HOLOMORPHIC VECTOR FIELDS AND PERTURBED EXTREMAL KÄHLER METRICS

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Abstract. We prove a theorem which asserts that the Lie algebra of all holomorphic vector fields on a compact Kähler manifold with a perturbed extremal metric has the structure similar to the case of an unperturbed extremal Kähler metric proved by Calabi.

1. Introduction

Let $M$ be a compact symplectic manifold with symplectic form $\omega$. On the space $\mathcal{J}$ of all $\omega$-compatible complex structures $J$ there is a natural symplectic form with respect to which the scalar curvature $S(J)$ of the Kähler manifold $(M, \omega, J)$ becomes a moment map for the action of the group of all Hamiltonian diffeomorphisms of $(M, \omega)$ acting on $\mathcal{J}$ (c.f. [3], [4]). This means that the problem of finding extremal Kähler metrics can be set in the framework of stability in the sense of geometric invariant theory. It was shown in [7] that, perturbing the symplectic form on $\mathcal{J}$ and the scalar curvature incorporating with the higher Chern classes and with a small real parameter $t$, the perturbed scalar curvature $S(J, t)$ becomes a moment map with respect to the perturbed symplectic form on $\mathcal{J}$. Note that the unperturbed scalar curvature is the trace of the first Chern class. See section 2 for the precise definitions.

Recall that a Kähler metric $g$ is called an extremal Kähler metric if the $(1,0)$-part of the gradient vector field of the scalar curvature $S$

$$\text{grad}^g S = g^{ij} \frac{\partial S}{\partial \bar{z}^i} \frac{\partial}{\partial z^j}$$

is a holomorphic vector field. Extremal Kähler metrics are critical points of two functionals. One is the so-called the Calabi functional. This is a functional $\Psi$ on the space $\mathcal{K}_{\omega_0}$ of all Kähler forms in a fixed de Rham class $\omega_0$ with fixed complex structure $J$. If $\omega \in \mathcal{K}_{\omega_0}$ and $S(\omega)$ denotes the scalar curvature of $\omega$ then

$$\Psi(\omega) = \int_M S(\omega)^2 \omega^m$$

where $m = \dim \mathbb{C} M$. Calabi originally defined extremal Kähler metrics to be the critical points of $\Psi$. The other functional $\Phi$ is defined on $\mathcal{J}$. If $S(J)$ denotes the scalar curvature of the Kähler manifold $(M, \omega, J)$ for $J \in \mathcal{J}$ then

$$\Phi(J) = \int_M S(J)^2 \omega^m.$$
It is easy to see that the extremal Kähler metrics are exactly the critical points of \( \Phi \) from the fact that the scalar curvature is the moment map on \( J \) for the action of Hamiltonian diffeomorphisms as mentioned above.

Inspired by a work of Bando \([1]\) the author defined in \([7]\) perturbed extremal Kähler metrics as follows: the Kähler metric \( g \) for \((M, \omega, J)\) is called a perturbed extremal Kähler metric if the \((1,0)\)-part of the gradient vector field
\[
\text{grad}' S(J,t) = g^{ij} \frac{\partial S(J,t)}{\partial z^j} \frac{\partial}{\partial z^i}
\]
is a holomorphic vector field. From the fact that \( S(J,t) \) becomes a moment map on \( J \) with respect to the perturbed symplectic structure, one can see that the critical points of the functional
\[
\Phi(J) = \int_M S(J,t)^2 \omega^m
\]
are \( J \)'s for which the Kähler metric of \((M, \omega, J)\) is a perturbed extremal Kähler metric. However it is not true for \( t \neq 0 \) that perturbed extremal Kähler metrics are the critical points of the functional \( \Psi \) on \( K_{\omega_t} \) defined by
\[
\Psi(\omega) = \int_M S(\omega, t)^2 \omega^m
\]
where \( S(\omega, t) \) is the perturbed scalar curvature of \((M, \omega, J)\), see Remark 3.3 in \([7]\). This is the significant difference between the perturbed case and the unperturbed case.

In \([9]\) Xiaowei Wang explains how one gets the decomposition theorem of Calabi \([2]\) for the structure of the Lie algebra of all holomorphic vector fields on compact Kähler manifolds with extremal Kähler metrics in the finite dimensional setting of the framework of the moment maps, see also \([6]\). On the other hand Lijing Wang \([8]\) explains how one gets the Hessian formulae for the Calabi functional and the functional \( \Phi \) in the finite dimensional setting of the framework of moment maps. Recall that the Hessian formula for the Calabi functional plays the key role for the proof of Calabi’s decomposition theorem of the Lie algebra of all holomorphic vector fields on compact Kähler manifolds with extremal Kähler metrics. Because of the above mentioned difference between the perturbed case and the unperturbed case, one can not expect that the same proof as the unperturbed case by Calabi can be applied to the perturbed case. The purpose of this paper is to see L.-J. Wang’s finite dimensional arguments provide us a rigorous proof of Calabi’s decomposition theorem for compact Kähler manifolds with perturbed extremal Kähler metrics. Thus we obtain a similar statement of the decomposition theorem:

**Theorem 1.1.** Let \( M \) be a compact Kähler manifold with a perturbed extremal Kähler metric. Let \( \mathfrak{h}(M) \) be the Lie algebra of all holomorphic vector fields and \( \mathfrak{k} \) be the real Lie algebra of all Killing vector fields of \( M \). Then

(a) \( \mathfrak{h}_0(M) := \mathfrak{k} \otimes \mathbb{C} \) is the maximal reductive subalgebra of \( \mathfrak{h}(M) \).

(b) The \((1,0)\)-part of the gradient vector field
\[
\text{grad}' S(J,t) = g^{ij} \frac{\partial S(J,t)}{\partial z^j} \frac{\partial}{\partial z^i}
\]
of \( S(J,t) \) belongs to the center of \( \mathfrak{h}_0(M) \).
(c) \( h(M) \) has the structure of semi-direct decomposition

\[
h(M) = h_0(M) + \sum_{\lambda \neq 0} h_\lambda(M)
\]

where \( h_\lambda(M) \) is the \( \lambda \)-eigenspace of the adjoint action of \( \text{grad}' S(J,t) \).

We will follow the arguments of L.-J. Wang almost word for word.

Throughout this paper Hermitian inner products are anti-linear in the first component and linear in the second component.

2. Perturbed extremal Kähler metric

Let \( M \) be a compact symplectic manifold of dimension \( 2m \) with symplectic form \( \omega \), \( J \) the space of all \( \omega \)-compatible complex structures on \( M \). Then for each \( J \in \mathcal{J} \), \((M,\omega,J)\) becomes a Kähler manifold. For a pair \((J,t)\), \( t \) being a small real number, we define a smooth function \( S(J,t) \) on \( M \) by

\[
S(J,t) \omega^m = c_1(J) \wedge \omega^{m-1} + tc_2(J) \wedge \omega^{m-2} + \cdots + t^{m-1}c_m(J)
\]

where \( c_i(J) \) is the \( i \)-the Chern form defined by

\[
\det(I + \frac{i}{2\pi}t(\Theta)) = 1 + tc_1(J) + \cdots + t^m c_m(J),
\]

\( \Theta \) being the curvature form with respect to \( \omega \). Note that we use \( S(J,t) \) in place of \( S(J,T)/2\pi m \) in [7] to avoid clumsy constant \( 1/2\pi m \).

**Definition 2.1.** The Kähler metric \( g \) of the Kähler manifold \((M,J,\omega)\) is called a \( t \)-perturbed extremal Kähler metric or simply perturbed extremal metric if

\[
\text{grad}' S(J,t) = \sum_{i,j=1}^{m} g^{ij} \frac{\partial S(J,t)}{\partial z^j} \frac{\partial}{\partial z^i}
\]

is a holomorphic vector field.

The following was proved in [7], Proposition 3.2.

**Proposition 2.2.** The critical points of the functional \( \Phi \) on \( \mathcal{J} \) defined by

\[
\Phi(J) = \int_M S(J,t)^2 \omega^m
\]

are the perturbed extremal Kähler metrics.

The proof of this proposition essentially follows from the fact that the perturbed scalar curvature \( S(J,t) \) gives the moment map for the action of the group of Hamiltonian diffeomorphisms with respect to a perturbed symplectic structure on \( \mathcal{J} \). This perturbed symplectic structure is described as follows. The tangent space of \( \mathcal{J} \) at \( J \) is identified with a subspace of \( \text{Sym}(\otimes^2 T^{\ast \ast} M) \). For a small real number \( t \), we define an Hermitian structure on \( \text{Sym}(\otimes^2 T^{\ast \ast} M) \) by

\[
(\nu,\mu)_t = \int_M mc_m(\mathcal{F}_{jk} \mu^j_I \nu^k_I \frac{\sqrt{-1}}{2\pi} dz^j \wedge dz^k, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \cdots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta)
\]

for \( \mu \) and \( \nu \) in the tangent space \( T_J \mathcal{J} \), where \( c_m \) is the polarization of the determinant viewed as a \( GL(m,\mathbb{C}) \)-invariant polynomial, i.e. \( c_m(A_1,\cdots,A_m) \) is the coefficient of \( ml t_1 \cdots t_m \) in \( \det(t_1 A_1 + \cdots + t_m A_m) \), where \( I \) denotes the identity matrix and \( \Theta = \overline{\partial}(g^{-1}\overline{\partial}g) \) is the curvature form of the Levi-Civita connection, and
where $u_{jk}\mu^j_i$ should be understood as the endomorphism of $T^*_jM$ which sends $\partial/\partial z^j$ to $u_{jk}\mu^j_i\partial/\partial z^i$. When $t = 0$, (5) gives the usual $L^2$-inner product. The perturbed symplectic form $\Omega_{J,t}$ at $J \in \mathcal{J}$ is then given by

$$\Omega_{J,t}(\nu, \mu) = \Re(\nu, -\Im(\mu)),$$

$$= \Re \int_M m c_m(\tau_{jk} \sqrt{-1}\mu^j_i \sqrt{-1}/2\pi d\bar{z}^k \wedge d\bar{z}^i, \omega \otimes I + \sqrt{-1}/2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1}/2\pi t\Theta),$$

where $\Re$ means the real part. In [7] we proved the following:

**Theorem 2.3** ([7]). If $\delta J = \mu$ then

(7) $$\delta \int_M u \cdot S(J, t) \omega^m = \Omega_{J,t}(2\sqrt{-1}\nabla''u, \mu).$$

Namely the perturbed scalar curvature $S(J, t)$ gives a moment map with respect to the perturbed symplectic form $\Omega_{J,t}$ for the action of the group of Hamiltonian diffeomorphisms on $\mathcal{J}$.

Now we can prove Proposition 2.2. From (7) we have

(8) $$\delta \int_M S(J, t)^2 \omega^m = 2 \int_M S(J, t) \delta S(J, t) \omega^m = 2 \Omega_{J,t}(2\sqrt{-1}\nabla''S(J, t), \mu).$$

This shows that $J$ is a critical point if and only if

(9) $$\nabla''\text{grad}^{L} S(J, t) = 0,$$

i.e. the Kähler metric of $(M, \omega, J)$ is a perturbed extremal Kähler metric.

Let $\mathfrak{g}$ be the complexification of the Lie algebra of the group of Hamiltonian diffeomorphisms. Then $\mathfrak{g}$ is simply the set of all complex valued smooth functions $u$ with the normalization

$$\int_M u \omega^m = 0$$

with the Lie algebra structure given by the Poisson bracket. The infinitesimal action of $u$ on $\mathcal{J}$ is given by $2i\sqrt{-1}\nabla''u$, see Lemma 10 in [3] or Lemma 2.3 in [7]. Define $L : \mathcal{C}^\infty(M) \otimes \mathbb{C} (\cong \mathfrak{g}) \to \mathcal{C}^\infty(M) \otimes \mathbb{C}$ by

(10) $$(v, Lu)_L = (\nabla''v, \nabla''u)_L$$

$$= \int_M m c_m(v_{jk} u^j_i \sqrt{-1}/2\pi d\bar{z}^k \wedge d\bar{z}^i, \omega \otimes I + \sqrt{-1}/2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1}/2\pi t\Theta).$$

More explicitly $L$ is expressed as

(11) $$Lu = m c_m(u_{jk} \sqrt{-1}/2\pi d\bar{z}^k \wedge d\bar{z}^i, \omega \otimes I + \sqrt{-1}/2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1}/2\pi t\Theta) \omega^m.$$

We define $\mathcal{L} : \mathcal{C}^\infty(M) \otimes \mathbb{C} \to \mathcal{C}^\infty(M) \otimes \mathbb{C}$ by $\mathcal{L}u := \mathcal{L}u_L$. Then $\mathcal{L}$ satisfies

(12) $$(v, \mathcal{L}u)_L = (\nabla''v, \nabla''u)_L$$

$$= \int_M m c_m(u_{jk} \sqrt{-1}/2\pi d\bar{z}^k \wedge d\bar{z}^i, \omega \otimes I + \sqrt{-1}/2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1}/2\pi t\Theta).$$
\( \mathcal{L}u = mc_m(u_{jk} \frac{\sqrt{-1}}{2\pi} dz^k \wedge dz^\ell, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta, \ldots, \omega \otimes I + \frac{\sqrt{-1}}{2\pi} t\Theta) / \omega^m. \)

**Lemma 2.4.** If \( v \) is a real smooth function and \( \delta J = \nabla'' \nabla'' v \) then
\[
\delta S(J, t) = Lv + \mathcal{L}v.
\]

**Proof.** Let \( u \) be also a real smooth function. Then by (7)
\[
\int_M u \delta S(J, t) \omega^m = \Re(2\sqrt{\mp} \nabla'' \nabla'' u, \sqrt{\mp} \mu)_t
\]
\[
= (\nabla'' \nabla'' u, \nabla'' \nabla'' v)_t + (\nabla'' \nabla'' v, \nabla'' \nabla'' u)_t
\]
\[
= (u, Lv)_{L^2} + (u, \mathcal{L}v)_{L^2}.
\]

\( \square \)

**Lemma 2.5.** Let \( u \) and \( v \) be real smooth functions and put \( X_u = 2\sqrt{\mp} \nabla'' \nabla'' u \) and \( X_v = 2\sqrt{\mp} \nabla'' \nabla'' v \). Then we have
\[
\Omega_{J,t}(X_u, X_v) = \langle \{u, v\}, S(J, t) \rangle_{L^2}.
\]

**Proof.** Consider \( X_u \) and \( X_v \) as the infinitesimal action of real Hamiltonian functions \( u \) and \( v \) on \( J \). Since \( S(J, t) \) gives an equivariant moment map
\[
\int_M u S(\sigma J, t) \omega^m = \int_M (\sigma^{-1} u) S(J, t) \omega^m
\]
for a Hamiltonian diffeomorphism \( \sigma \). If \( \sigma \) is generated by the Hamiltonian vector field of a Hamiltonian function \( v \) then (7) and (14) show
\[
\Omega_{J,t}(2\sqrt{\mp} \nabla'' \nabla'' u, 2\sqrt{\mp} \nabla'' \nabla'' v) = -\int_M S(J, t) \{v, u\} \omega^m.
\]

\( \square \)

**Lemma 2.6.** For any smooth complex valued function \( u \) we have
\[
(\mathcal{L} - L)u = -\frac{1}{2} (S(J, t) a_u - u^a S(J, t) a)_a
\]
where \( a^a \)’s are local holomorphic coordinates.

**Proof.** It is sufficient prove when \( u \) is a real valued function. Let \( v \) be also a real valued smooth function. From (10) and (12) we have
\[
(v, \mathcal{L}u - Lu)_{L^2} = (\nabla'' \nabla'' v, \nabla'' \nabla'' u)_{t} - (\nabla'' \nabla'' v, \nabla'' \nabla'' u)_{t}
\]
\[
= (\nabla'' \nabla'' v, \nabla'' \nabla'' u)_{t} - (\nabla'' \nabla'' v, \nabla'' \nabla'' u)_{t}.
\]

It follows from this that
\[
2 \Re(\nabla'' \nabla'' v, i \nabla'' \nabla'' u)_{t} = i(\nabla'' \nabla'' v, \nabla'' \nabla'' u)_{t} + i(\nabla'' \nabla'' v, \nabla'' \nabla'' u)_{t}
\]
\[
= -i(v, (\mathcal{L} - L)u)_{L^2}.
\]
Let $X_u$ denote the Hamiltonian vector field of $u$: $i(X_u)\omega = du$. Then $X_u = J\text{grad} u$ and $\{u, S\} = X_u S$. It then follows that

$$(v, (\overline{T} - L)u)_{L^2} = 2i \Re(\nabla''\nabla'' v, i\nabla''\nabla'u)_t$$

$$= \frac{i}{2} \Re(X_v, iX_u) = \frac{i}{2} \Omega_{J,t}(X_v, X_u)$$

$$= -\frac{i}{2}(\{u, v\}, S(J, t))_{L^2} = \frac{i}{2}(v, \{u, S(J, t)\})_{L^2}$$

$$= \frac{i}{2}(v, X_u S(J, t))_{L^2} = \frac{i}{2} \omega(v, g(X_u, J\text{grad} S(J, t)))_{L^2}$$

$$= \frac{i}{2}(v, \delta u(J\text{grad} S(J, t)))_{L^2} = -\frac{1}{2}(v, S(J, t)^\alpha u_\alpha - u^\alpha S(J, t)_\alpha)_{L^2}.$$

\[\square\]

**Lemma 2.7.** Let $u$ be a real smooth function and suppose $\delta J = \nabla''\nabla'' u$. Then

$$\delta \int_M S(J, t)^2\omega^m = 4(u, LS(J, t))_{L^2} = 4(u, \overline{T}S(J, t))_{L^2}.$$

**Proof.** By \[\mathfrak{S}\]

$$\delta \int_M S(J, t)^2\omega^m = 2\Omega_{J,t}(2i\nabla''\nabla'' S(J, t), \nabla''\nabla'' u)$$

$$= 4\Re(\nabla''\nabla'' S(J, t), \nabla''\nabla'' u)_t$$

$$= 2(\nabla''\nabla'' S(J, t), \nabla''\nabla'' u)_t + 2(\nabla''\nabla'' u, \nabla''\nabla'' S(J, t))_t$$

$$= 2(u, LS(J, t))_{L^2} + 2(u, \overline{T}S(J, t))_{L^2}.$$

But from Lemma \[\mathfrak{2.6}\] we have

$$\overline{T}S(J, t) = LS(J, t),$$

from which the lemma follows. \[\square\]

**Lemma 2.8.** Suppose that $(\omega, J)$ is a perturbed extremal Kähler metric and thus that the gradient vector field of $S(J, t)$ is a holomorphic vector field. If $\delta J = \nabla''\nabla'' u$ for a real smooth function $u$ then

$$(\delta L)S(J, t) = \frac{1}{2} L(S(J, t)^\alpha u_\alpha - u^\alpha S(J, t)_\alpha) = L(\overline{T} - L)u.$$

**Proof.** Recall that by Lemma 2.3 in \[\mathfrak{7}\]

$$\mathcal{L}_X J = 2i \nabla''_j X' - 2i \nabla''_j X''.$$

Therefore

$$\mathcal{L}_{JX} J = 2i \nabla''_j iX' - 2i \nabla''_j (-i)X''$$

$$= -2(\nabla''_j X' - \nabla''_j X'').$$

This shows that $\mathcal{L}_{JX} J \in T_J J$ corresponds to $-2\nabla''\nabla'' u \in \text{Sym} \otimes^2 T^{\ast\ast} M$ via the identification $T_J J \cong \text{Sym} \otimes^2 T^{\ast\ast} M$. Thus $\mathcal{L}_{-\frac{1}{2} JX_\alpha} J$ corresponds to $\nabla''\nabla'' u$. On the other hand

$$\mathcal{L}_{\frac{1}{2} JX_\alpha} \omega = d(i \frac{1}{2} JX_u)\omega$$

(16)
and
\begin{equation}
(i \frac{1}{2} JX_u) \omega(Y) = \omega(\frac{1}{2} JX_u, Y) = \omega(-\frac{1}{2} \text{grad} \, u, Y) = \omega(-\frac{1}{2} u, JY) = -\frac{1}{2} du \circ J = (d^c u)(Y)
\end{equation}
where $d^c = \frac{i}{2} (\partial - \bar{\partial})$. From (16) and (17) it follows that
\begin{equation}
\mathcal{L}_{\frac{1}{2} JX_u} \omega = dd^c u = i \partial \bar{\partial} u.
\end{equation}

Let $f_s$ be a flow generated by $-\frac{1}{2} JX_u$. Suppose that $S$ is a smooth function such that grad$^2 \, S$ is a holomorphic vector field and that $S_s$ is a function such that \begin{equation}
\text{grad}^2 \, S_s = \text{grad}^2 \, S, \quad \int_M S_s (f_s^* \omega)^m = \int_M S \omega^m
\end{equation}
where grad$^2 \, S_s$ is the $(1,0)$-part of the gradient vector field of $S_s$ with respect to $f_s^* \omega$. It is easy to see that if $f_s^* \omega = \omega + i \partial \bar{\partial} \varphi$ then $S_s = S + S^a \varphi_a$. Then (18) shows
\begin{equation}
S_s = S + s S^a u_a + O(s^2).
\end{equation}
We have
\begin{equation}
L(f_s J, \omega) f_s^* S_s = f_s^* (L(J, f_s^* \omega) S_s) = 0.
\end{equation}
Taking the derivative of (20) with respect to $t$ at $t = 0$ we obtain
\begin{equation}
\delta L \cdot S + L(-\frac{1}{2} (JX_u) S + S^a u_a) = 0.
\end{equation}
On the other hand
\begin{equation}
JX_u \cdot S = g(JX_u, \text{grad} \, S) = \omega(X_u, \text{grad} \, S) = du(\text{grad} \, S) = (\partial u + J\partial u)(\nabla' S + \nabla'' S) = u^a S_a + S^a u_a.
\end{equation}

It follows from (21) and (22) that
\begin{equation}
\delta L \cdot S = -L(-\frac{1}{2} (u^a S_a + S^a u_a)) + S^a u_a)
\end{equation}
\begin{equation}
= -L(\frac{1}{2} (S^a u_a - u^a S_a)).
\end{equation}
Applying this with $S = S(J, \omega)$ and using Lemma 2.4 complete the proof of Lemma 2.8.

\textbf{Theorem 2.9.} Let \( J \) be a critical point of $\Phi$, i.e. \((\omega, J)\) gives a perturbed extremal Kähler metric and $u$ be a real smooth function on $M$. Then the Hessian of $\Phi$ at $J$ in the direction of $\nabla'' \nabla'' u$ and $\nabla'' \nabla'' v$ is given by
\begin{equation}
\text{Hess}(\Phi)_{J}(\nabla'' \nabla'' u, \nabla'' \nabla'' v) = 8(u, L \overline{L} v) = 8(u, \overline{L} L v).
\end{equation}

\textbf{Proof.} Let $\delta J = \nabla'' \nabla'' v$. By using Lemma 2.7, Lemma 2.8 and Lemma 2.4 successively one obtains
\begin{equation}
\text{Hess}(\Phi)_{J}(\nabla'' \nabla'' u, \nabla'' \nabla'' v) = 4 \delta(u, L S(J, t))
\end{equation}
\begin{equation}
= 4(u, \delta L \cdot S(J, t) + L S(J, t))
\end{equation}
\begin{equation}
= 4(u, L (\overline{L} - L) v + L (L + \overline{L}) v)
\end{equation}
\begin{equation}
= 8(u, L \overline{L} v).
\end{equation}
If one uses the third term in Lemma 2.7 and \( \delta \mathcal{L} = L - \mathcal{L} \) then one gets the third term of Theorem 2.9. This completes the proof. □

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Suppose that \( g \) is a perturbed extremal Kähler metric on \((M, \omega, J)\). Let \( X \) be a holomorphic vector field and \( \alpha \) be the dual 1-form to \( X \), that is
\[
\alpha(Y) = g(X, Y), \quad \alpha = \alpha_{\pi}dz^i = g_{\pi}X^i dz^i.
\]
Since \( X \) is a holomorphic vector field
\[
\partial \alpha = (\nabla_i \alpha_j - \nabla_j \alpha_i)dz^i \wedge dz^j = 0.
\]
Let \( \alpha = H\alpha + \overline{\partial} \psi \) be the harmonic decomposition where \( H\alpha \) denotes the harmonic part. Then
\[
L\psi = mc_m(\psi_{\pi} \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
\[
= mc_m((X^i - (H\alpha)^i) \overline{\tau}_{jk} \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
\[
= -mc_m((H\alpha)^i \overline{\tau}_{jk} \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
\[
= -mc_m((H\alpha)^i j \overline{\tau}_{jk} + (R_{\overline{j}p}(H\alpha)^p) \overline{k} \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
Note that being \( \overline{\partial} \)-harmonic and being \( \partial \)-harmonic are equivalent on compact Kähler manifolds, and thus
\[
(H\alpha)_{\overline{j}k} = \nabla_j (H\alpha)_{\overline{k}} = 0.
\]
This implies \((H\alpha)^i_j = 0\). It follows that
\[
(23) \quad L\psi = -mc_m(R_{\overline{j}i p,k}(H\alpha)^p \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
\[
= -mc_m(R_{\overline{j}i p,k}(H\alpha)^p \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
\[
= -mc_m(R_{\overline{j}i p,k}(H\alpha)^p \sqrt{-1} 2\pi dz^k \wedge dz^l, \omega \otimes I + \sqrt{-1} 2\pi t\Theta, \cdots, \omega \otimes I + \sqrt{-1} 2\pi t\Theta)
\]
\[
= -(H\alpha)^p \nabla_p S(J, t) = -(H\alpha)_{\overline{p}} \overline{\nabla_p} S(J, t)
\]
where we have used the second Bianchi identity \( R_{j\ell}^{\ i\ k} = R_{j\ell}^{\ i\ k} \) and

\[
\nabla_p S(J, t) = \nabla_p \left( c_m (\omega \otimes I + \frac{i}{2\pi} t \Theta) - \omega^m \right) \\
= \frac{1}{t} \nabla_p c_m (\omega \otimes I + \frac{i}{2\pi} t \Theta) \\
= mc_m (R_{j\ell}^{\ i\ k} \sqrt{-1} dz^k \wedge dz^\ell, \omega \otimes I + \sqrt{-1} \frac{1}{2\pi} t \Theta, \\
\cdots, \omega \otimes I + \sqrt{-1} \frac{1}{2\pi} t \Theta).
\]

Note that \( \nabla_q S(J, t) \frac{\partial}{\partial z^q} \) is a conjugate holomorphic vector field and that \( (H\alpha)^q \frac{\partial}{\partial z^q} \) is a conjugate holomorphic 1-form because \( H\alpha \) is a \( \partial \)-harmonic \((0, 1)\)-form. It follows from (23) that \( L\psi = \text{constant} \). But since \( \int_M L\psi \omega^m = 0 \) by (10) we obtain \( L\psi = 0 \). This implies that \( \text{grad}'\psi \) is a holomorphic vector field. Then \( (H\alpha)^q \frac{\partial}{\partial z^q} = X - \text{grad}'\psi \) is also holomorphic. It then follows that

\[
\nabla_{\overline{\partial}} (H\alpha)^q = 0.
\]

But since \( (H\alpha) \) is \( \partial \)-harmonic we also have \( \nabla_k (H\alpha)^q = 0 \). Thus \( H\alpha \) is parallel.

This proves the direct sum decomposition as a vector space

\[
h(M) = a(M) + h'(M)
\]

where

\[
h'(M) = \{ X \in h(M) \mid X = \text{grad}'u \text{ for some } u \in C_\infty^\infty(M) \}.
\]

It is easy to see

\[
[a(M), a(M)] = 0; \\
[a(M), b'(M)] \subset b'(M); \\
[b'(M), b'(M)] \subset b'(M).
\]

Now by Theorem 2.9 we have \( \overline{\mathcal{L}} = \overline{LL} \). Thus \( \overline{L} \) preserves \( \text{Ker} \ L \). Let \( E_\lambda \) denote the \( \lambda \)-eigenspace of \( 2\overline{L}|_{\text{Ker} \ L} \). If \( u \in E_\lambda \) then \( \text{grad}'u \in b'(M) \) and

\[
\lambda u = 2\overline{L} u = 2(\overline{L} - L) u = S(J, t)^\alpha u_\alpha - u^\alpha S(J, t)_\alpha.
\]

This implies \( [\text{grad}'S(J, t), \text{grad}'u] = \lambda \text{grad}'u \). We put

\[
\text{grad}'(E_\lambda) := h_\lambda(M) \text{ for } \lambda \neq 0, \\
\text{grad}'(E_0) := b'_0(M), \\
b_0 = a(M) + b'_0(M).
\]

Then we obtain the decomposition

\[
h(M) = \sum_\lambda h_\lambda(M)
\]

where \( h_\lambda(M) \) is the \( \lambda \)-eigenspace of \( \text{ad}(\text{grad}'S(J, t)) \). Note that the real and imaginary parts of an element of \( a(M) \) are parallel and Killing and hence \( [\text{grad}'S(J, t), a(M)] = 0 \).

Finally since \( E_0 = \text{Ker} \ L \cap \text{Ker} \overline{L} \), the real and imaginary parts are respectively in \( E_0 \), that is \( E_0 \) is the complexification of the purely imaginary functions \( u \) such that \( \text{grad}'u \) is holomorphic. The real parts of such \( \text{grad}'u \)'s are Killing vector fields,
see Lemma 2.3.8 in [5]. The real parts of the elements of $\mathfrak{a}(M)$ are also Killing vector fields. Thus $\mathfrak{h}_0(M)$ is reductive. This completes the proof of Theorem 1.1.

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