Some Identities Related to the Second-Order Eulerian Numbers

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Abstract: We express the Norlund polynomials in terms of the second-order Eulerian numbers. Based on this expression, we derive several identities related to the Bernoulli numbers. In particular, we present a short proof of the problem raised by Rządkowski and Urlińska.

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1 Introduction

The Stirling permutations were introduced by Gessel and Stanley [6]. For some related results on this subject, we refer to [1, 4, 7, 9]. Let $Q_n$ be the multiset $\{1,1,2,2,\ldots,n,n\}$. A Stirling permutation of order $n$ is a permutation of $Q_n$ such that for each $1 \leq m \leq n$, the elements lying between two occurrences of $m$ are greater than $m$.

The second-order Eulerian numbers $C_{n,k}$ count the Stirling permutations of order $n$ with $k$ decents, which satisfy the recurrence relation:

$$C_{n,k} = kC_{n-1,k} + (2n-k)C_{n-1,k-1}$$

(1.1)

with $C_{1,1} = 1$ and $C_{1,0} = 0$.

By exhibiting a bijection between the set of partitions of $[n + m]$ with $m$ blocks and the bar permutations on the elements of $Q_n$ with $m$ bars, Gessel and Stanley [6] proved that

$$\sum_{m=0}^{\infty} S(n+m,m)x^m = \sum_{k=1}^{n} C_{n,k}x^k/(1-x)^{2n+1},$$

(1.2)

where $S(n,m)$ are the Stirling numbers of the second kind.

The Norlund polynomials $B_n^{(z)}$ can be defined by the exponential generating function:

$$\sum_{n=0}^{\infty} B_n^{(z)} x^n/n! = \left(\frac{x}{e^x - 1}\right)^z.$$

(1.3)

Note that for fixed $n$, $B_n^{(z)}$ are polynomials in $z$ with degree $n$. If $z = 0$, then we have $B_0^{(0)} = 1$ and $B_n^{(0)} = 0$ for $n \geq 1$. If $z = 1$, then $B_n^{(1)}$ are the classical Bernoulli numbers $B_n$, i.e.,

$$\sum_{n=0}^{\infty} B_n^{(1)} x^n/n! = \frac{x}{e^x - 1}.$$

(1.4)

If $z = n$, we have $B_n^{(n)} = (-1)^n c_n^{(2)}$, which are the Cauchy numbers of the second kind [3]:

$$\sum_{n=0}^{\infty} c_n^{(2)} x^n/n! = \frac{-x}{(1-x) \ln(1-x)}.$$

(1.5)
In [2], Carlitz showed that the Nörlund polynomials and the Stirling numbers of the second kind satisfy the relation:
\[
S(m + n, m) = \binom{m + n}{n} B_n(-m).
\] (1.6)

Let \( \langle z \rangle_n \) be the rising factorial defined by \( \langle z \rangle_n = z(z + 1) \cdots (z + n - 1) \). The following result expresses the Nörlund polynomials in terms of the second-order Eulerian numbers.

**Theorem 1.1** We have
\[
B_n^{(z)} = \frac{n!}{(2n)!} \sum_{k=1}^{n} (-1)^{k} C_{n,k} \langle z \rangle_k \langle -z + n + 1 \rangle_{n-k}.
\] (1.7)

Based on Theorem 1.1, we obtain several identities involving the convolutions of Bernoulli numbers. In particular, we give a short proof of the problem raised by Rządkowski and Urlińska:
\[
\int_0^1 \sum_{k=0}^{n-1} C_{n,k+1} u^{k+1} (u-1)^{2n-k} du = \frac{B_{n+1}}{n+1}.
\] (1.8)

By computing the integral
\[
\int_0^1 u^{k+1} (u-1)^{2n-k} du = \frac{(-1)^k}{2(n+1) \binom{2n+1}{k}},
\]
we may restate (1.8) as follows.

**Theorem 1.2** We have
\[
\sum_{k=1}^{n} (-1)^{k-1} \binom{2n+1}{k}^{-1} C_{n,k} = 2B_{n+1}.
\] (1.9)

Notice that, by considering the partial derivative equation \( \partial_t^n \psi(t) = v_n(\psi) \), where \( \psi \) is defined by the Lambert W-function \( \psi(t, x) = W(x e^{x+t}) \) and \( v_n(x) \) is given by
\[
v_n(x) = -x(1 + x)^{-2n+1} \sum_{k \geq 1} (-1)^k C_{n,k} x^k,
\]
an alternative proof of (1.9) was discussed on Mathoverflow [11] recently.

Let \( H_n \) be the harmonic number defined by \( H_n = \sum_{i=1}^{n} 1/i \). By using the p-adic method, Miki [8] proved that
\[
\sum_{k=2}^{n} \frac{B_k}{k} \frac{B_{n-k}}{n-k} = 2H_n \frac{B_n}{n} + \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k}{k} \frac{B_{n-k}}{n-k}.
\] (1.10)

By computing the second derivative of the both sides of (1.7), then employing Miki’s identity (1.10), we derive the following the result involving the harmonic numbers.
Theorem 1.3 We have
\[
\sum_{k=1}^{n} (-1)^k \binom{2n-1}{k-1}^{-1} (H_{2n-k} - H_k) C_{n,k} = \frac{n^2}{n-1} B_{n-1} + n \sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k} n - k. \tag{1.11}
\]

For \( n \geq N \), Dilcher [5] proved that
\[
\sum_{k_1 + k_2 + \cdots + k_N = n} \binom{n}{k_1, k_2, \ldots, k_N} B_{k_1} B_{k_2} \cdots B_{k_N} = N \left( \binom{n}{N} \sum_{k=0}^{N-1} (-1)^{N-k} s(N, N-k) \frac{B_{n-k}}{n-k} \right), \tag{1.12}
\]
where \( s(n, k) \) is the Stirling number of the first kind. By setting \( z = N \) in (1.7), then employing (1.12), we obtain the following identity.

Theorem 1.4 Let \( N, n \) be nonnegative integers with \( n \geq N \). We have
\[
\sum_{k=1}^{n} (-1)^k \binom{2n-1}{N+k-1}^{-1} C_{n,k} = 2n \sum_{k=0}^{N-1} (-1)^{N-k} s(N, N-k) \frac{B_{n-k}}{n-k}. \tag{1.13}
\]
In particular, if we let \( n = N \) in (1.13), the Cauchy numbers can be related by the second-order Eulerian numbers:
\[
2c^{(2)}_n = \sum_{k=1}^{n} (-1)^{n-k} \binom{2n-1}{n+k-1}^{-1} C_{n,k}. \tag{1.14}
\]

2 Proofs

If two polynomials in a single variable \( z \) agree for every nonnegative integer \( z \), then they agree as polynomials. Therefore, to prove Theorem 1.1, it suffices to prove the following lemma.

Lemma 2.5 Given a nonnegative integer \( m \), we have
\[
B_{n}^{(-m)} = \frac{n!}{(2n)!} \sum_{k=1}^{n} (-1)^k C_{n,k} \binom{m}{k} \binom{m+n+1}{n-k}. \tag{2.1}
\]

Proof: Equating the coefficients of \( x^m \) on the both sides of (1.2) leads to
\[
S(m+n, m) = \sum_{k=1}^{n} \binom{2n+m-k}{2n} C_{n,k}. \tag{2.2}
\]
Comparing (1.6) and (2.2) for any integer \( m \geq 0 \), we have
\[
B_{n}^{(-m)} = \frac{m!n!}{(m+n)!} \sum_{k=1}^{n} \binom{2n+m-k}{2n} C_{n,k} = \frac{n!}{(2n)!} \sum_{k=1}^{n} (-1)^k C_{n,k} \binom{m}{k} \binom{m+n+1}{n-k}, \tag{2.3}
\]
as desired. We complete the proof of Lemma 2.5, and the proof of Theorem 1.1 as well.
Observe that
\[
\frac{d}{dx}\ln\left(\frac{e^x-1}{x}\right) = \frac{1}{x} \left(\frac{-x}{e^{-x} - 1} - \frac{x}{x} \right) = \sum_{n=1}^{\infty} (-1)^n \frac{B_n x^{n-1}}{n!}.
\]

Thus,
\[
\ln\left(\frac{e^x-1}{x}\right) = \sum_{n=1}^{\infty} (-1)^n \frac{B_n x^n}{n n!} = \frac{x}{2} + \sum_{n=2}^{\infty} \frac{B_n x^n}{n n!}.
\]

Note that \(B_1 = -\frac{1}{2}\) and \(B_n = 0\) when \(n\) is odd and greater than 1.

Since
\[
\frac{d^\ell}{dz^\ell} \left(\frac{x}{e^x - 1}\right)^z = \frac{d^\ell}{dz^\ell} e^{z \ln\left(\frac{x}{e^x - 1}\right)} = (-1)^\ell e^{z \ln\left(\frac{x}{e^x - 1}\right)} \left[\ln\left(\frac{e^x - 1}{x}\right)\right]^\ell,
\]
we have
\[
\frac{d}{dz} B^{(z)}_n|_{z=0} = \frac{-B_n}{n}
\]
and
\[
\frac{d^2}{dz^2} B^{(z)}_n|_{z=0} = \frac{n}{n-1} B_{n-1} + \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k n - k}.
\]

Using Miki’s identity (1.10), we can rewrite (2.5) as
\[
\frac{d^2}{dz^2} B^{(z)}_n|_{z=0} = \frac{n}{n-1} B_{n-1} + \sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k n - k} - 2H_n \frac{B_n}{n}.
\]

**Proof of Theorem 1.2:** Applying the derivation \(d/dz\)|\(_{z=0}\) to both sides of (1.7), then using (2.4), we have
\[
-\frac{B_n}{n} = \frac{1}{2n} \sum_{k=1}^{n} (-1)^k \frac{(2n-1)}{k-1} C_{n,k}.
\]

Mutiplying both sides by \(-2n\), then replacing \(n\) by \(n + 1\), we have
\[
\sum_{k=1}^{n+1} (-1)^{k-1} \frac{(2n+1)}{k-1} C_{n+1,k} = 2B_{n+1}.
\]

By the recurrence relation (1.1), the left-hand side of (2.8) can be rewritten as
\[
\sum_{k=1}^{n+1} (-1)^{k-1} \frac{(2n+1)}{k-1} \left(C_{n,k} + (2n + 2 - k)C_{n,k-1}\right)
= \sum_{k=1}^{n} (-1)^{k-1} \frac{k!(2n+2-k)!}{(2n+1)!} C_{n,k} - \sum_{k=1}^{n} (-1)^{k-1} \frac{k!(2n+1-k)!}{(2n+1)!} \frac{C_{n,k}}{C_{n,k-1}}
= \sum_{k=1}^{n} (-1)^{k-1} \frac{k!(2n+1-k)!}{(2n+1)!} C_{n,k} = \sum_{k=1}^{n} (-1)^{k-1} \frac{(2n+1)}{k} C_{n,k},
\]
as desired. We complete the proof of Theorem 1.2.
Proof of Theorem 1.3: Apply the derivation \( \frac{d^2}{dz^2} \big|_{z=0} \) to both sides of (1.7). By (2.7), we have

\[
\frac{d^2}{dz^2} B_n(z) \big|_{z=0} = \frac{1}{n} \sum_{k=1}^{n} (-1)^k \binom{2n-1}{k-1} C_{n,k} (H_{k-1} - (H_{2n-k} - H_n))
\]

Combining with (2.6), after some arrangements, we complete the proof of Theorem 1.3.

We conclude this paper by the proof of Theorem 1.4. Setting \( z = N \) in (1.3) leads to

\[
\sum_{k=0}^{\infty} B_k(N) x^k \frac{k!}{k!} = \left( \frac{x}{e^x - 1} \right)^N = \left( \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \right)^N
\]

where \( k_1, k_2, \ldots, k_N \) are nonnegative integers.

Hence,

\[
B_n^{(N)} = \sum_{k_1 + k_2 + \cdots + k_N = n} \binom{n}{k_1, k_2, \ldots, k_N} B_{k_1} B_{k_2} \cdots B_{k_N}.
\]

By Theorem 1.1, for \( 1 \leq N \leq n \), we have

\[
B_n^{(N)} = \frac{n!}{(2n)!} \sum_{k=1}^{n} (-1)^k C_{n,k} \binom{N}{k} (-N + n + 1)_{n-k}
\]

Combining with Dilcher’s identity (1.12), we have

\[
\frac{N}{2n} \binom{n}{N} \sum_{k=1}^{n} (-1)^k C_{n,k} \binom{2n-1}{N+k-1} = \frac{N}{2n} \binom{n}{N} \sum_{k=0}^{N-1} (-1)^{N-1-k} s(N, N-k) \frac{B_{n-k}}{n-k}.
\]

Divided both sides by \( \frac{N}{2n} \binom{n}{N} \), we complete the proof of Theorem 1.4.

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