HÉNON-LIKE FAMILIES AND BLENDER-HORSESHOES AT NON-TRANSVERSE HETERODIMENSIONAL CYCLES

LORENZO J. DÍAZ AND SEBASTIÁN A. PÉREZ

ABSTRACT. In dimension three and under certain regularity assumptions, we construct a renormalisation scheme at the heterodimensional tangency of a non-transverse heterodimensional cycle associated with a pair of saddle-foci whose limit dynamic is a center-unstable Hénon-like family displaying blender-horseshoes. As a consequence, the initial cycle can be approximated in higher regularity topologies by diffeomorphisms having blender-horseshoes.

1. Introduction

Homoclinic tangencies and heterodimensional cycles are the two main sources of non-hyperbolic chaotic dynamics. Palis’ density conjecture claims the these two types of bifurcations are the only obstructions to hyperbolicity: any non-hyperbolic system can be approximated by diffeomorphisms displaying some of these configurations, see [37]. In the terminology of [10] (see Preface), homoclinic tangencies are in the core of the so-called critical dynamics while heterodimensional cycles occur in non-critical settings. Here heterodimensional refers to the fact that the cycle involves saddles whose unstable manifolds have different dimensions. A crucial point is that critical and non-critical dynamics may occur simultaneously in some bifurcation scenarios and their effects may superimpose. This is precisely the setting of this paper. We consider three-dimensional diffeomorphisms having simultaneously a heterodimensional cycle and a heterodimensional tangency (see part (b) of Figure 1) and study a renormalisation scheme leading to blender-horseshoes. Our results continues the line of research started in [14], where renormalisation schemes were introduced in this heterodimensional critical setting. The main difference with [14] is that we consider here cycles associated to a pair of saddle-foci instead a pair of saddles with real multipliers. We note that the dynamical setting of this paper is in some sense a “heterodimensional version” of the equidimensional (meaning that all the saddles in the bifurcation have unstable manifolds of the same dimension) configuration studied in [27], see part (a) of Figure 1. We now proceed to discuss more precisely the setting and the concepts involved in this paper.

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Homoclinic bifurcations and renormalisation. Homoclinic bifurcations may occur only in dimension two or higher. Some important dynamical phenomena associated to homoclinic tangencies are the existence of horseshoes whose invariant sets exhibit persistent non-transverse homoclinic intersections, \cite{34, 36}, the generic coexistence of infinitely many sinks/sources, \cite{35}, and the occurrence of Hénon-like attractors/repellers, \cite{2, 32}. Initially, these phenomena were studied independently, but the study in \cite{40} allows a unifying approach via renormalisation methods, see \cite[Chapter 3.4]{40}. Roughly, the quadratic family can be viewed as a limit dynamics after a re-scaling process at a homoclinic tangency point. This approach enables to translate persistent dynamical behaviours present in the quadratic family to the setting of homoclinic bifurcations. This is an important outcome of the renormalisation approach that can be used in another bifurcating settings that is explored in this paper. We emphasise that the renormalisation methods have two parts: a (limit) family exhibiting some persistent dynamical “interesting” features and a sequence of “dynamically defined” maps approaching the initial configuration whose “returns” (obtained by composing the diffeomorphisms) tend to this family. In this way, properties of the limit family are translated to the considered dynamical systems. Let us postpone for a while the discussion about renormalisation and analyse heterodimensional cycles.

Heterodimensional cycles. First note that heterodimensional cycles can occur only on manifolds of dimension three or higher. To keep our presentation as simple as possible, while describing all the essential complexity of these cycles, in what follows we will assume that the dimension of the ambient manifold is three. We recall that the index of a hyperbolic periodic point $P$ of a diffeomorphism $f$ is the number of expanding eigenvalues of $Df^\pi(P)$, where $\pi$ is the period or $P$. We say that a diffeomorphism $f$ has a heterodimensional cycle associated to two saddles of different indices (here, by dimension constrains, the indices are necessarily two and one, or vice-versa) $P$ and $Q$ if the their invariant manifolds meet cyclically. In what follows, again for simplicity of the presentation, we will assume that $P$ and $Q$ are both fixed points and that $P$ has index two and $Q$ has index one. Key aspects in the analysis of these cycles is the shape of the intersections between these invariant sets.
as well as the type of expanding eigenvalues of $Df(P)$ and contracting eigenvalues of $Df(Q)$. First, due to dimension deficiency the one-dimensional invariant sets $W^s(P)$ and $W^u(Q)$ cannot have transverse intersections. Second, due to dimension sufficiency the two-dimensional invariant manifolds $W^u(P)$ and $W^s(P)$ may have transverse and non-transverse intersections (and this two types of intersections may occur simultaneously). The combination of these aspects (types of eigenvalues of the saddles and types and shapes of the intersection of the invariant manifolds) leads to a plethora of dynamical configurations and features (going from hyperbolic to a completely non-dominated dynamics) that we do not aim to discuss in their full complexity. Instead, we prefer to outline two “antipodal” cases leading to two very different dynamical settings: partial hyperbolicity and non-dominated dynamics. Before presenting these scenarios let us observe that we follow the approach proposed in [38, Section 3], we fix a neighbourhood of the cycle (roughly, an open set containing the saddles of the cycle and intersections of their invariant manifolds) and study the relative dynamics in such a set. In what follows, we assume that the intersection of the one-dimensional invariant sets $W^s(P)$ and $W^u(Q)$ is quasi-transverse (i.e., their tangent vectors at the intersection are not colinear).

**Partial hyperbolicity versus non-domination.** A first (and certainly the simplest) configuration occurs when all the eigenvalues of $P$ and $Q$ are real and different in modulus and the intersection between the two-dimensional invariant sets $W^u(P)$ and $W^s(Q)$ is transverse and defines a heteroclinic curve $\gamma$ with endpoints $P$ and $Q$. The property of the eigenvalues allows to define the one-dimensional strong unstable foliation of $W^u(P)$ and the one-dimensional strong unstable foliation of $W^s(Q)$. The “simplest” case occurs when $\gamma$ is transverse to these two foliations. This dynamical configuration is depicted in part (a) in Figure 2.

This type of configuration leads to partially hyperbolic dynamics with a one-dimensional center: in the neighbourhood of the cycle there is a $Df$-splitting with three one-dimensional invariant directions $E^s \oplus E^c \oplus E^u$, $E^s$ uniformly contracting,

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1 An interesting illustration of the complexity of these configurations can be done following the lines in the discussion in [39, Section 2]. Here the complexity is much higher: the shapes of the intersections of the two-dimensional manifolds need to be considered.
$E^u$ uniformly expanding, and $E^c$ has contracting and expanding regions. For instance, this bifurcation can generate a pair of disjoint hyperbolic sets (horseshoes) each one containing one of the saddles in the cycle \cite{17} or a transitive set containing both saddles in the cycle and/or the phenomenon of intermingled homoclinic classes\footnote{A homoclinic class of a saddle is the closure of the set of its transverse homoclinic points. In this particular case, the homoclinic classes of $P$ and $Q$ coincide, since these saddles have different indices one has a non-hyperbolic transitive set.} see \cite{13, 12, 18}, but this list does not exhaust all the possibilities of this configuration, see \cite{19}.

A second configuration occurs when either (i) the two-dimensional manifolds $W^u(P)$ and $W^s(Q)$ have non-transverse intersections or (ii) the saddles in the cycle have both a pair of non-real eigenvalues ($Df^s(P)$ a pair of non-real expanding eigenvalues and $Df^s(Q)$ a pair of non-real contracting eigenvalues), see part (b) in Figure 2. These configurations lead to non-dominated dynamics: there is no (non-trivial) bundle defined on the neighbourhood of the cycle that is $Df$-invariant. The second type of dynamics was studied in \cite{4} (to get infinitely many sinks/sources) and in \cite{5} (to get universal dynamics), and the first type in \cite{15} (where heterodimensional tangencies were introduced). In this paper, we consider heterodimensional tangencies and heterodimensional cycles associated to saddles with non-real eigenvalues, continuing the study started in \cite{14}, where a similar configuration (with saddles having real eigenvalues) was considered.

Let us emphasise that in our setting the dynamics in the cycle is non-dominated, thus finding relevant hyperbolic sets and seeing how these sets are embedded in the global dynamics is a complicated task. Let us now make this sentence a bit more precise. In the case of homoclinic bifurcations, relevant sets are the so-called thick horseshoes which are the main responsible of the persistent non-hyperbolic features of these bifurcations (for an ample discussion see \cite{40} Chapter 5). In the case of heterodimensional cycles, it is proved in \cite{6, 8} that these cycles yield blenders (or more precisely, blender-horseshoes) and that these blenders are in the core of most of the non-hyperbolic features related to these cycles (as for instance, intermingled homoclinic classes and occurrence of robust cycles and tangencies). Our result state that, in our setting, the bifurcating diffeomorphisms $f$ can be approximated by diffeomorphisms exhibiting blender-horseshoes, see Theorem 1 and Corollary 1. This approximation is done using a renormalisation scheme converging to a center-unstable Hénon-like family (see equations (2.1) and (2.2)) and using the fact that this family exhibits blender-horseshoes, see \cite{16} and \cite{14} for a preliminary version. In this approach, the Hénon-like family plays the role of the quadratic family in the homoclinic setting.

**Blenders and Blender-horseshoes.** Although blenders do not appear explicitly in our paper, they are a fundamental object behind our constructions: the center-unstable Hénon-like family provides these blenders as a side effect. Let us say a few words about them. A blenders is just a transitive hyperbolic set that occurs in dimension three or higher and whose (local) stable set geometrically behaves as a set of dimension larger than the one of its stable bundle. For an informal introduction to blenders we refer to \cite{3}, for some discussions on the notion and properties of blenders see \cite{11} Chapter 6.2. Blender-horseshoes are a special kind of blenders, they are locally maximal hyperbolic sets conjugate to the complete shift in two symbols, for the precise definition of a blender-horseshoe see \cite{8}. Comparing with
the blenders in [7], an important advantage of blender-horseshoes is that they can
be used to get robust cycles and tangencies, as in [6, 8]. An important property
of blender-horseshoes, that we will use here in Corollary 1 is their $C^r$-persistence,
see [8, Lemma 3.9].

**Regularity.** So far, we have deliberately omitted (for simplicity of the discussion)
any reference to the regularity of the perturbations of the systems. This will be
a key point in the following discussion. To put the statements as simple as possi-
ble (and slightly oversimplifying), the occurrence of thick horseshoes in homoclinic
bifurcations demands $C^2$-regularity while the occurrence of blenders at heterodi-
mensional cycles is obtained for $C^1$-perturbations. This means that (partially) the
two theories are developed considering different degrees of differentiability. Here it
is important to recall the recent paper [1] where robust homoclinic tangencies are
obtained in some specific partially hyperbolic settings. A natural goal is to get
the occurrence of robust heterodimensional cycles and homoclinic tangencies in $C^2$-
settings. The rough idea (that we will explore) is that one can go from $C^1$-regularity
to higher regularity if the *dynamics of the intersection is rich enough and there are plenty of heteroclinic intersections* between the saddles of the cycle. It seems that
the sort of bifurcations that we consider here provide such a rich dynamics (this is
an ongoing project, see [11]). Here two problems arise: the generation of blenders
and their insertion in the global dynamics. One aims to “relate” these blenders
to the two saddles in the initial cycle, but this is not always possible, see [9]. A
part of this problem (concerning the generation of blenders) was considered and
solved in [14], but the embedding of these blenders in the dynamics in [14] is still
not completely adequate. The results in [14] have three parts, it is proved that: (i)
There is a center-unstable Hénon-like family displaying (for appropriate range of
parameters) blender-horseshoes (see also [16]): (ii) There is renormalisation scheme
converging to that Hénon-like family; (iii) Some heteroclinic-like relations between
the blender and the saddles in the cycle are obtained for appropriate $C^{1+\alpha}$, $\alpha < 1$,
perturbations. Here we see how (i) and (ii) hold in our setting and in a forthcom-
ing paper we deal with the problem of the dynamical embedding of the obtained
blender and the heteroclinic relations, see also [41].

**Final comments.** We now discuss briefly some results in the literature related to
our setting. As observed, there are interesting scenarios where critical and non-
critical dynamics are intermingled. In the equidimensional context let us recall the
homoclinic and heteroclinic two-dimensional non-transverse cycles in [43, 25, 28]
and, in dimension three, the series of papers [42, 20, 21, 22, 27], where homo-
clinic and heteroclinic non-transverse cycles were studied. These papers involve
renormalisation-like schemes leading to families of Hénon-like maps or some varia-
tions: the logistic map in [43], generalisations of the Hénon map in [25], Mira maps
in [20, 21, 52], and three-dimensional Hénon maps in [23]. See also the discussion

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3This is a higher-dimensional result where the ambient space is four-dimensional: to get the
tangencies one needs a “central direction” of dimension at least two (besides some stable and
unstable directions). It is also important to recall that in dimension two there are no $C^1$-robust
homoclinic tangencies, see [33].
We observe that there are families of three-dimensional Hénon-like maps that exhibit new types of strange attractors, as for example the so-called wild-hyperbolic attractors\(^4\), for details see [44, 23, 24].

On the other hand, renormalisation methods do not seem to be sufficiently exploited in the heterodimensional context. As far as we know, the first example in this direction was given in [14]. Here we provide a renormalisations scheme for a non-transverse cycle associated to pair of saddle-foci points in the spirit of [14]. Finally, since the configurations in this paper can be viewed as a “heterodimensional version” of [27], a natural question is if they lead to families of Hénon-like maps with wild-hyperbolic attractors.

Finally, let us recall that in the heterodimensional setting, critical bifurcations were explored in [31], where heterodimensional tangencies and non-transverse cycles lead to \(C^2\)-robust heterodimensional tangencies (again, this is related to non-dominated dynamics). In [30] it is shown how these tangencies lead to strange attractors.

2. Statements of results

2.1. Center-unstable Hénon-like families. We consider the center-unstable Hénon-like family of endomorphisms \(G_{\xi,\mu,\kappa_1,\kappa_2}: \mathbb{R}^3 \to \mathbb{R}^3\) defined by

\[
G_{\xi,\mu,\kappa_1,\kappa_2}(x,y,z) \overset{\text{def}}{=} (y,\mu + y^2 + \kappa_1 z^2 + \kappa_2 yz, \xi z + y).
\]

By [16, Theorem 1], the family \(G_{\xi,\mu,\kappa,\eta}\) has a blender-horseshoe for all \((\xi,\mu,\kappa,\eta)\) in the open set \(O = O_\varepsilon \overset{\text{def}}{=} (1.18, 1.19) \times (-10, -9) \times (-\varepsilon, \varepsilon)^2\), for some \(\varepsilon > 0\). See [14] for a version of this result for blenders (instead of blender-horseshoes) and [29] for a complete numerical analysis of this family including the study of the creation and annihilation of blenders.

Our results also involve the endomorphisms family \(E_{\xi,\mu,\zeta}: \mathbb{R}^3 \to \mathbb{R}^3\) defined by

\[
E_{\xi,\mu,\zeta}(x,y,z) \overset{\text{def}}{=} (\xi x + s_1 y, \mu + s_2 y^2 + s_3 x^2 + s_4 xy, s_5 y),
\]

where \(\zeta \overset{\text{def}}{=} (s_1, s_2, s_3, s_4, s_5)\).

Observe that if \(s_1 s_2 s_3 \neq 0\) then the families of endomorphisms

\[(\mu, E_{\xi,\mu,\zeta}) \quad \text{and} \quad (\mu, G_{\xi,\mu,\kappa,\eta}), \quad \text{where} \quad \kappa = s_2^{-1} s_3 s_5^{-1}, \quad \eta = s_1 s_4 s_5^{-1},
\]

are conjugate by the change of coordinates

\[
\Theta: \mathbb{R}^4 \to \mathbb{R}^4, \quad \Theta(\mu, x, y, z) = (s_2^{-1} \mu, s_2^{-1} s_1, s_2^{-1} y, s_2^{-1} s_5 x).
\]

2.2. Bifurcation setting and main result. In what follows, \(M\) denotes a closed boundaryless three manifold and \(\text{Diff}^r(M), r \geq 1\), the space of \(C^r\)-diffeomorphisms of \(M\) endowed with the norm \(\| \cdot \|_{C^r}\) of the uniform \(C^r\)-convergence.

We consider bifurcations of diffeomorphisms \(f \in \text{Diff}^r(M)\) having simultaneously a heterodimensional cycle and a heterodimensional tangency. We assume that \(f\) has a pair of (periodic) saddles \(P\) and \(Q\) of indices (dimension of the unstable bundle) two and one, respectively, such that conditions (A)-(C) below it holds:

\(^4\)Roughly, a wild-hyperbolic attractors possess the following two distinctive properties: (1) wild-hyperbolic attractors allow homoclinic tangencies; (2) every such attractor and all nearby attractors (in the \(C^r\)-topology with \(r \geq 2\)) have no stable periodic orbits.
(A) **Saddle-focus periodic points:** Let $\pi(P)$ and $\pi(Q)$ be the periods of $P$ and $Q$. Then $Df^{\pi(P)}(P)$ has a pair of non-real expanding eigenvalues and $Df^{\pi(Q)}(Q)$ has a pair of non-real contracting eigenvalues. We assume that the restrictions of $f^{\pi(P)}$ and $f^{\pi(Q)}$ in small neighbourhoods of $P$ and $Q$ are $C^r$-linearisable. Some open and non-degenerate relations involving the eigenvalues of $Df^{\pi(P)}(P)$ and $Df^{\pi(Q)}(Q)$ are assumed, see \[3.2\].

(B) **Quasi-transverse intersection:** The one-dimensional invariant manifolds of $P$ and $Q$ intersect quasi-transversely along the orbit of a heteroclinic point $X$, i.e., $X \in \mathcal{W}^s(P,f) \cap \mathcal{W}^u(Q,f)$ and

$$T_X \mathcal{W}^s(P,f) + T_X \mathcal{W}^u(Q,f) = T_X \mathcal{W}^s(P,f) \oplus T_X \mathcal{W}^u(Q,f).$$

Associated to the heteroclinic point $X$ there is a *transition map* corresponding to some iterate of $f$, going from a neighbourhood $U_Q$ of $Q$ to a neighbourhood $U_P$ of $P$ following the orbit of $X$. Some conditions on this transition are given in \[3.6\].

(C) **Heterodimensional tangency:** The two-dimensional invariant manifolds of $P$ and $Q$ intersect along the orbit of a heteroclinic point $Y$ that is a heterodimensional tangency, i.e., the orbit of $Y$ is contained in the set

$$\left(\mathcal{W}^u(P,f) \cap \mathcal{W}^s(Q,f)\right) \setminus \left(\mathcal{W}^u(P,f) \cap \mathcal{W}^s(Q,f)\right).$$

Associated to the heteroclinic point $Y$ there is a *transition map* corresponding to some iterate of $f$, going from a neighbourhood $U_P$ of $P$ to a neighbourhood $U_Q$ of $Q$ following the orbit of $Y$. Some conditions on this transition are given in \[3.10\].

We are ready to state our main result.

**Theorem 1.** Let $f : M \to M$ be a $C^r$-diffeomorphism, $r \geq 2$, having a cycle associate to a pair of saddle-foci periodic points $P$ and $Q$ satisfying conditions (A)-(C). Then there is a unfolding family $\mathcal{F} = \{ f_\epsilon \}_{\epsilon \in \mathbb{R}^8}$ in $\text{Diff}^r(M)$ with $f_0 = f$ satisfying the following properties:

For every $\xi > 0$ there exists a renormalisation scheme $\mathcal{R}(\xi, \mathcal{F}, f)$ consisting of

- two sequences of natural numbers $(m_k)$ and $(n_k)$ with $(m_k), (n_k) \to +\infty$ as $k \to +\infty$;
- a sequence of $C^r$-parameterisations $\Psi_k : \mathbb{R}^3 \to M$;
- a sequence of $C^\infty$-functions $\bar{v}_k : \mathbb{R} \times [-\pi, \pi]^2 \to \mathbb{R}^8$ parameterising the bifurcation parameter of the family $(f_\epsilon)_{\epsilon \in \mathbb{R}^8}$;
- a sequence of rescaled diffeomorphisms $\mathcal{R}_k(f_{\bar{v}_k}) : M \to M$, defined by

$$\mathcal{R}_k(f_{\bar{v}_k}) \overset{\text{def}}{=} (f_{\bar{v}_k})^{N_2 + m_k + N_1 + n_k},$$

where $N_1$ and $N_2$ are natural numbers independent of $k$ and $\xi$, satisfying the following convergence properties:

- for every pair of compact sets $\Delta \subset \mathbb{R}^3$ and $L \subset \mathbb{R} \times [-\pi, \pi]^2$ it holds

$$\Psi_k(\Delta) \to \{ Y \}, \quad \bar{v}_k(L) \to \{ 0 \} \quad \text{when} \quad k \to +\infty,$$

where $Y$ is the heterodimensional tangency point of $f$ in Condition (C);
- if $\varphi_\epsilon$ is the argument of the non-real eigenvalue of $Df^{\pi(R)}(R)$, $R = P, Q$, then the renormalised sequence

$$\Psi_k^{-1} \circ \mathcal{R}_k(f_{\bar{v}_k}) \circ \Psi_k, \quad \text{where} \quad \bar{v}_k = \bar{v}_k(\mu, \varphi_P, \varphi_Q), \quad \mu \in \mathbb{R},$$

converges in the $C^r$-topology and on compact sets of $\mathbb{R}^3$ to the endomorphism $E_{\xi, \mu, \zeta}$ in \[2.2\], where $\zeta$ depends smoothly on $f$ and $\xi$. 


Equation (8.1) provides the explicit formula for $\varsigma = \varsigma(f, \xi)$.

The family of endomorphisms $E_{\xi, \mu, \varsigma}$ is called the limit dynamics of the renormalisation scheme $R(\xi, \mathcal{F}, f)$.

**Remark 2.1.** The unfolding family in Theorem 1 involves eight parameters. Three parameters for each non-transverse heteroclinic orbit and other two parameters to control the arguments of the saddle-foci.

**Remark 2.2.** An adequate choice of $\varsigma = \varsigma(f, \xi)$ guarantees that the diffeomorphisms $f_{\varsigma_k}$ have blender-horseshoes for every $k$ large enough. More precisely, recalling the observations in Section 2.1, it follows that if $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5)$ is such that

$$\varsigma_1 \varsigma_2 \varsigma_5 \neq 0 \quad \text{and} \quad (\xi, \mu, \varsigma_1^2 \varsigma_2 \varsigma_5^{-1}, \varsigma_1 \varsigma_5 \varsigma_2^{-1}) \in \mathcal{O},$$

then the initial cycle is $C^r$-approximated by diffeomorphisms having blender-horseshoes arbitrarily close to the heterodimensional tangency of $f$.

Let us restate Remark 2.2 using the $C^r$-persistence of blender-horseshoes in [8, Lemma 3.9].

**Corollary 1.** Let $f : M \to M$ be a $C^r$ diffeomorphism as in Theorem 1. For each $\xi \in (1.18, 1.19)$ consider the renormalisation scheme $R(\xi, \mathcal{F}, f)$ of $f$ and its limit dynamics $E_{\xi, \mu, \varsigma}$. If $\varsigma$ satisfies equation (2.3) then $f_{\varsigma_k}$ has a blender-horseshoe nearby the heterodimensional tangency of $f$ for every $k$ large enough. In particular, for every $k$ large enough, there is a $C^r$-open set $\mathcal{U}_k$ of diffeomorphisms $C^r$-close to $f$ such that each $g \in \mathcal{U}_k$ has a blender-horseshoe nearby the heterodimensional tangency of $f$.

We observe that there are diffeomorphisms $f$ satisfying conditions (A)-(C) whose vector $\varsigma = \varsigma(f, \xi)$ satisfies the condition (2.3) for every $\xi \in (1.18, 1.19)$. This point will be proved in Lemma 9.1 after introducing the appropriate terminology.

**Organisation of the paper.** Conditions (A)-(C) are precisely stated in Section 3. In Section 4, we describe the perturbations in the renormalisation scheme of Theorem 1. The unfolding family $\mathcal{F}$ is defined in Section 5. In Section 6, we analyse the dynamics of the returns in the renormalisation scheme (i.e., compositions of the form $f_{\varsigma_k}^{N_k} \circ f_{\varsigma_k}^{m_k} \circ f_{\varsigma_k}^{N_k} \circ f_{\varsigma_k}^n$ defined nearby the heterodimensional tangency). The convergence of the renormalised sequence is obtained in Section 8 after obtaining an explicit formula for these maps in Section 7. The existence of diffeomorphisms satisfying Corollary 1 is proved in Section 9.

3. Description of the cycle: Conditions (A)-(C)

In this section, we describe in precise form the conditions (A)-(C) in Section 2.

3.1. (A) Local dynamics at $P$ and $Q$. Without loss of generality, let us assume that $P$ and $Q$ are fixed points of $f$. We assume the existence of local $C^r$-linearising charts $U_P, U_Q = (-10, 10)^3$ at the saddles $P$ and $Q$ (here the corresponding saddle is identified with the origin) such that the expression of $f$ in these neighbourhoods is of the form
\[ f|_{U_P} = \begin{pmatrix} \lambda_P & 0 & 0 \\ 0 & \sigma_P \cos(2\pi \varphi_P) & -\sigma_P \sin(2\pi \varphi_P) \\ 0 & \sigma_P \sin(2\pi \varphi_P) & \sigma_P \cos(2\pi \varphi_P) \end{pmatrix}, \quad \text{and} \]

\[ f|_{U_Q} = \begin{pmatrix} \lambda_Q \cos(2\pi \varphi_Q) & 0 & -\lambda_Q \sin(2\pi \varphi_Q) \\ 0 & \sigma_Q & 0 \\ \lambda_Q \sin(2\pi \varphi_Q) & 0 & \lambda_Q \cos(2\pi \varphi_Q) \end{pmatrix}, \]

where \( \varphi_P, \varphi_Q \in [0, 1], \varphi_P \neq \varphi_Q, \) and \( \lambda_P, \lambda_Q, \sigma_P, \sigma_Q \in \mathbb{R} \) are such that

\[ 0 < |\lambda_P|, |\lambda_Q| < 1 < |\sigma_P|, |\sigma_Q|. \]

We also assume the following condition called \textit{spectral condition of the cycle},

\[ 0 < \log(\frac{\lambda^-1}{\sigma}) < 1, \quad \text{where} \quad \eta = \frac{\log|\lambda^{-1}|}{\log|\sigma|}. \]

This condition plays a key role in the convergence of the renormalisation scheme. Lemma 3.1 claims that this spectral condition is open and non-empty.

3.1.1. \textit{Spectral conditions on the cycle.} Taking the square of \( f \), if necessary, we can assume that \( \lambda_P, \sigma_P, \lambda_Q \) and \( \sigma_Q \) are all positive. With this in mind, in the next proposition we show that the condition (3.3) is non-degenerate. As consequence the set of diffeomorphisms \( f \) such that its eigenvalues satisfy the spectral condition (3.3) is an open set in \( \text{Diff}^r(M) \).

\textbf{Lemma 3.1.} Let \( \mathcal{P} \) be the set of points \((\lambda, \tilde{\lambda}, \lambda, \sigma) \in \mathbb{R}^4 \) such that \( 0 < \lambda, \tilde{\lambda} < 1 \) and \( \tilde{\lambda}, \sigma > 1 \) and

\[ 0 < (\tilde{\lambda}^{\frac{1}{2}} \sigma)^\sigma < 1, \quad \text{where} \quad \eta = \frac{\log \lambda^{-1}}{\log \sigma}. \]

The set \( \mathcal{P} \) is non-empty and open.

\textit{Proof.} The set \( \mathcal{P} \) is open by definition. Thus it remains to show the existence of numbers satisfying these inequalities. For this, consider the set

\[ \tilde{Z} \overset{\text{def}}{=} \{(\hat{\lambda}, \tilde{\sigma}) \in (0, 1) \times (1, +\infty) : 0 < \hat{\lambda}^{\frac{1}{2}} \tilde{\sigma} < 1 \} \subset \mathbb{R}^2. \]

The lemma follows the next claim.

\textbf{Claim 3.2.} For \((\hat{\lambda}, \tilde{\sigma}, \lambda) \in \tilde{Z} \times (0, 1) \) there is an interval \( I_{(\lambda, \hat{\lambda}, \tilde{\sigma})} \), such that every \((\hat{\lambda}, \tilde{\sigma}, \lambda) \in \tilde{Z} \times (0, 1) \times I_{(\lambda, \hat{\lambda}, \tilde{\sigma})} \) satisfies (3.3).

\textit{Proof.} The equation (3.3) is equivalent to

\[ \frac{\log \lambda^{-1}}{\log \tilde{\sigma}} \log(\hat{\lambda}^{\frac{1}{2}} \tilde{\sigma}) + \log \sigma < 0. \]

Note that every vector \((\hat{\lambda}, \tilde{\sigma}, \lambda) \) in \( \tilde{Z} \times (0, 1) \) satisfies the inequality

\[ \frac{\log \lambda^{-1}}{\log \tilde{\sigma}} \log(\hat{\lambda}^{\frac{1}{2}} \tilde{\sigma}) < 0. \]

Thus for every \( \sigma > 1 \) such that

\[ \frac{\log \lambda^{-1}}{\log \tilde{\sigma}} \log(\hat{\lambda}^{\frac{1}{2}} \tilde{\sigma}) < - \log \sigma, \]
we get that \((\hat{\lambda}, \hat{\sigma}, \lambda, \sigma)\) satisfies (3.3). Now it is enough to take \(I^{(\lambda,\hat{\lambda},\hat{\sigma})} = (1, \sigma^{*}_{(\lambda,\hat{\lambda},\hat{\sigma})})\), where \(\sigma^{*}_{(\lambda,\hat{\lambda},\hat{\sigma})}\) is the supremum of the \(\sigma > 1\) satisfying (3.3). □

The proof of the lemma is now complete. □

3.2. (B) Quasi-transverse intersection and transition from \(Q\) to \(P\). We now describe the transition from \(Q\) to \(P\) along the orbit of the quasi-transverse intersection point \(X \in W^{s}(P, f) \cap W^{u}(Q, f)\).

Using Condition (A), we can consider the local invariant manifolds of \(P\) and \(Q\) defined by

\[
W_{loc}^{s}(P, f) \overset{\text{def}}{=} (-10, 10) \times \{(0, 0)\}, \quad W_{loc}^{u}(P, f) \overset{\text{def}}{=} \{(0)\} \times (-10, 10)^{2}
\]

and

\[
W_{loc}^{s}(Q, f) \overset{\text{def}}{=} (-10, 10) \times \{(0)\} \times (-10, 10), \quad W_{loc}^{u}(Q, f) \overset{\text{def}}{=} \{(0)\} \times (-10, 10) \times \{(0)\}.
\]

Replacing \(X\) by some backward iterate, if necessary, we can assume that \(X \in W_{loc}^{u}(Q, f)\). Rescaling the segment \(W_{loc}^{u}(Q, f)\), if necessary, we can assume that \(X = (0, 1, 0) \in U_{Q}\). By hypothesis, there exists \(N_{1} \in \mathbb{N}\) such that \(\hat{X} = f^{N_{1}}(X) \in W_{loc}^{s}(P, f) \subset U_{P}\). We choose \(N_{1}\) so that \(f^{N_{1} - 1}(X) \notin U_{P}\) (i.e. \(\hat{X}\) is the first iterated of the orbit of \(X\) meeting \(U_{P}\)). Arguing as before, we can take \(\hat{X} = (1, 0, 0) \in U_{P}\).

We assume that there are small neighbourhoods \(U_{X}\) of \(X\) in \(U_{Q}\) and \(U_{\hat{X}}\) of \(\hat{X}\) in \(U_{P}\) such that the map \(f^{N_{1}}|_{U_{X}} : U_{X} \to U_{\hat{X}}\) is given by

\[
f^{N_{1}} \begin{pmatrix} x \\ y + 1 \\ z \end{pmatrix} = \begin{pmatrix}
1 + \alpha_{1} x + \alpha_{2} y + \alpha_{3} z + \tilde{H}_{1}(x, y, z) \\
\beta_{1} x + \beta_{2} y + \beta_{3} z + \tilde{H}_{2}(x, y, z) \\
\gamma_{1} x + \gamma_{2} y + \gamma_{3} z + \tilde{H}_{3}(x, y, z)
\end{pmatrix},
\]

where \(\alpha_{i}, \beta_{i}, \gamma_{i}, i = 1, 2, 3\) are constants such that

\[
(3.7) \quad \beta_{1} = \beta_{3} = \gamma_{1} = \gamma_{2} = 0
\]

and the maps \(\tilde{H}_{i}, i = 1, 2, 3\), are higher order terms satisfying

\[
(3.8) \quad \tilde{H}_{i}(0) = \frac{\partial}{\partial x} \tilde{H}_{i}(0) = \frac{\partial}{\partial y} \tilde{H}_{i}(0) = \frac{\partial}{\partial z} \tilde{H}_{i}(0) = 0.
\]

Note that as \(f^{N_{1}}\) is a (local) diffeomorphism it holds that

\[
(3.9) \quad \alpha_{1} \beta_{2} \gamma_{3} \neq 0.
\]

We call \(f^{N_{1}}\) and \(N_{1}\) transition map and transition time from \(Q\) to \(P\), respectively.

**Remark 3.3.** The vector \(\tilde{u} = (0, 1, 0)\) spans \(T_{X}W^{u}(Q, f)\) and \(Df^{N_{1}}(X)(\tilde{u}) = (\alpha_{2}, \beta_{2}, 0)\), Since, by (3.9), \(\tilde{u} \neq 0\), this vector does not belong to \(T_{X}W^{s}(P, f)\) (which is spanned by \((1, 0, 0)\)). Thus \(X\) is a quasi-transverse heteroclinic point.

3.3. (C) Heterodimensional tangency and transition from \(P\) to \(Q\). We now describe the transition from \(P\) to \(Q\) along the orbit of the heterodimensional tangency point \(Y \in W^{u}(P, f) \cap W^{s}(Q, f)\). By replacing \(Y\) by some backward iterate we can assume that \(Y \in W_{loc}^{u}(P, f)\). By hypothesis, there is \(N_{2} \in \mathbb{N}\) such that \(\hat{Y} = f^{N_{2}}(Y) \in W_{loc}^{s}(Q, f)\). We choose \(N_{2}\) so that \(f^{N_{2} - 1}(Y) \notin U_{Q}\). By some linear coordinate change in \(U_{P}\) and in \(U_{Q}\), one can assume \(Y = (0, 1, 1) \in U_{P}\) and \(\hat{Y} = (1, 0, 1) \in U_{Q}\). Note that this coordinate change can be done without changing the previous choice of \(X\) and \(\hat{X}\). We assume that there are small neighbourhoods
Note that the choices of \( f \) and their associated \( \bar{f} \) and \( \bar{y} \) in \( U_Q \) such that the map \( f^{N_2}|_{U_Q} : U_Y \to U_{\bar{Y}} \) is given by

\[
(3.10) \quad f^{N_2}\left(\begin{array}{c} x \\ 1 + y \\ 1 + z \end{array}\right) = \left(\begin{array}{c} 1 + a_1 x + a_2 y + a_3 z + H_1(x, y, z) \\ b_1 x + b_2 y^2 + b_3 z^2 + b_4 y z + H_2(x, y, z) \\ 1 + c_1 x + c_2 y + c_3 z + H_3(x, y, z) \end{array}\right),
\]

where \( a_i, b_i, c_i, i = 1, 2, 3 \), are constants with

\[
(3.11) \quad c_2 = c_3
\]

and the maps \( H_i, i = 1, 2, 3 \), are higher order terms satisfying the following conditions:

\[
(3.12) \quad H_i(0) = \frac{\partial}{\partial x} H_i(0) = \frac{\partial}{\partial y} H_i(0) = \frac{\partial}{\partial z} H_i(0) = 0,
\]

\[
\frac{\partial^2}{\partial y^2} H_2(0) = \frac{\partial^2}{\partial z^2} H_2(0) = \frac{\partial^2}{\partial y \partial z} H_2(0) = 0.
\]

Note that since \( f^{N_2} \) is a (local) diffeomorphism it follows that

\[
(3.13) \quad b_1 c_2 (a_3 - a_2) \neq 0.
\]

We call \( f^{N_2} \) and \( N_2 \) transition map and transition time from \( P \) to \( Q \), respectively.

Finally, we assume the following condition on the parameters \( a_2, a_3, \) and \( \gamma_3 \) above

\[
(3.14) \quad \gamma_3 (a_3 - a_2) > 0.
\]

Note that the choices of \( a_2, a_3 \), and \( \gamma \) are compatible with \( (3.13) \) and \( (3.9) \).

**Remark 3.4.** The vectors \( \bar{v} = (0, 1, 0) \) and \( \bar{w} = (0, 0, 1) \) span \( T_Y W^s(P, f) \) and \( Df^{N_2}(Y)(\bar{v}) = (a_2, 0, c_2) \) and \( Df^{N_2}(Y)(\bar{w}) = (a_3, 0, c_3) \). Since these vectors belong to \( T_Y W^s(Q, f) \) (which is spanned by \( (1, 0, 0) \) and \( (0, 0, 1) \)), the point \( Y \) corresponds to a heterodimensional tangency.

### 4. Translation and Rotation-like Perturbations

In this section, we describe the two types of perturbations (translation and rotation like) involved in the renormalisation scheme of Theorem 3.

We consider auxiliary \( C^r \)-bump functions \( b_\rho : \mathbb{R} \to \mathbb{R}, \rho > 0 \), satisfying

\[
\begin{cases}
  b_\rho(x) = 0, & \text{if } |x| \geq \rho, \\
  0 < b_\rho(x) < 1, & \text{if } \frac{\rho}{2} < |x| < \rho, \\
  b_\rho(x) = 1, & \text{if } |x| \leq \frac{\rho}{2}.
\end{cases}
\]

and their associated \( C^r \)-bump functions \( B_\rho : \mathbb{R}^3 \to \mathbb{R} \) defined by

\[
B_\rho(x, y, z) \overset{\text{def}}{=} b_\rho(x) b_\rho(y) b_\rho(z).
\]

**4.1. Translation-like Perturbations.** For \( Z_0 \in \mathbb{R}^3 \) denote by \( \mathbb{B}_\rho(Z_0) \subset \mathbb{R}^3 \) the open ball of radius \( \rho \) and center \( Z_0 \). Given \( Z_0 \in \mathbb{R}^3 \), consider the family of \( C^r \)-maps

\[
(4.1) \quad T_{Z_0, \bar{w}} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \bar{w} \in \mathbb{R}^3,
\]

defined by

- if \( Z + Z_0 \in \mathbb{B}_\rho(Z_0) \), then \( T_{Z_0, \bar{w}}(Z + Z_0) = Z + Z_0 + B_\rho(Z) \bar{w} \),
- if \( Z \not\in \mathbb{B}_\rho(Z_0) \), then \( T_{Z_0, \bar{w}}(Z) = Z \).
Note that
\[ \|T_{Z_0, \bar{w}} - \text{id}\|_{C^r} \leq \|B_p\|_{C^r} \cdot \|\bar{w}\|. \]
Thus, for every \( \|\bar{w}\| \) small enough, the map \( T_{Z_0, \bar{w}} \) is a \( C^r \)-perturbation of the identity supported in \( B_p(Z_0) \). Note also that, by construction,
\[ T_{Z_0, \bar{w}}(B_p(Z_0)) = B_p(Z_0). \]
We call \( T_{Z_0, \bar{w}} \) a translation-like perturbation of the identity.

4.2. Rotation-like perturbations. Consider the families of linear maps
\[ I^x_\omega, I^y_\omega : \mathbb{R}^3 \to \mathbb{R}^3, \quad \omega \in [-\pi, \pi], \]
given by
\[
I^x_\omega \defeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi\omega) & -\sin(2\pi\omega) \\ 0 & \sin(2\pi\omega) & \cos(2\pi\omega) \end{pmatrix}, \quad I^y_\omega \defeq \begin{pmatrix} \cos(2\pi\omega) & 0 & -\sin(2\pi\omega) \\ 0 & 1 & 0 \\ \sin(2\pi\omega) & 0 & \cos(2\pi\omega) \end{pmatrix}.
\]
Observe that if \( \omega = 0 \) then \( I^x_\omega = I^y_\omega = \text{id} \).

Consider the families of \( C^r \)-diffeomorphisms
\[
R^x_{\omega, \rho} : \mathbb{R}^3 \to \mathbb{R}^3, \quad R^x_{\omega, \rho}(W) = I^x_\omega b_3(||W||) \cdot W^T,
\]
\[
R^y_{\omega, \rho} : \mathbb{R}^3 \to \mathbb{R}^3, \quad R^y_{\omega, \rho}(W) = I^y_\omega b_3(||W||) \cdot W^T,
\]
where \( W^T \) denotes the transpose of the vector \( W \in \mathbb{R}^3 \).

Note that the restriction of \( R^x_{\omega, \rho} \) to the set \([ -\frac{\rho}{2}, \frac{\rho}{2} ]^3 \) coincides with \( I^x_\omega \) and \( R^x_{\omega, \rho} \) is the identity map in the complement of \([ -\rho, \rho ]^3 \). Analogously, \( R^y_{\omega, \rho} = I^y_\omega \) in \([ -\frac{\rho}{2}, \frac{\rho}{2} ]^3 \) and \( R^y_{\omega, \rho} = \text{id} \) in \( \mathbb{R}^3 \setminus [ -\rho, \rho ]^3 \). Moreover,
\[ R^x_{\omega, \rho}([ -\rho, \rho ]^3) = R^y_{\omega, \rho}([ -\rho, \rho ]^3) = [ -\rho, \rho ]^3. \]

It is not hard to see that there are constants \( C_\rho \) and \( C'_\rho \) with
\[ \|R^x_{\omega, \rho} - \text{id}\|_{C^r} < C_\rho|\omega| \quad \text{and} \quad \|R^y_{\omega, \rho} - \text{id}\|_{C^r} < C'_\rho|\omega|. \]
Thus, for every \( \omega \) small enough, the maps \( R^x_{\omega, \rho} \) and \( R^y_{\omega, \rho} \) are \( C^r \)-perturbations of identity supported in \([ -\rho, \rho ]^3 \).

We call \( R^x_{\omega, \rho} \) and \( R^y_{\omega, \rho} \) rotation-like perturbations of the identity.

5. The unfolding family \( \mathcal{F} \)

We now describe the 8-parameter family \( \mathcal{F} = \{ f_\bar{v} \}_0 \) of \( C^r \)-diffeomorphisms, \( r \geq 2 \), unfolding the cycle of \( f \) at \( \bar{v} = 0 \in \mathbb{R}^8 \) (i.e., \( f_0 = f \)) in Theorem 1. The cycle of \( f \) has two parts with (say) “independent” unfoldings, one associated to the heterodimensional tangency and another one associated to the quasi-transverse heteroclinic point. Besides the unfolding of these heteroclinic non-transverse orbits we need to consider slight adjustments on the arguments of the eigenvalues of the saddles \( P \) and \( Q \). These adjustments are given by the rotation-like perturbations in Section 4. In summary, the family \( \mathcal{F} \) is obtained considering translation-like perturbations nearby the heteroclinic points \( X \) and \( Y \) and rotation-like perturbations corresponding to small changes of the arguments \( \varphi_P \) and \( \varphi_Q \). More precisely, the unfolding family is of the form
\[
f_{\bar{v}} = \Gamma_{\bar{v}} \circ f, \quad \text{with} \quad \bar{v} = (\bar{\mu}, \bar{\nu}, \alpha, \beta) \in [ -\epsilon, \epsilon ]^8,
\]
where $\Gamma_\varphi$ is the perturbation of the identity obtained as follows. For $R = P, Q, \bar{X}, \bar{Y}$ we take pairwise disjoint neighbourhoods $V_R \subset U_P \cup U_Q$ and let

\[
\begin{align*}
\Gamma'_\alpha(Z) & \overset{\text{def}}{=} R_{\alpha,\rho}(Z), \quad \text{if } Z \in V_P, \\
\Gamma^\varphi_\beta(Z) & \overset{\text{def}}{=} T_{\bar{X},\rho}(Z), \quad \text{if } Z \in V_\bar{X}, \\
\Gamma^Q_\beta(Z) & \overset{\text{def}}{=} R_{\beta,\rho}(Z), \quad \text{if } Z \in V_Q, \\
\Gamma^\varphi_\mu(Z) & \overset{\text{def}}{=} T_{\bar{Y},\mu}(Z), \quad \text{if } Z \in V_\bar{Y}, \\
id(Z), & \quad \text{if } Z \notin V_P \cup V_Q \cup V_X \cup V_Y,
\end{align*}
\]

(5.2) $\Gamma_{\vec{v}=(\vec{\mu},\vec{\nu},\alpha,\beta)}(Z) =$

where $\rho > 0$ is small $T_{Z,\bar{\varphi}}$, $R_{\omega,\rho}$ and $R_{\omega,\rho}$ are as in Section 4. The precise choice of these neighbourhoods $V_R$ is done below and the choice of $\rho$ in Sections 5.1 and 5.2 Properties of the map $\Gamma_{\vec{v}}$ are discussed in Section 5.3. Finally, we call $\Gamma^\varphi_\mu$ and $\Gamma^\varphi_\beta$ unfolding perturbations and $\Gamma^Q_\beta$ and $\Gamma^Q_\beta$ rotating perturbations.

We recall in Conditions (A)-(C), the definition of the neighbourhoods

$$U_R = (-10,10)^3 \ni R \quad \text{and} \quad U_Z \ni \bar{Z}, \quad \text{where} \quad R = P, Q, \quad Z = X, Y.$$ 

We recall that by the choice of $N_1$ and $N_2$ in Conditions (B)-(C), the points $\bar{X}$ and $\bar{Y}$ satisfies $f^{-1}(\bar{X}) \notin U_P$ and $f^{-1}(\bar{Y}) \notin U_Q$.

Consider the neighbourhoods $V_R = [-8,8]^3 \subset U_R$ of $R = P, Q$.

5.1. **The unfolding perturbations** $\Gamma^\varphi_\mu$ and $\Gamma^\varphi_\beta$. Consider a small enough $\rho > 0$ such that

$$\mathbb{B}_\rho(\bar{X}) \subset U_\bar{X}, \quad f^{-1}\left(\text{closure}(\mathbb{B}_\rho(\bar{X}))\right) \cap U_P = \emptyset, \quad \text{and}$$

$$\mathbb{B}_\rho(\bar{Y}) \subset U_\bar{Y}, \quad f^{-1}\left(\text{closure}(\mathbb{B}_\rho(\bar{Y}))\right) \cap U_Q = \emptyset.$$ 

In particular,

$$f(\mathbb{B}_\rho(\bar{Z})) \cap \mathbb{B}_\rho(\bar{Z}) = \emptyset, \quad Z = X, Y;$$

and

$$P \notin \text{closure}(\mathbb{B}_\rho(\bar{X})) \quad \text{and} \quad Q \notin \text{closure}(\mathbb{B}_\rho(\bar{Y})).$$

For $\bar{\mu}$ and $\bar{\nu}$ in $\mathbb{R}^3$ we define the unfolding perturbations as

$$\Gamma^\varphi_\mu : M \rightarrow M, \quad \Gamma^\varphi_\mu \overset{\text{def}}{=} T_{\bar{Y},\bar{\mu}} \quad \text{and} \quad \Gamma^\varphi_\beta : M \rightarrow M, \quad \Gamma^\varphi_\beta \overset{\text{def}}{=} T_{\bar{X},\bar{\nu}},$$

where $T_{Z,\varphi}$ are the translation-like perturbations in (4.1). See Figure 3

As was observed, if $||\bar{\mu}||$ and $||\bar{\nu}||$ are small enough the maps $\Gamma^\varphi_\mu$ and $\Gamma^\varphi_\beta$ are $C^\varphi$-perturbations of the identity supported in $\mathbb{B}_\rho(\bar{X})$ and $\mathbb{B}_\rho(\bar{Y})$, respectively, with

$$\Gamma^\varphi_\mu(\mathbb{B}_\rho(\bar{X})) = \mathbb{B}_\rho(\bar{X}) \quad \text{and} \quad \Gamma^\varphi_\mu(\mathbb{B}_\rho(\bar{Y})) = \mathbb{B}_\rho(\bar{Y}).$$

5.2. **The rotating perturbations** $\Gamma^P_\alpha$ and $\Gamma^Q_\beta$. For $\alpha$ and $\beta$ in $\mathbb{R}$, we define the rotating perturbations

$$\Gamma^P_\alpha : M \rightarrow M, \quad \Gamma^P_\alpha \overset{\text{def}}{=} R_{\alpha,\beta}^x \quad \text{and} \quad \Gamma^Q_\beta : M \rightarrow M, \quad \Gamma^Q_\beta \overset{\text{def}}{=} R_{\beta,\beta}^y,$$

where $R_{\omega,\beta}^x$ and $R_{\omega,\beta}^y$ are the rotation like-perturbations in (4.2). As was observed, if $\alpha$ and $\beta$ are small enough the maps $\Gamma^P_\alpha$ and $\Gamma^Q_\beta$ are $C^\varphi$-perturbations of the identity supported in $V_P$ and $V_Q$, respectively, such that

$$\Gamma^P_\alpha(V_P) = V_P \quad \text{and} \quad \Gamma^Q_\beta(V_Q) = V_Q.$$
5.3. Properties of the unfolding family $\mathcal{F}$. We now list some relevant properties satisfied by the family $\mathcal{F}$.

Remark 5.1 (Properties of $\mathcal{F}$). By construction, $f_0 = f$ and for every small (in norm) $\bar{\nu} = (\bar{\mu}, \bar{\nu}, \alpha, \beta)$ we have that $f_{\bar{\nu}}$ is a diffeomorphism $C^r$-close to $f$ having a pair of saddle points $P_{\bar{\nu}} = P$ and $Q_{\bar{\nu}} = Q$ such that (see Figure 3):

1. For every $W \in U_P \cap f^{-1}(V_P)$, it holds $f_{\bar{\nu}}(W) = \Gamma_P^\alpha \circ f(W)$.
2. For every $W \in U_Q \cap f^{-1}(V_Q)$, it holds $f_{\bar{\nu}}(W) = \Gamma_Q^\beta \circ f(W)$.
3. $f_{\bar{\nu}}(f^{-1}(B_\rho(\tilde{X}))) = B_\rho(\tilde{X})$ and $f^{N_1}_{\bar{\nu}}(X) = f^\alpha_{\bar{\nu}}(f^{-1}(\tilde{X})) = \tilde{X} + \bar{\nu}$.
4. $f_{\bar{\nu}}(f^{-1}(B_\rho(\tilde{Y}))) = B_\rho(\tilde{Y})$ and $f^{N_2}_{\bar{\nu}}(X) = f_{\bar{\nu}}(f^{-1}(\tilde{Y})) = \tilde{Y} + \bar{\mu}$.
5. $f_{\bar{\nu}} = f$ in $M \setminus V_P \cup V_Q \cup f^{-1}(B_\rho(\tilde{X})) \cup f^{-1}(B_\rho(\tilde{Y}))$.

6. Return maps at the heterodimensional tangency

The renormalisation scheme of $f$ in Theorem 5 involves return maps (defined on a small neighbourhood of the heterodimensional tangency point $\tilde{Y}$) of the form

$$F_{\bar{\nu}}^{m,n} \overset{\text{def}}{=} f^{N_2}_{\bar{\nu}} \circ f^m_{\bar{\nu}} \circ f^{N_1}_{\bar{\nu}} \circ f^n_{\bar{\nu}},$$

where $N_1$ and $N_2$ are the transition times between neighbourhoods of the saddles in the cycle in Conditions B and C (see Sections 3.2 and 3.3). For the next discussion recall the definitions of the quasi-transverse heteroclinic point $\tilde{X}$ and of the points $X \in U_Q$ with $f^{N_1}(X) = \tilde{X} \in U_P$ and $Y \in U_P$ with $f^{N_2}(Y) = \tilde{Y} \in U_Q$. 

Figure 3. The unfolding family $\mathcal{F}$. 

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For suitable choices of \( n \) and \( m \), we are interested in iterations of points close to heterodimensional tangency point \( \tilde{Y} \) by the map \( f_\tilde{Y} \) that after \( n \) iterates in \( U_Q \) land in a neighbourhood of \( X \). These points are mapped by \( f_\tilde{Y}^N \) close to \( \tilde{X} \) in \( U_P \). Thereafter, they remain in \( U_P \) during \( m \) iterates of \( f_\tilde{Y} \), landing in a neighbourhood of \( Y \). Finally, they return nearby \( \tilde{Y} \) by the transition \( f_\tilde{Y}^N \). The specially selected times \( n \) and \( m \) are called sojourn times in the neighbourhoods \( U_P \) and \( U_Q \). These times “determine” the set nearby \( \tilde{Y} \) where this return is defined.

The composition \((6.1)\) also demands another cares. For instance, to guarantee that after \( m \) iterations certain points nearby \( \tilde{X} \) are mapped in the domain of the transition from \( P \) to \( Q \) (i.e., in a small neighbourhood of \( Y \)) it is necessary a small change \( \varphi_P + \alpha_m(\varphi_P) \) of the argument \( \varphi_P \) of \( Df(P) \), where \( \alpha_m(\varphi_P) \) depends on the sojourn time \( m \). For the sojourn times \( n \), we need to consider similar adjustments \( \beta_n(\varphi_Q) \) of the argument \( \varphi_Q \) of \( Df(Q) \). The arguments \( \varphi_P + \alpha_m(\varphi_P) \) and \( \varphi_Q + \beta_n(\varphi_Q) \) are called adapted arguments.

We divide the study of the renormalisation scheme in two parts: (i) choice of adequate sojourn times \( m \) and \( n \) and adapted arguments, see Sections 6.1 and 6.2; (ii) choice of suitable sequences of unfolding parameters \( \bar{v}_{m,n} \) and charts \( \Psi_{m,n} : \mathbb{R}^3 \rightarrow M \), see Section 6.3. The convergence of the renormalised sequence \( \Psi_{m,n}^{-1} \circ F_{\bar{v}_{m,n}} \circ \Psi_{m,n} \) will be studied in Sections 6 and 7.

6.1. Sojourn times. Consider the set

\[ \mathcal{Z} \stackrel{\text{def}}{=} \{ (\sigma, \lambda) \in \mathbb{R}^2 : 0 < \lambda < 1 < \sigma \}. \]

An application of the next lemma will provide the desired sojourn times.

**Lemma 6.1** (Lemma 5.1 in [14]). There is a residual subset \( \mathcal{R} \) of \( \mathcal{Z} \) consisting of points \( (\sigma, \lambda) \) satisfying the following property. For every \( \epsilon > 0, \xi > 0, \tau > 0 \) and \( N_0 > 0 \), with \( \epsilon < \xi \), there exist integers \( m, n > N_0 \) such that

\[ |\tau \sigma^m \lambda^n - \xi| < \epsilon \quad \text{and} \quad |m - n \eta - \tilde{\eta}| < 1, \]

where

\[ \eta = \frac{\log \lambda^{-1}}{\log \sigma} \quad \text{and} \quad \tilde{\eta} = \frac{\log(\tau \xi^{-1})}{\log \sigma}. \]

In particular, there exit sequences \((m_k), (n_k) \rightarrow +\infty \) as \( k \rightarrow +\infty \) such that

\[ \lim_{k \rightarrow +\infty} \sigma^{m_k} \lambda^{n_k} = \tau^{-1} \xi. \]

**Remark 6.2** (Choice of sojourn times). A sequence of sojourn times (adapted to \( \sigma_P, \lambda_Q, \tau \) and \( \xi \)) is any sequence \((s_k) \), with \( s_k = (m_k, n_k) \in \mathbb{N}^2 \), obtained by applying Lemma 6.1 to:

- \((\sigma_P, \lambda_Q) \in \mathcal{Z} \), where \( \lambda_Q, \sigma_P \) satisfying the spectral condition of the cycle in \( (3.3) \);
- \( \tau = \tau(a_2, a_3, \gamma_3) \stackrel{\text{def}}{=} \frac{\gamma_3(a_3 - a_2)}{\sqrt{2}} \), where \( a_2, a_3 \) and \( \gamma_3 \) are the constants in the definition of the transition maps between \( P \) and \( Q \), see \((3.6)\) and \((3.10)\).

Recall that by \((3.14)\) it holds \( \tau(a_2, a_3, \gamma_3) > 0 \);

- \( \xi > 0 \) is arbitrary but fixed.

\[ ^5 \text{As the spectral condition } (3.3) \text{ is open, we can suppose that } (\sigma_P, \lambda_Q) \in \mathcal{R}. \]
As a consequence, for a sequence of sojourn times \( s_k = (m_k, n_k) \) it holds
\[
\alpha^P_k \lambda^Q_k \to \tau^{-1} \xi = \left( \frac{\gamma_3(a_3 - a_2)}{\sqrt{2}} \right)^{-1} \xi, \quad k \to +\infty.
\]

We say that the sequence \( s_k \) is adapted to \( \tau^{-1} \xi \).

### 6.2. Adapted arguments

We now discuss the choice of the “adjusting arguments”. Given a sequence \( s_k = (m_k, n_k) \) of sojourn times consider a sequence \( \Theta_k = (\zeta_{m_k}, \theta_{n_k}) \) in \( \mathbb{R}^2 \) such that \( \Theta_k \to (0, 0) \) as \( k \to +\infty \). We call the pair \( \Theta_k = (s_k, \Theta_k) \) a sequence of sojourn times with associated arguments. We define the sequences \( \alpha^P_k = \alpha^P_{\Theta_k} \) and \( \beta^Q_k = \beta^Q_{\Theta_k} \) of argument adjustment maps as follows,
\[
\alpha^P_k : [-\pi, \pi] \to [-\pi, \pi], \quad \alpha^P_k(\theta) = \frac{1}{2\pi m_k} \left( \frac{\pi}{4} - 2\pi m_k \theta + 2\pi [m_k \theta] + \zeta_{m_k} \right);
\]
\[
\beta^Q_k : [-\pi, \pi] \to [-\pi, \pi], \quad \beta^Q_k(\omega) = \frac{1}{2\pi n_k} \left( \frac{\pi}{2} - 2\pi n_k \omega + 2\pi [n_k \omega] + \theta_{n_k} \right),
\]
where \( [x] \) denotes the integer part of \( x \in \mathbb{R} \). Since \( x - [x] \in [0, 1) \) it follows that
\[
\alpha^P_k(\theta) \to 0, \quad \beta^Q_k(\omega) \to 0, \quad k \to +\infty,
\]
for every fixed \( \theta, \omega \in [-\pi, \pi] \).

The pair of sequences
\[
\theta^P_k \overset{\text{def}}{=} \theta + \alpha^P_k(\theta) \to \theta \quad \text{and} \quad \omega^Q_k \overset{\text{def}}{=} \omega + \beta^Q_k(\omega) \to \omega, \quad k \to +\infty,
\]
is called sequence of arguments adapted to \( \theta \) and \( \omega \) associated to \( \Theta_k = (s_k, \Theta_k) \).

### 6.3. Elements of the renormalisation scheme

Consider \( 0 < \lambda_Q < 1 < \sigma_P \) satisfying (3.3) and \( \xi > 0 \). If \( (\sigma_P, \lambda_Q) \in \mathbb{Z} \), Lemma 6.1 provides a sequence of sojourn times \( s_k = (m_k, n_k) \in \mathbb{N}^2 \) adapted to \( \tau^{-1} \xi \), where \( \tau \) is as Remark 6.2.

Consider a sequence of sojourn times with associated arguments \( \Theta_k = (s_k, \Theta_k) \), where \( \Theta_k = (\zeta_{m_k}, \theta_{n_k}) \).

The renormalisation scheme \( \mathcal{R}(\xi, \Theta) \) of \( f \) consists of the following elements:

(i) \( \Psi_k = \Psi_{s_k} : \mathbb{R}^3 \to M \) a sequence of parameterisations on the manifold \( M \);
(ii) \( \bar{v}_k = \bar{v}_{s_k} : \mathbb{R} \times [-\pi, \pi]^2 \to \mathbb{R}^8 \) a sequence of bifurcation parameter maps of the family \( \bar{v} \to f_{\bar{v}} \) in (5.1);
(iii) \( \mathcal{R}_k(f) = \mathcal{R}_{\bar{v}_k}(f) \) is the sequence in \( \text{Diff}^r(M) \) defined by
\[
f_{\bar{v}_k}^N \circ f_{\bar{v}_k}^m \circ f_{\bar{v}_k}^N \circ f_{\bar{v}_k}^m.
\]

This composition is the renormalised sequence of \( f \).

We now give the precise definitions of the objects in the renormalisation scheme. In what follows, we fix a sequence of sojourn times with arguments \( \Theta_k = (s_k, \Theta_k) \), where \( s_k = (m_k, n_k) \) and \( \Theta_k = (\theta_{m_k}, \theta_{n_k}) \).

### 6.3.1. The parameterisations \( \Psi_k = \Psi_{s_k} \)

Given \( s = (m, n) \in \mathbb{N}^2 \), consider the map \( \Psi_s : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
\[
\Psi_s(x, y, z) \overset{\text{def}}{=} (1 + \sigma_P^{-m} \sigma_Q^{-n} x, \sigma_Q^{-n} + \sigma_P^{-2m} \sigma_Q^{-2n} y, 1 + \sigma_P^{-m} \sigma_Q^{-n} z).
\]

**Remark 6.3.** Let \( K \subset \mathbb{R}^3 \) be any compact set. The sequence of compact sets \( (\Psi_{s_k}(K))_k \) in \( M \) satisfies
\[
\Psi_{s_k}(K) \to \bar{Y} = (1, 0, 1) \subset U_Q, \quad \text{as} \ k \to +\infty,
\]
where the convergence is in the Hausdorff distance. In particular, there is \( k_0 = k_0(K) \) such that \( \Psi_{s_k}(K) \subset U_Q \) for every \( k \geq k_0 \). In what follows, for notational simplicity, we write \( \Psi_k = \Psi_{s_k} \).

6.3.2. \textit{The bifurcation parameters} \( \tilde{v}_k = \tilde{v}_{\Theta_k} \). The sequence of maps \( \tilde{v}_{\Theta_k} : \mathbb{R}^3 \rightarrow \mathbb{R}^8 \) is of the form

\[
\tilde{v}_{\Theta_k}(\mu, \theta, \omega) = (\tilde{\mu}_{s_k}(\mu), \tilde{\nu}_{\Theta_k}, \alpha_{\Theta_k}(\theta), \beta_{\Theta_k}(\omega))
\]

where:

- \( \tilde{\mu}_{s_k} : \mathbb{R} \rightarrow \mathbb{R}^3 \) is defined by

\[
\tilde{\mu}_{s_k}(\mu) \overset{\text{def}}{=} (-\lambda_P^{m_k} a_1, \sigma_Q^{-n_k} + \sigma_Q^{-2n_k} \sigma_P^{-2m_k} \mu - \lambda_P^{m_k} b_1, -\lambda_P^{m_k} c_1),
\]

where \( a_1, b_1, c_1 \) are given in (3.10). Note that \( \tilde{\mu}_{s_k}(\mu) \rightarrow (0, 0, 0) \) as \( k \rightarrow +\infty \) for every fixed \( \mu \in \mathbb{R} \). In what follows, for notational simplicity, we write \( \tilde{\mu}_k = \tilde{\mu}_{s_k} \).

- The maps \( \alpha^P(\theta) = \alpha^P(\Theta_k)(\theta) \) and \( \beta^Q(\omega) = \beta^Q(\Theta_k)(\omega) \) are defined in (6.3).

- The sequence \( \tilde{\nu}_{\Theta_k} \in \mathbb{R}^8 \) is defined as follows. For the sequence of arguments adapted to \( \varphi^P \) and \( \varphi^Q \) (associated to \( \Theta_k \)), we write

\[
\varphi_{P,k} \overset{\text{def}}{=} (\varphi_{P})_k^P = \varphi_P + \alpha^P(\varphi_P) \quad \text{and} \quad \varphi_{Q,k} \overset{\text{def}}{=} (\varphi_{Q})_k^Q = \varphi_Q + \beta^Q(\varphi_Q)
\]

and define the following sequences

\[
\begin{align*}
\tilde{c}_k & \overset{\text{def}}{=} \cos (2\pi m_k(\varphi_{P,k})), & \tilde{s}_k & \overset{\text{def}}{=} \sin (2\pi m_k(\varphi_{P,k})), \\
\tilde{\sigma}_k & \overset{\text{def}}{=} \cos (2\pi n_k(\varphi_{Q,k})), & \tilde{\varphi}_k & \overset{\text{def}}{=} \sin (2\pi n_k(\varphi_{Q,k})), \end{align*}
\]

and

\[
\begin{align*}
\tilde{\rho}_{2,k} & \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{H}_2(0)(\tilde{c}_k - \tilde{s}_k)^2 + \frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{H}_2(0)(\tilde{s}_k + \tilde{c}_k)^2, \\
\tilde{\rho}_{3,k} & \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{H}_3(0)(\tilde{s}_k - \tilde{c}_k)^2 + \frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{H}_3(0)(\tilde{c}_k + \tilde{s}_k)^2.
\end{align*}
\]

We now let

\[
\tilde{v}_{\Theta_k} \overset{\text{def}}{=} \left( -\lambda_P^{n_k}(\alpha_1(\tilde{c}_k - \tilde{s}_k) + \alpha_3(\tilde{s}_k + \tilde{c}_k)), \sigma_P^{-m_k}(\tilde{c}_k + \tilde{s}_k) - \lambda_Q^{2n_k} \tilde{\rho}_{2,k}, \\
\sigma_P^{-m_k}(\tilde{c}_k - \tilde{s}_k) - \lambda_Q^{n_k} \gamma_3(\tilde{c}_k + \tilde{s}_k) - \lambda_Q^{2n_k} \tilde{\rho}_{3,k}. \right)
\]

In what follows, for notational simplicity, we write \( \tilde{v}_k = \tilde{v}_{\Theta_k} \).

\textbf{Claim 6.4.} \( \tilde{v}_k \rightarrow (0, 0, 0) \in \mathbb{R}^3 \) as \( k \rightarrow +\infty \).

\textbf{Proof.} Note that

\[
\begin{align*}
\tilde{c}_k & \overset{\text{def}}{=} \cos \left( \frac{\pi}{4} + \psi_{m_k} \right) \rightarrow \frac{1}{\sqrt{2}}, & \tilde{s}_k & \overset{\text{def}}{=} \sin \left( \frac{\pi}{4} + \psi_{m_k} \right) \rightarrow \frac{1}{\sqrt{2}}, \\
\tilde{\sigma}_k & \overset{\text{def}}{=} \cos \left( \frac{\pi}{2} + \psi_{n_k} \right) \rightarrow 0, & \tilde{\varphi}_k & \overset{\text{def}}{=} \sin \left( \frac{\pi}{2} + \psi_{n_k} \right) \rightarrow 1,
\end{align*}
\]

when \( k \rightarrow +\infty \). Therefore,

\[
\begin{align*}
\tilde{\rho}_{2,k} & \rightarrow \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{H}_2(0) + \frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{H}_2(0), & \tilde{\rho}_{3,k} & \rightarrow \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{H}_3(0) + \frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{H}_3(0).
\end{align*}
\]

The claim follows immediately from \( |\lambda_P|, |\lambda_Q| < 1 \) and \( |\sigma_P|, |\sigma_Q| > 1 \). \( \square \)
Remark 6.5 (Expression of \( f_{\vec{v}_k}(\mu, \theta, \omega) = f_{\vec{v}_k}(\mu, \theta, \omega) \)). Recall that \( f_{\vec{v}} = \Gamma_0 \circ f \), see (5.1). Recalling the definitions of \( \Gamma_0 \) in (5.2), of \( \Gamma_{P}^P \) and \( \Gamma_{P}^{ht} \) in Section 5.1 and of \( \Gamma_{\alpha}^P \) and \( \Gamma_{\beta}^Q \) in Section 5.2 we have that

\[
\begin{align*}
\Gamma_{\alpha}^P(\theta) & = f_{\alpha}^P(\theta), \quad \text{in } V_P, \\
\Gamma_{\mu}^{ht}(\mu) & = f_{\mu}(\mu), \quad \text{in } V_{f-1}(\overline{\gamma}), \\
\Gamma_{\beta}^Q(\omega) & = f_{\beta}^Q(\omega), \quad \text{in } V_Q, \\
\Gamma_{\nu}^* & = f_{\nu}^*, \quad \text{in } V_{f-1}(\overline{\chi}), \\
f & = M \setminus V_P \cup V_Q \cup V_{f-1}(\overline{\chi}) \cup V_{f-1}(\overline{\gamma}),
\end{align*}
\]

where \( V_{f-1}(\overline{Z}) \) is \( f^{-1}(\overline{B}_P(\overline{Z})) \), \( Z = X, Y \). See Remark 5.1.

Remark 6.6. By definition of the arguments adapted to \( \varphi_P \) and \( \varphi_Q \) (see Subsection 6.2) it follows that the argument of \( f_{\alpha_k(\varphi_P)} \) in the neighbourhoods \( V_P \) of \( U_P \) is given by \( 2\pi(\varphi_P + \alpha_k^{\prime}(\varphi_P)) = 2\pi(\varphi_{P,k}) \rightarrow 2\pi(\varphi_P) \). Thus, recalling the local form of \( f \) in the neighbourhoods of \( P \) and \( Q \) in equation (5.1),

\[
(f_{\alpha_k(\varphi_P)}|_{V_P})^{mk} = \begin{pmatrix}
\lambda_P^{mk} & 0 & 0 \\
0 & \sigma_P^{mk} \tilde{c}_k & -\sigma_P^{mk} \tilde{s}_k \\
0 & \sigma_P^{mk} \tilde{s}_k & \sigma_P^{mk} \tilde{c}_k
\end{pmatrix},
\]

where \( \tilde{c}_k, \tilde{s}_k \), are as in [6.5]. Analogously, it holds

\[
(f_{\beta_k(\varphi_Q)}|_{V_Q})^{nk} = \begin{pmatrix}
\lambda_Q^{nk} c_k & 0 & -\lambda_Q^{nk} s_k \\
0 & \sigma_Q^{nk} c_k & 0 \\
\lambda_Q^{nk} s_k & 0 & \lambda_Q^{nk} c_k
\end{pmatrix}.
\]

Remark 6.7 (Convergence to \( f \)). By construction, given any \( (\mu, \theta, \omega) \in \mathbb{R}^3 \), the sequence \( f_{\vec{v}_k, \Theta_k} \) with \( \vec{v}_k = \vec{v}_k(\mu, \theta, \omega) \) converges to \( f \) in the \( C^r \)-topology.

7. THE RENORMALISED SEQUENCE OF MAPS

Fixed the renormalisation scheme \( \mathcal{R}(\xi, \overline{y}) \) of \( f \) associated to the sequence \( (\Theta_k) = (\Theta_k) \), where \( \Theta_k = (\vartheta_{mk}, \vartheta_{nk}) \), and a compact set \( K \subset \mathbb{R}^3 \), we now study the sequence of maps

\[
\Psi_k^{-1} \circ \mathcal{R}_{\vec{v}_k}(f) \circ \Psi_k : K \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \vec{v}_k = \vec{v}_k(\mu, \theta, \omega).
\]

Using the notation of the previous section, we begin by considering parameters of the form

\[
\vec{v}_k(\mu, \varphi_P, \varphi_Q) = (\vec{v}_k(\mu), \vec{v}_k, \alpha_k^{\prime}(\varphi_P), \beta_k^{\prime}(\varphi_Q)).
\]

For this choice of parameters, the renormalisation scheme involves the adapted arguments \( \varphi_P \) and \( \varphi_Q \), the parameter \( \vec{v}_k \) (which depends on the choice of these arguments, see (6.7)), and a “free” parameter \( \mu \). For notational simplicity, write

\[
\alpha_k,P = \alpha_k^{\prime}(\varphi_P), \quad \beta_k,Q = \beta_k^{\prime}(\varphi_Q), \quad \vec{v}_k(\mu) = \vec{v}_k(\mu, \varphi_P, \varphi_Q),
\]

and for \( \overline{X} = (x, y, z) \in K \)

\[
(7.1) \quad \Psi_k^{-1} \circ \mathcal{R}_{\vec{v}_k(\mu)} \circ \Psi_k(x, y, z) = \overline{X}_k(x, y, z) \overset{\text{def}}{=} (\tilde{x}_k, \tilde{y}_k, \tilde{z}_k).
\]

The goal of this section is to determine the coordinates of \( \overline{X}_k \), this is done in equations (7.15)-(7.17). With these coordinates at hand, we will obtain the convergence
of the renormalisation scheme. The calculation of these coordinates involves three intermediate steps, corresponding to the compositions $\Psi_k(\bar{X}), \mathcal{R}_{\bar{\upsilon}_k(\mu)} \circ \Psi_k(\bar{X})$, and $\Psi^{-1}_k \circ \mathcal{R}_{\bar{\upsilon}_k(\mu)} \circ \Psi_k(\bar{X})$.

7.1. Coordinates of $\Psi_k(\bar{X})$. Write $\bar{X}_k \overset{\text{def}}{=} \Psi_k(\bar{X})$ where

$$\Psi_k(\bar{X}) = (1 + \sigma^{-m_k} \sigma_Q^{-n_k} x, \sigma^{-2m_k} \sigma_Q^{-2n_k} y, 1 + \sigma^{-m_k} \sigma_Q^{-n_k} z).$$

By the compactness of the set $K$, $\bar{X}_k \rightarrow \bar{Y} = (1, 0, 1) \in U_Q$, as $k \rightarrow +\infty$. Note that this step does not depend on $\mu$.

7.2. Coordinates of $\mathcal{R}_{\bar{\upsilon}_k(\mu)} \circ \Psi_k(\bar{X})$. Recall the definition of $\tilde{f}_{\bar{\upsilon}_k(\mu, \theta, \omega)}$ in (6.9). The application of $\mathcal{R}_{\bar{\upsilon}_k(\mu)}$ involves the following four "independent" intermediate steps (see Figure 4):

- **Step A**: $n_k$ iterations nearby $Q$ given by $\tilde{f}_{\bar{\upsilon}_k(\mu)}^{n_k} = \tilde{f}_{\beta_k, Q}^{n_k}$;
- **Step B**: the transition from $Q$ to $P$ along the orbit of $\bar{X}$ given by $\tilde{f}_{\bar{\upsilon}_k(\mu)}^{N_1} = f_{\bar{\upsilon}_k(\mu)} \circ f^{N_1-1}$;
- **Step C**: $m_k$ iterations nearby $P$ given by $\tilde{f}_{\bar{\upsilon}_k(\mu)}^{m_k} = \tilde{f}_{\alpha_k, P}^{m_k}$;
- **Step D**: the transition from $P$ to $Q$ along the orbit of $\bar{Y}$ given by $\tilde{f}_{\bar{\upsilon}_k(\mu)}^{N_2} = f_{\bar{\upsilon}_k(\mu)} \circ f^{N_2-1}$. This last step is the only one depending on $\mu$.

**Step A**: Recalling (6.9) and (7.2), write

$$\tilde{f}_{\beta_k, Q}^{n_k}(\bar{X}_k) \overset{\text{def}}{=} (x_k, y_k + 1, z_k) = X_k \in U_Q.$$
The following equalities follow from Remark 6.6
\[ x_k = \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} (\xi_k x - s_k z) + \lambda_Q^{n_k} (\xi_k - s_k), \]
\[ y_k = \sigma_P^{-2m_k} \sigma_Q^{-n_k} y, \]
\[ z_k = \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} (s_k x + \xi_k z) + \lambda_Q^{n_k} (\xi_k + s_k). \]
Since \( K \) is a compact set, we have that
\[ x_k = O(\lambda_Q^{n_k}), \quad y_k = O(\sigma_P^{-2m_k} \sigma_Q^{-n_k}), \quad z_k = O(\lambda_Q^{n_k}), \]
where \( O(\cdot) \) denotes the symbol of Landau.\(^6\) Therefore
\[ X_k = (x_k, y_k + 1, z_k) \to X = (0, 1, 0) \in U_X, \quad \text{as} \ k \to +\infty. \]
(7.6)

**Step B:** The transition from a neighbourhood of \( X \) to \( U_P \) given by \( f_{v_0}^{-1} (x, y, z) = f_{v_0}^{-1} (x, y, z) \in U_P \).

Using the definitions of \( f_{v_0} \) in (3.9) (recall also (3.7)), of \( f_{v_0} \) in (6.9), and of \( X_k \) in (7.3), we get
\[ \hat{x}_k = \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} \alpha_1 (\xi_k x - s_k z) + \sigma_P^{-2m_k} \sigma_Q^{-n_k} \alpha_2 y + \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} \alpha_3 (s_k x + \xi_k z) + \tilde{H}_1 (x_k), \]
\[ \hat{y}_k = \sigma_P^{-2m_k} \sigma_Q^{-n_k} \beta_2 y + \sigma_P^{-m_k} (\xi_k + s_k) - \lambda_Q^{n_k} \tilde{P}_{2,k} + \tilde{H}_2 (x_k), \]
\[ \hat{z}_k = \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} \gamma_3 (s_k x + \xi_k z) + \sigma_P^{-m_k} (\xi_k - s_k) - \lambda_Q^{2n_k} \tilde{P}_{3,k} + \tilde{H}_3 (x_k). \]

For simplicity, in what follows, we write
\[ \tilde{H}_i (x_k) \overset{\text{def}}{=} \hat{H}_i (x_k) - \lambda_Q^{2n_k} \tilde{P}_i, \quad i = 2, 3. \]

**Step C:** Recalling (6.9), Remark (6.6) and equation (7.7) writing
\[ f_{v_k}^{-1} (\hat{x}_k, 1 + \hat{y}_k, 1 + \hat{z}_k) = \hat{X}_k \in U_P, \]
we have (after some straightforward simplifications and using that \( \sigma_1^2 + \sigma_2^2 = 1 \)) the following equalities:
\[ \hat{x}_k = \lambda_P^{m_k} + \lambda_P^{m_k} \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} (\alpha_1 (\xi_k x - s_k z) + \alpha_3 (s_k x + \xi_k z)) + \lambda_P^{m_k} \sigma_P^{-2m_k} \sigma_Q^{-n_k} \alpha_2 y + \lambda_P^{m_k} \tilde{H}_1 (x_k), \]
\[ \hat{y}_k = \sigma_P^{-m_k} \sigma_Q^{-n_k} \xi_k \beta_2 y - \lambda_Q^{n_k} \sigma_Q^{-n_k} \xi_k \gamma_3 (s_k x + \xi_k z) + \sigma_P^{m_k} (\xi_k \tilde{H}_2 (x_k) - s_k \tilde{H}_3 (x_k)), \]
\[ \hat{z}_k = \sigma_P^{-m_k} \sigma_Q^{-n_k} \xi_k \beta_2 y + \lambda_Q^{n_k} \sigma_Q^{-n_k} \xi_k \gamma_3 (s_k x + \xi_k z) + \sigma_P^{m_k} (s_k \tilde{H}_2 (x_k) + \xi_k \tilde{H}_3 (x_k)). \]

**Lemma 7.1.** \( \hat{x}_k = O(\lambda_P^{m_k}), \hat{y}_k = O(\sigma_P^{-m_k} \sigma_Q^{-n_k}), \) and \( \hat{z}_k = O(\sigma_P^{-m_k} \sigma_Q^{-n_k}). \)

\(^6\)Given two real valued functions \( f \) and \( g \), one says that the symbol of Landau of \( f(x) \) is \( O(g(x)) \) (as \( x \to +\infty \)) if and only if there are \( M > 0 \) and \( x_0 \) with \( |f(x)| \leq M |g(x)| \) for all \( x > x_0 \).
\textbf{Proof.} Note that \( \hat{y}_k \) and \( \hat{z}_k \) have the same symbol of Landau. The conditions in (3.6) and (3.8) and the definition of \( x_k \) in (7.6) imply that the higher order terms \( H_i, i = 1, 2, 3 \), are dominated by quadratic terms. Thus, for \( i = 1, 2, 3 \), it holds

\[
\hat{H}_i(x_k) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \hat{H}_i(0) x_k^2 + \frac{\partial^2}{\partial x \partial y} \hat{H}_i(0) x_k y_k + \frac{\partial^2}{\partial x \partial z} \hat{H}_i(0) x_k z_k +
\]

\[
+ \frac{1}{2} \frac{\partial^2}{\partial y^2} \hat{H}_i(0) y_k^2 + \frac{\partial^2}{\partial y \partial z} \hat{H}_i(0) y_k z_k + \frac{1}{2} \frac{\partial^2}{\partial z^2} \hat{H}_i(0) z_k^2 + \text{h.o.t.} \quad (7.10)
\]

Since \( x_k \) and \( z_k \) have the same Landau symbol \( O(\lambda_Q^{2n_k}) \) and \( y_k \) has Landau symbol \( O(\sigma_P^{-2m_k} \sigma_Q^{-n_k}) \) (see (7.5)), it follows that

\[
\hat{H}_i(x_k) = O(\lambda_Q^{2n_k}) + O(\sigma_P^{-2m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k}) + O(\sigma_P^{-4m_k} \sigma_Q^{-2n_k}), \quad i = 1, 2, 3.
\]

Finally, as the set \( K \) is compact, it follows from (7.11) that \( \hat{x}_k = O(\lambda_P^{m_k}) \).

We now determine the symbol of \( \hat{y}_k \) (and hence one of \( \hat{z}_k \)). For this, its is enough to estimate the terms \( \sigma_P^{m_k} \hat{H}_2(x_k) \) and \( \sigma_P^{m_k} \hat{H}_3(x_k) \) (as the other terms in the expression of \( \hat{y}_k \) are larger). Using the Taylor formula in (7.10), the definition of the coordinates (7.4), and the definition of \( \hat{\rho}_{i,k} \) in (6.6), it follows from (7.5) and (7.8) that

\[
\hat{H}_i(x_k) = \hat{H}_i(x_k) - \lambda_Q^{2n_k} \hat{\rho}_{i,k} = O(\sigma_P^{-2m_k} \lambda_Q^{2n_k} \sigma_Q^{-2n_k}) + O(\sigma_P^{-m_k} \lambda_Q^{2n_k} \sigma_Q^{-n_k}) +
\]

\[
+ O(\sigma_P^{-2m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k}) + O(\sigma_P^{-4m_k} \sigma_Q^{-2n_k}), \quad i = 2, 3.
\]

For the next estimate, recall that by (6.2) the sequence \( \lambda_Q^{n_k} \sigma_P^{m_k} \) converges to some number different from 0. Hence \( \lambda_Q^{n_k} \sigma_P^{m_k} \) and \( \lambda_Q^{-n_k} \sigma_P^{-m_k} \) are bounded from above by some \( K_0 > 0 \). This bound and \( \sigma_P, \sigma_Q > 1 \) provide the following estimates,

- \( \sigma_P^{-2m_k} \lambda_Q^{2m_k} \sigma_Q^{-2n_k} < \sigma_P^{-m_k} \lambda_Q^{2n_k} \sigma_Q^{-n_k} \),
- \( \sigma_P^{-2m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} = (\lambda_Q^{2n_k} \sigma_P^{-m_k}) \sigma_P^{-m_k} \lambda_Q^{-2m_k} \sigma_Q^{-n_k} < K_0 \sigma_P^{-m_k} \lambda_Q^{n_k} \sigma_Q^{-n_k} \),
- \( \sigma_P^{-4m_k} \sigma_Q^{-2n_k} = (\lambda_Q^{2n_k} \sigma_P^{-2m_k})(\sigma_P^{m_k} \sigma_Q^{-n_k}) \sigma_P^{-m_k} \lambda_Q^{2n_k} \sigma_Q^{-n_k} < K_0^{2} \sigma_P^{-m_k} \lambda_Q^{2n_k} \sigma_Q^{-n_k} \).

These inequalities imply that

\[
\hat{H}_i(x_k) = \hat{H}_i(x_k) - \lambda_Q^{2n_k} \hat{\rho}_{i,k} = O(\sigma_P^{-m_k} \lambda_Q^{2n_k} \sigma_Q^{-n_k}), \quad i = 2, 3.
\]

Therefore

\[
\sigma_P^{m_k} \hat{H}_i(x_k) = O(\lambda_Q^{2n_k} \sigma_Q^{-n_k}), \quad i = 2, 3.
\]

Finally, observing that

\[
\lambda_Q^{2n_k} \sigma_Q^{-n_k} = (\sigma_P^{m_k} \lambda_Q^{n_k}) \lambda_Q^{n_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} < K_0 \sigma_P^{-m_k} \sigma_Q^{-n_k}
\]

we get \( \hat{y}_k = O(\sigma_P^{-m_k} \sigma_Q^{-n_k}) \), proving the lemma.

\[
\square
\]

Lemma 7.1 implies that

\[
\hat{X}_k = (\hat{x}_k, 1 + \hat{y}_k, 1 + \hat{z}_k) \to Y = (0, 1, 1) \in V_Y, \quad k \to +\infty.
\]

For notational simplicity we write

\[
\hat{X}_k \overset{\text{def}}{=} (\hat{x}_k, \hat{y}_k, \hat{z}_k).
\]
Step D: The transition from a neighbourhood of $Y$ to $U_Q$ is given by $f_{\tilde{\mu}_k(\mu)}^{N_2} \circ f_{N_2}^{-1} = \tilde{X}_k$. Write

$$(7.14) \quad f_{\tilde{\mu}_k(\mu)}^{N_2} \circ f_{N_2}^{-1}(\tilde{X}_k) \overset{\text{def}}{=} (1 + \tilde{x}_k, \tilde{y}_k, 1 + \tilde{z}_k) = \tilde{X}_k \in U_Q$$

Using equation (3.10), the expression of $f_{\tilde{\mu}_k(\mu)}^{N_2} \circ f_{N_2}^{-1}$ (see Remark 5.1), equation (6.9), the definition $\tilde{X}_k$ in (7.9), and the choice of $\tilde{\mu}_k(\mu)$ in (6.4) we have

$$\tilde{x}_k = a_1 \lambda \sigma^{-n_k} \lambda Q \left( \alpha_1 (\alpha_k x - \tilde{z}_k z) + \alpha_3 (\tilde{z}_k x + \tilde{c}_k z) \right) + a_1 \lambda \sigma^{-2n_k} \lambda Q \left( \alpha_1 (\alpha_k x - \tilde{z}_k z) + \alpha_3 (\tilde{z}_k x + \tilde{c}_k z) \right)$$

$$+ \lambda \sigma^{-n_k} \lambda Q \gamma_3 \left( \tilde{\mu}_k a_3 - \tilde{z}_k \alpha_2 \right) \left( \tilde{z}_k x + \tilde{c}_k z \right) + \text{h.o.t.} \star,$$

$$\tilde{y}_k = \lambda \sigma^{-n_k} \lambda Q \left( \alpha_1 (\alpha_k x - \tilde{z}_k z) + \alpha_3 (\tilde{z}_k x + \tilde{c}_k z) \right) + \lambda \sigma^{-2n_k} \lambda Q \left( \alpha_1 (\alpha_k x - \tilde{z}_k z) + \alpha_3 (\tilde{z}_k x + \tilde{c}_k z) \right) + \text{h.o.t.} \star,$$

$$\tilde{z}_k = a_1 \lambda \sigma^{-n_k} \lambda Q \left( \alpha_1 (\alpha_k x - \tilde{z}_k z) + \alpha_3 (\tilde{z}_k x + \tilde{c}_k z) \right) + a_1 \lambda \sigma^{-2n_k} \lambda Q \left( \alpha_1 (\alpha_k x - \tilde{z}_k z) + \alpha_3 (\tilde{z}_k x + \tilde{c}_k z) \right) + \text{h.o.t.} \star \star \star,$$

where

$$\text{h.o.t.} \overset{\text{def}}{=} a_1 \lambda \sigma_{P}^{n_k} \tilde{H}_1(x_k) + a_2 \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) + \lambda \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) + H_1(x_k).$$

$$\text{h.o.t.} \overset{\text{def}}{=} b_1 \lambda \sigma_{P}^{n_k} \tilde{H}_1(x_k) + b_2 \left[ \lambda \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) \right]^2 + 2 \lambda \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) \left( \lambda \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) \right) + \lambda \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) \left( \lambda \sigma_{P}^{n_k} \left( \tilde{\mu}_k \tilde{H}_2(x_k) - \tilde{z}_k \tilde{H}_3(x_k) \right) \right) + H_2(x_k).$$

This concludes the computations in Steps A-D.
7.3. Coordinates of $\Psi_k^{-1} \circ R_{\phi_k(\mu)} \circ \Psi_k(\bar{X})$. To conclude the renormalisation calculations, it remains to apply $\Psi_k^{-1}$ to the points $\bar{X}_k$ in (7.14). Note that, by the definition of $\bar{X}_k$ in (7.1) and by construction,

$$\Psi_k^{-1}(\bar{X}_k) = (\bar{x}_k, \bar{y}_k, \bar{z}_k) = \bar{X}_k.$$ 

Recalling the expression of $\Psi_k$ in (7.2) we get

$$\Psi_k^{-1}(1 + \bar{x}_k, \bar{y}_k, 1 + \bar{z}_k) = (\sigma_{Pk}^{mk} \sigma_Q^{nk} \bar{x}, \sigma_{Pk}^{2mk} \sigma_Q^{2nk} (\bar{y} - \sigma_{Qk}^{-nk}), \sigma_{Pk}^{mk} \sigma_Q^{nk} \bar{z}).$$

Applying the corresponding substitutions for $\bar{x}_k$, $\bar{y}_k$, and $\bar{z}_k$, we have:

$$\bar{x}_k = a_1 \lambda_P^{mk} \lambda_Q^{nk} (\alpha_1 (c_k x - s_k z) + \alpha_3 (s_k x + c_k z)) +$$

$$+ a_1 \lambda_P^{mk} \sigma_{Pk}^{mk} \alpha_2 y + (\xi_k a_2 + \tilde{s}_k a_3) \beta_2 y +$$

$$+ \sigma_{Pk}^{mk} \lambda_Q^{nk} \gamma_3 (\xi_k a_3 - \tilde{s}_k a_2) (s_k x + c_k z) +$$

$$+ \sigma_{Pk}^{mk} \sigma_{Qk}^{nk} \text{h.o.t.},$$

$$\bar{y}_k = \mu + b_1 \lambda_P^{mk} \sigma_Q^{nk} \alpha_2 y +$$

$$+ b_1 \lambda_P^{mk} \sigma_{Pk}^{mk} \lambda_Q^{nk} \sigma_{Qk}^{nk} \alpha_1 (c_k x - s_k z) + \alpha_3 (s_k x + c_k z)) +$$

$$+ \left(\tilde{\xi}_k b_2 + \tilde{s}_k b_3 + \tilde{c}_k \tilde{s}_k b_4\right) \beta_2 y +$$

$$+ \sigma_{Pk}^{2mk} \lambda_Q^{2nk} \left(\tilde{\xi}_k b_2 + \tilde{s}_k b_3 - \tilde{c}_k \tilde{s}_k b_4\right) \gamma_3 (s_k x + c_k z)^2 +$$

$$+ \sigma_{Pk}^{mk} \lambda_Q^{nk} \left(2\tilde{c}_k \tilde{s}_k (b_3 - b_2) + (\tilde{c}_k^2 - \tilde{s}_k^2) b_4\right) \beta_2 \gamma_3 (s_k x + c_k y z) +$$

$$+ \sigma_{Pk}^{mk} \sigma_{Qk}^{nk} \text{h.o.t.},$$

$$\bar{z}_k = c_1 \lambda_P^{mk} \lambda_Q^{nk} (\alpha_1 (c_k x - s_k z) + \alpha_3 (s_k x + c_k z)) +$$

$$+ c_1 \lambda_P^{mk} \sigma_{Pk}^{mk} \alpha_2 y + (\xi_k c_2 + \tilde{s}_k c_3) \beta_2 y +$$

$$+ \sigma_{Pk}^{mk} \lambda_Q^{nk} \gamma_3 (\xi_k c_3 - \tilde{s}_k c_2) (s_k x + c_k z) +$$

$$+ \sigma_{Pk}^{mk} \sigma_{Qk}^{nk} \text{h.o.t.}.$$ 

This completes the calculations of $\Psi_k^{-1} \circ R_{\phi_k(\mu)} \circ \Psi_k(\bar{X})$.

8. Convergence of the renormalised sequence: end of the proof of Theorem 1

Recall the choices of $\xi > 0$ and $\tau = \frac{\gamma_3 (a_3 - a_2)}{\sqrt{2}} > 0$ in Remark 6.2 satisfying

$$\sigma_{Pk}^{mk} \lambda_Q^{nk} \rightarrow \tau^{-1} \xi, \quad k \rightarrow +\infty.$$ 

Consider the vector $\zeta = \bar{\zeta}(\xi, f) \overset{\text{def}}{=} (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) \in \mathbb{R}^5$ where

$$\zeta_1 \overset{\text{def}}{=} \frac{\beta_2 (a_2 + a_3)}{\sqrt{2}}, \quad \zeta_2 \overset{\text{def}}{=} \frac{\beta_2 (b_2 + b_3 + b_4)}{2}, \quad \zeta_3 \overset{\text{def}}{=} \xi^2 \left(\frac{b_2 + b_3 - b_4}{(a_3 - a_2)^2}\right),$$

$$\zeta_4 \overset{\text{def}}{=} \xi \sqrt{2} \left(\frac{\beta_2 (b_3 - b_2)}{a_3 - a_2}\right), \quad \zeta_5 \overset{\text{def}}{=} \frac{\beta_2 (c_2 + c_3)}{\sqrt{2}}.$$ 

Note that the coordinates of $\zeta$ depend on $\xi > 0$ and the numbers $a_1, \ldots, c_3$ in the definition of $f^N_k$ (see (3.10)).

The next proposition implies Theorem 1.
Proposition 8.1. Consider any compact set \( K \subset \mathbb{R}^3 \). Then
\[
\lim_{k \to +\infty} \left\| (\Psi_k^{-1} \circ R_{\xi_k(\mu)} \circ \Psi_k - E_{(\xi,\mu,\zeta(\xi,f))}) \right\|_{C^r} = 0,
\]
where \( E_{(\xi,\mu,\zeta)} \) is the endomorphism in (2.2).

Proof. To make more transparent our calculations, let us first consider the leading terms (low order terms) of the coordinates \((\hat{x}_k, \hat{y}_k, \hat{z}_k)\) in (7.15)-(7.17). Write
\[
\hat{X}_k' \overset{\text{def}}{=} (\hat{x}_k', \hat{y}_k', \hat{z}_k'),
\]
and that
\[
\sigma_{m_k}^P \sigma_Q^{n_k} \mathbb{h}.o.t. \rightarrow 0, \quad \sigma_{m_k}^P \sigma_Q^{2n_k} \mathbb{h}.o.t. \rightarrow 0, \quad \sigma_{m_k}^P \sigma_Q^{n_k} \mathbb{h}.o.t. \rightarrow 0,
\]
where the converge occurs in the \( C^r \)-topology. This is done in the next two lemmas.

Lemma 8.2. \( \lim_{k \to +\infty} \left\| (\hat{X}_k' - E_{(\xi,\mu,\zeta(\xi,f))}) \right\|_{C^r} = 0. \)

Proof. This follows directly from \( \hat{c}_k, \hat{s}_k \to \frac{1}{\sqrt{2}} \), \( c_k \to 0 \), \( s_k \to 1 \) (see (6.8), and \( c_2 = c_3 \) (see (3.11)). \( \square \)

Lemma 8.3. The terms \( \sigma_{m_k}^P \sigma_Q^{n_k} \mathbb{h}.o.t. \), \( \sigma_{m_k}^P \sigma_Q^{2n_k} \mathbb{h}.o.t. \), and \( \sigma_{m_k}^P \sigma_Q^{n_k} \mathbb{h}.o.t. \) converge to zero in the \( C^r \)-topology on compact sets.

Proof. We begin with some preliminary observations. Since \( \hat{H}_2(x_k) \) and \( \hat{H}_3(x_k) \) have the same symbol of Landau \( O(\sigma_{m_k}^P \lambda_{Q_k}^{2n_k} \sigma_{Q_k}^{-n_k}) \), see (7.12), and \( \hat{c}_k, \hat{s}_k \to \frac{1}{\sqrt{2}} \), see (6.8), the terms
\[
\hat{c}_k \hat{H}_2(x_k) - \hat{s}_k \hat{H}_3(x_k), \quad \hat{s}_k \hat{H}_2(x_k) + \hat{c}_k \hat{H}_3(x_k)
\]
have both symbol of Landau\(^7\) equal to \( O(\sigma_{m_k}^P \lambda_{Q_k}^{2n_k} \sigma_{Q_k}^{-n_k}). \) With this in mind and using the multiplicative property of symbols of Landau\(^8\), we get that
\begin{enumerate}
\item \( \mathbb{h}.o.t. \) = \( O(\lambda_{m_k}^P \hat{H}_1(x_k)) + O(\sigma_{m_k}^P \hat{H}_2(x_k)) + O(H_1(x_k)) \).
\item \( \mathbb{h}.o.t. \) = \( O(\lambda_{m_k}^P \hat{H}_1(x_k)) + O(\sigma_{m_k}^P \hat{H}_2(x_k)) + O(H_1(x_k)) \) and \( O(\sigma_{m_k}^P \hat{H}_2(x_k)) + O(H_2(x_k)) \).
\item \( \mathbb{h}.o.t. \) = \( O(\lambda_{m_k}^P \hat{H}_1(x_k)) + O(\sigma_{m_k}^P \hat{H}_2(x_k)) + O(H_3(x_k)) \).
\end{enumerate}

Observing that \( O(H_1(x_k)) = O(H_3(x_k)) \) have the same symbol of Landau (see (3.10)), the proof of lemma is reduced to the following claim.

Claim 8.4. The following terms
\begin{enumerate}
\item \( \lambda_{m_k}^P \sigma_{m_k}^P \sigma_Q^{2n_k} \hat{H}_1(x_k) \),
\item \( \sigma_{m_k}^P \sigma_Q^{n_k} \hat{H}_2(x_k) \),
\item \( \sigma_{m_k}^P \sigma_Q^{n_k} \hat{H}_3(x_k) \),
\item \( \sigma_{m_k}^P \sigma_Q^{2n_k} \hat{H}_2(x_k) \),
\end{enumerate}
\(^7\)Here we use the following property: if \( g_1 = O(f_1) \) and \( g_2 = O(f_2) \) then \( g_1 + g_2 = O(|f_1| + |f_2|) \).
\(^8\)i.e., \( O(f \cdot g) = O(f) \cdot O(g) \).
Thus, arguing as in the previous cases,

\[
\lambda(8.3) \quad (8.2) \quad C
\]

Arguing as in (a) we get

\[
\kappa \leq \sigma_P^{-m_k} \lambda_Q^{-n_k} \sigma_Q^{-n_k}.
\]

It is easy to see that this convergence also holds for the derivatives of order 1 (see (3.3)). Thus, on any compact set \( K \subset \mathbb{R}^3 \), we have the convergence

\[
\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} \rightarrow 0, \quad k \rightarrow +\infty.
\]

Lemma 6.1 provides the constant \( C \) defined by \( (\lambda_P^{-\frac{1}{2}} \sigma_P)^2 \) such that

\[
\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} = \left( \lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} \right)^2 < C \left( \left( \lambda_P^{-\frac{1}{2}} \sigma_P \sigma_Q \right)^n \right)^2,
\]

where

\[
\eta = \log \frac{\lambda_Q^{-1}}{\log \sigma_P} \quad \text{and} \quad \tilde{\eta} = \frac{\log (\tau - 1)}{\log \sigma_P}.
\]

The convergence \( \lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} \rightarrow 0 \) as \( k \rightarrow +\infty \) follows from \( \lambda_P^{-\frac{1}{2}} \sigma_P \sigma_Q < 1 \) (see (3.3)). Thus, on any compact set \( K \subset \mathbb{R}^3 \), we have the convergence

\[
\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} \rightarrow 0, \quad k \rightarrow +\infty.
\]

It is easy to see that this convergence also holds for the derivatives of order 1 \( \leq k \leq r \). Therefore

\[
\lim_{k \rightarrow +\infty} \left\| \lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k} \left( H_1(x_k) \right) \right\|_{C^r} = 0.
\]

To prove (b), observe that (8.2) and (6.2) imply that

\[
\sigma_P^{-m_k} \sigma_Q^{-n_k} H_2(x_k) = O(\sigma_P^{-m_k} \sigma_Q^{-n_k}) = O(\lambda_Q^{-n_k}) \rightarrow 0, \quad k \rightarrow +\infty.
\]

Arguing as in (a) we get

\[
\lim_{k \rightarrow +\infty} \left\| \sigma_P^{-m_k} \sigma_Q^{-n_k} \left( H_2(x_k) \right) \right\|_{C^r} = 0.
\]

To prove (c), observe that (8.2) and (8.3) imply that

\[
\sigma_P^{-m_k} \sigma_Q^{-n_k} H_1(x_k) = O(\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k}) + O(\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-n_k})
\]

\[
\leq O(\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-m_k}) + O(\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-m_k})
\]

\[
+ O(\lambda_P^{-m_k} \sigma_P^{-m_k} \sigma_Q^{-m_k}) \rightarrow 0, \quad k \rightarrow +\infty.
\]

Thus, arguing as in the previous cases,

\[
\lim_{k \rightarrow +\infty} \left\| \sigma_P^{-m_k} \sigma_Q^{-n_k} \left( H_1(x_k) \right) \right\|_{C^r} = 0.
\]
Finally, to prove (d), observe that (8.2) and (8.3) imply that
\[
\sigma_P^{2m_k} \sigma_Q^{2n_k} H_2(\mathbf{x}_k) = O(\lambda_P^{2m_k} \sigma_P^{2m_k} \sigma_Q^{2n_k}) + O(\lambda_P^{m_k} \sigma_P^{m_k} \sigma_Q^{n_k})
\]
\[
\leq O(\lambda_P^{m_k}) \cdot O(\lambda_P^{m_k} \sigma_P^{2m_k} \sigma_Q^{2n_k}) + O(\lambda_P^{m_k} \sigma_P^{2m_k} \sigma_Q^{2n_k}) \to 0.
\]
Thus, as in the previous cases,
\[
\lim_{k \to +\infty} \left\| \sigma_P^{2m_k} \sigma_Q^{2n_k} H_2(\mathbf{x}_k) \right\|_{C^r} = 0.
\]
This proves the claim. \(\square\)

The proof of the lemma is now complete. \(\square\)

The proof of the proposition is now complete. \(\square\)

This completes the proof of Proposition 8.1.

9. **Non-transverse cycles leading blender-horseshoes**

We discuss here the existence of diffeomorphisms \(f\) satisfying the hypotheses of Corollary 1.

**Lemma 9.1.** There are diffeomorphisms \(f\) satisfying conditions (A)-(C) whose vector \(\zeta = \zeta(f, \xi)\) satisfies equation (2.3) for every \(\xi \in (1.18, 1.19)\).

**Proof.** Consider the vector 
\[
v = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \in \mathbb{R}^{10},
\]
where \(a_1, \ldots, c_3\) are the constants defining the transition map from \(P\) to \(Q\) in (3.10). The precise formula for the vector \(\zeta\) is given in equation (8.1) and is of the form (with a slight abuse of notation) \(\sigma(f, \xi) = \zeta(v, \xi) = (\zeta_1(v, \xi), \ldots, \zeta_5(v, \xi))\), where we interpret the coefficients \(\zeta_i\) as maps depending on \((v, \xi)\).

Consider the family of maps \(\gamma_\xi : \mathbb{R}^{11} \to \mathbb{R}^2\) defined by
\[
\gamma_\xi : \mathbb{R}^{11} \to (-\varepsilon, \varepsilon)^2, \quad \gamma_\xi(v) \triangleq (\kappa(v, \xi), \eta(v, \xi)),
\]
where \(\kappa, \eta : \mathbb{R}^{11} \to \mathbb{R}\) are given by
\[
\kappa(v, \xi) \triangleq \frac{\zeta_1^2(v, \xi) \zeta_3(v, \xi)}{\zeta_2(v, \xi)}, \quad \eta(v, \xi) \triangleq \frac{\zeta_2(v, \xi) \zeta_5(v, \xi)}{\zeta_2(v, \xi)}.
\]
It is immediate to verify that given any \(\xi \in (1.18, 1.19)\) then every \((\kappa_0, \eta_0) \in (-\varepsilon, \varepsilon)^2\) is a regular value of \(\gamma_\xi\). This implies the lemma. \(\square\)

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**Departamento de Matemática PUC-Rio, Marquês de São Vicente 225, Gávea, Rio de Janeiro 22545-900, Brazil**

**E-mail address:** lodiaz@mat.puc-rio.br

**Centro de Matemática da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal**

**E-mail address:** sebastian.opazo@fc.up.pt