Sequential Composition of Propositional Horn Theories

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Abstract

Rule-based reasoning is an essential part of human intelligence prominently formalized in artificial intelligence research via Horn theories. Describing complex objects as the composition of elementary ones is a common strategy in computer science and science in general. Recently, the author introduced the sequential composition of Horn logic programs for syntactic program composition and decomposition in the context of logic-based analogical reasoning and learning. This paper contributes to the foundations of logic programming, knowledge representation, and database theory by studying the sequential composition of propositional Horn theories. Specifically, we show that the notion of composition gives rise to a family of finite magmas and algebras, baptized Horn magmas and Horn algebras in this paper. On the semantic side, we show that the van Emden-Kowalski immediate consequence operator of a theory can be represented via composition, which allows us to compute its least model semantics without any explicit reference to operators. This bridges the conceptual gap between the syntax and semantics of a propositional Horn theory in a mathematically satisfactory way. Moreover, it gives rise to an algebraic meta-calculus for propositional Horn theories. In a broader sense, this paper is a first step towards an algebra of rule-based logical theories and in the future we plan to adapt and generalize the methods of this paper to wider classes of theories, most importantly to first-, and higher-order logic programs, and non-monotonic logic programs under the stable model or answer set semantics and extensions.

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1. Introduction

Rule-based reasoning is an essential part of human intelligence prominently formalized in artificial intelligence research via Horn theories with important applications in artificial intelligence and computer science in general (cf. [1, 2]). Its rule-like structure naturally induces the compositional structure of such theories as we are going to demonstrate in this paper.

Describing complex objects as the composition of elementary ones is a common strategy in computer science and science in general. In [3], the author introduced the sequential composition of Horn logic programs for syntactic program composition and decomposition in the context of logic-based analogical reasoning and learning. In this paper, we contribute to the foundations of logic programming, knowledge representation, and database theory by studying the sequential composition of propositional Horn theories as a binary operation on theories with unit. We show that interesting transformations of propositional Horn theories can be computed via composition. Moreover, we show that the composition of propositional Krom-Horn theories, which consists only of rules with at most one body atom, is associative and gives rise to the algebraic structure of a monoid. We also show that the restricted class of proper propositional Krom-Horn theories, which only contain rules with exactly one body atom, yields an idempotent semiring (Theorem 8). On the semantic side, we show that the van Emden-Kowalski immediate consequence operator of a theory can be represented via composition (Theorem 31), which allows us to compute its least model semantics without any explicit reference to operators (Theorem 35). This bridges the conceptual gap between the syntax and semantics of a propositional Horn theory in a mathematically satisfactory way. We then proceed by studying decompositions of theories (Section 5). As the main result in this
regard, we show that acyclic theories can be decomposed into a product of single-rule theories (Theorem 27). Finally, we provide some results on decompositions of general propositional Horn theories (Section 5.2).

From a logical point of view, we obtain a meta-calculus for reasoning about propositional Horn theories. From an algebraic point of view, this paper establishes a bridge between propositional Horn theories and algebra, which enables us to transfer algebraic concepts to the setting of propositional Horn theories and extensions thereof.

In a broader sense, this paper is a first step towards an algebra of rule-based logical theories and in the future we plan to adapt and generalize the methods of this paper to wider classes of theories, most importantly for first-, and higher-order logic programs [4, 5, 6, 7], and non-monotonic logic programs under the stable model [8] or answer set semantics and extensions thereof (cf. [9, 10, 11, 12]).

2. Preliminaries

In this section, we recall the syntax and semantics of propositional Horn theories, and the algebraic structures occurring in the rest of the paper.

2.1. Algebraic Structures

We recall some basic algebraic notions and notations by mainly following the lines of [13].

Given two sets $A$ and $B$, we write $A \subseteq_k B$ in case $A$ is a subset of $B$ with $k$ elements, for some non-negative integer $k$, and given an object $o$ of size $k$, we will write $A \subseteq_o B$ instead of $A \subseteq_{\text{size}(o)} B$. We denote the identity function on a set $A$ by $\text{Id}_A$. A permutation of a set $A$ is any mapping $A \rightarrow A$ which is one-to-one and onto.

A partially ordered set (or poset) is a set $L$ together with a reflexive, transitive, and anti-symmetric binary relation $\leq$ on $L$. A prefixed point of an operator $f$ on a poset $L$ is any element $x \in L$ such that $f(x) \leq x$; moreover, we call any $x \in L$ a fixed point of $f$ if $f(x) = x$. 

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A magma is a set $M$ together with a binary operation $\cdot$ on $M$. We call $(M,\cdot,1)$ a unital magma if it contains a unit element 1 such that $1x = x1 = x$ holds for all $x \in M$. A semigroup is a magma $(S,\cdot)$ in which $\cdot$ is associative. A monoid is a semigroup containing a unit element 1 such that $1x = x1 = x$ holds for all $x$. A group is a monoid which contains an inverse $x^{-1}$ for every $x$ such that $xx^{-1} = x^{-1}x = 1$. A left (resp., right) zero is an element 0 such that $0x = 0$ (resp., $x0 = 0$) holds for all $x \in S$. An ordered semigroup is a semigroup $S$ together with a partial order $\leq$ that is compatible with the semigroup operation, meaning that $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $x, y, z \in S$. An ordered monoid is defined in the obvious way. A non-empty subset $I$ of $S$ is called a left (resp., right) ideal if $SI \subseteq I$ (resp., $IS \subseteq I$), and a (two-sided) ideal if it is both a left and right ideal. An element $x \in S$ is idempotent if $x + x = x$.

A seminearring is a set $S$ together with two binary operations $+$ and $\cdot$ on $S$, and a constant $0 \in S$, such that $(S,+,0)$ is a monoid and $(S,\cdot)$ is a semigroup satisfying the following laws:

1. $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x, y, z \in S$ (right-distributivity); and

2. $0 \cdot x = 0$ for all $x \in S$ (left zero).

We say that $S$ is idempotent if $x + x = x$ holds for all $x \in S$. A semiring is a seminearring $(S,+,\cdot,0)$ such that $+$ is commutative and additionally to the laws of a seminearring the following laws are satisfied:

1. $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in S$ (left-distributivity); and

2. $x \cdot 0 = 0$ for all $x \in S$ (right zero).

2.2. Propositional Horn Theories

We recall the syntax and semantics of propositional Horn theories.

2.2.1. Syntax

In the rest of the paper, $A$ denotes a finite alphabet of propositional atoms.
A \textit{(propositional Horn) theory} over \(A\) is a finite set of \textit{rules} of the form
\[ a_0 \leftarrow a_1, \ldots, a_k, \quad k \geq 0, \tag{1} \]
where \(a_0, \ldots, a_k \in A\) are propositional atoms. It will be convenient to define, for a rule \(r\) of the form (1), \(\text{head}(r) = \{a_0\}\) and \(\text{body}(r) = \{a_1, \ldots, a_k\}\) extended to theories by \(\text{head}(P) = \bigcup_{r \in P} \text{head}(r)\) and \(\text{body}(P) = \bigcup_{r \in P} \text{body}(r)\). In this case, the \textit{size} of \(r\) is \(k\). A \textit{fact} is a rule with empty body and a \textit{proper rule} is a rule which is not a fact. We denote the facts and proper rules in \(P\) by \(\text{facts}(P)\) and \(\text{proper}(P)\), respectively. We call a rule \(r\) of the form (1) \textit{Krom} if it has at most one body atom, and we call \(r\) \textit{binary} if it contains at most two body atoms. A \textit{tautology} is any Krom rule of the form \(a \leftarrow a\), for \(a \in A\). We call a theory \textit{Krom} if it contains only Krom rules, and we call it \textit{binary} if it consists only of binary rules. A theory is called a \textit{single-rule theory} if it contains exactly one non-tautological rule. We call a theory \textit{minimalistic} if it contains at most one rule for each rule head.

For two propositional Horn theories \(P\) and \(R\), we say that \(P\) \textit{depends on} \(R\) if the intersection of \(\text{body}(P)\) and \(\text{head}(R)\) is not empty. We call \(P\) \textit{acyclic} if there is a mapping \(\ell : A \rightarrow \{0, 1, 2, \ldots\}\) such that for each rule \(r \in P\), we have \(\ell(\text{head}(r)) > \ell(\text{body}(r))\), and in this case we call \(\ell\) a \textit{level mapping} for \(P\). Of course, every level mapping \(\ell\) induces an ordering on rules via \(r \leq_\ell s\) if \(\ell(\text{head}(r)) \leq_\ell (\text{head}(s))\). We can transform this ordering into a total ordering by arbitrarily choosing a particular linear ordering within each level, that is, if \(\ell(a) = \ell(b)\) then we can choose between \(a <_\ell b\) or \(b <_\ell a\).

Define the \textit{dual} of \(P\) by
\[ P^d = \text{facts}(P) \cup \{b \leftarrow \text{head}(r) \mid r \in \text{proper}(P) : b \in \text{body}(r)\}. \]
Roughly, we obtain the dual of a theory by reversing all the arrows of its proper rules.

\footnote{Krom rules where first introduced and studied by \[14\].}
2.2.2. Semantics

An interpretation is any set of atoms from $A$. We define the entailment relation, for every interpretation $I$, inductively as follows: (i) for an atom $a$, $I \models a$ if $a \in I$; (ii) for a set of atoms $B$, $I \models B$ if $B \subseteq I$; (iii) for a rule $r$ of the form (1), $I \models r$ if $I \models \text{body}(r)$ implies $I \models \text{head}(r)$; and, finally, (iv) for a propositional Horn theory $P$, $I \models P$ if $I \models r$ holds for each rule $r \in P$. In case $I \models P$, we call $I$ a model of $P$. The set of all models of $P$ has a least element with respect to set inclusion called the least model of $P$ and denoted by $LM(P)$. We say that $P$ and $R$ are equivalent if $LM(P) = LM(R)$.

Define the van Emden-Kowalski operator [15] of $P$, for every interpretation $I$, by

$$T_P(I) = \{\text{head}(r) \mid r \in P : I \models \text{body}(r)\}.$$  

We have the following well-known operational characterization of models in terms of the van Emden-Kowalski operator [15].

**Proposition 1.** An interpretation $I$ is a model of $P$ iff $I$ is a prefixed point of $T_P$.

We call an interpretation $I$ a supported model of $P$ if $I$ is a fixed point of $T_P$. We say that $P$ and $R$ are subsumption equivalent if $T_P = T_R$.

The following constructive characterization of least models is due to [15].

**Proposition 2.** The least model of a propositional Horn theory coincides with the least fixed point of its associated van Emden-Kowalski operator, that is, for any theory $P$ we have

$$LM(P) = \text{lfp}(T_P).$$  

3. Composition

This is the main section of the paper in which we define the sequential composition of propositional Horn theories.
Notation 3. In the rest of the paper, $P$ and $R$ denote propositional Horn theories over some joint alphabet $A$.

We are ready to introduce the main notion of the paper.

Definition 4. We define the (sequential) composition of $P$ and $R$ by

$$P \circ R = \{\text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq_r R : \text{head}(S) = \text{body}(r)\}.$$ 

We will write $PR$ in case the composition operation is understood.

Roughly, we obtain the composition of $P$ and $R$ by resolving all body atoms in $P$ with ‘matching’ rule heads of $R$. Notice that we can reformulate sequential composition as

$$P \circ R = \bigcup_{r \in P} (\{r\} \circ R), \tag{3}$$

which directly implies right-distributivity of composition, that is,

$$(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R) \quad \text{holds for all propositional Horn theories } P, Q, R. \tag{4}$$

However, the following counter-example shows that left-distributivity fails in general:

$$\{a \leftarrow b, c\} \circ (\{b\} \cup \{c\}) = \{a\} \quad \text{and} \quad (\{a \leftarrow b, c\} \circ \{b\}) \cup (\{a \leftarrow b, c\} \circ \{c\}) = \emptyset.$$

We can write $P$ as the union of its facts and proper rules, that is,

$$P = \text{facts}(P) \cup \text{proper}(P). \tag{5}$$

Hence, we can rewrite the composition of $P$ and $R$ as

$$P \circ R = (\text{facts}(P) \cup \text{proper}(P))R \tag{6} \equiv \text{facts}(P)R \cup \text{proper}(P)R \tag{7} \equiv \text{facts}(P) \cup \text{proper}(P)R,$$ \tag{8}
which shows that the facts in $P$ are preserved by composition, that is, we have

$$\text{facts}(P) \subseteq \text{facts}(P \circ R).$$  \hfill (9)

Define the unit theory (over $A$) by the propositional Krom-Horn theory

$$1_A = \{a \leftarrow a \mid a \in A\}.$$

In the sequel, we will often omit the reference to $A$.

We are now ready to state the main structural result of the paper.

**Theorem 5.** The space of all propositional Horn theories over some fixed alphabet forms a finite unital magma with respect to composition ordered by set inclusion with the neutral element given by the unit theory. Moreover, the empty theory is a left zero and composition distributes from the right over union, that is, for any propositional Horn theories $P, Q, R$ we have

$$(P \cup R) \circ Q = (P \circ Q) \cup (R \circ Q).$$  \hfill (10)

**Proof.** The space of all propositional Horn theories is obviously closed under composition, which shows that it forms a magma.

We proceed by proving that 1 is neutral with respect to composition. By definition of composition, we have

$$P \circ 1 = \{\text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq r \, 1 : \text{head}(S) = \text{body}(r)\}.$$

Now, by definition of 1 and $S \subseteq r$, we have $\text{head}(S) = \text{body}(S)$ and therefore $\text{body}(S) = \text{body}(r)$. Hence,

$$P \circ 1 = P.$$

Similarly, we have

$$1 \circ P = \{\text{head}(r) \leftarrow \text{body}(S) \mid r \in 1, S \subseteq 1 P : \text{head}(S) = \text{body}(r)\}.$$
As $S$ is a subset of $P$ with a single element, $S$ is a singleton $S = \{s\}$, for some rule $s$ with $\text{head}(s) = \text{body}(r)$ and, since $\text{body}(r) = \text{head}(r)$ holds for every rule in $1$, $\text{head}(s) = \text{head}(r)$. Hence,

$$1 \circ P = \{ \text{head}(r) \leftarrow \text{body}(s) \mid r \in 1, s \in P : \text{head}(s) = \text{body}(r) \}$$

$$= \{ \text{head}(s) \leftarrow \text{body}(s) \mid s \in P \}$$

$$= P.$$

This shows that 1 is neutral with respect to composition. That composition is ordered by set inclusion is obvious. We now turn our attention to the operation of union. In [1] we argued for the right-distributivity of composition. That the empty set is a left zero is obvious. As union is idempotent, the set of all propositional Horn theories forms the structure of a finite idempotent seminearring.

The following example shows that, unfortunately, composition is not associative (but see Theorem 7).

**Example 6.** Consider the rule

$$r := a \leftarrow b, c$$

and the theories

$$P := \begin{cases} b \leftarrow b \\ c \leftarrow b, c \end{cases} \quad \text{and} \quad R := \begin{cases} b \leftarrow d \\ b \leftarrow e \\ c \leftarrow f \end{cases}.$$ 

Simple computations yield

$$\{r\}(PR) = \begin{cases} a \leftarrow d, f \\ a \leftarrow e, f \\ a \leftarrow d, e, f \end{cases} \neq \begin{cases} a \leftarrow d, f \\ a \leftarrow e, f \end{cases} = (\{r\}P)R.$$

### 3.1. Minimalistic Theories

Recall that we call a theory minimalistic if it contains at most one rule for each head atom. Theorem 7 shows that minimalistic theories are relevant.
Given a minimalistic theory $M$, in the computation of $P \circ M$ we can omit the reference to $r$ in $\subseteq_r$ in Definition 4, that is, we have

$$P \circ M = \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq M : \text{head}(S) = \text{body}(r) \}.$$  

In Example 6 we have seen that composition is not associative in general. The situation changes for minimalistic theories.

**Theorem 7.** For any propositional Horn theories $P, M, N$ if $M$ and $N$ are minimalistic, then

$$(PM)N = P(MN).$$  

**Proof.** As a consequence of (4), a rule $r$ is in $(PM)N$ iff

$$\{r\} = \{a_0 \leftarrow a_1, \ldots, a_k\} \circ \left( \begin{array}{l} a_1 \leftarrow B_1 \\ \vdots \\ a_k \leftarrow B_k \end{array} \right) \circ \left( \begin{array}{l} b_1 \leftarrow C_1 \\ \vdots \\ b_m \leftarrow C_m \end{array} \right),$$  

for some rule $a_0 \leftarrow a_1, \ldots, a_k \in P$, $k \geq 0$, $b_1 \leftarrow B_1, \ldots, b_k \leftarrow B_k \in M$, $b_1 \leftarrow C_1, \ldots, b_m \leftarrow C_m \in N$, and $B_1 \cup \ldots \cup B_k = \{b_1, \ldots, b_m\}$, $m \geq 0$. Since

$$\left( \begin{array}{l} a_1 \leftarrow B_1 \\ \vdots \\ a_k \leftarrow B_k \end{array} \right) \text{ and } \left( \begin{array}{l} b_1 \leftarrow C_1 \\ \vdots \\ b_m \leftarrow C_m \end{array} \right)$$  

are minimalistic, we can rewrite (12) as

$$\{r\} = \{a_0 \leftarrow a_1, \ldots, a_k\} \circ \left( \begin{array}{l} a_1 \leftarrow B_1 \\ \vdots \\ a_k \leftarrow B_k \end{array} \right) \circ \left( \begin{array}{l} b_1 \leftarrow C_1 \\ \vdots \\ b_m \leftarrow C_m \end{array} \right),$$  

which shows $r \in P(MN)$. The proof that every rule in $P(MN)$ is in $(PM)N$ is analogous. 

3.2. **Propositional Krom-Horn Theories**

Recall that we call a propositional Horn theory **Krom** if it contains only rules with at most one body atom. This includes interpretations, unit theories, and permutations.
For a propositional Krom-Horn theory $K$ and a propositional Horn theory $P$, sequential composition simplifies to

$$K \circ P = \text{facts}(K) \cup \{a \leftarrow B \mid a \leftarrow b \in K, b \leftarrow B \in P, a, b \in A, B \subseteq A\}.$$  

We have the following structural result as a specialization of Theorem 5.

**Theorem 8.** The space of all propositional Krom-Horn theories forms a monoid with respect to composition with the neutral element given by the unit theory, and it forms a seminearring with respect to sequential composition and union. More generally, we have

$$K(PR) = (KP)R,$$

(13)

for arbitrary propositional Horn theories $P$ and $R$. Moreover, Krom theories distribute from the left, that is, for any propositional Horn theories $P$ and $R$, we have

$$K(P \cup R) = KP \cup KR.$$

(14)

This implies that the space of proper propositional Krom-Horn theories forms a finite idempotent semiring.

**Proof.** We first prove (13), which implies associativity for propositional Krom-Horn theories. A rule $r$ is in $K(PR)$ iff

$$\{r\} = \{a \leftarrow b\} \circ \left(\{b \leftarrow b_1, \ldots, b_k\} \circ \left\{\begin{array}{c} b_1 \leftarrow B_1 \\ \vdots \\ b_k \leftarrow B_k \end{array}\right\}\right),$$

for some $a \leftarrow b \in K$, $b \leftarrow b_1, \ldots, b_k \in P$, and $b_1 \leftarrow B_1, \ldots, b_k \leftarrow B_k \in R$, in which case we have $r = a \leftarrow B_1 \cup \ldots \cup B_k$. A simple computation shows

$$\{r\} = (\{a \leftarrow b\} \circ \{b \leftarrow b_1, \ldots, b_k\}) \circ \left\{\begin{array}{c} b_1 \leftarrow B_1 \\ \vdots \\ b_k \leftarrow B_k \end{array}\right\}.$$  

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The proof that every rule in $(KPR)$ is in $KPR$ is analogous.

We now want to prove (14). For any proper Krom rule $r = a \leftarrow b$, we have

$$\{a \leftarrow b\} \circ (P \cup R) = \{a \leftarrow B \mid b \leftarrow B \in P \cup R, B \subseteq A\}$$

$$= \{a \leftarrow B \mid b \leftarrow B \in P\} \cup \{a \leftarrow B \mid b \leftarrow B \in R\}$$

$$= \{(a \leftarrow b) \circ P\} \cup \{(a \leftarrow b) \circ R\}.$$

Hence, we have

$$K(P \cup R) = \text{facts}(K) \cup \text{proper}(K)(P \cup R)$$

$$= \text{facts}(K) \cup \bigcup_{r \in \text{proper}(K)} \{r\}(P \cup R)$$

$$= \text{facts}(K) \cup \bigcup_{r \in \text{proper}(K)} (\{r\}P \cup \{r\}R)$$

$$= \text{facts}(K) \cup \bigcup_{r \in \text{proper}(K)} \{r\}P \cup \bigcup_{r \in \text{proper}(K)} \{r\}R$$

$$= \text{facts}(K) \cup \text{proper}(K)P \cup \text{proper}(K)R$$

$$= (\text{facts}(K) \cup \text{proper}(K)P) \cup (\text{facts}(K) \cup \text{proper}(K)R)$$

$$= KP \cup KR$$

where the sixth identity follows from the idempotency of union. This shows that Krom theories distribute from the left which implies that the space of all proper Krom theories forms a semiring (cf. Theorem 5). □

3.2.1. Interpretations

Formally, interpretations are propositional Krom-Horn theories containing only rules with empty bodies (i.e., facts), which gives interpretations a special compositional meaning.

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3If $K$ contains facts, then $K \circ \emptyset$ violates the last axiom of a semiring (cf. Section 2.1).
Proposition 9. Every interpretation $I$ is a left zero, that is, for any propositional Horn theory $P$, we have

$$I \circ P = I.$$ 

Consequently, the space of interpretations forms a right ideal.

Proof. We compute using reducts (see the forthcoming Section 4.1)

$$IP = I^{\text{head}(P)} \circ I^{\text{body}(I)} P = I \circ \emptyset P = I \circ \emptyset = \text{facts}(I) = I.$$ 

Corollary 10. A propositional Horn theory $P$ commutes with an interpretation $I$ iff $I$ is a supported model of $P$, that is,

$$PI = IP \iff I \in \text{Supp}(P).$$

Proof. A direct consequence of Proposition 9 and the forthcoming Theorem 31.

3.2.2. Permutations

With every permutation $\pi : A \to A$, we associate the propositional Krom-Horn theory

$$\pi = \{ \pi(a) \leftarrow a \mid a \in A \}.$$ 

We adopt here the standard cycle notation for permutations. For instance, we have

$$\pi_{(a,b)} = \begin{cases} a \leftarrow b \\ b \leftarrow a \end{cases} \quad \text{and} \quad \pi_{(a,b,c)} = \begin{cases} a \leftarrow b \\ b \leftarrow c \\ c \leftarrow a \end{cases}.$$ 

The inverse $\pi^{-1}$ of $\pi$ translates into the language of theories as

$$\pi^{-1} = \pi^d.$$ 

4See Corollary 33.
Interestingly, we can rename the atoms occurring in a theory via permutations and composition by

\[ \pi \circ P \circ \pi^d = \{ \pi(\text{head}(r)) \leftarrow \pi(\text{body}(r)) \mid r \in P \}. \]

We have the following structural result as a direct instance of the more general result for permutations.

**Proposition 11.** The space of all permutation theories forms a group.

### 3.3. Idempotent Theories

Recall that we call \( P \) idempotent if \( P^2 = P \). We investigate here the most basic properties of idempotent theories.

**Proposition 12.** A propositional Horn theory \( P \) is idempotent iff

\[
\text{proper}(P) \subseteq \text{facts}(P) \quad \text{and} \quad \text{proper}(P) = \text{proper}(\text{proper}(P) P).
\]

(17)

**Proof.** The theory \( P \) is idempotent iff we have (i) \( \text{facts}(P^2) = \text{facts}(P) \) and (ii) \( \text{proper}(P^2) = \text{proper}(P) \). For the first condition in (17), we compute

\[
\text{facts}(P^2) \overset{25}{=} P^2\emptyset = P(P\emptyset) = P \circ \text{facts}(P) \overset{5}{=} (\text{facts}(P) \cup \text{proper}(P)) \text{facts}(P) \overset{\text{H. 15}}{=} \text{facts}(P) \cup \text{proper}(P) \text{facts}(P).
\]

For the second condition in (17), we compute

\[
\text{proper}(P^2) = \text{proper}(\text{proper}(P) P) \overset{41, 46}{=} \text{proper}(\text{facts}(P) \cup \text{proper}(P) P) = \text{proper}(\text{proper}(P) P).
\]

\[ \square \]

**Corollary 13.** Every interpretation is idempotent.

**Proof.** A direct consequence of Proposition 12 and of 15.

\[ \square \]
3.4. Proper Theories

By the forthcoming equation (25) we can extract the facts of a propositional Horn theory $P$ by computing the right reduct of $P$ with respect to the empty set, $P^\emptyset$, and by sequentially composing $P$ with the empty set, $P \circ \emptyset$. Unfortunately, there is no analogous characterization of the proper rules in terms of composition.

The proper rules operator satisfies the following identities, for any propositional Horn theories $P$ and $R$, and interpretation $I$:

\[
\text{proper} \circ \text{proper} = \text{proper} \quad (18)
\]
\[
\text{proper} \circ \text{facts} = \emptyset \quad (19)
\]
\[
\text{proper}(1) = 1 \quad (20)
\]
\[
\text{proper}(I) = \emptyset \quad (21)
\]
\[
\text{proper}(P) \circ \emptyset = \text{proper}(P^\emptyset) = \emptyset \quad (22)
\]
\[
\text{proper}(P \cup R) = \text{proper}(P) \cup \text{proper}(R). \quad (23)
\]

Of course, we have $\text{proper}(P) = P$ iff $P$ contains no facts, that is, if $P^\emptyset = \emptyset$. The last identity (23) says that the proper rules operator is compatible with union; however, the following counter-example shows that it is not compatible with sequential composition:

\[
\text{proper} \left( \{a \leftarrow b, c\} \circ \left\{ \begin{array}{l}
b \leftarrow b \\
c
\end{array} \right\} \right) = \{a \leftarrow b\}
\]

whereas

\[
\text{proper}(\{a \leftarrow b, c\}) \circ \text{proper} \left( \left\{ \begin{array}{l}
b \leftarrow b \\
c
\end{array} \right\} \right) = \emptyset.
\]

Proposition 14. The space of all proper propositional Horn theories forms a subsemiring of the space of all propositional Horn theories with zero given by the empty set.

Proof. The space of proper theories is closed under composition. It remains to show that the empty set is a zero, but this follows from the fact that it is a left
zero by (15) and from the observation that for any proper theory $P$, we have $P \circ \emptyset = \text{facts}(P) = \emptyset$.

4. Algebraic Transformations

In this section, we study algebraic operations on theories expressible via composition and other operators.

Our first observation is that we can compute the heads and bodies via

$$\text{head}(P) = PA \quad \text{and} \quad \text{body}(P) = \text{proper}(P)^d A.$$  \hspace{1cm} (24)

Moreover, we have

$$\text{head}(PR) \subseteq \text{head}(P) \quad \text{and} \quad \text{body}(PR) \subseteq \text{body}(R).$$

4.1. Reducts

Reducing the rules of a theory to a restricted alphabet is a fundamental operation on theories which can be algebraically computed via composition (Proposition 16).

**Definition 15.** We define the left and right reduct of a propositional Horn theory $P$, with respect to some interpretation $I$, respectively by

$$I^P = \{ r \in P \mid I \models \text{head}(r) \} \quad \text{and} \quad P^I = \{ r \in P \mid I \models \text{body}(r) \}.$$  

Our first observation is that we can compute the facts of $P$ via the right reduct with respect to the empty set, that is, we have

$$\text{facts}(P) = P^\emptyset = P \circ \emptyset.$$  \hspace{1cm} (25)

On the contrary, computing the left reduct with respect to the empty set yields

$$\emptyset^P = \emptyset.$$  \hspace{1cm} (26)

Moreover, for any interpretations $I$ and $J$, we have

$$J^I = I \cap J \quad \text{and} \quad I^J = I.$$  \hspace{1cm} (27)
Notice that we obtain the reduction of $P$ to the atoms in $I$, denoted $P|_I$, by

$$P|_I = ^I(P^I) = ( ^I P )^I.$$  \hfill (28)

As the order of computing left and right reducts is irrelevant, in the sequel we will omit the parentheses in (28). Of course, we have

$$A^P = P^A = ^A P^A = P.$$  

Moreover, we have

$$1^I = ^I 1 = ^I 1_I = 1|_I$$

(29)

$$1^I \circ 1^J = 1^{I \cap J} = 1^I \circ 1^I = 1^I \cap 1^J$$

(30)

$$1^I \cup 1^J = 1^{I \cup J}.$$  \hfill (31)

We now want to relate reducts to composition and union.

**Proposition 16.** For any propositional Horn theory $P$ and interpretation $I$, we have

$$^I P = 1^I \circ P \quad \text{and} \quad P^I = P \circ 1^I.$$  \hfill (32)

Moreover, we have

$$P|_I = 1^I \circ P \circ 1^I.$$  \hfill (33)

Consequently, for any propositional Horn theory $R$, we have

$$^I (P \cup R) = ^I P \cup ^I R \quad \text{and} \quad ^I (P \circ R) = ^I P \circ R$$

(34)

$$(P \cup R)^I = P^I \cup R^I \quad \text{and} \quad (P \circ R)^I = P \circ R^I.$$  \hfill (35)

**Proof.** We compute

$$(^I P) = \{ \text{head}(r) \leftarrow \text{body}(s) \mid r \in 1^I, s \in P : \text{head}(s) = \text{body}(r) \}$$

$$= \{ \text{head}(r) \leftarrow \text{body}(s) \mid r \in 1, \ s \in P : \text{head}(s) = \text{body}(r), \text{head}(r) = \text{body}(r) \in I \}$$

$$= \{ \text{head}(s) \leftarrow \text{body}(s) \mid s \in P : \text{head}(s) \in I \}$$

$$= \{ s \in P \mid I \models \text{head}(s) \}$$

$$= ^I P.$$  \hfill (36)
Similarly, we compute

\[ P \circ 1^I = \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq_r 1^I : \text{head}(S) = \text{body}(r) \} \]  

(36)

\[ = \left\{ \begin{array}{l}
\text{head}(r) \leftarrow \text{body}(S) \\
r \in P, S \subseteq_r 1 \\
\text{head}(S) = \text{body}(r) \\
\text{head}(S) = \text{body}(S) \subseteq I
\end{array} \right\}. \]  

(37)

By definition of 1, we have \( \text{body}(S) = \text{head}(S) \) and therefore \( \text{body}(S) = \text{body}(r) \)
and \( \text{body}(r) \subseteq I \). Hence, (36) is equivalent to

\[ \{ \text{head}(r) \leftarrow \text{body}(r) \mid r \in P : I \models \text{body}(r) \} = P^I. \]

Finally, we have

\[ I(P \cup R) = I^1(P \cup R) = I^1P \cup I^1R = I^1P \cup I^1R \]

and

\[ I(PR) = (I^1)(PR) = ((I^1)P)R = I^1PR \]

with the remaining identities in (35) holding by analogous computations.

Consequently, we can simplify the computation of composition as follows.

**Corollary 17.** For any propositional Horn theories \( P \) and \( R \), we have

\[ P \circ R = P^{\text{head}(R)} \circ \text{body}(P)R. \]  

(38)

4.2. Adding and Removing Body Atoms

We now want to study algebraic transformations of rule bodies. For this, first notice that we can manipulate rule bodies via composition on the right. For example, we have

\[ \{a \leftarrow b, c\} \circ \left\{ \begin{array}{l}
b \leftarrow b \\
c \end{array} \right\} = \{a \leftarrow b\}. \]

The general construction here is that we add a tautological rule \( b \leftarrow b \) for every body atom \( b \) of \( P \) which we want to preserve, and we add a fact \( c \) in case we want
to remove $c$ from the rule bodies in $P$. We therefore define, for an interpretation $I$, the minimalistic theory

$$I^\ominus = 1^{A-I} \cup I.$$ 

Notice that $I^\ominus$ is computed with respect to some fixed alphabet $A$. For instance, we have

$$A^\ominus = A \quad \text{and} \quad \emptyset^\ominus = 1.$$ 

The first equation yields another explanation for (24), that is, we can compute the heads in $P$ by removing all body atoms of $P$ via

$$\text{head}(P) = PA^\ominus = PA.$$ 

Interestingly enough, we have

$$I^\ominus I = (1^{A-I} \cup I) I \equiv 1^{A-I} I \cup I^2 \equiv ((A-I) \cap I) \cup I = I$$

and

$$I^\ominus P = A-I P \cup I.$$ 

Moreover, in the example above, we have

$$\{c\}^\ominus = \left\{ \begin{array}{l} a \leftarrow a \\ b \leftarrow b \\ c \end{array} \right\} \quad \text{and} \quad \{a \leftarrow b, c\} \cap \{c\}^\ominus = \{a \leftarrow b\}$$

as desired. Notice also that the facts of a theory are, of course, not affected by composition on the right, that is, we cannot expect to remove facts via composition on the right (cf. (9)).

We have the following general result.

**Proposition 18.** For any propositional Horn theory $P$ and interpretation $I$, we have

$$PI^\ominus = \{\text{head}(r) \leftarrow \text{body}(r) - I \mid r \in P\}.$$
In analogy to the above construction, we can add body atoms via composition on the right. For example, we have

\[ \{a \leftarrow b\} \circ \{b \leftarrow b, c\} = \{a \leftarrow b, c\}. \]

Here, the general construction is as follows. For an interpretation \( I \), define the minimalistic theory

\[ I^\oplus = \{a \leftarrow (\{a\} \cup I) \mid a \in A\}. \]

For instance, we have

\[ A^\oplus = \{a \leftarrow A \mid a \in A\} \quad \text{and} \quad \emptyset^\oplus = 1. \]

Interestingly enough, we have

\[ I^\oplus I^\ominus = I^\ominus \quad \text{and} \quad I^\oplus I = I. \]

Moreover, in the example above, we have

\[ \{c\}^\oplus = \begin{cases} a \leftarrow a, c \\ b \leftarrow b, c \\ c \leftarrow c \end{cases} \quad \text{and} \quad \{a \leftarrow b\} \circ \{c\}^\oplus = \{a \leftarrow b, c\} \]

as desired. As composition on the right does not affect the facts of a theory, we cannot expect to append body atoms to facts via composition on the right. However, we can add arbitrary atoms to all rule bodies simultaneously and in analogy to Proposition 18, we have the following general result.

**Proposition 19.** For any propositional Horn theory \( P \) and interpretation \( I \), we have

\[ PI^\oplus = facts(P) \cup \{\text{head}(r) \leftarrow (\text{body}(r) \cup I) \mid r \in \text{proper}(P)\}. \]

We now want to illustrate the interplay between the above concepts with an example.
Example 20. Consider the propositional Horn theories

\[ P = \left\{ \begin{array}{l}
c \\
a \leftarrow b, c \\
b \leftarrow a, c 
\end{array} \right\} \quad \text{and} \quad \pi_{(a,b)} = \left\{ \begin{array}{l}
a \leftarrow b \\
b \leftarrow a 
\end{array} \right\}. \]

Roughly, we obtain \( P \) from \( \pi_{(a,b)} \) by adding the fact \( c \) to \( \pi_{(a,b)} \) and to each body rule in \( \pi_{(a,b)} \). Conversely, we obtain \( \pi_{(a,b)} \) from \( P \) by removing the fact \( c \) from \( P \) and by removing the body atom \( c \) from each rule in \( P \). This can be formalized as:

\[ P = \{c\}^* \pi_{(a,b)} \{c\}^\oplus \quad \text{and} \quad \pi_{(a,b)} = 1^{\{a,b\}} P \{c\}^\oplus. \]

5. Decomposition

In this section, we study sequential decompositions of theories. Notice that it is not possible to add arbitrary rules to a theory via composition, so the goal is to develop techniques for the algebraic manipulation of theories.

The following construction will be useful.

Definition 21. Define the closure of \( P \) with respect to \( A \) by

\[ c\text{l}_A(P) = 1_A \cup P. \]

Roughly, we obtain the closure of a theory by adding all possible tautological rules of the form \( a \leftarrow a \), for \( a \in A \). Of course, the closure operator preserves semantical equivalence.

Notation 22. We make the convention that for any \( n \geq 3 \),

\[ P^n := (((PP)P)\ldots P)P \quad \text{(n times)}. \]

\[^5\text{Here, we have}\ \{c\}^\ast = 1 \cup \{c\} \text{ and } \{c\}^\ast \pi_{(a,b)} = \pi_{(a,b)} \cup \{c\} \text{ by the forthcoming equation \[21\].} \]
We first start by observing that every rule $r = a_0 \leftarrow a_1, \ldots, a_k$, $k \geq 3$, can be decomposed into a product of binary theories by

$$\{r\} = \{a_0 \leftarrow a_1, \ldots, a_k\} = \{a_0 \leftarrow a_1, a_2\} \circ \prod_{i=2}^{k-1} cl_{\{a_i, \ldots, a_{i-1}\}}(\{a_i \leftarrow a_i, a_{i+1}\}).$$

In the next two subsections we study decompositions of acyclic and arbitrary theories.

5.1. Acyclic Theories

We first want to study decompositions of acyclic theories which have the characteristic feature that their rules can be linearly ordered via a level mapping, which means that there is an acyclic dependency relation between the rules of the theory.

**Lemma 23.** For any propositional Horn theories $P$ and $R$, if $P$ does not depend on $R$, we have

$$P(Q \cup R) = PQ.$$  \hspace{1cm} (39)

**Proof.** We compute

$$P(Q \cup R) = P_{\text{head}(Q \cup R)} \circ_{\text{body}(P)} (Q \cup R) = P_{\text{head}(Q \cup R)} \circ (\text{body}(P)Q \cup \text{body}(P)R).$$ \hspace{1cm} (40)

Since $P$ does not depend on $R$, we have $\text{body}(P) \cap \text{head}(R) = \emptyset$ and, hence,

$$\text{body}(P)R = \{r \in R \mid \text{body}(P) \models \text{head}(r)\} = \{r \in R \mid \text{head}(r) \in \text{body}(P)\} = \emptyset.$$

Moreover, we have

$$P_{\text{head}(Q \cup R)} = P_{\text{head}(Q) \cup \text{head}(R)}$$

$$= \{r \in P \mid \text{body}(r) \subseteq \text{head}(Q) \cup \text{head}(R)\}$$

$$= \{r \in P \mid \text{body}(r) \subseteq \text{head}(Q)\}$$

$$= P_{\text{head}(Q)}.$$
Hence, (40) is equivalent to
\[ P^{\text{head}(Q) \circ \text{body}(P)} Q \overset{\text{38}}{=} PQ. \]

\[ \square \]

**Lemma 24.** For any propositional Horn theories \( P \) and \( R \), in case \( P \) does not depend on \( R \), we have
\[ P \cup R = cl_{\text{head}(R)}(P) cl_{\text{body}(P)}(R) \quad \text{and} \quad cl_A(P \cup R) = cl_A(P) cl_A(R). \quad (41) \]
Moreover, for any alphabets \( A \) and \( B \), we have
\[ cl_A(cl_B(P)) = cl_{A \cup B}(P). \]

**Proof.** We compute
\[ cl_{\text{head}(R)}(P) cl_{\text{body}(P)}(R) = (1_{\text{head}(R)} \cup P)(1_{\text{body}(P)} \cup R) \]
\[ = 1_{\text{head}(R)} 1_{\text{body}(P)} \cup P \]
\[ = 1_{\text{head}(R)} 1_{\text{body}(P)} \cup 1_{\text{head}(R)} R \cup P \]
\[ = 1_{\text{head}(R)} 1_{\text{body}(P)} \cup R \cup P \]
\[ = R \cup P \]

where the fourth equality follows from \( 1_{\text{head}(R)} 1_{\text{body}(P)} = \emptyset \) and \( \text{head}(R) R = R \),
the fifth equality holds since \( P \) does not depend on \( R \) (Lemma 23), and the last equality follows from \( P^{\text{body}(P)} = P \).

For the second equality, we compute
\[ cl_A(P) cl_A(R) = (1 \cup P)(1 \cup R) \]
\[ = 1 \cup R \cup P \]
\[ = cl_A(P \cup R) \]

where the third identity follows from the fact that \( P \) does not depend on \( R \) (Lemma 23).
Lastly, we compute

\[ cl_A(cl_B(P)) = 1_A \cup 1_B \cup B = 1_{A \cup B} \cup P = cl_{A \cup B}(P). \]

**Example 25.** We define a family of acyclic theories over \( A \), which we call elevator theories, as follows: given a sequence \((a_1, \ldots, a_n) \in A^n\), \( 1 \leq n \leq |A| \), of distinct atoms, let \( E_{(a_1, \ldots, a_n)} \) be the acyclic theory

\[ E_{(a_1, \ldots, a_n)} = \{a_1\} \cup \{a_i \leftarrow a_{i-1} \mid 2 \leq i \leq n\}. \]

So, for instance, \( E_{(a,b,c)} \) is the theory

\[ E_{(a,b,c)} = \begin{cases} a \\ b \leftarrow a \\ c \leftarrow b \end{cases}. \]

The mapping \( \ell \) given by \( \ell(a) = 1 \), \( \ell(b) = 2 \), and \( \ell(c) = 3 \) is a level mapping for \( E_{(a,b,c)} \) and we can decompose \( E_{(a,b,c)} \) into a product of single-rule theories by

\[ E_{(a,b,c)} = \left( cl_{\{b,c\}}(\{a\}) \right) \left( cl_{\{c\}}(\{b \leftarrow a\}) \right) \left( cl_{\{c\}}(\{c \leftarrow b\}) \right). \]

We now want to generalize the reasoning pattern of Example 25 to arbitrary acyclic theories. For this, we will need the following lemma.

**Lemma 26.** For any propositional Horn theories \( P \) and \( R \) and alphabet \( B \), if \( P \) and \( 1_B \) do not depend on \( R \), we have

\[ cl_B(cl_{head(R)}(P)cl_{body(P)}(R)) = cl_{B \cup head(R)}(P)cl_{B \cup body(P)}(R). \] (42)

**Proof.** Since \( P \) does not depend on \( R \), as a consequence of Lemma 24 we have

\[ cl_{head(R)}(P)cl_{body(P)} = P \cup R. \]
Hence,
\[
\begin{align*}
\text{cl}_B(\text{cl}_{\text{head}}(R)(P)\text{cl}_{\text{body}}(P)(R))
&= \text{cl}_B(P \cup R) \\
&= 1_B \cup P \cup R \\
&\overset{\text{(41)}}{=} \text{cl}_{\text{head}}(R)(1_B \cup P)\text{cl}_{\text{body}}(1_B \cup P)(R) \quad (\text{head}(R) \cap B = \emptyset) \\
&= \text{cl}_{B \cup \text{head}}(R)(P)\text{cl}_{B \cup \text{body}}(P)(R).
\end{align*}
\]

We further will need the following auxiliary construction. Define, for any linearly ordered rules \(r_1 < \ldots < r_n\), \(n \geq 2\),
\[
\text{bh}_i \{(r_1 < \ldots < r_n)\} = \text{body} \{\{r_1, \ldots, r_{i-1}\}\} \cup \text{head} \{\{r_{i+1}, \ldots, r_n\}\}.
\]

We are now ready to prove the following decomposition result for acyclic theories.

**Theorem 27.** We can sequentially decompose any acyclic propositional Horn theory \(P = \{r_1 <_\ell \ldots <_\ell r_n\}, n \geq 2\), linearly ordered by a level mapping \(\ell\), into single-rule theories as
\[
P = \prod_{i=1}^{n} \text{cl}_{\text{bh}_i}(P) \{\{r_i\}\}.
\]
This decomposition is unique up to reordering of rules within a single level.\(^6\)

**Proof.** The proof is by induction on the number \(n\) of rules in \(P\). For the induction hypothesis \(n = 2\) and \(P = \{r_1 <_\ell r_2\}\), we proceed as follows. First, we have
\[
\text{cl}_{\text{bh}_1}(P) \{\{r_1\}\} \text{cl}_{\text{bh}_2}(P) \{\{r_2\}\} = (1_{\text{head}}(\{r_2\}) \cup \{r_1\})(1_{\text{body}}(\{r_1\}) \cup \{r_2\}) \quad (43)
\]
\[
= 1_{\text{head}}(\{r_2\})(1_{\text{body}}(\{r_1\}) \cup \{r_2\}) \cup \{r_1\}(1_{\text{body}}(\{r_1\}) \cup \{r_2\}) \quad (44)
\]

\(^6\)See the construction of the total ordering \(<_\ell\) in Section 2.2.1.
Since Krom theories distribute from the left (Theorem 8), we can simplify (43), by applying (32) and (31), into

\[ 1_{\text{head}}(r_2) \cap \text{body}(r_1) \cup 1_{\text{head}}(r_2) \cup \{r_1\} \text{body}(r_1) = 1_{\text{head}}(r_2) \cap \text{body}(r_1) \cup \{r_1, r_2\}. \] (45)

Now, since \( r_1 \) does not depend on \( r_2 \), that is, \( \text{head}(r_2) \cap \text{body}(r_1) = \emptyset \), the first term in (45) equals \( \emptyset = \emptyset \) which implies that (45) is equivalent to \( P \) as desired.

For the induction step \( P = \{ r_1 < \ell \ldots < \ell r_{n+1} \} \), we proceed as follows.

First, by definition of \( bh_i \), we have

\[
\prod_{i=1}^{n+1} (cl_{bh_i}(P)) = \prod_{i=1}^{n+1} (1_{bh_i}(P) \cup \{r_i\})
\]

(46)

\[
= \left[ \prod_{i=1}^{n} (1_{bh_i}(r_1, \ldots, r_n)) \cup 1_{\text{head}}(r_{n+1}) \cup \{r_i\} \right] (1_{\text{body}}(r_1, \ldots, r_n) \cup \{r_{n+1}\}).
\] (47)

Second, by idempotency of union we can extract the term

\[ 1_{\text{head}}(r_{n+1}) = 1_{\text{head}}(r_{n+1}) \ldots 1_{\text{head}}(r_{n+1}) \ (n \ times) \]

occurring in (47) thus obtaining

\[
\left[ 1_{\text{head}}(r_{n+1}) \cup \prod_{i=1}^{n} (1_{bh_i}(r_1, \ldots, r_n)) \cup 1_{\text{head}}(r_{n+1}) \cup \{r_i\} \right] (1_{\text{body}}(r_1, \ldots, r_n) \cup \{r_{n+1}\}).
\] (48)

Now, since \( r_i \) and \( 1_{bh_i}(r_1, \ldots, r_n) \) do not depend on \( r_{n+1} \), we can simplify (48) further to

\[
\left[ 1_{\text{head}}(r_{n+1}) \cup \prod_{i=1}^{n} (1_{bh_i}(r_1, \ldots, r_n)) \cup \{r_i\} \right] (1_{\text{body}}(r_1, \ldots, r_n) \cup \{r_{n+1}\}).
\] (49)

By applying the induction hypothesis to (49), we obtain

\[
(1_{\text{head}}(r_{n+1}) \cup \{r_1, \ldots, r_n\})(1_{\text{body}}(r_1, \ldots, r_n) \cup \{r_{n+1}\})
\]

which, by right-distributivity of composition, is equal to

\[ 1_{\text{head}}(r_{n+1}) (1_{\text{body}}(r_1, \ldots, r_n) \cup \{r_{n+1}\}) \cup \{r_1, \ldots, r_n\} (1_{\text{body}}(r_1, \ldots, r_n) \cup \{r_{n+1}\}). \] (50)
Now again since \( r_i \) does not depend on \( r_{n+1} \), for all \( 1 \leq i \leq n \), as a consequence of (14), (32), (31), and (39), the term in (50) equals
\[
1_{\text{head}(r_{n+1}) \cap \text{body}(r_1, \ldots, r_n)} \cup \text{head}(r_{n+1}) \{ r_{n+1} \} \cup \{ r_1, \ldots, r_n \} \text{body}((r_1, \ldots, r_n)). \tag{51}
\]
Finally, since \( 1_{\text{head}(r_{n+1}) \cap \text{body}(r_1, \ldots, r_n)} = 1_\emptyset = \emptyset \), (51) equals \( \{ r_1, \ldots, r_{n+1} \} \), which proves our theorem. \( \square \)

5.2. General Decompositions

We now wish to generalize Lemma 24 to arbitrary propositional Horn theories. For this, we define, for every interpretation \( I \) and disjoint copy \( I' = \{ a' \mid a \in I \} \) of \( I \), the minimalistic theory
\[
[I \leftarrow I'] = \text{cl}_{A-I}(\{ a \leftarrow a' \mid a \in I \}) = \{ a \leftarrow a' \mid a \in I \} \cup \{ a \leftarrow a \mid a \in A - I \}.
\]
We have
\[
[A \leftarrow A'] = \{ a \leftarrow a' \mid a \in A \}
\]
and therefore
\[
P[A \leftarrow A'] = \{ \text{head}(r) \leftarrow \text{body}(r)' \mid r \in P \}
\]
and
\[
[A' \leftarrow A]P = \{ \text{head}(r)' \leftarrow \text{body}(r) \mid r \in P \}.
\]
Moreover, we have
\[
[A \leftarrow A'][A' \leftarrow A] = 1_A. \tag{52}
\]

We are now ready to prove the main result of this subsection which shows that we can represent the union of theories via composition.

**Theorem 28.** For any propositional Horn theories \( P \) and \( R \), we have
\[
P \cup R = \text{cl}_{\text{head}(R)}(P[A \leftarrow A']) \text{cl}_{\text{body}(P[A \leftarrow A'])}(R) \text{cl}_A([A' \leftarrow A]).
\]
Proof. Since \( P[A \leftarrow A'] \) does not depend on \( R \), we have, as a consequence of Lemma 24,

\[
P[A \leftarrow A'] \cup R = \text{cl}_{\text{head}}(P[A \leftarrow A']) \text{cl}_{\text{body}}(P[A \leftarrow A'])(R).
\]

Finally, we have

\[
(P[A \leftarrow A'] \cup R)\text{cl}_A([A' \leftarrow A]) = P[A \leftarrow A']\text{cl}_A([A' \leftarrow A]) \cup R\text{cl}_A([A' \leftarrow A])
\]

\[
= P[A \leftarrow A'] \cup (1_A \cup [A' \leftarrow A]) \cup R(1_A \cup [A' \leftarrow A])
\]

\[
= P[A \leftarrow A'] \cup A \cup R
\]

where the third identity follows from the fact that \( P[A \leftarrow A'] \) does not depend on \( 1_A \) and \( R \) does not depend on \( [A' \leftarrow A] \) (apply Lemma 23).

Example 29. Let \( A = \{a, b, c\} \) and consider the theory \( P = R \cup \pi_{(b,c)} \) where

\[
R = \left\{\begin{array}{l}
a \leftarrow b, c \\
a \leftarrow a, b \\
b \leftarrow a
\end{array}\right\} \quad \text{and} \quad \pi_{(b,c)} = \left\{\begin{array}{l}
b \leftarrow c \\
c \leftarrow b
\end{array}\right\}.
\]

We wish to decompose \( P \) into a product of \( R \) and \( \pi_{(b,c)} \) according to Theorem 28. For this, we first have to replace the body of \( R \) with a distinct copy of its atoms, that is, we compute

\[
R[A \leftarrow A'] = \left\{\begin{array}{l}
a \leftarrow b', c' \\
a \leftarrow a', b' \\
b \leftarrow a'
\end{array}\right\}
\]

Notice that \( R[A \leftarrow A'] \) no longer depends on \( \pi_{(b,c)} \). Next, we compute the composition of

\[
\text{cl}_{\text{head}}(\pi_{(b,c)})(R[A \leftarrow A']) = \left\{\begin{array}{l}
a \leftarrow b', c' \\
a \leftarrow a', b' \\
b \leftarrow a' \\
b \leftarrow b \\
c \leftarrow c
\end{array}\right\}
\]
and

\[
\text{cl}_{\text{body}(R[A \leftarrow A'])}(\pi(b, c)) = \begin{cases} 
b \leftarrow c \\
c \leftarrow b \\
a' \leftarrow a' \\
b' \leftarrow b' \\
c' \leftarrow c' 
\end{cases}
\]

by

\[
R' = \text{cl}_{\text{head}(\pi(b, c))}(R[A \leftarrow A']) \text{cl}_{\text{body}(R[A \leftarrow A'])}(\pi(b, c)) = \begin{cases} 
a \leftarrow b', c' \\
a \leftarrow a', b' \\
b \leftarrow a' \\
b \leftarrow c \\
c \leftarrow b 
\end{cases}
\]

Finally, we need to replace the atoms from \( A' \) in the body of \( R' \) by atoms from \( A \), that is, we compute

\[
R' \text{cl}_A([A' \leftarrow A]) = \begin{cases} 
a \leftarrow b', c' \\
a \leftarrow a', b' \\
b \leftarrow a' \\
b \leftarrow c \\
c \leftarrow b 
\end{cases} \circ \begin{cases} 
a \leftarrow a \\
b \leftarrow b \\
c \leftarrow c \\
a' \leftarrow a \\
b' \leftarrow b \\
c' \leftarrow c 
\end{cases} = P
\]

as expected.

Todo 30. Unfortunately, at the moment we have no decomposition result of arbitrary theories analogous to Theorem 27 which remains as future work (cf. Section 7).

6. Algebraic Semantics

We now want to reformulate the fixed point semantics of propositional Horn theories in terms of sequential composition. For this, we first show that we can
express the van Emden-Kowalski operators via composition (Theorem 31) and we then show how propositional Horn theories can be iterated bottom-up to obtain their least models (Theorem 35).

6.1. The van Emden-Kowalski Operator

Recall from Proposition 2 that the least model of a theory can be computed by a least fixed point iteration of its associated immediate consequence or van Emden-Kowalski operator. The next results show that we can simulate the van Emden-Kowalski operators on a syntactic level without any explicit reference to operators.

The next result shows that we can represent the van Emden-Kowalski operator of a theory via composition on a syntactic level.

**Theorem 31.** For any propositional Horn theory $P$ and interpretation $I$, we have

$$T_P(I) = P \circ I.$$  

**Proof.** We compute

$$T_P(I) = \text{head}(P^I) = P^IA = (P(1^I))A = P((1^I)A) = P^I(A) = P(I \cap A)$$

where the last equality follows from $I$ being a subset of $A$. 

As a direct consequence of Proposition 1 and Theorem 31, we have the following algebraic characterization of (supported) models.

**Corollary 32.** An interpretation $I$ is a model of $P$ iff $P \circ I \subseteq I$, and $I$ is a supported model of $P$ iff $P \circ I = I$.

**Corollary 33.** The space of all interpretations forms an ideal.

**Proof.** By Proposition 9, we know that the space of interpretations forms a right ideal and Theorem 31 implies that it is a left ideal—hence, it forms an ideal.  

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6.2. Least Models

We interpret propositional Horn theories according to their least model semantics and since least models can be constructively computed by bottom-up iterations of the associated van Emden-Kowalski operators (cf. Proposition 2), we can finally reformulate the fixed point semantics of propositional Horn theories in terms of sequential composition (Theorem 35).

**Definition 34.** Define the unary *Kleene star* and *plus* operations by

\[ P^* = \bigcup_{n \geq 0} P^n \quad \text{and} \quad P^+ = P^* P. \]  

Moreover, define the *omega* operation by

\[ P^\omega = P^+ \circ \emptyset \overset{\text{(50)}}{=} \text{facts}(P^+). \]

Notice that the unions in (53) are finite since \( P \) is finite. For instance, for any interpretation \( I \), we have as a consequence of Corollary 13,

\[ I^* = 1 \cup I \quad \text{and} \quad I^+ = I \quad \text{and} \quad I^\omega = I. \]  

(54)

Interestingly enough, we can add the atoms in \( I \) to \( P \) via

\[ P \cup I \overset{(15)}{=} P \cup IP \overset{(4)}{=} (1 \cup I)P \overset{(54)}{=} I^* P. \]

(55)

Hence, as a consequence of (5) and (55), we can decompose \( P \) as

\[ P = \text{facts}(P)^* \circ \text{proper}(P), \]

which, roughly, says that we can sequentially separate the facts from the proper rules in \( P \).

We are now ready to characterize the least model of a theory via composition as follows.

**Theorem 35.** For any propositional Horn theory \( P \), we have

\[ \text{LM}(P) = P^\omega. \]

**Proof.** A direct consequence of Proposition 2 and Theorem 31.

**Corollary 36.** Two propositional Horn theories \( P \) and \( R \) are equivalent iff \( P^\omega = R^\omega \).
7. Conclusion

This paper contributed to the foundations of logic programming, knowledge representation, and database theory by studying the (sequential) composition of propositional Horn theories. We showed in our main structural result (Theorem 5) that the space of all propositional Horn theories forms a finite unital magma with respect to composition ordered by set inclusion, which distributes from the right over union. We called the magmas induced by sequential composition *Horn magmas*, and we called the algebras induced by sequential composition and union *Horn algebras*. Moreover, we showed that the restricted class of propositional Krom-Horn theories is distributive and therefore its proper instance forms an idempotent semiring (Theorem 8). From a logical point of view, we obtained an algebraic meta-calculus for reasoning about propositional Horn theories. Algebraically, we obtained a correspondence between propositional Horn theories and finite unital magmas, which enables us in the future to transfer concepts from the algebraic literature to the logical setting. In a broader sense, this paper is a further step towards an algebra of logical theories and we expect interesting concepts and results to follow.

*Future Work*

In the future, we plan to extend the constructions and results of this paper to wider classes of logical theories as, for example, first-, and higher-order logic programs [4, 5], disjunctive datalog [16], and non-monotonic logic programs under the stable model [8] or answer set semantics (cf. [9, 10, 11, 12]). The first task is non-trivial as function symbols require the use of most general unifiers in the definition of composition and give rise to infinite magmas, whereas the non-monotonic case is more difficult to handle algebraically due to negation as failure [17] occurring in rule bodies (and heads). Even more problematic, disjunctive rules yield non-deterministic behavior which is more difficult to handle algebraically. Nonetheless, we expect interesting results in all of the aforementioned cases to follow.
Another major line of research is to study (sequential) decompositions of various theories (cf. Todo [30]). Specifically, we wish to compute decompositions of arbitrary propositional Horn theories (and extensions thereof) into “prime” theories in the vein of Theorem [27] where we expect permutation theories (Section 3.2.2) to play a fundamental role in such decompositions. For this, it will be necessary to resolve the issue of “prime” or indecomposable theories. From a practical point of view, a mathematically satisfactory theory of theory decompositions is relevant to modular knowledge representation and optimization of reasoning.

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