ON THE ONE DIMENSIONAL SPECTRAL HEAT CONTENT FOR STABLE PROCESSES

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Abstract. This paper provides the second term in the small time asymptotic expansion of the spectral heat content of a rotationally invariant $\alpha$-stable process, $0 < \alpha \leq 2$, for the bounded interval $(a, b)$. The small time behavior of the spectral heat content turns out to be linked to the distribution of the supremum and infimum processes.

1. Introduction

Let $0 < \alpha \leq 2$ and consider $X = \{X_s\}_{s \geq 0}$ a rotationally invariant $\alpha$-stable process in $\mathbb{R}^d$ where $d \geq 1$ and whose transition densities, denoted along the paper by $p_t^{(\alpha)}(x, y)$, are known to satisfy the following properties.

(i) $p_t^{(\alpha)}(x, y)$ is radial. Namely, $p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(|x - y|)$.

(ii) Scaling property: $p_t^{(\alpha)}(|x - y|) = t^{-\alpha/d} p_t^{(\alpha)}(t^{-\frac{1}{\alpha}} |x - y|)$.

(iii) $p_t^{(\alpha)}(x, y)$ satisfies the following two sided estimates for all $0 < \alpha < 2$. There exists a constant $c_{\alpha, d} > 0$ such that

$$\lim_{t \downarrow 0} \frac{p_t^{(\alpha)}(|x - y|)}{t} = \frac{A_{\alpha, d}}{|x - y|^{d + \alpha}},$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$. See [10] for further details.

(iv) According to [7, Theorem 2.1], for all $0 < \alpha < 2$, we have

$$A_{\alpha, d} = \alpha 2^{\alpha - 1} \pi^{-\frac{d}{2}} \sin \left(\frac{\pi \alpha}{2}\right) \Gamma \left(\frac{d + \alpha}{2}\right) \Gamma \left(\frac{\alpha}{2}\right).$$

Before continuing, we introduce the following standard notation. $\mathbb{E}^x$ and $\mathbb{P}^x$ will denote the expectation and probability of any process started at $x$, respectively. Also for simplicity, we will connote $\mathbb{P} = \mathbb{P}^0$, $\mathbb{E} = \mathbb{E}^0$ and write $Z \overset{d}{=} Y$ for two random variables $Z, Y$ with values in $\mathbb{R}^d$ to mean that they are equal in distribution or have the same law.

Let us at this point establish the following convention which is needed to provide some references and motivation. When $d = 1$, $\Omega$ will stand for an interval $(a, b)$ with finite length $b - a$ denoted by $|\Omega|$. For $d > 1$, $\Omega$ will be a bounded domain with smooth boundary $\partial \Omega$ and volume denoted by $|\Omega|$. In addition, we set

$$|\partial \Omega| = \begin{cases} \# \{x \in \mathbb{R} : x \in \partial \Omega\}, & \text{if } d = 1, \\ \text{surface area of } \Omega, & \text{if } d \geq 2. \end{cases}$$
Given $\Omega \subseteq \mathbb{R}^d$ as above, we consider for $X = \{X_s\}_{s \geq 0}$ the first exit time upon $\Omega$. That is,

$$\tau_\Omega^{(\alpha)} = \inf \{s \geq 0 : X_s \in \Omega^c \}.$$ 

The purpose of the paper is to investigate the small time behavior of the following function, which is called the spectral heat content over $\Omega$ and it is given by

$$Q^{(\alpha)}_\Omega(t) = \int_\Omega dx \mathbb{P}^x \left( \tau_\Omega^{(\alpha)} > t \right), \quad t > 0,$$

when $\Omega = (a, b)$ with $|\Omega| = b - a < \infty$. Of course, $Q^{(\alpha)}_\Omega(t)$ makes sense even in the higher dimensional setting when $\Omega$ is taken according to our convention about $\Omega$.

It is worth noting that the spectral heat content of $\Omega$ takes an alternative form. In fact,

$$Q^{(\alpha)}_\Omega(t) = \int_\Omega dx \int_\Omega dy p_t^{\Omega,\alpha}(x, y),$$

where $p_t^{\Omega,\alpha}(x, y)$ is the transition density for the stable process killed upon exiting $\Omega$. An explicit expression is given by

$$p_t^{\Omega,\alpha}(x, y) = p_t^{(\alpha)}(x, y) \mathbb{P} \left( \tau_\Omega^{(\alpha)} > t \mid X_0 = x, \ X_t = y \right).$$

The name spectral heat content given to $Q^{(\alpha)}_\Omega(t)$ comes from the fact that $p_t^{\Omega,\alpha}(x, y)$ can be written in terms of the eigenvalues and eigenfunctions of the underlying domain $\Omega$. That is, it is known (see [12]) that there exists an orthonormal basis of eigenfunctions $\{\phi_n\}_{n \in \mathbb{N}}$ for $L^2(\Omega)$ with corresponding eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$ and $\lambda_n \to \infty$ as $n \to \infty$ such that

$$p_t^{\Omega,\alpha}(x, y) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \phi_n(x) \phi_n(y).$$

Notice that due to (1.5) and the last equality, we obtain an expression for $Q^{(\alpha)}_\Omega(t)$ involving both the spectrum $\{\lambda_n\}_{n \in \mathbb{N}}$ and eigenfunctions $\{\phi_n\}_{n \in \mathbb{N}}$. That is,

$$Q^{(\alpha)}_\Omega(t) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \left( \int_\Omega dx \phi_n(x) \right)^2.$$

The interest in investigating the spectral heat content is to obtain geometry features of the underlying set $\Omega$ such as volume, surface area, mean curvature, etc. The spectral heat content has been deeply studied so far for the Brownian motion ($\alpha = 2$) and we refer the interested reader to [2, 3, 4, 5] for further details concerning asymptotics of the spectral heat content corresponding to the Brownian motion for different kind of domains. As for the cases $0 < \alpha < 2$, the author of this paper in [1] has provided estimates of $Q^{(\alpha)}_\Omega(t)$ involving the volume of $\Omega$, its surface area and the fractional $\alpha$–perimeter (see (1.7) below). In fact, the following conjecture about the asymptotic expansion of $Q^{(\alpha)}_\Omega(t)$ is provided in [1].

**Conjecture:** Let $\Omega$ be an interval with finite length ($d = 1$) or a bounded domain with smooth boundary ($d \geq 2$). Then,

(i) for $1 < \alpha < 2$, there exists $C_{d,\alpha} > 0$ such that

$$Q^{(\alpha)}_\Omega(t) = |\Omega| - C_{d,\alpha} |\partial \Omega| t^{ \frac{\alpha}{2} } + \mathcal{O}(t), \ t \downarrow 0,$$

(ii) for $\alpha = 1$, there exists $C_d > 0$ such that

$$Q^{(1)}_\Omega(t) = |\Omega| - C_d |\partial \Omega| t \ln \left( \frac{1}{t} \right) + \mathcal{O}(t), \ t \downarrow 0,$$
(iii) for $0 < \alpha < 1$, there exists $\gamma_{d,\alpha} > 0$ such that
\[
Q_{\Omega}^{(\alpha)}(t) = |\Omega| - \gamma_{d,\alpha} \mathcal{P}_\alpha(\Omega) + o(t), \quad t \downarrow 0,
\]
where
\[
(1.7) \quad \mathcal{P}_\alpha(\Omega) = \int_{\Omega} \int_{\Omega} \frac{dxdy}{|x-y|^{d+\alpha}}
\]
is known as the fractional $\alpha$–perimeter. (We refer the reader to [14, 15] for further details.)

The study of $Q_{\Omega}^{(\alpha)}(t)$ when $\Omega$ is an interval with finite length is motivated by the aforementioned conjecture and our main result provides the exact constants $C_{1,\alpha}$, $C_1$ and $\gamma_{1,\alpha}$. However, we do not give any estimates concerning the remainders.

The preceding conjecture was proved bearing in mind the small time behavior of the function
\[
(1.8) \quad \mathcal{H}_{\Omega,\Gamma}^{(\alpha)}(t) = \int_\Omega dx \int_\Omega dy p^{(\alpha)}_t(x,y) = |\Omega| - \int_\Omega dx \int_\Omega dy p^{(\alpha)}_t(x,y),
\]
since for small values of $t$, it is expected that the heat kernels $p^{(\alpha)}_t(x,y)$ and $p^{\Omega,\alpha}_t(x,y)$ behave alike.

The function $\mathcal{H}_{\Omega,\Gamma}^{(\alpha)}(t)$ will play an important role in the proof of the main result of this paper, so that we need the following two facts about $\mathcal{H}_{\Omega,\Gamma}^{(\alpha)}(t)$ whose proofs can be found in [1]. First, by appealing to (1.6), we see that $P^{\Omega,\alpha}_t(x,y) \leq p^{(\alpha)}_t(x,y)$. Thus, we easily obtain
\[
(1.9) \quad \mathcal{H}_{\Omega,\Gamma}^{(\alpha)}(t) \leq |\Omega| - Q_{\Omega}^{(\alpha)}(t).
\]
Secondly, according to Theorem 1.1 in [1], we have for $\Omega = (a,b)$ with $|\Omega| = b-a < \infty$ the following.

(i) For $1 < \alpha \leq 2$,
\[
(1.10) \quad \lim_{t \downarrow 0} \frac{\mathcal{H}_{\Omega,\Gamma}^{(\alpha)}(t)}{t^{\frac{\alpha}{2}}} = \frac{2}{\pi} \Gamma \left( 1 - \frac{1}{\alpha} \right).
\]

(ii) For $\alpha = 1$, the following equality holds.
\[
(1.11) \quad \lim_{t \downarrow 0} \frac{\mathcal{H}_{\Omega,\Gamma}^{(1)}(t)}{t \ln \left( \frac{1}{t} \right)} = \frac{2}{\pi}.
\]

(iii) Let $0 < \alpha < 1$. Then,
\[
(1.12) \quad \lim_{t \downarrow 0} \frac{\mathcal{H}_{\Omega,\Gamma}^{(\alpha)}(t)}{t} = A_{\alpha,1} \mathcal{P}_\alpha(\Omega),
\]
with $A_{\alpha,1}$ and $\mathcal{P}_\alpha(\Omega)$ as given in (1.3) and (1.7), respectively.

Before proceeding with the statement of our main result, we introduce two stochastic processes associated with the one dimensional $\alpha$–stable process $X$. For $t \geq 0$ and $X$ started from 0, we set
\[
(1.13) \quad \overline{X}_t = \sup \{X_s : 0 \leq s \leq t\}, \quad \underline{X}_t = \inf \{X_s : 0 \leq s \leq t\}.
\]
The stochastic processes $\overline{X} = \{\overline{X}_s\}_{s \geq 0}$ and $\underline{X} = \{\underline{X}_s\}_{s \geq 0}$ are called the supremum and infimum process, respectively. The distribution of the process $\overline{X}$ has been widely investigated by many authors (see for example [8], [9] and [11]) because of its connections with fluctuation and excursion theory and major interest in stochastic modelling, such as queuing and risk theories.

The main result of this paper is the following.

**Theorem 1.1.** Consider $\Omega = (a,b)$ with $|\Omega| = b-a < \infty$. 

a) Let $1 < \alpha \leq 2$. Then, we have
\[ \lim_{t \downarrow 0} \frac{|\Omega| - Q^{(\alpha)}_{\Omega}(t)}{t^{\frac{2}{\alpha}}} = 2 \mathbb{E} [X_1]. \]
b) For $\alpha = 1$, we obtain
\[ \lim_{t \downarrow 0} \frac{|\Omega| - Q^{(1)}_{\Omega}(t)}{t \ln \left( \frac{1}{t} \right)} = \frac{2}{\pi}. \]
c) Let $0 < \alpha < 1$. Then,
\[ \lim_{t \downarrow 0} \frac{|\Omega| - Q^{(\alpha)}_{\Omega}(t)}{t} = A_{\alpha,1} \mathcal{P}_\alpha(\Omega), \]
with $A_{\alpha,1}$ and $\mathcal{P}_\alpha(\Omega)$ as given in (1.3) and (1.7), respectively.

We remark that the main idea in the proof of Theorem 1.1 consists in expressing $Q^{(\alpha)}_{\Omega}(t)$ as quantities involving the distribution of the supremum and infimum processes related to the $\alpha$-stable process $X$. The idea of employing the distribution of the supremum comes from the paper of van den Berg and le Gall [2], where they use in the case of the Brownian motion ($\alpha = 2$) the supremum process, the unique a.s. time $\tau$ satisfying $X_{\tau} = \underline{X}_{\tau}$ and the independence among coordinates to prove for smooth domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$ that
\begin{equation}
Q^{(2)}_{\Omega}(t) = |\Omega| - \frac{2}{\sqrt{\pi}} |\partial \Omega| t^{1/2} + \left( 2^{-1} (d - 1) \int_{\partial \Omega} \mathcal{M}(s) ds \right) t + O(t^{3/2}),
\end{equation}
as $t \downarrow 0$. Here, $\mathcal{M}(s)$ represents the mean curvature at the point $s \in \partial \Omega$.

We point out that when $\alpha = 2$ and using that for an interval $|\partial \Omega| = 2$ (see (1.4)), we obtain by Theorem 1.1, the one dimensional analogue to (1.14). That is,
\[ \lim_{t \downarrow 0} \frac{|\Omega| - Q^{(2)}_{\Omega}(t)}{t^{\frac{2}{3}}} = \frac{2}{\sqrt{\pi}} |\partial \Omega|, \]
since, according to (4.4) below, we have $\mathbb{E} [X_1] = \frac{2}{\sqrt{\pi}}$.

The rest of the paper is organized as follows. In §2, we establish properties concerning the distribution of the supremum and infimum processes. In §3, we rewrite $Q^{(\alpha)}_{\Omega}(t)$ in terms of integrals involving the distribution of $\underline{X}$ and $\overline{X}$. In §4, we provide the proof of Theorem 1.1, where the proof of a crucial proposition is postponed until §5.

2. PROPERTIES OF THE SUPREMUM AND INFIMUM PROCESSES

As we have mentioned in the introduction, the proof of our main result depends on the properties of the distribution of the supremum and infimum processes. For this reason, this section is dedicated to establishing properties about $\underline{X}$ and $\overline{X}$ only when $0 < \alpha < 2$ since for $\alpha = 2$ these properties are already well known.

To begin with, consider $A \subseteq \mathbb{R}$ a Borel-set and the first hitting time for $A$ defined by
\[ T_A = \inf \{ s > 0 : X_s \in A \}. \]
We recall that $x$ is regular for $A$ if $\mathbb{P}^x (T_A = 0) = 1$. That is, the process starting from $x$ meets $A$ for arbitrarily small strictly positive times with probability one.

Next, due to the fact that an $\alpha$-stable process $X$ satisfies by symmetry that $\mathbb{P} (X_t > 0) = \frac{1}{2}$, we have
\[ \int_0^1 dt \frac{1}{t} \mathbb{P} (X_t > 0) = \infty. \]

The divergence of the last integral guarantees by Theorem 6.5 in [16] that $0$ is regular for $(0, \infty)$ which in turn implies that $X_t > 0$ a.s. due to the fact that the process $X$ starting from $0$ will
hit \((0, \infty)\) for arbitrarily small strictly positive times with probability one. Since \(\overline{X}_t \overset{D}{=} -X_t\), we also obtain \(\overline{X}_t < 0\) a.s.

The next proposition will be very useful in the following. Roughly speaking, it says that the distribution of the random variables \(X_t\) and \(\overline{X}_t\) are comparable.

**Proposition 2.1.** For every \(u > 0\) and \(t > 0\), we have

\[
P(u \leq X_t) \leq P(u \leq X_t) \leq 2 P(u \leq X_t).
\]

**Proof.** We start by noticing that \(P(u \leq X_t) = P(u \leq \overline{X}_t, u \leq X_t) \leq P(u \leq \overline{X}_t)\). It remains to show that

\[
P(u \leq \overline{X}_t) \leq 2 P(u \leq X_t).
\]

Denote \(T_u = T(\infty, u) = \inf \{s \geq 0 : X_s \geq u\}\), the first passage time over \(u\) and observe that

\[
\{T_u \leq t\} = \{u \leq \overline{X}_t\}.
\]

Therefore, by a conditioning argument, we arrive at

\[
(2.1) \quad P(u \leq X_t) = P(u \leq \overline{X}_t, u \leq X_t)
\]

\[
= P(u \leq X_t | u \leq \overline{X}_t) P(u \leq \overline{X}_t)
\]

\[
= P(u \leq X_t | T_u \leq t) P(u \leq \overline{X}_t)
\]

\[
= P(u - X_{T_u} \leq X_{(t - T_u) + T_u} - X_{T_u} | T_u \leq t) P(u \leq \overline{X}_t).
\]

Now, by the definition of \(T_u\), we have \(u - X_{T_u} \leq 0\). Thus,

\[
(2.2) \quad P(u - X_{T_u} \leq X_{(t - T_u) + T_u} - X_{T_u} | T_u \leq t) \geq P(0 \leq X_{(t - T_u) + T_u} - X_{T_u} | T_u \leq t).
\]

Finally, by the strong Markov property and the symmetric property of the \(\alpha\)-stable process, we obtain

\[
(2.3) \quad P(0 \leq X_{(t - T_u) + T_u} - X_{T_u} | T_u \leq t) = \frac{1}{2}.
\]

Hence, the desired result is proved by combining (2.1), (2.2) and (2.3). \(\square\)

The following lemma provides estimates about the tail behavior of the \(\alpha\)-stable process \(X\) and its proof is omitted because it consists in a elementary integration problem.

**Lemma 2.1.** Let \(\psi(z) = \min \left\{ 1, \frac{1}{|z|^{(1+\alpha)}} \right\} \) with \(0 < \alpha < 2\). Then, for all \(u > 0\), we have

\[
\int_u^\infty dz \psi(z) = \mathbb{1}_{(0,1)}(u) \left(1 + \frac{1}{\alpha} - u\right) + \mathbb{1}_{[1,\infty)}(u) \left(\frac{u - \alpha}{\alpha}\right).
\]

In particular, due to (1.1), it follows for \(t < u^\alpha\) that

\[
(2.4) \quad c_\alpha^{-1} \frac{t}{u^\alpha} \leq P(u \leq X_t) = P\left(\frac{u}{t^{\frac{1}{\alpha}}} \leq X_1\right) \leq c_\alpha \frac{t}{u^\alpha}.
\]

As a consequence of combining Lemma 2.1 together with Proposition 2.1, we conclude the following result.

**Corollary 2.1.**

(i) For all \(t > 0\),

\[
\lim_{u \to +\infty} P(u \leq X_t) = \lim_{u \to +\infty} P(u \leq \overline{X}_t) = 0.
\]
\[ (ii) \quad \int_0^\infty du \mathbb{P}(u \leq X_1) < \infty \]

if and only if

\[ \int_0^\infty du \mathbb{P}(u \leq X_1) < \infty \]

if and only if \( 1 < \alpha < 2 \).

\[ (iii) \] For all \( t > 0 \),

\[ \mathbb{P}(X_t = \infty) = \mathbb{P}(X_t = -\infty) = 0. \]

Proof. We only need to prove part \((iii)\). It is a basic fact in probability theory that

\[ \mathbb{P}(X_t = \infty) = \lim_{n \to \infty} \mathbb{P}(n \leq X_t). \]

Hence, by Proposition 2.1 and Lemma 2.1, we arrive by part \((i)\) at

\[ \mathbb{P}(X_t = \infty) \leq 2 \lim_{n \to \infty} \mathbb{P}(n \leq X_t) = 0. \]

On the other hand, using that \( X_t \overset{D}{=} -X_t \), we have

\[ \mathbb{P}(X_t = \infty) = \mathbb{P}(X_t = -\infty) \]

and this completes the proof. \( \square \)

3. SPECTRAL HEAT CONTENT IN TERMS OF THE SUPREMUM AND INFIMUM PROCESSES

We start this section by expressing the spectral heat content \( Q^{(\alpha)}_\Omega(t) \) in a more convenient form. Henceforth, \( \Omega = (a, b) \) with \( |\Omega| = b - a < \infty \) and \( 0 < \alpha \leq 2 \).

Lemma 3.1. For \( x \in \Omega \) and \( t > 0 \), we set \( \Omega_t(x) = \left( (x - b)t^{-\alpha}, (x - a)t^{-\alpha} \right) \). Then,

\[ \mathbb{P}^x \left( \tau^{(\alpha)}_\Omega > t \right) = \mathbb{P} \left( \tau^{(\alpha)}_{\Omega_t(x)} > 1 \right). \]

Proof. The proof is based on the scaling property \( X_{\ell t} \overset{D}{=} t^\alpha X_\ell \) and the translation property of the rotationally invariant \( \alpha \)-stable process.

\[ \mathbb{P}^x \left( \tau^{(\alpha)}_\Omega > t \right) = \mathbb{P}^x \left( a \leq X_s \leq b, \forall 0 \leq s \leq t \right) \]

\[ = \mathbb{P} \left( a \leq x - t^{\alpha} X_\ell \leq b, \forall 0 \leq \ell \leq 1 \right) \]

\[ = \mathbb{P} \left( (x - b)t^{\frac{\alpha}{\ell}} \leq X_\ell \leq (x - a)t^{\frac{\alpha}{\ell}}, \forall 0 \leq \ell \leq 1 \right) \]

\[ = \mathbb{P} \left( \tau^{(\alpha)}_{\Omega_t(x)} > 1 \right). \]

\( \square \)

The upcoming proposition allows us to decompose \( Q^{(\alpha)}_\Omega(t) \) as a sum involving the distribution of the supremum and infimum processes.

Proposition 3.1. For every \( t > 0 \), we have

\[ Q^{(\alpha)}_\Omega(t) = |\Omega| - 2t^{\frac{\alpha}{\ell}} \int_0^{r^{(\alpha)}_\Omega(t)} du \mathbb{P}(u \leq X_1) + t^{\frac{\alpha}{\ell}} r^{(\alpha)}_\Omega(t), \]

where

\[ r^{(\alpha)}_\Omega(t) = \int_0^{\frac{\alpha}{\ell}} du \mathbb{P}(u \leq X_1, X_1 \leq u - |\Omega| t^{-\frac{\alpha}{\ell}}). \]
Proof. Because of Lemma 3.1, we know that
\[ Q_\Omega^{(\alpha)}(t) = \int_a^b dx \mathbb{P} \left( \tau_{\Omega_t(x)}^{(\alpha)} > 1 \right) . \]
Now, the change of variable \( x = t^\frac{1}{\alpha} u + a \) yields
\[ \Omega_\alpha(x) = \left( (x - b)t^\frac{1}{\alpha}, (x - a)t^\frac{1}{\alpha} \right) = (u - |\Omega| t^\frac{1}{\alpha}, u) \]
and
\[
Q_\Omega^{(\alpha)}(t) = t^\frac{1}{\alpha} \int_0^{|\Omega| t^\frac{1}{\alpha}} du \mathbb{P} \left( \tau_{(u-|\Omega| t^{-\frac{1}{\alpha}}, u)}^{(\alpha)} > 1 \right) 
= |\Omega| - t^\frac{1}{\alpha} \int_0^{|\Omega| t^\frac{1}{\alpha}} du \mathbb{P} \left( \tau_{(u-|\Omega| t^{-\frac{1}{\alpha}}, u)}^{(\alpha)} \leq 1 \right) .
\]

The key step towards the desired decomposition is based on the fact that
\[
\mathbb{P} \left( \tau_{(u-|\Omega| t^{-\frac{1}{\alpha}}, u)}^{(\alpha)} \leq 1 \right) = \mathbb{P} \left( \{u \leq X_1\} \cup \{X_1 \leq u - |\Omega| t^{-\frac{1}{\alpha}} \} \right) = 
\mathbb{P} \left( u \leq X_1 \right) + \mathbb{P} \left( X_1 \leq u - |\Omega| t^{-\frac{1}{\alpha}} \right) - \mathbb{P} \left( u \leq X_1, X_1 \leq u - |\Omega| t^{-\frac{1}{\alpha}} \right) .
\]
Next, integrate the last expression from 0 to \( |\Omega| t^{-\frac{1}{\alpha}} \) and replace it into (3.1). To finish the proof, it suffices to observe that the change of variable \( v = u - |\Omega| t^{-\frac{1}{\alpha}} \) and the fact \( \overline{X}_1 \equiv -X_1 \) yield
\[
\int_0^{|\Omega| t^{-\frac{1}{\alpha}}} du \mathbb{P} \left( X_1 \leq u - |\Omega| t^{-\frac{1}{\alpha}} \right) = \int_0^0 dv \mathbb{P} \left( X_1 \leq v \right) 
= \int_0^0 dv \mathbb{P} \left( -X_1 \leq v \right) = \int_0^{|\Omega| t^{-\frac{1}{\alpha}}} du \mathbb{P} \left( u \leq \overline{X}_1 \right) .
\]

\[ \square \]

4. PROOF OF THEOREM 1.1

Let us set for \( t > 0 \),
\[
L^{(\alpha)}(t) = \int_0^{|\Omega| t^{-\frac{1}{\alpha}}} du \mathbb{P} \left( \overline{X}_1 \leq u \right) ,
\]
and
\[
L^{(\alpha)}(0) = \int_0^\infty du \mathbb{P} \left( \overline{X}_1 \leq u \right) .
\]

According to Lemma 2 in [9] (see also [13, 8, 11]), it is known that there exists a strictly positive and continuous density \( f_\alpha(x) \) defined over \((0, \infty)\) such that
\[
P \left( \overline{X}_1 \in A \right) = \int_A dx f_\alpha(x) ,
\]
for any Borel set \( A \subseteq [0, \infty) \).

Remark 4.1. We point out that \( L^{(\alpha)}(0) < \infty \) only for \( 1 < \alpha \leq 2 \). To see this, we observe that Proposition 2.1 ensures the finiteness of \( L^{(\alpha)}(0) \) for \( 1 < \alpha < 2 \). Regarding the case \( \alpha = 2 \), \( X \) represents a Brownian Motion at twice velocity. It is well known that for every \( u > 0 \), the following equality holds.
\[
P \left( X_t \leq -u \right) = P \left( u \leq \overline{X}_t \right) = 2P \left( u \leq X_t \right) .
\]
Therefore, we have by interchanging the order of integration that
\begin{equation}
L^{(2)}(0) = 2 \int_0^\infty du \, \mathbb{P}(u \leq X_1) = 2 \int_0^\infty du \int_{-\frac{x}{\sqrt{4\pi}}}^\infty dz \, p_1^{(2)}(z)
= 2 \int_0^\infty dz \, z^2 \exp \left( -\frac{z^2}{4} \right) = \frac{2}{\sqrt{\pi}}.
\end{equation}

It is a basic fact in probability (see [17]) that the existence of the density \( f_\alpha(x) \) and the finiteness of \( L^{(\alpha)}(0) \) imply that \( L^{(\alpha)}(0) = \mathbb{E} [X_1] \).

We also mention that Proposition 2.1 yields \( \mathbb{E} [X_1, X_1 > 0] \leq \mathbb{E} [X_1] \leq 2 \mathbb{E} [X_1, X_1 > 0] \). In addition, it is proved in [1, page 14], by subordination of the Gaussian kernel that
\[
\mathbb{E} [X_1, X_1 > 0] = \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{\alpha} \right),
\]
whenever \( 1 < \alpha \leq 2 \).

The following proposition provides the small time behavior of \( L^{(\alpha)}(t) \) as \( t \downarrow 0 \) with a remainder term as long as \( 1 < \alpha \leq 2 \).

**Proposition 4.1.** Consider \( 1 < \alpha \leq 2 \). Then,
\[
L^{(\alpha)}(t) = \mathbb{E} [X_1] + O(t \cdot \mathbb{1}_{\{2\}}(\alpha) + t^{1-\frac{1}{\alpha}} \cdot \mathbb{1}_{\{1, 2\}}(\alpha)), \text{ as } t \downarrow 0.
\]

In particular,
\[
\lim_{t \downarrow 0} L^{(\alpha)}(t) = \mathbb{E} [X_1].
\]

**Proof.** It is clear by (4.1) and remark 4.2 that
\[
L^{(\alpha)}(t) = \mathbb{E} [X_1] - \int_{|\Omega| t^{-\frac{1}{\alpha}}}^\infty du \, \mathbb{P}(u \leq X_1).
\]

Now, by Proposition 2.1, we have
\[
\int_{|\Omega| t^{-\frac{1}{\alpha}}}^\infty du \, \mathbb{P}(u \leq X_1) \leq 2 \int_{|\Omega| t^{-\frac{1}{\alpha}}}^\infty du \, \mathbb{P}(u \leq X_1).
\]

Let us denote the integral term at the right hand side of the previous inequality by \( R^{(\alpha)}(t) \). To complete the proof of the proposition, it suffices to estimate how fast \( R^{(\alpha)}(t) \) tends to 0 as \( t \downarrow 0 \).

We proceed to consider cases.

**Case \( \alpha = 2 \):** By interchanging the order of integration, we obtain
\[
R^{(2)}(t) = \int_{|\Omega| t^{-\frac{1}{2}}}^\infty du \int_{-\frac{x}{\sqrt{4\pi}}}^\infty dz \, p_1^{(2)}(z) = \int_{|\Omega| t^{-\frac{1}{2}}}^\infty dz \, p_1^{(2)}(z) \int_{|\Omega| t^{-\frac{1}{2}}}^\infty du
= \mathbb{E} [X_1, |\Omega| t^{-\frac{1}{2}} < X_1] - |\Omega| t^{-\frac{1}{2}} \mathbb{P} \left( |\Omega| t^{-\frac{1}{2}} < X_1 \right).
\]

Let us denote \( j(t) = \mathbb{E} [X_1, |\Omega| t^{-\frac{1}{2}} < X_1] \) and observe that
\begin{equation}
(4.5) \quad j(t) \geq |\Omega| t^{-\frac{1}{2}} \mathbb{P} \left( |\Omega| t^{-\frac{1}{2}} < X_1 \right).
\end{equation}

Thus, the remainder function \( R^{(2)}(t) \) satisfies \( 0 \leq R^{(2)}(t) \leq 2 j(t) \). It follows from (4.5) that
\[
j(t) = (4\pi)^{-1/2} \int_{|\Omega| t^{-1/2}}^\infty dz \exp \left( -\frac{z^2}{4} \right) = \pi^{-1/2} \exp \left( -\frac{|\Omega|^2}{4t} \right).
\]
Next, by applying the elementary inequality
\[ \exp(-x) \leq x^{-1}, \ x > 0, \]
we conclude that \( j(t) \leq 4\pi^{-1/2} |\Omega|^{-2} t. \)

**Case 1 < \alpha < 2:** By inequality (2.4), we have for \( |\Omega| t^{-\frac{\alpha}{2}} > 1 \) that
\[
0 \leq R^{(\alpha)}(t) \leq \frac{c_{\alpha,1}}{\alpha} \int_{|\Omega| t^{-\frac{\alpha}{2}}}^{\infty} \, du \, u^{-\alpha} = \left( \frac{c_{\alpha,1} |\Omega|^{-\frac{\alpha}{2}}}{\alpha (\alpha - 1)} \right) t^{1-\frac{\alpha}{2}}.
\]

The last ingredient in the proof of part (a) of our main result is to show that \( r^{(\alpha)}(t) \to 0 \) as \( t \downarrow 0. \)

**Proposition 4.2.** Consider \( 1 < \alpha \leq 2 \) and
\[
r^{(\alpha)}(t) = \int_{[0,|\Omega|^{-\frac{\alpha}{2}}]} \, du \, \mathbb{P} \left( u \leq X_1, X_1 < u - |\Omega| t^{-\frac{\alpha}{2}} \right).
\]
Then,
\[
\lim_{t \downarrow 0} r^{(\alpha)}(t) = 0.
\]

**Proof.** Define
\[
G_{\alpha}(t, u) = \mathbb{1}_{(0,|\Omega|^{-\frac{\alpha}{2}})}(u) \cdot \mathbb{P} \left( u \leq X_1, X_1 < u - |\Omega| t^{-\frac{\alpha}{2}} \right),
\]
which satisfies that
\[
G_{\alpha}(t, u) \leq \mathbb{P} \left( u \leq X_1 \right),
\]
where according to Corollary 2.1 (for \( 1 < \alpha < 2 \)) and equality (4.4) (for \( \alpha = 2 \)), we have \( \mathbb{P} \left( u \leq X_1 \right) \in L^1((0, \infty)) \). Furthermore,
\[
G_{\alpha}(t, u) \leq \mathbb{P} \left( X_1 \leq u - |\Omega| t^{-\frac{\alpha}{2}} \right),
\]
which in turn implies by appealing once more to Corollary 2.1 and equality (4.3) that
\[
\lim_{t \downarrow 0} G_{\alpha}(t, u) \leq \mathbb{P} (X_1 = -\infty) = 0.
\]
Thus, the result follows from an application of the Dominated Convergence Theorem.

**Proof of part a) of Theorem 1.1:**

**Proof.** Let \( 1 < \alpha \leq 2 \). By Proposition 3.1, we obtain
\[
\lim_{t \downarrow 0} \frac{|\Omega| - G^{(\alpha)}(t)}{t^{-\frac{\alpha}{2}}} = \lim_{t \downarrow 0} 2L^{(\alpha)}(t) - r^{(\alpha)}(t).
\]
Thus, the desired result follows from Proposition 4.1 and Proposition 4.2.

As for the rest of the proof of our main result, the following proposition concerning how fast \( L^{(\alpha)}(t) \) diverges to \( \infty \) as \( t \downarrow 0 \) when \( 0 < \alpha \leq 1 \) will be a crucial ingredient. It is remarkable that the behavior of \( L^{(\alpha)}(t) \) for small values of \( t \) has been deduced from the function \( \mathbb{P}^{(\alpha)}_{\Omega, \Omega^c}(t) \) given in (1.8) and the limits described in (1.11) and (1.12).

**Proposition 4.3.**

(i) Let \( \alpha = 1 \). Then,
\[
\lim_{t \downarrow 0} \frac{L^{(1)}(t)}{\ln \left( \frac{1}{t} \right)} = \frac{1}{\pi}.
\]
(ii) For $0 < \alpha < 1$, we have
\[
\lim_{t \downarrow 0} \frac{t^{\frac{\alpha}{2}} L^{(\alpha)}(t)}{t} = \frac{1}{2} A_{\alpha,1} \mathcal{P}_\alpha(\Omega).
\]
Here, $A_{\alpha,1}$ and $\mathcal{P}_\alpha(\Omega)$ are the constants defined in (1.3) and (1.7), respectively.

In order to make this proof as clear as possible, we first proceed to complete the proof of the Theorem 1.1 and we postpone the proof of Proposition 4.3 until the last section.

Proof of part (b) and (c) of Theorem 1.1

Proof. To begin with, we notice that by the inequality (1.9) and Proposition 3.1, we arrive at the following inequality.

\[
H^{(\alpha)}(\Omega)(t) \leq |\Omega| - Q^{(\alpha)}_\Omega(t) \leq 2t^{\frac{\alpha}{2}} L^{(\alpha)}(t),
\]

with $Q^{(\alpha)}_\Omega(t)$ and $L^{(\alpha)}(t)$ as given in (1.5) and (4.1), respectively. Therefore, the desired result is obtained by applying the Squeeze Theorem to the inequality (4.6) since by (1.11), (1.12) and Proposition 4.3, we have

\[
\lim_{t \downarrow 0} \frac{H^{(1)}(\Omega)(t)}{t} = \lim_{t \downarrow 0} \frac{2L^{(1)}(t)}{t} = \frac{2}{\pi},
\]

\[
\lim_{t \downarrow 0} \frac{H^{(\alpha)}(\Omega)(t)}{t} = \lim_{t \downarrow 0} \frac{2t^{\frac{\alpha}{2}} L^{(\alpha)}(t)}{t} = A_{\alpha,1} \mathcal{P}_\alpha(\Omega), \text{ for } 0 < \alpha < 1.
\]

5. Proof of Proposition 4.3

Case $\alpha = 1$: For this case, $X$ is called Cauchy process. It was proved by Darling in [11] that the density of the supremum corresponding to the Cauchy process (see (4.2)) is given by

\[
f_1(x) = \frac{1}{\pi(1 + x^2)} F\left(\frac{1}{x}\right), \quad x > 0,
\]

where

\[
F(z) = \exp\left(\frac{1}{\pi} \int_0^\infty \ln(z + y) \cdot \frac{dy}{1 + y^2}\right).
\]

Remark 5.1. An elementary calculation shows that for $x > 0$

\[
\int_0^x \ln(y) \cdot \frac{dy}{1 + y^2} = - \int_x^\infty \ln(y) \cdot \frac{dy}{1 + y^2},
\]

which in turn implies by taking $x = 1$ that $\int_0^\infty \ln(y) \cdot \frac{dy}{1 + y^2} = 0$ and $F(0) = 1$.

We have then

\[
L^{(1)}(t) = \int_0^{\max|\Omega|^{-1}} du \mathbb{P}(u \leq X_1) = \int_0^{\max|\Omega|^{-1}} du \int_u^\infty dx f_1(x).
\]

Proof of part (i) of Proposition 4.3:
Proof. We start by observing that due to the Fundamental Theorem of Calculus and the facts that $P \left( |\Omega| t^{-1} \leq X_1 \right) \rightarrow 0$ and $L^{(i)}(t) \rightarrow \infty$ as $t \downarrow 0$ (see Corollary 2.4), we obtain by applying L'Hôpital's rule twice that

$$
\lim_{t \downarrow 0} \frac{L^{(i)}(t)}{\ln \left( \frac{1}{t} \right)} = \lim_{t \downarrow 0} \frac{|\Omega| P \left( |\Omega| t^{-1} \leq X_1 \right)}{t} = \lim_{t \downarrow 0} \frac{|\Omega|^2 t^{-2} f_1 \left( |\Omega| t^{-1} \right)}{t} = \lim_{t \downarrow 0} \frac{|\Omega|^2 F \left( t |\Omega|^{-1} \right)}{\pi (t^2 + |\Omega|^2)} = F(0) = 1.
$$

In the last equality above, we have used according to remark 5.1 that $F(0) = 1$.

Proof of part (ii) of Proposition 4.3:

Proof. Along the proof, we assume $0 < \alpha < 1$. For $\Omega = (a, b)$ with $|\Omega| = b - a < \infty$, the fractional $\alpha$-perimeter $P_\alpha(\Omega)$ defined in (1.7) is equal by simple integrations to

$$
P_\alpha(\Omega) = \int_a^b \int_{a \leq x < y \leq b} dx \, dy = \int_a^b dx \left( \int_{-\infty}^a (x - y)^{1+\alpha} \, dy + \int_b^\infty (y - x)^{1+\alpha} \, dy \right) = \frac{1}{\alpha} \int_a^b dx \left( (x - a)^{-\alpha} + (b - x)^{-\alpha} \right) = \frac{2}{\alpha(1-\alpha)} |\Omega|^{1-\alpha}.
$$

Now, for $0 < \alpha < 1$, we know that $L^{(\alpha)}(t)$ and $t^{1-\frac{\alpha}{2}}$ both tend to $\infty$ as $t \downarrow 0$ by Corollary 2.4. Thus, we have by applying L'Hôpital’s rule that

$$
\lim_{t \downarrow 0} \frac{t^{\frac{\alpha}{2}} L^{(\alpha)}(t)}{t} = \lim_{t \downarrow 0} \frac{L^{(\alpha)}(t)}{t^{1-\frac{\alpha}{2}}} = \frac{|\Omega|}{1-\alpha} \lim_{t \downarrow 0} \frac{P \left( |\Omega| t^{-\frac{\alpha}{2}} \leq X_1 \right)}{t}. \tag{5.2}
$$

Next, according to Proposition 4, in [6, page 221], it is known that

$$
\lim_{t \downarrow 0} \frac{P \left( |\Omega| t^{-\frac{\alpha}{2}} \leq X_1 \right)}{t} = 1.
$$

Hence, we deduce that

$$
\lim_{t \downarrow 0} \frac{P \left( |\Omega| t^{-\frac{\alpha}{2}} \leq X_1 \right)}{t} = \lim_{t \downarrow 0} \frac{P \left( |\Omega| t^{-\frac{\alpha}{2}} \leq X_1 \right)}{t}, \tag{5.3}
$$

and by applying L'Hôpital’s rule one more time, we arrive at

$$
\lim_{t \downarrow 0} \frac{P \left( \Omega t^{-\frac{\alpha}{2}} \leq X_1 \right)}{t} = \frac{|\Omega|}{\alpha} \lim_{t \downarrow 0} t^{1-\frac{\alpha}{2}} p^{(\alpha)}_1 \left( |\Omega| t^{-\frac{\alpha}{2}} \right). \tag{5.4}
$$

We point out that the transition densities of the one dimensional $\alpha$–stable process $X$ are known to satisfy that

$$
p^{(\alpha)}_1(x) = p^{(\alpha)}_1(|x|) = t^{-\frac{\alpha}{2}} p^{(\alpha)}_1(t^{-\frac{\alpha}{2}} |x|),
$$

which in turn implies by (1.2) that

$$
\lim_{t \downarrow 0} t^{1-\frac{\alpha}{2}} p^{(\alpha)}_1 \left( |\Omega| t^{-\frac{\alpha}{2}} \right) = \lim_{t \downarrow 0} \frac{p^{(\alpha)}_1(|\Omega|)}{t} = \frac{A_{\alpha,1}}{|\Omega|^{1+\alpha}}. \tag{5.5}
$$

Then, by combining (5.2), (5.3), (5.4) and (5.5), we obtain that

$$
\lim_{t \downarrow 0} \frac{t^{\frac{\alpha}{2}} L^{(\alpha)}(t)}{t} = A_{\alpha,1} |\Omega|^{1-\alpha} = \frac{A_{\alpha,1} P_\alpha(\Omega)}{2},
$$

where the last equality is a consequence of (5.1) and this completes the proof. □
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