REAL CONSTITUENTS OF PERMUTATION CHARACTERS

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ABSTRACT. We prove a broad generalization of a theorem of W. Burnside on real characters using permutation characters. Under the necessary hypothesis of $O^2(G) = G$, we can give some control on multiplicities (a result that needs the Classification of Finite Simple Groups). Along the way, we also give a new characterization of the 2-closed finite groups using odd-order real elements of the group. All this can be seen as a contribution to Brauer’s Problem 11.

1. Introduction

This paper began with an elementary generalization of an old theorem of W. Burnside, which we have not found in the literature.

Theorem A. Let $G$ be a finite group, and let $H$ be a subgroup of $G$. Then $(1_H)^G$ has a unique real-valued + type irreducible constituent if and only if $[G : \text{core}_G(H)]$ is odd.

Proof. Write $K = \text{core}_G(H)$. If $G/K$ has odd order, then the trivial character $1_G$ is the only real-valued irreducible constituent of $(1_H)^G$, by Burnside’s Theorem. Conversely, if $|G/K|$ is not odd, then let $xK$ be an involution of $G/K$. Then $|G : H| = (1_H)^G(x^2) > (1_H)^G(x)$, since $\ker((1_H)^G) = K$. Therefore, using this and that $(1_H)^G$ is a permutation character, we have that

$$\frac{1}{|G|}\sum_{g \in G}(1_H)^G(g^2) > \frac{1}{|G|}\sum_{g \in G}(1_H)^G(g) = [(1_H)^G, 1_G] = 1.$$ 

Then

$$1 < \frac{1}{|G|}\sum_{g \in G}(1_H)^G(g^2) = \sum_{\chi \in \text{Irr}(G)}[(1_H)^G, \chi]\nu_2(\chi),$$

where $\nu_2(\chi)$ is the Frobenius-Schur indicator of $\chi$. By the Frobenius-Schur theorem (see Theorem 4.5 of [Is]), $\nu_2(\chi)$ is 1, −1 or 0, and the proof easily follows. □

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There is a great deal of literature on permutation characters, many inspired by Richard Brauer’s Problem 11 of his celebrated list: Given the character table of a group $G$, how much information about the existence of subgroups can be obtained? Notice that Theorem A adds a new condition to check for a given character of a finite group if it is a permutation character or not. (See Theorem 5.18 of [Is].)

In most the cases that we have observed, it turns out that there is a non-trivial real-valued odd multiplicity irreducible constituent $\chi$ in $(1_H)^G$, as long as $[G : \text{core}_G(H)]$ is not odd. (As we shall point out in Lemma 4.4 below, this implies that $\chi$ has $+$ type.) Multiplicities of permutation characters correspond to the dimensions of the representations of the Hecke algebra $\text{End}((1_H)^G)$, so they are of interest. But unfortunately this is not always the case:

Suppose that $p \equiv 3 \mod 4$ is a prime, and let $q = p^a$, where $a > 1$ is odd. Let $G = \text{AGL}(1, q)$, the semidirect product of the Galois field $F = \text{GF}(q)$ with the multiplicative group $\text{GF}(q)^\times$. If $H \leq G$ has order $2p$, then $G/FH$ is a cyclic group of odd order $q - 1 - 2\theta$, and

$$(1_H)^G = \frac{q^{a-1}-1}{2} \theta + \sum_{\lambda \in \text{Irr}(G/FH)} \lambda,$$

where $\theta \in \text{Irr}(G)$ is the unique irreducible character of order $q - 1$. Also, notice that $\frac{q^{a-1}-1}{2}$ is even and $\theta$ is rational-valued.

Our main result in this paper is that this does not happen under the additional hypothesis that $\text{O}^2(G) = G$. For the proof, we require the Classification of Finite Simple Groups.

**Theorem B.** Let $H$ be a proper subgroup of $G$. If $\text{O}^2(G) = G$, then $(1_H)^G$ has a non-trivial real-valued irreducible constituent of odd multiplicity.

As we shall see, a key idea in the proof of Theorem B is that if $1_G$ is the only real-valued irreducible constituent of $(1_H)^G$ with odd multiplicity, then $[G : H]$ is odd, and every real conjugacy class of $G$ meets $H$. (See Lemma 4.1.) If $H$ is, say, a non-normal Sylow 2-subgroup of $G$, then an elementary argument (using Baer’s theorem and involutions, see Proposition 6.4 of [DNT]) produces a non-trivial odd order real element, hence proving Theorem B. In general, however, there are examples of proper subgroups $H$ meeting all the real classes of groups $G$. And in order to prove Theorem B, we shall need to classify, essentially, all the cases whenever $H$ is a maximal subgroup of odd index in a simple group $G$. We believe that this result might have some interest on its own. Indeed, we classify all primitive permutation groups in which every real element has a fixed point. We only state the result for $n$ odd since that is what we use. See Theorem 2.6 for the corresponding result for $n$ even.
Theorem C. Let $G$ be a primitive permutation group acting on a set $X$ of odd cardinality. Assume that every real element of the socle of $G$ fixes a point of $X$. Let $M$ be a point stabilizer. Then $G$ has a unique minimal normal subgroup $A$ and either $G$ has odd order or $A = L_1 \times \ldots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Also, $M \cap A = D_1 \times \ldots \times D_t$, $D_i$ is maximal in $L_i$ and intersects every real conjugacy class of $G$ which intersects $L_i$. Furthermore $G/A$ has odd order and $(1_M)^G$ has a nontrivial $+\text{-type}$ irreducible constituent of multiplicity 1. Indeed, $L_i \cong M_{22}, M_{23}$ or $\text{PSL}_n(q)$ with $n$ odd and $D_i$ is as given in Theorem 2.1.

In the special case where $H$ is a proper 2-Sylow normalizer of $G$, a case with particular interest, we shall prove (again using the Classification) that not only $H$ does not meet some real class of $G$, but something stronger.

Theorem D. Let $G$ be a finite group, and let $P \in \text{Syl}_2(G)$. Then the following are equivalent.

(i) $P$ is normal in $G$.
(ii) There are no nontrivial real elements in $G$ of odd order.
(iii) The 2-Brauer character $(1_{N_G(P)})^G$ has no nontrivial real-valued 2-Brauer irreducible constituent with odd multiplicity.
(iv) Every real element of odd order of $G$ normalizes a $G$-conjugate of $P$.

We remark that the equivalence of the first three conditions is elementary, while the fourth is proved using the Classification.

Theorem D(iii) naturally suggests a question: is Theorem B true for Brauer characters? And the answer is “no.” For instance $G = \text{PSL}_3(2)$ has $H = S_4$ as a subgroup and the 2-Brauer character $(1_{N_G(P)})^G$ decomposes as the trivial 2-Brauer character of $G$ plus two non-real-valued irreducible 2-Brauer characters of degree 3. Indeed, Theorem B fails for $p$-Brauer characters for any $p$. Let $G = \text{PSL}_n(p)$ with $n$ odd. By [ZS, 1.8], the composition factors in characteristic $p$ of $(1_H)^G$, with $H$ the stabilizer of 1-space, are the trivial module and $(p - 1)\omega_i$ where the $\omega_i$ are the fundamental dominant weights for $\text{SL}_n(p)$. In particular, since $n$ is odd, none of the nontrivial composition factors are self dual and so none of the nontrivial irreducible $p$-Brauer characters are real.

On the other hand, in Section 5 below, we shall prove that Theorems A and B do hold subgroups of odd index and for 2-Brauer characters if certain composition factors are not involved.

It is often the case that theorems on real-valued characters can be improved to rational-valued characters. This is clearly not the case with the results in this paper. For instance, if $G = D_{10}$ is the dihedral group and $P \in \text{Syl}_2(G)$, then $(1_P)^G$ has a unique rational-valued irreducible constituent with odd multiplicity.
2. Primitive Groups and Real Elements

In this section, we classify the primitive permutation groups such that all real elements have fixed points. We apply the result in the case that \(n\) is odd but the methods to deal with the general case are only a bit more onerous than the odd case and there are very few examples with \(n\) even.

The critical case is when the group is almost simple and the next theorem deals with that. The existence of the Mathieu group example shows that it is unlikely that there is a proof without the classification of finite simple groups.

It is convenient to work with a slightly weaker assumption. We shall use in several parts of this paper an elementary result (for instance, Lemma 3.1(d) of [DMN]): if a 2-group \(S\) acts non-trivially on a group of odd order \(T\), then there exists \(t \in T\) and \(s \in S\) such that \(t^s = t^{-1}\).

We state the results for \(n\) even and \(n\) odd separately.

**Theorem 2.1.** Let \(L\) be a finite nonabelian simple group. Let \(M\) be a maximal subgroup of \(L\) such that \([L : M]\) is odd and \(M\) intersects every \text{Aut}(L)-orbit of real \(L\)-classes. Then one of the following holds:

(i) \(L = M_{22}, M = 2^4 : \text{Alt}_6, [L : M] = 77\) and \((1_M)^L = 1a + 21a + 55a;
(ii) \(L = M_{22}, M = 2^4 : \text{Sym}_5, [L : M] = 231\) and \((1_M)^L = 1a + 21a + 55a + 154a;
(iii) \(L = M_{23}, M = M_{22}, [L : M] = 23\) and \((1_M)^L = 1a + 22a;
(iv) \(L = M_{23}, M = \text{PSL}_3(4), 2, [L : M] = 253\) and \((1_M)^L = 1a + 22a + 230a;
(v) \(L = M_{23}, M = 2^4 : \text{Alt}_7, [L : M] = 253\) and \((1_M)^L = 1a + 22a + 230a;
(vi) \(L = M_{23}, M = 2^4 : (3 \times \text{Alt}_5).2, [L : M] = 1771\) and \((1_M)^L = 1a + 22a + 230a + 253a + 1035a.
(vii) \(L = \text{PSL}_n(q)\) with \(n \geq 3\), \(M\) is the stabilizer of 1-space or hyperplane, \([L : M] = \frac{q^n - 1}{q-1}\) and \(L\) acts 2-transitively on the cosets of \(M\).

**Theorem 2.2.** Let \(L\) be a finite nonabelian simple group. Let \(M\) be a maximal subgroup of \(L\) such that \([L : M]\) is even and \(M\) intersects every \text{Aut}(L)-orbit of real \(L\)-classes. Then one of the following holds:

(i) \(L = M_{11}, M = \text{Sym}_5, [L : M] = 66\) and \((1_M)^L = 1a + 10a + 11a + 44a;
(ii) \(L = M_{23}, M = M_{11}, [L : M] = 1288\) and \((1_M)^L = 1a + 22a + 230a + 1035a;
(iii) \(L = M_{24}, M = M_{12}.2, [L : M] = 1288\) and \((1_M)^L = 1a + 252a + 1035a.

**Proof.** If \(L\) is a sporadic simple group, then this is quickly computed in GAP. (We thank Thomas Breuer for doing the computation [Br].) In the statement of Theorem 2.2, 230aa, for instance, means that the unique irreducible character of degree 230 of \(M_{23}\) appears with multiplicity 2.

Suppose that \(L = \text{Alt}_n\) with \(n \geq 5\). First suppose that \(n\) is odd and \(n \geq 9\). Note that the following \(\text{Sym}_n\)-conjugacy classes consist of real elements in \(\text{Alt}_n\): \(n - 2\) cycles, elements with exactly three orbits of size 3, 3 and \(n - 6\) and 3-cycles. Since
the first two classes intersect \( M \), it follows that \( M \) is primitive and since it contains a 3-cycle, \( M = L \), a contradiction.

Next suppose that \( n \) is even and \( n \geq 10 \). Then consider the following \( \text{Sym}_n \) real classes in \( \text{Alt}_n \): \( n - 3 \) cycles, elements with two cycles of size 4 and \( n - 4 \), elements with orbits of size 1, 3, 3 and \( n - 7 \) and 3-cycles. Again it follows that \( D \) is primitive and contains 3-cycles, a contradiction.

If \( n < 9 \), one can compute using GAP (or check by hand) that there are no possibilities.

So \( L \) is a finite simple group of Lie type in characteristic \( p \). The first cases where the adjoint form of the group is not simple require separate handling and can be checked in GAP (or they arise in different contexts, e.g. \( \text{Sp}_4(2) \cong \text{Sym}_6 \)) and so we assume that the adjoint group is simple.

First consider the case that \( L = \text{PSL}_2(q) \) with \( q \geq 11 \) and \( q \) odd (if \( q = 5, 7 \) or 9, these groups arise in other forms). Then elements of order \((q \pm 1)/2\) are real and no proper subgroup contains elements of both orders. If \( L = \text{PSL}_2(q) \) with \( q \geq 4 \) and even, then all elements are real.

Next suppose that \(-1\) is in the Weyl group of \( L \neq \text{PSL}_2(q) \). Then all semisimple elements are real. Since there are real unipotent elements, this implies that every prime dividing the order of \( L \) also divides the order of \( M \). By \([\text{LPS}, \text{10.7}]\), this can only occur in the following cases:

(i) \( L \cong \Omega_{2n+1}(q) \), \( n \geq 2 \), \( n \) even and \( M \) is the stabilizer of a nondegenerate hyperplane of \(-\) type (when \( q \) is even, \( L = \text{Sp}_{2n}(q) \));
(ii) \( L \cong \Omega_{2n}^+(q) \), \( n \geq 4 \), even and \( M \) is the stabilizer of a nondegenerate hyperplane;
(iii) \( L \cong \text{PSp}_4(7) \) and \( M = \text{Alt}_7 \);
(iv) \( L \cong \text{Sp}_4(8) \) and \( M = 2B_2(8) \);
(v) \( L \cong \text{Sp}_6(2) \) and \( M = \text{Sym}_8 \);
(vi) \( L \cong \Omega_8^+(2) \) and \( M \) a maximal end node parabolic or \( M = \text{Alt}_9 \);
(vii) \( L \cong G_2(3) \) and \( M = \text{PSL}_2(13) \); or
(viii) \( L \cong 2F_4(2)' \) and \( M = \text{PSL}_2(25) \).

In all but the first two cases, the result follows easily by inspection of the character tables. Suppose that \( L \cong \Omega_{2n+1}(q) \), \( n \geq 2 \) with \( n \) even and \( M \) the stabilizer of a nondegenerate hyperplane of \(-\) type. Let \( x \) be a real element stabilizing two complementary totally singular subspaces each of dimension \( n \) which generate a hyperplane of \(+\) type. Then \( x \) does not fix a hyperplane of \(-\) type and so no \( \text{Aut}(L) \)-conjugate of \( x \) is in \( M \).

Suppose that \( L \cong \Omega_{2n}^+(q) \), \( n \geq 4 \), even and \( M \) is the stabilizer of a nondegenerate hyperplane. Let \( x \) be a real element stabilizing two complementary totally singular subspaces each of dimension \( n \). Then \( x \) does not preserve any hyperplane and in particular is not conjugate to an element of \( M \).
So we may assume that $-1$ is not in the Weyl group of $L$.

First consider the case the $L = \Omega_{2n}$, $n \geq 5$ and odd. Consider the following real conjugacy classes. The first is an element of order $(q^n + 1)$ acting irreducibly on a nondegenerate subspace of codimension 2 (of $-$ type) and trivial on its perpendicular complement. The second is an element of order $q^{n-1} - 1$ acting irreducibly on each of a pair of totally singular spaces of dimension $n - 1$ which generate a nondegenerate subspace of dimension $2n - 2$ of $+$ type and again acting trivially on the 2 dimensional orthogonal complement.

Then $M$ must act irreducibly or is irreducible on a nondegenerate hyperplane.

Suppose $q$ is odd. Consider a unipotent element $x$ with two Jordan blocks of size $n$. Note that this commutes with an element in the coset of a graph automorphism and this class is also fixed by the Frobenius automorphism over the prime field. We can view this element in $\text{SO}_n \times \text{SO}_n$. By [LS2, Theorem 3.1], a regular unipotent element in $\text{SO}_n$ has connected centralizer and so is conjugate to its inverse in $\text{SO}_n$. Thus, $x$ is conjugate to its inverse by an element of spinor norm 1 and so $x$ is real and its fixed space is a two dimensional totally singular space. This shows that $M$ is irreducible and by considering a unipotent element with a Jordan block of size 3, it is absolutely irreducible. Since root elements are real, the result now follows either by [Ka, Theorem 1] or [GS, Theorem 7.1] that $M$ does not exist.

Suppose that $q$ is even. Decompose the space as an orthogonal sum of nondegenerate spaces with one of dimension 4 of $+$ type and the rest two dimensional. Let $x$ be a unipotent element consisting of a single Jordan block of size 4 in the four dimensional space and a single Jordan block of size 2 for each of the other blocks. It is trivial to check that $x$ is real and we can arrange it so the $x \in \Omega_{n}(q)$ (by choosing the type of each 2-space). The fixed space of $x$ is totally singular of dimension $n - 1$ and so $M$ acts irreducibly. Since $n$ is odd and our first element does not live in a unitary group, $M$ is absolutely irreducible. As above, $M$ cannot be irreducible (again as it contains root elements).

Next consider $L = E_6^{'}(q)$. Then long root elements are real as well as elements in the unipotent class $E_6(a_1)$ [LS1, Chapter 22] The latter class rules out $M = F_4$. Also every semisimple conjugacy class which intersects $F_4(q)$ is real. In particular, $M$ contains elements of order $q^4 \pm 1$ and $q^4 - q^2 + 1$. In particular, this implies that $M$ is not parabolic.

We claim that there exists a conjugacy class of real regular unipotent elements. If $p \geq 3$, this follows from the fact that a regular unipotent element is real in the algebraic group and the number of connected components of the centralizer is odd. If $p = 2$, it is still true that there is a class of real regular unipotent elements by inspection of the character table of $F_4(2)$ (and noting that regular unipotent elements in $F_4$ are also regular unipotent in $E_6$). A list of all possible maximal subgroups containing a regular unipotent element in an exceptional group is given in [GM] and none of them contain all the real elements described above.
Next suppose that \( L = \text{PSL}_n(q) \) with \( n \geq 3 \). Note that applying an outer automorphism does not affect semisimple classes (taking inverses or applying Frobenius preserves the conjugacy classes of cyclic subgroups and diagonal automorphisms preserve all semisimple classes). Thus, we can work with \( L \)-classes of real semisimple elements.

We first show that if \( M \) is reducible, then \( n \) is odd and \( M \) must be the stabilizer of a 1-space or hyperplane.

Suppose that \( n \) is odd. Then there exists \( x \) real acting irreducibly on a hyperplane (take an irreducible element in \( \text{Sp}_{n-1}(q) \)). Thus, if \( M \) is reducible, the only possibility is that \( M \) is the stabilizer of a hyperplane or 1-space. Note that since \( \text{SL}_n(q) \) has odd center, real elements lift to real elements in \( \text{SL} \). Any real element of \( \text{SL}_n(q) \) must have 1 as eigenvalue and so fixes a 1-space and a hyperplane and so this example is allowed in the conclusion. Suppose that \( n \geq 4 \) is even. Then there is a real element \( x \) that acts irreducibly and so is not in any parabolic.

Essentially the same proof works when \( q \) is even aside from a very few small cases. Note that given a Jordan form for a unipotent element, there is a real element with that Jordan form. It suffices to prove this for a regular unipotent element and we argue as for \( E_6 \). The centralizer of a regular unipotent element in \( \text{SL}_n(q) \) is \( Z \times U \) where \( U \) is a connected abelian unipotent group of dimension \( n - 1 \) and \( Z \) is the center. Since \( q \) is even, \( Z \) has odd order. Since unipotent elements are real (indeed rational) in the algebraic group, the result follows. Now argue as above using either \[ \text{Ka} \] or for \( d \geq 6 \) \[ \text{GS} \] Theorem 7.1] to conclude there are no such groups. If \( d < 6 \), a straightforward inspection of the maximal subgroups yields the result \[ \text{BHR} \].

Finally suppose that \( L = \text{PSU}_n(q) \) with \( n \geq 3 \). If \( n = 3 \), then an element of order \( q+1 \) with eigenvalues 1, \( a, a^{-1} = a^q \) is real and is contained in no parabolic. If \( n = 4 \), then similarly as long as \( q > 2 \), we can find a real element of order \( q+1 \) with distinct eigenvalues not contained in any parabolic subgroup. For \( L = \text{PSU}_4(2) \), one checks directly in GAP that there are no examples.

So assume that \( n \geq 5 \). Write \( n = 2m + \delta \) with \( m \) odd and \( 0 \leq \delta \leq 3 \). Choose an element \( x \) that on a nondegenerate \( 2m \) space has two nonisomorphic irreducible invariant subspaces each of dimension \( m \) with one the dual of the other. On the orthogonal complement, let \( x \) with distinct eigenvalues (1 if \( \delta = 1 \), \( a \) and \( a^{-1} \) if \( \delta = 2 \)
and $1, a, a^{-1}$ if $\delta = 3$). Then $x$ preserves no totally singular space and is real and is not in any parabolic subgroup.

Suppose that $L = \text{PSU}_n(q)$, $n > 2$ with $q$ odd. Then transvections are real and so if $M$ is irreducible, it is either linear, unitary or a symplectic group. We can rule out symplectic groups by considering real unipotent elements. If $n = 2m$ with $m$ odd, then there is a real element of order $q^m + 1$ preserving precisely 2 totally singular hyperplanes and this rules out the possibilities for linear subgroups or proper unitary subgroups. If $n = 2m + 1$, then there exists a real element of order $q^m + 1$ and again this eliminates the possibility of linear groups or proper unitary subgroups.

Suppose that $M$ is reducible. Let $y$ be a unipotent element with one Jordan block of size 3 and all other Jordan block sizes have size 2. Then $y$ is real and preserves no nondegenerate 1-space. A unipotent with a Jordan block of size $n - 1$ is real and so $M$ preserves no nondegenerate space. Thus, $M$ would be parabolic and we have ruled out that possibility above.

Suppose that $L = \text{PSU}_n(q)$, $n > 2$ with $q$ even. As for the case of $\text{PSL}_n(q)$, there are real classes of unipotents with any prescribed Jordan form and again arguing in the same manner as for $\text{PSL}_n(q)$ and using [GS] and [BHR], we see that there are no irreducible subgroups intersecting all classes of real elements. □

Next we consider subgroups $H$ which are not necessarily maximal but still satisfy the hypotheses of Theorem 2.1. Since we only use the result when $n$ is odd, we assume that and leave the case $n$ even to the interested reader.

**Corollary 2.3.** Let $L$ be a finite nonabelian simple group and $H$ a proper subgroup of $L$ such that $H$ intersects every $\text{Aut}(L)$ conjugacy class of real elements in $L$ and $[L : H]$ is odd. Then either $H$ is maximal in $L$ or one of the following holds:

(i) $L = M_{23}$ and there are three possible conjugacy classes of subgroups $H$ each self normalizing. In each case, $(1_H)^L$ contains a real-valued nontrivial constituent of multiplicity 1; or

(ii) $L = \text{PSL}_n(q)$ with $n$ odd, $H$ is contained in a unique maximal subgroup $M$ which is the stabilizer of a 1-space or hyperplane and $[M, M] \leq H$. Thus $M = N_L(H)$. Also, in all cases $(1_H)^L$ contains the irreducible real character $\tau \in \text{Irr}(L)$ of degree $\frac{q^n - 1}{q - 1} - 1$ with multiplicity 1.

**Proof.** If $L$ is a sporadic group, then by the previous result $L \cong M_{22}$ or $M_{23}$ and one computes using GAP (again, we thank Thomas Breuer). Otherwise, $L \cong \text{PSL}_n(q)$ with $n$ odd. Let $M$ be a maximal subgroup containing $H$ and we assume that $H \neq M$. Then $M$ is the stabilizer of a 1-space or hyperplane. As we noted in the proof of the previous result, $H$ contains a real element acting irreducibly on a hyperplane and so $M$ is the unique maximal subgroup containing $H$.

Let $U$ be the unipotent radical of $M$ and so $H = UJ$ where $J$ is a subgroup of a Levi subgroup of $M$. If $q$ is even and $H$ does not contain $[M, M]$, then $J$ would be
contained in a proper parabolic subgroup of $L$, a contradiction. So $H \geq [M, M]$ in this case.

Now suppose that $q$ is odd. Then $J$ contains transvections as well as an element of order $q^{(n-1)/2} + 1$. It follows by [Ka] that $J$ contains $[L, L]$ and the result follows.

For the last statement, if $L$ is sporadic, this is a straightforward computation. If $L = \text{PSL}_n(q)$, then we see that $H$ has precisely two orbits on 1-spaces whence $[1_H]^L \cdot (1_M)^L = 2$ and so the nontrivial constituent of $(1_M)^L$ occurs with multiplicity 1 in $(1_H)^L$. \hfill \square

The almost simple case also follows easily:

**Theorem 2.4.** Let $G$ be a finite almost simple group with socle $L$ (a nonabelian simple group). Let $M$ be a maximal subgroup of $G$ such that $[G : M]$ is odd and $D := L \cap M$ intersects all real $G$-classes contained in $L$. Then one of the following holds:

(i) $G = L = M_{22}, \; M = 2^4 : \text{Alt}_6$, $[G : M] = 77$ and $(1_M)^L = 1a + 21a + 55a$;
(ii) $G = L = M_{22}, \; M = 2^4 : \text{Sym}_5$, $[G : M] = 231$ and $(1_M)^L = 1a + 21a + 55a + 154a$;
(iii) $G = L = M_{23}, \; M = M_{22}, \; [G : M] = 23$ and $(1_M)^L = 1a + 22a$;
(iv) $G = L = M_{23}, \; M = \text{PSL}_3(4).2_2, \; [G : M] = 253$ and $(1_M)^L = 1a + 22a + 230a$;
(v) $G = L = M_{23}, \; M = 2^4 : \text{Alt}_7$, $[G : M] = 253$ and $(1_M)^L = 1a + 22a + 230a$;
(vi) $G = L = M_{23}, \; M = 2^4 : (3 \times \text{Alt}_5).2$, $[G : M] = 1771$ and $(1_M)^L = 1a + 22a + 230aa + 253a + 1035a$.
(vii) $L = \text{PSL}_n(q)$ with $n \geq 3$, $M$ is the stabilizer of 1-space or hyperplane, $[G : M] = \frac{q^{n-1}}{q-1}$ and $L$ acts 2-transitively on the cosets of $M$. Moreover, $[G : L]$ is odd.

**Proof.** If $G = L$, the previous result applies.

Suppose that $G > L$. If $L$ is sporadic, then $L = M_{22}$ and if an outer automorphism is present, then elements of order 11 in $L$ are real and we see there are no proper maximal subgroups of $L$ containing all real elements (the only possibilities are the $D$ already mentioned and they do not contain elements of order 11).

The only other possibility is that $L = \text{PSL}_n(q)$ with $n$ odd and by the case $G = L$ we see that $D$ is contained in a parabolic subgroup. Since $[\text{PGL}_n(q) : \text{PSL}_n(q)] = \gcd(n, q - 1)$ is odd, if $G/L$ has even order, $G$ contains an outer involution that is either a graph, field or graph-field automorphism. The graph automophism inverts all semisimple classes and so all semisimple classes in $L$ would then be real in $G$ and there are semisimple elements not contained in any parabolic subgroup. If $G$ contains a field or field-graph automorphism of order 2, then $q = 3$ and the field automorphism inverts all classes intersecting $\text{PSU}_n(q)$ and there are such classes which do not preserve a 1-space or hyperplane. If $G$ contains a field-graph automorphism of order 2, then any semisimple class defined over $q_0$ is inverted and there are irreducible such
elements. This shows that $[G : L]$ is odd as claimed. Since $D$ is contained in a unique maximal subgroup of $L$ (a parabolic), we see that $M = N_G(D)$ normalizes this maximal subgroup, whence the maximality of $M$ implies that $D$ is parabolic.  \[\]

We next consider general primitive groups of odd degree. This is Theorem C of the introduction. Recall that real-valued irreducible characters are either of $+$ type (or orthogonal) if they can afforded by a representation with real entries, or of $-$ type (or symplectic), if they cannot. The following is Theorem C of the introduction.

**Theorem 2.5.** Let $G$ be a primitive permutation group acting on a set $X$ of odd cardinality. Assume that every real element of the socle of $G$ fixes a point of $X$. Let $M$ be a point stabilizer. Then $G$ has a unique minimal normal subgroup $A$ and either $G$ has odd order or $A = L_1 \times \ldots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Also, $M \cap A = D_1 \times \ldots \times D_t$, $D_i$ is maximal in $L_i$ and intersects every real conjugacy class of $G$ which intersects $L_i$. Furthermore $G/A$ has odd order and $(1_M)^G$ has a nontrivial orthogonal constituent of multiplicity 1. Indeed, $L_i \cong M_{22}, M_{23}$ or $\text{PSL}_n(q)$ with $n$ odd and $D_i$ is as given in Theorem 2.1.

**Proof.** We have that $M$ is maximal in $G$ with core$_G(M) = 1$. Since $M$ contains a Sylow 2-subgroup of $G$, we have that $O_2(G) = 1$. Suppose that $O_p(G) \neq 1$ with $p$ odd. Let $A$ be a minimal normal subgroup contained in $O_p(G)$. Then $G = AM$ and $A \cap M = 1$. If $G$ does not have odd order, then $S$ cannot centralize $A$, and therefore there is $1 \neq a \in A$ which is inverted by some element of $S$ (see for instance Lemma 3.1(d) of [DMN]). By hypothesis, some $G$-conjugate of $a$ is in $M$, but this is impossible since $A \cap M = 1$.

Let $A$ be a minimal normal subgroup of $G$. Thus, $A$ is a direct product $L_1 \times \ldots \times L_t$ with $L_i \cong L$ a nonabelian group. Since $L_i$ contains real elements, $D_i = M \cap L_i \neq 1$ for some $i$. If $B$ is another minimal normal subgroup of $G$, then $[A, B] = 1$ and thus $M \cap B$ is normal in $AM = G$. Hence $M \cap B = 1$ and therefore $|B| = |G : M|$ is odd. But this cannot happen. Hence $A$ is the unique minimal normal subgroup of $G$.

Since $M$ acts transitively on the $L_i$, it follows that $D_i$ intersects any real class of $G$ that intersects some $L_i$. Let $G_i = N_G(L_i)/C_G(L_i)$. This is an almost simple group with socle $L_i$ and $N_{G_i}(D_i)$ is a maximal subgroup of $G_i$. Our assumptions imply that $D_i$ intersects every real class of $L_i$ (in $G_i$) whence $(G_i, L_i, N_{G_i}(D_i), D_i)$ satisfies the conclusions of Theorem 2.1. In particular, $D_i$ is maximal in $L_i$ and $G_i/L_i$ has odd order.

We next show that $G/A$ has odd order. Suppose that there exists $g \in G \setminus A$ an outer automorphism with $g^2 \in A$. We may assume that $g$ has order a power of 2. Then $g$ cannot normalize all components $L_i$ (as the automizer in $G$ of any component has odd order). Thus, we may assume that $g$ interchanges $L_1$ and $L_2$. Replacing $g$ by $gu$ with $u \in L_1 \times L_2$ allows us to assume that $g$ interchanges the coordinates of $L_1$ and $L_2$. Choose $x \in L_1$ that has
no fixed points. Then \( g \) inverts \( c := (x, x^{-1}, 1, \ldots, 1) \) with \( b \in L \). Then \( c \) has no fixed points but is real, a contradiction. \(\square\)

Here is the analog for \( n \) even.

**Theorem 2.6.** Let \( G \) be a primitive permutation group acting on a set \( X \) of even cardinality \( n \). Assume that every real element of the socle of \( G \) fixes a point of \( X \). Let \( M \) be a point stabilizer. Then \( G \) has a unique minimal normal subgroup \( A \) and \( A = L_1 \times \ldots \times L_t \) with \( L_i \cong L \) a nonabelian simple group. Also, \( M \cap A = M_1 \times \ldots \times M_t \), \( M_i \) is maximal in \( L_i \) and intersects every real conjugacy class of \( G \) which intersects \( L_i \). Furthermore \( G/A \) has odd order and \((L_i, M_i)\) are described in Theorem 2.3.

**Proof.** Let \( A \) be a minimal normal subgroup. Then \( A \) acts transitively whence \( A \) has even order. If \( A \) acts regularly, then some involution of \( A \) has no fixed points, a contradiction. Thus, \( G \) has no regular normal subgroups and so \( A = L_1 \times \ldots \times L_t \) with \( L_i \cong L \), a nonabelian simple group. Since \( M \cap L_i \neq 1 \) for some \( i \) (since \( L_j \) contains real elements), it follows that \( M \cap A = M_1 \times \ldots \times M_t \) where \( M_i \) are maximal in the almost simple group \( N_G(M_i)/C_G(M_i) \). It follows by Theorem 2.2 that \( L_i \) is a Mathieu group and has trivial outer automorphism group, whence \( N_G(M_i)/C_G(M_i) = M_i \) and \( M_i \) is maximal in \( L_i \). Arguing as in the case that \( n \) is even shows that \( G/A \) has odd order. \(\square\)

We close this section by proving a result about transitive groups which we require for 2-Brauer characters.

**Corollary 2.7.** Let \( G \) be a (faithful) transitive permutation group of odd degree \( n \). Let \( A = L_1 \times \ldots \times L_t \) be a normal subgroup of \( G \) such that each \( L_i \cong L \), a nonabelian simple not isomorphic to any of \( M_{22}, M_{23} \) or \( PSL_n(q) \) with \( n \) odd. Then some real element (in \( G \)) of \( A \) is fixed point free.

**Proof.** Let \( H \) be a point stabilizer and set \( B = H \cap A \neq A \). Since \( n \) is odd, \( B \geq S_1 \times \ldots \times S_1 \) where \( S_i \in \text{Syl}_2(L_i) \). Let \( Q_i = \pi_i(B) \) where \( \pi_i \) is the projection from \( A \) onto \( L_i \). If \( Q_i = L_i \), then \( L_i = [Q_i, S_i] = [B, S_i] \leq B \). Thus (reordering if necessary), \( B \leq M_1 \times M_2 \times \ldots \times M_t \). Let \( a \in L \) be a real element. Then \( y := (a, \ldots, a) \) has a fixed point, whence \( y^2 \cap H \) is nonempty. This implies that \( M_1 \geq Q_1 \) contains an \( \text{Aut}(L_1) \) conjugate of \( a \) for every real element in \( L_1 \). By Theorem 2.1 and the fact that we are excluding the simple groups occurring in the conclusion of that theorem, we obtain a contradiction. \(\square\)

3. **Sylow 2-subgroups and Theorem D**

In this section we prove Theorem D. For Brauer characters we use the notation in \( \text{[N]} \). If \( \Psi \) is a Brauer character of \( G \), and we write \( \Psi = \sum_{\varphi \in \text{IBr}(G)} a_\varphi \varphi \), we call \( a_\varphi \) the multiplicity of \( \varphi \) in \( \Psi \).
Recall that the equivalence of (i) and (ii) in Theorem D is elementary and well-known. (See for instance Proposition 6.4 in [DNT].) We start with condition (iii).

**Theorem 3.1.** Let \( G \) be a finite group and assume that \( P \in \text{Syl}_2(G) \) is not normal in \( G \). Then the 2-Brauer character \((1_{N_G(P)}^0)^G\) contains a non-trivial irreducible real 2-Brauer character with odd multiplicity.

**Proof.** Arguing by contradiction, suppose that 
\[
(1_{N_G(P)}^0)^G = a_1^G + \sum_i b_i (\varphi_i + \overline{\varphi_i}) + \sum_j c_j \mu_j,
\]
where \( \varphi_i \in \text{IBr}(G) \) are non-real, \( \mu_j \in \text{IBr}(G) \) are real but non-trivial, \( a \) is a positive integer, and \( c_j \) is even for every \( j \). Now, by Theorem 2.30 of [N], we have that \( \mu_j(1) \) is even for every \( j \). Since \([G : N_G(P)]\) is odd, we conclude that \( a \) is odd.

Let \( 1 \neq x \) be a real element of odd order (using that \( P \) is not normal and Proposition 6.4 in [DNT]). Suppose that \( x \) normalizes some Sylow 2-subgroup \( Q \). Let \( y \in G \) invert \( x \) with \( y \) a 2-element. Let \( \Omega \) be the set of Sylow 2-subgroups of \( G \) normalized by \( x \). Then notice that \( y \) acts on \( \Omega \) and has no fixed point on it. Indeed, if \( Q \in \Omega \) and \( Q^y = Q \), then \( y \in Q \). Then \( y^x \in Q^x = Q \). However, \( y^x = x^{-1} \), a contradiction.

We conclude that \( |\Omega| \) is even. Therefore \( b = (1_{N_G(P)}^0)^G(x) \) is even. Thus
\[
b = (1_{N_G(P)}^0)^G(x) = a + 2\alpha
\]
for some algebraic integer \( \alpha \). We deduce that \(-a/2\) is an algebraic integer, but this is not possible because \( a \) is odd.

We briefly digress to recall that, unlike ordinary characters, the trivial 2-Brauer character does not necessarily appear with multiplicity 1 in the 2-Brauer character \(((1_P)^0)^G\), even if \( P \) is self-normalizing. In the following, \( \Phi_\varphi \) is the projective indecomposable character associated with \( \varphi \in \text{IBr}(G) \).

**Lemma 3.2.** If \( p \) is a prime, \( G \) is a finite group and \( P \in \text{Syl}_p(G) \), then
\[
(1_P^0)^G = \sum_{\varphi \in \text{IBr}(G)} \frac{\Phi_\varphi(1)}{|P|} \varphi.
\]

**Proof.** Use the induction formula and compare the character \((1_P^0)^G\) with the restriction of the regular character of \( G \) to the \( p \)-regular elements of \( G \). (Notice that Lemma 3.2 reproves the well-known fact that \(|G|_p\) divides the degrees of the projective indecomposable characters.) Now, if \( G = A_6 \), then \( P = N_G(P) \) and the 2-Brauer trivial character of \( G \) enters with multiplicity 5 in the Brauer character \((1_P^0)^G\).

Next we show that condition (iv) of Theorem D does not happen in simple groups.
**Theorem 3.3.** Let $G$ be a finite nonabelian simple group and let $P$ be a Sylow 2-subgroup of $G$. Then there exists $g \in G$ with $g$ real and of odd order with $g$ not normalizing any $G$-conjugate of $P$.

**Proof.** Let us remark that if $P = N_G(P)$, it follows by the Baer-Suzuki theorem that there is a real element $g \in G$ of odd order, whence the result follows. This handles 21 of the 26 sporadic groups. In the remaining 5 cases, there is a real element of order 5 and 5 does not divide the order of $N_G(P)$ (these are the cases $G = J_1, J_2, J_3, Sz$ and $HN$).

If $G = A_n$, $n \geq 6$, then $P = N_G(P)$ and the result follows. If $n = 5$, then 5-cycles are real and do not normalize a Sylow 2-subgroup.

Now suppose that $G$ is a simple group of Lie type. Since the proof is quite similar to that of Theorem 2.1, we just indicate the modifications required.

Suppose first that $G$ is a group in characteristic 2. In this case, the proof given in Theorem 2.1 shows that there exist odd order real elements not contained in any parabolic subgroup aside from the cases with $G = PSL_n(q)$ with $n$ odd. The proof shows that the only parabolics which intersect every real class are the two maximal end node parabolics and in particular not the Borel subgroup.

Finally, suppose that $G$ is a simple group of Lie type in characteristic not 2. If $G = PSL(2, q)$, $q \geq 11$, then either $P$ is self normalizing and the result follows as above or $|P| = 4$ and $|N_G(P)| = 12$. Note that there is a real element of odd order either $(q + 1)/2$ or $(q - 1)/2$ and this element does not normalize a Sylow 2-subgroup.

If $G = PSL_n(q)$ or $PSU_n(q)$ with $n \geq 3$, then transvections are real. Note that a transvection normalizing a Sylow 2-subgroup would have to preserve each irreducible constituent of $P$ and so we can restrict to the case that $P$ is irreducible. However, by [Ka], the irreducible subgroups containing transvections are classified and $N_G(P)$ is not a possibility.

Next assume that $-1$ is in the Weyl group. Then every semisimple element is real. Note that there exist unipotent real elements (since a Borel subgroup $B$ has even order and $O_2(B) = 1$, this follows by the Baer-Suzuki theorem). It follows that any prime dividing $|G|$ also divides $|N_G(P)|$. By [LPS 10.7], the only possibilities for $(G, M)$ with $M$ containing $N_G(P)$ are $G = \Omega_{2m+1}(q)$ and $M$ the stabilizer of a hyperplane of $-1$ type. Let $x$ be an element whose order is the odd part of $q^m - 1$. Then the only irreducible subspaces of $x$ are two totally singular hyperplanes of dimension $m$ and a nondegenerate 1-space. Note that $x$ is real and not conjugate to an element of $M$, whence the result.

The only remaining possibility for $G$ a classical group is $\Omega_{2m}^\pm(q)$ with $m \geq 5$ odd. Note that in the case $P$ has a unique two dimensional irreducible submodule and so we can work in $\Omega_{2m-2}^\pm(q)$ and use the previous result.

This leaves only the case that $G = E_6^\pm(q)$. It is easy to see that $N_G(P)/P$ embeds in the Weyl group and so in particular contains no odd order prime elements of order...
greater than 5. On the other hand, if \( g \in F_1(q) < G \) is an element whose order is a primitive prime divisor of \( q^{12} - 1 \), then \( g \) is a real and has order at least 13, whence \( g \) normalizes no conjugate of \( P \). \( \square \)

In order to prove the next result, which easily closes the proof of Theorem D, we use that odd order real elements of factor groups can be lifted to real elements of the group. This is key in an inductive hypothesis.

**Theorem 3.4.** Suppose that \( G \) is a finite group, \( P \in \text{Syl}_2(G) \). If every odd order real element of \( G \) lies in some \( G \)-conjugate of \( N_G(P) \), then \( P \) is normal in \( G \).

**Proof.** Argue by induction on \(|G|\). Suppose that \( 1 < N \) is normal in \( G \), and let \( Nx \in G/N \) of odd order. By Lemma 3.2 of \([NST]\), there is \( y \in G \) real such that \( Nx = Ny \). Since \( Nx \) has odd order, then by taking \( y^2 \), we may assume that \( y \) has odd order. Then \( y^g \in N_G(P) \) for some \( g \in G \), so \( Nx^g \in N_{G/N}(PN/N) \), and by induction, we may assume that \( PN \) is normal in \( G \).

In particular, assume that \( N \) is a normal minimal normal subgroup, and let \( K = PN \). If \( N \) is a 2-group, then \( N \leq P \) and so \( P \) is normal in \( G \). If \( N \) is an elementary abelian subgroup of odd order, then either \( P \) centralizes \( N \) whence \( P \) is characteristic in \( K \) and so normal in \( G \) or \( C_P(N) = 1 \) and \( P = N_K(P) \). Then Baer-Suzuki implies that there exist a real element \( g \) of odd order in \( N \). Note that \( g \) does not normalize a Sylow 2-subgroup of \( K \). Since \( K \) is normal in \( G \), any Sylow 2-subgroup of \( G \) is contained in \( K \) and we have a contradiction.

So \( N \) is a direct product of nonabelian simple groups. Let \( L \) be a component of \( N \). By Theorem 3.3 there is an element \( g \in L \) that is real and of odd order which does not normalize any Sylow 2-subgroup of \( L \). Then \( g \) normalizes no Sylow 2-subgroup of \( N \) (since if \( g \) normalizes some Sylow 2-subgroup \( T \) of \( G \), then \( g \) also normalizes \( T \cap L \), a Sylow 2-subgroup of \( L \)). \( \square \)

### 4. Proof of Theorem B

Our notation for characters follows \([IS]\). If \( N \trianglelefteq G \) and \( \theta \in \text{Irr}(N) \), then \( \text{Irr}(G|\theta) \) is the set of irreducible constituents of the induced character \( \theta^G \). By the Frobenius reciprocity, these are the characters \( \chi \) such that \( \theta \) is a constituent of the restriction \( \chi_N \). Also, \( \nu_2(\chi) \) denotes the Frobenius-Schur indicator of \( \chi \in \text{Irr}(G) \). If \( \chi \) is real valued, then \( \nu_2(\chi) = \pm 1 \), according to its type.

**Lemma 4.1.** Let \( G \) be a finite group, and let \( H \) be a subgroup of \( G \). If \((1_H)^G\) has a unique real-valued irreducible constituent with odd multiplicity, then \( x^G \cap H \neq \emptyset \) for every real element \( x \in G \). Furthermore, \([G : H]\) is odd.

**Proof.** Write

\[
(1_H)^G = 1_G + 2 \sum_i a_i \nu_i + \sum_j b_j (\tau_j + \bar{\tau}_j),
\]

where \( \nu_i \) and \( \tau_j \) are the irreducible constituents of \( 1_H \) in \( G \) and \( H \) respectively.
where \( \nu_i \in \text{Irr}(G) \) are real-valued, and \( \tau_j \in \text{Irr}(G) \) are not. If \( x \in G \) is real and \( x^G \cap H = \emptyset \), then by evaluating in \( x \) the expression above, we conclude that
\[
0 = 1 + 2\alpha,
\]
where \( \alpha \) is an algebraic integer. This is impossible. For the second part, evaluate the character \((1_H)^G\) in the trivial element.

Indeed, in Lemma 4.1, the conclusion \( x^G \cap H \neq \emptyset \) can be replaced by the stronger condition \((1_H)^G(x)\) is odd for every real element \( x \in G \). Unfortunately, this does not seem to shorten the proofs much—in all the cases with \( G \) primitive of odd degree, every real elements has a fixed point turns out to be equivalent to every real element having an odd number of fixed points.

The following is well-known.

Lemma 4.2. Suppose that \( G/N \) has odd order.

(i) If \( \chi \in \text{Irr}(G) \) is real valued, then every irreducible constituent of \( \chi_N \) is real-valued.

(ii) If \( \theta \in \text{Irr}(N) \) is real-valued, then there exists a unique \( \chi \in \text{Irr}(G|\theta) \) real-valued.

Proof. Let \( \theta \in \text{Irr}(N) \) be under \( \chi \). By Clifford’s theorem, we have that \( \bar{\theta} \) and \( \theta \) are \( G \)-conjugate by some \( g \in G \). Then \( g^2 \) fixes \( \theta \), and therefore \( g \) fixes \( \theta \), using that \( G/N \) has odd order. Hence, \( \theta = \theta g = \bar{\theta} \). The second part is Corollary 2.2 of [NT].

We should warn the reader that if \( [G:H] \) is odd and \( \theta \in \text{Irr}(H) \) is real valued, even of \( + \) type, it is not necessarily true that \( \theta^G \) contains a real-valued irreducible constituent. The SmallGroup \( (48,30) = C_3 : Q_8 \) is an example. Also, it is often the case, that odd-degree real-valued irreducible constituents of \((1_H)^G\) appear with odd multiplicity whenever \( [G:H] \) is odd. But not always, as shown by \( G = \text{PSL}_2(25) \) and \( H = D_{24} \).

The following includes Theorem B of the introduction.

Theorem 4.3. Let \( G \) be a finite group and let \( H \) be a proper subgroup of \( G \). Assume that \( G = KH \) where \( K \) is the smallest normal subgroup of \( G \) with \( G/K \) of odd order. Then \((1_H)^G\) contains a non-trivial real-valued irreducible constituent with odd multiplicity.

Proof. Let \( G \) be a counterexample minimizing \( |G| \). By Lemma 4.1, we have that \( [G:H] \) is odd and that \( H \) intersects every real conjugacy class of \( G \). Let \( 1 < V \) be a normal subgroup of \( G \). By working in \( G/V \), we may assume that \( HV = G \) (since if \( \phi \) is an irreducible constituent of \((1_{HV})^G\), then its multiplicity in \((1_{HV})^G\) is the same as in \((1_H)^G\)). In particular, \( \text{core}_G(H) = 1 \).

Note that \( O_2(G) \leq H \) and so \( O_2(G) = 1 \). Let \( A \) be a minimal normal subgroup of \( G \), so that \( G = HA \). Suppose that \( A \) is an elementary abelian \( p \)-group for some
odd prime $p$. Then $H \cap A \triangleleft G$ and thus $H \cap A = 1$ and $C_H(A) = 1$. In particular, if $P$ is a Sylow 2-subgroup of $H$, then $P$ does not centralize $A$. Hence, $A$ contains a non-trivial real element $x$ (see for instance Lemma 3.1(d) of [DMN]). However, $x^G \cap H \subseteq A \cap H = 1$, a contradiction.

We claim that $A$ is the unique minimal normal subgroup of $G$. Let $B$ be another such. Then $G = HB$ and $B$ centralizes $H \cap A$, whence $H \cap A$ is normal, a contradiction.

It follows that $A = L_1 \times \ldots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Let $C_i = H \cap L_i$. If $x \in L_i$ is real, then $x^g \in H$. Using that $g = ah$ for some $a \in A$ and $h \in H$, and that $[L_i, L_j] = 1$ for $i \neq j$, we have that $x^{l_i} \in C_i$ for some $l_i \in L_i$. Therefore $L_i$ satisfies the hypotheses of Corollary 2.3 with respect to the subgroup $C_i$.

Let $M$ be a maximal subgroup of $G$ containing $A \cap H$. Notice that $\text{core}_G(M) = 1$, since otherwise, $A \leq M$ but $G = AH$. By Theorem 2.5, we have that $K = A$. In particular, $H$ acts transitively on the $L_i$.

Let $C = C_1 \times \ldots \times C_t$ and note that $H \leq N_G(C)$. Also, $N_L(C) = B_1 \times \ldots \times B_t$, where $B_i = N_L(C_i)$. Let $\pi_i : H \cap A \to L_i$ be the projection map. So $C_i \leq \pi_i(H \cap C) \leq B_i$. We now apply Corollary 2.3. If $L = M_{22}$, then $B_i = C_i$ is maximal in $L_i$. If $L = M_{23}$, then $B_i = C_i$ since $C_i$ is self-normalizing. Thus, in these cases $A \cap H = C$.

In the case that $L = PSL_n(q)$ with $n$ odd, $B_i$ is the maximal parabolic containing $C_i$.

By Corollary 2.3 we have that the characters

$$(1_{C_1})^{L_1} \times \ldots \times (1_{C_t})^{L_t}$$

and

$$(1_{B_1})^{L_1} \times \ldots \times (1_{B_t})^{L_t}$$

have a non-trivial real-valued constituent $\eta \in \text{Irr}(K)$ of multiplicity 1. Since $C \leq H \cap K \leq B$, in all cases we conclude that

$$((1_H)^G)_K = (1_{H \cap K})^K$$

contains a non-trivial irreducible constituent $\eta$ with multiplicity 1. (Here we are using that $(1_{C_i})^{L_i}$ and $(1_{B_i})^{L_i}$ contain the same non-trivial irreducible constituent $\tau_i$ with multiplicity 1, by Corollary 2.3.)

Since $G/K$ is odd, let $\nu \in \text{Irr}(G)$ be the unique real-valued irreducible constituent lying over $\eta$, by Lemma 4.2. Now, if we write

$$(1_H)^G = a\nu + \sum_i a_i(\nu_i + \bar{\nu}_i) + \Delta$$
where $\Delta$ is a character such that $[\Delta_K, \eta] = 0$ and $\nu_i \in \text{Irr}(G)$ are non-real and lie over $\eta$, we have that

$$1 = [(1_H^G)_K, \eta] = a + 2 \sum_i a_i[\nu_i, \eta],$$

and we deduce that $a = 1$. □

Notice that Lemma 4.4 below implies that the odd-multiplicity irreducible character in Theorem 4.3 is of $+$ type.

**Lemma 4.4.** Suppose that $G$ is a finite group $H$ is a subgroup of $G$. Let $\rho$ be a character of $G$ which admits a $G$-invariant non degenerate quadratic form. If $\chi \in \text{Irr}(G)$ is real-valued and $[\chi_H, \rho]$ is odd, then $\chi$ has $+$ type.

**Proof.** Let $V$ be a $CG$-module affording $\rho$. Let $W$ be the homogeneous component of $W$ corresponding to $\chi$. Note that $W$ is orthogonal to every other homogeneous component (since it is self dual), whence $W$ is non degenerate.

So we may assume that $\rho = m\chi$ with $m$ odd. We induct on $m$. If $m = 1$, the result is clear. Otherwise, let $U$ be an irreducible submodule of $W$. If $\chi$ has $-1$ type, then $U$ must be totally singular. Then $U^\perp/U$ is non degenerate with respect to the quadratic form with character $(m - 2)\chi$ and the result follows. □

In particular, Lemma 4.4 applies to the case where $\rho = (1_H^G)$ or more generally if $\rho = \psi^G$ with $\psi \in \text{Irr}(G)$ irreducible of $+$ type under the assumption that $[\rho_H, \psi]$ is odd. One can also give a proof of the previous result using Schur indices. The proof above shows that the Schur index over $\mathbb{R}$ is odd and divides 2 whence the result.

To summarize the results in this section, we have shown the following.

**Theorem 4.5.** Let $H$ be a subgroup of $G$. Then $(1_H^G)$ has a non-trivial real-valued irreducible constituent of odd multiplicity under any of the following hypotheses.

(i) $[G : H]$ is even.
(ii) $H \cap x^G = \emptyset$ for some real element $x$ of $G$.
(iii) $(1_H^G)(x)$ is even for some real element $x$ of $G$.
(iv) $H$ is maximal in $G$ and $G/\text{core}_G(H)$ has even order.
(v) $G = O^2(G)H$ and $H$ does not contain $O^2(G)$.

**Proof.** The first three results follow from Lemma 4.3 and its proof, while (iv) and (v) follow from Theorem 4.3. □

5. More on 2-Brauer characters

In this final section, we prove that there are versions of Theorems A and B for 2-Brauer characters. First we need some results for $p$-Brauer characters, where $p$ is any prime, which are essentially well-known. If $\Psi = \sum_{\varphi \in \text{IBr}(G)} a_\varphi \varphi$ is a Brauer character of $G$, with certain abuse of notation, let us write $a_\varphi = [\Psi, \varphi]$. 
Lemma 5.1. Suppose that $G/N$ has odd order.

(i) If $\chi \in \text{IBr}(G)$ is real valued, then every irreducible constituent of $\chi_N$ is real-valued.

(ii) If $\theta \in \text{IBr}(N)$ is real-valued, then there exists a unique $\chi \in \text{IBr}(G|\theta)$ real-valued. Furthermore, $[\chi_N, \theta] = 1$.

Proof. Part (i) follows precisely as in the proof of Lemma 4.2(i). The first part of (ii) follows from Lemma 5 and Corollary 1 of [NT06]. Now, notice, in that proof, that $\chi = \psi^G$, where $\psi \in \text{Irr}(T)$ is the unique real extension of $\theta$ to the stabilizer $T$ of $\theta$ in $G$. By the Clifford correspondence for Brauer characters (Theorem 8.9 of [N]), we have that $[\chi_N, \theta] = [\psi_N, \theta] = 1$. \quad \Box

We also need the following result, which is well-known for ordinary characters. We have been unable to find a proof without using projective representations.

Lemma 5.2. Suppose that $G = NH$, where $N \triangleleft G$ and $H \leq G$. Let $\theta \in \text{IBr}(N)$ be $G$-invariant such that $\theta_D = \varphi \in \text{IBr}(D)$. Then restriction defines a bijection $\text{IBr}(G|\theta) \rightarrow \text{IBr}(H|\varphi)$.

Proof. Let $\mathcal{P}$ be a projective representation of $G$ associated with $\theta$ with factor set $\alpha$. (That is, $\mathcal{P}_N$ affords $\theta$, $\mathcal{P}(g n) = \mathcal{P}(g)\mathcal{P}(n)$ and $\mathcal{P}(n g) = \mathcal{P}(n)\mathcal{P}(g)$ for $g \in G$, $n \in N$, see Theorem 8.14 of [N].) We know that $\alpha(x n, y m) = \alpha(x, y)$ for $x, y \in G$ and $n, m \in N$. View $\alpha \in Z^2(G/N, \mathbb{C}^\times)$. Then notice that $\mathcal{P}_H$ is a projective representation of $H$ associated with $\varphi$.

Now, let $\mu \in \text{IBr}(G|\theta)$. By Theorem 8.16 in [N], there is a projective representation $\mathcal{Q}$ of $G/N$ with factor set $\alpha^{-1}$ such that $\mathcal{Q} \otimes \mathcal{P}$ affords $\mu$.

By Theorem 8.18 of [N], we have that $\mathcal{Q}$ is irreducible. Now, notice that $\mathcal{Q}_H$ is irreducible since $N h \mapsto D h$ is an isomorphism $G/N \rightarrow H/D$. By Theorem 8.18 of [N], we have that the representation $\mathcal{Q}_H \otimes \mathcal{P}_H = (\mathcal{Q} \otimes \mathcal{P})_H$ is irreducible. Therefore $\mu_H$ is irreducible.

Suppose now that $\tau \in \text{IBr}(G|\theta)$, and let $\mathcal{Q}'$ as before such that $\mathcal{Q}' \otimes \mathcal{P}$ affords $\tau$. If $\tau_H = \mu_H$ then $\mathcal{Q}'_H \otimes \mathcal{P}_H$ and $\mathcal{Q}_H \otimes \mathcal{P}_H$ are similar. By Theorem 8.16 of [N], $\mathcal{Q}'_H$ and $\mathcal{Q}_H$ are similar. By the isomorphism, $\mathcal{Q}'$ and $\mathcal{Q}$ are similar, and therefore so are $\mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{Q}' \otimes \mathcal{P}$. Hence $\tau = \mu$. Surjectivity is proved similarly. \Box

Lemma 5.3. Suppose that $N \leq H \leq G$, where $N$ is normal in $G$, $G/N$ has odd order, and $\theta \in \text{IBr}(H)$ is real-valued. Then there exists a unique $\chi \in \text{IBr}(G)$ real-valued such that $\theta$ is an irreducible constituent of $\chi_H$. Furthermore, $[\chi_H, \theta] = 1$. If $p = 2$, then $[\theta^G, \chi] = 1$.

Proof. By Lemma 5.1(i), let $\nu \in \text{Irr}(N)$ be real-valued under $\theta$. First of all notice that if $\chi$ exists necessarily is unique. Indeed, if $\chi_i \in \text{Irr}(G)$ are real-valued over $\theta$, then let $\chi_i$ lies over $\nu$, and therefore $\chi_1 = \chi_2$ by Lemma 5.1(ii). Also, the fact that if $\chi$ exists then $[\chi_H, \theta] = 1$, follows from the second part of Lemma 5.1.
In this paragraph, we prove that if \( p = 2 \), then \( \chi \) is necessarily an irreducible constituent of the \( \theta^G \). By the second part of Corollary 8.7 of [N], we have that \([x_N, \nu] = [\nu^G, \chi]\), that is, Frobenius reciprocity holds for \( N \triangleleft G \) if the characteristic does not divide \(|G/N|\). Hence, by Lemma 5.1, we can write \( \nu^G = \theta + \sum_i a_i(\eta_i + \bar{\eta}_i) \), where \( \eta_i \in \text{IBr}(H) \) are not real-valued. Now \( \nu^G = \theta^G + \sum_i a_i(\eta_i + \bar{\eta}_i) \). Since \( \chi \) is real-valued and appears with multiplicity one in \( \nu^G \), necessarily \( \chi \) is a constituent of \( \theta^G \), and appears with multiplicity 1.

Hence, we only need to prove that there exists some \( \chi \in \text{Irr}(G) \) real-valued, lying over \( \theta \). Suppose that \( (G, H, N) \) is a counterexample minimizing first \(|G|\), then \(|G : N|\), and finally \(|G : H|\). Using our inductive hypotheses, it is straightforward to check that we may assume that \( H \) is maximal and that \( N \) is the core of \( H \) in \( G \). If \( H = N \), then the lemma is Lemma 5.1. Now, let \( K/N \) be a chief factor of \( G \). Then \( KH = G \) and \( K \cap H = N \), using that \( G/N \) is solvable. Let \( I \) be the stabilizer of \( \nu \) in \( H \) and let \( \tau \in \text{Irr}(I) \) be the Clifford correspondent of \( \theta \) over \( \nu \). By the uniqueness in the Clifford correspondence, we have that \( \tau \) is real-valued. If \( IK < G \), then by induction, there is a real-valued \( \mu \in \text{Irr}(IK) \) over \( \tau \). Since \([G : K] < [G : N]\), by induction, there is \( \chi \in \text{Irr}(G) \) real-valued over \( \mu \). Then \( \chi \) lies over \( \tau \). Thus there is \( \epsilon \in \text{IBr}(H) \) under \( \chi \) such that \( \epsilon \) contains \( \tau \). By the Clifford correspondence for Brauer characters, necessarily \( \epsilon = \tau^H = \theta \), and we are done in this case. Hence, we may assume that \( IK = G \) and therefore \( I = H \). If \( H \) is the stabilizer of \( \nu \) in \( G \), then \( \theta^G \) is irreducible and lies over \( \theta \) by the Clifford correspondence, and we are done. Hence, we may assume that \( \nu \) is \( G \)-invariant. By Lemma 5.1(ii), there is a unique \( \hat{\nu} \in \text{Irr}(G) \) real-valued over \( \nu \), that also extends \( \nu \). By uniqueness, notice that \( \hat{\nu} \) is \( G \)-invariant. By Lemma 5.2, we have that restriction defines a bijection \( \text{Irr}(G|\hat{\nu}) \rightarrow \text{Irr}(H|\nu) \). Now, if \( \gamma \in \text{Irr}(G|\hat{\nu}) \) extends \( \theta \), then \( \gamma \) is real-valued by the uniqueness of the restriction map. \( \square \)

We have mentioned in the introduction that Theorem B does not hold for Brauer characters, and we have given some counterexamples. However, using similar arguments as those used before, we can prove that essentially these are all the counterexamples. We only sketch a proof.

**Theorem 5.4.** Let \( G \) be a group with a proper subgroup \( H \) of odd index. Set \( K = O^2(G) \) and assume that \( H \) does not contain \( K \). If \( K \) does not surject onto \( M_{23} \) or \( PSL_n(q) \) with \( n \) odd and \( q \) even, then the 2-Brauer character \(((1_H)^0)^G \) has a nontrivial real-valued irreducible constituent \( \phi \).

**Proof.** First assume that \( G = KH \).

Arguing by induction on \(|G|\) and then on the index \(|G : H|\), it suffices to assume that \( H \) is maximal in \( G \) (note that this maximal subgroup does not contain \( K \)). In this case, we will prove that there is a real constituent of odd multiplicity.

We may assume that \( H \) is corefree. Since \( H \) has odd index and every nontrivial self dual irreducible Brauer character has even degree, it follows that the trivial character
occurs an odd number of times in \(((1_H)^0)^G\). Arguing as in the proof of Theorem 3.1 shows that the result follows unless every odd order real element has an odd number of fixed points on \(\{Hx \mid x \in G\}\). Since \([G : H]\) is odd, this is implies the same for any element whose odd part is real (because if \(g \in G\), then \((g_2)^0\) acts on the set of points fixed by \(g_2\)). Hence \(H\) must intersect every conjugacy class of real elements. Now by Theorem C, \(K\) is a direct product of copies of a simple nonabelian group \(L\) and \(L\) is isomorphic to \(M_{22}, M_{23}\) or \(\text{PSL}_n(q)\) with \(n\) odd.

Since we are excluding the possibility that \(L = M_{23}\) or \(\text{PSL}_n(q)\) with \(q\) even, it follows by Theorem C that \(L \cong M_{22}\) or \(\text{PSL}_n(q)\) with \(n\) odd and \(q\) odd and moreover the structure of \(M \cap K\) is given in Theorem C. Arguing as in the case of ordinary characters, the result follows once we know that \(((1_H)_{\mathcal{L}})^0\) has a nontrivial real constituent. If \(L = M_{22}\), then the result follows by inspection. If \(G = \text{PSL}_n(q)\) with \(q\) odd, then the permutation module in characteristic 2 is \(k \oplus W\) with \(W\) irreducible (and so self dual). This follows from the fact that the smallest representation of \(\text{PSL}_n(q)\) with \(nq\) odd in characteristic 2 has dimension \(q^{n-1} - 1\) \([\text{GPPS}]\).

The result follows in this case. Now consider the general case. Set \(J = HK\). Then we know that \(((1_H)^0)^J\) has a nontrivial real constituent \(\theta\). By Lemma 5.3, then there exists a nontrivial real irreducible Brauer character \(\phi\) such that \([\theta^G, \phi] = 1\) whence \(\phi\) is a real nontrivial constituent of \(((1_H)^0)^G\). □

If we strengthen the hypotheses a bit, we can prove the analog of Theorem B for 2-Brauer characters for subgroups of odd index. We first state an elementary lemma.

**Lemma 5.5.** Let \(G\) be a finite group, \(H\) a subgroup and \(A\) a normal subgroup with \(|A|\) not a multiple of the prime \(p\). Let \(\phi\) be an irreducible \(p\)-Brauer character with \(A \leq \ker(\phi)\). Then \(\left((1_H)^0\right)^G, \phi = \left((1_H)^0\right)^{HA}, \phi\).

**Proof.** Let \(k\) be an algebraically closed field of characteristic \(p\) and let \(M = k[G/H]\) be the permutation module. Since \(|A|\) has order prime to \(p\), \(M = [A, M] \oplus C_M(A)\). Also, note that there is a surjection \(f\) of \(kG\)-modules from \(k[G/H]\) onto \(k[G/J]\). Since \(A\) acts trivially on \(k[G/J]\) and since the dimension of the fixed points of \(A\) on \(k[G/H]\) is \([G : J]\), we that if \(A \leq \ker(\phi)\), then \(\phi\) is not a composition factor of the kernel of \(f\) whence the claim. □

**Theorem 5.6.** Let \(G\) be a group with a proper subgroup \(H\) of odd index. Set \(K = \text{O}_2^G(G)\) and assume that \(G = HK\). Assume that \(K\) has no composition factor isomorphic to \(M_{22}, M_{23}\) or \(\text{PSL}_n(q)\) with \(n\) odd. Then the 2-Brauer character \(((1_H)^0)^G\) has a nontrivial real-valued irreducible constituent \(\phi\) of odd multiplicity.

**Proof.** We induct on \(|G|\) and then on \([G : H]\). Consider a minimal counterexample. So \(H\) is corefree and \(\text{O}_2(G) = 1\). We use induction on the index. If there exists \(x \in G\) real of odd order with \(((1_H)^G(x)\) even, then as we have seen before the result follows. So we may assume that \(((1_H)^G(x)\) is odd for such \(x\), whence the same is true for all real \(x\). In particular, we may assume that \(H \cap x^G\) is nonempty for all real \(x\).
Suppose there exists a minimal normal abelian subgroup $A$ of $G$ contained in $K$.

First suppose that $J := HA \neq G$. By minimality, it follow that there exists a nontrivial real constituent $\phi$ of $((1_J)^0)^G$ of odd multiplicity. By Lemma 5.5, $[((1_H)^0)^G, \phi] = [((1_J)^0)^G, \phi]$ is odd, a contradiction.

Suppose that $G = J$. Then $H \cap A$ is normal in $G$, whence $H \cap A = 1$. If a Sylow 2-subgroup centralizes $A$, then $[K, A] = 1$, whence $A \leq Z(G)$ and so $G = H \times A$ and $K \leq H$, a contradiction. If not, then $A$ contains real elements but if $1 \neq x \in A$ is real, then $x^G \subseteq A$ is disjoint from $H$, a contradiction.

So we may assume that $F(K) = 1$. Let $A$ be a minimal normal subgroup of $G$ contained in $K$. So $A = L_1 \times \ldots \times L_t$ where $L_i \cong L$ is a nonabelian simple group (and not isomorphic to one of the forbidden composition factors). Since every real element of $A$ is conjugate to an element of $H$, Corollary 2.7 implies the result.

We close by noting that if $[G : H]$ is even, it is easy to see that the trivial 2-Brauer character occurs with even multiplicity in $((1_H)^0)^G$ and so in particular there are at least two real valued irreducible constituents.

References

[BHR] J. Bray, D. Holt and C. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups, London Mathematical Society Lecture Note Series, 407. Cambridge University Press, Cambridge, 2013.

[Br] T. Breuer, private communication.

[DMN] S. Dolfi, G. Malle, G. Navarro, The finite groups with no real $p$-elements. Israel J. Math. 192 (2012), no. 2, 831–840.

[DNT] S. Dolfi, G. Navarro and Pham Huu Tiep, Primes dividing the degrees of the real characters, Math. Z. 259 (2008), no. 4, 755–774.

[GM] R. Guralnick and G. Malle, Rational rigidity for $E_8(p)$, Compos. Math. 150 (2014), 1679–1702.

[GPPS] R. Guralnick, T. Penttila, C. Praeger and J. Saxl, Linear groups with orders having certain large prime divisors, Proc. London Math. Soc. 78 (1999), 167–214.

[GS] R. Guralnick and J. Saxl, Generation of finite almost simple groups by conjugates, J. Algebra 268 (2003), 519–571.

[Is] I. M. Isaacs, ‘Character Theory of Finite Groups’, AMS-Chelsea, Providence, 2006.

[Ka] W. Kantor, Subgroups of classical groups generated by long root elements, Trans. Amer. Math. Soc. 248 (1979), 347–379.

[LPS] M. Liebeck, C. Praeger and J. Saxl, Transitive subgroups of primitive permutation groups, J. Algebra 234 (2000), 291–361.

[LS1] M. Liebeck and G. Seitz, A survey of maximal subgroups of exceptional groups of Lie type. Groups, combinatorics & geometry (Durham, 2001), 139–146, World Sci. Publ., River Edge, NJ, 2003.

[LS2] M. Liebeck and G. Seitz, Unipotent and nilpotent classes in simple algebraic groups and Lie algebras, Mathematical Surveys and Monographs, 180. American Mathematical Society, Providence, RI, 2012. xii+380 pp.

[N] G. Navarro, Characters and Blocks of Finite Groups, Cambridge University Press, 1998.
[NT06] G. Navarro and Pham Huu Tiep, Rational Brauer characters. Math. Ann. 335 (2006), no. 3, 675–686.
[NT] G. Navarro and Pham Huu Tiep, Rational irreducible characters and rational conjugacy classes in finite groups, Trans. Amer. Math. Soc. 360 (2008), no. 5, 2443–2465.
[NST] G. Navarro, L. Sanus and Pham Huu Tiep, Real characters and degrees, Israel J. Math. 171 (2009), 157–173.
[ZS] A. E. Zalesskii and I. D. Suprunenko, Permutation representations and a fragment of the decomposition matrix of symplectic and special linear groups over a finite field, Sibirsk. Mat. Zh. 31 (1990), 46–60, 213; translation in Siberian Math. J. 31 (1990), 7440–755.

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