INSERTION OF CONTINUOUS SET-VALUED MAPPINGS

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Dedicated to the memory of Mitrofan Choban

Abstract. An interesting result about the existence of “intermediate” set-valued mappings between pairs of such mappings was obtained by Nepomnyashchii. His construction was for a paracompact domain, and he remarked that his result is similar to Dowker’s insertion theorem and may represent a generalisation of this theorem. In the present paper, we characterise the $\tau$-paracompact normal spaces by this set-valued “insertion” property and for $\tau = \omega$, i.e. for countably paracompact normal spaces, we show that it is indeed equivalent to the mentioned Dowker’s theorem. Moreover, we obtain a similar result for $\tau$-collectionwise normal spaces and show that for normal spaces, i.e. for $\omega$-collectionwise normal spaces, our result is equivalent to the Katětov-Tong insertion theorem. Several related results are obtained as well.

1. Introduction

All spaces in this paper are infinite Hausdorff topological spaces. For a set $Y$, we use $2^Y$ to denote the family of all nonempty subsets of $Y$. For a space $Y$, let

$$\mathcal{F}(Y) = \{ S \in 2^Y : S \text{ is closed} \}.$$ 

The following subfamilies of $\mathcal{F}(Y)$ will play an important role in this paper:

$$\mathcal{C}(Y) = \{ S \in \mathcal{F}(Y) : S \text{ is compact} \} \text{ and } \mathcal{C}'(Y) = \mathcal{C}(Y) \cup \{ Y \}.$$ 

Moreover, we will use the subscript “c” to denote the connected members of any one of the above families; namely, $\mathcal{F}_c(Y)$ for the connected members of $\mathcal{F}(Y)$; $\mathcal{C}_c(Y)$ for those of $\mathcal{C}(Y)$; and $\mathcal{C}'_c(Y) = \mathcal{C}_c(Y) \cup \{ Y \}$ provided that $Y$ is itself connected. Let us remark that in the setting of a topological vector space $Y$, some authors have used the subscript “c” in a more restricted sense, namely for the convex members of any of the above families. In this paper, convex sets will not be used explicitly, hence it will not cause any misunderstandings.

For spaces $X$ and $Y$, a set-valued mapping $\Phi : X \to 2^Y$ is lower semi-continuous, or l.s.c., if the set $\Phi^{-1}(U) = \{ x \in X : \Phi(x) \cap U \neq \emptyset \}$ is open in $X$. 

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X for every open \( U \subseteq Y \); and \( \Phi \) is upper semi-continuous, or u.s.c., if \( \Phi^{-1}(F) \) is closed in \( X \) for every closed \( F \subseteq Y \). Equivalently, \( \Phi \) is u.s.c. if

\[
\Phi^\#(U) = X \setminus \Phi^{-1}(Y \setminus U) = \{ x \in X : \Phi(x) \subseteq U \}
\]

is open in \( X \), for every open \( U \subseteq Y \). A set-valued mapping is continuous if it is both l.s.c. and u.s.c. A mapping \( \theta : X \to 2^Y \) is a selection (or, a set-valued selection) for \( \Phi : X \to 2^Y \) if \( \theta(x) \subseteq \Phi(x) \) for all \( x \in X \). To designate that \( \theta \) is a selection for \( \Phi \), we will often simply write \( \theta \leq \Phi \). Moreover, to bring the analogy closer to usual functions, we will also write \( \theta < \Phi \) provided that \( \theta \leq \Phi \) and \( \Phi(x) \setminus \theta(x) \neq \emptyset \), for all \( x \in X \). In [40], Nepomnyashchii showed that if \( X \) is a paracompact space, \( (Y, \rho) \) is a complete metric space, \( \Phi : X \to F_c(Y) \) is l.s.c. such that \( \{ \Phi(x) : x \in X \} \) is uniformly equi-\( LC^0 \) and \( \theta : X \to C(Y) \) is u.s.c. with \( \theta \leq \Phi \), then there exists a continuous mapping \( \varphi : X \to C_c(Y) \) such that \( \theta \leq \varphi \leq \Phi \). For the definition of “uniformly equi-\( LC^0 \), see the next section. In this paper, we will show that such intermediate set-valued mappings can characterise paracompactness and collectionwise normality. To state our results, we briefly recall some terminology. For an infinite cardinal number \( \tau \), a space \( X \) is \( \tau \)-paracompact [34, 35] if each open cover of \( X \) of cardinality \( \leq \tau \), has a locally finite open refinement. If \( \omega \) is the first infinite ordinal, then \( \omega \)-paracompactness is nothing else but countable paracompactness. By definition, a space \( X \) is paracompact if it is \( \tau \)-paracompact for any cardinal number \( \tau \). Paracompactness of \( X \) implies normality of \( X \). However, there are many examples of countably paracompact spaces which are not normal. Also, there are simple examples of \( \tau \)-paracompact spaces which are not \( \tau^+ \)-paracompact [35], where the cardinal \( \tau^+ \) is the immediate successor of \( \tau \). Finally, for a set \( \mathcal{A} \), we will use \( J(\mathcal{A}) \) to denote the metrizable hedgehog of spininess \( |\mathcal{A}| \) obtained from \( \mathcal{A} \) (see \([8, \text{Example 4.1.5}])\). Recall that the point set of \( J(\mathcal{A}) \) is \( \{ 0 \} \cup (0,1] \times \mathcal{A} \), while the metric \( d \) on \( J(\mathcal{A}) \) is defined by

\[
d(0, \langle s, \alpha \rangle) = s \quad \text{and} \quad d(\langle s, \alpha \rangle, \langle t, \beta \rangle) = \begin{cases} |s - t|, & \text{if } \alpha = \beta, \\ s + t, & \text{if } \alpha \neq \beta. \end{cases}
\]

Throughout this paper, \( J(\mathcal{A}) \) will be always endowed with this metric. The following theorem will be proved.

**Theorem 1.1.** For an infinite cardinal number \( \tau \) and a space \( X \), the following conditions are equivalent:

(a) \( X \) is normal and \( \tau \)-paracompact.

(b) If \( \Phi : X \to F_c(J(\tau)) \) is l.s.c., \( \theta : X \to C(J(\tau)) \) is u.s.c. and \( \theta < \Phi \), then there exists a continuous mapping \( \varphi : X \to C_c(J(\tau)) \) such that \( \theta < \varphi \leq \Phi \).

Theorem 1.1 is not only similar, but also represents a natural generalisation of Dowker’s insertion theorem [6], see Lemma 3.4. Moreover, it is still valid
in the setting of not necessarily connected-valued mappings, in which case the intermediate mapping \( \varphi \) in (b) is only supposed to be u.s.c., see Proposition 3.5.

If \( X \) is normal and \( \tau \)-paracompact, \( \Phi : X \to \mathcal{F}_c(\mathcal{J}(\tau)) \) is l.s.c., \( \theta : X \to \mathcal{C}(\mathcal{J}(\tau)) \) is u.s.c. and \( \theta \leq \Phi \), then there also exists a continuous mapping \( \varphi : X \to \mathcal{C}_c(\mathcal{J}(\tau)) \) such that \( \theta \leq \varphi \leq \Phi \), see Theorem 2.2 and Proposition 3.1. In this regard, the author would like to know, but doesn’t, whether this relaxed insertion property may still characterise the \( \tau \)-paracompact spaces. The answer is “Yes” in the special case of \( \tau = \omega \) (Corollary 5.3). Moreover, this property implies both countable paracompactness and \( \tau \)-collectionwise normality, see Theorem 1.2 below.

For an infinite cardinal number \( \tau \), a space \( X \) is called \( \tau \)-collectionwise normal if every discrete family \( \mathcal{F} \) of closed subsets of \( X \), with \( |\mathcal{F}| \leq \tau \), admits a discrete family \( \{U_F : F \in \mathcal{F}\} \) of open subsets of \( X \) such that \( F \subseteq U_F \) for each \( F \in \mathcal{F} \). If, in this definition, “discrete” is changed to “locally finite”, we get another important class of spaces. Namely, a space \( X \) is called \( \tau \)-expandable [22] if every locally finite family \( \mathcal{F} \) of closed subsets of \( X \), with \( |\mathcal{F}| \leq \tau \), admits a locally finite family \( \{U_F : F \in \mathcal{F}\} \) of open subsets of \( X \) such that \( F \subseteq U_F \) for each \( F \in \mathcal{F} \). A space \( X \) is collectionwise normal if it \( \tau \)-collectionwise normal for every \( \tau \), and \( X \) is expandable if it is \( \tau \)-expandable for every \( \tau \). A space \( X \) is normal if and only if it is \( \omega \)-collectionwise normal. However, for every \( \tau \) there exists a \( \tau \)-collectionwise normal space which is not \( \tau^+ \)-collectionwise normal [41]. Similarly, a space \( X \) is \( \omega \)-expandable precisely when it is countably paracompact [22, Theorem 2.5]. Finally, let us explicitly remark that a normal space \( X \) is \( \tau \)-expandable if and only if it is countably paracompact and \( \tau \)-collectionwise normal [7, 20]. Our next result deals with the following characterisation of \( \tau \)-expandable spaces.

**Theorem 1.2.** For an infinite cardinal number \( \tau \) and a space \( X \), the following conditions are equivalent:

(a) \( X \) is normal and \( \tau \)-expandable.

(b) If \( \Phi : X \to \mathcal{C}_c(\mathcal{J}(\tau)) \) is l.s.c., \( \theta : X \to \mathcal{C}(\mathcal{J}(\tau)) \) is u.s.c. and \( \theta \leq \Phi \), then there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c(\mathcal{J}(\tau)) \) such that \( \theta \leq \varphi \leq \Phi \).

For a space \( Y \) and \( k \in \mathbb{N} \), let \( \mathcal{C}_k(Y) = \{ S \in 2^Y : |S| \leq k \} \subseteq \mathcal{C}(Y) \). Regarding the role of countable paracompactness as a component of \( \tau \)-expandable spaces, we will also prove the following theorem which is complementary to Theorem 1.2.

**Theorem 1.3.** For an infinite cardinal number \( \tau \) and a space \( X \), the following conditions are equivalent:

(a) \( X \) is \( \tau \)-collectionwise normal.

(b) If \( \Phi : X \to \mathcal{C}_c^k(\mathcal{J}(\tau)) \) is l.s.c., \( k \in \mathbb{N} \) and \( \theta : X \to \mathcal{C}_k(\mathcal{J}(\tau)) \) is u.s.c. with \( \theta \leq \Phi \), then there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c(\mathcal{J}(\tau)) \) such that \( \theta \leq \varphi \leq \Phi \).
Regarding the role of the metrizable hedgehog, let us remark that the set-valued insertion property in Theorems 1.1, 1.2 and 1.3 remains valid if \( J(\tau) \) is replaced by any (connected) complete metric space \((Y, \rho)\) of topological weight \( w(Y) \leq \tau \), and it is required that the family \( \{ \Phi(x) : x \in X \} \) is uniformly equi-LC\(^0\). Furthermore, one can only require that \( Y \) is completely metrizable and \( \{ \Phi(x) : x \in X \} \) is equi-“locally connected” in the sense of Nepomnyashchii [39, 40]. Finally, let us also remark that some of these results we announced in [10].

The paper is organised as follows. In the next section, we use a general construction “\( \theta \preceq \Phi \rightarrow \mathcal{C}[\theta, \Phi] \)” assigning a set-valued mapping \( \mathcal{C}[\theta, \Phi] : X \rightarrow 2^{\mathcal{Y}(Y)} \) in the hyperspace \( \mathcal{C}(Y) \) corresponding to a pair of mappings \( \theta, \Phi : X \rightarrow 2^Y \) with \( \theta \preceq \Phi \), see (2.2) and (2.3). In this setting, each selection \( \Psi : X \rightarrow 2^{\mathcal{C}(Y)} \) for \( \mathcal{C}[\theta, \Phi] \) can be transformed into an intermediate mapping \( \theta \preceq \bigcup \Psi \preceq \Phi \) by taking the union of the point-images of \( \Psi \). Furthermore, the operation “\( \Psi \rightarrow \bigcup \Psi \)” preserves properties of semi-continuity. The prototype of this construction can be found in [13], it is also implicitly present in [40]. Based on this construction, we obtain a general result for the existence of intermediate continuous mappings, see Theorem 2.2. In fact, this result is a consequence of a previous result of the author [11, Theorem 7.1], and works for set-valued mappings whose domain is a Tychonoff space. However, in contrast to Theorems 1.1, 1.2 and 1.3, it deals with the so called “metric”-lower semi-factorizable mappings (abbreviated “metric”-s.l.s.f.) rather than l.s.c. mappings. Section 3 contains the proof of Theorem 1.1, which is now obtained as a consequence of the general result in Theorem 2.2. An interesting element in this proof is the special case of countably paracompact normal spaces, Lemma 3.4, which is shown to be equivalent to Dowker’s insertion theorem [6]. This is subsequently used in the proof of the general case of \( \tau \)-paracompact normal spaces. The section also contains several other related observations. Theorems 1.2 and 1.3 are obtained in a similar way. Section 4 contains the essential preparation to apply Theorem 2.2 in the proofs of these theorems, see Theorem 4.1. Finally, Theorems 1.2 and 1.3 are proved in Section 5. Just like before, an interesting element in these proofs is a special case — that of normal spaces, see Lemma 5.2. It is now equivalent to the Katětov-Tong insertion theorem, see [18, 19, 44, 45]. Several related results are obtained as well.

2. Continuous Intermediate Mappings

In this section, \((Y, \rho)\) is a fixed metric space. For \( \varepsilon > 0 \), we will use \( O_\varepsilon(p) \) for the open \( \varepsilon \)-ball centred at a point \( p \in Y \); and \( O_\varepsilon^\rho(S) = \bigcup_{p \in S} O_\varepsilon(p) \) whenever \( S \subseteq Y \). In what follows, we will consider the set \( \mathcal{C}(Y) \) as a topological space equipped with the Hausdorff topology, i.e. the topology generated by the Hausdorff distance \( H(\rho) \) on \( Y \) associated to \( \rho \). Recall that \( H(\rho) \) is defined by

\[
H(\rho)(S, T) = \inf \{ \varepsilon > 0 : S \subseteq O_\varepsilon(T) \text{ and } T \subseteq O_\varepsilon(S) \}, \quad S, T \in \mathcal{C}(Y).
\]
A mapping \( \varphi : X \to \mathcal{F}(Y) \) is continuous precisely when it is continuous with respect to the Vietoris topology on \( \mathcal{F}(Y) \), see [24, Corollary 9.3]. However, on the hyperspace \( \mathcal{C}(Y) \), the Hausdorff topology coincides with the Vietoris one, see [24, Theorem 3.3]. Thus, a mapping \( \varphi : X \to \mathcal{C}(Y) \) is continuous precisely when it is continuous with respect to the Hausdorff distance \( H(\rho) \) on \( \mathcal{C}(Y) \).

Motivated by the so-called s.f.s.c. mappings in [9], a mapping \( \Phi : X \to 2^Y \) was said to be lower semi-factorizable with respect to \( \rho \), or \( \rho\text{-l.s.f.} \), [12] if for every closed \( F \subseteq X \), every \( \varepsilon > 0 \) and every (not necessarily continuous) single-valued selection \( s : F \to Y \) for \( \Phi \upharpoonright F \), there exists a locally finite cozero-set (in \( F \)) covering \( \mathcal{U} \) of \( F \) and a map \( \pi : \mathcal{U} \to F \) such that \( |\mathcal{U}| \leq w(Y) \) and

\[
\mathcal{S}(\pi(U)) \subseteq \mathcal{O}_x(\Phi(x)), \quad \text{for every } x \in U \subseteq \mathcal{U}.
\]

The lower semi-factorizable mappings were very successful in [12] to obtain several selection theorems from a common point of view. Regarding set-valued continuous selections, the following refined version of these mappings was introduced in [11].

A mapping \( \Phi : X \to 2^Y \) is strongly lower semi-factorizable with respect to \( \rho \), or \( \rho\text{-s.l.s.f.} \), [11] if for every closed \( F \subseteq X \), every \( \varepsilon > 0 \), and every selection \( \sigma : F \to \mathcal{C}(Y) \) for \( \Phi \upharpoonright F \), there exists a locally finite cozero-set (in \( F \)) covering \( \mathcal{U} \) of \( F \) and a map \( \pi : \mathcal{U} \to F \) such that \( |\mathcal{U}| \leq w(Y) \) and

\[
\mathcal{S}(\pi(U)) \subseteq \mathcal{O}_x(\Phi(x)), \quad \text{for every } x \in U \subseteq \mathcal{U}.
\]

For the proper understanding of these mappings, let us point out the following relationship between them, it was obtained in [11, Corollary 2.2].

**Proposition 2.1** ([11]). For a space \( X \), a metric space \((Y, \rho)\) and \( \Psi : X \to 2^Y \), define a mapping \( \mathcal{C}[\Psi] : X \to 2^{\mathcal{C}(Y)} \) by

\[
\mathcal{C}[\Psi](x) = \{ S \in \mathcal{C}(Y) : S \subseteq \Psi(x) \}, \quad \text{for every } x \in X.
\]

Then \( \Psi \) is \( \rho\text{-s.l.s.f.} \) if and only if \( \mathcal{C}[\Psi] \) is \( H(\rho)\text{-l.s.f.} \).

Here, we will show that one of the main results of [11] implies the existence of continuous intermediate mappings between special pairs of set-valued mappings. To this end, following [13], for \( \Phi : X \to \mathcal{F}(Y) \) and \( \theta : X \to \mathcal{C}(Y) \) with \( \theta \leq \Phi \), we will associate the mappings \( \mathcal{C}[\theta, \Phi], \mathcal{C}_c[\theta, \Phi] : X \to 2^{\mathcal{C}(Y)} \cup \{ \emptyset \} \) defined by

\[
\mathcal{C}[\theta, \Phi](x) = \{ S \in \mathcal{C}(Y) : \theta(x) \subseteq S \subseteq \Phi(x) \} \quad \text{and}
\]

\[
\mathcal{C}_c[\theta, \Phi](x) = \{ S \in \mathcal{C}(Y) : \theta(x) \subseteq S \subseteq \Phi(x) \}, \quad x \in X.
\]

Evidently, \( \mathcal{C}[\theta, \Phi] : X \to \mathcal{F}(\mathcal{C}(Y)) \). If the point-images of \( \Phi \) are connected and locally path-connected, then we also have that \( \mathcal{C}_c[\theta, \Phi] : X \to \mathcal{F}_c(\mathcal{C}(Y)) \). Indeed, in this case, it follows from [5, Lemma 1.3] that each element of \( \mathcal{C}[\theta, \Phi](x) \) is contained in some element of \( \mathcal{C}_c[\theta, \Phi](x) \). Therefore, \( \mathcal{C}_c[\theta, \Phi](x) \) is also nonempty. Since \( \mathcal{C}_c[\theta, \Phi](x) \) is closed, it is a growth hyperspace in the sense of Curtis [4, 5]. Hence, by a result of Kelley [21, Lemma 2.3], see also [5, Lemma 1.1], \( \mathcal{C}_c[\theta, \Phi](x) \)
is path-connected as well, i.e. \( C_\varepsilon[\theta, \Phi] : X \to C_c(\mathcal{C}_\varepsilon(Y)) \). In the sequel, we will freely rely on this fact without any explicit reference.

A family \( \mathcal{S} \subseteq 2^Y \) is uniformly equi-LC\(^0\) [28] if for every \( \varepsilon > 0 \) there is \( \delta(\varepsilon) > 0 \) such that for every \( S \in \mathcal{S} \), every two points \( y_0, y_1 \in S \) with \( \rho(y_0, y_1) < \delta(\varepsilon) \), can be joined by a path in \( S \) of diameter \( < \varepsilon \). Based on strongly lower semifactorizable mappings and uniformly equi-LC\(^0\) families of sets, we now have the following general result about continuous intermediate mappings.

**Theorem 2.2.** Let \( X \) be a Tychonoff space, \( (Y, \rho) \) be a complete metric space and \( \Phi : X \to \mathcal{F}_c(Y) \) be such that \( \{ \Phi(x) : x \in X \} \) is uniformly equi-LC\(^0\). If \( \theta : X \to \mathcal{C}(Y) \) is a selection for \( \Phi \) such that \( C_\varepsilon[\theta, \Phi] : X \to \mathcal{F}_c(\mathcal{C}_\varepsilon(Y)) \) is \( H(\rho) \)-s.l.s.f, then there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c(Y) \) with \( \theta \leq \varphi \leq \Phi \).

**Proof.** According to [40, Lemma 5], see also [5, Lemma 1.4], \( \{ C_\varepsilon[\theta, \Phi](x) : x \in X \} \) is uniformly equi-LC\(^0\) in \( \mathcal{C}_c(Y) \). Since \( \mathcal{C}_c(Y) \) is closed in \( \mathcal{C}(Y) \), it is also complete with respect to \( H(\rho) \). Hence, by [11, Theorem 7.1], \( C_\varepsilon[\theta, \Phi] \) has a continuous selection \( \Psi : X \to \mathcal{C}_c(\mathcal{C}_\varepsilon(Y)) \). Finally, define \( \varphi : X \to 2^Y \) by \( \varphi(x) = \bigcup \Psi(x) \), \( x \in X \). Then by [21, Lemma 1.2], see also [24, Theorem 2.5] and [43, Lemma 2''], \( \varphi : X \to \mathcal{C}_c(Y) \). Moreover, \( \varphi \) is continuous, see [24, Theorem 5.7]. Since \( \theta(x) \subseteq \varphi(x) \subseteq \Phi(x), x \in X \), the proof is complete. \( \square \)

Regarding the proof of Theorem 2.2, let us explicitly remark that the original formulation of [11, Theorem 7.1] is in terms of controlled extensions of continuous set-valued selections. Moreover, it contains the extra condition that \( Y \) itself must be uniformly equi-LC\(^0\). However, one can always embed \( (Y, \rho) \) isometrically into a Banach space and use this Banach space instead of \( Y \).

As we will see in Proposition 3.1 of the next section, Theorem 2.2 contains a result previously obtained by Nepomnyashchii [40, Theorem B], it is in the special case when \( X \) is paracompact, \( \Phi \) is l.s.c. and \( \theta \) is u.s.c. In the same setting, an alternative proof of Nepomnyashchii’s theorem was given by Michael [30]. In fact, the proof of Theorem 2.2 is essentially the same as that of Michael. In this regard, let us also remark that in the setting of the hyperspace \( (\mathcal{C}(Y), H(\rho)) \), the construction (2.2) was previously used in [13, Example 3.11] where the idea was the same, namely to define a set-valued mapping in \( \mathcal{C}(Y) \) and take a set-valued selection of this mapping.

We conclude this section with another general result about continuous intermediate mappings. It is complementary to Theorem 2.2 and deals with the case when the domain \( X \) has a covering dimension \( \dim(X) = 0 \). The result is valid for Tychonoff spaces, but for simplicity we state it for normal spaces. Let us recall that \( X \) is a normal space with \( \dim(X) = 0 \) precisely when every two disjoint closed subsets are contained in disjoint clopen subsets.
**Theorem 2.3.** Let $X$ be a normal space with $\dim(X) = 0$, $(Y, \rho)$ be a complete metric space and $\Phi : X \to \mathcal{F}(Y)$. If $\theta : X \to \mathcal{C}(Y)$ is a selection for $\Phi$ such that $\mathcal{C}[\theta, \Phi] : X \to \mathcal{F}(\mathcal{C}(Y))$ is $H(\rho)$-l.s.f., then there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ with $\theta \leq \varphi \leq \Phi$.

**Proof.** Since $\mathcal{C}[\theta, \Phi]$ is $H(\rho)$-l.s.f., by [12, Theorem 5.1] and [36, Lemma 2.2], there is a metrizable space $Z$ with $\dim(Z) = 0$, an l.s.c. mapping $\Psi : Z \to \mathcal{F}(\mathcal{C}(Y))$ and a continuous map $g : X \to Z$ such that $\Psi(g(x)) \subseteq \mathcal{C}[\theta, \Phi](x)$, for every $x \in X$. Since $Z$ is paracompact, by a result of Michael [26], see also [31], the l.s.c. mapping $\Psi$ has a continuous selection $\psi : Z \to \mathcal{C}(Y)$. Then the composite mapping $\varphi = \psi \circ g : X \to \mathcal{C}(Y)$ is as required. □

**3. Intermediate Selections and Paracompactness**

In this section, we finalise the proof of Theorem 1.1. In the one direction, it is based on Theorem 2.2. To prepare for this, let us state explicitly the following example of strongly lower semi-factorizable mappings.

**Proposition 3.1.** Let $X$ be normal and $\tau$-paracompact, $(Y, \rho)$ be a metric space with $w(Y) \leq \tau$, and $\Phi : X \to \mathcal{F}_c(Y)$ be l.s.c. such that $\{\Phi(x) : x \in X\}$ is uniformly $\text{eqi-LC}^0$. If $\theta : X \to \mathcal{C}(Y)$ is a u.s.c. selection for $\Phi$, then the associated mapping $\mathcal{C}_c[\theta, \Phi] : X \to \mathcal{F}_c(\mathcal{C}(Y))$ is $H(\rho)$-s.l.s.f.

**Proof.** By [30, Lemma 5.3], $\mathcal{C}_c[\theta, \Phi]$ is l.s.c. Then by [11, Example 2.6], it is also $H(\rho)$-s.l.s.f. because $w(\mathcal{C}_c(Y)) \leq \tau$ and $X$ is normal and $\tau$-paracompact. □

The inequality “$\theta < \varphi \leq \Phi$” in Theorem 1.1 is based on the following general property of intermediate mappings.

**Proposition 3.2.** Let $X$ be normal and $\tau$-paracompact, $Y$ be a completely metrizable space with $w(Y) \leq \tau$, $\Phi : X \to \mathcal{F}(Y)$ be l.s.c. and $\theta : X \to \mathcal{C}(Y)$ be u.s.c. with $\theta < \Phi$. Then there exists a u.s.c. mapping $\psi : X \to \mathcal{C}(Y)$ with $\theta < \psi \leq \Phi$.

**Proof.** Let $\mathcal{B}$ be an open base for the topology of $Y$ with $|\mathcal{B}| \leq \tau$. Next, for every $B \in \mathcal{B}$ set $V_B = \Phi^{-1}(B) \cap \theta^\#(Y \setminus B)$. In this way, we get an open subset $V_B$ of $X$ such that $B \cap \Phi(x) \neq \emptyset$ and $B \cap \theta(x) = \emptyset$ for every $x \in V_B$. Hence, $\{V_B : B \in \mathcal{B}\}$ is an open cover of $X$ because $\mathcal{B}$ is a base and by condition, $\Phi(x) \setminus \theta(x) \neq \emptyset$ for every $x \in X$. Since $X$ is $\tau$-paracompact, it has a locally finite open cover $\{U_B : B \in \mathcal{B}\}$ such that $U_B \subseteq V_B$, for every $B \in \mathcal{B}$. Since $X$ is also normal, by the Lefschetz lemma [23], it has a closed cover $\{F_B : B \in \mathcal{B}\}$ with $F_B \subseteq U_B$ for every $B \in \mathcal{B}$. Finally, for every $B \in \mathcal{B}$, define a set-valued mapping $\Phi_B : F_B \to \mathcal{F}(Y)$ by $\Phi_B(x) = \Phi(x) \cap B$, $x \in F_B$. Obviously, each $\Phi_B$ is an l.s.c. selection for $\Phi \upharpoonright F_B$ such that

$$\Phi_B(x) \cap \theta(x) = \emptyset, \quad \text{for every } x \in F_B.$$
By a result of Choban [2, Theorem 11.2], see also Michael [29, Theorem 1.1], each \( \Phi_B \) admits a u.s.c. selection \( \varphi_B : F_B \to \mathcal{C}(Y) \). Since \( \{F_B : B \in \mathcal{B}\} \) is a locally finite closed cover of \( X \), we may now define a u.s.c. selection \( \varphi : X \to \mathcal{C}(Y) \) for \( \Phi \) by \( \varphi(x) = \bigcup \{\varphi_B(x) : x \in F_B \text{ and } B \in \mathcal{B}\}, x \in X \). Then by (3.1), we get that \( \varphi(x) \cap \theta(x) = \emptyset \) for all \( x \in X \). Hence, we may define the required intermediate u.s.c. mapping \( \psi : X \to \mathcal{C}(Y) \) by \( \psi(x) = \theta(x) \cup \varphi(x), x \in X \). \( \square \)

In this section, and what follows, \( J(2) \) is the hedgehog with two spines. By identifying the two spines with the intervals \([-1,0]\) and \([0,1]\), it follows from (1.1) that \( (J(2),d) \) is isometric to the usual interval \([-1,1]\) when this interval is equipped with the Euclidean distance \( d(s,t) = |s-t|, s,t \in [-1,1]\). This implies the following immediate property of the connected subsets of \( J(\tau) \).

**Proposition 3.3.** If \( \tau \) is a cardinal number, then the family \( \mathcal{F}_c(J(\tau)) \) is uniformly equi-LC\(^0\) in \( (J(\tau),d) \).

A function \( \xi : X \to \mathbb{R} \) is lower (upper) semi-continuous if the set
\[
\{x \in X : \xi(x) > r\} \quad \text{(respectively, } \{x \in X : \xi(x) < r\}\}
\]
is open in \( X \) for every \( r \in \mathbb{R} \). For functions \( \xi, \eta : X \to \mathbb{R} \), we write that \( \xi < \eta \) (\( \xi \leq \eta \)), if \( \xi(x) < \eta(x) \) (respectively, \( \xi(x) \leq \eta(x) \)), for every \( x \in X \). In these terms, we have the following set-valued interpretation of Dowker’s insertion theorem [6].

**Lemma 3.4.** For a space \( X \), the following are equivalent:

(a) \( X \) is countably paracompact and normal.
(b) If \( \Phi : X \to \mathcal{F}_c(J(2)) \) is l.s.c., \( \theta : X \to \mathcal{C}(J(2)) \) is u.s.c. and \( \theta < \Phi \), then there exists a continuous mapping \( \varphi : X \to \mathcal{C}(J(2)) \) such that \( \theta < \varphi \leq \Phi \).
(c) If \( \xi : X \to \mathbb{R} \) is upper semicontinuous, \( \eta : X \to \mathbb{R} \) is lower semicontinuous and \( \xi < \eta \), then there exists a continuous function \( f : X \to \mathbb{R} \) such that \( \xi < f < \eta \).

**Proof.** To show that (a) \( \Rightarrow \) (b), suppose that \( X \) is countably paracompact and normal, \( \Phi : X \to \mathcal{F}_c([-1,1]) \) is l.s.c. and \( \theta : X \to \mathcal{C}([-1,1]) \) is u.s.c. with \( \theta < \Phi \). By Proposition 3.2, there exists a u.s.c. mapping \( \psi : X \to \mathcal{C}([-1,1]) \) with \( \theta < \psi \leq \Phi \). Finally, since \( \Phi \) is convex-valued, it follows from Proposition 3.1 that \( \mathcal{C}_c[\psi,\Phi] : X \to \mathcal{F}_c(\mathcal{C}_c([-1,1])) \) is \( H(\rho)\)-s.l.s.f. Therefore, by Theorem 2.2, there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c([-1,1]) \) such that \( \theta < \varphi \leq \Phi \).

Suppose that (b) holds, and take functions \( \xi, \eta : X \to \mathbb{R} \) such that \( \xi \) is upper semicontinuous, \( \eta \) is lower semicontinuous and \( \xi < \eta \). Using the order preserving homeomorphism \( h(t) = \frac{t}{1+|t|}, t \in \mathbb{R} \), of the real line \( \mathbb{R} \) onto the interval \((-1,1)\), we may assume that \( \xi, \eta : X \to (-1,1) \). Define mapping \( \Phi_\xi, \theta_\xi : X \to \mathcal{C}_c([-1,1]) \) by \( \Phi_\xi(x) = [-1, \eta(x)] \) and \( \theta_\xi(x) = [-1, \xi(x)] \), \( x \in X \). Then \( \theta_\xi < \Phi_\xi \). Moreover, \( \Phi_\xi \) is l.s.c. and \( \theta_\xi \) is u.s.c. [27, Example 1.2*], see also [14, Proposition 5.4]. Hence, by
(b), there exists a continuous mapping \( \varphi_\xi : X \to \mathcal{C}_c([-1,1]) \) with \( \theta_\xi < \varphi_\xi \leq \Phi_\xi \). We may now define a continuous function \( f_\xi : X \to (-1,1) \) by \( f_\xi(x) = \max \varphi_\xi(x) \), \( x \in X \), see e.g. [14, Proposition 5.4]. According to the definitions of \( \Phi_\xi \) and \( \theta_\xi \), and the property of \( \varphi_\xi \), it follows that \( \xi < f_\xi \leq \eta \). Similarly, using the function \( \eta \), there also exists a continuous function \( f_\eta : X \to (-1,1) \) such that \( \xi \leq f_\eta < \eta \). Then \( f = \frac{f_\xi + f_\eta}{2} \) is as required in (c). Since the implication (c) \( \Rightarrow \) (a) is a part of Dowker’s insertion theorem [6, Theorem 4], the proof is complete. \( \square \)

Regarding the proper place of Lemma 3.4, it is evident from its proof that the countable paracompactness of \( X \) is only essential for the relation \( \theta < \varphi \leq \Phi \), compare with Lemma 5.2 in the last section.

**Proof of Theorem 1.1.** Let \( \tau \) be an infinite cardinal number, \( X \) be a \( \tau \)-paracompact normal space, \( \Phi : X \to \mathcal{F}_c(\mathcal{J}(\tau)) \) be l.s.c. and \( \theta : X \to \mathcal{G}(\mathcal{J}(\tau)) \) be u.s.c. with \( \theta < \Phi \). By Proposition 3.2, there exists a u.s.c. mapping \( \psi : X \to \mathcal{G}(\mathcal{J}(\tau)) \) such that \( \theta < \psi \leq \Phi \). By Proposition 3.3, the family \( \{ \Phi(x) : x \in X \} \) is uniformly equi-LC\(^0\). Hence, by Proposition 3.1, the mapping \( \mathcal{G}_c[\psi, \Phi] : X \to \mathcal{F}_c(\mathcal{G}_c(\mathcal{J}(\tau))) \) is \( H(\rho) \)-s.l.s.f. Thus, according to Theorem 2.2, there exists a continuous mapping \( \varphi : X \to \mathcal{G}_c(Y) \) such that \( \theta < \psi \leq \varphi \leq \Phi \). This shows (b) of Theorem 1.1.

Conversely, assume that \( X \) is as in (b) of Theorem 1.1. Since \( \mathcal{J}(2) \subseteq \mathcal{J}(\tau) \), it follows from Lemma 3.4 that \( X \) is countably paracompact and normal. To see that \( X \) is also \( \tau \)-paracompact, take an open cover \( \mathcal{U} \) of \( X \) with \( |\mathcal{U}| \leq \tau \). Next, for every \( x \in X \), let \( \mathcal{U}_x = \{ U \in \mathcal{U} : x \in U \} \). When \( \mathcal{U} \) is equipped with the discrete topology, this defines an l.s.c. mapping \( x \to \mathcal{U}_x, x \in X \), see the proof of [2, Theorem 11.2]. In our case, we may use the corresponding sub-hedgehog \( \mathcal{J}(\mathcal{U}_x) \) to get an l.s.c. mapping into the subsets of \( \mathcal{J}(\mathcal{U}) \) corresponding to the cover \( \mathcal{U} \). Namely, define \( \Phi : X \to \mathcal{F}_c(\mathcal{J}(\mathcal{U})) \) by \( \Phi(x) = \mathcal{J}(\mathcal{U}_x), x \in X \). Then \( \Phi \) is l.s.c. which follows easily from the fact that \( \Phi(x) \setminus \{0\} = (0, 1) \times \mathcal{U}_x, x \in X \). Moreover, \( \theta(x) = \{0\}, x \in X \), is clearly a u.s.c. selection for \( \Phi \) with \( \Phi(x) \setminus \theta(x) \neq \emptyset \) for each \( x \in X \). Hence, by assumption, there exists a continuous mapping \( \varphi : X \to \mathcal{G}_c(\mathcal{J}(\mathcal{U})) \) such that \( \theta < \varphi \leq \Phi \). For convenience, for each \( U \in \mathcal{U} \) and \( n \in \mathbb{N} \), set \( O[U,n] = (2^{-n}, 1] \times \{ U \} \) which is an open subset of \( \mathcal{J}(\mathcal{U}) \). Then for each \( n \in \mathbb{N} \), it follows from (1.1) that the family \( \Omega_n = \{ O[U,n] : U \in \mathcal{U} \} \) is discrete in \( \mathcal{J}(\mathcal{U}) \). Accordingly, the family

\[
\forall_n = \varphi^{-1}(\Omega_n) = \{ \varphi^{-1}(O[U,n]) : U \in \mathcal{U} \}
\]

is locally finite and open in \( X \) because the mapping \( \varphi \) is compact-valued and both l.s.c. and u.s.c. Moreover, \( \forall_n \) refines \( \mathcal{U} \) because \( \varphi(x) \cap O[U,n] \neq \emptyset \) implies \( \Phi(x) \cap O[U,n] \neq \emptyset \) and, therefore, \( x \in U \). Finally, \( \bigcup_{n \in \mathbb{N}} \forall_n \) is a cover of \( X \) because \( \varphi(x) \setminus \{0\} = \varphi(x) \setminus \theta(x) \neq \emptyset \) for all \( x \in X \). This shows that \( X \) is \( \tau \)-paracompact being countably paracompact and normal, see [35, Theorem 1.1]. \( \square \)

We conclude this section with some related observations.
**Proposition 3.5.** For an infinite cardinal number $\tau$ and a space $X$, the following conditions are equivalent:

(a) $X$ is normal and $\tau$-paracompact.

(b) If $Y$ is a completely metrizable space with $w(Y) \leq \tau$, $\Phi : X \to \mathcal{F}(Y)$ is l.s.c., $\theta : X \to \mathcal{C}(Y)$ is u.s.c. and $\theta < \Phi$, then there exists a u.s.c. mapping $\psi : X \to \mathcal{C}(Y)$ with $\theta < \psi \leq \Phi$.

**Proof.** The implication (a) $\implies$ (b) is Proposition 3.2. Conversely, assume that (b) holds and take an l.s.c. mapping $\Phi : X \to \mathcal{F}(Y)$, where $Y$ is a completely metrizable space with $w(Y) \leq \tau$. Next, let $Y^*$ be the space obtained from $Y$ by adding an isolated point $*$ to $Y$. We may now define another l.s.c. mapping $\Phi^* : X \to \mathcal{F}(Y^*)$ by $\Phi^*(x) = \Phi(x) \cup \{\ast\}, x \in X$. Also, let $\theta : X \to \mathcal{C}(Y^*)$ be the constant mapping $\theta(x) = \{\ast\}, x \in X$. Then $\theta$ is a u.s.c. selection for $\Phi^*$ with $\Phi^*(x) \setminus \theta(x) = \Phi(x) \neq \emptyset, x \in X$. Hence, by (b), $\Phi^*$ admits a u.s.c. selection $\psi : X \to \mathcal{C}(Y^*)$ such that $\varphi(x) = \psi(x) \setminus \theta(x) \neq \emptyset$ for every $x \in X$. Evidently, this defines a u.s.c. selection $\varphi : X \to \mathcal{C}(Y)$ for $\Phi$. According to [2, Theorem 11.2], $X$ is normal and $\tau$-paracompact.

Proposition 3.5 is complementary to similar characterisations of $\tau$-expandable spaces and $\tau$-collectionwise normal spaces obtained in [32]. Regarding this proposition, let us also remark that upper semi-continuity of the intermediate selection $\psi$ cannot be strengthened to continuity. In fact, in this case, continuity of the intermediate selection is equivalent to $X$ having a covering dimension zero.

**Proposition 3.6.** For an infinite cardinal number $\tau$ and a space $X$, the following conditions are equivalent:

(a) $X$ is a $\tau$-paracompact normal space with $\dim(X) = 0$.

(b) If $Y$ is a completely metrizable space with $w(Y) \leq \tau$, $\Phi : X \to \mathcal{F}(Y)$ is l.s.c., $\theta : X \to \mathcal{C}(Y)$ is u.s.c. and $\theta < \Phi$, then there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ with $\theta < \varphi \leq \Phi$.

**Proof.** To show that (a) $\implies$ (b), let $Y$ be a completely metrizable space with $w(Y) \leq \tau$, $\Phi : X \to \mathcal{F}(Y)$ be l.s.c. and $\theta : X \to \mathcal{C}(Y)$ be u.s.c. with $\theta < \Phi$. By Proposition 3.2, there exists a u.s.c. mapping $\psi : X \to \mathcal{C}(Y)$ such that $\theta < \psi \leq \Phi$. Next, consider the mapping $\mathcal{C}[\psi, \Phi] : X \to \mathcal{F}(\mathcal{C}(Y))$ which is l.s.c., see [13, Example 3.11]. Since $w(\mathcal{C}(Y)) \leq \tau$, it follows from [2, Theorem 11.1], see also [26, Theorem 2], that $\mathcal{C}[\psi, \Phi]$ has a continuous selection $\varphi : X \to \mathcal{C}(Y)$. According to (2.2), this $\varphi$ is as required. To show the converse, assume that (b) holds. Then by Proposition 3.5, $X$ is normal and $\tau$-paracompact. To show that $\dim(X) = 0$, take disjoint closed sets $F_1, F_2 \subseteq X$ and, for convenience, set $Y = \{0, 1, 2\}$. Next, define an l.s.c. mapping $\Phi : X \to 2^Y$ by $\Phi(x) = \{0, i\}$ if $x \in F_i$, and $\Phi(x) = Y$ otherwise. Also, just like before, consider the constant mapping $\theta(x) = \{0\}, x \in X$. Then $\theta < \Phi$ and by (b), there exists a continuous
mapping $\varphi : X \to 2^Y$ with $\theta < \varphi \leq \Phi$. In particular, $\varphi(x) \neq \{0\}$, $x \in X$, and we may define the disjoint sets $U_i = \varphi^{-1}(\{0, i\})$, $i = 1, 2$. Since $Y$ is discrete and $\varphi$ is continuous, $U_1$ and $U_2$ are clopen sets. Moreover, by the definition of $\Phi$, we have that $F_i \subseteq U_i$, $i = 1, 2$. This shows that $\dim(X) = 0$. □

In setting of paracompact spaces, the implication (a) $\implies$ (b) of Proposition 3.6 was stated by Nepomnyashchii [40].

4. Collectionwise Normality and “Metric”-s.l.s.f. Mappings

The proofs of Theorems 1.2 and 1.3 are similar to that of Theorem 1.1 being based on a reduction to Theorem 2.2. However, in the setting of $C'(Y)$-valued mappings, there is no simple version of Proposition 3.1. The reason is that for $\Phi : X \to C'(Y)$ and a selection $\theta : X \to C(Y)$ for $\Phi$, the associated mapping $C[\theta, \Phi]$ in (2.2) is not necessarily $C'(C(Y))$-valued; similarly, for the construction in (2.3). This requires an extra property of the selection $\theta : X \to C(Y)$.

**Theorem 4.1.** Let $X$ be $\tau$-collectionwise normal, $(Y, \rho)$ be a connected metric space with $w(Y) \leq \tau$, and $\Phi : X \to C'_c(Y)$ be l.s.c. such that $\{\Phi(x) : x \in X\}$ is uniformly $\text{equiv-LC}^0$. Also, let $Z$ be a metrizable space, $g : X \to Z$ be continuous and $\Theta : Z \to C(Y)$ be u.s.c. such that $\psi = \Theta \circ g \leq \Phi$. Then the mapping $C_c[\psi, \Phi] : X \to P_c(C_c(Y))$ is $H(\rho)$-s.l.s.f.

The proof of Theorem 4.1 is separated into two parts — the case of the constant mapping $\Phi(x) = Y$, $x \in X$, and that of $\Phi(x)$ being compact for each $x \in X$.

**Proposition 4.2.** Under the conditions of Theorem 4.1, suppose that $\Phi(x) = Y$ for every $x \in X$. Then $C_c[\psi, \Phi] : X \to P_c(C_c(Y))$ is $H(\rho)$-s.l.s.f.

**Proof.** Let $Z$, $\Theta : Z \to C(Y)$, $g : X \to Z$ and $\psi = \Theta \circ g : X \to C(Y)$ be as in Theorem 4.1. Set $\Psi(z) = Y$, $z \in Z$, and consider the associated mapping $C_c[\Theta, \Psi]$ defined as in (2.3). If $F \subseteq X$, then $g(F)$ is paracompact being metrizable. Hence, by Proposition 3.1, $C_c[\Theta, \Psi] | g(F)$ is $H(\rho)$-s.l.s.f. for every $F \subseteq X$. Since $C_c[\psi, \Phi](x) = C_c[\Theta, \Psi](g(x))$ for every $x \in X$, this implies that $C_c[\psi, \Phi]$ is also $H(\rho)$-s.l.s.f. □

It will be useful to state the case of compact-valued mappings in a little bit more general setting. To this end, let us recall that for an infinite cardinal number $\tau$, a space $X$ is called $\tau$-$\text{PF-normal}$ (see [42]) if every point-finite open cover of $X$ of cardinality $\leq \tau$ is normal. Every $\tau$-collectionwise normal space is $\tau$-$\text{PF-normal}$ [25] (see also Kandô [17] and Nedev [38]), and $\omega$-$\text{PF-normality}$ coincides with normality [33]. However, PF-normality is neither identical to collectionwise normality (see Bing's example [1] and [25, Example 1]), nor to normality ([25, Example 1]). For some properties of PF-normal spaces, the interested reader is referred to [15, Section 3] and [25].
Proposition 4.3. Let $X$ be $\tau$-PF-normal, $(Y, \rho)$ be a metric space with $w(Y) \leq \tau$, and $\Phi : X \to \mathcal{C}(Y)$ be l.s.c. such that $\{\Phi(x) : x \in X\}$ is uniformly equi-\(\text{LC}_0\). If $\psi : X \to \mathcal{C}(Y)$ is a u.s.c. selection for $\Phi$, then $\mathcal{C}_c[\psi, \Phi] : X \to \mathcal{P}_c(\mathcal{C}_c(Y))$ is $H(\rho)$-s.l.s.f.

Proof. Set $E = \mathcal{C}_c(Y)$ and $d = H(\rho)$. Then $(E, d)$ is a metric space with $w(E) \leq \tau$. Moreover, by [30, Lemma 5.3], $\Psi = \mathcal{C}_c[\psi, \Phi] : X \to \mathcal{C}_c(E)$ is l.s.c. Consider the mapping $\mathcal{C}[\Psi]$ defined as in (2.1), which remains l.s.c., see [39, Lemma 3.4']. In Neden’s terminology [38], $X$ is $\tau$-PF-normal precisely when it is $\tau^+$-pointwise-$\mathcal{N}_0$-paracompact space. Hence, by [38, Lemma 3.5], $\mathcal{C}[\Psi]$ has the Selection Factorisation Property in the sense of [3, 38] because $w(\mathcal{C}(E)) \leq \tau < \tau^+$. According to the proof of [12, Example 4.2], this implies that $\mathcal{C}[\Psi]$ is $H(\rho)$-l.s.f. Therefore, by Proposition 2.1, $\Psi = \mathcal{C}_c[\psi, \Phi]$ is $d = H(\rho)$-s.l.s.f. \qed

Finally, we also need the following property of $H(\rho)$-s.l.s.f. mappings defined on collectionwise normal spaces.

Proposition 4.4. Let $X$ be $\tau$-collectionwise normal, $(E, d)$ be a metric space and $\Psi : X \to 2^E$ be an l.s.c. mapping. Suppose that $F \subseteq X$ is a closed set, $\varepsilon > 0$, $\mathcal{V}$ is a point-finite open in $F$ cover of $F$ with $|V| \leq \tau$, and $K_V \subseteq \mathcal{C}(E)$, $V \in \mathcal{V}$, are compact sets such that $K_V \subseteq \mathcal{O}^d(\Psi(x))$, for every $x \in V$. Then there exists a locally finite open in $X$ cover $\{U_V : V \in \mathcal{V}\}$ of $F$ such that

$$U_V \cap F \subseteq V \quad \text{and} \quad K_V \subseteq \mathcal{O}^d(\Psi(x)), \quad \text{for every } x \in U_V \text{ and } V \in \mathcal{V}.$$ 

Proof. Take $V \in \mathcal{V}$. Since $K_V$ is compact and $\Psi$ is l.s.c., by [28, Lemma 11.3], there exists an open set $W_V \subseteq X$ such that $W_V \cap F = V$ and $K_V \subseteq \mathcal{O}^d(\Psi(x))$, for every $x \in W_V$. Thus, we get an open in $X$ and point-finite in $F$ cover $\{W_V : V \in \mathcal{V}\}$ of $F$. Since $X$ is $\tau$-collectionwise normal and $|\mathcal{V}| \leq \tau$, by [37, Lemma 1], see also [38, Lemma 1.6], there exists a locally finite open in $X$ cover $\{U_V : V \in \mathcal{V}\}$ of $F$ with $U_V \subseteq W_V$, $V \in \mathcal{V}$. \qed

Proof of Theorem 4.1. Let $X$, $(Y, \rho)$, $\Phi$ and $\psi$ be as in that theorem. Take $\varepsilon > 0$, a closed set $F \subseteq X$ and a selection $\sigma : F \to \mathcal{C}(\mathcal{C}_c(Y))$ for $\mathcal{C}(\mathcal{C}_c[\psi, \Phi]) \mid F$. Set $\Psi(x) = Y$, $x \in X$. Then by Proposition 4.2, $\mathcal{C}_c[\psi, \Phi]$ is $H(\rho)$-s.l.s.f. Hence, by Proposition 4.4 applied with $E = \mathcal{C}(Y)$ and $d = H(\rho)$, there exists a locally finite open in $X$ cover $\mathcal{V}_0$ of $F$ and a map $\pi_0 : \mathcal{V}_0 \to F$ such that $|\mathcal{V}_0| \leq \tau$ and

\begin{equation}
\sigma(\pi_0(V)) \subseteq \mathcal{O}^H(\mathcal{C}_c[\psi, \Phi])(x), \quad \text{for every } x \in V \in \mathcal{V}_0.
\end{equation}

Next, as in the proof of Proposition 4.4, for each $V \in \mathcal{V}$, define an open subset $U_V$ of $V$ by $U_V = \{x \in V : \sigma(\pi_0(V)) \subseteq \mathcal{O}^H(\mathcal{C}_c[\psi, \Phi])(x)\}$. Thus, we get a locally finite open in $X$ family $\mathcal{V}_0 = \{U_V : V \in \mathcal{V}_0\}$ such that

\begin{equation}
\sigma(\pi_0(V)) \subseteq \mathcal{O}^H(\mathcal{C}_c[\psi, \Phi])(x), \quad \text{for every } x \in U_V \text{ and } V \in \mathcal{V}_0.
\end{equation}
Moreover, if $\Phi(x) = Y$ for some $x \in V \in \mathcal{V}_0$, then $\mathcal{C}_c[\psi, \Phi](x) = \mathcal{C}_c[\psi, \Psi](x)$ and by (4.1), $x \in U_V$. Therefore, $\Phi$ is compact-valued on the closed set $F_1 = F \setminus \bigcup \mathcal{V}_0$. Hence, by Propositions 4.3 and 4.4, there is a locally finite open in $X$ cover $\mathcal{U}_1$ of $F_1$ and a map $\pi_1 : \mathcal{U}_1 \to F_1$ such that $|\mathcal{U}_1| \leq \tau$ and

$$
(4.3) \quad \sigma(\pi_1(U)) \subseteq O_\epsilon^{S_{\rho}}(\mathcal{C}_c[\psi, \Phi](x)), \quad \text{for every } x \in U \in \mathcal{U}_1.
$$

We can now take $\mathcal{U}$ to be the disjoint union of $\mathcal{U}_0$ and $\mathcal{U}_1$, and $\pi : \mathcal{U} \to F$ to be defined by $\pi | \mathcal{U}_i = \pi_i$, $i = 0, 1$. Evidently, by (4.2) and (4.3), this implies that $\mathcal{C}_c[\psi, \Phi]$ is $H(\rho)$-s.l.s.f. $\square$

Let us explicitly remark that in Theorem 4.1, the point-images of $\Phi$ were required to be connected and $\{\Phi(x) : x \in X\}$ to be uniformly equi-$LC^0$ only to make sure that $\mathcal{C}_c[\psi, \Phi] : X \to \mathcal{F}_c(\mathcal{C}_c(Y))$. However, this played no other role in the proof Theorem 4.1. If $Y$ is a metrizable space, $\Phi : X \to \mathcal{F}(Y)$ and $\theta : X \to \mathcal{C}(Y)$ is a u.s.c. selection for $\Phi$, then the mapping $\mathcal{C}[\theta, \Phi]$ defined as in (2.2), takes values in $\mathcal{F}(\mathcal{C}(Y))$ and is l.s.c., see the proof of [13, Example 3.11]. Hence, we have the following further result from the proof of Theorem 4.1.

**Theorem 4.5.** Let $X$ be $\tau$-collectionwise normal, $(Y, \rho)$ be a metric space with $w(Y) \leq \tau$, and $\Phi : X \to \mathcal{C}(Y)$ be l.s.c. Also, let $Z$ be a metrizable space, $g : X \to Z$ be continuous and $\Theta : Z \to \mathcal{C}(Y)$ be u.s.c. such that $\psi = \Theta \circ g \leq \Phi$. Then $\mathcal{C}[\psi, \Phi] : X \to \mathcal{F}(\mathcal{C}(Y))$ is $H(\rho)$-s.l.s.f. and, in particular, also $H(\rho)$-l.s.f.

5. Intermediate Selections and Collectionwise Normality

A mapping $\theta : X \to 2^Y$ has the locally finite lifting property [13, (3.3)] (see also [15, 16, 32]) if for every locally finite family $\mathcal{F}$ of closed subsets of $Y$, there is a locally finite family $\{U_F : F \in \mathcal{F}\}$ of open subsets of $X$ such that $\theta^{-1}(F) \subseteq U_F$ for each $F \in \mathcal{F}$. The following key observation will be used in the proofs of both Theorem 1.2 and Theorem 1.3.

**Proposition 5.1.** Let $X$ be $\tau$-collectionwise normal, $(Y, \rho)$ be a connected complete metric space with $w(Y) \leq \tau$, and $\Phi : X \to \mathcal{C}(Y)$ be l.s.c. such that $\{\Phi(x) : x \in X\}$ is uniformly equi-$LC^0$. Also, let $\theta : X \to \mathcal{C}(Y)$ be a u.s.c. selection for $\Phi$ which has the locally finite lifting property. Then there exists a continuous mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\theta \leq \varphi \leq \Phi$.

**Proof.** Since $\theta : X \to \mathcal{C}(Y)$ has the locally finite lifting property, by [16, Theorem 3.1], there exists a metrizable space $Z$, a continuous map $g : X \to Z$ and a u.s.c. mapping $\Theta : Z \to \mathcal{C}(Y)$ such that $\theta \leq \psi = \Theta \circ g \leq \Phi$. Accordingly, by Theorem 4.1, the mapping $\mathcal{C}_c[\psi, \Phi] : X \to \mathcal{F}_c(\mathcal{C}_c(Y))$ is $H(\rho)$-s.l.s.f. Thus, by Theorem 2.2, there exists a continuous mapping $\varphi : X \to \mathcal{C}_c(Y)$ such that $\theta \leq \psi \leq \varphi \leq \Phi$. $\square$

The other key element in these proofs is the following set-valued interpretation of the Katětov-Tong insertion theorem, see Tong [44, 45] and Katětov [18, 19].
Lemma 5.2. For a space $X$, the following are equivalent:

(a) $X$ is normal.
(b) If $\Phi : X \to \mathcal{C}_c(\mathcal{J}(2))$ is l.s.c., $\theta : X \to \mathcal{C}(\mathcal{J}(2))$ is u.s.c. and $\theta \leq \Phi$, then there exists a continuous mapping $\varphi : X \to \mathcal{C}_c(\mathcal{J}(2))$ such that $\theta \leq \varphi \leq \Phi$.
(c) If $\xi : X \to \mathbb{R}$ is upper semicontinuous, $\eta : X \to \mathbb{R}$ is lower semicontinuous and $\xi \leq \eta$, then there exists a continuous function $f : X \to \mathbb{R}$ such that $\xi \leq f \leq \eta$.

Proof. As in the proof of Lemma 3.4, we identify $\mathcal{J}(2)$ with the interval $[-1,1]$. To show that (a) $\Longrightarrow$ (b), suppose that $X$ is normal, $\Phi : X \to \mathcal{C}_c([-1,1])$ is l.s.c. and $\theta : X \to \mathcal{C}([-1,1])$ is a u.s.c. selection for $\Phi$. Since $\Phi$ is compact-convex-valued, by Proposition 4.3, the associated mapping $\mathcal{C}_c[\theta, \Phi]$ is $H(d)$-s.l.s.f. Hence, just like before, the required $\varphi : X \to \mathcal{C}_c([-1,1])$ is given by Theorem 2.2.

To see that (b) $\Longrightarrow$ (c), following the proof of Lemma 3.4, take functions $\xi, \eta : X \to (-1,1)$ such that $\xi$ is upper semicontinuous, $\eta$ is lower semicontinuous and $\xi \leq \eta$. Next, define mappings $\Phi, \theta : X \to \mathcal{C}_c([-1,1])$ by $\Phi(x) = [-1, \eta(x)]$ and $\theta(x) = [-1, \xi(x)]$, $x \in X$. Then $\Phi$ is l.s.c. and $\theta$ is a u.s.c. selection for $\Phi$. Hence, by (b), there exists a continuous mapping $\varphi : X \to \mathcal{C}_c([-1,1])$ such that $\theta \leq \varphi \leq \Phi$. The function $f : X \to (-1,1)$, defined by $f(x) = \max \varphi(x), x \in X$, is as required in (c). Since the implication (c) $\Longrightarrow$ (a) is a part of the Katetov-Tong insertion theorem, the proof is complete. \hfill $\Box$

Regarding the implication (a) $\Longrightarrow$ (b) of Lemma 5.2, let us remark that we used Theorem 2.2 which is behind the framework of all results of this paper. However, an alternative proof follows using the Katetov-Tong insertion theorem.

Proof of Theorem 1.2. Let $X$ be normal and $\tau$-expandable, $\Phi : X \to \mathcal{C}_c'(\mathcal{J}(\tau))$ be l.s.c. and $\theta : X \to \mathcal{C}(\mathcal{J}(\tau))$ be a u.s.c. selection for $\Phi$. By [13, Example 3.9], $\theta$ has the locally finite lifting property. Moreover, by Proposition 3.3, $\{\Phi(x) : x \in X\}$ is uniformly equi-$LC^0$. Hence, the required intermediate mapping $\varphi$ is now given by Proposition 5.1. This shows the implication (a) $\Longrightarrow$ (b) in Theorem 1.2.

To see the inverse implication, assume that $X$ is as in (b) of Theorem 1.2. Since $\mathcal{J}(2) \subseteq \mathcal{J}(\tau)$, it follows from Lemma 5.2 that $X$ is normal. Take a locally finite family $\mathcal{F}$ of closed subsets of $X$ with $|\mathcal{F}| \leq \tau$. By adding $X$ to $\mathcal{F}$, if necessary, we may assume that $\mathcal{F}$ is a cover of $X$. Next, define a u.s.c. mapping $\theta : X \to \mathcal{C}(\mathcal{J}(\mathcal{F}))$ by $\theta(x) = \{(1, F) : x \in F \text{ and } F \in \mathcal{F}\}$, $x \in X$. Also, let $\Phi(x) = \mathcal{J}(\mathcal{F})$, $x \in X$, be the constant mapping. Evidently, $\Phi : X \to \mathcal{C}_c'(\mathcal{J}(\mathcal{F}))$ is l.s.c. Hence, by assumption, there exists a continuous mapping $\varphi : X \to \mathcal{C}_c(\mathcal{J}(\mathcal{F}))$ with $\theta \leq \varphi \leq \Phi$. For convenience, set $V_F = O_{\tau/2}((1, F))$, $F \in \mathcal{F}$. Thus, we get an open and discrete in $\mathcal{J}(\mathcal{F})$ family $\{V_F : F \in \mathcal{F}\}$ with $(1, F) \in V_F$, $F \in \mathcal{F}$. Moreover, by the definition of $\theta$, we also have that $(1, F) \in \varphi(x)$, whenever $x \in F \in \mathcal{F}$. Since $\varphi$ is compact-valued and both l.s.c. and u.s.c., this implies that
the sets \( U_F = \varphi^{-1}(V_F) \), \( F \in \mathcal{F} \), form a locally finite open family in \( X \) such that \( F \subseteq U_F, F \in \mathcal{F} \). Accordingly, \( X \) is also \( \tau \)-expandable.

The proof of Theorem 1.3 is almost identical to that of Theorem 1.2, so we will briefly point out only the necessary changes. To this end, let us recall that the order \( \text{Ord}(\mathcal{W}) \) of a cover \( \mathcal{W} \) of \( X \) is the smallest cardinal number \( \kappa \) such that \(|\{W \in \mathcal{W} : x \in W\}| \leq \kappa \), for every \( x \in X \).

**Proof of Theorem 1.3.** The implication (a) \( \implies \) (b) in Theorem 1.3 is identical to that of Theorem 1.2. The only difference is in the verification of the locally finite lifting property of the u.s.c. selection \( \theta : X \to \mathcal{C}_c(\mathcal{J}(\tau)) \), where \( k \in \mathbb{N} \). This now follows from [13, Example 3.10] because \( \mathcal{J}(\tau) \) is finite-dimensional. The inverse implication is also very similar to that of Theorem 1.2. The essential difference is about normality of \( X \). To see this, take disjoint closed sets \( A, B \subseteq X \) and define an l.s.c. mapping \( \Phi : X \to \mathcal{C}_c([-1,1]) \) by \( \Phi(x) = [-1,0] \) if \( x \in A \), \( \Phi(x) = [0,1] \) if \( x \in B \) and \( \Phi(x) = [-1,1] \) otherwise. Also, define a u.s.c. mapping \( \theta : X \to \mathcal{C}_c([-1,1]) \) by \( \theta(x) = \{-1,0\} \) if \( x \in A \), \( \theta(x) = \{0,1\} \) if \( x \in B \) and \( \theta(x) = \{0\} \) otherwise. Then \( \theta \leq \Phi \) and by (b) of Theorem 1.3, there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c([-1,1]) \) with \( \theta \leq \varphi \leq \Phi \). Since \( \varphi \) is connected-valued, this implies that \( \varphi(x) = [-1,0] \) for \( x \in A \) and \( \varphi(x) = [0,1] \) for \( x \in B \). We may now use that \([{-1,0}] \) and \([0,1] \) are two different points in the metrizable space \( \mathcal{C}([-1,1]) \). Hence, there are disjoint open sets \( \Omega_A, \Omega_B \subseteq \mathcal{C}([-1,1]) \) such that \([{-1,0}] \in \Omega_A \) and \([0,1] \in \Omega_B \). Since \( \varphi : X \to \mathcal{C}_c(Y) \subseteq \mathcal{C}(Y) \) is continuous as a usual map in this hyperspace \( \mathcal{C}([-1,1]) \), it follows that

\[
U_A = \{x \in X : \varphi(x) \in \Omega_A \} \quad \text{and} \quad U_B = \{x \in X : \varphi(x) \in \Omega_B \}
\]

are disjoint open subsets of \( X \) with \( A \subseteq U_A \) and \( B \subseteq U_B \). Accordingly, \( X \) is a normal space. The rest of the proof is identical to that of Theorem 1.2. Namely, take a locally finite closed cover \( \mathcal{T} \) of \( X \) with \(|\mathcal{T}| \leq \tau \) and \( \text{Ord}(\mathcal{T}) \leq k \) for some \( k \in \mathbb{N} \). Next, as before, define a u.s.c. mapping \( \theta : X \to \mathcal{C}(\mathcal{J}(\mathcal{T})) \) by \( \theta(x) = \{1, F \} : x \in F \in \mathcal{T} \} \), \( x \in X \), and set \( \Phi(x) = \mathcal{J}(\mathcal{T}), x \in X \). Then \( \theta : X \to \mathcal{C}_c(\mathcal{J}(\mathcal{T})) \) because \( \text{Ord}(\mathcal{T}) \leq k \). Hence, by (b) of Theorem 1.3, there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c(\mathcal{J}(\mathcal{T})) \) with \( \theta \leq \varphi \leq \Phi \). Just like in the previous proof, this implies that the existence of a locally finite open cover \( \{U_F : F \in \mathcal{T}\} \) of \( X \) such that \( F \subseteq U_F, F \in \mathcal{T} \). According to a result of Katétov [20], \( X \) is \( \tau \)-collectionwise normal.

Regarding the difference between Theorems 1.1 and 1.2, let us point out the following special case of these theorems.

**Corollary 5.3.** For a space \( X \), the following are equivalent:

(a) \( X \) is normal and countably paracompact.

(b) If \( \Phi : X \to \mathcal{F}_c(\mathcal{J}(\omega)) \) is l.s.c., \( \theta : X \to \mathcal{C}(\mathcal{J}(\omega)) \) is u.s.c. and \( \theta \leq \Phi \), then there exists a continuous mapping \( \varphi : X \to \mathcal{C}_c(\mathcal{J}(\omega)) \) such that \( \theta \leq \varphi \leq \Phi \).
Proof. To see that (a) \(\implies\) (b), let \(X\) be normal and countably paracompact, \(\Phi : X \to \mathcal{F}_c(J(\omega))\) be l.s.c. and \(\theta : X \to \mathcal{C}(J(\omega))\) be a u.s.c. selection for \(\Phi\). By Proposition 3.3, \(\{\Phi(x) : x \in X\}\) is uniformly equi-\(LC^0\). Hence, by Proposition 3.1 and Theorem 2.2, there exists a continuous mapping \(\varphi : X \to \mathcal{C}_c(J(\omega))\) with \(\theta \leq \varphi \leq \Phi\). Since (b) \(\implies\) (a) follows from Theorem 1.2, the proof is complete. \(\square\)

Regarding the role of the family \(\mathcal{C}_c(J(\tau))\) in Theorems 1.2 and 1.3, let us point out the following characterisation of \(\tau\)-PF-normal spaces.

**Theorem 5.4.** For an infinite cardinal number \(\tau\) and a space \(X\), the following conditions are equivalent:

(a) \(X\) is \(\tau\)-PF-normal.

(b) If \(\Phi : X \to \mathcal{C}_c(J(\tau))\) is l.s.c., \(\theta : X \to \mathcal{C}(J(\tau))\) is u.s.c. and \(\theta \leq \Phi\), then there exists a continuous mapping \(\varphi : X \to \mathcal{C}_c(J(\tau))\) such that \(\theta \leq \varphi \leq \Phi\).

Proof. Suppose that \(X\) is \(\tau\)-PF-normal and \(\Phi, \theta : X \to \mathcal{C}(J(\tau))\) are as in (b). Since \(\{\Phi(x) : x \in X\}\) is uniformly equi-\(LC^0\), by Proposition 4.3, the associated mapping \(\mathcal{C}_c[\theta, \Phi]\) is \(H(d)\)-s.l.s.f. and we may now apply Theorem 2.2 to get the required intermediate mapping \(\varphi : X \to \mathcal{C}_c(Y)\).

The inverse implication is also almost identical to that of (b) \(\implies\) (a) in Theorem 1.2. Namely, by Lemma 5.2, \(X\) is normal. Take a locally finite closed cover \(\mathcal{F}\) of \(X\) with \(|\mathcal{F}| \leq \tau\), and a point-finite open cover \(\{O_F : F \in \mathcal{F}\}\) such that \(F \subseteq O_F\), \(F \in \mathcal{F}\). For every \(x \in X\), let \(\mathcal{F}_x = \{F \in \mathcal{F} : x \in O_F\}\) which is a finite set because \(O_F : F \in \mathcal{F}\) is point-finite. Hence, as in the proof of Theorem 1.1, we may define an l.s.c. mapping \(\Phi : X \to \mathcal{C}_c(J(\mathcal{F}))\) by \(\Phi(x) = J(\mathcal{F}_x), x \in X\). Next, as in the proof of Theorem 1.2, define a u.s.c. selection \(\theta : X \to \mathcal{C}(J(\mathcal{F}))\) for \(\Phi\) by \(\theta(x) = \{(1, F) : x \in F \text{ and } F \in \mathcal{F}\}\), \(x \in X\). Thus, by (b), there exists a continuous mapping \(\varphi : X \to \mathcal{C}_c(J(\mathcal{F}))\) with \(\theta \leq \varphi \leq \Phi\). Finally, just like before, there exists a locally finite open cover \(\{U_F : F \in \mathcal{F}\}\) of \(X\) such that \(F \subseteq U_F \subseteq O_F, F \in \mathcal{F}\). According to [15, Theorem 3.1], \(X\) is \(\tau\)-PF-normal. \(\square\)

In the setting of arbitrary \(\mathcal{C}(Y)\)-valued l.s.c. mappings, Theorems 1.2 and 1.3 are somewhat known. Namely, it was shown in [32, Theorem 1.3] that a space \(X\) is normal and \(\tau\)-expandable if and only if for every completely metrizable space \(Y\) with \(w(Y) \leq \tau\), every l.s.c. mapping \(\Phi : X \to \mathcal{C}(Y)\) and every u.s.c. selection \(\theta : X \to \mathcal{C}(Y)\) for \(\Phi\), there are mappings \(\varphi, \psi : X \to \mathcal{C}(Y)\) such that \(\varphi\) is l.s.c., \(\psi\) is u.s.c. and \(\theta \leq \varphi \leq \psi \leq \Phi\). Such a pair \((\varphi, \psi)\) of mappings is often called a Michael pair. So, the pair \((\theta, \Phi)\) admits an intermediate Michael pair \((\varphi, \psi)\). Similarly, it was shown in [32, Theorem 1.4] that a space \(X\) is \(\tau\)-collectionwise normal if and only if for every finite-dimensional completely metrizable space \(Y\) with \(w(Y) \leq \tau\), every l.s.c. mapping \(\Phi : X \to \mathcal{C}(Y)\) and its u.s.c. selection \(\theta : X \to \mathcal{C}_k(Y)\), where \(k \in \mathbb{N}\), there exists an intermediate Michael pair \((\varphi, \psi) : X \to \mathcal{C}(Y)\) for \((\theta, \Phi)\). The results in [32] were not stated in terms of a cardinal number \(\tau \geq \omega\), but the
above characterisations follow easily from the corresponding proofs. Moreover, the characterisation in [32, Theorem 1.4] was stated in the realm of normal spaces, but in the proof of [32, Theorem 1.3] was shown that the intermediate selection property implies normality. These results can be extended to \( \tau \)-PF-normal spaces as well. In fact, all these characterisations can be obtained using the framework of this paper and [12, Theorem 5.1].

Finally, let us briefly point out that Theorems 1.2, 1.3 and 5.4 also have natural zero-dimensional versions, see Proposition 3.6. Namely, we have the following consequences of known results.

**Corollary 5.5.** A space \( X \) is \( \tau \)-expandable, normal and has \( \dim(X) = 0 \) if and only if for every completely metrizable space \( Y \) with \( w(Y) \leq \tau \), every l.s.c. mapping \( \Phi : X \to \mathcal{C}'(Y) \) and every u.s.c. selection \( \theta : X \to \mathcal{C}(Y) \) for \( \Phi \), there exists a continuous mapping \( \varphi : X \to \mathcal{C}(Y) \) with \( \theta \leq \varphi \leq \Phi \).

**Proof.** If \( X \) is a \( \tau \)-expandable normal space with \( \dim(X) = 0 \) and \( Y \) is a metrizable space with \( w(Y) \leq \tau \), then every u.s.c. mapping \( \theta : X \to \mathcal{C}(Y) \) has the locally lifting property [16, Proposition 2.2]. Hence, the insertion property follows from [16, Theorem 3.1] and Theorems 2.3 and 4.5. Conversely, as in the proof of Proposition 3.6, we may show that \( X \) is a normal space with \( \dim(X) = 0 \). Namely, take disjoint closed sets \( F_1, F_2 \subseteq X \) and, for convenience, set \( Y = \{0, 1, 2\} \). Next, define an l.s.c. mapping \( \Phi : X \to 2^Y \) by \( \Phi(x) = \{0, i\} \) if \( x \in F_i \), and \( \Phi(x) = Y \) otherwise. In contrast to the proof of Proposition 3.6, define a u.s.c. selection \( \theta : X \to 2^Y \) for \( \Phi \) by \( \theta(x) = \{0, i\} \) if \( x \in F_i \), and \( \theta(x) = \{0\} \) otherwise. Then by condition, there exists a continuous mapping \( \varphi : X \to 2^Y \) such that \( \theta \leq \varphi \leq \Phi \). In particular, \( \varphi(x) \neq \{0\} \), \( x \in F_1 \cup F_2 \), and we may define the disjoint sets \( U_i = \varphi^{-1}(\{i\}) \cap \varphi^\#(\{0, i\}) \), \( i = 1, 2 \). Since \( Y \) is discrete and \( \varphi \) is continuous, \( U_1 \) and \( U_2 \) are disjoint clopen sets with \( F_i \subseteq U_i \), \( i = 1, 2 \), so \( X \) is a normal space with \( \dim(X) = 0 \). Finally, for each completely metrizable space \( Y \) with \( w(Y) \leq \tau \), by taking \( \Phi(x) = Y \), \( x \in X \), the selection property implies that each u.s.c. mapping \( \theta : X \to \mathcal{C}(Y) \) has the locally finite lifting property being a selection for some continuous mapping \( \varphi : X \to \mathcal{C}(Y) \). Applying [16, Proposition 2.2] once again, this implies that \( X \) is also \( \tau \)-expandable. \( \square \)

**Corollary 5.6.** A space \( X \) is \( \tau \)-collectionwise normal and has \( \dim(X) = 0 \) if and only if for every finite-dimensional completely metrizable space \( Y \) with \( w(Y) \leq \tau \), every l.s.c. \( \Phi : X \to \mathcal{C}'(Y) \) and every u.s.c. selection \( \theta : X \to \mathcal{C}_k(Y) \) for \( \Phi \), where \( k \in \mathbb{N} \), there exists a continuous mapping \( \varphi : X \to \mathcal{C}(Y) \) with \( \theta \leq \varphi \leq \Phi \).

**Proof.** Apply the same proof as in Corollary 5.5, but now use [16, Proposition 2.3] instead of [16, Proposition 2.2]. \( \square \)

Similarly, using [16, Proposition 2.5] instead of [16, Propositions 2.2 and 2.3], we also get the following result in the setting of \( \tau \)-PF-normal spaces.
Corollary 5.7. A space $X$ is $\tau$-PF-normal and has $\dim(X) = 0$ if and only if for every completely metrizable space $Y$ with $w(Y) \leq \tau$, every l.s.c. mapping $\Phi : X \to C(Y)$ and every u.s.c. selection $\theta : X \to C(Y)$ for $\Phi$, there exists a continuous mapping $\varphi : X \to C(Y)$ with $\theta \leq \varphi \leq \Phi$.

References

[1] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186.
[2] M. Choban, *Many-valued mappings and Borel sets. I, II*, Tr. Mosk. Mat. Obs. **22** (1970), 229–250; ibid. 23 (1970), 277–301 (in Russian).
[3] M. Choban and S. Nedev, *Factorization theorems for set-valued mappings, set-valued selections and topological dimension*, Math. Balkanica **4** (1974), 457–460, (in Russian).
[4] D. W. Curtis, *Growth hyperspaces of Peano continua*, Trans. Amer. Math. Soc. **238** (1978), 271–283.
[5] ———, *Hyperspaces of noncompact metric spaces*, Compositio Math. **40** (1980), no. 2, 139–152.
[6] C. H. Dowker, *On countably paracompact spaces*, Canad. J. of Math. **3** (1951), 291–224.
[7] ———, *Homotopy extensions theorems*, Proc. London Math. Soc. **6** (1956), 100–116.
[8] R. Engelking, *General topology, revised and completed edition*, Heldermann Verlag, Berlin, 1989.
[9] V. Gutev, *Unified selection and factorization theorems*, C. R. Acad. Bulgare Sci. **40** (1987), no. 5, 13–15.
[10] ———, *Continuous selections between set-valued mappings*, Abstracts of International Conference on Topology, Varna, Bulgaria, (1990), p. 24.
[11] ———, *Continuous selections for continuous set-valued mappings and finite-dimensional sets*, Set-Valued Anal. **6** (1998), 149–170.
[12] ———, *Weak factorizations of continuous set-valued mappings*, Topology Appl. **102** (2000), 33–51.
[13] ———, *Generic extensions of finite-valued u.s.c. selections*, Topology Appl. **104** (2000), 101–118.
[14] ———, *Extending paired multifunctions*, Topology Appl. **159** (2012), 1187–1194.
[15] V. Gutev, H. Ohta, and K. Yamazaki, *Selections and sandwich-like properties via semi-continuous Banach-valued functions*, J. Math. Soc. Japan **55** (2003), no. 2, 499–521.
[16] V. Gutev and T. Yamauchi, *Factorising usco mappings*, Topology Appl. **159** (2012), no. 9, 2423–2433.
[17] T. Kandô, *Characterization of topological spaces by some continuous functions*, J. Math. Soc. Japan **6** (1954), 45–54.
[18] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91.
[19] ———, *Correction to “On real-valued functions in topological spaces”* (Fund. Math. **38** (1951), pp. 85–91), Fund. Math. **40** (1953), 203–205.
[20] ———, *On the extension of locally finite coverings*, Colloq. Math. **6** (1958), 145–151, (in Russian).
[21] J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36.
[22] L. L. Krajewski, *On expanding locally finite collections*, Canad. J. Math. **23** (1971), 58–68.
[23] S. Lefschetz, *Algebraic Topology*, American Mathematical Society Colloquium Publications, v. 27, American Mathematical Society, New York, 1942.
[24] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.
[25] ———, *Point-finite and locally finite coverings*, Canad. J. Math. **7** (1955), 275–279.
Selected selections theorems, Amer. Math. Monthly 63 (1956), 233–238.

Continuous selections. I, Ann. of Math. (2) 63 (1956), 361–382.

Continuous selections II, Ann. of Math. 64 (1956), 562–580.

A theorem on semi-continuous set-valued functions, Duke Math. J 26 (1959), 647–651.

A theorem of Nepomnyashchii on continuous subset-selections, Topology Appl. 142 (2004), no. 1-3, 235–244.

E. Michael and G. Pixley, A unified theorem on continuous selections, Pacific J. Math. 87 (1980), 187–188.

K. Miyazaki, Characterizations of paracompact-like properties by means of set-valued semi-continuous selections, Proc. Amer. Math. Soc. 129 (2001), 2777–2782.

K. Morita, Star-finite coverings and star-finite property, Math. Japonicae 1 (1948), 60–68.

Note on paracompactness, Proc. Japan Acad. 37 (1961), 1–3.

Paracompactness and product spaces, Fund. Math. 50 (1961/1962), 223–236.

Čech cohomology and covering dimension for topological spaces, Fund. Math. 87 (1975), no. 1, 31–52.

S. Nedev, Four theorems of E. Michael on continuous selections, B”lgar. Akad. Nauk Izv. Mat. Inst. 15 (1974), 389–393 (in Russian).

Selection and factorization theorems for set-valued mappings, Serdica Math. J. 6 (1980), no. 4, 291–317.

G. Nepomnyashchii, Continuous set-valued selections for l.s.c. mappings, Sibirsk. Mat. Zh. 26 (1985), 111–119, (in Russian; Engl. Transl. in Siberian Math. J. 26 (1986), 566–572.).

About the existence of intermediate continuous multivalued selections, vol. xiii, pp. 111–122, Latv. Gos. Univ., Riga, 1986, (in Russian).

T. Przymusiński, Collectionwise normality and absolute retracts, Fund. Math. 98 (1978), 61–73.

Properties of expandable spaces, General topology and its relations to modern analysis and algebra, III (Proc. Third Prague Topological Sympos., 1971), Academia, Prague, 1972, pp. 405–410.

U. Tašmetov, Connectedness and local connectedness of certain hyperspaces, Sibirsk. Mat. Zh. 15 (1974), 1115–1130 (in Russian).

H. Tong, Some characterizations of normal and perfectly normal spaces, Bull. Amer. Math. Soc. 54 (1948), no. 1, 65.

Some characterizations of normal and perfectly normal spaces, Duke Math. J. 19 (1952), 289–292.

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