DIVERGING PERIOD AND VANISHING DISSIPATION: FAMILIES OF PERIODIC SINKS IN THE QUASI-CONSERVATIVE CASE

CORRADO FALCOLINI AND LAURA TEODESCHINI-LALLI
Dipartimento di Architettura, Università Roma Tre
Via della Madonna dei Monti 40, I-00184 Rome, Italy

Abstract. Hénon map is a well-studied classical example of area-contracting maps, modelling dissipative dynamics. The rich phenomena of coexistence of stable islands and their separatrices is typical of area-preserving maps, modelling conservative dynamics. In this paper we use the Hénon map to ascertain that coexistence of sinks is greater and greater approaching the conservative case, and that part of it can be organized following a renormalization argument. Using a numerical continuation that we devised, and called “dribbling method” [5], one can follow bifurcation paths from the area-preserving case into the dissipative one, organizing families of coexisting attractive periodic orbits with diverging period. When the dissipation parameter goes to zero, we will give numerical evidence of the increasing coexistence of such periodic orbits, in the coordinate and parameter space values. Vanishing dissipation and diverging period constitute a double limit that we study as such, giving evidence of a singularity in the limit. The families we study all appear as homoclinic bifurcation, and the fixed point causing the homoclinic onset also structures the renormalization scheme. One of the goals of this paper is to improve the results obtained by looking to higher periods, and to approach dissipation down to an area-contraction factor of $1 - 10^{-8}$. Using the same dribbling method, as further promising application, we also deal with the dissipative Standard map.

1. Introduction. We deal with the Hénon system of maps of the plane $(x, y)$, depending on two parameters $a$ and $b$:

$$T_{(a,b)}(x,y) := (a - x^2 - b y , x)$$

for values of the constant Jacobian $b$ in the quasi-conservative cases: orientation-preserving $b = 1 - \varepsilon$, and orientation-reversing $b = -1 + \varepsilon$, for very small values of $\varepsilon$. The “historical” Hénon model [9] is orientation reversing ($b = -0.3$), but physicists have found it important to deal with orientation preserving. Mathematically, the two cases differ in various aspects concerning possible bifurcations and their continuation in parameter plane $(b, a)$, and their singularities [10]. Moreover, one should bear in mind that the Standard map is orientation preserving, so we present both cases, for the Hénon model, warning the reader that some basic tools of the Standard map, such as rotation number, can be adapted easily to areas of the dynamical plane in the orientation-preserving case, but require more work, if any, in the orientation-reversing case.

2010 Mathematics Subject Classification. Primary: 37J25, 37J45; Secondary: 37F25.

Key words and phrases. Periodic orbits, bifurcation curves, Hénon map, standard map, renormalization schemes.

This research was partially supported by GNFM-INdAM.
The goals of the present paper are:

- to undertake a limiting approach as the dissipation $\varepsilon \to 0$, opposite or complementary to a perturbative approach. In both cases $\varepsilon$ is small.
- to give evidence of a singularity in the double limit of vanishing dissipation and diverging period.
- to improve the results obtained in previous papers about increasing coexistence of periodic sinks as $\varepsilon \to 0$, by looking at orbits of much higher periods, and moving toward the conservative case up to $\varepsilon = 10^{-8}$. In the orientation-reversing case this allows the further result that the boundaries of the stability ranges accumulate in an oscillatory fashion as the period $p$ diverges.
- to extend a full analysis of the orientation–reversing to the orientation–preserving case and compare the two cases. To compare the results as $\varepsilon \to 0$.
- to discuss in detail the phenomenological arguments of our (converging) renormalization algorithm that allows results for diverging periods, hoping for future theoretical insights.
- to give a first example of application of the same techniques to the Standard map, by entering the highly dissipative regime.

A periodic orbit of period $p$ solves

$$(x_p, y_p) = (x, y)$$

where $(x_p, y_p) := T^p_{a,b}(x, y) := T_{a,b}(T_{a,b}^{p-1}(x, y))$ is the $p$-th iteration of $T$. The periodic orbit is linearly attracting if the eigenvalues of its final Jacobian matrix lie inside the unit circle. The matrix $J$ is product of the Jacobian matrix along the orbit $(x_i, y_i), i = 0, ..., p-1$:

$$J = \prod_{i=0}^{p-1} \begin{pmatrix} -2x_i & -b \\ 1 & 0 \end{pmatrix}.$$

At a saddle-node bifurcation an eigenvalue of $J$ is exactly 1 whereas it is -1 for a period-doubling bifurcation.

In our paper [5] we introduce a numerical method to continue bifurcation values $a(b)$ from the conservative case into the dissipative one, thus gauging the stability range of each orbit in the parameter plane. In that paper we discuss the coexistence of an increasing number of periodic attractors moving closely towards the singular case $b = -1$; the starting point was a method to “dribble” around the singularities of the bifurcation curves at $b = -1$. We choose the periodic solutions arising via a homoclinic bifurcation, due to a heteroclinic connection of the two fixed points.

In [6] we performed a similar study for the orientation–preserving case by applying our dribbling method from $b = 1$, and moreover continued all bifurcation curves (arising from homoclinic or heteroclinic bifurcations) onto $b = 0$. This way we connected continuously area-preserving maps with the one-dimensional logistic map. This in turn allowed us to discuss some global unexpected features and organizations of bifurcation curves in the entire $[b, a]$ parameter plane. The bifurcation curves we follow are organized in sequences of curves of diverging period; as the period diverges, the two sequences converge to smooth monotone curves, that we called backbones, which can be seen as representing homoclinic (respectively heteroclinic) onset. To the ends of the present paper, it is important to notice that either backbone shows a cusp singularity in $|b| = 1$ (i.e. $\varepsilon = 0$). In fact, in the present paper we stay in the vicinities of $|b| = 1$ and delve into the limiting approach to this singularity, both in $p$ and in $\varepsilon$. (In the 2016 paper we were rather interested...
in showing that such backbones also organize precisely a number of other features, variously defined in literature along 40 years, such as the rotation number of accessible saddles, the boundary of parameter areas that have positive probability of a blowing up of the solutions ...)

The present paper is organized as follows: in Section 2 we recall in detail our “dribbling” method for following bifurcation curves in plane \((b, a)\), by starting at a saddle of fixed period \(p\) at \(|b| = 1\); in Section 3, for a fixed \(\varepsilon = 10^{-5}\), we discuss a renormalization algorithm in \(p\), to detect and evaluate the stability intervals for higher and higher periods \(p\). In Section 4 we discuss the double limit in \(b\) and \(p\).

We compare results for \(b = 1 - \varepsilon\) and \(b = -1 + \varepsilon\) cases (with very small values of \(\varepsilon > 0\)) concerning the number of coexisting periodic attractors and the behavior of the stability intervals depending on \(p\) and collect the results in two tables clearly pointing at trends for \(|b| \to 1\); in particular, we show that the order of two limits \(p \to \infty\) and \(\varepsilon \to 0\) cannot be exchanged. In Section 5 we study, in the dynamical plane \((b, a)\), the coexistence of sinks comparing two quasi-conservative cases with \(a\) inside their range of maximal coexistence. In Section 6 we present a preliminary application of similar techniques in the case of the dissipative Standard map, by initializing our dribbling method on periodic saddles inside resonant horns. Finally in Section 7 the conclusions.

![Diagram of a possible singularity at \(|b| = 1\)](image)

**Figure 1.** “Dribbling” a possible singularity at \(|b| = 1\), in 4-dimensional space \((x, y, b, a)\) to be able to then approach it from elsewhere. Starting at step 1 on a period-7 saddle in the conservative case, follow branches of continuation curves in \((x, y, a, b)\), by switching independent variable at each turning point 2.3.4. At steps 3 and 4 one can follow a bifurcation curve with known continuation methods. Upper and lower curves are respectively period-doubling and saddle-node bifurcation curves.

2. The “dribbling method” for studying bifurcation curves in the limit into the conservative case. We know that a saddle-node bifurcation value at \(|b| = 1\) is in general rather singular, or hard to continue as such in parameter
space. We devised a method to “dribble” this singularity (possible or actual). The crucial observation is that saddle periodic points in dynamic plane are robust under perturbation, so they can always be continued into the dissipative regime. Once in the dissipative regime, one can look for the companion node, that necessarily exists (see [5] for details and arguments), as explained in the following.

Given an initial saddle periodic orbit of period \( p \) in the conservative cases, continue it into \( |b| < 1 \), keeping the parameter \( a \) fixed. At each \( \bar{b} \) where the \( p \)-saddle orbit still exists, we detect the value of \( a_{sn}(\bar{b}) \) at which the saddle and a node appear, and the value of \( a_{pd}(\bar{b}) \) for which the corresponding periodic node first doubles its period (see Fig.1). Such values lie on curves in space \((b, a)\), which can then be evaluated with a continuation method in the four variables \((x, y, a, b)\), see for instance [12]. This way we avoid the local effects of the possible singularity of the saddle-node bifurcation at \( |b| = 1 \): we enter the dissipative regime by “riding” a saddle periodic orbit, and then run back toward the conservative case “riding” the period-doubling and saddle-node bifurcation curves, to see what happens. This is why we called the method “dribbling”.

Now all is set, to study the singularity of the curves \( a_{sn}(b) \), \( a_{pd}(b) \) as \( |b| \to 1 \), which we did in the previously cited papers.

![Figure 2. The two conservative cases to initialize a dribbling method. Left: for \( b = -1 \) and \( a = 0.244 \) big dots represent a period 14 saddle. Right: for \( b = 1 \) and \( a = -0.719 \) big dots represent a period 7 saddle.](image)

We want to study families of orbits sharing a common geometry. So we initialize the search of a family of periodic orbits with increasing period. We start with a saddle orbit of short period \( p \) at one of the area preserving cases, \( |b| = 1 \). The advantage of starting with a low period is that orbits and their spatial patterns are usually easier to detect (see Fig.2).

We follow the method described above to detect bifurcation curves depending on \( p \). We follow the periodic orbit into \( |b| < 1 \) till a value of \( b = \bar{b} \) for which the orbit still exists. We then find the precise \( a \) values, depending on \( \bar{b} \) and \( p \), at which the orbit appears \((a_{sn}(\bar{b}))\) and at which it first doubles its period \((a_{pd}(\bar{b}))\) (these values exist in this case, as for [17]). For each \( \bar{b} \) the stability range (in \( a \)) of the
orbit of period $p$ is therefore the interval $[a_{sn}(\bar{b}), a_{pd}(\bar{b})]$. We finally use a classical continuation method to follow the curves $a_{sn}(b)$ and $a_{pd}(b)$, as $b$ varies, back toward the conservative case, in order to study the stability range of the periodic orbit for $|b| \to 1$. In parameter plane $[b, a]$, we obtain, in fact, an entire region of stability for each periodic orbit. As each curve also depends on period $p$, the parameters that allow for coexistence of sinks lie on the overlapping of the regions of stability for different periods $p$ in the $(b, a)$ plane.

3. Large periods: a renormalization scheme in $p$. We study families of bifurcation curves $\{a_{sn}^{(p)}(b)\}$, $\{a_{pd}^{(p)}(b)\}$ concerning sinks of period $p$ that arise via homoclinic bifurcations; all sinks are therefore geometrically organized in families according to the homoclinic connection. These periodic orbits can be described in terms of their period, or their rotation number, in the orientation–preserving case. Here we present detailed results on increasing $p$, pointing to a renormalization argument as $p \to \infty$, $b$ fixed. In this paper we improve the results obtained earlier, by looking to much higher periods and accuracy. By doing so, we were able to observe a non-monotone behavior of the bifurcation curves as $p$ grows, which will need further study.

The algorithm we used for a fixed $b = \bar{b}$ is a Newton method in three variables $(x, y, a)$ which can be stated as requirements on the periodicity in the space variables $(x, y)$ and an extra requirement on the eigenvalue of its Jacobian matrix: to be 1 for the saddle-node case, therefore solving for $a_{sn}$:

$$
\begin{align*}
(x, y) - T_p^{(a_{sn}, b)}(x, y) &= 0 \\
1 + b^p - tr(J) &= 0
\end{align*}
$$

and respectively to be $-1$ in the period-doubling case, therefore converging at $a_{pd}$.

![Figure 3](image.png)

**Figure 3.** Adjustments in renormalization scheme of the saddle node bifurcation value. If Newton’s method initialized using linear prediction seems not to converge, adjust initial guess only in one variable $y$, and try again. This figure illustrates dependence of the final error in the calculation of $a_{sn}$ from $y$–adjusted guess.
Figure 4. Blow-up of same error function of Fig.3 but with $y$-range $2 \cdot 10^{-4}$. The minimal error can now be detected using an automatic algorithm in the suitable interval where the error function is observed to be smooth.

As usual in Newton algorithms, the starting point is crucial for the convergence and has to be guessed in a very precise way, especially in this case of high non-linearity of iterated maps. The usual way around this initial value problem is the "shooting method", i.e. to take random initial conditions in a certain neighborhood of the variables $(x, y, a)^{(p)} := (x, y, a_{sn})^{(p)}(\bar{b})$ independently step by step in $p$. We prefer to search for a renormalization approach in $p$ which can predict a good guess exploiting $p$.

We describe the method for the orientation-preserving case and the saddle-node value: an analogous computation is valid for the period-doubling value and the orientation-reversing case.

Given a fixed value of $b = \bar{b} = 1 - 10^{-5}$, we start with three initial data $(x, y, a)^{(p-3)}$, $(x, y, a)^{(p-2)}$, $(x, y, a)^{(p-1)}$, for small consecutive values of the periods (say 6, 7, 8) detected and followed from the conservative case using our dribbling method. Later, on Section 4, we illustrate other values of $b$ close to the orientation preserving and orientation reversing conservative limits.

We now search for the triple $(x, y, a)^{(p)}$ for the next value of the period (say $p = 9$). Remind that $\{(x, y, a)^{(p)}\}$ is a homoclinic sequence, so as $p \to \infty$ the final trajectory is bi-asymptotic to the fixed point.

We make as first guess for $(x, y, a)^{(p)}$ an interpolation, independently on each of the three coordinates $x$, $y$, $a$. Compute an error function for the first step of a Newton method to check if the guess is good enough for the method to have hope to converge.

The guess for the solution of the system (2) will produce in this case an error function: $Err((x, y, a)^{(p)}) := ||(x, y)^{(p)} - P_{(a_{sn}, \bar{b})}^{(p)}(x, y)^{(p)}|| + |1 + \bar{b}^p - tr(J)|$.

While we noticed that the guess on $a$ is always good, the guess on the spatial coordinates $(x, y)$ (the dynamical plane) sometimes produces an initial error preventing the convergence of the Newton scheme.

Now there can be two different situations. Either a relatively simple "manual tuning" of initial values in the neighborhood of the guess is enough to trigger a
converging Newton scheme (which happens for relatively low periods), or not. The second case is the most interesting as it involves the dynamics, and choosing a different representative of the same periodic orbit.

First case: for low periods stay far from fixed point. Each initial point \((x, y)^{(p)}\) lies on a periodic orbit of period \(p\), therefore each periodic orbit has \(p - 1\) possible other representatives. For the first few values of \(p\) we choose as the initial representative, the position along the orbit that are at the maximum distance from the fixed point, because of their almost straight spatial position. We first allow a change only in the coordinate \(y\) to try to lower the error \(\text{Err}((\bar{x}, y, \bar{a})^{(p)}) = \text{Err}(y)\): see Fig.3 and Fig.4 for a plot of the function. This manual tuning of the \(y\) variable near the interpolation guess actually allows the calculation of the sequence \(\{(x, y, a)^{(p)}\}\) for periods \(p\) up to \(p^* = 50\).

Second case: for higher and higher periods move toward the hyperbolic fixed point. For \(p > p^*\), when the Newton method fails to converge, we move the three initial points \((x, y)^{(p-3)}, (x, y)^{(p-2)}, (x, y)^{(p-1)}\) along their corresponding orbits by \(k\) iterates, until we see that the guess on \(T^k(x, y)^{(p-3)}, T^k(x, y)^{(p-2)}, T^k(x, y)^{(p-1)}\) makes the Newton method converge. We find that this convergence is triggered for few values of \(k\) \((k = 34 in the example)\) when the \(k\)-th iterates enter the region where the dynamics is conjugate to its linear part. From this \(k\)-th iterate on, we repeat the renormalization procedure on \(p\).

After this empiric triggering, the initial guess yields a nice convergence in all variables \(a, x, y, p\) and allowed us to reach period as high as \(p = 400\). All computations have been done with Wolfram Mathematica 11.0 with 1000 digits of accuracy. The time spent on most of the algorithm, when convergence is attained, was few minutes of CPU time or less.

![Figure 5](image_url)

**Figure 5.** Stability ranges in the \(a\) parameter for \(b = 1 - 10^{-5}\) and periodic orbits of period \(p\) between 6 and 50. For \(a = \bar{a}_{max}(\varepsilon) = -0.93294\) there are a maximum of 10 coexisting attractors of period between \(p = 11\) and \(p_1 = 20\).
In the following subsections 3.1-3.3, we fix the value of the Jacobian $b = 1 - \varepsilon$ at $\varepsilon = 10^{-5}$ in order to illustrate our quantitative results within the families of periodic orbits. In Section 4, we then compare these results with changing $\varepsilon$.

3.1. Within the renormalized family: Coexistence of sinks. The careful study of bifurcation curves in parameter plane allows the detection of parameter values at which several sinks coexist. On the other hand, at $b$ fixed, our renormalization method yields two sequences \((a_{sn}^{(p)}, x_{sn}, y_{sn})\) and \((a_{pd}^{(p)}, x_{pd}, y_{pd})\) in $p$, respectively the saddle-node and the period-doubling bifurcation values for a sink of period $p$. In principle and in fact, the width of the stability range \((a_{pd}^{(p)} - a_{sn}^{(p)})\) of a periodic sink is smaller as its period $p$ is larger. In the following we describe a tool to study, at $b$ fixed, the decrease of the range with $p$. A similar study allows to detect the amount of coexistence of sinks of the same family. In the following section we use these same tools to study how the vanishing of $\varepsilon$, i.e. the amount of dissipation, affects both the amount of coexistence and the regime of convergence in $p$.

Considering, as an example, $b = 1 - \varepsilon$ at $\varepsilon = 10^{-5}$ fixed, we look for the endpoints of the $a$-stability range of a periodic sink of period $p$, for varying $p$, and obtain a sequence of intervals \((a_{pd}, a_{sn})\); to check for coexistence of sinks we look for the intersection of some of these intervals. In Fig.5 each stability range is a vertical segment; a horizontal line $a = \bar{a}$ intersecting several vertical segments signals coexisting sinks for that value $\bar{a}$. By sliding the horizontal line up and down, we were able to ascertain the maximum coexistence (that is the maximum integer $n$ such that $a_{sn}^{(p)} < \bar{a} < a_{pd}^{(p+n)}$) within the family under study, for the chosen $b$ value: here we define the value of $p_1$ as the averaged maximum $p$ belonging to an interval of maximal coexistence.

In this example (see Fig.5) the maximum number of coexisting periodic attractors is 10 as can be shown for instance at $\bar{a} = -0.93294$ and we have $p_1 = 20$. Later in the paper we will compare results at different values of $b$.

\[ \text{Figure 6. Width of the stability ranges in the } a \text{ parameter for } b = 1 - 10^{-5} \text{ and periodic orbits of period } p \text{ between 6 and 60. The rate of fast exponential convergence starts around } p = p_2 = 25 \]

3.2. Within the renormalized family: Convergence of stability ranges. The width of the stability range is expected to decrease in $p$. This is one of the aspects interestingly affected by the vicinity of the conservative map. Therefore we
have computed the rate of change in $p$ for the width of successive stability ranges $(a_{pd}^{(p)} - a_{sn}^{(p)})$, and we are able to remark that while they always decrease in $p$, as expected, we find two different rates, characterizing low and high periods. The rate decreases slowly for small $p$'s and much faster for larger values of $p$.

Here we define the value of $p_2$ at which the faster rate regime seems to start, as the following $p$ values well fit a line in logarithmic scale.

In Fig. 6 we show a log-plot of the width of the stability range that shows the two different regimes: for small values of $p < 20$ we see a smaller rate and for values of $p > p_2 \equiv 25$ we see a larger rate which can be seen as two different linear slopes in a logarithmic scale. We will see in the following section that, as $b \to 1$, the slower regime takes over and over, as might be expected in general for a small (constant Jacobian) dissipation.

3.3. **Within the renormalized family: Convergence of bifurcation values.** Sequences $a_{sn}^{(p)}$, and $a_{pd}^{(p)}$ are (perhaps surprisingly, work is in progress for generalization) non-monotone in $p$. Both sequences $a_{sn}^{(p)}$, and $a_{pd}^{(p)}$ initially decrease with growing $p$. Our computations show that this is the case as long as $(a_{pd}^{(p)} - a_{sn}^{(p)}) > (a_{sn}^{(p+1)} - a_{sn}^{(p)})$. For values of $p$ much larger than $p_2$, so well inside the fast convergence regime, both sequences $a_{sn}^{(p)}$ and $a_{pd}^{(p)}$ reach a minimum value: let us call $a_{min}$ the minimum value of $a_{sn}^{(p)}$ and $p_3$ the value of $p$ at which it is obtained ($a_{min} = a_{sn}^{(p_3)}$). Here $p_1 = 38$ and we also notice that $p = 36$ is the largest period such that $a_{sn}^{(p)} \leq a_{sn}^{(36)}$ for any $p \geq 36$.

![Graph showing stability ranges in the $a - a_{min}$ parameter for $b = 1 - 10^{-5}$ and periodic orbits of period $p$ between 34 and 70. The values of $a_{min}$ is the minimum value of $a_{sn}^{(p)}$ (reached for $p = p_3 = 38$). Observe that $p = 36$ is the largest period such that $a_{sn}^{(p)} \leq a_{sn}^{(36)}$ for any $p \geq 36$.

At $p = p_3$, both sequences $a_{sn}^{(p)}$ and $a_{pd}^{(p)}$ reach a minimum and start to increase in $p$ up to a limit (see Fig. 7).
\[ k \quad \varepsilon = 10^{-k} \quad b = -1 + \varepsilon \quad p_1(\varepsilon) \quad p_2(\varepsilon) \quad p_3(\varepsilon) \]

\[
\begin{array}{cccc}
5 & 10^{-5} & -0.99999 & 68 & 86 & 144 \\
6 & 10^{-6} & -0.999999 & 96 & 114 & 188 \\
7 & 10^{-7} & -0.9999999 & 128 & 146 & 236 \\
8 & 10^{-8} & -0.99999999 & 164 & 182 & 288 \\
\end{array}
\]

\textbf{Table 1.} Values of \( p_1(\varepsilon) < p_2(\varepsilon) < p_3(\varepsilon) \) for \( b = -1 + \varepsilon \) and decreasing \( \varepsilon \).

4. Discussion of the double limit: \( p \to \infty \) and \( \varepsilon \to 0 \). We would like to follow such study both increasing the period \( p \), keeping the parameter \( b \) fixed (i.e. with a renormalization argument in the period), and also varying \( b = -1 + \varepsilon \), and keeping the period \( p \) fixed. Bear in mind that diverging \( p \) is guaranteed by an homoclinic bifurcations structure and that the homoclinic onset curve is the limit in \( p \) of the bifurcation curves. By varying \(|b| \to 1\), for \( p \) fixed, and then letting \( p \to \infty \) one can see the cusp singularity of two curves characterizing homoclinic (respectively heteroclinic) onset, attained at the conservative case, and how the bifurcation curves are organized in the neighborhood of these curves and their singularity. The two singularities were carefully observed and studied in \([6]\); these observations are consistent with the computations in \([7]\). Such cusp singularities are attained at \((b = -1, a = 0)\) and \((b = 1, a = -1)\) respectively for the orientation–reversing and the orientation–preserving cases. In particular, there is a singularity not allowing the exchange in the two limits, in \( p \) and \( \varepsilon \). So we performed careful computations (necessarily in both limits), for all bifurcation curves, and for related quantities such as stability ranges and amount of coexistence of sinks.

We remark that the families of sinks we study arise via homoclinic bifurcations, created by a homoclinic onset and the “family” is characterized by a rotation number. Such sinks are called “simple” in \([16]\) and, later, they have been named differently by different authors (see, for example, \([8]\)).

We summarize our results with \( \varepsilon \to 0 \) in four Tables. To this end we define several quantities (already introduced in Section 3) depending on \( \varepsilon \).

First we define \( N_{\text{max}}(\varepsilon) \) as the maximum number of coexisting sinks, \( a_{\text{max}}(\varepsilon) \) as a fixed value of \( a \) in the range of maximum coexistence, and \( p^*(\varepsilon) \) the renormalization threshold at which the automatic procedure described in Section 3 applies. Moreover we define three special values of \( p(\varepsilon) \): \( p_1(\varepsilon) \) defined as the averaged maximum period of the coexisting sinks in the largest interval of coexistence (see Fig.5), \( p_2(\varepsilon) \) as the value of the period such that \( \forall p > p_2(\varepsilon) \) convergence of the stability ranges \( (a_{pd}^{(p)} - a_{sn}^{(p)}) \) is fast and \( p_3(\varepsilon) \) defined as the period at which a minimum is reached for \( a_{pd}^{(p)} \) and \( a_{sn}^{(p)} \). Obviously \( p_2(\varepsilon) \) is not uniquely defined, but we see it in a small range. Similarly not uniquely defined, and belonging to the same range, is \( p^*(\varepsilon) \) the triggering of the renormalization method. Thus \( p^*(\varepsilon) \sim p_2(\varepsilon) \), not surprisingly.

The interesting one, to the ends of the double limit, is \( p_2(\varepsilon) \): the onset of fast convergence. We notice \( p_2(\varepsilon) \to \infty \) approaching the conservative case, as fast convergence is typical of dissipative case. We remark that our findings always indicate \( p_1(\varepsilon) < p_2(\varepsilon) < p_3(\varepsilon) \) so that coexistence \( (p_1(\varepsilon)) \) pertains to periods low enough for the \( p \)-iterated map to remain quasi-conservative.
The unexpected minimum reached in \( p_3(\varepsilon) \) is an obvious obstacle to the double limit. We find also \( p_3(\varepsilon) \to \infty \) so that such obstacle is removed as one move toward the conservative case.

Taking \( \varepsilon = 10^{-k} \), our data for \( p_1(k), p_2(k) \) and \( p_3(k) \) (see Table 1, 2) appear as few terms of six quadratic sequences in \( k \): a single linear recurrence rule for all our data is then \( p(k + 2) = 3p(k + 1) - 3p(k) + p(k - 1) \). Taking this rule as valid for all \( k \), the six sequences are asymptotic as \( p(k) \sim 2k^2 \) for \( b = -1 + \varepsilon \) and as \( p(k) \sim k^2/2 \) for \( b = 1 - \varepsilon \). In fact all our data are explicitly obtained if we take, for \( b = -1 + \varepsilon \), the three sequences \( p_1(k) = 2(k^2 - 3k - 6) \), \( p_2(k) = 2(k^2 - 3k + 3) \), \( p_3(k) = 2(k^2 - 11k - 8) \) and, for \( b = 1 - \varepsilon \), \( p_1(k) = (k^2 - k + 20)/2 \), \( p_2(k) = (k^2 - k + 10)/2 \) and \( p_3(k) = (k^2 + 5k + 26)/2 \).

Following more detailed study on coexistence and convergence rates of stability ranges.

\[
k \quad \varepsilon = 10^{-k} \quad b = 1 - \varepsilon \quad p_1(\varepsilon) \quad p_2(\varepsilon) \quad p_3(\varepsilon)
\]

| \( k \) | \( \varepsilon = 10^{-k} \) | \( b = 1 - \varepsilon \) | \( p_1(\varepsilon) \) | \( p_2(\varepsilon) \) | \( p_3(\varepsilon) \) |
|---|---|---|---|---|---|
| 5 | 10^{-5} | 0.99999 | 20 | 25 | 38 |
| 6 | 10^{-6} | 0.999999 | 25 | 30 | 46 |
| 7 | 10^{-7} | 0.99999999 | 31 | 36 | 55 |
| 8 | 10^{-8} | 0.999999999 | 38 | 43 | 65 |

Table 2. Values of \( p_1(\varepsilon) < p_2(\varepsilon) < p_3(\varepsilon) \) for \( b = 1 - \varepsilon \) and decreasing \( \varepsilon \).

| \( \varepsilon \) | \( b = -1 + \varepsilon \) | \( N_{\text{max}}(\varepsilon) \) | \( \tilde{a}_{\text{max}}(\varepsilon) \) | Periods |
|---|---|---|---|---|
| 10^{-3} | -0.999 | 4 | 0.151600 | 20 - 30 |
| 10^{-4} | -0.9999 | 9 | 0.098750 | 28 - 46 |
| 10^{-5} | -0.99999 | 16 | 0.076900 | 30 - 74 |
| 10^{-6} | -0.999999 | 27 | 0.054900 | 42 - 98 |
| 10^{-7} | -0.9999999 | 39 | 0.045000 | 46 - 134 |
| 10^{-8} | -0.99999999 | 54 | 0.035794 | 54 - 166 |

Table 3. Coexistence of attractors for \( b = -1 + \varepsilon \) with \( \varepsilon \) ranging from \( 10^{-3} \) to \( 10^{-8} \).

| \( \varepsilon \) | \( b = 1 - \varepsilon \) | \( N_{\text{max}}(\varepsilon) \) | \( \tilde{a}_{\text{max}}(\varepsilon) \) | Periods |
|---|---|---|---|---|
| 10^{-5} | 0.99999 | 10 | -0.93294 | 11 - 20 |
| 10^{-6} | 0.999999 | 14 | -0.95400 | 12 - 25 |
| 10^{-7} | 0.9999999 | 18 | -0.96490 | 12 - 29 |
| 10^{-8} | 0.99999999 | 23 | -0.97420 | 13 - 35 |

Table 4. Coexistence of attractors for \( b = 1 - \varepsilon \) with \( \varepsilon \) ranging from \( 10^{-5} \) to \( 10^{-8} \).
4.1. Coexistence of sinks, comparison as $\varepsilon \to 0$. We have analyzed the coexistence of attractors for $b = 1 - \varepsilon$ and $b = -1 + \varepsilon$, showing that $N_{\text{max}}(\varepsilon)$ increases as $\varepsilon$ goes to zero.

It is here confirmed (with larger $p$, smaller value of $\varepsilon$ and more accuracy) a conjecture advanced in [5]: as $\varepsilon \to 0$ we have respectively $\lim_{b \to -1} \tilde{a}_{\text{max}}(b) \to 0$ and $\lim_{b \to 1} \tilde{a}_{\text{max}}(b) \to -1$, i.e. the parameter value of maximum coexistence tends to the cusps while $N_{\text{max}}(\varepsilon) \to \infty$ (see Table 3, 4).

As in Sec.3 we dealt with the cases $b = 1 - \varepsilon$ for $\varepsilon = 10^{-6}, 10^{-7}$ and $10^{-8}$ and we reached a value of $p = 180$ using the automatic Newton-method procedure which applies smoothly for $p > p_2(\varepsilon)$. We also compare with case $b = -1 + \varepsilon$ (see Fig.8).

4.2. Convergence of stability ranges, changes as $\varepsilon \to 0$. We studied the convergence of width of the stability range $a_{pd}(\varepsilon) - a_{sn}(\varepsilon)$ for different values of $\varepsilon$, which tends to zero (as expected), with a rate that depends on $\varepsilon$. We think this is interesting.

We show that these width have two different rate of change varying $p$ (see Fig.9 and a similar figure for $b = -1 + \varepsilon$ in [5]): for small values of $p$ they decrease slowly...
up to a certain threshold and then, at $p_2(\varepsilon)$, they reach a renormalization-like regime where the convergence becomes exponentially fast. We have also indicated, in Fig.9, with dashed vertical lines the chosen values of $p_2(\varepsilon)$ for all values of $\varepsilon$ considered.
Figure 11. Coexistence of periodic attractors (isolated points), all other points are orbits of a discretized segment: on the unstable manifold (and inside a $10^{-8}$ neighborhood) of the fixed point (rightmost big dot). $b = -1 + \varepsilon$. Up: $\bar{a}_{\max}(10^{-5}) = 0.0769$. Down: for $\bar{a}_{\max}(10^{-6}) = 0.0549$ the scale is five times smaller.

Notice the positions of $p_d(\varepsilon)$ at different $\varepsilon$, similar in both cases $b = -1 + \varepsilon$ and $b = 1 - \varepsilon$ up to rescaling.

The two different regimes and the increasing number of coexisting attractors, as $\varepsilon$ decreases, can be analyzed looking at the changes of $a^{(p)}_{pd}(\varepsilon)$ and $a^{(p)}_{sn}(\varepsilon)$ separately for increasing $p$. The space $s$ between consecutive elements of the same sequence
has to be compared to the value of the difference \( a^{(p)}_{pd}(\varepsilon) - a^{(p)}_{sn}(\varepsilon) \) when such value becomes negligible with respect to \( s \).

In Fig.9 we also see how the rate of convergence, with respect to period \( p \), of the stability range \( a^{(p)}_{pd}(\varepsilon) - a^{(p)}_{sn}(\varepsilon) \) changes varying \( \varepsilon \). The negative value of the slope \( m_\varepsilon \) in logarithmic scale increases as \( \varepsilon \) decreases which means that the stability ranges of a given \( p \)-periodic orbit becomes larger and larger approaching the conservative regime.

4.3. Convergence of bifurcation values, changes as \( \varepsilon \to 0 \). Studying the convergence of sequences \( a^{(p)}_{sn}(\varepsilon) \) and \( a^{(p)}_{pd}(\varepsilon) \), we ascertained that they are both non-monotone in \( p \), \( \forall \varepsilon \), much as described in Section 3.3.

In all cases \( p_3(\varepsilon) > p_2(\varepsilon) \), so increasing \( p \) both sequences start decreasing, at \( p = p_2(\varepsilon) \) enter the fast convergence regime, then at \( p = p_3(\varepsilon) \) reach a minimum value \( a_{min}(\varepsilon) \) and finally start to increase in \( p \) up to a limit (see Fig.7,10). We stress that both sequences, above \( p_3(\varepsilon) \), remain definitely monotone in the orientation–preserving case and keep oscillating toward two different limit values in the orientation–reversing case.

5. Where are they? One can then study in dynamical plane the coexistence of these sinks, so as to ascertain the interaction of their basins of attraction, their boundary, and the role of the unstable manifold of the fixed points with respect to all this.

In Fig.11 we compute the unstable manifold of the fixed point for values of \( \bar{a}_{max}(\varepsilon) \) of maximum coexistence, which as we saw, depends on the amount of dissipation. We thus compared the cases of \( b = -1 + 10^{-5} \) and \( b = -1 + 10^{-6} \), i.e. orientation reversing. This plot, in turn, revealed how the coexistence of these periodic attractors is structured in dynamical plane. For the sake of inquiry into the geometrical coexistence, we selected a frame size in the neighborhood of the fixed point, containing the outermost 15 attractive periodic orbits. In the case at hand, for \( b = -1 + 10^{-5} \) such periods \( p \) are the even periods between 32 and 60 and for \( b = -1 + 10^{-6} \) the periods \( p \) are the even periods between 66 and 94 (see also Fig.5, 8). What is common and peculiar to both images is that the plotted points all belong to the unstable manifold of the fixed point. Such unstable manifold, thus, appear to be on the boundaries of all basins of attraction of these sinks. The other feature shared by the two cases represented, is that we have chosen a frame size, to represent only the 15 sinks closer to the fixed point (and therefore outermost). Note that the two images, which appear quite similar at different dissipation, have different scale: we do hope this suggests a possible renormalization in \( \varepsilon \).

For the coexisting attractive periodic orbits (some of which are shown in Fig.11), for \( b = -1 + 10^{-6} \) and \( a = 0.0549 \), we list as an example the period, the first 10 digits of the space coordinates of initial point, the corresponding eigenvalue of the Jacobian matrix along their orbit and the precision accuracy of the calculation (see Table 5). We remark that all calculations are done starting with 1000 digits of accuracy. These 27 orbits, with period from 42 to 94, are attractors with complex eigenvalues (see also Fig.8).

In the orientation–preserving case \( b = 1 - \varepsilon \), the dynamic structure is quite similar.

6. The dissipative Standard map. The dissipative Standard map (see for example [14], [11] and [1], [15] for a study on Arnoi’d tongues) on the cylinder, using
Table 5. $b = -1 + 10^{-6}$, $\bar{a} = 0.0549$. Orbits in Fig.11 with period from 42 to 94 are attractors with complex eigenvalues.

\[
\begin{array}{cccc}
p & x_0 & y_0 & \text{first eigenvalue} \\
42 & -0.02471212033 & -0.004353351966 & 0.999666+0.025007 i \\
44 & 0.003893968151 & 0.01400370243 & 0.998106+0.061164 i \\
46 & 0.0369238615 & 0.089124+0.146904 i \\
48 & 0.1254134421 & 0.9559+0.258534 i \\
50 & 0.1458234306 & 0.927268+0.374309 i \\
52 & 0.1607392037 & 0.849555+0.527432 i \\
54 & 0.1733182271 & 0.758004+0.652192 i \\
56 & 0.1838069083 & 0.693593+0.720313 i \\
58 & 0.1924697936 & 0.612348+0.790538 i \\
60 & 0.1995865673 & 0.509870+0.860204 i \\
62 & 0.2053481559 & 0.380610+0.924690 i \\
64 & 0.2100289888 & 0.217565+0.976002 i \\
66 & 0.2138038071 & 0.011907+0.999885 i \\
68 & 0.2162131915 & -0.247504+0.968841 i \\
70 & 0.2192688855 & -0.574717+0.818296 i \\
72 & 0.2212131915 & -0.987457+0.157593 i \\
74 & 0.2227652522 & -0.379187 \\
76 & 0.2240024479 & 0.2180138071 \\
78 & 0.2249875438 & 0.2340620792 \\
80 & 0.2257712060 & 0.2350621607 \\
82 & 0.2263941791 & 0.2372032327 \\
84 & 0.2272821942 & 0.2375022270 \\
86 & 0.2280383657 & 0.2377395373 \\
\end{array}
\]

the same notation of (1), can be written as:

\[
T_{(a,b,\nu)}(x,y) := (a + x + \nu \sin x + b y) \pmod{2\pi}, \quad a + \nu \sin x + b y
\]

with $x$ on a circle $(\pmod{2\pi})$, constant Jacobian $0 < b \leq 1$ and the extra parameter $\nu$ as the usual perturbation parameter of the conservative Standard map. The parameter $a$ is the so-called drift parameter.

For the unperturbed map $\nu = 0$, and $b = 1$, the orbit with rotation number $\omega$ is invariant for changes in $b$ if the parameter $a$ satisfy the relation $y = \omega = \frac{a}{1-b}$ and can be followed as $\nu$ varies. For different values of $\nu$ and a given $b$ there is an interval range of values of $a$ at which the invariant orbit is attractive.

The differences with Hénon map (1) are great: the topology of the space where they are defined, the number of parameters; we show an application to the study of the region of stability of periodic orbits also in this case.

A periodic orbit of rotation number $\omega = l/p$ (i.e. such that $x_p = x+l \pmod{2\pi}$) is linearly stable if the eigenvalues $\gamma_{1,2}$ of the matrix $M$

\[
M = \prod_{i=0}^{p-1} \begin{pmatrix} 1 + \nu \cos x_i & b \\ \nu \cos x_i & b \end{pmatrix}
\]
lie inside the unit circle. The condition on the stability range of a $p$-periodic orbit can then be stated also in terms of the residue $R$ defined as

\[
R = \frac{1 + \det(M) - tr(M)}{2(1 + \det(M))} = \frac{1 + b^p - tr(M)}{2(1 + b^p)}
\]

so that an equivalent condition for an orbit of period $p$ to be stable is

\[
|\gamma_{1,2}| < 1 \iff 0 < R < 1.
\]
Figure 12. Stability range in \((a, \nu)\), \(b = 0.5\), for two periodic orbits with rotation number \(\omega = \frac{3}{5}\) and \(\omega = \frac{8}{13}\). The border curves correspond to values of \(a_{sn}(\nu)\) for which the residue is 0. The middle curve corresponds, given \(b\) and \(\nu\), to values of \(a\) for which the residue is maximum.

In a seminal paper Greene proposed, in the conservative case, a method to detect the \(\nu\) threshold at which an invariant curve of irrational rotation number \(\omega\) breaks down: if we let \(l_j/p_j\) be the sequence of rational approximants of \(\omega\) an \(\omega\)-invariant circle exists if and only if the residue \(R(l_j/p_j)\) is bounded.

In the paper [4] a partial version of the Greene’s method was proved. As an example of application of the method to the dissipative Standard map see [2]; for a comparative study of the conservative Hénon map, both in the orientation-reversing and orientation-preserving cases, with the conservative Standard map on a torus see [13].

In a more recent paper [3] a partial justification of Greene’s criterion for conformally symplectic systems, including the dissipative Standard map, has been given.

We show as an example in Fig.12, for \(b\) fixed and periodic orbits of rotation number \(3/5\) and \(8/13\), the region of stability of the two orbits as a function of \(a\) and \(\nu\).

The border curves of Fig.12 are curves of 0 residue, that by (5) and (2) correspond to curves \(a_{sn}(\nu)\). Such curves are evaluated using our dribbling method, starting at \(b = 1\), \(\nu = 0\) and fixing the coordinates \((x, y)\) of one of the points of the \(p\)-periodic orbit. For values of the parameter \(0 < \nu < 1\) all orbits are stable with residue \(0 < R < 1\). We have looked for different values of \(b\) closer to the conservative case and the behavior of \(a_{sn}(\nu)\) and \(a_{pd}(\nu)\) of more periodic orbits approaching \(b = 1\) is in progress.

7. Conclusions. For the Hénon map, with very small dissipation, we have shown numerical evidence of coexisting attractive periodic orbits with diverging period: they accumulate toward the conservative case. Our approach is different from the perturbative analysis also used in literature, in the sense that we rather follow dissipative phenomena in the limit as the dissipation goes to zero.
We have studied the rate of convergence, with respect to the period $p$, of the stability range $a^{(p)}_{pd} - a^{(p)}_{sn}$ and how it depends on $\varepsilon$.

As a new result, in the quasi-conservative cases, we have illustrated the dependence on $p$ of the width of the stability intervals ($a^{(p)}_{sn}$ and $a^{(p)}_{pd}$), for periodic orbits of increasing period $p$, and shown that it is not monotone. The length of this intervals decreases until it reaches a minimum, depending on $b$, and then it remains bounded seeming to reach an asymptote from below. Moreover, it seems to be oscillating in the orientation–reversing case, and eventually monotone in the area–preserving case.

We have illustrated the renormalization scheme allowing a study for large $p$.

Using the same algorithms we have discussed, as an example of applications, the region of stability of periodic orbits in the dissipative Standard map.

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Received October 2017; revised June 2018.

E-mail address: falco@mat.uniroma3.it
E-mail address: tedeschi@mat.uniroma3.it