A note on Hedetniemi’s conjecture, Stahl’s conjecture and the Poljak-Rödl function

Claude Tardif
Royal Military College of Canada
Canada
Claude.Tardif@rmc.ca

Xuding Zhu*
Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang, China
xdzhu@zjnu.edu.cn

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Abstract

We prove that $\min\{\chi(G), \chi(H)\} - \chi(G \times H)$ can be arbitrarily large, and that if Stahl’s conjecture on the multichromatic number of Kneser graphs holds, then we can have $\chi(G \times H) / \min\{\chi(G), \chi(H)\} \leq 1/2 + \epsilon$ for large values of $\min\{\chi(G), \chi(H)\}$.

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1 Introduction

The categorical product $G \times H$ of graphs $G$ and $H$ has vertex set $V(G \times H) = \{(x, y) : x \in V(G), y \in V(H)\}$, in which two vertices $(x, y)$ and $(x', y')$ are adjacent if and only if $xx' \in E(G)$ and $yy' \in E(H)$. A proper colouring $\phi$ of $G$ can be lifted to a proper colouring $\Phi$ of $G \times H$ defined as $\Phi(x, y) = \phi(x)$. So $\chi(G \times H) \leq \chi(G)$, and similarly $\chi(G \times H) \leq \chi(H)$. Hedetniemi conjectured in 1966 that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ for all finite graphs $G$ and $H$ [6]. The conjecture received a lot of attention [7, 10, 13, 14] and remained open for more than half century. It is known that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ whenever $\min\{\chi(G), \chi(H)\} \leq 4$ [1] and that the fractional version is true, i.e., for any graphs $G$ and $H$, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$ [15]. However, Shitov refuted this conjecture recently [12]. Yet, some problems concerning the chromatic number of product graphs remain open.

The Poljak-Rödl function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$f(n) = \min\{\chi(G \times H) : \chi(G), \chi(H) \geq n\}.$$
Hedetniemi’s conjecture is equivalent to the statement that \( f(n) = n \) for all \( n \). Shitov proved that for sufficiently large \( n \), \( f(n) \leq n - 1 \). Still, very little is known about the behavior of the function \( f(n) \). In particular, it is unknown whether \( f(n) \) is bounded by a constant. However it is known that if \( f(n) \) is bounded by a constant, then \( f(n) \leq 9 \) for all \( n \) (see [10, 14]). In this note, we prove the following facts.

**Proposition 1.**

(i) \( \lim_{n \to \infty} (n - f(n)) = \infty \),

(ii) If Stahl’s conjecture on the multichromatic number of Kneser graphs [11] holds, then \( \lim_{n \to \infty} f(n)/n \leq 1/2 \).

Proposition 1 (i) will be proved in Section 2. Proposition 1 (ii) will be proved in Section 3, where a presentation of Stahl’s conjecture is also given.

# 2 Discussion and extensions of Shitov’s results

For a positive integer \( c \), the *exponential graph* \( K^H_c \) has vertices all the mappings \( f : V(H) \to \{1, 2, \ldots, c\} \), in which \( f, g \) are adjacent in \( K^H_c \) if \( f(u) \neq g(v) \) for every edge \( e = uv \) of \( H \). It is well known and easy to verify that \( \Phi(v, f) = f(v) \) is a proper \( c \)-colouring of \( H \times K^H_c \). Thus the way to find counterexamples to Hedetniemi’s conjecture is to find an integer \( c \) and a graph \( H \) such that both \( H \) and \( K^H_c \) have chromatic number larger than \( c \).

The *lexicographic product* \( G[H] \) of \( G \) and \( H \) is the graph with vertex set \( V(G[H]) = \{(x, y) : x \in V(G), y \in V(H)\} \), in which two vertices \((x, y)\) and \((x', y')\) are adjacent if and only if \( xx' \in E(G) \), or \( x = x' \) and \( yy' \in E(H) \).

Shitov’s construction of counterexamples to Hedetniemi’s conjecture is based on the following result.

**Theorem 2** ([12], Claim 3). *For any graph \( G \) with girth at least six, for all but finitely many values of \( q \), we have \( \chi(K^G[K^H_q]) \geq c + 1 \), with \( c = \lceil 3.1q \rceil^1 \).*

Finding such a lower bound on chromatic numbers of some exponential graphs was the key part of Shitov’s refutation of Hedetniemi’s conjecture. Finding lexicographic products \( G[H] \) with \( \chi(G[K^H_q]) > c \) is standard theory. Indeed the fractional chromatic number \( \chi_f(H) \) of a graph \( H \) is a standard lower bound for its chromatic number, and it is well known that \( \chi_f(G[H]) = \chi_f(G)\chi_f(H) \) (see [3]). Erdős’ classic probabilistic proof [2] shows that there are graphs with girth at least 6 and fractional chromatic number at least 3.1. For such a graph \( G \), we have \( \chi(G[K^H_q]) \geq \lceil \chi_f(G[K^H_q]) \rceil = \lceil \chi_f(G) \cdot q \rceil \geq \lceil 3.1q \rceil \), and by Theorem 2, this yields a counterexample to Hedetniemi’s conjecture.

Remarkably, replacing the condition \( \chi_f(G) \geq 3.1 \) by \( \chi_f(G) \geq B \) for \( B \geq 3.1 \) readily gives counterexamples to Hedetniemi’s conjecture where the chromatic number of at least

\(^1\)Technically, Shitov refers to the “strong product” rather than the lexicographic product of graphs, but with \( K^H_q \) as a second factor, the strong product coincides with the lexicographic product (see [4]).
one factor is arbitrarily larger than the chromatic number of the product. Also, the proof of Theorem 2 only uses a small subgraph of $K_{c}^{G[K_{q}]}$. Therefore it is possible that Shitov’s construction already gives examples that show that $\lim_{n \to \infty} f(n)/n = 0$. On the other hand, since $\chi_f(G[K_{q}]) > c$, the fractional version of Hedetniemi’s conjecture [15] implies that $\chi_f(K_{c}^{G[K_{q}]}), c$. Thus it is also reasonable to think that $\chi(K_{c}^{G[K_{q}]}))c$ may be bounded, and that the identity $\lim_{n \to \infty} f(n)/n = 0$, if true, can only be witnessed by a different construction.

Proof of Proposition 1 (i). Fix a positive integer $d$. We shall prove that if $n$ is sufficiently large, then $f(n + d) \leq n$. Let $G_d$ be a graph with girth at least 6 and fractional chromatic number at least $8d$. Then by Theorem 2, for sufficiently large $q$ and $c = [3.1q]$, we have $\chi(K_{c}^{G_d[K_{q}]}))c + 1$ while $\chi(G_d[K_{q}])) 2cd$. Now consider the graph $K_{cd}^{G_d[K_{q}]}$. For $i = 0, 1, \ldots, d - 1$, let $Q_i$ be the subgraph of $K_{cd}^{G_d[K_{q}]}$ induced by the functions with image in $\{ic + 1, ic + 2, \ldots, ic + c\}$. Each $Q_i$ is isomorphic to $K_{cd}^{G_d[K_{q}]}$ and hence at least $c + 1$ colours are needed for each copy. For $i \neq j$, each function in $Q_i$ is adjacent to each function in $Q_j$. Hence, $\chi(K_{cd}^{G_d[K_{q}]})) d(c + 1)$. As $\chi(K_{cd}^{G_d[K_{q}]})) dc$ and $\chi(G_d[K_{q}])) 2cd \geq cd + d$, it follows that $f(dc + d) \leq dc$.

Thus for every $d$ there exist infinitely many values of $n$ (of the form $dc + d$) such that $n - f(n)) d$. It only remains to show that the gap between $n$ and $f(n)$ will not close while going from one value of $c$ to the next. Note that $c = [3.1q]$, where $q$ is any value above a fixed threshold, and $[3.1(q + 1)] - [3.1q] \leq 4$. Thus it suffices to examine the values $n = dc + d + i$ where $i \leq 4d$, and we can suppose that $c \geq 5$. The graph $K_{cd+1}^{G_d[K_{q}]}$ contains a copy of $K_{cd}^{G_d[K_{q}]}$ induced by the functions with image in $\{1, 2, \ldots, cd\}$. For $j = cd + 1, cd + 2, \ldots, cd + i$, the constant functions $g_j$ with image $j$ are pairwise adjacent and each is adjacent to all the functions in $K_{cd}^{G_d[K_{q}]}$. Hence $\chi(K_{cd+1}^{G_d[K_{q}]})) \chi(K_{cd}^{G_d[K_{q}]})) + i \geq cd + d + i$. For $i \leq (c - 1)d$, we also have $\chi(G_d[K_{q}])) cd + d + i$, so that $f(cd + d + i) \leq cd + i$. Altogether, the inequality $f(n + d) \leq n$ is established for all but finitely many values of $n$. Thus, $\lim_{n \to \infty} n - f(n) = \infty$.

The gap between $n$ and $f(n)$ proved in this section depends on the minimum number $p$ of vertices of a girth 6 graph with fractional chromatic number at least $8d$. The best known upper bound for $p$ to our knowledge is $p = O((d \log d)^{2})$, which follows from a result of Krivelevich [8]. Using this result, one can show that for any $\epsilon > 0$, there is a constant $a$ such that for sufficiently large $n$, $f(n) \leq n - a(\log n)^{1/2 - \epsilon}$. Very recently, He and Wigderson [5] proved that for some $\epsilon \simeq 10^{-9}$, $f(n) < (1 - \epsilon)n$ for sufficiently large $n$. The examples are again cases of Shitov’s construction.

3 Stahl’s conjecture

In the proof of Proposition 1(i), based on the fact that $\chi(K_{c}^{G_d[K_{q}]})) c + 1$, we have shown that $\chi(K_{cd}^{G_d[K_{q}]})) cd + d$. In this section, we show that if a special case of a conjecture
of Stahl on the multichromatic number of Kneser graphs is true, then \( \chi(K_{cd}^{G_d[K_q]}) \) is much larger.

Consider a proper colouring \( \phi \) of the graph \( K_{cd}^{G_d[K_q]} \) with \( x \) colours. Let \( A \) be a subset of \( \{1, \ldots, cd\} \) of cardinality \( c \). Let \( R_A \) be the subgraph of \( K_{cd}^{G_d[K_q]} \) induced by the functions with image contained in \( A \). Then \( R_A \) is isomorphic to \( K_{\chi(K_{cd}^{G_d[K_q]})}^{G_d[K_q]} \), so \( \phi \) uses at least \( c + 1 \) colours on \( R_A \). Let \( \psi(A) \) be a subset of exactly \( c+1 \) colours used by \( \phi \) on \( R_A \). We have \( \psi(A) \) disjoint from \( \psi(B) \) whenever \( A \) is disjoint from \( B \), because \( R_A \) is totally joined to \( R_B \) in \( K_{cd}^{G_d[K_q]} \). This property can be formulated in terms of homomorphisms of Kneser graphs. Recall that the vertices of the Kneser graph \( K(n, \alpha) \) are the \( n \)-subsets of \( \{1, \ldots, \alpha\} \), and two of these are joined by an edge whenever they are disjoint. Thus the colouring \( \phi : K_{cd}^{G_d[K_q]} \rightarrow K_x \) induces a homomorphism \( \psi : K(cd, c) \rightarrow K(x, c + 1) \). The question is how large does \( x \) need to be for such a homomorphism to exist.

Stahl’s conjecture deals with the latter question. For an integer \( n \), the \( n \)-th multichromatic number \( \chi_n(H) \) of a graph \( H \) is the least integer \( m \) such that \( H \) admits a homomorphism to \( K(m, n) \). In particular \( \chi_1(H) = \chi(H) \). Lovász [9] proved that \( \chi_1(K(m, n)) = \chi(K(m, n)) = m - 2n + 2 \). Stahl [11] investigated the general multichromatic numbers of Kneser graphs, and observed the following.

\[ \begin{align*}
\textbf{a.} & \quad \text{For } 1 \leq k \leq n, \chi_k(K(m, n)) = m - 2(n - k), \\
\textbf{b.} & \quad \chi_{kn}(K(m, n)) = km, \\
\textbf{c.} & \quad \chi_{k+k'}(K(m, n)) \leq \chi_k(K(m, n)) + \chi_{k'}(K(m, n)).
\end{align*} \]

Based on this he conjectured the following.

**Conjecture 3** ([11]). If \( k = an + b \), \( a \geq 1, 0 \leq b \leq n - 1 \), then for \( m \geq 2n \),

\[ \chi_k(K(m, n)) = \chi_{an}(K(m, n)) + \chi_b(K(m, n)) = (a + 1)m - 2(n - b). \]

**Proof of Proposition 1 (ii).** For a fixed \( d \), let \( G_d \) have girth at least 6 and fractional chromatic number at least \( 8d \). For any \( q \) above a given threshold \( q_d \) and for \( c = \lfloor 3.1q \rfloor \), we have \( \chi(G_d[K_q]) \geq 2cd \) and \( \chi(K_{cd}^{G_d[K_q]}) \geq \chi_{c+1}(K(cd, c)) \), as explained in the first three paragraphs of this section. If Stahl’s conjecture holds, then \( \chi(K_{cd}^{G_d[K_q]}) \geq 2cd - 2c + 2 \). Since \( f \) is monotonic, this gives \( f(2cd - 2c + 2) \leq cd \). Therefore

\[ n \in [2(c - 4)d - 2(c - 4) + 2, 2cd - 2c + 2] \text{ implies } f(n) \leq \frac{cd}{2(c - 4)d - 2(c - 4) + 2}. \]

The intervals \( [2(c - 4)d - 2(c - 4) + 2, 2cd - 2c + 2], c \in \mathbb{N} \) cover all but a finite part of \( \mathbb{N} \). Hence

\[ \limsup \frac{f(n)}{n} \leq \lim_{c \to \infty} \frac{cd}{2(c - 4)d - 2(c - 4) + 2} = \frac{d}{2d - 2}. \]

Since this holds for arbitrarily large \( d \), \( \limsup \frac{f(n)}{n} \leq \frac{1}{2}. \)
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