THE SPLITTING PRINCIPLE AND SINGULARITIES

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Abstract. The splitting principle states that morphisms in a derived category do not “split” accidentally. This has been successfully applied in several characterizations of rational, DB, and other singularities. In this article I prove a general statement, Theorem 2.5, that implies many of the previous individual statements and improves some of the characterizations in the process. See Theorem 3.3 for the actual statement.

1. Introduction

The main guiding force of this article is the following principle.

The Splitting Principle. Morphisms in a derived category do not split accidentally.

I will recall several theorems that justify this principle and make it precise in their own context. For the necessary definitions please see the end of the introduction.

Remark 1.1. It is customary to casually use the word “splitting” to explain the statements of the theorems that follow. However, the reader should be warned that one has to be careful with the meaning of this, because these “splittings” take place in a derived category, and derived categories are not abelian. For this reason, in the statements of the theorems below I use the terminology that a morphism admits a left inverse. In an abelian category this condition is equivalent to “splitting” and being a direct summand. With a slight abuse of language I labeled these as “Splitting theorems” cf. (1.2), (1.6) and (1.7).

The first splitting theorem is a criterion for a singularity to be rational.

Theorem 1.2 [Kov00, Theorem 1] (Splitting Theorem I). Let \( \phi : Y \to X \) be a proper morphism of varieties over \( \mathbb{C} \) and \( g : \mathcal{O}_X \to R\phi_*\mathcal{O}_Y \) the associated natural morphism. Assume that \( Y \) has rational...
singularities and \(\varphi\) has a left inverse, i.e., there exists a morphism (in the derived category of \(\mathcal{O}_X\)-modules) \(\varphi': R\varphi_*\mathcal{O}_Y \to \mathcal{O}_X\) such that \(\varphi' \circ \varphi\) is a quasi-isomorphism of \(\mathcal{O}_X\) with itself. Then \(X\) has only rational singularities.

**Remark 1.3.** Note that \(\varphi\) in the theorem does not have to be birational or even generically finite. It follows from the conditions that it is surjective.

**Corollary 1.4.** Let \(X\) be a complex variety and \(\varphi: Y \to X\) a resolution of singularities. If \(\mathcal{O}_X \to R\varphi_*\mathcal{O}_Y\) has a left inverse, then \(X\) has rational singularities.

**Corollary 1.5.** Let \(X\) be a complex variety and \(\varphi: Y \to X\) a finite morphism. If \(Y\) has rational singularities, then so does \(X\).

Using this criterion it is quite easy to prove that log terminal singularities are rational \([\text{Kov00}, \text{Theorem 4}]\). For related statements see \([\text{KM98}, \text{5.22}]\) and the references therein.

The next several splitting theorems concern DB singularities:

**Theorem 1.6** \([\text{Kov99}, \text{2.3}]\) (Splitting Theorem II). Let \(X\) be a complex variety. If \(\mathcal{O}_X \to \Omega^0_X\) has a left inverse, then \(X\) has DB singularities.

This criterion has several important consequences. It implies directly that rational singularities are DB and it was used in \([\text{KK10}]\) to prove that log canonical singularities are DB as well. In fact it is used in the proof of the next splitting theorem.

**Theorem 1.7** \([\text{KK10}, \text{1.6}]\) (Splitting Theorem III). Let \(\varphi: Y \to X\) be a proper morphism between reduced schemes of finite type over \(\mathbb{C}\). Let \(W \subseteq X\) be a closed reduced subscheme with ideal sheaf \(\mathcal{I}_{W \subseteq X}\) and \(F := \varphi^{-1}(W) \subseteq Y\) with ideal sheaf \(\mathcal{I}_{F \subseteq Y}\). Assume that the natural map \(\varphi\)

\[
\mathcal{I}_{W \subseteq X} \xrightarrow{\varphi'} R\varphi_*\mathcal{I}_{F \subseteq Y}
\]

admits a left inverse \(\varphi'\), that is, \(\varphi' \circ \varphi = \text{id}_{\mathcal{I}_{W \subseteq X}}\). Then if \(Y, F,\) and \(W\) all have DB singularities, then so does \(X\).

This criterion forms the cornerstone of the proof of the following theorem:

**Theorem 1.8** \([\text{KK10}, \text{1.5}]\). Let \(\varphi: Y \to X\) be a proper surjective morphism with connected fibers between normal varieties. Assume that \(Y\) has log canonical singularities and \(K_Y \sim_{\mathbb{Q}, \phi} 0\), that is, \(K_Y\) is a \(\varphi\)-relatively numerically trivial \(\mathbb{Q}\)-divisor. Then \(X\) is DB.
Corollary 1.9 [KK10, 1.4]. Log canonical singularities are DB.

For the proofs and more general statements, please see [KK10].

Remark 1.9.1. Notice that in (1.7) it is not required that $\phi$ be birational. On the other hand the assumptions of the theorem and [Kov00, Thm 1] imply that if $Y \setminus F$ has rational singularities, e.g., if $Y$ is smooth, then $X \setminus W$ has rational singularities as well.

This theorem is used in [KK10] to derive various consequences, some of which regard stable families and have strong consequences for moduli spaces of canonically polarized varieties. The interested reader should look at the original article to obtain the full picture.

Finally, the newest splitting theorem is a generalization of (1.6) to the case of pairs:

Theorem 1.10 [Kov11, 5.4] (Splitting theorem IV). Let $(X, \Sigma)$ be a reduced generalized pair. Assume that the natural morphism $\mathcal{R}_{\Sigma \subseteq X} \to \Omega^0_{X, \Sigma}$ has a left inverse. Then $(X, \Sigma)$ is a DB pair.

The main goal of this article is to prove a general splitting theorem that provides a unified proof of (1.6), (1.7), and (1.10). For the special definitions see §2.

Theorem 1.11 (The Splitting Principle). Let $\text{Sch} = \text{Sch}_k$ be the category of schemes of finite type over a fixed algebraically closed field $k$, $\mathcal{F}$ and $\mathcal{G} : \text{Sch} \to \text{DSch}$ be two consistent ordinary functors, and $\eta : \mathcal{F} \to \mathcal{G}$ a consistent cohomologically surjective natural transformation as defined in (2.2) and (2.3). Let $Y$ be a generically reduced quasi-projective scheme of finite type over $k$ and $V \subseteq Y$ a dense open subset such that $\eta_Y : \mathcal{F}(V) \cong \to \mathcal{G}(V)$ is a quasi-isomorphism. Assume that $\eta_Y : \mathcal{F}(Y) \to \mathcal{G}(Y)$ has a left inverse. Then it is a quasi-isomorphism.

As a corollary of this theorem we obtain a more general statement that does not only imply these three theorems, but it also strengthens [KK10, 1.6] (see (1.7)) by changing a simple one way implication to an equivalence. For the precise statement please see (3.3). It may also be of interest that this constitutes a new proof of [KK10, 1.6] that is considerably simpler than the original one.

Finally, let me address the point that the reader have probably noticed. I have listed four splitting theorems and the abstract theorem proved in this article implies three of them. Considering the nature of the four theorems this is not surprising, but the abstract theorem (1.10) may actually be used to derive criteria similar to (1.2) that implies that certain singularities are rational. I will leave figuring out these possibilities for the reader. I would also like to issue a challenge to generalize (1.10) to a statement that implies (1.2) as well.
1.A. Definitions and Notation

If \( \phi : Y \to Z \) is a birational morphism, then \( \text{Exc}(\phi) \) will denote the exceptional set of \( \phi \). For a closed subscheme \( W \subseteq X \), the ideal sheaf of \( W \) is denoted by \( \mathcal{I}_W \subseteq \mathcal{O}_X \) or if no confusion is likely, simply by \( \mathcal{I}_W \). For a point \( x \in X \), \( \kappa(x) \) denotes the residue field of \( \mathcal{O}_{X,x} \).

For morphisms \( \phi : X \to B \) and \( \vartheta : T \to B \), the symbol \( X_T \) will denote \( X \times_B T \) and \( \varphi_T : X_T \to T \) the induced morphism. In particular, for \( b \in B \) I write \( X_b = \phi^{-1}(b) \). Of course, by symmetry, we also have the notation \( \varphi_X : T_X \cong X_T \to X \) and if \( \mathcal{F} \) is an \( \mathcal{O}_X \)-module, then \( \mathcal{F}_T \) will denote the \( \mathcal{O}_{X_T} \)-module \( \varphi_X^* \mathcal{F} \).

Let \( X \) be a scheme. Let \( D_{\text{filt}}(X) \) denote the derived category of filtered complexes of \( \mathcal{O}_X \)-modules with differentials of order at most \( 1 \) and \( D_{\text{filt},\text{coh}}(X) \) the subcategory of \( D_{\text{filt}}(X) \) of complexes \( K \), such that for all \( i \), the cohomology sheaves of \( Gr_i^{\text{filt}} K^* \) are coherent cf. \cite{DB81, GNPP88}. Let \( D(X) \) and \( D_{\text{coh}}(X) \) denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts \( +, -, b \) carry the usual meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by \( \cong_{\text{qis}} \). A sheaf \( \mathcal{F} \) is also considered as a complex \( \mathcal{F} \) with \( \mathcal{F}^0 = \mathcal{F} \) and \( \mathcal{F}^i = 0 \) for \( i \neq 0 \). If \( A \) is a complex in any of the above categories, then \( h^i(A) \) denotes the \( i \)-th cohomology sheaf of \( A \). The support of \( A \) is the union of the supports of its cohomology sheaves: \( \text{supp } A := \bigcup_i \text{supp } h^i(A) \).

The right derived functor of an additive functor \( F \), if it exists, is denoted by \( RF \) and \( RF^i \) is short for \( h^i \circ F \). Furthermore, \( \mathbb{H}^i, \mathbb{H}^i_c, \mathbb{H}^i_Z \), and \( \mathcal{H}^i_Z \) will denote \( R\Gamma, R\Gamma_c, R\Gamma_Z \), and \( R\mathcal{H}_Z \) respectively, where \( \Gamma \) is the functor of global sections, \( \Gamma_c \) is the functor of global sections with proper support, \( \Gamma_Z \) is the functor of global sections with support in the closed subset \( Z \), and \( \mathcal{H}_Z \) is the functor of the sheaf of local sections with support in the closed subset \( Z \). Note that according to this terminology, if \( \phi : Y \to X \) is a morphism and \( \mathcal{F} \) is a coherent sheaf on \( Y \), then \( R\phi_* \mathcal{F} \) is the complex whose cohomology sheaves give rise to the usual higher direct images of \( \mathcal{F} \).

I will often use the notion that a morphism \( f : A \to B \) in a derived category has a left inverse. This means that there exists a morphism \( f^\ell : B \to A \) in the same derived category such that \( f^\ell \circ f : A \to A \) is the identity morphism of \( A \). I.e., \( f^\ell \) is a left inverse of \( f \).

I will also make the following simplification in notation. First observe that if \( \iota : \Sigma \to X \) is a closed embedding of schemes then \( \iota_* \) is exact and hence \( R\iota_* = \iota_* \). This allows one to make the following harmless abuse of notation: If \( A \in \text{Ob } D(\Sigma) \), then, as usual for sheaves, I will drop
ι* from the notation of the object ι*A. In other words, I will, without further warning, consider A an object in D(X).

A generalized pair \((X, \Sigma)\) consists of an equidimensional variety (i.e., a reduced scheme of finite type over a field \(k\)) \(X\) and a closed subscheme \(\Sigma \subseteq X\). A morphism of generalized pairs \(\phi : (Y, \Gamma) \to (X, \Sigma)\) is a morphism \(\phi : Y \to X\) such that \(\phi(\Gamma) \subseteq \Sigma\). A reduced generalized pair is a generalized pair \((X, \Sigma)\) such that \(\Sigma\) is reduced.

The log resolution of a generalized pair \((X, W)\) is a proper birational morphism \(\pi : Y \to X\) such that \(\text{Exc}(\pi)\) is a divisor and \(\pi^{-1}W + \text{Exc}(\pi)\) is an snc divisor.

Let \(X\) be a complex scheme and \(\Sigma\) a closed subscheme whose complement in \(X\) is dense. Then \((X, \Sigma)\) is a good hyperresolution if \(X, \to X\) is a hyperresolution, and if \(U, = X, x_X (X \setminus \Sigma)\) and \(\Sigma, = X, \setminus U,\), then for all \(\alpha\) either \(\Sigma_\alpha\) is a divisor with normal crossings on \(X_\alpha\) or \(\Sigma_\alpha = X_\alpha\). Notice that it is possible that \(X,\) has some components that map into \(\Sigma\). These components are contained in \(\Sigma,\). For more details and the existence of such hyperresolutions see [DB81, 6.2] and [GNPP88] IV.1.21, IV.1.25, IV.2.1]. For a primer on hyperresolutions see the appendix of [KS11].

Let \((X, \Sigma)\) be a reduced generalized pair and let \(\Omega_{X, \Sigma}^0\) denote the Deligne-Du Bois complex of \((X, \Sigma)\). The 0th associated graded quotient of this will be denoted by \(\Omega_{X, \Sigma}^0\). If \(\Sigma = \emptyset\), it will be dropped from the notation: \(\Omega_X^0 := \Omega_{X, \Sigma}^0\). For more details see Steenbrink [Ste85, §3] and [Kov11] 3.9] and the relevant references in the latter article.

2. The Abstract Splitting Principle

In this section I will introduce a few new notions to generalize the conditions needed to prove the desired abstract theorem and prove a few general statements leading to the main theorem.

First we need a definition mainly for simplifying notation and terminology.

**Definition 2.1.** Let \(\text{Sch}\) be a category of schemes and \(D\text{Sch}\) the following associated category of pairs: An object of \(D\text{Sch}\) is a pair \((X, A)\) consisting of a scheme \(X \in \text{Ob}\text{Sch}\) and an object \(A \in \text{Ob} D(X)\); and a morphism \(\phi : (X, A) \to (Y, B)\) consist of a morphism of schemes \(\phi : X \to Y\) (denoted by the same symbol unless confusion is possible) and a morphism in \(D(Y)\), \(\phi^# : B \to R\phi_*A\). Observe that there exists a natural embedding of \(\text{Sch}\) into \(D\text{Sch}\) by mapping any \(X \in \text{Ob}\text{Sch}\) to the pair \((X, \mathcal{O}_X) \in \text{Ob} D\text{Sch}\). Note that \(\text{Sch}\) is a category of schemes, not necessarily the category of schemes. In particular, especially in
applications, we will often assume that $\mathbf{Sch}$ is the category of schemes of finite type over an algebraically closed field, for instance $\mathbb{C}$.

A functor $\mathcal{F} : \mathbf{Sch} \to \mathbf{DSch}$ will be called ordinary if $\mathcal{F}(X) = (X, \mathcal{O}_X(X))$ for any $X \in \text{Ob} \mathbf{Sch}$, i.e., the scheme part of the pair $\mathcal{F}(X)$ is equal to the original scheme $X$. In this case we will identify $\mathcal{F} = (\text{id}_{\mathbf{Sch}}, \mathcal{O})$ with $\mathcal{O}$.

Next we consider a condition that can be reasonably expected from any geometrically defined functors.

**Definition 2.2.** Let $\mathbf{Sch} = \mathbf{Sch}_k$ be the category of schemes of finite type over a fixed algebraically closed field $k$ and $\mathcal{F} : \mathbf{Sch} \to \mathbf{DSch}$ an ordinary functor. $\mathcal{F}$ will be called a consistent functor if for any quasi-projective generically reduced scheme $X \in \text{Ob} \mathbf{Sch}$ and any general hyperplane section $H \subseteq X$ there exist a natural isomorphism $\mathcal{F}(X) \otimes L \mathcal{O}_H \simeq \text{qis} \mathcal{F}(H)$.

If $\mathcal{F}$ and $\mathcal{G} : \mathbf{Sch} \to \mathbf{DSch}$ are two consistent (ordinary) functors, then a natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is called a consistent natural transformation if for any quasi-projective generically reduced scheme $X \in \text{Ob} \mathbf{Sch}$ and any general hyperplane section $H \subseteq X$ there exists a commutative diagram:

$$
\begin{aligned}
\mathcal{F}(X) \otimes L \mathcal{O}_H & \xrightarrow{\simeq \text{qis}} \mathcal{F}(H) \\
\eta_X \otimes \text{id} \mathcal{O}_H & \downarrow \quad \eta_H \\
\mathcal{G}(X) \otimes L \mathcal{O}_H & \xrightarrow{\simeq \text{qis}} \mathcal{G}(H).
\end{aligned}
$$

The next definition is an abstract way to grasp a condition implied by the Hodge decomposition of singular cohomology that plays a key role in the proof of (1.6). The fact that the Hodge-to-de Rham spectral sequence for a smooth complex projective variety $X$ degenerates at $E_1$ implies that the natural map on cohomology

$$
H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathcal{O}_X)
$$

is surjective for all $i$. For not necessarily smooth projective schemes the target of the equivalent of this surjectivity is the corresponding hypercohomology of $\Omega^n_X$. For our purposes this implies that for an arbitrary complex projective scheme of finite type there exists a natural map

$$
H^i(X, \mathcal{O}_X) \twoheadrightarrow H^i(X, \Omega^n_X)
$$

which is surjective for all $i$. This surjectivity comes from singular cohomology and Hodge theory, but once we have it in this form the rest of the proof of (1.6) does not require either one of those, in particular,
it does not require us to work over the complex numbers or even in characteristic zero (except for the definition of $\Omega^0_X$).

**Definition 2.3.** Let $\text{Sch} = \text{Sch}_k$ be the category of schemes of finite type over a fixed algebraically closed field $k$, $\mathcal{F}$ and $\mathcal{G} : \text{Sch} \to \text{DSch}$ be two consistent ordinary functors, and $\eta : \mathcal{F} \to \mathcal{G}$ a consistent natural transformation.

Then $\eta$ will be called *cohomologically surjective* if for any generically reduced affine scheme $X \in \text{Ob Sch}$ there exists an $\overline{X} \in \text{Ob Sch}$ such that $X \subseteq \overline{X}$ is an open set and

$$H^i(\eta_{\overline{X}}) : H^i(\overline{X}, \mathcal{F}(\overline{X})) \twoheadrightarrow H^i(\overline{X}, \mathcal{G}(\overline{X}))$$

is surjective for all $i$.

The following is a key ingredient of the overall argument. The main point of this statement is to relay the surjectivity obtained for projective schemes to quasi-projective ones. In order to avoid losing important information this is done by using local cohomology.

**Theorem 2.4.** Let $\mathcal{F}, \mathcal{G} : \text{Sch} \to \text{DSch}$ be two ordinary functors and $\eta : \mathcal{F} \to \mathcal{G}$ a natural transformation. Further let $\overline{X}$ be a scheme, $X \subseteq \overline{X}$ an open subscheme, and $P \subset \overline{X}$ a closed subscheme. Assume $P \subseteq X$ and let $U := X \setminus P$. Further assume that

\begin{align}
(2.4.1) \quad & H^i(\eta_{\overline{X}}) : H^i(\overline{X}, \mathcal{F}(\overline{X})) \twoheadrightarrow H^i(\overline{X}, \mathcal{G}(\overline{X})) \text{ is surjective for all } i,

(2.4.2) \quad & H^i(\eta_U) : H^i(U, \mathcal{F}(U)) \cong H^i(U, \mathcal{G}(U)) \text{ is an isomorphism for all } i.
\end{align}

Then $H^i_P(\eta_{\overline{X}}) : H^i_P(X, \mathcal{F}(X)) \to H^i_P(X, \mathcal{G}(X))$ is surjective for all $i$.

**Proof.** Let $Q = \overline{X} \setminus X$, $Z = P \cup Q$, and $U = \overline{X} \setminus Z = X \setminus P$. Consider the exact triangle of functors,

\begin{align}
(2.4.3) \quad & H^0_Z(\overline{X}, \underline{\_}) \longrightarrow H^0(\overline{X}, \underline{\_}) \longrightarrow H^0(U, \underline{\_}) \longrightarrow H^1(U, \underline{\_}) \\
\end{align}

and apply it to the morphism $\eta_{\overline{X}} : \mathcal{F}(\overline{X}) \to \mathcal{G}(\overline{X})$. One obtains a morphism of two long exact sequences:

$$\cdots \longrightarrow \text{H}^{i-1}(U, \mathcal{F}(U)) \longrightarrow \text{H}^i_Z(\overline{X}, \mathcal{F}(\overline{X})) \longrightarrow \text{H}^i(\overline{X}, \mathcal{F}(\overline{X})) \longrightarrow \text{H}^i(U, \mathcal{F}(U)) \longrightarrow \text{H}^{i+1}(U, \mathcal{F}(U)) \longrightarrow \cdots$$

$$\downarrow \alpha_i^{-1} \quad \downarrow \beta_i \quad \downarrow \gamma_i \quad \downarrow \alpha_i$$

$$\cdots \longrightarrow \text{H}^{i-1}(U, \mathcal{G}(U)) \longrightarrow \text{H}^i_Z(\overline{X}, \mathcal{G}(\overline{X})) \longrightarrow \text{H}^i(\overline{X}, \mathcal{G}(\overline{X})) \longrightarrow \text{H}^i(U, \mathcal{G}(U)) \longrightarrow \cdots$$

By assumption, $\alpha_i$ is an isomorphism and $\gamma_i$ is surjective for all $i$. Then by the 5-lemma, $\beta_i$ is also surjective for all $i$. 

By construction $P \cap Q = \emptyset$ and hence
\[
\begin{align*}
\mathbb{H}^i_Z(X, \mathcal{F}(X)) &\simeq \mathbb{H}^i_P(X, \mathcal{F}(X)) \oplus \mathbb{H}^i_Q(X, \mathcal{F}(X)) \\
\mathbb{H}^i_Z(X, \mathcal{G}(X)) &\simeq \mathbb{H}^i_P(X, \mathcal{G}(X)) \oplus \mathbb{H}^i_Q(X, \mathcal{G}(X))
\end{align*}
\]
It follows that the natural map (which is also the restriction of $\beta_i$),
\[
\mathbb{H}^i_P(X, \mathcal{F}(X)) \to \mathbb{H}^i_P(X, \mathcal{G}(X))
\]
is surjective for all $i$. Now, by excision on local cohomology one has that
\[
\begin{align*}
\mathbb{H}^i_P(X, \mathcal{F}(X)) &\simeq \mathbb{H}^i_P(X, \mathcal{G}(X)) \quad \text{and} \quad \mathbb{H}^i_P(X, \mathcal{G}(X)) \simeq \mathbb{H}^i_P(X, \mathcal{G}(X)),
\end{align*}
\]
and so the desired statement follows. \qed

The next theorem is the main result of this article. It generalizes the statement and proof of those theorems mentioned in the introduction to a quite general level. In the next section I will explain how this implies almost immediately those three results and strengthens one of them. However, it seems reasonable to expect that this form will be used later to prove similar statements in different situations.

**Theorem 2.5.** Let $\text{Sch} = \text{Sch}_k$ be the category of schemes of finite type over a fixed algebraically closed field $k$, $\mathcal{F}$ and $\mathcal{G} : \text{Sch} \to D\text{Sch}$ be two consistent ordinary functors, and $\eta : \mathcal{F} \to \mathcal{G}$ a consistent cohomologically surjective natural transformation as defined in (2.2) and (2.3). Let $Y$ be a generically reduced quasi-projective scheme of finite type over $k$ and $V \subseteq Y$ a dense open subset such that $\eta_V : \mathcal{F}(V) \xrightarrow{\simeq} \mathcal{G}(V)$ is a quasi-isomorphism. If for any general complete intersection $X \subseteq Y$ and any closed subscheme $Z \subseteq X \setminus V$,
\[
(2.5.1) \quad \mathbb{H}^i_Z(\eta_Y) : \mathbb{H}^i_Z(X, \mathcal{F}(X)) \to \mathbb{H}^i_Z(X, \mathcal{G}(X))
\]
is injective for all $i$, then $\eta_Y : \mathcal{F}(Y) \xrightarrow{\simeq} \mathcal{G}(Y)$ is a quasi-isomorphism.

**Proof.** Let $\mathcal{D}(Y)$ be an object in $D\text{Sch}$ that completes the morphism $\eta_Y : \mathcal{F}(Y) \to \mathcal{G}(Y)$ to a distinguished triangle:
\[
\begin{array}{c}
\mathcal{F}(Y) \\
\mathcal{G}(Y) \\
\mathcal{D}(Y)
\end{array} \xrightarrow{+1} \begin{array}{c}
\mathcal{F}(Y) \\
\mathcal{G}(Y) \\
\mathcal{D}(Y)
\end{array}.
\]

Let $T = \text{supp } \mathcal{D}(Y) \subseteq Y \setminus V$, a closed subset of $Y$. We need to prove that $\mathcal{D}(Y) \simeq_{\text{qis}} 0$, that is, that $T = \emptyset$. Suppose that $T \neq \emptyset$ and we will derive a contradiction.

By assumption $Y \setminus T \supset V$ a dense open subset of $Y$. It follows that if $X \subseteq Y$ is a general complete intersection of $Y$ of the appropriate codimension, then $P := X \cap T$ is a finite closed non-empty subset.
Since $\eta$ is consistent, cf. (2.2), setting $\mathcal{D}(X) := \mathcal{D}(Y) \otimes \mathcal{O}_X$ one obtains a distinguished triangle:

$$\mathcal{D}(X) \rightarrow \mathcal{D}(X) \rightarrow \mathcal{D}(X) + 1,$$

such that $P = \text{supp} \mathcal{D}(X)$. We will prove that $P = \emptyset$ which is a contradiction to the way $P$ was defined.

As $P$ is finite we may assume that $X$ is affine. Consider $X \subseteq \overline{X}$ given by the fact that $\eta$ is cohomologically surjective, cf. (2.3). Again, since $P$ is a finite set, it follows that $P \subset \overline{X}$ is also closed and then it follows by (2.4) and the assumption in (2.5.1) that

$$(2.5.2) \quad H^i_P(\eta_\overline{X}) : H^i_P(X, \mathcal{F}(X)) \stackrel{\sim}{\rightarrow} H^i_P(X, \mathcal{G}(X))$$

is an isomorphism for all $i$, and then it follows that

$$H^i_P(X, \mathcal{D}(X)) = 0 \quad \text{for all } i.$$

Since $\text{supp} \mathcal{D}(X) = P$ it also follows that

$$H^i(X \setminus P, \mathcal{D}(X)) = 0$$

for all $i$ as well, and then

$$(2.5.3) \quad H^i(X, \mathcal{D}(X)) = 0$$

for all $i$ by the long exact sequence induced by (2.4.3) applied with $\overline{X} \leftrightarrow X$ and $Z \leftrightarrow P$.

Since $X$ is affine, the spectral sequence that computes hypercohomology from the cohomology of the cohomology sheaves of the complex $\mathcal{D}(X)$ degenerates and gives that $H^i(X, \mathcal{D}(X)) = H^0(X, h^i(\mathcal{D}(X)))$ for all $i$. It follows by (2.5.3) that $h^i(\mathcal{D}(X)) = 0$ for all $i$. Therefore $\mathcal{D}(X) \simeq_{\text{qis}} 0$ and hence $P = \emptyset$. We arrived to our promised contradiction, so the desired statement is proven. \hfill $\square$

The following is a straightforward corollary of (2.5), its main value is in that its conditions may be easier to verify.

**Corollary 2.6 (The Splitting Principle).** Let $\text{Sch} = \text{Sch}_k$ be the category of schemes of finite type over a fixed algebraically closed field $k$, $\mathcal{F}$ and $\mathcal{G} : \text{Sch} \rightarrow D\text{Sch}$ be two consistent ordinary functors, and $\eta : \mathcal{F} \rightarrow \mathcal{G}$ a consistent cohomologically surjective natural transformation as defined in (2.2) and (2.3). Let $Y$ be a generically reduced quasi-projective scheme of finite type over $k$ and $V \subseteq Y$ a dense open subset such that $\eta_Y : \mathcal{F}(V) \simeq \mathcal{G}(V)$ is a quasi-isomorphism. Assume that $\eta_Y : \mathcal{F}(Y) \rightarrow \mathcal{G}(Y)$ has a left inverse. Then it is a quasi-isomorphism.
Proof. If $\eta_Y : \mathcal{F}(Y) \to \mathcal{G}(Y)$ has a left inverse, then the same holds for $\eta_X$ for any general complete intersection $X \subseteq Y$, and so

$$H^i_Z(\eta_X) : H^i_Z(X, \mathcal{F}(X)) \to H^i_Z(X, \mathcal{G}(X))$$

is injective for any closed subset $Z \subseteq X$. Then the statement follows from (2.5). □

3. Applications

In this sections I show how (2.5) implies (1.6), (1.7), and (1.10).

3.A. DB singularities

The first application is one of the first appearances of the splitting principle:

**Theorem 3.1** [Kov99, 2.3] see (1.6). Let $X$ be a scheme of finite type over $\mathbb{C}$. If the natural map $\mathcal{O}_X \to \Omega^0_X$ admits a left inverse, then $X$ has DB singularities.

**Proof.** Let $\text{Sch} = \text{Sch}_k$ be the category of schemes of finite type over $\mathbb{C}$, $\mathcal{F}(\underline{\phantom{X}}) = \mathcal{O}(\underline{\phantom{X}})$ and $\mathcal{G}(\underline{\phantom{X}}) = \Omega^0(\underline{\phantom{X}})$. These define two ordinary functors $\text{Sch} \to D\text{Sch}$. They are both consistent as defined in (2.2) by [Kov11, 2.6] and there exists a consistent cohomologically surjective natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ by [DB81, 4.5]. Let $V := X \setminus \text{Sing} X$. Then $\eta_V : \mathcal{F}(V) \cong \mathcal{G}(V)$ is a quasi-isomorphism. Then the statement follows from (2.6). □

3.B. DB pairs

**Theorem 3.2** [Kov11, 5.4] see (1.10). Let $(X, \Sigma)$ be a reduced generalized pair. Assume that the natural morphism $\mathcal{F}_{\Sigma \subseteq X} \to \Omega^0_{X, \Sigma}$ has a left inverse. Then $(X, \Sigma)$ is a DB pair.

**Proof.** Let $\text{Sch} = \text{Sch}_k$ be the category of subschemes $\Sigma$ of $X$ of finite type over $\mathbb{C}$, $\mathcal{F}(\Sigma) = \mathcal{I}_{\Sigma \subseteq X}$ and $\mathcal{G}(\Sigma) = \Omega^0_{X, \Sigma}$. These define two ordinary functors $\text{Sch} \to D\text{Sch}$. They are both consistent as defined in (2.2) by [Kov11, 3.18] and there exists a consistent cohomologically surjective natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ by [Kov11, 4.2]. Let $V := (X \setminus \text{Sing} X) \setminus \text{supp} \Sigma$. Then $\eta_V : \mathcal{F}(V) \cong \mathcal{G}(V)$ is a quasi-isomorphism. Then the statement follows from (2.6). □

3.C. The Kollár-Kovács DB criterion

**Theorem 3.3.** Let $f : Y \to X$ be a proper morphism between reduced schemes of finite type over $\mathbb{C}$, $W \subseteq X$ an arbitrary subscheme, and $F := f^{-1}(W)$, equipped with the induced reduced subscheme structure.
Assume that the natural map \( q \) allows a left inverse \( q' \). Then if \((Y, F)\) is a DB pair, then so is \((X, W)\). In particular, if \((Y, F)\) is a DB pair, then \( X \) is DB if and only if \( W \) is DB.

**Proof.** By functoriality one obtains a commutative diagram

\[
\begin{array}{ccc}
\mathcal{I}_{W \subseteq X} & \xrightarrow{q} & Rf_* \mathcal{I}_{F \subseteq Y} \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
\Omega^0_{W \subseteq X} & \xrightarrow{\beta} & Rf_* \Omega^0_{Y,F}.
\end{array}
\]

Since \((Y, F)\) is assumed to be a DB pair, it follows that \( \gamma \) is a quasi-isomorphism and hence \( q' \circ \gamma^{-1} \circ \beta \) is a left inverse to \( \alpha \). Then the statement follows by (3.2). \( \square \)

**Corollary 3.4** [KK10, 1.6] see (1.7). Let \( f : Y \to X \) be a proper morphism between reduced schemes of finite type over \( \mathbb{C} \), \( W \subseteq X \) an arbitrary subscheme, and \( F := f^{-1}(W) \), equipped with the induced reduced subscheme structure. Assume that the natural map \( q \) allows a left inverse \( q' \), that is, \( q' \circ q = \text{id}_{\mathcal{I}_{W \subseteq X}} \). Then if \( Y, F, \) and \( W \) all have DB singularities, then so does \( X \).

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