SOME KERNELS ON A RIEMANN SURFACE

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Abstract

We discuss certain kernels on a Riemann surface constructed mainly via Baker-Akhiezer (BA) functions and indicate relations to dispersionless theory.

1 INTRODUCTION

A preliminary sketch of some of this material was given in [13], and in view of recent developments in [14, 15, 21, 50, 51, 52, 55], we have expanded and rewritten that manuscript into [17] plus the present paper. We develop here, on a Riemann surface, a version of a kernel constructed in [14] for dispersionless KP (dKP), which is in fact equivalent to the dispersionless differential Fay identity (cf. also [41, 42, 62] for related material). There is also some discussion of other kernels and relations among them.

2 BACKGROUND

2.1 BA functions

For completeness we recall certain information about Riemann surfaces and BA functions following [13, 17]. The use of BA functions goes back to [7] for example but was developed in modern times by the Russian school (see e.g. [8, 9, 20, 22, 23, 24, 25, 46, 48, 49, 56, 61]). Thus take an arbitrary Riemann surface Σ of genus \( g \), pick a point \( Q \) ∼ \( P \)∞ and a local variable \( 1/k \) near \( Q \) such that \( k(Q) = \infty \), and, for illustration, take \( q(k) = kx + ky + kt \). Let \( D = P_1 + \cdots + P_g \) be a non-special divisor of degree \( g \) and write \( \psi \) for the (unique up to a constant multiplier by virtue of the Riemann-Roch theorem) Baker-Akhiezer (BA) function characterized by the properties (A) \( \psi \) is meromorphic on \( \Sigma \) except for \( Q \) where \( \psi(P)exp(-q(k)) \) is analytic and (*) \( \psi \sim exp(q(k))[1 + \sum_{j=1}^{\infty}(\xi_j/k^j)] \) near \( Q \). (B) On \( \Sigma/Q \), \( \psi \)
is meromorphic with a second order pole at \( \infty \).

A \( x,y,t \) \( Q \) \( D \) with zero divisor \( A \) pole at \( Q \) is the null divisor of a meromorphic differential \( d\psi \) from the uniqueness of BA functions with the same essential singularity and pole divisors. One has (\( \sim \) by nonspecial divisors or equivalently by points in general position on the Jacobian variety \( \xi \) given the \( A \) normalized.

\( \psi(x,y,t) = \exp[\int_{P_0}^P (xd\Omega^1 + yd\Omega^2 + td\Omega^3)] \cdot \frac{\Theta(A(P) + xU + yV + tW + z_0)}{\Theta(A(P) + z_0)} \quad (2.1) \)

where \( d\Omega^1 = dk + \cdots \), \( d\Omega^2 = d(k^2) + \cdots \), \( d\Omega^3 = d(k^3) + \cdots \), \( U_j = \int_{B_j} d\Omega^1 \), \( V_j = \int_{B_j} d\Omega^2 \), \( W_j = \int_{B_j} d\Omega^3 \) (\( j = 1, \cdots, g \)), \( z_0 = -A(D) - K \), and \( \Theta \) is the Riemann theta function. The symbol \( \sim \) will be used generally to mean “corresponds to” or “is associated with”; occasionally it also denotes asymptotic behavior and this should be clear from the context. Here the \( d\Omega_j \) are meromorphic differentials of second kind normalized via \( \int_{A_k} d\Omega_j = 0 \) (\( A_j, B_j \) are canonical homology cycles) and we note that \( xd\Omega^1 + yd\Omega^2 + td\Omega^3 \sim dq(k) \) normalized. \( A \) is the Abel-Jacobi map \( A(P) = (\int_{P_0}^P d\omega_k) \), where the \( d\omega_k \) are normalized holomorphic differentials, \( k = 1, \cdots, g \). \( \int_{A_k} d\omega_k = \delta_{jk} \), and \( K(K_j) \sim \) Riemann constants \( (2K = -A(K_\Sigma) \) where \( K_\Sigma \) is the canonical class of \( \Sigma \sim \) equivalence class of meromorphic differentials). Thus \( \Theta(A(P) + z_0) \) has exactly \( g \) zeros (or vanishes identically. The paths of integration are to be the same in computing \( \int_{P_0}^P d\Omega^i \) or \( A(P) \) and it is shown in [7, 8, 13, 23] that \( \psi \) is well defined (i.e. path independent). Then the \( \xi_j \) in (*) can be computed formally and one determines Lax operators \( L \) and \( A \) such that \( \partial_x \psi = L \psi \) with \( \partial_y \psi = A \psi \). Indeed, given the \( \xi_j \) write \( u = -2\partial_x \xi_1 \) with \( w = 3\xi_1 \partial_x \xi_1 - 3\partial^2_x \xi_1 - 3\partial_x \xi_2 \). Then formally, near \( Q \), one has \((-\partial_y + \partial^2_x + u)\psi = O(1/k)exp(q) \) and \((-\partial_y + \partial^2_x + (3/2)u\partial_x + w)\psi = O(1/k)exp(q) \) (i.e. this choice of \( u \), \( w \) makes the coefficients of \( k^n \exp(q) \) vanish for \( n = 0, 1, 2, 3 \)). Now define \( L = \partial^2_x + u \) and \( A = \partial^2_x + (3/2)u\partial_x + w \) so \( \partial_y \psi = L \psi \) and \( \partial_t \psi = A \psi \). This follows from the uniqueness of BA functions with the same essential singularity and pole divisors (Riemann-Roch). Then we have, via compatibility \( L_t - A_y = [A, L] \), a KP equation \( (3/4)u_{yy} = \partial_x [u_t - (1/4)(6uu_x + u_{xxx})] \) and therefore such KP equations are parametrized by nonspecial divisors or equivalently by points in general position on the Jacobian variety \( J(\Sigma) \). The flow variables \( x, y, t \) are put in by hand in \( A \) via \( q(k) \) and then miraculously reappear in the theta function via \( xU + yV + tW \); thus the Riemann surface itself contributes to establish these as linear flow variables on the Jacobian. The pole positions \( P_i \) do not vary with \( x, y, t \) and \((1) u = 2\partial_x^2 \log \Theta(xU + yV + tW + z_0) + c \) exhibits \( \Theta \) as a tau function.

We recall also that a divisor \( D^* \) of degree \( g \) is dual to \( D \) (relative to \( Q \)) if \( D + D^* \) is the null divisor of a meromorphic differential \( d\hat{\Omega} = dk + (\beta/k^2)dk + \cdots \) with a double pole at \( Q \) (look at \( \zeta = 1/k \) to recognize the double pole). Thus \( D + D^* - 2Q \sim K_\Sigma \) so \( A(D^*) - A(Q) + K = -[A(D) - A(Q) + K] \). One can define then a function \( \psi^*(x, y, t, P) = \exp(-kx - k^2y - k^3t)[1 + \xi^*_1/k + \cdots] \) based on \( D^* \) (dual BA function) and a differential \( d\hat{\Omega} \) with zero divisor \( D + D^* \), such that \( \phi = \psi^*d\hat{\Omega} \) is meromorphic, having for poles only a double pole at \( Q \) (the zeros of \( d\hat{\Omega} \) cancel the poles of \( \psi^* \)). Thus \( \psi^*d\hat{\Omega} \sim \psi^*(1 + (\beta/k^2 + \cdots)dk \) is meromorphic with a second order pole at \( \infty \), and no other poles. For \( L^* = L \) and \( A^* = -A + 2w - (3/2)u_x \) one has then \((\partial_y + L^*)\psi^* = 0 \) and \((\partial_t + A^*)\psi^* = 0 \). Note that
the prescription above seems to specify for \( \psi^s \) \( (\vec{U} = xU + yV + tW, \ z^*_0 = -A(D^*) - K) \)

\[
\psi^s \sim e^{-\int_{\rho_0}^{t}(x d\Omega^1 + y d\Omega^2 + t d\Omega^3)} \cdot \frac{\Theta(A(P) - \vec{U} + z^*_0)}{\Theta(A(P) + z^*_0)} \tag{2.2}
\]

In any event the message here is that for any Riemann surface \( \Sigma \) one can produce a BA function \( \psi \) with assigned flow variables \( x, y, t, \cdots \) and this \( \psi \) gives rise to a (nonlinear) KP equation with solution \( u \) linearized on the Jacobian \( J(\Sigma) \). For averaging with KP (cf. \([13, 17, 29, 46]\)) we can use formulas (cf. (2.1) and (2.2))

\[
\psi = e^{px+Ey+\Omega t} \cdot \phi(Ux + Vy + Wt, P) \tag{2.3}
\]

\[
\psi^* = e^{-px-Ey-\Omega t} \cdot \phi^*(-Ux - Vy - Wt, P) \tag{2.4}
\]

to isolate the quantities of interest in averaging (here \( p = p(P), \ E = E(P), \ \Omega = \Omega(P), \) etc.) We think here of a general Riemann surface \( \Sigma \) with holomorphic differentials \( d\omega_k \) and quasi-momenta and quasi-energies of the form \( dp = d\Omega^1, \ dE = d\Omega^2, \ d\Omega = d\Omega^3, \cdots \) \( (p = \int_{\rho_0}^{P} d\Omega^1 \) etc.) where the \( d\Omega^j = \Omega_j = j(\lambda^j + O(\lambda^{-1})) \) are meromorphic differentials of the second kind. Following \([46]\) one could normalize now via \( \Re \int_{A_j} d\Omega^k = \Re \int_{B_j} d\Omega^k = 0. \) Then write e.g. \( U_k = (1/2\pi i) \int_{A_k} dp \) and \( U_{k+g} = -(1/2\pi i) \int_{B_k} dp \) \( (k = 1, \cdots, g) \) with similar stipulations for \( V_k \sim \oint d\Omega^2, \ W_k \sim \oint d\Omega^3, \) etc. This leads to real \( 2g \) period vectors and evidently one could also normalize via \( \oint_{A_m} d\Omega^k = 0 \) or \( \oint_{B_m} d\Omega^k = 0 \) (further we set \( B_{jk} = \oint_{B_k} d\omega_j \)).

### 2.2 KP and dKP

We follow here \([10, 12]\) (cf. also \([41, 42]\)) and begin with two pseudodifferential operators \( (\partial = \partial/\partial x) \),

\[
L = \partial + \sum_{1}^{\infty} u_{n+1} \partial^{-n} ; \ W = 1 + \sum_{1}^{\infty} w_{n} \partial^{-n} , \tag{2.5}
\]

called the Lax operator and gauge operator respectively, where \( L = W \partial W^{-1} \). The KP hierarchy then is determined by the Lax equations \( (\partial_n = \partial/\partial t_n) \),

\[
\partial_n L = [B_n, L] = B_n L - LB_n , \tag{2.6}
\]

where \( B_n = L^n_+ \) is the differential part of \( L^n = L^n_+ + L^n_- = \sum_{0}^{\infty} \ell^n_i \partial^i + \sum_{-\infty}^{1} \ell^n_{-i} \partial^i \). One can also express this via the Sato equation \( \partial_n W W^{-1} = -L^n_- \) which is particularly well adapted to the dKP theory. Now define the wave function via

\[
\psi = W e^{\xi} = w(t, \lambda)e^{\xi} ; \ \xi = \sum_{1}^{\infty} t_{n} \lambda^{n} ; \ w(t, \lambda) = 1 + \sum_{1}^{\infty} w_{n}(t) \lambda^{-n} , \tag{2.7}
\]

where \( t_1 = x \). There is also an adjoint wave function \( \psi^s = W^{s-1} \exp(-\xi) = w^s(t, \lambda) \exp(-\xi) \), with \( w^s(t, \lambda) = 1 + \sum_{1}^{\infty} w_{n}^s(t) \lambda^{-n} \) and one has equations \( L \psi = \lambda \psi ; \ \partial_n \psi = B_n \psi ; \ L^s \psi^s = \psi^s \).
\[ \lambda \psi^*; \quad \partial_n \psi^* = -B_n^* \psi^*. \] Note that the KP hierarchy (2.4) is then given by the compatibility conditions among these equations, treating \( \lambda \) as a constant. Next one has the fundamental tau function \( \tau(t) \) and vertex operators \( X, X^* \) satisfying

\[
\psi(t, \lambda) = \frac{X(\lambda)\tau(t)}{\tau(t)} = \frac{e^{\xi} G_-(\lambda)\tau(t)}{\tau(t)} = \frac{e^{\xi} \tau(t - [\lambda^{-1}])}{\tau(t)}; \tag{2.8}
\]

\[
\psi^*(t, \lambda) = \frac{X^*(\lambda)\tau(t)}{\tau(t)} = \frac{e^{-\xi} G_+(\lambda)\tau(t)}{\tau(t)} = \frac{e^{-\xi} \tau(t + [\lambda^{-1}])}{\tau(t)}
\]

where \( G_{\pm}(\lambda) = \exp(\pm \xi(\tilde{\partial}, \lambda^{-1})) \) with \( \tilde{\partial} = (\partial_1, (1/2)\partial_2, (1/3)\partial_3, \cdots) \) and \( t \pm [\lambda^{-1}] = (t_1 \pm \lambda^{-1}, t_2 \pm (1/2)\lambda^{-2}, \cdots) \). One writes also

\[
e^{\xi} = \exp\left( \sum_{1}^{\infty} t_n \lambda^n \right) = \sum_{0}^{\infty} \chi_j(t_1, t_2, \cdots, t_j) \lambda^j \tag{2.9}
\]

where the \( \chi_j \) are the elementary Schur polynomials, which arise in many important formulas (cf. below).

We mention now the famous bilinear identity which generates the entire KP hierarchy. This has the form \( \mathbf{H} \) \( \oint_{\infty} \psi(t, \lambda)\psi^*(t', \lambda)d\lambda = 0 \) where \( \oint_{\infty} \cdot d\lambda \) is the residue integral about \( \infty \), which we also denote \( \text{Res}_{\lambda}[\cdot]d\lambda \). Using (2.8) this can also be written in terms of tau functions as

\[
\oint_{\infty} \tau(t - [\lambda^{-1}])\tau(t' + [\lambda^{-1}])e^{\xi(t, \lambda) - \xi(t', \lambda)}d\lambda = 0 \tag{2.10}
\]

This leads to the characterization of the tau function in bilinear form expressed via \( (t \to t - y, \ t' \to t + y) \)

\[
\left( \sum_{0}^{\infty} \chi_n(-2y)\chi_{n+1}(\tilde{\partial})e^{\sum_{1}^{\infty} y_i \partial_i} \right) \tau \cdot \tau = 0 \tag{2.11}
\]

where \( \partial^n a \cdot b = (\partial^n / \partial^m_j)a(t_j + s_j)b(t_j - s_j)\big|_{s=0} \) and \( \tilde{\partial} = (\partial_1, (1/2)\partial_2, (1/3)\partial_3, \cdots) \). In particular, we have from the coefficients of \( y_n \) in (2.11), \( \mathbf{HE} \) \( \partial_1 \partial_n \tau \cdot \tau = 2\chi_{n+1}(\tilde{\partial}) \tau \cdot \tau \), which are called the Hirota bilinear equations. One has also the Fay identity via (cf. \( \mathbf{II} \) - c.p. means cyclic permutations)

\[
\sum_{c.p.} (s_0 - s_1)(s_2 - s_3)\tau(t + [s_0] + [s_1])\tau(t + [s_2] + [s_3]) = 0 \tag{2.12}
\]

which can be derived from the bilinear identity (2.10). Differentiating this in \( s_0 \), then setting \( s_0 = s_3 = 0 \), then dividing by \( s_1s_2 \), and finally shifting \( t \to t - [s_2] \), leads to the differential Fay identity,

\[
\tau(t)\partial \tau(t + [s_1] - [s_2]) - \tau(t + [s_1] - [s_2])\partial \tau(t)
= (s_1^{-1} - s_2^{-1}) \left[ \tau(t + [s_1] - [s_2])\tau(t) - \tau(t + [s_1])\tau(t - [s_2]) \right] \tag{2.13}
\]
The Hirota equations after (2.11) can be also derived from (2.13) by taking the limit \( s_1 \to s_2 \). The identity (2.13) will play an important role later.

Now for the dispersionless theory (dKP) one can think of fast and slow variables, etc., or averaging procedures, but simply one takes \( t_n \to \epsilon t_n = T_n \) \( (t_1 = x \to \epsilon x = X) \) in the KP equation \( u_t = (1/4)u_{xxx} + 3au_x + (3/4)\partial^{-1}uyy \), \( (y = t_2, t = t_3) \), with \( \partial_n \to \epsilon \partial / \partial T_n \) and \( u(t_n) \to U(T_n) \) to obtain \( \partial_T U = 3UU_X + (3/4)\partial^{-1}UY_Y \) when \( \epsilon \to 0 \) \( (\partial = \partial / \partial X \text{ now}) \). Thus the dispersion term \( u_{xxx} \) is removed. In terms of hierarchies we write (L) : \( L_{\epsilon} = \epsilon \partial + \sum_{n=1}^{\infty} u_{n+1}(T/\epsilon)(\epsilon \partial)^{-n} \) and think of \( u_n(T/\epsilon) = U_n(T) + O(\epsilon) \), etc. One takes then a WKB form for the wave function with the action \( S \)

\[
\psi = \exp \left[ \frac{1}{\epsilon} S(T, \lambda) \right]
\]  

Replacing now \( \partial_n \) by \( \epsilon \partial_n \), where \( \partial_n = \partial / \partial T_n \) now, we define \( P = \partial S = S_X \). Then \( \epsilon \partial^i \psi \to \partial^i \psi \) as \( \epsilon \to 0 \) and the equation \( L_{\psi} = \lambda \psi \) becomes \( \lambda \psi \) which is the dKP hierarchy. We also note from \( \partial_n \psi = B_n \psi = \sum b_{nm} (\epsilon \partial)^m \psi \) that one obtains \( \partial_n S = B_n(P) = \lambda^+_n \) where the subscript (+) refers now to powers of \( P \) (note \( \epsilon \partial_n \psi / \psi \to \partial_n S \)). Thus \( B_n = L^+_n \to B_n(P) = \lambda^+_n = \sum b_{nm} P^m \) and the KP hierarchy goes to \( \partial_n S = B_n \Rightarrow \partial_n P = \partial B_n \). The action \( S \) in (2.14) can be computed from (2.8) in the limit \( \epsilon \to 0 \) as

\[
S = \sum_{n=1}^{\infty} T_n \lambda^n - \sum_{m=1}^{\infty} \partial_m F \lambda^{-m}
\]  

(2.15)

where the function \( F = F(T) \) (free energy) is defined by

\[
\tau = \exp \left[ \frac{1}{\epsilon} F(T) \right]
\]  

(2.16)

The formula (2.15) then solves the dKP hierarchy (\( \bullet \diamond \bullet \)), i.e. \( P = B_1 = \partial S \) and

\[
B_n = \partial_n S = \lambda^n - \sum_{m=1}^{\infty} \frac{F_{nm}}{m} \lambda^{-m}
\]  

(2.17)

where \( F_{nm} = \partial_n \partial_m F \) which play an important role in the theory of dKP.

Now following [62] one writes the differential Fay identity (2.13) with \( \epsilon \partial_n \) replacing \( \partial_n \), looks at logarithms, and passes \( \epsilon \to 0 \) (using (2.16)). Then only the second order derivatives survive, and one gets the dispersionless differential Fay identity

\[
\sum_{m,n=1}^{\infty} \mu^{-m} \lambda^{-n} F_{mn} \frac{m}{n} = \log \left( 1 - \sum_{m=1}^{\infty} \frac{\mu^{-m} - \lambda^{-m} F_{1m}}{m - \lambda} \frac{1}{n} \right)
\]  

(2.18)

Although (2.18) only uses a subset of the Plücker relations defining the KP hierarchy it was shown in [62] that this subset is sufficient to determine KP; hence (2.18) characterizes the
function $F$ for dKP. Following [10, 12], we now derive a dispersionless limit of the Hirota bilinear equations (HE), which we call the dispersionless Hirota equations. We first note from (2.15) and (●●●) that

$$F_{1n} = nP_{n+1}$$

so

$$\sum_{i} \lambda^{-i} F_{1n} = \sum_{i} P_{n+1} \lambda^{-n} = \lambda - P(\lambda)$$

(2.19)

Consequently the right side of (2.18) becomes \(\log \frac{P(\mu) - P(\lambda)}{\mu - \lambda}\) and for \(\mu \to \lambda\) with \(\dot{P} = \partial_{\lambda} P\) we have

$$\log \dot{P}(\lambda) = \sum_{m,n=1}^{\infty} \lambda^{-m-n} F_{mn} = \sum_{j=1}^{\infty} \left( \sum_{n+m=j} F_{mn} \right) \lambda^{-j}$$

(2.20)

Then using the elementary Schur polynomial defined in (2.9) and (●●●), we obtain

$$\dot{P}(\lambda) = \sum_{0}^{\infty} \chi_{j}(Z_{2}, \cdots, Z_{j}) \lambda^{-j} = 1 + \sum_{1}^{\infty} F_{1j} \lambda^{-j-1}; \ Z_{i} = \sum_{m+n=i} F_{mn} \ (Z_{1} = 0)$$

(2.21)

Thus we obtain the dispersionless Hirota equations,

$$F_{1j} = \chi_{j+1}(Z_{1} = 0, Z_{2}, \cdots, Z_{j+1})$$

(2.22)

These can be also derived directly from the Hirota equations (HE) with (2.16) in the limit \(\epsilon \to 0\) or by expanding (2.20) in powers of \(\lambda^{-n}\) as in [10, 12]). The equations (2.22) then characterize dKP.

It is also interesting to note that the dispersionless Hirota equations (2.22) can be regarded as algebraic equations for “symbols” \(F_{mn}\), which are defined via (2.17), i.e.

$$B_{n} := \lambda_{+}^{n} = \lambda^{n} - \sum_{i=1}^{\infty} F_{mn} \lambda^{-m}$$

(2.23)

and in fact

$$F_{nm} = F_{mn} = Res_{P}[\lambda^{m} d\lambda_{+}^{n}]$$

(2.24)

Thus for \(\lambda, P\) given algebraically as in (●●●), with no a priori connection to dKP, and for \(B_{n}\) defined as in (2.23) via a formal collection of symbols with two indices \(F_{mn}\), it follows that the dispersionless Hirota equations (2.22) are nothing but polynomial identities among \(F_{mn}\). In particular one has from [10]

- (2.24) with (2.22) completely characterizes and solves the dKP hierarchy.

Now one very natural way of developing dKP begins with (●●●) and (●●●) since eventually the \(P_{j+1}\) can serve as universal coordinates (cf. here [3] for a discussion of this in connection with topological field theory = TFT). This point of view is also natural in terms of developing a Hamilton-Jacobi theory involving ideas from the hodograph – Riemann invariant approach (cf. [11, 30, 41, 43, 44] and in connecting NKdV ideas to TFT,
strings, and quantum gravity. It is natural here to work with $Q_n = (1/n)B_n$ and note that $\partial_n S = B_n$ corresponds to $\partial_n P = \partial B_n = n\partial Q_n$. In this connection one often uses different time variables, say $T'_n = nT_n$, so that $\partial_n P = \partial B_n$ corresponds to $\partial_n P = \partial B_n = n\partial Q_n$ since one will be connecting a number of formulas to standard KP notation. Now given ($\bullet\bigcirc\bullet$) and ($\bullet\Diamond\bullet$) the equation $\partial_n P = n\partial Q_n$ corresponds to Benney’s moment equations and is equivalent to a system of Hamiltonian equations defining the dKP hierarchy (cf. [11, 41]); the Hamilton-Jacobi equations are $\partial_n S = nQ_n$ with Hamiltonians $nQ_n(X, P = \partial S)$. There is now an important formula involving the functions $Q_n$ from [41], namely the generating function of $\partial P Q_n(\lambda)$ is given by

$$K(\mu, \lambda) = \frac{1}{P(\mu) - P(\lambda)} = \sum_{n=1}^{\infty} \partial P Q_n(\lambda) \mu^{-n}$$  \hspace{1cm} (2.25)$$

In particular one notes

$$\int_\infty^\infty \frac{\mu^n}{P(\mu) - P(\lambda)} d\mu = \partial P Q_{n+1}(\lambda) ,$$  \hspace{1cm} (2.26)$$

which gives a key formula in the Hamilton-Jacobi method for the dKP [11]. Also note here that the function $P(\lambda)$ alone provides all the information necessary for the dKP theory. It is proved in [10] that

- The kernel formula (2.25) is equivalent to the dispersionless differential Fay identity (2.18).

The proof uses

$$\partial P Q_n = \chi_{n-1}(Q_1, \cdots, Q_{n-1})$$  \hspace{1cm} (2.27)$$

where $\chi_n(Q_1, \cdots, Q_n)$ can be expressed as a polynomial in $Q_1 = P$ with the coefficients given by polynomials in the $P_{j+1}$. Indeed one shows that $\chi_n = \partial P Q_{n+1}$ via a determinant construction and this leads to the observation that the $F_{mn}$ can be expressed as polynomials in $P_{j+1} = F_{ij}/j$. Thus the dispersionless Hirota equations can be solved totally algebraically via $F_{mn} = \Phi_{mn}(P_2, P_3, \cdots, P_{m+n})$ where $\Phi_{mn}$ is a polynomial in the $P_{j+1}$ so the $F_{1n} = nP_{n+1}$ are generating elements for the $F_{mn}$, and serve as universal coordinates. Indeed formulas such as (2.27) indicate that in fact dKP theory can be characterized using only elementary Schur polynomials since these provide all the information necessary for the kernel (2.25) or equivalently for the dispersionless differential Fay identity. This amounts also to observing that in the passage from KP to dKP only certain Schur polynomials survive the limiting process $\epsilon \to 0$. Such terms involve second derivatives of $F$ and these may be characterized in terms of Young diagrams with only vertical or horizontal boxes.

This is also related to the explicit form of the hodograph transformation where one needs only $\partial P Q_n = \chi_{n-1}(Q_1, \cdots, Q_{n-1})$ and the $P_{j+1}$ in the expansion of $P$ (cf. [10]). Given KP and dKP theory we can now discuss nKdV or dnKdV easily although many special aspects of nKdV for example are not visible in KP. In particular for the $F_{ij}$ one will have
\[ F_{nj} = F_{jn} = 0 \text{ for } \text{dnKdV}. \] We note also (cf. [12]) that from (2.27) one has
\[
\frac{1}{P(\mu) - P(\lambda)} = \sum_{1}^{\infty} \partial P Q_n \mu^{-n} = \sum_{0}^{\infty} \chi_n(Q) \mu^{-n} = \exp\left(\sum_{1}^{\infty} Q_m \mu^{-m}\right) \tag{2.28}
\]

3 CAUCHY TYPE KERNELS

3.1 The background kernel

One aim of this paper is to produce a Riemann surface analogue of the kernel \( K(\mu, \lambda) \) of \((2.24) \sim (2.25)\), namely
\[
K(\mu, \lambda) = \frac{1}{P(\mu) - P(\lambda)} = \sum_{1}^{\infty} \partial P Q_n(\lambda) \mu^{-n} \tag{3.1}
\]
where
\[
\partial P Q_{n+1} = \frac{1}{2\pi i} \oint_{\infty} \frac{\mu^n d\mu}{P(\mu) - P(\lambda)} \tag{3.2}
\]
Here \( nQ_n = B_n = \partial_n S \) so the desired analogy would naturally arise from
\[
\partial_n dS = d\Omega_n \tag{3.3}
\]
(cf. [17] for an extensive discussion). On the other hand \( \partial P Q_n \) suggests using \( p \sim P \) (recall \( P = S_X \) and \( dp = d\Omega_1 \) with \( d\partial S = d\Omega_1 \) where \( \partial \sim \partial_X \) with \( X \sim T_1 \)). Thus, denoting by \( \gamma \in \Sigma_g \) a point on the Riemann surface, one has heuristically for \( \Omega_n(\gamma) \sim \int_{\gamma} d\Omega_n \), the correspondences \( \Omega_n \sim B_n = nQ_n \) and
\[
\frac{\partial Q_n}{\partial \mu} \sim \frac{1}{n} \frac{d\Omega_n}{dp} \sim \frac{1}{n} \frac{\partial \Omega_n/\partial \gamma}{\partial \mu/\partial \gamma} = \frac{1}{n} \frac{d\Omega_n}{dp} = \frac{1}{n} \frac{d\Omega_n}{d\Omega_1} \tag{3.4}
\]
Note that in referring to \( Q_n(\lambda) \) for example one is thinking of \( \lambda \) near \( \infty \) and the expressions (3.1) - (3.2) involve then both \( \lambda \) and \( \mu \) near \( \infty \); we think of \( \lambda \sim k \) near \( \infty \) and \( z = \lambda^{-1} \sim k^{-1} \).

The analogue to (3.1) is now
\[
\mathcal{K}(\mu, \lambda) = \sum_{1}^{\infty} \frac{d\Omega_1}{d\Omega_1}(\lambda) \mu^{-j} \tag{3.5}
\]
It will be seen that (3.5) can be related to a local Cauchy kernel on a Riemann surface expressible in terms of \( d\zeta \log E(\zeta, z) \) where \( E \) is the Fay-Klein prime form, and this will be our first project.
3.2 Theta functions and the prime form

In [13] we organized this material in two forms, one from a physics viewpoint following [3, 4, 5] and another following [27, 28, 31, 32, 33, 38, 39, 53, 54, 57] for information on Riemann surfaces, theta functions, half differentials, etc. Thus begin with \( \Sigma_g \) a compact Riemann surface of genus \( g \) with a canonical homology basis \((A_i, B_i)\) and a complex atlas \((U_\alpha, z_\alpha)\). Divisors have the form \( \sum n_i P_i \) with \( \deg(D) = \sum n_i \) and \( D \sim D' \) if \( D - D' \) is the divisor of a meromorphic function (principal divisor). One wants to consider spinors of the form \( 1 \times 1 \) with other notations).

In \([13]\) we organized this material in two forms, one from a physics viewpoint following \([3, 4, 5]\) and another following \([27, 54]\). In a sense \([27]\) is surely the basic reference here but we will develop this material in a somewhat shortened approach following \([2, 4]\) (the notation of \([2, 4, 8, 9, 13, 23, 24, 27, 28, 31, 32, 33, 38, 39, 53, 54, 57]\) is equivalent and this gives us recourse to a number of formulas immediately, without worrying about factors of \(2\pi i\) etc.). We refer generally to \([2, 4, 8, 9, 13, 23, 24, 27, 28, 31, 32, 33, 38, 39, 53, 54, 57]\) for information on Riemann surfaces, theta functions, half differentials, etc.

Now the equivalence classes of \( 1/2 \) differentials turn out to be characterized by \( D_{\alpha}^{1/2} = \Delta - B\alpha_1 - \alpha_2 \) and to \( D_{\alpha}^{1/2} \) one associates a theta function

\[
\Theta[\alpha](z) = \sum_{n \in \mathbb{Z}^g} \exp[i\pi(n + \alpha_1)B(n + \alpha_1)^T + 2\pi i(z + \alpha_2)(n + \alpha_1)^T]
\]

(3.6)

This is holomorphic on \( C^g \) and quasiperiodic via (sometimes we write \( z \) for \( z^T \))

\[
\Theta[\alpha](z + Bn + m) = \exp \left( -i\pi n Bn^T - 2\pi n(z + \alpha_2) + 2\pi im\alpha_1 \right) \Theta[\alpha](z)
\]

(3.7)

Further (\( \bullet \)) \( \Theta[\alpha](-z) = (-1)^{\alpha_1\alpha_2} \Theta[\alpha](z) \) and \( \alpha \) is called even or odd depending on whether \( \Theta[\alpha](z) \) is even or odd. We recall next by the Riemann vanishing theorem that the zeros of \( \Theta[\alpha](z) \) form the set of \( z \in C^g \) represented via

\[
z = A \left( D_{\alpha}^{1/2} - \sum_{i=1}^{g-1} P_i \right) = \Delta - B\alpha_1 - \alpha_2 - \sum_{i=1}^{g-1} A(P_i)
\]

(3.8)
(\(P_1, \cdots, P_{g-1}\) being arbitrary). For odd \(\alpha\), \(\Theta[\alpha](0) = 0\), so by (18) there exists \(g-1\) points \(P_{\alpha,i} \in \Sigma_g\) such that \(D_\alpha^{1/2} = \sum_{i=1}^{g-1} P_{\alpha,i}\). In fact a 1/2 differential \(h_\alpha\) with divisor \(\sum_{i=1}^{g-1} P_{\alpha,i}\) can be obtained via

\[
h_\alpha(P)^2 = \sum \left( \frac{\partial \Theta[\alpha]}{\partial z_j} \right) (0) d\omega_j(P) \tag{3.9}\]

Then the (Fay-Klein) prime form of \(\Sigma_g\) is defined as the multivalued \(-1/2 \times -1/2\) differential on \(\Sigma_g \times \Sigma_g\) given by

\[
E(P, P') = -E(P', P) = \frac{\Theta[\alpha](A(P - P'))}{h_\alpha(P) h_\alpha(P')} \tag{3.10}
\]

(note \(A(P - P') = A(P) - A(P') = (\int_P^{P'} d\omega_j)\) and \(\alpha \sim\) arbitrary odd characteristic). This has the following properties:

- \(E(P, P')\) has a single zero at \(P = P'\)
- If \(z_1, z_2\) are the coordinates of \(P_1, P_2\) then near \(P_1 = P_2\)

\[
E(P_1, P_2) = \frac{z_1 - z_2}{\sqrt{dz_1} \sqrt{dz_2}} (1 + O(z_1 - z_2)^2))
\]

- As a function of \(P\) the prime form is single valued around the \(A\) cycles but multivalued around the \(B\) cycles; upon going around \(B_j\) one has \(E(P, P') \rightarrow E(P, P') \exp[-i\pi B_{jj} - 2\pi i A_j (P - P')]\)

One can also use \(E\) to construct a differential of third kind with first order poles at \(P_1\) and \(P_2\) obtained via

\[
d\omega(P_1, P_2) = dP \log \left( \frac{E(P_1, P)}{E(P, P_2)} \right) \tag{3.11}\]

Further, given that \(D = P_1 + \cdots + P_n - Q_1 - \cdots - Q_n\) is the divisor of a meromorphic function one can express this function via

\[
f(z) = \prod_{i=1}^n E(z, P_i) \prod_{i=1}^n E(z, Q_i) \tag{3.12}\]

**REMARK 3.1** The notation in [27] is slightly different in that \(\theta(x)\) is defined via (we use small \(\theta\) and small \(b\) for [27])

\[
\theta(z') = \theta[0](z') = \sum e^{\frac{1}{2} m b m^T + z' m^T} : \theta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z') = \sum e^{\frac{1}{2} (m + \alpha b (m + \alpha)^T + (z' + 2\pi i \beta)(m + \alpha)^T} = e^{\frac{1}{2} \alpha \alpha^T + (z' + 2\pi i \beta) \alpha^T} \theta(z' + e')
\]
where \( e' = 2\pi i \beta + ab = (\beta, \alpha)(2\pi i) \) and the period matrix \( b \) is symmetric with \( \Re b < 0 \). In this context

\[
E'(x', y') = \frac{\theta[\delta](y' - x')}{h'_b(x')h'_b(y')}; \quad (h'_b)^2(x') = \sum_1^g \left( \frac{\partial\theta[\delta](0)}{\partial z_i'} \right) d\omega'(x')
\]

Comparing with the theta function of (3.6) etc. we see that \( b \sim 2\pi i B \) and \( z' \sim 2\pi i z \) with \( \alpha \sim \alpha_1 \) and \( \beta \sim \alpha_2 \). We will want to use later a formula from [27] involving \( d_x \log E'(x', y') \) as a local Cauchy kernel and since the formulas connected with this have an invariant meaning on \( \Sigma_g \) we will simply transplant the notation to \( E \) and \( \Theta \) as in (3.6) and (3.10) (note e.g. \( z' = \alpha z = f_z dz = f_x dz z' \)). A tedious calculation could of course be made using the comparisons just indicated but that seems unnecessary.

Thus keeping our standard notation with \( \Im B > 0 \) one notes from [27] that \( Z_p(x) = (d/dx) \log E(x, p) \) has a pole of residue 1 at \( x = p \) and \( [Z_b(x) - Z_a(x)] dx = d\omega_{b-a}(x) = \oint_{(3.18)} d_x d_y \log E(x, y) dy \) (cf. (3.24)). It is shown in [27] that for \( f \) holomorphic in a neighborhood \( U \) of \( p \) one has

\[
f(p) = \frac{1}{2\pi i} \int_{\partial U} f(x) Z_p(x) dx \quad (3.15)
\]

so \( Z_p \) is a local Cauchy kernel. Next recall for \( s \geq 1 \), \( d\Omega_s = (-s z^{-s-1} - \sum_1^\infty q_{ms} z^{m-1}) dz \) with \( q_{ms} = q_{sm} \) (via the Riemann bilinear relations) or alternatively, \( d\Omega_s = (s\lambda^{s-1} + \sum_1^\infty q_{ms} \lambda^{-m-1}) d\lambda \) \((\lambda \sim k)\). Similarly \( d\omega_j = -\sum_1^\infty \sigma_{jm} z^{-m-1} dz = \sum_1^\infty \sigma_{jm} \lambda^{-m-1} d\lambda \). Consider now \( \omega_2(z, \zeta) = \partial_z \partial_{\zeta} \log E(z, \zeta) = \partial_z(E_z/E) = (E_{\zeta}/E) - (E_z E_{\zeta}/E^2) \) and via \( E_z \neq 0, E_{\zeta} \neq 0 \) at \( z = \zeta \) the last term looks like \( 1/(z - \zeta)^2 \) (note also \( \omega_2 d\zeta \) has zero \( A_1 \) periods - cf. [54]). This is a standard way of generating a differential with a second order pole. From this one picks up differentials of the second kind with poles of order \( n + 1 \) at \( z = \zeta \) via \( \omega_{n+1} = \partial_z^{n+1} \omega_2(z, \zeta)/n! \) \((n = 2, \ldots)\) for example and we note that (cf. [3, 6])

\[
\int_{B_1} \omega_{n+1}(z, \zeta) dz = \frac{2\pi i}{n!} D^{n-1}_{\zeta} f_i(\zeta) \quad (3.16)
\]

where \( d\omega_i = f_i d\zeta \). Further (cf. (3.11) with \( z \sim P \))

\[
\int_B d\omega(z, P, \P) = 2\pi i \int_{\P} d\omega \quad (3.17)
\]

where \( B \sim (B_1, \ldots, B_g) \) and \( d\omega \sim (d\omega_1, \ldots, d\omega_g) \) represents the standard holomorphic differentials. Here the relation (3.17) follows from standard bilinear identities as does

\[
\int d\omega_2 dz = 2\pi i \frac{d\omega_k}{d\zeta} \quad (3.18)
\]

and (3.16) follows from (3.18).

Now one can write \((z \sim 1/k)\)

\[
\omega_2(z, \zeta) dz d\zeta = d_x d\zeta \log E(z, \zeta) = -\sum_1^\infty d\Omega_s(z) \zeta^{s-1} d\zeta \quad (3.19)
\]
\[
\begin{align*}
&= - \sum_{1}^{\infty} \left[ -sz^{-s-1} - \sum_{m,s=1}^{\infty} q_{ms}z^{-m-1} \right] \zeta^{s-1} d\zeta = \sum_{1}^{\infty} sz^{-s-1} \zeta^{s-1} dzd\zeta + \\
&+ \sum_{m,s=1}^{\infty} z^{m-1} \zeta^{s-1} d\zeta = \frac{dzd\zeta}{(\zeta - z)^2} + \sum_{m,s=1}^{\infty} q_{ms}z^{m-1} \zeta^{s-1} dzd\zeta
\end{align*}
\]

This leads to

\[
\begin{align*}
\hat{\omega}_{n+1}(z,\zeta) dzd\zeta &= \frac{dzd\zeta}{n!} \partial_{\zeta}^{n-1} \hat{\omega}_{2}(x,\zeta)|_{\zeta=0} dzd\zeta = \frac{(n - 1)!}{n!} d\Omega_{n}(z) + \sum_{n+1}^{\infty} \left( \frac{s - 1}{n} \right) d\Omega_{s}(z) \zeta^{s-n} dzd\zeta
\end{align*}
\]

Now to apply this to the kernel $K$ of (3.5) we write $\zeta \sim (1/\lambda)$ and $z \sim (1/\mu)$ with $\lambda, \mu \to \infty$. One can write $K$ in the form

\[
K(\mu, \lambda) = \sum_{1}^{\infty} \frac{d\Omega_{j}(\lambda)}{d\Omega_{1}(\lambda)} \frac{\mu^{-j}}{j} \equiv \sum_{1}^{\infty} \frac{d\Omega_{j}(\zeta)}{d\Omega_{1}(\zeta)} \frac{z^{j}}{j} = \tilde{K}(z, \zeta)
\]

We can also write from above

\[
Z_{z}(\zeta) = d\zeta \log E(\zeta, z); \quad \left[ Z_{z}(\zeta) - Z_{w}(\zeta) \right] d\zeta =
\]

\[
= d\omega_{z-w} = \int_{w}^{z} \partial_{\zeta} \partial_{y} \log E(\zeta, y) dy
\]

From (3.19) we have

\[
\begin{align*}
\left( \int_{w}^{z} \partial_{\zeta} \partial_{y} \log E(\zeta, y) dy \right) d\zeta &= \left[ \int_{w}^{z} \left( \frac{dy}{(y - \zeta)^2} + \sum_{m,s=1}^{\infty} q_{ms} \zeta^{m-1} y^{s-1} \right) dy \right] d\zeta = \\
&= \left[ \frac{1}{w - \zeta} - \frac{1}{z - \zeta} + \sum_{m,s=1}^{\infty} \frac{q_{ms}}{s} \zeta^{m-1} (z^{s} - w^{s}) \right] d\zeta
\end{align*}
\]

This implies

\[
Z_{z}(\zeta) = \left[ \frac{1}{\zeta - z} + \sum_{m,s=1}^{\infty} \frac{q_{ms}}{s} \zeta^{m-1} z^{s} \right] d\zeta
\]
Now write out $\tilde{K}(z, \zeta)$ as

$$
\tilde{K}(z, \zeta) = \frac{1}{d\Omega_1(\zeta)} \sum_{j=1}^{\infty} \frac{z^j}{j} \left[ -j\zeta^{-j-1} - \sum_{m,j=1}^{\infty} q_{mj} \zeta^{m-1} \right] d\zeta =
$$

$$
= \frac{1}{d\Omega_1(\zeta)} \left[ \frac{1}{\zeta} - \sum_{m,j=1}^{\infty} \frac{q_{mj}}{j} \zeta^{m-1} z^j \right] d\zeta = \frac{1}{d\Omega_1(\zeta)} \left[ \frac{d\zeta}{\zeta} - Z_z(\zeta) \right]
$$

(3.27)

(note $(1/\zeta) - (1/(\zeta - z)) = - \sum_1^\infty (z^j/\zeta^{j+1})$). This leads to

**THEOREM 3.2.** The kernel $K$ of (3.3) or equivalently $\tilde{K}$ of (3.23) can be written in the form (3.27) in terms of the local Cauchy kernel $Z_z(\zeta)$.

**REMARK 3.3.** From (3.1) we have $K(\mu, \lambda) = 1/(P(\mu) - P(\lambda))$ where $P \sim d\Omega_1$ on the Riemann surface. Then formally and heuristically on the Riemann surface we are looking at

$$
\sum d\Omega_j(\lambda) \frac{\mu^{j-\frac{1}{2}}}{j} \sim \frac{d\Omega_1(\lambda)}{\Omega_1(\mu) - \Omega_1(\lambda)} = \frac{dp(\lambda)}{p(z) - p(\zeta)}
$$

(3.28)

In terms of $z$, $\zeta$ this leads to

$$
\sum \left( \frac{d\zeta}{\zeta} \right) \frac{z^j}{j} = \frac{dp(\zeta)}{p(z) - p(\zeta)} = \frac{d\zeta}{\zeta} - Z_z(\zeta)
$$

(3.29)

Note here for consistency $\Upsilon = (d\zeta/\zeta) - (d\zeta/(\zeta - z)) = - (zd\zeta/\zeta(\zeta - z))$ and for $\zeta = 1/\lambda$, $z = 1/\mu$, and $d\zeta = - (1/\lambda^2) d\lambda$ we obtain $\Upsilon = d\lambda/(\mu - \lambda)$ as expected from (3.1).

**4 KERNELS BASED ON $\psi\psi^*$**

We go back to the BA function for KP as in (2.1) so that $\psi(\tilde{t}, k)$ is meromorphic in $\Sigma_g/\infty$ with simple poles at $D \sim P_1, \ldots, P_g$ and no other singularities in $\Sigma_g/\infty$ (here $\tilde{t} = (t_i)$ with $t_1 = x$, $t_2 = y$, $t_3 = t$, etc.). We use $k^{-1}$ as the local coordinate at $\infty$ so $\psi \sim \exp[\xi(\tilde{t}, k)] \cdot (1 + \sum_{k=1}^{\infty} \chi_i k^{-1})$ for $|k|$ large where $\xi(\tilde{t}, k) = \sum_{k=1}^{\infty} t_{ki} k^i \sim q(k)$. For $t_{2i} = 0$ this is a KdV situation. The BA conjugate differential $\psi^*(\tilde{t}, \mu)$ can be defined via $\psi^* = \psi^* d\hat{\Omega} \sim \exp(-\xi(\tilde{t}, k)) (1 + \sum_{k=1}^{\infty} \chi_i k^{-1})(1 + (\beta/k^2) + \cdots) dk$ where $d\hat{\Omega}$ is the meromorphic differential indicated in Section 2 with zeros at $D + D^*$ and a double pole at $\infty$. Then one has

$$
\oint_C \psi(k, \tilde{t}) \psi^*(k, \tilde{t}) dk = \oint_C \psi(k, \tilde{t}) \psi^*(k, \tilde{t}) d\hat{\Omega} = 0
$$

(4.1)

for $C$ a small contour around $\infty$ (Hirota bilinear identity) and

$$
\int_{-\infty}^{\infty} \psi(k, \tilde{t}) \psi^*(k', \tilde{t}) dx = 2\pi i \delta(k - k')
$$

(4.2)

for $\Re p(k) = \Re p(k')$. In (4.2) the contour could apparently be a closed curve through $\infty$ for example (which would become a straight line in the - degenerate - scattering situation).
This point is however consistently overlooked and should be further examined (cf. also [18, 19]). Next the Cauchy-Baker-Akhiezer (CBA) kernel \( \omega(k, k', \tilde{t}) \) is defined via (we give an expanded form later and cf. also [38, 64] for kernels)

\[
d\omega(k, k', x, y, t, \cdots) = \frac{1}{2\pi i} \int_{\pm \infty}^x \psi(k, x', y, t, \cdots)\psi^\dagger(k', x', y, t, \cdots)dx'
\]  

(4.3)

According to [22, 33, 34] this kernel is to have the following properties: (A) \( d\omega(k, k', \tilde{t}) \) is a function in \( k \) and a one form in \( k' \) (B) \( d\omega(k, k', \tilde{t}) \) is meromorphic in \( k \) in \( \Sigma_{g/\infty} \) with simple poles at \( P_1, \cdots, P_g, k' \) (C) As a function of \( k' \), \( d\omega \) is meromorphic in \( \Sigma_{g/\infty} \) with one pole \( k \) and zeros at \( P_1, \cdots, P_g \) (D) \( d\omega(k, k', \tilde{t}) = O(exp[\xi(k, \tilde{t})]) \) as \( k \to \infty \) (E) \( \omega(k, k', \tilde{t}) = O(exp[-\xi(k, \tilde{t})]) \) as \( k' \to \infty \) (F) \( \omega(k, k', \tilde{t}) \sim (dk' / 2\pi i(k' - k)) \) as \( k \to k' \).

The material on the prime form is well discussed in [27, 54] for example (cf. also [39]). Formula (4.1) with \( \psi^* \) in place of \( \psi^\dagger \) (Hirota bilinear identity) can be proved in various traditional manners (cf. [1, 18]) so we omit comment. Now look at the integrands in (4.2) and (4.3) for large \( k \), \( k' \) and \( \Im(k) = \Re(k') \), namely, \( \psi(k, \tilde{t})\psi^\dagger(k', \tilde{t}) \sim exp[\sum t_n(k^n - k'^n)](1 + \sum \xi_k k^{-1})(1 + \sum \xi_{k'} k'^{-1})(1 + (\beta/k) + \cdots)dk'. \) As \( x = t_1 \) varies one has a multiplier \( exp[x(k - k')] \) with \( |exp[x(k - k')]| = exp[x\Re(k - k')] \) (note \( dp \sim -idk \) so for large \( k \), \( k' \), \( \Im(k') = \Re(k') \)). Thus \( exp[x(k - k')] \sim exp[x(\Re(k - k'))] \) and (4.2) has a Fourier integral flavor. On the other hand from \( \partial_n \psi = B_n \psi \) and \( \partial_n \psi^\dagger = -B_n^* \psi^\dagger \) one obtains \( \Im(p(k') = \Re(p(k)) \)

\[
\partial_n \int_{-\infty}^\infty \psi\psi^\dagger dx = \int_{-\infty}^\infty [(B_n\psi)\psi^\dagger - \psi(B_n^*\psi^\dagger)]dx = 0
\]  

(4.4)

provided integration by parts is permitted (not perhaps obviously valid but one can assume it under reasonable circumstances). Note that no \( x, y, t \) dependence is introduced in passing from \( \psi^* \) to \( \psi^\dagger \) so \( \partial_n \psi^\dagger = -B_n^* \psi^\dagger \) follows from \( \partial_n \psi^* = -B_n^* \psi^* \). Further for large \( t_n \) and \( k' \neq k \), \( |\psi^*| \sim O(|exp[t_n(k^n - k'^n)]|) = exp[t_n\Re(k^n - k'^n)] \to 0 \) for some \( n \) when \( t_n \to \infty \) or \( -\infty \) (with \( \Im(p(k') = \Re(p(k)) \) or not). This (with (4.4)) implies that the integral in (4.2) is 0 for \( k' \neq k \) and \( \Im(p(k') = \Re(p(k)) \). To show that (4.4) actually gives a delta function for \( \Im(p(k') = \Re(p(k)) \) one is referred in [32] to [47] for a discrete version whose proof is said to be extendable (the result is of course natural, up to normalization). Finally it is clear that the \( \pm \infty \) limit of the integral in (4.3) may change when \( k \) crosses the path \( \Im(p(k) = \Re(p(k)) \). In this regard recall \( |exp[x(k' - k)]| = exp[x(\Re(k') - \Re(k)] \) and \( \Re(p(k) = -\Re(k) \) for large \( k \). Hence \( \omega \) in (4.3) can have a jump discontinuity across the curve \( C : \Im(p(k) = \Re(p(k') = \Re(p(k)) \). To see this and to show that \( \omega(k, k') \) is nevertheless continuous for \( k \neq k' \) let \( I \subset C \) be a small arc segment and \( D \supset I \) a small open set with boundary \( \partial D \). Then \( \int_D \partial \omega dk = \int_{\partial D} \omega dk \) by Stokes theorem. Think of \( I \) as a small straight line segment with \( D \) shrunk down around \( I \) to be lines above and below with little end curves. One obtains in an obvious notation

\[
\int_D \partial \omega dk = \int_I (\partial \omega_+ - \partial \omega_-) dk = \int_I d\omega_+ - \int_I d\omega_-
\]  

(4.5)
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi(k) \psi^*(k') dk = \begin{cases} 
0 & \text{for } k' \notin I \\
1 & \text{for } k' \in I 
\end{cases}
\]

via (4.2) and (4.3) (using the change in integration limit in (4.3)). The other properties in (A) - (E) are more or less natural. Property (F) is suggested by e.g. (1/2\pi i) \int_{-\infty}^{\infty} \psi \psi^* dx'dk' \sim [1/2\pi i(k - k')] \cdot \int_{-\infty}^{\infty} \exp[x'(k - k')] \cdot O(1) dx'dk' when say \Re k > \Re k', k' \to k and x > 0 with the expectation here that 1/(k - k') will emerge from the integration (cf. Theorem 4.3). We will indicate other formulas for \(d\omega\) and \(\psi^*\) below which accord with (4.1), (4.2), and (A) - (F).

Now consider from (4.3) \(\partial_x d\omega(k, k', t) = (1/2\pi i) \psi(k, t) \psi^* (k, t)\) where \(\psi^*\) is given via \(\psi^* = \psi^* d\hat{\Omega}\) as indicated above. Let us rephrase the construction of BA functions etc. now in the notation of [2] (cf. also [13, 40]). In (2.1) for example we pick \(d\Omega^1 \sim d\Omega_1 = dk + \cdots, d\Omega^j \sim d\Omega_j = dk^j + \cdots\) with \(\oint A_i d\Omega_j = 0\) (or sometimes \(\Re \oint A_i d\Omega_j = 0\) = \(\Re \oint_{B_k} d\Omega_j\), and recall \(dp \sim d\Omega_1\). The flow variables arise via \(q(k)\) in Section 2, but the Riemann surface contributes via the argument \(xU + yV + yW + \cdots \sim \sum t_j(\Omega_{jk})\), where \(\Omega_{jk} = \oint_{B_k} d\Omega_j\), in the theta function, to establish linear flows on the Jacobian \(J(\Sigma_g)\) (note from (3.2)) that our \(d\Omega_n \sim \omega_n\) in [2]. For background here we note also that the zeros of \(\Theta[0](z)\) are the points \(z = \Delta - A(\sum_{g-1} P_i)\) with arbitrary \(P_1, \cdots, P_{g-1}\). Then given a positive divisor \(D = P_1 + \cdots + P_g\) the multivalued function of \(P \in \Sigma_g\) defined by \(f(P) = \Theta(A(P)) + \Delta - A(D)\) vanishes at \(P = P_1, \cdots, P_g\). Note that for divisors satisfying a relation \((\bullet \bullet) A(D) = 2\Delta - A(Q_1 + \cdots + Q_{g-2})\) with arbitrary \(Q_i\), we have \(f(P) = \Theta(\Delta - A(Q_1 + \cdots + Q_{g-2} + P) \equiv 0\). Such divisors satisfying \((\bullet \bullet)\) are called special divisors.

We can write the BA function now in the form

\[
\psi = \exp(\int_{P_0}^{P} \sum t_n d\Omega_n) \cdot \frac{\Theta(A(P)) + \sum (t_j/2\pi i)(\Omega_{jk}) + z_0}{\theta(A(P) + z_0)}
\]

where \(z_0 = -K - A(D) = \Delta - A(D) \sim e(D)\) in [3] and \(\Theta(z) \sim \Theta[0](z)\); the only change from (2.1) is a factor of 1/2\pi i in front of the \((\Omega_{jk})\) (cf. [17] where this is also done in adjusting an argument of \([3]\)). We recall as before in Section 2.1 that the path of integration \(\int_{P_0}^{P}\) is to be the same for all factors in (4.7) and then \(\psi\) is a single valued function of \(P \in \Sigma_g\). For the dual BA function one uses a dual divisor \(D^*\) as before in Section 2.1 with \(D + D^* - 2Q \sim K\Sigma\) where \(Q \sim P_\infty\). Then one can write \(\psi^*\) as in (2.2) with \(\hat{U}\) replaced by \(\sum (t_j/2\pi i)(\Omega_{jk})\), i.e.

\[
\psi^* \sim e^{-\int_{P_0}^{P} \sum t_n d\Omega_n} \cdot \frac{\Theta(A(P) - \sum (t_j/2\pi i)(\Omega_{jk}) + z_0^*)}{\Theta(A(P) + z_0^*)}
\]

The differential \(d\hat{\Omega}\) can be written as in [2], namely

\[
d\hat{\Omega}(P') = \frac{\Theta(A(P') + z_0)\Theta(A(P') + z_0^*)}{E(P, P_\infty)^2}
\]
This is shown to be single valued in \([2]\) and replaces here an earlier (multivalued) version of \([13]\). Note also that the \(1/E^2\) term can be computed as in \([2]\) and gives rise to a \(dk'\) in the numerator as needed (see Remark 4.2). Then (4.3) can be written as

\[
d\omega(P, P', x, y, t, \cdots) = \frac{1}{2\pi i} \int_{\pm\infty}^{x} \psi(k, x', y, t, \cdots)\psi^*(k', x', y, t, \cdots)d\Omega'(k')dx' \tag{4.9}
\]

and simplified as

\[
d\omega_x(P, P', \vec{t}) = \frac{e^{(\int_{P}^{P'} - \int_{P}^{P'}^d)\Omega}}{2\pi i} \cdot \left[\Theta(A(P) + \sum(t_j/2\pi i)(\Omega_jk) + z_0)\Theta(A(P') - \sum(t_j/2\pi i)(\Omega_jk) + z_0^*)\Theta(A(P') + z_0)\right] \quad \frac{\theta(A(P) + z_0^*)E(P', P_{\infty})^2}{\theta(A(P) + z_0)E(P', P_{\infty})^2} \tag{4.10}
\]

Hence we have

**THEOREM 4.1.** A path independent expression for \(d\omega_x\) having the essential poles and singularities stipulated in (A) - (F) can be written as in (4.10).

**REMARK 4.2.** A definition of \(d\omega\) via \(\psi\psi^*\) (instead of \(\psi\psi^\dagger\)) is appropriate in dealing with dispersionless limits but in order to have a Cauchy kernel the poles from \(\psi\psi^*\) should be eliminated (hence \(\psi^\dagger\)). This leads to \(\psi^\dagger\) as in (♣) and \(d\omega_x\) as in (4.10). The pole in \(d\omega\) at \(k = k'\) will emerge from the integration as indicated before (cf. also Theorem 4.3) and we will have a discontinuous Cauchy kernel analogue in the spirit of \([64]\) (cf. also \([13]\)). The expression for \(1/E^2\) in \([2]\) is given as

\[
\frac{1}{E(k^{-1}, P_{\infty})^2} = -exp \left( \sum_{m,n=1}^{\infty} C_{mn} \frac{k^{-m-n}}{mn} \right) dk \tag{4.11}
\]

where

\[
C_{mn} = -\frac{1}{(n-1)!(m-1)!} \partial_y^m \partial_{\zeta}^n \log \left( \frac{E(z, z')}{z - z'} \right) \bigg|_{z = z'} \tag{4.12}
\]

Let us also note here, using (3.25) that

\[
C_{mn} = \frac{1}{(m-1)!(n-1)!} \partial_y^{m-1} \partial_{\zeta}^{n-1} \left[ \frac{1}{(\zeta - y)^2} - \partial_y \partial_{\zeta} \log E(\zeta, y) \bigg|_{y = \zeta = 0} \right] = \frac{-1}{(m-1)!(n-1)!} \partial_y^{m-1} \partial_{\zeta}^{n-1} \left[ \sum_{p,s=1}^{\infty} q_{ps} \zeta^{p-1} y^{s-1} \right] \bigg|_{y = \zeta = 0} \tag{4.13}
\]

leading to \(C_{mn} = -q_{mn}\) for \(m, n \geq 1\). We recall now from \([1]\) that the differential Fay identity leads to

\[
\psi^*(\vec{r}, \lambda)\psi(\vec{r}, \mu) = \frac{1}{\mu - \lambda} \partial^{\{X(\vec{r}, \lambda, \mu)\tau(\vec{r})\}} = \frac{1}{\mu - \lambda} \partial^{\{X(\vec{r}, \lambda, \mu)\tau(\vec{r})\}} \tag{4.14}
\]
Given the equivalence of the dispersionless differential Fay identity with the kernel expansion (2.24) or (3.1) one expects (4.14) to be at least formally useful in studying \( d\omega \) in (4.3) since

\[
(4.14) \implies d\omega \sim \frac{1}{2\pi i} \frac{1}{\mu - \lambda} \partial \{ e^{\sum t_j (\mu_j - \lambda_j)} \frac{\tau (\vec{t} + \lfloor \lambda^{-1} \rfloor - \lfloor \mu^{-1} \rfloor)}{\tau (\vec{t})} \}
\]

modulo integration or normalization factors (note (4.15) implies \( d\omega \sim d\lambda / 2\pi i (\mu - \lambda) \) as \( \mu \to \lambda \)). Hence one should be able to determine easily a dispersionless limit for \( d\omega \) (thinking of asymptotic expansions around \( \infty \)). Thus as in [10, 63] we express \( \tau \) via

\[
\tau (\vec{t}) \sim \exp \left( \frac{F (\vec{T})}{\epsilon^2} \right)
\]

and write

\[
d\omega_\epsilon \sim \frac{1}{2\pi i} \frac{1}{\mu - \lambda} \frac{d\bar{\Omega} (\lambda)}{\epsilon^2 \sum T_i (\mu_i - \lambda_i) \cdot \frac{\tau (\vec{T} + \epsilon [\lambda^{-1} - \epsilon [\mu^{-1}])}{\tau (\vec{T})}}
\]

Take logarithms to obtain then

\[
\log d\omega_\epsilon \sim \log \frac{1}{2\pi i} - \log (\mu - \lambda) + \frac{1}{\epsilon} \sum T_i (\mu_i - \lambda_i) + \frac{1}{\epsilon^2} \left\{ F (\vec{T} + \epsilon [\lambda^{-1} - \epsilon [\mu^{-1}]) - F (\vec{T}) \right\} + \log d\bar{\Omega}
\]

The next to last term can be written as \( (x_n \sim \text{elementary Schur functions}) \)

\[
\frac{1}{\epsilon^2} \left\{ e^{\sum \lambda^{-1} \vec{\partial}} \cdot e^{-\sum \mu^{-1} \vec{\partial}} F - F \right\} =
\]

\[
= \frac{1}{\epsilon^2} \left( \sum_0^\infty x_n (\epsilon \vec{\partial}) \lambda^{-n} \cdot \sum_0^\infty \lambda_m (\epsilon \vec{\partial}) \mu^{-m} F - F \right) =
\]

\[
= \frac{1}{\epsilon^2} \left( \sum_1^\infty ( \ )_n + \sum_1^\infty ( \ )_m + \sum_1^\infty \sum_1^\infty ( \ )_n ( \ )_m \right)
\]

Now we have (cf. [10, 63])

\[
\frac{1}{\epsilon^2} \sum_1^\infty \sum_1^\infty ( \ )_n ( \ )_m \rightarrow - \sum_1^\infty \sum_1^\infty \frac{F_{m n}}{n m} \lambda^{-n} \mu^{-m}
\]

and we recall here from [10]

\[
\sum_1^\infty \sum_1^\infty \frac{F_{m n}}{n m} \lambda^{-n} \mu^{-m} = - \log (1 - \frac{\mu}{\lambda}) - \sum_1^\infty \frac{Q_n (\mu)}{\lambda^n} =
\]
\[= -\log(\lambda - \mu) + \log \lambda - \sum_{1}^{\infty} Q_n(\mu)\lambda^{-n} = \log\left[\frac{P(\lambda) - P(\mu)}{\lambda - \mu}\right]\]  

(c.f. also Section 2). Now try an expression \(d\omega_\epsilon = f(\lambda, \mu)\exp(R/\epsilon)d\hat{\Omega}\) which will entrain

\[
\frac{R}{\epsilon} + \log f \sim \log(\frac{1}{2\pi i}) - \log(\mu - \lambda) + \frac{1}{\epsilon} \sum T_1(\mu^i - \lambda^i) + \frac{1}{\epsilon^2} (\sum \lambda_n + \sum \mu_n - \sum \sum \lambda_m \mu_m) 
\]

Multiply by \(\epsilon\) and let \(\epsilon \to 0\) to obtain

\[
R \sim \sum T_1(\mu^i - \lambda^i) + \sum \frac{F_n}{n} \lambda^{-n} - \sum \frac{F_m}{m} \mu^{-m} 
\]

which leads to

\[
\log f \sim \log(\frac{1}{2\pi i}) - \log(\mu - \lambda) - \log(\frac{P(\mu) - P(\lambda)}{\mu - \lambda}) 
\]

This implies via [10] and Section 2

\[
\partial_X R = (\mu - \lambda) + \sum \frac{F_n}{n} (\lambda^{-n} - \mu^{-n}) = P(\mu) - P(\lambda) 
\]

and

\[
\log f = -\log[P(\mu) - P(\lambda)] + \log(\frac{1}{2\pi i}) 
\]

Thus \(d\omega_\epsilon \sim (1/2\pi i)[1/(P(\mu) - P(\lambda))]\exp(1/\epsilon)[S(\mu) - S(\lambda)]d\hat{\Omega}\) and \(\partial_x \log d\omega_\epsilon \sim P(\mu) - P(\lambda) (\partial_x = \epsilon \partial_X)\). Note \(\partial_X S = P\) in the notation of [10] implies

\[
\partial_X R = \partial_X [S(\mu) - S(\lambda)] \Rightarrow R \sim S(\mu) - S(\lambda) 
\]

One can therefore state

**THEOREM 4.3.** A dispersionless kernel analogous to \(\omega\) can be modeled on (4.13), to be extracted from

\[
2\pi i d\omega_\epsilon \sim \frac{d\hat{\Omega}(\lambda)}{P(\mu) - P(\lambda)} \exp(\frac{1}{\epsilon}[S(\mu) - S(\lambda)]) 
\]

(recall also \(\psi \sim \exp(S/\epsilon)\) and \(\psi^* \sim \exp(-S/\epsilon)\) in the dispersionless theory). Thus \(2\pi i d\omega_\epsilon \sim \psi \psi^* d\Omega\) in (1.27) gives (♣♣) \(\partial_x (\log d\omega_\epsilon) \sim \partial_x \frac{1}{\epsilon}[S(\mu) - S(\lambda)] \sim P(\mu) - P(\lambda)\). On the other hand (following [17]), given \(p(\mu) = < \log \psi(\hat{t}, \mu) >\) (ergodic averaging), we have from (1.13) (♠♠) \(< \partial_x \log d\omega \geq p(\mu) - p(\lambda)\). This indicates an interesting relation between \(K(\mu, \lambda) \sim K(\mu, \lambda)\) and \(d\omega\) (the latter being based on \(\psi \psi^*\)), as well as a connection between e.g. \(p(\lambda)\) and \(P(\lambda)\).

**REMARK 4.4.** Referring now to [10] and Section 2, we recall that (●♥●) and \(\partial_n P = \)
\( \partial \mathcal{B}_n \) represent the dKP hierarchy, which is characterized by the dispersionless differential Fay identity, or equivalently by the kernel formula (2.25). Thus the slow variables are inserted by hand with \( \tau \) subsequently represented via (2.16), and (2.25) arises to characterize dKP (along with \( \partial_m \partial_n F = F_{mn} \) corresponding to \( \partial_n P = \partial \mathcal{B}_n \)). The connection to Riemann surfaces involves then \( \partial_n dS = d\Omega_n \) as in (3.3) with Whitham equations based on \( \partial_n dp = d\Omega_n \) or \( \partial_n p = \partial \Omega_n \), leading formally to \( \partial^{-1} \partial_{mn} p = \partial_m \Omega_n = \partial_n \Omega_m \) by compatibility (cf. [17]). One can also argue via zero curvature equations as in [17, 49, 52]. Now in [2] (cf. also [17, 55]) it is shown that the Hirota bilinear identity for \( \psi \) of the form (4.6) leads automatically to \( \tau \) of the form

\[
\tau = \exp \left( - \frac{1}{2} \sum C_{mn} t_n t_m \right) \times \Theta \left( A(P_{\infty}) + z_0 + \sum \frac{t_j}{2\pi i} (\Omega_{jk}) \right)
\]

(4.28)

\((C_{mn} = -q_{mn})\) satisfying the corresponding Hirota bilinear identity. Then as in [17, 55] one arrives at an asymptotic form \( \tau = \exp[(F/\varepsilon^2) + O(1/\varepsilon)] \) with the quadratic part of \( F \) equal to \((1/2) \sum q_{mn} T_n T_m \). Hence one obtains \( F_{mn} \sim q_{mn} \) and a Riemann surface version of dKP corresponds to the Whitham equations. Indeed we note that from \( d\Omega_n \sim z^{-n} - \sum (q_{mn}/m) z^m \sim z^{-n} - \sum (F_{mn}/m) z^m \) one has

\[
\partial_k \Omega_n = - \sum_{m,n=1}^{\infty} \frac{F_{mkn}}{m} z^m = \partial_n \Omega_k = - \sum_{m,n=1}^{\infty} \frac{F_{mkn}}{m} z^m
\]

(4.29)
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