NON-GEODESIC SPHERICAL FUNK TRANSFORMS WITH ONE AND TWO CENTERS

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Abstract. We study non-geodesic Funk-type transforms on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ associated with cross-sections of $S^n$ by $k$-dimensional planes passing through an arbitrary fixed point inside the sphere. The main results include injectivity conditions for these transforms, inversion formulas, and connection with geodesic Funk transforms. We also show that, unlike the case of planes through one common center, the integrals over the plane sections through two distinct centers provide the corresponding reconstruction problem a unique solution. A reconstruction procedure is given. The main tools are Möbius-type automorphisms, analytic families of cosine transforms, and the billiard-like dynamics of $S^n$ generated by reflections about the centers.

1. Introduction

Let $B^{n+1}$ be the open unit ball in $\mathbb{R}^{n+1}$, $S^n$ its boundary, $\text{Gr}_a(n+1,k)$ the Grassmann manifold of $k$-dimensional affine planes in $\mathbb{R}^{n+1}$ passing through a fixed point $a \in B^{n+1}$; $1 \leq k \leq n$. We consider the generalized Funk transform

$$(F_a f)(L) = \int_{S^n \cap L} f(x) \, d\sigma(x), \quad L \in \text{Gr}_a(n+1,k),$$

where $d\sigma(x)$ stands for the corresponding surface area measure. This transform assigns to a continuous function $f$ on $S^n$ the integrals of $f$ over $(k-1)$-dimensional sub spheres $S^n \cap L$. The classical case $F_a = F$, when $a = o$ is the origin, goes back to the pioneering works by Funk [1, 2] ($n = 2$), which were inspired by Minkowski [8]. A generalization of the Funk transform $F$ to arbitrary $1 \leq k \leq n$ is due to Helgason [6]; see also [7, 15, 18] and references therein. Operators of this kind play an important role in convex geometry, spherical tomography, and various branches of Analysis [3, 4, 5, 18, 11, 12].

The case when $a$ differs from the origin is relatively new in the modern literature, though Funk-type transforms on $S^2$ for noncentral
plane sections were considered by Gindikin, Reeds, and Shepp [5] in the framework of the kappa-operator theory. One should also mention non-geodesic Funk-type transforms studied by Palamodov [12, Section 5.2]. Inversion formulas for these transforms were obtained in terms of delta functions and differential forms. Operators (1.1) with \( a \neq o \) are non-geodesic too, however, they differ from those in [12]. In particular, they are non-injective.

The case \( a \neq o \) with \( k = n \) was considered by Salman; see [21] for \( n = 2 \) and [22] for any \( n \geq 2 \). To avoid non-uniqueness, he imposed restrictions on the support of the functions under reconstruction. The stereographic projection method of [21, 22] makes it possible to reduce inversion of Salman’s operator to the similar problem for a certain Radon-like transform over spheres in \( \mathbb{R}^n \). The next step was made by Quellmalz [13] for \( n = 2 \), who expressed \( F_a \) through the totally geodesic Funk transform \( F \) and thus explicitly inverted this operator on a certain subclass of continuous functions (if \( a = o \) this subclass consists of even functions on \( S^n \)). The results from [13] were generalized by Quellmalz [14] and Rubin [19] to any \( n \geq 2 \) with \( k = n \). The paper [19] also contains an alternative inversion method of Salman’s operator.

Our aim in the present article is two-fold. First, we generalize the results from [19] and obtain inversion formulas for \( F_a \) for planes of arbitrary dimension 1 \( \leq k \leq n \). We also characterize the kernel of \( F_a \) and the subclass of continuous functions on which \( F_a \) is injective.

Second, to achieve uniqueness in the reconstruction problem, we consider sections by planes through two distinct centers. To the best of our knowledge, this approach is new. We shall prove that for any pair of distinct points \( a \) and \( b \) in \( \mathbb{R}^{n+1} \), the kernels of the corresponding transforms \( F_a \) and \( F_b \) have trivial intersection. The latter means that, unlike the case of one center, when (1.1) is non-injective, the collection of two Grassmannians \( \text{Gr}_a(n + 1, k) \) and \( \text{Gr}_b(n + 1, k) \) provides the corresponding reconstruction problem a unique solution. We also develop an analytic procedure of the reconstruction.

**Basic Ideas and Plan of the Paper.** The basic idea is to express the \( a \)-centered Funk transform \( F_a \) through the \( o \)-centered transform \( F \), for which the theory is well developed. To realize this idea, we need appropriate transformations of the unit sphere. However, in general, the corresponding Jacobians in each cross-section may not agree on the intersections of these cross-sections. If so, the whole idea may not work. Fortunately, this is not the case, as can be shown by a certain indirect way. Specifically, we introduce a new complex parameter \( \lambda \) and regard (1.1) as a member of a certain meromorphic family of
λ-cosine transforms, in which integration is performed over the entire sphere $\mathbb{S}^n$, rather than over cross-sections. Changing variables in the integral over $\mathbb{S}^n$ (which is technically much easier than over individual cross-sections!) and passing to the limit in the parameter $\lambda$, we obtain the desired connection between $F_a$ and $F$. This idea goes back to Semyanistyi [23] and has proved to be useful in diverse integral-geometric considerations; see, e.g., [15, 18, 19].

The recovery of functions from their two-center Funk transform is realized by the series, which converges in the pointwise sense and in the $L^p$-norm with $1 \leq p < n/(k - 1)$, where the bound for $p$ is sharp. An interesting feature of the reconstructing algorithm is that it gives rise to a certain billiard-like dynamical system on the sphere generated by reflections about the centers. Dynamical systems of this kind might be useful in some other reconstruction problems of integral geometry and tomography, where the parity issues occur.

The paper is organized as follows. Section 2 contains preliminaries. We introduce spherical automorphisms, which mimic reflection maps and Moebius-type transformations. In Section 3 we show that the Funk-type transform (1.1) can be factorized as $F_a = N_a F M_a$, where $F$ is the classical totally geodesic transform corresponding to (1.1) with $a = a$, $N_a$ and $M_a$ are the suitable bijections. The proof of the technical Lemma 3.1 is moved to Appendix. In Section 4 we give a description of the kernel (the null subspace) of the operator $F_a$, acting on the space $C(\mathbb{S}^n)$ of continuous functions, and characterize the class of functions on which $F_a$ is injective. We also obtain an explicit inversion formula for $F_a$. Section 5 deals with the system of two equations, $F_a f = g$ and $F_b f = h$, corresponding to distinct centers $a$ and $b$ inside the unit ball. It is shown that, unlike the case of one common center, such a system determines $f$ uniquely and the function $f$ can be reconstructed by a certain pointwise convergent series. Norm convergence of this series is studied in Section 6. It turns out that the series does not converge uniformly on $\mathbb{S}^n$ in spite of the assumption that $f$ is continuous. However, it converges in the $L^p$-norm for all $1 \leq p \leq p_0$, $p_0 = n/(k - 1)$, and this bound is sharp.

The main results are stated in Theorems 4.5, 4.8, 5.2, 6.2, and 6.4.

2. Preliminaries

2.1. Notation. The notation $C(\mathbb{S}^n)$ and $L^p(\mathbb{S}^n)$ for the corresponding spaces of continuous and $L^p$ functions on $\mathbb{S}^n$ is standard. If $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$, then $dx$ stands for the Riemannian measure on $\mathbb{S}^n$;
\[ \sigma_n \equiv \int_{S^n} dx = 2\pi^{(n+1)/2}/\Gamma((n+1)/2) \] is the area of \( S^n \); \( x \cdot y \) means the usual dot product.

We write \( \mathfrak{m}_{n,m} \) for the space of real matrices having \( n \) rows and \( m \) columns. In the following, \( M' \) denotes the transpose of the matrix \( M \), \( I_m \) is the identity \( m \times m \) matrix. For \( n \geq m \), \( \text{St}(n,m) = \{ M \in \mathfrak{m}_{n,m} : M'M = I_m \} \) denotes the Stiefel manifold of orthonormal \( m \)-frames in \( \mathbb{R}^n \); \( \text{Gr}(n,m) \) stands for the Grassmann manifold of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \). All points in \( \mathbb{R}^{n+1} \) are identified with the corresponding column vectors.

2.2. Spherical Automorphisms. We recall that the main idea of our work is to build a bridge between the non-geodesic transform \( F_a \) and the classical geodesic transform \( F \) by constructing an intertwining automorphism that moves the origin \( o \) to the point \( a \), preserves the unit ball \( \mathbb{B}^{n+1} \), and maps \( S^n \) onto \( S^n \). This map should also transform \( k \)-dimensional linear subspaces of \( \mathbb{R}^{n+1} \) to affine \( k \)-planes, passing through \( a \). Having such an automorphism at hands, we can establish in Sections 3 and 4 an intimate connection between \( F_a, F \), and the corresponding integrating measures.

To define the desired map, we use the spherical coordinates

\[
\begin{align*}
    x &= \sqrt{1 - u^2} \psi + u\tilde{a}, \quad \tilde{a} = \frac{a}{|a|}, \quad |u| \leq 1, \quad \psi \in S^n \cap a^\perp, \\
    a &\neq o,
\end{align*}
\]

and set

\[
\mu_a x = \sqrt{1 - v^2} \psi + v\tilde{a}, \quad v = \frac{u - |a|}{1 - |a|u}. \tag{2.1}
\]

The map (2.1) moves points on the sphere along meridians that connect the poles \( \tilde{a} \) and \( -\tilde{a} \), keeping these poles fixed. Because the derivative \( dv/du \) is positive for \( |a| < 1 \), the function \( u \to v \) is a monotonically increasing and maps \([-1, 1]\) onto itself. The inverse map \( \mu_a^{-1} \) takes the point \( y = \sqrt{1 - v^2} \psi + v\tilde{a} \) to the point \( x \) by the rule

\[
\begin{align*}
    x &= \mu_a^{-1} y = \sqrt{1 - u^2} \psi + u\tilde{a}, \quad u = \frac{v + |a|}{1 + |a|v}. \tag{2.2}
\end{align*}
\]

Thus \( \mu_a \) is an automorphism of \( S^n \).

Using projection maps

\[
P_a : x \to \frac{a \cdot x}{|a|^2} a, \quad Q_a = I_{n+1} - P_a,
\]
we can write (2.1) in the coordinate-free form 1

\[
\mu_a x = \frac{P_a(x - a) + s_a Q_a(x - a)}{1 - x \cdot a}, \quad s_a = \sqrt{1 - |a|^2}. \tag{2.3}
\]

Equivalently,

\[
\mu_a x = s_a x - a + (1 - s_a) P_a x. \tag{2.4}
\]

Similarly,

\[
\mu_a^{-1} y = \frac{P_a(y + a) + s_a Q_a(y + a)}{1 + y \cdot a}, \tag{2.5}
\]

\[
= \frac{s_a y + a + (1 - s_a) P_a y}{1 + y \cdot a}. \tag{2.6}
\]

We also define the reflection \( \tau_a : S^n \to S^n \) about the point \( a \in B^{n+1} \):

\[
\tau_a x = \frac{(|a|^2 - 1) x + 2(1 - x \cdot a) a}{|x - a|^2}, \quad x \in S^n. \tag{2.7}
\]

It assigns to \( x \) the antipodal point \( \tau_a x \in S^n \) that lies on the line passing through \( x \) and \( a \). A similar reflection map about the origin \( o \) is denoted by \( \tau_o \), so that \( \tau_o x = -x \).

**Lemma 2.1.** Formula (2.3) extends the automorphism \( \mu_a : S^n \to S^n \) as an automorphism of the ball \( B^{n+1} \) with the following properties.

(a) \( \mu_o a = o, \mu_o o = -a \).

(b) For any affine subset \( E \subset B^{n+1} \), the images \( \mu_a(E) \) and \( \mu_a^{-1}(E) \) are affine subsets of \( B^{n+1} \).

(c) \( \mu_a \) is an intertwining mapping between the reflection \( \tau_a \) about the point \( a \in B^{n+1} \) and the ordinary reflection \( \tau_o x = -x \) about the origin:

\[
\mu_a \tau_a = \tau_o \mu_a. \tag{2.8}
\]

**Proof.** The following identity 2 easily follows from (2.3) due to orthogonality of the vectors \( P_a(x - a) \) and \( Q_a(x - a) \):

\[
1 - |\mu_a x|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{(1 - x \cdot a)^2}. \tag{2.9}
\]

It implies that \( \mu_a \) maps \( B^{n+1} \) onto \( B^{n+1} \). In fact, \( \mu_a \) is a homeomorphism of the closed unit ball.

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1 An analogue of (2.4) for the unit ball in the complex space \( C^n \) is defined in the book by Rudin [20, Section 2.2.1, formula (2)]

2 Cf. [20, Section 2.2.2, formula (iv)] for the automorphism of the complex ball.
The property (a) is obvious. To prove (b), observe that $\mu_ax$ has the form

$$y = \mu_ax = \frac{Ax + b}{1 - x \cdot a},$$

where $A$ is a linear operator on $\mathbb{R}^{n+1}$ and $b$ is a vector. Hence any linear equation

$$\alpha_1 y_1 + \ldots + \alpha_{n+1} y_{n+1} = \beta,$$

defining a hyperplane, transforms under the substitution $y = \mu_ax$ into equation for $x_1, \ldots, x_{n+1}$ of exactly the same form. This reasoning extends to systems of linear equations. Therefore, the mappings $\mu_a$ and $\mu_a^{-1}$ transform solutions of the systems of linear equations to similar solutions. The latter means that $\mu_a$ and $\mu_a^{-1}$ preserve affine structure of the plane sections of the ball $\mathbb{B}^{n+1}$.

The property (c) immediately follows from (b). Indeed, $\mu_a$ maps chords of the ball onto chords. Hence, for any $x \in \mathbb{S}^n$, the segment $[x, \tau_o x]$ is mapped onto the segment $[\mu_ax, \mu_a \tau_o x]$. Since the first segment contains $a$, the second one contains $\mu_a(a) = o$. The latter means that the points $\mu_ax$ and $\mu_a \tau_o x$ are symmetric with respect to the origin, i.e., $\mu_a \tau_o x = \tau_o \mu_a x$. □

The following equality can be checked by straightforward calculation:

$$\frac{1 + a \cdot \mu_ax}{1 - a \cdot \mu_ax} = \frac{1 - |a|^2}{|a - x|^2}, \quad x \in \mathbb{S}^n. \quad (2.10)$$

**Lemma 2.2.** If $f \in L^1(\mathbb{S}^n)$, then for any $a \in \mathbb{B}^{n+1}$,

$$\int_{\mathbb{S}^n} f(x) \, dx = (1 - |a|^2)^{n/2} \int_{\mathbb{S}^n} \frac{(f \circ \mu_a^{-1})(y)}{(1 + a \cdot y)^n} \, dy. \quad (2.11)$$

**Proof.** We make use of the spherical polar decomposition according to which

$$\int_{\mathbb{S}^n} f(x) \, dx = \int_{-1}^1 (1 - u^2)^{(n-2)/2} \, du \int_{\mathbb{S}^{n-1} \cap a^\perp} f \left( \sqrt{1 - u^2} \psi + u \hat{a} \right) \, d\psi; \quad (2.12)$$

cf., e.g., [18, Lemma 1.34]. Changing variable

$$u = \frac{v + |a|}{1 + |a|v},$$

(cf.(2.2)) and taking into account that

$$\frac{du}{dv} = \frac{1 - |a|^2}{(1 + |a|v)^2}, \quad 1 - u^2 = \frac{(1 - |a|^2)(1 - v^2)}{(1 + |a|v)^2},$$

...
we obtain
\[
\text{l.h.s.} = (1 - |a|^2)^{n/2} \int_{-1}^{1} \frac{(1-v^2)^{(n-2)/2}}{(1+|a|v)^n} dv \\
\times \int_{S^n \cap a^\perp} f \left( \sqrt{1-|a|^2} \sqrt{1-v^2} \frac{\psi + v + |a|v \tilde{a}}{1+|a|v} \right) d\psi
\]
\[
= (1 - |a|^2)^{n/2} \int_{S^n} \frac{\left( f \circ \mu_a^{-1}\right)(y)}{(1 + a \cdot y)^n} dy = \text{r.h.s.}
\]

Lemma 2.3. If \( f \in L^1(S^n) \) and \( a \in B^{n+1} \), then
\[
\int_{S^n} f(\tau_ax) \, dx = \int_{S^n} f(x) \left( \frac{1 - |a|^2}{|a - x|^2} \right)^n \, dx, \quad (2.13)
\]
\[
\int_{S^n} f(x) \, dx = \int_{S^n} f(\tau_ax) \left( \frac{1 - |a|^2}{|a - x|^2} \right)^n \, dx. \quad (2.14)
\]

Proof. By (2.8) and (2.11),
\[
\int_{S^n} f(\tau_ax) \, dx = \int_{S^n} f(\mu_a^{-1}\tau_0 \mu_ax) \, dx \quad \text{(set } x = \mu_a^{-1}\tau_0 y)\]
\[
= (1 - |a|^2)^{n/2} \int_{S^n} \frac{\left( f \circ \mu_a^{-1}\right)(y)}{(1 + a \cdot y)^n} \left( \frac{1 + a \cdot y}{1 - a \cdot y} \right)^n dy
\]
\[
= \int_{S^n} f(x) \left( \frac{1 + a \cdot \mu_ax}{1 - a \cdot \mu_ax} \right)^n \, dx.
\]
It remains to apply (2.10). The second equality follows from the first one: just replace \( f(x) \) by \( f(\tau_ax) \) and use \( \tau_a \tau_ax = x \). \qed

3. Reduction of \( F_a \) to the Totally Geodesic Funk Transform

In this section we show that the study of the Funk-type transform \( F_a \) (see (1.1)) can be reduced to the study of the similar transform with \( a = \circ \), the properties of which are well known. For technical reasons, it is convenient to switch from the Grassmannian language, as in (1.1), to the corresponding language of Stiefel manifolds.
Given a continuous function $f$ on $S^n$, we define the totally geodesic Funk transform by the formula

$$ (Ff)(\xi) = \int_{\{x \in S^n : \xi'x = 0\}} f(x) d\sigma(x), \quad \xi \in \text{St}(n+1, n+1-k), \quad (3.1) $$

where $d\sigma(x)$ stands for the corresponding surface area measure. This transform mimics (1.1) with $a = 0$.

Here and on throughout the paper, we associate planes in $\mathbb{R}^{n+1}$ with their normal frames, which are the elements the corresponding Stiefel manifold. This identification is not one-to-one, however, it reflects the essence of the matter and does not cause any confusion.

A modification of (3.1) for non-central cross-sections by $k$-planes through an arbitrary fixed point $a \in \mathbb{B}^{n+1}$ can be accordingly defined by the formula

$$ (F_a f)(\xi) = \int_{\{x \in S^n : \xi'(x-a) = 0\}} f(x) d\sigma(x), \quad \xi \in \text{St}(n+1, n+1-k), \quad (3.2) $$

which agrees with (1.1). The operators (3.1) and (3.2) are intimately related to the normalized $\lambda$-cosine transforms

$$ (C^\lambda f)(\xi) = \gamma_{n,k}(\lambda) \int_{S^n} f(x) |\xi'|^\lambda |x\|^\lambda d\sigma, \quad (3.3) $$

$$ (C^\lambda_a f)(\xi) = \gamma_{n,k}(\lambda) \int_{S^n} f(x) |\xi'(x-a)|^\lambda d\sigma, \quad (3.4) $$

where, as above, $\xi \in \text{St}(n+1, n+1-k)$ and

$$ \gamma_{n,k}(\lambda) = \frac{\Gamma(-\lambda/2)}{2^{\lambda+n+1-k}(n-k+1)^{\lambda/2} \Gamma((\lambda+n+1-k)/2)}, \quad (3.5) $$

$$ \text{Re} \lambda > k - n - 1, \quad \lambda \neq 0, 2, 4, \ldots. $$

The integrals (3.1)-(3.4) can be regarded as functions on the Grassmann manifold $\text{Gr}(n+1, k)$ because they remain unchanged if we replace $\xi \in \text{St}(n+1, n+1-k)$ by $\xi \gamma$, $\gamma \in O(n+1-k)$. More general versions of (3.3) were studied in [10, 16], where one can find further references.

**Lemma 3.1.** Let $a \in \mathbb{R}^{n+1}$, $2 \leq k \leq n$. If $f$ is a continuous function on $S^n$, then for every $\xi \in \text{St}(n+1, n+1-k)$ satisfying $|\xi'| < 1$ the following equality holds:

$$ \lim_{\lambda \to k-n-1} (C^\lambda_a f)(\xi) = (1 - |\xi'|^2)^{-1/2} (F_a f)(\xi). \quad (3.6) $$
In the cases $|\xi'|a| > 1$ and $|\xi'|a| = 1$, $k > 2$, the above limit is zero. If $|\xi'|a| = 1$ and $k = 2$, then $\lim_{\lambda \to k-n-1} (C^\lambda_a f)(\xi) = \sigma_{k-1} f(P_\xi a)$, where $\sigma_{k-1}$ is the area of the $(k - 1)$-dimensional unit sphere and $P_\xi a$ is the orthogonal projection of $a$ onto the subspace spanned by $\xi$, so that $P_\xi a \in S^n \cap \text{span}(\xi)$.

This Lemma covers a general situation when the point $a$ does not necessarily lie in the ball $B^{n+1}$. If $a \in B^{n+1}$, the condition $|\xi'|a| < 1$ holds automatically for all $\xi$.

The proof of the lemma is given in Appendix.

Lemma 3.2. Let $C^\lambda_a$ be the $\lambda$-cosine transform (3.4). If $f \in L^1(S^n)$, then

$$(C^\lambda_a f)(\xi) = \gamma_{n,k}(\lambda)(1 - |a|^2)^{(n+\lambda)/2} \int_{S^n} \frac{(f \circ \mu^{-1}_a)(y)}{(1 + a \cdot y)^{n+\lambda}} |\xi' A y|^\lambda dy,$$

where $\mu_a$ is the automorphism (2.4),

$$A = s_a P_a + Q_a, \quad s_a = \sqrt{1 - |a|^2}. \quad (3.8)$$

Proof. By (2.11),

$$(C^\lambda_a f)(\xi) = \gamma_{n,k}(\lambda)(1 - a^2)^{n/2} \int_{S^n} \frac{(f \circ \mu^{-1}_a)(y)}{(1 + a \cdot y)^n} |\xi'(\mu_a^{-1}y - a)|^\lambda dy,$$

where, by (2.5),

$$\mu_a^{-1}y - a = \frac{P_a y + s_a Q_a y - a(y \cdot a)}{1 + y \cdot a} = \frac{s_a}{1 + y \cdot a} (s_a P_a + Q_a)y.$$

This gives the result. \hfill \Box

In the following, we will need a matrix polar decomposition, according to which every matrix $M \in \mathfrak{M}_{n,m}$ of rank $m$ can be uniquely decomposed as $M = \omega \rho^{1/2}$, where $\omega \in \text{St}(n,m)$ and $\rho = M'M$ is a positive definite $m \times m$ matrix; see, e.g., [9, pp. 66, 591].

Lemma 3.3. Let $1 < k < n$, $k - n - 1 < \lambda < k - n$. Given a matrix $M \in \mathfrak{M}_{n+1,n+1-k}$ of rank $n + 1 - k$, we define

$$I_\vartheta(\lambda, M) = \gamma_{n,k}(\lambda) \int_{S^n} \vartheta(\lambda, y) |M'y|^\lambda dy,$$

where $\gamma_{n,k}(\lambda)$ is the constant (3.5) and $\vartheta$ is a continuous function on the product space $(k - n - 1, k - n) \times S^n$. If the limit $\lim_{\lambda \to k-n-1} \vartheta(\lambda, y) = \vartheta_0(y)$

Operators $F_a$ with $a \notin B^{n+1}$ will be studied in our future publication.
exists, then
\[ \lim_{\lambda \to k-n-1} I_\theta(\lambda, M) = c_M (F\theta_0)(\omega), \] (3.10)
where \( \omega = M(M'M)^{-1/2} \in \text{St}(n+1, n+1-k) \),
\[ c_M = \frac{1}{\sigma_{n-k}} \int_{S^{n-k}} |(M'M)^{1/2}\psi|^{k-n-1} d\psi, \]
and \( F\theta_0 \) is the Funk transform (3.1).

**Proof.** According to the polar decomposition \( M = \omega \rho_{1/2} \) with \( \omega \in \text{St}(n+1, n+1-k) \), \( \rho = M'M \), we choose \( r_\omega \in O(n+1) \) that takes \( \xi_0 = \left[ \begin{array}{c} 0 \\ I_{n-k+1} \end{array} \right] \) to \( \omega \). Changing variable \( y = r_\omega x \), we obtain
\[ I_\theta(\lambda, M) = \gamma_{n,k}(\lambda) \int_{S^n} \vartheta(\lambda, r_\omega x) |\rho^{1/2}\xi_0 x|^{\lambda} dx. \]
Then we pass to bispherical coordinates (see, e.g., [18, p. 31])
\[ x = \left[ \begin{array}{c} \varphi \sin \theta \\ \psi \cos \theta \end{array} \right], \quad \varphi \in S^{k-1}, \quad \psi \in S^{n-k}, \quad 0 \leq \theta \leq \pi/2, \]
\[ dx = \sin^{k-1} \theta \cos^{n-k} \theta d\theta d\varphi d\psi, \]
and set \( s = \cos \theta \). This gives
\[ I_\theta(\lambda, M) = \gamma_{n,k}(\lambda) \int_{S^{k-1}} \vartheta(\lambda, r_\omega \varphi \sqrt{1-s^2} / s\psi) d\varphi \int_{S^{n-k}} \vartheta(\lambda, r_\omega \varphi \sqrt{1-s^2} / s\psi) |\rho^{1/2}\psi|^{\lambda} d\psi; \]
or
\[ I_\theta(\lambda, M) = \frac{1}{\Gamma(n-k+\lambda+1)} \int_0^1 s^{n-k+\lambda} \Phi_\lambda(s) ds, \] (3.11)
where
\[ \Phi_\lambda(s) = \frac{\Gamma((n-k+\lambda+2)/2) \Gamma(-\lambda/2)}{2^{k+2-n-\lambda} \pi^{n/2+1}} (1-s^2)^{(k-2)/2} \int_{S^{k-1}} d\varphi \times \int_{S^{n-k}} \vartheta(\lambda, r_\omega \varphi \sqrt{1-s^2} / s\psi) |\rho^{1/2}\psi|^{\lambda} d\psi. \]
The integral (3.11) falls into the scope of Lemma 5.1 from [19], according to which the integrals of the form
\[
\lim_{\alpha \to 0} \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} s^{\alpha-1} u(\alpha, s) \, ds
\]
converge to \( u_0 = \lim_{(\alpha,s) \to (0,0)} u(\alpha, s) \) if the function \( u(\alpha, s) \) is good enough.

Applying this lemma to our case with \( \alpha = n - k + \lambda + 1 \) and \( u(\alpha, s) \) replaced by \( \Phi_\lambda(s) \), we obtain
\[
\lim_{\lambda \to k - n - 1} I_{\vartheta}(\lambda, M) = \frac{1}{\sigma_{n-k}} \int_{\mathbb{S}^{k-1}} \vartheta_0 \left( r_\omega \left[ \begin{array}{c} \varphi \\ 0 \end{array} \right] \right) \, d\varphi
\times \int_{\mathbb{S}^{n-k}} |\rho^{1/2} \tilde{\psi}|^{k-n-1} d\omega = c_M \int_{\omega' x = 0} \psi_0(x) \, d\sigma(x).
\]
This completes the proof. \( \square \)

Lemmas 3.4 and 3.3 yield the desired factorization of the non-geodesic transform \( F_a \) in terms of the totally geodesic transform \( F \). To formulate the result we set
\[
A = s_a P_a + Q_a = I_{n+1} - (1 - s_a) aa', \quad (3.12)
\]
where \( s_a, P_a \) and \( Q_a \) have the same meaning as in (2.3);
\[
w(\xi) = s_a^{k-1} (1 - |\xi'| a^2)^{1/2} \int_{\mathbb{S}^{n-k}} |(\xi' A^2 \xi)^{1/2} \omega|^{k-n-1} d\omega, \quad (3.13)
\]
\[
\eta(\xi) = A \xi (\xi' A^2 \xi)^{-1/2}. \quad (3.14)
\]

Consider the map
\[
\text{St}(n+1, n+1-k) \ni \xi \mapsto \eta \in \text{St}(n+1, n+1-k), \quad (3.15)
\]
and set
\[
(M_a f)(y) = \frac{(f \circ \mu_a^{-1})(y)}{(1 + a \cdot y)^{k-1}}, \quad (N_a \Phi)(\xi) = w(\xi)(\Phi \circ \nu)(\xi), \quad (3.16)
\]
where \( \mu_a \) is defined by (2.3) and \( w \) has the form (3.13).

**Theorem 3.4.** Let \( a \in \mathbb{B}^{n+1}, 1 < k \leq n \). If \( f \in C(\mathbb{S}^n) \), then
\[
(F_a f)(\xi) = (N_a F M_a f)(\xi), \quad \xi \in \text{St}(n+1, n+1-k). \quad (3.17)
\]
Proof. Owing to (3.7) and (3.9),
\[
(C^λ_α f)(ξ) = (1 - |a|^2)^{(n+λ)/2} I_0(λ, M), \tag{3.18}
\]
where
\[
ϑ(λ, y) = (f ∘ μ_α^{-1})(y) \quad (1 + a · y)^{n+λ}, \quad M = Aξ.
\]
The limit of the left-hand side of (3.18) as \( λ → k - n - 1 \) is
\[
(1 - |ξ'|^2)^{-1/2} (Fa f)(ξ); \tag{3.19}
\]
see Lemma 3.1. By (3.10), the limit of the right-hand side is
\[
c(ξ)(1 - |a|^2)^{(k-1)/2}(Fϑ_0)(η(ξ)), \tag{3.20}
\]
where \( η(ξ) = Aξ(ξ'A^2ξ)^{-1/2} \),
\[
c(ξ) = \frac{1}{σ_{n-k}} \int_{S^{n-k}} |(ξ'A^2ξ)^{1/2}ψ|^k-n-1 dψ,
\]
\[
ϑ_0(y) = \lim_{λ→k-n-1} ϑ(λ, y) = \frac{(f ∘ μ_α^{-1})(y)}{(1 + a · y)^{k-1}} = (M₀ f)(y).
\]
This gives the result. \( □ \)

Remark 3.5. The map \( ν \) defined by (3.15) is one-to-one. To obtain explicit expression of the inverse map \( ν^{-1} \), let \( η = ν(ξ) = Aξ(ξ'A^2ξ)^{-1/2} \). We set \( ζ = A^{-1}η \). Then \( ζ = ξ(ξ'A^2ξ)^{-1/2} \) and we have
\[
ζ(ζ')^{-1/2} = ξ(ξ'A^2ξ)^{-1/2} \left[ (ξ'A^2ξ)^{-1/2}ξ(ξ'A^2ξ)^{-1/2} \right]^{-1/2} = ξ(ξ'A^2ξ)^{-1/2} \left[ (ξ'A^2ξ)^{-1} \right]^{-1/2} = ξ.
\]
This gives
\[
ξ = (A^{-1}η)[η'(A^{-1})^2η]^{-1/2}, \quad A = s_P + Q_a. \tag{3.21}
\]
In other words, the frame \( ξ ∈ St(n+1, n+1-k) \) can be reconstructed from \( η = ν(ξ) \) as the angular component of the matrix \( ζ = (s_P + Q_a)^{-1}η ∈ \mathfrak{M}_{n+1,n+1-k} \).

4. Reconstruction From One Center

Let \( F_a \) be the Funk-type transform (3.2), acting on the space \( C(S^n) \). In this section we describe the kernel of the operator \( F_a \) and the subspace of \( C(S^n) \) on which \( F_a \) is injective. We also obtain inversion formulas for this operator.
4.1. The case $a = 0$. We recall some known facts about the totally geodesic transform $F = F_0$. Because $F$ is non-injective on $C(S^n)$, it is important to make the concept of its inversion precise. More information can be found in [7, 15, 18].

Let $C(S^n) = C^+(S^n) \oplus C^-(S^n)$, $f = f^+ + f^-$, be the direct decomposition of $C(S^n)$ into the subspaces of even and odd functions, respectively. It is easy to see that the kernel of $F$ in $C(S^n)$ consists of all odd functions. Indeed, the transform $F$ annihilates odd functions. Conversely, if $Ff = 0$, then $Ff^+ = F[f^+ + f^-] = Ff = 0$ and $f^+ = 0$ due to the injectivity of $F$ on even functions. Hence $f = f^-$ is odd.

Thus we have the following statement.

Proposition 4.1. Let $F$ be the totally geodesic transform Funk transform acting on the space $C(S^n)$. Then

$$\ker(F) = C^-(S^n) = \{f \in C(S^n) : f^+ = 0\}. \quad (4.1)$$

The restriction of $F$ onto the subspace $C^+(S^n)$ of even functions is injective.

We define an operator $F^{-1} : F(C(S^n)) \to C^+(S^n)$ by setting

$$F^{-1}g = f^+, \quad g = Ff \in F(C(S^n)). \quad (4.2)$$

This definition does not depend on the choice of the representative $f$ in $g = Ff$, since, if $g = Ff_1 = Ff_2$, then $f_1 - f_2 \in \ker F$ and therefore

$$0 = Ff_1 - Ff_2 = F[f_1^+ - f_2^+] + F[f_1^- - f_2^-] = F[f_1^+ - f_2^+].$$

Because the restriction of $F$ onto $C^+(S^n)$ is injective, it follows that $f_1^+ = f_2^+$.

We can also write (4.2), as

$$F^{-1}F = P^+, \quad (4.3)$$

where $P^+ : C(S^n) \to C^+(S^n)$ is the projection operator. For the same reason, if $g \in F(C(S^n))$, that is, $g = Ff$ for some $f \in C(S^n)$, then $F(F^{-1}g) = Ff^+ = Ff = g$, so that $F^{-1}$ is the right inverse of the operator $F$, when the latter acts on $C(S^n)$.

Recall that for functions $f \in C^+(S^n)$, we have $F^{-1}Ff = f$, which means that $F^{-1}$ can also be considered as the left inverse of the restriction of $F$ onto $C^+(S^n)$.

Explicit formulas for $F^{-1}$ can be found in different sources; see, e.g., [7, 15, 17]. These formulas may have different analytic form. In particular, to reconstruct $f \in C^+(S^n)$ from $\varphi(\xi) = (Ff)(\xi)$, $\xi \in$...
St(n+1, n+1−k), one can introduce the so-called shifted dual transform

\[(F^*_x \varphi)(r) = \int_{\{x \in S^n : |\xi \cdot x| = r\}} \varphi(\xi) \, dm(\xi), \quad 0 < r < 1, \quad (4.4)\]

where integration is performed with respect to the relevant probability measure. To understand the geometrical meaning of this integral, we regard a right \(O(n + 1 - k)\)-invariant function \(\varphi(\xi)\) as a function \(\varphi_0\) on the Grassmannian \(\text{Gr}(n+1, k)\), so that \(\varphi(\xi) = \varphi_0(\xi^\perp)\). Then (4.4) is the average of \(\varphi_0\) over all \(k\)-dimensional subspaces \(\xi^\perp\) whose intersection with \(S^n\) has geodesic distance \(\cos^{-1} r\) from the point \(x\).

**Theorem 4.2.** (cf. [17, Theorem 5.3]) An even continuous function \(f\) on \(S^n\) can be recovered from \(\varphi = Ff\) by the formula

\[(F^{-1} \varphi)(x) = \lim_{s \to 1} \left( \frac{1}{2s} \frac{\partial}{\partial s} \right)^k \frac{\pi^{-k/2}}{\Gamma(k/2)} \int_0^s (s^2 - r^2)^{k/2 - 1} (F^*_x \varphi)(r) \, r^k \, dr \right].\]

\[(4.5)\]

In particular, for \(k\) even,

\[(F^{-1} \varphi)(x) = \lim_{s \to 1} \frac{1}{2\pi^{k/2}} \left( \frac{1}{2s} \frac{\partial}{\partial s} \right)^{k/2} \left[ s^{k-1}(F^*_x \varphi)(s) \right].\]

\[(4.6)\]

The limit in these formulas is understood in the sup-norm.

**4.2. The case of any \(a \in B^{n+1}\).** Our next aim is to extend Proposition 4.1, the equality (4.3), and Theorem 4.2 to non-geodesic Funk-type transforms \(F_a\) acting on \(C(S^n)\). This can be done by making use of Theorem 3.4. We denote

\[\rho_a(x) = \left(1 - \frac{|a|^2}{|a - x|^2}\right)^{k-1}, \quad (W_a f)(x) = \rho_a(x) f(\tau_a x), \quad (4.7)\]

where \(\tau_a\) is the reflection (2.7). The operator \(W_a\), which takes a function \(f \in C(S^n)\) to a function \(f \circ \tau_a\) of reflected argument with subsequent multiplication by the weight \(\rho_a(x)\), plays an important role in our consideration.

**Lemma 4.3.** The operator \(W_a\) is an involution, i.e., \(W_a W_a f = f\).

**Proof.** The statement is obvious for \(a = a\), when \((W_0 f)(x) = f(-x)\). It is also obvious for any \(a \in B^n\) if \(k = 1\). In the general case, taking into account that \(\tau_a \tau_a x = x\), we have

\[(W_a W_a f)(x) = \left[1 - |a|^2 \frac{1 - |a|^2}{|a - x|^2} \frac{1 - |a|^2}{|a - \tau_a x|^2}\right]^{k-1} f(x).\]
By (2.10) and (2.8), the expression in square brackets can be written as
\[
\frac{(1 + a \cdot \mu_0 x) (1 + a \cdot \mu_0 \tau_0 x)}{(1 - a \cdot \mu_0 x) (1 - a \cdot \mu_0 \tau_0 x)} = \frac{(1 + a \cdot \mu_0 x) (1 - a \cdot \mu_0 x)}{(1 - a \cdot \mu_0 x) (1 + a \cdot \mu_0 x)} = 1.
\]
This gives the result. \qed

We set
\[
f_a^+(x) = \frac{f(x) + (W_a f)(x)}{2}, \quad f_a^-(x) = \frac{f(x) - (W_a f)(x)}{2},
\]
so that
\[
f = f_a^+ + f_a^-.
\]

Definition 4.4. A continuous function \(f\) on \(S^n\) is called \(a\)-even (or \(a\)-odd) \footnote{Here we simplify the terminology, keeping in mind that \(W_a\) also depends on \(k\).} if \(f(x) = (W_a f)(x)\) (or \(f(x) = -(W_a f)(x)\)) for all \(x \in S^n\). The subspaces of all \(a\)-even and \(a\)-odd continuous functions on \(S^n\) will be denoted by \(C^+_a(S^n)\) and \(C^-_a(S^n)\), respectively.

The functions \(f_a^+\) and \(f_a^-\) in (4.8) can be regarded as an \(a\)-even and \(a\)-odd parts of \(f\), respectively. Further, if \(f\) is \(a\)-even (or \(a\)-odd), then \(f = f_a^+\) (or \(f = f_a^-\)).

The following statement is a generalization of Proposition 4.1.

Theorem 4.5. Let \(F_a\) be the Funk-type transform (3.2) acting on the space \(C(S^n)\). Then
\[
\ker(F_a) = C^-_a(S^n) = \{ f \in C(S^n) : f_a^+ = 0 \}.
\]
The restriction of \(F_a\) onto the subspace \(C^+_a(S^n)\) is injective.

Proof. By Remark 3.5, \(f \in \ker(F_a)\) if and only if \(M_a f\) is an odd function on \(S^n\), that is, \((M_a f)(y) = -(M_a f)(-y)\) for all \(y \in S^n\). The latter gives
\[
(f \circ \mu^{-1}_a)(y) = -\left(\frac{1 + a \cdot y}{1 - a \cdot y}\right)^{k-1} (f \circ \mu^{-1}_a)(-y).
\]
Setting \(y = \mu_0 x\) and making use of the equality \(\tau_a = \mu_0^{-1} \tau_0 \mu_0\) (see Lemma 2.1), we obtain
\[
f(x) = -\left(\frac{1 + a \cdot \mu_0 x}{1 - a \cdot \mu_0 x}\right)^{k-1} f(\tau_0 x).
\]
The last expression yields the first statement of the lemma in view of (2.10). The second statement follows similarly from the equality \((M_a f)(y) = (M_a f)(-y)\). \qed
Now we proceed to inversion of the Funk-type transform $F_a$. Let first $1 < k \leq n$ and suppose that $F_a$ acts on the space $C(S^n)$, as in Theorem 3.4. By this theorem, $F_a f = N_a F M_a f$, where the operators $M_a$ and $N_a$ are defined by (3.16) and $F$ is the totally geodesic Funk transform as in Subsection 4.1. Hence, formally, $F_a^{-1} = M_a^{-1} F^{-1} N_a^{-1}$.

Each component in this formula must be specified.

To obtain an explicit formula for $M_a^{-1}$, suppose that

$$(M_a f)(y) \equiv \frac{(f \circ \mu_a^{-1})(y)}{(1 + a \cdot y)^{k-1}} = \tilde{f}(y)$$

and let $y = \mu_a x$. Then

$$(M_a^{-1} \tilde{f})(x) = (1 + a \cdot \mu_a x)^{k-1} \tilde{f}(\mu_a x). \quad (4.11)$$

To find an expression for $N_a^{-1}$, we invoke (3.16) and let

$$(N_a \Phi)(\xi) \equiv w(\xi) (\Phi \circ \nu)(\xi) = \tilde{\Phi}(\xi), \quad \nu(\xi) = A \xi (\xi' A^2 \xi)^{-1/2},$$

where the weight $w$ has the form (3.13) and $A = s_a P_A + Q_a$. Then, by (3.21), we obtain an inversion formula

$$(N_a^{-1} \tilde{\Phi})(\eta) = [w(\xi)]^{-1} \tilde{\Phi}(\xi), \quad (4.12)$$

in which

$$\xi = (A^{-1} \eta) [\eta' (A^{-1})^2 \eta]^{-1/2}, \quad A^{-1} = (A = s_a P_A + Q_a)^{-1}.$$

Next we define an operator $F_a^{-1} : F_a(C(S^n)) \to C_a^+(S^n)$ by setting

$$F_a^{-1} g = f_a^+, \quad g = F_a f \in F_a(C(S^n)). \quad (4.13)$$

As in the case $a = o$ (see (4.2)), this definition does not depend on the choice of the representative $f \in C(S^n)$ satisfying $g = F_a f$. We can write (4.13) in the form

$$F_a^{-1} F_a f = f_a^+ = \frac{f + W_a f}{2}, \quad (4.14)$$

(see (4.8)) or

$$F_a^{-1} F_a f = P_a^+ f, \quad (4.15)$$

where $P_a^+ : C(S^n) \to C_a^+(S^n)$ is the projection operator; cf. (4.3) for $a = o$.

Remark 4.6. If $g \in F_a(C(S^n))$, $g = F_a f$, then

$$F_a F_a^{-1} g = F_a f_a^+ = F_a f = g,$$

and therefore $F_a^{-1}$ is the right inverse operator for the operator $F_a : C(S^n) \to F(C(S^n))$. On the other hand, if $f$ is $a$-even, then $F_a^{-1} F_a f = f$ and hence $F_a^{-1}$ serves as the inverse, both right and left, for the restricted operator $F_a : C_a^+(S^n) \to F_a(C_a^+(S^n))$. At the same time, the
ranges of both operators, restricted and non-restricted, coincide, i.e., \( F_a(C(\mathbb{S}^n)) = F_a(C_{\mathbb{S}^n}(\mathbb{S}^n)) \).

**Proposition 4.7.** The operator (4.13) can be represented in the form

\[
F^{-1}_a g = M^{-1}_a F^{-1} N^{-1}_a g, \quad g \in F_a(C(\mathbb{S}^n)),
\]

(4.16)

\( M^{-1}_a, F^{-1} \) and \( N^{-1}_a \) are defined by (4.11), (4.2), and (4.12), respectively.

**Proof.** Let \( g = F_a f, f \in C(\mathbb{S}^n) \). By (4.13), \( F^{-1}_a g = f_a^+ \). On the other hand, by Theorem 3.4 and (4.2),

\[
M^{-1}_a F^{-1} N^{-1}_a g = (M^{-1}_a F^{-1} N^{-1}_a)(N_a F M_a f) = \frac{1}{2} \left[ (f \circ \mu_a^{-1})(y) + (f \circ \mu_a^{-1})(-y) \right] \left[ (1 + a \cdot y)k^{-1} + (1 - a \cdot y)k^{-1} \right].
\]

By (4.11),

\[
(M_a^{-1} \varphi)(x) = (1 + a \cdot \mu_a x)^k \varphi(\mu_a x).
\]

Hence

\[
(M_a^{-1} F^{-1} N^{-1}_a g)(x) = \frac{1}{2} \left[ (f \circ \mu_a^{-1})(\mu_a x) + (f \circ \mu_a^{-1})(-\mu_a x) \right] \left[ (1 + a \cdot \mu_a x)^k - (1 - a \cdot \mu_a x)^k \right].
\]

Using Lemma 2.1, the latter can be written as

\[
(M_a^{-1} F^{-1} N^{-1}_a g)(x) = \frac{1}{2} \left[ f(x) + \left( \frac{1 - |a|^2}{|a - x|^2} \right)^{k-1} f(\tau_a x) \right] = f_a^+(x).
\]

This completes the proof. \( \square \)

The above reasoning yields the following inversion result for the operator \( F_a \) defined only on \( a \)-even functions.

**Theorem 4.8.** Let \( 1 < k \leq n, a \in \mathbb{B}^n \). An \( a \)-even function \( f \in C(\mathbb{S}^n) \) can be uniquely reconstructed from \( g = F_a f \) by the formula \( f = F_a^{-1} g \), where \( F_a^{-1} \) has the form (4.16).

**Proof.** If \( g = F_a f \), then, by Proposition 4.7 and (4.13), \( F_a^{-1} F_a f = f_a^+ \). Because \( f \) is \( a \)-even, we have \( f_a^+ = f \). Hence \( F_a^{-1} F_a f = f \). \( \square \)
**Remark 4.9.** In the case \( k = 1 \), which is not included in Theorem 4.8, the integral (1.1) is a sum of the values of \( f \) at the points, where the line \( L \) mentioned in (1.1) intersects the sphere. If \( x \) is one of these points and \( L = L_{a,x} \) is the line through \( a \) and \( x \), then

\[
(F_a f)(L_{a,x}) = f(x) + f(\tau ax).
\]

(4.17)

The \( a \)-odd functions, for which \( f(x) = -f(\tau ax) \), form the kernel of the operator (4.17). An \( a \)-even function \( f \), satisfying \( f(x) = f(\tau ax) \), can be reconstructed from \((F_a f)(L_{a,x})\) by the formula

\[
f(x) = \frac{1}{2} (F_a f)(L_{a,x}).
\]

(4.18)

5. **Reconstruction from Two Centers**

As we have seen in the precious section, a function \( f \in C(S^n) \) cannot be uniquely reconstructed from the equation \( F_a f = g \). In fact, we can reconstruct only \( f^+_a \), the \( a \)-even part of \( f \); cf. (4.14). The corresponding inverse operator was denoted by \( F^{-1}_a \). Our aim is to show that this non-uniqueness can be overcome if we consider two centers instead of one, in other words, if we want to reconstruct \( f \) from the system of two equations

\[
F_a f = g, \quad F_b f = h,
\]

(5.1)

where \( a \neq b \) are any fixed points in \( B^n \). Setting

\[
g_1 = 2F^{-1}_a g, \quad h_1 = 2F^{-1}_b h,
\]

and using (4.14), we obtain

\[
f = g_1 - W_a f, \quad f = h_1 - W_b f,
\]

(5.2)

where

\[
(W_a f)(x) = \rho_a(x)f(\tau_a x), \quad (W_b f)(x) = \rho_b(x)f(\tau_b x).
\]

(5.3)

Then we substitute \( f \) from the second equation into the right-hand side of the first one to get \( f = g_1 - W_a[h_1 - W_b f] \), or

\[
f = Wf + q, \quad W = W_a W_b, \quad q = g_1 - W_a h_1.
\]

(5.4)

Iterating (5.4), we obtain

\[
f = W^m f + \sum_{j=0}^{m-1} W^j q; \quad m = 1, 2, \ldots.
\]

(5.5)

This equation gives rise to a dynamical system on \( S^n \).
Lemma 5.1. Let \( a^* \) and \( b^* \) be the points on \( \mathbb{S}^n \) that lie on the straight line through \( a \) and \( b \). Suppose that \( a \) is closer to \( a^* \) than \( b \). If \( W = W_a W_b \), then \( \lim_{m \to \infty} (W^m f)(x) = 0 \) for all \( x \in \mathbb{S}^n \setminus \{a^*\} \) and \( 1 < k \leq n \).

If \( k = 1 \) and \( x \in \mathbb{S}^n \setminus \{a^*\} \), then \( \lim_{m \to \infty} (W^m f)(x) = f(b^*) \).

Proof. We observe that

\[
(W f)(x) = (W a W_b f)(x) = \rho_a(x) \rho_b(\tau_a x) f(\tau_b \tau_a x).
\]

Denote

\[
\rho(x) = \rho_a(x) \rho_b(\tau_a x) = \left[ \frac{(1 - |a|^2)(1 - |b|^2)}{|a - x|^2 |b - \tau_a x|^2} \right]^{k-1}, \quad T = \tau_b \tau_a.
\]

Then \( (W f)(x) = \rho(x) f(T x) \) and, by iteration,

\[
(W^m f)(x) = \omega_m(x) f(T^{m+1} x), \quad \omega_m(x) = \prod_{j=0}^{m} \rho(T^j x).
\]

For any \( x \neq a^* \), the mapping \( T \) preserves the circle \( C_{x,a,b} \) in the 2-plane spanned by \( x, a \) and \( b \), and leaves the points \( a^* \) and \( b^* \) fixed. A simple geometric consideration in the 2-plane shows that the distance from the points \( T^j x \in C_{x,a,b} \) to \( b^* \) monotonically decreases, and therefore, the sequence \( T^j x \) has a limit. This limit must be a fixed point of the mapping \( T \), and hence \( T^j x \to b^* \) as \( j \to \infty \). Because \( \rho \) is continuous, it follows that

\[
\lim_{j \to \infty} \rho(T^j x) = \rho(b^*).
\]

Using this fact, let us show that if \( k > 1 \), then

\[
\lim_{m \to \infty} \omega_m(x) = 0.
\]

Once (5.10) is proved, the statement of the lemma for \( k > 1 \) will follow because the factor \( f(T^{m+1} x) \) has the finite limit \( f(b^*) \).

To prove (5.10), it suffices to show that

\[
\rho(b^*) < 1,
\]

where, by (5.7),

\[
\rho(b^*) = \left[ \frac{(1 - |a|^2)(1 - |b|^2)}{|a - b^*|^2 |a^* - b|^2} \right]^{k-1}.
\]

Let

\[
a = a^* + t(b^* - a^*), \quad b = a^* + s(b^* - a^*), \quad 0 < t < s < 1.
\]

Taking into account that \( |a^*| = |b^*| = 1 \) and using (5.13), we obtain

\[
1 - |a|^2 = 2t(1-t)(1-a^* \cdot b^*), \quad 1 - |b|^2 = 2s(1-s)(1-a^* \cdot b^*),
\]
\[ |a - b^*|^2 = 2(1 - t)^2(1 - a^* \cdot b^*), \quad |a^* - b|^2 = 2s^2(1 - a^* \cdot b^*). \]

Hence
\[ \rho(b^*) = \left[ \frac{t(1 - s)}{s(1 - t)} \right]^{k-1} < 1. \tag{5.15} \]

The last inequality is an immediate consequence of the assumption \(0 < t < s < 1\).

The case \(k = 1\) is simpler. In this case \(\rho(x) = 1\), and therefore, \((W^m f)(x) = f(T^m x) \to f(b^*)\) as \(m \to \infty\), \(x \in \mathbb{S}^n \setminus \{a^*\}\). \(\square\)

Lemma 5.1 implies the following result. We recall that \(a^*\) and \(b^*\) denote the endpoints of the chord through \(a\) and \(b\).

**Theorem 5.2.** Let \(W_a\) and \(W_b\) be the involutions (5.3), \(1 < k \leq n\). If the system of equations \(F_a f = g\) and \(F_b f = h\) has a solution \(f \in C(\mathbb{S}^n)\), then this solution is unique and can defined by the pointwise convergent series
\[ f(x) = \sum_{j=0}^{\infty} W^j q(x), \quad x \neq a^*, \tag{5.16} \]

where \(W = W_a W_b\), \(q = 2 [F_a^{-1}g - W_a F_b^{-1}h]\), \(F_a^{-1}\) and \(F_b^{-1}\) being the operators of the form (4.16). Alternatively,
\[ f(x) = \sum_{j=0}^{\infty} \tilde{W}^j r(x), \quad x \neq b^*, \tag{5.17} \]

where \(\tilde{W} = W_b W_a\), \(r = 2 [F_b^{-1}h - W_b F_a^{-1}g]\).

**Proof.** To prove (5.16), it suffices to pass to the limit in (5.5), taking into account that, by Lemma 5.1, the remainder \((W^m f)(x)\) of the series (5.16) converges to zero for every \(x \neq a^*\). An alternative formula (5.17) then follows if we interchange \(a\) and \(b\), \(g\) and \(h\). \(\square\)

**Remark 5.3.** In the case \(k = 1\), a function \(f \in C(\mathbb{S}^n)\) can be reconstructed from the system \(F_a f = g\), \(F_b f = h\) as follows. By Lemma 5.1, \((W^m f)(x) \to f(b^*)\) as \(m \to \infty\). Hence
\[ f(x) = \sum_{j=0}^{\infty} q(T^j x) + f(b^*), \quad x \neq a^*, \quad T = \tau_b \tau_a, \tag{5.18} \]

where \(q(x) = 2 [(F_a^{-1}g)(x) - (F_b^{-1}h)(\tau_a x)]\). By (4.18),
\[ (F_a^{-1}g)(x) = \frac{1}{2} g(L_{a,x}), \quad (F_b^{-1}h)(\tau_a x) = \frac{1}{2} h(L_{b,\tau_a x}), \]
where the line $L_{a,x}$ passes through $a$ and $x$ and $L_{b,\tau_{a,b}}$ passes through $b$ and $\tau_{a,b}$. It follows that
\[ q(x) = g(L_{a,x}) - h(L_{b,\tau_{a,b}}). \] (5.19)

Similarly, $(\tilde{W}^m f)(x) \to f(a^*)$, and we have
\[ f(x) = \sum_{j=0}^{\infty} r(\tilde{T}^{j+1}x) + f(a^*), \quad x \neq b^*, \quad \tilde{T} = \tau_a \tau_b, \] (5.20)
\[ r(x) = h(L_{b,x}) - g(L_{a,\tau_{a,b}}). \] (5.21)

The series (5.18) and (5.20) reconstruct $f$ up to unknown additive constants $f(a^*)$ or $f(b^*)$, where $a^*$ and $b^*$ are the endpoints of the chord through $a$ and $b$. However, complete reconstruction is still possible, if we apply symmetrization, by summing (5.18) and (5.20). This gives the following result.

**Theorem 5.4.** Let $k = 1$. Then
\[ 2f(x) = \sum_{j=0}^{\infty} q(T^{j+1}x) + \sum_{j=0}^{\infty} r(\tilde{T}^{j+1}x) + F_a(L_{a,b}), \quad x \neq a^*, b^*, \] (5.22)
where $q$ and $r$ are defined by (5.19) and (5.21), respectively, $L_{a,b}$ is the line through $a$ and $b$, and $F_a(L_{a,b}) = f(a^*) + f(b^*) (= F_b(L_{a,b}))$ is known. The values of $f$ at the points $a^*$ and $b^*$ can be reconstructed by continuity.

### 6. Norm Convergence of the Reconstructing Series

Reconstruction of $f$ by the pointwise convergent series (5.16) and (5.17) gives a little possibility to control the accuracy of the reconstruction because the rate of the pointwise convergence depends on the point. Therefore, it is natural to look at the convergence in certain normed spaces. In this section, we explore such convergence in the spaces $C(S^n)$ and $L^p(S^n)$. As above, we keep the notation $a^*$ and $b^*$ for the endpoints of the chord through $a$ and $b$.

Consider the most interesting case $k > 1$. By (5.5), the convergence of the series (5.16) to $f$ is equivalent to convergence of its remainder $W^m f$ to 0 as $m \to \infty$. Thus in the following, we confine ourselves to the behavior of $W^m f$.

We first note that the series (5.16) may diverge at the point $a^*$. Indeed, because $(W^m f)(a^*) = f(T^{m+1}a^*) \prod_{j=0}^{m} \rho(T^j a^*)$ and $a^*$ is a fixed
point of the mapping $T$, we have

$$(W^mf)(a^*) = \rho(a^*)^{m+1} f(a^*), \quad \rho(a^*) = \left[ \frac{(1 - |a|^2)(1 - |b|^2)}{|a - a^*||b - b^*|} \right]^{k-1}.$$ 

Suppose that $a$ and $b$ are symmetric with respect to the origin and $|a| = |b| = 1/2$. Then

$$\rho(a^*) = \left[ \frac{(1 + |a|)(1 + |b|)}{(1 - |a|)(1 - |b|)} \right]^{k-1} = 9^{k-1},$$

and therefore $(W^mf)(a^*) = 9^{(k-1)(m+1)} f(a^*) \to \infty$ as $m \to \infty$ whenever $f(a^*) \neq 0$. The latter means that if $f(a^*) \neq 0$, then the series (5.16) diverges at $a^*$ and its uniform convergence on the entire sphere fails. Below it will be shown that the uniform convergence of this series fails for any $a, b \in \mathbb{B}^{n+1}$.

To understand the type of convergence, we need to take a deeper look at the dynamics of involved reflections.

6.1. Dynamics of the Double Reflection Mapping $T = \tau_b \tau_a$. We know that the trajectory $\{T^mx : m = 0, 1, 2, \ldots\}$ of any point $x \in S^n \setminus \{a^*\}$ converges to the point $b^*$, which is the endpoint of the chord containing $a$ and $b$. Let us specify the character of this convergence.

**Lemma 6.1.** The mapping $T = \tau_b \tau_a$ maps the punctured sphere $S^n_a = S^n \setminus \{a^*\}$ onto itself. The point $b^*$ is the attracting point of the dynamical system $T^m : S^n_a \to S^n_a$ uniformly on compact subsets, that is, for any open neighborhood $U \subset S^n_a$ of $b^*$ and any compact set $K \subset S^n_a$ there exists $\overline{m}$ such that $T^m K \subset U$ for all $m \geq \overline{m}$.

**Proof.** The first statement is obvious, because $Ta^* = a^*$ and $T^{-1}a^* = \tau_a \tau_b a^* = a^*$. The second statement follows by a standard argument for monotone pointwise convergence on compacts. In fact, it suffices to prove this statement for the sets $U$ and $K$ having the form

$$U = U_\varepsilon = B(b^*, \varepsilon), \quad K = K_\delta = S^n \setminus B(a^*, \delta),$$

where $B(a^*, \varepsilon)$ and $B(b^*, \delta)$ are geodesic balls in $S^n$ of sufficiently small radii.

The pointwise convergence yields that for any fixed $x_0 \in K_\delta$ there exists a number $m_0$ such that $T^{m_0}x_0 \in U_\varepsilon$. By continuity, the same is true for every $x$ in some neighborhood $V_{x_0}$ of $x_0$. Thus, the compact $K_\delta$ is covered by open sets $V_x$, $x \in K_\delta$, and therefore we can cover $K_\delta$ by a finite family $\{V_{x_1}, \ldots, V_{x_M}\}$. For each $x_i$, there is a number $m_i$ such that $T^{m_i}x_i \in U_\varepsilon$. Setting $\overline{m} = \max\{m_1, \ldots, m_M\}$, we have

$$T^{\overline{m}}K_\delta \subset U_\varepsilon.$$
A simple geometric consideration shows that the sequence of compacts \( T^{m+1}_K \delta \) monotonically decreases, i.e., \( T^{m+1}_K \delta \subset T^m_K \delta \). Hence
\[
T^m_K \delta \subset U_\varepsilon \quad \text{for all } m \geq \overline{m}.
\]

\[\square\]

6.2. Uniform Convergence on Compact Subsets of the Punctured Sphere.

**Theorem 6.2.** If \( f \in C(S^n) \), then the series (5.16) converges to \( f \) on the punctured sphere \( S^n \setminus \{a^*\} \) uniformly on compact subsets.

**Proof.** We need to prove that the remainder \( (W^m f)(x) \) of the series (5.16) uniformly converges to 0 on compact subsets of \( S^n \setminus \{a^*\} \). By (5.8),
\[
(W^m f)(x) = \omega_m(x) f(T^{m+1}_K x), \quad \omega_m(x) = \prod_{j=0}^{m} \rho(T^j x).
\]
Because \( \rho(b^*) < 1 \) (see (5.15)), for a fixed \( \gamma \) satisfying \( \rho(b^*) < \gamma < 1 \), there is an open neighborhood \( U \subset S^n \setminus \{a^*\} \) of the point \( b^* \) such that \( 0 < \rho(y) < \gamma \) for all \( y \in U \). On the other hand, Lemma 6.1 says that there exists \( \overline{m} \) such that \( T^m_K \delta \subset U \) for \( m \geq \overline{m} \) and hence \( 0 < \rho(T^m x) < \gamma \) for all \( x \in K_\delta \) and \( m \geq \overline{m} \). Thus
\[
\omega_m(x) \leq \gamma^{m-\overline{m}} \max_{x \in K_\delta} \overline{m} \prod_{j=0}^{m} \rho(T^j x)
\]
for all \( m \geq \overline{m} \) and all \( x \in K_\delta \). It follows that \( \omega_m(x) \to 0 \) as \( m \to \infty \) uniformly on \( K_\delta \). Since \( |f(T^m x)| \leq \|f\|_{C(S^n)} \), we conclude that \( W^m f \to 0 \) uniformly on \( K_\delta \). This gives the result. \(\square\)

6.3. \( L^p \)-Convergence.

**Lemma 6.3.** The operators \( W_a, W_b, W = W_a W_b \), and \( \tilde{W} = W_b W_a \) are isometries of the space \( L^{p_0}(S^n) \) with \( p_0 = n/(k - 1) \).

**Proof.** The statement about \( W_a \) follows from (2.13), which reads
\[
\int_{S^n} (\rho_a(x))^{p_0} f(\tau_a x) dx = \int_{S^n} f(x) dx.
\]
The equality holds for any \( f \in L^1(S^n) \) and therefore, if \( f \in L^{p_0}(S^n) \), then, using \( |f(x)|^{p_0} \) instead of \( f \), we obtain \( \|W_a f\|_{p_0} = \|f\|_{p_0} \). The statement for \( W_b \) follows analogously. The operators \( W \) and \( \tilde{W} \) are also isometries, as the products of two isometries. \(\square\)
Theorem 6.4. Let $f \in C(S^n)$, $p_0 = n/(k - 1)$. The series (5.16) and (5.17) converge to $f$ in the norm of $L^p(S^n)$ for any $1 \leq p < p_0$. The convergence to $f$ fails in any space $L^p(S^n)$ with $p_0 \leq p \leq \infty$.

Proof. It is clear that $f$ belongs to $L^p(S^n)$ for any $1 \leq p \leq \infty$. Fix $\delta > 0$ and consider the function $W^m f = (W_a W_b)^m f$. Suppose that $p < p_0$ and set $r = p_0/p > 1$. We write

$$\|W^m f\|_p = \int_{B(a^*, \delta)} |(W^m f)(x)|^p \, dx + \int_{K_\delta} |(W^m f)(x)|^p \, dx$$

$$= I_1(m, \delta) + I_2(m, \delta), \quad (6.1)$$

where, as above, $K_\delta = S^n \setminus B(a^*, \delta)$. By Hölder’s inequality,

$$I_1(m, \delta) \leq \left( \int_{B(a^*, \delta)} \left( |(W^m f)(x)|^p \right)^r \, dx \right)^{1/r} \left( \int_{B(a^*, \delta)} \, dx \right)^{r/(r-1)}$$

Owing to Lemma 6.3, the operator $W^m$ preserves the $L^{p_0}$-norm, and therefore

$$\left( \int_{B(a^*, \delta)} \left( |(W^m f)(x)|^p \right)^r \, dx \right)^{1/r} = \left( \int_{B(a^*, \delta)} |(W^m f)(x)|^{p_0} \, dx \right)^{1/r}$$

$$\leq \|W^m f\|_p^{p_0/r} = \|f\|_p^{p_0}.$$ 

Hence

$$I_1(m, \delta) \leq A(\delta)^{r/(r-1)} \|f\|_p^{p_0}, \quad (6.2)$$

where $A(\delta)$ is the $n$-dimensional surface area of the geodesic ball $B(a^*, \delta)$. For the second integral in (6.1) we have

$$I_2(m, \delta) < \sigma_n \sup_{x \in K_\delta} |(W^m f)(x)|^p, \quad (6.3)$$

where $\sigma_n$ is the area of the unit sphere $S^n$.

Now we fix $\varepsilon > 0$. Using (6.2), let us choose $\delta > 0$ so that $I_1(m, \delta) < \varepsilon/2$ for all $m \geq 0$. By Theorem 6.2, the inequality (6.3) implies that there exists $\tilde{m} = \tilde{m}(\delta)$ such that $I_2(m, \delta) < \varepsilon/2$ for all $m \geq \tilde{m}$. Hence, by (6.1), $\|W^m f\|_p^{p_0} < \varepsilon$ for $m \geq \tilde{m}$, and therefore $W^m f$ tends to 0 as $m \to \infty$ in the $L^{p_0}$-norm. The latter gives the desired convergence of the series (5.16).

On the other hand, if $p > p_0$, then, by Hölder’s inequality, $\|f\|_p = \|W^m f\|_p \leq c \|W^m f\|_p$, $c = \text{const} > 0$. It follows that the $L^p$-norm of the remainder $W^m f$ of the series does not tend to 0 as $m \to \infty$, unless $f = 0$.

The proof for $\tilde{W} f$ is similar. \qed
Remark 6.5. The iterative method in terms of the series (5.16) and (5.17) does not provide a uniformly convergent reconstruction of continuous functions $f$. The reconstruction is guaranteed only in the $L^p$-norm with $1 \leq p < p_0 = n/(k-1)$. The case $p = 1$ works for all $1 < k \leq n$. The critical exponent $p_0$ is always greater than 1, but it is close to 1 (if $n$ is big enough) for operators $F_a$ over hyperplane sections, that is, when $k = n$ and $p_0 = 1 + 1/(n-1)$. In this case, for instance, the $L^2$-convergence fails because $p_0$ never exceeds 2. On the other hand, the less is the dimension of the sections, the greater exponent $p$ of the $L^p$-convergence can be chosen.

7. Appendix. Proof of Lemma 3.1

We recall that our aim is to prove the equality

$$\lim_{\lambda \to k-n-1} (C_\lambda f)(\xi) = (1 - |\xi'|^2)^{-1/2} (F_a f)(\xi), \quad |\xi'| < 1,$$

(7.1)

where $f \in C(S^n)$, $C_\lambda$ and $F_a$ are defined by (3.4) and (3.2), respectively. For the sake of completeness and future purposes, the case $|\xi'| \geq 1$ will also be investigated.

We set $\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^{n-k+1}$,

$$\mathbb{R}^k = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_k, \quad \mathbb{R}^{n-k+1} = \mathbb{R}e_{k+1} \oplus \ldots \oplus \mathbb{R}e_{n+1},$$

e_i being coordinate unit vectors; $S^{k-1}$ is the unit sphere in $\mathbb{R}^k$, $S^{n-k}$ is the unit sphere in $\mathbb{R}^{n-k+1}$,

$$\xi_0 = \begin{bmatrix} 0 \\
I_{n-k+1} \end{bmatrix}.$$

Given $\xi \in \text{St}(n+1, n+1-k)$, we denote by $r_\xi$ an arbitrary rotation satisfying $r_\xi \xi_0 = \xi$ and set $f_\xi(x) = f(r_\xi x)$. Changing variable $x \to r_\xi x$, we have

$$(C_\lambda f)(\xi) = \gamma_{n,k}(\lambda) \int_{S^n} f_\xi(x) |\xi_0' x - h|^\lambda dx, \quad h = \xi' a.$$

Then we pass to bispherical coordinates

$$x = \begin{bmatrix} \varphi \sin \theta \\
\psi \cos \theta \end{bmatrix}, \quad \varphi \in S^{k-1}, \quad \psi \in S^{n-k}, \quad 0 \leq \theta \leq \pi/2,$$

(7.2)

$$dx = \sin^{k-1} \theta \cos^{n-k} \theta d\theta d\varphi d\psi,$$
and set $s = \cos \theta$. This gives

$$
(C_\lambda^a f)(\xi) = \gamma_{n,k}(\lambda) \int_0^1 s^{n-k} (1 - s^2)^{(k-2)/2} ds \int_{S^{k-1}} d\varphi
$$

$$
\times \int_{S^{n-k}} f_\xi \left( \left[ \varphi \sqrt{1 - s^2} \right] \right) |s\psi - h|^{\lambda} d\psi
$$

$$
= \gamma_{n,k}(\lambda) \int_{\mathbb{R}^{n-k+1}} H(y) |y - h|^{\lambda} dy,
$$

(7.3)

where

$$
H(y) = (1 - |y|^2)^{(k-2)/2} \int_{S^{k-1}} f_\xi \left( \left[ \varphi \sqrt{1 - |y|^2} \right] \right) d\varphi
$$

if $|y| \leq 1$ and $H(y) = 0$, otherwise. The integral (7.3) is the well-known Riesz potential in $\mathbb{R}^{n-k+1}$. We recall (see, e.g., [18, Chapter 3]) that the classical Riesz potential of a function $g$ on $\mathbb{R}^n$ is defined by

$$
(I_\alpha^n g)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} g(y) \frac{dy}{|x - y|^{n-\alpha}}; \quad \gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n - \alpha)/2)}.
$$

(7.4)

It is known that $\lim_{\alpha \to 0} (I_\alpha^n g)(x) = g(x)$ if $g$ is good enough; see, e.g., [18, Lemma 3.2]. A generalization of this fact can be found in [19, Lemma 5.2]. Thus we have

$$
(C_\lambda^a f)(\xi) = (I_{n-k+1}^{\lambda + n-k+1} H)(\xi'), \quad \lim_{\lambda \to k-n-1} (C_\lambda^a f)(\xi) = H(\xi),
$$

where

$$
H(\xi') = (1 - |\xi'|^2)^{(k-2)/2} \int_{S^{k-1}} f_\xi \left( \left[ \varphi \sqrt{1 - |\xi'|^2} \right] \right) d\varphi
$$

(7.5)

if $|\xi'| \leq 1$ and $H(\xi') = 0$, otherwise.

Suppose that $|\xi'| < 1$. If

$$
x = r_\xi \left[ \varphi \sqrt{1 - |\xi'|^2} \right], \quad \varphi \in S^{k-1},
$$

then...
then $x - a$ lies in the subspace perpendicular to $\xi$. Indeed, because $\varphi$ is orthogonal to $\xi_0$, we have

$$
\xi'(x - a) = \xi' r_\xi \left[ \varphi \begin{bmatrix} \sqrt{1 - |\xi'|a|^2} & 0 \\ 0 & 0 \end{bmatrix} \right] + \xi' r_\xi \left[ \begin{bmatrix} 0 \\ \xi' a \end{bmatrix} \right] - \xi' a 
$$

$$
= \xi_0' r_\xi r_\xi \left[ \varphi \begin{bmatrix} \sqrt{1 - |\xi'|a|^2} & 0 \\ 0 & 0 \end{bmatrix} \right] + \xi_0' r_\xi r_\xi \left[ \begin{bmatrix} 0 \\ \xi' a \end{bmatrix} \right] - \xi' a 
$$

$$
= \xi_0' \left[ \varphi \begin{bmatrix} \sqrt{1 - |\xi'|a|^2} \\ 0 \end{bmatrix} \right] + \xi_0' \left[ \begin{bmatrix} 0 \\ \xi' a \end{bmatrix} \right] - \xi' a = 0.
$$

Further, the integration in (7.5) is performed over the $(k-1)$-dimensional sphere of radius $\sqrt{1 - |\xi'|a|^2}$. Switching to surface area measure, we can write (7.5) as

$$
H(\xi'|a) = (1 - |\xi'|a|^2)^{-1/2} \int_{\{x \in S^n: \xi'(x-a) = 0\}} f(x) \, d\sigma(\eta),
$$

which gives (7.1).

If $|\xi'|a| > 1$, then $H(\xi'|a) = 0$ and $\lim_{\lambda \to k\cdot n-1} (C_\lambda f)(\xi) = 0$. In this case, the plane $\xi'(x - a) = 0$ does not meet the sphere $S^n$. If $|\xi'|a| = 1$, the plane $\xi'(x - a) = 0$ is tangent to $S^n$ and the result depends on $k$. If $k > 2$ our limit is zero. If $k = 2$ then

$$
H(\xi'|a) = \int_{S^{k-1}} f(\left[ r_\xi \begin{bmatrix} 0 \\ \xi' a \end{bmatrix} \right]) \, d\varphi = \sigma_{k-1} f(P_\xi a),
$$

where $P_\xi a$ is the orthogonal projection of $a$ onto the subspace spanned by $\xi$, i.e., $P_\xi a \in S^n \cap \text{span}(\xi)$.

The proof is complete.

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