Sparsity Prior Regularized Q-learning for Sparse Action Tasks

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Abstract

In many decision-making tasks, some specific actions are limited in their frequency or total amounts, such as “fire” in the gunfight game and “buy/sell” in the stock trading. We name such actions as “sparse action”. Sparse action often plays a crucial role in achieving good performance. However, their Q-values, estimated by classical Bellman update, usually suffer from a large estimation error due to the sparsity of their samples. The greedy policy could be greatly misled by the biased Q-function and takes sparse action aggressively, which leads to a huge sub-optimality. This paper constructs a reference distribution that assigns a low probability to sparse action and proposes a regularized objective with an explicit constraint to the reference distribution. Furthermore, we derive a regularized Bellman optimality operator and a regularized optimal policy. The regularized Bellman optimality operator holds the same basic properties as the classical ones. Besides, the regularized way to derive policy implicitly ensures that only when the Q-values of sparse action have a significant advantage over the remaining actions can the sparse action be taken with a high probability. This technique guides the agent to take sparse action more carefully. Based on the regularized Bellman operator and the regularized optimal policy, we propose a new algorithm, namely Sparsity Prior Regularized Q-learning (SPRQ), for solving sparse action tasks.

1 INTRODUCTION

In recent years, reinforcement learning (RL) has achieved much success in various decision-making tasks, such as games [Mnih et al. 2015, Silver et al. 2016], robotic control [Fujimoto et al. 2018, Haarnoja et al. 2018], etc. In many real-world decision-making tasks, some actions are sparse and cannot be arbitrarily executed, such as the “buy/sell” in the stock trading, the “shooting” in the football game, and the “firing” in the gunfight game. In this paper, we name such actions as “sparse action”. Sparse action often plays a crucial role in obtaining a high reward, even the only reward. For instance, the agent can get a positive reward only if it scores by taking the “shooting” action in the football game. Most existing RL methods [Mnih et al. 2015, Lillicrap et al. 2016, Fujimoto et al. 2018] first estimate a Q-function via classical Bellman optimality update [Puterman 2014] and then derive a policy that is greedy with regard to the estimated Q-function. With limited training samples containing sparse action, the Q-values of sparse action are inaccurate. Even worse, the maximum operator [Brim 2020, Lan et al. 2020] in classical Bellman update magnifies the error. As a consequence, the estimated Q-values of sparse action suffer a huge over-estimation error. Besides, the greedy way to derive policy could be greatly misled by the biased Q-function. The greedy policy exploits the over-estimation error on sparse action and aggressively takes sparse action, leading to a huge sub-optimality.

To guide the agent to take sparse action more cautiously, we construct a reference distribution that assigns a low probability to sparse action and then propose a regularized objective with an explicit constraint to the reference distribution. Under the regularized objective, we derive a regularized Bellman optimality operator and a regularized optimal policy. The regularized Bellman optimality operator holds the same basic properties as the classical ones. Besides, the regularized Bellman operator can slow down the propagation of over-estimation error and results in a more accurate Q-function. Furthermore, the regularized way to derive policy implicitly ensures that only when the Q-values of sparse action have a significant advantage over the remaining actions can the sparse action be taken with a high probability. This technique guides the agent to take sparse action more carefully. Based on the regularized Bellman operator and the regularized optimal policy, we propose a new algorithm, namely Sparsity Prior Regularized Q-learning (SPRQ), for solving sparse action tasks.

We evaluate SPRQ on three sparse action tasks and compare SPRQ with previous successful RL algorithms (e.g., SAC [Haarnoja et al. 2018], Rainbow [Hessel et al. 2018], con-
strained policy optimization method (CPO [Achiam et al., 2017] and PPO [Schulman et al., 2017]). Empirical results on three environments show that SPRQ achieves a substantial performance improvement. Ablation studies are conducted further to illustrate the importance of each component in SPRQ.

2 BACKGROUND

2.1 MARKOV DECISION PROCESS

We consider the infinite horizon discounted Markov Decision Process (MDP) [Puterman, 2014; Sutton and Barto, 2011], which is described as a tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, r, \gamma, d_0)$. Here $\mathcal{S}$ is the state space, $\mathcal{A}$ is the finite action space where $\mathcal{A} = \{a_0, a_1, \ldots, a_{|A|-1}\}$. Without loss of generality, we use $a_0$ to denote the sparse action. $P$ denotes the transition probability and $r$ denotes the reward function. $\gamma$ is the discount factor which determines the weights of future rewards and $d_0$ specifies the initial state distribution. A (stationary) policy $\pi: \mathcal{S} \rightarrow \Delta(\mathcal{A})$ is a mapping from state space to the probability space over action space. The agent interacts with the environment as follows: at time step $t$, the agent observes a state $s_t$ from the environment and executes an action $a_t \sim \pi(\cdot|s_t)$, then the agent receives a reward $r(s_t, a_t)$ and transits to the next state $s_{t+1} \sim P(\cdot|s_t, a_t)$.

Given a policy $\pi$, the value function is defined as the cumulative rewards it receives when starting with state $s$ and following $\pi$.

$$ V^\pi(s) := \mathbb{E}_t \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) | s_0 = s, a_t \sim \pi(\cdot|s_t) \right]. $$

(1)

Correspondingly, the state-action value function is defined as the cumulative rewards it receives when starting with state $s$, taking action $a$, and following policy $\pi$.

$$ Q^\pi(s, a) := r(s, a) + \gamma \mathbb{E}_{s', r'} \pi(\cdot|s, a) \left[ V^\pi(s') \right]. $$

(2)

An important connection between $V^\pi(s)$ and $Q^\pi(s, a)$ is shown as follows.

$$ V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ Q^\pi(s, a) \right]. $$

(3)

Then we define the optimal value function and state-action value function, $V^* (s) := \max_{\pi \in \Delta(\mathcal{A})} Q^\pi(s, a)$, $Q^* (s, a) := \max_a Q^\pi(s, a)$. Note that $V^*(s)$ and $Q^*(s,a)$ also hold the connection shown in Eq.(2) and Eq.(3). The optimal policy $\pi^*$ can be a deterministic policy which acts greedily with $Q^*$ (i.e., $\pi^*(s) = \arg\max_a Q^*(s, a)$).

Given a policy $\pi \in \Delta(\mathcal{A})$, the associated Bellman operator $\mathcal{T}^\pi$ is defined as, for any function $Q: \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$,

$$ \mathcal{T}^\pi Q(s, a) := r(s, a) + \gamma \mathbb{E}_{s', r'} \pi(\cdot|s, a) Q(s', a'). $$

From Eq.(2) and Eq.(3), we have that $Q^\pi(s, a)$ is the fixed point of $\mathcal{T}^\pi$ (i.e., $\mathcal{T}^\pi Q^\pi = Q^\pi$). Based on the Bellman operator, we can define the Bellman optimality operator $\mathcal{T}^*$. For any function $Q: \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$, $\mathcal{T}^* := \max_{\pi \in \Delta(\mathcal{A})} \mathcal{T}^\pi Q$. It is known that $Q^*(s, a)$ is the fixed point of $\mathcal{T}^*$ and $\mathcal{T}^*$ is a $\gamma$-contraction with respect to the infinity norm (i.e., $\forall Q_1, Q_2 \in \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}, ||\mathcal{T}^* Q_1 - \mathcal{T}^* Q_2||_\infty \leq \gamma ||Q_1 - Q_2||_\infty$). With these properties, (approximate) value iteration applies the Bellman optimality operator repeatedly to attain the optimal state-action value function [Bertsekas and Tsitsiklis, 1996; Munos, 2005].

2.2 REGULARIZED MARKOV DECISION PROCESS

In recent years, there emerged many successful RL methods [Haarnoja et al., 2018, 2017; Schulman et al., 2015, 2017] which utilize regularization for many purposes (e.g., encouraging exploration [Fox et al., 2016; Haarnoja et al., 2017, 2018] and monotonous policy improvement [Schulman et al., 2015, 2017]). Regularized Markov Decision Process (RMDP) [Geist et al., 2019; Yang et al., 2019] is a general framework that integrates such a big class of RL methods. Under RMDP, the reward function is augmented with a regularization term.

$$ r_{\Omega}(s, a) = r(s, a) - \lambda \Omega(\pi(\cdot|s)),$$

where $\lambda \in \mathbb{R}$ and $\Omega: \Delta(\mathcal{A}) \rightarrow \mathbb{R}$ is a regularization function. Under RMDP, we aim to find a policy to maximize the cumulative regularized rewards.

$$ \max_{\pi} \mathbb{E}_t \left[ \sum_{t=0}^{\infty} \gamma^t r_{\Omega}(s_t, a_t) \right]. $$

When the regularization function is chosen as $\Omega(\pi(\cdot|s)) = \mathbb{E}_{a \sim \pi(\cdot|s)} \log \pi(a|s)$, one can recover the objective of maximum-entropy RL [Fox et al., 2016; Haarnoja et al., 2017, 2018].

Like in MDP, we can define the regularized value function $V^\Omega_{\Omega}(s)$ and the regularized state-action value function $Q^\Omega_{\Omega}(s, a)$ in RMDP by replacing $r(s, a)$ with $r_{\Omega}(s, a)$. Under RMDP [Geist et al., 2019], the connection between $V^\pi_{\Omega}(s)$ and $Q^\pi_{\Omega}(s, a)$ is modified as

$$ Q^\pi_{\Omega}(s, a) := r(s, a) + \gamma \mathbb{E}_{s', r'} \pi(\cdot|s, a) \left[ V^\pi_{\Omega}(s') \right], $$(4)$$ V^\pi_{\Omega}(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ Q^\pi_{\Omega}(s, a) \right] - \lambda \Omega(\pi(\cdot|s)). $$

The regularized optimal value function and the regularized optimal state-action value function are defined as, $V^*_{\Omega}(s) = \max_{\pi} V^\pi_{\Omega}(s)$, $Q^*_{\Omega}(s, a) = \max_{\pi} Q^\pi_{\Omega}(s, a)$. $V^*_{\Omega}$ and $Q^*_{\Omega}$ also hold the connection in Eq.(4).
3 SPARSITY PRIOR REGULARIZED Q-LEARNING

3.1 THE KEY IDEA

In sparse action tasks, the opportunities of taking sparse action are limited, and thus, the agent needs to decide when to execute sparse action more carefully. As the training samples with sparse action are scant, the estimated Q-value on sparse action could suffer a large estimation error [Fu et al., 2019; Lan et al., 2020]. Therefore, the conventional way that the agent selects actions greedily with the estimated Q-function [Mnih et al., 2015] is inefficient. Concretely, the following Lemma provides the sub-optimality of the greedy policy.

Lemma 3.1 (Corollary 2 in [Singh and Yee 1994]). Given an estimated Q-function $Q$ with $\|Q - Q^*\|_\infty \leq \epsilon$, let $\pi_Q$ denote the greedy policy regarding $Q$, we have that

$$\|V^* - V^{\pi_Q}\|_\infty \leq \frac{\epsilon}{1 - \gamma}.$$  \hspace{1cm} (5)

Note that the error of Q-function is measured by the maximum difference over all state-action pairs. With scant training samples containing sparse action, the Q-function error could be dominated by the error on sparse action (i.e., $\|Q - Q^*\|_\infty = \|Q - Q(s, a_0) - Q(s, a_0)\|$). Furthermore, deriving policy in a greedy way could be greatly misled by the biased Q-function since the greedy policy may exploit the over-estimation error on sparse action and takes it aggressively. Such a way may cause the largest value loss ($\frac{1}{\gamma^T}$ in the RHS of Eq. (5)) as sparse action often plays a decisive role in obtaining the largest reward, even the only reward.

To guide the agent to take sparse action more carefully, we propose a regularized objective with an explicit constraint to the reference distribution.

$$\max_{\pi} \mathbb{E}_{s \sim P, a \sim \pi(s|s)} \left[ \sum_{t=0}^{\infty} \gamma^t \left( r(s, a_t) - \lambda D_{\text{KL}}(\pi(\cdot|s_t), \pi_{\text{ref}}(\cdot)) \right) d_0, \pi \right],$$  \hspace{1cm} (6)

where $\pi_{\text{ref}}$ is the reference distribution which assigns a small probability $\delta$ to sparse action and an identical probability of $\frac{1-\delta}{|A|-1}$ to the other actions \(\left(\text{i.e., } \pi_{\text{ref}}(a) = \left(\delta, \frac{1-\delta}{|A|-1}, \ldots, \frac{1-\delta}{|A|-1}\right), \forall s \in S\right)\). $D_{\text{KL}}$ is the Kullback-Leibler (KL) divergence. Intuitively, adding the regularization term could prevent that the optimized policy is far from the reference distribution and thus, the optimized policy could be more careful when taking sparse action. The soft q-learning [Haarnoja et al., 2017, 2018] selects the uniform distribution as the reference distribution to encourage exploration. Different from the soft q-learning, the constructed reference distribution here provides a prior information regarding sparse action, which prevents the agent from taking sparse action too aggressively. Notice that the reference distribution is not a good policy and the KL divergence term does not serve as the main objective as in imitation learning where the target is to imitate the reference distribution [Pomerleau 1991; Torabi et al., 2018]. The regularization term here aims to guide the policy to take sparse action more cautiously.

3.2 OUR METHOD

Under the regularized objective of Eq. (6), we first derive the regularized Bellman optimality operator, which serves as a fundamental brick for our algorithm.

Proposition 3.1. Under the regularized objective shown in Eq. (6), the regularized Bellman optimality operator $\mathcal{T}_{\Omega}$ is defined as, for any function $Q : S \times A \rightarrow \mathbb{R}$,

$$\mathcal{T}_{\Omega} Q(s, a) = r(s, a) + \gamma \lambda \mathbb{E}_{d' \sim P(\cdot|s, a)} \left[ \exp \left( \frac{Q(s', a')}{\lambda} \right) \right].$$  \hspace{1cm} (7)

The regularized optimal state-action value function $Q^*_{\Omega}$ is the fixed point of $\mathcal{T}_{\Omega}$. The regularized optimal policy $\pi^*_{\Omega}$ can be obtained by

$$\pi^*_{\Omega}(s, a) = \frac{p_{\text{ref}}(s, a)}{Z(s)},$$  \hspace{1cm} (8)

where $Z(s) = \sum_{a \in A} p_{\text{ref}}(s, a) \exp \left( \frac{Q^*_{\Omega}(s, a)}{\lambda} \right)$ is the partition function.

Proof. We first derive $\pi^*_{\Omega}$ from $Q^*_{\Omega}$. From Eq. (4), $\forall s \in S$,

$$\pi^*_{\Omega}(\cdot|s) = \arg\max_{\pi} \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ Q^*_{\Omega}(s, a) \right] - \lambda D_{\text{KL}}(\pi(\cdot|s), \pi_{\text{ref}}(\cdot)).$$

Differentiate the right side with respect to $\pi(a|s)$ and let the derivative equal zero, we obtain that

$$\sum_{a \in A} Q^*_{\Omega}(s, a) - \lambda \log \left( \frac{\pi^*(a|s)}{p_{\text{ref}}(a)} \right) - \lambda = 0.$$  \hspace{1cm} (9)

Solving this formula yields

$$\pi^*_{\Omega}(a|s) \propto p_{\text{ref}}(a) \exp \left( \frac{Q^*_{\Omega}(s, a)}{\lambda} \right).$$

Let $\sum_{a \in A} \pi^*_{\Omega}(a|s) = 1$ and we can obtain the final result. Next, we prove that $Q^*_{\Omega}$ is the fixed point of $\mathcal{T}_{\Omega}$. From Eq. (4), we can obtain that

$$Q^*_{\Omega}(s, a) = r(s, a) + \gamma \mathbb{E}_{d' \sim P(\cdot|s, a)} \left[ V^*_{\Omega}(s') \right],$$

with $V^*_{\Omega}(s) = \mathbb{E}_{a \sim \pi^*_{\Omega}(\cdot|s)} \left[ Q^*_{\Omega}(s, a) \right] - \lambda D_{\text{KL}}(\pi^*_{\Omega}(\cdot|s), \pi_{\text{ref}}(\cdot)).$
Substitute $\pi^*_\Omega(a|s)$ with $p_{\text{ref}}(a)$ and $\exp\left(\frac{Q_{\text{ref}}(s,a)}{\lambda}\right)/Z(s)$ and we obtain that

$$V^*_\Omega(s) = \log \left( \mathbb{E}_{a \sim p_{\text{ref}}(\cdot)} \left[ \exp\left(\frac{Q^*_\Omega(s,a)}{\lambda}\right) \right] \right).$$

Then we have that

$$Q^*_\Omega(s,a) = r(s,a) + \gamma \lambda \mathbb{E}_{a' \sim p_{\text{ref}}(\cdot)} \left[ \log \left( \mathbb{E}_{a'' \sim p_{\text{ref}}(\cdot)} \left[ \exp\left(\frac{Q^*_\Omega(s',a'')}{\lambda}\right) \right] \right) \right],$$

which shows that $Q^*_\Omega$ is the fixed point of $T^*_\Omega$.

**Remark 3.1.** When $p_{\text{ref}}$ is the greedy policy with respect to $Q$, $T^*_\Omega$ is reduced to the original Bellman optimality operator. When $p_{\text{ref}}$ is the uniform distribution, $T^*_\Omega$ is reduced to the soft Bellman optimality operator in [Haarnoja et al., 2017, 2018].

Compared to the classical Bellman update, the proposed regularized Bellman optimality operator and regularized optimal policy have two key advantages for solving sparse action tasks. 1) Observe that the target value is averaged according to $p_{\text{ref}}$ (i.e., $\mathbb{E}_{a \sim p_{\text{ref}}(\cdot)} \exp\left(\frac{Q(s',a')}{\lambda}\right)$ in Eq.(7)). When the estimated Q-function on sparse action suffers a error due to limited samples, $T^*_\Omega$ could reduce the error propagated from the next state as $p_{\text{ref}}$ puts a low probability mass on sparse action. This approach can slow down the propagation of over-estimation error and results in a more accurate Q-function. 2) Different from acting greedily regarding the estimated Q-function, the way of deriving policy in Eq.(8) ensures that when the estimated Q-values on sparse action have a great advantage over the other actions, the sparse action could be taken with a high probability. To be more specific, only if $Q^{\text{ref}}_\Omega(s,a_0) - Q^*_\Omega(s,a_0) \geq \log\left(\frac{1}{\gamma\lambda}\right)$ for all $a \in A \setminus \{a_0\}$, $\pi^*_\Omega(a_0|s)$ has the highest probability. This technique guides agent to take sparse action more carefully.

We verify that the proposed regularized Bellman optimality operator satisfies the same properties as the classical ones to design algorithms. For simplicity, we treat any $Q$, $T^*_\Omega Q \in S \times A \rightarrow \mathbb{R}$ as a vector whose size is $|S||A|$.

**Proposition 3.2.** The regularized Bellman optimality operator satisfies the following properties.

- **Monotonicity:** for any $Q_1, Q_2 \in S \times A \rightarrow \mathbb{R}$ such that $Q_1 \geq Q_2$, then we have

  $$T^*_\Omega Q_1 \geq T^*_\Omega Q_2,$$

  where $\geq$ means element-wise greater or equal.

- **Contraction:** for any $Q_1, Q_2 \in S \times A \rightarrow \mathbb{R}$, we have

  $$\|T^*_\Omega Q_1 - T^*_\Omega Q_2\|_\infty \leq \gamma\|Q_1 - Q_2\|_\infty.$$

See Appendix [A.1] for the detailed proof. As the proposed regularized Bellman optimality operator is a $\gamma$-contraction, we can repeatedly apply $T^*_\Omega$ to attain the optimal regularized state-action value function. In particular,

$$Q^{k+1}(s,a) = r(s,a) + \gamma \lambda \mathbb{E}_{a' \sim p_{\text{ref}}(\cdot)} \left[ \log \left( \mathbb{E}_{a'' \sim p_{\text{ref}}(\cdot)} \left[ \exp\left(\frac{Q^k(s',a'')}{\lambda}\right) \right] \right) \right].$$

Following [Mnih et al., 2015, Haarnoja et al., 2017], we can approximate the above update in an off-policy manner. In particular, we parameterize Q-function with $\theta$. Given a replay buffer $D = \{(s_i,a_i,r_i,s'_i)\}_{i=1}^m$, our target is to minimize the following empirical risk.

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^m (Q\theta(s_i,a_i) - y)^2, \quad (9)$$

where $y = r_i + \gamma \log \left( \mathbb{E}_{a' \sim p_{\text{ref}}(\cdot)} \left[ \exp\left(\frac{Q\theta(s'_i,a')}{\lambda}\right) \right] \right)$ and $\bar{\theta}$ are the target parameters updated by the Polyak averaging [Polyak and Juditsky, 1992].

Besides the original relay buffer used in [Mnih et al., 2015, Hessel et al., 2018], we also maintain a replay buffer $D_S$ to save the transitions containing sparse action. In each update, we additionally draw samples from $D_S$ to optimize the objective shown in Eq.(9). This technique improves the training on samples with sparse action, which is crucial in reducing Q-values’ error on sparse action. The experimental results in Section 5 validate that this technique can lead to an improvement. Following [Fujimoto et al., 2018, Haarnoja et al., 2018], we utilize two Q-functions and set the target to the minimum of Q-functions in Eq.(9) to mitigate the over-estimation error.

Given an estimated Q-function $Q_\theta$, one can compute the regularized optimal policy analytically by Eq.(8),

$$\pi(a|s) = \frac{p_{\text{ref}}(a) \exp\left(\frac{Q_\theta(s,a)}{\lambda}\right)}{Z(s)}$$

where $Z(s) = \sum_{a \in A} p_{\text{ref}}(a) \exp\left(\frac{Q_\theta(s,a)}{\lambda}\right)$. The final algorithm is formally described in Algorithm [1].

### 3.3 THEORETICAL IDENTIFICATION

In general, adding the regularization function to the original objective changes the MDP and the corresponding optimal policy. Following the careful selection of reference distribution, the regularized optimal policy also performs well on the original MDP.

**Proposition 3.3.** Let $\pi^*_\Omega$ denote the regularized optimal policy and $V^*_\Omega$ denote the value of $\pi^*_\Omega$ evaluated in the original MDP. Then we have that, for all $s \in S$

$$V^*_\Omega(s) \geq V^*(s) - \frac{C}{1 - \gamma}.$$
Algorithm 1 Sparsity Prior Regularized Q-Learning (SPRQ)

1: Input: the reference distribution \( p_{\text{ref}} \), total iterations \( N \), the number of environment steps \( N_E \), the number of gradient steps \( N_G \), the learning rate \( \eta \), the Polyak update stepsize \( \tau \).
2: Initialize Q-functions \( Q_{\theta_1} \) and \( Q_{\theta_2} \), synchronize the target Q-functions \( Q_{\bar{\theta}_1} \) and \( Q_{\bar{\theta}_2} \) with \( \bar{\theta}_1 \leftarrow \theta_1 \), \( \bar{\theta}_2 \leftarrow \theta_2 \), and empty replay buffer \( D \) as well as \( D_S \).
3: for \( N \) iterations do
4:   for \( N_E \) iterations do
5:     Sample action, \( a_t \sim \pi(\cdot|s_t) \) where \( \pi \) is derived by Eq. (10) with \( Q_{\theta_1} \).
6:     Receive reward and next state from the environment, \( r_t = r(s_t, a_t), s_{t+1} \sim p(\cdot|s_t, a_t) \).
7:     Store the transition into the replay buffer, \( D \leftarrow D \cup \{(s_t, a_t, r_t, s_{t+1})\} \).
8:     if \( a_t \) is sparse action then
9:       Store the transition into the replay buffer with sparse action, \( D_S \leftarrow D_S \cup \{(s_t, a_t, r_t, s_{t+1})\} \).
10:  end if
11:  for \( N_G \) iterations do
12:    Sample transitions from \( D \) and \( D_S \) together, update the Q-function parameters, \( \theta_i \leftarrow \theta_i - \eta \nabla_{\theta_i} J(\theta_i) \) for \( i \in \{1, 2\} \).
13:    Update the target parameters, \( \bar{\theta}_i \leftarrow \tau \theta_i + (1 - \tau) \bar{\theta}_i \) for \( i \in \{1, 2\} \).
14:  end for
15: end for
16: Output: \( Q_{\theta_1}, Q_{\theta_2} \) and the derived policy \( \pi \).

where \( C = \lambda \max \left( \log \left( \frac{1}{\phi} \right), \log \left( \frac{|A| - 1}{1 - \phi} \right) \right) \).

We refer the readers to Section A.2 in Appendix for detailed proof. Proposition 3.3 indicates that the difference of policy value is controlled by both \( \lambda \) and \( \delta \). As \( \lambda \) getting smaller, the value of \( \pi^* \) is approaching the optimal value. However, the restriction to the reference distribution is weaker and the agent takes sparse action more aggressive. The optimizing policy may not converge to the optimal policy because of the aggressive strategy. In Section 5.3 we show the influence of \( \lambda \) on the empirical performance.

4 RELATED WORK

Regularized RL Methods. Regularized Markov Decision Process [Geist et al., 2019] is a unified framework that contains many successful RL approaches. [Schulman et al., 2015, 2017] [Haarnoja et al., 2017, 2018]. For different purposes, different regularizers and different reference distributions are utilized. In [Schulman et al., 2015, 2017], they choose the policy obtained in the previous iteration as the reference distribution to achieve monotonic policy improvement. In soft q-learning [Haarnoja et al., 2017, 2018], the uniform distribution acts as the reference distribution, and the regularizer can be regarded as a reward bonus to encourage exploration. In this paper, we construct a new reference policy that encodes prior information regarding sparse action, and the regularizer is to guide the agent to take sparse action more carefully.

Constrained Policy Optimization. Sparse action can be regarded as an implicit constraint to be considered while searching for the optimal policy. Previous works have studied optimizing a policy under constraints [Uchibe and Doya, 2007] [Achiam et al., 2017] [Tessler et al., 2019] [Chow et al., 2017]. A standard and well-studied formulation for RL with constraints is the constrained Markov Decision Process (CMDP) [Altman and Shwartz, 1991], where the agent must satisfy constraints on expectations of additional costs. Constrained policy optimization (CPO) [Achiam et al., 2017] is the first deep RL method for solving CMDP, which provides guarantees for monotonic improvement in both reward and satisfaction of constraints on other costs. Nevertheless, we cannot directly apply CPO to solve sparse action tasks since it is hard to transform a sparse action task into a CMDP equivalently. In experiments, we run CPO by sending agents a cost when it takes sparse action. We refer the readers to Section 5.4 for the detailed CPO performance on sparse action tasks.

Self-Imitation Learning. Self-imitation learning (SIL) [Oh et al., 2018] [Blundell et al., 2016] [Lengyel and Dayan, 2007] [Pritzel et al., 2017] improves policy learning by focusing on good experiences in the past. For example, episodic control [Blundell et al., 2016] [Lengyel and Dayan, 2007] [Pritzel et al., 2017] and the nearest neighbor policy [Mansimov and Chen, 2018] construct a non-parametric policy directly from the past experience by retrieving observed similar states and choosing the best action made in the past. Inspired by the idea of SIL, SPRQ reuses experiences containing sparse action to improve the accuracy of Q-function estimation.

5 EXPERIMENTS

We evaluate SPRQ on three typical sparse action tasks: football shooting, gunfight, and stock trading as shooting opportunities, bullet capacity, and the number of stock transactions are limited in three tasks, respectively. Those tasks cover observation space from low dimension (\( \mathbb{R}^{16} \)) to high dimension (\( \mathbb{R}^{161} \)), and the degree of difficulty from easy (20%)
Appendix B.1. 

This section will first introduce comparison methods described as follows. (1) SAC [Haarnoja et al., 2018]: SAC is related to SPRQ as it is also based on the regularized Markov decision process and adds an entropy regularizer to reward function for improving exploration capacity. (2) Rainbow [Hessel et al., 2018]: To the best of our knowledge, Rainbow is the state-of-the-art RL algorithm for solving reinforcement learning tasks with discrete action space. In our implementation, Rainbow is DQN equipped with techniques of double Q-network, prioritized replay buffer, dueling network, and noisy layers. (3) CPO [Achiam et al., 2017]: CPO searches a policy under constraints. It has guarantees for near-constraint satisfaction on the given costs. In sparse action tasks, we set the cost in CPO as the consumption of sparse action opportunities. (4) PPO [Schulman et al., 2017]: PPO is a popular policy optimization method, and we compare the clipping version of PPO. We utilize Rainbow, SAC, and PPO implemented in stable baselines [Hill et al., 2018] and open-source CPO implementation to conduct comparison experiments. The hyper-parameters of the comparison methods are selected carefully by tuning.

Experiment details. All policy/critic networks in experiments use the same network architecture. In each experiment, the agent interacts with the environment for 6 million timesteps, and 40 episodes’ evaluation obtains each data point. The solid lines are the mean of results, and the shaded region corresponds to the standard deviation over 3 random seeds. Further descriptions of hyper-parameters and network architecture are shown in Appendix B.3.

5.2 EXPERIMENT RESULTS

To illustrate the performance of SPRQ, experimental results will be shown in the following themes:

1) episodic reward during training process on different sparse action tasks;
2) policy distribution under various optimization manners;
3) sparse action frequency during the training process;
4) comparison of SPRQ in different evaluation manners;
5) ablation studies on sparse action buffer and regularizer coefficient.

Episodic reward during training process. Figure 1 shows the episodic reward curves during the training process. Overall, SPRQ outperforms all comparison methods in three tasks in terms of the final performance and shows stable improvement during the training process. SAC performs poorly in all sparse action tasks, indicating that the maximum entropy regularized objective is not suitable for solving sparse action problems. Rainbow fails to make any progress in football and gunfight, and suffers from high variance between different random trails. Such a result is caused by epsilon-greedy exploration mode, in which the opportunities of executing sparse action are wasted in the early stage of one trajectory. CPO performs better in the gunfight task but hardly learns anything in the stock task. The policy learned by PPO has a good performance on all three tasks. Among all methods, only SPRQ obtains a positive score in stock-
test. The stock-test task is somewhat more difficult since the agent cannot interact with it during the training process.

Figure 2: Policy distributions in the stock trading task. Each axis in the figure represents the probability of executing specific action, where three axes stand for “no-op”, “buy-in”, and “sell-out” respectively. The “buy-in” and “sell-out” are sparse actions in this task. Each point stands for policy distribution at one decision-making step.

Analysis of policy distribution. After the training process, we use the policies trained by SPRQ, SAC, and PPO, to sample 600 environment steps in the stock trading task and record the policy distribution to present the policy’s tendency to perform each action. Results in Figure 2 show the policy distribution of different policies in the way of space rectangular coordinate system. Policy learned by SPRQ prefers no-op (this action is not a sparse action) at most decision-making time while inclined to take sparse actions at a small portion of decision steps, which are crucial steps during the decision process. PPO learns a policy that converges to the same policy distribution where the probability of executing each action is the identity on both train and test environments. The policy of SAC is relatively random in the stock trading task as its policy distribution takes place in a large space in the space rectangular coordinate system.

Figure 3: Episodic reward and frequency of executing sparse action during SPRQ training. The x-axis represents timesteps and two y-axes represent episodic reward and sparse action frequency, respectively.

Analysis of the sparse action frequency and reward. In sparse action tasks, most positive returns come from executing sparse action at the decisive step. Nevertheless, reference distribution in SPRQ encourages the agent to reduce the execution of sparse action. To examine whether the agent avoids executing sparse action too much due to the regularized objective, we conduct experiments to show the sparse action frequency during the training process. Results are shown in Figure 3. At the beginning of the training process, sparse action frequency is fairly low given that the randomly initialized Q-network does not assign a much greater value on sparse action than other actions. Thus, the agent will not execute sparse action based on the analysis in Section 3.2 Sparse action frequency is growing as the Q-function gets updated, and meanwhile, the policy’s performance is getting better, illustrating that the policy learns to execute sparse action at the proper states.

5.3 ABLATION STUDY

The results in section 5.2 suggest that SPRQ can outperform state-of-the-art RL methods on typical sparse action tasks and show the policy distribution as well as sparse action frequency during the SPRQ training process. In this section, we conduct ablation studies on evaluation manners, sparse action replay buffer, and regularizer coefficient to further examine the influence of each component in SPRQ on the final performance.

Evaluation of stochastic and deterministic policy. In conventional RL methods [Schulman et al., 2015, 2017 Haarnoja et al., 2018], the derived deterministic policy (i.e., $\pi_{det}(s) = \arg\max_{a \in A} \pi_{det}(a|s)$) often achieves a better performance than the stochastic policy. When it comes to SPRQ, nevertheless, the optimal policy (Eq. 10) is a stochastic policy. Here we conduct experiments to test the empirical performance of the stochastic and deterministic policy in SPRQ. The first row in Figure 4 shows the evaluation results during the training process. The stochastic policy performs comparably to the deterministic policy on Gunfight and Stock-train, and outperforms it on Football and Stock-test. The stochastic policy in SPRQ is more recommended than the deterministic policy from the perspective of empirical performance. As discussed in Section 5.2, the stochastic policy also has theoretical advantages for sparse action tasks. It is worth mentioning that the stochastic policy is far better than the deterministic policy in stock-test, which indicates that the stochastic policy in SPRQ also has a good generalization.

Ablation study on sparse action replay buffer. In the original replay buffer, the samples containing sparse action are far less than other actions as sparse action is less executed. We propose a sparse action replay buffer to strengthen the learning of Q-values on sparse action. The second row in Figure 4 shows the episodic reward of SPRQ trained with and without sparse action replay buffer, where “w/” and “w/o”
Ablation study on regularizer coefficient. The regularizer coefficient ($\lambda$) is an essential component in SPRQ as it controls the degree of restriction to the reference distribution. When $\lambda$ is getting closer to 0, the update rule of SPRQ turns into the update rule in DQN [Mnih et al., 2015]. With a too large $\lambda$, the policy cannot learn from the original reward and converges to the reference distribution. The third row in Figure 4 shows the episodic reward curves of the policies optimized under various regularizer coefficients from 0.01 to 1. Results show that SPRQ with $\lambda = 0.05$ performs well in all tasks. In football shooting tasks, smaller $\lambda$ produces a better policy performance. The policy improvement is getting slower or even stops at the early training stage when $\lambda$ is too large, corresponding to a strong restriction to the reference distribution. This conclusion can be verified from experiment results that policy fails to learn anything in all experiment tasks when $\lambda = 1$.

6 CONCLUSION

This paper presents sparse action problem in reinforcement learning and proposes an approach to solving it, namely Sparsity Prior Regularized Q learning (SPRQ). As the opportunities of executing sparse action are limited, we hope agents take sparse action more cautiously. To realize such an idea, a reference distribution that assigns sparse action a low probability is constructed to assist in policy optimization by the regularized Bellman optimality operator. Under such an operator, the agent searches for a policy to maximize the expected reward and simultaneously minimize the reference distribution distance. Furthermore, we obtain the regularized Bellman update rule and the optimal policy derived from the Q-value, which formulate SPRQ. SPRQ is compared with several state-of-the-art RL methods on three typical sparse action tasks and achieves a better performance. Besides, SPRQ also enjoys several theoretical properties.

To the best of our knowledge, this is the first work to study sparse action problem formally. Note that the proposed regularized Bellman update rule in SPRQ can generalize to...
an arbitrary reference distribution. When it comes to other
decision-making tasks except for sparse action tasks, SPRQ
can be applied if prior knowledge is available as the refer-
dence distribution can be modified. We hope this work could
inspire researchers a new idea of policy optimization in rein-
fforcement learning and provides a new thinking to introduce
prior knowledge into policy optimization.

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A OMITTED PROOFS

A.1 PROOF OF PROPOSITION 3.2

Proof of Proposition 3.2. We first prove the monotonicity. For any \( Q_1, Q_2 \in S \times A \to \mathbb{R} \) such that \( Q_1 \geq Q_2 \), for any \((s,a) \in S \times A\), we prove that

\[
T^*_Q Q_1(s,a) \geq T^*_Q Q_2(s,a).
\]

Recall that

\[
T^*_Q Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \log \left( E'_{s' \sim P(\cdot|s,a)} \left[ \exp \left( \frac{Q(s',a')}{\lambda} \right) \right] \right) \right].
\]

Since \( Q_1 \geq Q_2 \) and \( \lambda > 0 \), we have that

\[
\mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \log \left( E'_{s' \sim P(\cdot|s,a)} \left[ \exp \left( \frac{Q_1(s',a')}{\lambda} \right) \right] \right) \right] \geq \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \log \left( E'_{s' \sim P(\cdot|s,a)} \left[ \exp \left( \frac{Q_2(s',a')}{\lambda} \right) \right] \right) \right].
\]

From the monotonicity of the logarithm function, we show that

\[
T^*_Q Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \log \left( E'_{s' \sim P(\cdot|s,a)} \left[ \exp \left( \frac{Q_1(s',a')}{\lambda} \right) \right] \right) \right] \geq \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[ \log \left( E'_{s' \sim P(\cdot|s,a)} \left[ \exp \left( \frac{Q_2(s',a')}{\lambda} \right) \right] \right) \right] \geq T^*_Q Q_2(s,a).
\]

Then it is direct to obtain that for any \((s,a) \in S \times A\),

\[
\| T^*_Q Q_1 - T^*_Q Q_2 \| = \max_{(s,a) \in S \times A} \| T^*_Q Q_1(s,a) - T^*_Q Q_2(s,a) \|.
\]

Given \((s,a) \in S \times A\), without loss of generality, we assume that \( T^*_Q Q_1(s,a) \geq T^*_Q Q_2(s,a) \). Then we have \( \| T^*_Q Q_1(s,a) - T^*_Q Q_2(s,a) \| = T^*_Q Q_1(s,a) - T^*_Q Q_2(s,a) \). Let \( \pi_1(\cdot|s) = \arg \max_\pi \left\{ E_{a \sim \pi_1(\cdot|s)}[Q_1(s,a)] \right\} \), and

\[
\pi_2(\cdot|s) = \arg \max_\pi \left\{ E_{a \sim \pi_2(\cdot|s)}[Q_2(s,a)] \right\} - \lambda D_{KL}(\pi(\cdot|s), \pi_{\text{ref}}(\cdot)).
\]

Then we obtain that

\[
\begin{align*}
T^*_Q Q_1(s,a) - T^*_Q Q_2(s,a) & = \gamma E'_{s' \sim P(\cdot|s,a)}[V_1(s')] - V_1(s) \\
& \leq \gamma E'_{s' \sim P(\cdot|s,a)} \left[ E_{a' \sim \pi_2(\cdot|s,a')} [Q_1(s',a')] - \pi_1(\cdot|s) - \lambda D_{KL}(\pi(\cdot|s), \pi_{\text{ref}}(\cdot)) \right] - V_2(s) \\
& \leq \gamma E'_{s' \sim P(\cdot|s,a,a')} [Q_1(s',a') - Q_2(s',a')] - V_2(s) \leq \gamma \| Q_1 - Q_2 \|_\infty.
\end{align*}
\]

Inequality (1) holds since \( V_1(s') = \max_\pi \left\{ E_{a \sim \pi_2(\cdot|s,a')} [Q_1(s,a)] - \lambda D_{KL}(\pi(\cdot|s), \pi_{\text{ref}}(\cdot)) \right\} \). Thus, we prove that the regularized Bellman optimality operator is a \( \gamma \)-contraction.

A.2 PROOF OF PROPOSITION 3.3

Proof of Proposition 3.3. First, we show that for any policy \( \pi \) and any state \( s \),

\[
0 \leq D_{KL}(\pi(\cdot|s), \pi_{\text{ref}}(\cdot)) \leq \max_\pi \left\{ \log \left( \frac{1}{\delta} \right), \log \left( \frac{|A| - 1}{|A| - \delta} \right) \right\},
\]

where \( \pi_{\text{ref}}(\cdot) = \left( \delta, \frac{1-\delta}{|A|}, \cdots, \frac{1-\delta}{|A|} \right) \). The first inequality holds as KL divergence is non-negative. For the second inequality, we have that

\[
\begin{align*}
\max_\pi \left\{ \log \left( \frac{1}{\delta} \right), \log \left( \frac{|A| - 1}{|A| - \delta} \right) \right\} & \leq \sum_{a \in A} \pi(a) \log(1/p_{\text{ref}}(a)) \\
& \leq \sum_{a \in A} \pi(a) \log \left( \frac{1}{\delta} \right) + \sum_{a \in A \setminus \{a_0\}} \pi(a) \log \left( \frac{|A| - 1}{|A| - \delta} \right) \leq \max_\pi \left\{ \log \left( \frac{1}{\delta} \right), \log \left( \frac{|A| - 1}{|A| - \delta} \right) \right\}.
\end{align*}
\]

Inequality (1) holds from the non-negativity of entropy. Inequality follows as \( 0 \leq \pi(a) \leq 1 \), \( \forall a \in A \). Let \( \mathcal{C} = \lambda \max_\pi \left\{ \log \left( \frac{1}{\delta} \right), \log \left( \frac{|A| - 1}{|A| - \delta} \right) \right\} \). Then we show that for any policy \( \pi \),

\[
V^\pi(s) - \frac{C}{1-\gamma} \leq V^\pi(\pi(\cdot|s)) \leq V^\pi(s), \quad \forall s \in S.
\]

By the definition, we have

\[
\begin{align*}
V^\pi_\Omega(s) & = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t (r(s_t, a_t) - \lambda D_{KL}(\pi(\cdot|s_t), \pi_{\text{ref}}(\cdot))) | s_0 = s \right] \\
& \leq \sum_{t=0}^{\infty} \gamma^t D_{KL}(\pi(\cdot|s_t), \pi_{\text{ref}}(\cdot)) | s_0 = s \left| s = s \right. \\
& \geq V^\pi(s) - \frac{C}{1-\gamma}.
\end{align*}
\]

Thus, we prove that the regularized Bellman optimality operator is a \( \gamma \)-contraction.
Then we show the sub-optimality of \( \pi^*_\Omega \).

\[
V_{\Omega^*}^{(1)}(s) \geq V_{\Omega^*}^{(2)}(s) \geq V_{\Omega^*}^{(3)}(s) \geq V^* - \frac{C}{1-\gamma} = V^* - \frac{C}{1-\gamma}.
\]

Inequality (1) and (3) follow Eq. (11). Inequality (2) holds as \( \pi^*_\Omega \) is the regularized optimal policy. Therefore, we finish the proof.

**B EXPERIMENT DETAILS**

**B.1 DETAILED INFORMATION ABOUT TASKS**

All tasks are extended observation space by 1 dimension which represents the remaining opportunities to execute sparse action. Detailed observation and action space of experiment tasks is shown in Table 1 where opportunities, \( |O| \), \( |A| \) and \( |A_s| \) denote the maximum times of executing sparse action, observation space, full action number and sparse action number respectively. The visualization of three sparse action tasks is shown in Figure 5.

Figure 5: Visualization of experiment sparse action tasks.

- **Football shooting** (Figure 5(a)). The football shooting task is based on HFO benchmark [Hausknecht et al. 2016] and the task objective is to train a football offender to score under the interfering of a non-player character (NPC) defender and an NPC goalie. The agent gets +1 of reward when scoring and -1, if the ball is caught by the opponent or out of time or sparse action opportunity, is exhausted. Action space of football shooting task is one-dimensional with five actions: [no operation, go to the ball, move, dribble, shoot], where "shoot" is the sparse action or 20% of the action space is sparse action.

- **Gunfight** (Figure 5(b)). Gunfight task is based on Atari game “Carnival-ram-v0” in Gym benchmark [Brockman et al. 2016]. In origin, Carnival task agent is unlimited to shoot as long as task time is not used up, whereas it is a sparse action task when bullet capacity is limited. In our experiments, bullet capacity is limited to 4. The single-step reward is normalized to \([-2, 2]\). Note that such modification does not affect the optimal policy. The gunfight task’s action space is one-dimensional with six actions: [no operation, move to the left, move to the right, fire straightly, fire to the left, fire to the right], where "fire" is the sparse action.

- **Stock trading** (Figure 5(c)). The stock trading task is constructed from real-world tick-level price and volume data on stocks in the Chinese A-share market. The Chinese A-share market is a T+1 market, which means the stock shares bought cannot be sold on the same day. We remove this limitation and enable short sales by allowing borrowing stocks. In this experiment, we only consider the intra-day trading, which means the positions have to be closed before the market’s closure. In addition to the raw price and volume, we add many market indicators such as momentum, volume, volatility, and trend indicators with different window sizes into the observations. The final state space is comprised of the market indicators (150-dim), the open/high/low/current best bid/current best ask price (5-dim), the differenced trading volumes (2-dim), the remain opportunities for executing sparse action (1-dim) and the trader’s positions (3-dimensional one-hot coding representing the short, neutral and long positions), adding up to 161 dimensions. The action space is only one-dimensional and contains three high-level abstract actions: short, neutral, and long. The reward is the difference between the current price and the price k ticks later (A tick equals 3 seconds in the Chinese A-share market), subtracting the transaction cost.

\[
r_t = a_t \cdot (p_{t+k} - p_t) - c \cdot |a_t - a_{t-1}| \cdot p_t \cdot v,
\]

where \( a \) is the position (can be 0, 1 or 2), \( p \) is the price, \( c \) is the transaction cost coefficient, \( v \) is the number of stock shares you owned (can be negative for short sale) and the subscript \( t \) represents the time \( t \). We set \( k \) to be 5, \( c \) to be 3e-4 if \( a_t \) is greater than \( a_{t-1} \) and 1.3e-3 if \( a_t \) is smaller than \( a_{t-1} \) respectively. Stock trading is a real-world scene especially suitable for sparse action study given that agents should learn the best opportunity to buy or sell the stock. We choose the stock whose code is “XSHE: 000025” and separate the data into a training set (data from January 10, 2017 to April 28, 2017) and test set (data from April 29, 2017 to May 26, 2017) and the corresponding stock trading environment is called after “stock-train” and “stock-test”. The policy will be trained only on stock-train while evaluated on both stock-train and stock-test.

| Task             | Opportunities | \( |O| \) | \( |A| \) | \( |A_s| \) |
|------------------|---------------|--------|--------|--------|
| Gunfight         | 4             | 131    | 6      | 3      |
| Stock trading    | 30            | 161    | 3      | 2      |
| Football shooting| 4             | 16     | 5      | 1      |

**B.2 ADDITIONAL EXPERIMENTS OF SAC**

Figure 6 shows the episodic reward of SAC policy under stochastic and deterministic evaluation manners.
Figure 6: Episodic reward of the SAC policy that is evaluated in different manners.

B.3 HYPER-PARAMETERS

The hyper-parameters of SPRQ, Rainbow, SAC, PPO and CPO in experiments are listed in the following table, where $p_{\text{ref}}(A_s | ·)$ represents the probability that reference distribution executes sparse action.

| Hyper-parameter          | Value                  |
|--------------------------|------------------------|
| **SPRQ**                 |                        |
| update frequency         | 6                      |
| batch size               | 256                    |
| learning rate            | 0.001                  |
| $\tau$ of target smoothing | 0.005                 |
| buffer size              | 200000                 |
| sparse action buffer size | 100000                |
| training data from $D_s$ | 10%                    |
| $p_{\text{ref}}(A_s | ·)$ in football task | 0.01                   |
| $p_{\text{ref}}(A_s | ·)$ in gunfight task | 0.1                    |
| $p_{\text{ref}}(A_s | ·)$ in stock task | 0.05                   |
| $\lambda$               | 0.01~0.05              |
| Q network                | Adam, [256,128], relu  |

| **Rainbow**              |                        |
| update frequency         | 6                      |
| batch size               | 256                    |
| learning rate            | 0.001                  |
| buffer size              | 200000                 |
| Q network                | Adam, [256,128], relu  |

| **SAC**                  |                        |
| update frequency         | 6                      |
| batch size               | 256                    |
| learning rate            | 0.001                  |
| target smoothing ratio   | 0.005                  |
| buffer size              | 200000                 |
| reward scale             | 20                     |
| Q-network                | Adam, [256,128], relu  |

| **Hyper-parameter**      | **Value**              |
|--------------------------|------------------------|
| **PPO**                  |                        |
| clip ratios              | 0.2                    |
| epochs                   | 10                     |
| mini-batch number        | 10                     |
| batch size               | 1200                   |
| sample number per update | 12000                  |
| learning rate            | 0.0003                 |
| policy network           | Adam, [256,128], relu  |
| value network            | Adam, [256,128], relu  |

| **CPO**                  |                        |
| learning rate            | 0.001                  |
| value epochs             | 10                     |
| max KL                   | 0.01                   |
| cost for sparse action   | 0.1                    |
| sample number per update | 20000                  |
| policy network           | Adam, [256,128], relu  |
| value network            | Adam, [256,128], relu  |