Linear Stochastic Approximation: Constant Step-Size and Iterate Averaging

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Abstract

We consider $d$-dimensional linear stochastic approximation algorithms (LSAs) with a constant step-size and the so called Polyak-Ruppert (PR) averaging of iterates. LSAs are widely applied in machine learning and reinforcement learning (RL), where the aim is to compute an appropriate $\theta^* \in \mathbb{R}^d$ (that is an optimum or a fixed point) using noisy data and $O(d)$ updates per iteration. In this paper, we are motivated by the problem (in RL) of policy evaluation from experience replay using the temporal difference (TD) class of learning algorithms that are also LSAs. For LSAs with a constant step-size, and PR averaging, we provide bounds for the mean squared error (MSE) after $t$ iterations. We assume that data is i.i.d. with finite variance (underlying distribution being $P$) and that the expected dynamics is Hurwitz. For a given LSA with PR averaging, and data distribution $P$ satisfying the said assumptions, we show that there exists a range of constant step-sizes such that its MSE decays as $O(\frac{1}{t})$.

We examine the conditions under which a constant step-size can be chosen uniformly for a class of data distributions $P$, and show that not all data distributions ‘admit’ such a uniform constant step-size. We also suggest a heuristic step-size tuning algorithm to choose a constant step-size of a given LSA for a given data distribution $P$. We compare our results with related work and also discuss the implication of our results in the context of TD algorithms that are LSAs.

1 Introduction

Linear stochastic approximation algorithms (LSAs) of the form

$$\theta_t = \theta_{t-1} + \alpha_t (b_t - A_t \theta_{t-1}),$$

(1)

with $(\alpha_t)_t$ a positive step-size sequence chosen by the user and $(b_t, A_t) \in \mathbb{R}^d \times \mathbb{R}^{d \times d}$, $t \geq 0$, a sequence of identically distributed random variables is widely used in machine learning, and in particular in reinforcement learning (RL), to compute the solution of the equation $E[b_t] - E[A_t] \theta = 0$, where $E$ stands for mathematical expectation. Some examples of LSAs include the stochastic gradient descent algorithm (SGD) for the problem of linear least-squares estimation (LSE) [4, 5], and the temporal difference (TD) class of learning algorithms in RL [14, 17, 6, 15, 16, 11].

The choice of the step-size sequence $(\alpha_t)_t$ is critical for the performance of LSAs such as (1). Informally speaking, smaller step-sizes are better for noise rejection and larger step-sizes lead to faster forgetting of initial conditions (smaller bias). At the same time, step-sizes that are too large might result in instability of (1) even when $(A_t)_t$ has favourable properties. A useful choice has been the diminishing step-sizes [16, 11, 17], where $\alpha_t \to 0$ such that $\sum_{t \geq 0} \alpha_t = \infty$. Here, $\alpha_t \to 0$ circumvents the need for guessing the magnitude of step-sizes that stabilize the updates, while the second condition ensures that initial conditions are forgotten. An alternate idea, which we call LSA with constant step-size and Polyak-Ruppert averaging (LSA with CS-PR, in short), is to run (1)
by choosing $\alpha_t = \alpha > 0 \forall t \geq 0$ with some $\alpha > 0$, and output the average $\hat{\theta}_t = \frac{1}{t} \sum_{i=0}^{t} \theta_i$. Thus, in LSA with CS-PR, $\theta_i$ is an internal variable and $\hat{\theta}_t$ is the output of the algorithm (see Section 3 for a formal definition of LSA with CS-PR). The idea is that the constant step-size leads to faster forgetting of initial conditions, while the averaging on the top reduces noise. This idea goes back to Ruppert [13] and Polyak and Juditsky [12] who considered it in the context of stochastic approximation that LSA is a special case of.

**Motivation and Contribution:** Recently, Dieuleveut et al. [4] considered stochastic gradient descent (SGD)\(^1\) with CS-PR for LSE and i.i.d. sampling. They showed that one can calculate a constant step-size from only a bound on the magnitude of the noisy data so that the leading term as $t \to \infty$ in the mean-squared prediction error after $t$ updates is at most $\frac{C}{t}$ with a constant $C > 0$ that depends only on the bound on the data, the dimension $d$ and is in particular independent of the eigenspectrum of $E[A_i]$, a property which is not shared by other step-size tunings and variations of the basic SGD method\(^1\).

In this paper, we study LSAs with CS-PR (thereby extending the scope of prior work by Dieuleveut et al. [4] from SGD to general LSAs) in an effort to understand the effectiveness of the CS-PR technique beyond SGD. Our analysis for the case of general LSA does not use specific structures, and hence cannot recover entirely, the results of Dieuleveut et al. [4] who use the problem specific structures in their analysis. Of particular interest is whether a similar result to that Dieuleveut et al. [4] holds for the TD class of LSA algorithms used in RL. For simplicity, we still consider the i.i.d. case. Our restrictions on the common distribution is that the “noise variance” should be bounded (as we consider squared errors), and that the matrix $E[A_i]$ must be Hurwitz, i.e., all its eigenvalues have positive real parts. One setting that fits our assumption is policy evaluation\(^2\) using linear value function approximation from experience replay [10] in a batch setting [8] in RL using the TD class of algorithms [14, 17, 15, 16, 11].

Our main contributions are as follows:

- **Finite-time Instance Dependent Bounds** (Section 4): For a given $P$, we measure the performance of a given LSA with CS-PR in terms of the mean square error (MSE) given by $E_P \left[ \|\hat{\theta}_t - \theta_i\|^2 \right]$. For the first time in the literature, we show that (under our stated assumptions) there exists an $\alpha > 0$ such that for any $\alpha \in (0, \alpha_P)$, the MSE is at most $C_{P,\alpha}/t + C_{P,\alpha}^{'}/t^{2}$ with some positive constants $C_{P,\alpha}, C_{P,\alpha}^{'}$ that we explicitly compute from $P$.

- **Uniform Bounds** (Section 5): It is of major interest to know whether for a given class $P$ of distributions one can choose some step-size $\alpha$ such that $C_{P,\alpha}$ from above is uniformly bounded (i.e., replicating the result of Dieuleveut et al. [4]).\(^3\) We show via an example that in general this is not possible. In particular, the example applies to RL, hence, we get a negative result for RL, which states that from only bounds on the data one cannot choose a step-size $\alpha$ to guarantee that $C_{P,\alpha}$ of CS-PR is uniformly bounded over $P$. We also define a subclass $P_{SPD,B}$ of problems, related to SGD for LSE, that does “admit” a uniform constant step-size, thereby recovering a part of the result by Dieuleveut et al. [4]. Our results in particular shed light on the precise structural assumptions that are needed to achieve a uniform bound for CS-PR. For further details, see Section 6.

- **Automatic Step-Size** (Section 7): The above negative result implies that in RL one needs to choose the constant step-size based on properties of the instance $P$ to avoid the explosion of the MSE. To circumvent this, we propose a natural step-size tuning method to guarantee instance-dependent boundedness. We experimentally evaluate the proposed method and find that it is indeed able to achieve its goal on a set of synthetic examples where no constant step-size is available to prevent exploding MSE.

In addition to TD(0), our results directly can be applied to other off-policy TD algorithms such as GTD/GTD2 with CS-PR (Section 6). In particular, our results show that the GTD class of algorithms

\(^1\)SGD is an LSA of the form in [1].

\(^2\)See Section 6 for further discussion of the nature of these results.

\(^3\)Of course, the term $C_{P,\alpha}^{'}/t^{2}$ needs to be controlled, as well. Just like Dieuleveut et al. [4], here we focus on $C_{P,\alpha}$, which is justified if one considers the MSE as $t \to \infty$. Further justification is that we actually find a negative result. See above.
We denote the sets of real and complex numbers by \( \mathbb{R} \) and \( \mathbb{C} \), respectively. For \( x \in \mathbb{C} \) we denote its modulus and complex conjugate by \( |x| \) and \( \bar{x} \), respectively. We denote \( d \)-dimensional vector spaces over \( \mathbb{R} \) and \( \mathbb{C} \) by \( \mathbb{R}^d \) and \( \mathbb{C}^d \), respectively, and use \( \mathbb{R}^{d \times d} \) and \( \mathbb{C}^{d \times d} \) to denote \( d \times d \) matrices with real and complex entries, respectively. We denote the transpose of \( C \) by \( C^\top \) (and of course the same notation applies to vectors, as well). We will use \( \langle \cdot, \cdot \rangle \) to denote the inner products: \( \langle x, y \rangle = x^* y \). We use \( \|x\| = \langle x, x \rangle^{1/2} \) to denote the 2-norm. For \( x \in \mathbb{C}^d \), we denote the general quadratic norm with respect to a positive definite (see below) Hermitian matrix \( C \) (i.e., \( C = C^* \)) by \( \|x\|_C^2 = x^* C x \). The norm of the matrix \( A \) is given by \( \|A\| = \sup_{x \in \mathbb{C}^d : \|x\|=1} \|Ax\| \). We use \( \kappa(A) = \|A\| \|A^{-1}\| \) to denote the condition number of matrix \( A \). We denote the identity matrix in \( \mathbb{C}^d \times \mathbb{C}^d \) by \( I \) and the set of invertible \( d \times d \) complex matrices by \( \text{GL}(d) \). For a positive real number \( B > 0 \), we define \( C_B^d = \{ b \in \mathbb{C}^d \mid \|b\| \leq B \} \) and \( C_{d \times d}^B = \{ A \in \mathbb{C}^{d \times d} \mid \|A\| \leq B \} \) to be the balls in \( \mathbb{C}^d \) and \( \mathbb{C}^{d \times d} \), respectively, of radius \( B \). We use \( Z \sim P \) to denote the fact that \( Z \) (which can be a number, or vector, or matrix) is distributed according to probability distribution \( P \); \( E \) denotes mathematical expectation.

Let us now state some definitions that will be useful for presenting our main results.

**Definition 1.** For a probability distribution \( P \) over \( \mathbb{C}^d \times \mathbb{C}^{d \times d} \), we let \( P^V \) and \( P^M \) denote the respective marginals of \( P \) over \( \mathbb{C}^d \) and \( \mathbb{C}^{d \times d} \). By abusing notation we will often write \( P = (P^V, P^M) \) to mean that \( P \) is a distribution with the given marginals. Define
\[
A_P = \int M \, dP^M(M), \quad C_P = \int M^* M \, dP^M(M), \quad b_P = \int v \, dP^V(v),
\]
and
\[
\rho_d(\alpha, P) = \inf_{x \in \mathbb{C}^d : \|x\| = 1} \langle x, (A_P + A_P^*) - \alpha A_P^* A_P \rangle x,
\]
\[
\rho_s(\alpha, P) = \inf_{x \in \mathbb{C}^d : \|x\| = 1} \langle x, (A_P + A_P^*) - \alpha C_P \rangle x.
\]

Note that \( \rho_d(\alpha, P) \geq \rho_s(\alpha, P) \). Here, subscripts \( s \) and \( d \) stand for stochastic and deterministic respectively.

**Definition 2.** Let \( P = (P^V, P^M) \) as in Definition 1. \( b \sim P^V \) and \( A \sim P^M \) be random variables distributed according to \( P^V \) and \( P^M \). For \( U \in \text{GL}(d) \) define \( U_P \) to be the distribution of \( (U^{-1}b, U^{-1}AU) \). We also let \( (P^V_U, P^M_U) \) denote the corresponding marginals.

**Definition 3.** We call a matrix \( A \in \mathbb{C}^{d \times d} \) **Hurwitz** (\( H \)) if all eigenvalues of \( A \) have positive real parts. We call a matrix \( A \in \mathbb{C}^{d \times d} \) **positive definite** (\( PD \)) if \( \langle x, Ax \rangle > 0 \), \( \forall x \neq 0 \in \mathbb{C}^d \). If \( \inf_x \langle x, Ax \rangle \geq 0 \) then \( A \) is **positive semi-definite** (\( PSD \)). We call a matrix \( A \in \mathbb{R}^{d \times d} \) to be **symmetric positive definite** (\( SPD \)) if it is symmetric i.e., \( A^\top = A \) and \( PD \).

Note that \( SPD \) implies that the underlying matrix is real.

**Definition 4.** We call the distribution \( P \) in Definition 1 to be \( H/PD/SPD \) if \( A_P \) is \( H/PD/SPD \).

Though \( \rho_s(\alpha, P) \) and \( \rho_d(\alpha, P) \) depend only on \( P^M \), we use \( P \) instead of \( P^M \) to avoid notational clutter.

**Example 1.** The matrices \[
\begin{bmatrix} 0.1 & -1 \\ 1 & 0.1 \end{bmatrix}, \quad \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}
\]
are examples of \( H \), \( PD \) and \( SPD \) matrices, respectively, and they show that while \( SPD \) implies \( PD \), which implies \( H \), the reverse implications do not hold.

**Definition 5.** Call a set of distributions \( \mathcal{P} \) over \( \mathbb{C}^d \times \mathbb{C}^{d \times d} \) **weakly admissible** if there exists \( \alpha_\mathcal{P} > 0 \) such that \( \rho_s(\alpha, P) > 0 \) holds for all \( P \in \mathcal{P} \) and \( \alpha \in (0, \alpha_\mathcal{P}) \).

**Definition 6.** Call a set of distributions \( \mathcal{P} \) over \( \mathbb{C}^d \times \mathbb{C}^{d \times d} \) **admissible** if there exists some \( \alpha_\mathcal{P} > 0 \) such that \( \inf_{P \in \mathcal{P}} \rho_s(\alpha, P) > 0 \) holds for all \( \alpha \in (0, \alpha_\mathcal{P}) \). The value of \( \alpha_\mathcal{P} \) is called a witness.

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4 Projections can be problematic since they assume knowledge of \( \|\theta_*\| \), which is not available in practice.
It is easy to see that $\alpha \rightarrow \rho_\epsilon(\alpha, P)$ is decreasing, hence if $\alpha_P > 0$ witnesses that $P$ is (weakly) admissible then any $0 < \alpha' \leq \alpha_P$ is also witnessing this.

3 Problem Setup

We consider linear stochastic approximation algorithm (LSAs) with constant step-size (CS) and Polyak-Ruppert (PR) averaging of the iterates given as below:

\begin{align}
\text{LSA:} & \quad \theta_t = \theta_{t-1} + \alpha (b_t - A_t \theta_{t-1}), \quad (2a) \\
\text{PR-Average:} & \quad \hat{\theta}_t = \frac{1}{t+1} \sum_{i=0}^{t} \theta_i. \quad (2b)
\end{align}

The algorithm updates a pair of parameters $\theta_t, \hat{\theta}_t \in \mathbb{R}^d$ incrementally, in discrete time steps $t = 1, 2, \ldots$ based on data $b_t \in \mathbb{R}^d, A_t \in \mathbb{R}^{d \times d}$. Here $\alpha > 0$ is a positive step-size parameter; the only tuning parameter of the algorithm besides the initial value $\theta_0$. The iterate $\theta_t$ is treated as an internal state of the algorithm, while $\hat{\theta}_t$ is the output at time step $t$. The update of $\theta_t$ alone is considered a form of constant step-size LSA. Sometimes $A_t$ will have a special form and then the matrix-vector product $A_t \theta_{t-1}$ can also be computed in $O(d)$ time, a scenario common in reinforcement learning\cite{14, 17, 15, 16, 11}. This makes the algorithm particularly attractive in large-scale computations when $d$ is in the range of thousands, or millions, or more, as may be required by modern applications (e.g., \cite{9}). In what follows, for $t \geq 1$ we make use of the $\sigma$-fields $\mathcal{F}_{t-1} = \{ \theta_0, A_1, \ldots, A_{t-1}, b_1, \ldots, b_{t-1} \}$; $\mathcal{F}_1$ is the trivial $\sigma$ algebra. We are interested in the behaviour of (2) under the following assumption:

Assumption 1.
1. $(b_t, A_t) \sim P, t \geq 0$ is an i.i.d. sequence. We let $A_P$ be the expectation of $A_t$, $b_P$ be the expectation of $b_t$, as in Definition\cite{1} We assume that $P$ is Hurwitz.
2. The martingale difference sequence\footnote{That is, $E[M_t | \mathcal{F}_{t-1}] = 0$ and $E[N_t | \mathcal{F}_{t-1}] = 0$ and $M_t, N_t$ are $\mathcal{F}_t$ measurable, $t \geq 0$.} $M_t = A_t - A_P$ and $N_t = b_t - b_P$ associated with $A_t$ and $b_t$ satisfy the following

$$E \left[ \|M_t\|^2 | \mathcal{F}_{t-1} \right] \leq \sigma_{A_P}^2,$$

$$E \left[ \|N_t\|^2 | \mathcal{F}_{t-1} \right] \leq \sigma_{b_P}^2.$$

with some $\sigma_{A_P}^2$ and $\sigma_{b_P}^2$. Further, we assume $E [M_t N_t] = 0$.
3. $A_P$ is invertible and thus the vector $\theta_* = A_P^{-1} b_P$ is well-defined.

Performance Metric: We are interested in the behavior of the mean squared error (MSE) at time $t$ given by $E \left[ \|\hat{\theta}_t - \theta_*\|^2 \right]$. More generally, one can be interested in $E_P \left[ \|\hat{\theta}_t - \theta_*\|^2_C \right]$, the MSE with respect to a PD Hermitian matrix $C$. Since in general it is not possible to exploit the presence of $C$ unless it is connected to $P$ in a special way, here we restrict ourselves to $C = I$. For more discussion, including the discussion of the case of SGD for linear least-squares when $P$ and $C$ are favourably connected see Section\cite{6}.

4 Main Results and Discussion

In this section, we derive instance dependent bounds that are valid for a given problem $P$ (satisfying Assumption\cite{1}) and in the Section\cite{5} we address the question of deriving uniform bounds $\forall P \in P$, where $P$ is a class of distributions (problems). Here, we only present the main results followed by a discussion. The detailed proofs can be found in Appendix\cite{3} In what follows, for the sake of brevity, we drop the subscript $P$ in the quantities $E_P [\cdot], \sigma^2_{A_P}, \sigma^2_{b_P}$. We start with a lemma, which is needed to meaningfully state our main result:

Lemma 1. Let $P$ be a distribution over $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ satisfying Assumption\cite{1} Then there exists an $\alpha_{P_U} > 0$ and $U \in \text{GL}(d)$ such that $\rho_\epsilon(\alpha, P_U) > 0$ and $\rho_\epsilon(\alpha, P_U) > 0$ holds for all $\alpha \in (0, \alpha_{P_U})$.
Theorem 1. Let $P$ be a distribution over $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ satisfying Assumption 1. Then, for $U \in \text{GL}(d)$ and $\alpha_{P_U} > 0$ as in Lemma 1 for all $\alpha \in (0, \alpha_{P_U})$ and for all $t \geq 0$, \[ E \left[ \| \hat{\theta}_t - \theta_* \|^2 \right] \leq \nu \left\{ \| \theta_0 - \theta_* \|^2 + \frac{\nu^2}{(t+1)^2} \right\}, \] where $\nu = \left( 1 + \frac{\alpha^2}{\alpha_{P_U}} \right) \frac{\alpha(U)^2}{\alpha_{P_U}}, \alpha(U)$ and $\nu^2 = \alpha^2 (\sigma^2_{\alpha} \| \theta_* \|^2 + \sigma^2_b) + \alpha (\sigma^2_{\alpha} \| \theta_* \|) \| \theta_0 - \theta_* \|.$

Note that $\nu$ depends on $P_U$ and $\alpha$, while $\nu^2$ in addition also depends on $\theta_0$. The dependence, when it is essential, will be shown as a subscript.

Theorem 2 (Lower Bound). There exists a distribution $P$ over $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ satisfying Assumption 1 such that there exists $\alpha_P > 0$ so that $\rho_s(\alpha, P) > 0$ and $\rho_d(\alpha, P) > 0$ hold for all $\alpha \in (0, \alpha_P)$ and for any $t \geq 1$, \[ E \left[ \| \hat{\theta}_t - \theta_* \|^2 \right] \geq \frac{1}{\alpha^2 \rho_s(\alpha, P) \rho_d(\alpha, P)} \left\{ \beta_t \| \theta_0 - \theta_* \|^2 + \nu^2 \sum_{s=1}^t \beta_{t-s} \right\}, \] where $\beta_t = (1 - (1 - \alpha \rho_s(\alpha, P))^t)$ and $\nu^2$ is as in Theorem 1.

Note that $\beta_t \to 1$ as $t \to \infty$. Hence, the lower bound essentially matches the upper bound. In what follows, we discuss the specific details of these results.

Role of $U$: $U$ is helpful in transforming the recursion in $\theta_t$ to $\gamma_t = U^{-1} \theta_t$, which helps in ensuring $\rho_s(\alpha, P_U) > 0$. Such similarity transformation have also been considered in analysis of RL algorithms. More generally, one can always take $U$ in the result that leads to the smallest bound.

Role of $\rho_s(\alpha, P)$ and $\rho_d(\alpha, P)$: When $P$ is positive definite, we can expand the MSE as
\[ E \left[ \| \hat{\theta}_t \|^2 \right] = \frac{1}{(t+1)^2} \left\langle \sum_{s=0}^t e_s, \sum_{s=0}^t e_s \right\rangle, \tag{3} \]
where $\hat{\theta}_t = \hat{\theta}_t - \theta_*$ and $e_t = \theta_t - \theta_*$. The inner product in (3) is a summation of diagonal terms $E \left[ \langle e_s, e_s \rangle \right]$ and cross terms of $E \left[ \langle e_s, e_q \rangle \right], s \neq q$. The growth of the diagonal terms and the cross terms depends on the spectral norm of the random matrices $H_t = I - \alpha A_t$ and that of the deterministic matrix $H_P = I - \alpha A_P$, respectively. These are given by justifying the appearance of $\rho_s(\alpha, P)$ and $\rho_d(\alpha, P)$. For the MSE to be bounded, we need to norm the spectral near to be less than unity, implying the conditions $\rho_s(\alpha, P) > 0$ and $\rho_d(\alpha, P) > 0$. If $P$ is Hurwitz, we can argue on similar lines by first transforming $P$ into a positive definite problem $P_U$ and replacing $\rho_s(\alpha, P)$ and $\rho_d(\alpha, P)$ by $\rho_s(\alpha, P_U)$ and $\rho_d(\alpha, P_U)$, and introducing $\kappa(U)$ to account for the forward ($\gamma = U^{-1} \theta$) and reverse ($\theta = U \gamma$) transformations using $U^{-1}$ and $U$ respectively.

Constants $\alpha$, $\rho_s(\alpha, P)$ and $\rho_d(\alpha, P)$ do not affect the exponents $\frac{1}{2}$ for variance and $\frac{1}{4}$ for bias terms. This property is not enjoyed by all step-size schemes, for instance, step-sizes that diminish at $O\left( e^{-t} \right)$ are known to exhibit $O\left( \frac{1}{t^2} \right)$ (where $\mu$ is the smallest real part of eigenvalue of $A_P$), and hence the exponent of the rates are not robust to the choice of $c > 0$.

Bias and Variance: The MSE at time $t$ is bounded by a sum of two terms. The first bias term is given by $B = \nu \| \theta_0 - \theta_* \|^2$, bounding how fast the initial error $\| \theta_0 - \theta_* \|^2$ is forgotten. The second variance term is given by $V = \nu^2 \sum_{s=0}^t$ and captures the rate at which noise is rejected.

Behaviour for extreme values of $\alpha$: As $\alpha \to 0$, the bias term blows up, due to the presence of $\alpha^{-1}$ there. This is unavoidable (see also Theorem 2) and is due to the slow forgetting of initial conditions for small $\alpha$. Small step-sizes are however useful to suppress noise, as seen from that in our bound $\alpha$ is seen to multiply the variances $\sigma^2_{\alpha} \| \theta_* \|$ and $\sigma^2_b$. In quantitative terms, we can see that the $\alpha^{-2}$ and $\alpha^2$ terms are trading off the two types of errors. For larger values of $\alpha$ with $\alpha_P$ chosen so that $\rho_s(\alpha, P) \to 0$ as $\alpha \to \alpha_P$ (or $\alpha_{P_U}$, as the case may be), the bounds blow up again.

The lower bound of Theorem 2 shows that the upper bound of Theorem 1 is tight in a number of ways. In particular, the coefficients of both the $1/t$ and $1/t^2$ terms inside $\{ \cdot \}$ are essentially matched. Further, we also see that the $\rho_s(\alpha, P) \rho_d(\alpha, P)^{-1}$ appearing in $\nu = \nu_{P_U, \alpha}$ cannot be removed from the upper bound. Note however that there are specific examples, such as SGD for linear least-squares, where this latter factor can in fact be avoided (for further remarks see Section 6).
5 Uniform bounds

If \( \mathcal{P} \) is weakly admissible, then one can choose some step-size \( \alpha_\mathcal{P} > 0 \) solely based on the knowledge of \( \mathcal{P} \) and conclude that for any \( P \in \mathcal{P} \), the MSE will be bounded as shown in Theorem 1. When \( \mathcal{P} \) is not weakly admissible but rich enough to include the examples showing Theorem 2, no fixed step-size can guarantee bounded MSE for all \( P \in \mathcal{P} \). On the other hand, if \( \mathcal{P} \) is admissible then the error bound stated in Theorem 1 becomes independent of the instance, while when \( \mathcal{P} \) is not admissible, but “sufficiently rich”, this does not hold. Hence, an interesting question to investigate is whether a given set \( \mathcal{P} \) is (weakly) admissible.

A reasonable assumption is that \((b_t, A_t) \in \mathbb{R}^d \times \mathbb{R}^{d \times d}\) with some \( B > 0 \) (i.e., the data is bounded with bound \( B \)) and that \( A_P \) is positive definite for \( P \in \mathcal{P} \). Call the set of such distributions \( \mathcal{P}_B \). Is positive definiteness and boundedness sufficient for weak admissibility? The answer is no:

**Proposition 1.** For any fixed \( B > 0 \), the set \( \mathcal{P}_B \) is not weakly admissible.

Consider now the strict subset of \( \mathcal{P}_B \) that contains distributions \( P \) such that for any \( A \) in the support of \( P \), \( A \) is PSD. Call the resulting set of distributions \( \mathcal{P}_{PSD, B} \). Note that the distribution of data originating from linear least-squares estimation with SGD is of this type. Is \( \mathcal{P}_{PSD, B} \) weakly admissible? The answer is yes in this case:

**Proposition 2.** For any \( B > 0 \), the set \( \mathcal{P}_{PSD, B} \) is weakly admissible and in particular any \( 0 < \alpha < 2/B \) witnesses this.

However, admissibility does not hold for the same set:

**Proposition 3.** For any \( B > 0 \), the set \( \mathcal{P}_{PSD, B} \) is not admissible.

6 Related Work

We first discuss the related work outside of RL setting, followed by related work in the RL setting. In both cases, we highlight the insights that follows from the results in this paper.

**SGD for LSE:** As mentioned in the previous section, distributions underlying SGD for LSE with bounded data is a subset of \( \mathcal{P}_{PSD, B} \) and hence is weakly admissible under a fixed constant step-size choice. However, we also noted that \( \mathcal{P}_{PSD, B} \) is not admissible. This seems to be at odds with the result of Dieuleveut et al. [4] who prove that the MSE of SGD with CS-PR with an appropriate constant is bounded by \( \frac{C}{\alpha} \) where \( C > 0 \) only depends on \( B \). The apparent contradiction is resolved by noting that (i) in SGD the natural loss is \( \mathbb{E} \left[ \| \hat{\theta}_t - \theta_* \|^2_{A_P} \right] \) with \( A_P \) SPD, and (ii) the noise (arising due to the residual error) is “structured”, i.e., its variance is bounded by \( R A_P \) for some constant \( R > 0 \) (see [4], [3]).

**Additive vs. multiplicative noise:** Analysis of LSA with CS-PR goes back to the work by Polyak and Juditsky [12], wherein they considered the additive noise setting i.e., \( A_t = A \) for some deterministic Hurwitz matrix \( A \in \mathbb{R}^{d \times d} \). A key improvement in our paper is that we consider the ‘multiplicative’ noise case, i.e., \( A_t \) is non-constant random matrix. To tackle the multiplicative noise we use newer analysis introduced by Dieuleveut et al. [4]. However, since the general LSA setting (with Hurwitz assumption) does not enjoy special structures of the SGD setting of Dieuleveut et al. [4], we make use of Jordan decomposition and similarity transformations in a critical way to prove our results, thus diverging from the line of analysis of Dieuleveut et al. [4].

**Results for RL:** We are presented with data in the form of an i.i.d. sequence \((\phi_1, \phi_2, r_1) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \). For a fixed constant \( \gamma \in (0, 1) \) define \( \Delta_t = \phi_t \phi_t^\top - \gamma \phi_t \phi_t^\top, C_t = \phi_t \phi_t^\top \) and \( b_t = \phi_t r_t \). In what follows, \( \mu_t > 0 \) is an importance sampling factor whose aim is to correct for mismatch in the (behavior) distribution with which the data was collected and the (target) distribution with respect to which one wants to learn. A factor \( \mu_t = 1, \forall t \geq 0 \) will mean that no correction is required. The various TD class of algorithms that can be cast as LSAs are given in Table 1. The TD(0) algorithm is the most basic of the class of TD algorithms. An important shortcoming of TD(0) was its instability in the off-policy case, which was successfully mitigated by the generative temporal difference learning

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6 This is known as the on-policy case where the behavior is identical to the target. The general setting where \( \mu_t > 0 \) is known as off-policy.
GTD algorithm \cite{Sutton16}. GTD was proposed by Sutton et al. \cite{Sutton15}; its variants, namely GTD2 and TDC, were proposed later by Sutton et al. \cite{Sutton16}. The initial convergence analysis for GTD/GTD2/TDC was only asymptotic in nature \cite{Sutton15, Sutton16} with diminishing step-sizes.

The most relevant to our results are those by Korda and Prashanth \cite{Korda7} in TD(0) and by Liu et al. \cite{Liu11} were proposed later by Sutton et al. \cite{Sutton16}. The initial convergence analysis for GTD/GTD2/TDC was only asymptotic in nature \cite{Sutton15, Sutton16} with diminishing step-sizes.

It is straightforward to see from (1) that $\alpha_t$ cannot be asymptotically increasing. We now present some heuristic arguments in favour of a constant step-size over asymptotically diminishing step-sizes in (1). It has been observed that when the step-sizes of form $\alpha_t = \frac{c}{t}$ or $\alpha_t = \frac{c}{t^{\lambda}}$ (for some $c > 0$) are used, the MSE, $E[|\theta_t - \theta^*|^2]$, is not robust to the choice of $c > 0$ \cite{Sutton15, Sutton16}. In particular only a $O(\frac{1}{\sqrt{T}})$ decay can be achieved for the MSE, where $\mu$ is the smallest positive part of the eigenvalues of $A_P$ \cite{Sutton16}. Note that, in the case of LSA with CS-PR, Theorem\cite{Sutton16} guarantees a $O(\frac{1}{T})$ rate of decay for the MSE and the problem dependent quantities affect only the constants and not the exponent. Also, in the case of important TD algorithms such as GTD/GTD2/TDC, while the theoretical analysis uses diminishing step-sizes, the experimental results are with a constant step-size or with CS and PR averaging \cite{Sutton16, Sutton11}. Independently, Dann et al. \cite{Dann16} also observe in their experiments that a constant step-size is better than diminishing step-sizes.

We would like to remind that in Section 5 we showed that weak admissibility might not hold for all problem classes, and hence a uniform choice for the constant step-size might not be possible. However, motivated by Theorem\cite{Sutton16}and also by the usage of constant step-size in practice \cite{Sutton15, Sutton16, Sutton11}, we suggest a natural algorithm to tune the constant step-size, shown as Algorithm\cite{Sutton16}.

In Algorithm\cite{Sutton16} $T > 0$ is a time epoch and $k$ is a given integer and $\alpha_{max} > 0$ is the maximum step-size that is allowable. From the Gronwall-Bellman lemma it follows that in Algorithm\cite{Sutton16}$|\theta_t| \leq C(1 + e^{\beta t})$ with some $C > 0$, where the sign of $\beta$ determines whether the iterates are bounded. Using this fact, we observe that the sequence $r_i = \frac{\|\theta_{kT+(i-1)T}\|}{\|\theta_{kT+(i-1)T}\|}, i = 1, \ldots, k$ should be...
Algorithm 1 Automatic Tuning of Constant Step-Size

1: Initialize: $\theta_0$, $\alpha = \alpha_{\text{max}}$, $k$, $T$
2: for $t = 1, 2, \ldots$ do
3: $\theta_t = \theta_{t-1} + \alpha (b_t - A_t \theta_{t-1})$, $\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t+1} (\theta_t - \hat{\theta}_{t-1})$
4: if $\text{IsUnstable}(\|\hat{\theta}_t\|, \ldots, \|\hat{\theta}_{(t-kT)}\|, 0) = \text{True}$ then
5: $\alpha = \alpha / 2$
6: end if
7: end for

Figure 1: The left plot shows comparison of the constant step-size $\alpha_P$ as function of $\sigma_A$ found by Algorithm 1 versus the constant step-size computed in closed form. The right plot shows the performance of LSA with CS-PR (with the step-size chosen by Algorithm 1) for various $\sigma_A$ values. The errors were insignificant and hence error bars are not shown in the right plot.

“roughly” (making allowance for the persistent noise) decreasing and converge to 1 when the step-size is large enough so that the iterates stay bounded and eventually converge. The idea is that the $\text{IsUnstable}()$ routine in Algorithm 1 calculates $\{r_i\}_i$ based on its input and returns true when any of these is larger than a preset constant $c > 1$. By choosing a larger the constant $c$, the probability of false detection of a run-away event decreases rapidly, while still controlling for the probability of altogether missing a run-away event.

We ran numerical experiments on the class with $A_P = \begin{bmatrix} 1 & -10 \\ 10 & 1 \end{bmatrix}$, $\sigma_b = 0$ and $b_t = b, \forall t \geq 0$ (chosen such that $\theta_\star = (1, 1)^T$) and $M_x, t \geq 0$ with varying $\sigma_A$’s. This problem class does not admit an apriori step-size (due to the unknown $\sigma_A$ and the dependence of step-size on $\sigma_A$) that prevents the explosion of MSE. The results (see Figure 1) show that Algorithm 1 does find a problem dependent constant step-size (within a factor of the best possible hand computed step-size) that avoids the MSE blow up. We chose $k = 2$ and $T = 5$, the preset constant was chosen to be 1.025 and the results are for $\sigma_A = 0, 2, 5, 10, 20$. Algorithm 1 is oblivious of the data distribution, and the hand computed step-size is based on full problem information (i.e., $\sigma_A$). Further, the results (in the right plot of Figure 1) also confirm our expectation that higher step-sizes lead to faster convergence.

8 Conclusion

We presented a finite time performance analysis of LSAs with CS-PR and showed that the MSE decays at a rate $O(\frac{1}{t})$. Our results extended the analysis of Dieuleveut et al. [4] for SGD with CS-PR for the problem of linear least-squares estimation and i.i.d. sampling to general LSAs with CS-PR. Due to the lack of special structures, our analysis for the case of general LSA cannot recover entirely the results of Dieuleveut et al. [4] who use the problem specific structures in their analysis. Our results also improved the rates in the case of the GTD class of algorithms. We presented conditions under which a constant step-size can be chosen uniformly for a given class of data distributions. We showed a negative result in that not all data distributions ‘admit’ such a constant step-size. This is a
negative result from the perspective of TD algorithms in RL. We also argued that a problem dependent constant step-size can be obtained in an automatic manner and presented numerical experiments on a synthetic LSA.

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A Linear Algebra Preliminaries

A.1 Additional Notations

For \( x = a + ib \in \mathbb{C} \), we denote its real and imaginary parts by \( re(x) = a \) and \( im(x) = b \) respectively. Given a \( x \in \mathbb{C}^d \), for \( 1 \leq i \leq d \), \( x(i) \) denotes the \( i \)th component of \( x \). For any \( x \in \mathbb{C} \) we denote its modulus \( |x| = \sqrt{re(x)^2 + im(x)^2} \) and its complex conjugate by \( \bar{x} = a - ib \). We use \( A \geq 0 \) to denote that the square matrix \( A \) is Hermitian and positive semidefinite (HPSD): \( A = A^* \), \( \inf_x x^*Ax \geq 0 \). We use \( A \succ 0 \) to denote that the square matrix \( A \) is Hermitian and positive definite (HPD): \( A = A^* \), \( \inf_x x^*Ax > 0 \). For \( A, B \) HPD matrices, \( A \succeq B \) holds if \( A - B \succeq 0 \). We also use \( A \succ B \) similarly to denote that \( A - B \succ 0 \). We also use \( \preceq \) and \( \prec \) analogously. We denote the smallest eigenvalue of a real symmetric positive definite matrix \( A \) by \( \lambda_{\text{min}}(A) \).

We now present some useful results from linear algebra.

Let \( B \) be a \( d \times d \) block diagonal matrix given by

\[
B = \begin{bmatrix}
B_1 & 0 & 0 & \cdots & 0 \\
0 & B_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_k
\end{bmatrix},
\]

where \( B_i \) is a \( d_i \times d_i \) matrix such that \( d_i < d \), \( \forall i = 1, \ldots, k \) (w.l.o.g.) and \( \sum_{i=1}^{k} d_i = d \). We also denote \( B \) as

\[
B = B_1 \oplus B_2 \oplus \ldots B_k = \oplus_{i=1}^{k} B_i
\]

A.2 Results in Matrix Decomposition and Transformation

We will now recall Jordon decomposition.

**Lemma 2.** Let \( A \in \mathbb{C}^{d \times d} \) and \( \{\lambda_i \in \mathbb{C}, i = 1, \ldots, k \leq d\} \) denote its \( k \) distinct eigenvalues. There exists a complex matrix \( V \in \mathbb{C}^{d \times d} \) such that \( A = V \bar{A} V^{-1} \), where \( \bar{A} = \bar{A}_1 \oplus \ldots \oplus \bar{A}_k \), where each \( \bar{A}_i, i = 1, \ldots, k \) can further be written as \( \bar{A}_i = \bar{A}_i^1 \oplus \ldots \oplus \bar{A}_i^{l(i)} \). Each of \( \bar{A}_j, j = 1, \ldots, l(i) \) is a \( d_j^1 \times d_j^1 \) square matrix such that \( \sum_{j=1}^{l(i)} d_j^1 = d_i \) and has the special form given by

\[
\bar{A}_j = \begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & 1 & \cdots & 0 & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda_i
\end{bmatrix}.
\]

**Lemma 3.** Let \( A \in \mathbb{C}^{d \times d} \) be a Hurwitz matrix. There exists a matrix \( U \in \text{GL}(d) \) such that \( A = U \bar{A} U^{-1} \) and \( \bar{A}^* + \bar{A} \) is a real symmetric positive definite matrix.

**Proof.** It is trivial to see that for any \( \Lambda \in \mathbb{C}^{d \times d} \), \( (\Lambda^* + \Lambda) \) is Hermitian. We will use the decomposition of \( A = V \bar{A} V^{-1} \) in Lemma 2 and also carry over the notations in Lemma 2. Consider the diagonal matrices \( D_j^l = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ re(\lambda_i) & 0 & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & re(\lambda_i) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \forall j = 1, \ldots, l(i), \)

\( D^l = D_1^l \oplus \ldots \oplus D_{l(i)}^l, \forall i = 1, \ldots, k \) and \( D = D_1^1 \oplus \ldots \oplus D^k \). It follows that \( A = (VD)\Lambda(VD)^{-1} \), where \( \Lambda \) is a matrix such that \( \Lambda = \Lambda_1 \oplus \ldots \oplus \Lambda_k \), where each \( \Lambda_i, i = 1, \ldots, k \) can further be written as \( \Lambda_i = \Lambda_i^1 \oplus \ldots \oplus \Lambda_i^{l(i)} \). Each of \( \Lambda_j^l \) is a \( d_j^1 \times d_j^1 \) square matrix with the special form given by

\[
\Lambda_j^l = \begin{bmatrix}
\lambda_i & re(\lambda_i) & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & re(\lambda_i) & 0 & \cdots & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i & re(\lambda_i) \\
0 & 0 & 0 & \cdots & 0 & \lambda_i
\end{bmatrix}.
\]
Now we have \( \frac{(\Lambda^* + \Lambda)}{2} = \sum_{i=1}^{k} \frac{t(i)}{j=1} \frac{\Lambda_j^* + \Lambda_j}{2} \), where \( \frac{\Lambda_j^* + \Lambda_j}{2} = \)

\[
\begin{bmatrix}
\text{re}(\lambda_i) & \text{re}(\lambda_i) & \ldots & \text{re}(\lambda_i) \\
\text{re}(\lambda_i) & \text{re}(\lambda_i) & \ldots & \text{re}(\lambda_i) \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{bmatrix}
\]

Then for any \( x = (x(i), i = 1, \ldots, d) \in \mathbb{C}^d \), we have

\[
x^* \left( \frac{(\Lambda^* + \Lambda)}{2} \right) x = \text{re}(\lambda_i) \left( \sum_{i=1}^{d} \bar{x}(i)x(i) + \sum_{i=1}^{d-1} \bar{x}(i)x(i+1) + x(i)x(i+1) \right)
\]
\[
= \frac{\text{re}(\lambda_i)}{2} \left( |x(1)|^2 + |x(d)|^2 \right) + \frac{\text{re}(\lambda_i)}{2} \left( \sum_{i=1}^{d-1} |x(i)|^2 + \bar{x}(i)x(i+1) + x(i)x(i+1) + |x(i+1)|^2 \right)
\]
\[
> \frac{\text{re}(\lambda_i)}{2} \left( \sum_{i=1}^{d} |x(i)| + x(i+1) \right) > 0
\]

\[\square\]

**B Proofs**

**B.1 LSA with CS-PR for Positive Definite Distributions**

In this subsection, we re-write (2) and Assumption 1 to accommodate complex number computations and in addition assume that \( P \) is positive definite. To this end,

\[
\text{LSA:} \quad \theta_t = \theta_{t-1} + \alpha(b_t - A_t\theta_{t-1}), \quad (4a)
\]
\[
\text{PR-Average:} \quad \hat{\theta}_t = \frac{1}{t+1} \sum_{i=0}^{t} \theta_i, \quad (4b)
\]

where \( \hat{\theta}_t, \theta_t \in \mathbb{C}^d \). We now assume,

**Assumption 2.**

1. \( (b_t, A_t) \sim (P^b, P^A), t \geq 0 \) is an \( i.i.d. \) sequence, where \( P^b \) is a distribution over \( \mathbb{C}^d \) and \( P^A \) is a distribution over \( \mathbb{C}^{d \times d} \). We assume that \( P \) is positive definite.

2. The martingale difference sequence\(^7\) \( M_t = A_t - A_P \) and \( N_t = b_t - b_P \) associated with \( A_t \) and \( b_t \) satisfy the following

\[
\mathbb{E} \left[ \|M_t\|^2 \mid F_{t-1} \right] \leq \sigma_{A,P}^2, \quad \mathbb{E} \left[ N_t^* N_t \right] = \sigma_{b_P}^2.
\]

3. \( A_P \) is invertible and there exists a \( \theta_* = A_P^{-1}b_P \).

We now define the error variables and present the recursion for the error dynamics. In what follows, definitions in Section 2 and Section 3 continue to hold.

**Definition 7.**

- Define error variables \( e_t = \theta_t - \theta_* \) and \( \hat{e}_t = \hat{\theta}_t - \theta_* \).

- Define \( \forall t \geq 0 \) random vectors \( \zeta_t = b_t - b - (A_t - A_P)\theta_* \).

- Define constants \( \sigma_2^2 = \sigma_A^2 \| \theta_* \|^2 + \sigma_b^2 \) and \( \sigma_2^2 = \sigma_A^2 \| \theta_* \|^2 \). Note that \( \mathbb{E} \left[ \| \zeta_t \|^2 \right] \leq \sigma_2^2 \) and \( \mathbb{E} \left[ \| M_t \|^2 \right] \leq \sigma_2^2 \).

- Define \( \forall i \geq j \), the random matrices \( F_{i,j} = (I - \alpha A_i) \ldots (I - \alpha A_j) \) and \( \forall, i < j \ F_{i,j} = I \).

\(^7\) \( \mathbb{E} \left[ M_t | F_{t-1} \right] = 0 \) and \( \mathbb{E} \left[ N_t | F_{t-1} \right] = 0 \)
Error Recursion. Let us now look at the dynamics of the error terms defined by

\[ \begin{align*}
\theta_t &= \theta_{t-1} + \alpha \left( b_t - A_t \theta_{t-1} \right) \\
\theta_t - \theta_* &= \theta_{t-1} - \theta_* + \alpha \left( b_t - A_t (\theta_{t-1} - \theta_*) \right) \\
e_t &= (I - \alpha A_t)e_{t-1} + \alpha (b_t - (A_t - A) \theta_*) \\
e_t &= (I - \alpha A_t)e_{t-1} + \alpha \zeta_t
\end{align*} \]

(5)

Lemma 4. Let \( P \) be a distribution over \( \mathbb{C}^d \times \mathbb{C}^{d \times d} \) satisfying Assumption 2 then there exists an \( \alpha_P > 0 \) such that \( \rho_d(\alpha, P) > 0 \) and \( \rho_x(\alpha, P) > 0, \forall \alpha \in (0, \alpha_P) \).

Proof.\[
\rho_x(\alpha, P) = \inf_{x: ||x||=1} x^* (A_p^* + A_P) x - \alpha x^* E[A_p^* A_t] x
\]

(6)

By the tower-rule for conditional expectations and our measurability assumptions, Lemma 6.

\[
E[x^* F_{t,i+1} | \mathcal{F}_i] = x^* (I - \alpha A_P)^{t-i} y.
\]

Continuing this way we get

\[
E[x^* F_{t,i+1} | \mathcal{F}_{t-j}] = x^* (I - \alpha A_P)^{t-j} F_{t-j,i+1} y, \quad j = 1, 2, \ldots, t - i.
\]

Specifically, for \( j = t - i \) we get

\[
E[x^* F_{t,i+1} | \mathcal{F}_i] = x^* (I - \alpha A_P)^{t-i} y.
\]

Lemma 5 (Product unroll lemma). Let \( t > i \geq 1, x, y \in \mathbb{C}^d \) be \( \mathcal{F}_t \)-measurable random vectors. Then,

\[
E[x^* F_{t,i+1} | \mathcal{F}_i] = x^* (I - \alpha A_P)^{t-i} y.
\]

Lemma 6. Let \( t > i \geq 1 \) and let \( x \in \mathbb{C}^d \) be a \( \mathcal{F}_{t-1} \)-measurable random vector. Then, \( E[x^* F_{t,i+1} \zeta_i] = 0 \).

Proof. By Lemma 5.

\[
E[x^* F_{t,i+1} \zeta_i | \mathcal{F}_i] = x^* (I - \alpha A_P)^{t-i} \zeta_i.
\]

Using the tower rule,

\[
E[x^* F_{t,i+1} \zeta_i | \mathcal{F}_{t-1}] = x^* (I - \alpha A_P)^{t-i} E[\zeta_i | \mathcal{F}_{t-1}] = 0.
\]

Lemma 7. For all \( t > i \geq 0 \), \( E(e_t, F_{t,i+1} e_i) = E(e_t, (I - \alpha A_P)^{t-i} e_i) \).
**Proof.** The lemma follows directly from Lemma 5. Indeed, \( \theta_i \) depends only on \( A_1, \ldots, A_i, b_1, \ldots, b_i, \theta_i \) and so is \( \varepsilon_i, F_i \)-measurable. Hence, the lemma is applicable and implies that

\[
\mathbb{E} \left[ \langle e_i, F_{t,i+1} \rangle | F_i \right] = \mathbb{E} \left[ \langle e_i, (I - \alpha A_P)^{t+1} e_i \rangle | F_i \right].
\]

Taking expectation of both sides gives the desired result. \( \square \)

**Lemma 8.** Let \( i > j \geq 0 \) and let \( x \in \mathbb{R}^d \) be an \( F_j \)-measurable random vector. Then,

\[
\mathbb{E} \langle F_{i,j+1} x, F_{i,j+1} x \rangle \leq (1 - \alpha \rho_s(\alpha, P))^{i-j} \mathbb{E} \| x \|^2.
\]

**Proof.** Note that \( S_i = \mathbb{E} \left[ (I - \alpha A_i)^* (I - \alpha A_i) | F_{i-1} \right] = I - \alpha (A_i^* P + A_P) + \alpha^2 \mathbb{E} [A_i^* A_i | F_{i-1}]. \)
Since \( (b_t, A_t)_t \) is an independent sequence, \( \mathbb{E} [A_i^* A_i | F_{i-1}] = \mathbb{E} [A_i^* A_i] \). Now, using the definition of \( \rho_s(\alpha, P) \) from Definition 7, \( \sup_{x \in \mathbb{R}^d} \| x \|^2 S_i x = 1 - \alpha \inf_{x \in \mathbb{R}^d} \| x \|^2 (A_i^* P + A_P - \alpha \mathbb{E} [A_i^* A_i]) \). Hence,

\[
\mathbb{E} \left[ \langle F_{i,j+1} x, F_{i,j+1} x \rangle | F_{i-1} \right] \\
= \mathbb{E} \left[ x^* F_{i-1,j+1}^* (I - \alpha A_i) F_{i-1,j+1} x | F_{i-1} \right] \\
= \langle x F_{i-1,j+1}^* S_i F_{i-1,j+1} x \rangle \\
\leq (1 - \alpha \rho_s(\alpha, P)) \langle F_{i-1,j+1}, F_{i-1,j+1} \rangle \\
\leq (1 - \alpha \rho_s(\alpha, P))^{i-j} \| x \|^2.
\]

**Theorem 3.** Let \( \hat{e}_i \) be as in Definition 7. Then

\[
\mathbb{E} [\| \hat{e}_i \|^2] \leq \left( 1 + \frac{2}{\alpha \rho_d(\alpha, P)} \right) \frac{1}{\alpha \rho_s(\alpha, P)} \left( \frac{\| e_0 \|^2}{(t+1)^2} + \frac{\alpha^2 (\sigma_1^2) + \alpha \sigma_2^2}{t+1} \right). \quad (6)
\]

**Proof.**

\[
e_t = (I - \alpha A_t) (I - \alpha A_{t-1}) e_{t-2} \\
+ \alpha (I - \alpha A_t) \zeta_{t-1} + \alpha \zeta_t \\
\vdots \\
= (I - \alpha A_t) \cdots (I - \alpha A_1) e_0 \\
+ \alpha (I - \alpha A_t) \cdots (I - \alpha A_2) \zeta_1 \\
+ \alpha (I - \alpha A_t) \cdots (I - \alpha A_3) \zeta_2 \\
\vdots \\
+ \alpha \zeta_t,
\]

which can be written compactly as

\[
e_t = F_{t,1} e_0 + \alpha (F_{t,2} \zeta_1 + \cdots + F_{t,t+1} \zeta_t), \quad (7)
\]

\[
\hat{e}_t = \frac{1}{t+1} \sum_{i=0}^t e_i = \frac{1}{t+1} \left\{ \sum_{i=0}^t F_{i,1} e_0 + \alpha \sum_{i=1}^t \left( \sum_{j=i}^t F_{j,t+1} \right) \zeta_j \right\},
\]

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where in the second sum we flipped the order of sums and swapped the names of the variables that the sum runs over. It follows that

$$E[||\hat{e}_t||^2] = E(\hat{e}_t, \hat{e}_t) = \frac{1}{(t+1)^2} \sum_{i,j=0}^{t} E(e_i, e_j).$$

Hence, we see that it suffices to bound $E[\{e_i, e_j\}]$. There are two cases depending on whether $i = j$. When $i < j$,

$$E(e_i, e_j) = E(e_i, [F_{j+1}e_i + \alpha \sum_{k=i+1}^{j} F_{j+k+1}z_k]) = E(e_i, F_{j+1}e_i) (from \text{Lemma } 6) = E(e_i, (I - \alpha A)^{-1}e_i) (from \text{Lemma } 7)$$

and therefore

$$\sum_{i=0}^{t-1} \sum_{j=i+1}^{t} E(e_i, e_j) = \frac{1}{\alpha \rho_d(\alpha, P)} \sum_{i=0}^{t} E(e_i, e_i) \leq \frac{2}{\alpha \rho_d(\alpha, P)} \sum_{i=0}^{t} E(e_i, e_i).$$

Since $\sum_{i,j} = \sum_{i=j} + 2 \sum_{i,j>i}$,

$$\sum_{i=0}^{t} \sum_{j=0}^{t} E(e_i, e_j) = \left(1 + \frac{2}{\alpha \rho_d(\alpha, P)}\right) \sum_{i=0}^{t} E(e_i, e_i).$$

Expanding $e_i$ using (7) and then using Lemma 8 and Assumption 2

$$E(e_i, e_i) = E(F_{i,1}e_0, F_{i,1}e_0) + \alpha^2 \sum_{j=1}^{i} E(F_{i,j+1}z_j, F_{i,j+1}z_j) + \alpha \sum_{j=1}^{i} E(F_{i,1}e_0, F_{i,j+1}z_j) \leq (1 - \alpha \sigma_s(\alpha, P))^i ||\varepsilon_0||^2 + \alpha^2 \frac{\eta_\sigma^2}{\alpha \rho_s(\alpha, P)} + \alpha \frac{\sigma_2^2}{\alpha \rho_s(\alpha, P)} ||\varepsilon_0||,$$

and so

$$\sum_{i=0}^{t} \sum_{j=0}^{t} E(e_i, e_j) \leq \left(1 + \frac{2}{\alpha \rho_d(\alpha, P)}\right) \frac{1}{\alpha \rho_s(\alpha, P)} \left(\frac{||\varepsilon_0||^2}{(t+1)^2} + \frac{\alpha^2(\eta_\sigma^2) + \alpha \sigma_2^2}{t+1} \frac{||\varepsilon_0||}{t+1}\right).$$

Putting things together,

$$E[||\hat{e}_t||^2] \leq \left(1 + \frac{2}{\alpha \rho_d(\alpha, P)}\right) \frac{1}{\alpha \rho_s(\alpha, P)} \left(\frac{||\varepsilon_0||^2}{(t+1)^2} + \frac{\alpha^2(\eta_\sigma^2) + \alpha \sigma_2^2}{t+1} \frac{||\varepsilon_0||}{t+1}\right). \tag{8}$$

\textbf{Proof of Lemma 1}

\textbf{Lemma 9.} Let $P$ be a distribution over $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ satisfying Assumption 1, then there exists an $\alpha_{P_U} > 0$ and $U \in \text{GL}(d)$ such that $\rho_d(\alpha, P_U) > 0$ and $\rho_s(\alpha, P_U) > 0, \forall \alpha \in (0, \alpha_P)$.

\textbf{Proof.} We know that $A_P$ is Hurwitz and from Lemma 3 it follows that there exists an $U \in \text{GL}(d)$ such that $\Lambda = U^{-1}A_PU$ and $\Lambda^* + \Lambda$ is real symmetric and positive definite. Using Definition 2 we have $A_{P_U} = \Lambda$ and from Lemma 4 we know that there exists an $\alpha_{P_U}$ such that $\rho_d(\alpha, P_U) > 0$ and $\rho_s(\alpha, P_U) > 0, \forall \alpha \in (0, \alpha_{P_U})$. \hfill \Box

\textbf{Lemma 10 (Change of Basis).} Let $P$ be a distribution over $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ as in Assumption 1 and let $U$ be chosen according to Lemma 1. Define $\gamma_t = U^{-1}\theta_t, \gamma_* = U^{-1}\theta_*$, then

$$E \left[||\gamma_t - \gamma_*||^2\right] \leq \left(1 + \frac{2}{\alpha \rho_d(\alpha, P_U)}\right) \frac{||U^{-1}||^2}{\alpha \rho_s(\alpha, P_U)} \left(\frac{||\theta_0 - \theta_*||^2}{(t+1)^2} + \frac{\alpha^2(\eta_\sigma^2) + \alpha \sigma_2^2}{t+1} \frac{||\theta_0 - \theta_*||}{t+1}\right), \tag{9}$$

where $\gamma_t = \frac{1}{t+1} \sum_{s=0}^{t} \gamma_s$.  

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Thus, \( z_{\Lambda} \).

Now applying Theorem 3 to \((12)\) and taking \( \Lambda t = U^{-1}A_t U \) and \( H_t = U^{-1}\zeta_t \). Note that the error recursion in \( z_t \) might involve complex computations (depending on whether \( U \) has complex entries or not), hence (3) and Assumption 2 are useful in analyzing \( z_t \). We know that

\[
\mathbb{E} \[ \|H_t\|_2^2 \] \leq \|U^{-1}\|_2^2 \mathbb{E} \[ \| \zeta_t \|_2^2 \]
\]

and

\[
\mathbb{E} \[ \|H_t A_t \|_2 \] = \mathbb{E} \[ \|U^{-1}A_t U \| \| \zeta_t \|_2 \] = \mathbb{E} \[ \|U^{-1}A_t \| \| \zeta_t \|_2 \] = \|U^{-1}\|_2 \mathbb{E} \[ \|A_t \| \| \zeta_t \|_2 \] = \|U^{-1}\|_2 \sigma_{\zeta_t}.
\]

Now applying Theorem 3 to \( z_t = \frac{1}{t+1} \sum_{i=0}^{t} z_i \), we have

\[
\mathbb{E} \[ \|z_t\|_2^2 \] \leq \left( 1 + \frac{2}{\alpha \rho_\delta(\alpha, P)} \right) \frac{1}{\alpha \rho_{\zeta}(\alpha, P)} \left( \frac{\|z_0\|_2^2}{(t+1)^2} + \frac{\alpha^2(\|U^{-1}\|_2^2 \sigma_{\zeta_t}^2 + \alpha(\|U^{-1}\|_2 \| \zeta_t \|_2)}{(t+1)} \right)
\]

\[
\leq \left( 1 + \frac{2}{\alpha \rho_\delta(\alpha, P)} \right) \frac{1}{\alpha \rho_{\zeta}(\alpha, P)} \left( \frac{\|U^{-1}\|_2^2 \|c_0\|_2^2}{(t+1)^2} + \frac{\alpha^2(\|U^{-1}\|_2 \| \zeta_t \|_2)}{(t+1)} \right).
\]

Proof of Theorem 1. Follows by substituting \( \theta_t = U \gamma_t \) in Lemma 10.

Proof of Theorem 2. Consider the LSA with \( b_t, A_t \sim P \) such that \( b_t = (N_t, 0)^T \in \mathbb{R}^2 \) is a zero mean \( i.i.d. \) random variable with variance \( \sigma_{b_t}^2 \), and \( A_t = A, \forall t \geq 0 \), where \( A = A_P = \begin{bmatrix} \lambda_{\min} & 0 \\ 0 & \lambda_{\max} \end{bmatrix} \), for some \( \lambda_{\max} > \lambda_{\min} > 0 \). Note that in this example \( \theta_0 = 0 \). By choosing \( \alpha < \frac{2}{\lambda_{\max}} \), in this case it is straightforward to write the expression for \( \hat{e}_t \) explicitly as below:

\[
\hat{e}_t = \frac{1}{t+1} \sum_{s=0}^{t} e_t = \frac{1}{t+1} \sum_{s=0}^{t} (I - \alpha A_P)^{t-s} e_0 + \sum_{s=1}^{t} \sum_{i=s}^{t} (I - \alpha A_P)^{t-i} b_s
\]

\[
= \frac{1}{t+1} (\alpha A_P)^{-1} \left[ (I - (I - \alpha A_P)^{t+1}) e_0 + \sum_{s=1}^{t} (I - (I - \alpha A_P)^{t+1-s}) b_s \right].
\]

Thus,

\[
\mathbb{E} \[ \|\hat{e}_t\|_2^2 \] \leq \frac{1}{(t+1)^2} \mathbb{E} \[ \|\alpha A_P^{-1} (I - (I - \alpha A_P)^{t+1}) e_0 \|_2^2 + \sum_{s=1}^{t} \| (\alpha A_P^{-1} (I - (I - \alpha A_P)^{t+1-s}) b_s \|_2^2 \],
\]

and hence

\[
\mathbb{E} \[ \|\hat{e}_t\|_2^2 \] \geq \mathbb{E} \[ (\hat{e}_t)^2 \] \geq \frac{1}{(t+1)^2} \mathbb{E} \[ \| (I - (I - \alpha A_P)^{t+1}) e_0 \|_2^2 + \sum_{s=1}^{t} \| (I - (I - \alpha A_P)^{t+1-s}) b_s \|_2^2 \].
\]

Here (a) and (b) follows from the \( i.i.d. \) assumption. Note that in this example, \( \rho_\delta(\alpha, P) = \rho_\delta(\alpha, P) = 2 \lambda_{\min} - \alpha \lambda_{\max}^2 = \lambda_{\min} (2 - \alpha \lambda_{\max}) \), and \( \| \theta_0 \| = 0 \) and \( \sigma_{\zeta_t}^2 = 0 \). Further, the result follows by noting the fact that \( \|b_t\|_2^2 = b_t(1)^2 \) and \( \|b_t\|_2^2 = \theta_t(1)^2 \).

Proof of Proposition 1. Fix an arbitrary \( \alpha > 0 \). We show that there exists \( P \in \mathcal{P} \) such that \( \rho_\delta(P) < 0 \). For \( \epsilon \in (0, 1/2) \) let \( P = (P^\epsilon, P^M) \) be the distribution such that \( P^M \) is supported on \( \{-I, I\} \) and takes on the value of \( I \) with probability \( 1/2 + \epsilon \). Then \( A_P = 2I > 0 \), hence \( P \in \mathcal{P}_1 \). Further, \( Q_P = I \). Hence, \( \rho_\delta(P) = 4 \epsilon - \alpha \). Hence, if \( \epsilon < \alpha/4, \rho_\delta(P) < 0 \).
Proof of Proposition 2 Since $P_{\text{PSD},B}$ is supported on the set of positive semi-definite matrices, we know for any $A \in \mathbb{R}^{d \times d}$ that is PSD, we can consider the SVD of $A$: $A = U \Lambda U^\top$ where $U$ is orthonormal and $\Lambda$ is diagonal with nonnegative elements. Note that $\Lambda \preceq B I$ and thus $\Lambda^2 \preceq B \Lambda$. Then for any $x \in \mathbb{R}^d$, $x^\top A^\top A x = x^\top U \Lambda^2 U^\top x \leq B x^\top U \Lambda U^\top x = B x^\top A x$. Taking expectations we find that $x^\top C P x \leq B x^\top A P x$. Hence, $\rho_\alpha(P_{\text{PSD},B}) = 2 x^\top A P x - \alpha x^\top C P x \geq (2 - \alpha B) x^\top A P x$. Thus, for any $\alpha < 2/\beta$, $\rho_\alpha(P) > 0$.

Proof of Proposition 3 Consider the case when the smallest eigenvalue of $A P$ is 0.