Operadic structure on the Gerstenhaber–Schack complex for prestacks

Hoang Dinh Van¹ · Lander Hermans² · Wendy Lowen³,⁴

Accepted: 31 December 2021 / Published online: 28 February 2022
© The Author(s) 2022

Abstract
We introduce an operad Patch which acts on the Gerstenhaber–Schack complex of a prestack as defined by Dinh Van and Lowen, and which in particular allows us to endow this complex with an underlying $L_\infty$-structure. We make use of the operad Quilt which was used by Hawkins in order to solve the presheaf case. Due to the additional difficulty posed by the presence of twists, we have to use Quilt in a fundamentally different way (even for presheaves) in order to allow for an extension to prestacks.

Keywords Prestack · Hochschild cohomology · Gerstenhaber–Schack complex · Operad · Brace operations · $L_\infty$-algebra

Mathematics Subject Classification 18F20 · 18M60

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant Agreement No. 817762). The second named author holds a Ph.D. fellowship of the Research Foundation—Flanders (FWO). The third named author was partially supported by the HSE University Basic Research Program.

Wendy Lowen
wendy.lowen@uantwerpen.be
Hoang Dinh Van
hoangdv@hcmute.edu.vn
Lander Hermans
lander.hermans@uantwerpen.be

¹ Faculty of Applied Sciences, University of Technology and Education, 1 Vo Van Ngan, Thu Duc, Ho Chi Minh City, Vietnam
² Departement Wiskunde, Universiteit Antwerpen, Middelheimcampus, Middelheimlaan 1, 2020 Antwerp, Belgium
³ Departement Wiskunde, Universiteit Antwerpen, Middelheimcampus, Middelheimlaan 1, 2020 Antwerp, Belgium
⁴ Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow, Russian Federation
1 Introduction

The deformation theory of algebras due to Gerstenhaber furnishes the guiding example for algebraic deformation theory. For an algebra $A$, the Hochschild complex $C(A)$ is a dg Lie algebra governing the deformation theory of $A$ through the Maurer–Cartan formalism. This dg Lie structure is the shadow of a richer operadic structure, which can be expressed by saying that $C(A)$ is a homotopy G-algebra [6]. This structure, which captures both the brace operations and the cup product, is a special case of a $B_\infty$-structure [7]. Importantly, this purely algebraic structure constitutes a stepping stone in the proof of the Deligne conjecture, proving $C(A)$ to be an algebra over the chain little disk operad [10,13].

The deformation theory of algebras was later extended to presheaves of algebras by Gerstenhaber and Schack, who in particular introduced a bicomplex computing the natural bimodule Ext groups [4,5]. However, this GS-complex $C(\mathcal{A})$ of a presheaf $\mathcal{A}$ does not control deformations of $\mathcal{A}$ as a presheaf, but rather as a twisted presheaf, see for instance [2,11]. From this point of view, it is more natural to develop deformation theory at once on the level of twisted presheaves or, more generally prestacks, that is, pseudofunctors taking values in the 2-category of linear categories (over some fixed commutative ground ring). In [2], Dinh Van and Lowen established a Gerstenhaber–Schack complex for prestacks, involving a differential which features an infinite sequence of higher components in addition to the classical simplicial and Hochshild differentials. Further, for a prestack $\mathcal{A}$, they construct a homotopy equivalence $C_{GS}(\mathcal{A}) \cong CC(\mathcal{A})$ between the Gerstenhaber–Schack complex $C_{GS}(\mathcal{A})$ and the Hochschild complex $CC(\mathcal{A})$ of the Grothendieck construction $\mathcal{A}$ of $\mathcal{A}$. Through homotopy transfer, this endows the GS-complex with an $L_\infty$-structure. This result improves upon the existence of a quasi-isomorphism, which is a consequence of the Cohomology Comparison Theorem due to Gerstenhaber and Schack for presheaves [5] and to Lowen and Van den Bergh for prestacks [12].

Although the GS-complex does not possess a $B_\infty$-structure, its elements - linear maps involving different levels of the prestack - can be composed in an operadic fashion. As such, it makes sense to investigate this higher structure in its own right, and use it directly in order to establish an underlying $L_\infty$-structure. For particular types of presheaves, explicit $L_\infty$-structures on the GS-complex have been established by Frégier et al. [3] and by Barmeier and Frégier [1].

Let Brace be the brace operad and $F_2S$ the homotopy G-operad. In the case of a presheaf $(\mathcal{A}, m, f)$, in [8], Hawkins introduces an operad Quilt $\subseteq F_2S \otimes H$ Brace which he later extends to an operad mQuilt acting on the GS-complex. These operads are naturally endowed with $L_\infty$-operations as desired. The action of Quilt on the GS-complex considered by Hawkins only involves the restriction functors $f$ of the presheaf, the multiplication $m$ being incorporated later on in mQuilt. Unfortunately, the way in which functoriality of $f$ is built into these actions, does not allow for an extension to twisted presheaves or prestacks.

The goal of this paper is to solve the problem of establishing a natural operadic structure with underlying $L_\infty$-structure on $C_{GS}(\mathcal{A})$ in the case of a general prestack $(\mathcal{A}, m, f, c)$ with twists $c$. As part of our solution, we use Quilt in a fundamentally different way in relation to the GS-complex, but still allowing us to make use of the
naturally associated $L_\infty$-structure from [8]. In Sect. 3, we capture the higher structure of $C_{GS}(\mathcal{A})$ by introducing the new operad $\text{Patch} \subseteq \text{mNSOp} \otimes_H \text{NSOp}$ over which $C_{GS}(\mathcal{A})$ is shown to be an algebra (see Theorem 3.24). Here, $(\text{m})\text{NSOp}$ is the operad of nonsymmetric operads (with multiplication).

In [6], Gerstenhaber and Voronov obtain a brace algebra structure on an operad and a homotopy G-algebra structure on an operad with multiplication. Based upon the expression of these results in terms of the underlying operads $\text{NSOp}$ and $\text{mNSOp}$ in Sect. 2, we construct a morphism $\text{Quilt} \rightarrow \text{Patch}_s$ (see Proposition 3.27) as a restriction of

$$F_2S \otimes_H \text{Brace} \rightarrow \text{mNSOp}_{st} \otimes_H \text{NSOp}_s,$$

where the operads with subscript denote the (uncolored) graded operads associated to the unsubscripted colored operads. This gives rise to the composition

$$R : \text{Quilt} \rightarrow \text{Patch}_s \rightarrow \text{End}(sC_{GS}(\mathcal{A}))$$

which incorporates the multiplication $m$ and the restrictions $f$ of $\mathcal{A}$.

In Sect. 4, we extend the action $R$ to

$$R_c : \text{Quilt}_b[[c]] \rightarrow \text{End}(sC_{GS}(\mathcal{A}))$$

in order to incorporate the twists (see Theorem 4.17). Here, $\text{Quilt}_b[[c]]$ is obtained from an operad of formal power series. Further, we establish $L_\infty$-operations on $\text{Quilt}_b[[c]]$ extending those on $\text{Quilt}$ from [8] (see Theorem 4.10) by adding an infinite series of higher components containing twists. Under the action of $\text{Quilt}$ this neatly corresponds to and extends the differential on $C_{GS}(\mathcal{A})$ obtained in [2]. In the final Sect. 4.5 we briefly discuss the relation of this $L_\infty$-structure with the deformation theory of the presheaf $\mathcal{A}$.

The present work naturally grew out of [2], and at the time when [8] appeared large parts of an operadic approach to the GS complex of a presheaf had already been developed independently by us. Given the efficient way in which Hawkins’ description of $\text{Quilt}$ gives rise to an $L_\infty$-structure, we decided it was worthwhile to build on this approach to the presheaf case, albeit in a way which “flips and refines” the action of $\text{Quilt}$ in order to make it useful for general prestacks. As a consequence, when we follow through Hawkins’ approach, in comparison we manage to incorporate not only the restrictions $f$, but also the multiplications $m$ in an initial action of $\text{Quilt}$ on the GS complex. In analogy with the way in which Hawkins extends his action from $\text{Quilt}$ to $\text{mQuilt}$ in order to incorporate the multiplications $m$, we establish an extension from $\text{Quilt}_b[[c]]$ in order to incorporate the twists $c$.

The current paper is part of a larger project in which it is our goal to understand the homotopy equivalence $C_{GS}(\mathcal{A}) \cong \text{CC}(\mathcal{A})!$ from [2] operadically, showing in particular that the $L_\infty$-structure from [2] and the one established in the present paper actually coincide.
2 Gerstenhaber–Voronov operadically

In the seminal paper [6], Gerstenhaber and Voronov define a brace-algebra structure on the totalisation of a non-symmetric operad $O$. Moreover, in presence of a multiplication, they define a homotopy G-algebra structure on $O$ incorporating both the cup product and the Gerstenhaber-bracket.

In this section we describe the morphisms of operads underlying these results. To this end, in Sect. 2.3, we recall the colored operad $NSOp$ encoding non-symmetric operads, and we describe the natural extension $mNSOp$ which adds a multiplication. Let $NSOp$ and $mNSOp$ be their totalised graded (uncolored) operads with suspended, respectively standard degree (see Sects. 2.5 and 2.6). Let Brace be the brace operad (see Sect. 2.1) and $F_2S$ the Gerstenhaber–Voronov operad encoding homotopy G-algebras (see Sect. 2.2). The main goal of this section is the definition of morphisms of dg-operads

$$\phi : \text{Brace} \longrightarrow NSOp$$

and

$$\bar{\phi} : F_2S \longrightarrow mNSOp$$

(see Theorems 2.16 and 2.34 respectively). In these definitions, we have to pay particular attention to the choice of signs. For this, we will make use of morphisms of operads $(m)NSOp \longrightarrow \text{Multi}/\Delta^1$ landing in the multicategory associated to the simplex category $\Delta$ (see Proposition 2.11).

For both uncolored as colored operads, we use the term morphism of operads. In case confusion may arise, we add a subscript to differentiate the uncolored operads from their colored counterparts.

2.1 The operad Brace

Throughout, we work over a fixed commutative ground ring $k$.

The operad Brace encoding brace algebras is defined using trees, that is, planar rooted trees. Following the presentation from [8, §2.2], for a tree $T$ we denote the set of vertices by $VT$, the set of edges by $ET$, the “vertical” partial order on $VT$ generated by $ET$ by $\leq_T$, and the “horizontal” partial order on $VT$ by $\sqsubseteq_T$. For $(u, v) \in ET$ we call $u$ the parent of $v$ and $v$ a child of $u$.

For $n \in \mathbb{N}$, put $[n] := \{0, \ldots, n\}$ and $\langle n \rangle := \{1, \ldots, n\}$.

Let $\text{Tree}(n)$ denote the set of trees with vertex set $\langle n \rangle$ and let $\text{Brace}(n)$ be the free $k$-module on $\text{Tree}(n)$ endowed with the $S_n$-action given by permuting the vertices, i.e., $T^\sigma$ is the tree defined by replacing vertex $i$ in $T$ by $\sigma^{-1}(i)$. The operadic composition on Brace is based upon substitution of trees, as follows. For trees $T \in \text{Tree}(m)$, $T' \in \text{Tree}(n)$ and $1 \leq i \leq m$, we denote by $\text{Ext}(T, T', i) \subseteq \text{Tree}(m + n - 1)$ the set of trees extending $T$ by $T'$ at $i$ (that is, $U \in \text{Ext}(T, T', i)$ has $T'$ as a subtree which
upon removal reduces to the vertex \( i \) of \( T \). We then define

\[
T \circ_i T' := \sum_{U \in Ext(T, T', i)} U.
\]

Underlying every such extension lie two maps \( \langle n \rangle \prec \langle n + m - 1 \rangle \xrightarrow{\alpha} \langle m \rangle \) acting on the vertices, where \( \alpha \) embeds \( n \) vertices consecutively and \( \beta \) contracts the image of \( \alpha \) to the vertex \( i \). We call the pair \( (\alpha, \beta) \) the extension of \( m \) by \( n \) at \( i \). We refer to [8, §2.2] for more details.

### 2.2 The operad \( F_2S \)

The operad \( F_2S \) encodes homotopy G-algebras [6]. Again, we largely follow the exposition from [8, §2.3]. Given a set \( A \), a word over \( A \) is an element of the free monoid on \( A \). For a word \( W = a_1a_2 \ldots a_k \), corresponding to the function \( W : \langle k \rangle \to A : i \mapsto a_i \), the \( i \)-th letter of \( W \) is the couple \((i, a_i)\). We will often identify a word with its graph \( W = [(i, a_i) \mid i \in \langle k \rangle] \subseteq \langle k \rangle \times A \), writing \((i, a_i) \in W\).

For \( a \in A \), a letter \((i, a) \in W \) is called an occurrence of \( a \) in \( W \). The letter \((i, a)\) is a caesura if there is a later occurrence of \( a \) in \( W \), that is, a letter \((j, a)\) with \( i < j \). We say that \( a \in A \) is interposed in \( W \) if \( W = \cdots ba \cdots b \ldots \). The length of \( W : \langle k \rangle \to A \) is \(|W| = k\).

By definition, \( F_2S(n) \) is the free \( k \)-module generated by the words \( W \) over \( \langle n \rangle \) such that:

1. \( W : \langle k \rangle \to \langle n \rangle \) is surjective,
2. \( W \neq \cdots uu \ldots \) (nondegeneracy), and
3. For any \( u \neq v \in \langle n \rangle \), \( W \neq \cdots u \ldots v \ldots u \ldots v \ldots \) (no intercacing).

The set \( F_2S(n) \) is graded by setting \( \deg(W) := |W| - n \) and naturally carries a \( S_n \)-action by permuting letters, i.e. \( W^\sigma = \sigma^{-1}W \).

For a word \( W \in F_2S(n) \) and \( u \in \langle n \rangle \), let \((i_u, u)\) be the first occurrence of \( u \) in \( W \). Then we obtain a total order \( u \downarrow v \iff i_u \leq i_v \) on \( \langle n \rangle \).

The operadic composition on \( F_2S \) is based upon merging of words, as follows. For words \( W \in F_2S(m) \), \( W' \in F_2S(n) \) and \( 1 \leq i \leq m \), we denote by \( Ext(W, W', i) \subseteq F_2S(m + n - 1) \) the set of extensions of \( W \) by \( W' \) at \( i \) (that is, \( X \in Ext(W, W', i) \) if up to relabelling and deleting repetitions, \( W' \) is a subword of \( X \) and upon collapsing the letters from \( W \) to \( i \), relabelling and deleting repetitions, we recover \( W \)).

In order to define the composition, we need the sign of an extension. 

**Sign of Extension.** Let \( W \in F_2S(m) \) and let \( \text{int}(W) \) be the set of interposed elements of \( \langle m \rangle \) ordered by their first occurrence in \( W \). For \( X \in Ext(W, W', i) \) the relabelling gives rise to two maps \( \alpha : \text{int}(W') \to \text{int}(X) \) and \( \gamma : \text{int}(W) \to \text{int}(X) \) where \( \gamma := \beta^{-1} \) except if \( i \) is interposed in \( W \), then \( \gamma(i) := \alpha(a) \) for \((1, a)\) the first letter of \( W' \).
As \(|\text{int}(W)| = \text{deg}(W)\), an extension \(X\) defines a unique \((\text{deg}(W), \text{deg}(W'))\) — shuffle \(\chi\) and we define
\[
\text{sgn}_{W, W', i}(X) := (-1)^X
\]

Moreover, it is possible to talk about the boundary of a word, inducing a differential. **Boundary.** Given a word \(W \in F_2S(n)\) and a letter \((i, a)\) of \(W\) for which \(a\) is repeated in \(W\), then define \(\partial_i W \in F_2S(n)\) as the word obtained by deleting the letter \((i, a)\) from \(W\) (and relabelling). If \(a\) is not repeated, then set \(\partial_i W = 0\).

**Sign of Deletion.** Given a word \(W \in F_2S(n)\) of length \(k\), then we define \(\text{sgn}_W : \langle k \rangle \rightarrow \{ -1, 1 \} \) by setting \(\text{sgn}_W(i) = (-1)^k\) if \((i, a_i)\) is the \(k\)-th caesura of \(W\), and otherwise \(\text{sgn}_W(i) = (-1)^{k+1}\) if it is the last occurrence, but the previous occurrence is the \(k\)-th caesura of \(W\).

The \(S\)-module \(F_2S\) defines a dg-operad with operadic composition given by
\[
W \circ_i W' := \sum_{X \in \text{Ext}(W, W', i)} \text{sgn}_{W, W', i}(X) X
\]
and boundary given by
\[
\partial W := \sum_{i \in \langle|W|\rangle} \text{sgn}_W(i) \partial_i W
\]

The following lemma, which we include for the convenience of the reader, shows how \(F_2S\) encodes the algebraic operations of a homotopy \(G\)-algebra. **Notations.** To avoid too large expressions, we leave out certain bracketings by setting as default the bracketing
\[
a \circ_i b \circ_j c := (a \circ_i b) \circ_j c
\]
Moreover, we compress the following
\[
a(\circ_{i_1} a_1)_{i_2} := a \circ_{i_1} a_1 \circ_{i_2} \cdots \circ_{i_n} a_n
\]

**Lemma 2.1** Let \(M_2 := 12, M_{1,0} = 1\) and \(M_{1,k} := 121 \cdots 1(k + 1)1\) for \(k \geq 1\), then \(F_2S\) is generated by these elements and the following holds
\[
(1) \quad \partial(M_2) = 0
\]
\[
(2) \quad \partial(M_{1,k}) = - (M_2^{(12)} \circ_2 M_{1,k-1}) + \sum_{i=2}^{k} (-1)^i M_{1,k-1} \circ_i M_2 + (-1)^{k+1} M_2 \circ_1 M_{1,k-1}
\]

**Proof** It is a straightforward computation to determine that \(M_2\) and \(M_{1,k}\) satisfy these relations.

Let \(W \in F_2S(n)\), we then show that it lies in the suboperad generated by \(M_2, (M_{1,k})_{k \geq 1}\) using only the above relations. We prove this by induction on \(n\). If
\( n = 1 \), then \( W = 1 \). So assume \( n > 1 \) and apply a permutation such that the first letter of \( W \) is 1, then \( W \) is of the form

\[
W = 1W_1 \ldots 1W_k 1W_{k+1}
\]

where \( W_i \) is the image of a non-empty word \( W'_i \in F_2S(n_i) \) under the map \( \gamma_i : \langle n_i \rangle \rightarrow \langle n \rangle \), except \( W_{k+1} \) which is possibly empty. Due to no interlacing we also know that the images \( \text{Im}(\gamma_i) \) are pair-wise disjoint. Hence, we can apply a permutation to assume that \( \max \text{Im}(\gamma_i) < \min \text{Im}(\gamma_{i'}) \) holds for every \( i < i' \). In this case, we have that

\[
W = M_{1,k} \circ_{k+1} W'_k \circ_k \cdots \circ_2 W'_1
\]

if \( W_{k+1} = \emptyset \), and

\[
W = (M_2 \circ_2 W'_{k+1}) \circ_1 (M_{1,k} \circ_{k+1} W'_k \circ_k \cdots \circ_2 W'_1)
\]

otherwise. By induction, this shows that \( W \) is generated by \( M_2 \) and \( (M_{1,k})_{l \geq 0} \). \( \square \)

2.3 The operads NSOp and mNSOp

It is well-known that non-symmetric operads can be encoded using a colored operad NSOp which can be defined using indexed trees, that is, for \( q_1, \ldots, q_n \in \mathbb{N} \) and \( q' = 1 + \sum_{i=1}^{n} (q_i - 1) \), NSOp\((q_1, \ldots, q_n; q')\) is the set of pairs \((T, I)\) where \( T \in \text{Tree}(n) \) and \( I : E_T \rightarrow \mathbb{N} \) a function such that

- For \((u, v) \in E_T, 1 \leq I(u, v) \leq q_u \)
- \((t, u), (t, v) \in E_T \) and \( u <_T v \Rightarrow I(t, u) < I(t, v) \)

We will often write \( I \) to denote the indexed tree \((T, I)\). Moreover, NSOp is generated by those trees with a single edge, that is,

\[
E_i := \begin{array}{c}
2 \\
\hline
1 \\
\end{array}
\]

for every \( q_1, q_2 \) and \( 1 \leq i \leq q_1 \), with the following pair of relations

\[
(\text{I)} \begin{array}{ccc}
2 & 2 & 3 \\
i & o_2 & j \\
1 & 1 & 1 \\
\end{array} = \begin{array}{cc}
2 & 2 \\
i - 1 + j & o_1 \\
1 & 1 \\
\end{array} \quad \text{for } 1 \leq j \leq q_2 \quad \text{and}
\]

\[
1 \leq i \leq q_1,
\]

\[
(\text{II)} \begin{array}{ccc}
2 & 2 & 2 \\
i & o_1 & k \\
1 & 1 & 1 \\
\end{array} = \begin{array}{c}
3 \\
1 \\
\end{array} \begin{array}{c}
2 \\
1 \\
\end{array} \begin{array}{c}
k - 1 + q_2 \\
1 \\
\end{array} \begin{array}{c}
o_1 \\
1 \\
\end{array} \begin{array}{c}
i \\
1 \\
\end{array} \quad (23)
\]

for \( 1 \leq i < k \leq q_1 \).
Note that these are the well-known associativity relations for non-symmetric operads.

**Definition 2.2** Let $m\text{NSOp}$ be the $N$-colored operad generated by $\text{NSOp}$ and an element $m \in m\text{NSOp}(; 2)$ satisfying the relation

\[
\begin{array}{c|c|c}
2 & \circ_1 m & \circ_1 m \\
\hline
1 & \circ_1 m & \circ_1 m \\
1 & 2 & 2
\end{array}
\]

**Remark 2.3** We often write $m$ where we have already filled in the plugged in $m$'s.

More explicitly, every representative of an element $X \in m\text{NSOp}(q_1, \ldots, q_n; q)$ is of the form $I \circ_{i_1} m \circ_{i_2} \cdots \circ_{i_k} m$ for $I \in \text{NSOp}$ and appropriate $i_1, \ldots, i_k \in \mathbb{N}$. Due to equivariance, we can always consider a representative of $X$ of the form

\[I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m\]

for $I \in \text{NSOp}(q_1, \ldots, q_n, 2, \ldots, 2; q)$.

**Lemma 2.4** Let $X = [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in m\text{NSOp}(q_1, \ldots, q_n; q)$, the partial orders $<_I$ and $<_I$ on $\langle n \rangle$ are independent of the representative of $X$. We denote them by $<_X$ and $<_X$.

**Proof** We proceed by induction on $k$ the number of $m$’s in $X$. For $k = 0$ or $k = 1$, there is nothing to show, so assume $k > 1$. It is clear that if the lemma holds for $X$, then the relations that hold for $<$ and $<_I$ for trees, also hold for $<_X$ and $<_X$. In particular, if the lemma holds for $X$ and $X'$ and $(\alpha, \beta)$ is the extension of $n$ by $m$ at $i$, then for $a, b \notin \text{Im}(\alpha)$ we have

\[
a <_{X, X'} b \iff \beta a <_X \beta b \quad \text{and} \quad a <_{X, X'} b \iff \beta a <_X \beta b
\]

Now, let $X_{0,i} := X_0 \circ_{n+1} m \circ_{n+1} m$, then we have by induction that the lemma holds for $X_0$. Moreover, we have for $a, b \in \langle n \rangle$ that

\[
a <_{X, X'} b \iff a <_{X_0} b \iff a <_{X_0} b \iff a <_{X_0} b
\]

which proves the lemma for $X$. \qed
2.4 The morphisms \((m)\text{NSOp} \rightarrow \text{Multi}\Delta\)

Let \(C\) be a small category. We denote by \(\text{Multi}C\) the \(\text{Ob}(C)\)-colored operad for which \(\text{Multi}C(c_1, \ldots, c_n; c)\) is freely generated as a \(k\)-module by \(n\)-tuples \((\zeta_1, \ldots, \zeta_n)\) of \(C\)-morphisms with \(\zeta_i : c_i \rightarrow c\), \(S_n\) acts by permutating labels, and composition is defined in the obvious way.

Let \(\Delta\) be the simplex category. Next, we construct a morphism of operads \(\text{NSOp} \rightarrow \text{Multi}\Delta\), by associating to every indexed tree \(I\) in \(\text{NSOp}(q_1, \ldots, q_n; q)\) a \(n\)-tuple \(\zeta_I\) in \(\text{Multi}\Delta(q_1, \ldots, q_n; q)\) which assigns to each vertex \(a\), considered as an \(q_a\)-corolla, a numbering denoting where its inputs are amongst the inputs of the indexed tree as a whole.

It suffices to define the morphism on the generators \(E_i \in \text{NSOp}\) and show that it respects the relations.

**Construction 2.5** Let \(E_i \in \text{NSOp}(q_1, q_2; q_1 + q_2 - 1)\) for \(1 \leq i \leq q_1\), then we define

\[
\zeta_{E_i, 1}(t) := \begin{cases} 
t & \text{if } t < i \\
t + q_2 - 1 & \text{if } t \geq i 
\end{cases}
\quad \text{and} \quad \zeta_{E_i, 2}(t) := t + i - 1
\]

Then, \(\zeta_{E_i} \in \text{Multi}\Delta(q_1, q_2; q_1 + q_2 - 1)\), that is, it is a tuple of non-decreasing maps. Moreover, if \(q_2 > 0\), then these are strictly increasing.

We will employ it as in the following example.

**Example 2.6** Let \(\mathcal{A}\) be a \(k\)-linear category, then its Hochschild complex is defined as

\[
\mathbf{C}^n(\mathcal{A}) = \prod_{A_0, \ldots, A_n \in \mathcal{A}} \text{Hom}_k(\mathcal{A}(A_1, A_0) \otimes \cdots \otimes \mathcal{A}(A_n, A_{n-1}), \mathcal{A}(A_n, A_0))
\]

For a Hochschild cochain \(\phi \in \mathbf{C}^n(\mathcal{A})\) and a \(n\)-simplex \(A_0 \leftarrow a_1 \leftarrow A_1 \leftarrow \cdots \leftarrow a_{n-1} \leftarrow A_{n-1} \leftarrow a_n \leftarrow A_n\) in \(\mathcal{A}\), we have that \(\phi_{A_0 \ldots A_n}(a_1, \ldots, a_n) \in \mathcal{A}(A_n, A_0)\).

Let \(\phi_1 \in \mathbf{C}^{q_1}(\mathcal{A})\) and \(\phi_2 \in \mathbf{C}^{q_2}(\mathcal{A})\), then each \(E_i \in \text{NSOp}(q_1, q_2; q_1 + q_2 - 1)\) determines a cochain \(\phi_1 \circ_i \phi_2 \in \mathbf{C}^{q_1 + q_2 - 1}(\mathcal{A})\) as follows

\[
(\phi_1 \circ_i \phi_2)_{A_0, \ldots, A_{q_1 + q_2 - 1}} = \phi_1^{A_{\zeta_{E_i, 1}(0)}, \ldots, A_{\zeta_{E_i, 1}(q_1)}} \circ_i \phi_2^{A_{\zeta_{E_i, 2}(0)}, \ldots, A_{\zeta_{E_i, 2}(q_2)}}
\]
which we can visualize using \( n \)-corollas

\[
\begin{align*}
A_{i-1} & \quad \phi_2 \quad A_{i-1+q_2} \\
A_0 & \quad A_1 \quad \cdots \quad A_{i-1} \quad \phi_1 \quad A_{i-1+q_2} \quad A_{q_1+q_2-1} \\
A_0 & \quad \cdots \quad A_{q_1+q_2-1}
\end{align*}
\]

**Lemma 2.7** Construction 2.5 extends to a morphism of operads

\[
\text{NSOp} \longrightarrow \text{Multi} \Delta : (T, I) \longrightarrow \zeta_I
\]

**Proof** It suffices to verify the relations (I) and (II) of NSOp. These are two simple computations and thus we only verify the first relation (I) as an example. Let \( \zeta := \xi_{E_i} \circ \xi_{E_j} \) denote the left-hand side, then we compute

\[
\begin{align*}
\zeta_1(t) &= \xi_{E_i,1}(t) = \begin{cases} 
  t & t < i \\
  t + q_2 + q_3 - 2 & t \geq i
\end{cases} \\
\zeta_2(t) &= \xi_{E_i,2} \circ \xi_{E_j,1}(t) = \begin{cases} 
  t + i - 1 & t < j \\
  t + i - 1 + q_3 - 1 & t \geq j
\end{cases} \\
\zeta_3(t) &= \xi_{E_i,2} \circ \xi_{E_j,2}(t) = t + i - 1 + j - 1
\end{align*}
\]

On the other hand, we compute the right-hand side \( \zeta' := \xi_{E_{i-1+j}} \circ \xi_{E_i} \) and obtain

\[
\begin{align*}
\zeta'_1(t) &= \xi_{E_{i-1+j},1} \circ \xi_{E_i,1}(t) = \begin{cases} 
  t & t < i \\
  t + q_2 - 1 + q_3 - 1 & t \geq i
\end{cases} \\
\zeta'_2(t) &= \xi_{E_{i-1+j},1} \circ \xi_{E_i,2}(t) = \begin{cases} 
  t + i - 1 & t < j \\
  t + i - 1 + q_3 - 1 & t \geq j
\end{cases} \\
\zeta'_3(t) &= \xi_{E_{i-1+j},1}(t) = t + i - 1 + j - 1
\end{align*}
\]

\( \square \)

**Remark 2.8** In appendix A we have added a generator-free description of this morphism and an alternative proof of Lemma 2.7, which we consider insightful and valuable, especially for concrete computations of signs in later sections.
Example 2.9 Consider the indexed tree

$$I = \begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
2 \\
\downarrow \\
1
\end{array} \in \text{NSOp}(3, 0, 4, 1, 0; 4)$$

then we compute $\zeta_I$ and obtain

$$\begin{array}{cccccc}
\zeta_{I,1} & \zeta_{I,2} & \zeta_{I,3} & \zeta_{I,4} & \zeta_{I,5} \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4
\end{array}$$

It is also possible to associate to an element of $m\text{NSOp}$ an element of $\text{Multi} \Delta$.

Lemma 2.10 Let $X = [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in m\text{NSOp}(q_1, \ldots, q_n; q)$, then $\zeta_{I,t} : [q_t] \rightarrow [q]$ for $t \in \langle n \rangle$ is independent of the representative $I$ of $X$.

In this case, we write $\zeta_X$.

Proof We prove the lemma by induction on $k$ the number of occurrences of $m$. The cases $k = 0$ and $k = 1$ are trivial, so assume $k > 1$. Let $X_{0,i} := X_0 \circ_{n+1} \bigg|_i$ such that $X = X_{0,1} \circ_{n+1} m \circ_{n+1} m = X_{0,2} \circ_{n+1} m \circ_{n+1} m$, then by induction and Lemma 2.7 we have for $t \in \langle n \rangle$ that

$$\zeta_{X_{0,1},t} = \zeta_{X_0,t} = \zeta_{X_{0,2},t}$$

which proves the lemma. \qed

Proposition 2.11 We have morphisms of operads

$$\text{NSOp} \rightarrow \text{Multi} \Delta : I \mapsto \zeta_I$$

and its extension

$$\text{mNSOp} \rightarrow \text{Multi} \Delta : X \mapsto \zeta_X.$$
Example 2.12 Consider the indexed 2-corolla \( I := \begin{array}{c} 1 \\ \frac{2}{3} \\ 1 \end{array} \) and its permuted form \( I(2,3) = \begin{array}{c} 1 \\ \frac{3}{2} \\ 1 \end{array} \) as elements of NSOp(2, 0; 0, 0), then they have the same image in Multi\( \Delta \). Note that this example holds for both NSOp and mNSOp.

Hence, we can consider mNSOp as a finer operad than Multi\( \Delta \) and thus encoding more information.

2.5 The morphism Brace \( \longrightarrow \) NSOp\( _s \)

In order to define the morphism \( \phi : \text{Brace} \longrightarrow \text{NSOp}\_s \) properly we need to compile the colored operad NSOp into a graded non-colored operad

\[
\text{NSOp}\_s(n) \subseteq \prod_{p_1, \ldots, p_n} \text{NSOp}(p_1, \ldots, p_n; p).
\]

where an element \( x \in \text{NSOp}(p_1, \ldots, p_n; p) \) is graded as \( |x| = \sum_{i=1}^{q} (p_i - 1) - (p - 1) = 0 \) (this is the suspended grading, whence the subscript) and \( \text{NSOp}\_s(n) \) is the subspace generated by sequences of elements with constant grading. The composition on \( \text{NSOp}\_s \) is derived from the composition of \( \text{NSOp} \) where it is set to 0 when the colors do not match. Note in particular that the \( S_n \)-action on \( \text{NSOp}\_s(n) \) is affected by this grading: permuting two vertices \( i \) and \( j \) introduces the signs \((-1)^{(p_i - 1)(p_j - 1)}\).

Definition 2.13 Let \((T, I) \in \text{NSOp}(p_1, \ldots, p_n; p)\), then \((T, I)\) is a coloring of \( T \) and we write \( \text{Clr}(T, p_1, \ldots, p_n) \) as the set of all such colorings of \( T \).

In order to define the sign \( \text{sgn}_T(I) \) for \( T \in \text{Brace}(n) \), we use the morphism of operads

\[
\text{NSOp} \longrightarrow \text{Multi}\Delta : (T, I) \mapsto \zeta_I
\]

and base this definition on the sign \( \text{sgn}_Q(\zeta, I) \) from [8, Def. 4.20].

Construction 2.14 We work with the following alphabet

\[
i_1, \ldots, (p_i - 1)i
\]

for \( i = 1, \ldots, n \) and define the word

\[
J^s(p_1, \ldots, p_n) = i_1 \ldots (p_1 - 1)i_1 \ldots i_n \ldots (p_n - 1)i_n
\]

We define a second word \( J^s_T(I) \) having in the \( \zeta_{I,k}(i)\)-th position \( i_k \) for \( 1 \leq i \leq q_k - 1 \). Note that we start from position 1 for \( J^s_T(I) \) (instead of 0).
**Definition 2.15** For \( I \in \text{NSOp}(p_1, \ldots, p_n; p) \) where we replace those \( p_i = 0 \) by 2, we define \( \text{sgn}_T(I) \) as the sign of the shuffle transforming \( J^S(p_1, \ldots, p_n) \) to \( J^S_I(I) \).

**Theorem 2.16** We have a morphism of operads

\[
\phi : \text{Brace} \longrightarrow \text{NSOp}_s : T \longmapsto \left( \sum_{I \in \text{Clr}(T, p_1, \ldots, p_n)} \text{sgn}_T(I)(T, I) \right)_{p_1, \ldots, p_n}
\]

**Proof** Per definition of \( \text{sgn}_T(I) \) we see that \( \phi \) is equivariant. Hence, we only need to verify that \( \text{sgn}_{T \circ_1 T'}(I \circ_1 I') = \text{sgn}_T(I) \text{sgn}_{T'}(I') \) for \( T \in \text{Brace}(n) \), \( T' \in \text{Brace}(m) \) and \( I \in \text{Clr}(T, p_1, \ldots, p_n) \) and \( I' \in \text{Clr}(T', p'_1, \ldots, p'_m) \). This equation holds as we can decompose the shuffle \( \chi'' : J^S(p'_1, \ldots, p'_m, p_2, \ldots, p_n) \xrightarrow{\phi} J^S_I(T, I') \) into two shuffles

\[
J^S(p'_1, \ldots, p'_m, p_2, \ldots, p_n) \xrightarrow{\chi''} J^S_T(I')J^S(p_2, \ldots, p_m) \xrightarrow{\chi'} J^S_I(T, I')
\]

where \( \chi \) and \( \chi' \) are the corresponding shuffles determining \( \text{sgn}_T(I) \) and \( \text{sgn}_{T'}(I') \). \( \square \)

### 2.6 The morphism \( F_2S \longrightarrow \text{mNSOp}_{st} \)

In order to define the morphism \( \tilde{\phi} : \text{F}_2S \longrightarrow \text{mNSOp}_{st} \) properly, we again need to compile the colored operad \( \text{mNSOp} \) to obtain a graded non-colored operad

\[
\text{mNSOp}_{st}(n) \subseteq \prod_{q_1, \ldots, q_n} \text{mNSOp}(q_1, \ldots, q_n; q)
\]

where an element \( x \in \text{mNSOp}(q_1, \ldots, q_n; q) \) is graded as \( \text{deg}(x) = \sum_{i=1}^r q_i - q \) (standard grading) and \( \text{mNSOp}_{st}(n) \) is generated by the sequences of constant grading. The composition on \( \text{mNSOp}_{st} \) is derived from the composition of \( \text{mNSOp} \) where it is set to 0 when the colors do not match. Note in particular that the \( \mathfrak{S}_n \)-action on \( \text{mNSOp}_{st}(n) \) is affected by this grading: permuting two vertices \( i \) and \( j \) introduces the sign \( (-1)^{q_i q_j} \).

#### 2.6.1 Colorings

**Definition 2.17** Let \( X := [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \text{mNSOp}(q_1, \ldots, q_n; q) \) for \( I \) having \( n + k \) vertices, then \( X \) is a coloring of \( W \in \text{F}_2S(n) \) if

- Each vertex \( n + 1, \ldots, n + k \) has exactly two children in \( I \)
- For \( u, v \in \langle n \rangle \) holds

\[
(1) \ u <_X v \iff W = \ldots u \ldots v \ldots u
\]

\[
(2) \ u <_X v \iff \text{every occurrence of } u \text{ in } W \text{ is left of every occurrence of } v \text{ in } W.
\]

We write \( \text{Clr}(W, q_1, \ldots, q_n) \) for the set of all such colorings for \( W \).
Remark 2.18 An element \( X \in \text{Clr}(W, q_1, \ldots, q_n) \) has \( n - \text{deg}(W) - 1 \) many \( m \)'s plugged in. Hence, \( X \in \text{mNSOp}(q_1, \ldots, q_n; \sum_{i=1}^{n}(q_i - 1) + n - \text{deg}(W)) \).

We give some examples.

Example 2.19 The following three elements of \( \text{mNSOp} \)

\[
\begin{array}{c|c}
1 & 2 \\
\hline 2 & m \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
2 & 3 \\
\hline 1 & i \\
\end{array}
\in \text{mNSOp}(q_1, q_2; q_1+q_2), \quad 1 \in \text{mNSOp}(q_1, q_2, q_3; q_1+q_2+q_3-1)
\]

for \( 1 \leq i \leq q_1 \)

and

\[
\begin{array}{c|c}
4 & 5 \\
\hline 3 & 2 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
5 & 1 \\
\hline i & 1 \\
\end{array}
\in \text{mNSOp}(q_1, q_2, q_3; q_1+q_2+q_3-1)
\]

for \( 1 \leq j < k \leq q_1 \)

color respectively the words

\[12 \in \text{F}_2\text{S}(2), \quad 1231 \in \text{F}_2\text{S}(3) \quad \text{and} \quad 1213451 \in \text{F}_2\text{S}(5).\]

Note however that not all elements of \( \text{mNSOp} \) color a word of \( \text{F}_2\text{S} \): the following set of elements

\[
\begin{array}{c|c}
1 & r \\
\hline m & \in \text{mNSOp}(q_1; q_1 + 1)
\end{array}
\]

for \( r \in \{1, 2\} \), colors no word in \( \text{F}_2\text{S} \) because the vertex plugged by \( m \) does not have two children.

Lemma 2.20 Definition 2.17 is well-defined, that is, it is independent of the chosen representative \( I \) of \( X \).

Proof Due to Lemma 2.4, both \( <_X \) and \( \triangleleft_X \) are well-defined. We show that the condition stipulating that all vertices of \( I \) that are plugged by \( m \)'s have exactly two children, is independent of the representative of \( X \). Thus, suppose \( I = I_0 \circ_k 1 \) for some \( I_0 \in \text{NSOp} \), such that both vertices \( k \) and \( k + 1 \) of \( I \) are plugged by \( m \)'s in \( X \). Due to
the relations in mNSOp, $X$ can equivalently be represented using $I' := I_0 \circ 1^2$. In this case, we have that vertices $k$ and $k + 1$ each have exactly 2 children in $I$ iff vertex $k$ has exactly 3 children in $I_0$ iff vertices $k$ and $k + 1$ each have exactly 2 children in $I'$. □

We construct a word for every element of mNSOp satisfying the above criteria.

**Construction 2.21** Let $X := [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \text{mNSOp}(q_1, \ldots, q_n; q)$ such that each vertex $a > n$ in $I$ has exactly two children, then we construct a word $W_X \in F_2S(n)$ such that $X \in \text{Clr}(W_X, q_1, \ldots, q_n, q)$.

- To every tree $T$ we can associate a word $W_T \in F_2S(n + k)$ (see [8, §2.3]).
- Suppose for $X_0 \in \text{mNSOp}$ such that $X_0 \circ_{n+1} m = X$ we have an associated word $W_{X_0} \in F_2S(n + 1)$, then let $W_X$ be the word given by deleting all occurrences of $n + 1$. Then $W_X \in F_2S(n)$ because $n + 1$ had two children, so no degeneracy can occur.

We consider an example of this procedure.

**Example 2.22** We consider the element

\[
\begin{array}{c}
2 \\
1 \\
m
\end{array}
\begin{array}{c}
3 \\
2 \\
i
\end{array}
\in \text{mNSOp}(q_1, q_2, q_3; q_1 + q_2 + q_3 - 1)
\]

for some $1 \leq i \leq q_1$ from Example 2.19 and show how Construction 2.21 assigns a word. First, we associate to the indexed tree

\[
\begin{array}{c}
2 \\
1 \\
4
\end{array}
\begin{array}{c}
3 \\
2 \\
i
\end{array}
\in \text{NSOp}(q_1, q_2, q_3, 2; q_1 + q_2 + q_3 - 1)
\]

the word 1424341 and then delete all occurrences of 4 as it is plugged by an instance of $m$. As a result, we obtain the word 1231.

**Lemma 2.23** For $X \in \text{mNSOp}(q_1, \ldots, q_n; q)$ we have $X \in \text{Clr}(W_X, q_1, \ldots, q_n)$ and if $X \in \text{Clr}(W, q_1, \ldots, q_n)$, then $W = W_X$.

**Proof** This clearly holds for $X = I \in \text{NSOp}(q_1, \ldots, q_n; q)$. Assume the lemma holds for $X_0 \in \text{mNSOp}$ and $X = X_0 \circ_{n+1} m$, then $W_{X_0} = W_0(n + 1)W_1(n + 1)W_2(n + 1)W_3$ for $W_0$ and $W_3$ possibly empty. In this case, $W_X = W_0W_1W_2W_3$ and it is easy to see that $X \in \text{Clr}(W_X, q_1, \ldots, q_n)$.

Now reversely, if $X \in \text{Clr}(W, q_1, \ldots, q_n)$ and $a \prec b$ are the two children of $n + 1$ in $X_0$, then $W = W_0W_1W_2W_3$ where $W_1 = a \cdots a, W_2 = b \cdots b$ and $W_0$
and $W_3$ are possibly empty. In that case, $X_0 \in \text{Clr}(W_0(n + 1)W_1(n + 1)W_2(n + 1)W_3, q_1, \ldots, q_n, 2)$ and thus by induction $W_{X_0} = W_0(n + 1)W_1(n + 1)W_2(n + 1)W_3$. Hence, $W_X = W_0W_1W_2W_3 = W$. □

**Lemma 2.24** Let $X \in \text{Clr}(V, q_1, \ldots, q_n, q)$ and $Y \in \text{Clr}(W, q_1', \ldots, q_m', q_i)$, then there exists a unique $U \in \text{Ext}(V, W, i)$ such that $X \circ_i Y \in \text{Clr}(U, q_1, \ldots, q_1', \ldots, q_m', \ldots, q_n, q)$.

**Proof** By Construction 2.21 we obtain a word $U \in F_2S(n + m - 1)$ such that $Z := X \circ_i Y \in \text{Clr}(U, \ldots)$. We show that $U \in \text{Ext}(V, W, i)$: let $U_\alpha$ be the word obtained from deleting from $U$ occurrences of vertices not in the image of $\alpha$ and eliminating consecutive repetitions $(uu \mapsto u)$. It is easy to check that $W = \cdots u \cdots v \cdots u \cdots$ iff $U_\alpha = \cdots \alpha(u) \cdots \alpha(v) \cdots \alpha(u) \cdots$, and that all occurrences of $u$ are left to those of $v$ in $W$ iff the same holds for $\alpha(u)$ and $\alpha(v)$ in $U_\alpha$.

Let $U_\beta$ be the word obtained from $U$ by relabelling by $\beta$ and eliminating consecutive repetitions. To verify that $U_\beta = V$ is straight forward, except in the following case: $U_\beta = \cdots i \cdots \beta(u) \cdots \beta(u) \cdots$ where $\beta(u) \neq i$. In this case, there exist $v, v'$ such that $U = \cdots \alpha(v) \cdots u \cdots \alpha(v') \cdots$. Now, we argue that there exists a vertex $v''$ such that $\alpha(v'') <_Z u$. If neither $\alpha(v)$ nor $\alpha(v')$ do, then $U$ implies that $\alpha(v) <_Z u <_Z \alpha(v')$. As $\alpha(v)$ and $\alpha(v')$ are part of the same subtree $\alpha(Y)$, i.e. the image of $Y$ under $\alpha$ in $Z$, there must be some vertex $a$ in the tree underlying $\alpha(Y)$ (possibly plugged by $m$) such that $a$ lies underneath $u$. As $Y$ is a coloring of a word, the conditions imply that $a$ is not plugged by an instance of $m$ (otherwise it would not have two children in $Y$). As a result, there is some vertex $v''$ in $Y$ such that $\alpha(v'') = a <_Z u$. Thus, $i <_X \beta(u)$ which verifies that $V = \cdots i \cdots \beta(u) \cdots \beta(u) \cdots$. This clearly also holds reversibly. □

**Lemma 2.25** Let $U \in \text{Ext}(V, W, i)$ and $Z \in \text{Clr}(U, q_1, \ldots, q_1', \ldots, q_m, \ldots, q_n, q)$, then there exist unique colorings $X \in \text{Clr}(V, q_1, \ldots, q_n, q)$ and $Y \in \text{Clr}(W, q_1', \ldots, q_m', q_i)$ such that $Z = X \circ_i Y$.

**Proof** Let $Z = [I'' \circ_{n+m} m \circ_{n+m} \cdots \circ_{n+m} m]$ with $l + k$ added $m$’s. The word $W$ can be uniquely written as

$$W = W_1 \cdots W_l$$

where two subwords $W_j$ and $W_j'$ do not share any occurrence of the same number, and $W_j$ is of the form $a_j \ldots a_j$. As $Z$ is a coloring of $U$, we have that $\alpha(a_1) <_Z \cdots <_Z \alpha(a_1)$ and no vertex of $\text{Im}(\alpha)$ lies under any $a_j$. In this case, there exists some vertex $a \in \{n + m, \ldots, n + m + k + l - 1\}$ such that $a <_{I''} \alpha(a_j)$ which is $\leq_{I''}$-maximal for these conditions (otherwise when applying $\beta$ to $U$ we will not obtain $V$).

Let $I'$ be the minimal subtree of $I''$ on the root $a$ containing $\text{Im}(\alpha)$. By contracting this subtree to a point we obtain a tree $I$ such that, after permutation of some vertices, we obtain $I \circ_i I' = I''$. Consider also the permutation such that $Z = [I'' \circ_{m+i} m \circ_{m+i} \cdots \circ_{m+i} m \circ_{n+m} m \circ_{n+m} \cdots \circ_{n+m} m]$.

It now suffices to show that $X := [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m] \in \text{Clr}(V, \ldots)$ and $Y := [I' \circ_{m+1} m \circ_{m+1} \cdots \circ_{m+1} m] \in \text{Clr}(W, \ldots)$, which is a straightforward computation using the facts $X \circ_i Y = Z, Z \in \text{Clr}(U, \ldots)$ and $U \in \text{Ext}(V, W, i)$. □
2.6.2 Signs

In order to define a sign $\text{sgn}_W(X)$ for $W \in F_2S(n)$ and $X \in \text{Clr}(W, q_1, \ldots, q_n)$ we use the morphism of operads

$$\text{NSOp} \longrightarrow \text{Multi} : (T, I) \longmapsto \zeta_I$$

which extends to

$$\text{mNSOp} \longrightarrow \text{Multi} : X \longmapsto \zeta_X$$

We base this definition of the sign on the sign $\text{sgn}_Q(\zeta, I)$ defined in [8, §4.7].

**Lemma 2.26** Let $X \in \text{Clr}(W, q_1, \ldots, q_n; q)$ for $W \in F_2S(n)$ and $q > 0$, then $\zeta_X$ is a coloring of $W$ in the sense of [8, Def. 4.13], that is,

- $\zeta_X \in \text{Multi}_{\Delta^+}$,
- $\bigcup_{i=1}^n \text{Im}(\zeta_X, i) = [q]$,
- For each $a \in \langle n \rangle$ there exists a function $\pi_a : [q_a] \longrightarrow W$ such that
  
  - The image of $\pi_a$ is the set of occurrences of $a$ in $W$,
  - For an $(i, a) \in W$, $\zeta_{X,a}(\pi_a^{-1}(i, a))$ is an interval
  - If $W = \cdots ab\ldots$, then
    $$\max \zeta_{X,a}(\pi_a^{-1}(i, a)) = \min \zeta_{X,b}(\pi_b^{-1}(i + 1, b))$$

**Proof** We first show it holds for $X = (T, I) \in \text{NSOp}$ by induction on the number of vertices: let

$$I = \begin{array}{c|c|c} I^1 & \cdots & I^k \\ \hline \hline i_1 & & i_k \\ \hline u & & u \end{array}$$

be its decomposition into its root $u$ with maximal subtrees $I^i$. In this case, we have $W = uW^1u \ldots uW^ku$ where the subwords $W_i$ represent the subtrees $I^i$. Let $\gamma_i : \langle k_i \rangle \hookrightarrow \langle n \rangle$ be the maps embedding the tree $I^i_0$ onto $I^i$ in $I$ and $I^i_0 \in \text{Clr}(W^i_0, \ldots)$, then they extend to a map $\overline{\gamma}_i : W^i_0 \hookrightarrow W$ inserting $W^i_0$ as $W^i$ into $W$. By induction, the lemma holds for $I^i_0 \in \text{Clr}(W^i_0, \ldots)$. Let $(p_1, u), \ldots, (p_{k+1}, u)$ be all the occurrences of $u$ in $W$, then we define

$$\pi_{\gamma_i(a)} = \overline{\gamma}_i \circ \pi_a^0$$

and $\pi_a(t) = \begin{cases} (p_1, u) & t < i_1 \\ (p_j, u) & i_{j-1} \leq t < i_j \\ (p_{k+1}, u) & i_k \leq t \end{cases}$

then it is easy to verify that these satisfy the above conditions.

Now assume $X = [X_0 \circ_{n+1} m]$ such that the lemma holds for $X_0 \in \text{Clr}(W_{X_0}, \ldots)$, then $W_X$ is obtained from $W_{X_0}$ by deleting all occurrences of $(n + 1)$. As the vertex
$n+1$ has exactly two children in $X_0$ and $q_{n+1} = 2$, we still have that $\bigcup_{i=1}^{n} \text{Im}(\xi_{X,i}) = \bigcup_{i=1}^{n+1} \text{Im}(\xi_{X_0,i}) = [q]$ and that
\[
\max \xi_{X_0,a}(\pi_a^{-1}(i,a)) = \min \xi_{X_0,b}(\pi_b^{-1}(i,b))
\]
for $W = \cdots ab \ldots$. Hence, $\xi_X$ satisfies the lemma. $\square$

Let us define analogously the sign corresponding to the horizontal part of $\text{sgn}_Q(\xi_X, I)$.

**Construction 2.27** We work with the following alphabet
\[
0_i, \ldots, (q_i - 1)_i
\]
for $i = 1, \ldots, n$ and define the word
\[
J(q_1, \ldots, q_n) = 0_1 \ldots (q_1 - 1)_1 \ldots 0_n \ldots (q_n - 1)_n
\]
The second word $J_W(X)$ is the concatenation of two words $J_{0,W}(X)$ and $J_{1,W}(X)$ defined as follows

- $J_{0,W}(X)$ consists of all $0_k$ for $k$ interposed in $W$, put in reverse $\downarrow$-order.
- $J_{1,W}(X)$ has in the $\xi_X, k(i)$-th position $i_k$ for $1 \leq i \leq q_k - 1$ for $k$ interposed, and in the $\xi_I, k(i)$ position $i_k$ for $0 \leq i \leq q_k - 1$ for $k$ not interposed. Note that we start from position 0.

**Definition 2.28** For $X \in \text{mNSOp}(q_1, \ldots, q_n; q)$ where we replace those $q_i = 0$ by 2, we define $\text{sgn}_W(X)$ as the sign of the shuffle transforming $J(q_1, \ldots, q_n)$ to $J_W(X)$.

**Example 2.29** Consider the words
\[
W = 13121, \quad W' = 1231
\]
and colorings
\[
X = \begin{array}{c}
3 \downarrow \\
2 \downarrow \\
1 \\
\end{array}, \quad X' = \begin{array}{c}
m \\
2 \\
j \downarrow \\
1 \\
\end{array}
\]
for which we calculate the words $J_W(X)$ and $J_{W'}(X')$ and their corresponding signs. In the first case, we have
\[
J_W(X) = 0_3 0_2 0_1 \ldots (i - 1)_1 1_3 \ldots (q_3 - 1)_3 i_1 \ldots (j - 1)_1 1_2 \ldots (q_2 - 1)_2 j_1 \ldots (q_1 - 1)_1
\]
which corresponds to the sign \( \text{sgn}_W(X) = (-1)^{(q_2-1)(q_1-j)+(q_3-1)(q_1-i)+q_3q_2+q_3-1} \).
For the second case, we calculate
\[
J_W'(X') = 0_20_1 \ldots (k-1)_11_2 \ldots (q_2-1)_20_3 \ldots (q_3-1)_3k_1 \ldots (q_1-1)_1
\]
which corresponds to the sign \( \text{sgn}_W(X') = (-1)^{(q_2+q_3)(q_1-k)+q_1} \). Note in particular that in \( J_W'(X') \) the letter 0_3 is not taken to the front of the word as 3 is not interposed in \( W' \).

**Lemma 2.30** Let \( X \in \text{Clr}(W, q_1, \ldots, q_n) \) and \( X' \in \text{Clr}(W', q'_1, \ldots, q'_m) \), and \( W'' \in \text{Ext}(W, W', 1) \), then \( \text{sgn}_W(X) \text{sgn}_W(X') = \text{sgn}_W(W' \circ_1 X') \text{sgn}_W(W, W', 1)(W''). \)

**Proof** We can assume that all \( q_i \) and \( q'_j \) are not zero. We can decompose \( \text{sgn}_W(X) \) in three components
- Sign of the shuffle \( \sigma \) shuffling \( J_{0,W}(X) \) to \( 0_{v_1} \ldots 0_{v_k} \) for \( v_1 < \ldots < v_k \) the interposed vertices of \( X \).
- Sign of the shuffle \( \tau \) shuffling \( J_{1,W}(X) \) to concatenation of \( l_1 \ldots (q_i - 1)_i \) for \( i \) interposed and \( 0_1 \ldots (q_i - 1)_i \) for \( i \) not interposed. We call this latter sequence \( J_W^{\text{int}}(q_1, \ldots, q_n) \).
- Sign of the shuffle \( \rho \) shuffling
\[
0_{v_1} \ldots 0_{v_k} J_W^{\text{int}}(q_1, \ldots, q_n) \leadsto J(q_1, \ldots, q_n)
\]

We add ‘ and ” to denote the correspondings shuffles for \( X' \) and \( X'' := X \circ_1 X' \).

First we have that \( (-1)^{\sigma''} = (-1)^{\sigma + \sigma'} \text{sgn}_W(W, W', 1)(W'') \) per definition of \( \text{sgn}_W(W, W', 1)(W'') \). Further we clearly have \( (-1)^{\tau''} = (-1)^{\tau + \tau'} \) by simply applying one after the other and renaming using \( \alpha \) and \( \beta^{-1} \)
\[
J_{1,W''}(X) \leadsto J_{1,W''}J_W(q_2, \ldots, q_n) \leadsto J_W(q'_1, \ldots, q'_m)J_W(q_2, \ldots, q_n)
\]
as \( \xi_{X''} = \xi_X \circ_1 \xi_{X'} \). We also have that \( (-1)^{\rho''} = (-1)^{\rho + \rho'} \) because the length of the sequence \( J_W^{\text{int}}(q'_1, \ldots, q'_m) \) is \( q_1 - 1 \) if \( 1 \) is interposed, and \( q_1 \) if \( 1 \) is not interposed. \( \square \)

2.6.3 The morphism

**Proposition 2.31** We have a morphism of graded operads

\[
\varphi : F_2S \rightarrow \text{mNSOp}_{st} : W \mapsto \left( \sum_{X \in \text{Clr}(W, q_1, \ldots, q_n)} \text{sgn}_W(X)X \right)_{q_1, \ldots, q_n}
\]

**Proof** By definition of \( \text{sgn}_W(X) \) the above linear maps are equivariant. By Lemmas 2.24, 2.25 and 2.30 they define a morphism of graded operads. \( \square \)

We make \( \text{mNSOp}_{st} \) into a dg-operad with the hochschild differential, then \( \phi \) will be a morphism of dg-operads.
Definition 2.32 Define for every \( q \in \mathbb{N} \) the element

\[
D_q = \begin{vmatrix} 1 & m+1 \\ m & 1 \end{vmatrix} \sum_{i=1}^{q} (-1)^i \begin{vmatrix} m \\ i \end{vmatrix} + (-1)^{q+1} \begin{vmatrix} 1 \\ m \end{vmatrix} \in \text{mNSOp}(q; q+1) \tag{1}
\]

which compile into an element of degree \(-1\)

\[
D := (D_q)_{q \geq 0} \in \text{mNSOp}_{st}(1).
\]

We consider the associated derivation

\[
\partial_D(X) := D \circ_1 X - (-1)^{\deg(X)} \sum_{i=1}^{n} X \circ_i D
\]

for \( X \in \text{mNSOp}_{st}(n) \).

Proposition 2.33 \( \partial_D \) defines a differential making \( \text{mNSOp} \) into a dg-operad, for which holds

\[
\partial_D(\bar{\phi}(W)) = \bar{\phi}(\partial(W))
\]

Proof The first part follows directly if \( D \circ_1 D = 0 \) which is an easy computation (see [6, Prop. 2]).

In order to prove the second part we only need to show this for the generators of \( W \), i.e. \( 12 \) and \( 121 \ldots 1k1 \) for \( k \geq 1 \).

- For \( q_1, q_2 \in \mathbb{N} \), it is easy to compute that

\[
\partial_D(\bar{\phi}(12)) = \partial_D \left( \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \right) = 0
\]

in \( \text{mNSOp}(q_1, q_2; q_1 + q_2) \).

- For \( q_1, \ldots, q_k \in \mathbb{N} \), we also have

\[
\bar{\phi}(121 \ldots 1k1) = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq q_1} (-1)^\epsilon \begin{vmatrix} 2 \\ i_1 \end{vmatrix} \cdots \begin{vmatrix} k \\ i_{k-1} \end{vmatrix}
\]

where \( \epsilon = \sum_{j=2}^{k} (q_j - 1)(q_1 - i_j + \sum_{l>j}(q_l - 1)) \)

and

\[
\partial(121 \ldots 1k1) = -2131 \ldots 1 + \sum_{i=2}^{k-1} (-1)^i 1 \ldots 1i(i+1)1 \ldots 1 + (-1)^k 1 \ldots 1k
\]
for which it is also a standard computation to see that $\partial_D(\bar{\phi}(121\ldots1k1)) = \bar{\phi}(\partial(121\ldots1k1))$ (see [6, Thm 3]).

\[\square\]

**Theorem 2.34** We have a morphism of dg-operads

$$\bar{\phi} : F_2S \longrightarrow mNSOp_{st}$$

**Proof** This is the direct consequence of Propositions 2.31 and 2.33. \(\square\)

### 3 The Gerstenhaber–Schack complex for prestacks

Let $(A, m, f, c)$ be a prestack over a small category $\mathcal{U}$ and let $C_{GS}(A)$ be the associated Gerstenhaber–Schack complex as defined in [2] (see Sect. 3.1). In loc. cit., a homotopy equivalence $C_{GS}(A) \cong CC(A!)$ is constructed with the Hochschild complex $CC(A!)$ of the Grothendieck construction $A!$ of $A$. Through homotopy transfer, this allows to endow the GS-complex with an $L_\infty$-structure. However, it is desirable to have a direct description available of this structure, without reference to transfer.

In the case of a presheaf, originally considered by Gerstenhaber and Schack, in [8], Hawkins introduces an operad $\text{Quilt} \subseteq F_2S \otimes H \text{Brace}$ which he later extends to an operad $m\text{Quilt}$ acting on the GS-complex. These operads are naturally endowed with $L_\infty$-operations as desired. The action of $\text{Quilt}$ on the GS-complex considered by Hawkins only involves the restriction functors $f$ of the presheaf, the multiplication $m$ being incorporated later on in $m\text{Quilt}$. Unfortunately, the way in which functoriality of $f$ is built into these actions, does not allow for an extension to twisted presheaves or prestacks.

In our solution for the prestack case, we propose to use $\text{Quilt}$ in a fundamentally different way in relation to the GS-complex, but still allowing us to make use of the naturally associated $L_\infty$-structure. In this section we capture the higher structure of $C_{GS}(A)$ by introducing the operad $\text{Patch} \subseteq m\text{NSOp} \otimes_H \text{NSOp}$ (see Sect. 3.3) over which the bicomplex $C^{*,*}(A)$, of which $C_{GS}(A)$ is the totalisation, is shown to be an algebra (see Theorem 3.24). Next, we construct a morphism $\text{Quilt} \longrightarrow \text{Patch}_s$ (see Proposition 3.27) as a restriction of

$$\bar{\phi} \otimes_H \phi : F_2S \otimes H \text{Brace} \longrightarrow m\text{NSOp}_{st} \otimes_H \text{NSOp}_s.$$

This morphism is such that the resulting composition

$$R : \text{Quilt} \longrightarrow \textbf{End}(sC_{GS}(A))$$

incorporates the multiplications $m$ and the restrictions $f$. Note that in Hawkins’ approach to the presheaf case, the initial action of $\text{Quilt}$ on $\textbf{End}(sC_{GS}(A))$ only incorporates the restrictions. As far as the structure of both approaches goes, the auxiliary operad $\text{Patch}$ we use is the counterpart of the operad $\text{ColorQuilt}$ from [8, Def. 4.6].

In Sect. 4, we will further extend the action $R$ in order to incorporate the twists.
3.1 The GS complex

In this section, we recall the notions of prestack and its associated Gerstenhaber–Schack complex, thus fixing terminology and notations. We use the same terminology as in [2,12].

A prestack is a pseudofunctor taking values in \( k \)-linear categories. Let \( \mathcal{U} \) be a small category.

**Definition 3.1** A prestack \( \mathcal{A} = (\mathcal{A}, m, f, c) \) over \( \mathcal{U} \) consists of the following data:

- For every object \( U \in \mathcal{U} \), a \( k \)-linear category \( (\mathcal{A}(U), m^U, 1^U) \) where \( m^U \) is the composition of morphisms in \( \mathcal{A}(U) \) and \( 1^U \) encodes the identity morphisms of \( \mathcal{A}(U) \).
- For every morphism \( u : V \rightarrow U \) in \( \mathcal{U} \), a \( k \)-linear functor \( f^u = u^* : \mathcal{A}(U) \rightarrow \mathcal{A}(V) \). For \( u = 1_U \) the identity morphism of \( U \) in \( \mathcal{U} \), we require that \( (1_U)^* = 1_{\mathcal{A}(U)} \).
- For every couple of morphisms \( v : W \rightarrow V, u : V \rightarrow U \) in \( \mathcal{U} \), a natural isomorphism \( c^{u,v} : v^* u^* \rightarrow (uv)^* \).

Given such a prestack \( \mathcal{A} \), we have an associated Gerstenhaber–Schack complex \( C_{GS}(\mathcal{A}) \). In [2] this is defined as the totalisation of a bicomplex \( C^{\bullet, \bullet}(\mathcal{A}) \). We first review some notations.

**Notations.** Let \( \sigma = (U_0 \overset{u_1}{\rightarrow} U_1 \rightarrow \ldots \overset{u_p}{\rightarrow} U_p) \) be a \( p \)-simplex in the category \( \mathcal{U} \), then we have two functors \( \mathcal{A}(U_p) \rightarrow \mathcal{A}(U_0) \), namely

\[
\sigma^\# := u_1^* \circ \cdots \circ u_p^* \quad \text{and} \quad \sigma^* := (u_p \circ \cdots \circ u_1)^*
\]

For each \( 1 \leq k \leq p - 1 \), denote by \( L_k(\sigma) \) and \( R_k(\sigma) \) the following simplices

\[
L_k(\sigma) = (U_0 \overset{u_1}{\rightarrow} U_1 \rightarrow \ldots \overset{u_k}{\rightarrow} U_k) \quad \text{and} \quad R_k(\sigma) = (U_k \overset{u_{k+1}}{\rightarrow} U_{k+1} \rightarrow \ldots \overset{u_p}{\rightarrow} U_p)
\]

and we consider the following natural isomorphisms

\[
c^{\sigma,k} = c^{u_k \ldots u_1 \ldots u_k+1} : (L_k(\sigma))^* (R_k(\sigma))^* \rightarrow \sigma^*
\]

\[
\epsilon^{\sigma,k} = u_1^* \ldots u_{k-1}^* c^{u_k \ldots u_{k+1}} u_{k+2}^* \ldots u_p^* : \sigma^\# \rightarrow u_1^* \ldots (u_{k+1} u_k)^* \ldots u_p^*
\]
We write $e^{\sigma,k,A} = e^{\sigma,k}(A)$ and $e^{\sigma,k,A} = e^{\sigma,k}(A)$ for $A \in A(U_p)$.

We also define a set $P(\sigma)$ of formal paths from $\sigma^\#$ to $\sigma^*$ inductively. A formal path is finite sequence of couples $(\tau, i)$ consisting of a simplex $\sigma$ and a natural number $i$. We set

$$P(u_1, u_2) := \{((u_1, u_2), 1)\}$$

and

$$P(\sigma) := \{(r_1, \ldots, r_{p-2}, (\sigma, i)) : 1 \leq i \leq p - 1 \text{ and } (r_1, \ldots, r_{p-2}) \in P(\partial_i \sigma)\}$$

where $\partial_i$ denotes the $i$th face-operator of the nerve $N_p(U)$. Given such a formal path $r = (r_1, \ldots, r_{p-1})$ we define its sign

$$(−1)^r = \prod_{i=1}^{p-1} (−1)^{r_i} \text{ where } (−1)^{σ,i} = (−1)^i.$$

By interpreting the data $(\sigma, i)$ as the natural isomorphism $e^{σ,i}$, every formal path $r \in P(\sigma)$ induces a sequence of natural isomorphisms $\bar{r} \in N_{p-1}(\text{Fun}(A(U_p), A(U_0)))$. Note that $e^{a_1,a_2,1} = e^{a_1,a_2}$ and its associated sign is $−1$.

Let $S_{t,p-1}$ denote the set of $(t, p - 1)$-shuffles, then given a formal path $r \in P(\sigma)$, a shuffle $β \in S_{t,p-1}$ and a tuple $a = (A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_t) \in N_t(A(U_p))$ let $β(a, r) \in N_{p-1+t}(A(U_0))$ be its shuffle product with respect to evaluation of functors as defined in [2, Ex. 3.2, Ex. 3.4].

Here, we give a more explicit definition of $β(a, r)$: first we construct inductively a sequence $(b_1, \ldots, b_{t+p−1})$ which formally represents a sequence of morphisms in $A(U_0)$. Every $b_t$ is either of the form $(\tau, a_t, A_{t-1})$ or $(r_t, A_j)$ for $\tau$ a simplex, $a_t$ and $A_j$ respectively a morphism and an object occurring in $a$, and $r_t$ an element of the formal path $r$. Define

$$b_{t+p−1} = \begin{cases} (\sigma, a_t, A_{t-1}) & \text{if } β(t) = t + p - 1 \\ (r_{p-1}, A_t) & \text{if } β(t + p - 1) = t + p - 1 \end{cases}$$

then for $1 < i \leq t + p - 1$, we have two cases: if $b_{i+1} = (\tau, a_j, A_{j-1})$ for some $j$, then define

$$b_i = \begin{cases} (\tau, a_{j-1}, A_{j-2}) & \text{if } β(j - 1) = i \text{ for } j \leq t \\ (r_k, A_{j-1}) & \text{if } β(t + k) = i \end{cases}$$

If $b_{i+1} = (r_k, A_j)$ for $r_k = (\tau, t)$, then define

$$b_i = \begin{cases} (\partial_t \tau, a_j, A_{j-1}) & \text{if } β(j) = i \text{ for } j < t \\ (r_{k-1}, A_j) & \text{if } β(t + k - 1) = i \end{cases}$$
Finally, we define $\beta(a, r)$ as the realization $\overline{b} = (B_0 \overleftarrow{b_1} B_1 \leftarrow \cdots \overleftarrow{b_i} B_{p+r})$ of $b$ where $b_i = \tau^a j$ if $b_i = (\tau, a_j, A_j)$ and $b_i = \tau^r_k(A_j)$ if $b_i = (r_k, A_j)$.

**Definition 3.2** Let $p, q \geq 0$, then define

$$C^{p, q}(A) = \prod_{\sigma \in N_p(U)} \prod_{A \in A(U)} \Hom(\bigotimes_{i=1}^q A(U_p)(A_i, A_{i-1}), A(U_0)(\sigma^a A_q, \sigma^a A_0))$$

and set

$$C^n_{GS}(A) = \bigoplus_{p+q=n} C^{p, q}(A)$$

The differential $d$ on the GS-complex is defined for $\theta \in C^{p, q}(A)$ as

$$d(\theta) = \sum_{j=0}^{q+1} d_j(\theta)$$

where $d_j : C^{p, q}(A) \rightarrow C^{p+j, q+1-j}(A)$ is defined as

- $d_0(\theta)^\sigma(A)(a) = m^{U_0}(\theta^a, A_0, \ldots, A_{q+1})(a_1, \ldots, a_{q+1})$
  \begin{align*}
  &+ \sum_{i=1}^q (-1)^i \theta^a(A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{q+1}) \\
  &\quad (a_1, \ldots, m^{U_p}(a_i, a_{i+1}), \ldots, a_1) \\
  &\quad + (-1)^{q+1} m^{U_0}(\theta^a(A_0, \ldots, A_q)(a_1, \ldots, a_q), \sigma^a(a_{q+1}))
  \end{align*}

- $d_1(\theta)^\sigma(A)(a) = (-1)^{p+q+1} m^{U_0}(c^{\sigma, p, A_0}, u_i^*(\theta^{\beta_0}(A)(a)))$
  \begin{align*}
  &+ \sum_{i=1}^p (-1)^{p+q+1+i} m^{U_0}(\theta^{\beta_i}(A)(a), \epsilon^\sigma(A)) \\
  &\quad + (-1)^q m^{U_0}(c^{\sigma, p, A_0}, \theta^{\beta_{p+1}}(A)(u_{p+1} a_1, \ldots, u_{p+1} a_q))
  \end{align*}

- $d_j(\theta)^\sigma(A)(a) = \sum_{r \in P(R_p(\sigma))} (-1)^{r+\beta+q-j+1} m^{U_{p+j}}(c^{\sigma, p, A_0}, \theta^{L_p}(B)(\beta(a, r)))$

\[ \beta \in \mathbb{S}_{q-j+1, j+1} \]
for \( \sigma = (u_1, \ldots, u_{p+j}) \in N_{p+j}(\mathcal{U}(U_0, U_{p+j}), a = (a_1, \ldots, a_{q-j+1}) \) where \( a_i \in \mathcal{A}(U_{p+j})(A_i, A_{i-1}) \) and such that \( B \) is the sequence of objects underlying \( \beta(a, r) \).

We will also be interested in the subcomplex \( \overline{C}_{GS}(\mathcal{A}) \subseteq C_{GS}(\mathcal{A}) \) of normalized and reduced cochains which is shown to be quasi-isomorphic to the GS complex (see [2, Prop. 3.16]). A simplex \( \sigma = (u_1, \ldots, u_p) \) is reduced if \( u_i = 1_{U_i} \) for some \( 1 \leq i \leq p \). A cochain \( \theta = (\theta^\sigma(A))_{\sigma, A} \in C_{GS}(\mathcal{A}) \) is reduced if \( \theta^\sigma(A) = 0 \) for every reduced simplex \( \sigma \). A simplex \( a = (a_1, \ldots, a_q) \) in \( \mathcal{A}(U) \) is normal if \( a_i = 1_U \) for some \( 1 \leq i \leq q \). A cochain \( \theta \) is normalized if \( \theta^\sigma(A)(a) = 0 \) for every normal simplex \( a \) in \( \mathcal{A}(U_p) \). We come back to this in Sect. 4.

Elements of the GS complex have a neat geometric interpretation as rectangles: for \( \theta \in C^{p,q}(\mathcal{A}) \) and the data \( (\sigma, A, a) \) from above, we can represent \( \theta^\sigma(A)(a) \) as the rectangle of data

\[
\begin{array}{cccccc}
A_0 & \cdots & A_{q-1} & \cdots & A_q \\
& \sigma^* & & \sigma^* & \\
\theta^\sigma(A) & & & & \\
& \sigma^* A_0 & \cdots & \sigma^* A_q \\
\end{array}
\]

Similarly, we can draw different components of the differential \( d \) using rectangles, providing more insight in its rather technical definition. For the hochschild component \( d_0 \) we have

\[
d_0(\theta^\sigma(A)) = \sigma^* \]

\[
\begin{array}{cccccc}
a_1 & \cdots & a_{q+1} \\
& \sigma^* & & \sigma^* & \\
\theta^\sigma(A) & & & & \\
& m U_0 \\
\end{array}
\]

\[
+ \sum_{i=1}^{q} (-1)^i \sigma^*
\]

\[
\begin{array}{cccccc}
a_1 & \cdots & a_{i-1} & a_i & a_{i+1} & \cdots & a_{q+1} \\
& \sigma^* & & \sigma^* & & \sigma^* & \\
\theta^\sigma(A) & & & & & & \\
& m U_p \\
\end{array}
\]
The first component $d_1$ can similarly be drawn as

$$d_1(\theta)^{\sigma}(A) = (-1)^{p+q+1}\sigma^*$$

Finally, we will draw $d_2$ as an example from which it is easy to deduce the higher components $d_j$ for $j > 2$. Namely, we have
Operadic structure on the Gerstenhaber–Schack complex...

$$d_2(\theta) = \sum_{i=1}^{q} (-1)^{r+\beta+q-2+1}$$

\[
\begin{array}{cccc}
A_0 & a_1 & A_{i-1} & a_{q-1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
(u_{p+2} u_{p+1})^* & c_{u_{p+1} u_{p+2}}(A_{i-1}) & u_{p+1}^* u_{p+2}^* & u_{p+1}^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_0 & b_1 & B_{i-1} & b_q \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_1 & c_{u_{p+1} u_{p+2}}(A_s) & u_{p+1}^* u_{p+2}^* & u_{p+1}^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_q & c_{u_{p+1} u_{p+2}}(A_{i-1}) & u_{p+1}^* u_{p+2}^* & u_{p+1}^* \\
\end{array}
\]

for shuffle $\beta(q) = i$, $\beta(s) = s$ for $s < i$ and $\beta(s) = s + 1$ for $s \geq i$, and formal path $r = ((u_p, u_{p+1}), 1)$. Note in particular that we can draw $\beta(a, r)$ as follows:

\[
\begin{array}{cccc}
A_0 & a_1 & A_{i-1} & a_{q-1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
(u_{p+2} u_{p+1})^* & c_{u_{p+1} u_{p+2}}(A_{i-1}) & u_{p+1}^* u_{p+2}^* & u_{p+1}^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_0 & b_1 & B_{i-1} & b_q \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_1 & c_{u_{p+1} u_{p+2}}(A_s) & u_{p+1}^* u_{p+2}^* & u_{p+1}^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_q & c_{u_{p+1} u_{p+2}}(A_{i-1}) & u_{p+1}^* u_{p+2}^* & u_{p+1}^* \\
\end{array}
\]

where $b_s = a_{\beta^{-1}(s)}$ for $s \neq i$, and $b_i = c_{u_{p+1} u_{p+2}}(A_{i-1})$.

We will use this rectangular interpretation as a guide in the next sections.

### 3.2 Endomorphism operad of a prestack

Although the GS-complex does not have partial compositions $\circ_i$, its elements $\theta = (\theta^\sigma(A))_{(\sigma, A)}$ consist of parts that lie in the endomorphism operad $\text{End}(A)$.

**Definition 3.3** Let $\text{Ob}(U, A)$ be the set consisting of the triples $(U, A, A')$ for $U \in U$ and $A, A' \in A(U)$, then we define the $\text{Ob}(U, A)$-colored operad $\text{End}(A)$ as

$$\text{End}(A)((U_1, A_1, A'_1), \ldots, (U_n, A_n, A'_n); (U, A, A')) := \text{Hom}(\bigotimes_{i=1}^n A(U_i)(A_i, A'_i), A(U)(A, A'))$$

with partial compositions defined by composition of linear maps.

**Remark 3.4** Note that $\theta^\sigma(A) \in \text{End}(A)((U_p, A_1, A_0), \ldots, (U_p, A_q, A_{q-1}); (U_0, \sigma^\# A_q, \sigma^* A_0))$. 
3.3 The operad Patch

In this section we define an \( \mathbb{N} \times \mathbb{N} \)-colored operad \( \text{Patch} \subseteq \text{mNSOp} \times \text{NSOp} \). Its elements encode concrete (planar) patchworks of rectangles of size \((p_i, q_i)\) to form a rectangle of size \((p, q)\).

**Definition 3.5** Let \( \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p)) \) consists of the elements \((X, J) \in \text{mNSOp}(q_1, \ldots, q_n; q) \times \text{NSOp}(p_1, \ldots, p_n; p)\) such that

1. \( a <_J b \implies a <_X b \)
2. \( a <_X b \implies b <_J a \)

**Remark 3.6** Note that in order for \( \text{Patch} \) not to be empty, we need to allow a multiplication in one of its coordinates which is not present in the other coordinate.

This has a neat geometric interpretation as well: a \((p, q)\)-rectangle has \(p\) inputs on the right-hand side, \(q\) inputs on top and a single output on respectively the bottom and the left-hand side.

We then interpret a patchwork \((X, J)\) as an ordering of these rectangles: the first coordinate \(X\) represent the vertical ordering (from top to bottom) and the second coordinate \(J\) the horizontal ordering (from right to left). The multiplications \(m\) form a single exception: they appear only vertically, thus we draw them as flat rectangles, that is, having no horizontal input and output. From this perspective, the conditions impose planarity on the patchwork such that we have

\[
\begin{array}{ccc}
1 & \cdots & q \\
\vdots & \ddots & \vdots \\
\vdots & & \vdots \\
1 & & \vdots \\
p & & \\
\end{array}
\]

We then interpret a patchwork \((X, J)\) as an ordering of these rectangles: the first coordinate \(X\) represent the vertical ordering (from top to bottom) and the second coordinate \(J\) the horizontal ordering (from right to left). The multiplications \(m\) form a single exception: they appear only vertically, thus we draw them as flat rectangles, that is, having no horizontal input and output. From this perspective, the conditions impose planarity on the patchwork such that we have

\[
\begin{array}{ccc}
\text{below} <_X \text{above} & \text{above} <_J \text{below} \\
\text{left} <_X \text{right} & \text{left} <_J \text{right} \\
\end{array}
\]

Note that when we write down a patchwork using rectangles, possible ‘open spaces’ can appear. Moreover, it is possible that multiple rectangles are vertically the ‘lowest’ elements due to the insertion of multiplication elements \(m\). However, horizontally there can only appear a single most left rectangle which is (horizontally) connected to all other rectangles. We give an example.
**Example 3.7** The following pair determines an element in Patch((3, 5), (3, 2), (2, 1), (0, 2); (6, 7))

$$(X, J) = \begin{pmatrix} \begin{array}{ccc} 3 & 4 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \\ 2 & 3 & 2 \\ 4 & 2 & \end{array} \end{pmatrix}$$

which we can draw as the following patchwork of rectangles

where the grey areas denote the open spaces.

**Lemma 3.8** Patch is a suboperad of mNSOp $\otimes_H$ NSOp.

**Proof** Let $$(X, J) \in \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$$ and $$(X', J') \in \text{Patch}((q'_1, p'_1), \ldots, (q'_m, p'_m); (q, p))$$ and we set $X'' := X \circ_i X'$ and $J'' := J \circ_i J'$.

Let $(\alpha, \beta)$ be the extension of $n$ by $m$ at $i$, then for $a, b \in \langle m \rangle$ we compute

$$\alpha(a) <_{X''} \alpha(b) \iff a <_{X'} b \implies b \triangleleft_{J'} a \iff \alpha(b) \triangleleft_{J''} \alpha(a)$$

and for $c, d \notin \text{Im}(\alpha)$ we compute

$$c <_{X''} d \iff \beta c <_X \beta d \implies \beta d \triangleleft_{J} \beta c \iff d \triangleleft_{J''} \beta c$$

For $c \notin \text{Im}(\alpha)$ and $b \in \langle m \rangle$, we have

$$c <_{X''} \alpha(b) \implies \beta c <_X i \implies i = \beta \alpha(b) \triangleleft_{J} \beta \alpha(b) \implies \alpha(b) \triangleleft_{J''} c$$

and the same reasoning shows $\alpha(b) <_{X''} c \implies c \triangleleft_{J''} \alpha(b)$. Completely symmetrically, this also shows that $c <_{J''} d \implies c \triangleleft_{X''} d$ for $c, d \in \langle n + m - 1 \rangle$.  

$\square$
We again compile the colored operad $\text{Patch}$ to obtain a graded non-colored operad

$$\text{Patch}_s(n) \subseteq \prod_{q_1, \ldots, q_n, p_1, \ldots, p_n} \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$$

where an element $x \in \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$ is graded as

$$|x| = \sum_{i=1}^n (q_i + p_i - 1) - (q + p - 1)$$

and $\text{Patch}_s(n)$ is generated as a $k$-module by the sequences of constant degree. Its composition is derived from $\text{Patch}$ where it is set to 0 when the colors do not match. Note in particular that the $\mathbb{S}_n$-action on $\text{Patch}(n)$ is affected by this grading: permuting two vertices $i$ and $j$ introduces a sign $(-1)^{(q_i+p_i-1)(q_j+p_j-1)}$.

**Lemma 3.9** $\text{Patch}_s$ is a dg-suboperad of $(\text{mNSOp}_{st} \otimes_H \text{NSOp}_s, (\partial_D, \text{Id}))$.

**Proof** It suffices to see that the elements $(D_q, 1) \in \text{Patch}((q, p); (q+1, p))$ for every $p, q \in \mathbb{N}$.

### 3.4 The morphism $\text{Patch}_s \rightarrow \text{End}(sC_{GS}(\mathcal{A}))$

In this section we make the GS-complex $C_{GS}(\mathcal{A})$ of a prestack $\mathcal{A}$ into a $\text{Patch}_s$-algebra. We do so by making its underlying bicomplex $C^\bullet\bullet(\mathcal{A})$ into a Patch-algebra. We first fix some notations.

**Definition 3.10** Let $\sigma = (U_0 \xrightarrow{u_1} U_1 \xrightarrow{} \ldots \xrightarrow{u_p} U_p)$ be a $p$-simplex in the category $\mathcal{U}$ and $\xi : [p'] \rightarrow [p]$ a non-decreasing map (or equivalently a non-decreasing sequence), then let $\overline{\xi}$ be the reflection of $\xi$, that is,

$$\overline{\xi}(t) := p - \xi(p' - t)$$

and define

$$\xi(\sigma) := N_\bullet(\mathcal{U})(\overline{\xi})(\sigma)$$

a $p'$-subsimplex of $\sigma$, where $N_\bullet(\mathcal{U})$ denotes the nerve construction on $\mathcal{U}$.

**Remark 3.11** Note that we apply the reflection as we count the horizontal inputs of a patchwork from top to bottom (see Example 3.7) instead of bottom to top (see further, Example 3.22).

Given a patchwork $(X, J) \in \text{Patch}$, we now determine which simplices we need to fill in the ‘open spaces’ in between the rectangles. We first sketch the idea.

Given a simplex $\sigma$ in $\mathcal{U}$ and a vertex $a$, we want to determine two sorts of simplices: for every vertical input $i = I(a, b)$ for some vertex $b$, we want to determine a simplex
\(\sigma(a, b)\) that we place between them. For the other vertical inputs \(1 \leq i \leq q_a\), we determine a simplex \(\sigma_a(i)\) to place on top of \(a\) at input \(i\). To do so, we determine the set of left-most vertices which do not “surpass” the \(i\)th input and that lie higher than vertex \(a\). In the drawing below, this set consists of the vertices \(e_1, e_2\) and \(e_3\). To calculate \(\sigma(a, b)\), we restrict this set to those vertices that still lie below vertex \(b\), in this case, the vertices \(e_2\) and \(e_3\).

\[
\begin{array}{c}
\sigma(a, b) \\
\downarrow \\
\sigma_a(i) \\
\downarrow \\
b \\
\downarrow \\
e_1 \\
\downarrow \\
e_2 \\
\downarrow \\
e_3 \\
\end{array}
\]

We observe that each element of the GS complex composes in \(\mathcal{U}\) the subsimplex corresponding to its horizontal inputs. Hence, using our auxiliary set, we contract the corresponding subsimplices and obtain \(\sigma(a, b)\) and \(\sigma_a(i)\).

Note that we have not yet treated the multiplications \(m\). In order to do so, we have to add the following complexity. Let \(X = [I \circ_{n+1} m \circ_{n+1} \cdots \circ_{n+1} m]\) where \(I\) is an indexed tree with \(n + k\) vertices, then we call a vertex \(a\) of \(I\) non-plugged in \(X\) if in \(X\) it is not inserted by a multiplication element \(m\). We continue with the above chosen representation of \(X\) where \(a\) non-plugged is equivalent to stating \(a \leq n\).

**Definition 3.12** We define a function \(\downarrow: \langle n + k \rangle \rightarrow [n]\) on the vertices of \(I\) which associates to every vertex \(a\) the closest non-plugged vertex in \(X\) under or equal to \(a\), or 0 if no such vertex exists. Concretely,

\[
\downarrow a := \max_{y \in \langle n \rangle} \{y \leq I a\} \text{ and } \downarrow a = 0 \text{ if the set is empty}
\]

We also set \(a \triangleleft I 0\) for every vertex \(a\) and define

\[
\zeta_{J,a} := \zeta_{J,\downarrow a} \text{ and } \zeta_{J,0} := p
\]

where \(p\) is the total number of inputs of \(J\).

**Remark 3.13** This is clearly independent of the representative \(I\) of \(X\). Moreover, \(\downarrow\) is for the given representative \(I\) the identity on \(\langle n \rangle\).

Next, we determine the auxiliary set.
Definition 3.14 Consider a vertex $a$ of $I$ and let $b_1 \prec_I \ldots \prec_I b_t$ be the children of $a$ in $I$ lying in $\langle n \rangle$ with $i_s := I(a, b_s)$. We then define

\[ L_a(i) = \begin{cases} 
\{ e \in \langle n \rangle : e \not\prec_I a \text{ and } e \prec_J a \} & i < i_1 \\
\{ e \in \langle n \rangle : b_s \prec_I e \text{ and } e \prec_J a \} & i_s \leq i < i_{s+1} \\
\{ e \in \langle n \rangle : b_t \prec_I e \text{ and } e \prec_J a \} & i_t \leq i 
\end{cases} \]

for $i \in [q_a]$, and

\[ L(a, b_s) := \{ e \in \langle n \rangle : b_s \prec_I e \text{ and } b_s \prec_J e \prec_J a \} \]

and let $\min L_a(i)$ and $\min L(a, b_s)$ be respectively the set of $<_J$-minimal elements of $L_a(i)$ or $L(a, b_s)$.

Remark 3.15 Remark that $L_a(i), L(a, b_s) \subseteq \langle n \rangle$ and thus that it contains only vertices which are not plugged by $m$. By default, we will set the subsimplex underneath the plugged children of $a$ as empty (see Definition 3.18).

Remark 3.16 Note that the condition $e \not\prec_I a$ appearing in the first case becomes superfluous in the others.

Definition 3.17 Let $a$ be a vertex of $I$ and $\min L_a(i) = \{ e_1 \prec_J \ldots \prec_J e_l \}$, then we have the sequence of inequalities

\[ 0 \leq \zeta_{J,e_1}(0) \leq \zeta_{J,e_1}(p_{e_1}) \leq \cdots \leq \zeta_{J,e_l}(p_{e_l}) \leq \zeta_{J,a}(0) \]

and thus the non-decreasing sequence

\[ (0, 1, \ldots, \zeta_{J,e_1}(0), \zeta_{J,e_1}(p_{e_1}), \ldots, \zeta_{J,e_l}(p_{e_l}), \ldots, \zeta_{J,a}(0)) \]

from which we delete $\zeta_{J,e_l}(p_{e_l})$ if $\zeta_{J,e_l}(p_{e_l}) = \zeta_{J,e_{l+1}}(0)$ or $\zeta_{J,e_l}(p_{e_l}) = \zeta_{J,a}(0)$. This defines a subsimplex $\sigma_a(i)$ of $\sigma$ by Definition 3.10.

Definition 3.18 Let $a, b$ be vertices of $I$ such that $b$ is a child of $a$, and $\min L(a, b) = \{ e_1 \prec_J \ldots \prec_J e_l \}$, then we have the sequence of inequalities

- If $i = I(a, b)$ for some vertex $b \in \langle n \rangle$ (non-plugged)

  \[ \zeta_{J,b}(p_b) \leq \zeta_{J,e_1}(0) \leq \zeta_{J,e_1}(p_{e_1}) \leq \cdots \leq \zeta_{J,e_l}(p_{e_l}) \leq \zeta_{J,a}(0) \]

- If $i = I(a, b)$ for some vertex $b > n$ (plugged)

  \[ \zeta_{J,b}(p_b) = \zeta_{J,a}(0) \]

and thus the non-decreasing sequence

\[ (\zeta_{J,b}(p_b), \ldots, \zeta_{J,e_1}(0), \zeta_{J,e_1}(p_{e_1}), \ldots, \zeta_{J,e_l}(0), \zeta_{J,e_l}(p_{e_l}), \ldots, \zeta_{J,a}(0)) \]
from which we delete $\zeta_{J,e_i}(p_{e_i})$ if $\zeta_{J,e_i}(p_{e_i}) = \zeta_{J,e_{i+1}}(0)$ or $\zeta_{J,e_i}(p_{e_i}) = \zeta_{J,a}(0)$. We also delete $\zeta_{J,b}(p_b)$ if it equals $\zeta_{J,e_1}(0)$. This defines a subsimplex $\sigma(a, b)$ of $\sigma$ by Definition 3.10.

We consider an example.

**Example 3.19** Given the simplex $\sigma = (u_1, \ldots, u_8)$ and the following patchwork of rectangles

we analyse the case for rectangle 6: we have

- $L_6(0) = L_6(1) = \{1, 3, 4, 5, 7\}$ and $\min L_6(0) = \min L_6(1) = \{1, 3, 5, 7\}$,
- $L_6(2) = L_6(3) = \{3, 4, 5\}$ and $\min L_6(2) = \min L_6(3) = \{3, 5\}$
- $L(6, 1) = \{3\}$ and $\min L(6, 1) = \{3\}$

and thus

- $\sigma_6(0) = \sigma_6(1) = (u_3u_2, u_5u_4, u_6, u_8u_7)$,
- $\sigma_6(2) = \sigma_6(3) = (u_3u_2, u_4, u_5, u_6, u_8u_7)$,
- $\sigma(6, 1) = (u_3u_2)$.

Now, we can assemble for every element of Patch a concrete patchwork of elements of $C_{GS}(A)$ where the first coordinate determines a vertical patching using the operadic structure and the second component determines the horizontal patching to fill in and align the corresponding simplices.

**Construction 3.20** Given $(X, J) \in \text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p))$ and $\theta_i \in C_{p_i,q_i}(A)$, then we set $\theta_s = m \in C^{0,2}(A)$ for $s = n + 1, \ldots, n + k$. 
Consider the patching Example 3.22. Then, given $\mathcal{C}$ $\mathcal{L}$

All these compositions together define $C$.

Proof

Construction 3.20 is independent of the representative $I$ of $X$.

Let us work out an example.

Example 3.22 Consider the patching $(X, J)$ from Example 3.7. Let $\theta_1 \in C^{5,3}(\mathcal{A})$, $\theta_2 \in C^{2,3}(\mathcal{A})$, $\theta_3 \in C^{1,2}(\mathcal{A})$ and $\theta_4 \in C^{2,0}(\mathcal{A})$, then we compute $\mathcal{L}(X, J)(\theta_1, \theta_2, \theta_3, \theta_4) \in C^{7,6}(\mathcal{A})$. Given the simplex $(u_1, \ldots, u_7) \in N_7(\mathcal{U})(U_0, U_p)$ and the objects $(A_0, \ldots, A_6) \in \mathcal{A}(U_p)$, we first compute

$$\Theta_1 = \theta_1^{(u_1, u_2, u_3, u_4, u_5, u_6, u_7)}(A_0, A_1, A_2, A_3)$$

$$\Theta_2 = \theta_2^{(u_2, u_3)}(u_4^*(u_6 u_5)^* u_7^* A_3, u_4^*(u_6 u_5)^* u_7^* A_5, u_4^*(u_6 u_5)^* u_7^* A_6, A_2, A_3)$$

$$\Theta_3 = \theta_3^{(u_6, u_5)}(u_7^* A_3, u_7^* A_4, u_7^* A_5)$$

$$\Theta_4 = \theta_4^{(u_5, u_6)}(u_7^* A_6)$$

Then, given $(a_1, \ldots, a_6)$ where $a_i \in \mathcal{A}(U_p)(A_i, A_{i-1})$, we finally compute

$$m^U_0(\Theta_1(a_1, a_2, a_3), u_1^* \Theta_2(u_4^* \Theta_3(u_7^*(a_4), u_7^*(a_5)), u_4^*(u_6 u_5)^* u_7^*(a_6), u_4^* \Theta_4)$$
which we can draw as follows

$$\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
If $x$ does not lie in the image of $(X', J')$, then $\sigma_{x'}(i) = \sigma''_x(i)$ because if $n \in \min L_{\beta x}(i)$ then it is replaced by $\alpha(r')$ for $r'$ the root of $J'^i$ for which holds $\zeta_{J'^i, \alpha(r')} = \zeta_{J, n}\zeta_{J', r'}$.

As a result, in both (2) and (3) we have the term $\Theta_{\beta x} \circ_i \sigma_{\beta x}(i)$.

If $x$ is part of $(X', J')$, i.e. $x = \alpha(x')$ for some vertex $x'$, then $\min L''_{\alpha(x')}(i)$ is the union of $\min L'_{\alpha(x')}(i)$ and $\min L_n(i')$ for some $i'$. Hence, we obtain the concatenation of $\sigma'_{x'}(i)$ for $\sigma' = \zeta_{J, n}(\sigma)$ and $\sigma_n(i')$. As $\zeta''_{J', r'} = \zeta_{J, n}\zeta_{J', r'}$, this corresponds exactly to $\sigma''_x(i)$.

Hence, the corresponding term in both calculations agrees.

Next, we calculate $\sigma''(x, b)$ for $b$ a child of $x$ in $(X'', J'')$ that is not plugged. We again have three cases

- If both $x$ and $b$ lie either outside or inside the image of $(X', J')$, then clearly $\sigma''(x, b) = \sigma(\beta x, \beta b)$ or $\sigma'(\alpha^{-1} x, \alpha^{-1} b)$ for $\sigma' = \zeta_{J, n}(\sigma)$ due to the previous reasoning and thus the terms agree.
- If $b$ lies in the image of $(X', J')$, i.e. $b = \alpha(b')$, and $x$ does not, then $b'$ is clearly the root of $X'$. As a result, $\sigma''(x, b) = \sigma(\beta x, n)$ and thus the terms agree.
- If $x$ lies in the image of $(X', J')$, i.e. $x = \alpha(x')$, and $b$ does not, then $\min L''_{\alpha(x')}(i)$ is the union of $\min L'_{\alpha(x')}(i)$ and $\min L_n(i')$ for some $i'$.

Hence, we obtain in (2) the concatenation of $\sigma(n, \beta b)$ and $\sigma'_{x'}(i)$ for $\sigma' = \zeta_{J, n}(\sigma)$, which corresponds exactly to $\sigma''(x, b)$. 
In case either $x$ or $b$ is plugged, we possibly have to apply the functorial property of the restrictions, i.e. $u^* \circ m_U^V = m_V^V \circ (u^* \otimes u^*)$ for $u : V \to U$ in $\mathcal{U}$, to pull down $\Theta_{\beta x} = m_{U_{\xi,\beta x}}^{0}$ or $\Theta_{\alpha^{-1} x} = m_{U_{\xi,\alpha^{-1} x}}^{0}$. Specifically, in the following cases

- Let $\beta x$ lie on top of $n$ in $(X, J)$ and $\downarrow x = \alpha(y)$ for some vertex $y$ of $(X', J')$.

In this case, $\Theta_{\beta x} = m_{U_{\xi,n}}^{0}$ occurs in (2) and $\Theta''_x = m_{U_{\xi,n'}}^{0}$ occurs in (3).

Using functoriality, in (2) we equivalently have $\tau^# \circ m_{U_{\xi,n}}^{0} = m_{U_{\xi,n'}}^{0} \circ (\tau^# \otimes \tau^#)$ for an appropriate simplex $\tau$. As a result they agree.

Next, it is clear from the drawing that $\sigma''_x(j)$ is the concatenation of $\sigma_{\beta x}(j)$ and $\tau$. Moreover, for some vertex $b$, we have $\sigma''(x, b)$ as the concatenation of $\sigma(\beta x, \beta b)$ and $\tau$, except in the case that $b$ is plugged as well. In the latter case, we can also pull $\Theta_{\beta b}$ in (2) down to $\Theta_{\beta x}$ and obtain $m_{U_{\xi,n'}}^{0} = \Theta''_x = \Theta''_b$ as in (3).

- Let $\beta x$ lie on top of $n$, but $\downarrow x \notin \text{Im}(\alpha)$.

Again, we can pull down $\Theta_{\beta x}$ in (2) past both functors $\tau^#$ and $\sigma(y, n)^#$ and obtain $m_{U_{\xi,b,y}}^{0} = \Theta''_x$. The same reasoning as before also holds for the functors $\sigma''_x(j)$ and $\sigma''(x, b)$ in (3) and its counterparts $\sigma_{\beta x}(j)$ and $\sigma(\beta x, \beta b)$ in (2).

- The case where $x$ lies in the image of $(X', J')$ such that $\downarrow x \notin \text{Im}(\alpha)$, is analogous to the previous one.

This finishes the proof. \qed

**Theorem 3.24** We obtain a morphism of dg-operads

$$\mathcal{L} : \text{Patch}_x \longrightarrow \text{End}(sC_{GS}(\mathcal{A}), d_0).$$
The map \( \mathcal{L} : \text{Patch} \to \text{End}((C^\bullet, \cdot)(A)) \) is clearly equivariant and thus it is a morphism of operads due to Proposition 3.23. Hence, the induced map \( \mathcal{L} : \text{Patch}_s \to \text{End}(s\, C_{GS}(A)) \) is a morphism of graded operads. Moreover, it is a morphism of dg-operads as \( \mathcal{L}(D, 1) = 0. \)

3.5 The morphism \( \text{Quilt} \to \text{Patch}_s \)

In [8], Hawkins defines a suboperad \( \text{Quilt} \subseteq F_2S \otimes_H \text{Brace} \) for which \( \text{Quilt}(n) \) is the free \( k \)-module generated by \( (W, T) \in F_2S(n) \times \text{Tree}(n) \) such that

1. \( W = \ldots u \ldots v \ldots \Rightarrow u \not<_{T} v; \)
2. \( W = \ldots u \ldots v \ldots u \ldots \Rightarrow v \not<_{T} u. \)

Here, \( \text{deg}(W, T) := \text{deg}(W) \) and the boundary operator is \( \partial(W, T) := (\partial W, T). \)

Insightfully, elements of Quilt can also be drawn as a stacking of rectangles in the plane, as extensively explained in [8, §3.2]. We will use Quilt in a fundamentally different way by switching the roles of its first and second component, and thus flipping the rectangles on their side. As such, we also draw the elements of Quilt on their side. We give an example.

**Example 3.25** We consider an example from [8, Ex. 3.2] and flip it on its side as follows

\[
\begin{pmatrix}
14234, & 2 \\
1 & 3 \\
4 &
\end{pmatrix} = 
\begin{pmatrix}
2 & 3 \\
1 & 4 \\
\end{pmatrix}
\]

Note the double line above rectangle 4: this reflects the fact that 3 is not interposed, otherwise the corresponding word would be 14234.

By definition, we have \( \text{Patch}_s \subseteq \text{mNSOp}_s \otimes_H \text{NSOp}_{st}. \) In this section, we will construct a morphism of operads \( \text{Quilt} \to \text{Patch}_s \) as a restriction of

\[ \bar{\phi} \otimes_H \phi : F_2S \otimes_H \text{Brace} \to \text{mNSOp}_s \otimes_H \text{NSOp}_{st}. \]

**Lemma 3.26** Let \( Q = (W, T) \in \text{Quilt}, X \in \text{Clr}(W, q_1, \ldots, q_n) \) and \( I \in \text{Clr}(T, p_1, \ldots, p_n) \), then \( (X, I) \in \text{Patch}. \)

**Proof** Let \( u, v \in \langle n \rangle \), if \( u <_I v \), then \( u <_{T} v \) and thus \( W \neq \cdots v \cdots u \ldots \) and thus every occurrence of \( u \) in \( W \) is left of every occurrence of \( v \) in \( W \). Hence, \( u <_{X} v. \)

The other way around, if \( u <_{X} v \), then \( W = \cdots u \cdots v \cdots u \cdots \) and thus \( v <_{T} u \) which is equivalent to \( v <_{I} u. \)

We obtain a morphism of graded operads

\[ \bar{\phi} \otimes_H \phi : \text{Quilt} \to \text{Patch}_s \]

defined as
\[(\bar{\phi} \otimes_H \phi)((q_1, p_1), \ldots, (q_n, p_n))(W, T) = \sum_{X \in \text{Clr}(W,q_1,\ldots,q_n)} \sum_{I \in \text{Clr}(T,p_1,\ldots,p_n)} \text{sgn}_W(X) \text{sgn}_T(I)(-1)^{\sigma(X,I)}(X, I)\]

where the sign \((-1)^{\sigma}\) is defined as the Koszul sign obtained from switching
\[q_1, \ldots, q_n, p_1 - 1, \ldots, p_n - 1 \mapsto q_1, p_1 - 1, \ldots, q_n, p_n - 1\]

This is the consequence of the Hadamard product of two graded operads
\[\text{Patch}((q_1, p_1), \ldots, (q_n, p_n); (q, p)) \subseteq \text{mNSOp}(q_1, \ldots, q_n; q) \otimes \text{NSOp}(p_1, \ldots, p_n; p)\]

where we have switched the order of the inputs.

Note in particular that this sign corresponds to the sign defined in [8, §4.7] and that we write \(\text{sgn}_Q(X, J) := \text{sgn}_W(X) \text{sgn}_T(J)(-1)^{\sigma}\).

As a direct consequence of Lemma 3.9 we have the following.

**Proposition 3.27** We have a morphism of dg-operads
\[\bar{\phi} \otimes_H \phi : (\text{Quilt}, \partial) \rightarrow (\text{Patch}_s, \partial_{(D,1)})\]

**Corollary 3.28** We have a morphism of dg-operads
\[R := \mathcal{L} \circ (\bar{\phi} \otimes_H \phi) : \text{Quilt} \rightarrow \text{End}(sC_{GS}(A), d_0)\]

**Proof** Immediate from Theorem 3.24 and Proposition 3.27. \(\square\)

Our action of Quilt on the GS-complex of a prestack is orthogonal to the action constructed in [8, Thm. 4.26] in the case of presheaves, and thus also new for the latter case. This can be interpreted in a geometric sense: our action encodes a quilt \(Q = (W, T)\) as a vertical patchwork according to \(W\) and a horizontal patchwork according to \(T\). In Hawkins’ action their roles are reversed, where the role of the multiplication is filled in by the identity \(1^u,v : v^*u^* = (uv)^*\). This does not translate to the case of prestacks due to the occurring twists \(c^u,v : v^*u^* \rightarrow (uv)^*\).

### 4 Incorporating twists

The morphism \(R : \text{Quilt} \rightarrow \text{End}(sC_{GS}(A))\) from Corollary 3.28 only involves the multiplication \(m\) and the functors \(f\) of the data of a prestack \((A, m, f, c)\). In this section, we will incorporate the twists \(c\) by adding a formal element with certain relations, resulting in the bounded powerseries operad \(\text{Quilt}_{b[[c]]}\). In Sect. 4.4, we extend \(R\) above to a morphism \(R_c : \text{Quilt}_{b[[c]]} \rightarrow \text{End}(sC_{GS}(A))\) (see Theorem 4.17). In Hawkins’ approach to the presheaf case, the initial action of Quilt on \(\text{End}(sC_{GS}(A))\), which only involves the restriction maps \(f\), is later extended in order to incorporate...
the multiplications \( m \). As far as the structure of both approaches goes, our operad \( \text{Quilt}_b([c]) \) is the counterpart of the operad \( m\text{Quilt} \) from [8, Def. 5.2].

In [8, §7.1], Hawkins constructs a morphisms \( L_\infty \rightarrow \text{Quilt} \) (see Sect. 4.2). In Sect. 4.2, we establish a more involved morphism \( L_\infty \rightarrow \text{Quilt}_b([c]) \) (see Theorem 4.10) by extending to an infinite series of higher components incorporating the element \( c \).

Putting Theorems 4.10 and 4.17 together, we have thus endowed \( s\mathcal{C}_{GS}(\mathcal{A}) \) with an \( L_\infty \)-structure. In the case of presheaves, this coincides on reduced and normalised cochains with the \( L_\infty \)-structure from [8, Thm. 7.13].

In the final Sect. 4.5 we briefly discuss the relation of this structure with the deformation theory of the prestack \( \mathcal{A} \).

4.1 Powerseries operads

In order to obtain an \( L_\infty \)-structure incorporating twists, we will make use of operads of formal power series.

**Definition 4.1** Let \( \mathcal{O} \) be a graded operad, then define \( \mathcal{O}[x] \) as the graded operad generated by \( \mathcal{O} \) and an element \( x \) of degree \( t \) and define the subspaces

\[
\mathcal{O}[x](n, r) := \{ \gamma \in \mathcal{O}[x](n) : \gamma \text{ has } r \text{ occurrences of } x \} \subseteq \mathcal{O}[x](n)
\]

which is well-defined as there are no relations on \( x \). Define

\[
\mathcal{O}[[x]](n) := \prod_{r \geq 0} \mathcal{O}[x](n, r)
\]

with component-wise \( S_n \)-action and write their elements as \( \sum_{r \geq 0} Q_r \) for \( Q_r \in \mathcal{O}[x](n, r) \). For every \( 1 \leq k \leq n \) the composition of \( \mathcal{O}[x] \) descends to a map

\[
\mathcal{O}[x](n, r) \otimes \mathcal{O}[x](n, s) \rightarrow \mathcal{O}[x](n, r + s)
\]

which extends to a composition map

\[
(\sum_{r \geq 0} Q_r) \circ_k (\sum_{s \geq 0} P_s) := \sum_{t \geq 0} (\sum_{i+j=t} Q_i \circ_k P_j)
\]

We call an element \( \sum_{r \geq 0} Q_r \in \mathcal{O}[[x]] \) *bounded* if the set \( \{\deg(Q_r) : r \geq 0\} \subseteq \mathbb{Z} \) is bounded. Let \( \mathcal{O}_b[[x]] \) be the \( S \)-submodule of bounded series which is graded by the series with coefficients of constant degree.

**Lemma 4.2** (1) \( \mathcal{O}[[x]] \) is an operad.
(2) \( \mathcal{O}_b[[x]] \) is a graded suboperad of \( \mathcal{O}[[x]] \).
(3) We have a sequence of injective operad morphisms

\[
\mathcal{O} \leftrightarrow \mathcal{O}[x] \leftrightarrow \mathcal{O}_b[[x]] \leftrightarrow \mathcal{O}[[x]]
\]
**Proof** These are straight-forward computations. □

We call $\mathcal{O}[[x]]$ the *operad of powerseries with coefficients in $\mathcal{O}$* and $\mathcal{O}_b[[x]]$ the *operad of bounded powerseries with coefficients in $\mathcal{O}$*.

**Definition 4.3** Consider $\text{Quilt}_x$, $\text{Quilt}_b[[x]]$ and $\text{Quilt}[[x]]$ for $x$ a 0-ary element $x \in \text{Quilt}(0)$ of degree $-1$, then let $\text{Quilt}[c]$, $\text{Quilt}_b[[c]]$ and $\text{Quilt}[[c]]$ be their respective quotients under the following relations

1. $\partial(c) = 0$
2. $(12, 1 \rightarrow 2) \circ_1 c \circ_1 c = 0$
3. $(W, T) \circ_i c = 0$ if $i$ has more than two children in $T$ or $i$ is repeated in $W$ (that is, it has a child in $W$).

**Remark 4.4** In Definition 4.3, (1) determines that $c$ encodes a natural transformation, (2) embodies the cocycle condition and (3) determines the form of $c$. The letter $c$ will always stand for the twist subject to its relations, and should not be confused with a free variable.

On inspection of $\text{mQuilt}$ from [8, Def. 5.2], we see that our conditions on $c$ are a subset of those imposed on $m$ in $\text{mQuilt}$. Hence, we obtain a morphism $\text{Quilt}[c] \rightarrow \text{mQuilt}$ sending $c$ to $m$.

### 4.2 The morphism $L_\infty \rightarrow \text{Quilt}$

In [8, Thm 7.8] a morphism $L_\infty \rightarrow \text{Quilt} : l_n \mapsto L_0^0$ is defined by setting

$$L_0^0 := \sum_{Q \in \text{Quilt}(n)} \text{deg}(Q) = n-2 \text{sgn}(Q) \cdot Q$$

In particular, this means that for every $n \geq 2$ the equation

$$0 = \partial L_0^0 + \sum_{p+q=n+1} \sum_{\sigma \in \text{Sh}_{p-1,q}} (-1)^{(p-1)q} (-1)^\sigma (L_0^0 p \circ_p L_0^0 q)^{-1}$$

holds.

An important feature which we will need, is that we can write $L_0^0$ as the antisymmetrization of elements $P_0^0$. Namely, we set

$$P_0^0 := \sum_{Q \in \text{Quilt}(n)} \text{deg}(Q) = n-2 \text{sgn}(Q) \cdot Q \text{ labelled in ↓ order}$$

then we have

$$L_0^0 = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma (P_0^0)^\sigma.$$
In fact, the $L_\infty$-relations translate to the following

$$\partial P_0^n + \sum_{p+q=n+1}^{p} \sum_{j=1}^{p} (-1)^{(p-1)q + (p-j)(q-1)} P_j^0 \circ P_q^0 \in (Q - Q^\sigma | \sigma \in S_n) \subseteq \text{Quilt}(n)$$

(5)

where $(-)$ denotes ‘free $k$-module generated by $-$’.

### 4.3 The morphism $L_\infty \longrightarrow \text{Quilt}_b[[c]]$

Next we will define more involved $L_\infty$-operations incorporating $c$.

**Definition 4.5** For $n + r \geq 2$ we define

$$P_r^n \equiv \sum_{1 \leq y_1 < \cdots < y_r \leq n+r} (-1)^{y_1, \ldots, y_r} P_{n+r}^0 \circ y_1 \circ y_2 \ldots \circ y_r - r + 1 c$$

where $(-1)^{y_1, \ldots, y_r}$ denotes the sign of the $(r, n)$-shuffle defined by $(y_1, \ldots, y_r)$. Using these we set

$$L_r^n := \sum_{\sigma \in S_n} (-1)\sigma \left( P_r^n \right)^\sigma$$

**Remark 4.6** Note that $L_r^n$ live in $\text{Quilt}[c]$.

Let us compute $P_r^n$ for some low $n$ and $r$.

**Example 4.7** In case no elements $c$ are added, we obtain the original $P_0^n$

$$P_2^0 = \begin{array}{c} 1 \\ 2 \end{array}, \quad P_3^0 = \begin{array}{c} 1 \\ 3 \\ 2 \end{array}$$

$$P_4^0 = - \begin{array}{c} 1 \\ 3 \\ 4 \\ 2 \end{array} - \begin{array}{c} 1 \\ 3 \\ 4 \\ 2 \end{array} - \begin{array}{c} 1 \\ 3 \\ 4 \\ 2 \end{array}$$

Similar to how we drew elements from $m\text{NSOp}$ as trees with vertices plugged by $m$, elements of $\text{Quilt}[c]$ can be drawn as quilts with rectangles plugged by $c$. For example, we have

$$P_1^1 = \begin{array}{c} c \\ 1 \end{array} - \begin{array}{c} 1 \\ c \end{array}$$
or

\[ P_2^2 = \begin{array}{ccc} c & c & 2 \\ 1 \end{array} - \begin{array}{ccc} c & 2 & c \\ 1 \end{array} - \begin{array}{ccc} 1 & c & c \\ 2 \end{array} - \begin{array}{ccc} 1 & c & c \\ 2 \end{array} \]

Note that depending on where we plug the elements \( c \) a sign is added.

This enables us to define the following.

**Definition 4.8**

\[
L_n := \sum_{r \geq 0} L_n^r \quad \text{for} \quad n \geq 2 \quad \text{and} \quad L_1 := \sum_{r \geq 1} L_1^r
\]

Set

\[
\partial' := \partial + \partial_{L_1}
\]

where

\[
\partial_{L_1}(A) := L_1 \circ_1 A - (-1)^{|A|} \sum_i A \circ_i L_1
\]

(a derivation by an element) which will be the new differential.

**Remark 4.9** Note that \( L_n \) are bounded because their components have constant degree \( n - 2 \). Hence, \( L_n \) live in Quilt_{b[[c]]}.

The main theorem of this section is the following.

**Theorem 4.10** The map

\[
L_\infty \longrightarrow (\text{Quilt}_{b[[c]]}, \partial') : l_n \longmapsto L_n
\]

defines a morphism of dg-operads.

First we need some lemmas.

**Lemma 4.11** For \( r \geq 2 \) we have \( L_0^r = 0 \).

**Proof** Due to relation (3) in Definition 4.3 of c, there cannot be a vertical composition of \( c \). Hence, \( L_0^r = 0 \) for \( r \geq 3 \). \( L_0^2 = 0 \) due to condition 2 of the definition of \( c \). \( \square \)

**Remark 4.12** Note that we have used 2 out of the three conditions on \( c \) to prove this lemma.

The following lemma extends the \( L_\infty \)-equation of \( (L_n^0)_n \) for higher \( L_n^r \).
Lemma 4.13 For $n \geq 1$ and $r \geq 0$, we have

$$-\partial(L_n^r) = \sum_{i+j=r} \sum_{\chi \in S_{h-1,l}} (-1)^{\chi} (-1)^{(k-1)l} \left( L_k^i \circ_k L_l^j \right)^{\chi^{-1}}.$$

**Proof** By applying $\partial$ and using Eq. (5) and $\partial(c) = 0$, we deduce

$$-\partial(L_n^r) = \sum_{p+q=n+r+1} \sum_{\sigma \in S_{h-1,l}} \sum_{1 \leq y_1 < \ldots < y_r \leq n+r} \sum_{x=1}^{p} \sum_{\tau \in S_{j+l}} \sum_{\chi \in S_{h-1,l}} (-1)^{\chi} (-1)^{(p-1)q} (-1)^{(p-x)(q-1)} \left( P_p^0 \circ_x P_q^0 \left( \sigma_{y-e+1} \right)_e \right)^{\sigma} \right).$$

Given $1 \leq y_1 < \ldots < y_r \leq n+r$ and $1 \leq x \leq p$, we have a subdivision into two groups where $\sigma_{y-e+1} \in S_n$ is inserted into either $P_p^0$ or $P_q^0$. Hence, if we are also given a permutation $\sigma \in S_{h-1,l}$, then we show that there exists unique integers $i + j = r$, $k + l = n + 1$, indices $1 \leq z_1 < \ldots < z_i \leq k + i$ and $1 \leq z_1' < \ldots < z_{l'}' \leq j + l$, permutations $\tau \in S_{k+i}$, $\tau' \in S_{j+l}$ and a shuffle $\chi \in S_{h-1,l}$ such that

$$(1)^{y_1, \ldots, y_r} \cdot (-1)^{\sigma} (-1)^{(p-1)q} (-1)^{(p-x)(q-1)} \left( P_p^0 \circ_x P_q^0 \left( \sigma_{y-e+1} \right)_e \right)^{\sigma}$$

$$= (-1)^{(z_1, \ldots, z_i)+(z_1', \ldots, z_{l'})}\left( L_k^i \circ_k L_l^j \right)^{\chi+\tau+\tau'}\left( L_k^{i+j} \circ_k \left( P_{l+j}^0 \left( \sigma_{z'-f+1} \right)_f \right) \right)^{\chi^{-1}}.$$

In this case, we obtain

$$-\partial(L_n^r) = \sum_{i+j=r} \sum_{\chi \in S_{h-1,l}} (-1)^{\chi} (-1)^{(k-1)l} \left( P_k^0 \circ_i P_l^0 \left( \sigma_{z-e+1} \right)_e \right)^{\tau} \circ_k \left( P_{l+j}^0 \left( \sigma_{z'-f+1} \right)_f \right)^{\tau'} \right)^{\chi^{-1}}.$$

By applying the definition of $L_n$, this proves the lemma.

We show that Eq. (6) holds: the set $y_1 < \ldots < y_r$ splits into three subsets

- $z_1^1 \ldots < z_i^1$ such that $z_i^1 < x$,
- $z_1^2 \ldots < z_{l'}^2$ such that $z_{l'}^2 > x + q - 1$,
We then compute
\[ z_t := \begin{cases} 
  z^1_t & t \leq i_1 \\
  z^2_{t-i_1} & t > i_1 
\end{cases} \quad z'_t := z^0_t - x + 1 \\
i := i_1 + i_2 \text{ and } j := r - i, \quad k := p - i \text{ and } l := q - j \]

We then compute

\[ P^0_p \circ P^0_q \left( o_{y_e-e+1} \right)_e = (-1)^{i(l+j)+i_2j} \left( P^0_{k+i} \left( o_{z_e-e+1} \right)_e \circ x_{-i_1} \left( P^0_{l+j} \left( o_{z_j-e+1} \right)_e \right) \right) \]

where the sign appears because we move the \( c \)'s corresponding to \( z^0_t \) past \( j \) \( c \)'s corresponding to \( z^0_i \) and also \( i \) \( c \)'s past \( P^0_q \). Note that if we know \( x \) and \( i_1 \), then given \( i, j, k, l, z_1, \ldots, z_i, z'_i, \ldots, z'_j \) we can uniquely determine \( p, q, r, n, y_1, \ldots, y_r \).

We also compute the sign of \( y_1, \ldots, y_r \): let \( \theta \) and \( \theta' \) be the shuffles such that

\[
\begin{align*}
  z^1_1, \ldots, z^1_{i_1} & \rightsquigarrow 1, \ldots, i_1 \\
  z^0_1, \ldots, z^0_j & \rightsquigarrow x, \ldots, x + j, \quad z^1_1, \ldots, z^1_{i_1} \rightsquigarrow x + q, \ldots, x + q + i_2 - 1 \\
  z^2_1, \ldots, z^2_{i_2} & \rightsquigarrow x + q, \ldots, x + q + i_2 - 1 \rightsquigarrow i_1 + j + 1, \ldots, r
\end{align*}
\]

then we obtain that

\[ (-1)^{y_1 \ldots y_r} = (-1)^{\theta + (z^0_1, \ldots, z^0_j) + j(x-i_1-1) + \theta' + i_2(l+x-i_1-1)} \]

On the other hand,

\[ z_1, \ldots, z_{i_1}, z_{i_1+1}, \ldots, z_i \rightsquigarrow \theta + \theta' \rightsquigarrow 1, \ldots, i_1, x + 1, \ldots, x + i_2 \]

and thus we have

\[ (-1)^{y_1 \ldots y_r} = (-1)^{(z_1, \ldots, z_i) + (z^0_1, \ldots, z^0_j) + j(x-i_1-1) + i_2(l+1)} \]

Given \( \sigma \in \mathbb{S}_n \), let \( b_1 < \ldots < b_l \subseteq \langle k + l - 1 \rangle \) such that \( \sigma(b_l) \in \langle x - i_1, \ldots, x - i_1 + l - 1 \rangle \) and let \( a_1 < \ldots < a_{k_1} := \langle k + l - 1 \rangle \setminus \{ b_1 < \ldots < b_l \} \). We then define the \( (k-1,l) \)-shuffle \( \chi = (a_1 < \ldots < a_{k-1}, b_1 < \ldots < b_l) \) and

\[ \tau(t) = \begin{cases} 
  \sigma \chi(t) & t < k \\
  x - i_1 & t = k
\end{cases} \quad \text{and} \quad \tau'(t) = \sigma(b_l) - (x - i_1 - 1) \]
It is then easy to see that
\[
\left( p_{k+l}^0 \circ (\varphi_i^{e-(r+1)c}) \right) = \left( p_{k+l}^0 \circ (\varphi_i^{e-(r+1)c}) \right) = \left( p_{k+l}^0 \circ (\varphi_i^{e-(r+1)c}) \right) = \left( p_{k+l}^0 \circ (\varphi_i^{e-(r+1)c}) \right) = \left( p_{k+l}^0 \circ (\varphi_i^{e-(r+1)c}) \right)
\]
Note that \( \tau \) and \( z_i, \ldots, z_i \) determine \( x \) and \( i_1 \) uniquely. Hence, given \( z_i, \ldots, z_i, z_i', \ldots, z_i', k, l \) we can uniquely determine \( y_1, \ldots, y_r, p, q, n, r \) in the above manner. In order to show that Eq. (6) holds, we only need to verify the corresponding signs: let \( \tau \) correspond to
\[
\sigma \chi(1), \ldots, \sigma \chi(k-1) \xrightarrow{\tau_0} 1, \ldots, x - i_1 - 1, x - i_1 + l, \ldots, k + l - 1 \]
\[
\sigma \chi(k), \ldots, \sigma \chi(k + l - 1) \xrightarrow{\tau'} x - i_1, \ldots, x - i_1 + l
\]
and thus we have that \(-1)^{\sigma+\chi} \xrightarrow{(-1)^{\tau_0+\tau'+l(k-x+i_1)}} 1, \ldots, x - i_1 - 1, x - i_1 + 1, \ldots, k - 1, \tau(k) \xrightarrow{\kappa^{-1+i_1}} 1, \ldots, k - 1
\]
and thus we have
\[
(-1)^{\sigma+\chi} \equiv (-1)^{\tau+\tau'+(l+1)(k-x+i_1)}
\]
As such, we can compute
\[
(-1)^{z_1, \ldots, z_i} \xrightarrow{(z_1', \ldots, z_i')} (-1)^{x+\tau'} \xrightarrow{(-1)^{(k-1)i(l+j)+i_2j}} (-1)^{y_1, \ldots, y_r+\sigma_2(l+1)+j(x-i_1-1)+(l+1)(k-x+i_1)+(k-1)l+i(l+j)+i_2j}
\]
and
\[
i_2(l+1) + j(x - i_1 - 1) + (l+1)(k-x+i_1) + (k-1)l + i(l+j) + i_2j = (l+1)(k-x+i) + j(x - i - 1) + (k-1)l + i(l+j)
\]
\[
= (l+1)(p-x) + j(p-x+(k-1)) + (k-1)l + iq = (q-1)(p-x) + (k-1)q + iq = (q-1)(p-x) + (p-1)q
\]
which completes the proof.

\begin{lemma}
\textbf{Lemma 4.14} \( L_n \) are skew symmetric and \( \partial' \) is a differential making Quiltb[[c]] into a dg-operad.
\end{lemma}

\begin{proof}
It is clear from the definition of \( L_n' \) that they are skew symmetric and thus also \( L_n' \).
Per definition \( \partial_{L_n} \) is a derivation by construction and so is \( \partial \), and thus so is \( \partial' \).
\end{proof}
It is clear from the definition of $c$ that $\partial'(c) = 0$. Hence, we only need to show that $\partial'\partial'(Q) = 0$ for every $Q \in \text{Quilt}$. Using Lemmas 4.11 and 4.13 we first prove that $-\partial L_1 = L_1 \circ_1 L_1$. Namely, we compute

$$-\partial(L_1) = \sum_{r \geq 1} \sum_{i+j=r} \sum_{\chi \in Sh_{k-1}, l} (-1)^k (-1)^{k-1} (L_i \circ k L_j)^{\chi-1}$$

$$= \sum_{i+j=r \atop i,j \geq 1 \atop r \geq 1} L_i \circ_1 L_j = L_1 \circ_1 L_1$$

Now let us compute

$$\partial'\partial' Q = \partial \partial Q + \partial \partial L_1 Q + \partial L_1 \partial Q + \partial L_1 \partial L_1 Q$$

we compute separately

$$\partial L_1 \partial Q = L_1 \circ_1 \partial Q - (-1)^{|Q|-1} \sum_i \partial Q \circ_i L_1$$

and

$$\partial \partial L_1 Q = \partial (L_1 \circ_1 Q) - (-1)^{|Q|} \sum_i \partial (Q \circ_i L_1)$$

$$= \partial L_1 \circ_1 Q + (-1)^{|Q|} L_1 \circ_1 \partial Q - (-1)^{|Q|} \sum_i \partial Q \circ_i L_1 - (-1)^{|Q|-1} \sum_i \partial L_1$$

adding them gives

$$\partial L_1 \partial Q + \partial \partial L_1 Q = \partial L_1 \circ_1 Q - (-1)^{|Q|-1} \sum_i Q \circ_i \partial L_1$$

Next, we compute $\partial L_1 \partial L_1 Q$. As $L_1$ has only a single input and has degree $-1$, we have for $i \neq j$ that $Q \circ_i L_1 \circ_j L_1 = -Q \circ_j L_1 \circ_i L_1$. Hence, by also using using Eq. (9), we obtain

$$\partial L_1 \partial L_1 Q = L_1 \circ_1 (L_1 \circ_1 Q) - (-1)^{|Q|} \sum_i L_1 \circ_1 (Q \circ_i L_1)$$

$$+ (-1)^{|Q|} \sum_i (L_1 \circ_1 Q) \circ_i L_1 - \sum_{i,j} Q \circ_i L_1 \circ_j L_1$$

$$= L_1 \circ_1 L_1 \circ_1 Q - \sum_i Q \circ_i (L_1 \circ_1 L_1)$$

$$= \partial L_1 \partial Q + \partial \partial L_1 Q$$
As \( \partial \partial Q = 0 \) we thus obtain \( \partial' \partial' Q = 0 \). \( \square \)

**Proof of Theorem 4.10** We need to show for every \( n \geq 2 \) that the equation

\[
0 = \partial' (L_n) + \sum_{k+l=n+1} \sum_{k,l \geq 2} (-1)^{(k-1)(-1)} \left( L_k \circ L_l \right)^{\sigma^{-1}}
\]

holds, which is equivalent to

\[
0 = \partial (L_n) + \sum_{k+l=n+1} \sum_{k,l \geq 1} (-1)^{(k-1)(-1)} \left( L_k \circ L_l \right)^{\sigma^{-1}}
\]

This is equivalent to showing for every \( r \geq 0 \) that the equation

\[
0 = \partial (L^r_n) + \sum_{i+j=r} \sum_{k,l \geq 1} (-1)^{(k-1)(-1)} \left( L^i_k \circ L^j_l \right)^{\sigma^{-1}}
\]

holds, which follows from Lemma 4.13 and \( L^0_1 = 0 \) for \( i \geq 0 \) (Lemma 4.11). \( \square \)

**Remark 4.15** Under the natural morphism \( \text{Quilt} [c] \rightarrow \text{mQuilt} \) sending \( c \) to \( m \) our \( L_\infty \)-structure corresponds to the \( L_\infty \)-structure from [8, Thm. 7.13], that is, we have the commutative diagram

\[
\begin{array}{ccc}
\text{Quilt}_b[[c]] & \rightarrow & \text{Quilt}_b[[m]] \\
\downarrow & & \downarrow \\
L_\infty & \rightarrow & \text{mQuilt}
\end{array}
\]

where \( \text{Quilt}_b[[m]] \) denotes the quotient of the operad of bounded power series by the relations on \( m \) in \( \text{mQuilt} \).

**4.4 The morphism \( \text{Quilt}_b[[c]] \rightarrow \text{End}(sC_{GS}(A)) \)**

In this section, we make (the suspension of) \( C_{GS}(A) \) into a \( \text{Quilt}_b[[c]] \)-algebra.

The morphism \( R : \text{Quilt} \rightarrow \text{End}(sC_{GS}(A)) \) naturally extends to a morphism of graded operads \( R_c : \text{Quilt}[c] \rightarrow \text{End}(sC_{GS}(A)) \) by sending \( c \) to \( c \in C_{GS}(A) \) as the axioms of \( \text{Quilt}[c] \) correspond respectively to \( c \) being a natural transformation, the cocycle condition of \( c \) and \( c \in C^{2,0}(A) \). Next, we will show that it further extends to the operad of bounded power series.

**Lemma 4.16** Let \( \theta_1, \ldots, \theta_n \in C_{GS}(A) \) where \( \theta_i \in C^{p_i,q_i}(A) \) and \( Q \in \text{Quilt}[c](n, r) \) of degree \( t \), then \( R_c(Q)(\theta_1, \ldots, \theta_n) = 0 \) if \( r > \sum_{i=1}^{n} q_i - t \).
Proof: This follows from $C_{GS}(\mathcal{A})$ having only non-negative bidegree. Namely, given $Q = Q' \circ_{i_1} c \circ_{i_2} \ldots \circ_{i_r} c$ for a certain $Q' \in \text{Quilt}(n + r)$, the bidegree of $R_c(Q')$ is $(\sum_{i=1}^{n} p_i + 2r - (n + r - 1), \sum_{i=1}^{n} q_i - \deg(Q'))$. As $\deg(Q') = \deg(Q) + r = t + r$, we have that
\[
\sum_{i=1}^{n} q_i - \deg(Q') < 0 \iff r > \sum_{i=1}^{n} q_i - t
\]
proving the lemma.

\[\square\]

Theorem 4.17 For $\mathcal{A} = (A, m, f, c)$ a linear prestack over $\mathcal{U}$, the map

$$R_c : (\text{Quilt}_b[[c]], \partial') \longrightarrow \text{End}(sC_{GS}(\mathcal{A}), d)$$

defined as $R_c(Q) = R(Q)$ for $Q \in \text{Quilt}$ and $R_c(c) = c$, is a morphism of dg-operads.

Proof: The representation $R_c$ extends to $\text{Quilt}_b[[c]]$ due to Lemma 4.16.

We verify that $R_c \partial' Q = \partial d R_c Q$ for $Q \in \text{Quilt}$. It suffices to verify that $R_c L^j_1 = d_j$ as $\partial$ corresponds to $\partial_{d_0}$. Let $\theta \in C^{p+q}(\mathcal{A})$ and $\sigma \in N_{p+q}(\mathcal{U})$ and note that we write $|\theta|$ for the degree $p + q$ of $\theta$ in $C_{GS}(\mathcal{A})$.

Step 2 $R_c L^1_1 = d_1$

We can write $d_1 = \sum_{s=0}^{p+1} (-1)^{p+q+1+s} d_1^s$ where $d_1^s$ names the $s$th component of $d_1$. We write out the left-hand side

$$R_c L^1_1(\theta) = R(P^0_1)(\theta, c) - (-1)^{|\theta|-1} R(P^0_2)(\theta, c) \tag{11}$$

where $P^0_2 = (-1)^{21+1}(12, 1 \rightarrow 2)$. There exists two colorings $(X_0, J_0)$ and $(X_{p+1}, J_{p+1})$ of the quilt $Q = (W, T)$ in $P^0_2(c, \theta)$ given by the patchworks

\[
(X_1, J_1) = \begin{array}{|c|c|}
1 & q \\
\hline 1 & \ddots \\
\hline \end{array} \quad (X_{p+1}, J_{p+1}) = \begin{array}{|c|c|}
1 & q \\
\hline 1 & \ddots \\
\hline \end{array}
\]

which correspond respectively to $d_1^0$ and $d_1^{p+1}$. We verify their signs: the sign $\text{sgn}_Q(X_0, J_0) = \text{sgn}_W(X_0) \text{sgn}_T(J_0)(-1)^q$ is determined by two shuffles

$$0_11_10_2 \ldots (q - 1)_2 \sim 0_11_10_2$$

$$1_2 \ldots (p - 1)_21_1 \sim 1_11_2 \ldots (p - 1)_2$$
and thus \( \text{sgn}_Q(X_0, J_0) = (-1)^{p-1+q} \). In the second case, we have \( \text{sgn}(X_{p+1}, J_{p+1}) = \text{sgn}_W(X_{p+1}) \text{sgn}_T(J_{p+1})(-1)^q \) determined by the two shuffles

\[
0_11_10_2 \ldots (q-1)_2 \sim 0_11_10_2 \\
1_11_2 \ldots (p-1)_2 \sim 1_11_2 \ldots (p-1)_2
\]

Hence, \( \text{sgn}_Q(X_{p+1}, J_{p+1}) = (-1)^q \).

There exists \( p \) colorings \( (X_i, J_i)_{i=1}^p \) of \( P_2^0 \) occuring in the second term of (11), given by the patchworks

\[
(X_i, J_i) = \begin{array}{c}
\begin{array}{cccc}
1 & \cdots & q \\
& & 1 \\
& & & \vdots \\
& & & i \\
& & & \vdots \\
& & & i+1 \\
& & & \vdots \\
& & & p+1 \\
\end{array}
\end{array}
\]

for \( i = 1, \ldots, p \). They correspond to \( d_i^{p+1-i} \) and we verify their signs: we have \( \text{sgn}_Q(X_i, J_i) = \text{sgn}_W(X_i) \text{sgn}_T(J_i) \) determined by the two shuffles

\[
0_1 \ldots (q-1)_1 0_2 1_2 \sim 0_1 \ldots (q-1)_1 0_2 1_2 \\
1_1 \ldots (i-1)_1 1_2 1_2 \ldots (p-1)_1 \sim 1_1 \ldots (p-1)_1 1_2
\]

Hence, we obtain the sign

\[
-(-1)^{p+q-1} \text{sgn}_Q(X_i, J_i) = (-1)^{q-i}
\]

Step 2 \( R_c L_j^1 = d_j \)

First, we name the terms of \( d_j \) and write

\[
d_j(\theta^\sigma) := \sum_{\beta \in S_{q+1-j}} \sum_{r \in P(R_p(\sigma))} (-1)^{r+\beta+q-j+1} d_j(\theta^\sigma)(\beta, r)
\]

such that

\[
d_j(\theta^\sigma)(\beta, r) \in \prod_{A \in \mathcal{A}(U_p)}^{q} \text{Hom}(\bigotimes_{i=1}^{q} \mathcal{A}(U_p)(A_i, A_{i-1}), \mathcal{A}(U_0)(\sigma^A q_{-j+2}, \sigma^* A_0))
\]

Next, we note that the only non-vanishing term of \( L_j^1 \) is given by \((-1)^{j-1} P_{j+1}^0 \circ_{01} c \circ_{02} c \circ_{02} \ldots \circ_{02} c \) as \( c \) cannot have any children vertically. Moreover, the only non-
vanishing quilts are of the form

\[ Q_T = (W', T') = \begin{pmatrix} \text{12324} \ldots \text{2} (j+1) \text{2} \end{pmatrix} \]

for any binary tree \( T \) on \( j - 1 \) vertices whose numbering is compatible with the word component \( W \) of \( Q_T \). Hence, the possible colorings of \( Q_T \) are

\[ (X, J') = \begin{pmatrix} \text{3} \ldots \text{j+1} \\ \text{2} \\ \text{1} \\ \text{m} \\ \text{1} \\ \text{2} \\ \text{2} \end{pmatrix} \]

for \( t_1 < \ldots < t_{j-1} \) in \( \langle q \rangle \) and \( J \) an indexed tree coloring \( T \).

As a result, we have that

\[ R \sigma L_1^j (\theta) = (-1)^{(j-1)+j(p+q-1)} R P_{j+1}^0 (c, \theta, c, \ldots, c) \]

which sums over the terms

\[ \text{sgn}_{Q_T} (X, J') (-1)^{(j+1)j+1} L (X, J') \]

We finish the proof by showing that a formal path \( r \) and a shuffle \( \beta \in S_{q+1-j,j-1} \) correspond uniquely to such a binary tree \( T \) and a coloring \( (X, J') \) of \( Q_T \) such that

\[ (-1)^{r+\beta+(q+1-j)} d_j (\theta)^\sigma (\beta, r) = \text{sgn}_{Q} (X, J') (-1)^{j(j+1)j+1} (-1)^{(j-1)+(p+q-1)} L (X, J') (c, \theta, c, \ldots, c) (\theta)^\sigma \]

Given a formal path \( r = (r_1, \ldots, r_{j-1}) = ((\tau_1, i_1), \ldots, (\tau_{j-1}, i_{j-1})) \) we first define \( T \) and its coloring \( J \) as trees with vertex set \{3, \ldots, j + 1\} inductively:

- In the degenerate case \( j = 2 \), \( r \) is uniquely determined and we set \( T \) to be the one vertex tree and \( J \) the empty function.
- For \( j > 2 \), let \((T_0, J_0)\) be the indexed tree corresponding to \((\tau_1, i_1), \ldots, (\tau_{j-2}, i_{j-2})\), in order to add vertex \( j + 1 \) we have three cases
  1. If \( i_{j-2} < i_{j-1} \), then set \( j \prec J \) and start over with \((i_{j-1}, i_{j+1})\)
(2) If \( i_{j-1} = i_{j-2} \) or \( i_{j-1} = i_{j-2} + 1 \), then let \( (j, j+1) \in E_T \) and set resp. \( J(j, j+1) = 2 \) or \( = 1 \).

(3) If \( i_{j-1} > i_{j-2} + 1 \), then set \( (j+1) \prec_j j \) and start over with \( (i_{j-1}, i_{j+1} - 1) \) which we can draw as follows

```
(1) \[
\begin{array}{c}
\; j \\
\downarrow \quad u_{i_{j-2}+2}^*
\; j + 1 \\
\downarrow \quad u_{i_{j-1}+1}^*
\end{array}
\]
(2) \[
\begin{array}{c}
\; j \\
\downarrow \quad u_{i_{j-1}+2}^*
\; j + 1 \\
\downarrow \quad u_{i_{j-1}+1}^*
\end{array}
\]
(3) \[
\begin{array}{c}
\; j + 1 \\
\downarrow \quad u_{i_{j-1}+1}^*
\; j \\
\downarrow \quad u_{i_{j-1}+1}^*
\end{array}
\]
```

Clearly, this process is reversible: given \( (T, J) \) we obtain a unique sequence \( r \in P(\mathcal{R}_p(\sigma)) \). By identifying the shuffle \( \beta \in Sh_{q+1-j, j-1} \) and \( X \) via \( t_1 = \beta(q-j+1+l) \), we clearly obtain that \( L(X, J') \theta^\sigma = \mathcal{L}(X, J') \theta^\sigma = d_j(\theta)^\sigma(\beta, r) \).

The remaining work is to verify the signs: \( \text{sgn}_{Q_T}(X, J') \) consists of three components

\[
\text{sgn}_{W^*}(X) \text{sgn}_{T^*}(J')(\mathbf{1})^\sigma
\]

where \( (\mathbf{1})^\sigma = (\mathbf{1})^q \) as it corresponds to the shuffle

\[
0, q, 0, \ldots, 0, 1, p - 1, 1, \ldots, 1 \sim 0, 1, q, p - 1, 0, 1, \ldots, 0, 1.
\]

The sign \( \text{sgn}_{W^*}(X) \) corresponds to the shuffle transforming the word

\[
0_{(j+1)} \ldots 0_{(j+1)} 0_{(j+1)} \ldots \beta(t+1) - 1)_{(j+1)} \beta(t+1) \ldots 1_{(j+1)} \beta(t+1) - 1)_{(j+1)} (q - 1)_{(j+1)}.
\]

for \( t = q - j + 1 \), into the word

\[
0_1 \ldots (q_1 - 1)_1 \ldots 0_{(j+1)} \ldots (q_{j+1} - 1)_{(j+1)}.
\]

We observe that shuffling the second part (2)

\[
0_2 \ldots \beta(t+1) - 1)_{(j+1)} l_{2} \beta(t+1) \ldots 1_{(j+1)} \beta(t+1) - 1)_{(j+1)} (q - 1)_{(j+1)} \sim 0_2 \ldots (q_2 - 1)_{(j+1)} l_{2} \beta(t+1) \ldots 1_{(j+1)}
\]

almost corresponds to the shuffle \( \beta \). However, there is in every interval \( \beta(s)_{(j+1)} \), (\( \beta(s+1) - 1)_{(j+1)} \) exactly one element too many. We remedy this by moving \( 1_{(j-2)} \) one
place to the right, then $1_{j-3}$ two place, and so on. As such, its corresponding sign is 
\((-1)^{\beta+\sum_{i=1}^{j-2} i} = (-1)^{\beta+\frac{(j-2)(j-1)}{2}}\). Next, we shuffle 
\[
0_{(j+1)} \ldots 0_3 0_1 1_1 0_2 \ldots (q_2 - 1) 1_3 \ldots 1_{(j+1)} \\
\leadsto 0_1 1_1 0_2 \ldots (q - 1) 2_3 1_3 \ldots 0_{(j+1)} 1_{(j+1)}
\]
whose sign is $(-1)^{(j-1)(q+2)}$. Hence, we obtain that 
\[
\text{sgn}_{W'}(X) = (-1)^{\beta+\frac{(j-1)(j-2)}{2}+j-1}.
\]
Next, we determined $\text{sgn}_{T'}(J')$ as the sign of the shuffle 
\[
[C_J]_1 1_2 \ldots (p - 1)_2 \leadsto 1_1 1_2 \ldots (p - 1)_2 1_3 \ldots 1_{(j+1)},
\]
where $[C_J]$ denotes the word obtained from the indexed tree $J$. We will show that 
the sign corresponding to the shuffle $\chi : [C_J] \leadsto 1_3 \ldots 1_{(j+1)}$ is $(-1)^{r+j-1}$. As a consequence, we obtain that 
\[
\text{sgn}_{T'}(J') = (-1)^{r+j-1+p(j-1)},
\]
and thus that 
\[
\text{sgn}(X, J') = (-1)^{\beta+r+q+j-1}(-1)^{\frac{(j-1)j}{2}+(j-1)(p+q-1)}.
\]
Hence, we have 
\[
\text{sgn}_O(X, J')(1)^{j(j+1)}(-1)^{\frac{1(j+1)}{2}+1}(-1)^{j(j-1)+(j-1)(p+q-1)} = (-1)^{\beta+r+q-j+1}
\]
which completes the proof.

We compute $\chi$ inductively: we have that $[C_J] = AI_{(j+1)}B$ for certain words $A$ and 
$B$ and let $\chi_0$ denote the shuffle $AB \leadsto 1_3 \ldots 1_j$. By induction we know that $(-1)^{\chi_0} = 
(-1)^{r_0+j-2}$ for the formal path $r_0 = (r_1, \ldots, r_{j-2})$ and $(-1)^{l'} = (-1)^{r_0+j-1}$ where 
$r_{j-1} = (\sigma, i_{j-1})$. Moreover, we have $(-1)^{\chi} = (-1)^{\chi_0+|B|}$ where $|B|$ denotes the 
length of $B$. We determine $|B|$. First, observe that the sequence associated to two 
indexed trees 
\[
3 \quad 1 \quad 4 \quad 3 \quad 2 \quad 4
\]
is respectively $1_{h_4}1_{h_3}$ and $1_{h_3}1_{h_4}$. Thus, let $3 = v_1 <_T \ldots <_T v_t = j + 1$ be the 
unique chain of vertices from the root of $T$ to $j + 1$, then we can define the numbers 
- $l$ as the the number of vertices to the right of vertex $j + 1$ in $T$, 
- $k$ as the number of vertices in the above chain such that $J(v_k, v_{k+1}) = 1$.

We then easily compute that
• The height \( i_{j-1} \) of \( r_{j-1} = (\partial_{i_{j-1}}, \sigma, i_{j-1}) \) is exactly \((1 + l) + k\),
• Length of \( B \) is \( l + k \)

and thus by induction we obtain

\[
(-1)^x = (-1)^{x_0 + |B|} = (-1)^{x_0 + i_{j-1} + 1} = (-1)^{r + (j-1)}
\]

where the last equality follows from induction. \(\square\)

**Remark 4.18** In the case of presheaves, when looking at the subcomplex \( \overline{C}_{GS}(A) \) of normalised and reduced cochains, the map \( R_c \) factorises through Quilt,\([m]\) as \( m \) is sent to the identity \( 1_u \cdot v^* = (vu)^* \). Only in the case of normalised and reduced cochains does the identity satisfy all the relations on \( m \).

Note however that our induced morphism \( m_{\text{Quilt}} \rightarrow \text{Quilt},[[m]] \rightarrow \text{End}(\overline{C}_{GS}(A)) \) does not correspond to the mQuilt-algebra structure [8, Thm. 5.6].

However, the resulting \( L_{\infty} \)-algebra structure, in the case of presheaves, does correspond to the one obtained from [8, Thm. 5.6, Thm. 7.13]. This is essentially due to the multiplication of the prestack being unital. A mQuilt-algebra structure also induces a Gerstenhaber-algebra structure on cohomology [8, Thm. 6.11]. Writing down the relevant quilts, it is also easy to see that both mQuilt-algebra structures (ours and [8, Thm. 5.6]) induce the same Gerstenhaber-algebra structure on cohomology.

### 4.5 Deformations of prestacks

Let \( k \) be a field of characteristic 0. Let \((C, d, (L_n)_{n \geq 2})\) be an \( L_{\infty} \)-algebra. By definition, the Maurer–Cartan equation for \( \theta \in C \) is given by

\[
MC(\theta) = d(\theta) + \sum_{n=1}^{\infty} \frac{1}{n!} L_n(\theta, \ldots, \theta).
\]

We consider the set of (degree 1) Maurer–Cartan elements \( MC(C) = \{ \theta \in C^1 \mid MC(\theta) = 0 \} \) and for the appropriate notion of gauge equivalence (see [9]), we consider

\[
\overline{MC}(C) = \{ \theta \in C^1 \mid MC(\theta) = 0 \}/\sim.
\]

This gives rise to a functor \( \overline{MC} : \text{Art} \rightarrow \text{Set} : (R, m) \mapsto \overline{MC}(m \otimes C) \) on the category Art of Artin local \( k \)-algebras.

Consider the GS-complex \((C_{GS}(A), d)\) of a prestack \((A, (m + f + c))\). In [2, Thm. 3.19] it is shown that normalised reduced 2-cocycles in \( C_{GS}(A) \) correspond to first order deformations of \( A \), that is, deformations in the direction of \( R = k[\epsilon] \). More precisely, for \((m', f', c') \in C_{GS}^2(A)\), we have that \((A[\epsilon], m+m', f + f'\epsilon, c+c'\epsilon)\) is a first-order deformation of \((A, m, f, c)\) if and only if \( d(m', f', c') = 0 \) and \((m', f', c')\) is normalised and reduced. Further, it is shown in loc. cit. that the cohomology of the GS-complex classifies the first-order deformations of \( A \) up to equivalence.
Putting Theorems 4.10 and 4.17 together, $sC_{GS}(A)$ is endowed with an $L_\infty$-structure which can be used to obtain higher order versions of these results.

For normalised, reduced cochains, it will be convenient to express the MC-equation in terms of the unsymmetrised components $(P_n)_{n \geq 1}$ from Definition 4.5. The following characteristic-free expression of the MC-equation should be seen as the counterpart of the equation $MC(\theta) = d(\theta) + \theta \{ \theta \}$ for the first brace operation (or “dot product”) on the Hochschild complex of an algebra. Note that we omit writing $R_c$ and consider everything as elements of $\text{End}(sC_{GS}(A))$.

**Proposition 4.19** For a reduced and normalised cochain $\theta = (m', f', c') \in sC^1_{GS}(A)$, we have

$$MC(\theta) = d(\theta) + P_1(\theta) + P_2(\theta, \theta) + P_3(\theta, \theta, \theta) + P_4(\theta, \theta, \theta, \theta).$$

In particular, the MC-equation is quartic.

**Proof** First note that as $|\theta| = 1$, that $Q^\sigma(\theta, \ldots, \theta) = (-1)^\sigma Q(\theta, \ldots, \theta)$. Moreover, as we can consider $c \in C^{2,0}(A)$, the MC-equation consists of $d_0(\theta')$ and summations of $Q(\theta_1, \ldots, \theta_n)$ for quilts $Q$ of degree $n - 2$ and $\theta_i \in C^2_{GS}(A)$. Let $Q = (W, T)$ be such a quilt, part of some $P^0_n$, then $W = 12 \ldots 2$. For $n \geq 3$, we know that 3 is also a child in $T$ of 1. As 1 can have at most two children in $T$, for $n \geq 4$, we thus have 4 and 3 are children of 2 in $W$, i.e. $W = 12324 \ldots 2$.

In this case, only the elements $c'$ or $c$, and $m'$ can be inserted in $Q$ respectively in 1 and 2, with 2 a child of 1 in $T$. As $c'$ is reduced, this means that $Q(\theta, \ldots, \theta) = 0$ for $n \geq 4$.

As $c$ is not necessarily reduced, more quilts are possible. Note that we write $\theta[i]$ to refer to the element $\theta$ inserted in vertex $i$. The above reasoning still applies and we can once more apply this reasoning to obtain for $n \geq 6$ that $W = 12324546 \ldots 42$ and 1 and 3 have two children in $T$.

This means that 2 and 4 need be inserted by $m'$ and hence 3 and 1 by either $c'$ or $c$. In case either one is $c'$, we already know it is zero as $c'$ is reduced. Thus, consider vertex 1 and $c$ inserted by $c$, then, as $m'[4]$ is a child of $c[3]$ in $T$, $c[3]$ is the unit of the corresponding category $A(U)$ and it is plugged into $m'[2]$ which is normalised, whence we obtain that $Q(\theta_1, \ldots, \theta_n) = 0$ for $n \geq 6$. Hence, $P^0_n(\theta_1, \ldots, \theta_n) = 0$ for $n \geq 6$. 
Combining the above reasonings, we have that $P_n(\theta, \ldots, \theta) = 0$ for $n > 5$. In the case $n = 5$ there has to be at least one $c$ present and thus the non-zero terms are contained in $P_4$. \hfill \Box

**Proposition 4.20** For a prestack $(A, m, f, c)$ and $\theta = (m', f', c') \in sC^1_{GS}(A)$, we have that $(A, m + m', f + f', c + c')$ is a prestack if and only if $MC(\theta) = 0$ and $\theta$ is normalised and reduced.

**Proof** Using the fact that $\theta = (m', f', c') \in C^2_{GS}(A)$ is reduced and normalised, we compute $MC(\theta)$ and look at each component $MC(\theta)_{[p,q]} \in C^{p,q}(A)$ for $p + q = 3$. We will use that $c^{1,-} = 1 = c^{-1}, m$ is unital and that $(m, f, c)$ satisfy the axioms of a prestack. Note that for a cochain $\theta = (\theta^\sigma(A))_{\sigma,A}$, we omit writing the set of objects $A$ explicitly where possible in order to lighten the equations below.

For $(p, q) = (0, 3)$, let $U \in \mathcal{U}$ and $A = (A_0, A_1, A_2, A_3)$ objects in $\mathcal{A}(U)$, then we compute

$$MC(\theta)^U_{[0,3]}(A) = d\theta^U_{[0,3]}(A) + P_2(\theta, \theta)^U_{[0,3]}(A) + P_3(\theta, \theta)^U_{[0,3]}(A) + P_4(\theta, \theta, \theta)^U_{[0,3]}(A)$$

$$= d_0(m') + P_3^0(c, m', m')$$

$$= m^U \circ (1^U, m'^U) - m^U \circ (m^U, 1^U) + m^U \circ (1^U, m^U) - m^U \circ (m'^U, 1^U)$$

$$- m^U \circ (c_1^{1U,1U}(A_0), m'^U \circ (m^U, 1^U)) + m^U \circ (c_1^{1U,1U}(A_0), m'^U \circ (1^U, m'^U))$$

$$= (m^U + m'^U) \circ (1^U, (m^U + m'^U)) - (m^U + m'^U) \circ (m^U + m'^U, 1^U)$$

For $(p, q) = (1, 2)$, let $U \in \mathcal{U}$ and $A = (A_0, A_1, A_2)$ objects in $\mathcal{A}(U)$, then we compute

$$d\theta^U_{[1,2]}(A) = d_0(f')^u(A) + d_1(m')^u(A)$$

$$= m^U_0 \circ (f^u, f'^u) - f'^u \circ m^U_1 + m^U_0 \circ (f'^u, f^u)$$

$$+ m^U_0 \circ (c^{u,1U_0}(A_0), m'^U_0 \circ (f'^u, f^u))$$

$$- m^U_0 \circ (c^{1U_1,1U_0}(A_0), f^u \circ m'^U_1)$$

$$P_2(\theta, \theta)^U_{[1,2]}(A) = P_2^0(f', f'^u)(A) + P_3^0(c, m', f')^u(A) + P_3^0(c, f', m')^u(A)$$

$$= m^U_0 \circ (f'^u, f'^u) + m^U_0 \circ (c^{u,1U_0}(A_0), m'^U_0 \circ (f'^u, f^u))$$

$$+ m^U_0 \circ (c^{u,1U_0}(A_0), m'^U_0(f^u, f'^u))$$

$$- m^U_0 \circ (c^{1U_1,1U_0}(A_0), f'^u \circ m'^U_1)$$

$$P_3(\theta, \theta, \theta)^U_{[1,2]}(A) = P_4^0(c, m', f', f'^u)(A)$$

$$= m^U_0 \circ (c^{u,1U_0}(A_0), m'^U_0 \circ (f'^u, f'^u))$$

$$P_4(\theta, \theta, \theta, \theta)^U_{[1,2]}(A) = 0$$
and thus

\[ MC(\theta)_{[1,2]}(A) = -(f^u + f^u) \circ (m^U_0 + m^U_0) + (m^U_0 + m^U_0) \circ \left( (f^u + f^u) \otimes (f^u + f^u) \right) \]

For \((p, q) = (2, 1)\), let \(\sigma = (U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} U_2)\) be a 2-simplex in \(\mathcal{U}\) and \(A = (A_0, A_1)\) objects in \(\mathcal{A}(U_2)\), then we compute

\[
d\theta^\sigma_{[2,1]}(A) = d_0(c')^\sigma(A) + d_1(f')^\sigma(A) + d_2(m')^\sigma(A)
\]

\[
= m^U_0 \circ \left( f^u_2, c' \sigma(A_1) \right) - m^U_0 \circ \left( c' \sigma(A_0), f^u_2 \circ f^u_2 \right)
\]

\[
- m^U_0 \circ \left( c' \sigma(A_0), f^u_1 \circ f^u_2 \right)
\]

\[
+ m^U_0 \circ \left( f^u_2, c' \sigma(A_1) \right) - m^U_0 \circ \left( c' \sigma(A_0), f^u_1 \circ f^u_2 \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, c' u_2(A_0), m^U_0 \circ \left( c'^u_1, u_2(A_0), f^u_2 \circ f^u_1 \right) \right)
\]

\[
+ m^U_0 \circ \left( c'^u_1, c' u_2(A_0), m^U_0 \circ \left( f^u_2, c' u_2(A_1) \right) \right)
\]

\[
P^2(\theta, \theta)^\sigma_{[2,1]}(A) = P^0_2(f', c')^\sigma(A) + P^0_2(c', f')^\sigma(A)
\]

\[
+ P^0_3(c, f', f')^\sigma(A) + P^0_3(c, m', c')^\sigma(A)
\]

\[
+ P^0_4(c, m', f', c)^\sigma(A) + P^0_4(c, m', c', f)^\sigma(A)
\]

\[
= m^U_0 \circ \left( f^u_2, c'^u_1, u_2(A_1) \right) - m^U_0 \circ \left( c'^u_1, u_2(A_0), f^u_1 \circ f^u_2 \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, u_2(A_0), f^u_1 \circ f^u_2 \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, u_2(A_0), f^u_1 \circ f^u_2 \right)
\]

\[
+ m^U_0 \circ \left( c'^u_1, u_2(A_0), m^U_0 \circ \left( f^u_2, c'^u_1, u_2(A_1) \right) \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, u_2(A_0), m^U_0 \circ \left( f^u_2, c'^u_1, u_2(A_1) \right) \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, u_2(A_0), m^U_0 \circ \left( f^u_2, c'^u_1, u_2(A_1) \right) \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, u_2(A_0), m^U_0 \circ \left( c'^u_1, u_2(A_0), f^u_2 \right) \right)
\]

\[
- m^U_0 \circ \left( c'^u_1, u_2(A_0), m^U_0 \circ \left( c'^u_1, u_2(A_0), f^u_2 \right) \right)
\]
\[ P_3(\theta, \theta, \theta)_{[2,1]}(A) = P_3^0(c', f', f')^\sigma(A) + P_4^0(c, m', f', c')^\sigma(A) \]
\[ + P_4^0(c, m', c', f')^\sigma(A) + P_5^0(c, m', c, f', f')^\sigma(A) \]
\[ = -mU_0 \circ (c^{u_1, u_2}(A_0), f^{u_1} \circ f^{u_2}) \]
\[ + mU_0 \circ (c^{1U_0, u_2 u_1}(A_0), mU_0 \circ (f^{u_2 u_1}, c^{u_1, u_2}(A_1))) \]
\[ - mU_0 \circ (c^{1U_0, u_2 u_1}(A_0), mU_0 \circ (c^{u_1, u_2}(A_0), f^{u_1} \circ f^{u_2})) \]
\[ - mU_0 \circ (c^{1U_0, u_2 u_1}(A_0), mU_0 \circ (c^{u_1, u_2}(A_0), f^{u_1} \circ f^{u_2})) \]
\[ - mU_0 \circ (c^{1U_0, u_2 u_1}(A_0), mU_0 \circ (c^{u_1, u_2}(A_0), f^{u_1} \circ f^{u_2})) \]

\[ P_4(\theta, \theta, \theta)_{[2,1]}(A) = P_3^0(c, m', c', f', f')^\sigma(A) \]
\[ = -mU_0 \circ (c^{1U_0, u_2 u_1}(A_0), mU_0 \circ (c^{u_1, u_2}(A_0), f^{u_1} \circ f^{u_2})) \]

and thus

\[ MC(\theta)_{[2,1]}^\sigma(A) = (mU_0 + mU_0) \circ ((f^{u_2 u_1} + f^{u_2 u_1}), (c^{u_1, u_2}(A_0) + c^{u_1, u_2}(A_0))) \]
\[ - (mU_0 + mU_0) \circ ((c^{u_1, u_2}(A_0) + c^{u_1, u_2}(A_0)), (f^{u_1} + f^{u_1}) \circ (f^{u_2} + f^{u_2})) \]

Lastly, for \((p, q) = (3, 0)\), let \(\sigma = (U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} U_2 \xrightarrow{u_3} U_3)\) be a 3-simplex in \(U\) and \(A_0\) an object in \(A(U_2)\), then we compute

\[ d\theta_{[3,0]}^\sigma(A_0) = d_3(m')^\sigma(A_0) + d_2(f')^\sigma(A_0) + d_1(c')^\sigma(A_0) \]
\[ = mU_0 \circ (c^{1U_0, u_3 u_2 u_1}(A_0), mU_0 \circ (c^{u_2 u_1, u_3}(A_0), c^{u_1, u_2}(u_3^2 A_0))) \]
\[ - mU_0 \circ (c^{u_1, u_3, u_2}(A_0), f^{u_1} \circ c^{u_2, u_3}(A_0)) \]
\[ - mU_0 \circ (c^{u_1, u_3, u_2}(A_0), f^{u_1} \circ c^{u_2, u_3}(A_0)) \]
\[ + mU_0 \circ (c^{u_1, u_3, u_2}(A_0), c^{u_1, u_2}(A_0)) \]
\[ + mU_0 \circ (c^{u_2 u_1, u_3}(A_0), c^{u_1, u_2}(u_3^2 A_0)) \]
\[ - mU_0 \circ (c^{u_1, u_3, u_2}(A_0), f^{u_1} \circ c^{u_2, u_3}(A_0)) \]

\[ P_2(\theta, \theta)_{[3,0]}^\sigma(A_0) = P_2^0(c', c')^\sigma(A_0) + P_3^0(c, f', c')^\sigma(A_0) \]
\[ + P_3^0(c, f', c')^\sigma(A_0) + P_4^0(c, m', c', f')^\sigma(A_0) \]
\[ + P_4^0(c, m', c, f')^\sigma(A_0) + P_5^0(c, m', c, f', c')^\sigma(A_0) \]
\[ = mU_0 \circ (c^{u_2 u_1, u_3}(A_0), c^{u_1, u_2}(u_3^2 A_0)) \]
Operadic structure on the Gerstenhaber–Schack complex...

Page 59 of 63

\[- m^{U_0} \circ \left( c^{u_1,u_2} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \]
\[- m^{U_0} \circ \left( c^{u_1,u_2} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \]
\[- m^{U_0} \circ \left( c^{u_1,u_2} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \]
\[+ m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), c^{u_1,u_2} (u_3^A_0) \right) \right) \]
\[- m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) \]
\[- m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) \]
\[- m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) \]

\[P_3(\theta, \theta, \theta)_{[3,0]} = P_3^0(c', f', c')^\sigma (A_0) + P_4^0(c, m', c', c')^\sigma (A_0) + P_5^0(c, m', c', f', c')^\sigma (A_0) = - m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) + m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), c^{u_1,u_2} (u_3^A_0) \right) \right) - m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) - m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) \]

\[P_4(\theta, \theta, \theta)_{[3,0]} (A_0) = P_5^0(c, m', c', f', c')^\sigma (A_0) = - m^{U_0} \circ \left( c^{u_1,u_3,u_1} (A_0), m^{U_0} \circ \left( c^{u_2,u_1,u_3} (A_0), f^{u_1} \circ c^{u_2,u_3} (A_0) \right) \right) \]

and thus

\[MC(\theta)_{[3,0]} (A_0) = \left( m^{U_0} + m^{U_0} \right) \circ \left( c^{u_2,u_1,u_3} (A_0), c^{u_1,u_2} (u_3^A_0) \right) \]
\[= \left( m^{U_0} + m^{U_0} \right) \circ \left( c^{u_1,u_3,u_2} (A_0), c^{u_1,u_2} (u_3^A_0) \right) \]
\[= \left( m^{U_0} + m^{U_0} \right) \circ \left( c^{u_1,u_3,u_2} (A_0), c^{u_1,u_2} (u_3^A_0) \right) \]

These computations show that $MC(\theta) = 0$ if and only if $(A, m + m', f + f', c + c')$ is a prestack.

**Remark 4.21** Note that for the functor condition we only need the cubic part and for the twists the full quartic part of the equation.
Based upon Proposition 4.20, with some more work taking gauge equivalence into account, one may extend [2, Thm. 3.19] to higher order deformations. Recall that for \((R, m) \in \text{Art}\) an \(R\)-deformation of a \(k\)-linear prestack \((A, m, f, c)\) is an \(R\)-linear prestack \((R \otimes_k A, \bar{m}, \bar{f}, \bar{c})\) of which the algebraic structure reduces to that of \(A\) modulo \(m\), and an equivalence of deformations is an isomorphism between the deformed prestacks which reduces to the identity morphism. Let Def\(_A : \text{Art} \rightarrow \text{Set}\) be the deformation functor of \(A\) with Def\(_A(R, m)\) the set of \(R\)-deformations of \(A\) up to equivalence of deformations. The following theorem, of which the proof will appear elsewhere, expresses that the deformation theory of \(A\) is controlled by the \(L_\infty\)-algebra \(sC_{GS}(A)\).

**Theorem 4.22** Let \((A, m, f, c)\) be a prestack. There is a natural isomorphism of functors \(\text{Art} \rightarrow \text{Set}\):

\[
\text{Def}_A \cong MC_{sC_{GS}(A)}.
\]

**Acknowledgements** The authors are grateful to an anonymous referee for their meticulous reading of an earlier version of the manuscript, and in particular for several corrections and valuable suggestions which greatly helped improve the paper. The third named author thanks Severin Barmeier and Eli Hawkins for bringing the preprints [1], respectively [8] to her attention.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**Appendix A: Generator-free description of the morphism**\(\text{NSOp} \rightarrow \text{Multi}\Delta\)

In this appendix, we provide a generator-free description of the morphism \(\text{NSOp} \rightarrow \text{Multi}\Delta\) from Lemma 2.5. Although per construction we have a morphism of operads induced from the generators \(E_i\) of \(\text{NSOp}\), we consider it valuable in practice to have an explicit definition.

For \(I\) an indexed tree in \(\text{NSOp}(q_1, \ldots, q_n; \sum_{i=1}^n q_i - n + 1)\), we call \(l_I = \sum_{i=1}^n q_i - n + 1\) the number of leaves of \(I\). Given a number \(m \in \mathbb{N}\), we call the interval

\([m, m + l_I] = \{m, m + 1, \ldots, m + l_I\}\)

the numbering set of leaves of \(I\).

**Construction A.1** For \(I\) an indexed tree in \(\text{NSOp}(q_1, \ldots, q_n; \sum_{i=1}^n q_i - n + 1)\), let \(I_j\) be the maximal subtree of \(I\) with root \(j\). Let \(u\) be the root of \(I\) with children
Operadic structure on the Gerstenhaber–Schack complex...

Given a number \( m \in \mathbb{N} \), we define a non-decreasing map \( \zeta(I, m) : [q_u] \rightarrow [m, m + l_I] \) as follows

\[
\zeta(I, m)(t) = \begin{cases} 
  m + t & \text{for } 0 \leq t < i_1 \\
  m + \sum_{j=1}^{s} l_{I_j} + t & \text{for } i_s \leq t < i_{s+1} \\
  m + \sum_{j=1}^{s} l_{I_j} + t & \text{for } i_s \leq t \leq q_u 
\end{cases}
\]

which determines where the leaves of the root of \( I \) are placed in its numbering set of leaves \([m, m + l_I] \).

In order to define the tuple \( \zeta = (\zeta_{I,1}, \ldots, \zeta_{I,n}) \in \text{Multi}\Delta(q_1, \ldots, q_n; \sum_{i=1}^{n} q_i - n + 1) \), we run inductively through the tree \( I \) from root \( u \) to leaves setting

\[
\zeta_{I,u} := \zeta(I, 0) : [q_u] \rightarrow [l_I]
\]

and for vertex \( a \) with child \( b \) and index \( i := I(a, b) \) we set

\[
\zeta_{I,b} := \zeta(I_b, \zeta_{I,a}(i - 1)) : [q_b] \rightarrow [\zeta_{I,a}(i - 1), \zeta_{I,a}(i - 1) + l_{I_b}] \hookrightarrow [l_I]
\]

with \( l_I = \sum_{i=1}^{n} q_i - n + 1 \).

Note that Construction 2.5 corresponds to the above construction applied to the generators \( E_i \). As such, if the generator-free description defines a morphism of operads, they coincide.

**Proposition A.2** Construction A.1 defines a morphism of operads \( \text{NSOp} \rightarrow \text{Multi}\Delta \).

**Proof** Let \( I \in \text{NSOp}(q_1, \ldots, q_n; q) \) and \( I' \in \text{NSOp}(q'_1, \ldots, q'_m; q') \) and consider their composition \( I'' := I \circ_i I' \). We will show that

\[
\zeta_{I''} = \zeta_I \circ_i \zeta_{I'}
\]

As in Construction A.1, let \( I_j, I'_j \) and \( I''_j \) be the maximal subtrees of respectively \( I, I' \) and \( I'' \) with root \( j \). Due to equivariance, we can assume that a vertex \( j \) belongs to the subtree \( I_i \) iff \( j \geq i \). We then have three cases to consider which we depict
diagrammatically as follows

\[ (1) \quad \text{For } j < i, \text{ if we contract the subtree } I_i \text{ and } I''_i \text{ to a single vertex with number of leaves } l_{I_i} = l_{I''_i}, \text{ we obtain the same indexed tree } I \setminus I_i = I'' \setminus I''_i \text{ in which } I_j \text{ and } I''_j \text{ coincide. Thus, it is easy to see that } \\
\zeta_{I''_j, j} = \zeta_{I, j} \\

(2) \quad \text{If } j \geq i \text{ and } j \text{ lies in the image } \text{Im}(I') \text{ of } I' \text{ in } I'', \text{ then } I''_j \text{ consists of the subtree } I'_{j-i+1} \text{ with a sequence of subtrees } I_t, \ldots, I_{t'} \text{ placed on top.} \\
\]

\[ (2) \quad \text{If } j \geq i \text{ and } j \text{ lies in the image } \text{Im}(I') \text{ of } I' \text{ in } I'', \text{ then } I''_j \text{ consists of the subtree } I'_{j-i+1} \text{ with a sequence of subtrees } I_t, \ldots, I_{t'} \text{ placed on top.} \]

\[ \zeta_{I''_j, j} = \zeta_{I, j} \circ \zeta_{I', j-i+1} \]

which determines where the leaves of } j \text{ are put in } I''. \]

(3) \quad \text{If } j \geq i \text{ and } j \text{ does not lie in the image of } I' \text{ in } I'', \text{ then } j - m + 1 \text{ lies in } I \text{ above } i. \text{ In this case, the subtrees } I_{j-m+1} \text{ and } I''_j \text{ are equal. If the parent } a \text{ of } j \text{ in } I'' \text{ does not lie in } \text{Im}(I'), \text{ then we clearly have} \\
\zeta_{I'', j} = \zeta_{I, j-m+1}. \]
If \( a \) lies in \( \text{Im}(I') \), then \( \zeta_{I'',a} = \zeta_{I',a} \circ \zeta_{I',a-i+1} \) due to the previous case.

Moreover, the index \( I''(a, j) \) then equals \( I(i, j) - \zeta_{I',a-i+1}(0) \). Hence, we have that

\[
\zeta_{I'',j} = \zeta_{I,j-m+1}
\]

References

1. Barmeier, S., Frégier, Y.: Deformation-obstruction theory for diagrams of algebras and applications to geometry. J. Noncommut. Geom. 14(3), 1019–1047 (2020)
2. Dinh Van, H., Lowen, W.: The Gerstenhaber–Schack complex for prestacks. Adv. Math. 330, 173–228 (2018)
3. Frégier, Y., Markl, M., Yau, D.: The \( L_\infty \)-deformation complex of diagrams of algebras. N. Y. J. Math. 15, 353–392 (2009)
4. Gerstenhaber, M., Schack, S.D.: On the deformation of algebra morphisms and diagrams. Trans. Am. Math. Soc. 279(1), 1–50 (1983)
5. Gerstenhaber, M., Schack, S.D.: Algebraic cohomology and deformation theory. In: Hazewinkel, M., Gerstenhaber, M. (eds.) Deformation Theory of Algebras and Structures and Applications (II Ciocco, 1986). NATO ASI Series (Series C: Mathematical and Physical Sciences), vol. 247, pp. 11–264. Kluwer Academic Publishers, Dordrecht (1988)
6. Gerstenhaber, M., Voronov, A.A.: Homotopy \( G \)-algebras and moduli space operad. Int. Math. Res. Not. 1995(3), 141–153 (1995). (electronic)
7. Getzler, E., Jones, J.D.S.: Operads, homotopy algebra and iterated integrals for double loop spaces. Preprint, arxiv: hep-th/9403055
8. Hawkins, E.: Operations on the Hochschild bicomplex of a diagram of algebras. arXiv:2002.00886
9. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. 66(3), 157–216 (2003)
10. Kontsevich, M., Soibelman, Y.: Deformations of Algebras Over Operads and the Deligne Conjecture. Conférence Moshé Flato 1999, Volume 1 (Dijon). Mathematical Physics Studies, vol. 21, pp. 255–307. Kluwer Academic Publishers, Dordrecht (2000)
11. Lowen, W.: Algebroid prestacks and deformations of ringed spaces. Trans. Am. Math. Soc. 360(9), 1631–1660 (2008). Preprint, arxiv: math.AG/0511197v2
12. Lowen, W., van den Bergh, M.: A Hochschild cohomology comparison theorem for prestacks. Trans. Am. Math. Soc. 363(2), 969–986 (2011)
13. McClure, J.E., Smith, J.H.: A solution of Deligne’s Hochschild cohomology conjecture. Recent Progress in Homotopy Theory (Baltimore, MD, 2000). Contemporary Mathematics, vol. 293, pp. 153–193. American Mathematical Society, Providence, RI (2002)