NONNEGATIVE $C^2(\mathbb{R}^2)$ INTERPOLATION

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Abstract. We give an alternative proof of the finiteness principle for interpolation of data by nonnegative $C^2(\mathbb{R}^2)$ functions. We show that $k^u = 17$ is sufficient as a finiteness constant, which substantially improves the one given in [3, 4]. Moreover, we give an explicit construction for nonnegative $C^2(\mathbb{R}^2)$ interpolants.

1. Introduction

We fix positive integers $m, n$. We write $C^m(\mathbb{R}^n)$ to denote the Banach space of all real-valued locally $C^m$ functions $F$ on $\mathbb{R}^n$, for which the norm

$$\|F\|_{C^m(\mathbb{R}^n)} := \left( \sum_{\|\alpha\| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|^2 \right)^{\frac{1}{2}}$$

is finite. If $X$ is a finite set, we write $\#(X)$ to denote the number of elements in $X$.

In [4], the authors studied the following problem.

Problem (Whitney’s extension problem). Let $E \subset \mathbb{R}^n$ be a finite set. Suppose we are given a nonnegative function $f : E \to [0, \infty)$. Decide whether there exists a nonnegative $F \in C^m(\mathbb{R}^n)$ such that $F|_E = f$ and $\|F\|_{C^m(\mathbb{R}^n)} \leq C(m, n)\cdot \inf \left\{ \|\tilde{F}\|_{C^m(\mathbb{R}^n)} : \tilde{F} \in C^m(\mathbb{R}^n), \tilde{F} \geq 0, \tilde{F}|_E = f \right\}$, where $C(m, n)$ is a constant depending only on $m$ and $n$.

To solve this problem, they proved the following theorem.

Finiteness Principle ([4]). For large enough $k^u$ and $C^u$, depending only on $m$ and $n$, the following holds.

Let $f : E \to [0, \infty)$ with $E \subset \mathbb{R}^n$ finite. Suppose that for each $S \subset E$ with $\#(S) \leq k^u$, there exists $F^S \in C^m(\mathbb{R}^n)$ with norm $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, such that $F^S = f$ on $S$ and $F^S \geq 0$ on $\mathbb{R}^n$.

Then there exists $F \in C^m(\mathbb{R}^n)$ with norm $\|F\|_{C^m(\mathbb{R}^n)} \leq C^u$, such that $F = f$ on $E$ and $F \geq 0$ on $\mathbb{R}^n$.

The proof of the above theorem given in [4] depends on a finiteness principle for shape fields proven in [3]. As such, the construction of the interpolant is not very explicit, and the finiteness constant $k^u$ is larger than it is necessary. For example, for $m = 2, n = 2$, [4] gives $k^u \geq 100 + 5^l + 100$ for some $l \geq 100$.

The purpose of this paper is two-fold. First, we show that for $m = 2, n = 2, k^u = 17$ is sufficient. Although not proven sharp here, it is a substantial improvement over the one given by [4]. Second, we give an explicit construction of the nonnegative interpolants for $C^2(\mathbb{R}^2)$. Our approach is inspired by [6]. However, we will need new ingredients to apply the machinery adapted from [6].

We restrict ourselves to the case where $E$ is finite, for otherwise, the finiteness principle is not true for $C^2(\mathbb{R}^2)$.

Our approach can easily be adapted to prove the finiteness principle for nonnegative $C^{1,1}(\mathbb{R}^2)$ interpolation for an arbitrary closed set $E$.

We will give a sketch of our proof of the finiteness principle for interpolation by nonnegative $C^2(\mathbb{R}^2)$ functions, sacrificing accuracy for ease of understanding. We begin with interpolation in one dimension. For $C^2(\mathbb{R})$, we will show that, if one can interpolate three points, then one can interpolate any finite set of
points by patching consecutive three-point interpolants together. Thus, a finiteness constant for nonnegative $C^2(\mathbb{R})$ interpolation is 3. To prove the finiteness principle for $C^2(\mathbb{R}^2)$, we will reduce matter to the one dimensional situation. To illustrate the idea, we suppose that $E \subset Q_0 := [0,1] \times [0,1]$. We would like to decompose $E$ into disjoint subsets $E_1, \ldots, E_{\max}$ such that each $E_v$ lies on a nice $C^2$ curve (up to an orientation of the axes).

We accomplish this by means of a Calderón-Zygmund decomposition. To do this, we introduce convex sets $\sigma^h(x,k)$ to keep track of the freedom we have in modifying local nonnegative $C^2(\mathbb{R}^2)$ interpolants: Given $E \subset \mathbb{R}^2$ finite, $x \in \mathbb{R}^2$, we define $\sigma^h(x,k)$ to consist of all first order Taylor polynomials $P$ such that for all subsets $S \subset E$ with $|S| \leq k$, there exist $F^S_1, F^S_2 \in C^2(\mathbb{R}^2)$ with $F^S_1, F^S_2 \geq 0$ such that $(F^S_1 - F^S_2) \big|_S = 0$ and $\|F^S_1 - F^S_2\|_{C^2(\mathbb{R}^2)} \leq 1$. We can view $\sigma^h(x,k)$ as a subset of $\mathbb{R}^3$. We may therefore speak about the diameter of $\sigma^h(x,k)$. We say a cube $Q$ is “nice” if $\text{diam} (\sigma^h(x,k)) \geq C\delta_Q$ for all $x \in E \cap 2Q$, where $2Q$ is the two times concentric dilation of $Q$, $C$ is some large universal constant to be determined, and $\delta_Q$ is the sidelength of $Q$. To bisect a cube $Q$ means to subdivide $Q$ into four congruent cubes. We start with the unit cube $Q_0$, and repeatedly bisect $Q_0$ into ever smaller cubes, stopping at $Q_v$ when $2Q_v$ is nice. This procedure terminates after a finite number of steps because our set $E$ is finite. We then define $E_v = Q_v \cap E$. The Calderón-Zygmund decomposition accomplishes two things at once. First of all, it reduces the $C^2(\mathbb{R}^2)$ interpolation problem to a one dimensional problem. Indeed, since $\text{diam} (\sigma^h(x,k)) \geq C\delta_Q$, by the definition of $\sigma^h$, we know that either the $x$ or the $y$ directional derivative (up to reorientation of the axes) is big on the scale of $\delta_Q$. Therefore, the implicit function theorem tells us that $E_v$ lies on a $C^2$ curve with controlled norm. We can straighten the curve by a $C^2$ diffeomorphism with controlled norm and hence reduce the interpolation problem on $E_v$ to an interpolation problem on the real line. Second, the neighboring cubes are comparable in size. As such, the local nonnegative interpolants can be made compatible to allow patching by a partition of unity.

We have given a layman’s summary of our proof. The full account is to be found in the sections below.

This paper is part of a literature on extension and interpolation, going back to the seminal works of H. Whitney [8–10]. We refer the interested readers to [3, 4] and references therein for the history and related problems.

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2. Statement of results

First we set up notations. Let $n = 1,2$. $C^2_+(\mathbb{R}^n)$ denotes the collection of all functions $F : \mathbb{R}^n \to [0, \infty)$ whose derivatives up to the second order are continous and bounded. Let $E \subset \mathbb{R}^n$ be a finite set. $C^2_+(E)$ denotes the collection of functions $f : E \to [0, \infty)$. We use $\|f\|_{C^2_+(E)}$ to denote the infimum of the norm of all functions $F \in C^2_+(\mathbb{R}^n)$ such that $F$ and $f$ agree on $E$.

To see the statement for our results for $C^2_+(\mathbb{R}^2)$, it is easier to present the analogues for $C^2_+(\mathbb{R})$.

In one-dimension, our main results read as follows.

**Theorem 1 (1-D Finiteness Principle).** There exists a constant $C^h > 0$ such that the following holds.

Let $E = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ be a finite set with $x_1 < \cdots < x_N$ and $N \geq 3$. Let $f : E \to [0, \infty)$. Suppose

- For every consecutive three points $E_j = \{x_j, x_{j+1}, x_{j+2}\}$ ($j = 1, \ldots, N-2$) there exists a function $F_j \in C^2_+(\mathbb{R})$ such that $F_j|_{E_j} = f$; and
- $\|F_j\|_{C^2_+(\mathbb{R})} \leq 1$.

Then there exists $F \in C^2_+(\mathbb{R})$ with

- $F|_E = f$, and
• \( \|F\|_{C^2(\mathbb{R})} \leq C^\# \).

**Remark 2.1.** We may take a crude bound, say \( C^\# = 50 \). One can clearly do better even with the technology provided here, but we will not pursue the best \( C^\# \) in the present work.

**Remark 2.2.** The proof for **Theorem 1** does not rely on any other ingredients other than those presented in section 7. The same proof also shows the following 1-D Finiteness Principle without nonnegative constraint.

For future reference (see the proof of **Lemma 6.1**), we include more explicit control on the derivatives in the following statement.

**Theorem 1'**. There exists a constant \( C^\# > 0 \) such that the following holds.

Let \( E = \{x_1, \ldots, x_N\} \subset \mathbb{R} \) be a finite set with \( x_1 < \cdots < x_N \) and \( N \geq 3 \). Let \( f : E \to \mathbb{R} \). Suppose

- For every consecutive three points \( E_j = \{x_j, x_{j+1}, x_{j+2}\} \) \( (j = 1, \ldots, N - 2) \), there exists a function \( F_j \in C^2(\mathbb{R}) \) such that \( F_j|_{E_j} = f \);
- \( \|\partial^m F\|_{C^m(\mathbb{R})} \leq A_m \) for \( m = 0, 1, 2 \). Here and below, \( \partial^m \) denotes the \( m \)-th derivative.

Then there exists \( F \in C^2(\mathbb{R}) \) such that

- \( F|_{E} = f \);
- \( \|\partial^m F\|_{C^m(\mathbb{R})} \leq C^\# A_m \) for \( m = 0, 1, 2 \).

**Theorem 2 (1-D Extension Operator).** Let \( E \subset \mathbb{R} \) be a finite set. There exists an extension operator \( \mathcal{E}_+: C^2_+(E) \to C^2_+(\mathbb{R}) \) such that

- \( \mathcal{E}_+(f)|_E = f \) for all \( f \in C^2_+(E) \), and
- \( \|\mathcal{E}_+(f)\|_{C^2(\mathbb{R})} \leq C \|f\|_{C^2_+(E)} \).

Here, \( C \) is a universal constant. In particular, \( C \) is independent of \( f \) and \( E \).

We are now ready to state our main results for \( C^2_+(\mathbb{R}^2) \) that mirror the statements above.

**Theorem 3 (2-D Finiteness Principle).** There exists a constant \( C^\# > 0 \) such that the following holds.

Let \( f : E \to [0, +\infty) \) with \( E \subset \mathbb{R}^2 \) finite. Suppose for each \( S \subset E \) with \( \#(S) \leq 17 \), there exists \( F^S \in C^2_+(\mathbb{R}^2) \) such that

- \( \|F^S\|_{C^2(\mathbb{R}^2)} \leq 1 \), and
- \( F^S|_S = f \).

Then there exists \( F \in C^2_+(\mathbb{R}^2) \) such that

- \( F|_E = f \), and
- \( \|F\|_{C^2(\mathbb{R}^2)} \leq C^\# \).

**Theorem 4 (2-D Extension Operator).** Let \( E \subset \mathbb{R}^2 \) be a finite set. There exists an extension operator \( \mathcal{E}_+: C^2_+(E) \to C^2_+(\mathbb{R}^2) \) such that

- \( \mathcal{E}_+(f)|_E = f \) for all \( f \in C^2_+(E) \), and
- \( \|\mathcal{E}_+(f)\|_{C^2(\mathbb{R}^2)} \leq C \|f\|_{C^2_+(E)} \).

Here, \( C \) is a universal constant. In particular, \( C \) is independent of \( f \) and \( E \).

We will present the proofs of these results in Section 7 - Section 10 below.

3. **Conventions and Preliminaries**

3.1. **Constants.** Constants \( c_0, c_1, C_0, C_1 \in \mathbb{R}_{>0} \), etc., denote “controlled” universal constants. They may be different quantities in different instances. We will label them to avoid confusion when necessary.
3.2. Coordinates and $C^2$ norm. We assume that we are given an ordered orthogonal coordinate system $(s, t)\text{standard}$ on $\mathbb{R}^2$ a priori.

Let $n = 1, 2, 3$ and let $E \subset \mathbb{R}^n$ be finite.

For $m = 0, 1, 2$, $C^m(\mathbb{R}^n)$ denotes the vector space of $m$-times continuously differentiable real-valued functions such that the following norm is finite:

\[
\|F\|_{C^m(\mathbb{R}^n)} := \left( \sum_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|^2 \right)^{\frac{1}{2}}.
\]

$C^2(\mathbb{R}^n)$, as a convex subset of $C^2(\mathbb{R}^n)$, is the collection of nonnegative functions $F \in C^2(\mathbb{R}^n)$. This is not a vector space.

We define the following trace classes

\[
C^2(E) = \{ F|_E : F \in C^2(\mathbb{R}^n) \}, \quad C^2_+(E) = \{ F|_E : F \in C^2_+(\mathbb{R}^n) \}.
\]

$C^2(E)$ is a vector space that can be equipped with a seminorm

\[
\|f\|_{C^2(E)} = \inf \left\{ \|F\|_{C^2(\mathbb{R}^n)} : F \in C^2(\mathbb{R}^n), F|_E = f \right\}.
\]

$C^2_+(E)$ is a convex subset of $C^2(E)$, and we define

\[
\|f\|_{C^2_+(E)} = \inf \left\{ \|F\|_{C^2_+(\mathbb{R}^n)} : F \in C^2_+(\mathbb{R}^n), F|_E = f \right\}.
\]

We say that $F \in C^2_+(\mathbb{R}^n)$ is a C-optimal nonnegative interpolant of $f$ over $E$ if

- $F|_E = f$ and
- $\|F\|_{C^2(\mathbb{R}^n)} \leq C \|f\|_{C^2_+(E)}$.

Here, $C$ is a universal constant.

We will consider alternative coordinate systems in Section 6 to exploit the rotation-invariance of (3.1).

3.3. Jets. $\mathcal{P}$ denotes the space of degree one polynomials on $\mathbb{R}^2$. It is a three-dimensional vector space.

For $x_0 = (s_0, t_0) \in \mathbb{R}^2$ and a continuously differentiable function $F$ on $\mathbb{R}^2$, the 1-jet of $F$ at $x_0 \in \mathbb{R}^2$ is given by

\[
\mathcal{J}_{x_0}(F)(x) := F(x_0) + \nabla F(x_0) \cdot (x - x_0).
\]

$\mathcal{R}_{x_0}$ denotes the vector space of 1-jet at $x_0 \in \mathbb{R}^2$. $\mathcal{R}_{x_0}$ inherits a norm from $\mathbb{R}^3$ via the identification

\[
I_{x_0} : a(s - s_0) + b(t - t_0) + c \mapsto (a, b, c).
\]

3.4. Calderón-Zygmund Cubes. A cube $Q \subset \mathbb{R}^2$ is of the form $Q = [s_0, s_0 + \delta] \times [t_0, t_0 + \delta]$ where $\delta > 0$ and $s_0, t_0 \in \mathbb{R}$.

For a cube $Q \subset \mathbb{R}^2$, $\lambda Q$ denotes the concentric dilation of $Q$ by a factor of $\lambda > 0$. Let $Q^* = 2Q$. $\delta_Q$ denotes the side length of $Q$.

For a cube $Q_0 \subset \mathbb{R}^2$, by a dyadic decomposition of $Q_0$, we mean dividing $Q_0$ into four mutually disjoint congruent cubes $Q_1, Q_2, Q_3, Q_4$ such that $Q_0 = \bigcup_{i=1}^4 Q_i$. $Q_0$ is called the dyadic parent of $Q_1, \ldots, Q_4$. In this case, we write $Q_i^* = Q_0$ for $i = 1, \ldots, 4$. Dyadic parents are unique if they exist.

Two cubes $Q$ and $Q'$ are neighbors if one of the following holds.

- $Q = Q'$;
- $\text{closure}(Q) \cap \text{closure}(Q') \neq \emptyset$, but $\text{interior}(Q) \cap \text{interior}(Q') = \emptyset$.

If $Q$ and $Q'$ are neighbors, we write $Q \leftrightarrow Q'$. We write $\delta_i$ for $\delta_{Q_i}$.

A collection of mutually disjoint cubes $\Lambda = \{Q_i\}$ is a Calderón-Zygmund (CZ) covering of $\mathbb{R}^2$ if both of the following hold.
(CZ0) \( \mathbb{R}^2 = \bigcup_i Q_i \),

(CZ1) if \( Q_i \leftrightarrow Q_j \), then \( \frac{1}{4} \delta_i \leq \delta_j \leq 4 \delta_i \).

(CZ1) implies that a CZ covering satisfies the \textbf{bounded intersection property}: if \( Q \in \Lambda \), then

\[
\# \left( \left\{ Q' \in \Lambda : \frac{9}{8} Q' \cap \frac{9}{8} Q \neq \emptyset \right\} \right) \leq 21.
\]

We will only consider \textit{nonnegative} (smooth) cutoff functions and partition of unity. A \( C^2 \)-partition of unity \( \{ \theta_i \} \) subordinate to a CZ covering \( \{ Q_i \} \) of \( \mathbb{R}^2 \) is \textbf{CZ-compatible with} \( \{ Q_i \} \) if

\[
\theta_i \geq 0, \ \text{supp} (\theta_i) \subset \frac{9}{8} Q_i, \ |\partial^\beta \theta_i| \leq C \delta_i^{-|\beta|} \forall |\beta| \leq 2, \text{ and } \sum_i \theta_i = 1.
\]

Here \( C \) is some universal constant.

\section*{4. Basic properties of convex sets}

Let \( E \subset \mathbb{R}^2 \) be a finite set. Let \( f : E \to [0, +\infty) \). For a point \( x \in \mathbb{R}^2 \), a subset \( S \subset E \), and a real number \( M > 0 \), we introduce the following objects:

\[
\Gamma_+(x, S, M) := \left\{ P \in \mathcal{P} : \| F^S \|_{C^2(\mathbb{R}^2)} \leq M, F^S \big|_S = f, \text{ and } J_x(F^S) = P \right\}
\]

\[
\sigma(x, S, M) := \left\{ P \in \mathcal{P} : \| (F_1^S - F_2^S)|_S = 0, \| F_1^S - F_2^S \|_{C^2(\mathbb{R}^2)} \leq M \right\}.
\]

Given an integer \( k \geq 0 \), we define

\[
\Gamma_+^k(x, k, M) := \bigcap_{S \subset E, \#(S) \leq k} \Gamma_+(x, S, M), \ \sigma^\#(x, k, M) := \bigcap_{S \subset E, \#(S) \leq k} \sigma(x, S, M).
\]

For \( \sigma \) and \( \sigma^\# \), we will be mainly interested in the normalized

\[
\sigma(x, S) := \sigma(x, S, 1) \text{ and } \sigma^\#(x, k) := \sigma^\#(x, k, 1),
\]

since \( \sigma(x, S, M) = M \cdot \sigma(x, S, 1) \) and \( \sigma^\#(x, k, M) = M \cdot \sigma^\#(x, k, 1) \) by definition.

\textbf{Remark 4.1.} Since \( \#(E) < \infty \), for sufficiently large \( M > 0 \) depending on \( E \) and \( f \), \( \Gamma_+(x, S, M) \neq \emptyset \) for any \( S \subset E \). As a consequence, for a specific \( k \), \( \Gamma_+^k(x, k, M) \neq \emptyset \) if \( M \) is sufficiently large.

\textbf{Remark 4.2.} \( \Gamma_+, \Gamma_+^k, \sigma \), and \( \sigma^\# \) are convex and compact (as subsets of \( \mathbb{R}^3 \) via the identification (3.2)). \( \sigma \) and \( \sigma^\# \) are symmetric about the origin.

\textbf{Remark 4.3.} \( \sigma \) and \( \sigma^\# \) are never empty, since they contain the zero polynomial. Moreover, they depend solely on the set \( E \). We will use \( \sigma^\# \) to detect the local geometry of \( E \). See Section 6.

\textbf{Remark 4.4.} Both \( \sigma^\# \) and \( \Gamma_+^k \) are monotone decreasing (with respect to set inclusion \( \subset \)) in \( k \). \( \Gamma_+^k \) is monotone increasing in \( M \).

We state a basic property of \( \sigma^\# \), which follows from definition.

\textbf{Proposition 4.1.} \textit{For} \( x \in E \) \textit{and} \( k \geq 1 \), if \( P \in \sigma^\#(x, k) \), \textit{then} \( P(x) = 0 \).

The next lemma relates \( \Gamma_+^k \) with \( \sigma^\# \).

\textbf{Lemma 4.1.} \( \Gamma_+^k(x, k, M) - \Gamma_+^k(x, k, M) \subset 2M \cdot \sigma^\#(x, k) \). The minus sign denotes vector subtraction.
Lemma 4.2. Let $P_1, P_2 \in \Gamma^S_+(x, k, M)$. For each $S \subseteq E$ with $#(S) \leq k$, there exist $F^S_1, F^S_2 \in C^2_+ (\mathbb{R}^2)$ such that for $i = 1, 2$,

- $F^S_i \big|_S = f$,
- $\|F^S_i\|_{C^2(\mathbb{R}^2)} \leq M$, and
- $\mathcal{J}_S(F^S_i) = P_i$.

Then

- $(F^S_1 - F^S_2) \big|_S = 0$; and
- $\|F^S_1 - F^S_2\|_{C^2(\mathbb{R}^2)} \leq 2M$; and
- $\mathcal{J}_S(F^S_1 - F^S_2) = P_1 - P_2$.

Since $S$ is arbitrary, $P_1 - P_2 \in \sigma^S(x, k, 2M) = 2M \cdot \sigma^S(x, k)$. \hfill $\Box$

We recall a classical result by Helly.

Helly’s Theorem. Let $n, l$ be positive integers. Let $K_1, \ldots, K_l \subseteq \mathbb{R}^n$ be convex sets. Suppose

$$\bigcap_{i=1}^{n+1} K_i \neq \emptyset$$

for any $i_1, \ldots, i_{n+1} \in \{1, \ldots, l\}$, then

$$\bigcap_{i=1}^l K_i \neq \emptyset.$$ 

See [7].

The following lemma states that we can control polynomials in $\Gamma^S_+$ based at some point by polynomials that are based at a different point but are “less universal” (in the sense that it is the jet for an interpolant for fewer points).

Lemma 4.2. Let $x, x' \in E$. Let $k_1 \geq 4k_2 + 1$. If $P \in \Gamma^S_+(x, k_1, M)$, then there exists $P' \in \Gamma^S_+(x', k_2, M)$ such that for $0 < \beta \leq 1$,

$$|\partial^\beta(P - P')(x)|, |\partial^\beta(P - P')(x')| \leq CM|x - x'|^{2-\beta}$$

Proof. We proceed as Lemma 5.6 in [1].

Fix $P$ as in the hypothesis of the lemma. For each $S \subseteq E$ with $#(S) \leq k_1$, let $S'' = S \cup \{x, x'\}$. Put

$$\Gamma^\text{temp}_+(S) : = \left\{ P' \in \mathcal{P} : \begin{array}{c}
\text{There exists } F^S \in C^2_+ (\mathbb{R}^2) \text{ such that } \|F^S\|_{C^2(\mathbb{R}^2)} \leq M, \\
F^S \big|_{S''} = f, \mathcal{J}_S(F^S) = P, \text{ and } \mathcal{J}_{S'}(F^S) = P'.
\end{array} \right\}.$$ 

Then $\Gamma^\text{temp}_+$ is a convex and compact subset of $\mathcal{P}$, which we have identified with $\mathbb{R}^3$. Notice that $S \subseteq S''$ implies $\Gamma^\text{temp}_+(S) \subseteq \Gamma^\text{temp}_+(S'')$.

Claim 4.1. If $#(S') \leq k_1 + 1$, then $\Gamma^\text{temp}_+(S) \neq \emptyset$.

Proof of Claim 4.1. Recall that $S' = S \cup \{x, x'\}$ and $#(S') \leq k_1$. Since $#(S') \leq k_1 + 1$, $#(S' \setminus \{x'\}) \leq k_1$. Let $S'' = S' \setminus \{x'\}$. Since $P \in \Gamma^S_+(x, k_1, M)$, there exists $F^S'' \in C^2_+ (\mathbb{R}^2)$ such that

- $\|F^S''\|_{C^2(\mathbb{R}^2)} \leq M$,
- $F^S'' \big|_{S''} = f$, and
- $\mathcal{J}_{S'}(F^S'') = P$.

Then $\mathcal{J}_{S'}(F^S'') \in \Gamma^\text{temp}_+(S)$, and the claim follows. \hfill $\Box$
Let $S_1, \ldots, S_4 \subset E$ be given with $\#(S_i) \leq k_2$ for each $i$. Let $S = \bigcup_{i=1}^4 S_i$. Then $\#(S') \leq 2 + 4k_2 \leq k_1 + 1$. By our claim, $\Gamma^\text{temp}_+(S) \neq \emptyset$. Since $S_i \subset S$, $\Gamma^\text{temp}_+(S_i) \subset \Gamma^\text{temp}_+(S)$. Then

$$\bigcap_{i=1}^4 \Gamma^\text{temp}_+(S_i) \supset \Gamma^\text{temp}_+(S) \neq \emptyset.$$ 

Since $\{S_i \}_{i=1}^4$ are arbitrary, applying Helly’s Theorem to the convex sets $\Gamma^\text{temp}_+(S_i) \subset \mathbb{R}^2$, we have

$$\Gamma^\text{temp}_+(k_2) := \bigcap_{S \in E, \#(S) \leq k_2} \Gamma^\text{temp}_+(S) \neq \emptyset.$$

Let $P' \in \Gamma^\text{temp}_+(k_2)$. By definition, $P' \in \Gamma^+(x', k_2, M)$.

By letting $S = \emptyset$, we see that there exists $F^\emptyset \in C^2_+(\mathbb{R}^2)$ with

- $\|F^\emptyset\|_{C^1(\mathbb{R}^2)} \leq M$; and
- $\mathcal{J}_x(F^\emptyset) = P$ and $\mathcal{J}_{x'}(F^\emptyset) = P'$.

By Taylor’s theorem,

$$|\partial^\emptyset(P - P')(x)| = |\partial^\emptyset(\mathcal{J}_x(F^\emptyset) - \mathcal{J}_{x'}(F^\emptyset))(x)|$$

$$\leq |\partial^\emptyset(\mathcal{J}_x(F^\emptyset) - F^\emptyset)(x)| + |\partial^\emptyset(F^\emptyset - \mathcal{J}_{x'}(F^\emptyset))(x)|$$

$$\leq CM |x - x'|^{2-|\beta|}.$$ 

The estimate for $|\partial^\emptyset(P - P')(x')|$ is similar. \hfill \Box

Next, we prove a technical lemma that will give us a procedure to recover a $C^2_+$ function from a given jet in $\Gamma^+$. 

**Lemma 4.3.** Let $\hat{x} \in \mathbb{R}^2$. Let $B = B(\hat{x}, r)$ denote the ball of radius $r$ centered at $\hat{x}$. Then there exists a mapping

$$\mathcal{R}_B : \Gamma^+(\hat{x}, 0, M) \to C^2_+(\mathbb{R}^2), \; P \mapsto \mathcal{R}_B(P)$$

that satisfies the following:

(a) $\|\mathcal{R}_B(P)\|_{C^2(\mathbb{R}^2)} \leq C(r) M$,

(b) $\text{supp} \mathcal{R}_B(P)) \subset B(\hat{x}, r/2)$, and

(c) $\mathcal{J}_x(\mathcal{R}_B(P)) = P$.

The quantity $C(r)$ depends only on $r$.

**Proof:** We follow the construction in [4].

Fix $P \in \Gamma^+(\hat{x}, 0, M)$. Without loss of generality, we may assume that $\hat{x} = 0$. For $0 \leq r_1 < r_2 < \infty$, we use $A(r_1, r_2)$ to denote the closed annulus $\{r_1 \leq |x| \leq r_2\}$.

By Taylor’s theorem, $P$ satisfies the global and local “correctibility” conditions:

- (GC) there exists a constant $C > 0$ such that $P(x) + CM |x|^2 \geq 0$ for all $x \in \mathbb{R}^2$, and
- (LC) for each $\epsilon > 0$, there exists $\eta > 0$ such that $P(x) + \epsilon |x|^2 \geq 0$ for all $|x| \leq \eta$.

Let $\psi$ be a nonnegative $C^2$ cutoff function such that

- $\|\psi\|_{C^2(\mathbb{R}^2)} \leq C(r)$;
- $\psi = 1$ on $A(2^{-3}r, 2^{-2}r)$, and $\psi = 0$ outside $A(2^{-4}r, 2^{-1}r)$.

For each $m \geq 0$, let

$$\psi_m(x) := \psi \left(2^m x \right).$$
Also let
\[ b_m := \begin{cases} 
0 & \text{if } P(x) \geq 0 \text{ on } A(0, 2^{-m}r) \\
- \min \{ P(x) : x \in A(0, 2^{-m}r) \} & \text{otherwise} 
\end{cases} \]

Some basic properties of \( \{ \psi_m \} \) are immediate:

(i) \( \psi_m \geq 0 \);
(ii) \( \| \psi_m \|_{C^2(\mathbb{R}^2)} \leq C(r)2^{2m} \);
(iii) \( \psi_m = 1 \) on the annulus \( A(2^{-3-m}r, 2^{-2-m}r) \), and \( \psi_m = 0 \) away from the annulus \( A(2^{-1-m}r, 2^{-2-m}r) \);
(iv) \( \# \{ m : \psi_m(x) \neq 0 \} \leq N_0 \), where the number \( N_0 \) is universal.

Some basic properties of \( \{ b_m \} \) are in place.

(i) \( P + b_m \geq 0 \) on \( A(0, 2^{-m}r) \);
(ii) \( 0 \leq b_m \leq C2^{-2m}M \) for all \( m \), due to (GC);
(iii) \( 2^{2m}b_m \to 0 \) as \( m \to \infty \), due to (LC).

We first construct a function \( F \) on \( B \) by setting
\[ F(x) := P(x) + \sum_{m=0}^{\infty} b_m \psi_m(x). \]

Thanks to property (iv) of \( \{ \psi_m \} \), the sum is finite for each \( x \in B \), so \( F \) is well defined.

We will check that
(a’) \( F \geq 0 \) on \( B \),
(b’) \( \| F \|_{C^2(B)} \leq C(r)M \), and
(c’) \( f_0(F) = P \).

We check (a’) first. Since \( \psi_m = 0 \) in a neighborhood of the origin for each \( m \), \( F(0) = P(0) \geq 0 \). For a particular \( x \in B \setminus \{ 0 \} \), there exists \( m(x) \in \mathbb{Z}^+ \) such that \( x \in A(2^{-3-m(x)}r, 2^{-2-m(x)}r) \), so \( \psi_{m(x)}(x) = 1 \). By property (i) of \( \{ \psi_m \} \), \( P(x) + b_{m(x)}\psi_{m(x)}(x) \geq 0 \). By property (ii) of \( \{ b_m \} \), we have
\[ F(x) = (P(x) + b_{m(x)}\psi_{m(x)}) + \sum_{m \neq m(x)} b_m \psi_m(x) \geq 0. \]

For (b’) and (c’), we consider the partial sum
\[ F_N(x) := P(x) + \sum_{m=0}^{N} b_m \psi_m(x). \]

Since \( P \in \Gamma_+^\#(0, 0, M) \), we have
\[ \| P \|_{C^2(\bar{B})} \leq C(r)M. \] (4.1)

It follows from properties (ii) - (iv) of \( \{ \psi_m \} \) and property (ii) of \( \{ b_m \} \) that
\[ \| b_m \psi_m \|_{C^2(\bar{B})} \leq \sum_{n=0}^{N} b_m \psi_m \leq C(r)M. \] (4.2)

(4.1) and (4.2) imply
\[ \| F_N \|_{C^2(\bar{B})} \leq C(r)M. \] (4.3)

Since \( \psi_m \) vanishes in a neighborhood of the origin for each \( m \),
\[ \mathcal{J}_0(F_N) = P \text{ for each } N. \] (4.4)
Let $\epsilon > 0$. Property (ii) of $\psi_m$ and property (iii) of $b_m$ implies that for sufficiently large $N, N'$ with $N < N'$, we have

$$\max_{N < m \leq N'} ||b_m \psi_m||_{C^2(B)} \leq C(r) \epsilon.$$ 

It follows from property (iv) of $\{\psi_m\}$ that for such $N, N'$,

$$\|F_N - F_{N'}\|_{C^2(B)} = \left\| \sum_{n=N}^{N'} b_m \psi_m \right\|_{C^2(B)} \leq C(r) \epsilon.$$ 

This means that $(F_N)_{N \in \mathbb{Z}^d}$ is a Cauchy sequence in $C^2(B)$, which is a Banach space. Thus, the sequence converges to some $F_\infty \in C^2(B)$ with respect to the $C^2$ norm. (4.3) and (4.4) then imply

\begin{itemize}
  \item[(b'')] $\|F_\infty\|_{C^2(B)} \leq C(r) M$
  \item[(c'')] $\mathcal{J}_0(F_\infty) = P.$
\end{itemize}

On the other hand, property (iv) of $\{\psi_m\}$ implies that $F_N$ converges to $F$ pointwise, so in particular, $F_\infty = F$. (b') and (c') follow from (b'') and (c'').

Finally, let $\chi$ be a cutoff function that satisfies

\begin{itemize}
  \item[(i)] $\|\chi\|_{C^2(\mathbb{R}^d)} \leq C(r)$;
  \item[(ii)] $\text{supp} (\chi) \subset B(0, r/2)$, and
  \item[(iii)] $\chi \equiv 1$ in a neighborhood of the origin.
\end{itemize}

Define $\mathcal{R}_B(P) := \chi F$. Then $\mathcal{R}_B(P)$ is nonnegative and twice continuously differentiable. (a')-(c') follow from (a'')-(c'') and properties (i) - (iii) of $\chi$.

5. Calderón–Zygmund decomposition of the plane

**Definition 5.1.** Let $C_{\text{nice}} > 0$ and $k \geq 1$. We say a dyadic cube $Q$ is $k$-nice if for all $x \in E \cap Q^*$,

\begin{equation}
\text{diam} (\sigma^#(x, k)) \geq C_{\text{nice}} \delta_Q.
\end{equation}

Consider the following

**Algorithm.** Let $Q$ be a unit cube.

- If $Q$ is $k$-nice, then return $\Lambda_Q^{(k)} = \{Q\}$;
- otherwise, return

$$\Lambda_Q^{(k)} := \bigcup \left\{ \Lambda_Q^{(k)} : Q' \text{ dyadic and } (Q')^+ = Q \right\}.$$

**Remark 5.1.** The algorithm terminates after finitely many steps for each unit cube. To see this, notice that $E$ is finite, and for specific $k$ and $C_{\text{nice}}$, (5.1) clearly holds for each sufficiently small cube containing no more than one point.

**Remark 5.2.** Since $\sigma^#$ does not depend on $f$, the complexity of our algorithm depends solely on the set $E$.

For a particular choice of $C_{\text{nice}} > 0$ and $k \geq 1$, we use $\Lambda_{\text{nice}}^{(k)} = \{Q_i\}$ to denote the collection of $k$-nice cubes obtained from the procedure above, ranging from all the unit cubes with their vertices on the integer lattice.

**Lemma 5.1.** $\Lambda_{\text{nice}}^{(k)}$ is a CZ covering of $\mathbb{R}^2$.

**Proof.** Since we obtain $\Lambda_{\text{nice}}^{(k)}$ by applying the algorithm to each cube of the unit grid, (CZ0) is satisfied.

Suppose (CZ1) fails, that there exist some $Q, Q' \in \Lambda_{\text{nice}}^{(k)}$ with $Q \leftrightarrow Q'$ but

$$\delta_Q \leq \frac{1}{8} \delta_{Q'}.$$
Then $(Q^+)^* \subset (Q')^*$. Since $Q^+$ is not $k$-nice, there exists $\hat{x} \in E \cap (Q^+)^* \setminus Q^*$ such that
\[ \text{diam} \left( \sigma^\#(\hat{x}, k) \right) < 2C_{\text{nice}}\delta_Q. \]

On the other hand,
\[ C_{\text{nice}}\delta_Q \leq \text{diam} \left( \sigma^\#(\hat{x}, k) \right) \]
A contradiction is reached once we combine all the inequalities above, because $Q'$ is $k$-nice.

Our main goal is to construct a $C$-optimal interpolant for each $k$-nice cube and then to patch these local solutions together. We need several lemmas that guarantee the consistency of our operation.

The following lemma states that polynomials in $\Gamma^\#_x$ with the same base control each other in the Whitney sense after our decomposition.

**Lemma 5.2.** Let $C_{\text{nice}}, k \geq 1, Q \in \Lambda_{\text{nice}}^{(k)}$, $x \in E \cap Q^*$, and $0 \leq |\beta| \leq 1$. If $P_1, P_2 \in \Gamma^\#_x(x, k, M)$, then
\begin{equation}
|\partial^\beta(P_1 - P_2)(x)| \leq 14C_{\text{nice}}M\delta^2_Q|\beta|.
\end{equation}

**Proof.** Let $\delta = \delta_Q$. (5.2) is immediate if $\delta = 1$ or $\beta = 0$. Therefore, we only need to consider the case when $\delta < 1$ and $|\beta| = 1$. The assumption $\delta < 1$ implies that there exists $y \in E \cap (Q^+)^*$ such that $\text{diam} \left( \sigma^\#(y, k) \right) < 2C_{\text{nice}}\delta$. Fix such $y$.

Suppose towards a contradiction, that we can find a point $x \in E \cap Q^*$ and $P_1, P_2 \in \Gamma^\#_x(x, k, M)$ such that (5.2) is false for some $|\beta| = 1$. Fix such $\beta$.

By Lemma 4.1, $P_1 - P_2 \in 2M : \sigma^\#(x, k)$. By definition, for any $S \subset E$ with $|S| \leq k$, there exists $F^S \in C^2(\mathbb{R}^2)$ such that
\begin{itemize}
  \item $F^S|_S = 0$,
  \item $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 2M$, and
  \item $\partial^\beta \mathcal{F}_y(F^S) = \partial^\beta(P_1 - P_2)$.
\end{itemize}

By assumption, $|\partial^\beta(F^S(x))| > 14C_{\text{nice}}M\delta$. Since, $x, y \in (Q^+)^*$, we have $|x - y| < 6\delta$. Therefore,
\[ |\partial^\beta \mathcal{F}_y(F^S(y))| = |\partial^\beta F^S(y)| \geq |\partial^\beta F^S(x)| - \|F^S\|_{C^2(\mathbb{R}^2)} |x - y| \geq 2C_{\text{nice}}M\delta. \]

Since $S$ is arbitrary, we have $\text{diam} \left( \sigma^\#(y, k) \right) \geq 2C_{\text{nice}}\delta$. A contradiction.

**Lemma 5.3.** Let $C_{\text{nice}}, k \geq 1, Q_v \in \Lambda_{\text{nice}}^{(k)}$ for $v = i, j$, with $Q_i \leftrightarrow Q_j$, $x_v \in Q_v$, and $P_v \in \Gamma^\#_x(x_v, 4k + 1, M)$ for some $M > 0$. Then for all $x \in \frac{9}{8}Q_i \cup \frac{9}{8}Q_j$ and $0 \leq |\beta| \leq 1$,
\begin{equation}
|\partial^\beta(P_i - P_j)(x)| \leq CM\delta^2_i|\beta|.
\end{equation}

**Proof.** Write $\delta = \delta_i$. By (CZ1), $|x_v - x|, |x_i - x_j|, \delta_j \leq C_1\delta$ for $v = i, j$ and $x \in \frac{9}{8}Q_i \cup \frac{9}{8}Q_j$.

Lemma 4.2 produces a $P_{\text{temp}} \in \Gamma^\#_x(x_j, k, M)$ with
\begin{equation}
|\partial^\beta(P_i - P_{\text{temp}})(x_j)| \leq C_2 M|x - x_i|^{2-|\beta|} \leq C_3 M\delta^2_i|\beta|.
\end{equation}

Since $P_j \in \Gamma^\#_x(x_j, 4k + 1, M) \subset \Gamma^\#_x(x_j, k, M)$, Lemma 5.2 applied to $P_j$ and $P_{\text{temp}}$ gives
\begin{equation}
|\partial^\beta(P_j - P_{\text{temp}})(x_j)| \leq C_4 M\delta^2_j|\beta| \leq C_5 M\delta^2_i|\beta|.
\end{equation}

Combining (5.4) and (5.5), we have
\begin{equation}
|\partial^\beta(P_i - P_j)(x_j)| \leq C_6 M\delta^2_i|\beta|.
\end{equation}

Since $P_i$ and $P_j$ are affine polynomials, (5.3) follows from (5.6) in the case $|\beta| = 1$. By the fundamental theorem of calculus,
\begin{equation}
(P_i - P_j)(x) = (P_i - P_j)(x_j) + \int_{[x_j, x]} \nabla(P_i - P_j)
\end{equation}
where \([x_j, x] \) is the straight line segment connecting \(x_j\) and \(x\). Taking absolute value of (5.7) and applying (5.3) and (5.6) with \(|\beta| = 1\), we can conclude that (5.3) holds for \(\beta = 0\).

\[\Box\]

6. Local geometry

The goal of this section is to show that according to our decomposition, we have partitioned the data points into clusters with nice geometry. To proceed, we introduce some notations.

Recall that the \(C^2\) norm in (3.1) is rotationally invariant. Let \(\omega \in [-\pi/2, \pi/2]\). We associate with \(\omega\) a coordinate system obtained by rotating the plane counterclockwise about the origin by an angle of \(\omega\). Thus, for \(x \in \mathbb{R}^2\),

\[x = (s, t)_{\text{standard}} = (x_{\omega}^{(1)}, x_{\omega}^{(2)})_{\omega}\]

where \(x_{\omega}^{(1)} = s \cos \omega + t \sin \omega\) and \(x_{\omega}^{(2)} = -s \sin \omega + t \cos \omega\). When the choice of \(\omega\) is clear, we write \(\partial_1, \partial_2\) to denote the partial derivatives with respect to the first, second variable, respectively. They coincide with the directional derivatives along \(\omega\) and \(\omega^{-1}\), if we also treat \(\omega\) as a unit vector. If \(\phi : I \to \mathbb{R}\) is a function defined on \(I \subset \mathbb{R}\), we denote by \(\text{Graph}(\phi; I, \omega)\) the graph of \(\phi\) over \(I\) (with respect to the standard coordinate system) rotated by the angle \(\omega\).

**Lemma 6.1.** Let \(k \geq 4\) and let \(C_{\text{nice}}\) be sufficiently large. Suppose \(Q \in \Lambda_{\text{nice}}^{(k)}\), then there exist \(\omega \in [-\pi/2, \pi/2]\) and a twice continuously differentiable function \(\phi : \mathbb{R} \to \mathbb{R}\) such that

- \(E \cap Q^* \subset \text{Graph}(\phi; \mathbb{R}, \omega)\);
- \(\|\phi'\|_{C^0(\mathbb{R})} \leq C\), and
- \(\|\phi''\|_{C^0(\mathbb{R})} \leq C\delta^{-1}\).

The constant \(C\) depends only on \(C_{\text{nice}}\).

**Proof.** If \(E \cap Q^* = \emptyset\), there is nothing to prove. From now on, we assume \(E \cap Q^* \neq \emptyset\).

Fix \(x_0 \in E \cap Q^*\). Let \(\delta = \delta_Q\). Since \(Q \in \Lambda_{\text{nice}}^{(k)}\),

\[(6.1) \quad \text{diam} (\sigma^\#(x_0, k)) \geq C_{\text{nice}} \delta.\]

From (6.1) and the symmetry of \(\sigma^\#\), we learn that there exist \(P^{x_0} \in \sigma^\#(x_0, k)\) and \(\omega = \frac{I_{x_0}(P^{x_0})}{\|I_{x_0}(P^{x_0})\|}\) (where \(I_{x_0}\) is the identification map in (3.2) and \(\|\cdot\|\) is the Euclidean norm) such that

\[(6.2) \quad |\partial_2 P^{x_0}(x_0)| \geq C_{\text{nice}} \delta/2, \text{ and}\]
\[(6.3) \quad \partial_1 P^{x_0}(x_0) = 0.\]

Here, \(\partial_i = \partial_{x_i}^{x_0}\) for \(i = 1, 2\).

**Claim 6.1.** For any \(S \subset E \cap Q^*\) containing \(x_0\) with \(#(S) \leq k\), there exists \(\phi^S \in C^2(\mathbb{R})\) such that

- \((i) \quad S \subset \text{Graph}(\phi^S; F^S, \omega),\]
- \((ii) \quad \|\phi^S'\|_{C^0(F^S)} \leq C, \text{ and}\]
- \((iii) \quad \|\phi^S''\|_{C^0(F^S)} \leq C\delta^{-1}.\]

The constant \(C\) depends only on \(C_{\text{nice}}\).

**Proof of Claim 6.1.** Let \(S \subset E \cap Q^*\) be such that \(x_0 \in S\) and \(#(S) \leq k\).

Since \(P^{x_0} \in \sigma^\#(x_0, k)\), there exists \(F^S \in C^2(\mathbb{R}^2)\) such that

- \((i) \quad F^S|_S = 0,\]
- \((ii) \quad \|F^S\|_{C^0(\mathbb{R}^2)} \leq 1,\]
- \((iii) \quad \mathcal{J}_{x_0}(F^S) = P^{x_0}.\]
By (6.2), we have
\[ |\partial_2 F^S(x_0)| = |\partial_2 P^{x_0}| \geq C_{\text{nice}} \delta / 2. \]

Now, for all \( x \in Q^* \), we have \( |x_0 - x| \leq 3\delta \). Hence, for all \( x \in Q^* \), by (6.3) and property (ii) of \( F^S \), we have
\[ |\partial_1 F^S(x)| \leq \| F^S \|_{C^1(\mathbb{R}^2)} |x_0 - x| \leq 3\delta; \]

from (6.4), we also have, for all \( x \in Q^* \),
\[ |\partial_2 F^S(x)| \geq |\partial_2 F^S(x_0)| - \| F^S \|_{C^2(\mathbb{R}^2)} |x_0 - x| \geq C_{\text{nice}} \delta / 2 - 3\delta. \]

Therefore, if \( C_{\text{nice}} \) is sufficiently large, the implicit function theorem yields a function \( \phi^S \in C^2(I^S) \) for some open interval \( I^S \) such that \( S \subset \text{Graph}(\phi^S; I^S, \omega) \).

It remains to estimate the derivatives of \( \phi^S \).

We compute the derivatives of \( \phi^S \):
\[
(\phi^S)'(x_0^{(1)}) = \frac{\partial_1 F^S(x)}{\partial_2 F^S(x)}
\]
\[
(\phi^S)''(x_0^{(1)}) = -\left(\frac{\partial_2 F^S(x)}{\partial_2 F^S(x)}\right)^2 \frac{\partial^2 F^S(x)}{\partial_1 F^S(x) \partial_2^2 F^S(x)} - \left(\frac{\partial_1 F^S(x)}{\partial_2 F^S(x)}\right)^2 \frac{\partial_2^2 F^S(x)}{3 \partial_2 F^S(x)^3}.
\]

From (6.5) - (6.8), we conclude that
- \( \| (\phi^S)' \|_{C^1(I^S)} \leq C \), and
- \( \| (\phi^S)'' \|_{C^1(I^S)} \leq C\delta^{-1}. \)

The constant \( C \) depends only on \( C_{\text{nice}} \).

This concludes the proof of the claim.

Next, we define the projections \( \pi_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) by \( \pi_i((x_0^{(1)}, x_0^{(2)})) = x_0^{(i)} \), for \( i = 1, 2 \). By Claim 6.1, we know that \( \pi_i|_{E \cap Q^*} \) is a one-to-one map. Therefore, \( E \cap Q^* \) lies on a graph with respect to the \( x_0^{(1)} \)-axis.

It remains to see that the graph can be taken to have controlled derivatives.

For simplicity of notation, we suppress \( \omega \) in the subscript.

Let \( x_0 = (x_0^{(1)}, x_0^{(2)}) \). We may assume without loss of generality that \( \pi_1(E \cap Q^*) = \left\{ x_0^{(1)}, x_1^{(1)}, \ldots, x_L^{(1)} \right\} \) such that \( x_0^{(1)} < x_1^{(1)} < \cdots < x_L^{(1)} \), where \( L = \#(E \cap Q^*) \). Let \( \pi_2(E \cap Q^*) = \left\{ x_0^{(2)}, x_1^{(2)}, \ldots, x_L^{(2)} \right\} \), where \( x_i^{(2)} = \pi_2 \circ \pi_1^{-1}(x_i^{(1)}) \) for \( i = 1, \ldots, L - 1 \).

Let \( E_j = \left\{ x_j^{(1)}, x_{j+1}^{(1)}, x_{j+2}^{(1)} \right\} \) for \( j = 1, \ldots, L - 3 \). Let \( S_j = \pi_1^{-1}(E_j) \cup \{ x_0 \} \). By Claim 6.1, we know that there exist \( \phi^{S_j} \in C^2(I_j) \) and a constant \( C \), depending only on \( C_{\text{nice}} \), such that
- \( \phi^{S_j}|_{E_j} = \pi_2 \circ \pi_1^{-1} \),
- \( |(\phi^{S_j})'(x^{(1)})| \leq C \) for all \( x^{(1)} \in [x_j^{(1)}, x_{j+2}^{(1)}] \), and
- \( |(\phi^{S_j})''(x^{(1)})| \leq C\delta^{-1} \) for all \( x^{(1)} \in [x_j^{(1)}, x_{j+2}^{(1)}] \).

Therefore, by Theorem 1', we conclude that there exists \( \phi \in C^2(\mathbb{R}) \) such that
- \( \phi|_{E \cap Q^*} = \pi_2 \circ \pi_1^{-1} \),
- \( \| \phi' \|_{C^1(\mathbb{R})} \leq C \), and
- \( \| \phi'' \|_{C(\mathbb{R})} \leq C\delta^{-1} \).

This completes the proof of the lemma.

For future reference, we make the following definition.
Definition 6.1. A pair \((k, C_{\text{n}})\) guarantees good geometry if the following hold:
- \(k \geq 4\); and
- \(C_{\text{n}}\) is sufficiently large such that Lemma 6.1 holds.

Lemma 6.2. Let \((k, C_{\text{n}})\) guarantee good geometry. Let \(Q \in \Lambda^{(k)}_{\text{n}}\). There exists a diffeomorphism \(\Phi \in C^2(\mathbb{R}^2, \mathbb{R}^2)\), depending on \(Q\), such that
(a) \(\Phi(Q) \subset \mathbb{R}^n \setminus \{0\}\),
(b) \(\|\nabla \Phi\|, \|\nabla \Phi^{-1}\| \leq C\), and
(c) \(\|\nabla^2 \Phi\|, \|\nabla^2 \Phi^{-1}\| \leq C\delta_Q^{-1}\),
where \(\|\cdot\|\) denotes the Euclidean norm. The constant \(C\) depends only on \(C_{\text{n}}\).

Proof. We may compose on the right by a rotation \(\omega\) if necessary, and assume \(\omega = 0\). Such rotation will not affect the Euclidean norm. Let \(\phi\) be as in Lemma 6.1. Put
\[
\Phi(x, t) := (s, t - \phi(s)) \quad \text{and} \quad \Phi^{-1}(s', t') := (s', t' + \phi(s')).
\]
They are clearly inverses of each other and are twice continuously differentiable. (a) follows from how we construct \(\phi\). (b) and (c) follow from the derivative estimates of \(\phi\) in Lemma 6.1. \(\square\)

Lemma 6.3. Let \((k, C_{\text{n}})\) guarantee good geometry. Let \(Q \in \Lambda^{(k)}_{\text{n}}\). Then there exists \(x^Q \in Q\) with \(\text{dist}(x^Q, E) \geq c_0\delta_Q\). The constant \(c_0 > 0\) does not depend on \(Q\).

Proof. If \(E \cap \frac{1}{2}Q = \emptyset\), we may pick \(x^Q\) to be the center of \(Q\) and let \(c_0 = 1/4\).

Suppose \(E \cap \frac{1}{2}Q \neq \emptyset\). Fix \(\hat{x} \in E \cap \frac{1}{2}Q\). Write \(\delta := \delta_Q\). There exists universal constant \(c_1 > 0\) such that \(B(\hat{x}, c_1\delta) \subset Q\), where \(B(\hat{x}, c_1\delta)\) is the ball of radius \(c_1\delta\) centered at \(\hat{x}\). Let \(\Phi\) be as in Lemma 6.2. (Again, we may assume \(\omega = 0\).) By (b) of Lemma 6.2, there exists a constant \(c_2 > 0\), depending only on \(C_{\text{n}}\), such that \(B(\Phi(\hat{x}), c_2\delta) \subset \Phi(B(\hat{x}, c_1\delta))\). Recall that \(\Phi(E \cap Q^*) \subset \mathbb{R}^n \setminus \{0\}\). Let \(\hat{x}^Q := \Phi(\hat{x}) + (0, c_2\delta/2)\). Then \(\text{dist}(\hat{x}^Q, \Phi(E \cap Q^*)) \geq c_2\delta/2\). Let \(x^Q = \Phi^{-1}(\hat{x}^Q)\). By (b) of Lemma 6.2 again, \(\text{dist}(x^Q, E \cap Q^*) \geq c_3\delta_Q\) for some \(c_3 > 0\) depending only on \(C_{\text{n}}\). Finally, since \(x^Q \in Q\), \(\text{dist}(x^Q, E \setminus Q^*) \geq \delta/2\). This concludes the lemma. \(\square\)

From now on, let \(0 < \delta \leq 1\) be a constant. Its precise value will not matter. It will be our reference scale.

Lemma 6.4. Let \((k, C_{\text{n}})\) guarantee good geometry. Let \(Q \in \Lambda^{(k)}_{\text{n}}\). Suppose \(\delta_Q = \delta\). Let \(S = E \cap Q^*\). Let \(\Phi\) be as in Lemma 6.2. Let \(f \in C^2_0(S)\). We put \(\Phi_* f := f \circ \Phi^{-1}\). Then
\[
\widehat{C}^{-1} \|\Phi_* f\|_{C^2_\delta(\Phi(S))} \leq \|f\|_{C^2_\delta(S)} \leq \widehat{C} \|\Phi_* f\|_{C^2_\delta(\Phi(S))}.
\]
The constant \(\widehat{C}\) depends only on \(\delta\).

Proof. This follows from the chain rule and the gradient estimates in Lemma 6.2. \(\square\)

7. Proof of Theorem 1

In this section, we prove the finiteness principle for one-dimension. We will use \(x, y\) to denote points on \(\mathbb{R}\) and \(\partial^k\) to denote the \(k\)-th derivative of a single-variable function. When \(k = 1\), we simply write \(\partial\) instead of \(\partial^1\).

Proof of Theorem 1. For \(N \geq 3\), let \(I_1 = (-\infty, x_3], I_2 = [x_2, x_4], \ldots, I_{N-3} = [x_{N-3}, x_{N-1}], \) and \(I_{N-2} = [x_{N-2}, +\infty)\). By assumption, for each \(j\), there exists \(F_j \in C^2_\delta(\mathbb{R})\) with \(F_j|_{I_j} = f\) and
\[
\|F_j\|_{C^2_\delta(\mathbb{R})} \leq 1.
\]
We introduce a partition of unity \(\{\theta_j\}\) that satisfies
Clearly, \( F \in C^2_+ (\mathbb{R}) \), \( F|_E = f \), and
\[
\|F\|_{C^0(\mathbb{R})} \leq 2.
\]

Observe that \((7.1)\) and condition (ii) of \( \{\theta_j\} \) imply
\[
\|F\|_{C^2((-\infty, x_2] \cup [x_{N-1}, +\infty))} \leq 1.
\]

Suppose \( x \in (x_2, x_{N-1}) \). Let \( j \) be the least integer such that \( x \in I_j \). The only partition functions possibly nonzero at \( x \) are \( \theta_j \) and \( \theta_{j+1} \). Since \( \theta_j(x) + \theta_{j+1}(x) \equiv 1 \), we have \( \partial^k \theta_j(x) = -\partial^k \theta_{j+1}(x) \) for \( k = 1, 2 \). Thus,
\[
\partial^k F(x) = \partial^k F_j(x) \theta_j(x) + \partial^k F_{j+1}(x) \theta_{j+1}(x) + \sum_{l=0}^{k-1} \binom{k}{l} \partial^l (F_j - F_{j+1})(x) \partial^{k-l} \theta_j(x).
\]

**Claim 7.1.** For \( l = 0, 1 \) and \( x \in I_j \cap I_{j+1} \),
\[
|\partial^l (F_j - F_{j+1})(x)| \leq \left\{ \begin{array}{ll}
2 |x_{j+2} - x_{j+1}|^{2-l} & \text{if } |x_{j+2} - x_{j+1}| \leq 1 \\
| \partial^l (F_j - F_{j+1})(x)| & \text{if } |x_{j+2} - x_{j+1}| > 1
\end{array} \right.
\]

**Proof of Claim 7.1.** The estimate for the case when \( |x_{j+2} - x_{j+1}| > 1 \) follows from \((7.1)\).

Assume \( |x_{j+2} - x_{j+1}| \leq 1 \). Note that by construction, \( I_j \cap I_{j+1} = [x_{j+1}, x_{j+2}] \).

Observe that \( (F_j - F_{j+1})(x_{j+1}) = (F_j - F_{j+1})(x_{j+2}) = 0 \). By Rolle’s theorem, there exists \( \hat{x}_j \in (x_j, x_{j+1}) \) such that \( \partial (F_j - F_{j+1})(\hat{x}_j) = 0 \). By the fundamental theorem of calculus and triangle inequality,
\[
|\partial (F_j - F_{j+1})(x)| \leq \int_{\hat{x}_j}^{x} |\partial^2 (F_j - F_{j+1})(y)| \, dy
\]
\[
\leq 2 |x_{j+2} - x_{j+1}| \text{ for all } x \in I_j \cap I_{j+1}.
\]

Similar calculations yield the case \( l = 0 \). \(7.7)\) is proven.

From \((7.1)\), \((7.2)\), and \((7.4) - (7.7)\), we conclude that
\[
\|F\|_{C^2(\mathbb{R})} \leq 50.
\]

**Remark 7.1.** The 50 in the last estimate is crude. One can improve the estimate by patching consecutive \( F_j \)'s carefully. See, e.g., “gentle partition of unity” [2, 8].

\(^1\)We may locally construct each \( \theta \) by concatenating polynomials. For instance, on the unit interval \([0, 1]\), we may use \( \theta(x) = 2x^3 - 3x^2 + 1 \) for \( 0 \leq x \leq 1/2 \) and \( \theta(x) = -2x^3 + 3x^2 \) for \( 1/2 < x \leq 1 \).
8. Proof of Theorem 2

In this section, we will temporarily use \( \mathcal{P} \) to denote the vector space of affine polynomials of one variable.

8.1. Whitney field. Let \( S \subset \mathbb{R} \) be nonempty. A Whitney field \( \mathcal{P} \) on \( S \) is a family of polynomials \( \mathcal{P} = (P^x)_{x \in S} \) such that \( P^x \in \mathcal{P} \) for all \( x \in S \). We denote the vector space of all Whitney fields on \( S \) by \( \text{Wh}(S) \). This vector space can be equipped with a seminorm:

\[
\| \mathcal{P} \|_{\text{Wh}(S)} := \max_{x, y \in S, x \neq y, 0 \leq \beta \leq 1} \frac{|\partial^\beta (P^x - P^y)(x)|}{|x - y|^{2 - 2\beta}}.
\]

Lemma 8.1. Let \( f \in C^2_c(S) \) where \( S \subset \mathbb{R} \) is a nonempty finite set. There exists a Whitney field \( \mathcal{P} \in \text{Wh}(S) \) such that

- \( P^x \in \Gamma_+(x, \{x\}, C \|f\|_{C^2_c(S)}) \); and
- \( \|\mathcal{P}\|_{\text{Wh}(S)} \leq C \|f\|_{C^2_c(S)} \).

The constant \( C \) is universal.

Proof. This is a direct consequence of Taylor’s theorem. \( \square \)

The following lemma provides the converse.

Lemma 8.2 (Whitney Extension Theorem for Finite Sets). There exists a constant \( C > 0 \) such that for any finite \( E \subset \mathbb{R} \) and \( \mathcal{P} \in \text{Wh}(E) \) with

- \( P^x \in \Gamma^E_+(x, 0, 1) \) for all \( x \in E \), and
- \( \|\mathcal{P}\|_{\text{Wh}(S)} \leq 1 \),

there exists \( F \in C^2_c(\mathbb{R}) \) with

- \( \|F\|_{C^2(\mathbb{R})} \leq C \); and
- \( \beta_x(F) = P^x \) for each \( x \in E \).

See Lemma 2 in [4].

In light of Lemma 8.1, Lemma 8.2, and Theorem 1, in order to construct a \( C \)-optimal global interpolant in \( C^2_c(\mathbb{R}) \), it suffices to produce a \( C \)-optimal Whitney field over every consecutive three points. We will focus on this task in the next two subsections.

8.2. A computable convex set. In this section, we will use \( \mathcal{P}^r \) to denote the vector space of polynomials of one variable up to degree 2. Let \( \pi : \mathcal{P}^r \to \mathcal{P} \) be the natural projection.

Consider

\[
\Gamma_+^0 := \left\{ P \in \mathcal{P}^r : \begin{array}{l}
|\partial^\beta P(0)| \leq 1 \ \forall |\beta| \leq 2, \\
P(x) + x^2 \geq 0 \ \forall x \in \mathbb{R}, \\
\forall \epsilon > 0, \exists \delta > 0 \ \text{such that} \\
P(x) + \epsilon x^2 \geq 0 \ \forall |x| \leq \delta
\end{array} \right\},
\]

and put

\[
\Gamma_+^{\text{new}}(x, M) := \{ P(\cdot - x) : P \in M \cdot \Gamma_+^0 \}.
\]

It is known [4] that there exist universal constants \( c, C > 0 \) such that

\[
\Gamma_+^0(x, 0, cM) \subset \pi \Gamma_+^{\text{new}}(x, M) \subset \Gamma_+^0(x, 0, CM).
\]

In fact, the argument is the same as the one employed in Lemma 4.3.

Thanks to \( (8.2) \), we only need to consider candidates from \( M \cdot \Gamma_+^0 \) for suitable \( M \) to find a \( C \)-optimal Whitney field.
Let \( P \in M \cdot \Gamma^0_+ \). Write \( P = ax^2 + bx + c \). The nonnegative criteria in (8.1), after dilation, read
\[
\begin{align*}
0 \leq c & \leq M \\
|b| & \leq M \\
|2a| & \leq M \\
b^2 - 4(a + M)c & \leq 0 \\
\text{if } c = 0, \text{ then } a \geq 0, b = 0
\end{align*}
\]

We immediately have (8.3) where \( \minimize \)
\[
Q = Q(P) = Q_1(P) + Q_2(P)
\]
subject to the constraints specified in (NC) for a suitable \( M \). Here,
\[
\begin{align*}
Q_1(P) & = \sum_{i=1}^3 \sum_{k=0}^2 \frac{|\delta^k(P_1 - P_j)(x_i)|^2}{2(2-k)} \\
Q_2(P) & = \sum_{i=1}^3 \sum_{k=0}^2 |\delta^kP_i(x_i)|^2.
\end{align*}
\]

Put
\[
\begin{align*}
\delta_{ij} & = x_i - x_j, \\
\Delta_{ij} & = f(x_i) - f(x_j), \text{ and} \\
X & = (a_1, ..., a_3, b_1, ..., b_3)\).
\end{align*}
\]

We can view \( Q, Q_1, Q_2 \) as a function of \( X \), given by
\[
\begin{align*}
Q_1(X) & = \sum_{i \neq j} \left\{ (-a_j \delta_{ij}^2 - b_j \delta_{ij} + \Delta_{ij})^2 \delta_{ij}^2 + (-2a_j \delta_{ij} + (b_j - b_j))^2 \delta_{ij}^2 + 4(a_i - a_j)^2 \right\} \\
Q_2(X) & = \sum_{i=1}^3 \left\{ 4a_i^2 + b_i^2 + (f(x_i))^2 \right\}.
\end{align*}
\]

Furthermore, we can write
\[
\begin{align*}
Q_1(X) & = X^tSX + V^tX + \kappa \\
Q_2(X) & = X^tS_0X,
\end{align*}
\]
where \( S, S_0 \) are 6 \times 6 symmetric matrices whose entries will be determined shortly, \( V \) is a column vector, and \( \kappa \) is a positive quantity. They are all determined by the initial data. The constraints (NC) can be put into the form
\[
C_m(X) \leq 0.
\]

For each \( m \), \( C_m \) is at most quadric in its argument, and the matrix associated with its quadratic form (if exists) is positive semidefinite. We will show that
\[
\begin{align*}
\minimize \quad & Q(X) \\
\text{subject to} \quad & C_m(X) \leq 0 \text{ for } m = 1, ...
\end{align*}
\]
is a semidefinite programming problem, which is known to be solvable. See, e.g., [5].

It suffices to verify that both \( S_0 \) and \( S \) are positive semidefinite. We will do this by hand.

We immediately have \( S_0 = \text{diag}(4, 4, 4, 1, 1, 1) \). Thus, \( S_0 \) is positive definite.
Theorem 2.8.4. E Lemma 6.3 applies. Suppose $P, Q, \ldots$ there exists our estimate.

\[ \|A \| \text{ where } A \leq M, \text{Lemma 9.1.} \]

Proof. By re-scaling, we may assume that $B - M \leq M', \mathcal{M}^{-1} \mathcal{M}$ are positive definite by computing the determinant of each of their principle minors. For concreteness, we provide the expression of the determinant of $S/\mathcal{A}$ here:

\[
\det S/\mathcal{A} = \frac{5s^2T\left(85428(s^6 + t^6) + 256284(s^5t + st^5) + 496439(s^4t^2 + s^2t^4) + 565738s^3t^3\right)}{2312(s + t)^4}.
\]

The function above is strictly positive away from the axes $s = 0$ and $t = 0$.

By Schur’s complement criterion, $M_\mathcal{A}$ is positive definite.

8.4. Proof of Theorem 2.

Proof. Write $E = \{x_1, \ldots, x_N\}$ with $x_1 < \cdots < x_N$. For each $E_j := \{x_j, x_{j+1}, x_{j+2}\}, 1 \leq j \leq N-2$, let $P_j$ be a $C$-optimal Whitney field over $E_j$ as found in Section 8.3 by solving (8.4). By Lemma 8.2, for each $j, P_j$ extends to $F_j \in C^2_k(\mathbb{R})$ that is a $C$-optimal nonnegative interpolant of $f$ over $E_j$. Define $\{\theta_j\}$ as in Section 7, and define $\mathcal{E}_\mathcal{A}(f)$ as in (7.3). By Theorem 1, $\mathcal{E}_\mathcal{A}(f)$ is a $C$-optimal nonnegative interpolant of $f$ over $E$. □

9. Proof of Theorem 3

First we need a local version of the two dimensional finiteness principle which takes into account a prescribed jet based at a point sufficiently far away from the set $E$. We will use these jets as “transitions” in our estimate.

Recall Definition 6.1.

Lemma 9.1. Let $(k, C_{\text{nice}})$ guarantee good geometry. Let $Q \in \Lambda^{(k)}_\text{nice.}$ Let $x^0 \in Q$ with dist $(x^0, E) \geq c_0\delta_Q$ with $c_0$ as in Lemma 6.3. Suppose $P \in \Gamma^4_k(x^0, k, 1)$. Then there exists $F_{x^0, P} \in C^2_k(\mathbb{R}^2)$ with

(a) $F_{x^0, P}|_{E \cap Q^*} = f$.

(b) $\|F_{x^0, P}\|_{C^2(\mathbb{R}^2)} \leq C$, and

(c) $\mathcal{F}_{x^0}(F_{x^0, P}) \approx P$.

Proof. By re-scaling, we may assume that $\delta := \delta_Q = \hat{\delta}$. Let $\Phi$ be as in Lemma 6.2 (again we may assume $\omega = 0$).

By assumption, $\Gamma^4_k(x^0, k, 1) \neq \emptyset$.

Let $S_1 \in \Phi(E \cap Q^*) \subset \mathbb{R} \times [0] \approx \mathbb{R}$ be such that $\#(S_1) \leq k$. Recall that $k \geq 4$ by assumption. Put $S_2 = \Phi^{-1}(S_1)$. Since $P \in \Gamma^4_k(x^0, k, 1)$, there exists $F_{S^2} \in C^2_k(\mathbb{R}^2)$ such that $\mathcal{F}_{x^0}(F_{S^2}) = P$ with $F_{S^2}|_{S_2} \approx f$, and $\|F_{S^2}\|_{C^2(\mathbb{R}^2)} \leq C$. Put $f^{S^1} = F_{S^2} \circ \Phi^{-1}|_{E \times [0]} \in C^2_k(\mathbb{R})$. Then $f^{S^1}|_{S_1} = f \circ \Phi^{-1}$, and $\|f^{S^1}\|_{C^2(\mathbb{R})} \leq C$.

By Theorem 1, there exists $F_1 \in C^2_k(\mathbb{R}^2)$ with $F_1|_{E \cap Q^*} = f \circ \Phi^{-1}$, and $\|F_1\|_{C^2(\mathbb{R})} \leq C$. (Theorem 1 applies since 3 is the finiteness constant in one dimension, and in our case, $k \geq 4$.)

Put $F_2(s, t) = F_1(s)$ and $F = F_2 \circ \Phi$. Then $F \in C^2_k(\mathbb{R}^2)$, $F|_{E \cap Q^*} = f$, and

\[\|F\|_{C^2(\mathbb{R}^2)} \leq C\]
Recall that our choice of \( x^Q \) satisfies \( \text{dist}(x^Q, E) \geq c_0 \delta_Q = c_0 \hat{c} =: c_1 \). Let \( B := B(x^Q, c_1/2) \) denote the closed ball of radius \( c_1/2 \) centered at \( x^Q \). Then
\[
B \cap E = \emptyset.
\]

Let \( \chi \) be a \( C^2 \) cutoff function that satisfies
(i) \( \text{supp} \ (\chi) \subset B \),
(ii) \( \| \chi \|_{C^2(\mathbb{R}^2)} \leq C \), the constant \( C \) depends on \( c_1 \), and
(iii) \( \chi = 1 \) in a neighborhood of \( x^Q \).

Recall the map \( R_B \) defined in Lemma 4.3. Define
\[
F_{Q,P} = (1 - \chi) F + \chi R_B(P).
\]

(a) follows from property (i) of \( \chi \), (9.2), and how we construct \( F \). (b) and (c) follow from properties (ii) and (iii) of \( \chi \), (9.1), and Lemma 4.3.

\( \square \)

Before proceeding to the proof of Theorem 3, we make a brief comment on the finiteness constant \( 17 \). Lemma 4.2 and Lemma 5.3 state that jets of \((4k + 1)\)-point interpolants based in neighboring cubes from \( \Lambda_{\text{nice}}^{(k)} \) are compatible in the Whitney sense (see (5.3)); Lemma 6.1 states that the geometry of data points in each cube of \( \Lambda_{\text{nice}}^{(k)} \) is sufficiently nice when \( k \geq 4 \); Lemma 9.1 states that in such case, a local version of the extension problem is readily solved. Hence, if we pick \( k = 4 \), we may use the jets of \( 4 \cdot 4 + 1 = 17 \)-point interpolants (if they exist) to guarantee compatibility of nearby local extensions. We will examine such compatibility in the following proof.

**Proof of Theorem 3.** Set \( k = 4 \). Pick \( C_{\text{nice}} \) so that \((4, C_{\text{nice}})\) guarantees good geometry.

By Lemma 5.1 \( \Lambda_{\text{nice}}^{(4)} = \{Q_i\} \) is a CZ covering of \( \mathbb{R}^2 \).

By our hypothesis, \( \Gamma^\sharp(x, 17, 1) \neq \emptyset \) for any \( x \in \mathbb{R}^2 \).

We distinguish three types of cubes.

**Type 1** Suppose \( E \cap Q_i^* \neq \emptyset \). Pick \( \hat{x}_i = x_i^Q \in Q_i \) with \( \text{dist}(\hat{x}_i, E) \geq c_0 \delta_i \). Pick \( P_i \in \Gamma^\sharp(\hat{x}_i, 17, 1) \) and set \( F_i = F_{Q_i,P_i} \), where \( F_{Q_i,P_i} \) is as in Lemma 9.1.

**Type 2** Suppose \( E \cap Q_i^* = \emptyset \) but \( \delta_i < 1 \). This means that \( E \cap (Q_i^+)^* \neq \emptyset \). For some arbitrary \( x \in E \cap (Q_i^+)^* \), there exists \( Q_j \in \Lambda_{\text{nice}}^{(k)} \) of type 1 such that \( x \in E \cap Q_j^* \). Put \( \hat{x}_i = \hat{x}_j \), where \( \hat{x}_j \) is picked in the previous step. Pick \( P_j \in \Gamma^\sharp(\hat{x}_j, 17, 1) \), and set \( F_i = F_{Q_j,P_j} \).

**Type 3** Suppose \( E \cap Q_i^* = \emptyset \) and \( \delta_i = 1 \). Set \( F_i = 0 \).

To wit, the local interpolants only receive information from Type 1 cubes.

By Lemma 9.1, \( F_i \in C^2_\sharp(\mathbb{R}^2) \), \( F_i|_{E \cap Q_i^*} = f \), and
\[
||F_i||_{C^2(\mathbb{R}^2)} \leq C.
\]

**Claim 9.1.** If \( Q_i \leftrightarrow Q_j \), then for each \( x \in \frac{2}{3} Q_i \cup \frac{2}{3} Q_j \) and \( 0 \leq |\theta| \leq 1 \),
\[
|\partial^\theta (F_i - F_j)(x)| \leq C \delta_i^{2-|\theta|}.
\]

The constant \( C \) is universal.

**Proof of Claim 9.1.** Temporarily fix \( x \in \frac{2}{3} Q_i \cup \frac{2}{3} Q_j \) for \( Q_i \leftrightarrow Q_j \).

Assume that either \( Q_i \) or \( Q_j \) is of type 3, then (9.4) follows from (CZ1) and (9.3).

Suppose neither \( Q_i \) nor \( Q_j \) is of type 3. Write \( \delta = \delta_i \). Thanks to (CZ1) and our choice of \( \hat{x}_i \), \( \nu = i, j \), we have \( |\hat{x}_i - x|, |\hat{x}_i - \hat{x}_j| \leq C \delta \).

Recall that \( J_{\hat{x}_i} F_{\nu} = P_{\nu} \in \Gamma_\sharp^\#(\hat{x}_\nu, 17, 1) \) for \( \nu = i, j \).
By Taylor’s theorem,
\begin{equation}
\left| \partial^\beta (F_v - P_v) (x) \right| \leq C \delta^{2-|\beta|} \text{ for } v = i, j.
\end{equation}

By Lemma 5.3,
\begin{equation}
\left| \partial^\beta (P_i - P_j) (x) \right| \leq C \delta^{2-|\beta|}.
\end{equation}

Now, (9.4) follows from (9.5) and (9.6).

Let \{\theta_i\} be a partition of unity that is CZ-compatible with \(\Lambda^{(4)}\). Define
\[ F(x) := \sum_i F_i(x)\theta_i(x). \]
It is clear that \( F \geq 0, F|_E = f \), and \( F \) is twice continuously differentiable. For \( |\alpha| \leq 2 \) and \( x \in Q_i \),
\begin{equation}
\partial^\alpha F(x) = \sum_j \partial^\alpha F_j(x)\theta_j(x) + \sum_{Q_i \supset Q_j \cap 0 \leq \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \partial^\beta (F_j - F_i) (x) \partial^\alpha \theta_j(x).
\end{equation}
Applying (3.3), (3.4), (9.3), and (9.4) to (9.7), we can conclude
\[ \| F \|_{C^2(\mathbb{R}^2)} \leq C^\#. \]

10. Proof of Theorem 4

First we prove an analogue of Lemma 9.1. Recall Definition 6.1.

Lemma 10.1. Let \((k, C_{\text{nice}})\) guarantee good geometry. Let \( Q \in \Lambda^{(k)}_{\text{nice}} \). Pick \( x^Q \in Q \) such that \( \text{dist} (x^Q, E) \geq c_0 \delta_Q \) with \( c_0 \) as in Lemma 6.3. Let \( M > 0 \) be such that there exists \( F \in C^2_+(\mathbb{R}^2) \) such that \( F|_E = f \) and \( \| F \|_{C^2(\mathbb{R}^2)} \leq M \). Then there exists a mapping
\[ E^Q_+ : C^2_+(E) \times \Gamma^\#_+ (x^Q, 17, M) \to C^2_+ (\mathbb{R}^2), \quad (f, P) \mapsto E^Q_+ (f, P) \]
such that
\begin{enumerate}[(a)]
\item \( E^Q_+ (f, P) \Big|_{E \cap Q} = f \),
\item \( \| E^Q_+ (f, P) \|_{C^2(\mathbb{R}^2)} \leq CM \), and
\item \( T_{\Phi}(E^Q_+ (f, P)) = P \).
\end{enumerate}

Proof. By assumption, \( \Gamma^\#_+ (x^Q, 17, M) \neq \emptyset \). Fix \( P \in \Gamma^\#_+ (x^Q, 0, M) \). By definition, \( \| f \|_{C^2_+(E)} \leq M \).

By a re-scaling argument, we may assume \( \delta_Q = \tilde{c} \). Let \( \Phi \) be as in Lemma 6.2 (assuming \( \omega = 0 \)). Then \( \Phi (E \cap Q^+) \subset \mathbb{R} \times \{ 0 \} \). We identify the latter set with the real line \( \mathbb{R} \). Put \( \Phi_* f := f \circ \Phi^{-1} \), which we can think of as defined on \( \Phi (E \cap Q^+) \subset \mathbb{R} \). Let \( \vec{E}_+ \) be as in Theorem 2. Then
\begin{itemize}
\item \( \vec{E}_+ (\Phi_* f) \in C^2_+(\mathbb{R}) \),
\item \( \vec{E}_+ (\Phi_* f) \Big|_{\Phi (E \cap Q^+)} = \Phi_* f \),
\item \( \| \vec{E}_+ (\Phi_* f) \|_{C^2(\mathbb{R})} \leq C \| \Phi_* f \|_{C^2_+(\Phi (E \cap Q^+))}. \)
\end{itemize}
The trace norm can be understood as either one or two dimensional.

Put \( \vec{T} (s, t) := \vec{E}_+ (\Phi_* f) (s) \), and \( T := \Phi^* \vec{T} := \Phi \circ T \). Lemma 6.2 and Lemma 6.4 imply
\begin{enumerate}[(i)]
\item \( F \in C^2_+ (\mathbb{R}^2) \),
\item \( F|_{E \cap Q^+} = f \), and
\item \( \| F \|_{C^2(\mathbb{R}^2)} \leq C \| f \|_{C^2_+(E \cap Q^+)} \leq CM \).
\end{enumerate}
Recall that our choice of $x^Q$ satisfies $\text{dist} (x^Q, E) \geq c_0 \delta_Q = c_0 \hat{c} =: c_1$. Let $B := B(x^Q, c_1/2)$ denote the closed ball of radius $c_1/2$ centered at $x^Q$. Then

\begin{equation}
B \cap E = \emptyset.
\end{equation}

Let $\chi$ be a $C^2$ cutoff function that satisfies

- (i) $\text{supp} (\chi) \subset B$,
- (ii) $\| \chi \|_{C^2 (\mathbb{R}^2)} \leq C$, the constant $C$ depends on $c_1$, and
- (iii) $\chi = 1$ in a neighborhood of $x^Q$.

Recall the map $R_B$ defined in Lemma 4.3. Define

\[ E^Q_\chi (f, P) = (1 - \chi) F + \chi R_B (P). \]

$E^Q_\chi (f, P)$ is nonnegative and twice continuously differentiable. (a) follows from property (i) of $\chi$ and (10.1). (b) and (c) follow from properties (i) - (iii) of $F$, properties (ii) and (iii) of $\chi$, and Lemma 4.3.

The proof of Theorem 3 actually provides a construction for an extension operator. We will provide the details below for completeness.

**Proof of Theorem 4.** Theorem 3 produces a number $M \leq C \| f \|_{C^2_c (E)}$, where $C$ is some universal constant, such that $\Gamma^4_k (x, E, M) \neq \emptyset$. Fix $k = 4$ and let $(4, C_{\text{nice}})$ guarantee good geometry.

By Lemma 5.1, $\Lambda^{(4)}_{\text{nice}} = \{ Q_i \}$ is a CZ covering of $\mathbb{R}^2$.

We consider three types of cubes.

**Type 1** Suppose $E \cap Q_i^+ \neq \emptyset$. Pick $\hat{x}_i = x^Q_i \in Q_i$ with $\text{dist} (\hat{x}_i, E) \geq c_0 \delta_i$. Pick $P_i \in \Gamma^4_k (\hat{x}_i, 17, M)$ and set $\bar{E}^i_\chi (f) = E^Q_\chi (f, P_i)$, where $E^Q_\chi$ is as in Lemma 10.1.

**Type 2** Suppose $E \cap Q_i^+ = \emptyset$ but $\delta_i < 1$. This means that $E \cap (Q_i^+)^c \neq \emptyset$. For some arbitrary $x \in E \cap (Q_i^+)^c$, there exists $Q_j \in \Lambda^{(4)}_{\text{nice}}$ of type 1 such that $x \in E \cap Q_j^+$. Put $\hat{x}_j = \hat{x}_j$ where $\hat{x}_j$ is picked in the previous step. Pick $P_i \in \Gamma^4_k (\hat{x}_j, 17, M)$, and set $\bar{E}^i_\chi (f) = E^Q_\chi (f, P_i)$.

**Type 3** Suppose $E \cap Q_i^+ = \emptyset$ and $\delta_i = 1$. Set $\bar{E}^i_\chi (f) = 0$.

To wit, we only extend from Type 1 cubes.

By Lemma 10.1, $\bar{E}^i_\chi (f) \in C^2_{\text{c}} (\mathbb{R}^2)$, $\bar{E}^i_\chi (f) \big|_{E \cap Q_i^+} = f$, and

\begin{equation}
\| \bar{E}^i_\chi (f) \|_{C^2_c (\mathbb{R}^2)} \leq CM.
\end{equation}

The same proof for Claim 9.1 in Section 9 shows the following

**Claim 10.1.** If $Q_i \leftrightarrow Q_j$, then for each $x \in \frac{9}{8} Q_i \cup \frac{9}{8} Q_j$ and $0 \leq |\beta| \leq 1$,

\begin{equation}
| \partial^\beta \left( \bar{E}^j_\chi (f) - \bar{E}^i_\chi (f) \right) (x) | \leq CM \delta_i^{2-|\beta|}.
\end{equation}

The constant $C$ is universal.

Let $\{ \theta_i \}$ be a partition of unity that is CZ-compatible with $\Lambda^{(4)}_{\text{nice}}$. Define

\begin{equation}
E_\chi (f) := \sum_i \bar{E}^i_\chi (f) (x) \theta(x).
\end{equation}

It is clear that $E_\chi (f)$ is a nonnegative $C^2$-interpolant of $f$ over $E$. For $|\alpha| \leq 2$ and $x \in Q_i$, we can write

\begin{equation}
\partial^\alpha E_\chi (f) (x) = \sum_j \partial^\alpha \bar{E}^j_\chi (f) (x) \theta_j (x) + \sum_{Q_i \leftrightarrow Q_j} \sum_{0 \leq |\beta| \leq |\alpha|} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \left( \bar{E}^j_\chi (f) - \bar{E}^i_\chi (f) \right) (x) \partial^\beta \theta_j (x).
\end{equation}
Applying (3.3), (3.4), (9.3), and (10.3) to estimate (10.5), we have
\[ \|E_\star(f)\|_{C^2(\mathbb{R}^2)} \leq CM. \]
This proves optimality and concludes the proof of the theorem. \qed

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