Abstract. Given a surjective morphism $\pi: X \to Y$ between normal varieties satisfying some regularity hypotheses we prove how to recover a Cox ring of the generic fiber $X_\eta$ of $\pi$ from the Cox ring of $X$. We also prove that in some particular cases it is possible to recover the Cox ring of a very general fiber of $\pi$.

Introduction

Consider two normal varieties $X$ and $Y$ with finitely generated divisor class group and having only constant invertible global sections and let $\pi: X \to Y$ be a morphism. Under some regularity hypotheses on $\pi$ it is possible to give relations between the Cox rings of $X$ and $Y$. For instance, on one direction, in [12] it is proved that if $\pi$ is surjective and the Cox ring of $X$ is finitely generated (i.e. $X$ is a Mori dream space), then the Cox ring of $Y$ is finitely generated too. In the same vein, in [3] and [9], the case in which $\pi$ is a good quotient is studied and the Cox ring of $Y$ is described in terms of the one of $X$. On the other direction, if the Cox ring of $Y$ is known, it is possible to say something about the Cox ring of $X$ only in some particular cases: for instance in [1] the case of a cyclic cover is considered, while in [8] the case of a birational morphism.

In the present paper we consider the problem of determining the Cox ring of the generic fiber $X_\eta$ of the morphism $\pi$ from the Cox ring of $X$ and from the vertical classes of $\pi$, i.e. classes of divisors whose image in $Y$ is not dense. Observe that $X_\eta$ is defined over the function field $k := \mathbb{C}(Y)$, which is not algebraically closed so that we follow [6] in order to define a Cox ring for $X_\eta$.

In order to describe our results let us denote by $\text{Cl}_\pi(X)$ the subgroup of $\text{Cl}(X)$ generated by classes of vertical divisors, or equivalently the kernel of the surjection $\text{Cl}(X) \to \text{Cl}(X_\eta)$ induced by the pull-back. Let $\mathcal{R}_\pi(X)$ be the localization of $\mathcal{R}(X)$ by the multiplicative subsystem generated by the non-zero homogeneous elements $f \in \mathcal{R}_\eta(X_\eta)$. Then the following holds.

**Theorem 1.** Let $\pi: X \to Y$ be a proper surjective morphism of normal complex varieties having only constant invertible global sections, such that $\text{Cl}(X)$ is finitely generated and $\text{Cl}(Y)$ is torsion free. Let us suppose that the very general fiber of $\pi$ is irreducible and that $\pi$ admits a rational section. Then there exists a Cox ring $\mathcal{R}(X_\eta)$ of the generic fiber $X_\eta$ such that the canonical morphism $\iota: X_\eta \to X$ induces an isomorphism of $\text{Cl}(X_\eta)$-graded $\mathbb{C}(Y)$-algebras

$$\mathcal{R}_\pi(X)/(1 - u(w) : w \in \text{Cl}_\pi(X)) \cong \mathcal{R}(X_\eta),$$

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where $u: \mathcal{C}l_\pi(X) \to \mathcal{R}_\pi(X)^*$ is any homomorphism satisfying $u(w) \in \mathcal{R}_\pi(X)^* - w$ for each $w$.

As a consequence we are able to recover a subring of the Cox ring of a very general fiber of $\pi$ and in some cases the full Cox ring. More precisely, if we denote by $\bar{X}_\eta = X_\eta \times_k \bar{k}$ the base change of the generic fiber $X_\eta$ over the algebraic closure $\bar{k}$ of $k$, we have the following.

**Corollary 2.** Let $\pi: X \to Y$ satisfy the hypotheses of Theorem 1 and suppose in addition that the geometric divisor class group $\mathcal{C}l(\bar{X}_\eta)$ is isomorphic to $\mathcal{C}l(X_\eta)$. Then the Cox ring of a very general fiber of $\pi$ is isomorphic to $\mathcal{R}(X_\eta) \otimes_k \bar{k}$ as a graded ring.

We remark that the isomorphism of the corollary above is not an isomorphism of graded algebras, since one of them is defined over $\mathbb{C}$ while the other one over $\bar{k}$.

The paper is structured as follows. In Section 1 we recall some facts about varieties defined over a perfect field not necessarily closed and we construct a Cox sheaf for such varieties, following [6]. In Section 2 we collect some results about the generic fiber $X_\eta$ of a proper surjective morphism $\pi: X \to Y$, whose very general fiber is irreducible. Section 3 deals with the relation between sheaves of divisorial algebras on $X$ and $X_\eta$ and with some lemmas that we are going to use in Section 4 in order to prove Theorem 1. Finally, in the last section we prove Corollary 2 and as an application we study the case of toric varieties.

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1. **Preliminaries**

We begin by recalling some known facts about algebraic varieties defined over a perfect field $k$ and by fixing some notation. Given an algebraic variety $X$ defined over $k$ we will denote by $\bar{X}_k$ the base change of $X$ over the algebraic closure $\bar{k}$ of $k$. From now on we assume that any variety $X$ has only constant invertible global sections, i.e.

\[(1.1) \quad \bar{k}[X]^* = \bar{k}^*,\]

where $\bar{k}[X]$ denotes the ring of global sections of the structure sheaf of $X$. Let us denote by $G = \text{Gal}(\bar{k}/k)$ the absolute Galois group of $k$ and let

$$\text{WDiv}(X) := \{ D \in \text{WDiv}(X_\bar{k}) : \sigma(D) = D \text{ for any } \sigma \in G \}$$

be the group of $G$-invariant Weil divisors of $X_\bar{k}$. We will denote by $\text{PDiv}(X)$ the subgroup of $\text{WDiv}(X)$ consisting of principal divisors of the form $\text{div}(f)$ with $f \in k(X)$. By [10, Proposition A.2.2.10 (ii)] the equality $\text{PDiv}(X) = \text{WDiv}(X) \cap \text{PDiv}(X_\bar{k})$ holds (observe that the hypothesis in the proposition asks $X$ to be projective but it actually only makes use of the weaker condition (1.1)). Thus, if we denote by $\text{Cl}(X)$ the quotient group $\text{WDiv}(X)/\text{PDiv}(X)$ we get inclusions

$$\text{Cl}(X) \subseteq \text{Cl}(X_\bar{k})^G \subseteq \text{Cl}(X_\bar{k}).$$

The first inclusion can be strict, for example if $X$ is a conic without $k$-rational points. Given a divisor $D \in \text{WDiv}(X)$ and a Zariski open subset $U$ of $X$, the space of sections $\mathcal{O}_{X_\bar{k}}(D)(U_\bar{k})$ is a $\bar{k}$ vector space acted by $G$, since both $U$ and $X$ are defined over $k$, and thus it is a $G$-module. Observe that a $G$-invariant element...
$f \in \mathcal{O}_{X_k}(D)(U_k)^G$ is a rational function of $X_k$ which is defined over $k$ (see for instance [13, Exercise 1.12]). If we set

$$(1.2)\quad \mathcal{O}_X(D)(U) := \mathcal{O}_{X_k}(D)(U_k)^G,$$

by [10, Proposition A.2.2.10 (i)] we have that the $\bar{k}$ vector space $\mathcal{O}_X(D)(U) \otimes_k \bar{k}$ is isomorphic to $\mathcal{O}_{X_k}(D)(U_k)$. 

1.1. **Cox sheaf.** Let us now construct a sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras which turns out to be a *Cox sheaf of type $\lambda$* according to [6, Definition 2.2]. Let us suppose that $\text{Cl}(X)$ is finitely generated and let $K$ be a finitely generated subgroup of $\text{WDiv}(X)$ whose image via the class map $\omega: K \to \text{Cl}(X)$ is $\text{Cl}(X)$. Let us consider the $K$-graded sheaf of $\mathcal{O}_X$-algebras

$$S = \bigoplus_{D \in K} \mathcal{O}_X(D).$$

Denote by $K^0 \subseteq K$ the kernel of the class map and let $\mathcal{X}: K^0 \to k(X)^*$ be a homomorphism of groups such that $\text{div} \circ \mathcal{X} = \text{id}$ (such a $\mathcal{X}$ exists again by [10, Proposition A.2.2.10 (ii)]). Let $\mathcal{I}$ be the ideal sheaf of $S$ locally generated by sections of the form $1 - \mathcal{X}(D)$, where $D \in K^0$. Denote by $\mathcal{R}$ the presheaf $S/\mathcal{I}$ and by $\pi: S \to \mathcal{R}$ the quotient map.

**Proposition 1.1.** The presheaf $\mathcal{R}$ defined above is a Cox sheaf of type $\lambda$, where $\lambda: K/K^0 \to \text{Cl}(X_k)$ is induced by the class map.

**Proof.** Consider the sheaf of divisorial algebras

$$\bar{S} = \bigoplus_{D \in K} \mathcal{O}_{X_k}(D)$$

together with the $G$-invariant character $\mathcal{X}: K^0 \to k(X)^* \subseteq \bar{k}(X)^*$, where $G = \text{Gal}(\bar{k}/k)$ as before and let $\mathcal{I}$ be the ideal sheaf of $\bar{S}$ defined by $\mathcal{X}$. Let us denote by $\phi: X_k \to X$ the base change map. According to the proof of [6, Proposition 3.13] the quotient sheaf $\mathcal{R} = \bar{S}/\mathcal{I}$ is a $G$-equivariant Cox sheaf of type $\lambda$ and the push forward of the sheaf of invariants $\phi_* \mathcal{R}^G$ is a Cox sheaf of $X$ of type $\lambda$. Given an open subset $U \subseteq X$ and a divisor $D \in K$ the following holds

$$(\phi_* \mathcal{R}^G_D)(U) = (\mathcal{R}^G_D(U_k))^G$$

$$= (\bar{S}/\mathcal{I})_{|D}(U_k))^G$$

$$\cong (\bar{S}_D(U_k)/\mathcal{I}_D(U_k))^G$$

$$\cong S_D(U)/\mathcal{I}_D(U),$$

where the first isomorphism is by [6, Construction 2.7], and the second one is by (1.2). The above shows that $\mathcal{R} = S/\mathcal{I}$ is a sheaf, being isomorphic to $\phi_* \mathcal{R}^G$. 

**Lemma 1.2.** Let $L/k$ be a Galois extension of fields with Galois group $G$. Let $V_1 \subseteq V_2$ be $k$ vector spaces and let $\overline{V}_i = V_i \otimes_k L$. Then $\overline{V}_1$ is $G$-invariant and the homomorphism

$$j: V_2/V_1 \to (\overline{V}_2/\overline{V}_1)^G \quad v + V_1 \mapsto v + \overline{V}_1$$

is an isomorphism.
Write the generic point of about $X$ general fiber is irreducible. In this section we are going to summarise some results of

By hypothesis there is a $T$-invariant vector spaces, where $T$ is obtained by completing a basis of $V_1$ to a basis of $V_2$. Since $V_1 \cap V_2 = V_1^j = V_1$ the map $j$ is injective. To prove the surjectivity of $j$ let $v + V_1 \in (V_2/V_1)^G$, that is $gv + V_1 = v + V_1$ for any $g \in G$, or equivalently $gv - v \in V_1$.

Write $v = v_1 + t$ with $v_1 \in V_1$ and $t \in T$ and observe that $gv - v \in V_1$ implies $gt - t \in V_1 \cap T = 0$, so that $t$ is $G$-invariant, that is $t \in V_2$. Thus $v + V_1 = t + V_1 = j(t + V_1)$.

\[\square\]

2. Divisors on the generic fiber

Let $X$ and $Y$ be normal algebraic varieties satisfying (1.1) and let denote by $\eta$ the generic point of $Y$. Let $\pi: X \to Y$ be a proper surjective morphism whose very general fiber is irreducible. In this section we are going to summarise some results about $X_\eta$.

The morphism $i : X_\eta \to X$ induces a pullback isomorphism $i^*: \mathbb{C}(X) \to k(X_\eta)$, where $k = (i \circ \pi)^*(\mathbb{C}(Y)) \cong \mathbb{C}(Y)$. We remark that the complementary of the smooth locus $X_{\text{sm}}$ has codimension at least two in $X$ and the same holds for the generic fiber of the restriction $\pi|_{X_{\text{sm}}}$ in $X_\eta$. Therefore $i^*$ induces a surjective homomorphism

\[\text{WDiv}(X) \to \text{WDiv}(X_\eta)\]

that by abuse of notation we denote by the same symbol $i^*$. Let $\text{WDiv}_\pi(X)$ the kernel of the above map.

Proposition 2.1. The following hold:

(i) the diagram

\[
\begin{array}{ccc}
\mathbb{C}(X) & \xrightarrow{i^*} & k(X_\eta) \\
\text{div} & & \text{div} \\
\text{WDiv}(X) & \xrightarrow{i^*} & \text{WDiv}(X_\eta).
\end{array}
\]

is commutative;

(ii) if $D \in \text{WDiv}(X)$ is effective on an open subset $U$ of $X$ then $i^*(D)$ is effective on the corresponding open subset $U_\eta$ of $X_\eta$;

(iii) the group $\text{WDiv}_\pi(X)$ is freely generated by the prime divisors $D$ which do not dominate $Y$;

(iv) the map (2.1) induces a surjective homomorphism $\text{Cl}(X) \to \text{Cl}_k(X_\eta)$ whose kernel $\text{Cl}_\pi(X)$ is generated by the classes of divisors in $\text{WDiv}_\pi(X)$;

(v) for any $D \in \text{WDiv}(X)$ the pullback induces a map $i^*: \mathcal{O}_X(D) \to \iota_*\mathcal{O}_{X_\eta}(i^*D)$.

Proof: Recall that the generic fiber $X_\eta$ is limit of the family of open subsets $\pi^{-1}(V) \subseteq X$, where $V$ varies through the open subsets of $Y$. Let $V = \text{Spec}(B)$ and $U = \text{Spec}(A)$ be an affine open subset of $\pi^{-1}(V)$. The morphism $\pi|_U : U \to V$ is induced by an injective homomorphism $B \to A$ of $\mathbb{C}$-algebras. Identifying $B$ with a subalgebra of $A$ we have that the affine open subset $U_\eta \subseteq X_\eta$, obtained by
base change over $U$, is the spectrum of the localization $S^{-1}_BU$, whose multiplicative system is $S_B = B \setminus \{0\}$. The pullback

$$\iota^*: \mathcal{O}_U(U) \to \iota_*\mathcal{O}_{\pi_1}(U_\eta)$$

is thus defined on $U$ by the injection $A \to S^{-1}_BA$. This shows that a prime divisor $D$ defined by a prime ideal $\mathfrak{p} \subseteq A$ survives in the generic fiber if and only if $\mathfrak{p} \cap B = 0$, that is $D$ has non-empty intersection with $\pi^{-1}(V)$. This proves (ii) and (iii). In order to prove (i) recall that the order of a rational function $f \in \mathbb{C}(X)$ at $D$ is the length of the $\mathcal{O}_{X,\mathfrak{p}}$-module $\mathcal{O}_{X,\mathfrak{p}}/(f)$, but the local rings $\mathcal{O}_{X,\mathfrak{p}}$ and $\mathcal{O}_{X,\mathfrak{p}}$ are isomorphic if $\mathfrak{p} \cap B = 0$. In order to prove (iv) observe that the morphism

$$\text{Cl}(X) \to \text{Cl}(X_\eta), \quad [D] \mapsto [\iota^*(D)]$$

is well defined by (i). Let us fix a divisor $D \in \text{WDiv}(X)$ such that $[D]$ is in the kernel $\text{Cl}_\eta(X)$ of the above map. By definition this implies that $\iota^*(D)$ is principal on $X_\eta$, so that we can write $\iota^*(D) = \text{div}(g)$, with $g \in k(X_\eta)$. Since $\iota^*: \mathbb{C}(X) \to k(X_\eta)$ is an isomorphism we have $g = \iota^*(f)$, with $f \in \mathbb{C}(X)$. We conclude that

$$0 = \iota^*(D) - \text{div}(g) = \iota^*(D - \text{div}(f))$$

and in particular $D - \text{div}(f)$ is a divisor of $\text{WDiv}_\pi(X)$, linearly equivalent to $D$.

Finally, to prove (v) let $f \in \mathcal{O}_X(D)(U)$ so that the divisor $\text{div}(f) + D$ is effective on $U$. Then

$$\text{div}(\iota^*(f)) + \iota^*D = \iota^*(\text{div}(f) + D)$$

is effective on $U_\eta$ by (ii). \qed

In what follows we will refer to the elements of $\text{WDiv}_\pi(X)$ as vertical divisors and similarly to the elements of $\text{Cl}_\pi(X)$ as vertical classes.

3. Pullback of divisorial algebras

We begin by recalling the definition of sheaf of divisorial algebras on a normal variety $X$. Let $K$ be a finitely generated subgroup of the group of Weil divisors $\text{WDiv}(X)$ of $X$. The sheaf of divisorial algebras defined by $K$ is

$$S_K := \bigoplus_{D \in K} \mathcal{O}_X(D).$$

**Lemma 3.1.** Let $\varphi: K_1 \to K_2$ be an isomorphism between two finitely generated subgroups of $\text{WDiv}(X)$ such that $\varphi(D) \sim D$ for any $D \in K_1$. Then there exists an isomorphism of graded algebras

$$\Gamma(X, S_{K_1}) \to \Gamma(X, S_{K_2}).$$

**Proof.** Let us fix a basis $D_1, \ldots, D_s$ of $K_1$ and for any $i = 1, \ldots, s$ let us fix an element $f_i \in \mathbb{C}(X)^*$ such that $\text{div}(f_i) = D_i - \varphi(D_i)$. Given a divisor $D = a_1D_1 + \cdots + a_sD_s \in K_1$, multiplication by $f_1^{a_1} \cdots f_s^{a_s}$ induces an isomorphism between the degree-$D$ part of the first algebra and the degree-$\varphi(D)$ part of the second one. We conclude since it is compatible with multiplication of sections. \qed

**Lemma 3.2.** Let $\pi: X \to Y$ be a proper surjective morphism of normal varieties whose very general fiber is irreducible and such that the pull-back is defined. Given a subgroup $\Lambda$ of $\text{WDiv}(Y)$ we have an isomorphism of graded algebras

$$\Gamma(Y, S_{\Lambda}) \to \Gamma(X, S_{\pi^*\Lambda}), \quad f \mapsto f \circ \pi.$$
Proof. By applying the Stein factorization [7, §III, Cor. 11.5] to the proper morphism \( \pi: X \to Y \) we get \( \pi = \rho \circ \pi' \) where \( \pi': X \to Y' \) has connected fibers, \( \rho: Y' \to Y \) is finite and \( \pi'_* \mathcal{O}_X = \mathcal{O}_{Y'}. \) Since the very general fiber of \( \pi \) is irreducible and we are in characteristic zero, we deduce that \( \rho \) has degree one. Being \( Y \) normal we conclude that \( \rho \) is an isomorphism and in particular \( \pi_* \mathcal{O}_X = \mathcal{O}_Y. \) Thus, given a divisor \( D \in \text{WDiv}(Y) \), by the projection formula [7, §II, Ex. 5.1] and the fact that the pullback is defined, we get the isomorphism \( \mathcal{O}_Y(D) \simeq \pi_* \pi^* \mathcal{O}_Y(D) \) given by \( f \mapsto f \circ \pi. \) Since this map preserves the multiplication, we get the statement by taking global sections. 

Given a subgroup \( \Lambda \) of \( \text{WDiv}(Y) \) and a divisor \( D \in \Lambda \) we denote by \( \Gamma(Y, S_\Lambda)_D \) the degree \( D \) part of the ring of global sections. In what follows, whenever we need to keep trace of the degree of an element in \( \Gamma(Y, S_\Lambda)_D \), we will use the notation \( f_D^D \), where \( f \) is in the Riemann Roch space of \( D \). If we denote by \( \text{Frac}_0(\Gamma(Y, S_\Lambda)) \) the field of fractions of homogeneous sections having the same degree, we have a field homomorphism

\[
\mu_\Lambda : \text{Frac}_0(\Gamma(Y, S_\Lambda)) \to \mathbb{C}(Y), \quad f_D^D / g_D^D \mapsto f/g.
\]

Lemma 3.3. If the class map \( \Lambda \to \text{Cl}(Y) \) is surjective then \( \mu_\Lambda \) is an isomorphism.

Proof. Given \( h \in \mathbb{C}(Y) \) we can write \( \text{div}(h) = A - B \), with \( A \) and \( B \) effective divisors in \( \text{WDiv}(Y) \). Therefore there exists \( D \in \Lambda \) and \( g \in \Gamma(Y, S_\Lambda)_D \) such that \( \text{div}(g) + D = B \). The fraction \( h g D^D / g D^D \) is then an element in \( \text{Frac}_0(\Gamma(Y, S_\Lambda)) \) whose image via \( \mu_\Lambda \) is \( h \). This proves the surjectivity of \( \mu_\Lambda \) and since it is a homomorphism of fields the statement follows.

Let us suppose now that \( X \) and \( Y \) are normal varieties having only constant invertible global sections and let \( \pi: X \to Y \) be a proper surjective morphism whose very general fiber is irreducible and such that the pull-back is defined. Let \( K \) and \( \Lambda \) be a finitely generated subgroups of \( \text{WDiv}(X) \) and \( \text{WDiv}(Y) \) respectively, such that the class maps \( K \to \text{Cl}(X) \) and \( \Lambda \to \text{Cl}(Y) \) are surjective. Let us consider the localisation

\[
\Gamma_\pi(X, S_K) = S_{\pi}^{-1} \Gamma(X, S_K),
\]

where \( S_\pi \) is the multiplicative system consisting of the non-zero homogeneous elements whose degree \( D \in K \) is such that \([D] \in \text{Cl}_\pi(X)\).

Lemma 3.4. The \( \mathbb{C} \)-algebra \( \Gamma_\pi(X, S_K) \) has a structure of \( \mathbb{C}(Y) \)-algebra.

Proof. By Lemma 3.3, \( \mathbb{C}(Y) \) is isomorphic to \( \text{Frac}_0(\Gamma(Y, S_\Lambda)) \) and hence it is enough to give a homomorphism from the latter to \( \Gamma_\pi(X, S_K) \). Observe that by Lemma 3.2 we have that \( \Gamma(Y, S_\Lambda) \) is isomorphic to \( \Gamma(X, S_{\pi^* \Lambda}) \) and by Lemma 3.1 the latter is isomorphic to the ring of sections of the sheaf of divisorial algebras on \( X \), defined by a subgroup of \( K \) isomorphic to \( \pi^* \Lambda \). We conclude by observing that \( \pi^* \) induces a map from \( \text{Frac}_0(\Gamma(X, S_{\pi^* \Lambda})) \) to \( \Gamma_\pi(X, S_K) \), since the class of a pull-back divisor is in \( \text{Cl}_\pi(X) \).

Given the map \( \iota^* : \text{WDiv}(X) \to \text{WDiv}(X_\eta) \), we use the following notation

\[
(3.1) \quad K_\eta = \iota^*(K).
\]

By Proposition 2.1 (v) we have a morphism of sheaves of divisorial algebras \( \iota^* : S_K \to \iota_\eta^* S_{K_\eta} \) and passing to global sections we obtain a homomorphism of rings

\[
(3.2) \quad \iota^* : \Gamma(X, S_K) \to \Gamma(X_\eta, S_{K_\eta}).
\]
Remark 3.5. If the subgroup $K$ does not contain vertical divisors, that is $K \cap \text{WDiv}_\pi(X) = 0$, then the restriction of $i^*$ is an isomorphism between $K$ and $K_\eta := i^*(K)$. In this case the map $i^*$ defined in (3.2) is an injection since $i^*$ induces also an isomorphism between the fields of rational functions.

Proposition 3.6. If the subgroup $K$ does not contain vertical divisors, then the map $i^*$ defined in (3.2) extends to an isomorphism of $\mathbb{C}(Y)$-algebras

$$i^* : \Gamma_\pi(X, S_K) \to \Gamma(X_\eta, S_{K_\eta}), \quad \frac{f}{g} \mapsto i^*(f)/i^*(g).$$

Proof. By Remark 3.5 we already know that the map (3.2) is injective and hence we are now going to prove that the image of an element in the multiplicative system $S_\pi$ is invertible. Let us fix $g \in S_\pi$, i.e. $g \in \Gamma(X, S_K)_D$, and $[D] \in \text{Cl}_\pi(X)$. Then $i^*(g) \in \Gamma(X_\eta, S_{K_\eta}, \ast(D)$, where $i^*(D)$ is a principal divisor, being its class trivial. Thus

$$i^*(\text{div}(g) + D) = \text{div}(i^*(g)) + i^*(D) = 0,$$

where the first equality is by Lemma 2.1 and the second is due to the fact that the generic fiber $X_\eta$ is complete, being $\pi$ proper by hypothesis. In particular $i^*(g)$ is invertible with inverse $i^*(g^{-1}) \in \Gamma(X_\eta, S_{K_\eta}, \ast(-D))$. This shows that the map defined in the statement is an injective homomorphism of $\mathbb{C}(Y)$-algebras.

In order to prove the surjectivity, it suffices to show that any homogeneous $s \in \Gamma(X_\eta, S_{K_\eta}, \ast(D)$, with $D \in K$, is in the image. At the level of rational functions we have $s = i^*(f)$, where $f \in \mathbb{C}(X)$, with

$$i^*(\text{div}(f) + D) = \text{div}(i^*(f)) + i^*(D) = E_\eta,$$

where $E_\eta$ is effective too and $i^*(E) = E_\eta$. Then the above formula implies that $\text{div}(f) + D = E + V$, where $V \in \text{WDiv}_\pi(X)$. Write $V = A - B$, with $A$ and $B$ effective. Let $B' \in K$ linearly equivalent to $B$ and let $h \in \Gamma(X, S_K)_{B'}$ be such that $\text{div}(h) + B' = B$. By the equality

$$\text{div}(fh) + D + B' = E + A$$

we deduce that $fh \in \Gamma(X, S_K)_{D+B'}$ and thus $\frac{fh}{h} \in \Gamma_\pi(X, S_K)$ is a preimage of $g$. □

4. Proof of the main theorem

From now on we suppose that the subgroup $K$ of $\text{WDiv}(X)$ does not contain vertical divisors and that $\pi : X \to Y$ is a proper surjection whose very general fiber is irreducible and that admits a rational section $\sigma : Y \to X$. We also assume the divisor class group $\text{Cl}(Y)$ to be torsion-free. If we denote by $\text{PDiv}_\pi(X)$ the subgroup of $\text{WDiv}_\pi(X)$ consisting of the principal vertical divisors of $X$, we have the following.

Lemma 4.1. The subgroup $\text{PDiv}_\pi(X)$ equals $\pi^* \text{PDiv}(Y)$ and it is primitive in $\text{WDiv}_\pi(X)$.

Proof. We recall, by Section 2, that the field of rational functions of $X_\eta$ is $k(X_\eta)$ where $k \simeq \mathbb{C}(Y)$ is the image of $\pi^* \mathbb{C}(Y)$ via the isomorphism $i^* : \mathbb{C}(X) \to k(X_\eta)$. The inclusion $\pi^* \text{PDiv}(Y) \subseteq \text{PDiv}_\pi(X)$ is obvious. In order to prove the opposite
inclusion let $D \in \text{PDiv}_\pi(X)$ be a principal vertical divisor and let $f \in \mathbb{C}(X)$ be a rational function such that $\text{div}(f) = D$. By Proposition 2.1 we have

$$\text{div}(\iota^*(f)) = \iota^*(D) = 0,$$

and thus $\iota^*(f)$ must be constant, being $X_\eta$ complete by the properness hypothesis on $\pi$. In particular $\iota^*(f)$ is an element of $\bar{k} \cap k(X_\eta)$, where $\bar{k}$ is the algebraic closure of $k$. By [11, Example 2.1.12] the following equality holds, so that $\iota^*(f) \in k$. In particular $f \in \pi^*(\mathbb{C}(Y))$ and thus $D = \text{div}(f)$ lies in $\pi^* \text{PDiv}(Y)$, which proves the first statement.

In order to prove the primitivity statement, let us take a divisor $V \in \text{WDiv}_\pi(X)$ such that $nV$ belongs to $\text{PDiv}_\pi(X)$ for some integer $n > 1$. Since $\text{PDiv}_\pi(X) = \pi^* \text{PDiv}(Y)$ we can write $nV = \pi^*D$, with $D \in \text{PDiv}(Y)$ and hence

$$D = (\pi \circ \sigma)^*(D) = \sigma^*(\pi^*D) = \sigma^*(nV) = n\sigma^*(V).$$

In particular $n\sigma^*(V) \in \text{PDiv}(Y)$ and since we are assuming that $\text{Cl}(Y)$ is torsion free we conclude that $\sigma^*(V) \in \text{PDiv}(Y)$. By applying $\pi^*$ to both sides of the equation above we deduce $nV = n\pi^*(\sigma^*(V))$ so that $V = \pi^*(\sigma^*(V))$ holds since $\text{WDiv}(X)$ is free abelian. In particular we conclude that $V \in \pi^* \text{PDiv}(Y) = \pi^* \text{PDiv}_\pi(X)$, which proves the statement.\[\square\]

**Remark 4.2.** In order to see why in Lemma 4.1 we need the existence of a section, let $S$ be an Enriques surface. It is well known [4, Chapter VII, §16] that $S$ admits an elliptic fibration $\pi: S \to \mathbb{P}^1$ having two double fibers, say $2F$ and $2F'$, so that $\pi$ does not admit any rational section. The difference $F - F'$ is a vertical divisor for $\pi$ which is not principal, while $2(F - F')$ is.

On the other hand, to see why $\text{Cl}(Y)$ must be torsion free, consider the trivial fibration $S \times \mathbb{P}^1 \to S$, where $S$ is still an Enriques surface. In this case the class group $\text{Cl}(S)$ is not torsion free and $\text{PDiv}_\pi(S \times \mathbb{P}^1)$ is not primitive in $\text{WDiv}_\pi(S \times \mathbb{P}^1)$.

**Lemma 4.3.** Let $K_\eta$ be as in (3.1) and let us denote by $K_\eta^0$ the kernel of the surjection $K_\eta \to \text{Cl}(X_\eta)$. Then the groups $\text{Cl}_\pi(X)$ and $K_\eta^0/\iota^*(K^0)$ are isomorphic and torsion free.
Proof. A diagram chasing in the following commutative diagram with exact rows and columns establishes the claimed isomorphism

\[
\begin{array}{ccccccc}
0 & \to & K^0 & \to & K & \to & \text{Cl}(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & K^0_\eta & \to & K_\eta & \to & \text{Cl}(X_\eta) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & 0 \\
K^0_\eta/\iota^*(K^0) & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
\]

where we denote by \(\iota^*_K\) the restriction of \(\iota^*\) to \(K\). Let us now prove that \(\iota^*(K_0)\) is primitive in \(K_\eta\). Let us fix \(D' \in K^0_\eta\) such that \(nD'\) is in \(\iota^*(K^0_\eta)\). Since \(\iota^*_K\) is an isomorphism, we can write \(D' = \iota^*_K(D)\). Moreover \(\iota^*_K(nD) = nD' \in \iota^*_K(K^0_\eta)\), so that by the injectivity of \(\iota^*_K\), \(nD\) is in \(K^0_\eta\) and in particular \(nD = \text{div}(f)\), where \(f\) is a rational function on \(X\). On the other hand we have that

\[
\iota^*(D) = \text{div}(\iota^*(g)) = \iota^*(\text{div}(g)),
\]

where the first equality follows from the fact that \(\iota^*(D) \in K^0_\eta\) and \(\iota^*: \mathbb{C}(X) \to k(X_\eta)\) is an isomorphism, while the second equality is a consequence of Proposition 2.1. In particular the divisor \(V = D - \text{div}(g)\) is in the kernel of \(\iota^*\) and hence, again by Proposition 2.1, it is a vertical divisor (observe that \(\text{div}(g)\) does not necessarily lie in \(K\)). Therefore

\[
nV = nD - \text{div}(g^n) = \text{div}(f/g^n)
\]

is a principal vertical divisor, i.e. \(nV \in \text{PDiv}_{\pi}(X)\) and by Lemma 4.1 also \(V\) is in \(\text{PDiv}_{\pi}(X)\). This implies that \(D = \text{div}(g) + V\) is principal and hence \(D \in K^0\). The statement follows.

Lemma 4.4. Given a character \(\chi: K^0 \to \mathbb{C}(X)^*\) such that \(\text{div} \circ \chi = \text{id}\), there exists a character \(\chi_\eta: K^0_\eta \to k(X_\eta)^*\) such that \(\text{div} \circ \chi_\eta = \text{id}\) and the following diagram commutes

\[
\begin{array}{ccc}
K^0 & \xrightarrow{\chi} & \mathbb{C}(X)^* \\
\downarrow^{\iota^*_K} & & \downarrow^{\iota^*_\eta} \\
K^0_\eta & \xrightarrow{\chi_\eta} & k(X_\eta)^*
\end{array}
\]

In particular \(\iota^*: \mathcal{S}_K \to \mathcal{S}_K_\eta\) maps the ideal sheaf \(\mathcal{I}\) to \(\mathcal{I}_\eta\). Moreover the set of such characters \(\chi_\eta\) is in bijection with \(\text{Hom}(\text{Cl}_{\pi}(X), k^*)\).
Proof. Observe that since $i^{\ast}_{K_0}$ is an isomorphism on its image, there exists a unique homomorphism of groups $\varphi: i^{\ast}(K^0) \to k(X_\eta)^*$ which makes the following diagram commutative

\[
\begin{array}{ccc}
K^0 & \xrightarrow{i^\ast} & \mathcal{C}(X)^* \\
\downarrow{i^\ast} & & \downarrow{i^\ast} \\
i^{\ast}(K_0) & \xrightarrow{\varphi} & k(X_\eta)^*
\end{array}
\]

Therefore for any divisor $D \in K^0$ we have

\[
\operatorname{div}(\varphi(i^{\ast}(D))) = \operatorname{div}(i^{\ast}(\mathcal{X}(D))) = i^{\ast}(\operatorname{div}(\mathcal{X}(D))) = i^{\ast}(D),
\]

where the second equality follows from Proposition 2.1. Now by Lemma 4.3 the subgroup $i^{\ast}(K^0)$ is primitive in $K_\eta^0$ so that $\varphi$ can be extended to a homomorphism $\mathcal{X}_\eta: K_\eta^0 \to k(X_\eta)^*$. Such an extension is uniquely determined by the values of $\mathcal{X}_\eta$ on a basis of $K_\eta^0$ and thus can be chosen in such a way that the equality $\operatorname{div} \circ \mathcal{X}_\eta = \operatorname{id}$ holds. Given two such extensions $\mathcal{X}_\eta, \mathcal{X}_\eta'$ the map

\[
K_\eta^0 \to k(X_\eta)^* \quad D \mapsto \mathcal{X}_\eta(D)/\mathcal{X}_\eta'(D)
\]

is a homomorphism which is trivial on $i^{\ast}(K^0)$. Thus the above homomorphism descends to a homomorphism $\gamma: \operatorname{Cl}_\pi(X) \to k(X_\eta)^*$ by Lemma 4.3. Since

\[
\operatorname{div}(\mathcal{X}_\eta(D)/\mathcal{X}_\eta'(D)) = \operatorname{div}(\mathcal{X}_\eta(D)) - \operatorname{div}(\mathcal{X}_\eta'(D)) = D - D = 0
\]

and $X_\eta$ is complete, we deduce that $\mathcal{X}_\eta(D)/\mathcal{X}_\eta'(D) \in \bar{k}^* \cap k(X_\eta)^* = k^*$, where the last equality is by [11, Example 2.1.12]. In particular $\gamma \in \operatorname{Hom}(\operatorname{Cl}_\pi(X), k^*)$. On the other hand given such a $\gamma$, the product $\gamma \mathcal{X}_\eta$ is a character satisfying as in the statement of the lemma.

$\square$

Proof of Theorem 1. By Proposition 3.6, Lemma 4.4, the characterization of Cox rings in [2, Lemma 1.4.3.5] and [6, Construction 2.7] we have a commutative diagram

\[
\begin{array}{ccccccc}
0 & \xrightarrow{} & \Gamma(X, \mathcal{I}) & \xrightarrow{} & \Gamma(X, \mathcal{S}_K) & \xrightarrow{} & \mathcal{R}(X) & \xrightarrow{} & 0 \\
0 & \xrightarrow{} & \Gamma_\pi(X, \mathcal{I}) & \xrightarrow{} & \Gamma_\pi(X, \mathcal{S}_K) & \xrightarrow{} & \mathcal{R}_\pi(X) & \xrightarrow{} & 0 \\
0 & \xrightarrow{} & \Gamma(X_\eta, \mathcal{I}_\eta) & \xrightarrow{} & \Gamma(X_\eta, \mathcal{S}_{K_\eta}) & \xrightarrow{} & \mathcal{R}(X_\eta) & \xrightarrow{} & 0
\end{array}
\]

with exact rows, where $\Gamma_\pi(X, \mathcal{I})$ is the localization of the ideal $\Gamma(X, \mathcal{I})$. In particular the morphism $i_\pi: \mathcal{R}_\pi(X) \to \mathcal{R}(X_\eta)$ is surjective. Define the homomorphism of groups

\[
u: \operatorname{Cl}_\pi(X) \to \mathcal{R}_\pi(X)^*, \quad [D] \to i^\ast-1(\mathcal{X}_\eta(D)) + \Gamma_\pi(X, \mathcal{I}),
\]
where $D \in K^0_\eta$. By Lemma 4.4 the map $u$ is well defined since if $D' \in K^0_\eta$ is another representative of the same class, we have $D' - D = \iota^\ast(L)$ with $L \in K^0$ so that

$$\iota^\ast \left( \mathcal{X}_\eta(D + \iota^\ast(L)) \right) = \iota^\ast \left( \mathcal{X}_\eta(D) \cdot \mathcal{X}_\eta(\iota^\ast(L)) \right)$$

$$= \iota^\ast \left( \mathcal{X}_\eta(D) \right) \cdot \iota^\ast \left( \mathcal{X}(L) \right)$$

$$\equiv \iota^\ast \left( \mathcal{X}_\eta(D) \right) \mod \Gamma_\pi(X, I).$$

Observe that the homomorphism $u$ satisfies the condition $u(w) \in \mathcal{R}_\pi(X)^{\ast-w}$ for each $w$. Moreover given any two such homomorphisms $u, u'$, reasoning as in the proof of Lemma 4.4 we see that $u'/u \in \text{Hom}(\mathcal{Cl}_\pi(X), k^\ast)$. Thus $u'$ is defined by a character $\mathcal{X}_\eta'$ by the same Lemma 4.4. We conclude by describing the kernel of $\iota_R$:

$$\ker(\iota_R) = \{ s + \Gamma_\pi(X, I) : \iota^\ast(s) \in \Gamma(X_\eta, I_\eta) \}$$

$$= \{ \iota^\ast \left( s_\eta \right) + \Gamma_\pi(X, I) : s_\eta \in \Gamma(X_\eta, I_\eta) \}$$

$$= \langle 1 - \iota^\ast \left( \mathcal{X}_\eta(D) \right) + \Gamma_\pi(X, I) : D \in K^0_\eta \rangle$$

$$= \langle 1 - \iota((D)) : [D] \in \mathcal{Cl}(X) \rangle. \qedhere$$

5. Applications

In this section we give two applications of our main theorem to the Cox ring of a very general fiber and to toric fibrations, respectively.

5.1. Very general fibers. Let $\pi: X \to Y$ be a proper surjective morphism of normal varieties whose very general fiber is irreducible. Assume moreover that $\pi$ admits a rational section, that $\mathcal{Cl}(X)$ is finitely generated, so that the same holds for $\mathcal{Cl}(Y)$, and $\mathcal{Cl}(Y)$ is torsion-free.

Lemma 5.1. Let $X_i$, with $i \in \{1, 2\}$ be a normal variety defined over algebraically closed field $k_i$ of characteristic 0. Assume that each $\mathcal{Cl}(X_i)$ is finitely generated and that $k_i[X_i]^\ast = k_i^\ast$ holds for any $i$. If there is an isomorphism of fields $\varphi: k_2 \to k_1$ and an isomorphism of schemes $f: X_1 \to X_2$ such that the following diagram commutes

$\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow & & \downarrow \\
\text{Spec}(k_1) & \xrightarrow{\varphi^*} & \text{Spec}(k_2)
\end{array}$

then $f$ induces an isomorphism of graded rings $f^*: \mathcal{R}(X_2) \to \mathcal{R}(X_1)$ such that $f^*|_{k_2} = \varphi$.

Proof. Observe that $f$ induces the pullback isomorphism on the fields of rational functions $k_2(X_2) \to k_1(X_1)$. Given a prime divisor $D$ of $X_2$ the restriction $D \cap X_2^0$ to the smooth locus $X_2^0$ of $X_2$ is a Cartier non-trivial divisor, because $X_2 \setminus X_2^0$ has codimension at least two by the normality of $X_2$. Since $f$ is an isomorphism the pullback $f^*(D \cap X_2^0)$ is contained in the smooth locus $X_1^0$ of $X_1$ and it has a unique closure by the normality of $X_1$. By linearity the pullback map extends to an isomorphism $f^*: \mathcal{WDiv}(X_2) \to \mathcal{WDiv}(X_1)$ of the groups of Weil divisors, which maps principal divisors to principal divisors and thus gives also an isomorphism of divisor class groups $\mathcal{Cl}(X_2) \to \mathcal{Cl}(X_1)$. By the above discussion, given a Weil
divisor $D$ of $X_2$ and an open subset $U \subseteq X_2$, the pullback induces an isomorphism of Riemann-Roch spaces $\Gamma(U, \mathcal{O}_{X_2}(D)) \rightarrow \Gamma(f^{-1}(U), \mathcal{O}_{X_1}(f^*D))$. Thus, given a finitely generated subgroup $K \subseteq \text{WDiv}(X_2)$ which surjects onto $\text{Cl}(X_2)$, the pullback gives an isomorphism of sheaves of divisorial algebras $\mathcal{S}_K \rightarrow f_*\mathcal{S}_{f^*K}$, which induces an isomorphism of Cox sheaves. By taking global sections we get an isomorphism of sheaves of divisorial algebras.

Proof of Corollary 2. By [14, Lemma 2.1] there exists a subset $W \subseteq Y$ which is a countable intersection of non empty Zariski open subsets such that for each $b \in W$ there is an isomorphism of rings $\mathbb{C} \rightarrow \bar{k}$ which induces an isomorphism of schemes $X_b \rightarrow \bar{X}_b$. Therefore by Lemma 5.1 and Theorem 1 we obtain that the Cox ring of the very general fiber $X_b$ is isomorphic to the Cox ring of the geometric generic fiber $\bar{X}_b$. The isomorphism between $\text{Cl}(\bar{X}_b)$ and $\text{Cl}(X_b)$ implies that the former can be generated by classes of divisors in $\text{WDiv}(\bar{X}_b)$. We get the statement since by Section 1 the Cox ring $\mathcal{R}(\bar{X}_b)$ is obtained from $\mathcal{R}(X_b)$ by a base change.

5.2. Toric varieties. Let $X = X(\Sigma)$ and $Y = Y(\Sigma')$ be smooth complete toric varieties defined by fans $\Sigma \subseteq N_\mathbb{Q}$ and $\Sigma' \subseteq N'_\mathbb{Q}$ respectively, with lattices of one parameter subgroups $N$ and $N'$ and character groups $M = \text{Hom}(N, \mathbb{Z})$ and $M' = \text{Hom}(N', \mathbb{Z})$. Assume there is a toric proper surjective morphism $\pi: X \rightarrow Y$ with connected fibers, induced by a $\mathbb{Z}$-linear map $\alpha: N \rightarrow N'$.

The isomorphism between $\text{Cl}(\bar{X}_b)$ and $\text{Cl}(X_b)$ implies that the former can be generated by classes of divisors in $\text{WDiv}(\bar{X}_b)$. We get the statement since by Section 1 the Cox ring $\mathcal{R}(\bar{X}_b)$ is obtained from $\mathcal{R}(X_b)$ by a base change.

Proof. If we denote by $T_{N_0}$ the big torus acting on $Y$, by [5, §3.3] we have that $\pi^{-1}(T_Y) \cong T_Y \times X_0$, where $X_0 = X(\Sigma_0)$ is the toric variety associated to the fan $\Sigma_0 \subseteq (N_0)_\mathbb{Q}$. In particular each fiber over $T_Y$ has the same defining fan $\Sigma_0$, which coincides with that of the generic fiber $X_\eta$.

As a consequence of Corollary 2 the very general fiber $X_t$ of $\pi$ has Cox ring isomorphic to the Cox ring of the generic fiber (as rings but not as $\mathbb{C}$-algebras). Since $X_t$ is isomorphic to $X_0$ for any $t \in T_N$, we deduce the isomorphism

$$\mathcal{R}(X_\eta) \cong \mathcal{R}(X_0) \otimes_{\mathbb{C}} \mathbb{C}(Y).$$

Proposition 5.2 provides a description of the fan of $X_\eta$ and thus also of the Gale dual of its degree matrix. Another way to get the same combinatorial data is to compute the degree matrix of the toric variety $X_\eta$ and then compute its Gale dual. Let $Q$ be the degree matrix of $X$ and let $r^*: \text{Cl}(X) \rightarrow \text{Cl}(X_0)$ be the quotient map.
Write $Q = (Q'_0 \mid Q_π)$, where $Q_π$ consists of all the columns of $Q$ mapped to zero by $ι^*$ and $Q'_0$ are the remaining ones. Define the matrix

$$Q_0 := (ι^*(w) : w \text{ is a column of } Q'_0).$$

The matrix $Q_0$ so defined is the degree matrix of $X_0$ by Theorem 1 and thus it is also the degree matrix of $X_0$ by our previous observation. The Gale dual of $Q_0$ can be determined as follows: compute the Gale $P$ dual of $Q$ and write it as $P = (P'_0 \mid P_π)$, where the column indices of $P$ correspond to those of $Q$. Then $P_0$ is the unique solution of the linear equation

$$j \cdot P_0 = P'_0,$$

by Proposition 5.2.

**Example 5.3.** Let $n \geq 3$ and let $X = \mathbb{F}(a_1, \ldots, a_n)$ be the scroll over $Y = \mathbb{P}^1$ defined by the vector $(a_1, \ldots, a_n)$. The Cox ring of $X$ is $\mathbb{C}[T_1, \ldots, T_{n+2}]$, with grading matrix and Gale dual that can be written in the following way

$$Q = \begin{bmatrix} -a_1 & \cdots & -a_n & 1 & 1 \\ 1 & \cdots & 1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & \cdots & 0 & 1 & -1 \\ 1 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & -1 & 0 \end{bmatrix},$$

so that $Q_0 = [1 \ 1 \ \ldots \ 1]$ while $P_0$ is obtained from $P'_0$ by removing the first row since here $j: \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ is defined by $v \mapsto (0, v)$.

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Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

E-mail address: alaface@udec.cl

Dipartimento di Matematica e Informatica, Università degli studi di Palermo, Via Archirafi 34, 90123 Palermo, Italy

E-mail address: luca.ugaglia@unipa.it