Ulam stability of solutions of a discrete boundary value problem with fractional order

A George Maria Selvam1* and R Dhineshbabu1
1Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur - 635601, Vellore Dist., Tamil Nadu, India.
E-mail: 1*agmshc@gmail.com

Abstract. In this present work, we investigate Ulam - Hyers stability for the following nonlinear discrete fractional order boundary value problem of the form

\[ C_0^{\delta} \Delta_k^\delta v(k) = \Psi (k + \delta - 1, v(k + \delta - 1)), \]

for \( k \in [0, L]_{N_0} = \{0, 1, \ldots, L\} \), with boundary conditions \( v(\delta - 2) = 0 = v(\delta + L) \), where \( \Psi : [\delta - 1, \delta + L]_{N_0} \times R \to R \) is a continuous and \( C_0^{\delta} \Delta_k^\delta \) is the Caputo fractional difference operator with \( 1 < \delta \leq 2 \).

Keywords: Discrete fractional calculus, Boundary value problem, Caputo fractional derivative, Ulam - Hyers stability.

1. Introduction

Fractional calculus is one of the branches of study which involves integrals and derivatives of an arbitrary order. Moreover, students of engineering, economics, biology and physical sciences encounter differential operators \( \frac{d}{dx}, \frac{d^2}{dx^2}, \) etc., but only few of them debated whether it is required for the derivative to be only of an integer order. The fundamental concept of fractional calculus is widely believed to have arisen from a question raised by Marquis to Leibniz in the year 1695. Fractional calculus is the one of the most novel types of tools applied to problems in engineering, science and technology [1, 2].

In the past few decades, fractional calculus has played a key role of improve the specialized research in mathematical modeling of several phenomena such as enzyme kinetics, a nonlinear oscillation of earthquake, blood flow problems, aerodynamics, cancer modeling being among the popular ones, control theory, the fluid-dynamic traffic model, regular variation in thermodynamics, polymer rheology, arterial and heart disease modeling, etc [3, 4, 5, 6]. Miller and Ross [7] initiated the development of the study on discrete fractional calculus and the author [8, 9] have been developing the theoretical results for fractional difference equations in the case of forward difference operator. The initial definitions for fractional order sums and difference were developed in the works of [10]. The theory of Discrete Fractional Calculus is elaborated by Atici and Eloe et al [11, 12] and they obtained important properties and results.

A boundary value problem is an initial value problem with additional set of constraints known as the boundary conditions. This boundary condition defines a relationship between the
unknown solution and one or more of its derivatives. Now a days Homotopy analysis method and Hadamard, etc. are the various techniques applied to solve the fractional order derivatives likes Riemann - Liouville and Caputo differential / difference operators. Recently, many authors have shown great interest in the theoretical study of nonlinear fractional boundary value problems (FBVPs) for both differential equations and difference equations supplemented with different boundary conditions. These results have found applications in various fields such as thermoplastic, fluid flows, chemical physics, underground water flow models, electrical networks and viscoelasticity [1, 4]. For additional data in discrete fractional calculus or fractional calculus, especially for boundary value problems refer papers or books [2, 4]. In the past few decades, many researchers have focussed their attention on the study of stability of solutions, existence and uniqueness and multiplicity of solutions for nonlinear fractional differential equations by using various fixed point techniques such as Krasnoselskii fixed point theorem, Brower fixed point theorm, contraction mapping principle [13, 14, 15].

In 19th century, the study of stability analysis has emerged with several types of stability such as Mittage - Leffler, exponential, etc. The development of the theory on stability analysis of fractional differential equations has moved forward slowly and in recent years several mathematicians have published many research articles and books in this area based on the new concept of generalized Hyers stability for the definitions of the Hyers - Ulam stability and Hyers - Ulam - Rassias stability (see [16, 17]). Further a Wang et al. [18] studied the pioneering work on the solutions of Ulam stability, existence and uniqueness results and data dependence to a class of Cauchy boundary value problem for Caputo fractional differential equations. Also Fulai Chen and Yong Zhou [19] proved the existence and Ulam stability results for discrete FBVPs with Caputo derivative. In this present work, we establish the Ulam - Hyers stability and Ulam - Hyers - Rassias stability for the following discrete FBVPs of the form

\[ \Delta_{\delta}^\nu v(k) = \Psi(k + \delta - 1, v(k + \delta - 1)), \]  

for \( k \in [0, L] \cap \mathbb{N} = \{0, 1, ..., L\} \), subject to the conditions

\[ (i) \ v(\delta - 2) = 0 \quad \text{and} \quad (ii) \ v(\delta + L) = 0, \]  

where \( \Psi : [\delta - 1, \delta + L] \cap \mathbb{N} \times \mathbb{R} \to \mathbb{R} \) is a continuous function and \( \Delta_{\delta}^\nu \) is the Caputo fractional difference operator which satisfies \( 1 < \delta \leq 2 \) and \( L \in \mathbb{N}_0 \).

The plan of this paper is as follows. Some basic preliminaries which are useful to prove the main results are provided in section 2. In section 3, solution of the given FBVP is obtained and Ulam stability for the discrete FBVP of (1) with boundary conditions (2) is investigated in section 4.

2. Preliminaries

Now we present some useful definitions and basic lemmas of discrete fractional calculus that are to be used throughout this paper.

**Definition 1** (see [11, 12]). Let \( \delta > 0 \). The \( \delta \)th fractional sum of a function \( \Psi \), is defined as

\[ \Delta_{\delta}^\nu \Psi(k) = \frac{1}{\Gamma(\delta)} \sum_{\xi = a}^{k-\delta} (k - \xi - 1)^{(\delta - 1)} \Psi(\xi), \]  

for all \( k \in \{a + \delta, a + \delta + 1, \ldots\} = \mathbb{N}_{a+\delta} \) and \( k^{(\delta)} := \frac{\Gamma(k+1)}{\Gamma(k+1-\delta)} \).

**Definition 2** (see [9, 19]). Let \( \delta > 0 \) and set \( \mu = n - \delta \). The \( \delta \)th fractional Caputo difference
operator is defined as
\[ C_0^\Delta_k \Psi(k) = \Delta^{-\mu} (\Delta^n \Psi(k)) = \frac{1}{\Gamma(\mu)} \sum_{\xi=a}^{k-\mu} (k - \xi - 1)^{(\mu-1)} \Delta^n \Psi(\xi), \] (4)
for all \( k \in \{a + \mu, a + \mu + 1, \ldots\} = N_{a + \mu} \) and \( n - 1 < \delta \leq n \), where \( n = \lceil \delta \rceil \), \( \lceil . \rceil \) ceiling of number.

Now we give another important lemma which shall be used to prove the results.

**Lemma 3** (see [13, 19]). Suppose that \( \delta > 0 \) and \( \Psi \) is defined on \( N_a \). Then
\[ C_0^\Delta_k \Delta^{-\delta} \Psi(k) = \sum_{j=0}^{n-1} \frac{(k - a)^{(j)}}{j!} \Delta^j \Psi(a), \]
\[ = \Psi(k) + A_0 + A_1 k + \ldots + A_{n-1} k^{(n-1)}, \] (5)
for some \( A_i \in R \), where \( i = 1, 2, \ldots, n - 1 \).

3. Main results
In this section, we now determine the solution of a discrete FBVP (1) - (2), provided that the solution exists.

**Theorem 4.** Let \( \Psi : [\delta - 1, \delta + L]_{N_{\delta-1}} \times R \to R \) and \( 1 < \delta \leq 2 \) be given. A function \( v(k) \) is a solution of a discrete FBVP (1) - (2) iff \( v(k) \), for \( k \in [\delta - 2, \delta + L]_{N_{\delta-2}} \) has the form
\[ v(k) = \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1)) + \frac{(\delta - k - 2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1)). \] (6)

**Proof.** Suppose that \( v(k) \) is defined on \([\delta - 2, \delta + L]_{N_{\delta-2}} \) is a solution of (1) - (2). From Lemma 3, we obtain a general solution for (1) as
\[ v(k) = \Delta^{-\delta} \Psi(k + \delta - 1, v(k + \delta - 1)) + A_0 + A_1 k, \quad k \in [\delta - 2, \delta + L]_{N_{\delta-2}}, \]
where \( A_0, A_1 \in R \). Hence, by Definition 1, we get
\[ v(k) = \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1)) + A_0 + A_1 k. \]

Now we find \( A_0 \) and \( A_1 \) by assuming the boundary conditions (2) hold. By the first boundary condition (i) \( v(\delta - 2) = 0 \) and we get
\[ \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{-2} (\delta - \xi - 3)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1)) + A_0 + A_1 (\delta - 2) = 0. \] (7)
By the standard convention on sums, so in summary, we find that (7) implies that
\[ A_0 + A_1 (\delta - 2) = 0. \] (8)
On the other hand, by applying the second boundary condition (ii) \( v(\delta + L) = 0 \), we have

\[
\frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1)) + A_0 + A_1(\delta + L) = 0.
\] (9)

Substituting equation (8) in (9), we get

\[
A_1 = -\frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1))
\] (10)

Now to find \( A_0 \) by using equation (10). So equation (8) becomes

\[
A_0 = \frac{(\delta - 2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1))
\]

Using \( A_0 \) and \( A_1 \) in \( v(k) \), we obtain

\[
v(k) = \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1))
\]

\[
+ \frac{(\delta - k - 2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{(\delta-1)} \Psi(\xi + \delta - 1, v(\xi + \delta - 1))
\]

for \( k \in [\delta - 2, \delta + L]_{N_{\delta-2}} \). This show that if (1) - (2) has a solution, then it can be represented by (6).

Conversely, every function of the form (6) is a solution of (1) - (2). The proof is completed.

We now establish the following result, that will be useful in the next section 4.

**Lemma 5.** One has

\[
\sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} = \frac{\Gamma(k + 1)}{\delta \Gamma(k - \delta + 1)}.
\] (11)

**Proof.** For \( z > l, z, l \in R, l > -1, z > -1 \), we have [9]

\[
\frac{\Gamma(z + 1)}{\Gamma(l + 1)\Gamma(z - l + 1)} = \frac{\Gamma(z + 2)}{\Gamma(l + 2)\Gamma(z - l + 1)} - \frac{\Gamma(z + 1)}{(l + 2)\Gamma(z - l)}.
\] (12)

That is

\[
\frac{\Gamma(z + 1)}{\Gamma(z - l + 1)} = \frac{1}{l + 1} \left[ \frac{\Gamma(z + 2)}{\Gamma(z - l + 1)} - \frac{\Gamma(z + 1)}{\Gamma(z - l)} \right].
\] (13)

Then,

\[
\sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} = \sum_{\xi=0}^{k-\delta-1} (k - \xi - 1)^{(\delta-1)} + \Gamma(\delta).
\] (14)

Now we obtain

\[
\sum_{\xi=0}^{k-\delta-1} (k - \xi - 1)^{(\delta-1)} = \frac{1}{\delta} \left[ \frac{\Gamma(k + 1)}{\Gamma(k - \delta + 1)} - \frac{\Gamma(\delta + 1)}{\Gamma(1)} \right].
\] (15)
We suppose that:

\[ \text{there exists a solution} \]

be a solution of the following discrete FBVP:

\[ \text{Assume that (19). The nonlinear fractional difference equation (1) is Hyers-Ulam stable if for} \]

\[ \text{Definition 7} \]

\[ \parallel E \parallel \text{stablity definitions for fractional difference equation.} \]

\[ \text{discrete FBVP (1) - (2). Now we introduce the Hyers-Ulam stablity and Hyers-Ulam-Rassias} \]

\[ \text{4. The ulam stability} \]

In this section, we obtain Ulam-Hyers stability and Ulam-Hyers-Rassias stability results for the discrete FBVP (1) - (2). Now we introduce the Hyers-Ulam stability and Hyers-Ulam-Rassias stability definitions for fractional difference equation.

For our purpose, let \( E \) be the set of all real sequences \( v = \{v(k)\}_{k=\delta-2}^{\delta+L} \) with norm \( \|v\| = \max |v(k)| \) for \( k \in [\delta - 2, \delta + L]_{N_{\delta-2}} \). Then \( E \) is a Banach space.

**Definition 7** [19]. The nonlinear fractional difference equation (1) is Hyers-Ulam stable if for every \( \epsilon > 0 \), there is a constant \( M > 0 \) and for every solution \( u \in E \) of

\[ \Delta_k^\delta u(k) - \Psi(k + \delta - 1, u(k + \delta - 1)) \leq \epsilon, \ k \in [0, L]_{N_0}, \]  

there exists a solution \( v \in E \) of (1) such that

\[ |u(k) - v(k)| \leq M \epsilon, \ k \in [\delta - 2, \delta + L]_{N_{\delta-2}}. \]

**Definition 8** [19]. The nonlinear fractional difference equation (1) is Hyers-Ulam-Rassias stable if for every \( \epsilon > 0 \), there is a constant \( M > 0 \) and for every solution \( u \in E \) of

\[ \Delta_k^\delta u(k) - \Psi(k + \delta - 1, u(k + \delta - 1)) \leq \epsilon \Phi(k + \delta - 1), \ k \in [0, L]_{N_0}, \]  

there exists a solution \( v \in E \) of (1) such that

\[ |u(k) - v(k)| \leq M \epsilon \Phi(t), \ k \in [\delta - 2, \delta + L]_{N_{\delta-2}}. \]

We suppose that:

(H1) There exists a constant \( \Omega > 0 \) such that \( |\Psi(k, u) - \Psi(k, v)| \leq \Omega |u - v| \) for each \( k \in [\delta - 2, \delta + L]_{N_{\delta-2}} \) and all \( u, v \in E \).

(H2) Let \( \Phi \in [\delta - 2, \delta + L]_{N_{\delta-2}} \rightarrow R^+ \) be an increasing function. Then there exists a constant \( \lambda > 0 \) such that

\[ \frac{\epsilon}{\Gamma(\delta)} \sum_{k=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} \Phi(\xi + \delta - 1) \leq \lambda \epsilon \Phi(k + \delta - 1), \ k \in [0, L]_{N_0}. \]

**Theorem 9.** Assume that (H1) holds. Let \( u \in E \) be a solution of inequality (17) and let \( v \in E \) be a solution of the following discrete FBVP:

\[ \Delta_k^\delta v(k) = \Psi(k + \delta - 1, v(k + \delta - 1)), \ k \in [0, L]_{N_0}, \ 1 < \delta \leq 2, \]

\[ v(\delta - 2) = 0 = v(\delta + L). \]
Then, (1) is Ulam-Hyers stable provided that

$$\Omega < \frac{\Gamma(\delta + 1)\Gamma(L + 1)}{2\Gamma(\delta + L + 1)}. \quad (23)$$

**Proof.** From inequality (17), for \( k \in [\delta - 2, \delta + L]_{N_{\delta - 2}} \), it follows that

$$\left| u(k) - \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)(\delta-1)\Psi(\xi + \delta - 1, u(\xi + \delta - 1)) \right.$$  

$$- \frac{(\delta - k - 2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)(\delta-1)\Psi(\xi + \delta - 1, u(\xi + \delta - 1)) \right|$$

$$\leq \frac{\epsilon}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)(\delta-1)$$

$$\leq \frac{\epsilon}{\Gamma(\delta)} \frac{\Gamma(k + 1)}{\delta \Gamma(k + 1 - \delta)}$$

$$\leq \frac{\epsilon \Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1)} \quad (24)$$

From (6) and (24), for \( k \in [\delta - 2, \delta + L]_{N_{\delta - 2}} \), we have

$$|u(k) - v(k)| \leq \left| u(k) - \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)(\delta-1)\Psi(\xi + \delta - 1, v(\xi + \delta - 1)) \right.$$  

$$- \frac{(\delta - k - 2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)(\delta-1)\Psi(\xi + \delta - 1, v(\xi + \delta - 1)) \right|$$

$$\leq u(k) - \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)(\delta-1)\Psi(\xi + \delta - 1, u(\xi + \delta - 1))$$

$$- \frac{(\delta - k - 2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)(\delta-1)\Psi(\xi + \delta - 1, u(\xi + \delta - 1))$$

$$+ \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)(\delta-1)$$

$$\times |\Psi(\xi + \delta - 1, u(\xi + \delta - 1)) - \Psi(\xi + \delta - 1, v(\xi + \delta - 1))|$$

$$+ \frac{|\delta - k - 2|}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)(\delta-1)$$

$$\times |\Psi(\xi + \delta - 1, u(\xi + \delta - 1)) - \Psi(\xi + \delta - 1, v(\xi + \delta - 1))|$$

$$|u(k) - v(k)| \leq \frac{\epsilon \Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1)} + \frac{\Omega}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)(\delta-1) |u(\xi + \delta - 1) - v(\xi + \delta - 1)|$$

$$+ \frac{\Omega |\delta - k - 2|}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)(\delta-1) |u(\xi + \delta - 1) - v(\xi + \delta - 1)|$$
Then

\[ |u(k) - v(k)| \leq \frac{e\Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1)} + \frac{\Omega \|u - v\|}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{\delta-1}
\]

\[ + \frac{\Omega \|u - v\| \Omega}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{\delta-1} \]

From the above inequality, we have

\[ \|u - v\| \leq \frac{e\Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1)} + \frac{2\Omega \Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1)} \|u - v\|. \tag{25} \]

From the above inequality, we have

\[ \|u - v\| \leq \frac{\Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1) - 2\Omega \Gamma(\delta + L + 1)} \epsilon, \tag{26} \]

where \( \frac{\Gamma(\delta + L + 1)}{\Gamma(\delta + 1)\Gamma(L + 1) - 2\Omega \Gamma(\delta + L + 1)} > 0 \). Thus equation (1) is Ulam - Hyers stable.

**Theorem 10.** Assumptions \((H_1)\) and \((H_2)\) hold. Let \( u \in E \) be a solution of (19) and let \( v \in E \) be a solution of the following discrete FBVP (22). Then, (1) is the Ulam-Hyers-Rassias stable provided that (23) holds.

**Proof.** From inequality (19), for \( k \in [\delta - 2, \delta + L]_{\mathbb{N}_{\delta-2}} \), it follows that

\[ |u(k) - \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{\delta-1}\Psi(\xi + \delta - 1, u(\xi + \delta - 1)) - \frac{\Omega - (\delta-k-2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{\delta-1}\Psi(\xi + \delta - 1, u(\xi + \delta - 1))| \]

\[ \leq \frac{\epsilon}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{\delta-1}\Phi(\xi + \delta - 1) \]

\[ \leq \lambda \epsilon \Phi(k + \delta - 1). \tag{27} \]

From (6) and (27), for \( k \in [\delta - 2, \delta + L]_{\mathbb{N}_{\delta-2}} \), we have

\[ |u(k) - v(k)| \leq |u(k) - \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{\delta-1}\Psi(\xi + \delta - 1, v(\xi + \delta - 1)) - \frac{\Omega - (\delta-k-2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{\delta-1}\Psi(\xi + \delta - 1, v(\xi + \delta - 1))| \]

\[ \leq |u(k) - \frac{1}{\Gamma(\delta)} \sum_{\xi=0}^{k-\delta} (k - \xi - 1)^{\delta-1}\Psi(\xi + \delta - 1, u(\xi + \delta - 1)) - \frac{\Omega - (\delta-k-2)}{\Gamma(\delta)(L + 2)} \sum_{\xi=0}^{L} (\delta + L - \xi - 1)^{\delta-1}\Psi(\xi + \delta - 1, u(\xi + \delta - 1))| \]
where

\[ \lambda \]

From the above inequality, we have

\[ u(k) - v(k) | \leq \lambda \varepsilon \Phi(k + \delta - 1) + \frac{\Omega}{\Gamma(\delta)} \sum_{k=0}^{k-\delta} (k - \xi - 1)^{(\delta-1)} |u(\xi + \delta - 1) - v(\xi + \delta - 1)| \]

Then

\[ \|u - v\| \leq \lambda \varepsilon \Phi(k + \delta - 1) + \frac{2\Omega \|u - v\| \Gamma(\delta + L + 1)}{\Gamma(\delta + 1) \Gamma(L + 1)} \]

From the above inequality, we have

\[ \|u - v\| \leq \lambda \frac{\Gamma(\delta + 1) \Gamma(L + 1)}{\Gamma(\delta + 1) \Gamma(L + 1) - 2\Omega \Gamma(\delta + L + 1)} \varepsilon \Phi(k + \delta - 1), \]

where \( \lambda \frac{\Gamma(\delta + 1) \Gamma(L + 1)}{\Gamma(\delta + 1) \Gamma(L + 1) - 2\Omega \Gamma(\delta + L + 1)} > 0 \). Thus, equation (1) attains Ulam - Hyers - Rassias stability.

References

[1] Oldham K B and Spanier J 1974 The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order (California: Academic Press)
[2] Podlubny I 1999 Fractional Differential Equations, Mathematics in Science and Engineering (New York: Academic Press)
[3] Selvam A G M and Dhineshbabu R 2019 AIP Conference Proceedings 2095 7
[4] A A Kilbas H M S and Trujillo J J 2016 Theory and Applications of Fractional Differential Equations (Amsterdam: North-Holland Math. Studies, Elsevier)
[5] Norton L A 1988 Cancer Research 48 7067
[6] Selvam A G M and Dhineshbabu R 2019 American International Journal of Research in Science, Technology, Engineering and Mathematics 2019 4
[7] Miller K S and Ross B 1993 An Introduction to The Fractional Calculus and Fractional Differential Equations (New York, USA: John Wiley and Sons, INC)
[8] Anastassiou G A 2010 Mathematical and Computer Modelling 51 562
[9] Anastassiou G A 2010 Intelligent Mathematics: Computational Analysis (India: Springer)
[10] Diaz J B and Osler T J 1974 Mathematics of Computation 28 185
[11] FMAtici and Eloe P 2009 Proceedings of American Mathematical Society 137 981
[12] FMAtici and Eloe P 2007 *International Journal of Differential Equations* **2** 165
[13] Abdeljawad T 2011 *Computers and Mathematics with Applications* **62** 1602
[14] Goodrich C S 2010 *International Journal of Difference Equations* **5** 195
[15] Goodrich C S 2011 *International Journal of Dynamical Systems and Differential Equations* **3** 145
[16] Abbas S and Benchohra M 2014 *Appl. Math. E-Notes* **14** 20
[17] Gavruta P 1994 *J. Math. Anal. Appl.* **184** 431
[18] J Wang L L and Zhou Y 2011 *Electronic Journal of Qualitative Theory of Differential Equations* **63** 10
[19] Chen F and Zhou Y 2013 *Discrete Dynamics in Nature and Society* **2013** 7