NAVIER-STOKES-OSEEN FLOWS IN THE EXTERIOR OF A ROTATING AND TRANSLATING OBSTACLE

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Dedicated to Professor Dang Dinh Chau on the occasion of his 70th birthday

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ABSTRACT. In this paper, we investigate Navier-Stokes-Oseen equation describing flows of incompressible viscous fluid passing a translating and rotating obstacle. The existence, uniqueness, and polynomial stability of bounded and almost periodic weak mild solutions to Navier-Stokes-Oseen equation in the solenoidal Lorentz space $L^{3}_{3,\sigma,w}$ are shown. Moreover, we also prove the unique existence of time-local mild solutions to this equation in the solenoidal Lorentz spaces $L^{3}_{3,q}$.

1. Introduction and preliminaries. Consider a rigid body $\mathcal{R}$ moving through an incompressible viscous fluid that fills the whole three-dimensional space $\mathbb{R}^{3}$ exterior to $\mathcal{R}$. We assume that with respect to a frame attached to $\mathcal{R}$, the translational velocity $u_{\infty}$ and the angular velocity $\omega$ of $\mathcal{R}$ are both constant vectors. Without loss of generality we may assume that $\omega = a \mathbf{e}_{3}, \mathbf{e}_{3} = (0, 0, 1)^{T}$. If the flow is non-slip at the boundary, then the motion of fluid can be described by the following equation:

\[
\begin{aligned}
D_{t}v + (v \cdot \nabla)v - \Delta v + \nabla \pi &= \text{div } G, \quad \text{div } v = 0 \quad \text{in } \Omega(t) \ (t > 0), \\
v(y,t)|_{\partial \Omega(t)} &= \omega \times y|_{\partial \Omega(t)}, \quad \lim_{|y| \to \infty} v(y,t) = u_{\infty}, \quad v(y,0) = v_{0}(y)
\end{aligned}
\]

in the time-dependent exterior domain $\Omega(t) = \mathcal{O}(at)\Omega$, where $\mathcal{O}$ is a fixed exterior domain complemented to $\mathcal{R}$ at the time $t = 0$ with smooth boundary, and $\mathcal{O}(t)$ denotes the orthogonal matrix:

\[
\mathcal{O}(t) = \begin{pmatrix}
\cos t & \sin t & 0 \\
-\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Here $D_{t} = \partial / \partial t$, $v = v(y,t)$ is the velocity field of the liquid; $\pi = \pi(y,t)$ is the pressure field; and $G = G(y,t)$ is a second-order tensor field.
To study (1) in the time-independent exterior domain \(\Omega(t)\), the following change of variables and unknown functions have been introduced

\[
x = O(at)^T y, \quad u(x,t) = O(at)^T (v(y,t) - u_\infty),
p(x,t) = \pi(y,t), \quad F(x,t) = O(at)^T G(y,t) O(at).
\]

Then we obtain the Navier-Stokes-Oseen equation on the fixed domain \(\Omega\):

\[
\begin{cases}
  D_t u + (u \cdot \nabla) u - \Delta u + ((O(at)^T u_\infty) \cdot \nabla) u \\
  -((\omega \times x) \cdot \nabla) u + \omega \times u + \nabla p = \text{div} F & \text{in } \Omega \times (0, \infty), \\
  \text{div} u = 0 & \text{in } \Omega \times (0, \infty), \\
  u(x,t) = \omega \times x - O(at)^T u_\infty & \text{on } \partial\Omega \times (0, \infty), \\
  u(x,0) = u_0(x) := v_0(x) - u_\infty & \text{in } \Omega, \\
  \lim_{|x| \to \infty} u(x,t) = 0 & \text{for all } t \in (0, \infty).
\end{cases}
\]

In this paper, we consider only the case where \(u_\infty = k\mathbf{e}_3\). Therefore, the Eq. (2) leads to the system:

\[
\begin{cases}
  D_t u + (u \cdot \nabla) u - \Delta u + kD_3 u \\
  -((\omega \times x) \cdot \nabla) u + \omega \times u + \nabla p = \text{div} F & \text{in } \Omega \times (0, \infty), \\
  \text{div} u = 0 & \text{in } \Omega \times (0, \infty), \\
  u(x,t) = \omega \times x - k\mathbf{e}_3 & \text{on } \partial\Omega \times (0, \infty), \\
  u(x,0) = u_0(x) := v_0(x) - u_\infty & \text{in } \Omega, \\
  \lim_{|x| \to \infty} u(x,t) = 0 & \text{for all } t \in (0, \infty).
\end{cases}
\]

The Eq. (3) have been investigated by many authors, e.g., Shibata [26, 27], Galdi and Silvestre [7, 8, 9], Geissert, Heck and Hieber [11], and many others (see [2, 3, 5, 10, 15, 16, 18, 20, 23, 24]). Shibata [27] has shown the unique existence of solution to the linearized equation corresponding to Eq. (3) and proved the stability of the stationary solution. Meanwhile, Galdi and Silvestre [9] and Geissert, Heck and Hieber [11] have shown the existence and uniqueness of the steady solution. In the case of moving or rotating obstacle, the unique existence of time-local mild solution have been proved by Geissert, Heck and Hieber [11]. However, almost authors study the existence and properties of solution to Eq. (3) in the spaces \(L^p\) (see (4) below). The reader is also referred to [2, 14, 15, 24, 26, 27] for the treatment of the existence and uniqueness of solution to Eq. (3) in various cases.

In this paper we consider the case of rotating and translating obstacle. Thus, the form of Eq. (3) is more general than the equations before. Our purpose is to prove the existence, uniqueness, and polynomial stability of bounded and almost periodic weak mild solutions to the Eq. (3) in the solenoidal Lorentz space \(L^3_{\sigma, w}\), and the unique existence of time-local mild solutions to this equation in the solenoidal Lorentz spaces \(L^3_{\sigma, q}\) (see (5) below). When \(q > 3\), the space \(L^3_{\sigma, q}\) is bigger than the space \(L^3_{\sigma}\). Thus, the initial value can be chosen in the bigger data class. For the first results, our approach relies on the interpolation spaces combined with the smoothing properties of semigroup corresponding to linearized equation. For the latter result we use the Kato-iteration scheme as in [12, 13, 17, 29].

The keys of our strategy are lying on the duality estimates, the smoothing properties and interpolation factors for semigroup corresponding to linearized equation, and the technique of cut-off function to translate velocity of the liquid on boundary \(\partial\Omega\) of Eq. (3) to zero. This paper is organized as follows. In Section 2 and 3, we prove the existence, uniqueness, and polynomial stability of bounded and almost periodic weak mild solutions. In Section 4 we investigate the unique existence of
time-local mild solutions. The main results are contained in Theorem 2.1, Theorem 3.4 and Theorem 4.1.

We now recall some preliminaries that will be used in the next sections. Throughout this paper, the following spaces will be used

\[ C^\infty_0(\Omega) := \{ v \in C^\infty_0 : \text{div} v = 0 \text{ in } \Omega \}, \]

\[ L^p_\sigma(\Omega) := \left[ C^\infty_0(\Omega) \right]^{\| p}. \]

We also need the notion of Lorentz space \( L^{r,q}(\Omega) \), \( 1 \leq r \leq \infty, \ 1 \leq q \leq \infty \), defined in as \([1, 4, 19, 22, 28]\), and note that \( L^{r,r}(\Omega) = L^r(\Omega) \) and for \( q = \infty \) then the space \( L^{r,\infty}(\Omega) \) is called the weak-\( L^r \) space and is denoted by \( L^\infty_0(\Omega) := L^{r,\infty}(\Omega) \). Note that the Lorentz spaces can be described by using interpolation pairs as follows

\[ L^{r,q}(\Omega) = (L^p_0(\Omega), L^p_\infty(\Omega))_{\theta,q} \quad \text{for} \quad \frac{1}{r} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{with} \ 1 < r < \infty \text{ and } 0 < \theta < 1. \]

Some usefulness properties of Lorentz spaces such as embedding property, Hölder type inequality are given in the following lemma. 

**Lemma 1.1.** Let \( 1 \leq p, p_1, p_2 \leq \infty, \ 1 \leq q, q_1, q_2 \leq \infty \) satisfy \( \frac{1}{p} + \frac{1}{p_2} = \frac{1}{p}, \ \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \).

(i) If \( q_1 < q_2 \) and \( f \in L^{p,q_1}(\Omega) \) then \( f \in L^{p,q_2}(\Omega) \) and

\[ \| f \|_{p,q_2} \leq \left( \frac{q_1}{p} \right)^\frac{1}{q_2} \| f \|_{p,q_1}. \]

(ii) If \( f \in L^{p_1,q_1}(\Omega), \ g \in L^{p_2,q_2}(\Omega) \) then \( fg \in L^{p,q}(\Omega) \) and

\[ \| fg \|_{p,q} \leq 2^\frac{1}{2} \| f \|_{p_1,q_1} \| g \|_{p_2,q_2}. \]

**Proof:** (i) is proved respectively in [4, Theorem 6.3], so the proof is omitted here. 
(ii) is shown as follows.

The case \( p = \infty \) is evident, since spaces \( L^{\infty,q}(\Omega) = \{ 0 \} \) with \( q < \infty \) and \( L^{\infty,\infty}(\Omega) = L^\infty(\Omega) \). Suppose that \( 1 \leq p < \infty \), if \( q < \infty \) then we have

\[ \| fg \|_{p,q}^q = \int_0^\infty \left[ \frac{t^\frac{q}{p}(fg)^*(t)}{t} \right]^q dt \quad \text{(use [4, Theorem 4.11])} \]

\[ \leq \int_0^\infty t^{\frac{q}{p} - 1} f^*(2^{-1}t)^q g^*(2^{-1}t)^q dt \]

\[ = \int_0^\infty t^\frac{p}{p_1} f^*(2^{-1}t)^q g^*(2^{-1}t)^q dt \quad \text{(use Hölder inequality)} \]

\[ \leq \left( \int_0^\infty \left( \int_0^\infty t^{\frac{p}{p_1} - 1} f^*(2^{-1}t)^q dt \right)^\frac{q}{q_2} \right)^\frac{q_2}{q} \]

\[ = 2^\frac{1}{2} \| f \|_{p_1,q_1} \| g \|_{p_2,q_2}. \]

Otherwise, \( q = \infty \). Then, \( q_1 = q_2 = \infty \) and

\[ \| fg \|_{p,w} = \sup_{t \geq 0} t^\frac{q}{p} (fg)^*(t) \leq \sup_{t \geq 0} t^\frac{q}{p} f^*(2^{-1}t) g^*(2^{-1}t) \]

\[ \leq \sup_{t \geq 0} \left( t^\frac{q}{p_1} f^*(2^{-1}t) \right) \sup_{t \geq 0} \left( t^\frac{q}{p_2} g^*(2^{-1}t) \right) = 2^\frac{1}{2} \| f \|_{p_1,w} \| g \|_{p_2,w}. \]

\[ \square \]
Proposition 1.

Let estimates for that semigroup on the space $C_0^\alpha$ - the norm in is not analytic). By interpolation theory, the above Helmholtz defines a bounded projection $\mathbb{P} = \mathbb{P}_{r,q}$ on Lorentz space $L^{r,q}(\Omega)$ and

$$L^{r,q}(\Omega) = L^{r,q}_a(\Omega) \oplus \{ v \in L^{r,q}(\Omega) : p \in L^{r,q}_{\text{loc}}(\Omega) \}.$$  

We need also the solenoidal Lorentz spaces $[2]$, which are defined by

$$L^{r,q}_{\sigma}(\Omega) := (L^r_\sigma(\Omega), L^q_\sigma(\Omega))_{\theta,q}$$  

where $1 < r_0 < r < r_1 < \infty$, $1 \leq q \leq \infty$ and $\frac{1}{r} = \frac{1-q}{r_0} + \frac{q}{r_1}$. By interpolation theory, the above Helmholtz defines a bounded projection $\mathbb{P} = \mathbb{P}_{r,q}$ on Lorentz space $L^{r,q}(\Omega)$ and

$$L^{r,q}_{\sigma}(\Omega) = \text{Im} \mathbb{P}_{r,q}.$$  

We then have (see [2, Theorem 5.2])

$$L^{r,q}(\Omega) = L^{r,q}_a(\Omega) \oplus \{ \nabla p \in L^{r,q}(\Omega) : p \in L^{r,q}_{\text{loc}}(\Omega) \}.$$  

In the case $q = \infty$, we denote $L^{r,q}_a(\Omega) := L^{r,\infty}_a(\Omega)$. We denote by $\| \cdot \|_{r,q}$ the norm in the space $L^{r,q}(\Omega)$. (note that, generally, this semigroup is not analytic). By interpolation theory, $(e^{-t\mathcal{L}_{\sigma,k}})_{t>0}$ is also the bounded $C_0$-semigroup on the space $L^{r,q}_a(\Omega)$.

Using interpolation theory, we can transfer the $L^p - L^q$ decay estimates obtained by Shibata in [27], Theorem 3] for $(e^{-t\mathcal{L}_{\sigma,k}})_{t>0}$ on $L^{r,q}_a(\Omega)$ to the $L^{r,q} - L^{p,q}$ decay estimates for that semigroup on the space $L^{r,q}_a(\Omega)$ in the following proposition.

**Proposition 1.** Let $1 < r < \infty$, $1 \leq q \leq \infty$ and denote by $\| f \|_{r,q}$ the norm in the spaces $L^{r,q}(\Omega)$. Then, the following inequalities hold.

(i) For $1 < p \leq r < \infty$

$$\| e^{-t\mathcal{L}_{\sigma,k}} f \|_{r,q} \leq M t^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \| f \|_{p,q}. \tag{7}$$  

(ii) Furthermore, when $1 < p \leq r \leq 3$ and $1 < q < \infty$ we have

$$\| \nabla e^{-t\mathcal{L}_{\sigma,k}} f \|_{r,q} \leq M t^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \| f \|_{p,q}. \tag{8}$$  

(iii) For $1 < p \leq r < \infty$, $1 \leq q < \infty$ then

$$\| e^{-t\mathcal{L}_{\sigma,k}} f \|_{r,q} \leq M t^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \| f \|_{p,q} \tag{9}.$$  

(iv) Moreover, when $1 < p \leq r \leq 3$ and $1 < q < \infty$ we have

$$\| \nabla e^{-t\mathcal{L}_{\sigma,k}} f \|_{r,q} \leq M t^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \| f \|_{p,q} \tag{10}.$$
(v) For each \( f \in L_{1}^{p,q}(\Omega) \), \( 1 \leq q < \infty \), for \( t \to 0 \) then we have
\[
t^{\frac{1}{2}}(\frac{1}{p} - \frac{1}{r})\|e^{-t\mathcal{L}_{a,k}}f\|_{r,q} \to 0, \quad 1 < p < r < \infty, \tag{11}
\]
\[
t^{\frac{1}{2}}\|\nabla e^{-t\mathcal{L}_{a,k}}f\|_{p,q} \to 0, \quad 1 < p \leq 3. \tag{12}
\]

Proof. Using the interpolation theorem and \( L^{p} - L^{q} \) decay estimates in Shibata [27, Theorem 3] we obtain the estimate (7), the estimate (8) when \( r < 3 \). At \( r = 3 \), the estimate (8) holds since \( C_{0,\sigma}^{\infty}(\Omega) \) is dense in \( L_{r}^{r,q}(\Omega) \) for \( 1 < r < \infty, 1 \leq q < \infty \).

To prove (9) and (10) we just use the following interpolation relations
\[
\left( L_{\sigma}^{p,q}(\Omega), L_{\sigma}^{r,q}(\Omega) \right)_{\theta, \frac{1}{p} - \frac{1}{r}} = L_{\sigma}^{\theta,q}(\Omega),
\]
\[
\left( L_{\sigma}^{p^{0},q}(\Omega), L_{\sigma}^{p^{1},q}(\Omega) \right)_{\theta, \frac{1}{p^{0}} - \frac{1}{p^{1}}} = L_{\sigma}^{\theta,q}(\Omega)
\]
with \( \frac{1}{p} = \frac{1 - \theta}{p_{0}} + \frac{\theta}{p_{1}} \) and \( 1 < p_{0} < p < p_{1} < r \). Applying interpolation theorem for operators \( e^{-t\mathcal{L}_{a,k}}, e^{-t\mathcal{L}_{a,k}} \) and \( \nabla e^{-t\mathcal{L}_{a,k}}, \nabla e^{-t\mathcal{L}_{a,k}} \) we obtain (9) and (10).

In order to prove (11), we take \( g \in C_{0,\sigma}^{\infty}(\Omega) \). Using triangle inequality and (7) we have
\[
t^{\frac{1}{2}}(\frac{1}{p} - \frac{1}{r})\|e^{-t\mathcal{L}_{a,k}}f\|_{r,q} \leq M\|f - g\|_{p,q} + Mt^{\frac{1}{2}}(\frac{1}{p} - \frac{1}{r})\|g\|_{r,q}.
\]
The next step, approximating \( f \) by \( g \) and sending \( t \to 0 \) we get (11). The proof of (12) follows similar manner. \( \square \)

2. Boundedness and polynomial stability of weak mild solutions to Navier-Stokes-Oseen equation. In this section, we will show the unique existence and polynomial stability of bounded weak mild solutions to the Navier-Stokes-Oseen equation. We put
\[
L_{a,k}u = -\Delta u + kD_{3}u - ((\omega \times x) \cdot \nabla)u + \omega \times u.
\]

Then, the Eq. (3) is rewritten as
\[
\begin{cases}
D_{t}u + (u \cdot \nabla)u + L_{a,k}u + \nabla p = \text{div} F & \text{in } \Omega \times (0, \infty), \\
\text{div } u = 0 & \text{in } \Omega \times (0, \infty), \\
u(x,t) = \omega \times x - ke_{3} & \text{on } \partial \Omega \times (0, \infty), \\
u(x,0) = 0 & \text{in } \Omega, \\
\lim_{|x| \to \infty} u(x,t) = 0 & \text{for all } t \in (0, \infty). \tag{13}
\end{cases}
\]

To transform the boundary condition to the case of the zero vector-field on \( \partial \Omega \). We perform as in [25], take a cut-off function \( \varphi \in C_{c}^{\infty}(\mathbb{R}^{3}) \) such that \( \varphi \geq 0, \varphi \equiv 1 \) on the neighborhood of \( \mathcal{R} = \Omega^{c} \) and \( \text{supp} \varphi \subset B(0, r) \) for some \( r > 0 \) and define
\[
b_{\omega} = -\frac{1}{2} \nabla \times (\varphi(x)|x|^{2}\omega - 2k\varphi(x)x_{2}e_{1}), \quad e_{1} = (1, 0, 0)^{T}. \tag{14}
\]

Then, \( \text{div } b_{\omega} = 0, \ b_{\omega}|_{\partial \Omega} = \omega \times x - ke_{3} \) and \( b_{\omega} \in C_{c}^{\infty}(\mathbb{R}^{3}) \).

We now put \( z(x,t) = u(x,t) - b_{\omega}(x) \). Then, equation (13) is equivalent to the following equation with zero-boundary condition:
\[
\begin{cases}
D_{t}z + (z \cdot \nabla)z + L_{a,k}z + L_{a,k}b_{\omega} + (b_{\omega} \cdot \nabla)b_{\omega} \\
+ (z \cdot \nabla)b_{\omega} + (b_{\omega} \cdot \nabla)z + \nabla p = \text{div} F & \text{in } \Omega \times (0, \infty), \\
\text{div } z = 0 & \text{in } \Omega \times (0, \infty), \\
z(x,t) = 0 & \text{on } \partial \Omega \times (0, \infty), \\
z(x,0) = \text{z}_{0}(x) & \text{in } \Omega, \\
\lim_{|x| \to \infty} z(x,t) = 0 & \text{for all } t \in (0, \infty). \tag{15}
\end{cases}
\]
Applying the Helmholtz projection to (15) and considering initial value in the solenoidal Lorentz space $L^3_{\sigma,w}(\Omega)$, we obtain the following operator equation

\[
\begin{aligned}
D_t z + L_{a,k} z &= P \text{div}(F - z \otimes z - b_\omega \otimes z - z \otimes b_\omega - b_\omega \otimes b_\omega) - L_{a,k} b_\omega, \\
|z|_{t=0} &= z_0 \in L^3_{\sigma,w}(\Omega),
\end{aligned}
\]

where $L_{a,k}$ is defined as in (6).

By straightforward computations, we have

\[
\begin{aligned}
b_\omega \times \omega &= \text{div} F_\omega := \text{div} \begin{pmatrix}
\frac{2}{r} \varphi(x)|x|^2 & 0 & 0 \\
0 & \frac{2}{r} \varphi(x)|x|^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\end{aligned}
\]

and

\[
-L_{a,k} b_\omega = P \text{div} (\nabla b_\omega + b_\omega \otimes (\omega \times x - k_3) + F_\omega).
\]

Then, the Eq. (16) has form

\[
\begin{aligned}
D_t z + L_{a,k} z &= P \text{div} G(z), \\
|z|_{t=0} &= z_0 \in L^3_{\sigma,w}(\Omega),
\end{aligned}
\]

where

\[
G(z) = F + \nabla b_\omega + b_\omega \otimes (\omega \times x - k_3) + F_\omega - z \otimes z - b_\omega \otimes z - z \otimes b_\omega - b_\omega \otimes b_\omega.
\]

It is now convenient to give the definition of weak mild solutions to the above Navier-Stokes-Oseen equation, namely, by a weak mild solution to (17) that means the solution to equation

\[
\langle z(t), \varphi \rangle = \langle e^{-tL_{a,k}} z_0, \varphi \rangle - \int_0^t \langle G(z)(\tau), \nabla e^{-(\tau-t)L_{a,k}} \varphi \rangle \, d\tau
\]

for all $\varphi \in L^3_{\sigma,w}(\Omega)$ and $t \in \mathbb{R}_+$.

We next come to our first main result on the unique existence and polynomial stability of bounded weak mild solutions to (17) in the following theorem.

**Theorem 2.1.** Suppose that $F \in C_b(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)^{3 \times 3})$. Then, the following assertions hold true.

(a) If the norm $\|F\|_{\infty,3,w}, \|b_\omega\|_{3,w}$ and $\|z_0\|_{3,w}$ are small enough, Eq. (17) has a unique bounded weak mild solution in a small closed ball of $C_b(\mathbb{R}_+, L^3_{\sigma,w}(\Omega))$.

(b) The bounded weak mild solution $\hat{z}$ of Eq. (17) is polynomially stable in the sense that for any other weak mild solution $z \in C_b(\mathbb{R}_+, L^3_{\sigma,w}(\Omega))$ to Eq. (17) such that $\|z(0) - \hat{z}(0)\|_{3,w}$ is small enough, we have

$$
\|z(t) - \hat{z}(t)\|_{r,w} \leq Ct^{-\frac{3}{2}+\frac{r}{2}}
$$

for all $t > 0$.

where $r$ is any fixed number satisfying $r > 3$.

**Proof.** (a): For each $z_0 \in L^3_{\sigma,w}(\Omega)$, denoting $\hat{z}_0(t) = e^{-tL_{a,k}} z_0$. Applying the estimate (7) for $p = r = 3$, we have

$$
\|\hat{z}_0\|_{\infty,3,w} \leq M \|z_0\|_{3,w}
$$

for all $t > 0$.

Therefore, $\hat{z}_0 \in C_b(\mathbb{R}_+, L^3_{\sigma,w}(\Omega))$ for each $z_0 \in L^3_{\sigma,w}(\Omega)$.

Fixed $z_0 \in L^3_{\sigma,w}(\Omega)$, denoted

$$
B_\rho(z_0) = \{u \in C_b(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)) : \|u - \hat{z}_0\|_{\infty,3,w} \leq \rho\}.
$$
is closed ball in $C_0(\mathbb{R}_+, L^3_{σ,w}(Ω))$ centered at $\hat{z}_0$ with radius $ρ$. For $z ∈ B_ρ(\hat{z}_0)$ we define the map $T$ as follows $z → Tz$ with

$$\langle (Tz)(t), ϕ \rangle = \langle e^{-tL_{u,k}}z_0, ϕ \rangle - \int_0^t \langle G(z)(τ), \nabla e^{-(t-τ)L_{u,k}}ϕ \rangle dτ$$

for all $ϕ ∈ L^3_ω(Ω)$.

We show that $T : B_ρ(\hat{z}_0) → B_ρ(\hat{z}_0)$ and is a contractive mapping. Firstly, we prove that the function $Tz ∈ C_0(\mathbb{R}_+, L^3_{σ,w}(Ω))$. Indeed, for fixed $t > 0$ we have

$$|\langle (Tz)(t) - \hat{z}_0(t), ϕ \rangle| ≤ \int_0^t |\langle -G(z)(τ), \nabla e^{-(t-τ)L_{u,k}}ϕ \rangle| dτ ≤ \int_0^t \|G(z)(τ)\|_{L^2_{σ,w}}\|\nabla e^{-(t-τ)L_{u,k}}ϕ\|_{3,1} dτ.$$

By the Hölder type inequality in Lemma 1.1 (ii), we have $z(t) ⊗ z(t) ∈ L^3_{σ,w}(Ω)^{3×3}$ and

$$\|z(t) ⊗ z(t)\|_{L^2_{σ,w}} ≤ 2\frac{2}{3}\|z\|_{L^{∞,3}_{σ,w}} ≤ 2\frac{2}{3}(ρ + \|z_0\|_{L^{∞,3}_{σ,w}})^2 ≤ 2\frac{2}{3}(ρ + M\|z_0\|_{L^{3}_{σ,w}})^2.$$

This yields

$$\|z \otimes z\|_{L^∞,3_{σ,w}} ≤ 2\frac{2}{3}\|z\|_{L^{∞,3}_{σ,w}} \leq 2\frac{2}{3}(ρ + \|z_0\|_{L^{∞,3}_{σ,w}})^2 \leq 2\frac{2}{3}(ρ + M\|z_0\|_{L^{3}_{σ,w}})^2.$$

Hence,

$$\|G(z)\|_{L^∞,3_{σ,w}} \leq \|F\|_{L^∞,2_{σ,w}} + \|\nabla b_0 + b_ω ⊗ (ω × x - k e_3) + F_ω - b_ω ⊗ b_ω\|_{L^2_{σ,w}} + 2\frac{2}{3}(ρ + M\|z_0\|_{L^{3}_{σ,w}})^2 + 2\frac{2}{3}\|b_ω\|_{L^{3}_{σ,w}}(ρ + M\|z_0\|_{L^{3}_{σ,w}}).$$

Thus

$$\|z \otimes z\|_{L^∞,3_{σ,w}} ≤ 2\frac{2}{3}\|z\|_{L^{∞,3}_{σ,w}} \leq 2\frac{2}{3}(ρ + \|z_0\|_{L^{∞,3}_{σ,w}})^2 \leq 2\frac{2}{3}(ρ + M\|z_0\|_{L^{3}_{σ,w}})^2.$$

We now use $L^p \rightarrow L^q$ estimate and interpolation technique to prove that

$$\int_0^\infty \|\nabla e^{-tL_{u,k}}ϕ\|_{3,1} dτ \leq M\|ϕ\|_{L^3_{σ,w}}.$$

We choose real numbers $p_1$ and $p_2$ such that $1 < p_1 < \frac{3}{2} < p_2 < 3$. Consider the sublinear operator $A$ which maps a function $ϕ ∈ L^{p_1}_{σ,w}(Ω) + L^{p_2}_{σ,w}(Ω)$ to a function $v(·)$ defined on $(0, \infty)$ by $v(t) = \|\nabla e^{-tL_{u,k}}ϕ\|_{3,1}$. Then, by estimate (10) we have

$$v(t) ≤ Mt^{-\frac{3}{2p}}\|ϕ\|_{p_j, w} \quad \text{for } j = 1, 2.$$

Putting $\frac{1}{s_j} = \frac{3}{2p_j}$, we obtain that $v(·) ∈ L^{s_j}_{w}((0, \infty))$ and $\|v\|_{s_j, w} ≤ M\|ϕ\|_{p_j, w}$ for $j = 1, 2$. Moreover, the constant $θ ∈ (0, 1)$ such that $\frac{2}{3} = \frac{1-θ}{p_1} + \frac{θ}{p_2}$ also satisfies the equality $1 = \frac{1-θ}{s_1} + \frac{θ}{s_2}$. Therefore, we have the following real interpolation relations

$$(L^{p_1}_{σ,w}(Ω), L^{p_2}_{σ,w}(Ω))_{θ,1} = L^{\frac{3}{2}}_{σ}(Ω)$$

and

$$(L^{s_1}_{w}((0, \infty)), L^{s_2}_{w}((0, \infty)))_{θ,1} = L^{1}((0, \infty)).$$
Thus, applying real interpolation theorem for the operator $A$ (see for example, [1, Section 5.3], [19]) we obtain $v \in L^1((0, \infty))$ and $\|v\|_{L^1} \leq M\|\varphi\|_{\frac{3}{2}, 1}$. So,
\[
\int_0^{\infty} \|\nabla e^{-\tau \mathcal{L} \varphi}\|_{3, 1} d\tau \leq M\|\varphi\|_{\frac{3}{2}, 1}
\]
with the constant $M$ independent of $\varphi$. This yields
\[
|\langle (Tz)(t) - \bar{z}_0(t), \varphi \rangle| \leq M\|G(z)\|_{\infty, \frac{3}{2}} \|\varphi\|_{\frac{3}{2}, 1}
\]
for all $t > 0$ and $\varphi \in L^3_{\omega}(\Omega)$. Hence, we have $(Tz)(t) \in L^3_{\omega}(\Omega)$ for all $t > 0$ and
\[
\|(Tz)(t) - \bar{z}_0(t)\|_{3, w} \leq M\|G(z)\|_{\infty, \frac{3}{2}, w} \quad \text{for all } t > 0.
\]
Combining the above inequality with (19), we obtain
\[
\|Tz - \hat{z}_0\|_{\infty, 3, w} \leq M\|F\|_{\infty, \frac{3}{2}} + \|\nabla b_\omega + b_\omega \otimes (\omega \times x - k e_3) + F_\omega - b_\omega \otimes b_\omega\|_{\frac{3}{2}, w} + 2^{\frac{2}{3}}(\rho + M\|z_0\|_{3, w})^2 + 2^{\frac{2}{3}}|b_\omega|_{3, w}(\rho + M\|z_0\|_{3, w}). \tag{21}
\]
Note that we have
\[
\|\nabla b_\omega + b_\omega \otimes (\omega \times x - k e_3) + F_\omega - b_\omega \otimes b_\omega\|_{\frac{3}{2}, w} \leq C_0\|b_\omega\|_{3, w} \leq C_0(|a| + |k|). \tag{22}
\]
For $t_2 > t_1 > 0$, we have
\[
\langle (Tz)(t_2) - (Tz)(t_1), \varphi \rangle = \langle e^{-t_2 \mathcal{L} \varphi}z_{(0)} - e^{-t_1 \mathcal{L} \varphi}z_{(0)}, \varphi \rangle
\]
\[
- \int_0^{t_1} \langle G(z)(t_2 - \tau) - G(z)(t_1 - \tau), \nabla e^{-\tau \mathcal{L} \varphi} \rangle d\tau
\]
\[
- \int_{t_1}^{t_2} \langle G(z)(t_2 - \tau), \nabla e^{-\tau \mathcal{L} \varphi} \rangle d\tau
\]
By (20), we obtain
\[
\|(Tz)(t_2) - (Tz)(t_1)\|_{3, w} \leq \|z_{(0)}(t_2) - z_{(0)}(t_1)\|_{3, w}
\]
\[
+ t_1 M \sup_{\tau \in [0, t_1]} \|G(z)(t_2 - \tau) - G(z)(t_1 - \tau)\|_{\frac{3}{2}, w}
\]
\[
+ M\|G(z)\|_{\infty, \frac{3}{2}, w} |t_2 - t_1|.
\]
By the functions $z_0$ and $G(z)$ are continuous on $\mathbb{R}_+$, so the function $Tz$ is also continuous. Thus, $Tz \in C_b(\mathbb{R}_+ L^3_{\sigma, w}(\Omega))$.

Next, we prove that $T$ is a contractive mapping. Indeed,
\[
\langle (Tz)(t) - (Tu)(t), \varphi \rangle = - \int_0^t \langle G(z)(\tau) - G(u)(\tau), \nabla e^{-(t-\tau) \mathcal{L} \varphi} \rangle d\tau.
\]
By arguing similarly as above, we obtain
\[
\|Tz - Tu\|_{\infty, 3, w} \leq 2^{\frac{2}{3}} M(\rho + M\|z_0\|_{3, w} + \|b_\omega\|_{3, w}) \|z - u\|_{\infty, 3, w}. \tag{23}
\]
By (21), (23) and (22), if $\rho, \|F\|_{\infty, \frac{3}{2}, w}$, $\|b_\omega\|_{3, w}$ and $\|z_0\|_{3, w}$ are small enough then the mapping $T$ acts from $B_\rho(z_0)$ into itself and is a contractive mapping. So, Eq. (17) has a unique bounded weak mild solution in a small closed ball of $C_b(\mathbb{R}_+ L^3_{\sigma, w}(\Omega))$.

(b): Let $\hat{z}$ be bounded weak mild solution of Eq. (17) and $z \in C_b(\mathbb{R}_+ L^3_{\sigma, w}(\Omega))$ be any other weak mild solutions of Eq. (17) such that $\|z(0) - \hat{z}(0)\|_{3, w}$ is small enough.
Putting $u = z - \hat{z}$ we obtain that $u$ satisfies the equation
\[
\langle u(t), \varphi \rangle = \langle e^{-t\mathcal{L}_{a,k}}(z(0) - \hat{z}(0)), \varphi \rangle - \int_0^t \langle H(u)(\tau), \nabla e^{-(t-\tau)\mathcal{L}_{a,k}} \varphi \rangle \, d\tau
\]  
(24)
for all $\varphi \in L^3_0(\Omega)$, where
\[
H(u) = -u \otimes (u + \hat{z}) - \hat{z} \otimes u - b_w \otimes u - u \otimes b_w.
\]

For any fixed $r > 3$ we set
\[
\mathcal{M} = \left\{ v \in C_b(\mathbb{R}_+, L^3_{3,w}(\Omega)) : \sup_{t > 0} t^{\frac{2}{3}} \| v(t) \|_{r,w} < \infty \right\}
\]
edowed with the norm $\| v \|_{\mathcal{M}} := \| v \|_{\infty,3,w} + \sup_{t > 0} t^{\frac{1}{3} - \frac{2}{3}} \| v(t) \|_{r,w}$.

We will prove that if $\| b_w \|_{3,w}$, $\| z(0) - \hat{z}(0) \|_{3,w}$ and $\| \hat{z} \|_{\infty,3,w}$ are small enough then the Eq. (24) has a unique solution in a small closed ball of $\mathcal{M}$.

Indeed, for $u \in \mathcal{M}$ consider the mapping $\Phi$ defined as follows
\[
\langle (\Phi u)(t), \varphi \rangle = \langle e^{-t\mathcal{L}_{a,k}}(z(0) - \hat{z}(0)), \varphi \rangle - \int_0^t \langle H(u)(\tau), \nabla e^{-(t-\tau)\mathcal{L}_{a,k}} \varphi \rangle \, d\tau.
\]

Let $B_\rho$ be a closed ball in $\mathcal{M}$ centered at 0 with radius $\rho$. We then prove that if $\rho$, $\| b_w \|_{3,w}$, $\| z(0) - \hat{z}(0) \|_{3,w}$ and $\| \hat{z} \|_{\infty,3,w}$ are small enough then the mapping $\Phi$ acts from $B_\rho$ into itself and is a contraction. Arguing similarly as in the proof of Assertion (a) we have $\Phi u \in C_b(\mathbb{R}_+, L^3_{3,w}(\Omega))$. Furthermore,
\[
\langle t^{\frac{2}{3}} - \frac{2}{3} \Phi u(t), \varphi \rangle = \langle t^{\frac{2}{3}} - \frac{2}{3} e^{-t\mathcal{L}_{a,k}}(z(0) - \hat{z}(0)), \varphi \rangle - \int_0^t \langle H(u)(\tau), \nabla e^{-(t-\tau)\mathcal{L}_{a,k}} \varphi \rangle \, d\tau.
\]

By $L^{r,\infty} - L^{3,\infty}$ estimate for semigroup $e^{-t\mathcal{L}_{a,k}}$ (see (7)) we obtain that
\[
\| t^{\frac{2}{3}} - \frac{2}{3} e^{-t\mathcal{L}_{a,k}}(z(0) - \hat{z}(0)) \|_{r,w} \leq M \| z(0) - \hat{z}(0) \|_{3,w}.
\]

For each $t > 0$, we have
\[
\left| \int_0^t \langle -H(u)(t - \tau), \nabla e^{-\tau\mathcal{L}_{a,k}} \varphi \rangle \, d\tau \right| \leq \int_0^t \left| \langle -H(u)(t - \tau), \nabla e^{-\tau\mathcal{L}_{a,k}} \varphi \rangle \right| \, d\tau \leq \int_0^t \left| \langle H(u)(t - \tau), \nabla e^{\tau\mathcal{L}_{a,k}} \varphi \rangle \right| \, d\tau + \int_0^t \left| \langle -H(u)(t - \tau), \nabla e^{-\tau\mathcal{L}_{a,k}} \varphi \rangle \right| \, d\tau.
\]  
(25)

We have
\[
\int_0^t \left| \langle -H(u)(t - \tau), \nabla e^{-\tau\mathcal{L}_{a,k}} \varphi \rangle \right| \, d\tau \leq \int_0^t \| H(u)(t - \tau) \|_{\frac{3}{3r-1},w} \| \nabla e^{\tau\mathcal{L}_{a,k}} \varphi \|_{\frac{3r}{3r-1}} \, d\tau \leq 2^{\frac{3}{3r-1}}
\]
\[
\int_0^t \| u(t - \tau) \|_{3,w} + 2 \| b_w \|_{3,w} + 2 \| \hat{z}(t - \tau) \|_{3,w} \| u(t - \tau) \|_{r,w} \| \nabla e^{-\tau\mathcal{L}_{a,k}} \varphi \|_{\frac{3r}{3r-1}} \, d\tau
\]

Almost periodic weak mild solutions to Navier-Stokes-Oseen equation.

Assuming that the second-order tensor field $\phi$ is almost periodic on the whole line, we have unique solution in $B^1$ mapping $\Phi$ acts from $L^r_t\mathfrak{X}$ into itself and is a contraction. Then the equation $\Phi r, w \leq \frac{1}{t^2 + \frac{1}{t}}$ for all $r, w$. Therefore, we obtain the polynomial stability of $\Phi$.

Combining (25), (26) and (27) we obtain

$$\int_0^t \left| - H(u)(t - \tau), \nabla e^{-\tau \mathcal{L}} \phi \right| d\tau \leq M \left( \frac{t}{2} \right)^{-\frac{1}{2} + \frac{1}{2r}} (\| u \|_M + 2 \| b_\omega \|_{3, \infty} + 2 \| \hat{z} \|_{3, \infty, 3, w}) \| u \|_M \| \phi \|_{\frac{t}{t^2 + \frac{1}{t}}, 1}.$$  

We also have

$$\int_0^t \left| - H(u)(t - \tau), \nabla e^{-\tau \mathcal{L}} \phi \right| d\tau \leq \int_0^t \| H(u)(t - \tau) \|_{\frac{1}{2}, 3, w} \| \nabla e^{-\tau \mathcal{L}} \phi \|_{3, 1} d\tau \leq M_2 (\| u \|_M + 2 \| b_\omega \|_{3, \infty} + 2 \| \hat{z} \|_{3, \infty, 3, w}) \| u \|_M \int_0^\infty \tau^{-\frac{1}{2} + \frac{1}{2r}} \| \phi \|_{\frac{t}{t^2 + \frac{1}{t}}, 1} d\tau \leq M_2 t^{-\frac{1}{2} + \frac{1}{2r}} (\| u \|_M + 2 \| b_\omega \|_{3, \infty} + 2 \| \hat{z} \|_{3, \infty, 3, w}) \| u \|_M \| \phi \|_{\frac{t}{t^2 + \frac{1}{t}}, 1}. \quad (27)$$

Combining (25), (26) and (27) we obtain

$$\int_0^t \left| - H(u)(t - \tau), \nabla e^{-\tau \mathcal{L}} \phi \right| d\tau \leq \bar{M} t^{-\frac{1}{2} + \frac{1}{2r}} (\| u \|_M + 2 \| b_\omega \|_{3, \infty} + 2 \| \hat{z} \|_{3, \infty, 3, w}) \| u \|_M \| \phi \|_{\frac{t}{t^2 + \frac{1}{t}}, 1}$$

for all $\phi \in C^\infty_{0, \sigma}$. It follows that

$$\| \Phi u \|_M \leq \bar{M} \| \hat{z}(0) - \hat{z}(0) \|_{3, \infty} + \bar{M} \left( \| u \|_M + 2 \| b_\omega \|_{3, \infty} + 2 \| \hat{z} \|_{3, \infty, 3, w} \right) \| u \|_M.$$

Similar calculations lead to

$$\| \Phi u_1 - \Phi u_2 \|_M \leq \bar{M} \left( \| u_1 \|_M + \| u_2 \|_M + 2 \| b_\omega \|_{3, \infty} + 2 \| \hat{z} \|_{3, \infty, 3, w} \right) \| u_1 - u_2 \|_M$$

for all $u_1, u_2 \in M$.

Therefore, if $\rho, \| b_\omega \|_{3, \infty}, \| z(0) - \hat{z}(0) \|_{3, \infty}$ and $\| \hat{z} \|_{3, \infty, 3, w}$ are small enough then the mapping $\Phi$ acts from $B_\rho$ into itself and is a contraction. Then the equation $\Phi u = u$ has unique solution in $M$. Since $u = \hat{z} - \hat{z}$, we obtain the polynomial stability of the bounded weak mild solution $\hat{z}$ and the inequality

$$\| z(t) - \hat{z}(t) \|_{r, w, e} \leq Ct^{-\frac{1}{2} + \frac{1}{2r}}$$

for all $t > 0$.

\[\square\]

3. Almost periodic weak mild solutions to Navier-Stokes-Oseen equation. Assuming that the second-order tensor field $F$ is almost periodic on the whole line, we then show the existence and uniqueness of an almost periodic weak mild solutions for the Navier-Stokes-Oseen equation. Firstly, we recall the following definition of almost periodic functions.
**Definition 3.1.** Let $X$ be a Banach space. A continuous function $f : \mathbb{R} \to X$ is called almost periodic if for all $\epsilon > 0$ there exists a real number $L_\epsilon > 0$ such that for every $a \in \mathbb{R}$, we can find $T \in [a, a + L_\epsilon]$ such that $\|f(t + T) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$.

**Remark 1.** From definition we easily see that an almost periodic function is bounded and uniformly continuous. Details on almost periodic functions can be found in [21].

To prove the uniqueness of an almost periodic weak mild solution, we need the following lemma (can see [25, Lemma 3.2]).

**Lemma 3.2.** If the function $f : \mathbb{R} \to X$ is almost periodic and $\lim_{t \to \pm \infty} f(t) = 0$ then $f \equiv 0$ on $\mathbb{R}$.

**Proof.** Since $\lim_{t \to \pm \infty} f(t) = 0$ so for all $\epsilon > 0$ there exists a real number $M = M(\epsilon) > 0$ such that

$$\|f(t)\| < \epsilon \text{ for all } t \geq M.$$  

Pick up fixed $t_1 \in \mathbb{R}$. Choose $a = M + |t_1|$, there exists $T \in [a, a + L_\epsilon]$ such that $\|f(t + T) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$. Then, we have

$$\|f(t_1) - f(0)\| \leq \|f(t_1) - f(t_1 + T)\| + \|f(T) - f(0)\| + \|f(t_1 + T)\| + \|f(T)\| < 4\epsilon.$$  

Therefore, $f(t_1) = f(0)$ for all $t_1 \in \mathbb{R}$. So, $f \equiv 0$ on $\mathbb{R}$. \hfill \Box

We then consider the inhomogeneous equation

$$D_t u + L_{a,k} u = P \div F(t), \quad t \in \mathbb{R}. \quad (28)$$

Similarly to the case of semilinear equation, a weak mild solution to (28) that means the solution to the integral equation

$$\langle u(t), \varphi \rangle = \langle e^{-(t-s)L_{a,k}} u(s), \varphi \rangle - \int_s^t \langle F(\tau), \nabla e^{-(t-\tau)L_{a,k}} \varphi \rangle \, d\tau \quad \text{for} \quad t \geq s.$$

**Lemma 3.3.** Suppose that the second-order tensor field $F \in C_b(\mathbb{R}, L^3_{3,\infty}(\Omega)^{3 \times 3})$ is almost periodic. Then, Eq. (28) has a unique almost periodic weak mild solution in $C_b(\mathbb{R}, L^3_{3,\infty}(\Omega))$, and this solution has form

$$\langle u(t), \varphi \rangle = \int_0^\infty \langle -F(t-\tau), \nabla e^{-\tau L_{a,k}} \varphi \rangle \, d\tau = \int_{-\infty}^t \langle -F(\tau), \nabla e^{-(t-\tau)L_{a,k}} \varphi \rangle \, d\tau$$

for all $t \in \mathbb{R}$.

**Proof.** Indeed, for each $\varphi \in L^3_{3,1}(\Omega)$, we have

$$\left| \int_0^\infty \langle -F(t-\tau), \nabla e^{-\tau L_{a,k}} \varphi \rangle \, d\tau \right| \leq \|F\|_{3,\infty} \int_0^\infty \|\nabla e^{-\tau L_{a,k}} \varphi\|_{3,1} \, d\tau.$$  

On the other hand,

$$\int_0^\infty \|\nabla e^{-\tau L_{a,k}} \varphi\|_{3,1} \, d\tau \leq M \|\varphi\|_{3,1} \quad (\text{see (20)}).$$

Therefore, we have

$$\left| \int_0^\infty \langle -F(t-\tau), \nabla e^{-\tau L_{a,k}} \varphi \rangle \, d\tau \right| \leq M \|F\|_{3,\infty} \|\varphi\|_{3,1}.$$
Thus, $u(t)$ is uniformly bounded on $\mathbb{R}$, and
\[ \|u\|_{\infty, 3, w} \leq M\|F\|_{\infty, \frac{3}{2}, w}. \] (29)
It is straightforward to check that $u(t)$ is continuous on $\mathbb{R}$ and that for fixed $s \in \mathbb{R}$, $u(t)$ satisfies the equation
\[ \langle u(t), \varphi \rangle = \langle e^{-(t-s)\mathcal{L}_{a,k}}u(s), \varphi \rangle - \int_s^t \langle F(\tau), \nabla e^{-(t-\tau)\mathcal{L}_{a,k}} \varphi \rangle d\tau \quad \text{for} \quad t \geq s, \]
i.e., $u(t)$ is a weak mild solution to (28).

Since $F$ is almost periodic so for all $\epsilon > 0$ there exists a real number $L_\epsilon > 0$ such that for every $a \in \mathbb{R}$, we can find $T \in [a, a + L_\epsilon]$ such that $\|F(t + T) - F(t)\|_{\frac{3}{2}, w} < \epsilon$ for all $t \in \mathbb{R}$. Then, we have
\[ \|u(t + T) - u(t)\|_{3, w} \]
\[ = \sup_{\|\varphi\|_{\frac{3}{2}, 1} \leq 1} \left| \int_0^\infty \langle F(t + T - \tau) - F(t - \tau), \nabla e^{-\tau \mathcal{L}_{a,k}} \varphi \rangle d\tau \right| \]
\[ \leq M\|F(\cdot + T) - F(\cdot)\|_{\infty, \frac{3}{2}, w} \]
\[ \leq \epsilon M \text{ for all } t \in \mathbb{R}. \]
Therefore, $u(t)$ is almost periodic weak mild solution of Eq. (28).

Next we prove the uniqueness of the almost periodic weak mild solution. To do this, assume that $v(t) \in C_b(\mathbb{R}, \mathcal{L}^3_{\sigma,w}(\Omega))$ is another almost periodic weak mild solution to (28). Then, $z(t) = u(t) - v(t)$ is almost periodic function in $C_b(\mathbb{R}, \mathcal{L}^3_{\sigma,w}(\Omega))$, and for any fixed $s \in \mathbb{R}$ it holds that $z(t) = e^{-(t-s)\mathcal{L}_{a,k}}z(s)$ for $t \geq s$. By the estimate (7), for $r > 3$ then we have
\[ \|z(t + T) - z(t)\|_{r, w} = \|e^{-k|T|\mathcal{L}_{a,k}}(z(t + T - k|T|) - z(t - k|T|))\|_{r, w} \]
\[ \leq M(k|T|)^{-\frac{3}{2}}(k^{-\frac{3}{2}}) \|z(t + T - k|T|) - z(t - k|T|)\|_{3, w}. \]
With $T \neq 0$, there exists $k \geq 1$ such that $k|T| \geq 1$. Thus, $z(t)$ is also almost periodic function in the space $C_b(\mathbb{R}, \mathcal{L}^3_{\sigma,w}(\Omega))$. By the estimate (7), we have
\[ \lim_{t \to +\infty} \|z(t)\|_{r, w} = 0. \]
By Lemma 3.2, we have $z(t) = 0$ on $\mathbb{R}$. So, $u \equiv v$. \hfill \Box

We will now prove the existence and uniqueness of an almost periodic weak mild solution to the Navier-Stokes-Oseen equation on the whole line:
\[ D_t z + \mathcal{L}_{a,k} z = \mathbb{P} \text{div} G(z), \quad t \in \mathbb{R}, \] (30)
where
\[ G(z) = F + \nabla b_\omega + b_\omega \otimes (\omega \times x - ke_3) + F_\omega - z \otimes z - z \otimes b_\omega - b_\omega \otimes b_\omega. \]

**Theorem 3.4.** Suppose that the second-order tensor field $F \in C_b(\mathbb{R}, \mathcal{L}^3_{\sigma,w}(\Omega)^{3 \times 3})$ is almost periodic. Then, the following assertions hold true.
(a) If the norm $\|F\|_{\infty, \frac{3}{2}, w}$ and $\|b_\omega\|_{3, w}$ are small enough then Eq. (30) has a unique almost periodic weak mild solution in a small closed ball of $C_b(\mathbb{R}, \mathcal{L}^3_{\sigma,w}(\Omega))$.
(b) The almost periodic weak mild solution $\tilde{z}$ of Eq. (30) is polynomial stability in the sense that for any other weak mild solution $z \in C_b(\mathbb{R}^+, \mathcal{L}^3_{\sigma,w}(\Omega))$ of Eq. (30) such that $\|z(0) - \tilde{z}(0)\|_{3, w}$ is small enough, we have
\[ \|z(t) - \tilde{z}(t)\|_{r, w} \leq C t^{-\frac{1}{2} + \frac{1}{2r}} \text{ for all } t > 0, \]
where $r$ is any fixed number satisfying $r > 3$.

Proof. Since the almost periodic weak mild solution $\hat{z}$ is bounded, assertion (b) follows directly from the part (b) of Theorem 2.1. Hence, we need only to prove assertion (a).

Consider the following closed set

$$B_\rho = \{ v \in C_b(\mathbb{R}, L^3_{3,ω}(Ω)) : v \text{ is almost periodic and } ||v||_{3,3,w} \leq \rho \}.$$ 

Since $F \in C_b(\mathbb{R}, L^3_{3,ω}(Ω)^{3×3})$ is almost periodic it follows that for each $v \in B_\rho$, $G(v)$ belongs to the space $C_b(\mathbb{R}, L^3_{3,ω}(Ω)^{3×3})$ and is almost periodic. For each $v \in B_\rho$ we define the map $T$ as follows

$$Tv = z$$

where $z \in C_b(\mathbb{R}, L^3_{3,ω}(Ω))$ is the unique almost periodic weak mild solution to the equation

$$D_t z + \mathcal{L}_{a,k}z = P \text{ div } G(v). \quad (31)$$

The existence and uniqueness of $z$ are guaranteed by Lemma 3.3. Moreover, by (29) we have $||z||_{3,3,w} \leq M||G(v)||_{3,3,w}$. In a same way as in (19), we obtain

$$||G(v)||_{3,3,w} \leq ||F||_{3,3,w} + ||\nabla b_ω + b_ω \otimes (ω × x - ke_3) + F_ω - b_ω \otimes b_ω||_{3,3,w}
+ 2^\frac{5}{2} \rho^2 + 2^\frac{5}{2} \rho ||b_ω||_{3,3,w}.$$ 

Note that

$$||\nabla b_ω + b_ω \otimes (ω × x - ke_3) + F_ω - b_ω \otimes b_ω||_{3,3,w} \leq C_0 ||b_ω||_{3,3,w} \leq C_0 (|a| + |k|).$$

Therefore, if the norm $||F||_{3,3,w}$ and $||b_ω||_{3,3,w}$ are small enough then $T$ acts from $B_\rho$ into itself.

For $v_1, v_2 \in B_\rho$, by Lemma 3.3 for $Tv_1 = z_1$ and $Tv_2 = z_2$ we obtain

$$||z_1 - z_2||_{3,3,w} \leq M||G(v_1) - G(v_2)||_{3,3,w}
\leq 2^\frac{5}{2} M(\rho + ||b_ω||_{3,3,w})||v_1 - v_2||_{3,3,w}.$$ 

So, if $\rho, ||F||_{3,3,w}$ and $||b_ω||_{3,3,w}$ are small enough then the mapping $T$ acts from $B_\rho$ into itself and is a contractive mapping. Therefore, Eq. (30) has uniqueness of almost periodic weak mild solution in a small closed ball of $C_b(\mathbb{R}, L^3_{3,ω}(Ω))$. \hfill \Box

If the function $F(t)$ is periodic then the solution $u(t)$ in the Lemma 3.3 is also periodic and has the same period. Moreover, a periodic function on $\mathbb{R}_+$ can be extended to become a periodic function on $\mathbb{R}$. Thus without loss of generality, we assume that $F$ is periodic on $\mathbb{R}$. Then, we obtain the following corollary for the unique existence and polynomial stability of periodic weak mild solution to Eq. (30).

**Corollary 1.** Suppose that the second-order tensor field $F \in C_b(\mathbb{R}, L^3_{3,ω}(Ω)^{3×3})$ is periodic. Then, the following assertions hold true.

(a) If the norm $||F||_{3,3,w}$ and $||b_ω||_{3,3,w}$ are small enough then Eq. (30) has an uniquely periodic weak mild solution in a small closed ball of $C_b(\mathbb{R}, L^3_{3,ω,w}(Ω))$.

(b) The periodic weak mild solution $\hat{z}$ of Eq. (30) is polynomial stability in the sense that for any other weak mild solution $z \in C_b(\mathbb{R}_+, L^3_{3,ω,w}(Ω))$ of Eq. (30) such that $||z(0) - \hat{z}(0)||_{3,3,w}$ is small enough, we have

$$||z(t) - \hat{z}(t)||_{3,3,w} \leq Ct^{-\frac{1}{2} + \frac{1}{2\rho}} \text{ for all } t > 0.$$
where \( r \) is any fixed number satisfying \( r > 3 \).

The under viewpoint, a stationary solution is considered as periodic solution with arbitrary period. On the other hand, by Corollary 1 for each the periodic function \( F(t) \) then Eq. (30) having unique solution \( u(t) \) is periodic and has the same period. So, if the function \( F(t) \) is time-independent then Eq. (30) has an uniquely weak stationary solution and this solution is polynomial stability.

**Corollary 2.** Let second-order tensor field \( F \in L_{3,w}^{3,3}(\Omega) \). Then, the following assertions hold true.

(a) If the norm \( \| F \|_{3,w} \) and \( \| b_w \|_{3,w} \) are small enough then Eq. (30) has an uniquely weak stationary solution in a small closed ball of \( L_{3,w}^{3,3}(\Omega) \).

(b) The weak stationary solution \( \hat{z} \) of Eq. (30) is polynomial stability in the sense that if \( x(0) \) is any fixed number satisfying \( x \) is small enough, we have

\[
\| z(t) - \hat{z} \|_{3,w} \leq C t^{\frac{3}{2} + \frac{r}{w}} \quad \text{for all} \quad t > 0,
\]

where \( r \) is any fixed number satisfying \( r > 3 \).

4. **Time-local mild solutions to Navier-Stokes-Oseen equation.** In order to easily track, we restate Navier-Stokes-Oseen equation (see (17)) under the following form

\[
\begin{align*}
D_t z + \mathcal{L}_{a,k} z &= \mathbb{P} \div G_1(z) + \mathbb{P} f(t), \quad t > 0, \\
|z|_{t=0} &= z_0 \in L_{\sigma}^{3,q}(\Omega), \quad 1 < q < \infty,
\end{align*}
\]

(32)

where

\[
G_1(z) = \nabla b_\omega + b_\omega \otimes (\omega \times x - k e_3) + F_\omega - z \otimes z - b_\omega \otimes z - z \otimes b_\omega - b_\omega \otimes b_\omega.
\]

In this section we prove that Eq. (32) has time-local mild solution in the solenoidal Lorentz space \( L_{\sigma}^{3,q} \), \( 1 < q < \infty \), that means \( z(t) \) is solution of the integral equation

\[
z(t) = e^{-t \mathcal{L}_{a,k}} z_0 + \int_0^t e^{-(t-\tau) \mathcal{L}_{a,k}} \mathbb{P} [f(\tau) + \div G_1(z(\tau))] d\tau
\]

(33)

on \([0, T]\), with \( T > 0 \) is small enough. Here, we use Kato iteration scheme combining with interpolation theory and duality estimates to prove the existence of time-local mild solution.

**Theorem 4.1.** Assume that \( f \in C(\mathbb{R}_+, L_{\sigma}^{3,q}(\Omega)) \). Then, for each \( r > 3 \) there exists \( T = T(r, q) > 0 \) such that \( z(t) \in C([0, T], L_{\sigma}^{3,q}(\Omega)) \) is unique solution of Eq. (33) on \([0, T]\) and satisfies

\[
t^{\frac{3}{2}} \| z(t) \|_{L_{\sigma}^{3,q}(\Omega)} \in C([0, T], L_{\sigma}^{3,q}(\Omega)).
\]

**Proof.** Since \( b_\omega \in C_{c}^{\infty}(\mathbb{R}^3) \) so we have

\[
H_\omega := [\nabla b_\omega + b_\omega \otimes (\omega \times x - k e_3) + F_\omega - b_\omega \otimes b_\omega] \in C_{c}^{\infty}(\mathbb{R}^3)^{3 \times 3}.
\]

In the functional space \( C(\mathbb{R}_+, L_{\sigma}^{3,q}(\Omega)) \), we set the following iterative sequence

\[
z_1(t) = e^{-t \mathcal{L}_{a,k}} z_0,
\]

\[
z_j(t) = z_1(t) + \int_0^t e^{-\tau \mathcal{L}_{a,k}} \mathbb{P} [f(\tau) + \div G_1(z_{j-1}(\tau))] d\tau, \quad j \geq 2.
\]
Lemma 1.1 we have
\[ \phi \]
where
\[ \| \cdot \|_{3,q} \]
and
\[ \int_0^t \frac{2}{r^3} t^{\frac{1}{2} + \frac{q}{r}} \sigma d\tau \]
Using Proposition 1 (i)-(iv), we obtain the following estimates.
\[ 0 \leq M \| \|_{3,q} 2 \frac{r}{r+3} t^{\frac{1}{2} + \frac{q}{r}}. \]
For \( \varphi \in C^\infty_{0,a}(\mathbb{R}^3) \), using Hölder type inequality and embedding property in Lemma 1.1 we have
\[ \langle G_1(z_j-1)(\tau), \nabla e^{-(t-\tau)}L_{a,k} \varphi \rangle \leq \| G_1(z_j-1)(\tau) \|_{\frac{3}{r+3},q} \| \nabla e^{-(t-\tau)}L_{a,k} \varphi \|_{\frac{3}{r+3},q} \]
\[ \leq \left( \frac{q}{3} \right)^{\frac{1}{q}} \| z_j-1(\tau) \|_{r+3,q} (\| z_j-1(\tau) \|_{3,q} + 2 \| b_\omega \|_{3,q}) + \| H_\omega \|_{\frac{3}{r+3},q} \]
\[ M \| \varphi \|_{\frac{3}{r+3},q} (t - \tau)^{-\frac{1}{2} - \frac{q}{r}}, \]
and
\[ \langle G_1(z_j-1)(\tau), \nabla e^{-(t-\tau)}L_{a,k}^r \varphi \rangle \leq \| G_1(z_j-1)(\tau) \|_{\frac{3}{r+3},q} \| \nabla e^{-(t-\tau)}L_{a,k}^r \varphi \|_{\frac{3}{r+3},q} \]
\[ \leq \left( \frac{q}{r} \right)^{\frac{1}{q}} \| z_j-1(\tau) \|_{r+3,q} (\| z_j-1(\tau) \|_{r+3,q} + 2 \| b_\omega \|_{r+3,q}) + \| H_\omega \|_{\frac{3}{r+3},q} \]
\[ M \| \varphi \|_{\frac{3}{r+3},q} (t - \tau)^{-\frac{1}{2} - \frac{q}{r}}. \]
Therefore,
\[ \| \int_0^t e^{-(t-\tau)L_{a,k}} \mathbb{P} \text{div} G_1(z_j-1)(\tau) \|_{3,q} \]
\[ \leq K_j' \| K_j' + \| z_0 \|_{3,q} + 2 \| b_0 \|_{3,q} \| M \left( \frac{q}{3} \right)^{\frac{1}{q}} \]
\[ \int_0^t \tau^{\frac{1}{2} + \frac{q}{r}} (t - \tau)^{-\frac{1}{2} - \frac{q}{r}} d\tau + M \left( \frac{q}{3} \right)^{\frac{1}{q}} \| H_\omega \|_{\frac{3}{r+3},q} \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{q}{r}} d\tau \]
\[ = M \left( \frac{q}{3} \right)^{\frac{1}{q}} B \left( \frac{1}{2} + \frac{3}{2r}, \frac{1}{2} - \frac{3}{2r} \right) K_j' \| K_j' + \| z_0 \|_{3,q} + 2 \| b_\omega \|_{3,q} \| \]
\[ + M \left( \frac{q}{3} \right)^{\frac{1}{q}} \| H_\omega \|_{\frac{3}{r+3},q} \frac{2r}{r+3} t^{\frac{1}{2} + \frac{q}{r}}, \]
and
\[ \| \int_0^t e^{-(t-\tau)L_{a,k}} \mathbb{P} \text{div} G_1(z_j-1)(\tau) \|_{r+3,q} \leq M \left( \frac{q}{3} \right)^{\frac{1}{q}} K_j'^2 \| K_j' + \| z_0 \|_{3,q} + 2 \| b_\omega \|_{3,q} \| \]
\[ + 2M \left( \frac{q}{3} \right)^{\frac{1}{q}} \| b_\omega \|_{r+3,q} \int_0^t \tau^{\frac{1}{2} + \frac{q}{r}} (t - \tau)^{-\frac{1}{2} - \frac{q}{r}} d\tau \]
\[ + M\left(\frac{q}{r}\right)^{\frac{r}{q}}\|H_\omega\|_{L^q,q}\int_0^t (t-\tau)^{-\frac{\gamma}{q}} d\tau \]
\[ = M\left(\frac{q}{r}\right)^{\frac{r}{q}} B\left(\frac{3}{r}, \frac{1}{2}, \frac{3}{2r}\right) K_{j-1}^q t^{-\frac{\gamma+\frac{R}{q}}{r}} + 2 M\left(\frac{q}{r}\right)^{\frac{r}{q}} \|b_\omega\|_{r,q} B\left(\frac{3}{2r}, \frac{1}{2}, \frac{3}{2r}\right) K_{j-1}^q + M\left(\frac{q}{r}\right)^{\frac{r}{q}} \|H_\omega\|_{L^q,q} \frac{2r}{r-3} t^{\frac{1}{2}-\frac{\gamma}{r}}, \]
where \(B(\cdot, \cdot)\) is the Beta functions. So,
\[ K_j \leq K_1 + A_1 T + A_2 K_{j-1}^q (K_{j-1} + A_3) + A_4 T^{\frac{1}{2} - \frac{\gamma}{r}}, \]
and
\[ K_{j}^q \leq K_1^q + C_1 T + C_2 K_{j-1}^{qq} + C_3 K_{j-1}^{q} T^{\frac{1}{2} - \frac{\gamma}{r}} + C_4 T^{1 - \frac{r}{2}}, \]
in which
\[ A_1 = M\left(\frac{q}{r}\right)^{\frac{r}{q}} \|P\|_{L^\infty,3,q}, \quad A_2 = M\left(\frac{q}{r}\right)^{\frac{r}{q}} B\left(\frac{3}{r}, \frac{1}{2}, \frac{3}{2r}\right), \]
\[ A_3 = \left(\frac{q}{r}\right)^{\frac{r}{q}} \|z_0\|_{3,q} + 2 \|b_\omega\|_{3,q}, \quad A_4 = \frac{2Mr}{r-3} \left(\frac{q}{r}\right)^{\frac{r}{q}} \|H_\omega\|_{L^q,q}, \]
\[ C_1 = \frac{2Mr}{r+3} \left(\frac{q}{r}\right)^{\frac{r}{q}} \|P\|_{L^\infty,3,q}, \quad C_2 = M\left(\frac{q}{r}\right)^{\frac{r}{q}} B\left(\frac{3}{r}, \frac{1}{2}, \frac{3}{2r}\right), \]
\[ C_3 = \frac{2Mr}{r-3} \left(\frac{q}{r}\right)^{\frac{r}{q}} \|b_\omega\|_{r,q} B\left(\frac{3}{2r}, \frac{1}{2}, \frac{3}{2r}\right), \quad C_4 = \frac{2Mr}{r-3} \left(\frac{q}{r}\right)^{\frac{r}{q}} \|H_\omega\|_{L^q,q}. \]
By (11) and \(e^{-tC_{a+}}\) is \(C_0\)-semigroup, for every \(\lambda > 0\) there exists \(T_0 > 0\) such that
\[ K_1^q, K_1 < \lambda \text{ for all } T \leq T_0. \]
Pick up fixed \(\lambda < \min\{\frac{1}{2C_0}, \frac{1}{2C_1}\}\) and then choose \(T \leq T^*\), in which
\[ T^* = \min\left\{T_0, \frac{\lambda}{2C_1}, \left(\frac{\lambda}{2C_4}\right)^{\frac{r-\gamma}{r}}, \left(\frac{1}{3C_3}\right)^{\frac{2r}{r-\gamma}}\right\}. \]
The next step, we prove that \(\{z_j(t)\}\) is Cauchy sequence in the Banach space \(M_{T^*}\). In which,
\[ M_{T^*} = \left\{ v \in C([0,T], L^{3,q}_r(\Omega)) : t^{\frac{1}{2} - \frac{\gamma}{r}} v(t) \in C([0,T], L^{r,q}_o(\Omega)) \right\} \]
edowed with the norm \(\|v\|_{M_{T^*}} := \max_{t \in [0,T]} \|v\|_{3,q,q} + \max_{t \in [0,T]} t^{\frac{1}{2} - \frac{\gamma}{r}} \|v(t)\|_{r,q,q}. \) Indeed, setting
\[ L_j = \max\{\|z_{j+1}(t) - z_j(t)\|_{3,q}, t \in [0,T]\}, \]
\[ L_j' = \max\{t^{\frac{1}{2} - \frac{\gamma}{r}} \|z_{j+1}(t) - z_j(t)\|_{r,q}, t \in [0,T]\}, \quad j \geq 1. \]
The same argument as above, we obtain
\[ L_j \leq \left(8\lambda A_2 + \frac{4Mr}{r-3} \|b_\omega\|_{r,q} \left(\frac{q}{r}\right)^{\frac{r}{q}} T^{\frac{1}{2} - \frac{\gamma}{r}} \right) L_j, \]
\[ L_j' \leq (8\lambda C_2 + 4\lambda C_3 T^{\frac{1}{2} - \frac{\gamma}{r}}) L_j', \quad j \geq 2. \]
Therefore,
\[ \|z_{j+1} - z_j\|_{M_{T^*}} \leq d \|z_j - z_{j-1}\|_{M_{T^*}}, \]
with
\[ d = \max \left\{ 8\lambda A_2 + \frac{4Mr\|b_\omega\|_{r,q}}{r - 3} \left( \frac{q}{r} \right)^{\frac{1}{2}} T^{\frac{1}{2} - \frac{3}{2r}} + 8\lambda C_2 + 4\lambda C_3 T^{\frac{1}{2} - \frac{3}{2r}} \right\} < \frac{1}{2} \]
if pick up fixed \( \lambda < \min\left\{ \frac{1}{24C_2}, \frac{1}{32A_2}, \frac{1}{3} \right\} \) and then choose \( T \) such that
\[ T \leq \min \left\{ T^*, \left( \frac{r - 3}{16Mr\|b_\omega\|_{r,q}} \frac{T}{q} \right)^{\frac{3}{2r}} \right\}. \]
So, \( \{ z_j(t) \} \) is Cauchy sequence in the Banach space \( M_T \). Hence, there exists \( z(t) \in M_T \) such that \( z_j(t) \) converges to \( z(t) \).

The remainder, we show that \( z(t) \) is solution of Eq. (33). To do this, we will prove that \( \{ t^{\frac{1}{2}} \nabla z_j(t) \} \) is Cauchy sequence in the Banach space \( C([0, T], L^3, \Omega)^{3 \times 3} \).

Indeed, we observe that
\[ \text{div} G_1(z_{j-1})(\tau) = \text{div} H_\omega - (z_{j-1} \cdot \nabla) b_\omega - (b_\omega \cdot \nabla) z_{j-1} - (z_{j-1} \cdot \nabla) z_{j-1}, \quad j \geq 2, \]
\[ t^{\frac{1}{2}} \| \nabla z_1(t) \|_{3,q} = t^{\frac{1}{2}} \| e^{-t\mathcal{L}_{a,k}} z_0 \|_{3,q} \leq M \| z_0 \|_{3,q}, \quad \text{for all } t \in [0, T]. \]

Put
\[ Q_j = \max \{ t^{\frac{1}{2}} \| \nabla z_j(t) \|_{3,q}, \quad t \in [0, T] \}, \]
\[ P_j = \max \{ t^{\frac{1}{2}} \| \nabla z_{j+1}(t) - \nabla z_j(t) \|_{3,q}, \quad t \in [0, T] \}, \quad j \geq 1. \]

Using the estimate (8) and Lemma 1.1, we have
\[ \int_0^t \| \nabla e^{-t(\tau - \tau)\mathcal{L}_{a,k}} \mathbb{P} \left( (z_{j-1} \cdot \nabla) b_\omega + (b_\omega \cdot \nabla) z_{j-1} \right) \|_{3,q} d\tau \leq 2 \| \mathbb{P} (\| f \|_{\infty,3,q} + \| \text{div} H_\omega \|_{3,q}) t^{\frac{1}{2}}, \]
\[ \int_0^t \| \nabla e^{-t(\tau - \tau)\mathcal{L}_{a,k}} \mathbb{P} \left( z_{j-1} \cdot \nabla b_\omega \right) \|_{3,q} d\tau \leq 2 \| \mathbb{P} \| \| z_{j-1} \|_{\infty,3,q} \| \nabla b_\omega \| \infty t^{\frac{1}{2}}, \]
and
\[ \int_0^t \| \nabla e^{-t(\tau - \tau)\mathcal{L}_{a,k}} \mathbb{P} \left( (z_{j-1} \cdot \nabla) z_{j-1} + (b_\omega \cdot \nabla) z_{j-1} \right) \|_{3,q} d\tau \]
\[ \leq \| \mathbb{P} \| \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2r}} \| (z_{j-1} \cdot \nabla) z_{j-1} + (b_\omega \cdot \nabla) z_{j-1} \|_{r,q} d\tau \]
\[ \leq \| \mathbb{P} \| \left( \frac{q}{r} \right)^{\frac{1}{2}} \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2r}} \| \nabla z_{j-1}(\tau) \|_{3,q} \| z_{j-1}(\tau) \|_{r,q} d\tau \]
\[ + \| \mathbb{P} \| \| b_\omega \|_{r,w} \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2r}} \| \nabla z_{j-1}(\tau) \|_{3,q} d\tau \]
\[ \leq \| \mathbb{P} \| \left( \frac{q}{r} \right)^{\frac{1}{2}} K_{j-1} Q_{j-1} \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2r}} \| z_{j-1}(\tau) \|_{3,q} d\tau \]
\[ + \| \mathbb{P} \| \| b_\omega \|_{r,w} Q_{j-1} \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2r}} \| z_{j-1}(\tau) \|_{3,q} d\tau \]
\[ = \| \mathbb{P} \| \left( \frac{q}{r} \right)^{\frac{1}{2}} K_{j-1} Q_{j-1} B \left( \frac{3}{2r}, \frac{1}{2}, \frac{3}{2r} \right) t^{-\frac{1}{2}} + \| \mathbb{P} \| \| b_\omega \|_{r,w} Q_{j-1} B \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2r} \right) t^{-\frac{3}{2}}. \]
Therefore,
\[ Q_j \leq Q_1 + D_1 + (4D_2 \lambda + D_3 T^{\frac{1}{2} - \frac{3}{2r}}) Q_{j-1}, \quad j \geq 2, \]
where
\[ D_1 = 2\|P\|T(\|f\|_{\infty,3,q} + \|\text{div}H_u\|_{3,q}), \]
\[ + 4\|P\|\|\nabla b_\omega\|_\infty T(\lambda + A_1T + 4\lambda A_2A_3 + A_4T^{1/2} + \|z_0\|_{3,q}), \]
\[ D_2 = \|P\|\left(\left(\frac{q}{r}\right)^{1/2} P \left(\frac{3}{2r}, 1 - \frac{3}{2r}\right)\right), \quad D_3 = \|P\|\|b_\omega\|_{r,u} P \left(\frac{1}{2}, 1 - \frac{3}{2r}\right). \]

The now, we choose \( \lambda < \frac{1}{16D_2} \) and \( T \leq \left(\frac{1}{16D_2}\right)^{\frac{1}{2r}} \). Then, we get
\[ Q_j \leq 2(M\|z_0\|_{3,q} + D_1) \quad \text{for all} \quad j \geq 1. \]

The same argument as \( Q_j \), we obtain
\[ P_j \leq 2\|\nabla b_\omega\|_\infty T \|z_j - z_{j-1}\|_{\infty,3,q} + D_2(K_jP_{j-1} + L_jQ_j) + D_4T^{1/2} + \frac{1}{2} P_{j-1} \]
\[ \leq 2\|\nabla b_\omega\|_\infty T + MD_2\|z_0\|_{3,q} + D_1D_2\|z_j - z_{j-1}\|_{M_T} \]
\[ + (4D_2\lambda + D_4T^{1/2}) P_{j-1} \]
\[ \leq 2\|\nabla b_\omega\|_\infty T + MD_2\|z_0\|_{3,q} + D_1D_2d^{j-2}\|z_2 - z_1\|_{M_T} + \frac{1}{2} P_{j-1}, \quad j \geq 2. \]

This yields to
\[ P_j \leq \frac{1}{2^{j-1}} \left( \frac{\|\nabla b_\omega\|_\infty T + MD_2\|z_0\|_{3,q} + D_1D_2\|z_2 - z_1\|_{M_T} + P_1}{1 - 2d} \right)^{j/2}, \quad j \geq 2. \]

Therefore, \( \{t^{\frac{1}{2}} \nabla z_j(t)\} \) is Cauchy sequence in the Banach space \( C([0,T], L^{3,q}(\Omega)^{3 \times 3}) \) if we take \( \lambda < \min\left\{ \frac{1}{16D_2}, \frac{1}{24C_2}, \frac{1}{32A_2}, \frac{1}{8} \right\} \) and
\[ T \leq \min \left\{ T^*, \left( \frac{r - 3}{16Mr\|b_\omega\|_{r,q}} \left(\frac{r}{q}\right)^{1/2} \right)^{\frac{1}{2r}}, \left(\frac{1}{4D_3}\right)^{\frac{1}{2r}} \right\}. \]

So, \( t^{\frac{1}{2}} \nabla z_j(t) \) converges to \( t^{\frac{1}{2}} \nabla z(t) \) in the Banach space \( C([0,T], L^{3,q}(\Omega)^{3 \times 3}) \). Thus, \( z(t) \) is solution of Eq. \((33)\) on \([0,T]\).

Finally, the uniqueness of time-local mild solution follows as in [13] from Gronwall’s inequality. So, \( z(t) \) is unique solution of Eq. \((33)\) on \([0,T]\). \( \square \)

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