The free energy singularity of the asymmetric 6–vertex model and the excitations of the asymmetric $XXZ$ chain

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Abstract

We consider the asymmetric six–vertex model, i.e. the symmetric six–vertex model in an external field with both horizontal and vertical components, and the relevant asymmetric $XXZ$ chain. The model is widely used to describe the equilibrium shape of a crystal. By means of the Bethe Ansatz solution we determine the exact free energy singularity, as function of both components of the field, at two special points on the phase boundary. We confirm the exponent $\frac{3}{2}$ (already checked experimentally), as the antiferroelectric ordered phase is reached from the incommensurate phase normally to the phase boundary, and we determine a new singularity along the tangential direction. Both singularities describe the rounding off of the crystal near a facet. The hole excitations of the spin chain at this point on the phase boundary show dispersion relations with the striking form $\Delta E \sim (\Delta P)^{\frac{3}{2}}$ at small momenta, leading to a finite size scaling $\Delta E \sim N^{-\frac{1}{2}}$ for the low–lying excited states, where $N$ is the size of the chain. We conjecture that a Pokrovskii–Talapov phase transition is replaced at this point by a transition with diverging correlation length, but not classified in terms of conformal field theory.

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1 Introduction

After the pioneering paper of Yang, Yang and Sutherland [1] the asymmetric six-vertex model, i.e. the symmetric six-vertex model in a field, was recently rediscovered because of its connection to a number of physically interesting problems, first among them the determination of the shape of a crystal at equilibrium with its vapor phase [2, 3]. This can be achieved by mapping the asymmetric 6–vertex onto, say, the (001) facet of a bcc crystal under the condition that no overhangs or voids are allowed (see e.g. [4] and [7] for details on the mapping). Excitations in the vertex model correspond to small tilts away from the (001) facet [4] and it can be shown [3] that the free energy as function of the two components of the field gives exactly the equilibrium shape of the crystal.

In its own, the asymmetric six-vertex model provides an interesting 2-dimensional system of interacting dipoles in an external field with horizontal and vertical components \((h, v)\). Fluctuating two-valued variables (dipoles) are attached to the links of a two-dimensional square lattice, and the model is defined by assigning a set of Boltzmann weights (equivalently, interaction energies) to each allowed vertex configuration (see fig. 1). The transfer matrix can be diagonalized exactly by the Bethe-Ansatz in its coordinate or algebraic version [6, 7]. The phase diagram and the nature of the phase transitions are well understood when \(h = v = 0\) (symmetric six-vertex), or when \(h = 0\) and \(v \neq 0\) [8, 11]. If \(h, v \neq 0\), some general features of the phase diagram have been described in [12] and the details of the calculation spelled out in [13], but a few questions have remained unanswered. It is known that, in the antiferroelectric regime, with which we will be concerned in this paper, the free energy \(f(h, v)\) remains constant as function of the field (‘flat phase’) in a bounded region of the \((h, v)\) plane containing \(h = v = 0\). This corresponds to the flat (001) crystal plane. Beyond this region, bounded by a curve \(\Gamma\), the field is sufficiently strong to destroy the antiferroelectric order of the system, but not strong enough to impose ferroelectric order, and an incommensurate phase appears where the polarization (zero in the flat phase) changes continuously with the
field. Here the spectrum of the transfer matrix is gapless with finite size corrections typical of the gaussian model [10]. The curve $\Gamma$ has been investigated only at a few points and the nature of the phase transition was found to be of a Pokrovskii–Talapov (PT) type [11].

In this paper we determine the exact free energy singularity when approaching $\Gamma$ from the incommensurate phase, which determines the curvature of the crystal near the (001) facet. We find two special points $(h_c, v_c)$ (by symmetry under arrow-reversal the same holds for $(-h_c, -v_c)$, also on $\Gamma$), where

\[
\begin{align*}
  f(h_c + \delta h, v_c) &= f(h_c, v_c) - \text{const} (\delta h)^{3/2} \\
  f(h_c, v_c + \delta v) &= f(h_c, v_c) - \text{const} |\delta v|^3 
\end{align*}
\]

These are points on $\Gamma$ where the tangent to $\Gamma$ is parallel to the $v-$axis. The exponent $3/2$ measures the rounding off of the edges of the (001) facet.

Even though our calculation has been carried out only at these points of $\Gamma$, the technique we present should work in general, and our result, which generalizes Lieb and Wu’s method and complements the finite-size techniques of Kim [10], strengthens the long held belief that the exponent $3/2$ should govern the free energy singularity at every point of the phase boundary [2, 9]. This exponent has been measured [12] in some experiments with Pb crystals some years ago. Our results however show that along some tangential direction the exponent 3 should dominate, and this fact should be observable experimentally.

However, something more can be said about the nature of the phase transition along $\Gamma$.

The method of mapping a 2d statistical system into a 1d quantum spin chain has been fruitful and widely used in the past [13]. We pursue it here, regardless of the fact that the relevant spin chain which turns out to be the asymmetric $XXZ$ spin chain in a vertical field $V$, is not hermitian [14]. We find that the flat phase corresponds to a region in the $(h, V)$ plane where the ground state energy does not depend on the fields and where excitations are massive. Along the transition line, analogue of $\Gamma$, the excitations become massless, but the point $(h_c, V = 0)$ (with its symmetric $(-h_c, V = 0)$) is singled out by the fact that dispersion
relations obey the striking law, at small momenta

\[ \Delta E \simeq (\Delta P)^{1/2} \quad (1.2) \]

and finite-size corrections for low-lying excitations scale like

\[ \Delta E \simeq N^{-1/2} \quad (1.3) \]

if \( N \) is the length of the chain. At this point, analogue of \((h_c, v_c)\) for the statistical model, the vanishing of the mass gap exhibits an exponent 1/2 which does not appear at any other point of the transition line in the \((h, V)\) plane. In section 5 we propose an explanation of these results, arguing that at the point \((h_c, v_c)\) the transition occurs with a divergence of the correlation length in the correlator between two vertical arrows, while everywhere else the transition is induced by level-crossing (Pokrovskii-Talapov) which prevents the divergence of the same correlation length.

The paper is divided in 5 sections. In section 2 we give definitions and summarize previously known results. In section 3 hole excitations and the spectrum of the spin chain are studied and in section 4 the method of Lieb and Wu is suitably extended to determine the free energy singularity when both \( h \) and \( v \) are nonzero. Section 5 contains an interpretation of the results.

\section{2 Definitions}

The model is a natural generalization of the well-known symmetric six-vertex model. Arrows are placed on the edges of an \( N \times M \) square lattice and Boltzmann weights \( R^{\beta\beta'}_{\alpha\alpha'}(u) \) are assigned to the vertices (see Fig. 1) so that the row–to–row transfer matrices

\[ T(u)_{\{\alpha\},\{\alpha'\}} = \sum_{\{\beta\}} \prod_{k=1}^{N} R^{\beta_k\beta_{k+1}}_{\alpha_k\alpha'_{k+1}}(u) \quad (2.1) \]

form a commuting family

\[ [T(u), T(u')] = 0 \]
for any two values \( u, u' \) of the spectral parameter \([13]\). The associated (integrable) spin chain

\[
\mathcal{H} = \sum_{j=1}^{N} \left[ \frac{\cosh \gamma}{2} (1 + \sigma_j^z \sigma_{j+1}^z) - e^{2h} \sigma_j^+ \sigma_{j+1}^- - e^{-2h} \sigma_j^- \sigma_{j+1}^+ \right] - V \sum_{j=1}^{N} \sigma_j^z
\]  

(2.2)
is obtained from (2.1) by taking the so-called extremely anisotropic limit \((u \to 0)\)

\[
T(u) = e^{(v+h) \sum_{j=1}^{N} \sigma_j^+} T(u) \quad \mathcal{H} = -V \frac{d}{du} \log(e^{(v+h) \sum_{j=1}^{N} \sigma_j^+}) - \sinh \gamma \frac{d}{du} \log T(u) \bigg|_{u=0}
\]

\( V \) breaks the \( Z_2 \) symmetry of spin reversal while \( h \) breaks parity invariance (see App. A for a complete discussion of symmetries).

By means of the Bethe–Ansatz, eigenvalues of (2.1) and (2.2) are found from the solution of a set of coupled equations

\[
\left[ \frac{\sinh \left( \frac{\gamma}{2} + \frac{i\alpha_k}{2} \right)}{\sinh \left( \frac{\gamma}{2} - \frac{i\alpha_k}{2} \right)} \right]^N = (-1)^{n+1} e^{2hN} \prod_{l=1}^{n} \frac{\sinh(\gamma + \frac{i}{2}(\alpha_k - \alpha_l))}{\sinh(\gamma - \frac{i}{2}(\alpha_k - \alpha_l))} \quad k = 1, 2, \ldots, n
\]  

(2.3)

and given respectively by

\[
\Lambda(u) = e^{v(N-2n)} e^{hN} \left[ \frac{\sinh(\gamma - u)}{\sinh \gamma} \right]^N \prod_{j=1}^{n} \frac{\sinh(\frac{\gamma}{2} + u - \frac{i\alpha_j}{2})}{\sinh(\frac{\gamma}{2} - u + \frac{i\alpha_j}{2})} \\
+ e^{v(N-2n)} e^{-hN} \left[ \frac{\sinh u}{\sinh \gamma} \right]^N \prod_{j=1}^{n} \frac{\sinh(\frac{-3\gamma}{2} + u - \frac{i\alpha_j}{2})}{\sinh(\frac{-\gamma}{2} - u + \frac{i\alpha_j}{2})}
\]

\[
= \Lambda_R(u) + \Lambda_L(u)
\]

(2.4)

\[
E = N \cosh \gamma - \sum_{k=1}^{n} \frac{2 \sinh^2 \gamma}{\cosh \gamma - \cos \alpha_k} - V(N - 2n)
\]

\[
= N \cosh \gamma + \sum_{k=1}^{n} e(\alpha_k) - V(N - 2n)
\]  

(2.5)

Here \( n \) stands for the number of reversed spins (arrows) with respect to the reference ferromagnetic state \( |\uparrow \uparrow \cdots \uparrow \rangle \). It is a conserved quantity since \( S^z = \frac{1}{2} \sum_{j=1}^{N} \sigma_j^z \) commutes with \( \mathcal{H} \) and \( T(u) \).

Beside their energy, given by (2.3), the momentum can also be computed. \( T^{-1}(0) \) yields the right–shift operator \( S = e^{-iP} \)

\[
S = |\alpha_1, \alpha_2, \ldots, \alpha_M\rangle = |\alpha_M, \alpha_1, \alpha_2, \ldots, \alpha_{M-1}\rangle
\]  

(2.6)
and from (2.1) and (2.4) one gets
\[ e^{-iP} = e^{-2\hbar n} \prod_{j=1}^{n} \frac{\sinh(\frac{\gamma}{2} + \frac{i\alpha_j}{2})}{\sinh(\frac{\gamma}{2} - \frac{i\alpha_j}{2})} \]
\[ P = -2i\hbar - \sum_{j=1}^{n} p_0^{0}(\alpha_j) \]  
(2.7)
with
\[ p_0^{0}(\alpha) = -i \ln \left[ \frac{\sinh(\frac{\gamma}{2} + \frac{i\alpha}{2})}{\sinh(\frac{\gamma}{2} - \frac{i\alpha}{2})} \right] \]

Unlike most integrable spin chains studied before, (2.2) is not hermitian for \( h \neq 0 \), even though the statistical model has physically sensible positive Boltzmann weights. The question arises whether (2.1) and (2.2) have a complete set of eigenvectors. By numerically diagonalizing the transfer matrix on small chains it appears that, even though a few eigenvalues are degenerate in some charge sectors, the eigenvectors are linearly independent. We will assume that a complete set of eigenvectors exists and it is given by the Bethe Ansatz.

We first summarize, for the sake of completeness, some already known facts that have been published elsewhere, beginning with the spin chain because, although the Bethe Ansatz equations are identical, the form of the energy contribution from each rapidity \( \alpha_k \) is simpler.

Taking the logarithm of (2.3) in the usual way we get
\[ p_0^{0}(\alpha_k) - \frac{1}{N} \sum_{l=1}^{n} \Theta(\alpha_k - \alpha_l) + 2i\hbar = \frac{2\pi}{N} I_k \quad k = 1, 2, \ldots, n \]  
(2.8)
where \( I_k \) is half-odd (integer) if \( n \) is even (odd) and
\[ \Theta(\alpha) = -i \ln \left[ \frac{\sinh(\gamma + \frac{i\alpha}{2})}{\sinh(\gamma - \frac{i\alpha}{2})} \right] \]
We define \( p_0^{0}(0) = \Theta(0) = 0 \) and cuts are chosen to run from \( i\gamma \) to \( i\infty \) and from \( -i\gamma \) to \( -i\infty \) for \( p_0^{0}(\alpha) \); from \( 2i\gamma \) to \( i\infty \) and from \( -2i\gamma \) to \( -i\infty \) for \( \Theta(\alpha) \). Notice from (2.3) that \( \text{Re}(\alpha) \in [-\pi, \pi] \) so \( \text{Re}(\alpha_k - \alpha_l) \in [-2\pi, 2\pi] \). The cuts are chosen so that \( \Theta(\alpha) \) is analytical in \( -2\gamma < \text{Im}(\alpha) < 2\gamma \) and real, monotonically increasing when \( \alpha \in [-2\gamma, 2\gamma] \). With these conventions, the ground state at \( h = 0 \) in each sector of fixed \( S^z = \frac{N}{2} - n \) corresponds to a sequence of \( n \) consecutive numbers \( \{I_k\} \) in (2.8), from \( -\frac{n-1}{2} \) to \( \frac{n-1}{2} \) symmetric around 0 [10]. The rapidities \( \{\alpha_j\} \) are real and distributed symmetrically around \( \alpha = 0 \) too. As \( h \neq 0 \), the
rapidities move into the complex plane along a curve $C$. In a standard way \[9\], one gets in the thermodynamic limit from (2.8)

$$p^0(\alpha) - \frac{1}{2\pi} \int_C d\beta \Theta(\alpha - \beta) R(\beta) + 2i h = 2\pi x \quad -\frac{1-y}{4} \leq x \leq \frac{1-y}{4} \tag{2.9}$$

$x$ is the real parameter of the curve, $y$ is the polarization defined through

$$y = \lim_{N \to \infty} \frac{2S^z}{N} = \lim_{N \to \infty} (1 - \frac{2n}{N}) \tag{2.10}$$

and the rapidity density $R(\alpha_l) = \lim_{N \to \infty} 2\pi \frac{N}{N(\alpha_{l+1} - \alpha_l)}$ is determined by solving the integral equation

$$\xi(\alpha) - \frac{1}{2\pi} \int_C d\beta K(\alpha - \beta) R(\beta) = R(\alpha) \tag{2.11}$$

where

$$\xi(\alpha) = \frac{dp^0(\alpha)}{d\alpha}; \quad K(\alpha) = \frac{d\Theta(\alpha)}{d\alpha}$$

The energy and the polarization are thus given by

$$\lim_{N \to \infty} \frac{E}{N} = \cosh \gamma + \frac{1}{2\pi} \int_C d\alpha e(\alpha) R(\alpha) - V y \tag{2.12}$$

$$\frac{1-y}{2} = \frac{1}{2\pi} \int_C d\alpha R(\alpha) \tag{2.13}$$

Some preliminary information about the shape of $C$ and its location in the complex plane can be obtained by solving (2.3) numerically. We take as initial solution that composed of real roots and corresponding to the ground state, at fixed $S^z$, for $h = 0$. By the Perron–Frobenius theorem \[16\] \[17\], the relevant eigenstate remains the ground–state at fixed $S^z$ even when $h \neq 0$, and it is real. It appears that for $h > 0 (h < 0)$ the rapidities move into the lower (upper) half-plane as shown in fig. 2.

The curve $C$ is invariant under $\alpha \to -\alpha^*$, which is to be expected, being this transformation a symmetry of (2.3), so we will set $A = -a + ib$ and $B = a + ib$ to be the endpoints of the curve. Note that this property makes $E$ real, as it should.
Strictly speaking \( R(\alpha) \) is defined on \( C \) only, but (2.11) can be used to define it outside of \( C \). If \( C \) is contained in the strip \(-\gamma < \text{Im}(\alpha) < \gamma\), \( R(\alpha) \) is analytic in \(-\gamma < \text{Im}(\alpha) < \gamma\), but it inherits the poles of \( \xi(\alpha) \) at \( \pm i\gamma \). Let us consider the curve for which \( a = \pi \). In this case, since \( R(\alpha) \) is 2\( \pi \)-periodic, (2.11) can be solved straightforwardly by Fourier transform. The solution is

\[
R(\alpha) = \sum_n \frac{e^{-in\alpha}}{2 \cosh \gamma n} \quad -\gamma < \text{Im}(\alpha) < \gamma
\]  

(2.14)

Here and in the following, sums are understood to run from \(-\infty \) to \( \infty \) unless otherwise stated. Beyond this strip (2.14) can be expressed using elliptic functions. Introducing the complete elliptic integrals of the first kind \( I \) (\( I' \)) of modulus \( k \) (\( k' \)), with \( k'^2 + k^2 = 1 \) \[18\], related to \( \gamma \) by

\[
\frac{I'(k)}{I(k)} = \frac{\gamma}{\pi}
\]

the solution of (2.11) in a wider domain reads \[19,18\]

\[
R(\alpha) = \frac{I(k)}{\pi} \text{dn} \left( \frac{I(k)\alpha}{\pi}; k \right)
\]  

(2.15)

Notice the presence of a pole at \( \alpha = \pm i\gamma \), inherited from \( \xi(\alpha) \), which prevents the convergence of (2.14) beyond the smaller domain. The energy remains constant at its value for \( h = 0 \)

\[
e_0 = \lim_{N \to \infty} \frac{E_0}{N} = \cosh \gamma - 2 \sinh \gamma \sum_n \frac{e^{-\gamma |n|}}{2 \cosh \gamma n}
\]

and from (2.13) \( y = 0 \). In fact the solution considered here has \( n = \frac{N}{2} \) rapidities (\( S^z = 0 \)) and, as it will be shown in the next section, \( E_0 \) is the ground state energy for \( h \) and \( V \) sufficiently close to 0.

As to the precise position of the curve, one has to revert to (2.9). Since \( \Theta(\alpha + 2\pi) = \Theta(\alpha) + 2\pi \), we use the expansion

\[
\Theta(\alpha) = \alpha + i \sum_{n \neq 0} \frac{e^{-in\alpha - 2\gamma |n|}}{n} \quad -2\gamma < \text{Im}(\alpha) < 2\gamma
\]
and
\[ p^0(\alpha) = \alpha + i \sum_{n \neq 0} e^{-in\alpha - \gamma|n|} n \quad (2.16) \]
and we introduce
\[ p(\alpha) = \frac{\alpha}{2} + i \sum_{n \neq 0} \frac{e^{-in\alpha}}{2n \cosh \gamma n} = am\left(\frac{I(k)\alpha}{\pi}; k\right) \quad (2.17) \]

The series in (2.16) and (2.17) are certainly convergent when \(-\gamma < \text{Im}(\alpha) < \gamma\) but they converge also at \(\alpha = \pm \pi \pm i\gamma\), because of the alternating sign. Eq. (2.9) reduces to
\[ p(\alpha) + \frac{ib}{2} + i \sum_{n \neq 0} (-)^n \frac{e^{nb}}{2n \cosh \gamma n} + 2ih = 2\pi x - \frac{1}{4} \leq x \leq \frac{1}{4} \]

Specialization to the endpoints permits relating the value of \(h\) to \(b\) \([3, 4]\)
\[ h(b) = -\frac{b}{2} - \sum_{n=1}^{\infty} (-)^n \frac{\sinh nb}{n \cosh n\gamma} \quad (2.18) \]
so that the final equation of the curve is
\[ p(\alpha) + ih = 2\pi x - \frac{1}{4} \leq x \leq \frac{1}{4} \]

Notice that the points on the curve are characterized by
\[ \text{Im}(p(\alpha)) + h = 0 \quad (2.19) \]

We set \(h_c = h(b = -\gamma)\). One might suspect that, when \(h > h_c\), the endpoints would remain at \(a = \pi\) but with \(b < -\gamma\) (or \(b > \gamma\) if \(h < -h_c\)). If \(C\) does not cross the point \(-i\gamma\) where \(\xi(\alpha)\) has a pole, it can always be deformed to the real axis in (2.11)
\[ R(\alpha) + \frac{1}{2\pi} \int_{-\pi}^{\pi} du K(\alpha - u)R(u) = \xi(\alpha) \quad (2.20) \]
so that the solution is still given by \(\frac{I(k)}{\pi}dn\left(\frac{I(k)}{\pi}\alpha; k\right)\), but the expansion (2.14) is no longer useful. To find the \(h(b)\) relation, we close \(C\) in (2.9) to the real axis, and take \(\alpha = A\)
\[ p^0(A) - \frac{1}{2\pi} \int_{-\pi}^{\pi} du \Theta(A - u)R(u) + \int_{-\pi}^{A} d\beta R(\beta) + 2ih = -\frac{\pi}{2} \quad (2.21) \]
The integral of $R(\alpha)$ is obtained by integrating both sides of (2.20), and from (2.21) we conclude

$$2h(b) = b + 2 \ln \frac{\cosh\left(\frac{\gamma}{2} - \frac{b}{2}\right)}{\cosh\left(\frac{\gamma}{2} + \frac{b}{2}\right)} + 2 \sum_{n>0} \frac{(-)^n e^{-2n\gamma \sinh nb}}{n \cosh n\gamma}$$

(2.22)

which reduces to (2.18) when $|b| \leq \gamma$. Eq. (2.22) though cannot give the right dependence $h(b)$ at $|b| > \gamma$, because $h(b)$ decreases when $b < -\gamma$ and increases for $b > \gamma$, going back to the range of values it had as $b \in [-\gamma, \gamma]$. Clearly the initial assumption $a = \pi$ cannot be correct.

To gain more insight we resort as usual to the numerical solution of (2.3) which shows that, as $h > h_c$, the curve with $y = 0$ has endpoints at (see table 2. Note that all extrapolations presented in the tables have been done using data up to 80 sites. We give however only the first values, up to $N = 40$, for they already show clearly that the values are converging towards a limit. All tables have been calculated for $\cosh \gamma = 21$, where $\exp(2h_c) = 10.51787$.

$$b = -\gamma \quad a < \pi$$

This (new) result will be used in the calculation of the free energy singularity in Section 4.

Turning next to the statistical model, the largest eigenvalue of the transfer matrix yields the free energy per site (we drop here the inessential factor $\beta$)

$$f(u, \gamma, h, v) = -\lim_{N \to \infty} \frac{\ln \Lambda_0(u, \gamma, h, v)}{N}$$

whose value, as $N \to \infty$, is dominated by the largest of the two limits

$$\lim_{N \to \infty} \frac{1}{N} \Lambda_R(u) = F_R(u, \gamma, h, y) + vy = h + \ln \frac{\sinh(\gamma - u)}{\sinh \gamma} + vy$$

$$+ \frac{1}{2\pi} \int_C d\alpha R(\alpha) f_R(\alpha; u)$$

$$\lim_{N \to \infty} \frac{1}{N} \Lambda_L(u) = F_L(u, \gamma, h, y) + vy = -h + \ln \frac{\sinh(u)}{\sinh \gamma} + vy$$

$$+ \frac{1}{2\pi} \int_C d\alpha R(\alpha) f_L(\alpha; u)$$
where we have defined

\[ f_R(\alpha; u) = \ln \frac{\sinh(\frac{\gamma}{2} + u - \frac{i\alpha j}{2})}{\sinh(\frac{\gamma}{2} - u + \frac{i\alpha j}{2})} \]

\[ f_L(\alpha; u) = \ln \frac{\sinh(-\frac{3\gamma}{2} + u - \frac{i\alpha j}{2})}{\sinh(\frac{\gamma}{2} - u + \frac{i\alpha j}{2})} \]

If \( F_R, F_L \) are known, and we call \( F \) the dominant one, the equilibrium value of \( y \) and the free energy are determined by the minimum condition

\[ f(u, \gamma, h, v) = \min_{-1 \leq y \leq 1} \{-F(u, \gamma, h, y) - vy\} \quad (2.23) \]

When \(-h_c < h \leq h_c\) and for small enough values of \( v\), the state defined by \( C \) \((a = \pi; -\gamma \leq b \leq \gamma)\) also yields the largest eigenvalue of the transfer matrix. The free energy \( f(u, \gamma, h, v) = -2 \sum_{n=1}^{\infty} \frac{e^{-2\gamma n}}{n \cosh \gamma} \sinh(nu) \sinh(n(\gamma - u)) \quad (2.24) \)

is constant in a whole region of the \((h, v)\) plane bounded by a curve \( \Gamma \) ('flat' phase). The parametric equation \((h(b), v(b))\) of \( \Gamma \) is given by \((2.20)\) and by

\[ v(b) = -\frac{\partial F}{\partial y} \bigg|_{h \text{ fixed, } y=0} \]

which can be explicitly computed

\[ 2v(b) = \gamma - |\gamma - 2u + b| + 2 \sum_{n=1}^{\infty} \frac{(-)^n \sinh[n(\gamma - |\gamma - 2u - b|)]}{n \cosh n\gamma} \quad -\gamma \leq b \leq \gamma \quad (2.25) \]

The other half of the curve \( \Gamma \) (see fig. 3) can be recovered from the symmetry \( f(-h, -v) = f(h, v) \) (see Appendix A). Most of these results have been obtained elsewhere and we have presented them here only for the sake of completeness. It should be pointed out that the fact that the ground state energy does not depend on \( h \) is simply a consequence of the analogous property of the free energy.

The only part in which our analysis differs from \([9]\) is that \( \Lambda_R \) is exponentially larger for \( d < \gamma - 2u \) and \( \Lambda_L \) for \( d > \gamma - 2u \) where \( id \) is the point in which \( C \) crosses the imaginary axis. A comparison of \((2.24)\) with \([9]\) should take into account the different normalization of the Boltzmann weights.
3 Hole excitations of the spin chain

To understand the nature of the phase transition along $\Gamma$ we turn to the calculation of the excitation energies, and we set $V = 0$, since the role of $V$ is simply to shift the spectra at $S^z \neq 0$. As proven in appendix A it is sufficient to consider $h \geq 0$.

A complete treatment of the spectrum should rely on the classification of all possible solutions of (2.3). This is usually done in the framework of the string hypothesis, according to which complex rapidities (at $h = 0$) have an imaginary part which tends to well defined values in the thermodynamic limit. Exceptional solutions other than strings, but that still appear in complex conjugate pairs, can be handled in a similar way [20]. Yet, from the numerical analysis of (2.3), it appears that strings do not survive at moderately strong values of $h$. Therefore we shall limit our calculation to the so–called hole excitations, that is holes in the ground state distribution of rapidities, occurring in sectors with $S^z > 0$ ($n < n_0 = \frac{N}{2}$).

We introduce the counting function

$$Z(\alpha, \{\alpha_j\}) = \frac{p^0(\alpha)}{2\pi} - \frac{1}{2\pi N} \sum_{j=1}^n \Theta(\alpha - \alpha_j) + i\frac{h}{\pi}$$

so that (2.8) is rewritten

$$Z(\alpha_k) = \frac{I_k}{N}$$

Set $\text{vac} = \text{number of vacancies available for the quantum numbers } \{I_k\}$. With the usual hypothesis that $Z(\alpha)$ be monotonically increasing, we have

$$\Delta Z \equiv Z(\alpha)|_{\text{Re}(\alpha) = \pi} - Z(\alpha)|_{\text{Re}(\alpha) = -\pi} = \frac{\text{vac}}{N} \quad (3.2)$$

On the other hand, from (3.1), we find $\Delta Z = 1 - \frac{n}{N}$. For the ground state $n = n_0 = \frac{N}{2}$ and so $\text{vac} = n_0$, i.e. the available vacancies are all filled. For $n$ rapidities, $n = n_0 - r$ where $r = 1, 2, \cdots$ we have

$$\Delta Z = 1 - \frac{n_0 - r}{N} \quad \text{vac} = n_0 + r$$

so that $n_0 + r$ vacancies are partially filled with $(n_0 - r)$ $I_k$’s, leaving $N_h = 2r$ holes.
We will resort to the 'backflow method' \cite{19, 21} in dealing with \eqref{2.3}, \eqref{3.1} and \eqref{2.7} in the limit $N \to \infty$. The calculation differs slightly for the 2 cases $r = \text{even or odd}$, but the results are identical and we will present the case $r = \text{even}$ only. If this is the case, then $n = n_0 \pmod{2}$ and the quantum numbers $\{I_k\}$ of the excited state have the same oddness of the quantum numbers $\{I_0^0\}$ of the ground state. We assume that the $r$ additional vacancies for $\{I_k\}$ are placed $\frac{r}{2}$ to the left and $\frac{r}{2}$ to the right of the sequence (see fig. 4).

We call $\{\beta_j^{(1)}\}$ the $\frac{r}{2}$ additional rapidities at the left edge, $\{\beta_j^{(2)}\}$ the $\frac{r}{2}$ additional rapidities at the right edge and $\{\alpha_j^{(h)}\}$ the $N_h$ holes. Then, for the ground state

$$Z_0(\alpha) = p^0(\alpha) \frac{1}{2\pi} - \frac{1}{2\pi N} \sum_{j=1}^{n_0} \Theta(\alpha - \alpha_j^0) + \frac{ih}{\pi}$$

and for the excited state, adding and subtracting the holes

$$Z(\alpha) = p^0(\alpha) \frac{1}{2\pi} - \frac{1}{2\pi N} \sum_{j=1}^{n_0} \Theta(\alpha - \alpha_j) + \frac{1}{2\pi N} \sum_{j=1}^{N_h} \Theta(\alpha - \alpha_j^{(h)})$$

$$- \frac{1}{2\pi N} \sum_{j=1}^\frac{r}{2} \left[ \Theta(\alpha - \beta_j^{(1)}) + \Theta(\alpha - \beta_j^{(2)}) \right] + \frac{i}{\pi}$$

$$Z(\alpha_k) = \frac{I_k}{N}, \quad Z(\alpha_j^{(h)}) = \frac{I_j^{(h)}}{N}$$

As $N \to \infty \{\beta_j^{(1)}\} \to A$ and $\{\beta_j^{(2)}\} \to B$. Subtracting \eqref{3.4} from \eqref{3.6} and retaining terms of order $\frac{1}{N}$ (this is a standard Bethe Ansatz calculation) \cite{21} we find that

$$j(\alpha_l) = \lim_{N \to \infty} \frac{\alpha_l - \alpha_l^0}{\alpha_{l+1}^0 - \alpha_l^0}$$

satisfies

$$j(\alpha) + \frac{1}{2\pi} \int_C d\beta K(\alpha - \beta) j(\beta) = -\frac{1}{2\pi} \sum_{j=1}^{N_h} \Theta(\alpha - \alpha_j^{(h)})$$

$$+ \frac{r}{4\pi} \left[ \Theta(\alpha - B) + \Theta(\alpha - A) \right]$$

with

$$\Delta E = \int_C dae'(\alpha) j(\alpha) - \sum_{j=1}^{N_h} e(\alpha_j^{(h)}) + \frac{r}{2} \left( e(A) + e(B) \right)$$
\[ \Delta P = i\hbar N_h - \int_C d\alpha \xi(\alpha) j(\alpha) + \sum_{j=1}^{N_h} p^0(\alpha_j^{(h)}) - \frac{r}{2} \left( p^0(A) + p^0(B) \right) \]

Eq. (3.7) defines the analytical properties of \( j(\alpha) \) in the complex plane. Since for \( 0 \leq \hbar \leq h_c \) the curve \( C \) is contained in \( -\gamma \leq \text{Im}(\alpha) \leq 0 \), \( j(\alpha) \) is certainly analytic in \( -2\gamma < \text{Im}(\alpha) < \gamma \) and the curve can be closed to the real axis. Noticing, from (3.7), that \( j(\alpha + 2\pi) - j(\alpha) = -\frac{N_h}{2} \), we get, with \( u \in [-\pi, \pi] \)

\[ j(u) + \frac{1}{2\pi} \int_{-\pi}^\pi dv K(u - v) j(v) = \frac{N_h}{4\pi} \Theta(u + \pi) - \frac{1}{2\pi} \sum_{j=1}^{N_h} \Theta(u - \alpha_j^{(h)}) - \frac{r}{2} \] (3.8)

\[ \Delta E = \int_{-\pi}^\pi du e'(u) j(u) + \frac{N_h}{2} e(\pi) - \sum_{j=1}^{N_h} e(\alpha_j^{(h)}) \] (3.9)

\[ \Delta P = i\hbar N_h - \int_{-\pi}^\pi du \xi(u) j(u) + \sum_{j=1}^{N_h} p^0(\alpha_j^{(h)}) \] (3.10)

Equation (3.8) can be solved by Fourier transform, paying attention to the fact that \( j(u) \) is not periodic, but obeys the quasiperiodicity condition \( j(u + 2\pi) - j(u) = -\frac{N_h}{2} \). Alternatively, and with identical results, the symmetric integral operator \( (1 + \frac{1}{2\pi} K) \) at the left side of (3.8) can be formally inverted and the solution plugged in (3.9) and (3.10) [21]. The result has the usual additive form

\[ \Delta E = 2 \sinh \gamma \sum_{j=1}^{N_h} \epsilon(\alpha_j^{(h)}) \] (3.11)

where the dressed energy \( \epsilon(u) \) satisfies

\[ \epsilon(u) + \frac{1}{2\pi} \int_{-\pi}^\pi dv K(u - v) \epsilon(v) = \xi(u) \]

and therefore coincides with \( R(u) = \frac{I(k)}{\pi} \text{dn} \left( \frac{I(k)u}{\pi}, k \right) \). As to the momentum, one has

\[ \Delta P = \sum_{j=1}^{N_h} \left[ p(\alpha_j^{(h)}) + i\hbar \right] \] (3.12)

where \( p(\alpha) \) has been defined in (2.17). The calculation for \( r \) odd differs slightly in the intermediate steps but also yields (3.11) and (3.12), which are then true for \( N_h \) arbitrary (but obviously even). The fact that \( N_h \) is even was missed in [22] where, following the same
assumption made in [19], one hole was kept fixed at the edge. In other words, only a subset of the 2–hole band of states was dealt with.

Eq. (3.11) and (3.12) are simple generalizations of their limit at \( h = 0 \), since \( h \) appears only additively in \( \Delta P \) and, implicitly, in the position of the hole which is bound to be on the curve \( C \). This simple dependence could not be derived immediately from (2.5) and (2.7) because the rapidities \( \{ \alpha \} \) depend on \( h \) in a nontrivial way through (2.3), and only the explicit calculation guarantees that (3.11) and (3.12) are correct.

Several comments are in order. Unlike the energy, which being the eigenvalue of a non–hermitian operator can, and indeed does have an imaginary part, the momentum must be real. This is guaranteed by (2.19), since \( \alpha^{(h)} \in C \). It is well known that, from (2.7), (2.8) and the oddness of \( \Theta(\alpha) \), the momentum can be obtained by summing (2.8) over \( k \)

\[
P = -\frac{2\pi}{N} \sum_{k=1}^{n} I_k
\]

The momentum of the ground state \( P^0 \) is therefore always zero, while the momentum of an excited state is

\[
P = \frac{2\pi}{N} \sum_{k=1}^{N_h} I_k^{(h)}
\]

from which one sees that, as \( N \to \infty \), \(-\frac{\pi}{2} \leq \Delta P(\text{hole}) \leq \frac{\pi}{2}\). This is also confirmed by (2.17) and (3.12). The dispersion relations are obtained by eliminating \( \alpha^{(h)} \) in (3.11) and (3.12) using

\[
dn(\alpha; k) = \sqrt{1 - k^2 \sin^2(2am(\alpha; k))}
\]

which yields

\[
\Delta E(\Delta P) = m_0 \sqrt{1 - k^2 \sin^2 \left( \Delta P - ih \right)} \quad m_0 = 2 \sinh \gamma \frac{I(k)}{\pi}
\]

An apparent discrepancy with Gaudin’s result (for the case \( h = 0 \)) is clarified in appendix B. The \( dn(\alpha) \) function (and consequently \( \epsilon(\alpha) \)) has a non negative real part in the rectangle

\[\text{One of the authors (GA) is grateful to prof. C. Destri for pointing this out.}\]
\([-\pi; -\pi - i\gamma; \pi - i\gamma; \pi]\) and this lifts the ambiguity in the sign of (3.14). It also confirms that the choice of the ground state was correct, because the real part of the energy, at least under 'small' variations (a countable number of holes), increases. The minimum of \(\text{Re}(\Delta E)\) is reached when \(\alpha^{(h)} = A\) or \(B\). When this happens \(\Delta P = \pm \frac{\pi}{2}\) and the gap in the spectrum is (remember that 2 holes are present in the lowest excited state)

\[
\Delta E(\text{gap}) = 2m_0 \sqrt{1 - k^2 \cosh^2(h)}
\]  
(3.15)

which guarantees that the 'mass gap' is real (see table 3 for a comparison with numerical results). In particular it vanishes at \(b = -\gamma\), that is when \(A = -\pi - i\gamma\), \(B = \pi - i\gamma\) and

\[
h = h_c = \frac{\gamma}{2} + \sum_{n \neq 0} (-)^n \frac{\sinh(n\gamma)}{n \cosh(n\gamma)}
\]

This is most easily seen from (3.11)

\[
\Delta E(\text{gap}) = 2m_0 \text{dn} \left( \frac{I(k)}{\pi} (\pm \pi - i\gamma); k \right) = 2m_0 \text{dn} (\pm I(k) - iI'(k); k) = 0
\]

Therefore, from (3.15) an alternative equation for \(h_c\) is

\[
\cosh(h_c) = \frac{1}{k}
\]

It is particularly interesting to see how the mass gap vanishes as \(h \to h_c\)

\[
\Delta E(\text{gap}) \sim 2^{\frac{3}{2}} m_0 \sqrt{k'(h_c - h)^{\frac{1}{2}}} + O(h_c - h)
\]  
(3.16)

The vanishing with an exponent \(\frac{1}{2}\) is peculiar of the point under consideration, as it will appear clear from the general case, to be discussed later, which includes the vertical field \(V\).

Finally, we specialize (3.13) at \(h = h_c\). Then the hole excitations are massless and if we set \(\Delta P = -\frac{\pi}{2} + \epsilon\) or \(\Delta P = \frac{\pi}{2} - \epsilon\), \(0 < \epsilon << 1\), we get, respectively

\[
\Delta E(\Delta P) \sim 2m_0 \sqrt{2ik'} \epsilon^{\frac{1}{2}}
\]  
(3.17)

This dispersion relation is certainly surprising and reflects itself in the peculiar behavior of the finite size corrections of the low-lying energy gaps at \(h = h_c\). In marked contrast with
the $O(N^{\frac{1}{2}})$ scaling typical of spin chains which describe conformally invariant models in the continuum limit $[23]$, we find
\[ \Delta E \sim \frac{c}{N^{\frac{1}{2}}} + O(N^{-1}) \]
where $c$ depends on the state under consideration. The momentum being quantized in units of $\frac{2\pi}{N}$ on the finite lattice (3.13), this behavior is well in agreement with (3.17).

The sector $S^z = 0$ deserves a special comment. Not knowing what takes the place of strings, the analysis of the excitations has been necessarily numerical. Two things have been determined. Setting $E_0(S^z = 0, N, h)$ and $E_1(S^z = 0, N, h)$ to be respectively the ground state and the lowest–lying of the first band of excited states in the sector $S^z = 0$, on a chain of $N$ sites, and at fixed horizontal field $h \leq h_c$, we found
\[ \Delta E(S^z = 0, h) = \lim_{N \to \infty} \left[ E_1(S^z = 0, N, h) - E_0(S^z = 0, N, h) \right] \]
to be positive, non–zero for $h < h_c$ and
\[ \lim_{h \to h_c} \Delta E(S^z = 0, h) = 0 \]
so that, even in this sector, the spectrum becomes massless at $h = h_c$ (see table 4). Secondly, the $O(N^{-\frac{1}{2}})$ scaling is preserved at $h_c$
\[ E_1(S^z = 0, N, h_c) - E_0(S^z = 0, N, h_c) \sim \frac{c}{N^{\frac{1}{2}}} \quad N \gg 1 \]
$E_1$ must not be confused with the other (degenerate in the thermodynamic limit) ground state that appears in this sector at momentum $P = \pi$ and is responsible for the spontaneous breaking of the arrow–reversal symmetry in the symmetric 6–vertex model $[8, 19]$. See table 5 for a comparison with numerical results. Since we do not study the order parameter (staggered polarization) this state will not be discussed here.

We can now reintroduce $V$, whose effect is to shift the spectra at $S^z \neq 0$. The mass gap for a state with $n = n_0 \pm r$, and consequently $2r$ holes is easily read from (2.5) and (3.13)
\[ \Delta E(\text{gap}; n) = 2rm_0\sqrt{1 - k^2\cosh^2(h)} - 2V(\pm r) \quad (3.18) \]
An alternative way to reach the boundary with the massless phase is to have a sufficiently large $|V|$. From (3.18) the crossing occurs at

$$V = \pm m_0 \sqrt{1 - k^2 \cosh^2(h)}$$

and moves the ground state to sectors of $S^z > 0$ (i.e. $n < n_0$) if $V > 0$, and to sectors of $S^z < 0$ (i.e. $n > n_0$) if $V < 0$, as it was intuitively predictable from (2.3). Notice that, unlike what happens in (3.16), the mass gap goes to zero linearly in $V$ or linearly in $h$ if $V \neq 0$ were kept fixed and $h \to h(V)$, where $h(V)$ is defined by (3.19). The point $V = 0$, $h = h_c$ (or equivalently $h = -h_c$), where the exponent $\frac{1}{2}$ of (3.16) appears, is clearly special. Even if it were approached by changing $h$ and $V$ simultaneously, the term $(h_c - h)\frac{1}{2}$ would dominate over the linear term in $V$. There is no way to erase this effect because it is impossible to reach $(h_c, V = 0)$ by changing $V$ only: the line $h = h_c$ in the $(h, V)$ plane is tangent to the phase boundary curve defined by (3.19).

This result may look odd, because the energy difference between sectors of different $S^z$ corresponds to the step free energy for the statistical model, and from (2.4) it is hard to see how it could vanish other than linearly. Yet it is readily seen that the phenomenon is not an artifact of the spin chain. An explicit calculation of the step free energy

$$f_{\text{step}} = -\left[\ln \Lambda_{\text{max}}(S^z = 1) - \ln \Lambda_{\text{max}}(S^z = 0)\right]$$

can be bypassed observing that the vanishing of $f_{\text{step}}$ signals the transition to the incommensurate phase and therefore must be given by (2.23)

$$f_{\text{step}} = 2v - \gamma + |\gamma - 2u - b| - 2 \sum_{n=1}^{\infty} \frac{(-)^n \sinh[n(\gamma - |\gamma - 2u - b|)]}{n \cosh n\gamma} \quad -\gamma \leq b \leq \gamma$$

The points $(h_c, v_c)$ and $(-h_c, -v_c)$, reached on $\Gamma$ when $b = -\gamma$ (or $\gamma$) are the equivalent of $(\pm h_c, V = 0)$ in the spin chain phase diagram. They, again, cannot be approached from the flat phase by changing $v$ only, since the line $h = h_c$ in the $(h, v)$ plane is tangent to $\Gamma$. But,
from (2.18), near $b = -\gamma$

$$h_c - h \sim \frac{1}{2} \left( \frac{I(k)}{\pi} \right)^2 k'(b + \gamma)^2$$

and since $f_{\text{step}}$ is linear in $b$ near $b = -\gamma$ (unless $u=0$)

$$f_{\text{step}} \sim \text{const}(h_c - h)^{\frac{3}{2}} + \text{const}(v - v_c)$$

at $(h_c, v_c)$. This shows that, like for the spin chain, the exponent $\frac{3}{2}$ dominates and signals that the points $(h_c, v_c)$ and $(-h_c, -v_c)$ are essentially different from the other points of $\Gamma$.

As to the sector $S^z = 0$, we have to extend the numerical analysis carried out for the spin chain. If $\Lambda_0(S^z = 0, N, h, v_c)$ and $\Lambda_1(S^z = 0, N, h, v_c)$ are the largest and next–to–largest eigenvalues on the finite lattice in the sector under consideration, we find that

$$\Delta \Lambda(S^z = 0, h, v_c) = \lim_{N \to \infty} \left[ \Lambda_1(S^z = 0, N, h, v_c) - \Lambda_0(S^z = 0, N, h, v_c) \right]$$

is positive for $h < h_c$ and vanishes when $h = h_c$. Furthermore

$$- \left[ \ln \Lambda_1(S^z = 0, N, h_c, v_c) - \ln \Lambda_0(S^z = 0, N, h_c, v_c) \right] \sim \frac{c'}{N^{\frac{3}{2}}} \quad N \gg 1$$

in perfect correspondence with the spin chain scaling of low–lying excitations (see table 6).

4 The exponent $\frac{3}{2}$ of the free energy singularity

As the field crosses the critical value of the $\Gamma$ line (2.18), (2.23) the system enters a phase where horizontal and vertical polarizations change continuously. This is an incommensurate phase belonging to the universality class of the gaussian model [10]. It is interesting to determine the singularity of the free energy as $(h, v)$ approach $\Gamma$ from the incommensurate regime. It is widely believed [3, 8] that the free energy singularity should be governed by an exponent $\frac{3}{2}$, but an exact calculation has been done by Lieb and Wu when $h = 0$ only [8], in which case

$$f \sim c(\gamma, u) \left[ v - v_c(\gamma, u, b = 0) \right]^{\frac{3}{2}}$$
Our calculation is an extension of Lieb’s and Wu’s method. We will apply it first to the
ground state energy of the spin chain, and later extend it to the free energy of the statistical
model.

Eqs. (2.9) and (2.12)–(2.13) determine, through the solution of (2.11), \( e_0, y \) and \( h \) as
functions of \( A \) and \( B \). We suppose that such dependence is analytic and \( e_0, y \) and \( h \) can be
expanded in powers of \( \delta A, \delta B \) as \( A \to A + \delta A, B \to B + \delta B \). Making explicit the dependence
of \( R(\alpha) \) on \( A, B \) by writing \( R(\alpha; A, B) \) we have, from (2.12)

\[
\partial_A y(A, B) = -\frac{2}{2\pi} \int_A^B d\alpha \partial_A R(\alpha; A, B) + \frac{2}{2\pi} R(A; A, B)
\]

\[
\partial_B y(A, B) = -\frac{2}{2\pi} \int_A^B d\alpha \partial_B R(\alpha; A, B) - \frac{2}{2\pi} R(B; A, B)
\]

\[
\delta y = \partial_A y(A, B) \delta A + \partial_B y(A, B) \delta B + O(\delta A^2, \delta B^2, \delta A \delta B)
\]

(4.1)

Likewise, the energy per site

\[
e_0(A, B) = \cosh \gamma + \frac{1}{2\pi} \int_A^B d\alpha e(\alpha) R(\alpha; A, B) - V y = \cosh \gamma - V y + e_0^{(1)}(A, B)
\]

yields the derivatives

\[
\partial_A e_0^{(1)}(A, B) = \frac{1}{2\pi} \int_A^B d\alpha e(\alpha) \partial_A R(\alpha; A, B) - \frac{1}{2\pi} e(A) R(A; A, B)
\]

\[
\partial_B e_0^{(1)}(A, B) = \frac{1}{2\pi} \int_A^B d\alpha e(\alpha) \partial_B R(\alpha; A, B) + \frac{1}{2\pi} e(B) R(B; A, B)
\]

\[
\delta e_0(A, B) = \partial_A e_0^{(1)}(A, B) \delta A + \partial_B e_0^{(1)}(A, B) \delta B + O(\delta A^2, \delta B^2, \delta A \delta B)
\]

(4.2)

e etc. Similar equations can be obtained for \( h \), specializing (2.9) to the endpoints of the curve
and taking the symmetric form

\[-4i h(A, B) = p^0(A) + p^0(B) - \frac{1}{2\pi} \int_A^B d\beta \Theta(\beta; A, B) \left[ \Theta(A - \beta) + \Theta(B - \beta) \right]
\]

hence

\[-4i \partial_A h(A, B) = -\frac{1}{2\pi} \int_A^B d\beta \partial_A R(\beta; A, B) \left[ \Theta(A - \beta) + \Theta(B - \beta) \right] + R(A; A, B) \left[ 1 + \frac{1}{2\pi} \Theta(B - A) \right]
\]

\[-4i \partial_B h(A, B) = -\frac{1}{2\pi} \int_A^B d\beta \partial_B R(\beta; A, B) \left[ \Theta(A - \beta) + \Theta(B - \beta) \right] + R(B; A, B) \left[ 1 + \frac{1}{2\pi} \Theta(B - A) \right]
\]
etc.. Equations for the derivatives of \( R(\alpha; A, B) \) are readily obtained from
\[
\begin{align*}
\partial_A R(\alpha; A, B) + \frac{1}{2\pi} \int_A^B d\beta K(\alpha - \beta) \partial_A R(\beta; A, B) &= \frac{1}{2\pi} K(\alpha - A) R(A; A, B) \\
\partial_B R(\alpha; A, B) + \frac{1}{2\pi} \int_A^B d\beta K(\alpha - \beta) \partial_B R(\beta; A, B) &= -\frac{1}{2\pi} K(\alpha - B) R(B; A, B)
\end{align*}
\]
e etc..

We have carried out these expansions to the third order in \( \delta A, \delta B \). In principle they can be used for any \( A, B \) with \( a = \pi, |b| \leq \gamma \), and the integrals computed by Fourier transform. However, as it is already evident from the first order terms, the expansions simplify considerably when carried out around \( A_0 = -\pi \pm i\gamma, B_0 = \pi \pm i\gamma \) which are zeros of the \( dn \) function
\[
R(A_0; A_0, B_0) = R(B_0; A_0, B_0) = 0
\]
The details of the expansion are lengthy but straightforward, so only the final form is of interest. Writing
\[
\delta A = -\delta a + i\delta b \quad \delta B = \delta a + i\delta b
\]
and considering first the expansion around \( A_0 = -\pi - i\gamma, B_0 = \pi - i\gamma \), we have
\[
\begin{align*}
\delta e_0 &= 2c_2[(-\delta a)^3 - 3\delta a(\delta b)^2] - V\delta y + \ldots \quad (4.3) \\
\delta y &= -\frac{c_1}{\pi}\delta a\delta b + \frac{c_3}{\pi}\delta b(\delta a)^2 + \ldots \quad (4.4) \\
\delta h &= \frac{c_1}{2}[(\delta a)^2 - (\delta b)^2] + \frac{c_3}{3}(\delta a)^3 + \ldots \quad (4.5)
\end{align*}
\]
with
\[
\begin{align*}
c_1 &= k'(\frac{I(k)}{\pi})^2 > 0 \\
c_2 &= \sinh \gamma \left[ \frac{1}{4} + \sum_{n>0} \frac{(-1)^n \exp(-n\gamma)}{\cosh \gamma} \right] > 0 \\
c_3 &= \frac{1}{\pi} \sum_{n} \frac{\exp(-|n|\gamma)}{2 \cosh \gamma n} > 0
\end{align*}
\]
Notice that \( \delta y = 0 = \delta e_0 \) when \( \delta a = 0 \), as it should, since by taking \( \delta a = 0 \) and \( \delta b > 0 \) we reenter the ‘flat phase’. It is also important to check that if \( \delta a = 0 \) there is no way to increase
by changing $b$, as already discussed in section 2. An increase in $h$, when keeping $y$ fixed at $y = 0$, can instead be achieved by $\delta a < 0, \delta b = 0$, which confirms the numerical findings presented in section 2. A variation $\delta a > 0$ is ruled out a priori, because the periodicity of (2.3) in the real direction implies that rapidities are contained in the strip $-\pi \leq Re(\alpha) \leq \pi$ and $a$ cannot exceed $\pi$.

Another point to discuss is the reliability of (4.3)–(4.5) when $n > N/2$, that is $\delta y < 0$. The Bethe-ansatz equations for the symmetric six vertex model are always discussed keeping $n \leq N/2$, since the $Z_2$ symmetry of arrow reversal guarantees that the spectrum is the same when $N/2 < n \leq N$. It is not immediately clear what happens to (2.3) when $n > N/2$. As an example, consider the one dimensional sector $S^z = -N$ ($n = N$), whose only eigenstate is $|\downarrow\downarrow\cdots\downarrow\rangle$. It is not obvious that (2.3) should have only one solution when the number of unknowns is $N$. To be on the safe side we will trust (4.3)–(4.5) only for $n \leq N/2$ ($y \geq 0$). In this case $\delta b \geq 0$ and $\delta a \leq 0$. To deal with the states at $n > N/2$ ($y < 0$), one must resort to (5.4) which implies

$$e_0(\gamma, h_c + \delta h, -y) = e_0(\gamma, -h_c - \delta h, -V, y)$$

(4.6)

Hence it is necessary to consider also an expansion around $h = -h_c, y = 0$. This can be done evaluating (4.1) and the following equations at the endpoints $A_0' = -\pi + i\gamma, B_0' = \pi + i\gamma$. The result is that (4.3)–(4.5) still hold, with $\delta b > 0$ ($< 0$) if $\delta y > 0$ ($< 0$). A final observation about (4.3)–(4.5) is that it is legitimate to neglect higher order terms in (4.4) and (4.3). The parameters $\delta a$ and $\delta b$ are independent and there is no control over their relative magnitude, but the second order term in (4.3) is dominant unless $\delta a \simeq \pm \delta b$, in which case the third order term is certainly larger than all possible fourth order terms. Likewise, in (4.4), no term $\delta a^n$ or $\delta b^n$ is allowed since we know that $\delta y = 0$ if $\delta a = 0$ or $\delta b = 0$. Consequently, all higher order terms can certainly be neglected and one can further limit the expansion to

$$\delta y = -\frac{c_1}{\pi} \delta a \delta b$$
Suppose now that \( h \) is kept fixed at \( h_c \) and \( V \neq 0 \). Then, from (4.5),
\[
\delta b = \mp \delta a
\]
where the upper (lower) sign holds for \( \delta y > 0 \) \((<0)\). Consider first \( \delta y > 0 \). Then
\[
\delta y = \frac{c_1}{\pi} \delta a^2
\]
\[
\delta e_0(\delta a) = -4c_2 \delta a^3 - \frac{V c_1}{\pi} \delta a^2
\]
which has a minimum, when \( V > 0 \), at
\[
\delta a_0 = -\frac{V c_1}{6\pi c_2}
\]
that yields
\[
\delta e_0(h = h_c, V > 0) = -\frac{2}{c_2^3} \left( \frac{V c_1}{6\pi} \right)^3
\]
Notice that no minimum occurs if \( V < 0 \). Instead, if we consider \( \delta y < 0 \), one has a minimum at
\[
\delta a_0 = \frac{V c_1}{6\pi c_2}
\]
that yields
\[
\delta e_0(h = h_c, V < 0) = \frac{2}{c_2^3} \left( \frac{V c_1}{6\pi} \right)^3
\]
when \( V < 0 \). Consequently,
\[
e_0(h = h_c, V) = e_0(h = h_c, V = 0) - \frac{2}{c_2^3} \left( \frac{c_1}{6\pi} \right)^3 |V|^3
\]
\[
\delta y = \text{sgn}(V) \frac{c_1}{\pi} \left( \frac{V c_1}{6\pi c_2} \right)^2
\]
is the ground state energy singularity as one approaches the point \((h_c, V = 0)\) along the \( V \) direction.
The case $V = 0, \delta h \neq 0$ is more involved. We want $\delta h > 0$, in order to move into the incommensurate phase. From (4.3)–(4.5)

$$
\delta b = \pm \sqrt{f(\delta a)} \quad f(\delta a) = \delta a^2 - \frac{2}{3} \frac{c_3}{c_1} \delta a^3 - \frac{2}{c_1} \delta h
$$

(4.7)

$$\delta y = \frac{-c_1}{\pi} \delta a \left( \pm \sqrt{f(\delta a)} \right)
$$

(4.8)

$$\delta e_0(\delta a) = 2c_2\delta a \left( -2(\delta a)^2 - \frac{2}{c_1} (\delta a)^3 + \frac{6}{c_1} \delta h \right)
$$

(4.9)

where the sign in (4.8) depends on whether we want $\delta y > 0$ or $\delta y < 0$. The variation $\delta a$ must be negative and contained in a range where $f(\delta a)$ is non negative, so if

$$f(\delta a_0) = 0 \quad \delta a_0 = -\sqrt{\frac{2}{c_1} (\delta h)^{1/2} + O(\delta h)}
$$

we consider

$$\delta a \leq \delta a_0
$$

It is not difficult to see that there is a left neighborhood of $\delta a_0$ (of the order $\delta h^{1/2}$) where

1. $f(\delta a)$ is positive

2. $\delta y(\delta a)$ is monotonic

3. $\delta e_0(\delta a)$ is decreasing

Consequently, regardless of the sign in (4.8), $\delta a = \delta a_0$ is a local minimum of $\delta e_0(\delta a)$, which, incidentally, corresponds to $\delta y = 0$. Inserting $\delta a_0$ in (4.9)

$$e_0(h_c + \delta h, V = 0) = e_0(h_c, V = 0) - 2c_2 \left( \frac{2}{c_1} \delta h \right)^{3/2}
$$

(4.10)

Although it is not obvious from the previous proof, $\delta y = 0$ is actually a stationary point for $\delta e_0(\delta y)$. In fact

$$\frac{\partial \delta e_0}{\partial \delta y} = \frac{\partial \delta e_0}{\partial \delta a} \bigg|_{\delta a_0}
$$

(4.10)
and
\[
\frac{\partial \delta y}{\partial \delta a} = \pm \left( -\frac{c_1}{\pi} \sqrt{f(\delta a)} - \frac{c_1}{2\pi} \delta a \frac{f'(\delta a)}{\sqrt{f(\delta a)}} \right)
\]
becomes infinite at \( \delta a_0 \).

The calculation of the free energy singularity is a simple extension of this method. The variation of the ground state energy is now replaced by (see (2.23))
\[
-\delta(F(u, \gamma, h, y) + vy) = -\delta F - \delta y(v_c + \delta v)
\]
The variation of \( F \) is computed by means of an expansion analogous to (4.1)–(4.2). To keep things simple we consider \( d < \gamma - 2u \), which is certainly true for \( u \) sufficiently small, so that \( \Lambda_R \) dominates over \( \Lambda_L \). The surprisingly simple result is that, like for the spin chain, the first nonzero contribution comes at the third order. The quantity to minimize is
\[
2c'_2(\delta a^3 - 3\delta a\delta b^2) - \delta y\delta v
\]
where
\[
c'_2 = \frac{c_1}{6\pi} \left[ \frac{1}{2} + \sum_{n>0} \frac{(-1)^n \cosh(\gamma(n-2u))}{\cosh(\gamma n)} \right] > 0
\]
which looks exactly like (4.3)–(4.5) provided \( c_2 \rightarrow c'_2 \) and \( V \rightarrow \delta v \). The conclusions are therefore the same and the free energy leading singularities approaching \( (h_c, v_c) \) from the incommensurate phase are
\[
\begin{align*}
f(u, \gamma, h_c + \delta h, v_c) &= f(u, \gamma, h_c, v_c) - 2c'_2 \left( \frac{2}{c_1} \delta h \right)^{3/2} \\
f(u, \gamma, h_c, v_c + \delta v) &= f(u, \gamma, h_c, v_c) - \frac{2}{c_2^2} \left( \frac{c_1}{6\pi} \right)^3 |\delta v|^3
\end{align*}
\]

5 Discussion

It is interesting to speculate about the nature of the phase transition at \( (h_c, v_c) \) and compare it with what happens at the other points of \( \Gamma \).
We have always worked in the assumption that a complete set of eigenstates exists for the transfer matrix. It is well known then that the correlation function of 2 vertical arrows can be analysed through a spectral decomposition \[24\]. For the correlator of two vertical arrows along the same column one has, on a \( N \times M \) lattice
\[
\langle \alpha_{0,0} | \alpha_{0,n} \rangle = \text{Tr} \left( \sigma_z^0 T^n \sigma_z^0 T^{M-n} \right) \xrightarrow{M \to \infty} \sum_k \left| \langle 0 | \sigma_z^0 | k \rangle \right|^2 \left( \frac{\Lambda_k}{\Lambda_0} \right)^n
\]
and for the correlation function along the horizontal direction
\[
\langle \alpha_{0,0} | \alpha_{n,0} \rangle = \text{Tr} \left( \sigma_z^0 \sigma_z^n \sigma_z^0 \sigma_z^n \right) \xrightarrow{M \to \infty} \langle 0 | \sigma_z^0 \sigma_z^n | 0 \rangle
\]
where we have denoted with \(|0\rangle\) the eigenstate of the largest eigenvalue of the transfer matrix on a finite lattice of width \( N \). Here \( \sum_k \) denotes the sum over a complete set of eigenvectors of the transfer matrix. It is useful to consider first what happens at \( h = 0 \). Since, obviously
\[
\left[ \sigma_z^n, \sum_{j=1}^N \sigma_z^j \right] = 0
\]
and the ground state lies in the sector \( S^z = 0 \), only this sector contributes to (5.1) and (5.2). Furthermore, \( v \) does not change the spectrum within a sector of fixed \( S^z \), and it does not modify the eigenvectors which depend (as explicitly seen from the Bethe Ansatz \[3, 7\]) on \( \gamma \) and \( h \) only. We conclude that \( v \) has no effect whatsoever on the correlators, which remain those of the symmetric six–vertex model, until the level–crossing transition takes place at \( v = v(\gamma, u, b = 0) \) (as given by (2.25)). Here the system moves into the gaussian phase, and the ground state even for small \( \delta v = v - v(\gamma, u, b = 0) \) falls into a sector at \( y \neq 0 \) \[3\], hence with \( S^z \) of order \( N \). The transition occurs without divergence of the correlation length which jumps from the (finite) value of the symmetric six–vertex to infinity.

It is our conjecture that this picture does not change when \( -h_c < h < h_c \), and the level crossing transition persists along all \( \Gamma \), up to the points \((h_c, v_c)\) or \((-h_c, -v_c)\). Here the correlation length should diverge according to the following argument. The fact that in the sector \( S^z = 0 \), as found numerically in section 3,
\[
\lim_{h \to h_c} \Delta \ln \Lambda(S^z = 0, h, v_c) = 0
\]
is, by itself, not sufficient to prove the divergence of the correlation length $\xi_1$ in the vertical direction. The fact that $\Lambda_k$ in (3.1) are generally complex forces one to sum over a whole band of them because oscillations can affect the behavior of $\xi_1$ [24]. This is not doable until it is clarified what takes the place of strings in the sector under consideration. Nevertheless, one can look at the horizontal correlation function in (5.2). As shown in section 4, it is possible to enter the gaussian phase at $(h_c, v_c)$ keeping the ground state at $y = 0$. From (2.10) $y = 0$ does not necessarily imply that $S^z = 0$ ($n = \frac{N}{2}$), since $S^z$ can be nonzero but remain finite in the limit $N \to \infty$ and $y$ would still be zero. Still it is tempting to conjecture that the ground state remains at $S^z = 0$, and this is confirmed by preliminary numerical results on the spin chain. Hence no level crossing occurs here, because the Perron–Frobenius theorem prevents if from happening in a sector of fixed $S^z$, and the expectation value in (5.2) is taken with the same Bethe–Ansatz eigenvector in the 2 phases. Since in the gaussian phase the horizontal correlation length $\xi_2$ is infinite, $\xi_2(h)$ must diverge as $h \to \pm h_c$.

Although these conclusions are rather speculative, they seem to warrant a further investigation of the phase transition at issue. It should be recalled that, if indeed $(h_c, v_c)$ is critical in the sense of diverging correlation length, it cannot be classified in terms of conformal field theory as shown by the anomalous scaling discussed in section 3.

**Acknowledgements**

We wish to thank Prof. C. Destri, Prof. G. von Gehlen and Prof. P. Pearce for useful discussions. Special thanks go to Prof. V. Rittenberg for many discussions and constant encouragement. One of us (GA) has been supported by the EC under the program 'Human Capital and Mobility.'
Appendix A

We discuss here the symmetries of $\mathcal{H}$ and of the transfer matrix. Under the action of the (unitary) charge conjugation operator

$$C = \prod_{k=1}^{N} \sigma_k^x \quad C = C^\dagger = C^{-1}$$

$\mathcal{H}$ transforms as

$$C\mathcal{H}(\gamma, h, V)C = \mathcal{H}(\gamma, -h, -V)$$

(5.3)

Since

$$C \left( \sum_{j=1}^{N} \sigma_j^z \right) C = - \sum_{j=1}^{N} \sigma_j^z$$

the spectra of the sectors at fixed $S^z$ are related by

$$\mathcal{S}(\gamma, h, V, S^z) = \mathcal{S}(\gamma, -h, -V, -S^z)$$

(5.4)

The same symmetry operation can be applied to the transfer matrix, using the matrix form of $C$

$$\langle \alpha \mid C \mid \alpha' \rangle = \prod_{k=1}^{N} \delta_{\alpha_k, -\alpha'_k}$$

From (2.1) and the definition of the Botzmann weights it is elementary to see that

$$C T(\gamma, u, h, v) C \bigg|_{\alpha, \alpha'} = T(\gamma, u, -h, -v) \bigg|_{\alpha, \alpha'}$$

(5.5)

which implies the relation between spectra

$$\mathcal{S}_{TM}\{h, v, S^z\} = \mathcal{S}_{TM}\{-h, -v, -S^z\}$$

(5.6)

This symmetry manifests itself in the fact that the partition function

$$Z_{PF}(h, v) = Z_{PF}(-h, -v)$$

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As far as $\mathcal{H}$ is concerned, another symmetry is in effect, implemented by the space inversion operator (not to be confused with the momentum)

$$P |\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N\rangle = |\alpha_N, \alpha_{N-1}, \ldots, \alpha_2, \alpha_1\rangle \quad P^2 = 1$$

$$P \mathcal{H}(\gamma, h, V) P = \mathcal{H}(\gamma, -h, V)$$

Hence

$$C P \mathcal{H}(\gamma, h, V = 0) P C = \mathcal{H}(\gamma, h, V = 0)$$

So at $V = 0$ the spin chain recovers the $Z_2$ symmetry under spin reversal and the spectra at $S^z$ and $-S^z$ are identical. It is noteworthy that the same is not true for the transfer matrix.

**Appendix B**

Our Hamiltonian, at $h = 0, V = 0$, can be written, neglecting a constant additive term

$$\mathcal{H} = -\frac{1}{2} \sum_{j=1}^{N} \left[ \Delta \sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right] \quad \text{with} \quad \Delta < 1 \quad (5.7)$$

and it is mapped onto the Hamiltonian (see, e.g. [19])

$$\mathcal{H}_G = \frac{1}{2} \sum_{j=1}^{N} \left[ \Delta \sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right] \quad \text{with} \quad \Delta > 1 \quad (5.8)$$

through a unitary transformation

$$\mathcal{H}_G = U \mathcal{H} U^{-1} \quad U = \prod_{j=1}^{N/2} \sigma_{2j-1}^z$$

Consider the shift operator $S$ of (2.6). Let $|\Psi_n, P\rangle$ be an eigenstate of $\mathcal{H}$ with momentum $P$ and $S^z = \frac{N}{2} - n$

$$\frac{1}{2} \sum_{j=1}^{N} \sigma_j^z \ |\Psi_n, P\rangle = \left( \frac{N}{2} - n \right) \ |\Psi_n, P\rangle$$

$$S \ |\Psi_n, P\rangle = \exp(-iP) \ |\Psi_n, P\rangle$$
Define $U_0$ as

$$U_0 \left| \Psi_n, P \right> := \prod_{j=1}^{N} \sigma_j^z \left| \Psi_n, P \right>.$$ 

Then, an eigenstate of $\mathcal{H}_G$ with the same energy is $U \left| \Psi_n, P \right>$. Since

$$S \ U \left| \Psi_n, P \right> = \exp(-iP) \ S \ U \ S^{-1} \left| \Psi_n, P \right>$$

$$= \exp(-iP) \prod_{j=1}^{N/2} \sigma_{2j}^z \left| \Psi_n, P \right>$$

$$\exp(-iP) \ U_0 \left| \Psi_n, P \right> = \exp(-i(P + n\pi)) \ U \left| \Psi_n, P \right>$$

$$= \exp(-i\mathcal{P}) \ U \left| \Psi_n \right>$$

$U \left| \Psi_n, P \right>$ has momentum $\mathcal{P} = P + n\pi$. Hence the ground state of (5.8) has momentum $\mathcal{P}_0 = n_0 \pi = \frac{N}{2} \pi$, while $P_0 = 0$. As to the excitations

$$\Delta \mathcal{P} = \Delta P + (n - n_0)\pi = \Delta P - \frac{N_0 \pi}{2}$$

Each hole carries an additional momentum $-\pi/2$ which implies

$$\Delta E = m_0 \sqrt{1 - k^2 \sin^2(\Delta P)} \quad \rightarrow \quad \Delta E = m_0 \sqrt{1 - k^2 \cos^2(\Delta P)}$$

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6. Exponent for the scaling of the free energy gap in the sector $S^z = 0$
| Lattice size | $a$         | $b$         |
|-------------|-------------|-------------|
| 8           | 2.226901585 | -1.186380535|
| 12          | 2.522418777 | -1.215745628|
| 16          | 2.674453171 | -1.227403536|
| 20          | 2.766810718 | -1.23071106 |
| 24          | 2.828778340 | -1.236225697|
| 28          | 2.873206253 | -1.238154458|
| 32          | 2.906605731 | -1.239417236|
| 36          | 2.932624180 | -1.240288031|
| 40          | 2.953462078 | -1.240913442|
| Extrap. $\infty$ | 3.141592659(9) | -1.2436093723(1) |

Table 1: Position of the endpoints $\pm a + ib$ of the curve $C$ for $h < h_c$; $\exp(2h) = 3$
| Lattice size | $a$            | $b$            |
|--------------|----------------|----------------|
| 8            | 1.345603261    | −3.0497549071  |
| 12           | 1.479730321    | −3.236686985   |
| 16           | 1.535932819    | −3.3472122872  |
| 20           | 1.564274277    | −3.4190538006  |
| 24           | 1.580405494    | −3.4690826450  |
| 28           | 1.590409443    | −3.5057653322  |
| 32           | 1.597021761    | −3.5337453681  |
| 36           | 1.601612009    | −3.5557588049  |
| 40           | 1.604924714    | −3.5735136386  |
| Extrap. ⋄ ∞  | 1.6193362(3)   | −3.737102232(7)|

$\pi = 3.141592653$, $\gamma = −3.737102242$

Table 2: Position of the endpoints $±a + ib$ of the curve $C$ for $h > h_c$; $\exp(2h) = 18$
| Lattice size | $e^{2h} = 1.0$  | $e^{2h} = 9.0$ |
|--------------|----------------|----------------|
| 8            | 38.8549946261240 | 27.3592680025296 |
| 12           | 38.4729554492734  | 23.5353424072427 |
| 16           | 38.3095546699746  | 21.4216001709441 |
| 20           | 38.2251709688714  | 20.0937869680560 |
| 24           | 38.1760618666305  | 19.1948069770300 |
| 28           | 38.145012541243   | 18.5549980007490 |
| 32           | 38.1241500821837  | 18.0828136699350 |
| 36           | 38.1094640015976  | 17.7244227601235 |
| 40           | 38.0987382252780  | 17.4461598430727 |
| Extrap. $\infty$ | 38.04991359(1)  | 15.8884(8) |
| Exact eq.    | 38.04991361       | 15.8887254      |

Table 3: Energy gap of the spin chain for the first excited state for different values of $h$ compared to the analytical result (3.15)
| Lattice size | $e^{2h} = 9.0$ | $e^{2h} = 9.5$ | $e^{2h} = 10.0$ | $e^{2h_c} = 10.51787$ |
|-------------|----------------|----------------|----------------|-------------------|
| 8           | 40.28463586426577 | 40.60665243638542 | 41.00560893816288 | 41.50186071099847 |
| 12          | 32.50806556257382  | 32.33242547913501  | 32.27369006467646  | 32.34509918439876  |
| 16          | 28.03119022066448  | 27.47527007849106  | 27.06291429820305  | 26.80804074773090  |
| 20          | 25.23278497081201  | 24.37422080701744  | 23.67516261790179  | 23.15771126310622  |
| 24          | 23.35422673956813  | 22.24638140499817  | 21.30438602193786  | 20.56522041772249  |
| 28          | 22.02474072486461  | 20.7073242225853  | 19.55383243467817  | 18.62033939778466  |
| 32          | 21.0455297155199   | 19.54981005828927  | 18.20747202515815  | 17.10061549106331  |
| 36          | 20.3041022559842   | 18.65287555243093  | 17.14104807658379  | 15.87553491911761  |
| 40          | 19.72145517781357  | 17.94130164070793  | 16.27579201027032  | 14.86352304583235  |
| Extrap. $\infty$ | 15.946(3)   | 13.054(9)    | 9.37499(4)  | $-1.3(3) \cdot 10^{-5}$ |

Table 4: Mass gap of the spin chain for different values of $h \leq h_c$. 
| Lattice size | Exponent                   |
|--------------|----------------------------|
| 8            | −0.6147896371667922       |
| 12           | −0.6526671750014708       |
| 16           | −0.6559635919155932       |
| 20           | −0.6511929107450443       |
| 24           | −0.6444786397317260       |
| 28           | −0.6376031966136601       |
| 32           | −0.6311198906745727       |
| 36           | −0.6251786621408798       |
| 40           | −0.6197881465618827       |
| Extrap. ∞    | −0.5031(1)                |
| expected value | −0.5                   |

Table 5: Exponent of $N$ for the finite-size corrections of the energy gap in the sector $S^z = 0$ on the critical line $h = h_c$
| Lattice size | Exponent          |
|--------------|-------------------|
| 8            | -0.6696077334198590 |
| 12           | -0.6362863326795953 |
| 16           | -0.6165741467010993 |
| 20           | -0.603244670912252  |
| 24           | -0.5934852417528157  |
| 28           | -0.5859536198128706  |
| 32           | -0.5799199948938484 |
| 36           | -0.5749497141578801 |
| 40           | -0.5707659045338731 |
| Extrap. ∞    | -0.499998(4)       |

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