The Pair Contact Process in Two Dimensions

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(October 24, 2018)

Abstract

We study the stationary properties of the two-dimensional pair contact process, a nonequilibrium lattice model exhibiting a phase transition to an absorbing state with an infinite number of configurations. The critical probability and static critical exponents are determined via Monte Carlo simulations, as well as order-parameter moment ratios and the scaling of the initial density decay. The static critical properties are consistent with the directed percolation universality class.

PACS numbers: 05.50.+q, 02.50.-r, 05.70.Ln

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I. INTRODUCTION

Critical phenomena at absorbing-state phase transitions (i.e., between an active state and one in which the dynamics is frozen), are of longstanding interest in statistical physics, and have enjoyed renewed attention due to connections with epidemics [1], catalytic kinetics [2,3], surface growth [4], self-organized criticality [5–8], and issues of scaling and universality [9,10]. In these systems, conflict between two opposing processes (e.g., creation and annihilation), typically leads to a continuous transition at a critical parameter value. Such transitions are known to fall generically in the universality class of directed percolation (DP) [11–13], although the critical behavior is modified in the presence of local parity conservation [14–17].

Another interesting case (without a conservation law), appears when the dynamics can become trapped in one of an infinite number (in the thermodynamic limit) of absorbing configurations (INAC). Systems of this sort were introduced in catalysis modeling [18–20]; their critical properties have been studied in detail by various workers [21–28]. In one dimension, the pair contact process (PCP) [22], and other models with INAC exhibit static critical behavior in the DP class [23,29], but the critical exponents associated with the spread of activity from a localized seed are nonuniversal, varying continuously with the particle density in the environment [23,28], and follow a generalized hyperscaling relation [24,30]. The anomalous spreading can be traced to a long memory in the dynamics of the order parameter, \( \rho \), arising from coupling to an auxiliary field that remains frozen in regions where \( \rho = 0 \) [25,27,28]. A field theory (i.e., a stochastic partial differential equation for \( \rho(x,t) \)), incorporating this memory term reproduces the nonuniversal exponents observed in simulations [31].

In two dimensions the situation is much less clear. Simulation results for a microscopic model with INAC [26] conflict with studies of models exhibiting the aforementioned long memory [27,28]. (In particular, it seems possible that for one range of densities the spreading dynamics is that of dynamic percolation, while for another range there is compact growth, perhaps with nonuniversal exponents [27], or without well defined scaling behavior [28].) In hopes clarifying the nature of critical spreading in the presence of INAC, we propose to study the pair contact process in two dimensions. Our interest in the PCP is motivated by its simplicity, compared with the model studied in Ref. [26]. The present work is devoted to static critical behavior, and provides the critical parameter value and the “natural density” in the absorbing state (defined below) needed for a detailed study of spreading. These results confirm that the static behavior falls in the DP class; analyses of moment ratios and the initial decay of the order parameter provide further support. The balance of this paper is devoted to defining the model and simulation algorithm (Sec. II), simulation results (Sec. III), and a brief summary (Sec. IV).

II. MODEL

The pair contact process (PCP) is an interacting particle system: a Markov process whose state space is a set of particle configurations on a lattice [32,33]. Each nearest-neighbor pair of particles has a rate \( p \) of mutual annihilation, and a rate \( 1 − p \) of attempted creation. In a creation attempt on the square lattice, a new particle may appear (with equal likelihood) at any of the six sites neighboring the pair, provided the chosen site is vacant. (Attempts to
place a new particle at an occupied site fail.) The PCP exhibits an active phase for \( p < p_c \); above this value the system falls into an absorbing configuration that typically contains a substantial density, \( \phi \), of particles. (Any arrangement of particles devoid of nearest-neighbor pairs is absorbing.)

In our simulations, we maintain a list of the \( N_p \) current nearest-neighbor pairs. At each step we choose a pair at random from the list, and a process (annihilation with probability \( p \), creation with probability \( 1 - p \)). In case of annihilation, the two particles are simply removed. For creation, we choose a site \( x \) at random from among the six neighbors of the pair, and place a new particle there if \( x \) is currently vacant. (If \( x \) is occupied the configuration remains the same.) The time increment associated with this step is \( \Delta t = 1/N_p \), corresponding to one transition per pair per unit time, in agreement with the transition rates that define the process. Following each change we update the list of pairs. We use a square lattice of \( L \times L \) sites, with periodic boundaries; in the studies reported here, all sites are initially occupied.

### III. CRITICAL PROPERTIES

To locate the critical point \( p_c \), we study the size-dependence of the (quasi) stationary pair density \( \rho \), i.e., the fraction of nearest neighbors harboring a pair of particles, in surviving trials, following a transient during which \( \rho(t) \) relaxes from its initial value of unity. \( \rho \) is the order parameter for the PCP, and as such we expect that at the critical point,

\[
\rho(p_c, L) \sim L^{-\beta/\nu_{\perp}},
\]

while off-critical values of \( p \) should yield deviations from the power law. We studied the pair density in systems of size \( L = 10, 20, 40, 80 \) and \( 160 \), for times \( t_m \) ranging from \( 10^3 \) for \( L = 10 \) to \( 5 \times 10^4 \) for \( L = 160 \), with sample sizes ranging from \( 10^6 \) trials (\( L = 10 \)) to \( 10^4 \) trials (\( L = 160 \)). The results (see Fig. 1) show \( \rho(p, L) \) following a power law for \( p = 0.2005 \), but clearly not for \( p = 0.200 \) or \( 0.201 \). We conclude that \( p_c = 0.2005(2) \), the figure in parentheses denoting the uncertainty of our estimate. The data for \( p = 0.2005 \) yield the exponent ratio \( \beta/\nu_{\perp} = 0.793(5) \), in good agreement with the value of 0.799(2) for DP in 2+1 dimensions \(^3\).

We also determined the survival probability, \( P(t, p, L) \), i.e., the probability that the system contains at least one nearest-neighbor pair. For finite \( L \), this decays asymptotically as \( P \sim e^{-t/\tau_P} \), with the lifetime showing a power-law dependence on the system size at the critical point:

\[
\tau_P(p_c, L) \sim L^{\nu_{\parallel}/\nu_{\perp}}.
\]

The data for \( p = 0.2005 \) (see Fig. 2) yield \( \nu_{\parallel}/\nu_{\perp} = 1.79(1) \), reasonably close to the DP value, \( 1.766(2) \). We find that \( \rho \) and the particle density \( \phi \) also approach their stationary values exponentially: \( \rho(t) - \rho_s \sim e^{-t/\tau} \) (similarly for \( \phi \)), but on a much shorter time scale than that of \( P(t) \): \( \tau \simeq \tau_P/10 \). Analysis of the data for \( \tau \) at the critical point yields \( \nu_{\parallel}/\nu_{\perp} = 1.69(3) \).

For \( p \leq p_c \) the process always falls into the absorbing state. The properties of this state are determined by the probability distribution (induced by the dynamics) on the set of absorbing configurations for system size \( L \). Of interest is the particle density \( \phi \) in the
absorbing state, in particular, the “natural” density, defined as the limiting value at the critical point:

\[ \phi_{\text{nat}} \equiv \lim_{L \to \infty} \phi(p_c, L). \]  

(In one dimension, it is only for this particle density that the spreading exponents take DP values [23].) In our simulations at \( p_c \) virtually all of the trials end in an absorbing configuration before \( t_m \); the final particle density in the absorbing state yields an estimate for \( \phi(p_c, L) \).

Since the dynamics of \( \phi \) is tied to that of the order parameter, \( \rho \), and since the excess particle density \( \phi(p) - \phi_{\text{nat}} \) in a related one-dimensional model is known to be governed by the order-parameter exponent \( \beta \) [23], we expect that the leading finite-size correction to the particle density to be \( \sim L^{-\beta/\nu_\perp} \), just as for \( \rho \). This is confirmed in Fig. 3. Linear fits to the data for \( L \geq 40 \) yield \( \phi \simeq 0.1480 + aL^{-\beta/\nu_\perp} \) and \( \rho \simeq bL^{-\beta/\nu_\perp} \) (with \( a = 0.9662 \) and \( b = 1.426 \)), suggesting that the linear combination \( \phi - (a/b)\rho \) will be essentially independent of \( L \). This is indeed so for \( L \geq 20 \), as shown in the inset of Fig. 3, from which we obtain our final estimate, \( \phi = 0.1477(1) \).

Order parameter moment ratios provide another tool for assigning a model a universality class; in equilibrium spin systems Binder’s reduced fourth cumulant has been widely used for this purpose [35]. A variety of ratios, involving both odd and even moments, have been determined for several one dimensional models with absorbing-state transitions (including the PCP), as well as for the two-dimensional contact process [29]. We determined the stationary order-parameter moments \( m_1, \ldots, m_4 \) \( (m_j \equiv \langle \rho^j \rangle) \) in order to evaluate various ratios; the results are listed in Table I. In Fig. 4 we plot several of the moment ratios versus \( L^{-1} \); linear fits yield the infinite-\( L \) estimates given in Table I. The latter agree quite well with the results for the contact process, providing further support for the PCP belonging to the DP universality class. Curiously, the moment ratios for the PCP appear to approach their limiting values monotonically (with \( L \)), while the two-dimensional CP exhibits a nonmonotonic \( L \)-dependence (see Fig. 5 of Ref. [29]).

Finally, we analyzed the initial decay of the pair density at the critical point. In general, we expect the order parameter to decay as a power law, \( \rho \sim t^{-\delta} \) in a critical system at short times (i.e., \( t < \tau \sim L^{\nu_\parallel/\nu_\perp} \), for which the correlation length \( \xi < L \)). Our results, plotted in Fig. 5, show a power-law decay with an exponent of 0.443(5). This is somewhat smaller than, but still consistent with, recent estimates of \( \delta \) for DP in 2+1 dimensions, which range from 0.4505(10) [36] to 0.452(1) [34].

IV. SUMMARY

We studied the stationary critical properties of the pair contact process in two dimensions. On the basis of the exponent ratios \( \beta/\nu_\perp \) and \( \nu_\parallel/\nu_\perp \), moment ratios, and the initial decay of the order parameter, we can assign the PCP to the directed percolation universality class, generic for absorbing-state transitions without a conservation law or special symmetry. We have noted several minor discrepancies between our results and the standard DP values, but expect that these are due to finite-size effects and/or a small error in \( p_c \), and do not reflect non-DP universality. The issue of spreading dynamics will be addressed in future work.
JKLS acknowledges partial support by CNPq.
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**TABLE I.** Ratios of order-parameter moments in the critical PCP. Entries for $L = \infty$ represent linear extrapolations; data for the CP from Ref. [29]. Numbers in parentheses denote uncertainties in the last figure.

| $L$  | $m_2/m_1^2$  | $m_3/m_1^3$  | $m_3/(m_1m_2)$ | $m_4/m_2^2$ |
|------|--------------|--------------|----------------|-------------|
| 20   | 1.362(3)     | 2.300(6)     | 1.638(5)       | 2.247(10)   |
| 40   | 1.343(3)     | 2.147(7)     | 1.599(5)       | 2.159(9)    |
| 80   | 1.334(3)     | 2.111(9)     | 1.582(6)       | 2.116(10)   |
| 160  | 1.327(4)     | 2.086(9)     | 1.571(8)       | 2.093(13)   |
| $\infty$ | 1.323(3)  | 2.067(9)     | 1.56(1)        | 2.07(1)     |
| CP   | 1.326(1)     | 2.080(1)     | 1.569(1)       | 2.093(8)    |
Figure Captions

FIG. 1. Stationary pair density versus system size for $p = 0.21, 0.202, 0.201, 0.2005, 0.200$, and $0.195$ (left to right).

FIG. 2. Relaxation times versus system size at the critical point. Upper set: $\tau_P$, the mean lifetime; lower set: $\tau$, associated with the relaxation of $\rho$ and $\phi$.

FIG. 3. Particle density in absorbing configurations at the critical point versus $L^{-\beta/\nu_\perp}$. The inset is a plot of $\phi - 0.6776\rho$.

FIG. 4. Order-parameter moment ratios in the critical PCP. Upper set: $m_4/m_2^2$; middle: $m_3/(m_1m_2)$; lower: $m_2/m_1^2$.

FIG. 5. Decay of the order parameter in the critical PCP. $+$: $L = 160$; $\bullet$: $L = 320$; line: $L = 640$. 
