Rao–Blackwellization in the MCMC era

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Abstract

Rao–Blackwellization is a notion often occurring in the MCMC literature, with possibly different meanings and connections with the original Rao–Blackwell theorem (Blackwell, 1947; Rao, 1945), including a reduction of the variance of the resulting Monte Carlo approximations. This survey reviews some of the meanings of the term.

Keywords: Monte Carlo, simulation, Rao–Blackwellization, Metropolis-Hastings algorithm, Gibbs sampler, importance sampling, mixtures, parallelisation,

This paper is dedicated to Professor C.R. Rao in honour of his 100th birthday.

1 Introduction

The neologism Rao–Blackwellization\textsuperscript{1,2} stems from the famous Rao–Blackwell theorem (Blackwell, 1947; Rao, 1945), which states that replacing an estimator by its conditional expectation given a sufficient statistic improves estimation under any convex loss. This is a famous mathematical statistics result, both dreaded and appreciated by our students for involving conditional expectation and for producing a constructive improvement, respectively. While Monte Carlo approximation techniques cannot really be classified as estimation, since they operate over controlled simulations, rather than observations, with the ability to increase the sample size if need be, and since there is rarely a free and unknown parameter involved, hence almost never a corresponding notion of sufficiency, seeking improvement in Monte Carlo approximation via partial conditioning has nonetheless been named after this elegant theorem. As shown in Figure 1, the use of the expression Rao–Blackwellization has considerably increased in the 1990’s, once the foundational paper popularising MCMC techniques refered to this technique to reduce Monte Carlo variability.

\textsuperscript{*}The work of the first author was partly supported in part by the French government under management of Agence Nationale de la Recherche as part of the “Blanc SIMI 1” program, reference ANR-18-CE40-0034 and in part by the French government under management of Agence Nationale de la Recherche as part of the “Investissements d’avenir” program, reference ANR19-P3IA-0001 (PRAIRIE 3IA Institute). The first author also acknowledges the support of l’Institut Universitaire de France through two consecutive senior chairs.

\textsuperscript{1}We will use the American English spelling of the neologism as this version is more commonly used in the literature.

\textsuperscript{2}Berkson (1955) may have been the first one to use (p.142) this neologism.
The concept indeed started in the 1990 foundational paper by Gelfand and Smith (“foundational” as it launched the MCMC revolution, see Green et al., 2015). While this is not exactly what is proposed in the paper, as detailed in the following section, it is now perceived\(^3\) that the authors remarked that, given a Gibbs sampler whose component \(\theta_1\) is simulated from the conditional distribution, \(\pi(\theta_1|\theta_2, x)\), the estimation of the marginal \(\pi(\theta_1|x)\) is improved by considering the average of the (full) conditionals across iterations,

\[
\frac{1}{T} \sum_{t=1}^{r} \pi(\theta_1|\theta_2^{(t)}, x)
\]

which provides a parametric, unbiased and \(O(1/\sqrt{T})\) estimator. Similarly, the approximation to \(\mathbb{E}[\theta_1|x]\) based on this representation

\[
\frac{1}{T} \sum_{t=1}^{r} \mathbb{E}[\theta_1|\theta_2^{(t)}, x]
\]

is using conditional expectations with lesser variances than the original \(\theta_1^{(t)}\) and may thus lead to a reduced variance for the estimator, if correlation does not get into the way. (In that specific two-step sampler, this is always the case Liu et al., 1994.

We are thus facing the difficult classification task of separating what is Rao–Blackwellization from what is not Rao–Blackwellization in simulation and in particular MCMC settings.

The difficulty resides in setting the limits as

- there is no clear notion of sufficiency in simulation and, further, conditioning may increase the variance of the resulting estimator or slow down convergence;

- variance reduction and unbiasedness are not always relevant (as shown by the infamous harmonic mean estimator, Neal, 1999; Robert and Wraith, 2009), as for instance in infinite variance importance sampling (Chatterjee and Diaconis, 2018; Vehtari et al., 2019);

- there are (too) many forms of potential conditioning in simulation settings to hope for a ranking (see, e.g., the techniques of partitioning, antithetic or auxiliary variables, control variates as in Berg et al., 2019, delayed acceptance as in Banterle et al., 2019; Beskos et al., 2006a, adaptive mixtures as in Cornuet et al., 2012; Elvira et al., 2019; Owen and Zhou, 2000, the later more closely connected to Rao–Blackwellization);

\(^3\)See e.g. the comment in the Introduction Section of Liu et al. (1994).
the large literature on the approximation of normalising constants and Bayes factors (Marin and Robert, 2010, 2011; Robert and Marin, 2008) contains many proposals that relate to Rao-Blackwellisation, as, e.g., through the simulation of auxiliary samples from instrumental distributions as initiated in Geyer (1993) and expanded into bridge sampling by Chopin and Robert (2010) and noise-constrastive estimation by Gutmann and Hyvärinen (2012);

in connection with the above, many versions of demarginalization such as slice sampling (Mira et al., 2001; Roberts and Rosenthal, 1999) introduce auxiliary variables that could be exploited towards bringing a variance reduction;\(^4\)

there is no optimal solution in simulation as, mathematically, a quantity such as an expectation is uniquely and exactly defined once the distribution is known: if computation time is not accounted for, the exact value is the optimal solution;

while standing outside a probabilistic framework, quasi-Monte Carlo techniques (Liao, 1998) can also be deemed to constitute an ultimate form of Rao–Blackwellization, with the proposal of Kong et al. (2003) being an intermediate solution;\(^5\)

but we will not cover any further these aspects here.

The rest of this review paper discusses Gibbs sampling in Section 2, other MCMC settings in 3, particle filters and SMC in 5, and conclude in Section 6.

## 2 Gibbs sampling

Let us recall that a Gibbs sampler (Geman and Geman, 1984) is a specific way of building a Markov chain with stationary density \(\pi(\cdot)\) through the iterative generation from conditional densities associated with the joint \(\pi(\cdot)\). Its simplest version consists in partitioning the argument \(\theta\) into \(\theta = (\theta_1, \theta_2)\) and generating alternatively from \(\pi_1(\theta_1|\theta_2)\) and from \(\pi_2(\theta_2|\theta_1)\). This binary version is sometimes called data augmentation in reference to Tanner and Wong (1987), who implemented an algorithm related to the Gibbs sampler for latent variable models.

When proposing this algorithm as a way to simulating from marginal densities and (hence) posterior distributions, Gelfand and Smith (1990) explicitely relate to the Rao–Blackwell theorem, as shown by the following quote\(^6\)

\[\text{...we consider the problem of calculating a final form of marginal density from the final sample produced by either the substitution or Gibbs sampling algorithms. Since for any estimated marginal the corresponding full conditional has been assumed available, efficient inference about...}\]

\(^4\)This may however be seen as a perversion of Rao–Blackwellization in that the dimension of the random variable used in the simulation is increased, with the resulting estimate being obtained by the so-called Law of the Unconscious Statistician.

\(^5\)As mentioned by the authors, the “group-averaged estimator may be interpreted as Rao–Blackwellization given the orbit, so group averaging cannot increase the variance” (p. 592)

\(^6\)The text has been retyped and may hence contains typos. The notations are those introduced by Gelfand and Smith (1990) and used for a while in the literature, see e.g. Spiegelhalter et al. (1995) with \([X \mid Y]\) denoting the conditional density of \(X\) given \(Y\). The double indexation of the sequence is explained below.
the marginal should clearly be based on using this full conditional distribution. In the simplest case of two variables, this implies that \([X \mid Y]\) and the \(y_j^{(i)}\)’s \((j = 1, \ldots, m)\) should be used to make inferences about \([X]\), rather than imputing \(X_j^{(i)}\) \((j = 1, \ldots, nm)\) and basing inference on these \(X_j^{(i)}\)’s. Intuitively, this follows, because to estimate \([X]\) using \(y_j^{(i)}\)’s requires a kernel density estimate. Such an estimate ignores the known form \([X \mid Y]\) that is mixed to obtain \([X]\). The formal argument is essentially based on the Rao–Blackwell theorem. We sketch a proof in the context of the density estimator itself. If \(X\) is a continuous \(p\)-dimensional random variable, consider any kernel density estimator of \([X]\) based on the \(X_j^{(i)}\)’s (e.g., see Devroye and Györfi, 1985) evaluated at \(x_0\): 

\[
\Delta_{x_0}^{(i)} = (1/h_m^p) \sum_{j=1}^{m} K[(X_0 - X_j^{(i)})/h_m],
\]

say, where \(K\) is a bounded density on \(\mathbb{R}^p\) and the sequence \(\{h_m\}\) is such that as \(m \to \infty\), \(h_m \to 0\), whereas \(mh_m \to \infty\). To simplify notation, set \(Q_{m,x_0}(X) = (1/h_m^p)K[(X - X_j^{(i)})/h_m]\) so that \(\Delta_{x_0}^{(i)} = (1/m) \sum_{j=1}^{m} Q_{m,x_0}(X_j^{(i)}).\) Define \(\gamma_{x_0}^{(i)} = (1/m) \sum_{j=1}^{m} \mathbb{E}[Q_{m,x_0}(X) \mid Y_j^{(i)}].\) By our earlier theory, both \(\Delta_{x_0}^{(i)}\) and \(\gamma_{x_0}^{(i)}\) have the same expectation. By the Rao–Blackwell theorem, 

\[
\var\mathbb{E}[Q_{m,x_0}(X) \mid Y] \leq \var Q_{m,x_0}(X),
\]

and hence \(\text{MSE}(\gamma_{x_0}^{(i)} \leq \text{MSE}(\Delta_{x_0}^{(i)}),\) where \(\text{MSE}\) denotes the mean squared error of the estimate of \([X]\).

This Section 2.6. of the paper calls for several precisions:

- the simulations \(x_j^{(i)}\) and \(y_j^{(i)}\) are double-indexed because the authors consider \(m\) parallel and independent runs of the Gibbs sampler, \(i\) being the number of iterations since the initial step, in continuation of Tanner and Wong (1987),

- the Rao–Blackwell argument is more specifically a conditional expectation step,

- as later noted by Geyer (1994), the conditioning argument is directed at (better) approximating the entire density \([X]\), even though the authors mention on the following page that the argument is “simpler for estimation of” a posterior expectation,

- they compare the mean squared errors of the expected density estimate rather than the rates of convergence of a non-parametric kernel estimator (in \(n^{-1/4+d}\)) versus an unbiased parametric density estimator (in \(n^{-1/2}\)), which does not call for a Rao–Blackwellization argument,

- they do not (yet) mention “Rao–Blackwellization” as a technique,

- and they do not envision (more) ergodic averages across iterations, possibly fearing the potential impact of the correlation between the terms for a given chain.

A more relevant step in the use of Rao–Blackwellization techniques for the Gibbs sampler is found in Liu et al. (1994). This later article establishes in particular that, for the two-step Gibbs sampler, Rao–Blackwellization always produces a decrease in the variance of the empirical averages. This is established in a most elegant manner by showing that each extra conditioning (or further lag) decreases the correlation,
which is always positive. The proof relies on the associated notion of interleaving and expresses the above correlation as the variance of a multiply conditioned expectation:

\[
\text{cov}(h(\theta_1^{(0)}), h(\theta_1^{(n)})) = \text{var} \left( \mathbb{E}(\ldots \mathbb{E}[h(\theta_1) \mid \theta_2] \ldots) \right),
\]

where the number of conditional expectations on the rhs is \( n \). The authors also warn that a “fast mixing scheme gains an extra factor in efficiency if the mixture estimate can be easily computed” and give a counter-example when Rao–Blackwellization increases the variance. This counter-example is exploited in a contemporary paper by Geyer (1994) where a necessary and sufficient but highly theoretical condition is given for an improvement. As the author puts it in his conclusion,

\[
The point of this article is not that Rao–Blackwellized estimators are a good thing or a bad thing. They may be better or worse than simple averaging of the functional of interest without conditioning. The point is that, when the autocorrelation structure of the Markov chain is taken into account, it is not a theorem that Rao–Blackwellized estimators are always better than simple averaging. Hence the name Rao–Blackwellized should be avoided, because it brings to mind optimality properties that these estimators do not really possess. Perhaps “averaging a conditional expectation” is a better name.
\]

but his recommendation was not particularly popular, to judge from the subsequent literature resorting to this denomination.

Another connection between Rao–Blackwellization and Gibbs sampling can be found in Chib (1995), where his approximation to the marginal likelihood

\[
m(x) = \frac{\pi(\theta^*) f(x|\theta^*)}{\hat{\pi}(\theta^*|x)}
\]

is generally based on an estimate of the posterior density using a latent (or auxiliary) variable, as in Gelfand and Smith (1990),

\[
\hat{\pi}(\theta^*|x) = \frac{1}{T} \sum_{t=1}^{T} \pi(\theta^*|x, z(t))
\]

The stabilisation brought by this parametric approximation is notable when compared with kernel estimates, even though it requires that the marginal distribution on \( z \) is correctly simulated (Neal, 1999).

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7The “always” qualification applies to every transform of the chain and to every time lag.

8The function leading to the counter-example is however a function of both \( \theta_1 \) and \( \theta_2 \), which may be considered as less relevant in latent variable settings.

9Geyer (1994) also points out that a similar Rao–Blackwellization was proposed by Pearl (1987).

10Chib (1995) mentions this connection (p.1314) but seems to restrict it to the two-stage Gibbs sampler. In the earlier version known as “the candidate’s formula”, due to a Durham student coming up with it, Besag (1989) points out the possibility of using an approximation such as a Laplace approximation, rather than an MCMC estimation.

11A question found on the statistics forum Cross-Validated illustrates the difficulty with understanding demarginalisation and joint simulation: “Chib suggests that we can insert the Gibbs sampling outputs of \( \mu \) into the summation [of the full conditionals]. But aren’t the outputs obtained from Gibbs about the joint posterior \( p(\mu, \phi|y) \)? Why suddenly can we use the results from joint distribution to replace the marginal distribution?”
3 Markov chain Monte Carlo methods

In the more general setting of Markov chain Monte Carlo (MCMC) algorithms (Robert and Casella, 2004), further results characterise the improvement brought by Rao–Blackwellization. Let us briefly recall that the concept behind MCMC is to create a Markov sequence \( \theta^{(n)} \) of dependent variables that converge (in distribution) to the distribution of interest (also called target). One of the most ubiquitous versions of an MCMC algorithm is the Metropolis–Hastings algorithm (Green et al., 2015; Hastings, 1970; Metropolis et al., 1953)

One direct exploitation of the Rao–Blackwell theorem is found in McKeague and Wefelmeyer (2000), who show in particular that, when estimating the mean of \( \theta \) under the target distribution, a Rao–Blackwellized version based on \( \mathbb{E}[h(\theta^{(n)})|\theta^{(n-1)}] \) will improve the asymptotic variance of the ordinary empirical estimator when the chain \( \theta^{(n)} \) is reversible. While the setting may appear quite restrictive, the authors manage to recover data augmentation with a double conditional expectation (when compared with Liu et al., 1994) as well as reversible Gibbs and Metropolis samplers of the Ising model. The difficulty in applying the method resides in computing the conditional expectation, since a replacement with a Monte Carlo approximation cancels its appeal.

Casella and Robert (1996) consider an altogether different form of Rao–Blackwellization for both accept-reject and Metropolis–Hastings samples. The core idea is to integrate out via a global conditional expectation the Uniform variates used to accept or reject the proposed values.

A sample produced by the Metropolis–Hastings algorithm, \( \theta^{(1)}, \ldots, \theta^{(T)} \), is in fact based on two simulated samples, the sample of proposed values \( \eta_1, \ldots, \eta_T \) and the sample of decision variates \( u_1, \ldots, u_T \), with \( \eta_t \sim q(y|\theta^{(t-1)}) \) and \( u_t \sim U([0,1]) \). Since \( \theta^{(t)} \) is equal to one of the earlier proposed values, an empirical average associated with this sample can be written

\[
\delta_{\text{MH}} = \frac{1}{T} \sum_{t=1}^{T} h(\theta^{(t)}) = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{t} \mathbb{I}_{\theta^{(i)} = \eta_i} h(\eta_i).
\]

Therefore, taking a conditional expectation of the above by integrating the decision variates,

\[
\delta_{\text{RB}} = \frac{1}{T} \sum_{i=1}^{T} h(\eta_i) \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}_{\theta^{(t)} = \eta_i} \big| \eta_1, \ldots, \eta_T \right] = \frac{1}{T} \sum_{i=1}^{T} h(\eta_i) \sum_{t=1}^{T} \mathbb{P}(\theta^{(t)} = \eta_i | \eta_1, \ldots, \eta_T),
\]

leads to an improvement of the empirical average, \( \delta_{\text{MH}} \), under convex losses.

While, for the independent Metropolis–Hastings algorithm, the conditional probability can be obtained in closed form (see also Atchadé and Perron, 2005 and Jacob et al., 2011), the general case, based on an arbitrary proposal distribution \( q(\cdot|\theta) \) is such that \( \delta_{\text{RB}} \) is less tractable but Casella and Robert (1996) derive a tractable recursive expression for the weights of \( h(\eta_i) \) in \( \delta_{\text{RB}} \), with complexity of order \( O(T^2) \). Follow-up papers are Perron (1999) and Casella et al. (2004).

\[\text{In order to avoid additional notations, we assume a continuous model where all } \eta_i \text{'s are different with probability one.}\]
While again attempting at integrating out the extraneous uniform variates exploited by the Metropolis–Hastings algorithm, Douc and Robert (2011) derive another Rao–Blackwellized improvement over the regular Metropolis–Hastings algorithm by following a different representation of $\delta^{MH}$, using the accepted chain $(\xi_i)_i$ instead of the proposed sequence of the $\eta_t$’s as in Casella and Robert (1996). The version based on accepted values is indeed rewritten as

$$\delta^{MH} = 1/T \sum_{i=1}^M n_i h(\xi_i),$$

where the $\xi_i$’s are the accepted $\eta_j$’s, $M$ is the number of accepted $\eta_j$’s till iteration $T$, and $n_i$ is the number of times $\xi_i$ appears in the sequence $(\theta^{(t)})_t$. This representation is also exploited in Gásemýr (2002); Sahu and Zhigljavsky (1998, 2003), and Malefaki and Iliopoulos (2008). The Rao–Blackwellisation construct of Douc and Robert (2011) exploits the following properties:

1. $(\xi_i, n_i)_i$ is a Markov chain;
2. $\xi_{i+1}$ and $n_i$ are independent given $\xi_i$;
3. $n_i$ is distributed as a Geometric random variable with probability parameter

$$p(\xi_i) := \int \alpha(\xi_i, \eta) q(\eta|\xi_i) \, d\eta; \quad (1)$$

4. $(\xi_i)_i$ is a Markov chain with transition kernel $\tilde{Q}(\xi, d\eta) = \tilde{q}(\eta|\xi) d\eta$ and stationary distribution $\tilde{\pi}$ such that

$$\tilde{q}(\cdot|\xi) \propto \alpha(\xi_i, \cdot) q(\cdot|\xi) \quad \text{and} \quad \tilde{\pi}(\cdot) \propto \pi(\cdot)p(\cdot).$$

Since the Metropolis–Hastings estimator $\delta^{MH}$ only involves the $\xi_i$’s, i.e. the accepted $\eta_t$’s, an optimal weight for those random variables is the importance weight $1/p(\xi_i)$, leading to the corresponding importance sampling estimator

$$\delta^{IS} = 1/N \sum_{i=1}^M h(\xi_i) / p(\xi_i),$$

but this quantity is almost invariably unavailable in closed form and need be estimated by an unbiased estimator. The geometric $n_i$ is the de facto solution that is used in the original Metropolis-Hastings estimate, but solutions with smaller variance also are available, based on the property that (if $\alpha(\xi, \eta)$ denotes the Metropolis–Hastings acceptance probability)

$$\hat{\zeta}_i = 1 + \sum_{j=1}^{\infty} \prod_{\ell \leq j} \{1 - \alpha(\xi_i, \eta_\ell)\} \eta_\ell \sim q(\eta|\xi_i)$$

is an unbiased estimator of $1/p(\xi_i)$ whose variance, conditional on $\xi_i$, is lower than the conditional variance of $n_i$, $\{1 - p(\xi_i)\}/p^2(\xi_i)$. For practical implementation, in the event $\alpha(\xi, \eta)$ is too rately equal to one, the number of terms where the indicator function is replaced with its expectation $\alpha(\xi_i, \eta_t)$ may be limited, without jeopardising the variance domination.
4 Retrospective: Continuous time Monte Carlo methods

Retrospective simulation (Beskos et al., 2006a) is an attempt to take advantage of the redundancy inherent in modern simulation algorithms (particularly MCMC, rejection sampling) by subverting the traditional order of algorithm steps. It is connected to demarginalisation and pseudo-marginal (Andrieu and Roberts, 2009) techniques in that it replaces a probability of acceptance with an unbiased estimation of the said probability, hence creating an auxiliary variable in the process. In the case of the Metropolis-Hastings algorithm, this means substituting the ratio

$$\frac{\pi(\theta') q(\theta^t | \theta')}{\pi(\theta^t) q(\theta' | \theta^t)}$$

with

$$\frac{\hat{\pi}' q(\theta^t | \theta')}{\hat{\pi}(\theta^t) q(\theta' | \theta^t)}$$

where $\hat{\pi}'$ is an auxiliary variable such that

$$\mathbb{E}[\hat{\pi}' | \theta'] = \kappa \pi(\theta') \quad \mathbb{E}[\hat{\pi}(\theta^t) | \theta^t] = \kappa \pi(\theta^t)$$

Retrospective simulation is most powerful in infinite dimensional contexts, where its natural competitors are approximate and computationally expensive. The solution advanced by Beskos et al. (2006a) and Beskos et al. (2006b) to simulate diffusions in an exact manner (for a finite number of points) relies on an auxiliary and bounding Poisson process. The selected points produced this way actually act as a random sufficient statistic in the sense that the stochastic process can be generated from Brownian bridges between these points and closed form estimators conditional of these points may be available and with a smaller variance. See also Fearnhead et al. (2017) for related results on continuous-time importance sampling (CIS). This includes a sequential importance sampling procedure with a random variable whose expectation is equal to the importance weight.\(^\text{13}\)

5 Rao–Blackwellized particle filters

Also known as particle filtering\(^\text{14}\) sequential Monte Carlo (Del Moral et al., 2006; Doucet et al., 1999; Liu and Chen, 1998) is another branch of the Monte Carlo methodology where the concept of Rao–Blackwellisation has had an impact. We briefly recall here that sequential Monte Carlo is used in state-space and other Bayesian dynamic models where the magnitude of the latent variable prevents the call to traditional Monte Carlo (and MCMC) techniques. It is also relevant for dealing with complex static problems by creating a sequence of intermediate and artificial models, a technique called tempering (Marinari and Parisi, 1992).

Doucet et al. (2000) introduce a general version of the Rao–Blackwellized particle filter by commenting on the inherent inefficiency of particle filters in large dimensions, compounded by the dynamic nature of the sampling scheme. The central

\(^{13}\)One difficulty with the approach is the possible occurrence of negative importance weights (Jacob and Thiery, 2015).

\(^{14}\)An early instance, called bootstrap filter (Gordon et al., 1993), involved one of the authors of Gelfand and Smith (1990), who thus contributed to the birth of two major advances in the field.
filtering equation is a Bayesian update of the form
\[
p(z_{1:t}|y_{1:t} \propto p(z_{1:(t-1)}|y_{1:(t-1)})p(y_t|z_t)p(z_t|z_{t-1})
\] (2)
in a state-space formulation where \((z_t)_t\) (also denoted \(z_{1:T}\)) is the latent Markov chain and \((y_t)_t\) the observed sequence. In this update, the conditional densities of \(z_{1:t}\) and \(z_{1:(t-1)}\) are usually unavailable and need be approximated by sampling solutions.

If some marginalisation of the sampling is available for the model at hand, this reduces the degeneracy phenomenon at the core of particle filters. The example provided in Doucet et al. (2000) is one where \(z_t(x_t, r_t)\), with
\[
p(x_{1:t}, r_{1:t}|y_{1:t}) = p(x_{1:t}|y_{1:t}, r_{1:t})p(r_{1:t}|y_{1:t})
\]
and \(p(x_{1:t}|y_{1:t}, r_{1:t}) \) available in closed form. This component can then be used in the approximation of the filtering distribution (2), instead of weighted Dirac masses, which improves its precision if only by bringing a considerable decrease in the dimension of the particles (Doucet et al., 2000, Proposition 2). It is indeed sufficient to resort only to particles for the intractable part.

See Andrieu et al. (2001); Johansen et al. (2012); Lindsten (2011); Lindsten et al. (2011) for further extensions on this principle. In particular, the PhD thesis of Lindsten (2011) contains the following and relevant paragraph:

Moving from [the particle estimator] to [its Rao–Blackwellized version] resembles a Rao–Blackwellisation of the estimator (see also Lehmann, 1983). In some sense, we move from a Monte Carlo integration to a partially analytical integration. However, it is not clear that the Rao–Blackwellized particle filter truly is a Rao-Blackwellisation of [the original], in the factual meaning of the concept. That is, it is not obvious that the conditional expectation of [the original] results in the [its Rao–Blackwellized version]. This is due to the nontrivial relationship between the normalised weights generated by the [particle filter], and those generated by [its Rao–Blackwellized version]. It can thus be said that [it] has earned its name from being inspired by the Rao–Blackwell theorem, and not because it is a direct application of it.

Nonetheless, any exploitation of conditional properties that does not induce a (significant) bias is bound to bring stability and faster convergence to particle filters.

6 Conclusion

The term of Rao–Blackwellisation is therefore common enough in the MCMC literature to be considered as a component of the MCMC toolbox. As we pointed out in the introduction, many tricks and devices introduced in the past can fall under the hat of that term and, while a large fraction of them does not come with a demonstrated improvement over earlier proposals, promoting the concepts of conditioning and demarginalising as central to the field should be seen as essential for researchers and students alike. Linking such concepts, shared by statistics and Monte Carlo, with an elegant and historical result like the Rao–Blackwell theorem stresses both the universality and the resilience of the idea.

\footnote{Doucet et al. (2000) provide a realistic illustration for a neural network where the manageable part is obtained via a Kalman filter.}
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