Emergent 4D Gravity from Matrix Models

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Abstract

Recent progress in the understanding of gravity on noncommutative spaces is discussed. A gravity theory naturally emerges from matrix models of noncommutative gauge theory. The effective metric depends on the dynamical Poisson structure, absorbing the degrees of freedom of the would-be $U(1)$ gauge field. The gravity action is induced upon quantization.

1 Background and motivation

There is a fundamental conflict between quantum mechanics and general relativity at the Planck scale. This lead to the development of quantum field theory on noncommutative spaces [1], as a first step to overcome this problem. More recently, it was suggested that gravity emerges naturally from noncommutative gauge theory, without having to introduce any new degrees of freedom such as an explicit metric. Earlier forms of this idea [2, 3] can be cast in concise form for matrix models of noncommutative gauge theory [4], which describe dynamical noncommutative spaces. We discuss basic results of this approach. This also provides a new understanding of gravity in similar string theoretical matrix models [5].

2 Matrix models and effective geometry

Consider the matrix model with action

$$S_{YM} = -\text{Tr}[Y^{a}, Y^{b}][Y^{a'}, Y^{b'}]g_{aa'}g_{bb'}$$

for

$$g_{aa'} = \delta_{aa'} \quad \text{or} \quad g_{aa'} = \eta_{aa'}$$

in the Euclidean resp. Minkowski case. Here the "covariant coordinates" $Y^{a}$ for $a = 1, 2, 3, 4$ are hermitian matrices or operators acting on some Hilbert space $\mathcal{H}$. We will denote their commutator as

$$[Y^{a}, Y^{b}] = i\theta^{ab}$$

so that $\theta^{ab} \in L(\mathcal{H})$ is an antisymmetric matrix, which is not assumed to be constant here. We focus on configurations $Y^{a}$ (not necessarily solutions of the equation of motion) which can be interpreted as quantizations of a Poisson manifold $(\mathcal{M}, \theta^{ab}(y))$ with general Poisson structure $\theta^{ab}(y)$. This defines the geometrical background under consideration, and conversely essentially any (local) Poisson manifold provides after quantization a possible background $Y^{a}$. More formally, this means that there is a map of vector spaces ("quantization map")

$$\mathcal{C}(\mathcal{M}) \rightarrow \mathcal{A} \subset L(\mathcal{H})$$

where $\mathcal{C}(\mathcal{M})$ denotes the space of functions on $\mathcal{M}$, and $\mathcal{A}$ is interpreted as quantized algebra of functions on $\mathcal{M}$. The map (4) can be used to define a star product on $\mathcal{C}(\mathcal{M})$. Furthermore, we can then write

$$[f, g] \sim i\{f(y), g(y)\}$$

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2 the matrices are meant to be infinite-dimensional, but “regularized” by $N \times N$ matrices for $N \rightarrow \infty$. 

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for \( f, g \in \mathcal{A} \), where \( \sim \) denotes the leading term in a semi-classical expansion in \( \theta \), and \( \{ f, g \} \) the Poisson bracket defined by \( \theta^{ab}(y) \). \( Y^a \) can be interpreted as quantization of a classical coordinate function \( y^a \) on \( \mathcal{M} \). More importantly, \( Y^a \) defines a derivation on \( \mathcal{A} \) via

\[
[Y^a, f] \sim i\theta^{ab}(y)\partial_b f(y), \quad f \in \mathcal{A}.
\]  

Consider first the "irreducible" case i.e. assume that the centralizer of \( \mathcal{A} \) in \( \mathcal{H} \) is trivial. Then any matrix ("function") in \( L(\mathcal{H}) \) can be well approximated by a function of \( Y^a \). From the gauge theory point of view discussed in section 4.1 it means that we restrict ourselves to the \( U(1) \) case; this is also the sector where the UV/IR mixing occurs. For the general case see section 4.2.

Let us now couple a scalar field \( \Phi \in \mathcal{A} \) to the matrix model (11). The only possibility to write down kinetic terms for matter fields is through commutators \( [Y^a, \Phi] \sim i\theta^{ab}(y)\partial_b \Phi \) using (6). This leads to the action

\[
S[\Phi] = -\text{Tr} g_{ab}[Y^a, \Phi][Y^b, \Phi] \sim \int d^4y \rho(y) G^{ab}(y) \frac{\partial}{\partial y^a} \Phi(y) \frac{\partial}{\partial y^b} \Phi(y). \tag{7}
\]

Here

\[
G^{ab}(y) = \theta^{ac}(y)\theta^{bd}(y) g_{cd} \tag{8}
\]

is the effective metric for the scalar field \( \Phi \) [4]. Hence the Poisson manifold naturally acquires a metric structure \( (\mathcal{M}, \theta^{ab}(y), G^{ab}(y)) \), which is determined by the Poisson structure and the constant background metric \( g_{ab} \). We also used \( \text{Tr} \sim \int d^4y \rho(y) \), where

\[
\rho(y) = \sqrt{\text{det} G_{ab}(y)}^{1/4} = (\det \theta^{ab}(y))^{-1/2} \tag{9}
\]

is the symplectic measure on \( (\mathcal{M}, \theta^{ab}(y)) \). Notice that the action (7) is invariant under Weyl rescaling of \( \theta^{ab}(y) \) resp. \( G^{ab}(y) \). We can therefore write the action as

\[
S[\Phi] = \int d^4y \tilde{G}^{ab}(y) \partial_a \Phi(y) \partial_b \Phi(y) = \int d^4y \sqrt{\tilde{G}_{ab}} \Phi(y) \Delta_{\tilde{G}} \Phi(y) \tag{10}
\]

where \( \tilde{G}^{ab} \) is the unimodular metric

\[
\tilde{G}^{ab}(y) = (\text{det} G_{ab})^{1/4} G^{ab}(y), \quad \text{det} \tilde{G}^{ab}(y) = 1 \tag{11}
\]

and \( \Delta_{\tilde{G}} \) is the Laplacian of a scalar field on the classical Riemannian manifold \( (\mathcal{M}, \tilde{G}^{ab}(y)) \).

The main point here is that any kinetic term will always involve the metric \( G^{ab}(y) \) resp. \( \tilde{G}^{ab}(y) \), possibly with additional density factors which remain to be understood. Therefore this metric should indeed be interpreted as gravitational metric. For gauge fields this is discussed in section 4.2 and the case of fermions will be discussed elsewhere. Note also that \( \theta^{ac}(y) \) can be interpreted as a preferred frame or vielbein, which is however gauge-fixed and does not admit the usual local Lorentz resp. orthogonal transformations.

A linearized version of (11) was obtained in [2]. Related (but inequivalent) metrics were discussed in the context of the DBI action [3]; note that our metric \( G^{ab} \) which governs the matrix model is not the pull-back of \( g^{ab} \) using the change of coordinates (10), and it is indeed curved in general.

It is easy to see that in 4 dimensions, one cannot obtain the most general geometry from metrics of the form (8). Therefore the gravity theory which emerges here will not reproduce all (off-shell) degrees of freedom of general relativity. However, one does obtain a class of metrics which is sufficiently rich to describe the propagating ("on-shell") degrees of freedom of gravity, as well as e.g. the Newtonian limit for an arbitrary mass distribution [4]. On noncommutative spaces, the 2 physical helicities of gravitons can indeed be expressed in terms of the 2 physical helicities of photons.
Equations of motion. So far we considered arbitrary background configurations $Y^a$ as long as they admit a geometric interpretation. The equations of motion derived from the action (1) select on-shell geometries among all possible backgrounds, such as the Moyal-Weyl quantum plane (17). These amount to Ricci-flat spaces (26) at least in the near-flat case [2]. However we allow the most general off-shell configurations here.

3 Quantization and induced gravity

Now consider the quantization of the matrix model (1) coupled to a scalar field. In principle, the quantization is defined in terms of a (“path”) integral over all matrices $Y^a$ and $\Phi$. In 4 dimensions, only perturbative computations can be performed for the gauge sector encoded by $Y^a$, while the scalar $\Phi$ can be integrated out formally in terms of a determinant. Let us focus here on the effective action obtained by integrating out the scalar,

$$e^{-\Gamma_{\Phi}} = \int d\Phi e^{-S[\Phi]}, \quad \text{where} \quad \Gamma_{\Phi} = \frac{1}{2} \text{Tr} \log \Delta_G$$

for a non-interacting scalar field with action (10). A standard argument using the heat kernel expansion of $\Delta_G$ leads to

$$\Gamma_{\Phi} = \frac{1}{16\pi^2} \int d^4y \left(-2L^4 + \frac{1}{6} R[\tilde{G}] L^2 + O(\log L) \right)$$

$$= \frac{1}{16\pi^2} \int d^4y \left(-2L^4 + \frac{1}{6} \rho(y) \left(R[\tilde{G}] - 3\Delta_G\sigma - \frac{3}{2} G^{ab} \partial_a \sigma \partial_b \sigma \right) L^2 + O(\log L) \right)$$

where

$$\Delta_G\sigma = G^{ab} \partial_a \partial_b \sigma - \Gamma^c \partial_c \sigma, \quad \Gamma^a = G^{bc} \Gamma_{bc}^a,$$

$$e^{-\sigma(y)} = \rho(y) = (\det G_{ab})^{1/4}$$

This is essentially the mechanism of induced gravity [7], and it suggests to identify the gravitational constant with the cutoff $\lambda \sim L^2$. Note that the term $\int d^4y \sqrt{G} L^4$ is usually interpreted as cosmological constant, and its scaling with $L^4$ presents a major problem for induced gravity. However $\det \tilde{G} = 1$ here, which suggests that this term is essentially trivial in the present context. One finds indeed that flat space (17) is a solution even at one loop, in sharp contrast with general relativity. These are strong hints that the notorious cosmological constant problem is absent or at least much milder in this NC gravity theory.

A further very remarkable point is that the above gravitational effective action provides an understanding of the UV/IR mixing in NC gauge theory: (14) gives the physical content of the “strange” IR behavior of NC gauge theory in a suitable regime. This will be elaborated in detail elsewhere.

4 Gauge theory point of view

In this section we discuss the alternative (more conventional, up to now) interpretation of (11), (17) in terms of NC gauge theory. This will also set the stage for the extension to $SU(n)$ gauge theory coupled to gravity.
4.1 Geometry from U(1) gauge fields

Let us now rewrite the actions (1), (7) in terms of the U(1) gauge fields on the flat Moyal-Weyl background \( \mathbb{R}^4_\theta \) with generators \( \bar{X}^a \). This means that we consider “small fluctuations”

\[
Y^a = \bar{X}^a + A^a
\]

around the generators \( \bar{X}^a \) of the Moyal-Weyl quantum plane, which satisfy

\[
[\bar{X}^a, \bar{X}^b] = i \bar{\theta}^{ab}.
\]

Here \( \bar{\theta}^{ab} \) is a constant antisymmetric non-degenerate tensor. More precisely, we assume that the hermitian matrices \( \mathcal{A}^a = \mathcal{A}^a(\bar{X}) \sim \mathcal{A}^a(x) \) can be interpreted (at least “locally”) as smooth functions on \( \mathbb{R}^4_\theta \). Note that the effective geometry \( \mathbb{R}^4 \) for the Moyal-Weyl plane is indeed flat, given by

\[
\bar{g}^{ab} = \bar{g}^{ac} \bar{g}^{bd} g_{cd}, \quad \bar{\rho} \equiv (\det \bar{g})^{1/4} = (\det \bar{\theta})^{-1/2} \equiv L_{NC}^4.
\]

Consider now the change of variables

\[
A^a(x) = -\bar{\theta}^{ab} A_b(x)
\]

where \( A_a \) is hermitian. Using

\[
[\bar{X}^a + \mathcal{A}^a, f] = i \bar{\theta}^{ab} \left( \frac{\partial}{\partial x^b} f + i [A_b, f] \right) \equiv i \bar{\theta}^{ab} D_b f,
\]

the actions (1), (7) can be written as

\[
S[\Phi] = \text{Tr} \, \bar{\theta}^{ab} \bar{\theta}^{ac} g_{aa'} D_a \Phi D_b \Phi = \int d^4x \, \bar{g}^{ab} D_a \Phi(x) D_b \Phi(x),
\]

\[
S_{YM} = \int d^4x \, \bar{\rho} (\bar{g}^{aa'} \bar{g}^{bb'} F_{ab} F_{a'b'} + \bar{g}^{ab} g_{ab})
\]

where \( F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \) is the U(1) field strength. These formulas are exact (up to boundary terms) if interpreted as noncommutative gauge theory on \( \mathbb{R}^4_\theta \). In the geometrical interpretation (10) the gauge field U(1) gauge field \( A_a(x) \) is completely absorbed in the metric \( G^{ab}(y) \) resp. \( \theta^{ab}(y) \).

4.2 Nonabelian gauge fields

We now discuss the extension of the above model to nonabelian gauge fields

\[
S_{YM} = -\text{Tr} [X^a, X^b] [X^{a'}, X^{b'}] g_{aa'} g_{bb'}.
\]

The action is formally the same as (1), but we use different letters for the matrices hoping to avoid possible confusions. Consider the new vacuum given by the reducible solution \( X^a = \bar{X}^a \otimes \mathbb{1}_n \) of the equation of motion (12). The most general matrix near this vacuum can be written as

\[
X^a = \bar{X}^a \otimes \mathbb{1}_n + A^a = Y^a \otimes \mathbb{1}_n + \mathcal{A}^{a,\alpha}(Y) \otimes \lambda_\alpha
\]

where \( Y^a = \bar{X}^a + A^{a,0} \) denotes the trace-U(1) sector, and \( \lambda_\alpha \) the SU(n) Gell-mann matrices. It is well-known that this can be interpreted as U(n) gauge fields on the Moyal-Weyl quantum plane; in particular, \( \mathcal{A}^{a,0} \) is usually interpreted as U(1) gauge field. However, generalizing the argument in section 2 it is more natural to absorb the \( \mathcal{A}^{a,0} \) in the geometry. Indeed it was shown in [4] that the model (22) can be interpreted as SU(n) gauge fields coupled to gravity, with the same effective metric \( \mathbb{R}^4_\theta \) as above. This explains why the U(1) sector cannot be disentangled from the SU(n) gauge fields in the noncommutative case.

Technically speaking, the analysis of the semiclassical limit of (22) requires the use of the Seiberg-Witten map [6] for general noncommutativity \( \theta^{ab}(y) \). This allows to express the fluctuations \( \mathcal{A}^{a,\alpha}(Y) \)
through commutative $SU(n)$ gauge fields on $\mathcal{M}$, and ensures that the resulting semi-classical action is gauge invariant. The $SU(n)$ field strength is contracted with the effective metric $G^{ab}(y)$ as expected, up to a density factor (which remains to be understood). One finds after considerable effort \[4\]

\[S_{YM} \sim S_{\text{eff}} = \int d^4 y \rho(y) \mathop{tr} \left( 4\eta(y) + G_{\alpha\beta} G^{\alpha\beta} F_{a'b'} F_{a'b'} \right) - 2 \int \eta(y) \mathop{tr} F \wedge F \] (24)

up to higher-order corrections in $\theta$, where $F_{a'b'}$ is the $SU(n)$ field strength on $\mathcal{M}$, and

\[\eta(y) = \frac{1}{4} G^{ab}(y) g_{ab} \] (25)

Remarkably, this involves a “would-be topological term” $\int \eta(y) \mathop{tr} F \wedge F$. This may have implications for the strong CP problem. Furthermore, note that $S_{\text{eff}}$ appears to be generally covariant and invariant under local Lorentz transformations, if we consider $\eta(y), \rho(y)$ as a scalar functions. This is remarkable, because Lorentz-invariance would appear to be violated from the Moyal-plane point of view. However, these are not fundamental symmetries because $g_{ab}$ is fixed in (22). It is also remarkable that the (“would-be” $U(1)$) term $\int d^4 y \rho(y) \eta(y)$ implies that the vacuum geometries are Ricci-flat,

\[R_{ab} [\tilde{G}] = 0 + O(\theta^2) \] (26)

at least in linearized gravity $\tilde{G}_{ab} = \tilde{g}_{ab} + h_{ab}$ [2]. Finally, as shown in [4] the class of metrics (8) is rich enough to describe correctly the Newtonian limit of gravity, with metric

\[ds^2 = -c^2 dt^2 \left( 1 + 2 \frac{U}{c^2} \right) + dx^2 \left( 1 + O \left( \frac{1}{c^2} \right) \right). \] (27)

Here $\Delta (3) U = 4 \pi G \rho$ and $\rho$ is the mass density. Combined with [11], we see that a reasonable candidate for physical gravity emerges. It promises advantages over GR for quantization and the cosmological constant problem. The most exciting aspect is that it provides an extremely simple and intrinsically noncommutative mechanism for gravity. On the other hand the constrained class of metrics (8) makes the theory very restrictive (and falsifiable); adding extra dimensions might extend this class of metrics. Of course much more work is required to obtain a complete understanding and judgement.

References

[1] for basic reviews see M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” Rev. Mod. Phys. 73 (2001) 977 [arXiv:hep-th/0106048]; R. J. Szabo, “Quantum field theory on noncommutative spaces,” Phys. Rept. 378, 207 (2003) [arXiv:hep-th/0109162].

[2] V. O. Rivelles, “Noncommutative field theories and gravity,” Phys. Lett. B 558 (2003) 191 [arXiv:hep-th/0212262].

[3] H. S. Yang, “On The Correspondence Between Noncommutative Field Theory And Gravity,” Mod. Phys. Lett. A 22 (2007) 1119; H. S. Yang, “Emergent gravity from noncommutative spacetime,” arXiv:hep-th/0611174.

[4] H. Steinacker, “Emergent Gravity from Noncommutative Gauge Theory,” JHEP 12, (2007) 049; arXiv:0708.2426 [hep-th]]

[5] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A large-N reduced model as superstring,” Nucl. Phys. B 498 (1997) 467 arXiv:hep-th/9612115

[6] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909 (1999) 032 arXiv:hep-th/9908142.

[7] A. D. Sakharov, “Vacuum quantum fluctuations in curved space and the theory of gravitation,” Sov. Phys. Dokl. 12 (1968) 1040 [Dokl. Akad. Nauk Ser. Fiz. 177 (1967)].