A KIND OF COMPACT QUANTUM SEMIGROUPS

MAYSAM MAYSAMI SADR

ABSTRACT. We show that the quantum family of all maps from a finite space to a finite dimensional compact quantum semigroup has a canonical quantum semigroup structure.

1. Introduction

Gelfand’s duality Theorem says that the category of locally compact Hausdorff spaces and continuous maps and the category of commutative C*-algebras and *-homomorphisms are dual to each other. In this duality any space $X$ corresponds to $C_0(X)$, the $C^*$-algebra of all continuous complex valued maps on $X$ vanishing at infinity (note that $X$ is compact if and only if $C_0(X) = C(X)$ is unital). Thus one can consider a non commutative $C^*$-algebra $A$ as the algebra of functions on a symbolic quantum space $QA$. In this correspondence, *-homomorphisms $\Phi : A \rightarrow B$ interpret as symbolic continuous maps $Q\Phi : QB \rightarrow QA$.

Woronowicz [5] and Soltan [3] have defined a quantum space $QC$ of all maps from $QB$ to $QA$ and showed that $C$ exists under appropriate conditions on $A$ and $B$. In [2], we considered the functorial properties of this notion. In this short note, we show that if $QA$ is a compact finite dimensional (i.e. $A$ is unital and finitely generated) quantum semigroup, and if $QB$ is a finite commutative quantum space (i.e. $B$ is a finite dimensional commutative $C^*$-algebra), then $QC$ has a canonical quantum semigroup structure. In the other words, we construct the non commutative version of semigroup $F(X,S)$ described as follows:

Let $X$ be a finite space and $S$ be a compact semigroup. Then the space $F(X,S)$ of all maps from $X$ to $S$, is a compact semigroup with compact-open topology and pointwise multiplication.

2. Quantum families of maps and quantum semigroups

For any $C^*$-algebra $A$, $I_A$ denotes the identity homomorphism from $A$ to $A$. If $A$ is unital then $1_A$ denotes the unit of $A$. For $C^*$-algebras $A,B$, $A \otimes B$ denotes the spatial tensor product of $A$ and $B$. If $\Phi : A \rightarrow B$ and $\Phi' : A' \rightarrow B'$ are *-homomorphisms, then $\Phi \otimes \Phi' : A \otimes A' \rightarrow B \otimes B'$ is a *-homomorphism defined by $\Phi \otimes \Phi'(a \otimes a') = \Phi(a) \otimes \Phi'(a')$ ($a \in A, a' \in A'$).

Let $X,Y$ and $Z$ be three compact Hausdorff spaces and $C(Y,X)$ be the space of all continuous maps from $Y$ to $X$ with compact open topology. Consider a continuous map $f : Z \rightarrow C(Y,X)$. Then the pair $(Z,f)$ is a continuous family of maps from $Y$ to $X$ indexed by $f$ with parameters in $Z$. On the other hand, by

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topological exponential law we know that \( f \) is characterized by a continuous map \( \tilde{f} : Y \times Z \rightarrow X \) defined by \( \tilde{f}(y, z) = f(z)(y) \). Thus \((Z, \tilde{f})\) can be considered as a family of maps from \( Y \) to \( X \). Now, by Gelfand’s duality we can simply translate this system to non commutative language:

**Definition 1.** ([5], [3]) Let \( A \) and \( B \) be unital \( C^* \)-algebras. By a quantum family of maps from \( \mathfrak{Q}B \) to \( \mathfrak{Q}A \), we mean a pair \((C, \Phi)\), containing a unital \( C^* \)-algebra \( C \) and a unital *-homomorphism \( \Phi : A \rightarrow B \otimes C \).

Now, suppose instead of parameter space \( Z \) we use \( C(Y, X) \) (note that in general this space is not locally compact). Then the family

\[
\text{Id} : C(Y, X) \rightarrow C(Y, X) \quad (\text{Id} : C(Y, X) \times Y \rightarrow X)
\]

of all maps from \( Y \) to \( X \) has the following universal property:

For every family \( \tilde{f} : Z \times Y \rightarrow X \) of maps from \( Y \) to \( X \), there is a unique map \( f : Z \rightarrow C(Y, X) \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
Z \times Y & \xrightarrow{\tilde{f}} & X \\
\downarrow f \times \text{Id}_Y & & \downarrow \text{Id} \\
C(Y, X) \times Y & \xrightarrow{\text{Id}} & X
\end{array}
\]

Thus, we can make the following Definition in non commutative setting:

**Definition 2.** ([5], [3]) With the assumptions of Definition 1, \((C, \Phi)\) is called a quantum family of all maps from \( \mathfrak{Q}B \) to \( \mathfrak{Q}A \) if for every unital \( C^* \)-algebra \( D \) and any unital *-homomorphism \( \Psi : A \rightarrow B \otimes D \), there is a unique unital *-homomorphism \( \Gamma : C \rightarrow D \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
A & \xrightarrow{\Phi} & B \otimes C \\
\downarrow \cong & & \downarrow \cong \\
A & \xrightarrow{\Psi} & B \otimes D
\end{array}
\]

By the universal property of Definition 2, it is clear that if \((C, \Phi)\) and \((C', \Phi')\) are two quantum families of all maps from \( \mathfrak{Q}B \) to \( \mathfrak{Q}A \), then there is a *-isometric isomorphism between \( C \) and \( C' \).

**Proposition 3.** Let \( A \) be a unital finitely generated \( C^* \)-algebra and \( B \) be a finite dimensional \( C^* \)-algebra. Then the quantum family of all maps from \( \mathfrak{Q}B \) to \( \mathfrak{Q}A \) exists.

**Proof.** See [5] or [3].

**Definition 4.** ([4], [6]) A pair \((A, \Delta)\) consisting of a unital \( C^* \)-algebra \( A \) and a unital *-homomorphism \( \Delta : A \rightarrow A \otimes A \) is called a compact quantum semigroup if \( \Delta \) is a coassociative comultiplication: \( (\Delta \otimes I_A) \Delta = (I_A \otimes \Delta) \Delta \).

Let \( S \) be a compact Hausdorff semigroup. Using the canonical identity

\[
C(S) \otimes C(S) \cong C(S \times S)
\]
defined by
\[ f \otimes f'(s, s') = f(s)f'(s') \quad (f, f' \in \mathcal{C}(S), s, s' \in S), \]
we let the \(*\)-homomorphisms \( \Delta : \mathcal{C}(S) \to \mathcal{C}(S) \otimes \mathcal{C}(S) \) be defined by
\[ \Delta(f)(s, s') = f(ss') \quad (f \in \mathcal{C}(S), s, s' \in S). \]

Then \( \Delta \) is a coassociative comultiplication on \( \mathcal{C}(S) \) and thus \( (\mathcal{C}(S), \Delta) \) is a compact quantum semigroup. Conversely, if \( (A, \Delta) \) is a compact quantum semigroup with commutative \( A \), then there is a compact Hausdorff semigroup such that its corresponding compact quantum semigroup is \( (A, \Delta) \), [4].

3. The result

Now, we state and prove the main result.

**Theorem 5.** Let \( (A, \Delta) \) be a compact quantum semigroup with finitely generated \( A \), \( B \) be a finite dimensional commutative \( C^*\)-algebra, and \( (C, \Phi) \) be the quantum family of all maps from \( \Omega B \) to \( \Omega A \). Consider the unique unital \(*\)-homomorphism \( \Gamma : C \to C \otimes C \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\Phi} & B \otimes C \\
\downarrow{\Delta} & & \downarrow{1_B \otimes \Gamma} \\
A \otimes A & \xrightarrow{\Phi \otimes \Phi} & B \otimes C \otimes C \\
\downarrow{\Phi \otimes \Phi} & & \downarrow{m \otimes I_C \otimes C} \\
B \otimes C \otimes B \otimes C & \xrightarrow{1_B \otimes F \otimes I_C} & B \otimes B \otimes C \otimes C
\end{array}
\]

is commutative, where \( F : C \otimes B \to B \otimes C \) is the flip map, i.e. \( c \otimes b \mapsto b \otimes c \) \((b \in B, c \in C)\), and \( m : B \otimes B \to B \) is the multiplication \(*\)-homomorphism of \( B \), i.e. \( m(b \otimes b') = bb' \) \((b, b' \in B)\). Then \( (C, \Gamma) \) is a compact quantum semigroup.

**Proof.** We must prove that \( (I_C \otimes \Gamma)\Gamma = (\Gamma \otimes I_C)\Gamma \), and for this, by the universal property of quantum families of maps, it is enough to prove that

\[ (I_B \otimes I_C \otimes \Gamma)(I_B \otimes \Gamma)\Phi = (I_B \otimes \Gamma \otimes I_C)(I_B \otimes \Gamma)\Phi. \]

Note that by the commutativity of \( (\Gamma) \), we have

\[ (I_B \otimes \Gamma)\Phi = (m \otimes I_C \otimes C)(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta. \]
Let us begin from the left hand side of (2):

\[(I_B \otimes I_C \otimes \Gamma)(I_B \otimes \Gamma)\Phi\]
\[= (I_B \otimes I_C \otimes \Gamma)(m \otimes I_{C \otimes C})(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_C \otimes \Gamma)(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes \Gamma})(\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes \Gamma})(\Phi \otimes I_{B \otimes C})(I_A \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(\Phi \otimes I_{B \otimes C})(I_A \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(\Phi \otimes (I_B \otimes \Gamma)\Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(\Phi \otimes (m \otimes I_{C \otimes C})(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes \Gamma})(m \otimes I_{C \otimes C})(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)(I_A \otimes \Delta)\Delta\]

For the right hand side of (2), we have

\[(I_B \otimes \Gamma \otimes I_C)(I_B \otimes \Gamma)\Phi\]
\[= (I_B \otimes \Gamma \otimes I_C)(m \otimes I_{C \otimes C})(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_B \otimes \Gamma \otimes F \otimes I_C)(I_B \otimes \Gamma \otimes I_B \otimes I_C)(\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes F \otimes I_C})([I_B \otimes \Gamma)\Phi \otimes \Phi)\Delta\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes F \otimes I_C})([m \otimes I_{C \otimes C})(I_B \otimes F \otimes I_C)(\Phi \otimes \Phi)\Delta)\otimes \Phi)\Delta,\]

and thus if \(W = (I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes F \otimes I_C})\), then

\[(I_B \otimes \Gamma \otimes I_C)(I_B \otimes \Gamma)\Phi =\]
\[= (m \otimes I_{C \otimes C \otimes C})W(m \otimes I_{C \otimes C \otimes B \otimes C})(I_B \otimes F \otimes I_{C \otimes B \otimes C})(\Phi \otimes \Phi \otimes \Phi)(\Delta \otimes I_A)\Delta.\]

Thus, since \((I_A \otimes \Delta)\Delta = (\Delta \otimes I_A)\Delta\), to prove (2), it is enough to show that

\[(3) (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes F \otimes I_C}) = (m \otimes I_{C \otimes C \otimes C})W(m \otimes I_{C \otimes C \otimes B \otimes C})(I_B \otimes F \otimes I_{C \otimes B \otimes C}).\]

Let \(b_1, b_2, b_3 \in B\) and \(c_1, c_2, c_3 \in C\). Then for the left hand side of (3), we have,

\[(m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes F \otimes I_C})(b_1 \otimes c_1 \otimes b_2 \otimes c_2 \otimes b_3 \otimes c_3)\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(I_{B \otimes C \otimes B \otimes F \otimes I_C})(b_1 \otimes c_1 \otimes b_2 \otimes b_3 \otimes c_2 \otimes c_3)\]
\[= (m \otimes I_{C \otimes C \otimes C})(I_B \otimes F \otimes I_{C \otimes C})(b_1 \otimes c_1 \otimes (b_2 b_3) \otimes c_2 \otimes c_3)\]
\[= (m \otimes I_{C \otimes C \otimes C})(b_1 \otimes (b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3)\]
\[= b_1(b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3\]
\[= (b_1 b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3.\]
and for the right hand side of (3),
\[
(m \otimes I_{C \otimes C})W(m \otimes I_{C \otimes B \otimes C})(I_B \otimes F \otimes I_{C \otimes B \otimes C})(b_1 \otimes c_1 \otimes b_2 \otimes c_2 \otimes b_3 \otimes c_3)
\]
\[
=(m \otimes I_{C \otimes C \otimes C})W(m \otimes I_{C \otimes C \otimes B \otimes C})(b_1 \otimes b_2 \otimes c_1 \otimes c_2 \otimes b_3 \otimes c_3)
\]
\[
=(m \otimes I_{C \otimes C \otimes C})W(b_1 b_2 \otimes c_1 c_2 \otimes b_3 c_3)
\]
\[
=(b_1 b_2 b_3) \otimes c_1 c_2 c_3
\]
Therefore, (3) is satisfied and the proof is complete. \(\square\)

Now, we consider a class of examples. Let \(A = \mathbb{C}^n\) be the C*-algebra of functions on the commutative finite space \(\{1, \cdots, n\}\), and let \((C, \Phi)\) be the quantum family of all maps from \(\Omega A\) to \(\Omega A\). Then, as is indicated in Section 7 of \([3]\), \(C\) is the universal C*-algebra generated by \(n^2\) self-adjoint elements \(\{c_{ij} : 1 \leq i, j \leq n\}\) that satisfy the relations
\[
c_{ij}^2 = c_{ij}, \quad \sum_{i=1}^{n} c_{ij} = 1_C, \quad \sum_{j=1}^{n} c_{ij} = 1_C,
\]
for \(1 \leq i, j \leq n\), and \(\Phi : A \longrightarrow A \otimes C\) is defined by \(\Phi(e_k) = \sum_{i=1}^{n} e_i \otimes c_{ik}\), where \(e_1, \cdots, e_n\) is the standard basis for \(A\). Suppose that
\[
\xi : \{1, \cdots, n\} \times \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}
\]
is a semigroup multiplication. Then \(\xi\) induces a comultiplication \(\Delta : A \longrightarrow A \otimes A\)
\[
\Delta(e_k) = \sum_{r,s=1}^{n} \Delta_{rs}^{rs} e_r \otimes e_s,
\]
declared by \(\Delta_{rs}^{rs} = \delta_{k \xi(r,s)}\), where \(\delta\) is the Cronecker delta. Now, we compute the comultiplication \(\Gamma : C \longrightarrow C \otimes C\) that \(\Delta\) induce as in Theorem \([5]\) we have,
\[
(\Phi \otimes \Phi)\Delta(e_k) = (\Phi \otimes \Phi)\left(\sum_{r,s=1}^{n} \Delta_{rs}^{rs} e_r \otimes e_s\right) = \sum_{r,s=1}^{n} \Delta_{rs}^{rs} \Phi(e_r) \otimes \Phi(e_s)
\]
\[
= \sum_{r,s=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \Delta_{rs}^{rs} e_j \otimes c_{jr} \otimes e_i \otimes c_{is},
\]
and therefore
\[
(m \otimes I_{C \otimes C})(I_B \otimes F \otimes I_{C})(\Phi \otimes \Phi)\Delta(e_k)
\]
\[
=(m \otimes I_{C \otimes C})(\sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{r,s=1}^{n} \Delta_{rs}^{rs} e_j \otimes e_i \otimes c_{jr} \otimes c_{is})
\]
\[
= \sum_{i=1}^{n} \sum_{r,s=1}^{n} \Delta_{rs}^{rs} e_l \otimes c_{tr} \otimes c_{ts} = \sum_{l=1}^{n} e_l \otimes \left(\sum_{r,s=1}^{n} \Delta_{rs}^{rs} c_{tr} \otimes c_{ts}\right).
\]
This equals to \((I_A \otimes \Gamma) \Phi(e_k) = (I_A \otimes \Gamma) \sum_{i=1}^{n} e_i \otimes c_{ik} = \sum_{l=1}^{n} e_l \otimes \Gamma(c_{lk})\). Thus \(\Gamma\) is defined by

\[
\Gamma(c_{lk}) = \sum_{r,s=1}^{n} \Delta^r_{ls} c_{lr} \otimes c_{ls}.
\]

Some natural questions arise about relations between properties of the compact quantum semigroups \((A, \Delta)\) and \((C, \Gamma)\) described in Theorem 5:

(i) Suppose that \((A, \Delta)\) has a counite or antipode \([6]\). Then is \((C, \Gamma)\) so?
(ii) Let \((A, \Delta)\) be a compact quantum group \([6]\). Is \((C, \Gamma)\) a compact quantum group?
(iii) Are the converse of (i) and (ii) satisfied?

It seems that it is not easy to answer these questions even in special case of the above example. We end with two remarks:

1) In the above example, if \(A = \mathbb{C}^2\), then one can consider \(C\) as the pointwise multiplication C*-algebra of all continuous maps \(f\) from closed interval \([0,1]\) to the matrix algebra \(\mathcal{M}_2(\mathbb{C})\), such that \(f(0)\) and \(f(1)\) are diagonal matrix, see section II.2.\(\beta\) of \([1]\). This is one of the basic examples of noncommutative spaces.
2) There is another quantum semigroup structure on quantum families of all maps from any finite quantum space to itself introduced by Soltan \([3]\).

REFERENCES

[1] A. Connes, Noncommutative geometry, Academic Press, 1994.
[2] M. M. Sadr, Quantum functor \(\mathfrak{Mor}\), Mathematica Pannonica, 21/1 (2010), 1–12.
[3] P. M. Soltan, Quantum families of maps and quantum semigroups on finite quantum spaces, J. Geom. Phys., 59 (2009), 354–368.
[4] S. Vaes and A. Van Daele, Hopf C*-algebras, Proc. London Math. Soc., 82 (2001), 337–384.
[5] S. L. Woronowicz, Pseudogroups, pseudospaces and Pontryagin duality. Proceedings of the International Conference on Mathematical Physics, Lausanne 1979, Lecture Notes in Physics, 116, 407–412.
[6] S. L. Woronowicz , Compact quantum groups, Les Houches, Session LXIV, 1995, Quantum Symmetries, Elsevier, (1998).

DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES (IASBS), ZANJAN 45195-1159 IRAN.

E-mail address: sadr@iasbs.ac.ir