Instantaneous optimal control for vehicle nonlinear suspension system with time delay

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Abstract: This paper deals with the problem of instantaneous optimal control for vehicle suspension system with control time delay. The control objective is to improve vehicle ride comfort. Firstly, the dynamic equation of nonlinear suspension system is established. Secondly, to reduce the effect of control time delay, a formula based on integral transformation is introduced to obtain instantaneous optimal control strategy. Finally, the effectiveness of instantaneous optimal control is verified by numerical simulation. The analysis results reflect that the instantaneous optimal control could ensure the stability of the control system no matter how the time delay changes. And it is also found that the control effect varies with the control time delay. It lays a foundation for the study of how to add intentional time delay in the control loop to make the control effect better. This paper provides an effective way for nonlinear vibration control, which has important theoretical and engineering application value.

1. Introduction
Vehicle suspension system is an important part of the vehicle chassis. It can affect the safety and comfort of the passengers directly[1,2]. So during the past decades, vehicle suspension system has been studied extensively. Now there are three types of vehicle suspension system: passive, semi-active and active suspension. Because the parameters of the passive suspension system are constant, it is impossible to adjust the structure of the system in real time to adapt to different road conditions. The active suspension system is extensively accepted because of its more elastic and efficient in improving the performance of vehicle suspension system than the other two types. Great achievements have been made on active suspension system in recent years. However, it has complex structure and high cost. And the time delay in the control loop is inevitable in the system[3], in addition, it has been shown that even a very small time delay may lead to the decrease of control efficiency and even the instability of the system[4,5]. For this reason, many scholars have studied the elimination and compensation technology of time delay to solve the problem of time delay and improve the control effect, such as Smith prediction, phase-shift and recursive response [6-9]. But in vibration control, these methods are generally suitable for short time delay only. The controller will be failed when the time delay is too long. Moreover, the influence of nonlinear parameters makes the delay processing more difficult. Therefore, this paper aims to propose a control method for nonlinear suspension system with time delay, which is applicable regardless of the size of time delay.

The paper is organized as follows. Section 1 is the introduction. Then the vehicle nonlinear suspension system motion equation is modeled and the control strategy is designed in section 2. After
that, the efficiency of the optimal control strategy is illustrated by simulation in section 4. The conclusion is given in the final section.

2. Dynamic model of nonlinear suspension system

In order to verify the effectiveness of instantaneous optimal control for nonlinear suspension system, the control model of nonlinear suspension system is established. Because the rubber bush element is mostly used in the connection of suspension and frame, and the rubber bush has the hysteresis nonlinear characteristic. At present, the widely used hysteresis nonlinear models include bilinear hysteresis nonlinear model, Bouc-Wen model, polynomial model, Davidenkov model and asymmetric hysteresis nonlinear model. In this article, the cubic nonlinear polynomial model is used, that is, the polynomial function of system displacement and velocity is used to fit the hysteretic nonlinear force. What’s more, the vertical vibration of the vehicle body is the major factor influencing the ride comfort of the vehicle. And according to the structure of the vehicle is quite complex. The paper neglecting the pitching and rolling motion of the vehicle body, the passive suspension element and MR damper are used as actuators to establish a quarter of the car suspension model, as shown in Figure 1. In this model, \(m_s\) is the sprung mass and \(m_u\) is the unsprung mass, which denote the vehicle chassis and the mass of wheel assembly, respectively; \(F_d\) is the nonlinear force, \(k_1\) is the linear stiffness, \(k_2\) is the nonlinear stiffness, \(c_1\) is the linear damping, \(c_2\) is the nonlinear damping, \(k_t\) and \(c_t\) are the spring and damping coefficient of pneumatic tire, respectively, \(x_s\) and \(x_u\) represent the displacement of the sprung and unsprung masses, respectively, \(u(t)\) is the force to eliminate the vertical vibration of sprung mass, \(\tau\) is the time delay in the whole control loop, \(x_g\) is the road displacement input.

![Figure 1. 2-DOF simplified vehicle model](image)

The mathematical equation of the suspension model is obtained as follows:

\[
\begin{align*}
    m_s \ddot{x}_s + F_d - u(t - \tau) & = 0 \\
    m_u \ddot{x}_u - F_d + k_1 (x_u - x_s) + c_1 (\dot{x}_u - \dot{x}_s) + u(t - \tau) & = 0 \\
    F_d = k_2 (x_s - x_u) + k_t (x_s - x_u)^3 + c_2 (\dot{x}_s - \dot{x}_u) + c_t (\ddot{x}_s - \ddot{x}_u)^3 & = 0
\end{align*}
\]

(1)

The main control objectives of active suspension system are ride comfort, suspension deflection and road holding. In fact, the ride comfort of vehicle can be quantitatively analyzed by sprung mass acceleration. This is why \(\ddot{x}_s\) is chosen as the most important control output \(y_1\). Therefore, we hope it can be controlled as small as possible. Due to the characteristics of suspension structure, the relative displacement between the sprung and unsprung masses, \((x_s - x_u)\), must be within a certain range, which is also the reason why this paper takes it as the second control output \(y_2\). In addition, in order
to keep the wheel in contact with the road surface, the tire dynamic load \( k_i(x_u - x_e) + c_i(x_{\dot{e}} - \dot{x}_e) \) as another control output \( y_i \) must be less than the static tire load \( (m_i + m_w)g \).

Now define the state variables as \( x_1 = x, \ x_2 = \dot{x}, \ x_3 = x_u, \ x_4 = \dot{x}_u \). Define the state vector \( x = [x_1, x_2, x_3, x_4]^T \) and the control output vector \( y = [y_1, y_2, y_3]^T \). Then the dynamics of (1) can be rewritten as

\[
\begin{align*}
\dot{x} &= Ax + Bu(t - \tau) + f + EW \\
y &= Cx + Du(t - \tau) + \hat{f} + GW
\end{align*}
\]

(2)

Where

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-k_i & c_i & k_i & c_i \\
0 & 0 & 0 & 1 \\
k_i & c_i & -k_i + k_i & -c_i + c_i \\
\end{bmatrix}, \quad f = \begin{bmatrix}
0 \\
-k_i(m_i - x_i)^3 - c_i(m_x - x_x)^3 \\
0 \\
k_i(m_i - x_i)^3 + c_i(m_x - x_x)^3 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
\frac{1}{m_i} \\
0 \\
0 \\
-1 \\
\end{bmatrix}, \quad W = \begin{bmatrix}
\dot{x}_s \\\n\end{bmatrix}
\]

About road excitation, there are many models to simulate road random excitation, such as white noise model, harmonic superposition model, power spectrum model and so on. In this paper, the white noise model is used to abstract the road roughness into white noise satisfying certain conditions, and then the transformed colored noise random excitation model is fitted. The approximation of rational function can be expressed as

\[
\hat{x}_s(t) = -\alpha v(t) + w(t)
\]

(3)

Where \( w(t) \) respects white noise, \( v \) respects speed of vehicle, \( \alpha \) respects coefficient of road, such as class A road \( \alpha = 0.132 \), class B road \( \alpha = 0.1303 \), class C road \( \alpha = 0.12 \), class D road \( \alpha = 0.1007 \), class E road \( \alpha = 0.09 \)[10].

3. Instantaneous optimal controller design

In this paper, based on the integral transformation method and optimal control theory, the optimal control of LQR is designed [11,12]. In view of the state equation (2), the integral transformation method is used to transform the state equation of the system as follows:

\[
Z(t) = x(t) + \int_{-\infty}^{t} e^{\lambda(t-\eta)}Bu(t-\eta)d\eta
\]

(4)

Therefore, \( \dot{Z} = \dot{x} + A \int_{-\infty}^{t} e^{\lambda(t-\eta)}Bu(t-\eta)d\eta - Bu(t+\eta) + e^{\lambda t}u(t) \). Then the output state equation can be reformulated into equation (5) as following

\[
\dot{Z} = AZ(t) + \bar{B}u(t) + f + EW
\]

(5)

Where \( \bar{B} = e^{-\lambda t}B \), the equation (5) can be solved numerically step by step using the regular fourth order Runge-Kutta method. The solution of the equation (5) can be written in the following form by the mentioned method.
\[ Z(t) = Z(t - 2\Delta t) + \frac{1}{6}(A_h + 2A_t + 2A_j + A_s) \]  \tag{6}

Where \( \Delta t \) is the integration time step, \( A_h, A_t, A_j \) and \( A_s \) is a function of \( t - 2\Delta t, t - \Delta t \) and \( t \).

\[ A_h = 2\Delta t[AZ(t - 2\Delta t) + 2\Delta t u(t - 2\Delta t) + f(t - 2\Delta t) + EW(t - 2\Delta t)] \]

\[ A_t = 2\Delta t[AZ(t - 2\Delta t) + 0.5A_h + 2\Delta t u(t - \Delta t) + f(t - \Delta t) + EW(t - \Delta t)] \]

\[ A_j = 2\Delta t[AZ(t - 2\Delta t) + 0.5A_h + 2\Delta t u(t) + f(t) + EW(t)] \]

To simplify the formula, defining

\[ D(t - 2\Delta t, t - \Delta t) = Z(t - 2\Delta t) + \frac{1}{3}\Delta t \left\{ \begin{array}{c} AZ(t - 2\Delta t) + 2\Delta t u(t - 2\Delta t) + f(t - 2\Delta t) + EW(t - 2\Delta t) \\ + 4EW(t - \Delta t) + A[Z(t - 2\Delta t) + A_s] \end{array} \right\} \]  \tag{7}

Then the equation (6) can be rewritten into

\[ Z(t) = D(t - 2\Delta t, t - \Delta t) + \frac{1}{3}\Delta t[B_h u(t) + f(t) + EW(t)] \]  \tag{8}

It should be pointed out that the computational time step of the fourth order Runge-Kutta method for solving differential equations on MATLAB should be half of the time step in equation (6).

Take the optimal control objective function as

\[ J(t) = Z^T(t)QZ(t) + u^T(t)Ru(t) \]  \tag{9}

Where \( Q \) is a semi-positive definite coefficient matrix, and \( R \) is an positive definite coefficient matrix to represent the relative importance of the system response vector \( Z(t) \) and the control input vector \( u(t) \), respectively. To minimize the objective function \( J(t) \), the optimal control force can be obtained as following

\[ u(t) = -\frac{1}{3}A\overrightarrow{R^{-1}}B_h QZ(t) \]  \tag{10}

From the equations (4) and (10), the control law contains the time delay \( \tau \) and the integral term \( T(t) = \int_0^T e^{-\lambda(T-\tau)}Bu(t+\theta)d\theta \). And \( T(t) \) can be got by calculation below.

Set the data sampling period to \( T \), any time delay can be expressed as \( \tau = lT - m \), where \( l > 0 \) is a positive integer, \( 0 \leq m < T \). And use the integration time step to identify the sampling period. In this way, the integral term can be transformed into the following form when \( t = kT \).

\[ T(t) = \int_0^T e^{-\lambda(T-\tau)}Bu(kT + \theta)d\theta = e^{-\lambda(T-\tau)} \int_0^T e^{-\lambda\theta}Bu(kT + \theta)d\theta + \int_{T}^{2T} e^{-\lambda\theta}Bu(kT + \theta)d\theta + \cdots + \int_{(n-1)T}^{nT} e^{-\lambda\theta}Bu((n-1)T + \theta)d\theta + \cdots \]  \tag{11}

Defining \( D(t) = e^{\lambda T}, G(t) = \int_{0}^{T} e^{-\lambda\theta}d\theta \), equation (11) can expressed as the following equation.

\[ T(t) = 1_{m=0} G(T - m) Bu((k-l)T) + D(m(T - l)) G(T) Bu((k-l+1)T) + \cdots + D(m-l) G(T) Bu((k-1)T) \]  \tag{12}

In this paper, \( m = 0 \) is used for analysis, the time delay is an integer multiple of the sampling period. Therefore, equation (12) can be written as

\[ T(t) = 1_{m=0} G(T) Bu((k-l)T) + D(T - m) G(T) Bu((k-l+1)T) + \cdots + D(l-m) G(T) Bu((k-1)T) \]  \tag{13}

And \( G(t) \) in equation (13) can be obtained through the following equation.

\[ G(t) = \sum_{n=0}^{\infty} (-A)^{n}\left(\frac{T}{\Delta t}\right)^n \]  \tag{14}
When $t$ is given, $G(t)$ will converge to constant matrix after finite step iterative calculation[13].

4. Simulation studies

To demonstrate the effectiveness and the feasibility of the proposed method, simulation analysis is carried out. According to the parameters of a vehicle suspension system, choosing $m_s = 843$kg, $m_w = 98$kg, $k_i = 45482$N/m, $c_i = 2546.5$N·s/m, $k_j = 111324$N/m³, $c_j = 4102$N·s/m³, $k_j = 604685$N/m, $c_j = 370$N·s/m. The weighting matrix in equation (9) is given by $Q = \text{diag}[1e8,1e8,1e8,1e8]$, $R = 1$, respectively. Therefore, the time-delay control output response characteristics of the nonlinear suspension system under the random excitation of the road are shown in Figure 2.

![Figure 2](image)

From Figure 2, it can be seen that the time delay changes and the system output response changes. But no matter how the time delay changes, the system is always stable. According to the response, fill the root mean square (RMS) values of sprung mass acceleration, suspension dynamic travel and tire dynamic load amplitude into Table 1.

| Table 1 | RMS value of output response of optimal control for nonlinear suspension system |
|---------|--------------------------------------------------------------------------------|
|         | $\tau = 0$s | $\tau = 0.03$s | $\tau = 0.05$s | $\tau = 0.065$s | $\tau = 0.1$s | $\tau = 0.18$s |
| sprung mass acceleration (m/s²) | 1.03 | 0.86 | 0.90 | 0.95 | 0.87 | 0.86 |
| suspension dynamic displacement (m) | 0.008 | 0.007 | 0.007 | 0.009 | 0.008 | 0.008 |
| Tire Dynamic Load (N) | 1508.3 | 1627.1 | 1566.6 | 1691.0 | 1638.8 | 1637.3 |

From Table 1, it can be found that the RMS value of sprung mass acceleration changes with the control time delay. The proposed method guarantees the control stability of nonlinear suspension system with control time delay. It can also be found that the increase of time delay will make the acceleration of sprung mass larger or smaller. Therefore, based on the inherent time delay of the system, the artificial active time delay can be used as a parameter to adjust the damping performance of the system. Therefore, the instantaneous optimal control is effective.

5. Conclusions

In this paper, the instantaneous optimal control for vehicle nonlinear suspension system is carried out by using integral transformation method and optimal control theory. The performance of suspension
system is analyzed. The results show that the instantaneous optimal control can ensure the stability of suspension system. In addition, the control strategy can effectively improve the damping performance of suspension. The simulation results show that the method has a good effect on the performance regulation of the nonlinear vehicle suspension system. Moreover, the method provides a theoretical basis for the effective use of time delay.

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