BERNSTEIN THEOREMS FOR SPACE-LIKE GRAPHS WITH PARALLEL MEAN CURVATURE AND CONTROLLED GROWTH

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ABSTRACT. In this paper, we obtain an Ecker-Huisken type result for entire space-like graphs with parallel mean curvature.

1. Introduction

In 1914, Bernstein proved that the only entire minimal graph in $\mathbb{R}^3$ is a plane. This result was generalized to $\mathbb{R}^{m+1}$ for $m \leq 7$, and higher dimensions and codimensions under various growth conditions, see [EH], [SXW], [Wa3] and their references. In 1965, Chern [Ch] showed that the only entire graphic hypersurface in $\mathbb{R}^{m+1}$ with constant mean curvature must be minimal. Therefore we have the corresponding Bernstein type results for constant mean curvature hypersurfaces. Bernstein type results for submanifolds in $\mathbb{R}^{m+n}$ with parallel mean curvature were also obtained by some authors (cf. [HJW], [JX1] and [Do]).

In 1968, Calabi [Ca] raised a similar problem for extremal hypersurfaces in Lorentz-Minkowski space $\mathbb{R}^{m+1}$ and he proved that the Bernstein result is true for $2 \leq m \leq 4$. Later, Cheng and Yau [CY] extended Calabi’s result to all $m$ as follows: The only complete extremal space-like hypersurfaces in $\mathbb{R}^{m+1}_1$ are space-like hyperplanes. Recently, Jost and Xin [JX2] generalized this result to higher codimensional case.

On the other hand, it is important to investigate space-like constant mean curvature hypersurfaces in $\mathbb{R}^{m+1}_1$, which have interest in relative theory (cf. [MT]). In [Tr], Treibergs showed that there are many entire space-like graphs with constant mean curvature besides hyperboloids. Thus Chern type result is no longer true in this case. It is known that the Gauss map of a constant mean curvature space-like hypersurface $M$ is a harmonic map to hyperbolic space. Xin [Xi1] got a Bernstein result by assuming the boundedness of the Gauss map. Later, [XY] and [CSZ] extended this result by proving that $M$ must be a space-like hyperplane if its Gauss image lies in a horoball in the hyperbolic space. Another natural generalization is to consider a space-like submanifold in pseudo-Euclidean space $\mathbb{R}^{m+n}$ with parallel mean curvature. In [Xi2] the author extended the previous mentioned result in [Xi1] to higher codimensional case under the same boundedness assumption on Gauss map.

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In this paper, we consider a space-like graphic submanifold $M = \{(x, f(x)) : x \in \mathbb{R}^m\}$ in $\mathbb{R}^{m+n}_n$ with parallel mean curvature. Since $M$ is space-like, the induce metric 
\[ (g_{ij}) = (\delta_{ij} - \sum_{s=1}^{n} f_s^i f_s^j) \] is positive definite. Set 
\[ *\Omega = \sqrt{\det(I - \sum_{s=1}^{n} f_s^i f_s^j)}^{-1} \]

Our main result is the following:

**Theorem** Let $M^m = (x, f(x))$ be an entire space-like graph in $\mathbb{R}^{m+n}_n$ with parallel mean curvature. If the function $*\Omega$ has growth 
\[ *\Omega = o(r) \quad \text{as} \quad r \to \infty \]
where $r = \sqrt{\sum_{i=1}^{m} x_i^2}$, then $M$ is a space-like $m$-plane. 

Our strategy is to establish a Chern-type result for an entire space-like graph with parallel mean curvature under the growth condition of $*\Omega$. Then the result follows immediately from [CY] and [JX2]. Notice that if $n = 1$, we have 
\[ *\Omega = \frac{1}{\sqrt{1 - \|\nabla f\|^2}} \]

Therefore the growth condition of $*\Omega$ is similar to that one given by Ecker-Husken [EH] for minimal graphic hypersurfaces in $\mathbb{R}^{n+1}$. The above result may be regarded as an Ecker-Huisken type result for space-like graphs with parallel mean curvature. By calculating the quantity $*\Omega$ of the hyperboloid, we will see that the growth condition is optimal. In [Do], the author uses a similar method to establish some Bernstein type results for submanifolds in Euclidean space with parallel mean curvature.

### 2. Preliminaries

In this section, we will generalize Chern’s method [Ch] to our setting. Let $R^{m+n}_n$ be an $(m + n)$-dimensional pseudo-Euclidean space of index $n$, namely the vector space $\mathbb{R}^{m+n}$ endowed with the metric 
\[ (, ) = (dx_1)^2 + \cdots + (dx_m)^2 - (dx_{m+1})^2 - \cdots - (dx_{m+n})^2 \]

The standard Euclidean metric of $R^{m+n}$ will be denoted by $(, )_E$. For a vector $v$ in $R^{m+n}$, we will use the notations $|v|$ and $|v|_E$ to denote the norms of $v$ with respect to $(, )$ and $(, )_E$ respectively.

Let $z : M^m \to R^{m+n}_n$ be a space-like immersion of an oriented $m$-dimensional manifold into $R^{m+n}_n$. We will regard $z$ as a vector-valued function on $M$. Choose a local Lorentzian frame field $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}\}$ such that $\{e_{m+1}, \ldots, e_{m+n}\}$ is a normal frame field of $M$. Througth this paper, we agree with the following indices:

\[ 1 \leq A, B, C, \ldots \leq m + n \]
\[ 1 \leq i, j, k, \ldots \leq m, \quad m + 1 \leq \alpha, \beta, \gamma, \ldots \leq m + n \]
Write

\[ dz = \sum_A \omega_A e_A \]
\[ de_A = \sum_B \omega_{AB} e_B \]

Therefore \( \{\omega_i\} \) is a dual frame field of \( \{e_i\} \) and \( \omega_\alpha = 0 \) on \( M \). The induced Riemannian metric of \( M \) is then given by \( ds^2_M = \sum_i \omega_i^2 \). By Cartan’s lemma, we have

\[ \omega_{\alpha i} = \sum_k h_{\alpha ij} \omega_j, \quad h_{\alpha ij} = h_{\alpha ji} \]

where \( h_{\alpha ij} \) are components of the second fundamental form of \( M \) in \( R_n^{m+n} \). The mean curvature vector of \( M \) is defined by

\[ \vec{H} = \frac{1}{m} \sum_{\alpha,\gamma} h_{\alpha \gamma \gamma} e_\alpha \]

If \( \nabla^+ \vec{H} = 0 \), \( M \) is said to have parallel mean curvature. If \( \vec{H} = 0 \), \( M \) is called an extremal spacelike submanifold.

Now we consider a space-like graph \( M = \{(x, f(x)) : x \in D \subseteq R^m\} \) in \( R_n^{m+n} \) with parallel mean curvature \( \vec{H} \), where \( D \) is a compact domain with smooth boundary \( \partial D \). Obviously \( H = \sqrt{-(\vec{H}, \vec{H})} \) is a nonnegative constant.

Let \( \Omega = dx^1 \wedge \cdots \wedge dx^m \) be the parallel \( m \)-form on \( R_n^{m+n} \) and let \( \{a_1, ..., a_{m+n}\} \) be an oriented Lorentzian basis of \( R_n^{m+n} \) such that \( \{a_i\}_{i=1}^m \) is an oriented orthonormal basis of \( R^m \). If \( H > 0 \), we have a global future-directed normal vector field \( e_H = H^{-1} \vec{H} \). Therefore we may define a global \( m \)-form on \( M \) as follows:

\[ \Phi = (m-1)! \sum_{i=1}^m d(a_{i+1}, z) \wedge \cdots \wedge d(a_i, z) \wedge (d(a_i, e_H) \wedge d(a_{i+1}, z) \wedge \cdots \wedge d(a_m, z) \wedge \cdots) \]

Clearly \( \Phi \) is independent of the choice of the oriented orthogonal basis \( \{a_i\}_{i=1}^m \) in \( R^m \).

For any \( p \in M \) then the differential of \( f \) is a linear map from \( R^m \) to \( R^n \). As in [Wa1], we can use singular value decomposition to find orthonormal bases \( \{a_i\}_{i=1}^m \) for \( R^m \) and \( \{a_\alpha\}_{\alpha=m+1}^{m+n} \) for \( R^m \) such that

\[ df(a_i) = \lambda_i a_{m+i} \]

for \( i = 1, ..., m \). Notice that \( \lambda_i = 0 \) if \( i > \min\{m, n\} \). Then we have

\[ (a_i, a_j) = \delta_{ij}, \quad (a_i, a_\alpha) = 0, \quad (a_\alpha, a_\beta) = -\delta_{\alpha\beta} \]

Therefore we have a Lorentzian basis \( \{e_A\} \) at \( p \) given by

\[ \left\{ e_i = \frac{1}{\sqrt{1 - \lambda_i^2}} (a_i + \lambda_i a_{m+i}) \right\}_{i=1, ..., m} \in T_p M \]
and

\begin{equation}
\left\{ \frac{1}{\sqrt{1 - \lambda^2_{\alpha-m}}} (a_\alpha + \lambda_{\alpha-m} a_{\alpha-m}) \right\}_{\alpha=m+1}^{m+n} \in T_p^\perp M
\end{equation}

By definition \(*\Omega = \Omega(e_1, ..., e_m)\), and thus we have

\begin{equation}
*\Omega = \frac{1}{\sqrt{\prod_{i=1}^m (1 - \lambda_i^2)}}
\end{equation}

**Lemma 1.** Under the above notations, we have

\[ \Phi = m! H(*\Omega) \omega^1 \wedge \cdots \wedge \omega^m \]

where \(\omega^1 \wedge \cdots \wedge \omega^m\) is volume form of \(M\).

**Proof.** Using\((3)\), \((8)\) and \((9)\), we have from \((6)\) the following:

\[
\Phi = (m-1)! \sum_{i=1}^m (a_1, dz) \wedge \cdots \wedge (a_{i-1}, dz) \wedge (a_i, dH) \wedge (a_{i+1}, dz) \wedge \cdots \wedge (a_m, dz)
\]

\[
= (m-1)! \sum_{i=1}^m \frac{1}{\sqrt{1 - \lambda_i^2}} \omega^1 \wedge \cdots \wedge \frac{1}{\sqrt{1 - \lambda_{i-1}^2}} \omega^{i-1} \wedge \left( \frac{-h_{i1}^H}{\sqrt{1 - \lambda_i^2}} \omega^i \right) \wedge \frac{1}{\sqrt{1 - \lambda_{i+1}^2}} \omega^{i+1} \wedge \cdots \wedge \frac{1}{\sqrt{1 - \lambda_m^2}} \omega^m
\]

\[
= - (m-1)!(*\Omega)(\sum_i h_{i1}^H) \omega^1 \wedge \cdots \wedge \omega^m
\]

\[
= m!(*\Omega) H \omega^1 \wedge \cdots \wedge \omega^m
\]

where \(\sum_i h_{i1}^H = (\sum_{\alpha} h_{\alpha i} e_{\alpha}, e_H) = -mH\). This proves the Lemma. \(\square\)

We may write

\begin{equation}
\Phi = (m - 1)! d\alpha
\end{equation}

where

\begin{equation}
\alpha = \sum_i (-1)^{i-1} (a_i, e_H) d(a_1, z) \wedge \cdots \wedge d(a_{i-1}, z) \wedge d(a_{i+1}, z) \wedge \cdots \wedge d(a_m, z)
\end{equation}

Applying the Stokes Theorem to \((11)\), we get

\begin{equation}
mH \int_M (*\Omega) \omega^1 \wedge \cdots \wedge \omega^m = \int_{\partial M} \alpha.
\end{equation}
We project \( z(M) \) orthogonally into the \( m \)–plane spaned by \( \{ a_i \}_{i=1}^{m} \). If \( z'(p) \) is the image point of \( z(p), p \in M \), under this orthogonal projection, we have

\[
(14) \quad z' = z + \sum_{\alpha=m+1}^{m+n} (a_\alpha, z)a_\alpha
\]

Let \( \Psi \) be a non-zero differential form on \( \partial M = \{(x, f(x)) : x \in \partial D\} \), defined locally. Using this form, the elements of volume of \( z'(\partial M), z(\partial M) \) may be expressed respectively as \( P\Psi, Q\Psi \) with \( P \geq 0 \) and \( Q \geq 0 \). We write

\[
(15) \quad \omega_{i_1} \wedge \cdots \wedge \omega_{i_{m-1}} = p_{i_1,\ldots,i_{m-1}} \Psi
\]
on \( \partial M \).

By a direct computation, we have

\[
\frac{1}{(m-1)!} dz \wedge \cdots \wedge dz = \frac{1}{(m-1)!} (\sum \omega_{i_1} e_{i_1}) \wedge \cdots \wedge (\sum \omega_{i_{m-1}} e_{i_{m-1}})
\]

\[
= \sum_i p_{1,\ldots,i-1,i+1,\ldots,m} (e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_m) \Psi
\]

so that

\[
(16) \quad Q^2 = \sum_i p_{1,\ldots,i-1,i+1,\ldots,m}^2.
\]

Using (8), (9) and (10), we get:

\[
\alpha = \sum_{i=1}^{m} (-1)^{i-1} (a_i, e_H)(a_1, dz) \wedge \cdots \wedge (a_{i-1}, dz) \wedge (a_{i+1}, dz) \wedge \cdots \wedge (a_m, dz)
\]

\[
= \sum_{i=1}^{m} (-1)^{i-1} (a_i, e_H) \prod_{j \neq i} \frac{1}{\sqrt{1 - \lambda_j^2}} \omega^1 \wedge \cdots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \cdots \wedge \omega^m
\]

\[
= \sum_{i=1}^{m} (-1)^{i-1} (a_i, e_H)p_{1,\ldots,i-1,i+1,\ldots,m} \Psi
\]

\[
= (\ast\Omega) \sum_{i=1}^{m} (-1)^{i} \lambda_i \xi_{m+i} p_{1,\ldots,i-1,i+1,\ldots,m} \Psi
\]

where \( \xi_m = < e_H, e_m > \) if \( k \leq \min \{m, n\} \) and \( \xi_{m+k} = 0 \) if \( k > \min \{m, n\} \). Obviously \( \sum_{i=1}^{m} \xi_{m+i}^2 \leq 1 \). Since \( |df| < 1 \), we get from (17) and the Cauchy-Schwarz inequality that

\[
(18) \quad |R| \leq (\ast\Omega)Q
\]

Next, we will show that if \( M \) is a space-like hypersurface, there is a nice formula relating the quantities \( P, Q \) and \( R \). When \( n = 1 \), (12) is simplified to

\[
(19) \quad \alpha = v_{p_2,\ldots,m}
\]
where \( v = \lambda_1 / \sqrt{1 - \lambda_1^2} = \lambda_1 (\ast \Omega) \); and thus

\[
R^2 = v^2 p_2, \ldots, m
\]

Write \( a = a_{m+1} \). Then (14) becomes

\[
(21) \quad x' = x + (a, x)a
\]

From (9) and (10), we easily derive

\[
(22) \quad (a, e_i) = -\delta_{i1} v, \quad (a, e_{m+1}) = - \ast \Omega
\]

and

\[
(23) \quad a = -ve_1 + \ast \Omega e_{m+1}
\]

To determine \( P \), we compute \( \frac{1}{(m-1)!} \underbrace{dz' \wedge \cdots \wedge dz'}_{m-1} \) as follows:

\[
\frac{1}{(m-1)!} \underbrace{dz' \wedge \cdots \wedge dz'}_{m-1} = \frac{1}{(m-1)!} \left\{ \sum_{i_1} e_i \omega_{i_1} - v\omega_1 a \right\} \wedge \cdots \wedge \left\{ \sum_{i_{m-1}} e_{i_{m-1}} \omega_{i_{m-1}} - v\omega_1 a \right\}
\]

\[
= \frac{1}{(m-1)!} \sum_i \omega_i \wedge \cdots \wedge \omega_{i_{m-1}} (e_i \wedge \cdots \wedge e_{i_{m-1}}) - \frac{v}{(m-1)!} \sum_{1 \leq s \leq m-1} \omega_{i_{s-1}} \wedge \cdots \wedge \omega_{i_{s+1}} \wedge \cdots \wedge \omega_{i_{m-1}} (e_{i_1} \wedge \cdots \wedge e_{i_{m-1}})
\]

\[
= \frac{1}{(m-1)!} \sum_i \omega_i \wedge \cdots \wedge \omega_{i_{m-1}} (e_i \wedge \cdots \wedge e_{i_{m-1}}) + \frac{v^2}{(m-1)!} \sum_{1 \leq s \leq m-1} (-1)^{s-1} \omega_1 \wedge \cdots \wedge \omega_{i_{s-1}} \wedge \omega_{i_{s+1}} \wedge \cdots \wedge \omega_{i_{m-1}} (e_{i_1} \wedge \cdots \wedge e_{i_{m-1}})
\]

\[
- \frac{v \ast \Omega}{(m-1)!} \sum_{1 \leq s \leq m-1} (-1)^{s-1} \omega_1 \wedge \cdots \wedge \omega_{i_{s-1}} \wedge \omega_{i_{s+1}} \wedge \cdots \wedge \omega_{i_{m-1}} (e_{i_1} \wedge \cdots \wedge e_{i_{m-1}})
\]

So the coefficient of \( \Psi \) in \( \frac{1}{(m-1)!} \underbrace{dz' \wedge \cdots \wedge dz'}_{m-1} \) is

\[
\sum_{i_1 < \cdots < i_{m-1}} p_{i_1, \ldots, i_{m-1}} e_{i_1} \wedge \cdots \wedge e_{i_{m-1}} + v^2 \sum_{1 < i_2 < \cdots < i_{m-1}} p_{1, i_2, \ldots, i_{m-1}} e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_{m-1}}
\]

\[
- (-1)^m v (\ast \Omega) \sum_{i_2 < \cdots < i_{m-1}} p_{i_1, \ldots, i_{m-1}} e_{i_2} \wedge \cdots \wedge e_{i_{m-1}} \wedge e_{m+1}
\]
It follows that
\[
P^2 = \sum_{1 < i_1 < \ldots < i_{m-1}} p^2_{i_1 \ldots i_{m-1}} + (1 + v^2)^2 \sum_{1 < i_2 < \ldots < i_{m-1}} p^2_{1i_2 \ldots i_{m-1}}
- v^2 (\ast \Omega)^2 \sum_{1 < i_2 < \ldots < i_{m-1}} p^2_{i_1 \ldots i_{m-1}}
= \sum_{i_1 < \ldots < i_{m-1}} p^2_{i_1 \ldots i_{m-1}} + v^2 \sum_{1 < i_2 < \ldots < i_{m-1}} p^2_{1i_2 \ldots i_{m-1}}
= \sum_{i_1 < \ldots < i_{m-1}} p^2_{i_1 \ldots i_{m-1}} + v^2 (\sum_{i_1 < \ldots < i_{m-1}} p^2_{i_1 \ldots i_{m-1}} - p^2_{e_{2m}})
= (1 + v^2)Q^2 - R^2
\]
since \((e_{m+1}, e_{m+1}) = -1\) and \(v^2 - (\ast \Omega)^2 = -1\). Thus we have
\[
(24) \quad P^2 + R^2 = (\ast \Omega)^2 Q^2
\]

3. Bernstein-Type Theorems

In this section, we take \(D = \{ x \in \mathbb{R}^m : \sum_{i=1}^m x_i^2 \leq r \} \). As before, let
\[
M = \{(x, f(x) : x \in D \}
\]
be a space-like graph in \(\mathbb{R}_n^{m+n}\) with paraller mean curvature.

**Lemma 1.** On \(\partial M\), we have

\[
(25) \quad Q \leq P
\]

In particular, if \(n = 1\), i.e., \(M\) is a space-like hypersurface, then we have
\[
(26) \quad Q \leq \sqrt{(\ast \Omega)^{-2} + |df(\eta_m)|^2} P
\]

where \( |df(\eta_m)| \) may be regarded as the radial singular value of the map \(f\).

**Proof.** Choose an orthonormal basis \(\{\eta_1, \ldots, \eta_m\} \) at \(q \in \partial D\) in \(\mathbb{R}^m\) such that \(\eta_m\) is a normal vector of \(\partial D\). We have the corresponding tangent vectors of the graph \(M\) at \((q, f(a))\\)
\[
\xi_i = (\eta_i, df(\eta_i)), \quad i = 1, \ldots, m
\]
It is easy to see
\[
(27) \quad |\xi_1 \wedge \cdots \wedge \xi_m| = (\ast \Omega)^{-1}
\]
and
\[
(28) \quad |\xi_1 \wedge \cdots \wedge \xi_{m-1}| = Q/P
\]
Write
\[ \xi_i = \tilde{\eta}_i + \tilde{d}f(\eta_i) \]
where \( \tilde{\eta}_i = (\eta_i, 0) \) and \( \tilde{d}f(\eta_i) = (0, df(\eta_i)) \). Therefore
\[ |\xi_i|^2 = |\tilde{\eta}_i|^2_E - |\tilde{d}f(\eta_i)|^2_E = 1 - |\tilde{d}f(\eta_i)|^2_E \leq 1 \]
and thus
\[ (29) \quad |\xi_1 \wedge \cdots \wedge \xi_{m-1}| \leq 1 \]
From (28) and (29), we have (25).

Now assume that \( n = 1 \). Obviously \( \xi_i, i = 1, \ldots, m - 1 \), and \( \tilde{d}f(\eta_m) \) are tangent to the cylinder
\[ C_D = \{(x_1, \ldots, x_{m+n}) \in \mathbb{R}^{m+n} : \sum_{i=1}^{m} x_i^2 = r\} \]
and the horizontal vector \( \tilde{\eta}_m \) is orthogonal to \( C_D \) at the point \((q, f(q))\). It follows that
\[ (\ast \Omega)^{-2} = |\xi_1 \wedge \cdots \wedge \xi_m|^2 \]
\[ = |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \tilde{\eta}_m + \xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \tilde{d}f(\eta_m)|^2 \]
\[ = |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \tilde{\eta}_m|^2 + |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \tilde{d}f(\eta_m)|^2 \]
\[ = |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \tilde{\eta}_m|^2 + |\tilde{\eta}_m| \wedge |\tilde{d}f(\eta_m)|^2 \]
\[ \geq |\xi_1 \wedge \cdots \wedge \xi_{m-1}|^2 - |\tilde{d}f(\eta_m)|^2_E \]
i.e.,
\[ Q^2/P^2 = |\xi_1 \wedge \cdots \wedge \xi_{m-1}|^2 \leq (\ast \Omega)^{-2} + |\tilde{d}f(\eta_m)|^2_E \]
This gives (26). \(\square\)

We recall the following

Theorem A. ([CY], [JX]) Let \( M \) be an extremal space-like \( m \)-submanifold in \( \mathbb{R}^{m+n} \). If \( M \) is closed with respect to the Euclidean topology, then \( M \) has to be a space-like \( m \)-plane.

Remark. Obviously, an entire graph is closed with respect to the Euclidean topology. Hence we know that any entire extremal space-like graph must be a space-like \( m \)-plane.

Theorem 1. Let \( M^m = (x, f(x)) \) be an entire space-like graph in \( \mathbb{R}^{m+n} \) with parallel mean curvature. If \( \ast \Omega \) has the following growth
\[ (30) \quad \ast \Omega = o(r) \quad \text{as } r \to \infty \]
where \( r = \sqrt{\sum_{i=1}^{m} x_i^2} \), then \( M \) is a space-like \( m \)-plane.
Proof. Let \( \mathcal{M}_r = \{(x, f(x)) : x \in D_r \subseteq \mathbb{R}^m\} \), where \( D_r \) denotes the closed ball of radius \( r \) centered at the origin in \( \mathbb{R}^m \). From (13), (18) and Lemma 1, we have

\[
mH \int_{\mathcal{M}_r} (*\Omega)\omega^1 \wedge \cdots \wedge \omega^m \leq \int_{\partial \mathcal{M}_r} R \Psi \\
\leq \int_{\partial \mathcal{M}_r} *\Omega P \Psi \\
\leq \sup_{\partial D_r} (*\Omega \{Vol(\partial D_r)
\]

i.e.,

\[
mHV ol(D_r) \leq \sup_{\partial D_r} (*\Omega \{Vol(\partial D_r)
\]

Thus

\[
H \leq C \frac{\sup_{\partial D_r} (*\Omega)}{r}
\]

where \( C \) is a universal constant. Let \( r \to \infty \). It follows that \( H \equiv 0 \). Hence we may complete the proof by Theorem A. \( \square \)

For space-like hypersurfaces, we may give a more delicate growth condition to ensure the above result.

**Proposition 2.** Let \( \mathcal{M}^m = (x, f(x)) \) be an entire space-like hypersurface in \( \mathbb{R}^{m+1} \) with constant mean curvature. If

\[
\sup_{\partial D_r} \{|df(\eta_m)|_E * \Omega\} = o(r)
\]

where \( r = \sqrt{\sum_{i=1}^m x_i^2} \), then \( \mathcal{M} \) is a space-like \( m \)-plane.

**Proof.** From (13), (25) and Lemma 1, we have

\[
mH \int_{\mathcal{M}_r} (*\Omega)\omega^1 \wedge \cdots \wedge \omega^m \leq \int_{\partial \mathcal{M}_r} R \Psi \\
\leq \int_{\partial \mathcal{M}_r} \sqrt{(*\Omega)^2 Q^2 - P^2} \Psi \\
\leq \int_{\partial \mathcal{M}_r} |df(\eta_m)|_E (*\Omega) P \Psi \\
\leq \sup_{\partial D_r} \{|df(\eta_m)|_E * \Omega\} Vol(\partial D_r)
\]

By the same argument as in Theorem 1, we prove the proposition. \( \square \)

Let’s consider a typical example of space-like graphs in \( \mathbb{R}^{m+1} \) with constant mean curvature.
**Example 1.** The hyperboloid is defined by

\[ H^m_{-1} = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{m+1}_1 : \sum_{i=1}^{m} x_i^2 - x_{m+1}^2 = -1, x_{m+1} \geq 0\} \]

\[ = \{(x, f(x)) : f = \sqrt{1 + \sum_{i=1}^{m} x_i^2}, x \in \mathbb{R}^m\} \]

By a direct computation, we have

\[ \Omega = 1 \]

\[ \sqrt{1 - |\nabla f|^2} = \sqrt{1 + \sum_{i=1}^{m} x_i^2} = O(r) \]

From (31), we see that the growth condition in Theorem 1 is optimal.

**Theorem B. ([XY],[CSZ])** Let \( M = (x, f(x)) \) be a complete space-like hypersurface in \( R^{m+1}_1 \) with constant mean curvature. If the image of the Gauss map \( \gamma : M \to H^m_{-1} \) lies in a horoball in \( H^m_{-1} \), then \( M \) must be a space-like hyperplane.

It is known that every complete spacelike hypersurface in \( R^{m+1}_1 \) is spatially entire (cf. [AM]). To compare Theorem 1 with Theorem B, we hope to find the equivalent restriction on the function \( \Omega \), if the image of \( \gamma \) lies in a horoball.

Let \( M = (x, f(x)) \) be a space-like graphic hypersurface in \( R^{m+1}_1 \). Its Gauss map \( \gamma \) is given by

\[ \gamma : M \to H^m_{-1} \]

\[ x \mapsto \frac{1}{\sqrt{1 - |\nabla f|^2}}(f_{x_1}, \ldots, f_{x_m}, 1) = \Omega(f_{x_1}, \ldots, f_{x_m}, 1) \]

where \( H^m_{-1} \) is the hyperboloid endowed with the induced metric from \( R^{m+1}_1 \). Obviously the Gauss image of \( M \) is bounded in \( H^m_{-1} \) if and only if \( \Omega \) is bounded. This also holds true for higher codimensional case (cf. [Xi2]).

It is easier to use the upper half-space model \( H^m \) of the hyperbolic space for describing horoballs. We consider the following maps

\[ h_1 : H_{-1}^m \to B^m \]

\[ (x_1, \ldots, x_m, x_{m+1}) \mapsto \left( \frac{x_1}{1 + x_{m+1}}, \ldots, \frac{x_m}{1 + x_{m+1}} \right) \]

and

\[ h_2 : B^m \to H^m = \{(y_1, \ldots, y_m) \in \mathbb{R}^m : y_m > 0\} \]

\[ p \mapsto 2\frac{p - p_0}{|p - p_0|^2} - (0, \ldots, 0, 1) \]

where \( p_0 = (0, \ldots, -1) \) and \( H^m \) is endowed with the metric \( g = y_m^{-2}(dy_1^2 + \cdots + dy_m^2) \). The set \( \{(y_1, \ldots, y_m) \in H^m : y_m > c > 0\} \) for any positive constant \( c \) is a horoball in
It is known that $h_2 \circ h_1 : H^m_1 \to H^m$ is an isomorphism. From (32), (33) and (34), we may get the $m$–th component of $h_2 \circ h_1 \circ \gamma$ as follows:

$$(h_2 \circ h_1 \circ \gamma)_m = \frac{1}{(1 + f_{x_m}) \ast \Omega}$$

So the condition $y_m > c > 0$ is equivalent to

$$(35) \quad (1 + f_{x_m}) \ast \Omega < \frac{1}{c}$$

Note that $f_{x_m}$ may be replaced by any $f_{x_i}$ or $v(f)$ which denotes the derivative in any fixed unit direction $v$ in $R^m$. Obviously, if there exists a sequence of points $\{p_k\}$ such that $* \Omega(p_k) \to \infty$, then $(f_{x_m})(p_k) \to -1$. Therefore (35) implies that all ‘bad singular directions’ approach one direction, i.e., $\partial/\partial x_m$.

Since $* \Omega = (\sqrt{1 - |\nabla f|^2})^{-1}$, we see that the growth condition in Theorem 1 is very much like that one given by Ecker-Huisken in [EH] for a minimal graphic hypersurface in the Euclidean space $R^{m+1}$. Hence Theorem 1 may be regarded as an Ecker-Huisken type result.

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