How to recognize a Leonard pair

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Abstract

Let $V$ denote a vector space with finite positive dimension. We consider an ordered pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy (i) and (ii) below.

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

We call such a pair a Leonard pair on $V$. In the literature, there are some parameters that are used to describe Leonard pairs called the intersection numbers $\{a_i\}_{i=0}^d$, $\{b_i\}_{i=0}^{d-1}$, $\{c_i\}_{i=1}^d$, and the dual eigenvalues $\{\theta_i^*\}_{i=0}^d$. In this paper, we provide two characterizations of Leonard pairs. For the first characterization, the focus is on the $\{a_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$. For the second characterization, the focus is on the $\{b_i\}_{i=0}^{d-1}$, $\{c_i\}_{i=1}^d$, and $\{\theta_i^*\}_{i=0}^d$.

Keywords. Leonard pair, tridiagonal matrix, distance-regular graph, intersection numbers, orthogonal polynomials.

2010 Mathematics Subject Classification. Primary: 15A21. Secondary: 05E30.

1 Introduction

We begin by recalling the notion of a Leonard pair [6,7]. We will use the following terms. A square matrix $X$ is called tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume $X$ is tridiagonal. Then $X$ is called irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper, $\mathbb{K}$ will denote a field.

Definition 1.1 [7, Definition 1.1] Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a Leonard pair on $V$, we mean an ordered pair of $\mathbb{K}$-linear maps $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy (i) and (ii) below.

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.
There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

**Note 1.2** In a common notational convention, $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$, the linear transformations $A$ and $A^*$ are arbitrary subject to (i), (ii) above.

The concept of a Leonard pair originated in the study of $Q$-polynomial distance-regular graphs [11, p. 260], [6, Definition 2.3]. Since that time, Leonard pairs have found application in a variety of contexts, such as special functions/orthogonal polynomials [7,9,13] and representation theory [9,11]. Motivated by these applications, a number of characterizations of Leonard pairs have been discovered. For instance, there are characterizations of Leonard pairs in terms of orthogonal polynomials [10, Theorem 19.1] [12, Theorem 4.1], parameter arrays [7, Theorem 1.9], upper/lower bidiagonal matrices [12, Theorem 3.2] [13, Theorem 17.1], tridiagonal/diagonal matrices [13, Theorem 25.1], the notion of a tail [2, Theorem 5.1] [4, Theorem 10.1], and the intersection numbers $\{a_i\}_{i=0}^d$ [3, Theorem 5.1].

In this paper, we consider the following situation. Fix an integer $d \geq 1$ and consider matrices $A$ and $A^*$ over $\mathbb{K}$ that have the following form:

$$
A = \begin{pmatrix}
    a_0 & b_0 & 0 \\
    c_1 & a_1 & b_1 \\
    & \vdots & \ddots \\
    0 & \cdots & c_d & a_d
\end{pmatrix},
$$

$$
A^* = \begin{pmatrix}
    \theta_0^* & 0 \\
    \theta_1^* & \ddots \\
    & \ddots \\
    0 & \cdots & \theta_d^*
\end{pmatrix}.
$$

It is desirable to have attractive necessary and sufficient conditions for $A, A^*$ to form a Leonard pair. In the literature, there exist two kinds of results along this line. For the first kind of result, the focus is on the parameters $\{a_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ [3, Theorem 5.1]. For the second kind of result, the focus is on the parameters $\{b_i\}_{i=0}^{d-1}, \{c_i\}_{i=1}^d$, and $\{\theta_i^*\}_{i=0}^d$ [13, Theorem 25.1]. Each of these results has its drawbacks which we will describe shortly. The present paper has two main theorems, the first of which improves on [3, Theorem 5.1] and the second of which improves on [13, Theorem 25.1]. We now describe the drawbacks of [3, Theorem 5.1] and [13, Theorem 25.1], and how our results are an improvement. One shortcoming of [3, Theorem 5.1] is that it assumes $A$ is diagonalizable. Our improvement requires no such assumption. The result [13, Theorem 25.1] involves some equations containing the products $\{b_i c_i\}_{i=1}^d$, and checking the equations becomes cumbersome. Our improvement avoids this difficulty by treating the $\{b_i\}_{i=0}^{d-1}$ and $\{c_i\}_{i=1}^d$ separately. Our two main results are Theorem 3.1 and Theorem 4.1.

## 2 Leonard systems

When working with a Leonard pair, it is often convenient to consider a related object called a Leonard system. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. From now on, fix an integer $d \geq 0$. Let Mat$_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra
consisting of all \( d+1 \) by \( d+1 \) matrices with entries in \( \mathbb{K} \). We index the rows and columns by \( 0,1,\ldots,d \). Let \( \mathbb{K}^{d+1} \) denote the vector space over \( \mathbb{K} \) consisting of all \( d+1 \) by 1 matrices with entries in \( \mathbb{K} \). We index the rows by \( 0,1,\ldots,d \). The algebra \( \text{Mat}_{d+1}(\mathbb{K}) \) acts on \( \mathbb{K}^{d+1} \) by left multiplication. Let \( V \) denote a vector space over \( \mathbb{K} \) with dimension \( d+1 \). Let \( \text{End}(V) \) denote the \( \mathbb{K} \)-algebra consisting of \( \mathbb{K} \)-linear maps from \( V \) to \( V \). The identity of \( \text{End}(V) \) will be denoted by \( I \). The \( \mathbb{K} \)-algebra \( \text{End}(V) \) is isomorphic to \( \text{Mat}_{d+1}(\mathbb{K}) \). Let \( \{v_i\}_{i=0}^d \) denote a system of mutually orthogonal idempotents in \( \text{End}(V) \). For \( 0 \leq i \leq d \), let \( \theta_i \) denote a system of mutually orthogonal idempotents from Lemma 2.1. One checks that \( \theta \in \mathbb{K} \) such that \( \theta_i \) is the eigenvalue of \( \theta \). We say that \( \theta_i \) is diagonalizable whenever \( \theta_i \) is spanned by the eigenspaces of \( \theta_i \). We say that \( \theta_i \) is multiplicity-free whenever \( \theta_i \) is diagonalizable and each eigenspace of \( \theta_i \) has dimension one. By a system of mutually orthogonal idempotents in \( \text{End}(V) \), we mean a sequence \( \{E_i\}_{i=0}^d \) of elements in \( \text{End}(V) \) such that

\[
E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d),
\]

\[
\text{rank}(E_i) = 1 \quad (0 \leq i \leq d).
\]

By a decomposition of \( V \), we mean a sequence \( \{U_i\}_{i=0}^d \) of one-dimensional subspaces of \( V \) such that

\[
V = \sum_{i=0}^{d} U_i \quad \text{(direct sum)}.
\]

The following lemmas are routinely verified.

**Lemma 2.1** Let \( \{U_i\}_{i=0}^d \) denote a decomposition of \( V \). For \( 0 \leq i \leq d \), define \( E_i \in \text{End}(V) \) such that \( (E_i - I)U_i = 0 \) and \( E_i U_j = 0 \) if \( j \neq i \) \( (0 \leq j \leq d) \). Then \( \{E_i\}_{i=0}^d \) is a system of mutually orthogonal idempotents in \( \text{End}(V) \). Conversely, let \( \{E_i\}_{i=0}^d \) denote a system of mutually orthogonal idempotents in \( \text{End}(V) \). Define \( U_i = E_i V \) for \( 0 \leq i \leq d \). Then \( \{U_i\}_{i=0}^d \) is a decomposition of \( V \).

**Lemma 2.2** Let \( \{E_i\}_{i=0}^d \) denote a system of mutually orthogonal idempotents in \( \text{End}(V) \). Then \( I = \sum_{i=0}^{d} E_i \).

Let \( A \) denote a multiplicity-free element of \( \text{End}(V) \) and let \( \{\theta_i\}_{i=0}^d \) denote an ordering of the eigenvalues of \( A \). For \( 0 \leq i \leq d \), let \( V_i \) denote the eigenspace of \( A \) for \( \theta_i \). Then \( \{V_i\}_{i=0}^d \) is a decomposition of \( V \); let \( \{E_i\}_{i=0}^d \) denote the corresponding system of mutually orthogonal idempotents from Lemma 2.1. One checks that \( A = \sum_{i=0}^{d} \theta_i E_i \) and \( AE_i = E_i A = \theta_i E_i \) for \( 0 \leq i \leq d \). Moreover,

\[
E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d).
\]

We refer to \( E_i \) as the primitive idempotent of \( A \) corresponding to \( V_i \) (or \( \theta_i \)).

We now define a Leonard system.
Definition 2.3 [7, Definition 1.4] By a Leonard system on $V$, we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

(i) Each of $A, A^*$ is a multiplicity-free element of $\text{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $A$.

(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of $A^*$.

(iv) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

(v) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases}$ $(0 \leq i, j \leq d)$.

The Leonard system $\Phi$ is said to be over $\mathbb{K}$.

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on $V$. Then the pair $A, A^*$ is a Leonard pair on $V$ said to be associated with $\Phi$. See [7, pp. 4–5] for the precise connection between Leonard pairs and Leonard systems.

Definition 2.4 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on $V$. For $0 \leq i \leq d$, let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $E_i V$ (resp. $E_i^* V$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$.

Definition 2.5 Let $A, A^*$ denote a Leonard pair on $V$. By an eigenvalue sequence (resp. dual eigenvalue sequence) of $A, A^*$, we mean the eigenvalue sequence (resp. dual eigenvalue sequence) of an associated Leonard system.

For the remainder of this section, let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. To avoid trivialities, we assume $d \geq 1$. By construction, $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{K}$. Similarly, $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{K}$. By [7, Theorem 12.7], the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2} - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$. Define $\beta \in \mathbb{K}$ as follows. For $d \geq 3$, let $\beta + 1$ be the common value of $\{\theta_i\}$. For $d \leq 2$, let $\beta$ be arbitrary. By (1), $\theta_{i-1} - \beta \theta_i + \theta_{i+1}$ is independent of $i$ for $1 \leq i \leq d - 1$. Let $\gamma$ denote this common value, so

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1} = \gamma \quad (1 \leq i \leq d - 1).$$
For notational convenience, define $\theta_{-1}$ (resp. $\theta_{d+1}$) such that (2) holds at $i = 0$ (resp. $i = d$). Similarly, there exists $\gamma^* \in \mathbb{K}$ such that
\[
\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* = \gamma^* \quad (1 \leq i \leq d-1).
\] (3)

For notational convenience, define $\theta_{-1}^*$ (resp. $\theta_{d+1}^*$) such that (3) holds at $i = 0$ (resp. $i = d$). Choose $0 \neq u \in E_0V$. By [8, Lemma 5.1], $E_i^*u$ is a basis for $E_i^*V$ ($0 \leq i \leq d$). Moreover, \{${E_i^*u}_{i=0}^d$\} is a basis for $V$. By Lemma 2.2,
\[
u = \sum_{i=0}^d E_i^*u.
\] (4)

With respect to the basis \{${E_i^*u}_{i=0}^d$\}, the matrices representing $A$ and $A^*$ take the form
\[
A = \begin{pmatrix}
a_0 & b_0 \\
c_1 & a_1 & b_1 \\
c_2 & \cdots & \cdots & b_{d-1} \\
0 & \cdots & c_d & a_d
\end{pmatrix}
\quad A^* = \begin{pmatrix}
\theta_0^* & 0 \\
\theta_1^* & \cdots \\
0 & \cdots \\
0 & \cdots & \cdots & \theta_d^*
\end{pmatrix},
\]

for some scalars $a_i, b_i, c_i \in \mathbb{K}$ with $c_ib_{i-1} \neq 0$ for $1 \leq i \leq d$. We call the scalars \{${a_i}_{i=0}^d$\}, \{${b_i}_{i=0}^{d-1}$\}, and \{${c_i}_{i=1}^d$\} the intersection numbers of $\Phi$. By (1) and since $Au = \theta_0u$,
\[
c_i + a_i + b_i = \theta_0 \quad (0 \leq i \leq d),
\] (5)

where $c_0 = b_d = 0$. By [10, Definition 7.1 and Lemma 7.2],
\[
a_i = \text{tr}(E_i^*A) \quad (0 \leq i \leq d),
\]

where tr denotes trace. The next equation involves the intersection number $a_0^*$ for the Leonard system $(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$. By [12, Lemma 9.2],
\[
c_i(\theta_{i-1}^* - \theta_i^*) - b_i(\theta_i^* - \theta_{i+1}^*) = (\theta_1 - \theta_0)(\theta_i^* - a_0^*) \quad (0 \leq i \leq d).
\] (6)

By [14, Theorem 5.3], there exist $\omega, \eta^* \in \mathbb{K}$ such that
\[
a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = \gamma \theta_i^{*2} + \omega \theta_i^* + \eta^* \quad (0 \leq i \leq d).
\] (7)

Using (7), we obtain
\[
a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_{i-2}^*) - \gamma(\theta_i^* + \theta_{i-1}^*) = \omega \quad (1 \leq i \leq d).
\] (8)

**Proposition 2.6** With the above notation,
\[
c_i(\theta_{i-1}^* - \theta_{i+1}^*) - b_{i-1}(\theta_{i-2}^* - \theta_i^*) - (\theta_0 - \theta_{-1})(\theta_{i-1}^* + \theta_{i-1}^*) = \Omega \quad (1 \leq i \leq d),
\] (9)

where $\Omega = 2\theta_0(a_0^* - \gamma^*) - 2\theta_1a_0^* - \omega$. 

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Proof: Let the integer \( i \) be given. In (6), eliminate \( c_i \) using (5) to obtain

\[
b_i(\theta^*_i - \theta^*_{i-1}) = \theta_1(\theta^*_i - a_0^*) + \theta_0(a_0^* - \theta^*_i - \theta^*_{i-1}) - a_i(\theta^*_i - \theta^*_{i-1}). \tag{10}
\]

In (10), replace \( i \) by \( i - 1 \) to obtain

\[
b_{i-1}(\theta^*_i - \theta^*_{i-2}) = \theta_1(\theta^*_i - a_0^*) + \theta_0(a_0^* - \theta^*_i - \theta^*_{i-2}) - a_{i-1}(\theta^*_i - \theta^*_{i-2}). \tag{11}
\]

In (6), eliminate \( b_i \) using (5) to obtain

\[
c_i(\theta^*_i - \theta^*_{i+1}) = \theta_1(\theta^*_i - a_0^*) + \theta_0(a_0^* - \theta^*_i - \theta^*_{i+1}) - a_i(\theta^*_i - \theta^*_{i+1}). \tag{12}
\]

Adding (11) to (12) and simplifying the result using (2), (3), and (8), we routinely obtain (9). □

3 The first main theorem

In this section, we obtain our first main result. It involves the following setup. Fix an integer \( d \geq 1 \). Let \( V \) denote a vector space over \( \mathbb{K} \) of dimension \( d + 1 \). Let \( \{E^*_i\}_{i=0}^d \) denote a system of mutually orthogonal idempotents in \( \text{End}(V) \). Define \( A \in \text{End}(V) \) such that

\[
E^*_iAE^*_j = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d). \tag{13}
\]

Let \( \{\theta^*_i\}_{i=0}^d \) denote scalars in \( \mathbb{K} \) and define

\[
A^* = \sum_{i=0}^d \theta^*_i E^*_i. \tag{14}
\]

Let \( \{\theta_i\}_{i=0}^d \) denote any scalars in \( \mathbb{K} \).

Theorem 3.1 With the above notation, suppose the following (i)–(vi) hold.

(i) \( \theta_i \neq \theta_j \) if \( i \neq j \) (\( 0 \leq i, j \leq d \)).

(ii) \( \theta^*_i \neq \theta^*_0 \) (\( 1 \leq i \leq d \)).

(iii) There exist \( \beta, \gamma^* \in \mathbb{K} \) such that

\[
\gamma^* = \theta^*_{i-1} - \beta \theta^*_i + \theta^*_{i+1} \quad (1 \leq i \leq d - 1). \tag{15}
\]

Define \( \theta^*_{i-1} \) (resp. \( \theta^*_d \)) such that (15) holds at \( i = 0 \) (resp. \( i = d \)).

(iv) There exist nonzero vectors \( v_0, v_1 \in V \) such that

\[
Av_0 = \theta_0 v_0, \quad Av_1 = \theta_1 v_1, \quad A^* v_0 - v_1 \in \mathbb{K} v_0. \tag{16}
\]
Recall the scalars \( \theta \)
To verify (21), in the right-hand side, replace \( \delta \)
We claim that the scalar
We now show that
Using (14) and the fact that
In this equation, the right-hand side equals 0 by (18). Consequently,
For notational convenience, we introduce a 2-variable polynomial
(v) There exist \( \gamma, \omega, \eta^* \in \mathbb{K} \) such that for \( 0 \leq i \leq d \),
    \[
    a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = \gamma \theta_i^{*2} + \omega \theta_i^* + \eta^*,
    \]  
    where \( a_i = \text{tr}(E_i^* A) \).
(vi) \( \theta_{i-1} - \beta \theta_i + \theta_{i+1} = \gamma \) (1 \( \leq i \leq d - 1 \)).
Then \( A, A^* \) is a Leonard pair on \( V \) with eigenvalue sequence \( \{ \theta_i \}_{i=0}^d \) and dual eigenvalue sequence \( \{ \theta_i^* \}_{i=0}^d \).

Proof: By [13] and [2] Corollary 3.4, the elements \( A \) and \( E_0^* \) together generate \( \text{End}(V) \). Using (14) and the fact that \( \{ E_i^* \}_{i=0}^d \) are mutually orthogonal idempotents, we obtain

\[
E_0^* = \prod_{j=1}^d \frac{A^* - \theta_j^* I}{\theta_0^* - \theta_j^*}.
\]

Consequently, \( \text{End}(V) \) is generated by \( A \) and \( A^* \). The vector space \( V \) is irreducible as an \( \text{End}(V) \)-module, so \( V \) is irreducible as a module for \( A, A^* \).

By [2] Lemma 3.5], there exists a unique antiautomorphism \( \dagger \) of \( \text{End}(V) \) such that \( A^\dagger = A \) and \( E_i^{\dagger} = E_i^* \) for \( 0 \leq i \leq d \). By this and (14), \( A^\dagger = A^* \).

Recall the scalars \( \theta_i^* \) and \( \theta_{i+1}^* \) from below [15]. By construction,

\[
\gamma^* = \theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* \quad (0 \leq i \leq d).
\]  
We claim that the scalar

\[
\theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*)
\]

is independent of \( i \) for \( 0 \leq i \leq d + 1 \). Denote this scalar by \( p_i \). For \( 0 \leq i \leq d \),

\[
p_i - p_{i+1} = (\theta_{i-1}^* - \theta_i^*)(\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* - \gamma^*).
\]

In this equation, the right-hand side equals 0 by (18). Consequently, \( p_i \) is independent of \( i \) for \( 0 \leq i \leq d + 1 \). The claim is now proven. Let \( \delta^* \) denote the common value of (19), so

\[
\theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) = \delta^* \quad (0 \leq i \leq d + 1).
\]  
We now show that

\[
(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = (2 - \beta) \theta_i^{*2} - 2 \gamma^* \theta_i^* - \delta^* \quad (0 \leq i \leq d).
\]  
To verify (21), in the right-hand side, replace \( \delta^* \) by (19) and eliminate both occurrences of \( \gamma^* \) in the resulting expression using (18). We have now verified (21).

For notational convenience, we introduce a 2-variable polynomial

\[
P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma^* (\lambda + \mu) - \delta^*.
\]  

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We now claim that
\[ A^{*2}A - \beta A^*A + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^*A = \gamma A^{*2} + \omega A^* + \eta^* I. \] (23)

In (23), let \( C \) denote the left-hand side minus the right-hand side. We show \( C = 0 \). Using
\[ I = \sum_{i=0}^{d} E_i^* \], we obtain
\[ C = (E_0^* + E_1^* + \cdots + E_d^*)C(E_0^* + E_1^* + \cdots + E_d^*) \]
\[ = \sum_{i=0}^{d} \sum_{j=0}^{d} E_i^* CE_j^*. \]

For \( 0 \leq i, j \leq d \), we show \( E_i^* CE_j^* = 0 \). Using \( E_i^* A^* = \theta_i^* E_i^* \) and \( A^* E_i^* = \theta_j^* E_j^* \),
\[ E_i^* CE_j^* = E_i^* A E_j^* P(\theta_i^*, \theta_j^*) - \delta_{i,j}(\gamma \theta_i^{*2} + \omega \theta_i^* + \eta^*) E_i^*. \] (24)

To further examine (24), we consider two cases. First assume \( i \neq j \). In this case, (24) becomes
\[ E_i^* CE_j^* = E_i^* A E_j^* P(\theta_i^*, \theta_j^*). \]

If \(|i - j| > 1\), then \( E_i^* A E_j^* = 0 \) by (13). If \(|i - j| = 1\), then \( P(\theta_i^*, \theta_j^*) = 0 \) by (20). Therefore, \( E_i^* CE_j^* = 0 \) under our present assumption that \( i \neq j \). Next assume \( i = j \). In this case, (24) becomes
\[ E_i^* CE_i^* = E_i^* A E_i^* P(\theta_i^*, \theta_i^*) - (\gamma \theta_i^{*2} + \omega \theta_i^* + \eta^*) E_i^*. \] (25)

By (21) and (22), we find \( P(\theta_i^*, \theta_i^*) = (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) \). By [3, Proposition 3.6], \( E_i^* A E_i^* = a_i E_i^* \). Evaluating the right-hand side of (25) using these comments, we find that it equals \( E_i^* \) times
\[ a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma \theta_i^{*2} + \omega \theta_i^* - \eta^*. \] (26)

The scalar (26) is equal to 0 by (17), so \( E_i^* CE_i^* = 0 \). We have now shown \( E_i^* CE_j^* = 0 \) for \( 0 \leq i, j \leq d \). Therefore, \( C = 0 \). We have now verified (23).

We now claim that for \( 1 \leq i \leq d \), there exists a nonzero vector \( v_i \in V \) such that both
\[ Av_i = \theta_i v_i, \quad A^* v_{i-1} - v_i \in \text{span}(v_0, \ldots, v_{i-1}), \] (27)
where \( v_0 \) is from (16). We prove the claim by induction on \( i \). The case \( i = 1 \) follows by condition (iv). Next assume \( i \geq 2 \). Note that \( v_0, v_1, \ldots, v_{i-1} \) are linearly independent, because they are eigenvectors for \( A \) with distinct eigenvalues. For \( 0 \leq j \leq i - 1 \), define \( W_j = \text{span}(v_0, \ldots, v_j) \). By construction,
\[ W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{i-1}. \] (28)

By induction,
\[ AW_j \subseteq W_j \quad (0 \leq j \leq i - 1), \] (29)
\[ A^* W_j \subseteq W_{j+1} \quad (0 \leq j \leq i - 2). \] (30)
We apply both sides of (23) to \( v_{i-2} \) and evaluate the result using \( Av_{i-2} = \theta_{i-2}v_{i-2} \). This gives
\[
(A + \theta_{i-2} - \gamma)A^2v_{i-2} - (\gamma^* A + \theta_{i-2}\gamma + \omega)A^*v_{i-2} - \beta A^*AA^*v_{i-2} - (\delta^* \theta_{i-2} + \eta^*)v_{i-2} = 0. \tag{31}
\]
For notational convenience, define
\[
w_{i-2} = A^*v_{i-2} - v_{i-1}. \tag{32}
\]
Evaluate (31) using (32), and simplify the result using (32), \( Av \) gives
\[
(A - \theta_i)A^*v_{i-1} + (A + \theta_{i-2} - \gamma)A^*w_{i-2} - (\gamma^* \theta_{i-1} + \gamma^* \theta_{i-2} + \omega)v_{i-1} - \beta A^*Aw_{i-2} - (\gamma^* A + \gamma^* \theta_{i-2} + \omega)w_{i-2} - (\delta^* \theta_{i-2} + \eta^*)v_{i-2} = 0. \tag{33}
\]
By (27), (32), and induction, \( w_{i-2} \in W_{i-2} \). Using (29) and (30), \( Aw_{i-2} \in W_{i-2} \) and \( A^*Aw_{i-2} \in W_{i-1} \). Using these comments to simplify (33), we obtain
\[
(A - \theta_i)A^*v_{i-1} \in W_{i-1}. \tag{34}
\]
We now show that \( A^*v_{i-1} \not\in W_{i-1} \). Suppose \( A^*v_{i-1} \in W_{i-1} \). By this, together with (28) and (30), \( A^*W_{i-1} \subseteq W_{i-1} \). By (29), \( AW_{i-1} \subseteq W_{i-1} \). Comparing the dimensions of \( W_{i-1} \) and \( V \), we obtain \( W_{i-1} \neq V \). This contradicts the fact that \( V \) is irreducible as a module for \( A, A^* \). We have shown that \( A^*v_{i-1} \not\in W_{i-1} \). Let \( H \) denote the subspace of \( V \) spanned by \( W_{i-1} \) and \( A^*v_{i-1} \). The vectors \( v_0, \ldots, v_{i-1}, A^*v_{i-1} \) form a basis for \( H \). Recall that \( Av = \theta_j v_j \) for \( 0 \leq j \leq i - 1 \). By this and (34), \( AH \subseteq H \) and the action of \( A \) on \( H \) has characteristic polynomial \((\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_i)\). By condition (i), the roots of this characteristic polynomial are mutually distinct, so \( A \) is diagonalizable on \( H \) with eigenvalues \( \theta_0, \ldots, \theta_i \). Let \( 0 \neq v_i \in H \) denote an eigenvector for \( A \) with eigenvalue \( \theta_i \). So \( Av_i = \theta_i v_i \). Note that \( v_i \not\in W_{i-1} \), so there exists \( 0 \neq \epsilon \in \mathbb{K} \) such that \( A^*v_{i-1} - \epsilon v_i \in W_{i-1} \). Replacing \( v_i \) by \( \epsilon v_i \), we may assume \( \epsilon = 1 \). We have shown \( A^*v_{i-1} - v_i \in \text{span}(v_0, \ldots, v_{i-1}) \). The claim is proven.

By construction and since \( \{\theta_i\}_{i=0}^d \) are mutually distinct, \( \{v_i\}_{i=0}^d \) is a basis for \( V \) consisting of eigenvectors for \( A \). It follows that \( A \) is multiplicity-free. For \( 0 \leq i \leq d \), let \( E_i \) denote the primitive idempotent of \( A \) corresponding to \( \theta_i \). We now show that \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) is a Leonard system on \( V \). To do this, we verify conditions (i)–(v) of Definition 2.3. Definition 2.3(ii) holds by construction and Definition 2.3(iv) holds by (13). It is convenient to check the remaining conditions in a nonstandard order. Consider Definition 2.3(v). By (27),
\[
E_i A^*E_j = \begin{cases} 
0, & \text{if } i - j > 1; \\
\neq 0, & \text{if } i - j = 1 
\end{cases} \quad \text{for } 0 \leq i, j \leq d. \tag{35}
\]
Applying \( \dagger \),
\[
E_i A^*E_j = \begin{cases} 
0, & \text{if } j - i > 1; \\
\neq 0, & \text{if } j - i = 1 
\end{cases} \quad \text{for } 0 \leq i, j \leq d. \tag{36}
\]
Definition 2.3(v) holds by (35) and (36). To obtain Definition 2.3(i), we show that $A^*$ is multiplicity-free. The map $A^*$ is given in (14). By assumption, $\{E_i^*\}_{i=0}^d$ are mutually orthogonal idempotents in $\text{End}(V)$. Therefore, by Lemma 2.1, the sum $V = \sum_{i=0}^d E_i^* V$ is direct and $E_i^* V$ has dimension 1 for $0 \leq i \leq d$. By these comments, $A^*$ is diagonalizable. To show that $A^*$ is multiplicity-free, we show that $\{\theta_i^*\}_{i=0}^d$ are mutually distinct. Define a polynomial $\psi(\lambda) = \prod_{i=0}^d (\lambda - \theta_i^*)$ and note that $\psi(A^*) = 0$. The elements $\{A_i^*\}_{i=0}^d$ are linearly independent by Definition 2.3(v) and [2, Lemma 3.1], so the minimal polynomial of $A^*$ has degree $d + 1$. Therefore, the minimal polynomial of $A^*$ is precisely $\psi(\lambda)$. Because $A^*$ is diagonalizable, the roots $\{\theta_i^*\}_{i=0}^d$ of $\psi(\lambda)$ are mutually distinct. Therefore, $A^*$ is multiplicity-free as desired. We have established Definition 2.3(i).

By (14) and since $A^*$ is multiplicity-free, we see that $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of $A^*$. This gives Definition 2.3(iii). By these comments, $\Phi$ is a Leonard system on $V$. Consequently, $A, A^*$ is a Leonard pair on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$.

\[\square\]

4 The second main theorem

In this section, we obtain our second main result.

**Theorem 4.1** Fix an integer $d \geq 1$. Suppose there exist scalars $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, and $\{a_i\}_{i=0}^d$, $\{b_i\}_{i=0}^d$, $\{c_i\}_{i=1}^d$ in $\mathbb{K}$ such that the following (i)–(viii) hold.

(i) $\theta_i \neq \theta_j$ if $i \neq j$ ($0 \leq i, j \leq d$).

(ii) $\theta_i^* \neq \theta_0^*$ ($1 \leq i \leq d$).

(iii) There exist $\beta, \gamma^* \in \mathbb{K}$ such that

\[\gamma^* = \theta_i^* - \beta \theta_i^* + \theta_{i+1}^* \quad (1 \leq i \leq d - 1).\]  

(37)

Define $\theta_{i-1}^*$ (resp. $\theta_{d+1}^*$) such that (37) holds at $i = 0$ (resp. $i = d$).

(iv) $b_{i-1}c_i \neq 0$ for $1 \leq i \leq d$.

(v) $c_i + a_i + b_i = \theta_0$ for $0 \leq i \leq d$, where $b_d = c_0 = 0$.

(vi) There exists $a_0^* \in \mathbb{K}$ such that

\[c_i(\theta_{i-1}^* - \theta_i^*) - b_i(\theta_i^* - \theta_{i+1}^*) = (\theta_0 - \theta_0)(\theta_i^* - a_0^*) \quad (0 \leq i \leq d).\]  

(38)

(vii) There exists $\theta_{-1} \in \mathbb{K}$ such that

\[c_i(\theta_{i-1}^* - \theta_{i+1}^*) - b_{i-1}(\theta_{i-2}^* - \theta_i^*) - (\theta_0 - \theta_{-1})(\theta_{i-1}^* + \theta_i^*) \quad (0 \leq i \leq d).\]  

(39)

is independent of $i$ for $1 \leq i \leq d$. 

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We now show that the conditions stated above Theorem 3.1. Let $v$ denote the vector with every component equal to 1. By condition (v) in the present theorem, $A v = \theta_0 v$. Combining conditions (v) and (vi) in the present theorem, we obtain

$$c_i(\theta^*_i - a^*_0) + a_i(\theta^*_i - a^*_0) + b_i(\theta^*_i - \theta^*_0) = \theta_1(\theta^*_i - a^*_0) \quad (0 \leq i \leq d).$$

Let $v_i$ denote the vector with $i$th component $\theta^*_i - a^*_0$ for $0 \leq i \leq d$. By condition (ii) in the present theorem, $v_i \neq 0$. By (11) and (12), we obtain $A v_i = \theta_1 v_i$ and $A^* v_0 - v_1 = a^*_0 v_0$. This implies Theorem 3.1(iv).

We now show Theorem 3.1(v). Evaluating (38) using condition (v), we obtain

$$(\theta_0 - a_i)(\theta^*_i - \theta^*_0) - b_i(\theta^*_i - \theta^*_0) = (\theta_1 - \theta_0)(\theta^*_i - a^*_0) \quad (0 \leq i \leq d).$$

Rearranging the terms in (43), we obtain

$$b_i(\theta^*_i - \theta^*_0) = \theta_0(\theta^*_i - a^*_0) + \theta_0(a^*_0 - \theta^*_0) - a_i(\theta^*_i - \theta^*_0) \quad (0 \leq i \leq d).$$

Evaluating (38) using condition (v), we similarly obtain

$$c_i(\theta^*_i - \theta^*_0) = \theta_0(a^*_0 - \theta^*_0) - a_i(\theta^*_i - \theta^*_0) \quad (0 \leq i \leq d).$$

For $1 \leq i \leq d$, consider the equation obtained from (44) by replacing $i$ with $i - 1$. Add this to (45) to obtain

$$c_i(\theta^*_i - \theta^*_0) + b_{i-1}(\theta^*_i - \theta^*_0) = \theta_1(\theta^*_i + \theta^*_0 - 2a^*_0) + \theta_0(2a^*_0 - \theta^*_0 - \theta^*_i) - a_i(\theta^*_i - \theta^*_0) - a_{i-1}(\theta^*_0 - \theta^*_i) \quad (0 \leq i \leq d).$$

Define $\gamma = \theta_1 - \beta \theta_0 + \theta_1$. Then

$$\theta_i - \beta \theta_{i+1} + \theta_i = \gamma \quad (1 \leq i \leq d - 1).$$

Then there exists a Leonard system over $\mathbb{K}$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$, and intersection numbers $\{a_i\}_{i=0}^d$, $\{b_i\}_{i=0}^d$, $\{c_i\}_{i=1}^d$.

Proof: Define the vector space $V = \mathbb{K}^{d+1}$. We identify $\text{End}(V)$ with $\text{Mat}_{d+1}(\mathbb{K})$. Define $A, A^* \in \text{Mat}_{d+1}(\mathbb{K})$ as follows:

$$A = \begin{pmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ & \ddots & \ddots & \ddots \\ & & c_{d-1} & b_{d-1} \\ 0 & \cdots & c_d & a_d \end{pmatrix} \quad A^* = \begin{pmatrix} \theta^*_0 \\ \theta^*_1 \\ \vdots \\ 0 \end{pmatrix}.$$

For $0 \leq i \leq d$, define $E_i^* \in \text{Mat}_{d+1}(\mathbb{K})$ with $(i, i)$-entry 1 and all other entries 0. The elements $\{E_i^*\}_{i=0}^d$ are mutually orthogonal idempotents. Note that $A, A^*$ and $\{E_i^*\}_{i=0}^d$ satisfy the conditions stated above Theorem 3.1.
for $1 \leq i \leq d$.

Let $\Omega$ denote the common value of (39). By (46) and condition (vii) in the present theorem,

$$
\begin{align*}
\theta_1(\theta_i^* + \theta_i^* - 2a_i^0) + \theta_0(2a_0^* - \theta_{i+1}^* - \theta_i^* - \theta_{i-1}^* - \theta_{i-2}^*) \\
-a_i(\theta_i^* - \theta_{i+1}^*) - a_{i-1}(\theta_{i-1}^* - \theta_i^* + \theta_{i-1}^* + \theta_i^*) = \Omega
\end{align*}
$$

(47)

for $1 \leq i \leq d$. In (47), eliminate $\theta_{i-1}$ using $\gamma = \theta_{i-1} - \beta \theta_0 + \theta_1$. Evaluating the results using (37), we obtain

$$
a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_i^* - \theta_{i-1}^*) - \gamma(\theta_{i-1}^* + \theta_i^*) = 2\theta_0(a_0^* - \gamma^*) - 2\theta_1a_0^* - \Omega
$$

(48)

for $1 \leq i \leq d$. Let $\omega$ denote the right-hand side of (48). So,

$$
a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_i^* - \theta_{i-1}^*) - \gamma(\theta_{i-1}^* + \theta_i^*) = \omega \quad (1 \leq i \leq d).
$$

(49)

For $1 \leq i \leq d$, we multiply each side of (49) by $\theta_i^* - \theta_{i-1}^*$. After some rearranging, we obtain

$$
a_i(\theta_i^* - \theta_{i+1}^*)(\theta_i^* - \theta_{i-1}^*) - \gamma\theta_i^{2*} - \omega\theta_i^* = a_{i-1}(\theta_{i-1}^* - \theta_i^* + \theta_{i-1}^*) - \gamma\theta_i^{2*} - \omega\theta_i^* \quad (1 \leq i \leq d).
$$

Consequently, the scalar

$$
a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma\theta_i^{2*} - \omega\theta_i^*
$$

(50)

is independent of $i$ for $0 \leq i \leq d$. Let $\eta^*$ denote the common value of (50). So,

$$
a_i(\theta_i^* - \theta_{i+1}^*)(\theta_i^* - \theta_{i-1}^*) = \gamma\theta_i^{2*} + \omega\theta_i^* + \eta^* \quad (0 \leq i \leq d).
$$

(51)

By the equation on the left in (41) and by the definition of $E_i^*$ following (41), we routinely obtain $a_i = \text{tr}(E_i^* A) \quad (0 \leq i \leq d)$. This establishes Theorem 3.1(v). Theorem 3.1(vi) follows from (40). We have established the conditions of Theorem 3.1. Therefore, the pair $A, A^*$ is a Leonard pair on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. For $0 \leq i \leq d$, $E_i^*$ is the primitive idempotent of $A^*$ associated with $\theta_i^*$. For $0 \leq i \leq d$, let $E_i$ denote the primitive idempotent of $A$ associated with the eigenvalue $\theta_i$. By construction, the sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By the equation on the left in (41), $\Phi$ has intersection numbers $\{a_i\}_{i=0}^d$, $\{b_i\}_{i=0}^{d-1}$, and $\{c_i\}_{i=1}^d$. \hfill \Box

## 5 Three applications of Theorem 4.1

In this section, we illustrate Theorem 4.1 with three examples.

**Proposition 5.1** Fix an integer $d \geq 1$. Assume that the characteristic of $\mathbb{K}$ is zero or an odd prime greater than $d$. Define

$$
\begin{align*}
\theta_i &= d - 2i \quad (0 \leq i \leq d), \\
\theta_i^* &= d - 2i \quad (0 \leq i \leq d), \\
b_i &= d - i \quad (0 \leq i \leq d - 1), \\
c_i &= i \quad (1 \leq i \leq d), \\
a_i &= 0 \quad (0 \leq i \leq d).
\end{align*}
$$

(51)

(52)

(53)

(54)

(55)
Then the conditions of Theorem 4.1 are satisfied with
\begin{align*}
\beta &= 2, \quad \gamma = 0, \quad \gamma^* = 0, \quad \theta_{-1} = d + 2, \\
\theta^*_{-1} &= d + 2, \quad \theta^*_{d+1} = -d - 2, \quad a_0^* = 0.
\end{align*}

Proof: Using the data (51)–(57), one routinely verifies that each of conditions (i)–(viii) from Theorem 4.1 holds.

Note 5.2 Referring to Proposition 5.1, the corresponding Leonard system from Theorem 4.1 is said to have Krawtchouk type; see [11, Section 24].

Proposition 5.3 Let \( K \) be arbitrary and fix an integer \( d \geq 1 \). Let \( a, b, c, \) and \( q \) denote nonzero scalars in \( K \) such that each of the following hold.

- \( q^{2i} \neq 1 \) for \( 1 \leq i \leq d \).
- Neither of \( a^2, b^2 \) is among \( q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d} \).
- None of \( abc, a^{-1}bc, ab^{-1}c, abc^{-1} \) is among \( q^{d-1}, q^{d-3}, \ldots, q^{1-d} \).

Define
\begin{align*}
\theta_i &= aq^{2i-d} + a^{-1}q^{d-2i} \quad (0 \leq i \leq d), \\
\theta^*_i &= bq^{2i-d} + b^{-1}q^{d-2i} \quad (0 \leq i \leq d), \\
b_0 &= \frac{(q^d - q^{-d})(cq - a^{-1}b^{-1}q^d)(q^{-1} - abc^{-1}q^{-d})}{bq^{2i-d} - b^{-1}q^{d-2i}}, \\
b_i &= \frac{(q^{2i-d} - q^{-2i})(bq^{2i-d} - b^{-1}q^{2d-2i})(cq^{i+1} - a^{-1}b^{-1}q^{d-i})(q^{-i-1} - abc^{-1}q^{-d-i})}{(bq^{2i-d} - b^{-1}q^{d-2i+1})(bq^{2i-d} - b^{-1}q^{d-2i-1})} \quad (1 \leq i \leq d - 1), \\
c_i &= \frac{(q^i - q^{-i})(bq^i - b^{-1}q^{-i}b^{-1}c^{-1}q^{-d})(bq^i - acq^{d-i+1})}{(bq^{2i-d} - b^{-1}q^{d-2i+1})(bq^{2i-d} - b^{-1}q^{d-2i})} \quad (1 \leq i \leq d - 1), \\
c_d &= \frac{(q^d - q^{-d})(a^{-1}q^{-1} - b^{-1}c q^{-d})(bq^{d} - ac)q^{-d}}{bq^{d-1} - b^{-1}q^{1-d}}, \\
a_i &= \theta_0 - b_i - c_i \quad (0 \leq i \leq d),
\end{align*}

where \( b_d = c_0 = 0 \). Then the conditions of Theorem 4.1 are satisfied with
\begin{align*}
\beta &= q^2 + q^{-2}, \quad \gamma = 0, \quad \gamma^* = 0, \quad \theta_{-1} = aq^{-d-2} + a^{-1}q^{d+2}, \\
\theta^*_{-1} &= bq^{-d-2} + b^{-1}q^{d+2}, \quad \theta^*_{d+1} = bq^{d+2} + b^{-1}q^{-d-2}, \\
a_0^* &= \frac{(b + b^{-1})(aq - a^{-1}q^{-1}) - (c + c^{-1})(q^{d} - q^{-d})}{aq^{1-d} - a^{-1}q^{d-1}}.
\end{align*}
Proof: Using the data (58)–(67), one routinely verifies that each of conditions (i)–(viii) from Theorem 4.1 holds. In this calculation, it is useful to note that

\[ \theta_i - \theta_j = (aq^{i+j-d} - a^{-1}q^{d-i-j})(q^{i-j} - q^{j-i}), \]
\[ \theta_i^* - \theta_j^* = (bq^{i+j-d} - b^{-1}q^{d-i-j})(q^{i-j} - q^{j-i}), \]

for \(0 \leq i, j \leq d\). In Theorem 4.1(vii), expression (39) is equal to

\[ (q^2 - q^{-2})((q^{d+1} - q^{-d-1})(c + c^{-1}) - (a - a^{-1})(b + b^{-1})) \]

for \(1 \leq i \leq d\). \(\square\)

Note 5.4 Referring to Proposition 5.3, the corresponding Leonard system from Theorem 4.1 is said to have \(q\)-Racah type; see [5, Section 5].

In applications, we are often presented with a tridiagonal matrix and a diagonal matrix, each with numerical entries, and we wish to know whether this is a Leonard pair. In our next example, we illustrate how to proceed using Theorem 4.1.

Proposition 5.5 Assume that the characteristic of \(\mathbb{K}\) is zero. Define \(d = 5\) and

\[
\begin{align*}
\theta_0^* & = 3, \quad \theta_1^* = \frac{93}{35}, \quad \theta_2^* = \frac{69}{35}, \quad \theta_3^* = \frac{33}{35}, \quad \theta_4^* = -\frac{3}{7}, \quad \theta_5^* = -\frac{15}{7}, \\
b_0 & = 3, \quad b_1 = \frac{64}{35}, \quad b_2 = \frac{243}{175}, \quad b_3 = \frac{48}{49}, \quad b_4 = \frac{11}{21}, \\
c_1 & = 1, \quad c_2 = \frac{192}{175}, \quad c_3 = \frac{243}{245}, \quad c_4 = \frac{16}{21}, \quad c_5 = \frac{3}{7}, \\
a_0 & = 0, \quad a_1 = \frac{6}{35}, \quad a_2 = \frac{18}{35}, \quad a_3 = \frac{36}{35}, \quad a_4 = \frac{12}{7}, \quad a_5 = \frac{18}{7}.
\end{align*}
\]

Then the conditions of Theorem 4.1 are satisfied with

\[
\begin{align*}
\theta_i & = \theta_i^* \quad (0 \leq i \leq 5), \\
\beta & = 2, \quad \gamma = \gamma^* = -\frac{12}{35}, \quad \theta_{-1} = \theta_{-1}^* = 3, \quad \theta_6^* = -\frac{21}{5}, \quad a_6^* = 0.
\end{align*}
\]

Proof: We now verify conditions (i)–(viii) in Theorem 4.1. Theorem 4.1(ii) holds by (68). Theorem 4.1(iv) holds by (69) and (70). Concerning Theorem 4.1(iii), using the data (68), we evaluate (37) at \(i = 1\) and \(i = 2\) to compute \(\beta\) and \(\gamma^*\). We then verify (37) and compute \(\theta_{-1}^*\) and \(\theta_6^*\). We have now verified Theorem 4.1(iii). Using the data (69) (71), we verify that Theorem 4.1(v) holds with \(\theta_0 = 3 = \theta_0^*\). Concerning Theorem 4.1(vi), using the data (68) (70), we evaluate (38) at \(i = 0\) and \(i = 1\). We routinely solve for \(\theta_1\) and \(a_0^*\), and verify (38). We have now verified Theorem 4.1(vi). Concerning Theorem 4.1(vii), using the data (68) (70), we evaluate (39) at \(i = 1\) to obtain \(\theta_{-1}\) and, using that value, we routinely verify (39). We have now verified Theorem 4.1(vii). We obtain \(\gamma\) using the first equation in Theorem 4.1(viii). We define \(\theta_2, \theta_3, \theta_4,\) and \(\theta_5\) so that (40) holds. We obtain \(\theta_i = \theta_i^*\) \((0 \leq i \leq 5)\). Note that Theorem 4.1(i) is satisfied. We have now verified each of conditions (i)–(viii) from Theorem 4.1. \(\square\)

Note 5.6 Referring to Proposition 5.5, the corresponding Leonard system from Theorem 4.1 is said to have Racah type; see [11, Example 35.9].
6 The first and second split sequence

Consider the Leonard system from Definition 2.3. In [7], this Leonard system was described using a sequence of scalars called its parameter array. A parameter array takes the form \( \{\theta_i\}_{i=0}^d, \{\phi_i\}_{i=1}^d \), where \( \{\theta_i\}_{i=0}^d \) is the eigenvalue sequence and \( \{\phi_i\}_{i=0}^d \) is the dual eigenvalue sequence. The sequences \( \{\varphi_i\}_{i=1}^d \) and \( \{\phi_i\}_{i=1}^d \) are called the first and second split sequences, respectively [12, p. 5]. It follows from [11, Definition 23.1 and Theorem 23.5] that for \( d \geq 1 \),

\[
\varphi_1 = b_0(\theta_1^* - \theta_0^*), \tag{72}
\]

\[
\varphi_i = b_{i-1} \frac{(\theta_i^* - \theta_0^*) \cdots (\theta_i^* - \theta_{i-1}^*)}{(\theta_{i-1}^* - \theta_0^*) \cdots (\theta_{i-1}^* - \theta_{i-2}^*)} \quad (2 \leq i \leq d), \tag{73}
\]

\[
\phi_i = c_i \frac{(\theta_i^* - \theta_d^*) \cdots (\theta_i^* - \theta_{i-1}^*)}{(\theta_i^* - \theta_d^*) \cdots (\theta_i^* - \theta_{i+1}^*)} \quad (1 \leq i \leq d - 1), \tag{74}
\]

\[
\phi_d = c_d(\theta_d^* - \theta_d^*). \tag{75}
\]

Assume that our Leonard system is the one from Proposition 5.1. Using (52)–(54) to simplify (72)–(75), we obtain

\[
\varphi_i = -2i(d - i + 1) \quad (1 \leq i \leq d),
\]

\[
\phi_i = 2i(d - i + 1) \quad (1 \leq i \leq d).
\]

This matches the data presented in [8, Section 16].

Next, assume that our Leonard system is the one from Proposition 5.3. Using (59)–(63) to simplify (72)–(75), we find that for \( 1 \leq i \leq d \),

\[
\varphi_i = a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - abcq^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1}),
\]

\[
\phi_i = ab^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - a^{-1}bcq^{i-d-1})(q^{-i} - a^{-1}bc^{-1}q^{i-d-1}).
\]

This matches the data presented in [5, Definition 6.1].

Finally, assume that our Leonard system is the one from Proposition 5.5. Using (68)–(70) to simplify (72)–(75), we find that

\[
\varphi_1 = -\frac{36}{35}, \quad \varphi_2 = -\frac{4608}{1225}, \quad \varphi_3 = -\frac{8748}{1225}, \quad \varphi_4 = -\frac{2304}{245}, \quad \varphi_5 = -\frac{396}{49},
\]

\[
\phi_1 = -\frac{36}{49}, \quad \phi_2 = -\frac{2304}{1225}, \quad \phi_3 = -\frac{2916}{1225}, \quad \phi_4 = -\frac{2304}{1225}, \quad \phi_5 = -\frac{36}{49}.
\]

7 Acknowledgment

The author thanks Paul Terwilliger for providing many valuable ideas and detailed suggestions, and John Caughman for providing the data for the example from Proposition 5.5.
References

[1] E. Bannai, T. Ito, *Algebraic Combinatorics I: Association Schemes*. Benjamin/Cummings, London, 1984.

[2] E. Hanson, A characterization of Leonard pairs using the notion of a tail. *Linear Algebra Appl.* 435 (2011) 2961–2970; [arXiv:0911.0098v1](#).

[3] E. Hanson, A characterization of Leonard pairs using the parameters $\{a_i\}_{i=0}^d$. *Linear Algebra Appl.* 438 (2013) 2289–2305; [arXiv:1205.4368v1](#).

[4] E. Hanson, A characterization of bipartite Leonard pairs using the notion of a tail. *Linear Algebra Appl.* 452 (2014) 46–67; [arXiv:1308.3826v1](#).

[5] H. Huang, The classification of Leonard triples of QRacah type. *Linear Algebra Appl.* 436 (2012) 1442–1472; [arXiv:1108.0458v1](#).

[6] P. Terwilliger, The subconstituent algebra of an association scheme I. *J. Algebraic Combin.* 1 (1992) 363–388.

[7] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* 330 (2001) 149–203; [arXiv:math/0406555v1](#).

[8] P. Terwilliger, Leonard pairs from 24 points of view. *Rocky Mountain J. Math.* 32(2) (2002) 827–888; [arXiv:math/0406577v1](#).

[9] P. Terwilliger, Introduction to Leonard pairs, OPSFA Rome 2001, *J. Comput. Appl. Math.* 153 (2003) 463–475.

[10] P. Terwilliger, Leonard pairs and the $q$-Racah polynomials. *Linear Algebra Appl.* 387 (2004) 235–276; [arXiv:math/0306301v2](#).

[11] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials, Lecture notes for the summer school on orthogonal polynomials and special functions, Universidad Carlos III de Madrid, Leganes, Spain. July 8–July 18, 2004; [arXiv:math/0408390v3](#).

[12] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array. *Des. Codes Cryptogr.* 34(2-3) (2005) 307–332; [arXiv:math/0306291v1](#).

[13] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the TD-D canonical form and the LB-UB canonical form, *J. Algebra* 291 (2005) 1–45; [arXiv:math/0304077v1](#).

[14] P. Terwilliger, R. Vidunas, Leonard pairs and the Askey-Wilson relations. *J. Algebra Appl.* 3 (2004) 411–426; [arXiv:math/0305356v1](#).

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