EMBEDDING ASYMPTOTICALLY EXPANSIVE SYSTEMS

DAVID BURGUET

Abstract. We prove a Krieger like embedding theorem for asymptotically expansive systems with the small boundary property. We show that such a system \((X, T)\) embeds in the \(K\)-full shift with \(h_{\text{top}}(T) < \log K\) and \(\sharp \text{Per}_n(X, T) \leq \sharp \text{Per}_n(\{1, \ldots, K\}, \sigma)\) for any integer \(n\). The embedding is in general not continuous (unless the system is expansive and \(X\) is zero-dimensional) but the induced map on the set of invariant measures is a topological embedding. It is shown that this property implies asymptotical expansiveness. We prove also that the inverse of the embedding map may be continuously extended to a faithful principal symbolic extension.

1. Introduction

Symbolic dynamics play since the pioneer work of Hadamard \cite{15} a crucial role in the theory of dynamical systems. Here we investigate the problem of embedding a dynamical system in a shift with a finite alphabet.

For measure preserving ergodic systems the celebrated Krieger generator theorem gives a complete answer in terms of measure theoretical entropy:

Theorem 1.1. \cite{11} Let \(T\) be an ergodic automorphism of a standard probability space \((X, \mathcal{U}, \mu)\) then \(T\) embeds measure theoretically in the shift with \(K\) letters, i.e. there exists an isomorphism \(\psi : X \to \{1, \ldots, K\}^\mathbb{Z}\) with \(\sigma \circ \psi = \psi \circ T\) if and only if:

either \(h_\mu(T) < \log K\),

or \(h_\mu(T) = \log K\) and \(T\) is Bernoulli.

For topological dynamical systems (i.e. continuous maps on a compact metrizable space) this question was also solved by Krieger. The obstructions are now of three kinds: topological (\(X\) is zero-dimensional), set theoretical (the number of \(n\)-periodic points of \((X, T)\) is less than or equal to the number of \(n\)-periodic points of the shift) and dynamical ((\(X, T)\) is expansive and its topological entropy is less than the entropy of the shift). For any integer \(n\) we let \(\text{Per}_n(X, T)\) be the set of periodic points of \((X, T)\) with least period equal to \(n\).

Theorem 1.2. \cite{12} Let \((X, T)\) be a topological dynamical system, then \((X, T)\) embeds topologically in the shift \(\sigma\) with \(K\) letters, i.e. there exists a continuous injective map \(\psi : X \to \{1, \ldots, K\}^\mathbb{Z}\) with \(\sigma \circ \psi = \psi \circ T\) if and only if:

either we have

- \(X\) is zero-dimensional,
- \(\sharp \text{Per}_n(X, T) \leq \sharp \text{Per}_n(\{1, \ldots, K\}^\mathbb{Z}, \sigma)\) for any integer \(n\),
- \(h_{\text{top}}(T) < \log K\),
- \(T\) is expansive,
or \((X,T)\) is topologically conjugated to \((\{1,\ldots,K\}^\mathbb{Z},\sigma)\) (and thus \(h_{\text{top}}(T) = \log K\)).

Krieger only deals with the case \(h_{\text{top}}(T) < \log K\), however the case of equality may be proved as in Theorem 1.1 (See the Appendix).

To embed topologically more general systems (in particular of arbitrarily topological dimension) into shift spaces one has to consider the shift with alphabet in \([0,1]^k\), \(k \in \mathbb{N} \cup \{\mathbb{N}\}\). The existence of an embedding in the shift over \([0,1]^k\) with finite \(k\) is one important question in the theory of mean dimension, see e.g. [19] (for \(k = \mathbb{N}\) it always exists by considering an embedding of \(X\) into the Hilbert cube \([0,1]^{\mathbb{N}}\)). If we want still work with the shift with a finite alphabet we have to consider a larger class of embedding maps. Recently Hochman gives a Borel version of Krieger theorem for Borel dynamical systems, where the embedding is a Borel map. In this setting, after forgetting periodic points, the only constraint is the supremum of the entropy of Borel ergodic invariant probability measures:

**Theorem 1.3.** [16][5] Let \(T\) be a measurable automorphism of a standard Borel space \((X,\mathcal{U})\) then \((X,T)\) embeds almost Borel in the shift with \(K\) letters, i.e. there exists a Borel subset \(E\) of \(X\) of full measure with respect to any ergodic \(T\)-invariant aperiodic measure and a Borel injective map \(\psi : E \to \{1, \ldots, K\}^{\mathbb{Z}}\) with \(\sigma \circ \psi = \psi \circ T\), if and only if:

- either we have \(h_\mu(T) < \log K\) for any Borel ergodic \(T\)-invariant probability measure \(\mu\),
- or \((X,T)\) admits a unique measure \(\mu\) of maximal entropy, \(\mu\) is Bernoulli and \(h_\mu(T) = \log K\).

In the present paper we give a version of Krieger theorem for asymptotically expansive topological dynamical system with the small boundary property. A dynamical system is asymptotically expansive when entropy and periodic points grows subexponentially at arbitrarily small scales. The small boundary property is satisfied by many aperiodic systems, i.e. dynamical systems without periodic points. In particular any aperiodic system in finite dimension satisfies this property. Precise definitions and further known facts related to these two properties are given in Section 2.

Our embedding is not continuous but it is in some sense precipised later a limit of essentially continuous functions. However the map induced by the embedding on the set of invariant measures is continuous. Conversely we prove that a topological dynamical system embedding in such a way in a shift with a finite alphabet is asymptotically expansive.

Finally we relate for asymptotically expansive the present Krieger embedding theorem with the theory of symbolic extensions. More precisely the inverse of the Krieger embedding defines a principal faithful symbolic extension. In this way we build a bridge between the theory of symbolic extensions developed by M. Boyle and T. Downarowicz [3][2][7] and Krieger embedding like problems, for asymptotically expansive systems. In these previous works the symbolic extension is built as the intersection of decreasing subshifts whereas we build here the extension as the
In Section 2 we introduce the main notions. We then state our main theorem. Section 4 is devoted to the proof of our embedding Theorem for zero-dimensional asymptotically expansive systems. In Section 5 we deal with the general case by reduction to the zero-dimensional one. Finally we will see that systems embeddable in our sense are asymptotically expansive. In this way we get a new characterization of asymptotical expansiveness.

2. Background

We will always consider topological dynamical systems \((X, T)\), i.e. \(X\) is a compact metrizable space and \(T : X \to X\) is an invertible continuous map. We also always assume that \((X, T)\) has finite topological entropy. We denote by \(\mathcal{M}(X, T)\) the set of \(T\)-invariant Borel probability measures and we endow this set with the weak \(\ast\)-topology.

2.1. Small boundary property and essential partitions. We recall here some facts about the small boundary property developed in [19]. A Borel subset of \(X\) is called a null (resp. full) set if it has null (resp. full) measure for any \(T\)-invariant ergodic Borel probability measure.

A subset of \(X\) is said to have a small boundary when its boundary is a null set. A partition of \(X\) is called essential when any of its element have a small boundary. For a zero-dimensional compact space \(Y\), a Borel map \(\psi : X \to Y\) is said to be essentially continuous if there exists a basis of clopen sets \(\mathcal{B}\) of \(Y\) such that for any \(B \in \mathcal{B}\) the set \(\psi^{-1}(B)\) has a small boundary. Observe that it easily implies that the map induced by \(\psi\) on \(\mathcal{M}(X, T)\) is continuous. We also say finally that \((X, T)\) has the small boundary property if \(X\) admits finite essential partitions with arbitrarily small diameter.

This property has been investigated in [18], [19], [14] and used in the theory of symbolic extensions and entropy structures developed in [2], [9]. In particular any dynamical system of finite topological entropy with an aperiodic minimal factor [19] or any finite dimensional aperiodic system [18] satisfies the small boundary property. In fact in Theorem 3.3 of [18] it is proven that any finite dimensional system admits a basis of neighborhoods whose boundaries have zero measure for any aperiodic invariant measures. When \(\text{Per}(X, T)\) is a zero-dimensional subset of \(X\) one may easily arrange the construction of [18] to ensure that any element of the basis has a small boundary. Thus any finite dimensional system with a zero-dimensional set of periodic points has the small boundary property. This result may also follow from Lemma 3.7 of [13]. In [21] it is proven that \(C^r\) dynamical systems on a compact manifold with \(r \geq 1\) have generically the small boundary property (the proof is independent, but it may be deduced from the previous fact by Kupka-Smale Theorem).

2.2. Asymptotic \(h\)-expansiveness. Given two finite open covers \(\mathcal{U}\) and \(\mathcal{V}\) of \(X\), we define the topological conditional entropy \(h(\mathcal{V} \| \mathcal{U})\) of \(\mathcal{V}\) given \(\mathcal{U}\) as

\[
h(\mathcal{V} \| \mathcal{U}) := \lim_{n \to \infty} \frac{1}{n} \sup_{U^n \in \mathcal{U}^n} \log \min \{ \mathcal{F}_n \subset \mathcal{V}^n, U^n \subset \bigcup_{V^n \in \mathcal{F}_n} V^n \}.
\]
A map is said to be **asymptotically** $h$-**expansive** when for any decreasing sequence of open covers $(\mathcal{U}_k)_k$ whose diameter goes to zero we have

$$\lim_{k} \sup_{\mathcal{V}} h(\mathcal{V}|\mathcal{U}_k) = 0,$$

or equivalently when we have

$$\inf_{\mathcal{U}} \sup_{\mathcal{V}} h(\mathcal{V}|\mathcal{U}) = 0.$$

We refer to [7] for basic properties of the topological conditional entropy and different characterizations of asymptotically $h$-expansiveness.

These notions were introduced by Misiurewicz in the seventies. One important consequence of the asymptotically $h$-expansiveness property is the upper semicontinuity of the measure theoretical entropy function and thus the existence of a measure of maximal entropy. A large class of dynamical systems satisfies this property, e.g. continuous piecewise monotone interval maps [20], endomorphisms on compact groups, $C^\infty$ maps on compact manifolds [23],...

A topological extension $\pi : (Y, S) \to (X, T)$ is called **principal** when it preserves the entropy of invariant measures, i.e. $h_\nu(S) = h_\mu(T)$ for any $T$-invariant measure $\mu$ and for any $S$-invariant measure $\nu$ projecting on $\mu$. Ledrappier [17] proved that if $\pi$ is a principal extension then $T$ is asymptotically $h$-expansive if and only if the same holds for $S$.

2.3. **Asymptotic per-expansiveness.** Similarly we introduce now the new notion of asymptotically per-expansiveness. With the previous notations we let

$$\text{per}(T|\mathcal{U}) := \lim_{n} \frac{1}{n} \sup_{U_n \in \mathcal{U}} \sup_{U_n \in U^n} \log |\text{Per}_n(X, T) \cap U^n|$$

and we say $(X, T)$ is **asymptotically per-expansive** when we have

$$\lim_{k} \text{per}(T|\mathcal{U}_k) = 0,$$

or equivalently when we have

$$\inf_{\mathcal{U}} \text{per}(T|\mathcal{U}) = 0.$$

Obviously aperiodic systems and expansive systems are asymptotically per-expansive.

A topological extension is said to be **faithful** if the induced map between the sets of invariant probability measures is an homeomorphism. As any system has a principal faithful aperiodic extension (even zero-dimensional, see [6]), the factor of a asymptotically per-expansive map is not necessarily asymptotically per-expansive, even when the entropy of measures is preserved and the extension is faithful. Thus asymptotically per-expansiveness is not preserved under principal faithful extensions.

A topological extension $\pi : (Y, S) \to (X, T)$ is said to be **strongly faithful** if it is faithful and if for any integer $n$ we have $\pi(\text{Per}_n(Y, S)) = \text{Per}_n(X, T)$. In Subsection [6.3] we will show that a dynamical system with a principal strongly faithful asymptotically per-expansive extension is also asymptotically per-expansive.
We say that a topological dynamical system is **asymptotically expansive** when it is both asymptotically $h$- and $\text{per}$-expansive.

### 2.4. Symbolic extensions

A **symbolic extension** of $(X,T)$ is a topological extension by a subshift over a finite alphabet. The question of the existence of (principal, faithful) symbolic extension has led to a deep theory of entropy (we refer to [7] for an introduction to the topic). One first positive result appeared in this area is the existence of principal symbolic extensions for asymptotically $h$-expansive systems [3], [8]. More recently Serafin [22] proved that such an extension could be chosen to be faithful when $(X,T)$ is aperiodic. Here we give a new proof of these results that we relate with our Krieger like embedding theorem (Main Theorem below): the symbolic extension is just given by the inverse of the Krieger embedding.

### 3. Krieger embedding for asymptotically expansive systems

We state now our main result. For two topological dynamical systems $(X,T)$ and $(Y,S)$, a map $\phi : X \rightarrow Y$ is called **equivariant** when it semi-conjugates $T$ with $S$, i.e. $\phi \circ T = S \circ \phi$. Moreover we say $\phi : X \rightarrow Y$ is $\epsilon$-**injective** for some $\epsilon > 0$, if there exists $\delta > 0$, such that for any set $Z \subset Y$ with diameter less than $\delta$ the preimage $\phi^{-1}(Z)$ has diameter less than $\epsilon$. Finally we will denote by $\phi^* : \mathcal{M}(X,T) \rightarrow \mathcal{M}(Y,S)$ the map induced by $\phi$ on the set of probability invariant measures.

**Main Theorem.** Let $(X,T)$ be a topological dynamical system with the following properties:

- $(X,T)$ has the small boundary property,
- $\sharp\text{Per}_n(X,T) \leq \sharp\text{Per}_n(\{1,\ldots,K\}^\mathbb{Z},\sigma)$ for any integer $n$,
- $h_{\text{top}}(T) < \log K$,
- $(X,T)$ is asymptotically expansive.

Then there exists an equivariant injective Borel map $\psi : X \rightarrow \{1,\ldots,K\}^\mathbb{Z}$ such that:

- $\psi$ is a pointwise limit of a sequence of equivariant essentially continuous $\epsilon_k$-injective maps $\psi_k$, where $(\epsilon_k)_k$ is going to zero,
- the induced map $\psi^*$ is the uniform limit of $(\psi_k^*)_k$ on $\mathcal{M}(X,T)$, in particular $\psi^*$ is a topological embedding,
- $\psi^{-1}$ is uniformly continuous on $\psi(X)$ and the continuous extension $\pi$ of $\psi^{-1}$ on the closure $Y = \overline{\psi(X)}$ of $\psi(X)$ is a principal strongly faithful symbolic extension of $(X,T)$ with $\psi^* = (\pi^*)^{-1}$.

As previously discussed in Subsection 2.1 the above theorem applies to any asymptotically expansive finite dimensional system with finite topological entropy and finite exponential growth of periodic points, since in this case the set of periodic points is zero-dimensional. Observe also that the asymptotic $\text{per}$-expansiveness and the inequality $h_{\text{top}}(T) < \log K$ implies that the exponential growth of periodic point $\text{per}(T) := \limsup_n \frac{\log \sharp\text{Per}_n(X,T)}{n}$ also satisfies $\text{per}(T) < \log K$.

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1. This was extended by Downarowicz and Huczek to any asymptotically $h$-expansive systems [9]
The dynamical consequences of our statement characterizes the asymptotic expansiveness. More precisely we have as a converse of our Main Theorem:

**Proposition 3.1.** Let \((X, T)\) be a topological dynamical system. Assume one of the two following properties:

1. there exists a Borel equivariant injective map \(\psi : X \to \{1, ..., K\}^\mathbb{Z}\) such that \(\psi^*\) is a topological embedding,
2. there exists a strongly faithful principal extension \(\pi : (Y, \sigma) \to (X, T)\) with \(Y\) a closed \(\sigma\)-invariant subset of \(\{1, ..., K\}^\mathbb{Z}\).

Then \((X, T)\) satisfies the following properties:

- \(\sharp\text{Per}_n(X, T) \leq \sharp\text{Per}_n(\{1, ..., K\}^\mathbb{Z}, \sigma)\) for any integer \(n\),
- \(h_{\text{top}}(T) \leq \log K\),
- \((X, T)\) is asymptotically expansive.

Together with the Main Theorem we obtain in particular that when \(h_{\text{top}}(T) < \log K\) and \(X\) has the small boundary property then the assumptions (1) and (2) in Proposition 3.1 are equivalent.

We did not manage to deal with the general equality case, \(h_{\text{top}}(T) = \log K\), as in the Krieger like Theorems presented in the introduction.

**Question 3.1.** Let \((X, T)\) be a topological dynamical system with a Borel equivariant injective map \(\psi : X \to \{1, ..., K\}^\mathbb{Z}\) such that \(\psi^*\) is a topological embedding, and with \(h_{\text{top}}(T) = \log K\). Is \(\psi^*\) in fact an homeomorphism?

Observe that, when replacing in the above Question the condition on the topological entropy \(h_{\text{top}}(T) = \log K\) by the following condition on periodic points

\[ \sharp\text{Per}_n(X, T) = \sharp\text{Per}_n(\{1, ..., K\}^\mathbb{Z}, \sigma) \] for any integer \(n\)

then \(\psi^*\) is an homeomorphism as periodic measures are dense in the set of invariant measures of the full shift.

In Krieger Theorem for topological dynamical systems \((X, T)\) recalled in Theorem 1.2, the existence of a topological embedding from \(X\) to a shift space with finite alphabet forces the space \(X\) to be zero-dimensional. Analogously we have now:

**Proposition 3.2.** Let \((X, T)\) be a topological dynamical system and let \(Y\) be a zero-dimensional topological space. Assume that for any \(\epsilon > 0\) there exists an essentially continuous \(\epsilon\)-injective map from \(X\) to \(Y\). Then \((X, T)\) has the small boundary property.

### 4. The case of zero-dimensional systems

We first consider zero-dimensional dynamical systems. Then in Section 5 we will deal with general systems with the small boundary property by reduction to the zero-dimensional case.

We will prove the following strong version of our Main theorem for zero-dimensional systems:

Proposition 4.1. Let \( (X,T) \) be a zero-dimensional topological dynamical system with the following properties:

- \( \#\text{Per}_n(X,T) \leq \#\text{Per}_n(\{1,\ldots,K\},\sigma) \) for any integer \( n \),
- \( h_{\text{top}}(T) < \log K \),
- \( (X,T) \) is asymptotically expansive.

Then there exists an equivariant injective Borel map \( \psi : X \to \{1,\ldots,K\}^\mathbb{Z} \) such that:

- \( \psi \) is a pointwise limit of a sequence of equivariant continuous \( \epsilon_k \)-injective maps \( \psi_k \), where \( (\epsilon_k)_k \) is going to zero,
- the induced map \( \psi^* \) is the uniform limit of \( (\psi_k^*)_k \) on \( \mathcal{M}(X,T) \), in particular \( \psi^* \) is a topological embedding,
- \( \psi^{-1} \) is uniformly continuous on \( \psi(X) \) and the continuous extension \( \pi \) of \( \psi^{-1} \) on the closure \( Y = \overline{\psi(X)} \) of \( \psi(X) \) is a principal strongly faithful symbolic extension of \( (X,T) \) with \( \psi^* = (\pi^*)^{-1} \).

In the next Subsections 4.1, 4.2 and 4.3 we develop some tools used later in the proof of Proposition 4.1.

4.1. Asymptotic expansiveness for zero-dimensional systems. Let \( (X,T) \) be a zero-dimensional system. In the definition of asymptotic \( h \)-expansiveness given in Subsection 2.2 we may choose the open covers \( U \) and \( V \) to be finite clopen partitions such that \( V > U \), i.e. any element \( V \) of \( V \) is contained in a element \( U \) of \( U \). The topological conditional entropy \( h(V|U) \) may be then rewritten as follows

\[
h(V|U) := \lim_{n \to \infty} \frac{1}{n} \sup_{U_n \in U^n} \log \# \{ V_n \in V^n, V_n \subset U^n \}.
\]

The following lemma used later in the proof of Proposition 4.1 follows directly from the definition of asymptotically expansiveness independently from the above expression of the topological conditional entropy.

Lemma 4.1. Let \( (X,T) \) be an asymptotically expansive system zero-dimensional system. For all \( \alpha > 0 \) there exists a decreasing sequence \( (V_k)_k \) of clopen partitions whose diameter goes to zero, such that for all \( k \geq 1 \):

- \( h(V_{k+1}|V_k) < \alpha/2^k \),
- \( \text{per}(T|V_k) < \alpha/2^k \).

4.2. The nested marker property for zero-dimensional systems. One key tool in our construction is the following ”marker property” (see [9]). A similar approach is used in the theorem of symbolic extensions of Boyle and Downarowicz [2] where the product with an odometer is used to mark the shift space in the same way.

Lemma 4.2. Let \( (X,T) \) be a zero-dimensional dynamical system. Then for every sequence of positive integers \( (n_k)_k \) and for every sequence of positive real numbers \( (\epsilon_k)_k \) there exists a sequence \( (U_k)_k \) of clopen sets such that we have for each \( k \):

- \( T^n U_k \) are pairwise disjoint for \( i = 0,\ldots,n_k - 1 \),
- \( \bigcup_{i \leq n'_{k}} T^n U_k = X \setminus \text{Per}_{n'_{k}}(X,T) \), where \( n'_{k} \) is the integer given by \( n'_{k} := \sum_{l=1}^{k} n_l \) and \( \text{Per}_{n'_{k}}(X,T) \) denotes a clopen \( \epsilon_k \)-neighborhood of the set \( \bigcup_{n \leq n'_{k}} \text{Per}_n(X,T) \) of periodic points with least period less than or equal to \( n'_{k} \).
\[ U_{k+1} = U_{k+1}' \cup U_{k+1}'' \text{ with } U_{k+1}' \subset U_k \text{ and } U_{k+1}'' \subset \Per_{n_k}^{x_k} . \]

By the first item \( U_k \) do not contain any \( n_k \)-periodic point. Observe also that when \((X,T)\) is aperiodic the clopen sets \( (U_k)_k \) are then nested, i.e. \( U_{k+1} \subset U_k \) for any \( k \).

**Proof.** We follow the lines of the construction of [9] with emphasis on the nested property, which is not presented in this reference. We argue by induction on \( k \). We can take \( U_0 = X \) and \( n_0 = 0 \) to initialize the induction. Assume there exists a clopen set \( U_{k-1} \) satisfying the two first items of the lemma. Let \( \Per_{n_k}^{x_k} \) be a clopen \( \epsilon_k \)-neighborhood of the set of periodic points with period less than or equal to \( n_k \). We let \( \{W_1^1, \ldots, W_{M_k}^{M_k-1}\} \) and \( \{W_k^N, \ldots, W_k^N\} \) be finite clopen partitions of \( U_{k-1} \backslash \bigcup_{i<n} T^i \Per_{n_k}^{x_k} \) and \( \Per_{n_k}^{x_k} \backslash \Per_{n_k}^{x_k} \) respectively such that for any \( 1 \leq l \leq N_k \), the \( n_k \) first iterates of \( W_k^l \) are pairwise disjoint. Then for any \( k \) we define \( U_k, U_k', U_k'' \) with the following procedure:

\[
\begin{align*}
F_k^0 &= \emptyset, \\
F_k^1 &= W_k^1, \\
\text{and } F_k^{l+1} &= F_k^l \cup \left( W_k^l \setminus \bigcup_{|m|<n_k} T^m F_k^l \right) \text{ for any } l \leq N_k.
\end{align*}
\]

All the sets \((F_k^l)\) are clopen. We let \( U_k = F_k^{N_k}, U_k' = F_k^{M_k} \) and \( U_k'' = U_k \setminus U_k' \). Clearly we have \( U_k'' \subset \Per_{n_k-1}^{x_k} \) and \( U_k' \subset U_{k-1} \).

Finally let us prove the pairwise disjointness of \( T^i U_k \) for \( i = 0, \ldots, n_k - 1 \) and the covering property \( \bigcup_{|i|<n_k} T^i U_k \supset X \setminus \Per_{n_k}^{x_k} \). Assume by contradiction that \( T^i U_k \) and \( U_k \) are not disjoint for some \( i \) with \( 0 < |i| < n_k \) and let \( x \) be in their intersection. We consider the smallest \( l \) (resp. \( l' \)) for which \( x \) (resp. \( T^i x \)) belongs to \( F_k^l \) (resp. \( F_k^{l'} \)), in particular \( x \in W_k^l \) (resp. \( T^i x \in W_k^{l'} \)). By inverting the role of \( x \) and \( T^i x \) and replacing \( i \) by \(-i \), we can suppose \( l' \leq l \). Then \( T^i x \) belongs to \( F_k^l \) too. As \( x \) and \( T^i x \) can not belong to the same \( W_k^l \) the point \( T^i x \) is necessarily in \( F_k^{l-1} \) but then \( x \) can not be in \( F_k^l \) as it belongs to \( \bigcup_{|m|<n_k} T^m F_k^{l-1} \).

Now by induction hypothesis, any point \( x \) in the set \( X \setminus \Per_{n_k}^{x_k} \) lies either in \( \Per_{n_k-1}^{x_k} \) and \( x \in W_k^l \) with \( l \geq M_k \) or \( x \) lies in \( T^i U_{k-1} \) for some \( |i| < n_k-1 \) and \( T^i x \in W_k^l \) with \( l < M_k \). In the first case when \( x \) is not in \( F_k^l \) \( \subset U_k \) then it belongs to \( \bigcup_{|m|<n_k} T^m F_k^{l-1} \subset \bigcup_{|m|<n_k} T^m U_k \). For the second case one proves similarly that \( T^{-i} x \in \bigcup_{|m|<n_k} T^m U_k \) and thus \( x \in \bigcup_{|m|<n_k} T^m U_k \). This proves the covering property. As the clopen set \( U_k \) does not contain any periodic point with period less than \( n_k \) the complementary set \( \Per_{n_k}^{x_k} \) of \( \bigcup_{|i|<n_k} T^i U_k \) satisfy the conclusion of the lemma.

\[ \square \]

4.3. **Some tools on shift spaces.** We consider in this section a finite set \( \Lambda \) and a compact metrizable space \( X \). We let \( P_0 \) be the zero coordinate partition of \( \Lambda^\mathbb{Z} \).
4.3.1. **Decreasing metrics on \( \Lambda^\mathbb{Z} \).** The product space \( \Lambda^\mathbb{Z} \) will be endowed with different metrics. We first let \( d \) be a metric inducing the usual product topology (we consider the discrete topology on \( \Lambda \)), for example the Cantor metric defined as

\[
\forall x, y \in \Lambda^\mathbb{Z}, \; d(x, y) := 2^{-k} \text{ where } k = \min \{i \geq 0, \; x_i \neq y_i \}.
\]

Also for any integer \( N \) we consider the metric \( d_N \) defined as

\[
\forall x, y \in \Lambda^\mathbb{Z}, \; d_N(x, y) := \sup_{n \geq N} \left\{ \frac{1}{2n+1} \sharp \{i, |i| \leq n, \; x_i \neq y_i \} \right\}.
\]

For any \( N \) the topology given by \( d_N \) is stronger than the product topology. Observe now that the sequence of metrics \( (d_N)_N \) is nonincreasing, i.e. \( (d_N(x, y))_N \) is non-increasing in \( N \) for any \( x, y \in \Lambda^\mathbb{Z} \). We let \( d_{\infty} \) be the (shift-invariant) pseudometric on \( \Lambda^\mathbb{Z} \) given by the limit of \( (d_N)_N \)

\[
d_{\infty}(x, y) := \lim_N d_N(x, y).
\]

The pseudometric \( d_{\infty} \) is called the Besicovitch pseudometric on the shift space. Topological properties of the induced quotient metric space and dynamical properties of cellular automata on this space were studied in [1]. In particular, although we do not used here directly, this metric space is known to be complete.

4.3.2. **Convergence of functions taking values in \( \Lambda^\mathbb{Z} \).** We consider now maps from \( X \) to \( \Lambda^\mathbb{Z} \) and we define different kinds of convergence. A sequence of such maps \( (\psi_k)_k \) is said to converge to \( \psi \):

- **pointwisely with respect to** \( d \), when for all \( x \in X \),

\[
d(\psi_k x, \psi x) \xrightarrow{k \to +\infty} 0,
\]

- **uniformly with respect to** \( d_{\infty} \), when

\[
\sup_{x \in X} d_\infty(\psi_k x, \psi x) \xrightarrow{k \to +\infty} 0,
\]

- **uniformly with respect to** \( (d_N)_N \), when

\[
\sup_{x \in X} d_N(\psi_k x, \psi x) \xrightarrow{N,k \to +\infty} 0.
\]

Observe that if \( (\psi_k)_k \) is converging to \( \psi \) uniformly with respect to the decreasing sequence of pseudometrics \( (d_N)_N \), then it is also converging to \( \psi \) uniformly with respect to its limit \( d_{\infty} \).

We let \( \mathcal{M}(\Lambda^\mathbb{Z}, \sigma) \) be the set of probability Borel measures endowed with the following metric \( d_* \) inducing the weak \(*\)-topology : \( \forall \mu, \nu \in \mathcal{M}(\Lambda^\mathbb{Z}, \sigma) \),

\[
d_*(\mu, \nu) = \sum_n \frac{|\mu(A_n) - \nu(A_n)|}{2^n}
\]

where \( (A_n)_n \) is a given enumeration of \( \bigcup_N P_0^N \). We also consider the space \( \mathcal{K}(\Lambda^\mathbb{Z}, \sigma) \) of compact subsets of \( \mathcal{M}(\Lambda^\mathbb{Z}, \sigma) \) endowed with the Hausdorff metric \( d_H \) associated to \( d_* \). For all \( x \in \Lambda^\mathbb{Z} \) we let \( \phi(x) \in \mathcal{K}(\Lambda^\mathbb{Z}, \sigma) \) be the set of limits of the empirical measures, i.e. the accumulation points of the sequence \( \left( \sum_{0 \leq k < n} \delta_{\sigma^k x} \right)_n \) for the weak \(*\)-topology.
Lemma 4.3. Let $\psi : X \to \Lambda^Z$ be a uniform limit with respect to $d_\infty$ of $(\psi_k)_k$. Then $(\phi \circ \psi_k)_k$ converge uniformly to $\phi \circ \psi$ with respect to $d_H$.

When $X = \Lambda^Z$ and $\psi = \psi_k = Id_{\Lambda^Z}$, then we get in particular that two points $x, y \in \Lambda^Z$ with $d_\infty(x,y) = 0$ satisfy $\phi(x) = \phi(y)$.

Proof. It is enough to prove that for any $A \in \bigcup_N P_0^N$ we have

$$\lim \sup_k \lim \sup_{x \in X} \frac{1}{n} \sum_{0 \leq l < n} \delta_{\sigma^l \circ \psi_k(x)}(A) - \delta_{\sigma^l \circ \psi(x)}(A) = 0.$$  

Fix $N \in \mathbb{N}$, $A \in P_0^N$, $x \in X$ and $k \in \mathbb{N}$. Then

$$\lim \sup_n \frac{1}{n} \sum_{0 \leq l < n} \delta_{\sigma^l \circ \psi_k(x)}(A) - \delta_{\sigma^l \circ \psi(x)}(A) \leq \lim \sup_n \frac{1}{n} \sum_{0 \leq l < n} |\delta_{\sigma^l \circ \psi_k(x)}(A) - \delta_{\sigma^l \circ \psi(x)}(A)|,$$

$$\leq \lim \sup_n \frac{N}{n} \sum_{0 \leq l < n} |\sigma^l \circ \psi_k(x) \neq \sigma^l \circ \psi(x)|,$$

$$\leq N d_\infty(\psi_k(x), \psi(x)).$$

By uniform convergence of $(\psi_k)_k$ to $\psi$ with respect to $d_\infty$, this last term goes to zero uniformly in $x$ when $k$ goes to infinity. This concludes the proof the lemma.

For a subset $Y$ of $\Lambda^Z$ we let $\overline{Y}$ be the closure of $Y$ for the product topology and we let $Y^\infty$ be the $d_\infty$-saturated set of $Y$, i.e. $Y^\infty = \{x \in \Lambda^Z, \exists y \in Y, d_\infty(x,y) = 0\}$.

Lemma 4.4. Let $\psi : X \to \Lambda^Z$ be both a pointwise limit with respect to $d$ and a uniform limit with respect to $(d_N)_N$ of $(\psi_k)_k$. Assume moreover the maps $(\psi_k)_k$ are continuous (for the product topology on $\Lambda^Z$), then

$$\overline{\psi(X)} \subset \psi(X)^\infty.$$

Proof. Let $(y_n = \psi(x_n))_n$ be a sequence converging with respect to $d$ in $\psi(X)$ to say $y$. We consider a sequence of continuous maps $(\psi_k)_k$ converging to $\psi$ uniformly with respect to $(d_N)_N$. By definition for all $\epsilon > 0$ there exist integers $K$ and $M$ such that for all $k \geq K$ and for all $N \geq M$ we have $d_N(\psi(x_n), \psi_k(x_n)) \leq \epsilon$. Observe the function $(x,y) \in (X,d)^2 \to d_N(x,y)$ is lower semicontinuous as a supremum of continuous functions. Up to extract a subsequence we may assume by compacity of $X$ that $(x_n)_n$ is converging to $x \in X$. Therefore by taking the limit in $n$ in the previous inequality we obtain by continuity of $\psi_k$ for all $k \geq K$ and for all $N \geq M$

$$d_N(y,\psi_k(x)) \leq \epsilon.$$  

and then by pointwise convergence of $(\psi_k)_k$ to $\psi$ in $(\Lambda^Z,d)$ we have for all $N \geq M$

$$d_N(y,\psi(x)) \leq \epsilon.$$  

Finally we let $N$ go to infinity to get :

$$d_\infty(y,\psi(x)) \leq \epsilon.$$  

This concludes the proof of the lemma as $\epsilon > 0$ may be chosen arbitrarily small.
4.3.3. Dynamical consequences. We let now $T$ be an invertible map acting continuously on $X$. For a Borel map $\xi : X \to \Lambda^Z$ we let $\xi^* : \mathcal{M}(X,T) \to \mathcal{M}(\Lambda^Z,\sigma)$ be the map induced by $\xi$ on the set of invariant Borel probability measures. As done for the full shift in the previous subsection we denote for any $x \in X$ by $\phi(x) \subset \mathcal{M}(X,T)$ the set of limits of empirical measures at $x$, i.e. the set of accumulation points of the sequence $(\frac{1}{n}\sum_{l=0}^{n-1} \delta_{T^lx})_n$ for the weak-* topology.

**Lemma 4.5.** Let $\psi : X \to \Lambda^Z$ be a uniform limit with respect to $d_\infty$ of $(\psi_k)_k$. Assume moreover $\psi$ and $(\psi_k)_k$ are Borel and equivariant maps.

Then the induced maps $(\psi_k^*)_k$ converge uniformly to $\psi^*$ with respect to $d_\ast$.

**Proof.** By the ergodic decomposition it is enough to consider only ergodic $T$-invariant measures $\mu$. By Birkhoff ergodic theorem we have for $\mu$ almost $x$ and for all finite cylinders $A$ in $\Lambda^Z$:

$$
\mu(\psi^{-1}A) = \lim_{n} \frac{1}{2n-1} \sum_{|l|<n} \delta_{T^lx}(\psi^{-1}A).
$$

By equivariance we have

$$
\mu(\psi^{-1}A) = \lim_{n} \frac{1}{2n-1} \sum_{|l|<n} \delta_{\psi \circ T^l(x)}(A),
$$

$$
= \lim_{n} \frac{1}{2n-1} \sum_{|l|<n} \delta_{\psi \circ \phi(x)}(A).
$$

Since this holds for any $A$ the sequence of empirical measures

$$
(\frac{1}{2n-1} \sum_{|l|<n} \delta_{\psi \circ \phi(x)})_n
$$

is converging to $\psi^* \mu$ in the weak-*topology, in others terms we have $\phi \circ \psi(x) = \psi^* \mu$ for $x$ in a set $E^\mu$ of full $\mu$-measure. Similarly for any integer $k$ we have $\phi \circ \psi_k(x) = \psi_k^* \mu$ for $x$ in a set $E_k^\mu$ of full $\mu$-measure.

As by Lemma 4.3 the sequence $(\phi \circ \psi_k)_k$ converges uniformly to $\phi \circ \psi$ with respect to $d_H$ we conclude that $\psi_k^*$ converges to $\psi^*$ uniformly with respect to $d_\ast$. Indeed for any $\mu$ we take $x_\mu \in E^\mu \cap \bigcap_k E_k^\mu$ and we conclude that:

$$
\sup_{\mu} d_\ast(\psi_k^* \mu, \psi^* \mu) = \sup_{x_\mu} d_\ast(\phi \circ \psi_k(x_\mu), \phi \circ \psi(x_\mu))
$$

$$
\leq \sup_x d_H(\phi \circ \psi_k(x), \phi \circ \psi(x)),
$$

$$
k \to +\infty \to 0.
$$

\[\square\]

Assuming moreover continuity of the maps $(\psi_k)_k$ we prove the identity $\psi^* \phi(x) = \phi \circ \psi(x)$ for every $x \in X$.

**Lemma 4.6.** Let $\psi : X \to \Lambda^Z$ be a uniform limit with respect to $d_\infty$ of $(\psi_k)_k$. Assume moreover $\psi$ (resp. $(\psi_k)_k$) are Borel (resp. continuous) equivariant.

Then for all $x \in X$,

$$
\psi^* \phi(x) = \phi \circ \psi(x).
$$

**Proof.** The maps $\psi_k$ being continuous the associated maps $\psi_k^*$ induced on the set of Borel probability measures on $X$ endowed with the weak-* topology are also
continuous and therefore we have the equality $\psi_k^* \phi = \phi \circ \psi_k$. By the above Lemma 4.3 the left member goes uniformly to $\psi^* \phi$ (with respect to $d_H$), whereas the right member goes uniformly to $\phi \circ \psi$ when $k$ goes to infinity according to Lemma 4.3.

**Lemma 4.7.** Let $\psi : X \to \Lambda^\mathbb{Z}$ be both a pointwise limit with respect to $\mathbb{d}$ and a uniform limit with respect to $(d_N)_N$ of $(\psi_k)_k$. Assume moreover the maps $(\psi_k)_k$ are continuous equivariant.

Then any $\sigma$-invariant measure $\mu$ on $\overline{\psi(X)}$ is supported on $\psi(X)$, i.e. $\mu(\psi(X)) = 1$.

**Proof.** Let $\mu$ be a $\sigma$-invariant ergodic measure on $\overline{\psi(X)}$. According to Birkhoff ergodic theorem there is $y \in \overline{\psi(X)}$ with $\phi(y) = \{\mu\}$. By Lemma 4.4 one can find $x \in X$ such that $d_{\infty}(y, \psi(x)) = 0$. As previously discussed after Lemma 4.3 it implies that $\phi \circ \psi(x) = \phi(y) = \mu$. Finally by Lemma 4.6 we have $\psi^* \phi(x) = \phi \circ \psi(x) = \mu$. This concludes the proof as $\psi^* \phi(x)$ is supported on $\psi(X)$. □

4.4. **Proof of Proposition 4.1.** We first deal with aperiodic systems. For general systems with periodic points the proof is a little more involved as we need to encode periodic points. We will adapt the construction of Krieger [12] (Theorem 1.2) using the per-expansiveness property and the nested marker property given in Lemma 4.2.

Observe first that in any case one only needs to embed our system in a shift space with some finite alphabet. Indeed the topological entropy and the cardinality of periodic points with least period $n$ for any $n$ will be preserved by our Borel embedding so that by applying Krieger Theorem for topological expansive systems (Theorem 1.2) we get after a composition an embedding in the shift space with the desired number of letters (but also in any subshift of finite type with the suitable lower bound on the topological entropy and cardinality of periodic points).

4.4.1. **Construction of the embedding, the aperiodic case.** We consider an aperiodic dynamical system $(X, T)$. Let $K$ be an integer with $\log K > h_{\text{top}}(T)$. We let $\alpha > 0$ be the difference $\alpha = \log K - h_{\text{top}}(T)$.

0. **Marker structure.** Let $n_k = (n_{k})_k$ be a nondecreasing sequence of positive integers with $\alpha n_k >> 2^k$ and $n'_k = \sum_{i=1}^{k} n_i < 2n_k$. We consider a nondecreasing sequence $\Lambda = (\Lambda_k)_k$ of partitions of $\mathbb{Z}$ into intervals such that for any $k$ the length of any interval in $\Lambda_k$, by which we mean the number of integers inside this interval, is larger than or equal to $n_k$ and less than $4n_k$. We define the boundary $\partial \Lambda_k$ of $\Lambda_k$ as the set of integers $l$ such that $l$ and $l - 1$ are not in the same element of the partition $\Lambda_k$. The sequence $(\partial \Lambda_k)_k$ is nondecreasing as $(\Lambda_k)_k$. For any integer $k > 0$ a empty $k$-block associated to $(\Lambda_k)_k$ will be a finite word $u^k$ in $\{|, \|, *, \circ\}$ indexed over an element of $\Lambda_k$: it starts and finishes at consecutive integers $i_k, j_k \in \partial \Lambda_k$, that is $u^k := (u_{i_k}, \ldots, u_{j_k-1})$. The length of such a block $u^k$ is denoted by $|u^k|$. The empty $k$-blocks and their $k$-marker, -filling, -free positions are then defined by induction as follows. Firstly a empty 1-block is a word $u^1$ such that

- the first coordinate of $u^1$, the 1-marker position, coincides with $|$
- the next $\lfloor(1 - \alpha/2)|u^1|\rfloor$ coordinates of $u^1$, called the 1-filling positions all take the value $*$
- the remaining coordinates, called the 1-free positions all take the value $\circ$

$[x]$ denotes the integer part of a real number $x$
For $k > 1$ an empty $k$-block $u^k$ is obtained from a concatenation of consecutive empty $(k - 1)$-blocks, where we change the first $(k - 1)$-free position of the first $(k - 1)$-block to $\|$. This position in $u^k$ defines the $k$-marker position and allows to detect the entry in the $k$-block, i.e. that $i_k$ belongs to $\partial \Lambda_k$. The $[\alpha|u^k|/2^k]$ next $(k - 1)$-free positions are the $k$-filling positions of $u^k$ and the remaining $(k - 1)$-free positions of the concatenation define the $k$-free positions of $u^k$. The empty $k$-code $\psi^k$ with respect to $\Lambda$ is then just the infinite word in $\{1,1,*,0\}^\mathbb{Z}$ obtained by concatenation of all empty $k$-blocks.

1. Dynamical Markers. We let now $(\mathcal{V}_k)_k$ be a sequence of nested clopen partitions as in Lemma 4.1. We let $n_1$ be such that for $n > n_1$:

$$\mathcal{V}_1^n \leq e^{\alpha h_{\text{top}}(T)+\alpha/4} < K^{n(1-\alpha/2)-1}$$

and for $k > 1$ we let $n_k$ be such that for $n > n_k$:

$$\sup_{U^n \in \mathcal{V}_k^n} \mathcal{V}_n \leq \mathcal{V}_k^{n \in \mathcal{V}_{k+1}} \text{ with } V^n \subset U^n < K^{n\alpha/2^k-1}.$$

We may also ensure that $(n_k)_k$ satisfy the conditions at the beginning of the previous paragraph. We will build an injective map from $X$ to $\{1,0,*,1,...,K\}^\mathbb{Z}$ which conjugates $T$ to the shift as in the ergodic generator Krieger theorem [12] (Theorem 1.1 of the Introduction) by first encoding the dynamic with respect to the covers $\mathcal{V}_1, \mathcal{V}_2, ... \mathcal{V}_k,...$ on finite pieces of orbits of length larger than or equal to $n_1, n_2, ..., n_k, ...$. These pieces of orbits are given by the nested topological Rokhlin towers $(U_k)_k$ given by Lemma 4.2 with respect to the sequences $(n_k)_k$.

We consider more precisely for any $x \in X$ the partition $\Lambda_k(x)$ of $\mathbb{Z}$ into intervals with boundary corresponding to the return times of the orbit of $x$ to $U_k$, i.e. $\partial \Lambda_k(x) := \{l \in \mathbb{Z}, f^l x \in U_k\}$. Observe that any interval in $\Lambda_k(x)$ has length larger than or equal to $n_k$ and less than $2n_k - 1 < 4n_k$. We let $y$ be the sequence of integers $(n_k)_k$ and $\Lambda(x)$ be the sequence of partitions $(\Lambda_k(x))_k$. Finally, for any $k$, we let $\tau_k$ be the first return time of $y$ in $U_k$, that is $\tau_k(y) := \inf\{n > 0, T^n y \in U_k\}$.

2. First scale encoding. For any $n > n_1$ we consider an injective map from $\mathcal{V}_1^n$ to the $[(1-\alpha/2)n]$-words of $\{1,...,K\}^\mathbb{Z}$. Then to any $y \in U_1$ we associate such the word corresponding to $\mathcal{V}_1^{\tau_k}(y)$. The 1-code $\psi_1(x)$ of $x$ is then obtained by replacing in the empty 1-blocks in $\psi_1^n(x)$ associated to $\Lambda(x)$ the symbols $*$'s in the 1-filling positions by these words for $y = T^l x$ with $l$ the return times in $U_1$, which corresponds to the first index of the empty 1-blocks.

3. Higher scales encoding. We will now encode the dynamics with respect to $\mathcal{V}_2$ conditionally to $\mathcal{V}_1$. For any $n > n_2$ and for any $V_1^n \in \mathcal{V}_1^n$ we consider an injective map from $V_1^n \subset \mathcal{V}_2^n$ to the $[\alpha n/4]$-words of $\{1,...,K\}^\mathbb{Z}$. Then in the empty 2-block of $\psi_2^n(x)$ we replace the symbols $*$'s at the 2-filling positions by the $[\alpha n/4]$-words of $\{1,...,K\}^\mathbb{Z}$ associated to $\mathcal{V}_1^{\tau_2}(y)$ conditionally to $\mathcal{V}_1^{\tau_2}(y)$ for $y = T^l x$ with $l$ the first index of the empty 2-blocks. In a similar way we build $\psi_k$ for any $k > 2$. The sequence $(\psi_k(x))_k$ is converging pointwisely for the product topology in $\{1,...,K\}^\mathbb{Z}$ as for any $i$ the $i^{th}$ coordinate of $\psi_k(x)$ is constant after some rank. We let $\psi(x)$ be the pointwise limit of $\psi_k(x)$. 
4. Decoding. The \( k \)-code \( \psi_k(x) \) of \( x \) may be deduced from \( \psi(x) \) by replacing the \( l \)-marker and \( l \)-filling positions in \( \psi(x) \) for \( l > k \) by \( k \)-free positions. Thus we get in this way a sequence of continuous maps \( \pi_k : \psi(X) \to \{1, \ldots, K\}^\mathbb{Z} \) such that \( \pi_k \circ \psi(x) = \psi_k(x) \). We can then identify \( \mathcal{V}_n^\mathbb{Z}(x) \) for any integer \( n \) by reading the words in the \( 1 \)-filling positions of \( \psi_1(x) \) and finally \( \mathcal{V}_n^\mathbb{Z}(x) \) inductively on \( l \leq k \) by reading in \( \psi_k(x) \) the words in \( l \)-filling positions. Consequently any fiber of \( \psi_k \) is contained in a unique element of \( \mathcal{V}_k^\mathbb{Z} = \{ \bigcap_{n \in \mathbb{Z}} T^{-n}V_k^n, \ V_k^n \in \mathcal{V}_k \} \). In particular \( \psi_k \) is \( \epsilon_k \)-injective with \( \epsilon_k \) being the diameter of \( V_k \). As the diameter of \( V_k \) goes to zero as \( k \) goes to infinity it follows that the limit \( \psi \) is injective.

5. Continuity and convergence of \( (\psi_k)_k \). The maps \( (\psi_k)_k \) are continuous because \( (V_k)_k \) are open covers and \( \tau_k \) are continuous maps (as the set \( U_k \) are clopen). Moreover \( \psi_k(x) \) and \( \psi(x) \) differs on any \( k \)-block only at the \( k \)-marker, \(-\)-free and \(-\)-filling positions. Recall that these positions represent a proportion of at most \( \alpha/2^k \) positions in \( k \)-blocks. Moreover for any \( N > n_k^2 \) the segment \([-N,N]\) is the union of at most \( N/n_k \) \( k \)-blocks and 2 \( k \)-subblock (whose length is less than or equal to \( 4n_k \)). Then we have

\[
\sharp \{i, |i| \leq N \text{ and } (\psi_k)^i(x) \neq (\psi)^i(x) \} \leq N\alpha/2^k - N/n_k - 8n_k.
\]

Thus we conclude that \( d_N (\{i, (\psi_k)^i(x) \neq (\psi)^i(x) \}) \leq \alpha/2^k \) for any \( x \in X \) and any \( N > n_k^2 \) because we took \( \alpha n_k > 2^k \). Therefore \( \psi_k \) goes uniformly to \( \psi \) w.r.t. \( (d_N)_N \).

6. Principal strongly faithful symbolic extension as the inverse of Krieger embedding. The inverse of \( \psi \) is (uniformly) continuous on \( \psi(X) \). Indeed if \( \psi(x) \) and \( \psi(y) \) coincide on their \([-4n_k,4n_k]\) coordinates then they belong to the same \( l \)-block for any \( l \leq k \) and in particular \( x \) and \( y \) belong to the same element of \( V_k \).

Lemma 4.7 applies to \( \psi \), thus any \( T \)-invariant measure on \( \psi(x) \) is supported on \( \psi(x) \). Finally as \( \psi : (X,T) \to (\psi(X),\sigma) \) is a Borel isomorphism, the induced map on the set of invariant measures is bijective and preserve the measure theoretical entropy. Moreover \( \psi \) preserves periodic points. This proves \( \pi \) is strongly faithful and principal.

Remark 4.1. Observe that \( \psi X \) is in general not closed. Indeed let \((X,T)\) be the odometer to base \((p_k)_k = (2^k)_k\). For any positive integer \( n \) we let \( x_n \) be the point in \( X \) given by \( x_n = (\ldots,0,\ldots,2^n,0,\ldots) \) where \( 2^n \) is at the \((n+1)\)-th coordinate and \( U_n \) be the clopen set of points whose \( n \)-th first coordinates are zero. The sequence \((U_n)_n\) satisfies the properties of the Marker Lemma (Lemma 4.7). Then we consider the Borel embedding \( \psi : X \to \{0,1\}^\mathbb{Z} \) given by the previous construction. Let \( y \) be a limit point of \( \psi(x_n) \). If it belongs to \( \psi(X) \) the point \( y \) is necessarily \( \psi(0) \) as \( \pi_k(y) = \lim_n \pi_k \circ \psi(x_n) = \lim_n \psi_k(x_n) = \psi_k(0) \). Now the \(-1 \)-position of \( \psi(0) \) is a free position whereas for any \( n \) the \(-1 \)-position of \( \psi(x_n) \) is a filling position.

4.4.2. Construction of the embedding, the periodic case. We consider now the general case, i.e. we consider asymptotically \( \alpha \)-expansive system not necessarily aperiodic.

0. Marker structure. Let \( \underline{n} = (n_k)_k \) be a nondecreasing sequence of positive integers with \( \alpha n_k > 2^k \) and \( n_k > n_{k-1} \) for any \( k \). We consider a sequence
\( \Lambda = (\Lambda_k)_k \) of open subsets of \( \mathbb{R} \) with boundaries in \( \mathbb{N} \) such that for any \( k \) the diameter of any connected component in \( \Lambda_k \) (resp. \( \Lambda_1 \)) is larger than \( n_k/2 \) and less than \( 10n_k \) (resp. \( 2n_1 \)). Moreover any connected component of the complementary set of \( \Lambda_k \) (resp. \( \Lambda_1 \)) has length larger than or equal to \( n_k \) (resp. \( 2n_1 \)). We denote by \( \partial^+ \Lambda_k \) (resp. \( \partial^- \Lambda_k \)) the set of integers \( i \) in the boundary of \( \Lambda_k \) with \( i+t \in \Lambda_k \) (resp. \( i-t \)) for arbitrarily small \( t \) and we also let \( \partial^+ \Lambda_k := \partial^+ \Lambda_k \setminus \partial^- \Lambda_k \), \( \partial^- \Lambda_k := \partial^- \Lambda_k \setminus \partial^+ \Lambda_k \) and \( \partial^{\pm} \Lambda_k := \partial^+ \Lambda_k \cap \partial^- \Lambda_k \).

For any integer \( k > 0 \) a empty regular \( k \)-block associated to \( (\Lambda_k)_k \) is a finite word \( u^k \in \{[\cdot],[\cdot],[\cdot],*,o\}^\mathbb{Z} \) indexed over a connected component of \( \Lambda_k \). More precisely it starts and finishes at consecutive integers \( i_k, j_k \in \partial \Lambda_k \) with \( i_k \in \partial^+ \Lambda_k \) and \( j_k \in \partial^- \Lambda_k \), i.e. \( u^k := (u_{i_k}, \ldots, u_{j_k-1}) \). Empty singular \( k \)-blocks are finite words indexed over a connected component of the complementary set of \( \Lambda_k \). Among singular \( k \)-blocks we will distinguish special singular \( k \)-blocks. The set of special singular \( k \)-blocks is nested with \( k \) and their intersection over \( k \) is either the empty set or the whole set of integers.

The empty \( k \)-blocks and their \( k \)-marker, \(-filling\), \(-free\) positions are then defined by induction as follows.

**Empty 1-blocks.** Firstly a empty regular 1-block is a word \( u^1 \) such that

- the first and last coordinate of \( u^1 \), resp. called the left and right 1-marker positions, coincide with \([\cdot]\),
- the next \( (1-\alpha)n \) coordinates of \( u^1 \), called the 1-filling positions all take the value \(*\),
- the remaining coordinates, called the 1-free positions, all take the value \( o \).

The 2-marker position of a 1-regular block is the first 1-free position of this block. An empty singular 1-block is just given by 1-filling positions, it is a word over a connected component of the complementary set of \( \Lambda_1 \) with the letter \(*\) repeated. All singular 1-blocks are special. All the positions of a 1-singular block are 2-marker positions (in fact for any \( k \) all the positions of a special singular \( k \)-block are \( k+1 \)-marker positions).

**Empty \( k \)-blocks from empty \( k-1 \)-blocks.** As already mentioned any special singular \( k \)-block is a subblock of a special singular \( k-1 \)-block. An empty \( k \)-block is obtained from a concatenation of empty \( k-1 \)-blocks and at most two empty singular \( k-1 \)-subblocks at the boundary after the following modifications which will ensure that all \( k \)-blocks except special singular ones have a proportion \( \alpha/2^k \) of free positions. In this way the \( k \)-codes defined below will converge uniformly with respect to \((d_N)_N\).

Let \( u^k := (u_{i_k}, \ldots, u_{j_k-1}) \) be an empty regular \( k \)-block. Either \( i_k \) corresponds to the first coordinate of a regular \( k-1 \)-block and we change to \([\cdot]\) (resp. \([\cdot]\)) the \( k \)-marker coordinate of this block when \( i_k \in \partial^+ \Lambda_k \) (resp. when \( i_k \in \partial^- \Lambda_k \)) or \( i_k \) lies at the \( k \)-marker position of a \( k-1 \)-singular block and then we change this coordinate to \([\cdot]\) or \([\cdot]\) as above. We argue similarly for \( j_k \), replacing the left marker \([\cdot]\) by the right marker \([\cdot]\). We change the values in the empty (bounded) special singular \( k-1 \)-block or -subblocks \( v_k \) in \( u^k \) by making free all the \( l \)-positions in \( v_k \) with \( l \) larger than \( n_1 \) of the form \( l = \lfloor \alpha/2^k \rfloor p, p \in \mathbb{N}\setminus\{0\} \) (we change their value to \( o \)).
For empty special $k-1$-singular blocks or subblocks $v_k$ in a empty singular $k$-block $w_k$ we proceed in a similar way. To any such $w_k$ we will associate an integer $m = m(w_k)$ less than or equal to $n_k$. When $v_k$ is one-sided bounded to the left we do not change the $n_1$ first coordinates of $w_k$ but then we make free the $l$-positions in $v_k$ where $l$ is of the form $l = n_1 + km - r$ with $k \geq 1$ and $r = 1, ..., \lfloor am/2^k \rfloor$. We argue similarly when $v_k$ is only one-sided bounded to the right. If $v_k$ is unbounded (in both sides) we make free the $l$-positions in $v_k$ where $l$ is of the form $l = km - r$ with $k \in \mathbb{Z}$ and $r = 1, ..., \lfloor am/2^k \rfloor$. These unbounded $k$-singular blocks $w_k$ will be $m(w_k)$ periodic. We may arrange in this case the construction to be equivariant, by applying the previous construction to one element in the periodic orbit and then by shifting it.

$k+1$-marker position of a empty $k$-block. Now the $k+1$-marker position of a regular $k$-block is the first $k$-free position in this block. For non special singular $k$-blocks one-sided bounded to the left (resp. to the right) we let the first (resp. last) $k$-free position, say $l$, be a $k+1$-marker but also all the positions of the form $l + pm'$ where $m'$ is the smallest multiple of $m$ larger than or equal $n_k$ and $p \in \mathbb{N}$ (resp. $-p \in \mathbb{N}$). Observe that $m' \leq 2n_k$. When the singular $k$-block is unbounded we argue similarly by letting $l$ be the first $k$-free positive position (ensuring equivariance as above).

The empty $k$-code $\psi_k^0$ with respect to $\Lambda$ is then just the infinite word in $\{[,|,*,o]\}^\mathbb{Z}$ obtained by concatenation of all empty $k$-blocks.

1. Dynamical Markers. We consider a sequence of clopen partitions $(\mathcal{V}_k)_k$ given by Lemma 4.1 and we let $\underline{n} = (n_k)_k$ be a sequence of integers satisfying for any $k$ and for all $n > n_k$

$$\sup_{U^n \in \mathcal{V}^n_k} \sharp \{V^n, V^n \in \mathcal{V}^{n+1}_{k+1} \text{ with } V^n \subset U^n \} \leq e^{n_\alpha/2^k}$$

and moreover

$$\sup_{V^n \in \mathcal{V}^n_k} \log \sharp \{\text{Per}_n(X,T) \cap V^n_k \} < \alpha/2^k. \tag{1}$$

We also choose $n_1$ large enough so that $\sharp \text{Per}_{n_1}(X,T) < K^{n_1-1}$. We may also ensure that $(n_k)_k$ satisfy the conditions at the beginning of the previous paragraph. Far from the $n_1, n_2, ..., n_k, ...$-periodic points we will encode the dynamics with respect to the covers $\mathcal{V}_1, \mathcal{V}_2, ... \mathcal{V}_k, ...$ on finite pieces of orbits of length larger than $n_1, n_2, ..., n_k, ...$ These pieces of orbits are given again by the topological Rokhlin towers over $(U_k)_k$ given in Lemma 4.2 with respect to the sequences $\underline{n} = (n_k)_k$ and $(\epsilon_k)_k$. This last sequence will be precised latter on, a first condition being that $\epsilon_k$ is less than the Lebesgue number of the open cover $T^i \mathcal{V}_k$ for any $k$ and any $|i| \leq n_k$. We consider for any $x \in X$ the open subset $\Gamma_k(x)$ of $\mathbb{R}$ such that the boundary of its connected components corresponds to return times in $U_k$ less than $2n_k$, i.e. $\partial^+ \text{Per}_{n_k}(X,T) := \{l \in \mathbb{Z}, f^l x \in U_k \text{ and } \exists m < 2n_k \text{ (resp. } 0 < -m < 2n_k) \}$. Observe that any connected component of $\Gamma_k(x)$ corresponds to a piece of the orbit of $x$ between two consecutive return times in $U_k$, which does not enter in $\text{Per}_{n_k}^{\epsilon_k}$. In particular its length is larger than $n_k$ and
Assume that \( v \) is a periodic point. Thus we may associate to any 1-singular block \( n \) a word in \( \{ 1 \} \) so small that if \( \epsilon_1 \) and \( n_1 \) are less than \( 2 \epsilon_1 \) or equal to \( n_1 \) then there is a unique periodic point \( p_y \) with period less than or equal to \( n_1 \) which is \( \epsilon_1 \)-close to \( y \). We may also ensure that when \( TP_{n_1} \) have non empty intersection then this intersection contain a unique such periodic point. Thus we may associate to any 1-singular block \( v \) a single periodic point \( p \) of least period \( n \leq n_1 \). The integer \( m(v_1) \) mentioned in the construction of blocks is given by the period \( n \). We will choose the name \( w(p) \) of this periodic point, which is a word in \( \{ 1, ..., K \}^n \), so that we can identify the periodic point by reading the \( n_1 \) first letters of \( w(p)w(p)...w(p)... \):

**Lemma 4.8.** Assume that

- \( \limsup_n \frac{\log \#Per_n(X,T)}{n} < \log K \);
- \( \#Per_n(X,T) \leq \#Per_n(\{ 1, ..., K \}^2, \sigma) \) for any integer \( n \).

Then for large enough \( n_1 \) there exist an injective equivariant map \( \psi : \bigcup_{n \leq n_1} Per_n(X,T) \rightarrow \{ 1, ..., K \}^{n_1} \) such that the map \( \psi^{n_1} : \bigcup_{n \leq n_1} Per_n(X,T) \rightarrow \{ 1, ..., K \}^{n_1} \) given by the \( n_1 \) first coordinates of \( \psi \) is also injective.

**Proof.** For \( n_1 \) large enough and for \( n_1 > n > \sqrt{n_1} \) and \( x \in Per_n(X,T) \) we can take \( \psi(x) \) so that \( \psi^{n_1}(x) \) is not of the form \( uu...w \) for some word of length less than \( n \) and \( \overline{w} \) is some truncation of \( w \). Indeed the number \( \frac{\sum_{n=0}^{n-1} K^l} {K^n} \) is less than or equal to \( \frac{K^{n_1}} {K^n} \). One can show directly this number is less than \( 2^n - 2^{n-3} \) and we have supposed \( \limsup_n \frac{\log \#Per_n(X,T)}{n} < \log K \). Then for \( n < \sqrt{n_1} \) we let \( \psi|_{Per_n(X,T)} \) be any equivariant injection from \( Per_n(X,T) \) to \( \{ 1, ..., K \}^{n_1} \).

We claim now that the associated map \( \psi^{n_1} \) is injective. Let \( a \) be in the image of \( \psi^{n_1} \). We shall prove that there is a unique point \( x \) with \( \psi^{n_1}(x) = a \). There is a unique word \( w \) with minimal length satisfying \( uu...w \) for \( a \). The length of \( w \) is thus less than or equal to the period of any \( y \) with \( \psi^{n_1}(y) = a \). When this length is larger than \( \sqrt{n_1} \) then one conclude directly as by our construction \( \psi^{n_1} \) is injective on \( \bigcup_{n \leq n_1} Per_n(X,T) \). When \( |w| < \sqrt{n_1} \) then the above construction also ensures that the least period \( n_y \) of \( y \) is less than \( \sqrt{n_1} \). Now by Bezout Theorem there are integers \( b \) and \( c \) with \( |b| < n_y, |c| < |w| \) and \( |b|w| + cn_y = d \) with \( d \) being the smallest common divisor of \( n_y \) and \( |w| \). Then one easily sees that \( a \) is also of the form \( rr...r \) with \( r \) of length \( d \) (writing \( a = (a_1, ..., a_{n_1}) \) and assuming for example \( b > 0 \) and \( c < 0 \) we have \( a_{k+d} = a_{k+b|w|+cn_y} = a_{k+b|w|} = a_k \) for any \( k \leq |w| \) as \( k + b|w| \leq n_1 \).
Therefore \( r = w \) and \( \psi(y)^n \) is then a integer concatenation of \( r \). Thus \( y \) is \( |w| \)-periodic. It concludes the lemma as \( \psi \) is one-to-one on \( \text{Per}_w(X,T) \).

4. Higher scale encoding. Regular \( k \)-blocks are encoded by writing in the \( k \)-filling positions of \( \psi_k^n(x) \), defined as in the aperiodic case, the code of \( \mathcal{V}_k \) conditionally to \( \mathcal{V}_{k-1} \). We explain now the procedure for a singular \( k \)-block. This block correspond to a visit in \( \text{Per}_{n_k}^k \). We choose again \( \epsilon_k \) so small that singular \( k \)-blocks correspond to visit the \( \epsilon_k \)-neighborhood of a single \( n \)-periodic orbit with \( n \leq n_k \). By continuity of \( \psi_{k-1} \) we may also ensure that the \( \psi_{k-1} \)-image on the singular \( k \)-block is given by the image of the associated periodic orbit which is the repetition of a word of length \( n \). We distinguish then two cases: either the period \( n \) is less than or equal to \( n_{k-1} \) and we let \( \psi_k \) be equal to \( \psi_{k-1} \) on this block, or the period \( n \) is larger than \( n_{k-1} \) and in the first \( \alpha n/2^k \)-free positions of the repeated word we write the code of \( \mathcal{V}_k^n \) conditionally to \( \mathcal{V}_{k-1}^n \) and in the next \( \alpha n/2^k \)-free positions we specify the periodic point by asymptotic \( \per \)-expansiveness according to Inequality (1). The sequence \( (\psi_k(x))_k \) is converging pointwisely for the product topology in \( \{1,\ldots,K\}^\mathbb{Z} \) as for any \( i \) the \( i^{th} \) coordinate is changed at most two times. We let \( \psi(x) \) be the pointwise limit of \( \psi_k(x) \).

5. Decoding. The \( k \)-code \( \psi_k(x) \) may be deduced on \( k \)-regular blocks from \( \psi(x) \) by identifying \( k \)-blocks by induction as follows. Firstly regular 1-block correspond to finite words in \( \psi(x) \) delimited by two letters \( | \) with length \( n \) less than 2\( n_1 \). In these blocks we may read the associated element of \( \mathcal{V}_1^n \) by looking at the filling 1-positions. Now in a singular bounded 1-block the first symbol \( | \) lies in the \( k^{th} \) letter with \( k \geq n_1 \). As the \( n_1 \)-first coordinates have not been changed by higher scale encoding we may read the periodic point associated to this 1-singular block according to Lemma 4.8. Then regular 2-blocks are identified as words between consecutive markers \( [ \) and \( ] \), where these markers occur at the first 1-free position of a 1-regular block or at a 2-marker position inside a 1-singular block. Singular 2-blocks outside 2-free positions are given by periodic words and we may identify the associated periodic point in the 2-filling positions. Similarly one may extract \( k \)-blocks from \( \psi_k(x) \) by induction on \( k \) and then read either the associated periodic point closed to the piece of orbit for singular \( k \)-block or the orbit with respect to \( \mathcal{V}_k \) for regular ones. In particular this proves that any two points with the same image under \( \phi_k \) belongs to the same element of \( \mathcal{V}_k \) according to the choice of \( \epsilon_k \). Thus \( \psi_k \) is \( \epsilon_k \)-injective with \( \epsilon_k \) being the diameter of \( \mathcal{V}_k \).

6. Continuity and convergence of \( (\psi_k)_k \). One proves as in the aperiodic case the codes \( (\psi_k)_k \) are continuous by the clopen property of \( \mathcal{V}_k \), \( (U_k)_k \) and \( \text{Per}^{n_k}_n \). As we change at each step a proportion \( \alpha/2^k \) of coordinates on subblocks of size larger than \( n_k/2 \) and less than \( 10n_k \) the sequence \( (\psi_k)_k \) is converging again uniformly with respect to \( (d_N)_N \).

7. Principal strongly faithful symbolic extension as the inverse of Krieger embedding. The inverse of \( \psi \) is (uniformly) continuous on \( \psi(X) \). Indeed if \( \psi(x) \) and \( \psi(y) \) coincide on their \([-10n_k,10n_k]\) coordinates then the points \( x \) and \( y \) belong either to the same regular \( l \)-block for any \( l < k \) or are \( \epsilon_k \)-close to the same \( n_k \)-periodic point. In both cases \( x \) and \( y \) belong to the same element of \( \mathcal{V}_k \). The other desired properties of the extension are shown as in the aperiodic case.
5. The general case

Now we will deduce the general case from the zero-dimensional one by using the small boundary property. More precisely this last property allows us to embed the system in a zero-dimensional one as follows:

5.1. Embedding systems with the small boundary property in zero-dimensional ones.

Proposition 5.1. (p. 73 [10]) Let \((X, T)\) be an (resp. aperiodic, resp. asymptotically per-expansive) topological dynamical system with the small boundary property. Then there exists a zero dimension (resp. aperiodic, resp. asymptotically per-expansive) system \((Y, S)\) and a Borel equivariant injective map \(\psi : X \to Y\) such that

- \(\psi\) is continuous on a full set,
- the induced map \(\psi^* : \mathcal{M}(X, T) \to \mathcal{M}(Y, S)\) is an homeomorphism,
- \(\psi^{-1}\) is uniformly continuous on \(\psi(X)\) and extends to a continuous principal strongly faithful extension \(\pi\) on \(Y = \overline{\psi(X)}\) of \((X, T)\).

For completeness we sketch the proof of Proposition 5.1. We consider a topological dynamical system with the small boundary property. Let \((P_k)_{k}\) be a nonincreasing sequence of essential partitions of \(X\) whose diameter goes to zero. We consider the shift \(\sigma\) over \(\prod_k P_k^\mathbb{Z}\). Let \(\psi : X \to \prod_k P_k^\mathbb{Z}\) be defined by \(\psi(x) = (P_k(T^n x))\), it is injective and semi-conjugates \(T\) with the shift, \(\sigma \circ \psi = \psi \circ T\). Clearly \(\psi\) is continuous on the full subset of points whose orbit does not intersect the boundary of the \(P_k\)'s. Moreover the induced map \(\psi^*\) is a topological embedding. Indeed to prove the continuity of \(\psi^*\) it is enough to see that \(\psi^* \nu(A)\) converges to \(\psi^* \mu(A)\) when \(\nu\) goes to \(\mu\) for any finite cylinder \(A\) in \(\prod_k P_k^\mathbb{Z}\). But for such \(A\) the closure and the interior of \(\psi^{-1} A\) have the same measure for any invariant measure so that the previous property of continuity holds.

The inverse \(\psi^{-1}\) satisfies \(\psi^{-1}((A_n)_{n,k}) = \bigcap_{n,k} \overline{P_{n,k}^{-n} A_{n,k}}\) and is thus clearly uniformly continuous. The continuous extension \(\pi\) of \(\psi^{-1}\) maps \(\psi(X)\) \(\psi(X)\) on the boundaries of the \(\overline{P_{n,k}^{-n} A_{n,k}}\)'s and thus any \(\sigma\)-invariant measure on \(\overline{\psi(X)}\) is supported on \(\psi(X)\). As \(\psi(\text{Per}_n(X, T)) = \text{Per}_n(\psi(X), S)\) for any integer \(n\) we conclude that \(\pi\) is a strongly faithful principal extension of \((X, T)\).

Principal extensions preserves the asymptotic \(h\)-expansiveness property and strongly faithful extensions preserves the aperiodicity. Finally if \(n\)-periodic points in \(Y = \overline{\psi(X)}\) lie in the same \(P_k^n\) for some \(k\) their \(\pi\)-image are \(n\)-periodic points for \(T\) in the same \(T_k^n\). As these periodic points are in fact in \(\psi(X)\) and \(\pi\) is one-to-one on \(\psi(X)\), we conclude asymptotic per-expansiveness is also preserved by this extension.

5.2. Reduction to the zero-dimensional case. By applying the above Proposition 5.1 our Main Theorem follows from Proposition 4.1. Indeed the zero-dimensional extension \((Y, S)\) of \((X, T)\) obtained in the above proposition satisfies the assumption of our Main Theorem. The desired embedding is then given by the composition of the embedding in the zero-dimensional system \((Y, S)\) and the Krieger embedding of \((Y, S)\) in the shift with the desired finite alphabet. The property of convergence and continuity of the embedding and its inverse follows then from basic properties of composition.
6. Proof of Proposition 3.1 and 3.2

6.1. Topological embedding in the set of invariant measures ⇒ asymptotic $h$-expansiveness. Assume $\psi : X \rightarrow \Lambda^\mathbb{Z}$ is a Borel equivariant embedding such that $\psi^*$ is a topological embedding. Recall that $P_0$ denotes the zero coordinate partition of $\Lambda^\mathbb{Z}$. Then for any $\mu \in \mathcal{M}(X,T)$ we have $h(\mu) = h(\psi^* \mu) = h(\psi^* \mu, P_0) = h(\mu, \psi^{-1} P_0)$. In the present paper such a partition $\psi^{-1}(P_0)$ is said to be generating. Let $B \in P_0$ and $A = \psi^{-1} B \in \psi^{-1} P_0$ and let $\mu$ be $T$-invariant measure. As $\psi^*$ is continuous and $B$ is a clopen set we have

$$
\lim \sup_{\nu \rightarrow \mu} \nu(A) = \lim \sup_{\nu \rightarrow \mu} \psi^* \nu(B),
$$

$$
\leq \lim_{\xi \rightarrow \psi^* \mu} \xi(B),
$$

$$
\leq \psi^* \mu(B) = \mu(A).
$$

Similarly we prove $\lim \inf_{\nu \rightarrow \mu} \nu(A) \geq \mu(A)$. Therefore for any $A \in \psi^{-1} P_0$, the function $\mu \mapsto \mu(A)$ is continuous on $\mathcal{M}(X,T)$. In general it does not imply that $A$ has a small boundary, for example when there is an attracting fixed point in the boundary of $B \in \psi^{-1} P_0$ (the Dirac measure at this point is then isolated in $\mathcal{M}(X,T)$). Finally asymptotic $h$-expansiveness follows from the following lemma.

**Lemma 6.1.** Let $(X,T)$ be a topological dynamical system admitting a finite generating partition $P$ such that for any $A \in P$, the function $\mu \mapsto \mu(A)$ is continuous on $\mathcal{M}(X,T)$, then $(X,T)$ is asymptotically $h$-expansive.

**Proof.** Let $(h_k)_k$ be the Lebesgue entropy structure [9]. For any integer $k$ the function $h_k$ is defined as the entropy of $\mu \times \lambda$ with respect to an essential partition $P_k$ of $(X \times S^1, T \times R)$, where $R$ is an irrational circle rotation and $(P_k)_k$ is a nonincreasing sequence of essential partitions. By the tail variational principle[10] $T$ is asymptotically $h$-expansive if and only if $(h_k)_k$ is converging uniformly to $h$. By assumption on $P$, the maps $\mu \mapsto h(\mu \times \lambda, P \times S^1 \lor P_k)$ are upper semicontinuous on $\mathcal{M}(X,T)$. Moreover they are nonincreasing in $k$ and converging to zero. It is well known that the convergence is in fact uniform for such sequences of functions. As $h(\mu) - h_k(\mu) \leq h(\mu \times \lambda, P \times S^1 \lor P_k)$ for any $T$-invariant measure $\mu$ the sequence $h - h_k$ is also converging uniformly to zero and thus $T$ is asymptotically $h$-expansive. $\square$

6.2. Topological embedding in the set of invariant measures ⇒ asymptotic per-expansiveness. Let $\psi : (X,T) \rightarrow (\{1, \ldots, K\}^\mathbb{Z}, \sigma)$ be a Borel equivariant embedding such that $\psi^*$ is a topological embedding and let $(U_k)_k$ be a sequence of open covers such that the diameter of $U_k$ goes to zero. We will show that the exponential growth in $n$ of the set $U_k^n \cap \text{Per}_n(X,T)$ goes to zero uniformly in $U_k^n \in U_k^\mathbb{Z}$ when $k$ goes to infinity. Assume first $(X,T)$ has the small boundary property. Let $(P_l)_l$ be a sequence of essential partitions with $\text{diam}(P_l) \xrightarrow{l \rightarrow 0} 0$. Let $\epsilon > 0$. For any $l$ there exist $k_l$ and $n_l$ such that any $U_k^n$ with $k > k_l$ and $n > n_l$ meets at most $\epsilon^n$ elements of $P_l^n$ (see [2]). Thus it is enough to prove that the exponential growth in $n$ of the set $A^n \cap \text{Per}_n(X,T)$ goes to zero uniformly in $A^n \in P_l^n$ when $l$ goes to infinity. Take $A^n$ so that the previous intersection has maximal cardinality. We consider the associated empirical measures $\mu_n := \frac{1}{\text{Per}_n(X,T)} \sum_{x \in A^n \cap \text{Per}_n(X,T)} \delta_x$ and $\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} T^* \mu_n$ for any integer $n$. Let $P = \psi^{-1}(P_0)$. Following [4] we get
for any $m < n$

\[
\frac{H_{\nu_n}(P^m|P^m)}{m} = \frac{H_{\psi^*\nu_n}(P^m_0|\psi(P)^m)}{m} \geq \frac{H_{\psi^*\mu_n}(P^m_0|\psi(P)^m) - 3m \log K}{n} \geq \frac{\log \sharp A^n \cap \text{Per}_n(X,T) - 3m \log K}{n}.
\]

The last inequality follows from the injectivity of $\psi$, the inclusion $\psi_*(\text{Per}_n(X,T)) \subset \text{Per}_n(\Lambda^2, S)$ and the inequalities $\sharp \text{Per}_n(\Lambda^2, S) \cap B^n \leq 1$ for any given $B^n \in P^m_0$. Observe also that $\nu_n$ is invariant: it is the barycenter of the periodic measures associated to the periodic points in $A^n \cap \text{Per}_n(X,T)$. Let $\nu$ be a weak limit of $(\nu_n)_n$. As seen above $(\nu_n(A))_n$ converges to $\nu(A)$ for any $A \in P$. By taking the limit in $n$ and then $m$ we get thus by upper semicontinuity:

\[
\bar{h}(\nu, P|P) \geq \limsup_n \frac{\log \sharp A^n \cap \text{Per}_n(X,T)}{n}.
\]

However by asymptotic $h$-expansiveness, which follows from Subsection 6.1, the left member goes to zero uniformly in $\nu$ when $l$ goes to infinity (see [12]). This conclude the proof when $(X,T)$ has the small boundary property. For general systems one may consider the product with an irrational circle rotation and apply the previous proof by replacing $\mu_n$ with is product with a Dirac measure of the circle, $\psi$ by its product with the circle identity and $P$ by its product with the circle.

6.3. Principal strongly faithful asymptotic per-expansive extension ⇒ asymptotic per-expansiveness. We prove now the second item of Proposition 3.1. We only have to prove asymptotic per-expansiveness as principal extensions preserves asymptotical $h$-expansiveness as proved by Ledrappier [17]. Let $\pi : (Y,S) \to (X,T)$ be a principal extension. Equivalently the topological conditional entropy $h(S|T)$ of $S$ w.r.t. $T$ vanishes (see Section 6.3 of [14] for precise definitions and basic properties). In particular, for any $\epsilon > 0$ there exists an open cover $U$ of $X$ such that for any open cover $V$ and for any $U^n \in U^n$ with large $n$, the set $\pi^{-1}U^n$ may be covered by $\epsilon^{n/2}$ element of $V^n$. If we assume moreover the extension to be strongly faithful it preserves periodic orbits. In particular the number of $n$-periodic points for $T$ in $U^n$ is equal to the number of $n$-periodic points for $S$ in $\pi^{-1}U^n$. Therefore if $(Y,S)$ is itself asymptotically per-expansive, one may choose $\epsilon$ so that the number $n$-periodic points for $S$ in any $V^n$ is less than $\epsilon^{n/2}$. One concludes that there at most $\epsilon^n n$-periodic points of $(X,T)$ in $U^n$. This proves $(X,T)$ is asymptotically per-expansive.

6.4. $\epsilon$-injective essentially continuous map ⇒ small boundary property. Let $(X,T)$ be a topological dynamical system and let $Y$ be a zero-dimensional space. Assume that for any $\epsilon > 0$ there exists a $\epsilon$-injective essentially continuous map $\psi$. Let $\delta > 0$ be such that for any set $Y$ with diameter less than $\delta$ the set $\psi^{-1} Y$ has diameter less than $\epsilon$. We let $B$ be a basis of clopen sets such that $\psi^{-1} B$ has a small boundary for any $B \in B$. Consider then a finite clopen partition $P$ of $Y$ with diameter less than $\delta$ such that any element of $P$ is a finite intersection of elements of $B$. In particular, for any $A \in P$, the set $\psi^{-1} A$ is a finite intersection of small boundary sets and thus it has itself small boundary. As the diameter of $\psi^{-1} A$ is less than $\epsilon$, it concludes the proof of Proposition 3.2.
Appendix A. Appendix : Equality in Krieger topological embedding problem

Proposition A.1. Let \((X,T)\) be a topological dynamical system with \(h_{\text{top}}(T) = \log K\). Assume also there is a topological equivariant embedding \(\psi : X \to \{1, \ldots, K\}^\mathbb{Z}\). Then \(\psi\) is a topological conjugacy.

Proof. It is enough to prove that \(\psi\) is onto. Let \(P := \psi^{-1} P_0\). Then for \(\delta > 0\) and \(n \in \mathbb{N}\), any \((\delta, n)\) separated set \(E\) in \(A^n \in P^n\) has cardinality bounded by a constant. Indeed there is \(\delta' > 0\) depending only on \(\delta\) and \(\psi\) such that \(\psi(E)\) is \((\delta', n)\) separated in \(\psi(A^n) \in P^n_0\). In particular \(h_{\text{top}}(T) = h_{\text{top}}(T, P) := \inf_n \log \# P_n^n\). But as \(\# P = K\) we have \(\# P^n = K^n\) for all \(n \in \mathbb{N}\). Fix \(y \in \{1, \ldots, K\}^\mathbb{Z}\) and let us show there exists \(x \in X\) with \(\psi(x) = y\). Let \(n \in \mathbb{N}\). As \(\# P^{2n+1} = K^{2n+1}\) there is \(x_n \in X\) such that the \(k\)th coordinates of \(\psi(x_n)\) with \(|k| \leq n\) coincide with those of \(y\). Then if \(x\) is an accumulation point of \((x_n)_n\) we get \(\psi(x) = y\) by continuity of \(\psi\). \(\square\)

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LPMA - CNRS UMR 7599, Universite Paris 6, 75252 Paris Cedex 05 FRANCE,
david.burguet@upmc.fr