Fano 3-folds and double covers by half elephants

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Abstract
We construct a deformation family for each of the 34 Hilbert series of Fano 3-folds in codimension 4 having Fano index 2. In 18 cases we construct two different families, distinguished by the topology of their general members.

Keywords Fano 3-fold · Fano index · Equivariant unprojection

Mathematics Subject Classification 14E05 · 14E25 · 14J30 · 14J45

1 Introduction

We work over the field of complex numbers $\mathbb{C}$. A $\mathbb{Q}$-Fano 3-fold $X$ is a normal projective 3-dimensional variety with ample anticanonical divisor $-K_X$ and $\mathbb{Q}$-factorial terminal singularities. The Fano index of $X$ is

$$\iota_X := \max \left\{ q \in \mathbb{Z}_{\geq 1} : -K_X \sim qA \text{ for some } A \in \text{Cl}(X) \right\}.$$ 

The Hilbert series $P_{X,A}$ for $(X, A)$ is the one defined by the graded ring $R(X, A)$ (see [5, Section 3.1]). The landscape of Fano 3-folds having Fano index 2 is partially understood: [7] gives a list of possible Hilbert series of these varieties, although that paper does not confirm the existence of any particular one. We call this list $\mathcal{H}_{BS}^2$. By elephant we mean a general member of the anticanonical linear system $|-K_X|$ (cf. [22]).
1.1 Main Theorem

We consider Fano 3-folds that are embedded primitively into weighted projective spaces in codimension 4. The aim of this paper is to systematically build at least one deformation family for each 34 of the Hilbert series in $H^2_{BS}$. The strategy is to retrieve them from just as many deformation families of codimension 4 Fano 3-folds $X$ of Fano index 1 by performing a quotient by a $\mathbb{Z}/2\mathbb{Z}$ action $\gamma$ on the ambient spaces of these $X$’s.

The construction we achieve is summarised in this diagram:

```
codim 4          codim 3
```

```
index 1

\[
\begin{array}{ccccccccc}
X & \leftrightarrow & \text{unprojection} & \rightarrow & Z
\end{array}
\]

\[
\begin{array}{ccccccccc}
Z/2Z & \downarrow & \gamma & \rightarrow & Z/2Z
\end{array}
\]

\[
\begin{array}{ccccccccc}
\text{index 2} & \rightarrow & \tilde{X} & \rightarrow & Z
\end{array}
\]
```

The Main Theorem we prove is:

**Theorem 1.1** For each of the 34 power series $P \in H^2_{BS}$ there exists at least one deformation family of codimension 4 Fano 3-folds $\tilde{X}$ having Fano index 2 such that $P_{\tilde{X}, A} = P$ for $A \in \{-\frac{1}{2}K_{\tilde{X}}\}$. For 18 of the Hilbert series in $H^2_{BS}$ there are at least two distinguished deformation families.

In practice, we prove that for each $P \in H^2_{BS}$ there exists at least one deformation family of Fano index 1 Fano 3-folds $X \subset wP^7$ with a $\mathbb{Z}/2\mathbb{Z}$ action $\gamma$ such that $\tilde{X} := X/\gamma$ is a deformation family of quasi-smooth Fano 3-folds having Fano index 2, and the Hilbert series of $(\tilde{X}, A)$ is equal to $P$. We produce $X$ by Type I [15, 19] and Type II [16] unprojections from particular Fano 3-folds $Z$ in codimension 3 and from Fano hypersurfaces respectively. The key is to have $X$ invariant under $\gamma$. We discuss how and when this is possible in Sects. 4, 5, 7. The details are summarised in Table 1.

Our construction echoes those in the smooth case: for instance, the double cover of a smooth cubic 3-fold $X_3 \subset P^4$, having Fano index 2, with branch locus a smooth anticanonical surface is the complete intersection $X_{2,3} \subset P^5$ having Fano index 1 (see also [2, Theorem 3.1 and Examples in Section 3] for a more extensive discussion on finite morphisms between smooth Fano 3-folds).

1.2 Framework

Some constructions of index 2 Fano 3-folds already exist in the literature. In [21], Prokhorov and Reid have constructed one example of an index 2 Fano 3-fold in codimension 3 and one in codimension 4 performing divisorial extractions of curves in $P^3$ and in a quadric in $P^4$ respectively. Their argument is concluded by running the Sarkisov Program initiated by such extractions. Their construction was further generalised by Ducat in [11], who obtained one new family in codimension 4, and one each in codimension 5 and 6.
| Index 2 | Index 1 | #1 | Tom | Jerry | Failures |
|---------|---------|----|-----|-------|----------|
| 39557  | 327     | 2  | $T_3$ | $J_{24}$ | none |
| 39560  | 512     | n/a | no Type I projection. $\#_{II} \geq 1$ | none |
| 39576  | 569     | 1  | $T_1$ | none | $J_{25}$ |
| 39578  | 574     | 2  | $T_1$ | $J_{24} \cdot 12$ | $J_{45}$ |
| 39605  | 869     | 2  | $T_4$ | $J_{13}$ | none |
| 39607  | 872     | n/a | no Type I projection. $\#_{II} \geq 1$ | none |
| 39660  | 1158    | 2  | $T_5$ | none | none |
| 39675  | 1395    | 2  | $T_5$ | none | $J_{12}$ |
| 39676  | 1401    | 1  | $T_2$ | none | $T_4$ | $J_{12}$ |
| 39678  | 1405    | 1  | $T_1$ | none | $J_{24}$ |
| 39890  | 4810    | 2  | $T_3$ | $J_{24}$ | $J_{14} \cdot 13$ |
| 39898  | 4896    | 2  | $T_3$ | $J_{24}$ | $J_{14} \cdot 13$ |
| 39906  | 4925    | 1  | $\frac{1}{4}(1, 6): T_1$ | none | $\frac{1}{4}(1, 1, 6): J_{34}$ | $\frac{1}{4}(1, 3, 4): J_{35}$ |
| 39912  | 4938    | 1  | $T_2$ | none | $J_{24}$ |
| 39913  | 4939    | 2  | $\frac{1}{3}: T_1$ | none | $T_1$ | $J_{23} \cdot 13$ | $J_{34}$ |
| 39928  | 4987    | 1  | $T_5$ | none | $J_{12}$ |
| 39929  | 5000    | 1  | $\frac{1}{3}: T_2$ | none | $\frac{1}{3}: J_{45}$ | $\frac{1}{3}: J_{24}$ |
| 39934  | 5052    | 2  | $T_1$ | $J_{23} \cdot 13$ | $J_{34}$ |
| 39961  | 5176    | 1  | $\frac{1}{2}: T_2$ | none | $\frac{1}{2}: J_{35}$ | $\frac{1}{2}: J_{25}$ |
| 39968  | 5260    | 2  | $T_3$ | none | $\frac{1}{2}: J_{35}$ | $J_{25}$ |
| 39969  | 5266    | 2  | $\frac{1}{3}: T_3$ | $J_{13} \cdot 13$ | $\frac{1}{3}: J_{45}$ | $\frac{1}{3}: J_{34}$ |
| 39970  | 5279    | 1  | $\frac{1}{3}: T_1$ | none | $\frac{1}{3}(1, 1, 4): T_1$ | $\frac{1}{3}(1, 2, 3): T_1$ | $\frac{1}{3}(1, 1, 4): J_{34}$ | $\frac{1}{3}(1, 2, 3): J_{35}$ |
| 39991  | 5516    | 1  | $\frac{1}{2}: T_1$ | none | $\frac{1}{2}: J_{45}$ | $\frac{1}{2}: J_{13}$ |
| 39993  | 5519    | 2  | $\frac{1}{2}: T_1$ | $J_{34}$ | $\frac{1}{2}: J_{35}$ | $\frac{1}{2}: J_{13}$ |
| 40360  | 10963   | 2  | $T_3$ | $J_{24}$ | $J_{14} \cdot 13$ |
| 40370  | 11004   | 1  | $T_2$ | none | $J_{12}$ |
| 40371  | 11005   | 2  | $\frac{1}{2}: T_1$ | $J_{25} \cdot 13$ | $\frac{1}{2}: J_{35}$ | $\frac{1}{2}: J_{24}$ |
| 40399  | 11104   | 1  | $T_5$ | none | $J_{24}$ |
| 40400  | 11123   | 1  | $\frac{1}{2}: T_3$ | none | $\frac{1}{2}: J_{24} \cdot 25$ | $\frac{1}{2}: J_{12} \cdot 15$ | $J_{24}$ |
| 40407  | 11222   | 2  | $T_1$ | $J_{23} \cdot 13$ | $J_{34}$ |
| 40663  | 16206   | 2  | $T_4$ | $J_{23}$ | $J_{12} \cdot 15$ |
| 40671  | 16227   | 1  | $T_2$ | none | $J_{12}$ |
| 40672  | 16246   | 2  | $T_2$ | $J_{15} \cdot 14$ | $J_{25}$ |
| 40933  | 24078   | 2  | $T_5$ | $J_{12}$ | $T_1$ |
| 41028  | n/a     | n/a | none | none | none |
There are 35 Hilbert series in \( H^2_{\text{BS}} \) associated to codimension 4 Fano 3-folds with Fano index 2 as in the Graded Ring Database (GRDB) [1, 4], of which each realise one using their constructions. A further case\(^1\) is the Hilbert series of the smooth Fano 3-fold with GRDB ID # 41028, constructed in [20, Section 12, Table 12.3, No. 32], which is a divisor of bidegree \((1, 1)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \). We added it to Table 1 for completeness.\(^1\)

Of the 34 Hilbert series relative to codimension 4 index 2 Fano 3-folds constructed with our method, 32 admit a double cover \( X \) obtained via Type I unprojection from \( Z \) in codimension 3. There are between two and four possible distinguished deformation families of \( X \), associated to just as many formats (Tom and Jerry) of the antisymmetric matrix \( M \) defining the equations of \( Z \) [5]. Our construction is applicable to one or more of these deformation families: the number changes case-by-case, and the full extent of the result is outlined in Table 1. Accordingly, \( \tilde{X} \) has one or more distinguished deformation families. We give criteria to find deformation families for \( \tilde{X} \) in Sects. 4 and 5. In a similar fashion to the phenomena occurring in [5], some formats do not give rise to deformation families with the desirable features (terminality, for instance); we refer to this as to failure. Fundamentally, most information is included in the geometry of \( Z \).

On the other hand, two codimension 4 index 2 Fano 3-folds in the GRDB have a double cover \( X \) obtained via Type II\(_2\) unprojection. Performing the unprojection in this case is more complicated, and we refer to [16, 17, 24]. The double cover construction still works in this case: this is the content of Sect. 7.

The Hilbert series with GRDB ID # 40933, # 40663 are in the overlap with the results of [11, 21]. In particular, Ducat finds two different deformation families of # 40663, and we retrieve them here with our method (cf. [11, Section 3]).

A different approach to the construction of Fano 3-folds is given by Coughlan and Ducat in [10], where the authors employ rank 2 cluster algebras to find families of Fano 3-folds in codimension 4 and 5, also in index 2. In addition, in certain cases they are able to determine different deformation families associated to the same Hilbert series (what they call cluster formats). Cluster formats mimic the Tom and Jerry formats of [5]. Our work finds more deformation families that are not realised by cluster formats, which could possibly have higher Picard rank (see [9, 11]).

1.3 Details of the construction

The following table summarises our construction. In the first column we have the GRDB ID of each Hilbert series in \( H^2_{\text{BS}} \). In the second column there are the GRDB IDs of the Hilbert series corresponding to the index 1 double covers. These are index 1 Fano 3-folds \( X \) in codimension 4, whose construction of many deformation families has been achieved in [5] (whose extensive results are collected in [6]) and in [16, 18]. In particular, the former are in correspondence with the Tom and Jerry formats of the antisymmetric matrix defining \( Z \). The third column \#1 denotes the total number

\(^1\) The Hilbert series with ID #40367 and #40378 also appear in the GRDB in index 2 and codimension 4. However, the Fano 3-folds associated to these two Hilbert series cannot embed in codimension 4 in the weighted projective space suggested by the Graded Ring Database. These two Fano 3-folds may exist embedded in higher codimension. They also do not have a candidate double cover \( X \) in codimension 4 having Fano index 1. Indeed, neither our method nor the method in [10] construct them.
of distinguished deformation families constructed by Theorems 4.3 and 5.2, that is, the total number of Tom and Jerry deformation families that descend from index 1 to index 2. Columns four and five describe which are such Tom and Jerry deformation families; for the notation $T_k$, $J_{ij}$, $J_{ij} · k_l$ see Sect. 2 below and [5, 6]. The last column reports the formats inducing deformation families in index 1 that do not descend to index 2, following Sects. 5, 5.1 and Theorem 5.2. Note that when the same index 2 Fano 3-fold admits more than one format of the same type, we specify the centre for each format.

Lastly, we write “n/a” in the cases in which the Tom and Jerry construction is not applicable. The symbol #II represents the total number of distinguished deformation families when there is no Type I projection. We construct one in Sect. 7, but there could possibly be others.

### 2 Background

The lack of structure theorems for Fano 3-folds in codimension greater than 3 has forced the search for new approaches to produce their equations explicitly. Unprojections are a technique to retrieve equations for Fano 3-folds in codimension 4 from Fano 3-folds in lower codimension. They were firstly studied by Kustin and Miller [14], and later on by Reid and Papadakis [15, 19]. There are different kinds of unprojections: the most widely employed are called Type I unprojections.

Type I unprojections are initiated by the following type of data:

- A fixed projective plane $D := \mathbb{P}^2(a, b, c) \subset \mathbb{P}^6(a, b, c, d_1, \ldots, d_4)$ with coordinates $x_1, x_2, x_3, y_1, \ldots, y_4$ respectively, and defined by the ideal $I_D := \langle y_1, y_2, y_3, y_4 \rangle$.
- A family $Z$ of codimension 3 Fano 3-folds $Z \subset w\mathbb{P}^6$, each defined by the five maximal Pfaffians of a skew-symmetric $5 \times 5$ syzygy matrix $M$ whose entries $(a_{ij})$ have weights

$$
\begin{pmatrix}
    m_{12} & m_{13} & m_{14} & m_{15} \\
    m_{23} & m_{24} & m_{25} \\
    m_{34} & m_{35} \\
    m_{45}
\end{pmatrix}.
$$

In this context, two kinds of formats arise for $M$, based on conditions on its entries. These are the so-called Tom and Jerry formats. If $M$ is in one of these formats, then $D \subset Z \subset w\mathbb{P}^6$.

**Definition 2.1** ([5, Definition 2.2]) A $5 \times 5$ skew-symmetric matrix $M$ is in Tom$_k$ format if and only if each entry $a_{ij}$ for $i, j \neq k$ is in the ideal $I_D$.

It is in Jerry$_{kl}$ format if and only if $a_{ij} \in I_D$ for either $i$ or $j$ equals $k$ or $l$. If $M$ is in Jerry$_{kl}$ format, we call pivot entry the entry $a_{kl} \in I_D$.

Note that if a codimension 3 Fano 3-fold $Z$ is in either Tom or Jerry format, then it is nodal on $D$ and quasi-smooth elsewhere (see [5, Section 7 and Lemma 7.1]).
A Type I unprojection is a birational transformation of a codimension 3 Fano 3-fold \( Z \) in either Tom or Jerry format into a codimension 4 Fano 3-fold \( X \). For \( D := \mathbb{P}^2(a, b, c) \subset Z \), the unprojection map factorises as (see [5, Section3.2]):

- the partial resolution (small extraction) of the nodes on \( D \subset Z \), followed by
- the blow-down (divisorial contraction) to a cyclic quotient singularity \( p \sim \frac{1}{r}(a, b, c) \) inside a codimension 4 Fano 3-fold \( X \).

Practically, the unprojection contracts the divisor \( D := \mathbb{P}^2(a, b, c) \subset Z \) to the cyclic quotient singularity \( p \sim \frac{1}{r}(a, b, c) \in X \).

Recall the definition of Fano 3-fold of Tom type (respectively, of Jerry type).

**Definition 2.2** ([8, Definition 2.2]) Let \( X \) be a codimension 4 index 1 Fano 3-fold \( X \) listed in the table [6]. We say \( X \) is of Tom (Jerry) type if it is obtained as Type I unprojection of the codimension 3 pair \( Z \supset D \) in a Tom (Jerry) family [5, 19]. The image of \( D \subset Z \) in \( X \) is called Tom (Jerry) centre: it is a cyclic quotient singularity \( p \in X \). In the unprojection setup \( D \subset Z \), \( D \) is a complete intersection of four linear forms of weight \( d_1, \ldots, d_4 \). Such \( X \) of Tom (Jerry) type is said to be general if \( Z \supset D \) is general in its Tom (Jerry) family.

Note that \( X \) is quasi-smooth [5, Theorem 3.2] and Gorenstein [15, Theorems 5.6 and 5.14]; our arguments begin within this framework.

The formats denoted by \( \cdot_{ij} \) in Table 1 are those where: either, the matrix \( M \) can be manipulated by row/column operations so that the entry \( m_{ij} \) is 0; or, there is no polynomial that fits entry \( m_{ij} \) such that it satisfies the Tom and Jerry constraints.

### 3 Double covers

Let \( X \) be a quasi-smooth \( \mathbb{Q} \)-Fano 3-fold in codimension 4 anticanonically embedded in a weighted \( w \mathbb{P}^7 \) having Fano index 1 and such that \( h^0(X, -K_X) \geq 1 \). Thus we assume that \( w \mathbb{P}^7 = \mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r) \) with homogeneous coordinates \( x_1, x_2, x_3, y_1, \ldots, y_4, s \). Consider the \( \mathbb{Z}/2\mathbb{Z} \) action on \( w \mathbb{P}^7 \) that changes the sign of \( x_1 \) of weight \( wt(x_1) = 1 \).

\[
\gamma : (x_1, x_2, x_3, y_1, y_2, y_3, y_4, s) \mapsto (-x_1, x_2, x_3, y_1, y_2, y_3, y_4, s). \tag{3.1}
\]

First we describe the quotient of the ambient space of \( X \) by the \( \mathbb{Z}/2\mathbb{Z} \) action \( \gamma \).

**Lemma 3.1** The \( \mathbb{Z}/2\mathbb{Z} \) quotient of \( \mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r) \) via \( \gamma \) is the weighted projective space \( \mathbb{P}^7(2, b, c, d_1, \ldots, d_4, r) \) with coordinates \( \xi, x_2, x_3, y_1, \ldots, y_4, s \) respectively, where \( \xi := x_1^2 \).

**Proof** Consider the affine patches of

\[
\mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r) = \text{Proj} \mathbb{C}(x_1, x_2, x_3, y_1, \ldots, y_4, s).
\]

For instance,

\[
\mathcal{U}_{x_2} := \{x_2 \neq 0\} = \text{Spec} \mathbb{C}[x_1, x_3, y_1, \ldots, y_4, s]^{\mu_b} \cong \mathbb{A}^7/\mu_b
\]
where $\mu_b$ is the finite cyclic group of order $b$. The affine patches relative to the coordinates $x_3, y_1, \ldots, y_4, s$ are analogous. They are invariant under $\gamma$ if and only if $x_1$ appears with even powers. Such affine patches in which powers of $\xi$ appear exactly the affine patches of $\mathbb{P}^7(2, b, c, d_1, \ldots, d_4, r)$, with the new coordinate $\xi$. So,

$$\mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r)/\gamma = \text{Proj } \mathbb{C}(x_1, x_2, x_3, y_1, \ldots, y_4, s)$$

where $F$ is a Fano 3-fold and double covers... Page 7 of 20 [98]

Section 5, pp. 31–32]. Define the variety $V_X$ exactly the affine patches of $\mathbb{P}^3$. Moreover, since $X$ is quasismooth, then $-K_X$ is ample. Suppose $V_X$ empty. Then, $f_i$ is a Fano 3-fold: its anticanonical divisor $-K_X$ is a multiple of an ample divisor (see proof of Lemma 3.2), it has terminal singularities (see Lemma 3.4), it is $\mathbb{Q}$-factorial.

Let $\mathbb{P}_{\text{even}}$ be the weighted projective space defined by the vanishing of all the coordinates of $w \mathbb{P}^7$ with odd weight, except for $x_1$. In this section, let us assume that it is possible to realise $X \subset \mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r)$ as invariant under $\gamma$. To fix ideas, we call the defining equations of $X = \{f_i = 0\}_{i=1}^9$ and the equations of $\tilde{X} = \{f_i = 0\}_{i=1}^9$, where $f_i = f_i(\xi, \ldots, s)$. We can therefore draw the following conclusions.

**Lemma 3.2** The 3-fold $\tilde{X}$ has Fano index 2, and $-K_{\tilde{X}}$ is ample.

**Proof** Consider the quotient map $\varphi: X \to \tilde{X}$. Then, the anticanonical divisor of $X$ is given by $-K_X = -\varphi^*K_{\tilde{X}} - R$ where $R$ is the ramification divisor. In our case, $-K_X = \{x_1 = 0\} \sim \mathcal{O}(1)$, and the ramification divisor is $R = \{x_1 = 0\}$. Therefore, $-\varphi^*K_{\tilde{X}} = 2\{x_1 = 0\}$. This implies that $-K_{\tilde{X}} = \{\xi = 0\} \sim \mathcal{O}(2)$: thus, $\tilde{X}$ has index 2. Moreover, since $-K_X$ is ample, so is $-K_{\tilde{X}}$. \hfill $\square$

**Lemma 3.3** If $X$ is quasismooth, then $\tilde{X}$ is quasismooth.

**Proof** For $p \in w \mathbb{P}^7$ consider the Jacobian matrix $J_X$ of $X$ at the point $p$ as in [13, Section 5, pp. 31–32]. Define the variety $V_X$ as

$$V_X := \{\ p \in w \mathbb{P}^7: \text{rk}(J_X|_p) < \text{codim}(X)\}.$$ 

The affine cone of $V_X$ is the singular locus of the affine cone of $X$. By definition, if $V_X$ empty, then $X$ is quasismooth. Suppose $V_X$ empty. For each equation $f_i$ of $X$,

$$\frac{\partial f_i}{\partial x_1} = \frac{\partial f_i}{\partial \xi} \frac{\partial \xi}{\partial x_1} \quad \text{and} \quad \frac{\partial \xi}{\partial x_1} = 2x_1.$$ 

The difference between the Jacobian matrices $J_X$ and $J_{\tilde{X}}$ lies in the column relative to the derivative by $x_1$. Suppose $x_1 \neq 0$; then, the rank of $J_X$ is equal to the rank of $J_{\tilde{X}}$. \hfill $\square$
If instead $x_1 = 0$, certain entries of the $\partial/\partial x_1$ column of $J_X$ might vanish for $x_1 = 0$, whereas they would be just constant in $J_{\tilde{X}}$. Thus, for $x_1 = 0$ we have that $\text{rk } J_X \leq \text{rk } J_{\tilde{X}}$; so, $\tilde{X}$ is quasismooth if $X$ is. \hfill \Box

**Lemma 3.4** The 3-fold $\tilde{X}$ has terminal singularities.

**Proof** The fixed locus of the group action $\gamma$ is $\text{Fix}(\gamma) = \{x_1 = 0\} \cup \mathbb{P}_{\text{even}}$. We want to study the intersection $X \cap \text{Fix}(\gamma)$.

From [15, Theorems 5.6 and 5.14] we have that $X$ is Gorenstein. Moreover, since the coordinate point $P_{x_1} \in X$ is smooth then all cyclic quotient singularities of $X$ lie at coordinate points different from $P_{x_1}$. Thus, they all lie inside the locus $\{x_1 = 0\}$.

On the other hand, $X \cap \mathbb{P}_{\text{even}} = \emptyset$. Suppose that $X \cap \mathbb{P}_{\text{even}} \neq \emptyset$. If instead $\dim X \cap \mathbb{P}_{\text{even}} \geq 1$ we reach a contradiction, because $X$ is terminal. If instead $\dim X \cap \mathbb{P}_{\text{even}} = 0$, that is, $X \cap \mathbb{P}_{\text{even}}$ is a finite number of points, such points would be singularities of type $\frac{1}{r}(1, a, b)$ with even $r$, in accordance to the basket of singularities of $X$. However, none of the codimension 4 index 1 candidate double covers $X$ listed in the second row of Table 1 has a basket that contains cyclic quotient singularities of even order. Therefore, we conclude that $X \cap \mathbb{P}_{\text{even}} = \emptyset$. Thus, the quotient $\tilde{X}$ only contains the cyclic quotient singularities inherited from $X$. In particular, it does not contain non-isolated singularities. So, $\tilde{X}$ has terminal singularities. \hfill \Box

The above lemmas prove the following theorem, which will be crucial in the rest of this paper to prove a criterion for finding the deformation families of $\tilde{X}$.

**Theorem 3.5** If $X \subset \mathbb{P}^7(1, b, c, d_1, \ldots, d_4, r)$ has an invariant realisation under $\gamma$, then $\tilde{X}$ is a terminal $\mathbb{Q}$-factorial Fano 3-fold. Hence, it is in the Graded Ring Database.

In contrast, our method does not produce the index 2 Fano 3-folds in codimension 3 in the Graded Ring Database. Restrict the action $\gamma$ in (3.1) to the ambient space $w \mathbb{P}^6$ of $Z$. With a little abuse of notation, we still call the restriction $\gamma$.

**Proposition 3.6** The index 2 Fano 3-fold $\tilde{Z} := Z/\gamma$ in codimension 3 is not terminal.

**Proof** Consider the fixed locus $\text{Fix}(\gamma) = \{x_1 = 0\} \cup \mathbb{P}_{\text{even}}$ of $\gamma$, and its intersection with $Z$. The plane $D \cong \mathbb{P}(1, b, c)$ is contracted via unprojection to the terminal singularity of type $\frac{1}{r}(1, b, c)$ in $X$, where $c = r - b$ and $(b, r) = 1$. Thus, either $b$ or $c$ is even. In the quotient we have that $\tilde{D} := D/\gamma \cong \mathbb{P}(2, b, c)$. Therefore, $\tilde{Z}$ and $\mathbb{P}_{\text{even}}$ meet on $D$ along a line constituted by 1/2 singularities. \hfill \Box

The phenomenon described in Proposition 3.6 was already anticipated in [21, Sections 6.4, 6.5]. There are only two Hilbert series corresponding to terminal index 2 codimension 3 Fano 3-folds: one is smooth, and is constructed in [20] and [12]. The other one was constructed by Ducat in [11].

We would like to stress that $X$ is general in its Tom or Jerry formats, as in Sects. 4, 5 (cf. [8, Section 3], [5, Section 4]). Thus, provided the additional condition of invariance of $X$ under $\gamma$, $\tilde{X}$ is general.
4 Tom families

Let $X$ be a codimension 4 $\mathbb{Q}$-Fano 3-fold having at least one Type I centre, and suppose that $X$ is obtained by the corresponding Type I unprojection from a divisor $D$ inside a codimension 3 $\mathbb{Q}$-Fano 3-fold $Z$ in Tom format.

In this section we show a criterion to determine which Tom formats of $Z$ induce a double cover $X$ for the corresponding $\tilde{X}$ of Fano index 2. This produces deformation families for $\tilde{X}$. To this purpose, it is crucial to understand the geometry of $Z$, and of the nodes lying on the divisor $D \subset Z$. The following statement holds for both Tom and Jerry formats, and will also be used in Sect. 5 below.

**Lemma 4.1** If a general codimension 3 Fano 3-fold $Z$ in either Tom or Jerry format is invariant under the $\mathbb{Z}/2\mathbb{Z}$ action $\gamma$, then the nodes on the divisor $D \subset Z$ are not fixed by $\gamma$. In particular, they are pairwise-identified in the quotient $\tilde{Z}$, and $\# \{\text{nodes on } Z\} = 2 \cdot \# \{\text{nodes on } \tilde{Z}\}$.

**Proof** From [5, Theorem 3.2, Lemma 3.1], the nodes of $Z$ only lie on the divisor $D$. They are given by the $3 \times 3$ minors of the Jacobian matrix $J|_Z$ restricted to $D$, i.e. $\Lambda^3 J|_D = 0$. Their equations are not $\gamma$-invariant, because the entries of the $\partial/\partial x_1$ column of $J_Z$ are not $\gamma$-invariant. Thus, the nodes are not fixed by the action. In addition, for $\tilde{D} := D/\gamma$, the nodes on $\tilde{D} \subset \tilde{Z}$ are given by $\Lambda^3 \tilde{J}|_{\tilde{D}} = 0$. Such equations depend on $\xi$. As a consequence, the nodes on $D \subset Z$ are pairwise identified by $\gamma$ in the quotient and the number of nodes on $Z$ is twice the number of nodes on $\tilde{Z}$. $\Box$

There are either one or two Tom deformation families for each $X$ in index 1 [5, 6]. For each of the 32 index 1 Fano 3-folds $X$ in the GRDB that are candidates to be double covers of just as many $\tilde{X}$ in index 2, there is exactly one Tom deformation family that realises the double cover. The Tom families coming from codimension 3 Tom formats having odd number of nodes are automatically excluded by Lemma 4.1.

Conversely, it is also possible to prove the following lemma.

**Lemma 4.2** Let $Z$ be a codimension 3 terminal Fano 3-fold in Tom format having Fano index 1 such that the number of nodes on $D \subset Z$ is even. If a Type I unprojection $X$ of $Z$ has basket of singularities formed only by cyclic quotient singularities with odd order, then it is possible to realise $Z$ as $\gamma$-invariant in such a way that $\tilde{X}$ is terminal.

**Proof** In [5] the authors provide a Magma code (available at [4] → Downloads) that automatises the check for failure of Tom and Jerry formats in the sense of [5, Section 5]. The code essentially follows the steps of the proof of [5, Theorem 3.2], as illustrated in [5, Section 8]. We run the code on the codimension 3 Fano 3-folds $Z$ from which we obtain candidate double covers $X$ for $\tilde{X}$, imposing that $x_1$ appears only with even powers, i.e. replacing $x_1$ with $\xi$.

This shows that only the Tom formats having even number of nodes on $D \subset Z$ descend to Fano index 2 in codimension 4. Moreover, it is needed that $Z$ is in Tom format simply because these hypotheses are not enough when considering Jerry formats. Indeed, there are Jerry formats that satisfy the conditions on basket and nodes of Lemma 4.2 that do not descend to Fano index 2 (see for instance #11104 Jerry12
and the other families mentioned in Sect. 5.1). Once a successful format is found, any general choice of entries of the matrix \( M \) would still work (cf. [5, Section 8]).

The hypotheses on the basket of singularities of \( Z \) ensures that \( \tilde{X} \) is terminal (cf. proof of Lemma 3.4 and [23, Lemma 1.2 (3)]). \( \square \)

Recall from the proof of Lemma 3.4 that \( \text{Fix}(\gamma) = \{ x_1 = 0 \} \cup \mathbb{P} \text{even} \), and that \( X \cap \{ x_1 = 0 \} \) contains all the cyclic quotient singularities of \( X \). Thus, the basket of \( X \) is fixed by \( \gamma \). Therefore, it is important to observe that the following situations do not occur (cf. [23, Lemma 1.2 (3)]): a singularity of even order \( 2n \) on \( X \) descends to a singularity of order \( 4n \) on \( \tilde{X} \); the singularities of even order on \( X \) are pairwise identified on \( \tilde{X} \). This, together with Lemmas 4.1, 4.2, therefore proves the following theorem.

**Theorem 4.3** A general codimension 3 terminal Fano 3-fold \( Z \) of index 1 in Tom format can be realised as invariant under \( \gamma \) if and only if the number of nodes on \( D \subset Z \) is even and the basket of its Type I unprojection \( X \) exclusively contains cyclic quotient singularities of odd order.

As summarised in Table 1, each of the 32 codimension 4 index 1 Fano 3-folds that are candidates to be double covers, have one and only one Tom deformation family as described in Theorem 4.3. Except for the Hilbert series with GRDB ID #24078, the only Tom deformation family that admits the quotient by \( \gamma \) is the first Tom family as in [6]. Instead, the Fano #24078 only admits its second Tom family Tom\(_{5}\); the deformation family Tom\(_{5}\) of #24078 has Picard rank 2 by [3].

### 5 Jerry families

In this section we examine codimension 4 terminal Fano 3-folds \( X \) of Jerry type, and we determine which Jerry families give rise to a deformation family for \( \tilde{X} \) in index 2.

Suppose that \( Z \) is a terminal codimension 3 Fano 3-fold in Jerry format having at least one Type I centre, and call \( P \) the weight of the pivot entry (see Definition 2.1). The criterion to determine which Jerry deformation families of \( X \) descend to \( \tilde{X} \) depends on whether the following condition is satisfied or not.

**Condition 5.1** There exists a generator \( y_k \) of the ideal \( I_D \) whose weight is equal to \( P \).

What has been discussed for Tom families still holds here, but some care is needed with regards to some features of the Jerry formats. In particular, the structure of the proof of Theorem 4.3 can also be replicated in the Jerry case. However, there are some Jerry families, listed in Sect. 5.1, that satisfy the hypotheses of Theorem 4.3 but do not descend to Fano index 2. We therefore have the following theorem, proved in the next subsection.

**Theorem 5.2** A general codimension 3 terminal Fano 3-fold \( Z \) of index 1 in Jerry format (except for #24077 Jerry\(_{12}\)) can be realised as invariant under \( \gamma \) if and only if the following statements are simultaneously satisfied: the number of nodes on \( D \subset Z \) is even; the basket of its Type I unprojection \( X \) exclusively contains cyclic quotient singularities of odd order; Condition 5.1 is fulfilled; the weight \( P \) of the pivot entry is even.
5.1 Proof of Theorem 5.2 and failure of Jerry formats

The first check we do in order to exclude a Jerry format is to verify that it has even number of nodes and the basket of singularities of $Z$ exclusively contains singularities of odd order. Afterwards, we proceed with checking the properties of the pivot entry. The check on the nodes excludes the majority of the failing Jerry formats. Only a few are left, which present different reasons for failure.

The presence of a pivot entry with even weight is crucial for having reduced singularities on $D$. If $P$ is odd, of the three entries of $M$ that are not in $I_D$ two have odd weight and one has even weight for homogeneity of the maximal Pfaffians. Imposing that $x_1$ appears only with even powers constitutes a further constraint that is often not compatible with the Jerry format: in other words, it can happen that, for a certain entry of $M$ not in $I_D$, it is not possible to find a polynomial not in $I_D$ of the right degree; this therefore forces a 0 in that entry (see below). Furthermore, row/column operations on $M$ show that other entries can be made 0. Such configurations of entries of $M$ cause $Z$ to have non-reduced singularities (cf. [5, Section 5]).

We say that a certain format for a codimension 3 Fano 3-fold $Z$ fails if the deformation family of $Z$ associated to that format does not enjoy the following properties: terminal, nodal on $D$ with reduced singularities. The reasons for formats’ failures is extensively described in [5, Section 5]. In this paper, we have additional reasons for failure: not having even number of nodes on $D$, not satisfying Condition 5.1 (in the Jerry case), not having the pivot entry with even weight. Except for #24078, all Jerry formats for which $Z$ has even number of nodes on $D$ also satisfy Condition 5.1. Here we want to discuss the Jerry formats for which Condition 5.1 and the even nodes are satisfied, but do not have the pivot entry with even weight.

There are exactly seven formats that fall in this case. We run an explicit check on all of them using the reasons for failure in [5, Section 5]. These failing formats all give rise to non-reduced singularities on $D$. We specify the Type I centre in case of ambiguity, and we refer to [6] for the grading of the matrix $M$.

### 1401, 17 (1, 2, 5), Jerry24
The ambient space of $Z$ in Jerry24 format is $\mathbb{P}^6(1, 2, 3, 3, 4, 5, 5)$ with homogeneous coordinates $x_1, x_2, y_1, y_2, y_3, y_4, x_3$ respectively, and the ideal $I_D$ is generated by $y_1, y_2, y_3, y_4$. The entry $a_{13}$ has degree 3, and we want the coordinate $x_1$ to appear only with even powers in order to perform the double cover construction in Sect. 3. Thus, there is no way to fill entry $a_{13}$ with an homogeneous polynomial of degree 3 not in $I_D$, so $a_{13}$ must be 0. On the other hand, the entries $a_{14}, a_{23}$ must both be equal to $y_3$, as there is no other polynomial of degree 4 in $I_D$ in the available coordinates. Thus, the format Jerry24 fails for the reason in [5, Section 5.2(4)].

### 1401, Jerry45
For $Z$ in Jerry45 the ambient space is $\mathbb{P}^6(1, 2, 3, 3, 4, 5, 7)$ with homogeneous coordinates $x_1, x_2, x_3, y_1, y_2, y_3, y_4$ respectively. The entries $a_{12}, a_{13}$ of $M$ are both equal to $x_3$ because there is no other polynomial of degree 3 not in the ideal $I_D$ that is only in $x_1, x_2, x_3$. In addition, the entry $a_{34}$ can be made 0 via row/column operations on $M$. Thus, Jerry45 fails because of [5, Section 5.2(4)].
#1405, Jerry_24. Since there are only two generators of $I_D$ in degree 5, either $a_{25}$ or $a_{34}$ can be made 0 via row/column operations on $M$. Also $a_{14} = a_{23}$ because there is only one choice for a degree 4 polynomial in $I_D$. Hence, Jerry_24 fails due to [5, Section 5.2 (4)].

#4987, Jerry_12. The entry $a_{23}$ is 0 after row/column operations, and $a_{34} = 0$ because there is no degree 7 polynomial not in $I_D$. So, Jerry_12 fails due to [5, Section 5.2 (3)].

#11104, Jerry_12. The entry $a_{23}$ is 0 after row/column operations, and $a_{34} = 0$ because there is no degree 3 polynomial not in $I_D$. So, Jerry_12 fails due to [5, Section 5.2 (3)].

#11123, Jerry_{12;15}. The entry $a_{15}$ is already 0 in this format. Moreover, $a_{34} = a_{35}$ because there is only one choice for a degree 3 polynomial in $I_D$. Thus, Jerry_{12;15} fails due to [5, Section 5.2 (4)].

#11123, Jerry_{24;25}. The entry $a_{25}$ is already 0. In addition, $a_{15}$ can be made 0 by row/column operations. Thus, Jerry_{24;25} fails due to [5, Section 5.2 (3)].

The Jerry formats above that do not satisfy all the conditions of Theorem 5.2 simultaneously: for all of them it is possible to find a reason for failure as in [5, Section 5]. All the other Jerry formats satisfying the conditions of Theorem 5.2 simultaneously do not present any reason for failure. Running the Magma code implemented in [5] and mentioned in the proof of Lemma 4.2 confirms this. This concludes the proof of Theorem 5.2.

**Remark 5.3** The Fano 3-fold with GRDB ID #24078 of Jerry type is the only family that does not follow Theorem 5.2. Indeed, its codimension 3 source #24077 in Jerry_12 format has even number of nodes and satisfies Condition 5.1, but the weight of its pivot is 1, and its basket is $B_Z := \{(1, 1, 1)\}$. However, employing a code analogous to the ones in Sect. 8, it is still possible to produce explicit equations for its corresponding index 2 Fano #40933. The ambient space of #24077 is $\mathbb{P}^6(16, 2)$, and suppose that $\gamma$ changes sign to the coordinate $x_1$, so $\xi := x_1^2$. This Fano 3-fold is peculiar because it is the only index 1 double cover whose ambient space has enough coordinates of weight 1 to fill the entries of $M$ in in Jerry_12 format appropriately. Suppose $I_D := \langle x_3, x_4, x_5, x_6 \rangle$. The grading of $M$ is the following, and, for $q_1, q_2, q_3 \in I_D$ homogeneous polynomials of degree 1, 2, 2 respectively, it can be filled as

$$
\begin{pmatrix}
1 & 1 & 1 & 2 \\
1 & 1 & 2 \\
1 & 2 \\
2 \\
\end{pmatrix}
\rightsquigarrow
\begin{pmatrix}
x_3 & x_4 & x_5 & q_1 \\
x_6 & q_2 & q_3 \\
x_2 & y \\
\xi \\
\end{pmatrix}
= M.
$$

This shows that only $q_2$ could be made 0 after row/column operations on $M$, and that there is no further row/column elimination possible. Thus, #24077 does not fall into the failure description of [5, Section 5].
6 Proof of Main Theorem: type I unprojections

The proof of Theorem 1.1 relies on the possibility to realise $X$ in codimension 4 and index 1 as invariant under the action $\gamma$. The construction of $X$ starts from $Z$ in codimension 3. We gave necessary and sufficient conditions to realise $Z$ as invariant under $\gamma$ in Theorems 4.3 and 5.2 for both types of deformation families of $Z$. It is possible to verify that the invariance under $\gamma$ is maintained in $X$, that is, performing the unprojection does not affect the $\gamma$-invariance.

The equations of $X$ are of the form

$$X = \{ \text{Pf}_i(M) = s y_j - g_j = 0 : i = 0, \ldots, 4 \text{ and } j = 1, \ldots, 4 \}$$

(6.1)

for $g_j = g_j(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$ a homogeneous polynomial of degree $r + d_j$. We call $s y_j - g_j = 0$ the four unprojection equations (cf. [19, Definition 1.2]). In the following, we refer to [15, Section 5] and [8, Appendix] for notation and definitions.

First, let us consider the case of Tom formats and, to fix ideas, suppose that the matrix $M$ is in Tom 1 format (the other Tom formats are analogous). Then, $M$ looks like

$$M = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ a_{23} & a_{24} & a_{25} \\ a_{34} & a_{35} \\ a_{45} \end{pmatrix}$$

where the $a_{ij} \in I_D$ are polynomials of the form $a_{ij} := \sum_{k=1}^{4} \alpha_{ij}^k y_k$ for some polynomial coefficients $\alpha_{ij}^k$, and $p_j \notin I_D$. Assume that all the entries of $M$ are $\gamma$-invariant. Here we calculate $\text{Pf}_i$ by excluding the $(i + 1)$-th row and the $(i + 1)$-th column of $M$ for $i \in \{0, 1, 2, 3, 4\}$. For $i \neq 0$, we can define the matrix $Q = (\text{Pf}_i(N_j))_{i,j=1..4}$ where

$$Q_i = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ \alpha_{23}^i & \alpha_{24}^i & \alpha_{25}^i \\ \alpha_{34}^i & \alpha_{35}^i \\ \alpha_{45}^i \end{pmatrix}$$

and $\alpha_{kl}^i$ is the coefficient of $y_i$ in $a_{kl}$. The polynomial $g_1$, and analogously the other $g_2, g_3, g_4$, is defined as

$$g_1 = \frac{1}{p_1} \det \begin{pmatrix} \text{Pf}_2(N_2) & \text{Pf}_2(N_3) & \text{Pf}_2(N_4) \\ \text{Pf}_3(N_2) & \text{Pf}_3(N_3) & \text{Pf}_3(N_4) \\ \text{Pf}_4(N_2) & \text{Pf}_4(N_3) & \text{Pf}_4(N_4) \end{pmatrix}$$

and the determinant on the right-hand side is divisible by $p_1$ (cf. [15, Lemma 5.3]).

Now, if $Z$ in Tom 1 format is $\gamma$ invariant then all the entries of $M$ are $\gamma$-invariant. Therefore, the same holds for the matrices of coefficients $N_i$. Also, it is clear that $\text{Pf}_i(N_j)$ and the determinant in (6.2) are all sums and multiplications of $\gamma$-invariant
polynomials. Thus, \(X\) is \(\gamma\)-invariant, and it is possible to perform the \(\mathbb{Z}/2\mathbb{Z}\) quotient of \(X\) to obtain \(\tilde{X}\).

By Lemmas 3.2, 3.4 and Theorem 3.5, we have just constructed a Tom type deformation family of a codimension 4 Fano 3-fold having Fano index 2.

For \(M\) of Jerry format, we proceed analogously to the Tom case by retracing the construction of the unprojection equations in [15, Section 5.7]. In this case, to fix ideas suppose that \(M\) is in Jerry 12 format, so it is of the form

\[
M = \begin{pmatrix}
c & a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
p_1 & p_2 \\
p_3 
\end{pmatrix}
\]

where \(a_i = \sum_{k=1}^{4} \alpha_i^k y_k, b_i = \sum_{k=1}^{4} \beta_i^k y_k, \) and \(c = \sum_{k=1}^{4} \gamma^k y_k\) are all polynomials in \(I_D\), and \(p_i \notin I_D\) for \(i = 1, 2, 3\) (see [15, (5.9), Section 5.7]). Again, assume that \(Z\) is \(\gamma\)-invariant, so each of the entries of \(M\) are too.

Define the \(3 \times 4\) matrix \(Q\) to have entries

\[
Q_{1k} := \beta_1^k p_3 - \beta_2^k p_2 + \beta_3^k p_1, \\
Q_{2k} := \alpha_1^k p_3 - \alpha_2^k p_2 + \alpha_3^k p_1, \\
Q_{3k} := \gamma^k p_3 - \gamma^k p_2 + \gamma^k p_1
\]

and define the polynomials \(h_i\) as the determinant of the \(3 \times 3\) minor of \(Q\) after removing the \(i\)-th column. Observe that each \(h_i\) is \(\gamma\)-invariant because it is a determinant of a \(\gamma\)-invariant matrix.

By [15, Lemma 5.11] we have that for each \(h_i\) there exist two polynomials \(K_i, L_i\) such that \(h_i = p_3 K_i + (a_2 p_2 - a_3 p_1) L_i\). These two polynomials are defined implicitly by considering the matrix \(Q\) restricted to \(p_3 = 0\). The polynomial \(L_i\) is the determinant of the rightmost matrix appearing in the proof of [15, Lemma 5.11], and it is \(\gamma\)-invariant because it is a determinant of a \(\gamma\)-invariant matrix; the polynomial \(K_i\) is the remainder. Since \((a_2 p_2 - a_3 p_1), p_3, L_i, h_i\) are \(\gamma\)-invariant, so is \(K_i\).

By [15, Equation (5.12) and Lemma 5.12] we have that the polynomials \(g_i = K_i + a_1 L_i\) are there right-hand side of the unprojection equations in (6.1); in particular, they are \(\gamma\)-invariant. Thus, as before, \(X\) is \(\gamma\)-invariant, and we can obtain \(\tilde{X}\) taking the quotient of \(X\) by the action \(\gamma\). Such \(\tilde{X}\) is in GRDB by Lemmas 3.2, 3.4 and Theorem 3.5.

The discussion about the two double cover Fano 3-folds arising from Type II2 unprojections is contained in the next Sect. 7. In this case we explicitly build the \(\gamma\)-invariant equations of \(\tilde{X}\), and the conclusion follows analogously as the Tom and Jerry cases.
7 Proof of Main Theorem: type II2 unprojections

The two Hilbert series #39569 and #39607 have candidate double covers, #512 and #872 respectively, that do not have any Type I centre. Indeed, #512 and #872 are obtained as Type II2 unprojections of the corresponding Fano hypersurfaces $Y$ [24, Section 5.2]. The Tom and Jerry construction is not appropriate in this case, but different deformation families can still arise. For the theory of Type II unprojections we follow ideas contained in [16–19, 24] and in the paper in preparation [25]. In particular, in this section we follow closely [16] for the general strategy, and [24] for part of the explicit approach. The calculations presented in this section are supported by computer algebra and inspired by the above works; as for now, there is no general theory of Type II

7.1 #39607

The index 1 codimension 4 Fano 3-fold $X \subset \mathbb{P}^7(1, 3^2, 4, 5^2, 6, 7)$ is a candidate to be a double cover of #39607 of index 2. Its basket is

$$B_X = \left\{ 5 \times 1^3(1, 1, 2), 1^5(1, 1, 4) \right\},$$

and $1^5(1, 1, 4)$ is its Type II centre. It comes from the Type II2 unprojection of the hypersurface $Y_{15} \subset \mathbb{P}^4(1, 3^2, 4, 5)$ having basket $B_Y = \left\{ 5 \times 3^2(1, 1, 2), 1^4(1, 1, 3) \right\}$. Following [24], $X$ is obtained by unprojecting the divisor $D$ defined as the image of $\mathbb{P}^2(1, 1, 4)_{a,b,c}$ inside $Y$ via the embedding

$$\phi: \mathbb{P}^2(1, 1, 4) \longrightarrow \mathbb{P}^4(1, 3, 3, 4, 5)$$

$$(a, b, c) \longmapsto (a, b^3, b^3, c, bc).$$

In this way, $D := \text{Im}(\phi)$ is a divisor inside the general hypersurface $Y_{15} \subset \mathbb{P}^4(1, 3^2, 4, 5)$ with homogeneous coordinates $x, u, z, y, v$. In addition, $\text{Im}(\phi)$ can be written as the $3 \times 3$ minors of the following $3 \times 6$ matrix ([24, Example 5.2.2])

$$N = \begin{pmatrix}
  u & v & 0 & 0 & x^2z - yz \\
  x^2 - y & u & v & 0 & 0 \\
  0 & 0 & x^2 - y & u & v 
\end{pmatrix}.$$

The equation defining $Y_{15}$ is an homogeneous polynomial of degree 15 of the form

$$F = \sum_{i=1}^{20} A_i N_i \quad (7.1)$$

where $N_i$ is the $i$-th $3 \times 3$ minor of $N$, and $A_i$ is a general homogeneous polynomial of degree $15 - \deg(N_i)$. Note that here both $N_{15}$ and $N_{20}$ have already degree 15, so $A_{15}$ and $A_{20}$ are both non-zero constants. Since $x$ appears already as a square in $N$,
we need that \( x \) only appears with even powers in the polynomials \( A_i \); in this way, \( Y_{15} \) is invariant under \( \gamma \).

We follow closely [16, 17] and we adapt their general construction to our specific case of #872. For Type II_2 unprojections, there will be three new unprojection variables: indeed, the codimension increases by 3. We call them \( s_1, s_2, s_3 \), and they have weights 5, 6, 7 respectively. Finding the equations for \( X \) consists in finding the linear and quadratic relations between the unprojection variables \( s_1, s_2, s_3 \) (in the spirit of [16, Subsection 2.2 and Lemma 2.5]).

Consider the polynomial ring \( R \) generated by the coordinates of \( \mathbb{P}^4(1, 3^2, 4, 5) \), the monomials \( p_1 := -a^2, p_2 := c \), and the polynomials \( A_i \) (cf. [16, Definition of \( O_{\amb} \))]. The equations of \( X \) are retrieved by looking at the resolution of \( R/(F = 0) \), and they consist in sums and products of the coordinates of \( \mathbb{P}^4(1, 3^2, 4, 5) \), the monomials \( p_1 := -a^2, p_2 := c \), and the polynomials \( A_i \). Thus, \( x \) still retains its property of appearing only with even powers, and \( X \) is defined by nine equations invariant under \( \gamma \). So, we can quotient \( X \) by the action \( \gamma \), and obtain \( \tilde{X} \) analogously to Sect. 3. This constructs #39607.

Note that it is possible to check with a Macaulay2 routine that the ring extension \( R[ s_1, s_2, s_3 ] / I \) is Gorenstein as follows. First insert the input data: field, polynomial ring \( R \) (including the general polynomials \( A_i \) and the unprojection variables \( s_1, s_2, s_3 \)), matrix \( N \). Then, define \( J \) to be the ideal generated by the \( 3 \times 3 \) minors of \( N \). The ideal of the hypersurface \( Y_{15} \) is

\[
J = \text{minors}(3, N) \\
I = \text{ideal}(F = A0*J_(0) + A1*J_(1) + A2*J_(2) + A3*J_(3) + A4*J_(4) \\
+ A5*J_(5) + A6*J_(6) + A7*J_(7) + A8*J_(8) + A9*J_(9) + A10*J_(10) \\
+ A11*J_(11) + A12*J_(12) + A13*J_(13) + J_(14) + A15*J_(15) \\
+ A16*J_(16) + A17*J_(17) + A18*J_(18) + J_(19) )
\]

By [16, Definition 2.2] we have that the unprojection of the ideal \( J \) of \( R \) is \( O_{Y_{15}}[ J^{-1} ] \) where \( O_{Y_{15}} := R/I \) and \( J^{-1} \) is defined as the elements \( f \) in the field of fractions of \( O_{Y_{15}} \) such that \( fJ \subset O_{Y_{15}} \). Moreover, \( O_{Y_{15}}[ J^{-1} ] \cong \text{Hom}_{O_{Y_{15}}}(J, O_{Y_{15}}) \) (see [16, Equation (2.3), Proposition 2.6, and Theorem 2.15]).

To construct the unprojection ideal \( \text{unprI} \) and to check that it is Gorenstein we then look at

\[
M = \text{Hom}(J, R^1/I)
\]

and we therefore give a presentation of \( O_{Y_{15}}[ J^{-1} ] \) following the ideas in [16, Section 2.2]. From [16, Proposition 2.16] we have that

\[
O_{Y_{15}}[ J^{-1} ] \cong R[ s_1, s_2, s_3 ] / I + (\text{linear and quadratic relations between } s_1, s_2, s_3 ).
\]

The following command finds exactly the linear relations.

\[
\text{tempI} = \text{ideal}(\text{matrix}([1, s1, s2, s3]) \ast (\text{presentation} M)) + \text{ideal}(F)
\]

The the colon ideal finds exactly the missing quadratic relations.

\[
\text{unprI} = \text{tempI} : \text{ideal}(y)
\]

To check that \( \text{unprI} \) is Gorenstein, it is enough to look at the resolution of \( O_{Y_{15}}[ J^{-1} ] \) and check its Betti numbers. We do
and verify that its output gives a palindromic list of dimensions at the various stages of the resolution (part of the output has been omitted).

\[
\begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
\text{total:} & 1 & 9 & 16 & 9 & 1
\end{array}
\]

The routine is analogous for the example below in Sect. 7.2.

We are not aware of any work in the literature regarding the question of whether there are multiple deformation families induced by Type II 2 unprojections, and if there are, how many. This procedure constructs at least one.

7.2 # 39569

We proceed analogously for #39569. Its double cover candidate is \( \#512 X \subset \mathbb{P}^7(1, 3, 5, 6, 7^2, 8, 9) \), whose basket is \( \mathcal{B}_X := \{3 \times \frac{1}{3}(1, 1, 2), \frac{1}{2}(1, 2, 3), \frac{1}{7}(1, 1, 6)\} \), and \( \frac{1}{7}(1, 1, 6) \) is one of its Type II 2 centres. It is induced by the Type II 2 unprojection of the hypersurface \( \#508 Y_{21} \subset \mathbb{P}^4(1, 3, 5, 6, 7) \) with basket \( \mathcal{B}_Y := \{3 \times \frac{1}{3}(1, 1, 2), \frac{1}{7}(1, 2, 3), \frac{1}{6}(1, 1, 5)\} \). The divisor \( D \) is defined as the image of \( T \) via the embedding

\[
\phi: \mathbb{P}^2(1, 1, 6) \longrightarrow \mathbb{P}^4(1, 3, 5, 6, 7) \\
(a, b, c) \longmapsto (a, b^2, b^5, c, bc).
\]

The divisor \( D := \text{Im}(\phi) \) is given by the vanishing of the \( 3 \times 3 \) minors of the \( 3 \times 6 \) matrix

\[
N = \begin{pmatrix}
u & -z^2 & 0 & 0 & -yz \\
u & 0 & -y & 0 & -z^2 \\
-z & 0 & 0 & -y & u & v
\end{pmatrix}.
\]

It sits inside the general hypersurface \( Y_{21} \subset \mathbb{P}^4(1, 3, 5, 6, 7) \) with homogeneous coordinates \( x, z, u, y, v \) respectively. The equation of \( Y_{21} \) is again given by (7.1), and we can impose that the general polynomials \( A_i \) contain the variable \( x \) only with even powers. As before, \( A_{15} \) and \( A_{20} \) are constants. The three new unprojection variables are \( s_1, s_2, s_3 \) with weights 7, 8, 9 respectively. The unprojection of \( Y \) and the quotient of \( X \) by \( \gamma \) constructs \( \tilde{X} \# 39569 \).

8 Examples

Here we give two examples of our construction, one of a Tom family and one of a Jerry family. We explicitly construct the deformation families relative to the Hilbert series with GRDB ID #39660. This is \( \tilde{X} \subset \mathbb{P}^7(2, 2, 3, 5, 5, 7, 12, 17) \), whose basket of singularities is \( \mathcal{B}_{\tilde{X}} := \{1 \times \frac{1}{17}(2, 5, 12)\} \). The codimension 4 Fano 3-fold \( X \subset \mathbb{P}^7(1, 2, 3, 5, 5, 7, 12, 17) \) in index 1 with Hilbert series \#1158 is the candidate to be the double cover for \#39660. The coordinates of the ambient space of \( X \)
are \( x, y, z, u_1, u_2, v, w, s \), and its basket of singularities is \( \mathcal{B}_X := \{ \frac{1}{17} (1, 5, 12) \} \). In this case, the grading of \( M \) is

\[
\text{wts } M = (m_{ij}) = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 5 & 7 & 9 \\ 8 & 10 \\ 12 \end{pmatrix}.
\]

The family of the codimension 3 Fano 3-fold \( Z \) with GRDB ID # 1157 is composed of 3-folds sitting inside \( \mathbb{P}^6(1, 2, 3, 5, 7, 12) \) and whose equations are the five maximal pfaffians of \( M \) with the above grading. The divisor \( D \) is \( \mathbb{P}^2(2, 5, 12) \). The Tom and Jerry formats of \( M \) that admit an embedding \( D \subset Z \) are \( \text{Tom}_5 \) and \( \text{Jerry}_{12} \) each with 4 and 6 nodes on \( D \) respectively. In addition, the pivot entry of the \( \text{Jerry}_{12} \) format is \( a_{12} \), which has weight 2. Therefore, by Lemmas 4.1, 4.2 we have two possible deformation families for \( \tilde{X} \), one coming from the \( \text{Tom}_5 \) format, the other from the \( \text{Jerry}_{12} \) format. It remains to exhibit the equations for these two families. These calculations can be checked using the \( t\j \) package for \texttt{Magma} that can be found on the Graded Ring Database website [4]. Instructions on how to fill the matrix \( M \) can be found in [5, Section 4] and [8, Section 3.2].

### 8.1 Finding equations for # 39660 of \( \text{Tom}_5 \) type

Here we want to build \( \tilde{Z} \) produced from \( Z \) in \( \text{Tom}_5 \) format. We define the ambient space \( \mathbb{P}^6(2, 2, 3, 5, 7, 12) \) with coordinates as above, where \( x \) has been replaced by \( \xi \) of weight 2. The divisor \( D \cong \mathbb{P}^2(2, 5, 12) \) is defined by the vanishing of the coordinates \( y, z, u_2, v \). Here the matrix \( M \) in \( \text{Tom}_5 \) format is filled following [5, Section 6.2].

As a rule of thumb, we place homogeneous coordinates in the entries with matching degrees where possible according to the format. The rest of the entries are occupied by general polynomials in the given degrees, still maintaining the format. The matrix can be tidied up by row/column operations. For instance, we can fill the entries of \( M \) as follows (cf. [8, Sections 3.2 and 5.1] on how to populate \( M \))

\[
M = (a_{ij}) = \begin{pmatrix} y & z & u_2 & u_2 & -v \\ -u_2 & v & -z^3 + \xi^2 u_1 \\ \xi^3 y + y^4 & -\xi^5 + u_1^2 \\ w \end{pmatrix}.
\]

The equations of the 3-fold \( \tilde{Z} \) are the maximal Pfaffians of \( M \), and it is possible to check with a \texttt{Magma} routine that the number of nodes on \( D \) is 2, in accordance to Lemma 4.1. The next step is to perform the Type I unprojection from \( D \subset \tilde{Z} \) as in [19]. The nine equations of \( \tilde{X} \) are

\[
\xi^3 y^2 + y^5 - u_2^2 - zv = 0,
\]
\[
\xi^5 y - z^4 + \xi^2 z u_1 - y u_1^2 - u_2 v = 0,
\]
\[
z^3 u_2 - \xi^2 u_1 u_2 - v^2 + yw = 0,
\]

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\[\xi^5 u_2 - \xi^3 y v - y^4 v - u_1^2 u_2 + z w = 0,\]
\[\xi^3 y - y^4 z - \xi^2 y u_1 - \xi^5 v - z^4 u_1 + \xi^7 z u_1^2 - y u_1^3 + u_1^2 v - u_1 u_2 v + u_2 w = 0,\]
\[\xi^5 z_3 - \xi^7 u_1 - z^3 u_1^2 + \xi^2 u_1 - v w + y s = 0,\]
\[-\xi^{10} + 2 \xi^5 u_1^2 + \xi^3 v + y^3 v^2 - u_1^4 - z s = 0,\]
\[\xi^3 z^3 v + y^3 z^3 v - \xi^5 u_1 v - \xi^2 y^3 u_1 v - \xi^5 w + u_1^2 w + u_2 s = 0,\]
\[-\xi^8 y z^2 - \xi^5 y^4 z^2 + \xi^5 z^3 u_1 + \xi^2 y^3 z^3 u_1 - \xi^7 u_1^2 - \xi^4 y^3 u_1^2 + \xi^3 y z^2 u_1^2 + y^3 z^2 u_2 v + y^3 z^2 u_2 v + w^2 - vs = 0.\]

### 8.2 Finding equations for # 39660 of Jerry\(_{12}\) type

In a similar fashion to the Tom case above, here the matrix \(M\) in Jerry\(_{12}\) is defined as

\[M = (a_{ij}) = \begin{pmatrix} y & z & u_2 & v \\ -u_2 & v & \xi^3 z + z^3 & \xi^4 + y^4 \xi y^4 - u_1^2 \\ \xi^4 y + y^5 - u_2^2 - z v = 0, \\ \xi^3 y z^2 - z^4 - y u_1^2 - u_2 v = 0, \\ \xi^3 z u_2 + z^3 u_2 - v^2 - y w = 0, \\ \xi^4 u_2 - \xi^4 v - y^4 v - u_1^2 u_2 - z w = 0, \\ \xi^7 z + \xi^3 y^4 z + \xi^4 z^3 + y^4 z^3 - \xi y^4 v + u_1^3 v - u_2 w = 0, \\ \xi^7 u_2 + \xi^4 z^2 u_2 - \xi^3 u_1^2 - z^2 u_1^2 + v + w + y s = 0, \\ \xi^4 u_1^2 - \xi^3 z^2 u_1^2 + y^3 v^2 - \xi^4 w - u_1^4 - z s = 0, \\ \xi^{11} + \xi^3 y^4 + \xi^8 z^2 + \xi^4 y^4 z^2 + \xi^3 y^3 z v + y^3 z^3 v - \xi y^3 v^2 - u_1^2 w + u_2 s = 0, \\ \xi^7 u_1^2 + \xi^3 y^4 u_1^2 - \xi^4 y^3 u_1^2 + \xi^4 z^2 u_1^2 + y^4 z^2 u_1^2 - \xi y^3 z^2 u_2^2 + \xi^3 y^3 z u_2 v + y^3 z^2 u_2 v + w^2 - vs = 0. \end{pmatrix}\]

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