Well-Posedness for Degenerate Schrödinger Equations

Massimo Cicognani - University of Bologna

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Outline

Introduction

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Change of variable

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Concluding remarks and open problems
Joint work with

Michael Reissig - TU Bergakademie Freiberg
Schrödinger equations with time-dependent Hamiltonian

We consider the Schrödinger operator

\[ S := \frac{1}{i} \partial_t - H(t) \]

with a time-dependent Hamiltonian

\[ H(t) = a(t) \Delta_x - \sum_{j=1}^{n} b_j(t, x) \partial_{x_j}, \quad a(t) \geq 0, \]

\( t \in [0, T], x \in \mathbb{R}^n \), with coefficients which are continuous in time and smooth and bounded in the space variables.
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\( t \in [0, T], x \in \mathbb{R}^n, \) with coefficients which are continuous in time and smooth and bounded in the space variables.

We are interested in the Cauchy problem

\[
\begin{cases}
  Su = 0, & t > 0, \\
  u(0, x) = u_0(x).
\end{cases}
\]
Well-posedness

We take Cauchy data $u_0$ in a Sobolev space $H^m$, $m \in \mathbb{R}$, or in a Gevrey-Sobolev space $H_{\varrho}^{m,s}$, $s > 1$, $\varrho > 0$, where

$$H_{\varrho}^{m,s} := e^{-\varrho \langle D_x \rangle^{1/s}} H^m, \quad H^{m,s} := \bigcup_{\varrho > 0} H_{\varrho}^{m,s}.$$
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We take Cauchy data $u_0$ in a Sobolev space $H^m$, $m \in \mathbb{R}$, or in a Gevrey-Sobolev space $H^{m,s}_\varrho$, $s > 1$, $\varrho > 0$, where

$$H^{m,s}_\varrho := e^{-\varrho \langle D_x \rangle^{1/s}} H^m, \quad H^{m,s} := \bigcup_{\varrho > 0} H^{m,s}_\varrho.$$ 

We say that the Cauchy problem is well-posed in

$$\begin{cases}
L^2 \\
H^\infty \\
H^{\infty,s} := \bigcap_m H^{m,s}
\end{cases} \quad \text{when for every given } u_0 \in \begin{cases}
H^m \\
H^m \\
H^{m,s}_\varrho
\end{cases} \quad \text{there exists a unique solution } u \in \begin{cases}
C([0, T]; H^m) \\
C([0, T]; H^{m-\delta}) \\
C([0, T]; H^{m',s}_\varrho)
\end{cases}.$$
Decay conditions

When the coefficients $b_j(t, x)$ are pure imaginary we have well-posedness without loss of derivatives by the energy method and Gronwall inequality since $H(t)$ is the sum of a self-adjoint operator and of a bounded operator in $L^2$. 
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$$\frac{1}{i} \partial_t u = \partial_x^2 u + \partial_x u$$

by Fourier transform.
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Decay conditions as \( x \to \infty \) for the real parts \( \Re b_j(x) \) have been proved to be necessary in the case of a time-independent Hamiltonian \( H = \Delta_x + \sum_{j=1}^n b_j(x) \partial_{x_j} \), Ichinose et al.
Sufficient conditions

Still in the time-independent case, the condition

$$|\Re b_j(x)| \leq C \langle x \rangle^{-\sigma}, \quad \sigma > 0, \quad \langle x \rangle = \sqrt{1 + |x|^2},$$

is sufficient for the well-posedness in

$$\begin{cases} 
L^2, & \sigma > 1, \\
H^\infty, & \sigma = 1, \\
H^\infty, s, & s < \frac{1}{1-\sigma}, \quad \sigma < 1,
\end{cases}$$

Kajitani-Baba et al. These results are optimal.
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After minor changes, the same proof works for time-dependent $H(t)$ provided that the coefficient $a(t)$ of the Laplacian never vanishes so that

$$|\Re b_j(t, x)| \leq Ca(t) \langle x \rangle^{-\sigma}.$$
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As far as we know, there are no well-posedness results for time-dependent Hamiltonians with $a(t)$ that may vanish.
We consider a real coefficient $a(t) \geq 0$ vanishing of finite order $\ell$ at $t = 0$, that is,

$$ct^\ell \leq a(t) \leq Ct^\ell,$$

for $\ell \in \mathbb{R}_+$ and positive constants $c, C$. 
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The other coefficients are complex-valued and satisfy
\[ |\Re b_j(t, x)| \leq Ct^k \langle x \rangle^{-\sigma}, \quad 0 < k \leq \ell, \quad \sigma > 0. \]

We have new effects for $k < \ell$ when $|\Re b_j(t, x)| \leq Ca(t)\langle x \rangle^{-\sigma}$ does not hold true.
Main Result

Theorem

The Cauchy problem is well-posed in

\[
\begin{cases}
    L^2 & \text{if } k = \ell, \sigma > 1, \\
    H^{\infty} & \text{if } k = \ell, \sigma = 1, \\
    H^{\infty, s} & \text{with } s < \frac{\ell+1}{\ell-k} \text{ if } k < \ell, \sigma \geq 1, \\
    H^{\infty, s} & \text{with } s < \frac{(\ell-k)\sigma+k+1}{(\ell-k)\sigma+(k+1)(1-\sigma)} \text{ if } k \leq \ell, \sigma < 1.
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\end{aligned}
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For any \( \ell \geq 0 \) and \( k = \ell \) we have the same optimal spaces of well-posedness as in the time-independent case.
Transforming $iH(t)$ into a bounded from above operator

We get the well-posedness of the Cauchy problem after performing a change of variable $v(t, x) = e^{\Lambda}(t, x, D_x)u(t, x)$, where $e^{\Lambda}(t, x, D_x), \ D = \frac{1}{i}\partial$, is an invertible pseudo-differential operator with symbol $e^{\Lambda(t, x, \xi)}, \ \Lambda(t, x, \xi)$ real-valued of order $q, \ 0 \leq q < 1$. 

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$$\|v(t)\|_{L^2} \leq C\|v(0)\|_{L^2}$$

for any solution of the transformed equation

$$S_\Lambda v = 0, \ S_\Lambda := e^\Lambda S(e^\Lambda)^{-1}.$$
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The energy estimate (without any loss of regularity) follows by Gronwall’s lemma if we find $\Lambda$ such that

$$iS_\Lambda = \partial_t - ia(t)\Delta_x - A(t, x, D_x), \quad 2\Re(Av, v) \leq C \| v \|_{L^2}^2.$$
The crucial inequality for the symbol $\Lambda$

We seek for a function $\Lambda$ that solves

$$
\partial_t \Lambda(t, x, \xi) + 2 a(t) \sum_{j=1}^n \xi_j \partial_{x_j} \Lambda(t, x, \xi) + \Re \sum_{j=1}^n b_j(t, x) \xi_j \leq 0,
$$

for all $|\xi| \geq h$, and such that $\partial_t \Lambda(t, x, \xi)$ has the order 1 and $a(t)\partial_{x_j} \Lambda$ has the order zero.
The crucial inequality for the symbol $\Lambda$

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This means that we make $A(t)$ an operator of order 1 with negative principal symbol. In view of the sharp Gårding inequality, this leads to a bounded from above operator.
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for all $|\xi| \geq h$, and such that $\partial_t \Lambda(t, x, \xi)$ has the order 1 and $a(t) \partial_{x_j} \Lambda$ has the order zero.

This means that we make $A(t)$ an operator of order 1 with negative principal symbol. In view of the sharp Gårding inequality, this leads to a bounded from above operator. As it is well-known, then the energy estimate gives the well-posedness in $L^2$ of the Cauchy problem for the operator $S_\Lambda$. 
The transformation carries the loss

If $X$ is a suitable Banach or Frechet space of functions on $\mathbb{R}^n_X$ such that the operators

$$e^{\Lambda(t)} : X \to H^m, \quad (e^{\Lambda(t)})^{-1} : H^m \to X,$$

are continuous, then we have (at least locally in time) a unique solution $u \in C([0, T]; X)$ of the original Cauchy problem for any given initial data $u_0 \in X$. The order of $e^{\Lambda}$ corresponds to the loss of derivatives and determines the space $X$. 
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We obtain spaces of well-posedness from the following estimates:

$$|\Lambda(t, x, \xi)| \leq \begin{cases} C\langle \xi \rangle^{\frac{\ell-k}{\ell+1}} & \text{if } \sigma > 1, \\ C\langle \xi \rangle^{\frac{\ell-k}{\ell+1}} \log(1 + \langle \xi \rangle) & \text{if } \sigma = 1, \\ C\langle \xi \rangle^{\frac{(\ell-k)\sigma+(k+1)(1-\sigma)}{(\ell-k)\sigma+k+1}} & \text{if } \sigma < 1. \end{cases}$$
Degeneracy leads to solvability in Gevrey spaces

In particular, when $k = \ell$ the operator $e^{\Lambda}$ is:

- of order zero for $\sigma > 1$, $X$ is the Banach space $H^m$;
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- of a finite positive order $\delta$ for $\sigma = 1$, $X$ is the Frechet space $H^\infty$ (with a loss of $\delta$ derivatives);
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- of infinite order described by the symbol $e^{\rho \langle \xi \rangle^{1-\sigma}}$ for $\sigma < 1$, $X$ is the Frechet space $H^{m,s}$, $s = 1/(1 - \sigma)$. 
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In this case, we have the same spaces of well-posedness as in in the time-independent case.
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In this case, we have the same spaces of well-posedness as in the time-independent case.

- For $k < \ell$ the operator $e^{\Lambda}$ is of infinite order described by $e^{\varrho \langle \xi \rangle^q}$, $0 < q < 1$, $q = q(\ell, k, \sigma)$, even with a fast decay $\sigma > 1$. A strong degeneracy leads to well-posedness only in Gevrey classes of index $s \leq 1/q$. 
Solving modulo a prescribed order

Let us devote to the inequality which is to be satisfied by Λ. Let \( w(\xi) \) be a weight function corresponding to a possible order of solutions. It is sufficient to find \( \lambda(t, x, \xi) \) of the same order as that of \( w(\xi) \) such that

\[
\partial_t \lambda(t, x, \xi) + 2a(t) \sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda(t, x, \xi) + \Re \sum_{j=1}^{n} b_j(t, x) \xi_j \leq Kw(\xi).
\]
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Let us devote to the inequality which is to be satisfied by $\Lambda$. Let $w(\xi)$ be a weight function corresponding to a possible order of solutions. It is sufficient to find $\lambda(t, x, \xi)$ of the same order as that of $w(\xi)$ such that

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$$

In fact, if we define $\Lambda(t, x, \xi)$ by

$$
\Lambda(t, x, \xi) = \varrho(t)w(\xi) + \lambda(t, x, \xi),
$$

then we have a solution of still of the order of $w(\xi)$ taking $\varrho(t)$ such that $\varrho'(t) \leq -K$. 
Absorbing lower order terms

It is natural to absorb an error of the order of $w(\xi)$ because terms of such an order appear under the principal part in the asymptotic expansion of the operator $A(t)$ in any case. If $w(\xi)$ is not of order zero, then we also need to control them in the application of the Gårding inequality.
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The symbol of this part of the order of $w(\xi)$ will be bounded by $N|\varrho(t)| + N, \ N \geq K$, so we will choose $\varrho(t)$ as a solution of

$$\varrho'(t) + N(\varrho(t) + 1) = 0, \ \varrho(t) > 0.$$
Splitting the phase-space

The study of the inequality for $\lambda(t, x, \xi)$ is crucial in the zone

$$\{(x, \xi) \in \mathbb{R}_{x, \xi}^{2n} : \langle x \rangle \leq \langle x_\xi \rangle \text{ with } \langle x_\xi \rangle = \langle \xi \rangle^{(1-q)/\sigma} \}$$

of the phase-space $\mathbb{R}_{x, \xi}^{2n}$ since we have in the other part

$$\sum_{j=1}^{n} |\Re b_j(t, x)\xi_j| \leq Ct^k \langle \xi \rangle^q, \quad \langle x \rangle \geq \langle x_\xi \rangle.$$

So, we can use here the above absorbing argument.
The solution for mild degeneracy

For $k = \ell$ we have

$$a(t) M_0 |\xi(\langle x \rangle)^{-\sigma} \geq \sum_{j=1}^{n} |\Re b_j(t, x) \xi_j|. $$
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$$a(t)M_0|\xi|\langle x \rangle^{-\sigma} \geq \sum_{j=1}^{n} |\Re b_j(t, x)\xi_j|.$$ 

In this case we can take a time-independent solution $\lambda_0(x, \xi)$

$$\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda_0(x, \xi) + M|\xi|\langle x \rangle^{-\sigma} \chi(\langle x \rangle/\langle x\xi \rangle) \leq 0,$$

where $\chi(y)$ is a cut-off function.
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$$\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda_0(x, \xi) + M|\xi|\langle x \rangle^{-\sigma} \chi\left(\langle x \rangle/\langle x\xi \rangle\right) \leq 0,$$

where $\chi(y)$ is a cut-off function.

The equation $\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda(x, \xi) + |\xi|g(x, \xi) = 0$ is solved by

$$\lambda(x, \xi) = -\int_{0}^{x \cdot \omega} g(x - \tau \omega, \xi) d\tau, \quad \omega = \xi/|\xi|.$$
The order of the time-independent solution

We have

\[ |\lambda_0(x, \xi)| \leq \begin{cases} 
C_0 \langle \xi \rangle^{(1-q)(1-\sigma)/\sigma}, & \sigma < 1, \\
C_0 \log(1 + \langle \xi \rangle), & \sigma = 1, \\
C_0, & \sigma > 1.
\end{cases} \]
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\end{cases} \]

For \( \ell = k \) the optimal choice of the order \( q \), together with the related Gevrey index \( s < 1/q \) for \( q > 0 \), follows from

\[ \begin{cases}
(1 - q)(1 - \sigma)/\sigma = q, & \sigma \in (0, 1), \\
q = 0, & \sigma \geq 1.
\end{cases} \]

The first line gives \( q = 1 - \sigma \) for \( \sigma < 1 \).
Strong degeneracy - Splitting the extended phase-space

For $k < \ell$ we split the extended phase-space $(t, x, \xi)$ into two zones. Defining $t_{\xi} = \langle \xi \rangle^{-1/(k+1)}$ we introduce the
• pseudo-differential zone: $t \leq t_{\xi}$; evolution zone: $t \geq t_{\xi}$.
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• pseudo-differential zone: $t \leq t_\xi$; evolution zone: $t \geq t_\xi$.

We put in the construction of a solution $\lambda(t, x, \xi)$ a first term

$$
\lambda_\psi(t, \xi) = - M \langle \xi \rangle \int_0^t \tau^k \chi(\tau / t_\xi) d\tau
$$

which is localized to the pseudo-differential zone. The symbol $\lambda_\psi(t, \xi)$ is of order $q$ by the definition of $t_\xi$. 

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Taking a sufficiently large $M$ it follows

$$\partial_t \Lambda_\psi(t, \xi) - \chi(t/t_\xi) \Re \sum_{j=1}^n b_j(t, x) \xi_j \leq 0$$

since

$$\sum_{j=1}^n |\Re b_j(t, x) \xi_j| \leq M_0 t^k \langle x \rangle^{-\sigma} |\xi| \leq M_0 t^k |\xi|.$$
The solution in the evolution zone

In the evolution zone we define

\[ \lambda_e(t, x, \xi) = \lambda_{e,0}(t, x, \xi) + \lambda_{e,1}(t, \xi) \]

with

\[ \lambda_{e,0}(t, x, \xi) = (1 - \chi(t/t_\xi)) t^{k-\ell} \lambda_0(x, \xi), \]

\[ \lambda_{e,1}(t, \xi) = -C_1 Mw(\xi) \int_0^t \tau^{k-\ell-1} (1 - \chi(2\tau/t_\xi)) d\tau \]

where \( \lambda_0(x, \xi) \) is the time independent solution for \( k = \ell \) and the weight function

\[ w(\xi) = \begin{cases} \langle \xi \rangle^{(1-q)(1-\sigma)} \frac{1}{\sigma}, & \sigma < 1, \\ \log \langle \xi \rangle, & \sigma = 1, \\ 1, & \sigma > 1, \end{cases} \]

gives its order before fixing \( q \).
Fixing the order

We have a solution in the evolution zone as soon as $\partial_t \lambda_e(t, x, \xi) \leq 0$. We have this fixing a large constant $C_1$ in the correction term $\lambda_{e, 1}(t, \xi)$. 
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Using the definitions of $t_\xi$ and the order of the time-independent term $\lambda_0(x, \xi)$, the symbol $\lambda_e(t, x, \xi)$ can be estimated by

$$
\begin{cases}
\langle \xi \rangle (1-q)((\ell-k)/(k+1)+(1-\sigma)/\sigma), & \sigma < 1, \\
\langle \xi \rangle (1-q)(\ell-k)/(k+1) \log \langle \xi \rangle, & \sigma = 1, \\
\langle \xi \rangle (1-q)((\ell-k)/(k+1), & \sigma > 1.
\end{cases}
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Fixing the order

We have a solution in the evolution zone as soon as \( \partial_t \lambda_e(t, x, \xi) \leq 0 \). We have this fixing a large constant \( C_1 \) in the correction term \( \lambda_{e,1}(t, \xi) \).

Using the definitions of \( t_\xi \) and the order of the time-independent term \( \lambda_0(x, \xi) \), the symbol \( \lambda_e(t, x, \xi) \) can be estimated by

\[
\begin{cases}
\langle \xi \rangle^{(1-q)((1-k)/(k+1)+(1-\sigma)/\sigma)}, & \sigma < 1, \\
\langle \xi \rangle^{(1-q)(1-k)/(k+1) \log \langle \xi \rangle}, & \sigma = 1, \\
\langle \xi \rangle^{(1-q)(1-k)/(k+1)}, & \sigma > 1.
\end{cases}
\]

In order to have also \( \lambda_e \) of order \( q \) (or \( q \log \) for \( \sigma = 1 \)) we choose

\[
q = \begin{cases}
\frac{(\ell-k)\sigma+(k+1)(1-\sigma)}{(\ell-k)\sigma+k+1}, & \sigma < 1, \\
\frac{\ell-k}{\ell+1}, & \sigma \geq 1.
\end{cases}
\]
The transformed Cauchy problem

For operators of infinite order of Gevrey type, we use the calculus of Kajitani and Nishitani. Localizing the support of $\lambda(t, x, \xi)$ for $|\xi| \geq h$ with a sufficiently large $h$, we can make the change of variable $v = e^\Lambda u$ invertible with $(e^\Lambda)^{-1}$ given by a Neumann series of operators.
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\[
iS_{\Lambda} = \partial_t - ia(t)\Delta_x - A(t, x, D_x), \quad 2\Re(Av, v) \leq C\|v\|^2_{L^2}.
\]

This gives the energy estimate without loss of derivatives hence the well-posedness in \( L^2 \) of the Cauchy problem for \( S_{\Lambda} \).
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$$i S_{\Lambda} = \partial_t - i a(t) \Delta_x - A(t, x, D_x), \quad 2 \Re(Av, v) \leq C \|v\|^2_{L^2}.$$

This gives the energy estimate without loss of derivatives hence the well-posedness in $L^2$ of the Cauchy problem for $S_{\Lambda}$. Taking the order of $e^{\Lambda}$ into account (the transformation carries the loss) we have the results of well-posedness for the operator $S$. 
General degeneracy

Let us consider a general coefficient $a(t)$ increasing and such that $a(t) = 0$ (also of infinite order) and let us assume

$$|\Re b_j(t, x)| \leq Ca(t)\mu(t)\langle x\rangle^{-\sigma}, \quad \sigma > 0,$$

with $\mu(t)$ decreasing and such that $\lim_{t \to +0} \mu(t) = \infty$. 
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The separating line $t = t_{\xi}$ in the extended phase-space is now defined by

$$B(t_{\xi}) = \langle \xi \rangle^{q-1}, \text{ where } B(t_{\xi}) := \int_0^{t_{\xi}} a(\tau) \mu(\tau) d\tau, \ q < 1.$$
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Computing the lowest possible order of \(\lambda = \lambda_\psi + \lambda_e\) we can get results of well-psedness.
A model with degeneracy of infinite order

Let us take

\[ a(t) = \frac{\alpha}{t^{\alpha+1}} \exp\left(-\frac{1 + c_0}{t^\alpha}\right), \]

\[ \mu(t) = \exp\left(\frac{c_0}{t^\alpha}\right), \]

and let us assume, consequently, that

\[ |\Re b_j(t, x)| \leq C \frac{\alpha}{t^{\alpha+1}} \exp\left(-\frac{1}{t^\alpha}\right) \langle x \rangle^{-\sigma} \text{ with } \sigma \in (0, 1). \]
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The critical order follows from the balance

\[ q = \frac{(1-q)(1-\sigma)}{\sigma} + (1 - q)c_0. \]

Then the Cauchy problem is well-posed in the Gevrey spaces \( H^{\infty,s} \) with \( s < \frac{1+c_0\sigma}{1-\sigma+c_0\sigma} \).
Vibrating plates

Let us consider the vibrating plate equation $Pu = 0$

$$Pu := u_{tt} + a^2(t)\Delta_x^2 u + \sum_{|\alpha|\leq 3} b_\alpha(t, x) \partial_x^\alpha u$$

with $a(t) \geq 0$ vanishing at $t = 0$ of finite order $\ell$ and with real-valued $b_\alpha(t, x)$ with $|\alpha| = 3$ satisfying

$$|b_\alpha(t, x)| \leq Ct^j \langle x \rangle^{-\sigma} \text{ with } \ell \leq j < 2\ell.$$
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The operator $P$ can be formally factorized in the product of two (pseudo-differential) Schrödinger operators

$$P = S_+ S_-, \quad S_\pm = \partial_t \pm ia(t)\Delta_x \pm b(t, x, \partial_x)$$

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Performing a complete diagonalization in the evolution zone one should obtain for $P$ the same results as for $S$ with $k = j - \ell$. 
Necessity

An interesting problem is the optimality of the results, a subject widely studied for non-degenerate models. One can not find “better” spaces of well-posedness in the case $\ell = k$ in view of the necessary decay conditions as $x \to \infty$ obtained for $\ell = k = 0$. 
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THANKS FOR YOUR ATTENTION!!!