Complex order fractional derivatives in viscoelasticity

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Abstract

We introduce complex order fractional derivatives in models that describe viscoelastic materials. This can not be carried out unrestricly, and therefore we derive, for the first time, real valued compatibility constraints, as well as physical constraints that lead to acceptable models. As a result, we introduce a new form of complex order fractional derivative. Also, we consider a fractional differential equation with complex derivatives, and study its solvability. Results obtained for stress relaxation and creep are illustrated by several numerical examples.

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1 Introduction

Fractional calculus is a powerful tool for modeling various phenomena in mechanics, physics, biology, chemistry, medicine, economy, etc. Last few decades have brought a rapid expansion of the non-integer order differential and integral calculus, from which both the theory and its applications benefit significantly. However, most of the work done in this field so far has been based on the use of real order fractional derivatives and integrals. It is worth to mention that there are several authors who also applied complex order fractional derivatives to model various phenomena, see the work of Machado or Makris. [14, 16, 17]. In all of these papers, restrictions on constitutive parameters that follow from the Second Law of Thermodynamics were not examined. In the analysis that follows this issue will be addressed.

The main goal of this paper is to motivate and explain basic concepts of fractional calculus with complex order fractional derivatives. Throughout the paper we will investigate constitutive equations in the dimensionless form, for all independent (\( t \) and \( x \)) and dependent (\( \sigma \)) variables, and the differentiability of functions.
and $\varepsilon$) variables. Thus, consider a constitutive equation, given by (1), connecting the stress $\sigma(t, x)$ at the point $x \in \mathbb{R}$ and time $t \in \mathbb{R}_+$, with the strain $\varepsilon(t, x)$:

$$\sum_{n=0}^{N} a_n \partial_t^\alpha \sigma(t, x) = \sum_{m=0}^{M} b_m \partial_t^\beta \varepsilon(t, x),$$  \hspace{1cm} (1)

that contains fractional derivatives of complex order $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_M$. The precise definition of the operator $\partial_t^\alpha$ of fractional differentiation with respect to $t$ is given below. In order to make a useful framework for the study of (1) we involve two types of conditions: 1. real valued compatibility constraints, and 2. thermodynamical constraints. Since this paper deals only with the well-posedness of constitutive equations of type (1) and their solvability for strain if stress is prescribed, we may, without loss of generality, assume that both $\sigma$ and $\varepsilon$ are functions only of $t$. Also, equation (1) can be seen as a generalization of different models considered in the literature so far (see e.g. [11, 13, 15, 18]), since by taking all $\alpha_n$ and $\beta_m$ to be real numbers, the problem is reduced to the real case studied in the mentioned papers.

Fractional operators of complex order are introduced as follows (see [13, 19]): For $y \in AC([0, T])$ coincides with the case of real $\eta$, i.e.,

$$\partial_t^\eta y(t) := \frac{1}{\Gamma(\eta)} \int_0^t \frac{y(\tau)}{(t-\tau)^{1-\eta}} d\tau, \quad t \in [0, T],$$

where $\Gamma$ is the Euler gamma function. If $\eta = i\theta$, $\theta \in \mathbb{R}$, then the latter integral diverges, and hence one introduces the fractional integration of imaginary order as

$$\partial_t^{i\theta} y(t) := \frac{d}{dt} \partial_t^{1+i\theta} y(t) = \frac{1}{\Gamma(1+i\theta)} \frac{d}{dt} \int_0^t (t-\tau)^{i\theta} y(\tau) d\tau, \quad t \in [0, T].$$

However, in both cases the left Riemann-Liouville fractional derivative of order $\eta \in \mathbb{C}$ with $0 \leq \Re \eta < 1$ is given by

$$\partial_t^\eta y(t) := \frac{d}{dt} \partial_t^{1-\eta} y(t) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_0^t \frac{y(\tau)}{(t-\tau)^{\eta}} d\tau, \quad t \in [0, T].$$

The basic tool for our study will be the Laplace and Fourier transforms. In order to have a good framework we will perform these transforms in $S^r(\mathbb{R})$, the space of tempered distributions. It is the dual space for the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$. In particular, we are interested in the space $S^r_+(\mathbb{R})$ whose elements are of the form $y = P(D) Y_0$, where $Y_0$ is a locally integrable polynomial bounded function on $\mathbb{R}$ that vanishes on $(-\infty, 0)$, and $P(D)$ denotes a partial differential operator.

The Fourier transform of $y \in L^1(\mathbb{R})$ (or $y \in L^2(\mathbb{R})$) is defined as

$$\mathcal{F} y(\omega) = \hat{y}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} y(x) dx, \quad \omega \in \mathbb{R}. \hspace{1cm} (2)$$

In the distributional setting, one has $\langle \mathcal{F} y, \varphi \rangle = \langle y, \mathcal{F} \varphi \rangle$, $y \in S^r(\mathbb{R})$, $\varphi \in S(\mathbb{R})$, where $\mathcal{F} \varphi$ is defined by (2). For $y \in L^1(\mathbb{R})$ with $y(t) = 0$, $t < 0$, and $|y(t)| \leq Ae^{at}$, $a, A > 0$, the Laplace transform is given by

$$\mathcal{L} y(s) = \tilde{y}(s) = \int_0^{\infty} e^{-st} y(t) dt, \quad \Re s > a.
If $y \in S'_+(\mathbb{R})$ then $a = 0$ (since $y$ is bounded by a polynomial). Then $Ly$ is a holomorphic function in the half plane $\text{Re} s > 0$ (see e.g. [20]).

Let $Y(s), \text{Re} s > 0$, be a holomorphic function bounded by a polynomial in that domain. Then, for a suitable polynomial $P$, $Y(s)/P(s)$ is integrable along the line $\Gamma = (a-i\infty, a+i\infty)$, and the inverse Laplace transform of $Y$ is a tempered distribution $y(t) = P(d/dt)Y_0(t)$, where $Y_0(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{2\pi i} \int_{\Gamma} Y(s) e^{st} ds$.

Let $y \in S'_+$. Recall:

$$\mathcal{F}\left[ \frac{d^n}{dx^n}y \right](\omega) = (i\omega)^n \mathcal{F}y(\omega) \quad (\omega \in \mathbb{R}), \quad \mathcal{L}\left[ \frac{d^n}{dt^n}y \right](s) = s^n Ly(s) \quad (\text{Re} s > 0), \quad n \in \mathbb{N},$$

$$\mathcal{F}[0D_\alpha^\nu y](\omega) = (i\omega)^\alpha \mathcal{F}y(\omega) \quad (\omega \in \mathbb{R}), \quad \mathcal{L}[0D_\alpha^\nu y](s) = s^\alpha Ly(s) \quad (\text{Re} s > 0), \quad \alpha \in \mathbb{C}.$$
Theorem 2.1 Let \( \varepsilon \in AC([0, T]) \) be real-valued, for all \( T > 0 \), \( 0 < A < 1 \) and \( b \neq 0 \). Then function \( \sigma \) defined by (3) is real valued if and only if \( \beta \in \mathbb{R} \).

Proof. It follows from (3), with \( \beta = A + iB \), and \( 1/\Gamma(1 - \beta) = h + ir \), that
\[
\text{Im} \sigma(t) = b \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A} \left( r \cos(B \ln \tau) - h \sin(B \ln \tau) \right) d\tau, \quad t \geq 0.
\]
Denoting by \( \phi_h := \text{tg} \phi \), for \( h \neq 0 \), we obtain
\[
\text{Im} \sigma(t) = \frac{bh}{\cos \phi} \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A} \sin(\phi - B \ln \tau) d\tau, \quad t \geq 0.
\]
In the case \( h = 0 \), the imaginary part of \( \sigma \) reduces to \( bh \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A} \cos(B \ln \tau) d\tau \).

If \( B = 0 \) then \( r = 0 \) and \( \phi = 0 \), hence \( \text{Im} \sigma = 0 \) and \( \sigma \) is a real-valued function.

Next, suppose that \( B \neq 0 \). But then one can find a subinterval \((t_1, t_2)\) of \((0, T)\) where \( \sin(\phi - B \ln \tau) \), respectively \( \cos(B \ln \tau) \), is positive (or negative), and choose \( \varepsilon \in AC([0, T]) \) which is compactly supported in \((t_1, t_2)\) and strictly positive. This leads to a contradiction with the assumption \( \text{Im} \sigma = 0 \).

The previous theorem implies that equations of form (3) with \( \beta \in \mathbb{C} \setminus \mathbb{R} \) can not be a constitutive equation for a viscoelastic body.

Next, consider the equation
\[
\sigma(t) = b_1 \, D_t^{\beta_1} \varepsilon(t) + b_2 \, D_t^{\beta_2} \varepsilon(t), \quad t \geq 0,
\]
where \( b_1, b_2 \in \mathbb{R} \), and \( \beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{R} \), i.e., \( \beta_k = A_k + iB_k \) and \( B_k \neq 0 \) (\( k = 1, 2 \)). Suppose again that \( \varepsilon \in AC([0, T]) \) is a real-valued function, for every \( T > 0 \).

Remark 2.2 Note that dimension \([0D_t^{\beta_1}]\) is \( T^{-\beta_1} \), where \( T \) is time unit. Therefore, (4) makes sense if \([b_1] = T^{\beta_1}\) and \([b_2] = T^{\beta_2}\).

Theorem 2.3 Function \( \sigma \) given by (4) is real-valued for all real-valued positive \( \varepsilon \in AC([0, T]) \) if and only if \( b_1 = b_2 \) and \( \beta_2 = \beta_1 \).

Proof. We continue with the notation of Theorem 2.1. Let \( t \geq 0 \). Then
\[
\sigma(t) = \frac{b_1}{\Gamma(1 - \beta_1)} \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-\beta_1} d\tau + \frac{b_2}{\Gamma(1 - \beta_2)} \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-\beta_2} d\tau
\]
Denote by \( h_k + ir_k := 1/\Gamma(1 - \beta_k) \), \( k = 1, 2 \). Then the imaginary part of the right hand side reads:
\[
\text{Im} \sigma(t) = \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A_1} \left( b_1 r_1 \cos(B_1 \ln \tau) - b_1 h_1 \sin(B_1 \ln \tau) \right) d\tau
\]
\[
+ \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A_2} \left( b_2 r_2 \cos(B_2 \ln \tau) - b_2 h_2 \sin(B_2 \ln \tau) \right) d\tau
\]
Using the identity $\Gamma(z) = \Gamma(z)$, it is straightforward to check that $b_1 = b_2$ and $\beta_2 = \beta_1$ imply that $\text{Im} \sigma = 0$, and hence $\sigma$ is a real-valued function on $[0, T]$, for every $T > 0$.

Conversely, we want to find conditions on parameters which yield a real-valued function $\sigma$. Thus, we look at (5), with the change of variables $p = \ln \tau$, $\tau \in (0, t), t \leq T$, and solutions of the equation

$$\frac{d}{dt} \left( \int_{-\infty}^{\ln t} e^{p(t-e^{p})} \left( b_1 e^{-A_1 p} \cos(B_1 p) - b_1 h_1 e^{-A_1 p} \sin(B_1 p) + b_2 e^{-A_2 p} \cos(B_2 p) - b_2 e^{-A_2 p} \sin(B_2 p) \right) dp \right) = 0, \quad t \in [0, T].$$

Set $r_k := \tan \phi_k, k = 1, 2$, for $h_1, h_2 \neq 0$. (For $h_k = 0$ set $\phi_k := \frac{\pi}{2}$, $k = 1, 2$.) Then (6) gives

$$\frac{d}{dt} \left( \int_{-\infty}^{\ln t} e^{p(t-e^{p})} \left( \frac{b_1 h_1}{\cos \phi_1} e^{-A_1 p} \sin(\phi_1 - B_1 p) + \frac{b_2 h_2}{\cos \phi_2} e^{-A_2 p} \sin(\phi_2 - B_2 p) \right) dp \right) = 0.$$

Assume first that $|B_1| \neq |B_2|$, say $|B_1| > |B_2|$. Since the basic period of $\sin(\phi_1 - B_1 p)$ ($T_0 = 2\pi/|B_1|$) is smaller than for $\sin(\phi_2 - B_2 p)$, it follows that for every $k \in \mathbb{N}$, $k > k_0$, where $k_0$ depends on $\ln t$, the function $\sin(\phi_1 - B_1 p)$ changes its sign at least three times in the interval $(k\pi, k\pi + 2\pi/|B_2|)$. Thus, on that interval, there exist at least two intervals where $\sin(\phi_1 - B_1 p)$ and $\sin(\phi_2 - B_2 p)$ have the same sign, and two intervals where they have opposite signs. We conclude that there exists an interval $[a, b] \subseteq (k\pi, k\pi + 2\pi/|B_2|) \subset (-\infty, \ln t)$, so that

$$\frac{b_1 h_1}{\cos \phi_1} e^{-A_1 p} \sin(\phi_1 - B_1 p) + \frac{b_2 h_2}{\cos \phi_2} e^{-A_2 p} \sin(\phi_2 - B_2 p) > 0, \quad p \in [a, b].$$

Choose $\delta > 0$ and $k \in \mathbb{N}$ so that

$$\{t-e^{p}; t \in (T/2 - \delta, T/2 + \delta), p \in [a, b]\} = (T/2 - \delta - e^b, T/2 + \delta - e^a) = I \subset (0, T).$$

Now, we choose a non-negative function $\varepsilon \in AC((0, T])$ with the following properties: $\text{supp} \varepsilon \subseteq I$, so that the function $p \mapsto \varepsilon(t-e^{p})$, $p \in [a, b]$, is strictly positive on some $[a_1, b_1] \subseteq (a, b)$. This implies that for $t \in (T/2 - \delta, T/2 + \delta)$,

$$\int_a^b \varepsilon(t-e^{p}) \left( \frac{b_1 h_1}{\cos \phi_1} e^{-A_1 p} \sin(\phi_1 - B_1 p) + \frac{b_2 h_2}{\cos \phi_2} e^{-A_2 p} \sin(\phi_2 - B_2 p) \right) dp$$

is not a constant function. This is in contradiction with (6).

Therefore, in order to have (6), one must have $|B_1| = |B_2|$. Moreover, arguing as above, one concludes that $|\phi_1 - B_1 p| = |\phi_2 - B_2 p|$ must hold, for all $p \in (0, T), t \geq 0$. Then we examine the equation

$$b_1 h_1 e^{-A_1 p} - b_2 h_2 e^{-A_2 p} = 0, \quad \text{or} \quad b_1 h_1 e^{-A_1 p} + b_2 h_2 e^{-A_2 p} = 0, \quad p \in [k\pi, k\pi + 2\pi/|B_1|].$$

Now in both cases, $b_1 b_2 > 0$ or $b_1 b_2 < 0$, it is easy to conclude that $A_1 = A_2$, $b_1 = b_2$ and $B_1 = -B_2$ have to be satisfied. This proves the theorem.

\[\Box\]

\textbf{Remark 2.4} (i) Theorem 2.3 states that a real-valued compatibility constraint for constitutive equations of form (4) holds if they contain complex fractional derivatives of strain, whose
orders have to be complex conjugated numbers. Therefore, we may assume in the sequel, without loss of generality, that $B > 0$.

(ii) According to the above analysis, one can take arbitrary linear combination of pairs of complex conjugated fractional derivatives of strain. Moreover, one can also allow the same type of fractional derivatives of stress. Thus, one can consider the most general stress-strain constitutive equation with fractional derivatives of complex order:

$$
\sigma(t) + \sum_{i=1}^{N} c_i \left( 0 D_t^{\gamma_i} + 0 D_t^{\bar{\gamma}_i} \right) \sigma(t) = \varepsilon(t) + \sum_{j=1}^{M} b_j \left( 0 D_t^{\beta_j} + 0 D_t^{\bar{\beta}_j} \right) \varepsilon(t),
$$

where $c_i, b_j \in \mathbb{R}$ and $\gamma_i, \beta_j \in \mathbb{C}$, $i = 1, \ldots, N$, $j = 1, \ldots, M$.

(iii) As a consequence one has that stress-strain relations can also contain arbitrary real order fractional derivatives, without any additional restrictions. This fact has already been known from previous work.

(iv) The same conclusions can also be obtained using a different approach. One can apply the result from [8, p. 293, Satz 2], which tells that a function $F$ is real-valued (almost everywhere), if its Laplace transform is real-valued, for all real $s$ in the half-plane of convergence right from some real $x_0$, in order to show that an admissible fractional constitutive equation (ii) may be of complex order only if it contains pairs of complex conjugated fractional derivatives of stress and strain.

2.2 Thermodynamical restrictions

We start with the constitutive equation

$$
\sigma(t) = 2b_0 D_t^{\beta} \varepsilon(t), \quad D_t^{\beta} := \frac{1}{2} \left( D_t^{\beta} + 0 D_t^{\bar{\beta}} \right), \quad t \geq 0, \quad (8)
$$

where we assume that $b > 0$ and $\beta = A + iB$, $0 < A < 1$, $B > 0$. Note that [8] generalizes the Hooke law in the complex fractional framework. In the case $\beta \in \mathbb{R}$, this new complex fractional operator $D_t^{\beta}$ coincides with the usual left Riemann-Liouville fractional derivative.

We apply the Fourier transform to (8):

$$
\hat{\sigma}(\omega) = b ((i \omega)^{\beta} + (i \omega)^{\bar{\beta}}) \hat{\varepsilon}(\omega), \omega \in \mathbb{R}.
$$

Then, define the complex modulus $\hat{E}$ such that

$$
\hat{E}(\omega) := b ((i \omega)^{\beta} + (i \omega)^{\bar{\beta}}) = b \omega A \left( e^{-\frac{B \pi}{2}} e^{i \left( \frac{A \pi}{2} + B \ln \omega \right)} + e^{\frac{B \pi}{2}} e^{i \left( \frac{A \pi}{2} - B \ln \omega \right)} \right), \quad \omega \in \mathbb{R}.
$$

Thermodynamical restrictions involve, for $\omega \in \mathbb{R}_+$,

$$
\text{Re} \hat{E}(\omega) = b \omega A \left( e^{-\frac{B \pi}{2}} \cos \left( \frac{A \pi}{2} + B \ln \omega \right) + e^{\frac{B \pi}{2}} \cos \left( \frac{A \pi}{2} - B \ln \omega \right) \right) \geq 0, \quad (9)
$$

$$
\text{Im} \hat{E}(\omega) = b \omega A \left( e^{-\frac{B \pi}{2}} \sin \left( \frac{A \pi}{2} + B \ln \omega \right) + e^{\frac{B \pi}{2}} \sin \left( \frac{A \pi}{2} - B \ln \omega \right) \right) \geq 0. \quad (10)
$$

But this is in contradiction with $B > 0$, since for $\omega > 0$, (9) and (10) imply $B = 0$.

In order not to confront the real valued compatibility requirement and the Second Law of Thermodynamics for [8], one may require that (9) and (10) hold for $\omega$ in some bounded interval instead of in all of $\mathbb{R}$. Alternatively, as we shall do in the sequel, one can modify (8) by adding additional terms, in order to preserve the Second Law of Thermodynamics.

Thus, we proceed by proposing the following constitutive equation

$$
\sigma(t) = a_0 D_t^{\alpha} \varepsilon(t) + 2b_0 D_t^{\beta} \varepsilon(t), \quad t \geq 0, \quad (11)
$$
where $a, \alpha \in \mathbb{R}$, $0 < \alpha < 1$, and $b, \beta, \delta \in \mathbb{R}$ are as in (8). Again we follow the procedure described above for deriving thermodynamical restrictions: $\sigma(\omega) = [a(i\omega)^\alpha + b((i\omega)^\beta + (i\omega)^\delta)] \hat{\epsilon}(\omega)$, $\omega \in \mathbb{R}$. Consider the complex module ($\omega \in \mathbb{R}$)

$$E(\omega) = a \omega^\alpha e^{i \frac{\alpha \pi}{2}} + b \omega^\beta \left(e^{-\frac{B \pi}{2}} e^{i \left(\frac{A \pi}{2} + B \ln \omega\right)} + e^{\frac{B \pi}{2}} e^{i \left(\frac{A \pi}{2} - B \ln \omega\right)}\right);$$

(12)

$$\text{Re} \hat{E}(\omega) = a \omega^\alpha \cos \frac{\alpha \pi}{2} + b \omega^\beta \left(e^{-\frac{B \pi}{2}} \cos \left(\frac{A \pi}{2} + B \ln \omega\right) + e^{\frac{B \pi}{2}} \cos \left(\frac{A \pi}{2} - B \ln \omega\right)\right),$$

(13)

$$\text{Im} \hat{E}(\omega) = a \omega^\alpha \sin \frac{\alpha \pi}{2} + b \omega^\beta \left(e^{-\frac{B \pi}{2}} \sin \left(\frac{A \pi}{2} + B \ln \omega\right) + e^{\frac{B \pi}{2}} \sin \left(\frac{A \pi}{2} - B \ln \omega\right)\right).$$

(14)

We will investigate conditions $\text{Re} \hat{E}(\omega) \geq 0$ and $\text{Im} \hat{E}(\omega) \geq 0$ on $\mathbb{R}_+$. The assumption $\alpha > A$ leads to a contradiction, since for $\omega \searrow 0$ the sign of the second term in (13) determines the sign of $\text{Re} \hat{E}$, and it can be negative. Thus, we must have $\alpha \leq A$. If $\alpha < A$, then for $\omega \to \infty$, the second term in (13) could be negative. This together yields that the only possibility is $A = \alpha$. (The same conclusion is obtained if one considers $\text{Im} \hat{E}(\omega) \geq 0$, $\omega > 0$.)

Therefore, with $A = \alpha$, (13) and (14) become:

$$\text{Re} \hat{E}(\omega) = a \omega^\alpha \cos \frac{\alpha \pi}{2} + 2b \omega^\alpha f(\omega), \quad \omega > 0,$$

(15)

$$\text{Im} \hat{E}(\omega) = a \omega^\alpha \sin \frac{\alpha \pi}{2} + 2b \omega^\alpha g(\omega), \quad \omega > 0,$$

(16)

with

$$f(\omega) := \cos \frac{\alpha \pi}{2} \cos(\ln \omega^B) \cosh \frac{B \pi}{2} + \sin \frac{\alpha \pi}{2} \sin(\ln \omega^B) \sinh \frac{B \pi}{2}, \quad \omega > 0,$$

(17)

$$g(\omega) := \sin \frac{\alpha \pi}{2} \cos(\ln \omega^B) \cosh \frac{B \pi}{2} - \cos \frac{\alpha \pi}{2} \sin(\ln \omega^B) \sinh \frac{B \pi}{2}, \quad \omega > 0.$$
We obtain the thermodynamical restrictions for (11) by requiring $\text{Re} \tilde{E}(\omega) \geq 0$ and $\text{Im} \tilde{E}(\omega) \geq 0$, for $\omega \in \mathbb{R}_+$:

$$a \geq 2b \cosh \frac{B\pi}{2} \sqrt{1 + \left( \frac{\text{ctg} \frac{\alpha\pi}{2}}{2} \tgh \frac{B\pi}{2} \right)^2}, \quad \text{if } \alpha \in \left(0, \frac{1}{2}\right],$$

and

$$a \geq 2b \cosh \frac{B\pi}{2} \sqrt{1 + \left( \frac{\text{ctg} \frac{\alpha\pi}{2}}{2} \tgh \frac{B\pi}{2} \right)^2}, \quad \text{if } \alpha \in \left[\frac{1}{2}, 1\right).$$

Notice that both restrictions further imply $a \geq 2b$.

**Remark 2.5** (i) Fix $a$ and $\alpha$. Inequalities (19) and (20) imply that as $B$ increases, the constant $b$ has to decrease, i.e., the contribution of complex fractional derivative of strain in the constitutive equation (11) is smaller if its imaginary part is larger.

(ii) Also, inequalities (19) and (20) lead to the same restrictions on parameters $a, b, \alpha$ and $B$, since for $\alpha \in (0, \frac{1}{2}]$ one has $1 - \alpha \in [\frac{1}{2}, 1)$, and the values of (19) and (20) coincide.

(iii) Under the same conditions, constitutive equation (11) can be extended to $\sigma(t) = \varepsilon(t) + a_0 D^\alpha_0 \varepsilon(t) + 2b_0 D^\beta_1 \varepsilon(t), \; t \geq 0$, which will be investigated in the next section.

### 3 Complex order fractional Kelvin-Voigt model

Consider the constitutive equation involving the complex order fractional derivative

$$\sigma(t) = \varepsilon(t) + a_0 D^\alpha_0 \varepsilon(t) + 2b_0 D^\beta_1 \varepsilon(t), \quad t \geq 0.$$  

We assume $a, b, E > 0$, $0 < \alpha < 1$, $B > 0$, $\beta = \alpha + iB$, and $\sigma$ and $\varepsilon$ are real-valued functions. The Laplace transform of (21) is $\tilde{\sigma}(s) = E(1 + a s^\alpha + b (s^\beta + s^\beta)) \tilde{\varepsilon}(s)$, $\text{Re} s > 0$, and hence

$$\tilde{\varepsilon}(s) = \frac{1}{1 + a s^\alpha + b (s^\beta + s^\beta)} \tilde{\sigma}(s), \quad \text{Re} s > 0.$$  

In order to determine $\varepsilon$ from (22) we need to analyze zeros of

$$\psi(s) = 1 + a s^\alpha + b (s^\beta + s^\beta), \quad s \in \mathbb{C}.$$  

Note that if we put $s = i\omega$, $\omega \in \mathbb{R}_+$, in (23), it becomes the complex modulus:

$$\psi(i\omega) = 1 + \hat{\varepsilon}(\omega) = 1 + a (i\omega)^\alpha + b ((i\omega)^\beta + (i\omega)^\beta), \quad \omega > 0,$$

where $\hat{\varepsilon}$ is given in (12).

Let $s = re^{i\varphi}$, $r > 0$, $\varphi \in [0, 2\pi]$. Then (with $\beta = \alpha + iB$)

$$\psi(s) = 1 + ar^\alpha e^{i\alpha \varphi} + br^\alpha (e^{-B\varphi} e^{i(ln r^B + \alpha \varphi)} + e^{B\varphi} e^{-i(ln r^B - \alpha \varphi)}),$$

and

$$\text{Re} \psi(s) = 1 + ar^\alpha \cos(\alpha \varphi) + 2br^\alpha \left( \cos(ln r^B) \cos(\alpha \varphi) \cosh(B\varphi) + \sin(ln r^B) \sin(\alpha \varphi) \sinh(B\varphi) \right),$$

$$\text{Im} \psi(s) = ar^\alpha \sin(\alpha \varphi) + 2br^\alpha \left( \cos(ln r^B) \sin(\alpha \varphi) \cosh(B\varphi) - \sin(ln r^B) \cos(\alpha \varphi) \sinh(B\varphi) \right).$$

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3.1 Thermodynamical restrictions

In the case of (24), using (25) and (26) we obtain:

\[
\text{Re } \psi(i\omega) = 1 + \text{Re } \hat{E}(\omega) \geq 1 + a\omega^\alpha \cos \frac{\alpha \pi}{2} + 2b\omega^\alpha \min_{x \in \mathbb{R}} f(x), \quad x = \ln \omega^B, \ \omega > 0,
\]

\[
\text{Im } \psi(i\omega) = \text{Im } \hat{E}(\omega) \geq a\omega^\alpha \sin \frac{\alpha \pi}{2} + 2b\omega^\alpha \min_{x \in \mathbb{R}} g(x), \quad x = \ln \omega^B, \ \omega > 0,
\]

where \( f \) and \( g \) are as in (17) and (18). This leads to the same thermodynamical restrictions (19) and (20), as in Section 2.2.

Therefore, from now on, we shall assume (19) and (20) to hold true. Now we shall examine the zeros of \( \psi \).

3.2 Zeros of \( \psi \) and solutions of (21)

**Theorem 3.1** Let \( \psi \) be the function given by (23). Then

(i) \( \psi \) has no zeros in the right complex half-plane \( \text{Re } s \geq 0 \).

(ii) \( \psi \) has no zeros in \( \mathbb{C} \) if the coefficients \( a, b, \alpha \) and \( B \) satisfy

\[
a \geq 2b \cosh(B\pi) \sqrt{1 + \left( \tan(\alpha \pi) \tgh(B\pi) \right)^2}, \quad \text{for } \alpha \in \left[ \frac{1}{4}, \frac{3}{4} \right] \setminus \left\{ \frac{1}{2} \right\}.
\]

\[
a \geq 2b \cosh(B\pi) \sqrt{1 + \left( \cotan(\alpha \pi) \tgh(B\pi) \right)^2}, \quad \text{for } \alpha \in \left( 0, \frac{1}{4} \right) \cup \left( \frac{1}{2}, \frac{3}{4}, 1 \right).
\]

**Proof.** First, we notice that if \( s_0 \) is a solution to \( \psi(s) = 0 \), then \( \bar{s}_0 \) (the complex conjugate of \( s_0 \)) is also a solution, since \( \psi(\bar{s}) = 1 + a \bar{s}^\alpha + b \left( \bar{s}^\beta + \bar{s}^{\bar{\beta}} \right) = \bar{\psi}(s) \). Thus, we restrict our attention to the upper complex half-plane \( \text{Im } s \geq 0 \), i.e., \( \varphi \in [0, \pi] \).

Using (25) and (26) we have, with \( s = re^{i\varphi}, \ r > 0, \ \varphi \in [0, \pi], \) and \( x = \ln r^B, \)

\[
\text{Re } \psi(s) \geq 1 + ar^\alpha \cos(\alpha \varphi) + 2br^\alpha \min_{x \in \mathbb{R}} f(x), \quad \text{(28)}
\]

\[
\text{Im } \psi(s) \geq ar^\alpha \sin(\alpha \varphi) + 2br^\alpha \min_{x \in \mathbb{R}} g(x), \quad \text{(29)}
\]

where

\[
f(x) = \cos(x) \cos(\alpha \varphi) \cosh(B \varphi) + \sin(x) \sin(\alpha \varphi) \sinh(B \varphi), \quad x \in \mathbb{R}, \quad \text{(30)}
\]

\[
g(x) = \cos(x) \sin(\alpha \varphi) \cosh(B \varphi) - \sin(x) \cos(\alpha \varphi) \sinh(B \varphi), \quad x \in \mathbb{R}. \quad \text{(31)}
\]

The critical points \( x_f \) and \( x_g \) of \( f \) and \( g \), respectively, satisfy

\[
tg x_f = \tan(\alpha \varphi) \tgh(B \varphi) \geq 0 \quad \text{and} \quad tg x_g = -\cotan(\alpha \varphi) \tgh(B \varphi) \leq 0, \quad \text{(32)}
\]

The proof of (i) and (ii) will be given by the argument principle.

(i) Consider \( \psi \) in the case \( \text{Re } s, \text{Im } s > 0 \). Choose a contour \( \Gamma = \gamma_{R1} \cup \gamma_{R2} \cup \gamma_{R3} \cup \gamma_{R4}, \) as it is shown in Fig. 1.

\( \gamma_{R1} \) is parametrized by \( s = x, \ x \in (\varepsilon, R) \) with \( \varepsilon \to 0 \) and \( R \to \infty \), so that (25) and (26) yield

\[
\text{Re } \psi(x) \geq 1 + x^\alpha (a + 2b \cos(\ln x^\beta)) \geq 1 + x^\alpha (a - 2b) \geq 0,
\]

\[
\text{Im } \psi(x) = 0,
\]
since both (19) and (20) imply $a \geq 2b$. Moreover, we have $\lim_{x \to 0} \psi(x) = 1$ and $\lim_{x \to \infty} \psi(x) = \infty$.

Along $\gamma_{R2}$ one has $s = Re^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2}]$, with $R \to \infty$. By (32) we have $\tan x_{f} \geq 0$, so that

$$
\min_{x \in \mathbb{R}} f(x) = -\cos(\alpha\varphi) \cosh(B\varphi) \sqrt{1 + (\tan(\alpha\varphi) \tanh(B\varphi))^2},
$$

and therefore (28) becomes

$$
\Re \psi(s) \geq 1 + R^a \cos(\alpha\varphi)(a - 2b \cosh(B\varphi)) \sqrt{1 + (\tan(\alpha\varphi) \tanh(B\varphi))^2} \geq 0.
$$

The previous inequality holds true, since for $\varphi \in [0, \frac{\pi}{2}]$ we have that

$$
p(\varphi) = \cosh(B\varphi) \sqrt{1 + (\tan(\alpha\varphi) \tanh(B\varphi))^2} \leq \cosh \frac{B\varphi}{2} \sqrt{1 + \left(\frac{\alpha\varphi}{2} \frac{B\varphi}{2}\right)^2},
$$

because of the fact that the function $p$ monotonically increases on $[0, \frac{\pi}{2}]$. Moreover, by (25) and (26), we have

$$
\Re \psi(s) \to \infty \quad \text{and} \quad \Im \psi(s) = 0, \quad \text{for} \quad \varphi = 0, \quad R \to \infty,
$$

$$
\Re \psi(s) \to \infty, \quad \text{for} \quad \varphi = \frac{\pi}{2}, \quad R \to \infty.
$$

The next segment is $\gamma_{R3}$, which is parametrized by $s = i\omega$, $\omega \in [R, \varepsilon]$, with $\varepsilon \to 0$ and $R \to \infty$. Then (25) and (26) yield

$$
\Re \psi(i\omega) = 1 + \Re \hat{E}(\omega) \geq 0 \quad \text{and} \quad \Im \psi(i\omega) = \Im \hat{E}(\omega) \geq 0, \quad \omega \in (\varepsilon, R),
$$

due to the thermodynamical requirements. Moreover, by (25) and (26), we have

$$
\Re \psi(\omega) \to 1 \quad \text{and} \quad \Im \psi(\omega) \to 0, \quad \text{as} \quad \omega \to 0,
$$

$$
\Re \psi(\omega) \to \infty \quad \text{and} \quad \Im \psi(\omega) \to \infty, \quad \text{as} \quad \omega \to \infty.
$$

The last part of the contour $\Gamma$ is the arc $\gamma_{R4}$, with $s = \varepsilon e^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2}]$, with $\varepsilon \to 0$. Using the same arguments as for the contour $\gamma_{R2}$, we have

$$
\Re \psi(s) \geq 1 + \varepsilon^a \cos(\alpha\varphi)(a - 2b \cosh(B\varphi)) \sqrt{1 + (\tan(\alpha\varphi) \tanh(B\varphi))^2} \geq 1.
$$
Also, by (26) and (33), we have

\[ \text{Re } \psi(s) \to 1 \quad \text{and} \quad \text{Im } \psi(s) \to 0, \quad \text{as} \quad \varepsilon \to 0. \]

We conclude that \( \Delta \arg \psi(s) = 0 \) so that, by the argument principle, there are no zeroes of \( \psi \) in the right complex half-plane \( \text{Re } s \geq 0 \).

(ii) In order to discuss the zeros of \( \psi \) in the left complex half-plane, we use the contour \( \Gamma_L = \gamma_{L1} \cup \gamma_{L2} \cup \gamma_{L3} \cup \gamma_{L4} \), shown in Fig. 1. The contour \( \gamma_{L1} \) has the same parametrization as the contour \( \gamma_{R3} \), so the same conclusions as for \( \gamma_{R3} \) hold true.

The parametrization of the contour \( \gamma_{L2} \) is \( s = Re^{i\varphi}, \varphi \in [\frac{\pi}{2}, \pi] \), with \( R \to \infty \). Let us distinguish two cases.

a1. \( \alpha \varphi \in (0, \frac{\pi}{2}) \)

Then \( \sin(\alpha \varphi) > 0, \cos(\alpha \varphi) > 0 \), so the critical points of \( f \) and \( g \), see (30) and (31), given by (32), satisfy \( \tan g > 0 \) and \( \tan x_g < 0 \). For the minimums of \( f \) and \( g \) in (28) and (29) we have

\[
\begin{align*}
\min_{x \in \mathbb{R}} f(x) &= -\cos(\alpha \varphi) \cosh(B \varphi) \sqrt{1 + (\tan(\alpha \varphi) \tgh(B \varphi))^2}, \\
\min_{x \in \mathbb{R}} g(x) &= -\sin(\alpha \varphi) \cosh(B \varphi) \sqrt{1 + (\cot(\alpha \varphi) \tgh(B \varphi))^2},
\end{align*}
\]

respectively, so that (28) and (29) become

\[
\begin{align*}
\text{Re } \psi(s) &\geq 1 + R^\alpha \cos(\alpha \varphi) \left( a - 2b \cosh(B \varphi) \sqrt{1 + (\tan(\alpha \varphi) \tgh(B \varphi))^2} \right), \\
\text{Im } \psi(s) &\geq R^\alpha \sin(\alpha \varphi) \left( a - 2b \cosh(B \varphi) \sqrt{1 + (\cot(\alpha \varphi) \tgh(B \varphi))^2} \right).
\end{align*}
\]

Function \( H_f(\varphi) = \cosh(B \varphi) \sqrt{1 + (\tan(\alpha \varphi) \tgh(B \varphi))^2} \) is monotonically increasing for \( \varphi \in [\frac{\pi}{2}, \pi] \), since \( \alpha \varphi \in (0, \frac{\pi}{2}) \), thus

\[
\text{Re } \psi(s) \geq 1 + R^\alpha \cos(\alpha \varphi) \left( a - 2b \cosh(B \varphi) \sqrt{1 + (\tan(\alpha \varphi) \tgh(B \varphi))^2} \right) \geq 0,
\]

if (27) is satisfied. Note that \( \text{Re } \psi(s) \to \infty \) and \( \text{Im } \psi(s) \to \infty \), for \( \varphi = \pi, R \to \infty \).

b1. \( \alpha \varphi \in [\frac{\pi}{2}, \pi] \)

Then \( \sin(\alpha \varphi) > 0, \cos(\alpha \varphi) \leq 0 \), so the critical points of \( g \), see (31), given by (32), satisfy \( \tan x_g \geq 0 \). For the minimum of \( g \) in (31) we have

\[
\min_{x \in \mathbb{R}} g(x) = -\sin(\alpha \varphi) \cosh(B \varphi) \sqrt{1 + (\cot(\alpha \varphi) \tgh(B \varphi))^2},
\]

so that (29) becomes

\[
\text{Im } \psi(s) \geq R^\alpha \sin(\alpha \varphi) \left( a - 2b \cosh(B \varphi) \sqrt{1 + (\cot(\alpha \varphi) \tgh(B \varphi))^2} \right).
\]

Function \( H_g(\varphi) = \cosh(B \varphi) \sqrt{1 + (\cot(\alpha \varphi) \tgh(B \varphi))^2} \) is monotonically increasing for \( \varphi \in [\frac{\pi}{2}, \pi] \), since \( \alpha \varphi \in [\frac{\pi}{2}, \pi] \), and (27) implies

\[
\text{Im } \psi(s) \geq R^\alpha \sin(\alpha \varphi) \left( a - 2b \cosh(B \varphi) \sqrt{1 + (\cot(\alpha \varphi) \tgh(B \varphi))^2} \right) \geq 0.
\]

Note that \( \text{Im } \psi(s) \to \infty \), for \( \varphi = \pi, R \to \infty \).
Now we discuss possible situations for $\alpha \in (0, 1)$ and $\varphi \in \left[\frac{\pi}{2}, \pi\right]$.

If $\alpha \in (0, \frac{1}{2})$ then a1. holds so that $\Re \psi(s) \geq 0$.

If $\alpha \in \left[\frac{1}{2}, 1\right)$ then we distinguish two cases. For $\varphi \in \left[\frac{\pi}{2}, \frac{\pi}{\alpha}\right]$ case a1. holds, so $\Re \psi(s) \geq 0$. For $\varphi \in \left[\frac{\pi}{2\alpha}, \pi\right)$ case b1. holds, and $\Im \psi(s) \geq 0$.

Parametrization of the contour $\gamma_{L3}$ is $s = xe^{i\varphi}, x \in (\varepsilon, R)$, with $\varepsilon \to 0$ and $R \to \infty$.

Again, we have two cases.

a2. $\alpha \in (0, \frac{1}{2})$

Then, for $x \in (\varepsilon, R)$, using the same argumentation as in case a1, we have, by (34) and (35) (with $R = x$), $\Re \psi(s) \geq 1$ and $\Im \psi(s) \geq 0$, due to (27). Thus, looking at (34) and (35) (with $R = x$), we conclude

$$\Re \psi(s) \to \infty \text{ and } \Im \psi(s) \to \infty, \text{ for } x \to \infty,$$

$$\Re \psi(s) \to 1 \text{ and } \Im \psi(s) \to 0, \text{ for } x \to 0.$$

b2. $\alpha \in \left[\frac{1}{2}, 1\right)$

Then, for $x \in (\varepsilon, R)$, using the same argumentation as in case b1, we have (by (35)) that $\Im \psi(s) \geq 0$, and

$$\Im \psi(s) \to \infty, \text{ for } x \to \infty \text{ and } \Im \psi(s) \to 0, \text{ for } x \to 0.$$

The parametrization of the contour $\gamma_{L4}$ is $s = \varepsilon e^{i\varphi}, \varphi \in \left[\pi, \pi\right], \varepsilon \to 0$. From (25) and (26), for sufficiently small $\varepsilon$, we have

$$\Re \psi(s) \to 1 \text{ and } \Im \psi(s) \to 0, \text{ for } \varepsilon \to 0, \varphi \in \left[\frac{\pi}{2}, \pi\right].$$

Summing up all results from cases a1, a2, b1 and b2, we obtain the following:

- For $\alpha \in (0, \frac{1}{2})$, $\Re \psi(s) \geq 0$, for $s \in \Gamma_{L}$, which implies that $\Delta \arg \psi(s) = 0$. Therefore, using the argument principle, we conclude that in this case $\psi$ has no zeros in the left complex half-plane.

- If $\alpha \in \left[\frac{1}{2}, 1\right)$, then for $s \in \gamma_{L1}$ and $s \in \{z \in \gamma_{L2} | \arg z \leq \frac{\pi}{2\alpha}\}$, we have $\Re \psi(s) \geq 0$, while for $s \in \{z \in \gamma_{L2} | \arg z > \frac{\pi}{2\alpha}\}$ and $s \in \gamma_{L3}$, we have $\Im \psi(s) \geq 0$. For $s \in \gamma_{L4}$ we again have $\Re \psi(s) \geq 0$. Hence, we conclude that $\Delta \arg \psi(s) = 0$, and therefore, using the argument principle, neither in case $\alpha \in \left[\frac{1}{2}, 1\right)$ function $\psi$ has zeros in the left complex half-plane.

This completes the proof. \square

Rewrite (22) as

$$\tilde{e}(s) = \tilde{K}(s)\tilde{\sigma}(s), \quad \tilde{K}(s) = \frac{1}{1 + as^\alpha + b(s^\beta + s\bar{\beta})}, \quad \Re s > 0. \quad (36)$$

**Theorem 3.2** Let $\tilde{e}$ be given by (36). Then

$$e(t) = K(t) \ast \sigma(t), \quad t \geq 0. \quad (37)$$
Moreover, if (27) holds, then

\[ K(t) = K_I(t) = \frac{1}{2\pi i} \int_0^\infty \left( \frac{e^{-qt}}{1 + q^{\alpha} e^{i\alpha\pi} [a + b(e^{i\ln q^B e^{-B\pi}} + e^{-i\ln q^B e^{B\pi}})]} \right. \]

\[ \left. - \frac{e^{-qt}}{1 + q^{\alpha} e^{-i\alpha\pi} [a + b(e^{i\ln q^B e^{B\pi}} + e^{-i\ln q^B e^{-B\pi}})]} \right) dq. \] (38)

If condition (27) is violated, then \( \psi \) has at most finite number of zeros in the left complex half-plane, and

\[ K = K_I \quad \text{or} \quad K = K_I + K_R, \]

where

\[ K_R(t) = \sum_{\psi(\tilde{s}_i)=0, i=1,2,\ldots,n} \left( \text{Res}(\tilde{K}(s) e^{st}, s_i) + \text{Res}(\tilde{K}(s) e^{st}, \tilde{s}_i) \right), \] (39)

with \( \tilde{K} \) given by (36).

**Proof.** The first part is clear. We invert now \( \tilde{K} \), given in (36), by the use of the Cauchy residues theorem

\[ \oint_{\tilde{\Gamma}} \tilde{K}(s) e^{st} ds = 2\pi i \sum_{\psi(\tilde{s})=0} \text{Res}(\tilde{K}(s) e^{st}, \tilde{s}) \] (40)

and the contour \( \tilde{\Gamma} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_r \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0 \) shown in Fig. 2

![Figure 2: Contour \( \tilde{\Gamma} \).](image)

If condition (27) is satisfied, then, by Theorem 3.1 the residues equal zero. One can show that the integrals over the contours \( \Gamma_1, \Gamma_r \) and \( \Gamma_4 \) tend to zero when \( R \to \infty \) and \( r \to 0 \). The
remaining integrals give:

\[
\lim_{R \to \infty, \ r \to 0} \int \Gamma_2 \tilde{K}(s)e^{st} ds = -\int_0^\infty \frac{e^{-qt}}{1 + q^{\alpha}e^{i\alpha\pi}[a + b(e^{i\ln q^B} e^{-B\pi} + e^{-i\ln q^B} e^{B\pi})]} dq,
\]

\[
\lim_{R \to \infty, \ r \to 0} \int \Gamma_3 \tilde{K}(s)e^{st} ds = \int_0^\infty \frac{e^{-qt}}{1 + q^{\alpha}e^{-i\alpha\pi}[a + b(e^{i\ln q^B} e^{B\pi} + e^{-i\ln q^B} e^{-B\pi})]} dq,
\]

\[
\lim_{R \to \infty, \ r \to 0} \int \gamma_0 \tilde{K}(s)e^{st} ds = 2\pi i K_I(t),
\]

which, by the Cauchy residues theorem (40), leads to (38).

If condition (27) is violated, then, by Theorem 3.1, the denominator of \( \tilde{K} \) either has no zeros in the complex plane, and so \( K = K_I \), or it has zeros in the left complex half-plane, which comes in pairs with complex conjugates. We show now that \( \psi \) (which is the denominator of \( \tilde{K} \)) can have at most finite number of zeros for \( \text{Im} s \leq 0 \). Rewrite \( \psi(s) = 0 \) as

\[
a + b(s^{iB} + s^{-iB}) = s^{-\alpha}. \tag{41}
\]

If \( s = re^{i\varphi} \), \( \varphi \in \left[\frac{\pi}{2}, \pi\right] \), then we have

\[
a + b(r^{iB}e^{-\varphi B} + r^{-iB}e^{\varphi B}) = r^{-\alpha}e^{-\alpha\varphi i}. \tag{42}
\]

When \( r \to \infty \) the right hand side tends to zero, while the left hand side tends to \( a \). As \( r \to 0 \) we see that the left hand side is bounded, while the right hand side is not bounded. Thus, the zeros in the left half-plane of function \( \psi \), if exist, have to be bounded both from above and below. In that case we have \( K = K_I + K_R \), where \( K_R \) is given by (39).

\[\square\]

4 Numerical verifications

Here we present several examples of the proposed constitutive equation. We shall treat stress relaxation, creep and periodic loading cases.

4.1 Stress relaxation experiment

We take (21) with \( \varepsilon(t) = H(t) \), \( H \) is the Heaviside function, and regularize it as \( H_\varepsilon(t) = 1 - \exp(-t/k) \), \( k \to 0 \). In order to determine \( \sigma \) we calculate

\[
\sigma(t) = H_\varepsilon(t) + a_0D_\gamma^\alpha H_\varepsilon(t) + b_0D_\delta^\beta H_\varepsilon(t), \quad t \geq 0, \tag{41}
\]

with \( \sigma(0) = 0 \), using the expansion formula (see [2, 3]),

\[
a_0D_\gamma^\gamma y(t) \approx \frac{y(t)}{t^\gamma} \frac{A(N, \gamma)}{t^{p+\gamma}} - \sum_{p=1}^N C_{p-1}(\gamma) \frac{V_{p-1}(y)(t)}{t^{p+\gamma}}, \tag{42}
\]

where

\[
A(N, \gamma) = \frac{\Gamma(N + 1 + \gamma)}{\Gamma(1 - \gamma)\Gamma(\gamma)N!}, \quad C_{p-1}(\gamma) = \frac{\Gamma(p + \gamma)}{\Gamma(1 - \gamma)\Gamma(\gamma)(p - 1)!},
\]

and

\[
V_{p-1}(y)(t) = t^{p-1}y(t), \quad V_{p-1}(y)(0) = 0, \quad p = 1, 2, 3, \ldots \tag{43}
\]

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Inserting (42) into (41) we obtain

$$\sigma(t) \approx \left\{ 1 + \left[ \frac{a A(N, \alpha)}{t^\alpha} + b \left( \frac{A(N, \alpha + iB)}{t^{\alpha+iB}} + \frac{A(N, \alpha - iB)}{t^{\alpha-iB}} \right) \right] \right\} H_\varepsilon(t)$$

$$- \sum_{p=1}^{N} \left\{ \frac{C_{p-1}(\alpha)}{t^{p+\alpha}} + \left[ \frac{C_{p-1}(\alpha + iB)}{t^{p+\alpha+iB}} + \frac{C_{p-1}(\alpha - iB)}{t^{p+\alpha-iB}} \right] \right\} V_{p-1}^{(1)}(H_\varepsilon(t)), \quad (44)$$

where

$$V_{p-1}^{(1)}(H_\varepsilon)(t) = t^{p-1} H_\varepsilon(t), \quad p = 1, 2, 3, \ldots \quad (45)$$

We will compare (44) with the stress $\sigma$ obtained by (41) and by the definition of fractional derivative (8):

$$\sigma(t) = H_\varepsilon(t) + a \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t H_\varepsilon(\tau) d\tau \left( \frac{t-\tau}{(t-\tau)^\alpha} \right)$$

$$+ b \frac{d}{dt} \left[ \frac{1}{\Gamma(1-\alpha + iB)} \int_0^t H_\varepsilon(\tau) d\tau + \frac{1}{\Gamma(1-\alpha - iB)} \int_0^t H_\varepsilon(\tau) d\tau \left( \frac{t-\tau}{(t-\tau)^{\alpha+iB}} \right) \right]. \quad (46)$$

In Fig. 3 we show results obtained by determining $\sigma$ from (46) for small times and different values of $B$. In the same figure we show, by dots, the values of $\sigma$, in several points, obtained by using (44), (45) for $k = 0.01$, $N = 100$. As could be seen from Fig. 3, the results obtained from (46) and (44), (45) agree well. The stress relaxation curves are shown in Fig. 4 for the same set of parameters and for larger times. As could be seen, regardless of the values of $B$, we have $\lim_{t \to \infty} \sigma(t) = 1$. Note that in all cases of $B$, the restriction which follows from the dissipation inequality is satisfied.

![Stress relaxation curves](image)

Figure 3: Stress relaxation curves for $\alpha = 0.4$ and $B \in \{0.4, 0.6, 0.8, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 0.25]$.

### 4.2 Creep experiment

Suppose that $\sigma(t) = H(t)$, i.e.,

$$H(t) = (1 + a_0 D_t^\alpha + 2b_0 D_t^\beta) \varepsilon(t), \quad t \geq 0. \quad (47)$$
Figure 4: Stress relaxation curves for $\alpha = 0.4$ and $B \in \{0.4, 0.6, 0.8, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 10]$.

By (42), we obtain

$$\varepsilon(t) \approx \frac{H(t) + \sum_{p=1}^{N} \left\{ \frac{C_{p-1}(\alpha)}{t^p} + \frac{C_{p-1}(\alpha+iB)}{t^p+iB} + \frac{C_{p-1}(\alpha-iB)}{t^p-iB} \right\} V_{p-1}(\varepsilon)(t)}{1 + a A(N, \alpha) + 2b \left( \frac{A(N, \alpha+iB)}{t^\alpha+iB} + \frac{A(N, \alpha-iB)}{t^\alpha-iB} \right)},$$

or

$$\varepsilon(t) \approx \frac{H(t) t^\alpha + \sum_{p=1}^{N} \left\{ \frac{C_{p-1}(\alpha)}{t^p} + \frac{C_{p-1}(\alpha+iB)}{t^p+iB} + \frac{C_{p-1}(\alpha-iB)}{t^p-iB} \right\} V_{p-1}(\varepsilon)(t)}{t^\alpha + a A(N, \alpha) + 2b \left( \frac{A(N, \alpha+iB)}{t^\alpha+iB} + \frac{A(N, \alpha-iB)}{t^\alpha-iB} \right)}. \tag{48}$$

By using (48) in (43) we obtain

$$V_{p-1}(\varepsilon)(t) \approx t^{p-1} H(t) t^\alpha + \sum_{p=1}^{N} \left\{ \frac{C_{p-1}(\alpha)}{t^p} + \frac{C_{p-1}(\alpha+iB)}{t^p+iB} + \frac{C_{p-1}(\alpha-iB)}{t^p-iB} \right\} V_{p-1}(\varepsilon)(t)$$

$$V_{p-1}(\varepsilon)(0) = 0, \quad p = 1, 2, 3, \ldots$$

Equation (47) may also be solved by contour integration

$$\varepsilon(t) = K(t) * H(t), \quad t \geq 0, \tag{49}$$

where $K$ is given by (38), see (37). Finally, the values of $\varepsilon$, at discrete points, could be determined directly from $\varepsilon(s) = \frac{1}{s} \tilde{K}(s)$, Re $s > 0$, see (36), by the use of Post inversion formula, see [7]. Thus,

$$\varepsilon(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left[ s^{n+1} \frac{d^n}{ds^n} \left( \frac{1}{s(1 + as^\alpha + b(s^\beta + s^\gamma))} \right) \right]_{s=\frac{\alpha}{\beta}}, \quad t \geq 0. \tag{50}$$

In Fig. [5] [6] and [7] we show $\varepsilon$ for several values of parameters determined form (49). In Fig. [5] for specified values of $t$ we present values of $\varepsilon$, determined from (48), with $N = 7$, denoted by dots, as well as the values of $\varepsilon$, determined by (50), with $n = 25$, denoted by squares. As could be seen the agreement of results determined by different methods is significant.

From Fig. [6] and [7] one sees that, regardless of the value of $B$, creep curves tend to $\varepsilon = 1$. In Fig. [6] the creep curves are monotonically increasing, while in Fig. [7] creep curves have oscillatory character, characteristic for the case when the mass of the rod is not neglected. Note that in all cases of $B$, the restriction determined by the dissipation inequality is satisfied.
Figure 5: Creep curve for $\alpha = 0.4$ and $B = 0.4$, $a = 0.8$, $b = 0.1$, $t \in [0, 100]$. 

Figure 6: Creep curves for $\alpha = 0.4$ and $B \in \{0.2, 0.4, 0.6\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 200]$. 

Figure 7: Creep curves for $\alpha = 0.4$ and $B \in \{0.7, 0.8, 0.9, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 200]$. 
5 Conclusion

In this work, we proposed a new constitutive equation with fractional derivatives of complex order for viscoelastic body of the Kelvin-Voigt type. The use of fractional derivatives of complex order, together with restrictions following from the Second Law of Thermodynamics, represent the main novelty of our work. Our results can be summarized as follows.

1. In order to obtain real stress for given real strain, we used two fractional derivatives of complex order that are complex conjugated numbers, see Theorem 2.3.

2. The restrictions that follow from the Second Law of Thermodynamics for isothermal process implied that the constitutive equation must additionally contain a fractional derivative of real order. Thus, the simplest constitutive equation that gives real stress for real strain and satisfies the dissipativity condition is given by (21).

3. We provided a complete analysis of solvability of the complex order fractional Kelvin-Voigt model given by (21) (see Theorems 3.1 and 3.2).

4. We studied stress relaxation and creep problems through equation (21). An increase of $B$ implied that the stress relaxation decreases more rapidly to the limiting value of stress, i.e., $\lim_{t \to \infty} \sigma(t) = 1$.

5. We presented numerical experiments when the dissipation inequality is satisfied. The creep experiment showed that the increase in the imaginary part of the complex derivative $B$ changes the character of creep curve from monotonic to oscillatory form, see Fig. 6 and 7. However, the creep curves never cross the value equal to 1. The creep curve resembles the form of creep curve when the mass of the rod is taken into account, see [4, p. 124].

6. Our further study will be directed to the problems of vibration and wave propagation of a rod with finite mass and constitutive equation (21), along the lines presented in [4], see also [10].

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