On the relation between two quantum group invariants of 3-cobordisms

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Abstract: We prove in the context of quantum groups at even roots of unity that a Turaev-Viro type invariant of a 3-dimensional cobordism \( M \) equals the tensor product of the Reshetikhin-Turaev invariants of \( M \) and \( M^* \), where the latter denotes \( M \) with orientation reversed.

1 Introduction

According to [At] a 3-dimensional topological quantum field theory (TQFT) associates a finite dimensional vector space \( V_\Sigma \) to each compact closed oriented 2-dimensional surface \( \Sigma \) and a vector (partition function) \( Z(M) \in V_\Sigma \) to each compact oriented 3-dimensional manifold \( M \) with boundary \( \Sigma \), satisfying a certain set of axioms. Of particular relevance for the following discussion are the following: 1) \( V_{\Sigma^*} \) is the dual space of \( V_\Sigma \) for each surface \( \Sigma \), where \( \Sigma^* \) denotes \( \Sigma \) with orientation reversed, 2) given an orientation preserving diffeomorphism \( f : \Sigma \to \Sigma' \) between oriented surfaces, there exists an isomorphism \( U(f) : V_\Sigma \to V_{\Sigma'} \) fulfilling \( U(f_1f_2) = U(f_1)U(f_2) \) for any pair of diffeomorphisms that can be composed, and 3) if \( M \) is obtained by gluing two 3-manifolds \( M_1 \) and \( M_2 \) along \( \Sigma \in \partial M_1 \) and \( \Sigma^* \in \partial M_2 \) then \( Z(M) \) is obtained by contracting \( Z(M_1) \otimes Z(M_2) \) with respect to \( V_\Sigma \). In addition, the vector space associated to the empty surface is assumed to be the complex numbers, and if \( \Sigma \) is the disjoint union of two surfaces \( \Sigma_1 \) and \( \Sigma_2 \), the partition function \( Z(\Sigma_1 \cup \Sigma_2) \) is obtained by applying the previous property to the disjoint union of \( \Sigma_1 \) and \( \Sigma_2^* \).
\( \Sigma_2 \) then \( V_\Sigma = V_{\Sigma_1} \otimes V_{\Sigma_2} \). In particular, if \( M \) is a closed manifold \( Z(M) \) is a complex number which is a topological invariant of \( M \).

Alternatively, the gluing property 3) can be reformulated in terms of operators as follows. Viewing \( M_1 \) and \( M_2 \) as cobordisms with \( \partial M_1 = \Sigma_1 \cup \Sigma \) and \( \partial M_2 = \Sigma' \cup \Sigma_2 \) we can correspondingly consider the state sums as operators \( Z(M_1) : V_{\Sigma_1}^* \to V_\Sigma \) and \( Z(M_2) : V_{\Sigma'} \to V_{\Sigma_2} \) by 1). Given an orientation preserving diffeomorphism \( f : \Sigma \to \Sigma' \) and letting \( M \) denote the manifold obtained by gluing \( M_1 \) to \( M_2 \) along \( f \), property 3) is equivalent to

\[
Z(M) = Z(M_2)U(f)Z(M_1).
\] (1.1)

Note that the symmetry of the gluing w.r.t. \( M_1 \) and \( M_2 \) requires that

\[
U(f^*) = (U(f)^t)^{-1},
\] (1.2)

where \( f^* : \Sigma^* \to (\Sigma')^* \) denotes \( f \) with orientations on \( \Sigma \) and \( \Sigma' \) switched, and the superscript \( t \) indicates transposition. There now exists in the literature a variety of rigorous constructions of 3-dimensional TQFT’s. In this note we shall consider the constructions by Reshetikhin-Turaev [RT] and the one by Turaev-Viro [TV] and their generalizations (see [T], [DJN], [KS], [BD]). These are all based on the algebraic structure of the representation theory of quantum groups with deformation parameter equal to a root of unity, and are known to be related to Chern-Simons theory with an arbitrary compact gauge group.

In [BD] we have proven that for closed manifolds the invariant \( Z_{TV}(M) \) of the Turaev-Viro construction equals the modulus squared of the invariant \( \tau(M) \) obtained by the Reshetikhin-Turaev construction for a general quantum group at simple even roots of unity (see also [Wa], [T] and [R]). The purpose of this paper is to extend this result to manifolds with boundary, i.e. we show that

\[
Z_{TV}(M) = \tau(M) \otimes \tau(M^*)
\]

for any 3-cobordism \( M \). Here \( Z_{TV}(M) \) and \( \tau(M) \) denote the cobordism invariants defined in [BD] and [T], respectively. In section 2 we recall briefly the basic elements of the Turaev-Viro construction as developed in [BD] and
refer the reader to that paper for fuller details. We then prove a basic lemma which yields certain isomorphisms from the state spaces of the theory onto certain explicitly realizable spaces. This result is used in Section 3 to obtain an equivalent TQFT for which the announced factorization property is then proven.

2 Turaev-Viro TQFT

In this section we briefly recall the formulation and basic properties of TQFT of the Turaev-Viro type (for more details see [BD]). The corresponding state sum will be denoted by \( Z(M) \) (omitting the index \( TV \) in the following).

Originally, the Turaev-Viro invariant was defined for a compact connected closed oriented 3-manifold \( M \) as follows [TV]: Choose a triangulation of \( M \) and associate to each 1-simplex of the triangulation an index (or a colour) from a finite set \( \mathcal{I} \) of so-called “admissible” representations of a quantum group. To each coloured tetrahedron one then associates a 6j-symbol, which is possible due to the invariance of 6j-symbols under the tetrahedral symmetry group. In addition, to each coloured link one attaches a factor \( \omega_2^i \), which equals the quantum dimension of the corresponding colour \( i \), and to each vertex one attaches a factor \( \omega^{-2} \), where

\[
\omega^2 = \sum_{i \in \mathcal{I}} \omega_i^4.
\]

The invariant \( Z(M) \) is then obtained as the sum over all colourings of the triangulation of the product of all factors associated to tetrahedra, links and vertices. It can be shown (using the Biedenharn-Elliott relations for 6j-symbols) that the resulting quantity is independent of the particular choice of triangulation.

We have here assumed that the 6j-symbols are scalars, i.e. that the multiplicity of any representation \( i \in \mathcal{I} \) in a tensor product of two representations in \( \mathcal{I} \) is always 0 or 1, which e.g. is the case for \( SU_q(2) \). For more general quantum groups the 6j-symbols are tensors. To be specific we associate to each oriented, coloured triangle \( t \) in \( \Sigma = \partial M \) with oriented boundary...
links as indicated in Fig.1 (where the orientation of the plane is assumed to be counter clock-wise) the vector space \( V^k_{ij} \) of Clebsch-Gordan coefficients defined by

\[
H_i \otimes H_j = \sum_{k \in I} V^k_{ij} \otimes H_k,
\]

where \( H_i \) denotes the vector space of the representation \( i \).

Fig.1 An oriented \( \{i, j, k\} \)-coloured 2-simplex

The canonically dual vector space \( (V^k_{ij})^* = V^ij \) will be associated to the oppositely oriented triangle. For other configurations of arrows than that on Fig.1 the corresponding spaces are defined by requiring that reversing an arrow on a 1-simplex is equivalent to replacing its colour by the dual one (i.e. replacing the corresponding representation by its adjoint).

Moreover, the 6j-symbol associated to an oriented coloured tetrahedron with oriented links belongs to the tensor product of the vector spaces associated to the triangles in its boundary. Thus, we may define \( Z(M) \) by replacing above the product of 6j-symbols by the corresponding tensor product and contracting with respect to the dual pairs of spaces associated to triangles (with some fixed orientation of links), and the result is again independent of the choice of triangulation as well as of the chosen orientation of links. In fact, this definition is easily extended to non-closed, oriented manifolds \( M \) by simply contracting only with respect to dual pairs of spaces associated to interior triangles of the triangulation. One then obtains a tensor \( Z'(M) \) in the vector space \( V'_{\partial M} \) defined as the direct sum over all colourings of the links in \( \partial M \) of the tensor product of the spaces associated to the triangles in \( \partial M \). This space, of course, depends on the triangulation of \( \partial M \). However, any two such triangulations may be connected by a triangulation of the cylinder \( \partial M \times [0, 1] \) in the obvious sense, and \( Z'(\partial M \times [0, 1]) \) defines a cylinder map between the corresponding spaces. In particular, choosing the same
triangulation at the two ends of the cylinder the map becomes a projection, and the supports of the projections so obtained may be canonically identified by the cylinder maps thus defining the vector space $V_{\partial M}$, and at the same time the partition functions $Z'(M)$ are also identified with a unique vector $Z(M) \in V_{\partial M}$ fulfilling the required properties.

Exploiting ideas of Turaev [Tu] an effective calculational tool was developed in [KS] by introducing coloured graphs $G_{\underline{x}}$ on the boundary of the manifold $M$ and defining an associated state sum $Z(M, G_{\underline{x}})$ generalizing $Z(M)$. Here a coloured graph $G_{\underline{x}}$ is a closed 1-dimensional simplicial complex, whose 0-simplexes have order at most 3 and whose lines (i.e. maximal sequences of 1-simplexes joined by vertices of order 2 are oriented and coloured (by elements in $I$), the collection of colours being indicated by $\underline{x}$. The graph is assumed to be embedded into $\partial M$ such that over- and undercrossings are distinguished. The definition of $Z(M, G_{\underline{x}})$ proposed in [KS] has the following geometrical interpretation (see [BD]). One glues to the boundary $\Sigma$ of $M$ a certain pseudo-manifold $P_G$ whose boundary consists partly of one copy of $\Sigma^*$ (triangulated as $\Sigma$) and partly of a surface on which the dual graph of $G$ determines a cell decomposition into triangles (corresponding to 3-vertices) and rectangles (corresponding to over- and undercrossings) and whose edges inherit a colouring from $\underline{x}$. The state sum $Z(M, G_{\underline{x}})$ then equals $Z(M_{G_{\underline{x}}})$, where $M_{G_{\underline{x}}}$ is the resulting pseudo-manifold with fixed colouring of boundary links given by $\underline{x}$. Actually, the construction requires a slight modification in case rectangles are present in the boundary (see [BD]). Suffice here to mention that $Z(M, G_{\underline{x}})$ in all cases belongs to the tensor product of the vector spaces associated to the triangles dual to the 3-vertices in $G_{\underline{x}}$ and is a homotopy invariant of the coloured graph $G_{\underline{x}}$ in $\Sigma$.

In case $G$ is empty the pseudo-manifold $P_G$ is the cone over $\Sigma$ and the boundary of the resulting manifold degenerates to a point. On the other hand, if $G$ is sufficiently "large" so that $P_G$ is homeomorphic to the cylinder $\Sigma \times [0, 1]$, then $M_{G_{\underline{x}}}$ is homeomorphic to $M$, and if $G$ in addition has no over- or undercrossings it follows that $\oplus_{\underline{x}} Z(M, G_{\underline{x}})$ equals $Z'(M)$ with $\partial M$ triangulated by the dual graph to $G$.

The gluing axiom described at the beginning of section 1 can now be
reformulated in the language of graphs as follows. If $M$ is obtained by gluing $M_1$ and $M_2$ along $\Sigma$ we have

$$Z(M) = \frac{1}{\omega^2} \sum_{x} \omega_x^2 Z(M_1, G^F_x) Z(M_2, G^F_x)$$

(2.1)

for any canonical graph $G$ without over- or undercrossings, and where $F$ is the gluing homeomorphism and $G^F$ denotes the image of $G$ under $F$.

The state sums $Z(M, G_x)$ satisfy a number of simple relations under certain elementary changes of the graph $G_x$, which together with (2.1) can be used to show that the dimension of $V_{\Sigma_g}$, where $\Sigma_g$ is a connected surface of genus $g \geq 1$, is given by the square of the Verlinde formula:

$$\dim V_{\Sigma_g} = tr \ id_{V_{\Sigma_g}} = tr Z(\Sigma_g \times I) = Z(\Sigma_g \times S^1) = (tr N^2(g-1))^2$$

(2.2)

where $\tilde{N}^2 = \sum_a (N^a)^2$ and $(N^a)$ is the multiplicity matrix given by

$$(N^a)_{bc} = N^a_{bc} = \dim V^{a}_{bc}$$

(2.3)

for $a, b, c \in \mathcal{I}$.

It is even possible to realize the space $V_{\Sigma_g}$ explicitly as follows. Consider a handle body $M_g$ of genus $g$ in $\mathbb{R}^3$ with $\partial M_g = \Sigma_g$ and introduce two copies $c^L$ and $c^R$ of the graph depicted below such that they are deformation retracts of $\Sigma$ in $M_g$ and such that they are disjoint (and not linked).

![Diagram]

Clearly $c^L$ and $c^R$ then possess tubular neighborhoods that are disjoint and diffeomorphic to $M_g$ and whose boundaries are homotopic to $\Sigma_g$ in $M_g$. Removing two such tubular neighborhoods from $M_g$ yields a manifold $\tilde{M}_g$ with three boundary components $\Sigma_g$, $(\Sigma^L_g)^*$ and $(\Sigma^R_g)^*$ all of genus $g$. We now choose the coordinates so that the cores $c^L$ and $c^R$ lie in the $xy$-planes and
their $z$-components are equal to 1 and -1 respectively. We will call the part of $\Sigma^L_g$ (resp. $\Sigma^R_g$) where $z > 1$ (resp. $z > -1$) the upper side and the other part where $z < 1$ (resp. $z < -1$) the back side of $\Sigma^L_g$ (resp. $\Sigma^R_g$).

Next, we embed a copy $G^L$ of the graph (2.4) on the upper side of $\Sigma^L_g$ in such a way that the graph is homotopic to the core $c^L$. Analogously, we embed the second copy $G^R$ of the graph (2.4) on the back side of $\Sigma^R_g$.

Finally, we make $G^L$ lefthanded and $G^R$ righthanded, i.e. we introduce meridians on each of the tubes corresponding to the lines of $c^L$, resp. $c^R$, which undercross, resp. overcross, the lines of $G^L$ on $\Sigma^L_g$, resp. $G^R$ on $\Sigma^R_g$.

We then define

$$K_{\underline{e},\underline{f}} = \sum_{\underline{x},\underline{y}} \prod_{i=1}^{3g-3} \frac{\omega_{x_i}^2 \omega_{y_i}^2}{\omega_2^2 \omega_2} Z(\tilde{M}_g, G^L_m \cup m_x^L \cup G^R_m \cup m_y^R \cup G^g)$$

where $\underline{e}$, resp. $\underline{f}$, is a colouring of $G^L$, resp. $G^R$, the product is over meridians and the sum is over colourings $\underline{x}$ and $\underline{y}$ of the meridians $m^L$ and $m^R$, on $\Sigma^L_g$ and $\Sigma^R_g$, respectively, and $G^g$ is some canonical graph on $\Sigma_g$ without over- or undercrossings.

We denote by $V^L_g$, resp. $V^R_g$, the vector space associated to $G^L$, resp. $G^R$, regarded as embedded into $\Sigma^L_g$, resp. $\Sigma^R_g$, i.e.

$$V^L_g = \oplus_{\underline{e}} V^L_g(\underline{e}),$$

where $V^L_g(\underline{e})$ is the tensor product of vector spaces associated to the coloured 3-vertices of $G^L$ taking into account the orientation of $\Sigma^L_g$ and similarly for $G^R$. Then

$$\dim V^L_g = \dim V^R_g = \text{tr}(N^2)^{(g-1)}$$

by a simple counting, and hence

$$\dim (V^L_g \otimes V^R_g) = \dim V_{\Sigma_g}.$$

Moreover, with the chosen orientation convention we have (see [BD]) $K_{\underline{e},\underline{f}} \in V^L_g(\underline{e})^* \otimes V^R_g(\underline{f})^* \otimes V_{\Sigma_g}$ and hence (2.5) defines an operator

$$K_{\underline{e},\underline{f}} : V^L_g(\underline{e}) \otimes V^R_g(\underline{f}) \rightarrow V_{\Sigma_g}.$$
in an obvious way. We intend to show that the direct sum over $e, f$ of these operators yields an isomorphism between $V_g^L \otimes V_g^R$ and $V_{\Sigma g}$. This was proven for the case $g = 1$ in [BD]. In the general case it is a consequence of Lemma 1 below in which, however, we have found it convenient first to rewrite $K_{e,f}$, up to a factor $\omega^{2(-g+1)}$, as

$$K_{e,f} = \sum_{\mathcal{L}} \prod_{i=1}^{3g-3} \frac{\omega_{2L}^2}{\omega_{2}^2} Z(M'_g, G_{\mathcal{L}} \cup m \cup G^g), \quad (2.9)$$

where $M'_g$ is the manifold with boundary components $\Sigma_g$ and $\Sigma'_g$ obtained by removing one tubular neighborhood instead of two as above and where $G_{\mathcal{L}}$ is the coloured graph on $\Sigma'_g$ indicated on the figure below together with a system $m$ of meridians (of which there are $3g - 3$ for $g \geq 1$, and 1 for $g = 1$), and $G^g$ is as above.

The equivalence of (2.5) and (2.9) follows by merging $\Sigma_g^L$ and $\Sigma_g^R$ as in the proof of Lemma 4.4 ii) in [BD]; see also the proof of Lemma 1 below, where the same technique is used. We shall henceforth take (2.9) as the definition of $K_{e,f}$.

We now introduce an operator

$$L_{e,f} : V_{\Sigma_g} \to V_g^L(e) \otimes V_g^R(f) \subseteq V_g^L \otimes V_g^R$$
as a mirror image of $K_{e,f}$ w.r.t. a plane parallel to the $z$-axis and not intersecting the handlebody $M_g$. More precisely,

$$L_{e,f} = \sum \prod_{i=1}^{3g-3} \frac{\omega_i}{\omega_i^2} Z(M''_g, \bar{G}_{e,f} \cup m \cup \bar{G}^g),$$

(2.11)

where $M''_g$ is the mirror image of $M'_g$ and $\partial M''_g = \Sigma^*_g \cup \Sigma''_g$. The graphs $G_{e,f} \in \Sigma''_g$ and $G_g \in \Sigma^*_g$ are the mirror images of $G_{e,f} \in \Sigma^*_g$ and $G_g \in \Sigma_g$ respectively.

Gluing $(M'_g, G_{e,f} \cup m_e)$ and $(M''_g, \bar{G}_{e,f} \cup m_e)$ along $\Sigma_g$ we obtain $(N_g, G_{e,f} \cup m_e, \bar{G}_{e,f} \cup m_e)$ where $N_g$ is diffeomorphic to $\Sigma_g \times [0,1]$ with boundary $\Sigma''_g \cup \Sigma^*_g$. The graph $\tilde{G}_{e,f} \cup m_e \in \Sigma'_g$ can be obtained from the standard graph $G_{e,f} \cup m_e \in \Sigma'_g$ depicted in (2.10) by changing the colourings $e \rightarrow e', f \rightarrow f', x \rightarrow y$ and replacing all overcrossings by undercrossings and vice versa.

Eq. (2.1) implies that

$$L_{e',f'} K_{e,f} = \sum \prod_{i=1}^{3g-3} \frac{\omega_i}{\omega_i^2} \frac{\omega_{i'}}{\omega_{i'}} Z(N_g, G_{e,f} \cup m_e \cup \tilde{G}_{e',f'} \cup m_e').$$

(2.12)

We are now in position to state the announced lemma.

**Lemma 1** The operator $L_{e',f'} K_{e,f} : V^L_g(e) \otimes V^R_g(f) \rightarrow V^L_g(e') \otimes V^R_g(f')$ satisfies

$$\omega^{2g-2} \omega_{e,f} \omega_{e',f'} L_{e',f'} K_{e,f} = \delta_{e,e'} \delta_{f,f'} 1_{V^L_g(e) \otimes V^R_g(f)},$$

(2.13)

where we have introduced the notation $\omega_{e,f} = \prod_{i=1}^{3g-3} \omega_i$, and $\omega_{e,f'} = \prod_{i=1}^{3g-3} \delta_{e_i',e'}$.

**Proof:** The idea of the argument is the following. By introducing tubes between $\Sigma'_g$ and $\Sigma''_g$ we step by step lift the lines of $G_{e,f} \in \Sigma'_g$ on $\Sigma''_g$ and cut the handles traversed by these lines. Applying the technique developed in [BD] and [KS] we will arrive on (2.13).

Due to Lemma 3.3 in [BD] introduction of a tube with an $a$-coloured meridian (which is not normalized by $\omega^{-2}$) does not change the state sum. Pictorially this looks as follows:
Fig. 2 A part of the manifold $N_g$ where the boundary component $\Sigma'_g$ of the tube is connected to $\Sigma''_g$ by a tube with an a-coloured meridian on it

where we do not draw the $e$, $f$- and $e'$, $f'$-coloured lines. Applying Lemma 4.2 ii) in [BD] (or the Wigner-Eckart type relation (A.15) in [KS]) to the meridians $m_1$, $m'_1$ and $a$ we can change the graph so that the handle $(ABC) \times I$ will be traversed by a single line only. According to Remark 3.6 in [BD] the colour of this line can be set to zero and the handle cut. This yields a manifold $N'_g$ as depicted on Fig. 3.

Fig. 3 A part of the manifold $N'_g$ with associated graph on it

Using lemma 4.2 ii) in [BD] once more (see also example 5.8 iii) in [KS])
one can cut the handle traversed by \( e'_1, e_1, f'_1 \) and \( f_1 \)-coloured lines. After that the state sum of the resulting \((g - 1)\)-cylinder becomes multiplied by

\[
\omega_{e_1}^{-2} \omega_{f_1}^{-2} \delta_{e_1 e'_1} \delta_{f_1 f_1}.
\]

Continuing this procedure analogously we obtain the desired result:

\[
L_{e'}_{f'} K_{e'f'} = \omega^{2g+2} \delta_{e'e'_1} \delta_{f'f_1} (\omega_{e_1}^2 \omega_{f_1}^2)^{-1} \mathbf{1}_{V_g^g \otimes V_g^g}. \]

\[
\square
\]

Defining the operators \( K : V_g^g \otimes V_g^g \to V_{\Sigma g} \) and \( L : V_{\Sigma g} \to V_g^g \otimes V_g^g \) by

\[
K = \omega^{g-1} \oplus_{e'_1} \omega_{e'_1} K_{e'f'}, \quad L = \omega^{g-1} \oplus_{f'_1} \omega_{f'_1} L_{e'f'},
\]

it follows from (2.13) that \( LK = \mathbf{1}_{V_g^g \otimes V_g^g} \) and hence by (2.8) \( K \) and \( L \) are isomorphisms and

\[
L = K^{-1}. \tag{2.14}
\]

Although we shall strictly speaking not use them in the following let us introduce the left- and righthanded counterparts \( K_L \) and \( K_R \) of \( K_{e'f'} \) by replacing in eq. (2.9) the graph \( G_{e'f'} \) by its left- and righthanded parts \( G_{e'f'}^L \) and \( G_{e'f'}^R \), respectively, and similarly \( L_L \) and \( L_R \) by replacing \( G_{e'f'} \) in eq. (2.11) by \( G_{e'f'}^L \) and \( G_{e'f'}^R \), respectively. The proof of Lemma 1 then yields

\[
\omega^{2g-2} \omega_{e'_1} \omega_{f'_1} L_L^L K_L^L = \delta_{e'_1 e_1} \mathbf{1}_{V_g^g},
\]

and

\[
\omega^{2g-2} \omega_{f'_1} \omega_{e'_1} L_R^R K_R^R = \delta_{f'_1 f_1} \mathbf{1}_{V_g^g},
\]

and consequently

\[
L^L K^L = \mathbf{1}_{V^g_L}, \quad L^R K^R = \mathbf{1}_{V^g_R},
\]

where \( K^L : V^g_L \to V_{\Sigma g} \) and \( L^L : V_{\Sigma g} \to V^g_L \) are defined by

\[
K^L = \omega^{g-1} \oplus_{e'_1} \omega_{e'_1} K_{e'f'}, \quad L^L = \omega^{g-1} \oplus_{f'_1} \omega_{f'_1} L_{e'f'}, \tag{2.15}
\]

and similarly for \( K^R : V^g_R \to V_{\Sigma g} \) and \( L^R : V_{\Sigma g} \to V^g_R \).
3 Factorization of state sums

For each genus \( g \geq 0 \) we fix once and for all manifolds \( M'_g \) and \( M''_g \) as defined in Section 2 with \( \partial M'_g = \Sigma_g \cup \Sigma'_g \ast \) and \( \partial M''_g = \Sigma'_g \cup \Sigma''_g \), where \( \Sigma_g, \Sigma'_g \) and \( \Sigma''_g \) are fixed oriented surfaces of genus \( g \) and where fixed graphs \( G^g_g \) and \( G''^g_g \) are embedded in \( \Sigma'_g \) and \( \Sigma''_g \ast \) respectively, together with the associated sets of meridians. We have here made the dependence of the graphs and meridians on the genus explicit, and will do so likewise for the associated operators \( K^g_g, L^g_g \) etc.

By a parametrized surface of genus \( g \) we mean a pair \((\Sigma, \phi)\), where \( \Sigma \) is a compact, connected, oriented surface of genus \( g \) and \( \phi : \Sigma \to \Sigma_g \) is a diffeomorphism. We call \( \phi \) a parametrization of \( \Sigma \) and set
\[
\tilde{V}_{\Sigma}(\phi) = V^L_g \otimes V^R_g .
\]

Let us consider a 3-dimensional cobordism \( M \) whose boundary \( \partial M = \Sigma_1^g \cup \Sigma_2^g \) consists of two compact, connected, oriented surfaces of genus \( g_1 \) and \( g_2 \), respectively, which are parametrized by \( \phi_1 \) and \( \phi_2 \). An operator \( \tilde{Z}(M) : \tilde{V}_{\Sigma_1} (\phi_1) \to \tilde{V}_{\Sigma_2} (\phi_2) \) can be defined as follows:
\[
\tilde{Z}(M) = L(\phi_2) Z(M) K(\phi_1) ,
\]
where
\[
K(\phi_1) = U(\phi_1) K^{g_1}_{\Sigma_1}, \quad L(\phi_2) = L^{g_2} U(\phi_2)
\]
and \( U(\phi) : V_{\Sigma} \to V_{\Sigma_g} \) satisfying (1.2).

More generally, given a compact, oriented cobordism \( M \) with boundary components \( \Sigma_1^g, ..., \Sigma_m^g, \Sigma_{m+1}^g, ..., \Sigma_n^g \) and parametrization \( \phi_i \) of \( \Sigma_i^g \) we set
\[
\tilde{Z}(M) = L(\phi_{m+1}, ..., \phi_n) Z(M) K(\phi_1, ..., \phi_n) \quad (3.1)
\]
where
\[
K(\phi_1, ..., \phi_k) = \otimes_{i=1}^k K(\phi_i)
\]
and \( L(\phi_1, ..., \phi_k) \) is defined analogously.

Equivalently, (3.1) can be expressed as follows. Let \( \tilde{M} \) denote the manifold obtained by gluing \( M'_g \) onto \( M \) along \( \phi \) for \( 1 < i < m \), and gluing
$M''_g$ onto $M$ along $\phi_i$ in case $i > m$. Then, clearly, $\bar{M}$ is diffeomorphic to $M$ and has boundary components $(\Sigma'_1)^*, \ldots, (\Sigma'_{gm+1})^*, \Sigma''_m, \ldots, \Sigma''_n$, with embedded graphs $G^{\Sigma}_1, \ldots, G^{\Sigma}_m, G^{\Sigma}_{m+1}, \ldots, G^{\Sigma}_n$, respectively. With the notation $\tilde{e} = (e^1_1, \ldots, e^n)$ and

$$\omega_{\tilde{e}} = \prod_{i=1}^n \omega_{\tilde{e}_i}$$

we then have

$$\tilde{Z}(M) = \bigoplus_{\tilde{e}, \tilde{j}} \tilde{Z}_{\tilde{e}, \tilde{j}}(M),$$

(3.2)

where the coloured state sum $\tilde{Z}_{\tilde{e}, \tilde{j}}(M)$ is defined by

$$\tilde{Z}_{\tilde{e}, \tilde{j}}(M) = \omega^{g_1+\ldots+g_m-n} \omega_{\tilde{e}} \omega_{\tilde{j}} \sum_{\tilde{x}} \prod_{i,j} \frac{\omega_{e_i}}{\omega_{e_j}} Z(M, G_{\tilde{e}, \tilde{j}} \cup \mathcal{M}_{\tilde{x}})$$

(3.3)

where

$$G_{\tilde{e}, \tilde{j}} = G^{\Sigma}_1 f_1 \cup \ldots \cup G^{\Sigma}_n f_n$$

and

$$\mathcal{M}_{\tilde{x}} = m_{\tilde{x}}^1 \cup \ldots \cup m_{\tilde{x}}^n.$$ 

Finally, we define an isomorphism $\tilde{U}(f) : \tilde{V}_{\Sigma}(\phi) \to \tilde{V}_{\Sigma'}(\phi')$ by

$$\tilde{U}(f) = L(\phi') U(\phi) K(\phi),$$

(3.4)

for any orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ between parametrized surfaces $(\Sigma, \phi)$ and $(\Sigma', \phi')$ of genus $g$. This definition is extended in an obvious way to orientation preserving diffeomorphisms between arbitrary compact, oriented surfaces in terms of tensor products.

The objects $\tilde{V}, \tilde{U}, \tilde{Z}$ define a TQFT on compact, oriented 3-manifolds with parametrized boundary. This can be easily verified using the definition of these objects and eq. (2.14). The TQFT based on $\tilde{V}, \tilde{U}$ and $\tilde{Z}$ is equivalent to the theory defined in the previous section. The equivalence is given by the $K$ and $L$-operators (see [T] or [DJ]).

We are now ready to state and prove the main result of this paper.
Theorem 2 Let $M$ be a compact, oriented 3-manifold. For any colouring $(\tilde{e}, \tilde{f})$ as defined above we have

$$\tilde{Z}_{\tilde{e}, \tilde{f}}(M) = \tau_{\tilde{e}}(M) \otimes \tau_{\tilde{f}}(M^*)$$  \hspace{1cm} (3.5)

where the invariant $\tau_{\tilde{e}}$ is given by eq. (3.8) below and coincides with the invariant introduced in [T] up to normalization.

Proof:
As remarked earlier, we can replace each tube in $\bar{M}$ defined above with graph $G^g_{g^1 \tilde{e}^1 \cup m_{\tilde{x}^1}}$ by two tubes with graphs $(G^g_{g^1 \tilde{e}^1})_L \cup (m_{\tilde{x}^1})_L$ and $(G^g_{g^1 \tilde{e}^1})_R \cup (m_{\tilde{x}^1})_R$, respectively, at the cost of a factor $\omega^{2(g_i - 1)}$. Let us assume we have done so for each $i = 1, \ldots, n$ and denote the resulting manifold also by $\bar{M}$. As is well known, the closed manifold obtained from $\bar{M}$ by filling all $2n$ tubes has a representation by surgery on $S^3$ along a set of links $l_1, \ldots, l_N$ which, of course, may be assumed not to intersect the filled tubes. Using Lemma 1 for the case $g = 1$ as in the proof of Theorem 5.2 in [BD] one obtains

$$Z(\bar{M}, G^g_{g_1 \tilde{e}^1 \cup m_{\tilde{x}^1}} \cup \ldots \cup G^g_{g_n \tilde{e}^n \cup m_{\tilde{x}^n}}) =$$

$$\omega^{2(g_1 + \ldots + g_n - n - N)} \sum_{\tilde{a}, \tilde{b}, \tilde{z}, \tilde{z}'} \omega_{\tilde{a}}^2 \omega_{\tilde{b}}^2 \frac{\omega_{\tilde{z}'}^2}{\omega_{\tilde{z}}^2} \frac{\omega_{\tilde{z}'}^2}{\omega_{\tilde{z}}^2}$$

$$Z(\tilde{S}^3, \mathcal{L}^L_{\tilde{a}} \cup (\mathcal{M}_{\tilde{x}}^L)^L \cup \mathcal{L}^R_{\tilde{b}} \cup (\mathcal{M}_{\tilde{x}}^R)^R \cup \mathcal{G}_{\tilde{e}}^L \cup \mathcal{M}_{\tilde{z}}^L \cup \mathcal{G}_{\tilde{f}}^R \cup \mathcal{M}_{\tilde{y}}^R)$$  \hspace{1cm} (3.6)

where we have introduced the shorthand notation

$$\mathcal{G}_{\tilde{e}}^L = (G^g_{g_1 \tilde{e}^1})_L \cup \ldots \cup (G^g_{g_n \tilde{e}^n})_L$$

and similarly for the righthanded part and the meridians. Furthermore, $\tilde{S}^3$ denotes the manifold obtained from $\bar{M}$ by removing two disjoint tubular neighborhoods $T_i^L$ and $T_i^R$ for each $i = 1, \ldots, N$. We define $T_i^L$ and $T_i^R$ by splitting a tubular neighborhood of $l_i$ into two nearby ones as was done previously for the graphs $G^g_1, \ldots, G^g_n$. Finally, $\mathcal{L}^L = L^L_1 \cup \ldots \cup L^L_N$ (together with associated meridians $\mathcal{M}_L = m^L_1 \cup \ldots \cup m^L_N$) is a collection of lefthanded graphs on the boundary components $\partial T^L_1, \ldots, \partial T^L_N$ of $\tilde{S}^3$, where the graphs are determined by the surgery prescription, and similarly for $\mathcal{L}^R$ and $\mathcal{M}^R$. 

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Next we recall from [BD] (see also [KS]) that two tubes with left- and righthanded lines, respectively, have trivial braiding, i.e. they may be deformed through each other. Using this and the fact that $\tilde{S}^3$ is a 3-sphere with a collection of $2(n + N)$ tubes removed, together with the factorization property of $Z(M, G)$ w.r.t. connected sums (see Lemma 3.2 in [BD]), we obtain by substituting (3.6) into (3.3) that

$$\tilde{Z}_{e, f}(M) = \omega^{2(g_1 + ... + g_n - n - N + 1)} \sum_{a, b} Z(\tilde{S}^3, \mathcal{L}_a^L \cup \mathcal{G}_e^L) \otimes Z(\tilde{S}^3, \mathcal{L}_b^R \cup \mathcal{G}_f^R),$$

(3.7)

where we have introduced

$$Z(\tilde{S}^3, \mathcal{L}_a^L \cup \mathcal{G}_e^L) = \omega^{g_m + ... + g_n - (n - m)} \omega_z \omega_a^2 \sum_{i, j} \frac{\omega^2_{x_{i,j}}}{\omega^{2n}_{z}} \prod_{i, j} \frac{\omega^2_{x_{i,j}}}{\omega^{2n}_{z}},$$

and

$$Z(\tilde{S}^3, \mathcal{L}_b^R \cup \mathcal{G}_f^R) = \omega^{g_1 + ... + g_m - m} \omega_{\tilde{z}} \omega_{\tilde{y}}^2 \sum_{i, j} \frac{\omega^2_{y_{i,j}}}{\omega^{2n}_{\tilde{z}}} \prod_{i, j} \frac{\omega^2_{y_{i,j}}}{\omega^{2n}_{\tilde{z}}},$$

$$Z((\tilde{S}^3)^L, \mathcal{L}_a^L \cup (\mathcal{M}_e')^L \cup \mathcal{G}_e^L \cup \mathcal{M}_e^L),$$

and

$$Z((\tilde{S}^3)^R, \mathcal{L}_b^R \cup (\mathcal{M}_f')^R \cup \mathcal{G}_f^R \cup \mathcal{M}_f^R).$$

where $(\tilde{S}^3)^L$ is defined in analogy with $\tilde{S}^3$ except that only tubes with lefthanded graphs or links are removed from $S^3$ and $(\tilde{S}^3)^R$ is defined similarly.

Finally, setting

$$\Delta_L = \sum_{c \in \mathcal{I}} q_c^2 \omega_c^4,$$

we define

$$\tau_e(M) = \omega^{g_1 + ... + g_n - n - N + 1} (\Delta_L \omega^{-1})^\sigma(\mathcal{L}) \sum_a Z((\tilde{S}^3)^L, \mathcal{L}_a^L \cup \mathcal{G}_e^L),$$

(3.8)

where $\sigma(\mathcal{L})$ is the signature of a certain 4-manifold whose boundary is $\mathcal{M}$ with tubes filled in. Similarly, the righthanded counterpart $\tau_f^R$ is defined with $\Delta_R$ given by the same formula as $\Delta_L$ except that $q_c$ should be replaced by $q_c^{-1}$. Then

$$\Delta_L \Delta_R = \omega^2.$$
(see [T]) and hence (3.7) can be rewritten as

\[ Z_{\tilde{e},\tilde{f}}(M) = \tau_{\tilde{e}}(M) \otimes \tau_{\tilde{f}}^R(M). \]

By arguments identical to those in [BD] one shows that

\[ \tau_{\tilde{f}}^R(M) = \tau_{\tilde{f}}(M^*) \]

thus proving (3.5). Likewise the argument that \( \tau_{\tilde{e}}(M) \) equals the ribbon graph invariant introduced in [T] follows as in [BD] by projecting the tubes in \((\tilde{\mathcal{S}}^3)^L\) with graphs and links onto a plane. \( \square \)

4 Concluding remarks

The proof of Theorem 2 can be extended in a straightforward manner to the case where punctures are introduced on the boundary components of \( M \). We shall, however, not elaborate on that case here (see also [T]).

It should be mentioned that the equivalence of the TQFT defined in section 2 and the one defined in terms of \( \tilde{V}, \tilde{U}, \tilde{Z} \) follows from the equality of the corresponding state sums of closed manifolds, shown in [BD] and [T], once it is known that the two theories are non-degenerate (see e.g. [T]). The method of this paper gives the equivalence explicitly and at the same time prepares the ground for the proof of (3.5).

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