Test of the multiquark structure of $a_1(1420)$ in strong two-body decays

Thomas Gutsche,1 Mikhail A. Ivanov,2 Jürgen G. Körner,3 Valery E. Lyubovitskij,1,4,5,6 and Kai Xu1,7

1Institut für Theoretische Physik, Universität Tübingen, Kepler Center for Astro and Particle Physics, Auf der Morgenstelle 14, D-72076, Tübingen, Germany
2Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
3PRISMA Cluster of Excellence, Institut für Physik, Johannes Gutenberg-Universität, D-55099 Mainz, Germany
4Departamento de Física y Centro Científico Tecnológico de Valparaíso-CCTVal, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
5Department of Physics, Tomsk State University, 634050 Tomsk, Russia
6Laboratory of Particle Physics, Tomsk Polytechnic University, 634050 Tomsk, Russia
7School of Physics and Center of Excellence in High Energy Physics & Astrophysics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand

We present an analysis of strong two-body decays of the $a_1(1420)$ with $J^{PC} = 1^{++}$ recently reported by the COMPASS Collaboration at CERN. Following the interpretation of the COMPASS Collaboration that the $a_1$ is an unusual state with a four-quark $q\bar{q}s\bar{s}$ structure we consider two possible configurations for this state — hadronic molecular and color diquark-antidiquark structures. We find that the dominant decay mode of the $a_1$ is the decay into $K$ and $K^*$. In particular, we calculate that the four decay modes $a_1 \to VP$ with $VP = K^{\mp}K^*$, $K^{\ast0}K^*$, $K^{*0}K^*$ together give a dominant contribution to the measured total width of about 150 MeV. The observational mode $a_1 \to f_0(980) + \pi^0$ is significantly suppressed by one order of magnitude.

PACS numbers: 13.25.Jx,14.40.Be,14.40.Rt,14.65.Bt
Keywords: quark model, confinement, exotic states, tetraquarks, decay widths

I Introduction

Recently the COMPASS Collaboration at CERN reported $a_1(1420)$ with quantum numbers $J^{PC} = 1^{++}$ recently reported by the COMPASS Collaboration at CERN. Following the interpretation of the COMPASS Collaboration that the $a_1$ is an unusual state with a four-quark $q\bar{q}s\bar{s}$ structure we consider two possible configurations for this state — hadronic molecular and color diquark-antidiquark structures. We find that the dominant decay mode of the $a_1$ is the decay into $K$ and $K^*$. In particular, we calculate that the four decay modes $a_1 \to VP$ with $VP = K^{\mp}K^*$, $K^{\ast0}K^*$, $K^{*0}K^*$ together give a dominant contribution to the measured total width of about 150 MeV. The observational mode $a_1 \to f_0(980) + \pi^0$ is significantly suppressed by one order of magnitude.

In Ref. [2] some of us applied holographic QCD to the study of the $a_1(1420)$ state. In this analysis we confirmed its four-quark structure and predicted a mass of about 1414$^{+15}_{-13}$ MeV and a width of 153$^{+8}_{-23}$ MeV this unusual state was interpreted by the collaboration as a tetraquark bound state with the quark content $q\bar{q}s\bar{s}$ ($q = u$ or $d$). The $a_1(1420)$ was observed in the $f_0(980)\pi^0$ decay channel, hinting towards the large $s\bar{s}$ component.

In Ref. [3] the $a_1(1420)$ was considered to be a peak in the $f_0(980)$ spectrum. In particular, the results of the QCD sum rule analysis of Ref. [3] showed that the axial-vector tetraquark assignment to the $a_1(1420)$ is disfavored. Instead, the author argues that the axial-vector $a_1(1420)$ is a mixed state of the $a_1(1260)$ meson and a tetraquark state with the configuration $[u\bar{u}]_{S=1}[d\bar{d}]_{S=0} + [u\bar{d}]_{S=0}[d\bar{u}]_{S=1}$. Quite differently the QCD sum rule analysis performed in Ref. [3] confirmed the existence of the $a_1(1420)$ as a tetraquark state. Questions related to the nature of the $a_1(1420)$ meson have also been addressed in other papers. In particular, it has been proposed that this state is a consequence of rescattering effects, and in fact in Ref. [3] the $a_1(1420)$ was interpreted as a dynamical effect due to a singularity (branch point) in the triangle diagram formed by the processes $a_1(1260) \to K^*K$, $K^* \to K\pi$ and $KK \to f_0(980)$. In Ref. [3] it was shown that a single $I = 1$ spin-parity $J^{PC} = 1^{++}$ $a_1$ resonance can manifest itself as two separated mass peaks. One mass peak decays into an $S$-wave $\rho\pi$ final state and the second one decays into a $P$-wave $f_0(980)\pi$ system, hence the tetraquark interpretation remains uncertain. In Ref. [3] it was claimed that a resonance such as the $a_1(1420)$ could be produced because of an “anomalous triangle singularity”, if the resonance is located in a specific kinematical region. In Ref. [3] the $a_1(1420)$ state was considered to be a peak in the $a_1(1260) \to \pi f_0(980)$ decay mode. Finally, in Ref. [3] it was proposed to test the possible rescattering nature of the $a_1(1420)$ in heavy meson decays.

In this paper we test the four-quark structure interpretation of the $a_1(1420)$ state in the study of its strong two-body decays $a_1(1420) \to f_0(980) + \pi$ and $a_1(1420) \to K^* + K$ using the covariant confined quark model (CCQM) proposed in Refs. [10, 12]. The CCQM has been successfully applied to the description of the properties of the exotic $X(3872)$, $Z_c(3900)$, $Z(4430)$, $X(5568)$, $Z_b(10610)$, $Z_b'(10650)$ states [11, 13]. Our approach is based on the use of the compositeness condition [10-13] originally formulated in terms of hadronic [10, 12] and later on in terms of quark constituents [15]. This procedure provides an opportunity to analyze both tetraquark (color diquark-antidiquark)
and hadronic molecular configurations of the four constituent quarks forming the exotic state. In Refs. [14, 15] we have shown that the four-quark picture with molecular-type interpolating currents are favored for the $Z_b(3900)$, $Z_b(10610)$ and $Z_b^*(10650)$ states. In the case of the $Z_b(10610)$ and $Z_b^*(10650)$ states there is strong experimental confirmation [15] and theoretical [21] justification that these exotic states are four-quark states with molecular-type interpolating currents. In particular, since the masses of the $Z_b^+(10610)$ and $Z_b^*(10650)$ resonances are very close to the respective $B^* B$ ($10604$ MeV) and $B^* B^*$ ($10649$ MeV) thresholds, in Ref. [21] it was also suggested that they have molecular-type binding structures.

We observe a similar manifestation of the hadronic molecular picture in the case of the $a_1$ state. We find that the dominant decay modes of the $a_1$ are the $KK^*$ channels and that the hadronic molecular configuration is preferred when compared to the color-diquark-antidiquark structure interpretation. In particular, we estimate that the four decay modes $a_1 \to V P$ with $VP = K^{\pm} K^{\mp}$, $K^{*0} K^0$, $K^{*0} K^0$ together nearly make up the total width of the $a_1$ of about $150$ MeV. The decay mode $a_1 \to f_0(980) + \pi^0$, observed by the COMPASS Collaboration, is significantly suppressed by one order of magnitude.

The paper is organized as follows. In Sec. II we discuss the basic notions of our approach for the description of the composite structure of mesons and tetraquark states: the choice of interpolating quark-antiquark and four-quark currents with the quantum numbers of the respective states, the Lagrangians that describe the coupling of the bound states with their constituents, and the choice of parameters. We give model independent formulas for the matrix elements with the quantum numbers of the respective states, and for the decays rates in the framework of our covariant quark model. In Sec. IV we discuss the numerical results obtained in our approach and compare them with available experimental data. Finally, in Sec. V we summarize our findings.

II Formalism

According to the quantum number assignment of the COMPASS Collaboration [1] we consider the new exotic state $a_1(1420)$ as an axial-vector meson state with $J^{PC} = 1^{++}$, in a minimal configuration composed of a nonstrange and strange quark-antiquark pair. For the internal structure of the $a_1(1420)$ we test two possible four-quark configurations — hadronic molecular (HM) and color-diquark-antidiquark (CD). The respective HM and CD currents for the $a_1(1420)$ state read

$$HM : J_{a_1}^\mu = \frac{1}{2} \left[ (\bar{u}^a \gamma^\mu s^a) (\bar{s}^b \gamma^5 u^b) - (\bar{u}^a \gamma^5 s^a) (\bar{s}^b \gamma^\mu u^b) - (\bar{d}^a \gamma^5 s^a) (\bar{s}^b \gamma^\mu d^b) + (\bar{d}^a \gamma^\mu s^a) (\bar{s}^b \gamma^5 d^b) \right], \tag{1}$$

$$CD : J_{a_1}^\mu = \frac{\sqrt{3}}{4} \epsilon^{abcd} \epsilon^{cde} \left[ (u^a C \gamma^\mu s^b) (\bar{u}^c \gamma^5 C s^e) + (u^a C \gamma^5 s^b) (\bar{u}^c \gamma^\mu C s^e) \right. - \left. (d^a C \gamma^\mu s^b) (\bar{d}^c \gamma^5 C s^e) - (d^a C \gamma^5 s^b) (\bar{d}^c \gamma^\mu C s^e) \right], \tag{2}$$

where $C = \gamma^0 \gamma^2$ is the charge conjugation matrix, and $a, b, c, d, e$ refer to color.

The quark currents of the other hadrons relevant in the decays are specified as follows. We consider the $f_0(980)$ also as a four-quark state with $J^{PC} = 0^{++}$, applying the two possible structures — HM and CD:

$$HM : J_{f_0} = \frac{1}{\sqrt{2}} \left[ (\bar{u}^a \gamma^5 s^a) (\bar{s}^b \gamma^\mu u^b) + (\bar{d}^a \gamma^5 s^a) (\bar{s}^b \gamma^\mu d^b) \right], \tag{3}$$

$$CD : J_{f_0} = \sqrt{3} \epsilon^{abcd} \epsilon^{cde} \left[ (u^a C \gamma^5 s^b) (\bar{u}^c \gamma^\mu C s^e) + (d^a C \gamma^\mu s^b) (\bar{d}^c \gamma^5 C s^e) \right]. \tag{4}$$

In the case of the CD currents we have added an additional factor $\sqrt{3}/2$ resulting in identical expressions for the mass operators of the four-quark states in the HM and CD versions (see below). In Table II we further specify the $J^P$ quantum numbers and the interpolating currents of the nonexotic quark-antiquark mesons $\pi^0$, $K$ and $K^*$ that appear in the final decay channel.

In the following we discuss the spin kinematics for the two decay modes $1^+ \to 0^+ + 0^-$ ($a_1(1420) \to f_0(980) + \pi^0$) and $1^+ \to 1^- + 0^-$ ($a_1(1420) \to K^{*\pm} + K^0$, $a_1(1420) \to K^{*0}(K^{*0}) + K^0(K^0)$):

- The decay $1^+ \to 0^+ + 0^-$,

the 4-momenta and the Lorentz index of the polarization four-vector of the $a_1(1420)$ state are labeled in the transition matrix element as

$$M = \langle 0^+(q_1), 0^- (q_2) | T | 1^+(p; \mu) \rangle. \tag{5}$$
The product of the intrinsic parities of the two final states meson is \((-1)\) which is opposite to the parity to the initial state \((+1)\). Therefore the two final states mesons must have odd relative orbital momenta, which in the present case must be \(L = 1\). The spins \(s_1 = 0\) and \(s_2 = 0\) of the two final state mesons couple to the total spin \(S = 0\). Thus one has a single \((LS)\) amplitude with \((L = 1, S = 0)\). The covariant transition matrix element in terms of this amplitude is given by

\[
M = A q_1^\mu \varepsilon_\mu ,
\]

where \(\varepsilon_\mu\) is the polarization vector of the \(a_1(1420)\) state. The amplitude \(A\) is related to the helicity amplitude \(H_{\lambda=0}\) by

\[
H_0 = -A |q_1| ,
\]

where the particles of the initial and final states are on their mass-shells with \(p^2 = M^2\), \(q_1^2 = M_1^2\), \(q_2^2 = M_2^2\) and \(p^\mu \varepsilon_\mu = 0\); \(|q_1| = \lambda^{1/2}(M^2, M_1^2, M_2^2)/2M\) is the magnitude of the final state three-momentum in the rest frame of the initial particle. Here \(\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\) is the Källen kinematical triangle function.

The decay rate of \(1^+(p) \rightarrow 0^+(q_1) + 0^-(q_2)\) finally reads

\[
\Gamma = \frac{|q_1|^3}{24\pi M^2} A^2 = \frac{|q_1|}{24\pi M^2} |H_0|^2 .
\]

- The decay \(1^+ \rightarrow 1^- + 0^-\), momenta and Lorentz indices of the polarization four-vectors in the decay are labeled according to the transition matrix element

\[
M = (1^- (q_1; \delta), 0^- (q_2)) T |1^+ (p; \mu)\rangle.
\]

Parity conservation implies even relative orbital momenta in the final state with \(L = 0, 2\). The spins \(s_1 = 1\) and \(s_2 = 0\) of the two final state mesons couple to the total spin \(S = 1\). Thus one has the two \((LS)\) amplitudes \((L = 0, S = 1)\) and \((L = 2, S = 1)\). The covariant expansion of the transition matrix is then given by

\[
M = (B g^\mu \gamma^\delta + C q_1^\mu q_2^\delta) \varepsilon_\mu \varepsilon^{*}_\delta ,
\]

with \(p^\mu \varepsilon_\mu = 0\) and \(q_1^\mu q_2^\mu = 0\). Alternatively one may describe the transition amplitude by the helicity amplitudes \(H_{\lambda\lambda_1}\) which can be expressed as a linear superposition of the invariant amplitudes \(B\) and \(C\). One has

\[
H_{00} = -\frac{E_1}{M_1} B - \frac{M}{M_1} |q_1|^2 C, \quad H_{+1+1} = H_{-1-1} = -B .
\]

The decay rate of \(1^+(p) \rightarrow 1^- (q_1) + 0^- (q_2)\) finally reads

\[
\Gamma = \frac{|q_1|}{24\pi M^2} \left\{ \left( 3 + \frac{|q_1|^2}{M_1^2} \right) B^2 + (M^2 + M_1^2 - M_2^2) \frac{|q_1|^2}{M_1^2} BC + \frac{M^2}{M_1^2} |q_1|^4 C^2 \right\}
\]

or

\[
\Gamma = \frac{|q_1|}{24\pi M^2} \left\{ |H_{+1+1}|^2 + |H_{-1-1}|^2 + |H_{00}|^2 \right\} .
\]

| name | \(J^P\) | quark current | mass (MeV) |
|------|--------|--------------|------------|
| \(\pi^0\) | 0\(^-\) | \(j_{\pi^0} = \frac{1}{2}(\bar{u}i\gamma^5 u - \bar{d}i\gamma^5 d)\) | 143.977 ± 0.0005 |
| \(K^+\) | 0\(^-\) | \(j_{K^+} = \bar{s}i\gamma^5 u\) | 493.677 ± 0.016 |
| \(K^-\) | 0\(^-\) | \(j_{K^-} = \bar{u}i\gamma^5 s\) | 493.677 ± 0.016 |
| \(K^0\) | 0\(^-\) | \(j_{K^0} = \bar{s}i\gamma^5 d\) | 497.611 ± 0.013 |
| \(\bar{K}^0\) | 0\(^-\) | \(j_{\bar{K}^0} = \bar{d}i\gamma^5 s\) | 497.611 ± 0.013 |
| \(K^{*+}\) | 1\(^-\) | \(j_{K^{*+}}^\mu = \bar{s}\gamma^\mu u\) | 891.76 ± 0.25 |
| \(K^{*-}\) | 1\(^-\) | \(j_{K^{*-}}^\mu = \bar{u}\gamma^\mu s\) | 891.76 ± 0.25 |
| \(K^{*0}\) | 1\(^-\) | \(j_{K^{*0}}^\mu = \bar{s}\gamma^\mu d\) | 895.55 ± 0.20 |
| \(K^{*0}\) | 1\(^-\) | \(j_{K^{*0}}^\mu = \bar{d}\gamma^\mu s\) | 895.55 ± 0.20 |
III Strong two-body decays of \( a_1(1420) \) in the covariant quark model

The nonlocal versions of the quark-antiquark currents of the \( \pi^0 \), \( K \) and \( K^* \) mesons written down in Table II are given by

\[
\begin{align*}
\pi^0: \quad J_{\pi^0}(x) &= \int dy \Phi_{\pi^0}(y^2) J_{\pi^0}(x, y), \\
J_{\pi^0}(x, y) &= \frac{1}{\sqrt{2}} \left\{ \bar{u}^a(x + y/2)i\gamma^5u^a(x - y/2) - \bar{d}^a(x + y/2)i\gamma^5d^a(x - y/2) \right\}, \\
K: \quad J_K(x) &= \int dy \Phi_K(y^2) J_K(x, y), \\
J_{K+}(x, y) &= \bar{s}^a(x + \hat{w}y)i\gamma^5u^a(x - \hat{w}s,y), \\
J_{K-}(x, y) &= \bar{u}^a(x + \hat{w}s y)i\gamma^5s^a(x - \hat{w}y), \\
J_{K0}(x, y) &= \bar{s}^a(x + \hat{w}y)i\gamma^5d^a(x - \hat{w}s,y), \\
J_{K0}(x, y) &= \bar{d}^a(x + \hat{w}s y)i\gamma^5s^a(x - \hat{w}y), \\
K^*: \quad J_{K^*}^\mu(x) &= \int dy \Phi_{K^*}(y^2) J_{K^*}^\mu(x, y), \\
J_{K^*+}^\mu(x, y) &= \bar{s}^a(x + \hat{w}y)\gamma^\mu u^a(x - \hat{w}s,y), \\
J_{K^*-}^\mu(x, y) &= \bar{u}^a(x + \hat{w}s y)\gamma^\mu s^a(x - \hat{w}y), \\
J_{K^*0}^\mu(x, y) &= \bar{s}^a(x + \hat{w}y)\gamma^\mu d^a(x - \hat{w}s,y), \\
J_{K^*0}^\mu(x, y) &= \bar{d}^a(x + \hat{w}s y)\gamma^\mu s^a(x - \hat{w}y),
\end{align*}
\]

where \( \hat{w} = m_\ell/(m + m_s) \) and \( \hat{w} = m/(m + m_s) \) are the fractions of the masses of nonstrange \( m = m_u = m_d \) (we work in the isospin limit) and strange \( m_s \) quark obeying the condition \( \hat{w} + \hat{w}_s = 1 \). The \( \Phi_{\pi^0}(y^2) \), \( \Phi_K(y^2) \), \( \Phi_{K^*}(y^2) \) denote the set of vertex functions of the \( \pi^0 \), \( K \), and \( K^* \) states, respectively.

The nonlocal extensions of the four-quark currents of the \( a_1(1420) \) and \( f_0(980) \) states written down in Eqs. (I)-(II) are given by

\[
\begin{align*}
a_1(1420), \text{ HM:} \quad J_{a_1;HM}^\mu(x_1, \ldots, x_4) &= \int dx_1 \ldots dx_4 \delta \left( x - \sum_{i=1}^4 \frac{w_i x_i}{\Phi_{a_1} \left( \sum_{i<j} (x_i - x_j)^2 \right)} \right) J_{a_1;HM}^\mu(x_1, \ldots, x_4), \quad (16) \\
J_{a_1;HM}^\mu(x_1, \ldots, x_4) &= \frac{1}{2} \left\{ (\bar{u}^a(x_3)\gamma^\mu s^a(x_1))(\bar{s}^b(x_2)\gamma^5 u^b(x_4)) - (\bar{u}^a(x_3)\gamma^5 s^a(x_1))(\bar{s}^b(x_2)\gamma^\mu u^b(x_4)) \right. \\
&\quad \left. - (\bar{d}^a(x_3)\gamma^\mu s^a(x_1))(\bar{s}^b(x_2)\gamma^5 d^b(x_4)) + (\bar{d}^a(x_3)\gamma^5 s^a(x_1))(\bar{s}^b(x_2)\gamma^\mu d^b(x_4)) \right\}, \\
f_0(980), \text{ HM:} \quad J_{f_0;HM}(x_1, \ldots, x_4) &= \int dx_1 \ldots dx_4 \delta \left( x - \sum_{i=1}^4 \frac{w_i x_i}{\Phi_{f_0} \left( \sum_{i<j} (x_i - x_j)^2 \right)} \right) J_{f_0;HM}(x_1, \ldots, x_4), \quad (17) \\
J_{f_0;HM}(x_1, \ldots, x_4) &= \frac{1}{\sqrt{2}} \left\{ (\bar{u}^a(x_3)\gamma^5 s^a(x_1))(\bar{s}^b(x_2)\gamma^5 u^b(x_4)) + (\bar{d}^a(x_3)\gamma^5 s^a(x_1))(\bar{s}^b(x_2)\gamma^5 d^b(x_4)) \right\}, \\
a_1(1420), \text{ CD:} \quad J_{a_1;CD}^\mu(x) &= \int dx_1 \ldots dx_4 \delta \left( x - \sum_{i=1}^4 \frac{w_i x_i}{\Phi_{a_1} \left( \sum_{i<j} (x_i - x_j)^2 \right)} \right) J_{a_1;CD}^\mu(x_1, \ldots, x_4), \quad (18) \\
J_{a_1;CD}^\mu(x_1, \ldots, x_4) &= \frac{\sqrt{3}}{4} \epsilon^{a b c d} \epsilon^{e c d} \left\{ (u^a(x_4)C^\gamma^5 s^b(x_1))\bar{u}^c(x_3)\gamma^5 C s^e(x_2) \right. \\
&\quad + (u^a(x_4)C^\gamma^5 s^b(x_1))\bar{u}^c(x_3)\gamma^5 C s^e(x_2) \\
&\quad - (d^a(x_4)C^\gamma^5 s^b(x_1))\bar{d}^c(x_3)\gamma^5 C s^e(x_2) \\
&\quad - (d^a(x_4)C^\gamma^5 s^b(x_1))\bar{d}^c(x_3)\gamma^5 C s^e(x_2) \right\},
\end{align*}
\]
where \( w_i = m_i / \sum_{i=1}^{4} m_i \) is the fraction of constituent masses in the case of four-quark states. Again, the vertex functions of \( a_1 \) and \( f_0 \) are denoted by \( \Phi_H, H = f_0, a_1 \).

The effective interaction Lagrangians describing the coupling of the \( \pi^0, K, K^*, a_1, f_0 \) states to their constituent quarks is written in the form

\[
\mathcal{L}_{\text{int}, \pi^0}(x) = g_{\pi^0} \pi^0(x) J_{\pi^0}(x),
\]

\[
\mathcal{L}_{\text{int}, K}(x) = g_K K(x) J_K(x),
\]

\[
\mathcal{L}_{\text{int}, K^*}(x) = g_{K^*} K^*(x) J_{K^*}(x),
\]

\[
\mathcal{L}_{\text{int}, a_1}(x) = g_{a_1} a_1(x) J_{a_1}(x),
\]

\[
\mathcal{L}_{\text{int}, f_0}(x) = g_{f_0} f_0(x) J_{f_0}(x).
\]

The coupling constants \( g_H, H = \pi^0, K, K^*, a_1, f_0 \) in Eqs. \([20]-[24]\) are determined by the normalization condition called the compositeness condition \([16]-[18]\)

\[
Z_H = 1 - g_H^2 \Pi_H(M_H^2) = 0,
\]

where \( \Pi_H(p^2) \) is the mass operator for the \( \pi^0, K, f_0 \) mesons and the scalar part of the \( K^* \) and \( a_1 \) mass operator

\[
\Pi_H^{\mu\nu}(p) = g^{\mu\nu} \Pi_H(p^2) + p^\mu p^\nu \Pi_H^{(1)}(p^2),
\]

\[
\Pi_H(p^2) = \frac{1}{3} \left( g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) \Pi_H^{\mu\nu}(p).
\]

The Fourier-transforms of the \( \pi^0, K, K^*, a_1, f_0 \) mass operators are given by

\[
\Pi_{\pi^0}(p) = \frac{3}{(2\pi)^4} \int d^4 k_i \Phi_{\pi^0}(-k^2) \text{tr} \left[ \gamma_5 S(k + p/2) \gamma_5 S(k - p/2) \right],
\]

\[
\Pi_K(p) = \frac{3}{(2\pi)^4} \int d^4 k_i \Phi_K(-k^2) \text{tr} \left[ \gamma_5 S(k + pw) \gamma_5 S_k(k - pw) \right],
\]

\[
\Pi_{K^*}(p) = \frac{3}{(2\pi)^4} \int d^4 k_i \Phi_{K^*}(-k^2) \text{tr} \left[ \gamma_5 S(k + pw) \gamma_5 S_k(k - pw) \right],
\]

and for the \( a_1 \) as

\[
\Pi_{a_1}(p) = \frac{9}{2} \prod_{i=1}^{3} \int \frac{d^4 k_i}{(2\pi)^4} \Phi_{a_1}(-\omega^2)
\]

\[
\times \left\{ \text{tr} \left[ \gamma_5 S_1(\hat{k}_1) \gamma_5 S_3(\hat{k}_3) \right] \text{tr} \left[ \gamma_5 S_4(\hat{k}_4) \gamma_5 S_2(\hat{k}_2) \right] 
+ \text{tr} \left[ \gamma_5 S_1(\hat{k}_1) \gamma_5 S_2(\hat{k}_2) \right] \text{tr} \left[ \gamma_5 S_4(\hat{k}_4) \gamma_5 S_3(\hat{k}_3) \right] \right\}
\]

and for the \( f_0 \)

\[
\Pi_{f_0}(p) = 9 \prod_{i=1}^{3} \int \frac{d^4 k_i}{(2\pi)^4} \Phi_{f_0}(-\omega^2)
\]

\[
\times \text{tr} \left[ \gamma_5 S_1(\hat{k}_1) \gamma_5 S_3(\hat{k}_3) \right] \text{tr} \left[ \gamma_5 S_4(\hat{k}_4) \gamma_5 S_2(\hat{k}_2) \right].
\]
In previous equations we use the notation
\[\hat{k}_1 = k_1 - pw_1, \quad \hat{k}_2 = k_2 - pw_2, \quad \hat{k}_3 = k_3 + pw_3, \quad \hat{k}_4 = k_1 + k_2 - k_3 + pw_4,\]
\[\omega^2 = \frac{1}{2} \left( k_1^2 + k_2^2 + k_3^2 + k_1 k_2 - k_1 k_3 - k_2 k_3 \right)\]
(32)
and \(S_i(k) = 1/(m_i - k)\) is the free quark propagator with constituent mass \(m_i\). In particular, \(S(k)\) and \(S_a(k)\) are the nonstrange and strange quark propagators. \(\hat{\Phi}_H(-k^2) = \exp(k^2/\Lambda_H^2)\) is the Fourier-transform of the vertex function of the hadron \(H\), where \(\Lambda_H\) is the hadronic size parameter. Note that the mass operators of the four-quark states \(a_1\) and \(f_0\) are formally identical for both the HM and CD currents.

Next we list the matrix elements of the two-body decays of the \(a_1(1420)\) state. In case of the transition \(a_1(1420) \to f_0(980) + \pi^0\) there are four cases depending on the structure assumptions: (1) \(a_1\) and \(f_0\) are the HM states (HM \(\to\) HM transition), (2) \(a_1\) and \(f_0\) are the CD states (CD \(\to\) CD transition), (3) \(a_1\) is the HM and \(f_0\) is the CD state (HM \(\to\) CD transition), (4) \(a_1\) is the CD and \(f_0\) is the HM state (CD \(\to\) HM transition). The matrix elements for the HM \(\to\) HM and CD \(\to\) CD transitions are apart from the spatial correlations contained in the vertex function identical and are given by
\[
M^{(1)}(a_1(p, \mu) \to f_0(q_1) + \pi^0(q_2)) = 9 g_{a_1} g_{f_0} g_{\pi^0} \\
\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \Phi_{a_1}(-\omega_{a_1}^2) \Phi_{f_0}(-\omega_{f_0}^2) \Phi_{\pi^0}(-\omega_{\pi^0}^2) \\
\times \text{tr} \left[ 3^5 S_3(\hat{k}_4) S_3(\hat{k}_3) S_3(\hat{k}_1) S_1(\hat{k}_1) S_1(\hat{k}_2) \right] \quad \text{tr} \left[ 3^5 \hat{S}_4(\hat{k}_4) \hat{S}_2(\hat{k}_2) \right] \\
= A_{a_1 f_0 \pi}^{(1)} q^\mu.
\]
(33)
The matrix elements for the HM \(\to\) CD and CD \(\to\) HM transitions are also the same and are given by
\[
M^{(2)}(a_1(p, \mu) \to f_0(q_1) + \pi^0(q_2)) = -3\sqrt{3} g_{a_1} g_{f_0} g_{\pi^0} \\
\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \Phi_{a_1}(-\omega_{a_1}^2) \Phi_{f_0}(-\omega_{f_0}^2) \Phi_{\pi^0}(-\omega_{\pi^0}^2) \\
\times \left\{ \text{tr} \left[ 3^5 S_3(\hat{k}_4) S_3(\hat{k}_3) S_3(\hat{k}_1) S_1(\hat{k}_1) S_4(\hat{k}_4) S_2(\hat{k}_2) \right] \right\} \\
+ \text{tr} \left[ 3^5 \hat{S}_4(\hat{k}_4) S_3(\hat{k}_3) S_3(\hat{k}_1) S_1(\hat{k}_1) S_4(\hat{k}_4) S_2(\hat{k}_2) \right] \\
= A_{a_1 f_0 \pi}^{(2)} q^\mu.
\]
(34)
In both Eqs. (33) and (34) we use the same notation as in the expression for the mass operators with the additional quantities
\[\hat{k}_3' = \hat{k}_3 - q_2,\]
(35)
and
\[
\omega_{a_1}^2 = \frac{1}{2} \left( k_1^2 + k_2^2 + k_3^2 + k_1 k_2 - k_1 k_3 - k_2 k_3 \right), \]
\[
\omega_{f_0}^2 = \frac{1}{2} \left( k_1^2 + k_2^2 + k_3^2 + k_1 k_2 - k_1 k_3 - k_2 k_3 \right), \]
\[
\omega_{\pi^0}^2 = \left( k_3 + pw_3 - \frac{q_2^2}{2} \right)^2.
\]
(36)
The matrix elements for the \(a_1(1420) \to K^* + K\) transitions obey the following conditions due to charge conjugation symmetry
\[
M^{\mu \delta}(a_1(p, \mu) \to K^{*+}(q_1, \delta) + K^-(q_2)) = -M^{\mu \delta}(a_1(p, \mu) \to K^{*-}(q_1, \delta) + K^+(q_2)), \]
\[
M^{\mu \delta}(a_1(p, \mu) \to K^{*0}(q_1, \delta) + \bar{K}^0(q_2)) = -M^{\mu \delta}(a_1(p, \mu) \to K^{*0}(q_1, \delta) + K^0(q_2)).
\]
(37)
The matrix elements of the decays $a_1 \to K^{*+} + K^-$ and $a_1 \to K^{*0} + \bar{K}^0$ read

\[
HM : \quad M^{\mu\delta} (a_1(p, \mu) \to K^{*+}(q_1, \delta) + K^-(q_2)) = \frac{9}{2} g_{a_1} g_{K^{*+}+g_{K^-}} \\
\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \bar{\Phi}_a (\delta_\delta^a) \Phi_{K^*} (\delta_{K^*}) \Phi_K (\delta_K) \\
\times \text{tr} [\gamma^\mu S_1(k_1) \gamma^K S_3(k_1 + q_1)] \text{tr} [\gamma_5 S_2(k_2) \gamma_5 S_4(k_2 + q_2)] \\
= B_{a_1K^{*+}K^-} g^{\mu\delta} + C_{a_1K^{*+}K^-} q_1^\mu q_2^\delta,
\]

\[\text{(38)}\]

\[
CD : \quad M^{\mu\delta} (a_1(p, \mu) \to K^{*0}(q_1, \delta) + \bar{K}^0(q_2)) = \frac{3\sqrt{3}}{2} g_{a_1} g_{K^{*0}+g_{K^0}} \\
\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \bar{\Phi}_a (\delta_\delta^a) \Phi_{K^*} (\delta_{K^*}) \Phi_K (\delta_K) \\
\times \left\{ \text{tr} [\gamma^K S_1(k_1) \gamma^K S_3(k_1 + q_1)] \text{tr} [\gamma^K S_2(k_2) \gamma^K S_4(k_2 + q_2)] \\
+ \text{tr} [\gamma^K S_1(k_1) \gamma^K S_3(k_1 + q_1)] \text{tr} [\gamma^K S_2(k_2) \gamma^K S_4(k_2 + q_4)] \right\} \\
= B_{a_1K^{*0}K^0} g^{\mu\delta} + C_{a_1K^{*0}K^0} q_1^\mu q_2^\delta,
\]

\[\text{(40)}\]

\[
\;
\]

where

\[
\delta_\delta^a = \frac{1}{8} \left( k_1 - k_2 + q_1 (w_1 - w_2) + q_2 (w_1 - w_2) \right)^2 \\
+ \frac{1}{8} \left( k_1 - k_2 + q_1 (1 + w_4 - w_3) + q_2 (-1 + w_4 - w_3) \right)^2 \\
+ \frac{1}{4} \left( k_1 + k_2 + q_1 (1 + w_1 + w_2) + q_2 (1 + w_1 + w_2) \right)^2,
\]

\[
\delta_{K^*} = (k_1 + q_1 \bar{w}_1)^2, \quad \bar{w}_1 = \frac{m_1}{m_1 + m_3},
\]

\[
\delta_K^2 = (k_2 + q_2 \bar{w}_2)^2, \quad \bar{w}_2 = \frac{m_2}{m_2 + m_4}.
\]

\[\text{(42)}\]

The quark masses are specified as $m_1 = m_2 = m_s$, $m_3 = m_4 = m_d = m_u$. 
IV Numerical results

First of all, most of the adjustable parameters of our model (constituent quark masses $m_u = m_d$ and $m_s$, infrared cutoff $\lambda$ and size parameters $\Lambda_\pi$, $\Lambda_K$, and $\Lambda_{K^*}$ of the $\pi$, $K$, and $K^*$ mesons) have been fixed in previous studies (see, e.g., Refs. [21, 22]) by a global fit to a multitude of data [23]:

$$m_u = m_d = 241.29 \text{ MeV}, \quad m_s = 428.20 \text{ MeV}, \quad \lambda = 181 \text{ MeV},$$
$$\Lambda_\pi = 870.77 \text{ MeV}, \quad \Lambda_K = 1014.20 \text{ MeV}, \quad \Lambda_{K^*} = 804.82 \text{ MeV}.$$  \hspace{1cm} (43)

For the hadronic masses in our calculations we use the central data values [23] with:

$$M_{\pi^0} = 134.9766 \text{ MeV}, \quad M_{f_0(980)} = 990 \text{ MeV}, \quad M_{a_1(1420)} = 1414 \text{ MeV},$$
$$M_{K^+} = 493.677 \text{ MeV}, \quad M_{K^0} = M_{\bar{K}^0} = 497.611 \text{ MeV},$$
$$M_{K^{*+}} = 891.76 \text{ MeV}, \quad M_{K^{*0}} = M_{\bar{K}^{*0}} = 895.55 \text{ MeV}.$$  \hspace{1cm} (44)

The only two new parameters are the size parameters of the $a_1(1420)$ and $f_0(980)$ states $\Lambda_{a_1}$ and $\Lambda_{f_0}$. According to our experience in the description of light hadronic systems they should be of the order of 1 GeV.

First, we analyze the decay mode $a_1(1420) \rightarrow f_0(980) + \pi^0$. Since this decay proceeds in a $p$-wave in the final state, this mode is suppressed by a factor of $|q_1|^{2L+1}$ with $L = 1$ where $q_1$ is the three-momentum in the decay channel. We remind the reader that the corresponding decay width is given by $\Gamma = |q_1|^3/(24\pi M^2) A^2$, where $|q_1|^3/(24\pi M^2) \simeq 0.26$ MeV. To obtain a total width of $\Gamma_{a_1} \sim 100$ MeV the dimensionless amplitude $A$ (which in fact is an effective $a_1|f_0|\pi^0$ coupling) should be relatively large, i.e. of the order of 20. Varying the parameters $\Lambda_{a_1}$ and $\Lambda_{f_0}$ from 0.5 to 1.6 GeV we find that the decay width of $a_1 \rightarrow f_0 + \pi^0$ changes in the cases where both $a_1$ and $f_0$ are hadronic molecular or color diquark-antidiquark four quark states as $\Gamma(a_1 \rightarrow f_0 + \pi^0) = 5.1^{+1.0}_{-1.5}$ MeV. The central value corresponds to the case $\Lambda_{a_1} = \Lambda_{f_0} = 1$ GeV, while the maximal value results in $\Lambda_{a_1} = \Lambda_{f_0} = 0.5$ GeV, and for the minimal one in $\Lambda_{a_1} = 0.5$ GeV and $\Lambda_{f_0} = 1.6$ GeV. In the case when one of these two states has a hadronic molecular structure and the other — color diquark-antidiquark — the result for the decay width reads $\Gamma(a_1 \rightarrow f_0 + \pi^0) = 1.2^{+1.6}_{-1.1}$ MeV. For transparency, in Fig. 1 we present three-dimensional plots for the decays rates $\Gamma(a_1 \rightarrow f_0 + \pi^0)$ as function of the dimensional parameters $\Lambda_{a_1}$ and $\Lambda_{f_0}$ running from 0.5 to 1.6 GeV for HM (CD) → HM (CD) transition (upper plot) and HM (CD) → CD (HM) transition (lower plot).

Next we look at the results for the $a_1 \rightarrow K^* + K$ decays. Even without doing an explicit calculation it is clear that the rate for the color diquark-antidiquark configuration should be suppressed in comparison with the hadronic molecular configuration. This expectation is supported by our experience in the study of decay properties of the heavy four-quark states $Z_c$ and $Z_b$ [14, 15]. Our calculations for the $a_1 \rightarrow K^* + K$ decay rate also confirm this expectation. The rates in the color diquark-antidiquark scenario are suppressed by one order of magnitude in comparison with the hadronic molecular scenario. We found that results for the partial $a_1 \rightarrow K^* + K$ decay rates are very sensitive to a choice of the size parameter $\Lambda_{a_1}$ and decrease when the size parameter increases. Second, the results for the HM interpretation for $a_1(1420)$ are overestimated in the region of $\Lambda_{a_1}$ from 0.5 to 1.5 GeV. For convenience, we show our numerical results in the two regions — from 0.5 to 1.5 GeV and 1.5 to 1.6 GeV. The results for some specific values of the size parameter $\Lambda_{a_1} = 0.5 - 1.5$ GeV and $\Lambda_{a_1} = 1.5 - 1.6$ GeV are shown in Tables III and IV respectively. We display results for the hadronic molecular scenario of the $a_1$ state, while the results in case of the color diquark-antidiquark configuration are shown in brackets. We present the results for four partial decay modes and for the sum of these four modes (last column). For a size parameter of $\Lambda_{a_1} = 1.56 - 1.58$ GeV we are able to describe the data for the total width of the $a_1(1420)$ assuming that the four modes $a_1 \rightarrow VP$ with $VP = K^{*\pm}K^\mp$, $K^{*0}\bar{K}^0$, $K^{*0}\bar{K}^0$ make up the dominant part of the total width of the $a_1$, i.e. about 150 MeV (as measured by the COMPASS Collaboration [1]). If the same value of $\Lambda_{a_1}$ is used in the evaluation of the decays $a_1(1420)$ into $f_0(980)\pi^0$ and $K^*K$, the $a_1(1420) \rightarrow f_0(980) + \pi^0$ decay width is suppressed in comparison with the $a_1(1420) \rightarrow K^* + K$ mode by one order of magnitude.

Finally, we would like to discuss how the obtained results may be used for interpretation of the observed $a_1(1420)$ meson which is supposed to be $q\bar{q}s\bar{s}$ state. As was pointed out in the review [24] there is no one to one correspondence between the current and the state since the CD current can be rewritten in terms of a sum over molecular type currents through the Fierz transformation. However, they have shown that the molecular components appear with the color and Dirac suppression factors. This means that if the physical state is a molecular state, it would be best to choose the HM current, and vice versa, for a tetraquark state it would be better to choose a tetraquark current. In some sense we follow this strategy. We are using both the HM and CD currents to evaluate the observed decay widths. Then we simply choose the current which is more suitable from the point of view of the experimental data.
FIG. 1: $\Gamma(a_1 \rightarrow f_0 + \pi^0)$ as function of the dimensional parameters $\Lambda_{a_1}$ and $\Lambda_{f_0}$ running from 0.5 to 1.6 GeV for HM (CD) → HM (CD) transition (upper plot) and HM (CD) → CD (HM) transition (lower plot).

### Table II: $a_1(1420) \rightarrow K^* + K$ decay widths (in MeV) for $\Lambda_{a_1} = 0.5 - 1.5$ GeV. Results for the HM structure of the $a_1$ are given first while the ones for the CD structure assumption are quoted in brackets.

| $\Lambda_{a_1}$ (GeV) | $\Gamma(a_1 \rightarrow K^{*+} + K^-)$ (MeV) | $\Gamma(a_1 \rightarrow K^{*0}(K^+) + K'^0(K'^-))$ (MeV) | $\Gamma_{\text{sum}}(a_1 \rightarrow K^* + K)$ (MeV) |
|-----------------------|--------------------------------------------|----------------------------------------------------------|------------------------------------------------|
| 0.50                  | 1004.86 (143.57)                           | 880.54 (125.14)                                          | 3770.80 (537.42)                                     |
| 0.75                  | 539.84 (69.53)                             | 468.02 (60.04)                                           | 2015.72 (259.14)                                     |
| 1.00                  | 242.90 (29.80)                             | 209.67 (25.64)                                           | 905.14 (110.88)                                     |
| 1.25                  | 107.25 (12.82)                             | 92.37 (11.01)                                            | 399.24 (47.66)                                     |
| 1.50                  | 49.30 (5.81)                               | 42.40 (4.98)                                             | 183.40 (21.58)                                     |

### Table III: $a_1(1420) \rightarrow K^* + K$ decay widths (in MeV) for $\Lambda_{a_1} = 1.5 - 1.6$ GeV. Results for the HM structure of the $a_1$ are given first while the ones for the CD structure assumption are quoted in brackets.

| $\Lambda_{a_1}$ (GeV) | $\Gamma(a_1 \rightarrow K^{*+} + K^-)$ (MeV) | $\Gamma(a_1 \rightarrow K^{*0}(K^+) + K'^0(K'^-))$ (MeV) | $\Gamma_{\text{sum}}(a_1 \rightarrow K^* + K)$ (MeV) |
|-----------------------|--------------------------------------------|----------------------------------------------------------|------------------------------------------------|
| 1.50                  | 49.30 (5.81)                               | 42.40 (4.98)                                             | 183.40 (21.58)                                     |
| 1.51                  | 47.84 (5.63)                               | 41.15 (4.83)                                             | 177.98 (20.92)                                     |
| 1.52                  | 46.43 (5.44)                               | 39.94 (4.67)                                             | 172.74 (20.22)                                     |
| 1.53                  | 45.07 (5.30)                               | 38.77 (4.55)                                             | 167.68 (19.70)                                     |
| 1.54                  | 43.76 (5.14)                               | 37.63 (4.41)                                             | 162.78 (19.10)                                     |
| 1.55                  | 42.48 (4.99)                               | 36.54 (4.28)                                             | 158.04 (18.54)                                     |
| 1.56                  | 41.25 (4.85)                               | 35.47 (4.15)                                             | 153.44 (18.00)                                     |
| 1.57                  | 40.05 (4.70)                               | 34.45 (4.03)                                             | 149.00 (17.46)                                     |
| 1.58                  | 38.90 (4.57)                               | 33.45 (3.91)                                             | 144.70 (16.96)                                     |
| 1.59                  | 37.78 (4.43)                               | 32.49 (3.80)                                             | 140.54 (16.46)                                     |
| 1.60                  | 36.70 (4.30)                               | 31.55 (3.69)                                             | 136.50 (15.98)                                     |

### V Summary

We have tested the possible four-quark configuration of the $a_1(1420)$ state by studying its strong two-body decay rates for the modes $a_1(1420) \rightarrow f_0(980) + \pi^0$ and $a_1(1420) \rightarrow K^* + K$. For both four-quark states $a_1(1420)$ and $f_0(980)$ we have considered two possible structure scenarios — the hadronic molecular and the color diquark-antidiquark current structure. We have found that the $a_1(1420) \rightarrow f_0(980) + \pi^0$ decay width is significantly suppressed in comparison to the $a_1(1420) \rightarrow K^* + K$ mode by one order of magnitude. Studying the decay $a_1(1420) \rightarrow K^* + K$ and using data of the COMPASS Collaboration \[1\] we have shown that the hadronic molecular configuration is preferred when compared to the compact colored diquark-antidiquark state. In particular, for values of the size parameter $\Lambda_{a_1} = 1.56 - 1.58$ GeV we are able to describe the available data for the total decay width of the $a_1$ state in terms of the decay modes $a_1(1420) \rightarrow K^* + K$ alone: $\Gamma_{\text{sum}}(a_1 \rightarrow K^* + K) = 144.70 - 153.44$ MeV.
Acknowledgments

This work was supported by the German Bundesministerium für Bildung und Forschung (BMBF) under Project 05P2015 - ALICE at High Rate (BMBF-FSP 202): “Jet- and fragmentation processes at ALICE and the parton structure of nuclei and structure of heavy hadrons”, by CONICYT (Chile) PIA/Basal FB0821, by Tomsk State University Competitiveness Improvement Program and the Russian Federation program “Nauka” (Contract No. 0.1764.GZB.2017), by Tomsk Polytechnic University Competitiveness Enhancement Program (Grant No. VIU-FTI-72/2017). M.A.I. acknowledges the support from PRISMA cluster of excellence (Mainz Univ.). M.A.I. and J.G.K. thank the Heisenberg-Landau Grant for partial support. K.X. was supported by the SUT-PhD/13/2554 Scholarship of Suranaree University of Technology and by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission.

[1] C. Adolph et al. (COMPASS Collaboration), Phys. Rev. Lett. 115, 082001 (2015) [arXiv:1501.05732 [hep-ex]].
[2] T. Gutsche, V. E. Lyubovitskij and I. Schmidt, Phys. Rev. D 96, 034030 (2017) [arXiv:1706.07716 [hep-ph]].
[3] Z. G. Wang, [arXiv:1401.1134 [hep-ph]].
[4] H. X. Chen, E. L. Cui, W. Chen, T. G. Steele, X. Liu and S. L. Zhu, Phys. Rev. D 91, 094022 (2015) [arXiv:1503.02597 [hep-ph]].
[5] M. Mikhasenko, B. Ketzer and A. Sarantsev, Phys. Rev. D 91, 094015 (2015) [arXiv:1501.07023 [hep-ph]].
[6] J. L. Basdevant and E. L. Berger, Phys. Rev. Lett. 114, 192001 (2015) [arXiv:1504.05955 [hep-ph]].
[7] X. H. Liu, M. Oka and Q. Zhao, Phys. Lett. B 753, 297 (2016) [arXiv:1507.01674 [hep-ph]].
[8] F. Aceti, L. R. Dai and E. Oset, Phys. Rev. D 94, 094012 (2016) [arXiv:1606.06893 [hep-ph]].
[9] W. Wang and Z. X. Zhao, Eur. Phys. J. C 76, 59 (2016) [arXiv:1511.06998 [hep-ph]].
[10] T. Branz, A. Faessler, T. Gutsche, M. A. Ivanov, J. G. Körner and V. E. Lyubovitskij, Phys. Rev. D 81, 034010 (2010) [arXiv:0912.3710 [hep-ph]].
[11] S. Dubnicka, A. Z. Dubnickova, M. A. Ivanov and J. G. Körner, Phys. Rev. D 81, 114007 (2010) [arXiv:1004.1291 [hep-ph]].
[12] S. Dubnicka, A. Z. Dubnickova, M. A. Ivanov, J. G. Körner, P. Santorelli and G. G. Saidullaeva, Phys. Rev. D 84, 014006 (2011) [arXiv:1104.3973 [hep-ph]].
[13] T. Gutsche, M. A. Ivanov, J. G. Körner and V. E. Lyubovitskij, Phys. Rev. D 94, 094012 (2016) [arXiv:1608.00415 [hep-ph]].
[14] F. Goerke, T. Gutsche, M. A. Ivanov, J. G. Körner, V. E. Lyubovitskij and P. Santorelli, Phys. Rev. D 94, 094017 (2016) [arXiv:1608.04656 [hep-ph]].
[15] F. Goerke, T. Gutsche, M. A. Ivanov, J. G. Körner and V. E. Lyubovitskij, Phys. Rev. D 96, 054028 (2017) [arXiv:1707.00539 [hep-ph]].
[16] S. Weinberg, Phys. Rev. 130, 776 (1963); A. Salam, Nuovo Cim. 25, 224 (1962).
[17] K. Hayashi, M. Hirayama, T. Muta, N. Seto and T. Shirafuji, Fortsch. Phys. 15, 625 (1967).
[18] G. V. Efimov and M. A. Ivanov, The Quark Confinement Model of Hadrons, (IOP Publishing, Bristol & Philadelphia, 1993).
[19] A. Garmash et al. (Belle Collaboration), Phys. Rev. Lett. 116, 212001 (2016) [arXiv:1512.07419 [hep-ex]].
[20] A. E. Bondar, A. Garmash, A. I. Milstein, R. Mizuk and M. B. Voloshin, Phys. Rev. D 84, 054010 (2011) [arXiv:1105.4473 [hep-ph]].
[21] T. Gutsche, M. A. Ivanov, J. G. Körner, V. E. Lyubovitskij, P. Santorelli and N. Haby, Phys. Rev. D 91, 074001 (2015), Phys. Rev. D 91, 119907(E) (2015) [arXiv:1502.04864 [hep-ph]].
[22] A. Issadykov, M. A. Ivanov and S. K. Sakhiev, Phys. Rev. D 91, 074007 (2015) [arXiv:1502.05280 [hep-ph]].
[23] C. Patrignani et al. [Particle Data Group], Chin. Phys. C 40, 100001 (2016).
[24] M. Nielsen, F. S. Navarra and S. H. Lee, Phys. Rept. 497, 41 (2010) [arXiv:0911.1958 [hep-ph]].