A Class of Lower Bounds for Bayesian Risk with a Bregman Loss

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Abstract—A general class of Bayesian lower bounds when the underlying loss function is a Bregman divergence is demonstrated. This class can be considered as an extension of the Weinstein–Weiss family of bounds for the mean squared error and relies on finding a variational characterization of Bayesian risk. The approach allows for the derivation of a version of the Cramér–Rao bound that is specific to a given Bregman divergence. The new generalization of the Cramér–Rao bound reduces to the classical one when the loss function is taken to be the Euclidean norm. The effectiveness of the new bound is evaluated in the Poisson noise setting.

I. INTRODUCTION

Finding lower bounds on a Bayesian risk is an important aspect in signal estimation as such bounds provide fundamental limits on signal recovery. Moreover, they can contribute useful insights and guidelines for algorithm design in data-driven applications, where Bayesian analysis is of ever-increasing importance. A plethora of such bounds are known for the mean squared error (MSE). Loosely speaking, these bounds can be divided into three families. The first family, termed Weinstein–Weiss, works by using the Cauchy–Schwarz inequality [1], and includes the prevalent Cramér–Rao (CR) bound (also known as the van-Trees bound [2]) as a special case. The second family, termed Ziv–Zakai, is derived by connecting estimation and binary hypothesis testing [3]. The third family uses a variational approach and works by minimizing the MSE subject to a constraint on a suitably chosen divergence measure, for example, the Kullback–Leibler divergence [4].

This paper is concerned with a generalization of the Weinstein–Weiss family of bounds beyond the MSE. Specifically, the aim is to provide a generalization of this family to a larger class of Bayesian risks, where the loss functions are taken to be Bregman divergences (BDs). The difficulty with such a generalization is that it is not immediately clear how the Cauchy–Schwarz inequality can be applied to a Bregman divergence, which is, in general, not a metric and, at a first glance, does not have a natural norm associate with it.

In this paper, by using elementary techniques such as Taylor’s remainder theorem, it is shown that the Weinstein–Weiss approach can be generalized to the Bayesian risks where the loss function is taken to be a BD. Furthermore, this generalization makes it possible to derive a version of the CR bound that is specific to a given BD. The new generalization of the CR bound reduces to the classical CR bound when the loss function is taken to be the Euclidean norm.

The paper outline and contributions are as follows:

• Section III reviews properties of Bregman divergences and of the corresponding Bayesian risk.
• Section IV presents a variational characterization of the Bregman risk and the new family of bounds.
• Section V discusses some applications of the variational representation. In particular, Theorem 3 presents a generalization for the CR that is specific to a given BD and reduces to the classical CR bound when the Bayesian risk corresponds to the MSE.
• Section VI in order to show the utility of the new CR bound, evaluates the CR bound for the Poisson noise case with a BD natural for this setting. In particular, it is shown that the CR bound has the same behavior as the Bayesian risk when the scaling parameter of the Poisson noise is taken to be large.

Notation: Random variables are denoted by upper case letters, and their realizations are denoted by lower case letters. The inner product operator is denoted by $\langle \cdot, \cdot \rangle$. The identity matrix is denoted by $I$. For two symmetric matrices $A$ and $B$, we say that $A \prec B$ if $B - A$ is positive-definite. For $0 < A$ we define the Mahalanobis metric as $\|x\|_A = x^T A x$, where $\|x\|$ denotes the Euclidean metric. The expected value operator is denoted by $E[\cdot]$. For a random variable $X \in \mathbb{R}^n$ with a probability density function (pdf) $f_X$, the score function is defined as $\rho_X(x) = \nabla f_X(x) / f_X(x)$, where $\nabla$ is the gradient operator.

II. BREGMAN DIVERGENCE AND BAYESIAN RISK

In order to define a Bayesian risk or estimation error one needs to select a loss function. The family of loss functions considered in this paper is defined next.

Definition 1. (Bregman Divergence.) Let $\phi : \Omega \to \mathbb{R}$ be a continuously-differentiable and strictly convex function defined on a closed convex set $\Omega \subseteq \mathbb{R}^n$. The Bregman divergence between $u$ and $v$ associated with the function $\phi$ is defined as

$$\ell_\phi(u, v) = \phi(u) - \phi(v) - \langle u - v, \nabla \phi(v) \rangle.$$ (1)

BDs have been introduced in [5] in the context of convex optimization. There exists several extensions of the BD definition such as an extension to functional spaces [6], an extension to submodular set functions [7], and a matrix extension [8].

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In [9], BDs, together with f-divergences, where characterized axiometrically. A thorough investigation of BDs was undertaken in [10], where it was shown that many commonly used loss functions are members of this family. Moreover, the authors of [10] have shown that every regular exponential distribution has a unique BD associated with it.

Now consider the problem of estimating a random variable X from a noisy observation Y, where the loss function is according to (1). The smallest Bayesian risk associated with this estimation problem is defined next.

**Definition 2.** (Minimum Bayesian Risk with Respect to BD.) For a joint distribution \( P_{Y,X} \) we denote the minimum Bayesian risk with respect to loss function \( \ell_\phi(u,v) \) as

\[
R_\phi(X|Y) = \inf_{f: f \text{ is measurable}} E[\ell_\phi(X, f(Y))].
\]

\( R_\phi(X|Y) \) is also referred to as Bayesian Bregman risk in what follows.

**Remark 1.** The most prominent example of \( R_\phi(X|Y) \) is the minimum mean squared error (MMSE), which is induced by choosing \( \phi(u) = \|u\|^2 \) and will be denoted by

\[
\text{mmse}(X|Y) = R_\phi(X|Y).
\]

The structure of the optimal estimator in (2) was studied in [11], where it was shown that the conditional expectation is the unique minimizer. Moreover, the authors of [11] have also demonstrated the converse result, namely that the conditional expectation is an optimal estimator only when the loss function is a Bregman divergence.

**A. Fundamental Properties of Bregman Divergences**

We now, for completeness, review the most important properties of BDs and the associated Bayesian risks.

**Theorem 1.** (Fundamental Properties of Bregman Divergences and Bayesian Bregman Risks.)

1) (Non-Negativity) \( \ell_\phi(u,v) \geq 0, \forall u, v \in \Omega, \) with equality if and only if \( u = v; \)

2) (Convexity) \( \ell_\phi(u,v) \) is convex in \( u; \)

3) (Linearity) \( \ell_\phi(u,v) \) is linear in \( \phi; \)

4) (Generalized Law of Cosines): For \( u, v, w \in \Omega \)

\[
\ell_\phi(u,v) = \ell_\phi(u,w) + \ell_\phi(w,v) - \langle u - w, \nabla \phi(v) - \nabla \phi(w) \rangle; \]

5) (Orthogonality Principle and Pythagorean Identity) For every random variable \( X \in \Omega \) and every \( u \in \Omega \)

\[
E[\ell_\phi(X,u)] = E[\ell_\phi(X,E[X])] + \ell_\phi(E[X],u).
\]

Moreover, for any measurable \( f(Y) \)

\[
E[\ell_\phi(X,f(Y))] = E[\ell_\phi(X,E[X|Y])] + E[\ell_\phi(E[X|Y],f(Y))].
\]

6) (Conditional Expectation is the Unique Minimizer) Suppose that \( E[X] < \infty \) and \( E[\phi(X)] < \infty. \) Then,

\[
\inf_{f: f \text{ is measurable}} E[\ell_\phi(X,f(Y))] = E[\ell_\phi(X,E[X|Y])].
\]

The optimizer in (7) is unique \( Y \)-almost surely.

7) (Coupling between Conditional Expectation and Bregman Divergence) Let \( F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be a non-negative function such that \( F(x,x) = 0 \) and assume that all partial derivatives \( F_{x_i,x_j} \) are continuous. If for all random variables \( X \in \mathbb{R}^n \) it holds that

\[
\inf_{u \in \mathbb{R}^n} E[F(X,u)] = E[F(X,E[X])],
\]

with \( E[X] \) being the unique minimizer, then \( F(u,v) = \ell_\phi(u,v) \) for some strictly convex and differentiable function \( \phi : \mathbb{R}^n \to \mathbb{R}. \)

**B. Notable Examples of BDs in Estimation Theory**

BDs and their associated Bayesian risks appear naturally in connection with information measures. For example, the mutual information between \( Y \) and \( X \) can be represented as an integral of the BD induced by

- \( \phi(u) = \|u\|^2 \) (i.e., the MMSE) when \( P_{Y|X} \) is a Gaussian distribution [12], [13];

- \( \phi(u) = u \log u \) when \( P_{Y|X} \) is a Poisson distribution [14], [15];

- \( \phi(u) = u \log \frac{u}{1-u} \) when \( P_{Y|X} \) is a Binomial distribution [16]; and

- \( \phi(u) = u \log \frac{u}{1+u} \) when \( P_{Y|X} \) is a Negative Binomial distribution [16].

Table I summarizes the above examples together with the corresponding BDs. The latter will be referred to as the natural BDs in what follows. For a more detailed treatment of connections between information measures and BDs, the interested reader is referred to [17].

**III. A VARIATIONAL REPRESENTATION OF BREGMAN RISK**

As one might expect, the variational characterization of the Bayesian Bregman risk requires an application of the Cauchy–Schwarz inequality. However, a priori, it is not immediately clear how the Cauchy–Schwarz inequality can be applied to
frequently complicated expressions (see Table I) of BDs. The
approach, however, becomes clear after an elementary application
of Taylor’s remainder theorem, which allows representing
a BD as a weighted squared error.

**Lemma 1.** (\(\ell_2\)-representation of Bregman Divergences) For
every \(\phi\) in Definition [1] it holds that
\[
\ell_\phi(u, v) = \|((u - v)^T \Delta_\phi(u, v))\|^2,
\]
where
\[
\Delta_\phi(u, v) = \int_0^1 (1 - t)H_\phi((1 - t)u + tv)dt,
\]
and \(x \mapsto H_\phi(x)\) is the Hessian matrix of \(\phi\) evaluated at \(x\).

**Proof:** Recall that given a twice differentiable function
\(\phi : \mathbb{R}^n \rightarrow \mathbb{R}\), Tailor’s remainder theorem [18] asserts that
\[
\phi(u) = \phi(v) + \langle u - v, \nabla \phi(v) \rangle + (u - v)^T \left[ \int_0^1 (1 - t)H_\phi(u + t(v - u))dt \right] (u - v),
\]
where \(x \mapsto H_\phi(x)\) is the Hessian matrix of \(\phi\) evaluated at \(x\).

Observe that the BD \(\ell_\phi(u, v)\) is the remainder of the first
order Tailor series expansion of \(\phi(u)\) around \(v\). Therefore, by
the integral representation of the Taylor expansion remainder
in (11), it follows that
\[
\ell_\phi(u, v) = (u - v)^T \left[ \int_0^1 (1 - t)H_\phi(u + t(v - u))dt \right] (u - v) = (u - v)^T \Delta_\phi(u, v)(u - v).
\]
This concludes the proof.

**Remark 2.** In the scalar case [13] simplifies to
\[
\ell_\phi(u, v) = \ell_{x^2}(x, u, v)\Delta_\phi(u, v),
\]
with \(\Delta_\phi\) being strictly positive.

**Remark 3.** It can be argued that (9) and (14) trivially hold
true since any two functions with identical support can be
expressed as weighted versions of one another by simply
choosing the weight to be their ratio. Here, however, it is
important to note that \(\Delta_\phi\) can be obtained directly from \(\phi\) by
evaluating the integral on the right hand side of (10). In other
words, \(\Delta_\phi\) can be calculated without evaluating \(\ell_\phi\).

With Lemma [1] in our disposal, we are now ready to derive
a variational characterization of the Bayesian Bregman risk.

**Theorem 2.** (Variational Characterization of Bayesian Breg-
man Risk.) Let \(g : \mathbb{R}^k \rightarrow \mathbb{R}^n\). Then,
\[
\mathbb{E}[\ell_\phi(X, g(Y))] = \sup_{\psi: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n} \mathbb{E}\left[ \frac{\|\Delta_\phi^{\frac{1}{2}}(X, g(Y))\psi(X, Y)\|^2}{\|\Delta_\phi^{\frac{1}{2}}(X, g(Y))\|^2} \right]
\]
and equality in (15) is attained if and only if
\[
\psi(X, Y) = \Delta_\phi(X, g(Y))(X - g(Y)).
\]

**Proof:** By using Lemma [1] we have that
\[
\mathbb{E}[\ell_\phi(X, g(Y))] = \mathbb{E}\left[ ||(X - g(Y))^T \Delta_\phi^{\frac{1}{2}}(X, g(Y))||^2 \right] 
\geq \mathbb{E}\left[ \frac{\|\Delta_\phi^{\frac{1}{2}}(X, g(Y))\|^2}{\|\Delta_\phi^{\frac{1}{2}}(X, g(Y))\|^2} \right]
\]
where the last step follows from the Cauchy–Schwarz inequality
for some arbitrary function \(h : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n\). Next, we
rescale the expression by choosing
\[
h(x, y) = \Delta_\phi^{\frac{1}{2}}(x, g(y))\psi(x, y),
\]
for some arbitrary function \(\psi : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n\), which leads to
the expression on right side of (15). The proof of the equality
condition follows by inspection. This concludes the proof.

The variational characterization in (15) is a generalization of
the Weinstein–Weiss representation of the MSE, which is
included as the special case when \(\Delta_\phi = I\).

Setting \(g(Y) = \mathbb{E}[X\mid Y]\) yields a variational characterization
of the minimum Bayesian risk with respect to a BD, which is
an important corollary of the above result.

**Corollary 1.**
\[
R_\phi(X\mid Y) = \sup_{\psi: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n} \mathbb{E}\left[ \frac{\|\Delta_\phi^{\frac{1}{2}}(X, \mathbb{E}[X\mid Y])\psi(X, Y)\|^2}{\|\Delta_\phi^{\frac{1}{2}}(X, \mathbb{E}[X\mid Y])\|^2} \right]
\]

**IV. Applications**

In this section, we first show a small application of the
alternative representation of the BD in Lemma [1]. Second, we
present a generalized version of the CR that is specific to a
given BD bound.

**A. Comparing Bregman Risk and the MMSE**

Our first application shows how Bregman risks can be
connected to the ubiquitous case of a risk with a squared error
loss.

**Proposition 1.** Suppose that \(\kappa_1 \leq H_\phi \leq \kappa_u I\) for some
constants \(\kappa_1, \kappa_u \geq 0\). Then,
\[
\kappa_1 \text{mmse}(X\mid Y) \leq R_\phi(X\mid Y) \leq \kappa_u \text{mmse}(X\mid Y).
\]

**Proof:** The proof follows by using Lemma [1] that is under
the hypothesis of the theorem we have that
\[
\kappa_1 I \preceq \Delta_\phi(u, v) \preceq \kappa_u I.
\]
Hence,
\[
\kappa_1 \ell_{x^2}(u, v) \leq \ell_\phi(u, v) \leq \kappa_u \ell_{x^2}(u, v).
\]
This concludes the proof.
B. A Generalization of the Bayesian CR Bound

The classic CR bound allows for lower bounding the MMSE with the Fisher information: for $X \in \Omega \subseteq \mathbb{R}^n$
\[
\text{mmse}(X|Y) \geq \frac{n^2}{\mathbb{E}[(\nabla_X \log f_{Y,X}(Y,X))^2]} \tag{25}
\]
where the above holds under the regularity conditions
\[
\mathbb{E}[(\nabla_X \log f_{Y,X}(Y,X)|Y=y) = 0, \forall y; \text{ and } x f_{Y,X}(y,x) = 0, \forall y, \forall x \in \partial \Omega, \tag{26a}
\]
where $\partial \Omega$ denotes the boundary of the set $\Omega$. The quantity $\mathbb{E}[(\nabla_X \log f_{Y,X}(Y,X))^2]$ is known as Fisher information.

The next theorem proposes a generalization of the CR bound.

**Theorem 3.** (Generalized CR-Bound.) Suppose that conditions in (26) hold. Then,
\[
\mathbb{E}[\ell_\phi(X, g(Y))] \geq \frac{n^2}{\mathbb{E}[(\nabla_\phi \nabla_X \log f_{Y,X}(Y,X))^2]} \tag{27}
\]
\[
\text{Proof: } \text{The proof follows by choosing } \psi(x,y) = \nabla_X \log f_{Y,X}(y,x) \text{ in (15). Now observe that}
\]
\[
\mathbb{E}[(X - g(Y))^2] = \mathbb{E}[X^T \nabla_X \log f_{Y,X}(Y,X)]
\]
\[
\mathbb{E}[g(Y) \nabla_X \log f_{Y,X}(Y,X)] = 0 \text{ from the assumption in (26)}. \text{ To conclude the proof note that}
\]
\[
\mathbb{E}[X^T \nabla_X \log f_{Y,X}(Y,X)]
\]
\[
= \int \int x^T \nabla_X f_{Y,X}(y,x) f_{Y,X}(y,x) \mathrm{d}x \mathrm{d}y \tag{30}
\]
\[
= \int \int x^T \nabla_X f_{Y,X}(y,x) \mathrm{d}x \mathrm{d}y \tag{31}
\]
\[
= \sum_{i=1}^n \int x_i \frac{\partial}{\partial x_i} f_{Y,X}(y,x) \mathrm{d}x \mathrm{d}y \tag{32}
\]
\[
= -n, \tag{33}
\]
where the last step we have used integration by parts and that $x f_{Y,X}(y,x) = 0$ for $x \in \partial \Omega$.

Observe that the quantity in (27) is a generalization of the Fisher information that takes into account the corresponding BD.

**Remark 4.** An interesting feature of the CR lower bound in (27) is that it depends on the estimator $g(Y)$. Note that in the case of the MSE the CR bound in (25) does not depend on the estimator in question and is uniform over all estimators. On the one hand, a benefit of this dependence is that one can adapt the lower bound to the estimator in use and potentially get a tighter bound. On the other hand, a drawback might be the computability of such a bound. However, the letter can be addressed by using the CR bound corresponding to $R_\phi(X|Y)$, which is uniform over all estimators, i.e.,
\[
\mathbb{E}[\ell_\phi(X,g(Y))] \geq R_\phi(X|Y) \geq \frac{n^2}{\mathbb{E}[(\nabla_\phi \nabla_X \log f_{Y,X}(Y,X))^2]} \tag{34}
\]
In the next section, using an example of estimation in Poisson noise, we outline a procedure for applying the CR bound in Theorem 3.

V. **Evaluation of the CR Bound for the Poisson Noise Case**

In this section, we seek to compute the CR bound in Theorem 3 for a non-trivial case. Specifically, we consider $\phi(u) = u \log u$ with $\Omega = \mathbb{R}_+$ so that
\[
\ell_\phi(u; v) = u \log \frac{u}{v} - (u - v), \tag{36}
\]
which is natural for Poisson noise. Note that $\phi''(u)$ is unbounded and the results of Proposition 1 do not apply. Therefore, it is non-trivial to compare the Bayesian risk corresponding to (36) and the MMSE.

Next, consider the problem of denoising a non-negative random variable in Poisson noise, which will serve as a running example in this section. The random transformation of a non-negative real-valued input random variable $X$ to a non-negative integer-valued output random variable $Y$ will be denoted by
\[
Y = \mathcal{P}(aX), \tag{37}
\]
where $a > 0$ is the scaling factor. Concretely, the Poisson noise channel is dictated by the following probability mass function (pmf)
\[
P_{Y|X}(y|x) = \frac{1}{y!} (ax)^y e^{-(ax)}, \tag{38}
\]
where $y = 0, 1, \ldots$ and $x \geq 0$. In words, conditioned on a non-negative input $X$, the output of the Poisson channel is a non-negative integer-valued random variable $Y$ that is distributed according to (38). Note that in (38) we use the convention that $0^0 = 1$. Poisson noise models are an important family of models with a wide range of applications, including laser communications [19], [20].

The first result of this section provides a condition under which the CR bound in (27) holds. Note that the output of the Poisson noise channel is discrete. However, there is no issue in applying the CR bound in (27) as differentiability is only required in the $X$ variable, while the $Y$ variable can have an arbitrary support.

**Proposition 2.** Let $X \sim f_X$ and $Y = \mathcal{P}(aX)$. The conditions in (26) hold if
\[
\lim_{x \to 0^+} f_X(x) = 0. \tag{39}
\]
Proof: \[
\begin{align*}
\mathbb{E} \left[ \nabla_x \log \left( P_{Y|X}(y|x) f_X(x) \right) \right | Y = y & = \int_0^\infty \nabla_x \left( \frac{P_{Y|X}(y|x) f_X(x)}{P_{Y|X}(y|x)} \right) f_{X|Y}(x|y) dx \\
& = \frac{1}{P_Y(y)} \int_0^\infty \nabla_x \left( P_{Y|X}(y|x) f_X(x) \right) dx \\
& = \frac{1}{P_Y(y)} \lim_{x \to 0^+} P_{Y|X}(y|x) f_X(x) = 0, \\
\end{align*}
\]
where the last step follows from the assumption in (39). This verifies that the CR bound applies.

A. Estimation with the Square Distance

In this section, we evaluate the tightness of the CR bound for the MMSE. In order to do this, we also need an upper bounds on the MMSE. Therefore, we begin by finding the best linear estimator for the Poisson noise and the corresponding MSE error, which we will use to upper bound the MMSE.

Lemma 2. Let \( Y = \mathcal{P}(aX), \phi(u) = u^2, \) and let
\[
f(c, d) = \mathbb{E} [\phi(X ; cY + d)].
\]
Then,
\[
\text{mmse}(X|Y) \leq \min_{c,d} f(c, d)
\]
\[
= f \left( \frac{\nabla(X)}{a\nabla(X) + \mathbb{E}[X]}, \frac{\mathbb{E}[X]^2}{a\nabla(X) + \mathbb{E}[X]} \right)
\]
\[
= \frac{\nabla(X)}{a\nabla(X) + \mathbb{E}[X]} + 1.
\]

Proof: The first inequality in (45) follows directly from the definition of the MMSE. It is well-known that the minimizers of (44) are given by
\[
c^* = \mathbb{E} [ (X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) ] \mathbb{V}(Y),
\]
\[
d^* = \mathbb{E}[X] - c\mathbb{E}[Y],
\]
and
\[
f(c^*, d^*) = \nabla(X) - (c^*)^2\nabla(Y).
\]
The proof is complete by observing that
\[
\mathbb{E}[Y] = \mathbb{E}[aX] = a\mathbb{E}[X],
\]
\[
\nabla(Y) = a^2\nabla(X) + a\mathbb{E}[X],
\]
and
\[
\mathbb{E} [(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = a\nabla(X).
\]

Next, we evaluate the CR lower bound for the MMSE.

Theorem 4. Suppose (39) holds. Then,
\[
\text{mmse}(X|Y) \geq \frac{1}{a\mathbb{E}[X] + \mathbb{E}[\mathbb{E}[Y]]}.
\]

Proof: Observe that the score function is readily computed to be
\[
\nabla_x \log \left( P_{Y|X}(y|x) f_X(x) \right) = \frac{\nabla_x P_{Y|X}(y|x)}{P_{Y|X}(y|x)} + \frac{\nabla_x f_X(x)}{f_X(x)}
\]
\[
= \frac{y}{x} - a + \rho_X(x).
\]
Therefore, the denominator in the CR bound is given by
\[
\begin{align*}
\mathbb{E} \left[ \left( \frac{Y}{X} - a + \rho_X(X) \right)^2 \right] & = \mathbb{E} \left[ \left( \frac{Y}{X} - a \right)^2 \right] + 2\mathbb{E} \left[ \left( \frac{Y}{X} - a \right) \rho_X(X) \right] + \mathbb{E} \left[ \rho_X^2(X) \right] \\
& = \mathbb{E} \left[ \left( \frac{Y}{X} - a \right)^2 \right] + \mathbb{E} \left[ \rho_X^2(X) \right] \\
& = a\mathbb{E} \left[ \frac{1}{X} \right] + \mathbb{E} \left[ \rho_X^2(X) \right],
\end{align*}
\]
where in (58) we have used that
\[
\mathbb{E} \left[ \left( \frac{Y}{X} - a \right) | X \right] = \mathbb{E} \left[ (Y - aX) | X \right] = 0;
\]
and in (59) we have used the variance of the Poisson distribution, so that
\[
\mathbb{E} \left[ (Y - aX)^2 | X \right] = aX.
\]
This concludes the proof. \( \square \)

From Theorem 4 and the upper bound in (47) we conclude that under the aforementioned regularity conditions
\[
\text{mmse}(X|Y) = \Theta \left( \frac{1}{a} \right).
\]
In other words, the CR bound is an effective lower bound for large values of \( a \).

We now proceed to evaluate the bounds in Lemma 2 and Theorem 4 for the case when \( X \) is distributed according to a gamma distribution, whose pdf is given by
\[
f(x) = \frac{\alpha^\theta}{\Gamma(\theta)} x^{\theta - 1} e^{-\alpha x}, \quad x \geq 0,
\]
where \( \theta > 0 \) is the shape parameter and \( \alpha > 0 \) is the rate parameter. We denote the distribution with the pdf in (63) by \( \text{Gam}(\alpha, \theta) \). The choice of a gamma distribution is dictated by the fact that it is a conjugate prior of the Poisson distribution (21). The next lemma compiles several properties of the gamma distribution needed in this section and the next.
Lemma 3. Suppose that $X \sim \text{Gam}(\alpha, \theta)$ and that $Y = \mathcal{P}(aX)$. Then,

\[
\begin{align*}
E[X] &= \frac{\theta}{\alpha}, \\
V(X) &= \frac{\theta}{\alpha^2}, \\
\mathbb{E}\left[ \frac{1}{X^n} \right] &= \begin{cases} 
\frac{\Gamma(n+\theta-\alpha)}{\Gamma(\theta)}, & \theta < 0, \\
\frac{\Gamma(n+\theta)}{\Gamma(\theta)}, & \theta > 0, \\
\frac{\Gamma(n+\theta-\alpha)}{\Gamma(\theta)}, & \theta = 0.
\end{cases} \\
\mathbb{E}[\rho_\alpha^2(X)|X|] &= \begin{cases} 
\infty, & \theta \leq 2, \\
\frac{\theta^2}{2}, & \theta > 2.
\end{cases}
\end{align*}
\]

Proof: See Appendix A.

The next result evaluates the MMSE and the CR lower bound for the case when $X \sim \text{Gam}(\alpha, \theta)$.

**Proposition 3.** Suppose that $X \sim \text{Gam}(\alpha, \theta)$ and $Y = \mathcal{P}(aX)$. Then, the following statements hold:

- The upper bound in (47) is tight, that is

\[
\text{mmse}(X|Y) = \frac{\mathbb{V}(X)}{a^2 \mathbb{E}(X)} + \frac{\mathbb{E}[\rho_\alpha^2(X)]}{\alpha^2} = \frac{\theta}{\alpha(\alpha + \alpha)}
\]

with $\mathbb{V}(X) = \frac{\theta}{\alpha^2}$ and $\mathbb{E}(X) = \frac{\theta}{\alpha}$.

- The CR regularity condition in (39) holds for $\theta > 1$. Moreover, the bound in (54) reduces to

\[
\text{mmse}(X|Y) \geq \begin{cases} 
0, & 1 < \theta \leq 2, \\
\frac{\theta}{\alpha(\alpha + \alpha)} y, & \theta > 2
\end{cases}
\]

Proof: Observe that the optimal MMSE estimator for $X \sim \text{Gam}(\alpha, \theta)$ is given by (69), while the estimator that achieves the upper bound in (47) is given by (66) by

\[
\mathbb{V}(X) - \mathbb{E}[\rho_\alpha^2(X)] = \frac{1}{\alpha} y + \frac{\theta}{\alpha + \alpha},
\]

where we have used that $\mathbb{E}[X] = \frac{\theta}{\alpha}$ and $\mathbb{V}(X) = \frac{\theta}{\alpha^2}$. Since the two estimators agree, the upper bound is achieved with equality.

The fact that the regularity condition in (39) holds for $\theta > 1$ follows from the limit

\[
\lim_{x \to 0^+} x^{\theta-1} e^{-\alpha x} = \begin{cases} 
\infty, & \theta < 1, \\
1, & \theta = 1, \\
0, & \theta > 1
\end{cases}
\]

The proof of (71) now follows by inserting (66) and (67) into (54).

Fig. 1 compares the exact MMSE and the CR lower bound evaluated in Proposition 3.

**B. Estimation with the Natural Bregman Divergence**

Before evaluating the CR bound in Theorem 3 for $\phi(u) = u \log u$, we present two ancillary lemmas. The first result provides bounds on $\Delta_\phi(u, v)$.

**Lemma 4.** Let $\phi(u) = u \log u$. Then,

\[
u \leq \frac{1}{\Delta_\phi(u, v)} \leq \frac{4u}{3} + \frac{2v}{3}.
\]

Proof: We first show the upper bound. To that end, let $T$ be a random variable on $[0, 1]$ with a pdf given by $f_T(t) = 2(1 - t)$. Then,

\[
\frac{1}{\Delta_\phi(u, v)} = \int_0^1 \frac{1}{(1-t)u + tv} dt 
\]

where in (77) we have used Jensen’s inequality. The proof of the lower bound follows since $v \geq 0$ and

\[
\int_0^1 \frac{1}{(1-t)u + tv} dt \leq \frac{1}{u}.
\]

This concludes the proof.

We will also need the following result that has been shown in [22, Theorem 10].

**Lemma 5.** Let $Y = \mathcal{P}(aX)$. Then, for every distribution on $X$ there exist non-negative constants $c_1$ and $c_2$ (possibly dependent on $a$) such that

\[
\mathbb{E}[X|Y = y] \leq \frac{c_1 y + c_2}{c_1 y + c_2}, \\
y = 0, 1, \ldots
\]
We now proceed to evaluate the CR lower bound in Theorem 3 for the specific choice of Poisson noise with the Bregman divergence in (36). The next result provides upper and lower bounds on the modified Fisher information in the denominator of (27) in Theorem 5.

**Theorem 5.** Suppose \( \eqref{phi} \) holds. Then, for \( \phi(u) = u \log u \) and \( Y = \mathcal{P}(aX) \),

\[
L \leq E \left[ \frac{\big(\frac{d}{dx} \log f_{Y,X}(Y,X)\big)^2}{\Delta_{\phi}(X,E[X|Y])} \right] \leq D, \tag{82}
\]

where

\[
D = \frac{4}{3} \left(a + E \left[ \phi'(X)X \right] \right) + \frac{2c_1}{3} \left(a^2 + aE \left[ \frac{1}{X} \right] + aE \left[ \phi'(X)X \right] \right) + \frac{2c_2}{3} \left(aE \left[ \frac{1}{X} \right] + E \left[ \phi''(X) \right] \right), \tag{83}
\]

with \( c_1, c_2 \) are given in Lemma 5 and

\[
L = a + E \left[ \phi'(X)X \right]. \tag{84}
\]

Consequently,

\[
R_{\phi}(X|Y) \geq \frac{1}{D}. \tag{85}
\]

**Proof:** See Appendix B.

Using Lemma 3 it is now straightforward to evaluate the bound in Theorem 5 for \( X \sim \text{Gam}(\alpha, \theta) \).

**Proposition 4.** Let \( \phi(u) = u \log u \), \( Y = \mathcal{P}(aX) \), and \( X \sim \text{Gam}(\alpha, \theta) \). Then, the CR regularity condition holds for \( \theta > 1 \).

Moreover, the bounds in \( \eqref{82} \) reduces to

\[
L = a + \alpha, \tag{86}
\]

and

\[
D = \frac{4}{3} (a + \alpha)
+ \frac{2}{3} \frac{1}{\alpha + a} \left(a^2 + a\frac{\alpha}{\theta - 1} + a\alpha\right)
+ \frac{2}{3} \frac{\theta}{\alpha + a} \left(a\frac{\alpha}{\theta - 1} + a\right), \tag{87}
\]

for \( \theta > 2 \) and \( D = \infty \) for \( \theta \in [1, 2] \).

The result in Proposition 4 implies that the modified Fisher information evaluated with a Gamma distribution satisfies

\[
E \left[ \frac{\big(\frac{d}{dx} \log f_{Y,X}(Y,X)\big)^2}{\Delta_{\phi}(X,E[X|Y])} \right] = \Theta(a) \tag{88}
\]

for \( \theta > 2 \).

To evaluate the tightness of the CR bound we also need the following upper and lower bounds on the \( R_{\phi}(X|Y) \) with \( \phi(u) = u \log u \), which are amenable to numerical computations.

**Proposition 5.** Let \( \phi(u) = u \log u \), \( Y = \mathcal{P}(aX) \) and \( X \sim \text{Gam}(\alpha, \theta) \). Then,

\[
R_{\phi}(X|Y) = E[X \log X] - B \tag{89}
\]

where

\[
E \left[ X \log \left( \frac{aX}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
\geq B \geq E \left[ \left( \frac{aX}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \log \left( \frac{aX}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right]. \tag{90}
\]

Moreover,

\[
E[X \log X] = \frac{\theta \left(\log \left(\frac{1}{\alpha} \right) + \psi(\theta + 1)\right)}{\alpha}, \tag{91}
\]

where \( \psi \) is the digamma function, and

\[
B = E \left[ \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right], \tag{92}
\]

and where \( Y \) a negative binomial distribution with pmf given by

\[
P_Y(y) = \frac{a^y \alpha^y}{(\alpha + a)^{y+1} y}, \quad y = 0, 1, \ldots \tag{93}
\]

**Proof:** See Appendix C.

Fig. 2 compares the CR bound in \( \eqref{87} \) to the exact value and bounds on \( R_{\phi}(X|Y) \) computed in Proposition 5.

It is also instructive to loosen the lower bound in \( \eqref{90} \) to

\[
B \geq E \left[ \left( \frac{aX}{\alpha + a} \right) \log \left( \frac{aX}{\alpha + a} \right) \right], \tag{94}
\]

which implies that

\[
R_{\phi}(X|Y) \leq \frac{1}{\alpha + a} E[X \log X] - \frac{aE[X]}{\alpha + a} \log \left( \frac{a}{\alpha + a} \right). \tag{95}
\]
we use the following well-known integral:

\[ \int_0^\infty \rho(x,x) \, dx = \frac{\theta - 1}{\alpha x} - \alpha, \quad \Theta \left( \frac{1}{\alpha} \right), \quad (96) \]

This conclusion demonstrates that the new CR bound is effective. Note, that both \( \Phi(X|Y) \) and the MMSE, as shown in (70), have the same order:

Fig. 3 concludes this section by comparing the MMSE in (70), the CR bound in (71) specific to the MMSE, the Bayesian Bregman risk in (89) and the CR bound in (87).

VI. CONCLUSION

This paper has proposed a general class of Bayesian lower bounds for the case when the underlying loss function is a BD. The approach allows for deriving a version of the CR bound that is specific to a given BD. To show the applicability of the new CR bound it has been evaluated for the Poisson noise case and been shown to have the same behaviors as Bregman risk when the scaling parameter is large, hence, demonstrating the effectiveness of the new CR bound.

APPENDIX A

PROOF OF LEMMA [3]

The expressions for \( \mathbb{E}[X] \) and \( \mathbb{V}(X) \) are standard. To show (66) we use the following well-known integral:

\[ \int_0^\infty x^k e^{-\alpha x} \, dx = \begin{cases} \infty, & k \leq -1 \\ \frac{\Gamma(k+1)}{\alpha^{k+1}}, & k > -1 \end{cases} \]

Therefore,

\[ \mathbb{E} \left[ \frac{1}{X^n} \right] = \frac{\alpha^n}{\Gamma(n)} \quad \mathbb{E} \left[ x^{\theta-1-n} e^{-\alpha x} \right] \]

\[ = \frac{\alpha^n}{\Gamma(n)} \frac{\Gamma(\theta-n)}{\alpha^{\theta-n}} \]

\[ = \frac{\alpha^n \Gamma(\theta-n)}{\Gamma(\theta)}, \quad (98) \]

\[ = \frac{\alpha^n \Gamma(\theta-n)}{\Gamma(\theta)}, \quad (99) \]

\[ = \frac{\alpha^n \Gamma(\theta-n)}{\Gamma(\theta)}, \quad (100) \]

\[ \text{for } \theta > n \text{ and infinity otherwise.} \]

To show (97) observe that

\[ \mathbb{E}[X] = \frac{\theta - 1}{\alpha x} - \alpha, \quad (101) \]

and, hence,

\[ \mathbb{E} \left[ \left( \frac{\theta - 1}{X} - \alpha \right)^2 \right] = \mathbb{E} \left[ \frac{(\theta - 1)}{X^2} - 2\alpha \frac{\theta - 1}{X} + \alpha^2 \right] \]

\[ = \begin{cases} \infty, & \theta \leq 2 \\ (\theta-1)^2 \theta^2 - 2\alpha^2 + \alpha^2, & \theta > 2 \end{cases} \]

(102)

(103)

where we have used that \( \mathbb{E} \left[ \frac{1}{X^2} \right] = \alpha^2 \mathbb{E}[\theta-2] = \frac{\alpha^2}{(\theta-2)(\theta-1)} \)

for \( \theta > 2 \) and \( \mathbb{E} \left[ \frac{1}{X} \right] = \frac{\alpha^2}{\theta} = \frac{\alpha}{\theta-1} \) for \( \theta > 1 \). Moreover,

\[ \mathbb{E} \left[ \left( \frac{\theta - 1}{X} - \alpha \right)^2 \right] = \mathbb{E} \left[ \frac{(\theta - 1)}{X} - 2\alpha(\theta - 1) + \alpha^2 \right] \]

\[ = \begin{cases} \infty, & \theta \leq 1 \\ \alpha, & \theta > 1 \end{cases} \]

(104)

(105)

where we have used that \( \mathbb{E} \left[ \frac{1}{X} \right] = \frac{\alpha^2}{\theta} = \frac{\alpha}{\theta-1} \) for \( \theta > 1 \).

The proof of (69) can be found in [22]. This concludes the proof.

APPENDIX B

PROOF OF THEOREM [5]

We begin with the proof of the upper bound

\[ \mathbb{E} \left[ \frac{4 \log f_{Y|X}(Y,X)}{\Delta \phi(X,E[Y|X])} \right] \]

\[ = \mathbb{E} \left[ \frac{4 \log f_{Y|X}(Y,X)}{\Delta \phi(X,E[Y|X])} \right] \]

\[ \leq \mathbb{E} \left[ \left( \frac{Y}{X} - a + \rho_X(X) \right)^2 \left( \frac{4X}{3} + \frac{2E[X|Y]}{3} \right) \right], \quad (108) \]

where in the last step we have used the inequality in (24). The first term in (108) can now be computed as follows:

\[ \mathbb{E} \left[ \left( \frac{Y}{X} - a + \rho_X(X) \right)^2 \right] \]

\[ = \mathbb{E} \left[ \left( \frac{Y}{X} - a \right)^2 \right] + 2 \mathbb{E} \left[ \left( \frac{Y}{X} - a \right) \rho_X(X) \right] \]

\[ + \mathbb{E} \left[ \rho_X^2(X)X \right] \]

\[ = \mathbb{E} \left[ \left( \frac{Y}{X} - a \right)^2 \right] + \mathbb{E} \left[ \rho_X^2(X)X \right] \]

\[ = a + \mathbb{E} \left[ \rho_X^2(X)X \right], \quad (111) \]

where in the last step we have used that \( \mathbb{E} \left[ (Y-a)^2 | X \right] = aX \).
The second term in (108) can be bounded by using the inequality in (51)

\[
E \left( \frac{Y}{X} - a + \rho_X(X) \right)^2 \mathbb{E}[X|Y] \
\leq E \left( \frac{Y}{X} - a + \rho_X(X) \right)^2 (c_1 Y + c_2).
\]

(112)

Now observe that

\[
E \left( \frac{Y}{X} - a + \rho_X(X) \right)^2 \\
= E \left( \frac{Y}{X} - a \right)^2 Y + 2E \left( \frac{Y}{X} - a \right) \rho_X(X) Y \\
+ E \left[ \rho_X^2(X) Y \right] \\
= E \left( \frac{Y}{X} - a \right)^2 Y + E \left[ \rho_X^2(X) Y \right] \\
= E \left[ \frac{Y^3}{X^2} - 2a \frac{Y^2}{X} + 2a Y \right] \\
+ aE \left[ \rho_X^2(X) X \right] \\
= \frac{a^3 X - 3a^2 X^2 + a X}{X^2} - 2a \frac{a X + a^2 X^2}{X} + a X \\
+ aE \left[ \rho_X^2(X) X \right],
\]

(113)

(114)

(115)

(116)

(117)

(118)

(119)

(120)

(121)

where we have used that

\[
E \left[ \frac{Y}{X} \rho_X(X) Y \right] = E \left[ \frac{a X + a^2 X^2}{X} \rho_X(X) \right] = 0 - a^2,
\]

(122)

Furthermore, it has been shown in (59) that

\[
E \left[ \frac{Y}{X} - a + \rho_X(X) \right)^2 \mathbb{E}[X|Y] = E \left[ \frac{a}{X} \right] + E \left[ \rho_X^2(X) \right].
\]

(123)

Finally, combining (108), (111), (119) and (122) leads to the desired bound. This concludes the proof of the upper bound.

To show the lower bound we use the inequality in (74)

\[
E \left[ \frac{Y}{X} - a + \rho_X(X) \right)^2 \mathbb{E}[X|Y] \geq E \left[ \frac{Y}{X} - a + \rho_X(X) \right)^2 X \\
= a + E \left[ \rho_X^2(X) X \right],
\]

(124)

where in the last step we have used (111). This concludes the proof.

**APPENDIX C**

**PROOF OF PROPOSITION 5**

From the structure of the Bregman divergence we have that

\[
R_\phi(X|Y) = E[X \log X - E[X|Y]] \\
- E \left[ X \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
- E \left[ X - \frac{Y}{\alpha + a} - \frac{\theta}{\alpha + a} \right]
\]

(125)

Now by using Lemma 3 we have that

\[
E \left[ X - \frac{Y}{\alpha + a} - \frac{\theta}{\alpha + a} \right] = 0,
\]

(126)

and

\[
E \left[ X \log X \right] = \frac{\alpha^\theta}{\Gamma(\theta)} \int_0^\infty x \log(x) x^{\theta-1} e^{-ax} dx \\
= \theta \left( \log \left( \frac{1}{\alpha} \right) + \psi(\theta + 1) \right),
\]

(127)

(128)

where \( \psi(t) \) is the digamma function. Next, observe that \( x \log x \) is a convex function, and hence,

\[
E \left[ X \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
= E \left[ E[X|Y] \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
= E \left[ \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
= E \left[ \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \log \left( E[X|Y] \right) \right] \\
\geq E \left[ \left( \frac{E[Y]}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \log \left( \frac{a X}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
= E \left[ \left( \frac{a X}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \log \left( \frac{a X}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right],
\]

(129)

(130)

(131)

(132)

where in (132) we have used Jensen’s inequality. Observe that (132) leads to (22), the fact that \( Y \) is according to a negative-binomial was show in (22).

To show the upper bound in (20) observe that

\[
E \left[ X \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
= E \left[ X \log \left( \frac{Y}{\alpha + a} + \frac{\theta}{\alpha + a} \right) | X \right] \\
\leq E \left[ X \log \left( \frac{E[Y]}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right] \\
= E \left[ X \log \left( \frac{a X}{\alpha + a} + \frac{\theta}{\alpha + a} \right) \right],
\]

(133)

(134)

(135)

(136)

where the last step is a consequence of Jensen’s inequality. This concludes the proof.
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