(A)dS EXCHANGES AND PARTIALLY-MASSLESS HIGHER SPINS

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ABSTRACT

We determine the current exchange amplitudes for free totally symmetric tensor fields $\varphi_{\mu_1...\mu_s}$ of mass $M$ in a $d$-dimensional $dS$ space, extending the results previously obtained for $s = 2$ by other authors. Our construction is based on an unconstrained formulation where both the higher-spin gauge fields and the corresponding gauge parameters $\Lambda_{\mu_1...\mu_{s-1}}$ are not subject to Fronsdal’s trace constraints, but compensator fields $\alpha_{\mu_1...\mu_{s-3}}$ are introduced for $s > 2$. The free massive $dS$ equations can be fully determined by a radial dimensional reduction from a ($d + 1$)-dimensional Minkowski space time, and lead for all spins to relatively handy closed-form expressions for the exchange amplitudes, where the external currents are conserved, both in $d$ and in ($d+1$) dimensions, but are otherwise arbitrary. As for $s = 2$, these amplitudes are rational functions of $(ML)^2$, where $L$ is the $dS$ radius. In general they are related to the hypergeometric functions $3F_2(a, b, c; d, e; z)$, and their poles identify a subset of the “partially-massless” discrete states, selected by the condition that the gauge transformations of the corresponding fields contain some non-derivative terms. Corresponding results for $AdS$ spaces can be obtained from these by a formal analytic continuation, while the massless limit is smooth, with no van Dam-Veltman-Zakharov discontinuity.

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1 Introduction

In [1] we determined the current exchange amplitudes for massless spin-$s$ totally symmetric higher-spin fields $\phi_{\mu_1...\mu_s}$ in a $d$-dimensional Minkowski space time, and we also extended the analysis to massless fields in $(A)dS$ backgrounds. In this analysis we resorted to an unconstrained formulation for higher-spin fields, introduced and developed in [3] [4] [1], where Fronsdal’s trace constraints [5]
on the gauge parameters $\Lambda_{\mu_1...\mu_{s-1}}$ and the corresponding double-trace constraints on the gauge fields are eliminated via the introduction, for $s \geq 3$, of spin-$(s-3)$ compensators $\alpha_{\mu_1...\mu_{s-3}}$. This unconstrained formalism has the peculiar feature of containing some higher-derivative terms, and was motivated by the previous unconstrained geometric non-local formalism of [3]. It is far simpler, however, than the previous local constructions of Buchbinder, Pashnev, Tsulaia and others [6], the first where Fronsdal’s constraints were overcome, starting from a BRST [7] construction, but at the price of introducing $O(s)$ additional fields. Further developments along these lines, leading to an action with only a few additional fields, are contained in [8], while the extension of the non-local formalism to the massive case and the reduction of the local formalism of [3, 4, 1] to two derivatives are described in [9]. The extension of the non-local geometric equations to higher-spin fields of mixed symmetry is also available, and is discussed in [10]. In this work we focus on the local formulation for massive unconstrained higher-spin bosons in an $(A)dS$ background. The corresponding analysis for the geometric, non local massive theory, along the lines on [9], as well as the extension to massive $(A)dS$ fermions, are interesting issues that we leave for future work.

The key property of the external currents determining the exchanges of [1] was the fact that they were conserved but otherwise arbitrary, whereas in Fronsdal’s formulation they would only need to be partly conserved, while their double traces would be constrained to vanish. Our flat-space result can be subsumed in the relatively handy expression,

$$\langle J, \Box \varphi \rangle = \sum_n \frac{1}{n! 2^n (3 - \zeta)_n} \langle J^{[n]} , J^{[n]} \rangle , \quad \text{(1.1)}$$

that identifies the residue of the pole at the physical mass shell of the exchanged higher-spin particle, where $J^{[n]}$ denotes the $n$-th trace of the external current, the coefficient $\zeta$ is defined as

$$\zeta = \frac{d}{2} + s , \quad \text{(1.2)}$$

$(a)_n$ denotes the $n$-th Pochhammer symbol of $a$, defined in Appendix A, and the brackets denote a suitably normalized inner product, so that

$$\langle \varphi, \varphi \rangle = \frac{1}{s!} \varphi_{\mu_1...\mu_s} \varphi^{\mu_1...\mu_s} . \quad \text{(1.3)}$$

In [1] the amplitudes of eq. (1.1) were shown to propagate the correct numbers of on-shell degrees of freedom, exactly like in Fronsdal’s formulation, even though the currents involved were not doubly traceless. The analysis was also generalized to the case of massless fields in maximally symmetric $(A)dS$ space times, where a very similar result was obtained. The main difference was the replacement of the flat-space D’Alembertian with the Lichnerowicz operator [11], that in the index-free notation of [1] is such that

$$\Box_L \varphi = \Box_{AdS} \varphi + \frac{1}{L^2} \left[ s(d + s - 2)\varphi - 2g \varphi' \right] . \quad \text{(1.4)}$$

In this expression $\Box_{AdS}$ is the D’Alembertian for the $AdS$ background with metric $g$, $L$ is the $AdS$ radius and $\varphi'$ denotes the trace of $\varphi$. The Lichnerowicz operator has the nice property of commuting with traces, covariant derivatives and contractions, and the corresponding result for the $dS$ background can be directly obtained from the previous expressions via the formal continuation of $L^2$ to $-L^2$.

The extension of these results to massive fields in Minkowski space times is straightforward, essentially since the little groups for a massive particle in $d$ dimensions and for a massless particle
in $(d+1)$ dimensions coincide. This simple correspondence underlies the familiar description of massive $d$-dimensional fields as Kaluza-Klein modes of massless $(d+1)$-dimensional fields. As a result, the flat-space massive exchange can be simply obtained from eq. (1.1) replacing $d$ with $(d+1)$ and taking into account that the residue now refers to the physical mass shell of the particle of mass $M$, and reads

$$\langle J, (\square - M^2) \varphi \rangle = \sum_{n} \frac{1}{n! 2^{2n}} \left( \frac{1}{2 - \zeta} \right)^n \langle J^{[n]}, J^{[n]} \rangle .$$  

(1.5)

The key property of this residue is that its massless limit differs markedly from the massless result of eq. (1.1). It should be appreciated that the difference between the two expressions generalizes to spin-$s$ fields the van Dam-Veltman-Zakharov (vDVZ) discontinuity originally found in [12] for the spin-two exchange in Minkowski space.

A naive attempt to extend this result to $(A)dS$ backgrounds would readily run into difficulties. This can somehow be anticipated from the very nature of the corresponding unitary representations [13, 14]: aside from massive fields, describing states of the continuous series, and massless fields, whose degrees of freedom are maximally reduced, these backgrounds allow a number of intermediate options, usually termed partially massless fields [16]. The corresponding states belong to the discrete series of unitary $SO(1, d)$ representations, whose degrees of freedom are intermediate, in number, between those of massless and massive fields. For a spin-$s$ field there are $s - 1$ special values for the mass $M$. They all lie below the lower end $(ML)^2 = (s - 1)(d + s - 4)$ of the continuous series, and feature the emergence of surprising gauge symmetries. The corresponding parameters have spins $0, 1, \ldots, s - 2$, all of which are smaller than $s - 1$, the spin of the more familiar massless gauge parameters $\Lambda_{\mu_1 \ldots \mu_{s-1}}$. For one matter, discrete states should somehow show up in the current exchanges, as they do for $s = 2$ [17, 18], and this explains the relative complexity of these quantities in $(A)dS$ backgrounds. Alternatively, one can observe, in analogy with the known $s = 2$ result, that the exchange amplitudes depend on $(ML)^2$, and exhibit two distinct limiting behaviors in the flat massless case, $(ML)^2 \to 0$, and in the flat massive case, $(ML)^2 \to \infty$. As such, in accord with Liouville’s theorem, they must have zeros and poles, and the latter, as we shall see, can be associated in general to a subset of the partially massless points.

As is well known, de Sitter space times can be represented as quadratic surfaces in higher-dimensional Minkowski space times, and this suggests to perform a “polar” decomposition, much along the lines of what is usually done for spheres in a Euclidean space. Hence, a massless field in a $(d+1)$-dimensional Minkowski space time with a suitable “radial” dependence should give rise to an effective massive $dS$ field. This interesting correspondence, discussed in [19] and recently reconsidered in [20], will be reviewed in Section 3. As a byproduct, the procedure will also establish a neat correspondence between discrete states and partially massless fields with residual gauge symmetries. The unconstrained formulation of [3, 4, 11] is particulary convenient in this context, since for generic values of the mass $M$ it allows a gauge choice eliminating all tensor fields with at least one radial index. The remaining difficulty is related to the compensator fields, that do not vanish in the presence of an external current and must be disentangled, while a key novelty with respect to the flat case is that a conserved current must have a well-defined pattern of non-vanishing radial components. This is due to the combination of two facts, the conservation equation in the $(d+1)$-dimensional Minkowski space and the condition that the $dS$ current be covariantly conserved. Notice, however, that the first property is forced upon by the construction while the second is not inevitable, but is added here in analogy with what was done for the massive fields.

Alternative treatments of massive totally symmetric $(A)dS$ fields can be found in [21].
flat-space construction in [1]. With this proviso, the full current is constructed in Section 4, while its relation to the compensators is spelled out in Section 5. The intermediate expressions will be somewhat lengthy, but the final result for the current exchanges, as we shall see in Section 7, can be presented in a rather compact closed form, where the \((3 - \zeta)_n\) Pochhammer symbols of the flat-space expression are replaced by interesting analytic functions of \((ML)^2\), that can be related to the generalized hypergeometric function \(3 F_2(a, b, c; d, e; z)\). The final expressions are rational functions of \((ML)^2\), although this property is hardly evident at first sight, with poles that identify a subset of the discrete states. These particular discrete states are selected because the gauge invariance of the corresponding Lagrangians would require that the external currents satisfy conservation conditions involving some of their own traces. In Section 6 we shall also discuss the massless limit of the \(dS\) exchanges, that strictly speaking should be taken for the \(AdS\) case, after a suitable analytic continuation, not to cross the unitarity gap present in \(dS\) space times. As we shall see, the result obtained in [17, 18], namely that for linearized gravity the \(vDVZ\) discontinuity disappears in the presence of a cosmological constant, generalizes to all higher-spin cases. Let us stress that our results concern free fields: the arguments of [22], that ascribe the disappearance of the discontinuity for \(s = 2\) to the classical approximation, or the considerations of [23], are clearly out of reach at the present time for higher spins. With this proviso, our conclusion will be that the flat limit of all spin-\(s\) \(AdS\) current exchanges is smooth, with no \(vDVZ\) discontinuity, and coincides with the massless flat-space result of [1]. We end in Section 8 with a brief discussion of our results, while the three Appendices fill some technical gaps of our derivations.

2 Massive current exchanges in Minkowski space times

In this Section we review how the massless current exchanges of [1] also encode the relevant information on massive exchanges in Minkowski space times. This discussion is also meant to review briefly the unconstrained formulation of [3, 4, 1] and to stress its advantages for obtaining explicit expressions for the current exchanges.

2.1 Massive spin-\(s\) fields from massless \((d + 1)\)-dimensional fields

Our starting point is the Fronsdal equation,

\[
\mathcal{F}_{\mu_1...\mu_s} \equiv \square \varphi_{\mu_1...\mu_s} - (\partial_{\mu_1} \partial \cdot \varphi_{\mu_2...\mu_s} + \ldots) + (\partial_{\mu_1} \partial_{\mu_2} \varphi'_{\mu_3...\mu_s}) = 0,
\]

where the omitted terms in the “Fronsdal operator” \(\mathcal{F}\) complete the symmetrization in the \(s\) indices \(\mu_1, \ldots, \mu_s\) and the “prime” denotes a trace. The key feature of this expression is that it is not fully gauge invariant, but rather transforms as

\[
\delta \mathcal{F}_{\mu_1...\mu_s} = \frac{1}{2} \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Lambda'_{\mu_4...\mu_s} + \ldots ,
\]

where the remaining, omitted terms, are \(s! - 1\), under the natural gauge transformation

\[
\delta \varphi_{\mu_1...\mu_s} = \partial_{\mu_1} \Lambda_{\mu_2...\mu_s} + \ldots .
\]

In the compact, index-free notation of [3, 4, 1], these expressions read simply

\[
\mathcal{F} = \square \varphi - \partial \partial \cdot \varphi + \partial^2 \varphi', \quad \delta \varphi = \partial \Lambda,
\]

\[5\]
and

\[ \delta F = 3 \partial^3 \Lambda'. \quad (2.5) \]

For \( s \geq 3 \), in [3, 4] we thus introduced a compensator \( \alpha_{\mu_1...\mu_s-3} \), such that

\[ \delta \alpha = \Lambda', \quad (2.6) \]

and in [1] we combined \( \mathcal{F} \) and \( \alpha \) into the gauge-invariant tensor

\[ \mathcal{A} = \mathcal{F} - 3 \partial^3 \alpha. \quad (2.7) \]

In this index-free notation, the general gauge invariant flat-space Lagrangian takes the relatively simple form

\[ \mathcal{L} = \frac{1}{2} \varphi \left( \mathcal{A} - \frac{1}{2} \eta \mathcal{A}' \right) - \frac{3}{4} \begin{pmatrix} s \\ 3 \end{pmatrix} \alpha \partial \cdot \mathcal{A}' + 3 \left( \begin{pmatrix} s \\ 4 \end{pmatrix} \beta \left[ \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' \right] \right) \]

\[ - J \varphi, \quad (2.8) \]

where the coefficients are expressed in terms of the binomial coefficients

\[ \binom{n}{k} = \frac{n!}{k! (n-k)!}. \quad (2.9) \]

We have also added, for future convenience, an external current \( J \), which is to be conserved in order that \( \mathcal{L} \) be gauge invariant up to total derivatives. Notice that, aside from the gauge field \( \varphi \) and the compensator \( \alpha \), this Lagrangian also involves a spin-(\( s - 4 \)) Lagrange multiplier \( \beta \), that first presents itself for \( s = 4 \) and whose gauge transformation is

\[ \delta \beta = \partial \cdot \partial \cdot \partial \cdot \Lambda'. \quad (2.10) \]

The resulting field equation for \( \varphi \) is

\[ \mathcal{A} - \frac{1}{2} \eta \mathcal{A}' + \eta^2 \mathcal{B} = J, \quad (2.11) \]

where

\[ \mathcal{B} \equiv \beta - \frac{1}{2} (\partial \cdot \partial \cdot \varphi' - 2 \Box \partial \cdot \alpha - \partial \partial \cdot \partial \cdot \alpha), \quad (2.12) \]

is the gauge-invariant completion of the Lagrange multiplier \( \beta \).

In the absence of an external current, eq. (2.11) can be turned into

\[ \mathcal{A} = 0, \quad (2.13) \]

and eventually into the Fronsdal equation (2.1), after a partial gauge fixing using the trace \( \Lambda' \) of the gauge parameter. On the other hand, the field equation for \( \beta \) replaces, in this formulation, Fronsdal’s double trace constraint, and relates the double trace of \( \varphi \) to the compensator \( \alpha \) according to

\[ \varphi'' = 4 \partial \cdot \alpha + \partial \alpha'. \quad (2.14) \]

One can also show that the double trace of \( \mathcal{A} \) vanishes identically if one makes use of eq. (2.14).

Let us now reconsider these results in a slightly different notation, to which we shall resort in this paper since it will prove particularly convenient for some of our derivations. Let us therefore
introduce a *constant* auxiliary vector, $u^\mu$, so as to replace the fully symmetric tensor $\varphi_{\mu_1\ldots\mu_s}$ with the index-free expression

$$
\varphi(x, u^\mu) = \frac{1}{s!} \varphi_{\mu_1\ldots\mu_s} u^{\mu_1} \ldots u^{\mu_s}.
$$

The Fronsdal equation (2.1) then takes the form \(^3\)

$$
\mathcal{F}(x, u^\mu) \equiv \left[ \Box - (u \cdot \partial) (\partial \cdot \partial u) + \frac{1}{2} (u \cdot \partial)^2 (\partial u \cdot \partial u) \right] \varphi(x, u^\mu) = 0,
$$

while the corresponding equation (2.13) of the unconstrained formalism reads

$$
\left[ \Box - (u \cdot \partial) (\partial \cdot \partial u) + \frac{1}{2} (u \cdot \partial)^2 (\partial u \cdot \partial u) \right] \varphi(x, u^\mu) - \frac{1}{2} (u \cdot \partial)^3 \alpha(x, u^\mu) = 0.
$$

One more specification is needed, since our aim here is connecting massive $d$-dimensional higher-spin fields to massless fields in a $(d+1)$-dimensional Minkowski space time. The starting point is thus a spin-$s$ field in a $(d+1)$-dimensional Minkowski space time, with a corresponding $(d+1)$-dimensional auxiliary vector $U^A$, or equivalently with two types of auxiliary variables, a $d$-dimensional vector $u^\mu$ and a scalar $v$, the internal component of $U^A$ along the extra dimension. The dependence on the additional Kaluza-Klein coordinate $y$ is simply chosen to be $e^{iMy}$, with $M$ the resulting mass in $d$ dimensions, so that one is led to the expansions

$$
\varphi(X, U^A) \equiv e^{iMy} \varphi(x, U^A) = e^{iMy} \sum_{r=0}^{s} \frac{1}{r!} \varphi_{s-r}(x, u^\mu) v^r,
$$

$$
\Lambda(X, U^A) \equiv e^{iMy} \Lambda(x, U^A) = e^{iMy} \sum_{r=0}^{s-1} \frac{1}{r!} \Lambda_{s-1-r}(x, u^\mu) v^r,
$$

for the field $\varphi$ and the gauge parameter $\Lambda$, where $\varphi_r$ and $\Lambda_r$ denote $d$-dimensional spin-$r$ quantities. In the following we shall factor out the $y$ dependence whenever possible, for brevity. In this notation, the gauge transformation in $(d+1)$ dimensions reads

$$
\delta \varphi(x, U^A) = u \cdot \partial \Lambda(x, U^A) + iv M \Lambda(x, U^A),
$$

that in terms of the $d$-dimensional component fields becomes

$$
\delta \varphi_r(x) = \partial \Lambda_{r-1}(x) + i M (s-r) \Lambda_r(x), \quad (r = 0, \ldots s).
$$

In a similar fashion, the spin-$(s-3)$ compensator $\alpha_{A_1\ldots A_{s-3}}$ present in $(d+1)$ dimensions transforms as

$$
\delta \alpha(x, U^A) = (\partial_u \cdot \partial_u + \partial^2_v) \Lambda(x, U^A),
$$

that in terms of $d$-dimensional components becomes

$$
\delta \alpha_r(x) = \Lambda_{r+2}(x) + \Lambda_r(x), \quad (r = 0, \ldots s-3).
$$

The two fields $\varphi$ and $\alpha$ determine, as we have seen, the gauge invariant modification $\mathcal{A}$ of the Fronsdal operator, that in this notation reads

$$
\mathcal{A}(x, U^A) = \mathcal{F}(x, U^A) - \frac{1}{2} (U \cdot \partial)^3 \alpha(x, U^A).
$$

\(^3\)Here $u \cdot \partial$ computes the gradient, $\partial \cdot \partial_u$ the divergence and $\partial_u \cdot \partial_u$ the trace.
As we have seen, in this unconstrained formalism the double trace of $\varphi$ does not vanish identically, but can be fully expressed in terms of the compensator. Moreover, as we have stressed the double trace $A''$ of $A$ vanishes identically after using eq. (2.14).

Notice that the gauge transformation of eq. (2.20) involves shifts that clearly allow the elimination of the Stueckelberg modes $\varphi_r$ with $r = 0, \ldots, s-1$. In the resulting gauge $\partial_v \varphi(x, U^A) = 0$, so that the free equations reduce to

$$\Box - M^2) \varphi - (u \cdot \partial + i M v) \partial \cdot \partial_u \varphi + \frac{1}{2} (u \cdot \partial + i M v)^2 \partial_u \cdot \partial_u \varphi - \frac{1}{2} (u \cdot \partial + i M v)^3 \alpha = 0.$$  (2.24)

This determines, in particular, the compensator field, whose components $\alpha_0, \ldots, \alpha_{s-3}$ all vanish, as can be seen considering, recursively, the coefficients of $v^s$ for $s \geq 3$. The coefficient of $v^2$ then gives the trace condition

$$\partial_u \cdot \partial_u \varphi = 0,$$  (2.25)

while the coefficient of $v$ gives the divergence condition

$$\partial \cdot \partial_u \varphi = 0.$$  (2.26)

Finally the terms independent of $v$ give the mass shell condition

$$(\Box - M^2) \varphi = 0.$$  (2.27)

In other words, one thus recovers the Fierz-Pauli conditions, summarized in eqs. (2.25)-(2.27), that identify a traceless and divergence-free tensor $\varphi_{\mu_1 \ldots \mu_s}$ describing an irreducible set of massive spin-$s$ modes in a $d$-dimensional Minkowski space time.

### 2.2 Coupling to an external current and the vDVZ discontinuity

We can now turn to a brief review of the coupling between $(d + 1)$-dimensional massless fields and conserved external currents. In the unconstrained formulation of [3, 4, 1], the relevant equations in $(d + 1)$ dimensions are

$$A - \frac{1}{2} \eta A' + \eta^2 B = J.$$  (2.28)

The condition $A'' = 0$ still holds, after making use of eq. (2.14), so that taking successive traces eq. (2.28) can be turned into

$$\langle J, A \rangle = \sum_{n=0}^{N} \rho_n (d-1, s) \frac{1}{n!} \frac{1}{2^n} \langle J^{[n]}, J^{[n]} \rangle ,$$  (2.29)

where $\langle \varphi, \varphi \rangle$ is defined in eq. (1.3).

In [1] the coefficients $\rho_n (d-1, s)$ were related to a difference equation, whose solution can be cast in the form

$$\rho_n (d-1, s) = \frac{1}{2^n (\frac{s}{2} - \zeta)^n},$$  (2.30)

with $(a)_n$ the $n$-th Pochhammer symbol of $a$, defined in Appendix A, and $\zeta$ is defined in eq. (1.2).

Let us now choose again for the dependence on the extra dimension of fields and currents the exponential $e^{i M y}$, where $M$ will be the resulting mass for the spin-$s$ field. Current conservation in $(d + 1)$ dimensions then reads

$$\partial_v J = \frac{i}{M} \partial \cdot \partial_u J ,$$  (2.31)
so that, integrating this equation, the $v$ dependence is fully encoded in
\[ J(x, U^A) = e^{\frac{iv}{2M}} \partial_u J_s(x, u^\mu). \] (2.32)

Let us also add the condition that the $d$-dimensional current be conserved, so that
\[ \partial \cdot \partial_u J_s(x, u^\mu) = 0. \] (2.33)

Strictly speaking, this further condition would appear not fully motivated for massive fields, but by explicitly manipulating the field equations for the first few low-spin examples one can convince oneself that, even starting from elementary non-conserved currents, the field equations can be conveniently recast in a form such that the exchanges involve effectively conserved currents. For instance, for spin $s = 1$ the Maxwell-Proca equation reads
\[ \Box A_\mu - \partial_\mu \partial \cdot A - M^2 A_\mu = J_\mu, \] (2.34)
whose divergence implies that
\[ \partial \cdot A = - \frac{1}{M^2} \partial \cdot J, \] (2.35)
so that eq. (2.34) can be turned into
\[ (\Box - M^2) A_\mu = \tilde{J}_\mu, \] (2.36)
where the effective current is
\[ \tilde{J}_\mu = J_\mu - \frac{1}{M^2} \partial_\mu \partial \cdot J. \] (2.37)

Notice that, on shell, the effective current $\tilde{J}_\mu$, defined as the residue at the physical pole, is actually conserved, as advertised, while working from the beginning with a conserved current has the effect of hiding the singularity of this expression at $M = 0$, and thus allows for a massless limit that is clearly smooth.

Once one makes the choice of working with conserved currents, in flat space the complete current does not depend on $v$ or, equivalently, all of its components with indices along the additional dimension vanish, so that $J$ reduces to $J_s$. The contraction of the spin-$s$ field equation (2.28) with a conserved current then gives
\[ \langle J, (\Box - M^2) \varphi \rangle = \sum_{n=0}^{N} \rho_n(d - 1, s) \frac{1}{n!} \frac{1}{2^n} \langle J_s^{[n]} \cdot J_s^{[n]} \rangle. \] (2.38)

As we anticipated, the massless limit of this expression is indeed regular, but does not coincide with the massless current exchange, that has a similar form but for the key replacement of $\rho_n(d - 1, s)$ with $\rho_n(d - 2, s)$. It is important to stress that the existence of a non singular massless limit is guaranteed by the conservation of the $d$-dimensional current: without this crucial condition, the limit would be singular, as can be foreseen from eq. (2.32), or from eq. (2.36).

We can end this section by showing how the form of the current exchange amplitude of eq. (2.38) makes unitarity manifest. In general, the issue at stake is whether the residue of the propagator pole is positive for the generic conserved currents allowed in the exchange. The key observation to this effect is that, due to current conservation, only the spatial components of the current enter the contractions, giving rise to terms of the type
\[ \tilde{J}_{a_1...a_s} \tilde{J}^{a_1...a_s}, \] (2.39)
where the \( a_i \) indices are transverse to the momentum, that is time like for \( M^2 > 0 \), and \( \tilde{J} \) is a traceless and conserved current, determined by the projection of eq. (2.38). Let us stress that this argument actually proves the positivity of the current exchange only for \( M^2 > 0 \), since only in this case is the on-shell momentum time like, while for \( M^2 < 0 \) it would be space like. One thus recovers a well-known fact: both the positivity of current exchanges in Minkowski space and the unitarity of the coupling require a positive squared mass for all spins \( s \geq 1 \), although the restriction on the sign of \( M^2 \) is not necessary for \( s = 0 \).

3 Massive \((A)dS\) fields via a radial dimensional reduction

Let us now consider a curved space-time, with \( e^a(x) = e^a_\mu(x) dx^\mu \) a moving basis, where \( e^a_\mu(x) \) is the vielbein, and let \( e^a_\mu(x) \partial_\mu \) be the dual vector fields, where \( e^a_\mu(x) \) is the inverse vielbein, such that

\[
e^a_\mu(x) e^\mu_b(x) = \delta^a_b. \tag{3.1}
\]

Under an infinitesimal Lorentz transformation

\[
\delta e^a = e^a_b(x) e^b(x), \tag{3.2}
\]

the spin connection \( \omega^{ab}(x) \) transforms as

\[
\delta \omega^{ab}(x) = \epsilon^{ac}(x) \omega^{bc}(x) + \epsilon^{bc}(x) \omega^{ac}(x) - d\epsilon^{ab}(x), \tag{3.3}
\]

so that the corresponding curvature two-form is

\[
R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c^b, \tag{3.4}
\]

with

\[
R^{ab} = \frac{1}{2} R_{\mu\nu}^{\ ab} dx^\mu \wedge dx^\nu. \tag{3.5}
\]

In the moving basis, a rank-\( s \) fully symmetric tensor field takes the form

\[
\varphi(x) = \frac{1}{s!} \varphi_{a_1...a_s}(x) e^{a_1}(x) \otimes \cdots \otimes e^{a_s}(x). \tag{3.6}
\]

As in the previous section, let us also introduce a fixed and coordinate-independent auxiliary vector \( u^a \), or, better, a collection of auxiliary constants, in order to associate to the tensor a function \( \varphi(x, u^a) \) that is homogeneous of degree \( s \) in \( u \):

\[
\varphi(x, u^a) = \frac{1}{s!} \varphi_{a_1...a_s}(x) u^{a_1} \cdots u^{a_s}. \tag{3.7}
\]

Under the infinitesimal local Lorentz transformation (3.2) the tensor components also rotate, and the same is true for the composite function \( \varphi(x, u^a) \), since \( u^a \) is a fixed vector. Nonetheless, the effect of the transformation can be conveniently mimicked by

\[
\delta \varphi(x, u^a) = \frac{1}{2} \epsilon^{ab} S_{ab} \varphi(x, u^a), \tag{3.8}
\]
with
\[ S_{ab} = u_a \frac{\partial}{\partial u^b} - u_b \frac{\partial}{\partial u^a} \equiv u_a \partial_{u^b} - u_b \partial_{u^a}. \] (3.9)

This is the starting point for a convenient algorithm to derive all the subsequent results. For instance, it leads directly to define the covariant derivative
\[ D_\mu \varphi(x, u^a) = \left( \partial_\mu + \frac{1}{2} \omega^a_{\mu b} S_{ab} \right) \varphi(x, u^a). \] (3.10)

In our applications to \((A)dS\) space times the commutator of a pair of covariant derivatives \((3.10)\) will thus recover the curvature tensor
\[ [D_\mu, D_\nu] \chi(x, u^a) = \frac{1}{2} R_{\mu\nu ab} S_{ab} \chi(x, u^a), \] (3.11)
since in Einstein’s gravity the torsion tensor vanishes identically.

Let us now define
\[ e^a_\mu D_\mu = D_a, \quad \text{and} \quad D = u^a D_a. \] (3.12)

Just like the function \(\varphi(x, u^a)\) is associated to the tensor \(\varphi_{\mu_1...\mu_s}\), the function \(D\varphi\) is then associated to the tensor whose components in the moving basis are the symmetrized gradient of \(\varphi_{\mu_1...\mu_s}\),
\[ D_{a_1} \varphi_{a_2...a_{s+1}} + D_{a_2} \varphi_{a_1...a_{s+1}} + \ldots. \] (3.13)

Notice that, with respect to the natural scalar product
\[ \int d^d x \det e \langle \varphi, \varphi \rangle, \] (3.14)
where
\[ \langle \varphi, \varphi \rangle = \frac{1}{s!} \varphi_{a_1...a_s} \varphi^{a_1...a_s}, \] (3.15)
the adjoint of \(D\) is
\[ D^\dagger = -\partial_{u^a} D_a. \] (3.16)

We can conclude this section by displaying the useful commutator
\[ [D, D^\dagger] = D^a D_a - \frac{1}{4} R_{ab}^{cd} S^{ab} S_{cd}. \] (3.17)

As we shall see, this notation will prove particularly useful in the following sections.

### 3.1 Foliating a flat \((d + 1)\)-dimensional space time by \(dS\) sections

Let us now consider a \((d + 1)\)-dimensional Minkowski space time with coordinates \(X^\hat{\mu}\), and let us foliate it by “de Sitter” sections with constant \(R^2 = X^\hat{\mu} X^\hat{\nu} \eta_{\hat{\mu}\hat{\nu}}\) [20]. If the \(dS\) coordinates are denoted by \(x^\mu\), the flat metric can thus be presented in the “polar” decomposition
\[ ds^2 = dR^2 + R^2 g_{\mu\nu}(x) dx^\mu dx^\nu, \] (3.18)
where \(g_{\mu\nu}(x)\) is the metric for a \(dS\) space of unit radius. A more convenient parametrization obtains letting
\[ z = L \log \left( \frac{R}{T} \right), \] (3.19)
with $L$ a length scale to be identified with the $dS$ radius, since eq. (3.18) then becomes

$$ds^2 = e^{\frac{2z}{L}}(dz^2 + ds^2_{dS}),$$

(3.20)

where

$$ds^2_{dS} = L^2 g_{\mu\nu}(x) dx^\mu dx^\nu$$

(3.21)

is the metric for a $dS$ space of radius $L$. Notice that this foliation actually covers half of the original Minkowski space-time, the region with $X^\mu X_\mu > 0$, where the radial coordinate, or the corresponding $z$, are real. Notice also that the $z = -\infty$ hypersurface corresponds to the light cone.

A moving frame for the $(d+1)$-dimensional flat space is then given by

$$\tilde{e}^A = (\tilde{e}^z, \tilde{e}^a) = e^z L (e^z, e^a),$$

(3.22)

where $e^z$ is simply $dz$ and $e^a$ is a $dS$ moving frame. As a result, the complete spin connection for the $(d+1)$-dimensional flat space decomposes as

$$\tilde{\omega}^{ab} = \omega^{ab}, \quad \tilde{\omega}^{az} = \frac{1}{L} e^{-\frac{z}{L}} e^a,$$

(3.23)

where $\omega^{ab}$ is the corresponding $dS$ spin connection.

A spin-$s$ field in the $(d+1)$-dimensional flat space-time can thus be described starting from

$$\varphi(X) = \frac{1}{s!} \varphi_{A_1...A_s}(X) \tilde{e}^{A_1}(X) \otimes \cdots \otimes \tilde{e}^{A_s}(X),$$

(3.24)

and introducing the fixed and constant auxiliary vector $U^A = (u^a, v)$ one can again turn the attention to a function that is homogeneous of degree $s$ in $U$,

$$\varphi(X, U^A) \equiv \varphi(x, z, u^a, v) = \frac{1}{s!} \varphi_{A_1...A_s}(X) U^{A_1} \cdots U^{A_s}.$$  

(3.25)

Expanding in powers of $v$ then gives

$$\varphi(X, U^A) = \sum_{r=0}^s \frac{v^r}{r!} \varphi_{s-r}(x, z, u^a).$$

(3.26)

The symmetrized gradient of $\varphi$ is a spin-$(s+1)$ field $\tilde{D}\varphi$, that can also be turned into a function of the auxiliary variable letting

$$(\tilde{D}\varphi)(X, U^A) = \tilde{D}\varphi(x, z, u^a, v),$$

(3.27)

which in its turn defines $\tilde{D}$ as

$$\tilde{D} = e^{-\frac{z}{L}} \left[ D + v \partial_z + \frac{1}{L} \left( u^2 \partial_v - v u \cdot \partial_u \right) \right],$$

(3.28)

with

$$D = u^a e^\mu_a \left( \partial_\mu + \frac{1}{2} \omega^{ab}_\mu S_{ab} \right).$$

(3.29)

The corresponding adjoint with respect to the scalar product $[3.14]$ in the $(d+1)$-dimensional flat space time is then

$$\tilde{D}^\dagger = e^{-\frac{z}{L}} \left[ D^\dagger - \partial_u \partial_z - \frac{1}{L} \left( d \partial_v + \partial_v u \cdot \partial_u - v \partial_u \cdot \partial_u \right) \right],$$

(3.30)
where the relation between $D$ and $D^\dagger$ was displayed in eq. \[3.16\].

Let us now consider a massless spin-$s$ field in the $(d + 1)$-dimensional flat space with the metric \[3.20\]. In an arbitrary coordinate system, its equation of motion in the unconstrained formalism of \[3.11\] reads

$$\hat{\mathcal{F}}(X, U^A) - \frac{1}{2} \hat{D}^3 \alpha(X, U^A) = 0 ,$$  \tag{3.31}

where the flat Fronsdal operator in $(d + 1)$ dimensions is

$$\hat{\mathcal{F}}(X, U^A) = \left\{ [\hat{D}, \hat{D}^\dagger] + \hat{D} \hat{D}^\dagger + \frac{1}{2} (\hat{D})^2 \partial_U \partial_U \right\} \varphi(X, U^A).$$ \tag{3.32}

The resulting equations are invariant under the gauge transformations

$$\delta \varphi(X, U^A) = \hat{D} \Lambda(X, U^A), \quad \delta \alpha(X, U^A) = \partial_U \Lambda(X, U^A),$$ \tag{3.33}

that for the different components of the higher-spin field $\varphi$ imply the relations

$$\delta \varphi_r(X) = e^{-\frac{v}{L}} \left[ DA_{r-1}(x) + (s - r) \left( \partial_z - \frac{r}{L} \right) \Lambda_r(x) + 2 \frac{1}{L} g(x) \Lambda_{r-2}(x) \right].$$ \tag{3.34}

When expressed in terms of $dS_d$ quantities, the flat-space Fronsdal operator \[3.32\] becomes, after a lengthy but straightforward calculation,

$$\hat{\mathcal{F}}(X, U^A) = e^{-\frac{2v}{L}} \left\{ \mathcal{F} + \left[ \partial_z^2 + \frac{d - 1}{L} \partial_z - \frac{s}{L^2} - \frac{u^2}{2} \left( \frac{s - 4}{L^2} - \frac{1}{L} \partial_z \right) \partial_U \cdot \partial_U \right] \varphi \\
- \frac{v}{L} \left( s - 2 - L \partial_z \right) \left( D^\dagger + D \partial_U \partial_U \right) \varphi \\
+ \frac{v^2}{2L^2} \left[ (s - 2)(s - 3) + L^2 \partial_U^2 + (5 - 2s)L \partial_z \right] \partial_U \cdot \partial_U \varphi \\
+ \frac{v}{L} \partial_U \left[ (-2s - d - 1)L \partial_z + (d + 1)u \cdot \partial_U + (u \cdot \partial_U)^2 - u^2 \left( \frac{1}{2} + u \cdot \partial_U - L \partial_z \right) \partial_U \cdot \partial_U \right] \varphi \\
+ \frac{1}{2L^2} \partial_U^2 \left[ 2u^2 \left( 1 - d - \frac{1}{2} \right) L \partial_z - \frac{3}{2} u \cdot \partial_U + \frac{1}{2} u^2 \partial_U \partial_U \right] + L^2 D^2 \right] \varphi \\
+ \frac{1}{2L} \partial_U \left[ 2L \partial_U \partial_D - D(2 + u \cdot \partial_U) \right] \varphi \\
+ \frac{v}{2L} \partial_U^2 \left[ 2L \partial_U - 2u \cdot \partial_U \right] \varphi \right\},$$ \tag{3.35}

where we have left implicit, for brevity, the argument $(X, U^A)$ of the field $\varphi$, and the Fronsdal operator for the given $dS$ background is

$$\mathcal{F} = \left( \Box_{dS} + DD^\dagger + \frac{1}{2} D^2 \partial_U \cdot \partial_U \right) \varphi .$$ \tag{3.36}

Here $\Box_{dS}$ denotes the $dS$ d’Alembertian, and we used eq. \[3.17\], that for the $dS$ background becomes

$$[D, D^\dagger] = D^a D_a + \frac{1}{L^2} \left[ s(s + d - 2) - u^2 \partial_U \cdot \partial_U \right].$$ \tag{3.37}
Taking into account the homogeneity of the composite field, one can simplify to some extent these expressions making the two replacements
\[ v \partial_v \longrightarrow s - u \cdot \partial_u, \]
\[ v^2 \partial_v^2 \longrightarrow (s - u \cdot \partial_u)(s - 1 - u \cdot \partial_u). \] (3.38)

4 Free massive $dS$ fields

One can now fix the gauge in such a way that $\varphi(X, U^A)$ has no $v$-dependent radial components,
\[ \partial_v \varphi(X, U^A) = 0, \] (4.1)
thus eliminating all internal Stueckelberg modes, as in Section 2. The free wave equations then determine completely the compensator, setting $\alpha = 0$, and also imply the two conditions
\[ \partial_u \cdot \partial_u \varphi(X, u^a) = 0, \quad D^\dagger \varphi(X, u^a) = 0. \] (4.2)
Comparing the resulting expression with the expected form of the massive spin-$s$ equation (see, e.g., eq. (3.67) of [1], but with $L^2 \rightarrow -L^2$, since here we are discussing the $dS$ case),
\[ \Box_L \varphi + \frac{2}{L^2} (s - 1)(d + s - 3) \varphi - M^2 \varphi = 0, \] (4.3)
where $\Box_L$ is the Lichnerowicz operator, defined for $AdS$ in eq. (1.4), one can read the resulting mass-shell condition,
\[ \left[ -M^2 + \frac{1}{L^2} (3 - d - s)(2 - s) \right] \varphi(X, u^a) = \left( \partial^2_z + \frac{d - 1}{L} \partial_z \right) \varphi(X, u^a). \] (4.4)
This simple differential equation determines the $z$-dependence of $\varphi(X, u^a)$,
\[ \varphi(X, u) \sim e^{\mu z}, \] (4.5)
with
\[ (ML)^2 = (s - 2 - \mu)(d + s - 3 + \mu). \] (4.6)

The solutions of this mass-shell condition are then
\[ \mu_{\pm} = -\frac{d - 1}{2} \pm \frac{i}{2} \sqrt{(2ML)^2 - (2s + d - 5)^2}. \] (4.7)
Notice that when $2ML < (2s + d - 5)$ they are both real, and
\[ \mu_{\pm} = -\frac{d - 1}{2} \pm \frac{1}{2} \sqrt{(2s + d - 5)^2 - (2ML)^2}. \] (4.8)
Taking into account the measure $\sqrt{-g} g^{zz} = e^{(d-1)z}$, however, one can see that only the first solution, $\mu_{+}$, is well behaved on the light cone, $z = -\infty$ or $R = 0$, of the original $(d+1)$-dimensional flat space, and is thus acceptable, while the other, $\mu_{-}$, should be rejected. On the other hand, when $2ML \geq (2s + d - 5)$ the two solutions are related to one another by complex conjugation, and therefore are both acceptable. This pattern reflects the more familiar one found in the more conventional toroidal reduction. Even in that case a single periodic wave function, the constant, determines the massless mode, while pairs of periodic wave functions, characterized by opposite momenta, determine the massive ones.
4.1 Discrete states and “partially massless” $dS$ fields

As we anticipated, for generic values of $\mu$, fixing the Stueckelberg gauge symmetries one can set $\varphi_r=0$ for $r=0,1,\ldots,s-1$, and this fixes the gauge completely. Strictly speaking, however, this procedure is not quite possible when $\mu+1=k$, with $k=s-1,s-2,\ldots,0$. In these cases a residual gauge invariance emerges\cite{16}, that is associated with spin-$k$ gauge parameters $\Lambda_k$, together with a consequent reshuffling of the propagating degrees of freedom. This can be seen extracting the $z$ dependence while taking into account the overall factor in eq. (3.34),

$$\Lambda_r \sim e^{(\mu+1)\frac{z}{L}} \forall r,$$

(4.9)

so that the gauge transformations can be simplified and become

$$\delta \varphi_r = e^{-\frac{z}{L}} D\Lambda_{r-1} + (s-r) \left( \frac{\mu + 1 - r}{L} \right) \Lambda_r + \frac{1}{2L} g \Lambda_{r-2}, \quad (r=0,\ldots,s).$$

(4.10)

Notice that, if $\mu = k - 1$,\hspace{1cm} (4.11)

the middle coefficient in eq. (4.10) vanishes for $r=k$, so that the corresponding parameter, $\Lambda_k$, has no effect on $\delta \varphi_k$. This allows to determine, recursively, corresponding values for the other parameters $\Lambda_l$, for $l > k$, capable of keeping all the $\varphi_r$ for $r \neq k$ fixed at their vanishing values, for $r=0,\ldots,s-1$. The resulting residual gauge transformation of $\varphi_s$ finally reads

$$\delta \varphi_s = e^{-\frac{z}{L}} D\Lambda_{s-1} + \frac{1}{L} u^2 \Lambda_{s-2},$$

(4.12)

where $\Lambda_{s-1}$ and $\Lambda_{s-2}$ are determined from $\Lambda_k$ according to

$$\Lambda_{k+1} = \frac{L}{s-k-1} D\Lambda_k,$$

(4.13)

and

$$\Lambda_t = \frac{L}{(t-k)(s-t)} D\Lambda_{t-1} + \frac{1}{L} u^2 \Lambda_{t-2}, \quad (t=k+2,k+3,\ldots,s-1).$$

(4.14)

The corresponding masses are

$$(ML)^2 = (s-1-r)(d+s+r-4), \quad (r=s-1,\ldots,0),$$

(4.15)

and in particular $r=s-1$ corresponds to the massless case, while the full sequence of values of $(ML)^2$ for the discrete states is $0,d+2s-6,2(d+2s-7),3(d+2s-8),\ldots,(s-1)(d-4+s)$.

Let us stress that, from the $(d+1)$-dimensional vantage point, what happens is a mere redistribution of degrees of freedom: for the special masses above what is usually a Stueckelberg field becomes a propagating field, which takes up precisely the degrees of freedom lost by the “partially massless” field $\varphi$. Nothing special happens, in fact, in the $(d+1)$-dimensional flat space, for these special values of the mass. This is to be contrasted with the conventional way of looking at partial masslessness directly in $d$ dimensions, where one starts with a massive field and discovers that a residual gauge symmetry emerges for special values of $(ML)^2$.

In four dimensions, the discrete values for the mass $M$ that we have thus identified correspond to the unitary representations $\pi_{p,q}$, with $1 \leq q \leq p$, determined by Dixmier in \cite{13}, with the choices
$p = s$ and $q = k + 1$. In order to compare with the results of that paper, let us note that the second-order Casimir operator $I_2$ of the de Sitter group is related to the masses here defined by

$$I_2 + 2(s^2 - 1) - (ML)^2 = 0.$$  

(4.16)

The additional representations called $\pi_{p,0}$ by Dixmier [13] are special scalar representations, minimally massless for $p = 0$ and tachyonic for the other values. The other bosonic unitary representations in [13] are characterized by an integer $p$, the spin $s$ and a continuous positive label $\sigma$, and are called $\nu_{p,\sigma}$: their masses are given by

$$(ML)^2 = (s^2) - s + \sigma.$$  

(4.17)

Notice that these values are bounded from below by the masses of the discrete states, the last of which is also determined in four dimensions by this expression for $\sigma = 0$. This last type of representation corresponds to the unitary massive fields. The representations of the higher-dimensional de Sitter groups are more complicated, since they also involve mixed-symmetry fields, and were studied in [14]. The construction of $dS$ quantum field theories starting from unitary irreducible representations of $SO(1, D)$ was recently considered in [15].

5 Coupling to a $d$-dimensional conserved current

We can now turn to the central topic of this paper, the coupling of massive higher-spin fields to external currents in a de Sitter background.

5.1 Conserved $dS$ currents from flat $(d + 1)$-dimensional currents

The $(d + 1)$-dimensional external current $J$ couples to a massless field of the unconstrained formulation of [3, 4, 11], and therefore should be conserved, a condition that in the present notation reads

$$\hat{D}^\dagger \hat{J}(X, U^A) = 0,$$  

and in $d$-dimensional terms becomes

$$\left[ D^\dagger - \partial_\nu \partial_z - \frac{1}{L} (d\partial_\nu + \partial_\nu u \cdot \partial_u - v \partial_u \cdot \partial_u) \right] J(X, U^A) = 0.$$  

(5.2)

Notice that, with the chosen exponential $z$-dependence, that we are factoring out of all expressions,

$$\partial_z \rightarrow \frac{1}{L} (\mu - 2)$$  

(5.3)

when acting on $J$, where the “shift” is due to the pre-factor in eq. (3.35). The $v$-dependence of the current can be made explicit integrating eq. (5.2), and is fully encoded in the expression

$$J(x, U^A) = e^{(\mu + d + u \cdot \partial_u - 2)} \left( L^2 D^\dagger + \frac{v^2}{2} \partial_\nu \partial_u \right) J_s(x, u^a).$$  

(5.4)

If one adds the further condition that the $d$-dimensional current be also conserved, so that

$$D^\dagger J_s(x, u^a) = 0,$$  

(5.5)
eq. (5.4) reduces to
\[
J(x, U^A) = \left[1 + \sum_{n=1}^{[\frac{s}{2}]} \frac{(v^2 \partial_u \cdot \partial_u)^n}{2^n n! (\mu + d + s - 4)(\mu + d + s - 6) \ldots (\mu + d + s - 2n - 2)} \right] J_s ,
\]
and letting
\[
\nu = \frac{1}{2} (\mu + d + s) ,
\]
it can be recast in the more convenient form
\[
J(x, U^A) = \sum_{n=0}^{[\frac{s}{2}]} \frac{(-1)^n}{2^{2n} n! (2 - \nu)_n} (v^2 \partial_u \cdot \partial_u)^n J_s (x, u^a) .
\]
The \(\nu\)-dependence of this current reflects the fact that its radial components do not vanish identically, and is an important novelty with respect to the flat Kaluza-Klein reduction reviewed in Section 2.

The \(m\)-th trace of the current \(J\) in \((d+1)\)-dimensions will also be useful later. It can be derived from the previous expression by manipulations similar to those illustrated in Appendix B, or by an inductive argument, and is given by
\[
(\partial_U \cdot \partial_U)^m J(x, U^A) = \left(\frac{3}{2} - \nu\right) \sum_{m=0}^{[\frac{s}{2}] - m} \frac{(-1)^n}{2^{2n} n! (2 - \nu)_n} (v^2 \partial_u \cdot \partial_u)^n (\partial_u \cdot \partial_u)^m J_s (x, u^a) .
\]

### 5.2 The field equations

In terms of the gauge-invariant extension of the Fronsdal operator,
\[
\mathcal{A}(x, U^A) = \tilde{F}(x, U^A) - \frac{1}{2} \tilde{D}^3 \alpha(x, U^A) ,
\]
proceeding as in [2] the field equation with an external current in the \((d+1)\)-dimensional Minkowski space can be turned into
\[
\mathcal{A}(x, U^A) = \sum_{n=0}^{[\frac{s}{2}]} \frac{\rho_n (d - 1, s)}{2^n n!} (U^2)^n (\partial_U \cdot \partial_U)^n J \equiv f(x, U^A) ,
\]
where
\[
(\partial_U \cdot \partial_U)^2 \mathcal{A}(x, U^A) = 0 ,
\]

since \(\mathcal{A}\) is doubly-traceless in \((d+1)\)-dimensions, after using eq. (2.14). Here \(J\) is the \((d+1)\)-dimensional conserved current, and as we have seen
\[
\rho_n (d - 1, s) = \frac{1}{2^n \left(\frac{3}{2} - \zeta\right)_n} ,
\]
where \(\zeta\) is defined in eq. (1.12).

Substituting for \(J\) its expression in terms of the de Sitter current \(J_s (x, u^a)\) then gives
\[
f(x, U^A) = \sum_{n=0}^{[\frac{s}{2}]} \frac{(u^2 + v^2)^n}{2^{2n} n! \left(\frac{3}{2} - \nu\right)_n} \sum_{p=0}^{[\frac{s}{2}] - n} \frac{(-1)^p}{2^{2p} p! (2 - \nu)_{p+n}} (v^2)^p (\partial_u \cdot \partial_u)^{n+p} J_s .
\]
After rearranging this expression as in Appendix B, one can recast eq. (5.14) in the rather compact form

\[ f(x, U^A) = \sum_{r=0}^{[\frac{s}{2}]} \sum_{m=0}^{[\frac{d}{2}] - r} b(r, m) \frac{(u^2)^r}{(2r)!} (u^2)^m (\partial_u \cdot \partial_u)^{r+m} J_s(x, u^a), \]

(5.15)

where the coefficients \( b(r, m) \) are

\[ b(r, m) = (-1)^r \frac{(2r)!}{2^{r+m} r! m!} \frac{(\frac{s}{2} - \nu)_m (\nu - \zeta + 1)_r}{(2 - \nu)_{r+m} (\frac{s}{2} - \zeta)_{r+m}}. \]

(5.16)

5.3 Determining the compensator \( \alpha \)

As in the flat case, let us work in the gauge \( \partial_v \varphi = 0 \), with no radial components for the spin-\( s \) gauge field. Our goal is to determine the current-exchange amplitude, and to this end we should first determine the compensator \( \alpha \), the trace and the divergence of \( \varphi \), up to \( d \)-dimensional gradients, in terms of the external de Sitter current. The \( v \)-independent part of the field equation then determines the effective kinetic operator relevant for the current exchanges,

\[ \mathcal{K}(x, u^a) \equiv \left( \Box_L + \frac{2}{L^2} (s - 1)(d + s - 3) - M^2 \right) \varphi(x, u^a). \]

(5.17)

In this subsection we begin by expressing

\[ \alpha(x, U^A) = \sum_{i=0}^{s-3} \alpha_i(x, u^a) \frac{v^i}{i!} \]

(5.18)

in terms of \( J_s(x, u^a) \), and we shall see shortly that \( \alpha \) contains only odd powers of \( v \). In fact, we shall first express the components \( \alpha_i \) in terms of those of

\[ f(x, U^A) = \sum_{r=0}^{[\frac{s}{2}]} f_{2r}(x, u^a) \frac{v^{2r}}{(2r)!}, \]

(5.19)

that we already related to \( J_s(x, u^a) \) in eq. (5.15), so that

\[ f_{2r}(x, u^a) = \sum_{m=0}^{[\frac{d}{2}] - r} b(r, m) (u^2)^m (\partial_u \cdot \partial_u)^{r+m} J_s(x, u^a), \]

(5.20)

with \( b(r, m) \) given in eq. (5.16).

As can be seen from eqs. (3.35) and (5.11), \( F \) does not involve any terms of order \( v^3 \) or higher, and therefore the field equation (5.11) implies the conditions

\[ \partial_v^3 \left( \frac{1}{2} \tilde{D}^3 \alpha + f(x, U^A) \right) = 0. \]

(5.21)

We shall see shortly that this determines completely \( \tilde{D}^3 \alpha \), and consequently, as we anticipated, \( \alpha \) will only contain odd powers of \( v \). This is due to the fact that \( f(x, U^A) \) is even in \( v \), which will
then be the case also for \( \hat{D}^3 \alpha \) that, up to \( d \) dimensional gradients, is given by

\[
\hat{D}^3 \alpha(X, U^A) = e^{\frac{(\mu - 2)\alpha}{L}} \left\{ \frac{v^3}{L^3} (\mu - 1 - u \cdot \partial_u)(\mu - u \cdot \partial_u)(\mu + 1 - u \cdot \partial_u) \\
+ \frac{3v}{L} u^2 [(s - 3 - u \cdot \partial_u)(\mu - 1 - u \cdot \partial_u)^2 + (\mu + 1 - u \cdot \partial_u)(\mu - 1 - u \cdot \partial_u)] \\
+ \frac{3\partial_v}{L^3} (u^2)^2 [(s - 3 - u \cdot \partial_u)(\mu - 2 - u \cdot \partial_u) + 2] + \frac{\partial^3}{L^3} (u^2)^3 \right\} \alpha(x, U^A), \tag{5.22}
\]

where different arguments, \( x \) and \( X \), are present on the two sides of this equation since we have made the dependence on \( z \) fully explicit. In deriving this and the following expressions we are using repeatedly the further condition

\[
\left( v \partial_v + u \cdot \partial_u \right) \alpha(x, U^A) = (s - 3) \alpha(x, U^A), \tag{5.23}
\]

which reflects the degree of homogeneity, and thus the spin, of the compensator \( \alpha_{A_1...A_{n-3}} \). As a result, letting

\[
g = -2L^3 \partial_v^3 f(x, U^A), \tag{5.24}
\]

the equation for \( \alpha(x, U^A) \) can finally be written in the form

\[
\left[ A_0 + A_2 \partial_v^2 + A_4 \partial_v^4 + A_6 \partial_v^6 \right] \alpha(x, U^A) = g, \tag{5.25}
\]

with

\[
A_0 = (s - u \cdot \partial_u)(s - 1 - u \cdot \partial_u)(s - 2 - u \cdot \partial_u)(\mu - 1 - u \cdot \partial_u)(\mu + 1 - u \cdot \partial_u), \\
A_2 = 3u^2(s - 2 - u \cdot \partial_u)^2 [(s - 3 - u \cdot \partial_u)(\mu - 1 - u \cdot \partial_u)^2 + (\mu + 1 - u \cdot \partial_u)(\mu - 1 - u \cdot \partial_u)], \\
A_4 = 3(u^2)^2 [(s - 3 - u \cdot \partial_u)(\mu - 2 - u \cdot \partial_u) + 2], \\
A_6 = (u^2)^3. \tag{5.26}
\]

In order to determine \( \alpha \) explicitly, let us take into account eq. (5.23) so as to replace \( u \cdot \partial_u \) with \( v \partial_v \), and let us define

\[
a = \mu - s \equiv 2(\nu - \zeta). \tag{5.27}
\]

Eq. (5.22) can then be fully expressed in terms of the four quantities

\[
E_0(t) = (1 + t)(2 + t)(3 + t)(a + 2 + t)(a + 3 + t)(a + 4 + t), \\
E_2(t) = 3(1 + t)[t(a + 2 + t)^2 + (a + 4 + t)(a + 2 + t)], \\
E_4(t) = 3t(a + 1 + t) + 6, \\
E_6(t) = 1, \tag{5.28}
\]

and let us also define

\[
\tilde{E}_2(t) = -\frac{E_2(t)}{E_0(t - 2)}, \quad \tilde{E}_4(t) = -\frac{E_4(t)}{E_0(t - 4)}, \quad \tilde{E}_6(t) = -\frac{1}{E_0(t - 6)}, \tag{5.29}
\]

and

\[
\gamma = \frac{1}{E_0(v \partial_v)} g. \tag{5.30}
\]
Here \( g \) is given in eq. (5.24), and \( t \) will be shortly identified with \( v \partial_u \). The equation for \( \alpha \) thus reads
\[
\left[ 1 - \sum_{i=1}^{3} (u^2)^i \partial_v^i \tilde{E}_{2i}(v \partial_v) \right] \alpha(x, U^A) = \gamma, \tag{5.31}
\]
where we have made use of the relation
\[
v \partial_v (\partial_v)^i = (\partial_v)^i (v \partial_v - i). \tag{5.32}
\]

One can now invert eq. (5.31) and write the solution in the form
\[
\alpha(x, U^A) = \sum_{n \geq 0} \left[ \sum_{i=1}^{3} (u^2)^i \partial_v^i \tilde{E}_{2i}(v \partial_v) \right]^n \gamma
= \sum_{n \geq 0} \sum_{i_1, \ldots, i_n = 1} (u^2 \partial_v)^{i_1+i_2+\cdots+i_n} \tilde{E}_{2i_1}(v \partial_v - 2i_2 - \cdots - 2i_n) \ldots \tilde{E}_{2i_n}(v \partial_v) \gamma. \tag{5.33}
\]

This result embodies explicit expressions for all the \( \alpha_i \),
\[
\alpha_i = \gamma_i + \sum_{n \geq 1} (u^2)^n \sum_{p \geq 1} \sum_{i_1+i_2+\cdots+i_p = n} \tilde{E}_{2i_1}(i + 2i_1) \tilde{E}_{2i_2}(i + 2i_1 + 2i_2) \ldots \tilde{E}_{2i_p}(i + 2n) \gamma_{i+2n} \tag{5.34}
= \gamma_i + \sum_{n \geq 1} (u^2)^n \sum_{p \geq 1} \sum_{i_1+i_2+\cdots+i_p = n} (-1)^p \tilde{E}_{2i_1}(i + 2i_1) \tilde{E}_{2i_2}(i + 2i_1 + 2i_2) \ldots \tilde{E}_{2i_p}(i + 2n) \frac{E_{0}(i) E_{0}(i + 2i_1) \ldots E_{0}(i + 2i_1 + \cdots + 2i_{n-1})}{E_{0}(i) E_{0}(i + 2i_1) \ldots E_{0}(i + 2i_1 + \cdots + 2i_{n-1})} \gamma_{i+2n},
\]
where, from eq. (5.30),
\[
\gamma_i = -2L^3 \frac{f_{i+3}}{E_{0}(i)}. \tag{5.35}
\]

Having determined \( \alpha(x, U^a) \) from the previous equation, one can now obtain the zeroth-order and second-order contributions to \( \hat{D}^3 \alpha \), that enter the \( \varphi \) equations at the same order in \( v \), making use of eq. (5.22), and the end result is
\[
L^3 \hat{D}^3 \alpha = 3(u^2)^2(a + 4) \alpha_1 + (u^2)^2 \alpha_3
+ [12u^2(a + 3)(a + 3) \alpha_1 + 3(u^2)^2(3a + 14) \alpha_3 + (u^2)^2 \alpha_5] \frac{v^2}{2} + \mathcal{O}(v^4). \tag{5.36}
\]

6 The \( d \)-dimensional equation

We are now ready to analyze the three remaining equations, the \((d + 1)\)-dimensional equations at order \( v^k \) with \( k \leq 2 \). The final aim will be to relate \( \mathcal{K}(x, u^a) \) of eq. (5.17) to the external current \( J_s(x, u^a) \), up to \( d \)-dimensional gradients. We shall proceed in steps, obtaining first the expression in terms of \( f(x, U^A) \), in eq. (6.6), and then, in the next section, the explicit dependence on \( J_s(x, u^a) \).

The order-\( v^2 \) terms in the equation of motion determine the trace of \( \varphi \), so that, making use of eq. (3.35),
\[
\frac{(a + 2)(a + 3)}{L^2} \partial_u \cdot \partial_u \varphi(x, u^a) = f_2 + \frac{u^2}{2L^3} \left[ 12(a + 3)(a + 3) \alpha_1 + 3(u^2)(3a + 14) \alpha_3 + (u^2)^2 \alpha_5 \right], \tag{6.1}
\]
while the terms of order $v$ in the field equation lead to the condition

$$\left(D^\dagger + D \partial_u \cdot \partial_u\right) \varphi(x, u^a) = 0,$$

that determines the divergence of $\varphi$.

Finally, the order-$v^0$ part of the field equation, after using the previous results, is the effective $d$-dimensional equation we are after, that up to pure gradients, which do not contribute to the current exchange, reads

$$\mathcal{K}(x, u^a) + \left[ \frac{u^2}{2L^2} (a + 2) \partial_u \cdot \partial_u \right] \varphi(x, u^a) = \begin{cases} f_0 + \frac{3}{2L^3} (u^2)^2(a + 4)\alpha_1 + \frac{(u^2)^3}{2L^3} \alpha_3, & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases},$$

where $\mathcal{K}$ is defined in eq. (6.1). Replacing the trace $\partial_u \cdot \partial_u \varphi$ by its explicit form obtained from eq. (6.1) gives

$$\mathcal{K}(x, u^a) = f_0 - u^2 \frac{f_2}{2(a + 3)} - \frac{3}{2L^3} (u^2)^2(a + 4)\alpha_1 - \frac{(u^2)^3(7a + 36)\alpha_3 + (u^2)^4\alpha_5}{4(a + 3)L^3}.$$

Notice that the $\alpha_i$ vanish for $s < 3 + i$, so that all the three contribute only for $s \geq 8$. Replacing the $\alpha_i$ with their expressions (5.34) in terms of the $f_i$ yields

$$\mathcal{K}(x, u^a) = \begin{cases} f_0 - u^2 \frac{f_2}{2(a + 3)} + 3(u^2)^2(a + 4) \left[ \frac{f_4}{E_0(1)} + \mathcal{K}_0 \right] \\ + (u^2)^3 \frac{(7a + 36)}{2(a + 3)} \left[ \frac{f_6}{E_0(3)} + \mathcal{K}_1 \right] + \frac{(u^2)^4}{2(a + 3)} \left[ \frac{f_8}{E_0(5)} + \mathcal{K}_2 \right], & \text{if } n = 1, \end{cases}$$

with

$$\mathcal{K}_j = \sum_{n \geq 1, p \geq 1, i_1 + \ldots + i_p = n} \frac{(u^2)^n}{E_0(1 + 2j)E_0(1 + 2j + 2i_1)E_0(1 + 2j + 2i_1 + 2i_2)\ldots E_0(1 + 2j + 2i_1 + 2i_2) \ldots E_0(1 + 2j + 2n) f_4 + 2j + 2n}.$$

This expression looks indeed rather clumsy. Remarkably, however, it can be largely simplified, since the sums over all the $i_i$ can be performed explicitly. The final result, proved in Appendix C, is simply

$$\mathcal{K}(x, u^a) = \sum_{r=0}^{[s]} (u^2)^r \lambda_r f_{2r},$$

where the coefficients are

$$\lambda_r = (-1)^r \frac{1}{2^{2r}r! \left( \frac{a + 3}{2} \right)_r} = (-1)^r \frac{1}{2^{2r}r! (\nu - \zeta + \frac{3}{2})_r}.$$

This is the main result of this Section.

Before considering the general case, let us now examine the massless and flat limits of the propagator. First, the massless limit is attained for $\mu \rightarrow s - 2$, which implies that $\zeta - \nu \rightarrow 1$. From eq. (5.10), we see that in this limit all the $b(r, m)$ with $r \neq 0$ vanish, while

$$\lim_{\mu \rightarrow s - 2} b(0, m) = \frac{1}{2^{2m}m!(3 - \zeta)m}.$$
so that
\[
\lim_{\mu \to s-2} \mathcal{K}(x, u^a) = \sum_{m=0}^{\left[ \frac{s}{2} \right]} \frac{1}{2^mm! (3-\zeta)_m} (u^2)^m (\partial_u \cdot \partial_u)^m J_s(x, u^a),
\] (6.9)
which is precisely the massless propagator of [1]. In other words, the vDVZ discontinuity is absent for all \( s \), exactly as was found to be the case in [18] for \( s = 2 \). Notice that, strictly speaking, this limit should be taken after a proper continuation to AdS not to leave the region of unitarity.

On the other hand, the flat limit is obtained for \( L \to \infty \) with finite \( M \), which implies that \( \nu \to i\infty \). From eq. (5.16) one can deduce the limiting behavior of the coefficients \( b(r, m) \),
\[
\lim_{\nu \to i\infty} b(r, m) = \frac{(2r)!}{2^{2(r+m)} r! m! (\frac{5}{2} - \zeta)_{r+m}},
\] (6.10)
while eq. (6.7) shows that all the \( \lambda_r \) coefficients tend to zero for \( r \neq 0 \). One can then readily deduce the limiting behavior of the propagator
\[
\lim_{\nu \to i\infty} \mathcal{K}(x, u^a) \to \sum_{m=0}^{\left[ \frac{s}{2} \right]} \frac{1}{2^mm! \left( \frac{5}{2} - \zeta \right)_m} (u^2)^m (\partial_u \cdot \partial_u)^m J_s(x, u^a),
\] (6.11)
which is precisely the massless \((d+1)\)-dimensional propagator in Minkowski space that one can also derive letting \( M \to 0 \) in eq. (1.5).

7 The \((A)dS\) current exchanges

We can finally express the coefficients \( f_{2r} \) in eq. (6.6) in terms of the \( d \)-dimensional current \( J_s(x, u^a) \) and obtain the \( dS \) current-exchange amplitudes for all spin-\( s \) bosonic fields.

7.1 Some useful identities

The expressions that were obtained so far, and in particular eq. (6.6), depend on \( \nu \), that in its turn determines the mass via eq. (4.7) and we rewrite for convenience in the form
\[
\nu = \frac{2\zeta + 1}{4} \pm \frac{i}{2} \sqrt{(ML)^2 - \left( \zeta - \frac{5}{2} \right)^2},
\] (7.1)
where \( \zeta \) is defined in eq. (1.2). As a result, both the higher dimensional current \( J \) and the compensator field \( \alpha \) are not analytic in \((ML)^2\). On the other hand, as we shall see, the propagator \( \mathcal{K} \) is always a rational function of \((ML)^2\), as was shown to be the case for \( s = 2 \) in [17] [18]. Proving this result, however, requires some intermediate steps, to which we now turn.

Let us begin by displaying a few expressions that will prove useful in exhibiting the mass dependence of the propagator. To this end, let us recall that the mass shell condition reads
\[
(ML)^2 = 2(2\nu - 3)(\zeta - \nu - 1) = -4\nu^2 + 2\nu(2\zeta + 1) - 6(\zeta - 1),
\] (7.2)
and implies that
\[
4 \left( \frac{3}{2} - \nu + c \right) (\nu - \zeta + 1 + c) = (ML)^2 + 2c(2c + 5 - 2\zeta),
\] (7.3)
for any $c$. One can therefore deduce the useful relations

$$
\left( \frac{3}{2} - \nu \right)_n (\nu - \zeta + 1)_n = \frac{1}{2^{2n}} \prod_{j=0}^{n-1} (ML)^2 + 2j(2j + 5 - 2\zeta) \equiv h_n(\nu, \zeta),
$$

(7.4)

$$
(2 - \nu)_n \left( \nu - \zeta + \frac{3}{2} \right)_n = \frac{1}{2^{2n}} \prod_{j=0}^{n-1} (ML)^2 + 2(2j + 1)(j + 3 - \zeta) \equiv g_n(\nu, \zeta),
$$

(7.5)

where $h_n$ and $g_n$ are manifestly polynomials in $(ML)^2$ and $\zeta$. This result can also be understood from a different vantage point. To this end, let us define the new variables

$$
\gamma = \frac{1}{2} \left( \frac{5}{2} - \zeta \right),
$$

$$
\delta = \pm \frac{i}{2} \sqrt{(ML)^2 - \left( \frac{5}{2} - \zeta \right)^2},
$$

(7.6)

so that the arguments of the Pochhammer symbols in eqs. (7.4) and (7.5) can be conveniently expressed as

$$
\frac{3}{2} - \nu = \gamma - \delta, \quad \nu - \zeta + 1 = \gamma + \delta,
$$

$$
2 - \nu = \gamma - \delta + \frac{1}{2}, \quad \nu - \zeta + \frac{3}{2} = \gamma + \delta + \frac{1}{2}.
$$

(7.7)

This makes it possible to rewrite $h_n$ and $g_n$ in the form

$$
h_n = (\gamma - \delta)_n (\gamma + \delta)_n,
$$

$$
g_n = \left[ (\gamma + \frac{1}{2}) - \delta \right]_n \left[ (\gamma + \frac{1}{2}) + \delta \right]_n.
$$

(7.8)

Eqs. (7.4) and (7.5) then follow as a consequence of a generalisation of the standard formula for the difference of two squares,

$$
(\gamma - \delta)_n (\gamma + \delta)_n = \prod_{i=0}^{n-1} \{ (\gamma + i)^2 - \delta^2 \},
$$

(7.9)

where the explicit dependence only on even powers of $\delta$ is again manifest. Let us stress that the roots of $h_n$ are the masses of the odd discrete states, while the roots of $g_n$ are the masses of the even discrete states.

### 7.2 Expansion of the propagator: the first few cases

From eqs. (6.6), (6.7) and (5.20) one can see that the propagator $\mathcal{K}$ takes form

$$
\mathcal{K}(x, u^a) = \sum_{n=0}^{[\frac{1}{2}]} k_n(u^2)^n (\partial_{u^a}\partial_{u^b})^n J_n(x, u^a),
$$

(7.10)

with

$$
k_n = \sum_{r=0}^{n} (-1)^r b(r, n - r) \frac{1}{2^{2r} r! (\nu - \zeta + \frac{3}{2})_r}.
$$

(7.11)
Using the expression for the coefficients $b(r, m)$ in (5.16), one can obtain

$$
k_n = \frac{1}{2^{2n}(2 - \nu)_n (\frac{3}{2} - \zeta)_n} \sum_{r=0}^{n} \frac{(2r)!}{2^{2r}(r!)^2(n-r)!} \frac{(\frac{3}{2} - \nu)_{n-r} (\nu - \zeta + 1)_r}{(\nu - \zeta + \frac{3}{2})_r}.
$$

(7.12)

We shall examine shortly the general expression for the coefficients $k_n$. As we shall see, they are all rational fractions of $(ML)^2$, although this is clearly not evident at first sight, but let us begin by examining the first four terms. The first two are

$$
k_0 = 1, \quad k_1 = \frac{4h_1 + 5 - 2\zeta}{8g_1 (5 - 2\zeta)},
$$

(7.13)

where $h_1$ and $g_1$ were defined respectively in (7.4) and (7.5). As we have seen in the previous subsection, $h_1$ is linear in $(ML)^2$, so that

$$
k_1 = -\frac{1}{2(2\zeta - 5)} \frac{(ML)^2 - 2\zeta + 5}{(ML)^2 - 2\zeta + 6},
$$

(7.14)

is indeed a rational function of $(ML)^2$. In a similar fashion,

$$
k_2 = \frac{1}{2^4 (2 - \nu)_2 (\frac{3}{2} - \zeta)_2} \left[ \frac{1}{2} \left( \frac{3}{2} - \nu \right) + \frac{1}{2} \left( \frac{3}{2} - \nu \right) \frac{(\nu - \zeta + 1)}{(\nu - \zeta + \frac{3}{2})} + \frac{3 (\nu - \zeta + 1)_2}{8 (\nu - \zeta + \frac{3}{2})_2} \right],
$$

(7.15)

but after some rearrangements, using also eqs. (7.4) and (7.5), one can turn this expression into

$$
k_2 = \frac{(ML)^4 - 4(ML)^2(2\zeta - 7) + 3(2\zeta - 5)(2\zeta - 7)}{8(2\zeta - 5)(2\zeta - 7)((ML)^2 - 2\zeta + 6)((ML)^2 - 2\zeta + 6)},
$$

(7.16)

that is again a rational function of $(ML)^2$.

The next case is more involved, and therefore let us examine it in detail, since

$$
k_3 = b(0, 3) - \frac{b(1, 2)}{2(2\nu - 2\zeta + 3)} + \frac{b(2, 1)}{8(2\nu - 2\zeta + 3)(2\nu - 2\zeta + 5)} - \frac{b(3, 0)}{48 (2\nu - 2\zeta + 3)(2\nu - 2\zeta + 5)(2\nu - 2\zeta + 7)}.
$$

(7.17)

We would like to show that this expression is also a function of $(ML)^2$ and $\zeta$. After replacing the $b(r, m)$ by their expressions, $k_3$ can be written in the form

$$
k_3 = \frac{N_3}{D_3},
$$

(7.18)

with

$$
D_3 = 2^9 3! \left( \frac{5}{2} - \zeta \right)_3 (2 - \nu)_3 \left( \nu - \zeta + \frac{3}{2} \right)_3 = 2^9 3! \left( \frac{5}{2} - \zeta \right)_3 g_3,
$$

(7.19)

and

$$
N_3 = 8 \left( \frac{3}{2} - \nu \right)_3 \left( \nu - \zeta + \frac{3}{2} \right)_3 + 12 \left( \frac{3}{2} - \nu \right)_2 \left( \nu - \zeta + \frac{5}{2} \right)_2 (\nu - \zeta + 1) + 18 \left( \frac{3}{2} - \nu \right) (\nu - \zeta + \frac{7}{2}) (\nu - \zeta + 1)_2 + 15 (\nu - \zeta + 1)_3.
$$

(7.20)
One can now use the binomial identity (A.6) for Pochhammer symbols to obtain

\[
\left(\nu - \zeta + \frac{3}{2}\right)_3 = \sum_{k=0}^{3} \frac{6}{k!(3-k)!} \left(\frac{1}{2}\right)_k (\nu - \zeta + 1)_{3-k},
\]

\[
\left(\nu - \zeta + \frac{5}{2}\right)_2 = \sum_{k=0}^{2} \frac{2}{k!(2-k)!} \left(\frac{1}{2}\right)_k (\nu - \zeta + 2)_{2-k},
\]

(7.21)

so that the numerator becomes

\[
N_3 = 8h_3 + 12\left(\frac{7}{2} - \nu\right) h_2 + 18\left(\frac{5}{2} - \nu\right) h_1 + 15\left(\frac{3}{2} - \nu\right) + 12 (\nu - \zeta + 3) h_2
\]

\[+ 12h_2 + 9\left(\frac{5}{2} - \nu\right) h_1 + 18(\nu - \zeta + 2) h_1 + 9(\nu - \zeta + 2) h_1 + 15(\nu - \zeta + 1) h_3.
\]

The next step is to use eq. (A.6) to obtain

\[
\left(\frac{5}{2} - \nu\right)_2 + (\nu - \zeta + 2)_2 = \left(\frac{9}{2} - \zeta\right)_2 - 2\frac{h_2}{h_1}
\]

\[
\left(\frac{3}{2} - \nu\right)_3 + (\nu - \zeta + 1)_3 = \left(\frac{5}{2} - \zeta\right)_3 - 3\left(\frac{9}{2} - \zeta\right) h_1,
\]

(7.23)

(7.24)

so that the numerator finally becomes

\[
N_3 = 8h_3 + 12\left(\frac{9}{2} - \zeta\right) h_2 + 18\left(\frac{7}{2} - \zeta\right) h_1 + 15\left(\frac{5}{2} - \zeta\right)_3,
\]

(7.25)

that indeed depends only on \((ML)^2\) and \(\zeta\).

This explicit analysis of the first three \(k_n\) makes it plausible that the propagator be analytic in \((ML)^2\) for all spins, a fact that we can now prove in full generality.

### 7.3 General form of the current exchanges

The general expression for the \(k_n\), obtained in (7.12), can be cast in the form

\[
k_n = \frac{N_n}{D_n},
\]

(7.26)

where the denominator,

\[
D_n = 2^{3n} n! \left(\frac{5}{2} - \zeta\right)_n g_n(\nu, \zeta),
\]

(7.27)

with \(g_n\) defined in eq. (7.5), is manifestly a polynomial in \((ML)^2\) and the constant \(\zeta\) was defined in eq. (1.2). Notice that the poles are manifestly at the even discrete states. We would like to show that the numerator \(N_n\), given by

\[
N_n = 2^n \sum_{r=0}^{n} \binom{n}{r} \left(\frac{1}{2}\right)_r \left(\frac{3}{2} - \nu\right)_{n-r} (\nu - \zeta + 1)_r (\nu - \zeta + \frac{3}{2} + r)_{n-r},
\]

(7.28)
is also a polynomial in \((ML)^2\). The sum in eq. (7.28) can be expressed in terms of the generalized hypergeometric function \(3F_2(a, b; c; d, e; z)\) (see Appendix A), and indeed after some rearrangements one finds
\[
N_n = 2^n \left(\frac{3}{2} - \nu\right) n \left(\nu - \zeta + \frac{3}{2}\right) n ~ 3F_2 \left(-n, \frac{1}{2}, \nu - \zeta + 1; \nu - n - \frac{1}{2}, \nu - \zeta + \frac{3}{2}; 1\right). \tag{7.29}
\]

Although not manifestly, this form actually defines a polynomial in \((ML)^2\). This can be proved in several ways. To begin with, a more symmetric expression obtains via the binomial identity (A.6),
\[
\left(\nu - \zeta + \frac{3}{2} + r\right) \frac{1}{n - r} \sum_{k=0}^{n-r} \binom{n-r}{k} \left(\nu - \zeta + 1 + r\right)_{n-r-k} \left(\frac{1}{2}\right)_k,
\]
which leads to
\[
N_n = 2^n n! \sum_{r+k \leq n} \frac{1}{r! k! (n-r-k)!} \left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_k \left(\frac{3}{2} - \nu\right)_{n-r} (\nu - \zeta + 1)_{n-k}. \tag{7.31}
\]

Now the \(\nu\)-dependent sum is manifestly invariant under the transformation
\[
\left(\nu, \zeta\right) \rightarrow \left(-\nu + \zeta + \frac{1}{2}, \zeta\right), \tag{7.32}
\]
that interchanges the two factors \(\left(\frac{3}{2} - \nu\right)\) and \((\nu - \zeta + 1)\). Hence, it is an even polynomial in \(\left(\nu - \frac{3}{2} - \frac{1}{4}\right)\), and thus a polynomial in
\[
\left(\nu - \zeta - \frac{1}{2}\right)^2 = \frac{1}{4} \left[\left(\zeta - \frac{5}{2}\right)^2 - (ML)^2\right], \tag{7.33}
\]
where we have made use of eq. (7.1). Let us also notice that the invariance of the numerator under the above transformation and the expression in terms of hypergeometric functions make it possible to turn eq. (7.29) into yet another expression:
\[
N_n = 2^n (2 - \nu)_n (\nu - \zeta + 1)_n ~ 3F_2 \left(-n, \frac{1}{2}, \frac{3}{2} - \nu; \zeta - \nu - n, 2 - \nu; 1\right). \tag{7.34}
\]

Alternatively, one could rewrite (7.31) in terms of the variables \(\gamma\) and \(\delta\) introduced in (7.6) as
\[
N_n = 2^n n! \sum_{r+k \leq n} c_{n; r, k} (\gamma - \delta)_{n-r} (\gamma + \delta)_{n-k}, \tag{7.35}
\]
where the coefficients are those of eq. (7.31), and read
\[
c_{n; r, k} = \frac{1}{r! k! (n-r-k)!} \left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_k. \tag{7.36}
\]

Next, separating in the sum the terms with \(r = k\) from those where \(r \neq k\) gives
\[
N_n = 2^n n! \left[\sum_{j=0}^{[\frac{n}{2}]} c_{n; j, j} h_{n-j} + \sum_{i>j=0}^{n} c_{n; i, j} h_{n-i} \left[(\gamma + \delta)_{i-j} + (\gamma - \delta)_{i-j}\right]\right]. \tag{7.37}
\]
In order to show that this expression only involves even powers of $(ML)^2$, we need to analyze the sum

$$(\gamma + \delta)_{i-j} + (\gamma - \delta)_{i-j}$$

that, making use of (A.6), can be written as

$$(\gamma + \delta)_{i-j} + (\gamma - \delta)_{i-j} = \sum_{k=0}^{i-j} \binom{i-j}{k} (\gamma)_{i-j-k} [(\delta)_k + (-\delta)_k].$$

Finally, it is possible to show by induction that the quantity

$$(\delta)_k + (-\delta)_k$$

defines an even polynomial in $\delta$, whose explicit form is discussed in Appendix A. As a result $N_n$ is indeed analytic, and actually a polynomial in $(ML)^2$ of degree $n$.

In conclusion, for all spin-$$s$$ totally symmetric $dS$ tensors of mass $M$ the portions of the propagator not involving gradients, which suffice to determine the current exchanges, take the form

$$K = \sum_{n=0}^{p} \frac{(2 - \nu)_n (\nu - \zeta + 1)_n}{(\frac{3}{2} - \zeta)_n n!} \frac{3F_2(-n, \frac{1}{2}; \frac{3}{2} - \nu; \zeta - \nu - n; 2 - \nu; 1)}{\prod_{j=0}^{n-1} [(ML)^2 + 2(2j + 1)(j + 3 - \zeta)]} (u)^n (\partial_u \cdot \partial_u)^n J_s,$$

so that the current exchange amplitudes are finally

$$\sum_{n=0}^{p} \frac{(2 - \nu)_n (\nu - \zeta + 1)_n}{(\frac{3}{2} - \zeta)_n n!} \frac{3F_2(-n, \frac{1}{2}; \frac{3}{2} - \nu; \zeta - \nu - n; 2 - \nu; 1)}{\prod_{j=0}^{n-1} [(ML)^2 + 2(2j + 1)(j + 3 - \zeta)]} \langle J_s^{[n]} \rangle,$$

where the product in the denominator is simply to be read as equal to 1 for $n = 0$. This is the main result of this paper. Let us stress that, as we have shown, this expression is actually a rational function of $(ML)^2$, although this key property is not manifest in this form.

A more explicit form for the propagator, and thus for the current exchange, can be obtained computing the numerator at values of $\nu$ corresponding to the odd discrete states, that is for $\zeta - \nu = p + 1$ with an integer $p$, which defines the quantities

$$N_n^p = 2^n \left( \frac{5}{2} + p - \zeta \right)_n \left( \frac{1}{2} - p \right)_n \frac{3F_2(-n, \frac{1}{2}; -p; \zeta - p - n; 3 - \frac{1}{2}; \frac{1}{2} - p; 1)}{\prod_{p' \neq p} [(ML)^2 - 2p' (2p' - 2\zeta + 5)]}.$$

Using the identity principle for polynomials, $N_n$ can indeed be expressed as a sum of $n$ terms, via the $n$ polynomials obtained from $h_n$ removing one of the factors in eq. (7.44). More in detail, it can be recast in the form

$$N_n = \sum_p N_n^p \frac{\prod_{p' \neq p} [(ML)^2 - 2p' (2p' - 2\zeta + 5)]}{\prod_{p' \neq p} [2p' (2p - 2\zeta + 5) - 2p' (2p' - 2\zeta + 5)]},$$

since both sides take the same values at the $n$ odd discrete points. Notice that, in terms of the function $h_n$ defined in eq. (7.33), this expression becomes the partial-fraction expansion

$$N_n = 2h_n \sum_p N_n^p \frac{(-1)^p (\frac{5}{2} + 2p - \zeta)}{[(ML)^2 - 2p (2p - 2\zeta + 5)] p! (n - p - 1)! (\frac{5}{2} + p - \zeta)}.$$
which is manifestly a rational function of $\langle ML \rangle^2$.

There is actually a better way of presenting our results. The idea is to expand the numerator directly in terms of the $h_n$ polynomials of eq. (7.4). In fact, we have found out that identities satisfied by $3F_2$ allow to turn $\mathcal{N}$ into the expression

$$
\mathcal{N}_n^{(J)} = \frac{1}{2n} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right)_{n-k} \left( \frac{5}{2} - \zeta + k \right) \prod_{j=0}^{k-1} \left[ \langle ML \rangle^2 + 2j(2j + 5 - 2\zeta) \right]. \quad (7.46)
$$

This form has the advantage of being manifestly expressed in terms of $\langle ML \rangle^2$, and was originally guessed by one of us after an explicit analysis of a few cases of low $n$. In fact, eq. (7.46) can be expressed in terms of the generalized hypergeometric function $3F_2$, as

$$
\mathcal{N}_n^{(J)} = 2^n \binom{3}{2 - \nu} \binom{\nu - \zeta + 1}{n} \left( \frac{1}{2} \right)_{n-k} \left( \frac{5}{2} - \zeta + k \right) \prod_{j=0}^{k-1} \left[ \langle ML \rangle^2 + 2j(2j + 5 - 2\zeta) \right], \quad (7.47)
$$

and coincides with $\mathcal{N}$ on account of the identity

$$
3F_2(-n, b, c; d, e; 1) = \frac{(d - b)_n}{(d)_n} 3F_2(-n, b - c; e, b - d - n + 1; 1), \quad (7.48)
$$

that is proved in Appendix A.

These results lead to a rather compact expression for $K$ that is also an explicitly rational function of $\langle ML \rangle^2$,

$$
\mathcal{K} = \sum_{n=0}^{[\frac{3}{2}]} \frac{(u^2)^n J_s^n}{2^n n! (\frac{5}{2} - \zeta)^n g_n} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right)_{n-k} \left( \frac{5}{2} - \zeta + k \right)_{n-k} h_k, \quad (7.49)
$$

where $h_n$ and $g_n$ were defined in eqs. (7.4) and (7.5) and the superscripts on the currents identify their successive traces, whose first few terms read

$$
\mathcal{K}(x, u^a) = J_s + \frac{u^2}{4(\frac{5}{2} - \zeta)} (\langle ML \rangle^2 + 2(\frac{5}{2} - \zeta)) J_s' + \frac{(u^2)^2}{32 (\frac{5}{2} - \zeta)_2] [(\langle ML \rangle^2 - 2(\zeta - 3)][(\langle ML \rangle^2 - 6(\zeta - 4))] J_s^{[2]} + \frac{(u^2)^3}{384 (\frac{5}{2} - \zeta)_3 [((\langle ML \rangle^2 - 2(\zeta - 3)][(\langle ML \rangle^2 - 6(\zeta - 4)][(\langle ML \rangle^2 - 10(\zeta - 5))] J_s^{[3]} + \cdots + (u^2)^n \frac{\mathcal{N}_n}{D_n} J_s^{[n]} + \ldots , \quad (7.50)
$$

and where in general

$$
\mathcal{N}_n = (\langle ML \rangle^{2n} + \cdots + \frac{(2n)!}{n!} \left( \frac{5}{2} - \zeta \right)^n. \quad (7.51)
$$

Therefore, as we already stressed, the poles of the current-exchange amplitudes correspond manifestly to the even discrete states, while the zeros are not as evident, since eq. (7.46) involves a sum of contributions related to the odd discrete states.\(^4\)

\(^4\)In order to compare with the notations of [1], one should make the substitution $(u^2/2)^n \rightarrow g^n n!$, where $g$ denotes the background $dS$ metric.
For the sake of clarity, let us display explicitly the first few current-exchange amplitudes fully implied by eq. (7.50), in the standard notation of [1], that for convenience are here rescaled by an overall factor \(s!\), for spin \(s\), with respect to eq. (7.50):

\[
\begin{align*}
\mathbf{s} &= 1: \quad (J_a)^2 \\
\mathbf{s} &= 2: \quad (J_{ab})^2 - \frac{1}{d-1} \frac{(ML)^2 - (d-1)}{(ML)^2 - (d-2)} (J')^2 \\
\mathbf{s} &= 3: \quad (J_{abc})^2 - \frac{3}{d+1} \frac{(ML)^2 - (d+1)}{(ML)^2 - d} (J'_{abc})^2 \\
\mathbf{s} &= 4: \quad (J_{abcd})^2 - \frac{6}{d+3} \frac{(ML)^2 - (d+3)}{(ML)^2 - (d+2)} (J'_{abcd})^2 \\
&\quad + \frac{3}{(d+1)(d+3)} \frac{(ML)^4 - 4(ML)^2(d+1) + 3(d+1)(d+3)}{[(ML)^2 - (d+2)][(ML)^2 - 3d]} (J'')^2 \\
\mathbf{s} &= 5: \quad (J_{abcde})^2 - \frac{10}{d+5} \frac{(ML)^2 - (d+5)}{(ML)^2 - (d+4)} (J'_{abcde})^2 \\
&\quad + \frac{15}{(d+3)(d+5)} \frac{(ML)^4 - 4(ML)^2(d+3) + 3(d+3)(d+5)}{[(ML)^2 - (d+4)][(ML)^2 - 3(d+2)]} (J''_{abcde})^2 \\
\mathbf{s} &= 6: \quad (J_{abcdef})^2 - \frac{15}{d+7} \frac{(ML)^2 - (d+7)}{(ML)^2 - (d+6)} (J'_{abcdef})^2 \\
&\quad + \frac{45}{(d+5)(d+7)} \frac{(ML)^4 - 4(ML)^2(d+5) + 3(d+5)(d+7)}{[(ML)^2 - (d+6)][(ML)^2 - 3(d+4)]} (J'')^2 \\
&\quad - \frac{15[(ML)^6 - (ML)^4(9d+31) + 23(ML)^2(9d+23)(d+5) - 15(9d+23)(d+5)(d+7)]}{(d+3)(d+5)(d+7)(d+6)(d+9)} \frac{(ML)^2 - (d+6)}{[(ML)^2 - 3(d+4)][(ML)^2 - 5(d+2)]} (J'')^2 \\
\mathbf{s} &= 7: \quad (J_{abcdefg})^2 - \frac{21}{d+9} \frac{(ML)^2 - (d+9)}{(ML)^2 - (d+8)} (J'_{abcdefg})^2 \\
&\quad + \frac{105}{(d+7)(d+9)} \frac{(ML)^4 - 4(ML)^2(d+7) + 3(d+7)(d+9)}{[(ML)^2 - (d+8)][(ML)^2 - 3(d+6)]} (J'')^2 \\
&\quad - \frac{105[(ML)^6 - (ML)^4(9d+49) + 23(ML)^2(9d+49)(d+7) - 15(9d+49)(d+7)(d+9)]}{(d+5)(d+7)(d+9)(d+8)(d+10)} \frac{(ML)^2 - (d+8)}{[(ML)^2 - 3(d+6)][(ML)^2 - 5(d+4)]} (J'')^2
\end{align*}
\]

Before ending this Section, let us finally mention yet another compact form of the propagator, that can be obtained applying the identity (7.48) to the expression (7.29) for the numerator. The result is also manifestly a rational function of \((ML)^2\), but this time is expressed solely in terms of the polynomials \(g_k\) of eq. (7.5), and reads

\[
\mathcal{K} = \sum_{n=0}^{[\frac{d}{2}]} \frac{(u^2)^n J_s^{[n]}}{n! (\frac{d}{2} - \zeta)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{1}{2} \right)_k^2 \frac{1}{g_k}.
\]

### 8 Discussion

In this paper we have generalized the previous massive flat-space results of [1] to the case of \((A)dS\) backgrounds. At the same time, we have generalized the previous \(s = 2\) \(dS\) current exchanges of [17, 18] to the case of symmetric tensors of arbitrary rank. Our results confirm the indications
obtained for \( s = 2 \) in [18]: as soon as a cosmological constant is turned on, the current exchange amplitudes become analytic functions of \( (ML)^2 \) and the vDVZ discontinuity disappears. As we stressed in the Introduction, however, our current grasp of higher-spin gauge theories confines our analysis to the case of free fields, so that we can not connect our findings with the observations of [22], or of [23], related to the interplay of discontinuities and interactions.

Even working with currents that are also conserved in \( d \) dimensions, as we have done in this paper, the resulting expressions, which we presented in their most compact form in eq. (7.49), display an interesting feature. Their poles correspond in fact to

\[
(ML)^2 = 2(\zeta - 3 - j)(2j + 1), \quad j = 0, 1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor - 1,
\]

and these values identify the even discrete states of Section 4.1, with \( r = s - 2 - 2j \). The reason for their emergence in this context is interesting, and is related to the form of the corresponding gauge transformations. The key observation is that the coupling to the external current,

\[
-\int d^d x \sqrt{-g} \phi \cdot J_\mathfrak{s}(x, u^a),
\]

is gauge invariant, for the even discrete states, only if the \( \frac{(s-r)}{2} \)-th trace of \( J_\mathfrak{s}(x, u^a) \) vanishes. Hence, the poles of the propagator accompanying the various traces of the current are just like the poles that non-conserved currents would give rise to in massless exchanges, an example of which was displayed in eq. (2.36)!

As we have seen, the \( dS \) current exchanges greatly simplify in the massless limit, where the numerator reduces to

\[
\mathcal{N}_n = 2^n \left( \frac{1}{2} \right)_n \left( \frac{5}{2} - \zeta \right)_n,
\]

while the denominator reduces to

\[
\mathcal{D}_n = 2^{3n} n! \left( \frac{5}{2} - \zeta \right)_n \left( \frac{1}{2} \right)_n (3 - \zeta)_n,
\]

so that \( k_n \) becomes

\[
k_n = \frac{1}{n! 2^{2n} (3 - \zeta)_n}.
\]

Hence, they tend smoothly to the massless result of eq. (1.1), with no vDVZ discontinuity.

One can similarly investigate the flat limit, that can be recovered in the \( \nu \to i\infty \) limit. In this case the dominant term in the numerator is the \( r = 0 \) term of the sum, since the various contributions depend on \( \nu^{2n-r} \). The denominator is also dominated by \( \nu^{2n} \), so that the ratio \( k_n \) does not depend on \( \nu \) in the limit and is given by

\[
k_n \to \frac{1}{n! 2^{2n} \left( \frac{5}{2} - \zeta \right)_n},
\]

so that one recovers the flat massive exchanges of eq. (1.5).

In general, we arrived at rather compact expressions for the spin-\( s \) current exchange amplitudes in terms of generalized hypergeometric functions of the type \( \mathbf{3F}_2(a, b, c; d, e; z) \),

\[
\sum_{n=0}^{\left\lfloor \frac{s}{2} \right\rfloor} \frac{(2 - \nu)_n (\nu - \zeta + 1)_n}{(\frac{5}{2} - \zeta)_n} \mathbf{3F}_2 \left( -n, \frac{1}{4}, \frac{3}{2} - \nu, \zeta - \nu - n, 2 - \nu \right) \prod_{j=0}^{n-1} \left( (ML)^2 + 2(2j + 1)(j + 3 - \zeta) \right) \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle,
\]

where

\[
\frac{1}{\nu^{2n}} \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle = \mathbf{3F}_2 \left( -n, \frac{1}{4}, \frac{3}{2} - \nu, \zeta - \nu - n, 2 - \nu \right) \prod_{j=0}^{n-1} \left( (ML)^2 + 2(2j + 1)(j + 3 - \zeta) \right) \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle,
\]

\[
\frac{1}{\nu^{2n}} \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle = \mathbf{3F}_2 \left( -n, \frac{1}{4}, \frac{3}{2} - \nu, \zeta - \nu - n, 2 - \nu \right) \prod_{j=0}^{n-1} \left( (ML)^2 + 2(2j + 1)(j + 3 - \zeta) \right) \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle,
\]

\[
\frac{1}{\nu^{2n}} \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle = \mathbf{3F}_2 \left( -n, \frac{1}{4}, \frac{3}{2} - \nu, \zeta - \nu - n, 2 - \nu \right) \prod_{j=0}^{n-1} \left( (ML)^2 + 2(2j + 1)(j + 3 - \zeta) \right) \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle,
\]

\[
\frac{1}{\nu^{2n}} \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle = \mathbf{3F}_2 \left( -n, \frac{1}{4}, \frac{3}{2} - \nu, \zeta - \nu - n, 2 - \nu \right) \prod_{j=0}^{n-1} \left( (ML)^2 + 2(2j + 1)(j + 3 - \zeta) \right) \langle J_\mathfrak{s}^{[n]}, J_\mathfrak{s}^{[n]} \rangle.
\]
where the poles are manifest while the zeros, or the very fact that the numerator is an analytic function of \((ML)^2\), are not. We have also shown that the numerator is indeed a polynomial in \((ML)^2\), and have complemented eq. (8.7) with other, manifestly rational forms, whose zeros, however, are still not manifest. The most compact expression of this type for the current exchange, given in eq. (7.49), is

\[
\sum_{n=0}^{\frac{5}{2}} \frac{\langle J^{[n]}_L, J^{[n]}_R \rangle}{2^{2n} n! (\frac{5}{2} - \zeta)} n g_n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{5}{2} - \zeta + k\right)^{n-k} h_k,
\]

where \(h_k\) and \(g_n\) are given in eqs. (7.4) and (7.5). The link between this result to the other expressions for \(K\) presented in Section 7 rests on an identity for the generalized hypergeometric function \(3F2(a, b, c; d, e; 1)\), given in eq. (7.48), that is proved in Appendix A.

Unitarity is another issue of utmost importance, on which unfortunately the current indications of our analysis are less conclusive. In flat space, as we have stressed, unitarity is reflected in the positivity of \(J \cdot K\) on shell. In de Sitter space one can try to proceed along the same lines, associating unitary exchanges to conserved currents of positive norm and \(K\) operators that are positive definite. While the second of these conditions is relatively easy to verify, however, the first appears less direct in curved space. Moreover, we can now show, with reference to the examples of spin two and three, that the positivity of \(K\) alone is not sufficient to identify the known regions of unitarity. Indeed, for spin two and three

\[
K = 1 + k_1 u^2 \partial_u \cdot \partial_u,
\]

so that its eigenvectors verify the condition

\[
(1 - \lambda)j + k_1 u^2 \partial_u \cdot \partial_u j = 0.
\]

Applying \(\partial_u \cdot \partial_u\) to this equation while taking into account the transversality of the current (so that in the resulting relations \(d\) is actually replaced with \(d - 1\)) gives

\[
[1 - \lambda + 2k_1(d - 1 + 2s - 4)]\partial_u \cdot \partial_u j = 0,
\]

and one can now distinguish two possibilities. The first is that \(\partial_u \cdot \partial_u j = 0\), which from eq. (8.10) leads to the eigenvalue 1, while the second is that \(\partial_u \cdot \partial_u j \neq 0\), that leads to the eigenvalue

\[
\lambda = 1 + 2k_1(d + 2s - 5).
\]

The positivity condition for the spin-2 and spin-3 propagators then reduces to

\[
1 + 2k_1(d + 2s - 5) > 0,
\]

and replacing in this expression \(k_1\) with its explicit form of eq. (7.14) finally gives

\[
(ML)^2 > d + 2s - 6.
\]

Under this condition, the operator \(K\) is thus positive on conserved currents. In four dimensions and for \(s = 2\), the bound of eq. (8.14) agrees precisely with the unitarity bound obtained from the group theoretical treatment. However, for \(s = 3\) Group Theory gives \((ML)^2 > 2d - 2\), corresponding to the mass of the second discrete state, to be contrasted with the result of eq. (8.14), that would identify the first discrete state. In a similar fashion, the special role of the discrete tachyonic scalar masses that we have mentioned at the end of Section 4 is not at all manifest in this type of analysis. Therefore, the positivity condition for a conserved current in a de Sitter background deserves a closer look, and we hope to return to this issue in the near future.
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Pochhammer symbols and hypergeometric functions

The Pochhammer symbols \((a)_n\) are defined, for \(n \in \mathbb{N}\), as
\[
(a)_n = \begin{cases} 
1 & \text{if } n = 0 \\
(a)(a+1)\ldots(a+n-1) & \text{if } n \neq 0
\end{cases},
\] (A.1)
and are simply related to the Euler \(\Gamma\) function according to
\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.
\] (A.2)

They are a convenient device to express a number of our results, so that, for instance
\[
(2n)!2^{2n} = \left(\frac{1}{2}\right)_n.
\] (A.3)

A few properties of the Pochhammer symbols are used repeatedly in the text, and are summarized below for convenience. To begin with, one can connect rather simply the two Pochhammer symbols \((a)_n\) and \((-a)_n\), according to
\[
(-a)_n = (-1)^n (a+1-n)_n.
\] (A.4)

In addition, the useful relation
\[
(a)_{n+k} = (a)_k(a+k)_n
\] (A.5)
can be proved by simply expanding both sides, while the binomial identity
\[
\sum_{k=0}^{n} \binom{n}{k} (a)_k(b)_{n-k} = (a+b)_n
\] (A.6)
follows from the comparison of two ways of expanding \((1-x)^{-(a+b)}\), obtained from
\[
(1-x)^{a} = \sum_{n=0}^{\infty} (a)_n \frac{x^n}{n!}.
\] (A.7)

The two sums
\[
(a)_n + (-a)_n,
\]
\[
(a)_n - (-a)_n,
\] (A.8)
can be shown by induction to define even and odd polynomials in \(a\), respectively:
\[
(a)_{k+1} + (-a)_{k+1} \equiv \lambda_{2k}(a) = \rho_{2}^{(2k)} a^2 + \rho_{4}^{(2k)} a^4 + \rho_{6}^{(2k)} a^6 + \ldots,
\] (A.9)
\[
(a)_{k+1} - (-a)_{k+1} \equiv \lambda_{2k+1}(a) = \rho_{1}^{(2k+1)} a + \rho_{3}^{(2k+1)} a^3 + \rho_{5}^{(2k+1)} a^5 + \ldots.
\] (A.10)

They satisfy the system of equations
\[
\lambda_{2k}(a) = a \lambda_{2k-1}(a) + k \lambda_{2(k-1)}(a),
\] (A.11)
\[
\lambda_{2k+1}(a) = a \lambda_{2(k-1)}(a) + k \lambda_{2k-1}(a),
\] (A.12)
whose solution can be found using the identity principle for polynomials. For the first few cases

\[
\begin{align*}
\rho_1^{(2k+1)} &= 2k!, \\
\rho_2^{(2k)} &= 2k! \sum_{i=1}^{k} \frac{1}{i}, \\
\rho_3^{(2k+1)} &= 2k! \sum_{i<j=1}^{k} \frac{1}{i \cdot j}, \\
\rho_4^{(2k)} &= 2k! \sum_{i_1<i_2<i_3=1}^{k} \frac{1}{i_1 \cdot i_2 \cdot i_3},
\end{align*}
\]

(A.13)

while in general, for both \(\lambda_{2k}\) and \(\lambda_{2k+1}\)

\[
\rho_l = 2k! \sum_{i_1<\ldots<i_l=1}^{k} \frac{1}{i_1 \cdot \ldots \cdot i_l}.
\]

(A.14)

The identity [24]

\[
\sum_{n=0}^{r} \binom{r}{n} (-1)^n \frac{(a)_n}{(b)_n} = \frac{(b-a)_r}{(b)_r}
\]

(A.15)

follows from the Gauss relation

\[
2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\]

(A.16)

that in its turn can be derived, up to an analytic continuation, from the Gauss integral

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \ t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a}
\]

(A.17)

for the hypergeometric function \(2F_1(a, b; c; z)\), that is valid for \(Re(c) > Re(b) > 0\). It is interesting to recall that this key result follows inserting eq. (A.7) in the series expansion

\[
2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}.
\]

(A.18)

defined for \(|z| < 1\), which turns the sum into

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \sum_{n=0}^{\infty} \frac{t^{b+n-1}}{n!} \frac{(a)_n}{(d)_n} z^n
\]

(A.19)

for \(|z| < 1\), via the series expansion

\[
3F_2(a, b, c; d, e; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n} \frac{z^n}{n!},
\]

(A.20)

In a similar fashion, the generalized hypergeometric function \(3F_2(a, b, c; d, e; z)\) can be defined, for \(|z| < 1\), via the series expansion
and similar steps lead to the integral representation

\[
3F_2(a, b, c; d, e; z) = \frac{\Gamma(e)}{\Gamma(c)\Gamma(e - c)} \int_0^1 dt \, t^{e-1} (1 - t)^{e-c-1} 2F_1(a, b; d tz).
\] (A.21)

Here we are particularly interested in the special case \(a = -n\), where hypergeometric series collapse to finite sums, so that, in particular,

\[
2F_1(-n, b; c) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(b)_k}{(c)_k} z^k,
\] (A.22)

\[
3F_2(-n, b, c; d, e; z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(b)(c)_k}{(d)(e)_k} z^k.
\] (A.23)

Eq. (7.48) follows if the integral representation (A.21) is combined with the relation

\[
2F_1(-n, b; d; t) = \frac{(d - b)_n}{(d)_n} 2F_1(-n, b - d + 1 - n; 1 - t),
\] (A.24)

which leads to

\[
3F_2(-n, b, c; d, e; z) = \frac{(d - b)_n\Gamma(e)}{(d - b)_n\Gamma(e - c)} \int_0^1 dt \, t^{e-c-1} (1 - t)^{c-1} 2F_1(-n, b; b - d + 1 - n; t),
\]

after the integration variable is redefined interchanging \(t\) and \(1 - t\). Notice indeed that, up to the two Pochhammer symbols, this last expression can be obtained from eq. (A.21) letting

\[
c \rightarrow e - c; \quad e \rightarrow e;
\]

\[
b \rightarrow b; \quad d \rightarrow b - d + 1 - n.
\] (A.25)

so that the end result is indeed eq. (7.48), albeit in the equivalent form

\[
3F_2(-n, b, c; d, e; 1) = \frac{(d - b)_n}{(d)_n} 3F_2(-n, b, e - c; b - d - n + 1, e; 1),
\] (A.26)

with the two (symmetric) lower arguments interchanged. Eq. (A.24) is a special case of the basic continuation formula

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c-a-b} 2F_1(c - a, c - b; c - a - b + 1; 1 - z)
\]

\[
+ \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} 2F_1(a, b; a + b + 1 - c; 1 - z),
\] (A.27)

that follows from Barnes’ lemma, the contour integral representation

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a + s)\Gamma(b + s)}{\Gamma(c + s)} \Gamma(-s) (-z)^s ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} 2F_1(a, b; c; z),
\] (A.28)

and is discussed, for instance, in [24].
Finally, the pair of summations over \( u^2 + v^2 \)^n using Newton’s binomial identity, obtaining

\[
f = \sum_{m=0}^{[\frac{1}{2}]} \sum_{p=0}^{[\frac{1}{2}] - m} \sum_{n=0}^{[\frac{1}{2}] - p - m} \frac{(u^2)^{n+p}(u^2)^m(-1)^p}{2^{2(n+p+m)} p! \ M! n!} \frac{(\frac{3}{2} - \nu)_{n+m}}{(\frac{3}{2} - \zeta)_{n+m} (2 - \nu)_{p+n+m}} (\partial_{\nu} \cdot \partial_u)^{p+n+m} J_s . \tag{B.1}
\]

In order to arrive at this expression from eq. (5.14), one actually needs to reorder the summations according to

\[
\sum_{n=0}^{[\frac{1}{2}] - n} \sum_{p=0}^{[\frac{1}{2}] - p} \sum_{m=0}^{n} = \sum_{p=0}^{[\frac{1}{2}] - p} \sum_{m=0}^{n} \sum_{n=0}^{[\frac{1}{2}] - m} \sum_{n=0}^{[\frac{1}{2}] - p} = \sum_{m=0}^{[\frac{1}{2}] - m} \sum_{p=0}^{[\frac{1}{2}] - p} \sum_{n=0}^{[\frac{1}{2}] - n'} \sum_{n'=0}^{M} , \tag{B.2}
\]

where \( n' = n - k \). One can then insert in the sum the factor

\[
\sum_{r=0}^{[\frac{1}{2}]} \delta_{r,n'+p} , \tag{B.3}
\]

which is unity on account of the fact that, in the complete summation, \( 0 \leq n' + p \leq [\frac{1}{2}] \), so that one of the Kronecker \( \delta \)’s is always effective. The order of the resulting \( p \) and \( n' \) summations can now be inverted, according to

\[
\sum_{m=0}^{[\frac{1}{2}] - m} \sum_{p=0}^{[\frac{1}{2}] - p} \sum_{n'=0}^{n'} \sum_{r=0}^{M} = \sum_{m=0}^{[\frac{1}{2}] - m} \sum_{p=0}^{[\frac{1}{2}] - p} \sum_{n'=0}^{n'} \sum_{r=0}^{M} , \tag{B.4}
\]

in order to begin from the summation over \( p \). This amounts to the replacement, induced by the \( \delta \)’s, of \( p \) with \( r - n' \), which also lowers the upper bound on \( n' \) to \( r \), so that

\[
f = \sum_{m=0}^{[\frac{1}{2}] - m} \sum_{p=0}^{[\frac{1}{2}] - p - m} \sum_{n=0}^{[\frac{1}{2}] - p - m} \frac{(u^2)^{n+p}(u^2)^m(-1)^p}{2^{2(n+p+m)} p! \ M! n!} \frac{(\frac{3}{2} - \nu)_{n+m}}{(\frac{3}{2} - \zeta)_{n+m} (2 - \nu)_{p+n+m}} (\partial_{\nu} \cdot \partial_u)^{p+n+m} J_s . \tag{B.5}
\]

Another identity needed to arrive at the final expressions of eqs. (5.15) and (5.16) is eq. (A.5). Finally, the pair of summations over \( m \) and \( r \) can effectively be cut to the region \( m + r \leq [\frac{1}{2}] \), since for larger values the corresponding traces of the current do not exist, and the differential operator \((\partial_{\nu} \cdot \partial_u)^{m+r}\) annihilates \( J_s(x, u^\nu) \).
C  Proof of eq. (6.6)

In this Appendix we would like to provide a proof of the expression (6.6) for the propagator. In order to simplify the notation, as in Section 5 it is convenient to define

\[ a = \mu - s = 2(\nu - \zeta). \]

(C.1)

In addition, let us recall that

\[
E_0(t) = (1 + t)(2 + t)(3 + t)(a + 2 + t)(a + 4 + t),
\]

\[
E_2(t) = 3(1 + t)[t(a + 2 + t)^2 + (a + 4 + t)(a + 2 + t)],
\]

\[
E_4(t) = 3t(a + 1 + t) + 6,
\]

\[
E_6(t) = 1,
\]

(C.2)

The first step in the proof is the explicit computation of the first four \( \lambda_i \), for \( i = 0, \ldots, 4 \), for which eq. (6.5) takes a special form. The result,

\[
\begin{align*}
\lambda_0 &= 1, \\
\lambda_1 &= -\frac{1}{2(a + 3)}, \\
\lambda_2 &= \frac{1}{8(a + 3)(a + 5)}, \\
\lambda_3 &= -\frac{1}{48(a + 3)(a + 5)(a + 7)}, \\
\lambda_4 &= \frac{1}{384(a + 3)(a + 5)(a + 7)(a + 9)},
\end{align*}
\]

(C.3)

displays a nice pattern, with poles at odd negative values of \( a \), in agreement with eq. (6.7). For \( n > 4 \), eq. (6.5) simplifies, since only the \( K_i \) contribute, and

\[
\begin{align*}
\lambda_n &= 3(a + 4) \sum_{i_1 + \ldots + i_p = n-2} (-1)^p \frac{E_{2i_1}(1 + 2i_1)E_{2i_2}(1 + 2i_1 + 2i_2)}{E_0(1 + 2i_1)E_0(1 + 2i_1 + 2i_2)} \ldots \frac{E_{2i_p}(2n - 3)}{E_0(2n - 3)} \\
&+ \frac{7a + 36}{2(a + 3)} \sum_{i_1 + \ldots + i_p = n-3} (-1)^p \frac{E_{2i_1}(3 + 2i_1)E_{2i_2}(3 + 2i_1 + 2i_2)}{E_0(3 + 2i_1)E_0(3 + 2i_1 + 2i_2)} \ldots \frac{E_{2i_p}(2n - 3)}{E_0(2n - 3)} \\
&+ \frac{1}{2(a + 3)} \sum_{i_1 + \ldots + i_p = n-4} (-1)^p \frac{E_{2i_1}(5 + 2i_1)E_{2i_2}(5 + 2i_1 + 2i_2)}{E_0(5 + 2i_1)E_0(5 + 2i_1 + 2i_2)} \ldots \frac{E_{2i_p}(2n - 3)}{E_0(2n - 3)}.
\end{align*}
\]

(C.4)

These are still rather cumbersome expressions, but performing explicitly the sums over \( i_p = 1, 2, 3 \), one can derive the relatively handy difference equation

\[
\lambda_n = -\frac{1}{E_0(2n - 3)} \left[ E_2(2n - 3)\lambda_{n-1} + E_4(2n - 3)\lambda_{n-2} + \lambda_{n-3} \right].
\]

(C.5)

A proof of eq. (6.7) can now be obtained by induction, starting from the observation that the same recursion relation holds for the first four coefficients in eq. (C.3), that can also serve as initial conditions. On the other hand, the relation underlying eq. (6.7) is

\[
\lambda_m = -\frac{\lambda_{m-1}}{2m(a + 2m + 1)},
\]

(C.6)

that as we have stressed is also a special solution of (C.5). Thus, if we assume that eq. (C.6) hold for \( m = n - 1 \), eq. (C.5) implies that it also does for \( m = n \).
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