CENTER STABLE MANIFOLDS AROUND LINE SOLITARY WAVES OF THE ZAKHAROV–KUZNETSOV EQUATION WITH CRITICAL SPEED

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Abstract. In this paper, we construct center stable manifolds around unstable line solitary waves of the Zakharov–Kuznetsov equation on two dimensional cylindrical spaces \( \mathbb{R} \times \mathbb{T}_L \) \( (\mathbb{T}_L = \mathbb{R}/2\pi L \mathbb{Z}) \). In the paper [39], center stable manifolds around unstable line solitary waves have been constructed excluding the case of critical speeds \( c \in \{4n^2/5L^2; n \in \mathbb{Z}, n > 1\} \). In the case of critical speeds \( c \), any neighborhood of the line solitary wave with speed \( c \) in the energy space includes solitary waves which depend on the direction \( \mathbb{T}_L \). To construct center stable manifolds including their solitary waves and to recover the degeneracy of the linearized operator around line solitary waves with critical speed, we prove the stability condition of the center stable manifold for critical speed by applying to the estimate of the 4th order term of a Lyapunov function in [37] and [38].

1. Introduction. We consider the two dimensional Zakharov–Kuznetsov equation
\[
     u_t + \partial_x (\Delta u + u^2) = 0, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}_L,
\]
where \( \Delta = \partial_x^2 + \partial_y^2 \), \( u = u(t, x, y) \) is an unknown real-valued function, \( \mathbb{T}_L = \mathbb{R}/2\pi L \mathbb{Z} \) and \( L > 0 \). The equation (1) preserves the mass and the energy:
\[
     M(u) = \int_{\mathbb{R} \times \mathbb{T}_L} |u|^2 dx dy,
\]
\[
     E(u) = \int_{\mathbb{R} \times \mathbb{T}_L} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{3} u^3 \right) dx dy,
\]
for \( u \in H^1(\mathbb{R} \times \mathbb{T}_L) \).

The Zakharov–Kuznetsov equation was introduced in [40] to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma. The rigorous derivation of the Zakharov–Kuznetsov equation was proved in [18]. The Cauchy problem for the Zakharov–Kuznetsov equation is extensively studied in the literature. The global well-posedness of the Zakharov–Kuznetsov equation in \( H^s(\mathbb{R} \times \mathbb{T}_L) \) for \( s > \frac{3}{2} \)

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has been proved by Linares, Pastor and Saut [19] to study the transverse instability of the N-soliton of the Korteweg–de Vries equation. Molinet and Pilod [26] showed the global well-posedness in $H^1(\mathbb{R} \times T_L)$ by proving a bilinear estimate in the context of Bourgain’s spaces $X^{s,b}$.

The Zakharov–Kuznetsov equation has solitary wave solutions $\varphi(x - ct, y)$ which are nontrivial solutions to the stationary equation

$$- \Delta \varphi + c \varphi - \varphi^2 = 0$$

(2)

for $c > 0$. de Bouard [9] proved the orbital stability of positive solitary waves in $H^1(\mathbb{R}^2)$ which are ground states defined by the action

$$S_c(u) = E(u) + \frac{c}{2} M(u)$$

of (2) on $\mathbb{R}^2$. Côte, Muñoz, Pilod and Simpson [7] proved the asymptotic stability of the positive solitary waves and multi solitary waves in $H^1(\mathbb{R}^2)$ by adapting the argument of Martel and Merle [22, 23] to the multidimensional model.

The Zakharov–Kuznetsov equation is one of the two dimensional extensions of the Korteweg–de Vries equation

$$u_t + \partial_x (\partial^2_x u + u^2) = 0 \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$

which is complete integral. Schiavomaggi [35] showed the Zakharov–Kuznetsov equation posses the Painlevé property. In a numerical result [10], Iwasaki, Toh and Kawahara showed that a collision of two pulses generates the radiation of ripples. This numerical result suggests that the Zakharov–Kuznetsov equation is not complete integrable and does not have the inverse scattering transform similar to that of the KdV equation. The Zakharov–Kuznetsov equation has the line solitary wave

$$Q_c(x - ct) = \frac{3c}{2} \cosh^{-2} \left( \frac{\sqrt{c}(x - ct)}{2} \right), \quad c > 0$$

which is also the one soliton of the KdV equation and is a solution to the equation

$$-\partial^2_x Q_c + cQ_c - Q^2_c = 0.$$
sufficiently large $L$. Using the linear instability of the line solitary waves in [5], Rousset and Tzvetkov [33] proved the orbital instability of the line solitary waves of the Zakharov–Kuznetsov equation on $\mathbb{R}^2$ by proving the general framework to show the transverse instability for Hamiltonian partial differential equations. Johnson [13] proved the linear instability of line periodic solitary waves of the generalized Zakharov–Kuznetsov equation on $\mathbb{T}_{L_1} \times \mathbb{T}_{L_2}$ with sufficiently large $L_2$ by applying the Evans function method. In [38], the author proved that the line solitary waves $Q_c(x - ct)$ of the Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$ are orbitally stable and asymptotically stable for $0 < c \leq \frac{4}{5L^2}$ and is orbitally unstable for $c > \frac{4}{5L^2}$. To show the asymptotic stability of the line solitary waves $Q_c(x - ct)$ for $H^1$-data, the author applied the argument in [7, 23] which relies on the Liouville type theorem for spatially decaying functions around line solitary waves. In the case of $c = \frac{4}{5L^2}$, to get the coerciveness of the virial type estimate around $Q_c(x - ct)$, the author introduced a correction term of the virial type estimate which is derived from bifurcated solitary waves. Pelinovsky [31] proved the asymptotic stability of the transversely modulated solitary waves of the Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$ with exponentially weighted spaces.

The linearized operator of line solitary waves with $Q_c(x - ct)$ has extra eigenfunctions with respect to the 0 eigenvalue for critical speed

$$c \in \text{CS} = \left\{ \frac{4a_0^2}{5L^2}; a_0 \in \mathbb{Z}, n > 1 \right\}.$$ 

The degeneracy of the kernel of the linearized operator of line solitary waves with $Q_c(x - ct)$ at $c \in \text{CS}$ involves a symmetry breaking bifurcation of line solitary waves, which is given by the following proposition. The instability of line solitary waves occurs at the first symmetry breaking bifurcation point $Q_{4/5L^2}$.

**Proposition 1.** Let $c^* = \frac{4a_0^2}{5L^2} \in \text{CS}$. There exist $C_*, \delta_0 > 0$, $\tilde{c} \in C^2((-\delta_0, \delta_0)^2, (0, \infty))$ and $\varphi_{c^*} \in C^2((-\delta_0, \delta_0)^2, H^2(\mathbb{R} \times \mathbb{T}_L))$ such that $\varphi_{c^*}(\mathbf{a})(x, y) = \varphi_{c^*}(\mathbf{a})(-x, y)$, $\varphi_{c^*}(\mathbf{a}) > 0$, $\tilde{c}(\mathbf{a}) = \tilde{c}(|\mathbf{a}|, 0)$,

$$-\Delta \varphi_{c^*}(\mathbf{a}) + \tilde{c}(\mathbf{a})\varphi_{c^*}(\mathbf{a}) - (\varphi_{c^*}(\mathbf{a}))^2 = 0,$$

$$\varphi_{c^*}(\mathbf{a}) = Q_{c^*} + a_0 Q_{c^*}^3 \cos \frac{ny_0y}{L} + a_1 Q_{c^*}^3 \sin \frac{ny_0y}{L} + O(|\mathbf{a}|^2) \text{ as } |\mathbf{a}| \to 0,$$

$$\|\varphi_{c^*}(\mathbf{a})\|_{L^2}^2 = \|Q_{c^*}\|_{L^2}^2 + \frac{C_{2,c^*}}{2} |\mathbf{a}|^2 + o(|\mathbf{a}|^2) \text{ as } |\mathbf{a}| \to 0,$$

and

$$\dot{\tilde{c}}(\mathbf{a}) = c^* + \frac{C_{2,c^*}}{2} |\mathbf{a}|^2 + o(|\mathbf{a}|^2) \text{ as } |\mathbf{a}| \to 0$$

for $\mathbf{a} = (a_0, a_1) \in (-\delta_0, \delta_0)^2$, where

$$C_{2,c^*} = \frac{3C_* \|Q_{c^*}\|_{L^2}^2}{2c^*} - \frac{5}{2} \left\|Q_{c^*}^3 \cos \frac{ny_0y}{L}\right\|_{L^2}^2 > 0.$$ 

**Remark 1.** In the case $n_0 = 1$, Proposition 1 was shown in Proposition 1.3 of [38]. By applying the Lyapunov–Schmidt reduction and the Crandall–Rabinowitz Transversality in [8, 17] and the proof of Proposition 1.3 in [38], we can prove Proposition 1 for $n_0 \in \mathbb{Z} \setminus \{-1, 0, 1\}$. We show the sketch of the proof of Proposition 1 in Appendix A.
In this paper, we construct center stable manifolds around unstable line solitary waves to the Zakharov–Kuznetsov equation on \( \mathbb{R} \times \mathbb{T}_L \). To introduce the main results, we define some notations. The orbit of the solitary wave \( Q_c \) is defined by

\[
S(c) = \{ \tau_q Q_c ; q \in \mathbb{R} \}
\]

and the tubular neighborhood of the orbit of the solitary wave \( Q_c \) is defined by

\[
\mathcal{N}_{\delta, c} = \{ u \in H^1(\mathbb{R} \times \mathbb{T}_L) ; \inf_{q \in \mathbb{R}} \| u - \tau_q Q_c \|_{H^1} < \delta \},
\]

where \((\tau_q u)(x, y) = u(x-q, y)\). By the global well-posedness result in [26], we define \( U(t) \) by the flow map of (1) at time \( t \). The following is the main theorem.

**Theorem 1.2.** Let \( c^* > \frac{4}{5L^2} \). Then, there exists a \( C^1 \) submanifold \( \mathcal{M}_{cs}(c^*) \) of \( H^1(\mathbb{R} \times \mathbb{T}_L) \) containing the orbit \( S(c^*) \) with the following properties:

(i) The codimension of \( \mathcal{M}_{cs}(c^*) \) in \( H^1(\mathbb{R} \times \mathbb{T}_L) \) equals \( \min \{ n \in \mathbb{Z} ; \frac{\sqrt{5}-1}{2} - 1 \leq n \} \) which is the total dimension of the eigenspaces of the linearized operator \( \partial_x (\Delta + c^* - 2Q_{c^*}) \) corresponding to eigenvalues with positive real part.

(ii) \( \tau_q U(t) \mathcal{M}_{cs}(c^*) \subset \mathcal{M}_{cs}(c^*) \) for \( q \in \mathbb{R} \) and \( t \geq 0 \).

(iii) \( \mathcal{M}_{cs}(c^*) \) is normal at \( Q_{c^*} \) to the eigenspaces corresponding to eigenvalues of \( \partial_x (\Delta + c^* - 2Q_{c^*}) \) with positive real part (see Corollary 3 for the detail).

(iv) For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( U(t)(\mathcal{M}_{cs}(c^*) \cap \mathcal{N}_{\delta, c^*}) \subset \mathcal{M}_{cs}(c^*) \cap \mathcal{N}_{\varepsilon, c^*} \) for \( t \geq 0 \).

(v) There is \( \varepsilon_0 > 0 \) such that for \( u_0 \in \mathcal{N}_{\varepsilon_0, c^*} \setminus \mathcal{M}_{cs}(c^*) \) there exists \( t_0 > 0 \) satisfying \( U(t_0)u_0 \notin \mathcal{N}_{\varepsilon_0, c^*} \).

**Remark 2.** In this paper, we only consider the solutions of (1) in [26] which are in the Bourgain spaces in local time. If the unconditional uniqueness of the solutions to (1) in \( C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L)) \) is proved, we can obtain the center stable manifold without the restriction of the class of solutions.

**Remark 3.** In [39], we obtain the existence of the center stable manifold \( \mathcal{M}_{cs}(c^*) \) for \( c^* \notin \text{CS} \). Since the linearized operator \( \partial_x (\Delta + c^* - 2Q_{c^*}) \) with critical speed \( c^* = 4n^2/5L^2 \) has extra eigenfunctions corresponding to the 0 eigenvalue for any positive integer \( n \), it is difficult to obtain Theorem 1.2 by applying the argument in [39] to the case of critical speed \( c^* = 4n^2/5L^2 \) directly.

**Remark 4.** If \( c \) is close to \( c^* \), then \( Q_c(x - ct) \) also close to the orbit \( S(c^*) \). By (v) of Theorem 1.2, the center stable manifold \( \mathcal{M}_{cs}(c^*) \) contains the orbit \( S(c) \) near by \( S(c^*) \). Particularly, in the case of critical speed \( c^* \in \text{CS} \), \( \mathcal{M}_{cs}(c^*) \) contains the orbit of solitary waves \( \varphi_{c^*}(\alpha)(x - \hat{c}(\alpha)t, y) \) near by \( S(c^*) \). Then, for \( 0 < c^* - c \ll 1 \) and \( |\alpha| \ll 1 \), \( Q_{c^*}(x - ct) \) and \( \varphi_{c^*}(\alpha)(x - \hat{c}(\alpha)t, y) \) are stable on \( \mathcal{M}_{cs}(c^*) \). In Appendix, we prove the stability of solitary waves near by \( Q_{c^*}(x - c^*t) \) on \( \mathcal{M}_{cs}(c^*) \) (see Corollary 4).

By developing the Hadamard method, Bates and Jones [1] proved a general theorem for the existence of invariant manifolds around equilibria of nonlinear partial differential equations. In [1], applying the general theorem and using a estimate of a Lyapunov function, Bates and Jones showed the existence of a Lipschitz center stable manifold of a stationary solution for the nonlinear Klein–Gordon equation on \( \mathbb{R}^n \) with the power nonlinearity \( |u|^pu \) \((0 < p < \frac{2}{n-2})\) under the radial symmetry restriction and the assumption which is the triviality of the null space of the linearized
operator. Nakanishi and Schlag [29] constructed center stable manifolds of ground states for the nonlinear Klein–Gordon equation on $\mathbb{R}^n$ with the $H^1$-subcritical power nonlinearity without the radial symmetry restriction. In [1], Bates and Jones assume the Lipschitz continuity of the function $|u|^p u : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Using the Strichartz norm to the contraction of center stable manifolds, Nakanishi and Schlag treated the $H^1$-subcritical power nonlinearity in [29]. In the case of the non-radial symmetry, the linearized operator around ground states has non-trivial null space which comes from the translation symmetry and yields a derivative loss in the Hadamard graph contraction argument. To treat the derivative loss term due to the translation, in [29] Nakanishi and Schlag introduce the mobile distance for the construction of center stable manifolds. By using the Strichartz estimate of the linear evolution around ground states, Schlag [34] constructed center stable manifolds in $W^{1,1} \cap W^{1,2}$ around ground states for the 3D cubic nonlinear Schrödinger equation and proved the asymptotic behavior of solutions on the center stable manifolds. Applying the argument in [34], Krieger and Schlag [14] constructed center stable manifolds around ground states for the 1D nonlinear Schrödinger equation with the $L^2$-supercritical nonlinearity. The result [34] was improved by Beceanu [2] who constructed center stable manifolds around ground states for the 3D cubic nonlinear Schrödinger equation on the critical space $\dot{H}^1(\mathbb{R}^3)$. By proving the trichotomy results which classifies initial datum near the one soliton by the asymptotic behavior of solutions, Martel, Merle, Nakanishi and Raphael [24] constructed center stable manifolds around the one soliton for the $L^2$-critical generalized KdV equation. By using a smooth bundle coordinate system instead of a translational parametrization, Jin, Lin and Zeng [12] constructed center stable manifolds around the one soliton for the $L^2$-supercritical generalized KdV equation. In [12], applying the smoothing estimate for solutions to the Airy equation, Jin, Lin and Zeng treated the nonlinearity of the generalized KdV equation with a loss of derivative. In [39], the author constructed center stable manifolds around line solitary waves $Q_{c_0}(x - c_0 t)$ with $c \notin \text{CS}$ for (1) by modifying the mobile distance in [29]. Using the bi-linear estimate on Fourier restriction spaces in [26], the author controlled a loss of derivative for the nonlinearity of (1). For more results of center stable manifolds around relative equilibria, we refer to the papers [3, 11, 15, 16, 28], and the references therein.

Applying the argument in [39] with a modulation of extra eigenfunctions corresponding to the 0 eigenvalue, we obtain the existence of Lipschitz invariant manifolds around $Q_{c^*}(x - c^*t)$ with critical speed $c^* \in \text{CS}$. For $c_0 \in \{4n^2/5L^2; n \in \mathbb{Z}, n \geq 1\}$, the linearized operator of (1) around $Q_{c_0}(x - c_0 t)$ has extra eigenfunctions corresponding to the 0 eigenvalue and $Q_{c_0}(x - c_0 t)$ is a bifurcation point of the stationary equation (2) with the bifurcation parameter $c$. We cannot control this degeneracy of the linearized operator by modulations for the translation symmetry of the equation (1). Therefore, we cannot use the coercivity of the linearized operator around line solitary waves $Q_{c^*}(x - c^*t)$ on the invariant manifold to show the orbital stability of line solitary waves $Q_{c^*}(x - c^*t)$ with critical speed $c^* \in \text{CS}$ on the invariant manifold. The orbital stability and the orbital instability of standing waves of the nonlinear Schrödinger equation with the degeneracy of the linearized operator of the evolution equation was proved by Comech and Pelinovsky [6], Maeda [21] and the author [37]. In the case that the linearized operator of the evolution equation has extra eigenfunctions corresponding to the 0 eigenvalue, the positivity of the higher order term of the Taylor expansion for Lyapunov functions derives
the orbital stability of standing waves. In Section 3, to prove the orbital stability of \( Q_{c^*} (x - c^* t) \) on the invariant manifold around \( Q_{c^*} (x - c^* t) \), we apply the fourth order estimate of the Lyapunov function \( S_{c^*} (u) \) for \( Q_{4/5L^2} (x - 4t/5L^2) \) in [37, 38].

By the spectral decomposition \( u = P_+ u + P_- u + P_0 u + P_\gamma u \) around line solitary wave \( Q_{c^*} (x - c^* t) \) (see Proposition 2 for the detail of the statement) and the estimate in [29, 39], we obtain that the order of unstable modes \( P_+ u \) on the invariant manifolds is controlled by the order of \( u \) and the order of stable modes \( P_- u \) on the invariant manifolds is controlled by the second order of \( u \). Since the dominant order of the positive term of the Lyapunov function \( S_{c^*} (u) \) around the line solitary wave is 4, by using the dominant order of the positive term of \( S_{c^*} (u) \), we cannot control the error term which is the \( L^2 \)-inner product of \( P_+ u \) and \( L_{c^*} P_- u \). To obtain a sharper estimate of the error term, we construct invariant manifolds which is a Lipschitz graph function on the stable invariant space of the linearized operator with the Hölder exponent \( \frac{3}{4} < \alpha < 2 \) at \( Q_{c^*} (x - c^* t) \) (see the definition of \( G_{l_1, l_2, \alpha, \delta, \kappa}^c \) in Section 3 for the detail). By the order of the invariant manifolds at \( Q_{c^*} (x - c^* t) \), we have that the order of unstable modes \( P_+ u \) on the invariant manifolds is \( \alpha \). Thus, the order of the error term is controlled by the \( 2 + \alpha \)-order of \( u \). Since \( 2 + \alpha < 4 \), to show that the order of the error term is controlled by the fourth order of \( u \), we apply the bootstrap argument for the estimate of the order of the error term by using the Lyapunov function \( S_{c^*} (u) \).

This paper is organized as following. In Section 2, we introduce a spectral decomposition with respect to the linearized operator of (1) and the estimate of the difference between solutions of a localized equation of (1) by a mobile distance. In Section 3, we construct invariant manifolds in \( G_{l_1, l_2, \alpha, \delta, \kappa}^c \) by applying the argument [29] and prove the orbital stability of line solitary waves on the invariant manifolds by using the fourth order estimate of the Lyapunov function in [38]. In Section 4, we prove the \( C^1 \) regularity of the center stable manifolds. In Appendix, we prove Proposition 1 and Corollary 4 which shows the stability of line solitary waves and solitary waves \( \varphi_{c^*} (a) (x - \tilde{c} (a), y) \) on the center stable manifold \( M_{cs} (c^*) \) for \( c^* \in CS \).

2. Linearized operator and localized equation. In this section, we show the properties of the linearized operator around line solitary waves and introduce the localized equation around line solitary waves. We define the linearized operator \( L_{c^*} \) of the stationary equation by

\[
L_{c^*} = -\Delta + c - 2Q_{c^*}.
\]

Let \( c^* > 4/5L^2 \) and \( n_0 \) be the integer with \( \frac{2(n_0 - 1)}{\sqrt{c^*}} < L \leq \frac{2n_0}{\sqrt{c^*}} \). The following proposition follows Proposition 3.1 in [38] and Proposition 2.2 and Proposition 2.3 in [39].

**Proposition 2.** Let \( c^* > 4/5L^2 \). The following holds.

(i) **Eigenvalues of** \( \partial_x L_{c^*} \) **with the positive real part are positive real numbers** \( \lambda_1, \lambda_2, \ldots, \lambda_{n_0 - 1} \).

(ii) **Eigenvalues of** \( \partial_x L_{c^*} \) **with the negative real part are negative real numbers** \( -\lambda_1, -\lambda_2, \ldots, -\lambda_{n_0 - 1} \).

(iii) There exist \( f_1, f_2, \ldots, f_{n_0 - 1} \in H^\infty (\mathbb{R}) \) such that for \( k \in \{1, 2, \ldots, n_0 - 1\} \),

\[
F_{k, 0} (x, y) = \pm f_k (\pm x) \cos \frac{ky}{L}, \quad F_{k, 1} (x, y) = \pm f_k (\pm x) \sin \frac{ky}{L}
\]
satisfy $\partial_x L_{c^*} F_k^{\pm,j} = \pm \lambda_k F_k^{\pm,j}$ and $(F_k^{\pm,j}, L_{c^*} F_k^{\mp,j})_{L^2} = 1$.

(iv) $\partial_x L_{c^*} \partial_c Q_{c^*} = -\partial_c Q_{c^*}$ and

$$\text{Ker}(\partial_x L_{c^*}) = \begin{cases} \text{Span}\{\partial_x Q_{c^*}\}, & c^* \notin \text{CS}, \\ \text{Span}\{\partial_x Q_{c^*}, Q_{c^*}^2 \cos \frac{n_0 y}{L}, Q_{c^*}^2 \sin \frac{n_0 y}{L}\}, & c^* = \frac{4n_0^2}{5L^2} \in \text{CS}, \end{cases}$$

where $\text{Span}A$ is the linear subspace which is spanned by elements in the set $A$.

(v) There exists $C > 0$ such that for $u \in H^1(\mathbb{R} \times T_L)$,

$$P_{\gamma} u = u - \sum_{k=1,2,\ldots,n_0-1} (A_k^{\pm,j} F_k^{\mp,j} + A_k^{-j} F_k^{-j}) - P_0 u$$

satisfies

$$\langle P_{\gamma} u, L_{c^*} P_{\gamma} u \rangle_{H^1, H^{-1}} \geq C \|P_{\gamma} u\|_{H^1}^2,$$

where $\partial_c Q_{c^*} = \frac{\partial Q_{c^*}}{\partial c}|_{c=c^*}$.

$$P_0 u = \begin{cases} \mu_1 \partial_x Q_{c^*} + \mu_2 \partial_c Q_{c^*}, & c^* \notin \text{CS}, \\ \mu_1 \partial_x Q_{c^*} + \mu_2 \partial_c Q_{c^*} + a_0 Q_{c^*}^2 \cos \frac{n_0 y}{L} + a_1 Q_{c^*}^2 \sin \frac{n_0 y}{L}, & c^* = \frac{4n_0^2}{5L^2} \in \text{CS}, \end{cases}$$

$$\Lambda_k^{\pm,j} = (u, L_{c^*} F_k^{\mp,j})_{L^2}, \quad \mu_1 = \frac{(u, \partial_x Q_{c^*})_{L^2}}{\|\partial_x Q_{c^*}\|_{L^2}^2}, \quad \mu_2 = \frac{(u, Q_{c^*})_{L^2}}{(\partial_c Q_{c^*}, Q_{c^*})_{L^2}}.$$

$$a_0 = \frac{(u, Q_{c^*}^2 \cos \frac{n_0 y}{L})_{L^2}}{\|Q_{c^*}^2 \cos \frac{n_0 y}{L}\|_{L^2}^2}, \quad a_1 = \frac{(u, Q_{c^*}^2 \sin \frac{n_0 y}{L})_{L^2}}{\|Q_{c^*}^2 \sin \frac{n_0 y}{L}\|_{L^2}^2}.$$

We define the spectral projection

$$P_{\pm} u = \sum_{j=0,1} \Lambda_k^{\pm,j} F_k^{\pm,j}, \quad P_1 u = \mu_1 \partial_x Q_{c^*}, \quad P_2 u = \mu_2 \partial_c Q_{c^*},$$

$$P_0 u = 1_{CS}(c^*) \left( a_0 Q_{c^*}^2 \cos \frac{n_0 y}{L} + a_1 Q_{c^*}^2 \sin \frac{n_0 y}{L} \right), \quad P_d u = u - P_{\gamma} u$$

and the norm

$$\|u\|_{E_{c^*}} = \left( \sum_{k=1,2,\ldots,n_0-1} \left( (\Lambda_k^{+j})^2 + (\Lambda_k^{-j})^2 + \kappa^2 \mu_1^2 + \mu_2^2 + \kappa^2 1_{CS}(c^*) (a_0^2 + a_1^2) \right) \right)^{\frac{1}{2}},$$

where $\kappa > 0, \Lambda_k^{+j}, \mu_1, \mu_2, a_0$ and $a_1$ are given in Proposition 2 and $1_A$ is the indicator function of $A$.

Let $u$ be a solution to the equation (1). Then, $v(t) = \tau_{-\rho(t)} u(t) - Q_{c(t)}$ satisfies

$$v_t = \partial_x L_{c^*} v + (\dot{\rho} - c) \partial_x Q_{c^*} - \ddot{c} \partial_c Q_{c^*} + N(v, c, \rho),$$

where

$$N(v, c, \rho) = \partial_x [-v^2 + (\dot{\rho} - c^*) v + 2(Q_{c^*} - Q_c) v + (\dot{\rho} - c)(Q_c - Q_{c^*})] - \ddot{c} (Q_c - Q_{c^*}).$$

In the following lemma, we choose the modulation parameters $\rho$ and $c$ satisfying the orthogonality condition

$$(v, \partial_x Q_{c^*})_{L^2} = (v, Q_{c^*})_{L^2} = 0.$$
Lemma 2.1. There exist $\delta_0, C_{\delta_0} > 0$ and smooth maps $\rho : \mathcal{N}_{\delta_0,c^*} \to \mathbb{R}$ and $c : \mathcal{N}_{\delta_0,c^*} \to (0, \infty)$ such that for $u \in \mathcal{N}_{\delta_0,c^*}$, $v = \tau_\rho(u) - Q_c(u)$ satisfies orthogonality condition (4) and

$$
\|v\|_{H^1} + |c(u) - c^*| < C_{\delta_0} \inf_{q \in \mathbb{R}} \|u - \tau_q Q_c\|_{H^1}.
$$

Proof. We define

$$
G(u, c, \rho) = \left(\left(\tau_\rho u - Q_c, \partial_x Q_c\right)_{L^2}, \left(\tau_\rho u - Q_c, Q_{c^*}\right)_{L^2}\right).
$$

Then, $G(Q_{c^*}, c^*, 0) = t(0, 0)$ and $\frac{\partial G}{\partial (c, \rho)}(Q_{c^*}, c^*, 0)$ is regular. By the implicit function theorem, we obtain the conclusion. \hfill \square

For solution $(v(t), c(t), \rho(t))$ to the equation (3) satisfying (4), $(c(t), \rho(t))$ satisfies

$$
\begin{pmatrix}
\dot{\rho} - c \\
\dot{c}
\end{pmatrix} = \left(\|\partial_x Q_{c^*}\|_{L^2}^{-2}(v, L_{c^*} \partial_x^2 Q_{c^*})_{L^2} \right) + \mathcal{N}(v, c),
$$

where

$$
\mathcal{N}(v, c) = \left(\left(\partial_x Q_x + \partial_x v, \partial_x Q_{c^*}\right)_{L^2}, 0, \left((v, L_{c^*} \partial_x^2 Q_{c^*})_{L^2}\right)\right)^{-1} \times \left(\left(v, L_{c^*} \partial_x^2 Q_{c^*} \right)_{L^2} + \left(\partial_x v^2 - (c - c^*)v - 2(Q_{c^*} - Q_c)v, \partial_x Q_{c^*}\right)_{L^2}\right)_{L^2} \left(0, \|\partial_x Q_{c^*}\|_{L^2}^{-2}(v, L_{c^*} \partial_x^2 Q_{c^*})_{L^2} \right)
$$

$$
= O(\|v\|^2_{L^2} + \|v\|_{L^2} |c - c^*|) \text{ as } \|v\|_{L^2} + |c - c^*| \to 0.
$$

On the tubular neighborhood $\mathcal{N}_{\delta_0,c^*}$, $u = \tau_\rho(v + Q_c)$ is a solution to the equation (1) satisfying the orthogonality condition (4) if and only if $v = \tau_\rho u - Q_c$ is a solution to the equation (3) with $(c, \rho)$ satisfying (5) and $(v(0), \partial_x Q_{c^*})_{L^2} = (v(0), Q_{c^*})_{L^2} = 0$.

Let $\chi \in C_0^\infty(\mathbb{R})$ be a smooth function with

$$
\chi(r) = \begin{cases} 
1, & |r| \leq 1 \\
0, & |r| \geq 2
\end{cases}, \quad 0 \leq \chi \leq 1
$$

and

$$
\chi_\delta = \chi_\delta(v, c) = \chi(\delta^{-2}(\|v\|_{H^1}^2 + |c - c^*|^2)).
$$

We define the localized system of the system (3) and (5) as

$$
v_t = \partial_x L_{c^*} v + (\dot{\rho} - c) \partial_x Q_{c^*} - i \partial_c Q_{c^*} + \chi_\delta(v, c) \mathcal{N}(v, c, \rho),
$$

$$
\begin{pmatrix}
\dot{\rho} - c \\
\dot{c}
\end{pmatrix} = \left(\|\partial_x Q_{c^*}\|_{L^2}^{-2}(v, L_{c^*} \partial_x^2 Q_{c^*})_{L^2} \right) + \mathcal{N}_\delta(v, c),
$$

as $|c - c^*| \to 0$.
Then, a solution \((v, c, \rho)\) to the system (6) and (7) satisfies that \((v(t), \partial_x Q_c \cdot L^2)\) does not have the advection term and the system (8)–(9) does not have the advection term \(\partial_x w\). To solve the system (8)–(9), we define the Bourgain space \(X^{s,b}_T\) related to the linear part of (1) as the completion of the Schwartz space under the norm

\[
\|\partial_x Q_c \cdot \|_{L^2}^2 (w, \tau, (L_x^c \partial^2_x Q_c)) L^2 \] + \(\tilde{N}_\delta(w, c, \rho),
\]

where

\[
\tilde{N}_\delta(w, c, \rho) = \partial_x [-w^2 + 2w\tau_{\rho s} (Q_c - Q_c) + (\hat{\rho} - c)\tau_{\rho s} (Q_c - Q_c)] - \hat{\rho}_{\rho s} \partial_c (Q_c - Q_c),
\]

\[
\tilde{N}_\delta(w, c, \rho) = N_\delta(\tau_{\rho s}, w, c) and
\]

\[
\rho_{\star}(w, c, \rho, t) = \rho_{\star}(t) = c^* t + \int_0^t \chi_{\delta}(w(s), c(s) - c^*) (\hat{\rho}(s) - c^*) ds.
\]

Then, we have \(v = \tau_{\rho s} w\) and the system (8)–(9) does not have the advection term \(\partial_x w\). To solve the system (8)–(9), we define the Bourgain space \(X^{s,b}_T\) related to the linear part of (1) as the completion of the Schwartz space under the norm

\[
\|\partial_x Q_c \cdot \|_{L^2}^2 (w, \tau, (L_x^c \partial^2_x Q_c)) L^2 \] + \(\tilde{N}_\delta(w, c, \rho),
\]

where

\[
\tilde{N}_\delta(w, c, \rho) = \partial_x [-w^2 + 2w\tau_{\rho s} (Q_c - Q_c) + (\hat{\rho} - c)\tau_{\rho s} (Q_c - Q_c)] - \hat{\rho}_{\rho s} \partial_c (Q_c - Q_c),
\]

\[
\tilde{N}_\delta(w, c, \rho) = N_\delta(\tau_{\rho s}, w, c) and
\]

\[
\rho_{\star}(w, c, \rho, t) = \rho_{\star}(t) = c^* t + \int_0^t \chi_{\delta}(w(s), c(s) - c^*) (\hat{\rho}(s) - c^*) ds.
\]

Then, we have \(v = \tau_{\rho s} w\) and the system (8)–(9) does not have the advection term \(\partial_x w\). To solve the system (8)–(9), we define the Bourgain space \(X^{s,b}_T\) related to the linear part of (1) as the completion of the Schwartz space under the norm

\[
\|\partial_x Q_c \cdot \|_{L^2}^2 (w, \tau, (L_x^c \partial^2_x Q_c)) L^2 \] + \(\tilde{N}_\delta(w, c, \rho),
\]

where

\[
\tilde{N}_\delta(w, c, \rho) = \partial_x [-w^2 + 2w\tau_{\rho s} (Q_c - Q_c) + (\hat{\rho} - c)\tau_{\rho s} (Q_c - Q_c)] - \hat{\rho}_{\rho s} \partial_c (Q_c - Q_c),
\]

\[
\tilde{N}_\delta(w, c, \rho) = N_\delta(\tau_{\rho s}, w, c) and
\]

\[
\rho_{\star}(w, c, \rho, t) = \rho_{\star}(t) = c^* t + \int_0^t \chi_{\delta}(w(s), c(s) - c^*) (\hat{\rho}(s) - c^*) ds.
\]

Then, we have \(v = \tau_{\rho s} w\) and the system (8)–(9) does not have the advection term \(\partial_x w\). To solve the system (8)–(9), we define the Bourgain space \(X^{s,b}_T\) related to the linear part of (1) as the completion of the Schwartz space under the norm

\[
\|\partial_x Q_c \cdot \|_{L^2}^2 (w, \tau, (L_x^c \partial^2_x Q_c)) L^2 \] + \(\tilde{N}_\delta(w, c, \rho),
\]

where

\[
\tilde{N}_\delta(w, c, \rho) = \partial_x [-w^2 + 2w\tau_{\rho s} (Q_c - Q_c) + (\hat{\rho} - c)\tau_{\rho s} (Q_c - Q_c)] - \hat{\rho}_{\rho s} \partial_c (Q_c - Q_c),
\]

\[
\tilde{N}_\delta(w, c, \rho) = N_\delta(\tau_{\rho s}, w, c) and
\]

\[
\rho_{\star}(w, c, \rho, t) = \rho_{\star}(t) = c^* t + \int_0^t \chi_{\delta}(w(s), c(s) - c^*) (\hat{\rho}(s) - c^*) ds.
\]
the solution \((v, c, \rho)\) to the system (6)–(7) with initial data \((v(0), c(0), \rho(0))\) satisfies
\[
\sup_{|t| \leq T^*} \|v(t)\|_{E_n} + \|\dot{c}\|_{L^2(-T^*, T^*)} + \|\dot{\rho} - c\|_{L^2(-T^*, T^*)} \leq C_n \|v(0)\|_{E_n},
\]
\[
\sup_{|t| \leq T^*} \|P_d(v(t) - e^{tA}v(0))\|_{E_n} \leq C_n \min\{\|v(0)\|_{E_n} (\|v(0)\|_{E_n} + |c(0) - c^*|)\}, \delta^2\},
\]
\[
\sup_{|t| \leq T^*} \|\|P_{\gamma}v(t)\|_{E_n}^2 - \|P_{\gamma}v(0)\|_{E_n}^2\| \leq C_n \min\{\|v(0)\|_{E_n}^2 (\|v(0)\|_{E_n} + |c(0) - c^*|)\}, \delta^3\},
\]
(10)

where the constants \(C_n\) and \(T^*\) do not depend on \(\delta\) and \((v(0), c(0), \rho(0))\) and \(A = (I - P_1)\partial_x L_c\), which is the linearized operator of the system (6)–(7).

To show the global well-posedness of the system (6)–(7) in Theorem 2.2, we use the estimates
\[
\|\partial_x (Qu)\|_{X^{1,0}} \lesssim \left(\|\partial_x Q\|_{L^\infty_{x,y}} W^{1,\infty}_{x,y} + \sum_{|\alpha| \leq 1} \|\partial^\alpha Q\|_{L^2_{x,y}}\right) \|u\|_{X^{1,\delta}}
\]
and
\[
\|\partial_x (uv)\|_{X^{1,\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{X^{1,\frac{1}{2}+\varepsilon}} \|v\|_{X^{1,\frac{1}{2}+\varepsilon}}
\]
by Molinet–Saut–Tzvetkov [27] and Molinet–Pilod [26], where \(b > 1/2, 0 < \varepsilon \ll 1,\) \(\alpha = (\alpha_1, \alpha_2)\) and \(\partial^\alpha = \partial^{\alpha_1}_x \partial^{\alpha_2}_y\). Moreover, combining the estimates (11), (12) and
\[
\left| \int_0^T \int_{\mathbb{R} \times T_L} (-\Delta u)vdt\,dx\,dy \right| \lesssim \|\tau_{\rho} u\|_{X^{1,\delta}} \|\tau_{\rho} v\|_{X^{1,-\beta}}
\]
(13)
for \(\tau_{\rho} u \in X^{1,\beta}, \tau_{\rho} v \in X^{1,-\beta}, \rho \in L^\infty, b > \frac{1}{2}\) and \(0 \leq \beta < \frac{1}{2}\), we obtain that for any solution \((v, c, \rho)\) to the system (6)–(7), solution \((w, c, \rho)\) to the system (8)–(9) corresponding to \((v, c, \rho)\) satisfies
\[
\sup_{|t| \leq T^*} \|\|P_{\gamma}w(t)\|_{E_n}^2 - \|P_{\gamma}w(0)\|_{E_n}^2\| \leq C\|w\|_{X^{1,\delta}}^2 (\|w\|_{X^{1,\delta}} + |c(0) - c^*|)
\]
\[
\lesssim \|v(0)\|_{H^1}^2 (\|v(0)\|_{H^1} + |c(0) - c^*|)
\]
(14)
and the estimate (10), where the constant \(C\) does not depend on \(\kappa\). Since the proof of Theorem 2.2 follows the proof of Theorem 3.4 in [39], we omit the detail of the proof.

To construct invariant manifolds from the flow map of the system (6)–(7), we show estimates of the difference between solutions to the system (6)–(7) and solutions to the linearized equation of (6)–(7). Theorem 2.2 yields the Lipschitz continuity of the flow map of the system (8)–(9) by the energy norm. However, Theorem 2.2 does not imply the Lipschitz continuity of the flow map of the system (6)–(7) by the energy norm. To show the estimates, we define the mobile distance which was introduce in [29]. Let \(C_2\) be a large real constant, \(\mathcal{H} = H^1(\mathbb{R} \times T_L) \times (0, \infty)\) and \(\phi\) be the smooth positive non-decreasing function with
\[
\phi(r) = \begin{cases} 1, & r \leq C_2, \\ r, & r \geq 2C_2. \end{cases}
\]
We define \(\phi_\delta\) by
\[
\phi_\delta(u) = \phi(\delta^{-1}\|P_{\gamma}u\|_{E_n}).
\]
Definition 2.3. Let \( \delta, \kappa > 0 \). We define the mobile distance \( m_{\delta, \kappa} : \mathcal{H}^2 \to [0, \infty) \) by

\[
m_{\delta, \kappa}(v_0, v_1) = \left[ \| P_d(v_0 - v_1) \|_{E^1}^2 + \inf_{q \in \mathbb{R}, j = 0, 1} (\| P_q v_j - \tau_q P_v v_{j-1} \|_{E^1} + \| q \|_1^2 \phi_q(v_{1-j})^2) \right]^{1/2} + |\log c_0 - \log c_1|^{1/2}
\]

for \( v_0 = (v_0, c_0), v_1 = (v_1, c_1) \in \mathcal{H} \).

In the following lemma, we recall the mobile distance \( m_{\delta, \kappa} \) is a quasi-distance on \( \mathcal{H} \).

Lemma 2.4. Let \( 0 < \delta, \kappa < 1 \). Then, \( m_{\delta, \kappa} \) satisfies the following.

(i) \( m_{\delta, \kappa}(v_0, v_1) = m_{\delta, \kappa}(v_1, v_0) \geq 0 \), where the equality holds iff \( v_0 = v_1 \).

(ii) \( m_{\delta, \kappa}(v_0, v_1) \leq C(m_{\delta, \kappa}(v_0, v_2) + m_{\delta, \kappa}(v_2, v_1)) \), for some absolute constant \( C > 0 \) which does not depend on \( \delta \).

(iii) If \( m_{\delta, \kappa}(v_n, v_m) \to 0 \) \( (n, m \to \infty) \), then \( \{v_n\} \) converges in \( \mathcal{H} \).

(iv) For \( v_0 = (v_0, c_0), v_1 = (v_1, c_1) \in \mathcal{H} \)

\[
\| v_0 \|_{H^1} - \| v_1 \|_{H^1} + \| v_0 - v_1 \|_{L^2} + |\log c_0 - \log c_1| \lesssim m_{\delta, \kappa}(v_0, v_1)
\]

where the implicit constants do not depend on \( \delta \).

Since

\[
|q| \| \nabla P_q v_1 - j \|_{L^2}^2 \lesssim \delta^{1/2} \| q \|_1^2 \phi_q(v_{1-j}),
\]

Lemma 2.4 follows Proposition 2.2 in [29].

In the following lemma, we show the estimates of the difference between solutions to the system (6)–(7) and solutions to the linearized equation of (6)–(7) on the quasi-metric space \( (\mathcal{H}, m_{\delta, \kappa}) \).

Lemma 2.5. Let \( 0 < \kappa < 1 \). There exists \( T^*, \delta^*, C_\kappa > 0 \) such that for any \( 0 < \delta < \delta^* \) and solutions \( (v_j, \rho_j) = (v_j, c_j, \rho_j) \) to the system (6)–(7) given in Theorem 2.2, we have

\[
\sup_{|t| \leq T^*} m_{\delta, \kappa}(v_0(t), v_1(t)) \leq C_\kappa m_{\delta, \kappa}(v_0(0), v_1(0)),
\]

\[
\sup_{|t| \leq T^*} \left( \| P_d(v_0(t) - v_1(t) - e^{-iA}(v_0(0) - v_1(0))) \|_{E^1}^2 + |I(v_0(t), v_1(t)) - I(v_0(0), v_1(0))| \right) \leq C_\kappa \delta^{1/2} m_{\delta, \kappa}(v_0(0), v_1(0))^2,
\]

where

\[
I(v_0, v_1) = \inf_{q \in \mathbb{R}, j = 0, 1} (\| P_q v_j - \tau_q P_v v_{j-1} \|_{E^1}^2 + |q|_1^2 \phi_q(v_{1-j})^2).
\]

In the case of \( c^* \notin CS \), Lemma 2.5 is same as Lemma 3.8 in [39]. Since the estimate of \( P_q(v_0(t) - v_1(t) - e^{-iA}(v_0(0) - v_1(0))) \) follows the local well-posedness in Theorem 2.2, for \( c^* \in CS \) the proof of Lemma 2.5 also follows the arguments in the proof of Lemma 3.2 in [29] and the proof of Lemma 3.8 in [39]. Therefore, we omit the proof of Lemma 2.5.
3. Construction of the center stable manifolds. In this section, we construct the center stable manifolds by applying the argument in [29].

For $l_1, l_2, \alpha, \delta, \kappa > 0$, we define the set of the graph function on the stable invariant space of the linearized operator $A$ with the Hölder exponent $\alpha$ at $(0, c^*)$ by

$$\mathcal{G}_{l_1, l_2, \alpha, \delta, \kappa} = \{ G : \mathcal{H} \to P_+ H^1(\mathbb{R} \times T_L); G = G \circ P_{\leq 0}, G(0, c^*) = 0, \|G(v)\|_{E_n} \leq l_1 \|v\|^2_{E_n} \text{ for } v \in \mathcal{H}, \|G(v_0) - G(v_1)\|_{E_n} \leq l_2 m_{\delta, \kappa}(v_0, v_1) \text{ for } v_0, v_1 \in \mathcal{H} \},$$

where $P_{\leq 0}(v, c) = (I - P_{+})v, c)$ and

$$\|(v, c)\|_{E_n} = \left(\|v\|^2_{E_n} + |\log c - \log c^*|^2 \right)^{1/2}.$$

We define the graph $[G]$ of $G \in \mathcal{G}_{l_1, l_2, \alpha, \delta, \kappa}$ as

$$\{(v, c) \in \mathcal{H}; P_{+}v = G(v, c) \}.$$

In the following lemma, we prove the upper estimate of the unstable eigen mode.

**Lemma 3.1.** There exist $T^* > 0$, $C_\kappa > 0$ and $C_L > 0$ such that if $l_2, \delta, \kappa > 0$ satisfy

$$l_2 + C_\kappa \delta + \kappa \ll 1 \text{ and } C_\kappa l_2^{-1} \delta^2 \ll 1,$$

then for any solutions $(v_j, c_j, \rho_j)$ to the system (6)–(7) $(j = 0, 1)$ satisfying

$$\|P_{+}(v_0(0) - v_1(0))\|_{E_n} \leq l_2 m_{\delta, \kappa}(v_0(0), v_1(0))\text{ (16)}$$

one has

$$\|P_{+}(v_0(t) - v_1(t))\|_{E_n} \leq \begin{cases} C_L l_2 m_{\delta, \kappa}(v_0(t), v_1(t)), & |t| \leq T^*, \\ l_2 m_{\delta, \kappa}(v_0(t), v_1(t)), & -T^* \leq t \leq -\frac{T^*}{2} \end{cases}. \text{ (17)}$$

**Proof.** In the case of $c^* \notin \text{CS}$, the conclusion follows Lemma 4.1 in [39]. Therefore, we consider the case $c^* = \frac{4n_1}{\lambda_2} \in \text{CS}$. Let

$$k_\kappa = \min_{k = 1, 2, \ldots, n_0 - 1} \lambda_k, \quad k^* = \max_{k = 1, 2, \ldots, n_0 - 1} \lambda_k. \text{ (18)}$$

By Lemma 2.5 and (16), we have

$$\|P_{+}(v_0(t) - v_1(t))\|_{E_n} \leq \left( \max\{e^{k_1 t}, e^{k_2 t}\}l_2 + C_\kappa \delta^2 \right)m_{\delta, \kappa}(v_0(0), v_1(0)), \text{ (19)}$$

and

$$m_{\delta, \kappa}(v_0(t), v_1(t))^2 - |\log c_0(0) - \log c_1(0)|^2 \lesssim \delta m_{\delta, \kappa}(v_0(0), v_1(0))^2, \text{ (20)}$$

and

$$m_{\delta, \kappa}(v_0(t), v_1(t))^2 - |\log c_0(0) - \log c_1(t)|^2 \geq \|P_{a}(e^{tA}(v_0(0) - v_1(0)))\|^2_{E_n} + I(v_0(0), v_1(0)) - C_\kappa^2 \delta^2 m_{\delta, \kappa}(v_0(0), v_1(0))^2 \quad \text{ (21)}$$

for sufficiently small $|t|$. Since $(P_1 + P_{a})L_{c^*} = 0$, for $|t| < 1$ there exists $C > 0$ such that

$$\|P_{a}(e^{tA}(v_0(0) - v_1(0)) - (v_0(0) - v_1(0)))\|_{E_n} \leq C m_{\delta, \kappa}(v_0(0), v_1(0)). \text{ (22)}$$

By the estimates (20)–(22), the assumption (16) and $e^{tA}(P_1 + P_2) = P_1 + P_2$, there exist $C, C_{\kappa}, T > 0$ such that

$$m_{\delta, \kappa}(v_0(t), v_1(t))^2 \geq \begin{cases} (1 - l_2^2 + e^{k_1 t} l_2^2 - C_\kappa - C_\kappa \delta^2)m_{\delta, \kappa}(v_0(0), v_1(0))^2, & -T \leq t \leq 0 \\ (e^{-2k_1 |t|} - C_K - C_\kappa \delta^2)m_{\delta, \kappa}(v_0(0), v_1(0))^2, & |t| \leq T. \end{cases} \text{ (23)}$$
The inequalities (19) and (23) yield the inequalities (17) for sufficiently small $\kappa$ and $\delta$.

In the following lemma, we prove the upper estimate of the unstable mode around 0.

**Lemma 3.2.** For $1 < \alpha < 2$, there exist $T^* > 0$, $C_\kappa > 0$ and $C_\alpha > 0$ such that if $l_1, l_2, \kappa, \delta > 0$ satisfy

$$l_1 + C_\kappa \delta + \kappa \ll 1 \text{ and } C_\kappa l_1^{-\frac{1}{2}} \delta^{\frac{1}{2}} + C_\kappa l_1^{-1} \delta^{2-\alpha} \ll 1,$$

(24) then for any solutions $(v, c, \rho)$ to the system (6)–(7) satisfying

$$\|P_+ v(0)\|_{E_n} \leq l_1 \min\{\|v(0)\|_{E_n}^\alpha, \|v(0)\|_{E_n}\}$$

one has

$$\|P_+ v(t)\|_{E_n} \leq \begin{cases} C_\alpha l_1 \|v(t)\|_{E_n}^\alpha, & |t| \leq T^*, \\ l_1 \|v(t)\|_{E_n}^\alpha, & -T^* \leq t \leq -\frac{T^*}{2}, \end{cases}$$

(26) where $v(t) = (v(t), c(t))$.

**Proof.** Let $t \in (-1, 1)$. In the case of $\|v(t)\|_{E_n} \geq 1$, the inequality (26) follows from Theorem 2.2 and the assumption (25). We consider the case of $\|v(t)\|_{E_n} < 1$. By Theorem 2.2 and the assumption (25), we have

$$\|P_+ v(t)\|_{E_n} \leq \max\{e^{k_1 t}, e^{k_2 t}\} l_1 \|v(0)\|_{E_n}^\alpha + C_\kappa \min\{\|v(0)\|_{E_n}^2, \delta^2\}$$

(27)

and

$$\|v(t)\|_{E_n}^2 \geq \|(P_0 + P_-) v(0)\|_{E_n}^2 + \min\{e^{2k_1 t}, e^{2k_2 t}\} \|P_+ v(0)\|_{E_n}^2 + \|P_\gamma v(0)\|_{E_n}^2 + |\log c(0) - \log c^*|^2 - C\kappa \|v(0)\|_{E_n}^2 - C_\kappa \min\{\|v(0)\|_{E_n}^2, \delta^3\}.$$  

(28)

By the inequality (28) and the concavity of $x^{\alpha/2}$, we obtain

$$\|v(t)\|_{E_n}^\alpha \geq \|v(0)\|_{E_n}^\alpha - C(t_1^\alpha \|v(0)\|_{E_n}^\alpha + (c_\kappa + C_\delta) \|v(0)\|_{E_n}^\alpha)$$

(29)

for $|t| < 1$. Combining (27) and (29), we have there exists $T^* > 0$ such that

$$\|v(t)\|_{E_n}^\alpha \geq l_1^{-1}(\max\{e^{k_1 t}, e^{k_2 t}\} + l_1^{-1} C_\kappa \delta^{2-\alpha})^{-1} \times (1 - Ct_1^\alpha - C\kappa^{1/2} - C_\kappa \delta^{1/2}) \|P_+ v(t)\|_{E_n}$$

$$\geq \begin{cases} C_l^{-1} \|P_+ v(t)\|_{E_n}, & |t| \leq T^*, \\ l_1^{-1} \|P_+ v(t)\|_{E_n}, & -T^* \leq t \leq -\frac{T^*}{2}, \end{cases}$$

which completes the proof. 

The following lemma shows that the flow map of the system (6)–(7) preserves the Lipschitz coefficients and the Hölder coefficients of graphs.

**Lemma 3.3.** Let $1 < \alpha < 2$. Assume that $l_1, l_2, \kappa, \delta > 0$ satisfy the conditions (15) and (24). There exists $T^* > 0$ such that the solution map $U_\delta(t)$ of the system (6)–(7) for $|t| \leq T^*$ defines a map $U_\delta(t) : \mathcal{G}_1 \to \mathcal{G}_1$ uniquely by the relation $U_\delta(t)(\{G\} \times \mathbb{R}) = \{U_\delta(t) G\} \times \mathbb{R}$. Moreover, if $-T^* \leq t \leq -T^*/2$, then $U_\delta(t)$ maps $\mathcal{G}_1$ into itself.

We can define $U_\delta(t) G$ satisfying $U_\delta(t)(\{G\} \times \mathbb{R}) = [U_\delta(t) G] \times \mathbb{R}$ and $U_\delta(t) G : P_{\leq 0} = U_\delta(t) G$ on the set $P_{\leq 0} U(t) [G] + P_+ H^1(\mathbb{R} \times \mathbb{T}_L)$. Using a topological invariant of the $(2n_0-2)$-sphere, we can show $P_{\leq 0} U(t) [G] + P_+ H^1(\mathbb{R} \times \mathbb{T}_L) = H$. Since the Lipschitz coefficients and the Hölder coefficients of graphs is preserved by the flow map of the system (6)–(7), we obtain the conclusion of Lemma 3.3. The detail of the proof of
Lemma 3.3 follows Lemma 3.1, Lemma 3.2 and the argument of the proof of Lemma 3.4 in [29]. We omit the detail of the proof of Lemma 3.3.

We define
\[ \|G\|_{\mathcal{F}^+} = \sup_{v \in \mathcal{H} \setminus \{(0, c')\}} \frac{\|G(v)\|_{E_\kappa}}{\|v\|_{E_\kappa}}. \]

Since
\[ \|G(v)\|_{E_\kappa} \leq I_2\|v\|_{E_\kappa} \]
for \( G \in \mathcal{F}_{l_1,l_2,\alpha,\delta,\kappa}^+ \) and \( v \in \mathcal{H} \), we have \( \|G\|_{\mathcal{F}^+} \leq I_2 \) for \( G \in \mathcal{F}_{l_1,l_2,\alpha,\delta,\kappa}^+ \). Therefore, \( \mathcal{F}_{l_1,l_2,\alpha,\delta,\kappa}^+ \) is a bounded complete metric space.

In the following lemma, we obtain the fix point of \( \mathcal{U}_G \).

**Lemma 3.4.** Let \( 1 < \alpha < 2 \). Assume that \( l_1, l_2, \delta, \kappa > 0 \) satisfy the conditions (15) and (24). Then, the mapping \( \mathcal{U}_G(t) \) is a contraction on \( \mathcal{F}_{l_1,l_2,\alpha,\delta,\kappa}^+ \) for \( t < -T^*/2 \). Moreover, there exists a unique \( G^*_+ \in \mathcal{F}_{l_1,l_2,\alpha,\delta,\kappa}^+ \) such that \( \mathcal{U}_G(t)G^*_+ = G^*_+ \) for all \( t < 0 \), where \( G^*_+ \) does not depend on \( \kappa \). In particular, the uniqueness holds for any fixed \( t < 0 \).

**Proof.** To show Lemma 3.4, we apply the argument in Lemma 3.5 and Theorem 3.6 in [29].

Firstly, we show that \( \mathcal{U}_G(t) \) is a contraction for \( t < -T^*/2 \). Let \( G_0, G_1 \in \mathcal{F}_{l_1,l_2,\alpha,\delta,\kappa}^+ \) and \( T \in [-T^*, -T^*/2] \). We define solutions \( (v_j, c_j, \rho_j) \) to the system (6)–(7) by
\[ (v_j(t), c_j(t), \rho_j(t)) = U_G(t - T)(P_{\leq 0}v + (\mathcal{U}_G(T)G_j)(\psi, \iota, \iota, q)) \]
for \( j \in \{0, 1\}, (\psi, \iota) \in \mathcal{H}, t \in \mathbb{R} \) and \( q \in \mathbb{R} \). Then, by Lemma 2.5 and \( P_{\leq 0}v_0(T) = P_{\leq 0}v_1(T) \), there exists \( C_{\kappa} > 0 \) such that
\[ \|(P_0 + P_-)(v_0(0) - v_1(0))\|_{E_\kappa} + I(v_0(0), v_1(0))^{\frac{1}{2}} \leq C_{\kappa} \delta^{\frac{1}{2}}\|P_+(v_0(T) - v_1(T))\|_{E_\kappa}. \]
(30)

By the inequality
\[ \|P_{\alpha}e^{-TA}v\|_{E_\kappa} \leq \|P_{\alpha}v\|_{E_\kappa} + C_{\kappa}(I - P_1 - P_\alpha)v\|_{E_\kappa} \leq (1 + C_{\kappa})\|v\|_{E_\kappa} \]
and Theorem 2.2, there exists \( C_{\kappa,1} > 0 \) such that
\[ \|(P_{\leq 0}v_0(0), c_0(0))\|_{E_\kappa} \leq (1 + C_{\kappa} + C_{\kappa,1} \delta)\|(P_{\leq 0}v_0(T), c_0(T))\|_{E_\kappa} \]
(31)
and
\[ + C_{\kappa,1} \delta\|P_+v_0(T)\|_{E_\kappa}. \]
(32)
The equality
\[ P_+(P_{\leq 0}v + \mathcal{U}_G(T)G_j(\psi, \iota)) = \mathcal{U}_G(T)G_j(P_{\leq 0}v + \mathcal{U}_G(T)G_j(\psi, \iota), \iota) \]
yields
\[ (P_{\leq 0}v + \mathcal{U}_G(T)G_j(\psi, \iota), \iota) \in [\mathcal{U}_G(T)G_j]. \]
Therefore, we have
\[ \mathcal{U}_G(-T)(P_{\leq 0}v + \mathcal{U}_G(T)G_j(\psi, \iota), \iota, \rho) \in \mathcal{U}_G(-T)([\mathcal{U}_G(T)G] \times \mathbb{R}) = [G] \times \mathbb{R} \]
which implies that
\[ P_+v_j(0) = G_j(U_\delta(-T)(P_{\leq 0}v + \mathcal{U}_G(T)G_j(\psi, \iota), \iota, q), c_j(0)) = G_j(v_j(0), c_j(0)). \]
(33)
By the inequalities (30)–(33), there exist $C_{\kappa,2}, C_{\kappa,3} > 0$ such that
\[
\|U(T)G_0(\psi, t) - U(T)G_1(\psi, t)\|_{E_\kappa} = \|P_+(v_0(T) - v_1(T))\|_{E_\kappa} \\
\leq (e^{k_T} + C_{\kappa,2}\delta^\frac{1}{2}) \|e^{-TA}P_+(v_0(T) - v_1(T))\|_{E_\kappa} \\
\leq (e^{k_T} + C_{\kappa,3}\delta^\frac{1}{2}) \|P_+(v_0(0) - v_1(0))\|_{E_\kappa} \\
\leq (e^{k_T} + C_{\kappa,3}\delta^\frac{1}{2}) (\|G_0 - G_1\|_{\bar{\mathcal{E}}} + \|P_{\leq 0}v_0(0), c_0(0)\|_{E_\kappa}) \\
+ \|G_1\|_{\mathcal{G}\mathcal{M}_\kappa}(P_{\leq 0}v_0(0), P_{\leq 0}v_1(0)) \\
\leq (e^{k_T} + C_{\kappa,3}\delta^\frac{1}{2}) ((1 + C\kappa + C_{\kappa,1}(1 + l_2)\delta)\|G_0 - G_1\|_{\bar{\mathcal{E}}} + l_2C\delta^\frac{1}{2}\|P_+(v_0(T) - v_1(T))\|_{E_\kappa}).
\]
Therefore, under the smallness condition (15) and (24), there exists $0 < \Lambda < 1$ such that for $(\psi, t) \in \mathcal{H}$,
\[
\|U(T)G_0(\psi, t) - U(T)G_1(\psi, t)\|_{E_\kappa} < \Lambda \|G_0 - G_1\|_{\bar{\mathcal{E}}}
\]
which implies that $U(T)$ is a contraction mapping for $T \in [-T^*, -T^*/2]$. Therefore, the equality $U(t) \circ U(s) = U(t+s)$ yields that $U(t)$ is a contraction mapping for $t < -T^*/2$.

Secondly, we show the uniqueness of the fixed points of $U_\delta(t)$ with respect to $t < 0$. Let $G_\delta^+ \subset \mathcal{G}_\kappa^+\mathcal{H}_\kappa$ be the fixed point of the contraction $U_\delta(T^*)$. Since $U_\delta(T^*)U_\delta(t)G_\delta^+ = U_\delta(t)U_\delta(T^*)G_\delta^+ = U_\delta(t)G_\delta^+$ for $t < 0$, we have that $U_\delta(t)G_\delta^+ = G_\delta^+$ which implies $G_\delta^+$ is a unique fixed point of $U_\delta(t)$ for $t < 0$.

Lastly, We show that $G_\delta^+$ does not depend on $\kappa$. Let $G_\delta^\kappa_{\mathcal{A}} \subset \mathcal{G}_\kappa\mathcal{H}_\kappa$ be the fixed point of $U_\delta$ on $\mathcal{G}^\kappa_{\mathcal{A}}$. By the definition of $\|\cdot\|_{E_\kappa}$, we have $\mathcal{G}^\kappa_{\mathcal{A}} \subset \mathcal{G}_{\kappa_1,\kappa_2}\mathcal{H}_{\kappa_1,\kappa_2}$ for $\kappa_1 < \kappa_2$. Therefore, by the uniqueness of the fixed point $G_\delta^\kappa_{\mathcal{A}}$ of $U_\delta$ on $\mathcal{G}^\kappa_{\mathcal{A}}$, we have $G_\delta^\kappa_{\mathcal{A}} = G_\delta^\kappa_{\mathcal{A}}$ for $(l_1, l_2, \alpha, \delta, \kappa_1)$ and $(l_1, l_2, \alpha, \delta, \kappa_2)$ satisfying (15), (24) and $\kappa_1 < \kappa_2$.

We define $g(w, c)$ by
\[
g(w, c) = w + C_\delta^+(w, c) + Q_c
\]
for $w \in H^1(\mathbb{R} \times \mathbb{T}_L), c > 0$. Let
\[
\mathcal{M}_\kappa^\delta(c^*, r) = \{ \tau_\rho g(w, c); w \in (P_+ + P_\kappa + P_\gamma)H^1(\mathbb{R} \times \mathbb{T}_L), |c - c^*| < c^*/2, \\
inf_{\rho \in \mathbb{R}} \|g(w, c) - \tau_\rho Q_{c^*}\|_{H^1} < r, \rho \in \mathbb{R} \}
\]
and
\[
\hat{\mathcal{M}}_\kappa^\delta(c^*, r) = \{ \tau_\rho g(w, c); w \in P_{\leq 0}H^1(\mathbb{R} \times \mathbb{T}_L), \|(P_1 + P_2)w\|_{H^1(\mathbb{R} \times \mathbb{T}_L)} < r^{1/2}, \\
|c - c^*| < c^*/2, \inf_{\rho \in \mathbb{R}} \|g(w, c) - \tau_\rho Q_{c^*}\|_{H^1} < r, \rho \in \mathbb{R} \}
\]
for $r > 0$.

We show the stability of $Q_{c^*}$ on $\mathcal{M}_\kappa^\delta$, by using the conservation law $S_\kappa(u) = E(u) + \frac{1}{2}M(u)$ as a Lyapunov function. To recover the degeneracy of the Lyapunov function $S_\kappa(u)$ around $Q_{c^*}$ for critical speed $c^* \in \mathcal{CS}$, we define the modulated solitary wave
\[
\Theta(a, c) = \frac{c}{c^*}\varphi_{c^*}(a)\left(\sqrt{\frac{c}{c^*}}x, y\right)
\]
and the modulation
\[ \beta(a, c) = \frac{c^* \|Q_c\|_{L^2}^{4/3}}{\|\varphi_{c^*}(a)\|_{L^2}^{4/3}} \]
for \( a \in \mathbb{R}^2 \) and \( c > 0 \). Then,
\[
\|\Theta(a, \beta(a, c))\|_{L^2} = \|Q_c\|_{L^2}
\]
and
\[
\beta(a, c) - c = -\frac{c^* \|\varphi_{c^*}(a)\|_{L^2}^{4/3}}{\|\varphi_{c^*}(a)\|_{L^2}^{4/3}} \left( \frac{2}{5} c^* C_{2,c^*} \right) = -\frac{c C_{2,c^*} \|Q_c\|_{L^2}^2}{\|Q_c\|_{L^2}^2} + o(|a|^2).
\]

In the following lemma, we investigate the fourth order term of the Lyapunov function.

**Lemma 3.5.** For \( c^* \in CS, c > 0 \) and \( a \in \mathbb{R}^2 \),
\[
S_c(\Theta(a, \beta(a, c))) - S_c(Q_c) = \left( \frac{c}{c^*} \right)^5 \frac{5 c^* C_{2,c^*}}{48 \|Q_c\|_{L^2}^2} \left( \|Q_c^{3/2} \cos \frac{\alpha}{2} \|_{L^2}^2 \right) |a|^4
\]
\[
+ \frac{c^* - c}{c^*} \|\partial_y \Theta(a, \beta(a, c))\|_{L^2}^2 + o(|a|^4).
\]

**Proof.** Since
\[
\lim_{|a| \to 0} \frac{S_c(\varphi_{c^*}(a)) - S_c(Q_c)}{c^* - c} = \frac{1}{2} M(Q_{c^*})
\]
and
\[
\lim_{|a| \to 0} \frac{M(\varphi_{c^*}(a)) - M(Q_{c^*})}{c^* - c} = \frac{C_{2,c^*}}{C_{c^*}},
\]
by (35) and Proposition 1 we have
\[
S_{c^*}(\Theta(a, \beta(a, c^*))) - S_{c^*}(Q_{c^*})
\]
\[
= S_{c^*}(\varphi_{c^*}(a)) - S_{c^*}(Q_{c^*}) + \frac{c^* - c}{2} M(Q_{c^*})
\]
\[
+ \frac{1}{2} (\beta(a, c^*) - c^*)^2 (S_{c^*}(Q_{c^*})\partial_x Q_{c^*}, \partial_x Q_{c^*})_{L^2} + o(|a|^4)
\]
\[
= \frac{C_{2,c^*} C_{c^*}}{16} |a|^4 - \frac{c^* C_{2,c^*}}{24 \|Q_c\|_{L^2}^2} |a|^2 + o(|a|^4)
\]
\[
= \frac{5 c^* C_{2,c^*}}{48 \|Q_c\|_{L^2}^2} \left( \|Q_{c^*}^{3/2} \cos \frac{\alpha}{2} \|_{L^2}^2 \right) |a|^4 + o(|a|^4).
\]
The equality \( M(\Theta(a, \beta(a, c))) = M(Q_{c^*}) \) yields
\[
S_{c^*}(\Theta(a, \beta(a, c^*))) - S_{c^*}(Q_{c^*})
\]
\[
= \left( \frac{c}{c^*} \right)^5 \left( S_{c^*}(\Theta(a, \beta(a, c^*))) - S_{c^*}(Q_{c^*}) \right) + \left( 1 - \frac{c}{c^*} \right) \|\partial_y \Theta(a, \beta(a, c^*))\|_{L^2}^2.
\]

Therefore, we obtain the conclusion of Lemma 3.5. \( \square \)

We define the orthogonality condition
\[
(v, \Theta(a, c))_{L^2} = (v, \partial_x \Theta(a, c))_{L^2} = (v, \partial_{a_0} \Theta(a, c))_{L^2} = (v, \partial_{a_1} \Theta(a, c))_{L^2} = 0.
\]

for \((v, c, a) \in L^2(\mathbb{R} \times T_L) \times (0, \infty) \times \mathbb{R}^2\)
Lemma 3.6. Let \(c^* \in \text{CS}\). There exist \(\delta, C > 0, c : N_{\delta, c^*} \to (0, \infty), \rho : N_{\delta, c^*} \to \mathbb{R}\) and \(a = (a_0, a_1) : N_{\delta, c^*} \to \mathbb{R}^2\) such that for \(u \in N_{\delta, c^*}\), \((v(u), c(u), a(u))\) satisfies the orthogonality condition (36) and
\[
\|v(u)\|_{H^1} + |c(u) - c^*| + |a(u)| \leq C \inf_{q \in \mathbb{R}} \|u - \tau_q Q_{c^*}\|_{H^1},
\]
where
\[
v(u) = \tau_{-\rho(u)}u - \Theta(a(u), c(u)).
\]

Proof. We define
\[
F_q(u, c, \rho, a) = \begin{pmatrix}
(\tau_{-\rho}u - \tau_q \Theta(a, c), \tau_q \Theta(a, c))_{L^2}, \\
(\tau_{-\rho}u - \tau_q \Theta(a, c), \tau_q \partial_x \Theta(a, c))_{L^2}, \\
(\tau_{-\rho}u - \tau_q \Theta(a, c), \tau_q \partial_{\alpha_1} \Theta(a, c))_{L^2}, \\
(\tau_{-\rho}u - \tau_q \Theta(a, c), \tau_q \partial_{\alpha_2} \Theta(a, c))_{L^2}
\end{pmatrix}
\]
for \(c > 0, q, \rho \in \mathbb{R}, a \in \mathbb{R}^2\) and \(u \in H^1(\mathbb{R} \times T_L)\). Then, \(F_q(\tau_q Q_{c^*}, c^*, 0, (0, 0)) = \mathcal{L}(0, 0, 0, 0)\) and
\[
\frac{\partial F_q}{\partial (c, \rho, \alpha_1, \alpha_2)}|_{(c, \rho, \alpha_1, \alpha_2) = (\tau_q Q_{c^*}, c^*, 0, (0, 0))} = \text{diag}(-Q_{c^*}, \partial_x Q_{c^*})_{L^2}, -\|\partial_x Q_{c^*}\|_{L^2}^2 - \left|Q_{c^*}^{3/2} \cos \frac{nqy}{L} \right|_{L^2}^2 - \left|Q_{c^*}^{3/2} \sin \frac{nqy}{L} \right|_{L^2}^2.
\]
By the implicit function theorem, there exist \(\delta, C > 0, c_q : B_\delta(\tau_q Q_{c^*}) \to (0, \infty), \rho_q : B_\delta(\tau_q Q_{c^*}) \to \mathbb{R}\) and \(a_q : B_\delta(\tau_q Q_{c^*}) \to \mathbb{R}^2\) such that \((\tau_{-q} v_q(u), c_q(u), a_q(u))\) satisfies the orthogonality condition (36),
\[
\|v_q(u)\|_{H^1} + |c_q(u) - c^*| + |a_q(u)| \leq C \|u - \tau_q Q_{c^*}\|_{H^1},
\]
and
\[
(c_{q_1}(u), \rho_{q_1}(u), a_{q_1}(u)) = (c_{q_2}(u), q_1 - q_2 + \rho_{q_2}(u), a_{q_2}(u))
\]
for \(q, q_1, q_2 \in \mathbb{R}\) and \(u \in B_\delta(\tau_q Q_{c^*}) \cap B_\delta(\tau_{q_1} Q_{c^*})\), where \(v_q(u) = \tau_{-\rho(u)}u - \tau_q \Theta(a(u), c(u))\) and \(B_\delta(f) = \{u \in H^1(\mathbb{R} \times T_L) ; \|u - f\|_{H^1} < \delta\}\). By the compatibility condition (38), we can define \((c, \rho, a) : N_{\delta, c^*} \to (0, \infty) \times \mathbb{R} \times \mathbb{R}^2\) by
\[
(c(u), \rho(u), a(u)) = (c_q(u), \rho_q(u) - q, a_q(u)), \quad u \in B_\delta(\tau_q Q_{c^*}).
\]
Then, \((v(u), c(u), a(u))\) satisfies the orthogonality condition (36) and (37). \(\square\)

In the following lemma, we show the estimate of the difference between the modulated solitary waves.

Lemma 3.7. Let \(c^* \in \text{CS}\). There exists \(\delta > 0\) such that for \(c_0 > 0\) and \(u \in N_{\delta, c^*}\) with \(\|u\|_{L^2} = \|Q_{c_0}\|_{L^2}\) and \(|c_0 - c^*| \ll \delta^{1/2}\),
\[
\|\Theta(a(u), \beta(a(u), c_0)) - \Theta(a(u), c(u))\|_{H^1} \lesssim \|v(u)\|_{L^2}^2,
\]
\[
|\beta(a(u), c_0) - c(u)| \lesssim \|v(u)\|_{L^2}^2,
\]
where \(a(u), c(u)\) and \(v(u)\) are defined in Lemma 3.6.

Proof. For \(u \in N_{\delta, c^*}\) with \(\|u\|_{L^2} = \|Q_{c_0}\|_{L^2}\), the equation (34) yields
\[
\|\Theta(a(u), \beta(a(u), c_0))\|_{L^2}^2 = \|Q_{c_0}\|_{L^2}^2 = \|v(u) + \Theta(a(u), c(u))\|_{L^2}^2 = \|v(u)\|_{L^2}^2 + \|\Theta(a(u), c(u))\|_{L^2}^2.
\]
Since
\[
|c_0 - c^*| + |\beta(a(u), c_0) - c^*| < \frac{c^*}{2}
\]
for sufficiently small $\delta > 0$, we have
\[
\|v(u)\|_{L^2}^2 = \|\Theta(a(u), \beta(a(u), c_0))\|^2_{L^2} - \|\Theta(a(u), c(u))\|^2_{L^2} = (\beta(a(u), c_0) - c(u))\|\varphi_c(a(u))\|^2_{L^2} \gtrsim \beta(a(u), c_0) - c(u) \geq 0.
\]
Therefore, we obtain
\[
\|\Theta(a(u), \beta(a(u), c_0)) - \Theta(a(u), c(u))\|_{H^1} \lesssim \|\beta(a(u), c_0) - c(u)\| \|\partial_c Q_{c^*}\|_{H^1} + o(\|\beta(a(u), c_0) - c(u)\|) \lesssim \|v(u)\|^2_{L^2}.
\]

In the following theorem, we prove the stability of the line solitary wave on the set $\mathcal{M}^\delta_{c^*}(c^*, \varepsilon)$.

**Theorem 3.8.** Let $3/2 < \alpha < 2$. Assume that $l_1, l_2, \delta, \kappa > 0$ satisfy the condition (15) and (24). For any $\varepsilon > 0$, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(c^*, \varepsilon) > 0$ such that for $u_0 \in \mathcal{M}^\delta_{c^*}(c^*, \tilde{\varepsilon})$ the solution $u$ to the equation (1) with the initial data $u(t) \in \mathcal{M}^\delta_{c^*}(c^*, \varepsilon)$ for all $t > 0$.

**Proof.** In the case of $c^* \notin CS$, the conclusion follows the proof of Theorem 4.6 in [39].

We show the case of $c^* \in CS$. Let $l_1, l_2, \delta, \kappa > 0$ satisfying (15) and (24). We prove the conclusion by contradiction. We assume there exists $0 < \varepsilon_0 \ll \delta^2$ such that for $0 < \bar{\varepsilon} \leq \varepsilon_0$ there exist $t_0 > 0$ and a solution $u$ to (1) with an initial data $\tau_{\rho_0}(w_0, c_0) \in \mathcal{M}^\delta_{c^*}(c^*, \varepsilon)$ satisfying $\|(P_1 + P_2)w_0\|_{H^1} \leq \bar{\varepsilon}$.

We define the solution $(v_1(t), c_1(t), \rho_1(t))$ to the system (6)-(7) with the initial data $(w_0 + G^\delta_{c^*}(w_0, c_0), c_0, \rho_0)$ and $c_2(t) = c_1(t) - c^*$, $\rho_2(t) = \rho_1(t)$, $a_2(t) = a_1(t)$ and $v_2(t) = \tau_{\rho_1(t)-\rho_2(t)}(v_1(0) + Q_{c_1(0)}) - \Theta(a_2(t), c_2(t))$, where $c(u), \rho(u)$ and $a(u)$ are defined in Lemma 3.6. Then, $(v_2(t), c_2(t), a_2(t))$ satisfies the orthogonality condition (36). Since
\[
\|\tau_{\rho_1} Q_c - \tau_{\rho} Q_{c^*}\|_{H^1} \lesssim \|(P_1 + P_2)(Q_c - \tau_{\rho_1} Q_{c^*})\|_{H^1} \lesssim \|(P_1 + P_2)v\|_{H^1} + \|\tau_{\rho_1} (v + Q_c) - \tau_{Q_{c^*}}\|_{H^1},
\]
we obtain
\[
\|v_1(t)\|_{H^1} + |c_1(t) - c^*| \lesssim \inf_{q \in \mathbb{R}} \|\tau_{\rho_1(t)}(v_1(t) + Q_{c_1(t)}) - \tau_{Q_{c^*}}\|_{H^1} + \|(P_1 + P_2)v_1(0)\|_{H^1}.
\]
Therefore, by the continuity of $u(t)$ and $(v_1(t), c_1(t), \rho_1(t))$ we have
\[
\|v_1(t)\|_{H^1} + |c_1(t) - c^*| \lesssim \varepsilon_0 + \bar{\varepsilon}^{1/2} \ll \delta
\]
and
\[
u(t) = \tau_{\rho_1(t)}(v_1(t) + Q_{c_1(t)}) = \tau_{\rho_2(t)}(v_2(t) + \Theta(a_2(t), c_2(t)))
\]
for $0 \leq t \leq t_0$. We define $c_u > 0$ by $\|u(0)\|_{L^2} = \|Q_{c_u}\|_{L^2}$. By Lemma 3.7, we have
\[
\|\Theta(a_2(t), \beta(a_2(t), c^*)) - \Theta(a_2(t), c_2(t))\|_{H^1} \lesssim |c^* - c_u| + \|v_2(t)\|^2_{L^2}.
\]
Therefore, we obtain
\[
(S_c^1 (\Theta (a_2, \beta (a_2, c^*))), v_2)_{H^{-1}, H^1}
= (S_c^1 (\varphi_c (a_2)), v_2)_{H^{-1}, H^1}
+ (S_c^1 (\varphi_c (a_2)) (\Theta (a_2, \beta (a_2, c^*)) - \varphi_c (a_2)), v_2)_{H^{-1}, H^1}
+ (c^* - \hat{c}(a_2)) (\Theta (a_2, \beta (a_2, c^*)) - \Theta (a_2, a_2), v_2)_{H^{-1}, H^1}
+ O(\|\Theta (a_2, \beta (a_2, c^*)) - \varphi_c (a_2)\|^2_{H^1} \|v_2\|_{H^1})
= O((|c_u - c^*| + |\beta (a_2, c^*) - c^*|^2 + \|v_2\|^2_{L^2}) \|v_2\|_{H^1})
\]
(42)

Thus, from (35), (36), (41) and (42) there exists \( C > 0 \) such that
\[
S_{c^*} (u) - S_{c^*} (Q_{c^*})
= S_{c^*} (\Theta (a_2, \beta (a_2, c^*)) - S_{c^*} (Q_{c^*}) + \frac{1}{2} (S_c^1 (Q_{c^*}) (I - P_0) v_2, (I - P_0) v_2)_{H^{-1}, H^1}
+ O(\|v_2\|^2_{H^1}) + o(|c^* - c_u| + |a_2|^2)
\geq C |a_2|^2 + \frac{1}{2} \langle L_{c^*} \gamma_2, \gamma_2 \rangle_{H^{-1}, H^1} + \sum_{j,k} \Lambda^{+, j}_{k, 2} \Lambda^{-, j}_{k, 2} + O(\|v_2\|^3_{H^1}) + o(|c^* - c_u| + |a_2|^2)
\]
(43)

where
\[
\Lambda^{+, j}_{k, 2} = \langle v_2, L_{c^*} F_k^{\pm, j} \rangle_{L^2}, \quad \gamma_2 = P_1 v_2
\]
for \( j \in \{0, 1\} \). By Lemma 3.6, we have
\[
\sup_{0 \leq t \leq \tau_0} \|A_c (t)\|_{H^1} + \|c_2 (t) - c^*\| + |a_2 (t)| \lesssim \varepsilon_0.
\]
(44)

The equation (40) yields
\[
\langle (\tau_{p_1 - \rho_2} Q_{c^*}, \partial_x Q_{c^*}) L^2 \rangle
\leq \langle (\tau_{p_1 - \rho_2} Q_{c^*}, \partial_x \Theta (a_2, c_2)) L^2 \rangle - (\tau_{p_1 - \rho_2} Q_{c^*}, \partial_x Q_{c^*}) L^2 \rangle + (v_1, \partial_x Q_{c^*}) L^2 \rangle
+ \langle (v_1, \tau_{\rho_2 - \rho_1} \partial_x \Theta (a_2, c_2)) L^2 \rangle - (v_1, \partial_x Q_{c^*}) L^2 \rangle
\lesssim (\varepsilon_0 + |p_1 - \rho_2|^2 + \tilde{\varepsilon}^{1/2})
\]
for sufficiently small \( |p_1 - \rho_2| \). Therefore, if \( |p_1 - \rho_2|, \varepsilon_0 \) and \( \varepsilon_0 \) are sufficiently small, we have
\[
|p_1 - \rho_2| \leq (\tau_{p_1 - \rho_2} Q_{c^*}, \partial_x Q_{c^*}) L^2 \rangle \lesssim \varepsilon_0^2 + \tilde{\varepsilon}^{1/2}
\]
which yields
\[
\sup_{0 \leq t \leq \tau_0} |p_1 (t) - \rho_2 (t)| \lesssim \varepsilon_0^2 + \tilde{\varepsilon}^{1/2}.
\]

Since \( P_1 v_1 (t) = G^\pm (v_1 (t), c_1 (t)) \), we obtain
\[
\sum_{j,k} |\Lambda^+_{k, 2} (t)|^2 = \|P_1 v_2 (t)\|_{E_n}^2
\lesssim \|P_1 v_1 (t)\|_{E_n}^2 + |p_1 (t) - \rho_2 (t)|^2 \|v_1 (t)\|_{E_n}^2
= \|G^\pm (v_1 (t), c_1 (t))\|_{E_n}^2 + |p_1 (t) - \rho_2 (t)|^2 \|v_1 (t)\|_{E_n}^2
\lesssim \varepsilon_0^2 + |c_1 (t) - c^*|^2 + |p_1 (t) - \rho_2 (t)|^2 \|v_1 (t)\|_{E_n}^2
\lesssim (\varepsilon_0 + \tilde{\varepsilon})^2 + |c_1 (t) - c^*|^2 + |p_1 (t) - \rho_2 (t)|^2 \|v_1 (t)\|_{E_n}^2.
\]
(45)
By the definition of $v_2$, we have
\[ \partial_t \Lambda^{-2}_{k,j} = -\lambda_k \Lambda^{-2}_{k,j} + (-\partial_x [(v_2)^2 - (\rho_2 - c^*) v_2 + 2v_2(\Theta(a_2, c_2) - Q_{c^*})], L_{c^*} F^{+j}_{k})_{L^2}. \]
(46)

Since $|\Lambda^{-2}_{k,j}(0)| \lesssim \tilde{\varepsilon}$, for sufficiently small $\tilde{\varepsilon}$ we have
\[ \sup_{0 \leq t \leq t_0} |\Lambda^{-2}_{k,j}(t)| \lesssim \tilde{\varepsilon} + \varepsilon_0^2. \]
(47)

Combining (43)–(47), (v) of Proposition 2 and Lemma 3.5, we obtain
\[ \sup_{0 \leq t \leq t_0} (\|(I - P_- - P_+) v_2(t)\|_{H^1}^2 + |a_2(t)|^4) \]
\[ \lesssim S_{c^*}(u) - S_{c^*}(Q_{c^*}) + l_1((\varepsilon_0 + \varepsilon_1^{1/2})^\alpha + (\varepsilon_0^2 + \varepsilon_1^2) + \varepsilon_0^2 + \varepsilon_1^2) \]
\[ + O((\|P_+ + P_-\|_{L^2}^2) + o(\|v_2\|_{L^2})) \]
\[ \lesssim l_1(\varepsilon_0^{\alpha+2} + \varepsilon_1^2 + \varepsilon_0^2). \]
(48)

Using the estimates (45), (47) and (48), we obtain
\[ \sup_{0 \leq t \leq t_0} |\Lambda^{-2}_{k,j}(t)| \lesssim \tilde{\varepsilon} + \varepsilon_0^{\alpha+1}. \]
(49)

By the bootstrap argument, we have
\[ \sup_{0 \leq t \leq t_0} (\|(I - P_- - P_+) v_2(t)\|_{H^1}^2 + |a_2(t)|^4) \lesssim l_1(\varepsilon_0^{2\alpha+1} + \varepsilon_1^{3\alpha}). \]
(50)

For $\tilde{\varepsilon} \ll \varepsilon_0$, the inequalities (45), (49) and (50) yield
\[ \sup_{0 \leq t \leq t_0} \inf_{q \in \mathbb{R}} \|u(t) - \tau_q Q_{c^*}\|_{H^1} \]
\[ \lesssim \sup_{0 \leq t \leq t_0} (\|v_2(t)\|_{H^1} + \|\Theta(a_2(t), c_2) - \Theta(a_2(t), \beta(a_2(t), c^*))\|_{H^1} + \|\varphi_{c^*}(a_2(t), c_2) - Q_{c^*}\|_{H^1} \]
\[ \lesssim \varepsilon_0^{\alpha+1} + \varepsilon_0^{3\alpha} + \varepsilon_1^{1/2} \ll \varepsilon_0 \]
which contradicts the inequality (39). Therefore, for any $\varepsilon > 0$ there exists $\tilde{\varepsilon} > 0$ such that for $u_0 \in \mathcal{M}_{\varepsilon_{\Lambda}}(c^*, \tilde{\varepsilon})$ the solution $u$ to the equation (1) with the initial data $u_0$ satisfies
\[ \sup_{t \geq 0} \inf_{q \in \mathbb{R}} \|u(t) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon. \]

By the same calculation to obtain the equality (40) and $P_+(v_1(t)) = G_{c^*}^2(v_1(t), c_1(t))$, we have $u(t) \in \mathcal{M}_{\varepsilon_{\Lambda}}(c^*, \tilde{\varepsilon})$ for all $t > 0$. \hfill \Box

In the following lemma, we show a lower estimate for the unstable mode.

**Lemma 3.9.** There exists $C_\kappa > 0$ such that if $\kappa, \delta, l_0 > 0$ satisfy
\[ (\kappa^{1/2} + C_\kappa \delta^{1/4})(1 + l_0) \ll \min\{1, l_0\}, \]
(51)
then there exists $T^* > 0$ such that for any solutions $(v_0, \omega_0, \rho_0)$ and $(v_1, c_1, \rho_1)$ to the system (6)–(7) satisfying
\[ m_{\omega,\kappa}(v_0(0), v_1(0))^2 - \|P_+(v_0(0) - v_1(0))\|_{E_{c^*}}^2 \leq l_0^2 \|P_+(v_0(0) - v_1(0))\|_{E_{c^*}}^2, \]
(52)
Lemma 3.10. Let $\delta, l_0, \kappa > 0$. Suppose the assumption (51). There exists $0 < \varepsilon_* = \varepsilon_*(\varepsilon^*, \delta, l_0, \kappa) < \delta$ such that for any $0 < \varepsilon < \varepsilon_*$ and solutions $u_0(t)$ and $u_1(t)$ to the equation (1) satisfying

$$\sup_{t \geq 0} \inf_{\mathcal{Q} \in \mathbb{R}} \| u_1(t) - \tau_\varepsilon Q_{c^*} \|_{H^1} < \varepsilon, \quad \inf_{\mathcal{Q} \in \mathbb{R}} \| u_0(0) - \tau_\varepsilon Q_{c^*} \|_{H^1} < \varepsilon$$

(55)

and

$$m_{\delta, \kappa}(v_0(t), v_1(t))^2 - \| P_+(v_0(t) - v_1(t)) \|_{E_n}^2 < l_0^2 \| P_+(v_0(0) - v_1(0)) \|_{E_n}^2,$$

(56)

one has

$$\inf_{\mathcal{Q} \in \mathbb{R}} \| u_0(t) - \tau_\varepsilon Q_{c^*} \|_{H^1} \geq \varepsilon$$

(57)

for some $t_0 > 0$, where $(v_0(0), c_0(0), \rho_0(0))$ and $(v_1(0), c_1(0), \rho_1(0))$ satisfy

$$u_j(0) = \tau_{\rho_j(0)}(v_j(0) + Q_{c_j(0)}), \quad \| (v_1(0), \partial_x Q_{c^*})_{L^2} \| + \| (v_1(0), Q_{c^*})_{L^2} \| < \varepsilon^{1/2}$$

(58)

for $j = 0, 1$.

Proof. Let $v_j$ be the solutions to (6)–(7) with the initial data $(v_j(0), c_j(0), \rho_j(0))$. We prove the inequality (57) by the contradiction. Assume for any $0 < \varepsilon_* < \delta^2$ there exist $0 < \varepsilon < \varepsilon_*$ and solutions $u_0(t)$ and $u_1(t)$ to the equation (1) satisfying (55), (56), (58) and

$$\sup_{t \geq 0} \inf_{\mathcal{Q} \in \mathbb{R}} \| u_0(t) - \tau_\varepsilon Q_{c^*} \|_{H^1} < \varepsilon.$$

(59)
By the inequality
\[ \|v_1(t)\|_{H^1} + |c_1(t) - c^*| \]
\[ \lesssim \inf_{q \in \mathbb{R}} \|u_1(t) - \tau_q Q_{c^*}\|_{H^1} + \|(v_1(0), \partial_x Q_{c^*})_{L^2}\| + \|(v_1(0), Q_{c^*})_{L^2}\| - \varepsilon \]
and \(u_1(t) = \tau_{\rho_1(t)}(v_1(t) + Q_{c_1(t)})\) as long as \(\|v_1(t)\|_{H^1}^2 + |c_1(t) - c^*|^2 < \delta^2\), we obtain \(u_1(t) = \tau_{\rho_1(t)}(v_1(t) + Q_{c_1(t)})\) for all \(t \geq 0\). From the inequality (56), applying Lemma 3.9 repeatedly, we have
\[ 0 < \frac{1}{2} e^{\frac{\varepsilon}{2}} \|P_+(v_0(0) - v_1(0))\|_{L^2} \leq \|P_+(v_0(t) - v_1(t))\|_{L^2} \]
and
\[ (m_{\delta,\kappa}(v_0(t), v_1(t))^2 - \|P_+(v_0(t) - v_1(t))\|_{L^2}^2) \lesssim 2l_0\|P_+(v_0(t) - v_1(t))\|_{L^2} \]
for all \(t > 0\). Since
\[ \|P_+(v_0(t) - v_1(t))\|_{L^2} \lesssim \|P_+(v_0(t) - v_1(t))\|_{L^2} \lesssim \inf_{q \in \mathbb{R}} \|v_0(t) + Q_{c_0(t)} - \tau_q Q_{c^*}\|_{L^2} + \inf_{q \in \mathbb{R}} \|v_1(t) + Q_{c_1(t)} - \tau_q Q_{c^*}\|_{L^2}, \]
by the assumption (55) and the inequality (60) we have
\[ \frac{1}{2} e^{\frac{\varepsilon}{2}} \|P_+(v_0(0) - v_1(0))\|_{L^2} \leq \|P_+(v_0(t) - v_1(t))\|_{L^2} \lesssim \inf_{q \in \mathbb{R}} \|u_0(t) - \tau_q Q_{c^*}\|_{L^2} + \varepsilon \]
for all \(t \geq 0\) and \(0 < \varepsilon \ll \delta\) as long as \(\|v_0(t)\|_{L^2}^2 + |c_0(t) - c^*|^2 < \delta^2\). By the assumption (55) and the inequality (61), we obtain
\[ \|v_0(t)\|_{L^2} + |c_0(t) - c^*|^2 \]
\[ \lesssim m_{\delta,\kappa}(v_0(t), (0, c^*))^2 \]
\[ \lesssim m_{\delta,\kappa}(v_0(t), v_1(t))^2 + \inf_{q \in \mathbb{R}} \|(I - P_0)v_1(t) + Q_{c_1(t)} - \tau_q Q_{c^*}\|_{L^2}^2 \]
\[ + \|(v_1(t), \partial_x Q_{c^*})_{L^2}\| + \|(v_1(t), Q_{c^*})_{L^2}\|^2 \]
\[ \lesssim (1 + l_0)^2\|P_+(v_0(t) - v_1(t))\|_{L^2}^2 + \varepsilon. \]
The inequality (60), (62) and (63) contradict the assumption (59) for sufficiently small \(\varepsilon > 0\). Therefore, the proof was completed.

In the next corollary, we show (v) of Theorem 1.2 which means that solutions with the initial data off \(M_{c^*}\) go out of the neighborhood of the line solitary waves.

**Corollary 1.** Let \(\delta, l_0, \kappa > 0\). Suppose the assumption (51). There exists \(\varepsilon^* = \varepsilon^*(c^*, \delta, l_0, \kappa) > 0\) such that for \(u(0) \in \mathcal{N}_{c^*} \setminus M_{c^*}(c^*, \varepsilon^*)\), the solution \(u\) of the equation (1) corresponding to the initial data \(u(0)\) satisfies
\[ \inf_{q \in \mathbb{R}} \|u(t_0) - \tau_q Q_{c^*}\|_{H^1} \gtrsim \varepsilon^* \]
for some \(t_0 \geq 0\).

**Proof.** By the Lipschitz continuity of \(G_{c^*}\), there exists \(C > 1\) such that for \(0 < \varepsilon < \min\{\bar{\varepsilon}, \varepsilon_\ast\}\) and \(u \in \mathcal{N}_{c^*} \setminus M_{c^*}(c^*, \varepsilon_\ast)\), the solution \(u\) of the equation (1) corresponding to the initial data \(u(0)\) satisfies
\[ \inf_{q \in \mathbb{R}} \|u(t_0) - \tau_q Q_{c^*}\|_{H^1} \gtrsim \varepsilon^* \]
for some \(t_0 \geq 0\).
Corollary 2. Let $\delta, \kappa, l > 0$. Assume the conditions (15) and (51). There exists $\varepsilon_1 = \varepsilon_1(c^*, \delta) > 0$ such that for $0 < \varepsilon < \varepsilon_1$

$$\mathcal{M}^\delta_{cs}(c^*, \varepsilon) = \mathcal{M}^\delta_{cs}(c^*, \varepsilon)$$

and

$$\{g(w, c^*); w \in P_{<0}H^1(\mathbb{R} \times \mathbb{T}_L), \|(P_1 + P_2)w\|_{H^1} < \varepsilon^{1/2}, \inf_{q \in \mathbb{R}} \|g(w, c^*) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon\}$$

$$= \{\{\tau_p g(w, c^*) \} \in (P_1 + P_2)(\mathbb{R} \times \mathbb{T}_L), |c - c^*| \leq c^*/2, \rho \in \mathbb{R}, \|\| (P_1 + P_2)(\tau_p g(w, c) - Q_{c'})\|_{H^1} < \varepsilon^{1/2}, \inf_{q \in \mathbb{R}} \|g(w, c) - \tau_q Q_{c'}\|_{H^1} < \varepsilon\}. $$

Moreover,

$$\mathcal{M}^\delta_{cs}(c^*, \varepsilon) = \bigcup_{\rho \in \mathbb{R}} \{\tau_p g(w, c^*) \in P_{<0}H^1(\mathbb{R} \times \mathbb{T}_L), \|\| (P_1 + P_2)w\|_{H^1} < \varepsilon^{1/2}, \inf_{q \in \mathbb{R}} \|g(w, c^*) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon\}. $$

Proof. By the definitions of $\mathcal{M}^\delta_{cs}(c^*, \varepsilon)$ and $\mathcal{M}^\delta_{cs}(c^*, \varepsilon)$, we have

$$\mathcal{M}^\delta_{cs}(c^*, \varepsilon) \subset \mathcal{M}^\delta_{cs}(c^*, \varepsilon).$$

Let $0 < \varepsilon < \min\{\bar{\varepsilon}(c^*, \varepsilon^*), \varepsilon^*\}$, where $\bar{\varepsilon}$ is defined in Theorem 3.8 and $\varepsilon^*$ is defined in Corollary 1. For any solutions $u(t)$ with initial data $u(0) \in \mathcal{M}^\delta_{cs}(c^*, \varepsilon)$, we have

$$\sup_{t \geq 0} \inf_{q \in \mathbb{R}} \|u(t) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon^*.$$ 

By Corollary 1, solutions $u(t)$ of (1) with initial data $u(0) \in \mathcal{N}_{c^*, c^*} \setminus \mathcal{M}^\delta_{cs}(c^*, \varepsilon)$ satisfy

$$\sup_{t \geq 0} \inf_{q \in \mathbb{R}} \|u(t) - \tau_q Q_{c^*}\|_{H^1} \geq \varepsilon^*.$$ 

Therefore, we obtain $\mathcal{M}^\delta_{cs}(c^*, \varepsilon) \subset \mathcal{M}^\delta_{cs}(c^*, \varepsilon)$. 

Equation (1) corresponding to the initial data $u(0)$, where $\varepsilon_*$ is defined by Lemma 3.10 and $\bar{\varepsilon} = \bar{\varepsilon}(c^*, \varepsilon^*)$ is defined by Theorem 3.8. We define the solution $(v_1, c_1, \rho_1)$ to the system (6)–(7) corresponding to the initial data $(P_+ + P_a + P_\rho)\varepsilon(0) + G_+((P_+ + P_a + P_\rho)c(0), c(u(0)), \rho(u(0)))$, where $v(0) = f_{\rho(u(0))}u(0) - Q_{c(u(0))}$. Since $(P_+ + P_a + P_\rho)c(0) + G_+((P_+ + P_a + P_\rho)c(0), c(u(0))) + Q_{c(u(0))} \in \mathcal{M}^\Delta_{cs}$.
Since \( \hat{\mathcal{M}}_{cs}^{\delta}(c^*, \varepsilon) \subset \mathcal{M}_{cs}^\delta(c^*, \varepsilon) \), we have
\[
\{ g(w, c^*); w \in P_{\leq 0}H^1(\mathbb{R} \times T_L), \|(P_1 + P_2)w\|_{H^1} < \varepsilon^{1/2}, \quad \inf_{q \in \mathbb{R}} \|g(w, c^*) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon \}
\]
\[
\subset \{ \tau_{\rho}g(c); w \in (P_+ + P_\alpha + P_\gamma)H^1(\mathbb{R} \times T_L), |c - c^*| \leq c^*/2, \rho \in \mathbb{R}, \quad \|(P_1 + P_2)(\tau_{\rho}g(w, c) - Q_{c^*})\|_{H^1} < \varepsilon^{1/2}, \inf_{q \in \mathbb{R}} \|g(w, c) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon \}.
\]

Let \( w \in (P_+ + P_\alpha + P_\gamma)H^1(\mathbb{R} \times T_L), c > 0 \) and \( \rho \in \mathbb{R} \) satisfying \( |c - c^*| \leq c^*/2 \),
\[
\|(P_1 + P_2)(\tau_{\rho}g(w, c) - Q_{c^*})\|_{H^1} < \varepsilon^{1/2} \quad \text{and} \quad \inf_{q \in \mathbb{R}} \|g(w, c) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon.
\]

We define the solution \( u_0(t) \) to the equation (1) with the initial data \( \tau_\rho g(w, c) \) and the solution \( u_1(t) \) to the equation (1) with the initial data \( g(P_{\leq 0}w_0, c^*) \), where
\[
w_0 = \tau_\rho g(w, c) - Q_{c^*}.
\]
By Theorem 3.8, there exists \( l_0 > 0 \) such that \( l_0 \) and \( \delta \) satisfy (51) and
\[
\sup_{t \geq 0} \inf_{q \in \mathbb{R}} \|u_j(t) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon_*(c^*, \delta, l_0)
\]
for \( j = 0, 1 \) and sufficiently small \( \varepsilon > 0 \), where \( \varepsilon_* \) is defined in Lemma 3.10. Since \( u_1(0) \) satisfy (58) in Lemma 3.10 for \( 0 < \varepsilon < \varepsilon_* \), we have
\[
l_0P_{\leq 0}(u_0(0) - u_1(0)) \leq \|P_{\leq 0}(u_0(0) - u_1(0))\|_{H^1} = 0
\]
and \( u_0(0) = u_1(0) \). Therefore, we obtain
\[
\{ g(w, c^*); w \in P_{\leq 0}H^1(\mathbb{R} \times T_L), \|(P_1 + P_2)w\|_{H^1} < \varepsilon^{1/2}, \quad \inf_{q \in \mathbb{R}} \|g(w, c^*) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon \}
\]
\[
\subset \{ \tau_{\rho}g(w, c); w \in (P_+ + P_\alpha + P_\gamma)H^1(\mathbb{R} \times T_L), |c - c^*| \leq c^*/2, \rho \in \mathbb{R}, \quad \|(P_1 + P_2)(\tau_{\rho}g(w, c) - Q_{c^*})\|_{H^1} < \varepsilon^{1/2}, \inf_{q \in \mathbb{R}} \|g(w, c) - \tau_q Q_{c^*}\|_{H^1} < \varepsilon \}.
\]

Since
\[
\inf_{\rho \in \mathbb{R}} \|(P_1 + P_2)(\tau_{\rho}g(w, c^*) - Q_{c^*})\|_{H^1} \leq \inf_{q \in \mathbb{R}} \|g(w, c^*) - \tau_q Q_{c^*}\|_{H^1}.
\]
for \( w \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \), we have the equality (66) for sufficiently small \( \varepsilon > 0 \). \( \square \)

4. The \( C^1 \) regularity of center stable manifolds. In this section, we prove that \( G_+^\delta \) is a \( C^1 \) function on \( P_{\leq 0}H^1(\mathbb{R} \times T_L) \times (0, \infty) \) by applying the argument in the section 2.3 in [16]. Let \( \varepsilon > 0 \) and \( \psi_0, \psi_1 \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \) with \( \|\psi_0\|_{H^1} < \varepsilon \).
We define a solution \( (v_0, c_0, \rho) \) to the system (6)–(7) such that \( v_0(0) = \psi_0 + G_+^\delta(\psi_0, c^*) \) and \( c_0(0) = c^* \). Let \( v_0 \) be a solution to the equation
\[
v_t = \partial_x \psi_0, v + (\dot{\rho} - c^*)\partial_x v + 2\partial_x(\dot{v} - \dot{Q}_c v) + (\dot{\rho} - c_0)\partial_x Q_{c_0} - \dot{\epsilon}_0 \partial_x Q_{c_0} - \partial_x(v^2)
\]
with initial data \( v_0(0) = \psi_0 + h\psi_1 + G_+^\delta(\psi_0 + h\psi_1, c^*) \). If \( \varepsilon > 0 \) is sufficiently small, then \( \tau_{\rho_0}(v_0 + Q_{c_0}) \) and \( \tau_{\rho_0}(v_0 + Q_{c_0}) \) are solutions to the equation (1). By the Lipschitz continuity of \( G_+^\delta \), for any sequence \( \{h_n\}_n \in \mathbb{R} \) with \( h_n \rightarrow 0 \) as \( n \rightarrow \infty \) there exist a subsequence \( \{h_n\}_n \subset \{h_n\}_n \) and \( \psi_* \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \) such that
\[
\frac{G_+^\delta(\psi_0 + h_n\psi_1, c^*) - G_+^\delta(\psi_0, c^*)}{h_n} \rightarrow \psi_* \quad \text{as} \quad n \rightarrow \infty.
\]
Let \( w_h = \tau_{p_0} v_h \) for \( h \geq 0 \). Then, \( w_h \) is the solution to the equation
\[
w_t = -\partial_x \Delta w - 2\partial_x (\tau_{p_0} Q_{c_0} w) + (\rho_0 - c_0) \partial_{x} \partial_x Q_{c_0} - \partial_{x} (w^2)
\]
with \( w_h(0) = v_h(0) \). By the well-posedness result of the equation (1) in [26], we have there exists \( b_0 > \frac{1}{2} \) such that for \( T > 0 \) and \( \frac{1}{2} < b < b_0 \) there exists \( C = C(T, b) > 0 \) satisfying
\[
\|w_0\|_{X^{1,b}_T} \leq C\|v_0(0)\|_{H^1}.
\] (68)

We define \( \xi \) as the solution to the equation
\[
\xi_t = -\partial_x \Delta \xi - 2\partial_x (\tau_{p_0} Q_{c_0} \xi) - 2\partial_x (w_0 \xi)
\] (69)
with the initial data \( \xi(0) = \psi_1 + \psi_2 \). By the smoothness of the flow map of equation (1) given by [26], we have that for \( T > 0 \)
\[
\left\| \frac{w_{h_n} - w_0}{H_n} - \xi \right\|_{L^\infty((-T,T), H^1)} \to 0 \text{ as } n \to \infty.
\] (70)

Let \( \eta = \tau_{-\rho_0} \xi \). Then, \( \eta \) satisfies the equation
\[
\eta_t = \partial_x L_{c^*} \eta - 2\partial_x ((Q_{c_0} - Q_{c^*}) \eta) + (\rho_0 - c^*) \partial_x \eta - 2\partial_x (v_0 \eta).
\] (71)

The following lemma shows the well-posedness of (71).

**Lemma 4.1.** Let \( \kappa > 0 \), \( b_0 > 1/2 \) and \((v_0, c_0, \rho_0)\) be a solution to the system (6)–(7) satisfying
\[
\sup_{t \geq 0}(\|v_0(t)\|_{H^1} + |c_0(t) - c^*|) \leq \kappa \text{ and } \tau_{\rho_0(t)}(v_0(t) + Q_{c_0(t)}) \text{ is a solution to the equation (1) on } [0, \infty) \text{ with }
\]
\[
\tau_{\rho_0(t)}(v_0(t) + Q_{c_0(t)}) \in X^{1,b}_{0,T}.
\]

Then, the Cauchy problem of the equation (71) is well-posed in \( H^1(\mathbb{R} \times \mathbb{T}_L) \). Precisely, there exists \( b > \frac{1}{2} \) such that for any \( \eta_0 \in H^1(\mathbb{R} \times \mathbb{T}_L) \) there exists a unique solution \( \xi \) to the equation (69) satisfying that \( \xi(0) = \tau_{-\rho_0(0)} \eta_0 \),
\[
\xi \in X^{1,b}_{0,T} \text{ for } T > 0
\]
and \( \tau_{-\rho_0} \xi \) is a solution to the equation (71) with initial data \( \eta_0 \). Then, the solution map \( \eta_0 \to \xi \in X^{1,b}_{0,T} \) is continuous for \( T > 0 \). Moreover, for any solutions \( \eta \) to the equation (71) and \( s \geq 0 \) there exists \( \xi_s \in X^{1,b} \) such that for \( t \in \{\min\{s-1, 0\}, s+1\} \) we have \( \eta(t) = \tau_{\rho_0(s)-\rho_0(t)} \xi_s(t) \) and
\[
\|\eta\|_{L^\infty((\min\{s-1, 0\}, s+1), H^1)} \lesssim \|\xi_s\|_{X^{1,b}} \lesssim \|\eta(s)\|_{H^{1}}.
\] (72)

Applying the estimate (11)–(13) to Duhamel’s formula for the equation (69), we obtain the well-posedness of the equation (71). We omit the detail of the proof of Lemma 4.1.

In the following lemma, we prove the criterion of the growth estimate of solutions to (71).

**Lemma 4.2.** Let \( K_0 > 0 \). There exist \( \iota_0 = \iota_0(K_0), K_1 = K_1(K_0) > 0 \) such that for \( 0 < \iota < \iota_0 \) and a solution \((v_0, c_0, \rho_0)\) to the system (6)–(7) satisfying that
\[
\sup_{t \geq 0}(\|v_0(t)\|_{H^1} + |c_0(t) - c^*|) \leq \iota \text{ and } \tau_{\rho_0(t)}(v_0(t) + Q_{c_0(t)}) \text{ is a solution to the equation (1) on } [0, \infty) \text{ with }
\]
\[
\tau_{\rho_0(t)}(v_0(t) + Q_{c_0(t)}) \in X^{1,b}_{0,T},
\]
the following holds. If a solution \( \eta \) to the equation (71) with initial data \( \eta(0) \in H^1(\mathbb{R} \times \mathbb{T}_L) \) satisfies
\[
K_0 t^\frac{2}{3} \| P_{\leq 0} \eta(t_0) \|_{E^{1/3}_{1/3}} < \| P_+ \eta(t_0) \|_{E^{1/3}_{1/3}} \quad (73)
\]
at some \( t_0 \geq 0 \), then for \( t \geq t_0 + 1/2 \)
\[
3 \| P_+ \eta(t) \|_{E^{1/3}_{1/3}} > e^{K_1 t_0^{1/6} t^{1/3}} (\| P_+ \eta(t_0) \|_{E^{1/3}_{1/3}} + K_0 t_0^{1/3} \| P_{\leq 0} \eta(t) \|_{E^{1/3}_{1/3}}) \quad (74)
\]
On the other hand, if (73) fails for \( t_0 \geq 0 \), then for \( t \geq 0 \)
\[
\| P_+ \eta(t) \|_{E^{1/3}_{1/3}} \leq K_0 t^{1/3} \| P_{\leq 0} \eta(t) \|_{E^{1/3}_{1/3}} \leq e^{K_1 t^{1/6} t^{1/3}} \| P_{\leq 0} \eta(0) \|_{E^{1/3}_{1/3}} \quad (75)
\]
Proof. By the inequality (13), (68) and (72), for \( t_1, t_2 \geq 0 \) with \( |t_1 - t_2| < 1 \) we have
\[
\| P_\gamma \eta(t_2) \|^2_{E^{1/3}_{1/3}} - \| P_\gamma \eta(t_1) \|^2_{E^{1/3}_{1/3}} \lesssim \| \eta(t_1) \|^2_{H^1} \lesssim t^{1/3} \| \eta(t_1) \|^2_{E^{1/3}_{1/3}} \quad (76)
\]
Since
\[
\| P_\gamma \partial_t \eta(t) - \partial_t L \epsilon, P_\gamma \eta(t) \|_{E^{1/3}_{1/3}} \lesssim \| \eta(t) \|_{H^1},
\]
we have
\[
\| P_\gamma \eta(t_2) \|_{E^{1/3}_{1/3}} - \| P_\gamma \eta(t_1) \|_{E^{1/3}_{1/3}} \lesssim t^{2/3} \| \eta(t_1) \|_{E^{1/3}_{1/3}} \quad (77)
\]
and
\[
\| P_- \eta(t_2) \|_{E^{1/3}_{1/3}} - e^{-k_+ (t_2 - t_1)} \| P_- \eta(t_1) \|_{E^{1/3}_{1/3}} \lesssim t^{2/3} \| \eta(t_1) \|_{E^{1/3}_{1/3}} \quad (78)
\]
for \( t_1, t_2 \geq 0 \) with \( |t_1 - t_2| < 1 \). By the inequality
\[
| (\partial_x Q_\epsilon, \partial_t \eta(t))_{L^2} | + \left| \left( Q_{\epsilon, c}' \cos \frac{n_0 y}{L}, \partial_t \eta(t) \right)_{L^2} \right| + \left| \left( Q_{\epsilon, c}' \sin \frac{n_0 y}{L}, \partial_t \eta(t) \right)_{L^2} \right| \lesssim \| P_\gamma \eta(t) \|_{H^1} + \| P_\gamma \eta(t) \|_{H^1},
\]
and (72)–(77) we obtain for \( t_1, t_2 \geq 0 \) with \( |t_1 - t_2| < 1 \)
\[
\| P_\gamma \eta(t_2) \|_{E^{1/3}_{1/3}} - \| P_\gamma \eta(t_1) \|_{E^{1/3}_{1/3}} \lesssim \| \eta(t_1) \|_{E^{1/3}_{1/3}} t^{1/3} (P_\gamma + P_0) \eta(t_1) \|_{E^{1/3}_{1/3}} \quad (79)
\]
for sufficiently small \( \epsilon > 0 \). By the inequality (76)–(79), there exists \( C > 0 \) such that
\[
\| P_{\leq 0} \eta(t_2) \|_{E^{1/3}_{1/3}} \leq (1 + C t^{1/6}) \| P_{\leq 0} \eta(t_1) \|_{E^{1/3}_{1/3}} + C t^{1/2} \| P_+ \eta(t_1) \|_{E^{1/3}_{1/3}} \quad (80)
\]
for \( t_1, t_2 \geq 0 \) with \( |t_1 - t_2| < 1 \). The inequality
\[
\| P_+ \partial_t \eta(t) - \partial_t L \epsilon, P_+ \eta(t) \|_{E^{1/3}_{1/3}} \lesssim \| \eta(t) \|_{H^1}
\]
implies that there exists \( C > 0 \) such that for \( t_1, t_2 \geq 0 \) with \( |t_1 - t_2| < 1 \)
\[
\partial_t \| P_+ \eta(t_2) \|_{E^{1/3}_{1/3}} \geq k_+ \| P_+ \eta(t_2) \|_{E^{1/3}_{1/3}} - C t^{2/3} \| \eta(t_1) \|_{E^{1/3}_{1/3}} \quad (82)
\]
Suppose (73) for some \( t_0 \). The inequality (82) yields
\[
\| P_+ \eta(t) \|_{E^{1/3}_{1/3}} \geq e^{k_+ (t-t_0)} \| P_+ \eta(t_0) \|_{E^{1/3}_{1/3}} - (e^{k_+ (t-t_0)} - 1) C t^{1/3} \| P_+ \eta(t_0) \|_{E^{1/3}_{1/3}}
\]
\[
\geq (1 - C t^{1/3}) e^{k_+ (t-t_0)} \| P_+ \eta(t_0) \|_{E^{1/3}_{1/3}} \quad (83)
\]
for $t_0 \leq t < t_0 + 1$. From the assumption (73) and the inequalities (80) and (83), we obtain
\[
\|P_+\eta(t)\|_{E_{1/3}} > (1 - 2Ct^{1/3})(1 + Ct^{1/6})^{-1} e^{k_i(t-t_0)} K_0 t^{1/3} \|P \omega \eta(t)\|_{E_{1/3}}
\]
(84)
for $t_0 \leq t < t_0 + 1$ and sufficiently small $\epsilon > 0$. Therefore, we have
\[
\|P_+\eta(t)\|_{E_{1/3}} > K_0 t^{1/3} \|P \omega \eta(t)\|_{E_{1/3}}
\]
for $t_0 + 1/2 \leq t < t_0 + 1$ and sufficiently small $\epsilon > 0$. Applying this manner repeatedly, we obtain the inequality (74) for $t \geq t_0 + 1/2$.

Suppose (73) fails for $t \geq 0$. Then, the inequality (80) yields the inequality (75) for all $t \geq 0$ and some $K_1 > 0$.

In the following lemma, we prove the uniqueness of unstable mode of the solution to (71) not satisfying the growth condition (73).

**Lemma 4.3.** Let $K_0 > 0$. Then, there exists $\iota_1 > 0$ such that for $0 < \iota < \iota_1$, $a_1 \in \mathbb{R}$ and a solution $(v_0, c_0, \rho_0)$ to the system (6)–(7) with $\sup_{t \geq 0} \|v_0(t)\|_{H^1} + |c_0(t) - c^*| \leq \iota$ and for the solutions $\eta_1$ and $\eta_2$ to the equation (71) with $P \omega \eta_1(t) = P \omega \eta_2(t)$ not satisfying that (73) for some $t \geq 0$, we have $P_+ \eta_1(t) = P_+ \eta_2(t)$.

**Proof.** Assume there exist $0 < \iota \ll \iota_0(K_0)$, a solution $(v_0, c_0, \rho_0)$ to the system (6)–(7) with $\sup_{t \geq 0} \|v_0(t)\|_{H^1} + |c_0(t) - c^*| \leq \iota$ and for the solutions $\eta_1$ and $\eta_2$ to the equation (71) such that $P \omega \eta_1(t) = P \omega \eta_2(t)$, $P_+ \eta_1(t) \neq P_+ \eta_2(t)$ and $\eta_1$ and $\eta_2$ do not satisfy that (73) for some $t \geq 0$. Let $\eta = \eta_1 - \eta_2$. Since
\[
\epsilon^{1/3} K_0 \|P \omega \eta(t)\|_{E_{1/3}} < \|P_+\eta(t)\|_{E_{1/3}},
\]
by Lemma 4.2 we have
\[
3\|P_+\eta(t)\|_{E_{1/3}} \geq e^{\epsilon^{1/4}} (\|P_+\eta(t)\|_{E_{1/3}} + \epsilon^{1/3} \|P \omega \eta(t)\|_{E_{1/3}})
\]
(85)
for $t \geq 1/2$. On the other hand, by (73) we have
\[
\|P_+\eta(t)\|_{E_{1/3}} \leq e^{K t^{1/3}} (\|P \omega \eta_1(t)\|_{E_{1/3}} + \|P \omega \eta_2(t)\|_{E_{1/3}})
\]
for $t \geq 0$ and $K t^{1/6} \ll k_*$, where $K_1 = K_1(K_0)$ is defined in Lemma 4.2. This inequality contradicts the inequality (85). Therefore, $P_+ \eta_1(t) = P_+ \eta_2(t)$. We show the Gâteaux differentiability of $G^b_\delta$. Let $\delta, \kappa, l, l_0 > 0$ satisfying (15) and (51). Since
\[
\|v_h(0)\|_{H^1} \lesssim \epsilon + h \|v_1\|_{H^1},
\]
Theorem 3.8 yields
\[
\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|\tau_{\rho_0}(t)(v_h(t) + Q_{c_h(t)}) - \tau_{\eta} Q_{c^*}\|_{H^1} < \min \{\epsilon_* \epsilon_*^{1/2}, \delta\}
\]
(86)
for sufficiently small $\epsilon, h > 0$, where $\epsilon_* = \epsilon_* (c^*, \delta, l_0, \kappa)$ is defined in Lemma 3.10. By $|v_0, \partial_x Q_{c^*}|_{L^2} + |v_0, Q_{c^*}|_{L^2} \lesssim \epsilon_*^{1/2}$ and Lemma 3.10, we obtain
\[
m_{\delta, \kappa}(P \omega v_h(t), (P \omega v_h(t), c_h(t))) > l_0 \|P_+ (v_h(t) - v_h(t))\|_{E_{\kappa}}
\]
for $t \geq 0$. Therefore, we have
\[
\frac{\|P_+(v_h(t) - v_0(t))\|_{E_{\kappa}}}{\|P \omega (v_h(t) - v_0(t))\|_{E_{\kappa}}} \leq \frac{l_0 m_{\delta, \kappa}(P \omega v_h(t), (P \omega v_h(t), c_h(t)))}{\|P \omega v_h(t) - v_0(t)\|_{E_{\kappa}}} \lesssim l_0
\]
(87)
for \( t \geq 0 \). On the other hand, the convergence (70) yields
\[
\frac{\| (h_n^\epsilon)^{-1} P_+ (v_n^\epsilon (t) - v_0(t)) \|_{E_n}}{\| (h_n^\epsilon)^{-1} P_{\leq 0} (v_n^\epsilon (t) - v_0(t)) \|_{E_n}} \rightarrow \frac{\| P_+ \eta(t) \|_{E_n}}{\| P_{\leq 0} \eta(t) \|_{E_n}}
\]
(88)
as \( n \rightarrow \infty \) for \( t \geq 0 \). By the inequality (87) and the convergence (88), we have
\[
\frac{\| P_+ \eta(t) \|_{E_n}}{\| P_{\leq 0} \eta(t) \|_{E_n}} \lesssim \eta_0
\]
which shows that the inequality (74) fails for sufficiently large \( t \geq 0 \). Thus, Lemma 4.2 yields \( \eta \) does not satisfy (73) for \( t \geq 0 \). By Lemma 4.3, we obtain the convergence
\[
\frac{G^\delta_+ (\psi_0 + h \psi_1, c^*) - G^\delta_+ (\psi_0, c^*)}{h} \rightarrow \psi_* \text{ as } h \rightarrow 0
\]
(89)
which means that \( G^\delta_+ \) is Gâteaux differentiable at \( (\psi_0, c^*) \). The linearity of the Gâteaux derivative of \( G^\delta_+ \) follows the linearity of solutions to the equation (71) with respect to the initial data. The boundedness of the Gâteaux derivative of \( G^\delta_+ \) follows the Lipschitz property of \( G^\delta_+ \).

Next, we prove the continuity of the Gâteaux derivative of \( G^\delta_+ \). Let
\[
0 < \varepsilon \ll \min \{ \delta, \iota_0(1), K_1(1)^{-6} \}
\]
and \( \{ \psi_n \}_{n=0}^\infty \subset P_{\leq 0} H^1 (\mathbb{R} \times \mathbb{T}_L) \) with \( \psi_n \rightarrow \psi_0 \) in \( H^1 (\mathbb{R} \times \mathbb{T}_L) \) as \( n \rightarrow \infty \) and \( \sup_{n \in \mathbb{Z}_{\geq 0}} \| \psi_n \|_{H^1} < \xi(c^*, \varepsilon) \), where \( \mathbb{Z}_{\geq 0} \) is the set of non-negative integers, \( \xi \) is defined in Theorem 3.8 and \( \iota_0 \) and \( K_1 \) are defined in Lemma 4.2. By Theorem 3.8, we obtain
\[
\sup_{t \geq 0, n \in \mathbb{Z}_{\geq 0}} (\| v_n(t) \|_{H^1} + |c_n(t) - c^*|) < \varepsilon,
\]
where \( (v_n, c_n, \rho_n) \) is the solution to the system (6)–(7) with initial data \( (\psi_n, c^*, 0) \). We define \( \eta^\psi_n \) as the solution of
\[
\partial_t \eta = \partial_x \left[ (Q_{c_n} - Q_{c^*}) \eta \right] + (\dot{\rho} - c^*) \partial_x \eta - 2 \partial_x (v_n \eta)
\]
(90)
with initial data \( \psi \). By the convergence of \( \{(\tau_{-\rho_n} v_n, c_n, \rho_n)\}_{n} \) local in time, for \( T, C > 0 \) we obtain the convergence
\[
\left\| \eta^\psi_n - \eta^\psi_0 \right\|_{L^\infty(0,T), H^1} \rightarrow 0
\]
as \( n \rightarrow \infty \) uniformly on \( \{ \psi \in H^1 (\mathbb{R} \times \mathbb{T}_L) : \| \psi \|_{H^1} < C \} \). For \( T > 0 \), by the boundedness of \( \{ \| \tau_{-\rho_n} v_n \|_{X_{T}^{1,\dot{b}}} + |c_n - c^*| \| \}_{L^\infty(0,T)} \), we have the convergence
\[
\sup_{n \geq 0} \left\| \eta^\psi_n - \eta^\psi_0 \right\|_{L^\infty((0,T), H^1)} \rightarrow 0
\]
(91)
as \( \varphi \rightarrow \psi \) in \( H^1 \). Let the Gâteaux derivative of \( G^\delta_+ \) at \( (\psi_n, c^*) \) be \( \partial G^\delta_+ \). Applying Lemma 4.2 to \( \eta^\psi_0 \), by the boundedness of \( \eta^\psi_0 \), we have for there exists \( C > 1 \) such that
\[
3 \left\| P_+ \eta^\psi_0 \right\|_{E_{1/3}} \leq 3 \varepsilon^{1/3} \left\| P_{\leq 0} \eta^\psi_0 \right\|_{E_{1/3}} \leq C \varepsilon^{1/6} \varepsilon^{1/3} \| \psi \|_{E_{1/3}}
\]
(92)
for \( t > 0 \), \( \psi \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \) with \( \|\psi\|_{E_{1/3}} \leq 1 \). On the other hand, applying Lemma 4.2 to \( \eta_+^\psi \) for \( \psi_+ \in P_+H^1(\mathbb{R} \times T_L) \setminus \{0\} \), we have

\[
3 \left\| P_+\eta_0^\psi(t) \right\|_{E_{1/3}} > e^{\frac{k_1t}{3}}\|\psi\|_{E_{1/3}} + e^{\frac{k_1t}{3}}\|\psi\|_{E_{1/3}} + 3 \left\| P_+\eta_0^\psi(t) \right\|_{E_{1/3}} - 3 \left\| P_+\eta_0^\psi(t) \right\|_{E_{1/3}}
\]

for \( t > 1/2 \). Combining the inequalities (92) and (93), we obtain

\[
3 \left\| P_+\eta_0^\psi+\partial G_+^{e,0}(\psi)+\psi_+(t) \right\|_{E_{1/3}} \geq 3 \left\| P_+\eta_0^\psi(t) \right\|_{E_{1/3}} - 3 \left\| P_+\eta_0^\psi(t) \right\|_{E_{1/3}} > e^{\frac{k_1t}{3}}\|\psi\|_{E_{1/3}} + e^{\frac{k_1t}{3}}\|\psi\|_{E_{1/3}} + 3 \left\| P_+\eta_0^\psi(t) \right\|_{E_{1/3}}
\]

for \( t > 1/2 \), \( \psi_+ \in P_+H^1(\mathbb{R} \times T_L) \setminus \{0\} \) and \( \psi \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \) with \( \|\psi\|_{E_{1/3}} \leq 1 \). Since

\[
\eta_0^\psi(t) \rightarrow \eta_0^\psi(t)
\]
as \( n \rightarrow \infty \) in \( L^\infty((0,T),H^1) \) uniformly on \( \varphi \in H^1 \) with \( \|\varphi\|_{E_{1/3}} \leq 1 \) for each \( T > 0 \), there exists \( n_T > 0 \) such that for \( n \geq n_T \) and \( 1/2 < t < T \),

\[
6 \left\| P_+\eta_n^\psi+\partial G_+^{e,0}(\psi)+\psi_+(t) \right\|_{E_{1/3}} \geq e^{\frac{k_1t}{3}}\|\psi\|_{E_{1/3}} - 2e^{\frac{k_1t}{3}}+2K_1\epsilon^{1/6}C_\epsilon^{1/3}\|\psi\|_{E_{1/3}}
\]

Thus, for \( \sigma > 0 \) there exists \( T_\sigma > 0 \) such that \( \eta_n^\psi+\partial G_+^{e,0}(\psi)+\psi_+ \) satisfies (73) with \( K_0 = e^{K_1\epsilon^{1/6}T_\sigma} \) at \( T_\sigma \) for \( n \geq n_{T_\sigma} \), \( \psi \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \) and \( \psi_+ \in P_+H^1(\mathbb{R} \times T_L) \) with \( \|\psi\|_{E_{1/3}} \leq 1 \) and \( \|\psi_+\|_{E_{1/3}} \geq \sigma \). Since \( \eta_n^\psi+\partial G_+^{e,n}(\psi) \) does not satisfy that (73) for some \( t \geq 0 \), for \( n \geq n_{T_\sigma} \) and \( \psi \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \) with \( \|\psi\|_{E_{1/3}} \leq 1 \) we obtain

\[
\left\| \partial G_+^{e,n}(\psi) - \partial G_+^{e,0}(\psi) \right\|_{E_{1/3}} < \sigma
\]

and the continuity of the Gâteaux derivative of \( G_+^e \) at \( \psi_0, c^* \) as the operator from \( P_{\leq 0}H^1(\mathbb{R} \times T_L) \) to \( P_+H^1(\mathbb{R} \times T_L) \). Therefore, \( G_+^e \) is \( C^1 \) class on \( P_{\leq 0}H^1(\mathbb{R} \times T_L) \times \{c^*\} \) in the sense of Fréchet differential. By the equation (66) in Corollary 2, we obtain the \( C^1 \) regularity of the manifold containing \( M_{\text{ex}}(c^*, \epsilon) \).

The following corollary shows (iii) of Theorem 1.2.

**Corollary 3.** Let \( \partial G_+^{e}(0, c^*) \) be the Gâteaux derivative of \( G_+^e \) at \( (0, c^*) \). Then, for \( \psi \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \),

\[
P_+\partial G_+^{e}(0, c^*)(\psi) = 0.
\]

**Proof.** Since \( (0, c^*, c^t) \) is the solution to (67), the equation (71) corresponding to the solution \( (0, c^*, c^t) \) is \( \eta_t = \partial_xL_{c^*}.\eta \). Therefore, for \( \psi \in P_{\leq 0}H^1(\mathbb{R} \times T_L) \),

\[
P_+\partial G_+^{e}(0, c^*)(\psi) = P_+\psi = 0.
\]

\[\square\]
Appendix A. Appendix.

A.1. Proof of Proposition 1. In this appendix, applying the argument of the proof of Proposition 1 and Theorem 1.3 in [37], we prove Proposition 1.

Let $n_0$ be a positive integer, $c^* = 4n_0^2/5L^2$. We define the function $F$ from $H^2_{sym}(\mathbb{R} \times \mathbb{T}_L) \times (0, \infty)$ to $L^2_{sym}(\mathbb{R} \times \mathbb{T}_L) \times (0, \infty)$ by

$$F(\varphi, c) = -\Delta \varphi + c \varphi - \varphi^2,$$

where $L^2_{sym}(\mathbb{R} \times \mathbb{T}_L) = \{ u \in L^2(\mathbb{R} \times \mathbb{T}_L) : u(x, y) = u(x, -y) = u(-x, y), (x, y) \in \mathbb{R} \times \mathbb{T}_L \}$, $H^2_{sym}(\mathbb{R} \times \mathbb{T}_L) = H^2(\mathbb{R} \times \mathbb{T}_L) \cap L^2(\mathbb{R} \times \mathbb{T}_L)$ and $L^2(\mathbb{R} \times \mathbb{T}_L)$ be the set of real valued $L^2$-function on $\mathbb{R} \times \mathbb{T}_L$. By (iv) of Proposition 2, we have that $\text{Ker}(\partial_c F(Q_{c^*}, c^*))$ is spanned by $Q_3^2 \cos \frac{n_0 y}{L}$. Applying the Lyapunov–Schmidt decomposition, we obtain that there exist $\delta_1 > 0$ and $h(c, a) \in C^2((c^* - \delta_1, c^* + \delta_1) \times (-\delta_1, \delta_1))$, $H^2_{sym, L}(\mathbb{R} \times \mathbb{T}_L)$ such that

$$P_L F(Q_{c^*} + aQ_3^2 \cos \frac{n_0 y}{L} + h(c, a), c) = 0,$$

where $H^2_{sym, L}(\mathbb{R} \times \mathbb{T}_L) = \{ u \in H^2_{sym}(\mathbb{R} \times \mathbb{T}_L) : (u, Q_3^2 \cos \frac{n_0 y}{L})_{L^2} = 0 \}$ and $P_L$ is the orthogonal projection onto $\{ u \in L^2_{sym}(\mathbb{R} \times \mathbb{T}_L) : (u, Q_3^2 \cos \frac{n_0 y}{L})_{L^2} = 0 \}$. Then, the problem $F(Q_{c^*} + aQ_3^2 \cos \frac{n_0 y}{L} + h(c, a), c) = 0$ is equivalent to the problem

$$F_{||}(c, a) = F(Q_{c^*} + aQ_3^2 \cos \frac{n_0 y}{L} + h(c, a), c), Q_3^2 \cos \frac{n_0 y}{L}) L^2 = 0.$$

We consider the problem $g(c, a) = 0$, where $g$ is a $C^2$ function defined by

$$g(c, a) = \begin{cases} F_{||}(c, a) - F_{||}(c, 0), & a \neq 0, \\ \frac{\partial F_{||}}{\partial a}(c, 0), & a = 0. \end{cases}$$

Then, for $a \neq 0$, $F_{||}(c, a) = 0$ if and only if $g(c, a) = 0$. Since

$$\frac{\partial g}{\partial c}(c^*, 0) = -\frac{5}{4} \left\| Q_3^2 \cos \frac{n_0 y}{L} \right\|_{L^2}^2,$$

by the implicit function theorem there exists a $C^2$ function $\tilde{c}(a) > 0$ such that $\tilde{c}(0) = c^*$ and $g(\tilde{c}(a), a) = 0$ for $|a| < 1$. Therefore, $\varphi_{c^*}(a) = Q_{c^*} + aQ_3^2 \cos \frac{n_0 y}{L} + h(\tilde{c}(a), a)$ is a solution of $F(\varphi_{c^*}(a), \tilde{c}(a)) = 0$ and $\tilde{c}$ satisfies

$$\frac{dc}{da}(0) = -\frac{\partial g}{\partial c}(c^*, 0) = 0$$

and

$$\frac{d^2 \tilde{c}}{da^2}(0) = \lim_{a \to 0} \frac{dc}{da}(a) = -\frac{16 \left( Q_3^2 \cos^2 \frac{n_0 y}{L}, (L, c^*)^{-1} Q_3^2, \cos^2 \frac{n_0 y}{L} \right)_{L^2}}{5 \left( Q_3^2 \cos \frac{n_0 y}{L} \right)_{L^2}^2}.$$

Let $A_n = -\Delta^2 + \frac{n^2 \pi^2}{L^2} + c^* - 2Q_{c^*}$. Then, we have

$$\left( Q_3^2 \cos^2 \frac{n_0 y}{L}, (L, c^*)^{-1} Q_3^2, \cos^2 \frac{n_0 y}{L} \right)_{L^2} = \frac{1}{2} (Q_3^2, (A_0)^{-1} Q_3^2)_{L^2} + \frac{1}{4} (Q_3^2, (A_2)^{-1} Q_3^2)_{L^2}.$$

By the equalities

$$A_0^{-1} Q_3^2 = \frac{3}{4} Q_3^2 - \frac{9}{4} c^* Q_{c^*},$$

$$A_2^{-1} Q_3^2 = \frac{3}{4} Q_3^2 - \frac{9}{4} c^* Q_{c^*},$$

$$A_0^{-1} Q_3^2 = \frac{3}{4} Q_3^2 - \frac{9}{4} c^* Q_{c^*},$$

$$A_2^{-1} Q_3^2 = \frac{3}{4} Q_3^2 - \frac{9}{4} c^* Q_{c^*},$$

and

$$A_0^{-1} Q_3^2 = \frac{3}{4} Q_3^2 - \frac{9}{4} c^* Q_{c^*},$$

we get

$$\frac{d^2 \tilde{c}}{da^2}(0) = \frac{16}{5} \frac{1}{\left( Q_3^2 \cos \frac{n_0 y}{L} \right)_{L^2}^2}.$$
and
\[ \int_{\mathbb{R}} Q_{c^*}^{a+1} \, dx = \frac{3ac^*}{2a + 1} \int_{\mathbb{R}} Q_{c^*}^{a} \, dx, \quad (95) \]
we obtain
\[ (Q_{c^*}^{a}, (A_0)^{-1}Q_{c^*}^{a})_{L^2} = - \frac{5c^*}{4} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy. \quad (96) \]
Since \( \| (A_2)^{-1} \|_{L^2 \to L^2} \leq \frac{4}{15c^*} \), by the equality (96) we have
\[ \| (Q_{c^*}^{a}, (A_2)^{-1}Q_{c^*}^{a}) \|_{L^2} \leq \frac{4}{15c^*} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy = \frac{16c^*}{33} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{4} \, dxdy = \frac{144(c^*)^2}{231} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy. \quad (97) \]
By (95)–(97), we have \( \frac{d^2 c}{da^2}(0) > 0 \). Combining (96), (97) and
\[ (Q_{c^*}, \partial_c Q_{c^*})_{L^2} = \frac{3}{4c^*} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy, \]
we obtain
\[ \frac{d^2}{da^2} \| \varphi_c(a) \|^2_{L^2(a=0)} = - \frac{5}{4} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy + 2 \frac{d^2 c}{da^2}(0)(Q_{c^*}, \partial_c Q_{c^*})_{L^2} \]
\[ = \frac{55}{28} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy - \frac{1}{(c^*)^2}(Q_{c^*}, (A_2)^{-1}Q_{c^*})_{L^2} \]
\[ \geq \frac{55}{28} - \frac{144}{231} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy = \frac{1239}{924} \int_{\mathbb{R} \times \mathbb{T}_L} Q_{c^*}^{2} \, dxdy > 0. \]
Thus, we obtain the conclusion of Proposition 1 on the symmetric space \( H^2_{\text{sym}}(\mathbb{R} \times \mathbb{T}_L) \). By the translation invariant in \( \mathbb{T}_L \) for the equation \(-\Delta \varphi + c \varphi - \varphi^2 = 0\), we obtain the conclusion of Proposition 1 on \( H^2(\mathbb{R} \times \mathbb{T}_L) \).

A.2. the continuity of center stable manifolds around solitary waves. In the following corollary, we show the continuity of center stable manifolds around solitary waves around \( Q_{c^*} \) with critical speed \( c^* \).

**Corollary 4.** Let \( c^* = \frac{4n_{\text{sym}}^2}{57} \in \text{CS} \) and \( 3/2 < \alpha < 2 \). Assume that \( l_0, l_1, l_2, \delta, \kappa > 0 \) satisfy the conditions (15), (24) and (51). Then, there exists \( \delta_1 > 0 \) such that for any \( c \in (c^* - \delta_1, c^*], \) \( a \in (-\delta_1, \delta_1)^2 \) and \( 0 < \varepsilon < \varepsilon^* \) there exists \( \delta_2 = \delta_2(\varepsilon, c, a) > 0 \) satisfying that for \( u_0 \in \mathcal{M}_{cs}(c^*, \varepsilon^* \rangle \cap \mathcal{N}_{\delta_2, c} \) and \( v_0 \in \mathcal{M}_{cs}(c^*, \varepsilon^* \rangle \cap \mathcal{N}_{\delta_2, a} \) the solutions \( u(t) \) and \( v(t) \) to the equation (1) with \( u(0) = u_0 \) and \( v(0) = v_0 \) satisfy \( u(t) \in \mathcal{M}_{cs}(c^*, \varepsilon^*) \cap \mathcal{N}_{\delta, c} \) and \( v(t) \in \mathcal{M}_{cs}(c^*, \varepsilon^*) \cap \mathcal{N}_{\delta, a} \), where \( \varepsilon^* = \varepsilon^*(c^*, \delta, l_0, \kappa) \) is defined in Corollary 1 and
\[ \mathcal{N}_{\delta, a} = \{ f \in H^1(\mathbb{R} \times \mathbb{T}_L); \inf_{(x,y) \in \mathbb{R} \times \mathbb{T}_L} \| f(\cdot, \cdot) - \varphi_c(a)(\cdot - x, \cdot - y) \|_{H^1} \leq \varepsilon \}. \]

**Proof.** There exists \( \delta_0 > 0 \) such that for \( c \in (c^* - \delta_0, c^* + \delta_0) \) and \( a \in (-\delta_0, \delta_0)^2 \) we have \( \| Q_c - Q_{c^*} \|_{H^1} < \varepsilon^* \) and \( \| \varphi_c(a) - Q_{c^*} \|_{H^1} < \varepsilon^* \). By Corollary 1, we obtain \( \tau_q Q_c, \tau_q \varphi_c(a) \in \mathcal{M}_{cs}^0(c^*, \varepsilon^*) \) for \( q \in \mathbb{R}, c \in (c^* - \delta_0, c^* + \delta_0) \) and \( a \in (-\delta_0, \delta_0)^2 \).

Firstly, we show the stability of \( Q_{c}(x - ct) \) on \( \mathcal{M}_{cs}^0(c^*, \varepsilon^*) \). Let \( 0 < \varepsilon < \frac{1}{5}(\varepsilon^* - \| Q_c - Q_{c^*} \|_{H^1}) \). By Theorem 3.8, there exists \( \delta^* > 0 \) such that for \( u_0 \in \mathcal{M}_{cs}^0(c, \delta^*) \) the solution \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies
\[ u(t) \in \mathcal{M}_{cs}^0(c, \varepsilon^*) \subset \mathcal{N}_{\delta, c} \]
for $t \geq 0$. Therefore, we obtain $u(t) \in N_{c,c^*}$ for $t \geq 0$. By Corollary 1, we have $u(t) \in \mathcal{M}_{cs}^\delta(c^*,c^*)$ for $t \geq 0$. Thus, $\mathcal{M}_{cs}^\delta(c^*,c^*) \subset \mathcal{M}_{cs}^\delta(c^*,c^*) \cap N_{c,c}$. Since $\text{codim}(\mathcal{M}_{cs}^\delta(c^*,c^*)) = \text{codim}(\mathcal{M}_{cs}^\delta(c^*,c^*))$, there exists $0 < \delta_2 < \delta'$ such that

$$\mathcal{M}_{cs}^\delta(c^*,c^*) \cap N_{c,c} = \mathcal{M}_{cs}^\delta(c^*,c^*) \cap N_{c,c}.$$

Therefore, for any $u_0 \in \mathcal{M}_{cs}^\delta(c^*,c^*) \cap N_{c,c}$, the solution $u(t)$ to the equation (1) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{M}_{cs}^\delta(c^*,c^*) \cap N_{c,c}.$

Lastly, we show the stability of $\varphi(a)$ on $\mathcal{M}_{cs}^\delta(c^*,c^*)$. We define $\Phi(\theta) : T_1 \to \mathbb{S}^1$ by $\Phi(\theta) = (\cos \theta, -\sin \theta)$ for $\theta \in T_1$. Then, $\varphi(c^*)(a) = \varphi(c^*)(|a|,0)$, and $\varphi(c^*)$ is $C^2$-function with respect to $a$. We define $P(a), P_+(a)$ and $P_-(a)$ by

$$P_+(a) = \frac{1}{2\pi i} \int_\Gamma (z - \partial_x \mathbb{L}(a))^{-1} dz, \quad P_-(a) = \frac{1}{2\pi i} \int_\Gamma (z + \partial_x \mathbb{L}(a))^{-1} dz.$$

and $P_0(a) = \mu_1 \partial_x \varphi(c^*)(a,0) + \mu_2 \partial_y \varphi(c^*)(a,0) + \mu_3 \partial_y \varphi(c^*)(a,0) \in H^1(\mathbb{R} \times T_L)$, where

$$\mu_1 = \frac{(u, \partial_x \varphi(c^*)(a,0))_{L^2}}{\|\partial_x \varphi(c^*)(a,0)\|^2_{L^2}}, \quad \mu_2 = \frac{(u, \varphi(c^*)(a,0))_{L^2}}{\|\partial_y \varphi(c^*)(a,0)\|^2_{L^2}}, \quad \mu_3 = \frac{(u, \partial_y \varphi(c^*)(a,0))_{L^2}}{\|\partial_y \varphi(c^*)(a,0)\|^2_{L^2}}.$$

Then, we have $P_{\pm}(0) = P_\pm$, $P_{-}(a)$ and $P_{+}(a)$ are continuous with respect to $a$, $\partial_x \mathbb{L}(a) P_\pm(a) = P_\pm(a) \partial_x \mathbb{L}(a)$ and

$$\dim(P_\pm(a)H^1(\mathbb{R} \times T_L)) = \dim(P_\pm H^1(\mathbb{R} \times T_L)) = 2n_0 - 2$$

for $|a| \ll 1$. The equality $\mathbb{L}(a)P_{+}(a) = -(P_{-}(a))^*\mathbb{L}(a)$ implies that

$$\mathbb{L}(a)P_+(a)u, P_+(a)u \in \mathbb{L}^2 \quad \text{for} \quad u \in \mathbb{L}^2 \times T_L.$$
By Proposition 1 and (99), for any \(b_1, b_2, b_3, b_4 \in \mathbb{R}\), \(u_+ \in P_+(a)H^1(\mathbb{R} \times \mathbb{T}_L)\) and \(|a| \ll 1\),

\[
(\mathbb{L}(a)(b_1 \partial_x \varphi_c - (a, 0) + b_2 \partial_\mu \varphi_c - (a, 0) + b_3 \partial_\gamma \varphi_c - (a, 0) + u_+ + b_4 u_0),
\]

\[
b_1 \partial_x \varphi_c - (a, 0) + b_2 \partial_\mu \varphi_c - (a, 0) + b_3 \partial_\gamma \varphi_c - (a, 0) + u_+ + b_4 u_0)_{H^{-1}, H^1}.
\]

Therefore, the dimension of the eigenspace of \(L(a)\) corresponding to non-positive eigenvalues is larger than \(2n_0 + 2\). This contradicts that the dimension of the eigenspace of \(L(a)\) corresponding to non-positive eigenvalues is \(2n_0 + 1\). Thus, for \(u \in P_+(a)H^1(\mathbb{R} \times \mathbb{T}_L), \langle L(a)u, u \rangle_{H^{-1}, H^1} > 0\). By Weyl’s theorem on essential spectrum, the essential spectrum of \((P_+(a))^*L(a)P_+(a)\) is \([c(a), \infty)\). Therefore, there exists \(C(a) > 0\) such that

\[
\langle L(a)P_+(a)u, P_+(a)u \rangle_{H^{-1}, H^1} \geq C(a)\|P_+(a)u\|_{H^1}^2. \tag{100}
\]

We define the norm

\[
\|u\|_{E_{\kappa, a}} = \left(\|P_+(a)u\|_{L^2}^2 + \|P_+(a)u\|_{L^2}^2 + \kappa^2 \mu_1^2 + \mu_2^2 + \kappa^2 \mu_3^2 \right)^{\frac{1}{2}} + \langle L(a)P_+(a)u, P_+(a)u \rangle_{H^{-1}, H^1}^{\frac{1}{2}}
\]

for \(\kappa > 0, |a| \ll 1\) and \(u \in H^1(\mathbb{R} \times \mathbb{T}_L)\), where \(\mu_1, \mu_2, \mu_3\) are defined by (98). Then, by the inequality (100), we have \(\|u\|_{E_{\kappa, a}} \simeq \|u\|_{H^1}\). Let \(C_2\) be a large real constant, \(0 < a^* \ll 1\) and \(\chi, \phi \in C^\infty(\mathbb{R})\) with \(0 \leq \chi \leq 1\) and

\[
\chi(r) = \begin{cases} 
1, & |r| \leq 1 \\
0, & |r| \geq 2 
\end{cases}, \quad \phi(r) = \begin{cases} 
1, & r \leq C_2 \\
r, & r \geq 2C_2 
\end{cases}.
\]

We define \(\chi_{\delta'} = \chi_{\delta'}(v, a) = \chi((\delta')^{-2}(|v|^2_{H^1} + |a - a_0|^2))\) and

\[
\phi_{\delta, a^*}(v) = \phi((\delta')^{-1}\|P_+(a^*)v\|_{E_{\kappa, a^*}})
\]

for \(\delta' > 0\). We consider the following system:

\[
v_t = \partial_x L(a^*)v + (\dot{\rho}_1 - \dot{c}(a^*, 0))\partial_x \varphi_c - (a^*, 0) + \dot{\rho}_2 \partial_\gamma \varphi_c - (a^*, 0) - \dot{\alpha} \partial_\gamma \varphi(a^*, 0) + \chi_{\delta'} N_{a^*},
\]

\[
\frac{d}{dt}(v, \partial_x \varphi_c - (a^*, 0))_{L^2} = \frac{d}{dt}(v, \partial_\gamma \varphi_c - (a^*, 0))_{L^2} = \frac{d}{dt}(v, \partial_\gamma \varphi_c - (a^*, 0))_{L^2} = 0, \tag{102}
\]

where

\[
N_{a^*} = N_{a^*}(v, a, \mu_1, \rho_2)
\]

\[
= (\dot{\rho}_1 - \dot{c}(a^*, 0))\partial_x v + (\dot{\rho}_1 - \dot{c}(a, 0))\partial_x (\varphi_c - (a, 0) - \varphi_c - (a^*, 0)) + \dot{\rho}_2 \partial_\gamma v
\]

\[
+ \dot{\rho}_2 \partial_\gamma (\varphi_c - (a, 0) - \varphi_c - (a^*, 0)) - \dot{\alpha} \partial_\gamma (\varphi_c - (a, 0) - \varphi_c - (a^*, 0))
\]

\[
+ 2\partial_x ((\varphi_c - (a^*, 0) - \varphi_c - (a, 0))v - \partial_x(v^2)).
\]

Then, by applying the argument to prove Theorem 3.4 in [39], we can prove that for small \(\delta' > 0\) the Cauchy problem of the system (101)–(102) is globally well-posed.
for initial data in \( H^1(\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R}^3 \). We define the mobile distance
\[
m_{\delta,\gamma,\alpha}(v_0, v_1) = \left( \|P_d(a^*)(v_0 - v_1)\|_E^2 + |a_0 - a_1|^2 \right)^{\frac{1}{2}} + \inf_{j=0,1} \|P_\gamma(a^*) v_j - \tau_{q_j,q_{j+1}} P_\gamma(a^*) v_{j-1} \|_E^2 + \delta' (q_1^2 + q_2^2) \phi_{\delta,\alpha}(v_{j-1})
\]
for \( v_0 = (v_0(a_0), v_1 = (v_1(a_1), a_1) \in H^1(\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R}^3 \), where \( \tau_{q_j,q_{j+1}}(u)(x,y) = u(x-q_j,y-q_{j+1}) \). Applying the argument in Section 3, we can obtain that for \( l > 0 \) there exist \( \delta'_1, \kappa > 0 \) and a Lipschitz function \( G_{\delta',\alpha} : H^1(\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R} \to P_+ (a^*) H^1(\mathbb{R} \times \mathbb{T}_L) \) such that \( G_{\delta',\alpha}(0,0) = 0 \), for \( 0 < \delta' < \delta'_1 \) and \( (v_0(a_0), v_1(a_1), a_1) \in H^1(\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R} \),
\[
\left\| G_{\delta',\alpha}(v_0(a_0), a_0) - G_{\delta',\alpha}(v_1(a_1), a_1) \right\|_E \leq \lim_{\delta' \to 0} \left\| v_0(a_0), v_1(a_1) \right\|
\]
and for \( (w, a(0), \rho_1(0), \rho_2(0)) \in (I \overline{\rho}_1, (a^*) \overline{H}^1(\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R}^3 \), the solution \( (v, a, \rho_1, \rho_2) \) to the system \((101)-(102)\) with initial data \( (w + G_{\delta',\alpha}^s(w, a(0)), a(0), \rho_1(0), \rho_2(0)) \) satisfies \( v(t) = (I \overline{\rho}_1, (a^*) \overline{H}^1(\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R}^3) \), for \( t = \tau_{\rho_1,\rho_2}(v + \varphi_{c^*}(a, 0)) \) in \( H^1(\mathbb{R} \times \mathbb{T}_L) \), \( S_{\delta,\alpha,0}(u) = S_{\delta,\alpha,0}(\varphi_{c^*}(a^*, 0)) \)
\[
= S_{\delta,\alpha,0}(\varphi_{c^*}(a, 0)) - S_{\delta,\alpha,0}(\varphi_{c^*}(a^*, 0))
\]
\[
\geq \frac{\partial \varepsilon}{\partial a_0}(a^*, 0)\frac{\partial \varphi_{c^*}(a, 0)}{\partial a_0} \left( \| \varphi_{c^*}(a_0, 0) \|_L^2 - \| \varphi_{c^*}(a, 0) \|_L^2 \right)
\]
\[
+ \left( \frac{\partial \varepsilon}{\partial a_0}(a^*, 0) \right)^2 \left( \| \varphi_{c^*}(a, 0) \|_L^2 - \| \varphi_{c^*}(a^*, 0) \|_L^2 \right)
\]
\[
\geq \frac{\partial \varepsilon}{\partial a_0}(a^*, 0)\frac{\partial \varphi_{c^*}(a, 0)}{\partial a_0} \left( \| \varphi_{c^*}(a_0, 0) \|_L^2 - \| \varphi_{c^*}(a, 0) \|_L^2 \right)
\]
\[
+ \left( \frac{\partial \varepsilon}{\partial a_0}(a^*, 0) \right)^2 \left( \| \varphi_{c^*}(a, 0) \|_L^2 - \| \varphi_{c^*}(a^*, 0) \|_L^2 \right)
\]
\[
\geq \frac{\partial \varepsilon}{\partial a_0}(a^*, 0)\frac{\partial \varphi_{c^*}(a, 0)}{\partial a_0} \left( \| \varphi_{c^*}(a_0, 0) \|_L^2 - \| \varphi_{c^*}(a, 0) \|_L^2 \right)
\]
\[
+ \left( \frac{\partial \varepsilon}{\partial a_0}(a^*, 0) \right)^2 \left( \| \varphi_{c^*}(a, 0) \|_L^2 - \| \varphi_{c^*}(a^*, 0) \|_L^2 \right) + (a - a^*)^2.
\]
Therefore, the solution \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies \( u(t) \) in \( \mathcal{M}_{\delta,\alpha}(a^*) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) the solution \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies \( u(t) \) in \( \mathcal{M}_{\delta,\alpha}(a^*) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \)
\[
\geq \frac{\partial \varepsilon}{\partial a_0}(a^*, 0)\frac{\partial \varphi_{c^*}(a, 0)}{\partial a_0} \left( \| \varphi_{c^*}(a_0, 0) \|_L^2 - \| \varphi_{c^*}(a, 0) \|_L^2 \right)
\]
\[
+ \left( \frac{\partial \varepsilon}{\partial a_0}(a^*, 0) \right)^2 \left( \| \varphi_{c^*}(a, 0) \|_L^2 - \| \varphi_{c^*}(a^*, 0) \|_L^2 \right) + (a - a^*)^2.
\]
and \( \mathcal{M}_{\delta,\alpha}(a^*) \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \subset \mathcal{N}_{\delta,\alpha}(a^*) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \). Therefore, \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies \( u(t) \) in \( \mathcal{M}_{\delta,\alpha}(a^*) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \)
\[
is C^1 \) manifolds satisfying that \( \text{codim}(\mathcal{M}_{\delta,\alpha}(a^*)) = 2n_0 - 2 \) and for \( 0 < \varepsilon < \frac{1}{2}(\varepsilon^* - \| \varphi_{c^*}(a^*, 0) - Q_{\delta,\alpha} \|_H^1) \) there exists \( 0 < \delta' \ll \delta'' \) such that for \( u_0 = \mathcal{M}_{\delta,\alpha}(a^*) \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \text{solution} \) \( u(t) \) to the equation (1) with \( u(0) = u_0 \) satisfies \( u(t) \) \( \in \mathcal{M}_{\delta,\alpha}(a^*) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \subset \mathcal{N}_{\delta,\alpha}(a^*) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \). Therefore, for all \( u_0 = \mathcal{M}_{\delta,\alpha}(a^*) \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \) \( \cap \mathcal{N}_{\delta,\alpha}(a, 0) \).

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