Statistical analysis of the best relay location in a random two-way relay network with multi-slope path loss

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Abstract
This paper presents an exact statistical analysis for the best relay location in a random two-way relay network whose channels are subject to multi-slope path loss and Rayleigh fading. In this network, relays are distributed randomly as a two-dimensional Poisson point process, and the best relay is selected according to a max–min criterion. The midpoint between the two sources is adopted as a reference point and the relays are indexed such that the closer to the reference point, the smaller is the index. An exact analytical expression is derived for the probability that the $k$th nearest relay to the reference point is the best relay. Then, a detailed statistical analysis for the location of the best relay with respect to the reference point is presented. In each case, a low-complexity approximation is also derived and its accuracy is verified by numerical simulation. The proposed analysis demonstrates the counter-intuitive fact that the best relay in a random two-way relay network is quite unlikely to be located on, or very close to the reference point. Numerical results also suggest that the proposed analysis can be used to restrict the search region for finding the best relay.

1 | INTRODUCTION

Two-way relay networks have received significant attention in recent years due to their superior spectral efficiency over one-way relay networks [1–3]. In a particular class of two-way relay networks, referred to as multiple access broadcast, data exchange between two source nodes takes place in two time slots [4, 5]. In the first time slot, the source nodes simultaneously broadcast their signals to all potential relays. The two signals are then combined in the relay and forwarded to both source nodes [4, 5]. This is different from the conventional one-way relaying in which data exchange between two source nodes takes place in four time slots. In references [6, 7], the error probability performance of a two-way relay network has been investigated. In [8], it is shown that two-way relaying is more spectrally efficient than one-way relaying as in the former the average sum rate is larger than in the latter.

The performance of two-way relay networks can be improved by appropriate selection of the best relay node [9]. In [9], the symbol error rates (SERs) of the signals transmitted by the two sources are evaluated at each relay, and the relay with the smallest maximum SER is chosen as the best relay, that is, a min–max criterion is used. Motivated by Jing [9], in [10] a relay selection scheme based on the max–min signal-to-noise ratio (SNR) criterion is presented. In this scheme, the SNRs of both sources are examined at each relay and the minimum of the two SNRs is determined. Then, the relay with the largest minimum SNR is selected for cooperation. In [11], a max-sum relay selection criterion is proposed in which the relay that maximises the instantaneous sum rates of the two sources is chosen as the best relay.

Due to mobility of the nodes, the topology of the wireless networks is not often static. Hence, it is reasonable to assume that the number of nodes as well as their locations are random [12–14]. In some studies, the wireless nodes are assumed to be distributed according to a two-dimensional (2D) Poisson point process (PPP) with constant intensity [13–16]. In [17], a max–min relay selection criterion is used to obtain the coverage probability of a millimeter wave random two-way relay network. In this reference, relays are assumed to be distributed as a 2D PPP.
It is shown that the coverage and spectral efficiency of the max–min relay selection criterion is significantly better than those of random relay selection scheme.

In two-way wireless relay networks, the random distribution of the nodes along with signal attenuation due to path loss, multi-path fading and shadowing, makes the best relay selection a complicated task [18]. In order to simplify the best relay selection process, it is convenient to find a subset of relays that contains the best relay with a high probability and search for the best relay in this subset. This is owing to the fact that in relay selection process, we need to estimate the channels between the relays and the source nodes. Therefore, the smaller this subset of relays, the lower is the computational complexity incurred by channel estimations. Clearly, finding this subset of relays requires some statistical information about the best relay’s location. In [19, 20], a thorough statistical analysis has been presented for the distance between a reference node and its communication best neighbour (CBN)\(^1\) and for the index of CBN. In both references, the nodes are assumed to constitute an infinite 2D PPP. In addition, a standard path loss model and a general fading model are considered for the communication channels. Interestingly, the statistics derived in [19, 20] place limits on the range and the index of CBN with a given certainty level.

Motivated by [19, 20], this paper considers a two-way relay network in which the relays are distributed as a 2D PPP with constant intensity. The communication channels are assumed to be subject to Rayleigh fading and multi-slope (N-slope) path loss [21]. Different from [19, 20], this paper considers a multi-slope path loss model which is more realistic than the standard single-slope model. In addition, the reference point is assumed to be halfway between the two source nodes and the locations of the nodes are represented in a polar coordinate system whose pole is the reference point, and whose polar axis is the line passing through the source nodes. The key contributions of this paper are as follows:

1. We assume that the nodes are randomly distributed according to a 2D PPP and thus, the number of nodes and their locations are random. This is in contrast to the majority of the works in the literature that assume the number of relays and their location are known and deterministic.

2. We consider a multi-slope path loss model, in which the path loss exponent is not fixed and depends on the distance between communicating nodes. This model is more general than the standard single-slope model and can better account for the effects of physical environment on system performance.

3. We conduct a detailed statistical analysis for the index of the best relay, its distance from the pole and its angle with the polar axis. In particular, we derive exact analytical expressions for
   - the probability mass function (PMF) and the cumulative distribution function (CDF) of the best relay’s index;
   - the joint probability density function (PDF) of the best relay’s index and its distance from the pole;
   - the joint PDF of the best relay’s index and its angle with the polar axis;
   - the complementary CDF (CCDF) and the average SNR experienced by each source node due to the signal transmitted by the \(j\)th best relay.
   - quantifies the SNR penalty for switching from the \(j\)th best relay to the \(k\)th best relay; this is especially useful when the best relay becomes unavailable and the relaying has to be resumed through a new relay.

The remainder of this paper is organised as follows. In Section 2, the system model is introduced. In Section 3, an exact analytical expression is derived for the PMF and CDF of the best relay’s index. A detailed statistical analysis for the best relay’s location is presented in Section 4. In Section 5, analytical expressions are derived for CCDF and the expected value of SNR received from the \(j\)th best relay by each source node. In Section 6, our analytical results are verified through numerical simulations. Conclusions are drawn in Section 7.

2 | SYSTEM MODEL

We consider a two-way relay network which is composed of two source nodes, \(\mathcal{S}_1\) and \(\mathcal{S}_2\), and a number of relays that are randomly distributed in a region \(\mathcal{C}\), as shown in Figure 1. This paper

\(\text{FIGURE 1} \quad \text{A typical two-way relay network with two source nodes and a number of randomly distributed relays}\)
assumes that the relay deployment region, \( \mathcal{C} \), is arbitrarily large. Hence, relays that are far from \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \), are very unlikely to be selected for cooperation owing to significant signal attenuation due to path loss [13]. We assume that the locations of the relays constitute a homogeneous 2D PPP with a fixed density \( \lambda \). A polar coordinate system whose pole, \( O \), is the midpoint of \( \mathcal{S}_1 \)-\( \mathcal{S}_2 \) line segment, and whose polar axis is \( O-\mathcal{S}_2 \) line segment, is used to represent the locations of the nodes. In this coordinate system, \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are represented as \((d, \pi)\) and \((d, 0)\), respectively. Hence, the distance between \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) equals 2d. Suppose that the relays are indexed based on their distances from \( O \) as \( R_1, R_2, \ldots \), where \( R_1 \) is the nearest relay to the pole. Then, the location of a given relay, for example \( \mathcal{R}_j \), is represented as \((r_j, \theta_j)\). All nodes are assumed to be equipped with a single antenna and the transmission channels between any two nodes in the system are subject to Rayleigh fading and multi-slope path loss. The instantaneous SNRs of the \( \mathcal{S}_1-\mathcal{R}_j \) and \( \mathcal{S}_2-\mathcal{R}_j \) links are then given by

\[
\begin{align*}
\Gamma_{1,j} & = r_{1,j} \Omega_{1,j} \quad (1a) \\
\Gamma_{2,j} & = r_{2,j} \Omega_{2,j}, \quad (1b)
\end{align*}
\]

where \( \Omega_{1,j} \) and \( \Omega_{2,j} \) are fading powers of the \( \mathcal{S}_1-\mathcal{R}_j \) and \( \mathcal{S}_2-\mathcal{R}_j \) links, respectively. We assume that \( \Omega_{1,j} \) and \( \Omega_{2,j} \) are jointly independent and identically distributed random variables (RVs) with unit mean and CDF \( F_{\Omega}(\cdot) \). This paper uses an N-slope path loss model in which \( \gamma_{1,j} \) and \( \gamma_{2,j} \) are defined as [21]

\[
\begin{align*}
\gamma_{1,j} & = \frac{G_0 P_0}{N_0} \mathcal{X}_i(\ell_+(r_j, \theta_j)) \quad (1c) \\
\gamma_{2,j} & = \frac{G_0 P_0}{N_0} \mathcal{X}_i(\ell_-(r_j, \theta_j)), \quad (1d)
\end{align*}
\]

where

\[
\mathcal{X}_i(x) = \begin{cases} 
\frac{x^{-\alpha_1}}{\mathcal{K}_1}, & 0 \leq x < R_{i1} \\
\frac{x^{-\alpha_2}}{\mathcal{K}_2}, & R_{i1} \leq x < R_{i2} \\
\vdots \\
\frac{x^{-\alpha_N}}{\mathcal{K}_N}, & R_{iN-1} \leq x < R_{iN} 
\end{cases}
\]

\[
\ell_{\pm}(\rho, \phi) = \sqrt{\rho^2 + d^2 \pm 2d \rho \cos \phi}.
\]

\( G_0 \) is a constant depending on the system parameters such as antenna gains, \( P_0 \) is the transmitted power by the source, and \( N_0 \) is the power spectral density of the noise at destination. It is assumed that \( G_0 \) is same for all \( \mathcal{S}_1-\mathcal{R}_j \) and \( \mathcal{S}_2-\mathcal{R}_j \) links, and \( \ell_+(r_j, \theta_j) \) and \( \ell_-(r_j, \theta_j) \) denote the distances between \( \mathcal{S}_1 \) and \( \mathcal{R}_j \), and \( \mathcal{S}_2 \) and \( \mathcal{R}_j \), respectively. Note that \( \{\alpha_i\}_{i=1}^N \) are the path loss exponents such that \( 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N \) and \( 2 < \alpha_N < 7 \). Note that in (1e), \( \{R_{iN}\}_{i=1}^N \) are the critical distances in the N-slope path loss model so that \( 0 < R_{i1} < \cdots < R_{iN} \), \( \mathcal{K}_1 = 1 \) and [21]:

\[
\mathcal{K}_k = \left( \prod_{j=2}^{k} \frac{R_{i,j-1}^{-\alpha_{j-1}}}{\mathcal{K}_{j-1}} \right)^{-1}, \quad k = 2, \ldots, N. \quad (2)
\]

Note that when \( \alpha_1 = \alpha_2 = \cdots = \alpha_N \), the multi-slope path loss model is simplified to the standard path loss model. We use a Rayleigh fading model for the transmission channels and thus

\[
F_{\Omega}(\infty) = 1 - e^{-\infty}, \quad \infty \geq 0. \quad (3)
\]

In our two-way relay system, each data exchange between \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) takes place in two time slots. In the first time slot, both source nodes broadcast their signals and the relays receive a combination of the two signals. In the second time slot, the best relay amplifies and forwards the received signals to the source nodes and the rest of relays remain idle. As mentioned earlier, there are several approaches for selecting the best relay in the context of two-way relaying systems. Here, we adopt a max–min SNR criterion [10] in which the index of the best relay is given by

\[
J = \arg \max_{j=1,2,\ldots} \{ \min(\Gamma_{1,j}, \Gamma_{2,j}) \}. \quad (4)
\]

In what follows we denote the polar coordinates of the best relay with respect to \( O \) as \( n_b \) and \( \theta_b \), respectively.

### 3 Statistical Analysis of the Best Relay’s Index

In this section, we derive the PMF and CDF of the best relay’s index (i.e. \( J \)), the joint PDF of \( J \) and \( n_b \), and the joint PDF of \( J \) and \( \theta_b \).

Denoting by \( p_{J/J} \) the probability that \( \mathcal{R}_j \) is the best relay, we can write

\[
p_{J/J} = \Pr \left\{ \bigcap_{j=1}^{\infty} \left\{ \min(\Gamma_{1,j}, \Gamma_{2,j}) \leq \min(\Gamma_{1,j}, \Gamma_{2,j}) \right\} \right\}
\]

\[
= \Pr \left\{ \bigcap_{j=1}^{\infty} \left\{ \Gamma_{j} \leq \Gamma_{j} \right\} \right\}, \quad (5a)
\]

where

\[
\Gamma_{j} \triangleq \min \left( \frac{G_0 P_0}{N_0} \mathcal{X}_i(\ell_+(r_j, \theta_j)) \Omega_{1,j}, \frac{G_0 P_0}{N_0} \mathcal{X}_i(\ell_-(r_j, \theta_j)) \Omega_{2,j} \right), \quad (5b)
\]
Thus, using total probability theorem, we obtain

\[
P_{J} [\gamma] = \int_{0}^{2\pi} \int_{0}^{\infty} \Pr \left( \bigcap_{j=1}^{\infty} \left\{ \Gamma_j \leq \gamma \right\} \middle| \eta = r, \theta = \theta \right) \times f_{\eta J}(r, \theta) \, dr \, d\theta.
\]

We now use the properties of PPP to obtain the joint PDF of \( \eta \) and \( \theta \) [20]. To this end, we first note that \( f_{\eta J}(r, \theta) \, dr \, d\theta \) is the probability that exactly \( j - 1 \) relays lie on a circle with centre \( O \) and radius \( r \), and one relay lies on an annular sector defined by the intervals \([r, r + dr]\) and \([\theta, \theta + d\theta]\). The above events are mutually independent due to the properties of PPP [22, Proposition 3.2]. Hence, we can write

\[
f_{\eta J}(r, \theta) = \frac{\lambda \pi r^2}{(J - 1)!} e^{-\lambda \pi r^2} \lambda r \, dr \, d\theta = e^{-\lambda \pi r^2} \lambda r \, dr \, d\theta.
\]

When \( dr \) and \( d\theta \) approach zero, the term \( e^{-\lambda \pi r^2} \) on the right of (7) approaches unity. Hence, we arrive at

\[
f_{\eta J}(r, \theta) = \frac{\lambda \pi r^2}{(J - 1)!} e^{-\lambda \pi r^2}.
\]

for \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \). Note that Equation (8) implies that \( \eta \) and \( \theta \) are independent RVs.

We now evaluate the first term in the integrand of (6). To this end, we use total probability theorem to obtain

\[
\Pr \left( \bigcap_{j=1}^{\infty} \left\{ \Gamma_j \leq \gamma \right\} \middle| \eta = r, \theta = \theta \right) = \int_{0}^{\infty} \Pr \left( \bigcap_{j=1}^{\infty} \left\{ \Gamma_j \leq \gamma \right\} \middle| \eta = r, \theta \right) f_{\gamma J}(\gamma \mid r, \theta) \, d\gamma,
\]

where \( \mathfrak{C} = \{ \eta = r, \theta = \theta, \Gamma_j = \gamma \} \) and (see Appendix A.1);

\[
f_{\gamma J}(\gamma \mid r, \theta) = \Psi_{1}(r, \theta) e^{-\gamma \Psi_{1}(r, \theta)}, \quad \gamma \geq 0
\]

where

\[
\Psi_{k}(r, \theta) = \frac{1}{k \pi} \mathcal{L}_{i} \left( \mathcal{E}_{+}(r, \theta) \right) + \frac{1}{\mathcal{L}_{i} \left( \mathcal{E}_{-}(r, \theta) \right)}.
\]

Observe that when \( r_j \) is known, the inner and outer regions of the annulus \( \mathcal{A} \) in Figure 1 (i.e. an annulus centred at \( O \) and bounded by circles of radii \( r_j \) and \( r_j + dr \)) are disjoint. Therefore, using the properties of PPP, the nodes are distributed independently over these two regions [22, Proposition 3.2]. Hence, we have

\[
\Pr \left( \bigcap_{j=1}^{\infty} \left\{ \Gamma_j \leq \gamma \right\} \middle| \mathfrak{C} \right) = \Pr \left( \bigcap_{j=1}^{J-1} \left\{ \Gamma_j \leq \gamma \right\} \middle| \mathfrak{C} \right) \times \Pr \left( \bigcap_{j=J}^{\infty} \left\{ \Gamma_j \leq \gamma \right\} \middle| \mathfrak{C} \right).
\]

In order to calculate \( P_{1} \), we first note that when the number of nodes in a given region are known, the nodes are distributed uniformly, identically and independently in that region [22, Definition 3.2(ii)]. Hence, we can write

\[
P_{1} = \prod_{j=1}^{J-1} \Pr \{ \Gamma_j \leq \gamma \},
\]

where

\[
\Pr \{ \Gamma_j \leq \gamma \} = \int_{0}^{r} \int_{0}^{2\pi} \Pr \{ \Gamma_j \leq \gamma \mid r_j = \varrho, \theta_j = \vartheta \} \times f_{r, \theta}(\varrho, \vartheta) \, d\varrho \, d\vartheta,
\]

\[
f_{r, \theta}(\varrho, \vartheta) = \begin{cases} \frac{1}{2\pi r^2}, & 0 \leq \varrho \leq r, \quad 0 \leq \vartheta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}
\]

and (12c) follows from [19, Equation (8)] and the independence of \( r_j \) and \( \theta_j \). Using the procedure explained in Appendix A.1, we can find the conditional CDF of \( \Gamma_j \) as

\[
\Pr \{ \Gamma_j \leq \gamma \mid r_j = \varrho, \theta_j = \vartheta \} = \Pr \{ \min \{ \Gamma_{1,j}, \Gamma_{2,j} \} \leq \gamma \mid r_j = \varrho, \theta_j = \vartheta \} = 1 - e^{-\gamma \Psi_{1}(\varrho, \vartheta)}, \quad j = 1, \ldots, J-1,
\]

where (12d) results from (3). Therefore, \( P_{1} \) can be written as

\[
P_{1} = \left[ 1 - \frac{1}{\pi r^2} \int_{0}^{r} \int_{0}^{2\pi} e^{-\gamma \Psi_{1}(\varrho, \vartheta)} \, d\vartheta \, d\varrho \right]^{J-1}.
\]

We now turn our attention to evaluating \( P_{0} \) in (11). To this end, we note that the relays located outside the circle with centre \( O \) and radius \( r \) are also distributed according to a 2D PPP with density \( \lambda \) [19]. Denoting the set of these relays as \( \Lambda_{o} \), one can
write
\[ P_\circ = \Pr \left\{ \bigcap_{j=J+1}^\infty \{ \Gamma_j \leq \gamma \} \bigg| \mathcal{G} \right\} \]
\[ = E_{\Lambda_\circ} \left\{ \prod_{j=J+1}^\infty F_{T_j}(\gamma|\Gamma_j, \Theta_j) \right\} \]
\[ = \exp \left[ -\lambda \int_0^\infty \int_0^{2\pi} \left( 1 - F_{T_j}(\gamma|\Gamma_j = \varphi, \Theta_j = \theta) \right) \varphi \, d\varphi \, d\theta \right] \]
\[ \approx \exp \left[ -\lambda \int_0^\infty \int_0^{2\pi} e^{-\gamma \Phi_\circ(\varphi, \theta)} \varphi \, d\varphi \, d\theta \right], \tag{14} \]
where \( E_{\Lambda_\circ} \{ \} \) denotes the ensemble average over all realisations of \( \Lambda_\circ \), (a) follows from Campbell's theorem [23, Equation (3.35)], and (14) results from (12d). Note that for the standard path loss model, the expressions given in Equations (13) and (14) can be approximated by adopting the approximation given in Equation (A8) of Appendix A.1 for \( \Psi_\circ(\varphi, \theta) \) as
\[ P_\circ \approx \left[ 1 - \frac{1}{\pi r^2} \int_0^r 2\pi \varphi e^{-\frac{1}{2} \Phi_\circ(\varphi, \alpha)} I_0 \left( \frac{\gamma}{2} \Phi_\circ(\varphi, \alpha) \right) \varphi \, d\varphi \right] \]
\[ P_0 \approx \exp \left[ -\lambda \int_0^\infty \int_0^{2\pi} e^{-\gamma \Phi_\circ(\varphi, \theta)} I_0 \left( \frac{\gamma}{2} \Phi_\circ(\varphi, \alpha) \right) \varphi \, d\varphi \, d\theta \right], \tag{16} \]
respectively, where
\[ \Phi_\circ(\varphi, \alpha) = |d + r|^{-\alpha} + |d - r|^{-\alpha} \pm 2\sqrt{(r^2 + d^2)^\alpha} \tag{17} \]
and \( I_0(\cdot) \) denotes the zeroth-order modified Bessel function of the first kind defined as \( I_0(\zeta) = \frac{1}{\pi} \int_0^\pi e^{\zeta \cos \theta} \, d\theta \). Finally, using the expressions given for \( P_1 \) and \( P_0 \), one can write Equation (6) as
\[ \Pr[Y] = \int_0^{2\pi} \int_0^\infty \left( \int_0^\infty P_1 P_0 f_{T_1}(\gamma|r, \theta) \, dy \right) f_{\theta_1, \Theta_1}(r, \theta) \, dr \, d\theta. \tag{18} \]
Clearly, the CDF of the best relay’s index is then given by
\[ F_{\theta_1, \Theta_1}(\eta, J) = \sum_{k=1}^J \Pr[Y = k], \quad J = 1, 2, \ldots. \tag{19} \]

In order to obtain the joint PDF of the best relay’s index and \( r_b \), that is, \( f_{\theta_1, \Theta_1}(\cdot, \cdot) \), and the joint PDF of the best relay’s index and \( \Theta_1 \), that is, \( f_{\theta_1, \Theta_1}(\cdot, \cdot) \), we first note from (18) that
\[ f_{\theta_1, \Theta_1}(r, \Theta_1) = \int_0^\infty P_1 P_0 f_{T_1}(\gamma|r, \Theta_1) f_{\theta_1, \Theta_1}(r, \Theta_1) \, dr, \]
\[ 0 \leq r, \quad 0 \leq \Theta_1 \leq 2\pi, \quad J = 1, 2, \ldots. \tag{20} \]
Taking the integral of (20) with respect to \( \Theta_1 \), we obtain
\[ f_{\theta_1, \Theta_1}(r, J) = \int_0^\infty \int_0^{2\pi} P_1 P_0 f_{T_1}(\gamma|r, \Theta_1) f_{\theta_1, \Theta_1}(r, \Theta_1) \, dy \, d\Theta_1, \]
\[ 0 \leq r, \quad J = 1, 2, \ldots. \tag{21} \]
Similarly, taking the integral of (20) with respect to \( r \), we arrive at
\[ f_{\theta_1, \Theta_1}(\Theta_1, J) = \int_0^\infty \int_0^{2\pi} P_1 P_0 f_{T_1}(\gamma|r, \Theta_1) f_{\theta_1, \Theta_1}(r, \Theta_1) \, dr \, d\Theta_1, \]
\[ 0 \leq \Theta_1 \leq 2\pi, \quad J = 1, 2, \ldots. \tag{22} \]
For the standard path loss model with \( \alpha = 2 \), we can simplify (18) and (21) to
\[ \Pr[Y] = \int_0^\infty \int_0^{2\pi} \left( 1 - e^{-2\gamma r^2} \right) \left( 1 - e^{-2\gamma r^2} \right)^{J-1} \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \]
\[ \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \times \int_0^\infty f_{\theta_1, \Theta_1}(r, \Theta_1) \, dr \, dr. \tag{23} \]
Using (8), Equation (23) can be written as
\[ \Pr[Y] = \int_0^\infty \int_0^{2\pi} \left( \int_0^\infty \frac{4\lambda \pi r^2 \gamma (r^2 + d^2)}{(J-1)!} \frac{\lambda \pi r^2 - \lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right)^{J-1} \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \, dr \, dr \]
\[ \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \, dy \, dr \tag{24} \]
and
\[ f_{\theta_1, \Theta_1}(r, \Theta_1) = \int_0^{2\pi} \int_0^{2\pi} \left( 1 - e^{-2\gamma r^2} \right) \left( 1 - e^{-2\gamma r^2} \right)^{J-1} \]
\[ \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \times \exp \left( -\frac{\lambda \pi e^{-2\gamma (r^2 + d^2)}}{2\gamma} \right) \, dy \, dr. \tag{25} \]
In this section, we derive the joint PDF of $r$, respectively. Note that in this case

$$f_{r,b}(r,\theta) = \frac{\lambda r^2 (r^2 + d^2)}{(j - 1)!} \left( \lambda r^2 - \frac{\lambda r e^{-2\gamma r^2}}{2\gamma} \right)^{j-1} \times \exp \left( \frac{-\lambda r e^{-2\gamma (r^2 + d^2)}}{2\gamma} - \frac{\lambda r^2 - 2\gamma (r^2 + d^2)}{2\gamma} \right) dy, \quad 0 \leq r, \quad (26)$$

respectively. Note that in this case

$$f_{\theta,b}(\theta,\gamma) = \frac{1}{2\pi} P_{J} [\gamma], \quad 0 \leq \theta \leq 2\pi. \quad (27)$$

Using (24), we can write (27) as

$$f_{\theta,b}(\theta,\gamma) = \frac{1}{2\pi} \int_{\theta}^{\pi+\theta} P_{J} [\gamma] \int_{0}^{\infty} 2\lambda r (r^2 + d^2) \left( \lambda r^2 - \frac{\lambda r e^{-2\gamma r^2}}{2\gamma} \right)^{j-1} \times \exp \left( \frac{-\lambda r e^{-2\gamma (r^2 + d^2)}}{2\gamma} - \frac{\lambda r^2 - 2\gamma (r^2 + d^2)}{2\gamma} \right) dy \ dr. \quad (28)$$

Therefore, using Equations (29) and (31)–(33), we can write the joint PDF of $\eta_b$ as

$$f_{\eta,b}(\eta, \theta) = \int_{0}^{\infty} \lambda r \exp \left[ -\lambda \int_{0}^{\infty} 2\pi \left(1 - F_{\gamma}(\gamma)\right) d\gamma \right] \times F_{\gamma}(\gamma | \eta, \theta) dy, \quad 0 \leq r, \quad 0 \leq \theta \leq 2\pi. \quad (34)$$

The marginal PDF of $\theta_b$ can then be obtained as

$$f_{\theta,b}(\theta) = \int_{0}^{\infty} \int_{0}^{\infty} \lambda r \exp \left[ -\lambda \int_{0}^{\infty} 2\pi \left(1 - F_{\gamma}(\gamma)\right) d\gamma \right] \times F_{\gamma}(\gamma | \eta, \theta) d\gamma \ dr, \quad 0 \leq \theta \leq 2\pi. \quad (35)$$

Using the properties of PPPs, we can express the probability on the right of (30) as the probability that exactly one relay falls in the annular sector defined by the intervals $[r, r + dr]$ and $[\theta, \theta + d\theta]$, and that for any arbitrary relay (e.g. $R_J$) we have $\Gamma_j < \gamma$.

Again, owing to the properties of PPPs, the event that one relay falls within the intervals $[r, r + dr]$ and $[\theta, \theta + d\theta]$ and the event that $\Gamma_j < \gamma$ are mutually independent. Therefore, we can write

$$f_{\eta,b}(\eta, \theta) = \lambda r dr d\theta \Pr \left\{ \bigcap_{\eta, \theta \in \Lambda} \left\{ \Gamma_j < \gamma \right\} \right\}, \quad (31)$$

where $\Lambda$ denotes the set of all relays except the best relay and the last probability in (31) can be written as

$$\Pr \left\{ \bigcap_{\eta, \theta \in \Lambda} \left\{ \Gamma_j < \gamma \right\} \right\} = \mathbb{E} \left\{ \prod_{\eta, \theta \in \Lambda} F_{\gamma} \right\}. \quad (32)$$

Note that $\mathbb{E}\{\cdot\}$ in (32) is over all realisations of $\Lambda$. Now, using Campbell's theorem [23, Equation (3.35)] the expectation on the right side of (32) is obtained as

$$\mathbb{E} \left\{ \prod_{\eta, \theta \in \Lambda} F_{\gamma} \right\} = \exp \left[ -\lambda \int_{0}^{\infty} \int_{0}^{2\pi} 1 - F_{\gamma}(\gamma) \ d\gamma \ d\phi \right] = \exp \left[ -\lambda \int_{0}^{\infty} \int_{0}^{2\pi} e^{-\gamma \varphi(\gamma, \phi)} \ d\gamma \ d\phi \right]. \quad (33)$$

Therefore, using Equations (29) and (31)–(33), we can write the joint PDF of $\eta_b$ and $\theta_b$ as

$$f_{\eta,b}(\eta, \theta) = \int_{0}^{\infty} \lambda r \exp \left[ -\lambda \int_{0}^{\infty} 2\pi \left(1 - F_{\gamma}(\gamma)\right) d\gamma \right] \times F_{\gamma}(\gamma | \eta, \theta) dy, \quad 0 \leq r, \quad 0 \leq \theta \leq 2\pi. \quad (34)$$

The marginal PDF of $\theta_b$ can then be obtained as

$$f_{\theta,b}(\theta) = \int_{0}^{\infty} \int_{0}^{\infty} \lambda r \exp \left[ -\lambda \int_{0}^{\infty} 2\pi \left(1 - F_{\gamma}(\gamma)\right) d\gamma \right] \times F_{\gamma}(\gamma | \eta, \theta) d\gamma \ dr, \quad 0 \leq \theta \leq 2\pi. \quad (35)$$
For the standard path loss model, we can use the approximation given in (A8) for \( S_1(\varphi, \theta) \) to simplify Equation (35) as

\[
f_{\delta, \theta}(\theta) \approx \int_0^\infty \lambda r \exp \left[ -\lambda \int_0^\infty 2\pi r e^{-\frac{\gamma}{2}} d\varphi \right] \times f_{\gamma | r, \theta}(\gamma) d\gamma dr, \quad 0 \leq \theta \leq 2\pi.
\]

Similarly, the marginal PDF of \( n_j \) can be evaluated from (34) as

\[
f_{n_j}(r) = \int_0^{2\pi} \int_0^\infty \lambda r \exp \left[ -\lambda \int_0^\infty 2\pi r e^{-\frac{\gamma}{2}} d\varphi \right] \times f_{\gamma | r, \theta}(\gamma) d\gamma d\theta, \quad 0 \leq r.
\]

Again, for the standard path loss model, the approximation of \( S_1(\varphi, \theta) \) (given in (A8)) can be used to obtain

\[
f_{n, \theta}(r, \theta) = \int_0^\infty 2\pi r \exp \left[ -\frac{\lambda}{2\pi} e^{-2\pi r^2/2} \right] \times (r^2 + d^2) e^{-2\gamma(r^2 + d^2)} d\gamma
\]

and

\[
f_{n}(r) = \int_0^\infty 4\pi r \exp \left[ -\frac{\lambda}{2\pi} e^{-2\pi r^2/2} \right] (r^2 + d^2) e^{-2\gamma(r^2 + d^2)} d\gamma,
\]

respectively. In addition, from Equations (39) and (40) it is easy to verify that in this case, \( f_{\delta, \theta}(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta \leq 2\pi \).

### 4.2 Application

In some fifth-generation (5G) wireless networks with high mobility users, relay selection procedure must be performed fast enough to avoid any disconnections. This procedure, however, can be quite complicated as the channel impulse responses between \( S_1 \) and \( S_2 \) and the potential relays should be estimated. When the number of potential relays is large, this procedure imposes significant computational overhead to the system.

A straightforward method to reduce the aforementioned overhead and make the relay selection procedure faster is to limit the relay selection region, that is, the region that is highly likely to contain the best relay. As will be seen in Section 6, our statistical analysis can significantly limit the surface area of the relay selection region and thus, speed up the relay selection procedure.

### 5 Statistical Analysis of Signal-to-Noise-Ratio

In this section, we derive analytical expressions for CCDF of SNR experienced by \( S_1 \) due to the signal transmitted by the \( j \)th best relay, that is, \( \Gamma_{(j)}^{S_1} \). We also obtain a tight upper bound for the average of \( \Gamma_{(j)}^{S_1} \).

#### 5.1 Mathematical derivation

It is shown in [10] that \( \Gamma_{(j)}^{S_1} \) is given by

\[
\Gamma_{(j)}^{S_1} = \frac{\sum_{j=1}^{K} \gamma_j}{\sum_{j=1}^{K} \gamma_j},
\]

where \( \gamma_j \) is the transmitted power by the \( j \)th best relay. We assume that \( P_k \) is same for all relays. When \( P_k = P_k \), one can use Equation (1) to write (41) as

\[
\Gamma_{(j)}^{S_1} = \frac{\Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}}{1 + \Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}}.
\]

Since the distribution of \( \Gamma_{(j)}^{S_1} \) is often intractable, we use a tight upper bound for \( \Gamma_{(j)}^{S_1} \) as

\[
\Gamma_{(j)}^{S_1} \leq \Gamma_{(j)}^{S_1} \triangleq \frac{\Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}}{2 \Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}}.
\]

Using the aforementioned upper bound for \( \Gamma_{(j)}^{S_1} \) and the total probability theorem, one can obtain an upper bound for the CCDF of \( \Gamma_{(j)}^{S_1} \) as

\[
\Pr\{\Gamma_{(j)}^{S_1} > \gamma_{th}\} \leq \Pr\left\{\frac{\Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}}{2 \Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}} > \gamma_{th} \right\}
\]

\[
= \int_0^\infty \int_0^{2\pi} \Pr\left\{\frac{\Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}}{2 \Gamma_{1,j}^{S_1} + \Gamma_{2,j}^{S_1}} > \gamma_{th} \right\} f_{r_j, \theta}(r, \theta) d\theta dr.
\]
The first term in the integrand of (44) can be written as [24, Equation (5.2.3)]

\[
\Pr\left\{ \frac{\Gamma_{1,j}\Gamma_{2,j}}{2\Gamma_{1,j} + \Gamma_{2,j}} > y_{th} \right| r_j = r, \theta_j = \theta \} = y_{th} \sqrt{\frac{8}{y_+ y_-}} e^{-\frac{(\frac{y_{th} + y}{r_+ + r_+})}{\frac{y_{th}}{r_+} + \frac{y}{r_-}}} K_1\left(y_{th} \sqrt{\frac{8}{y_+ y_-}} \right) .
\]

(45)

where \(K_1(\cdot)\) is the first-order modified Bessel function of the second kind. In addition, \(y_+\) and \(y_-\) are defined as

\[
y_+ = \frac{G_0 P_j S}{N_0} \mathcal{X}_i(\ell_+ (r, \theta))
\]

(46a)

\[
y_- = \frac{G_0 P_j S}{N_0} \mathcal{X}_i(\ell_- (r, \theta)).
\]

(46b)

Now using Equation (45) one can write Equation (44) as

\[
\Pr\left\{ \frac{\Gamma_{1,j}\Gamma_{2,j}}{2\Gamma_{1,j} + \Gamma_{2,j}} > y_{th} \right\} = \int_0^\infty \int_0^{2\pi} \int_0^\infty \sqrt{\frac{8}{y_+ y_-}} e^{-\frac{(y_{th} + y_+ + y_+)}{y_+ y_-}} K_1\left(y_{th} \sqrt{\frac{8}{y_+ y_-}} \right) \lambda r \left(\frac{\lambda \pi r^2}{(j-1)!}\right)^{j-1} e^{-\lambda \pi r^2} d\theta dr dz
\]

(47)

Using tail-sum formula for expectation the expected value of \(\Gamma^{(j)}_{\Phi_1}\) is given by [25]

\[
E\{\Gamma^{(j)}_{\Phi_1}\} = \int_0^\infty \Pr\{\Gamma^{(j)}_{\Phi_1} > z\} dz \leq \int_0^\infty \Pr\left\{ \frac{\Gamma_{1,j}\Gamma_{2,j}}{2\Gamma_{1,j} + \Gamma_{2,j}} > z \right\} dz.
\]

(48)

Using Equation (47), Equation (48) can be written as

\[
E\{\Gamma^{(j)}_{\Phi_1}\} \leq \int_0^\infty \int_0^{2\pi} \int_0^\infty \sqrt{\frac{8}{y_+ y_-}} e^{-\frac{(y_{th} + y_+ + y_+)}{y_+ y_-}} K_1\left(y_{th} \sqrt{\frac{8}{y_+ y_-}} \right) \lambda r \left(\frac{\lambda \pi r^2}{(j-1)!}\right)^{j-1} e^{-\lambda \pi r^2} d\theta dr dz.
\]

(49)

5.2 | Application

Suppose that two users, namely, \(U_1\) and \(U_2\), communicate with each other through the best relay in a wireless communica-

tion network. Also suppose that for some reason the best relay becomes out of service. Thus, it is reasonable to assume that \(U_1\) and \(U_2\) should now connect to the second best relay in order to continue their communication. Expectedly, when these users connect to the second best relay their received SNRs are decreased. Thus, in order to maintain the quality of service for these users, the transmission power of the relay should be increased. This increase in the transmission power is referred to as power penalty in the sequel. For example, for standard path loss model with \(\alpha = 4, \lambda = 1\) and \(\gamma_{th} = 3\) dB, the received SNR at \(U_1\) is greater than the threshold with probability 0.95, provided that \(P_k/N_0 \geq 34.15\) dB at the best relay. Now, if the users switch to the second best relay, \(P_k\) should be increased by approximately 1 dBW in order to ensure that the received SNR at \(U_2\) is greater than the threshold with probability 0.95.

6 | NUMERICAL RESULTS

In this section, we verify our analytical results presented in Sections 3 and 4 by means of computer simulation. Simulation results are obtained using 100 million independent realisations of the system. Table 1 shows the values of \(\lambda, \alpha\) and \(d\) that are used in Figures 2–8. The transmission channels between the sources and relays are assumed to follow a Rayleigh fading model, and the best relay is chosen according to a max–min criterion.

Figure 2 shows the CDF of the index of the best relay (Equation (19)) for 3-slope path loss model with \([\alpha_1, \alpha_2, \alpha_3] = [2, 4, 6]\), \([R_1, R_2] = [4, 8]\) and selected values of \(\lambda\). As seen in this figure, as \(\lambda\) increases, the slope of the CDF decreases. As a result, relays with larger index (viz., relays that are farther to the origin) are less likely to be chosen as the best relay. Moreover, the smaller the value of \(\lambda\), the faster the CDF grows with the relay index. Therefore, the probability that relays are farther to the origin chosen as the best relay becomes small. For
instance, when $\lambda = 1$ the probability that the best relay is among the first 10 relays approximately equals 0.76. However, for $\lambda = 5$ this probability is almost equal to 0.36. Also observe that in 3-slope path loss model, the probability that the best relay to be close to the origin is greater than that of the standard path loss model with $\alpha = 2$ and $\lambda = 1$. This is because in 3-slope path loss model, relays with larger index experience larger path loss. Thus, the signal attenuation due to path loss is quite pronounced for the relays that are far from the origin. The Figure 2 also shows that the simulation results match the analytical results quite well.

The joint PDF of index of the best relay and its distance to the origin, $f_{n,J}(r,J)$ (Equation (21)), for $\lambda = 1$, 2-slope path loss model with $[\alpha_1, \alpha_2] = [4, 6], R_1 = 2.5$, and selected values of $J$ is illustrated in Figure 3. As seen in this figure, the simulation results verify our analytical results in all the examined scenarios. Observe that for any fixed positive integer $J$, the area below $f_{n,J}(r,J)$ equals $p_J[J]$. Figure 3 also shows that when $J$ increases, $f_{n,J}(r,J)/p_J[J]$ resembles a Gaussian PDF whose mean and variance increase as $J$ increases. Note that for each value of $J$, there is a range of $r$-values for which the $J$th relay has the highest likelihood to be the best relay. This range for $J = 1, 2$ and 3 is $[0, 0.65], [0.65, 0.9]$ and $[0.9, 1.1]$, respectively. Note also that for all the examined values of $J$, the best relay is quite unlikely to be on, or very close to, the reference point. Similar to Figure 2, in 2-slope path loss model, the probability that the best relay is close to the origin is larger than that of the standard path loss model with $\alpha = 4$ and $\lambda = 1$.

The joint PDF of the best relay’s index and its phase, $f_{\theta,J}(\theta,J)$, as a function of $\theta$ is shown in Figure 4, for $\lambda = 1$, the standard path loss model with $\alpha = 4$ and different values of $J$. Similar to what we observed for $f_{n,J}(r,J)$, as $J$ increases, the area below $f_{\theta,J}(\theta,J)$ (i.e. indeed, $p_J[J]$) decreases. Note, importantly, that when $\theta$ is around 90° and 270° the joint PDF has a larger value for all the examined values of $J$. This means that the best relay is more likely to be on, or close to the perpendicular bisector of $\mathbb{S}_1-\mathbb{S}_2$ line segment. On the other hand, when the relays are on or close to the line passing $\mathbb{S}_1$ and $\mathbb{S}_2$, they are less likely to be the best relay. Note that when $J$ increases, $f_{\theta,J}(\theta,J)$ becomes more concentrated.
around $\theta = 90^\circ$ and $270^\circ$. Hence, relays with larger index (i.e. relays that are farther from the origin) are more likely to be the best relay when their corresponding phase, $\hat{\theta}$, is close to $90^\circ$ or $270^\circ$.

The PDF of $\theta_b$ given by Equation (35) is shown in Figure 5 for $\lambda = 1$, Rayleigh fading channel and selected values of the path loss exponent in the standard model, $\alpha$. As shown in this figure, for all the examined values of $\alpha$, the PDF has its maximum at $\theta = 90^\circ$ and $270^\circ$, and its minimum occurs at $\theta = 0^\circ$ and $180^\circ$. In addition, when $\alpha$ grows from 3 to 5 the PDF becomes more concentrated about $\theta = 90^\circ$ and $270^\circ$. This reflects the fact that when path loss becomes more severe the best relay is expected to be on, or close to the perpendicular bisector of $S_1 - S_2$ line segment. Also shown in this figure is fact that the larger $\alpha$, the smaller the probability that the best relay lies on, or very close to $S_1 - S_2$ line segment.

Figure 6 shows the joint PDF of $r_b$ and $\theta_b$ as contour plots for $\lambda = 1$ and selected values of the path loss exponent in the standard model. As seen in (39), when $\alpha$ equals 2, $f_{r_b,\theta_b}(r,\theta)$ does not depend on $\theta$. As a result, all contours in Figure 6(a) are in the form of circles centred at the origin. Observe that in Figure 6(a), the region that is very likely to include the best relay, referred to as relay-abundant (RA) region in the following, is an annulus centred at O. In contrast, when $\alpha = 4$ or 6 the RA region is no longer an annulus and is mostly concentrated around the perpendicular bisector of $S_1 - S_2$. These figures highlight the important fact that the best relay for $\alpha = 2$, 4 or 6 is very unlikely to be in on or very close to the reference point. Also note that these figures can be readily used to bound the RA region mentioned above. In fact, when $\alpha$ grows large, this region becomes smaller albeit its centre remains almost unchanged with respect to the pole. This is important as it can significantly decrease the computational overhead associated with the max–min criterion for selecting the best relay.

Figure 7 shows the joint PDF of $r_b$ and $\theta_b$ as contour plots for the standard path loss model with $\alpha = 3$ and selected values of $\lambda$. Clearly, for all the examined values of $\lambda$ the best relay is quite unlikely to be in on or very close to the origin. Also observed in Figure 7(a) through (c), is the fact that the larger the value of $\lambda$, the smaller and closer to the RA region. Again this fact is quite helpful as it can significantly restrict the search region for the best relay particularly when $\lambda$ is large.

Figure 8 shows the required $P_R/N_0$ to guarantee $\Pr[\Gamma^{(j)}_{S_1} > \gamma_{th}] = 0.95$ as a function of the best relay’s index for $\gamma_{th} = 3$.
FIGURE 7 The joint PDF of $r_b$ and $\theta_b$ as a contour plot in the $x$-$y$ plane, for $\alpha = 3$, and (a) $\lambda = \frac{1}{3}$, (b) $\lambda = 1$, and (c) $\lambda = 3$ dB. $G_0 = 1$ dB and selected values of $\alpha$ and $\lambda$. As shown in this figure, when $\alpha$ increases, the SNR penalty for switching from $j$th best relay to the $k$th best relay for $k > j \geq 1$ is increased.

For example, for $\alpha = 4$ and $\lambda = 1/3$, the SNR penalty for switching from the first best relay to the fifth best relay is 5.55 dB while for $\lambda = 3$ this SNR penalty is 1.35 dB.

7 | CONCLUSION

This paper presents a detailed statistical analysis of the best relay’s index and location in a two-way relay network with spatially random relays. All transmission channels were assumed to be subject to multi-slope path loss and Rayleigh fading. Our analysis showed that the best relay in a multiple access broadcast two-way relay network is quite unlikely to be located on or be very close to the midpoint between the two sources. Indeed, in all the examined scenarios except for the case where $\alpha = 2$, we observed that the best relay is highly likely to be on or close to the perpendicular bisector of $S_1$–$S_2$ line segment but offset from the midpoint between the two sources. The amount of this offset depends on the intensity of the nodes in PPP, and grows large when this parameter becomes small. We defined a RA region and observed that this region becomes smaller when the path loss exponent or relay density increases. Using these results, we can restrict the search region for finding the best relay and, thus, reduce the computational complexity which arises when estimating the channels between a potential relay and the source nodes.

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APPENDIX A
A.1 Proof of Equation (10a)

In order to obtain the conditional PDF of $\Gamma_j$, that is, $f_{\Gamma_j}(\gamma| r, \theta)$, we first obtain the conditional CDF as

$$f_{\Gamma_j}(\gamma| r, \theta) = \Pr(\Gamma_j \leq \gamma | r, \theta)$$

$$= \Pr(\min\{X_{\geq}(r, \theta), X_{\leq}(r, \theta)\} \Omega_1, J_f, X_{\geq}(r, \theta) \Omega_2) \leq \gamma | r, \theta)$$

$$= F_{\Omega_1}(\frac{\gamma}{X_{\geq}(r, \theta)}) + F_{\Omega_2}(\frac{\gamma}{X_{\leq}(r, \theta)})$$

$$- F_{\Omega_1}(\frac{\gamma}{X_{\geq}(r, \theta)}) F_{\Omega_2}(\frac{\gamma}{X_{\leq}(r, \theta)}).$$ (A1)

Recalling that the transmission channels are modeled by a Rayleigh fading model, we can simplify (A1) as

$$f_{\Gamma_j}(\gamma| r, \theta) = 2 - e^{-\frac{\gamma}{\Omega_1 + e^{J_f}} - e^{-\frac{\gamma}{\Omega_2 + e^{J_f}}}}$$

$$- (1 - e^{-\frac{\gamma}{\Omega_1 + e^{J_f}}})(1 - e^{-\frac{\gamma}{\Omega_2 + e^{J_f}}})$$

$$= 1 - e^{-\frac{\gamma}{\Omega_1 + e^{J_f}}} \left(1 + e^{-\frac{\gamma}{\Omega_2 + e^{J_f}}} \right).$$ (A2)

Taking the derivative of (A2) with respect to $\gamma$ we obtain

$$f_{\Gamma_j}(\gamma| r, \theta) = \left(1 - e^{-\frac{\gamma}{\Omega_1 + e^{J_f}}} \right) \left(1 + e^{-\frac{\gamma}{\Omega_2 + e^{J_f}}} \right)$$

$$\times e^{-\frac{\gamma}{\Omega_1 + e^{J_f}}} \left(1 + e^{-\frac{\gamma}{\Omega_2 + e^{J_f}}} \right).$$ (A3)

We now obtain two accurate approximations for $\Omega_1(r, \theta) = \frac{1}{\Omega_1 + e^{J_f}}$ for the standard path loss model with $\alpha = 4$ which are used in Equations (13), (14), (35) and (37). To this end, we first note from Equation (11) that

$$\Omega_1(r, \theta) = \left(r^2 + d^2\right)^{\frac{2}{\alpha}}$$

$$\times \left[1 + 2n d \cos \theta \left(r^2 + d^2\right) + \frac{2 d \cos \theta}{r^2 + d^2} \right].$$ (A4)

Using the fact that $0 \leq (r - d)^2$, one concludes that $2n d \cos \theta \leq 2rd \leq r^2 + d^2$ or equivalently

$$\frac{2r d \cos \theta}{r^2 + d^2} \leq 1.$$ (A5)

For an arbitrary real number $\eta$ such that $1$, the Taylor series expansions of $(1 + \eta)^{\frac{2}{\alpha}}$ and $(1 - \eta^{\frac{2}{\alpha}}$ around $\eta = 0$ can be written as

$$(1 \pm \eta)^{\frac{2}{\alpha}} = 1 \pm \frac{\alpha}{2} \eta + \frac{\alpha(\alpha - 2)}{4 \times 2!} \eta^2 \pm \frac{\alpha(\alpha - 2)(\alpha - 4)}{8 \times 3!} \eta^3 + \ldots.$$ (A6)
Now, using the first four terms of the expressions on the right of (A6) with $\eta = 2rd\cos\theta/(r^2 + d^2)$ we get

$$\mathcal{L}_1(r, \theta) \approx (r^2 + d^2)^\frac{\eta}{2} \left[ 2 + \frac{\alpha(\alpha - 2)}{4} \left( \frac{2rd\cos\theta}{r^2 + d^2} \right)^2 \right]. \quad (A7)$$

Using straightforward algebra, one can show that the maximum and minimum of (A4) are obtained at $\theta = 0$ and $\theta = \pi/2$, and are equal to $2(r^2 + d^2)^\frac{\eta}{2}$ and $|r - d|^\alpha + |r + d|^\alpha$, respectively. However, thorough inspection of the expression on the right of (A7) reveals that the maximum of this expression is not equal to $|r - d|^\alpha + |r + d|^\alpha$, albeit its minimum equals that of $\mathcal{L}_1(r, \theta)$. Therefore, the approximation on the right of (A7) can be slightly modified to account for the aforementioned mismatch between the maxima. To this end, we note that the expression on the right of (A7) is a first-order polynomial in $\cos(2\theta)$. This suggests that another approximation for $\mathcal{L}_1(r, \theta)$ can be obtained as

$$\mathcal{L}_1(r, \theta) \approx a_0 + a_1 \cos(2\theta), \quad (A8a)$$

where $a_0$ and $a_1$ are determined from the maximum and minimum of $\mathcal{L}_1(r, \theta)$ as

$$a_0 = \frac{|r - d|^\alpha + |r + d|^\alpha}{2} + \sqrt{(r^2 + d^2)^\alpha} \quad (A8b)$$

$$a_1 = \frac{|r - d|^\alpha + |r + d|^\alpha}{2} - \sqrt{(r^2 + d^2)^\alpha}. \quad (A8c)$$