The universal $\ln^2 s$ increase in total cross sections

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Abstract

While it has long been known that many models of high energy scattering give cross sections which rise as $\ln^2 s$, the determination of the coefficient of this term is rarely given. We show that in gaussian and exponential eikonal models an exact expression for the cross section can be obtained, which demonstrates the $\ln^2 s$ asymptotic behaviour and determines its coefficient. The coefficient is universal, as found empirically, and the value of the constant obtained from the gaussian model is in good agreement with the empirical value.

1 Introduction

Recent analyses of high energy forward scattering data have confirmed that the cross sections rise as $\ln^2 s$, a behaviour first suggested by Heisenberg, and proportional to Froissart-Martin Bound. Similar behaviour has been suggested by Giddings in non conformal gauge/gravity duality. He finds the high energy scattering cross section is $\sigma = M^{-2} \ln^2 s$, with $M$ the mass of the lightest Kaluza-Klein excitation. The coefficient $B$ of the $\ln^2 s$ term in the fit to the cross section is found to be independent of the

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hadronic reaction studied ($\bar{p}p, pp, \Sigma^-p, \pi^\pm p,$ and $K^\pm p$), while for $\gamma p$ and $\gamma\gamma$, the the equivalent coefficients are written as $B_\gamma = \delta B$ and $B_{\gamma\gamma} = \delta^2 B$ respectively with the same constant $\delta$ as expected from factorization of the $\gamma$ cross sections. From all of these cross-sections the value $B = 0.313(9)\text{mb}$ is determined. Note that this constant is two orders of magnitude smaller than the Lukaszuk-Martin limiting value, \cite{5}

$$B \leq \frac{\pi}{m^2_T} = 62\text{mb}. \quad (1)$$

This observation raises the question of identifying the physics responsible for the value of $B$.

Dosch, Gauron and Nicolescu \cite{6} have recently proposed that Heisenberg’s original model can be extended to obtain the value of $B \approx \pi/(4M^2)$, where $M$ is a glueball mass.

It has long been recognised \cite{7,8} that the eikonal representation of the total cross section can lead to cross sections rising as $\ln^2 s$. In this paper we study three models for the eikonal and determine the coefficient $B$ for these models.

The three models for the eikonal studied are

1. a factorisable exponential model, motivated by meson exchange \cite{7},
2. a non-factorisable gaussian model, motivated by multiperipheral models \cite{7},
3. the QCD motivated model of Block and co-workers \cite{9,10}.

We are able to obtain exact results for the cross section for cases 1 and 2 in terms of incomplete gamma functions and their derivatives. Examination of the asymptotic behaviour gives the expected $\ln^2 s$ rise at large $s$, and determines the coefficient $B$ in each case.

For case 3 we have been unable to find an exact solution in terms of known functions, but an approximate evaluation of the cross section gives $\ln^2 s$ behaviour, and determines its coefficient.

## 2 The eikonal representation of the total cross-section

In terms of the eikonal function $\chi(b,s)$ the scattering amplitude is represented as

$$f(s,t) = i \int_0^\infty \! bdbJ_0(b\sqrt{-t}) \left(1 - e^{-\chi(b,s)}\right) \quad (2)$$
and the total cross-section is thus

\[ \sigma_t = 4\pi \text{Im} f(s, 0) = 4\pi \int_0^\infty bdb \left(1 - e^{-\chi(b,s)}\right), \]  

(3)

where the eikonal is approximated as purely real corresponding to the black-disc collision at high energies.

Strictly, for \( pp \) and \( \bar{p}p \) scattering we should introduce eikonals \( \chi^\pm(b, s) = \frac{1}{2} (\chi_{pp}(b, s) \pm \chi_{\bar{p}p}(b, s)) \) to write crossing-even and crossing-odd forward amplitudes. For other processes similar \( \chi^\pm(b, s) \) eikonals are constructed. At high energies, we may neglect the \( \chi^- \) contribution, and so we use \( \chi \) to simply mean \( \chi^+ \), the crossing-even amplitude. Physically \( \chi(b, s) \) is made of all elementary inelastic collision cross section of parton-pairs \( \sigma_0 \) and overlaps of convoluted structure functions of parton pairs at impact parameter \( b \). Following the normalization conventions used in mini-jet models [9, 10], we parametrize the eikonal such that

\[ \int \chi(b, s) d^2 b = \sigma_0(s) \]  

(4)

In cases where \( \chi(b, s) \) is written as the product of a function of \( b \) and a function of \( s \), the eikonal is said to be factorisable, as it is in the Chou-Yang model [11].

Our technique for evaluating the integral is most straightforward for the gaussian case which we consider first in the following.

### 2.1 The Non-factorisable Gaussian Eikonal

The gaussian representation of the eikonal is motivated by the multi-Pomeron representation of multiperipheral ladder exchanges [7]. One writes

\[ \chi(b, s) = \frac{\pi \lambda}{B(s)^\Delta} s^{\Delta} \exp \left\{ -\frac{b^2}{B(s)} \right\}. \]  

(5)

The parameters \( \Delta \) and \( B(s) \) are determined by the Pomeron trajectory \( \alpha_P(t) = \alpha_P(0) + \alpha'_P(0)t \), consistent with shrinkage of the forward peak in differential cross sections,

\[ \Delta = \alpha_P(0) - 1 \quad \text{and} \quad B(s) = 2\alpha'_P(0) \ln s + k \]  

(6)

where \( k \) is a constant\(^1\). The logarithmic dependence of \( B(s) \) leads to a typical shrinkage of the forward peak in the differential cross-section.

\(^1\)There is an unfortunate conflict in notation between reference [1], which uses \( B \) for the coefficient of \( \ln^2 s \), and reference [7], which uses \( B(s) \) as the gaussian scale parameter. We follow this usage, warning the reader to note carefully whether \( B \) is a constant or a function of \( s \).
The use of $\chi$ as the variable of integration enables us to write the exact result for the total cross-section:

$$\sigma_t(s) = 2\pi B(s) \left( \ln C(s) + \gamma + E_1(C(s)) \right)$$  \hspace{1cm} (7)

where $C(s) = (\pi \lambda/B(s)) s^\Delta = \chi(b = 0, s)$ is the opacity of the eikonal, $\gamma$ is Euler’s constant, and $E_1(x) = \int_x^\infty e^{-t} dt/t$ is the exponential integral.

For large $x$, $E_1(x) \sim (e^{-x}/x)(1+O(x^{-1}))$, and so the large energy behaviour of the cross-section is

$$\sigma_t(s) \sim 2\pi B(s) \left( \ln C(s) + \gamma \right),$$  \hspace{1cm} (8)

and the leading term in this gives

$$\sigma_t(s) = 4\pi \alpha_P'(0) \Delta \ln^2 s + O(\ln s),$$  \hspace{1cm} (9)

defining the constant $B$ in terms of the properties of the Pomeron. Clearly $B$ is a universal constant. With $\alpha_P' = 0.25\text{GeV}^{-2}$ and with the value $\Delta$ in the range $(0.1, 0.4)$, determined from deep inelastic scattering data, we obtain

$$B \in (0.12, 0.48)\text{mb}$$  \hspace{1cm} (10)

values which compare favourably with the observed value of $0.31\text{mb}$.

This method also gives the $\ln s$ and the constant terms in the total cross section, but the coefficients of those terms are not process independent, so we do not quote them here.

### 2.2 The factorizable exponential eikonal

The eikonal in this case takes the form

$$\chi(b, s) = \frac{\lambda}{2\pi b_0^2} s^\Delta \exp \left\{ -\frac{b}{b_0} \right\},$$  \hspace{1cm} (11)

and corresponds to massive particle exchange, with $b_0$ being determined by the mass of the lightest exchanged meson. One can again use the eikonal as the variable of integration in the eikonal expression for the total cross section, which then becomes exactly

$$\sigma_t(s) = 2b_0^2 \left( \ln^2 C(s) + \left[ \frac{\pi^2}{6} + \gamma^2 + 2\ln C(s)\gamma \right] 
- \left[ \Gamma_1(0, C(s)) - 2\ln C(s)\Gamma(0, C(s)) \right] \right),$$  \hspace{1cm} (12)
where \( C(s) = \frac{\lambda}{2\pi b_0^2} s^\Delta \), \( \Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt \) is the upper incomplete gamma function, and \( \Gamma_1(\alpha, s) \) is its partial derivative with respect to \( \alpha \).

The terms involving \( \Gamma_1(0, C(s)) \) and \( \Gamma(0, C(s)) \) are exponentially small for large \( s \), and the high energy behaviour is again dominated by the \( \ln^2 s \) term

\[
\sigma_t(s) = 2\pi b_0^2 \Delta^2 \ln^2 s + O(\ln s) \quad (13)
\]

with the coefficient \( B = 2\pi b_0^2 \). In this case \( b_0^{-1} \) is the mass of the exchanged meson, which can be universally taken to be the lightest glueball. Then with \( b_0^{-1} = 1.4 \text{GeV} \), and the previous range of values for \( \Delta \), \( B \in (0.006, 0.1) \text{mb} \), rather smaller than the observed value of the coefficient, but, for the larger values of \( \Delta \), or \( 1/b_0 \), or both, the same order of magnitude for the coefficient can readily be obtained.

2.3 The QCD inspired eikonal

Block et al \[10\] have introduced the eikonal

\[
\chi = \sigma_{gg}(s)W(b; \mu_{gg}) + \sigma_{qg}W(b; \mu_{qg}) + \sigma_{qq}W(b; \mu_{qq}) \quad (14)
\]

where \( \sigma_{gg}, \sigma_{qg} \) and \( \sigma_{qq} \) represent gluon-gluon, quark-gluon and quark-quark interaction cross sections, and \( W(b; \mu) \) are overlap functions normalised so that \( 2\pi \int_0^\infty bdbW(b; \mu) = 1 \). The parameterisation adopted in \[9, 10\], which is the Fourier transform of a dipole form factor squared, is

\[
W(b; \mu) = \frac{\mu^2}{96\pi} (\mu b)^3 K_3(\mu b), \quad (15)
\]

where \( K_3(x) \) is the modified Bessel function of the second kind. Note the behaviour of \( x^3 K_3(x) \) for small \( x \)

\[
x^3 K_3(x) = 8 + O(x^2) \quad (16)
\]

and for large \( x \)

\[
x^3 K_3(x) = \sqrt{\frac{\pi}{2}} x^{5/2} e^{-x} (1 + O(x^{-1})). \quad (17)
\]

We approximate the eikonal with an exponential form,

\[
\chi_\alpha(b, s) = C(s)e^{-b/b_0(s)}, \quad (18)
\]

reducing the calculation to the previous case. \( C(s) \) is chosen to be the opacity of the eikonal we are approximating, and \( b_0(s) \) is determined so that

\[
\int_0^\infty bdb\chi(b, s) = \int_0^\infty bdb\chi_\alpha(b, s), \quad (19)
\]

5
\[ C(s) = \sigma_{gg}(s) \frac{\mu_{gg}^2}{12\pi} + \sigma_{qq}(s) \frac{\mu_{qq}^2}{12\pi} + \sigma_{qg}(s) \frac{\mu_{qg}^2}{12\pi}, \] 

(20)

and

\[ 2\pi b_0^2(s) = \frac{\sigma_{gg}(s) + \sigma_{qg}(s) + \sigma_{qq}(s)}{C(s)}. \] 

(21)

Note that if a single elementary collision process dominates over other terms \( b_0 \) becomes \( \sqrt{6}/\mu \), independent of \( s \), where \( \mu \) is the scale parameter of the dominant process.

Applying the analysis of the previous section, the leading large \( s \) behaviour of the total cross section is

\[ \sigma_t(s) = 2\pi b_0^2(s) \ln^2 C(s), \] 

(22)

Block et al [10] show that in the large \( s \) region, the eikonal is dominated by the gluon-gluon scattering term, which has the structure

\[ \sigma_{gg}(s) \approx \lambda \left( \frac{s}{m_0^2} \right)^\epsilon \ln \left( \frac{s}{m_0^2} \right), \] 

(23)

where \( \epsilon \) is obtained from the gluon structure function \( f_g(x) = N_g(1 - x)^5 x^{-(1+\epsilon)} \). The cross section again has the Heisenberg \( \ln^2 s \) dependence at high energies, with the universal coefficient

\[ B = 12\pi \mu_{gg}^{-2} \epsilon^2. \] 

(24)

Block et al use the numerical values \( \epsilon = 0.05, \mu_{gg} = 0.73\text{GeV} \), which give \( B = 0.069\text{mb} \), in the range of values given by the exponential eikonal, and rather smaller than the empirical value. However, recent results on \( \epsilon \) [17] give a range of values \( \epsilon \in (0.08, 0.4) \) which give \( B \in (0.18, 4.42)\text{mb} \), covering the empirical value.

### 3 Conclusion

In the light of the results of [11], it is no longer sufficient to develop models for high energy scattering which show cross sections rising as \( \ln^2 s \). It is now important to demonstrate that the coefficient, \( B \), of the \( \ln^2 s \) term is a universal constant, independent of the hadronic reaction under consideration.

We have been able to determine the cross section exactly for both gaussian and exponential eikonsals, and not only demonstrate the \( \ln^2 s \) asymptotic behaviour, but also determine its coefficient. Each of the three eikonal models we have considered give a universal \( B \). The numerical value of \( B \) is sensitive to both the choice of eikonal model, and the choice of parameters within that model. Both the gaussian eikonal and the QCD inspired eikonal give values of \( B \) consistent with the observed value, and the factorisable exponential eikonal value is of the correct order of magnitude.
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