DYNAMICS OF NON-AUTONOMOUS FRACTIONAL GINZBURG-LANDAU EQUATIONS DRIVEN BY COLORED NOISE

HONG LU

School of Mathematics and Statistics
Shandong University, Weihai, Shandong 264209, China

MINGJI ZHANG*

College of Mathematics and Systems Science
Shandong University of Science and Technology, Qingdao, Shandong 266590, China
Department of Mathematics
New Mexico Institute of Mining and Technology, Socorro, NM 87801, USA

(Communicated by María J. Garrido-Atienza)

Abstract. In this work, the existence and uniqueness of random attractors of a class of non-autonomous non-local fractional stochastic Ginzburg-Landau equation driven by colored noise with a nonlinear diffusion term is established. We comment that compared to white noise, the colored noise is much easier to handle in examining the pathwise dynamics of stochastic systems. Additionally, we prove the attractors of the random fractional Ginzburg-Landau system driven by a linear multiplicative colored noise converge to those of the corresponding stochastic system driven by a linear multiplicative white noise.

1. Introduction. In this work, we examine the asymptotic behavior of solutions of the following non-autonomous, non-local, fractional stochastic Ginzburg-Landau equations driven by a colored noise in a bounded domain $\Omega$:

$$\frac{\partial u}{\partial t} + (1 + i \nu)(-\Delta)^\alpha u + \rho u = f(t, x, u) + g(t, x, u)\zeta_\delta(\theta_t \omega), \quad x \in \Omega, \quad t > \tau, \quad (1)$$

where $u(x, t)$ is a complex-valued function on $\Omega \times [0, +\infty)$. In (1), $i$ is the imaginary unit, $\nu$ is a real constant, $\rho > 0$, $\alpha \in (1/2, 1)$, $g \in L^2_{\text{loc}}(\Omega, L^2(\Omega))$, $f : \mathbb{R} \times \Omega \times \mathbb{C} \to \mathbb{R}$ is a nonlinear function which satisfies certain dissipative conditions, and the process $\zeta_\delta$ ($0 < \delta \leq 1$) is an Ornstein-Uhlenbeck (O-U) process (also known as a colored noise).

The O-U process is a stationary Gaussian process with the mathematical expectation $\mathbb{E}(\zeta_\delta) = 0$, and so far, the O-U process is the only existing Markovian Gaussian colored noise (see [6, 26] for example). Furthermore, the O-U process is also called a colored noise due to the fact that its power spectrum is not flat compared to the white noise ([2, 9, 26, 30, 32]).

2020 Mathematics Subject Classification. Primary: 35B40, 35B41; Secondary: 37L30.

Key words and phrases. Fractional stochastic equation, random attractor, colored noise, white noise.

This work was supported by the NSF of China (No. 11601278 and No.11601274), the NSF of Shandong Province (No. ZR2019MA050) and MPS Simons Foundation of USA (No. 628308).

* Corresponding author: Mingji Zhang.
As we know, one can choose the Wiener process \( W \) as a stochastic process to represent the position of the Brownian particle, however, the velocity of the particle cannot be obtained from the Wiener process due to the nowhere differentiability of the sample paths of \( W \). However, the O-U process was originally constructed to approximately describe the stochastic behavior of the velocity ([30, 32]), therefore, one can further use it to determine the position of the particle. Moreover, as demonstrated in [26], in many complex systems, stochastic fluctuations are actually correlated and hence should be modeled by colored noise rather than white noise.

During studying stochastic dynamics, one of the most crucial issues arises from the modeling of random forcing. To study such random forcing, we need to consider both the time scale \( \tau_d \) of the deterministic system and the time scale \( \tau_r \) of the random forcing. The stochastic forcing is modeled in different ways based on the ratio of \( \tau_r/\tau_d \). If \( \tau_r/\tau_d \gg 1 \), the dynamical system is very slow with respect to the temporal variability of its random drivers, and hence the random forcing could be modeled as white noise. If \( \tau_r/\tau_d \simeq 1 \), then the dynamics of the system is sensitive to the autocorrelation of the random forcing, and hence the random forcing should be modeled by colored noise. Based on these considerations, the colored noise has been employed in many works to study the dynamics of physical and biological system (see [2, 9, 15, 16, 17, 26, 30, 32] and the reference therein).

In this work, we will consider the dynamics of system (1) driven by colored noise. We will prove the random system (1) is pathwise well-posed in \( L^2(\mathbb{I}) \) and hence generates a continuous non-autonomous cocycle. Moreover, this cocycle possesses a unique tempered random attractor in \( L^2(\mathbb{I}) \). This is in contrast with the corresponding stochastic system driven by a white noise:

\[
\frac{\partial u}{\partial t} + (1+i\nu)(-\Delta)\alpha u + pu = f(t, x, u) + g(t, x) + R(t, x, u) \circ \frac{dW}{dt}, \quad x \in \mathbb{I}, \ t > \tau, \ (2)
\]

where the symbol \( \circ \) indicates the system is understood in the sense of Stratonovich’s integration. Currently, we can only define a random dynamical system for (2) when the diffusion term \( R(\cdot, \cdot, u) \) is a linear function in \( u \in C \). In other words, we are unable to define a random dynamical system for (2) with a general nonlinear function \( R \) and hence cannot investigate the dynamics of the stochastic equations by the random dynamical system approach. Therefore, there is no result available in the literature on the existence of random attractors for (2) with a nonlinear function \( R \).

The differential equations involving the fractional Laplacian have a wide range of applications in physics, biology, chemistry and other fields of science, see [1, 8, 11, 12, 13, 14, 18]. The solutions of fractional deterministic equations have been studied extensively ([1, 4, 5, 7, 8, 11, 12, 13, 14, 18, 19, 22, 23, 27, 28, 29], and the references therein). Our main interest in this work is to establish the existence of random attractors of the fractional stochastic Ginzburg-Landau system (1).

We point out that the deterministic fractional Ginzburg-Landau equation ([19, 22, 23, 25]) and the random attractors of stochastic equations with the Wiener process ([20, 21]) have already been studied. In particular, in [20, 21], the author established the existence of random attractors for stochastic equations driven by white noise. However, there is no result available in the literature for the existence of random attractors for the fractional stochastic Ginzburg-Landau equation (1) driven by colored noise. The purpose of the present paper is to close this gap and prove system (1) driven by colored noise has a unique tempered random attractor for \( \alpha \in (\frac{1}{2}, 1) \).
The rest of the paper is organized as follows. In Section 2, we prove the existence and uniqueness of tempered random attractors for system \((1)\) with a nonlinear diffusion term \(R\). We then prove the existence of such attractors for the stochastic system \((2)\) when \(R(t, x, u) = u\) in Section 3. In Section 4, we prove the convergence of solutions and random attractors of system \((1)\) with \(R(t, x, u) = u\), as \(\delta \to 0\).

Hereafter, we will denote the inner product and norm of \(L^2(\mathbb{I})\) by \((\cdot, \cdot)\) and \(\| \cdot \|\), respectively. The letter \(c\) and \(c_i\) are used for positive constants whose values may change from line to line.

2. Attractors of random fractional Ginzburg-Landau systems. In this section, we study the dynamics of the random fractional Ginzburg-Landau systems driven by a colored noise, which consists of two steps. In the first step, we define a continuous non-autonomous cocycle for the system. Secondly, we prove the existence of pullback random attractors in \(L^2(\mathbb{I})\) for a wide class of nonlinear functions \(f\) and \(R\).

2.1. Definition of continuous cocycles. Let \(\mathbb{I}\) be a bounded interval in \(\mathbb{R}\) and \(\tau, \delta \in \mathbb{R}\) with \(0 < \delta \leq 1\). We consider the following non-autonomous fractional random Ginzburg-Landau equation:

\[
\frac{\partial u}{\partial t} + (1+i\nu)(-\Delta)^{\alpha} u + \rho u = f(t, x, u) + g(t, x) + R(t, x, u)\zeta_\delta(\theta, \omega), \quad x \in \mathbb{I}, \ t > \tau, \tag{3}
\]

with homogeneous Dirichlet boundary condition and initial condition

\[
u(t, x) = 0, \quad x \in \partial \mathbb{I}, \ t > \tau, \tag{4}
\]

\[
u(t, x) = u_\tau(x), \quad x \in \mathbb{I}, \tag{5}
\]

where \(g \in L^2_{\text{loc}}(\mathbb{I}, L^2(\mathbb{I}))\), \(\zeta_\delta(\theta, \omega)\) is a colored noise and \(f, R : \mathbb{R} \times \mathbb{I} \times \mathbb{C} \to \mathbb{R}\) are continuous functions such that for all \(t \in \mathbb{R}, x \in \mathbb{I}\),

\[
\operatorname{Re} f(t, x, u)\bar{u} \leq -\gamma |u|^p + \psi_1(t, x), \tag{6}
\]

\[
\left| \frac{\partial f}{\partial u}(t, x, u) \right| \leq \psi_2(t, x), \tag{7}
\]

\[
\left| \frac{\partial f}{\partial x}(t, x, u) \right| \leq \psi_3(t, x), \tag{8}
\]

\[
|\theta(t, x, u)| \leq \lambda |u|^{q-1} + \psi_4(t, x), \tag{9}
\]

\[
\left| \frac{\partial R}{\partial u}(t, x, u) \right| \leq \psi_5(t, x), \tag{10}
\]

\[
\left| \frac{\partial R}{\partial x}(t, x, u) \right| \leq \psi_6(t, x), \tag{11}
\]

where \(\gamma > 0\) and \(p \geq 2\) are constants, \(\psi_1 \in L^1_{\text{loc}}(\mathbb{R}, L^1(\mathbb{I}))\), \(\psi_2, \psi_5 \in L^\infty_{\text{loc}}(\mathbb{R}, L^\infty(\mathbb{I}))\), \(\psi_3, \psi_6 \in L^\infty_{\text{loc}}(\mathbb{R}, L^2(\mathbb{I}))\), and \(\psi_4 \in L^\infty_{\text{loc}}(\mathbb{R}, L^p(\mathbb{I}))\) with \(1/p + 1/q_1 = 1\).

From [3], there exists a \(\{\theta_t\}_{t \in \mathbb{R}}\)-invariant subset of full measure, which is still denoted by \(\Omega\), such that for all \(\omega \in \Omega\),

\[
\lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0. \tag{12}
\]

Throughout this paper, for every \(\omega \in \Omega\) and \(\delta \in (0, 1]\), we write

\[
\zeta_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^{0} e^{\frac{s}{\delta}} dW = -\frac{1}{\delta^2} \int_{-\infty}^{0} e^{\frac{s}{\delta}} \omega(s) ds. \tag{13}
\]
Then $\zeta_\delta(\theta_t \omega)$ is the so-called O-U process (also known as colored noise) with $\mathbb{E}(\zeta_\delta) = 0$. In addition, this process has the following properties from [10].

**Lemma 2.1.** For every $\omega \in \Omega$, the mapping $t \to \zeta_\delta(\theta_t \omega)$ is continuous, and for every $0 < \delta \leq 1$,

$$\lim_{t \to \pm \infty} \frac{|\zeta_\delta(\theta_t \omega)|}{|t|} = 0$$

and

$$\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \zeta_\delta(\theta_s \omega) ds = 0$$ uniformly for $0 < \delta \leq 1$.

**Lemma 2.2.** Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon)$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T],$

$$\left| \int_0^t \zeta_\delta(\theta_s \omega) ds - \omega(t) \right| < \varepsilon. \tag{16}$$

By Lemma 2.2 and the continuity of $\omega$, the following estimates hold immediately.

**Corollary 2.3.** Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then there exists $\delta_0 = \delta_0(\tau, \omega, T)$ and $M = M(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T],$

$$\left| \int_0^t \zeta_\delta(\theta_s \omega) ds \right| \leq M. \tag{17}$$

We would like to point out that (3)-(5) is a deterministic system parametrized by $\omega \in \Omega$. By the Galerkin's method and compactness argument as those in [25], we can prove that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in L^2(\Omega)$, system (3)-(5) possesses a unique solution $u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty), L^2(\Omega))$. Furthermore, the solution is continuous with to initial conditions in $L^2(\Omega)$ and is $(\mathcal{F}, \mathcal{B}(L^2(\Omega)))$-measurable in $\omega \in \Omega$. Then, we can define a continuous cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\Omega) \to L^2(\Omega)$ such that for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in L^2(\Omega),$

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau). \tag{18}$$

Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $L^2(\Omega)$ with the property: for every $\beta > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{\beta t} \|D(\tau + t, \theta_t \omega)\|_{L^2(\Omega)} = 0. \tag{19}$$

Such a family is called tempered in $L^2(\Omega)$. Let $D$ be the collection of all tempered families of bounded nonempty subsets of $L^2(\Omega)$, i.e.

$$D = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies } \lim_{t \to -\infty} e^{\beta t} \|D(\tau + t, \theta_t \omega)\|_{L^2(\Omega)} = 0. \tag{20}\}$$

Let $\rho'$ be a fixed number in $(0, \rho]$. Hereafter, we assume

$$\int_{-\infty}^0 e^{\rho' s} \|g(s + \tau, \cdot)\|^2 ds < \infty, \quad \text{for every } \tau \in \mathbb{R}; \tag{21}$$

and for every positive number $c$,

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^0 e^{\rho' s} \|g(s + t, \cdot)\|^2 ds = 0. \tag{22}$$

In addition, the following Gagliardo-Nirenberg inequality ([24]) is frequently used.
Lemma 2.4. Let $u$ belong to $L^q$ and its derivatives of order $m$, $D^m u$, belong to $L^r$, $1 \leq q, r \leq \infty$. For the derivatives $D^j u$, $0 \leq j < m$, the following inequalities hold
\[
||D^j u||_{L^p} \leq c||u||_{W^{m,r}}^{\frac{q}{r}} ||u||_{L^r}^{-\theta}.
\]
where
\[
\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r} - \frac{m}{n}) + (1 - \theta)\frac{1}{q},
\]
for all $\theta$ in the interval
\[
\frac{j}{m} \leq \theta \leq 1
\]
(the constant $c$ depending only on $n, m, j, q, r, \theta$), with the following exceptional case
1. If $j = 0$, $rm < n, q = \infty$ then we make the additional assumption that either $u$ tends to zero
   at infinite or $u \in L^q$ for some finite $\hat{q} > 0$.
2. If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then (23) holds only
   for $\theta$ satisfying
\[
\frac{j}{m} \leq \theta < 1.
\]

2.2. Existence of pullback random attractors. In this subsection, we prove the existence of tempered pullback random attractors for system (3)-(5). We first prove the existence of tempered absorbing sets and then show the pullback asymptotic compactness of solutions.

Lemma 2.5. Under the conditions of (6), (9) and (21), for every $\xi, \tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subset D$, there exists $T = T(\tau, \omega, D, \xi, \delta) > 0$ such that for all $t \geq T$, the solution of system (3)-(5) satisfies
\[
\begin{align*}
||u(\xi, \tau - t, \theta, \omega, u_{\tau + \xi})||^2 &+ \int_{\tau - t}^{\xi} e^{\rho(s-\xi)}||(-\Delta)^{\frac{\theta}{2}} u(s)||^2 ds \\
&+ \frac{\rho}{2} \int_{\tau - t}^{\xi} e^{\rho(s-\xi)}||u(s)||^2 ds \\
&\leq 1 + M_1 \int_{\tau - t}^{\xi} e^{\rho(s-\xi)} \left(1 + ||g(s + \tau, \cdot)||^2 + \eta_{\delta}(\theta, \omega)\right) ds.
\end{align*}
\]
where $u_{\tau + \xi} \in D(\tau - t, \theta, \omega, \eta_{\delta}(\theta, \omega) = ||\zeta_{\delta}(\theta, \omega)||^2 + ||\zeta_{\delta}(\theta, \omega)||^2$, and $M_1$ is a positive constant independent of $\tau, \omega, D$ and $\xi$.

Proof. Taking the inner product of (3) with $u$ in $L^2(\Omega)$ and taking the real part, we obtain
\[
\frac{d}{dt}||u||^2 + 2||(-\Delta)^{\frac{\theta}{2}} u||^2 + 2\rho||u||^2
\]
\[
= 2\text{Re} \int_{\Omega} f(t, x, u)\bar{u}dx + 2\text{Re} \int_{\Omega} g(t, x)\bar{u}dx + 2\zeta_{\delta}(\theta, \omega)\text{Re} \int_{\Omega} R(t, x, u)\bar{u}dx.
\]
Applying (6), (9) and Young’s inequality, we deduce that
\[
2\text{Re} \int_{\Omega} f(t, x, u)\bar{u}dx \leq -2\gamma||u||_p^p + 2||\psi_1||_1,
\]
\[
2\text{Re} \int_{\Omega} g(t, x)\bar{u}dx \leq \frac{\rho}{2}||u||^2 + \frac{2}{\rho}||g||^2
\]
and
\[
2\zeta_{\delta}(\theta, \omega)\text{Re} \int_{\Omega} R(t, x, u)\bar{u}dx \leq 2\lambda||\zeta_{\delta}(\theta, \omega)||\int_{\Omega} |u|^q dx + 2 \int_{\Omega} |\zeta_{\delta}(\theta, \omega)| \psi_1 ||u|| dx
\]
\[
\leq \gamma||u||_p^p + c||\zeta_{\delta}(\theta, \omega)||^{\frac{p}{p-q}} + c||\zeta_{\delta}(\theta, \omega)||^n.
\]
Thus, there exists $T$ such that for all $t > T$, we have
\[
\frac{d}{dt} \|u\|^2 + 2\|(-\Delta)^{\frac{r}{2}} u\|^2 + 3p \|u\|^2 + \gamma \|u\|^p \leq \frac{2}{p} \|g\|^2 + 2\|\psi_1\|_1 + c\eta_{\delta}(\theta_1\omega),
\] (29)

where $\eta_{\delta}(\theta_1\omega) = |\zeta_1(\theta_1\omega)|^{\frac{p}{p-1}} + |\zeta_{\delta}(\theta_1\omega)|^{q_1}$. Applying Gronwall’s Lemma to (29) over $(r, \xi)$ with $\xi \geq r$ and for every $\omega \in \Omega$, we infer
\[
\|u(r, \omega, u_r)\|^2 + \int_r^\xi e^{\rho(s-\xi)} \|(-\Delta)^{\frac{r}{2}} u(s)\|^2 ds + \frac{p}{2} \int_r^\xi e^{\rho(s-\xi)} \|u(s)\|^2 ds
\leq e^{\rho(r-\xi)} \|u_r\|^2 + c \int_r^\xi e^{\rho(s-\xi)} (\|\psi_1(s, \cdot)\|_1 + \|g(s, \cdot)\|^2) ds
+ c \int_r^\xi e^{\rho(s-\xi)} \eta_{\delta}(\theta_{s-\omega}) ds.
\] (30)

Now, replacing $r$ by $\tau - t$ and $\omega$ by $\theta - \tau \omega$ in (30), we get
\[
\|u(\xi, \tau - t, \theta - \tau \omega, u_{\tau - t})\|^2 + \int_{\tau - t}^{\xi} e^{\rho(s-\xi)} \|(-\Delta)^{\frac{r}{2}} u(s)\|^2 ds
+ \frac{p}{2} \int_{\tau - t}^{\xi} e^{\rho(s-\xi)} \|u(s)\|^2 ds
\leq e^{\rho(\tau - t - \xi)} \|u_{\tau - t}\|^2 + c \int_{\tau - t}^{\xi} e^{\rho(s-\xi)} (\|\psi_1(s, \cdot)\|_1 + \|g(s, \cdot)\|^2) ds
+ c \int_{\tau - t}^{\xi} e^{\rho(s-\xi)} \eta_{\delta}(\theta_{s-\tau \omega}) ds
\]
\[
= e^{\rho(\tau - t - \xi)} \|u_{\tau - t}\|^2 + c \int_{\tau - t}^{\xi - \tau} e^{\rho(s+\tau-\xi)} (\|\psi_1(s+\tau, \cdot)\|_1 + \|g(s+\tau, \cdot)\|^2) ds
+ c \int_{\tau - t}^{\xi - \tau} e^{\rho(s+\tau-\xi)} \eta_{\delta}(\theta_{s+\tau \omega}) ds
\]
\[
\leq e^{\rho(\tau - t - \xi)} \|u_{\tau - t}\|^2 + c \int_{-\infty}^{\xi - \tau} e^{\rho(s+\tau-\xi)} (1 + \|g(s+\tau, \cdot)\|^2) ds
+ c \int_{-\infty}^{\xi - \tau} e^{\rho(s+\tau-\xi)} \eta_{\delta}(\theta_{\omega}) ds.
\] (31)

Next, we estimate every term on the right-hand side of (31). For the first term, since $u_{\tau - t} \in D(\tau - t, \theta - \tau \omega)$ and $D \in D$, we have
\[
e^{\rho(\tau - t - \xi)} \|u_{\tau - t}\|^2 \to 0, \quad \text{as } t \to \infty.
\]
Thus, there exists $T = T(\tau, \omega, D, \xi, \delta) > 0$ such that for all $t \geq T$,
\[
e^{\rho(\tau - t - \xi)} \|u_{\tau - t}\|^2 \leq 1.
\]

For the second term, applying (21) we have
\[
c \int_{-\infty}^{\xi - \tau} e^{\rho(s+\tau-\xi)} (1 + \|g(s+\tau, \cdot)\|^2) ds
= ce^{\rho(\tau - \xi)} \int_{-\infty}^{\xi - \tau} e^{\rho s} (1 + \|g(s+\tau, \cdot)\|^2) ds < \infty.
\] (32)
Due to (14), the third term on the right-hand side of (31) is well-defined. Therefore, we obtain that for all $t \geq T$,

$$
\|u(\xi, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + \int_{\tau-t}^{\xi} e^{\rho(s-\xi)} \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|^2 ds \\
+ \frac{\rho}{2} \int_{\tau-t}^{\xi} e^{\rho(s-\xi)} \|u(s)\|^2 ds \\
\leq 1 + M_1 \int_{-\infty}^{\xi-\tau} e^{\rho(s+\tau-\xi)} \left(1 + \|g(s + \tau, \cdot)\|^2 + \eta_\delta(\theta_s\omega)\right) ds.
$$

(33)

□

As a consequence of Lemma 2.5, we obtain that problem (3)-(5) has a $D$-pullback absorbing set in $L^2(\Omega)$.

**Corollary 2.6.** Suppose (6), (9) and (21)-(22) hold. Then the cocycle $\Phi$ associated with the system (3)-(5) possesses a $D$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ in $L^2(\Omega)$ which is given by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$
K(\tau, \omega) = \{u \in L^2(\Omega) : \|u\|^2 \leq B(\tau, \omega)\},
$$

(34)

where

$$
B(\tau, \omega) = 1 + M_1 \int_{-\infty}^{0} e^{\rho s} \left(1 + \|g(s + \tau, \cdot)\|^2 + \eta_\delta(\theta_s\omega)\right) ds,
$$

with the same constant $M_1$ as in (24).

**Proof.** Let $\xi = \tau$ in Lemma 2.5. We obtain that there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,

$$
\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_t\omega)) = u(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_t\omega)) \subseteq K(\tau, \omega). \quad (35)
$$

Next, we prove $K \in D$. Let $\beta$ be an arbitrary positive constant and consider

$$
\lim_{t \to -\infty} e^{\beta t} \|K(\tau + t, \theta_{t}\omega)\|^2 = \lim_{t \to -\infty} e^{\beta t} B(\tau + t, \theta_{t}\omega)
$$

$$
= \lim_{t \to -\infty} e^{\beta t} + M_1 \lim_{t \to -\infty} e^{\beta t} \int_{-\infty}^{0} e^{\rho s} ds + M_1 \lim_{t \to -\infty} e^{\beta t} \int_{-\infty}^{0} e^{\rho s} \eta_\delta(\theta_{t+s}\omega) ds \\
+ M_1 \lim_{t \to -\infty} e^{\beta t} \int_{-\infty}^{0} e^{\rho s} \|g(s + \tau + t, \cdot)\|^2 ds \\
= \lim_{t \to -\infty} e^{\beta t} + \frac{M_1}{\rho} \lim_{t \to -\infty} e^{\beta t} + M_1 \lim_{t \to -\infty} e^{(\beta - \kappa)t} \int_{-\infty}^{T} e^{\rho s} \eta_\delta(\theta_s\omega) ds \\
+ M_1 e^{-\rho \tau} \lim_{t \to -\infty} e^{\beta t} \int_{-\infty}^{T} e^{\rho s} \|g(s + t, \cdot)\|^2 ds,
$$

(36)

which along with (22) and Lemma 2.1 implies

$$
\lim_{t \to -\infty} e^{\beta t} \|K(\tau + t, \theta_{t}\omega)\|^2 = 0.
$$

This completes the proof. □

Also, we can obtain the following estimates from Lemma 2.5 for later purpose.
Lemma 2.7. Under the conditions of (6), (9) and (21), for every \( \xi, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), there exists \( T = T(\tau, \omega, D, \xi, \delta) \geq 1 \) such that for all \( t \geq T \), the solution of system (3)-(5) satisfies
\[
\int_{\tau-1}^{\tau} \left( \| (-\Delta)^{\frac{3}{4}} u(s, \tau - t, \theta - \omega, u_{\tau - t}) \|^2 + \| u(s, \tau - t, \theta - \omega, u_{\tau - t}) \|^2 \right) ds \\
\leq M_2 + M_2 \int_{-\infty}^{0} e^{\rho s} \left( 1 + \| g(s + \tau, \cdot) \|^2 + \eta_\delta(\theta, \omega) \right) ds.
\]
where \( u_{\tau - t} \in D(\tau - t, \theta - \omega) \) and \( M_2 \) is a positive constant independent of \( \tau \) and \( D \).

Next, we deduce uniform estimates of \( u \) in \( H_0^2(\Omega) \).

Lemma 2.8. Under the conditions of (6)-(11) and (21), for every \( \xi, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), there exists \( T = T(\tau, \omega, D, \xi, \delta) \geq 1 \) such that for all \( t \geq T \), the solution of system (3)-(5) satisfies
\[
\| (-\Delta)^{\frac{3}{4}} u(\tau, \tau - t, \theta - \omega, u_{\tau - t}) \|^2 \\
\leq M_3 + M_3 \int_{-\infty}^{0} e^{\rho s} \left( 1 + \| g(s + \tau, \cdot) \|^2 + \eta_\delta(\theta, \omega) \right) ds.
\]
where \( u_{\tau - t} \in D(\tau - t, \theta - \omega) \) and \( M_2 \) is a positive constant dependent of \( \tau \) and \( \omega \).

Proof. Taking the inner product of (3) with \((-\Delta)^{\alpha} u \) in \( L^2(\Omega) \) and taking the real part, we obtain
\[
\frac{d}{dt} \| (-\Delta)^{\frac{3}{4}} u \|^2 + 2 \| (-\Delta)^{\alpha} u \|^2 + 2 \rho \| (-\Delta)^{\frac{3}{4}} u \|^2 = 2 \text{Re} \int_{\Omega} f(t, x, u)(-\Delta)^{\alpha} \bar{u} dx \\
+ 2 \text{Re} \int_{\Omega} g(t, x)(-\Delta)^{\alpha} \bar{u} dx + 2\xi_\delta(\theta, \omega) \text{Re} \int_{\Omega} R(t, x, u)(-\Delta)^{\alpha} \bar{u} dx.
\]
For the first term of the right-hand side of (37), by (7)-(8), Gagliardo-Nirenberg inequality and Young’s inequality, we have
\[
2 \text{Re} \int_{\Omega} f(t, x, u)(-\Delta)^{\alpha} \bar{u} dx \leq 2 \left| \int_{\Omega} (f_x + f_u u_x)(-\Delta)^{\alpha - \frac{1}{2}} \bar{u} dx \right| \\
\leq 2 \left| \int_{\Omega} |\psi(x)| (-\Delta)^{\alpha - \frac{1}{2}} u |dx \right| + 2 \left| \int_{\Omega} |\psi_x(x)||u_x||(-\Delta)^{\alpha - \frac{1}{2}} u |dx \right| \\
\leq \| \psi(x, \cdot) \|^2 \| (-\Delta)^{\alpha - \frac{1}{2}} u \|^2 + 2 \| \psi_x(x, \cdot) \|_{L^\infty(\Omega)} \| u_x \| \| (-\Delta)^{\alpha - \frac{1}{2}} u \| \\
\leq 2 \| (-\Delta)^{\alpha - \frac{1}{2}} u \|^2 + \left( \| \psi_x(x, \cdot) \|^2_{L^\infty(\Omega)} \| u_x \|^2 + \| \psi_\delta(x, \cdot) \|^2 \right) \\
\leq c(\| u \|^2 + \| (-\Delta)^{\alpha - \frac{1}{2}} u \|^2)^{2(2 - \frac{1}{\alpha})} \| u \|^{2(2 - \frac{1}{\alpha}) - 1} + c\| u \|^2 \\
+ c\| \psi_\delta(x, \cdot) \|^2_{L^2(\Omega)} (\| u \|^2 + \| (-\Delta)^{\alpha} u \|^2) \| u \|^{2 - \frac{1}{\alpha}} + \| \psi_\delta \|^2 \\
\leq \frac{\rho}{2} \| (-\Delta)^{\frac{3}{4}} u \|^2 + \frac{1}{4} \| (-\Delta)^{\alpha} u \|^2 \\
+ c \left( \| u \|^2 + \| \psi_\delta(x, \cdot) \|_{L^\infty(\Omega)} \| u \|^2 + \| \psi_\delta(x, \cdot) \|^2 \right) .
\]

For the second term of the right-hand side of (37), applying Young’s inequality, we get
\[
2 \text{Re} \int_{\Omega} g(t, x)(-\Delta)^{\alpha} \bar{u} dx \leq \frac{1}{2} \| (-\Delta)^{\alpha} u \|^2 + c\| g(t, \cdot) \|^2 .
\]
Similar to (38), for the third term of the right-hand side of (37), applying (10)-(11), Gagliardo-Nirenberg inequality and Young’s inequality, we obtain

\[
2 \zeta_3(\theta, \omega) \text{Re} \int_{\Omega} R(t, x, u)(-\Delta)^{\alpha/2} u dx \leq 2|\zeta_3(\theta, \omega)| \left| \int_{\Omega} (R_x + R_u u_x)(-\Delta)^{\alpha/2} u dx \right|
\]

\[
\leq 2|\zeta_3(\theta, \omega)| \int_{\Omega} |\psi_5||(-\Delta)^{\alpha/2} u dx| + 2|\zeta_3(\theta, \omega)| \int_{\Omega} |\psi_5||u_x||(-\Delta)^{\alpha/2} u dx|
\]

\[
\leq |\zeta_3(\theta, \omega)|^2 \|\psi_5(t, \cdot)\|^2 + \|(-\Delta)^{\alpha/2} u\|^2
\]

\[
+ 4|\zeta_3(\theta, \omega)| \|\psi_5(t, \cdot)\|_{L^\infty(\Omega)} \|u_x\| \|(-\Delta)^{\alpha/2} u\|
\]

\[
\leq c \|(-\Delta)^{\alpha/2} u\|^2 + c|\zeta_3(\theta, \omega)|\|\psi_5(t, \cdot)\|_{L^\infty(\Omega)}^2 \|u_x\|^2 + \|\psi_5(t, \cdot)\|^2
\]

\[
\leq c \|u\| + \|(-\Delta)^{\alpha/2} u\|^2 + \frac{1}{4} \|(-\Delta)^{\alpha/2} u\|^2
\]

By (37)-(40), we deduce

\[
\frac{d}{dt} \|(-\Delta)^{\alpha/2} u\|^2 + \|(-\Delta)^{\alpha/2} u\|^2 + \rho \|(-\Delta)^{\alpha/2} u\|^2
\]

\[
\leq c \left( 1 + \|\psi_2(t, \cdot)\|_{L^\infty(\Omega)}^{\frac{4}{1-\alpha}} + \left| \zeta_3(\theta, \omega) \|\psi_5(t, \cdot)\|_{L^\infty(\Omega)}^{\frac{4}{1-\alpha}} \right| \right) \|u\|^2
\]

\[
+ c \left( \|\psi_3(t, \cdot)\|^2 + \|\zeta_3(\theta, \omega)\|_{L^\infty(\Omega)} \|\psi_5(t, \cdot)\|^2 + \|g(t, \cdot)\|^2 \right).
\]

Given \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( s \in (\tau - 1, \tau) \), by integrating (41) on \( (s, \tau) \) we know that

\[
\|(-\Delta)^{\alpha/2} u(t, \tau - \omega, u_{\tau-\omega})\|^2 \leq \|(-\Delta)^{\alpha/2} u(s, \tau - \omega, u_{\tau-\omega})\|^2
\]

\[
+ c_1 \int_s^\tau \|u(r, \tau - \omega, u_{\tau-\omega})\|^2 dr + c_1 \int_s^\tau \|g(r, \cdot)\|^2 dr + c_1,
\]

(42)

where \( c_1 = c_1(\tau, \omega) > 0 \). We now integrate (42) with respect to \( s \) on \( (\tau - 1, \tau) \) and replace \( \omega \) by \( \theta_{-\tau} \omega \) to obtain

\[
\|(-\Delta)^{\alpha/2} u(t, \tau - \tau - \omega, u_{\tau-\omega})\|^2 \leq c_2 \int_{\tau-1}^\tau \|(-\Delta)^{\alpha/2} u(s, \tau - \tau - \omega, u_{\tau-\omega})\|^2 ds
\]

\[
+ c_3 \int_{\tau-1}^\tau \|u(s, \tau - \tau - \omega, u_{\tau-\omega})\|^2 ds + c_3,
\]

(43)

where \( c_2 = c_2(\tau, \omega) > 0 \) and \( c_3 = c_3(\tau, \omega, g) > 0 \), which along with Lemma 2.7 implies the desired result. \( \square \)

Now, we can obtain the \( D \)-pullback asymptotic compactness of \( \Phi \) in \( L^2(\Omega) \).

**Lemma 2.9.** Suppose the conditions of (6)-(11) and (21) hold. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), the sequence \( \Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n}) \) has a convergent subsequence in \( L^2(\Omega) \) provided \( t_n \to \infty \) and \( u_{0,n} \in D(\tau - t_n, \theta_{-t_n} \omega) \).
Proof. First, for $u_0 \in D(\tau - t, \theta_{-t}\omega)$, by Lemma 2.5, 2.7 and 2.8, there exist $T_1 = T_1(\tau, \omega, D, \delta) \geq 1$ and $c_1(\tau, \omega, \delta) > 0$ such that for all $t \geq T_1$,
\[ \|\Phi(t, \tau - t, \theta_{-t}\omega, u_0)\|_{H^s_0(I)}^2 \leq c_1. \] (44)

Let $N_1 = N_1(\tau, \omega, D, \delta) \geq 1$ be large enough such that $t_n \geq T_1$ for $n \geq N_1$. Then by (44) we find that, for all $n \geq N_1$,
\[ \|\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})\|_{H^s_0(I)}^2 \leq c_1. \] (45)

By the compactness of embedding $H^s_0(I) \hookrightarrow L^2(I)$, it follows from (45) that there is $\phi \in L^2(I)$ such that, up to subsequence
\[ \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) \to \phi \] strongly in $L^2(I)$,
as desired. \qed

We now prove the existence of $D$-pullback attractors of $\Phi$.

**Theorem 2.10.** Suppose the conditions of (6)-(11) and (21)-(22) hold. Then the cocycle $\Phi$ associated with problem (3)-(5) has a unique $D$-pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{D}$ in $L^2(I)$.

**Proof.** Since $\Phi$ has a closed measurable $D$-pullback absorbing set $K \subset \mathcal{D}$ by Corollary 2.6 and is $D$-pullback asymptotically compact in $L^2(I)$ by Lemma 2.9, then the existence and uniqueness of $D$-pullback attractor $\mathcal{A}$ of $\Phi$ follows immediately. \qed

3. **Attractors of stochastic fractional Ginzburg-Landau equations.** In this section, we study the dynamics of stochastic fractional Ginzburg-Landau equations driven by a linear multiplicative noise. More precisely, we will prove the existence and uniqueness of tempered random attractors for the system. The results of this section will be used for studying the limiting behavior of solutions of the random system (3)-(5) when $\delta \to 0$.

Given $\tau \in \mathbb{R}$, consider the stochastic fractional Ginzburg-Landau equations
\[ \frac{\partial u}{\partial t} + (1 + i\nu)(-\Delta)^{\alpha}u + pu = f(t, x, u) + g(t, x) + u \circ \frac{dW}{dt}, \quad x \in \mathbb{I}, \ t > \tau, \] (46)
with homogeneous Dirichlet boundary condition and initial condition
\[ u(t, x) = 0, \quad x \in \partial \mathbb{I}, \ t > \tau, \] (47)
\[ u(\tau, x) = u_\tau(x), \quad x \in \mathbb{I}, \] (48)
where $f$ and $g$ are the same as in the previous section.

As usual, to investigate the pathwise dynamics of (46)-(48), we need to transform the stochastic equations into random ones parametrized by $\omega \in \Omega$. Let $v(t, \tau, \omega) = e^{-\omega(t)}u(t, \tau, \omega)$. Then we obtain
\[ \frac{\partial v}{\partial t} + (1 + i\nu)(-\Delta)^{\alpha}v + pv = e^{-\omega(t)}f(t, x, e^{\omega(t)}v) + e^{-\omega(t)}g(t, x), \quad x \in \mathbb{I}, \ t > \tau, \] (49)
with homogeneous Dirichlet boundary condition and initial condition
\[ v(t, x) = 0, \quad x \in \partial \mathbb{I}, \ t > \tau, \] (50)
\[ v(\tau, x) = v_\tau(x) = e^{-\omega(\tau)}u_\tau(x), \quad x \in \mathbb{I}. \] (51)

As in [25], one can show that under condition (6)-(8), for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the problem (49)-(51) has a solution $v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), L^2(\mathbb{I}))$, which is continuous.
in \( v_\tau \) and measurable in \( \omega \in \Omega \). Based on the solution of problem (49)-(51), one can define a continuous cocycle \( \Psi \) for the stochastic system (46)-(48) in \( L^2(\Omega) \):

\[
\Psi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau) = e^{\omega(t) - \omega(\tau)} v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau).
\]

(52)

We will prove \( \Psi \) has a unique \( \mathcal{D} \)-pullback attractor \( A_0 \) in \( L^2(\Omega) \). First, we derive uniform estimates of solutions in \( L^2(\Omega) \).

### Lemma 3.1

Under the conditions of (6), for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \), there exists \( L = L(\tau, \omega, T) > 0 \) such that for all \( t \in [\tau, \tau + T] \), the solution of system (49)-(51) satisfies

\[
\|v(t, \tau, \omega, v_\tau)\|^2 + \int_\tau^t \|v(s, \tau, \omega, v_\tau)\|^2 ds \leq L \left( 1 + \|v_\tau\|^2 + \int_\tau^t \|g(s, \cdot, \cdot, \cdot)\|^2 ds \right). \tag{53}
\]

**Proof.** Taking the inner product of (49) with \( v \) in \( L^2(\Omega) \) and taking the real part, we obtain

\[
\frac{d}{dt} \|v\|^2 + 2\|(-\Delta)^{\frac{p}{2}} v\|^2 + 2\rho \|v\|^2 = 2e^{-\omega(t)} \text{Re} \int_1^t f(t, x, e^{\omega(t)} v) \bar{v} dx + 2e^{-\omega(t)} \text{Re} \int_1^t g(t, x) \bar{v} dx. \tag{54}
\]

Applying (6) and Young’s inequality, we deduce that

\[
2e^{-\omega(t)} \text{Re} \int_1^t f(t, x, e^{\omega(t)} v) \bar{v} dx \leq -2\gamma e^{(p-2)-\omega(t)} \|v\|^p_p + 2e^{-2\omega(t)} \|\psi_1(t, \cdot, \cdot)\|_1, \tag{55}
\]

\[
2e^{-\omega(t)} \text{Re} \int_1^t g(t, x) \bar{v} dx \leq \frac{\rho}{2} \|v\|^2 + \frac{2}{\rho} e^{-2\omega(t)} \|g(t, \cdot, \cdot)\|^2, \tag{56}
\]

which implies that

\[
\frac{d}{dt} \|v\|^2 + 2\|(-\Delta)^{\frac{p}{2}} v\|^2 + \frac{3\rho}{2} \|v\|^2 + 2\gamma e^{(p-2)-\omega(t)} \|v\|^p_p \leq \frac{2}{\rho} e^{-2\omega(t)} \|g(t, \cdot, \cdot)\|^2 + 2e^{-2\omega(t)} \|\psi_1(t, \cdot, \cdot)\|_1. \tag{57}
\]

For every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \), integrating (57) over \( (\tau, t) \) for \( t \in [\tau, \tau + T] \), we obtain (53) immediately. \( \square \)

### Lemma 3.2

Under the conditions of (6) and (21), for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D \in \mathcal{D} \), there exists \( T_1 = T_1(\tau, \omega, D) > 0 \) such that for all \( t \geq T_1 \), the solution of system (49)-(51) satisfies

\[
\|v(\tau - t, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})\|^2 + \int_{\tau - t}^\tau e^{\rho(s - \tau)} \left( \|(-\Delta)^{\frac{p}{2}} v(s)\|^2 + \|v(s)\|^2 \right) ds \leq \tilde{R}(\tau, \omega), \tag{58}
\]

where \( e^{\omega(t) - \omega(\tau)} v_{\tau - t} \in D(\tau - t, \theta_{-\tau} \omega) \) and

\[
\tilde{R}(\tau, \omega) = L_1 \int_{-\infty}^0 e^{\rho s - 2\omega(s) - 2\omega(\tau)} \left( 1 + \|g(s + \tau, \cdot, \cdot)\|^2 \right) ds
\]

with \( L_1 \) being a positive constant independent of \( \tau \) and \( \omega \).
Proof. Applying Gronwall’s inequality to (57) over \((\tau - t, \tau)\) with \(\omega\) replaced by \(\theta_{-\tau \omega}\), after computations, we deduce
\[
\|v(\tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})\|^2 + \int_{\tau - t}^\tau e^{\rho(s-t)}( -\Delta)^{\frac{7}{2}} v(s)\|^2 ds \\
+ \frac{\rho}{2} \int_{\tau - t}^\tau e^{\rho(s-t)}\|v(s)\|^2 ds \\
\leq e^{-\rho t}\|v_{\tau - t}\|^2 + c_1 \int_{-\infty}^0 e^{\rho s - 2\omega(s) -2\omega(-\tau)} \left( \|g(s + \tau, \cdot)\|^2 + \|v_1(s + \tau, \cdot)\|_1 \right) ds \\
\leq e^{-\rho t} e^{2\omega(\tau) - 2\omega(-t)} \|D(\tau - t, \theta_{-\tau \omega})\|^2 \\
+ c_2 \int_{-\infty}^0 e^{\rho s - 2\omega(s) -2\omega(-\tau)} \left( 1 + \|g(s + \tau, \cdot)\|^2 \right) ds,
\]
where \(c_2\) is independent of \(\tau, \omega\) and \(\delta\). Since \(D \in \mathcal{D}\), by (12) we obtain that there exists \(T_0 = T_0(\tau, \omega, D) > 0\) such that for all \(t \geq T_0\),
\[
e^{-\rho t} e^{2\omega(\tau) - 2\omega(-t)} \|D(\tau - t, \theta_{-\tau \omega})\|^2 \\
\leq \int_{-\infty}^0 e^{\rho s - 2\omega(s) -2\omega(-\tau)} \left( 1 + \|g(s + \tau, \cdot)\|^2 \right) ds.
\]
Then (58) follows from (59) and (60). This completes the proof.

By Lemma 3.2 we obtain the following estimates.

Lemma 3.3. Suppose (6) and (21) holds. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D \in \mathcal{D}\), there exists \(T = T(\tau, \omega, D) > 0\) such that for all \(t \geq T\), the solution of system (46)-(48) satisfies
\[
\|u(\tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})\|^2 \leq R_1(\tau, \omega),
\]
where \(u_{\tau - t} \in D(\tau - t, \theta_{-\tau \omega})\) and
\[
R_1(\tau, \omega) = L_1 \int_{-\infty}^0 e^{\rho s - 2\omega(s) -2\omega(-\tau)} \left( 1 + \|g(s + \tau, \cdot)\|^2 \right) ds
\]
with \(L_1\) being the same constant as in Lemma 3.2 which is independent of \(\tau\) and \(\omega\).

Proof. By (52), we have
\[
u(\tau, \tau - t, \theta_{-\tau \omega}, u_{\tau - t}) = e^{-\omega(-\tau)} v(\tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})
\]
with \(v_{\tau - t} = e^{\omega(-\tau) - \omega(-t)} u_{\tau - t}\). Then (61) follows from Lemma 3.2 immediately.

Next, we derive uniform estimates of \(v\) in \(H_0^1(\Omega)\).

Lemma 3.4. Suppose (6)-(8) and (21) holds. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\), there exists \(T = T(\tau, \omega, D) \geq 1\) such that for all \(t \geq T\), the solution of system (49)-(51) satisfies
\[
\|(-\Delta)^{\frac{7}{2}} v(\tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})\|^2 \\
\leq L_2 + L_2 \int_{-\infty}^0 e^{\frac{7}{2} s - 2\omega(s) -2\omega(-\tau)} \left( 1 + \|g(s + \tau, \cdot)\|^2 \right) ds,
\]
where \(e^{\omega(-t) - \omega(-\tau)} v_{\tau - t} \in D(\tau - t, \theta_{-t \omega})\) and \(L_2\) is a positive constant depending on \(\tau\) and \(\omega\).
Proof. Taking the inner product of (49) with \((-\Delta)^{\alpha}v\) in \(L^2(\Omega)\) and taking the real part, we obtain
\[
\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 + 2\|(-\Delta)^{\alpha}v\|^2 + 2\rho\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 = 2e^{-\omega(t)}\Re\int f(t, x, e^{\omega(t)}v)(-\Delta)^{\alpha}\bar{v}dx + 2e^{-\omega(t)}\Re\int g(t, x)(-\Delta)^{\alpha}\bar{v}dx.
\]
For the first term of the right-hand side of (64), by (7)-(8), Gagliardo-Nirenberg inequality and Young’s inequality, we have
\[
2e^{-\omega(t)}\Re\int f(t, x, e^{\omega(t)}v)(-\Delta)^{\alpha}\bar{v}dx \leq 2e^{-\omega(t)}\left|\int(f_x + f_uu_x)(-\Delta)^{\alpha-\frac{1}{2}}\bar{v}dx\right|
\]
\[
\leq 2e^{-\omega(t)}\int|\psi_3||(-\Delta)^{\alpha-\frac{1}{2}}v|dx + 2e^{-\omega(t)}\int|\psi_2||(-\Delta)^{\alpha-\frac{1}{2}}v|dx
\]
\[
\leq e^{-2\omega(t)}\|\psi_3(t, \cdot)\|^2 + \|(-\Delta)^{\alpha-\frac{1}{2}}v\|^2 + 2\|\psi_2(t, \cdot)\|_{L^\infty(\Omega)}\|v_x\|\|(-\Delta)^{\alpha-\frac{1}{2}}v\|
\]
\[
\leq 2\|(-\Delta)^{\alpha-\frac{1}{2}}v\|^2 + (\|\psi_2(t, \cdot)\|_{L^\infty(\Omega)}^2\|v_x\|^2 + e^{-2\omega(t)}\|\psi_3(t, \cdot)\|^2)
\]
\[
\leq c\left(\|v\| + \|(-\Delta)^{\frac{\alpha}{2}}v\|\right)^{\frac{2\alpha-1}{\alpha}}\|v\|^{\frac{2\alpha-1}{\alpha}} + e^{-2\omega(t)}\|\psi_3\|^2
\]
\[
\leq \frac{\rho}{2}\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 + \frac{1}{2}\|(-\Delta)^{\alpha}v\|^2 + c\left(\|v\|^2 + \|\psi_2(t, \cdot)\|_{L^\infty(\Omega)}^2\|v_x\|^2\right)
\]
\[+ e^{-2\omega(t)}\|\psi_3(t, \cdot)\|^2.\]
For the second term of the right-hand side of (64), applying Young’s inequality, we get
\[
2e^{-\omega(t)}\Re\int g(t, x)(-\Delta)^{\alpha}\bar{v}dx \leq \frac{1}{2}\|(-\Delta)^{\alpha}v\|^2 + 2e^{-\omega(t)}\|g(t, \cdot)\|^2.
\]
By (64)-(66), we deduce
\[
\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 + \|(-\Delta)^{\alpha}v\|^2 + \frac{3\rho}{2}\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 \leq c\left(1 + \|\psi_2(t, \cdot)\|_{L^\infty(\Omega)}\right)\|v\|^2 + e^{-2\omega(t)}\left(\|\psi_3(t, \cdot)\|^2 + 2\|g(t, \cdot)\|^2\right).
\]
Given \(t \in \mathbb{R}^+, \tau \in \mathbb{R}\) and \(s \in (\tau - 1, \tau)\), by integrating (67) on \((s, \tau)\) we know that
\[
\|(-\Delta)^{\frac{\alpha}{2}}v(t, \tau - t, \omega, v_{\tau-t})\|^2 \leq \|(-\Delta)^{\frac{\alpha}{2}}v(s, \tau - t, \omega, v_{\tau-t})\|^2
\]
\[+ c_1\int_s^\tau \|v(r, \tau - t, \omega, v_{\tau-t})\|^2 dr + \int_s^\tau e^{-2\omega(r)}\left(\|\psi_3(r, \cdot)\|^2 + 2\|g(r, \cdot)\|^2\right) dr.\]
We now integrate (68) with respect to \(s\) on \((\tau - 1, \tau)\) and replace \(\omega\) by \(\theta_{\tau-\tau}\omega\) to obtain
\[
\|(-\Delta)^{\frac{\alpha}{2}}v(\tau, \tau - t, \theta_{\tau-\tau}\omega, v_{\tau-t})\|^2 \leq c_2\int_{\tau-1}^\tau \|(-\Delta)^{\frac{\alpha}{2}}v(s, \tau - t, \theta_{\tau-\tau}\omega, v_{\tau-t})\|^2 ds
\]
\[+ c_1\int_{\tau-1}^\tau \|v(s, \tau - t, \theta_{\tau-\tau}\omega, v_{\tau-t})\|^2 ds + c_3,
\]
where \(c_2 = c_2(\tau, \omega) > 0\) and \(c_3 = c_3(\tau, \omega, g) > 0\), which along with Lemma 3.2 implies the desired result. \(\square\)

We now prove the \(D\)-pullback asymptotic compactness of solutions of (46)-(48).
Lemma 3.5. Suppose the conditions of (6)-(8) and (21) hold. Then the cocycle $\Psi$ associated with the stochastic system (46)-(48) is $D$-pullback asymptotically compact in $L^2(\Omega)$, that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence $\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ has a convergent subsequence in $L^2(\Omega)$ provided $t_n \to \infty$ and $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. The proof is similar to that of Lemma 2.9, and hence omitted here. □

We now in the position to show the existence of $D$-pullback attractors of $\Psi$.

Theorem 3.6. Suppose the conditions of (6)-(8) and (21) hold. Then the cocycle $\Phi$ associated with problem (46)-(48) has a unique $D$-pullback attractors $A_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\Omega)$.

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, define a subset $K_0(\tau, \omega)$ by

$$K_0(\tau, \omega) = \{u \in L^2(\Omega) : ||u||^2 \leq R_1(\tau, \omega)\},$$

where $R_1(\tau, \omega)$ is defined in (62). Then by Lemma 3.2 we know that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $DinD$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\Psi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) = u(\tau, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_0(\tau, \omega).$$

Moreover, by (22), one can verify that $K_0 \in \mathcal{D}$. Therefore, $K_0$ is a closed measurable $D$-pullback absorbing set, which together with Lemma 3.5 implies the existence and uniqueness of $D$-pullback attractors $A_0$ of $\Psi$. □

4. Convergence of random attractors. In this section, we study the limiting behavior of solutions of the random system (3)-(5) when $\delta \to 0$. Under certain conditions, we will show the solutions and attractors of system (3)-(5) converge to that of the corresponding stochastic system when $\delta \to 0$.

Consider the following random system

$$\frac{\partial u_\delta}{\partial t} + (1 + i\nu)(-\Delta)u_\delta + \rho u_\delta = f(t, x, u_\delta) + g(t, x) + u_\delta \zeta_\delta(\theta_t\omega), \quad x \in \mathbb{R}, \ t > \tau, \quad (72)$$

$$u_\delta(t, x) = 0, \quad x \in \partial \mathbb{R}, \ t > \tau, \quad (73)$$

$$u_\delta(\tau, x) = u_{\delta, \tau}(x), \quad x \in \mathbb{R}. \quad (74)$$

Note that system (72)-(74) is a special case of (3)-(5) and can be obtained by formally replacing $W(t)$ by $\int_0^t \zeta_\delta(\theta_{-t}\omega)dr$ in (46). We will establish the relations between the solutions of systems (46)-(48) and (72)-(74) and show that the limiting behavior of system (72)-(74) is governed by the stochastic system (46)-(48) as $\delta \to 0$.

As indicated in Section 2, for every $\delta \in (0, 1]$, system (72)-(74) generates a continuous cocycle $\Phi_\delta$ in $L^2(\Omega)$ which possesses a unique $D$-pullback attractor $A_\delta$. To compare the solutions of (72)-(74) and (46)-(48), we introduce a new variable $v_\delta$ given by

$$v_\delta(t, \tau, \omega) = e^{-\int_0^t \zeta_\delta(\theta_{-t}\omega)dr} u_\delta(t, \tau, \omega). \quad (75)$$

By (72)-(74) and (75), we deduce

$$\frac{\partial v_\delta}{\partial t} + (1 + i\nu)(-\Delta)v_\delta + \rho v_\delta = e^{-\int_0^t \zeta_\delta(\theta_{-t}\omega)dr} f(t, x, e^{\int_0^t \zeta_\delta(\theta_{-t}\omega)dr} v_\delta)$$

$$+ e^{-\int_0^t \zeta_\delta(\theta_{-t}\omega)dr} g(t, x), \quad x \in \mathbb{R}, \ t > \tau, \quad (76)$$

$$v_\delta(t, x) = 0, \quad x \in \partial \mathbb{R}, \ t > \tau, \quad (77)$$

$$v_\delta(\tau, x) = v_{\delta, \tau}(x), \quad x \in \mathbb{R}. \quad (78)$$
First, we derive the uniform estimates on the solutions of system (76)-(78) on the finite time intervals.

**Lemma 4.1.** Under the conditions of (6). For every \(\tau \in \mathbb{R}, \omega \in \Omega \) and \(T > 0\), there exists \(\delta_0 = \delta_0(\tau, \omega, T) > 0\) and \(L_3 = L_3(\tau, \omega, T) > 0\) such that for all \(0 < \delta < \delta_0\) and \(t \in [\tau, \tau + T]\), the solution of system (76)-(78) satisfies

\[
\|v_5(t, \tau, \omega, v_{\delta, \tau})\|^2 + \int_\tau^t \|v_5(s, \tau, \omega, v_{\delta, \tau})\|_p^p ds \leq L_3 \left( 1 + \|v_{\delta, \tau}\|^2 + \int_\tau^t \|g(s, \cdot)\|^2 ds \right).
\]  

(79)

**Proof.** Taking the inner product of (76) with \(v_5\) in \(L^2(\mathbb{I})\) and taking the real part, we obtain

\[
\frac{d}{dt}\|v_5\|^2 + 2\|(-\Delta)^{\frac{\delta}{2}} v_5\|^2 + 2\rho \|v_5\|^2 = 2e^{-\int_0^t \zeta_1(\theta, \omega) ds} \text{Re} \int_\mathbb{I} f(t, x, e^{\int_0^s \zeta_1(\theta, \omega) ds} v_5) \bar{v}_5 \, dx + 2e^{-\int_0^t \zeta_1(\theta, \omega) ds} \text{Re} \int_\mathbb{I} g(t, x) \bar{v}_5 \, dx.
\]

(80)

Applying (6) and Young’s inequality, we deduce that

\[
2e^{-\int_0^t \zeta_1(\theta, \omega) ds} \text{Re} \int_\mathbb{I} f(t, x, e^{\int_0^s \zeta_1(\theta, \omega) ds} v_5) \bar{v}_5 \, dx \leq -2\gamma e^{(p-2)\int_0^t \zeta_1(\theta, \omega) ds}\|v_5\|^p_p
\]

(81)

\[
2e^{-\int_0^t \zeta_1(\theta, \omega) ds} \text{Re} \int_\mathbb{I} g(t, x) \bar{v}_5 \, dx \leq \frac{\rho}{2} \|v_5\|^2 + 2e^{-\int_0^t \zeta_1(\theta, \omega) ds} \|g(t, \cdot)\|^2,
\]

(82)

which implies that

\[
\frac{d}{dt}\|v_5\|^2 + 2\|(-\Delta)^{\frac{\delta}{2}} v_5\|^2 + \frac{3\rho}{2} \|v_5\|^2 + 2\gamma e^{(p-2)\int_0^t \zeta_1(\theta, \omega) ds}\|v_5\|^p_p \leq \frac{2}{\rho} e^{-\int_0^t \zeta_1(\theta, \omega) ds} \|g(t, \cdot)\|^2 + 2e^{-\int_0^t \zeta_1(\theta, \omega) ds} \|\psi_1(t, \cdot)\|_1.
\]

(83)

For all \(\omega \in \Omega\) and \(t \geq \tau\) with \(\tau \in \mathbb{R}\), integrating (83) from \(\tau\) to \(t\), we obtain

\[
\|v_5(t, \tau, \omega, v_{\delta, \tau})\|^2 + \int_\tau^t e^{\rho(s-t)} \|(-\Delta)^{\frac{\delta}{2}} v_5(s)\|^2 ds
\]

(84)

\[
+ \int_\tau^t e^{\rho(s-t)} e^{(p-2)\int_0^s \zeta_1(\theta, \omega) ds} \|v_5(s)\|^p_p ds \leq c e^{\rho(t-\tau)} \|v_{\delta, \tau}\|^2 + c \int_\tau^t e^{\rho(s-t)-2\int_0^s \zeta_1(\theta, \omega) ds} \|g(s, \cdot)\|^2 + \|\psi_1(s, \cdot)\|_1 \, ds,
\]

which along with (17) implies (79).

Next, we derive uniform estimates of the solutions when \(t \to \infty\).

**Lemma 4.2.** Let (6) and (21) hold. Then for every \(0 < \delta \leq 1\), \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D\), there exists \(T = T(\tau, \omega, D, \delta) > 0\) such that for
all $t \geq T$, the solution of system (76)-(78) satisfies
\begin{equation}
\|v_\delta(\tau, \tau - t, \theta_\tau \omega, v_{\delta, \tau - t})\|_2^2 + \int_{\tau-t}^\tau e^{\rho(s-t)} \left( \left\| (-\Delta) \frac{\delta}{2} v_\delta(s) \right\|_2^2 + \|v_\delta(s)\|_2^2 \right) ds \\
\leq \tilde{R}_\delta(\tau, \omega),
\end{equation}
where $e^{f_{\tau-t} \zeta_\tau(\theta, \omega) ds} v_{\delta, \tau - t} \in D(\tau - t, \theta_\tau \omega)$ and
\begin{equation}
\tilde{R}_\delta(\tau, \omega) = L_4 \int_{-\infty}^0 e^{\rho s - 2 f_\tau^t \zeta_\tau(\theta, \omega) ds} \left( 1 + \|g(s + \tau, \cdot)\|_2^2 \right) ds,
\end{equation}
with $L_4$ being a positive constant independent of $\tau$, $\omega$ and $\delta$.

**Proof.** For $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, integrating (83) from $\tau - t$ to $\tau$, we deduce
\begin{align}
\|v_\delta(\tau, \tau - t, \theta_\tau \omega, v_{\delta, \tau - t})\|_2^2 &+ \int_{\tau-t}^\tau e^{\rho(s-t)} \left( \left\| (-\Delta) \frac{\delta}{2} v_\delta(s) \right\|_2^2 + \frac{\rho}{2} \|v_\delta(s)\|_2^2 \right) ds \\
&\leq e^{-\rho t} \|v_{\delta, \tau - t}\|_2^2 + c_1 \int_{\tau-t}^\tau e^{\rho s - 2 \int_{\tau-t}^s \zeta_\tau(\theta, \omega) ds} \left( \|g(s + \tau, \cdot)\|_2^2 + \|\psi_1(s + \tau, \cdot)\|_1 \right) ds \\
&\leq e^{-\rho t + 2 \int_{\tau-t}^\tau \zeta_\tau(\theta, \omega) ds} \|D(\tau - t, \theta_\tau \omega)\|_2^2 \\
&\quad + c_2 \int_{-\infty}^0 e^{\rho s - 2 \int_{\tau-t}^s \zeta_\tau(\theta, \omega) ds} \left( 1 + \|g(s + \tau, \cdot)\|_2^2 \right) ds,
\end{align}
where $c_2$ is a positive constant independent of $\tau$, $\omega$ and $\delta$. Due to (15), we have
\begin{equation}
\int_{-\infty}^0 e^{\rho s - 2 \int_{\tau-t}^s \zeta_\tau(\theta, \omega) ds} \left( 1 + \|g(s + \tau, \cdot)\|_2^2 \right) ds < \infty.
\end{equation}
On the other hand, since $D \in \mathcal{D}$, applying (15), we obtain that there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,
\begin{align}
e^{-\rho t + 2 \int_{\tau-t}^\tau \zeta_\tau(\theta, \omega) ds} \|D(\tau - t, \theta_\tau \omega)\|_2^2 \\
&\leq e^{-\rho t + 2 \int_{-\infty}^0 \zeta_\tau(\theta, \omega) ds} \left( 1 + \|g(s + \tau, \cdot)\|_2^2 \right) ds,
\end{align}
which together with (12) and (87) completes the proof. \qed

As an immediate consequence of Lemma 4.2, we obtain the following estimates.

**Lemma 4.3.** Let (6) and (21) hold. Then for every $0 < \delta \leq 1$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$, the solution of system (72)-(74) satisfies
\begin{equation}
\|u_\delta(\tau, \tau - t, \theta_\tau \omega, u_{\delta, \tau - t})\|_2^2 \leq R_\delta(\tau, \omega),
\end{equation}
where $u_{\delta, \tau - t} \in D(\tau - t, \theta_\tau \omega)$ and
\begin{equation}
R_\delta(\tau, \omega) = L_4 \int_{-\infty}^0 e^{\rho s - 2 \int_{\tau-t}^s \zeta_\tau(\theta, \omega) ds} \left( 1 + \|g(s + \tau, \cdot)\|_2^2 \right) ds,
\end{equation}
with $L_4$ being the same positive constant as in Lemma 4.2 which is independent of $\tau$, $\omega$ and $\delta$. 
Proof. Due to (75), we get
\[ \|u_\delta(\tau, \tau - t, \theta_{\tau - \omega}, u_\delta, \tau - t)\|^2 = e^{2 \int_0^\tau \zeta(s, \theta_{\tau - s}) ds} \|u_\delta(\tau, \tau - t, \theta_{\tau - \omega}, e^{-\int_0^{\tau - t} \zeta(s, \theta_{\tau - s}) ds} u_\delta, \tau - t)\|^2, \]
which along with Lemma 4.2 implies the desired estimates.

Based on Lemma 4.3, we find a $\mathcal{D}$-pullback absorbing set for system (72)-(74).

**Lemma 4.4.** Supposed (6) and (21)-(22) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the continuous cocycle $\Phi_\delta$ associated with system (72)-(74) possesses a $\mathcal{D}$-pullback absorbing set $K_\delta \in \mathcal{D}$ given by

\[ K_\delta(\tau, \omega) = \{ u_\delta \in L^2(\mathbb{I}) : \|u_\delta\|^2 \leq R_\delta(\tau, \omega) \}, \]
where $R_\delta(\tau, \omega)$ is given by (89), and

\[ \lim_{\delta \to 0} R_\delta(\tau, \omega) = L_4 \int_{-\infty}^0 e^{\rho s - 2\omega(s)} (1 + \|g(s, \cdot , \cdot)\|^2) ds. \]

Proof. As in Lemma 2.8, one can verify $K_\delta \in \mathcal{D}$, which along with Lemma 4.3 implies that $K_\delta$ given by (90) is a $\mathcal{D}$-pullback absorbing set of $\Phi_\delta$. We now prove (91). Given $s, \tau \in \mathbb{R}$, $\delta > 0$ and $\omega \in \Omega$, let

\[ R_\delta(\tau, \omega, s) = e^{\rho s - 2\omega(s)} (1 + \|g(s, \tau, \cdot)\|^2) ds. \]

By Lemma 2.2, we obtain

\[ \lim_{\delta \to 0} R_\delta(\tau, \omega, s) = e^{\rho s - 2\omega(s)} (1 + \|g(s, \tau, \cdot)\|^2) ds. \]

By (15), we deduce that there exists $T_1 = T_1(\omega) > 0$ such that for all $s \leq -T_1$ and $0 < \delta \leq 1$,

\[ 2 \left| \int_0^s \zeta(s, \theta_{\tau - s}) ds \right| < - (\rho - \rho') s. \]

By (92) and (94) we deduce, for all $s \leq -T_1$ and $0 < \delta \leq 1$,

\[ R_\delta(\tau, \omega, s) \leq e^{\rho s} (1 + \|g(s, \tau, \cdot)\|^2) ds. \]

By (21), (93), (95) and the Lebesgue Dominated Convergence Theorem, we obtain

\[ \lim_{\delta \to 0} \int_{-\infty}^{-T_1} R_\delta(\tau, \omega, s) ds = \int_{-\infty}^{-T_1} e^{\rho s - 2\omega(s)} (1 + \|g(s, \tau, \cdot)\|^2) ds. \]

On the other hand, by Lemma 2.2, we have

\[ \int_0^s \zeta(s, \theta_{\tau - s}) ds \to \omega(\tau) \text{ uniformly on } [-T_1, 0] \text{ as } \delta \to 0. \]

Hence we obtain

\[ \lim_{\delta \to 0} \int_{-T_1}^0 R_\delta(\tau, \omega, s) ds = \int_{-T_1}^0 e^{\rho s - 2\omega(s)} (1 + \|g(s, \tau, \cdot)\|^2) ds. \]

By (96) and (97) we obtain

\[ \lim_{\delta \to 0} \int_{-\infty}^0 R_\delta(\tau, \omega, s) ds = \int_{-\infty}^0 e^{\rho s - 2\omega(s)} (1 + \|g(s, \tau, \cdot)\|^2) ds, \]
which yield (91).
Next, we establish the convergence of solutions of (72)-(74) as $\delta \to 0$. For that purpose, we further that there exist $\phi \in L^{p_{1}}(\mathbb{R}, L^{p}(\mathbb{I})) (p > 2$ and $1/p_{1} + 1/p = 1)$ and a positive constant $\mu$ such that for all $t, s \in \mathbb{R}$ and $x \in I$,

$$|f(t, x, s)| \leq \phi(t, x) + \mu|s|^{p-1}. \quad (99)$$

**Lemma 4.5.** Supposed (6)-(7) and (99) hold and let $B$ be a bounded set in $L^{2}(\mathbb{I})$ with $u_{\delta, \tau}, u_{\tau} \in B$. If $u_{\delta}$ and $u$ are the solutions of (72)-(74) and (46)-(48) with initial data $u_{\delta, \tau}$ and $u_{\tau}$, respectively, then for every $\tau \in \mathbb{R}, \omega \in \Omega, T > 0$ and $\varepsilon \in (0, 1)$, there exists $\delta_{0} = \delta_{0}(\tau, \omega, T, \varepsilon)$ and $c = c(\tau, \omega, T, B) > 0$ such that for all $0 < \delta < \delta_{0}$ and $t \in [\tau, \tau + T]$,

$$\|u_{\delta}(t, \tau, \omega, u_{\delta, \tau}) - u(t, \tau, \omega, u_{\tau})\|^{2} \leq c\|u_{\delta, \tau} - u_{\tau}\|^{2} + c\varepsilon. \quad (100)$$

**Proof.** Let $\xi = v_{\delta} - v$. From (76) and (49), we infer

$$\frac{d}{dt}\|\xi\|^{2} + 2\|(\Delta)^{2}\xi\|^{2} + 2p\|\xi\|^{2} = 2\Re\int_{1}^{e^{\int_{0}^{t} \xi(t, \omega, u, \delta)dr} f(t, x, e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} v_{\delta}) - e^{-\omega(t)} f(t, x, e^{\omega(t)} v) \bar{\xi} dx \quad (101)

+ 2\left(e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} - e^{-\omega(t)}\right) \Re \int_{1}^{g(t, x)} \bar{\xi} dx.

By Lemma 2.1 we know that for $\varepsilon > 0$ and $T > 0$, there exists $\delta_{1} = \delta_{1}(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_{1}$ and $t \in [\tau, \tau + T]$,

$$\left|e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} - e^{-\omega(t)}\right| < \varepsilon \quad \text{and} \quad \left|e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} - e^{-\omega(t)} - 1\right| < \varepsilon. \quad (102)

Applying (7) and (99) to estimate the first term on the right-side hand of (101), we obtain that there exists $c_{1} = c_{1}(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_{1}$ and $t \in [\tau, \tau + T]$,

$$2\Re\int_{1}^{e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} f(t, x, e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} v_{\delta}) - e^{-\omega(t)} f(t, x, e^{\omega(t)} v) \bar{\xi} dx \quad (103)

= 2\Re\int_{1}^{e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} f(t, x, e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} v_{\delta}) - f(t, x, e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} v) \bar{\xi} dx

+ 2\Re\int_{1}^{e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} - e^{-\omega(t)} f(t, x, e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} v) \bar{\xi} dx

+ 2\Re\int_{1}^{e^{-\omega(t)} f(t, x, e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} v) - f(t, x, e^{\omega(t)} v) \bar{\xi} dx}

\leq c_{1}\|\xi\|^{2} + c_{1}\varepsilon(1 + \|u\|_{p}^{p} + \|u_{\delta}\|_{p}^{p} + \|\phi\|_{p_{1}}^{p_{1}}).

For the second term on the right-side hand of (101), we get

$$2\left(e^{\int_{0}^{t} \zeta(t, \omega, u, \delta)dr} - e^{-\omega(t)}\right) \Re \int_{1}^{g(t, x)} \bar{\xi} dx \leq \varepsilon\|\xi\|^{2} + \varepsilon\|g(t, \cdot)\|^{2}. \quad (104)

By (103),(104) and (101), we obtain that for all $\varepsilon \in (0, 1), 0 < \delta < \delta_{1}$ and $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\|\xi\|^{2} \leq c_{2}\|\xi\|^{2} + c_{2}\varepsilon(1 + \|u\|_{p}^{p} + \|u_{\delta}\|_{p}^{p} + \|g(t, \cdot)\|^{2} + \|\phi\|_{p_{1}}^{p_{1}}). \quad (105)$$
Integrating (105) from $\tau$ to $t$ with $t \in [\tau, \tau + T]$, we deduce

$$
\|\xi(t)\|^2 \leq e^{c_1 (t-\tau)} \|\xi(\tau)\|^2 \\
+ c_2 e^{c_1 (t-\tau)} \int_{\tau}^{t} (1 + \|v(s)\|_p^p + \|v_{\delta}(s)\|_p^p + \|g(s, \cdot)\|^2 + \|\phi(s, \cdot)\|^2_1) \, ds.
$$

By (106), (53) and Lemma 4.1 we find that there exist $\delta_2 \in (0, \delta_1)$ and $c_3 = c_3(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_2$ and $t \in [\tau, \tau + T],$

$$
\|v_\delta(t, \tau, \omega, v_{\delta, \tau}) - v(t, \tau, \omega, v_\tau)\|^2 \leq c_3 \|v_{\delta, \tau} - v_\tau\|^2 + c_3 \varepsilon.
$$

Note that

$$
u_\delta(t, \tau, \omega, u_{\delta, \tau}) - u(t, \tau, \omega, u_\tau) = e^\int_{\tau}^{t} \zeta_\delta(\theta, \omega) \, d\theta v_\delta(t, \tau, \omega, u_{\delta, \tau}) - e^{\omega(t)} v(t, \tau, \omega, v_\tau)
\quad = e^\int_{\tau}^{t} \zeta_\delta(\theta, \omega) \, d\theta (v_\delta(t, \tau, \omega, u_{\delta, \tau}) - v(t, \tau, \omega, v_\tau)) \\
+ \left(e^\int_{\tau}^{t} \zeta_\delta(\theta, \omega) \, d\theta - e^{\omega(t)}\right) v(t, \tau, \omega, v_\tau),
$$

where $u_{\delta, \tau} = v_{\delta, \tau} e^{\int_{\tau}^{t} \zeta_\delta(\theta, \omega) \, d\theta}$ and $u_\tau = e^{\omega(t)} v_\tau$. It follows from (17) and (108) that there exist $\delta_3 \in (0, \delta_2)$ and $c_4 = c_4(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_3$ and $t \in [\tau, \tau + T],$

$$
\|u_\delta(t, \tau, \omega, u_{\delta, \tau}) - u(t, \tau, \omega, u_\tau)\|^2 \leq c_4 \|u_{\delta, \tau} - u_\tau\|^2 + c_4 \varepsilon,
$$

which completes the proof. \(\Box\)

As an immediate consequence of Lemma 4.5, we obtain the convergence of solutions of (72)-(74) as $\delta \to 0$.

**Corollary 4.6.** Supposed (6)-(7) and (99) hold and $\delta_n \to 0$. Let $u_{\delta_n}$ and $u$ be the solutions of (72)-(74) and (46)-(48) with initial data $u_{\delta_n, \tau}$ and $u_\tau$, respectively. If $u_{\delta_n, \tau} \to u_\tau$ in $L^2(1)$ as $n \to \infty$, then for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$
u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \to u(t, \tau, \omega, u_\tau) \quad \text{in} \quad L^2(1) \quad \text{as} \quad n \to \infty.
$$

To establish the uniform compactness of random attractors, we need the following estimates.

**Lemma 4.7.** Supposed (6)-(7) and (21) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $\delta_0 = \delta_0(\tau, \omega) \in (0, 1)$ such that for all $0 < \delta < \delta_0$, there exists $T = T(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T$, the solution of system (76)-(78) satisfies

$$
\left\|(-\Delta)^{\frac{1}{2}} v_{\delta}(\tau, \tau - t, \theta_{\tau - t}, v_{\delta, \tau - t})\right\|^2 \leq L_5 \\
+ L_5 \int_{-\infty}^{\tau} e^{p_s - 2 \int_{s}^{\tau} \zeta_\delta(\theta, \omega) \, d\theta} \left(1 + \|g(s, \tau, \cdot)\|^2\right) \, ds,
$$

where $e^{\int_{-\infty}^{\tau} \zeta_\delta(\theta, \omega) \, d\theta} v_{\delta, \tau - t} \in D(\tau - t, \theta_{\tau - t})$ and $L_5$ is a positive constant depending on $\tau$ and $\omega$, but independent of $\delta$.  

**DYNAMICS OF FRACTIONAL GINZBURG-LANDAU EQUATIONS 3571**
Given \( t < \delta < \delta \) for \( 0 \), which along with Lemma 4.2 implies that there exists \( \delta \) such that for all \( 0 < \delta < \delta \),

\[
\begin{align*}
\text{Proof.} & \quad \text{Taking the inner product of (76) with } (-\Delta)^x v_{\delta} \text{ in } L^2(\mathbb{I}) \text{ and taking the real part, we obtain} \\
& \quad \quad \frac{d}{dt} \|(-\Delta)_{\alpha} v_{\delta}\|^2 + 2\|(-\Delta)^{x} v_{\delta}\|^2 + 2\rho\|(-\Delta)_{\alpha} v_{\delta}\|^2 \\
& = 2e^{-f^\prime_\alpha \zeta^{\alpha}(\theta,\omega)dr} \Re \int_{\mathbb{I}} f(t, x, e^{f_\alpha \zeta^{\alpha}(\theta,\omega)dr} v_{\delta})(-\Delta)^{x} v_{\delta}dx \\
& \quad + 2e^{-f^\prime_\alpha \zeta^{\alpha}(\theta,\omega)dr} \Re \int g(t, x)(-\Delta)^{x} v_{\delta}dx.
\end{align*}
\]

Following the proof of Lemma 3.4, we deduce from (111) that

\[
\frac{d}{dt} \|(-\Delta)_{\alpha} v_{\delta}\|^2 + \|(-\Delta)^{x} v_{\delta}\|^2 + \rho\|(-\Delta)_{\alpha} v_{\delta}\|^2 \\
\leq c \left( 1 + \|\psi_2(t, \cdot)\|_{L^\infty(\mathbb{I})}^{4\alpha} \right) \|v_{\delta}\|^2 + e^{-2f_\alpha \zeta^{\alpha}(\theta,\omega)dr} \left( \|\psi_3(t, \cdot)\|^2 + 2\|g(t, \cdot)\|^2 \right).
\]

Given \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( s \in (\tau - 1, \tau) \), by integrating (112) on \((s, \tau)\) we know that

\[
\|(-\Delta)_{\alpha} v_{\delta}(\tau, \tau - t, \omega, v_{\delta,\tau-t})\|^2 \leq \|(-\Delta)_{\alpha} v_{\delta}(s, \tau - t, \omega, v_{\delta,\tau-t})\|^2 \\
+ c_1 \int_s^\tau \|v_{\delta}(r, \tau - t, \omega, v_{\delta,\tau-t})\|^2 dr \\
+ \int_s^\tau e^{-2f_\alpha \zeta^{\alpha}(\theta,\omega)dr} \left( \|\psi_3(r, \cdot)\|^2 + 2\|g(r, \cdot)\|^2 \right) dr.
\]

We now integrate (113) with respect to \( s \) on \((\tau - 1, \tau)\) and replace \( \omega \) by \( \theta \cdot \omega \) to obtain

\[
\|(-\Delta)_{\alpha} v_{\delta}(\tau, \tau - t, \theta \cdot \omega, v_{\tau-t})\|^2 \\
\leq \int_{\tau-1}^0 \|(-\Delta)_{\alpha} v_{\delta}(s + \tau, \tau - t, \theta \cdot \omega, v_{\delta,\tau-t})\|^2 ds \\
+ c_1 \int_{\tau-1}^0 \|v_{\delta}(s + \tau, \tau - t, \theta \cdot \omega, v_{\delta,\tau-t})\|^2 ds \\
+ \int_{\tau-1}^0 e^{-2f_\alpha \zeta^{\alpha}(\theta,\omega)ds} \left( \|\psi_3(s + \tau, \cdot)\|^2 + 2\|g(s + \tau, \cdot)\|^2 \right) ds.
\]

By (17) and (114), we infer that there exist \( \delta_0 = \delta_0(\tau, \omega) > 0 \) and \( c_1 = c_1(\tau, \omega) > 0 \) such that for all \( 0 < \delta < \delta_0 \),

\[
\|(-\Delta)_{\alpha} v_{\delta}(\tau, \tau - t, \theta \cdot \omega, v_{\tau-t})\|^2 \leq c_1 \\
+ c_1 \int_{\tau-1}^0 \|(-\Delta)_{\alpha} v_{\delta}(s + \tau, \tau - t, \theta \cdot \omega, v_{\delta,\tau-t})\|^2 ds \\
+ c_1 \int_{\tau-1}^0 \|v_{\delta}(s + \tau, \tau - t, \theta \cdot \omega, v_{\delta,\tau-t})\|^2 ds
\]

which along with Lemma 4.2 implies that there exists \( T_1 = T_1(\tau, \omega, D, \delta) \geq 1 \) such that for \( 0 < \delta < \delta_0 \) and \( t \geq T_1 \),

\[
\|(-\Delta)_{\alpha} v_{\delta}(\tau, \tau - t, \theta \cdot \omega, v_{\tau-t})\|^2 \leq c_1 \\
+ c_1 L_4 \int_{-\infty}^0 e^{\rho s - 2f_\alpha \zeta^{\alpha}(\theta,\omega)ds} \left( 1 + \|g(s + \tau, \cdot)\|^2 \right) ds,
\]

which completes the proof.
Recall that for each $\delta > 0$, $\mathcal{A}_\delta$ is the unique $\mathcal{D}$-pullback attractor of $\Phi_\delta$ in $L^2(I)$. To obtain the uniform compactness of these attractors with respect to $\delta$, we need further estimates on $\Phi_\delta$ as given below.

**Lemma 4.8.** Suppose (6)-(8), (21) and (99) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega \} \subset \mathcal{D}$, there exists $\delta_0 = \delta_0(\tau, \omega) \in (0, 1)$ such that for all $0 < \delta < \delta_0$, there exists $T = T(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T$,

$$
\|\Phi_\delta(t, \tau - t, \theta_\omega, u_{\delta, \tau - t})\|_{H_0^2(I)}^2 \leq L_0 + L_6 \int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds,
$$

where $u_{\delta, \tau - t} \in D(\tau - t, \theta_\omega)$ and $L_7$ is a constant depending on $\tau$ and $\omega$, but independent of $\delta$.

**Proof.** By (75), we have

$$
\Phi_\delta(t, \tau - t, \theta_\omega, u_{\delta, \tau - t}) = e^{\int_0^t \zeta_{\delta}(\theta_\omega) dr} v_\delta(t, \tau - t, \theta_\omega, v_{\delta, \tau - t}),
$$

which along with Lemma 4.2 and 4.7 implies that there exists $T_1 = T_1(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T_1$,

$$
\|\Phi_\delta(t, \tau - t, \theta_\omega, u_{\delta, \tau - t})\|_{H_0^2(I)}^2 \leq c_1 e^{\int_0^t \zeta_{\delta}(\theta_\omega) dr} + c_1 \int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds,
$$

where $c_1 = c_1(\tau, \omega) > 0$. By (17), we have that there exist $\delta_2 = \delta_2(\tau, \omega) \in (0, \delta_1)$ and $c_2 = c_2(\tau, \omega) > 0$ such that for all $0 < \delta < \delta_2$,

$$
\left| \int_{-\infty}^0 \zeta_{\delta}(\theta_\omega) dr \right| \leq c_2.
$$

On the other hand, by the argument of (98), we obtain

$$
\lim_{\delta \to 0} \int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds
$$
$$
= \int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds,
$$

and hence there exists $\delta_3 = \delta_3(\tau, \omega) \in (0, \delta_2)$ such that for all $0 < \delta < \delta_3$,

$$
\int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds \leq 1 + \int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds.
$$

It follows from (116)-(118) that for all $0 < \delta < \delta_3$ and $t \geq T_1$, there exists $c_3 = c_3(\tau, \omega) > 0$ such that

$$
\|\Phi_\delta(t, \tau - t, \theta_\omega, u_{\delta, \tau - t})\|_{H_0^2(I)}^2 \leq c_3 + c_3 \int_{-\infty}^0 e^{\rho s - 2\omega(s)} \left(1 + \|g(s + \tau, \cdot)\|^2\right) ds.
$$

This completes the proof. □

Next, we establish the uniform compactness of random attractors with respect to $\delta$. 
Lemma 4.9. Suppose (6)-(8), (21) and (99) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $\delta_0 = \delta_0(\tau, \omega) \in (0, 1)$ such that the set $\bigcup_{0 < \delta < \delta_0} A_\delta(\tau, \omega)$ is precompact in $L^2(\mathbb{I})$.

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by Lemma 4.8, we know that there exists $\delta_1 = \delta_1(\tau, \omega) > 0$ such that for every $0 < \delta < \delta_1$ and $D \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D, \delta) \geq 1$ such that for all $t \geq T_1$,

$$||\Phi_\delta(t, \tau - t, \theta - t, u, w)||^2_{H^0_0(\mathbb{I})} \leq c_1,$$

(119)

where $u, w \in D(\tau - t, \theta - t)$ and $c_1 = c_1(\tau, \omega) > 0$. Let $t_n \to \infty$ and $W \in A_\delta(\tau, \omega)$ for some $\delta \in (0, \delta_1)$. By the invariance of $\mathcal{A}_\delta$, for each $n$, there exists $W_n \in A_\delta(\tau - t_n, \theta - t_n, \omega)$ such that

$$W = \Phi_\delta(t_n, \tau - t_n, \theta - t_n, W_n) = \Phi_\delta(t_n, \tau - t_n, \theta - t_n, W_n), c_1,$$

(120)

Since $A_\delta \in \mathcal{D}$ and $t_n \to \infty$, by (119), we obtain that there exists $N_1 = N_1(\tau, \omega, \delta) \geq 1$ such that for all $n \geq N_1$,

$$||\Phi_\delta(t_n, \tau - t_n, \theta - t_n, W_n)||^2_{H^0_0(\mathbb{I})} \leq c_1.$$

(121)

By (121), we infer that there exists $\tilde{W} \in H^0_0(\mathbb{I})$ such that, up to a subsequence

$$\Phi_\delta(t_n, \tau - t_n, \theta - t_n, W_n) \to \tilde{W} \text{ weakly in } H^0_0(\mathbb{I}), \text{ as } n \to \infty.$$  

(122)

Moreover, we have

$$||\tilde{W}||^2_{H^0_0(\mathbb{I})} \leq c_1.$$  

(123)

By (12) and (122), we obtain that $W = \tilde{W}$, which along with (123) shows that

$$||W||^2_{H^0_0(\mathbb{I})} \leq c_1 \text{ for all } W \in \mathcal{A}_\delta(\tau, \omega) \text{ with } 0 < \delta < \delta_1.$$  

(124)

By (124), the set $\bigcup_{0 < \delta < \delta_0} \mathcal{A}_\delta(\tau, \omega)$ is bounded in $H^0_0(\mathbb{I})$, and hence precompact in $L^2(\mathbb{I})$. This completes the proof. $\square$

We finally establish the upper semicontinuity of random attractors as $\delta \to 0$.

Theorem 4.10. Suppose the conditions of (6)-(8), (21)-(22) and (99) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta \to 0} \text{dist}_{L^2(\mathbb{I})}(\mathcal{A}_\delta(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0.$$  

(125)

Proof. Let $K_0 = \{K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ and $K_\delta = \{K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be the $\mathcal{D}$-pullback absorbing sets of $\Phi_0$ and $\Phi_\delta$ given by (70) and (90), respectively. By (91), we get

$$\lim_{\delta \to 0} ||K_\delta(\tau, \omega)|| = ||K_0(\tau, \omega)|| \text{ for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega.$$  

(126)

Then by (126), Corollary 4.6 and Lemma 4.9, we obtain (125) from Theorem 3.1 in [31] immediately. $\square$

Acknowledgments. The authors would like to thank the referee for his/her valuable comments and suggestions which greatly improved the paper. Special thanks to Dr. Bixiang Wang for his valuable comments and suggestions in the preparation of this manuscript.
REFERENCES

[1] S. Abe and S. Thurner, Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion, *Physica A*, 356 (2005), 403–407.

[2] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, Wiley-Interscience, New York, 1974.

[3] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

[4] L. Caffarelli, J. Roquejoffre and Y. Sire, Variational problems for free boundaries for the fractional Laplacian, *J. Eur. Math. Soc.*, 12 (2010), 1151–1179.

[5] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.

[6] J. Doob, The Brownian movement and stochastic equations, *Annals of Math.*, 43 (1942), 351–369.

[7] C. Gal and M. Warma, Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions, *Discrete Contin. Dyn. Syst. Ser. A*, 36 (2016), 1279–1319.

[8] A. Garroni and S. Muller, A variational model for dislocations in the line tension limit, *Arch. Ration. Mech. Anal.*, 181 (2006), 535–578.

[9] W. Gerster, W. Kistler, R. Naud and L. Paninski, *Neuronal Dynamics: From Single Neurons to Networks and Models of Cognition*, Cambridge University Press, Cambridge, 2014.

[10] Q. Guan and Z. Ma, Reflected symmetric α-stable processes and regional fractional Laplacian, *Probab. Theory Related Fields*, 134 (2006), 649–694.

[11] Q. Guan and Z. Ma, Boundary problems for fractional Laplacians, *Stoch. Dyn.*, 5 (2005), 385–424.

[12] M. Jara, Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps, *Comm. Pure Appl. Math.*, 62 (2009), 198–214.

[13] T. Jiang, X. Liu and J. Duan, Approximation for random stable manifolds under multiplicative correlated noise, *Discrete Contin. Dyn. Syst. Ser. B*, 21 (2016), 3163–3174.

[14] N. van Kampen, *Stochastic Processes in Physics and Chemistry*, Amsterdam-New York, 1981.

[15] M. Koslowski, A. Cuitino and M. Ortiz, A phasefield theory of dislocation dynamics, strain hardening and hysteresis in ductile single crystal, *J. Mech. Phys. Solids*, 50 (2002), 2597–2635.

[16] H. Lu, P. W. Bates, S. Lu and M. Zhang, Dynamics of 3D fractional complex Ginzburg-Landau equation, *J. Differential Equations*, 259 (2015), 5276–5301.

[17] H. Lu, P. W. Bates, J. Xin and M. Zhang, Asymptotic behavior of stochastic fractional power dissipative equations on $\mathbb{R}^n$, *Nonlinear Anal.*, 128 (2015), 176–198.

[18] H. Lu, P. W. Bates, S. Lu and M. Zhang, Dynamics of the 3D fractional Ginzburg-Landau equation with multiplicative noise on an unbounded domain, *Comm. Math. Sci.*, 14 (2016), 273–295.

[19] H. Lu, S. Lv and M. Zhang, Fourier spectral approximation to the dynamical behavior of 3D fractional Ginzburg-Landau equation, *Discrete Contin. Dyn. Syst. Ser. A*, 37 (2017), 2539–2564.

[20] H. Lu and M. Zhang, The spectral method for long-time behavior of a fractional power dissipative system, *Taiwanese J. Math.*, 22 (2018), 453–483.

[21] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, 13 (1959), 115–162.

[22] X. Pu and B. Guo, Well-posedness and dynamics for the fractional Ginzburg-Landau equation, *Applicable Analysis*, 92 (2013), 318–334.

[23] L. Ridolfi, P. D’Odorico and F. Laio, *Noise-Induced Phenomena in the Environmental Sciences*, Cambridge University Press, New York, 2011.

[24] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary, *Journal de Mathematiques Pures et Appliquees*, 101 (2014), 275–302.
[28] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators, *Proc. Roy. Soc. Edinburgh Sect. A*, 144 (2014), 831–855.

[29] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst. Ser. A*, 33 (2013), 2105–2137.

[30] G. Uhlenbeck and L. Ornstein, On the theory of Brownian motion, *Phys. Rev.*, 36 (1930), 823–841.

[31] B. Wang, Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, *Stoch. Dyn.*, 14 (2014), 1450009, 31pp.

[32] M. Wang and G. Uhlenbeck, On the theory of Brownian motion. II, *Rev. Modern Phys.*, 17 (1945), 323–342.

Received for publication August 2019.

E-mail address: ljuenling@163.com
E-mail address: mingji.zhang@nmt.edu, mzhang0129@gmail.com