SUPER RSK-ALGORITHMS AND SUPER PLACTIC MONOID

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Abstract. We construct the analogue of the plactic monoid for the super semistandard Young tableaux over a signed alphabet. This is done by developing a generalization of the Knuth’s relations. Moreover we get generalizations of Greene’s invariants and Young-Pieri rule. A generalization of the symmetry theorem in the signed case is also obtained. Except for this last result, all the other results are proved without restrictions on the orderings of the alphabets.

1. Introduction

Young tableaux are simple objects with a wide range of applications. A vast literature dealing with Young tableaux has been developing through the decades. In our paper, we shall be mainly concerning their relations with an algebraic structure called plactic monoid.

Starting from the beginning, in 1938 Robinson worked out an algorithm in order to compute the coefficients of products of Schur functions (Littlewood-Richardson rule). In 1961 Schensted ([15]) brought new life to Robinson algorithm, in a clearer form. Finally, in 1970 Knuth refined the Robinson-Schensted correspondence, by detecting its fundamental laws and giving it the form of a computer program. This, together with the further results obtained by Greene, revealed the inner structure of the correspondence, and are the key step in order to apply it as a combinatorial tool.

Knuth relations were the crucial point in the construction by Lascoux and Schützenberger ([12]). They turned the set of Young tableaux into a monoid structure, called the plactic monoid, taking into account most of their combinatorial properties and having a wide range of applications.

In these classical settings, the filling of the Young tableaux are from an alphabet, i.e. from a totally ordered set $L$. In our paper we shall use entries from a signed alphabet. A signed alphabet is a totally ordered set $L$, usually finite or countable, which is disjoint union of two subsets, say $L_0, L_1$. We denote $|x| = \alpha$ when $x \in L_\alpha$. Algebraic structures insisting on a signed alphabet have been conceived to provide combinatorial methods for important algebraic theories as the invariant theory in superalgebras [10] and the representation theory of general Lie superalgebras [1]. Let us mention that these theories can be applied in particular to study algebras satisfying polynomial identities (see for example [2, 6]). The combinatorics involved with this subject primarily concern with Young tableaux. A notion of semistandardness has to be properly defined for these Young tableaux. Many papers follow the approach introduced in [3] by defining these tableaux as “$(k,l)$-semistandard.

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tableaux” that is assuming \( L_0 < L_1 \). Following the setting suggested in [10] we define instead the notion of “super semistandard Young tableaux” (see Section 2), where the ordering of \( L \) is any. In fact, we are able to show that many results about Young tableaux do not rely on the condition \( L_0 < L_1 \).

In Section 3, for a signed alphabet, we develop in full generality a notion of “superplactic monoid” by means of analogues of the Knuth relations. We obtain as a by-product a generalization of the Greene’s results about his invariants. In Section 4 we recall a variant of the Robinson-Schensted-Knuth algorithms establishing a one-to-one correspondence between pairs of super semistandard tableaux and two-rowed arrays with entries in signed alphabets ([4]). A different correspondence for \((k, l)\)-semistandard tableaux also appeared in [9]. We give here a new account of the algorithms in [3] providing an “implementation-ready” description of them. We conclude with a super-analogue of the “symmetry theorem” [16], with the aid of some further assumptions.

One application of the RSK-correspondence consists in providing the linear independence in the celebrated “standard basis theorem”. Note that by means of the super RSK we obtain the same result for the “super standard basis” introduced in [10]. Let us explain this in more detail. Let \( X \) be a signed alphabet and \( F \) a field of characteristic zero. We denote by \( F(X) \) the free associative algebra generated by \( X \) that is the tensor algebra of the vector space \( FX \). This algebra is \( \mathbb{Z}_2 \)-graded if we put \(|w| := |x_1| + \cdots + |x_n|\) for any monomial \( w = x_1 \cdots x_n \). Note that a \( \mathbb{Z}_2 \)-graded algebra is called also a superalgebra. Denote by \( I \) the two-sided ideal of \( F(X) \) generated the binomials \( x_ix_j - (-1)^{|x_i||x_j|}x_jx_i \), where \( x_i, x_j \in X \). Clearly \( I \) is \( \mathbb{Z}_2 \)-graded ideal and we define \( \text{Super}[X] := F(X)/I \). This \( \mathbb{Z}_2 \)-graded algebra is called the free supercommutative algebra generated by \( X \). The identities of \( \text{Super}[X] \) defined by the above binomials are said in fact supercommutative identities. It is plain that an \( F \)-basis of \( \text{Super}[X] \) is given by the cosets of the monomials \( w = x_1 \cdots x_n \) such that \( x_i \leq x_{i+1} \), with \( x_i = x_{i+1} \) only if \(|x_i| = 0\). Note that the algebra \( \text{Super}[X] \) is isomorphic to the tensor product \( F[X_0] \otimes_F E[X_1] \) where \( F[X_0] \) is the polynomial ring in the commuting variables \( x_i \in X_0 \) and \( E[X_1] \) is the exterior (or Grassmann) algebra of the vector space \( FX_1 \).

Consider now \( L, P \) two signed alphabets and define for the set \( L \times P \) a structure of signed alphabet by ordering in the right lexicographic way and putting \(|[a, b]| = |a| + |b|\), for any \((a, b) \in L \times P \). We put \( \text{Super}[L|P] := \text{Super}[L \times P] \) and, according to [14], we call this supercommutative algebra the letter-place superalgebra. A variable \((a, b) \in L \times P \) will be written as \((a|b)\) where \( a \) is said a “letter” and \( b \) a “place”. This comes from the possibility to embed the tensor superalgebra \( F(L) \) into \( \text{Super}[L \times P] \) simply by putting \( w = a_1 \cdots a_n \mapsto m = (a_1|1) \cdots (a_n|n) \). Note that for the purposes of invariant theory, we do not have to assume \( \text{char}(F) = 0 \). Therefore, in characteristic free, a more general definition of signed alphabet and \( \text{Super}[L|P] \) is given which involves the notion of divided powers (see [10]). In this case, \( \text{Super}[L|P] \) is known as the four-fold algebra.

All the monomials \( m = (a_1|b_1) \cdots (a_n|b_n) \in \text{Super}[L|P] \) can be written so that \((a_i|b_i) \leq (a_{i+1}|b_{i+1})\), with \((a_i|b_i) = (a_{i+1}|b_{i+1})\) only if \(|a_i|b_i| = 0\). Therefore, we can denote them in a combinatorial way as two-rowed arrays with signed entries that is as \( S = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix} \). An important problem for the letter-place algebra consists in describing an \( F \)-linear basis given by polynomials that have some invariant
behavior. It was showed in [10] that we can attach to each pair \((T, U)\) of super semi-standard Young tableaux a suitable invariant polynomial of the algebra \(\text{Super}[L|P]\) which is usually denoted as \((T|U)\). Up to a sign, \((T|U)\) is a product of polynomials defined for each row of the tableaux \(T, U\). Let \(a_1,\ldots, a_k\) and \(b_1,\ldots, b_k\) denote the elements of some row of \(T\) and \(U\) respectively. In the non-signed case, that is for \(L = L_0, P = P_0\), by considering the matrix whose entries are the commuting variables \((a|b)\) we have that the row polynomial \((a_1,\ldots, a_k|b_1,\ldots, b_k)\) is defined simply as the determinant of the \(k \times k\) submatrix extracted by the rows \(a_1,\ldots, a_k\) and the columns \(b_1,\ldots, b_k\) (see [10] for the general definition in the signed case).

By the “straightening law” it can be proved that the polynomials \((T|U)\) span the letter-place superalgebra. Since the super RSK-correspondence implies a bijection between the elements of the monomial basis of \(\text{Super}[L|P]\) and the pairs of tableaux \((T, U)\), we conclude that the invariants \((T|U)\) are actually an \(F\)-linear basis of \(\text{Super}[L|P]\). This result is known as “super standard basis theorem” and it can be proved also with different techniques. Note finally that this theorem is not only a fundamental tool for invariant theory but it can been applied as well to the representation theory of general Lie superalgebras (see [5]).

2. Super semi-standard Young tableaux. Bumping

Let \(\mathbb{N}\) denote the set of positive integers and let \(n \in \mathbb{N}\). A partition of \(n\) is a sequence of integers \(\lambda = (\lambda_1, \ldots, \lambda_m)\) such that \(\lambda_1 \geq \cdots \geq \lambda_m > 0\) and \(\sum \lambda_i = n\). The integer \(m \in \mathbb{N}\) is called number of parts or height of the partition. We denote \(\lambda \vdash n\) if \(\lambda\) is a partition of \(n\).

The Ferrers-Young diagram of a partition \(\lambda = (\lambda_1, \ldots, \lambda_m)\) is defined as the set:

\[
D(\lambda) := \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}.
\]

We can visualize \(D(\lambda)\) by drawing a box for each pair \((i, j)\). For instance, the diagram of \(\lambda = (7, 7, 5, 3, 3, 1)\) is:

Note that the transposed diagram \(\{(j, i) \mid (i, j) \in D(\lambda)\}\) defines another partition \(\lambda \vdash n\) whose parts are the lengths of the columns of \(D(\lambda)\). The partition \(\lambda\) is called the conjugate partition of \(\lambda\). Finally, let \(\lambda, \mu\) be two partitions. We write \(\mu \subset \lambda\) if \(\lambda_i \leq \mu_i\) for all \(i\). In this case we put \(D(\lambda/\mu) := \{(i, j) \mid 1 \leq i \leq m, \mu_i < j \leq \lambda_i\}\).

Although Ferrers-Young diagrams are interesting combinatorial objects on their own right, we are mainly interested in Young tableaux. Denote by \(\mathbb{Z}_2 = \{0, 1\}\) the additive cyclic group of order 2.

**Definition 2.1.** Let \(L\) be a finite or countable set. Assume \(L\) totally ordered and let \(|·| : L \to \mathbb{Z}_2\) be any map. We call the ordered pair \((L, |·|)\) a signed alphabet and we put \(L_0 := \{a \in L \mid |a| = 0\}\) and \(L_1 := \{a \in L \mid |a| = 1\}\).

**Definition 2.2.** Let \(\lambda \vdash n\) be a partition and let \(L\) be a signed alphabet. A super semistandard Young tableau is a pair \(T := (\lambda, \tau)\) where \(\tau : D(\lambda) \to L\) is a map such that:
We say that $D(\lambda)$ is the frame and $\tau$ the filling of the tableau $T$. Moreover, $\lambda$ is called the shape of $T$. We say that $T$ is a standard Young tableau if the map $\tau$ is injective.

In the same way, we can define also a tableau with frame $D(\lambda/\mu)$ when $\mu \subset \lambda$. In this case we say that this tableaux has skew shape $\lambda/\mu$. Note that the classic notion of semistandard Young tableau is recovered when the map $|\cdot|$ has a constant value. More precisely, if $L = L_0$ we obtain row-strict semistandard tableaux and column-strict ones when $L = L_1$. If we assume the ordering on $L$ is such that $L_0 < L_1$ we obtain the notion of $(k,l)$-semistandard tableau introduced in [3, 1]. We do not choose to make this assumption because many results about tableaux on a signed alphabet do not depend on it. From now on, unless explicitly stated, we use the word “alphabet” for “signed alphabet” and “Young tableau” meaning “super semistandard Young tableau”.

Assume that a Young tableau has been given, and a letter from $L$ as well. A basic problem in algebraic combinatorics consists in finding a method for constructing a tableau that includes the new letter and yet is a Young tableau. In the non-signed case, the Schensted’s construction solves the problem in a fully satisfying way. It can be realized by two “dual” algorithms: the insertion by rows or columns. Here we give generalizations of Schensted’s algorithms that take into account the fact that we are dealing with signed letters. We start defining the row-insertion.

**Definition 2.3.** Let $T = (\lambda, \tau)$ be a Young tableau with $\lambda = (\lambda_1, \ldots, \lambda_m)$ and let $x \in L$. By $T \leftarrow x$ we mean the following procedure:

**Step 0** Put $\lambda_{m+1} := 0$ and $i := 1$.

**Step 1a** If $|x| = 0$ then put $J := \{1 \leq j \leq \lambda_i \mid x < \tau(i, j)\}$.

**Step 1b** If $|x| = 1$ then put $J := \{1 \leq j \leq \lambda_i \mid x \leq \tau(i, j)\}$.

**Step 2** If $J = \emptyset$ then go to Step 4.

**Step 3** Put $j := \min(J)$ and $y := \tau(i, j)$. Put $\tau(i, j) := x$, $x := y$, $i := i + 1$ and go to Step 1.

**Step 4** Put $\lambda_i := \lambda_i + 1$, $\tau(i, \lambda_i) := x$. Output $(\lambda, \tau)$ and $i$.

In the Step 3, we say the letter $x$ bumps the entry $y$ from the $i$-th row. We denote the Young tableau obtained by means of this procedure as $[T \leftarrow x]$.

**Example 2.4.** Let $L := \mathbb{N}$, with signature given by $L_0 := \{\text{odd numbers}\}$ and $L_1$ defined consequently. Consider the following tableau:

$$
\begin{array}{cccccc}
1 & 1 & 1 & 2 & 4 & 5 \\
2 & 3 & 3 & 4 \\
T := 2 & 4 \\
2 & 4 \\
2 \\
3
\end{array}
$$
and let us row-insert at first the letter 6. By Step 2, since 6 is greater than any entry of the first row, we go immediately to Step 4 and hence \([T \leftarrow 6]\) is:

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 2 & 4 & 5 & 6 \\
2 & 3 & 3 & 4 & & & \\
2 & 4 & & & & & \\
2 & 4 & & & & & \\
2 & & & & & & \\
2 & & & & & & \\
3 & & & & & & \\
\end{array}
\]

Now, let us insert the letter 1 in the tableau \(T' := [T \leftarrow 6]\). We go through Step 3 and this letter bumps the entry 2 which now we are trying to place in the second row. This can be done bumping the already-placed 2, and then we test its insertion in the third row. There, and in the rows below, the letter 2 continues to be displaced, until the last row, where we test the insertion of 2. Such insertion can be done by bumping the letter 3 and finally by Step 4, this letter forms a new row. Hence the tableau \([T' \leftarrow 1]\) is the following:

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 4 & 5 & 6 \\
2 & 3 & 3 & 4 & & & \\
2 & 4 & & & & & \\
2 & 4 & & & & & \\
2 & & & & & & \\
2 & & & & & & \\
3 & & & & & & \\
\end{array}
\]

Note that procedure of row-insertion \(T \leftarrow x\) can be reversed since we record the row of the new box added to \(T\). In fact, by the definition of row-insertion such box is added to \(T\) as the last in its row and column. Precisely, a procedure of row-deletion is defined as follows:

**Definition 2.5.** Let \(T = (\lambda, \tau)\) be a Young tableau with \(\lambda = (\lambda_1, \ldots, \lambda_m)\) and let \(1 \leq i \leq m\). We denote by \(i \leftarrow T\) the following procedure:

**Step 0**  Put \(x := \tau(i, \lambda_i)\) and \(\lambda_i := \lambda_i - 1\). Put \(\lambda_0 := 0\) and \(h := i - 1\).

**Step 1a** If \(|x| = 0\) then put \(J := \{1 \leq j \leq \lambda_i \mid \tau(i, j) < x\}\).

**Step 1b** If \(|x| = 1\) then put \(J := \{1 \leq j \leq \lambda_i \mid \tau(i, j) \leq x\}\).

**Step 2**  If \(J = \emptyset\) then go to Step 4.

**Step 3**  Put \(k := \max(J)\) and \(y := \tau(h, k)\). Put \(\tau(h, k) := x, x := y, h := h - 1\) and go to Step 1.

**Step 4**  Output \((\lambda, \tau)\) and \(x\).

We write \([i \leftarrow T]\) for the tableau we obtain.

Resembling the Definition 2.3, a column-insertion algorithm can also be defined.

**Definition 2.6.** Let \(T = (\lambda, \tau)\) be a tableau and let \(x \in L\). Denote by \(\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_r)\) the conjugate partition of \(\lambda\). We define \(x \rightarrow T\) the following procedure:
Step 0 Put $\tilde{\lambda}_{r+1} := 0$ and $j := 1$.

Step 1a If $|x| = 0$ then put $I := \{ 1 \leq i \leq \tilde{\lambda}_j \mid x \leq \tau(i, j) \}$.

Step 1b If $|x| = 1$ then put $I := \{ 1 \leq i \leq \tilde{\lambda}_j \mid x < \tau(i, j) \}$.

Step 2 If $I = \emptyset$ then go to Step 4.

Step 3 Put $k := \min(I)$ and $y := \tau(i, j)$. Put $\tau(i, j) := x, x := y, j := j + 1$ and go to Step 1.

Step 4 Put $\lambda_j := \lambda_j + 1, \tau(\lambda_j, j) := x$. Output $(\lambda, \tau)$ and $j$.

The tableau obtained by means of this procedure will be denoted as $[x \to T]$.

The definition of the column-deletion is left to the reader. Such procedure will be denoted as $T \leftarrow i$.

Remark 2.7. The two procedures of Definitions 2.3 and 2.6 are related by a simple duality involving the signature on $L$. Namely, let $L$ be an alphabet and define $\tilde{L}$ by putting $L_0 := L_1$ and $L_1 := L_0$. We call $\tilde{L}$ the conjugate alphabet of $L$. If $T = (\lambda, \tau)$ is a Young tableau, denote by $\tilde{T}$ the pair $(\tilde{\lambda}, \tilde{\tau})$ where $\tilde{\lambda}$ is the conjugate partition of $\lambda$ and $\tilde{\tau}(i, j) := \tau(j, i)$ for any $(i, j) \in D(\tilde{\lambda})$. Clearly $\tilde{T}$ is a Young tableau for the conjugate alphabet $L$. Then, from the definitions it follows easily that:

$$[x \to T] = [\tilde{T} \leftarrow x]$$

where in the right hand side the letter $x$ has to be considered as an element of the conjugate alphabet.

Proposition 2.8. Let $T$ be a Young tableau and $x \in L$. Then both $[T \leftarrow x]$ and $[x \to T]$ are Young tableaux.

Proof. Put $T' := [T \leftarrow x]$. Clearly, there is no loss of generality on assuming that $T'$ has just two rows, say $w'_1, w'_2$. Denote by $w_1, w_2$ the rows of $T$ with $w_2$ possibly empty. By Definition 2.3 of the row-insertion process, it is plain that $w'_1, w'_2$ verify condition (i) of Definition 2.2. Then, we have to check the shape and the columns of $T'$. If $x$ is appended at the end of $w_1$ there is nothing to prove. Otherwise, define $y$ the entry of $w_1$ bumped by $x$ and denote by $j, j'$ the columns where $y$ occurs in the rows $w_1, w'_2$ respectively. We claim that $j' \leq j$. This implies that if $y$ is appended at the end of the row $w_2$ then the length of $w_2$ is strictly less than the one of $w_1$. Moreover, we have that $x \leq y$, with $x = y$ only if $|x| = 1$ by Definition 2.3 and $z \leq x$ for any entry $z$ of $w'_1$ at the right or equal to $x$. By $j' \leq j$ this implies that the entry above $y$ in the row $w'_1$ satisfies condition (ii) of Definition 2.2. By contradiction, assume now that $j < j'$ and let $z$ be the entry of $w_2$ below $x$. Since $T$ is a Young tableau and $z$ is below $y$ in $T$ we have that $y \leq z$, with $y = z$ only if $|y| = 1$. But $z$ is at the left of $y$ in $w'_2$ and therefore $y \geq z$, with $y = z$ only if $|y| = 0$ which is impossible.

By Remark 2.7 it follows immediately that $[x \to T]$ is also a Young tableau. \hfill $\Box$

Let $T', i$ be the tableau and the row-index defined as the output of the procedure $T \leftarrow x$ and put $j$ the length of the row $i$ in $T'$. In other words, $(i, j)$ is the position of the new box added to the shape of $T$ for obtaining the shape of $T'$. Consider also $T'', i'$ the output of $T' \leftarrow x'$ and let $j'$ be the length of the row $i'$ in $T''$. We have the following result.

Proposition 2.9 (Row-bumping lemma). The following conditions are equivalent:

(a) $x \leq x'$, with $x = x'$ only if $|x| = 0$, 

(b) $x \leq x'$, with $x = x'$ only if $x = j'$, 

(c) $x \leq x'$, with $x = x'$ only if $x = j'$.

Proof. Suppose (a) holds. Then $x \leq x'$, with $x = x'$ only if $|x| = 0$. This follows easily from the definition of $T'$. Conversely, suppose (b) holds. Then $x \leq x'$, with $x = x'$ only if $|x| = 0$. This follows easily from the definition of $T'$. Conversely, suppose (c) holds. Then $x \leq x'$, with $x = x'$ only if $|x| = 0$. This follows easily from the definition of $T'$. \hfill $\Box$
(b) $j < j'$ (and hence $i \geq i'$).

Proof. For any $q$ ($1 \leq q \leq i$) we define an entry $x_q$ of the $q$-th row of $T'$ as follows:

(i) $x_1 := x$,

(ii) $x_{q+1}$ is the entry of the row $q$ bumped by the element $x_q$.

We call $x_1, x_2, \ldots, x_i$ the \textit{bumping sequence} defined by the row-insertion $T \leftarrow x$. By Definition 2.3 we have that $x_q \leq x_{q+1}$, with $x_q = x_{q+1}$ only if $|x_q| = 1$. Denote by $j_q$ the column where the entry $x_q$ occurs in the row $q$ of $T'$. As proved in the argument of Proposition 2.8 we have that $j_q \geq j_{q+1}$.

For the procedure $T'' \leftarrow x'$ we define in the same way the bumping sequence $x'_1, x'_2, \ldots, x'_i$, and denote by $j'_q$ the column where the entry $x'_q$ is placed on the $q$-th row of $T''$. The basic argument of the proof consists in observing that the following statements are equivalent:

$$(\alpha) x_q \leq x'_q, \text{ with } x_q = x'_q \text{ only if } |x_q| = 0,$$

$$(\beta) j_q < j'_q.$$ 

In fact, by Definition 2.3 of row-insertion from $(\alpha)$ it follows that $(\beta)$ holds. Moreover, since $T''$ is a Young tableau we have that $(\beta)$ implies $(\alpha)$.

Let us assume the condition $(\alpha)$ holds. Since $x \leq x'$, with $x = x'$ only if $|x| = 0$ we have that $j_1 < j'_1$. By induction on the row-index $q$, we prove that $j_q < j'_q$ for any $q$ ($1 \leq q \leq \min(i, i')$). In fact, it is sufficient to note that before we insert the elements $x_q, x'_q$ in the row $q$ of the tableau $T$, we have the elements $x_{q+1}, x'_{q+1}$ in those positions. By assuming $j_q < j'_q$ we have therefore $x_{q+1} \leq x'_{q+1}$, where $x_{q+1} = x'_{q+1}$ implies that $|x_{q+1}| = 0$. In the row $q + 1$ where the letters $x_{q+1}, x'_{q+1}$ are inserted, this means that $j_{q+1} < j'_{q+1}$.

If we show that $i' \leq i$ we can conclude that $j' = j'_i \geq j_i \geq j_1 = j$. Note that the entry $x_i$ is placed at the end of the $i$-th row of the tableau $T'$. The same happens to the entry $x'_i$ of $T''$. Moreover, the shapes of $T', T''$ differ only for the box of position $(i', j')$. If we suppose that $i \leq i'$, by $j = j_i < j'_i$ we have that $i' = i$ and $j' = j'_i = j + 1$. This implies that $i' \leq i$.

We assume now (b). From $j < j'$ it follows that $i \geq i'$. Since the entry $x'_i$ is the last one of the row $i'$ of $T''$ we have $j' < j'_i = j$. By backward induction on $q$ ($i' \geq q \geq 1$) it is proved that $j_q < j'_q$. In particular, one has $j_1 < j'_1$ and hence $x \leq x'$, with $x = x'$ only if $|x| = 0$.

Let $T', j$ be the tableau and the column-index that are the output of the procedure $x \rightarrow T$ and put $i$ the length of the column $j$ in $T'$. Moreover, denote by $T'''$, $j'$ the output of $x' \rightarrow T'$ and let $i'$ be the length of the row $j'$ in $T'''$. A “dual result” of Proposition 2.9 can be obtained immediately by Remark 2.7.

**Proposition 2.10** (Column-bumping lemma). \textit{We have the following equivalent conditions:}

(a) $x \leq x'$, with $x = x'$ only if $|x| = 1$,

(b) $i < i'$ (and hence $j \geq j'$).

3. \textsc{The super plactic monoid}

While Schensted's construction \cite{15} was originally made in order to count some invariants of sequences of symbols, it was Knuth \cite{11} who refined the insertion algorithm revealing the inner structure of it. Then, there were Lascoux and Schützenberger \cite{12} who connected it to a relevant algebraic structure they called the \textit{plactic}
monoid. Here, we are going to generalize that structure to the case of a signed alphabet.

Let $L$ be an alphabet and denote by $L$ the free monoid generated by $L$. It consists of all words $w := x_1 \cdots x_n \ (x_i \in L)$ endowed with juxtaposition product. We admit the empty word belonging to $L$ and behaving as the unity for the product.

**Definition 3.1.** A monoid $\mathcal{M}$ is said a $\mathbb{Z}_2$-graded monoid or a supermonoid if a map $|\cdot|: \mathcal{M} \to \mathbb{Z}_2$ is given such that $|u \cdot v| = |u| + |v|$, for any $u, v \in \mathcal{M}$. We call $|u|$ the $\mathbb{Z}_2$-degree of the element $u$.

If $L$ is a signed alphabet, note that the free monoid $L$ is $\mathbb{Z}_2$-graded simply by putting $|w| := |x_1| + \cdots + |x_n|$, for any word $w = x_1 \cdots x_n$.

Any Young tableau $T$ defines a word $w(T)$ of $L$ in a simple way: start collecting the rows of $T$ from its bottom upward and multiply those sequences of letters by juxtaposition. For instance, if we consider the tableau:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 4 & 5 \\
2 & 3 & 3 & 4 \\
2 & 4 \\
T := 2 & 4 \\
2 \\
3
\end{array}
\]

then it defines the word:

\[w(T) := 32224242334111145.\]

Moreover, any word $w = x_1 \cdots x_n$ defines a Young tableau $T(w)$ by inserting its letters progressively starting from the empty tableau. Choosing the insertion by rows, we define:

\[T(w) := [[[\emptyset \leftarrow x_1] \leftarrow x_2] \cdots \leftarrow x_n]\]

where $\emptyset$ denotes the empty tableau.

**Proposition 3.2.** Let $L$ be a signed alphabet. For any Young tableau $T$, it holds:

\[T(w(T)) = T\]

**Proof.** Denote by $w_i$ the $i$-th row of $T$ and let $T'$ be the tableau obtained by considering all the rows of $T$ but $w_1$. By induction on the number of rows we can assume that $T(w(T')) = T'$. Note that $w(T) = w(T') w_1$. Hence, if $w_1 = x_1 \cdots x_n$ we have:

\[T(w(T)) = [[[T' \leftarrow x_1] \leftarrow x_2] \cdots \leftarrow x_n].\]

We claim that the bumping sequence defined by the letter $x_i$ is all in the $i$-th column. This clearly implies that $T(w(T)) = T$. Let $T_i := [[[T' \leftarrow x_1] \leftarrow x_2] \cdots \leftarrow x_i]$ and say $w_2 = y_1 \cdots y_m \ (m \leq n)$. By induction on $i$, the first row of the tableau $T_i$ is $x_1 \cdots x_i y_{i+1} \cdots y_m$ if $i < m$ or $x_1 \cdots x_i$ otherwise. By applying the row-insertion $T_i \leftarrow x_{i+1}$ we have that $x_{i+1}$ bumps $y_{i+1}$ if $i < m$ or $x_{i+1}$ is appended at the end of the first row. In fact, since $T$ is a Young tableau we have that $x_i \leq x_{i+1}$, with $x_i = x_{i+1}$ only if $|x_i| = 0$ and $x_{i+1} \leq y_{i+1}$, where $x_{i+1} = y_{i+1}$ only if $|x_{i+1}| = 1$. If $i < m$ then $y_{i+1}$ has to be inserted in the second row. Similar arguments apply to this letter that has to be placed in the column $i + 1$ and so on for the remaining bumping sequence. \qed
It is almost a natural question to wonder when different words give rise to the same Young tableau. Note that there are words $w$ such that $w(T(w)) \neq w$. We show that any pair of words $w, w(T(w))$ is related by a congruence on the monoid $L$ which is compatible with its $\mathbb{Z}_2$-grading.

**Definition 3.3.** We say that $w = x_1 \cdots x_n$ is a row word if we have that $x_i \leq x_{i+1}$, with $x_i = x_{i+1}$ only if $|x_i| = 0$. In other words, we have $w = w(T)$ where $T$ is a Young tableau whose shape is a row. In a similar way we can define the notion of column word. Note that all the words of length $\leq 2$ are clearly row or a column words.

**Definition 3.4.** Denote by $\sim$ the equivalence relation on $L$ defined by the map $w \mapsto T(w)$, that is $w \sim w'$ if and only if $T(w) = T(w')$.

Clearly, if $w, w'$ are row or column words then $w \sim w'$ if and only if $w = w'$. This happens in particular if the length of these words is $\leq 2$. So, the first non-trivial relations can be found for words of length 3 that correspond to tableaux of shape $(2,1)$. In fact, a straightforward computation provides the following:

**Lemma 3.5.** Let $x \leq y \leq z$ be letters of the alphabet $L$. We have:

(i) $T(xzy) = T(zxy)$ with $x = y$ only if $|y| = 0$ and $y = z$ only if $|y| = 1$,

(ii) $T(yxz) = T(yzx)$ with $x = y$ only if $|y| = 1$ and $y = z$ only if $|y| = 0$.

**Definition 3.6.** Denote by $\equiv$ the congruence on the monoid $L$ generated by the following relations:

(K1) $xyz \equiv zyx$, with $x = y$ only if $|y| = 0$ and $y = z$ only if $|y| = 1$,

(K2) $yxz \equiv yzx$, with $x = y$ only if $|y| = 1$ and $y = z$ only if $|y| = 0$,

for any triple $x \leq y \leq z$ of elements of $L$. Note that in the classic case, that is for $L = L_0$, we have that (K1),(K2) are exactly the Knuth’s relations [11]. Then, we put:

$$\text{Pl}(L) := L/\equiv$$

and we call such monoid the super plactic monoid defined by the signed alphabet $L$. Since the relations (K1),(K2) are both $\mathbb{Z}_2$-homogeneous we have that the quotient $\text{Pl}(L)$ is actually a supermonoid.

We are going to show now that the monoid $\text{Pl}(L)$ coincide with the quotient set $L/\sim$. This allows to define on the set of super semistandard Young tableaux a structure of $\mathbb{Z}_2$-graded monoid which can be identified with $\text{Pl}(L)$ by means of the map $w \mapsto T(w)$. The arguments we use for obtaining this result follow closely the approach introduced in [13], chapter 5, for the non-signed case. We start with some preparatory results.

**Proposition 3.7.** Let $L$ be a signed alphabet. For all words $w \in L$, we have $w \equiv w(T(w))$.

**Proof.** We argue by induction on the length of the word $w$. To simplify the notation, we use here the same letter $T$ for denoting a tableau and its corresponding word
$w(T)$. If the length of $w$ is $\leq 3$ then either $w$ is a row or column word, or it occurs in the relations (K1),(K2). Hence, by Lemma 3.5 we conclude for this case that $w \equiv T(w)$.

Now, assume that $w \equiv T(w)$ and let $x$ be an element of $L$. We have to prove that $wx \equiv T(wx)$ that is $T(w)x \equiv T(wx)$. Note that:

$$T(wx) = [T(w) \leftarrow x].$$

Since the procedure $T(w) \leftarrow x$ is defined row-by-row and being $\equiv$ a congruence, we need only to consider the case when $w$ is a row and hence $T(w) = w$. Then, we have to prove that $wx$ is congruent to the word obtained by row-inserting $x$ in $T(w)$. Note immediately that if $wx$ is a row then $T(wx) = wx$. Otherwise, we have two cases depending on the value of $|x|$.

If $|x| = 1$ then $T(wx) = yu$, where $y$ is the leftmost entry of $w$ among those $\geq x$ and $u'$ is obtained from $w$ by replacing $y$ with $x$. By putting $w = uyv$ we have hence that $wx = uyvx$ and $T(wx) = yuvx$. Note that if $a, b$ are any letters of the words $u, v$ respectively then $a < x \leq y \leq b$. Moreover, one has $y = b$ only if $|y| = 0$ since $yu$ is a row. Then, by applying iteratively the congruence (K2) we get:

$$wx = uyvx \equiv uyv.$$

Making use of the congruence (K1) we are able to conclude that:

$$uyvx \equiv yuvx = T(wx).$$

In case $|x| = 0$, one has $a < x < y \leq b$, and in a similar way we argue that $wx \approx T(wx)$.

In general, $T(m)$ is a product $w_n w_{n-1} \cdots w_2w_1$ of rows $w_i$ and, since the insertion $[T(m) \leftarrow x]$ is through rows and $\equiv$ is a congruence, one can apply iteratively what has been done for one row and go through the general case. \hfill $\square$

Let $v, w$ be words of $L$ and let $w = x_1 \cdots x_n$. We say that the word $v$ is extracted by $w$ if $v = x_{i_1} \cdots x_{i_m}$ with $1 \leq i_1 < \cdots < i_m \leq n$.

**Definition 3.8.** For any $k \in \mathbb{N}$, we denote by $l_k(w)$ the maximal number which can be obtained as the sum of the lengths of $k$ row words that are disjoint and extracted by $w$. In the same way, we define $\tilde{l}_k(w)$ as the maximal value among the sums of the lengths of $k$ column words disjoint and extracted by $w$.

Clearly, the integers $l_k(w)$ and $\tilde{l}_k(w)$ are the super-analogues of Greene’s row and column invariants [8].

For instance, if $L_0 = \{1, 2, 4\}$, $L_1 = \{3, 5\}$ and $w = 1233455$ we have:

$$l_1(w) = 5, \ l_2(w) = l_3(w) = \ldots = 7, \ l_1(w) = 2, \ l_2(w) = 4, \ l_3(w) = 5, \ l_4(w) = 6, \ l_5(w) = \tilde{l}_6(w) = \ldots = 7.$$

The motivation to call the above numbers “invariants” is that they stay stable over the plactic classes.

**Proposition 3.9.** If $w \equiv w'$ then $l_k(w) = l_k(w')$ for any $k$.

**Proof.** We can assume that the words $w, w'$ are congruent by means of a single generating relation, say (K1). Then:

$$w = uxyzv, \ w' = uxyzv,$$
with $u, v$ words and $x \leq y \leq z$ letters such that $x = y$ only if $|y| = 0$ and $y = z$ only if $|y| = 1$ (hence $x \neq z$). Note that two row words can be extracted by the word $xzy$, precisely $xy$ and $xz$. Moreover, by the word $zyx$ we can extract just $xy$. Then, all the words extracted by $w'$ can be also extracted by $w$ and hence $l_k(w) \geq l_k(w')$.

Assume now that $(w_1, \ldots, w_k)$ is a $k$-tuple of row words that are disjoint and extracted by $w$. We want to prove that there is a $k$-tuple $(w_1', \ldots, w_k')$ of row words disjoint and extracted by $w'$ such that the sum of the lengths of the $w_i'$ is equal to the corresponding sum for the $w_i$. In this case, in fact, we get clearly $l_k(w) \leq l_k(w')$.

Note that the word $w_i$ is extracted also by $w'$ unless that $w_i = u'x z v'$ with $y \neq z$ and $u', v'$ words extracted by $u, v$. If the letter $y$ does not occur in any of the words $w_j$ ($i \neq j$) then we can obtain the $k$-tuple $(w_1', \ldots, w_k')$ from $(w_1, \ldots, w_k)$ simply by substituting the row word $w_i$ with $w_i' = u'x y v'$. Otherwise, if we have $w_j = w'' y v''$ then we replace the pair $(w_i, w_j)$ with $w_i' = u'x y v''$ and $w_j' = u'' y v'$. In a similar way we argue for the case when $w \equiv w'$ by means of (K2). □

**Theorem 3.10** (Greene’s theorem). Let $L$ be a signed alphabet. Let $w$ be a word and denote by $\lambda = (\lambda_1, \ldots, \lambda_r)$ the shape of the tableau $T(w)$. Moreover, consider $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_r)$ the conjugate partition of $\lambda$. For any $k \in \mathbb{N}$ we have:

$$l_k(w) = \lambda_1 + \ldots + \lambda_k, \quad \tilde{l}_k(w) = \tilde{\lambda}_1 + \ldots + \tilde{\lambda}_k.$$  

**Proof.** By using the previous propositions, it is sufficient to prove that $l_k(w(T)) = \lambda_1 + \ldots + \lambda_k$ for any Young tableau $T$. If we consider the first $k$ rows of $T$, it is clear that $l_k(w(T)) \geq \lambda_1 + \ldots + \lambda_k$. Conversely, if $w$ is a row word of $w(T)$ then $w$ cannot have two letters in the same column of $T$. Hence, a $k$-tuple of row words disjoint and extracted by $w(T)$ can be done with at most $\lambda_1 + \ldots + \lambda_k$ letters of $T$. In other words, we have that $l_k(w(T)) \leq \lambda_1 + \ldots + \lambda_k$.

A similar argument can be used for proving that $\tilde{l}_k(w) = \tilde{\lambda}_1 + \ldots + \tilde{\lambda}_k$. □

An analogous result has been proved in [3] for the language of “$(k, l)$-semistandard tableaux” that is under the assumption that $L_0 < L_1$. We can finally state the main result of this section.

**Theorem 3.11** (Cross-section theorem). Let $L$ be a signed alphabet. The equivalence relation $\sim$ coincide with the congruence $\equiv$ on the monoid $L$. Moreover, the maps $w \mapsto T(w)$ and $T \mapsto w(T)$ define a one-to-one correspondence between the super plactic monoid $Pl(L)$ and the set of super semistandard Young tableaux over $L$. In particular, a cross-section of the plactic classes is given by the words $w(T)$, as $T$ ranges over the set of Young tableaux.

**Proof.** If $T(w) = T(w')$ then by Proposition 3.7 we have:

$$w \equiv w(T(w) = w(T(w')) \equiv w'.$$

Now assume $w \equiv w'$. By Proposition 3.9 and Theorem 3.10 it follows that the tableaux $T(w), T(w')$ have the same shape. Denote by $z$ the greatest letter of the words (with same content) $w, w'$ and put:

$$w = uzv, \quad w' = u'z v'$$

where $u, v, u', v'$ words. Moreover, assume that if $|z| = 0$ then $z$ does not occur in $v, v'$ and if $|z| = 1$ then $z$ does not occur in $u, u'$.

We want to prove that $uv \equiv u'v'$. We can assume that $w \equiv w'$ by means of one of the generating relations (K1), (K2). If $z$ does not occur in such relation then clearly
either $u \equiv u'$ and $v = v'$ or $u = u'$ and $v \equiv v'$. Otherwise, note that by deleting the letter $z$ in (K1) or (K2) we have that $xy = xy$ or $yx = yx$. By the assumptions on the words $u, v, u', v'$ with respect to $|z|$ we have therefore that $w = u'v'$.

By induction on the length of $w$, we can assume hence that $T(uv) = T(u'v')$. Since $z$ is the greatest among the entries of $T(w) = T(uzv)$ and therefore it occurs at the end of a row and of a column, it is plain that if we delete $z$ by a suitable row of $T(w)$ then we get the tableau $T(uw)$. Since $T(uv) = T(u'v')$ and the shapes of $T(w), T(w')$ are the same, we conclude that $T(w) = T(w')$.

Note finally that Proposition 3.2 states that $T(w(T)) = T$ for any tableau $T$. Now, by Proposition 3.7 we have also that $[w(T(w))] = [w]$, for any plactic class $[w]$. □

Let us assume, in the rest of this section, $L$ being finite. It is a well known fact that it is possible to build a ring, the so called group ring, rising from a monoid. In our case, it is a $\mathbb{Z}$-algebra, whose linear generators are the monomials in $P_l(L)$ (or, to be more precise, the plactic classes). This is an associative and unitary (but not commutative) ring. Let us denote it by $R_L$. A generic element in it can be realized by a formal sum of classes with coefficients from $\mathbb{Z}$. Since every class can be represented by a tableau, a typical element in $R_L$ is a formal sum of tableaux. If $X := \{x_a \mid a \in L\}$ is a set of free variables, the map $a \in L \rightarrow x_a$ leads to an homomorphic image of $R_L$ in the free commutative $\mathbb{Z}$-algebra $\mathbb{Z}[X]$.

Traditionally, the element

$$S_\lambda = \sum_{T(w) \text{ has shape } \lambda} w$$

(with $\lambda \vdash n$) gives rise to the so called Schur function $s_\lambda(X)$. The properties of $R_L$ reflect themselves in the commutative ring $\mathbb{Z}[X]$ in several ways. A striking result is the Pieri-Young rule:

**Theorem 3.12.** Let $\lambda$ be a partition of $n$. Then, for $p \in \mathbb{N}$,

$$s_\lambda(X) \cdot s_{(p)} = \sum_\mu s_\mu$$

where the sum is on all partitions $\mu \vdash (n+p)$ such that $\lambda \subseteq \mu$ and the skew-diagram $\mu/\lambda$ does not contains two boxes in the same column.

Similarly,

$$s_\lambda(X) \cdot s_{(1^p)} = \sum_\mu s_\mu$$

where the sum is on all partitions $\mu \vdash (n+p)$ such that $\lambda \subseteq \mu$ and the skew-diagram $\mu/\lambda$ does not contains two boxes in the same row.

Now we are going to prove a similar result within our settings. So, let $L$ be a proper finite signed alphabet, and let us denote by $R_L$ the group ring associated to the superplactic monoid $M(L)/\approx$.

As a consequence of Proposition 2.9 we get

**Corollary 3.13.** Let $T$ be a tableau, $\lambda \vdash n$ be its frame, and let $w$ be a row word of length $p$. Then $[T \leftarrow w]$ is a tableau of frame $\mu \vdash (n+p)$ containing $\lambda$ and such that the skew diagram $\mu/\lambda$ does not contain any pair of boxes in the same column.

Conversely, if $U$ is a tableau of frame $\mu \vdash (n+p)$, and $\lambda$ is a diagram contained in $\mu$ and such that in the skew diagram $\mu/\lambda$ no two boxes occur in the same column,
then there exist a unique tableau $T$ with frame $\lambda$ and a row word $w$ of length $p$ such that $U = [T \leftarrow w]$.

Proof. Let $T$ be a tableau with shape $\lambda$ and let $w = x_1 \ldots x_p$. Let $(r_i, c_i)$ be the final extra box for the insertion $(T \leftarrow x_1 \ldots x_{i-1}) \leftarrow x_i$ in the tableau $[T \leftarrow x_1 \ldots x_{i-1}]$ for each $i \leq p$. Notice that $\mu/\lambda$ is the skew diagram consisting of the boxes $(r_1, c_1), \ldots, (r_p, c_p)$. Moreover, since $w$ is a row word, it holds $x_1 \leq x_2 \leq \cdots \leq x_p$ and, if $x_i = x_{i+1}$, then $|x_i| = 0$. Therefore, by Proposition 2.9, for all $i \leq p - 1$ the boxes $(r_i, c_i)$ and $(r_{i+1}, c_{i+1})$ satisfy $c_i < c_{i+1}$. Hence no two boxes of the skew-diagram $\mu/\lambda$ are in the same column.

Conversely, if $T_1$ is a tableau with shape $\mu$ and such that no two boxes of $\mu/\lambda$ are in the same column, then there exists a unique row word $w$ such that $[T \leftarrow w] = T_1$. The reason is the following: starting from the rightmost box of $\mu/\lambda$, say $(r_p, c_p)$, we may perform the row extraction algorithm, as in Definition 2.5, and we get a letter $x_p$ and a tableau with shape $\mu \setminus \{(r_p, c_p)\}$. Then we repeat the algorithm, at each step moving leftward. The word $w = x_1 \ldots x_p$ obtained this way is a row word and, of course, $[T \leftarrow w] = T_1$. The uniqueness is clear: if $w'$ a word with the same property of $w$ then $w$ and $w'$ must have the same content, because $[T \leftarrow w] = [T \leftarrow w']$. Then, since $w \approx w'$ and both must be row words, one has $w = w'$.

With other words, this means that

**Theorem 3.14.** Let $\lambda \vdash n$ and let $S_\lambda$ be the (formal) sum of all tableaux of shape $\lambda$ over the proper signed alphabet $L$. Let $S_{(p)}$ be the sum of all row words of length $p$. Then

$$S_\lambda \cdot S_{(p)} = \sum_\mu S_\mu$$

where $\mu$ runs over all partitions of $n + p$ such that $\lambda \subseteq \mu$ and no two boxes of $\mu/\lambda$ occur in the same column.

Proof. It holds

$$S_\lambda \cdot S_{(p)} = \sum T_\lambda T_{(p)}.$$ 

By applying Corollary 3.13, each tableau $T_\lambda T_{(p)}$ is of type $T_\mu$ with $\mu$ partition of $n + p$ containing $\lambda$ and satisfying the condition that no two boxes of $\mu/\lambda$ are in the same column. On the other hand, any tableau $T_\mu$ with these properties can be factorized uniquely as $T_\lambda T_{(p)}$. This proves the statement.

Similar considerations provide a result extending the Pieri-Young formula concerning the column words.

4. **Super RSK-correspondence**

If we want to count the number of words in each plactic class, we have to transform the map $w \mapsto T(w)$ into a bijective correspondence between words and pairs of Young tableaux. In each pair, one of the tableaux has to be standard since related to the positions of the letters in the corresponding word. If we consider instead any pair of Young tableaux, the correspondence is with the so-called “two-rowed arrays”. We explain here these phenomena within the setting of signed alphabets.
Let \((L, |\cdot|')\) and \((P, |\cdot|'')\) be two alphabets. On the product set \(L \times P\) we can define a structure of alphabet as follows. The total order is right lexicographic, that is:

\[(a_1, b_1) < (a_2, b_2), \text{ if } b_1 < b_2 \text{ or } b_1 = b_2, a_1 < a_2.\]

Moreover, the map \(|\cdot| : L \times P \to \mathbb{Z}_2\) is defined as

\[| (a, b) | := |a|' + |b|''.\]

To simplify the notation, from now on we will denote by \(|\cdot|\) all the mappings from the sets \(L, P\) and \(L \times P\) to \(\mathbb{Z}_2\).

**Definition 4.1.** Let \(L, P\) be two alphabets and let \(S := \begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix}\) be a 2 \(\times\) \(n\) matrix with \(a_i \in L, b_i \in P\). We call \(S\) a signed two-rowed array on the alphabets \(L, P\) if its entries satisfies the following condition:

\[(a_i, b_i) \leq (a_{i+1}, b_{i+1}), \text{ with } (a_i, b_i) = (a_{i+1}, b_{i+1}) \text{ only if } |(a_i, b_i)| = 0.\]

In Section 1 we have explained how to associate these combinatorial objects to the monomials of the letter-place superalgebra. They are related to the Young tableaux by means of the following result:

**Theorem 4.2.** [Super RSK-correspondence] Let \(L, P\) be signed alphabets. A one-to-one correspondence is given between signed two-rowed arrays and pairs of super semistandard Young tableaux on \(L, P\). Precisely, if we denote by \(S \mapsto (T, U)\) this mapping, we have that \(T, U\) are tableaux of the same shape whose entries are the ones of the first and the second row of \(S\) respectively.

This theorem is due to F. Bonetti, D. Senato and A. Venezia [4] and it is based on a variant of the Robinson-Schensted-Knuth algorithms. They were originally presented in the language of the four-fold algebra, and here we give an equivalent formulation in terms of signed two-rowed arrays. From now on, unless explicitly stated, we use the word “two-rowed array” for “signed two-rowed array”.

**Definition 4.3.** Let \(S := \begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix}\) be a two-rowed array on the alphabets \(L, P\).

The map \(S \mapsto (T, U)\) is defined by the following algorithm:

1. **Step 0** Put \(T := \emptyset\), \(U := \emptyset\) and \(k := 1\).
2. **Step 1** Put \(x := a_k\) and \(y := b_k\).
3. **Step 2a** If \(|y| = 0\) then put \(T, i\) the output of \(T \leftarrow x\) and append \(y\) at the end of row \(i\) of \(U\).
4. **Step 2b** If \(|y| = 1\) then put \(T, j\) the output of \(x \rightarrow T\) and append \(y\) at the end of column \(j\) of \(U\).
5. **Step 3** Put \(k := k + 1\).
6. **Step 4** If \(k \leq n\) then go to Step 1 else output \(T, U\).

**Definition 4.4.** Let \(T, U\) a pair of Young tableaux of the same shape on the alphabets respectively \(L, P\). The map \((T, U) \mapsto S\) is by definition the following algorithm:
Step 0  Put \( S := \begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix} \) and \( k := 1 \).

Step 1  Put \( y \) the maximal entry of \( U \).

Step 2a  If \(|y| = 0\) then put \( i \) the minimal index of a row of \( U \) containing \( y \).

Step 2b  If \(|y| = 1\) then put \( j \) the minimal index of a column of \( U \) containing \( y \).

Step 3  Put \( a_k := x, b_k := y \) and \( k := k + 1 \).

Step 4  If \( k \leq n \) then go to Step 1 else output \( S \).

The procedures \( S \mapsto (T, U) \) and \( (T, U) \mapsto S \) are both correct and one is the inverse of the other. The proof of these facts can be viewed in section 4 of [4] (more precisely, in Proposition 4.2 and 4.3). From this, Theorem 4.2 follows immediately.

The number of words in each plactic class can be computed in the following way. Put \( P = P_0 := \mathbb{N} \) and note that there is an injective map \( w \mapsto S \) from the set \( L \) into the set of two-rowed arrays, where if \( w := x_1 \ldots x_n \) then:

\[
S := \begin{bmatrix} x_1 \ldots x_n \\ 1 \ldots n \end{bmatrix}
\]

If \( S \mapsto (T, U) \), from Definition 1.5, it follows that \( T(w) = T \) since \( P = P_0 \). Moreover, if \( \lambda \vdash n \) is the shape of \( T \) then \( U \) is a standard tableau of shape \( \lambda \) whose entries are \( 1, 2, \ldots, n \). By Theorem 4.2, we get finally:

**Corollary 4.5.** Let \( L \) be a signed alphabet. Let \( w \in L \) and \( \lambda \) be the shape of the tableau \( T := T(w) \). The number of words in the plactic class \( [w] \) is equal to the number of standard Young tableaux of shape \( \lambda \).

A natural question is to give the right generalization of the so-called symmetry theorem in the case of signed alphabets. Consider the product alphabets \( L \times P \) and \( P \times L \). Recall that they are ordered by the right lexicographic ordering and \(|(a, b)| := |a| + |b|\) for any pair \((a, b)\) belonging to one of them. We can define an involution between the set of two-rowed arrays on the alphabets \( L, P \) and the corresponding set for \( P, L \) simply as:

\[
S = \begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix} \mapsto S' = \begin{bmatrix} b'_1 \ldots b'_n \\ a'_1 \ldots a'_n \end{bmatrix}
\]

where \( \{a_1, \ldots, a_n\} = \{a'_1, \ldots, a'_n\} \) and \( \{b_1, \ldots, b_n\} = \{b'_1, \ldots, b'_n\} \).

**Definition 4.6.** Let \( L, P \) signed alphabets. We say that a two-rowed array \( S \) has symmetry if \( S \mapsto (T, U) \) and \( S' \mapsto (U, T) \).

The symmetry theorem [16] [11] states that in the non-signed case \((L = L_0, P = P_0)\) or \((L = L_1, P = P_1)\) all two-rowed arrays have symmetry. We generalize this result to a particular case in the following way:

**Proposition 4.7.** Assume that the alphabets satisfy the conditions \( L_0 < L_1, P_0 < P_1 \) or \( L_1 < L_0, P_1 < P_0 \) and let \( S = \begin{bmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{bmatrix} \) be a two-rowed array on \( L, P \). If \(|(a_i, b_i)| = 0\) for \( i = 1, 2, \ldots, n \) then \( S \) has symmetry.

Before proving the theorem we need new notations. Assume \( L_0 < L_1, P_0 < P_1 \) and let \( S \) be a two-rowed array such that \(|(a_i, b_i)| = 0\) that is \(|a_i| = |b_i|\) for all \( i \).
Since \( b_1 \leq \ldots \leq b_n \), note that \( S \) splits as \( S = \left[ S_0 \ S_1 \right] \), where \( S_0 \) is a two-rowed array on \( L_0, P_0 \) and \( S_1 \) on \( L_1, P_1 \). Moreover, if \( |a_i| = |b_i| = 0 \) for \( 1 \leq i \leq k \) then clearly \( S_0 \) has \( k \) columns.

Suppose now that \( S \mapsto (T, U) \) and let \( \lambda \vdash n \) be the shape of the Young tableaux \( T, U \). Note that also \( T \) splits into two tableaux \( T_0, T_1 \) of content respectively \( \{a_1, \ldots, a_k\}, \{a_{k+1}, \ldots, a_n\} \), where \( T_0 \) has some shape \( \mu + \{k, \mu \subset \lambda \) and \( T_1 \) has the skew shape \( \lambda/\mu \). The same happens to the tableau \( U \) and it is clear that \( S_0 \mapsto (T_0, U_0) \).

**Example 4.8.** Let \( L_0 = P_0 := \{1,2\}, L_1 = P_1 := \{3,4,5,6\} \) and define:

\[
S := \left[ \begin{array}{cccccc}
2 & 1 & 1 & 1 & 6 & 5 \\
1 & 2 & 2 & 3 & 4 & 5 & 6
\end{array} \right].
\]

Then, we have:

\[
T = \begin{array}{cccc}
1 & 1 & 1 & 6 \\
2 & 4 & 5
\end{array}, \quad U = \begin{array}{ccc}
1 & 2 & 2 \\
4 & 5
\end{array}
\]

and hence:

\[
T_0 = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 4 & 5
\end{array}, \quad T_1 = \begin{array}{ccc}
6 \\
3
\end{array}, \quad U_0 = \begin{array}{ccc}
1 & 2 & 2 \\
4 & 5
\end{array}, \quad U_1 = \begin{array}{ccc}
6 \\
3
\end{array}
\]

Finally, for any partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \) we define:

\[
C_\lambda := \begin{array}{cccc}
1 & 2 & \ldots & \lambda_1 \\
1 & 2 & \ldots & \lambda_2 \\
& & \vdots & \\
1 & 2 & \ldots & \lambda_r
\end{array}
\]

Clearly \( C_\lambda \) is a Young tableau on the signed alphabet \( L = L_1 := \{1,2,\ldots, \lambda_1\} \).

**Proof of Proposition 4.7.** We argue for the case \( L_0 < L_1, P_0 < P_1 \) since the other case can be recovered by conjugating the alphabets. Let \( S \mapsto S' \), where \( S' = \left[ b_1' \ldots b_n' \right]_{a_1' \ldots a_n'} \), and let \( S \mapsto (T, U), S' \mapsto (T', U') \). Since \( |a_i| = |b_i| \) and therefore \( |b_i'| = |a_i'| \) for any \( i \), we have that the two-rowed arrays \( S, S' \) split respectively into \( S_\alpha, S'_\alpha \) (\( \alpha = 0,1 \)). Moreover, denote by \( T_\alpha, U_\alpha, T'_\alpha, U'_\alpha \) the tableaux obtained by splitting \( T, U, T', U' \). Clearly \( S_0 \) maps to \( S_0' \) under the involution and we have that \( S_0 \mapsto (T_0, U_0), S_0' \mapsto (T_0', U_0') \). By the symmetry theorem it follows that \( S_0 \) has symmetry that is \( T_0 = U_0, U_0' = T_0 \).

We claim that we have also \( T_1 = U_1, U_1' = T_1 \) and hence \( S \) has symmetry. Let \( \lambda \vdash n \) and \( \mu \subset \lambda \) be the shapes of the pairs of tableaux respectively \( T, U \) and \( T_0, U_0 \). Consider the tableau \( C_\mu \) defined on an alphabet \( \tilde{L} := \tilde{L}_1 \) such that \( \tilde{L}_1 < L_1 \) and put \( \tilde{w} := w(C_\mu) \). If \( \tilde{w} = \tilde{a}_1 \ldots \tilde{a}_k \) then we define also \( \tilde{S}_1 := \left[ \tilde{a}_1 \ldots \tilde{a}_k \right]_{\tilde{b}_1 \ldots \tilde{b}_k} \), where \( \tilde{b}_1 < \ldots < \tilde{b}_k \) are letters of an alphabet \( \tilde{P} := \tilde{P}_1 \) such that \( \tilde{P}_1 < P_1 \). Put \( \tilde{S} := \left[ \tilde{S}_1 \ S_1 \right] \) and let \( \tilde{S} \mapsto (\tilde{T}, \tilde{U}) \). The two-rowed array \( \tilde{S}_1 \) clearly maps to a pair of tableaux of shape \( \mu \) (the left one is \( C_\mu \)). From \( \tilde{L}_1 < L_1, \tilde{P}_1 < P_1 \) it follows that \( \tilde{T}, \tilde{U} \) have shape \( \lambda \) and are respectively the union of these tableaux with the tableaux \( T_1, U_1 \) of skew shape \( \lambda/\mu \). Since \( \tilde{S}_1, \tilde{S} \) have symmetry we conclude that the claim holds. \( \square \)
We remark that the condition in Proposition 4.7 is not necessary. As an instance, whatever the signature is, the two-rowed array
\[
\begin{bmatrix}
3 & 4 & 1 & 2 \\
1 & 2 & 3 & 4
\end{bmatrix}
\]
has symmetry. To the best of our knowledge, proving a symmetry Theorem taking into account a definition of symmetry more general than Definition 4.6 and leading to the classical statement as a particular case, is still an open problem.

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