{d_4^{(3)}}{2}, & B_3^{(3)} = \frac{d_4^{(3)}}{\sqrt{2}}, \\
  d_2 = \frac{i}{\sqrt{2}}, & d_3 = -\frac{i}{\sqrt{2}}.
\end{cases}
\end{align*}
\]
(Type III-3-8): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\alpha_4^{(1)} &= \frac{(2 + i \sqrt{2}) A_3^{(1)} + 2 A_2^{(2)}}{3}, \\
\beta_4^{(1)} &= \frac{(1 - i \sqrt{2}) A_3^{(1)} - \sqrt{3} A_2^{(2)}}{3}, \\
\beta_4^{(2)} &= \frac{(\sqrt{2} + i) A_3^{(1)} + 4i A_2^{(2)}}{3}, \\
B_4^{(1)} &= -\frac{(1 + \sqrt{2}) A_2^{(1)} (2 A_3^{(2)} - 3 A_3^{(3)})}{6}, \\
B_4^{(3)} &= \frac{i \alpha_4^{(3)}}{\sqrt{2}}, \\
B_4^{(3)} &= 1 + \frac{i \sqrt{2}}{3} - \frac{(2 + i \sqrt{2}) A_2^{(1)}}{3}.
\end{align*}
\]  

\text{(III-3-8)}

(Type III-3-9): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\alpha_4^{(1)} &= \frac{(1 - d_2^2)(A_3^{(2)} (1 - d_2 d_3) + A_1^{(2)} d_2 (1 + d_3))}{(1 - d_2^2)(1 - d_3^2)}, \\
\beta_4^{(1)} &= \frac{(1 - d_3^2)(A_3^{(1)} (1 - d_2 d_3) + A_1^{(1)} d_2 (1 + d_3))}{(1 - d_2^2)(1 - d_3^2)}, \\
\beta_4^{(3)} &= \frac{d_3 (1 - d_3^2) (A_2^{(1)} (1 - d_2 d_3) + A_1^{(1)} d_2 (1 + d_3))}{(1 - d_3^2)(1 - d_2^2)}, \\
B_4^{(1)} &= \frac{d_2 (1 - d_2^2) (A_2^{(2)} (1 - d_2 d_3) + A_1^{(2)} d_2 (1 + d_3))}{(1 - d_2^2)(1 - d_3^2)}, \\
B_4^{(3)} &= \frac{d_3 (1 - d_3^2) (A_2^{(2)} (1 - d_2 d_3) + A_1^{(2)} d_2 (1 + d_3))}{(1 - d_3^2)(1 - d_2^2)}, \\
B_4^{(2)} &= \frac{A_1^{(1)} d_2 (1 + d_3)}{1 - d_2^2} - A_2^{(1)} (d_2 - d_3), \\
B_4^{(3)} &= \frac{d_3 (1 - d_2^2)(1 - d_2^2)}{(d_2^2 - d_3^2)(1 - d_2^2)} - \frac{2 (1 - d_2^2)}{d_2^2 (1 - d_2^2)} \frac{A_1^{(1)} (1 - d_2 d_3)}{d_2 d_3 (1 + d_2)}, \\
B_4^{(2)} &= \frac{(2 + d_2) (1 - d_2)}{d_2^2 + d_3^2} + \frac{(1 - d_2) (d_2 d_3 (1 - d_2^2) - (1 - d_2^2))}{d_2^2 - d_2 d_3 (1 + d_2)}, \\
A_3^{(3)} &= \frac{d_2 d_3 (1 - d_2) (1 - d_3^2)}{1 - d_2^2 d_3^2} + \frac{(1 + d_3) A_2^{(1)} (1 - d_3^2)}{1 + d_2}, \\
B_3^{(3)} &= -\frac{A_1^{(2)} (d_2 - d_3) + A_2^{(3)} d_2 d_3 (1 + d_2)}{(1 - d_2^2)(1 - d_3^2)}.
\end{align*}
\]  

\text{(III-3-9)}

(Type III-3-10): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
\alpha_4^{(2)} &= \frac{A_4^{(1)} + A_2^{(2)} + B_2^{(1)} d_4^{(3)}}{A_3^{(1)} - 2 d_4^{(1)} + A_2^{(2)} + B_2^{(1)}}, \\
\beta_4^{(3)} &= \frac{A_4^{(1)} + 2 A_2^{(1)} - B_2^{(1)}}{A_3^{(1)} - 2 d_4^{(1)} + A_2^{(2)} + B_2^{(1)}}, \\
\beta_4^{(2)} &= \frac{1 + 2 A_1^{(1)} + 2 A_3^{(3)}}{2}, \\
B_4^{(1)} &= \frac{1 + 2 A_1^{(1)} + 2 A_3^{(3)}}{2}, \\
B_4^{(2)} &= \frac{A_1^{(2)} - A_2^{(3)} + (A_4^{(1)} + A_2^{(2)} + B_2^{(1)} d_4^{(3)})}{A_3^{(1)} - 2 d_4^{(1)} + A_2^{(2)} + B_2^{(1)}}, \\
d_2 &= d_3 = 1.
\end{align*}
\]  

\text{(III-3-10)}
Type III-3-11): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
da_4^{(2)} &= \frac{a_1^{(1)}a_4^{(3)}(A_3^{(1)} + A_2^{(2)} + B_2^{(1)})^2}{(A_3^{(1)} - a_4^{(1)} + A_2^{(2)} + B_2^{(1)} + b_2^{(1)})^2}, \\
b_4^{(3)} &= a_4^{(3)} - \frac{a_1^{(1)}a_4^{(3)}(a_4^{(1)} - b_2^{(1)})}{(A_3^{(1)} - a_4^{(1)} + A_2^{(2)} + B_2^{(1)} + b_2^{(1)})^2}, \\
B_3^{(1)} &= 1 - A_2^{(1)} - A_3^{(3)} - \frac{a_4^{(1)}A_3^{(1)} + A_3^{(2)} + B_2^{(1)} + b_2^{(1)}}{a_4^{(3)}}, \\
B_2^{(3)} &= -A_2^{(1)} - A_3^{(3)} - \frac{a_4^{(1)}A_3^{(1)} + A_3^{(2)} + B_2^{(1)} + b_2^{(1)}}{a_4^{(3)}}, \\
B_3^{(3)} &= -A_3^{(2)} - A_2^{(3)} - \frac{a_4^{(1)}a_4^{(3)}(A_3^{(1)} + A_3^{(2)} + B_2^{(1)})}{(A_3^{(1)} - a_4^{(1)} + A_2^{(2)} + B_2^{(1)} + b_2^{(1)})^2}, \\
d_2 &= d_3 = -1.
\end{align*}
\]

Type III-3-12): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
da_4^{(1)} &= \frac{d_2^2(1 - d_3)(1 - d_3^2)(A_2^{(1)}(1 + d_2) - B_2^{(1)}(1 + d_3))}{(1 - d_2^2 d_3^2)^2}, \\
b_4^{(2)} &= \frac{d_2^2(1 - d_2^2 d_3^2)(A_2^{(1)}(1 + d_2) - B_2^{(1)}(1 + d_3))}{d_2^2(1 - d_2^2 d_3^2)^2}, \\
a_4^{(3)} &= \frac{d_2^2(1 - d_2^2 d_3^2)(1 - d_2^2 d_3^2)}{d_2^2(1 - d_2^2 d_3^2)}, \\
b_4^{(1)} &= \frac{(1 + d_2)(1 - d_2^2 d_3^2)}{d_2^2(1 - d_2^2 d_3^2)}, \\
A_3^{(1)} &= \frac{1}{d_2^2(1 - d_2^2 d_3^2)}, \\
B_3^{(1)} &= \frac{1 - d_2^2 d_3^2}{A_3^{(1)}(1 - d_2 d_3)}, \\
A_3^{(2)} &= \frac{1}{d_2^2(1 - d_2^2 d_3^2)}, \\
A_3^{(3)} &= \frac{A_3^{(2)}(1 - d_2 d_3)}{d_2^2(1 - d_2^2 d_3^2)}, \\
B_2^{(3)} &= \frac{d_2(1 - d_2^2 d_3^2)}{d_2^2(1 - d_2^2 d_3^2)}, \\
A_3^{(2)} &= \frac{1}{d_2^2(1 - d_2^2 d_3^2)}, \\
B_3^{(3)} &= \frac{A_3^{(2)}(1 - d_2 d_3)}{d_2^2(1 - d_2^2 d_3^2)}, \\
B_3^{(3)} &= \frac{A_3^{(2)}(1 - d_2 d_3)}{d_2^2(1 - d_2^2 d_3^2)}.
\end{align*}
\]
(Type III-3-13): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
  a_4^{(1)} &= -\frac{(A_2^{(2)} - B_2^{(1)})d_2^2(1 + d_2^2)}{(1 - d_2^2)(1 + d_2^4)}, \\
  a_4^{(2)} &= -\frac{(A_2^{(2)} - B_2^{(1)})d_4^2(1 + d_4^4)}{(1 - d_2^2)^2d_2}, \\
  a_4^{(3)} &= b_4^{(3)}d_2(1 + d_2^2), \\
  b_4^{(1)} &= -\frac{(A_2^{(2)} - B_2^{(1)})d_2(1 + d_2^2)}{1 + d_2^4}, \\
  A_4^{(1)} &= -\frac{B_4^{(1)}(1 + d_4^2) - 2A_2^{(2)}d_2}{(1 - d_2)d_2^2}, \\
  B_4^{(1)} &= \frac{1}{(1 + d_2^2)} + \frac{d_2}{1 + d_2} - \frac{A_2^{(1)}(1 - d_2)}{d_2^2}, \\
  A_4^{(2)} &= \frac{b_4^{(2)}(1 - d_2^4)}{d_4^3(1 + d_2)(1 - d_2^2)^2} = \frac{A_2^{(2)}d_2(1 - d_2)}{d_2^4}, \\
  B_4^{(2)} &= -\frac{d_2}{(1 + d_2^2)} - \frac{A_4^{(3)}(1 - d_2)}{d_2^2}, \\
  A_4^{(3)} &= A_2^{(3)} - \frac{d_2^2(1 + d_2^2)}{(1 - d_2)(1 + d_2^2)^2}, \\
  B_4^{(3)} &= A_3^{(2)} \frac{b_4^{(3)}(1 + d_4^2)}{d_2(1 + d_2^3)}, \\
  d_3 &= d_2.
\end{align*}
\]

(Type III-3-14): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
  a_4^{(1)} &= d_2^2(1 + d_2^2)(B_2^{(1)}(1 - d_2) - A_2^{(2)}(1 + d_2)) \\
  a_4^{(2)} &= \frac{d_4^2(1 - d_2)(1 + d_2^2)^2}{(1 - d_2)(1 + d_2^2)(1 + d_2^4)}, \\
  a_4^{(3)} &= b_4^{(3)}d_2(1 + d_2^2), \\
  b_4^{(3)} &= -\frac{d_2(1 - d_2)(B_2^{(1)}(1 - d_2) - A_2^{(2)}(1 + d_2))}{1 + d_2^4}, \\
  A_3^{(1)} &= \frac{B_4^{(1)}(1 + d_2^2)}{d_2^2(1 + d_2)} = \frac{d_2^2(1 - d_2^2) + A_2^{(3)}(1 - d_2^2)}{d_2^2(1 + d_2^2)(1 - d_2)}, \\
  B_3^{(1)} &= \frac{d_2^2(1 - d_2^2)(1 - d_2^2)}{(1 + d_2^2)^2}, \\
  A_3^{(2)} &= d_2^2(1 + d_2^2) - \frac{d_2}{(1 - d_2)(1 + d_2^2)^2}, \\
  B_3^{(2)} &= \frac{A_3^{(2)}(1 + d_2^2)}{d_2^2(1 + d_2)^2}, \\
  A_3^{(3)} &= \frac{(1 - d_2)(1 + d_2^2)}{d_2^2(1 + d_2)^2} + \frac{1 + d_2}{1 + d_2^2}, \\
  B_3^{(3)} &= \frac{A_4^{(3)}d_2(1 - d_2^2) - b_4^{(3)}(1 + d_2^2)}{d_2(1 + d_2)^2}, \\
  d_3 &= -d_2.
\end{align*}
\]
(Type III-3-15): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
A_{2}^{(3)} & = \frac{d_{2}^{3}d_{3}^{3}}{d_{3}(1 - d_{3}^{2})^{2}(1 + d_{3}^{2})}, \\
B_{3}^{(3)} & = \frac{A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 + d_{2})d_{3}^{2}}, \\
A_{4}^{(3)} & = \frac{A_{2}^{(3)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{d_{2}(1 - d_{3})^{2}(1 + d_{3})}, \\
b_{4}^{(3)} & = \frac{A_{2}^{(3)}d_{3}^{2}}{d_{2}(1 - d_{3}^{2})}. \\
A_{4}^{(4)} & = \frac{A_{2}^{(3)}d_{3}^{2} + b_{4}^{(3)}(1 + d_{2})(1 + d_{3}^{2})}{d_{2}(1 - d_{3}^{2})}, \\
b_{4}^{(4)} & = \frac{A_{2}^{(3)}d_{3}^{2} + b_{4}^{(3)}(1 + d_{2})(1 + d_{3}^{2})}{d_{2}(1 - d_{3}^{2})}. \\
A_{4}^{(5)} & = \frac{A_{2}^{(3)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{d_{2}(1 - d_{3})^{2}(1 + d_{3})}, \\
b_{4}^{(5)} & = -\frac{A_{2}^{(3)}d_{3}^{2} + b_{4}^{(3)}(1 + d_{2})(1 + d_{3}^{2})}{d_{2}(1 - d_{3}^{2})}.
\end{align*}
\]  

\[
\begin{align*}
A_{2}^{(1)} & = \frac{A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{d_{2}(1 - d_{3})^{2}(1 + d_{3})}, \\
A_{2}^{(2)} & = \frac{A_{2}^{(2)}d_{3}^{2} + b_{4}^{(2)}(1 + d_{2})(1 + d_{3}^{2})}{d_{2}(1 - d_{3})^{2}(1 + d_{3})}, \\
b_{4}^{(2)} & = -\frac{A_{2}^{(2)}d_{3}^{2} + b_{4}^{(2)}(1 + d_{2})(1 + d_{3}^{2})}{d_{2}(1 - d_{3})^{2}(1 + d_{3})}.
\end{align*}
\]  

(III-3-15)

(III-3-16): the condition (III-3) and the following condition with (A.27):

\[
\begin{align*}
A_{2}^{(1)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{2}^{(2)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{4}^{(1)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{4}^{(2)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{4}^{(1)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{4}^{(2)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{4}^{(1)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{4}^{(2)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{4}^{(3)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{4}^{(4)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{4}^{(3)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{4}^{(4)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}.
\end{align*}
\]  

\[
\begin{align*}
A_{3}^{(1)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{3}^{(2)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{3}^{(1)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{3}^{(2)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{3}^{(3)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{3}^{(4)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{3}^{(3)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{3}^{(4)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{3}^{(5)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
A_{3}^{(6)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{3}^{(5)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}, \\
b_{3}^{(6)} & = \frac{d_{2}^{2}(1 + d_{3}^{2})^{2}(A_{2}^{(2)}(1 + d_{2}) - B_{2}^{(1)}(1 + d_{3})}{(1 - d_{3})(1 - d_{2})d_{3}^{2}}.
\end{align*}
\]
REFERENCES

[1] V. E. Adler, A. I. Bobenko, and Y. B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. Comm. Math. Phys., 233(3):513–543, 2003.
[2] V. E. Adler, A. I. Bobenko, and Y. B. Suris. Discrete nonlinear hyperbolic equations: classification of integrable cases. Funktional. Anal. i Prilozhen., 43(1):3–21, 2009.
[3] R. Boll. Classification and Lagrangian Structure of 3D Consistent Quad-Equations. Doctoral Thesis, Technische Universität Berlin, submitted August 2012.
[4] R. Boll. Classification of 3D consistent quad-equations. J. Nonlinear Math. Phys., 18(3):337–365, 2011.
[5] R. Boll. Corrigendum: Classification of 3D consistent quad-equations. J. Nonlinear Math. Phys., 19(4):1292001, 3, 2012.
[6] R. Fuchs. Sur quelques équations différentielles linéaires du second ordre. Comptes Rendus de l’Académie des Sciences Paris, 141(1):555–558, 1905.
[7] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est a points critiques fixes. Acta Math., 33(1):1–55, 1910.
[8] B. Grammaticos and A. Ramani. Discrete Painlevé equations: a review. In Discrete integrable systems, volume 644 of Lecture Notes in Phys., pages 245–321. Springer, Berlin, 2004.
[9] R. Hirota. Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation. J. Phys. Soc. Japan, 43(4):1424–1433, 1977.
[10] N. Joshi and N. Nakazono. Classification of quad-equations consistent around a cuboctahedron. In preparation.
[11] N. Joshi and N. Nakazono. Reduction of quad-equations consistent around a cuboctahedron. I: additive case. In preparation.
[12] N. Joshi and N. Nakazono. Lax pairs of discrete Painlevé equations: (A₂ + A₁)⁽(1) case. Proc. A., 472(2196):20160696, 14, 2016.
[13] N. Joshi, N. Nakazono, and Y. Shi. Geometric reductions of ABS equations on an n-cube to discrete Painlevé systems. J. Phys. A, 47(50):505201, 16, 2014.
[14] N. Joshi, N. Nakazono, and Y. Shi. Lattice equations arising from discrete Painlevé systems. I. (A₂ + A₁)⁽(1) and (A₁ + A₁)⁽(1) cases. J. Math. Phys., 56(9):092705, 25, 2015.
[15] N. Joshi, N. Nakazono, and Y. Shi. Lattice equations arising from discrete Painlevé systems: II. A₁⁽(1) case. J. Phys. A, 49(49):495201, 39, 2016.
[16] K. Kajiwara, M. Noumi, and Y. Yamada. Geometric aspects of Painlevé equations. J. Phys. A, 50(7):073001, 164, 2017.
[17] N. Nakazono. Reduction of lattice equations to the Painlevé equations: P₁V and Pᵥ. J. Math. Phys., 59(2):022702, 18, 2018.
[18] F. Nijhoff and H. Capel. The discrete Korteweg-de Vries equation. Acta Appl. Math., 39(1-3):133–158, 1995. KdV ‘95 (Amsterdam, 1995).
[19] F. W. Nijhoff, H. W. Capel, G. L. Wiersma, and G. R. W. Quispel. Bäcklund transformations and three-dimensional lattice equations. Phys. Lett. A, 105(6):267–272, 1984.
[20] F. W. Nijhoff, G. R. W. Quispel, and H. W. Capel. Direct linearization of nonlinear difference-difference equations. Phys. Lett. A, 97(4):125–128, 1983.
[21] P. Painlevé. Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale générale est uniforme. Acta Math., 25(1):1–85, 1902.
[22] G. R. W. Quispel, F. W. Nijhoff, H. W. Capel, and J. van der Linden. Linear integral equations and nonlinear difference-difference equations. Phys. A, 125(2-3):344–380, 1984.
[23] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. Comm. Math. Phys., 220(1):165–229, 2001.

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