Almost sure Assouad-like Dimensions of Complementary sets

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Abstract. Given a non-negative, decreasing sequence $a$ with sum 1, we consider all the closed subsets of $[0,1]$ such that the lengths of their complementary open intervals are given by the terms of $a$, the so-called complementary sets. In this paper we determine the almost sure value of the $\Phi$-dimensions of these sets given a natural model of randomness. The $\Phi$-dimensions are intermediate Assouad-like dimensions which include the Assouad and quasi-Assouad dimensions as special cases. The answers depend on the size of $\Phi$, with one size behaving like the Assouad dimension and the other, like the quasi-Assouad dimension.

1. Introduction

The upper and lower Assouad dimensions were introduced by Assouad in [1][2] and Larman in [20]. These dimensions were initially used in the theory of embeddings of metric spaces into $\mathbb{R}^n$ (see [23]) but, together with their less extreme versions, the quasi-Assouad dimensions introduced in [5][21], have recently been extensively used within the fractal geometry community; see, for example, [7][10][18][22] and the references cited in those papers. In [12], the authors further generalized these notions, introducing a range of dimensions that are intermediate between the box and Assouad dimensions. One topic explored in [12] were the dimensional properties of (deterministic) rearrangements of a Cantor set of zero Lebesgue measure. In this paper, we continue this investigation, studying the almost sure dimensional properties of random rearrangements. This extends the work of Hawkes [14] who studied the Hausdorff dimensions of random rearrangements of Cantor sets.

The Assouad dimensions can roughly be thought of as refinements of the box-counting dimensions where one “localizes” and takes the worst local behaviour. The localization is accomplished by choosing a window size, $R$, and then analyzing this window at a smaller scale $r$. The quasi-Assouad dimension requires, in addition, that $r \leq R^{1+\delta}$ for some fixed $\delta > 1$ and then lets $\delta \to 0$. Our refinement uses a dimension function $\Phi$ and requires $r \leq R^{1+\Phi(R)}$, so we can very precisely measure the local scaling behaviour of a set by varying $\Phi$ to be adapted to the set in question.

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The extreme values of the $\Phi$-dimensions are the box and Assouad dimensions. Both the quasi-Assouad and Assouad dimensions are special examples and when $\Phi(R) \to 0$ as $R \to 0$, the $\Phi$-dimensions lie between these two. An example is constructed in [12] of a set where the range of $\Phi$-dimensions is the full interval from the quasi-Assouad to Assouad dimensions. Fraser’s modified $\theta$-spectrum, studied in [8,9], is another example of a $\Phi$-dimension. More generally, if $\Phi$ stays bounded away from 0, then the $\Phi$-dimensions lie between the box and quasi-Assouad dimensions. The definitions and basic properties of these dimensions are given in Section 2.

Given $a = \{a_j\}$, a non-negative decreasing sequence with sum equal to one, we define the class $C_a$ to be the family of all closed subsets of $[0, 1]$ whose complement in $[0, 1]$ consists of disjoint open intervals with lengths given by the $a_j$. The sets in $C_a$ are called the complementary sets of $a$ and all have zero Lebesgue measure. Every compact subset of $[0, 1]$ of Lebesgue measure zero belongs to exactly one $C_a$ and each $C_a$ contains both countable and uncountable sets. Thus it is natural to ask about the possible dimensions in a given family.

Besicovitch and Taylor [3] were the first to study this problem for the case of Hausdorff dimension. Among other things, they proved that the set of attained Hausdorff dimensions for elements of $C_a$ is the closed interval $[0, \dim_H C_a]$, where $C_a$ is the Cantor set in $C_a$. Recent work produced similar results for the packing dimension where the set of attainable dimensions is $[0, \dim_P C_a]$, (see [13]) and the upper and lower $\Phi$-dimensions, where under natural technical assumptions on the sequence $a$, the sets of attainable dimensions are the intervals $[\underline{\dim}_\Phi C_a, 1]$ and $[0, \underline{\dim}_\Phi C_a]$ respectively, (see [11] for the Assouad dimensions and [12] for the more general $\Phi$-dimensions).

An alternative thread, started in [14], found the almost sure Hausdorff dimension for a random element of $C_a$, under a very natural model of randomness. This was extended in [15,16] where the exact almost sure Hausdorff and packing dimension functions were found for the same random model. Note that since the value of the dimension of a set depends on the asymptotics of very fine scales, any dimensional calculation will be a tail event and thus will have a constant value almost surely (at least if the randomness is given by appropriately independent choices). In this paper, we determine the upper and lower $\Phi$-dimensions of these random rearrangements. Surprisingly, the almost sure behaviour of the dimension depends on the asymptotic “size” of $\Phi$. In fact, it is this difference in the almost sure behaviour which motivated us to study the $\Phi$-dimensions.

Our main results, which can be found in Sections 4 and 5, can be summarized as follows:

**Theorem.** Let $a$ be a level comparable sequence and let $\Psi(x) = \log |\log x|/|\log x|$. (i) If $\Phi(x) >> \Psi(x)$ for $x$ near 0, then for a.e. $E \in C_a$ we have $$\underline{\dim}_\Phi E = \dim_\Phi C_a \quad \text{and} \quad \dim_\Phi E = \dim_\Phi C_a.$$ (ii) If $\Phi(x) << \Psi(x)$ for $x$ near 0, then for a.e. $E \in C_a$ we have $$\underline{\dim}_\Phi E = 1 \quad \text{and} \quad \dim_\Phi E = 0.$$ We do not know what happens if $\Phi \sim \Psi$. We note that the values 1, 0, that arise in the small case, are the upper and lower $\Phi$-dimensions of the countable decreasing set that belongs to $C_a$. 
Since the Assouad dimensions are examples of “small” $\Phi$-dimensions, while the quasi-Assouad dimensions are examples of “large” $\Phi$-dimensions, we immediately deduce:

**Corollary 1.1.** Let $a$ be a level comparable sequence. Then for a.e. $E \in C_a$ we have

$$\dim_{qA} E = \dim_{qA} C_a, \dim_{qL} E = \dim_{qL} C_a$$

and

$$\dim_A E = 1, \dim_L E = 0.$$  

The proofs of these results are very different from both the deterministic arguments and the earlier random results, and rely heavily upon probabilistic information about the tails of binomial distributions. A very loose interpretation of these results is that the quasi-Assouad dimensions require consideration of deep enough scales for the Central limit theorem to “reveal” itself so that the almost sure dimension coincides with the dimension of $C_a$, the “average” set.

2. **Background**

2.1. **Definition and examples of $\Phi$-dimensions.** Given a metric space $X$, we denote the ball centred at $x \in X$ with radius $R$ by $B(x, R)$. For a bounded set $E \subseteq X$, the notation $N_r(E)$ will mean the least number of balls of radius $r$ that cover $E$.

**Definition 2.1.** By a **dimension function**, we mean a map $\Phi : (0, 1) \to \mathbb{R}^+$ with the property that $R^{1+\Phi(R)}$ decreases as $R$ decreases to 0. Of course, $R^{1+\Phi(R)} \leq R$, so $R^{1+\Phi(R)} \to 0$ as $R \to 0$ for any dimension function $\Phi$. As will be seen, interesting examples of dimension functions include the constant functions, as well as $\Phi(x) = 1/|\log x|$ and $\Phi(x) = \log |\log x|/|\log x|$.

**Definition 2.2.** Let $\Phi$ be a dimension function and $X$ a metric space. The **upper and lower $\Phi$-dimensions** of $E \subseteq X$ are given by

$$\dim_{\Phi} E = \inf \left\{ \alpha : (\exists c_1, c_2 > 0) (\forall 0 < r \leq R^{1+\Phi(R)} \leq R < c_1) \sup_{x \in E} N_r(B(x, R) \cap E) \leq c_2 \left( \frac{R}{r} \right)^\alpha \right\}$$

and

$$\dim_{\Phi} E = \sup \left\{ \alpha : (\exists c_1, c_2 > 0) (\forall 0 < r \leq R^{1+\Phi(R)} \leq R < c_1) \sup_{x \in E} N_r(B(x, R) \cap E) \geq c_2 \left( \frac{R}{r} \right)^\alpha \right\}.$$ 

The $\Phi$-dimensions were first introduced in [12] where their basic properties were established. Some of these will be highlighted below.

Special examples of $\Phi$-dimensions include the following:

(i) The **upper Assouad** and **lower Assouad dimensions** of $E$, denoted $\dim_A E$ and $\dim_L E$ respectively. These are the special cases of the upper and lower $\Phi$-dimensions with $\Phi = 0$.

(ii) The (modified) **upper and lower $\theta$-spectrum**, $\dim_{A\theta} E$ and $\dim_{L\theta} E$, introduced by Fraser in [9], arise by taking the constant function $\Phi = 1/\theta - 1$. More
generally, it is shown in [12] that if $\Phi(x) \to 1/\theta - 1$ as $x \to 0$, then the $\Phi$-dimensions coincide with the $\theta$-spectrum.

(iii) The upper quasi-Assouad and lower quasi-Assouad dimensions, denoted $\dim_A E$ and $\dim_L E$, are defined as the limit as $\delta \to 0$ of the upper and lower $\Phi_\delta$ dimensions, respectively. For every set $E$ there are dimension functions $\Phi_1, \Phi_2$ such that $\overline{\dim}_{\Phi_1} E = \dim_A E$ and $\underline{\dim}_{\Phi_2} E = \dim_L E$, see [12, Proposition 2.11]. But the choice of dimension functions depends on the set $E$.

Remark 2.3. Since a set and its closure have the same $\Phi$-dimensions, unless we say otherwise we will assume all sets are compact. We will also assume the underlying metric space $X$ is doubling. This ensures, in particular, that $\dim_A X$ is finite.

2.2. Basic properties of $\Phi$-dimensions. The following relationships between these dimensions are known (see [6, 7, 12, 21]):

$$ \dim_L E \leq \dim_{qL} E \leq \dim_H E \leq \overline{\dim}_B E \leq \underline{\dim}_B E \leq \dim_A E $$

and

$$ \overline{\dim}_L E \leq \underline{\dim}_L E \leq \overline{\dim}_B E \leq \underline{\dim}_B E \leq \overline{\dim}_\Phi E \leq \dim_A E. $$

Here are some other facts which were shown in [12, Section 2].

Proposition 2.4. (i) If $\Phi(x) \to \infty$ as $x \to 0$, then $\overline{\dim}_B E = \overline{\dim}_\Phi E$. If, in addition, $\underline{\dim}_B E > 0$, then $\underline{\dim}_B E = \overline{\dim}_B E$. Without the additional assumption, the latter statement need not be true since any set with an isolated point will have $\overline{\dim}_\Phi E = 0$.

(ii) If $\Phi \leq \Psi$, then $\underline{\dim}_\Phi E \leq \underline{\dim}_\Psi E$ and $\overline{\dim}_\Phi E \geq \overline{\dim}_\Psi E$. In particular, if $\Phi(R) \to 0$ as $R \to 0$, then the $\Phi$-dimensions give a range of dimensions between the Assouad and quasi-Assouad type dimensions:

$$ \dim_L E \leq \underline{\dim}_\Phi E \leq \underline{\dim}_{qL} E \leq \dim_{qA} E \leq \overline{\dim}_{qA} E \leq \dim_A E. $$

(iii) If $\Phi(x) \leq c/|\log x|$ for all small $x$, then the $\Phi$-dimensions coincide with the Assouad dimensions.

Many examples have been constructed to illustrate strict inequalities between these dimensions. For instance, although the $\Phi$ and $\Psi$ dimensions coincide for all sets $E$ if $\Phi/\Psi \to 1$ as $x \to 0$, there will be sets where these dimensions differ if $\Phi$ is bounded above away from $\Psi$. Moreover, given $0 < \alpha < \beta < 1$, there is a set $E \subseteq \mathbb{R}$ such that

$$ \{\overline{\dim}_\Phi E : \Phi \to 0\} = [\alpha, \beta] = [\dim_{qA} E, \dim_A E]; $$

[12, Theorems 3.6, 3.7].

It is easy to see that the $\Phi$-dimensions are bi-Lipschitz invariant and give detailed geometric information about the structure of the underlying sets.

The following notation will be convenient for later in the paper.

Notation 2.5. We write $f \sim g$, and say $f$ is comparable to $g$, if there are positive constants $c_1, c_2$ such that $c_1 f \leq g \leq c_2 f$. The symbols $\geq$ and $\leq$ are defined similarly. When we write $f \ll g$, this means $f/g \to 0$ as either $x \to 0$ or $n \to \infty$, depending on the context.
3. Complimentary sets and the Random model

3.1. Complementary sets and the associated Cantor set. The focus of this paper will be on the relationship between the dimensions of compact subsets of \( \mathbb{R} \) whose complements are open intervals of the same length. We refer to these as complementary sets or rearrangements and begin by explaining precisely what we mean by that.

Every closed subset of the interval \( [0, 1] \) of Lebesgue measure zero is of the form \( E = [0, 1] \setminus \bigcup U_j \) where \( \{ U_j \} \) is a disjoint family of open subintervals of \( [0, 1] \) whose lengths sum to one. Let \( a_j \) be the length of \( U_j \). There is no loss of generality in assuming \( a = (a_j) \) is a decreasing sequence. We denote by \( C_a \) the collection of all such closed sets \( E \); the sets in \( C_a \) are called the \textbf{complementary sets of} \( a \). Every family, \( C_a \), contains a countable set, the decreasing rearrangement, \( D_a = \{ \sum_{i \geq k} a_i \}_{k=1}^{\infty} \).

Another complementary set in \( C_a \) is the so-called \textbf{Cantor set associated with} \( a \) and denoted by \( C_a \). It is constructed as follows: In the first step, we remove from \([0, 1]\) an open interval of length \( a_1 \), resulting in two closed intervals \( I^1_1 \) and \( I^1_2 \). Having constructed the \( k \)-th step, we obtain the closed intervals \( I^k_1, \ldots, I^k_{2^k} \) contained in \([0, 1]\). The intervals \( I^k_j, j = 1, \ldots, 2^k \), are called the Cantor intervals of step \( k \). The next step consists in removing from each \( I^k_j \) an open interval of length \( a_{2^k+j-1} \), obtaining the closed intervals \( I^k_{2^k-1} \) and \( I^k_{2^k} \). We define

\[
C_a := \bigcap_{k \geq 1} \bigcup_{j=1}^{2^k} I^k_j.
\]

This construction uniquely determines the set because the lengths of the removed intervals on each side of a given gap are known. For instance, the classical middle-third Cantor set is the Cantor set associated with the sequence \( a = \{ a_i \} \) where \( a_i = 3^{-n} \) if \( 2^{n-1} \leq i \leq 2^n - 1 \). This sequence \( a \) is \textbf{doubling}, meaning there is a constant \( \kappa \) such that \( a_n \leq \kappa a_{2n} \) for all \( n \). Whenever \( a \) is doubling, then \( C_a \) is bi-Lipschitz equivalent to the central Cantor set \( C_b \) where \( b_{2^n} = a_{2^n} \) and central means all interval (equivalently, gaps) on the same level have the same length.

3.2. Dimensional properties of complementary sets. All complementary sets have the same box dimensions (see [6]), however, this is not true for the other dimensions. For instance, \( \dim_H D_a = 0 \), but this is not true in general for \( C_a \). Thus it is of interest to study the dimensional properties of complementary sets. This investigation began with Besicovitch and Taylor in [3] where they proved that the Cantor set associated with \( a \) has the maximal Hausdorff dimension of all sets in \( C_a \). Moreover, they showed that given any \( s \in [0, \dim_H C_a] \), there was some \( E \in C_a \) with \( \dim_H E = s \). The analogous result was subsequently shown in [13] for packing dimension.

In [11], this problem was studied for the Assouad dimensions with the same result again true for the lower Assouad dimension. However, for the upper Assouad dimension, it was discovered that the associated Cantor set had the minimal Assouad dimension of all complementary sets and the decreasing set had the maximal dimension. Under the assumption that \( a \) is doubling, it was shown that set of attainable values for the upper Assouad dimension was the full interval \([\dim_A C_a, \dim_A D_a] = [\dim_A C_a, 1] \). Under a slightly stronger assumption, implied
by level comparable (defined below) the set of attainable lower Assouad dimensions was also shown to be the interval $[0, \dim C_n]$.

**Definition 3.1.** Given a decreasing sequence $a$, let $s_n = 2^{-n} \sum_{j \geq 2^n} a_j$, the average length of the Cantor intervals of $C_n$ of step $n$. We will say the doubling sequence $a$ is **level comparable** if there are constants $\tau$ and $\lambda$ with

$$0 < \tau \leq s_{j+1}/s_j \leq \lambda < 1/2.$$  

For a central Cantor set, level comparable simply means the ratios of dissection, $s_{j+1}/s_j$, are bounded away from 0 and 1/2. We should point out that the doubling condition already ensures the left hand inequality holds in (3.1). The level comparable condition is very helpful as it implies that $s_k \sim a_{2^k}$ because $s_k \geq a_{2^k+1} \gtrsim a_{2^k}$ and

$$1 - 2\lambda)s_k \leq s_k - 2s_{k+1} \leq a_{2^k}.$$  

In [12], the $\Phi$-dimensions of rearrangements were investigated. The results are similar to the Assouad dimensions (although new proofs were needed in some cases).

**Theorem 3.2.** [12] Cor. 4.2, Theorem 4.3, Cor. 4.4] If $\Phi$ is any dimension function and $a$ a decreasing, summable sequence, then $\dim \Phi E \leq \dim \Phi D_n$. If $a$ is level comparable, then $\dim \Phi E \geq \dim \Phi C_n$ and $\dim \Phi C_n \geq \dim \Phi E$. If, in addition, $\Phi \rightarrow p \in [0, \infty]$, then

$$\begin{align*}
\{\dim \Phi E : E \in C_n\} = [0, \dim \Phi C_n] \\
\{\underline{\dim} \Phi E : E \in C_n\} = [\underline{\dim} \Phi C_n, \underline{\dim} \Phi D_n].
\end{align*}$$  

One ingredient in the proof was a formula for computing the $\Phi$-dimensions of Cantor sets. For this, it is helpful to understand the comparison $r \leq R^{1+\Phi(R)}$ in terms of the sequence $(s_n)$.

**Notation 3.3.** Given a dimension function $\Phi(x)$ and a doubling sequence $a = (a_j)$, we define the **depth function** $\phi$ on $\mathbb{N}$ by the rule that $\phi(n)$ is the minimal integer $j$ such that $s_{n+j} \leq s_n^{1+\Phi(s_n)}$. In other words, $\phi(n)$ is the minimal integer with $s_{n+\phi(n)}/s_n \leq s_n^{\Phi(s_n)}$.

One can easily check that if $\phi(n) \geq 2$, then $\phi(n) \sim n\Phi(s_n)$. Thus if $\phi(n)/n \rightarrow \infty$, then $\dim \Phi E = \dim E$, while if $\phi$ is bounded, then the upper (or lower) $\Phi$-dimension coincides with the upper (resp., lower) Assouad dimension.

**Theorem 3.4.** [12] Theorem 3.3] Let $a$ be a doubling sequence and $C_n$ the associated Cantor set. The upper and lower $\Phi$-dimensions of $C_n$ are given by

$$\begin{align*}
\underline{\dim} \Phi C_n &= \inf \left\{ \beta : (\exists k_0, c_0 > 0) \ (\forall k \geq k_0, n \geq \phi(k)) \ (\frac{s_k}{s_{k+n}})^\beta \geq c_0 2^n \right\} \\
\dim \Phi C_n &= \sup \left\{ \beta : (\exists k_0, c_0 > 0) \ (\forall k \geq k_0, n \geq \phi(k)) \ (\frac{s_k}{s_{k+n}})^\beta \leq c_0 2^n \right\}.
\end{align*}$$
3.3. Random Model for Complementary sets. The goal of this paper is to study the almost sure dimensional properties of random rearrangements. We now describe the model that we use to generate a random ordering of $\mathbb{N}$ and thereby a random set belonging to the sequence $\{a_n\}$. Our approach is formally different from that of Hawkes in [14], but the resulting random ordering is the same, as we explain below.

The random order has two salient and defining features: 1) when it is restricted to any finite subset of $\mathbb{N}$ each possible ordering is equally likely, and 2) for any two disjoint $A, B \subset \mathbb{N}$, the random order restricted to $A$ is independent of the one restricted to $B$.

Our construction is inductive. We start the induction with the trivial order on the set $\{1\}$. Having constructed a random order on $\{1, 2, \ldots, 2^n - 1\}$, the induction step consists of two parts which are done independently:

1) Choose a (uniformly) random permutation of $\{2^n, 2^n + 1, \ldots, 2^n+1 - 1\}$, and

2) Randomly choose, independently and with replacement, a set of $2^n$ “locations” in which to insert the elements of $\{2^n, 2^n + 1, \ldots, 2^n+1 - 1\}$.

The idea is that extending a permutation of $\{1, 2, \ldots, 2^n - 1\}$ to a permutation of $\{1, 2, \ldots, 2^n - 1, \ldots, 2^n+1 - 1\}$ involves ordering $\{2^n, 2^n + 1, \ldots, 2^n+1 - 1\}$ and then inserting these elements into the already existing permutation (including left of the left-most or right of the right-most). Each of these “places” could contain zero or more “new” elements. This means that 2) above is equivalent to generating a sample from the multinomial distribution of $2^n$ trials with $2^n$ outcomes (the “locations”) which are all equally likely.

Let $\mathcal{O}$ be the set of all total orders on $\mathbb{N}$ and for a finite subset $F \subset \mathbb{N}$ and a total order $\prec$ on $F$, let $\mathcal{O}_F = \{\prec \in \mathcal{O} : \prec \mid F = \prec\}$ (these are the analogues in this situation of the “cylinder sets” from a countable product). The $\sigma$-algebra we use is generated by the sets $\mathcal{O}_F$ taken over all finite sets $F$ and over all total orders $\prec$ on $F$. Our probability measure on $\mathcal{O}$ is generated by the property that $\text{Prob}(\mathcal{O}_F) = (|F|!)^{-1}$.

Given a total order $\prec \in \mathcal{O}$ and $a = \{a_n\}$, we define, for each $i \in \mathbb{N}$, a random open interval of length $a_i$ by

$$J_i(\prec) := (\sum_{j < i} a_j, a_i + \sum_{j > i} a_j).$$

We define the random set $K_\prec \in \mathcal{C}_a$ by

$$K_\prec := [0, \sum_i a_i] \setminus \bigcup_i J_i(\prec).$$

(3.5)

Notice in particular that if $i \prec j$ then $J_i(\prec)$, the gap corresponding to $a_i$, is to the left of $J_j(\prec)$, the gap corresponding to $a_j$. The subintervals of $[0, 1]$ that are bounded by the gaps of lengths $a_1, \ldots, a_{2^n-1}$ (or the unbounded gaps) will be called the intervals of level $n$ for the set $K_\prec$. We remark that almost surely a set in $\mathcal{C}_a$ has no isolated points and hence there are $2^n$ closed intervals at step $n$ in this construction.

The model from [14] generates a random order on $\mathbb{N}$ by choosing an iid sequence, $\omega_n$, of $U[0, 1]$ random variables and defining $i \prec j$ if and only if $\omega_i \leq \omega_j$. Hawkes’ random set $K_\omega$ is our set $K_{\prec_\omega}$. Obviously, $K_\omega = K_{\prec_\omega}$ if and only if the corresponding total orders, $\prec_\omega$ and $\prec_\omega'$, agree. When restricted to a finite subset $F \subset \mathbb{N}$ each possible order is equally likely, so if we are given $\omega \in [0, 1]^F$ and let $\mathcal{O}_\omega = \{\omega' : \prec_\omega' \mid F = \prec_\omega\}$, then $\text{Prob}(\mathcal{O}_\omega) = (|F|!)^{-1}$. The $\sigma$-algebra on $\mathcal{O}$ is also
generated by the various $O_\omega$, $\omega \in [0, 1]^F$, taken over all finite sets $F$ since the product $\sigma$-algebra on $[0, 1]^\infty$ is generated by the cylinder sets. Thus the two random models are effectively the same.

Our proofs will rely heavily upon the following variation on the DeMoivre-Laplace theorem [4, p.13, Theorem 7].

**Theorem 3.5.** If $Y$ is a binomially distributed random variable with distribution $B(M, 2^{-N})$ and $\eta 2^{-N}(1 - 2^{-N})M \geq 12$ for some $\eta < 1/12$, then

$$\mathcal{P}(|Y - M2^{-N}| \geq \eta M2^{-N}) \leq \exp(-\eta^3 2^{-N}/3)(\eta \sqrt{M2^{-N}}).$$

Specifically, we will use the following corollary.

**Corollary 3.6.** There is a constant $c > 0$ such that if $Y$ is a binomially distributed random variable with distribution $B(M, 2^{-N})$ and $M2^{-N} \geq 200$, then

$$\mathcal{P}(Y << M2^{-N}), \mathcal{P}(Y >> M2^{-N}) \leq \exp(-cM2^{-N}).$$

**Proof.** These follow immediately from the theorem upon noting that the set $\{Y >> M2^{-N}\}$ is contained in $\{Y \geq (13/12)M2^{-N}\}$ and the set $\{Y << M2^{-N}\}$ is contained in $\{Y \leq (11/12)M2^{-N}\}$. □

### 4. Almost Sure Upper Dimensions for Complementary sets

**Terminology:** To study the dimensional properties of random rearrangements, it will helpful to introduce the following iterative construction of $E \in \mathcal{C}_a$ that we will refer to as the standard construction. We begin with the interval $[0, 1]$ and then remove the open interval (gap) $G_1 \subseteq E^c$, of length $a_1$, leaving $E_1 = [0, 1] \setminus G_1$, a union of at most two closed intervals called the intervals of step one. At step (or level) 2 we remove the two gaps of level two, $G_2, G_3$, of lengths $a_2$ and $a_3$, leaving $E_2 = E_1 \setminus (G_2 \cup G_3)$, a union of at most 4 closed intervals. Now repeat this process. Given $E_{n-1}$, to form $E_n$ we remove from $E_{n-1}$ the $2^{n-1}$ gaps of level $n$ of lengths $a_{2^{n-1}}$, ..., $a_{2^n}$, leaving $E_n$, a union of at most $2^n$ closed intervals, the intervals of step $n$. The set $E$ equals $\bigcap E_n$. Each set $E_n$ is the union of the finitely many closed intervals and isolated points that lie between the gaps that have been removed at levels 1, ..., $n$.

**4.1. Almost sure upper dimensions for “large” $\Phi$.**

**Theorem 4.1.** Let $a = \{a_n\}$ be a level comparable sequence. For almost every $E \in \mathcal{C}_a$ we have $\overline{\dim}_q E = \overline{\dim}_q C_a$ if

$$\Phi(x) >> \frac{\log|\log(x)|}{|\log x|} \text{ for } x \text{ near } 0,$$

equivalently, $\phi(n) >> \log n$.

**Corollary 4.2.** For almost all rearrangements $E \in \mathcal{C}_a$, $\dim_q A E = \dim_q A C_a$.

**Proof.** For each $\delta > 0$ and $\Phi(x) = \delta$, we have $\overline{\dim}_q E = \overline{\dim}_q C_a$ a.s. Letting $\delta \to 0$ gives the result for the quasi-Assouad dimension.

**Proof of Theorem 4.1.** As we noted in Theorem 3.2 in [12] it was shown that $\overline{\dim}_q E \geq \overline{\dim}_q C_a$ for all $E \in \mathcal{C}_a$. Thus it suffices to show the other inequality holds almost surely. Fix $d > \overline{\dim}_q C_a$ and we will show $\overline{\dim}_q E \leq d$ a.s.
Since the sequence $a$ is level comparable, there are constants $\tau, \lambda$ such that $0 < \tau \leq s_{j+1}/s_j \leq \lambda < 1/2$. Consider $N_r(B(x, R) \cap E)$ for

$$(1 - 2\lambda)s_{n+1} \leq R \leq (1 - 2\lambda)s_n, \ r \leq R^{1+\Phi(R)}$$

and $x \in E$. As $R \leq s_n$, we have $s_{n+m+1} \leq r < s_{n+m}$ for some $m \geq \phi(n)$.

Since $x \in E$, $x$ does not belong to any of the removed gaps and so $x$ must belong to some interval, $I_n(x)$, that arises at step $n$ in the standard construction of $E$. As pointed out in (4.2), $(1 - 2\lambda)s_n \leq a_{2^n}$. Since the gaps that bound $I_n(x)$ (one of which may be unbounded) have length at least $a_{2^n}$, we see that $B(x, R) \cap E \subseteq I_n(x)$, so

$$N_r(B(x, R) \cap E) \leq N_r(I_n(x) \cap E).$$

As $R/r \sim s_n/s_{n+m}$, it will be enough to prove that almost surely $N_r(I_n(x) \cap E) \leq 2\left(\frac{s_n}{s_{n+m}}\right)^d$ for $n$ sufficiently large. In other words, we want to prove that it is with probability zero that for infinitely many $n$ there are intervals $I_n$ of level $n$ and integers $m \geq \phi(n)$ such that

$$N_{s_{n+m}}(I_n \cap E) > 2\left(\frac{s_n}{s_{n+m}}\right)^d.$$

This will be a Borel Cantelli argument.

To begin, choose $L = L(n, m)$ so that

$$\sum_{j=2^{n+m+L}}^{\infty} a_j = 2^{n+m+L}s_{n+m+L} \leq s_{n+m+1}.$$

Note that $2^i s_i \leq (2\lambda)^i$ and $s_{n+m} \geq \tau^{n+m}$, so we may take $L = C(n+m)$ for a suitable constant $C > 0$.

Temporarily fix interval $I_n$. It is known that $N_r(I) \sim N_r(I')$ whenever $I, I'$ are rearrangements of the same set of gaps (see [12, Lemma 4.1]), hence there is no loss of generality in assuming the gaps which are placed in $I_n$ in the construction of $E$ at subsequent (deeper) levels, are placed in decreasing order.

The choice of $L$ ensures that the gaps placed in $I_n$ after level $n + m + L - 1$ have total length at most $r$ and thus one interval of length $r$ will cover these in totality.

Let

$$\Lambda_0(I_n) = \# \text{ gaps in } I_n \text{ from levels } n+1, \ldots, n+m$$

$$\Lambda_k(I_n) = \# \text{ gaps in } I_n \text{ from level } n+m+k \text{ for } k = 1, \ldots, L-1.$$

The gaps of level $n + m + k$ each have length comparable to $s_{n+m+k}$ and thus for each $k$, the totality of these gaps will be covered by

$$\Lambda_k(I_n)\frac{s_{n+m+k}}{s_{n+m}}$$

intervals of length $r$. The gaps of levels $n+1, \ldots, n+m$ can be covered by $\Lambda_0(I_n)$ intervals of length $r$, thus an upper bound on $N_r(I_n \cap E)$ is given by

$$N_r(I_n \cap E) \lesssim \Lambda_0(I_n) + \sum_{k=1}^{L-1} \Lambda_k(I_n)\frac{s_{n+m+k}}{s_{n+m}} + 1.$$
We wish to compare this to
\[
\left( \frac{R}{r} \right)^d \sim \left( \frac{s_n}{s_{n+m}} \right)^d
\]
If \( N_r(I_n \cap E) > 2 \left( \frac{s_n}{s_{n+m}} \right)^d \), then
\[
(4.1) \quad \Lambda_k(I_n) \frac{s_{n+m+k}}{s_{n+m}} > \frac{1}{L} \left( \frac{s_n}{s_{n+m}} \right)^d
\]
for some \( k = 0, 1, \ldots, L - 1 \).

There are \( 2^n \) such intervals \( I_n \) for each \( n \); temporarily label them as \( I_{n(i)} \), \( i = 1, \ldots, 2^n \). The probability that at least one of these intervals, \( I_{n(i)} \), satisfies condition (4.1) for some \( k = 0, 1, \ldots, L \), is at most \( \sum_{i=1}^{2^n} p_{i,k} \) where
\[
p_{i,k} = \mathcal{P} \left( \Lambda_k(I_{n(i)}) \geq \frac{1}{L} \left( \frac{s_{n+m}}{s_{n+m+k}} \right) \left( \frac{s_n}{s_{n+m}} \right)^d \right).
\]

Choose \( \varepsilon > 0 \) such that \( d - \varepsilon > \dim_4 C_a \). The choice of \( \phi(n) \) and the formula for the upper \( \Phi \)-dimension of a Cantor set (3.3) ensures that for large enough \( n \) and all \( m \geq \phi(n) \),
\[
\left( \frac{s_n}{s_{n+m}} \right)^{d-\varepsilon} \geq 2^m.
\]

Since \( s_i/s_{i+1} \geq 1/\lambda = b > 2 \),
\[
\frac{s_{n+m}}{s_{n+m+k}} \geq b^k \text{ and } \left( \frac{s_n}{s_{n+m}} \right)^{\varepsilon} \geq b^{\varepsilon m}.
\]

Thus for each \( i = 1, \ldots, 2^n \) and \( k = 0, \ldots, L - 1 \),
\[
p_{i,k} \leq \mathcal{P} \left( \Lambda_k(I_{n(i)}) \geq \frac{1}{L} 2^m b^k b^{\varepsilon m} \right)
\]
Let \( Y_{i,k} \) be the random variable that counts the number of gaps of level \( n+m+k \) for \( k \in \{1, \ldots, L - 1\} \) (or levels \( n+1, \ldots, n+m \) if \( k = 0 \)) in interval \( I_{n(i)} \) given that the gaps are placed uniformly among the \( 2^n \) such intervals. With this notation
\[
p_{i,k} \leq \mathcal{P} \left( Y_{i,k} \geq \frac{1}{C(n+m)} 2^m b^k b^{\varepsilon m} \right)
\]
(with the appropriate modification for \( k = 0 \)). The function \( Y_{i,k} \) is a binomially distributed random variable with distribution \( B(2n+m+k-1, 2^{-n}) \) (or \( B \left( \sum_{i=1}^{m} 2n+i \right. \left. , 2^{-n} \right) \) if \( k = 0 \)) as there are \( 2n+m+k-1 \) gaps (or \( \sum_{i=1}^{m} 2n+i-1 \) many gaps) to be placed in \( 2^n \) positions, thus our strategy to bound \( p_{i,k} \) is to use Corollary 3.6, taking the \( M \) in that corollary to be \( 2n+m+k-1 \) (with the obvious modification if \( k = 0 \)), and the \( N \) to be \( n \), so \( M^{2^{-N}} \sim 2^{m+k} \).

Since \( b > 2 \), incorporating this notation gives
\[
\frac{1}{C(n+m)} 2^m b^k b^{\varepsilon m} \geq \frac{1}{C(n+m)} M^{2^{-N}} b^{\varepsilon m}.
\]
The assumption \( \phi(n) \gg \log n \) guarantees that for large enough \( n \), depending on \( \varepsilon, b^{\varepsilon m} \gg n+m \). Consequently, Corollary 3.6 implies
\[
p_{i,k} \leq \exp \left( -c2^{m+k} \right)
\]
for a suitable constant $c > 0$. Hence

$$\sum_{i=1}^{2^n} \sum_{k=0}^{L} p_{i,k} \lesssim 2^n \exp(-c2^m).$$

If we write $m = \phi(n) + J$ for $J \in \mathbb{N}$, then the term $2^n \exp(-c2^m)$ is dominated by $\gamma^{n+J}$ for some $\gamma < 1$ because $\phi(n) > \log n$. Hence for large enough $n$, the probability that there is any interval $I_n^{(i)}$ at level $n$ and integer $m \geq \phi(n)$ with $N_{s_n+m}(I_n^{(i)} \cap E) > 2 \left(\frac{s_n}{s_{n+m}}\right)^d$ is at most

$$\sum_{m=\phi(n)}^{\infty} \mathcal{P}\left(\exists i \text{ with } N_{s_n+m}(I_n^{(i)} \cap E) > 2 \left(\frac{s_n}{s_{n+m}}\right)^d\right) \lesssim \sum_{m=\phi(n)}^{\infty} \sum_{i=1}^{2^n} \sum_{k=0}^{L-1} p_{i,k} \lesssim \sum_{J=0}^{\infty} \gamma^{n+J} \lesssim \gamma^n.$$

This shows that if $F_n$ is the event that there is any interval $I$ at level $n$ and any $m \geq \phi(n)$ with $N_{s_n+m}(I \cap E) > 2 \left(\frac{s_n}{s_{n+m}}\right)^d$, then

$$\sum_{n=1}^{\infty} \mathcal{P}(F_n) \lesssim \sum_{n=1}^{\infty} \gamma^n < \infty.$$

An application of the Borel Cantelli lemma proves that $\mathcal{P}(F_n \text{ i.o.}) = 0$ and that is what we desired to prove. Thus $\overline{\dim}_E \Phi \leq d$ almost surely. □

4.2. Almost sure upper dimensions for “small” $\Phi$.

**Theorem 4.3.** Let $a$ be a level comparable sequence. For a.e. $E \in \mathcal{C}_a$ we have $\overline{\dim}_E \Phi = 1$ if

$$\Phi(x) \ll \frac{\log |\log(x)|}{|\log x|} \text{ for } x \text{ near } 0,$$

equivalently, $\phi(n) \ll \log n$.

Notice that if we take $\phi = 0$ we get the result for the Assouad dimension.

**Corollary 4.4.** For almost all rearrangements $E \in \mathcal{C}_a, \dim_A E = 1$.

**Corollary 4.5.** The set of uniformly disconnected rearrangements in $\mathcal{C}_a$ is of measure zero.

**Proof.** This is immediate from the fact that a subset $E$ of $\mathbb{R}$ is uniformly disconnected (or porous) if and only if $\dim_A E < 1$ (cf. [23]). □

**Proof of Theorem 4.3.** This will require us to prove that almost surely there are $x_n \in E$, $R_n \to 0$ and $r_n \leq R_n^{1+\Phi(R_n)}$ satisfying

$$N_{r_n}(B(x_n, R_n) \cap E) \gtrsim \left(\frac{R_n}{r_n}\right)^{1-\varepsilon}$$

for each fixed $\varepsilon > 0$. 

Then the ball centred at an endpoint \( x \) of the gaps are placed in each of the intervals arising at step \( n \) of levels deeper than \( n \). The sequence \( a \) is decreasing.

Let \( I_n \) denote the interval of step \( n \) containing the maximum number of gaps of levels \( n+1, \ldots, n + \phi(n) \) and let \( L_n \) denote the sum of the lengths of the gaps of levels deeper than \( n + \phi(n) \) that are contained in \( I_n \). Since the sequence \( a \) is decreasing,

\[
L_n + M_n a_{2n+\phi(n)} \leq \text{length } I_n \leq L_n + M_n a_{2n}.
\]

Put

\[
R_n = L_n + M_n a_{2n} \quad \text{and} \quad r_n = a_{2n+\phi(n)+1}.
\]

Then the ball centred at an endpoint \( x_n \) of \( I_n \) and radius \( R_n \) contains \( I_n \). As noted in the proof of the previous theorem, there is no loss of generality in assuming the gaps are placed in \( I_n \) in decreasing order. Hence

\[
N_{r_n}(B(x_n, R_n) \cap E) \geq M_n + \frac{L_n}{r_n},
\]

while

\[
\frac{R_n}{r_n} = M_n \frac{a_{2n}}{a_{2n+\phi(n)}} + \frac{L_n}{r_n} \leq M_n \frac{\tau^{-\phi(n)}}{r_n} + \frac{L_n}{r_n},
\]

since \( a_{2j}/a_{2j+1} \sim s_j/s_{j+1} \leq 1/\tau \) by the level comparable assumption.

Fix \( \varepsilon > 0 \) and consider

\[
\frac{N_{r_n}(B(x_n, R_n) \cap E)}{\left( \frac{r_n}{R_n} \right)^{1-\varepsilon}} \geq \frac{M_n + \frac{L_n}{r_n}}{\left( M_n \frac{\tau^{-\phi(n)}}{r_n} + \frac{L_n}{r_n} \right)^{1-\varepsilon}} = \frac{M_n + \frac{L_n}{r_n}}{\left( M_n n g(n) |\log \tau| + \frac{L_n}{r_n} \right)^{1-\varepsilon}} \geq \frac{M_n + \frac{L_n}{r_n}}{\left( M_n n g(n) |\log \tau| + \frac{L_n}{r_n} \right)^{1-\varepsilon}}.
\]

If \( M_n \geq L_n/r_n \), then for large \( n \) this ratio is at least

\[
\frac{M_n}{2 n g(n) |\log \tau| (1-\varepsilon)} \geq \frac{M_n}{n \varepsilon/2}
\]

since \( g(n) \to 0 \). It follows that if \( M_n \geq K_n \) and \( n \) is sufficiently large, then

\[
\frac{N_{r_n}(B(x_n, R_n) \cap E)}{\left( \frac{r_n}{R_n} \right)^{1-\varepsilon}} \geq \frac{n^{\varepsilon/2}}{(\log n)^{1-\varepsilon}} \geq 1.
\]

Similar arguments give the same conclusion if, instead, \( K_n \leq M_n \leq L_n/r_n \).

We conclude that there are \( x_n, R_n, r_n \) as outlined above, with

\[
N_{r_n}(B(x_n, R_n) \cap E) \geq \left( \frac{R_n}{r_n} \right)^{1-\varepsilon}.
\]
whenever $M_n \geq K_n$. Since $M_n$ depends only on levels $n, n + 1, \ldots, n + \phi(n)$ and $K_n$ only on $n$, if we choose a sequence $n_k \to \infty$ such that $n_{k+1} > n_k + \phi(n_k)$, then the sets $\{M_n \geq K_n\}$ are independent events, each occurring with probability at least $1/2$. The Borel Cantelli lemma implies that these events occur infinitely often with probability one. Thus almost surely

$$N_{r_n}(B(x_n, R_n) \cap E) \gtrsim \left(\frac{R_n}{r_n}\right)^{1-\varepsilon}$$

for infinitely many $n$. Moreover, for such $n$ we have

$$R_n \gtrsim M_n a_2^n \gtrsim K_n s_n \gtrsim \frac{n}{\log n} s_n.$$  

Thus $(R_n)^{1+\Phi(R_n)} \gtrsim s_n^{1+\Phi(s_n)} \sim s_{n+\phi(n)} \sim r_n$. Also, since $M_n \lesssim 2^{n+\phi(n)}$,

$$R_n \lesssim \sum_{i>n+\phi(n)} 2^i a_2^i + M_n a_2^n \lesssim \sum_{i>n+\phi(n)} (2\lambda)^i + 2^{n+\phi(n)} \lambda^n$$

$$\lesssim (2\lambda)^n 2^{\phi(n)} = (2\lambda)^n n^{g(n) \log 2} \to 0 \text{ as } n \to \infty$$

since $\lambda < 1/2$.

This suffices to prove that with probability one, $\dim_\Phi E \geq 1 - \varepsilon$ for each $\varepsilon > 0$ and that completes the argument. \hfill \Box

5. Almost Sure Lower Dimensions for Complementary sets

The almost sure results for lower $\Phi$-dimensions will again make use of the probabilistic result, Corollary 3.6, but will also use the fact that it is “quite likely” that some interval at step $n$ will contain no gaps from levels $n + 1, \ldots, n + \log n$. (This will be made precise in the proof.) In addition, for the case of “large” $\Phi$, we will also use estimates on the size of the intervals created at step $n$ in the rearranged set.

5.1. Almost sure lower dimensions for “small” $\Phi$. We will begin with the “small” $\Phi$ case. We remark that any $E \in \mathcal{C}_a$ which admits an isolated point has lower $\Phi$-dimension zero. However, these form a null set in $\mathcal{C}_a$ and thus are not of interest to us.

**Theorem 5.1.** Let $a = \{a_n\}$ be a level comparable sequence. For almost every $E \in \mathcal{C}_a$ we have $\dim_\Phi E = 0$ if

$$\Phi(x) << \frac{\log |\log(x)|}{\log x} \text{ for } x \text{ near } 0,$$

equivalently, $\phi(n) << \log n$.

**Corollary 5.2.** For almost all rearrangements $E \in \mathcal{C}_a$, $\dim_L E = 0$ a.s.

**Corollary 5.3.** The set of uniformly perfect rearrangements in $\mathcal{C}_a$ is of measure zero.

**Proof.** This follows as a set $E$ is uniformly perfect if and only if $\dim_L E > 0$, by Lemma 2.1 in [17]. \hfill \Box
PROOF OF THEOREM 5.1. We will prove that for each \( \varepsilon > 0 \), \( \dim_{q} E \leq \varepsilon \) a.s. By definition, this is true if almost surely there are \( x_n \in E \), \( R_n \to 0 \) and \( r_n \leq R_n^{1+\Phi(R_n)} \) satisfying

\[
N_{r_n}(B(x_n, R_n) \cap E) \leq \left( \frac{R_n}{r_n} \right) ^\varepsilon .
\]

Choose \( J \geq 1 \) such that \( s_n \leq a_{2^{n-J}} \) for all \( n \). Put \( R_n = s_n \) and \( r_n = s_{n+\Phi(n)} \leq R_n^{1+\Phi(n)} \). Label the intervals arising at level \( n-J \) in the construction of \( E \) as \( I_n^{(j)} \), \( j = 1, \ldots, 2^{n-J} \) and let \( x_n = x_n(j) \) be an endpoint of interval \( I_n^{(j)} \). The gap sizes ensure that \( E \cap B(x_n, R_n) \) is contained in \( I_n^{(j)} \). Choose \( D \in \mathbb{N} \) such that

\[
\sum_{j=2^{n+\Phi(n)+Dn}}^{\infty} a_j = 2^{n+\Phi(n)+Dn}s_{n+\Phi(n)+Dn} \leq r_n.
\]

If the interval \( I_n^{(j)} \) admits no gaps from levels \( n-J+1, \ldots, n+\Phi(n)+A \log n \) (where \( A \) will be specified later) and \( \Lambda_k^{(j)} \) gaps at each of levels \( n+\Phi(n)+k \) for \( k = 1, A \log n, \ldots, Dn \), then

\[
N_{r_n}(B(x_n, R_n) \cap E) \leq N_{r_n}(I_n^{(j)} \cap E) \leq \sum_{k=1+\log n}^{Dn} \Lambda_k^{(j)} \frac{s_{n+\Phi(n)+k}}{s_{n+\Phi(n)}} + 1
\]

since the totality of the gaps of levels deeper than \( n+\Phi(n)+Dn \) can be covered by one interval of radius \( r_n \). We want to prove the quantity above is bounded by \( C (s_n/s_{n+\Phi(n)})^\varepsilon \).

Let \( F_n^{(j)} \) be the event that interval \( I_n^{(j)} \) contains no gaps from levels \( n-J+1, \ldots, n+\Phi(n)+A \log n \), but each interval \( I_n^{(i)} \) for \( i < j \) does contain at least one such gap. Let \( G_n^{(j)} \) be the event that

\[
(5.1) \sum_{k=1+\log n}^{Dn} \Lambda_k^{(j)} \frac{s_{n+\Phi(n)+k}}{s_{n+\Phi(n)}} \leq 2 \left( \frac{s_n}{s_{n+\Phi(n)}} \right) ^\varepsilon .
\]

If \( F_n^{(j)} \cap G_n^{(j)} \) is non-empty for some \( j \), then there is a “suitable” interval \( I_n^{(j)} \), meaning, an interval at level \( n-J \) which both admits no gaps from levels \( n-J+1, \ldots, n+\Phi(n)+A \log n \) and has property (5.1). As the events \( F_n^{(j)} \cap G_n^{(j)} \) are disjoint and the pairs \( F_n^{(j)}, G_n^{(j)} \) are independent (since the location of gaps at different levels are independent),

\[
\mathcal{P}(\exists \text{ suitable } I_n^{(j)}) \geq \sum_{j=1}^{2^{n-J}} \mathcal{P}(F_n^{(j)} \cap G_n^{(j)}) = \sum_{j} \mathcal{P}(F_n^{(j)}) \mathcal{P}(G_n^{(j)}).
\]

We first focus on \( G_n^{(j)} \). For an appropriate constant \( B > A+1 \), to be specified later,

\[
\mathcal{P} \left( \left( G_n^{(j)} \right)^c \right) \leq \sum_{k>B \log n}^{B \log n} \mathcal{P} \left( \Lambda_k^{(j)} \geq \left( \frac{s_n}{s_{n+\Phi(n)}} \right) ^\varepsilon \frac{s_{n+\Phi(n)}}{s_{n+\Phi(n)+k}} \frac{1}{(B-A) \log n} \right) + \sum_{k>B \log n}^{Dn} \mathcal{P} \left( \Lambda_k^{(j)} \geq \left( \frac{s_n}{s_{n+\Phi(n)}} \right) ^\varepsilon \frac{s_{n+\Phi(n)}}{s_{n+\Phi(n)+k} Dn} \right).
\]
If \( k > A \log n \) then, since \( s_i/s_{i+1} \geq 1/\tau > 2 \) for all \( i \), taking \( \gamma = (2\tau)^{-1} > 1 \) gives

\[
\frac{1}{2^{k+\phi(n)}} \left( \frac{s_n}{s_{n+\phi(n)}} \right)^\varepsilon \frac{1}{s_{n+\phi(n)+k}} \geq \frac{\gamma^k (2\tau^\varepsilon - g(n) \log n)}{\log n} \geq \frac{\gamma^k n^{-g(n) \log(2\tau^\varepsilon)}}{\log n} \geq \frac{n^A \log \gamma - g(n) \log(2\tau^\varepsilon)}{\log n} \to \infty
\]
as \( n \to \infty \). Thus if \( n \) is sufficiently large, then

\[
\left( \frac{s_n}{s_{n+\phi(n)}} \right)^\varepsilon \frac{1}{s_{n+\phi(n)+k}} \frac{1}{(B - A) \log n} \gg 2^{k+\phi(n)} = E(A^{(j)}_k).
\]

Similarly, if \( k > B \log n \), then

\[
\frac{1}{2^{k+\phi(n)}} \left( \frac{s_n}{s_{n+\phi(n)}} \right)^\varepsilon \frac{1}{s_{n+\phi(n)+k}} \frac{1}{Dn} \geq \frac{\gamma^k (2\tau^\varepsilon - g(n) \log n)}{B \log n} \geq \frac{n^A \log \gamma - g(n) \log(2\tau^\varepsilon)}{B \log n} \to \infty,
\]
provided we choose \( B \) so large that \( B \log \gamma > 1 \). Hence, again, we conclude that

\[
\left( \frac{s_n}{s_{n+\phi(n)}} \right)^\varepsilon \frac{1}{s_{n+\phi(n)+k}} \frac{1}{Dn} \gg 2^{k+\phi(n)} = E(A^{(j)}_k).
\]

Appealing to Corollary 3.6, we deduce that

\[
\mathcal{P} \left( \left( G^{(j)}_n \right)^c \right) \leq \sum_{k > A \log n} B^{\log n} \exp(-c2^{k+\phi(n)}) + \sum_{k > B \log n} D_n \exp(-c2^{k+\phi(n)}) \leq \exp(-c2^A \log n \log \phi(n)) \leq 1/2
\]

for large enough \( n \). Thus \( \mathcal{P}(G^{(j)}_n) \geq 1/2 \) for each \( j \).

Next, we observe that \( \sum_j \mathcal{P}(F^{(j)}_n) \) is the probability of there being an interval at level \( n - J \) with no gaps from levels \( n - J + 1, \ldots, n + \phi(n) + A \log n \). This is mathematically the same as the problem of equally distributing \( 2^{n+\phi(n)+A \log n} - 2^{n-J} \sim 2^{n+\phi(n)+A \log n} \) balls into \( 2^{n-J} \) bins and asking if one of the bins is empty. The expected number of balls in a bin is \( \sim 2^{\phi(n)+A \log n} \leq n^p \) for \( p < 1 \) if we choose \( A \) sufficiently small and \( n \) large. Thus

\[
2^n \exp(-\text{Expected \# balls}) \geq 2^n \exp(-C n^p) \to \infty.
\]

According to [19] p. 111, Theorem 4, the probability that there is an empty bin tends to 1 as \( n \to \infty \).

Thus for \( n \) sufficiently large, \( \sum_j \mathcal{P}(F^{(j)}_n) \geq 1/2 \) and hence

\[
\mathcal{P}(\exists \text{ suitable } I^{(j)}_n) \geq \sum_j \mathcal{P}(F^{(j)}_n) \mathcal{P}(G^{(j)}_n) \geq 1/4
\]

if \( n \) is large and \( A, B \) are chosen suitably. Furthermore, this probability depends only upon the placement of the gaps at levels \( n - J + 1, \ldots, Dn \), hence if we pick a subsequence \( \{n_k\} \to \infty \) with \( n_{k+1} - J >> Dn_k \), these events are independent. By the Borel Cantelli lemma the events occur infinitely often with probability one.
In other words, with probability one there are choices \( x_n \in E, R_n \to 0 \) and \( r_n \leq R_n^1 + \Phi(R_n) \) satisfying

\[
N_{r_n}(B(x_n, R_n) \cap E) \leq \left( \frac{R_n}{r_n} \right)^\varepsilon \text{ i.o.}
\]

and, as we observed at the beginning of the proof, this is sufficient to show \( \dim_{\Phi} E = 0 \text{ a.s.} \). \qed

### 5.2. Almost sure lower dimensions for “large” \( \Phi \)

Before turning to the “large” \( \Phi \) case, we first establish a bound on the almost sure length of the intervals of level \( n \). This lemma will be useful in determining the almost sure behaviour of the covering numbers of intervals from the construction, in place of covering numbers of the lower \( \Phi \)-dimensions because it will allow us to use lower bounds for the “large” \( \Phi \) case, we first establish a bound on the almost sure length of the intervals of level \( n \).

Since any gap of level \( L \) has length at most \( a_{2^{-i}} \leq C s_i \), the length of an interval \( I_n \) of level \( n \) is bounded by

\[
L_j = \text{length}(I_n^{(j)}) \leq C \left( \Lambda_0 s_n + \sum_{k=1}^{Dn-n(1+\varepsilon_n/2)} \Lambda_k s_n(1+\varepsilon_n/2)+k + 2 Dn s_{Dn} \right).
\]

Hence, if \( L_j > 3 C s_n^{1-\varepsilon_n} \), then since \( \sum_{k>Dn} 2^k s_k \leq s_n^{1-\varepsilon_n} \), either \( \Lambda_0 s_n > s_n^{1-\varepsilon_n} \), or for some \( k = 1, \ldots, Dn - n(1+\varepsilon_n/2) \),

\[
\Lambda_k s_n(1+\varepsilon_n/2)+k \geq \frac{s_n^{1-\varepsilon_n}}{Dn}.
\]

In other words, either

\[
\Lambda_0 \geq s_n^{-\varepsilon_n},
\]

or for some \( k \leq Dn - n(1+\varepsilon_n/2) \),

\[
\Lambda_k \geq \frac{s_n^{1-\varepsilon_n}}{Dn s_n(1+\varepsilon_n/2)+k}.
\]

In comparison, the expected value of \( \Lambda_0 \sim 2^{n \varepsilon_n/2} = n^{\log 4} << s_n^{-\varepsilon_n} \) and the expected value of \( \Lambda_k \sim 2^{k+n \varepsilon_n/2} \sim 2^k n^{\log 4} \), while

\[
\frac{s_n^{1-\varepsilon_n}}{Dn s_n(1+\varepsilon_n/2)+k} \geq \frac{s_n^{2n \varepsilon_n/2}}{Dn} = \frac{\varepsilon_n}{Dn} \sim 2^k n^{6 \log 2^2} >> 2^k n^{\log 4}.
\]

Appealing to Corollary \( \ref{corollary} \), we deduce that

\[
P \left( \Lambda_0 s_n \geq s_n^{1-\varepsilon_n} \right) \leq \exp(-c n^{\log 4})
\]
and
\[
\mathcal{P}\left(\lambda_k^{(j)} s_{n(1+\varepsilon_n/2)+k} \geq s_n^{1-\varepsilon_n}/Dn\right) \leq \exp(-c2^k n^{\log 4}).
\]
Thus the probability that \(L_j\) is more than \(3Cs_n^{1-\varepsilon_n}\) is at most
\[
\sum_{k=0}^{Dn} \exp(-c2^k n^{\log 4}) \lessapprox \exp(-cn^{\log 4}).
\]
Therefore the probability that any of the \(2^n\) intervals at level \(n\) has length exceeding
\(3Cs_n^{1-\varepsilon_n}\) is at most \(2^n \exp(-cn^{\log 4})\), and that decays exponentially in \(n\). Applying
the Borel Cantelli lemma it follows that for almost all \(E \in C_a\), all intervals of level
\(n\) have length at most \(3Cs_n^{1-\varepsilon_n}\), for large enough \(n\).

**Theorem 5.5.** Let \(a\) be a level comparable sequence. For a.e. \(E \in C_a\) we have \(\dim B E = \dim_B C_a\) if
\[
\Phi(x) >> \frac{\log |\log(x)|}{\log(x)} \text{ for } x \text{ near } 0,
\]
equivalently, \(\phi(n) >> \log n\).

By the same reasoning as Corollary 4.2 we have

**Corollary 5.6.** For almost all rearrangements \(E \in C_a\), \(\dim qL E = \dim qL C_a\).

**Proof of Theorem 5.5.** Choose \(d < \dim_B C_a\). We will show that \(\dim_B E \geq d\) a.s. Since it was already seen in Theorem 4.3 of [12] (see Theorem 3.2) that always \(\dim_B E \leq \dim_B C_a\), this will complete the proof.

From the previous lemma, we know that for all \(E \in \mathcal{O}\), a subset of the probability space \(\Omega\) with full measure, all intervals at level \(n\) (in the construction of \(E\)) have length at most \(3Cs_n^{1-\varepsilon_n}\) for \(\varepsilon_n = (4 \log n)/n\) and for \(n\) sufficiently large. Our task is to prove that for almost every \(E \in \mathcal{O}\), we have \(N_r(B(x, R) \cap E) \geq (\frac{R}{2})^d\) for all \(x \in E\), small \(R\) and \(r \leq R^{1+\Phi(R)}\).

It suffices to consider \(R = R_n = 3Cs_n^{1-\varepsilon_n}\) (where \(C\) is as in the previous lemma) since the definition of \(\varepsilon_n\) implies that there are positive constants \(a, b\) such that \(a \leq s_n^{1-\varepsilon_n}/s_{n+1}^{1-\varepsilon_n} \leq b\). Choose \(m\) such that \(s_{m+1} \leq R \leq s_m\). Notice that as \(s_n/s_m \in [\tau^{n-m}/\lambda^{m-n}], s_n^{1-\varepsilon_n} \in [\tau^{n-k}/\lambda^{m-n}], n - m \sim \log n\).

We may assume \(r = r_n = s_m + \phi(m+k)\) for some \(k \geq 0\).

If \(x\) belongs to the level \(n\) interval \(I_n^{(j)}\), then \(B(x, R) \supseteq I_n^{(j)}\). Hence it will be enough to show that
\[
N_r(I_n^{(j)} \cap E) \gtrapprox \left(\frac{s_m}{s_m + \phi(m+k)}\right)^d \sim \left(\frac{R}{r}\right)^d
\]
for all large \(n\).

From the formula for the lower \(\Phi\)-dimension of \(C_a\) [34], we know that for \(\delta > 0\) chosen such that \(d + \delta < \dim_B C_a\) and large enough \(m\),
\[
\left(\frac{s_m}{s_m + \phi(m+k)}\right)^{d+\delta} \leq 2^{\phi(m+k)},
\]
thus it will be enough to check that
\[
N_r(I_n^{(j)} \cap E) \gtrapprox 2^{\phi(m+k)d/(d+\delta)}.
\]
Choose $J$ such that $a_{2^J-j} \geq 2s_i$ for all $i$. Then $N_{r_n}(I_n^{(j)} \cap E)$ will be at least the number of gaps of level $m + \phi(m) + k - J$ contained in $I_n^{(j)}$ as such gaps have length at least $2r_n$. The expected number of gaps of level $m + \phi(m) + k - J$ contained in $I_n^{(j)}$ is at least

$$\frac{2^{m+\phi(m)+k-J}}{2^n} \sim 2^{\phi(m)+k-b_n \log n}$$

for some $b_n$ bounded above and below from 0. Since $\phi(m) = f(m) \log m$ where $f(m) \to \infty$,

$$\frac{2^{\phi(m)+k-b_n \log n}}{2^{\phi(m)+k)\delta/(d+\delta)} \geq 2^{-b_n \log n} 2^{\delta f(m) \log m)} \to \infty$$

as $m \to \infty$ (or equivalently, $n \to \infty$ since $m \sim n$). Corollary 3.6 implies that the probability that $N_{r}(I_n^{(j)} \cap E) < 2^{(\phi(m)+k)\delta/(d+\delta)}$ for some $k \geq 1$ and $j = 1, \ldots, 2^n$ is at most

$$\sum_{j=1}^{2^n} \sum_{k=1}^{\infty} \exp(-c2^{\phi(m)+k-b_n \log n}) \leq 2^n \exp(-c2^{\phi(m)-b_n \log n})$$

$$= 2^n \exp(-c2^{f(m) \log m-b_n \log n}) \leq \gamma^n$$

for some $\gamma < 1$ since $m \sim n$ and $f(m) \to \infty$.

Applying the Borel Cantelli lemma again, the probability that there are some $x_n$ with $N_{r_n}(B(x_n, R_n) \cap E) \leq \left(\frac{2s_i}{r_n}\right)^d$ i.o. is zero. That completes the proof. \( \square \)

**Remark 5.7.** It would be interesting to know what happens if $\phi(n)/\log n$ does not tend to either 0 or infinity. Even for the case $\Phi = \log \|x\|/\|\log x\|$ we do not know if the $\Phi$ dimensions almost surely coincide with the dimension of the Cantor set, the dimension of the decreasing set or something else altogether.

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