Lowering and raising operators for some special orthogonal polynomials

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This paper is dedicated to Ian Macdonald on the occasion of his 75th birthday.

Abstract. This paper discusses operators lowering or raising the degree but preserving the parameters of special orthogonal polynomials. Results for one-variable classical (q-)orthogonal polynomials are surveyed. For Jacobi polynomials associated with root system $BC_2$ a new pair of lowering and raising operators is obtained.

1. Introduction

Kirillov and Noumi [8] gave explicit $q$-difference lowering and raising operators for $A_{n-1}$ type Macdonald polynomials $J_\lambda(x;q,t) = c_\lambda(q,t) P_\lambda(x;q,t)$ (see [14], (VI.8.3)). These operators don’t change the parameter $t$, they only lower or raise $\lambda$. This is quite different from Opdam’s [15] shift operators acting on Jacobi polynomials associated with root systems, which do change parameters. In [8, Remark 5.3] the interesting problem is mentioned to find lowering and raising operators for Macdonald-Koornwinder polynomials (see [12]). As far as we know, such operators have not yet been given in literature until now, and neither in the corresponding $q = 1$ case of $BC_n$ type Jacobi polynomials (see [5, 6]).

The present paper makes only a minor step in the direction of a general answer to the problem raised in [8, Remark 5.3]. In the $BC_2$, $q = 1$ case, for parameters $(\alpha, \beta, \gamma)$ with $\alpha = \beta$, and for partition $\lambda = (n, 0)$ of length 1, a raising and lowering operator in explicit form are obtained (sections 5 and 6). In the earlier sections 2–4 lowering and raising formulas in the rank 1 case (continuous $q$-ultraspherical, ultraspherical and Jacobi polynomials, but not yet Askey-Wilson polynomials) are discussed. Some formulas scattered in the literature are brought here together, and also the explicit specialization of the Kirillov-Noumi operators to the $A_1$ case is given. Some of the formulas in sections 2–4 may be new in one-variable special function theory.

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2. Lowering and raising continuous $q$-ultraspherical polynomials

See \[11 \text{ or } 9 \, \S3.10.1\] for the definition of the continuous $q$-ultraspherical polynomials $C_n(x; t|q)$. A pleasant explicit formula for them is as a finite Fourier series:

\begin{equation}
C_n(\cos \theta; t|q) = \sum_{k=0}^{n} \frac{(t; q)_k (t; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \cos^{n-2k} \theta.
\end{equation}

(Throughout see \[4\] for definition of $(q)$-hypergeometric series and $(q)$-Pochhammer symbols.) The $A_1$ type Macdonald polynomials can be expressed in terms of continuous $q$-ultraspherical polynomials:

\begin{equation}
P_{m,n}(x, y; q, t) = \frac{(q; q)^{m-n}_m}{(t; q)_{m-n}} (xy)^{\frac{1}{2}(m+n)} C_{m-n} \left( \frac{x+y}{2(xy)}; t|q \right) \quad (m \geq n \geq 0).
\end{equation}

For the renormalized polynomials we have

\begin{equation}
J_{m,n}(x, y; q, t) = c_{m,n}(q, t) P_{m,n}(x, y; q, t),
\end{equation}

where $c_{m,n}(q, t) = (t^2 q^{m-n}; q)_n (t; q)_{m-n} (t; q)_n$, so:

\begin{equation}
J_{m,n}(x, y; q, t) = (t^2 q^{m-n}; q)_n (t; q)_n (q; q)_{m-n}
\end{equation}

\begin{equation}
\times (xy)^{\frac{1}{2}(m+n)} C_{m-n} \left( \frac{x+y}{2(xy)}; t|q \right) \quad (m \geq n \geq 0).
\end{equation}

In particular, if $(m, n)$ is replaced by $(n, 0)$, and if we use \(2.4\):

\begin{equation}
J_{n,0}(x, y; q, t) = (q; q)_n (xy)^{\frac{1}{2}n} C_n \left( \frac{x+y}{2(xy)}; t|q \right)
\end{equation}

\begin{equation}
= (q; q)_n \sum_{k=0}^{n} \frac{(t; q)_k (t; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} x^{n-k} t^k.
\end{equation}

Let us consider the Kirillov-Noumi operators \[8\] in the two-variable case while acting on the polynomials $J_{n,0}$, raising or lowering $n$. There are two variants $K_1^+$ and $K_1^-$ of the raising operator, and two variants $M_1^+$ and $M_1^-$ of the lowering operator. The expressions for these operators will involve the operators $T_{q,x}$ and $T_{q,y}$:

\begin{equation}
(T_{q,x} f)(x, y) := f(qx, y), \quad (T_{q,y} f)(x, y) := f(x, qy).
\end{equation}

Then $K_1^+$ and $M_1^+$ and their actions are given by:

\begin{align*}
K_1^+ &:= x \left( 1 - \frac{tx - y}{x - y} T_{q,x} \right) + y \left( 1 - \frac{ty - x}{y - x} T_{q,y} \right), \\
K_1^- &:= x \left( -t T_{q,x} T_{q,y} + \frac{x - ty}{x - y} T_{q,y} \right) + y \left( -t T_{q,x} T_{q,y} + \frac{y - tx}{y - x} T_{q,y} \right).
\end{align*}

\begin{equation}
K_1^+ J_{n,0}(x, y; q, t) = J_{n+1,0}(x, y; q, t);
\end{equation}
\[ M^+_1 := \frac{1}{x} \left( -t x - y \frac{T_{q,x}}{x - y} \right) + \frac{1}{y} \left( -t y - x \frac{T_{q,y}}{y - x} \right), \]
\[ M^-_1 := \frac{1}{x} \left( -t T_{q,x} - x \frac{T_{q,y}}{x - y} \right) + \frac{1}{y} \left( -t T_{q,y} + y \frac{T_{q,x}}{y - x} \right), \]
\[ (2.7) \quad M^\pm_1 J_{n,0}(x, y; q, t) = (1 - q^n)(1 - t^2 q^{n-1}) J_{n-1,0}(x, y; q, t). \]

For the moment I will take (2.6) and (2.7) for granted from [8]. At the end of this section I will prove these formulas independently.

We can write (2.6) and (2.7) more explicitly by substitution of (2.4) and the explicit expressions for the operators \( K^\pm \) and \( M^\pm_1 \). Then put \( x = z, y = z^{-1} \). In the resulting formulas it is convenient to assume \( t, q \) fixed and to use the notation
\[ C_n[z] := C_n(\frac{1}{2}(z + z^{-1}); t|q), \quad TC_n[z] := C_n[q^{\frac{1}{2}} z], \quad T^{-1}C_n[z] := C_n[q^{-\frac{1}{2}} z]. \]

We obtain:
\[ -\frac{t z^2 - 1}{z - z^{-1}} TC_n[z] + \frac{t z^{-2} - 1}{z - z^{-1}} T^{-1}C_n[z] + q^{-\frac{1}{2}} n(z + z^{-1}) C_n[z] \]
\[ = (q^{-\frac{1}{2}} n - q^{\frac{1}{2} n + 1}) C_{n+1}[z], \]
\[ -\frac{z^{-2} - t}{z - z^{-1}} TC_n[z] + \frac{z^2 - t}{z - z^{-1}} T^{-1}C_n[z] - q^{\frac{1}{2}} n t(z + z^{-1}) C_n[z] \]
\[ = (q^{-\frac{1}{2}} n - q^{\frac{1}{2} n + 1}) C_{n+1}[z], \]
\[ -\frac{t - z^{-2}}{z - z^{-1}} TC_n[z] + \frac{t z^2}{z - z^{-1}} T^{-1}C_n[z] + q^{-\frac{1}{2}} n(z + z^{-1}) C_n[z] \]
\[ = (q^{-\frac{1}{2}} n - t^2 q^{\frac{1}{2} n - 1}) C_{n-1}[z], \]
\[ -\frac{1 - t z^2}{z - z^{-1}} TC_n[z] + \frac{1 - t z^{-2}}{z - z^{-1}} T^{-1}C_n[z] - q^{\frac{1}{2}} n t(z + z^{-1}) C_n[z] \]
\[ = (q^{-\frac{1}{2}} n - t^2 q^{\frac{1}{2} n - 1}) C_{n-1}[z]. \]

If we subtract (2.9) from (2.8) or (2.10) from (2.11), and if we divide the resulting second order \( q^2 \)-difference formula by a suitable factor which all terms have in common, then we obtain
\[ (2.12) \quad \frac{1 - t z^2}{1 - z^2} TC_n[z] + \frac{1 - t z^{-2}}{1 - z^{-2}} T^{-1}C_n[z] - (q^{-\frac{1}{2}} n + q^{\frac{1}{2}} n t) C_n[z] = 0, \]
which is a known formula for continuous \( q \)-ultraspherical polynomials. Indeed, rewrite continuous \( q \)-ultraspherical polynomials as Askey-Wilson polynomials by
\[ C_n(x; q^\alpha, q^{\frac{1}{2}} |q) = \text{const.} P_n^{(\alpha, \alpha)}(x; q^{\frac{1}{2}}) \]
\[ = \text{const.} p_n(x; q^{\frac{1}{2}}, q^{\frac{1}{2} + \alpha}, q^{\frac{1}{2} + \alpha}, q^{\frac{1}{2}} |q^{\frac{1}{2}}), \]
(see [4] (7.5.34), (7.5.25), (7.5.1)), and then use the second order \( q \)-difference formula [9] (3.1.7). Thus (2.8) is equivalent with (2.9) modulo (2.12), and similarly for (2.10) and (2.11).
In addition to the operators $K^\pm$ and $L^\pm$ we introduce the operators $A$ and $\Omega$ given by:

$$A := T_{q,x} T_{q,y}, \quad \Omega := \frac{1}{xy} \left( 1 - \frac{tx - y}{x - y} T_{q,x} - \frac{x - ty}{x - y} T_{q,y} + t T_{q,x} T_{q,y} \right).$$

Since

$$K_1^+ - K_1^- = xy(x + y) \Omega, \quad M_1^+ - M_1^- = (x + y) \Omega,$$

we will no longer consider $K_1^-$ and $M_1^-$, but we will concentrate on $K_1^+, M_1^+, \Omega$ and $A$. We can derive relations

(2.13) \quad A \Omega = q^{-2} \Omega A, \quad A K_1^+ = q K_1^+ A, \quad A M_1^+ = q^{-1} M_1^+ A,$n

(2.14) \quad q^2 K_1^+ M_1^+ - M_1^+ K_1^+ = (q^2 - 1) + (1 - q)(q + t^2) A,$n

(2.15) \quad A K_1^+ = q K_1^+ A, \quad A M_1^+ = q^{-1} M_1^+ A.$

It does not seem that the relations (2.14) and (2.15) are equivalent to the familiar relations

(2.16) \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad KE = q^2EK, \quadKF = q^{-2}FK,$

for the generators of $U_q(sl(2))$ as given, for instance, in [7, Definition VI.1.1]. Indeed, relations (2.14) and (2.16) take after rescaling of the generators the form

(2.17) \quad q^2 K_1^+ M_1^+ - M_1^+ K_1^+ = A - 1, \quad A K_1^+ = q K_1^+ A, \quad A M_1^+ = q^{-1} M_1^+ A.$

while relations (2.16), after substitution of $\tilde{E} := EK$ and after rescaling, become

(2.18) \quad q^2 \tilde{E}F - F \tilde{E} = K^2 - 1, \quad \tilde{K}E = q^2 \tilde{E}K, \quad \tilde{K}F = q^{-2} \tilde{F}K.$

Relations (2.17) would match with relations (2.18) if the first relation in (2.17) would have been $q^2 K_1^+ M_1^+ - M_1^+ K_1^+ = A - 1$. Now I will give the promised independent proof of (2.16) and (2.17). Because of the first relation in (2.13), the symmetric polynomials annihilated by $\Omega$ have a basis of homogeneous polynomials. From $\Omega \left( \sum_{k=0}^{n} c_k (x^{n-k}y^k + x^k y^{n-k}) \right) = 0$ with $c_k = c_{n-k}$ one derives a recurrence relation for the $c_k$ which, on comparison with (2.10), shows that the polynomials $J_{n,0}(x, y; q, t) (n \in \mathbb{Z}_{\geq 0})$ given by (2.4) span the space of symmetric polynomials annihilated by $\Omega$. Thus, because $K_1^+$ resp. $M_1^+$ raise resp. lower the degree of a homogeneous symmetric polynomial by 1, we find (2.10) for $K_1^+$ and (2.7) for $M_1^+$ up to a constant factor. These constant factors are then obtained by comparing terms of highest degree on the left and on the right.
3. Lowering and raising ultraspherical polynomials

The *ultraspherical polynomials* \(C_n^{(\lambda)}(x)\) (see for instance [3] or [9]) can be obtained from (2.1) by putting \(t = q^\lambda\) and letting \(q \uparrow 1:\)

\[
C_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{n} \frac{(\lambda)_k}{k!(n-k)!} e^{i(n-2k)\theta}.
\]

They are special cases of Jacobi polynomials (see (3.1)):

\[
C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x).
\]

In principle, one could take limits for \(q \uparrow 1\) of all formulas in [2] but I prefer to present the results on lowering and raising operators for ultraspherical polynomials, more classical than the results in the \(q\)-case, here independently from [2].

The first form (see [3] 10.9(15)) is:

\[
(1 - x^2) \frac{d}{dx} + n x C_n^{(\lambda)}(x) = (n + 2\lambda - 1) C_{n-1}^{(\lambda)}(x), \tag{3.1}
\]

\[
(1 - x^2) \frac{d}{dx} - (n + 2\lambda - 1) C_n^{(\lambda)}(x) = -n C_{n+1}^{(\lambda)}(x). \tag{3.2}
\]

Note that substitution of (3.1) into (3.2) causes \(n\) to drop out from the terms with derivatives. There results \((1 - x^2)\) times the second order differential equation [3] 10.9(14) for \(C_n^{(\lambda)}(x)\). Also, if \(n\) is replaced by \(n + 1\) in (3.2) and if the term with first derivative is eliminated from the resulting equation together with (3.1), then we obtain the three-term recurrence relation [3] 10.9(13) for \(C_n^{(\lambda)}(x)\).

We get a second form of lowering and raising formulas by rewriting (3.1) and (3.2) into an equivalent form:

\[
\frac{d}{dx} \left( 1 + x^2 \right)^{\frac{1}{2} n} C_n^{(\lambda)} \left( \frac{x}{\sqrt{1 + x^2}} \right) = (n + 2\lambda - 1) \left( 1 + x^2 \right)^{\frac{1}{2} (n-1)} C_{n-1}^{(\lambda)} \left( \frac{x}{\sqrt{1 + x^2}} \right), \tag{3.3}
\]

\[
\frac{d}{dx} \left( 1 + x^2 \right)^{-\frac{1}{2} (n-1) - \lambda} C_n^{(\lambda)} \left( \frac{x}{\sqrt{1 + x^2}} \right) = -n \left( 1 + x^2 \right)^{-\frac{1}{2} n - \lambda} C_n^{(\lambda)} \left( \frac{x}{\sqrt{1 + x^2}} \right). \tag{3.4}
\]

Iteration of (3.4) yields the Rodrigues type formula

\[
C_n^{(\lambda)} \left( \frac{x}{\sqrt{1 + x^2}} \right) = \frac{(-1)^n}{n!} \left( 1 + x^2 \right)^{\frac{1}{2} n + \lambda} \frac{d^n}{dx^n} \left( (1 + x^2)^{-\lambda} \right). \tag{3.5}
\]

A formula equivalent to (3.5) (by analytic continuation) is given in [3] 10.9(37), where the formula is ascribed to F. Tricomi, Ann. Mat. Pura Appl. (4) 28 (1949), 283–300 (but I could not find the formula there).

Transformation of the generating function [3] 10.9(29) for \(C_n^{(\lambda)}(x)\) yields

\[
\frac{1}{(1 + (x - z)^2)^{\lambda}} = \sum_{n=0}^{\infty} (1 + x^2)^{-\frac{n}{2} - \lambda} C_n^{(\lambda)} \left( \frac{x}{\sqrt{1 + x^2}} \right) z^n \]

\((z \in \mathbb{C}, x \in \mathbb{R}, |z| < \sqrt{1 + x^2}).\)
Then \( H := 2 \) follows by considering Taylor coefficients in the above formula.

We obtain a third form of lowering and raising operators by rewriting (3.1) and (3.2) in an equivalent form as follows:

\[
(3.6) \quad \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left( (xy)^{\frac{1}{2}} C_n^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right) \right) = (n + 2\lambda - 1) (xy)^{\frac{1}{2}}(n-1) C_{n-1}^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right),
\]

\[
(3.7) \quad \left( x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \lambda(x+y) \right) \left( (xy)^{\frac{1}{2}}(n-1) C_{n-1}^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right) \right) = n (xy)^{\frac{1}{2}} C_n^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right).
\]

Iteration of (3.7) yields the Rodrigues type formula

\[
(3.8) \quad (xy)^{\frac{1}{2}} C_n^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right) = \frac{1}{n!} \left( x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \lambda(x+y) \right)^n (1).
\]

Formulas (3.6) and (3.7) may be rewritten in terms of the following special Jack polynomials in two variables:

\[
J_{n,0}^{1/\lambda}(x, y) = \frac{(\lambda)^n}{n^n} P_{n,0}^{1/\lambda}(x, y) = \frac{n!}{\lambda^n} (xy)^{\frac{1}{2}} C_n^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right)
\]

(see also 3.6, 3.4). Thus (3.1) and (3.3) can be seen to be special cases of formulas (5.14) resp. (2.16) in [8] (there put \( n = 2, m = 1 \)).

Formulas (3.6) and (3.7) are realizations of a representation of the Lie algebra \( sl(2) \). Indeed, put

\[
H := 2 \left( x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \lambda \right), \quad E := x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \lambda(x+y), \quad F := - \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right),
\]

\[
f_n(x, y) := (xy)^{\frac{1}{2}} C_n^{(\lambda)} \left( \left( \frac{1}{2}((x/y)^{\frac{1}{2}} + (y/x)^{\frac{1}{2}}) \right) \right).
\]

Then

\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H,
\]

\[
Ff_n = -(n + 2\lambda - 1)f_{n-1}, \quad Ef_{n-1} = nf_n, \quad Hf_n = 2(n + \lambda)f_n.
\]

4. Lowering and raising Jacobi polynomials

Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) and their normalized version \( R_n^{(\alpha, \beta)}(x) \) (see for instance [3] or [9]) can be defined in terms of hypergeometric functions by

\[
P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} R_n^{(\alpha, \beta)}(x)
\]

\[
= \frac{(\alpha + 1)_n}{n!} \binom{x}{\alpha + 1} \left( \left( 1 - x \right) \right).
\]

They satisfy the symmetry

\[
P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x),
\]

and the second order differential equation (see [3] 10.8(14))

\[
(1 - x^2) \frac{d^2}{dx^2} + (\beta - \alpha - (\alpha + 2)x) \frac{d}{dx} + n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x) = 0.
\]
As in (3) lowering and raising formulas can be given in three different forms. I start with analogues of (3.3) and (3.4) (the second form in (3):

\[
(4.4) \quad \left( \frac{d^2}{dx^2} + \frac{2 \alpha + 1}{x} \frac{d}{dx} \right) (1 + x^2)^n P_n^{(\alpha, \beta)} \left( \frac{1 - x^2}{1 + x^2} \right) = -4(n + \alpha)(n + \beta) (1 + x^2)^{n-1} P_{n-1}^{(\alpha, \beta)} \left( \frac{1 - x^2}{1 + x^2} \right),
\]

\[
(4.5) \quad \left( \frac{d^2}{dx^2} + \frac{2 \alpha + 1}{x} \frac{d}{dx} \right) (1 + x^2)^{-n-\alpha-\beta} P_{n-1}^{(\alpha, \beta)} \left( \frac{1 - x^2}{1 + x^2} \right) = -4(n + \alpha + \beta) (1 + x^2)^{-n-\alpha-\beta-1} P_n^{(\alpha, \beta)} \left( \frac{1 - x^2}{1 + x^2} \right).
\]

These formulas were first given in [101] (2.10), (2.11).

From (4.4) and (4.5) one can derive analogues of (3.1) and (3.2) (the first form in (3):

\[
(4.6) \quad \left( 2n + \alpha + \beta \right)(1 - x^2) \frac{d}{dx} + n(2n + \alpha + \beta)x + \beta - \alpha) P_n^{(\alpha, \beta)}(x)
\]

\[
= 2(n + \alpha)(n + \beta) P_n^{(\alpha, \beta)}(x).
\]

\[
(4.7) \quad \left( 2n + \alpha + \beta \right)(1 - x^2) \frac{d}{dx} - (n + \alpha + \beta)((2n + \alpha + \beta)x + \alpha - \beta) P_n^{(\alpha, \beta)}(x)
\]

\[
= -2n(n + \alpha + \beta) P_n^{(\alpha, \beta)}(x).
\]

The lowering formula (4.6) was earlier given in [3] 10.8(15)].

In order to obtain (4.6) from (4.4), first rewrite (4.4) as

\[
\left( 1 - x^2 \frac{d^2}{dx^2} - 2(x + \alpha) \frac{d}{dx} \right) (1 + x)^{-n} P_n^{(\alpha, \beta)}(x)
\]

\[
= -2(n + \alpha)(n + \beta)(1 + x)^{-n-1} P_{n-1}^{(\alpha, \beta)}(x),
\]

and next as

\[
(4.8) \quad \left( (1 + x)(1 - x^2) \frac{d^2}{dx^2} + 2(1 + x)((n - 1)x - n - \alpha) \frac{d}{dx}
\]

\[
+ n(-(n - 1)x + n + 2\alpha + 1) \right) P_n^{(\alpha, \beta)}(x) = -2(n + \alpha)(n + \beta) P_n^{(\alpha, \beta)}(x).
\]

Subtract (1 + x) times the second order differential equation (4.3) for Jacobi polynomials from identity (4.8) in order to remove its term with a second order derivative. Then we obtain (4.6).

The derivation of (4.7) from (4.5) is similar, with the two intermediate formulas

\[
\left( 1 - x^2 \frac{d^2}{dx^2} - 2(x + \alpha) \frac{d}{dx} \right) (1 + x)^{n+\alpha+\beta} P_{n-1}^{(\alpha, \beta)}(x)
\]

\[
= 2(n + \alpha + \beta)(1 + x)^{n+\alpha+\beta-1} P_n^{(\alpha, \beta)}(x),
\]
where they can be expressed in terms of classical Jacobi polynomials
Heckman and Opdam \[ 5 \] (the motivating examples) of the Jacobi polynomials associated with root systems of
(4.9)
\[ x \]
\[ P^{(\alpha,\beta)}_{n-1}(x) \]
\[ = -2n(n+\alpha+\beta)P^{(\alpha,\beta)}_{n}(x). \]

The third form of the lowering and raising formulas is:
(4.10)
\[ \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \right)^2 + \frac{4\beta+2}{z+w} \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \right) \left( (zw)^n P^{(\alpha,\beta)}_{n}(\frac{1}{2}(z/w + w/z)) \right) \]
\[ = 4(n+\alpha)(n+\beta)(zw)^{n-1}P^{(\alpha,\beta)}_{n-1}(\frac{1}{2}(z/w + w/z)), \]
(4.11)
\[ \left( z^2 \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w} \right)^2 + \left( (\alpha + \beta + 1)(z + w) - \frac{(2\beta+1)zw}{z+w} \right) \left( z^2 \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w} \right) \]
\[ + (\alpha + \beta + 1)((\alpha + \beta + 2)(z^2 + w^2) + 2(\alpha - \beta)zw) \left( (zw)^{n-1}P^{(\alpha,\beta)}_{n-1}(\frac{1}{2}(z/w + w/z)) \right) \]
\[ = 4n(n+\alpha+\beta)(zw)^nP^{(\alpha,\beta)}_{n}(\frac{1}{2}(z/w + w/z)). \]

The lowering formula (4.10) can be obtained by rewriting (4.9) (use the symmetry (4.2)). Similarly, the raising formula (4.11) is obtained from (4.10).

The cases \( \beta = \pm \frac{1}{2} \) of (4.10) correspond to iterated cases of (3.6) in view of the quadratic transformations
(4.12)
\[ C^{(\lambda)}_{2n}(x) = \binom{\lambda}{\tfrac{1}{2}}_{n} P^{(\lambda - \frac{1}{2}, -\frac{1}{2})}_{n}(2x^2 - 1), \]
(4.13)
\[ C^{(\lambda)}_{2n+1}(x) = \binom{\lambda}{\tfrac{1}{2}}_{n+1} x P^{(\lambda - \frac{1}{2}, \frac{1}{2})}_{n+1}(2x^2 - 1). \]

5. Jacobi Polynomials for Root System BC\(_2\)

Jacobi polynomials for root system BC\(_2\) are a very special case (in fact one of the motivating examples) of the Jacobi polynomials associated with root systems of Heckman and Opdam [5 Theorem 8.3], [15], [13]. (Of course, the BC\(_1\) case is given by the classical Jacobi polynomials of [10]) They were introduced by the author in [16], and further elaborated in [16] and in (my main reference) [13].

The BC\(_2\) Jacobi polynomials \( R^{\alpha,\beta,\gamma}_{n,k}(\xi, \eta) \) \( (n \geq k \geq 0) \) are obtained by orthogonalizing the sequence \( 1, \xi, \eta, \xi^2, \xi \eta, \eta^2, \xi^3, \ldots, \xi^n, \xi^{n-1} \eta, \ldots, \xi^{n-k} \eta^k, \ldots \) with respect to the weight function \( \eta^n(1 - \xi + \eta)^\gamma(\xi^2 - 4\eta)^\gamma d\xi d\eta \) on the region in the \( (\xi, \eta) \) plane bounded by the straight lines \( \eta = 0 \) and \( 1 - \xi + \eta = 0 \) and by the parabola \( \xi^2 - 4\eta = 0 \) (so the region has vertices \( (0,0), (1,0) \) and \( (2,1) \)). Furthermore, the polynomials are normalized such that \( R^{(\alpha,\beta,\gamma)}_{n,k}(0,0) = 1 \). In fact, it can be shown that \( R^{(\alpha,\beta,\gamma)}_{n,k}(\xi, \eta) \) is only a linear combination of the monomials \( \xi^{m-l} \eta^l \) for \( m \leq n \) and \( m + l \leq n + k \).

Important special cases of the BC\(_2\) Jacobi polynomials occur for \( \gamma = \pm \frac{1}{2} \), where they can be expressed in terms of classical Jacobi polynomials \( R^{(\alpha,\beta)}_{n}(x) \) (the
normalized form, see (4.1)):

\begin{equation}
R_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x + y, xy) = \frac{1}{2} \left( R_{n}^{(\alpha,\beta)}(1 - 2x) R_{k}^{(\alpha,\beta)}(1 - 2y) + R_{k}^{(\alpha,\beta)}(1 - 2x) R_{n}^{(\alpha,\beta)}(1 - 2y) \right),
\end{equation}

\begin{equation}
R_{n,k}^{\alpha,\beta,\frac{1}{2}}(x + y, xy) = \frac{-(\alpha + 1)}{(n - k + 1)(n + k + \alpha + \beta + 2)} (x - y)
\times \left( R_{n+1}^{(\alpha,\beta)}(1 - 2x) R_{k}^{(\alpha,\beta)}(1 - 2y) - R_{k}^{(\alpha,\beta)}(1 - 2x) R_{n+1}^{(\alpha,\beta)}(1 - 2y) \right).
\end{equation}

Many explicit formulas for $BC_2$ Jacobi polynomials with general values for the parameters $\alpha, \beta, \gamma$ were found in [10], [16], [13] by first deriving the desired formula for $\gamma = \pm \frac{1}{2}$, next guessing the formula for general $\gamma$ by interpolation between the two known cases ($\gamma = \pm \frac{1}{2}$), and finally proving the conjectured formula in some way. This method was for instance successful in the derivation of explicit second order differential operators raising or lowering some parameters (so-called shift operators, which were important motivating examples for Opdam [15]):

\begin{align*}
D_{-}^{\gamma}: & R_{n,k}^{\alpha,\beta,\gamma} \rightarrow R_{n-1,k-1}^{(\alpha+1,\beta+1,\gamma)}, & D_{+}^{\alpha,\beta,\gamma}: & R_{n+1,k-1}^{(\alpha+1,\beta+1,\gamma)} \rightarrow R_{n,k}^{(\alpha,\beta,\gamma)}, \\
E_{-}^{\alpha,\beta,\gamma}: & R_{n,k}^{(\alpha,\beta,\gamma)} \rightarrow R_{n-1,k}^{(\alpha,\beta,\gamma+1)}, & E_{+}^{\alpha,\beta,\gamma}: & R_{n+1,k}^{(\alpha,\beta,\gamma+1)} \rightarrow R_{n,k}^{(\alpha,\beta,\gamma)}.
\end{align*}

For instance,

\begin{equation}
D_{-}^{\alpha,\beta} = \frac{1}{2} \left( \partial_{\xi} \xi + \partial_{\eta} \eta \right).
\end{equation}

(Here and in the following I use notation $\partial_{z}$ for the partial derivative with respect to $z$, and similarly for other variables.)

For Jack polynomials in two variables I will use the notation

\begin{equation}
Z_{m,n}^{\gamma,-\frac{1}{2}}(x + y, xy) := P_{m,n}^{1/\gamma}(x, y),
\end{equation}

where the standard notation for Jack polynomials is used on the right-hand side. These polynomials can be expressed in terms of ultraspherical or Jacobi polynomials by

\begin{equation}
Z_{m,n}^{\gamma,-\frac{1}{2}}(x + y, xy) = \frac{(m - n)!}{(\gamma)_{m-n}} (xy)^{\frac{1}{2}(m+n)} C_{m-n}^{\gamma} \left( \frac{x + y}{2(xy)^{\frac{1}{2}}} \right)
\times \frac{(2\gamma)_{m-n}}{(\gamma)_{m-n}} (xy)^{\frac{1}{2}(m+n)} R_{m-n}^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})} \left( \frac{x + y}{2(xy)^{\frac{1}{2}}} \right).
\end{equation}

The $BC_2$ Jacobi polynomials can be explicitly expanded in terms of Jack polynomials in two variables (see [13] Corollary 6.6):

\begin{equation}
R_{n,k}^{(\alpha,\beta,\gamma)} = \sum_{l=0}^{k} \sum_{m=l}^{n} c_{n,k,m,l}^{\alpha,\beta,\gamma} Z_{m,l}^{\gamma},
\end{equation}
From (6.2) and (6.3) we obtain

$$R_{n,k;m,l} = \frac{(-k)_l(-n - \gamma - \frac{1}{2})_l}{(-n)_l(\alpha + 1)_l} \times \left( -n \right)_m(n + \alpha + \beta + \gamma + \frac{1}{2})_m \frac{(k + \alpha + \beta + 1)_l(\gamma + \frac{1}{2})_m}{(\alpha + \gamma + \frac{1}{2})_m(\gamma + \frac{1}{2})_m} \frac{l!}{(m-l)!} \times 4F_3(-m + l, -n + k, -n - k - \alpha - \beta - 1, \gamma + \frac{1}{2}; 1).$$

More generally, $BC_n$ Jacobi polynomials can be expanded in terms of Jack polynomials in $n$ variables with the expansion coefficients given combinatorially. For this formula, due to Macdonald, see [2] (5.12), (5.13). For $k = 0$ formulas [510], [530] simplify to:

$$R_{n,0}^{\alpha,\beta,\gamma} = \sum_{m=0}^{n} \frac{(-n)_m(n + \alpha + \beta + 2\gamma + 2)_m(\gamma + \frac{1}{2})_m}{(\alpha + \gamma + \frac{1}{2})_m(2\gamma + 1)_m m!} Z^{\gamma}_{m,0}.$$  

6. Lowering and raising $BC_2$ Jacobi polynomials in a special case

Let us now look for lowering and raising operators

$$M_{n}^{\alpha,\beta,\gamma}: R_{n,0}^{\alpha,\beta,\gamma} \rightarrow R_{n-1,0}^{\alpha,\beta,\gamma}, \quad K_{n}^{\alpha,\beta,\gamma}: R_{n,0}^{\alpha,\beta,\gamma} \rightarrow R_{n+1,0}^{\alpha,\beta,\gamma}.$$ 

In this paper we will restrict to the case that $\alpha = \beta$. Let us first try to find such operators acting on $R_{n}^{(\alpha,\alpha)}(x) \pm R_{n}^{(\alpha,\alpha)}(y)$ (slight variants of the case $\alpha = \beta$, $k = 0$ of [511] and [532]). From (6.1) we obtain

$$(1 - x^2)\partial_x + nx) R_{n}^{(\alpha,\alpha)}(x) = nR_{n-1}^{(\alpha,\alpha)}(x).$$

Then

$$(1 - x^2)\partial_x + nx + (1 - y^2)\partial_y + ny) R_{n}^{(\alpha,\alpha)}(x) + R_{n}^{(\alpha,\alpha)}(y)$$

$$= n \left( R_{n-1}^{(\alpha,\alpha)}(x) + R_{n-1}^{(\alpha,\alpha)}(y) \right) + n \left( xR_{n}^{(\alpha,\alpha)}(y) + yR_{n}^{(\alpha,\alpha)}(x) \right).$$

Here we cannot express all occurrences of $R_{m}^{(\alpha,\beta)}(x)$ and $R_{m}^{(\alpha,\beta)}(x)$ in terms of $R_{m}^{(\alpha,\beta)}(x) + R_{m}^{(\alpha,\beta)}(x)$. The following trick will help us.

Rewrite (6.1) as

$$(n + 2\alpha + 1)(1 - x^2)\partial_x + n(n + 2\alpha + 1)x) R_{n}^{(\alpha,\alpha)}(x) = n(n + 2\alpha + 1)R_{n}^{(\alpha,\alpha)}(x)$$

Then recognize $n(n + 2\alpha + 1)$ as the eigenvalue in the second order differential equation for $R_{n}^{(\alpha,\alpha)}(x)$ (see [534]):

$$(1 - x^2)\partial_x x - 2(\alpha + 1)x\partial_x) R_{n}^{(\alpha,\alpha)}(x) = -n(n + 2\alpha + 1)R_{n}^{(\alpha,\alpha)}(x).$$

From (6.2) and (6.3) we obtain

$$(1 - x^2)x\partial_x x - ((2\alpha + 2)x^2 + (n + 2\alpha + 1)(1 - x^2))\partial_x) R_{n}^{(\alpha,\alpha)}(x)$$

$$= -n(n + 2\alpha + 1)R_{n}^{(\alpha,\alpha)}(x).$$
This can be rewritten as

\[ (x(1 - x)(1 - 2x)\partial_{xx} + (\alpha + 1 + 2(n - 1)x - 2(n - 1)x^2)\partial_x) R_n^{(\alpha, \beta)}(1 - 2x) = -n(n + 2\alpha + 1)R_n^{(\alpha, \alpha)}(1 - 2x). \]

If we add or subtract \((6.4)\) and the same identity with \(x\) replaced by \(y\) then we obtain a lowering operator acting on \(R_n^{(\alpha, \alpha)}(1 - 2x) \pm R_n^{(\alpha, \alpha)}(1 - 2y)\). There is still a lot of freedom here, since we can add terms which end on \(\partial_{xy}\). Thus, there are many ways to write down lowering operators acting on \(R_n^{(\alpha, \beta), \pm \frac{1}{2}}(x + y, xy)\) and it will be hard to decide in this way on a possible interpolation with respect to the parameter \(\gamma\) of the lowering operators for \(\gamma = \pm \frac{1}{2}\).

We can do better by the following approach. Put

\[ \mathcal{D}_- := \partial_x + \partial_y, \quad \mathcal{D}_+^\gamma := x^2\partial_x + y^2\partial_y + (\gamma + \frac{1}{2})(x + y), \quad \mathcal{D}_0 := x\partial_x + y\partial_y. \]

From \((6.4)\), \((6.7)\) and the homogeneity of \(Z_{m,0}^\gamma(x + y, xy)\) in \(x, y\) we obtain:

\[ \mathcal{D}_- Z_{m,0}^\gamma(x + y, xy) = \frac{m(2\gamma + m)}{\gamma + m - \frac{1}{2}} Z_{m-1,0}^\gamma(x + y, xy), \]

\[ \mathcal{D}_+^\gamma Z_{m,0}^\gamma(x + y, xy) = (\gamma + m + \frac{1}{2})Z_{m+1,0}^\gamma(x + y, xy), \]

\[ \mathcal{D}_0 Z_{m,0}^\gamma(x + y, xy) = mZ_{m,0}^\gamma(x + y, xy). \]

Let us try to use the operators \((6.4)\) as building blocks for a lowering operator acting on \(R_n^{(\alpha, \beta), \gamma}(x + y, xy)\) such that it reduces for \(\gamma = \pm \frac{1}{2}\) to an operator we already know. I will work this out here only for the case \(\alpha = \beta\). The following conjectured lowering formula is obtained:

\[ \left( \mathcal{D}_- \mathcal{D}_0 - 3(\mathcal{D}_0)^2 + 2\mathcal{D}_+^\gamma \mathcal{D}_0 + (\alpha + \gamma + \frac{1}{2})\mathcal{D}_- + 2(n - 2\gamma - \frac{1}{2})\mathcal{D}_0 \right) R_n^{(\alpha, \gamma)}(x + y, xy) = -n(n + 2\alpha + 1)R_n^{(\alpha, \alpha, \gamma)}(x + y, xy). \]

Formula \((6.9)\) can indeed be verified by using \((6.5)\), \((6.6)\), \((6.7)\), \((6.8)\).

Similarly as for \((6.9)\), one can conjecture and next prove the following raising formula:

\[ \left( \mathcal{D}_- \mathcal{D}_0 - 3(\mathcal{D}_0)^2 + 2\mathcal{D}_+^\gamma \mathcal{D}_0 + (\alpha + \gamma + \frac{1}{2})\mathcal{D}_- - 2(n + 2\alpha + 4\gamma + \frac{3}{2})\mathcal{D}_0 \right) R_n^{(\alpha, \gamma)}(x + y, xy) + 2(n + 2\alpha + 2\gamma + 2)(\mathcal{D}_+^\gamma - \gamma - \frac{1}{2})\right) R_n^{(\alpha, \gamma)}(x + y, xy) = -(n + 2\gamma + 1)(n + 2\alpha + 2\gamma + 2)R_n^{(\alpha, \gamma)}(x + y, xy). \]

The computations to check \((6.9)\) and \((6.10)\) are feasible on paper, but I have also checked the results in Mathemtica.

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