Seeking a Game in which the standard model Group shall Win

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Abstract

It is attempted to construct a group-dependent quantity that could be used to single out the Standard Model group \( S(U(2) \times U(3)) \) as being the “winner” by this quantity being the biggest possible for just the Standard Model group. The suggested quantity is first of all based on the inverse quadratic Cassimir for the fundamental or better smallest faithful representation in a notation in which the adjoint representation quadratic Cassimir is normalized to unity. Then a further correction is added to help the wanted Standard Model group to win and the rule comes even to involve the abelian group \( U(1) \) to be multiplied into the group to get this correction be allowed. The scheme is suggestively explained to have some physical interpretation(s). By some appropriate procedure for extending the group dependent quantity to groups that are not simple we find a way to make the Standard Model Group the absolute “winner”. Thus we provide an indication for what could be the reason for the Standard Model Group having been chosen to be the realized one by Nature.

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1 Introduction

It is one of the great questions asked in connection with our Bled Conference: Why Nature has selected just those gauge groups, which we find? Of course so far the only gauge group found is that of the Standard Model. Thus it is a priori this gauge group, which we should attempt to explain; then the theory, we might invent for that purpose, may or may not suggest further gauge groups as for instance the hierarchy of gauge groups suggested in the model of Norma Mankoc et al.\cite{1}. One of us (H.B.N) and Rugh and Surlykke\cite{16} estimated quantitatively the amount of information contained in the knowledge of the gauge group, and with N. Brene\cite{15} we found that defining a quantitative concept of skewness - lack of automorphisms - appropriately we could declare the Standard Model Group to be characterized as essentially the most “skew”.

In the present article we should go for inventing a somewhat different group dependent quantity than the “skewness”\cite{15}, and then imagine that Nature for some reason has selected just that group, which, say, maximizes this group dependent quantity. This means that we strictly speaking in a phenomenological way attempt to adjust the rules of a competition between groups and seek to adjust the rules, so as to make an already selected winner, the Standard Model Group, to win the game. It is a bit as a great dictator seeking to make, say, his son become the winner of a sport game by cleverly adjusting the rules of the game, so that he wins. In an analogously “nepotisitic” way we shall seek to arrange the game so as to make the Standard Model group win the game.

It should be said though, that in inventing the game we also look at some physical model behind, much inspired indeed by our long ongoing project of Random Dynamics\cite{5,7,6}. Strictly speaking there may be even a couple of routes suggesting, what groups are “best” based on the ideas of Random Dynamics, that the fundamental laws of Nature are extremely complicated. That is to say, if indeed the laws of nature were fundamentally in some way random and very complicated, what would then be the characteristic property -the strength so to speak - of the group combination that would appear as the gauge group effectively as seen by relative to say Planck scale physicists working at low energy?:

a) The one route is based on, that the gauge symmetry appears at first by accident approximately, but that then quantum fluctuations take over and cause the gauge group to appear effectively as an exact gauge group\cite{6}. In such a philosophy the group with the best chance should be the group for which a gauge theory most easily can appear just by accident. Suggestively such a favoured group should be one for which, say a lattice gauge theory, could most easily turn out to appear with approximately gauge invariant action by accident\cite{14,6}. This in turn is at least suggestively argued to occur for a gauge group, for which the range in the configuration space, over which the action has not to vary according to the rule of gauge symmetry is in some way - that may be hard to make precise though - so small as possible. If namely the range over which the variation of the action shall be small is “small”, then there is the better chance to get it constant approximately there just by accident. This argumentation is then turned into saying, that the range of variation of the link variables caused by a gauge transformation say associated with a site in the lattice should be as “small” as possible in order to make the gauge group most likely to occur by accident. Now we typically imagine the lattice link variables to be or at least be represented as matrix elements of some representation of the gauge group. Well, at least we typically take the action contribution from one plaquette to be a trace of some representation of the gauge group. Normally we have the “intuitive” or conventional expectation that although the most general action contribution $S_{\square}$ should be a linear expansion on traces/characters for all the possible representations of the gauge group, the traces of the smallest representations would somehow dominate this expansion. Such an expected dominance
of the small representation trace in the action means that the variation of the action as function of the physical combination of the link variables - i.e. the plaquette variables - vary relatively slowly over the group. But if we can get the action in this sense vary relatively slowly over the gauge group, it may mean that it also suggestively varies relatively slowly, when we vary the gauge. If somehow we have a “setting” - meaning say that everything is written basically in terms of matrices in some low representation - so that the variation of the action along the group is relatively slow, then very likely one would think that also the variation in this “setting” of any possible candidate for a term as a function of some gauge transformation would a priori be relatively slow. In other words we say that a good ‘setting” for making the variation a priori along the gauge variation “small” is one in which the plaquette action is dominated by a “low” representation/character - meaning say low quadratic Casimir for it -.

It should be had in mind that the quadratic Casimir is crudely a measure for how much variation the representation matrix in the representation in question varies as function of the group element it represents. If then we imagine that in the lattice model, which supposed to be the fundamental model, the group is represented by a certain representation rather than directly as an abstract group element, the variation of the “fundamental” lattice model variables are in some sense - that may not be so clear though - more slowly varying as function of the group elements the smaller the quadratic Casimir for the representation functioning as the fundamental fields. But the slower this variation is the less extensive region is passed by the “fundamental” lattice variables and the easier it would therefore be that by accident under a gauge transformation the action that were at first not taken to be gauge invariant would be so by accident nevertheless. By this argumentation it is here argued in words that may really be meaningless that a small quadratic Casimir for the representation which is used by Nature as the “fundamental” lattice field degrees of freedom makes it more likely for the gauge group in question to occur by accident in an a priori random action theory. The point should then crudely be that we should look among all groups and seek which ones have the representations with smallest quadratic Casimir for representations that must still be faithful in order to at all represent the group in question. The smaller these faithful representations that could be used the better should be the chance for the group to be the one realized in Nature.

b) The second route - which were the one, we started working on - involves several assumptions which we have worked on before, but it may become too much for this route being trustable unless we somehow can get the number of assumptions somewhat reduced. Most importantly we assume “Multiple Point Principle” \[11\] on which we have worked much and which states that there are several degenerate vacua. That is to say the coupling constants get - mysteriously ? - adjusted so as to make the theory discussed just sit on a phase transition, where several phases meet. The next assumption then is that after such an adjustment of the lattice action coefficients - which are basically the coupling constants being adjusted by the “Multiple point principle” - we look for that group which gives us the numerically biggest value in the vacuum realized (we argue it is the Coulomb phase one) of the plaquette action \(S(U(\Box))\). For this latter assumption we may loosely say, well it means minimizing the energy density may be. Or we may involve the complex action model \[9\] and argue that a big contribution for the plaquette action may likely lead to a big contribution also numerically for the imaginary part of the action. Since now it is the main point of this complex action model to minimize the imaginary part of the action the best chance for a certain gauge theory to be realized should then be, if it can give the numerically biggest imaginary part. But assuming real and imaginary parts to depend in roughly the similar way on the variables this would then favour groups with that numerically large plaquette action. We shall go into this a bit complicated route to get a suggestive game for the groups in section \[8\].
The game proposed at the end in the present article is a somewhat new one, but actually one of the present authors (H.B.N.) and Niels Brene[15] long ago had a slightly different proposal for the game to be won by the Standard Model group, namely that it should be the most skew in certain sense, which we even made quantitative. Of course “skew” for a group means that it has relatively few automorphisms. Honestly speaking we did not yet publish what to do with the Abelian invariant subgroups so we strictly speaking took the competition between the groups having just one $U(1)$ invariant subgroup. The precise quality for the game to minimize were then the number $\#\text{Out}(G)$ of outer automorphisms divided by the logarithm of the rank $r$ of the group. In reality what comes to count a lot in this game about skewness turns out to be the division out of subgroups of the center, which is what distinguishes the various groups having the same Lie algebra. We shall see below that in order to finally adjust the game of the present article based in stead at first on the quadratic Casimir for a faithful and small representation to really get the Standard Model group win is again to allow this “division out of a subgroup of the center”. This means the distinction of the group (rather than the Lie algebra) is to give a lot of points in the game. So at the end we might be forced to let the game depend much on the property of the group rather than of the Lie algebra, and that may presumably be the main lesson that it is the group property rather than the Lie algebra properties, that really matters to select the Standard Model.

We shall therefore in the following section 2 review the seemingly so important distinction between group and Lie algebra, and call attention to that even though we can claim that the phenomenology of Standard Models gives us not only the Lie algebra but also the Lie group, so that this distinction really has a phenomenological significance in fact in terms of the representations of quarks and leptons. In the following section 4 we shall then discuss that at least reasonable notation independent quantities have to be chosen for the competition, so that the game will not vary unreasonably by varying notations and normalizations. This suggests essentially to use the Dynkin index which is precisely being an important index, because it is somewhat sensible with respect to independence of notation as a start. Then in the next section 5 we shall present the group theoretical values of interest for our proposed game, i.e. the Dynkin indices essentially and the corrections connected with the group rather than the Lie algebra for at first the simple groups. How to combine the simple groups by a kind of averaging may open up for a bit freedom and therefore nepotism to let the Standard Model Group win, but really there is not so much to do to help the Stand Model Group, it must essentially fight for itself. This discussion is put into section 6. The conclusive discussion of the game is put into section 7. The model behind of the somewhat more complicated nature involving “multiple point principle” is put in section 8. A by itself very interesting motivation for our a bit complicated multiple point principle route is, that it goes in connection with very old attempts of ours to fit the fine structure constants.

2 Phenomenological significance of Group rather than Lie Algebra

A priori one might say that it is only the gauge Lie algebra of the Yang Mills theory that matters, since the Yang Mills field theories are constructed alone from the knowledge of the Lie algebra of the gauge group. So from this point of view one can say that the Standard Model group (without now stressing the word group it means that we think of the Lie algebra) is $U(1) \times SU(2) \times SU(3)$. However, we can, and we shall in this article, assign a “phenomenological meaning” to the gauge group rather than just the Lie algebra by associating the choise of the
group (among the several groups having the same Lie algebra) with the system of representations under which the various matter fields - the Fermions and the Higgs fields - transform. The reader should have in mind that while all the possible representations for quarks and leptons and the Higgs or thinkable additions to the Standard Model are allowed a priori, we may prevent some by requiring representation of a certain group. Indeed it is only some of the representations of the Lie algebra of the Standard Model, as we might denote the Lie algebra of $U(1) \times SU(2) \times SU(3)$, which are also representations of the various groups having the same Lie algebra, such as $U(1) \times SU(2)/Z_2 \times SU(3)/Z_3$, $SU(2) \times U(3)$, $U(2) \times SU(3)$ etc. For example the group $SO(3) = SU(2)/Z_2$ has the same Lie algebra as $SU(2)$, but as is rather wellknown while $SU(2)$ has all the representations of the Lie algebra - it is indeed the covering group of $SO(3)$ - both with half integer and integer (weak iso)spin, the group $SO(3) = SU(2)/Z_2$ has only as true representations the (weak iso)spin integer ones. Since the left handed quarks and leptons belong to the weak isospin =1/2 representation of $SU(2)$, which is not allowed as true representation of $SO(3)$, we can conclude that a group with the same Lie algebra as the Standard Model using $SO(3)$ instead of $SU(2)$ would be an example of a group that could not be used in the Standard Model. It would e.g. not be allowed to claim that $U(1) \times SO(3) \times SU(3)$ were the Standard Model group, because it could not have the left handed quarks and leptons and the Higgs as representations.

So you see that there are many groups that are forbidden as Standard Model groups, but e.g. the covering group $R \times SU(2) \times SU(3)$ for which all representations of the Lie algebra are also allowed representations of the (covering) group could at first not be prevented as “the group for the Standard Model”.

However, it is our philosophy to impose a phenomenological extra requirement to select the group, which deserves to be called the Standard Model Group (SMG). The idea is to among the various groups with the Standard Model Lie algebra, which are allowed in the sense of having all the representations present in the Standard Model, we believe in, to select as the Standard Model Group to be that one (or several ?) which is most informative w.r.t. selecting, which representations are allowed, so that just knowing this group tells us as much as possible about, which representations occur in nature as presently known. With requirement of the most informative group about the representations in the Standard Model we should of course not accept the covering group $R \times SU(2) \times SU(3)$, which would give no information, provided we can at all find a group with the Standard Model Lie algebra which would exclude some representations (which of course should be some representations not found in nature so far). Such a more informative group giving correct information about representations found empirically is the group denoted $S(U(2) \times U(3)) = (U(1) \times SU(2) \times SU(3))/Z_6$. The symbol $U(2)$ in this symbol $S(U(2) \times U(3))$ alludes to it being constructed as a pair of a $2 \times 2$ unitary matrix (meaning one in the group $U(2)$) and the $U(3)$ symbol alludes to an $3 \times 3$ unitary matrix (i.e. one in $U(3)$) and then the extra condition being imposed by the $S$ in front that the product means that the determinant of the two unitary matrices put into a $5 \times 5$ matrix shall be unity. Seen in this way it is rather obvious that the here proposed “Standard Model Group” $S(U(2) \times U(3))$ is a subgroup of $SU(5)$ as a group and not only as far as the Lie algebra is concerned. One can even say that some of victories of $SU(5)$ concerning the weak hypercharges of the particles in the Standard Model can be ascribed to the information gotten out of the from $SU(5)$ surviving subgroup $S(U(2) \times U(3))$. The second way of denoting the same group $S(U(2) \times U(3))$ is $U(1) \times SU(2) \times SU(3)/Z_6$ and it describes it as first considering the group $U(1) \times SU(2) \times SU(3)$ and then divide out its center a certain subgroup isomorphic to the group of integers counted modulo 6, called here $Z_6$. This special subgroup is generated by the group element $(2\pi, -1, \exp(2\pi/3)1)$ of $U(1) \times SU(2) \times SU(3)$ and the elements generated by
it being divided out. This means that one divides out the invariant subgroup generated by this element \((2\pi, -1, \exp(2\pi/3))\) so as to construct the corresponding factor group. We here counted the length around of the \(U(1)\) as being \(6 \times 2\pi\), so that the sixth power of the generating element \((2\pi, -1, \exp(2\pi/3))\) becomes the unit element in \(U(1) \times SU(2) \times SU(3)\). One might also describe this group starting from the covering group \(\mathbb{R} \times SU(2) \times SU(3)\) dividing out the subgroup generated by essentially the same element as we just used \((2\pi, -1, \exp(2\pi/3))\).

It should be remarked that by this division out of group isomorphic to the integers modulo 6 we get the three invariant Lie algebras for respectively \(U(1)\), \(SU(2)\), and \(SU(3)\) linked together. While the Lie group \(U(1) \times SU(2) \times SU(3)\) is the cross product of three factors, the suggested phenomenological group for the Standard Model, or for nature we could almost say, \(S(U(2) \times U(3)) = (U(1) \times SU(2) \times SU(3))/\mathbb{Z}_6\) is not a cross product of any corresponding groups. This corresponds to that the rules for hypercharge quantisation which follows from the “phenomelogically” supported group \(S(U(2) \times U(3))\) are such that the hypercharge values \(y/2\) allowed by this group depends on the representations of the non-abelian Lie algebras \(SU(2)\) and \(SU(3)\).

It should be remarked immediately that this type of bringing an abelian group \(U(1)\) together with non-abelian groups by division out of a discrete subgroup is a rather characteristical property of the Standard Model Group \(S(U(2) \times U(3))\). That means then that it is “tallest for the Standard Model Group”, in the sense that among Lie groups with similar rank or similar dimension as this Standard Model Group there are not many that can claim to divide out in the nontrivial way a bigger discrete group than this \(\mathbb{Z}_6\), which is divided out in the Standard Model Group case. So if we want to “help” the “Standard Model Group” \(S(U(2) \times U(3))\) to win a game, we should give it many points to have such a “division out” with a relatively large group, so that \(S(U(2) \times U(3))\) can win on its \(\mathbb{Z}_6\).

For example in the article by one of us and N. Brene and one of us [15] in which we claimed that having few automorphisms was what singled out the Standard Model Group \(S(U(2) \times U(3))\) among other groups with the same number of abelian dimensions, it were in reality the division out of the discrete subgroup \(\mathbb{Z}_6\) causing a connection between SU(3) and U(1) that removed some separate automorphism acting on U(1) separately and one on SU(3) separately replacing it by only a common automorphism for them both that helped to make the Standard Model Group more skew so as to win the game for being “skewest”.

3 Introductory guidance for what game to propose

One could imagine several directions for speculations giving ideas about what type of games among groups one should attempt in order to seek a game suitable for the Standard Model Group to win.

Some such inspiration ways of thinking could be:

- One idea would be that the Standard Model Group is the end or close to the end of a perhaps long series of group break downs - you could think of Normas theory in which it comes after several break downs of some \(SO(N, 1)\) at higher energies- and thus one could almost in Darwinistic terms think about what would be the typical way for a group to break and under such a breaking, is there some property that gets enhanced by the breaking. By this we mean: Is there some property - expressed by number say - of the group surviving the break down that will typically or always be bigger than for the group that broke down to it. If we have such a quantity we would - if it is true that there are many breakings - expect it to be so big for the Standard Model Group that making a game
for such a quantity would likely make the Standard Model win or at least get close to win. There are of course some quantities that do get say smaller each time the group breaks, namely the dimension or the rank. So in such a many breaking philosophy we would expect that the Standard Model Group would have - in some sense - very low dimension and very low rank, say. But it is difficult to say what to compare. At least we must admit that some groups have smaller rank and/or smaller dimension than the Standard Model Group, so these simple ideas were not quite so useful.

One route though might be to require for instance that the gauge group we look for should have a system of Weyl fermions that are both mass protected and nevertheless leads to no anomalies in the gauge charges. Then one could even add (extra) assumptions about that the representations of the Weyl fermions be in some sense small or simple.

- Alternatively we could think somewhat in the direction of the landscape model (from string theory) \cite{20} that there are many a priori possible vacua having different gauge groups. Then we need some extra speculation or assumption about which of these vacua then have the best chance of be the one in which we come to live, or which gets realized at all. To selects such vacuum and thus the gauge group to be found, one might first think of the antropic principle: then it would we should speculate about which gauge group would be the most favourable for humans.

One could also say we need a theory for initial conditions to tell us which vacuum should be selected to be produced in the beginning and then likely survive. Here the complex action model of one of us (H.B.N.) and Ninomiya could come in as a candidate to select a vacuum. In fact the main point of this complex action model ends up being that the initial conditions get settled in such a way as to minimize the imaginary part of the action $S_I$ evaluated for the whole history of the Universe though both past and future. Since so enormously much of the universe is practically empty - i.e. vacuum - it is clear minimizing such an imaginary part of the action $S_I$ will in very first approximation mean that that vacuum should be selected to exist through most of time and space, which has the smallest imaginary part of the Lagrangian density $L_I$. Without knowing what the imaginary part of the Lagrangian density in the correct fundamental theory is it is of course somewhat difficult to guess how to get this imaginary part of the Lagrangian density minimized, but we could attempt the following loose argument: Suppose that the imaginary part of the Lagrangian density has a very similar form as the real part, just with different coefficients. Then we would guess that to find the minimal imaginary Lagrangian density we might instead seek an extremal real part, and then hope that this is where most likely the extreme imaginary part will also be. Such a search for an extremal real part of the Lagrange density might by itself be supported by other arguments without using complex action model. In fact we could speculate that somehow the most stable vacuum were one with smallest energy density. Ignoring or approximating away the kinetic part of the energy density extremizing the energy density would lead to extremizing the Lagrangian density among the possible vacua.

Such a search for a numerically largest plaquette action - if one thinks i a lattice gauge theory model terms - could thus be an idea that could be supported by several speculations; either our selection by complex action model or by some minimization of energy or Lagrangian density.

But the Lagrangian density in a certain vacuum of course depends on the coupling constants or equivalently on the coefficients to the various terms that may occur in say a lattice gauge theory. Therefore such a minimization of the plaquette action among the
different vacua requires that we have in addition a method for calculating these coefficients or coupling constants for all the different theories with their different gauge groups, which were what were to compete. Now at this point we propose our determination of the coupling constant by means of our principle of multiple point principle (MPP)\[11\]. This principle MPP means that the vacuum sits at a phase transition point as function of these coupling constants. But now it sounds, that we have really put too many unreliable assumptions on top of each other so that the chance of the all being true gets very low: existence of an imaginary action, vacuum being selected to have it minimal, the imaginary action of the vacuum being minimal just when real part is extremal too, the multiple point principle of couplings being chosen to just sit on the phase borders (at some multiple point, where several phases meet). And then to make use of this long series of assumptions we have to make the approximations to be used to estimate the size of the plaquette action under the MPP etc., assumptions. It is actually this series of ideas that were the point of the route of section\[8\]. But probably it can only be excused by saying, that doing such a series of speculations we have at least an attempt to a connected picture and should have a better chance of stumpling on to a correct proposal for the game, that is characterizing the Standard Model Group, because we should not make any totally stupid and wrog things, if we are in some at least thinkable scheme.

- One very attrakttive way to proceed would be a genuine Random Dynamics\[5\,7\,6\]. In principle we might imagine a quantum field theory, which instead of being assumed translational invariant is assumed to have a quenched random (glassy) Lagragian density\[14\] or action for the unit cell if we think of the model as regularized to let us say a lattice type theory. We may even take the number of degrees of freedom to vary in a quenched random way from cell to cell in the lattice. So we take it, that there is connected to each 4-cube in a lattice at random - quenched randomly - chosen a number of degrees of freedom. Next also in the quenched random way an action contribution expression is chosen, and that expression delivers then the action contribution from the cell considered, and it depends of course from assumed locality only on the degrees of freedom of that cell and the neighboring few cells. Such a model on a lattice and with locality and background geometry put in but otherwise with quenched random action and number of degrees of freedom could be considered a Random Dynamics model\[5\,7\,14\].

According to our old idea\[6\] there can in say a lattice model occur effectively gauge invariance without it being put in to the extend that a photon without mass can appear in a model with no exact gauge invariance. Let us though mention that this phenomenon of a gauge symmetry appearing by itself, as one might say, comes about that at a stage we write the theory as having a formal gauge symmetry looking at first as if it were Higgsed. Then it is the Higgs degrees of freedom in the formally gauge invariant description, that by quantum fluctuations wash out, so as to become ordinary massive particles or just an unimportant field not accessible at low energy. The idea should now be that the quenched random theory proposed here as a manifestation of the Random Dynamics project would in a way similar to the one described in\[6\] be rewritable into a theory with some formal gauge invariance, which then due to quantum fluctuations could appear at the end not Higgsed (although it looked at first Higgsed). Thus some gauge symmetries would come out as observable at long distances, giving rise to say massless photons gluons etc..

Thinking in terms of such a quenched random theory producing effective although at first formal gauge symmetries it becomes in principle a matter of a may be hard - but presumably doable - computer calculation to find out which gauge groups occur and how
often in this “by itself way” [6], provided though that we put in the definite rules for the quenched random distribution of the action and the number of degrees of freedom per cell. However, it might very likely turn out that this specific choice of a quenched random distribution of the degrees of freedom numbers per cell and the action per cell will be of little significance as to how the model will show up at long distances, and what gauge groups will appear.

Such an insensitivity to the details of the quenched random probability set up may though be just the wishful thinking of Random Dynamics, that at the end it is features of the theory determined by looking only at long distances (or in other regimes, where the “poor” physicist can get access), that determine the effective laws of nature which we see. In any case it would be a very important project to by computer or just theoretically find out which gauge groups preferentially would come out by themselves from such quenched random lattice theory with even a quenched random number of degrees of freedom (varying from cell to cell). In the spirit of the present article the idea of course would be a bit speculatively to figure out what properties of a group would make it likeliest that just that group in question would appear by itself.

How now to get an idea of which groups would most likely come out of such a quenched random theory? Well, in order that one can get the formally exact gauge symmetry to appear effectively so as to deliver massless gauge bosons effectively it is needed, that at the starting level the gauge symmetry is there approximately, because it is the rudiment of the fundamentally not present gauge symmetry being broken, that leads to the “Higgs” or Higgs like effects breaking at first seemingly spontaneously the global gauge symmetry, that has to be small enough for being destructed by quantum fluctuations. In other words we only can get sufficient quantum fluctuations to bring the formal gauge symmetry which we might invent to become physically effective at long distances provided the original gauge breaking were small enough to be beaten by the quantum fluctuations. So we are in fact asking for which gauge groups are likely to occur by quenched random accident in small regions of the lattice theory as approximate gauge symmetries. Now let us think of seeking such a locally accidentally approximate gauge symmetry by starting to look for it say near some starting point in the configuration space of the theory locally and then estimate the chance, that going further and further away from this point the action will by accident not change more than some limit corresponding to the limit for getting it finally appear as a long distance gauge symmetry. Now a gauge transformation in a lattice theory is to be thought about as if we locally have the possibility of transforming the configuration by means of any (gauge) group element. So we now ask for how to get the best chance for that we acting with any element in the group corresponding to approximate realization of the gauge symmetry of the action at a certain site. When we here talk about a site, it is just meant that in many places one can presumably find some way of transforming the even random number locally of degrees of freedom in a neighborhood so as to approximately (but approximately only) not change the action (contribution from that region). To begin with in asking for approximate symmetry of the action at first when the gauge group elements of the transformation are near to the unit element it is mainly the Lie algebra that must be relevant. The chance for having by accident the same action as one goes further and further away by transforming with elements which lie longer and longer away from the unit element gets of course smaller and smaller the longer away we go to ask for this accidental symmetry. So it makes it most likely to find an accidental symmetry for a given group, when the action of the group changes the variables in the quenched random theory as
little as possible. In the extreme case, when the variables of the quenched random theory were not transformed at all the invariance of the action would of course be guaranteed, but that would be a trivial case, that would of course at the end not lead to any effective gauge theory at long distances. So we must ask for a slow variation, but there should be some variation. To make it easy - or at least for start - we shall think of the degrees of freedom among the quenched random ones being roughly representation matrix elements. That is to say we may think of that there are among the quenched random number of degrees of freedom locally some we may think of as matrix elements and of the proposed transformation law as a linear representation of the group. In this way we allow ourselves to think of the speed with which the configuration moves when varying the group element in the (local) gauge transformation as motion speed for a representaion matrix. This latter speed is proportional to the square root of the quadratic Casimir $c_R$ for the representation in question $R$. So we see that the chance of getting an approximate symmetry under such a one point local gauge transformation is biggest, if the representation to which we relate it has the smallest quadratic Casimir, because then so to speak the speed of moving of the configuration - approximated by the matrix elements of the representation $R$ - when we move the group element is the smallest. Since we thought of starting at around the unit group element and got the normalization for the speed to consider specified by the Lie algebra, we would naturally count the quadratic Casimir $c_R$ normalized by setting the quadratic Casimir for the adjoint representation, which is the representation on the Lie algebra itself, equal to unity.

In this way we get from the Random Dynamics picture of looking for approximate gauge symmetry by accident the suggestion of selecting the game to be:

Which group has the smallest quadratic Casimir for its smallest faithful representation in a notation normalize to let the adjoint representation quadratic Casimir be normalized to be 1.

Typically of course it will be in the local cases, wherein the representation matrix that we can use as an approximation to the local variables is one with the smallest quadratic casimir that will be most important for finding approximate gauge symmetry by accident, because it is these cases that have the biggest chance. It is therefore we in practice must think of the relevant representation $\tilde{R}$ as being the one with the smallest quadratic casimir. The representation with smallest quadratic casimir is practically the same as the “fundamental" representation. Thus we arrive essetially to that the game to win for being the most likely group to appear approximately by accident is the one which has the smallest fundamental representation quadratic casimir $c_F$. The ratio of the two quadratic Casimirs, the fundamental and the adjoint, is actually such an interesting quantity group theoretically that it got essentially - i.e. apart from some dimension of representation factors - the name Dynkin-index.

In the spirit of the just above it is clear that if we could somehow “divide out” part of the center this would make the group smaller(in volume) and thus easier to get realized as approximately a good symmetry (i.e. an approximate symmetry of the action) by accident. We should therefore let such a divission out of the center count extra, enhancing the success of the group to win the bigger the subgroup divided out.

As is explained a bit more in the following section it is suggested that we should improve our quatity to be minimzed to $c_F/(\#\text{center divided out})^{2/d}$. We can namely crudely consider $c_F$ as proportional to the $2/d$th power of the volume of the group in the sense, that
since \( c_F \) is a quadratic form in the “distance” in the group the volume of a d-dimensional group gets by varying this \( c_R \) from representation to representation its volume changed proportional to the \( d \)th power of the square root of \( c_R \). If one therefore change the volume by some other effect effectively, namely by dividing out a subgroup of the center having \( \# \text{centerpart divided out} \) elements - which of course diminish the volume by a factor \( \frac{1}{\# \text{center divided out}} \), this would correspond to replacing \( c_F \) by an effective quadratic Casimir \( \frac{c_F}{\# \text{centerpart divided out}} \).

4 Requirements of correct behavior under group volume scalings

It is important to fix the precise quantity to be proposed as the one that the group winning should say maximize so that this quantity shall not be notation dependent but as stable under change of conventions as possible. It is therefore we had to take the ratio of two relatively easy to select representations. If we had namely not taken a ratio this way the quadratic Cassimir would depend on the notation for normalizing quadratic Cassimirs.

For giving a possible good physical sense to this ratio it is immediately obvious that a meaning of the type that this ratio denotes the square of the speed of motion of the group element in the two different representations discussed is called for. If now the true physical quantity to be argued for were indeed rather a total volume ratio we can see that a volume correction for say the “fundamental” representation would have to come in just the right power to combine in a physically consistent way with the speed ratio already being present in the proposed \( 1/c_F \). This considerations leads rather quickly to that our first proposal \( 1/c_F \) can only be corrected by a division out of a center subgroup of order \( \# \text{center} \) by the factor \( (\# \text{center})^{2/d} \), where \( d \) is the dimension of the group.

That is to say that the quantity to be say maximized would in order to combine the volume dependence correctly

\[
(\# \text{center(divided out)})^{2/d}/c_F.
\]

The to be minimized quantity could then be of course the inverse of this \( c_F/(\# \text{center divided out})^{2/d} \).

5 What Scores do Different (Simple) Groups get?

Before we in the next subsection \( 5.2 \) shall tell about how one extracts from the literature the values for the quantity \( 1/c_F \), we may put forward some features of how the competition goes by mentioning a few remarks:

- **Large rank behavior** As is wellknown the simple Lie algebras are classified into four infinite series and further some “exceptional” Lie algebras. For the infinite series it actually turns out that if we allow the smallest quadratic Casimir representation \( F \) to be the one making \( c_F \) smallest we get for the algebras for “large \( N \)” - meaning the late algebras in the infinite chain - that

\[
\frac{1}{c_F} \to 2 \text{ for the rank } r \to \infty.
\]

This is a very important property for our project because you could add a formally \( \infty \) to the rank \( r \) region and the function of the algebra \( 1/c_F \) would remain a continous function and that now on a compactified space of algebras. These means that there should exist one (or perhaps several) largest value for \( 1/c_F \). So we can really expect to find a presumably
single winner among the simple algebras - or we might have got an infinite limit, but that luckily does not happen -.

- **The front field** The winner number one among the simple Lie algebras turns out to be $SU(2) = A_1$, since it gets using the general formula for $A_1$

$$\frac{1}{c_F} = \frac{2}{1 - \frac{N}{2}} \text{ for } SU(N) = A_{N-1},$$

that

$$\frac{1}{c_F(SU(2))} = \frac{2}{1 - 1/4} = \frac{8}{3}. \tag{4}$$

This $8/3$ is the absolutely record for any simple Lie algebra, and so $SU(2) = A_1$ is the “gold medal winner” among simple Lie groups.

If we use the correction factor ($\frac{1}{\# \text{center-elements divided out}}$), which we mentioned above it happens that it is also bigger for $SU(2) = A_1$ than for any other simple Lie algebra. In fact it is for $SU(2)$ equal to $2^{2/3} = 1.587401052$, so that the full score with this factor included becomes for the gold winner $SU(2)$ equal to $\frac{8}{3} \times 1.587401052 = 4.233069472$ So the winner just even more certainly becomes $SU(2)$.

It is of course comforting for our model that this absolute winner among the simple Lie algebras is at least one of the invariant subliealgebras of the Standard Model Lie algebra $U(1) \times SU(2) \times SU(3)$.

But now comes for our scheme a problem: The silver winner among the simple Lie algebras using only the ratio of the quadratic Casimirs $\frac{1}{c_F}$ is not as we might hope for the Standard Model algebras to win the $A_2 = SU(3)$ algebra, but rather $SU(3)$ is beaten by $SO(5) = B_2 \approx Sp(4) = C_2$ which obtain the score

$$\frac{1}{c_F(C_2)} = 12/5 \tag{5}$$

obtained from the general formula $\frac{1}{c_F(B_r)} = \frac{4(r+1)}{2r+1} = \frac{2N+4}{N+1}$ where $N = 2r$ by putting $r = 2$ or equivalently $N = 4$. The $SO(5) = B_2$ Lie algebra is isomorphic to the symplectic one $C_2$; to get the fourdimensional representation, which is the vectorrepresentation $V$ for the symplectic $C_2$, we must for $SO(5) = B_2$ use the spinor representation.

Now the for our hoped for explanation of the Standard Model a bit unfortunate fact is that the $SU(3) = A_2$ algebra only reach the score $\frac{1}{c_F(A_2)} = \frac{2}{1 - 1/3} = \frac{6}{4} < \frac{12}{5}$. So in the pure use of $1/c_F$ the phenomenologically relevant $SU(3) = A_2$ lost and only obtained the bronce medal. Of course it is still promising that it got a medal at all, but we could have said that we got the two genuine simple Lie groups if the winning gold and silver to be the two phenomenologically found ones. But alas, it were not like that completely!

But now we have already mentioned the idea of the extra factor $(\# \text{center})^{\frac{2}{d}}$, where $d$ is the dimension of the algebra.

For $SU(3)$ and $SO(5) \approx Sp(4)$ the extra factor turns out:

For $SU(3) = A_2$ : $(\# \text{center})^{2/d} = 3^{1/4} = 1.316074013 \tag{6}$

For $Sp(4) = SO(5) = B_2 = C_2$ : $(\# \text{center})^{2/d} = 2^{1/5} = 1.148698355. \tag{7}$
Thus we get for the full scores when this factor is included:

For $SU(3) = A_2$ : \( \frac{(#\text{center})^{2/d}}{c_F} = 3^{1/4} \times 9/4 = 1.316074013 \times 9/4 = 2.961166529(8) \)

For $Sp(4) = C_2 = SO(5) = B_2$ : \( \frac{(#\text{center})^{2/d}}{c_F} = 2^{1/5} \times 12/5 = 1.148698355 \times 12/5 = 2.756876052(9) \)

So we see that the extra factor from dividing out the center just barely brought the $SU(3)$ algebra in front of $SO(5) \approx Sp(4)$ by .20 out of ca 2.9 meaning by 7%.

This looks extremely promising for the Standard Model indeed doing very well in the game provided we include “dividing out the center” factor $(#\text{center})^{2/d}$. The only two genuine simple Lie algebras in the Standard Model then come out with respectively gold and silver medals, $SU(2)$ with gold, $SU(3)$ with silver.

• The problem of $U(1)$ With the $U(1)$ there are several problems, which we must discuss:

  1. Since the adjoint representation should be considered either as non existing or as trivial we must consider the quadratic Casimir for the the Abelian $U(1)$ as either $C_A(U(1)) = 0$ or at best for our hopes for favouring the Standard Model ill-defined. Actually there is a possibility for making sense of the ratio $C_A/C_F$ if we could somehow arbitrarily select one representation of $U(1)$ given by some “charge” $q_A$ to be considered formally the “adjoint” $A$ and then another one with another “charge” $q_F$ to be the F-representation. Then one would naturally say that the Casimir is the square of the “charge” so that $C_A = q_A$ and $C_F = q_F$. In this case of course our competition quantity $\frac{1}{c_F} = \frac{C_A}{C_F} = 4q_A / q_F$. But what shall be considered $A$ and what $F$?

  2. The idea that one could “divide out the center” of one of the genuine simple Lie groups such as $SU(2)$ or $SU(3)$ were meant to mean that after having divided it out we got instead the groups $SU(2)/Z_2$ and $SU(3)/Z_3$ instead. But then we should only be allowed to use as $F$ the representations that are representations of these groups. But then the representations $F$ which we used in the construction of our $1/c_F$’s above are not allowed. That in turn would mean that we would have instead of the $F$ we used in the cases mentioned and actually typically to use rather the adjoint representation itself, so as to get for the competing quantity $1/c_F$ now replaced by $1/C_A = C_A/C_A = 1$. If we do that we loose a factor bigger than 2 for the algebras in the strong field. That is not compensated by the extra factor $(#\text{center})^{2/d}$ and if this extra factor is only achievable by paying the price that only the adjoint representation get allowed to be used as $F$, then it is better for winning the game to give up the extra factor.

  3. If the total group - the cross product say of several simple factors - has a $U(1)$-factor in it, one can divide out a subgroup of the center that could be e.g. $Z_2$, if we have $SU(2)$ and $Z_3$ if we have $SU(3)$ in such a way that this divided out subgroup of the center is not subgroup neither of the genuine simple Lie group nor of the $U(1)$ separately. If one divides say a $Z_3$ or a $Z_2$ out in such a way, then it does not prevent that there can be a representation which with respect to $SU(2)$ or $SU(3)$ corresponds to the $F$ we used in our above calculation and which managed to make these simple algebras win the game.

In this way we can claim that we have a way - by means of using a $U(1)$ - to both get the favourable $F$ representaion used to let our favourites win, and at the same time get “division out of the center” take place.
This situation seems so favorable and really needed to get win for a simple Lie algebra by the help of the extra factor, that unless it requires a very high price in form of some loss in the final score, it seems to be very needed to include a $U(1)$-factor in the total group.

So here we have essentially argued that unless the rules for the Abelian $U(1)$ get adjusted in detail to be very unfavourable for winning then because of the otherwise impossible combination of the extra factor and the representation $F$, it becomes needed to have a $U(1)$ included in the total group.

5.1 **Standard Model Group very promising, crude review**

Let us here argue how one with very little (extra) assumptions about the averaging, when having a team of Lie algebras, is to be taken, can argue for the Standard Model group being the winner among teams of Lie algebras:

We must of course have some rule for making a score for a group that is not simple from the score numbers for those simple invariant subgroups of the group. One can imagine several weightings such as e.g. weighting the individual simple group scores by the dimensions of the simple groups. But of course our derivation that the Standard Model wins the game would be most convincing if it could be done with so mild assumptions as possible concerning these rules of combining. Otherwise we could be accused for having adjusted the rule of weighting so as to favor the Standard Model (if we do not succeed in arguing that it does not matter much what rule we use, then of course we shall assume some rule that favours the Standard Model, so as to see if it is at least possible to make the Standard Model win in such a way.)

If we cannot get the “extra factor from dividing out center(subgroup)” $(\#\text{center})^{2/d}$, the largest achievable score for any simple Lie algebra and therefore also for any (sensible average, which of course can never be bigger than the quantities from which it is averaged) average over a “team” (a non-simple group) becomes the $8/3$ which is the biggest achievable value for $1/c_F$, being reached for the simple algebra $SU(2)$. To reach a score higher than these $8/3 = 2.6667$ we have to obtain the extra factor $(\#\text{center})^{2/d}$, but for that we need to have a $U(1)$. So we suggestively should have a $U(1)$ combined with an $SU(2)$ and then have a center $Z_2$ divided out in a way that is not a subgroup of neither $SU(2)$ nor $U(1)$. But that means we have now suggestively reached $U(2)$. It could now seemingly be possible that what would win would be just one or a cross product of several $U(2)$’s. We can namely with a sensible averaging not get a different score for a group and this group crossed with itself a number of times. But now we already argued that we needed the $U(1)$. So we ask, could we not apply this same $U(1)$ several times instead of just to help $SU(2)$ to get a high score? Actually we may use it again, but we cannot use it to help another algebra, which has again a $Z_2$ center to get divided out. The problem is that if we attempt that, we shall miss the allowance to use the representation $F$ we used for both the $SU(2)$ and the other algebra, that also has the center being $Z_2$. That would mean that the score for either $SU(2)$ or the other Lie algebra would miss more than a factor 2 in the score. If, however, we can divide out a center which is a $Z_n$ with an odd $n$ so that it has no common factor with the 2 in $Z_2$, there will be no such problem. So e.g. a $Z_3$ would be o.k.. Such an extension with a Lie algebra that had a group from which we could divide out e.g. $Z_3$ could be added without the need for any further $U(1)$. From what we already saw about the individual scores for the simple Lie algebras or rather groups the silver medal winner were already $SU(3)$. So now we must ask, if it so to speak would pay in terms of getting the best score average for the full group if we to our first suggestion $U(2)$ add/extend with the $SU(3)$. Because of the ambiguity comming from that we do not clearly have settled how to count the
we do not know, if the addition of the proposed $SU(3)$ will pay. It is namely so: If the averaging of $SU(2)$ with the $U(1)$ has brought this average from the $8/3 \ast 2^{2/3} = 4.233069472$ below the score-value of $SU(3)$ being $9/4 \ast 3^{1/4} = 2.961166529$, then it will pay to include the $SU(3)$. That might happen, but we must admit that it dependence on the exact averaging rule, as well as on what one puts the score for $U(1)$ in itself. So honestly we only got to, that it is possible to imagine an averaging proceedure, that would make the Standard Model win!

5.2 Extraction of the $1/c_F$

In [19] we find for the quadratic Casimir $C_A$

$$C_A = \eta g,$$

where $g$ is the dual Coxeter number, while $\eta$ is a notation-dependent normalization constant, which is defined via the formula

$$C_R = \frac{\eta}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} (a_i + 2)G_{ij}a_j$$

for the quadratic Casimir in the representation $R$. Here again the quantities $a_i$ for $i = 1, 2, ..., r$ are the Dynkin labels for this representation $R$. Finally the $r$ is the rank of the group, and $G_{ij}$ is the inverse of the Cartan matrix.

Using still [19] the “second index” $I_2(V)$ for the “vector” representation $V$ given as

$$I_2(V) = \frac{\eta}{2} \text{ for } SU(N) \text{ and } Sp(N),$$

$$I_2(V) = \eta \text{ for } SO(N).$$

and the relation

$$I_2(R) = \frac{N_R}{N_A}C_R,$$

where $N_R$ is the dimension of the general representation $R$ while $N_A$ is that of the adjoint representation $A$, we get

$$\frac{1}{c_F} = \frac{C_A}{C_F} = \frac{C_A}{C_V} = \frac{2N_V g}{N_A} \text{ for } SU(N) \text{ and } Sp(N),$$

$$\frac{1}{c_F} = \frac{C_A}{C_F} = \frac{C_A}{C_V} = \frac{N_V g}{N_A} \text{ for } SO(N).$$

(We here took it that the “smallest” representation $F$ were indeed the “vector” representation $V$, which is not always the case) Herein we shall then insert the dual Coxeter numbers $g$, which are

$$g_{A_r} = r + 1 = N \text{ for } A_r = SU(N) \text{ where } r = N - 1,$$

$$g_{B_r} = 2r - 1 = N - 2 \text{ for } B_r = SO(N) \text{ for } N \text{ odd and } r = \frac{N - 1}{2},$$

$$g_{C_r} = r + 1 \text{ for symplectic groups } C_r,$$

$$g_{D_r} = 2r - 2 = N - 2 \text{ for } N \text{ even and } D_r = SO(N) \text{ where } r = N/2,$$

$$g_{G_2} = 4 \text{ for } G_2,$$

$$g_{F_4} = 9 \text{ for } F_4,$$

$$g_{E_6} = 12 \text{ for } E_6,$$

$$g_{E_7} = 18 \text{ for } E_7,$$

$$g_{E_8} = 30 \text{ for } E_8.$$
and then we obtain e.g.

\[\frac{C_A}{C_V} = \frac{2g_A N_V}{N_A} = \frac{2(r+1)(r+1)}{(r+1) - 1} = \frac{2}{1 - 1/(r+1)} = \frac{2}{1 - 1/N}\] (26)

For \(B_r = SO(2r+1)\):

\[\frac{C_A}{C_V} = \frac{g_{B_r} N_V}{N_A} = \frac{(2r-1)(2r+1)}{r(2r+1)} = 2 - 1/r,\] (27)

For \(C_r = Sp(2r)\):

\[\frac{C_A}{C_V} = \frac{2g_{C_r} N_V}{N_A} = \frac{2(r+1)2r}{r(2r+1)} = \frac{4(r+1)}{2r+1} = \frac{2N+4}{N+1}\] (28)

For \(D_r = SO(2r)\):

\[\frac{C_A}{C_V} = \frac{g_{D_r} N_V}{N_A} = \frac{(2r+1)2r}{r(2r-1)} = \frac{4r+2}{2r-1} = \frac{2N+2}{N-1}\] (29)

But now we must admit that those “vector” representations \(V\) which we here used are not in all cases the smallest neither as concerns the quadratic Casimir nor w.r.t. dimensions. This is the case for the relatively low rank \(r SO(N)\) groups. They have namely spinor representations. In fact we have for an even \(N\) that \(SO(N) = SO(2r) = D_r\) have a spinor representation - we shall have the chiral irreducible representation - of dimension \(2^{r-1} = 2^{r-1}\); for odd-\(N\) we have \(SO(N) = SO(2r+1) = B_r\) a spinor representation of dimension \(2^r = 2^{(N-1)/2}\).

We may read this problem off in the list of what [19] propose as “reference representations”. Here the odd-\(N\) SO(\(N\)) algebras \(B_1, B_2, B_3,\) and \(B_4\) are proposed represented as reference representations by their spinor representations, while it is for \(r \geq 5\) the \(B_r\) have as their reference representations the vector representations \(V\). Similarly it is proposed to use as “reference representations” for the even SO(\(N\))-algebras in the case \(D_3\) meaning \(SO(6)\), while for \(r \geq 4\) we use the “vector” representation.

6 How to Combine Scores to Scores for Non-Simple Groups?

When we combine simple gauge groups into semisimple group we have to postulate some rule for combining in some way averaging our quantities for the various simple groups. We might think of more complicated rules but in the light of the “theory behind” the favoring of the groups

6.1 The \(U(1)\) problem, what to take for its \(C_A/C_F\)

Since the Lie algebra of \(U(1)\) has a trivial adjoint representation it has really no meaning to talk about \(C_A\) for \(U(1)\), or we might say it is zero, but a zero is not so usefull for our normalization. We could propose instead to replace the adjoint representation by the “unit charge” representation of the \(U(1)\) and use that as a normalization representation. Now we should ask as we did for the other groups: can we find a “smaller” representation? That should now be one with a smaller charge, but such a smaller charge would only be allowed if we used a bigger version of the \(U(1)\) circle. So keeping the group unchanged there are no smaller charges allowed. Looked upon this way we can say that the \(U(1)\) is analogous to the \(E_8\) algebra for which there is no smaller representation than the adjoint one. Therefore we get for \(E_8\) that \(C_A/C_F = 1\). Therefore we should by analogy also take 1 for the abelian group \(U(1)\). Then it may not matter so much whether we really have and use an adjoint representation.

6.2 Volume product weighting, a proposal

One way to combine into some average the scores of the different simple groups going into not simple Lie group is suggested by having in mind that
• a: We thought of the chance of getting symmetry by accident crudely being a good symmetry for some a priori "random" action suggesting that it is the volume of possible set of field configurations in which the group transformation brings a state around by transformation under the group that counts. (This volume should be minimalized to make the chance for having the accidental symmetry by accident maximized.)

• b: We should attempt to count in such a way that just putting some repetition of the group as a cross product should not change the chances; rather it should be the type and structure of the group occuring that we should get information about.

• c: When we have say a cross product of groups the image in the configuration space should also have the character of being a product, so that the volume of the combined group representation would become a product of the volumes of the components.

• d: The quantity, which we used $\frac{C_A}{C_F} \ast (\#_{\text{center}})^{2/d}$ were - by the accident of our notation - as going inversely as the $2/d$th power of the volume in the configuration space relative to some more crudely chosen group volume. (indeed we selected this group volume by means of the commutation rules so as make it given by the quadratic Casimir for the adjoint representation.)

The way suggested by this thinking is that we should use logarithms of our numbers used for scores and weight them by the dimensions of the groups. That is to say we propose the quantity:

$$T = \sum_S \frac{d_S \ln(\frac{C_A}{C_F} \ast (\#_{\text{center}})^{2/d} |_S)}{\sum_S d_S} = \frac{1}{\sum_S d_S} \ln \prod_S \frac{C_A}{C_F} |_S^{d_S} (\#_{\text{center}})^2.$$  (30)

Here $S$ symbolizes the various simple Lie algebras going into the non-simple group, we consider. So e.g. in the case of the Standard Model Group $S(U(2) \times U(3))$ this $S$ runs over the three Lie algebras, $S = U(1), SU(2), \text{ and } SU(3)$.

Having already found above the scores for $U(1)$ being in our way counted as 1 meaning a 0 when we take the logarithm (this were somewhat not quite clean, but the most reasonable), for $SU(2)$ the seemingly everyone beating $\frac{8}{3} \ast 2^{2/3} = 4.233069472$, and for the $SU(3)$ score $\frac{9}{2} \ast 2^{3/8} = 2.96116529$, we may as an example evaluate the by Nature beloved Standard Model Group:

$$T_{SMG=S(U(2)\times U(3))} = \frac{1 \ast 0 + 3 \ast \ln 4.233069472 + 8 \ast \ln 2.96116652}{1 + 3 + 8} = \frac{0 + 0.360731843 + 0.723722192}{1.084454035} = 1.084454035.$$  (31)

This means that the averaged score for the Standard Model group should be counted as having this averaged quantity as its logarithm, so that it becomes itself:

$$\exp T_{SMG=S(U(2)\times U(3))} = 2.95782451.$$  (33)

This score by the Standard Model shall be compared to other obviously competing candidates such as $U(2)$. We should remember that without the company of the $U(1)$ the $SU(2)$ is not allowed to gain its $2^{2/3}$-factor, so without this $U(1)$ it would not even have $8/3 = 2.666666667$(because we could not use the representation $F$ being the spin 1/2) and could not compete. With the inclusion of $SU(2)$ having to carry along the $U(1)$ - with only its 1 score - we get this 4.233069472 formally for $SU(2)$ cut down to its 3/4th power, meaning 2.951151786 for
It is really a very tight game but it is the Standard Model that wins over even the $U(2)$! That it must be like that is also signaled by that fact, that the number for $SU(3)$ when the center-factor is counted is 2.96116652 and brings the average for the Standard model group up. This makes us look for if $U(3)$ could now beat the Standard Model Group? Well $U(3)$ would score the $8/9$th power of these 2.96116652 giving 2.624690339, which is less than the score of the Standard Model Group 2.95782451.

It should be remembered, that the application of this formula should be done only, when there are sufficient $U(1)$’s to make the simple groups $S$ over which we sum get their $F$-representations used realized. It is really the importance of the $SU(2)$ and the $SU(3)$ groups sharing their $U(1)$. This is only possible because their center $Z_N$’s have mutually prime numbers $n$, namely 2 and 3. It is this collaboration between the two by sharing the burden of the $U(1)$ which they need for getting their center-factors $2^{2/3}$ and $3^{2/8}$ respectively, that brings the Standard Model Group $S(U(2) \times U(3))$ to win. All of the three simple groups collaborate to win.

We leave it to the reader to check that no other combination of groups can beat the Standard Model Group! Most of the competitors are soon losing out, because it is only the small rank simple groups that get the high scores.

7 Conclusion on the Game Found so far

Let us summarize the most important of the games discussed - the game between “teams” meaning Groups that are not necessarily simple, so that they appear as combinations of simple algebras. Here the proposal for game quantity is the w.r.t. to dimension averaged logarithm of the quantity originally proposed $C_F$ including - if allowed without spoiling the representation $F$ used - a center-factor $(\# center)^{2/d}$. To bring the total averaged logarithm $T$ for a group that is typically not simple to be compared to the previously discussed numbers it may be best to exponentiate it back by taking as the “team-score” (meaning score for groups, that are not necessarily simple) $\exp T$ for the combined group in question.

For this dimension averaged quantity we found that the Standard Model Group $S(U(2) \times U(3))$ (as suggested from the representations of the quarks and leptons found in nature) is the maximal score of

$$\exp T_{SMG=S(U(2)\times U(3))} = 2.95782451. \quad (34)$$

This $SMG = S(U(2) \times U(3))$ is extremely tightly followed by $U(2)$ which got

$$\exp T_{U(2)} = 2.951151786. \quad (35)$$

But it were the Standard Model, that won!

If the reader would accept that the rules of the game were chosen in a reasonable simple way, one would say, that it is very remarkable, that we have been able to present a game giving just the Standard Model Group the best score! It should be expected there were a reason, that should be found, to explain, why precisely this group with the highest score in our sense should be the realized one. So this finding should possibly bring us to get to an understanding of the question: Why the standard model group?

8 Our Early Model with MPP and Numerically Maximizing Plaquette action

We came into the ideas of the present article by in a lattice gauge theory speculating about some reason for that the energy per plaquette normalized in some way should be minimized. Of course
such an energy of a plaquette contribution to the energy depends on the couplings constants, the finest structure constants, and so we would have to combine such a looking for minimal energy (or a minimal action), with some assumption about what the coupling constants would be with different possibilities for the gauge group. As such a machinery to provide the gauge couplings we then had in mind to assume the idea of multiple point principle MPP, which means in a lattice gauge theory that the gauge coupling parameters shall be adjusted so as to get the lattice theory go to a “multiple point”, i.e. a point in coupling constant space where several phases meet.

We shall not too deeply into the calculations needed in the present article. What we have to do is to use the constraints on the coupling constants imposed by the requirement of the several phases just meet, that is to say the couplings are in this sense “critical”. In principle we can include several possibly only as lattice artifact relevant parameters among the here mentioned “coupling constants”. Using such contraints which in principle are constraints which we can calculate we should be able to have estimates of coupling constants even for groups, which are not realized, but only thought of as possibilities. In this way we become able to estimate questions such as what would the energy or action (whatever we ask for) per plaquette in the lattice theory be, if the group were say $G$. Thus we can in such a scheme ask for maximizing e.g. say the action of the plaquettes.

Now the question is, if we can make the details of the here proposed scheme so that we get the groups classified much the same way as we have in the present article proposed partly by phenomenological guessing. Indeed it seems that the scheme with use of MPP to restrict the coupling constants a then maximizing the plaquette action normalized in an appropriate way with the square of the dimension involved.

8.1 On our Finestructure Constant Fitting in New Light

One possibly great feature of using the scene with MPP and maximaizing plaquette action is that it together with the selection of the group also provide the coupling constants, so that we in addition to the prediction of the gauge group as we did in the present article get a related prediction about the fine structure constants. This can hopefully soon bring us to present a fit to the latter in such a combined scheme. That might open up for making interesting phenomenology on the details of the model-type proposed by fitting both the gauge group and the fine structure constants.

9 Further Speculations for a Reason for the Selection

9.1 What is Good for Prevention of Spontaneous Breaking of Gauge Symmetry

We should imagine a gauge glass or just a glassy structure in the sense that the action is given with terms which vary from point or lattice cell to point in a quenched random way. This is what we mean here in the abstract sense by “glass” that the theory or its action involves a lot of quenched random - meaning fixed randomly before you integrate to make the partition function or the Feynman path integral - variables, so that in a way one could think of it as, the theory itself being random. It is even random in a non-translational invariant way in as far as it varies from point to point or from little lattice neighborhood to the next little lattice neighborhood.

The main point, we now want to point out is that if we let the quenched random theory not a priori obey gauge symmetry and gauge symmetry has to come out the way suggested in
the gauge theory that we might formally think about is also in the danger of being broken - spontaneously - by the ground state not having the the plaquette variables driven to a center element - as is required for the invariance under a global gauge transformation of the vacuum - but to some non-central element. Honestly speaking: in the quenched random model it will almost certainly happen that here and there in space(time) will be plaquette variables, which actually will lead to the minimum energy density, say by standing at some non-central element. If it stands at a central element, it is not so serious, since we can essentially just think of all the elements being displaced by a right translation and that after such a transformation the central element at the bottom of the energy were transformed into the unit element so that we can really think of it as if it had the bottom at the unit element. But for a noncentral element being at first at the bottom we cannot transform it to the unit element without changing the system physically. So if truly a non-central value occurs for the vacuum field it means indeed that the global part of the gauge group in question has broken spontaneously.

10 Conclusion and Outlook

We have in the present contribution put up an attempt to by combined looking at some physical ideas behind and on the goal of making the Standard Model group win produce some function defined for compact Lie groups with the property that it singles out just the Standard Model group \( S(U(2) \times U(3) \) as being the Lie group for which this function has its biggest value. Indeed we managed - in an almost satisfactory way - to construct such function in a reasonable simple way. The procedure for evaluating our proposed function is like this:

- **A)** For each of the non-abelian simple Lie algebras of the Lie algebra we construct the quantity
  \[
  \frac{C_A}{C_F} \ast \left( \# \text{center} \right)^{2/d},
  \]  
  where \( C_A \) is the quadratic Casimir for the adjoint representation \( A \) of the simple non-abelian group in question. The quadratic Casimir \( C_F \) is for a “smaller” representation if possible and this representation \( F \) shall be chosen at the end with the purpose of making the quantity \( 36 \) as large as possible. Typically the representation \( F \) will be the “fundamental” representation. The quantity \( \# \text{center} \) is the number of elements in the center of the covering group for the Lie algebra in question, and \( d \) is its dimension.

- **B)** Next average the logarithm of this quantity over all the simple non-abelian Lie weighting with the dimensions \( d \) of the Lie algebras including the abelian components in the total Lie algebra for the whole group counted to give 0 in logarithm (as if the quantity \( 36 \) were 1 for \( U(1) \)). This average is presented in equation (30).

- **C)** There is, however, an important restriction forbidding, that unless there are enough \( U(1) \)’s included in the group and they have got to a sufficient degree some (discrete) center-subgroup divided out of the cross product of the covering groups and and the \( U(1) \)’s in a way connecting the groups to be no longer just a cross product of seperated groups, we cannot use the formula above. This restriction shall be understood to mean:

  Firstly: Under the division out of the (non-trivial) subgroup of the center of the cross product of the abelian and non-abelian simple groups we must not identify the center elements of any of the simple groups so that we obtain a factor group, which no longer has the representation \( F \) for that simple groups as a representation without allowing phase
Secondly: In order to obtain the factor $(\#_{\text{center}})^{2/d}$ for one of the simple Lie algebras averaged between - it is required that the discrete subgroup divided out has indeed a factor group in correspondence to the center of the simple group in question. (This requirement implies that the divided out discrete subgroup of the center of the product of the covering groups and the $U(1)$'s should (at least) has as many elements as the product of the numbers $\#_{\text{center}}$ for all the simple groups for which the factor $(\#_{\text{center}})^{2/d}$ in (36) is to be used.).

- D) The quantity - the score so to speak - which should be largest possible for the Lie group to be realized in Nature should under the restrictions in C) be the average constructed under B).

The really remarkable fact of the present article is that The Standard Model Group as phenomenologically defined partly under use of its physically realized representations of quarks and leptons and the Higgs turns out to be precisely that (compact) Lie group which gives the biggest value for the average constructed under B) with the restrictions C) imposed!

In fact one gets for the exponential of the average over the logarithms as told in B) the number $2.95782451$.

The Standard Model Group is, however, remarkably closely followed by the Lie group $U(2)$ for which the exponentiated average becomes: $2.951151786$. They only deviate on the fourth significant cipher, and difference is only of the order of 0.007 compared to almost 3.

In our opinion the procedure for constructing the function of the compact Lie groups, the score in the game so to speak, is so simple that one would say it is pretty remarkable that it should give just the Standard Model Group, which is realized in Nature at least for the energetically accessible physics in practice, to have the biggest average. after all there are many groups which nature could have chosen, if one did not impose the phenomenological or other restrictions. Of course the Standard Model Group is the only fitting if we do not include parts of the group, which are not at all seen experimentally at present. But that just means that the Standard Model Group is - we could say - measured to be the true model. In the present paper we search for some theoretical assumption as simple as possible, that could single out and point to just this special group $S(U(2) \times U(3))$, which is the by the representations of quarks and leptons group with the Standard Model Lie algebra, and we found the principle of maximizing the quantity $\exp T$ where $T$ is the average described! It singles out the right group for nature!

10.1 Taking serious that it is not an accident

The point of such an excursion as the present one is of course to get some hints as to what is the reason Nature has just chosen the Standard Model Group and not some other group among the after all pretty many groups she could have chosen between.

It were above suggested that the quantity in which the Standard Model Group is excellent is that compared to a normalization given by the quadratic Casimir $C_A$ for the adjoint representation $A$ the group has (a) very small representation(s) in terms of some quadratic Casimir $C_F$ for a representation, which we above have thought upon as a representation related to the fields in the e.g. lattice gauge theory model working in Nature. The thing that seems to be important is that compared to some “natural” distance measure on the group (related to the quadratic Casimir for the adjoint representation $C_A$) the way it is possible to make it move the fields in
Figure 1: The ratio $C_A/C_F$ for $E$ groups plotted as a function of rank. Here we have used that $E_5 = SO(10)$ and $E_4 = SU(5)$.

some appropriate representation $F$ is very slow. That is to say you may move the group element a lot but the fields only tinily for the groups having high scores in our game. Such a property of it being easy to push the feilds only little around for the group element moving much without making the transformation completely zero (i.e. still using a true representation $F$) seems to be what our result points to as the important principle used to select just the model, which nature has chosen.

We suggested that such a selection were likely to be the result, if the gauge group had in reality appeared by first getting an approximate gauge group by accident. Then the gauge symmetry should for practical purposes have become exact due to quantum fluctuations. But the important point to extract is that the choice of the Standard Model group suggests that the group that can be represented, on fields say, being most tinily moved around under a by some adjoint representation related normalization of distances in the group is the group most beloved by nature to be realized.

11 Acknowledgement

One of us (H.B.N.) thanks for being allowed to have room and stay as emeritus at the Niels Bohr Institute at Copenhagen University, since 1st of September, when the writing of this proceeding took place.
12 Appendix: A Pedagogical Calculation Procedure for our Purpose

Having in mind that one of the main ideas for why our proposed quantity $C_A/C_F$ - i.e. the ratio of the quadratic Casimir for the adjoint representation $C_A$ divided by some representation $F$ having the smallest quadratic Casimir $C_F$ among faithfull representations - is suggested to be that this quantity $C_A/C_F$, the bigger it, is favours the chance that the group in question should be the one realized in nature because a big $C_A/C_F$ means that varying the potential gauge an amount measured by the Killing form normalized by the adjoint representation Casimir being ut to say 1 makes the variation in the link variables supposedly in the representation $F$ minimal, we shall here present as a couple of examples a practical calculation of our quantity, almost making clear its physical significance for our purpose.

Remember that we in our speculative arguments for which group would most easily become a gauge group by accidentally being so near to being it that a quantum fluctuation effect might set in and make it practically an exact gauge symmetry we used the normalization of the Killing form to make the quadratic Casimir for the adjoint representation say 1 so that we thereby got a physically meaningfull distance concept on the group manifold. Under use of this distance concept we then asked how the field theory variables say the link variables in some lattice gauge theory formulation will vary for a given - unit- variation of the gauge group element. If one in the link variable space use the distance concept derived from the trace of the square of the difference of the couple of representation matrices correspon ding to the link variable, the ratio of the infinitesimal distance in the group manifold relative to the corresponding distance in the link variable will be given by the square root of the ratio $C_A/C_F$, where $F$ is the representation used to represent the link variable. The main idea were that the chance to find an approximate symmetry under the transformation of the various link variables as transformed under the gauge transformation will be bigger the bigger the ratio $C_A/C_F$ and so the group “to be realized by accident” with best chance is expected to be the group with largest $C_A/C_F$ value.

In this appendix we shall present a way to calculate this ratio $C_A/C_F$ by using just the Cartan algebra representaions in a way that allows us to calculte simply the ratio of the average of the square of the root vector length compared to the corresponding average of the weight vectors for the representation $F$.

Let us provide a couple of examples:

- $A_1 = SU(2)$:
  
  In this case the smallest represntation - w.r.t. say the quadratic Casimir - is $F = \mathbb{2}$. As is wellknown the root system for the $F = \mathbb{2}$ consists of two weights both of half length of the roots. Thus we find if we say roots have length $\sqrt{2}$ - as is usual -

  - Adjoint : Average of the squared roots $\frac{2 + 2 + 0}{3} = \frac{4}{3}$.
  - $F$: Average of the squared weights $\frac{1/2 + 1/2}{2} = \frac{1}{2}$.

  So we obtain by taking the ratio of these averages:

  $\frac{C_A}{C_F}|_{A_1} = \frac{4/3}{1/2} = \frac{8}{3}$ \hspace{1cm} (37)

- $G_2$ : The root system for the exceptional Lie group $G_2$ consists of two regular hexagons with centers in the zero point, the one rotated by $30^0$ w.r.t. the other one and one $1/\sqrt{3}$
times the other one. The Lie algebra of $G_2$ has an $SU(3) = A_2$ subgroup corresponding to the roots of the bigger one of the two hexagones. The “smallest” representation $F$ for the $G_2$ is sevendimensional and consists w.r.t. the $SU(3)$ subgroup of a triplet an antitriplet and a singlet. This means that the weight system for this representation $F$ consists of the six roots in the smaller of the two hexagons and in addition a weight at zero. Then we have for the average of the squares of the distances from zero for the weights

- Adjoint representation: $\text{average} = \frac{6 \times 2 + 6 \times 2/3}{14} = \frac{8}{7}$.
- $F$: $\text{average} = \frac{6 \times 2/3 + 0}{7} = \frac{4}{7}$.

We thus find that

$$\frac{C_A}{C_F}|_{G_2} = \frac{8/7}{4/7} = 2 \quad (38)$$

$• B_2 = SO(5) = C_2 = Sp(4)$: The root system for these isomorphic Lie algebras consists of the corners and the midpoints of the sides of a square (with side 2 say). There are thus 8 roots. The “smallest” representation $F$ is a four dimensional one with the roots with the weights sitting in the four centers for the four squares with side 1 into which the coordinate axes divide the mentioned square of side 2. Then we get for the averages of the squares of the distances from zero in the root and weight systems:

- Adjoint: $\text{average} = \frac{4 \times 1 + 4 \times 2 + 2 \times 0}{10} = \frac{6}{5}$.
- $F$: “quark”representation: $\text{average} = \frac{3 \times 2/3}{3} = \frac{2}{3}$.

So our competition number becomes

$$\frac{C_A}{C_F}|_{B_2} = \frac{6/5}{1/2} = \frac{12}{5} \quad (39)$$

$• A_2 = SU(3)$: For $SU(3)$ the root system is a regular hexagon around zero, and we take the length of the roots as usual to be $\sqrt{2}$. The “smallest” representation $F$ is the quark or we can equally well take the antiquark representation $\bar{3}$. The weight system for say the quark representation forms is a triangle centered around zero and having the side length $\sqrt{2}$ like we took the roots to have. Thereby the distances of the weights from zero become $\sqrt{2}$. So the averages of the squares of the distances from zero becomes:

- Adjoint: $\text{average} = \frac{6 \times 2 + 2 \times 0}{8} = \frac{3}{2}$.
- $F$: “quark”representation: $\text{average} = \frac{3 \times 2/3}{3} = \frac{2}{3}$.

So we obtain for our ratio

$$\frac{C_A}{C_F}|_{A_2} = \frac{3/2}{2/3} = \frac{9}{4} \quad (40)$$

$• F_4$: The root system for the exceptional Lie algebra $F_4$, $\Phi$, is described as contained in $V = \mathbb{R}^4$ and consisting of those vectors $\alpha$ with length 1 or $\sqrt{2}$ for which the coordinates obey that $2\alpha$ having all coordinates integer and that so that for each $2\alpha$ these coordinates are either all even or all odd. There are 48 roots in this system.

These 48 roots are easily seen to fall into one group of 16 of length 1 for which the coordinates are all $\pm 1/2$, one group of 24 of length $\sqrt{2}$ having two coordinates 0 and two $\pm 1$, and 8 roots have just one coordinate equal to $\pm 1$ and the other coordinates being 0.

The average square distance of these roots together with the 4 Cartan group basis vectors with 0 distance so to speak becomes $\frac{16 \times 1 + 3 \times 1 + 24 \times 2 + 8 \times 0}{48 + 4} = \frac{72}{52} = \frac{18}{13}$. 

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We used here the Cartan algebra only, but since these Cartan algebra elements can be transformed around to go into the non-Cartan algebra so at the end the average “charges” must be the same and thus this restriction would not matter.

13 Appendix 2: Calculation of $C_A/C_F$.

In order to calculate the ratios of quadratic Casimirs we shall here rewrite a list of the adjoint representations for the Lie algebras:

- $A_n$: $(1,0,0,...,0,0,1)$  $n(n+2)$ (41)
- $B_n$: $(0,1,0,...,0,0,0)$  $n(2n+1)$ (42)
- $C_n$: $(2,0,0,...,0,0,0)$  $n(2n+1)$ (43)
- $D_n$: $(0,1,0,...,0,0,0)$  $n(2n-1)$ (44)
- $G_2$: $(1,0)$  $14$ (45)
- $F_4$: $(1,0,0,0)$  $52$ (46)
- $E_6$: $(0,0,0,0,0,1)$  $78$ (47)
- $E_7$: $(1,0,0,0,0,0,0)$  $133$ (48)
- $E_8$: $(0,0,0,0,0,0,1,0)$  $248$ (49)

Here the orders of the Dynkin labels correspond to enumerating the being successive in the chain except for the $E$-algebras for which the largest number though is assigned to the node which is both an end node and attached to the node having three neighbours. In the cases of $B_n$ and $C_n$ it is the $n$th node that is respectively the short and the long simple roots. In cases $F_4$ and $G_2$ the short roots are numbered with the largest numbers.

In the same notation we also copy in what we can call Simple irreducible representations of the simple Lie algebras:

For $A_n$ $(1,0,...,0,0)$  $dim = n + 1$ (50)
or $(0,0,...,0,1)$  $dim = \frac{n + 1}{2}$ (51)
For $B_n$ $(1,0,...,0)$  $dim = 2n + 1$ (52)
and $(0,0,...,0,1)$  $2^n$ (53)
For $C_n$ $(1,0,...,0,0)$  $dim = 2n$ (54)
For $D_n$ $(1,0,...,0,0)$  $dim = 2n$ (55)
and $(0,0,...,0,1)$  $dim = 2^{n-1}$ (56)
or $(0,0,...,0,1,0)$  $dim = 2^{n-1}$ (57)
For $G_2$ $(0,1)$  $dim = 7$ (58)
For $F_4$ $(0,0,0,1)$  $dim = 26$ (59)
For $E_6$ $(1,0,0,0,0,0)$  $dim = 27$ (60)
or $(0,0,0,0,1,0)$  $dim = \frac{27}{2}$ (61)
For $E_7$ $(0,0,0,0,0,1,0)$  $dim = 56$ (62)
For $E_8$ $(0,0,0,0,0,1,0)$  $dim = 248$ (63)

In order to use the equation

$$C_R = \frac{\eta}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} (a_i + 2)G_{ij}a_j$$ (64)
for the quadratic Casimir $C_R$ of a representation $R$ being in our cases of interest, we must know the matrix elements of the “Metric tensors for the weight spaces” $G_{ij}$ (or the inverse Cartan matrix) at the relevant places: For $R$ being the adjoint representation $A$ we have for the $A_n$ Lie algebra both $a_1 = 1$ and $a_n = 1$ while the other Dynkin labels $a_i = 0$. For the other algebras than the $A_n$-series, we have only one Dynkin label different from zero, and that is

For Adjoint Representations

\[
B_n : \quad a_2(\text{Adj} B_n) = 1; \quad (65)
\]
\[
C_n : \quad a_1(\text{Adj} C_n) = 2; \quad (66)
\]
\[
D_n : \quad a_2(\text{Adj} D_n) = 1; \quad (67)
\]
\[
G_2 : \quad a_1(\text{Adj} G_2) = 1; \quad (68)
\]
\[
F_4 : \quad a_1(\text{Adj} F_4) = 1; \quad (69)
\]
\[
E_6 : \quad a_6(\text{Adj} E_6) = 1; \quad (70)
\]
\[
E_7 : \quad a_1(\text{Adj} E_7) = 1; \quad (71)
\]
\[
E_8 : \quad a_7(\text{Adj} E_8) = 1. \quad (72)
\]

For the simple representations mentioned in the list above we have correspondingly that the only non-zero Dynkin labels are

For simple representations:

For $A_n$ : \quad $a_1(A_n, n + 1) = 1, \quad (73)$
\quad or $a_n(A_n, n + 1) = 1; \quad (74)$
For $B_n$ : \quad $a_1(B_n, 2n + 1) = 1, \quad (75)$
and for the spinor rep. \quad $a_n(B_n, 2^n) = 1; \quad (76)$
For $C_n$ : \quad $a_1(C_n, 2n) = 1; \quad (77)$
For $D_n$ : \quad $a_1(D_n, 2n) = 1, \quad (78)$
and for spinors \quad $a_n(D_n, 2^{n-1}) = 1, \quad (79)$
\quad or $a_{n-1}(D_n, 2^{n-1},* ) = 1; \quad (80)$
For $G_2$ : \quad $a_2(G_2, 7) = 1; \quad (81)$
For $F_4$ : \quad $a_4(F_4, 26) = 1; \quad (82)$
For $E_6$ : \quad $a_1(E_6, 27) = 1, \quad (83)$
\quad or $a_5(E_6, 27) = 1; \quad (84)$
For $E_7$ : \quad $a_6(E_7, 56) = 1; \quad (85)$
For $E_8$ : \quad $a_7(E_8, 248) = 1 \quad (86)$

So except for the case of $A_n$, in which we need the $G_{1,n} = \frac{1}{n+1}$ and $G_{n,1} = \frac{1}{n+1}$ matrix elements also, we only need the diagonal elements and the sums of the elements in the columns of the metric tensor matrices for the weight or the inverse Cartan matrices. We therefore here present these diagonal series of elements:

Diagonal Elements of Weight Space Metric

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\[ A_n = \frac{1 \times n}{n + 1}, G_{2,2} = \frac{2(n-1)}{n + 1}, \ldots, G_{(n-1), (n-1)} = \frac{(n-1) \times 1}{n + 1}, G_{n,n} = \frac{n \times 1}{n + 1}; \quad (87) \]
\[ B_n = G_{1,1} = 1, G_{2,2} = 2, \ldots, G_{(n-1), (n-1)} = n - 1, G_{n,n} = \frac{n}{4}; \quad (88) \]
\[ C_n = \frac{1}{2}, G_{2,2} = 1, \ldots G_{(n-1), (n-1)} = \frac{n-1}{2}, G_{n,n} = \frac{n}{2}; \quad (89) \]
\[ D_n = G_{1,1} = 1, G_{2,2} = 2, \ldots, G_{n-2,n-2} = n - 2, G_{n-1,n-1} = \frac{n}{4}, G_{n,n} = \frac{n}{4}; \quad (90) \]
\[ G_2 = G_{1,1} = 2, G_{2,2} = \frac{2}{3}; \quad (91) \]
\[ F_4 = G_{1,1} = 2, G_{2,2} = 6, G_{3,3} = 3, G_{4,4} = 1; \quad (92) \]
\[ E_6 = G_{1,1} = \frac{4}{3}, G_{2,2} = \frac{10}{3}, G_{3,3} = 6, G_{4,4} = \frac{10}{3}, G_{5,5} = \frac{4}{3}, G_{6,6} = 2; \quad (93) \]
\[ E_7 = G_{1,1} = 2, G_{2,2} = 6, G_{3,3} = 12, G_{4,4} = \frac{15}{2}, G_{5,5} = 4, G_{6,6} = \frac{3}{2}, G_{7,7} = \frac{7}{2}; \quad (94) \]
\[ E_8 = G_{1,1} = 4, G_{2,2} = 14, G_{3,3} = 30, G_{4,4} = 20, G_{5,5} = 12, G_{6,6} = 6, G_{7,7} = 2, G_{8,8} = 8; (95) \]

In addition we needed the sums over the columns and thus we present these sums e.g.

\[ \text{Sum}(A_n) = \left( \sum_{i=1}^{n} G_{i,1}, \sum_{i=1}^{n} G_{i,2}, \ldots, \sum_{i=1}^{n} G_{i,n} \right) \quad (96) \]

and get the following:

\[ \text{Sums}(A_n) = \left( \frac{n}{2}, n - 1, \frac{3n}{2} - 3, \ldots, n - 1, \frac{n}{2} \right) \quad (97) \]
\[ \text{Sums}(B_n) = \left( 1(n-1/2), 2(n-1), 3(n-3/2), \ldots, \frac{(n+1)(n-1)}{2} = (n-1)(n-(n-1)/2), \frac{n}{4} \right) \quad (98) \]
\[ \text{Sums}(C_n) = \left( \frac{n}{2}, \frac{2n-1}{2}, \frac{3n-3}{2}, \ldots, \frac{(n+2)(n-1)}{4}, \frac{(n+1)n}{4} \right) \quad (99) \]
\[ \text{Sums}(D_n) = \left( n - 1, 2n - 3, 3n - 6, \ldots, \frac{(n-1)n}{2}, \frac{(n-1)n}{2} \right) \quad (100) \]
\[ \text{Sums}(G_2) = \left( 3, \frac{5}{3} \right) \quad (101) \]
\[ \text{Sums}(F_4) = \left( 8, 15, \frac{21}{2}, \frac{11}{2} \right) \quad (102) \]
\[ \text{Sums}(E_6) = \left( 8, 15, 21, 15, 8, 11 \right) \quad (103) \]
\[ \text{Sums}(E_7) = \left( 17, 33, 48, \frac{75}{2}, 26, \frac{27}{2}, \frac{49}{2} \right) \quad (104) \]
\[ \text{Sums}(E_8) = \left( 46, 91, 135, 110, 84, 57, 29, 68 \right) \quad (105) \]

Denoting the \( j \)th element in these Sums by an index \( j \) like e.g. \( \text{Sums}(G_2)_j \), we can then write the expression for the typical cases above of a “simple” representation where the \( a_j \) alone is different from zero:

\[ C_F = \frac{n}{2} (G_{j,j} + 2\text{Sums}_j) \quad (106) \]

We can thus by insertion obtain:

**Simple Quadratic Casimirs**
These quadratic Casimirs can be compared with e.g. the corresponding ones for the adjoint representations, and then the normalization - symbolized by the factor \( \eta \), which we thus avoid having to choose. Thereby we obtain the ratio which were our first proposal in the present article for quantity about which to hold the game. These adjoint representation quadratic Casimirs become:

**Quadratic Casimirs for Adjoint Representations**

\[
C_A(A_n) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(A_n)_1 + 2 \cdot \text{Sums}(A_n)_n + G_{1,1}(A_N) + G_{n,n}(A_n) + G_{n,1}(A_n) + G_{1,n}(A_n)\right)
\]

\[
C_A(B_n) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(B_n)_2 + G_{2,2}\right) = \frac{\eta}{2} \left(2 \cdot (n-1) + 2\right) = \eta \cdot (2n - 1)
\]

\[
C_A(C_n) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(C_n)_1 + 2^2 \cdot G_{1,1}\right) = \frac{\eta}{2} \left(4 \cdot \frac{n}{2} + 2 \cdot \frac{1}{2}\right) = \eta \cdot (n + 1)
\]

\[
C_A(D_n) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(D_n)_2 + G_{2,2}\right) = \frac{\eta}{2} \left(2 \cdot (n-3) + 2\right) = \eta \cdot 2(n - 1)
\]

\[
C_A(G_2) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(G_2)_1 + G_{1,1}\right) = \frac{\eta}{2} \left(2 \cdot 3 + 2\right) = \eta \cdot 4
\]

\[
C_A(F_4) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(F_4)_1 + G_{1,1}\right) = \frac{\eta}{2} \left(2 \cdot 8 + 2\right) = \eta \cdot 9
\]

\[
C_A(E_6) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(E_6)_1 + G_{1,1}(E_6)\right) = \frac{\eta}{2} \left(2 \cdot 11 + 2\right) = \eta \cdot 12
\]

\[
C_A(E_7) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(E_7)_7 + G_{7,7}(E_7)\right) = \frac{\eta}{2} \left(2 \cdot 17 + 2\right) = \eta \cdot 18
\]

\[
C_A(E_8) = \frac{\eta}{2} \left(2 \cdot \text{Sums}(E_8)_7 + G_{7,7}(E_8)\right) = \frac{\eta}{2} \left(2 \cdot 29 + 2\right) = \eta \cdot 30
\]
(These numbers are indeed $\eta * g$, where $g$ is the dual coxeter number, see above) Combining these calculations of quadratic casimirs we then finally obtain by taking the ratios our competition quantity $C_A/C_F$.

**Our Ratio of Adjoint to “Simplest” Quadratic Casimirs $C_A/C_F$**

\[
\frac{C_A}{C_F} \bigg|_{A_n} = \frac{2(n+1)^2}{n(n+2)} = \frac{2(n+1)^2}{(n+1)^2 - 1} = \frac{2}{1 - \frac{1}{(n+1)^2}} \quad (129)
\]

\[
\frac{C_A}{C_F} \bigg|_{B_n} = \frac{2n-1}{n} = 2 - \frac{1}{n} \quad (130)
\]

\[
\frac{C_A}{C_F} \bigg|_{B_n} = \frac{2n-1}{2n^2 + n} = \frac{16n - 8}{n(2n+1)} \quad (131)
\]

\[
\frac{C_A}{C_F} \bigg|_{C_n} = \frac{n+1}{n/2 + 1/4} = \frac{4(n+1)}{2n+1} \quad (132)
\]

\[
\frac{C_A}{C_F} \bigg|_{D_n} = \frac{2(n-1)}{n - 1/2} = \frac{4(n-1)}{2n-1} \quad (133)
\]

\[
\frac{C_A}{C_F} \bigg|_{D_n} = \frac{2n^2 - n}{2n^2 + n} = \frac{16(n-1)}{n(2n-1)} \quad (134)
\]

\[
\frac{C_A}{C_F} \bigg|_{G_2} = \frac{4}{2} = 2 \quad (135)
\]

\[
\frac{C_A}{C_F} \bigg|_{F_4} = \frac{9}{6} = \frac{3}{2} \quad (136)
\]

\[
\frac{C_A}{C_F} \bigg|_{E_6} = \frac{12}{20/3} = \frac{18}{13} \quad (137)
\]

\[
\frac{C_A}{C_F} \bigg|_{E_7} = \frac{18}{57} = \frac{72}{57} = \frac{24}{19} \quad (138)
\]

\[
\frac{C_A}{C_F} \bigg|_{E_8} = \frac{30}{30} = 1 \quad (139)
\]

**14 Appendix 3: Checks and overview of $C_A/C_F$**

It may be comforting that one can put the calculations in section 13, i.e. appendix 2, up to a few cross checkings, such as checking that isomorphic algebras give the same ratio $C_A/C_F$ as of course they shall for a notation independent quantity:

- $A_1 \approx B_1 \approx C_1$

\[
\frac{C_A}{C_F} \bigg|_{A_1} = \frac{8}{3} \quad (140)
\]

\[
\frac{C_A}{C_F} \bigg|_{B_1} = \frac{16 \times 1 - 8}{1 \times (2 \times 1 + 1)} = \frac{8}{3} \quad (141)
\]

\[
\frac{C_A}{C_F} \bigg|_{C_1} = \frac{4(1 + 1)}{2 \times 1 + 1} = \frac{8}{3} \quad (142)
\]

- $A_1 \times A_1 \approx D_2$
the spinor and vector representations, which are both 8-dimensional. Thus we should have

\[ \frac{C_A}{C_F} \bigg|_{A_1} = \frac{2}{1 - \frac{1}{(1+1)^2}} = \frac{8}{3} \]  

\[ \frac{C_A}{C_F} \bigg|_{D_2} = \frac{16(2 - 1)}{2 * (2 * 2 - 1)} = \frac{8}{3} \]  

- \( B_2 = SO(5) \approx C_2 = Sp(4) \)

\[ \frac{C_A}{C_F} \bigg|_{B_2} = \frac{16 * 2 - 8}{2 * (2 * 2 + 1)} = \frac{12}{5} \]  

\[ \frac{C_A}{C_F} \bigg|_{C_2} = \frac{4 * (2 + 1)}{2 * 2 + 1} = \frac{12}{5} \]  

- \( D_3 = SO(6) \approx A_3 = SU(4) \)

\[ \frac{C_A}{C_F} \bigg|_{D_3} = \frac{16(3 - 1)}{3 * (2 * 3 - 1)} = \frac{32}{15} \]  

\[ \frac{C_A}{C_F} \bigg|_{A_3} = \frac{2 * (3 + 1)^2}{(3 + 1)^2 - 1} = \frac{32}{15} \]  

Further we should note that for \( D_4 = SO(8) \) (w.r.t. Lie algebra) there is symmetry between the spinor and vector representations, which are both 8-dimensional. Thus we should have \( \frac{C_A}{C_F} \bigg|_{D_4} = \frac{C_A}{C_F} \bigg|_{D_4} \). Indeed we find

\[ \frac{C_A}{C_F} \bigg|_{D_4} = \frac{16(4 - 1)}{4 * (2 * 4 - 1)} = \frac{12}{7} \]  

\[ \frac{C_A}{C_F} \bigg|_{D_4} = \frac{4(4 - 1)}{2 * 4 - 1} = \frac{12}{7} \]  

We should also expect approximately the same large \( N \) behavior behavior for \( SO(N) \), whether it be for even \( N \) for which we have \( D_{N/2} \), or for odd \( N \) for which we have \( B_{(N-1)/2} \). Let us indeed formally consider these two Lie algebras:

\[ \frac{C_A}{C_F} \bigg|_{D_{N/2}} = \frac{4(n - 1)}{2n - 1} = \frac{4(N/2 - 1)}{2N/2 - 1} = \frac{2N - 4}{N - 1} \]  

\[ \frac{C_A}{C_F} \bigg|_{B_{(N-1)/2}} = \frac{2n - 1}{n} = \frac{(N - 1) - 1}{(N - 1)/2} = \frac{2N - 4}{N - 1} \]  

Remarkable we get even exactly the same formal expressions \( \frac{2N-4}{N-1} \).

Similarly we may compare the spinor representation for \( F \) using ratios \( \frac{C_A}{C_F} \bigg|_{B_{(N-1)/2}} \) and \( D_{N/2} \):

\[ \frac{C_A}{C_F} \bigg|_{B_{(N-1)/2}} = \frac{16n - 8}{n(2n + 1)} = \frac{16 * (N - 1)/2 - 8}{(N - 1)/2 * (2(N - 1)/2 + 1)} = \frac{8N - 16}{(N^2 - N)/2} = \frac{16(N - 2)}{N(N - 1)} \]  

\[ \frac{C_A}{C_F} \bigg|_{D_{N/2}} = \frac{16(n - 1)}{n(2n - 1)} = \frac{16(N/2 - 1)}{N/2 * (N - 1)} = \frac{16(N - 2)}{N(N - 1)} \]  

So in spite of the fact that the dimensionality of the spinor representations is not a smooth function of \( N \) but rather jumps up and down with the even or oddness of \( N \), we got formally the same formula for our ratio for competition becomes the same written as a function of the \( N \) of \( SO(N) \).
14.1 The speculation of the high rank groups almost giving same $C_A/C_F$

We have already seen that for large rank $r$ the infinite series of Lie algebras have our $C_A/C_F$ going to $2$. This is not so surprising from the thinking that as the rank goes up the root systems and the weight system for $F$ (the “smallest” representation) become more and more rich in number of roots and weights as the rank goes up. But then we might consider the root and weight distributions to be more and more statistical understandable. And if so then we might expect that the small details in the Dynkin diagram deviating from just a long chain of single line connected nodes like in the $A_n$’s would have less and less effect and so the approach to a single number common for all Lie algebras.

References

[1] Norma Mankoc et al. see many contributions to the present and previous Bled Proceedings.

[2] H. B. Nielsen and N. Brene, “Spontaneous Emergence Of Gauge Symmetry,” IN *KRAKOW 1987, PROCEEDINGS, SKYRMIIONS AND ANOMALIES*, 493-498 AND COPENHAGEN UNIV. - NBI-HE-87-28 (87,REC.JUN.) 6p H. B. Nielsen and N. Brene, “Skewness Of The Standard Model: Possible Implications,” Physicatia Magazine, The Gardener of Eden, 12 (1990) 157; NBI-HE-89-38; H. B. Nielsen and N. Brene, “What Is Special About The Group Of The Standard Model?,” Phys. Lett. B 223 (1989) 399.

[3] H.B. Nielsen, S.E. Rugh and C. Surlykke, Seeking Inspiration from the Standard Model in Order to Go Beyond It, Proc. of Conference held on Korfu (1992)

[4] Philippe Di Francesco, Pierre Mathieu, David Snchal, Conformal Field Theory, 1997 Springer-Verlag New York, ISBN 0-387-94785-X

[5] RANDOM DYNAMICS: H. B. Nielsen, “Dual Strings.”, “Fundamentals of quark models”, In: Proc. of the Seventeenth Scott. Univ. Summer School in Physics, St. Andrews, august 1976. I.M. Barbour and A.T. Davies(eds.), Univ. of Glasgow , 465-547 (publ. by the Scott.Univ. Summer School in Physics, 1977) (CITATION = NBI-HE-74-15;) H.B. Nielsen, Har vi brug for fundamentale naturlove(in Danish) (meaning:“Do we need laws of Nature?”) Gamma 36 page 3-16, 1978(1. part) and Gamma 37 page 35-46, 1978 (2. part) H.B. Nielsen and C. D. Froggatt, “Statistical Analysis of quark and lepton masses”, Nucl. Phys. B164(1979) 114 - 140.

[6] D. Førster, H.B. Nielsen, and M. Ninomiya, “Dynamical stability of local gauge symmetry. Creation of light from chaos.” Phys. Lett. B94(1980) 135 -140

[7] H.B. Nielsen, Lecture notes in Physics 181, “Gauge Theories of the Eighties” In: Proc. of the Arctic School of Physics 1982, Akaeslompolo, Finland, Aug. 1982. R. Raitio and J. Lindfors(eds.). Springer, Berlin, 1983,p. 288-354. H.B. Nielsen, D.L. Bennett and N. Brene: “The random dynamics project from fundamental to human physics”. In: Recent developments in quantum field theory. J. Ambjoern, B.J. Durhuus and J.L. Petersen(eds.), Elsvier Sci.Publ. B.V., 1985, pp. 263-351
H. B. Nielsen and D. L. Bennett, “The Gauge Glass: A short review”, Elaborated version of talk at the Conf. on Disordered Systems, Copenhagen, September 1984. Nordita preprint 85/23.

see for example some papers in this Bled Proceedings series: H. B. Nielsen and M. Ninomiya, arXiv:1008.0464 [physics.gen-ph].

H. B. Nielsen and M. Ninomiya, Int. J. Mod. Phys. A 24 (2009) 3945 [arXiv:0802.2991 [physics.gen-ph]].

H. B. Nielsen and M. Ninomiya, arXiv:0711.3080 [hep-ph].

H. B. Nielsen and M. Ninomiya, JHEP 0603 (2006) 057 [hep-th/0602020].

D. L. Bennett and H. B. Nielsen, “Predictions for nonAbelian fine structure constants from multicriticality,” Int. J. Mod. Phys. A 9 (1994) 5155 [arXiv:hep-ph/9311321]. (CITATION = IMPAE,A9,5155)

D. L. Bennett, “Multiple point criticality, nonlocality and fine tuning in fundamental physics: Predictions for gauge coupling constants gives alpha**(1) = 136.8 +- 9,” arXiv:hep-ph/9607341 (CITATION = hep-ph/9607341)

D. L. Bennett and H. B. Nielsen, “Gauge couplings calculated from multiple point criticality yield alpha**(1) = 137+-9: At last, the elusive case of U(1),” Int. J. Mod. Phys. A 14 (1999) 3313 [arXiv:hep-ph/9607278]. (CITATION = IMPAE,A14,3313);

D. L. Bennett, C. D. Froggatt and H. B. Nielsen, “Nonlocality as an explanation for fine tuning in nature,” (CITATION = C94-08-30);

D. L. Bennett, C. D. Froggatt and H. B. Nielsen, “Nonlocality as an explanation for fine tuning and field replication in nature,” arXiv:hep-ph/9504294 (CITATION = hep-ph/9504294).

H.B. Nielsen and N. Brene, Gauge Glass, Proc. of the XVIII International Symposium on the Theory of Elementary Particles, Ahrenshoop, 1985 (Institut fur Hochenergiphysik, Akad. der Wissenschaften der DDR, Berlin-Zeuthen, 1985);

H.B. Nielsen and N. Brene, Skewness of the Standard Model: Possible Implications, Physicalia Magazine, The Gardener of Eden, 12 (1990) 157; H.B. Nielsen and N. Brene, Phys. Lett. 223 (1989) 399.

H.B. Nielsen, S.E. Rugh and C. Surylykke, Seeking Inspiration from the Standard Model in Order to Go Beyond It, Proc. of Conference held on Korfu (1992)

D. L. Bennett, “Who is Afraid of the Past” (A resume of discussions with H.B. Nielsen during the summer 1995 on Multiple Point Criticality and the avoidance of Paradoxes in the Presence of Non-Locality in Physical Theories), talk given by D. L. Bennett at the meeting of the Cross-disciplinary Initiative at Niels Bohr Institute on September 8, 1995.
QLRC-95-2. D. L. Bennett, “Multiple point criticality, nonlocality and fine tuning in fundamental physics: Predictions for gauge coupling constants gives alpha**(-1) = 136.8 +/- 9,” arXiv:hep-ph/9607341. (CITATION = hep-ph/9607341) H. B. Nielsen and C. Froggatt, “Influence from the future,” arXiv:hep-ph/9607375. (CITATION = hep-ph/9607375)

[18] A J Macfarlane and Hendryk Pfeiffer, J. Phys. A: Math. Gen. 36 (2003) 2305–200–2232317 PII: S0305-4470(03)56335-1 Representations of the exceptional and other Lie algebras with integral eigenvalues of the Casimir operator

[19] T. van Ritbergen, A. N. Sellekens, J. A. M. Vermaseren, UM-TH-98-01 NIKHEF-98-004 Group theory factors for Feynman ... www.nikhef.nl/ form/maindir/oldversions/.../packages/.../color.ps

[20] See M. Douglas, "The statistics of string / M theory vacua", JHEP 0305, 46 (2003). arXiv:hep-th/0303194 S. Ashok and M. Douglas, "Counting flux vacua", JHEP 0401, 060 (2004). Frederik Denef; Douglas, Michael R. (2006). "Computational complexity of the landscape”. Annals of Physics 322 (5): arXiv:hep-th/0602072. Bibcode 2007AnPhy.322.1096D. doi:10.1016/j.aop.2006.07.013. S. Weinberg, "Anthropic bound on the cosmological constant”, Phys. Rev. Lett. 59, 2607 (1987). S. M. Carroll, "Is our universe natural?”, arXiv:hep-th/0512148 M. Tegmark, A. Aguirre, M. Rees and F. Wilczek, "Dimensionless constants, cosmology and other dark matters”, arXiv:astro-ph/0511774 F. Wilczek, "Enlightenment, knowledge, ignorance, temptation,” arXiv:hep-th/0512187. See also the discussion at [1]. See, e.g. Alexander Vilenkin (2006). "A measure of the multiverse”. Journal of Physics A: Mathematical and Theoretical 40 (25): arXiv:hep-th/0609193 Bibcode 2007JPhA...40.6777V. doi:10.1088/1751-8113/40/25/S22. Abraham Loeb (2006). "An observational test for the anthropic origin of the cosmological constant” (subscription required). JCAP 0605: 009. http://www.arxiv.org/astro-ph/0604242. Jaume Garriga and Alexander Vilenkin (2006). "Anthropic prediction for Lambda and the Q catastrophe” (subscription required). Prog. Theor.Phys. Suppl. 163: 245@57. arXiv:hep-th/0508005 Bibcode 2006PThPS.163.245G. doi:10.1143/PTPS.163.245. http://www.arxiv.org/hep-th/0508005. Delia Schwartz-Perlov and Alexander Vilenkin (2006). "Probabilities in the Bousso-Polchinski multiverse” (subscription required). JCAP 0606: 010. http://www.arxiv.org/hep-th/0601162. L. Smolin, "Did the universe evolve?,” Classical and Quantum Gravity 9, (1992). L. Smolin, The Life of the Cosmos (Oxford, 1997) L. Susskind, "The anthropic landscape of string theory”, arXiv:hep-th/0302219 L. Susskind, The cosmic landscape: string theory and the illusion of intelligent design (Little, Brown, 2005). M. J. Rees, Just six numbers: the deep forces that shape the universe (Basic Books, 2001). R. Bousso and J. Polchinski, "The string theory landscape”, Sci. Am. 291, (2004). Lubos Motl’s blog criticized the anthropic principle and Peter Woit’s blog frequently attacks the anthropic string landscape.