Towards A Categorical Approach of Transformational Music Theory

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Abstract:

Transformational music theory mainly deals with group and group actions on sets, which are usually constituted by chords. For example, neo-Riemannian theory uses the dihedral group $D_{24}$ to study transformations between major and minor triads, the building blocks of classical and romantic harmony. Since the developments of neo-Riemannian theory, many developments and generalizations have been proposed, based on other sets of chords, other groups, etc. However music theory also face problems for example when defining transformations between chords of different cardinalities, or for transformations that are not necessarily invertible. This paper introduces a categorical construction of musical transformations based on category extensions using groupoids. This can be seen as a generalization of a previous work which aimed at building generalized neo-Riemannian groups of transformations based on group extensions. The categorical extension construction allows the definition of partial transformations between different set-classes. Moreover, it can be shown that the typical wreath products groups of transformations can be recovered from the category extensions by "packaging" operators and considering their composition.
1 Introduction

After the pionneering work of David Lewin [1], music theory has seen developments which have relied heavily on the group structure, wherein group elements are seen as operations between some set elements which usually represent chords. In neo-Riemannian theory, the classical set of elements was originally constituted by the major and minor chords, and the typical corresponding group of transformations is isomorphic to the dihedral group $D_{24}$ of 24 elements, whether it acts through the famous L, R and P operations or through the transpositions and inversions operators [2, 3, 4, 5], or many others (see for example the Schritt-Wechsel group) [6].

Following its application to major/minor triads, generalizations have been actively researched. For example, transformational theory has also been applied to other sets of chords [7]. Different groups of transformations than the dihedral one have been proposed. Julian Hook’s UTT group is a much larger group of order 288 and has at its a core a wreath product construction [8, 9]. Wreath products were also studied by Robert Peck in a more general setting [10]. More recently, Robert Peck also introduced imaginary transformations [11], in which quaternion groups, dicyclic groups and other extraspecial groups appear. A different approach has been undertaken in [12], in an attempt to unify all these different groups, in which generalized neo-Riemannian groups of musical transformations are built as extensions.

However, the current group-based transformational theories raise multiple issues. One of them is that they sometimes fail to provide interesting groups of transformations for some sets of chords (an example will be given below). A second one is that transformational theories have also failed to provide a solution to the cardinality problem, namely finding transformations between chords of different cardinalities. While Childs [13] studied neo-Riemannian theory applied to seventh chords, his model does not include triads. Hook [14] introduced another approach, namely cross-type transformations, to circumvent this problem.

In this paper, we introduce a categorical approach to musical transformations with the aim of generalizing existing constructions. This work can be viewed as a generalization of the previous work on group extensions, by using groupoids instead of groups, and by building the corresponding groupoid extensions. Note that a categorical approach to music theory has been heavily investigated in the book *The Topos of Music* by G. Mazzola [15]. Mazzola deplores in particular that "Although the theory of categories has been
around since the early 1940s and is even recognized by computer scientists, no attempt is visible in AST (Atonal Set Theory) to deal with morphisms between pcsets, for example”. While this paper is rather technical and more mathematically-than musically-oriented, we nevertheless hope that it will provide useful leads for application to music analysis. The first section highlights some of the limitations of current transformational theories based on particular examples. The second part introduces a categorical construction for musical transformations. Finally, the third part explores the relation between the categories constructed in section 2 and the more familiar groups of musical transformations, showing in particular how wreath products are naturally recovered from the category extensions.

2 On some limitations of transformational theories

2.1 Groups of transformations acting on three set-classes

Consider the pitch-class sets [0,4,7], [0,2,5] and [0,4,5], as represented in Figure 1. In the rest of this paper, we will label these sets as M, α and β respectively.

These set-classes have a well-defined root, which can therefore take any value in $\mathbb{Z}_{12}$. In this paper we will denote by $n_t$ a chord of root $n$ and of
type \( t \). By analogy with the action of the \( T/I \) group on the set of major and minor triads, transposition operators \( T_i \) can be defined for \( M, \alpha \) and \( \beta \). The action of these transposition operators is straightforward as \( T_i \) takes a chord \( n_t \) to \( (n + i)_t \) (all operations are understood modulo 12).

There also exists voice-leading transformations \( VL \) between these set-classes. For example, if one represents a chord as an ordered set \( (x, y, z) \), where \( x \) is the root, we can define the \( VL \) transformation as

\[
VL : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z + 2 \\ x - 1 \\ y - 2 \end{pmatrix}
\]

Using the notation \( n_t \) for chords, this transformation is then defined as :

\[
VL : \begin{pmatrix} n_M \\ n_\alpha \\ n_\beta \end{pmatrix} \mapsto \begin{pmatrix} (n - 3)_\alpha \\ (n - 5)_\beta \\ (n - 5)_M \end{pmatrix}
\]

We can define another voice-leading transformation \( VL' \) with a similar action as

\[
VL' : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z + 4 \\ x + 1 \\ y \end{pmatrix}
\]

or equivalently :

\[
VL' : \begin{pmatrix} n_M \\ n_\alpha \\ n_\beta \end{pmatrix} \mapsto \begin{pmatrix} (n + 1)_\alpha \\ (n - 3)_\beta \\ (n - 3)_M \end{pmatrix}
\]

We can notice that \( VL^{-3} = VL'^{21} = T_1 \). The \( VL \) and \( VL' \) operations are clearly contextual \[^{[16]}\] since their action on the root depends on the type of the chord on which they act. Since their action switches the type of the chords, they can be seen as ”generalized inversions” similar to the \( I \) transformations of the \( T/I \) group, or the \( P, L \) or \( R \) operations of the \( PLR \) group. If we wish to build a group which includes both the transposition operators and these generalized inversions, we will obtain that \( \langle T_i, VL \rangle = \langle T_i, VL' \rangle \cong \mathbb{Z}_{36} \), as can be checked with any computational group theory software such as GAP.
The construction introduced in [12] aims at building generalized neo-Riemannian groups of musical transformations which include both transposition and inversion operators. These groups $G$ are built as extensions of $Z$ by $H$, where $Z$ is the group of transpositions and $H$ can be seen as a group of ”formal inversions”. In the present case, $Z$ would be isomorphic to $\mathbb{Z}_{12}$ whereas $H$ would be isomorphic to $\mathbb{Z}_3$ to reflect the inversions between the three different pitch-class sets. If one tries to apply this construction to build a group extension $G$ of simply transitive musical transformations as

$$1 \rightarrow \mathbb{Z}_{12} \rightarrow G \rightarrow \mathbb{Z}_3 \rightarrow 1$$

one ends up with only two abelian groups, namely $G = \mathbb{Z}_{12} \times \mathbb{Z}_3$ or $G = \mathbb{Z}_{36}$. The reason for this is that $\mathbb{Z}_{12}$ has too few automorphisms (remember that $\text{Aut}(\mathbb{Z}_{12}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$) and therefore there can be no action of $\mathbb{Z}_3$ on $\mathbb{Z}_{12}$ except for the trivial one. We thus see that group structures such as semidirect products, as is the case for the dihedral group $D_{24}$ used in neo-Riemannian theory, cannot exist for sets containing three different types of chords. The group extension structure determines a specific group operation based both on the action by automorphism of $H$ on $Z$, and on a 2-cocycle $H \times H \rightarrow Z$. As discussed in [12], this group operation directly determines whether left or right actions are contextual. Here, the case $G = \mathbb{Z}_{12} \times \mathbb{Z}_3$ corresponds to the trivial direct product, i.e the trivial 2-cocycle, hence both actions are non-contextual. There also exists a non-trivial 2-cocycle which leads to $G = \mathbb{Z}_{36}$. As shown in [12], non-trivial 2-cocycles give rise to contextual group actions on chords, the $VL$ and $VL'$ operations being such examples. However, since the group is abelian (i.e the 2-cocycle is symmetric) the left and right actions of these transformations coincide, and are thus both contextuals.

The case of semidirect products is particularly interesting since left ac-
tions are non-contextual whereas right actions can be. An important part of the literature about neo-Riemannian theory has focused on the duality between these left and right actions [17, 18, 19], and in particular their commuting property. However the present case does not allow for such richness. One could circumvent this problem by considering group extensions of the form:

\[ 1 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_{12} \to 1 \]

but in this case, the transpositions operators would not be well-defined anymore, since \( \mathbb{Z}_{12} \) would no longer be a normal subgroup of \( G \) in the general case.

Moreover, the consideration of group extensions of the form \( 1 \to \mathbb{Z}_{12} \to G \to \mathbb{Z}_3 \to 1 \) limits the contextual and/or voice-leading transformations that can be applied to this set of chords, even when the 2-cocycle is non-trivial. Consider for example the following transformations:

\[
I_{M \leftrightarrow \alpha} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ (2x - 3) - y \\ (2x - 3) - z \end{pmatrix}, \text{i.e} \left( \begin{array}{c} n_M \\ n_\alpha \end{array} \right) \mapsto \left( \begin{array}{c} n_\alpha \\ n_M \end{array} \right)
\]

\[
I_{M \leftrightarrow \beta} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (2z + 4) - y \\ (2z + 4) - x \\ z \end{pmatrix}, \text{i.e} \left( \begin{array}{c} n_M \\ n_\beta \end{array} \right) \mapsto \left( \begin{array}{c} n + 2\beta \\ n - 2M \end{array} \right)
\]

\[
I_{\alpha \leftrightarrow \beta} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (2y - 1) - z \\ y \\ (2y - 1) - x \end{pmatrix}, \text{i.e} \left( \begin{array}{c} n_\alpha \\ n_\beta \end{array} \right) \mapsto \left( \begin{array}{c} n - 2\beta \\ n + 2\alpha \end{array} \right)
\]

These inversion-like transformations are represented in Figure 3. Each one of them is an involution, just as the \( L, R \) and \( P \) operations are. However, they can only be applied to the indicated pair of set-classes. In other terms, these operations are partial and cannot form a group of transformations since the closure condition would not be satisfied. We propose a way to unify these transformations in the next section.
Figure 3: Graphical representation of three contextual operations acting on the pairs \((M, \alpha)\) (a), \((M, \beta)\) (b) and \((\alpha, \beta)\) (c) of pitch-class sets. No operation can be applied to all set classes altogether.
2.2 Transformations between chords of different cardinalities

The work of Childs [13] has shown that neo-Riemannian constructions can be applied to seventh chords. In view of [12] and since seventh chords have a well-defined root, it is indeed possible to envision a group extension acting on seventh chords. However Childs’ work does not include triads.

Since \( Aut(\mathbb{Z}_{12}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), one could consider that there is enough room for transformations of a set of four set-classes and their transpositions. However Hook [14] (see footnote 9 p.5) has argued against putting all set-classes in a single set on which transformations could be applied because (we paraphrase):

1. A transformation may not have the same meaning as its inverse, especially for transformations between different set-classes.

2. Transformations should be well-defined on the whole set of chords (this is the totality requirement for groups).

3. Some transformations may not have inverses at all.

4. Different sets of chords may not have the same cardinality and defining transformations between them would be problematic if not impossible. (Note that we differentiate between the cardinality of a set of chords, i.e the number of chords of the same set-class that constitutes the set, and the cardinality of a set-class or chord, i.e the number of pitch-classes that constitutes it.

We have seen in the previous part examples of transformations which do not apply on the whole set of \( M, \alpha \) and \( \beta \) chords, i.e closure is lost. On another level, Cohn’s model of triadic progression involves the transformation between major or minor chords, in each case a set of 12 elements, to augmented triads, a set of 4 elements. In that case these transformations are surjective and therefore have no formal inverse, although in Cohn’s model one can freely choose the major/minor image of a given augmented triad.

On the other hand, if we push the reasoning behind Hook’s objections one step further, we could wonder why major and minor chords are considered as a single set. Consider for example the usual neo-Riemannian \( P \) operation: if one views this operation as an inversion, it is then an involution, i.e \( P^2 = \)
meaning that this operation is formally equal to its inverse. However if one considers this operation as a voice-leading transformation, it then corresponds to:

1. A pitch down in the major-to-minor way.
2. A pitch up in the minor-to-major way.

In this view, the $P$ operation cannot be said to be equal to its inverse. In order to restore coherence in this point of view, one has to consider two different transformations, one from the set of major triads to the set of minor ones, the other one from the set of minor triads to the set of major ones. Notice however we can only do so at the expense of closure: the transformations thus defined only acts on a given set of chords, or in other terms these are partial transformations.

The $P$ operation can thus be viewed as a ”package” of two partial transformations. Notice that this is not a unique case: the $L$ and $R$ operations can also be viewed as ”packaged operators”. Moreover, the usual transposition operators in neo-Riemannian theories actually represents two partial transposition operators which apply respectively to the major and minor chords. In Hook’s notation of UTT (Uniform Triadic Transformations), these correspond to the two operations $<+,1,0>$ and $<+,0,1>$. Since they act very similarly, it is conceivable to package them into a single transposition operator $T_1$. The last section of this paper will provide a link between the construction we introduce next and packaged operators.

3 A categorical construction for musical transformations

3.1 Construction of the category of transformations by extension

In this section, we introduce a construction of musical transformations based on categories rather than groups. Notice that groups are themselves a particular case of categories as they can be viewed as single-object categories, where the morphisms are group elements under the usual composition. The construction we will use is based on a generalization of the construction of group extensions that was introduced in [12].
Recall first that the construction of generalized neo-Riemannian groups of transformations as extensions

\[ 1 \to Z \to G \to H \to 1 \]

involves a base-group \( Z \) and a shape group \( H \). Notice that the \( T/I \) group or the \( PLR \) group are both isomorphic to the dihedral group \( D_{24} \), which is an extension (and more precisely a semidirect product) of \( \mathbb{Z}_{12} \) by \( \mathbb{Z}_2 \). Inspired by this case, we can consider \( Z \) as a group of "generalized transpositions", whereas \( H \) can be considered as a group of "formal inversions" between different pitch-class sets.

Instead of considering \( H \) as a group, we now replace it with a groupoid \( H \). Recall that a groupoid is a category in which every morphism is invertible. Groupoids can be viewed as generalizations of groups in which closure (or totality) has been left out. Indeed, morphisms \( g_{XY} \) of the groupoid can be seen as partial transformations between objects \( X \) and \( Y \). In the rest of the paper, the maps \( s \) and \( t \) will refer to the source and target maps, i.e \( s(g_{XY}) = X \) and \( t(g_{XY}) = Y \). It has been suggested that groupoids are in some cases superior to groups in describing symmetries of objects. For a gentle introduction to groupoids, the reader is invited to refer to [20] and [21].

In our case, the objects of the groupoid \( H \) are the different pitch-class sets and the morphisms are the different formal transformations between these set-classes. Actual transformations of chords usually involve transpositions of the root as is the case for the \( L \) or \( R \) operation, which we will introduce below. By definition, these transformations are partial and the composition of two morphisms \( h_2 \cdot h_1 \) is only possible if the codomain of \( h_1 \) matches the domain of \( h_2 \). Figure [4] give two examples of such groupoids, corresponding respectively to transformations between major (M) and minor (m) chords, and M, \( \alpha \), and \( \beta \) chords.

We now introduce the definition of a category extension, following the work of Hoff [22, 23, 24]:

**Definition (Hoff)** An extension of the category \( Z \) by the category \( H \) is a category \( G \) such that there exists a sequence

\[ 1 \to Z \to G \to \mathcal{H} \to 1 \]

in which:
Figure 4: Two examples for categories $\mathcal{H}$ of formal transformations between major and minor chords \((a)\) and major, $\alpha$, and $\beta$ chords \((b)\)

1. $\mathcal{Z}, \mathcal{G}$ and $\mathcal{H}$ have the same number of objects.
2. $I$ is a functor injective on morphisms, while $P$ is a functor surjective on morphisms
3. $\forall g_1, g_2 \in \mathcal{G}, P(g_1) = P(g_2) \iff \exists ! z \in \mathcal{Z}, g_2 = I(z) \cdot g_1$

This definition closely follows the group extension one, and the third condition is actually similar to the $Im(I) = Ker(P)$ condition. Hoff has shown that if $\mathcal{H}$ is a category extension as defined above, then $\mathcal{Z}$ is a disjoint union of groups indexed by the objects of $\mathcal{H}$. The category $\mathcal{Z}$ thus plays the role of the transposition operators. We will assume in our case that each pitch-class set can be transposed in the same way, i.e there is a simply transitive group action of $\mathbb{Z}_n$ on each set of chords of the same type. The category $\mathcal{Z}$ is therefore built as such

1. The objects of $\mathcal{Z}$ are the same as in $\mathcal{H}$ and represent the pitch-class sets.
2. For each object $X \in \mathcal{Z}$, we have $Hom(X, X) \cong \mathbb{Z}_n$. We denote a morphism of $X$ as $z^X_\alpha$. These morphisms represent transpositions of the individual pitch-class sets.
3. For any two different objects $X$ and $Y$ of $\mathcal{Z}$, $\text{Hom}(X,Y) = \emptyset$.

With this knowledge, we see that the third condition in the definition of a category extension has a very concrete meaning from a musical point of view. Consider for example the $L$ and $R$ operations acting on the C major triad. It is clear that the images of this triad under $L$ and $R$ differ by a unique transposition. We thus axiomatize this fact by considering a construction, as a category extension, in which any two switching transformation, or partial inversion transformation, differ only by a unique transposition in the target pitch-class set.

The role of the functor $I$ is to introduce the transposition operators in the category $\mathcal{H}$, whereas the functor $P$ classifies in $\mathcal{H}$ the morphisms of $\mathcal{G}$ as transpositions or partial inversions. To sum up, the functors $I$ and $P$ are defined as:

1. $I$ and $P$ map objects in the natural way.

2. $I$ maps morphisms $z^p_X$ of $\mathcal{Z}$ to equivalent transposition morphisms $z^p_X$ in $\mathcal{G}$. By an abuse of terminology, $z^p_X$ will designate from now on a transposition morphism of $\mathcal{Z}$ or of $\mathcal{G}$ indifferently.

3. $P$ maps morphisms $z^p_X$ in $\mathcal{G}$ to $id_X$ in $\mathcal{H}$, and morphisms $m_{XY}$ in $\mathcal{G}$ to morphisms $h_{XY}$ in $\mathcal{H}$.

As shown by Hoff, the extension construction of $\mathcal{G}$ brings more structure with regards to morphism composition, which will allow us to define actions of $\mathcal{G}$ on sets of objects. Indeed, it can be proved that when the groups $\text{Hom}(X,X)$ of $\mathcal{Z}$ are abelian, all category extensions $1 \to \mathcal{Z} \to \mathcal{G} \to \mathcal{H} \to 1$ can be constructed as such:

1. $\mathcal{G}$ has the same objects as $\mathcal{H}$ or $\mathcal{Z}$.

2. Morphisms of $\mathcal{G}$ are of the form $(z,h)$, i.e. they are indexed by the morphisms from $\mathcal{H}$ or $\mathcal{Z}$, with $z$ being a transposition of the codomain of $h$.

3. Composition of two morphisms $g_1 = (z_1,h_1)$ and $g_2 = (z_2,h_2)$, whenever they are compatible (i.e. $s(g_2) = t(g_1)$) is given by the law:

$$(z_2,h_2) \cdot (z_1,h_1) = (z_2 \cdot \phi_{h_2}(z_1) \cdot \zeta(h_2,h_1), h_2 \cdot h_1)$$

where $\phi$ is an action of the category $\mathcal{H}$ on $\mathcal{Z}$, and $\zeta$ is a 2-cocycle.
The cohomology theory built by Hoff allows to classify all category extensions based on the second cohomology group \( H^2(\mathcal{H}, \mathcal{Z}) \), with the corresponding 1- and 2-cocycles. We now give the definitions for the terms involved in the composition law of morphism.

An action \( \phi \) of \( \mathcal{H} \) on \( \mathcal{Z} \) is a functor \( \phi : \mathcal{H} \to \text{Grp} \) where the images of the objects of \( \mathcal{H} \) are the groups associated to the corresponding objects in \( \mathcal{Z} \), i.e., for any object \( X \in \mathcal{H} \), \( \phi(X) \cong \text{Hom}_\mathcal{Z}(X, X) \). In other terms, this functor defines homomorphisms between the groups of \( \mathcal{Z} \) which are compatible with composition of morphisms in \( \mathcal{H} \).

Some examples of actions in the case of major and minor chords, or M, \( \alpha \) and \( \beta \) chords are given in Figure 5. Whereas figure 5(a) is reminiscent of the typical transformations which are used in neo-Riemannian theory, Figure 5(b) and Figure 5(c) show new structures.

A 2-cocycle is a function \( \zeta : \mathcal{H} \times \mathcal{H} \to \mathcal{Z} \) between two morphisms of \( \mathcal{H} \) which outputs a morphism from the appropriate object of \( \mathcal{Z} \) such that:

\[
\phi_{h_3}(\zeta(h_2, h_1)) \cdot \zeta(h_3, h_2 \cdot h_1) = \zeta(h_3, h_2) \cdot \zeta(h_3 \cdot h_2, h_1)
\]

whenever \( h_1 \), \( h_2 \) and \( h_3 \) are compatible.

We see that the terminology used for category extensions is very close to the one used for group extensions. In a similar approach, we will now define

\[
\phi_{h_3}(\zeta(h_2, h_1)) \cdot \zeta(h_3, h_2 \cdot h_1) = \zeta(h_3, h_2) \cdot \zeta(h_3 \cdot h_2, h_1)
\]
the actions of $\mathcal{G}$.

### 3.2 Construction of partial actions

A left (right) action of a group $G$ (considered as a single-object category) on a set of chords can be described as a covariant (contravariant) functor $F : G \to \text{Set}$. In particular, it is known that simply transitive left (right) group actions are equivalent to representable functors $F : G \to \text{Set}$, i.e. functors which are naturally isomorphic to $\text{Hom}(\bullet, -)$ (or $\text{Hom}(-, \bullet)$) where $\bullet$ represent the single object of $G$. Recall that in such a case, set elements can be put in bijection with group elements after a particular element has been identified to the identity element in the group. As shown in [12], this allows the determination of group actions, and this also determines a Generalized Interval System (GIS), since Kolman [?] has shown that GIS are equivalent to simply transitive group actions.

By analogy, we can build actions of $\mathcal{G}$ on the different sets of chords by using a representable functor $F : \mathcal{G} \to \text{Set}$ and the composition law of morphisms in $\mathcal{G}$. Notice that such a functor has multiple images in $\text{Set}$ (one for each object of $\mathcal{G}$), instead of just one in the case of a group. Therefore, we are actually building partial actions between sets of chords. We now show how to recover the partial actions described in Section 2.

The category $\mathcal{Z}$ we use has three objects $M, \alpha$ and $\beta$, with $\text{Hom}(M, M) = \text{Hom}(\alpha, \alpha) = \text{Hom}(\beta, \beta) = \mathbb{Z}_{12}$. The category $\mathcal{H}$ has the same objects with the formal inversions $h_{M\alpha}, h_{M\beta}$ and $h_{\alpha\beta}$. We build the category extension $\mathcal{G}$ with only an action $\phi$ of $\mathcal{H}$ on $\mathcal{Z}$ and no 2-cocycle. This action is depicted in Figure 6.

The partial transformations between pitch-class sets $\alpha$ and $\beta$, and $M$ and $\beta$ defined in Section 2 are contextual, and therefore we need a contravariant representable functor. We consider the functor $\text{Hom}(-, M) : \mathcal{G} \to \text{Set}$. This functor sends

1. the object $M$ to the set of morphisms $\{(z^n_M, id_M)\}$ which are identified bijectively with the chords $n_M$.
2. the object $\alpha$ to the set of morphisms $\{(z^n_M, h_{\alpha M})\}$ which are identified bijectively with the chords $n_\alpha$.
3. the object $\beta$ to the set of morphisms $\{(z^n_M, h_{\beta M})\}$ which are identified bijectively with the chords $n_\beta$. 

14
Figure 6: The action of $\mathcal{H}$ on $\mathcal{Z}$ used for building the partial transformations between pitch-class sets $M$, $\alpha$ and $\beta$. We show here the images of the functor $\phi: \mathcal{H} \to \text{Grp}$. The homomorphisms between groups are represented by their multiplicative action.

To compute the action of a morphism $g \in \mathcal{G}$ on a chord, we thus identify the morphism corresponding to the chord, compose with $m$ on the right and identify the chord of the resulting morphism. For example, the action of $(id_M, h_{\alpha M})$ on a chord $n_m$ results in

$$(z^n_M, id_M).(id_M, h_{\alpha M}) = (z^n_M \cdot \phi_{id_M}(id_M), h_{\alpha M}) = (z^n_M, h_{\alpha M})$$

which corresponds to the chord $n_{\alpha}$. Similarly, the action of $(id_\alpha, h_{M\alpha})$ on a chord $n_\alpha$ results in

$$(z^n_M, h_{\alpha M}).(id_\alpha, h_{M\alpha}) = (z^n_M \cdot \phi_{id_M}(id_\alpha), id_M) = (z^n_M, id_M)$$

which corresponds to the chord $n_M$. We thus recover the partial action between pitch-class sets $M$ and $\alpha$ described previously. If we consider similarly the action of $(z^2_M, h_{\beta M})$ on a chord $n_M$, we obtain

$$(z^n_M, id_M).(z^2_M, h_{\beta M}) = (z^n_M \cdot \phi_{id_M}(z^2_M), h_{\beta M}) = (z^{n+2}_M, h_{\beta M})$$

which corresponds to the chord $(n + 2)_\beta$. If we consider now the action of $(z^2_\beta, h_{M\beta})$ on a chord $n_\beta$, we obtain

$$(z^n_M, h_{\beta M}).(z^2_\beta, h_{M\beta}) = (z^n_M \cdot \phi_{h_{\beta M}}(z^2_\beta), id_M) = (z^{n-2}_M, id_M)$$

since $\phi_{h_{\beta M}}(z^2_\beta) = z^{10}_M$, which corresponds to the chord $(n - 2)_\beta$ and we thus recover the partial contextual action between pitch-class sets $M$ and $\beta$. The
partial contextual action between pitch-class sets \( \alpha \) and \( \beta \) can be computed in a similar way.

We see here that considering groupoids and their extensions allow for much richer structure than the group extension structure does. In particular, the interplay of group homomorphisms between set-classes, as shown in Figure 6, is a way to circumvent the limitations of group extensions when considering the automorphisms of the only group \( \mathbb{Z}_{12} \).

4 Forming groups of transformations from category extensions

Starting from a groupoid \( \mathcal{G} \) of musical transformations defined as a full extension, it is possible to revert back to a group-theoretical description by "packaging" partial operations.

**Definition** A packaged operator is a set of morphisms \( O = \{ \phi_1, \ldots, \phi_n \} \) from \( \mathcal{G} \) (\( n \) being the number of objects of \( \mathcal{G} \)) such that for all objects \( i \), \( i \) appear only once as the domain of a morphism from \( O \), and only once as the codomain of a morphism from \( O \).

Packaged operators can be composed according to:

**Definition** The composition \( O_1 \cdot O_2 \) of two packaged operators \( O_1 = \{ \phi_1, \ldots, \phi_n \} \), \( O_2 = \{ \phi'_1, \ldots, \phi'_n \} \) is the set of morphisms \( \{ \phi''_1, \ldots, \phi''_n \} \) obtained by composing all morphisms \( \phi_x \cdot \phi'_y \) from \( O_1 \) and \( O_2 \) whenever their domain and codomain are compatible. It can be verified that \( O_1 \cdot O_2 \) is also a packaged operator.

We then have:

**Proposition** Packaged operators form a group under composition.

**Proof** The identity packaged operator is the set of identity morphisms of each object. Closure is given by definition. Associativity is inherited from the category structure. Finally, since \( \mathcal{H} \) is a groupoid it is always possible to find inverses for each morphism of a packaged operator, thus giving the inverse packaged operator.
For example, one can define a packaged transposition operator of the form \( T_X = \{ z_X, id_Y \} \) with \( z_X \in G \), for all objects \( Y \neq X \). If we have inversion operators \( g_{XY} \) in \( G \), we can also form a packaged inversion operator of the form \( I_{XY} = \{ g_{XY}, g_{YX}, id_Z \} \) for all objects \( Z \neq X, Z \neq Y \), with \( g_{YX} \) being the inverse of \( g_{XY} \).

The next proposition makes the link between such the group generated by such packaged operators and the wreath products that appeared in the work of Hook and Peck. We assume here that the groups associated to each object of \( Z \) are isomorphic to \( \mathbb{Z}_n \). Consider on one hand the set \( N \) of all packaged transposition operators \( \{ T_X \} \), for all objects \( X \in G \). It can readily be seen that \( N \) is a group under the composition law defined above, and that it is isomorphic to a direct product of \( m \) copies of \( \mathbb{Z}_n \), where \( m \) is the number of objects of \( G \). Consider on the other hand the set of all packaged inversion operators \( K = \{ I_{XY} \} \), for all pairs of objects \( X \) and \( Y \) in \( G \), along with the identity element. It can also be seen that \( K \) is isomorphic to the symmetric group \( S_m \), since \( I_{2XY} = id \), \( I_{XY}I_{WZ} = I_{WZ}I_{XY} \) and \( I_{XY} \cdot I_{YZ} \cdot I_{XY} = I_{YZ} \cdot I_{XY} \cdot I_{YZ} \). We then have the following result:

**Proposition** The group \( G \) generated by the set \( \{ N, K \} \) of packaged operators is isomorphic to the wreath product \( \mathbb{Z}_n \wr S_m \).

**Proof** We first show that \( N \) is normal in \( G \). Let \( T_X^p \) be an element of \( N \) for some object \( X \). If \( g \in G \) is an element of the form \( T_Y^q \) then it is obvious that \( T_Y^q \cdot T_X^p \cdot T_Y^{-q} \in N \). If \( g \in G \) is of the form \( I_{YZ} \) for some pair \((Y, Z)\) of objects of \( G \) with \( Y \neq X, Z \neq X \) then we also have immediately \( I_{YZ} \cdot T_X^p \cdot I_{YZ}^{-1} \in N \). In the case \( Y = X \), we have

\[
I_{XZ} \cdot T_X^p \cdot I_{XZ}^{-1} = \{ g_{XZ} \cdot z_X^p \cdot g_{ZX}, id, \ldots \}
\]

which also belongs to \( N \) since \( g_{XZ} \cdot z_X^p \cdot g_{ZX} = z_X^q \) for some \( q \). If \( g \in G \) is a composite element of packaged transpositions and inversions, the relation \( g \cdot n \cdot g^{-1} \in N \) holds with the previous results.

We also have \( NK = G \) by definition and \( N \cap K = \{ id_1, ..., id_m \} \). Since \( G \) is not abelian (consider for example \( T_X \cdot I_{XY} \) and \( I_{XY} \cdot T_X \)) this shows that \( G \) is a semidirect product of \( N \) by \( K \).
It is not necessary to include all transpositions operators for each object $X$, as the composition of one $T_X$ with the packaged inversions will lead to the others. One can check for example that the packaged partial operations defined on $M$, $\alpha$ and $\beta$ chords in section 2.1, along with the packaged transposition operator $\{n_M \to n + 1_M, n_\alpha \to n_\alpha, n_\beta \to n_\beta\}$, generate a group of order 10368 which is isomorphic to $\mathbb{Z}_{12} \wr S_3$.

5 Conclusions

We have introduced in this paper a categorical construction for musical transformations based on groupoids which extend the precedent construction based on group extensions. It overcome its inherent limitations, in particular the limited choice of automorphims in the pc-set group. More importantly, this construction allows to define compatible set of partial transformations between pair of set-classes. We also saw how groups of transformations can be recovered from category extensions, based on packaged operators and their composition.

While this paper is more mathematical than musical, we hope it will provide foundations for building appropriate groups of transformations in musically-relevant domains. This could be applied for example to cardinality changes between chords (ex. major/minor to seventh chords), a very important problem in music theory as of now.

In this paper, we only considered the case of groupoids, and in particular the groupoid $\mathcal{H}$ which assume that partial and reversible transformations between set-classes always exist. As seen in the work of Cohn regarding major/minor and augmented triads, there are cases in music theory where partial transformations may not be reversible at all. It would therefore be interesting to consider category extensions in which $\mathcal{H}$ is a more general category. As well, it could also be interesting to investigate non-abelian category extensions, i.e in which the groups of $\mathcal{Z}$ are non-abelian.

References

[1] D. Lewin, *Generalized Musical Intervals and Transformations*, Yale University Press, New Haven, CT, 1987
[2] R. Cohn, *An Introduction to Neo-Riemannian Theory: A Survey and Historical Perspective*, Journal of Music Theory 42/2 (1998), pp. 167-180

[3] R. Cohn, *Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions*, Music Analysis 15/1 (1996), pp. 9-40

[4] R. Cohn, *Neo-Riemannian Operations, Parsimonious Trichords, and Their Tonnetz Representations*, Journal of Music Theory 41, pp. 166

[5] G. Capuzzo, *Neo-Riemannian Theory and the Analysis of Pop-Rock Music*, Music Theory Spectrum 26/2 (2004), pp. 177-200

[6] J. Douthett, *Flip-flop Circles and Their Groups*, in *Music Theory and Mathematics: Chords, Collections, and Transformations*, pp. 23-49, Eastman Studies in Music, J. Douthett, M. Hyde, and C. Smith, eds., University of Rochester Press, 2008.

[7] J.N. Straus, *Contextual-Inversion Spaces*, Journal of Music Theory 55/1 (2011), pp. 43-89

[8] J. Hook, *Uniform Triadic Transformation*, Journal of Music Theory 46/1-2 (2002), pp. 57-126

[9] J. Hook, *Signature Transformations*, in *Music Theory and Mathematics: Chords, Collections, and Transformations*, pp. 137-161, Eastman Studies in Music, J. Douthett, M. Hyde, and C. Smith, eds., University of Rochester Press, 2008.

[10] R. Peck, *Wreath Products in Transformational Music Theory*, Perspectives of New Music 47/1 (2009), pp. 193-211

[11] R. Peck, *Imaginary Transformations*, Journal of Mathematics and Music 4/3 (2010), pp. 157-171

[12] A. Popoff, *Building Generalized Neo-Riemannian Groups of Musical Transformations as Extensions*, submitted

[13] A. Childs, *Moving beyond Neo-Riemannian Triads: Exploring a Transformational Model for Seventh Chords*, Journal of Music Theory 42/2 (1998), pp. 181-193
[14] J. Hook, *Cross-Type Transformations and the Path Consistency Condition*, Music Theory Spectrum 29/1 (2007), pp. 1-40

[15] G. Mazzola, *The Topos of Music*, Birkhuser, Basel (2002), pp. 257

[16] J. Kochavi, *Some Structural Features of Contextually-Defined Inversion Operators*, Journal of Music Theory 42 (1998), pp. 307-320

[17] T.M. Fiore, R. Satyendra, *Generalized Contextual Groups*, Music Theory Online 11(3) (2005).

[18] A.S. Crans, T.M. Fiore, R. Satyendra, *Musical Action of Dihedral Groups*, American Monthly Mathematical, June/July 2009.

[19] T.M. Fiore, T. Noll, *Commuting Groups and the Topos of Triads*, Mathematics and Computation in Music, Third International Conference, MCM 2011, C. Agon, M. Andreatta, G. Assayag, E. Amiot, J. Bresson, J. Mandereau. eds., Springer Lecture Notes in Artificial Intelligence, 6726 (2011), pp 69-83.

[20] A. Weinstein, *Groupoids: Unifying Internal and External Symmetry*, Notices of the AMS 43/7 (1996), pp. 744-752.

[21] A. Guay, B. Hepburn, *Symmetry and Its Formalism: Mathematical Aspects*, Philosophy of Science 76/2 (2009), pp. 160-178.

[22] G. Hoff, *On the Cohomology of Categories*, Rend. Mat. 7 (1974), pp. 169-192.

[23] G. Hoff, *Extensions Multiples de Categories*, Rend. Mat. 26 (2006), pp. 351-365.

[24] G. Hoff, *Cohomologies et Extensions de Categories*, Math. Scand. 74 (1994), pp. 191-207.