The three-loop $\beta$-function of SU(N) lattice gauge theories with Wilson fermions.

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Abstract

We calculate the third coefficient of the lattice $\beta$-function associated with the Wilson formulation for both gauge fields and fermions. This allows us to evaluate the three-loop correction (linear in $g_0^2$) to the relation between the lattice $\Lambda$-parameter and the bare coupling $g_0$, which is important in order to verify asymptotic scaling predictions. Our calculation also leads to the two-loop relation between the coupling renormalized in the $\overline{\text{MS}}$ scheme and $g_0$.

The original version of this paper contained a numerical error in one of the diagrams, which has now been corrected. The calculations, as well as the layout of the paper have remained identical, but there are some important changes in the numerical results.

Keywords: Lattice QCD, Lattice gauge theory, Beta function, Asymptotic scaling, Lattice perturbation theory, Running coupling constant.

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I. INTRODUCTION

In lattice formulations of asymptotically free field theories, such as QCD, the relevant region for the continuum physics is that where scaling is verified. Scaling in the continuum limit requires that all RG-invariant dimensionless ratios of physical quantities approach their continuum value with non-universal corrections which get depressed by integer powers of the inverse correlation length. This can be equally stated by introducing a lattice scale $\Lambda_L$, which is a particular solution of the RG equation

$$\left(-a \frac{\partial}{\partial a} + \beta_L(g_0) \frac{\partial}{\partial g_0}\right) \Lambda_L = 0$$  \hspace{1cm} (1)

($a$ is the lattice spacing), i.e.

$$a\Lambda_L = \exp \left[ - \int_{g_0}^{g^*} \frac{dg}{\beta_L(g)} \right],$$  \hspace{1cm} (2)

and requiring that the ratio of any RG invariant quantity to the appropriate power of $\Lambda_L$ approach a constant in the continuum limit $g_0 \to 0$. $\beta_L(g_0)$ is the lattice $\beta$-function

$$\beta_L(g_0) = -a \frac{dg_0}{da} \mid_{\text{physical quantities}},$$  \hspace{1cm} (3)

which tells us how the bare coupling and the lattice spacing must be changed in order to keep physical quantities fixed.

In the asymptotic region $g_0 \to 0$ one may perform a weak coupling expansion of the $\beta$-function

$$\beta_L(g_0) = -b_0 g_0^3 - b_1 g_0^5 - b_2 g_0^7 + ...,$$  \hspace{1cm} (4)

where, in $SU(N)$ gauge theory with $N_f$ fermion species,

$$b_0 = \frac{1}{(4\pi)^2} \left( \frac{11}{3} N - \frac{2}{3} N_f \right),$$  \hspace{1cm} (5)

$$b_1 = \frac{1}{(4\pi)^4} \left[ \frac{34}{3} N^2 - N_f \left( \frac{13}{3} N - \frac{1}{N} \right) \right].$$  \hspace{1cm} (6)

Consequently one has the asymptotic relation

$$a\Lambda_L = \exp \left( - \frac{1}{2 b_0 g_0^2} \right) (b_0 g_0^2)^{-1/2b_0^2} \left[ 1 + q g_0^2 + O \left( g_0^4 \right) \right],$$  \hspace{1cm} (7)

where

$$q = \frac{b_1^2 - b_0 b_2}{2 b_0^3}.$$  \hspace{1cm} (8)

The verification of the two-loop behavior of $\Lambda_L$ is usually called asymptotic scaling. Knowledge of the first correction to the two-loop approximation of $\Lambda_L$ is important in order
to verify the asymptotic prediction (7). We recall that Monte Carlo simulations using the Wilson action are actually performed at \( g_0 \approx 1 \), and so deviations from the two-loop formula might not be negligible. \( b_L^2 \) has been recently calculated for the pure gauge theory \([1,2]\). In this paper we extend this calculation to the full theory including fermions in the Wilson formulation.

The knowledge of \( b_L^2 \) can be also used to improve the perturbative relation between the coupling \( g \) renormalized in the \( \overline{\text{MS}} \) (modified minimal subtraction renormalization) scheme and the bare lattice coupling \( g_0 \), which is useful in calculations such as studies concerning running couplings (see e.g. Refs. \([3–6]\)). Actually, as we shall see, in order to calculate \( b_L^2 \), we will first compute the two-loop relation between \( g \) and \( g_0 \). This, supplemented with knowledge of the three-loop coefficient of the \( \beta \)-function in the \( \overline{\text{MS}} \) scheme, allows us to derive \( b_L^2 \).

In Sec. II we describe the method we employed to calculate \( b_L^2 \). We present the results of our lattice perturbative calculations, from which we derive the three-loop coefficient \( b_L^2 \) associated with the Wilson formulation of both gauge fields and fermions. Sec. III describes in some detail the necessary lattice perturbative calculations we performed in the background gauge framework.

II. PERTURBATIVE CALCULATION OF THE \( \beta \)-FUNCTION.

The lattice \( \beta \)-function depends on the lattice formulation considered. We recall that only the first two coefficients of its perturbative expansion are universal, i.e. \( b_0 \) and \( b_1 \). We consider the Wilson formulation for both the pure gauge theory and fermions:

\[
S_L = \frac{1}{g_0^2} \sum_{x,\mu,\nu} \text{Tr} \left[ 1 - U_{\mu\nu}(x) \right] + \sum_f \sum_x (4r + m_{f,0}) \bar{\psi}_f(x) \psi_f(x) +
- \frac{1}{2} \sum_f \sum_x \left[ \bar{\psi}_f(x) (r - \gamma_\mu) U_\mu(x) \psi_f(x + \hat{\mu}) + \bar{\psi}_f(x + \hat{\mu}) (r + \gamma_\mu) U_\mu(x) \psi_f(x) \right],
\]

where \( U_{\mu\nu}(x) \) is the usual product of link variables \( U_\mu(x) \) along the perimeter of a plaquette originating at \( x \) in the positive \( \mu-\nu \) directions; \( r \) is the Wilson parameter; \( f \) is a flavor index.

The lattice \( \beta \)-function is independent of the renormalized fermionic masses. Indeed, since it is dimensionless, its mass dependence may come only through the combination \( x \equiv am \), where \( m \) is the renormalized fermionic mass. Since lattice gauge theories with Wilson fermions are renormalizable \([7]\), the lattice \( \beta \)-function must have a well defined zero lattice-spacing limit. Thus any dependence on \( x \), and therefore on \( m \), must disappear in that limit. In the presence of fermions with different masses, one may also assume a well defined chiral limit in order to complete this argument. So in order to calculate \( \beta_L(g_0) \) we can restrict ourselves to the case of \( N_f \) fermions with zero renormalized mass.

The relation between the bare lattice coupling \( g_0 \) and the renormalized coupling \( g \) defined in a generic renormalization scheme is

\[
g_0 = Z_g(g_0, a\mu)g,
\]

where \( \mu \) indicates a renormalization scale. \( Z_g(g_0, a\mu) \) depends on the renormalization scheme. The lattice \( \beta \)-function (9) can be derived from \( Z_g \) by
Thus the evaluation of $b_2^L$ would require the calculation of $Z_g(g_0, a\mu)$ up to three loops on the lattice.

The lattice calculation can be simplified by exploiting the following considerations. Let us choose the $\overline{\text{MS}}$ procedure as renormalization scheme in which the renormalized coupling $g$ is defined. The corresponding $\beta$-function

$$\beta(g) = \mu \frac{dg}{d\mu} \bigg|_{\text{bare quantities}} = -b_0 g^3 - b_1 g^5 - b_2 g^7 + ...$$

is known to three loops. By writing the $\overline{\text{MS}}$ $\beta$-function in the form

$$\beta(g) = -g_0 \mu \frac{d}{d\mu} \ln Z_g(g_0, a\mu) \bigg|_{a,g},$$

and by comparing it with Eq. (11), one can easily derive the relation

$$\beta^L(g_0) = \left( 1 - g_0^2 \frac{\partial}{\partial g_0} \ln Z_g^2 \right)^{-1} Z_g^2 \beta(g_0 Z_g^{-1}),$$

which is valid to all orders of perturbation theory. We write $Z_0^2$ as

$$Z_g(g_0, a\mu)^2 = 1 + L_0(a\mu) g_0^2 + L_1(a\mu) g_0^4 + O(g_0^6)$$

where $L_0(x) = 2b_0 \ln x + l_0$ and $L_1(x) = 2b_1 \ln x + l_1$. The constant $l_0$ is related to the ratio of the $\Lambda$ parameters associated with the particular lattice regularization and the $\overline{\text{MS}}$ renormalization scheme:

$$l_0 = 2b_0 \ln (\Lambda_L/\Lambda_{\overline{\text{MS}}}).$$

Its value as a function of $N$ and $N_f$ can be inferred from Refs. [8,9]. It has been obtained with improved accuracy in Ref. [11] (pure gluons) and in the present paper (fermionic case). For $r = 1$

$$l_0 = \frac{1}{8N} - 0.16995599N + 0.00669600N_f.$$

By expanding Eq. (14) in powers of $g_0^2$, one finds the well-known result that the coefficients $b_0$ and $b_1$ are the same in $\beta^L(g_0)$ and $\beta(g)$. Furthermore

$$b_2^L = b_2 - b_1 l_0 + b_0 l_1.$$  

Thus, since $b_2$ is known [10],

$$b_2 = \frac{1}{(4\pi)^6} \left[ \frac{2857}{54} N^3 + N_f \left( -\frac{1709N^2}{54} + \frac{187 N}{36} + \frac{1}{4N^2} \right) + N_f^2 \left( \frac{56N}{27} - \frac{11}{18N} \right) \right],$$

(19)
the evaluation of $b_2^L$ requires only a two-loop calculation on the lattice, i.e. the calculation of the constant $l_1$.

The computation of the $\overline{\text{MS}}$ renormalization constant $Z_g$ is easier in the background field gauge. In fact, this renormalization constant has a simple relationship with the background field renormalization constant $Z_A$ \cite{11},

$$Z_A(g_0, a \mu)Z_g(g_0, a \mu)^2 = 1.$$ \hfill (20)

Lüscher and Weisz \cite{12} have shown that pure lattice gauge theory with a background gauge field is renormalizable to all orders in perturbation theory. No additional counterterms are required besides those already needed in the absence of a background field. Their argument, based on renormalizability of pure lattice gauge theory, BRS, background gauge and shift symmetries of the lattice functional integral, can be extended to full lattice QCD in the Wilson formulation. An essential point is the renormalizability of lattice gauge theory with Wilson fermions proved by Reisz to all orders in perturbation theory \cite{7}.

As a consequence of the relation (20), in order to calculate $Z_g$ we need to compute only the two-loop self-energy of the background field on the lattice. In the background field formulation \cite{13} the links are written as

$$U_\mu(x) = U_\mu^q(x)U_\mu^{\text{cl}}(x),$$

$$U_\mu^q(x) \equiv e^{ig_0Q_\mu(x)},$$

$$U_\mu^{\text{cl}}(x) \equiv e^{iag_0A_\mu(x)},$$

\hfill (21)

where $Q_\mu(x) = T^cQ_\mu^c(x)$ and $A_\mu(x) = T^cA_\mu^c(x)$ are the quantum and background fields respectively. The following gauge-fixing term preserves gauge invariance of the background field

$$S_{gf} = \lambda_0 \sum_{\mu, \nu} \sum_x \text{Tr} \left( D^-_\mu Q_\mu(x)D^-_\nu Q_\nu(x) \right),$$

\hfill (22)

$$D^-_\mu Q_\nu(x) \equiv U_\mu^{\text{cl}}(x - \hat{\mu})Q_\nu(x - \hat{\mu})U_\mu^{\text{cl}}(x - \hat{\mu}) - Q_\nu(x).$$

\hfill (23)

We worked in the Feynman gauge, $\lambda_0 = 1$. This gauge fixing produces the following Fadeev Popov action for the ghosts fields $\omega$ and $\overline{\omega}$

$$S_{gh} = 2 \sum_{x} \sum_{\mu} \text{Tr} \left( D^+_\mu \omega(x) \right)\left[ D^+_\mu \omega(x) + ig_0 [Q_\mu(x), \omega(x)] + \frac{1}{2}ig_0 \left[ Q_\mu(x), D^+_\mu \omega(x) \right] \right] \left[ Q_\mu(x), D^+_\mu \omega(x) \right] + \cdots,$$

\hfill (24)

$$D^+_\mu \omega(x) \equiv U_\mu^{\text{cl}}(x)\omega(x + \hat{\mu})U_\mu^{\text{cl}}(x) - \omega(x).$$

\hfill (25)

Finally the change of integration variables from links to vector fields yields a jacobian that can be rewritten as a new term $S_m$ in the action:

$$S_m = \frac{1}{12} Ng_0^2 \sum_x \sum_{\mu} \text{Tr} Q_\mu(x)Q_\mu(x) + \cdots$$

\hfill (26)

In all the expansions above we have written only the relevant terms for our two-loop computation.
The full action is therefore

\[ S = S_L + S_{gf} + S_{gh} + S_m. \]  \hspace{1cm} (27)

The vertices needed in perturbation theory are obtained as usual by expanding the exponential of the action \( S \) in powers of the fields.

As in Ref. [1], we set the following notation. We rewrite the renormalized one-particle irreducible two-point functions of the background and quantum fields as

\[ \Gamma_{AA}^R(p)_{\mu\nu}^{ab} = -\delta^{ab} \left( \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) \left( 1 - \nu(p) \right) / g^2, \]  \hspace{1cm} (28)

\[ \Gamma_{QQ}^R(p)_{\mu\nu}^{ab} = -\delta^{ab} \left[ \left( \delta_{\mu\nu} p^2 - p_\mu p_\nu \right) \left( 1 - \omega(p) \right) + \lambda p_\mu p_\nu \right], \]  \hspace{1cm} (29)

\[ \nu(p) = \sum_{l=1}^\infty g^2 l \nu^{(l)}(p), \]  \hspace{1cm} (30)

\[ \omega(p) = \sum_{l=1}^\infty g^2 l \omega^{(l)}(p). \]  \hspace{1cm} (31)

Correspondingly on the lattice

\[ \sum_\mu \Gamma_{AA}^L(p)_{\mu\nu}^{ab} = -\delta^{ab} 3p^2 \left[ 1 - \nu(p) \right] / g_0^2, \]  \hspace{1cm} (32)

\[ \sum_\mu \Gamma_{QQ}^L(p)_{\mu\nu}^{ab} = -\delta^{ab} \left[ 3 \left( 1 - \omega(p) \right) + \lambda_0 \right], \]  \hspace{1cm} (33)

\[ \nu(p) = \sum_{l=1}^\infty g_0^2 \nu^{(l)}(p), \]  \hspace{1cm} (34)

\[ \omega(p) = \sum_{l=1}^\infty g_0^2 \omega^{(l)}(p). \]  \hspace{1cm} (35)

The bare and renormalized functions are related by

\[ [1 - \nu_R(p, \mu, g)] = Z_A [1 - \nu(p, a, g_0)], \]  \hspace{1cm} (36)

therefore

\[ Z_g^2 = \frac{1 - \nu(p, a, g_0)}{1 - \nu_R(p, \mu, g)}. \]  \hspace{1cm} (37)

In order to compare continuum against lattice expressions, one also needs to renormalize the gauge parameter according to \( \lambda = Z_Q \lambda_0 \) \hspace{1cm} [11], where \( Z_Q \) is the renormalization constant of the quantum field. \( Z_Q \) is needed only to one loop and can be evaluated by imposing

\[ [1 - \omega_R(p, \mu, g)] = Z_Q [1 - \omega(p, a, g_0)]. \]  \hspace{1cm} (38)

One can fix \( \lambda_0 = 1 \), but then one must take \( \lambda = Z_Q \) as corresponding renormalized gauge parameter.

The \( \overline{\text{MS}} \) renormalized functions necessary for the calculation of \( Z_g \) to two loops are
\[ \nu_R^{(1)}(p, \lambda) = \frac{N}{16\pi^2} \left[ -\frac{11}{3} \ln \frac{p^2}{\mu^2} + \frac{205}{36} + \frac{3}{2\lambda} + \frac{1}{4\lambda^2} \right] + \frac{N_f}{16\pi^2} \left[ \frac{2}{3} \ln \frac{p^2}{\mu^2} - \frac{10}{9} \right], \]  
(39)

\[ \omega_R^{(1)}(p, \lambda) = \frac{N}{16\pi^2} \left[ \left( -\frac{13}{6} + \frac{1}{2\lambda} \right) \ln \frac{p^2}{\mu^2} + \frac{97}{36} + \frac{1}{2\lambda} + \frac{1}{4\lambda^2} \right] + \frac{N_f}{16\pi^2} \left[ \frac{2}{3} \ln \frac{p^2}{\mu^2} - \frac{10}{9} \right], \]  
(40)

\[ \nu_R^{(2)}(p, \lambda = 1) = \frac{N^2}{(16\pi^2)^2} \left[ -8 \ln \frac{p^2}{\mu^2} + \frac{577}{18} - 6\zeta(3) \right] + \frac{N_f}{(16\pi^2)^2} \left[ N \left( 3 \ln \frac{p^2}{\mu^2} - \frac{401}{36} \right) + \frac{1}{N} \left( -\ln \frac{p^2}{\mu^2} + \frac{55}{12} - 4\zeta(3) \right) \right]. \]  
(41)

Details of the lattice calculations will be given in the next section. Here we just list the results. We report only the fermionic contributions for \( r = 1 \) which must be added to the pure gauge functions calculated in Refs. [1,2].

\[ \nu^{(1)}(p, \lambda_0 = 1) = \nu^{(1)}(p, \lambda_0 = 1) \mid_{N_f=0} + N_f \left[ \frac{1}{24\pi^2} \ln(a^2p^2) + k_{1f} \right], \]  
(42)

\[ \omega^{(1)}(p, \lambda_0 = 1) = \omega^{(1)}(p, \lambda_0 = 1) \mid_{N_f=0} + N_f \left[ \frac{1}{24\pi^2} \ln(a^2p^2) + k_{1f} \right], \]  
(43)

\[ \nu^{(2)}(p, \lambda_0 = 1) = \nu^{(2)}(p, \lambda_0 = 1) \mid_{N_f=0} + N_f \left[ \frac{1}{(16\pi^2)^2} \left( 3N - \frac{1}{N} \right) \ln(a^2p^2) + k_{2f} \frac{1}{N} + k_{3f}N \right]. \]  
(44)

For \( r = 1 \) we found

\[ k_{1f} = -0.013732194(5), \]
\[ k_{2f} = 0.0011877(14), \]
\[ k_{3f} = -0.0013617(16). \]  
(45)

The origin and meaning of the errors in the above results will be explained in the next section.

We have now what we need to calculate \( Z_g \) to two loops using Eq. (37). The resulting two-loop constant \( l_1 \), cf. Eq. (13), is given by

\[ l_1 = -\frac{3}{128N^2} + \frac{0.018127763}{N} - \frac{0.0079101185}{N^2} \]
\[ +N_f \left[ -0.0011967(14) \frac{1}{N} + 0.0009998(16)N \right]. \]  
(46)

Through Eq. (18) one can then obtain \( b^L_2 \). For example for \( N = 3 \) we find

\[ b^L_2 = -0.0015998323 + 0.0000799(4)N_f - 0.00000605(2)N^2_f. \]  
(47)

Knowledge of \( b^L_2 \) allows us to evaluate the coefficient \( q \) of the linear correction to asymptotic scaling, cf. Eq. (8). For \( N = 3 \), we obtain:

\[ \text{1} \] Here and below, results pertaining to the pure gluonic case can be obtained with higher precision from Ref. [3].
\[ q \simeq 0.1896 \quad (N_f = 0), \quad q \simeq 0.2160 \quad (N_f = 2), \quad q \simeq 0.2355 \quad (N_f = 3). \] (48)

These numbers show that for \( g_0 \simeq 1 \), which is a typical value of the bare coupling where simulations are nowadays performed, the linear correction to asymptotic scaling cannot be ignored.

Finally, the relation between the \( \overline{\text{MS}} \) coupling \( \alpha \equiv g^2/(4\pi) \) and \( \alpha_0 \equiv g_0^2/(4\pi) \) can be easily read from Eq. (14):

\[ \alpha = \alpha_0 + d_1(\mu^2)\alpha_0^2 + d_2(\mu^2)\alpha_0^3 + O\left(\alpha_0^4\right), \] (49)

where, for \( r = 1 \),

\[ d_1(x) = -4\pi L_0(x) \]
\[ = -\frac{1}{2\pi} \left( \frac{11}{3} N - \frac{2}{3} N_f \right) \ln x - \frac{\pi}{2N} + 2.13573007N - 0.08414443(8) N_f, \] (50)

and

\[ d_2(x) = (4\pi)^2 \left[ L_0(x)^2 - L_1(x) \right] \]
\[ = d_1(x)^2 - \frac{1}{24\pi^2} \left[ 34N^2 - N_f \left( 13N - \frac{3}{N} \right) \right] \ln x \]
\[ + \frac{3\pi^2}{8N^2} - 2.8626216 + 1.2491158N^2 + N_f \left[ 0.18898(22) \frac{1}{N} - 0.15789(26) N \right]. \] (51)

For comparison we have also estimated \( b_L^f \) for \( r = 0.5 \) and \( r = 2.0 \). For \( N = 3 \), we obtain

\[ b_L^f - b_L^2 \mid_{N_f=0} \simeq +0.000092N_f - 0.0000033N_f^2 \] (52)

for \( r = 0.5 \), and

\[ b_L^f - b_L^2 \mid_{N_f=0} \simeq +0.0000079N_f - 0.0000039N_f^2 \] (53)

for \( r = 2.0 \).

### III. THE CALCULATION IN LATTICE PERTURBATION THEORY

#### i) Preliminaries

In this Section we proceed to describe the technical aspects in our calculation of the quantities \( \nu^{(1)}(p) \), \( \nu^{(2)}(p) \). The fermionic contributions in \( \omega^{(1)}(p) \) are identical to those in \( \nu^{(1)}(p) \).

Two diagrams containing fermions contribute to \( \nu^{(1)}(p) \), shown in Figure 1. There are 18 two-loop fermionic diagrams contributing to \( \nu^{(2)}(p) \), as well as two diagrams containing an insertion of the one-loop fermion mass counterterm; these are shown in Figure 2.

The algebra involving lattice quantities was performed using a symbolic manipulation package in Mathematica, developed by us in recent years. For the purposes of the present work, this package was extended to include fermions.
A first, relatively brief, part in the evaluation of the diagrams is the contraction. This is done completely automatically for diagrams with an arbitrary number of loops, once the types of vertices, as well as the “incidence matrix” for the diagram are specified. This step also includes: Complete reduction of color structures, Dirac matrices and tensor structures; exploiting permutation symmetry and lattice rotational invariance to keep the size of the expression down to a minimum; use of trigonometry to arrive at a canonical form. The resulting expression is a rational function of sines and cosines of the external \((p)\) and internal \((q,k)\) momenta.

**ii) Extracting the external momentum**

The next task is to make explicit the functional dependence of each diagram on \(p\). The two-loop amplitude \(\nu^{(2)}(p)\) can be written as:

\[
\nu^{(2)}(p) = \nu^{(2)}(p)|_{N_f=0} + \sum_i \nu_i(p)
\]  

(the index \(i\) runs over diagrams with fermions shown in Fig.2), where, generally,

\[
\hat{a}^2 \nu_i(p) = c_{0,i} + c_{1,i} a^2 \sum_{\mu} \frac{p_{\mu}^4}{p^2} + a^2 p^2 \left\{ c_{2,i} \left( \frac{\ln a^2 p^2}{(4\pi)^2} \right)^2 + c_{3,i} \frac{\ln a^2 p^2}{(4\pi)^2} + c_{4,i} \right\} + O((\hat{a} p)^4) \tag{55}
\]

\((p^2 = 4 \sum_{\mu} \sin^2(p_{\mu}/2))\). The dependence of \(c_{n,i}\) of \(N, N_f\) is:

\[
c_{n,i} = \left[ c_{n,i}^{(-1)}/N + c_{n,i}^{(1)} N \right] N_f
\]  

We note in passing that certain diagrams are infrared convergent only when taken in pairs: (7, 11), (8, 18), (9, 17); we have evaluated these accordingly, with due care taken to avoid divergences in intermediate results.

To extract the \(p\)-dependence, we first isolate the superficially divergent terms; these are responsible for the double logarithms. There are relatively few such terms, and in the pure gluonic case they all have been tabulated [2]. We can use these tables also in diagrams with fermions, applying successive subtractions of the type:

\[
\frac{1}{\bar{q}^2} = \frac{1}{\hat{q}^2} + \left( \frac{1}{\bar{q}^2} - \frac{1}{\hat{q}^2} \right)
\]  

where \(\bar{q}^2\) is the inverse fermionic propagator: \(\bar{q}^2 = (\hat{q}^2 r/2) + \sum_{\mu} \sin^2 q_{\mu}\). This leads to the tabulated expressions plus a series of superficially convergent terms.

The rest of the expression may still contain subdivergences, which can give rise to single logarithms. These are handled using successive subtractions of the type:

\[
\frac{1}{q + k^2} = \frac{1}{k^2} + \left( \frac{1}{q + k^2} - \frac{1}{k^2} \right)
\]  

The first term on the right hand side then yields factorized one-loop expressions, whose \(p\)-dependence is easily extracted, while the second term is more convergent. Finally, all
remaining terms, containing no (sub-)divergences can be evaluated by Taylor expansion in $ap$.

In our code, judicious choices for the right set of subtractions are applied automatically to each term. While the extraction of divergences is not particularly complicated conceptually, it leads to a great proliferation in the resulting number of terms. As an example, diagram 16 by itself leads to $\sim 100$ types of expressions; each type must then be numerically evaluated on its own (because it contains a different set of subtracted propagators) and contains typically up to some hundreds of terms.

**iii) Numerical integration**

At this stage, all the above types of expressions no longer contain $p$ and must be numerically integrated over the internal momenta. The integration is done in momentum space over finite lattices; an extrapolation to infinite size is then performed (see below).

For expressions with only gluonic propagators a coordinate space method was proposed in [14], yielding high precision results. Recently, this method was also used for some study cases involving fermions [15]. For the present calculation, involving fermionic and bosonic propagators with multiple subtractions, the coordinate space method becomes a bit more complicated; we may return to it in the future, if the need arises for more precision than presented in the present paper.

Fortran code for the numerical evaluation of each type of expression is then created automatically by our programs. The code is highly optimized: It avoids redundant evaluation of common subexpressions, of symmetric regions in momentum space, of $r$-independent subexpressions, etc. The code can be run for different values of $r$ and lattice size $L$. For the present paper we used $L \leq 30$; we present $r = 1$ and, for comparison, we also estimated $r = 0.5, r = 2.0$.

Extrapolation to infinite lattice size is of course a source of systematic error. To estimate this error, our procedure carries out automatically the following steps: First, different extrapolations are performed using a broad spectrum of functional forms of the type:

$$\sum_{i,j} e_{i,j} L^{-i} (\ln L)^j$$

For the $k^{th}$ such extrapolation, a deviation $d_k$ is calculated using several criteria for quality of fit. Finally, these deviations are used to assign weights $d_k^{-2}/(\sum_k d_k^{-2})$ to each extrapolation, producing a final value together with the error estimate. We have checked the validity of these estimates in cases where the exact answer was known (see also below), finding the estimate to be always correct.

**iv) Cancellations and cross checks**

Several constraints exist on the coefficients $c_{n,i}$. We have used these constraints as verifications both on the algebraic expressions and on the numerical results:

$c_{0,i}$: Gauge invariance requires

$$\sum_i c_{0,i} = 0.$$  \hfill (60)

We checked this property in three ways: Firstly, a formal, Ward-identity type derivation was performed, in which vertices with background fields at zero momentum were written in
terms of appropriate derivatives of inverse propagators. From this we find not only Eq. (60), but also some additional constraints:

\[ 2c_{0,1} + c_{0,3} = 0 \]
\[ c_{0,1} + c_{0,3} + c_{0,6} + c_{0,8} = 0 \]
\[ (c_{0,4} + c_{0,5})|_{N^2 = 2} = 0 \]
\[ c_{0,7} + c_{0,9} + c_{0,11} + \frac{1}{2}c_{0,16} + c_{0,17} = 0 \]
\[ c_{0,13} = 0 \]
\[ c_{0,19} + c_{0,20} = 0 \]
\[ (c_{0,12} + c_{0,18})(N^2 - 2) + c_{0,14}(N^2 - 1) - c_{0,15}(N^2 - 1)(N^2 - 2) = 0 \]

The above are true for any value of the fermion mass \( m \) and Wilson parameter \( r \). Secondly, all the above identities were checked by algebraic manipulation of the full expression for each of the coefficients. Thirdly, we substituted in each identity the numerical results for the coefficients, finding in all cases zero within the error estimates.

\( c_{1,i} \): The sum of these terms must vanish if Lorentz invariance is to be recovered in the continuum limit. Only diagrams 9 and 17 give nonvanishing contributions, and we checked in the same ways as above that their sum is zero.

\( c_{2,i} \): These coefficients must coincide with those of the continuum. We checked that this is so:

\[ c_{2,15} = \frac{1}{3} \frac{1}{N} N_f , \quad c_{2,16} = \frac{4}{3} N N_f , \quad c_{2,17} = -\frac{5}{3} N N_f , \quad c_{2,18} = \frac{1}{3} \frac{N^2 - 1}{N} N_f . \]

For all other diagrams: \( c_{2,i} = 0 \).

\( c_{3,i} \): The total contribution for single logarithms must coincide with the continuum result:

\[ \sum_i c_{3,i} = \frac{1}{16\pi^2} (3N - \frac{1}{N}) N_f \]

Again, this was checked both algebraically and numerically.

The results for each diagram are presented in Tables 1 and 2. Diagrams not appearing in these Tables give vanishing contributions.

For the one-loop amplitude \( \hat{o}p^2 \nu^{(1)} (p) \), the individual contributions from the corresponding two diagrams in Figure 1 are, respectively:

\[ 0.040848920(4) - a^2 p^2 0.013732194(5) + \frac{1}{24\pi^2} a^2 p^2 \ln(a^2 p^2) , \quad -0.040848919(5) \]

In concluding this section, we would like to point out that the procedure outlined here applies unchanged to several other interesting cases, including: More complicated actions,

\[ ^2 \text{ Compared to the original version, only one entry is different, corresponding to diagram 19.} \]
matrix elements of different operators. We hope to address these issues in a future publication.

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## TABLE I. Coefficients $c_{0,i}^{(-1)}$, $c_{3,i}^{(-1)}$, $c_{4,i}^{(-1)}$, $r = 1$.

| $i$   | $c_{0,i}^{(-1)}$ | $c_{3,i}^{(-1)}$ | $c_{4,i}^{(-1)}$ |
|-------|------------------|------------------|------------------|
| 1     | -0.00158221542(13) | 0                | 0                |
| 3     | 0.00316444309(6)  | 0.051644463410   | -0.0010637877(8) |
| 4     | 0.00039273(5)     | 0                | 0                |
| 5     | -0.0005077(9)     | 0.00010879(5)    | 0.00005077(6)    |
| 6     | -0.0086173230(12) | 0                | 0                |
| 7 + 11| 0                | 0.25822317052    | -0.0031633707(22)|
| 8     | 0.0070351064(17)  | 0.00528244566(5) | 0.0000757(12)    |
| 12    | 0.00069739(6)     | 0                | 0                |
| 14    | 0.00022988(14)    | 0.019554587(16)  | 0.0001732(6)     |
| 15    | -0.00060983(5)    | -0.00007427(4)   | 0.00007427(4)    |
| 18    | -0.0002019(4)     | 0.023786891(14)  | -0.00007910(15)  |
| 19    | -0.0107211662(2)  | -0.325714118(9)  | 0.004443346(3)   |
| 20    | 0.0107211662(6)   | 0                | 0                |

## TABLE II. Coefficients $c_{0,i}^{(1)}$, $c_{3,i}^{(1)}$, $c_{4,i}^{(1)}$, $r = 1$.

| $i$   | $c_{0,i}^{(1)}$ | $c_{3,i}^{(1)}$ | $c_{4,i}^{(1)}$ |
|-------|------------------|------------------|------------------|
| 1     | 0.00158221542(13) | 0                | 0                |
| 3     | -0.00316444309(6) | -0.051644463410  | 0.0010637877(8)  |
| 4     | -0.00039273(5)    | 0                | 0                |
| 5     | 0.0004503(6)      | 0                | -0.00008893(4)   |
| 6     | 0.0086173230(12)  | 0                | 0                |
| 7 + 11| 0.0014560(8)      | 0.25822317052    | 0.0031633707(22)|
| 8     | -0.0020085(5)     | 0.08132606(7)    | -0.0011335(13)   |
| 12    | -0.00069739(6)    | 0                | 0                |
| 13    | 0                 | 0                | 0.00006908(5)    |
| 14    | -0.00011494(7)    | -0.00264122283(3)| -0.0000378(6)    |
| 16    | 0.0011052(4)      | -0.0517470(17)   | 0.0007092(7)     |
| 18    | 0.0002019(4)      | -0.023786891(14) | 0.00007910(15)   |
| 19    | 0.0107211662(2)   | 0.325714118(9)   | -0.004443346(3)  |
| 20    | -0.0107211662(6)  | 0                | 0                |
FIGURES

FIG. 1. Diagrams with fermionic lines contributing to the one-loop function $\nu^{(1)}(p)$. Dashed lines ending on a cross represent background gluons. Solid lines represent fermions.

FIG. 2. Diagrams with fermionic lines contributing to the two-loop function $\nu^{(2)}(p)$. Dashed lines represent gluonic fields; those ending on a cross stand for background gluons. Solid lines represent fermions. The filled circle is a one-loop fermion mass counterterm.
