Perfect and Quasi-Perfect Lattice Actions

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Abstract: Perfect lattice actions are exiting with several respects: they provide new insight into conceptual questions of the lattice regularization, and quasi-perfect actions could enable a great leap forward in the non-perturbative solution of QCD. We try to transmit a flavor of them, also beyond the lattice community.

1 Introduction

For many field theoretic models, in particular in $d = 4$, the lattice is the only regularization so far, which provides non-perturbative results. One discretizes the Euclidean space and introduces matter fields only on the lattice sites, and gauge fields on the links connecting them. By means of some Monte Carlo procedure one generates a number of configurations, and for large statistics we can numerically perform a functional integral (to some accuracy). However, even if the statistical error is under control, we still have to worry about systematic errors. Simulations take place at a finite lattice spacing $a$ and in a finite size $L$, so the final limits $\xi/a$, $L/\xi \to \infty$ require extrapolations of the simulation results ($\xi$ is the correlation length). The finiteness of these ratios causes artifacts, and in practice those due to $a > 0$ are most troublesome.

It is very expensive in computer time to approximate the continuum limit by "brute force", i.e. by using finer and finer lattices. The computational effort grows at least like $a^{-6}$, in some cases this factor can rise up to $a^{-10}$. When he was about to quit lattice physics, K. Wilson estimated that a convincing solution of QCD requires a lattice of $256^3 \times 512$ sites. This appears hopeless for generations of supercomputers as well as human beings; what the up-coming supercomputers ("teraflops") can process is around $24^4 \ldots 32^4$ for full QCD. (Larger lattices can be used in the so-called quenched approximation: there one sets the fermion determinant equal to a constant, which means physically that sea quark effect are neglected. Hence one brings in another systematic error, which is often difficult to control.)

However, Wilson referred to his standard lattice action, which handles derivatives by nearest lattice site differences (including a possible gauge variable living on the connecting link). The pure gauge part is described by a "plaquette action": from the paths around single plaquettes one constructs a (gauge invariant) lattice version of
Nowadays, there is a consensus that improved lattice actions are a ray of hope for a much faster progress in the near future, than the gradual increase in computer power provides. Of course, many lattice actions have the correct continuum limit, and a better choice could suppress the artifacts significantly. Non-standard lattice actions involve additional terms, which make the simulations more complicated and slower. Still the ratio gain/cost can be huge, if they allow us to, say, double or triple the lattice spacing, which has to be chosen typically \( \leq 0.1 \, fm \) for Wilson’s QCD action.

At present, there are essentially two improvement programs under investigation. One of them has been formulated by K. Symanzik [2], and tries to cancel the cut-off artifacts order by order in \( a \), by adding irrelevant operators to Wilson’s action. This is very similar to the Runge and Kutta procedure for solving differential equations. In QCD (with Wilson fermions) it has been realized to \( O(a) \), first on the classical level by adding a so-called “clover term” (a plaquette version of \( \sigma_{\mu\nu} F_{\mu\nu} \))[3]. More recently, the renormalization of the clover coefficient has been estimated by a mean-field approach [4], and finally the non-perturbative \( O(a) \) improvement has been completed by extensive simulations, taking the PCAC relation as a guide-line [5]. At present, many tests are being performed with the resulting action, but it is not clear yet to which extent it enables the use of coarse lattices [3]. As a preliminary impression, there is progress in the scaling on fine lattices, but it does not enable the use of really coarse lattices (it seems that this \( O(a) \) improvement accidently amplifies the \( O(a^2) \) artifacts).

These notes are devoted to the alternative improvement program, which goes under the name “perfect actions”. It is based on renormalization group concepts, and it is non-perturbative with respect to \( a \). It works beautifully in principle, as we know from a sequence of 2d toy models: the \( O(3) \) model, the Gross-Neveu model, the \( CP(3) \) model, and the Schwinger model. Moreover, it has theoretically fascinating properties. In particular, it allows us to reproduce symmetries of the continuum theory exactly on the lattice, even in cases where this is apparently impossible. Examples are the continuous Poincaré invariance, as well as chiral symmetry. It is even possible to introduce a perfect lattice topology. Hence the program is very attractive from the conceptual point of view, also apart from the practical aspect of reducing the computational effort in simulations.

However, the implementation in full QCD is very tedious. In practice, a crucial aspect is the struggle for an excellent locality of the action, so that the truncation of the couplings – which is required at some point – does not do too much harm to it. In this context, a good parameterization and truncation of the action are now major issues in this program, which have not yet been solved in a really satisfactory way. Still, the potential of this method is great; if it can really be applied, then it can do by far better than an \( O(a) \) improvement, although it requires more non-standard terms than just a clover term. In this program, the improvement can be extended to better and better approximations to perfection. In contrast, it is hardly feasible to carry on Symanzik’s program to \( O(a^2) \), so if \( O(a) \) should be insufficient with some respect, then that program is at a dead-lock.
2 Perfect and classically perfect actions

It has been known for a long time that there exist so-called renormalized trajectories in parameter space \([7]\), which represent perfect lattice actions. These are actions without any lattice artifacts; they display the continuum values of scaling quantities at any lattice spacing.

This can be understood from the consideration of block variable renormalization group transformations (RGTs). As a simple example, we start from a hypercubic lattice with spacing \(1/n\), and divide it into disjoint blocks of \(n^d\) sites, where \(d\) is the Euclidean space-time dimension, and \(n\) is going to be the blocking factor. The block centers form a coarse lattice of unit spacing. There we define new variables, collectively denoted by \(\phi'\), which we relate to the corresponding block averages of the fine lattice variables \(\phi\). The action on the coarse lattice is given by

\[
e^{-S[\phi']} = \int D\phi' K[\phi', \phi] e^{-S[\phi]}.
\]

The kernel \(K\) must be chosen such that the partition function, and all expectation values – hence the physical contents of the theory – remain invariant. This requires

\[
\int D\phi' K[\phi', \phi] = 1,
\]

which still leaves quite some freedom. The simplest choice,

\[
K[\phi', \phi] = \prod_{x'} \delta(\phi'_{x'} - \frac{1}{n} \sum_{x \in x'} \phi_x),
\]

determines the \(\delta\) function RGT (the sum \(x \in x'\) runs over the fine lattice sites \(x\) in the block with center \(x' \in \mathbb{Z}^d\)). An obvious generalization of the kernel (3) “smears” the \(\delta\) function to a Gaussian, so that the transformation term appears in the exponent. The RGT reduces the correlation length in lattice units, \(\xi\), by a factor \(n\), \(\xi' = \xi/n\).

Assume that we are on a “critical surface” in parameter space, where \(\xi = \infty\). For suitable parameters we arrive – after an infinite number of RGT iterations (which include permanent re-scaling to the newest lattice units) – at a finite fixed point action (FPA) \(S^*\), which is invariant under the considered RGT: \(S^{*'} = S^*\).

Let us now perform a tiny step away from the FPA – and from the critical surface – in a relevant direction, and then keep on iterating RGTs. Thus we follow a trajectory in parameter space, which takes us to shorter and shorter correlation length. This is a renormalized trajectory, each point of which is related to the vicinity of the FPA solely by the renormalization group. There is no way irrelevant operators can contaminate the actions represented by the points on such a trajectory, hence these actions are perfect. Scaling quantities extracted from such actions are therefore completely free of lattice spacing artifacts. Needless to say that the identification of (quasi-)perfect actions at moderate or even short \(\xi\) – where the simulations take

\[1\]One may think of the parameter space as being spanned by all possible couplings between the lattice variables.
place – is a dream of humanity (or at least of the lattice community). However, it is very difficult to find good approximations to perfect actions, which are tractable in simulations.

As an alternative description, one may start on a suitable point close to the critical surface, and after many RGTs a renormalized trajectory is approximated asymptotically. This property suggests an explicit construction by Monte Carlo RGTs, which has been tried for some time, but which did not prove very useful: such RGT steps are tedious to perform, and the iteration is restricted to very few (often hardly one) reliable steps, which do not suppress the artifacts dramatically.

![Figure 1: A caricature of the picture in parameter space.](image)

A few years ago, P. Hasenfratz and F. Niedermayer suggested a new trick to construct approximately perfect actions, which is particularly designed for asymptotically free theories. We write the RGT (in a slightly modified notation) as

\[
e^{-\frac{1}{g^2}S'[\phi']} = \int D\phi \ e^{-\frac{1}{g} \left( S[\phi] + T[\phi', \phi] \right)},
\]

where \( T \) is now the transformation term, which specifies the RGT. Here the critical surface – and hence the FPA – is situated at \( g = g' = 0 \), where the RGT is determined simply by minimization,

\[
S'[\phi'] = \min_\phi \{ S[\phi] + T[\phi', \phi] \}, \quad S^*[\phi'] = \min_\phi \{ S^*[\phi] + T[\phi', \phi] \}.
\]

Thus the search for a FPA simplifies enormously to a classical field theory problem; no (numeric) functional integral is needed. But we still need to proceed to moderate \( \xi \) in order to control the finite size effects. And here the authors of Ref. 2

\[2\text{In the literature, this is often called a “saddle point problem”, although one just deals with minima.}

\[3\text{In practice one may proceed as follows: make a parameterization ansatz for the action } S'; \text{ choose some configurations } \phi'; \text{ (numeric) minimization yields } S'[\phi']. \text{ For a sufficient number of configurations, the parameters in the ansatz for } S' \text{ are determined.}

\]
suggest to just switch on $g$ and use the “classically perfect action” $(1/g^2)S^*\{\phi\}$. Thus we follow the weakly relevant (in leading order marginal) direction away from the FPA. If we multiply $g^2$ by $\hbar$, we understand the notion “classically perfect”; in the limit $\hbar \to 0$ the minimization trick persists at finite $g$. In the full quantum theory, however, this is not the case; the renormalized trajectory deviates from its classically perfect approximation. The hope – and the one uncontrolled assumption in this program – is that the latter is still an approximately perfect action at moderate $\xi$. (Intuitively we may support this hope by associating the quantum deviations of renormalized trajectories with the $\beta$ functions, which tend to be smooth for asymptotically free theories.) In fact, toy model studies confirm the excellent quality of this approximation in a striking way: in a scaling test for the 2d $O(3)$ model (in a small volume $L \simeq \xi$) no lattice artifacts at all could be seen down to $\xi \simeq 5$. A study of the Gross-Neveu model revealed that in the large $N$ limit the classically perfect action is also quantum perfect ($N$ suppresses the quantum fluctuations) so that, for instance, the dynamically generated fermion mass divided by the chiral condensate is a constant, independent of $\xi$ [9]. The classically perfect 2d $O(3)$ action, together with a classically perfect topological charge, confirmed accurately the absence of scaling of the topological susceptibility [10]. Decent topological scaling was found, however, using a classically perfect action for the 2d $CP(3)$ model [11]. In the 1d XY model even a quantum perfect topology was worked out: for a given lattice configuration, we integrate over all possible continuum interpolations, each with a well defined topological charge. Thus we attach to a lattice configuration an ensemble of charges with appropriate Boltzmann weights [12]. Furthermore, the scaling artifacts of the classically perfect approximation could be studied: they become negligible around $\xi \simeq 3$, whereas the standard lattice formulation still suffers from severe artifacts at large $\xi$, see Fig. 2.

Figure 2: The scaling of the topological susceptibility $\chi$ (multiplied by the correlation length in lattice units, $\xi$) in the 1d XY model for the (quantum) perfect action, the classically perfect action and the standard lattice action.

In the Schwinger model with Wilson-type fermions, the classically perfect action was again very successful, in particular in view of the scaling of “meson” masses; also the dispersion relations and the rotational invariance of the correlation functions look good [13]. Approximations to FPAs were also constructed for non-Abelian gauge
fields in \( d = 4 \). They have been applied to studies of topology of \( SU(2) \) \cite{14}, and various types of strongly truncated FPAs have been suggested for \( SU(3) \) \cite{15} (but recently those authors reported certain problems with their actions).

Let us emphasize again that the improvement is designed to improve the *scaling*, i.e. scaling quantities (dimensionless ratios of observables) should converge to their continuum limit at smaller \( \xi \) than it is the case for the standard action. The impact on *asymptotic scaling* can not be predicted. However, it has been observed that also asymptotic scaling tends to set in much earlier for quasi-perfect actions \cite{9, 15, 16}. In particular, \( \Lambda_{QCD} \) is much larger – hence much closer to its continuum value – for the classically perfect action (compared to Wilson’s action) \cite{17}.

### 2.1 Classically perfect operators

For a given configuration \( \Phi \) on the coarse lattice, the first eq. in (5) singles out one configuration \( \phi_c \) on the fine lattice, which minimizes the right-hand side. This can be iterated to finer and finer interpolating lattices (in units of the original coarse lattice), until we arrive at a minimizing continuum field \( \varphi_c[\Phi] \), which we call the “classically perfect field” \cite{18}. It can be viewed as a particularly sophisticated interpolation of the initial lattice field. In contrast to ordinary interpolations, this inverse blocking process is based on the renormalization group, at least in the classical limit, and its artifacts tend to be suppressed exponentially. It can be used to define classically perfect topological objects on the lattice, by requiring their stability under inverse blocking. This implies a better foundation for long range stability than just smoothing the lattice field locally by hand.

Now consider an operator \( O[\varphi] \) given as a functional of the continuum fields \( \varphi \). We build a classically perfect version of such operators simply by the substitution

\[
O[\varphi] \rightarrow O[\varphi_c[\Phi]].
\]

Thus the operator is given in terms of lattice fields, but still defined in the continuum, again as a sophisticated interpolation.

As an application, this procedure has been applied to the free gauge field, referring to a lattice with unit spacing \cite{18}. From the classically perfect field, we constructed Polyakov loops, the correlation function of which yields a static quark-antiquark potential \( V(\vec{r}) \). Indeed, the classically perfect potential converges to the continuum value with increasing \( r = |\vec{r}| \) much faster than the potential arising from Wilson’s plaquette action, it is continuously defined and – most importantly – it has an amazing degree of rotational invariance, even at very short distances, see Fig. \[3\].

### 3 Perturbatively perfect actions

As another approximation to perfection, perfect actions can be calculated analytically for free and perturbatively interacting fields. As potential applications, one can hope that such actions are immediately useful in simulations, or – more modestly
that they single out the most important non-standard lattice terms, which gives a handle for a good parameterization. Moreover, they provide a promising starting point for the minimization program, leading to a classically perfect action. This is important, because even the minimization can not easily be iterated.

3.1 Free fermion and gauge fields

In cases where the blocking step can be performed analytically, it is more efficient to send the blocking factor \( n \to \infty \) and to perform only one step, which yields the perfect action directly, instead of tedious iterations. Hence (in coarse lattice units) the initial action lives in the continuum, so we don’t need to worry either which lattice action to use on the fine lattice.

Consider the free fermion. As described in Sec. 2, one usually relates \( \psi'_x \sim \frac{1}{n^d} \sum_{x' \in x} \psi_x \), where \( \psi', \psi \) are fermionic fields on the coarse resp. fine lattice. In the limit \( n \to \infty \) this relation turns into \( \Psi_x \sim \int_{C_x} dy \psi(y) \), where \( \Psi_x \) lives on a unit lattice, \( C_x \) is the unit hypercube with center \( x \), and \( \psi \) is a continuum field. In momentum space, this relation reads

\[
\Psi(p) \sim \sum_{l \in \mathbb{Z}^d} \psi(p + 2\pi l)\Pi(p + 2\pi l), \quad \Pi(p) = \prod_{\mu} \hat{p}_\mu, \quad \hat{p}_\mu = 2 \sin \frac{p_\mu}{2},
\]

where \( p \in B = [-\pi, \pi]^d \). If we compute the Gaussian type RGT \(^4\)

\[
e^{-S[\Psi, \Psi]} = \int D\bar{\Psi} D\Psi \exp \left\{ -\int \frac{dp}{(2\pi)^d} \bar{\psi}(-p)[i\hat{\gamma} + m]\psi(p)\right.
\]

\[
-\frac{1}{\alpha} \int_B \frac{dp}{(2\pi)^d} \left[ \bar{\Psi}(-p) - \sum_{l \in \mathbb{Z}^d} \bar{\psi}(-p - 2\pi l)\Pi(p + 2\pi l) \right] \times
\]

\(^4\)We ignore constant factors in the partition function. The RGT parameter \( \alpha \) is arbitrary. The mass \( m \) is given in lattice units, even in the continuum action.
we obtain the perfect lattice action

\[
S[\bar{\Psi}, \Psi] = \int_B \frac{dp}{(2\pi)^d} \bar{\Psi}(-p)G(p)^{-1}\Psi(p), \quad G(p) = \sum_{l \in \mathbb{Z}^d} \frac{\Pi^2(p + 2\pi l)}{i(p_\mu + 2\pi l_\mu)\gamma_\mu + m} + \alpha, \quad (9)
\]

which we write in coordinate space as

\[
S[\bar{\Psi}, \Psi] = \sum_{x,r \in \mathbb{Z}^d} \bar{\Psi}_x[\rho_\mu(r)\gamma_\mu + \lambda(r)]\Psi_{x+r} \quad (10)
\]

The couplings \(\rho_\mu(r), \lambda(r)\) have been evaluated numerically \([18]\).

The spectrum of this perfect lattice fermion can be read off from the pole structure of the propagator \(G(p)\),

\[
E(\vec{p})^2 = (\vec{p} + 2\pi \vec{l})^2 + m^2, \quad (11)
\]

hence it is the exact continuum spectrum (plus \(2\pi\) periodic copies, which are omnipresent on the lattice). This reveals the perfect character of this action: the continuous rotation and translation invariance is exactly present in the observables, though not in the form of the action itself. In the latter, the hypercubic structure of the lattice is visible – a perfect action on a triangular lattice, for instance, looks different \([19]\) – but what matters are the observables.

The procedure described above, which we call “blocking from the continuum”, is also applicable to the Abelian gauge field. Here we integrate over all straight connections between corresponding continuum points in two adjacent lattice cells, and relate this integral to the non-compact lattice link variable \(A_\mu\). (A RGT in terms of “compact” link variables \(U_\mu \in U(1)\) in \(d = 2\) has been carried out in Ref. \([20]\).)

### 3.2 Chiral symmetry

A major issue in fermionic actions is the doubling problem: according to the Nielsen Ninomiya “No Go theorem”, species doubling (unphysical extra fermions) always
occurs under some mild assumptions about a lattice action, like locality, hermiticity and chiral invariance \[21\]. For instance, in the naive fermion formulation with the inverse lattice propagator \(G^{-1}_{\text{naive}}(p) = i \sin p \gamma \mu\) (at \(m = 0\)), this is obvious from the occurrence of \(2d\) zeros in the first Brillouin zone \(B\). It is a notorious problem to put a single chiral fermion on the lattice.

It turns out that the perfect action \([4]\) is not plagued by doubling. Moreover, in the case \(m = \alpha = 0\), the action is chirally symmetric, but in this case it is non-local (\(|\rho \mu(r)|\) only decays like \(|r|^{1-d}\) at large distances) so there is no contradiction with the No Go theorem \([22]\). As soon as \(m > 0\) or \(\alpha > 0\) (or both), the action becomes local, i.e. the couplings in \(\rho \mu\) and \(\lambda\) decay exponentially in \(|r|\), and at the same time the chiral symmetry is explicitly broken in the action.

But this is not the end of the story. If we start from a chiral fermion in the continuum, then the chiral symmetry is supposed to be preserved under the RGT, due to its very nature, also for finite \(\alpha\). And this is in fact the case, as we see if we focus on the observables again: as an example, it has been shown explicitly in the Schwinger model that the axial anomaly is reproduced correctly, if we map the entire continuum theory in a consistently perfect way on the lattice \([23]\). This includes the blocking of the fermion and gauge field, of the fermion-gauge interaction term to the first order, and of the axial current. A perfect lattice current is identified by a procedure analogous to the treatment of the fields. The blocking scheme here integrates the continuum flux through the face between adjacent lattice cells, and a coupling to an external source incorporates the current in the RGT. \[6\] The axial charge is blocked from the continuum too, and all these perfect quantities match to reproduce the correct axial anomaly on the lattice.

By construction, the chiral symmetry comes out correctly to all orders in perturbation theory, where the blocking from the continuum can be carried out. So we can put the system on the lattice without doing any harm to it, i.e. the lattice regularization is – with this respect – as good as any regularization in the continuum. (This is in contrast to a wide-spread feeling that it has a particular weakness due to the fermionic doubling problem.)

The construction of local chiral fermions we sketched above – the fixed point action for \(\alpha > 0\) – is a nice way around the Nielsen Ninomiya theorem, which refers to a chiral symmetry of the lattice action itself. We don’t need it to be manifest in the action, \[7\] but we can still reproduce it in the observables, which really matter (like the continuous rotation and translation symmetry, which we recovered in the spectrum). However, since this construction by “blocking from the continuum” is perturbative, it does not directly imply any claim about a non-perturbative formulation of the lattice chiral fermion (which is not known in the continuum either).

While these notes are being written down, however, it has been shown that also

\[5\] The case \(\alpha = 0\) corresponds to the \(\delta\) function RGT.

\[6\] The perfect currents, which have been worked out in this context, gave also rise to a study of perfectly discretized hydrodynamics \([25]\).

\[7\] The violation in the form of the action is due to the non chirally symmetric transformation term for \(\alpha > 0\).
the Atiyah Singer index theorem holds for a FPA [24], confirming again that the
perfect fermion is perfect.

3.3 Truncation

The perfect actions mentioned above all include couplings over infinite distances. Even if the long range couplings are exponentially suppressed (locality), they are still needed to reproduce the continuum symmetries exactly. Since we can’t work with them in simulations, the nice properties described above may seem somewhat academic, rather far from application (is it comparable to the string theory talks at this symposium?).

To proceed to practical applications, we need to truncate the couplings, and this does necessarily some harm to the perfect properties. In the 2d applications, it was permissible to truncate only couplings, which are suppressed by various orders of magnitude [8, 13]. However, in $d = 4$ – and with non-Abelian gauge fields, where many more lattice paths have to be distinguished – we can not be so generous. The number of additional terms, that the practitioner can work with, is strongly limited.

Hence it is very important to choose the RGT such that the perfect action is as local as possible, i.e. the couplings decay as fast as possible. Then there is hope that perfection survives the truncation in a good approximation. In this sense, we have tuned the RGT parameter $\alpha$ in the RGT [8] in order to optimize locality and we could achieve that in the special case $p = (p_1, 0 \ldots 0)$ (mapping on $d = 1$) the couplings are restricted to nearest neighbors. This is a successful optimization criterion for the locality in the general 4d case, for fermions as well as scalars [18, 19]. For the gauge field we have chosen the RGT parameter, which is analogous to $\alpha$, to be momentum dependent. Thus we obtained the standard plaquette action in $d = 2$, and again the same RGT parameters also provide excellent locality in $d = 4$ [18].

The truncation itself was performed by means of periodic boundary conditions: we construct a perfect action in a small volume of $3^4$ sites, and then use the same couplings – which are restricted to a unit hypercube – in larger volumes too. This truncation keeps all normalizations exact. To check the quality of the truncated free actions, we considered the spectrum after truncation, which is not perfect any more, but still drastically improved compared to the standard action [20], see Fig. 1. The same holds for thermodynamic scaling quantities at finite temperature [26, 27] and at finite chemical potential [28], as illustrated in Fig. 3. A different truncation of the perfect free gauge field in terms of closed loops has been given in Ref. 26.

For an immediate application, the truncated perfect “hypercube fermion” can be “gauged by hand”: we connect the coupled sites by the link variables on the shortest lattice paths, and average over these paths. This is of course a drastic short-cut and not the consistently perfect construction, but with some respect it already leads to a remarkable progress. In particular, the meson dispersion relation reaches an impressive quality, as shown in Fig. 4. To handle such complicated actions efficiently, work is in progress for an adequate optimization of the algorithm on a parallel machine. In particular, we are working on a quick evaluation of the fermion
matrix $[29]$, which involves many more non-zero off-diagonal elements than it is the case for Wilson’s action.

### 3.4 Vertex function

As we proceed to perturbatively interacting fields, we can incorporate the interaction into the RGT and still block from the continuum. If we realize this to the first order in the gauge coupling $g$, we obtain a perfect quark-gluon and a 3-gluon vertex function. The $\bar{q}qq$ vertex has been evaluated in $d = 2$ $[26]$ and its couplings have been reproduced numerically from a multigrid $[13]$. In spite of good locality, we obtain in the 4d case still inconveniently many couplings, which seem to contribute significantly. By means of a rather tough truncation, their number was reduced to 5 short ranged extra terms, in addition to those present in the action of the “hypercube fermion gauged by hand” $[30]$.

The best candidate for an immediate use of this action is heavy quark physics. This optimism is supported from a consideration of the non-relativistic expansion

$$E = m_s + \frac{1}{2m_{\text{kin}}} \vec{p}^2 + \frac{1}{2m_B} \vec{\Sigma} \cdot \vec{B} + \ldots, \quad \Sigma_k = \epsilon_{ijk} \sigma_{ij} / 2.$$  

For the periodically truncated perfect fermion, the mass parameter $m$ coincides with the static mass $m_s$. The lattice parameters tend to deviate from the continuum relation $m_s = m_{\text{kin}} = m_B$. Fig. $\mathcal{F}$ shows that $m_s = m_{\text{kin}}$ is approximated well for the hypercube fermion, and if we gauge it by hand then $m_B(m_s)$ is somewhat improved. We can achieve a drastic improvement also for $m_B$ by including terms of the truncated perfect vertex function.

As a first experiment, we performed quenched simulations with the truncated perfect vertex function for the charmonium spectrum. The computational overhead
Figure 6: The thermodynamic scaling quantities pressure/temperature with \( N_t \) sites in Euclidean time (at \( \mu = 0 \)), and (baryon number density)/(chemical potential) at \( T = 0 \), for massless free fermions.

(compared to Wilson’s action) is around a factor 20, but we simulated on an coarse lattice of spacing \( a = 0.24\, fm \) (about 3 times larger than usual), which could clearly out-do this factor. Here a lattice of \( 8^3 \times 16 \) sites appears to be sufficient. The physical units were determined from the string tension, and the 1s \( \eta_c \) ground state was matched to the experimental value. The further states were predictions of the simulation. The 2s states of \( \eta_c \) and \( J/\psi \) were predicted successfully, but the gap to the 1s \( J/\psi \) state (hyperfine splitting) was clearly too small \[30\], see Fig. 9. A possible reason – except for quenching – is that the vertex couplings are still supposed to be renormalized (due to \( O(g^2) \)). As a rather ad hoc procedure, we have tested a “tadpole improvement” (mean field estimate of the renormalization à la Ref. \[4\]), which helps to some extent, but the hyperfine splitting – a quantity, which is extremely sensitive to lattice artifacts – is still too small. However, it is known from the clover action that the fermion part tends to be insufficiently renormalized by this method, so we are now incorporating the larger renormalization factor, translated from the result found by the ALPHA collaboration for the clover term \[5\].

The form of the perfect \( ggg \) vertex function is very similar to the \( \bar{q}qg \) vertex \[31\]. If we include it, then we obtain an action which is entirely perfect (before truncation) to \( O(g) \); any artifacts of \( O(a^n) \) and \( O(ga^n) \) are eliminated, so that the leading artifacts occur in \( O(g^2a) \). It is not obvious how this compares to a Symanzik improved action, which erases all artifacts in \( O(ag^n) \), but which is plagued for instance by cut-off effects in \( O(ga^2) \).

The climax of the improvement is of course a QCD action, which is improved non-perturbatively in both, \( a \) and \( g \). The classically perfect action, to be constructed from a multigrid minimization, is an action of this type, but it could not be worked out in an “ultimate” form yet. In practice one would perform inverse blocking steps with a small blocking factor (see footnote 3), but iteration rapidly leads to large lattices, which can not be handled any more. Of course one could put a well identified action,
say after a blocking factor 2 RGT, back to the fine lattice and block again, hence avoiding the use of large lattices. However, small lattices are only instructive if the action is extremely local. In any case, a good starting point for the iteration, as well as a good parameterization ansatz for the blocked action, are highly desirable, and the perfect vertex functions could help with this respect.

4 Status and outlook

Although we are still working on better charmonium results, it seems that the direct application of small field perfect actions is not as successful as we hoped. This is also confirmed by a study of the anharmonic oscillator [16]. Apparently, at some point the minimization trick must come into play. In simple cases one might think about extending it to a semi-classically perfect action à la WKB, i.e. taking the quadratic fluctuations around the classical solution into account. Also one full RGT step with a small blocking factor, starting from a classically perfect action at moderate $\xi$, could be very helpful.

One point that is still very important to work on, is the optimization of locality: in the case of the vertex functions, it seems that the interaction term in the RGT can be further optimized with this respect. In other cases, e.g. for so-called staggered fermions (see second and third Ref. in [1]), it turned out that one can do better even for the free particles by taking a blocking scheme different from the block average described in Sec. 2 [33]. The fine lattice (or continuum) variables can be blocked in many ways, not only by a piece-wise constant weight factor as in eq. (7). Such an optimization led to significant progress for the truncated perfect staggered fermion, which has been tested in simulations of the Schwinger model (with naive gauging plus “fat links”); in particular the “pion” mass gets strongly suppressed.

In thermodynamics, where lattice artifacts are especially bad, a remarkable im-
Figure 8: The kinetic and the magnetic mass as functions of the static mass for the Wilson fermion and the hypercube fermion. $m_B$ improves strongly only after including elements of the perfect vertex function.

Figure 9: The charmonium spectrum. The experimental values are dashed, and only the ground state of $\eta_c$ is fitted.

Improvement has been observed using approximately perfect actions [34]. As a further step, one could put those actions on anisotropic lattices, which hardly poses additional complications [19]. It has also been shown how to include a chemical potential in a perfect action, which is needed for simulations at finite baryon density [28] (a field, which had only little success so far).

In further experiments for QCD, pionic systems were investigated and some improvement in the scaling behavior was found [35], but that study also tends to involve more and more ad hoc elements, which are not related to the perfect program. From that side, there is no claim for an “ultimate” version of a quasi-perfect action either. A problem, which our attempts have in common, is a strong additive mass renormalization. This is a effect of truncation (and simplified gauging), which affects the reliability of the “perfect couplings”.

According to our experience, it is worthwhile proceeding in very small steps to higher degrees of complications, in order to elaborate a really clean treatment. All
attempts at short-cuts to QCD have led to half-cooked proposals, which have to be reconsidered afterwards. Thus the program is quite time-consuming, but if it really works one day, a quasi-perfect action (given by a set of couplings) can be used by the practitioner for any problem in QCD, without worrying about its lengthy derivation. Following these piecemeal tactics, we are now working on a FPA for 2d and 3d non-Abelian gauge theories, starting with $SU(2)$. As a scaling test, the 3d glueball spectrum can be measured.

![Figure 10: A schematic overview over the status of the improvement for QCD lattice actions with Wilson-type fermions.](image)

On the conceptual side, it is straightforward to apply the procedure of blocking from the continuum for instance to supersymmetry. This results in a SUSY lattice action, which is completely different from the one presented by I. Montvay at this symposium. Again it is more complicated, but it could represent all symmetries exactly. Even applications to quantum gravity are conceivable: also here it would be desirable to integrate out the short-range details between the lattice sites in a way, which is based on the renormalization group. Again, such a method would differ from those discussed by J. Ambjörn and B. Petersson in their talks.

At last, as another project in progress, it would be interesting to combine the perfect treatment of fermions with the domain wall concepts of lattice chiral fermions. So far, that construction has always been based on Wilson fermions, although this can be generalized. Inserting a truncated perfect fermion instead, one cumulates all sort of virtues: small lattice artifacts, an arbitrary number of flavors (unlike staggered fermions), and the absence of additive mass renormalization (unlike Wilson fermions). Note that the domain wall formalism for chiral fermions is related to the
subject, which was most celebrated at this symposium, the theory of D-branes (I thank J. Schwarz for this remark).

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