DING PROJECTIVE DIMENSION OF COMPLEXES*

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Abstract In this paper, we define and study a notion of Ding projective dimension for complexes of left modules over associative rings. In particular, we consider the class of homologically bounded below complexes of left $R$-modules, and show that Ding projective dimension has a nice functorial description.

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1. Introduction

In [AF], Avramov and Foxby defined the projective (resp. injective or flat) dimension for unbounded complexes by means of DG-projective (resp. DG-injective or DG-flat) resolutions. A complex $P$ of $R$-modules is called DG-projective if $\text{Hom}_R(P, -)$ transforms surjective quasi-isomorphisms into surjective quasi-isomorphisms, which is equivalent to saying that $P$ is a complex of projective $R$-modules and $\text{Hom}_R(P, X)$ is exact for every exact complex $X$ by [AF, 1.2.P]. A DG-projective resolution of $X$ is a quasi-isomorphism $P \to X$ with $P$ DG-projective. By [EJX, Corollary 3.10], every complex has a surjective DG-projective resolution $P \to X$. If $X$ is homologically bounded below, then $P$ can be chosen so that $\inf\{i | P_i \neq 0\} = \inf X$.

Over commutative local rings, Yassemi [Y] and Christensen [C] introduced a Gorenstein projective dimension for complexes with bounded below homology. In [V], Veliche defined and studied Gorenstein projective dimension for complexes of left $R$-modules over associative ring $R$. Not much later Gorenstein injective and Gorenstein flat dimension for complexes were introduced and studied in [AS, I]. These Gorenstein dimensions are related to the Gorenstein rings. General background materials about Gorenstein homological algebra can be found in [EJ, EL, Ga].

In [DLM], Ding, Li and Mao introduced and studied strongly Gorenstein flat modules, and several well-known classes of rings are characterized in terms of these modules. A left $R$-module $M$ is called strongly Gorenstein flat if there is an exact sequence

$$\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

of projective left $R$-modules with $M = \text{Coker}(P_0 \to P_{-1})$ such that $\text{Hom}(-, F)$ leaves the sequence exact, where $F$ stands for the class of all flat left $R$-modules. Since strongly Gorenstein flat modules have properties analogous to Gorenstein projective modules, Gillespie [Gi] called these modules Ding projective modules. For every left $R$-module $M$ over an associative ring $R$, Ding at al. also defined and investigated the strongly Gorenstein flat dimension for modules and rings.

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The main purpose of this paper is to introduce and study a concept of Ding projective dimension \( \text{Dpd}_R(X) \) associated to every complex \( X \) of left \( R \)-modules over an arbitrary associative ring \( R \). In particular, we consider the class of homologically bounded below complexes of left \( R \)-modules, and show that Ding projective dimension has a nice functorial description.

In this paper, \( R \) denotes an associative ring with unity. We consistently use the notation from the appendix of [C]. In particular, the category of \( R \)-complexes is denoted \( \text{C}(R) \), and use subscripts \( \sqsupseteq, \sqsubset \) and \( \Box \) to denote boundedness conditions, and use subscripts \( (\sqsupseteq), (\sqsubset) \) and \( (\Box) \) to denote homological boundedness conditions. For example, \( \text{C}_\sqsupseteq(R) \) is the full subcategory of \( \text{C}(R) \) of bounded below complexes; \( \text{C}_\sqsubset(R) \) is the full subcategory of \( \text{C}(R) \) of homologically bounded below complexes.

Given a complex \( C \) and an integer \( i \), \( \Sigma^i C \) denotes the complex such that \( (\Sigma^i C)_n = C_{n-i} \) and whose boundary operators are \( (-1)^i \delta^C_{n-i} \). The \( n \)th homology module of \( C \) is the module \( \text{H}_n(C) = Z_n(C)/B_n(C) \), where \( Z_n(C) = \text{Ker}(\delta^C_n) \), \( B_n(C) = \text{Im}(\delta^C_{n+1}) \); we set \( \text{H}_n(C) = \text{H}_{-n}(C) \). Given a left \( R \)-module \( M \), we will denote by \( S^n(M) \) the complex with \( M \) in the \( n \)th place and 0 in the other places. For more details of complexes used in this paper the reader can consult [Ha, M].

2. Ding projective dimension of complexes

In the section, \( \mathcal{F} \) stands for the class of flat modules.

**Definition 2.1.** A complex of \( R \)-modules \( T \) is said to be totally \( \mathcal{F} \)-acyclic if the following conditions are satisfied:

1. \( T_n \) is projective for every \( n \in \mathbb{Z} \).
2. \( T \) is exact.
3. \( \text{Hom}_R(T, F) \) is exact for every \( R \)-module \( F \in \mathcal{F} \).

An exact complex of projective \( R \)-modules \( T \) is said to be totally acyclic [V] if \( \text{Hom}_R(T, P) \) is exact for every projective \( R \)-module \( P \). By definitions, totally \( \mathcal{F} \)-acyclic complex is totally acyclic.

For totally \( \mathcal{F} \)-acyclic complex, we have the following two properties using the routine proof.

**Lemma 2.2.** Let \( T \) be a totally \( \mathcal{F} \)-acyclic complex. If \( Q \) is a complex of flat modules and \( n \) is an integer, then any morphism of complexes \( \varphi : T_n \to Q_n \) can be extended to morphism \( \varphi : T \to Q \) such that \( \varphi_n \sqsupseteq = \varphi_n \). Every morphism \( \varphi \) with this property is defined as unique up to homotopy.

**Lemma 2.3.** Let \( T \) be a totally \( \mathcal{F} \)-acyclic complex. If \( Q \) is a bounded above complex of flat modules, then

\[
\text{H}(\text{Hom}_R(T, Q)) = 0.
\]

An \( R \)-module \( M \) is called strongly Gorenstein flat [DLM] if there exists a totally \( \mathcal{F} \)-acyclic complex \( T \) such that \( C_0(T) = M \). Since strongly Gorenstein flat modules have properties analogous to Gorenstein projective modules, Gillespie [Gi] call these modules Ding projective modules. Note that every projective module is Ding projective, and every cokernel \( C_n(T) \) of totally \( \mathcal{F} \)-acyclic complex \( T \) is Ding projective.

Ding projective modules have the following properties.

**Lemma 2.4.** Let \( \text{DP}(R) \) stand for the class of Ding projective modules. The following assertions hold.
(1) If $M \in \text{DP}(R)$, then $\text{Ext}_R^i(M, L) = 0$ for all $i > 0$ and all module $L$ of finite flat or finite injective dimension.

(2) $\text{DP}(R)$ is a projectively resolving class, and closed under direct sums and direct summands.

(3) If $0 \to A \to B \to C \to 0$ is an exact sequence, and $A, B \in \text{DP}(R)$, $\text{Ext}^1_R(C, F) = 0$ for every flat module $F$, then $C \in \text{DP}(R)$.

**Proof.** (1) is obvious.

(2) It follows by analogy with the proof of Theorem 2.5 in [Ho].

(3) It follows by analogy with the proof of Corollary 2.11 in [Ho].

□

**Lemma 2.5.** Let $T$ be an exact complex of projective modules. Then the following are equivalent:

1. $T$ is totally $\mathcal{F}$-acyclic.
2. $C_i(T)$ is Ding projective for all $i \in \mathbb{Z}$.
3. $C_i(T)$ is Ding projective for infinitely many $i \leq 0$.

**Proof.** The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear, so we only argue (3) $\Rightarrow$ (1).

Let $Q$ be a flat $R$-module and fix $n \in \mathbb{Z}$. We need to show $H^n(\text{Hom}_R(T, Q)) = 0$. By (3), we choose an integer $m \geq 1$ such that $n - m$ is small enough for $C_{n-m}(T)$ to be Ding projective. Thus

$$H^n(\text{Hom}_R(T, Q)) = \text{Ext}_R^m(C_{n-m}(T), Q) = 0.$$  

□

**Lemma 2.6.** If $G$ is Ding projective and $\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\alpha} G \to 0$ is a projective resolution of $G$, then there exists a totally $\mathcal{F}$-acyclic complex $T$ such that $T_0 \sqcup = P$, where $P = \cdots \to P_2 \to P_1 \to P_0 \to 0 \to \cdots$.

**Proof.** By the definition of Ding projective modules, there is a totally $\mathcal{F}$-acyclic complex $T'$ such that $C_0(T') = G$. Set

$$T_i = \begin{cases} T'_i & \text{for } i < 0, \\ P_i & \text{for } i \geq 0, \end{cases}$$

and

$$\delta^T_i = \begin{cases} \delta^T_i' & \text{for } i < 0, \\ \beta \alpha & \text{for } i = 0, \\ \delta^P_i & \text{for } i > 0. \end{cases}$$

where $\alpha : P_0 \to G$ and $\beta : G \to T'_{-1}$ are the canonical maps. The complex $T$ is exact, $C_0(T) = G$ and $\sqcup_{-1} T = \sqcup_{-1} T'$, so $T$ is totally $\mathcal{F}$-acyclic by Lemma 2.5. □

According to [DLM], Ding projective dimension, or strongly Gorenstein flat dimension, of $M$ is defined by:

$$Dpd(M) = \inf \left\{ n \in \mathbb{N}_0 \left| \begin{array}{l} 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \text{ is exact, and } G_i \in \text{DP}(R) \end{array} \right. \right\}.$$  

**Lemma 2.7.** (1) Let $M$ be an $R$-module with finite Ding projective dimension, and let $n$ be an integer. Then the following conditions are equivalent:

(i) $\text{Dpd}(M) \leq n$.

(ii) $\text{Ext}_R^i(M, L) = 0$ for all $i > n$, and all R-modules $L$ of finite flat dimension.
(iii) $\operatorname{Ext}^i_R(M, F) = 0$ for all $i > n$, and all flat $R$-modules $F$.

(iv) For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$, where $G_i$ is Ding projective, then also $K_n$ is Ding projective.

(2) If $(M_i)_{i \in I}$ is a family of $R$-modules, then

$$\operatorname{Dpd}(\oplus_i M_i) = \sup \{ \operatorname{Dpd}(M_i) | i \in I \}.$$ 

Proof. They follow by analogy with the proofs of Theorem 2.20 and Proposition 2.19 respectively in [Ho].

**Definition 2.8.** An $\mathcal{F}$-complete resolution of $X$ is a diagram of morphisms of complexes $T \xrightarrow{\tau} P \xrightarrow{\pi} X$, where $\pi : P \to X$ is a DG-projective resolution, $T$ is a totally $\mathcal{F}$-acyclic complex and $\tau_i$ is bijective for all $i \geq 0$. An $\mathcal{F}$-complete resolution $T \xrightarrow{\tau} P \xrightarrow{\pi} X$ of $X$ is said to be surjective if $\tau_i$ is surjective for all $i \in \mathbb{Z}$.

**Lemma 2.9.** Let $T \xrightarrow{\tau} P \xrightarrow{\pi} X$ be an $\mathcal{F}$-complete resolution. If $g$ is an integer such that $\tau_i$ is bijective for all $i \geq g$, then there exists an $\mathcal{F}$-complete resolution $T' \xrightarrow{\tau'} P \xrightarrow{\pi'} X$ such that $\tau' = \tau \circ \alpha$ and $\alpha_i = \text{id}^{T_i}$ for all $i \geq g$.

Proof. Set $(T')_n = (T \oplus \bigcap_{g-1} P \oplus \bigcap_{g-1} P)_n$ as desired.

**Definition 2.10.** The Ding projective dimension of $X$ is defined by

$$\operatorname{Dpd}_R(X) = \inf \left\{ n \in \mathbb{Z} \mid \begin{array}{c} T \xrightarrow{\tau} P \xrightarrow{\pi} X \text{ is an } \mathcal{F}\text{-complete resolution} \\ \text{with } \tau_i : T_i \to P_i \text{ bijective for each } i \geq n \end{array} \right\}.$$ 

**Remark 2.11.** (1) For any complex $X$, $\operatorname{Dpd}_R(X) \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

(2) $\operatorname{Dpd}_R(X) = -\infty$ if and only if $X$ is exact.

(3) For any $k \in \mathbb{Z}$, $\operatorname{Dpd}_R(\Sigma^k X) = \operatorname{Dpd}_R(X) + k$.

**Theorem 2.12.** Let $X$ be a complex, $n$ an integer. Then the following assertions are equivalent:

1. $\operatorname{Dpd}_R(X) \leq n$.

2. $\sup X \leq n$ and there exists a DG-projective resolution $P \to X$ such that the module $C_n(P)$ is Ding projective.

3. $\sup X \leq n$ and for every DG-projective resolution $P' \to X$, the module $C_n(P')$ is Ding projective.

4. For every DG-projective resolution $P' \to X$, there exists a surjective $\mathcal{F}$-complete resolution $T' \to P' \to X$ such that $\tau'_i = \text{id}^{T'_i}$ for all $i \geq n$.

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists an $\mathcal{F}$-complete resolution $T \xrightarrow{\tau} P \xrightarrow{\pi} X$ such that $\tau_n : T_n \to P_n : \pi$ is an isomorphism of complexes. This yields isomorphisms $H_i(X) \cong H_i(P)$ for all $i \in \mathbb{Z}$, $H_i(P) \cong H_i(T)$ for all $i > n$, and $C_n(P) \cong C_n(T)$. Since the complex $T$ is totally $\mathcal{F}$-acyclic, we have $H_i(T) = 0$ for each $i \in \mathbb{Z}$ and $C_n(T)$ is Ding projective.

(2) $\Rightarrow$ (3) Let $P' \to X$ be a DG-projective resolution. Then $P \simeq P'$. Since $P'$ is DG-projective, there exists a quasi-isomorphism $P' \to P$. We can assume that $P' \to P$ is a surjective quasi-isomorphism (if not, let $F \to P$ be surjective with $F$ a projective complex, then $F \oplus P' \to P$ is a surjective quasi-isomorphism). Hence there exists an exact sequence

$$0 \to K \to P' \to P \to 0$$
with $K$ an exact complex. Both $P'$ and $P$ are DG-projective complexes, so $K$ is a DG-projective complex. Thus $K$ is exact and DG-projective, and so $K$ is a projective complex. In addition, we have an exact sequence

$$0 \rightarrow C_n(K) \rightarrow C_n(P') \rightarrow C_n(P) \rightarrow 0.$$ 

By (2), we get that $C_n(P)$ is Ding projective. But $C_n(K)$ projective, and so $C_n(K)$ is Ding projective. It follows that $C_n(P')$ is Ding projective by Lemma 2.4.

(3) $\Rightarrow$ (4) Let $P' \rightarrow X$ be a DG-projective resolution with $C_n(P')$ Ding projective and $H_i(P') = 0$ for all $i > n$. Then $\Sigma^{-n}P'_n \oplus C_n(P')$ is a projective resolution. By Lemma 2.6, there is a totally $F$-acyclic complex $T''$ such that $T''_n \oplus = P'_n \oplus$. So we obtain an $F$-complete resolution $T'' \rightarrowtail P' \rightarrowtail X$ with $\tau_i'' = \text{id}\tau''_i$ for all $i \geq n$ and $C_n(T'') \cong C_n(P')$ by Lemma 2.2. From Lemma 2.9, we get a surjective $F$-complete resolution $T' \rightarrowtail P' \rightarrowtail X$ with the desired properties.

(4) $\Rightarrow$ (1) is clear. $\square$

**Corollary 2.13.** For every family of complexes of $R$-modules $(X_i)_{i \in I}$ one has \n
$$\text{Dpd}_R(\oplus_i X_i) = \sup\{\text{Dpd}_R(X_i)|i \in I\}.$$ 

**Proof.** For each $i \in I$, there is a DG-projective resolution $P_i \rightarrowtail X_i$. Set $P = \oplus_i P_i$. Then $P \rightarrowtail \oplus_i X_i$ is a DG-projective resolution and $C_n(P) = \oplus_i C_n(P_i)$ for each $n \in \mathbb{Z}$. Thus the assertion follows from Theorem 2.12 and Lemma 2.7. $\square$

**Corollary 2.14.** Let $M$ be an $R$-module. Then $\text{Dpd}_R(S^0(M)) = \text{Dpd}(M)$.

**Proof.** Let \n
$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$ 

be a projective resolution of $M$. Then $P \rightarrowtail S^0(M)$ is a DG-projective resolution, where $P = \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$. If $\text{Dpd}(M) = \infty$, and $\text{Dpd}_R(S^0(M)) = l < \infty$, then $C_j(P)$ is Ding projective for any $j \geq l$ by Theorem 2.12. Since \n
$$0 \rightarrow C_i(P) \rightarrowtail P_{i-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is exact with $C_i(P)$ Ding projective and $P_j$ projective modules, it follows that $\text{Dpd}(M) \leq l$. This contradicts $\text{Dpd}(M) = \infty$. So $\text{Dpd}_R(S^0(M)) = \infty$. If $\text{Dpd}(M) = l < \infty$, then $C_l(P)$ is Ding projective, and so $C_j(P)$ is Ding projective for all $j \geq l$ by Lemma 2.4. Hence $P \rightarrowtail S^0(M)$ is a DG-projective resolution with $C_j(P)$ Ding projective and $H_j(P) = 0$ for all $j \geq l$. By Theorem 2.12, $\text{Dpd}_R(S^0(M)) \leq l$. Suppose that $\text{Dpd}_R(S^0(M)) \leq l - 1$. Then $C_{l-1}(P)$ is Ding projective. In the exact sequence \n
$$0 \rightarrow C_{l-1}(P) \rightarrowtail P_{l-2} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

$C_{l-1}(P)$ is Ding projective and every $P_j$ is projective, which yields $\text{Dpd}(M) \leq l - 1$. This contradicts $\text{Dpd}(M) = l$. Therefore, $\text{Dpd}_R(S^0(M)) = l$. $\square$

**Corollary 2.15.** Let $X$ be a complex. Then $\text{Gpd}_R(X) \leq \text{Dpd}_R(X) \leq \text{pd}_R(X)$, with equalities if $\text{pd}_R(X)$ is finite.

**Proof.** If $\text{pd}_R(X) = \infty$, then it is clear. If $\text{pd}_R(X) = -\infty$, then $X$ is exact, so $\text{Gpd}_R(X) = \text{Dpd}_R(X) = -\infty$. Let $\text{pd}_R(X) = g < \infty$. Then for any DG-projective resolution $P \rightarrowtail X$ we have $\sup P \leq g$ and $C_j(P)$ is projective for all $j \geq g$. So $C_j(P)$ is Ding projective for all $j \geq g$. 

By Theorem 2.12, we get $\text{Dpd}_R(X) \leq g$. In similar method, we obtain that $\text{Gpd}_R(X) \leq k$ if $\text{Dpd}_R(X) = k$. The final assertion follows from Theorem 3.7 in [V].

**Proposition 2.16.** Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of complexes. If two complexes have finite Ding projective dimension, then so does the third.

**Proof.** By [V, Proposition 1.3.8], there is an exact sequence of complexes $0 \to P^X \to P^Y \to P^Z \to 0$ with $P^X \to X$, $P^Y \to Y$ and $P^Z \to Z$ DG-projective resolutions. If two of the complexes $X, Y, Z$ have finite Ding projective dimension, then there is $n \in \mathbb{Z}$ such that $H_j(P^X) = H_j(P^Y) = H_j(P^Z) = 0$ for all $j \geq n$. For each $j \geq n$, we have an exact sequence

$$0 \to C_j(P^X) \to C_j(P^Y) \to C_j(P^Z) \to 0$$

in $R$-$\text{Mod}$. If $C_j(P^Z)$ is Ding projective, then $C_j(P^X)$ is Ding projective if and only if $C_j(P^Y)$ is Ding projective. If $C_j(P^X)$ and $C_j(P^Y)$ are Ding projective, then $\text{Dpd}(C_j(P^Z)) \leq 1$, and so $C_{j+1}(P^Z)$ is Ding projective by Lemma 2.7. Therefore the assertion follows from Theorem 2.12. □

**Proposition 2.17.** Let $X$ be a homologically bounded below complex. Then

$$\text{Dpd}_R(X) = \inf \left\{ \sup \left\{ l \in \mathbb{Z} \mid Q_l \neq 0 \right\} \left| Q \simeq X \text{ and } Q \text{ is a bounded below complex of Ding projective modules} \right. \right\}.$$ 

**Proof.** Since $X$ is a homologically bounded below complex, we can assume $\inf X = 0$. Let $P \to X$ be a DG-projective resolution of $X$ with $\inf\{ l \mid P_l \neq 0 \} = 0$. Set

$$\Omega = \inf \left\{ \sup \left\{ l \in \mathbb{Z} \mid Q_l \neq 0 \right\} \left| Q \simeq X \text{ and } Q \text{ is a bounded below complex of Ding projective modules} \right. \right\}.$$ 

If $\text{Dpd}_R(X) = n$, then $C_j(P)$ is Ding projective for every $j \geq n$ by Theorem 2.12. Let

$$P' = 0 \to C_n(P) \to P_{n-1} \to \cdots \to P_0 \to 0,$$

and

$$X' = 0 \to C_n(X) \to X_{n-1} \to X_{n-2} \to \cdots.$$ 

Since $P \simeq X$, we have $P \simeq X'$. From $P' \simeq X'$ and $X \simeq X'$, we get $P' \simeq X$. Each component of $P'$ is a Ding projective module, which implies that $\Omega \leq n$.

Now suppose that $\Omega = m < \infty$. We are going to show that $\text{Dpd}_R(X) \leq m$. By hypothesis, there exists a complex

$$Q = 0 \to Q_m \to Q_{m-1} \to \cdots \to Q_0 \to 0$$

of Ding projective modules such that $Q \simeq X$. Since $P \simeq X \simeq Q$ and $P$ is a DG-projective complex, there is a quasi-isomorphism $P \to Q$. In addition, $Q$ is bounded below, so there is a surjective morphism $P^* \to Q$ with $P^*$ a bounded below projective complex. Then $P \oplus P^* \to Q$ is a surjective quasi-isomorphism. Thus we have an exact sequence

$$0 \to K \to P \oplus P^* \to Q \to 0,$$

with $K$ exact, which implies that there is an exact sequence

$$0 \to K_j \to P_j \oplus P_j^* \to Q_j \to 0.$$
In particular, the induced morphisms of complexes of $A$ implies that a totally projective module $F \in P$ are quasi-isomorphisms. By Lemma 2.18(2) it follows that $\text{Hom}_R(V, W)$ and each module in $C(R)$ is a quasi-isomorphism. Quasi-isomorphism of Ding projective modules, then the induced morphism in $C(R)$ is a Ding projective module. Since $P \rightarrow X$ is a DG-projective resolution with sup $P \leq m$ and $C_j(P)$ is Ding projective for all $j \geq m$, it follows that $\text{Dpd}_R(X) \leq m$. By the above, $\text{Dpd}_R(X) = \infty$ if and only if $\Omega = \infty$; and note that $\text{Dpd}_R(X) = -\infty$ if and only if $\Omega = -\infty$.

Lemma 2.18. (1) ([CFH, Proposition 2.6(a)]) Let $U$ be a class of $R$-modules, and $\alpha : X \rightarrow Y$ be a morphism in $C(R)$ such that

$$\text{Hom}_R(U, \alpha) : \text{Hom}_R(U, X) \xrightarrow{\cong} \text{Hom}_R(U, Y)$$

is a quasi-isomorphism for every module $U \in U$. If $\overline{U} \in C_\omega(R)$ is a complex consisting of modules from $U$, then the induced morphism $\text{Hom}_R(U, \alpha) : \text{Hom}_R(\overline{U}, X) \xrightarrow{\cong} \text{Hom}_R(\overline{U}, Y)$ is a quasi-isomorphism.

(2) ([CFH, Proposition 2.7(a)]) Let $V$ be a class of $R$-modules, and $\alpha : X \rightarrow Y$ be a morphism in $C(R)$ such that

$$\text{Hom}_R(\alpha, V) : \text{Hom}_R(X, V) \xrightarrow{\cong} \text{Hom}_R(X, V)$$

is a quasi-isomorphism for every module $V \in V$. If $\overline{V} \in C_\omega(R)$ is a complex consisting of modules from $U$, then the induced morphism $\text{Hom}_R(\alpha, \overline{V}) : \text{Hom}_R(X, \overline{V}) \xrightarrow{\cong} \text{Hom}_R(Y, \overline{V})$ is a quasi-isomorphism.

Lemma 2.19. Let $V \xrightarrow{\cong} W$ be a quasi-isomorphism between $R$-complexes, where $V, W \in C_\omega(R)$ and each module in $V$ and $W$ has finite flat or finite injective dimension. If $A \in C_\omega(R)$ is a complex of Ding projective modules, then the induced morphism $\text{Hom}_R(A, V) \rightarrow \text{Hom}_R(A, W)$ is a quasi-isomorphism.

Proof. By Lemma 2.18(1), we may immediately reduce to the case, where $A$ is a Ding projective module. In this case we have quasi-isomorphisms $\alpha : P \xrightarrow{\cong} A$ and $\beta : A \xrightarrow{\cong} \overline{P}$ in $C(R)$, where $P \in C_\omega(R)$ and $\overline{P} \in C_\omega(R)$ are respectively the left half and right half of a totally $\mathcal{F}$-acyclic complex of $A$. Let $T$ be any $R$-module of finite flat or finite injective dimension. Lemma 2.4(1) implies that a totally $\mathcal{F}$-acyclic complex stays exact when the functor $\text{Hom}_R(\cdot, T)$ is applied to it. In particular, the induced morphisms

$$\text{Hom}_R(\alpha, T) : \text{Hom}_R(A, T) \xrightarrow{\cong} \text{Hom}_R(P, T),$$

and

$$\text{Hom}_R(\beta, T) : \text{Hom}_R(\overline{P}, T) \xrightarrow{\cong} \text{Hom}_R(A, T)$$

are quasi-isomorphisms. By Lemma 2.18(2) it follows that $\text{Hom}_R(\alpha, V)$ and $\text{Hom}_R(\alpha, W)$ are quasi-isomorphisms. In the commutative diagram

$$\text{Hom}_R(A, V) \xrightarrow{\cong} \text{Hom}_R(A, W)$$

$$\text{Hom}_R(\alpha, V) \approx \approx \text{Hom}_R(\alpha, V)$$

$$\text{Hom}_R(P, V) \xrightarrow{\cong} \text{Hom}_R(P, W)$$
the lower horizontal morphism is obviously a quasi-isomorphism, and this makes the induced morphism $\text{Hom}_R(A, V) \to \text{Hom}_R(A, W)$ a quasi-isomorphism as well. □

**Lemma 2.20.** If $X \simeq A$, where $A \in C_{\square}(R)$ is a complex of Ding projective modules, and $U \simeq V$, where $V \in C_{\square}(R)$ is a complex in which each module has finite flat or finite injective dimension, then

$$\text{RHom}_R(X, U) \simeq \text{Hom}_R(A, V).$$

**Proof.** Assume that $V \xrightarrow{\sim} I \in C_{\square}(R)$ is a DG-injective resolution of $V$. We have

$$\text{RHom}_R(X, U) \simeq \text{RHom}_R(A, V) \simeq \text{Hom}_R(A, I).$$

From Lemma 2.19 we get a quasi-isomorphism $\text{Hom}_R(A, V) \xrightarrow{\sim} \text{Hom}_R(A, I)$, and the result follows. □

**Lemma 2.21.** Let $F$ be a flat $R$-module. If $X \simeq A$, where $X \in C_{\square}(R)$ and $A \in C_{\square}(R)$ is a complex of Ding projective modules and $n \geq \sup X$, then

$$\text{Ext}^m_R(C^A_n, F) = H_{-(m+n)}(\text{RHom}_R(X, F)).$$

**Proof.** Since $n \geq \sup X = \sup A$ we have $A_n \sqsubseteq \simeq \Sigma^n C^A_n$, and since $F$ is flat it follows by Lemma 2.20 that $\text{RHom}_R(C^A_n, F)$ is represented by $\text{Hom}_R(\Sigma^{-n}A_n \sqsubseteq, F)$. For $m > 0$ the isomorphism class $\text{Ext}^m_R(C^A_n, F)$ is then represented by

$$H_{-n}(\text{Hom}_R(\Sigma^{-n}A_n \sqsubseteq, F)) = H_{-n}(\Sigma^n \text{Hom}_R(A_n \sqsubseteq, F))$$

$$= H_{-(m+n)}(\text{Hom}_R(A_n \sqsubseteq, F))$$

$$= H_{-(m+n)}(\Sigma^{-n} \text{Hom}_R(A, F))$$

$$= H_{-(m+n)}(\text{Hom}_R(A, F)).$$

It also follows from Lemma 2.20 that the complex $\text{Hom}_R(A, F)$ represents $\text{RHom}_R(X, F)$, so

$$\text{Ext}^m_R(C^A_n, F) = H_{-(m+n)}(\text{RHom}_R(X, F)).$$

□

**Theorem 2.22.** Let $X \in C_{\square}(R)$ of finite Ding projective dimension. For $n \in \mathbb{Z}$ the following are equivalent:

1. $\text{Dpd}_R(X) \leq n$.
2. $\inf U - \inf \text{RHom}_R(X, U) \leq n$ for all $U \in C_{\square}(R)$ of finite flat dimension with $H(U) \neq 0$.
3. $-\inf \text{RHom}_R(X, F) \leq n$ for all flat $R$-modules $F$.
4. $\sup X \leq n$ and for any bounded below complex $A \simeq X$ of Ding projective modules, the cokernel $C^A_n = \text{Coker}(A_{n+1} \to A_n)$ is a Ding projective module.

Moreover, the following hold:

$$\text{Dpd}_R(X) = \sup \{\inf U - \inf \text{RHom}_R(X, U) \mid \text{fd}_RU < \infty \text{ and } H(U) \neq 0\}$$

$$= \sup \{-\inf \text{RHom}_R(X, F) \mid F \text{ is flat}\}.$$

**Proof.** The proof of the equivalence of (1)-(4) is cyclic. Obviously (2) ⇒ (3). So this leaves us three implications to prove.

1. (1) ⇒ (2) Choose a complex $A \in C_{\square}(R)$ consisting of Ding projective modules, such that $A \simeq X$ and $A_l = 0$ for all $l > n$. First let $U$ be a complex of finite flat dimension with $H(U) \neq 0$. Set $i = \inf U$ and note that $i \in \mathbb{Z}$ as $U \in C_{\square}(R)$ with $H(U) \neq 0$. Choose a bounded complex $F \simeq U$
of flat modules with \( F_l \neq 0 \) for \( l < i \). By Lemma 2.20, the complex \( \text{Hom}_R(A,F) \) is equivalent to \( \text{RHom}_R(X,U) \); in particular, \( \inf \text{RHom}_R(X,U) = \inf \text{Hom}_R(A,F) \). For \( l < i - n \) and \( q \in \mathbb{Z} \), either \( q > n \) or \( q + l \leq n + l < i \), so the module

\[
\text{Hom}_R(A,F)_l = \prod_{q \in \mathbb{Z}} \text{Hom}_R(A_q,F_{q+l}) = 0.
\]

Hence, \( \text{H}_l(\text{Hom}_R(A,F)) = 0 \) for \( l < i - n \), and \( \inf \text{RHom}_R(X,U) \geq i - n = \inf U - n \) as desired.

(3) \( \Rightarrow \) (4) This part is divided into three steps. First we establish the inequality \( n \geq \sup X \), next we prove that the \( n \)th cokernel in a bounded complex \( A \simeq X \) of Ding projective modules is again Ding projective, and finally we give an argument that allows us to conclude the same for \( A \in C_{\cap}(R) \).

To see that \( n \geq \sup X \), it is sufficient to show that

\[
(\ast) \quad \sup\{-\inf \text{RHom}_R(X,F) | F \text{ is flat}\} \geq \sup X.
\]

By assumption, \( g = \text{Dpd}_R X \) is finite; That is, \( X \simeq A \) for some complex

\[
A = 0 \rightarrow A_g \xrightarrow{\delta^A_g} A_{g-1} \rightarrow \cdots \rightarrow A_i \rightarrow 0,
\]

and it is clear \( g \geq \sup X \) since \( X \simeq A \). For any flat module \( F \), the complex \( \text{Hom}_R(A,F) \) is concentrated in degrees \(-i \) to \(-g\).

\[
0 \rightarrow \text{Hom}_R(A_i,F) \rightarrow \cdots \rightarrow \text{Hom}_R(A_{g-1},F) \xrightarrow{\text{Hom}_R(\delta^A_{g-1},F)} \text{Hom}_R(A_g,F) \rightarrow 0.
\]

By Lemma 2.20, \( \text{Hom}_R(A,F) \) is equivalent to \( \text{RHom}_R(X,F) \) in \( C(\mathbb{Z}) \). First, consider the case \( g = \sup X \): The differential \( \delta^A_g : A_g \rightarrow A_{g-1} \) is not injective, as \( A \) has homology in degree \( g = \sup X = \sup A \). By the definition of Ding projective modules, there exists a projective (and so flat) module \( F \) and an injective homomorphism \( \varphi : A_g \rightarrow F \). Because \( \delta^A_g \) is not injective, \( \varphi \in \text{Hom}_R(A_g,F) \) cannot have the form \( \text{Hom}_R(\delta^A_g,F)(\psi) = \psi \delta^A_g \) for some \( \psi \in \text{Hom}_R(A_{g-1},F) \). That is, the differential \( \text{Hom}_R(\delta^A_g,F) \) is not surjective; Hence \( \text{Hom}_R(A,F) \) has nonzero homology in degree \(-g = -\sup X\), and (\( \ast \)) follows.

Next, assume that \( g > \sup X = s \) and consider the exact sequence

\[
0 \rightarrow A_g \rightarrow \cdots \rightarrow A_{s+1} \rightarrow A_s \rightarrow A_s \rightarrow C_s^A \rightarrow 0.
\]

It shows that \( \text{Dpd}_R C_s^A \leq g - s \), and it is easy to check that equality must hold; otherwise, we would have \( \text{Dpd}_R X < g \) by 2.17. By Lemma 2.21, it follows that for all \( m > 0 \), all \( n \geq \sup X \), and all flat modules \( F \) one has

\[
(\ast) \quad \text{Ext}_R^m(C_n^A,F) = H_{-(m+n)}(\text{RHom}_R(X,F)).
\]

By Lemma 2.7, we have \( \text{Ext}_R^{g-s}(C_s^A,F) \neq 0 \) for some flat \( F \), whence \( H_{-g}(\text{RHom}_R(X,F)) \neq 0 \) by (\( \ast \)) and (\( \ast \)) follows. We conclude that \( n \geq \sup X \).

It remains to prove that \( C_n^A \) is Ding projective for any bounded below complex \( A \simeq X \) of Ding projective modules. By assumption, \( \text{Dpd}_X Y \) is finite, so a bounded complex \( \widetilde{A} \simeq X \) of Ding projective modules does exist. Consider the cokernel \( C_n^A \). Since \( n \geq \sup X = \sup \widetilde{A} \), it fits in an exact sequence \( 0 \rightarrow \widetilde{A}_l \rightarrow \cdots \rightarrow \widetilde{A}_{n+1} \rightarrow \widetilde{A}_n \rightarrow C_n^A \rightarrow 0 \), where all the \( \widetilde{A}_l \)'s are Ding projective. By (\( \ast \)) and Lemma 2.4(3), it now follows that also \( C_n^A \) is Ding projective. With this, it is sufficient to prove the following:
If \( P, A \in C_{\triangle}(R) \) are complexes of, respectively, projective and Ding projective modules, and \( P \simeq X \simeq A \), then the cokernel \( C_n^P \) is Ding projective if and only if \( C_n^A \) is so.

Let \( A \) and \( P \) be two such complexes. As \( P \) consists of projective modules, there is a quasi-isomorphism \( \pi : P \xrightarrow{\cong} A \), which induces a quasi-isomorphism between the truncated complexes, \( \pi_n : \pi \circ \tau_n P \xrightarrow{\cong} \tau_n A \). The mapping cone

\[
\text{Cone}(\pi_n) = 0 \rightarrow C_n^P \rightarrow P_{n-1} \oplus C_n^A \rightarrow P_{n-2} \oplus A_{n-1} \rightarrow \cdots
\]
is a bounded exact complex, in which all modules but the two left-most ones are known to be Ding projective modules. It follows by the resolving properties of the class of Ding projective modules that \( C_n^P \) is Ding projective if and only if \( P_{n-1} \oplus C_n^A \) is so, which is equivalent to \( C_n^A \) being Ding projective.

(4) \Rightarrow (1) Choose a DG-projective resolution \( P \) of \( X \), by (4) the truncation \( \pi_n P \) is a complex of the desired type.

The last claim are immediate consequences of the equivalence of (1), (2) and (3).

\[ \square \]

**Corollary 2.23.** Let \( X \in C_{\triangle}(R) \) of finite Ding projective dimension. Then \( \text{Gpd}_R(X) = \text{Dpd}_R(X) \).

**Proof.** It follows from Theorem 2.22 and [CFH, Theorem 3.1]

\[ \square \]

**Lemma 2.24.** Let \( \varphi : R \rightarrow S \) be a homomorphism of rings.

(1) If \( M \) is a Ding projective \( S \)-module, then \( \text{Hom}_R(\widetilde{P}, M) \) is a Ding projective \( S \)-module for every finite projective \( R \)-module \( \widetilde{P} \).

(2) If \( M \) is a Ding projective \( S \)-module, then \( \widetilde{P} \otimes_R M \) is a Ding projective \( S \)-module for every projective \( R \)-module \( \widetilde{P} \).

**Proof.** (1) Let \( T \) be a totally \( \mathcal{F} \)-acyclic complex of \( M \). Then the complex \( \text{Hom}_R(\widetilde{P}, T) \) of projective \( S \)-modules is exact. For any flat \( S \)-module \( Q \) we have

\[
\text{Hom}_S(\text{Hom}_R(\widetilde{P}, T), Q) \cong \widetilde{P} \otimes_R \text{Hom}_S(T, Q).
\]

Since \( \text{Hom}_S(T, Q) \) is exact, and \( \widetilde{P} \) is finite projective, we obtain that \( \text{Hom}_S(\text{Hom}_R(\widetilde{P}, T), Q) \) is exact, and so \( \text{Hom}_R(\widetilde{P}, T) \) is a totally \( \mathcal{F} \)-acyclic complex of \( \text{Hom}_R(\widetilde{P}, M) \).

(2) Let \( T \) be a totally \( \mathcal{F} \)-acyclic complex of \( M \). Then the complex \( \widetilde{P} \otimes_R T \) of projective \( S \)-modules is exact. For any flat \( S \)-module \( Q \) we have the exactness of

\[
\text{Hom}_S(\widetilde{P} \otimes_R T, Q) \cong \text{Hom}_R(\widetilde{P}, \text{Hom}_S(T, Q))
\]
as \( \text{Hom}_S(T, Q) \) is exact. Hence \( \widetilde{P} \otimes_R T \) is a totally \( \mathcal{F} \)-acyclic complex of \( \widetilde{P} \otimes_R M \).

\[ \square \]

**Theorem 2.25.** Let \( \varphi : R \rightarrow S \) be a homomorphism of rings, \( X \in C_{\triangle}(S) \).

(1) If \( U \in P^{(i)}(R) \), then \( \text{Dpd}_S(\text{RHom}_R(U, X)) \leq \text{Dpd}_S(X) - \inf U \).

(2) If \( U \in P(R) \), then \( \text{Dpd}_S(U \otimes_R f^* X) \leq \text{Dpd}_S(X) + \text{pd}_R(U) \).

**Proof.** (1) We can assume that \( U \) is not exact, otherwise the inequality is trivial; and we set \( i = \inf U, \text{pd}_R(U) = n \). The inequality is also trivial if \( X \) is exact or not of finite Ding projective dimension, so we assume that \( X \) is not exact and set \( \text{Dpd}_R(X) = g \). We can now choose a complex \( A \in C_{\triangle}(S) \) of Ding projective modules which is equivalent to \( X \) and has \( A_l = 0 \) for \( l > g \); we set \( v = \inf \{ l \in \mathbb{Z} \mid A_l \neq 0 \} \). Since \( U \in P^{(i)}(R) \), \( U \) is equivalent to a complex \( P \) of finite projective
Let \( R \) be a subring of the ring \( S \), and assume that \( R \) and \( S \) have the same unity \( 1 \). The ring \( S \) is called an excellent extension of \( R \) if

(A) \( S \) is a free normalizing extension of \( R \) with a basis that includes \( 1 \); that is, there is a finite subset \( \{a_1, \ldots, a_n\} \) of \( S \) such that \( a_1 = 1 \), \( S = \sum_{i=1}^{n} a_i R \) and \( a_i R = Ra_i \) for all \( i = 1, \ldots, n \) and \( S \) is free with basis \( \{a_1, \ldots, a_n\} \) both as a right and left \( R \)-module, and

(B) \( S \) is \( R \)-projective; that is, if \( SN \) is a submodule of \( SM \), then \( RN \) is a left \( R \)-module for every \( SM \).

Excellent extensions were introduced by Passman [P1]. Examples include \( n \times n \) matrix rings, and crossed products \( R \ast G \) where \( G \) is a finite group with \( |G|^{-1} \in R \) [P2].

**Lemma 2.26.** Let \( S \) be an excellent extension of \( R \).

1. If \( N \) is a Ding projective \( R \)-module, then \( \text{Hom}_R(\hat{P}, N) \) is a Ding projective \( S \)-module for every finite projective \( S \)-module \( \hat{P} \).

2. If \( N \) is a Ding projective \( R \)-module, then \( \hat{P} \otimes_R N \) is a Ding projective \( S \)-module for every projective \( S \)-module \( \hat{P} \).

**Proof.** (1) Let \( T \) be a totally \( F \)-acyclic complex of \( N \). Then the complex \( \text{Hom}_R(\hat{P}, T) \) consists of projective \( S \)-modules, and it is exact as \( \hat{P} \) is a finite projective \( R \)-module. For any flat \( S \)-module \( Q \) we have

\[
\text{Hom}_S(\text{Hom}_R(\hat{P}, T), Q) \cong \hat{P} \otimes_R \text{Hom}_S(T, Q),
\]
which is exact as $Q$ is a flat $R$-module. Hence $\text{Hom}_R(\tilde{P}, T)$ is a totally $\mathcal{F}$-acyclic complex of $\text{Hom}_R(\tilde{P}, N)$.

(2) Let $T$ be a totally $\mathcal{F}$-acyclic complex of $N$. Then the complex $\tilde{P} \otimes_R T$ consists of projective $S$-modules, and it is exact as $\tilde{P}$ is a projective $R$-module. For any flat $S$-module $Q$, we have

$$\text{Hom}_S(\tilde{P} \otimes_R T, Q) \cong \text{Hom}_R(\tilde{P}, \text{Hom}_S(T, Q)),$$

which is exact as $Q$ is a flat $R$-module. Hence $\tilde{P} \otimes_R T$ is a totally $\mathcal{F}$-acyclic complex of $\tilde{P} \otimes_R N$. □

Theorem 2.27. Let $S$ be an excellent extension of $R$, $X \in C_{(\to)}(R)$.

(1) If $V \in \mathcal{P}^{(f)}(S)$, then $\text{Dpd}_S(\text{RHom}_R(V, X)) \leq \text{Dpd}_R(X) - \inf V$.

(2) If $V \in \mathcal{P}(S)$, then $\text{Dpd}_S(V \otimes^L_R X) \leq \text{Dpd}_R(X) + \text{pd}_S(V)$.

Proof. They follow by analogy with the proof of Theorem 2.25, only this time use Lemma 2.26. □

Corollary 2.28. Let $S$ be an excellent extension of $R$, $X \in C_{(\to)}(R)$. Then

(1) $\text{Dpd}_S(\text{RHom}_R(S, X)) \leq \text{Dpd}_R(X)$.

(2) $\text{Dpd}_S(S \otimes^L_R X) \leq \text{Dpd}_R(X)$.

In [MD], Mao and Ding introduced and studied Gorenstein FP-injective modules, and showed that there is a very close relationship between Gorenstein FP-injective modules and Gorenstein flat modules. A left $R$-module $N$ is called Gorenstein FP-injective if there is an exact sequence

$$\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow E_{-2} \rightarrow \cdots$$

of injective left $R$-modules with $N = \text{Coker}(E_0 \rightarrow E_{-1})$ such that $\text{Hom}(E, -)$ leaves the sequence exact whenever $E$ an FP-injective $R$-module. Since Gorenstein FP-injective modules have properties analogous to Gorenstein injective modules, Gillespie [Gi] called these modules Ding injective modules.

Remark 2.29. Above we have only mentioned the Ding projective dimension of $R$-complexes. Dually one can also define and study the Ding injective dimension for complexes of left $R$-modules over an associative ring $R$. All the results concerning Ding projective dimension have a Ding injective counterpart.

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