OPTIMAL $L^2$ EXTENSION OF SECTIONS FROM SUBVARIETIES IN WEAKLY PSEUDOCONVEX MANIFOLDS

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ABSTRACT. In this paper, we obtain optimal $L^2$ extension of holomorphic sections of a holomorphic vector bundle from subvarieties in weakly pseudoconvex Kähler manifolds. Moreover, in the case of line bundle the Hermitian metric is allowed to be singular.

1. INTRODUCTION AND MAIN RESULTS

The $L^2$ extension problem is an important topic in several complex variables and complex geometry. Many generalizations and applications have been obtained since the original work of Ohsawa and Takegoshi ([25]). A recent progress is about the optimal $L^2$ extension and its applications.

Most recently, several general $L^2$ extension theorems with optimal estimates were proved in [14] for holomorphic sections defined on subvarieties in Stein or projective manifolds. In [11], several $L^2$ extension theorems were obtained for holomorphic sections defined on subvarieties in weakly pseudoconvex Kähler manifolds.

In this paper, we prove an optimal $L^2$ extension theorem, which generalizes the main theorems in [14] to weakly pseudoconvex Kähler manifolds and optimizes a main theorem in [11] (cf. Theorem 2.8 and Remark 2.9 in [11]).

Let us recall some definitions in [11].

Definition 1.1. A function $\psi : X \rightarrow [-\infty, +\infty)$ on a complex manifold $X$ is said to be quasi-plurisubharmonic if $\psi$ is locally the sum of a plurisubharmonic function and a smooth function. In addition, we say that $\psi$ has neat analytic singularities if every point $x \in X$ possesses an open neighborhood $U$ on which $\psi$ can be written as

$$\psi = c \log \sum_{1 \leq j \leq J_0} |g_j|^2 + u,$$

where $c$ is a nonnegative number, $g_j \in \mathcal{O}_X(U)$ and $u \in C^\infty(U)$.

Definition 1.2. If $\psi$ is a quasi-plurisubharmonic function on a complex manifold $X$, the multiplier ideal sheaf $\mathcal{I}(\psi)$ is the coherent analytic subsheaf of $\mathcal{O}_X$ defined by

$$\mathcal{I}(\psi)_x = \{ f \in \mathcal{O}_{X,x} : \exists U \ni x, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \}.$$
where $U$ is an open coordinate neighborhood of $x$, and $d\lambda$ is the Lebesgue measure in the corresponding open chart of $\mathbb{C}^n$. We say that the singularities of $\psi$ are log canonical along the zero variety $Y = V(\mathcal{I}(\psi))$ if $\mathcal{I}((1-\varepsilon)\psi)|_Y = \mathcal{O}_X|_Y$ for every $\varepsilon > 0$.

If $\omega$ is a Kähler metric on $X$, we let $dV_{X,\omega} := \frac{\omega^n}{n!}$ be the corresponding Kähler volume element, where $n = \dim X$. In case $\psi$ has log canonical singularities along $Y = V(\mathcal{I}(\psi))$, one can associate in a natural way a measure $dV_{X,\omega}[\psi]$ on the set $Y^0 = Y_{\text{reg}}$ of regular points of $Y$ as follows.

**Definition 1.3.** If $g \in C_r(Y^0)$ is a compactly supported nonnegative continuous function on $Y^0$ and $\bar{g}$ is a compactly supported nonnegative continuous extension of $g$ to $X$ such that $(\text{supp} \bar{g}) \cap Y \subset Y^0$, then we set

$$
\int_{Y^0} g dV_{X,\omega}[\psi] = \lim_{t \to -\infty} \int_{\{x \in X : t < \psi(x) < t+1\}} \bar{g} e^{-\psi} dV_{X,\omega}.
$$

**Remark 1.1.** By Hironaka’s desingularization theorem 2.7, it is not hard to see that the limit in the above definition does not depend on the extension $\bar{g}$ and then $dV_{X,\omega}[\psi]$ is well defined on $Y^0$ (see Proposition 4.5 in [11] for a proof).

**Remark 1.2.** The definition of $dV_{X,\omega}[\psi]$ here has a slight difference with the one in [14]. In fact, if we denote the measure in [14] by $d\tilde{V}_{X,\omega}[\psi]$, the integral $\int_{Y^0} g dV_{X,\omega}[\psi]$ here is equal to

$$
\sum_{1 \leq j \leq n} \frac{\pi^{j}}{j!} \int_{Y_{n-j}} g d\tilde{V}_{X,\omega}[\psi],
$$

where $Y_{n-j}$ is the $(n-j)$-dimensional component of $Y_{\text{reg}}$.

We will define a class of functions before the statement of our main theorem.

**Definition 1.4.** Let $\alpha_0 \in (-\infty, +\infty]$ and $\alpha_1 \in [0, +\infty)$. When $\alpha_0 \neq +\infty$, let $\mathcal{R}_{\alpha_0, \alpha_1}$ be the class of functions defined by

$$
\{ R \in C^\infty(-\infty, \alpha_0) : R > 0, \text{ R is decreasing near } -\infty, \text{ and } \lim_{t \to -\infty} e^t R(t) < +\infty, \quad C_R := \int_{-\infty}^{\alpha_0} \frac{1}{R(t)} dt < +\infty \text{ and } \quad \int_{t_1}^{\alpha_0} \left( \frac{\alpha_1}{R(\alpha_0)} + \int_{t_2}^{\alpha_0} \frac{dt_1}{R(t_1)} \right) dt_2 + \frac{(\alpha_1)^2}{(R(\alpha_0))^2} < \int_{t_1}^{\alpha_0} \frac{dt_1}{R(t_1)} \right)^2
$$

for all $t \in (-\infty, \alpha_0)$.

When $\alpha_0 = +\infty$, we replace $R \in C^\infty(\alpha_0, \alpha_0]$ with $R \in C^\infty(-\infty, +\infty)$ and $R(+\infty) := \lim_{t \to +\infty} R(t) \in (0, +\infty]$ in the above definition of $\mathcal{R}_{\alpha_0, \alpha_1}$.

**Remark 1.3.** The number $\alpha_0$, $\alpha_1$, and the function $R(t)$ are equal to the number $A$, $\frac{1}{\gamma}$ and the function $\frac{1}{cA(-t)^{\gamma}}$ defined just before the main theorems in [14]. If $\alpha_0 \neq +\infty$ and $R$ is decreasing on $(-\infty, \alpha_0]$, the longest inequality in the definition of $\mathcal{R}_{\alpha_0, \alpha_1}$ holds for all $t \in (-\infty, \alpha_0)$. If $\alpha_0 = +\infty$, the longest inequality in the definition of $\mathcal{R}_{\alpha_0, \alpha_1}$ implies that $\int_{t_1}^{\alpha_0} \frac{\alpha_1}{R(t_1)} dt_2 < +\infty$ for all $t \in (-\infty, +\infty)$. Therefore, $\frac{\alpha_1}{R(+\infty)} = 0$, i.e., $\alpha_1 = 0$ or $R(+\infty) = +\infty$. 
Theorem 1.1 (The main theorem). Let $R \in \mathcal{R}_{\alpha_0, \alpha_1}$. Let $(X, \omega)$ be a weakly pseudoconvex complex $n$-dimensional manifold possessing a Kähler metric $\omega$, and $\psi$ be a quasi-plurisubharmonic function on $X$ with neat analytic singularities. Let $Y$ be the analytic subvariety of $X$ defined by $Y = V(I(\psi))$ and assume that $\psi$ has log canonical singularities along $Y$. Let $L$ (resp. $E$) be a holomorphic line bundle (resp. a holomorphic vector bundle) over $X$ equipped with a singular Hermitian metric $h = h_L$ (resp. a smooth Hermitian metric $h = h_E$), which is written locally as $e^{-\phi_L}$ for some quasi-plurisubharmonic function $\phi_L$ with respect to a local holomorphic frame of $L$. Assume that 

(i) $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi$ is semi-positive on $X \setminus \{\psi = -\infty\}$ in the sense of currents (resp. in the sense of Nakano),

and that there is a continuous function $\alpha < \alpha_0$ on $X$ such that the following two assumptions hold:

(ii) $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi + \frac{1}{\chi(t)}\sqrt{-1}\partial\bar{\partial}\psi$ is semi-positive on $X \setminus \{\psi = -\infty\}$ in the sense of currents (resp. in the sense of Nakano),

(iii) $\psi \leq \alpha$,

where $\chi(t)$ is the function

$$
\frac{\int_{t}^{\alpha_0} \left( \frac{R(\alpha_0)}{R(\alpha_1)} + \frac{dt}{R(\alpha_1)} \right) dt}{\frac{\alpha_1}{R(\alpha_0)} + \frac{dt}{R(\alpha_1)}}.
$$

Then for every section $f \in H^0(Y^0, (K_X \otimes L)|_{Y^0})$ (resp. $f \in H^0(Y^0, (K_X \otimes E)|_{Y^0})$) on $Y^0 = Y_{\text{reg}}$ such that 

$$
\int_{Y^0} |f|^2_{\omega, h} dV_{X, \omega}[\psi] < +\infty,
$$

there exists a section $F \in H^0(X, K_X \otimes L)$ (resp. $F \in H^0(X, K_X \otimes E)$) such that $F = f$ on $Y^0$ and 

$$
\int_X \frac{|F|^2_{\omega, h}}{e^{-\phi_L}} dV_{X, \omega} \leq \left( \frac{\alpha_1}{R(\alpha_0)} + C_R \right) \int_{Y^0} |f|^2_{\omega, h} dV_{X, \omega}[\psi].
$$

Remark 1.4. The case of Theorem 1.1 when $X$ is Stein or projective was proved in [14] (see also Proposition 4.1 in [31] for a simplified version). Hence Theorem 1.1 can be regarded as a generalization of the main theorems in [14] to weakly pseudoconvex Kähler manifolds. Then it is easy to see from Remark 1.2 and the main theorems in [14] that the constant $\frac{\alpha_1}{R(\alpha_0)} + C_R$ in (1.3) is optimal. Hence Theorem 1.1 gives an optimal version of a main theorem in [11] (cf. Theorem 2.8 and Remark 2.9 in [11]).

Remark 1.5. In [31], Theorem 1.1 was proved for $L$ in the special case when $\psi = m \log |s|^2$, $\alpha_0 = \alpha_1 = 0$ and $R$ is decreasing on $(-\infty, 0]$ where $s$ is a global holomorphic section of some holomorphic vector bundle of rank $m$ over $X$ equipped with a smooth Hermitian metric and $s$ is transverse to the zero section. Similarly as in [31], a global plurisubharmonic negligible weight can be added to Theorem 1.1 by adding another regularization process to Step 2 in Section 4.

Remark 1.6. In order to deal with the singular metric $h_L$ on the weakly pseudoconvex Kähler manifold $X$, not only the regularization theorem 2.2 and the error term method of solving $\partial\bar{\partial}$ equations (Lemma 2.1) are needed, but also a limit problem
about $L^2$ integrals with singular weights needs to be solved. We solve the limit problem in Proposition 3.2. Then by using Proposition 3.1, Proposition 3.2 and the strong openness property of multiplier ideal sheaves (Theorem 2.6) as the key tools, we construct a family of smooth extensions of $f$ satisfying some uniform estimates, and overcome the difficulty in dealing with the singular metric (see also [31] for the special case).

The rest sections of this paper are organized as follows. First, we give some results used in the proof of Theorem 1.1 in Section 2. Then, we prove two key propositions in Section 3 which will be used to deal with the singular metric $h_L$. Finally, we prove Theorem 1.1 in Section 4 by using the results in Section 2 and Section 3.

2. Some results used in the proof of Theorem 1.1

In this section, we give some results which will be used in the proof of Theorem 1.1.

**Lemma 2.1** ([9], [11]). Let $(X, \omega)$ be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric $\omega$, and let $(Q, h)$ be a holomorphic vector bundle over $X$ equipped with a smooth Hermitian metric $h$. Assume that $\tau$ and $A$ are smooth and bounded positive functions on $X$ and let

$$B := [\tau \sqrt{-1} \Theta_{Q,h} - \sqrt{-1} \partial \bar{\partial} \tau - \sqrt{-1} A^{-1} \partial \tau \wedge \bar{\partial} \tau, A].$$

Assume that $\delta \geq 0$ is a nonnegative number such that $B + \delta I$ is semi-positive definite everywhere on $\Lambda^n Q$ for some $q \geq 1$. Then given a form $g \in L^2(X, \Lambda^n Q)$ such that $D^\omega g = 0$ and

$$\int_X \langle (B + \delta I)^{-1} g, g \rangle_{\omega,h} dV_X < +\infty,$$

there exists an approximate solution $u \in L^2(X, \Lambda^n Q)$ and a correcting term $v \in L^2(X, \Lambda^n Q)$ such that $D'' u + \sqrt{\rho} v = g$ and

$$\int_X \frac{|u|^2_{\omega,h}}{\tau + A} dV_X + \int_X |v|^2_{\omega,h} dV_X \leq \int_X \langle (B + \delta I)^{-1} g, g \rangle_{\omega,h} dV_X.$$

**Theorem 2.2** (Theorem 6.1 in [8]). Let $(X, \omega)$ be a complex manifold equipped with a Hermitian metric $\omega$, and $\Omega \subset X$ be an open subset. Assume that $T = \bar{T} + \frac{\omega}{n+1} \partial \bar{\partial} \varphi$ is a closed $(1,1)$-current on $X$, where $\bar{T}$ is a smooth real $(1,1)$-form and $\varphi$ is a quasi-plurisubharmonic function. Let $\gamma$ be a continuous real $(1,1)$-form such that $T \geq \gamma$. Suppose that the Chern curvature tensor of $T_X$ satisfies

$$(\sqrt{-1} \Theta_{T_X} + \omega \otimes \text{Id}_{T_X})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_X \text{ with } (\kappa_1, \kappa_2) = 0)$$

for some continuous nonnegative $(1,1)$-form $\omega$ on $X$. Then there is a family of closed $(1,1)$-currents $T_{\varphi, \rho} = \bar{T} + \frac{\rho}{n+1} \partial \bar{\partial} \varphi_{\varphi, \rho}$ defined on a neighborhood of $\Omega$ ($\varphi \in (0, +\infty)$ and $\rho \in (0, \rho_1)$ for some positive number $\rho_1$) independent of $\gamma$, such that

(i) $\varphi_{\varphi, \rho}$ is quasi-plurisubharmonic on a neighborhood of $\Omega$, smooth on $\Omega \setminus E_\varphi(T)$, increasing with respect to $\varphi$ and $\rho$ on $\Omega$, and converges to $\varphi$ on $\Omega$ as $\rho \to 0$,

(ii) $T_{\varphi, \rho} \geq \gamma - \varsigma \omega - \delta_\rho \omega$ on $\Omega$,

where $E_\varphi(T) := \{ x \in X : \varphi(T, x) \geq \varsigma \} (\varsigma > 0)$ is the $\varsigma$-upperlevel set of Lelong numbers, and $\{ \delta_\rho \}$ is an increasing family of positive numbers such that $\lim_{\rho \to 0} \delta_\rho = 0.$
Remark 2.1. Although Lemma 2.2 is stated in [8] in the case $X$ is compact, almost the same proof as in [8] shows that Lemma 2.2 holds in the noncompact case while uniform estimates are obtained only on the relatively compact subset $\Omega$.

Lemma 2.3 (Theorem 1.5 in [7]). Let $X$ be a Kähler manifold, and $Z$ be an analytic subset of $X$. Assume that $\Omega$ is a relatively compact open subset of $X$ possessing a complete Kähler metric. Then $\Omega \setminus Z$ carries a complete Kähler metric.

Lemma 2.4 (Theorem 4.4.2 in [19]). Let $\Omega$ be a pseudoconvex open set in $\mathbb{C}^n$, and $\varphi$ be a plurisubharmonic function on $\Omega$. For every $w \in L^2_{(p,q)}(\Omega, e^{-\varphi})$ with $\bar{\partial}w = 0$ there is a solution $s \in L^2_{(p,q)}(\Omega, \text{loc})$ of the equation $\bar{\partial}s = w$ such that
\begin{align*}
\int_{\Omega} \frac{|s|^2}{(1 + |z|^2)^2} e^{-\varphi} d\lambda \leq \int_{\Omega} |w|^2 e^{-\varphi} d\lambda,
\end{align*}
where $d\lambda$ is the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$.

Lemma 2.5 (Lemma 6.9 in [7]). Let $\Omega$ be an open subset of $\mathbb{C}^n$ and $Z$ be a complex analytic subset of $\Omega$. Assume that $u$ is a $(p, q - 1)$-form with $L^1_{\text{loc}}$ coefficients and $g$ is a $(p, q)$-form with $L^1_{\text{loc}}$ coefficients such that $\bar{\partial}u = g$ on $\Omega \setminus Z$ (in the sense of currents). Then $\bar{\partial}u = g$ on $\Omega$.

Theorem 2.6 (Strong openness property of multiplier ideal sheaves, [15]). Let $\varphi$ be a negative plurisubharmonic function on the unit polydisk $\Delta^n \subset \mathbb{C}^n$. Assume that $F$ is a holomorphic function on $\Delta^n$ satisfying
\begin{align*}
\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda < +\infty.
\end{align*}
Then there exists $r \in (0, 1)$ and $\beta \in (0, +\infty)$ such that
\begin{align*}
\int_{\Delta^n_r} |F|^2 e^{-(1 + \beta)\varphi} d\lambda < +\infty,
\end{align*}
where $\Delta^n_r := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| < r, 1 \leq k \leq n\}$.

Theorem 2.7 (Hironaka’s desingularization theorem, [18], [4]). Let $X$ be a complex manifold, and $M$ be an analytic subvariety in $X$. Then there is a local finite sequence of blow-ups $\mu_j : X_{j + 1} \rightarrow X_j$ ($X_1 := X, j = 1, 2, \cdots$) with smooth centers $S_j$ such that:
\begin{enumerate}
\item Each component of $S_j$ lies either in $(M_j)_{\text{sing}}$ or in $M_j \cap E_j$, where $M_1 := M, M_{j+1}$ denotes the strict transform of $M_j$ by $\mu_j$, $(M_j)_{\text{sing}}$ denotes the singular set of $M_j$, and $E_{j+1}$ denotes the exceptional divisor $\mu_j^{-1}(S_j \cup E_j)$.
\item Let $M'$ and $E'$ denote the final strict transform of $M$ and the exceptional divisor respectively. Then:
\begin{enumerate}
\item The underlying point-set $|M'|$ is smooth.
\item $|M'|$ and $E'$ simultaneously have only normal crossings.
\end{enumerate}
\end{enumerate}

Remark 2.2. We say that $|M'|$ and $E'$ simultaneously have only normal crossings if, locally, there is a coordinate system in which $E'$ is a union of coordinate hyperplanes, and $|M'|$ is a coordinate subspace.
3. Key propositions used to deal with the singular metric $h_L$

In order to deal with the singular metric $h_L$, we will prove two key propositions in this section, which are generalizations of the key propositions in [31].

**Proposition 3.1.** Let $R$ be a positive continuous function defined on $(-\infty, 0]$ such that $\beta_R := \sup_{t \leq 0} (e^t R(t)) < +\infty$ and $\hat{\beta}_R := \inf_{t \leq 0} R(t) > 0$. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $\phi$ be a plurisubharmonic function on $\Omega$, and $\Upsilon$ be a quasi-plurisubharmonic function defined on a neighborhood of $\overline{\Omega}$. Assume that $\Upsilon$ has neat analytic singularities and the singularities of $\Upsilon$ are log canonical along the zero variety $\Upsilon = V(T(\Upsilon))$. Set

$$U = \{x \in \Omega : \Upsilon(x) < 0\}.$$ 

Furthermore, assume that

$$\sqrt{-1} \partial \bar{\partial} \Upsilon \geq -\gamma \sqrt{-1} \partial \bar{\partial}|z|^2$$

on $\Omega$ for some nonnegative number $\gamma$, where $z := (z_1, \cdots, z_n)$ is the coordinate vector in $\mathbb{C}^n$. Then for every $\beta_1 \in (0, 1)$ and every holomorphic $n$-form $f$ on $U$ satisfying

$$\int_U \frac{|f|^2 e^{-\phi}}{e^{R(\Upsilon)}} d\lambda < +\infty,$$

there exists a holomorphic $n$-form $F$ on $\Omega$ satisfying $F = f$ on $U$,

$$(3.1) \quad \int_U \frac{|F|^2 e^{-\phi}}{e^{R(\Upsilon)}} d\lambda \leq e^{2\gamma \sup_{\Omega} |z|^2} \left(2 + \frac{72\beta_R}{\beta_1 \hat{\beta}_R}\right) \int_U \frac{|f|^2 e^{-\phi}}{e^{R(\Upsilon)}} d\lambda,$$

and

$$(3.2) \quad \int_{\Omega} \frac{|F|^2 e^{-\phi}}{(1 + e^2 \gamma)^{1 + \beta_1}} d\lambda \leq e^{2\gamma \sup_{\Omega} |z|^2} \left(\beta_R + 36\beta_R \beta_1 2\gamma\right) \int_U \frac{|f|^2 e^{-\phi}}{e^{R(\Upsilon)}} d\lambda.$$

**Proof.** This proposition is a modification of a theorem in [12].

Since $\Omega$ is a pseudoconvex domain, there is a sequence of pseudoconvex subdomains $\Omega_k \subset \subset \Omega$ (for $k = 1, 2, \cdots$) such that $\bigcup_{k=1}^{+\infty} \Omega_k = \Omega$. Then for fixed $k$, by convolution we can get a decreasing family of smooth plurisubharmonic functions $\{\phi_j\}_{j=1}^{+\infty}$ defined on a neighborhood of $\overline{\Omega_k}$ such that $\lim_{j \to +\infty} \phi_j = \phi$.

Let $\theta : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\theta = 1$ on $(-\infty, \frac{1}{4})$, $\theta = 0$ on $(\frac{1}{4}, +\infty)$ and $|\theta'| \leq 3$ on $\mathbb{R}$.

Fix $k$ and $j$. Set $\tilde{f} = \theta(e^\Upsilon) f$. Then the construction of $\tilde{f}$ implies that $\tilde{f}$ is smooth on $\Omega$ and $\tilde{f} = f$ on $Y \cap \Omega$.

Set $g = \partial \tilde{f}$. Then $g = \theta'(e^\Upsilon) e^\Upsilon \partial \Upsilon \wedge f$ on $\Omega$.

Let $\Sigma := \{\Upsilon = -\infty\}$. Lemma 2.3 implies that $\Omega_k \setminus \Sigma$ is a complete K"ahler manifold. Let $\Omega_k \setminus \Sigma$ be endowed with the Euclidean metric and let $Q$ be the trivial line bundle on $\Omega_k \setminus \Sigma$ equipped with the metric

$$h := e^{-\theta_j - \Upsilon - \beta_1 \log(1 + e^\Upsilon) - 2\gamma |z|^2}.$$ 

Then we want to solve a $\partial$ equation on $\Omega_k \setminus \Sigma$ by applying Lemma 2.1 to the case $\tau = 1$, $A = 0$ and $\delta = 0$ (in fact, the case $\tau = 1$ and $A = 0$ is the non twisted version...
of Lemma 2.1). The key step in applying Lemma 2.1 is to estimate the term
\[ \int_{\Omega_k \setminus \Sigma} \langle B^{-1} g, g \rangle_h d\lambda, \]
where \( B := [\sqrt{-1} \Theta_h, \Lambda] \).
Set \( \nu = \partial \bar{\nu} \). Then \( g = \theta'(e^\nu) e^\nu \wedge f \) on \( \Omega \).
Since
\[ \sqrt{-1} \Theta_h|_{\Omega_k \setminus \Sigma} = \sqrt{-1} \partial \bar{\partial} \phi_j + \beta_1 \sqrt{-1} \partial \bar{\partial} \log(1 + e^\nu) + 2\gamma \sqrt{-1} \partial \bar{\partial} |z|^2 \]
\[ = \sqrt{-1} \partial \bar{\partial} \phi_j + \left( 1 + \frac{\beta_1 e^\nu}{1 + e^\nu} \right) \sqrt{-1} \partial \bar{\partial} \bar{\nu} + 2\gamma \sqrt{-1} \partial \bar{\partial} |z|^2 + \frac{\beta_1 e^\nu \sqrt{-1} \partial \bar{\partial} \bar{\nu}}{(1 + e^\nu)^2} \]
we get
\[ B \geq \frac{\beta_1 e^\nu}{(1 + e^\nu)^2} T_\nu T_\nu^* \]
on \( \Omega_k \setminus \Sigma \), where \( T_\nu \) denotes the operator \( \nu \wedge \bullet \) and \( T_\nu^* \) is its Hilbert adjoint operator.
Then we get \( \langle B^{-1} g, g \rangle_h |_{\Omega \setminus \nu} = 0 \) and
\[ \langle B^{-1} g, g \rangle = \langle B^{-1}(\theta'(e^\nu) e^\nu \wedge f), \theta'(e^\nu) e^\nu \wedge f \rangle_h \]
\[ \leq \frac{(1 + e^\nu)^2}{\beta_1} |\theta'(e^\nu) e^\nu f|^2 e^{-\phi_j - \gamma \bar{\nu}} \log(1 + e^\nu)^2 e^{-2\gamma |z|^2} \]
\[ = \frac{(1 + e^\nu)^2 - \beta_1}{\beta_1} |\theta'(e^\nu) f|^2 e^{-\phi_j - 2\gamma |z|^2} \]
\[ \leq \frac{36}{\beta_1} |f|^2 e^{-\phi_j - 2\gamma |z|^2}. \]
Hence it follows from Lemma 2.1 that there exists \( u_{k,j} \in L^2(\Omega_k \setminus \Sigma, K_\Omega \otimes Q, h) \)
such that \( \bar{\partial} u_{k,j} = g = \bar{\partial} \hat{f} \) on \( \Omega_k \setminus \Sigma \) and
\[ \int_{\Omega_k \setminus \Sigma} |u_{k,j}|^2 d\lambda \leq \int_{\Omega_k \setminus \Sigma} \langle B^{-1} g, g \rangle_h d\lambda. \]
Thus
\[ (3.3) \]
\[ \int_{\Omega_k \setminus \Sigma} |u_{k,j}|^2 e^{-\phi_j - 2\gamma |z|^2} \]
\[ \leq \frac{36}{\beta_1} \int_{U \cup \Omega_k} |f|^2 e^{-\phi_j - 2\gamma |z|^2} d\lambda \]
\[ \leq \frac{36}{\beta_1} \int_{U} |f|^2 e^{-R(Y)} d\lambda. \]
Hence we have \( u_{k,j} \in L^2(\Omega_k \setminus \Sigma, K_\Omega) \). Since \( g \in C^\infty(\Omega_k, \Lambda^{n,1}T_\Omega^*) \), Lemma 2.5 implies that \( \bar{\partial} u_{k,j} = g \) holds on \( \Omega_k \).
Let $F_{k,j} := \tilde{f} - u_{k,j}$. Then $\bar{\partial}F_{k,j} = 0$ on $\Omega_k$. Thus $F_{k,j}$ is holomorphic on $\Omega_k$. Hence $u_{k,j}$ is smooth on $\Omega_k$. Then the non-integrability of $e^{-Y}$ along $Y$ implies that $u_{k,j} = 0$ on $Y \cap \Omega_k$. Therefore, $F_{k,j} = f$ on $Y \cap \Omega_k$.

It follows from (3.3) that
\[
\int_{U \cap \Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^Y R(Y)} \, d\lambda \\
\leq \frac{2\beta_1}{\beta_R} \int_{U \cap \Omega_k} |u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2} \, d\lambda \\
\leq \frac{36\beta_R}{\beta_1 \beta_R} \int_{U} \frac{|f|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^Y R(Y)} \, d\lambda.
\]

Since
\[
|F_{k,j}|^2 |_{U \cap \Omega_k} \leq 2|\tilde{f}|^2 + 2|u_{k,j}|^2 \leq 2|f|^2 + 2|u_{k,j}|^2,
\]
we get
\[
(3.4) \quad \int_{U \cap \Omega_k} \frac{|F_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^Y R(Y)} \, d\lambda \\
\leq 2 \int_{U \cap \Omega_k} (|f|^2 + |u_{k,j}|^2) e^{-\phi_j - 2\gamma|z|^2} \, d\lambda \\
\leq \left( 2 + \frac{72\beta_R}{\beta_1 \beta_R} \right) \int_{U} \frac{|f|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^Y R(Y)} \, d\lambda.
\]

Since
\[
(3.5) \quad \langle \kappa_1 + \kappa_2, \kappa_1 + \kappa_2 \rangle \leq \langle \kappa_1, \kappa_1 \rangle + \langle \kappa_2, \kappa_2 \rangle + c \langle \kappa_1, \kappa_1 \rangle + \frac{1}{e} \langle \kappa_2, \kappa_2 \rangle
\]
for any inner product space $(H, \langle \cdot, \cdot \rangle)$, where $\kappa_1, \kappa_2 \in H$, we get
\[
|F_{k,j}|^2 |_{U \cap \Omega_k} \leq (|f|^2 + |u_{k,j}|^2) \leq (1 + e^Y)|f|^2 + (1 + \frac{1}{e^Y})|u_{k,j}|^2.
\]

Then
\[
\frac{|F_{k,j}|^2}{(1 + e^Y)^{1+\beta_1}} |_{U \cap \Omega_k} \leq |f|^2 + \frac{|u_{k,j}|^2}{e^Y (1 + e^Y)^{\beta_1}}.
\]

Since $|F_{k,j}|^2 |_{\Omega_k \setminus U} = |u_{k,j}|^2$, we get
\[
\frac{|F_{k,j}|^2}{(1 + e^Y)^{1+\beta_1}} |_{\Omega_k \setminus U} \leq \frac{|u_{k,j}|^2}{e^Y (1 + e^Y)^{\beta_1}}.
\]

Hence it follows from the two inequalities above and (3.3) that
\[
(3.6) \quad \int_{\Omega_k} \frac{|F_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{(1 + e^Y)^{1+\beta_1}} \, d\lambda \\
\leq \int_{U} |f|^2 e^{-\phi_j - 2\gamma|z|^2} \, d\lambda + \int_{\Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^Y (1 + e^Y)^{\beta_1}} \, d\lambda \\
\leq \left( \beta_R + \frac{36\beta_R}{\beta_1 \beta_1^{2\gamma_1}} \right) \int_{U} \frac{|f|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^Y R(Y)} \, d\lambda.
\]
Assume that \( e^{-2\gamma \sup_{|z|^2}} \leq e^{-2|z|^2} \leq 1 \) on \( \Omega \), the desired holomorphic \( n \)-form \( F \) on \( \Omega \) and the \( L^2 \) estimates (3.1) and (3.2) can be obtained from (3.4) and (3.6) by applying Montel’s theorem and extracting weak limits of \( \{F_{k,j}\}_{k,j} \), first as \( j \to +\infty \) and then as \( k \to +\infty \).

□

**Proposition 3.2.** Let \( X, \psi, Y \) and \( Y^0 \) be as in Theorem 1.1. Let \( U \subset V \subset \Omega \) be three local coordinate balls in \( X \), \( \phi \) be a plurisubharmonic function on \( \Omega \) such that \( \sup \phi < +\infty \), and \( v \) be a nonnegative continuous function on \( \Omega \) with \( \sup v \subset U \). Let \( C, \beta, c_1 \) and \( c_2 \) be positive numbers, and let \( \beta_1 \) be a small enough positive number. Assume that \( f \) is a holomorphic function on \( \Omega \cap Y \) satisfying

\[
\int_{\Omega \cap Y^0} |f|^2 e^{-\phi} d\lambda[\psi] < +\infty,
\]

and that \( f_t \in \mathcal{O}(\Omega) \) (\( t \in (-\infty,0) \)) are a family of holomorphic functions such that for all \( t \in (-\infty,0) \), \( f_t = f \) on \( \Omega \cap Y \),

\[
\sup_v |f_t| \leq Ce^{-\beta_1 t}
\]

and

\[
\frac{1}{e^t} \int_{\Omega \cap \{\psi < t + c_2\}} |f_t|^2 e^{-(1+\beta)\phi} d\lambda \leq C.
\]

Then

\[
\lim_{t \to -\infty} \int_{U \cap \{t-c_1 < \psi < t + c_2\}} \frac{e^t v f_t |f_t|^2 e^{-\phi}}{(e^v + e^t)^2} d\lambda \leq \int_{U \cap Y^0} v |f|^2 e^{-\phi} d\lambda[\psi].
\]

**Remark 3.1.** One of the key points in the proof of Proposition 3.2 is to verify that the upper limit in (3.10) produces the zero measure on the singular set of \( Y \), i.e., we have (3.16). Then the key uniform estimates in Step 2 of the proof are obtained.

In order to prove Proposition 3.2, we prove the following lemma at first.

**Lemma 3.3.** Let \( r_1, r_2 \) and \( \gamma \) be positive numbers such that \( r_1 < r_2 < \gamma \). Let \( \varphi \) be a bounded negative subharmonic function on \( \Delta_{r_2} \), where \( \Delta_r := \{w \in \mathbb{C} : |w| < r\} \). Assume that \( \{v_t\}_{t \in (-\infty,0)} \) are nonnegative continuous functions defined on \( \Delta_r \) such that

\[
\lim_{t \to -\infty} \sup_{w \in \mathbb{C}, \ e^t(r_1)^{2\alpha} < |w|^{2\alpha} < e^t(r_2)^{2\alpha}} |v_t(w) - v_0| = 0,
\]

where \( \alpha \in [1, +\infty) \) and \( v_0 \in [0, +\infty) \). Let

\[
P_t := \int_{\{w \in \mathbb{C}, \ e^t(r_1)^{2\alpha} < |w|^{2\alpha} < e^t(r_2)^{2\alpha}\}} \frac{e^t |w|^{2\alpha} v_t(w) e^{-\varphi(w)}}{(|w|^{2\alpha} + e^t)^2} d\lambda(w).
\]

Then

\[
\lim_{t \to -\infty} P_t \leq \frac{\pi v_0 e^{-\varphi(0)}}{\alpha}.
\]

**Proof.** Put

\[
S_{\delta,t} = \{z \in \Delta_r : \varphi(e^{-t}\omega z) < (1 + \delta)\varphi(0)\}, \quad \delta \in (0, +\infty), \quad t \in (-\infty,0).
\]

Denote by \( \lambda(S_{\delta,t}) \) the 2-dimensional Lebesgue measure of \( S_{\delta,t} \).
Since \( \varphi(w) \) is a negative upper semicontinuous function on \( \Delta_\gamma \) and \( \varphi(0) > -\infty \), we have that for every \( \varepsilon \in (0, 1) \), there exists \( t_\varepsilon \in (-\infty, 0) \) such that
\[
\varphi(e^{\frac{t}{\varepsilon}}z) \leq (1 - \varepsilon)\varphi(0)
\]
for all \( z \in \Delta_\gamma \) when \( t \in (-\infty, t_\varepsilon) \).

Since \( \varphi(e^{\frac{t}{\varepsilon}}z) \) is subharmonic on \( \Delta_\gamma \) with respect to \( z \) for any \( t \in (-\infty, t_\varepsilon) \), it follows from the mean value inequality that, for all \( t \in (-\infty, t_\varepsilon) \),
\[
\varphi(0) \leq \frac{1}{\pi \gamma^2} \int_{z \in \Delta_\gamma} \varphi(e^{\frac{t}{\varepsilon}}z) d\lambda(z) \leq \frac{1}{\pi \gamma^2} \int_{z \in \Delta_\gamma} \varphi(e^{\frac{t}{\varepsilon}}z) d\lambda(z) + \frac{1}{\pi \gamma^2} \int_{z \in \Delta_\gamma} \varphi(e^{\frac{t}{\varepsilon}}z) d\lambda(z) 
\leq \frac{1}{\pi \gamma^2} \varphi(0)(\pi \gamma^2 - \lambda(S_{\delta,t})) + \frac{1}{\pi \gamma^2} (1 + \delta) \varphi(0) \lambda(S_{\delta,t})
\leq \varphi(0) \left( 1 - \varepsilon + \frac{(\delta + \varepsilon)\lambda(S_{\delta,t})}{\pi \gamma^2} \right).
\]
Then \( \varphi(0) < 0 \) implies that
\[
\lambda(S_{\delta,t}) \leq \frac{\pi \gamma^2 \varepsilon}{\delta + \varepsilon} \leq \frac{\pi \gamma^2 \varepsilon}{\delta}
\]
when \( t \in (-\infty, t_\varepsilon) \). Hence
\[
(3.13) \lim_{t \to -\infty} \lambda(S_{\delta,t}) = 0, \quad \forall \delta \in (0, +\infty).
\]
Since \( \varphi \) is bounded, we have
\[
-\varphi \leq C_1
\]
for some positive number \( C_1 \).

(3.11) implies that
\[
\sup_{\{w \in C : t_{r_1} \leq \pi |w|^2 \leq t_{r_2} \}} v_t(w) \leq C_2
\]
for some positive number \( C_2 \) independent of \( t \) when \( t \) is small enough.

Then by the change of variables \( w = e^{\frac{t}{\varepsilon}z} \), we have
\[
P_t = \int_{\{z \in C : r_1 < |z| < r_2 \}} \frac{|z|^{2\alpha - 2} v_t(e^{\frac{t}{\varepsilon}z}) e^{-\varphi(e^{\frac{t}{\varepsilon}z})}}{|z|^{2\alpha} + 1} d\lambda(z)
\leq \int_{\{r_1 < |z| < r_2 \} \cap S_{\delta,t}} \frac{|z|^{2\alpha - 2} v_t(e^{\frac{t}{\varepsilon}z}) e^{-\varphi(e^{\frac{t}{\varepsilon}z})}}{|z|^{2\alpha} + 1} d\lambda(z)
+ \int_{\{r_1 < |z| < r_2 \} \setminus S_{\delta,t}} \frac{|z|^{2\alpha - 2} v_t(e^{\frac{t}{\varepsilon}z}) e^{-\varphi(e^{\frac{t}{\varepsilon}z})}}{|z|^{2\alpha} + 1} d\lambda(z)
\leq \frac{(r_2)^{2\alpha - 2} C e^{C_1}}{(r_1)^{2\alpha} + 1} \cdot \lambda(S_{\delta,t})
+ \left( \sup_{r_1 < |z| < r_2} v_t(e^{\frac{t}{\varepsilon}z}) \right) e^{-(1 + \delta)\varphi(0)} \int_{\{r_1 < |z| < r_2 \}} \frac{|z|^{2\alpha - 2}}{|z|^{2\alpha} + 1} d\lambda(z).
\]
Since
\[
\int_{\{r_1 < |z| < r_2 \}} \frac{|z|^{2\alpha - 2}}{|z|^{2\alpha} + 1} d\lambda(z) \leq \frac{\pi}{\alpha},
\]
we obtain from (3.11), (3.13) that
\[
\lim_{t \to -\infty} P_t \leq \frac{\pi \nu_0 e^{-(1+\delta)\varphi(0)}}{\alpha}.
\]
Since \(\delta\) is an arbitrary positive number, we get (3.12).

\[\square\]

Now we begin to prove Proposition 3.2.

**Proof.** Let \(\beta_v := \sup_U v\).

Without loss of generality, we may suppose that \(\phi\) is negative on \(\Omega\).

We will use Hironaka’s desingularization theorem (Lemma 2.7) to deal with the measure \(d\lambda[\psi]\). This idea comes from the work [11].

At first we use Lemma 2.7 on \(X\) to resolve the singularities of \(Y\) and we denote the corresponding proper modification by \(\mu_1\). Next, we make a blow-up \(\mu_2\) along \(|Y'|\). Then we use Lemma 2.7 again to resolve the singularities of \(\Sigma\) and we denote the corresponding proper holomorphic modification by \(\mu_3\), where \(\Sigma\) denote the strict transform of \(\{\psi = -\infty\}\) by \(\mu_1 \circ \mu_2\). Finally, we make a blow-up \(\mu_4\) along \(|\Sigma'|\). Thus we can get a proper holomorphic map \(\mu : \tilde{X} \to X\), which is locally a finite composition of blow-ups with smooth centers and is equal to \(\mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_4\). Moreover, \(\tilde{Y}\) and the divisor \(\mu^{-1}(\{\psi = -\infty\}) \setminus Y\) simultaneously have only normal crossings in \(\tilde{X}\), where \(\tilde{Y}\) denotes the strict transform of \(\mu^{-1}_2(\{Y'\})\) by \(\mu_3 \circ \mu_4\).

**Step 1:** we will represent the measure \(|f|^2 d\lambda[\psi]|\) on \(Y^0 \cap U\) explicitly as an integral on \(\tilde{Y}\) (see (3.15)).

For any \(\tilde{x} \in \mu^{-1}(U) \cap \mu^{-1}(\{\psi = -\infty\})\), there exists a relatively compact coordinate ball \((\tilde{W}; w_1, \cdots, w_n)\) contained in \(\mu^{-1}(V)\) centered at \(\tilde{x}\) such that \(w^b = 0\) is the zero divisor of the Jacobian \(J_\mu\), and \(\psi \circ \mu\) can be written on \(\tilde{W}\) as
\[
\psi \circ \mu(w) = c \log |w|^2 + \tilde{u}(w),
\]
where \(c\) is a positive number, \(w := (w_1, \cdots, w_n)\), \(\tilde{u} \in C^\infty(\tilde{W})\), \(w^s := \prod_{p=1}^n w_p^a_p\) and
\[
w^b := \prod_{p=1}^n w_p^{b_p}\text{ for some nonnegative integers } a_p\text{ and } b_p.
\]

Let \(D_p := \{w_p = 0\}\). Then as proved in [11], the multiplier ideal sheaf \(I(\psi)\) is given by the direct image formula
\[
I(\psi) = \mu_* \mathcal{O}_{\tilde{X}}(- \sum_{p=1}^n [ca_p - b_p]_+ D_p),
\]
where \([ca_p - b_p]_+\) denotes the minimal nonnegative integer bigger than \(ca_p - b_p - 1\).

Since \(\psi\) has log canonical singularities, by the construction of \(\mu\) and Lemma 2.7, one of the following cases is true on \(W\):

(A) \(\tilde{Y}\) is given on \(W\) precisely by \(D_{p_0}\) (if \(W\) is small enough) for some \(p_0\) satisfying \(ca_{p_0} - b_{p_0} = 1\), and \(ca_p - b_p \leq 1\) for \(p \neq p_0\);
(B) \(\tilde{Y} \cap W = \emptyset\), and \(ca_p - b_p \leq 1\).

By definition, the measure \(|f|^2 d\lambda[\psi]|\) can be defined as
\[
g \mapsto \lim_{t \to -\infty} \int \left\{t < \log |w|^2 + \tilde{u}(w) < t + 1\right\} \frac{|f \circ \mu|^2 (\tilde{g} \circ \mu) \xi e^{-\tilde{u}}}{|w^{ca-b}|^2} d\lambda(w),
\]
(3.14)
where \( d\lambda(w) := \) the Lebesgue measure with respect to the coordinate vector \( w \), \( \tilde{f} \) is a holomorphic extension of \( f \) to \( \Omega \), \( g \) and \( \tilde{g} \) are defined as in Definition 1.3, and \( \xi \) is the smooth positive function \( \frac{|f_d|^2}{|w|^2} \) (as stated in [11], one would still have to take into account a partition of unity on the various coordinate charts covering the fibers of \( \mu \), but we will avoid this technicality for the simplicity of notation).

In Case (A), let us denote \( w = (w', w_{p_0}) \in \mathbb{C}^{n-1} \times \mathbb{C} \), \( a = (a', a_{p_0}) \), \( b = (b', b_{p_0}) \) and \( d\lambda(w) = d\lambda(w')d\lambda(w_{p_0}) \). Then (3.14) becomes

\[
g \mapsto \lim_{t \to -\infty} \int_{\{ t < \log |w'|^2 + \tilde{u}(w) < t + 1 \}} \frac{|\tilde{f} \circ \mu|^2}{|w'|^{a - b'}|2} \cdot \frac{(\tilde{g} \circ \mu)\xi e^{-\tilde{u}}}{|w_{p_0}|^2} d\lambda(w).
\]

Since the domain of integration can be written as

\[
\{(t, w') \mid |(w')^a'| - 2c < |w_{p_0}|^{2a_{p_0}} < e^{t+1 - \tilde{u}(w)} |(w')^a'| - 2c\},
\]

(3.14) becomes

\[
g \mapsto \frac{\pi}{c a_{p_0}} \int_{w' \in D_{p_0}} \frac{|\tilde{f} \circ \mu|^2}{|w'|^{a - b'}|2} \cdot (\tilde{g} \circ \mu)\xi e^{-\tilde{u}} d\lambda(w').
\]

Set \( \kappa = \{ p : c a_p - b_p = 1 \} \).

If \( p \in \kappa \setminus \{ p_0 \} \), then Theorem 2.7 and the construction of \( \mu \) imply that an image of \( D_p \) under a finite sequence of blow-ups in the desingularization process must be contained in a smooth center contained in \( Y' \) or \( \mu_2^{-1}(\langle Y' \rangle) \). Hence the images of \( D_p \) and \( D_{p_0} \) coincide under the composition of these blow-ups.

Since it is implied from (3.7) and (3.15) that \( f \circ \mu \rvert_{D_{p_0} \cap D_{p_0}} = 0 \), we obtain that

\[
f \circ \mu \rvert_{D_p} = 0
\]

holds for all \( p \in \kappa \setminus \{ p_0 \} \) in Case (A).

Similarly, we can get that (3.16) holds for all \( p \in \kappa \) in Case (B). Then (3.14) is the zero measure in Case (B).

Therefore, we represent the measure \( |f|^2d\lambda[w'] \) on \( Y^0 \cap U \) explicitly as in (3.15).

**Step 2: we will obtain some uniform estimates for** \( f_t \circ \mu \).

By Cauchy’s inequality for holomorphic functions, it follows from (3.8) that

\[
\sup_{U_1} |\partial^{\gamma} f_t|^2 \leq C_1 \sup_{V} |f_t|^2 \leq C_1 Ce^{-\beta_t t} \tag{3.17}
\]

for any \( t \in (-\infty, 0) \) and any multi-index \( \gamma \) satisfying \( |\gamma| \leq n \), where \( U_1 \subset V \) is a neighborhood of \( U \), and \( C_1 \) is a positive number independent of \( t \) and \( \gamma \).

Let \( W_t := W \cap \mu^{-1}(U) \cap \{ \psi \circ \mu < t + c_2 \} \).

In Case (A), by applying the mean value theorem to \( f_t \circ \mu \) successively along the directions in \( \kappa \), we get from (3.17) and (3.16) that for any \( w = (w', w_{p_0}) \in W_t \),

\[
|f_t \circ \mu(w', w_{p_0}) - f_t \circ \mu(w', 0)|^2 \leq C_2 \prod_{\rho \in \kappa} |w_p|^2 \sup_{|\gamma| \leq |\rho|} \sup_{\mu^{-1}(U_1)} |\partial^{\gamma} f_t|^2 \leq C_3 e^{-\beta t} \prod_{\rho \in \kappa} |w_p|^2 \tag{3.18}
\]

and

\[
|f_t \circ \mu(w', 0)|^2 = |f \circ \mu(w', 0)|^2 \leq C_4 \prod_{\rho \in \kappa \setminus \{ p_0 \}} |w_p|^2 \tag{3.19}
\]
when \( t \) is small enough, where \( C_2, C_3 \) and \( C_4 \) are positive numbers independent of \( t \).

In Case (B), if \( \kappa \neq \emptyset \), take \( p_1 \in \kappa \) and denote \( w = (w', w_{p_1}) \). Since \( f_t \circ \mu(w', 0) = f \circ \mu(w', 0) = 0 \), by the similar method we have that

\[
|f_t \circ \mu(w', w_{p_1})|^2 \leq C_5 e^{-\beta_1 t} \prod_{p \in \kappa} |w_p|^2
\]

for any \( w = (w', w_{p_1}) \in W_i \) when \( t \) is small enough, where \( C_5 \) is a positive number independent of \( t \). If \( \kappa = \emptyset \), (3.8) implies that

\[
|f_t \circ \mu(w)|^2 \leq C e^{-\beta_1 t}
\]

for any \( w \in W_i \).

**Step 3: the proof of (3.10).**

Let \( j \) be a positive integer. Then (3.9) implies that

\[
\frac{1}{e^t} \int_{\{ \phi \leq -j \} \cap U \cap \{ \psi < t + c_2 \}} |f_t|^2 e^{-\phi} d\lambda \\
\leq \frac{1}{e^t} \int_{\{ \phi \leq -j \} \cap U \cap \{ \psi < t + c_2 \}} |f_t|^2 e^{-(1 + \beta) \phi - \beta j} d\lambda \\
\leq C e^{-\beta j}
\]

for any \( t \in (-\infty, 0) \).

Therefore, for every \( \epsilon \in (0, 1) \), there exists a positive integer \( j_\epsilon \) such that

\[
\int_{\{ \phi \leq -j \} \cap U \cap \{ t - c_1 < \psi < t + c_2 \}} e^t v |f_t|^2 e^{-\phi} d\lambda \\
\leq \frac{1}{(e^{-c_1} + 1)^2 e^t} \int_{\{ \phi \leq -j \} \cap U \cap \{ \psi < t + c_2 \}} v |f_t|^2 e^{-\phi} d\lambda \\
\leq \frac{\beta_1 C e^{-\beta j \epsilon}}{(e^{-c_1} + 1)^2} \\
< \frac{\epsilon}{2}
\]

for any \( t \in (-\infty, 0) \).

Set \( \phi_\epsilon = \max\{ \phi, -j_\epsilon \} \). We want to prove

\[
\lim_{t \to -\infty} \int_{U \cap \{ t - c_1 < \psi < t + c_2 \}} e^t v |f_t|^2 e^{-\phi_\epsilon} d\lambda \leq \int_{U \cap Y_0} v |f|^2 e^{-\phi_\epsilon} d\lambda[v].
\]

Set

\[
I_0 = \lim_{t \to -\infty} \int_{W \cap \mu^{-1}(U) \cap \{ t - c_1 < \psi, \mu < t + c_2 \}} e^t (v \circ \mu) |f_t \circ \mu|^2 e^{-\phi_\epsilon} \|J_\mu\|^2 d\lambda.
\]

Then by Step 1, it suffices to prove that

\[
I_0 \leq \frac{\pi}{c \alpha_{p_0}} \int_{W \cap \mu^{-1}(U) \cap D_{p_0}} \frac{(v \circ \mu) |f \circ \mu|^2 (\xi e^{-\bar{\psi} \circ \phi_\epsilon} \phi_\epsilon \xi)}{|(w') \alpha - \beta |^2} d\lambda(w')
\]

in Case (A) and \( I_0 = 0 \) in Case (B), where \( \xi \) is the smooth positive function \( \frac{|J_\mu|^2}{|w'|^2} \) defined in Step 1.
In Case (A), let
\[
\Phi_t(w') := \int_{W_{t,w'}} e^t (v \circ \mu)[f_t \circ \mu] e^{-\phi \circ \mu} |f_t| d\lambda(w_{p_0})
\]
and
\[
\Phi(w') := \pi \cdot \frac{v \circ \mu(w', 0) [f \circ \mu(w', 0)]^{2} e^{-\tilde{w}(w', 0) - \phi \circ \mu(w', 0)}}{|(w')^{c \alpha - b'}|^2},
\]
where \( W_{t,w'} \) is the 1-dimensional open set
\[
\{ e^t - c_1 - \tilde{w}(w', w_{p_0}) |(w')^a - 2c < |w_{p_0}|^{2c} \alpha < e^t + c_2 - \tilde{w}(w', w_{p_0}) |(w')^a - 2c \} \cap W \cap \mu^{-1}(U)
\]
for every fixed \( t \) and \( w' \in D_{p_0} \setminus \cup_{p \neq p_0} D_p \). Then

(3.25) \[ I_0 = \lim_{t \to -\infty} \int_{W \cap \mu^{-1}(U) \cap D_{p_0}} \Phi_t(w') d\lambda(w'). \]

Since \(-c_1 < \psi \circ \mu - t < c_2\) holds on \( W_{t,w'} \), we obtain from (3.18) and (3.19) that
\[
\Phi_t(w') \leq C_6 \int_{W_{t,w'}} \frac{(v \circ \mu)[f_t \circ \mu] e^{-\phi \circ \mu}|f_t|}{e^\psi \circ \mu} d\lambda(w_{p_0}) \leq C_7 \int_{W_{t,w'}} \frac{|f_t \circ \mu|^2}{|w_{p_0}|} d\lambda(w_{p_0}) \leq C_8 \int_{W_{t,w'}} \frac{\prod_{p \in \mathbb{K}} |w_p|^2}{|w_{p_0}|^{1 + \beta_1(c \alpha - b)}^2} d\lambda(w_{p_0}) + C_8 \int_{W_{t,w'}} \frac{|w_{p_0}|^2}{|w_{p_0}|^{1 + \beta_1(c \alpha - b)}^2} d\lambda(w_{p_0}),
\]
where \( C_7 \) and \( C_8 \) are positive numbers independent of \( t \).

Since it is easy to prove that the right-hand side of the above inequality is dominated by a function of \( w' \) which is independent of \( t \) and belongs to \( L^1(W \cap \mu^{-1}(U) \cap D_{p_0}) \) when
\[
\beta_1 < \min_{\{p: a_p \neq 0\}} \frac{1 - (ca_p - b_p) + |ca_p - b_p|}{ca_p},
\]
it follows from (3.25) and Fatou’s lemma that

(3.26) \[ I_0 \leq \int_{W \cap \mu^{-1}(U) \cap D_{p_0}} \lim_{t \to -\infty} \Phi_t(w') d\lambda(w'). \]

Since (3.18) implies that
\[
\lim_{t \to -\infty} \sup_{w' \in W_{t,w'}} |f_t \circ \mu(w', w_{p_0}) - f \circ \mu(w', 0)| = 0
\]
for every fixed \( w' \in (W \cap \mu^{-1}(U) \cap D_{p_0}) \setminus \cup_{p \neq p_0} (D_{p_0} \cap D_p) \) when \( \beta_1 < 1/|ca_{p_0}| \), it follows from Lemma 3.3.3 that
\[
\lim_{t \to -\infty} \Phi_t(w') \leq \Phi(w'), \quad \forall w' \in (W \cap \mu^{-1}(U) \cap D_{p_0}) \setminus \cup_{p \neq p_0} (D_{p_0} \cap D_p).
\]
Hence (3.24) follows from (3.26). Similarly, we can obtain from (3.20) and (3.21) that \( I_0 = 0 \) in Case (B) when
\[
\beta_1 < \min_{\{p: a_p \neq 0\}} \frac{1 - (ca_p - b_p) + |ca_p - b_p|}{ca_p}.
\]
Thus we get (3.23).

It is easy to see that (3.10) follows from (3.22) and (3.23). Thus we finish the proof of Proposition 3.2.

\[ \square \]

4. Proof of Theorem 1.1

Without loss of generality, we can suppose that \( f \) is not 0 identically.

Let \( h_0 \) be any fixed smooth metric of \( L \) on \( X \). Then \( h = h_0 e^{-\phi} \) for some global function \( \phi \) on \( X \), which is quasi-plurisubharmonic by the assumption in the theorem.

Since \( X \) is weakly pseudoconvex, there exists a smooth plurisubharmonic exhaustion function \( P \) on \( X \). Let \( X_k := \{ P < k \} \ (k = 1, 2, \ldots, \) we choose \( P \) such that \( X_1 \neq \emptyset \).

Our proof consists of several steps. We will discuss for fixed \( k \) until the end of Step 5.

We will give the proof for the vector bundle \( E \) in the first five steps, and we will give the proof for the vector bundle \( E \) in Step 6.

**Step 1:** construction of a family of special smooth extensions \( \tilde{f}_i \) of \( f \) to a neighborhood of \( X_k \cap Y \) in \( X \).

In order to deal with singular metrics of holomorphic line bundles on weakly pseudoconvex Kähler manifolds, we construct in this step a family of smooth extensions \( \tilde{f}_i \) of \( f \) satisfying some special estimates by using the results in Section 3.

Let \( \epsilon \in (0, \frac{1}{2}) \).

For the sake of clearness, we divide this step into four parts.

**Part I:** construction of local coordinate patches \( \{ \Omega_i \}_{i=1}^N \), \( \{ U_i \}_{i=1}^N \) and a partition of unity \( \{ \xi_i \}_{i=1}^{N+1} \).

For any point \( x \in Y \), we can find a local coordinate ball \( \Omega'_x \) in \( X \) centered at \( x \) such that there exists a local holomorphic frame of \( L \) on \( \Omega'_x \) and such that \( \psi \) can be written as a sum of a smooth function and a plurisubharmonic function on \( \Omega'_x \).

Moreover, we assume that \( \psi \) can be written on \( X \) as

\[
\psi = c_x \log \sum_{1 \leq j \leq j_0} |g_{x,j}|^2 + u_x,
\]

where \( c_x \) is a positive number, \( g_{x,j} \in \mathcal{O}_X(\Omega'_x) \) and \( u_x \in C^\infty(\Omega'_x) \).

Let \( U_x \subset \subset V_x \subset \subset \Omega_x \subset \subset \Omega'_x \) be three small coordinate balls.

Since \( \overline{X_k} \cap Y \) is compact, there exist points \( x_1, x_2, \ldots, x_N \in \overline{X_k} \cap Y \) such that \( \overline{X_k} \cap Y \subset \bigcup_{i=1}^N U_{x_i} \).

For simplicity, we will denote \( \Omega'_{x_1}, \Omega_{x_1}, U_{x_1}, V_{x_1} \) and \( u_{x_1} \) by \( \Omega'_i, \Omega_i, U_i, V_i \) and \( u_i \) respectively. We will write the local expression (4.1) on \( \Omega'_i \) by

\[
\psi = Y_i + u_i.
\]

Choose an open set \( U_{N+1} \) in \( X \) such that \( \overline{X_k} \cap Y \subset X \setminus \overline{U_{N+1}} \subset \bigcup_{i=1}^N U_i \). Set \( U = X \setminus \overline{U_{N+1}} \).

Let \( \{ \xi_i \}_{i=1}^{N+1} \) be a partition of unity subordinate to the cover \( \{ U_i \}_{i=1}^{N+1} \) of \( X \).

Then \( \text{supp} \xi_i \subset U_i \) for \( i = 1, \ldots, N \) and \( \sum_{i=1}^N \xi_i = 1 \) on \( U \).
**Part II: construction of local holomorphic extensions** \(\hat{f}_{i,t}\) \((1 \leq i \leq N)\) of \(f\) to \(\Omega_i \cap \{\psi < t + c_2\}\), where \(c_2\) will be defined in this part.

By Remark 1.4, \(f\) has local \(L^2\) extensions to local coordinate balls around every point in \(Y\). Hence \(f\) is indeed a holomorphic section well defined on \(Y\) (not only on \(Y^0\)). By Step 1 (see (3.15)) in the proof of Proposition 3.2, (1.2) is equivalent to

\[
\int_{D_{\rho_0}} \frac{|f \circ \mu_{\omega,h_0}^{2} \xi e^{-\bar{u} - \phi_0}|}{|(w')^{\alpha} - b|^2} d\lambda(w') < +\infty.
\]

Hence by Theorem 2.6, there exists a positive number \(\beta \in (0, 1)\) such that

\[
(4.2) \quad \int_{\Omega_i \cap Y^0} |f|_\omega,h_0^2 e^{-(1+\beta)\psi} dV_\omega \leq C (1 \leq i \leq N).
\]

Let \(\tilde{\alpha}_0 < \alpha_0\) be a fixed number such that \(R\) is decreasing on \((-\infty, \tilde{\alpha}_0]\). Then set \(R_0(t) = R(\tilde{\alpha}_0) e^{-\beta_2(t-\tilde{\alpha}_0)}\), \(t \in (-\infty, \tilde{\alpha}_0]\), where \(\beta_2\) is a positive number which will be determined later in Step 4. Let

\[
R_1(t) := \min \{R_0(t + \tilde{\alpha}_0), R(t + \tilde{\alpha}_0)\}, \quad t \in (-\infty, 0].
\]

Then \(R_1\) is decreasing and thereby satisfies all the requirements for the functions in \(\mathcal{N}_{0,\alpha_1}\) except that \(R_1\) is only continuous.

Let \(c_1 = c_2 := \log \frac{2\epsilon}{\epsilon} + 1\), \(m_i := \inf_{\Omega_i} u_i\) and \(M_i := \sup_{\Omega_i} u_i\).

For each fixed \(t \in (-\infty, 0)\), by Remark 1.4, we apply Theorem 1.1 to the Stein manifold \(\Omega_i \cap \{\tilde{Y}_i < t + c_2 - m_i\}\), to the negative plurisubharmonic function \(\tilde{Y}_i - t - c_2 + m_i\), to the holomorphic section \(f\) on \(\Omega_i \cap Y^0\) with the \(L^2\) condition (4.2) and to the function \(R_1\) (\(R_1\) is only needed to be continuous by the remark after Theorem 2.1 in [14]), and then we obtain \(L^2\) extensions of \(f\) from \(\Omega_i \cap Y^0\) to \(\Omega_i \cap \{\tilde{Y}_i < t + c_2 - m_i\}\), where we equip the line bundle \(L\) with the singular metric \(h_0 e^{-(1+\beta)\psi}\). More precisely, there exists a uniform positive number \(C_1\) (independent of \(t\)) and holomorphic extensions \(\tilde{f}_{i,t}\) \((1 \leq i \leq N)\) of \(f\) from \(\Omega_i \cap Y^0\) to \(\Omega_i \cap \{\tilde{Y}_i < t + c_2 - m_i\}\) such that

\[
(4.3) \quad \int_{\Omega_i \cap \{\tilde{Y}_i < t + c_2 - m_i\}} |\tilde{f}_{i,t}|_\omega,h_0^2 e^{-(1+\beta)\tilde{Y}_i} dV_\omega \leq C_1 \int_{\Omega_i \cap Y^0} |f|_\omega,h_0^2 e^{-(1+\beta)\tilde{Y}_i} dV_\omega \leq C_2 e^t \int_{\Omega_i \cap Y^0} |f|_\omega,h_0^2 e^{-(1+\beta)\tilde{Y}_i} dV_\omega
\]

where \(C_2\) is a positive number independent of \(t\). Furthermore, we get that \(f\) is in fact holomorphic on \(\Omega_i \cap Y\) and \(\tilde{f}_{i,t} = f\) on \(\Omega_i \cap Y\).

**Part III: construction of local holomorphic extensions** \(\hat{f}_{i,t}\) \((1 \leq i \leq N)\) of \(f\) to \(\Omega_i\).

For each fixed \(t\), applying Proposition 3.1 to the local extensions \(\tilde{f}_{i,t}\) \((1 \leq i \leq N)\) with the weight \((1 + \beta)\phi\) and to the case \(\Upsilon = \tilde{Y}_i - t - c_2 + m_i\), \(\Omega = \Omega_i\) and some small positive number \(\beta_1\) which will be determined later in Step 4, we obtain from (4.3) holomorphic sections \(\hat{f}_{i,t}\) \((1 \leq i \leq N)\) on \(\Omega_i\) satisfying \(\hat{f}_{i,t} = \hat{f}_{i,t} = f\) on
\[ \Omega_i \cap Y^0, \]

\[
\int_{\Omega_i \cap \{Y_i < t + c_2 - m_i\}} \frac{|\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-(1+\beta)t}}{e^{V_{i,t-c_2+m_i}R_i(1 + c_2 + m_i)}} dV_{X,\omega} \leq C_3 e^t,
\]

and

\[
\int_{\Omega_i} \frac{|\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-(1+\beta)t}}{(1 + e^{V_{i,t-c_2+m_i}+\beta t})} dV_{X,\omega} \leq C_3 e^t
\]

for some positive number \( C_3 \) independent of \( t \).

Since \( \sup_{t \leq 0} (e^t R_i(t)) < +\infty \), it follows from (4.4) that

\[
\int_{\Omega_i \cap \{\psi < t + c_2\}} |\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-(1+\beta)t} dV_{X,\omega} \leq C_4 e^t
\]

for any \( t \), where \( C_4 \) is a positive number independent of \( t \).

Since \( Y_i \) is bounded above on \( \Omega_i \), it follows from (4.5) that

\[
\int_{\Omega_i} |\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-(1+\beta)t} dV_{X,\omega} \leq C_5 e^{-\beta t}
\]

for any \( t \), where \( C_5 \) is a positive number independent of \( t \).

Since \( |f_{i,t}|^2 \) is subharmonic on \( \Omega_i \), by mean value inequality, we get from (4.7) that

\[
\sup_{\tilde{V}_i} |\tilde{f}_{i,t}|_{\omega,h_0}^2 \leq C_6 e^{-\beta t}
\]

for any \( t \), where \( C_6 \) is a positive number independent of \( t \).

Since (4.6) and (4.8) imply that the assumptions in Proposition 3.2 hold for \( \tilde{f}_{i,t} \), we apply Proposition 3.2 to \( \tilde{f}_{i,t} \) (1 \( \leq i \leq N \)) and get

\[
\lim_{t \to -\infty} \int_{U_i \cap \{t - c_1 < \psi < t + c_2\}} e^{t} |\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-\phi} \frac{(e^t + e^\phi)^2}{(e^{\psi + e^\phi})^2} dV_{X,\omega} \leq \int_{U_i \cap Y^0} \xi_i |\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-\phi} dV_{X,\omega} [\psi],
\]

which will be used in Step 4.

**Part IV:** construction of a family of smooth extensions \( \tilde{f}_t \) of \( f \) to a neighborhood of \( X_k \cap Y \) in \( X \).

Define \( \tilde{f}_t = \sum_{i=1}^{N} \xi_i \tilde{f}_{i,t} \) for all \( t \).

Since

\[
\tilde{f}_t|_{U_j} = \sum_{i=1}^{N} \xi_i \tilde{f}_{j,t} + \sum_{i=1}^{N} \xi_i (\tilde{f}_{i,t} - \tilde{f}_{j,t}) = \tilde{f}_{j,t} + \sum_{i=1}^{N} \xi_i (\tilde{f}_{i,t} - \tilde{f}_{j,t})
\]

for any \( j = 1, \cdots, N \), we have

\[
|D^n \tilde{f}_t|_{\omega,h_0}|_{U_j} = \sum_{i=1}^{N} \partial^\nu \xi_i \wedge (\tilde{f}_{i,t} - \tilde{f}_{j,t})|_{\omega,h_0}, \quad \forall t.
\]

Let \( \mu \) and \( W \) be as in the beginning of the proof of Proposition 3.2 (here \( W \) is centered at a point \( \bar{x} \in \mu^{-1}(U_i \cap U_j) \cap \{\psi = -\infty\} \)). For similar reasons as in
(3.18), (3.20) and (3.21), we get from (4.8) that

\[ |\tilde{f}_{i,t} \circ \mu - \tilde{f}_{j,t} \circ \mu|_{\tilde{\omega}_{i,h_0}}^2 \leq C_T e^{-\beta_{1,t}} \prod_{p \in \kappa} |u_p|^2 \]

when \( \kappa \neq \emptyset \) and \( t \) is small enough, and that

\[ |\tilde{f}_{i,t} \circ \mu - \tilde{f}_{j,t} \circ \mu|_{\tilde{\omega}_{i,h_0}}^2 \leq C_T e^{-\beta_{1,t}} \]

when \( \kappa = \emptyset \) and \( t \) is small enough, where

\[ W_{i,j,t} := W \cap \mu^{-1}(U_i \cap U_j) \cap \{ \psi \circ \mu < t + c_2 \} \]

and \( C_T \) is a positive number independent of \( t \).

**Step 2: singularity attenuation process for the current \( \sqrt{-1} \partial \bar{\partial} \phi \).**

Since the singularities of \( \sqrt{-1} \partial \bar{\partial} \psi \) obstruct the application of Lemma 2.2, we will work on \( \tilde{X} \) first and then go back to \( X \). Some ideas in this step come from [29].

Let \( \mu : \tilde{X} \to X \) be as in the beginning of the proof of Proposition 3.2. Let \( \tilde{X}_{k+1} := \mu^{-1}(X_{k+1}) \), \( \tilde{X}_k := \mu^{-1}(X_k) \) and \( \tilde{\Sigma}_0 := \mu^{-1}(\Sigma_0) \), where \( \Sigma_0 := \{ \psi = -\infty \} \). Then

\[ \gamma_1 := \sqrt{-1} \partial \bar{\partial} (\psi \circ \mu) - \sum_j q_j [D_j] \]

is a smooth real \((1,1)\)-form for some positive numbers \( q_j \), where \( (D_j) \) are the irreducible components of \( \tilde{\Sigma}_0 \). It is not hard to prove the following lemma and we won’t give its proof.

**Lemma 4.1.** There exists a positive number \( n_k \) such that

\[ \tilde{\omega}_{k+1} := n_k \mu^* \omega + \sqrt{-1} \partial \bar{\partial} (\psi \circ \mu) - \sum_j q_j [D_j] \]

is a Kähler metric on \( \tilde{X}_{k+1} \).

Since \( \mu : \tilde{X} \setminus \tilde{\Sigma}_0 \to X \setminus \Sigma_0 \) is biholomorphic and \( \sum_j q_j [D_j] \big|_{\tilde{X} \setminus \tilde{\Sigma}_0} = 0 \), the curvature assumptions \((i)\) and \((ii)\) in Theorem 1.1 implies that

\[ \sqrt{-1} \partial \bar{\partial} (\phi \circ \mu) \big|_{\tilde{X} \setminus \tilde{\Sigma}_0} + \gamma_2 \big|_{\tilde{X} \setminus \tilde{\Sigma}_0} \geq 0 \]

and

\[ \sqrt{-1} \partial \bar{\partial} (\phi \circ \mu) \big|_{\tilde{X} \setminus \tilde{\Sigma}_0} + \gamma_3 \big|_{\tilde{X} \setminus \tilde{\Sigma}_0} \geq 0 \]

hold on \( \tilde{X} \setminus \tilde{\Sigma}_0 \), where

\[ \gamma_2 := \sqrt{-1} \mu^* \Theta_{L,h_0} + \gamma_1, \quad \gamma_3 := \sqrt{-1} \mu^* \Theta_{L,h_0} + \left( 1 + \frac{1}{\chi(\alpha \circ \mu)} \right) \gamma_1. \]

Since \( \gamma_2 \) and \( \gamma_3 \) are continuous on \( \tilde{X} \), and \( \phi \circ \mu \) is quasi-plurisubharmonic on \( \tilde{X} \), we get that

\[ \sqrt{-1} \partial \bar{\partial} (\phi \circ \mu) + \gamma_2 \geq 0 \]

and

\[ \sqrt{-1} \partial \bar{\partial} (\phi \circ \mu) + \gamma_3 \geq 0 \]
hold on $\bar{X}$. Since there must exist a continuous nonnegative $(1,1)$-form $\varphi_{k+1}$ on the Kähler manifold $(\bar{X}_{k+1}, \bar{\varphi}_{k+1})$ such that

$$(\sqrt{-1} \Theta_{T_{\bar{X}_{k+1}}} + \varphi_{k+1} \circ \operatorname{Id}_{T_{\bar{X}_{k+1}}})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_{\bar{X}_{k+1}})$$

holds on $\bar{X}_{k+1}$, by Theorem 2.2, we obtain from (4.13) and (4.14) a family of functions $\{\phi_{\varsigma, \rho}\}_{\varsigma > 0, \rho \in (0, \rho_1)}$ on a neighborhood of the closure of $\bar{X}_k$ such that

(i) $\bar{\phi}_{\varsigma, \rho}$ is quasi-plurisubharmonic on a neighborhood of the closure of $\bar{X}_k$,

(ii) $\bar{\phi}_{\varsigma, \rho}$ converges to $\phi \circ \mu$ on $\bar{X}_k$ as $\rho \to 0$,

(iii) $\bar{\phi}_{\varsigma, \rho}$ is increasing with respect to $\varsigma$ and $\rho$ on $\bar{X}_k$, and

converges to $\phi \circ \mu$ on $\bar{X}_k$ as $\rho \to 0$.

Since $\bar{\varphi}_{k+1}$ is a Kähler metric on $\bar{X}_{k+1}$ by Lemma 4.1 and $\bar{X}_k$ is relatively compact in $\bar{X}_{k+1}$, there exists a positive number $n_k > 1$ such that $n_k \bar{\varphi}_{k+1} \geq \varphi_{k+1}$ holds on $\bar{X}_k$. Take $\varsigma = \delta_\rho$ and denote $\bar{\phi}_{\delta_\rho, \rho}$ simply by $\bar{\phi}_\rho$. Then $\bar{\phi}_\rho$ is quasi-plurisubharmonic on a neighborhood of the closure of $\bar{X}_k$, smooth on $\bar{X}_k \setminus E_{\delta_\rho, \rho}((\phi \circ \mu)$, increasing with respect to $\rho$ on $\bar{X}_k$, and converges to $\phi \circ \mu$ on $\bar{X}_k$ as $\rho \to 0$.

Furthermore,

$$\sqrt{-1} \bar{\partial} \bar{\partial} \bar{\phi}_\rho + \gamma_2 + 2\pi n_k \delta_\rho \bar{\varphi}_{k+1} \geq 0$$

and

$$\sqrt{-1} \bar{\partial} \bar{\partial} \bar{\phi}_\rho + \gamma_3 + 2\pi n_k \delta_\rho \bar{\varphi}_{k+1} \geq 0$$

hold on $\bar{X}_k$. Since $\mu : \bar{X}_k \setminus \Sigma_0 \to X_k \setminus \Sigma_0$ is biholomorphic, we get that

$$\sqrt{-1} \bar{\partial} \bar{\partial} (\bar{\phi}_\rho \circ \mu^{-1}) + (\mu^{-1})^* \gamma_2 + 2\pi n_k \delta_\rho (\mu^{-1})^* \bar{\varphi}_{k+1} \geq 0$$

and

$$\sqrt{-1} \bar{\partial} \bar{\partial} (\bar{\phi}_\rho \circ \mu^{-1}) + (\mu^{-1})^* \gamma_3 + 2\pi n_k \delta_\rho (\mu^{-1})^* \bar{\varphi}_{k+1} \geq 0$$

hold on $X_k \setminus \Sigma_0$. Then, replacing $\gamma_2, \gamma_3$ and $\bar{\varphi}_{k+1}$ with their definitions, we obtain that

(4.15) $\sqrt{-1} \bar{\partial} \bar{\partial} (\bar{\phi}_\rho \circ \mu^{-1}) + \sqrt{-1} \Theta_{L,h_0} + \left(1 + 2\pi n_k \delta_\rho \right) \sqrt{-1} \bar{\partial} \bar{\partial} \psi \geq -2\pi n_k \bar{\varphi}_{k} \delta_\rho \omega$

and

(4.16) $\sqrt{-1} \bar{\partial} \bar{\partial} (\bar{\phi}_\rho \circ \mu^{-1}) + \sqrt{-1} \Theta_{L,h_0} \left(1 + 2\pi n_k \delta_\rho + \frac{1}{\chi(\alpha)} \right) \sqrt{-1} \bar{\partial} \bar{\partial} \psi \geq -2\pi n_k \bar{\varphi}_{k} \delta_\rho \omega$

hold on $X_k \setminus \Sigma_0$.

Since $E_{\delta_\rho, \rho}((\phi \circ \mu)$ is an analytic set in $\bar{X}$, Remmert’s proper mapping theorem implies that

$$\Sigma_\rho := \mu(E_{\delta_\rho, \rho}((\phi \circ \mu))$$

is an analytic set in $X$. By Lemma 2.3, $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$ is a complete Kähler manifold.

It follows from the properties of $\bar{\phi}_\rho$ that $\bar{\phi}_\rho \circ \mu^{-1}$ is smooth on $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$, increasing with respect to $\rho$ on $X_k \setminus \Sigma_0$, uniformly bounded above on $X_k \setminus \Sigma_0$ with respect to $\rho$, and converges to $\phi$ on $X_k \setminus \Sigma_0$ as $\rho \to 0$. 

In Step 3, we will use $\tilde{\phi}_p \circ \mu^{-1}$ to construct a smooth metric of $L$ on $X_k \backslash (\Sigma_0 \cup \Sigma_p)$.

**Step 3: construction of additional weights and twist factors.**

Let $\zeta$, $\chi$ and $\eta$ be the solution to the following system of ODEs defined on $(-\infty, \alpha_0)$:

\begin{align*}
\chi(t)\zeta'(t) - \chi'(t) &= 1, \quad (4.17) \\
(\chi(t) + \eta(t))e^{\zeta(t)} &= \left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right)R(t), \quad (4.18) \\
\frac{(\chi'(t))^2}{\chi(t)\zeta''(t) - \chi''(t)} &= \eta(t), \quad (4.19)
\end{align*}

where we assume that $\zeta$, $\chi$ and $\eta$ are all smooth on $(-\infty, \alpha_0)$, and that $\inf_{t<\alpha_0} \zeta(t) = 0$, $\inf_{t<\alpha_0} \chi(t) = \alpha_1$, $\eta > 0$, $\zeta' > 0$ and $\chi' < 0$ on $(-\infty, \alpha_0)$. If $\alpha_0 = +\infty$, we replace the assumption $\inf_{t<\alpha_0} \chi(t) = \alpha_1$ by $\chi > 0$. By the similar calculation as in [14] or [31], we can solve the system of ODEs and the solution is

$$
\begin{align*}
\chi(t) &= \bar{\chi}(t), \\
\zeta(t) &= \log\left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right) - \log\left(\frac{\alpha_1}{R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)}\right), \\
\eta(t) &= R(t)\left(\frac{\alpha_1}{R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)}\right) - \bar{\chi}(t),
\end{align*}
$$

where $\bar{\chi}(t)$ is defined by (1.1).

Let $\epsilon \in (0, \frac{1}{2})$ be as in Step 1 and put $\sigma_t = \log(e^\psi + e^t) - \epsilon$. Then there exists a negative number $t_*$ such that $\sigma_t \leq \alpha - \frac{\epsilon}{2}$ on $X_k$ for any $t \in (-\infty, t_*)$.

Let $h_{p,t}$ be the new metric on the line bundle $L$ over $X_k \backslash (\Sigma_0 \cup \Sigma_p)$ defined by

$$
h_{p,t} := h_0e^{-\tilde{\phi}_p \circ \mu^{-1} - (1 + 2\pi n_k \rho) \psi - \zeta(\sigma_t)}.
$$

Let $\tau_t := \chi(\sigma_t)$ and $A_t := \eta(\sigma_t)$. Set $B_{p,t} := [\Theta_{p,t}, L]$ on $X_k \backslash (\Sigma_0 \cup \Sigma_p)$, where

$$
\Theta_{p,t} := \tau_t \sqrt{-1} \Theta_{L,h_{p,t}} - \sqrt{-1} \partial \bar{\partial} \tau_t - \sqrt{-1} \frac{\partial \tau_t \wedge \bar{\partial} \tau_t}{A_t}.
$$

Set $\nu_t = \partial \sigma_t$. We want to prove

$$
\Theta_{p,t} \mid_{X_k \backslash (\Sigma_0 \cup \Sigma_p)} \geq \frac{e^t}{e^\psi} \sqrt{-1} \nu_t \wedge \bar{\nu}_t - 2\pi n_k \bar{n}_k \chi(\sigma_t)\delta_{p,\omega}.
$$

It follows from (4.17) and (4.19) that

$$
\begin{align*}
\Theta_{p,t} \mid_{X_k \backslash (\Sigma_0 \cup \Sigma_p)} &= \chi(\sigma_t)\left(\sqrt{-1} \Theta_{L,h_{0}} + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_p \circ \mu^{-1} + (1 + 2\pi n_k \rho) \sqrt{-1} \partial \bar{\partial} \psi\right) \\
&\quad + \left(\chi(\sigma_t)\zeta'(\sigma_t) - \chi'(\sigma_t)\right) \sqrt{-1} \partial \bar{\partial} \tau_t \\
&\quad + \left(\chi(\sigma_t)\zeta''(\sigma_t) - \chi''(\sigma_t) - \left(\frac{\chi'(\sigma_t)}{\eta(\sigma_t)}\right)^2\right) \sqrt{-1} \partial \tau_t \wedge \bar{\partial} \tau_t \\
&= \chi(\sigma_t)\left(\sqrt{-1} \Theta_{L,h_{0}} + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_p \circ \mu^{-1} + (1 + 2\pi n_k \rho) \sqrt{-1} \partial \bar{\partial} \psi\right) + \sqrt{-1} \partial \bar{\partial} \tau_t \\
&= \chi(\sigma_t)\left(\sqrt{-1} \Theta_{L,h_{0}} + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_p \circ \mu^{-1} + (1 + 2\pi n_k \rho) \sqrt{-1} \partial \bar{\partial} \psi\right) + \sqrt{-1} \partial \bar{\partial} \tau_t \\
&= \frac{e^t}{e^\psi} \sqrt{-1} \nu_t \wedge \bar{\nu}_t + \frac{e^t}{e^\psi + e^t} \sqrt{-1} \partial \bar{\partial} \psi.
\end{align*}
$$
Since \( \chi \) is decreasing and \( \chi = \bar{\chi} \), it follows from (4.15) and (4.16) that
\[
\chi(\sigma_t)(\sqrt{-1}\Theta_{L,\rho t} + \sqrt{-1}\partial\bar{\partial}(\phi_\rho \circ \mu^{-1}) + (1 + 2\pi n_k \delta_\rho)\sqrt{-1}\partial\bar{\partial}\psi) \\
+ e^\psi + e^t \sqrt{-1}\partial\bar{\partial}\psi \\
= \chi(\sigma_t)(\sqrt{-1}\Theta_{L,\rho t} + \sqrt{-1}\partial\bar{\partial}(\phi_\rho \circ \mu^{-1}) + (1 + 2\pi n_k \delta_\rho)\sqrt{-1}\partial\bar{\partial}\psi + 2\pi n_k \bar{\nu}_k \delta_\rho) \\
- 2\pi n_k \bar{\nu}_k \chi(\sigma_t) \delta_\rho \omega + \frac{\chi(\alpha)e^\psi}{e^\psi + e^t} \cdot \frac{\sqrt{-1}\partial\bar{\partial}\psi}{\chi(\alpha)} \\
\geq \chi(\sigma_t)e^\psi \left(\sqrt{-1}\Theta_{L,\rho t} + \sqrt{-1}\partial\bar{\partial}(\phi_\rho \circ \mu^{-1}) + (1 + 2\pi n_k \delta_\rho)\sqrt{-1}\partial\bar{\partial}\psi \right) \\
+ 2\pi n_k \bar{\nu}_k \delta_\rho \omega + \frac{\sqrt{-1}\partial\bar{\partial}\psi}{\chi(\alpha)} - 2\pi n_k \bar{\nu}_k \chi(\sigma_t) \delta_\rho \omega \\
\geq -2\pi n_k \bar{\nu}_k \chi(\sigma_t) \delta_\rho \omega
\]
on \( X_k \setminus (\Sigma_0 \cup \Sigma_\rho) \). Hence we get (4.20) as desired.

Let \( \beta \) be as in Step 1. Let \( \beta_0 \) and \( \beta_3 \) be two positive numbers which will be determined later in Step 4. We choose an increasing family of positive numbers \( \{\rho_t\}_{t \in (-\infty, t_s)} \) such that \( \lim_{t \to -\infty} \rho_t = 0 \) and for any \( t \),
\[
(4.21) \\
2\pi n_k \bar{\nu}_k \chi(t-1) \delta_\rho_t < e^{\beta_0 t}, \\
(4.22) \\
2\pi n_k \delta_\rho_t < \beta_3,
\]
and
\[
(4.23) \\
\left(\frac{\epsilon}{2} - \epsilon \right)^{2\pi n_k \delta_\rho_t} > \frac{1}{1 + \epsilon}.
\]

Since \( \sigma_t \geq t - 1 \) on \( X_k \) and \( \chi \) is decreasing, we have \( \chi(\sigma_t) \leq \chi(t - 1) \) on \( X_k \). Then it follows from (4.20) and (4.21) that
\[
\Theta_{\rho_t, f}|_{X_k \setminus (\Sigma_0 \cup \Sigma_\rho)} \geq \frac{e^t}{e^\psi} \sqrt{-1}\nu_t \wedge \bar{\nu}_t - e^{\beta_0 t} \omega.
\]
Hence
\[
(4.24) \\
B_{\rho_t, f} + e^{\beta_0 t} I \geq \left[ \frac{e^t}{e^\psi} \sqrt{-1}\nu_t \wedge \bar{\nu}_t, \Lambda \right] = \frac{e^t}{e^\psi} T_{\rho_t} T^*_\rho \geq 0
\]
on \( X_k \setminus (\Sigma_0 \cup \Sigma_\rho) \) as an operator on \((n, 1)\)-forms, where \( T_{\rho_t} \) denotes the operator \( \bar{\nu}_t \wedge \bullet \) and \( T^*_\rho \) is its Hilbert adjoint operator.

**Step 4:** construction of suitably truncated forms and solving \( \partial_t \) globally with \( L^2 \) estimates.

In this step and Step 5, we will denote \( B_{\rho_t, f} \) and \( h_{\rho_t, f} \) simply by \( B_t \) and \( h_t \) respectively.

Let \( \epsilon \in (0, \frac{1}{2}) \) be as in Step 1. It is easy to construct a smooth function \( \theta : \mathbb{R} \to [0, 1] \) such that \( \theta = 0 \) on \((-\infty, \frac{\epsilon}{2}] \), \( \theta = 1 \) on \([1 - \frac{\epsilon}{2}, +\infty) \) and \( |\theta'| \leq \frac{1 + \epsilon}{1 - \epsilon} \) on \( \mathbb{R} \).

Define \( g_t = D''(\theta(\frac{e^t}{e^\psi} + e^t)\hat{f}_t) \), where \( \hat{f}_t \) is constructed in Step 1. Then \( D''g_t = 0 \) and
\[
g_t = -\theta'(\frac{e^t}{e^\psi} + e^t)(\frac{e^t}{e^\psi} + e^t)^2 \partial_t \hat{f}_t + \theta(\frac{e^t}{e^\psi} + e^t)D'' \hat{f}_t \\
= g_{1,t} + g_{2,t}, \]
where $g_{1,t}$ denotes $-\nu_t \wedge \theta'\frac{e^t}{e^{\tau t}} \frac{e^t}{e^{\tau t}} f_t$ and $g_{2,t}$ denotes $\theta(\frac{e^t}{e^{\tau t}})D^n f_t$. Then

$$\text{supp} \ g_{1,t} \subset \{ t - c_1 < \psi < t + c_2 \}$$

and

$$\text{supp} \ g_{2,t} \subset \{ \psi < t + c_2 \},$$

where $c_1$ and $c_2$ are defined as in Step 1.

It follows from (3.5) and (4.24) that

$$(4.25) \quad \langle (B_t + 2e^{\beta t}I)^{-1}g_{1,t}, g_{1,t} \rangle_{\omega,h_t} \leq \frac{1 + \epsilon}{\epsilon} \langle (B_t + 2e^{\beta t}I)^{-1}g_{2,t}, g_{2,t} \rangle_{\omega,h_t} + \frac{1 + \epsilon}{\epsilon} \langle (B_t + 2e^{\beta t}I)^{-1}g_{1,t}, g_{1,t} \rangle_{\omega,h_t} + \frac{1 + \epsilon}{\epsilon} \langle (B_t + 2e^{\beta t}I)^{-1}g_{2,t}, g_{2,t} \rangle_{\omega,h_t}.$$  

By (4.24), we have

$$\langle (B_t + 2e^{\beta t}I)^{-1}g_{1,t}, g_{1,t} \rangle_{\omega,h_t} \leq \frac{e^t}{e^t} e^t e^{\psi} \langle f_t, f_t \rangle_{\omega,h_t}.$$  

Then $\zeta > 0$ implies that

$$I_{1,t} := \int_{X_t \setminus (\Sigma_0 \cup \Sigma_{\mu'})} \langle (B_t + 2e^{\beta t}I)^{-1}g_{1,t}, g_{1,t} \rangle_{\omega,h_t} dV_{X,\omega} \leq \frac{(1 + \epsilon)}{1 - \epsilon} \int_{X_t \cap \{t - c_1 < \psi < t + c_2\}} e^t \frac{|f_t|_{\omega,h_0}^2 e^{-\phi_{\mu'}^t}}{(e^t + 2e^t)^2} dV_{X,\omega}.$$  

Since $\phi_{\mu'}^t \circ \mu^{-1} \geq \phi$ on $X_t \setminus \Sigma_0$, it follows from (4.23) that

$$I_{1,t} \leq \frac{(1 + \epsilon)}{1 - \epsilon} \int_{X_t \cap \{t - c_1 < \psi < t + c_2\}} e^t \frac{|f_t|_{\omega,h_0}^2 e^{-\phi}}{(e^t + 2e^t)^2} dV_{X,\omega}.$$  

Since

$$|f_t|_{2,\omega,h_0}^2 \leq \frac{N}{\epsilon} \sum_{i=1}^{N} \xi_i \sqrt{\sum_{i=1}^{N} \xi_i |f_i|_{2,\omega,h_0}^2} = \frac{N}{\epsilon} \sum_{i=1}^{N} \xi_i |f_i|_{2,\omega,h_0} = \sum_{i=1}^{N} \xi_i |f_i|_{2,\omega,h_0}$$

by the Cauchy-Schwarz inequality, we have

$$I_{1,t} \leq \frac{(1 + \epsilon)}{1 - \epsilon} \int_{X_t \cap \{t - c_1 < \psi < t + c_2\}} e^t \frac{\xi_i |f_i|_{2,\omega,h_0}^2 e^{-\phi}}{(e^t + 2e^t)^2} dV_{X,\omega}.$$  

Then it follows from (4.9) that

$$\lim_{t \to -\infty} I_{1,t} \leq \sum_{i=1}^{N} \int_{X_t \cap \{t - c_1 < \psi < t + c_2\}} e^t \frac{\xi_i |f_i|_{2,\omega,h_0}^2 e^{-\phi}}{(e^t + 2e^t)^2} dV_{X,\omega} \leq \frac{(1 + \epsilon)}{1 - \epsilon} \int_{X_t \cap \{t - c_1 < \psi < t + c_2\}} e^t \frac{\xi_i |f_i|_{2,\omega,h_0}^2 e^{-\phi}}{(e^t + 2e^t)^2} dV_{X,\omega}.$$  

Then

$$\langle (B_t + 2e^{\beta t}I)^{-1}g_{1,t}, g_{1,t} \rangle_{\omega,h_t} \leq \frac{e^t}{e^t} e^t e^{\psi} \langle f_t, f_t \rangle_{\omega,h_t}.$$
When $t$ is small enough.

Since $\zeta(\sigma_t) > 0$ and $\phi_{\mu_t} \circ \mu^{-1} \geq \phi$ on $X_k \setminus \Sigma_0$, by (4.22), we have

$$I_{2,t} := \frac{1}{e^{\beta_0 t}} \int_{X_k \setminus \{\psi \leq t + c_2\}} |D'_t \hat{f}_{i,t} |^2 e^{-\phi} \phi_{\mu_t} \circ \mu^{-1} e^{(1 + 2\pi k \delta_0) \psi} dV_{X,\omega}$$

Then it follows from (4.10) and the Cauchy-Schwarz inequality that $I_{2,t}$ is bounded by the sum of the terms

$$\frac{C_9}{e^{\beta_0 t}} \int_{U_i \cap U_j \setminus \{\psi \leq t + c_2\}} \left| \int_{U_i \cap U_j \setminus \{\psi < t + c_2\}} \frac{|\hat{f}_{i,t} - \hat{f}_{j,t} |^2 e^{-\phi}}{e^{(1 + \beta_0) \psi}} dV_{X,\omega} \right| \leq C_9$$

for some positive number $C_9$ independent of $t$ when $t$ is small enough. Then by the Hölder inequality, we get

$$\int_{U_i \cap U_j \setminus \{\psi < t + c_2\}} \frac{|\hat{f}_{i,t} - \hat{f}_{j,t} |^2 e^{-\phi}}{e^{(1 + \beta_0) \psi}} dV_{X,\omega} \leq C_{10} \left( \int_{U_i \cap U_j \setminus \{\psi < t + c_2\}} \frac{|\hat{f}_{i,t} - \hat{f}_{j,t} |^2 e^{\psi R_0(\psi)}}{e^{(1 + \beta_0) \psi / \psi}} dV_{X,\omega} \right)^{\frac{\gamma}{\gamma + \beta_0}}$$

when $t$ is small enough, where $C_{10}$ is a positive number independent of $t$.

We will estimate the last integral above by estimating its pull-back under $\mu$. We cover $\mu^{-1}(U_i \cap U_j \setminus \{\psi \circ \mu < t + c_2\})$ by a finite number of coordinate balls such as $W$ in Step 1 in the proof of Proposition 3.2. It follows from (4.11) and (4.12) that for each $W$,

$$\int_{W_{i,j,t}} \frac{|\hat{f}_{i,t} \circ \mu - \hat{f}_{j,t} \circ \mu |^2 e^{-\phi}}{e^{(1 + \beta_0) \psi \circ \mu}} d\lambda(w) \leq C_{11} \int_{W_{i,j,t}} \frac{1}{\prod_{p=1}^n |w_p|^{2\beta_{5,p}}} d\lambda(w),$$

where

$$\beta_{5,p} := \beta_4 c_a + (c_a - b_p) - |(c_a - b_p)|_+.$$
\[ \beta_4 := \beta_3 \cdot \frac{1 + \beta}{\beta} + \beta_2 \cdot \frac{1}{\beta} + \beta_1, \]

and \( C_{11} \) is a positive number independent of \( t \).

Since
\[ (W \cap \{ \psi \circ \mu < t + c_2 \}) \subseteq \bigcup_{p=1}^{n} \{ \{ |w_p| < e^{\frac{t+c_2-m}{2|\alpha|}} \} \cap W \}, \]

where \( m := \inf \bar{u}(w) \), we obtain
\[ \int_{W_{i;\omega,t}} \frac{1}{\prod_{p=1}^{n} |w_p|^{2\beta_{5,p}}} d\lambda(w) \leq \sum_{p=1}^{n} \int_{\{ |w_p| < e^{\frac{t+c_2-m}{2|\alpha|}} \} \cap W} \frac{1}{\prod_{p=1}^{n} |w_p|^{2\beta_{5,p}}} d\lambda(w) \]
\[ \leq C_{12} \sum_{p=1}^{n} e^{\frac{1-\beta_{5,p}}{e^{\frac{1}{\beta}} t}} \]

when \( \max_{1 \leq p \leq n} \beta_{5,p} < 1 \), where \( C_{12} \) is a positive number independent of \( t \).

Let \( \beta_1 \) be a positive number such that
\[ (4.28) \quad \beta_1 < \min_{\{p: a_p \neq 0\}} \frac{1 - (ca_p - b_p) + |ca_p - b_p| +}{3ca_p} \]

Take \( \beta_2 = \beta_1 \beta, \beta_3 = \frac{\beta_1 \beta}{1 + \beta} \). Then \( \beta_4 = 3\beta_1 \) and \( \max_{1 \leq p \leq n} \beta_{5,p} < 1 \).

Let \( \beta_0 \) be a positive number such that
\[ \beta_0 < \min_{1 \leq p \leq n} \frac{\beta(1 - \beta_{5,p})}{2(1 + \beta)|a|} \]

for every \( W \). Then we have
\[ (4.29) \quad I_{2,t} \leq C_{13} \cdot e^{\beta_0 t}, \]

where \( C_{13} \) is a positive number independent of \( t \).

Therefore, it follows from (4.25), (4.26) and (4.29) that
\[ \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_k})} \langle (B_t + 2e^{\beta_0 t})^{-1} g_t, g_t \rangle_\omega, h_t dV_{X,\omega} \leq (1 + \epsilon)I_{1,t} + \frac{1 + \epsilon}{\epsilon} I_{2,t} \leq C(t), \]

where
\[ C(t) := \frac{(1 + \epsilon)^5}{(1 - \epsilon)^2} \int_{X_0} |f|^2_{\omega, h_0} e^{-\phi} dV_{X,\omega}[\psi] + \frac{1 + \epsilon}{\epsilon} C_{13} \cdot e^{\beta_0 t}. \]

Then by Lemma 2.1, there exists \( u_{k,t} \in L^2(X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_k}), k_X \otimes L, h_t) \) and \( v_{k,t} \in L^2(X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_k}), k^{n+1}T^e_X \otimes L, h_t) \) such that
\[ (4.30) \quad D' u_{k,t} + \sqrt{2e^{\beta_0 t} v_{k,t}} = g_t \]
on \( X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_k}) \) and
\[ (4.31) \quad \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_k})} |u_{k,t}|^2_{\omega, h_0} e^{-\phi_{\rho_k} \mu^{-1} - (1+2\pi n_k \delta_{\rho_k}) \psi - \zeta(\sigma_t)} dV_{X,\omega} \]
\[ + \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_k})} |v_{k,t}|^2_{\omega, h_0} e^{-\phi_{\rho_k} \mu^{-1} - (1+2\pi n_k \delta_{\rho_k}) \psi - \zeta(\sigma_t)} dV_{X,\omega} \]
\[ \leq C(t). \]
Since \( \{ \tilde{\phi}_{p_t} \circ \mu^{-1} \} \) are uniformly bounded above on \( X_k \setminus \Sigma_0 \) with respect to \( t \) as obtained in Step 2, we have

\[
e^{-\tilde{\phi}_{p_t} \circ \mu^{-1}} \geq C_{14}
\]

for \( X_k \setminus \Sigma_0 \) for any \( t \), where \( C_{14} \) is a positive number independent of \( t \). Since \( t - \epsilon \leq \sigma_t \leq \alpha - \frac{\epsilon}{2} \) on \( X_k \) and \( \psi \) is upper semicontinuous on \( X \), we have that \( \psi \), \( \zeta(\sigma_t) \) and \( \tau_t + A_t \) are all bounded above on \( X_k \) for each fixed \( t \). Then it follows from (4.31) that \( u_{k,t} \in L^2 \) and \( v_{k,t} \in L^2 \). Hence it follows from (4.30) and Lemma 2.5 that

\[
\begin{align*}
D'' u_{k,t} + \sqrt{2e^{\beta_k}} v_{k,t} &= D'' \left( \frac{e^t}{\psi + e^t} \tilde{f}_t \right) \\
\int_{X_k} \left| u_{k,t} \right|^2 &\leq \int_{X_k} \left| \tilde{f}_t \right|^2 e^{-\tilde{\phi}_{p_t} \circ \mu^{-1}} + \int_{X_k} \left| u_{k,t} \right|^2 e^{-\tilde{\phi}_{p_t} \circ \mu^{-1}} - \psi - \zeta(\sigma_t) dV_{X,\omega}
\end{align*}
\]

holds on \( X_k \). Furthermore, (4.31) and (4.18) imply that

\[
\begin{align*}
\int_{X_k} \frac{|u_{k,t}|^2}{e^\sigma (t + C(R\sigma_t))} dV_{X,\omega} + \int_{X_k} \left| \tilde{f}_t \right|^2 - \psi - \zeta(\sigma_t) dV_{X,\omega} \\
\leq e^{2\pi k \delta_{\epsilon,2} M_\psi C(t)}
\end{align*}
\]

where \( M_\psi := \sup_{X_k} \psi \).

Define \( F_{k,t} := -u_{k,t} \cdot \theta(\frac{e^t}{\psi + e^t}) \tilde{f}_t \). Then (4.33) implies that \( D'' F_{k,t} = \sqrt{2e^{\beta_k}} v_{k,t} \) on \( X_k \). Since \( \tilde{\phi}_{p_t} \circ \mu^{-1} \geq \phi \) on \( X_k \setminus \Sigma_0 \), it follows from (3.5) and (4.34) that

\[
\begin{align*}
\int_{X_k} \frac{|F_{k,t}|^2}{e^\sigma \max\{R(\psi - \epsilon), R(\sigma_t)\}} dV_{X,\omega} \\
\leq (1 + \epsilon) \int_{X_k} \left| u_{k,t} \right|^2 e^{-\tilde{\phi}_{p_t} \circ \mu^{-1}} dV_{X,\omega} + \frac{1 + \epsilon}{\epsilon} \int_{X_k} \left| \tilde{f}_t \right|^2 e^{-\tilde{\phi}_{p_t} \circ \mu^{-1}} dV_{X,\omega} \\
\leq (1 + \epsilon) e^{2\pi k \delta_{\epsilon,2} M_\psi \left( \frac{\alpha_1}{R(\alpha_0)} + C \right) C(t) + \tilde{C}(t)}
\end{align*}
\]

when \( t \) is small enough, where

\[
\tilde{\tilde{C}}(t) := \frac{1 + \epsilon}{\epsilon} \int_{X_k \cap \{ \psi < \tau_{t+\epsilon} \}} \left| \tilde{f}_t \right|^2 e^{-\tilde{\phi}} dV_{X,\omega}.
\]

Now we want to prove

\[
\lim_{t \to -\infty} \tilde{\tilde{C}}(t) = 0.
\]

As in (4.27), we can obtain from (4.4) that for \( i = 1, \cdots, N \),

\[
\int_{\Omega_t \cap \{ \psi < t+\epsilon \}} \frac{\left| \tilde{f}_{i,t} \right|^2}{e^\sigma R(\psi - \epsilon)} dV_{X,\omega} \leq C_{15}
\]
for some positive number $C_{15}$ independent of $t$ when $t$ is small enough. Then by the H"older inequality, we have that
\[
\int_{U_i \cap X_i \cap \{\psi < t + c_2\}} \frac{|	ilde{f}_{i,t}|^2}{e^\psi R(\psi - \epsilon)} dV_{X,\omega} \
\leq \left( \int_{U_i \cap \{\psi < t + c_2\}} \frac{|	ilde{f}_{i,t}|^2}{e^\psi R(\psi - \epsilon)} dV_{X,\omega} \right)^{\frac{\alpha}{\alpha + \beta}} 
\times \left( \int_{U_i \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t}|^2}{e^\psi R(\psi - \epsilon)} dV_{X,\omega} \right)^{\frac{\beta}{\alpha + \beta}} 
\leq C_{15} \left( \int_{U_i \cap \{\psi < t + c_2\}} \frac{|	ilde{f}_{i,t}|^2}{e^\psi R(\psi - \epsilon)} dV_{X,\omega} \right)^{\frac{\beta}{\alpha + \beta}}
\]
when $t$ is small enough.

We cover $\mu^{-1}(U_i) \cap \{\psi \circ \mu < t + c_2\}$ by a finite number of coordinate balls such as $W$ in Step 1 in the proof of Proposition 3.2. Then, in order to prove $\lim_{t \to -\infty} C(t) = 0$, it suffices to prove
\[
\lim_{t \to -\infty} \int_{W_{i,t}} \frac{|	ilde{f}_{i,t} \circ \mu|_\omega^2 |J_\mu|^2}{e^{\psi \circ \mu} R(\psi \circ \mu - \epsilon)} d\lambda(w) = 0,
\]
where $W_{i,t} := W \cap \mu^{-1}(U_i) \cap \{\psi \circ \mu < t + c_2\}$.

Then by (3.18), (3.19), (3.20) and (3.21), it suffices to prove
\[
\text{(4.37) } \lim_{t \to -\infty} \int_{W_{i,t}} \frac{d\lambda(w)}{R(\psi \circ \mu - \epsilon) |w_{p_0}|^2 \prod_{\substack{1 \leq p \leq n, p \neq p_0 \atop 1 \leq r \leq m, r \neq \beta}} |w_p|^{2(\alpha_p - \beta_p) - 2(\alpha_p - b_p) + 2(\alpha_p - b_p)}} = 0
\]
in Case (A) and
\[
\text{(4.38) } \lim_{t \to -\infty} \int_{W_{i,t}} \frac{d\lambda(w)}{R(\psi \circ \mu - \epsilon) \prod_{p=1}^n |w_p|^{2(\beta_p \alpha_p + 2(\alpha_p - b_p) - 2(\alpha_p - b_p) + 2(\alpha_p - b_p))}} = 0
\]
in Case (A) and Case (B).

Applying Fubini's theorem with respect to $(w', w_{p_0})$ and then using change of variables, we can obtain that
\[
\lim_{t \to -\infty} \int_{W_{i,t}} \frac{d\lambda(w)}{R(\psi \circ \mu - \epsilon) |w_{p_0}|^2 \prod_{\substack{1 \leq p \leq n, p \neq p_0 \atop 1 \leq r \leq m, r \neq \beta}} |w_p|^{2(\alpha_p - \beta_p) - 2(\alpha_p - b_p) + 2(\alpha_p - b_p)}} \leq C_{16} \lim_{t \to -\infty} \int_{-\infty}^{t+c_2-m} \frac{ds}{R(s + M - \epsilon)} = 0,
\]
where $M := \sup_{W} \tilde{u}(w)$, $m := \inf_{W} \tilde{u}(w)$ and $C_{16}$ is a positive number independent of $t$. Hence we get (4.37).

Similarly, it is easy to see that (4.28) implies that (4.38).

Therefore, we obtain (4.36).
Let \( \widehat{\alpha}_k := \sup_{X_k} \alpha \). Then

\[
e^\psi \max \{ R(\psi - \epsilon), R(\sigma_t) \} \leq e' \sup_{t \leq \widehat{\alpha}_k} (e^t R(t)).
\]

Hence it follows from (4.32) and (4.35) that

\[
(4.39) \quad \int_{X_k} |F_{k,e,t}|^2_{\omega, h_0} dV_{X, \omega} \leq C_{17}
\]

for some positive number \( C_{17} \) independent of \( t \) when \( t \) is small enough.

Since the positive continuous function \( R \) is decreasing near \(-\infty\), it is easy to see that \( \max \{ R(\psi - \epsilon), R(\sigma_t) \} \) is equal to \( R(\psi - \epsilon) \) near \( \{ \psi = -\infty \} \) and converges uniformly to \( R(\psi - \epsilon) \) on \( X_k \) as \( t \to -\infty \).

Since \( \overline{\partial}_\mu \circ \mu^{-1} \) is increasing with respect to \( t \) and converges to \( \phi \) on \( X_k \setminus \Sigma_0 \) as \( t \to -\infty \), by extracting weak limits of \( \{ F_{k,e,t} \} \) as \( t \to -\infty \), we get from (4.39) and (4.35) a sequence \( \{ t_j \}_{j=1}^{\infty} \) and \( F_{k,e} \in L^2 \) such that \( \lim_{j \to +\infty} t_j = -\infty \), and \( F_{k,e,t_j} \to F_{k,e} \)

weakly in \( L^2 \) as \( j \to +\infty \) and

\[
(4.40) \quad \int_{X_k} |F_{k,e,t}|^2_{\omega, h_0} e^{-\phi} dV_{X, \omega} \leq \left( 1 + e \right) \frac{1}{(1 - e)^2} R(\alpha_0) C_R \int_{\Sigma_0} |f|^2_{\omega, h_0} e^{-\phi} dV_{X, \omega} [\psi].
\]

Since \( \sigma_t \leq \alpha - \frac{\epsilon}{2} \) on \( X_k \), \( \widehat{\alpha}_k := \sup_{X_k} \alpha \) and \( \zeta \) is increasing, we get

\[
e^{-\zeta(\sigma_t)} \geq e^{-\zeta(\widehat{\alpha}_k - \frac{\epsilon}{2})}
\]

on \( X_k \). Then (4.34), (4.32) and (4.41) imply that

\[
\int_{X_k} |v_{k,e,t}|^2_{\omega, h_0} dV_{X, \omega} \leq e^{\zeta(\widehat{\alpha}_k - \frac{\epsilon}{2}) + (1 + 2\pi n_k \delta_{\alpha_0}) M_c C_{14} C(t),
\]

Hence \( \sqrt{2 e^{\phi}} v_{k,e,t} \to 0 \) in \( L^2 \) as \( j \to +\infty \). Since \( D'' F_{k,e,t} = \sqrt{2 e^{\phi}} v_{k,e,t} \) on \( X_k \), we get \( D'' F_{k,e} = 0 \) on \( X_k \). Then \( F_{k,e} \) is a holomorphic section of \( K_X \otimes L \) on \( X_k \).

In Step 5, we will prove that \( F_{k,e} = f \) on \( X_k \cap Y^0 \) by solving \( \overline{\partial} \) locally.

**Step 5:** solving \( \overline{\partial} \) locally with \( L^2 \) estimates and the end of the proof for the line bundle \( L \).

For any \( x \in X_k \cap Y \), let \( \Omega_x \) be as in Step 1. Let

\[
\widehat{\Omega}_x \subset \subset (X_k \cap \Omega_x)
\]

be a coordinate ball with center \( x \). Since the bundle \( L \) is trivial on \( \Omega_x \), \( u_{k,e,t} \) and \( v_{k,e,t} \) can be regarded as forms on \( \Omega_x \) with values in \( \mathbb{C} \) and the metric \( h_0 \) of \( L \) on \( \Omega_x \) can be regarded as a positive smooth function.

It is easy to see that \( C(t) \leq C_{18} \) for some positive number \( C_{18} \) independent of \( t \) when \( t \) is small enough. Then it follows from (4.34), (4.41) and (4.32) that

\[
\int_{\widehat{\Omega}_x} |v_{k,e,t}|^2 e^{-\psi} d\lambda \leq C_{19} C_{18}
\]

for some positive number \( C_{19} \) independent of \( t \) when \( t \) is small enough.

Since \( \overline{\partial} v_{k,e,t} = 0 \) on \( \Omega_x \) by (4.33), applying Lemma 2.4 to the \((n,1)\)-form

\[
\sqrt{2 e^{\phi}} v_{k,e,t} \in L^2_{(n,1)}(\widehat{\Omega}_x, e^{-\psi}),
\]

we get an \((n,0)\)-form \( s_{k,e,t} \in L^2_{(n,0)}(\widehat{\Omega}_x, e^{-\psi}) \) such that

\[
\overline{\partial} s_{k,e,t} = \sqrt{2 e^{\phi}} v_{k,e,t}
\]
on $\hat{\Omega}_x$ and

$$\int_{\hat{\Omega}_x} |s_{k,\epsilon,t}|^2 e^{-\psi} d\lambda \leq C_{20} \int_{\hat{\Omega}_x} |\sqrt{2e^{i\beta t}} v_{k,\epsilon,t}|^2 e^{-\psi} d\lambda \leq 2C_{20} C_{19} e^{\beta t} t$$

for some positive number $C_{20}$ independent of $t$. Hence

$$\int_{\hat{\Omega}_x} |s_{k,\epsilon,t}|^2 d\lambda \leq C_{21} e^{\beta t}$$

for some positive number $C_{21}$ independent of $t$.

Now define $G_{k,\epsilon,t} = -u_{k,\epsilon,t} - s_{k,\epsilon,t} + \theta\left(\frac{e^\epsilon}{e^{\psi_i} + \epsilon}\right)f_0$ on $\hat{\Omega}_x$. Then $G_{k,\epsilon,t} = F_{k,\epsilon,t} - s_{k,\epsilon,t}$ and $\partial G_{k,\epsilon,t} = 0$. Hence $G_{k,\epsilon,t}$ is holomorphic in $\hat{\Omega}_x$. Therefore, $u_{k,\epsilon,t} + s_{k,\epsilon,t}$ is smooth in $\hat{\Omega}_x$. Furthermore, we get from (4.39) and (4.43) that

$$\int_{\hat{\Omega}_x} |G_{k,\epsilon,t}|^2 d\lambda \leq 2 \int_{\hat{\Omega}_x} |F_{k,\epsilon,t}|^2 d\lambda + 2 \int_{\hat{\Omega}_x} |s_{k,\epsilon,t}|^2 d\lambda \leq C_{22}$$

for some positive number $C_{22}$ independent of $t$ when $t$ is small enough. We get from (4.32) and (4.34) that

$$\int_{\hat{\Omega}_x} \frac{|u_{k,\epsilon,t}|^2 e^{-\psi}}{R(\sigma_i)} d\lambda \leq C_{23} C(t) \leq C_{23} C_{18}$$

for some positive number $C_{23}$ independent of $t$ when $t$ is small enough. Since $R(\sigma_i) \leq R(t - \epsilon)$ on $\hat{\Omega}_x$ when $t$ is small enough, we have that

$$\int_{\hat{\Omega}_x} |u_{k,\epsilon,t}|^2 e^{-\psi} d\lambda \leq C_{23} C_{18} R(t - \epsilon).$$

Therefore, combining the last inequality and (4.42), we obtain that

$$\int_{\hat{\Omega}_x} |u_{k,\epsilon,t} + s_{k,\epsilon,t}|^2 e^{-\psi} d\lambda \leq 2C_{23} C_{18} R(t - \epsilon) + 4C_{20} C_{19} e^{\beta t} t.$$

Then the non-integrability of $e^{-\psi}$ along $\hat{\Omega}_x \cap Y$ and the smoothness of $u_{k,\epsilon,t} + s_{k,\epsilon,t}$ in $\hat{\Omega}_x$ show that $u_{k,\epsilon,t} + s_{k,\epsilon,t} = 0$ on $\hat{\Omega}_x \cap Y$ for any $t$. Hence $G_{k,\epsilon,t} = f$ on $\hat{\Omega}_x \cap Y^0$ for any $t$.

Since $s_{k,\epsilon,t_j} \to 0$ in $L^2_{\Omega}(\hat{\Omega}_x)$ by (4.43) and $F_{k,\epsilon,t_j} \to F_{k,\epsilon}$ weakly in $L^2_{\Omega}(\hat{\Omega}_x)$ as $j \to +\infty$, $G_{k,\epsilon,t_j} \to F_{k,\epsilon}$ weakly in $L^2_{\Omega}(\hat{\Omega}_x)$ as $j \to +\infty$. Hence it follows from (4.44) and routine arguments with applying Montel’s theorem that a subsequence of $\{G_{k,\epsilon,t_j}\}_{j=1}^{+\infty}$ converges to $F_{k,\epsilon}$ uniformly on compact subsets of $\hat{\Omega}_x$. Then $F_{k,\epsilon} = f$ on $\hat{\Omega}_x \cap Y^0$ and thereby on $X_k \cap Y^0$.

Since the positive continuous function $R$ is decreasing near $-\infty$, $e^t R(t)$ is bounded above near $-\infty$ and $\phi$ is locally bounded above, applying Montel’s theorem and extracting weak limits of $\{F_{k,\epsilon}\}_{k,\epsilon}$, first as $\epsilon \to 0$, and then as $k \to +\infty$, we get from (4.40) a holomorphic section $F$ on $X$ with values in $K_X \otimes L$ such that $F = f$ on $Y^0$ and

$$\int_X \frac{|F|^2_{\omega_h}}{e^{vR(\psi)}} dV_{X,\omega} \leq \left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right) \int_{Y^0} |f|^2_{\omega,\psi} dV_{X,\omega}[\psi].$$

Theorem 1.1 is thus proved for the line bundle $L$.

**Step 6: the proof for the vector bundle $E$.**

The proof for $E$ is similar but simpler. We only point out the main modifications by examining the proof for $L$. 

In Step 1, we don’t need to construct a family of special smooth extensions $\tilde{f}_t$ of $f$ since $h_E$ is smooth. Hence the strong openness property and the key propositions are not needed. Delete Part II and Part III in Step 1 and replace the family of sections $\tilde{f}_{i,t}$ with a fixed local holomorphic extension $\tilde{f}_i$. Then $\tilde{f}_t$ becomes a fixed smooth extension $\tilde{f} = \sum_{i=1}^N \xi_i \tilde{f}_i$. Then it is easy to see that (4.9), (4.10), (4.11) and (4.12) hold for $\tilde{f}_{i,t} = \tilde{f}_i$, $\tilde{f}_t = \tilde{f}$ and $\beta_1 = 0$.

Step 2 is not needed since $h_E$ is already smooth.

In Step 3, the negative term will not appear on the right hand side of (4.20) since $\delta_\rho = 0$.

In Step 4, it is easy to prove the estimate (4.26) for $I_{1,t}$ by the modified (4.9). It is also not hard to prove the estimate (4.29) for $I_{2,t}$ by the modified (4.10), (4.11) and (4.12). (4.36) can be easily obtained since $h_E$ is smooth.

Step 5 for $E$ is almost the same and Theorem 1.1 is thus proved for the vector bundle $E$.

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