Kinematic Self-Similar Heat Conducting and Charge Solutions

M. Sharif *and Wajiha Javed

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus Lahore-54590, Pakistan.

Abstract

The objective of this paper is to study the plane symmetric kinematic self-similar heat conducting fluid and charge dust solutions of the Einstein field equations. These solutions are classified according to self-similarity of the first, second, zeroth and infinite kinds with different equations of state. We take the self-similar vector to be tilted, orthogonal and parallel to the fluid flow. For heat conducting fluid, it is found that there exist only one solution in parallel case. In all other possibilities, these solutions reduce to the perfect fluid kinematic self-similar solutions. For charge dust case, we also obtain only one kinematic self-similar solution.

Keywords: Self-similarity; Heat conducting fluid; Charge dust solutions.

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1 Introduction

General Relativity (GR) demonstrates the most dominant attribute of the universe, gravity, in terms of geometry of the curved spacetimes. The Einstein field equations (EFEs) are the core of GR that are highly non-linear,
second order coupled partial differential equations (PDEs). Due to the mathematical complexity of these equations, there does not exist any general solution. However, one can find particular solutions (exact solutions) by imposing certain symmetry assumptions on the concerned system. Self-similarity (SS) is one of the techniques which is very helpful in simplifying the field equations by reducing the number of variables.

Self-similarity is a scale invariant property which does not change the solution of the EFEs under any scale transformation. These scale invariant solutions of the field equations are called self-similar solutions (SSS). There are many applications of such solutions in astrophysics and cosmology. These solutions are used in the discussion of extreme physical complications. The astrophysical applications include gravitational collapse and the occurrence of naked singularities while the cosmological applications include features of gravitational clustering and cosmic voids.

The concept of SS in GR was defined for the first time by Cahill and Taub [1], who discussed SS of the first kind by the existence of homothetic vector field in the spacetime. Carter and Henriksen [2] defined SS of the second, zeroth and infinite kinds. Maeda et al. [3] worked on kinematic self-similar (KSS) perfect fluid and dust solutions of spherically symmetric spacetime for the tilted, parallel and orthogonal cases.

Carr et al. [4] described physical features of spherically symmetric self-similar perfect fluid models with EOS \( p = k \rho \). Further, they explored the KSS vector associated with critical behavior observed in gravitational collapse. Coley and Golaith [5] investigated self-similar spherically symmetric cosmological models with a perfect fluid and a scalar field with an exponential potential. Sintes et al. [6] considered plane, spherical and hyperbolical symmetric spacetimes to discuss the KSS perfect fluid solutions of the infinite kind.

The symmetries of plane symmetric spacetime has been considered by using different procedures [7]. Sharif and Sehar [8]-[10] investigated the KSS solutions for plane and cylindrical symmetric spacetimes. The analysis has been given for perfect fluid and dust cases with tilted, parallel and orthogonal vectors. The same authors [11] also discussed the physical properties of homothetic solutions for spherically, cylindrically and plane symmetric spacetimes. Sharif with his collaborators [12, 13] found the KSS solutions of the most general cylindrically symmetric spacetimes for perfect and dust fluid.

Recently, Misthry et al. [14] studied radiative collapse of the realistic
models (radiating stars) by using non-viscous heat conducting fluid. Herrera [15] found that heat produced inertia in the dissipative collapse. Nath et al. [16] have studied the gravitational collapse of heat conducting non-viscous fluid. They have found that electromagnetic field reduces the pressure at the boundary which is balanced by the heat flux.

In this paper, we explore the influence of heat flux and electromagnetism on KSS solutions. For this purpose, we study non-viscous heat conducting fluid and charge dust as a gravitating material. The paper has been organized as follows. In section 2, the EFEs are simplified by taking KSS vector of the first, second, zeroth and infinite kinds and the resulting system of ODEs is solved analytically for the plane symmetric spacetimes. We take non-viscous heat conducting fluid and dust fluid, when the KSS vector is tilted, orthogonal and parallel to the fluid flow. Sections 3 is devoted to study the KSS solutions by taking charge dust fluid. The last section gives an outlook of the results.

2 Kinematic Self-similar Heat Conducting Fluid Solutions

The most general plane symmetric spacetime is given in the form [17]

$$ds^2 = e^{2\nu(t,x)}dt^2 - e^{2\mu(t,x)}dx^2 - e^{2\lambda(t,x)}(dy^2 + dz^2),$$

(1)

where $\nu$, $\mu$ and $\lambda$ are arbitrary functions of $t$ and $x$. This metric has three isometries given as $\xi_1 = \partial_y$, $\xi_2 = \partial_z$, $\xi_3 = y\partial_z - z\partial_y$. In plane symmetric spacetime, the most general spacetime (1) can be simplified to the following form

$$ds^2 = e^{2\mu(t,x)}(dt^2 - dx^2) - e^{2\lambda(t,x)}(dy^2 + dz^2).$$

(2)

The energy-momentum tensor for a non-viscous heat conducting fluid is given by [18]

$$T_{ab} = [\rho(t, x) + p(t, x)]u_a u_b - p(t, x)g_{ab} + q_a u_b + q_b u_a, \quad (a, b = 0, 1, 2, 3)$$

(3)

where $\rho$ and $p$ are density and pressure respectively, $u_a = (e^\mu, 0, 0, 0)$ is the four-velocity of the fluid element in the co-moving coordinate system and $q^a = (0, q(t, x), 0, 0)$ is the heat flow vector. Notice that $q^a u_a = 0$ for $x$
directed heat flow. The EFEs for the line element (2) can be written as

\[
\begin{align*}
\kappa \rho &= e^{-2\mu}(-3\lambda_x^2 - 2\lambda_{xx} + \lambda_t^2 + 2\lambda_x \mu_x + 2\lambda_t \mu_t), \\
\kappa q &= 2e^{-3\mu}(\lambda_{tx} - \lambda_t \mu_x + \lambda_t \lambda_x - \mu_t \lambda_x), \\
\kappa p &= e^{-2\mu}(\lambda_x^2 - 2\lambda_x \mu_x - 2\lambda_t \mu_t - 3\lambda_t^2), \\
\kappa p &= e^{-2\mu}(\mu_{xx} + \lambda_x^2 + \lambda_{xx} - \lambda_t^2 - \mu_t^2).
\end{align*}
\]

(4) (5) (6) (7)

The conservation of energy-momentum tensor, \( T^{ab}_{;b} = 0 \), gives

\[
\begin{align*}
\mu_t &= -\frac{\rho_t}{(\rho + p)} - 2\lambda_t - \frac{e^{\mu}}{(\rho + p)}(q_x + 3q \mu_x + 2q \lambda_x), \\
\mu_x &= -\frac{p_x}{(\rho + p)} - \frac{1}{(\rho + p)}e^{\mu}(q_t + 2q \lambda_t + 3\mu_t q).
\end{align*}
\]

(8) (9)

The vector field \( \xi \) for a plane symmetric spacetime is

\[
\xi^a \frac{\partial}{\partial x^a} = h_1(t,x) \frac{\partial}{\partial t} + h_2(t,x) \frac{\partial}{\partial x},
\]

(10)

where \( h_1 \) and \( h_2 \) are arbitrary functions of \( t \) and \( x \). For \( h_2 = 0 \), this gives parallel case while for \( h_1 = 0 \), we have the orthogonal case. When both \( h_1 \) and \( h_2 \) are non-zero we have the most general case known as the tilted case.

A KSS vector \( \xi \) satisfies the following conditions

\[
\mathcal{L}_\xi h_{ab} = 2\delta h_{ab}, \quad \mathcal{L}_\xi u_a = \alpha u_a,
\]

(11)

where \( \alpha \) and \( \delta \) are dimensionless constants and \( h_{ab} = g_{ab} - u_a u_b \) is the projection tensor. The ratio, \( \frac{\alpha}{\delta} \), is called the similarity index which gives rise to the following two cases:

1. \( \delta \neq 0 \),
2. \( \delta = 0 \).

It is mentioned here that self-similar variables and the metric functions for all kinds of tilted, orthogonal and parallel flow remain the same as found earlier [9]. The first case gives self-similarity of the first, zeroth and second kinds while the second case gives infinite kind. We would like to omit the details here as it is given extensively in literature [9]. It is mentioned here that there does not exist self-similar variable and the corresponding metric functions of the tilted infinite kind for the line element (2). Also, these quantities do not
exist for the parallel (except for the second and infinite kinds) and orthogonal (except for the zeroth kind) cases.

In this paper, we focus on the following two kinds of polytropic equations of state (EOS) [19]. The first EOS, denoted by EOS(1), is $p = k\rho^\gamma$, where $k$ and $\gamma$ are constants. We assume that $k \neq 0$ and $\gamma \neq 0$.

Another EOS, denoted by EOS(2), is $p = k n^\gamma$, $\rho = m_b n + \frac{P}{\gamma-1}$, where the constants $m_b$ and $n(t, x)$ correspond to the baryon mass and baryon number density respectively. Here we assume that $k \neq 0$ and $\gamma \neq 0$.

We also treat an EOS, i.e., EOS(3) by $p = k\rho$, where we assume that $-1 \leq k \leq 1$ and $k \neq 0$.

2.1 Tilted Fluid Case

2.1.1 Self-similarity of the First Kind

It follows from the EFEs that the energy density $\rho$, pressure $p$ and heat flux $q$ must take the following form

$$\kappa \rho(t, x) = \frac{1}{x^2} \rho(\xi),$$

$$\kappa q(t, x) = \frac{1}{xt} q(\xi),$$

$$\kappa p(t, x) = \frac{1}{x^2} p(\xi),$$

where the self-similar variable $\xi$ is $\frac{t}{x}$. When the EFEs and the equations of motion for the matter field are satisfied, it yields a set of ODEs and hence Eqs.(4)-(9) reduce to

$$\dot{\rho} = - (\ddot{\mu} + 2\dot{\lambda})(\rho + p) + e^{2\mu}(\dot{q} + q + 3q\dot{\mu} + 2q\dot{\lambda}),$$

$$2p - \dot{p} = \mu(\rho + p),$$

$$e^{2\mu} \rho = -4\dot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda} - 1 + 2\dot{\mu} + 2\dot{\lambda}\mu,$$

$$0 = \dot{\lambda}^2 + 2\dot{\lambda}\mu,$$

$$e^{2\mu} q = -2(\ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\mu} - 2\dot{\lambda}\mu),$$

$$e^{2\mu} p = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\mu} + 2\dot{\lambda}\mu,$$

$$0 = 2\dot{\lambda}\mu - 2\dot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda},$$

$$e^{2\mu} \dot{p} = \ddot{\lambda} + \ddot{\lambda}^2 + \dot{\lambda} + \dot{\mu} - 2\dot{\lambda},$$

$$0 = -\dot{\lambda} - \dot{\lambda}^2 - \dot{\lambda} - \dot{\mu} - \ddot{\mu},$$

$$0 = \dot{q} + q + 2q\dot{\lambda} + 3\dot{\mu} q,$$
where dot (.) represents derivative with respect to ln ξ. Equation (18) gives either \( \dot{\lambda} = 0 \) or \( \dot{\lambda} = -2\dot{\mu} \). For \( \dot{\lambda} = 0 \), Eqs. (20)-(23) give contradiction. For \( \dot{\lambda} = -2\dot{\mu} \), Eqs. (20)-(22) yield contradiction.

### 2.1.2 Self-similarity of the Second Kind

The EFEs imply that \( \rho, p \) and \( q \) in terms of \( \xi \), i.e.,

\[
\kappa \rho(t, x) = \frac{1}{x^2}[\rho_1(\xi) + \frac{x^2}{t^2} \rho_2(\xi)],
\]

(25)

\[
\kappa q(t, x) = \frac{1}{xt} q(\xi),
\]

(26)

\[
\kappa p(t, x) = \frac{1}{x^2}[p_1(\xi) + \frac{x^2}{t^2} p_2(\xi)],
\]

(27)

where \( \xi = \frac{x}{(\alpha t)^{\frac{1}{\gamma}}} \). A set of ODEs is obtained when the EFEs and the equations of motion for the matter field are satisfied for the \( O[(\xi^0)] \) and \( O[(\xi)^2] \) terms separately. Equations (4)-(9) reduce to the following

\[
\dot{\rho}_1 = -(\dot{\mu} + 2\dot{\lambda})(\rho_1 + p_1) + e^\mu \alpha(\dot{q} + q + 3q\dot{\mu} + 2q\dot{\lambda}),
\]

(28)

\[
\dot{\rho}_2 + 2\alpha \rho_2 = -(\dot{\mu} + 2\dot{\lambda})(\rho_2 + p_2),
\]

(29)

\[
-\dot{p}_1 + 2p_1 = \dot{\mu}(\rho_1 + p_1),
\]

(30)

\[
-\dot{p}_2 = \dot{\mu}(\rho_2 + p_2) + \frac{e^\mu}{\alpha}(-\dot{q} - \alpha q - 2q\dot{\lambda} - 3\dot{\mu}q),
\]

(31)

\[
e^{2\mu} \rho_1 = -4\dot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda} - 1 + 2\dot{\mu} + 2\dot{\mu}\dot{\lambda},
\]

(32)

\[
\alpha^2 e^{2\mu} \rho_2 = \dot{\lambda}^2 + 2\dot{\mu}\dot{\lambda},
\]

(33)

\[
e^{3\mu} \alpha q = -2(\dot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - 2\dot{\lambda}\dot{\mu} - \ddot{\mu}),
\]

(34)

\[
e^{2\mu} p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\mu} + 2\dot{\lambda}\dot{\mu},
\]

(35)

\[
\alpha^2 e^{2\mu} p_2 = -2\dot{\lambda} - 3\dot{\lambda}^2 - 2\alpha \dot{\lambda} + 2\dot{\lambda}\dot{\mu},
\]

(36)

\[
e^{2\mu} p_1 = \dot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \ddot{\mu} - \ddot{\mu},
\]

(37)

\[
\alpha^2 e^{2\mu} p_2 = -\dot{\lambda} - \dot{\lambda}^2 - \alpha \dot{\lambda} - \ddot{\mu} - \ddot{\mu}.
\]

(38)

**EOS (1) and (2)**

If a fluid obeys EOS(1) for \( k \neq 0 \) and \( \gamma \neq 0, 1 \), we find from Eqs. (25) and (27) that

\[
\alpha = \gamma, \quad p_1 = 0 = p_2, \quad p_2 = \frac{k}{(8\pi G)^{\gamma-1} \gamma^2 \xi^{-2\gamma}} \rho_1 \gamma, \quad [Case \ I]
\]

(39)
\[ \alpha = 1, \quad p_2 = 0 = p_1, \quad p_1 = \frac{k}{(8\pi G)^{(\gamma - 1)\gamma^2} \xi^2} \gamma. \quad [Case \ II] \] (40)

A fluid satisfying EOS(2) for \( k \neq 0 \) and \( \gamma \neq 0, 1 \), then it follows from Eqs.(25) and (27) that

\[ \alpha = \gamma, \quad p_1 = 0, \quad p_2 = \frac{k}{m_b^\gamma(8\pi G)^{(\gamma - 1)\gamma^2}} \xi^{-2\gamma} \rho_1 \gamma = (\gamma - 1)\rho_2, \quad [Case \ III] \] (41)

or

\[ \alpha = 1, \quad p_2 = 0, \quad p_1 = \frac{k}{m_b^\gamma(8\pi G)^{(\gamma - 1)\gamma^2}} \xi^2 \rho_2 \gamma = (\gamma - 1)\rho_1. \quad [Case \ IV] \] (42)

For the Case I, Eq.(30) yields either \( \dot{\mu} = 0 \) or \( \rho_1 = 0 \). In both possibilities, we obtain contradiction. For the Case II, the general case could not be solved while the special choice leads to a contradiction. In the Case III, Eq.(30) yields either \( \dot{\mu} = 0 \) or \( \rho_1 = 0 \). If \( \dot{\mu} = 0 \), Eq.(35) implies that \( \dot{\lambda} = -1 \), Eqs.(36) and (38) provide \( \alpha = 2 \), while Eq.(32) gives \( \rho_1 = 0 \). If \( \gamma = 2 \), then Eq.(29) implies that \( \rho_2 = p_2 = constant = e^{-2c_0} \) and from Eq.(34), we obtain \( q = 0 \). This case leads to the perfect fluid case and we get the same solution as given in [9]. For \( \rho_1 = 0 \), Eq.(35), we solve set of ODEs either for \( \dot{\mu} = 0 \) or \( \dot{\mu} = 1 \). The case \( \dot{\mu} = 0 \) provides the same solution as we can obtain for the earlier case \( \dot{\mu} = 0 \), which leads to the perfect fluid case [9]. When \( \dot{\mu} = 1 \), Eq.(35) implies that either \( \dot{\lambda} = -1 \) or \( \dot{\lambda} = -3 \). Both the possibilities do not provide solution. Also, the Case IV gives the same behavior as the Case II.

**EOS (3)**

When a perfect fluid satisfies EOS(3), Eqs.(25) and (27) yield

\[ p_1 = k\rho_1, \quad p_2 = k\rho_2, \quad [Case \ V] \] (43)

where \(-1 \leq k \leq 1, k \neq 0\). We explore the solutions either for \( k = -1 \) or \( k \neq -1 \). When \( k = -1 \), basic equations contradict. For \( k \neq -1 \), the case \( \rho_1 = 0 = \rho_2 \) provides \( q = 0 \) which reduces to the perfect fluid case. The case, when \( \rho_1 = 0, \rho_2 = arbitrary \), Eq.(35) implies that either \( 1 + \dot{\lambda} + 2\dot{\mu} = 0 \) or
\[ \dot{\lambda} = -1. \] The first option gives no solution while for \[ \dot{\lambda} = -1, \] Eqs. (30)-(33) imply that \[ \dot{\mu} = 0 \] and \[ \alpha = -2 \] then Eq. (34) provides \[ q = 0. \] This leads to the perfect fluid case [9] and gives the same solution as we have for EOS(2). For \[ \rho_2 = 0, \] \[ \rho_1 = \text{arbitrary}, \] Eq. (33) implies that either \[ \dot{\lambda} = 0 \] or \[ \dot{\lambda} = -2 \dot{\mu} \] both yield contradiction. The case \[ \rho_1, \rho_2 = \text{arbitrary} \] gives contradiction in the basic equations. Also, for \[ k = 1, \] we have no solution.

2.1.3 Self-similarity of the Zeroth Kind

The EFEs show that the quantities \( \rho, p \) and \( q \) must be of the form

\[
\kappa \rho = \frac{1}{x^2} [\rho_1(\xi) + x^2 \rho_2(\xi)], \tag{44}
\]
\[
\kappa q = \frac{1}{x} q(\xi), \tag{45}
\]
\[
\kappa p = \frac{1}{x^2} [p_1(\xi) + x^2 p_2(\xi)], \tag{46}
\]

where the self-similar variable is \( \xi = \frac{t}{e}. \) A set of ODEs yield

\[
\dot{\rho}_1 = -(\dot{\mu} + 2\dot{\lambda})(\rho_1 + p_1) + e^\mu (\ddot{q} + q + 3q\dot{\mu} + 2q\dot{\lambda}), \tag{47}
\]
\[
\dot{\rho}_2 = -(\dot{\mu} + 2\dot{\lambda})(\rho_2 + p_2), \tag{48}
\]
\[
-\dot{p}_1 + 2p_1 = \dot{\mu}(\rho_1 + p_1), \tag{49}
\]
\[
-\dot{p}_2 = \dot{\mu}(\rho_2 + p_2) + e^\mu (-\ddot{q} - 2q\dot{\lambda} - 3q\dot{\mu}), \tag{50}
\]
\[
e^{2\mu} \rho_1 = -4\lambda - 3\dot{\lambda}^2 - 2\ddot{\lambda} - 1 + 2\dot{\mu} + 2\lambda \dot{\mu}, \tag{51}
\]
\[
e^{2\mu} \rho_2 = \dot{\lambda}^2 + 2\dot{\mu} \dot{\mu}, \tag{52}
\]
\[
e^{3\mu} q = 2(-\ddot{\lambda} - \dot{\lambda}^2 - \dot{\lambda} + 2\dot{\lambda} \dot{\mu} + \ddot{\mu}), \tag{53}
\]
\[
e^{2\mu} p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\mu} + 2\lambda \dot{\mu}, \tag{54}
\]
\[
e^{2\mu} p_2 = 2\dot{\lambda} \dot{\mu} - 2\dot{\lambda} - 3\dot{\lambda}^2, \tag{55}
\]
\[
e^{3\mu} p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \ddot{\mu} - \ddot{\mu}, \tag{56}
\]
\[
e^{2\mu} p_2 = -\ddot{\lambda} - \dot{\lambda}^2 - \ddot{\mu}. \tag{57}
\]

Proceeding in a similar way as we have done for the first and second kinds, we obtain either contradiction or \( q = 0. \) This reduces to the perfect fluid case already given in the literature [9].
2.2 Tilted Dust, Orthogonal Fluid and Dust and Parallel Dust Cases

For the dust case, we take \( p = 0 \) in the equations of the fluid case. Proceeding in a similar fashion as above, we ultimately arrive either at contradiction or \( q = 0 \) and hence reduces to the perfect fluid and dust cases [9].

2.3 Parallel Fluid Case

2.3.1 Self-similarity of the Second Kind

In this kind, the EFEs indicate that the quantities \( \rho, p \) and \( q \) must be of the following form

\[
\kappa \rho = t^{-2} \rho_1(\xi) + t^{-4} \rho_2(\xi), \quad (58)
\]
\[
\kappa q = t^{-4} q(\xi), \quad (59)
\]
\[
\kappa p = t^{-2} p_1(\xi) + t^{-4} p_2(\xi). \quad (60)
\]

For this kind, a set of ODEs will be

\[
e^{3\mu} q = -2\mu', \quad (61)
\]
\[
e^{2\mu} \rho_1 = 2\lambda'\mu' - 3\lambda'' - 2\lambda'', \quad (62)
\]
\[
e^{2\mu} \rho_2 = 3, \quad (63)
\]
\[
e^{2\mu} p_1 = \lambda'^2 + 2\lambda'\mu', \quad (64)
\]
\[
e^{2\mu} p_2 = 1, \quad (65)
\]
\[
e^{2\mu} m_1 = \lambda'' + \lambda^2 + \mu'', \quad (66)
\]
\[
\rho_1 + 3p_1 = -e^{\mu}(q' + 3q\mu' + 2q\lambda'), \quad (67)
\]
\[
3p_2 = \rho_2, \quad (68)
\]
\[
-p_1' = \mu'(p_1 + p_1), \quad (69)
\]
\[
-p_2' = \mu'(p_2 + p_2) + e^{\mu}q, \quad (70)
\]

where prime indicates derivative with respect to \( \xi = x \).

**EOS(1) and EOS(2)**

When a perfect fluid satisfies EOS(1), Eqs.(58) and (60) give that

\[
p_2 = 0 = \rho_1, \quad \alpha = 2, \quad \gamma = \frac{1}{2}, \quad p_1 = k(8\pi G)^{(1-\gamma)/\gamma} \rho_2^\gamma. \quad [Case I] \quad (71)
\]
For EOS(2), it turns out that 

\[ p_2 = 0, \quad \alpha = 2, \quad \gamma = \frac{1}{2}, \quad p_1 = \frac{k}{m_b^\gamma (8\pi G)^{\gamma - 1}} \rho_2^{\gamma} = (\gamma - 1)\rho_1. \]  

[Case II] (72)

For both cases, Eq. (65) gives a contradiction.

**EOS(3)**

For EOS(3), Eqs. (58) and (60) show that 

\[ p_1 = k\rho_1, \quad p_2 = k\rho_2. \]  

[Case III] (73)

Here Eq. (68) implies that \( k = \frac{1}{3}. \) Equations (62), (64) and (66) provide a relation \( \mu'' + 3\lambda'^2 + 2\lambda'' = 0. \) Now by taking \( \lambda = \text{const}, \) we get \( \mu'' = 0. \) Since \( \mu = \mu(\xi), \) Eq. (61) gives \( q = -2e^{-3\mu}\mu'. \) Eqs. (62) and (63) yield \( \rho_1 = 0 = p_1 \) and \( p_2 = e^{-2\mu} \) respectively. Finally, we arrive at the following solution

\[ \lambda = \text{constant}, \quad \mu = \mu(\xi), \quad \rho_1 = 0 = p_1, \quad \rho_2 = 3e^{-2\mu}, \quad p_2 = e^{-2\mu}, \quad k = \frac{1}{3}. \]  

(74)

This gives the following spacetime

\[ ds^2 = t^2e^{2\mu(\xi)}(dt^2 - dx^2) - t^2(dy^2 + dz^2). \]  

(75)

### 2.3.2 Self-similarity of the Infinite Kind

A set of ODEs is given by

\[ e^{2\mu}\rho = 2\lambda'\mu' - 3\lambda'^2 - 2\lambda'', \]  

(66)

\[ e^{2\mu}p = \lambda'^2 + 2\lambda'\mu', \]  

(67)

\[ e^{2\mu}p = \lambda'' + \lambda'^2 + \mu'', \]  

(68)

\[ -p' = \mu'(\rho + p), \]  

(69)

\[ q = 0. \]  

(70)

Equation (70) implies that \( q = 0 \) which reduces to the perfect fluid case and provides the same solution as in [9].
3 Kinematic Self-similar Charge Dust Solutions

Here we find KSS charge solutions of the plane symmetric spacetime given by Eq.(2). We restrict here to explore solutions for the dust case only. The energy-momentum tensor for the sum of dust and electromagnetic field can be written as

$$T_{ab} = \rho u_a u_b - \frac{1}{4\pi}(F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}).$$  \hspace{1cm} (81)

The Maxwell’s field tensor $F_{ab}$ is defined as

$$F_{ab} = \phi_{b,a} - \phi_{a,b},$$  \hspace{1cm} (82)

where $\phi_a$ is the four-potential. Since charge is assumed to be at rest with respect to the co-moving coordinate system, there is no magnetic field present in this system [20]. Thus we can write $\phi_a = (\phi(t, x), 0, 0, 0)$. We would like to mention here that $\phi(t, x)$ and $J$ are related by the Maxwell’s equation $F_{ab} ; b = -\frac{4}{8\pi} J_a$, where $F_{ab}$ is the Maxwell field tensor which involves potential. In order to find solution of the EFEs, we need to calculate the non-zero components of $T_{ab}$ which can be obtained by $F_{ab}$. The only non-zero components of the field tensor are \[F_{01} = -F_{10} = \phi_x e^{-4\mu}.\] Thus the EFEs become

$$\kappa \rho = e^{-2\mu} (-3\lambda_x^2 - 2\lambda_{xx} + \lambda_t^2 + 2\lambda_x \mu_t + 2\lambda_t \mu_x) - \frac{\kappa}{8\pi} \phi_x^2 e^{-4\mu},$$  \hspace{1cm} (83)

$$0 = \lambda_{tx} - \lambda_t \mu_x + \lambda_t \lambda_x - \lambda_x \mu_t,$$  \hspace{1cm} (84)

$$0 = e^{-2\mu}(\lambda_x^2 + 2\lambda_x \mu_x - 2\lambda_{tt} - 3\lambda_t^2 + 2\lambda_t \mu_t) + \frac{\kappa}{8\pi} \phi_x^2 e^{-4\mu},$$  \hspace{1cm} (85)

$$0 = e^{-2\mu}(\lambda_x^2 - \lambda_{tt} - \lambda_t^2 + \lambda_{xx} + \mu_{xx} - \mu_{tt}) - \frac{\kappa}{8\pi} \phi_x^2 e^{-4\mu}.$$  \hspace{1cm} (86)

The energy-momentum conservation provides

$$\mu_t = -\frac{\rho_t}{\rho} - 2\lambda_t - \frac{e^{-4\mu} \phi_x}{4\pi \rho} (-2\phi_x \mu_t + \phi_{tx} + 2\phi_x \lambda_t),$$  \hspace{1cm} (87)

$$\mu_x = -\frac{e^{-4\mu} \phi_x}{4\pi \rho} (2\phi_x \mu_x - \phi_{xx} - 2\phi_x \lambda_x).$$  \hspace{1cm} (88)
The above set of equations yields contradiction.

3.1.2 Self-similarity of the Second Kind

The self-similar variable for this kind is \( \xi = \frac{x}{\alpha t} \) and hence \( \phi = \phi(\xi) \). Thus the EFEs and equations of motion yield

\[
\begin{align*}
\dot{\rho}_1 &= -(\ddot{\mu} + 2\dot{\lambda})\rho_1 + \frac{K}{4\pi} e^{-4\mu(\xi)} \dot{\phi}(2\dot{\phi}\ddot{\mu} - \ddot{\phi} - 2\dot{\phi}\dot{\lambda}), \\
\dot{\rho}_2 + 2\alpha\rho_2 &= -(\dot{\mu} + 2\dot{\lambda})\rho_2, \\
\dot{\mu}\rho_1 &= -\frac{K}{4\pi} e^{-4\mu(\xi)} \dot{\phi}(2\dot{\phi}\ddot{\mu} - \ddot{\phi} - 2\dot{\phi}\dot{\lambda}), \\
0 &= \dot{\mu}\rho_2, \\
\rho_1 &= e^{-2\mu(\xi)}(-4\dot{\lambda} - 3\dot{\lambda}^2 - 2\ddot{\lambda} - 1 + 2\dot{\mu} + 2\dot{\lambda}) - \frac{K}{8\pi} e^{-4\mu(\xi)} \dot{\phi}^2, \\
0 &= \dot{\lambda} + \dot{\lambda}^2 + \dot{\mu} + \ddot{\mu}.
\end{align*}
\]
\[ \alpha^2 e^{2\mu} \rho_2 = \dot{\lambda}^2 + 2\dot{\lambda}\ddot{\mu}, \quad (103) \]
\[ 0 = -\ddot{\lambda} - \dot{\lambda}^2 - \dot{\lambda} + 2\dot{\lambda}\ddot{\mu} + \ddot{\mu}, \quad (104) \]
\[ 0 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\mu + 2\dot{\lambda}\ddot{\mu} + \frac{K}{8\pi} e^{-2\mu(\xi)} \dot{\phi}^2, \quad (105) \]
\[ 0 = 2\dot{\lambda} + 3\dot{\lambda}^2 + 2\alpha\dot{\lambda} - 2\dot{\lambda}\ddot{\mu}, \quad (106) \]
\[ 0 = \ddot{\lambda} + \dot{\lambda}^2 + \ddot{\lambda} - \dot{\mu} + \ddot{\mu} - \frac{K}{8\pi} e^{-2\mu(\xi)} \dot{\phi}^2, \quad (107) \]
\[ 0 = \dot{\lambda} + \dot{\lambda}^2 + \alpha\dot{\lambda} + \alpha\ddot{\mu} + \ddot{\mu}. \quad (108) \]

Equation (101) implies that either \( \dot{\mu} = 0 \) or \( \rho_2 = 0 \). The first possibility, \( \dot{\mu} = 0 \), yields contradiction while for the second possibility \( \rho_2 = 0 \), Eq. (103) gives either \( \dot{\lambda} = 0 \) or \( \dot{\lambda} = -2\dot{\mu} \), both yield contradiction.

### 3.1.3 Self-similarity of the Zeroth Kind

For \( \phi = \phi(\xi) \), where \( \xi = \frac{x}{e^t} \), Eqs. (83)-(88) yield the following set of ODEs

\[ \dot{\rho}_1 = -(\dot{\mu} + 2\dot{\lambda})\rho_1 + \frac{K}{4\pi} e^{-4\mu(\xi)} \dot{\phi}(2\dot{\phi}\ddot{\mu} - \ddot{\phi} - 2\dot{\phi}\dot{\lambda}), \quad (109) \]
\[ \dot{\rho}_2 = -(\dot{\mu} + 2\dot{\lambda})\rho_2, \quad (110) \]
\[ \dot{\mu}\rho_1 = -\frac{K}{4\pi} e^{-4\mu(\xi)} \dot{\phi}(2\dot{\phi}\ddot{\mu} - \ddot{\phi} - 2\dot{\phi}\dot{\lambda}), \quad (111) \]
\[ 0 = \dot{\mu}\rho_2, \quad (112) \]
\[ \rho_1 = e^{-2\mu(\xi)}(-4\dot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda} - 1 + 2\dot{\mu} + 2\dot{\lambda}\ddot{\mu}) - \frac{K}{8\pi} e^{-4\mu(\xi)} \dot{\phi}^2, \quad (113) \]
\[ e^{2\mu}\rho_2 = \dot{\lambda}^2 + 2\dot{\lambda}\ddot{\mu}, \quad (114) \]
\[ 0 = \ddot{\lambda} + \dot{\lambda}^2 + \ddot{\lambda} - 2\dot{\lambda}\ddot{\mu} - \ddot{\mu}, \quad (115) \]
\[ 0 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\mu} + 2\dot{\lambda}\ddot{\mu} + \frac{K}{8\pi} e^{-2\mu(\xi)} \dot{\phi}^2, \quad (116) \]
\[ 0 = -2\dot{\lambda}\ddot{\mu} + 2\dot{\lambda} + 3\dot{\lambda}^2, \quad (117) \]
\[ 0 = \ddot{\lambda} + \dot{\lambda}^2 + \ddot{\lambda} - \dot{\mu} + \ddot{\mu} - \frac{K}{8\pi} e^{-2\mu(\xi)} \dot{\phi}^2, \quad (118) \]
\[ 0 = -\ddot{\lambda} - \dot{\lambda}^2 - \ddot{\mu}. \quad (119) \]

Equation (112) implies that either \( \dot{\mu} = 0 \) or \( \rho_2 = 0 \), both give contradiction.

### 3.2 Orthogonal Dust Cases

In this case, set of basic equations contradict for the self-similarity of the zeroth kind.
3.3 Parallel Dust Case

3.3.1 Self-similarity of the Second Kind

For this kind, we have $\xi = x$, $\phi(t, x) = \phi(\xi)$ and $\rho$ given in Eq. (58). Thus a set of ODEs will be

\[
\begin{align*}
\mu' &= 0, \quad \text{(120)} \\
e^{2\mu} \rho_1 &= 2\lambda'\mu' - 3\lambda'^2 - 2\lambda'', \quad \text{(121)} \\
e^{2\mu} \rho_2 &= 3 - \frac{\kappa}{8\pi} e^{-2\mu(\xi)} \phi'^2, \quad \text{(122)} \\
0 &= 2\lambda'\mu' + \lambda'^2, \quad \text{(123)} \\
1 &= -\frac{\kappa}{8\pi} e^{-2\mu(\xi)} \phi'^2, \quad \text{(124)} \\
0 &= \lambda'' + \lambda'^2 + \mu'', \quad \text{(125)} \\
1 &= \frac{\kappa}{8\pi} e^{-2\mu(\xi)} \phi'^2, \quad \text{(126)} \\
\rho_1 &= 0, \quad \text{(127)} \\
\rho_2 &= 0, \quad \text{(128)} \\
\mu' \rho_2 &= -\frac{\kappa}{4\pi} e^{-4\mu(\xi)} \phi'(2\phi'\mu' - \phi'' - 2\phi'\lambda'), \quad \text{(129)} \\
\mu' \rho_1 &= 0, \quad \text{(130)}
\end{align*}
\]

Eqs. (124) and (126) clearly give contradiction.

3.3.2 Self-similarity of the Infinite Kind

A set of ODEs in terms of $\xi = x$ is

\[
\begin{align*}
e^{2\mu} \rho &= 2\lambda'\mu' - 3\lambda'^2 - 2\lambda'' - \frac{\kappa}{8\pi} \phi'^2 e^{-2\mu}, \quad \text{(131)} \\
0 &= \lambda'^2 + 2\lambda'\mu' + \frac{\kappa}{8\pi} \phi'^2 e^{-2\mu}, \quad \text{(132)} \\
0 &= \lambda'^2 + \lambda'' + \mu'' - \frac{\kappa}{8\pi} \phi'^2 e^{-2\mu}, \quad \text{(133)} \\
e^{2\mu} \mu' \rho &= -\frac{\kappa}{4\pi} \phi' e^{-2\mu}(2\phi'\mu' - \phi'' - 2\lambda'\phi'), \quad \text{(134)}
\end{align*}
\]

Equations (131) and (132) yield

\[
\rho = 2e^{-2\mu}(2\lambda'\mu' - \lambda'' - \lambda'^2), \quad \text{(135)}
\]
Now Eqs. (132) and (133) imply
\[
2\lambda^\prime\mu^\prime + 2\lambda^\prime 2 + \mu'' + \lambda'' = 0.
\] (136)
Assume that \(\mu = \text{constant}\) so that Eq. (136) gives \(\lambda'' = -2\lambda^2\) then Eq. (135) provides \(\rho = 2e^{-2\mu \lambda^2}\). Equation (134) can be written as \(\phi'(-\phi'' - 2\lambda \phi') = 0\), which yields either \(\phi' = 0\) or \(\phi'' = -2\lambda \phi'\). For \(\phi' = 0\), we get contradiction while for the second possibility \(\phi'' = -2\lambda \phi'\), we obtain a solution of the following form
\[
\lambda = \frac{1}{2} \ln c_1(2\xi - c), \quad \mu = \text{constant}, \quad \rho = 2e^{-2\mu \lambda^2},
\]
\[
\phi = \frac{c_2}{2} \ln c_3(2\xi - c), \quad \frac{\kappa}{8\pi} e^{-2\mu} = -\frac{1}{c_2^2}.
\] (137)
The corresponding metric is
\[
d\!s^2 = dt^2 - dx^2 - 2x(dy^2 + dz^2).
\] (138)

4 Outlook

This paper is devoted to study the effects of heat conduction and electromagnetic field on plane symmetric KSS solutions of the EFEs. We have found these solutions by using heat conducting and charge dust fluids. These types of fluid are real and exist in nature rather than imaginary and ideal as perfect fluid. We have investigated KSS solutions for the case when the KSS vector is tilted, orthogonal and parallel to the fluid flow with either EOS(1), EOS(2) or EOS(3).

Firstly, we have analyzed plane symmetric KSS heat conducting fluid solutions. The self-similar variables and the metric functions remain the same as for the perfect fluid case \([9]\). We find only one solution with non-zero heat flux \(q\) given by Eq. (74). The remaining cases either give contradiction or reduce to the perfect fluid solutions already discussed in \([9]\). The heat flux turns to be an arbitrary function with negative sign which indicates dissipation. Secondly, we have obtained one plane symmetric KSS charge dust solution in the parallel dust case with self-similarity of the infinite kind given by Eq. (137). Here we obtain potential in terms of \(x\) and its sign depends upon the arbitrary constants. We would like to mention here that energy density for both the solutions is positive.

It would be interesting to extend this analysis to find the KSS charge perfect solution and also for the most general plane symmetric spacetime.
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