ON TWO CONGRUENCE CONJECTURES OF Z.-W.
SUN INVOLVING FRANEL NUMBERS

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Abstract. In this paper, we mainly prove the following conjectures of Z.-W. Sun [17]: Let \( p > 2 \) be a prime. If \( p = x^2 + 3y^2 \) with \( x, y \in \mathbb{Z} \) and \( x \equiv 1 \pmod{3} \), then

\[
x \equiv \frac{1}{4} \sum_{k=0}^{p-1} (3k + 4) \frac{f_k}{2^k} \equiv \frac{1}{2} \sum_{k=0}^{p-1} (3k + 2) \frac{f_k}{(-4)^k} \pmod{p^2},
\]

and if \( p \equiv 1 \pmod{3} \), then

\[
\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \pmod{p^3},
\]

where \( f_n = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)^3 \) stands for the \( n \)th Franel number.

1. Introduction

In 1894, Franel [2] found that the numbers

\[
f_n = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)^3 \quad (n = 0, 1, 2, \ldots)
\]

satisfy the recurrence relation (cf. [12, A000172]):

\[
(n + 1)^2 f_{n+1} = (7n^2 + 7n + 2)f_n + 8n^2 f_{n-1} \quad (n = 1, 2, 3, \ldots).
\]

These numbers are now called Franel numbers. Callan [1] found a combinatorial interpretation of the Franel numbers. The Franel numbers play important roles in combinatorics and number theory. The sequence \( (f_n)_{n \geq 0} \) is one of the five sporadic sequences (cf. [21, Section 4]) which are integral solutions of certain Apéry-like recurrence equations and closely related to the theory of modular forms. In 2013, Sun [17] revealed some unexpected connections between the numbers.

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\[ f_n \text{ and representations of primes } p \equiv 1 \pmod{3} \text{ in the form } x^2 + 3y^2 \text{ with } x, y \in \mathbb{Z}, \text{ for example, Z.-W. Sun } [17] \text{ (1.2) showed that} \]

\[ \sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}, \quad (1.1) \]

and in the same paper, Sun proposed some conjectures involving Franel numbers, one of which is

**Conjecture 1.1.** Let \( p > 2 \) be a prime. If \( p = x^2 + 3y^2 \) with \( x, y \in \mathbb{Z} \) and \( x \equiv 1 \pmod{3} \), then

\[ x \equiv \frac{1}{4} \sum_{k=0}^{p-1} (3k + 4)f_k \equiv \frac{1}{2} \sum_{k=0}^{p-1} (3k + 2) \frac{f_k}{(-4)^k} \pmod{p^2}. \]

For more studies on Franel numbers, we refer the readers to \([3, 4, 6, 7, 8, 15, 18]\) and so on.

In this paper, our first goal is to prove the above conjecture.

**Theorem 1.1.** Conjecture 1.1 is true.

Z.-W. Sun \([17]\) also gave the following conjecture which is much difficult and complex.

**Conjecture 1.2.** Let \( p > 2 \) be a prime. If \( p \equiv 1 \pmod{3} \), then

\[ \sum_{k=0}^{p-1} f_k 2^k \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \pmod{p^3}. \]

Our last goal is to prove this conjecture.

**Theorem 1.2.** Conjecture 1.2 is true.

We are going to prove Theorem 1.1 in Section 2. Section 3 is devoted to proving Theorem 1.2. Our proofs make use of some combinatorial identities which were found by the package \textit{Sigma} \([11]\) via the software \textit{Mathematica} and the \( p \)-adic Gamma function. The proof of Theorem 1.2 is somewhat difficult and complex because it is rather convoluted. Throughout this paper, prime \( p \) always \( \equiv 1 \pmod{3} \), so in the following Lemmas \( p > 5 \) or \( p > 3 \) or \( p > 2 \) is the same, we mention it here first.

### 2. Proof of Theorem 1.1

For a prime \( p \), let \( \mathbb{Z}_p \) denote the ring of all \( p \)-adic integers and let \( \mathbb{Z}_p^\times := \{ a \in \mathbb{Z}_p : a \text{ is prime to } p \} \). For each \( \alpha \in \mathbb{Z}_p \), define the \( p \)-adic
order $\nu_p(\alpha) := \max\{n \in \mathbb{N} : p^n \mid \alpha\}$ and the $p$-adic norm $|\alpha|_p := p^{-\nu_p(\alpha)}$. Define the $p$-adic gamma function $\Gamma_p(\cdot)$ by

$$\Gamma_p(n) = (-1)^n \prod_{1 \leq k < n \atop (k,p)=1} k, \quad n = 1, 2, 3, \ldots,$$

and

$$\Gamma_p(\alpha) = \lim_{|\alpha-n|_p \to 0} \Gamma_p(n), \quad \alpha \in \mathbb{Z}_p.$$  

In particular, we set $\Gamma_p(0) = 1$. Following, we need to use the most basic properties of $\Gamma_p$, and all of them can be found in [9, 10]. For example, we know that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } |x|_p = 1, \\ -1, & \text{if } |x|_p > 1. \end{cases} \quad (2.1)$$

$$\Gamma_p(1-x)\Gamma_p(x) = (-1)^{a_0(x)}, \quad (2.2)$$

where $a_0(x) \in \{1, 2, \ldots, p\}$ such that $x \equiv a_0(x) \pmod{p}$. And a property we need here is the fact that for any positive integer $n$,

$$z_1 \equiv z_2 \pmod{p^n} \implies \Gamma_p(z_1) \equiv \Gamma_p(z_2) \pmod{p^n}. \quad (2.3)$$

**Lemma 2.1.** ([17, Lemma 2.2]) For any $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} \binom{n}{k}^3 z^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{3k} (2k) \binom{3k}{k} z^k (1+z)^{n-2k} \quad (2.4)$$

and

$$f_n = \sum_{k=0}^{n} \binom{n+2k}{3k} (2k) \binom{3k}{k} (-4)^{n-k}. \quad (2.5)$$

For $n, m \in \{1, 2, 3, \ldots\}$, define

$$H_n^{(m)} = \sum_{1 \leq k \leq n} \frac{1}{k^m},$$

these numbers with $m = 1$ are often called the classic harmonic numbers.

Recall that the Bernoulli polynomials are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots).$$
Lemma 2.2. ([13, 14]) Let \( p > 5 \) be a prime. Then

\[
H_{p-1}^{(2)} \equiv 0 \pmod{p}, \quad H_{p-1}^{(2)} \equiv 0 \pmod{p}, \quad H_{p-1} \equiv 0 \pmod{p^2},
\]

\[
\frac{1}{3} H_{\left\lfloor \frac{p}{3} \right\rfloor}^{(2)} \equiv H_{\left\lfloor \frac{p}{3} \right\rfloor}^{(2)} \equiv \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p},
\]

\[
H_{\left\lfloor \frac{p}{2} \right\rfloor} \equiv -2q_p(2) - \frac{3}{2} q_p(3) + p q_p^2(2) + \frac{3p}{4} q_p^2(3) - \frac{5p}{12} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^2},
\]

\[
H_{\left\lfloor \frac{p}{4} \right\rfloor} \equiv -\frac{3}{2} q_p(3) + \frac{3p}{4} q_p^2(3) - \frac{p}{6} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^2},
\]

\[
H_{\left\lfloor \frac{p}{6} \right\rfloor} \equiv -2q_p(2) + p q_p^2(2) \pmod{p^2}, \quad H_{\left\lfloor \frac{p}{4} \right\rfloor} \equiv (-1)^{p-1} 4E_{p-3} \pmod{p},
\]

\[
H_{\left\lfloor \frac{p}{4} \right\rfloor} \equiv -\frac{3}{2} q_p(3) + \frac{3p}{4} q_p^2(3) + \frac{p}{3} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^2},
\]

where \( q_p(a) = (a^{p-1} - 1)/p \) stands for the Fermat quotient.

Lemma 2.3. Let \( p > 2 \) be a prime and \( p \equiv 1 \pmod{3} \). If \( 0 \leq j \leq (p-1)/2 \), then we have

\[
\binom{3j}{j} \binom{p+j}{3j+1} \equiv \frac{p}{3j+1} (1 - pH_{2j} + pH_j) \pmod{p^3}.
\]

Proof. If \( 0 \leq j \leq (p-1)/2 \) and \( j \neq (p-1)/3 \), then we have

\[
\binom{3j}{j} \binom{p+j}{3j+1} = (p+j) \cdots (p+1) p(p-1) \cdots (p-2j) \quad \frac{j!(2j)!}{(3j+1)!}
\]

\[
\equiv \frac{pj!(1+pH_j)(-1)^{2j}(2j)!(1-pH_{2j})}{j!(2j)! (3j+1)}
\]

\[
\equiv \frac{p}{3j+1} (1 - pH_{2j} + pH_j) \pmod{p^3}.
\]

If \( j = (p-1)/3 \), then by Lemma [2.2], we have

\[
\binom{p-1}{\frac{p-1}{3}} \binom{p+\frac{p-1}{3}}{\frac{p-1}{3}}
\]

\[
\equiv \left( 1 - pH_{\frac{p-1}{3}} + \frac{p^2}{2} (H_{\frac{p-1}{3}}^2 - H_{\frac{p-1}{3}}) \right) \left( 1 + pH_{\frac{p+1}{3}} + \frac{p^2}{2} (H_{\frac{p+1}{3}}^2 - H_{\frac{p+1}{3}}) \right)
\]

\[
\equiv 1 - p^2 H_{\frac{p-1}{3}}^{(2)} \equiv 1 - \frac{p^2}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}
\]

and

\[
1 - pH_{\frac{p-1}{3}} + pH_{\frac{p+1}{3}} \equiv 1 - \frac{p^2}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}.
\]

Now the proof of Lemma 2.3 is complete. \( \Box \)
Proof of Theorem 1.1. With the help of (2.4), we have

\[
\sum_{k=0}^{p-1} (3k + 4) f_k = \sum_{k=0}^{p-1} \frac{3k + 4}{2^k} \sum_{j=0}^{[k/2]} \binom{k+j}{j} \binom{3j}{3j} 2^{k-2j}
\]

\[
= \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{3j}{3j}}{4^j} \sum_{k=2j}^{p-1} (3k + 4) \binom{k+j}{3j}. \tag{2.6}
\]

By loading the package \texttt{Sigma} in the software \texttt{Mathematica}, we find the following identity:

\[
\sum_{k=2j}^{n-1} (3k + 4) \binom{k+j}{3j} = 9nj + 3n + 9j + 5 \binom{n+j}{3j+1}.
\]

Thus, replacing \( n \) by \( p \) in the above identity and then substitute it into (2.6), we have

\[
\sum_{k=0}^{p-1} (3k + 4) f_k \equiv \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{3j}{3j}}{4^j} \frac{9p + 3p + 9j + 5}{3j+2} \binom{p+j}{3j+1}. \tag{2.7}
\]

Hence we immediately obtain the following result by Lemma 2.3

\[
\sum_{k=0}^{p-1} (3k + 4) f_k \equiv p \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} 9j + 5}{4^j (3j+1)(3j+2)} \mod p^2. \tag{2.8}
\]

It is easy to verify that

\[
p \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} 9j + 5}{4^j (3j+1)(3j+2)} \equiv S_1 + S_2 \mod p^2,
\]

where

\[
S_1 = p \sum_{j=0}^{(p-1)/2} \binom{(p-1)/2}{j} (-1)^j \left( \frac{2}{3j+1} + \frac{1}{3j+2} \right) \tag{2.9}
\]

and

\[
S_2 = \frac{3p + 2}{p + 1} \left( \frac{(2p - 2)/3}{(p - 1)/3} \right)^{4(p-1)/3} - \left( \frac{(p-1)/2}{(p-1)/3} \right).
\]

Applying the famous identity

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+x} = \frac{n!}{x(x+1)\cdots(x+n)} \tag{2.10}
\]
with \( x = 1/3, n = (p - 1)/2 \) and \( x = 2/3, n = (p - 1)/2 \), we may simplify (2.9) as
\[
S_1 = \frac{4p}{3p - 1} \left( \frac{1}{(1/3)^{(p-1)/2}} \right) + \frac{2p}{3p + 1} \left( \frac{1}{(2/3)^{(p-1)/2}} \right),
\]
where \( (a)_n = a(a + 1) \cdots (a + n - 1) \) is the rising factorial or the Pochhammer symbol.

In view of (2.1) and (2.2), we have
\[
\frac{4p}{3p - 1} \left( \frac{1}{(1/3)^{(p-1)/2}} \right) = \frac{4p}{3p - 1} \left( \frac{1}{\Gamma(\frac{1}{3} + \frac{p-1}{2})} \right) = \frac{4p}{3p - 1} \left( (-1)^{\frac{p-1}{2}} \Gamma_p(\frac{p+1}{2}) \Gamma_p(\frac{1}{3}) \right)
\]
\[
= \frac{12}{1 - 3p} \frac{\Gamma_p(\frac{p+1}{2}) \Gamma_p(\frac{1}{3})}{\Gamma_p\left(\frac{p+1}{2} - \frac{1}{3}\right)} = \frac{12(-1)^{\frac{p-1}{2}}}{1 - 3p} \Gamma_p\left(\frac{p+1}{2}\right) \Gamma_p\left(\frac{1}{3}\right) \Gamma_p\left(\frac{7}{6} - \frac{p}{2}\right),
\]
where \( \Gamma(\cdot) \) is the Gamma function. It is known that for \( \alpha, s \in \mathbb{Z}_p \), we have
\[
\Gamma_p(\alpha + ps) \equiv \Gamma_p(\alpha) + ps\Gamma'_p(\alpha) \pmod{p^2}
\]
and
\[
\frac{\Gamma'_p(\alpha)}{\Gamma_p(\alpha)} \equiv 1 + H_{p-\langle \alpha \rangle - 1} \pmod{p},
\]
where \( \Gamma'_p(x) \) denotes the \( p \)-adic derivative of \( \Gamma_p(x) \), \( \langle \alpha \rangle_n \) denotes the least non-negative residue of \( \alpha \) modulo \( n \), i.e., the integer lying in \( \{0, 1, \ldots, n - 1\} \) such that \( \langle \alpha \rangle_n \equiv \alpha \pmod{n} \).

Therefore modulo \( p^2 \), we have
\[
\frac{4p}{3p - 1} \left( \frac{1}{(1/3)^{(p-1)/2}} \right) \equiv \frac{12(-1)^{\frac{p-1}{2}}}{1 - 3p} \Gamma_p\left(\frac{p+1}{2}\right) \Gamma_p\left(\frac{1}{3}\right) \Gamma_p\left(\frac{7}{6} - \frac{p}{2}\right) \left(1 + \frac{p}{2} \left(H_{p-1} - H_{p-7/6}\right)\right).
\]

In view of (2.11) and (2.12), we have
\[
\frac{4p}{3p - 1} \left( \frac{1}{(1/3)^{(p-1)/2}} \right) \equiv \frac{2(1 + 3p)\Gamma_p\left(\frac{p+1}{2}\right) \Gamma_p\left(\frac{1}{3}\right) \Gamma_p\left(\frac{7}{6} - \frac{p}{2}\right) \left(1 + \frac{p}{2} \left(H_{p-1} - H_{p-7/6}\right)\right) \pmod{p^2}.
\]

And then by using [20, Proposition 4.1], we have
\[
\frac{\Gamma_p\left(\frac{p+1}{2}\right) \Gamma_p\left(\frac{1}{3}\right) \Gamma_p\left(\frac{7}{6} - \frac{p}{2}\right) \left(1 + \frac{p}{2} \left(H_{p-1} - H_{p-7/6}\right)\right) \pmod{p^2}.
\]

Then with the help of [20, Theorem 4.12] and Lemma 2.2, we have
\[
\frac{4p}{3p - 1} \left( \frac{1}{(1/3)^{(p-1)/2}} \right) \equiv 4x + 3pxq_p(3) - \frac{p}{x} \pmod{p^2}
\]
(2.13)
and
\[ \frac{2p}{3p + 1} \cdot \frac{(p-1)/2}{(2/3)(p-1)/2} \equiv \frac{p}{x} \pmod{p^2}. \]  
(2.14)
Hence
\[ S_1 \equiv 4x + 3pxq_p(3) \pmod{p^2}. \]  
(2.15)

**Lemma 2.4.** Let \( p > 3 \) be a prime. For any \( p \)-adic integer \( t \), we have
\[ \left( \frac{p-1}{2} + pt \right) \equiv \left( \frac{p-1}{3} \right) \left( 1 + pt \left( H_{p-1} - H_{p-1} \right) \right) \pmod{p^2}. \]  
(2.16)

**Proof.** Set \( m = (p-1)/2 \). It is easy to check that
\[ \left( \frac{m+pt}{(p-1)/3} \right) = \frac{(m+pt) \cdots (m+pt-(p-1)/3+1)}{(p-1)/3)! \]
\[ \equiv \frac{m \cdots (m-(p-1)/3+1)}{(p-1)/3)! (1 + pt(H_m - H_{m-(p-1)/3})) \]
\[ = \left( \frac{m}{(p-1)/3} \right) (1 + pt(H_m - H_{m-(p-1)/3})) \pmod{p^2}. \]
So Lemma 2.4 is finished. \( \Box \)

Now we evaluate \( S_2 \) modulo \( p^2 \). It is easy to obtain that
\[ S_2 \equiv 2 \left( \left( -\frac{1}{2} \right) - \left( \frac{(p-1)/2}{(p-1)/3} \right) \right) \]
\[ \equiv -p \left( \frac{(p-1)/2}{(p-1)/3} \right) (H_{(p-1)/2} - H_{(p-1)/6}) \]
\[ \equiv -3pxq_p(3) \pmod{p^2} \]  
(2.17)
with the help of Lemma 2.2, Lemma 2.4 and [20, Theorem 4.12].
Therefore, in view of (2.7), (2.8), (2.15) and (2.17), we immediately get the desired result
\[ \frac{1}{4} \sum_{k=0}^{p-1} (3k + 4) \frac{f_k}{2^k} \equiv x \pmod{p^2}. \]

On the other hand, we use the equation (2.5) to obtain that
\[ \sum_{k=0}^{p-1} (3k + 2) \frac{f_k}{(-4)^k} = \sum_{k=0}^{p-1} 3k + 2 \sum_{j=0}^k \binom{k + 2j}{3j} \binom{2j}{3j} (3j) (3j) (-4)^{k-j} \]
\[ = \sum_{j=0}^{p-1} \frac{(2j)}{(-4)^j} \sum_{k=j}^{p-1} (3k + 2) \binom{k + 2j}{3j}. \]
By using the package \texttt{Sigma} again, we find the following identity:
\[
\sum_{k=j}^{n-1} (3k + 2) \binom{k + 2j}{3j} = \frac{9nj + 3n + 1}{3j + 2} \binom{n + 2j}{3j + 1}.
\]

Thus,
\[
\sum_{k=0}^{p-1} (3k + 2) \frac{f_k}{(-4)^k} = \sum_{j=0}^{p-1} \binom{3j}{j} \frac{\binom{p+2j}{3j+1}}{(-4)^j} \frac{9pj + 3p + 1}{3j + 2}. 
\] (2.18)

\textbf{Lemma 2.5.} Let \( p > 3 \) be a prime and \( p \equiv 1 \) (mod 3). If \( 0 \leq j \leq (p - 1)/2 \) and \( j \neq (p - 1)/3 \), then
\[
\binom{3j}{j} \binom{p + 2j}{3j + 1} \equiv \frac{p(-1)^j}{3j + 1} (1 + pH_{2j} - pH_j) \pmod{p^3}.
\]
If \( (p + 1)/2 \leq j \leq p - 1 \), then
\[
\binom{3j}{j} \binom{p + 2j}{3j + 1} \equiv \frac{2p(-1)^j}{3j + 1} \pmod{p^2}.
\]

\textbf{Proof.} If \( 0 \leq j \leq (p - 1)/2 \) and \( j \neq (p - 1)/3 \), then we have
\[
\binom{3j}{j} \binom{p + 2j}{3j + 1} = \frac{(p + 2j) \cdots (p + 1)p(p - 1) \cdots (p - j)}{j!(2j)!(3j + 1)}
\]
\[
\equiv \frac{p(2j)!(1 + pH_{2j})(-1)^j(1 - pH_j)}{j!(2j)!(3j + 1)}
\]
\[
\equiv \frac{p(-1)^j}{3j + 1} (1 + pH_{2j} - pH_j) \pmod{p^3}.
\]
If \( (p + 1)/2 \leq j \leq p - 1 \), then
\[
\binom{3j}{j} \binom{p + 2j}{3j + 1}
\]
\[
= \frac{(p + 2j) \cdots (2p + 1)(2p) \cdots (p + 1)p(p - 1) \cdots (p - j)}{j!(2j)!(3j + 1)}
\]
\[
\equiv \frac{2p^2(2j) \cdots (p + 1)(p - 1)!(-1)^j(j)!}{j!(2j)!(3j + 1)} = \frac{2p(-1)^j}{3j + 1} \pmod{p^2}.
\]
Now the proof of Lemma 2.5 is complete. \( \Box \)

It is known that \( \binom{2k}{k} \equiv 0 \pmod{p} \) for each \( (p + 1)/2 \leq k \leq p - 1 \), and it is easy to check that for each \( 0 \leq j \leq (p - 1)/2 \),
\[
\binom{3j}{j} \binom{p + 2j}{3j + 1} \equiv \frac{p(-1)^j}{3j + 1} \pmod{p^2}.
\]
These, with (2.18) yield that

\[
\sum_{k=0}^{p-1} (3k + 2) f_k \equiv \sum_{j=0}^{\frac{p-1}{2}} \frac{(2j)}{j} \frac{p(-1)^j}{3j + 1} \frac{9pj + 3p + 1}{3j + 2} + \sum_{j=\frac{p+1}{2}}^{p-1} \frac{(2j)}{j} \frac{2p(-1)^j}{3j + 1} \frac{1}{3j + 2}
\]

\[
\equiv \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \frac{p(-1)^j}{3j + 1} \frac{1}{3j + 2} + S_3
\]

\[
= p \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} (-1)^j \left( \frac{1}{3j + 1} - \frac{1}{3j + 2} \right) + S_3 \pmod{p^2}, \quad (2.19)
\]

where

\[
S_3 = \binom{\frac{2p-2}{3}}{\frac{p-1}{3}} \frac{1}{p+1} \frac{p-1}{3} \frac{1}{p+1} - \binom{\frac{p-1}{2}}{\frac{p-1}{3}} \frac{1}{4} \frac{2p-2}{3}^2
\]

\[
= \frac{1}{p+1} \left( \frac{-1/2}{p-1/3} - \binom{\frac{p-1}{2}}{\frac{p-1}{3}} \right) - \frac{-1/2}{(2p-2)/3}
\]

In the same way of above, with (2.10), (2.13), (2.14), Lemma 2.2 and [20, Theorem 4.12], we have the following congruence modulo \(p^2\)

\[
p \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} (-1)^j \left( \frac{1}{3j + 1} - \frac{1}{3j + 2} \right) \equiv 2x + \frac{3px}{2} q_p(3) - \frac{3p}{2x}, \quad (2.20)
\]

Now we evaluate \(S_3\). It is easy to see that

\[
\left( -\frac{1}{2} \right) \left( -\frac{1}{2} - 1 \right) \cdots \left( -\frac{1}{2} - \frac{2p-2}{3} + 1 \right) = \frac{(-\frac{p-1}{2})!}{(2p-2)!} \frac{1}{(2p-2)!}
\]

\[
= \frac{\left( -\frac{p-1}{2} \right)! \cdots \left( -\frac{p-1}{2} - 1 \right)! \left( \frac{p}{2} \right)! \cdots \left( \frac{p}{2} + \frac{p-7}{6} \right)!}{(2p-2)!}
\]

\[
= \frac{\left( -\frac{p-1}{2} \right)! \cdots \left( -\frac{p-1}{2} - 1 \right)! \left( \frac{p}{2} \right)! \cdots \left( \frac{p}{2} + \frac{p-7}{6} \right)!}{(2p-2)!}
\]

\[
= (-1)^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right)! \left( \frac{p-7}{6} \right)! \equiv (-1)^{\frac{p-1}{2}} \frac{1}{3p} \frac{1}{(\frac{p-1}{2})!}
\]

\[
\equiv -3p(-1)^{(p-1)/2} \left( \frac{2p-2}{3} \right)^{(p-1)/2} \pmod{p^2}.
\]
In view of (2.17) and [20, Theorem 4.12], we immediately obtain that
\[ S_3 \equiv -\frac{3px}{2}q_p(3) + \frac{3p}{2x} \pmod{p^2}. \]
This, with (2.19) and (2.20) yields that
\[ \frac{1}{2} \sum_{k=0}^{p-1} (3k + 2) \frac{f_k}{(-4)^k} \equiv x \pmod{p^2} \]
Now the proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2 With the help of (2.4), we have
\[ \sum_{k=0}^{p-1} \frac{f_k}{2^k} = \sum_{k=0}^{p-1} \frac{1}{2^k} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k+j}{3j} \binom{2j}{j} \binom{3j}{j} 2^{k-2j} \]
\[ = \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{3j}{j}}{4^j} \sum_{k=2j}^{p-1} \binom{k+j}{3j}. \quad (3.1) \]

By loading the package Sigma in the software Mathematica, we have the following identity:
\[ \sum_{k=2j}^{n-1} \binom{k+j}{3j} = \binom{n+j}{3j+1}. \]
Thus, replace \( n \) by \( p \) in the above identity and then substitute it into (3.1), we have
\[ \sum_{k=0}^{p-1} \frac{f_k}{2^k} = \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{3j}{j}}{4^j} \binom{p+j}{3j+1}. \]
Hence we immediately obtain the following result by Lemma 2.3,
\[ \sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv p \sum_{j=0, j \neq \frac{p-1}{3}}^{(p-1)/2} \frac{\binom{2j}{j} \binom{3j}{j}}{4^j} \frac{1 - pH_{2k} + pH_k}{(3j+1)} + S_1 \pmod{p^3}, \quad (3.2) \]
where
\[ S_1 = \frac{\binom{2p-2}{p-1} \binom{p-1}{p-1}}{4^{\frac{p-1}{3}}} = \left( -\frac{1}{2} \right) \left( \frac{p-1}{p-1} \right) \left( \frac{p}{p-3} \right). \]
It is easy to verify that
\[
\begin{align*}
\frac{(p-1)/2}{4j} \frac{1}{3j+1} 
\end{align*}
\]
\[
\equiv \frac{p}{j=0} j \neq \frac{p-1}{2} \left( -1 \right)^j \left( 1 - pH_{2k} + pH_k \right) 
\end{align*}
\]
\[
\equiv \frac{p}{j=0} j \neq \frac{p-1}{2} \frac{(-1)^j}{3j+1} \left( 1 - p \sum_{r=1}^{j} \frac{1}{2r-1} \right) 
\end{align*}
\]
\[
\equiv \frac{p}{j=0} j \neq \frac{p-1}{2} \frac{(-1)^j}{3j+1} \left( 1 + \frac{p}{2} H_k \right) - S_2 \pmod{p^3},
\]
where
\[
S_2 = \frac{\left( \frac{p-1}{p-1} \right) \left( 1 + \frac{p}{2} H_{p-1} \right)}{p}.
\]
So
\[
\sum_{k=0}^{p-1} \frac{f_k}{2k} \equiv p \sum_{j=0}^{(p-1)/2} \frac{\left( \frac{p-1}{j} \right) \left( 1 + \frac{p}{2} H_k \right)}{3j+1} + S_1 - S_2 \pmod{p^3}. \quad (3.3)
\]

It is easy to see that
\[
\frac{2p}{3p-1} \left( \frac{1}{2} \right) \frac{\left( p-1 \right)!}{\left( \frac{p-1}{3} \right)!} = \frac{\left( \frac{p-1}{2} \right)!}{\frac{1}{3} \cdots \left( \frac{p-1}{3} \right) \left( \frac{p-1}{3} + 1 \right) \cdots \left( \frac{p-1}{3} + \frac{p-1}{6} \right)} \equiv \frac{\left( \frac{p-1}{3} \right)!}{\left( \frac{p-1}{2} \right)!} \pmod{p}.
\]
\[
(3.4)
\]
On the other hand, We have
\[
\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \sum_{k=0}^{p-1} \frac{1}{(-4)^k} \sum_{j=0}^{k} \binom{k+2j}{3j} \binom{2j}{j} \binom{3j}{j} (-4)^{k-j} 
\end{align*}
\]
\[
= \sum_{j=0}^{p-1} \frac{(-1)^j}{(-4)^j} \sum_{k=j}^{p-1} \binom{k+2j}{3j} \binom{2j}{j} \binom{3j}{j} = \sum_{j=0}^{p-1} \frac{(-1)^j}{(-4)^j} \binom{2j}{j} \binom{3j}{j} \binom{p+2j}{3j+1}.
\]
So by Lemma 2.5 and the fact that for each $0 \leq k \leq (p-1)/2$,
\[
\frac{\left( \frac{2k}{k} \right)}{(-4)^k} \equiv \frac{\left( \frac{p-1}{2} \right)!}{\left( 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} \right)^{2j-1}} \pmod{p^2}, j \frac{\binom{2j}{j}}{\left( \frac{2p-2j}{p-j} \right)} \equiv 2p \pmod{p^2}
\]
\[
(3.4)
\]
for each $(p+1)/2 \leq j \leq p-1$, we have the following congruence modulo $p^3$,

\[
\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} - S_3 \equiv p \sum_{j=0}^{p-1} \frac{(2j)(1 + pH_{2j} - pH_j)}{(3j + 1)4^j} + 2p \sum_{j=\frac{p+1}{2}}^{p-1} \frac{(2j)}{(3j + 1)4^j}
\]

\[
\equiv \sum_{j=0}^{p-1} \frac{p(-1)^j \left(\frac{p-1}{3} \right) (1 + 2pH_{2j} - \frac{3}{2}pH_j)}{3j + 1} + \sum_{j=\frac{p+1}{2}}^{p-1} \frac{4p^2}{4^j(3j + 1)j(\frac{2p-2j}{p-j})}
\]

\[
\equiv \sum_{j=0}^{p-1} \frac{p(-1)^j \left(\frac{p-1}{3} \right) (1 + 2pH_{2j} - \frac{3}{2}pH_j)}{3j + 1} + \sum_{j=1}^{\frac{p-1}{2}} \frac{p^24^j}{(3j - 1)j(\frac{2j}{j})} - S_4,
\]

where

\[
S_3 = \left(\frac{2p-2}{p-1}\right) \left(\frac{p-1}{3}\right) \left(\frac{p+2p-2}{p}\right) \left(-\frac{1}{(p-3)}\right) \left(\frac{p-1}{p-3}\right) \left(\frac{p+2p-2}{p}\right),
\]

\[
S_4 = \left(\frac{p-1}{p-3}\right) \left(1 + 2pH_{2\frac{p-1}{2}} - \frac{3}{2}pH_{\frac{p-1}{2}}\right).
\]

Hence we have

\[
\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} - \sum_{k=0}^{p-1} \frac{f_k}{2^k}
\]

\[
\equiv 2p^2 \sum_{j=0}^{p-1} \frac{(-1)^j(H_{2j} - H_j)}{3j + 1} + S_5 + \sum_{j=1}^{\frac{p-1}{2}} \frac{p^24^j}{(3j - 1)j(\frac{2j}{j})} \pmod{p^3},
\]

(3.5)

where

\[
S_5 = S_3 - S_4 + S_2 - S_1.
\]

By \textit{Sigma}, we can find and prove the following identity:

\[
\sum_{j=0}^{n} \frac{2(n)}{j}(-1)^j(H_{2j} - H_j)
\]

\[
\sum_{j=0}^{n} \frac{3k}{3j + 1}
\]

\[
= \frac{1}{3n + 1} \prod_{k=1}^{n} 3k - 2 \left(\sum_{k=1}^{n} \prod_{j=1}^{k} \frac{3j - 2}{3j} - \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{2(3j - 2)}{3(2j - 1)}\right)
\]

\[
= \frac{(1)_n}{(3n + 1) \left(\frac{1}{3}\right)_n} \left(\sum_{k=1}^{n} \left(\frac{1}{k(1)}\right)_k - \sum_{k=1}^{n} \left(\frac{1}{k} \cdot \left(\frac{1}{2}\right)\right)_k\right).
\]

(3.6)
In view of Lemma 3.1 and Lemma 2.2 we have
\[
\sum_{k=1}^{n-1} \frac{\binom{k}{3}}{k(1)_k} = \sum_{k=1}^{n-1} \frac{(-1/3)}{k(1)_k} = \frac{3}{2} q_2(3) - \frac{3p}{4} q_2^2(3) - \frac{p}{3} \sum_{k=1}^{n-1} \frac{4^k}{k^2(2k)_k} \quad (\text{mod } p^2). \tag{3.7}
\]

\[
\sum_{k=1}^{n-1} \frac{\binom{1}{3}}{k(1/2)_k} = \sum_{k=1}^{n-1} \frac{(-1/3)}{k(1/2)_k} \equiv \frac{4p}{3} (-1)^{p-3} E_{p-3} + \frac{3}{2} q_2(3) - \frac{3p}{4} q_2^2(3) - \frac{2p}{3} (-1)^{p-3} \sum_{k=1}^{n-1} \frac{4^k}{(2k-1)k(2k)_k} \quad (\text{mod } p^2). \tag{3.8}
\]

It is easy to check that
\[
\sum_{k=1}^{n-1} \frac{4^k}{(2k-1)k(2k)_k} = 2 \sum_{k=1}^{n-1} \frac{4^k}{(2k-1)(2k)_k} - \sum_{k=1}^{n-1} \frac{4^k}{k(2k)_k}. \tag{3.9}
\]

And by Lemma 2.2, we have
\[
\frac{1}{\binom{n+1+k}{k}} = (n+1) \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{1}{k + r + 1}. \tag{3.10}
\]

\[
2 \sum_{k=1}^{n-1} \frac{4^k}{(2k-1)(2k)_k} \equiv 2 \sum_{k=1}^{n-1} \frac{(-1)^k}{(2k-1)(\binom{p-1}{k})} \equiv (-1)^{p-1} \sum_{k=1}^{n-1} \frac{(-1)^k}{(k+1)(\binom{p-1}{k})} \equiv (-1)^{p-1} \binom{n}{p-1} \left( \sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{(k+1)(\binom{p-1}{k})} - \sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{(k+1)(\binom{p-1}{k})} \right) \quad (\text{mod } p). \tag{3.11}
\]

By Sigma, we find the following identity which can be proved by induction on \(n\):
\[
\sum_{k=0}^{n} \frac{(-1)^k}{(k+1)(\binom{n}{k})} = \frac{2(-1)^n - 1}{n+1} - (n+1) H_n^{(2)} - 2(n+1) \sum_{k=1}^{n} \frac{(-1)^k}{k^2}. \tag{3.12}
\]

So by setting \(n = (p - 1)/2\) in the above identity, we have
\[
\sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{(k+1)(\binom{p-1}{k})} = 2 \left( (-1)^{p-1} - 1 \right) - (-1)^{p-1} 2E_{p-3} \quad (\text{mod } p). \tag{3.12}
\]
And by (3.10), we have
\[
\sum_{k=0}^{\frac{p-1}{6}} \frac{(-1)^k}{(k+1)\left(\frac{k}{k}\right)} = \sum_{k=0}^{\frac{p-7}{6}} \frac{1}{(k+1)\left(\frac{k}{k}\right)}
\]
\[= \sum_{k=0}^{\frac{p-7}{6}} \frac{1}{k+1} \left\{ \sum_{r=0}^{\frac{p-3}{2}} \left(\frac{p-3}{2r}\right)(-1)^r \frac{1}{k+r+1} \right\}
\]
\[= -\frac{1}{2} \sum_{k=1}^{\frac{p-1}{6}} \frac{1}{k} \sum_{r=0}^{\frac{p-3}{2}} \left(\frac{p-3}{2r}\right)(-1)^r \frac{1}{k+r}
\]
\[= -\frac{1}{2} H_{\frac{p-1}{6}}^{(2)} - \frac{1}{2} \sum_{r=1}^{\frac{p-3}{2}} (-1)^r \left(\frac{p-3}{2r}\right) \sum_{k=1}^{\frac{p-1}{6}} \left(1 - \frac{1}{k + r}\right) \pmod{p}.
\]

It is easy to check that
\[
H_{\frac{p-1}{6}}^{(2)} - \sum_{k=1}^{\frac{p-1}{6}} \frac{1}{k + r} \equiv -\sum_{k=1}^{r} \frac{1}{k(6k-1)} \pmod{p}.
\]

And by Sigma again, we have
\[
\sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} \sum_{k=1}^{r} \frac{1}{k(6k-1)} = H^{(2)}_n - \sum_{k=1}^{n} \frac{(1)_k}{k \left(\frac{2}{6}\right)_k}
\]

So by Lemma (2.2) and [20], we have
\[
\sum_{k=0}^{\frac{p-7}{6}} \frac{(-1)^k}{(k+1)\left(\frac{k}{k}\right)}
\]
\[\equiv \frac{(-1)^{\frac{p-1}{x}}}{x} - 2 - \frac{5}{4} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) - \frac{1}{2} \sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k^2 \left(\frac{2}{6}\right)} \pmod{p}.
\]
Again, by (3.10), we have
\[
\sum_{k=0}^{p-1} \frac{(-1)^k}{k^2 \left( -\frac{5}{k} \right)} = - \frac{6}{5} \sum_{k=1}^{p-1} \frac{(-1)^k}{k \left( -\frac{5}{k-1} \right)} \equiv \frac{6}{5} \sum_{k=0}^{p-1} \frac{(-1)^k}{k+1} \left( \frac{\frac{p-1}{k}}{k+1} \right)
\]
\[
= \frac{6}{5} \sum_{k=0}^{p-3} \frac{1}{k+1} \left( \frac{k+5}{k+6} \right) = \frac{6}{5} \sum_{k=0}^{p-3} \frac{1}{k+1} - \frac{6}{6} \sum_{r=0}^{p-1} (-1)^r \left( \frac{\frac{p-1}{6}}{r} \right) \frac{1}{k+1+r}
\]
\[
\equiv \sum_{k=1}^{p-1} \frac{1}{k} \sum_{r=0}^{p-1} (-1)^r \left( \frac{\frac{p-1}{6}}{r} \right) \frac{1}{k+r}
\]
\[
= H_{\frac{p-1}{2}} + \sum_{r=1}^{p-1} \frac{(-1)^r}{r} \left( \frac{\frac{p-1}{6}}{r} \right) \sum_{k=1}^{p-1} \left( \frac{1}{k} - \frac{1}{k+r} \right) \pmod{p}.
\]
Also it is easy to see that
\[
H_{\frac{p-1}{2}} - \sum_{k=1}^{p-1} \frac{1}{k+r} \equiv - \sum_{k=1}^{p-1} \frac{1}{k(2k-1)} \pmod{p}.
\]
And by \textit{Sigma}, we have
\[
\sum_{r=1}^{n} \frac{(-1)^r}{r} \left( \frac{n}{r} \right) \sum_{k=1}^{r} \frac{1}{k(2k-1)} = H^{(2)}_n - \sum_{k=1}^{n} \frac{4^k}{k^2 \binom{2k}{k}}.
\]
So by Lemma 2.2, we have
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2 \left( -\frac{5}{k} \right)} \equiv \sum_{k=1}^{p-6} \frac{4^k}{k^2 \binom{2k}{k}} - \frac{5}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}.
\]
Hence
\[
\sum_{k=0}^{p-7} \frac{(-1)^k}{(k+1) \left( \frac{k+1}{k} \right)} \equiv \frac{(-1)^{p-1}}{x} - 2 - \frac{1}{2} \sum_{k=1}^{p-1} \frac{4^k}{k^2 \binom{2k}{k}} \pmod{p}.
\]
This, with (3.11) and (3.12) yields that
\[
2 \sum_{k=1}^{p-1} \frac{4^k}{(2k-1)}
\]
\[
\equiv - 2 + \frac{1}{x} + 2E_{p-3} - \frac{1}{2}(-1)^{p-2} \sum_{k=1}^{p-6} \frac{4^k}{k^2 \binom{2k}{k}} \pmod{p}.
\]
By Sigma, we have
\[
\sum_{k=1}^{n} \frac{4^k}{k(2k)} = -2 + 2 \frac{4^n}{(2n)}.
\] (3.14)

So by \([20]\), we have
\[
\sum_{k=1}^{\frac{p-1}{2}} \frac{4^k}{k(2k)} \equiv -2 + \frac{2}{\left(\frac{p-1}{2}\right)} \equiv -2 + \frac{1}{x} \pmod{p}.
\]

This, with \((3.9)\) and \((3.13)\) yields that
\[
\sum_{k=1}^{\frac{p-1}{2}} \frac{4^k}{(2k)} \equiv 2E_{p-3} - \frac{1}{2}(-1)^{\frac{p-1}{2}} \sum_{k=1}^{\frac{p-1}{2}} \frac{4^k}{k^2} \pmod{p}.
\]

This with \((3.8)\) yields that
\[
\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{3k} \equiv 3q_p(3) - \frac{3p}{4}q_p^2(3) + \frac{p}{3} \sum_{k=1}^{\frac{p-1}{2}} \frac{4^k}{k^2} \pmod{p^2}.
\] (3.15)

So by \((3.7)\), we have
\[
\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{3k} \equiv -p^2 \sum_{k=1}^{\frac{p-1}{2}} \frac{4^k}{k^2} \pmod{p^3}.
\] (3.16)

Now we evaluate the second sum in the right side of \((3.5)\). It is easy to see that
\[
\sum_{j=1}^{\left(\frac{p-1}{2}\right)} \frac{4^j}{(3j-1)j(2j)} = 3\sum_{j=1}^{\left(\frac{p-1}{2}\right)} \frac{4^j}{(3j-1)(2j)} - \sum_{j=1}^{\left(\frac{p-1}{2}\right)} \frac{4^j}{j(2j)}.
\] (3.17)

It is easy to see from \((3.14)\) that
\[
\sum_{j=1}^{\left(\frac{p-1}{2}\right)} \frac{4^j}{j(2j)} \equiv -2 + 2(-1)^{\frac{p-1}{2}} \pmod{p}.
\] (3.18)
Now we consider the first sum of the right side in (3.17).

\[
\sum_{j=1}^{p} \frac{4^j}{(3j-1)\binom{2j}{j}} = \sum_{j=1}^{\frac{p-1}{3}} \frac{4^j}{(3j-1)\binom{2j}{j}} + \sum_{j=\frac{p+2}{3}}^{\frac{p-1}{3}} \frac{4^j}{(3j-1)\binom{2j}{j}}.
\]

The following identity is very important to us:

\[
\sum_{k=1}^{n} \frac{4^k}{(k+n)\binom{2k}{k}} = -2 + 2 \frac{4^n}{\binom{2n}{n}} - n \frac{n}{4^n} \sum_{k=1}^{n} \frac{4^k}{k^2\binom{2k}{k}}.
\] (3.19)

This, with (20) yields that

\[
3 \sum_{j=1}^{\frac{p-1}{2}} \frac{4^j}{(3j-1)\binom{2j}{j}} \equiv \sum_{j=1}^{\frac{p-1}{3}} \frac{4^j}{(j + \frac{p-1}{3})\binom{2j}{j}}
\]

\[
\equiv -2 + \frac{2}{(-1/2)^{\frac{p-1}{3}}} + \frac{1}{3} \left(\frac{-1/2}{\frac{p-1}{3}}\right) \sum_{k=1}^{\frac{p-1}{3}} \frac{4^k}{k^2\binom{2k}{k}}
\]

\[
\equiv -2 + \frac{1}{3} \left(\frac{-1}{\frac{p-1}{3}}\right) \sum_{k=1}^{\frac{p-1}{3}} \frac{4^k}{k^2\binom{2k}{k}} \pmod{p}.
\] (3.20)

It is easy to check that And by (3.19), we have

\[
3 \sum_{j=\frac{p+2}{3}}^{\frac{p-1}{2}} \frac{4^j}{(3j-1)\binom{2j}{j}} \equiv 3 \sum_{j=0}^{\frac{p-7}{6}} \frac{4^j}{(6j+5)\binom{2j}{j}} \equiv (-1)^{\frac{p-1}{2} - j} \frac{(-1)^{\frac{p-1}{2} - j}}{\binom{2j}{j}}
\]

\[
\equiv 6(-1)^{\frac{p+1}{2}} \sum_{j=0}^{\frac{p-7}{6}} \frac{4^j}{(6j+5)\binom{2j}{j}} \equiv (-1)^{\frac{p+1}{2}} \sum_{j=0}^{\frac{p-7}{6}} \frac{(-1)^j}{(j + \frac{p+5}{6})\binom{2j}{j}}
\]

\[
\equiv 6 \frac{(-1)^{\frac{p+1}{2}}}{5} + (-1)^{\frac{p+1}{2}} \sum_{j=1}^{\frac{p+5}{6}} \frac{4^j}{(j + \frac{p+5}{6})\binom{2j}{j}} + \frac{3}{\binom{2\frac{p}{2}}{\frac{p}{2}}} \pmod{p}.
\] (3.21)

By (3.19) and (20), we have

\[
\sum_{j=1}^{\frac{p+5}{6}} \frac{4^j}{(j + \frac{p+5}{6})\binom{2j}{j}} \equiv -\frac{16}{5} + 5(-1)^{\frac{p-1}{6}} \frac{4^j}{2x} - (-1)^{\frac{p-1}{6}} \sum_{k=1}^{\frac{p-1}{6}} \frac{4^k}{k^2\binom{2k}{k}} \pmod{p}.
\]
This, with (3.21) yields that
\[
3 \sum_{j=p+1}^{p+2} \frac{4^j}{(3j-1) \binom{2j}{j}} \equiv 2(-1)^{p+1} - \frac{1}{x} + \frac{1}{3} \left( \frac{p-1}{3} \right) \sum_{k=1}^{p-1} \frac{4^k}{2k^2(2k)} \mod p.
\]
Combining this with (3.20), we have
\[
3 \sum_{j=1}^{\frac{p-1}{2}} \frac{4^j}{(3j-1) \binom{2j}{j}} \equiv -2 + 2(-1)^{p+1} + \frac{1}{3} \left( \frac{p-1}{3} \right) \left( \sum_{k=1}^{p-1} \frac{4^k}{k^2(2k)} + \sum_{k=1}^{p-1} \frac{4^k}{2k^2(2k)} \right) \mod p.
\]
Thus, by (3.17) and (3.18), we have
\[
\sum_{j=1}^{\frac{p-1}{2}} \frac{4^j}{(3j-1) \binom{2j}{j}} \equiv \frac{1}{3} \left( \frac{p-1}{3} \right) \left( \sum_{k=1}^{p-1} \frac{4^k}{k^2(2k)} + \sum_{k=1}^{p-1} \frac{4^k}{2k^2(2k)} \right) \mod p.
\]
This, with (3.5) and (3.16) yields that
\[
\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} - \sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv S_5 \mod p^3. \tag{3.22}
\]
While
\[
S_5 = \left( \frac{-1}{p-1} \right) \left( \frac{p-1}{3} \right) \left( \sum_{k=1}^{p-1} \frac{4^k}{k^2(2k)} \right) + \frac{2p}{(p-1)^3} \left( H_{\frac{p-1}{3}} - H_{\frac{2p-2}{3}} \right).
\]
It is easy to check that
\[
\left( \frac{p + 2p-2}{p-1} \right) \equiv 1 + pH_{\frac{2p-2}{3}} + \frac{p^2}{2} \left( H_{\frac{2p-2}{3}} - H_{\frac{2p-2}{3}} \right) \mod p^3
\]
and
\[
\left( \frac{p + p-1}{p-1} \right) \equiv 1 + pH_{\frac{p-1}{3}} + \frac{p^2}{2} \left( H_{\frac{p-1}{3}} - H_{\frac{p-1}{3}} \right) \mod p^3.
\]
So by Lemma 2.2 and the fact that $H_{p-1-k}^{(2)} \equiv -H_k^{(2)} \pmod{p}$ for each $0 \leq k \leq p-1$, we have

\[ \left( p + \frac{2p-2}{3} \right) \equiv p \left( H_{\frac{2p-2}{3}}^{(2)} - H_{\frac{p-1}{3}}^{(2)} \right) \equiv p^2 \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3} \]

and

\[ 2p \left( H_{\frac{p}{3}}^{(2)} - H_{\frac{2p-2}{3}}^{(2)} \right) \equiv -p^2 \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}. \]

So by \( \left( \frac{1}{p+1} \right) \equiv \left( \frac{p-1}{2} \right) \pmod{p} \) and \( \left( \frac{p-1}{2} \right) \equiv (-1)^{\frac{p-1}{2}} = 1 \pmod{p} \), we can immediately obtain that

\[ S_5 \equiv 0 \pmod{p^3}. \]

This, with (3.22) yields that

\[ \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{2k} \pmod{p^3}. \]

Now the proof of Theorem 1.2 is complete. \( \square \)

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