The theory of vector-valued modular forms for the modular group

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Abstract We explain the basic ideas, describe with proofs the main results, and demonstrate the effectiveness, of an evolving theory of vector-valued modular forms (vvmf). To keep the exposition concrete, we restrict here to the special case of the modular group. Among other things, we construct vvmf for arbitrary multipliers, solve the Mittag-Leffler problem here, establish Serre duality and find a dimension formula for holomorphic vvmf, all in far greater generality than has been done elsewhere. More important, the new ideas involved are sufficiently simple and robust that this entire theory extends directly to any genus-0 Fuchsian group.

1 Introduction

Even the most classical modular forms (e.g. the Dedekind eta $\eta(\tau)$) need a multiplier, but this multiplier is typically a number (i.e. a 1-dimensional projective representation of some discrete group like $\Gamma = \text{SL}_2(\mathbb{Z})$). Simple examples of vector-valued modular forms (vvmf) for $\text{SL}_2(\mathbb{Z})$ are the weight $-\frac{1}{2}$ Jacobi theta functions $\Theta(\tau) = (\theta_2(\tau), \theta_3(\tau), \theta_4(\tau))^t$, which obey for instance

$$\Theta(-1/\tau) = \sqrt{\frac{\tau}{i}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Theta(\tau),$$

(1)

and $P(\tau) = (\tau, 1)^t$, which has weight $-1$ and obeys for instance
\[
P(-1/\tau) = \tau^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P(\tau).
\]

Back in the 1960s Selberg [31] called for the development of the theory of vvmf, as a way to study growth of coefficients of (scalar) modular forms for noncongruence groups. Since then, the relevance of vvmf has grown significantly, thanks largely to the work of Borcherds (see e.g. [9]) and, in physics, the rise of rational conformal field theory (RCFT).

In particular, the characters of rational and logarithmic conformal field theories, or \(C_2\)-cofinite vertex operator algebras, form a weight-0 vvmf for \(I \Gamma_0(2)\) [29]. Less appreciated is that the 4-point functions (conformal blocks) on the sphere in RCFT can naturally be interpreted as vvmf for \(I \Gamma(2)\), through the identification of the moduli space of 4-punctured spheres with \(I \Gamma(2)\backslash \mathbb{H}\). Moreover, the 1- and 2-point functions on a torus are naturally identified with vector-valued modular resp. Jacobi forms for \(I\). Also, if the RCFT has additional structure, e.g. \(N = 2\) supersymmetry or a Lie algebra symmetry, then the 1-point functions on the torus can be augmented, becoming vector-valued Jacobi (hence matrix-valued modular) forms for \(I_0(2)\) or \(I_\Gamma(2)\) [19].

The impact in recent years of RCFT on mathematics makes it difficult to dismiss these as esoteric exotica. For example, 1-point torus functions for the orbifold of the Moonshine module \(V^\natural\) by subgroups of the Monster contain as very special cases the Norton series of generalized Moonshine, so a study of them could lead to extensions of the Monstrous Moonshine conjectures. For these less well-known applications to RCFT, the multiplier \(\rho\) will typically have infinite image, and the weights can be arbitrary rational numbers. Thus the typical classical assumptions that the weight be half-integral, and the modular form be fixed by some finite-index subgroup of \(I\), is violated by a plethora of potentially interesting examples. Hence in the following we do not make those classical assumptions (nor are they needed). In fact, there should be similar applications to sufficiently nice non-rational CFT, such as Liouville theory, where the weights \(w\) can be irrational.

In spite of its relevance, the general theory of vvmf has been slow in coming. Some effort has been made (c.f. [33, 17]) to lift to vvmf, classical results like dimension formulas and the ‘elementary’ growth estimates of Fourier coefficients. Moreover, differential equations have been recognised as valuable tools for studying vvmf, for many years (c.f. [1, 25, 26, 24] to name a few).

Now, an elementary observation is that a vvmf \(\mathcal{Z}(\tau)\) for a finite-index subgroup \(G\) of \(I\) can be lifted to one of \(I\), by inducing the multiplier. This increases the rank of the vvmf by a factor equal to the index. This isomorphism tells us that developing a theory of vvmf for \(I\) gives for free that of any finite-index subgroup. But more important perhaps, it also shows that the theory of vvmf for \(I\) contains as a small subclass the scalar modular forms for noncongruence subgroups. This means that one can only be so successful in lifting results from the classical (=scalar) theory to vvmf. We should be looking for new ideas!
Our approach is somewhat different, and starts from the heuristic that a vvmf for a Fuchsian group $\Gamma$ is a lift to $\mathbb{H}$ of a meromorphic section of a flat holomorphic vector bundle over the (singular) curve $\Gamma \backslash \mathbb{H}$ compactified if necessary by adjoining the cusps. The cusps mean we are not in the world of algebraic stacks. In place of the order-$d$ ODE on $\mathbb{H}$ studied by other authors, we consider a first order Fuchsian DE on the sphere. Fuchsian differential equations on compact curves, and vvmf for Fuchsian groups, are two sides of the same coin. Another crucial ingredient of our theory is the behaviour at the elliptic fixed-points. This has been largely ignored in the literature. For simplicity, this paper restricts to the most familiar (and important) case: $\Gamma = \text{SL}_2(\mathbb{Z})$, where $\Gamma \backslash \mathbb{H}^*$ is a sphere with 3 conical singularities (2 at elliptic points and 1 at the cusp). The theory for other Fuchsian groups is developed elsewhere [6, 14].

There are two aspects to the theory: holomorphic (no poles anywhere) and weakly holomorphic (poles are allowed at the cusps, but only there). We address both. We start with weakly holomorphic not because it is more interesting, but because it is easier, and this makes it more fundamental. There is nothing particularly special about the cusps from this perspective — the poles could be allowed at any finitely many $\Gamma$-orbits, and the theory would be the same.

Section 3 is the heart of this paper. There we establish existence, using Rõhrl’s solution to the Riemann–Hilbert problem. We obtain analogues of the Birkhoff–Grothendieck and Riemann–Roch Theorems. We find a dimension formula for holomorphic vvmf, and are able to quantify the failure of exactness of the functor assigning to multipliers $\rho$, spaces of holomorphic vvmf. Our arguments are simpler and much more general than others in the literature. In Section 4 we give several illustrations of the effectiveness of the theory.

### 2 Elementary Remarks

#### 2.1 The geometry of the modular group

Fix $\xi_n = \exp(2\pi i/n)$. Complex powers $z^w$ throughout the paper are defined by $z^w = |z|^w e^{i\arg(z)}$ for $-\pi \leq \arg(z) < \pi$. We write $\mathbb{C}[x]$ for the space of polynomials, $\mathbb{C}[[x]]$ for power series $\sum_{n=0}^{\infty} a_n x^n$, and $\mathbb{C}[x^{-1}, x]$ for Laurent expansions $\sum_{n=-N}^{\infty} a_n x^n$ for any $N$.

This paper restricts to the modular group $\Gamma := \text{SL}_2(\mathbb{Z})$. Write $T = \text{PSL}_2(\mathbb{Z})$. Throughout this paper we use

$$
S = \rho \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad T = \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \rho \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ST^{-1}.
$$

Write $\mathbb{H}^*$ for the extended half-plane $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Then $\Gamma \backslash \mathbb{H}^*$ is topologically a sphere. As it is genus 0, it is uniformised by a Hauptmodul, which can be chosen to be

$$
S = \rho \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad T = \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \rho \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ST^{-1}.
$$
\[ J(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 86429970q^3 + \cdots, \]  

(4)

where as always \( q = e^{2\pi i \tau} \). For \( k \geq 2 \) write \( E_k(\tau) \) for the Eisenstein series, normalised so that \( E_2(\tau) = 1 + \cdots \). For \( \Gamma \), the ring of \textit{weakly holomorphic} (i.e., poles allowed only at the cusp) modular functions is \( \mathbb{C}[J] \) and the ring of \textit{holomorphic} (i.e., holomorphic everywhere including the cusp) modular forms is \( \mathfrak{m} = \mathbb{C}[E_4, E_6] \).

In the differential structure induced by that of \( \mathcal{H} \), \( \Gamma \backslash \mathcal{H} \) will have a singularity for every orbit of \( \mathcal{T} \)-fixed-points. \( J(\tau) \) smooths out these 3 singularities. The important \( q \)-expansion is the local expansion at one of those singularities, but it is a mistake to completely ignore the other two, at \( \tau = i \) and \( \tau = \xi_6 \). These elliptic point expansions have been used for at least a century, even if they are largely ignored today. They play a crucial role in our analysis.

In particular, define \( \tau_2 = \epsilon_2(\tau - i)/(\tau + i) \) and \( j_2(w; \tau) = E_4(i)^{-w/4}(1 - \tau_2 \epsilon_2)^{-w} \), where \( \epsilon_2 = \pi \sqrt[3]{E_4(i)} \) and \( E_4(i) = 3! \Gamma(1/4)!/(2\pi)! \), and define \( \tau_3 = \epsilon_3(\tau - \xi_6)/(\tau - \xi_6^2) \) and \( j_3(w; \tau) = E_6(\xi_6)^{-w}/(1 - \tau_3 \epsilon_3)^{-w} \), where \( \epsilon_3 = \pi \sqrt[3]{E_6(\xi_6)^{1/3}} \) and \( E_6(\xi_6) = 27! \Gamma(1/12)!/(2^{9}\pi^{12}) \). The transformations \( \tau \mapsto \tau_i \) for \( i = 2, 3 \) map \( \mathcal{H} \) onto the discs \( |\tau_i| < e_i \) and send \( i \) and \( \xi_6 \) respectively to 0; \( j_2 \) and \( j_3 \) are proportional to the corresponding multipliers for a weight \( -w \) modular form. The elliptic fixed-point \( \tau = i \) is fixed by \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), which sends \( \tau_2 \mapsto -\tau_2 \). The elliptic fixed-point \( \tau = \xi_6 \) is fixed by \( \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \); it sends \( \tau_3 \mapsto \xi_6 \tau_3 \). If \( f \) is a scalar modular form of weight \( k \), and we write \( \tilde{f}_i(\tau) = j_i(k; \tau) f(\tau_i) \) for \( i = 2, 3 \), we get \( \tilde{f}_2(\tau_2) = i^k \tilde{f}(\tau_2) \) and \( \tilde{f}_3(\xi_6 \tau_3) = \xi_6^k \tilde{f}(\tau_3) \). We generalise this in Lemma 2.1 below.

We have rescaled \( \tau_2, \tau_3 \) by \( \epsilon_2, \epsilon_3 \) to clean up the expansions, but this isn’t important in the following. For instance, we have the rational expansions:

\[
\begin{align*}
  j_2(4; \tau) E_4(\tau) &= 1 + 10\tau_2^2/9 + 5\tau_4/27 + 4\tau_2^6/81 + 19\tau_2^8/5103 + \cdots, \\
  j_2(6; \tau) E_6(\tau) &= 2\tau_2 + 28\tau_4^2/27 + 56\tau_2^5/135 + 28\tau_2^7/405 + \cdots, \\
  J(\tau) &= 1728 + 6912\tau_2^2 + 11776\tau_4 + 1594112\tau_2^6/135 + \cdots. 
\end{align*}
\]  

Curiously, the expansion coefficients for \( J(\tau) \) at \( i \) are all \textit{positive} rationals, but infinitely many distinct primes divide the denominators. However, these denominators arise because of an \( n! \) that appears implicitly in these coefficients (see Proposition 17 in [11]). Factoring that off, the sequence becomes:

\[
1728, 10368, 158976, 3586752, 107057664, 4097780928, 193171879296, 1098718906592, \\
737967598470144, 57713234231210688, 5184724381875974016, \ldots
\]

Is there a Moonshine connecting these numbers with representation theory?

The multiplier systems (see Definition 2.1 below) of \( \Gamma \) at weight \( w \) are parametrised by the representations of \( \mathcal{T} \). In dimension \( d \), the space of \( \mathcal{T} \)-representations, — more precisely, the algebraic quotient of all group homomorphisms \( \mathcal{T} \to GL_d(\mathbb{C}) \) by the conjugate action of \( GL_d(\mathbb{C}) \), form a variety, and the completely reducible representations form an open subvariety. The connected components of this open subvariety correspond to ordered 5-tuples \( (\alpha_i; \beta_j) \) [35], where \( \alpha_i \) is the
eigenvalue multiplicity of $(-1)^i$ for $S$, and $\beta_j$ is that of $\zeta_j^i$ for $U$. Of course $a_0 + a_1 = \beta_0 + \beta_1 + \beta_2 = d$. When $d > 1$, the $(a_i; \beta_j)$ component is nonempty iff

$$\max\{\beta_j\} \leq \min\{a_i\},$$

in which case its dimension is $d^2 + 1 - \sum_i a_i^2 - \sum_j \beta_j^2$ [35]. The irreducible $T^\tau$-representations of dimension $d < 6$ are explicitly described in [34]. For irreducible representations, (8) is obtained using quiver-theoretic means in [35]; later we recover [3] within our theory.

### 2.2 Definitions

Write $1_d$ for the $d \times d$ identity matrix, and $e_j = (0, \ldots, 1, \ldots, 0)^T$ its $j$th column.

**Definition 2.1(a)** An admissible multiplier system $(\rho, w)$ consists of some $w \in \mathbb{C}$ called the weight and a map $\rho: \Gamma \rightarrow GL_d(\mathbb{C})$ called the multiplier, for some positive integer $d$ called the rank, such that:

(i) the associated automorphy factor

$$\tilde{\rho}_w(\gamma, \tau) = \rho(\gamma)(c\tau + d)^w$$

satisfies, for all $\gamma_1, \gamma_2 \in \Gamma$,

$$\tilde{\rho}_w(\gamma_1 \gamma_2, \tau) = \tilde{\rho}_w(\gamma_1, \gamma_2 \tau) \tilde{\rho}_w(\gamma_2, \tau);$$

(ii) $\rho(1_d)$ and $e^{-\pi i w} \rho(-1_d)$ both equal the identity matrix.

The conditions on $\rho(\pm 1_d)$ in (ii) are necessary (and sufficient, as we’ll see) for the existence of nontrivial vvmf at weight $w$. In practise modular forms are most important for their Fourier expansions. This is often true for vvmf, and this is why we take $\rho$ to be matrix-valued in Definition 2.1. Note that the multiplier $\rho(\gamma)$ need not be unitary, and the weight $w$ need not be real.

When $w \in \mathbb{Z}$, $(\rho, w)$ is admissible iff $\rho$ is a representation of $\Gamma$ satisfying $\rho(-1_d) = e^{\pi i w}$. When $w \not\in \mathbb{Z}$, $\rho$ is only a projective representation of $\Gamma$, and is most elegantly described in terms of the braid group $B_3$. More precisely, $B_3 = \langle \sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$ is a central extension by $\mathbb{Z}$ of $\Gamma$, where the surjection $B_3 \rightarrow \Gamma$ sends $\sigma_1 \rightarrow \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\sigma_2 \rightarrow \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. The kernel is $\langle (\sigma_1 \sigma_2 \sigma_1)^4 \rangle$, half of the centre of $B_3$. Then $(\rho, w)$ is admissible iff there is a representation $\tilde{\rho}$ of $B_3$ (necessarily unique) satisfying

$$\tilde{\rho}(\sigma_1) = T, \quad \tilde{\rho}((\sigma_1 \sigma_2 \sigma_1)^{-1}) = S, \quad \tilde{\rho}((\sigma_1 \sigma_2)^{-2}) = U$$

(10)

and also $\tilde{\rho}((\sigma_1 \sigma_2 \sigma_1)^2) = e^{\pi i w}$. Alternatively, we see in Lemma 3.1 below that for any $w \in \mathbb{C}$, there is an admissible system $(\nu_w, w)$ of rank 1; then $(\rho, w)$ is admis-
sible iff $\overline{\nu} \otimes \rho$ is a $\Gamma$-representation. From any of these descriptions, we see that a multiplier $\rho$ determines the corresponding weight $w$ modulo 2.

**Definition 2.2.** Let $(\rho, w)$ be an admissible multiplier system of rank $d$. A map $X : \mathbb{H} \rightarrow \mathbb{C}^d$ is called a vector-valued modular form (vvmf) provided

$$X(\gamma \tau) = \tilde{\mu}_w(\gamma, \tau) X(\tau)$$

for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$, and each component $X_i(\tau)$ is meromorphic throughout $\mathbb{H}^*$. We write $M^!(w)(\rho)$ for the space of all weakly holomorphic vvmf, i.e. those holomorphic throughout $\mathbb{H}$.

Meromorphicity at the cusps is defined as usual, e.g. by a growth condition or through the $q$-expansion given shortly.

Generic $\rho$ will have $T$ diagonalisable, which we can then insist is diagonal without loss of generality. For simplicity, we assume throughout this paper that $T$ is diagonal. The following theory generalises to $T$ a direct sum of Jordan blocks (the so-called ‘logarithmic’ case) without difficulty, other than notational awkwardness $[6,14]$.

Assume then that $T$ is diagonal. By an exponent $\lambda$ for $\rho$, we mean any diagonal matrix such that $e^{2\pi i \lambda} = T$, i.e. $T_{jj} = e^{2\pi i \lambda_j}$ for all $j$. An exponent is uniquely defined modulo 1. It is typical in the literature to fix the real part of $\lambda$ to be between 0 and 1. But we learn in Theorem 3.2 below that often there will be better exponents to choose.

For any vvmf $X$ and any exponent $\lambda$, $q^{-\lambda} X(\tau)$ will be invariant under $\tau \mapsto \tau + 1$, where we write $q^\lambda = \text{diag}(e^{2\pi i \lambda_1}, \ldots, e^{2\pi i \lambda_d})$. This gives us a Fourier expansion

$$X(\tau) = q^\lambda \sum_{n=-\infty}^{\infty} X(n) q^n,$$

where the coefficients $X(n)$ lie in $\mathbb{C}^d$. $X(\tau)$ is meromorphic at the cusp $\infty$ iff only finitely many coefficients $X(n)$, for $n < 0$, are nonzero.

### 2.3 Local expansions

We are now ready to describe the local expansions about any of the 3 special points $i\infty, 0, \xi_6$ (indeed, the same method works for any point in $\mathbb{H}^*$).

**Lemma 2.1.** Let $(\rho, w)$ be admissible and $T$ diagonal. Then any $X \in M_w^!(\rho)$ obeys

$$X(\tau) = q^\lambda \sum_{n=0}^{\infty} X(n) q^n = j_2(w; \tau)^{-1} \sum_{n=0}^{\infty} X(n) \tau_2^n = j_3(w; \tau)^{-1} \sum_{n=0}^{\infty} X(n) \tau_3^n,$$

for some exponent $\lambda$. These converge for $0 < |q| < 1$ and $|\tau| < \epsilon$. Also,
\[ e^{\frac{\pi i w}{2}} S[X_{(n)}] = (-1)^n X_{(n)} \quad \text{and} \quad e^{z + \frac{\pi i w}{3}} Y[X_{(n)}] = \frac{\xi}{3} X_{(n)}. \quad (14) \]

The existence of the \( q \)-series follows from the explanation at the end of last subsection. For \( \tau = \alpha (\tau = \xi_6 \text{ is identical}), j_2(w; \tau) X(\tau) \) is holomorphic in the disc \( |\tau_2| < \epsilon_2 \) and so has a Taylor expansion. The transformation \( (14) \) can be seen by direct calculation, but will be trivial once we know Lemma 3.1 below.

We label components by \( X_{(n)} \); etc. A more uniform notation would have been to define e.g. \( q_2 = \tau_2^2 \) find a matrix \( P \) and an ‘exponent matrix’ \( \lambda \) whose diagonal entries lie in \( \frac{1}{2} \mathbb{Z} \), such that \( P^2 \rho^{-1} = \xi_2 \) and \( \mathfrak{X} = j_2(w; \tau)^{-1} P_2^{-1} q_2^2 \sum_{n=0}^{\infty} X_{(n)} q_2^n \).

For most purposes the simpler \( (13) \) is adequate, but see \( (35) \) below.

2.4 Differential operators

Differential equations have played a large role in the theory of \( \text{vvmf} \). The starting point is always the modular derivative

\[ D_w f = \frac{1}{2\pi i} \frac{d}{d\tau} \frac{w}{12} E_2 = q \frac{d}{dq} \frac{w}{12} E_2, \quad (15) \]

where \( E_2(\tau) = 1 - 24q - 72q^2 - \cdots \) is the quasi-modular Eisenstein function. Note that \( D_{12} \) kills the discriminant form \( \Delta(\tau) = \eta(\tau)^{24} \). This \( D_w \) maps \( \mathcal{M}_w(\rho) \) to \( \mathcal{M}_{w+2}(\rho) \). It is a derivation in the sense that if \( f \in \mathcal{M} \) is weight \( k \) and \( X \in \mathcal{M}_w(\rho) \), then \( D_w(f)(X + fD_w(X)) \). We write \( D_w = D_{w+2j-2} \circ \cdots \circ D_{w+2} \circ D_w \).

There are several different applications of differential equations to modular forms — some are reviewed in \[ 11 \]. But outside of our work, the most influential for the theory of \( \text{vvmf} \) (see e.g. \[ 1, 28, 24, 21 \]) has been the differential equation coming from the Wronskian (see e.g. \[ 24 \] for the straightforward proof):

**Lemma 2.2(a)** Let \( (\rho, w) \) be admissible. For \( X \in \mathcal{M}_w(\rho) \), define

\[ \text{Wr}(X) := \det \begin{bmatrix} X_1 & d^{w} & \cdots & d^{w} \psi_{d-1} X_1 \\ \vdots & \vdots & \ddots & \vdots \\ X_{d} & d^{w} & \cdots & d^{w} \psi_{d-1} X_{d} \end{bmatrix} = \det \begin{bmatrix} X_1 & D_w X_1 & \cdots & D_{w}^{d-1} X_1 \\ \vdots & \vdots & \ddots & \vdots \\ X_{d} & D_w X_{d} & \cdots & D_{w}^{d-1} X_{d} \end{bmatrix}. \]

Then \( \text{Wr}(X)(\tau) \in \mathcal{M}_{(w+2j-2)}(\det \rho) \). If the coefficients of \( X \) are linearly independent over \( \mathbb{C} \), then the function \( \text{Wr}(X)(\tau) \) is nonzero.

**b)** Given admissible \( (\rho, w) \) and \( X \in \mathcal{M}_w(\rho) \), define an operator \( L_X \) on the space of all functions \( y \) meromorphic on \( \mathbb{H} \), by
\[
L_\mathfrak{X} = \det \begin{pmatrix}
\sigma_1 & \ldots & \sigma_{d-1} \\
\sigma_2 & \ldots & \sigma_d \\
\vdots & \ddots & \vdots \\
\sigma_d & \ldots & \sigma_d
\end{pmatrix}
= \sum_{l=0}^{d} h_l(\tau) D_l^i \sigma_1,
\]

where \( h_d = \text{Wr}(\mathfrak{X}) \) and each \( h_l \) is a (meromorphic scalar) modular form of weight \((d+1)\ell - 2\ell\) with multiplier \( \det \rho \). Then \( L_\mathfrak{X} \mathfrak{X}_i = 0 \) for all components \( \mathfrak{X}_i \) of \( \mathfrak{X} \). Conversely, when the components of \( \mathfrak{X} \) are linearly independent, the solution space to \( L_\mathfrak{X} y = 0 \) is \( \text{Span}_C \{ \mathfrak{X}_i \} \).

In our theory, the differential equation (16) plays a minor role. More important are the differential operators which don’t change the weight:

\[
\nabla_{1,w} = \frac{E_4}{2} D_w, \quad \nabla_{2,w} = \frac{E_4}{2} D_w^2, \quad \nabla_{3,w} = \frac{E_6}{3} D_w^3.
\]

Each \( \nabla_{i,w} \) operates on \( \mathcal{M}_w^{-1} (\rho) \), and an easy calculation shows that any \( f D_w^l \) for \( f \in \mathcal{M}_w^{-1} (\rho) \) (1) is a polynomial in these three \( \nabla_{i,w} \) with coefficients in \( \mathbb{C} [J] \). Conversely, \( \nabla_{3,w} \) is not in \( \mathbb{C} [J, \nabla_{1,w}, \nabla_{2,w}] \) and \( \nabla_{2,w} \) is not in \( \mathbb{C} [J, \nabla_{1,w}] \). The reason \( \nabla_{2,w} \) and \( \nabla_{3,w} \) are needed is because of the elliptic points of order 2 and 3 — this is made explicit in the proof of Proposition 3.2. It is crucial to our theory that \( \mathcal{M}_w^{-1} (\rho) \) is a module over \( \mathbb{C} [J, \nabla_{1,w}, \nabla_{2,w}, \nabla_{3,w}] \).

We sometimes drop the subscript \( w \) on \( D_w \) and \( \nabla_{i,w} \) for readability.

### 3 Our main results

#### 3.1 Existence of vvmf

The main result (Theorem 3.1) of this subsection is the existence proof of vvmf for any \((\rho, w)\). As a warm-up, let us show there is a (scalar) modular form of every complex weight, and compute its multiplier.

**Lemma 3.1.** For any \( w \in \mathbb{C} \), there is a weakly holomorphic modular form \( \Delta^w(\tau) = q^w (1 - 24wq + \cdots) \) of weight \( 12w \), nonvanishing everywhere except at the cusps. The multiplier \( \nu_{12w} \), in terms of the braid group \( B_3 \) (recall (10)), is

\[
\tilde{\nu}_{12w}(\sigma_1) = \tilde{\nu}_{12w}(\sigma_2) = \exp(2\pi i w).
\]

**Proof.** First note that the discriminant form \( \Delta(\tau) \) is holomorphic and nonzero in the simply-connected domain \( \mathbb{H} \), and so has a logarithmic derivative \( \text{Log} \Delta(\tau) \) there. Hence \( \Delta^w(\tau) = \exp(w \text{Log} \Delta(\tau)) \) is well-defined and holomorphic throughout \( \mathbb{H} \). It is easy to verify that \( \Delta^w(\tau) \) satisfies the differential equation

\[
\frac{1}{2\pi i} \frac{df}{d\tau} = w E_2(\tau) f(\tau)
\]
— indeed, this simply reduces to the statement that $E_2(\tau)$ is the logarithmic derivative of $\Delta(\tau)$. Therefore any solution to (13) is a scalar multiple of $\Delta^w(\tau)$.

Now, fix $\gamma \in \Gamma$. Then $f(\tau) = (c\tau+d)^{-12w}\Delta^w(\gamma \tau)$ exists and is holomorphic throughout $\mathbb{H}$ for the same reason. Note that $f(\tau)$ also satisfies the differential equation (19):

$$\frac{1}{2\pi i} \frac{d}{d\tau} f(\tau) = \frac{-6c w}{\pi i} (c\tau+d)^{-12w}(c\tau+d)^{-1} \Delta^w(\gamma \tau)$$

$$(c\tau+d)^{-12w}(c\tau+d)^{-2} w E_2(\gamma \tau) \Delta^w(\gamma \tau) = w E_2(\tau) f(\tau),$$

using quasi-modularity of $E_2(\tau)$, and thus $f(\tau) = \nu \Delta^w(\tau)$ throughout $\mathbb{H}$, for some constant $\nu = \nu(\gamma) \in \mathbb{C}$.

The final step needed to verify that $\Delta^w(\tau)$ is a modular form of weight $12w$ is that it behaves well at the cusps. By the previous paragraph it suffices to consider $\infty$. But there $\Delta^w(\tau)$ has the expansion $\Delta^w(\tau) = q^w (1 - 24wq + \cdots)$, up to a constant factor which we can take to be 1. As all cusps lie in the $I$-orbit of $\infty$, we see that $\Delta^w(\tau)$ is indeed holomorphic at all cusps.

In general, $\nu_{12w}$ can be interpreted as a representation $\tilde{\nu}$ of $B_3$. The expansion tells us that $\Delta^w(\tau + 1) = \exp(2\pi i w) \Delta^w(\tau)$, so we have $\tilde{\nu}_{12w} (\sigma_1) = \exp(2\pi i w).$ From the familiar $B_3$ presentation we see that any one-dimensional representation of $B_3$ takes the same value on both generators $\sigma_1$, so $\tilde{\nu}_{12w}$ is determined. \hfill \Box

For admissible $(\rho, w)$, Lemma 3.1 says the matrices $e^{\pi i w/2} S$ and $e^{2\pi i w/3} U$ have order 2 and 3 respectively. Write $a_j(\rho, w)$ for the multiplicity of $(-1)^j$ as an eigenvalue of $e^{\pi i w/2} S$, and $b_j(\rho, w)$ for the eigenvalue multiplicity of $\bar{e}_3^j$ for $e^{2\pi i w/3} U$.

**Theorem 3.1.** Let $(\rho, w)$ be admissible and $T$ diagonal. Then there is a $d \times d$ matrix $\Phi(\tau)$ and exponent $\lambda$ such that the columns of $\Phi$ lie in $\mathcal{M}_w^d(\rho)$, and

$$\Phi(\tau) = q^\lambda F(q) = q^\lambda \sum_{n=0}^\infty F(n) q^n,$$

where the matrix $F(q)$ is holomorphic and invertible in a neighbourhood of $q = 0$.

**Proof.** Consider any representation $\rho'$ of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \cong F_2$, the free group with 2 generators. Röhl's solution [30] to the Riemann–Hilbert problem [7,8] (see also [16]) says that there exists a Fuchsian differential equation

$$\frac{d}{dz} \psi(z) = \psi\left(\frac{A_1}{z} + \frac{A_2}{z-1} + \frac{B}{z-b}\right)$$

on the Riemann sphere $\mathbb{P}^1$ whose monodromy is given by $\rho'$ — i.e. the monodromy corresponding to a small circle about 0 and 1, respectively, equals the value of $\rho'$ at the corresponding loops in the fundamental group. There will also be a simple pole at $\infty$, with residue $A_\infty := -A_0 - A_1 - B$. The $B$ term in (21) corresponds to an apparent singularity; it can be dropped if the monodromies about 0 or 1 or $\infty$ have finite order [7,8] (which will happen for the $\rho'$ of interest to us).
About any of these 3 or 4 singular points \( c \in \{0, 1, \infty, b\} \), Levelt [20] proved that a solution \( \Psi(z) \) to such a differential equation has the form

\[
\Psi(z) = P_c^{-1} z^{N_c} \tilde{z}^{\lambda_c} F_c(z)
\]

where \( N_c \) is nilpotent, \( \lambda_c \) is diagonal, \( F_c(z) \) is holomorphic and holomorphically invertible about \( z = c \), and \( \tilde{z} \) is a coordinate on the universal cover of a small disc punctured at \( c \). By [19], we may take \( N_c + \lambda_c \) to be conjugate to \( A_c \).

Now suppose we have a representation of the free product \( F \cong \mathbb{Z}_2 * \mathbb{Z}_3 \). This is a homomorphic image of the free group \( F_2 \), so we can lift \( \rho' \) to \( \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \). For us, \( \mathbb{P}^1 \) is \( \mathbb{P}^1 \setminus \mathbb{P}^1 \), with (smooth) global coordinate \( z = J(\tau)/1728 \). The monodromy at \( z = 0 \) and \( z = 1 \) has finite order 2 and 3 respectively, so we won’t need the apparent singularity \( b \). The point \( \infty \) corresponds to \( \infty \) (or rather its \( \Gamma \)-orbit), where \( q_\infty = q \), and we have \( N_\infty = 0 \) and \( P_\infty = 1_d \) since \( T \) is diagonal; in this case Levelt’s equation (22) reduces to (20). The singularities 0 and 1 correspond to the order 2 and 3 elliptic points \( i \) and \( \xi_6 \); for them, \( N = 0 \), \( z = \tau_i \) locally looks like \((\tau - i)^2\) and \((\tau - \xi_6)^3\), and the diagonal elements of \( \lambda \) lie in \( \frac{1}{2}\mathbb{Z} \) and \( \frac{3}{2}\mathbb{Z} \). Then Levelt’s equation (22) says \( \Psi(J(\tau)/1728) \) is meromorphic at \( i \) and \( \xi_6 \) (hence their \( \Gamma \)-orbits). It is automatically holomorphic at all other points.

The desired matrix is \( \Phi(\tau) = \Phi(J(\tau)) \Delta_w^{12}(\tau) J(\tau)^m (J(\tau) - 1)^n \), where \( m, n \in \mathbb{Z}_{\geq 0} \) are taken large enough to kill any poles at \( z = 0 \) and 1 (i.e. at all elliptic points), and \( \Psi \) corresponds to the \( \overline{T} \)-representation \( \rho' = v^{-w} \otimes \rho \). \( \square \)

The proof generalises without change to nondiagonalisable \( T \), and to any other genus-0 Fuchsian group of the first kind [14]. Theorem 3.1 is vastly more general than previous vvmf existence proofs. Previously (see [17] [18]), existence of vvmf was only established for \( \Gamma \) with real weight \( w \), and requiring in addition that the eigenvalues of \( T \) all have modulus 1. Their proof used Poincaré series; the difficult step there is to establish convergence, and that is what has prevented their methods to be generalised. That analytic complexity was handled here by Röhrli’s argument.

### 3.2 Mittag-Leffler

Let \( (\rho, w) \) be admissible and \( T \) diagonal. In this subsection we study the principal part map and calculate its index. This is fundamental to our theory. As always in this paper, the generalisation to nondiagonalisable \( T \) and to other genus-0 Fuchsian groups is straightforward [6].

Given any exponent \( \lambda \) and any \( \mathcal{X}(\tau) \in \mathcal{M}_w'(\rho) \), we have the \( q \)-expansion [12]. Define the principal part map \( \mathcal{P}_\lambda : \mathcal{M}_w'(\rho) \to \mathbb{C}[q^{-1}] \) by

\[
\mathcal{P}_\lambda(\mathcal{X}) = \sum_{n \geq 0} \mathcal{X}_n q^n.
\]
When we want to emphasise the domain, we’ll write this $\mathcal{P}_\lambda(\rho, w)$.

**Theorem 3.2.** [5] Assume $(\rho, w)$ is admissible, and $T$ is diagonal. Recall the eigenvalue multiplicities $\alpha_j = \alpha_j(\rho, w)$ and $\beta_j = \beta_j(\rho, w)$ from Section 3.1.

(a) For any exponent $\lambda$, $\mathcal{P}_\lambda : \mathcal{M}^1_\wedge(\rho) \to \mathbb{C}^d[q^{-1}]$ has finite-dimensional kernel and cokernel, and the index is

$$\dim \ker \mathcal{P}_\lambda - \dim \text{coker} \mathcal{P}_\lambda = -\text{Tr} \lambda + c(\rho, w),$$

for

$$c(\rho, w) = \frac{(w - 7)d}{12} + \frac{e^{\pi i w / 2}}{4} \text{Tr} S + \frac{2}{3\sqrt{3}} \text{Re} \left( e^{-\pi i \frac{2j w}{3}} \text{Tr} U \right)$$

$$= \frac{wd}{12} \frac{\alpha_1}{2} \frac{\beta_1 + 2\beta_2}{3}.$$  \hspace{1cm} (24)

(b) There exist exponents $\lambda$ for which $\mathcal{P}_\lambda : \mathcal{M}^1_\wedge(\rho) \to \mathbb{C}^d[q^{-1}]$ is a vector space isomorphism.

By a bijective exponent we mean any exponent $\lambda$ for which $\mathcal{P}_\lambda : \mathcal{M}^1_\wedge(\rho) \to \mathbb{C}^d[q^{-1}]$ is an isomorphism. Of course by (24) its trace $\sum \lambda_{ij}$ must equal $c(\rho, w)$, but the converse is not true as we will see.

For example, for the trivial 1-dimensional representation, $T = 1$ so an exponent is just an integer. Here, $\mathcal{M}^1_\wedge(1) = \mathbb{C}[J]$ and $c(1, 0) = 0$. The map $\mathcal{P}_1$ is injective but not surjective (nothing has principal part 1), while $\mathcal{P}_1$ is surjective but not injective (it kills all constants). For another example, taking $\rho = \nu_{-2}$ (the multiplier of $\eta^{-4}$), we have $c(\nu_{-2}, 0) = -7/6$.

It is standard in the literature to restrict from the start to exponents satisfying $0 \leq \lambda_{ij} < 1$. However, such $\lambda$ are seldom bijective. It is rarely wise to casually throw away a freedom.

Theorem 3.2(b) first appeared in [5], though for restricted $(\rho, w)$, and with the erroneous claim that $\lambda$ is bijective iff $\text{Tr} \lambda = c(\rho, w)$. The deeper part of Theorem 3.2 is the index formula, which is new. We interpret it later as Riemann–Roch, and obtain from it dimensions of spaces of holomorphic vvmf.

The right-side of (25) is always integral. To see that, take $w = 0$ and note

$$\exp(2\pi i \text{Tr} \lambda) = \det T = \det S \det U^{-1} = (-1)^{\alpha_1} \xi^{-\beta_1 + \beta_2}.$$  \hspace{1cm} (26)

Fix an admissible $(\rho, w)$ with diagonal $T$, and a bijective exponent $\Lambda$. As a vector space over $\mathbb{C}$, $\mathcal{M}^1_\wedge(\rho)$ has a basis

$$\mathcal{M}^1_\wedge(\rho) = \mathcal{P}_{\Lambda}^{-1}(q^{-n} e_j) = q^n e_j + \sum_{m=1}^{\infty} \mathcal{M}^{(j;n)}(m),$$

where $e_j = (0, \ldots, 1, \ldots, 0)^t$ and $n \in \mathbb{Z}_{\geq 0}$. We describe next subsection an effective way to find all these $\mathcal{M}^{(j;n)}(\tau)$, given the $d^2$ coefficients.
\[ \chi_{ij} := \chi_{ni}^{(j)} \in \mathbb{C}. \] (28)

We learn there that \( \mathcal{M}_w^1(\rho) \) is under total control once a bijective exponent \( \lambda \) and its corresponding matrix \( \chi = \chi(\lambda) \) are found.

Recall that for fixed \( \rho \), the weight \( w \) is only determined mod 2, i.e. \( (\rho, w) \) is admissible iff \( (\rho, w + 2k) \) is, for any \( k \in \mathbb{Z} \). We find from the definition of the \( \alpha_i \) and \( \beta_j \) that

\[
\alpha_j(\rho, w + 2k) = \alpha_{j+k}(\rho, w), \quad \beta_j(\rho, w + 2k) = \beta_{j+k}(\rho, w). \quad (29)
\]

Plugging this into (25), we obtain the trace of a bijective exponent for \( (\rho, w + 2k) \):

\[
c_{(\rho, w + 2k + 12l)} = c_{(\rho, w)} + ld + \begin{cases} 
0 & \text{if } k = 0 \\
\alpha_1 - \beta_0 & \text{if } k = 1 \\
\beta_2 & \text{if } k = 2 \\
\alpha_1 & \text{if } k = 3 \\
\beta_1 + \beta_2 & \text{if } k = 4 \\
\alpha_1 + \beta_2 & \text{if } k = 5 
\end{cases}. \quad (30)
\]

**Sketch of proof of Theorem 3.2.** (see [3] for the complete proof and its generalisation). First of all, it is easy to show that if the real part of an exponent \( \lambda \) is sufficiently large, then \( \mathcal{P}_\lambda \) is necessarily injective. This implies that the kernel of any \( \mathcal{P}_\lambda' \) is finite-dimensional, in fact \( \dim \ker \mathcal{P}_\lambda' \leq \sum_j \max \{ \lambda_j - \lambda'_j, 0 \} \) for any \( \lambda \) with \( \mathcal{P}_\lambda \) injective.

Let \( M \) be the \( \mathbb{C}[f] \)-span of the columns of the matrix \( \Phi \) of Theorem 3.1. Let \( \lambda_M \) be the corresponding exponent appearing in (20). Then invertibility of \( F(q) \) implies invertibility of \( \mathcal{R}_{(0)} \), which in turn implies \( \mathcal{P}_{\lambda_M} : M \to \mathbb{C}^d[q^{-1}] \) is surjective, and hence so is \( \mathcal{P}_{\lambda_M} : \mathcal{M}_w^1(\rho) \to \mathbb{C}^d[q^{-1}] \). This implies any cokernel is also finite-dimensional.

A bijective exponent can be obtained by starting from \( \lambda_M \), and recursively increasing one of its entries by \( +1 \) so that something in the kernel of the previous \( \mathcal{P} \) is no longer in the kernel of the new \( \mathcal{P} \). This implies the range is unchanged. This process must terminate, by finiteness of the kernel.

The index formula (24), for some value of \( c_{(\rho, w)} \), is now computed by showing that, whenever \( \lambda' \geq \lambda \), the index of \( \mathcal{P}_\lambda \) minus that of \( \mathcal{P}_\lambda' \) equals \( \text{Tr}(\lambda' - \lambda) \). The constant \( c_{(\rho, w)} \) is computed next subsection.

**Corollary 3.1.** Suppose \( (\rho, w) \) is admissible, \( T \) is diagonal, and \( \rho \) is completely reducible with no 1-dimensional subrepresentation. Then setting \( \varepsilon = 0, 1 \) for \( d \) even, odd respectively, we obtain the bounds

\[
\frac{wd}{12} - d + \frac{\varepsilon}{4} \leq c_{(\rho, w)} \leq \frac{wd}{12} - \frac{5d}{12} - \frac{\varepsilon}{4}. \quad (31)
\]

**Proof:** Assume without loss of generality that weight \( w = 0 \). Then (25) tells us that \( c_{(\rho, w)} = -\alpha_1/2 - (\beta_1 + 2\beta_2)/3 \), where \( \alpha_1 = \alpha_1(\rho, w), \beta_1 = \beta_1(\rho, w) \).
First, let’s try to maximise $c_{(\rho, w)}$, subject to the inequalities \((\ref{ineq})\). Clearly, $c_{(\rho, w)}$ is largest when $a_0 \geq a_1 \geq \beta_0 \geq \beta_1 \geq \beta_2 \geq 0$. In fact, we should take $\beta_0$ as large as possible, i.e. $a_1 = \beta_0$. Then our formula for $c_{(\rho, w)}$ simplifies to $-\frac{d}{2} + \frac{\beta_1 - \beta_2}{6}$. It is now clear this is maximised by $\beta_2 = \epsilon$ and $\beta_0 = \beta_1 = \frac{d - \epsilon}{2}$, which recovers the upper bound in \((\ref{ineq})\). Similarly, the lower bound in \((\ref{ineq})\) is realised by $a_0 = \beta_2 = \beta_1 = (d - \epsilon)/2$. \(\Box\)

The question of which exponents $\lambda$ with the correct trace are bijective, can be subtle, though we see next that for generic $\rho$ the trace condition $\text{Tr} \lambda = c_{(\rho, w)}$ is also sufficient. With this in mind, define the $\infty \times \infty$ complex matrix $\mathcal{X} = (\mathcal{X}^{ij}_{\rho,n})$ built from the $m$th coefficient of the $i$th component of the basis \((\ref{basis})\). For any $\ell \in \mathbb{Z}^d$ define a $\left(\sum_i \max\{\ell_i, 0\}\right) \times \left(\sum_j \max\{-\ell_j, 0\}\right)$ submatrix $\mathcal{X}(\ell)$ of $\mathcal{X}$ by restricting to the rows $(i; m)$ with $0 < m \leq \ell_i$ and columns $(j; n)$ with $0 \leq n < -\ell_j$.

**Proposition 3.1.** Let $(\rho, w)$ be admissible, $T$ diagonal, and $\Lambda$ bijective. Then an exponent $\lambda$ is also bijective iff the matrix $\mathcal{X}(\lambda - \Lambda)$ is invertible.

The effectiveness of this test will be clear next subsection, where we explain how to compute the $\mathcal{X}^{ij}_{\rho,n}$ and hence the submatrices $\mathcal{X}(\ell)$. Of course, invertibility forces $\mathcal{X}(\lambda - \Lambda)$ to be square, i.e. $\sum_i \ell_i = 0$, i.e. that $\text{Tr} \lambda = \text{Tr} \Lambda$.

To prove Proposition 3.1, observe that the spaces $\ker \mathcal{P}_\rho$ and $\text{null} \mathcal{X}(\ell)$ are isomorphic, with $v \in \text{null} \mathcal{X}(\ell)$ identified with $\sum_j \sum_{n=0}^{\ell_j - 1} v_{j,n} \mathcal{X}^{ij}_{\rho,n}$ (the nullspace null$M$ of a $m \times n$ matrix $M$ is all $v \in \mathbb{C}^n$ such that $M v = 0$). Similarly, the spaces $\text{coker} \mathcal{P}_\rho$ and null $\mathcal{X}(\ell)^t$ are isomorphic.

For example, given a bijective exponent $\Lambda$, Proposition 3.1 says that $\chi_{ij} \neq 0$ iff $\Lambda + e_i - e_j$ is bijective, where $\chi$ is defined in \((\ref{chi})\).

### 3.3 Birkhoff–Grothendieck and Fuchsian differential equations

As mentioned in the introduction, one of our (vague) heuristics is to think of vvmf as meromorphic sections of holomorphic bundles over $\mathbb{P}^1$. But the Birkhoff–Grothendieck Theorem says that such a bundle is a direct sum of line bundles. If taken literally, it would say that, up to a change $P$ of basis, each component $\mathcal{X}_i(\tau)$ of any $\mathcal{X} \in \mathcal{H}_w(\rho)$ would be a scalar modular form of weight $w$ (and some multiplier) for $\Gamma$. This is absurd, as it only happens when $\rho$ is equivalent to a sum of 1-dimensional projective representations. The reason Birkhoff–Grothendieck cannot be applied here is that we do not have a bundle (in the usual sense) over $\mathbb{P}^1$ — indeed, our space $\Gamma \backslash \mathbb{H}^*$ has three singularities.

Nevertheless, Theorem 3.3(a) below says Birkhoff–Grothendieck still holds in spirit.

**Theorem 3.3.** Let $(\rho, w)$ be admissible, $T$ diagonal, and $\Lambda$ bijective.

(a) $\mathcal{H}_w(\rho)$ is a free $\mathbb{C}[J]$-module of rank $d = \text{rank}(\rho)$. Free generators are $\mathcal{X}^{ij}_{\rho,n}(\tau)$ (see \((\ref{basis})\)).
(b) Let \( \Xi(\tau) = q^4(1_d + \chi q + \sum_{n=2}^{\infty} \Xi(n)q^n) \) be the \( d \times d \) matrix whose columns are the \( \mathcal{X}^{(j,0)}(\tau) \). Then

\[
\frac{E_d(\tau) E_d(\tau)}{\Delta(\tau)} D_w \Xi(\tau) = \Xi(\tau) \{ (J(\tau) - 984)A_w + \chi_w + [A_w, \chi_w] \} ,
\]

where \( A_w := A - \frac{1}{4} l_d, \chi_w := \chi + 2w l_d \), and \([\cdot, \cdot]\) denotes the usual bracket.

(c) Assume weight \( w = 0 \). The multi-valued function \( \Xi(z) := \Xi(\tau(z)) \), where \( \tau(z) = f(\tau)/1728 \), obeys the Fuchsian differential equation

\[
\frac{d}{dz} \Xi(z) = \Xi(z) \left( \frac{\alpha_2}{z-1} + \frac{\alpha_3}{z} \right),
\]

\[
\alpha_2 = \frac{31}{72} A + \frac{1}{1728}(\chi + [A, \chi]), \quad \alpha_3 = \frac{41}{72} A - \frac{1}{1728}(\chi + [A, \chi])
\]

(recall (21)). Moreover, \( \alpha_2, \alpha_3 \) are diagonalisable, with eigenvalues in \([0, \frac{1}{3}]\) and \([0, \frac{1}{3}, \frac{2}{3}]\), respectively.

Sketch of proof (see [6] for the complete proof and generalisation). The basis vvmf \( \mathcal{X}^{(j,n)}(\tau) \) exist by surjectivity of \( \mathcal{P}_A \). To show \( \mathcal{M}_w^i(\rho) \) is generated over \( \mathbb{C}[J] \) by the \( \mathcal{X}^{(j,0)}(\tau) \), follows from an elementary induction on \( n \): if the \( \mathcal{X}^{(j,m)}(\tau) \) all lie in \( \sum_i \mathbb{C}[J] \mathcal{X}^{(i,0)}(\tau) \) for all \( i \) and all \( m < n \), then \( \mathcal{X}^{(j,n)}(\tau) = J(\tau) \mathcal{X}^{(j,n-1)}(\tau) + \sum_i \mathbb{C}[J] \mathcal{X}^{(i,0)}(\tau) \subseteq \sum_i \mathbb{C}[J] \mathcal{X}^{(i,0)}(\tau) \), using the fact that \( \mathcal{P}_A \) is injective. That these generators are free, follows by noting the determinant of \( \Xi(\tau) \) has a nontrivial leading term (namely \( q^4 \)) and so is nonzero.

The columns of \( \nabla_{1,w} \Xi(\tau) \) also lie in \( \mathcal{M}_w^i(\rho) \), and so \( \nabla_{1,w} \Xi(\tau) = \Xi(\tau) D(J) \) for some \( d \times d \) matrix-valued polynomial \( D(J) \), which can be determined by Theorem 3.2 by comparing principal parts. (33) follows directly from (32), by changing variables. Because \( e^{2\pi i \alpha_2} \) and \( e^{2\pi i \alpha_3} \) must be conjugate to \( S \) and \( U \), respectively, then \( \alpha_2 \) and \( \alpha_3 \) are diagonalisable with eigenvalues in \( \frac{1}{2} \mathbb{Z} \) and \( \frac{1}{2} \mathbb{Z} \). But (22) implies that none of these eigenvalues can be negative (otherwise holomorphicity at \( \tau = i \) or \( \tau = 5 \) would be lost). None of these eigenvalues can be \( \geq 1 \), as otherwise the corresponding column of \( \Xi(\tau) \) could be divided by \( J(\tau) - 1728 \) or \( J(\tau) \), retaining holomorphicity in \( \mathbb{H} \) but spanning over \( \mathbb{C}[J] \) a strictly larger space of weakly holomorphic vvmf. Since by (34) \( \alpha_2 + \alpha_3 = \lambda \), the trace of \( \lambda \) is the sum of the eigenvalues of \( \alpha_2 \) and \( \alpha_3 \), and we thus obtain (25). \( \Box \)

Theorem 3.3 first appeared in [5], though for restricted \( (\rho, w) \). It is generalised to arbitrary \( T \) and arbitrary genus-0 groups in [6].

Theorem 3.3(c) and (22) say that

\[
\Xi(\tau) = j_2(w; \tau)^{-1} P_2^{-1} q_2^N \sum_{n=0}^{\infty} \Xi(n)q_2^n = j_3(w; \tau)^{-1} P_3^{-1} q_3^N \sum_{n=0}^{\infty} \Xi(n)q_3^n
\]

(35)

where \( \Xi(0) \) and \( \Xi(0) \) are invertible, \( \lambda_2, \lambda_3 \) are diagonal, \( \lambda_2 \) has \( \alpha_1 \) diagonal values equal to \( i/2 \) for \( i = 0, 1, \lambda_3 \) has \( \beta_j \) diagonal entries equal to \( j/3 \) for \( j = 0, 1, 2, \lambda_2 \), \( P_2 S P_2^{-1} = e^{2\pi i \alpha_2} \), and \( P_3 U P_3^{-1} = e^{2\pi i \lambda_3} \). The key properties here are the bounds \( 0 \leq (\lambda_2)_{ii} < 1 \) and \( 0 \leq (\lambda_3)_{ij} < 1 \), and the invertibility of \( \Xi(0) \) and \( \Xi(0) \).
Given \( A \) and \( \chi \), it is easy to solve (32) recursively:

\[
[A_w, \Xi(n)] + n \Xi(u) = \sum_{l=0}^{n-1} \Xi(l) \left( f_{n-l} A_w + \frac{w}{12} t_{n-l} + g_{n-l}(\chi_w + [A_w, \chi_w]) \right)
\]

(36)

for \( n \geq 2 \), where we write \( E_2(\tau) = \sum_{n=0}^{\infty} f_n q^n = 1 - 24 q - \cdots \), \( (J(\tau) - 984) \Delta(\tau)/E_{10}(\tau) = \sum_{n=0}^{\infty} g_n q^n = 1 + 0 q + \cdots \) and \( \Delta(\tau) / E_{10}(\tau) = \sum_{n=0}^{\infty} g_n q^n = q + \cdots \). We require \( \Xi(0) = 1_d \). Note that the \( ij \)-entry on the left-side of (35) is \( (A_{ii} - A_{jj} + n) \Xi(n)_{ij} \), so (35) allows us to recursively identify all entries of \( \Xi(n) \), at least when all \( |A_{jj} - A_{ii}| \neq n \). Indeed, \( A_{jj} - A_{ii} \) can never lie in \( \mathbb{Z}_{\geq 2} \), thanks to this recursion, since then the value of \( \Xi(n)_{ij} \) would be unconstrained, contradicting uniqueness of the solution to (32) with \( \Xi(0) = 1_d \).

Theorem 3.3 tells us that \( \Xi(\tau) \in \mathcal{M}_w(\rho) \) iff \( \Xi(\tau) = \Xi(\tau)P(\tau) \), where \( P(\tau) \in \mathbb{C}^{d}[J] \). The basis \( \mathcal{X}(\tau) \) of (27) can be easily found recursively from this [5]. They can also be found as follows. Define the generating function

\[
\mathcal{X}_{ij}(\tau, \sigma) := q^{-A_{ii}-1} \sum_{n=0}^{\infty} \left[ \mathcal{X}(\tau^n) \right] q^n = \delta_{ij} q - z + \sum_{m=1}^{\infty} \mathcal{X}_{ij}(\tau^n) q^m z^n,
\]

where we write \( z = e^{2\pi i \tau} \). Then, writing \( J' = D_{\phi} J = -E_{4}E_{6}/\Delta \), we have [5]

\[
\mathcal{X}(\tau, \sigma) = J'(\sigma) q^{-A-1} \frac{\Xi(\tau)}{J(\tau) - J(\sigma)} \Xi(\tau) \Xi(\sigma)^{-1} z^A.
\]

(38)

We call \( \Xi(\tau) \) in Theorem 3.3(b) the fundamental matrix associated to \( A \). A \( d \times d \) matrix \( \Xi(\tau) \) is a fundamental matrix for \( \rho \) iff all columns lie in \( \mathcal{M}_w(\rho) \), and \( \Xi(\tau)q^A \) is a fundamental matrix for \( \rho \) in \( \mathbb{C}^{d}[q^{-1}] \). The reason is that \( \mathcal{M}_w(\rho) \rightarrow \mathbb{C}^{d}[q^{-1}] \) is then surjective, so by the index formula it must also be injective.

The determinant of any fundamental matrix is now easy to compute [5]:

\[
\det \Xi(\tau) = E_4(\tau)^{\alpha_1} + 2E_6(\tau)^{\alpha_1} \Delta(\tau)^{d w - 4\beta_1 + 8\beta_2 - 6\alpha_1}/12,
\]

(39)

where \( \alpha_i = \alpha_i(\rho, w), \beta_j = \beta_j(\rho, w) \) are the eigenvalue multiplicities of Section 3.1. Indeed, the determinant is a scalar modular form (with multiplier); use \( E_4(\tau) \) to factor off the zeros at the elliptic points (which we can read off from [35]), and note that the resulting modular form has no zeros in \( \mathbb{H} \) and hence must be a power of \( \Delta(\tau) \), where the power is determined by the weight.

A very practical way to obtain bijective \( A \) and \( \chi \), and hence a fundamental matrix \( \Xi(\tau) \), is through cyclicity:

**Proposition 3.2.** Suppose \( \rho \) is admissible, and \( T \) is diagonal. Suppose \( \Xi(\tau) \in \mathcal{M}_w(\rho) \), and the components of \( \Xi(\tau) \) are linearly independent over \( \mathbb{C} \). Then \( \mathcal{M}_w(\rho) = \mathbb{C}[J, \nabla_{1, w}, \nabla_{2, w}, \nabla_{3, w}, \Xi(\tau)] \).

*Proof.* Let \( \mathcal{M}_\chi := \mathbb{C}[J, \nabla_{1, w}] \Xi(\tau) \). Since \( \mathcal{M}_\chi \) is a module of a PID \( \mathbb{C}[J] \), it is a sum of cyclic submodules \( \mathbb{C}[J] \mathcal{Y}^{(i)}(\tau) \). Each \( \mathbb{C}[J] \mathcal{Y}^{(i)}(\tau) \) is torsion-free (by looking at
leading powers of \( q \). So \( \mathcal{M} \) must be free of some rank \( d' \). Because it is a sub-module of the rank \( d \) module \( \mathcal{M}^d_x(\rho) \) (and again using the fact that \( \mathbb{C}[J] \) is a PID), \( d' \leq d \). That \( d' = d \) follows by computing the determinant of the \( d \times d \) matrix with columns \( \nabla^{i-1} \mathcal{X}(\tau) \); that determinant equals \((2\pi i)^{1-d}(E_1E_0/\Delta)^{d-1}d/2 \) times the Wronskian of \( \mathcal{X}(\tau) \), which is nonzero by Lemma 2.2(a).

Let \( \mathcal{X}(\tau) \) be the matrix formed by those \( d \) generators of \( \mathcal{M} \). Because \( \nabla_1, w, \mathcal{M} \subseteq \mathcal{M} \), the argument of Theorem 3.3 applies and \( \mathcal{X}(\tau) \) satisfies analogues of (32) and hence (33). This means \( \mathcal{M} \) will have its own analogues \( \mathcal{A} \), \( \mathcal{A}_2 \), \( \mathcal{A}_3 \) (their exponentials \( e^{2\pi i \mathcal{X}} \) etc will be conjugate to \( e^{2\pi i \mathcal{X}} \) etc). The trace of \( \mathcal{A} \) will equal the trace of \( \mathcal{A}_2 + \mathcal{A}_3 \), for the same reason it did in Theorem 3.3. Now, \( \mathcal{A} \) is an isomorphism when restricted to \( \mathcal{M} \), so when extended to \( \mathcal{M}^d_x(\rho) \) will also have trivial cokernel. Thus the dimension of \( \mathcal{M}^d_x(\rho) \)/\( \mathcal{M} \) will equal the dimension of ker \( \mathcal{A} \), which, by the index formula (24), equals

\[
\text{Tr} \mathcal{A} - \text{Tr} A = (\text{Tr} \mathcal{A}_2 - \text{Tr} \mathcal{A}_3) - (\text{Tr} \mathcal{A}_3 - \mathcal{A}_3).
\]

This means that if \( \mathcal{M} \neq \mathcal{M}^d_x(\rho) \), then at least one eigenvalue of \( \mathcal{A}_2 \) or \( \mathcal{A}_3 \) is \( \geq 1 \). Suppose one of \( \mathcal{A}_2 \) is. Then by (33) every component of some row of \( \mathcal{X}(\tau) \) has order \( \geq 2 \) at \( \tau = i \). This means every vvmf in \( \mathcal{M} \) has some component with a zero at \( \tau = i \) of order \( \geq 2 \). Hit each column of \( \mathcal{X}(\tau) \) with \( \nabla_1, w, \mathcal{M} \subseteq \mathcal{M} \) and \( \Xi(\tau) \) will span a free rank-

To indicate the nontriviality of our theory, we get a 1-line proof of the solution (30) to the Deligne-Simpson problem for \( T \), at least for most \( \rho \) (the general case requires slightly more work). Let \( \rho \) be any \( T \)-representation with \( T \) diagonal, and let \( A \) be bijective for \( (\rho, 0) \) and \( \Xi(\tau) \) a corresponding fundamental matrix. Then as long as all \( A_{ij} \neq 0 \), the columns of the derivative \( D_0 \Xi \) will span a free rank-\( d \) submodule of \( \mathcal{M}_d^2(\rho) \) over \( \mathbb{C}[J] \), on which \( \mathcal{P} \) is surjective. Thus \( c_{(\rho, 0)} \leq c_{(\rho, 2)} \) and so by (30) we obtain \( a_1 \geq b_0 \). (It is clear that things are more subtle when some \( A_{ij} = 0 \), as this inequality fails for \( \rho = 1 \)!) The other inequalities \( a_i \geq b_j \) follow by comparing \( c_{(\rho, 2k)} \) and \( c_{(\rho, 2k+2)} \) in the identical way.

As mentioned earlier, \( S \) and \( U \) are conjugate to \( e^{2\pi i \mathcal{X}_1} \) and \( e^{2\pi i \mathcal{X}_3} \), respectively, but identifying precisely which conjugate is a transcendental and subtle question. For example, we see in Section 4.2 below that when \( d = 2 \), they are related by Gamma function values.
3.4 Holomorphic vvmf

Until this point in the paper, our focus has been on weakly holomorphic vvmf of fixed weight, i.e. vvmf holomorphic everywhere except at the \( \Gamma \)-orbit of \( \infty \). The reason is that structurally it is the simplest and most fundamental. For example, it is acted on by the ring of (scalar) modular functions holomorphic away from \( \Gamma \infty \), which for any genus-0 Fuchsian group is a PID. By contrast, the holomorphic vvmf (say of arbitrary even integral weight) is a module over the ring of holomorphic modular forms, which in genus-0 is usually not even polynomial. However, there is probably more interest in holomorphic vvmf, so it is to these we now turn. The two main questions we address are the algebraic structure (see Theorem 3.4 below), and dimensions (see Theorem 3.5 next subsection).

**Definition 3.1.** Let \( (\rho, w) \) be admissible, \( T \) diagonal, and \( \lambda \) any exponent. Define

\[
\mathcal{M}_w^\lambda(\rho) := \ker \mathcal{P}_{\lambda-1_d} = \left\{ \Xi(\tau) \in \mathcal{M}^\lambda_w(\rho) \mid \Xi(\tau) = q^\lambda \sum_{n=0}^{\infty} \zeta(n) q^n \right\}
\]

and \( \mathcal{M}^\lambda(\rho) = \prod_{k \in \mathbb{Z}} \mathcal{M}^\lambda_{w+2k}(\rho) \). We call any \( \Xi(\tau) \in \mathcal{M}^\lambda(\rho) \), \( \lambda \)-holomorphic.

For example, for the trivial representation, \( \mathcal{M}^1(1) \) are the modular forms \( m = \mathcal{C}[E_4, E_6] \), while \( \mathcal{M}^1(1) \) are the cusp forms \( m \Delta(\tau) \). More generally, define \( \lambda^\text{hol} \) to be the unique exponent with \( 0 \leq \text{Re} \lambda_{ii} < 1 \) for all \( i \). Then \( \mathcal{M}^{\lambda^\text{hol}}(\rho) \) coincides with the usual definition of holomorphic vvmf. Choosing \( 0 < \text{Re} \lambda_{ii} \leq 1 \) would give the vector-valued cusp forms.

Theorem 3.2 computes \( \dim \mathcal{M}^\lambda_{w+2k}(\rho) \) for all sufficiently large \(|k|\):

**Lemma 3.2.** Let \( (\rho, w) \) be admissible and \( T \) diagonal. Let \( \lambda \) be any exponent, and \( \Lambda \) any bijective exponent.

(a) For any \( k \in \mathbb{Z} \), \( \mathcal{M}_w^\lambda(\rho) \) is finite-dimensional, and obeys the bound

\[
\dim \mathcal{M}_{w+2k}^\lambda(\rho) \geq \max \left\{ 0, \frac{w+2k+2}{12} \frac{\alpha_k}{d} + \frac{\beta_k - \beta_{k+2}}{3} - \text{Tr} \lambda \right\}.
\]

(b) Choose any \( m, n \in \mathbb{Z} \) satisfying \( \text{Re}(\Lambda + m 1_d) \geq \text{Re} \lambda \geq \text{Re}(\Lambda - n 1_d) \) entrywise. Then \( \mathcal{M}_{w-2k}^\lambda(\rho) = 0 \) when \( k = 6n + 6 \) or \( k \geq 6n + 8 \), and equality holds in (41) whenever \( k = 6m - 6 \) or \( k \geq 6m - 4 \).

(c) Let \( w_0 \) be the weight with smallest real part for which \( \mathcal{M}_{w_0}^\lambda(\rho) \neq 0 \) and write \( \epsilon = 0, 1 \) for \( d \) even, odd respectively. Suppose \( \rho \) is irreducible and not 1-dimensional. Then \( w_0 \) satisfies the bounds

\[
\frac{12}{d} \text{Tr} \lambda + 1 - d \leq w_0 \leq \frac{12}{d} \text{Tr} \lambda - \frac{3\epsilon}{d}.
\]

**Proof.** Theorem 3.2(a) gives finite-dimensionality. The bound (41) follows from \( \dim \ker \mathcal{P}_{\lambda-1_d} \geq \text{index} \mathcal{P}_{\lambda-1_d} \), the index formula in Theorem 3.2(a), and (29). Note
that $\Delta(\tau)^k \mathcal{M}_w^1(\rho) = \mathcal{M}_{w+2k}^1(\rho)$ for any $k \in \mathbb{Z}$ so $A + k1_d$ is bijective for $(\rho, w + 12k)$. Hence $\mathcal{P}_{\lambda-1_d}$ is injective on $\mathcal{M}_{w-12(n+1)}^1(\rho)$ because $\mathcal{P}_{\lambda-(n+1)_d}$ is, while $\mathcal{P}_{\lambda-1_d}$ is surjective on $\mathcal{M}_{w+12(m-1)}^1(\rho)$ because $\mathcal{P}_{\lambda+(m-1)_d}$ is. This proves (b) for those weights. Now, for any $k \geq 2$ there is a scalar modular form $f(\tau) \in m$ of weight $2k$ with nonzero constant term, so $f(\tau)^{-1} \in \mathbb{C}[q]]$ and the surjectivity of $\mathcal{P}_{\lambda-1_d}$ on $\mathcal{M}_{w+12(m-1)}^1(\rho)$ implies the surjectivity on $\mathcal{M}_{w+12(m-1)+2k}^1(\rho)$. More directly, injectivity of $\mathcal{P}_{\lambda-1_d}$ on $\mathcal{M}_{w-12(n+1)}^1(\rho)$ implies the injectivity on $\mathcal{M}_{w-12(n+1)+2k}^1(\rho)$.

Now turn to (c). We know that $\dim \mathcal{M}_w^1(\rho) > 0$ for any $w$ for which $\text{Tr} \Lambda > \text{Tr} \lambda - d$, since $\text{dimker} \mathcal{P}_{\lambda-1_d} \geq \text{Index} \mathcal{P}_{\lambda-1_d} > 0$. Using (31), we obtain the upper bound of (42).

Choose any nonzero $X \in \mathcal{M}_w^\lambda(\rho)$. Then its components must be linearly independent over $\mathbb{C}$, because they span a subrepresentation of the irreducible $\rho$. Therefore, Lemma 2.2(a) says $\text{Wr}(X)(\tau) \in \mathcal{M}_{d(w+d-1)}^1(\det \rho)$ is nonzero, with leading power of $q$ in $\text{Tr} \lambda + Z_{\geq 0}$. Hence $\text{Wr}(X)(\tau)/\Delta^1(\tau)$ lies in $\mathcal{M}_{d(w+d-1)-12\text{Tr} \lambda}^0(\upsilon_U)$ for some $w \in \mathbb{Z}$, from which follows the lower bound of (42).

The lower bound in (42) is due to Mason [24] (he proved it for $\lambda = \lambda^{hol}$ but the generalisation given here is trivial). The $d \neq 1$ assumption in (c) is only needed for the upper bound.

Theorem 3.3(a) tells us the space $\mathcal{M}_w^\lambda(\rho)$ of weakly holomorphic vvmf is a free module of rank $d$ over $\mathbb{C}[J]$. The analogous statement for holomorphic vvmf, namely that $\mathcal{M}_{w}^{\lambda^{hol}}(\rho)$ is free of rank $d$ over $m$, is implicit in [13] (see the Remark there on page 98). It was also proved independently in [22], and independently but simultaneously we obtained the following generalisation. The proof of freeness given here is far simpler than in [22], is more general (as it applies to arbitrary $\lambda$), gives more information (see (b) below), and generalises directly to arbitrary $T$ and arbitrary genus-0 groups [14].

**Theorem 3.4.** Let $(\rho, w)$ be admissible and $T$ diagonal. Choose any exponent $\lambda$. Let $\alpha_i = \alpha_i(\rho, w)$ and $\beta_i = \beta_i(\rho, w)$.

(a) $\mathcal{M}_w^{\lambda}(\rho)$ is a free module over $m = \mathbb{C}E_4, E_6$, of rank $d$.

(b) Let $w_0 = w^{(1)} \leq w^{(2)} \leq \cdots \leq w^{(d)}$ be the weights of the free generators. Then precisely $\alpha_i$ of the $w^{(j)}$ are congruent mod $4$ to $w + 2i$, and precisely $\beta_i$ of them are congruent mod $6$ to $w - 2i$. Moreover, $\sum_i w^{(j)} = 12\text{Tr} \lambda$. Let $\Xi^j(\tau)$ be the matrix obtained by putting these $d$ generators into $d$ columns. Then (up to an irrelevant nonzero constant)

$$\det \Xi^j(\tau) = \Delta^{12}(\tau).$$

**Proof.** Let $w_0 \in w + 2\mathbb{Z}$ be the weight of minimal real part with $\mathcal{M}_{w_0}^{\lambda}(\rho) \neq 0$; this exists by Lemma 3.2(b). Write $\mathcal{M} = \mathcal{M}_w^{\lambda}(\rho), w_k = w_0 + 2k$ and $\mathcal{M}_k = \mathcal{M}_{w_k}^{\lambda}(\rho)$. For $\Xi(\tau) \in \mathcal{M}_{w_k}^{\lambda}(\rho)$, recall from [13] that the constant term $\Xi_{(0)}$ at $\tau = i$ is $\Xi(i)/E_4(i)^{u/4}$. Fix $S_0 = e^{\pi i u/2}S$; then $e^{\pi i u/2}S = (-1)^k S_0$ and so for any $\Xi \in \mathcal{M}_k$, its constant term satisfies $S_0 \Xi_{(0)} = (-1)^k \Xi_{(0)}$ thanks to [14].

Find $\Xi^{(0)}(\tau) \in \mathcal{M}_1$, with the property that, for any $k \geq 0$, the space of constant terms $\Xi_{(0)}$, as $\Xi(\tau)$ runs over all $\bigcup_{k=0}^\infty \mathcal{M}_k$, has a basis given by the constant terms.
\( X^{(i)}_{[0]} \) for those \( X^{(i)}(\tau) \in \cup_{i=0}^k M_i \) (i.e. for those \( i \) with \( l_i \leq k \)). This is done recursively with \( k \). We will show that these \( X^{(i)}(\tau) \) freely generate \( M \).

The key observation is the following. Consider any \( X(\tau) \in M_k \). Then by definition of the \( X^{(i)}(\tau) \), \( X_{[0]} = \sum_i c_i X^{(i)}_{[0]} \) where \( c_i = 0 \) unless \( l_i \leq k \) and \( l_i \equiv k \pmod{2} \). Note that the constant term \( X^{(i)}_{[0]} \) of \( X^{(i)}(\tau) = X(\tau) - \sum_i c_i E_i(\tau)^{l_i-k+1/2} X^{(i)}(\tau) \) is 0, so \( X(\tau)/E_0(\tau) \in M_{k-3} \).

One consequence of this observation is that, by an easy induction on \( k \), any \( X(\tau) \in M \) must lie in \( \sum_i m X^{(i)}(\tau) \). Another consequence is that there are exactly \( d \) of these \( X^{(i)}(\tau) \), in particular their constant terms form a basis for \( \mathbb{C}^d \). To see this, take any fundamental matrix \( \Xi(\tau) \) at weight \( w_0 \). Then for sufficiently large \( l \), each column of \( \Delta(\tau)^l \Xi(\tau) \) is in \( M_{12l} \). The constant terms of \( \Delta(\tau)^l \Xi(\tau) \) (which have \( S_0 \)-eigenvalues \( +1 \)) and of \( \Delta(\tau)^l D_{w_0} \Xi(\tau) \) (which have \( S_0 \)-eigenvalues \( -1 \)) must have rank \( a_0(\rho, w_0) \) and \( a_1(\rho, w_0) \), respectively, as otherwise the columns of \( \Delta(\tau)^l \Xi(\tau) \) would be linearly dependent over \( \mathbb{C} \), contradicting their linear independence over \( \mathbb{C} \). This means of course that exactly \( a_0(\rho, w_0) \) of these \( X^{(i)}(\tau) \) have \( l_i \) even, and exactly \( a_1(\rho, w_0) \) have \( l_i \) odd.

Thus these \( d \) \( X^{(i)}(\tau) \) generate over \( \mathbb{m} \) all of \( M \). To see they are linearly independent over \( \mathbb{m} \), suppose we have a relation \( \sum_i p_i(\tau) X^{(i)}(\tau) = 0 \), for modular forms \( p_i(\tau) \in \mathbb{m} \) which do not share a (nontrivial) common divisor. The constant term at \( i \) of that relation reads \( \sum_i p_i(0) X^{(i)}_{[0]} = 0 \) and hence each \( p_i(0) = 0 \), since the \( X^{(i)}_{[0]} \) are linearly independent by construction. This forces all \( p_i(\tau) \) to be 0, since otherwise we could divide them all by \( E_0 \), which would contradict the hypothesis that they share no common divisor.

The identical argument applies to the constant terms \( X_{[0]} = X(\xi_0)/E_0(\xi_0)^{w/6} \) at \( \tau = \xi_6 \); this implies that exactly \( \beta_i(\rho, w_0) \) generators \( X^{(i)}(\tau) \) have \( l_j \equiv i \pmod{3} \).

Form the \( d \times d \) matrix \( \Xi^{(i)}(\tau) \) from these \( d \) generators \( X^{(i)}(\tau) \) and call the determinant \( \delta(\tau) \). The linear independence of the constant terms of the generators at the elliptic fixed points, says that \( \delta(\tau) \) cannot vanish at any elliptic fixed point. \( \delta(\tau) \) also can’t have a zero anywhere else in \( \mathbb{H} \). To see this, first note that a zero at \( \tau^* \in \mathbb{H} \) implies there is nonzero row vector \( \nu \in \mathbb{C}^d \) such that \( \nu \Xi^{(i)}(\tau^*) = 0 \) and hence \( \nu X^{(i)}(\tau^*) = 0 \) for any \( X(\tau) \in M(\rho) \). But \( \text{det} \) says the determinant of any fundamental matrix \( \Xi(\tau) \) for \( (\rho, w_0) \) can only vanish at elliptic fixed points and cusps, and so \( \nu \Xi^{(i)}(\tau^*) \neq 0 \) for any fundamental matrix, and hence \( \nu \Xi(\tau) \neq 0 \) for some \( \nu(\tau) \in M_{w_0}(\rho) \). To get a contradiction, choose \( N \) big enough so that \( \Delta(\tau)^N \Xi(\tau) \in M \). This means \( \delta(\tau) \) is a scalar modular form (with multiplier) which doesn’t vanish anywhere in \( \mathbb{H} \). Hence \( \delta(\tau) \) must be a power of \( \Delta(\tau) \), and considering weights we see this must be \( \delta(\tau) = \Delta^{\sum w_i/12}(\tau) \). We compute that sum over \( i \), shortly.

Find the smallest \( \ell \) such that \( c(\rho, w_0 + 2\ell) > \text{Tr} \lambda \) (recall \( 30 \)) and put \( w'_0 := w_\ell \). Define \( n'_k = 0 \) for \( k < 0 \), and

\[
n'_k = \frac{w_0' + 2k + 2}{12} d + \frac{\alpha_k}{2} + \frac{\beta_k - \beta_{2+k}}{3} - \text{Tr} \lambda
\] (43)
for $k \geq 0$. Using (29), the numbers $n'_k$ are the values $\max\{c_1, c_2 + 2k\}$. From Lemma 3.2 we obtain the equality $n_k = n'_{k-1}$ for $|k|$ sufficiently large. The tight Hilbert–Poincaré series $H^{\lambda}_{I_1}(-\mathcal{M}; x) := \sum_k n'_k x^{w_{\lambda}(k)}$ equals

$$H^{\lambda}_{I_1}(-\mathcal{M}; x) = x^{w_0} + n'_1 x^2 + (n'_2 - n'_0) x^4 + (n'_3 - n'_1 - n'_0) x^6 + (n'_4 - n'_2 - n'_1) x^8, \quad (1 - x^4)(1 - x^6)$$

by a simple calculation (the significance of $H^{\lambda}_{I_1}(x)$ is explained in Proposition 3.3 below). From the numerator we read off the weights of the hypothetical 'tight' generators: write $w^{(i)} = w_0$ for $1 \leq i \leq n'_0$, ..., $w^{(i)} = w_0 + 8$ for $d - n'_1 + n'_2 + n'_3 < i \leq d$. We know that the actual Hilbert–Poincaré series, $H^{\lambda}_{\mathcal{M}}(x) = \sum_k n_k x^{w_k}$, minus the tight one, equals a finite sum of terms, since $n_k = n'_{k-1}$ for $|k|$ large. Therefore $(1 - x^4)(1 - x^6)(H^{\lambda}(x) - H^{\lambda}_{I_1}(x)) = \sum x^{w^{(i)}} - \sum x^{w^{(i)}}$ is simply a polynomial identity. Differentiating with respect to $x$ and setting $x = 1$ gives $\sum w^{(i)} = \sum w^{(i)}$, and the latter is readily computed to be $12 \text{Tr} \lambda$. \Box

This freeness doesn’t seem directly related to that of Theorem 3.3(a). The reason is not to look at local expansions about $\tau = i$ and $\tau = \zeta_6$ in the proof of part (a) is because if $X(\tau)$ is holomorphic and $X(i) = 0$, then $X(\tau)/E_4(\tau)$ is also holomorphic (similarly for $X(\zeta_6)$ and $X(\tau)/E_4(\tau)$).

Call an admissible $(\rho, w)$ tight if it has the property that for all $k \in \mathbb{Z}$, an exponent $\lambda$ is bijective for $w + 2k$ if $\text{Tr} \lambda \geq c_1(\rho, w)$ and if $\text{Tr} \lambda \leq c_1(\rho, w)$. In this case we also say $\rho$ is tight. Generic $\rho$ are tight. We learn in Theorem 4.1 that all irreducible $\rho$ in dimension $d < 6$ are tight. For tight $\rho$, most quantities can be easily determined:

**Proposition 3.3.** Suppose $(\rho, w)$ is admissible and tight, and $T$ is diagonal.

(a) Then $v_u \otimes \rho$ is also tight for any $u \in \mathbb{C}$, as is the contragredient $\rho^* = (\rho^t)^{-1}$.

(b) For any exponent $\lambda$ (not necessarily bijective), either $\ker \mathcal{P}_\lambda = 0$ (if $\text{Tr} \lambda \geq c_1(\rho, w)$) or $\text{coker} \mathcal{P}_\lambda = 0$ (if $\text{Tr} \lambda \leq c_1(\rho, w)$). Moreover,

$$\dim \mathcal{M}^{\lambda}(\rho) = \max\{0, c_1(\rho, w) + d - \text{Tr} \lambda\}, \quad (45)$$

and the Hilbert–Poincaré series $H^{\lambda}(x)$ of $\mathcal{M}^{\lambda}(\rho)$ equals the tight Hilbert–Poincaré series $H^{\lambda}_{I_1}(x)$ of (44).

The proof of (a) uses the equality $\mathcal{M}^{\lambda}_{w + \delta}(v_u \otimes \rho) = \Delta^{w + \delta}(\tau). \mathcal{M}^{\lambda}_{w + \delta}(\rho)$, as well as the duality in Proposition 3.4 below. To prove (b), let $\Lambda$ be the unique bijective exponent matrix for $(\rho, w)$ satisfying $\Lambda_{ii} = \lambda_{ii} - 1$ for all $i \neq 1$. Write $n = \lambda_{11} + 1 - \lambda_{11} = c_1(\rho, w) + d - \text{Tr} \lambda \in \mathbb{Z}$. If $n \geq 0$, $\mathcal{P}_\lambda$ inherits the surjectivity of $\mathcal{P}_\Lambda$, while if $n \leq 0$ it inherits the injectivity. The index formula (24) gives the dimension, which by (29) equals $n'_{k-1}$. This is why $H^{\lambda}_{I_1}$ arises.

As long as $(\rho, w)$ is tight, this argument tells us how to find a basis for $\mathcal{M}^{\lambda}_{w + \delta}(\rho)$. Let $\Lambda$ and $n$ be as above. For $n > 0$ a basis for $\mathcal{M}^{\lambda}_{w + \delta}(\rho)$ consists of the basis vectors $X^{(1)}$ (see Section 3.2) for $0 \leq i < n$. Incidentally, the name ‘tight’ refers to the fact that the numerator of $H^{\lambda}_{I_1}$ is maximally bundled together.

There are other constraints on the possible weights $w^{(i)}$ of Theorem 3.4(b). A useful observation in practice is that if $\rho$ is irreducible, then the set $\{w^{(1)}, \ldots, w^{(d)}\}$. 

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can’t have gaps, i.e. for \( n = (w(d) - w(1))/2 \),

\[
\{w(1), \ldots, w(d)\} = \{w(1), w(1) + 2, w(1) + 4, \ldots, w(1) + 2n\}. \tag{46}
\]

The reason is that when \( w(1) + 2k \) doesn’t equal any \( w(i) \), then when \( w(j) < w(1) + 2k \), \( D^{(w(j) - w(1))/2} \Xi(\tau)(\tau) = \sum_{m < w(i) + 2k} f_{ji}(\tau) \Xi(\tau)(\tau) \) for \( f_{ji}(\tau) \in \mathcal{M} \), where the sum is over all \( l \) with \( w(i) < w(1) + 2k \). If in addition \( w(1) + 2k < w(d) \) (i.e. \( w(1) + 2k \) is a gap), then the Wronskian \( \text{W}(\Xi(\tau)) \) would have to vanish, contradicting irreducibility.

Several papers (e.g. [24][21]) consider a ‘cyclic’ class of vvmf where the components of \( \Xi(\tau) \) span the solution space to a monic modular differential equation \( D_k^d + \sum_{l=0}^{d-1} f_{li}(\tau) D_k^l = 0 \), where each \( f_{li}(\tau) \in \mathfrak{n} \) is of weight 2\( d - 2l \). In this accessible case the free generators \( \Xi^{(l)}(\tau) \) can be taken to be \( \Xi(\tau), D_k \Xi(\tau), \ldots, D_k^{d-1} \Xi(\tau) \), i.e. the weights are \( a(l) = a(1) + (l-1)2 \). This means the corresponding \( \rho \) cannot be tight, when \( d \geq 6 \). Indeed, the multipliers of such vvmf are exceptional, requiring the multiplicities of \( \rho \) to satisfy \( |a_i - a_j| \leq 1 \) and \( |\beta_i - \beta_j| \leq 1 \). Recall that the connected components of the moduli space of \( \mathcal{T} \)-representations are parametrised by these multiplicities; these particular components are of maximal dimension. For example, when \( d = 6 \), such a representation \( \rho \) can lie in only 1 of the 12 possible connected components, and these \( \rho \) define a 6-dimensional subspace inside that 7-dimensional component.

### 3.5 Serre duality and the dimension formula

A crucial symmetry of the theory is called the adjoint in the language of Fuchsian equations, and shortly we reinterpret this as Serre duality.

**Proposition 3.4.** Let \( (\rho, w) \) be admissible, \( T \) diagonal, and \( \Lambda \) bijective, and let \( \Xi(\tau) \) be the associated fundamental matrix of \( \mathcal{M}_1(\rho) \). Let \( \rho^* \) denote the contragredient \( (\rho^{-1})^t \) of \( \rho \). Then \( \mathcal{M}_1(\rho) \) and \( \mathcal{M}_{2-w}(\rho^*) \) are naturally isomorphic as \( \mathbb{C}[J, \nabla_1, \nabla_2, \nabla_3] \)-modules. Moreover,

\[
\Xi^*(\tau) = E_4(\tau)^2 E_6(\tau) \Delta(\tau)^{-1} \left( \Xi(\tau)^t \right)^{-1}, \tag{47}
\]

\[
\mathcal{X}^*(\sigma, \tau) = -\mathcal{X}(\sigma, \tau)^t, \tag{48}
\]

where \( \Xi^* \) is the fundamental matrix of \( \mathcal{M}_{2-w}(\rho^*) \) corresponding to the bijective exponent \( \Lambda^* = -1_d - \Lambda \), and \( \mathcal{X}, \mathcal{X}^* \) are the generating functions \([47]\) for \( (\rho, w) \) and \( (\rho^*, 2-w) \) respectively. In other words the \( q^{n+1j} \) coefficient \( \mathcal{X}^{(j; m-1)}_{(m)} \) of the basis vector \( \Xi^{(j; m-1)}(\tau) \) is the negative of the \( q^{n+1j} \) coefficient \( \mathcal{X}^{(i; m-1)}_{(m)} \) of the basis vector \( \Xi^{(i; m-1)}(\tau) \), for all \( m, n \geq 1 \).

**Proof.** Define \( \Xi^*(\tau) \) by \([47]\); to show it is the fundamental matrix associated to the bijective exponent \( \Lambda^* = -1_d - \Lambda \), we need to show that \( \Xi^*(\tau) = q^{-1d-\Lambda}(1_d + \ldots, \ldots) \).
\[ \sum_{n=1}^{\infty} \Xi_n^i q^n \] (this is clear), that \(-1_d - \Lambda\) has trace \(c_{(\rho^*, 2-w)}\) (we'll do this next), and that the columns of \(\Xi^i(\tau)\) are in \(\mathcal{M}_{2-w}(\rho^*)\) (we'll do that next paragraph). Using \(29\), we find \(\alpha_i(\rho^*, 2-w) = a_{i+1}(\rho, w)\), and \(\beta_j(\rho^*, 2-w) = \beta_{2-j}(\rho, w)\), so we compute from \(25\) that \(c_{(\rho^*, 2-w)} = 1 - (1_d - \Lambda)\).

From \(\Xi(\gamma \tau) = \tilde{\rho}_w(\gamma, \tau) \Xi(\tau)\), we get \((\Xi(\gamma \tau))^i = \tilde{\rho}_w^* (\gamma, \tau)(\Xi(\tau))^i\). It thus suffices to show \(\Xi^i(\tau)\) is holomorphic in \(\mathbb{H}\). We see from \(39\) that \(\Xi(\tau)^{-1}\) is meromorphic everywhere in \(\mathbb{H}^*\), with poles possible only at the elliptic points and the cusp. Locally about \(\tau = i\), \(35\) tells us

\[
(\Xi(\tau)^i)^{-1} = j(-w; \tau) \frac{q_2^{-\lambda_2}}{(1 + \sum_{n=1}^{\infty} (\Xi^i_{n})^{-1} - (\Xi^i_{n})^{-1} q^n)^{-1} - (\Xi^i_{0})^{-1}}. \tag{49}
\]

The series in the middle bracketed factor is invertible at \(\tau = i\), because its determinant equals 1 there. So every entry of \((\Xi(\tau)^i)^{-1}\) at \(\tau = i\) has at worst a simple pole (coming from \(q_2^{-\lambda_2} = \tau^{-2}\lambda_2\)). But \(J = E_{1/2}^* E_6/\Delta\) has a simple pole at \(\tau = i\). Therefore \(\Xi^i(\tau)\) is holomorphic at \(\tau = i\). Likewise, at \(\tau = \xi_6\), the entries of \((\Xi(\tau)^i)^{-1}\) have at worst an order 2 pole (coming from \(q_3^{-\lambda_3} = \tau^{-3}\lambda_3\)), but \(J\) has an order 2 zero at \(\tau = \xi_6\), so \(\Xi^i(\tau)\) is also holomorphic at \(\tau = \xi_6\). Thus the columns of \(\Xi^i(\tau)\) lie in \(\mathcal{M}_{2-w}(\rho^*)\) (the 2 comes from \(J\)).

This concludes the proof that \(\Xi(\tau)\) is a fundamental matrix for \((\rho^*, 2-w)\). \(48\) now follows directly from \(38\). \(\square\)

A special case of equation \((47)\) was found in \(5\). An interesting special case of \(48\) is that the constant term of any weakly holomorphic modular form \(f(\tau) \in \mathcal{M}_2(1)\) is 0. To see this, recall first that \((\rho, w) = (1, 0)\) has \(\Lambda = 0\) and \(\Xi(\tau) = \chi_1^{(1;0)}(\tau) = 1\), i.e. the \(q^{m+o}\)-coefficient of \(\chi_1^{(1;0)}\) vanishes for all \(m \geq 1\). Then \(48\) says the \(q^{1-1}\), coefficient of all \(\chi_1^{(m-1)}(\tau)\) must also vanish. Since the \(\chi_1^{(1;\lambda)}(\tau)\) span (over \(\mathbb{C}\)) all of \(\mathcal{M}_2(1)\), the constant term of any \(f(\tau) \in \mathcal{M}_2(1)\) must vanish. The same result holds for any genus-0 group.

The final fundamental ingredient of our theory connects the index formula \(24\) to this duality:

**Theorem 3.5.** Let \((\rho, w)\) be admissible and \(T\) diagonal, and recall the quantity \(c_{(\rho, w)}\) computed in \(25\). Then for any exponent \(\lambda\),

\[
\text{coker} \mathcal{O}_{3,(\rho, w)} \cong \left(\mathcal{M}^{\lambda_2}_{2-w}(\rho^*)\right)^*, \tag{50}
\]

\[
\dim \mathcal{M}^{\lambda_2}_{2-w}(\rho^*) - \dim \mathcal{M}^{1-\lambda}_{2-w}(\rho^*) = c_{(\rho, w)} + d - \text{Tr} \lambda. \tag{51}
\]

**Proof.** Let \(\chi(\tau) \in \mathcal{O}^{d}[q^{-1}, q]\), i.e. \(\chi(\tau) = q^j \sum_{n=-N}^{\infty} \chi(n) q^n\) for some \(N = N(\chi)\), and let \(\chi(\tau) \in \mathcal{M}^{\lambda_{2-w}}(\rho^*)\), and define a pairing \(\langle \chi, \chi \rangle\) to be the \(q^0\)-coefficient \(f_0\) of \(\chi(\tau)^i \langle \chi, \chi \rangle\) \(= \sum_{i=1}^{d} \chi_i(\tau) \chi_i(\tau) = \sum_{n=-N}^{\infty} f_{n} q^n\). Note that \(\langle \chi, \chi \rangle = \sum_{N \leq n \leq 2g} \chi(n) \chi(-n)\) depends only on the coefficients \(\chi(n)\) for \(n \leq 0\), since by hypothesis \(\chi(k) = 0\) for \(k < 0\). In other words, the pairing \(\langle \chi, \chi \rangle\) depends only on \(\chi(\tau)\) and the principal part \(\mathcal{O}_3 \chi(q)\) of \(\chi(\tau)\).
If $\mathcal{X}(\tau) \in \mathcal{M}_\lambda^{\text{hol}}(\rho)$, then $\mathcal{X}(\tau)^\vee \mathcal{Y}(\tau)$ will lie in $\mathcal{M}_\lambda^J(1)$. Hence from the observation after Proposition 3.4, in that case $\langle \mathcal{X}, \mathcal{Y} \rangle$ will vanish. This means the pairing $\langle \mathcal{X}, \mathcal{Y} \rangle$ is a well-defined pairing between the cokernel of $\mathcal{P}_\lambda(\rho, w)$ and $\mathcal{M}_{2-w}^{\lambda}(\rho^*)$.

Let $\mathcal{Y}^1(\tau), \ldots, \mathcal{Y}^m(\tau)$ be a basis of $\mathcal{M}_{2-w}^{\lambda}(\rho^*)$. We can require that this basis be triangular in the sense that for each $1 \leq i \leq m$ there is an $n_i, k_i$ so that the coefficient $\mathcal{Y}^i_{(m)k} = \delta_{ij}$ for all $i, j$. Indeed, choose any $n_1, k_1$ such that $\mathcal{Y}^1_{(m)k} \neq 0$, and rescale $\mathcal{Y}^1(\tau)$ so that coefficient equals 1. Subtract if necessary a multiple of $\mathcal{Y}^1(\tau)$ from the other $\mathcal{Y}^i(\tau)$ so that $\mathcal{Y}^i_{(m)k} = 0$. Now, repeat: choose any $n_2, k_2$ such that $\mathcal{Y}^2_{(n_2)k_2} \neq 0$, etc. If we take $\mathcal{X}(\tau)$ to be $q^{-w_k} e_k$ (i.e. all coefficients vanish except one coefficient in one component), then $\langle \mathcal{X}^i, \mathcal{Y}^j \rangle = \delta_{ij}$. This form of nondegeneracy means that $\dim \mathcal{M}_{2-w}^{\lambda}(\rho^*) \leq \dim \text{coker} \mathcal{P}_\lambda(\rho, w)$.

Repeating this argument in the dual direction, more precisely replacing $\rho, w, \lambda$ with $\rho^*, 2 - w, -1_d - \lambda$ respectively, gives us the dual inequality $\dim \mathcal{M}_{w}^{\lambda+1_d}(\rho) \leq \dim \text{coker} \mathcal{P}_{1_d - \lambda}(\rho^*, 2 - w)$. However, from Proposition 3.4 we know that $c(\rho^*, 2 - w) = \text{Tr}(-1_d - \lambda) = -d - c(\rho, w)$, so the index formula (24) gives us both

$$\dim \mathcal{M}_{w}^{\lambda+1_d}(\rho) - \dim \text{coker} \mathcal{P}_\lambda(\rho, w) = c(\rho, w) - d - \text{Tr} \lambda,$$

$$\dim \mathcal{M}_{2-w}^{\lambda}(\rho^*) - \dim \text{coker} \mathcal{P}_{1_d - \lambda}(\rho^*, 2 - w) = -d - c(\rho, w) + d + \text{Tr} \lambda.$$

Adding these gives

$$\dim \mathcal{M}_{w}^{\lambda+1_d}(\rho) + \dim \mathcal{M}_{2-w}^{\lambda}(\rho^*) = \dim \text{coker} \mathcal{P}_\lambda(\rho, w) + \dim \text{coker} \mathcal{P}_{1_d - \lambda}(\rho^*, 2 - w).$$

Together with the two inequalities, this shows that the dimensions of $\mathcal{M}_{2-w}^{\lambda}(\rho^*)$ and coker $\mathcal{P}_{1_d}(\rho, w)$ match. Hence the pairing $\langle \mathcal{X}, \mathcal{Y} \rangle$ is nondegenerate and establishes the isomorphism (50). Equation (51) now follows immediately from the index formula (24). □

We suggest calling part (a) Serre duality because ker $\mathcal{P}_\lambda$ has an interpretation as $H^0(\Gamma; \mathcal{O})$ for some space $\mathcal{O} = \mathcal{O}_{\lambda}(\rho, w)$ of meromorphic functions on which $\Gamma$ acts by $(\rho, w)$, while coker $\mathcal{P}_{1_d}$ being the obstruction to finding meromorphic sections of our $(\rho, w)$-vector bundle, should have an interpretation as some $H^1$. The shifts by $1_d$ and 2 would be associated to the canonical line bundle. This interpretation is at this point merely a heuristic, however.

Compare (50) to Theorem 3.1 in [10]. There, Borcherds restricts attention to weight $w = k$ a half-integer, groups commensurable to $\text{SL}_2(\mathbb{Z})$, representations $\rho$ with finite image, and $\lambda = \lambda^{\text{hol}}$. The assumption of finite image was essential to his proof. Our proof extends to arbitrary $\Gamma$ and arbitrary genus-zero groups [14] (so by inducing the representation, it also applies to any finite-index subgroup of a genus-zero group).

The most important special case of the dimension formula (51) is $\lambda = \lambda^{\text{hol}}$, which relates the dimensions of holomorphic vvmf with those of vector-valued cusp forms. Compare (51) to [33], where Skoruppa obtained the formula assuming $2w \in \mathbb{Z}$, and that $\rho$ has finite image. His proof used the Eichler-Selberg trace formula. Once again we see that these results hold in much greater generality. Of course
course thanks to the induction trick, \([51]\) also gives the dimension formulas for spaces of e.g. holomorphic and cusp forms, for any finite index subgroup of \(\Gamma\).

Incidentally, it is possible to have nonconstant modular functions, holomorphic everywhere in \(\mathbb{H}^n\), for some multipliers with infinite image (see e.g. Remark 2 in Section 3 of [17]). On the other hand, Lemma 2.4 of [17] prove that for unitary multipliers \(\rho\), there are no nonzero holomorphic vvmf \(X(\tau) \in \mathcal{M}_{\rho}^1(\rho)\) of weight \(w < 0\). This implies there are no vector-valued cusp forms of weight \(w \leq 0\), for unitary \(\rho\).

### 3.6 The \(q\)-expansion coefficients

Modular forms — vector-valued or otherwise — are most important for their \(q\)-expansion coefficients. In this subsection we study these. A special case of Proposition 3.5(b) (namely, \(w = 0\) and \(Q_{\mathbb{R}} = \mathbb{Q}\) is given in [1], but our proof generalises without change to \(T\) nondiagonalisable and to any genus-0 group [13].

As a quick remark, note from Theorem 3.3(b) that the coefficients \(X_{(n)}\) of a vvmf \(X \in \mathcal{M}_{\rho}^1(\rho)\) will all lie in the field generated over \(\mathbb{Q}\) by the entries of \(\Lambda_w\) and \(\chi_w\), as well as the coefficients \(X_{(n)}\) of the principal part \(\mathcal{P}_\Lambda(X)\), where \(\Lambda\) is bijective and as always \(\chi = \Xi(1)\).

Let \(\mathcal{F}\) denote the span of all \(p(\tau)q^uh(q)\), where \(p(\tau) \in \mathbb{C}[\tau]\), \(u \in \mathbb{C}\), and \(h \in \mathbb{C}[[q]]\) is holomorphic at \(q = 0\). The components of any vvmf (including logarithmic', where \(T\) is nondiagonalisable) must lie in \(\mathcal{F}\). \(\mathcal{F}\) is a ring, where terms simplify in the obvious way (thanks to the series \(h\)'s converging absolutely). \(\mathcal{F}\) is closed under both differentiation and \(\tau \mapsto \tau + 1\). The key to analysing functions in \(\mathcal{F}\) is the fact that they are equal only when it is obvious that they are equal:

**Lemma 3.2.** Suppose \(\sum_{i=1}^n p_i(\tau)q^{u_i}h_i(q) = 0\) for all \(q\) in some sufficiently small disc about \(q = 0\), where each \(p_i(\tau) \in \mathbb{C}[\tau]\), \(u_i \in \mathbb{C}\), \(0 \leq \Re u_i < 1\), and \(h_i \in \mathbb{C}[[q]]\) is holomorphic at \(q = 0\), and all \(u_i\) are pairwise distinct. Then for each \(i\), either \(p_i(\tau)\) is identically 0 or \(h_i(q)\) is identically 0 (i.e. all coefficients of \(h_i(q)\) vanish).

**Proof.** For each \(v \in \mathbb{C}\), define an operator \(\mathcal{F}_v\) on \(\mathcal{F}\) by \((\mathcal{F}_v g)(\tau) = g(\tau + 1) - e^{2\pi i v} g(\tau)\). Note that \(p(\tau)q^uh(q)\) lies in the kernel of \(\mathcal{F}_v\) if (and as we will see only if) the degree of the polynomial \(p(\tau)\) is < \(k\).

Assume for contradiction that no \(p_i(\tau)\) nor \(h_i(q)\) are identically 0. Let \(k_i\) be the degree of \(p_i(\tau)\). Apply \(\mathcal{F}_{k_{i+1}}^k \circ \cdots \circ \mathcal{F}_{k_{i+1}}^k \circ \mathcal{F}_{k_1}^k\) to \(\sum_{i=1}^n p_i(\tau)q^{u_i}h_i(q) = 0\) to obtain \(aq^{u_1}h_1(q) = 0\) for some nonzero \(a \in \mathbb{C}\) and all \(q\) in that disc. This forces \(h_1(q) \equiv 0\), a contradiction. \(\Box\)

**Proposition 3.5.** Let \((\rho, w)\) be admissible and \(T\) diagonal, and choose any vvmf \(X(\tau) \in \mathcal{M}_{\rho}^1(\rho)\). Write \(X(\tau) = q^k \sum_{n=0}^\infty X(n)q^n\).

(a) Let \(\sigma\) be any field automorphism of \(\mathbb{C}\), and for each \(1 \leq i \leq d\) define \(X(\tau) = q^{\sigma(\lambda_i)} \sum_{n=0}^\infty \sigma(X(n))q^n\). Then \(X(\tau) \in \mathcal{M}_{\rho^\sigma}^1(\rho^\sigma)\), where \((\rho^\sigma, \sigma w)\) is admissible, \(T^\sigma = e^{2\pi i w}\), \((e^{2\pi i w}/2 S)\) is conjugate to \(e^{2\pi i w}/2 S\) and \(e^{2\pi i w/3 T} \) is conjugate to...
$e^{2\pi i w/3} U$. $X^\sigma(\tau)$ is $\sigma \lambda$-holomorphic iff $X(\tau)$ is $\lambda$-holomorphic. Choose any bijective exponent $\Lambda$ and fundamental matrix $\Xi(\tau)$ for $(\rho, w)$; then $(\rho^\sigma, \sigma w)$ has bijective exponent $\sigma \Lambda$ and fundamental matrix $\Xi^\sigma(\tau)$, where $\sigma$ acts on $\Xi(\tau)$ column-wise.

(b) Let $\mathbb{Q}_\infty$ be the field generated over $\mathbb{Q}$ by all Fourier coefficients $X_{\alpha}(\tau)$ of $X(\tau)$. Assume the components $X_{\alpha}(\tau)$ of $X(\tau)$ are linearly independent over $\mathbb{C}$. Then both the weight $w$ and all exponents $\lambda_j$ lie in $\mathbb{Q}_\infty$.

Proof. Start with (a), and consider first $w = 0$. Write $X = \Xi$. Then $\Xi(\tau)$ obeys the differential equation (32). Recall that $J_1, E_2, E_4, E_6, \Delta$ all have coefficients in $\mathbb{Z}$ and hence are fixed by $\sigma$. Then $\Xi^\sigma(\tau)$ formally satisfies (32) with $\lambda_0, \chi_0$ replaced with $\sigma(\lambda_0), \sigma(\chi_0)$ (recall that (32) is equivalent to the recursions (56)). Therefore $\Xi^\sigma(\tau)$ is a fundamental solution of that differential equation, with entries meromorphic in $\mathbb{H}^\sigma$.

In fact, thanks to the Fuchsian equation (32), the only possible poles of $\Xi^\sigma(\tau)$ are at the cusps or elliptic fixed points. The behaviour at the elliptic points is easiest to see from (53): in particular, $S^\rho$ and $U^\rho$ are conjugate to $e^{2\pi i \tau^2} \Xi$ and $e^{2\pi i \tau^3} \Xi$ respectively. But $\sigma^\rho = \sigma \rho$ entry-wise and both $\sigma^\rho$ are diagonalisable with rational eigenvalues, so $\sigma^\rho$ is conjugate to $\rho$ (as it has the identical eigenvalue multiplicities).

To generalise to arbitrary weight $w$, it’s clear from Lemma 3.1 that $(\Delta^{w/12})^\rho(\tau) = \Delta^{w/12}(\tau)$. A weight-$w$ fundamental matrix is $\Delta^{w/12}(\tau)$ times a weight-0 one.

Finally, if $X(\tau) \in \mathcal{A}_w^{\sigma}(\rho)$, then there exists a polynomial $p(J) \in \mathcal{C}(J)$ such that $X(\tau) = \Xi(\tau) p(J(\tau))$, so $X^\sigma(\tau) = \Xi^\sigma(\tau) p(J(\tau))^\rho$. Since $\sigma$ fixes the coefficients of $J(\tau)$, $\Xi^\sigma(\tau)$ manifestly lies in $\mathcal{A}_w^\sigma(\rho^\sigma)$.

Now turn to (b). Suppose first that there is some entry $A_{ij} \notin \mathbb{Q}_\infty$. Then there exists some field automorphism $\sigma$ of $\mathbb{C}$ fixing $\mathbb{Q}_\infty$ but with $\delta := \sigma A_{ij} - A_{ij}$ nonzero (see e.g. (11) for a proof of why such a $\sigma$ exists).

From part (a), $X^\sigma(\tau) \in \mathcal{A}_w^\sigma(\rho^\sigma)$. Then $X^\sigma(\tau) = q^{(\rho^\Lambda - \lambda) \chi(\tau)}$. Choose any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$ (because $\Gamma$ is Fuchsian of the first kind it will have many such $\gamma$). Then $\tau \rightarrow -1/\tau$ gives

\[ S^\rho X^\sigma(\tau) = \tau^{w-\sigma w} \exp(-2\pi i (\sigma \lambda - \lambda)/\tau) S X(\tau). \]  \hspace{1cm} (54)

Write $f(\tau), g(\tau) \in \mathcal{F}$ for the $i$th entry of $S^\rho$ and of $S^\rho X^\sigma(\tau)$, respectively. Because the entries of $X(\tau)$ are linearly independent, $f(\tau)$ (and also $g(\tau)$) are nonzero. Write $w^* = w - \sigma w$. By (54), $g(\tau) = \tau^{w^*} \exp(-2\pi i \delta /\tau) g(\tau)$. Like all entries of $S^\rho X^\sigma(\tau)$, $g(\tau)$ is killed by the order-$d$ differential operator

\[ L_{S^\rho X^\sigma} = \sum \tilde{h}_1(\tau) \left( \frac{d}{2\pi i d\tau} \right)^j \]

obtained from Lemma 2.2(b) by expanding out each $D_{\alpha, w}^f$; note that each $\tilde{h}_1(\tau) \in q^v \mathbb{C}[[q]]$ for some $v \in \mathbb{C}$, being a combination of the modular forms $h_1(\tau)$ of Lemma 2.2(b) and various derivatives of $E_2(\tau)$. The product rule and induction
on \(l\) gives
\[
\left(\frac{d}{2\pi i d\tau}\right)^l g(\tau) = \tau^w \exp(-2\pi i \delta/\tau) \sum_{k=0}^l \frac{p_{l,k}(\tau)}{\tau^{2l-2k}} \left(\frac{d}{2\pi i d\tau}\right)^k f(\tau),
\]
(55)
where \(p_{l,k}(\tau)\) is a polynomial in \(\tau\) of degree \(l-k\) and \(p_{l,0}(\tau)\) has nonzero constant term \((2\pi i \delta)^{l-k}\). Multiplying \(L_{S^\varrho \mathcal{X}^\varrho} g = 0\) by \(\tau^{2d} \tau^{-w} \exp(2\pi i \delta/\tau)\), we obtain
\[
\sum_{l=0}^d \tilde{h}_l(\tau) \sum_{k=0}^l \tau^{2k} p_{l,k}(\tau) \left(\frac{d}{2\pi i d\tau}\right)^k f(\tau) = 0.
\]
(56)
Now, the derivatives \(\left(\frac{d}{2\pi i d\tau}\right)^k f(\tau)\) are manifestly \(q\)-series (i.e. functions in \(\mathcal{X}\) whose denominator are all constant, so we see from Lemma 3.2 that (regarding (55) as a polynomial in \(\tau\) with \(q\)-series coefficients) the \(\tau^0\) coefficient of (56) must itself vanish. This is simply \(\tilde{h}_d(\tau)(2\pi i \delta)^d f(\tau) = 0\), where \(\tilde{h}_d(\tau) = \text{Wr}(\mathcal{X}^\varrho)(\tau)\neq 0\) by Lemma 2.2(a). This forces \(\delta = 0\), i.e. \(\sigma A_{ij} = A_{ij}\).

Therefore all entries \(A_{ij}\) must lie in \(\mathbb{Q}_{\mathcal{X}}\). Suppose now that the weight \(w\) doesn’t lie in \(\mathbb{Q}_{\mathcal{X}}\), and as above choose a field automorphism \(\sigma\) of \(\mathbb{C}\) fixing \(\mathbb{Q}_{\mathcal{X}}\) but with \(\sigma w - w \notin \mathbb{Z}\) (if \(w\) lies in an algebraic extension of \(\mathbb{Q}_{\mathcal{X}}\) then this is automatic, while if \(w\) lies in a transcendental extension we can select \(\sigma w\) to likewise be an arbitrary transcendental). The remainder of the argument is as above: \(\frac{d_{l,k}(\tau)}{\tau^{2l-k}}\) in (55) is now replaced with \(\frac{c_{l,k}}{\tau^l}\) for some \(c_{l,k} \in \mathbb{C}\), and \(c_{l,0} = w'(w'-1)\cdots(w'-l+1)\neq 0\).

Multiplying \(L_{S^\varrho \mathcal{X}^\varrho} g = 0\) by \(\tau^{d} \tau^{-w'}\), we see that \(\tilde{h}_d(\tau) c_{d,0} f(\tau) = 0\), likewise impossible unless \(w \in \mathbb{Q}_{\mathcal{X}}\). \(\square\)

The obvious Galois action on representations, namely \((\sigma p)(\gamma)_{ij} = \sigma(\rho(\gamma)_{ij})\), is unrelated to this \(\rho^\sigma\). It would be interesting though to understand the relation between the vvmf of \(\sigma p\) and those of \(\rho\).

For vvmf with rank-1 multipliers, we have \(T^\varrho = e^{2\pi i a^\lambda}, e^{2\pi i w S^\varrho} = e^{2\pi i w/2 S}\), and \(e^{2\pi i w^3/3 U^\varrho} = e^{2\pi i w^3/3 U}\). When the rank \(d\) is greater than 1, however, the precise formula for \(S^\varrho\) and \(U^\varrho\) is delicate. For example, we learn in Section 4.2 that when \(d = 2\), the explicit relation between \(S^\varrho\) and \(S\) involves the relation between the Gamma function values \(\Gamma(\sigma A_{11})\) and \(\Gamma(A_{11})\).

The next result is formulated in terms of certain modules \(\mathcal{X}\). One important example is \(\mathcal{X} = \mathbb{K}[[q^{-1}, q]]\) for any subfield \(\mathbb{K}\) of \(\mathbb{C}\). Another example is the subset \(\mathcal{X}'\) of \(f \in \mathbb{Q}[[q]]\) with bounded denominator, i.e. for which there is an \(N \in \mathbb{Z}_{>0}\) such that \(NF \in \mathbb{Z}[[q]]\). Both examples satisfy all conditions of Proposition 3.6. This latter example can be refined in several ways, e.g. by fixing from the start a set \(P\) of primes and requiring that the powers of the primes in \(P\) appearing in the denominators be bounded, but primes \(p \notin P\) be unconstrained.

**Proposition 3.6.** Suppose \((\rho, w)\) is admissible, \(\rho\) irreducible and \(T\) is diagonal, and choose any exponent \(\lambda\). Let \(\mathbb{K}\) be any subfield of \(\mathbb{C}\) and let \(\mathcal{X} \subset \mathbb{K}[[q^{-1}, q]]\) be any module over both \(\mathbb{K}\) and \(\mathbb{Z}[q^{-1}, q]\) (where both of these act by multiplication),
such that \( \frac{d}{dx} X \subseteq X \). Let \( \mathcal{H}_w^1(\lambda) \) denote the intersection \( \mathcal{H}_w^1(\lambda) \cap q^\lambda X \). Then the following are equivalent:

(i) \( \mathcal{H}_w^1(\lambda)(\rho) \neq 0 \);
(ii) \( \mathcal{H}_{w+2k}^1(\lambda)(\rho) \neq 0 \) for all \( k \in \mathbb{Z} \);
(iii) \( \text{span}_{\mathbb{C}} \mathcal{H}_w^1(\lambda)(\rho) = \mathcal{H}_w^1(\lambda)(\rho) \);
(iv) for any bijective exponent \( \Lambda \), all entries \( \Xi_{ij}(\tau) \) of the corresponding fundamental matrix \( \Xi(\tau) \) lie in \( q^{\lambda_1}X \);
(v) for any exponent \( \lambda \), the space \( \text{span}_{\mathbb{C}} \left( \mathcal{H}_w^1(\lambda)(\rho) \cap q^\lambda X \right) = \mathcal{H}_w^1(\lambda)(\rho) \).

Proof. Assume (i) holds, and choose any nonzero \( \Xi(\tau) \in \mathcal{H}_w^1(\lambda)(\rho) \). Then \( \Delta E_{ij}^\dagger E_{ij} \in q^\lambda X \) for any \( h \in \mathbb{Z} \) and \( i, j \in \mathbb{Z}_{\geq 0} \), which gives us (ii). That \( \rho \) is irreducible forces the components of \( \Xi(\tau) \) to be linearly independent over \( \mathbb{C} \). Then Proposition 3.5 would require all \( W, \lambda_i \in \mathbb{K} \), and hence \( q^\lambda X^d \) is mapped into itself by \( \nabla_{w'}, \forall \) for any \( w' \in w + \mathbb{Z} \). Moreover, Proposition 3.2 applies and \( \mathcal{H}_w(\rho) = \mathbb{C}[J, \nabla_1, \nabla_2, \nabla_3] \Xi(\tau) \), which gives us (iii). Let \( \Lambda \) be any bijective exponent. Then each column of \( \Xi(\tau) \) is a linear combination over \( \mathbb{C} \) of finitely many vvmf in \( \mathcal{H}_w^1(\lambda)(\rho) \); but all of these vectors are uniquely determined by their principal parts \( \mathcal{P}_\lambda \), which are all in \( \mathbb{K}^d[q^{-1}] \), so that linear combination must have a solution over the field \( \mathbb{K} \). This gives us (iv).

To get (v), note that in the proof of Theorem 3.4(a) we may choose our \( \Xi^{(1)}(\tau) \) to lie in \( q^\lambda X \), since all we require of them is a linear independence condition. \( \square \)

This proposition is relevant to the study of modular forms for noncongruence subgroups of \( \Gamma \). A conjecture attributed to Atkin–Swinnerton-Dyer \cite{12} states that a (scalar) modular form for some subgroup of \( \Gamma \) will have bounded denominator only if it is a modular form for some congruence subgroup. More generally, it is expected that a vvmf \( \Xi(\tau) \) for \( \Gamma \), with entries \( \Xi_{ij}(\tau) \) linearly independent over \( \mathbb{C} \) and with coefficients \( \Xi_{ij}(\tau) \) all in \( \mathbb{Q} \), will have bounded denominators only if its weight \( w \) lies in \( \frac{1}{2} \mathbb{Z} \) and the kernel of \( \psi_w \otimes \rho \) is a congruence subgroup. If the kernel is of infinite index, then it is expected that infinitely many distinct primes will appear in denominators of coefficients.

### 3.7 Semi-direct sums and exactness

If \( \rho \) is a direct sum \( \rho' \oplus \rho'' \), then trivially its vvmf are comprised of those of \( \rho' \) and \( \rho'' \). But what if \( \rho \) is a semi-direct sum?

Consider a short exact sequence

\[
0 \to U \xrightarrow{\iota} V \xrightarrow{\pi} W \to 0
\]

of finite-dimensional \( \mathbb{T} \)-modules. We can consider an action at arbitrary weight by tensoring with \( \psi_w \). Choose a basis for \( V \) in which \( \rho_V = \begin{pmatrix} \rho_U & x \\ 0 & \rho_W \end{pmatrix} \). In terms of this basis, \( \Xi_V \in \mathcal{H}_w^1(\rho_V) \) implies \( \Xi_V = \begin{pmatrix} \Xi_U \\ \Xi_W \end{pmatrix} \) where \( \Xi_W \in \mathcal{H}_w(\rho_W) \), and \( \begin{pmatrix} \Xi_U \\ \Xi_W \end{pmatrix} \in \mathcal{H}_w(\rho_V) \).
\[ \mathcal{M}_w^1(\rho_V) \] iff \( \mathcal{M}_w^1(\rho_U) \). For this basis we have the natural embedding \( \iota'(\mathcal{X}_U) = \begin{pmatrix} \mathcal{X}_U \\ 0 \end{pmatrix} \) and projection \( \pi'(\begin{pmatrix} \mathcal{X}_U \\ 0 \end{pmatrix}) = \mathcal{X}_W \). We write (57) as \( \rho_V = \rho_U \circ \rho_W \).

The fundamental objects in the theory of vvmf are the functors \( (\rho, w) \mapsto \mathcal{M}_w^1(\rho), (\rho, w; \lambda) \mapsto \mathcal{M}_w^1(\rho, \lambda) \), attaching spaces of vvmf to multipliers, weights and exponents. Marks–Mason (22) suggest considering the effects of these functors on (57). It is elementary that they are left-exact, i.e. that \( \rho_V = \rho_U \circ \rho_W \) trivially implies

\[ 0 \to \mathcal{M}_w^1(\rho_U) \to \mathcal{M}_w^1(\rho_V) \to \mathcal{M}_w^1(\rho_W) \to 0, \]

and similarly for \( \mathcal{M}_w^1 \) etc. However (22) found 1-dimensional \( U \) and \( W \) such that the functor \( \rho \mapsto \mathcal{M}_w^1(\rho) \) is not right-exact.

Thanks to Theorem 3.5, we can generalise and quantify this discrepancy. We thank Geoff Mason for suggesting the naturalness of explaining the failure of right-exactness with a long exact sequence.

**Theorem 3.6.** Write \( \rho_V = \rho_U \circ \rho_W \) as in (57), and suppose \( (\rho_V, w) \) is admissible and \( T \) \( \tau \) diagonal. Choose any exponent \( \lambda_V = \text{diag}(\lambda_U, \lambda_W) \). Then we obtain the exact sequences of \( C \)-spaces

\[ 0 \to \mathcal{M}_w^1(\rho_U) \to \mathcal{M}_w^1(\rho_V) \to \mathcal{M}_w^1(\rho_W) \to 0, \]

\[ 0 \\to \mathcal{M}_U \to \mathcal{M}_V \to \mathcal{M}_W \to \ker \mathcal{P}_U \cong (\mathcal{M}_V) \to (\mathcal{M}_V) \to (\mathcal{M}_W) \to 0 \]

where we write \( \mathcal{P}_U = \mathcal{P}_{\lambda_U^{-1} \lambda_U(\rho_U)} \), \( \mathcal{M}_U = \mathcal{M}_w^{1, \lambda_U}(\rho_U) \), \( \mathcal{M}_W = \mathcal{M}_w^{1, \lambda_W}(\rho_U) \) etc.

**Proof.** First let’s prove (59). Let \( \mathcal{M}_w^1, \mathcal{M}_w^1, \mathcal{M}_w^1 \) denote \( \mathcal{M}_w^1(\rho_V), \mathcal{M}_w^1(\rho_U), \mathcal{M}_w^1(\rho_W) \) respectively. Choose any bijective exponents \( \lambda_U, \lambda_W \) and define \( \lambda_V = \text{diag}(\lambda_U, \lambda_W) \), \( \mathcal{P}_V, \mathcal{P}_U, \mathcal{P}_W \) for \( \mathcal{P}_V, \mathcal{P}_U, \mathcal{P}_W \) etc. If \( \begin{pmatrix} \mathcal{X}_U \\ \mathcal{X}_W \end{pmatrix} \in \ker \mathcal{P}_V \), then \( \mathcal{X}_W(\tau) \in \ker \mathcal{P}_W \) and hence \( \mathcal{X}_W(\tau) = 0 \) because \( \mathcal{X}_W \) is bijective. This means \( \mathcal{X}_U(\tau) \in \mathcal{M}_U \), so also \( \mathcal{X}_U(\tau) \in \ker \mathcal{P}_U \) and \( \mathcal{X}_U(\tau) = 0 \) because \( \mathcal{X}_U \) is bijective. Therefore \( \mathcal{P}_V \) is injective. From the first line of (25) we know \( c(\rho_V, w) = c(\rho_U, w) \) (since \( T \mathcal{S}_V = T \mathcal{S}_U + T \mathcal{S}_W \) etc); the index formula (24) then implies \( \text{ker} \mathcal{P}_V = 0 \) and hence \( \lambda_V \) is bijective. In order to establish (59), only the surjectivity of \( \pi' \) needs to be shown, but this follows from the surjectivity of \( \mathcal{P}_V \) together with the injectivity of \( \mathcal{P}_W \).

Now let’s turn to (60). Most of this exactness again comes from (58): the dual of \( 0 \to \mathcal{M}_W \to \mathcal{M}_V \to \mathcal{M}_U \) gives the second half of (60). The connecting map \( \delta \) is defined as follows. Given any \( \mathcal{X}_W \in \mathcal{M}_W \), exactness of (59) says that there is an \( \mathcal{X}_U \), unique mod \( \mathcal{M}_U \), such that \( \begin{pmatrix} \mathcal{X}_U \\ \mathcal{X}_W \end{pmatrix} \in \mathcal{M}_V \). The connecting map \( \delta : \mathcal{M}_W \to \ker \mathcal{P}_U \) sends \( \mathcal{X}_W \) to \( \mathcal{P}_U(\mathcal{X}_U) + \text{Im} \mathcal{P}_U \). Exacness at \( \mathcal{M}_W \) is now clear: if \( \mathcal{X}_W \in \ker \delta \) then \( \mathcal{P}_U(\mathcal{X}_U) = \mathcal{P}_U(\mathcal{X}_U') \) for some \( \mathcal{X}_U' \in \mathcal{M}_U \), so \( \begin{pmatrix} \mathcal{X}_U' \\ 0 \end{pmatrix} \) and hence
Consider $\rho$, summed over all $w$. For $/M_{\cal V}$, these functionals can be expressed as $/Y_{/X_{/M_{\cal V}}}$ and $/$. The independence of the functional on $/Y_{/X_{/M_{\cal V}}}$ means $/$, so we can take $/P_{/X_{/M_{\cal V}}} = /P_{/X_{/M_{\cal V}}}$. That the functional on $/M_{/X_{/M_{\cal V}}}$ is 0 means (again from (50)) that $/$ is $/X_{/M_{\cal V}}$ for some $/$. Moreover, $/X_{/M_{\cal V}} \in /M_{/X_{/M_{\cal V}}}$, so $/P_{/X_{/M_{\cal V}}} = /X_{/M_{\cal V}}$, as desired. \( \square \)

Theorem 3.6 allows us to classify all bijective $\Lambda_{/M_{/X_{/M_{\cal V}}}}$. These are given by all exponents $\text{diag}(\lambda_{/X_{/M_{\cal V}}}, \lambda_{/X_{/M_{\cal V}}})$ such that $/P_{/X_{/M_{\cal V}}}$ is injective, $/P_{/X_{/M_{\cal V}}}$ is surjective, and the connecting map $/\delta : \text{ker} /P_{/X_{/M_{\cal V}}} \rightarrow \text{coker} /P_{/X_{/M_{\cal V}}}$ is an isomorphism.

We can now quantify the failure of $\mathcal{M}_{/M_{\cal V}}$ to be exact. For each fixed $w$, the discrepancy is

$$\dim \mathcal{M}_{/M_{/X_{/M_{\cal V}}}} + \dim \mathcal{M}_{/X_{/M_{\cal V}}} - \dim \mathcal{M}_{/X_{/M_{\cal V}}} = \dim \text{Im} /\delta .$$

For $w < 0$, $\mathcal{M}_{/M_{\cal V}}(\rho_{/X_{/M_{\cal V}}}) = 0$ so $/\delta = 0$ and the discrepancy is 0, while for $w \geq 2$, then $\mathcal{M}_{/M_{\cal V}}(\rho_{/X_{/M_{\cal V}}}) = 0$ so again $/\delta = 0$ and the discrepancy is 0. Thus the total discrepancy, summed over all $w$, is finite.

Let us recover in our picture the calculation in Theorem 4 of [22]. Take $\lambda = /\delta$. Consider $\rho_{/X_{/M_{\cal V}}}$ of the form $u_{2a} \circ u_{2b}$, where as always $u_{j}$ has $T = e^{2\pi ij/12}$. Then Theorem 4 of [22] says $\rho_{/X_{/M_{\cal V}}}$ can be indecomposable iff $|a-b| = 1$. As above, if $w < 0$ then $\mathcal{M}_{/X_{/M_{\cal V}}} = 0$ while if $w \geq 2$ then $\mathcal{M}_{/X_{/M_{\cal V}}} = 0$, so only at $w = 0$ can $/\delta \neq 0$. We find that $w = 0$, $\mathcal{M}_{/X_{/M_{\cal V}}} \neq 0$ forces $b = 0$ and $a = 5$, in which case $/\delta : \mathbb{C} \rightarrow \mathbb{C}$. Now, a bijective exponent for $(\rho_{/X_{/M_{\cal V}}}, 0)$ is diag $(-\lambda_{/X_{/M_{\cal V}}}, 0)$ by Theorem 3.6 (and Section 4.1 below), so there must be a $X_{/X_{/M_{\cal V}}} \in q^{-1/6} \sum_{n=0}^{\infty} c_n q^n$ such that $X_{/X_{/M_{\cal V}}} \in /X_{/M_{\cal V}}$, by surjectivity. Indeed, $/\delta(1) = /P_{/X_{/M_{\cal V}}}(X_{/X_{/M_{\cal V}}}) = c_0$. If $c_0 = 0$ then the Wronskian of $X_{/X_{/M_{\cal V}}}$ would be a nonzero holomorphic modular form $c_1 q^{2/6} + \cdots$ of weight 2 (the Wronskian is nonzero because $\rho_{/X_{/M_{\cal V}}}$ is indecomposable). This is impossible (e.g. $q^{-20}$ times it would also be holomorphic but with trivial multiplier and weight $-8$). Therefore $c_0 \neq 0$, so $/\delta \neq 0$ and the total discrepancy is 1-dimensional.

4 Effectiveness of the theory

Explicit computations within our theory are completely feasible. Recall from Theorem 3.3 that we have complete and explicit knowledge of the space $\mathcal{M}_{/M_{/X_{/M_{\cal V}}}}(\rho_{/X_{/M_{\cal V}}})$ of weakly holomorphic vmf, if we know the diagonal matrix $\Lambda$ and the complex matrix $\mathcal{X}$. We know $\text{Tr} /\Lambda$, and generically any matrix with the right trace and with $e^{2\pi i /\Lambda} = T$ is a bijective exponent. This $\mathcal{X}$ can be obtained in principle from $\rho$ and $\Lambda$ using the Rademacher expansion (see e.g. [12]). However this series expansion
for $\mathcal{X}$ converges notoriously slowly, and obscures any properties of $\mathcal{X}$ that may be present (e.g. integrality). Hence other methods are needed for identifying $\mathcal{X}$. In this section we provide several examples, illustrating some of the ideas available. See $[4, 5]$ for further techniques and examples.

### 4.1 One dimension

It is trivial to solve the $d = 1$ case $[3]$. Here, $\rho = \upsilon_u$, and $u \in \mathbb{C}$, $0 \leq \text{Re } u < 1$, and the weight $w$ is required to be $w \in 12u + 2\mathbb{Z}$. Write $w = 12u + 2j + 12n$ where $0 \leq j \leq 5$ and $n \in \mathbb{Z}$; then for $j = 0, 1, 2, 3, 4, 5$ resp., the fundamental matrix $\Xi(\tau)$ for $\mathcal{M}_{12u+2j+12n}(\upsilon_u)$ is (recall Lemma 3.1):

$$
\Delta^{u+n}(\tau), \quad \Delta^{u+n-5/6}(\tau) E_{10}(\tau), \quad \Delta^{u+n-2/3}(\tau) E_{8}(\tau),
$$

$$
\Delta^{u+n-1/2}(\tau) E_{6}(\tau), \quad \Delta^{u+n-1/3}(\tau) E_{4}(\tau), \quad \Delta^{u+n-7/6}(\tau) E_{14}(\tau),
$$

respectively, where $E_8 = E_2^2$, $E_{10} = E_4 E_6$ and $E_{14} = E_4^2 E_6$. The unique free generator of $\mathcal{M}_u^{\text{hol}}(\upsilon_u)$ is $\Delta^u(\tau)$. This means that $\dim \mathcal{M}_u^{\text{hol}}(\upsilon_u)$ equals the dimension of the weight $w - 12u$ subspace of $\mathfrak{m}$, assuming as above that $0 \leq \text{Re } u < 1$.

### 4.2 Two dimensions

Much more interesting is $d = 2$ (see e.g. $[34, 23]$). Let’s start with weakly holomorphic. As usual, it suffices to consider weight $w = 0$. We continue to require that $T$ be diagonal, although we give a ‘logarithmic’ example shortly.

The moduli space of equivalence classes of 2-dimensional representations of $\Gamma$ consists of 15 isolated points, together with 3 half-planes. Each half-plane has 4 singularities: 2 conical singularities and 2 triple points. The equivalence classes of irreducible representations with $T$ diagonalisable correspond bijectively with the regular points on the 3 half-planes. The 15 isolated points are all direct sums $\rho_1 \oplus \rho_2$ of 1-dimensional representations which violate the inequalities (8), and are handled by Section 4.1. Each conical singularity corresponds to an irreducible ‘logarithmic’ representation with $T$ nondiagonalisable. Each triple point consists of a direct sum $\rho \oplus \rho'$ as well as the semi-direct sums $\rho \varphi \rho'$ and $\rho' \varphi \rho$. In all cases except the 6 triple points, the set of eigenvalues of $T$ uniquely determine the representation. The 3 half-planes correspond to the 3 different choices of $(\alpha_i, \beta_j)$ possible at $d = 2$ (recall (8)).

More explicitly, one half-plane has det $T = \xi_6 =: \zeta$:

$$
T = \begin{pmatrix} z & 0 \\ 0 & \zeta z^{-1} \end{pmatrix}, \quad S = \begin{pmatrix} \frac{\zeta z}{z^2 - 1} & \frac{y}{z^2 - 1} \\ \frac{y}{z^2 - 1} & -\frac{\zeta z}{z^2 - 1} \end{pmatrix},
$$

(62)
for arbitrary $y, z \in \mathbb{C}$ provided $y \neq 0$ and $z \notin \{0, \pm \xi_{12}\}$. This is irreducible iff $z \notin \{\pm 1, \pm \zeta\}$. The redundant parameter $y$ is introduced for later convenience. Irreducible $\rho, \rho'$ with $z, z'$ related by $(zz')^2 = \zeta$ are naturally isomorphic. Here, $\alpha_1 = \beta_0 = \beta_1 = 1$, so $\text{Tr} \Lambda = -\frac{5}{6}$ and $\Lambda = \text{diag}(t, -\frac{5}{6} - t)$ where $z = e^{2\pi i t}$ for some $t \neq \frac{1}{12}, \frac{7}{12} (\text{mod } 1)$. Then Theorem 3.3(c) tells us

$$\mathcal{X} = \begin{pmatrix} 24 \frac{(60r-11)}{12r+5} & 10368x \frac{f(2r+1)(3r+1)(6r+5)}{(12r+1)(12r+3)(12r+5)} \\ \frac{10368x}{x(12r-1)} & -4 \frac{(6r+5)(60r+61)}{12r+5} \end{pmatrix}, \quad (63)$$

for some $x \neq 0$ to be determined shortly. Equation (63) is ideally suited to relate $x$ and $y$, since at $d = 2$ it reduces to the classical hypergeometric equation. We read off from it the fundamental matrix

$$\Xi(z(\tau)) = \begin{pmatrix} f(t; \frac{5}{6}; z(\tau)) & \mathcal{X}_{12}f(t + 1; \frac{5}{6}; z(\tau)) \\ \mathcal{X}_{21}f(t; -\frac{5}{6}; z(\tau)) & f(-\frac{5}{6} - t; \frac{5}{6}; z) \end{pmatrix} \quad (64)$$

for $z(\tau) = J(\tau)/1728$, where we write

$$f(a; c; z) = (-1728z)^{-a} F(a, a + \frac{1}{2}; 2a + c; z^{-1}) \quad (65)$$

for $F(a, b; c; z) = 1 + \frac{ab}{z} + \cdots$ the hypergeometric series. Substituting $z(\tau) = J(\tau)/1728$ directly into (65) gives the $q$-expansion of $\Xi(\tau)$. The parameters $x, y$ appearing in $S$ and $\mathcal{X}$ can be related by the standard analytic continuation of $F(a, b; c; z)$ from $z \approx 0$ to $z \approx \infty$, which implies

$$f(a; c; z) = (1728)^{-a} \left\{ \frac{\sqrt{\pi} \Gamma(2a+c)}{\Gamma(a+c)} \frac{2\sqrt{\pi} \Gamma(2a+c)}{\Gamma(a+c)} - \frac{2\sqrt{\pi} \Gamma(2a+c)}{\Gamma(a+c)} \right\} \quad (66)$$

for small $|z|$. Hence

$$y = \frac{\sqrt{3} \pi}{1728} \frac{2^{2/3}}{432^{2/3}} \frac{2^{2/3} \Gamma(2t + \frac{5}{6})^2}{\Gamma(2t) \Gamma(2t + \frac{5}{6})}. \quad (67)$$

In particular, we see that when $\rho$ is irreducible, $\Lambda$ is bijective for a $\mathbb{Z}$ worth of $t$’s (i.e. the necessary conditions $e^{2\pi i t} = T$ and $\text{Tr} \Lambda = -\frac{5}{6}$ are also sufficient) — at the end of Section 3.4 we call such $\rho$ tight. This fact is generalised in Theorem 4.1 below. There are 4 indecomposable but reducible $\rho$ here:

- $z = 1$: this $\rho$ is the semi-direct sum $1 \oplus v_2$ of 1 (the subrepresentation) with $v_2$ (the quotient); in this case $\Lambda = \text{diag}(t, -\frac{5}{6} - t)$ is bijective iff $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$;
- $z = -1$: here, $\rho = v_6 \oplus v_8$; its $\Lambda$ is bijective iff $t \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$;
- $z = \zeta$: here, $\rho = v_2 \oplus 1$; its $\Lambda$ is bijective iff $t \in \frac{1}{6} + \mathbb{Z}_{\geq 0}$;
- $z = -\zeta$: here, $\rho = v_8 \oplus v_6$; its $\Lambda$ is bijective iff $t \in \frac{2}{3} + \mathbb{Z}_{\geq 0}$.

These restrictions on $t$ are needed to avoid the Gamma function poles in (67). Nevertheless, the ‘missing’ values of $t$ (apart from the forbidden $t \equiv \frac{1}{12} (\text{mod } \frac{1}{2})$)
are all accounted for; for example $t = 0$ describes the direct sum $1 \oplus v_2$ (the limiting case where $y \to 0$ slowly compared with $t \to 0$), while $t \in \mathbb{Z}_{<0}$ recovers the $z^2 = \zeta$ solution given above.

The free generators over $m$ of $\mathcal{M}^{\lambda}_{hol}(\rho)$ for these $\rho$ are now easy to find. Fix $0 \leq \text{Re } t < 1$, and note that $\lambda^{\text{hol}} = \text{diag}(t, \frac{1}{t} - t)$ when $\text{Re } t \leq \frac{1}{2}$, and otherwise $\lambda^{\text{hol}} = \text{diag}(t, \frac{2}{t} - t)$. Consider first the case where $\rho$ is irreducible; then $\rho$ is tight and Proposition 3.3(b) tells us $\dim \mathcal{M}^{\lambda}_{hol}(\rho)$ for all $w < 0$. In particular, if $\text{Re } t \leq \frac{1}{2}$ then $w^{(1)} = 0$ and $w^{(2)} = 2$, and $\chi^{(1)}(\tau)$ is the first column of $\Xi(\tau)$ given above; if instead $\text{Re } t > \frac{1}{2}$ then $w^{(1)} = 6, 8, \chi^{(1)}(\tau)$ is the first column of $\Xi(\tau)$ at $w = 6$ for $\Lambda = \text{diag}(t, \frac{1}{t} - t)$. In both cases, $\chi^{(2)}(\tau) = D\chi^{(1)}(\tau)$.

The holomorphic vvmf for the 2-dimensional indecomposable representations was discussed at the end of Section 3.7, and we find that for our four such $\rho$, $\dim \mathcal{M}^{\lambda}_{hol}(v_{2a} \oplus v_{2b}) = \dim \mathcal{M}^{\lambda}_{hol}(v_{2a}) + \dim \mathcal{M}^{\lambda}_{hol}(v_{2b})$. The 1-dimensional case was worked out in Section 4.1, and we find that in all cases $\{w^{(1)}, w^{(2)}\} = \{2a, 2b\}$ with $\chi^{(j)}(\tau)$ given by the appropriate column of the fundamental matrix at $w = 2a$ and $2b$ respectively.

The choice $z = \pm \zeta$, i.e. $t \equiv \frac{1}{12} \pmod{\frac{1}{2}}$, corresponds here to two logarithmic representations. Consider for concreteness $z = \zeta_{12}$. A weakly holomorphic vvmf for it is $\eta(\tau)^2 \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)$. This generates all of $\mathcal{M}_0^{\eta}$, using $\mathbb{C}[J]$ and the differential operators $\nabla_i$; together with

$$a^{1/12} \left(\pi i q^{-1} - 242 \pi i + (-14965 \pi i - 55440 \pi)q + \cdots\right) - 55440q + \cdots$$

it freely generates $\mathcal{M}_0^{\eta}$ over $\mathbb{C}[J]$. The 5 other 2-dimensional logarithmic representations correspond to this one tensored with a $T$ character. The free basis for holomorphic vvmf is $\eta(\tau)^2 \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)_{\pm}$ and its derivative.

We can see the Galois action of Section 3.6 explicitly here: $\sigma$ takes $t \mapsto \sigma t$ and $x \mapsto \sigma x$, and it keeps one inside this connected component.

Another class of two-dimensional representations has det $T = -1$:

$$T = \begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}, \quad S = \begin{pmatrix} \frac{z}{z^2 + 1} & y \\ \frac{z}{y(z^2 + 1)^2} & z \end{pmatrix},$$

for arbitrary $y, z \in \mathbb{C}$ provided $y \neq 0, z \notin \{0, \pm i\}$. Irreducibility requires $z \notin \{\pm 1, \pm \zeta\}$. Irreducible $\rho$ with $zz' = -1$ are isomorphic. Here, $a_1 = \beta_1 = \beta_2 = 1$, so $\text{Tr } \Lambda = -\frac{3}{2}$ and $\Lambda = \text{diag}(t, -\frac{3}{2} - t)$ for $z = e^{2\pi i t}$ and some $t \notin \frac{1}{2} \pmod{1}$. Then as before
\[ \mathcal{X} = \left( \begin{array}{c}
24 (201^2 + 51r + 32) \\
384 (3r + 2) (3r + 1) (6r + 5) (6r + 7) \\
384 \\
(4r + 3) (4r + 1) \end{array} \right), \quad (69) \]

\[ \Xi(z) = (1 - z)^{1/3} \left( \begin{array}{c}
-1728z \left( f(t + \frac{1}{6}; \frac{3}{6}; z) \right) \mathcal{X}_{12} f(t + \frac{4}{5}; \frac{2}{5}; z) \\
\mathcal{X}_{21} f(-\frac{1}{6}; t; \frac{5}{6}; z) \\
f(-t; -\frac{5}{6}; z) \\
\end{array} \right), \quad (70) \]

\[ y = \frac{x \sqrt{3}}{6912 \ 432^2 \ (2t + \frac{3}{4}) \ (2t + \frac{5}{4})}, \quad (71) \]

Again, for irreducible \( \rho \), any possible \( t \) yields bijective \( \Lambda \). The 4 indecomposable but reducible \( \rho \) are:

\( z = \zeta \): here, \( \rho = \nu_2 \oplus \nu_4 \) and \( \Lambda \) is bijective iff \( t \in \frac{1}{6} + \mathbb{Z}_{\geq 0} \);

\( z = -\zeta \): here \( \rho = \nu_6 \oplus \nu_{10} \) and \( \Lambda \) is bijective iff \( t \in \frac{5}{6} + \mathbb{Z}_{\geq 0} \);

\( z = \zeta \): here, \( \rho = \nu_{10} \oplus \nu_6 \) and \( \Lambda \) is bijective iff \( t \in -\frac{1}{6} + \mathbb{Z}_{\geq 0} \);

\( z = -\zeta \): here \( \rho = \nu_4 \oplus \nu_2 \) and \( \Lambda \) is bijective iff \( t \in \frac{1}{3} + \mathbb{Z}_{\geq 0} \).

Again the ‘missing’ \( t \) correspond to other reducible \( \rho \).

The holomorphic analysis is identical to before. Here \( \text{Tr} \lambda^{hol} = \frac{1}{6} \) or \( \frac{1}{3} \), depending on whether or not \( \text{Re} \ t \leq \frac{1}{2} \). In the former case \( w^{(i)} = (2, 4) \), and in the latter it equals \( (8, 10) \). \( \Xi^{(i)}(\tau) \) is as before. The indecomposable \( \rho \) behave exactly as before.

The final class of two-dimensional representations has det \( T = \zeta \):

\[ T = \left( \begin{array}{cc}
z & 0 \\
0 & \overline{z} \zeta^{-1} \end{array} \right), \quad S = \left( \begin{array}{cc}
\overline{z} \zeta^{-1} & y \\
y & \overline{z} \zeta^{-1} \end{array} \right), \quad (72) \]

for arbitrary \( y, z \in \mathbb{C} \) provided \( y \neq 0, z \notin \{0, \pm e^{-m/6}\} \). Irreducibility requires \( z \notin \{\pm 1, \pm \zeta\} \); irreducible \( \rho \) related by \( zz' = \zeta \) are equivalent. Here, \( \alpha_1 = 0 = \beta_2 = 1 \), so \( \text{Tr} \lambda = -\frac{7}{6} \) and \( \Lambda = \text{diag}(t, -\frac{7}{6} - t) \) for some \( t \) satisfying \( z = e^{2\pi i t} \). Then

\[ \mathcal{X} = \left( \begin{array}{c}
24 (601 + 71) \\
10368 \ (2r + 1) (3r + 2) (6r + 7) \\
10368 \\
4 (6r + 7) (6r + 1) \end{array} \right), \quad (73) \]

\[ \Xi(z) = \left( \begin{array}{c}
f(t; \frac{7}{6}; z) \\
\mathcal{X}_{12} f(t + 1; \frac{2}{6}; z) \\
\mathcal{X}_{21} f(-\frac{1}{6}; t; \frac{5}{6}; z) \\
f(-t; -\frac{5}{6}; z) \end{array} \right), \quad (74) \]

\[ y = \frac{x \sqrt{3}}{10368 \ 432^2 \ (2t + \frac{3}{4}) \ (2t + \frac{5}{4})}, \quad (75) \]

Again, for irreducible \( \rho \), any possible \( t \) yields bijective \( \Lambda \). The 4 indecomposable but reducible \( \rho \) are:

\( z = 1 \): then \( \rho = 1 \oplus \nu_{10} \) and \( \Lambda \) is bijective iff \( t \in 1 + \mathbb{Z}_{\geq 0} \);

\( z = -1 \): then \( \rho = \nu_6 \oplus \nu_4 \) and \( \Lambda \) is bijective iff \( t \in \frac{5}{6} + \mathbb{Z}_{\geq 0} \);

\( z = \zeta \): then \( \rho = \nu_{10} \oplus 1 \) and \( \Lambda \) is bijective iff \( t \in -\frac{1}{6} + \mathbb{Z}_{\geq 0} \);

\( z = -\zeta \): then \( \rho = \nu_4 \oplus \nu_6 \) and \( \Lambda \) is bijective iff \( t \in \frac{1}{3} + \mathbb{Z}_{\geq 0} \).
Again the ‘missing’ $t$ correspond to the other reducible $\rho$.

The holomorphic story for irreducible $\rho$ is as before. Here $\text{Tr} \lambda^{\text{hol}} = \frac{5}{6}$ or $\frac{11}{6}$ depending on whether or not $\text{Re } t \leq \frac{5}{6}$. This means $w(i)$ will equal $(4,6)$ or $(10,12)$, respectively. The only new phenomenon here is the indecomposable at $z = \frac{5}{6}$; at the end of Section 3.7 we learned that $\dim \mathcal{H}_{0}^{\text{hol}}(v, 10 \oplus 1)$ is 0, not 1. We find $w(i) = (4,6)$.

The $\lambda^{\text{hol}}$-holomorphic two-dimensional theory is also studied in [23], though without quantifying the relation between Fourier coefficients and the matrix $S$ (i.e. his statements are only basis-independent), which as we see involves the Gamma function. Our two-dimensional story can be extended to any triangle group [3].

4.3 vvmf in dimensions < 6

Trivially, the spaces of vvmf for an arbitrary $\rho$ are direct sums of those for its indecomposable summands. Theorem 3.6 reduces understanding the vvmf for an indecomposable $\rho$, to those of its irreducible constituents. In this section we prove any admissible $(\rho, w)$ is tight, provided $\rho$ is irreducible and of dimension < 6 (recall the definition of tight at the end of Section 3.4). This means in dimension < 6 we get all kinds of things for free (see Proposition 3.3), including identifying the Hilbert–Poincaré series $H^{\lambda}((x); \rho)$. These series were first computed in [21], for the special case of exponent $\lambda = \lambda^{\text{hol}}$, when $T$ is unitary, and the representation $\rho$ is what Marks calls $T$-determined, which means that any indecomposable $\rho'$ with the same $T$-matrix is isomorphic to $\rho$. It turns out that most $\rho$ are $T$-determined.

We will see this hypothesis is unnecessary, and we can recover and generalise his results with much less effort. The key observation is the following, which is of independent interest:

**Theorem 4.1.** Let $(\rho, w)$ be admissible and $T$ diagonal. Assume $\rho$ is irreducible and the dimension $d < 6$. Then $\rho$ is tight: an exponent $\lambda$ is bijective for $(\rho, w + 2k)$ iff $\text{Tr } \lambda = c(\rho, w + 2k)$.

**Proof.** The case $d = 1$ is trivial, and $d = 2$ is explicit in Section 4.2, so it suffices to consider $d = 3, 4, 5$. Without loss of generality (by tensoring with $u_{-w}$) we may assume $\rho$ is a true representation of $T$, i.e. that $(\rho, 0)$ is admissible. Let $\alpha_i = a_i(\rho, 0)$, $\beta_j = b_j(\rho, 0)$. Let $x^{(i)}(\tau)$ be the free generators which exist by Theorem 3.4(a), and let $w^{(i)} \leq \cdots \leq w(d)$ be their weights. We will have shown that $\rho$ is tight, if we can show that these $w^{(i)}$ agree with those in the tight Hilbert–Poincaré series $\mathcal{H}_{\rho}$.

This is because this would require all $\dim \mathcal{H}_{\rho}(\omega)$ to equal that predicted by $H^{\lambda}(x)$, as given in Proposition 3.3, and that says $\lambda$ will be bijective iff $\lambda$ has the correct trace. In fact it suffices to verify that the values of $w^{(i)} - w^{(i)}$ match the numerator of [44], as the value of $\sum_{i} w^{(i)}$ would then also fix $w^{(i)}$.

Let $n_i$ be the total number of generators $x^{(i)}(\tau)$ with weight $w^{(i)} \equiv 2i$ (mod 12). By Theorem 3.4(b) we know $\sum_{i} n_i = d$, $n_i + n_{2+1} + n_{4+i} = a_i$, $n_j + n_{3+j} = \beta_j$ for all
i, j. These have solutions

\[(n_0, n_1, n_2, n_3, n_4, n_5) = (a_0 - \beta_2 + t - s, \beta_1 - s, \beta_2 - t, s - t + \alpha_1 - \beta_1, s, t)\]  

(76)

for parameters \(s, t, \).

Consider first \(d = 3\). The inequalities (8) force \(\beta_i = 1\) and \(\{\alpha_0, \alpha_1\} = \{1, 2\}\). By Proposition 3.3(a), we can assume without loss of generality (hitting with \(v_6\) if necessary) that \(\alpha_1 = 2\). Then the only nonnegative solutions to (76) are \((n_i) = (0, 1, 1, 0, 0), (1, 1, 0, 0, 0, 0), (0, 0, 1, 1, 1)\). From (25) we see \(L := \text{Tr} \lambda \in \mathbb{Z}\). Theorem 3.4(b) says \(\sum_i w^{(i)} = 12L\). The inequality (42) requires \(w^{(i)} = 4L - 2\), so the only possibility consistent with the given values of \(n_i\) together with \(\sum_i w^{(i)} = 12L\) is \((w^{(i)}) = (4L - 2, 4L, 4L + 2)\), which is the prediction of (44).

Consider next \(d = 4\). As before we may assume \((\beta_i) = (2, 1, 1)\) and \((\alpha_i) = (2, 2)\). Then (76) forces \((n_i) = (1, 1, 1, 0, 0), (1, 0, 0, 1, 1, 1), (2, 1, 0, 0, 0, 1), (0, 0, 1, 2, 1, 0)\). Again \(L := \text{Tr} \lambda \in \mathbb{Z}\) and \(\sum_i w^{(i)} = 12L\), and (42) forces \(w^{(i)} \in \{3L - 3, 3L - 1\}\) (if \(L\) is odd) or \(w^{(i)} \in \{3L - 2, 3L\}\) (if \(L\) is even).

We claim for each \(L\) there is a unique possibility for the \(w^{(i)}\) which is compatible with \(\sum_i w^{(i)} = 12L\), the listed possibilities for \((n_i)\), the 2 possible values for \(w^{(i)}\) given above, and the absence of a ‘gap’ in the sense of (46). When \(L\) is even this is \((w^{(i)}) = (3L - 2, 3L, 3L)\); when \(L\) is odd this is \((w^{(i)}) = (3L - 3, 3L - 1, 3L + 1, 3L + 3)\). These match (44).

Finally, consider \(d = 5\). We may take \((\beta_i) = (1, 2, 2)\) and \((\alpha_i) = (3, 2)\), so

\[(n_i) \in \{(1, 2, 2, 0, 0, 0), (1, 1, 1, 0, 1, 1), (1, 0, 0, 0, 2, 2), (0, 1, 2, 1, 1, 0), (0, 0, 1, 1, 2, 1)\}, \]  

(77)

\(L := \text{Tr} \lambda \in \mathbb{Z}\) and \(\sum_i w^{(i)} = 12L\).

If \(L = 5L' + 1\), then (42) forces \(w^{(i)} = 12L' + 2\) or \(12L' + 4\). We find the only possible value of \(w^{(i)}\) is \((12L', 12L' + 2, 12L' + 4, 12L' + 4)\).

If \(L = 5L' + 2\), then (42) forces \(w^{(i)} = 12L' + 4\). We find the only possible value of \(w^{(i)}\) is \((12L' + 2, 12L' + 4, 12L' + 4, 12L' + 4)\).

If \(L = 5L' + 3\), then (42) forces \(w^{(i)} = 12L' + 6\). We find the only possible value of \(w^{(i)}\) is \((12L' + 2, 12L' + 4, 12L' + 4, 12L' + 6)\).

If \(L = 5L' + 4\), then (42) forces \(w^{(i)} = 12L' + 8\). We find the only possible value of \(w^{(i)}\) is \((12L' + 2, 12L' + 4, 12L' + 4, 12L' + 8)\).

All of these agree with (44). \(\square\)

Proposition 3.3 gives some consequences of tightness.

### 4.4 Further remarks

Now let’s turn to more general statements. The simplest way to change the weight \(w\) has already been alluded to several times. Namely, suppose we are given any
admissible multiplier system \((\rho, w)\) with bijective \(\Lambda\) and fundamental matrix \(\Xi(\tau)\). Recall the multiplier \(v_w\) of \(\Delta^w(\tau)\). Then for any \(w' \in \mathbb{C}, (v_w \otimes \rho, w + 12w')\) is admissible, with bijective and fundamental matrices \(\Lambda + w'1_d\) and \(\Delta^w(\tau)\Xi(\tau)\).

Suppose bijective \(\Lambda, \Lambda'\) with corresponding fundamental matrices \(\Xi(\tau), \Xi'(\tau)\) are known for admissible \((\rho, w)\) and \((\rho', w')\). Then the \(d'd'\) columns of the Kronecker matrix product \(\Xi(\tau) \otimes \Xi'(\tau)\) will manifestly lie in \(\mathcal{M}_w(w + w')(\rho \otimes \rho')\), and will generate over \(\mathbb{C}[J]\) a full rank submodule of it. By Proposition 3.2 the differential operators \(\nabla_i\) then generate from that submodule all of \(\mathcal{M}_w(w + w')(\rho \otimes \rho')\). In that way, bijective exponents and fundamental matrices for tensor products (and their submodules) can be obtained.

The easiest and most important products involve the six one-dimensional \(T^i\) representations \(v_{2i}\). Here we can be much more explicit. Equivalently, we can describe the effect of changing the weights by even integers but keeping the same representation. For simplicity we restrict to even integer weights.

**Proposition 4.1.** Let \((\rho, 0)\) be admissible and \(T\) diagonal. Fix a bijective \(\Lambda\), with corresponding \(\Xi(\tau)\). Then for \(i = 2, 3, 4, 5\) respectively, the columns for a fundamental matrix for \((\rho, 2i)\) can be obtained as a linear combination over \(\mathbb{C}\) of the columns of, respectively:

1. \(E_2 \Xi = q^4(1_d + \cdots)\) and \(D^2 \Xi - E_4 \Xi \Lambda = q^4(1728.\mathcal{S}_5(\mathcal{S}_5 - \frac{1}{5})q + \cdots)\);
2. \(E_2 \Xi = q^4(1_d + \cdots)\) and \(D_4 \Xi - E_6 \Xi \Lambda = q^4(1728.\mathcal{S}_2 q + \cdots)\);
3. \(E_2 \Xi = q^4(1_d + \cdots)\) and \(E_6 \Xi - E_4 \Xi \Lambda = q^4(1728.\mathcal{S}_2 q + \cdots)\);
4. \(E_2 \Xi = q^4(1_d + \cdots)\) and \(E_6 \Xi - E_4 \Xi \Lambda = q^4(1728.\mathcal{S}_2 q + \cdots)\);
5. \(E_2 \Xi = q^4(1_d + \cdots)\) and \(E_6 \Xi - E_4 \Xi \Lambda = q^4(1728.\mathcal{S}_2 q + \cdots)\).

**Proof.** Let \(\alpha_i = \alpha_i(\rho, w)\) and \(\beta_i = \beta_i(\rho, w)\), and write \(A(i)\) for some to-be-determined bijective exponent at weight \(w + 2i\). Then \(\text{Tr}A(i) - \text{Tr}A\) can be read off from (30). Consider first the case \(i = 1\). Theorem 3.3(c) says \(\mathcal{S}_2(\mathcal{S}_5 - \frac{1}{5})\) has rank \(\beta_2\), while \(\mathcal{S}_2(\mathcal{S}_5 - \frac{1}{5})\) has rank \(\alpha_1\). The column spaces of the second and third matrices given above for \(i = 5\) have trivial intersection, since any \(\Xi(\tau)\) in their intersection is a vvmf for \((\rho, w + 2)\) which would be divisible by \(E_6 \Xi \Lambda\). This would mean \(E_14(\tau)^{-1}\Delta(\tau)\Xi(\tau)\in \mathcal{M}_w(\rho)\) would lie in the kernel of \(\mathcal{S}_1\), so by bijectivity \(\Xi(\tau)\) must be 0. The desired generators will be linear combinations of \(\alpha_1\) columns of the second matrix with \(\beta_2\) of the third and \(d - \beta_1 - \alpha_1\) of the first. These would define a matrix \(\Xi_3(\tau)\) whose \(A(3)\) has the correct trace, and therefore it must be a fundamental matrix (since its principal part map \(\mathcal{S}_5(\tau)\) is manifestly surjective).

The other cases listed are easier. \(\square\)

In all these cases \(2 \leq i \leq 6\), Proposition 4.1 finds a bijective exponent \(A(i)\) such that \(A(i) - \Lambda\) consists of 0’s and 1’s (for \(i = 6\), \(A(6) = \Lambda + 1_d\) always works).

The case \(i = 1\) is slightly more subtle. Note that a fundamental matrix for \(w + 2i + 12j\) is \(\Delta(\tau)^j\) times that for \(w + 2i\). So one way to do \(i = 1\) is to first find \(i = 4\), then find the \(i = 3\) from \(w + 8\), then divide by \(\Delta(\tau)\). Here is a more direct approach: almost always, the columns of the fundamental matrix for \(i = 1\) is a linear combination over \(\mathbb{C}\) of the columns of \(M_1(\tau) := D\Xi, M_2(\tau) := E_4^2(E_4 \Xi - E_6 \Xi \Lambda) / \Delta = q^4(1728.\mathcal{S}_2 + \cdots)\), and \(M_3(\tau) := (E_6^2 \Xi - E_4^2 \Xi \Lambda) / \Delta = q^4(-1728.\mathcal{S}_2 + \cdots)\). Indeed,
take any $v \in \text{Null}(\alpha^2 - \frac{1}{2}) \cap \text{Null}\left(\alpha^3 - \frac{1}{3}\right)$. That intersection has dimension at least $\alpha_1 - \beta_0$, because $\text{Null}(\alpha^2 - \frac{1}{2})$ has dimension $\alpha_1$ and $\text{Null}(\alpha^3 - \frac{1}{3})$ has dimension $\beta_1$. Then $M_2v = q^A(v/2 + \cdots)$ and $M_3v = q^A(jv/3 + \cdots)$ where $j = 1$ or $2$, so $2jM_2(\tau)v - 3M_3(\tau)v \in q^A + 1_d)C[[q]]$. Generically, this gives $\alpha_1 - \beta_0$ linearly independent vvmf, which together with $d - \alpha_1 + \beta_0$ columns of $M_1(\tau)$ will give the columns of the $i = 1$ fundamental matrix. This method doesn’t work for all $\rho$ however, e.g. it fails for $\rho = 1$.

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