Separability of Dirac equation in higher dimensional Kerr-NUT-de Sitter spacetime

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Abstract

It is shown that the Dirac equations in general higher dimensional Kerr-NUT-de Sitter spacetimes are separated into ordinary differential equations.
Recently, the separability of Klein-Gordon equations in higher dimensional Kerr-NUT-de Sitter spacetimes \[1\] was shown by Frolov, Krtoň and Kubizňák \[2\]. This separation is deeply related to that of geodesic Hamilton-Jacobi equations. Indeed, a geometrical object called conformal Killing-Yano tensor plays an important role in the separability theory \[3, 4, 5, 2, 6, 7, 8, 9\]. However, at present, a similar separation of the variables of Dirac equations is lacking, although the separability in four dimensional Kerr geometry was given by Chandrasekhar \[10\]. In this paper we shall show that Dirac equations can also be separated in general Kerr-NUT-de Sitter spacetimes.

The D-dimensional Kerr-NUT-de Sitter metrics are written as follows \[1\]:

\[(a) \quad D = 2n \]
\[
g^{(2n)} = \sum_{\mu=1}^{n} \frac{dx_{\mu}^2}{Q_\mu(x)} + \sum_{\mu=1}^{n} Q_\mu(x) \left( \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k \right)^2,
\]

\[(b) \quad D = 2n + 1 \]
\[
g^{(2n+1)} = \sum_{\mu=1}^{n} \frac{dx_{\mu}^2}{Q_\mu(x)} + \sum_{\mu=1}^{n} Q_\mu(x) \left( \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k \right)^2 + \frac{c}{A^{(n)}} \left( \sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2.
\]

The functions \(Q_\mu\) \((\mu = 1, 2, \cdots, n)\) are given by
\[
Q_\mu(x) = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^{n} \left( x_{\nu}^2 - x_{\nu}^2 \right),
\]
where \(X_\mu\) is a function depending only on the coordinate \(x_\mu\), and \(A^{(k)}_\mu\) and \(A^{(k)}_{\mu}\) are the elementary symmetric functions of \(\{x_{\nu}^2\}\) and \(\{x_{\nu}^2\}_{\nu \neq \mu}\) respectively:
\[
\prod_{\nu=1}^{n} (t - x_{\nu}^2) = A^{(0)} t^n - A^{(1)} t^{n-1} + \cdots + (-1)^n A^{(n)},
\]
\[
\prod_{\nu=1}^{n} (t - x_{\nu}^2) = A^{(0)}_\mu t^{n-1} - A^{(1)}_\mu t^{n-1} + \cdots + (-1)^{n-1} A^{(n-1)}_{\mu},
\]

The metrics are Einstein if \(X_\mu\) takes the form \[11\]

\[(a) \quad D = 2n \]
\[
X_\mu = \sum_{k=0}^{n} c_{2k} x_{\mu}^{2k} + b_\mu x_\mu,
\]

\[(b) \quad D = 2n + 1 \]
\[
X_\mu = \sum_{k=0}^{n} c_{2k} x_{\mu}^{2k} + b_\mu + \frac{(-1)^n c}{x_{\mu}^{2}},
\]
where \(c, c_{2k} \) and \(b_\mu\) are free parameters.

1. \[D = 2n\]

For the metric \[11\] we introduce the following orthonormal basis \(\{\epsilon^a\} = \{\epsilon^\mu, \epsilon^{n+\mu}\} \ (\mu = 1, 2, \cdots, n)\):
\[
\epsilon^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad \epsilon^{n+\mu} = \sqrt{Q_\mu} \sum_{k=0}^{n-1} A^{(k)} d\psi_k
\]

The dual vector fields are given by
\[
\epsilon_\mu = \sqrt{Q_\mu} \frac{\partial}{\partial x_\mu}, \quad \epsilon_{n+\mu} = \sum_{k=0}^{n-1} (-1)^k x_\mu^{2(n-1-k)} \frac{\partial}{\sqrt{Q_\mu U_\mu} \partial \psi_k}.
\]
The spin connection is calculated as \( [11] \)

\[
\omega_{\mu\nu} = -\frac{x_\nu \sqrt{Q_\mu}}{x_\nu - x_\mu} e^\mu - \frac{x_\mu \sqrt{Q_\nu}}{x_\mu - x_\nu} e^\nu, \quad (\mu \neq \nu)
\]

\[
\omega_{\mu, n+\mu} = -(\partial_\mu \sqrt{Q_\mu}) e^{n+\mu} - \sum_{\rho \neq \mu} \frac{x_\mu \sqrt{Q_\rho}}{x_\mu - x_\rho} e^{n+\rho}, \quad \text{(no sum over } \mu) ,
\]

\[
\omega_{\mu, n+\nu} = \frac{x_\mu \sqrt{Q_\nu}}{x_\mu - x_\nu} e^{n+\mu} - \frac{x_\mu \sqrt{Q_\mu}}{x_\mu - x_\nu} e^{n+\nu}, \quad (\mu \neq \nu)
\]

\[
\omega_{n+\mu, n+\nu} = -\frac{x_\mu \sqrt{Q_\nu}}{x_\mu - x_\nu} e^\mu - \frac{x_\nu \sqrt{Q_\mu}}{x_\nu - x_\mu} e^\nu, \quad (\mu \neq \nu).
\]

Then, the Dirac equation is written in the form

\[
(\gamma^a D_a + m)\Psi = 0,
\]

where \( D_a \) is a covariant differentiation,

\[
D_a = e_a + \frac{1}{4} \omega_{bc}(e_a)\gamma^b\gamma^c. \tag{12}
\]

From (9), (10) and (12), we obtain the explicit expression for the Dirac equation

\[
\sum_{\mu=1}^{n} \gamma^\mu \sqrt{Q_\mu} \left( \frac{\partial}{\partial x_\mu} + \frac{1}{2} \sum_{\nu=1}^{n} \frac{x_\mu}{x_\nu - x_\mu} \right) \Psi \\
+ \sum_{\mu=1}^{n} \gamma^{n+\mu} \sqrt{Q_\mu} \left( \sum_{k=0}^{n-1} (-1)^k x_\mu^{2(n-k)} \frac{\partial}{\partial x_\mu} + \frac{1}{2} \sum_{\nu=1}^{n} \frac{x_\nu}{x_\nu - x_\mu} \right) \Psi + m \Psi = 0. \tag{13}
\]

Let us use the following representation of \( \gamma \)-matrices: \( \{ \gamma^a, \gamma^b \} = 2\delta^{ab} \),

\[
\gamma^\mu = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes I \otimes \cdots \otimes I, \tag{14}
\]

\[
\gamma^{n+\mu} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \otimes I \otimes \cdots \otimes I,
\]

where \( I \) is the 2 \( \times \) 2 identity matrix and \( \sigma_i \) are the Pauli matrices. In this representation, we write the 2\( n \) components of the spinor field as \( \Psi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} \) (\( \epsilon_\mu = \pm 1 \)), and it follows that

\[
(\gamma^\mu \Psi)_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} = \prod_{\nu=1}^{n-1} \epsilon_\nu \Psi_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n}, \tag{15}
\]

\[
(\gamma^{n+\mu} \Psi)_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} = -i \epsilon_\mu \prod_{\nu=1}^{n-1} \epsilon_\nu \Psi_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n}.
\]

By the isometry the spinor field takes the form

\[
\Psi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x, \psi) = \hat{\Psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) \exp \left( \frac{i}{\hbar} \sum_{k=0}^{n-1} N_k \psi_k \right). \tag{16}
\]
with arbitrary constants $N_k$. Substituting (15) into (13), we obtain

$$\sum_{\mu=1}^{n} \sqrt{Q_\mu} \left( \prod_{\rho=1}^{\mu-1} \epsilon_\rho \right) \left( \frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X_\mu'}{X_\mu} + \frac{1}{2} \frac{1}{x_\mu - \epsilon_\mu x_\nu} \sum_{\nu \neq \mu}^{n} \right) \Psi_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n} \right) + m \Psi_{\epsilon_1 \cdots \epsilon_n} = 0,$$

(17)

where we have introduced the function

$$Y_\mu = \sum_{k=0}^{n-1} (-1)^k x_\mu^{2(n-1-k)} N_k,$$

(18)

which depends only on $x_\mu$.

Consider now the region $x_\mu - x_\nu > 0$ for $\mu < \nu$ and $x_\mu + x_\nu > 0$. Let us define

$$\Phi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) = \prod_{1 \leq \mu < \nu \leq n} \frac{1}{\sqrt{x_\mu + \epsilon_\mu x_\nu}}.$$

(19)

Then, one can obtain an equality

$$\psi_{\epsilon_1 \cdots \epsilon_n}(x) = \Phi_{\epsilon_1 \cdots \epsilon_n}(x) \prod_{\mu=1}^{n} \chi_{\epsilon_\mu}(x_\mu).$$

(21)

It should be noticed that

$$\frac{\partial}{\partial x_\mu} \log \hat{\psi}_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n}(x) = (-\epsilon_\mu)^{\mu-1} \prod_{\rho=1}^{\mu-1} \epsilon_\rho \sqrt{(-1)^{\mu-1} U_\mu} \prod_{\nu=1}^{n} \frac{1}{(x_\mu - \epsilon_\mu x_\nu)}.$$

(20)

Now we show that the Dirac equation allows a separation of variables by setting

$$\hat{\psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) = \Phi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) \prod_{\mu=1}^{n} \chi_{\epsilon_\mu}(x_\mu).$$

(21)

By using (20) and (22), the substitution of (21) into (17) leads to

$$\sum_{\mu=1}^{n} \prod_{\nu=1}^{n} \frac{P_{\epsilon_\mu}^{(\mu)}(x_\mu)}{(x_\mu - \epsilon_\mu x_\nu)} + m = 0,$$

(23)

where $P_{\epsilon_\mu}^{(\mu)}$ is a function of the coordinate $x_\mu$ only,

$$P_{\epsilon_\mu}^{(\mu)} = (-1)^{\mu-1} (\epsilon_\mu)^{n-\mu} \sqrt{(-1)^{\mu-1} X_\mu} \frac{1}{\chi_{\epsilon_\mu}} \left( \frac{d}{dx_\mu} + \frac{1}{2} \frac{X_\mu'}{X_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu} \right) \chi_{-\epsilon_\mu}.$$ 

(24)

Putting

$$Q(y) = -my^{n-1} + \sum_{j=0}^{n-2} q_j y^j$$

(25)
with arbitrary constants $q_j$, we find

$$P_{(\mu)}(x_\mu) = Q(\epsilon_\mu x_\mu).$$  \hfill (26)

Thus, the functions $\chi_{(\mu)}^{(i)}$ satisfy the ordinary differential equations

$$
\left(\frac{d}{dx_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu}\right) \chi^{(\mu)}_{\epsilon_\mu} - \frac{(1-\mu^{-1})(\epsilon_\mu)^{n-1}Q(\epsilon_\mu x_\mu)}{\sqrt{(1-\mu^{-1})X_\mu}} \chi^{(\mu)}_{\epsilon_\mu} = 0.
$$  \hfill (27)

2. $D=2n+1$

For the metric \[2\] we introduce the orthonormal basis \(\{e^a\} = \{\hat{\epsilon}^\mu, \; \hat{e}^{n+\mu}, \; \hat{e}^{2n+1}\} (\mu = 1, 2, \ldots, n)$:

$$
\hat{\epsilon}^\mu = e_\mu, \quad \hat{e}^{n+\mu} = e_{n+\mu}, \quad \hat{e}^{2n+1} = \sqrt{S} \sum_{k=0}^n A(k) \psi_k
$$  \hfill (28)

with $S = c/A(n)$. The 1-forms $e^\mu$ and $e^{n+\mu}$ are defined by \[3\]. The dual vector fields are given by

$$
\hat{\omega}_{\mu
u} = \omega_{\mu\nu}, \quad \hat{\omega}_{\mu,n+\nu} = \omega_{\mu,n+\nu} + \delta_{\mu\nu} \sqrt{S} \hat{e}^{2n+1}, \quad \hat{\omega}_{n+\mu,n+\nu} = \omega_{n+\mu,n+\nu},
$$  \hfill (29)

A similar calculation to the even dimensional case yields the following Dirac equation,

$$
\sum_{\mu=1}^n \gamma^\mu \sqrt{Q_\mu} \left( \frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{1}{2} \sum_{\nu=1}^n \frac{x_\mu}{x_\mu^2 - x_\nu^2} \right) \Psi
$$

$$
+ \sum_{\mu=1}^n \gamma^{n+\mu} \sqrt{Q_\mu} \left( \sum_{k=0}^{n-1} (-1)^k \frac{x_\mu^{2(n-1-k)}}{X_\mu} \frac{\partial}{\partial \psi_k} + \frac{(-1)^n}{x_\mu^2 X_\mu} \frac{\partial}{\partial \psi_n} + \frac{1}{2} \sum_{\nu=1}^n \frac{x_\nu}{x_\mu^2 - x_\nu^2} (\gamma^\nu \gamma^{n+\nu}) \right) \Psi
$$

$$
+ \gamma^{2n+1} \sqrt{S} \left( - \sum_{\mu=1}^n \frac{1}{2x_\mu} (\gamma^\mu \gamma^{n+\mu}) + \frac{1}{c} \frac{\partial}{\partial \psi_n} \right) \Psi + m \Psi = 0.
$$  \hfill (31)

We use the representation of $\gamma$-matrices given by \[14\] together with

$$
\gamma^{2n+1} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3.
$$  \hfill (32)

Thus, the spinor field $\hat{\Psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}$ defined by

$$
\Psi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x, \psi) = \hat{\Psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) \exp \left( i \sum_{k=0}^n N_k \psi_k \right)
$$  \hfill (33)

satisfies the equation

$$
\sum_{\mu=1}^n \sqrt{Q_\mu} \left( \prod_{\rho=1}^{\mu-1} \epsilon_\rho \right) \left( \frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{1}{2} \frac{\epsilon_\mu Y_\mu}{X_\mu} \right) \hat{\Psi}_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n}
$$

$$
+ \left( i \sqrt{S} \prod_{\rho=1}^n \epsilon_\rho \right) \left( - \sum_{\mu=1}^n \frac{\epsilon_\mu}{2x_\mu} + \frac{N_n}{c} \right) + m \right) \hat{\Psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} = 0,
$$  \hfill (34)
where
\[ \hat{Y}_\mu = \sum_{k=0}^{n} (-1)^k x_\mu 2^{(n-1-k)} N_k. \] (35)

We find that the Dirac equation above allows a separation of variables
\[ \hat{\Psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) = \Phi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) \prod_{\mu=1}^{n} \left( \frac{\chi_{(\mu)}(x_\mu)}{\sqrt{x_\mu}} \right) \] (36)

with \( \Phi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} \) defined by (19). Indeed, (34) becomes
\[ \sum_{\mu=1}^{n} P_{\epsilon_\mu}(x_\mu) + i\sqrt{c} \left( -\sum_{\mu=1}^{n} \epsilon_\mu x_\mu \right) + m = 0 \] (37)

with the help of (24). Let us introduce the function
\[ \hat{Q}(y) = \sum_{j=-2}^{n-1} q_j y^j \] (38)

where
\[ q_{n-1} = -m, \quad q_{-1} = \frac{i}{2} (-1)^{n-1} \sqrt{c}, \quad q_{-2} = \frac{i}{\sqrt{c}} (-1)^{n} N_n. \] (39)

Using the identities
\[ \sum_{\mu=1}^{n} \frac{1}{y_\mu} \prod_{\nu \neq \mu} \frac{1}{y_\mu - y_\nu} = \frac{(-1)^{n-1}}{n} \sum_{\mu=1}^{n} \frac{1}{y_\mu}, \] (40)
\[ \sum_{\mu=1}^{n} \frac{1}{y_\mu} \prod_{\nu \neq \mu} \frac{1}{y_\mu - y_\nu} = \frac{(-1)^{n-1}}{n} \sum_{\mu=1}^{n} \frac{1}{y_\mu}, \] (41)

we can confirm that the functions \( \chi_{(\mu)}^{(\nu)} \) satisfy the ordinary differential equations (27) by the replacements \( Y_\mu \rightarrow \hat{Y}_\mu \) and \( Q(\epsilon_\mu x_\mu) \rightarrow \hat{Q}(\epsilon_\mu x_\mu) \).

We have shown the separation of variables of Dirac equations in general Kerr-NUT-de Sitter spacetimes. An interesting problem is to investigate the origin of separability. In the case of geodesic Hamilton-Jacobi equations and Klein-Gordon equations we know that the existence of separable coordinates comes from that of a rank-2 closed conformal Killing-Yano tensor. However, we have no clear answer for Dirac equations. As another problem we can study eigenvalues of Dirac operators on Sasaki-Einstein manifolds. Indeed, as shown in [11, 12, 13, 14], the BPS limit of odd-dimensional Kerr-NUT-de Sitter metrics leads to Sasaki-Einstein metrics. Especially, the five-dimensional metrics are important from the point of view of AdS/CFT correspondence.

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