Convergence in KL Divergence of the Inexact Langevin Algorithm with Application to Score-based Generative Models

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Abstract

We study the Inexact Langevin Algorithm (ILA) for sampling using estimated score function when the target distribution satisfies log-Sobolev inequality (LSI), motivated by Score-based Generative Modeling (SGM). We prove a long-term convergence in Kullback-Leibler (KL) divergence under a sufficient assumption that the error of the score estimator has a bounded Moment Generating Function (MGF). Our assumption is weaker than $L^\infty$ (which is too strong to hold in practice) and stronger than $L^2$ error assumption, which we show not sufficient to guarantee convergence in general. Under the $L^\infty$ error assumption, we additionally prove convergence in Rényi divergence, which is stronger than KL divergence. We then study how to get a provably accurate score estimator which satisfies bounded MGF assumption for LSI target distributions, by using an estimator based on kernel density estimation. Together with the convergence results, we yield the first end-to-end convergence guarantee for ILA in the population level. Last, we generalize our convergence analysis to SGM and derive a complexity guarantee in KL divergence for data satisfying LSI under MGF-accurate score estimator.

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1 Introduction

Score-based Generative Modeling (SGM) is a family of sampling methods which have state-of-the-art performance in many applied areas [SE19, IJA20, CYAE+20, SE20, SSDK+21, CZZ+21, LWYL22]. The basic idea of SGM is a two-step procedure. The first step is to estimate the score function from the data (for example via score matching using neural network), the second step is to get new samples from the estimated score via Langevin algorithms. Despite the demonstrated empirical successes, the theoretical understanding of the methods is still lacking. Motivated by this, we study the Langevin dynamics and its discrete algorithms using estimated score in this paper.

1.1 Related work

In the case of knowing exact score function, there have been extensive studies on the convergence properties of the Langevin dynamics under various assumptions such as strong log-concavity or weaker isoperimetric inequality such as log-Sobolev inequality (LSI), which allows for some non-log-concavity [JKO98, OV00, VW19, BCE+22]. There are also convergence guarantees in discrete time with small bias under strong log-concavity or the weaker LSI and smoothness with small step size [DK17, CB18, VW19, MMS20, DMM19, BCE+22].

In the case where exact evaluation of score function is computationally costly or even not available, [HZ17, DK17, MMS20] studied Langevin algorithms using approximated score. When the score approximation has bounded bias and variance, the Wasserstein distance converges to a bias under strong log-concavity and smoothness with a small step size, see [HZ17, Theorem 3.4], [DK17, Theorem 4] and [MMS20, Theorem 1.4]. However, their assumptions on the error of score approximation are fundamentally different from ours and not applicable in the context of SGM, see Section 4 for more details.

More recently, there has been a surge of work in the analysis of SGM. [DBTHD21] studied the convergence in total variation (TV) that requires $L^\infty$ error assumption. Although $L^\infty$ is sufficient to guarantee the convergence, it is too strong to hold in practice since it requires a finite and uniform error at every point. [BMR20] provided the first convergence result under $L^2$ error assumption, the result is in Wasserstein distance of order 2 but the error bound suffers from curse of dimensionality. [LLT22a, DB22] also studied convergence under $L^2$ error assumption. Their results are in TV and Wasserstein distance of order 1 respectively, but the convergence can only be achieved after running the algorithm for a moderate amount of time since their error bounds blow up as the running time $T \to \infty$. The reason why one cannot run the algorithm for infinitely long time is that the stationary distribution of Langevin dynamics with an $L^2$-accurate score estimator can be far away from the desired distribution [BCE+22, LLT22a]. All the aforementioned work on SGM assumed either strong log-concavity or weaker isoperimetric inequality such as log-Sobolev inequality (LSI). More recently, [CCL+22, LLT22b] generalized the convergence in TV to a more general data assumption which doesn’t require LSI. Although the two papers get similar results, their proof techniques are
\[
\begin{array}{|c|c|c|c|}
\hline
\text{time} & \text{score} & \text{convergence under LSI} & \text{reference} \\
\hline
\text{continuous} & \text{exact} & H_\nu(\hat{\rho}_t) \leq e^{-2\alpha t} H_\nu(\rho_0) & [OV00] \\
& \text{inexact} & H_\nu(\rho_t) \lesssim e^{-\frac{\alpha}{2} t} H_\nu(\rho_0) + \varepsilon_{\text{mgf}} & \text{Theorem 1} \\
\text{discrete} & \text{exact} & H_\nu(\hat{\rho}_k) \lesssim e^{-\alpha h k} H_\nu(\rho_0) + h & [VW19] \\
& \text{inexact} & H_\nu(\rho_k) \lesssim e^{-\frac{\alpha h k}{2}} H_\nu(\rho_0) + h + \varepsilon_{\text{mgf}} & \text{Theorem 2} \\
\hline
\end{array}
\]

Table 1: Comparison between exact and inexact Langevin dynamics and algorithm

very different. We will discuss this in more detail in Section 6. However, their error bounds are not stable either and suffer from long running time \(T\).

### 1.2 Our contribution

In this paper, we study inexact Langevin dynamics (ILD) and inexact Langevin Algorithm (ILA). We show that \(L^p\) error assumption for any \(1 \leq p < \infty\) is not sufficient to guarantee long-term convergence in TV for ILD, see Example 1, necessitating a stronger assumption on the score error. Note that \(L^\infty\) is sufficient but unfortunately it hardly holds in practice. Even in Gaussian case, the \(L^\infty\) assumption can easily fail if one estimates score function of \(N(\mu, \sigma^2)\) by that of \(N(\mu, \sigma^2 + \epsilon)\) for an arbitrarily small \(\epsilon > 0\). This also suggests having an \(L^p\)-accurate score estimator is not sufficient to guarantee we will get a distribution close to the desired one after running the algorithm for a long time. In the context of neural network, our result indicates controlling mean squared loss in score matching is not enough, one may need a stronger control over the error when constructing the loss function, such as MGF of the error (mean of exponential of squared error).

We therefore introduce bounded Moment Generating Function (MGF) assumption, which is in between \(L^\infty\) and \(L^p\). We prove convergence in Kullback-Leibler (KL) divergence under this MGF assumption for both ILD and ILA, see a summary of our results in Table 1. We also give a proof of convergence in Rényi divergence under \(L^\infty\) error assumption, which is stronger than KL divergence, see Theorem 4. Contrary to previous results [LLT22a, DB22, CCL+22, LLT22b], our convergence result is stable, which means we can run the algorithm for an infinitely long time and our bound will not blow up as \(T \to \infty\), therefore is more desirable. We compare our results with the most recent two [CCL+22, LLT22b] in Table 2. [DB22, LLT22a] are not included since [DB22] studies the manifold setting which is not in the scope of our paper and [LLT22a] is generalized by [LLT22b].

We also partially answer an open question in [CCL+22, LLT22b] of how to get a provable accurate score estimator in an idealized setting where we have infinitely many samples. We show using Kernel Density Estimate (KDE)-based approach in LSI distribution will give us a score estimator which satisfies the bounded MGF assumption in the population level, thus also satisfies the weaker \(L^p\) assumption. Together with our convergence results in Theorem 1 and Theorem 2,
Table 2: Comparison of convergence results for SGM. Here $\rho_K$ is the distribution of output of SGM at $K$-th step, $\nu$ is the target distribution, $\gamma = \mathcal{N}(0, \alpha^{-1} I)$ is the target distribution of forward process and $h$ is step size.

we yield the first end-to-end guarantee for ILA, see Theorem 3. Last, we generalize our analysis from static ILA to SGM and derive a similar convergence result in KL divergence, see Theorem 5.

2 Problem Setting

Suppose we want to sample from a probability distribution $\nu$ on $\mathbb{R}^d$. We assume $\nu$ has a full support on $\mathbb{R}^d$ and it has a density function $\nu(x) \propto e^{-f(x)}$ with respect to the Lebesgue measure. We assume $f: \mathbb{R}^d \to \mathbb{R}$ is differentiable. The \textit{score function} of $\nu$ is the vector field $s_\nu: \mathbb{R}^d \to \mathbb{R}^d$ given by

$$s_\nu(x) = \nabla \log \nu(x) = -\nabla f(x).$$

How to sample from $\nu$ depends on what information on $\nu$ we are allowed to access.

2.1 Sampling with exact score function

\textbf{Langevin dynamics.} Suppose we know (can evaluate) the score function $s_\nu = \nabla \log \nu = -\nabla f$. Then we can run the Langevin dynamics in continuous time:

$$dX_t = s_\nu(X_t) \, dt + \sqrt{2} \, dW_t$$

where $W_t$ is the standard Brownian motion in $\mathbb{R}^d$. We know the Langevin dynamics converges to the target distribution $\nu$ exponentially fast in KL divergence and Wasserstain distance if $\nu$ satisfies some assumption such as isoperimetry (LSI or Poincaré inequality). We also have fast convergence rate in Rényi divergence under LSI, which is important for privacy applications [VW19, BCE+22]. If the target does not satisfy isoperimetry, then the Langevin dynamics has slow convergence (e.g. metastability phenomenon for a mixture of Gaussians).

In discrete time, a simple discretization is the Unadjusted Langevin Algorithm (ULA):

$$x_{k+1} = x_k + h \, s_\nu(x_k) + \sqrt{2h} \, z_k$$

where $h > 0$ is step size and $z_k \sim \mathcal{N}(0, I)$ is an independent standard Gaussian in $\mathbb{R}^d$. We have matching convergence rates with vanishing bias (scales with step size). We also have fast convergence rate in Rényi divergence under LSI, see [VW19, BCE+22].
2.2 Sampling with inexact score function

Inexact Langevin dynamics. Suppose we only have an estimate $s: \mathbb{R}^d \to \mathbb{R}^d$ of the score function $s_\nu$ of $\nu$. Then we can run the inexact Langevin dynamics (ILD) in continuous time:

$$dX_t = s(X_t) dt + \sqrt{2} dW_t. \quad (4)$$

If $s$ is a good estimator of $s_\nu$, then we might hope that the evolution of ILD (4) approximately converges to $\nu$. This is not always true, depending on how we measure the error (between $s$ and $s_\nu$) and the convergence (between $\rho_t$ along ILD and $\nu$).

In discrete time, we can run the Inexact Langevin Algorithm (ILA):

$$x_{k+1} = x_k + h s(x_k) + \sqrt{2h} z_k \quad (5)$$

where $h > 0$ is step size and $z_k \sim \mathcal{N}(0, I)$ is an independent standard Gaussian in $\mathbb{R}^d$. Under some error assumption between $\hat{s}_\nu$ and $s_\nu$, we derive a biased convergence rate of the ILA (5) to $\nu$.

2.3 Notations and definitions

In this section, we review notations and definitions of KL divergence, relative Fisher information, Rényi divergence and Rényi information. Let $\rho, \nu$ be two probability distributions in $\mathbb{R}^d$ denoted by their probability density functions w.r.t. Lebesgue measure on $\mathbb{R}^d$. Assume $\rho$ and $\nu$ have full support on $\mathbb{R}^d$, and they have differentiable log density functions.

Definition 1 (KL divergence). The Kullback-Leibler (KL) divergence of $\rho$ w.r.t. $\nu$ is

$$H_{\nu}(\rho) = \int_{\mathbb{R}^d} \rho \log \frac{\rho}{\nu} dx.$$

Definition 2 (Relative Fisher information). The relative Fisher information of $\rho$ w.r.t. $\nu$ is

$$J_{\nu}(\rho) = \int_{\mathbb{R}^d} \rho \| \nabla \log \frac{\rho}{\nu} \|^2 dx.$$

Definition 3 (Rényi divergence). For $q \geq 0$, $q \neq 1$, the Rényi divergence of order $q$ of $\rho$ w.r.t. $\nu$ is

$$R_{q,\nu}(\rho) = \frac{1}{q - 1} \log F_{q,\nu}(\rho)$$

where

$$F_{q,\nu}(\rho) = \mathbb{E}_{\nu} \left[ \left( \frac{\rho}{\nu} \right)^q \right].$$

Note that when $q \to 1$, Rényi divergence recovers the KL divergence. Furthermore, $q \mapsto R_{q,\nu}(\rho)$ is increasing, so Rényi divergence bounds are stronger.

Definition 4 (Rényi information). For $q \geq 0$, the Rényi information of order $q$ of $\rho$ w.r.t. $\nu$ is

$$G_{q,\nu}(\rho) = \mathbb{E}_{\nu} \left[ \left( \frac{\rho}{\nu} \right)^q \| \nabla \log \frac{\rho}{\nu} \|^2 \right] = \frac{4}{q^2} \mathbb{E}_{\nu} \left[ \| \nabla \left( \frac{\rho}{\nu} \frac{\dot{\mathbb{I}}}{\|\dot{\mathbb{I}}\|^2} \right) \|^2 \right].$$

When $q = 1$, Rényi information recovers the relative Fisher information.
3 Main results

3.1 Convergence of inexact Langevin dynamics

We first consider continuous time and compare the exact Langevin dynamics (2) and ILD (4). Before stating the result, we first introduce the following assumptions.

Assumption 1 (LSI). A probability distribution $\nu$ satisfies LSI with constant $\alpha > 0$, which means for any probability distribution $\rho$

$$H_\nu(\rho) \leq \frac{1}{2\alpha} J_\nu(\rho).$$

Assumption 2 (MGF error assumption). The score estimator $s$ has an error that has finite moment generating function of some order $r > 0$, $\varepsilon_{\text{mgf}} = \log E_\nu[\exp(r\|s - s_\nu\|^2)] < \infty$.

Recall that if the target distribution $\nu$ satisfies $\alpha$-LSI, then along Langevin dynamics (2), KL divergence is contracting exponentially fast: $H_\nu(\tilde{\rho}_t) \leq e^{-2\alpha t} H_\nu(\rho_0)$ [VW19]. When $s$ is an approximation of score function $s_\nu$ and its error satisfies Assumption 2, we show a matching convergence rate with an additional bias term induced by score estimation error.

Theorem 1 (Convergence of KL divergence for ILD). Suppose Assumptions 1 and 2 with $r \geq \frac{1}{\alpha}$ hold. Then for $X_t \sim \rho_t$ along the ILD (4),

$$H_\nu(\rho_t) \leq e^{-\frac{1}{2\alpha} t} H_\nu(\rho_0) + 2(1 - e^{-\frac{1}{2\alpha} t}) \varepsilon_{\text{mgf}}.$$

The proof of Theorem 1 uses Donsker–Varadhan variational characterization of KL divergence; see Appendix 8.1.

We observe that the bound above is stable, which means it has a finite limit as $t \to \infty$. In particular, it implies an estimate of the asymptotic bias: $H_\nu(\nu_s) \leq 2\varepsilon_{\text{mgf}}$ where $\nu_s$ is the biased limit of the ILD with score estimator $s$. We note this is actually a trivial bound in the following sense: If $s = \nabla \log \nu_s$, then the MGF assumption with $r = \frac{1}{\alpha}$ and the assumption that $\nu$ be $\alpha$-LSI already imply $\varepsilon_{\text{mgf}} \geq \frac{1}{\alpha} J_\nu(\nu_s) \geq 2H_\nu(\nu_s)$, which is better than the asymptotic bias estimate above by a factor of 4.

3.2 Convergence of inexact Langevin algorithms

We then consider discrete time and compare the ULA (3) with ILA (5). To prove convergence for discrete-time algorithm, we require the following Lipschitzness assumptions.

Assumption 3 (L-smoothness). $f = -\log \nu$ is $L$-smooth, which means $\nabla f$ is $L$-Lipschitz.

Assumption 4 (Lipschitz score estimator). The score estimator $s$ is $L_s$-Lipschitz.
Now we state our main result on the convergence for ILA.

**Theorem 2** (Convergence of KL divergence for ILA). Suppose Assumption 1, 2 with $r = \frac{\alpha}{\alpha}$, 3 and 4 hold. If $0 < h < \min\left(\frac{\alpha}{12L^2}, \frac{1}{2\alpha}\right)$, then after $k$ iterations of ILA (5),

$$H_\nu(\rho_k) \leq e^{-\frac{1}{2}\alpha h}H_\nu(\rho_0) + C_1 h + C_2 \varepsilon_{\text{mgf}},$$

where $C_1 = O\left(\frac{dL^2}{\alpha}\right)$ and $C_2 = \frac{8}{3}$.

Proof of Theorem 2 is in Appendix 8.2, via an extension of the interpolation technique of [VW19]. The convergence rate matches the state-of-the-art result in ULA (3):

$$H_\nu(\rho_k) \lesssim e^{-\alpha h}H_\nu(\rho_0) + \frac{dL^2}{\alpha} h,$$

see [VW19, Theorem 1]. But here we have an extra non-vanishing term induced by the error of score estimator. So in order to have a small asymptotic error, we need an accurate score estimator which can be obtained via Gaussian KDE with a small bandwidth, shown in the following lemma.

**Lemma 1.** For $t \geq 0$, let $\rho_t = \rho \ast \mathcal{N}(0, tI)$ and $s_t = \nabla \log \rho_t$. Assume $\rho$ is $L$-smooth and $\sigma$-sub-Gaussian for some $0 < \sigma < \infty$. For all $r > 0$ and for all $0 \leq t \leq \min\{\frac{d}{\|s(0)\|^2}, \frac{1}{2\sqrt{2}\sigma\sqrt{rL^2}}\}$,

$$\log \mathbb{E}_\rho[|s_t - s_\ast|^2] \lesssim trL^2(d + \sigma^2 tL^2).$$

Note that this result is in the population level where we can compute the estimator exactly. We will discuss it in more details in Section 5. Combining Lemma 1 and Theorem 2, we arrive the following complexity result of KDE-based ILA.

**Theorem 3.** Assume the target distribution $\nu$ is $\alpha$-LSI ($\alpha > 0$) and $L$-smooth. For any $\varepsilon > 0$, we estimate the score function $s_\nu$ by $s_t$ using Gaussian kernel with bandwidth $\sqrt{t} < \frac{\alpha^3 \varepsilon}{192L^2(2d\alpha^2 + L^2)}$.

Then running ILA (5) with $s_t$ and step size $h = O\left(\frac{\varepsilon \alpha}{dL^2}\right)$ for at least $k = \frac{4}{\alpha h} \log \frac{4H_\nu(\rho_0)}{\varepsilon}$ iterations reaches $H_\nu(\rho_k) \leq \varepsilon$.

### 4 Comparing different error assumptions

In this section we will compare Assumption 2 with the following.

**Assumption 5** ($L^\infty$ error assumption). The error of $s(x)$ has a finite $L^\infty$ norm at every $x$, i.e.

$$\varepsilon_\infty = \sup_{x \in \mathbb{R}^d} \|s_\nu(x) - s(x)\| < \infty.$$

**Assumption 6** ($L^p$ error assumption). The error of $s(x)$ has finite moment of order $p$ for some $1 \leq p < \infty$, i.e.

$$\varepsilon_{L^p} = \mathbb{E}_\nu[\|s_\nu(x) - s(x)\|^p] < \infty.$$
Comparing with $L^\infty$ error  Since Assumption 5 implies Assumption 2, the convergence results in Theorem 1 and Theorem 2 also hold under Assumption 5 in place of Assumption 2. In addition, under Assumption 5, we prove convergence in Rényi divergence of order $q \geq 1$, which is stronger than KL divergence. We conjecture the convergence holds under Assumption 2 and leave it for future work.

**Theorem 4** (Convergence of Rényi divergence under $L^\infty$ error). Suppose Assumption 1, 3, 4 and 5 hold. Let $q \geq 1$. If $0 < h < \min\{\frac{\alpha}{12Ls_q}, \frac{q}{4\alpha}\}$, then after $k$ iterations of ILA (5),

$$R_{q,\nu}(\rho_k) \leq e^{-\frac{1}{2}ahk}R_{q,\nu}(\rho_0) + C_1h + C_2\varepsilon^2_{\infty},$$

where $C_1 = O\left(\frac{dL^2_{s}\alpha^2}{\alpha}\right)$ and $C_2 = O\left(\frac{q^2}{\alpha}\right)$.

Proof of Theorem 4 is in Appendix 8.3.

Comparing with $L^p$ error  $L^\infty$-accurate score estimator requires a finite and uniform error at every point $x$, which is too strong and may not hold in practice. Even in Gaussian case, the $L^\infty$ assumption can easily fail if one estimates score function of $\mathcal{N}(\mu, \sigma^2)$ by that of $\mathcal{N}(\mu, \sigma^2 + \epsilon)$ for an arbitrarily small $\epsilon > 0$. In practice, score matching is a popular approach to estimate score function. By the construction of its loss function, the estimated score is only $L^2$-accurate with high probability, see Proposition 9 in [BMR20]. However, the stationary distribution of Langevin dynamics with $L^2$-accurate score can be far away from the true distribution in TV [LLT22a, BCE+22]. So having an $L^2$-accurate score estimator cannot guarantee the sampling algorithm will converge to the desired distribution. We further claim that any higher moment bound (Assumption 6) cannot guarantee the convergence either. This is to say, the stationary distribution of Langevin dynamics with an $L^p$-accurate score estimator where $1 \leq p < \infty$ can also be far away from the true distribution. We use the example in [BCE+22] to illustrate this.

**Example 1** ($L^p$ bound is not sufficient). Let $\nu = \frac{3}{4}\mathcal{N}(-m, 1) + \frac{1}{4}\mathcal{N}(m, 1)$ and $\mu = \frac{1}{2}\mathcal{N}(-m, 1) + \frac{1}{2}\mathcal{N}(m, 1)$. For all $m \geq \frac{1}{80}$ and $p \geq 1$, the following holds

$$\mathbb{E}_\nu\left[\|\nabla \log \nu - \nabla \log \mu\|^p\right] \leq 4^{p-1}m^{p}\exp\left(-\frac{m^2}{2}\right) \to 0 \quad \text{as} \quad m \to \infty. \quad (7)$$

However, the total variation between $\mu$ and $\nu$ is large,

$$\text{TV}(\mu, \nu) \geq \frac{1}{800}. \quad (8)$$

If the target distribution is $\nu$, but we run ILD with an $L^p$-accurate score estimator $s = \nabla \log \mu$, then the limiting distribution is $\mu$. One may expect as $m \to \infty$ the dynamics is asymptotically unbiased since the score estimation error vanishes exponentially fast, but our result shows the asymptotic bias $\text{TV}(\mu, \nu)$ is at least a constant. By Pinsker’s inequality, this implies $H_\nu(\mu)$ is also
at least a constant. Therefore, the result in Theorem 1 does not hold for non-LSI target distribution under Assumption 6 since as $t \to \infty$, $H_\nu(\rho_t) \to H_\nu(\mu) > \varepsilon_{L^p}$. We provide the derivation of Example 1 in Appendix 8.4. Therefore, in order to obtain a convergence guarantee with a minimal sufficient assumption, we need the error of score estimator in between $L^p$ and $L^\infty$.

Note that [HZ17, DK17, MMS20] also studied Langevin algorithms using estimated score. When the score estimator has bounded bias and variance, they showed convergence of Wasserstein distance under strong log-concavity and smoothness with a small step size. However, their assumptions require a bounded $L^2$ error w.r.t. all distributions along the Langevin dynamics, i.e. $E_{\rho_t}[\|s_\nu(x) - s(x)\|^2] < \infty$ for all $t \geq 0$. This is satisfied e.g. when we have an $L^\infty$-accurate score estimator; otherwise, this is not an easily verifiable condition. In fact, if we assume $E_{\rho_t}[\|s_\nu(x) - s(x)\|^2] \leq \varepsilon$ for all $\rho_t$ along the ILD or ILA, then we can also show the convergence in KL and Rényi divergence with an $\varepsilon$ bias. One of our contributions in this paper is to identify a weaker general assumption, namely the bounded MGF assumption, that is sufficient to guarantee convergence in KL divergence.

5 KDE-based score estimator

There are many different approaches to estimate the score function of a distribution, including score matching [KHR22] which is popular in practice, or classical techniques such as Maximum Likelihood Estimator (MLE). In this paper, we consider a simple score estimator based on kernel density estimation (KDE) with the Gaussian kernel. For simplicity, in this section we will work in the population level where we have infinitely many samples. We show that when the data distribution is sub-Gaussian, the KDE score estimator satisfies the bounded MGF assumption (Assumption 2), and thus also the $L^p$-error assumption (Assumption 6). This partially answers an open question in [CCL+22, LLT22b] for LSI case, in an idealized setting.

Recall that we say that $\rho$ is $\sigma$-sub-Gaussian for some $0 \leq \sigma < \infty$, which means:

$$E_\rho[\exp(\lambda \|X\|)] \leq \exp(\frac{\lambda^2 \sigma^2}{2}) \quad \forall \lambda \in \mathbb{R}.$$  

We show KDE score estimator for sub-Gaussian data satisfies Assumption 2 in the population level.

Lemma 1. For $t \geq 0$, let $\rho_t = \rho * N(0, tI)$ and $s_t = \nabla \log \rho_t$. Assume $\rho$ is $L$-smooth and $\sigma$-sub-Gaussian for some $0 < \sigma < \infty$. For all $r > 0$ and for all $0 \leq t \leq \min\{\frac{d}{\|s(0)\|}, \frac{1}{2\sqrt{2\sigma^2tL^2}}\}$,

$$\log E_\rho[e^{r\|s_t - s\|^2}] \lesssim trL^2(d + \sigma^2tL^2).$$

Proof of Lemma 1 is in Appendix 8.6. Moreover, since $\alpha$-LSI ($\alpha > 0$) implies $\frac{1}{\alpha}$-sub-Gaussian [Led99], together with Theorem 2 we obtain the end-to-end convergence guarantee in Theorem 3 for $\alpha$-LSI target distribution.

Lemma 2 ([Led99]). If $\rho$ satisfies $\alpha$-LSI where $\alpha > 0$, then $\rho$ is $\sigma$-sub-Gaussian with $\sigma = \frac{1}{\alpha}$.  

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Lemma 3. Assume $\rho \propto e^{-f}$ where $f$ is $L$-smooth. Let it evolves along the heat flow, then at time $t \in (0, \frac{1}{L})$, $\rho_t = \rho * \mathcal{N}(0, tI)$ is $L_t$-smooth where $L_t$ is decreasing in $t$.

Proof of Lemma 3 is in Appendix 8.7. Lemma 3 implies that smoothness is preserved when using Gaussian-KDE with a small bandwidth.

Our above result shows score estimator using Gaussian KDE works well in the population level, i.e. to estimate $s = \nabla \log \rho$ we can use $s_t = \nabla \log \rho_t$ for small enough $t$, where $\rho_t = \rho * \mathcal{N}(0, tI)$. So assuming we can compute $s_t(y)$, it is a good estimator. In practice, suppose we have i.i.d. samples $X_1, \ldots, X_n \sim \rho$. We define the empirical measure $\hat{\rho} = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$. Let $\hat{\rho}_t = \hat{\rho} * \mathcal{N}(0, tI) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{N}(X_i, tI)$. Then we can compute the empirical version of the KDE score function estimator:

$$\hat{s}_t(y) = \nabla \log \hat{\rho}_t(y) = \frac{\sum_{i=1}^{n}(X_i - y)e^{-\frac{|y - X_i|^2}{2t}}}{t \sum_{i=1}^{n} e^{-\frac{|y - X_i|^2}{2t}}}.$$  \hspace{1cm} (9)

which can be obtained for each $y$ and $t$, in $O(n)$ time. It is interesting to investigate the finite sample version of Lemma 1, i.e. error bound of estimator (9). We leave this for future work.

6 Application to SGM

In this section, we generalize our convergence analysis to SGM. We consider a specific type of SGM called denoising diffusion probabilistic modeling (DDPM) where the forward process is the Ornstein-Uhlenbeck (OU) process, a.k.a. variance preserving SDE [HJA20]. But our proof framework also works for other types of SGM, such as [SSDK+21, DVK22]. We first provide a brief introduction to DDPM. Here the goal is to sample from $\nu$.

**Forward process** For the forward process, we start from $X_0 \sim \nu_0 = \nu$ which is the data distribution satisfying LSI with constant $\alpha > 0$, and follow the OU process targeting $\gamma = \mathcal{N}(0, \alpha^{-1}I)$:

$$dX_t = -\alpha X_t dt + \sqrt{2}dW_t.$$ 

Let $X_t \sim \nu_t = P_t(\nu)$ be the evolution along the OU flow. Let $s_t = \nabla \log \nu_t$ be the score function at time $t$. As $t \to \infty$, $s_t(x) \to s_\infty(x) = -\alpha x$.

**Backward process** Suppose we run the forward process until time $T > 0$, ending at $\nu_T$. The true backward OU process (run forward in time) follows the SDE:

$$d\tilde{Y}_t = (\alpha \tilde{Y}_t + 2s_{T-t}(\tilde{Y}_t))dt + \sqrt{2}dW_t.$$ \hspace{1cm} (10)

We start from $\tilde{Y}_0 \sim \mu_0 = \nu_T$ to $\tilde{Y}_T \sim \mu_T = \nu$, so $\tilde{Y}_t \sim \mu_t = \nu_{T-t}$ for $0 \leq t \leq T$ [And82].

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Implementation and algorithm  Our algorithm is as follows. Let $h > 0$ be step size, and $K = \frac{T}{h}$ so $T = Kh$ (assume $K \in \mathbb{N}$). We construct a continuous-time process $(Y_t)_{0 \leq t \leq T}$ that starts from $Y_0 \sim \rho_0 = \gamma = \mathcal{N}(0, \alpha^{-1}I)$. In each step, from time $hk$ to $h(k+1)$, our process follows the SDE:

$$dY_{hk+t} = (\alpha Y_{hk+t} + 2\hat{s}_{h(K-k)}(Y_{hk}))dt + \sqrt{2}dW_t$$  \hspace{1cm} (11)

where $\hat{s}_{hk}(y)$ is an approximation to $s_{hk}(y)$ and $t$ is from 0 to $h$. Then we update $y_{k+1}$ as the solution of the SDE (11) at time $t = h$ starting from $Y_{hk} = y_k$, i.e.

$$y_{k+1} = e^{\alpha h}y_k + 2\left(\frac{e^{\alpha h} - 1}{\alpha}\hat{s}_{h(K-k)}(y_k) + \sqrt{\frac{e^{2\alpha h} - 1}{\alpha}}z_k\right)$$  \hspace{1cm} (12)

where $z_k \sim \mathcal{N}(0, I)$.

We then present our convergence result for DDPM under the bounded MGF assumption.

**Theorem 5.** Let $\nu$ be the data distribution. Suppose Assumption 1, 3 and 4 hold and the score estimator satisfies MGF assumption with $r \sim \alpha^{-1}$,

$$\varepsilon_{mgf} = \sup_{t \in \{kh, k \in [K]\}} \log \mathbb{E}_{\nu_t}[\exp\left(\frac{65}{6\alpha} \|\hat{s}_t - s_t\|^2\right)] < \infty.$$  

Let $\rho_k$ be the distribution of output of DDPM (12) at $k$-th step starting from $\rho_0 = \gamma = \mathcal{N}(0, \alpha^{-1}I)$. If $H_{\nu}(\gamma) < \infty$ and $0 < h \leq \frac{\alpha}{96LsL}$, then

$$H_{\nu}(\rho_K) \leq e^{\frac{-92hK}{4}}H_{\nu}(\gamma) + 3360\frac{hL^2d}{\alpha} + 8\varepsilon_{mgf}.$$  

To interpret the result, the first term is from initialization error. Note that the true backward process starts from $\nu_T$ (the terminal distribution of forward process) but in practice our algorithm starts from the nicer $\gamma = \mathcal{N}(0, \alpha^{-1}I)$. The second term is discretization error which scales with $h$. The third term is from the score estimation error and is non-vanishing (doesn’t go to 0 when step size goes to 0). Contrary to the results in [CCL+22, LLT22b], our convergence result is stable (the bound will not blow up as $T \to \infty$) though we do require the LSI data assumption and $\varepsilon_{mgf}$ score error assumption, see Table 2. Later in this section, we will provide more comparison between our proof techniques and those in [CCL+22, LLT22b].

Our proof is via extending the interpolation approach. Specifically, we compare the evolution of KL divergence along one step of the DDPM with the evolution along the true backward process in continuous time. For simplicity, suppose $k = 0$, so we start the backward process at $\tilde{y}_0 \sim \mu_0 = \nu_T$ and the DDPM at $y_0 \sim \rho_0 = N(0, \alpha^{-1}I)$. We can write one step of the DDPM

$$y_1 = e^{\alpha h}y_0 + 2\left(\frac{e^{\alpha h} - 1}{\alpha}\hat{s}_T(y_0) + \sqrt{\frac{e^{2\alpha h} - 1}{\alpha}}z_0\right),$$
as the output at time $t = h$ of the SDE
\[ dY_t = (\alpha Y_t + 2\hat{s}_T(Y_0))dt + \sqrt{2}dW_t \] (13)
where $W_t$ is the standard Brownian motion in $\mathbb{R}^d$ starting at $W_0 = 0$, and $t$ is from 0 to $h$. Then we have the following lemma on time derivative of the KL divergence between DDPM and true backward process.

**Lemma 4.** Suppose the assumptions in Theorem 5 hold. Let $\mu_t$ be the distribution at time $t$ along the true backward process (10) and $\rho_t$ be the distribution at time $t$ along the interpolation SDE (13). If $0 < h \leq \frac{\alpha}{96L_s}$, then
\[
\frac{d}{dt}H_{\mu_t}(\rho_t) \leq -\frac{\alpha}{4}H_{\mu_t}(\rho_t) + \frac{3}{4}\alpha \varepsilon_{mgf} + 315tL_s^2d.
\]

We provide the proof of Lemma 4 in Appendix 8.5.3. Following Lemma 4 we can get the following one step contraction.

**Lemma 5.** Suppose the assumptions in Theorem 5 hold. Let $\mu_k$ be the distribution at $t = hk$ along the true backward process (10) starting from $\mu_0 = \nu_T$, then $\mu_k = \nu_T - hk$ and $\mu_K = \nu_T$. Let $\rho_k$ be the distribution of output of DDPM (12) at $k$-th step starting from $\rho_0 = N(0, \alpha^{-1}I)$. Assume $0 < h \leq \frac{\alpha}{96L_s}$. Then for all $K > 0$ and for all $k = 0, 1, \ldots, K - 1$,
\[
H_{\mu_{k+1}}(\rho_{k+1}) \leq e^{-\frac{\alpha h}{4}}H_{\mu_k}(\rho_k) + 630h^2L_s^2d + \frac{3}{2}h\alpha \varepsilon_{mgf}.
\]

We provide the proof of Lemma 5 in Appendix 8.5.4. Recursively applying Lemma 5 will give us the contraction of KL divergence starting from $H_{\mu_0}(\rho_0)$, i.e. $H_{\nu_T}(\gamma)$. Note that $\nu_T$ is the distribution at time $T$ along OU process and $\gamma$ is the target distribution of the same OU process therefore we can further measure $H_{\nu_T}(\gamma)$ by analyzing the OU process. First we will need the following result which states how LSI constant evolves along the OU process.

**Lemma 6.** Let $X_0 \sim \nu_0 = \nu$ where $\nu$ is $\alpha$-LSI ($\alpha > 0$) evolve along the following OU process targeting $N(0, \beta^{-1}I)$:
\[ dX_t = -\beta X_t dt + \sqrt{2}dW_t. \] (14)
At time $t$, $X_t \sim \nu_t$ where $\nu_t$ is $\alpha_t$-LSI and $\alpha_t = \frac{\alpha \beta}{\alpha + (\beta - \alpha)e^{-2\beta t}}$. In particular, if $\beta = \alpha$, then $\alpha_t = \alpha$.

We provide the proof of Lemma 6 in Appendix 8.5.5. Theorem 5 then follows from Lemma 5 and Lemma 6, see complete proof in Appendix 8.5.6. Furthermore, Theorem 5 directly implies the following complexity result.

**Corollary 1.** Suppose the assumptions in Theorem 5 hold. For any $\varepsilon > 0$, if the error of score estimator $\hat{s}$ satisfies $\varepsilon_{mgf} \leq \frac{1}{2\varepsilon}$, then running the DDPM with step size $h = O\left(\frac{\varepsilon \alpha}{dL_s^2}\right)$ for $K = O\left(\frac{1}{\alpha h} \log \frac{3H_{\nu}(\gamma)}{\varepsilon}\right)$ iterations will reach a distribution $\rho_K$ with $H_{\nu}(\rho_K) \leq \varepsilon$. 

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Remark 1 Here we provide a brief overview of the proof techniques used in [CCL+22, LLT22b]. [CCL+22] uses data processing inequality to decompose $\text{TV}(\rho_K, \nu)$ into initialization error $\text{TV}(\rho_0, \nu_T)$ and $\text{TV}(\rho_T^\gamma, \mu_T^\nu)$ where $\rho_T^\gamma$ is path measure on the inexact backward SDE (11) starting from $\gamma$ and $\mu_T^\nu$ is path measure on the exact backward SDE (10) starting from $\nu_T$, the terminal distribution of forward process. The initialization error $\text{TV}(\rho_0, \nu_T)$ can be taken care of by the exponential convergence of OU process. Note that they bypass the LSI assumption on $\nu$ by using the fact that the target distribution $\gamma$ of OU process is $\alpha$-LSI. Then they bound the second error $\text{TV}(\rho_T^\gamma, \mu_T^\nu)$ by Girsanov theorem. [LLT22b] adopts a different approach; their proof is also via interpolation like ours. However, instead of directly considering the $L^2$ error assumption, they first consider the convergence of $\chi^2$-divergence under $L^\infty$ error assumption and then convert the result into TV convergence under $L^2$ assumption via a “bridge” theorem (see [LLT22b, Theorem 7.1]).

Remark 2 In the DDPM algorithm we described above, the target distribution of forward process is $\mathcal{N}(0, \alpha^{-1} I)$ where $\alpha$ is the LSI constant of the target distribution $\nu$. In fact, we can target a general Gaussian $\mathcal{N}(0, \beta^{-1} I)$ in the forward process, but some computation shows that the optimal choice of $\beta$ that gives the best convergence rate is $\alpha$.

7 Conclusion and discussion

In this paper we studied Inexact Langevin Dynamics (ILD) and Inexact Langevin Algorithm (ILA) where we don’t have access to the exact score function but we have an estimator of it. We proved convergence guarantees of ILD and ILA in KL divergence under a sufficient assumption that the score estimator error has a bounded moment generating function (MGF, Assumption 2). The bounded MGF assumption is weaker than $L^\infty$ but stronger than $L^p$ error assumption; we showed the latter is not sufficient to guarantee convergence when the target distribution is a mixture of two Gaussians. In the context of score-based generative models, our results suggest that controlling $L^2$ loss (mean squared error) as in score matching may not be enough to guarantee the upstream sampling algorithm will converge to the desired distribution, therefore in order to get a theoretical guarantee one may need a stronger control over the error when constructing the loss function, such as the MGF (mean of exponential of squared error) We also generalize our convergence analysis to SGM and derive a complexity guarantee in KL divergence for data satisfying LSI under MGF-accurate score estimator. Contrary to previous results [LLT22a, DB22, CCL+22, LLT22b], our convergence result is stable, which means the bound does not blow up as $T \to \infty$ and thus gives a bound on the asymptotic bias, albeit at the assumption that the target distribution satisfies LSI.

Although the bounded MGF assumption is quite strong, we show it is satisfied in the population level when using a simple KDE-based score estimator when the data distribution is smooth and sub-Gaussian. This partially answers an open question of how to get a provable accurate score estimator in [CCL+22, LLT22b], since the MGF assumption is stronger than $L^2$. Together with
our convergence results under MGF assumption, this yields the first end-to-end guarantee for ILA.

Our results leave many open questions. Our convergence result for ILA is under the bounded MGF assumption; is it possible to prove similar convergence result in KL divergence under LSI only with $L^2$ error assumption? The counterexample 1 shows it is not possible to obtain a stable KL result under $L^2$ error for non-LSI data such as a mixture of Gaussians; however, it does not rule out the combination of LSI and $L^2$-accurate score estimator. It may be reasonable to hope that $L^2$-accurate score estimators work for the nicer LSI target distribution. It would also be interesting to investigate whether the score estimators learned via neural networks satisfy the bounded MGF assumption, or the weaker $L^2$ assumption. Another intriguing question is to obtain a finite sample result of the KDE-based score estimator (9) and further one may ask what is the sample complexity of SGM via KDE score estimator? Finally, we can study the convergence of other sampling algorithms such as the underdamped Langevin dynamics with inexact score function.

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8 Appendix

8.1 Proof of Theorem 1

Proof. The Fokker-Planck equation of the Langevin dynamics (4) is

$$\frac{\partial \rho_t}{\partial t} = \nabla \cdot (-\rho_t s) + \Delta \rho_t.$$

Therefore the time derivative of KL divergence is

$$\frac{d}{dt} H_\nu(\rho_t) = \int_{\mathbb{R}^d} \frac{\partial \rho_t}{\partial t} \log \frac{\rho_t}{\nu} \, dx$$

$$= \int_{\mathbb{R}^d} \left( \nabla \cdot (-\rho_t s) + \Delta \rho_t \right) \log \frac{\rho_t}{\nu} \, dx$$

$$= \int_{\mathbb{R}^d} \left( -\nabla \cdot (\rho_t s) + \nabla \cdot (\rho_t \nabla \log \frac{\rho_t}{\nu}) + \nabla \cdot (\rho_t \nabla \log \nu) \right) \log \frac{\rho_t}{\nu} \, dx$$

$$= \int_{\mathbb{R}^d} \left( \nabla \cdot (\rho_t (\nabla \log \frac{\rho_t}{\nu} - s + \nabla \log \nu)) \right) \log \frac{\rho_t}{\nu} \, dx$$

$$= -\int_{\mathbb{R}^d} \rho_t (\nabla \log \frac{\rho_t}{\nu} - s + \nabla \log \nu, \nabla \log \frac{\rho_t}{\nu}) \, dx \quad \text{integration by parts}$$

$$= -\int_{\mathbb{R}^d} \rho_t \|\nabla \log \frac{\rho_t}{\nu}\|^2 \, dx + \int_{\mathbb{R}^d} \rho_t (s - \nabla \log \nu, \nabla \log \frac{\rho_t}{\nu}) \, dx$$

$$= -J_\nu(\rho_t) + \int_{\mathbb{R}^d} \rho_t (s - \nabla \log \nu, \nabla \log \frac{\rho_t}{\nu}) \, dx$$

$$\leq -J_\nu(\rho_t) + \mathbb{E}_{\rho_t} \left[ \|s - \nabla \log \nu\|^2 \right] + \frac{1}{4} \mathbb{E}_{\rho_t} \left[ \|\nabla \log \frac{\rho_t}{\nu}\|^2 \right] \quad \text{since } \langle a, b \rangle \leq \|a\|^2 + \frac{1}{4} \|b\|^2$$

$$= -\frac{3}{4} J_\nu(\rho_t) + \mathbb{E}_{\rho_t} \left[ \|s - \nabla \log \nu\|^2 \right].$$

By Assumption 2 and Donsker–Varadhan variational characterization of KL divergence $\mathbb{E}_P[f(x)] \leq \log \mathbb{E}_Q e^{f(x)} + H_Q(P)$,

$$\mathbb{E}_{\rho_t} \left[ \|s - \nabla \log \nu\|^2 \right] \leq \alpha \varepsilon_{mgf} + \alpha H_\nu(\rho_t).$$

Therefore,

$$\frac{d}{dt} H_\nu(\rho_t) \leq -\frac{3}{4} J_\nu(\rho_t) + \alpha \varepsilon_{mgf} + \alpha H_\nu(\rho_t)$$

$$\leq -\frac{3}{2} \alpha H_\nu(\rho_t) + \alpha \varepsilon_{mgf} + \alpha H_\nu(\rho_t) \quad \text{by } \alpha \text{-LSI}$$

$$= -\frac{1}{2} \alpha H_\nu(\rho_t) + \alpha \varepsilon_{mgf}.$$

This is equivalent to

$$\frac{d}{dt} e^{\frac{1}{2} \alpha t} H_\nu(\rho_t) \leq e^{\frac{1}{2} \alpha t} \alpha \varepsilon_{mgf}.$$
Integrating from 0 to $t$, we have

$$H_\nu(\rho_t) \leq e^{-\frac{1}{2}\alpha t} H_\nu(\rho_0) + 2 \left(1 - e^{-\frac{1}{2}\alpha t}\right) \epsilon_{mgf}.$$ 

\[\square\]

### 8.2 Proof of Theorem 2

We restate the full theorem here.

**Theorem 2.** Suppose the target distribution $\nu \propto e^{-f}$ satisfies LSI with constant $\alpha > 0$ and $f$ is $L$-smooth. Let $s(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an $L_s$-Lipschitz score estimator and the error satisfies Assumption 2 with $r = \frac{9}{\alpha}$. Let $0 < h < \min\left(\frac{\alpha}{12L_sL}, \frac{1}{2}\alpha\right)$, after $k$ iterations of ILA (5)

$$H_\nu(\rho_k) \leq e^{-\frac{1}{4}\alpha hk} H_\nu(\rho_0) + \frac{896 dL_s^2L}{\alpha} h \frac{8}{3} \epsilon_{mgf}.$$ 

We prove the theorem via an extension of interpolation approach in [VW19]. Note that one step of the ILA (5) is the solution $x_{k+1} = X_h$ of the following SDE at time $t = h$ starting from $X_0 = x_k$

$$dX_t = s(X_0)dt + \sqrt{2}dW_t \tag{15}$$

where $W_t$ is the standard Brownian motion in $\mathbb{R}^d$, and $t$ is from 0 to $h$. To prove Theorem 2, we will use the following auxiliary results.

**Lemma 7.** Suppose the assumptions in Theorem 2 hold. Let $\rho_t$ be the law of Eq. (15), then we have the following bound for time derivative of KL divergence

$$\frac{d}{dt} H_\nu(\rho_t) \leq -\frac{3}{4} J_\nu(\rho_t) + \mathbb{E}_{\rho_t} \left[\|s(x_0) - \nabla \log \nu(x_t)\|^2\right].$$

**Lemma 8.** If the score estimator $s(x)$ is $L_s$-Lipschitz and $t \leq \frac{1}{3L_s}$, then

$$\|s(x_t) - s(x_0)\|^2 \leq 18L_s^2 t^2 \|s(x_t) - \nabla \log \nu(x_t)\|^2 + 18L_s^2 t^2 \|\nabla \log \nu(x_t)\|^2 + 6L_s^2 t^2 \|z_0\|^2$$

where $x_t = x_0 + ts(x_0) + \sqrt{2t}z_0$ and $z_0 \sim \mathcal{N}(0, I)$.

**Lemma 9.** Suppose the assumptions in Theorem 2 hold, then along each step of ILA (5),

$$H_\nu(\rho_{k+1}) \leq e^{-\frac{1}{4}\alpha h} H_\nu(\rho_k) + 144dL_s^2Lh^3 + 24dL_s^2h^2 + \frac{\alpha h}{2} \epsilon_{mgf}.$$
8.2.1 Proof of Lemma 7

Proof of Lemma 7. The continuity equation corresponding to Eq. (15) is

\[ \frac{\partial \rho_t(x)}{\partial t} = -\nabla \cdot \left( \rho_t(x) \mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] \right) + \Delta \rho_t(x). \]

Therefore

\[ \frac{d}{dt} H_\nu(\rho_t) = \int_{\mathbb{R}^d} \frac{\partial \rho_t}{\partial t} \log \frac{\rho_t}{\nu} \, dx \]

\[ = \int_{\mathbb{R}^d} \left( -\nabla \cdot \left( \rho_t \mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] \right) + \Delta \rho_t \right) \log \frac{\rho_t}{\nu} \, dx \]

\[ = \int_{\mathbb{R}^d} \left( -\nabla \cdot \left( \rho_t (\nabla \log \frac{\rho_t}{\nu} - \mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] + \nabla \log \nu) \right) \right) \log \frac{\rho_t}{\nu} \, dx \]

\[ = -\int_{\mathbb{R}^d} \rho_t (\nabla \log \frac{\rho_t}{\nu} - \mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] + \nabla \log \nu, \nabla \log \frac{\rho_t}{\nu}) \, dx \quad \text{integration by parts} \]

\[ = -\int_{\mathbb{R}^d} \rho_t ||\nabla \log \frac{\rho_t}{\nu}||^2 \, dx + \int_{\mathbb{R}^d} \rho_t (\mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] - \nabla \log \nu, \nabla \log \frac{\rho_t}{\nu}) \, dx \]

\[ = -J_\nu(\rho_t) + \int_{\mathbb{R}^d} \rho_t (\mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] - \nabla \log \nu, \nabla \log \frac{\rho_t}{\nu}) \, dx \]

\[ = -J_\nu(\rho_t) + \mathbb{E}_{\rho_0} \left[ \langle s(x_0) - \nabla \log v(x_t), \nabla \log \frac{\rho_t}{\nu} \rangle \right] \quad \text{by renaming } x \text{ as } x_t \]

\[ \leq -J_\nu(\rho_t) + \mathbb{E}_{\rho_0} \left[ ||s(x_0) - \nabla \log v(x_t)||^2 \right] + \frac{1}{4} \mathbb{E}_{\rho_0} \left[ ||\nabla \log \frac{\rho_t}{\nu}||^2 \right] \quad \text{since } (a, b) \leq ||a||^2 + \frac{1}{4} ||b||^2 \]

\[ = -J_\nu(\rho_t) + \mathbb{E}_{\rho_0} \left[ ||s(x_0) - \nabla \log v(x_t)||^2 \right] + \frac{1}{4} J_\nu(\rho_t) \]

\[ = -\frac{3}{4} J_\nu(\rho_t) + \mathbb{E}_{\rho_0} \left[ ||s(x_0) - \nabla \log v(x_t)||^2 \right]. \]

\[ \square \]

8.2.2 Proof of Lemma 8

Proof of Lemma 8. By \( L_s \)-Lipschitzness

\[ ||s(x_t) - s(x_0)||^2 \leq L_s^2 ||x_t - x_0||^2 = L_s^2 ||ts(x_0) + \sqrt{2t}z_0||^2 \leq 2L_s^2t^2 ||s(x_0)||^2 + 4L_s^2t ||z_0||^2. \]

It is more convenient for our subsequent analysis to have a bound in terms of \( s(x_t) \) rather than \( s(x_0) \), so we use

\[ L_s ||x_t - x_0|| \geq ||s(x_t) - s(x_0)|| \geq ||s(x_0)|| - ||s(x_t)||. \]
Rearranging it gives

\[ \|s(x_0)\| \leq L_s\|x_t - x_0\| + \|s(x_t)\| \]

\[ = L_s\|ts(x_0) + \sqrt{2t}z_0\| + \|s(x_t)\| \text{ since } x_t = x_0 + ts(x_0) + \sqrt{2t}z_0 \]

\[ = L_s\|s(x_0)\| + L_s\sqrt{2t}\|z_0\| + \|s(x_t)\| \text{ by triangle inequality} \]

\[ \leq \frac{1}{3}\|s(x_0)\| + L_s\sqrt{2t}\|z_0\| + \|s(x_t)\| \text{ since } t \leq \frac{1}{3L_s} \iff t \leq h \leq \frac{\alpha}{12L_sL} \text{ and } \alpha \leq L \]

Therefore,

\[ \|s(x_0)\| \leq \frac{3}{2}\|s(x_t)\| + \frac{3}{\sqrt{2}}L_s\sqrt{t}\|z_0\| \]

\[ \iff \|s(x_0)\|^2 \leq \frac{9}{2}\|s(x_t)\|^2 + 9L_s^2t\|z_0\|^2 \quad \text{by } \|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2 \]

(16) So we can bound \(\|s(x_t) - s(x_0)\|^2\) as follows

\[ \|s(x_t) - s(x_0)\|^2 \leq 2L_s^2t^2\|s(x_0)\|^2 + 4L_s^2t\|z_0\|^2 \]

\[ \leq 2L_s^2t^2\left(\frac{9}{2}\|s(x_t)\|^2 + 9L_s^2t\|z_0\|^2\right) + 4L_s^2t\|z_0\|^2 \quad \text{by plugging in Eq. (16)} \]

\[ = 9L_s^2t^2\|s(x_t)\|^2 + (18L_s^4t^3 + 4L_s^2t)\|z_0\|^2 \]

\[ \leq 9L_s^2t^2\|s(x_t)\|^2 + 6L_s^2t\|z_0\|^2 \quad \text{since } t \leq \frac{1}{3L_s} \]

\[ = 9L_s^2t^2\|s(x_t) - \nabla \log \nu(x_t) + \nabla \log \nu(x_t)\|^2 + 6L_s^2t\|z_0\|^2 \]

\[ \leq 18L_s^2t^2\|s(x_t) - \nabla \log \nu(x_t)\|^2 + 18L_s^2t^2\|\nabla \log \nu(x_t)\|^2 + 6L_s^2t\|z_0\|^2. \]
8.2.3 Proof of Lemma 9

Proof of Lemma 9. Let \(M(x) = \|\nabla \log \nu(x) - s(x)\|^2\). By Lemma 7,

\[
\frac{\partial}{\partial t} H_\nu(\rho_t) \leq -\frac{3}{4} J_\nu(\rho_t) + E_{\rho_t}\left[\|s(x_0) - \nabla \log v(x_t)\|^2\right] \\
\leq -\frac{3}{4} J_\nu(\rho_t) + 2E_{\rho_t}\left[\|s(x_0) - s(x_t)\|^2\right] + 2E_{\rho_t}\left[\|s(x_t) - \nabla \log v(x_t)\|^2\right] \\
\leq -\frac{3}{4} J_\nu(\rho_t) + 2E_{\rho_t}\left[18L_s^2 t^2\|s(x_t) - \nabla \log v(x_t)\|^2\right] + 6L_s^2 t\|z_0\|^2 \\
+ 2E_{\rho_t}\left[M(x)\right] \quad \text{by Lemma 8}
\]

\[
= -\frac{3}{4} J_\nu(\rho_t) + (36L_s^2 t^2 + 2)E_{\rho_t}\left[M(x)\right] + 36L_s^2 t\|\nabla \log v(x_t)\|^2 + 12dL_s^2 t \\
\leq -\frac{3}{4} J_\nu(\rho_t) + \frac{9}{4} E_{\rho_t}\left[M(x)\right] + 36L_s^2 t\|\nabla \log v(x_t)\|^2 + 12dL_s^2 t \\
\leq -\frac{3}{4} J_\nu(\rho_t) + \frac{9}{4} E_{\rho_t}\left[M(x)\right] + 144L_s^2 t^2 L^2 \quad \text{by [VW19, Lemma 12]}
\]

\[
\leq -\frac{3}{4} J_\nu(\rho_t) + \frac{9}{4} E_{\rho_t}\left[M(x)\right] + \alpha H_\nu(\rho_t) + 72dL_s^2 t^2 L + 12dL_s^2 t \\
\leq -\frac{3}{4} J_\nu(\rho_t) + \frac{9}{4} E_{\rho_t}\left[M(x)\right] + \alpha H_\nu(\rho_t) + 72dL_s^2 t^2 L + 12dL_s^2 t \\
= -\frac{3}{4} J_\nu(\rho_t) + \frac{9}{4} E_{\rho_t}\left[M(x)\right] + 72dL_s^2 t^2 L + 12dL_s^2 t.
\]

We then bound the second term by applying Donsker–Varadhan variational characterizations of KL divergence \(E_P[f(x)] \leq \log E_Q e^{f(x)} + H_Q(P)\) for \(f(x) = \frac{a}{\alpha} M(x), P = \rho_t\) and \(Q = \nu\)

\[
E_{\rho_t}\left[\frac{a}{\alpha} M(x)\right] \leq \log E_{\nu}\left[e^{\frac{a}{\alpha} M(x)}\right] + H_\nu(\rho_t) \\
\iff E_{\rho_t}\left[M(x)\right] \leq \frac{\alpha}{9} \log E_{\nu}\left[e^{\frac{a}{\alpha} M(x)}\right] + \frac{\alpha}{9} H_\nu(\rho_t).
\]

Therefore

\[
\frac{\partial}{\partial t} H_\nu(\rho_t) \leq -\frac{1}{2} \alpha H_\nu(\rho_t) + \frac{\alpha}{4} \log E_{\nu}\left[e^{\frac{a}{\alpha} M(x)}\right] + \frac{1}{4} \alpha H_\nu(\rho_t) + 72dL_s^2 t^2 L + 12dL_s^2 t \\
\leq -\frac{1}{4} \alpha H_\nu(\rho_t) + \frac{\alpha}{4} \log E_{\nu}\left[e^{\frac{a}{\alpha} M(x)}\right] + 72dL_s^2 t^2 L + 12dL_s^2 t \\
\leq -\frac{1}{4} \alpha H_\nu(\rho_t) + \frac{1}{4} \alpha \epsilon_{mgf} + 72dL_s^2 t^2 L + 12dL_s^2 t \quad \text{by Assumption 2}
\]

\[
\leq -\frac{1}{4} \alpha H_\nu(\rho_t) + \frac{1}{4} \alpha \epsilon_{mgf} + 72dL_s^2 h^2 L + 12dL_s^2 h \quad \text{since } t \in (0, h)
\]

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This is equivalent to
\[ \frac{\partial}{\partial t} e^{4\alpha t} H_\nu(\rho_t) \leq e^{4\alpha t} \left( \frac{1}{4} \alpha \varepsilon_{mgf} + 72dL_s^2h^2L + 12dL_s^2h \right) \]

\[ \Rightarrow e^{4\alpha h} H_\nu(\rho_h) \leq H_\nu(\rho_0) + \frac{4(e^{4\alpha h} - 1)}{\alpha} \left( \frac{1}{4} \alpha \varepsilon_{mgf} + 72dL_s^2h^2L + 12dL_s^2h \right) \]

\[ \Rightarrow H_\nu(\rho_h) \leq e^{-\frac{1}{4}\alpha h} H_\nu(\rho_0) + 2h \left( \frac{1}{4} \alpha \varepsilon_{mgf} + 72dL_s^2h^2L + 12dL_s^2h \right). \]

where the last inequality uses \( e^{-\frac{1}{4}\alpha h} \leq 1 \) and \( e^c - 1 \leq 2c \) for \( c = \frac{1}{4}\alpha h \in (0, 1) \). This is satisfied since \( h < \frac{1}{2\alpha} \). Renaming \( \rho_0 \) as \( \rho_k \) and \( \rho_h \) as \( \rho_{k+1} \),

\[ H_\nu(\rho_{k+1}) \leq e^{-\frac{1}{4}\alpha h} H_\nu(\rho_k) + 144dL_s^2Lh^3 + 24dL_s^2h^2 + \frac{\alpha h}{2} \varepsilon_{mgf}. \]

\[ \square \]

8.2.4 Proof of Theorem 2

The proof of Theorem 2 then follows from Lemma 9.

Proof of Theorem 2. Applying the recursion contraction \( k \) times

\[ H_\nu(\rho_k) \leq e^{-\frac{1}{4}\alpha kh} H_\nu(\rho_0) + \sum_{i=0}^{k-1} e^{-\frac{1}{4}\alpha hi} \left( 144dL_s^2h^3L + 24dL_s^2h^2 + \frac{\alpha h}{2} \varepsilon_{mgf} \right) \]

\[ \leq e^{-\frac{1}{4}\alpha kh} H_\nu(\rho_0) + \frac{1}{1 - e^{-\frac{1}{4}\alpha h}} \left( 144dL_s^2h^3L + 24dL_s^2h^2 + \frac{\alpha h}{2} \varepsilon_{mgf} \right) \]

\[ \leq e^{-\frac{1}{4}\alpha kh} H_\nu(\rho_0) + \frac{16}{3\alpha h} \left( 144dL_s^2h^3L + 24dL_s^2h^2 + \frac{\alpha h}{2} \varepsilon_{mgf} \right) \]

\[ = e^{-\frac{1}{4}\alpha kh} H_\nu(\rho_0) + \frac{768dL_s^2L}{\alpha} h^2 + \frac{128dL_s^2}{\alpha} h + \frac{8}{3} \varepsilon_{mgf} \]

\[ \leq e^{-\frac{1}{4}\alpha kh} H_\nu(\rho_0) + \frac{896dL_s^2L}{\alpha} h + \frac{8}{3} \varepsilon_{mgf}. \]

\[ \square \]

8.3 Proof of Theorem 4

We restate the full theorem here.

Theorem 4. Assume Assumption 1-4. Let score estimator \( s(x) : \mathbb{R}^d \to \mathbb{R}^d \) satisfies Assumption 5. Let \( 0 < h < \min\left( \frac{\alpha}{12L_s q \sqrt{d}}, \frac{2}{15} \right) \), then after \( k \) iterations of ILA (5)

\[ R_{q, \nu}(\rho_k) \leq e^{-\frac{1}{4}\alpha kh} R_{q, \nu}(\rho_0) + \frac{224dLL_s^2q^2}{\alpha} h + \left( \frac{2L_s}{L} + \frac{16}{3} \right) \frac{q^2}{\alpha} \varepsilon_{\infty}. \]
We will use the following auxiliary results.

**Lemma 10.** Let \( \varphi_t = \frac{\rho_t}{v} \) and \( \psi_t = \frac{\varphi_t^{q-1}}{E_v[\varphi_t]} = \frac{\varphi_t^{q-1}}{F_{q,v}(\rho_t)} \), then

\[
\frac{\partial}{\partial t} R_{q,v}(\rho_t) \leq -\frac{3}{4}q \frac{G_{q,v}(\rho_t)}{F_{q,v}(\rho_t)} + q\mathbb{E}_{\rho_t} \left[ \psi_t(x_t) \left| s(x_0) - \nabla \log \nu(x_t) \right|^2 \right].
\]

This is a generalized version of [CEL+22, Proposition 15] to the setting of estimated score.

**Lemma 11.** Suppose the assumptions in Theorem 4 hold, then along each step of ILA (5)

\[
R_{q,v}(\rho_{k+1}) \leq e^{-\frac{1}{2}n\epsilon_1} R_{q,v}(\rho_{k}) + 144nLL_2^2qh^3 + 24nL_2^2qh^2 + (72L_2^2qh^3 + 4qh)\epsilon_2^2.
\]

### 8.3.1 Proof of Lemma 10

**Proof of Lemma 10.**

\[
\frac{\partial}{\partial t} R_{q,v}(\rho_t) = \frac{1}{q - 1} \int \frac{\partial}{\partial t} \frac{\rho_t^q}{F_{q,v}(\rho_t)} dx
\]

\[
= \frac{q}{q - 1} \int \frac{\rho_t^{q-1} \partial \rho_t}{F_{q,v}(\rho_t)} dx
\]

\[
= \frac{q}{q - 1} \int \frac{\rho_t^{q-1} \left( - \nabla \cdot \left( \rho_t \mathbb{E}_{\rho_t} [s(x_t) | x_t = x] \right) + \Delta \rho_t \right)}{F_{q,v}(\rho_t)} dx
\]

\[
= \frac{q}{q - 1} \int -\rho_t \left( \nabla \left( \frac{\rho_t}{v} \right)^{q-1}, \nabla \log \frac{\rho_t}{v} - \mathbb{E}_{\rho_t} [s(x_0) | x_t = x] + \nabla \log \nu \right) dx
\]

\[
= \frac{q}{q - 1} \int \left( -\rho_t \left( \nabla \left( \frac{\rho_t}{v} \right)^{q-1}, \nabla \frac{\rho_t}{v} \right) + \int \frac{\rho_t \left( \nabla \left( \frac{\rho_t}{v} \right)^{q-1}, \mathbb{E}_{\rho_t} [s(x_0) | x_t = x] - \nabla \log \nu \right) dx \right)
\]

Since

\[
\langle \nabla \left( \frac{\rho_t}{v} \right)^{q-1}, \nabla \frac{\rho_t}{v} \rangle = (q - 1) \langle \left( \frac{\rho_t}{v} \right)^{q-2} \nabla \frac{\rho_t}{v}, \nabla \frac{\rho_t}{v} \rangle
\]

\[
= (q - 1) \langle \left( \frac{\rho_t}{v} \right)^{q-2} \nabla \frac{\rho_t}{v}, \left( \frac{\rho_t}{v} \right)^{q-2} \nabla \frac{\rho_t}{v} \rangle
\]

\[
= (q - 1) \left\| \frac{1}{q} \nabla \left( \frac{\rho_t}{v} \right)^{\frac{q}{2}} \right\|^2
\]

\[
= \frac{4(q - 1)}{q^2} \left\| \nabla \left( \frac{\rho_t}{v} \right)^{\frac{q}{2}} \right\|^2,
\]

\[24\]
\[ A_1 = \frac{4(q-1)}{q^2} \mathbb{E}_\nu \left[ \| \nabla (\frac{\rho_t}{\nu}) \|_2^2 \right]. \]

And since \( \nabla (\frac{\rho_t}{\nu})^{q-1} = (q-1)(\frac{\rho_t}{\nu})^{q-2} \nabla \frac{\rho_t}{\nu} = (q-1)(\frac{\rho_t}{\nu})^{2-q} \frac{2-q}{q} \nabla \frac{\rho_t}{\nu} = (q-1)(\frac{\rho_t}{\nu})^{2-q} \frac{2-q}{q} \nabla (\frac{\rho_t}{\nu})^{q/2}, \)

\[ A_2 = \int \rho_t \langle \nabla (\frac{\rho_t}{\nu})^{q-1}, \mathbb{E}_{\rho_0|t} [s(x_0)|x_t = x] - \nabla \log \nu \rangle dx \]

\[ = 2^{q-1} \int \rho_t \langle \nabla (\frac{\rho_t}{\nu})^{q-1}, s(x_0) - \nabla \log \nu(x_t) \rangle dx \]

\[ = 2^{q-1} \int \rho_t \langle \nabla (\frac{\rho_t}{\nu})^{q-1}, \frac{2-q}{q} \nabla (\frac{\rho_t}{\nu})^{q/2} (s(x_0) - \nabla \log \nu(x_t)) \rangle dx \].

Applying \( \langle x, y \rangle \leq \frac{1}{2q} \| x \|^2 + \frac{q}{2} \| y \|^2 \) to \( x = (\frac{\rho_t}{\nu})^{-\frac{q-1}{2}} \nabla (\frac{\rho_t}{\nu})^{q/2} \) and \( y = (\frac{\rho_t}{\nu})^{\frac{q-1}{2}} (s(x_0) - \nabla \log \nu(x_t)) \),

\[ \langle \nabla (\frac{\rho_t}{\nu})^{q/2} (s(x_0) - \nabla \log \nu(x_t)) \rangle \leq \frac{1}{2q} \| \nabla (\frac{\rho_t}{\nu})^{q/2} \|^2 + \frac{q}{2} \| (\frac{\rho_t}{\nu})^{q-1} (s(x_0) - \nabla \log \nu(x_t)) \|^2. \]

Therefore,

\[ A_2 \leq 2^{q-1} \int \rho_t \langle \nabla (\frac{\rho_t}{\nu})^{q/2} \|^2 \rangle + \frac{q}{2} \| (\frac{\rho_t}{\nu})^{q-1} (s(x_0) - \nabla \log \nu(x_t)) \|^2 \]

\[ = 2^{q-1} \int \rho_t \langle \nabla (\frac{\rho_t}{\nu})^{q/2} \|^2 \rangle + (q-1) \| (\frac{\rho_t}{\nu})^{q-1} (s(x_0) - \nabla \log \nu(x_t)) \|^2 \].

Hence,

\[ \frac{\partial}{\partial t} R_{q,\nu}(\rho_t) = \frac{q}{(q-1) F_{q,\nu}(\rho_t)} (-A_1 + A_2) \]

\[ \leq \frac{q}{(q-1) F_{q,\nu}(\rho_t)} \left( - \frac{3(q-1)}{q^2} \mathbb{E}_\nu \left[ \| \nabla (\frac{\rho_t}{\nu}) \|_2^2 \right] + (q-1) \mathbb{E}_{\rho_0} \left[ (\frac{\rho_t}{\nu})^{q-1} \| s(x_0) - \nabla \log \nu(x_t) \|_2^2 \right] \right) \]

\[ = - \frac{1}{F_{q,\nu}(\rho_t)} \left( \frac{3}{q} \mathbb{E}_\nu \left[ \| \nabla (\frac{\rho_t}{\nu}) \|_2^2 \right] - q \mathbb{E}_{\rho_0} \left[ (\frac{\rho_t}{\nu})^{q-1} \| s(x_0) - \nabla \log \nu(x_t) \|_2^2 \right] \right). \]

Let \( \varphi_t = \frac{\rho}{\nu} \) and \( \psi_t = \frac{\varphi_t^{q-1}}{\mathbb{E}_\nu[\varphi_t]} = \frac{\varphi_t^{q-1}}{F_{q,\nu}(\rho_t)} \), then

\[ \frac{\partial}{\partial t} R_{q,\nu}(\rho_t) \leq - \frac{3}{4} \frac{G_{q,\nu}(\rho_t)}{F_{q,\nu}(\rho_t)} + q \mathbb{E}_{\rho_0} \left[ \psi_t(x_t) \| s(x_0) - \nabla \log \nu(x_t) \|_2^2 \right]. \]
### 8.3.2 Proof of Lemma 11

**Proof of Lemma 11.** Lemma 10 states

\[
\frac{\partial}{\partial t} R_{q,\nu}(\rho_t) \leq -\frac{3}{4} G_{q,\nu}(\rho_t) + q\mathbb{E}_{\nu}[\psi_t(x_t)||s(x_0) - \nabla \log \nu(x_t)||^2].
\]

All we need is to bound \(\mathbb{E}_{\rho_t}[\psi_t(x_t)||s(x_0) - \nabla \log \nu(x_t)||^2]\):

\[
\mathbb{E}_{\rho_t}[\psi_t(x_t)||s(x_0) - \nabla \log \nu(x_t)||^2] \leq 2 \mathbb{E}_{\rho_t}[\psi_t(x_t)||s(x_0) - s(x_t)||^2] + 2\mathbb{E}_{\rho_t}[\psi_t(x)||s(x) - \nabla \log \nu(x)||^2]
\]

where

\[
A_3 \leq \mathbb{E}_{\rho_t}[\psi_t(x_t)(18L^2s_t^2||s(x_t) - \nabla \log \nu(x)||^2 + 18L^2s_t^2||\nabla \log \nu(x)||^2 + 6L^2s_t||z_0||^2)
\]

by Lemma 8

\[
= 18L^2s_t^2\mathbb{E}_{\rho_t}[\psi_t(x)(||s(x) - \nabla \log \nu(x)||^2)] + 18L^2s_t^2\mathbb{E}_{\rho_t}[\psi_t(x)||\nabla \log \nu(x)||^2] + 6L^2s_ttd
\]

So we have

\[
\mathbb{E}_{\rho_t}[\psi_t(x_t)||s(x_0) - \nabla \log \nu(x)||^2] \leq (36L^2s_t^2 + 2)\mathbb{E}_{\rho_t}[||s(x) - \nabla \log \nu(x)||^2]
\]

\[
+ 36L^2s_t^2\mathbb{E}_{\rho_t}||\nabla \log \nu(x)||^2 + 12L^2s_ttd
\]

\[
\leq (36L^2s_t^2 + 2)\epsilon_{\infty}^2 + 36L^2s_t^2\mathbb{E}_{\rho_t}||\nabla \log \nu(x)||^2 + 12L^2s_ttd.
\]

By [CEL+22, Lemma 16] under the assumption of \(\nabla \log \nu\) being \(L\)-Lipschitz,

\[
A_5 \leq \mathbb{E}_{\rho_t}[\frac{\psi_t}{\nu}||\nabla \log \nu||^2] + 2dL
\]

\[
= \mathbb{E}_{\rho_t}[\frac{\nu}{\rho_t\psi_t}\nabla \frac{\psi_t}{\nu}||^2] + 2dL
\]

\[
= \mathbb{E}_{\rho_t}[\frac{\nu}{\rho_t\psi_t}\frac{1}{\psi_t}\nabla \varphi^2_t||^2] + 2dL
\]

\[
= \int \frac{\nu^2}{\rho_t\psi_t} \frac{1}{\psi_t^2} \nabla \varphi^2_t||^2 dx + 2dL
\]

\[
= \frac{\mathbb{E}_{\nu}[\frac{1}{\varphi_t^2}||\nabla \varphi^2_t||^2]}{F_{q,\nu}(\rho_t)} + 2dL
\]

\[
= \frac{4\mathbb{E}_{\nu}[||\nabla \varphi^2_t||^2]}{F_{q,\nu}(\rho_t)} + 2dL
\]

\[
= \frac{q^2 G_{q,\nu}(\rho_t)}{F_{q,\nu}(\rho_t)} + 2dL.
\]
Putting together,
\[
\frac{\partial}{\partial t} R_{q,\nu}(\rho_t) \leq (36L_s^2t^2q^2 - \frac{3}{4}q) \frac{G_{q,\nu}(\rho_t)}{F_{q,\nu}(\rho_t)} + (36L_s^2t^2 + 2)\varepsilon_\infty^2q + 72L_s^2t^2dLq + 12L_s^2tdq \\
\leq \frac{1}{2}q \frac{G_{q,\nu}(\rho_t)}{F_{q,\nu}(\rho_t)} + (36L_s^2t^2 + 2)\varepsilon_\infty^2q + 72L_s^2t^2dLq + 12L_s^2tdq \\
\leq \frac{\alpha}{q} R_{q,\nu}(\rho_t) + (36L_s^2t^2 + 2)\varepsilon_\infty^2q + 72L_s^2t^2dLq + 12L_s^2tdq
\]
where the last inequality is from \cite[VW19, Lemma 5]{VW19} under the assumption of \(\nu\) satisfying \(\alpha\)-LSI. Therefore
\[
\frac{\partial}{\partial t} e^{\frac{\alpha}{q}t} R_{q,\nu}(\rho_t) \leq e^{\frac{\alpha}{q}t} \left(72L_s^2dLqh^2 + 12L_s^2dqh + (36L_s^2t^2 + 2)\varepsilon_\infty^2q\right).
\]
Integrating from 0 to \(h\),
\[
e^{\frac{\alpha}{q}h} R_{q,\nu}(\rho_h) \leq R_{q,\nu}(\rho_0) + \frac{q(e^{\frac{\alpha}{q}h} - 1)}{\alpha} \left(72L_s^2dLqh^2 + 12L_s^2dqh + (36L_s^2t^2 + 2)\varepsilon_\infty^2q\right) \\
\leq R_{q,\nu}(\rho_0) + 2h \left(72L_s^2dLqh^2 + 12L_s^2dqh + (36L_s^2t^2 + 2)\varepsilon_\infty^2q\right).
\]
The last inequality uses \(e^c - 1 \leq 2c\) for \(c = \frac{\alpha}{q}h \in (0, 1)\). Rearranging and renaming \(\rho_0 \equiv \rho_k, \rho_h \equiv \rho_{k+1}\), we obtain the recursive contraction
\[
R_{q,\nu}(\rho_{k+1}) \leq e^{-\frac{\alpha}{q}h} R_{q,\nu}(\rho_k) + 144L_s^2dLqh^3 + 24L_s^2dqh^2 + (72L_s^2h^3 + 4h)\varepsilon_\infty^2q.
\]
\[
\square
\]

8.3.3 Proof of Theorem 4

Proof of Theorem 4. Applying the recursion in Lemma 11 \(k\) times, we have
\[
R_{q,\nu}(\rho_k) \leq e^{-\frac{1}{\alpha}ohk} R_{q,\nu}(\rho_0) + \sum_{i=1}^{k-1} e^{-\frac{\alpha}{q}hi} \left(144L_s^2dLqh^3 + 24L_s^2dqh^2 + (72L_s^2h^3 + 4h)\varepsilon_\infty^2q\right) \\
\leq e^{-\frac{1}{\alpha}ohk} R_{q,\nu}(\rho_0) + \frac{1}{1 - e^{-\frac{\alpha}{q}h}} \left(144L_s^2dLqh^3 + 24L_s^2dqh^2 + (72L_s^2h^3 + 4h)\varepsilon_\infty^2q\right) \\
\leq e^{-\frac{1}{\alpha}ohk} R_{q,\nu}(\rho_0) + \frac{4q}{3\alpha h} \left(144L_s^2dLqh^3 + 24L_s^2dqh^2 + (72L_s^2h^3 + 4h)\varepsilon_\infty^2q\right) \text{ since } 1 - e^{-c} \geq \frac{3}{4} c \text{ for } c \in (0, \frac{1}{4}] \\
\leq e^{-\frac{1}{\alpha}ohk} R_{q,\nu}(\rho_0) + \frac{192dLL_s^2q^2h^2}{\alpha} + \frac{32dLL_s^2q^2}{\alpha} h + \left(\frac{96L_s^2h^2q^2}{\alpha} + \frac{16q^2}{3\alpha}\right)\varepsilon_\infty^2 \\
\leq e^{-\frac{1}{\alpha}ohk} R_{q,\nu}(\rho_0) + \frac{224dLL_s^2q^2}{\alpha} h + \left(\frac{2L_s}{L} + \frac{16}{3}\right) q^2\varepsilon_\infty^2.
\]
\[
\square
\]

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8.4 Details of Example 1

Consider \( \nu \) as target distribution and we estimate \( \nabla \log \nu \) by \( \nabla \log \mu \). We will show \( L^p \)-accuracy goes to 0 as \( m \to \infty \) but the total variation between \( \nu \) and \( \mu \) is bounded below by a positive constant.

For convenience, let \( \nu_0 = N(-m,1), \nu_1 = N(m,1) \) and rewrite \( \nu = \frac{3}{4} \nu_0 + \frac{1}{4} \nu_1, \mu = \frac{1}{2} \nu_0 + \frac{1}{2} \nu_1 \).

The lower bound of \( TV(\nu,\mu) \) follows from [DMR18, BCE+22]. Moreover, [BCE+22, Proposition 1] calculates

\[
\nabla \log \nu - \nabla \log \mu = -\frac{m}{2} \frac{\nu_0}{\nu \mu}.
\]

Therefore,

\[
\mathbb{E}_\nu \left[ \| \nabla \log \nu - \nabla \log \mu \|^p \right] = \frac{m^p}{2^p} \int \frac{\nu_0^p \nu_1^p}{\nu^p \mu^p} \nu^p \nu_1^p \\
= \frac{m^p}{2^p} \int \frac{\nu_0^p \nu_1^p}{\nu_0^p \nu_1^p} \\
= \frac{m^p}{2^p} \int \frac{\nu_0^p \nu_1^p}{(\frac{3}{4} \nu_0 + \frac{1}{4} \nu_1)^p - (\frac{1}{2} \nu_0 + \frac{1}{2} \nu_1)^p} \\
= 4^{p-1} m^p \int \frac{\nu_0^p \nu_1^p}{(3 \nu_0 + \nu_1)^p - (\nu_0 + \nu_1)^p} \\
\leq 4^{p-1} m^p \int \frac{\nu_0^p \nu_1^p}{(\nu_0 + \nu_1)^{2p-1}} \text{ since } 3 \nu_0 + \nu_1 \geq \nu_0 + \nu_1 \\
\leq 4^{p-1} m^p \left( \int_{\mathbb{R}^-} \nu_0^p \nu_1^{p-1} + \int_{\mathbb{R}^+} \nu_1^p \nu_0^{p-1} \right).
\]

Since

\[
\int_{\mathbb{R}^-} \frac{\nu_0^p}{\nu_1^{p-1}} = \exp \left( \frac{2p(2p-1)m^2}{\sqrt{2\pi}} \right) \int_{\mathbb{R}^-} \exp \left( -\frac{1}{2} (x - (2p-1)m)^2 \right) dx \\
= \exp \left( 2p(2p-1)m^2 \right) P_{N(0,1)} \{ Z \leq (2p-1)m \} \\
= \exp \left( 2p(2p-1)m^2 \right) P_{N(0,1)} \{ Z \geq -(2p-1)m \} \\
\leq \frac{1}{2} \exp \left( 2p(2p-1)m^2 \right) \exp \left( -\frac{(2p-1)^2m^2}{2} \right) \text{ by Gaussian tail} \\
= \frac{1}{2} \exp \left( -\frac{m^2}{2} \right).
\]
and similarly
\[
\int_{\mathbb{R}^+} \frac{\nu_0^p}{\nu_1^{p-1}} = \frac{\exp \left( 2p(2p-1)m^2 \right)}{\sqrt{2\pi}} \int_{\mathbb{R}^-} \exp \left( -\frac{1}{2} (x + (2p-1)m)^2 \right) dx
\]
\[
= \exp \left( 2p(2p-1)m^2 \right) \mathbb{P}_{N(0,1)} \{ Z \geq -(2p-1)m \}
\]
\[
\leq \frac{1}{2} \exp \left( -\frac{m^2}{2} \right).
\]

Therefore,
\[
\mathbb{E}_\nu \left[ \| \nabla \log \nu - \nabla \log \mu \|^p \right] \leq 4^{p-1} m^p \exp \left( -\frac{m^2}{2} \right) \to 0 \quad \text{as} \quad m \to \infty.
\]

### 8.5 Proof of Theorem 5

To prove Theorem 5, we need first prove Lemma 4 and Lemma 5. We will need the following auxiliary results to prove the two lemmas.

**Lemma 12.** The continuity equation for the interpolation SDE (13) is
\[
\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t \alpha y) - 2\nabla \cdot (\rho_t \mathbb{E}_{\rho_{t|0}}[\hat{s}_T(Y_0) \mid Y_t = y]) + \Delta \rho_t.
\]

**Lemma 13.** Assume the score estimator \( \hat{s}_t \) is \( L_s \)-Lipschitz for \( 0 \leq t' \leq T \). If \( t \leq \frac{1}{12L_s} \), then
\[
\| \hat{s}_t(y_t) - \hat{s}_t(y_0) \|^2 \leq 36\alpha^2 t^2 L_s^2 \| y_0 \|^2 + 36L_s^2 t \| z \|^2 + 72L_s^2 t^2 \| \hat{s}_t(y_t) \|^2
\]
where \( y_t = e^{\alpha t} y_0 + 2\left( e^{\alpha t} - 1 \right) \hat{s}_t(y_0) + \sqrt{\frac{e^{2\alpha t} - 1}{\alpha}} z \) and \( z \sim \mathcal{N}(0, I) \).

#### 8.5.1 Proof of Lemma 12

**Proof of Lemma 12.** Conditioning on \( y_0 \), FP equation for the conditional distribution \( \rho_{t|0} \) is
\[
\frac{\partial \rho_{t|0}(y_t \mid y_0)}{\partial t} = -\nabla \cdot \left( \rho_{t|0}(y_t \mid y_0)(\alpha y_t + 2\hat{s}_T(y_0)) \right) + \Delta \rho_{t|0}(y_t \mid y_0).
\]
Therefore
\[
\frac{\partial \rho_t(y_t)}{\partial t} = \frac{\partial}{\partial t} \int \rho_{t|0}(y_t \mid y_0) \rho_0(y_0) dy_0
\]
\[
= \int \frac{\partial}{\partial t} \rho_{t|0}(y_t \mid y_0) \rho_0(y_0) dy_0
\]
\[
= \int [-\nabla \cdot \left( \rho_{t|0}(y_t \mid y_0)(\alpha y_t + 2 \hat{s}_T(y_0)) \right) + \Delta \rho_{t|0}(y_t \mid y_0)] \rho_0(y_0) dy_0
\]
\[
= \int [-\nabla \cdot \left( \rho_{t,0}(y_t, y_0)(\alpha y_t + 2 \hat{s}_T(y_0)) \right) + \Delta \rho_{t,0}(y_t, y_0)] dy_0
\]
\[
= \int [-\nabla \cdot \left( \rho_{t,0}(y_t, y_0) \alpha y_t \right) dy_0 - \nabla \cdot \left( 2 \rho_{t,0}(y_t, y_0) \hat{s}_T(y_0) \right)] + \int \Delta \rho_{t,0}(y_t, y_0) dy_0
\]
\[
= -\nabla \cdot \left( \rho_t(y_t) \alpha y_t \right) - 2 \nabla \cdot \left( \rho_t(y_t) \mathbb{E}_{\rho_{0|t}}[\hat{s}_T(y_0) \mid Y_t = y_t] \right) + \Delta \rho_t(y_t).
\]

\[\Box\]

8.5.2 Proof of Lemma 13

Proof of Lemma 13.

\[\| \hat{s}_t'(y_t) - \hat{s}_t'(y_0) \| \leq L_s \| y_t - y_0 \|
\]
\[
= L_s \| (e^{\alpha t} - 1)y_0 + 2 \left( \frac{e^{\alpha t} - 1}{\alpha} \right) \hat{s}_t'(y_0) + \sqrt{\frac{e^{2\alpha t} - 1}{\alpha}} z \|
\]
\[
\leq L_s (e^{\alpha t} - 1) \| y_0 \| + 2 L_s \left( \frac{e^{\alpha t} - 1}{\alpha} \right) \| \hat{s}_T(y_0) \| + L_s \sqrt{\frac{e^{2\alpha t} - 1}{\alpha}} \| z \|
\]
\[
\leq 2 L_s \alpha t \| y_0 \| + 4 L_s t \| \hat{s}_t'(y_0) \| + 2 L_s \sqrt{t} \| z \| \quad \text{since } e^{\alpha t} - 1 \leq 2 \alpha t.
\]

It is more convenient to work with a bound involving \( \hat{s}_t'(y_t) \) rather than \( \hat{s}_t'(y_0) \). Therefore we use
\[\| \hat{s}_t'(y_0) \| - \| \hat{s}_t'(y_t) \| \leq \| \hat{s}_t'(y_t) - \hat{s}_t'(y_0) \| \leq L_s \| y_t - y_0 \|
\]
to bound \( \hat{s}_t'(y_0) \).

\[\| \hat{s}_t'(y_0) \| \leq \| \hat{s}_t'(y_t) \| + L_s \| y_t - y_0 \|
\]
\[
\leq \| \hat{s}_t'(y_0) \| + 2 L_s \alpha t \| y_0 \| + 4 L_s t \| \hat{s}_t'(y_0) \| + 2 L_s \sqrt{t} \| z \|
\]
\[
\leq \frac{1}{3} \| \hat{s}_t'(y_0) \| + \| \hat{s}_t'(y_t) \| + 2 L_s \alpha t \| y_0 \| + 2 L_s \sqrt{t} \| z \| \quad \text{since } t \leq \frac{1}{12 L_s}.
\]

Rearrange gives
\[\| \hat{s}_t'(y_0) \| \leq \frac{3}{2} \| \hat{s}_t'(y_t) \| + 3 L_s \alpha t \| y_0 \| + 3 L_s \sqrt{t} \| z \|.
\]
Therefore
\[
\| \hat{s}_t(y_t) - \hat{s}_t(y_0) \| \leq (2L_s \alpha t + 12L_s^2 \alpha t^2) \| y_0 \| + 6L_s t \| \hat{s}_t(y_t) \| + (12L_s^2 t^{3/2} + 2L_s \sqrt{t}) \| z \|
\]
\[
\leq 3L_s \alpha t \| y_0 \| + 6L_s t \| \hat{s}_t(y_t) \| + 3L_s \sqrt{t} \| z \|
\]
since \( t \leq \frac{1}{12L_s} \).
Hence
\[
\| \hat{s}_t(y_t) - \hat{s}_t(y_0) \|^2 \leq 36 \alpha^2 t^2 L_s^2 \| y_0 \|^2 + 36t L_s^2 \| z \|^2 + 72t^2 L_s^2 \| \hat{s}_t(y_t) \|.
\]

8.5.3 Proof of Lemma 4

Proof of Lemma 4. By Lemma 12, we have
\[
\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t \alpha y) - 2\nabla \cdot (\rho_t \mathbb{E}_{\rho_0_t} [\hat{s}_T(y_0) \mid Y_t = y]) + \Delta \rho_t.
\]
On the other hand, the continuity equation of the true backward process (10) is
\[
\frac{\partial \mu_t}{\partial t} = -\nabla \cdot (\mu_t \alpha y) - \Delta \mu_t.
\]
Therefore
\[
\frac{d}{dt} H_{\mu_t}(\rho_t) = \frac{d}{dt} \int \rho_t \log \frac{\rho_t}{\mu_t} dy
\]
\[
= \int \left[ -\nabla \cdot (\rho_t \alpha y) - 2\nabla \cdot (\rho_t \mathbb{E}_{\rho_0_t} [\hat{s}_T(y_0) \mid Y_t = y]) + \Delta \rho_t \right] \log \frac{\rho_t}{\mu_t} dy - \int \left[ -\nabla \cdot (\mu_t \alpha y) - \Delta \mu_t \right] \frac{\rho_t}{\mu_t} dy
\]
\[
= \int \left[ \nabla \cdot (\rho_t \alpha y) \right] \frac{\rho_t}{\mu_t} dy - \int \left[ \nabla \cdot (\rho_t \alpha y) \right] \log \frac{\rho_t}{\mu_t} dy + \int \left[ \nabla \cdot (\mu_t \alpha y) \right] \log \frac{\rho_t}{\mu_t} dy - \int \Delta \rho_t \log \frac{\rho_t}{\mu_t} dy - \int \Delta \mu_t \frac{\rho_t}{\mu_t} dy
\]
\[
- 2 \int \nabla \cdot (\rho_t \mathbb{E}_{\rho_0_t} [\hat{s}_T(y_0) \mid Y_t = y]) \log \frac{\rho_t}{\mu_t} dy + 2 \int \Delta \mu_t \frac{\rho_t}{\mu_t} dy.
\]
Note that
\[
\int \left[ \nabla \cdot (\mu_t \phi) \right] \frac{\rho_t}{\mu_t} dy - \int \left[ \nabla \cdot (\rho_t \phi) \right] \log \frac{\rho_t}{\mu_t} dy
= - \int \langle \mu_t \phi, \nabla \frac{\rho_t}{\mu_t} \rangle dy + \int \langle \rho_t \phi, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy
\]
by integration by parts
\[
= - \int \langle \rho_t \phi, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy + \int \langle \rho_t \phi, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy
= 0,
\]
\[
\int \Delta \rho_t \log \frac{\rho_t}{\mu_t} dy - \int \Delta \mu_t \frac{\rho_t}{\mu_t} dy = - \int \langle \nabla \rho_t, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy + \int \langle \nabla \mu_t, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy
\]
by renaming as \( y \)
\[
\int \Delta \mu_t \frac{\rho_t}{\mu_t} dy = - \int \langle \nabla \rho_t, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy + \int \langle \nabla \mu_t, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy
\]
and
\[
2 \int \Delta \mu_t \frac{\rho_t}{\mu_t} dy = -2 \int \langle \nabla \mu_t, \nabla \log \frac{\rho_t}{\mu_t} \rangle dy = -2 \int \rho_t \langle \nabla \log \mu_t, \log \frac{\rho_t}{\mu_t} \rangle dy.
\]
Therefore,
\[
\frac{d}{dt} H_{\mu_t}(\rho_t) = -J_{\mu_t}(\rho_t) + 2 \int \rho_t (\mathbb{E}_{\rho_0|t}[\hat{s}_T(y_0) | Y_t = y] - \nabla \log \mu_t, \nabla \log \frac{\rho_t}{\mu_t}) dy
\]
\[
\leq -J_{\mu_t}(\rho_t) + 2 \mathbb{E}_{\rho_0|t} \left[ \| \hat{s}_T(y_0) - \nabla \log \mu_t(y_t) \|^2 \right] + \frac{1}{4} \mathbb{E}_{\rho_t} \left[ \| \nabla \log \frac{\rho_t}{\mu_t} \|^2 \right]
\]
\[
\leq -\frac{3}{4} J_{\mu_t}(\rho_t) + 4 \mathbb{E}_{\rho_0|t} \left[ \| \hat{s}_T(y_0) - \nabla \log \mu_t(y_t) \|^2 \right]
\]
\[
\leq -\frac{3}{4} J_{\mu_t}(\rho_t) + 8 \mathbb{E}_{\rho_0|t} \left[ \| \hat{s}_T(y_0) - \hat{s}_T(y_t) \|^2 \right] + 8 \mathbb{E}_{\rho_t} \left[ \| \hat{s}_T(y_t) - \nabla \log \mu_t(y_t) \|^2 \right]
\]
\[
\leq -\frac{3}{2} \mathbb{E}_{\rho_t} \left[ \| \hat{s}_T(y_0) - \hat{s}_T(y_t) \|^2 \right] + 8 \mathbb{E}_{\rho_t} \left[ \| \hat{s}_T(y_t) - \nabla \log \mu_t(y_t) \|^2 \right].
\]
By Lemma 13,
\[
\mathbb{E}_{\rho_t}[\|\hat{s}_T(y_0) - \hat{s}_T(y_t)\|^2] \leq \mathbb{E}_{\rho_t}[36\alpha^2L^2\|y_0\|^2 + 36L^2t\|z\|^2 + 72L^2t^2\|\hat{s}_T(y_t)\|^2] = 36\alpha^2L^2d + 36L^2td + 72L^2t^2\mathbb{E}_{\rho_t}[\|\hat{s}_T(y_t)\|^2] \
\leq 36\alpha^2L^2d + 36L^2td + 144L^2t^2(\mathbb{E}_{\rho_t}[\|\hat{s}_T(y) - \nabla \log \mu_t(y)\|^2] + \mathbb{E}_{\rho_t}[\|\nabla \log \mu_t(y)\|^2]).
\]
Since \(\mu_t = \nu_{t-l}\) is \(\alpha\)-LSI and \(L\)-smooth, by Lemma 12 in [VW19],
\[
\mathbb{E}_{\rho_t}[\|\nabla \log \mu_t(y)\|^2] \leq \frac{4L^2}{\alpha}H_{\mu_t}(\rho_t) + 2L.
\]
By Donsker–Varadhan variational characterizations of KL divergence \(\mathbb{E}_P[f(x)] \leq \log \mathbb{E}_Qe^{f(x)} + H_Q(P),\)
\[
\mathbb{E}_{\rho_t}[\|\hat{s}_T - \nabla \log \mu_t\|^2] \leq \frac{6}{65} \alpha \log \mathbb{E}_{\mu_t}[\exp(\frac{65}{6\alpha}||\hat{s}_t - \nabla \log \mu_t||^2)] + \frac{6}{65}\alpha H_{\mu_t}(\rho_t) \
\leq \frac{6}{65}\alpha \epsilon_{mgf} + \frac{6}{65}\alpha H_{\mu_t}(\rho_t).
\]
Putting together and using \(t^2 \leq h^2 \leq \alpha^2/96^2L^2L^2,\)
\[
\frac{d}{dt}H_{\mu_t}(\rho_t) \leq -\frac{\alpha}{4}H_{\mu_t}(\rho_t) + \frac{3}{4}\alpha \epsilon_{mgf} + 315L^2td.
\]
\[\square\]

8.5.4 Proof of Lemma 5

Proof of Lemma 5. Following Lemma 4, we have
\[
\frac{d}{dt}e^{\frac{\alpha}{4}H_{\mu_t}(\rho_t)} \leq e^{\frac{\alpha}{4}}\left(\frac{3}{4}\alpha \epsilon_{mgf} + 315hL^2d\right).
\]
Integrating from 0 to \(h,\)
\[
H_{\mu_0}(\rho_h) \leq e^{-\frac{\alpha}{4}h}H_{\mu_0}(\rho_0) + e^{-\frac{\alpha}{4}h}e^{\frac{\alpha}{4}} - 1 \left(\frac{3}{2}\alpha \epsilon_{mgf} + 315hL^2d\right) \
\leq e^{-\frac{\alpha}{4}h}H_{\mu_0}(\rho_0) + 2h\left(\frac{3}{2}\alpha \epsilon_{mgf} + 315hL^2d\right) \quad \text{by } e^{-\frac{\alpha}{4}} \leq 1, e^c - 1 \leq 2e \text{ for } 0 < c = \frac{\alpha h}{4} \leq 1 \\
\leq e^{-\frac{\alpha}{4}h}H_{\mu_0}(\rho_0) + \frac{3}{2}\alpha \epsilon_{mgf} + 630h^2L^2d.
\]
1 \(<\frac{\alpha h}{4} \leq 1\) holds because \(h \leq \frac{\alpha}{96\epsilon_{mgf}} \leq \frac{1}{96\alpha}.\) Renaming \(\rho_0 = \rho_k, \rho_h = \rho_{k+1}, \mu_0 = \mu_k\) and \(\mu_h = \mu_{k+1},\)
we get the desired contraction
\[
H_{\mu_{k+1}}(\rho_{k+1}) \leq e^{-\frac{\alpha}{4}h}H_{\mu_k}(\rho_k) + \frac{3}{2}\alpha \epsilon_{mgf} + 630h^2L^2d.
\]
\[\square\]

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8.5.5 Proof of Lemma 6

Proof. Eq. (14) is equivalent to \( d(e^{\beta t}X_t) = \sqrt{2}e^{\beta t}dt \), therefore we have

\[
X_t = e^{-\beta t}X_0 + \sqrt{\frac{1 - e^{-2\beta t}}{\beta}}Z,
\]

where \( Z \) is a standard Gaussian in \( \mathbb{R}^d \). By Lemma 16 in [VW19], the distribution of \( e^{-\beta t}X_0 \) satisfies LSI with constant \( \alpha e^{2\beta t} \). Therefore by Lemma 17 in [VW19], the LSI constant of \( \nu_t \) is

\[
(\alpha^{-1} e^{-2\beta t} + \frac{1 - e^{-2\beta t}}{\beta})^{-1} = \frac{\alpha \beta}{\alpha + (\beta - \alpha) e^{-2\beta t}} \in (\min(\alpha, \beta), \max(\alpha, \beta)).
\]

In particular, when \( \beta = \alpha \), \( \alpha_t \equiv \alpha \).

8.5.6 Proof of Theorem 5

Proof of Theorem 5. Applying the recursion in Lemma 5 \( K \) times, we obtain

\[
H_{\mu_K}(\rho_K) \leq e^{-\frac{ahK}{4}} H_{\mu_0}(\rho_0) + \sum_{i=0}^{K-1} e^{-\frac{ah}{4}} \left( \frac{3}{2} \alpha h \varepsilon_{mgf} + 630h^2 L_2^2 d \right).
\]

Since \( \sum_{i=0}^{K-1} e^{-\frac{ah}{4}} \leq \sum_{i=0}^{\infty} e^{-\frac{ah}{4}} = \frac{1}{1 - e^{-\frac{ah}{4}}} \) and \( 1 - e^{-c} \geq \frac{3}{4}c \) for \( 0 < c = \frac{ah}{4} \leq \frac{1}{4} \) which is satisfied because \( h \leq \frac{1}{96\alpha} \),

\[
H_{\nu}(\rho_K) = H_{\mu_K}(\rho_K) \leq e^{-\frac{ahK}{4}} H_{\mu_0}(\rho_0) + \frac{16}{3\alpha h} \left( \frac{3}{2} \alpha h \varepsilon_{mgf} + 630h^2 L_2^2 d \right)
\]

\[
\leq e^{-\frac{ahK}{4}} H_{\mu_0}(\rho_0) + \frac{16}{3\alpha h} \left( \frac{3}{2} \alpha h \varepsilon_{mgf} + 3360h L_2^2 d \right).
\] (17)

Note that \( H_{\mu_0}(\rho_0) = H_{\nu_T}(\gamma) \) and we can bound this by analyzing the OU process.

\[
\frac{d}{dt} H_{\nu_t}(\gamma) = -\int \frac{\gamma}{\nu_t} \cdot \nu_t \log \frac{\nu_t}{\gamma} dx
\]

\[
= -\int \frac{\gamma}{\nu_t} \nabla \cdot (\nu_t \nabla \log \frac{\nu_t}{\gamma}) dx
\]

\[
= \int \langle \nabla \frac{\gamma}{\nu_t}, \nu_t \nabla \log \frac{\nu_t}{\gamma} \rangle \quad \text{by integration by parts}
\]

\[
= \int \gamma (\nabla \log \frac{\gamma}{\nu_t}, \nu_t \nabla \log \frac{\nu_t}{\gamma})
\]

\[
= -\mathbb{E}_{\nu_t} [||\nabla \log \frac{\gamma}{\nu_t}||^2]
\]

\[
= -J_{\nu_t}(\gamma)
\]

\[
\leq -2\alpha H_{\nu_t}(\gamma) \quad \text{since } \nu_t \text{ is } \alpha\text{-LSI by Lemma 6.}
\]
Therefore we obtain

\[ H_{\nu_T}(\gamma) \leq e^{-2\alpha T} H_{\nu_0}(\gamma) = e^{-2\alpha T} H_{\nu}(\gamma). \]

Plugging this into Eq. (17) gives

\[ H_{\nu}(\rho_K) \leq e^{-\frac{3\alpha h K}{4}} H_{\nu}(\gamma) + 8\varepsilon_{mgf} + 3360 \frac{hL^2d}{\alpha}. \]

\[ \Box \]

8.6 Proof of Lemma 1

We will use the following alternative definition of sub-Gaussian to prove Lemma 1.

**Lemma 14** (Theorem 2.6 in [Wai19]). If \( \rho \) is \( \sigma \)-sub-Gaussian, then for all \( 0 \leq \lambda < 1 \):

\[ \mathbb{E}_\rho[\exp(\frac{\lambda \|X\|^2}{2\sigma^2})] \leq \frac{1}{(1 - \lambda)^{1/2}}. \]

**Proof of Lemma 1.** By [CCL+22, Lemma 13],

\[ \|s_t(x) - s(x)\| \lesssim L\sqrt{td} + Lt\|s(x)\| \]

\[ \leq L\sqrt{td} + Lt(\|s(0)\| + L\|x\|) \]

where \( \lesssim \) hides absolute constants. If \( t \leq d/\|s(0)\|^2 \), then

\[ \|s_t(x) - s(x)\| \lesssim 2L\sqrt{td} + L^2t\|x\| \]

Therefore,

\[ \|s_t(x) - s(x)\|^2 \lesssim 8L^2td + 2L^4t^2\|x\|^2 \]

Then

\[ \mathbb{E}_\rho[e^{r\|s_t - s\|^2}] \lesssim e^{8rL^2td} \mathbb{E}_\rho[e^{2rL^4t^2\|x\|^2}] \leq \frac{e^{8rL^2td}}{(1 - 4\sigma^2rL^4t^2)^{1/2}} \]

as long as \( 4\sigma^2rL^4t^2 < 1 \), or \( t < \frac{1}{2\sigma^4rL^2} \).

Note \( 1 - x \geq e^{-2x} \) for \( 0 \leq x \leq \frac{1}{2} \). Then if \( t \leq \frac{1}{2\sigma^4rL^2} \),

\[ \mathbb{E}_\rho[e^{r\|s_t - s\|^2}] \lesssim e^{8rL^2td + 4\sigma^2rL^4t^2} \]

Therefore,

\[ \log \mathbb{E}_\rho[e^{r\|s_t - s\|^2}] \lesssim 4rtL^2(2d + \sigma^2tL^2) \]

\[ \Box \]
8.7 Proof of Lemma 3

**Lemma 3.** Assume \( \rho \propto e^{-f} \) where \( f \) is \( L \)-smooth. Let it evolves along the heat flow, then at time \( t \in (0, \frac{1}{L}) \), \( \rho_t = \rho \ast \mathcal{N}(0, tI) \) is \( L_t \)-smooth where \( L_t \) is decreasing in \( t \).

**Proof.** First we show \( s_t(y) = \nabla \log \rho_t(y) = \frac{E_{\rho_0 \mid x \mid y}[X] - y}{t} \). Note that \( \rho_t(y \mid x) = (2\pi t)^{-d/2} \exp\left(-\frac{\|y-x\|^2}{2t}\right) \) and \( \nabla \rho_t(y \mid x) = \rho_t(y \mid x) \frac{(y-x)}{t} \). And since \( \rho_t = \rho \ast \mathcal{N}(0, tI) = \int \rho(x) \rho_t(y \mid x) \, dx \), the score function at time \( t \) is

\[
\begin{aligned}
    s_t(y) &= \nabla \log \rho_t(y) \\
    &= \nabla \int \rho(x) \rho_t(y \mid x) \, dx \\
    &= \int \rho(x) \nabla \rho_t(y \mid x) \, dx \\
    &= \int \rho(x) \rho_t(y \mid x) \frac{(y-x)}{t} \, dx \\
    &= \frac{E_{\rho_0 \mid x \mid y}[X] - y}{t}.
\end{aligned}
\]

Then we have

\[
-\nabla^2 \log \rho_t(y) = \frac{I_d}{t} - \frac{\text{Cov}_{\rho_0 \mid x \mid y}[X]}{t^2}.
\]

On the other hand, since \( \rho_0(x \mid y) \propto e^{-f(x) - \frac{1}{2t} \|y-x\|^2} \),

\[
-\nabla^2 \log \rho_0(x \mid y) = \nabla^2 f(x) + \frac{1}{2t} \|y-x\|^2 = \nabla^2 f(x) + \frac{1}{t} I_d.
\]

So we have

\[
-\nabla^2 \log \rho_0(x \mid y) \leq (L + \frac{1}{t}) I_d \quad \text{and} \quad -\nabla^2 \log \rho_0(x \mid y) \geq (-L + \frac{1}{t}) I_d \geq 0
\]

From [BL76, KP21] we know this implies \( \text{Cov}_{\rho_0 \mid x \mid y}[X] \leq \frac{1}{L+1/t} I_d \). Therefore,

\[
-\nabla^2 \log \rho_t(y) \leq \left(\frac{1}{t} - \frac{1}{t(tL+1)}\right) I_d = \frac{L}{tL+1} I_d.
\]

We conclude \( \rho_t \) is \( L_t \)-smooth, where \( L_t = \frac{L}{tL+1} \) is decreasing in \( t \). \( \square \)