Unitary modules for the twisted Heisenberg-Virasoro algebra

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Abstract

In this paper, the conjugate-linear anti-involutions and the unitary irreducible modules of the intermediate series over the twisted Heisenberg-Virasoro algebra are classified respectively. We prove that any unitary irreducible module of the intermediate series over the twisted Heisenberg-Virasoro algebra is of the form \(A_{a,b,c}\) for \(a \in \mathbb{R}, b \in \frac{1}{2} + \sqrt{-1}\mathbb{R}, c \in \mathbb{C}\).

2000 Mathematics Subject Classification: 17B10, 17B40, 17B68

Keywords: twisted Heisenberg-Virasoro algebra; conjugate-linear anti-involution; unitary module.

1 Introduction

The twisted Heisenberg-Virasoro algebra \(\mathcal{L}\) is defined to be a Lie algebra with \(\mathbb{C}\)-basis \(\{L_m, I_m, C_L, C_I, C_{LI} | m \in \mathbb{Z}\}\) subject to the following Lie brackets:

\[
\begin{align*}
[L_m, L_n] &= (n - m)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C_L, \\
[I_m, I_n] &= n\delta_{n+m,0} C_L, \\
[L_m, I_n] &= nI_{m+n} + \delta_{m+n,0} (m^2 - m) C_{LI}, \\
[\mathcal{L}, C_L] &= [\mathcal{L}, C_I] = [\mathcal{L}, C_{LI}] = 0.
\end{align*}
\]

This Lie algebra was first introduced by Arbarello et al. in Ref. [1]. It is the universal central extension of the Lie algebra of differential operators on a circle of order at most one:

\[
L_{HV} = \{f(t)\frac{d}{dt} + g(t) | f, g \in \mathbb{C}[t, t^{-1}]\}.
\]

Supported by the National Natural Science Foundation of China (No. 10931006).

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By the definition, both the Heisenberg algebra and the Virasoro algebra are subalgebras of the twisted Heisenberg-Virasoro algebra \( \mathfrak{L} \). It is known that the twisted Heisenberg-Virasoro algebra has some important applications in the representation theory of the toroidal Lie algebra which is the prime example of the generalizations of Kac-Moody algebra (see [6]). Moreover, \( \mathfrak{L} \) has some relations with the well known \( N = 2 \) Neveu-Schwarz superalgebra. In fact, the even part of the \( N = 2 \) Neveu-Schwarz superalgebra is essentially the twisted Heisenberg-Virasoro algebra (see [11]).

The representation theories of the Virasoro algebra and its related algebras play crucial roles in many areas of Mathematics and Physics and have been well developed. In particular, the unitary representations are significant. The unitary highest weight representations of Virasoro algebras are determined in [4,5,8-10]. The complete classification of the unitary Harish-Chandra modules over Virasoro algebra is shown in [3]. The irreducible Harish-Chandra modules of Virasoro algebras are classified in [15]. Recently, the representation theories over the twisted Heisenberg-Virasoro algebra were studied by several authors. When the central element of the Heisenberg subalgebra acts trivially, the highest weight modules for twisted Heisenberg-Virasoro algebra are studied in [1] and [2]. It is proved in [1] that the unitary highest weight modules for twisted Heisenberg-Virasoro algebra is just the unitary highest weight modules for Virasoro algebra when the central charge of the Heisenberg subalgebra is zero (Theorem 6.6 (I) in [1]). The irreducible Harish-Chandra modules, the module of the intermediate series, the irreducible weight modules with a finite dimensional weight space and the Whittaker modules over the twisted Heisenberg-Virasoro algebra are are also studied (see [1,2,7,12-14,16]).

The goal of the present paper is to study the conjugate-linear anti-involutions and the unitary irreducible modules of the intermediate series over the twisted Heisenberg-Virasoro algebra.

The paper is organized as follows. In Section 2, we study the conjugate-linear anti-involution of the twisted Heisenberg-Virasoro algebra \( \mathfrak{L} \). In Section 3, the unitary irreducible Harish-Chandra modules of the intermediate series are classified.

Throughout this paper we make a convention that the weight modules over twisted Heisenberg-Virasoro algebra and Virasoro algebra are all with finite dimensional weight spaces, i.e., the Harish-Chandra modules. The symbols \( \mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z} \) and \( S^1 \) represent for the complex field, real number field, the set of positive real number, the set of integers and the set of complex number of modulus one respectively.

2 Conjugate-linear anti-involutions of \( \mathfrak{L} \)

It is easy to see the following facts about \( \mathfrak{L} \):

1. \( \mathfrak{C} := \mathbb{C}C_L \oplus \mathbb{C}C_I \oplus \mathbb{C}C_{LI} \oplus \mathbb{C}I_0 \) is the center of \( \mathfrak{L} \).
2. If \( x \in \mathfrak{C} \) acts semisimply on \( \mathfrak{L} \) by the adjoint action, then

\[
x \in \mathfrak{h} := \text{span}_\mathbb{C}\{L_0, I_0, C_L, C_I, C_{LI}\},
\]
the Cartan subalgebra of \( \mathfrak{L} \).

(3) \( \mathfrak{L} \) has a weight space decomposition according to the Cartan subalgebra \( \mathfrak{h} \):

\[
\mathfrak{L} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{L}_n,
\]

where \( \mathfrak{L}_n = \text{span}_\mathbb{C} \{ L_n, I_n \} \) if \( n \neq 0 \) and \( \mathfrak{L}_0 = \mathfrak{h} \).

(4) The Heisenberg algebra \( H = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} I_n \oplus \mathbb{C} C_I \) and the Virasoro algebra \( \text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} C_L \) are subalgebras of \( \mathfrak{L} \).

**Lemma 2.1.** \( H \oplus \mathbb{C} C_L \oplus \mathbb{C} C_I \) is the unique maximal ideal of \( \mathfrak{L} \).

**Proof.** The proof is similar as that for Lemma 2.2 in [17], we omit the details. □

**Definition 2.2.** Let \( \mathfrak{g} \) be a Lie algebra and \( \theta \) be a conjugate-linear anti-involution of \( \mathfrak{g} \), i.e. \( \theta \) is a map \( \mathfrak{g} \rightarrow \mathfrak{g} \) such that

\[
\theta(x + y) = \theta(x) + \theta(y), \quad \theta(\alpha x) = \bar{\alpha} \theta(x),
\]

\[
\theta([x, y]) = [\theta(y), \theta(x)], \quad \theta^2 = \text{id}
\]

for all \( x, y \in \mathfrak{g}, \alpha \in \mathbb{C} \), where \( \text{id} \) is the identity map of \( \mathfrak{g} \). A module \( V \) of \( \mathfrak{g} \) is called unitary if there is a positive definite Hermitian form \( \langle \cdot, \cdot \rangle \) on \( V \) such that

\[
\langle xu, v \rangle = \langle u, \theta(x)v \rangle
\]

for all \( u, v \in V, x \in \mathfrak{g} \).

**Lemma 2.3.** (Proposition 3.2 in Ref. [3]) Any conjugate-linear anti-involution of \( \text{Vir} \) is one of the following types:

(i) \( \theta_\alpha^+(L_n) = \alpha^n L_{-n}, \quad \theta_\alpha^+(C_L) = C_L \), for some \( \alpha \in \mathbb{R}^* \), the set of nonzero real number.

(ii) \( \theta_\alpha^-(L_n) = -\alpha^n L_n, \quad \theta_\alpha^-(C_L) = -C_L \), for some \( \alpha \in S^1 \), the set of complex number of modulus one.

**Lemma 2.4.** Let \( \theta \) be an arbitrary conjugate-linear anti-involution of \( \mathfrak{L} \). Then

(i) \( \theta(\mathfrak{h}) = \mathfrak{h} \).

(ii)

\[
\theta(L_0) = \lambda L_0 + \lambda' C, \\
\theta(I_m) = \alpha_{m,-\lambda m} I_{-\lambda m} (m \neq 0), \\
\theta(C_I) = \alpha_{1,-\lambda} \alpha_{-1,\lambda} C_I,
\]

where \( \lambda \in \{ 1, -1 \}, \lambda' \in \mathbb{C}, \ C \in \mathbb{C}, \alpha_{m,m} \in S^1, \overline{\alpha_{m,-m}} \alpha_{-m,m} = 1 \).

(iii) \( \theta(H \oplus \mathbb{C} C_{LI}) = H \oplus \mathbb{C} C_{LI} \).

**Proof.** For (i), we see that

\[
\theta(\mathfrak{c}) = \mathfrak{c} \subset \mathfrak{h}.
\]
On the other hand, identities \([\theta(L_0), \theta(L_n)] = -n\theta(L_n)\) and \([\theta(I_0), \theta(I_n)] = -n\theta(I_n)\) imply that \(\theta(L_0)\) acts semisimply on \(L\). Thus \(\theta(L_0) \in \mathfrak{h}\).

For (ii), by (i) and \(\theta^2(L_0) = L_0\), we can assume
\[
\theta(L_0) = \lambda L_0 + \lambda' C
\]
for some \(\lambda \in S^1, \lambda' \in \mathbb{C}\) and \(C \in \mathfrak{c}\). On the other hand, we see that \(\theta(H \oplus \mathbb{C}C_{L_1})\) is an ideal of \(L\) since \(H \oplus \mathbb{C}C_{L_1}\) is an ideal. Thus we have \(\theta(H \oplus \mathbb{C}C_{L_1}) \subset H \oplus \mathbb{C}C_{L_1} \oplus \mathbb{C}C_L\) by Lemma 2.1, and we can assume
\[
\theta(I_m) = \sum_n \alpha_{m,n} I_n + \beta_m C_I + \gamma_m C_{LI} + \zeta_m C_L \text{ for } m \neq 0,
\]
where \(\alpha_{m,n}, \beta_m, \gamma_m, \zeta_m \in \mathbb{C}, x \in H\). By (2.1), (2.2) and \([\theta(I_m), \theta(L_0)] = m\theta(I_m)\), we deduce that
\[
\lambda = \pm 1,
\]
\[
\theta(I_m) = \begin{cases} 
\alpha_{m,m} I_m & \text{if } \lambda = -1 \\
\alpha_{m,-m} I_{-m} & \text{if } \lambda = 1 
\end{cases} = \alpha_{m,-m} I_{-m} \lambda m \text{ for } m \neq 0,
\]
where \(\alpha_{m,m} \in S^1, \overline{\alpha_{m,-m}} \alpha_{-m,m} = 1\) since \(\theta^2(I_m) = I_m\). Further more, we have
\[
\theta(C_I) = [\theta(I_1), \theta(I_{-1})] = \alpha_{1,-1} \alpha_{-1,1} \lambda \lambda C_I.
\]

For (iii), we see that
\[
\theta(I_0) = [\theta(L_1), \theta(L_{-1})] = [\theta(L_1), \alpha_{-1,1} I_\lambda] \in H \oplus \mathbb{C}C_{L_1}.
\]
\[
\theta(C_{LI}) = \frac{1}{2} ([\theta(I_1), \theta(L_{-1})] - \theta(I_0)) \in H \oplus \mathbb{C}C_{LI}.
\]
From (2.4)-(2.7), we have \(\theta(H \oplus \mathbb{C}C_{LI}) = H \oplus \mathbb{C}C_{LI}\).

**Proposition 2.5.** Any conjugate-linear anti-involution of \(L\) is one of the following types:

(i) : \(\theta^+_{\alpha,\gamma}(L_n) = \alpha^n L_{-n}\),
\[
\theta^+_{\alpha,\gamma}(C_L) = C_L,
\]
\[
\theta^+_{\alpha,\gamma}(I_m) = \alpha^m e^{ir} I_{-m} (m \neq 0),
\]
\[
\theta^+_{\alpha,\gamma}(I_0) = e^{ir} I_0 + 2e^{ir} C_{LI},
\]
\[
\theta^+_{\alpha,\gamma}(C_I) = e^{2ir} C_I,
\]
\[
\theta^+_{\alpha,\gamma}(C_{LI}) = -e^{ir} C_{LI}.
\]
where $\alpha = \mathbb{R}^*$, $\gamma \in \mathbb{R}$, $i = \sqrt{-1}$.

(ii): $\theta_{\alpha, 1, \beta_{-1}}^{-}(L_n) = -\alpha^n L_n + \left( \frac{n + 1}{2} \alpha^{n-1} \beta_1 - \frac{n - 1}{2} \alpha^{n+1} \beta_{-1} \right) I_n$

$\theta_{\alpha, 1, \beta_{-1}}^{-}(C_L) = -C_L - 12(\alpha^{-1} \beta_1 - \alpha \beta_{-1}) C_{LI} + (14 \alpha^{-1} \beta_1 - 3 \alpha^{-2} \beta_2 - 3 \alpha^2 \beta_{-1}^2) C_I$

$\theta_{\alpha, 1, \beta_{-1}}^{-}(I_0) = \alpha_1 \alpha^{-1} I_0 - \alpha_1 \alpha^{-2} \beta_1 C_I$

$\theta_{\alpha, 1, \beta_{-1}}^{-}(C_I) = -a_1^2 \alpha^{-2} C_I$

$\theta_{\alpha, 1, \beta_{-1}}^{-}(C_{LI}) = \mu \alpha^{-1} C_{LI} + \frac{1}{2}(\alpha_1 \alpha^{-2} \beta_1 - \alpha_1 \beta_{-1}) C_I$

where $\alpha, \alpha_n (n \neq 0) \in S_1, \beta_1, \beta_{-1} \in \mathbb{C}$, satisfying $\alpha_n = \alpha_1 \alpha^{n-1}, \alpha_1 \beta_1 = \alpha \beta_{-1}, \alpha \beta_{-1} = \alpha_1 \beta_{-1}, (\alpha_1 \alpha^{-1}) = 0$ and $\beta_{1, -1, 1} \alpha(1 + \alpha_1 \alpha^{-2}) = 0$.

**Proof.** Let $\theta$ be any conjugate-linear anti-involution of $\mathfrak{L}$. By Lemma 2.4 (iii), we have the induced conjugate-linear anti-involution of $\mathfrak{L}/(H \oplus \mathbb{C} C_L) \cong V i r :$

$$\tilde{\theta} : \mathfrak{L}/(H \oplus \mathbb{C} C_L) \rightarrow \mathfrak{L}/(H \oplus \mathbb{C} C_L).$$

Thus, by Lemma 2.3, we see that $\tilde{\theta}$ is one of the following types:

(a) $\tilde{\theta}^+_{\alpha}(L_n) = \alpha^n L_{-n}, \tilde{\theta}^+_{\alpha}(C_L) = C_L$, for some $\alpha \in \mathbb{R}^*$.

(b) $\tilde{\theta}^-_{\alpha}(L_n) = -\alpha^n L_n, \tilde{\theta}^-_{\alpha}(C_L) = -C_L$, for some $\alpha \in S_1$.

If $\theta$ is of type (a), then we can assume

$$\theta(L_n) = \alpha^n L_{-n} + \sum_i \beta_{n,i} I_i + \gamma_n C_I + \zeta_n C_{LI}, \quad (2.8)$$

where $\alpha \in \mathbb{R}^*, \beta_n, \gamma_n, \zeta_n \in \mathbb{C}$. By (2.8) and Lemma 2.4 (ii), we have

$$\theta(L_0) = L_0 + \beta_{0,0} I_0 + \gamma_0 C_I + \zeta_0 C_{LI}. \quad (2.9)$$

By (2.8)-(2.9) and $[\theta(L_n), \theta(L_0)] = n \theta(L_n)$, we can deduce easily that $\beta_{n,i} = 0$ unless $i = -n, \gamma_n = \zeta_n = 0$ for all $n \neq 0$. Thus

$$\theta(L_n) = \alpha^n L_{-n} + \beta_{n,-n} I_{-n} \quad \text{for } n \neq 0. \quad (2.10)$$

By $[\theta(L_{-1}), \theta(L_1)] = -2 \theta(L_0)$, we deduce that

$$\gamma_0 = \frac{1}{2} \beta_{-1,1} \beta_{1,-1}, \quad (2.11)$$

$$\beta_{0,0} = \frac{1}{2} (\beta_{1,-1} \alpha^{-1} + \beta_{-1,1} \alpha), \quad (2.12)$$

and

$$\zeta_0 = \beta_{-1,1} \alpha. \quad (2.13)$$
By (2.10) and $[\theta(L_n), \theta(L_m)] = (n - m)\theta(L_{m+n})$ ($m + n \neq 0$), we can get
\[
(n - m)\beta_{m+n,-(m+n)} = n\beta_{n,-n}\alpha^m - m\alpha^n\beta_{m,-m}(m + n \neq 0). \tag{2.14}
\]

By (2.10) and $[\theta(L_n), \theta(L_{-n})] = 2n\theta(L_0) + \frac{n-n^3}{12}\theta(C_L)$, we have
\[
\frac{n-n^3}{12}\theta(C_L) = \frac{n-n^3}{12}C_L + [(n^2 + n)\alpha^n\beta_{n,n} - (n^2 - n)\alpha^{-n}\beta_{n,-n} - 2n\zeta_0]C_{LI} - (n\beta_{n,-n}\beta_{-n,n} + 2n\gamma_0)C_I + (n\alpha^n\beta_{n,n} + n\alpha^{-n}\beta_{n,-n} - 2n\beta_{0,0})I_0. \tag{2.15}
\]

From (2.14) and (2.12), we can prove by induction that
\[
\beta_{m,-m} = \frac{m+1}{2}\alpha^{m-1}\beta_{1,-1} - \frac{m-1}{2}\alpha^{m+1}\beta_{-1,1}. \tag{2.16}
\]

Setting $n = 2$ in (2.15) and using (2.11), (2.13) and (2.16), we get that
\[
\theta(C_L) = C_L + 12(\alpha^{-1}\beta_{1,-1} - \alpha\beta_{-1,1})C_{LI} - (14\beta_{-1,1}\beta_{1,-1} - 3\alpha^{-2}\beta_{1,-1} - 3\alpha^2\beta_{-1,1})C_I. \tag{2.17}
\]

By Lemma 2.4 (ii) and (2.9), we see that $\lambda = 1$ and
\[
\theta(I_m) = \alpha_{m,-m}I_{-m} \text{ for } m \neq 0, \tag{2.18}
\]
\[
\theta(C_I) = \alpha_{-1,1}C_{-1,1}I_I. \tag{2.19}
\]

By (2.18) and identity $\theta^2(I_m) = I_m$, we have
\[
\alpha_{m,-m}\alpha_{-m,m} = 1. \tag{2.20}
\]

Setting $n = \pm 1$ in $[\theta(L_n), \theta(L_n)] = -n\theta(I_0) + (n^2 - n)\theta(C_{LI})$ respectively and combing with (2.10) and (2.18)-(2.19), we get that
\[
\theta(I_0) = \alpha_{-1,1}\alpha I_0 + 2\alpha_{-1,1}\alpha C_{LI} + \alpha_{-1,1}\beta_{1,-1}C_I, \tag{2.21}
\]

and
\[
\theta(C_{LI}) = -\alpha_{-1,1}\alpha C_{LI} + \frac{1}{2}(\alpha_{1,-1}\alpha^{-1} - \alpha_{-1,1}\alpha)I_0 + \frac{1}{2}(\alpha_{1,-1}\beta_{-1,1} - \alpha_{-1,1}\beta_{1,-1})C_I. \tag{2.22}
\]

By $\theta^2(I_0) = I_0$, we obtain that
\[
\alpha \in \mathbb{R}^*, \alpha\beta_{-1,1} + \beta_{1,-1}\alpha_{-1,1} = 0. \tag{2.23}
\]

By $[\theta(I_n), \theta(L_{-n})] = n\theta(I_0) + (n^2 + n)\theta(C_{LI})$, (2.16) and (2.21)-(2.23), we can deduce that
\[
\alpha_{n,-n} = \alpha_{-1,1}\alpha^{n+1}. \tag{2.24}
\]

Setting $m = 1$ in (2.20) and combing with (2.24), we see that
\[
\alpha_{1,-1}\alpha_{1,-1} = \alpha^2.
\]
Thus
\[ \alpha_{1,-1} = \alpha e^{i\gamma}, \quad \alpha_{-1,1} = \alpha^{-1} e^{i\gamma} \text{ for some } \gamma \in \mathbb{R}. \] (2.25)
From (2.23) and (2.25), we get that
\[ \overline{\beta}_{1,-1} = -\alpha^2 \beta_{-1,1} e^{-i\gamma}, \quad \overline{\beta}_{-1,1} = -\alpha^{-2} \beta_{1,-1} e^{-i\gamma}. \] (2.26)

By (2.17), (2.19), (2.22), (2.25)-(2.26) and identity \( \theta^2(C_L) = C_L \), we can deduce that
\[ \beta_{1,-1} = 0 = \beta_{-1,1}. \] (2.27)
So type (i) follows from (2.9)-(2.27).

If \( \bar{\theta} \) is of type (b), by a similar discussion as that in type (i), we can prove that
\[ \theta(L_n) = -\alpha^n L_n + \left( \frac{n + 1}{2} \alpha^{n-1} \beta_{1,-1} - \frac{n - 1}{2} \alpha^{n+1} \beta_{-1,1}\right) I_n. \] (2.28)
\[ \theta(C_L) = -C_L - 12(\alpha^{-1} \beta_{1,-1} - \alpha \beta_{-1,1}) C_L I + (14 \beta_{-1,1} \beta_{1,-1} - 3 \alpha^{-2} \beta_{1,-1}^2 - 3 \alpha^2 \beta_{-1,1}^2) C_I. \] (2.29)
where \( \alpha \in S^1, \beta_{1,-1}, \beta_{-1,1} \in \mathbb{C} \).
\[ \theta(I_n) = \alpha_n I_n \text{ for } n \neq 0, \] (2.30)
where \( \alpha_n = \alpha_{n,n} \in S^1(n \neq 0) \).
\[ \theta(C_I) = -\alpha_1 \alpha_{-1} C_I. \] (2.31)
\[ \theta(I_0) = \alpha_{-1} I_0 - \alpha_{1} \beta_{1,-1} C_I. \] (2.32)
\[ \theta(C_{LI}) = \alpha_1 \alpha^{-1} C_{LI} + \frac{1}{2} (\alpha_1 \alpha^{-1} - \alpha_{-1} \alpha) I_0 + \frac{1}{2} (\alpha_{-1} \beta_{1,-1} - \alpha_{1} \beta_{-1,1}) C_I. \] (2.33)
By \( \theta^2(I_0) = I_0 \) and \( \theta^2(C_{LI}) = C_{LI} \), we obtain that
\[ \alpha_1 \beta_{1,-1} = \overline{\alpha_{-1} \beta_{1,-1}}, \quad \alpha_{-1} \beta_{-1,1} = \overline{\alpha_1 \beta_{-1,1}}. \] (2.34)

By \( [\theta(I_n), \theta(L_{-n})] = n \theta(I_0) + (n^2 + n) \theta(C_{LI}) \), we can deduce that
\[ \alpha_n = \alpha_1 \alpha^{n-1}. \] (2.35)

By (2.31)-(2.35) and identity \( \theta^2(C_L) = C_L \), we can deduce that
\[ (\overline{\alpha} - \alpha) \beta_{-1,1} = 0, \quad \beta_{1,-1} \beta_{-1,1} (1 + \overline{\alpha} \alpha^2) = 0. \] (2.36)
Thus type (ii) follows from (2.28)-(2.36). \( \square \)

**Lemma 2.6.** Let \( \theta \) be a conjugate-linear anti-involution of \( \mathfrak{L} \).

(i) If \( \theta = \theta_{\alpha,\gamma}^+ \), then \( \theta(Vir) = Vir \).

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(ii) If \( \theta = \theta_{\alpha,\alpha_1,\beta_1,\beta_{-1}}^{-} \), we denote by \( Vir' \) the subalgebra of \( \mathfrak{L} \) generated by

\[
\{ C', L'_n := L_n + x_n I_n \mid n \in \mathbb{Z} \},
\]

where \( x_n \in \mathbb{C} \) is determined by \( \overline{x_n} \alpha_1 + \frac{n+1}{2} \beta_1 - \frac{n-1}{2} \alpha^2 \beta_{-1} = -\alpha x_n \), \( C' \) is determined by \( n^2 - C' = [L'_n, L'_n] + 2nL'_0 \). Then \( Vir' \cong Vir \) and \( \theta_{\alpha,\alpha_1,\beta_1,\beta_{-1}}^{-}(L'_n) = -\alpha^n L'_n, \theta_{\alpha,\alpha_1,\beta_1,\beta_{-1}}^{-}(C') = -C' \).

**Proof.** It can be checked directly, we omit the details. \( \square \)

**Lemma 2.7.** (Theorem 3.5 in Ref. [3]) Let \( V \) be a nontrivial irreducible weight \( Vir \)-module.

(i) If \( V \) is unitary for some conjugate-linear anti-involution \( \theta \) of \( Vir \), then \( \theta = \theta_\alpha^+ \) for some \( \alpha > 0 \).

(ii) If \( V \) is unitary for \( \theta_\alpha^+ \) for some \( \alpha > 0 \), then \( V \) is unitary for \( \theta_{1,\gamma}^+ \).

**Proposition 2.8.** Let \( V \) be a nontrivial irreducible weight \( \mathfrak{L} \)-module.

(i) If \( V \) is unitary for some conjugate-linear anti-involution \( \theta \) of \( \mathfrak{L} \), then \( \theta = \theta_{\alpha,\gamma}^+ \) for some \( \alpha \in \mathbb{R}^+ \).

(ii) If \( V \) is unitary for \( \theta_{\alpha,\gamma}^+ \) for some \( \alpha > 0 \), then \( V \) is unitary for \( \theta_{1,\gamma}^+ \).

**Proof.** Suppose \( V \) is unitary for some conjugate-linear anti-involution \( \theta \) of \( \mathfrak{L} \). By Lemma 2.6, \( V \) can be viewed as a unitary \( Vir' \)-module for the conjugate-linear anti-involution \( \theta|_{Vir'} \), where \( Vir' = Vir \) if \( \theta = \theta_\alpha^+ \) and \( Vir' \) is that defined in Lemma 2.6(ii) if \( \theta = \theta_{\alpha,\alpha_1,\beta_1,\beta_{-1}}^{-} \). Then \( V \) is a direct sum of irreducible unitary \( Vir' \)-modules since any unitary weight \( Vir' \)-module is complete reducible. We claim that \( V \) is a nontrivial \( Vir' \)-module. Otherwise, for any \( 0 \neq v \in V \), by \( [L'_1, I_n]v = 0 \) for \( n \neq 0, -1 \), we see that

\[
I_nv = 0, \forall n \neq 0, 1.
\]

By \([L'_2, I_{-1}]v = 0\), we have

\[
I_1v = 0.
\]

Since \( I_nv = 0 \) for \( n \neq 0 \), we have

\[
C_Iv = 0.
\]

By \([L'_1, I_{-1}]v = 0\), we have

\[
I_0v = 0.
\]

By \([L'_{-1}, I_1]v = 0\), we have

\[
C_{LI} = 0.
\]

So \( \mathfrak{L}V = 0 \), a contradiction. Thus there is a nontrivial irreducible unitary \( Vir' \)-submodule of \( V \) for conjugate-linear anti-involution \( \theta|_{Vir'} \). By Lemma 2.7, \( \theta|_{Vir'} = \theta_\alpha^+ \) for some \( \alpha > 0 \). Then by Proposition 2.5, we have \( \theta = \theta_{1,\gamma}^+ \) for some \( \alpha \in \mathbb{R}^+ \).

The proof of (ii) is similar as that for Theorem 3.5 in Ref. [3], we omit the details. \( \square \)
3 Unitary representations for $\mathfrak{L}$

In this section, we study the unitary modules for $\mathfrak{L}$. By Proposition 2.8, we see that the conjugate-linear anti-involution is of the form $\theta_{a,\gamma}^+$ for some $\alpha \in \mathbb{R}^+$. For the sake of simplicity, we write $\theta_{a,\gamma}^+$ by $\theta$. To prove the main result, we give some lemmas first.

The following result is well known:

**Lemma 3.1.** If $V$ is unitary weight module for $Vir$, then $V$ is completely reducible.

In [12], all indecomposable Harish-Chandra modules of intermediate series over the twisted Heisenberg-Virasoro algebra are classified:

**Lemma 3.2.** (Theorem 3.5 in [12]) Let $V = \sum_{m \in \mathbb{Z}} \mathbb{F}v_m$ be an indecomposable $\mathfrak{L}$-module such that $\mathfrak{L}_m v_n \in \mathbb{C}v_{m+n}$ for all $m, n \in \mathbb{Z}$. Then $V$ is isomorphic to one of the modules $A_{a,b,c}, B(a,c), U_d, V_d, \tilde{U}_d, \tilde{V}_c$ for appropriate $a, b, c, d \in \mathbb{F}$.

For more details, we refer the readers to Ref. [12].

It is well known that there are three types modules of the intermediate series over $Vir$, denoted respectively by $A_{a,b}, A_{\alpha, B_{\beta}}$, they all have basis $\{v_k \mid k \in \mathbb{Z}\}$ such that $C_L$ acts trivially and

- $A_{a,b}: L_n v_k = (a + k + nb)v_{n+k};$
- $A_{\alpha}: L_n v_k = (n+k)v_{n+k}$ if $k \neq 0$, $L_n v_0 = n(n+a)v_n;$
- $B_{\beta}: L_n v_k = kv_{n+k}$ if $k \neq -n$, $L_n v_{-n} = -n(n+a)v_0.$

for all $n, k \in \mathbb{Z}$. About the modules of the intermediate series of type $A_{a,b}$, we have the following facts:

1. $A_{a,b}$ is not simple if and only if $a \in \mathbb{Z}$ and $b = 0$ or 1.
2. $A_{a,b} \simeq A_{a',b'}$ if and only if (i) $a - a' \in \mathbb{Z}, b = b'$ or (ii) $a - a' \in \mathbb{Z}, a \notin \mathbb{Z}, \{b, b'\} = \{0, 1\}.$

**Lemma 3.3.** (Theorem 0.5 in [3]) Let $V$ be an irreducible unitary module of $Vir$ with finite-dimensional weight spaces. Then either $V$ is highest or Lowest weight, or $V$ is isomorphic to $A_{a,b}$ for some $a \in \mathbb{R}, b \in \frac{1}{2} + \sqrt{-1}\mathbb{R}$.

Now we prove the main theorem of this section.

**Theorem 3.4.** Any unitary irreducible Harish-Chandra module from intermediate series with conjugate-linear anti-involution $\theta_{a,\gamma}^+$ over twisted Heisenberg-Virasoro algebra $\mathfrak{L}$ is a unitary irreducible $\mathfrak{L}$-module of the intermediate series with form $A_{a,b,c}$ for $a \in \mathbb{R}, b \in \frac{1}{2} + \sqrt{-1}\mathbb{R}, c \in \mathbb{C}$ satisfying $c = \bar{c}e^{-i\gamma}$.

**Proof.** Let $V$ be a unitary irreducible $\mathfrak{L}$-module of intermediate series for a conjugate-linear anti-involution $\theta$. Then the central elements $C_L, C_I, C_{LI}, I_0$ are assigned zero by [12]. By Lemma 2.6(i) and Proposition 2.8, $V$ is also unitary for $Vir$ and then by Lemma 3.1 and Lemma 3.3, we can suppose that

$$V = A_{a_1,b_1} \oplus \cdots \oplus A_{a_K,b_K} \oplus W,$$
where \( a_i \in \mathbb{R}, b_i \in \frac{1}{2} + \sqrt{-1} \mathbb{R} \), \( W \) is a trivial \( \mathfrak{vir} \)-module.

**Case 1.** \( W = 0 \).

In this subcase

\[
V = A_{a_1, b_1} \oplus \cdots \oplus A_{a_K, b_K}.
\]

Let \( \{ v_k \mid k \in \mathbb{Z} \} \) be a basis of \( A_{a_1, b_1} \) such that \( L_n v_k = (a_1 + k + nb_1)v_{n+k} \). As a irreducible \( \mathfrak{L} \)-module, \( V \) is generated by the \( L_0 \)-eigenvector \( v_0 \) with eigenvalue \( a_1 \). Thus \( L_0 \)-eigenvalue on \( V \) are of the form \( a_1 + n, n \in \mathbb{Z} \). This means that \( a_i \in \{ a_1 + n \mid n \in \mathbb{Z} \}, i = 1, \cdots, K \). Recall that \( A_{a_1, b_1} \simeq A_{a, b} \) for any \( n \in \mathbb{Z} \).

So there exists \( 0 \leq a < 1 \) such that \( A_{a_i, b_i} \) are of the form \( A_{a, b} \), i.e.,

\[
V = A_{a, b_1} \oplus \cdots \oplus A_{a, b_K}, \tag{3.1}
\]

where \( 0 \leq a < 1, b_i \in \frac{1}{2} + \sqrt{-1} \mathbb{R} \).

**Claim 1.** \( V = A_{a, b} \).

**proof of claim 1.** By (3.1) we can choose a basis of \( V : \)

\[
\{ v_{k, l} \mid k \in \mathbb{Z}, 1 \leq l \leq K \}
\]

such that

\[
L_m v_{k, l} = (a + k + mb_l)v_{k+m, l}, \tag{3.2}
\]

for \( m \in \mathbb{Z}, 1 \leq l \leq K \). Suppose

\[
I_1 v_{k, l} = \sum_{l' = 1}^{K} \mu_{k, l}^{l'} v_{k+1, l'} \tag{3.3}.
\]

By (3.2), (3.3) and \([L_{-1}, I_1] = I_0 + 2C_{IJ}, \) we have

\[
I_0 v_{k, l} = \sum_{l' = 1}^{K} [(a + k + 1 - b_l') \mu_{k, l}^{l'} - (a + k - b_l) \mu_{k-1, l}^{l'}] v_{k, l'}.
\]

Considering \( I_0 V = cV \) for some \( c \in \mathbb{C} \) since \( I_0 \in \mathcal{C} \), we see that

\[
(a + k + 1 - b_l') \mu_{k, l}^{l'} = (a + k - b_l) \mu_{k-1, l}^{l'} \text{ for } l' \neq l, \tag{3.4}
\]

and

\[
I_0 v_{k, l} = [(a + k + 1 - b_l) \mu_{k, l}^{l} - (a + k - b_l) \mu_{k-1, l}^{l}] v_{k, l} = cv_{k, l}. \tag{3.5}
\]

If there exists \( k_0 \) such that \( a + k_0 - b_l = 0 \) or \( a + k_0 + 1 - b_{l'} = 0 \), noting that \( 0 \leq a < 1, b_l, b_{l'} \in \frac{1}{2} + \sqrt{-1} \mathbb{R} \), we can deduce recursively from (3.4) that \( \mu_{k, l}^{l'} = 0, \forall k \in \mathbb{Z}, l \neq l' \).
If $a + k - b_l \neq 0$ and $a + k + 1 - b_l \neq 0$ for all $k \in \mathbb{Z}$, then for $l \neq l'$,
\[
\mu_{k,l}' = \frac{a + k - b_l}{a + k + 1 - b_l'} \mu_{k-1,l}' \quad \mu_{k-1,l}' = \frac{a + k + 1 - b_l'}{a + k - b_l} \mu_{k,l}'.
\] (3.6)

By (3.2), (3.3) and $[L_{-2}, I_1] = I_{-1} + 6C_{L1}$, we have
\[
I_{-1}v_{k,l} = \sum_{l' = 1}^{K} \{ (a + k + 1 - 2b_{l'}) \mu_{k-1,l}' - (a + k + 1 - 2b_l) \mu_{k-2,l}' \} v_{k-1,l'}. \] (3.7)

By (3.2), (3.7) and $[L_1, I_{-1}] = -I_0$, we have
\[
I_0 v_{k,l} = \sum_{l' = 1}^{K} \{ (a + k + b_l)((a + k + 2 - 2b_{l'}) \mu_{k+1,l}' - (a + k + 1 - 2b_l) \mu_{k-1,l}'] - (a + k - 1 + b_{l'})((a + k + 1 - 2b_{l'}) \mu_{k,l}' - (a + k - 2b_l) \mu_{k-2,l}' ) \} v_{k,l'}. \] (3.8)

Comparing (3.5) and (3.8), we have
\[
(a + k + b_l)((a + k + 2 - 2b_{l'}) \mu_{k+1,l}' - (a + k + 1 - 2b_l) \mu_{k-1,l}'] - (a + k - 1 + b_{l'})((a + k + 1 - 2b_{l'}) \mu_{k,l}' - (a + k - 2b_l) \mu_{k-2,l}' ) = 0 \quad \text{for } l \neq l'.
\] (3.9)

By (3.6) and (3.9), we obtain that
\[
\begin{align*}
& [(a + k + b_l)(a + k - b_l)(a + k + 2 - 2b_{l'})(a + k + 1 - b_l')(a + k + 1 - b_l)(a + k - 1 - b_l) \\
& - (a + k + b_l)(a + k + 1 - 2b_{l'})((a + k + 1 - 2b_{l'}) (a + k + 2 - b_{l'})(a + k + 1 - b_l)(a + k + 2 - b_{l'})(a + k - b_l)(a + k - 1 - b_l) \\
& - (a + k - 1 + b_{l'})(a + k + 2b_{l'})(a + k + 1 - b_{l'})(a + k + 1 - b_{l'})(a + k + 2 - b_{l'})(a + k + 2 - b_{l'}) \mu_{k,l}' \\
& = 0. \end{align*}
\]

Computing the above equality by mathematical software, such as maple, we get that:
\[
\{ [-(b_l + 2 - b_{l'})(b_l + 1 - b_{l'})(b_l + b_{l'})(b_l + b_{l'} - 1) - 2b_{l'})] k + f(\mu_{k,l}') \} \mu_{k,l}' = 0, \quad \text{(3.10)}
\]
where $f(\mu_{k,l}') \in \mathbb{C}$ is a constant determined by $b_l$ and $b_{l'}$. Note that $b_l, b_{l'} \in \mathbb{Z}$. It is easy to deduce that the coefficient of $k$ in (3.10) is nonzero. Thus there exists $k \in \mathbb{Z}$ such that $\mu_{k,l}' = 0$. Then, by (3.6), we have
\[
\mu_{k,l}' = 0, \forall k \in \mathbb{Z}, l \neq l'.
\] (3.11)

From (3.2), (3.3) and (3.11), we see that each subspace $A_{a,b_l}(1 \leq l \leq K)$ in (3.1) is a submodule of $V$. Since $V$ is irreducible, we have $V = A_{a,b_l}$. Thus Claim 1 holds.
By Claim 1, for the sake of simplicity, we can rewrite (3.2), (3.3), (3.5) and (3.8) as following:

\[ L_m v_k = (a + k + mb)v_{k+m}, \quad (3.12) \]
\[ I_1 v_k = \mu_k v_{k+1}. \quad (3.13) \]
\[ I_0 v_k = [(a + k + 1 - b)\mu_k - (a + k - b)\mu_{k-1}]v_k = cv_k. \quad (3.14) \]
\[ I_0 v_k = \{(a + k + b)[(a + k + 2 - 2b)\mu_{k+1} - (a + k + 1 - 2b)\mu_{k-1}] \]
\[ - (a + k - 1 + b)[(a + k + 1 - 2b)\mu_k - (a + k - 2b)\mu_{k-2}]\}v_k = cv_k. \quad (3.15) \]

Claim 2. \( I_m v_k = cv_{m+k}, \forall m, k \in \mathbb{Z}. \)

Proof of Claim 2. If \( (a + k + 1 - b) = 0, \) we see that \( a = \frac{1}{2} = b, k = -1 \) since \( 0 \leq a < 1, b \in \frac{1}{2} + \sqrt{-1}\mathbb{R}. \) Thus by (3.14) we have \( \mu_{-2} = c. \) Then we can deduce recursively that \( \mu_k = c, \forall k \in \mathbb{Z}. \) Similarly, if \( (a + k - b) = 0, \) we can also deduce that \( \mu_k = c, \forall k \in \mathbb{Z}. \)

Now we suppose \( (a + k + 1 - b) \neq 0 \) and \( (a + k - b) \neq 0 \) for all \( k \in \mathbb{Z}. \) By (3.14) we have

\[ \mu_k = \frac{c + (a + k - b)\mu_{k-1}}{a + k + 1 - b}, \quad \mu_{k-1} = \frac{(a + k + 1 - b)\mu_k - c}{a + k - b}. \quad (3.16) \]

By (3.15) and (3.16) we get that

\[ [(a + k - b)(a + k - 1 - b)(a + k + b)(a + k + 2 - 2b)(a + k + 1 - b) \]
\[ -(a + k + 2 - b)(a + k - 1 - b)(a + k + b)(a + k + 1 - 2b)(a + k + 1 - b) \]
\[ -(a + k - 1 + b)(a + k + 1 - 2b)(a + k + 2 - b)(a + k - b)(a + k - 1 - b) \]
\[ +(a + k + 2 - b)(a + k - b)(a + k - 1 + b)(a + k - 2b)(a + k + 1 - b)]\mu_k \]
\[ = (a + k + 2 - b)(a + k - b)(a + k - 1 - b)c \]
\[ -(a + k - b)(a + k - 1 - b)(a + k + 2 - 2b)(a + k + b)c \]
\[ -(a + k + 2 - b)(a + k - 1 - b)(a + k + b)(a + k + 1 - 2b)c \]
\[ + 2(a + k + 2 - b)(a + k - b)(a + k - 1 + b)(a + k - 2b)c. \]

Resort to maple software again, we have

\[ [(8b - 8b^2)k + 4b + 8ab - 12b^2 + 8b^2 - 8ab^2](\mu_k - c) = 0. \]

Noting that \( b \in \frac{1}{2} + \sqrt{-1}\mathbb{R} \) and combing with (3.16), we can easily deduce that

\[ \mu_k = c, \forall k \in \mathbb{Z}. \quad (3.17) \]

By (3.12), (3.13) and (3.17), we have \( I_{m+1} v_k = [L_m, I_1] v_k = cv_{m+k+1}, \forall m, k \in \mathbb{Z}. \) Thus Claim 2 holds.
From Claim 1, Claim 2 and (3.12) we see that $V$ is a unitary irreducible $\mathfrak{L}$-module of the intermediate series with form $A_{a,b,c}$ in Case 1. Moreover, by

$$\langle I_0v_k, I_0v_k \rangle = c\langle v_k, v_k \rangle$$

and

$$\langle I_0v_k, I_0v_k \rangle = \langle v_k, e^{i\gamma}c^2v_k \rangle = t^2e^{-i\gamma}\langle v_k, v_k \rangle,$$

we see that

$$c = te^{-i\gamma}.$$  

**Case 2.** $W \neq 0$.

Choose an arbitrary nonzero element $w \in W$. We see that $V$ is generated by $w$ since $V$ is an irreducible $\mathfrak{L}$-module. If $I_1w = 0$, then $\mathfrak{L}W = 0$ and $V$ is a trivial $\mathfrak{L}$-module, a contradiction. Thus $I_1w \neq 0$. By $L_0I_1w = I_1w$, we see that

$$I_1w \in A_{a_1,b_1} \oplus \cdots \oplus A_{a_K,b_K}.$$

Moreover,

$$A_{a_i,b_i} \cong A_{0,b_i}, \forall i \in \{1, \cdots, K\}$$

since $V$ is generated by the eigenvector $w$ of $L_0$ with eigenvalue 0. Thus

$$V = A_{0,b_1} \oplus \cdots \oplus A_{0,b_K} \oplus W.$$  

Choose the standard basis $\{v_{k,i} \mid k \in \mathbb{Z}\}$ for each $A_{0,b_i}$. Suppose

$$I_1v_{k,i} = \sum_{t'=1}^{K} \mu_{k,i,t'}v_{k+1,t'} + w_{k,t},$$

where $w_{k,t} \in W$. By a similar calculation as that of Claim 1 in Case 1, we can deduce that

$$V = A_{0,b}.$$  

This contradicts with the assumption that $W \neq 0$. Thus Case 2 is impossible. This completes the proof of Theorem 3.4. \hfill \Box

**Acknowledgments**

The research was done during the visit of the first author to Chern Institute of Mathematics in 2011. The hospitality and financial support of Chern Institute of Mathematics are gratefully acknowledged. It is also a pleasure to thank Professor Chengming Bai for useful conversations.

**References**

[1] E. Arbarello, C. DeConcini, V. G. Kac, C. Procesi, Moduli spaces of curves and representation theory. Comm. Math. Phys. 117 (1988) 1-36.
[2] Y. Billig, Respresentations of the twisted Heisenberg-Virasoro algebra at level zero, Canad. Math. Bulletin, 46 (2003) 529-537.

[3] V. Chari, A. Pressley, Unitary representations of the Virasoro algebra and a conjecture of Kac, Compo. Math. 67 (1998) 315-342.

[4] D. Friedan, Z. Qiu and S. Shenker, Conformal invariance, unitarity and two-dimensional critical exponents, Vertex Operators in Mathematical and Physics, MSSRI Publications No. 4 Springer (1985) 419-449.

[5] D. Friedan, Z. Qiu and S. Shenker, Details of the non-unitarity proof for highest weight representations of the Virasoro algebras, Commun. Math. Phys. 107 (1986) 535-542.

[6] M. A. Fabbri, R. V. Moody, Irreducible representations of Virasoro-toroidal Lie algebras, Comm. Math. Phys. 159 (1994) 1C13.

[7] M. A. Fabbri, F. Okoh, Representations of Virasoro-Heisenberg algebras and Virasoro-toroidal algebras, Canad. J. Math. 51 (1999) 523-545.

[8] P. Goddard, A. Kent and D. Olive, Virasoro algebra and coset space model, Phys. Lett. 152 (1985) 88-93.

[9] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro and super-Virasoro algebras, Commun. Math. Phys. 103 (1986) 105-119.

[10] V. G. Kac, A.K. Raina, Bombay lectures on highest weight representations of infinite-dimensional Lie algebras, Word Scientific, 1987.

[11] V.G. Kac, J.W. van de Leur, On classification of superconformal algebras, Strings 88 (College Park, MD, 1988) 77C106.

[12] D. Liu, C. Jiang, Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra, J. Math Phys. 49 (2008) 012901.

[13] D. Liu, Y. Wu, L. Zhu, Whittaker modules for the twisted Heisenberg-Virasoro algebra, J. Math Phys. 51 (2010) 023524.

[14] R. Lu, K. Zhao, Classification of irreducible weight modules over the twisted Heisenberg- Virasoro algebra, Commu. Contem. Math. 12 (2010) 183-205.

[15] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro Lie algebra, Invent. Math. 107 (1992) 225-234.

[16] R. Shen, Y. Su, Classification of irreducible weight modules with a finite-dimensional weight space over twisted Heisenberg-Virasoro algebra, Acta Mathematica Sinica, English series, 23 (2007) 189-192.
[17] S. Tan, X. Zhang, Automorphisms and Verma modules for generalized Schrödinger-Virasoro algebras, J. Algebra 322 (2009) 1379-1394.

[18] X. Zhang, S. Tan, Unitary representations for the $W$-algebra $W(2, 2)$, submitted.

[19] X. Zhang, S. Tan, Unitary representations for the Schrödinger-Virasoro Lie algebra, to appear in Journal of algebra and its applications.