FREENESS CRITERIA VIA VANISHING OF Tor

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Abstract. We investigate some aspects of the module $L$ equipped with the property that $\text{Tor}_1^R(L, F) = 0$ implies that $F$ is free. This has some applications.

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§ 1. Introduction

Our initial motivation is to understand the following amusing question of Lichtenbaum:

Question 1.1. (See [23, Question 4]) For a given local ring $R$, which modules $L$ have the property that $\text{Tor}_1^R(L, F) = 0$ implies that $F$ is free?

Concerning to the previous item, we call such an $L$ a Lichtenbaum module. The classic example of Lichtenbaum modules is the residue field. We discuss about the abundance and basic properties of Lichtenbaum modules. For instance, we detect the following properties of the ring from Lichtenbaum modules:

(i) $\text{depth}_R(R) = 0$;
(ii) $R$ is a field;
(iii) $R$ is a DVR;
(iv) $R$ is regular.

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In contrast to Lichtenbaum modules, there are a lot of papers dealing with modules so called test modules, see [10] and references therein. Test modules behave both similar to and different from those of Lichtenbaum modules. Our goal, in this paper, is to present such behaviors. To record a difference, see Proposition 3.18 and compare it with [10, Corollary 3.7].

Various aspects of Litchenbaum modules are giving. As a sample, we characterize them over regular rings, see Corollary 3.15. This not only presents a reverse of [23, Corollary 6], but also gives a new proof of [23, Corollary 6] and has an advantage over hypersurface rings (see Corollary 1.5). It may be worth to mention that, this characterization determines the regularity condition (see Proposition 3.29). This has an application in Lazard-type properties. For a sample, see Corollary 1.5. Also, this enables us to recover a funny and old result of Levin-Vasconcelos by some different arguments. For example, in the statement of the next result there is no trace of Litchenbaum modules:

**Corollary 1.2.** Let $(R, \mathfrak{m})$ be a local ring. Assume that there exists a positive integer $n$ such that $R/\mathfrak{m}^n$ has finite injective dimension. The following assertions holds:

(i) $R$ is Gorenstein.

(ii) If $\dim R > 0$, then $R$ is regular.

(iii) If $\dim R = 0$, then $\mathfrak{m}^n = 0$.

Also, Lichtenbaum asked:

**Question 1.3.** (See [23, Question 3]) For which local ring $R$, and for which module $T$ have the property that $\text{Tor}_R^i(T, N) = 0$ implies that $\text{Tor}_R^{j>i}(T, N) = 0$?

Concerning to the previous item, we call such a module $T$ a tor-rigid module. Let $R_0 := k[[x, y]]/(x^2, xy)$, and recall that $R_0$ is of depth zero and dimension one. In the same paper, Lichtenbaum remarked that $R_0/yR_0$ is not tor-rigid. By using an idea taken from [13] and [12], we present the following connection between Litchenbaum’s questions:

**Observation 1.4.** Let $(R, \mathfrak{m})$ be a local ring of positive dimension. Then the following are equivalent:

(i) $\text{depth}_R R > 0$.

(ii) $R/\mathfrak{m}^n$ is a Lichtenbaum module for $R$ for all integers $n > 0$.

(iii) $R/\mathfrak{m}^n$ is tor-rigid for all $n > 0$.

This may regard as a converse part of the recent works [13] and [12]. We will construct some nontrivial examples fitting into the setting of the observation.

We present a connection from Lichtenbaum modules to the Burch modules. In this regard, we sharpening a celebrated result of Burch, see Observation 4.17. This is independent of [18], and gives a short and elementary proof of [18]. Due to [14] 4.8+4.7 we know that integral closure may be considered as a test module if we restrict ourselves to the category of finite length modules over a 1-dimensional complete local domain $(R, \mathfrak{m}, k)$ of prime characteristic with $k = \overline{k}$ . In §4, we show:
Corollary 1.5. Let \((R, m)\) be a 1-dimensional local integral domain and let \(M\) be a finitely generated module such that

\[\text{Tor}_i^R(R, M) = \text{Tor}_{i+1}^R(R, M) = 0,\]

for some \(i > 0\). Then \(\text{pd}_R(M) \leq 1\). In particular, \(M\) is free provided it is of positive depth.

It is easy to find examples for which \(R\) is not tor-rigid, as a sample see Observation 4.13. Despite this, we present a situation for which the vanishing of the corresponding tor is restricted only at a single spot, e.g., a situation for which \(R\) is tor-rigid. For more details, see Corollary 4.10. These lead us to study the length of syzygies of Burch modules. This was asked in [14]: Is \(\ell(\text{Syz}_i(M)) = \infty\) for all \(i > 0\)?

As an application, we compute the length of syzygies of certain Lichtenbaum modules that we constructed in §3. For a sample, see Corollary 5.4.

The next topic is about the associated prime ideals of \(\text{Tor}_i^R(M, N)\) in terms of \(M\) and \(N\). As far as we know, this rarely computed in the literature. In fact, as far as we know, \(\text{Ass} \text{Tor}_0^R(\cdot, \cdot)\) is difficult to compute even in some special forms, see [3] and references therein. We determine the associated prime ideals of \(\text{Tor}_i^R(M, N)\) in some cases. This may facilitate the study of higher Tor-modules. For instance, and as an application, we present a situation for which the corresponding Tor-module is Cohen-Macaulay. For more details, see Corollary 2.14. Also, we study the annihilator of \(\text{Tor}_i^R(M, N)\) in some nontrivial cases, and open some relevant questions.

Finally, we investigate the vanishing of \(\text{Tor}_1^R(\cdot, \cdot)\) and dealing with a question asked by Quy and others.

For all unexplained notation and definitions see the books [8] and [24].

§ 2. Associated Primes of Tor

In this note \((R, m, k)\) is a commutative noetherian local ring. The notation \(\text{pd}_R(\cdot)\) (resp. \(\text{id}_R(\cdot)\)) stands for the projective (resp. injective) dimension.

Question 2.1. How can compute depth and dimension of Tor-modules?

We start by recalling the following basic result of Auslander:

Fact 2.2. (See [4, Lemma 1.1]) Let \(M\) and \(N\) be finitely generated modules such that \(p := \text{pd}_R M < \infty\) and \(\text{depth}_R(N) = 0\). Then \(\text{Tor}_1^R(M, N) \neq 0\) and it is of depth zero.

Remark 2.3. i) The finitely generated assumption of \(M\) is really important. Here, we follow an idea of Lazard [20]: Let \(R := k[x, y, z]/xm\) and look at \(0 \rightarrow R(x) \xrightarrow{f} R(x, y) \rightarrow M := \text{Coker } f \rightarrow 0\) and
we set \( N := k \) which is of depth zero. From this flat resolution, \( \text{Tor}_t^R(M, N) = 0 \). Let \( t := \text{pd}_R(M) \). Since \( M \) is of finite flat dimension, it follows that \( t < \infty \). It remains to note that \( \text{Tor}_t^R(M, N) = 0 \).

ii) There is a way to drop the finite assumption of \( N \). In this regard we need to fix the notion of depth. We say a general module \( N \) is of E-depth zero if \( R/m \subset N \). This holds if any \( x \) is zero-divisor over \( N \). Having this in mind, the finitely generated assumption of \( N \) is not needed. In order to see this, it is enough to apply Auslander’s argument. We leave the details to the reader.

**Definition 2.4.** For every \( R \)-module \( L \), the non-flat locus of \( L \) is defined by

\[
\text{NF}(L) := \{ p \in \text{Spec } R \mid \text{the } R_p\text{-module } L_p \text{ is not flat}\}.
\]

In the case \( L \) is finitely generated, we call \( \text{NF}(L) \) the non-free locus, since over local rings a finitely generated flat module is free.

We denote the set of all associated prime ideals by \( \text{Ass}_R(-) \).

**Proposition 2.5.** Let \((R, m)\) be a local ring and \( M \) and \( N \) be two \( R \)-modules. If \( M \) has finite projective dimension \( t \geq 1 \), then \( \text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \text{Ass}_R N \cap \text{NF}(M) \).

**Proof.** Let \( F := 0 \rightarrow R^{n_t} \overset{d_t}{\rightarrow} R^{n_{t-1}} \overset{d_{t-1}}{\rightarrow} \cdots \overset{d_1}{\rightarrow} R^n \rightarrow 0 \) be a free resolution of \( M \) where \( n_i \in \mathbb{N} \cup \{ \infty \} \). Then

\[
\text{Tor}_t^R(M, N) = H_t(F \otimes_R N)
\]

\[
= \ker(d_t \otimes 1)
\]

\[
\cong \ker(N^{n_t} \rightarrow N^{n_{t-1}})
\]

\[
\subseteq N^{n_t},
\]

and so

\[
\text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \text{Ass}_R(N^{n_t}) = \text{Ass}_R N.
\]

Since \( t \geq 1 \), \( \text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \text{NF}(M) \). Thus

\[
\text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \text{Ass}_R N \cap \text{NF}(M),
\]

as claimed. \( \square \)

**Corollary 2.6.** Adopt the previous assumption and suppose in addition that \( M \) ad \( N \) are finitely generated. If \( M \) has projective dimension one, then \( \text{Ass}_R(\text{Tor}_1^R(M, N)) = \text{Ass}_R N \cap \text{NF}(M) \).

**Proof.** In view of Proposition 2.5, we only need to show that

\[
\text{Ass}_R N \cap \text{NF}(M) \subseteq \text{Ass}_R(\text{Tor}_1^R(M, N)).
\]

For a finitely generated \( R \)-module \((-)\), it is known that \( p \in \text{Ass}_R(-) \) if and only if \( \text{depth}_{R_p}((-)_p) = 0 \). Let \( p \in \text{Ass}_R N \cap \text{NF}(M) \). As \( p \in \text{Ass}_R N \), we get \( \text{depth}_{R_p}(N_p) = 0 \). Also, from \( p \in \text{NF}(M) \) and \( \text{pd}_R M = 1 \), we deduce that \( \text{pd}_{R_p}(M_p) = 1 \). Fact 2.2 yields that \( \text{depth}_{R_p}(\text{Tor}_1^{R_p}(M_p, N_p)) = 0 \), and so \( p \in \text{Ass}_R(\text{Tor}_1^R(M, N)) \). \( \square \)

**Notation 2.7.** By \( \text{Syz}_i(M) \), we mean the \( i \)th syzygy module of \( M \). Some times we set \( \Omega(-) := \text{Syz}_1(-) \).
Corollary 2.8. Let \((R, \mathfrak{m})\) be a local ring and \(M\) and \(N\) two finitely generated \(R\)-modules. Assume that \(M\) has finite projective dimension \(t \geq 1\). Then \(\text{Ass}_R(\text{Tor}_t^R(M, N)) = \text{Ass}_R N \cap \text{NF}(\text{Syz}_{t-1}(M))\).

Proof. Set \(L := \text{Syz}_{t-1}(M)\). By shifting, \(\text{pd}_R L = 1\) and \(\text{Tor}_t^R(M, N) = \text{Tor}_1^R(L, N)\). Now, apply Corollary 2.6. \(\square\)

Corollary 2.9. Let \((R, \mathfrak{m})\) be a local ring, \(M\) and \(N\) be finitely generated. If \(M\) has finite projective dimension \(t \geq 1\) and \(N\) is \(p\)-primary for some \(p \in \text{Spec}(R)\), then \(\text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \{p\}\). In particular, if \(\text{Tor}_t^R(M, N) \neq 0\), then \(\text{Ass}_R(\text{Tor}_t^R(M, N)) = \{p\}\).

Proof. The first part is in Proposition 2.8. To see the particular case note that

\[ \emptyset \neq \text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \{p\}, \]

and so \(\text{Ass}_R(\text{Tor}_t^R(M, N)) = \{p\}\). \(\square\)

Corollary 2.10. Assume that \((R, \mathfrak{m})\) is a local domain and \(M\) is a finitely generated \(R\)-module with \(\text{pd}_R M = 1\). Then \(\text{Ass}_R(\text{Tor}_t^R(M, M)) = \text{Ass}_R M \setminus \{0\}\).

Proof. Let \((-)_0\) be the localization with respect to 0 \(\in \text{Spec}(R)\). Since \(M_0\) is an \(R_0\)-vector space, it follow that \(0 \notin \text{NF}(M)\). Let \(0 \neq p \in \text{Ass}_R M\). Then \(\text{depth}_{R_p}(M_p) = 0\), and so by the Auslander-Buchsbaum formula, we deduce that

\[ 1 \leq \text{depth}(R_p) = \text{pd}_{R_p}(M_p) + \text{depth}_{R_p}(M_p) = \text{pd}_{R_p}(M_p) \leq \text{pd}_R M \leq 1, \]

i.e., \(\text{pd}_{R_p}(M_p) = 1\). In particular, \(M_p\) is not a free \(R_p\)-module. Hence \(\text{NF}(M) \cap \text{Ass}_R M = \text{Ass}_R M \setminus \{0\}\), and Corollary 2.10 completes the proof. \(\square\)

Corollary 2.11. Let \((R, \mathfrak{m})\) be a local ring, \(M\) and \(N\) be finitely generated. Assume that \(M\) is a locally free on the punctured spectrum, and it has finite projective dimension \(t \geq 1\). Then \(\text{Tor}_t^R(M, N) \neq 0\) if and only if \(\text{depth}_R N = 0\).

Proof. First, assume that \(\text{depth}_R N = 0\). Fact 2.2 implies that \(\text{Tor}_t^R(M, N) \neq 0\). Conversely, suppose \(\text{Tor}_t^R(M, N) \neq 0\). In view of Proposition 2.4 we observe that

\[ 0 \neq \text{Ass}_R(\text{Tor}_t^R(M, N)) \subseteq \text{NF}(M) \cap \text{Ass}_R N = \{\mathfrak{m}\} \cap \text{Ass}_R N. \]

Thus \(\mathfrak{m} \in \text{Ass}_R N\), and so \(\text{depth}_R N = 0\). \(\square\)

Corollary 2.12. Let \((R, \mathfrak{m})\) be a complete local ring and \(p\) a Cohen-Macaulay prime ideal of \(R\) of finite projective dimension. The following assertions are true:

(i) \(R\) is a Cohen-Macaulay integral domain.
(ii) \(\text{Ass}_R(\text{Tor}_{\text{ht}\, p}^R(R/p, R/p)) = \{p\}\).
(iii) \(\dim(\text{Tor}_{\text{ht}\, p}^R(R/p, R/p)) = \dim R/p\).
(iv) \(\text{Ann}_R(\text{Tor}_{\text{ht}\, p}^R(R/p, R/p)) = p\).
Proof. (i) Recall that $R$ is integral domain by intersection theorem. For more details, see [29]. In particular, $R$ is equidimensional and catenary. This allows us to apply [24] Page 250] and deduce that

$$\dim R - \dim_R(R/p) = \text{ht } p =: t.$$ 

The Auslander-Buchsbaum formula implies that

$$\text{pd}_R(R/p) = \text{depth}_R R - \text{depth}_R(R/p)$$

$$\leq \dim R - \dim_R(R/p)$$

$$= \text{ht } p$$

$$= t.$$ 

Since $\text{pd}_R(R/p) < \infty$, it follows that $\text{pd}_{R_p}(R_p/pR_p) < \infty$, and so the local ring $R_p$ is regular. Now,

$$\text{Tor}_i^R(R/p, R/p)_p \cong \text{Tor}_{\dim_{R_p}}^R(R_p/pR_p, R_p/pR_p) \neq 0.$$ 

Consequently, $\text{Tor}_i^R(R/p, R/p) \neq 0$. So, $\text{pd}_R(R/p) = t$. In particular, (*) is an equality, i.e.,

$$\dim R = \text{depth}_R R.$$ 

(ii) Since Ass$_R(R/p) = p$, and without loss of generality, we may and do assume that $t > 0$. According to Proposition 2.5 Ass$_R(\text{Tor}_i^R(R/p, R/p)) \subseteq \{p\}$. As $\text{Tor}_i^R(R/p, R/p) \neq 0$, it follows that Ass$_R(\text{Tor}_i^R(R/p, R/p)) \neq \emptyset$, and so Ass$_R(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)) = \{p\}.$

(iii) Thanks to part ii) we know $\text{Supp}(\text{Tor}_i^R(R/p, R/p)) = \text{Var}(p)$. From this, the desired claim is clear.

(iv) Clearly, $p \subseteq \text{Ann}_R(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p))$. If the inclusion were be strict, we should have

$$\dim(R/\text{Ann}_R(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p))) < \dim R/p,$$

because $p$ is prime. This is in contradiction with the third item. So, $\text{Ann}_R(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)) = p$, as claimed. 

\ \ [Fact 2.13. (See [4] Theorem 1.2)] Let $R$ be any local ring and $\text{pd}_R(M) < \infty$. Let $q$ be the largest number such that $\text{Tor}_q^R(M, N) \neq 0$. If $\text{depth}_R(\text{Tor}_q^R(M, N)) \leq 1$ or $q = 0$, then

$$\text{depth}_R(N) = \text{depth}_R(\text{Tor}_q^R(M, N)) + \text{pd}_R(M) - q.$$ 

\ \ [Corollary 2.14. Adopt the assumption of Corollary 2.12 and suppose in addition that $\dim R/p \leq 2$. Then $\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)$ is Cohen-Macaulay.]

Proof. We only deal with the case $\dim R/p = 2$. Thanks to Corollary 2.12 $\dim(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)) = 2$, and recall that $\text{depth}_R(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)) \leq \dim(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)) = 2$. Suppose on the way of contradiction that $\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)$ is not Cohen-Macaulay. Since $zd(-) = \bigcup_{P \in \text{Ass}(-)} P$, we observe that any $x \in m \setminus p$ is a regular sequence over $\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)$. It follows that $\text{depth}_R(\text{Tor}_{\dim_{R_p}}^R(R/p, R/p)) = 1$. This enables us to apply Fact 2.13. We combine this along with Corollary 2.12 to see
\[2 = \dim R/p = \text{depth}_R(R/p) = \text{depth}_R(\text{Tor}^R_{1p}(R/p, R/p)) + \text{pd}_R(R/p) - \text{ht} p = 1 + \text{depth} R - \text{depth}_R(R/p) - \text{ht} p = 1 + \dim R - \dim(R/p) - \text{ht} p = 1,\]

which is a contradiction. So, \(\text{Tor}^R_{ht}(R/p, R/p)\) is Cohen-Macaulay.

It may be nice to recover \(M\) from \(\text{Tor}^R_i(M, M)\). As an easy fact:

**Fact 2.15.** Let \((R, m)\) be a local ring and \(I\) an ideal of \(R\) which is generated by an \(R\)-regular sequence of length \(t\). Then \(\text{Tor}^R_i(R/I, R/I) = \oplus R/I\) for all \(1 \leq i \leq t\). In particular, the following holds.

(i) \(\text{Tor}^R_i(R/I, R/I)\) is Cohen-Macaulay iff \(R\) is Cohen-Macaulay.

(ii) \(\text{Ass}_R(\text{Tor}^R_i(R/I, R/I)) = \text{Ass}_R R/I\).

(iii) \(\text{Ann}_R(\text{Tor}^R_i(R/I, R/I)) = I\).

**Proof.** Suppose \(I\) is generated by an \(R\)-regular sequence \(x := x_1, \ldots, x_t\), and set \(M := R/I\). The Koszul complex of \(R\) with respect to the sequence \(x\) has the form

\[0 \rightarrow R \rightarrow R^t \rightarrow \cdots \rightarrow R \rightarrow 0,
\]

and it provides a free resolution of \(M\). Thus

\[\text{Tor}^R_i(M, M) = H(M^{n_{i+1}} \xrightarrow{0} M^{n_i} \xrightarrow{0} M^{n_{i-1}}) = M^{n_i},\]

and so the desired claims follow.

By the linearity, \(I \subseteq \text{Ann}_R(\text{Tor}^R_i(R/I, R/I))\). There are many examples for which the inequality is strict. A natural question arises: When is \(\text{Ann}_R(\text{Tor}^R_i(R/I, R/I)) = I\)? Let us ask more. Inspired by Vasconcelos (resp. Simis) \cite{31}, who asked the initial cases, we ask:

**Question 2.16.** Let \(I\) be an ideal and \(t > 0\). When is \(\text{Tor}^R_i(R/I, R/I)\) free as an \(R/I\)-module?

It may be nice to note that there are 1-dimensional prime ideals with high number of generators such as \(p\) in a 3-dimensional regular local rings. Thus, the assumption of Corollary \cite{2412} holds, and we have \(\text{Ann}(\text{Tor}^R_2(R/p, R/p)) = p\) but \(p\) is not generated by a regular sequences. However, \(\text{Tor}^R_2(R/p, R/p)\) is not free as an \(R/p\)-module.

§ 3. Freeness criteria via vanishing of \(\text{Tor}_1\)

We start with:

**Definition 3.1.** A nonzero \(R\)-module \(L\) is called (strong) Lichtenbaum if for every (not-necessarily) finitely generated \(R\)-module \(F\), the vanishing of \(\text{Tor}^R_1(L, F)\) implies that \(F\) is (flat) free.
Remark 3.2. In most of the applications, we assume \( F \) (resp. \( L \)) is finitely generated. Since the ring is local, any finitely generated flat module is free. 

Lemma 3.3. Let \( (R, m) \) be a local ring and let \( L \) be a module such that
\[
\text{Tor}_1^R(L, F) = 0 \implies F \text{ is faithful.}
\]
Then \( \text{depth}_R(L) = 0 \).

Proof. On the contrary, assume that \( \text{depth}_R(L) > 0 \). Then there is an \( L \)-regular element \( x \in m \). Consider the free resolution of the \( R \)-module \( R/xR \):
\[
\cdots \rightarrow R^a \rightarrow R \xrightarrow{x} R \rightarrow R/xR \rightarrow 0
\]
By definition,
\[
\text{Tor}_1^R(L, R/xR) = H(L^a \rightarrow L \xrightarrow{x} L) = 0.
\]
According to the assumption, it follows that \( R/xR \) is faithful, which is our desired contradiction. 

Remark 3.4. In Lemma 3.3 one can not replace the faithful assumption with the torsion-free. Indeed, let \( (R, m) \) be of depth zero and of positive dimension. This gives us a non maximal prime ideal \( p \). Let \( L := R/p \). Since \( \text{depth}(R) = 0 \), any module is torsion-free. In particular, the following implication holds:
\[
\text{Tor}_1^R(L, F) = 0 \implies F \text{ is torsion-free.}
\]
In order to see the left hand side is not empty, let \( F := R/xR \) where \( x \in m \setminus p \). It is easy to see \( \text{Tor}_1^R(L, F) = 0 \). But, \( \text{depth}(L) > 0 \), because \( x \) is regular over it.

Corollary 3.5. Let \( (R, m) \) be a local ring and \( L \) a finitely generated Lichtenbaum module for \( R \). Then \( \text{depth}_R(L) = 0 \).

Proof. This is in Lemma 3.3.

Remark 3.6. Adopt the notation of Corollary 3.5. The finitely generated assumption is not so important. In fact, we need to fix the notion of depth. This is defined to the supremum of the lengths of all weak regular sequences on \( L \). This is called the classical depth. Now, Corollary 3.5 can be extend, by the same proof, as follows: Let \( (R, m) \) be a local ring and \( L \) a Lichtenbaum module for \( R \). Then the classical depth of \( L \) is zero.

\*for the case \( F \) (resp. \( L \)) is not-necessarily finitely generated, see Remark 3.32 (resp. Proposition 3.20).
In one of the applications we deal with regular rings and we need to use an extension of Remark 3.6. Suppose $R$ is regular. In order to extend the previous item, we recall the concept of Koszul depth: Suppose $m$ is generated by a set $x := x_1, \ldots, x_d$, we denote the Koszul complex of $R$ with respect to $\underline{x}$ by $K(\underline{x})$. Koszul depth of $m$ on $M$ is defined by

$$\text{K. depth}_R(M) := \inf \{i \in \mathbb{N} \cup \{0\} | H^i(\text{Hom}_R(K(\underline{x}), M)) \neq 0 \}.$$ 

Note that by [8, Corollary 1.6.22] and [8, Proposition 1.6.10 (d)], this does not depend on the choice of generating sets of $m$. Now, Corollary 3.5 extends in the following sense:

**Corollary 3.7.** Let $(R, m)$ be a Cohen-Macaulay local ring and $L$ be a Lichtenbaum module for $R$. Then $\text{K. depth}_R(L) = 0$.

**Proof.** Let $d := \text{dim } R$. We may assume that $d > 0$, because $\text{K. depth}_R(L) \leq d$. Suppose on the way of contradiction that $\text{K. depth}_R(L) > 0$. Since $R$ is Cohen-Macaulay, $m$ is a radical an ideal generated by a set $x := x_1, \ldots, x_d$ of length $d$ which is a regular sequence. In this case, the Koszul complex with respect to $\underline{x}$ is a free resolution of $R/\underline{x}R$. We know that Koszul grade is unique up to radical. So, Koszul grade of $L$ on $\underline{x}R$ is also zero. Also, recall that the symmetry of Koszul cohomology and Koszul homology says that $H^i(K(\underline{x}) \otimes_R L) \cong H^{d-i}(\text{Hom}_R(K(\underline{x}), L))$, see [8, Proposition 1.6.10 (d)]. Thus,

$$\text{Tor}_2^R(\text{Syz}_{d-1}(R/\underline{x}R), L) \cong \text{Tor}_d^R(R/\underline{x}R, L) = H_d(\text{K}(\underline{x}, R) \otimes_R L) = H^0(\text{Hom}_R(K(\underline{x}), L)) = 0.$$ 

Since $L$ is Lichtenbaum, we deduce that $\text{Syz}_{d-1}(R/\underline{x}R)$ is free. In other words, $\text{pd}_R(R/\underline{x}R) \leq d-1$. By Auslander-Buchsbaum formula, $\text{pd}_R(R/\underline{x}R) = d$. This contradiction completes the proof. \qed

**Notation 3.8.** The notation $\text{Tr}(\cdot)$ stands for the Auslander’s transpose.

**Lemma 3.9.** Let $(R, m)$ be a local ring which isn’t a field and $L$ a finitely generated Lichtenbaum module for $R$. The following assertions are valid:

(i) $\text{pd}_R L < \infty$ if and only if $R$ is regular.

(ii) $L$ is not $e$-rigid, i.e. $\text{Ext}_2^R(L, L) \neq 0$.

**Proof.** (i) Assume that $d := \text{pd}_R L < \infty$. Then

$$0 = \text{Tor}_{d+1}^R(R/m, L) = \text{Tor}_d^R(L, \text{Syz}_d(R/m)).$$

As $L$ is Lichtenbaum, this implies that $\text{Syz}_d(R/m)$ is free, and so $\text{pd}_R R/m < \infty$. Hence, $R$ is regular. The reverse implication is obvious.

(ii) We suppose that $L$ is $e$-rigid, and look for a contradiction. Recall that $\Omega(-) := \text{Syz}_1(-)$. From the exact sequence (see [B 2.8])

$$\text{Tor}_2^R(\text{Tr}(\Omega L), L) \longrightarrow \text{Ext}_2^R(L, R) \otimes_R L \longrightarrow \text{Ext}_1^R(L, L) \longrightarrow \text{Tor}_1^R(\text{Tr}(\Omega L), L) \longrightarrow 0,$$
we conclude that \( \text{Tor}^1(\text{Tr}(\Omega L), L) = 0 \). As \( L \) is a Lichtenbaum module for \( R \), we deduce that the \( R \)-module \( \text{Tr}(\Omega L) \) is free. Recall that \( \text{Tr}(\text{Tr}(\cdot)) \cong (\cdot) \), and so \( \Omega L \cong \text{Tr}(\text{Tr}(\Omega L)) \) is free. Thus, \( \text{pd}_R L \leq 1 \). Next, we have
\[
\text{Ext}^1_R(L, R) \otimes_R L \cong \text{Ext}^1_R(L, L) = 0.
\]
This yields that \( \text{Ext}^1_R(L, R) = 0 \), and so
\[
\text{pd}_R L = \sup\{ n \in \mathbb{N}_0 \mid \text{Ext}^n_R(L, R) \neq 0 \} = 0.
\]
This is a contradiction because a free module can’t be Lichtenbaum unless the ring is field, and this is excluded from the assumption. \( \square \)

**Notation 3.10.** Suppose \( p := \text{char}(R) > 0 \), and let \( \varphi : R \to R \) denotes the Frobenius endomorphism given by \( \varphi(a) = a^p \) for \( a \in R \). Each iteration \( \varphi^n \) of \( \varphi \) defines a new \( R \)-module structure on the set \( R \), and this \( R \)-module is denoted by \( \varphi^n R \), where \( a \cdot b = a^{p^n} b \) for \( a, b \in R \).

We present some (non-) examples of Lichtenbaum modules.

**Example 3.11.** Let \((R, \mathfrak{m}, k)\) be local. The following holds:

(i) Let \( x \in \mathfrak{m} \) be an \( R \)-regular element. Then \( \mathfrak{m}/x \mathfrak{m} \) is a Lichtenbaum module for \( R \).

(ii) \( \mathfrak{m} \) is Lichtenbaum if and only if \( \text{depth}_R \mathfrak{m} = 0 \).

(iii) Suppose \( R \) is Cohen-Macaulay and \( p := \text{char}(R) > 0 \). Then \( \varphi^n R \) is Lichtenbaum for all \( n \gg 0 \) iff \( \dim(R) = 0 \).

(iv) Localization of a Lichtenbaum module is not necessarily Lichtenbaum, even over 2-dimensional (regular) rings.

**Proof.** (i) Suppose \( \text{Tor}^1_R(\mathfrak{m}/x \mathfrak{m}, F) = 0 \) for some finitely generated \( R \)-module \( F \). We are going to show \( F \) is free. Indeed, since \( x \) is regular over \( R \), it is also regular over \( \mathfrak{m} \). So, we have the short exact sequence
\[
0 \to \mathfrak{m} \to \mathfrak{m}/x \mathfrak{m} \to 0.
\]
It yields the exact sequence
\[
\text{Tor}^1_R(\mathfrak{m}, F) \to \text{Tor}^1_R(\mathfrak{m}, F) \to \text{Tor}^1_R(\mathfrak{m}/x \mathfrak{m}, F) = 0.
\]
By Nakayama’s lemma, we conclude that \( \text{Tor}^1_R(\mathfrak{m}, F) = 0 \). By shifting, we have \( \text{Tor}^2_R(R/\mathfrak{m}, F) \cong \text{Tor}^2_R(\mathfrak{m}, F) = 0 \), and so \( \text{pd}_R F \leq 1 \). It is easy to see that the map \( \psi : R/\mathfrak{m} \to \mathfrak{m}/x \mathfrak{m} \) defined by \( \psi(r + \mathfrak{m}) = xr + x \mathfrak{m} \) is an injective \( R \)-homomorphism. This fits in the following short exact sequence
\[
0 \to R/\mathfrak{m} \to \mathfrak{m}/x \mathfrak{m} \to \text{Coker}\psi \to 0,
\]
which yields the following exact sequence:
\[
\text{Tor}^2_R(\text{Coker}\psi, F) \to \text{Tor}^2_R(R/\mathfrak{m}, F) \to \text{Tor}^2_R(\mathfrak{m}/x \mathfrak{m}, F).
\]
Hence \( \text{Tor}^2_R(R/\mathfrak{m}, F) = 0 \), and so \( F \) is free.

(ii) It is enough to apply Corollary 3.5 along with the argument presented in part (i).
(iii) Let \( d := \dim R \). First assume that \( d = 0 \), and suppose \( \text{Tor}^R_1(\varphi^n R, F) = 0 \) for some finitely generated \( R \)-module \( F \). Denote the \( n \)-th Frobenius power of an ideal \((-\cdot)^n\). Let \( n \) be such that \( m^n = 0 \). This is possible, because \( m \) is nilpotent. In other words, \( m \varphi^n R = 0 \). Then \( \oplus \text{Tor}^R_1(k, F) = 0 \) for some finitely generated \( R \)-module \( F \). Let \( n \) be such that \( m[p^n] = 0 \). This is possible, because \( m \) is nilpotent. In other words, \( m \varphi^n R = 0 \). Then \( \oplus \text{Tor}^R_1(k, F) = 0 \). Since \( k \) is a test module and \( F \) is finitely generated, we deduce that \( F \) is free. So, \( \varphi^n R \) is Lichtenbaum. Now, assume \( d > 0 \) and suppose on the way of contraction that \( \varphi^n R \) is Lichtenbaum. According to Remark 3.6 we know that the classical depth of \( \varphi^n R \) is zero. In view of [24, Theorem 16.1] we observe that \( \text{depth}(R) = 0 \). This is in contradiction with the Cohen-Macaulay assumption.

(iv) Let \( R \) be any local ring of dimension at least two, e.g., \( R := k[[x, y]] \) and look at the following chain \( p_1 \subseteq p_2 \subseteq m \) of prime ideals. Let \( L := R/m \oplus R/p_1 \). As \( k \) is a direct summand of \( L \), we observe that \( L \) is Lichtenbaum. Since \( L_{p_2} \) is of positive depth over \( R_{p_2} \), and due to Corollary 3.5, we deduce that \( L_{p_2} \) is not Lichtenbaum over \( R_{p_2} \).

In §5 we will apply Example 3.11(iii) for not necessarily Cohen-Macaulay rings. Let us drop the Cohen-Macaulay assumption:

**Remark 3.12.** Suppose \( R \) is local and \( p := \text{char}(R) > 0 \). Then \( \varphi^n R \) is Lichtenbaum for all \( n \gg 0 \) iff \( \text{depth}(R) = 0 \).

**Proof.** Indeed, this is similar to Example 3.11(iii). It is enough to combine Remark 3.6 along with [25, Proposition 2.2.11]. □

**Definition 3.13.** (Auslander) A module \( T \) is said to be tor-rigid if there is a non-negative integer \( n \) such that for every finitely generated \( R \)-module \( M \), vanishing of \( \text{Tor}^R_n(T, M) \) implies \( \text{Tor}^R_{n+i}(T, M) \) vanishes for all \( i \geq 0 \).

**Remark 3.14.** i) Over regular rings any finitely generated module is tor-rigid (see [4, 2.2]).

ii) The finiteness assumption is important. Indeed, let \( (R, m) \) be a regular local ring of dimension \( d > 1 \). By local duality, \( \text{Tor}^R_1(E_R(k), k) = 0 \) and \( \text{Tor}^R_d(E_R(k), k) = k \).

The if part of the next result is in [23, Corollary 6]. We recover it by a new argument:

**Corollary 3.15.** Let \( (R, m) \) be a regular and local ring, and let \( L \) be finitely generated. Then \( L \) is Lichtenbaum if and only if \( \text{depth}_R(L) = 0 \).

**Proof.** Suppose \( \text{depth}_R(L) = 0 \) and \( \text{Tor}^R_1(L, F) = 0 \) for some finitely generated \( R \)-module \( F \). By tor-rigidity, we observe that \( \text{Tor}^R_1(L, F) = 0 \). By depth formula (see Fact 2.13),

\[
\text{depth}_R(L) \geq \text{depth}_R(F) = \text{depth}_R(L \otimes F) + \text{pd}_R(L) = \text{depth}_R(L \otimes F) + \text{depth}_R R \geq \text{depth}_R L.
\]

This yields that \( \text{depth}_R(L) = \text{depth}_R(L) \). By Auslander-Buchsbaum formula, \( F \) is free. Thus, \( L \) is Lichtenbaum. The reverse part is in Corollary 3.5. □

**Corollary 3.16.** Let \( (R, m) \) be a regular and local ring, and let \( L \) be Lichtenbaum. Then \( L \) is a directed limit of finitely generated Lichtenbaum modules.
**Proof.** Recall from Corollary 3.7 that \( \text{depth}_R(L) = 0. \) It turns out that

\[
\text{E. grade}_R(m, L) := \inf \{ i \in \mathbb{N} \cup \{ 0 \} | \text{Ext}_R^i(R/m, L) \neq 0 \} = 0.
\]

Let \( f : R/m \to L \) be any nonzero morphism, and recall that \( R/m \) is simple as an \( R \)-module. From these, \( R/m \) can be embedded into \( L \). There is a filtered system \( \{ L_i \}_{i \in I} \) of finitely generated submodules of \( L \) such that \( L = \bigcup_{i \in I} L_i \) and that \( R/m \subseteq L_i \). In particular, \( \text{depth}_R(L_i) = 0. \) In view of Corollary 3.15 we observe that \( L_i \) is Lichtenbaum, and this completes the proof. \( \square \)

**Notation 3.17.** Let \( (-)^\vee := \text{Hom}_R(-, E_R(R/m)) \) be the Matlis duality functor.

Also, there is a connection to field theory:

**Proposition 3.18.** Let \( (R, \mathfrak{m}) \) be a local ring. The following are equivalent:

(i) \( R \) is a field.

(ii) \( R \) is Cohen-Macaulay with canonical module \( \omega_R \) which is Lichtenbaum.

(iii) \( K_R \) is a Lichtenbaum module for \( R \).

**Proof.** (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii): These are easy.

(iii) \( \Rightarrow \) (i): Recall that \( K_R = H^d_{\mathfrak{m}}(R)^\vee \) where \( d := \dim R. \) As \( K_R \) is Lichtenbaum, and in view of Corollary 3.15, \( \text{depth}_R(K_R) = 0. \) Since it satisfies Serre’s condition \((S_2)\), it follows that \( d = 0. \) In particular, \( K_R = \omega_R. \) Let \( M \) be finitely generated and torsionless as an \( R \)-module. As \( M \) is torsionless, from the exact sequence (see [3, 2.6])

\[
0 \to \text{Ext}_R^1(\text{Tr} M, R) \to M \to M^* \to \text{Ext}_R^2(\text{Tr} M, R) \to 0,
\]

we deduce that \( \ker \varphi = 0, \) and so \( \text{Ext}_R^1(\text{Tr} M, R) = 0. \) Now, \( \text{Ext}_R^1(\text{Tr} M, R)^\vee = \text{Tor}_R^1(\text{Tr} M, E(R/m)) = \text{Tor}_R^1(\text{Tr} M, \omega_R) \approx \text{Tor}_R^1(\omega_R, \text{Tr} M). \)

As \( \omega_R \) is Lichtenbaum, it follows that \( \text{Tr}(M) \) is free. In order to see \( M \) is free, we apply another Auslander’s transpose and obtain \( M \approx \text{Tr}(\text{Tr}(M)) \) which is free. Next, let \( M \) be an arbitrary finitely generated \( R \)-module. Consider the short exact sequence

\[
0 \to \text{Syz}_1(M) \to R^n \to M \to 0.
\]

As \( \text{Syz}_1(M) \) is a submodule of a free \( R \)-module, it is torsionless. We repeat the above argument, and deduce that the \( R \)-module \( \text{Syz}_1(M) \) is free. In other words, \( \text{pd}_R M \leq 1. \) Now, Auslander-Buchsbaum formula asserts that \( \text{pd}_R M = 0. \) In sum, we proved that any finite generated \( R \)-module is free, and so \( R \) is a field. \( \square \)

**Corollary 3.19.** Let \( (R, \mathfrak{m}) \) be an artinian local ring which is not a field and \( L \) be a finitely generated Lichtenbaum module for \( R. \) Then \( L \) has infinite injective dimension.
Proof. On the contrary, suppose $\text{id}_R L < \infty$. It follows that $\text{pd}_R (L^\vee) < \infty$. By Auslander-Buchsbaum formula, we see that $\text{pd}_R (L^\vee) = 0$. Since the ring is local, $L^\vee$ is free. Now, $L \cong L^{\vee\vee}$ is injective. Due to the Matlis theory, we know

$$L \cong \oplus E_R (R/m) \cong \oplus \omega_R,$$

and so $\omega_R$ is a Lichtenbaum module for $R$. Now, in view of Proposition 3.18 we arrived at the desired contradiction. □

Proposition 3.20. Let $(R, m)$ be a local domain which is not a field and $L$ a finitely generated Lichtenbaum module for $R$. Then $\Omega L \otimes_R \Omega L$ is torsion-free if and only if $R$ is a discrete valuation ring.

Proof. First, assume that $R$ is a discrete valuation ring. Then $\text{gldim} R = 1$, and so $\Omega N$ is a free module for all $R$-modules $N$. In particular, it follows that $\Omega L \otimes_R \Omega L$ is free, and clearly, it is torsion-free. Conversely, suppose $\Omega L \otimes_R \Omega L$ is torsion-free. From the exact sequence

$$0 \to \Omega L \to R^n \to L \to 0,$$

we deduce the exact sequence

$$0 \to \text{Tor}_R^1 (L, \Omega L) \to \Omega L \otimes_R \Omega L \to (\Omega L)^n \to L \otimes_R \Omega L \to 0 \quad (*)$$

Set $S := R \setminus 0$. As $S^{-1} R$ is a field, it follows that

$$S^{-1} (\text{Tor}_R^1 (L, \Omega L)) \cong \text{Tor}_R^1 (S^{-1} L, S^{-1} (\Omega L)) = 0.$$

Hence, the $R$-module $\text{Tor}_R^1 (L, \Omega L)$ is torsion. As $\Omega L \otimes_R \Omega L$ is torsion-free, from the exact sequence $(*)$, we conclude that $\text{Tor}_R^1 (L, \Omega L) = 0$. Since $L$ is a Lichtenbaum module for $R$, it turns out that $\Omega L$ is free, and this implies that $\text{pd}_R L \leq 1$. Thus, by Lemma 3.9(i), it turns out that $R$ is regular. Now, the Auslander-Buchsbaum formula and Corollary 3.5 imply that $\text{dim} R \leq 1$. As $R$ is not a field, we get that $\text{dim} R = 1$, and so $R$ is a discrete valuation ring. □

We observed in Example 3.11(ii) that certain modules are Lichtenbaum iff the ring is of depth zero. In the same vein, we detect the DVR property of rings.

Observation 3.21. Let $(R, m)$ be a Gorenstein local ring which is not a field. Then $E_R (R/m)$ is a Lichtenbaum module for $R$ if and only if $R$ is a discrete valuation ring.

Proof. First, assume that $E := E_R (R/m)$ is a Lichtenbaum module for $R$. Let $d := \text{dim} R$ and $\underline{x} = x_1, \ldots, x_d$ be a system of parameters for $R$. Recall that the Čech complex of $R$ with respect to $\underline{x}$ has the form

$$\tilde{C} := 0 \to R \to \oplus R_{x_i} \to \cdots \to \oplus R_{x_1 \ldots x_d} \to R_{x_1 \ldots x_d} \to 0,$$

and it provides a flat resolution for the $R$-module $H^d_m (R)$. Hence,

$$\text{Tor}_i^R (E, M) \cong \text{Tor}_i^R (H^d_m (R), M) \cong H_i (\tilde{C} \otimes_R M) \cong H^d-m_i (M)$$
for every $R$-module $M$. If $d \geq 2$, then $\text{Tor}_1^R(E, R/m) \cong H^{d-1}_m(R/m) = 0$, while $R/m$ is not a free $R$-module. So, $d \leq 1$. If $d = 0$, then Corollary 3.19 indicates that $\text{id}_R E = \infty$, which is a contradiction. Thus $d = 1$. Next, we have

$$\text{Tor}_1^R(E, \text{Syz}_1(R/m)) \cong H^0_m(\text{Syz}_1(R/m)) = 0.$$ 

Since $L$ is Lichtenbaum, we deduce that $\text{Syz}_1(R/m)$ is free. This in turns is equivalent with $\text{pd}_R(k) \leq 1$. In other words, $R$ is a discrete valuation ring, because the ring is not a field.

Conversely, assume that $R$ is a discrete valuation ring. Let $F$ be a finitely generated $R$-module such that $\text{Tor}_1^R(E, F) = 0$. We need to show $F$ is free. As $H^0_m(F) \cong \text{Tor}_1^R(E, F) = 0$, we obtain $\text{depth}_R F \geq 1$. Over a discrete valuation ring, every module has finite projective dimension. Now, the Auslander-Buchsbaum formula yields that $F$ is free. □

**Notation 3.22.**

i) By $\mu(-)$ we mean the minimal number of elements that needs to generates a finitely generated module $(-)$.

ii) By $\ell(-)$ we mean the length function.

iii) Here, $e(R)$ is the Hilbert-Samuel multiplicity.

Let us connect to the multiplicity.

**Observation 3.23.** Let $(R, m)$ be a 1-dimensional local ring of depth zero. If $e(R) = 1$, then $R/m^n$ is not a Lichtenbaum module for all $n \gg 0$.

**Proof.** By definition, $\ell(R/m^n) = e(R)n - c = n - c$ for all $n \gg 0$, where $c$ is a constant. Hence,

$$1 = (n + 1) - c - (n - c) = \ell(R/m^{n+1}) - \ell(R/m^n) = \ell(m^n/m^{n+1}) = \mu(m^n),$$

where the last (resp. the third) equality is in [24, Theorem 2.3] (resp. follows from the short exact sequence $0 \to m^n/m^{n+1} \to R/m^{n+1} \to R/m^n \to 0$). Say $m^n = xR$ for some $x \in R$ and for all $n \gg 0$. In particular, $x$ is a system of parameter. Also, $x$ is not nilpotent. There is a minimal prime ideal $p$ such that $x \notin p$. This implies that the multiplication map $R/p \xrightarrow{x} R/p$ is injective.

Now, we look at

$$\begin{array}{cccccc}
R^n & \xrightarrow{x} & R & \xrightarrow{x} & R/xR & \to 0 \\
\pi & & \searrow & & \searrow & \\
(0 :_R x) & & & & & \\
\end{array}$$

where $\pi$ is the natural epic. We apply $- \otimes R/p$ to the displayed exact sequence and deduce the following:

$$\text{Tor}_1^R(R/m^n, R/p) = \text{Tor}_1^R(R/xR, R/p) = H((R/p)^n \xrightarrow{\varphi} R/p \xrightarrow{x} R/p) = \frac{\text{Ker}(R/p \xrightarrow{\varphi} R/p)}{\text{im}(\varphi)} = 0.$$
If $R/p$ were be free, then we should had $p = 0$, this is impossible, because depth$(R) = 0$. Since $R/p$ is not free, $R/m^n$ is not Lichtenbaum.

Here, we extend the previous observation to the general setting.

**Proposition 3.24.** Let $(R, m)$ be a local ring which is not artinian. Then the following are equivalent:

(i) depth$_R R > 0$.

(ii) $R/m^n$ is a Lichtenbaum module for $R$ for all integers $n > 0$.

(iii) $R/m^n$ is tor-rigid for all $n > 0$.

**Proof.** (i) ⇒ (ii): If $R$ is regular, then the assertion follows by Corollary 3.15. So, without loss of generality, we may assume that $R$ is not regular. Now, [12, Corollary 2.14] yields the claim.

(ii) ⇒ (i): Suppose $R/m^n$ is a Lichtenbaum module for all integers $n > 0$. We are going to show depth$_R R > 0$. On the contrary, assume that depth$_R R = 0$. Then $m \in \text{Ass}_R R$. By definition, there is some nonzero element $x$ of $R$ such that $m = (0 :_R x)$. Since $R$ is not a field, we get that $x \in m$. By Krull’s intersection theorem, we have $\bigcap_{n \in \mathbb{N}} m^n = 0$. As $x \neq 0$, there is a natural number $n$ such that $x / \in m^n$.

Let $y$ be in $\langle x \rangle \cap m^n$. Take $r \in R$ be such that $y = rx$. We have two possibilities: Either $r \in m$ or $r \notin m$. In the first case $rx = 0$, and so $y = 0$. Now, suppose that $r \notin m$. As the ring is local, $r$ should be a unit. From this, $x = r^{-1}y \in m^n$. By the choice of $n$, we get to a contradiction. In sum, we proved that $\langle x \rangle \cap m^n = 0$. In particular,

$$\text{Tor}^R_1(R/m^n, R/\langle x \rangle) = (\langle x \rangle \cap m^n)/xm^n = 0.$$ 

Now, recall that $R/\langle x \rangle$ is not free. In view of Definition 3.1 we conclude that $R/m^n$ is not Lichtenbaum. This contradiction shows that depth$_R R > 0$.

(i) ⇒ (iii): This holds by [13, Corollary 1.3].

(iii) ⇒ (i): On the contrary, suppose depth$_R R = 0$, it follows $m = (0 :_R x)$ for some nonzero $x \in R$. By Krull’s intersection theorem, there is a natural number $n$ such that $x \notin m^n$. In the previous argument, we observed that $\text{Tor}^R_1(R/m^n, R/\langle x \rangle) = 0$. As $R/m^n$ is tor-rigid, it turns out that $\text{Tor}^R_2(R/m^n, R/\langle x \rangle) = 0$. From $0 \to \langle x \rangle \to R \to R/\langle x \rangle \to 0$ we deduce that

$$\text{Tor}^R_1(R/m^n, \langle x \rangle) \cong \text{Tor}^R_2(R/m^n, R/\langle x \rangle) \quad (+)$$

Also, $0 \to (0 :_R x) = m \to R \xrightarrow{x} \langle x \rangle \to 0$ yields that

$$R/m \cong \langle x \rangle \quad (*)$$

Combining these together:

$$\text{Tor}^R_1(R/m^n, R/m) \cong \text{Tor}^R_1(R/m^n, \langle x \rangle) \cong \text{Tor}^R_2(R/m^n, R/\langle x \rangle) = 0.$$ 

As $R/m$ is Lichtenbaum, it follows that $R/m^n$ is free. By taking annihilator, $m^n = 0$. This is excluded by the assumptions, a contradiction. So, the desired claim follows. □
Remark 3.25. Adopt the above assumption.

(a) Proposition 3.24 may be considered as the reverse parts of [13, Corollary 1.3] and [12, Corollary 2.14].

(b) The non-artinian assumption is important. For example, let $R := k[x]/(x^2)$. It is easy to see $R/m^n$ is tor-rigid for all $n > 0$. But, depth$_R(R) = 0$.

(c) In order to see the item $b)$ is special, let $R := k[x]/(x^n)$ where $n > 2$. It is easy to see $R/m^{n-1}$ is not tor-rigid. Indeed, let $M := R/x^{n-1}R$, and look at its free resolution

$$\ldots \to R \to R^2 \to \ldots \to R \to M \to 0,$$

which give us Tor$_R^i(R/m^{n-1}, M) = H^i(\ldots \to M \to M \to M \to \ldots)$, and so

$$\text{Tor}_R^i(R/m^{n-1}, M) = \begin{cases} \frac{xR}{m^n} & \text{if } i \in 2\mathbb{N} + 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, $R/m^{n-1}$ is not tor-rigid.

(d) Concerning (ii) $\Leftrightarrow$ (i) we may reduce the assumption from non-artinian to the case that the ring is not field.

Suppose depth$(R) = 0$. It may be nice to find \{ $n : R/m^n$ is not Lichtenbaum \}. This may be large, as the next example suggests.

Example 3.26. In this item rings are of depth zero and of dimension bigger than zero.

(i) Let $R := k[[x,y]]/(x^2, xy)$ and let $n > 0$. Then

$R/m^n$ is Lichtenbaum $\iff n = 1.$

(ii) More general than i), suppose the ring is equipped with an element $x \in m^{\ell-1} \setminus m^\ell$ such that $xm = 0$ and $\ell > 0$. Then $R/m^n$ is not Lichtenbaum for all $n \geq \ell$.

(iii) Let $R := k[[x,y]]/(x^3, x^2y)$. Then $R/m^n$ is not Lichtenbaum for all $n \geq 3$.

(iv) More general than i), suppose $m^n$ is principal for all $n \geq \ell$. Then $R/m^n$ is not Lichtenbaum for all $n \geq \ell$.

(v) Let $R := k[[x,y]]/(x^3, x^2y, xy^2)$. Then $R/m^n$ is not Lichtenbaum for all $n \geq 3$.

Proof. i) Clearly, $R/m$ is Lichtenbaum. For the converse part, recall that $R$ is of depth zero, multiplicity one and dimension one. In fact, $m^n = (y^n)$ for all $n \geq 2$. Now, apply the proof of Observation 3.23 to see $R/m^n$ is not Lichtenbaum, when $n > 1$.

ii) See the proof of Proposition 3.24.

iii) This is in part (ii). Note that $x^2m = 0$ and that $x^2 \notin m^3$.

iv) See the proof of (i).

v) Since $m^n = (y^n)$ for all $n \geq 3$, it is enough to apply part (iv). □

Let us recover a funny result of Levin-Vasconcelos:

Corollary 3.27. Let $(R, m)$ be a non-artinian local ring. Then $R$ is regular if and only if $R/m^n$ has finite injective dimension for some natural number $n$. 

Proof. If $R$ is regular, then every $R$-module has finite injective dimension. Now, assume that there exists a natural number $n$ such that $R/m^n$ has finite injective dimension. Immediately, Bass theorem implies that $R$ is Cohen-Macaulay, and so $\text{depth}_R R = \dim R > 0$. Hence, $R/m^n$ is a Lichtenbaum $R$-module by Proposition 3.24. Set $d := \dim R$. Then $\text{id}_R(R/m^n) = d$. Let $M$ be any finitely generated $R$-module. As $\text{id}_R(R/m^n) = d$, we have

$$\text{Ext}_R^1(\text{Syz}_d(M), R/m^n) \cong \text{Ext}_R^{d+1}(M, R/m^n) = 0.$$ 

By applying this along from the 4-terms exact sequence

$$\text{Tor}_R^1(\text{Tr}(\Omega(\text{Syz}_d(M))), R/m^n) \rightarrow \text{Syz}_d(M) \otimes_R R/m^n \rightarrow \text{Tor}_R^1(\text{Tr}(\Omega(\text{Syz}_d(M))), R/m^n) \rightarrow 0,$$

we deduce that $\text{Tor}_R^1(\text{Tr}(\Omega(\text{Syz}_d(M))), R/m^n) = 0$. Since $R/m^n$ is Lichtenbaum, $\Omega(\text{Syz}_d(M))$ is also free. In other words, $\text{pd}_R M < \infty$. We proved every finitely generated has finite projective dimension. Thanks to a theorem of Auslander-Buchsbaum-Serre, this means that $R$ is regular. \qed

Suppose $S \subseteq F$ are finitely generated. By Artin-Rees, there is $k > 0$ such that $m^{n-k}(m^k F \cap S) = m^n F \cap S$ for all $n > k$. Why $k > 0$? In fact, the following stronger property holds:

**Corollary 3.28.** Let $(R, m)$ be a local ring such that $\text{depth}_R R > 0$ and $M$ be finitely generated which is not free. Let $F := R^{\text{pd}_R(M)} \rightarrow M \rightarrow 0$ be the natural map. Then $m^n \text{Syz}_1(M) \neq m^n F \cap \text{Syz}_1(M)$ for all $n > 0$.

**Proof.** Since $\text{depth}(R) > 0$, and in view of Proposition 3.24 we know $R/m^n$ is Lichtenbaum. Since $M$ is not free and as

$$\text{Tor}_R^1(R/m^n, M) = m^n F \cap \text{Syz}_1(M)/m^n \text{Syz}_1(M),$$

we get the desired claim. \qed

In the next section, we will revisit Corollary 3.27 from a different point of view. Here, we may talk more:

**Proposition 3.29.** Let $(R, m, k)$ be a local ring. The following assertions are equivalent:

(i) If $L$ is a finitely generated Lichtenbaum module, then $\text{id}_R(L) < \infty$.

(ii) $R$ is regular.

(iii) Any finitely generated module of depth zero is Lichtenbaum.

**Proof.** (i) $\Rightarrow$ (ii): We know $\text{id}_R(L) = \text{depth}_R(R) := d$. From this, $\text{Ext}_R^{d+1}(M, L) = 0$ for any finitely generated module $M$. This in turn is equivalent with $\text{Ext}_R^1(\text{Syz}_d(M), L) = 0$. Recall that

$$\cdots \rightarrow \text{Ext}_R^1(\text{Syz}_d(M), L) \rightarrow \text{Tor}_1(\text{Tr} \Omega \text{Syz}_d(M), L) \rightarrow 0,$$

which yields $\text{Tor}_1(\text{Tr} \Omega \text{Syz}_d(M), L) = 0$. Since $L$ is Lichtenbaum and by definition, $\text{Tr} \Omega \text{Syz}_d(M)$ is free. This yields that $\Omega \text{Syz}_d(M) \cong \text{Tr}(\text{Tr} \Omega \text{Syz}_d(M))$ is free. In other words, $\text{pd}_R(M) < \infty$ and so $R$ is regular.
apply
− ⊗

we need a little more, namely $x$

$R$

module. Since $\text{Tor}_1^R(R, -) = 0$ we deduce that any finitely generated module is free. Consequently, $R$ is a field. So, the desired claim in this case is trivial. Now, assume that $d := \text{depth}_R(R) > 0$.

Let $x := x_1, \ldots, x_d$ be a maximal $R$-sequence. It is well-known that $\text{depth}_R(\text{Syz}_d(k)) = d$. Here, we need a little more, namely $x_j$ is a $\text{Syz}_j(k)$-sequence, where $x_j := x_1, \ldots, x_j$ and $1 \leq j \leq i \leq d$.

Indeed, we proceed by induction. Let $1 \leq j \leq i$ and $\beta_i$ be the $i$-th Betti number of $R/m$. We apply $- \otimes_R R/x \!R$ to the exact sequence

$$0 \rightarrow \text{Syz}_{i+1}(k) \rightarrow R^{\beta_i} \rightarrow \text{Syz}_i(k) \rightarrow 0,$$

and deduce the following:

$$0 \rightarrow \text{Tor}_1^R(R/x \!R, \text{Syz}_i(k)) \rightarrow \frac{\text{Syz}_{i+1}(k)}{x_i \text{Syz}_{i+1}(k)} \rightarrow (R/x \!R)^{\beta_i} \rightarrow \frac{\text{Syz}_i(k)}{x_i \text{Syz}_i(k)} \rightarrow 0.$$

By the induction hypothesis, we know $x_i$ is regular over $\text{Syz}_i(k)$. We apply this to deduce that $\text{Tor}_1^R(R/x \!R, \text{Syz}_i(k)) = 0$, and consequently,

$$\text{Syz}_{i+1}(k)/x_i \text{Syz}_{i+1}(k) \subseteq (R/x \!R)^{\beta_i}.$$

Since $x_{j+1}$ is regular over $R/x \!R$ we conclude that $x_{j+1}$ is regular over $\frac{\text{Syz}_j(k)}{x_j \text{Syz}_j(k)}$. Therefore, $x_{j+1}$ is regular over $\text{Syz}_j(k)$. Recall that $K(\bar{x}, -)$ is the Koszul complex of $(-)$ with respect to $x$. Then

$$\text{Tor}_1^R(R/x \!R, \text{Syz}_d(k)) = H_1(K(\bar{x}, R) \otimes_R \text{Syz}_d(k)) = H_1(K(\bar{x}, \text{Syz}_d(k))) = 0.$$

By the assumption, $R/x \!R$ is Lichtenbaum. This yields that $\text{Syz}_d(k)$ is free. In other words, $\text{pd}_R(k) < \infty$. In the light of Auslander-Buchsbaum-Serre theorem, we observe that $R$ is regular. In particular, any (Lichtenbaum) module is of finite injective dimension.

\begin{corollary}
Let $(R, m)$ be a local ring. Assume that there exists a positive integer $n$ such that $R/m^n$ has finite injective dimension. The following assertions are valid:

(i) $R$ is Gorenstein.

(ii) If $\dim R > 0$, then $R$ is regular.

(iii) If $\dim R = 0$, then $m^n = 0$.

Proof. (i) It is enough to recall from [30] that a local ring is Gorenstein if and only if it admits a nonzero cyclic module.

(ii) This is immediate from Corollary 3.27.

(iii) By part (i) $R$ is Gorenstein. As $\dim R = 0$, we have id$_R R = \text{depth}_R R = 0$, and so $R$ is injective. As id$_R(R/m^n) < \infty$, by the same reasoning, we get $R/m^n$ is injective. From the short exact sequence

$$0 \rightarrow m^n \rightarrow R \rightarrow R/m^n \rightarrow 0 \quad (*),$$

we deduce id$_R(m^n) = 0$. Thus, $(*)$ splits, and so $R = m^n \oplus R/m^n$. Multiply both sides with $m^n$ yields $m^n = m^{2n}$. By Nakayama’s lemma $m^n = 0$, as claimed.

\end{corollary}
Remark 3.31. Let us give another proof of Corollary 3.30(iii): We know \( R \) is Gorenstien. Combine this with \( \text{id}_R(R/m^n) < \infty \) yields that \( \text{pd}_R(R/m^n) < \infty \), and so \( R/m^n \) is free. Taking annihilator, we see \( m^n = 0 \).

The monograph [3] is our reference for the concept of \( G \)-dimension. One may simplify [11] if assumes some restrictions:

Remark 3.32. Let \((R, m)\) be a local ring \( L \) is a finitely generated and strong Lichtenbaum module for \( R \). Then \( \text{Gdim}_R L < \infty \) if and only if \( R \) is Gorenstein.

Also, see Corollary 4.20.

Proof. The if part is obvious, because over Gorenstein local rings all modules have of finite \( G \)-dimension. Conversely, suppose \( d := \text{Gdim}_R L < \infty \). As \( \text{Gdim}_R L = \sup\{i \mid \text{Ext}^i_R(L, R) \neq 0\} \), it follows that \( \text{Ext}^{d+1}_R(L, R) = 0 \). Now, we have

\[
0 = \text{Ext}^{d+1}_R(L, R)^\vee = \text{Tor}^{R}_{d+1}(L, R^\vee) = \text{Tor}^{R}_{d+1}(L, E) = \text{Tor}^{R}_1(L, \text{Syz}_d(E)).
\]

Since \( L \) is a strong Lichtenbaum module, \( \text{Syz}_d(E) \) is flat, and so \( p := \text{pd}_R(E(R/m)) < \infty \). There is an exact sequence

\[
0 \rightarrow P_p \rightarrow \cdots \rightarrow P_0 \rightarrow E \rightarrow 0,
\]

in which each \( P_i \) is projective. By applying the exact functor \((-)^\vee\) on the above exact sequence, we see that \( R \cong E^\vee \) has finite injective dimension. Thus, \( \hat{R} \) is Gorenstein, and so \( R \) is as well.

Here, is the lifting property of Lichtenbaum modules:

Observation 3.33. Let \((R, m)\) be a local ring, \( x \in m \) and \( L \) be a finitely generated module such that \( xL = 0 \). Suppose \( L \) is a Lichtenbaum module over \( \overline{R} := R/xR \). Then \( L \) is Lichtenbaum as an \( R \)-module.

Proof. Recall that \( L = \frac{L}{xL} \), and it is indeed equipped with a structure of an \( R/xR \)-module. Let \( F \) be a finitely generated \( R \)-module such that \( \text{Tor}^1_R(L, F) = 0 \). By applying the standard argument we know that \( \text{Tor}^1_R(L, F/xF) = 0 \). Since \( L \) is Lichtenbaum over \( \overline{R} \), we deduce that \( \overline{F} := F/xF \) is free as an \( \overline{R} \)-module. Say \( \overline{F} = \bigoplus_n \overline{R} \). Let \( k := R/m = \overline{R}/m \overline{R} \). Thanks to [24] Theorem 2.3] we know

\[
n = \ell \left( \frac{\overline{F}}{m \overline{F}} \right) = \ell \left( \frac{F}{mF} \right) = \mu_R(F).
\]

Then there is an exact sequence \( 0 \rightarrow \text{Syz}_1(F) \rightarrow \overline{R}^n \rightarrow F \rightarrow 0 \). Tensor it with \(- \otimes_R \overline{R}\) we deduce the following

\[
\text{Syz}_1(F) \otimes_R \overline{R} \rightarrow \overline{R}^n \xrightarrow{\phi} \overline{F} \rightarrow 0,
\]

where \( \phi \) is an isomorphism. It turns out that \( \text{Syz}_1(F) \otimes_R \overline{R} = 0 \). In view of Nakayama’s lemma, \( \text{Syz}_1(F) = 0 \) (see [1] Ex. 2.3]). By definition, \( F = R^n \) which is free. So, \( L \) is Lichtenbaum when we view it as an \( R \)-module.
§ 4. Connections to a result of Burch

**Definition 4.1.** A nonzero $R$-module $L$ is called quasi Lichtenbaum if for every finitely generated $R$-module $N$, the vanishing of $\text{Tor}_1^R(L, F) = \text{Tor}_2^R(L, F) = 0$ implies that $F$ is free.

As a source of quasi Lichtenbaum modules, we recall the concept of Burch ideals. An ideal $I$ is called Burch, if $m(I : m) \neq Im$.

**Example 4.2.** There is a quasi-Lichtenbaum module which is not Lichtenbaum.

**Proof.** Let $R := k[[x, y]]/(x^2, xy)$. In order to see $I := y^3R$ is Burch, recall that $(I : m) = m$ and so $m(I : m) = m^2 = y^3R \neq y^3R = Im$.

Following definition, $I$ is Burch. Thanks to Observation 4.17 (see below), we deduce that $R/I$ is quasi-Lichtenbaum. Since $\text{Tor}_1^R(R/I, R/xR) = 0$ and $R/xR$ is not free, we conclude that $R/I$ is not Lichtenbaum. $\square$

**Proposition 4.3.** Let $(R, m)$ be a local ring and let $L$ be finitely generated such that $\text{pd}_R(L) < \infty$ and

$\text{Tor}_1^R(L, M) = \text{Tor}_2^R(L, M) = 0 \implies \text{pd}_R(M) \leq 1,$

where $M$ is finitely generated. Then $\text{depth}_R(L) \leq 1$.

**Proof.** On the contrary, assume that $\text{depth}_R(L) > 1$. Then there is an $L$-regular sequence $\underline{x} := x, y \in m$. According to Auslander’s zero-divisor, we know $\underline{x}$ is an $R$-sequence. From this, we compute the following homologies

$\text{Tor}_i^R(L, R/xR) = H_i(K(\underline{x}, R) \otimes_R L) = H_i(K(\underline{x}, L)) = 0,$

where $1 \leq i \leq 2$. By the assumption, we get to a contradiction. $\square$

**Example 4.4.** Let $R := k[[x]]$ and apply the previous result for $L := m$. This shows that the bound $\text{depth}_L(L) \leq 1$ achieves.

**Corollary 4.5.** Let $(R, m)$ be hypersurface, $L$ be finitely generated and $\text{depth}_R(L) = 0$. Suppose $\text{Tor}_1^R(L, M) = \text{Tor}_2^R(L, M) = 0$ where $M$ is finitely generated. Then $M$ is maximal Cohen-Macaulay.

**Proof.** By a result of Murthy [26, Theorem 1.6], we observe that $\text{Tor}_1^R(L, M) = 0$. By Huneke-Wiegand [16 Theorem 1.9], either $\text{pd}(L) < \infty$ or $\text{pd}(M) < \infty$. This allow us to apply depth formula. By depth formula,

$\text{depth}_R(M) = \text{depth}_R(L) + \text{depth}_R(M) = \text{depth}_R(L \otimes M) + \text{depth}_R(R) \geq \text{depth}(R) = \text{dim}_R,$

as claimed. $\square$

**Notation 4.6.** By $\overline{R}$ we mean the integral closure of a domain $R$ in its field of fractions.
Proposition 4.7. Let $(R, m)$ be a 1-dimensional local integral domain and let $M$ be a finitely generated module such that

$$\text{Tor}^1_R(\overline{R}, M) = \text{Tor}^2_R(\overline{R}, M) = 0 \quad (*)$$

Then $\text{pd}_R(M) < 2$.

Proof. Let $F := \ldots \to F_1 \to F_0 \to M \to 0$ be a free resolution of $M$. Tensor it with $- \otimes_R \overline{R}$, yielding the following complex

$$F_3 \otimes_R \overline{R} \to F_2 \otimes_R \overline{R} \to F_1 \otimes_R \overline{R} \to F_0 \otimes_R \overline{R} \to M \otimes_R \overline{R} \to 0,$$

which is exact, because we have the vanishing property $(*)$, and recall that the above complex can be used for a part of a free resolution of the $R$-module $M \otimes_R \overline{R}$. Now, tensor this with $- \otimes_R \overline{R}/m\overline{R}$, yields the following diagram of complexes:

$$\zeta_1 := (F_3 \otimes_R \overline{R}) \otimes_\overline{R} \frac{\overline{R}}{m\overline{R}} \to \ldots \to (F_0 \otimes_R \overline{R}) \otimes_\overline{R} \frac{\overline{R}}{m\overline{R}} \to (M \otimes_R \overline{R}) \otimes_\overline{R} \frac{\overline{R}}{m\overline{R}} \to 0,$$

$$\zeta_2 := F_3 \otimes_R \overline{R}/m\overline{R} \to \ldots \to F_0 \otimes_R \overline{R}/m\overline{R} \to M \otimes_R \overline{R}/m\overline{R} \to 0.$$

By this identification, the corresponding homology of $\zeta_1$ and $\zeta_2$ at the second spot coincides, i.e., the following holds:

$$\text{Tor}_2^R(M \otimes_R \overline{R}/m\overline{R}) = H^2(\zeta_1) \cong H^2(\zeta_2) = \text{Tor}_2^R(M, \overline{R}/m\overline{R}),$$

when we view them as $R$-modules. Recall that the integral closure of a noetherian domain of dimension one is noetherian. It turns out that $\overline{R}$ is a Dedekind domain. Since $\overline{R}$ is a Dedekind domain, its global dimension is one. This yields that $\text{Tor}_2^R(M \otimes_R \overline{R}/m\overline{R}) = 0$. Now recall that $\overline{R}/m\overline{R} = \bigoplus R/m$. We apply this at the previous displayed item to conclude that

$$\bigoplus \text{Tor}_2^R(M, R/m) = \text{Tor}_2^R(M, \overline{R}/m\overline{R}) = 0.$$

In other words, $\text{Tor}_2^R(M, R/m) = 0$. Since $M$ is finitely generated, $\text{pd}_R(M) \leq 1$. 

Corollary 4.8. Let $(R, m)$ be a 1-dimensional local integral domain and let $M$ be a finitely generated module such that

$$\text{Tor}_i^R(\overline{R}, M) = \text{Tor}_{i+1}^R(\overline{R}, M) = 0$$

for some $i > 0$. Then $\text{pd}_R(M) \leq 1$. In particular, $M$ is free provided it is of positive depth.

Proof. Without loss of generality, we may assume that $i = 1$. Indeed, let $j := i - 1$. We pass to $M' := \text{Syz}_j(M)$, the $j$-th syzygy module of $M$, and recall that $\text{pd}(M') \leq j + \text{pd}(M)$. Let us apply the previous result to see $\text{pd}_R(M') \leq 1$. So, $\text{pd}_R(M) \leq i$. In the light of Auslander-Buchsbaum formula we observe that $\text{pd}_R(M) \leq 1$. The particular case follows by Auslander-Buchsbaum formula.

Problem 4.9. Find conditions for which $\overline{R}$ is tor-rigid.
Here is a partial positive answer:

**Corollary 4.10.** Let \((R, \mathfrak{m})\) be a 1-dimensional complete local integral domain which is hypersurface. Let \(M\) be a finite length module such that \(\text{Tor}_{i}^{R}(\overline{R}, M) = 0\) for some \(i > 1\). Then \(\text{pd}_{R}(M) \leq 1\).

**Proof.** The integral closure of a noetherian domain of dimension one is noetherian, but not necessarily module-finite. The complete assumption implies that \(\overline{R}\) is module-finite. By a result of Huneke-Wiegand \[15, 2.3 \text{ Corollary}\], we observe that \(\text{Tor}_{i}^{R}(\overline{R}, M) = 0\). Now, apply Corollary 4.8. \(\Box\)

**Example 4.11.** The dimension restriction is needed. Indeed, let \(R := k[[x^2, xy, y^2]] = k[[u,v,w]]_{(u-v-w^2)}\). This is a 2-dimensional normal hypersurface integral domain. In particular, \(\text{Tor}_{i}^{R}(\overline{R}, M) = 0\) for all \(i > 0\) and all modules \(M\).

The next result when \(\ell(M) < \infty\) implicitly is in \[14, \text{ Theorem 4.7}\]:

**Fact 4.12.** Let \((R, \mathfrak{m})\) be a 1-dimensional complete local integral domain of prime characteristic with algebraically closed residue field. Let \(M\) be a finitely generated module such that \(\text{Tor}_{i}^{R}(\overline{R}, M) = 0\) for some \(i > 1\). Then \(\text{pd}_{R}(M) \leq 1\).

**Proof.** Recall that there is an \(n\), large enough, so that \(\oplus_{n} R = \mathfrak{v}^{\ast} R\) (see \[14, \text{ Lemma 4.5}\]). In particular, \(\text{Tor}_{i}^{R}(\mathfrak{v}^{\ast} R, M) = 0\) for some \(i > 1\). But, \(\mathfrak{v}^{\ast} R\) is tor-rigid over Cohen-Macaulay rings of dimension one, see \[25, 2.2.12\]. So, \(\text{pd}_{R}(M) \leq 1\). \(\Box\)

Here is a partial negative answer:

**Observation 4.13.** Let \((R, \mathfrak{m}, k)\) be a 2-dimensional complete local domain which is not Cohen-Macaulay (for example \(R = k[[x^4, y^4, x^3y, xy^3]]\)). Then \(\overline{R}\) is not tor-rigid.

The reader may skip parentheses if not interested in the example.

**Proof.** Recall that \(\overline{R}\) is finitely generated as an \(R\)-module, and also it is noetherian and local, because \(R\) is complete (for example \(R = k[[x^4, y^4, x^3y, xy^3, x^2y^2]]\)). Note that \((x^2y^2)_{+} = (x^3y)^2_{+}\) belongs to the fraction field of \(R\), and it is the root of \(f(T) = T^3 - x^4 T \in R[T]\). Since \(k[[x^4, y^4, x^3y, xy^3, x^2y^2]]\) is the invariant ring, it is normal, and so becomes the integral closure of \(R\). Suppose on the way of contradiction that \(\overline{R}\) is tor-rigid. This allows us to apply \[4, 4.3\] to deduce that each \(\overline{R}\)-regular sequence is an \(R\)-regular sequence. Now, let \(a, b\) be a system of parameter for \(R\), and so a parameter sequence for \(\overline{R}\) (for example, in the example set \(a := x^4, b := y^4\)). By Serre’s characterization of normality, see \[24, \text{ Theorem 23.8}\], we know \(\overline{R}\) satisfies Serre’s condition \((S_2)\) (for the definition, see \[24, \text{ page 183}\]). Thus, \(a, b\) is an \(\overline{R}\)-regular sequence. By the mentioned result of Auslander, \(a, b\) is an \(R\)-regular sequence. So, \(R\) is Cohen-Macaulay, a contradiction. \(\Box\)

**Remark 4.14.** It may be nice to give situations for which \(\text{pd}_{R}(\overline{R}) = \infty\) or even \(\text{Gdim}_{R}(\overline{R}) = \infty\).

(i) Let \((R, \mathfrak{m})\) be a 1-dimensional local domain which is not regular. Then \(\text{pd}_{R}(\overline{R}) = \infty\).
(ii) Let \((R, \mathfrak{m})\) be a 2-dimensional Cohen-Macaulay complete domain which is not normal. Then \(\text{pd}_R(\mathfrak{m}) = \infty\).

(iii) Let \((R, \mathfrak{m}, k)\) be a 1-dimensional complete local domain of prime characteristic with \(k = \mathbb{F}\). If \(\text{Gdim}_R(\mathfrak{m}) < \infty\), then \(R\) is Gorenstein.

(iv) Let \((R, \mathfrak{m})\) be a 2-dimensional Cohen-Macaulay complete domain which is not quasi-normal. Then \(\text{Gdim}_R(\mathfrak{m}) = \infty\).

Proof. (i): This is similar to ii).

(ii): Recall that \(R\) is noetherian and local. By definition, its normal. Suppose on the way of contradiction that \(\text{pd}_R(\mathfrak{m}) < \infty\). Recall that \(R\) and \(\mathfrak{m}\) both are Cohen-Macaulay. In particular, both of them satisfy Serre’s condition \((S_2)\). Due to Auslander-Buchsbaum formula we know \(R \to \mathfrak{m}\) is flat (in fact free). By Serre’s characterization of normality (see [24, Theorem 23.8]), we know \(R\) is not \((R_1)\) and \(\mathfrak{m}\) is \((R_1)\). This is in contradiction with [24, Theorem 23.9]. So, \(\text{pd}_R(\mathfrak{m}) = \infty\).

(iii): By the argument presented in [14, Theorem 4.7] we know that \(\text{Gdim}_R(\mathfrak{m}^n R) < \infty\) for some \(n\) large enough. Thanks to Auslander-Bridger formula, \(\text{Gdim}_R(\mathfrak{m}^n R) = 0\). In particular, \(\text{Ext}^1_R(\mathfrak{m}^n R, R) = 0\). This implies that \(R\) is Gorenstein, see e.g. [22, Corollary 2.7].

(iv): Recall that quasi-normal means \((S_2)+(G_1)\), here a ring \(A\) is called \((G_1)\) if \(A_p\) is Gorenstein for all prime ideal \(p\) of height at most one. Now, the desired conclusion is a slight modification of part (ii), and we leave details to the reader. \(\square\)

Proposition 4.15. Let \((R, \mathfrak{m})\) be a local integral domain and let \(L\) be such that

\[\text{Tor}^R_1(L, M) = \text{Tor}^R_2(L, M) = 0 \implies M\text{ is torsion-free}.\]

Then \(\text{depth}_R(L) = 0\).

One may replace the integral domain assumption with \(\text{pd}_R(L) < \infty\), and derives the same conclusion.

Proof. If not, then \(\text{depth}_R L > 0\), i.e., there is an \(L\)-regular element \(x\). It is easy to see, \(\text{Tor}^R_2(L, R/x R) = \text{Tor}^R_1(L, R/x R) = 0\). Since \(R/x R\) is not torsion-free, we get a contradiction. \(\square\)

Several years ago, Burch proved:

Theorem 4.16. Let \(M\) be finitely generated and \(I\) be Burch. If \(\text{Tor}^R_t(R/I, M) = \text{Tor}^R_{t+1}(R/I, M) = 0\) for some positive integer \(t\), then \(\text{pd}_R(M) \leq t\).

Despite its importance, this was proved very recently in [18, Theorem 1.2] that the above bound is not sharp. Our elementary and short proof is independent of them:

Observation 4.17. Adopt the above notation. Then \(\text{pd}_R(M) \leq t - 1\).

Proof. This is a combination of Fact 222 and the above result of Burch. \(\square\)

In fact [18, Theorem 1.2] presents the module version of Observation 4.17. By using some ideas taken from §3 we observed in the previous draft that if a Burch ideal is of finite injective dimension,
then the ring is regular. Also, this result is in [18 Corollary 3.20]. Since their argument is short compared to us, we skip the mentioned observation. Instead, we apply it for $mM$ and reconstruct some known results:

**Corollary 4.18.** (Levin-Vasconcelos) Let $M$ be finitely generated and such that $mM$ is nonzero and of finite injective (projective) dimension. Then $R$ is regular.

**Example 4.19.** The finitely generated assumption is needed. For example, let $R$ be any integral domain which is not regular. Since $E_R(k)$ is divisible, we have $mE_R(k) = E_R(k)$. Then $mE_R(k)$ is nonzero and injective. But, $R$ is not regular.

The following extends the main result of [2] by dropping an assumption on Castelnuovo-Mumford regularity$^*$.  

**Corollary 4.20.** Let $(R, m, k)$ be any local ring, $M$ be finitely generated of positive depth such that $m^i M$ is nonzero and of finite complete-intersection dimension for some $i > 0$. Then $R$ is complete-intersection.

Instead of $mM$ one may assume the module is Burch and derives the same conclusion.

**Proof.** Let $x \in m$ be regular over $M$. Clearly, $x$ is $m^i M$-sequence. We know from [7] that

$$\text{CIdim}_R\left(\frac{m^i M}{x m^i M}\right) = \text{CIdim}_R(m^i M) - 1 < \infty.$$ 

By [13 Lemma 3.7], $k$ is a direct summand of $\frac{m^i M}{x m^i M}$. This yields that $\text{CIdim}_R(k) < \infty$, and so $R$ is complete-intersection. \qed

§ 5. AN APPLICATION: DIMENSION OF SYZYGIES

Rings in this section are not artinian. Recall that $\ell(\cdot)$ is the length function.

**Question 5.1.** (See [14 Question 1.2]) Let $M$ be such that $\text{pd}_R(M) = \infty$ and $\ell(M) < \infty$. Is $\ell(\text{Syz}_i(M)) = \infty$ for all $i > \dim(R) + 1$?

**Proposition 5.2.** Let $L$ be Lichtenbaum, $\text{pd}_R(L) = \infty$ and $\ell(L) < \infty$. Then $\ell(\text{Syz}_i(L)) = \infty$ for all $i > 0$.

**Proof.** By the short exact sequence $0 \to \text{Syz}_1(L) \to R^n \to L \to 0$ and the facts that $\ell(L) < \infty = \ell(R^n)$ we see $\ell(\text{Syz}_1(L)) = \infty$. Suppose $\ell(\text{Syz}_{i+1}(L)) < \infty$ for some fixed $i > 0$. In the light of [3 Lemma 4.1] we observe

$$\text{Tor}_1^R(L, \text{Syz}_{i-1}(R/H_0^H(m)(R))) = \text{Tor}_1^R(L, R/H_0^H(m)(R)) = 0.$$ 

Since $M$ is Lichtenbaum, $\text{Syz}_{i-1}(R/H_0^H(m)(R))$ is free. In other words, $\text{pd}_R(H_0^H(m)(R)) < \infty$. Suppose $H_0^H(m)(R) \neq 0$. According to a celebrated result of Burch, $H_0^H(m)(R)$ contains a regular element $x$. 

$^*$we only focus on $\text{CIdim}_R(\cdot)$ (see [7] for its definition and basic properties) and other homological invariants such as $\text{Gdim}_R(\cdot)$ follow in the same vein.
By definition, $m^n x = 0$ for some $n > 0$. Since $x$ is regular, we observe that $H^n_m(R) = 0$, i.e., depth$_R(R) > 0$. In view of \[5\] Lemma 3.4], we see $\ell(Syz_i(L)) = \infty$. This completes the proof. □

Remark 5.3. Adopt the notation of Proposition 5.2. Instead of $\ell(L) < \infty$ we may assume that $L$ is locally free on the punctured spectrum. By the same proof $\ell(Syz_i(L)) = \infty$ for all $i > 1$.

We say a local ring $R$ is of isolated singularity if it is singular and $R_p$ is regular for all $p \in (\text{Spec}(R) \setminus \{m\})$.

Corollary 5.4. Let $(R, m)$ be a complete local ring of prime characteristic with perfect residue field. If $R$ is of isolated singularity, then $\ell(Syz_i(x^nR)) = \infty$ for all $i > 1$ and all $n \gg 0$.

Proof. By the assumption, $R$ is $F$-finite (see \[8\], Page 398). This means that $x^n R$ is finitely generated as an $R$-module. By a celebrated theorem of Kunz (see \[8\], Corollary 8.2.8) we know that $pd_R(x^n R) = \infty$, because $R$ is not regular. Recall from \[5\], Proposition 8.2.5] that $\varphi_i^\ast(R) \cong \varphi_i^\ast(R_p)$ for all $p \in \text{Spec}(R) \setminus \{m\}$. Again, another use of \[8\], Corollary 8.2.8] shows that $x^n R$ is locally free on the punctured spectrum, because $R_p$ is regular for all $p \in \text{Spec}(R) \setminus \{m\}$. Without loss of generality we may assume that depth$(R) = 0$, see the proof of \[5\], Lemma 3.4]. In view of Remark 3.12 we observe that $x^n R$ is Lichtenbaum for all $n \gg 0$. So, we are in the situation of Remark 5.3. We conclude from this remark that $\ell(Syz_i(x^n R)) = \infty$ for all $i > 1$. □

The Lichtenbaum assumption in Proposition 5.2 is important:

Example 5.5. Let $R := k[[x, y]]/(x^2, xy)$. Since $R \xrightarrow{x} R \xrightarrow{y} R \rightarrow R/yR \rightarrow 0$ is exact, $\text{Syz}_2(R/yR) = xR = R/(0 : x) = k$ is of length one. In order to see $R/yR$ is not Lichtenbaum, recall that $\text{Tor}_R^1(R/yR, R/xR) = 0$ and that $R/xR$ is not free.

In the previous example $m(y : m) = m^2 = y^2 R = ym$. So, $yR$ is not Burch. One may ask: Is Proposition 5.2 true for Burch modules? This is not the case as the next example indicates:

Example 5.6. Let $R := k[[x, y]]/(x^2, xy)$. Since $R \xrightarrow{x} R \xrightarrow{y^2} R \rightarrow R/y^2 R \rightarrow 0$ is exact, $\text{Syz}_2(R/y^2 R) = xR = R/(0 : x) = k$ is of length one, and recall that $I := y^2 R$ is Burch.

In the previous two examples $\text{Tor}_R^1(R/I, A) = 0$ where $I$ is generated by an $A$-regular sequence. Is this true in general? In fact, this was asked before than us:

§ 6. VANISHING AND NON-VANISHING OF $\text{Tor}_1$

Question 6.1. Let $(R, m)$ be a local ring and $M$ a finitely generated $R$-module. Let $x_1, \ldots, x_t$ be an $M$-regular sequence and $I = (x_1, \ldots, x_t)$. Is it true that $\text{Tor}_1^R(R/I^n, M) = 0$ for all $n \geq 1$?

Observation 6.2. Here, we collect a couple elementary observations:

(i) The case $M$ is cyclic is in \[17\], Ex. 11.12].
(ii) If $I$ is generated by regular sequence, the answer is positive. Indeed, we apply a routine induction on $n$, to reduce to the case $n = 1$. In this case, we have

$$\text{Tor}_1^R(R/I, M) = H_1(K(x, R) \otimes_R M) = H_1(K(x, M)) = 0.$$ 

So, the claim follows.

(iii) If $\text{pd}_R(M)$ is finite, the answer is positive. Indeed, in view of Auslander’s zero divisor theorem, $I$ is generated by a regular sequence $x$. In view of ii) we see $\text{Tor}_1^R(R/I^n, M) = 0$ for all $n \geq 1$.

(iv) The question is true over regular rings. Indeed, as over regular local rings modules are of finite projective dimension, the claim is in iii).

(v) The question is true if $I$ is principal.

(vi) The question is true over 1-dimensional rings. Indeed, recall that $1 \leq \text{grade}_R(I, M) \leq \text{depth}_R(M) \leq \dim(M) \leq \dim(R) = 1$.

This says that $\text{grade}(I, M) = 1$. Since $I$ is generated by an $M$-sequence, we deduce that $I$ is principal. Now the desired claim is in the previous item.

Here, is a nontrivial example:

**Example 6.3.** Let $R := k[[x^4, y^4, x^3y, xy^3]]$ and $I := (x^4, y^4)$. The following holds:

(a) $B := k[[x^4, y^4, x^3y, xy^3, x^2y^2]]$ is finitely generated as a $R$-module.

(b) $x^4, y^4$ is $B$-regular.

(c) $\text{Tor}_1^R(R/I, B) = 0$.

**Proof.** In view of [19, 42.4] we observe that $x := x^4, y^4$ is $B$-sequence and there is an exact sequence

$$0 \to R \to B \to k \to 0.$$ 

We apply $- \otimes_R R/\mathfrak{z}R$ to $\zeta$ and deduce the following exact sequence

$$0 \to \text{Tor}_1^R(R/I, B) \to \text{Tor}_1^R(R/I, k) \to R/I \otimes_R R \to R/I \otimes_R B \to R/I \otimes_R k \to 0.$$ 

Since $2 = \mu(I) = \beta_1(R/I) = \dim_k(\text{Tor}_1^R(R/I, k))$ we see $\text{Tor}_1^R(R/I, k) = k \otimes k$. Also, $R/I \otimes_R k = R/m + I = k$. Let $\ell := \ell(\text{Tor}_1^R(R/I, B))$. By plugging these in the previous sequence and taking the length, we obtain

$$\ell = 1 - \ell(R/I) + \ell(B/IB) \quad (\star)$$

On the one hand we have

$$0 = \frac{IB}{IB} \nRightarrow \frac{IB + (x^3y)B}{IB} \nRightarrow \frac{IB + (x^3y, xy^3)}{IB} \nRightarrow \frac{mB}{IB} \nRightarrow B/IB.$$ 

In order to see the factors are simple modules, we remark that $mxy^3 \in IB$ and $mx^3y \in I$. Indeed, for example we have $(x^3y)x^3y = x^6y^2$ and $x^2y^2 \in B$, i.e., $(x^3y)x^3y = (x^2y^2)x^4 \in IB$. It turns out that $\ell(B/IB) = 4$. On the other hand, we have the following composite sequence

$$0 = \frac{I}{I} \nRightarrow \frac{I + (x^3y)}{I} \nRightarrow \frac{I + (x^3y, xy^3)}{I} = \frac{mR}{I} \nRightarrow R/I.$$
In order to see the compute the factors, we remark that \( m x^3 y \notin I \) since \( x^2 y^2 \notin R \). However, the factors of \( 0 = \mathcal{I} \subseteq I + (x^3y) \subseteq I + (x^3a) \) are simple. In the same vein, the factor of

\[
\frac{IB + (x^3 y)B}{IB} \supsetneq \frac{IB + (x^3 y, xy^3)}{IB}
\]

is of length two. We proved that \( \ell(R/I) = 5 \). In view of (\(*\)) we have \( \ell = 0 \). In other words, \( \text{Tor}_1^R(R/I, B) = 0 \), as claimed.

\[\square\]

**Lemma 6.4.** (See \cite{28} Lemma 2.6) Let \( F \) be an \( R \)-module, \( K \) be a submodule of \( F \) and set \( M = F/K \). Let \( J = (a_1, \ldots, a_r) \) be an ideal generated by an \( M \)-regular sequence. Then \( J^n F \cap K = J^n K \) for all \( n > 0 \).

**Observation 6.5.** Answer to Question 6.1 is yes.

**Proof.** We look at \( 0 \rightarrow \text{Syz}_1(M) \rightarrow F := R^n \rightarrow M \rightarrow 0 \) (\(*\)), i.e. \( M \cong F/\text{Syz}_1(M) \). Let \( I := (a_1, \ldots, a_r) \). We tensor (\(*\)) with \( R/I \) gives us

\[
0 = \text{Tor}_1^R(F, R/I) \rightarrow \text{Tor}_1^R(M, R/I) \rightarrow \text{Syz}_1(M)/I \text{Syz}_1(M) \rightarrow F/IF \rightarrow M/IM \rightarrow 0.
\]

Then, \( \text{Tor}_1^R(M, R/I) \) is isomorphic to \( (\text{Syz}_1(M) \cap IF)/I \text{Syz}_1(M) \). Combining these with the previous lemma, by plugging \( K := \text{Syz}_1(M) \), it follows that if \( a_1, \ldots, a_r \) is an \( M \)-regular sequence, then \( \text{Tor}_1^R(M, R/(a_1, \ldots, a_r)^n) = 0 \) for all \( n > 0 \). \[\square\]

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