SHOT NOISE OF WEAK COTUNNELING CURRENT:  
NON-EQUILIBRIUM FLUCTUATION-DISSIPATION THEOREM

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We study the noise of the cotunneling current through one or several tunnel-coupled quantum dots in the Coulomb blockade regime. We consider the regime of weak (elastic and inelastic) cotunneling, and prove a non-equilibrium fluctuation-dissipation theorem which leads to a universal expression for the noise-to-current ratio (Fano factor).

1 Introduction

In recent years, there has been great interest in the shot noise in mesoscopic systems, because it contains additional information about correlations, which is not contained, e.g., in the linear response conductance. The shot noise is characterized by the Fano factor $F = S/eI$, the dimensionless ratio of the zero-frequency noise power $S$ to the average current $I$. While it assumes the Poissonian value $F = 1$ in the absence of correlations, it becomes suppressed or enhanced when correlations set in as e.g. imposed by the Pauli principle or due to interaction effects. In the present paper we study the shot noise of the cotunneling current. We consider the transport through a quantum-dot system (QDS) in the Coulomb blockade (CB) regime, in which the quantization of charge on the QDS leads to a suppression of the sequential tunneling current except under certain resonant conditions. We consider the transport away from these resonances and study the next-order contribution to the current (see Fig. 1). One might expect that the cotunneling, being a two-particle process, leads to strong correlations in the shot noise and to the deviation of the Fano factor from its Poissonian value $F = 1$. As it has been found recently, this is indeed the case for the regime of strong cotunneling, i.e. when the cotunneling rate $I/e$ is large compared to the intrinsic relaxation rate $w_{in}$ of the QDS to its equilibrium state due to the coupling to the environment, $I/e \gg w_{in}$. However we find here, that in the weak cotunneling regime, $I/e \ll w_{in}$, the zero-frequency noise takes on its Poissonian value, as first obtained for a special case. This result is generalized here, and we find a universal relation between noise and current for the QDS in the first nonvanishing order in the tunneling perturbation. Because of the universal character of this result (Eq. [2]) we call it the nonequilibrium fluctuation-dissipation theorem (FDT) in analogy with linear response theory.

2 Model system

In general, the QDS can contain several dots, which can be coupled by tunnel junctions, the double dot (DD) being a particular example. The QDS is assumed to be weakly coupled to external metallic leads which are kept at equilibrium with their associated reservoirs at the chemical potentials $\mu_l$, $l = 1, 2$, where the currents $I_l$ can be measured and the average current $I$ through the QDS is defined by Eq. [5]. Using a standard tunneling Hamiltonian approach, we write

$$ H = H_0 + V, \quad H_0 = H_L + H_S + H_{int}, $$

$$ H_L = \sum_{l=1,2} \sum_k \epsilon_k c_{lk}^\dagger c_{lk}, \quad H_S = \sum_p \epsilon_p d_p^\dagger d_p, $$

\( * \) The majority of papers on the noise of quantum dots consider the sequential tunneling regime, where a classical description (“orthodox” theory) is applicable. In this regime the noise is generally suppressed below its full Poissonian value $F = 1$. This suppression can be interpreted as being a result of the natural correlations imposed by charge conservation.
to be complemented by some other dissipative counterparts, such as differential conductances

The correlators

In this representation, the time dependence of all operators is governed by the unperturbed Hamiltonian

by replacing \( \rho \rightarrow (0) \) with one added electron \( (N + 1 \text{ electrons in total}) \) are indicated by their energies \( E_1, E_2, \ldots \). In the cotunneling regime there is a finite energy cost \( \Delta_\pm (l, N) > 0 \) for the electron tunneling from the Fermi level of the lead \( l \) to the QDS (+) and vice versa (−), so that only processes of second order in \( V \) (visualized by two arrows) are allowed.

\[
V = \sum_{l=1,2} (D_l + D_l^\dagger), \quad D_l = \sum_{k,p} T_{lk,p} c_{lk}^\dagger d_{lp},
\]

where the terms \( H_L \) and \( H_S \) describe the leads and QDS, respectively (with \( k \) and \( p \) from a complete set of quantum numbers), and tunneling between leads and QDS is described by the perturbation \( V \). The interaction term \( H_{\text{int}} \) does not need to be specified for our proof of the universality of noise. The \( N \)-electron QDS is in the cotunneling regime where there is a finite energy cost \( \Delta_\pm (l, N) > 0 \) for the electron tunneling from the Fermi level of the lead \( l \) to the QDS (+) and vice versa (−), so that only processes of second order in \( V \) are allowed.

To describe the transport through the QDS we apply standard methods and adiabatically switch on the perturbation \( V \) in the distant past, \( t = t_0 \to -\infty \). The perturbed state of the system is described by the time-dependent density matrix \( \rho(t) = e^{-iH(t-t_0)} \rho_0 e^{iH(t-t_0)} \), with \( \rho_0 \) being the grand canonical density matrix of the unperturbed system, \( \rho_0 = Z^{-1} e^{-K/kB T} \), where we set \( K = H_0 - \sum_l \mu_l N_l \). Because of tunneling the total number of electrons in each lead \( N_l = \sum_k \delta_{lk} c_{lk}^\dagger c_{lk} \) is no longer conserved. For the outgoing currents \( \hat{I}_l = e \hat{N}_l \) we have

\[
\hat{I}_l = ei [V, \hat{N}_l] = ei (D_l^\dagger - D_l).
\]

The observables of interest are the average current \( \bar{I} \equiv I_2 = -I_1 \) through the QDS, and the spectral density of the noise \( S_{I\nu}(\omega) = \int dt S_{I\nu}(t) \exp(i \omega t) \),

\[
\bar{I}_l = \text{Tr} \rho(0) \hat{I}_l, \quad S_{I\nu}(t) = \text{Re} \text{Tr} \rho(0) \delta I_l(t) \delta I\nu(0),
\]

where \( \delta I_l = \hat{I}_l - \bar{I}_l \). Below we will use the interaction representation where Eq. 3 can be rewritten by replacing \( \rho(0) \to \rho_0 \) and \( \hat{I}_l(t) \to U^\dagger(t) \hat{I}_l(t) U(t) \), with

\[
U(t) = T \exp \left[ -i \int_{-\infty}^t dt' V(t') \right].
\]

In this representation, the time dependence of all operators is governed by the unperturbed Hamiltonian \( H_0 \).

3 Non-equilibrium fluctuation-dissipation theorem

We note that the two currents \( \hat{I}_l \) are not independent, because \( [\hat{I}_1, \hat{I}_2] \neq 0 \), and thus all correlators \( S_{I\nu} \) are nontrivial. The charge accumulation on the QDS for a time of order \( \Delta_\pm^{-1} \) leads to an additional contribution to the noise at finite frequency \( \omega \). Thus, we expect that for \( \omega \sim \Delta_\pm \) the correlators \( S_{I\nu} \) cannot be expressed through the steady-state current \( I \) only and thus \( I \) has to be complemented by some other dissipative counterparts, such as differential conductances
$G_{12}$. On the other hand, at low enough frequency, $\omega \ll \Delta \pm$, the charge conservation on the QDS requires $\delta I_s = (\delta I_2 - \delta I_1)/2 \approx 0$. Below we concentrate on the limit of low frequency and neglect contributions of order of $\omega/\Delta \pm$ to the noise power. In the Appendix we prove that $S_{ss} \sim (\omega/\Delta \pm)^2$ (see Eq. 2), and this allows us to redefine the current and the noise power as $I \equiv I_d = (I_2 - I_1)/2$ and $S(\omega) \equiv S_{dd}(\omega)$. In addition we require that the QDS is in the cotunneling regime, i.e. the temperature is low enough, $k_BT \ll \Delta \pm$, although the bias $\Delta \mu$ is arbitrary as soon as the sequential tunneling to the dot is forbidden, $\Delta \pm > 0$. In this limit the current through a QDS arises due to the direct hopping of an electron from one lead to another (through a virtual state on the dot) with an amplitude which depends on the energy cost $\Delta \pm$ of a virtual state. Although this process can change the state of the QDS (inelastic cotunneling), the fast energy relaxation in the weak cotunneling regime, $w_m \gg I/e$, immediately returns it to the equilibrium state (for the opposite case, see Refs. 6). This allows us to apply a perturbation expansion with respect to tunneling $V$ and to keep only first nonvanishing contributions, which we do next.

It is convenient to introduce the notation $\tilde{D}_l(t) \equiv \int_{-\infty}^{t} dt' D_l(t')$. We notice that all relevant matrix elements, $\langle N| D_l(t)|N+1 \rangle \sim e^{-i\Delta \pm t}$, $\langle N-1| D_l(t)|N \rangle \sim e^{i\Delta \pm t}$, are fast oscillating functions of time. Thus, under the above conditions we can write $\tilde{D}_l(\infty) = 0$, and even more general, $\int_{-\infty}^{+\infty} dt \tilde{D}_l(t) e^{\pm i\omega t} = 0$ (note that we have assumed earlier that $\omega \ll \Delta \pm$). Using these equalities and the cyclic property of the trace we obtain the following results (for details of the derivation, see Appendix),

$$I = e \int_{-\infty}^{\infty} dt \langle \{A^\dagger(t), A(0)\} \rangle, \quad A = D_2 \tilde{D}_1^\dagger + \tilde{D}_1 D_2,$$

$$S(\omega) = e^2 \int_{-\infty}^{\infty} dt \cos(\omega t) \langle \{A^\dagger(t), A(0)\} \rangle,$$

where we have dropped a small contribution of order $\omega/\Delta \pm$ and used the notation $\langle \ldots \rangle = \text{Tr} \rho_0(\ldots)$.

Next we apply the spectral decomposition to the correlators Eqs. 7 and 8, a similar procedure to that which also leads to the equilibrium fluctuation-dissipation theorem. The crucial observation is that $[H_0, N_l] = 0$, $l = 1, 2$. Therefore, we are allowed to use for our spectral decomposition the basis $|n\rangle = |E_n, N_1, N_2\rangle$ of eigenstates of the operator $K = H_0 - \sum_{i} \mu_i N_i$, which also diagonalizes the grand-canonical density matrix $\rho_0$, $\rho_n = \langle n|\rho_0|n\rangle = Z^{-1} \exp[-E_n/k_BT]$. We introduce the spectral function,

$$A(\omega) = 2\pi \sum_{n,m} (\rho_n + \rho_m) |\langle m|A|n\rangle|^2 \delta(\omega + E_n - E_m),$$

and rewrite Eqs. 7 and 8 in the matrix form in the basis $|n\rangle$ taking into account that the operator $A$, which plays the role of the effective cotunneling amplitude, creates (annihilates) an electron in the lead 2 (1) (see Eqs. 3 and 6). We obtain following expressions

$$I(\Delta \mu) = e \tanh \left( \frac{\Delta \mu}{2k_BT} \right) A(\Delta \mu),$$

$$S(\omega, \Delta \mu) = \frac{e^2}{2} \sum_{\pm} A(\Delta \mu \pm \omega).$$

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*b* We note that charge fluctuations, $\delta Q(t) = 2 \int_{-\infty}^{t} dt' \delta \mathcal{L}_e(t')$, on a QDS are also relevant for device applications such as SET. While we focus on current fluctuations in the present paper, we mention here that in the cotunneling regime the noise power $\langle \delta Q^2 \rangle_{\omega=0}$ does not vanish at zero frequency, $\langle \delta Q^2 \rangle_{\omega=0} = 4\omega^2 S_{ss}(\omega)|_{\omega=0} \neq 0$. Our formalism is also suitable for studying such charge fluctuations; this will be addressed elsewhere.
We note that because of additional integration over time $t$ in the amplitude $A$ (see Eq. [7]), the spectral density $A$ depends on $\mu_1$ and $\mu_2$ separately. However, away from the resonances, $\omega \ll \Delta_{\pm}$, only $\Delta\mu$-dependence is essential, and thus $A$ can be regarded as being one-parameter function. Comparing Eqs. [10] and [11], we obtain

$$S(\omega, \Delta\mu) = e^2 \sum_{\pm} \coth \left[ \frac{\Delta\mu \pm \omega}{2k_B T} \right] I(\Delta\mu \pm \omega) \quad (12)$$

up to small terms on the order of $\omega/\Delta_{\pm}$. This equation represents our nonequilibrium FDT for the transport through a QDS in the weak cotunneling regime. A special case with $T, \omega = 0$, giving $S = eI$, has been derived earlier.

To conclude this section we would like to list again the conditions used in the derivation. The universality of noise to current relation Eq. [12] proven here is valid in the regime in which it is sufficient to keep the first nonvanishing order in the tunneling $V$ which contributes to transport and noise. This means that the QDS is in the weak cotunneling regime with $\omega, k_B T \ll \Delta_{\pm}$, and $I/e \ll \omega_{\text{th}}$.

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Appendix

In this Appendix we present the derivation of Eqs. [3] and [4]. In order to simplify the intermediate steps, we use the notation $\bar{O}(t) \equiv \int_{-\infty}^{t} dt'O(t')$ for any operator $O$, and $O(0) \equiv O$. We notice that, if an operator $O$ is a linear function of operators $D_l$ and $D_l^\dagger$, then $\bar{O}(\infty) = 0$ (see the discussion in Sec. [3]). Next, the currents can be represented as the difference and the sum of $\hat{I}_1$ and $\hat{I}_2$,

$$\hat{I}_d = (\hat{I}_2 - \hat{I}_1)/2 = ie(X^\dagger - X)/2, \quad (13)$$

$$\hat{I}_s = (\hat{I}_1 + \hat{I}_2)/2 = ie(Y^\dagger - Y)/2, \quad (14)$$

where $X = D_2 + D_1^\dagger$, and $Y = D_1 + D_2$. While for the perturbation we have

$$V = X + X^\dagger = Y + Y^\dagger. \quad (15)$$

First we concentrate on the derivation of Eq. [6] and redefine the average current Eq. [3] as $I = I_d$ (which gives the same result anyway, because the average number of electrons on the QDS does not change $I_s = 0$).

To proceed with our derivation, we make use of Eq. [3] and expand the current up to fourth order in $T_{lp}$:

$$I = i \int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' \langle \hat{I}_d V(t)V(t') \hat{V}(t') \rangle - i \int_{-\infty}^{0} dt \langle \hat{V} \hat{I}_d V(t) \hat{V}(t) \rangle + \text{c.c.} \quad (16)$$

Next, we use the cyclic property of trace to shift the time dependence to $\hat{I}_d$. Then we complete the integral over time $t$ and use $I_d(\infty) = 0$. This procedure allows us to combine first and second term in Eq. [16],

$$I = -i \int_{-\infty}^{0} dt \langle [\hat{I}_d V + \hat{V} \hat{I}_d] V(t) \hat{V}(t) \rangle + \text{c.c.} \quad (17)$$

\*To be more precise, we neglect small $\omega$-shift of the energy denominators $\Delta_{\pm}$, which is equivalent to neglecting small terms of order $\omega/\Delta_{\pm}$ in Eq. [4].
Now, using Eqs. 13 and 15 we replace operators in Eq. 17 with $X$ and $X^\dagger$ in two steps: $I = e \int_{-\infty}^{\infty} dt \langle [\hat{X}^\dagger X^\dagger - \hat{X} X] V(t) \bar{V}(t) \rangle + \text{c.c.}$, where some terms cancel exactly. Then we work with $V(t) \bar{V}(t)$ and notice that some terms cancel, because they are linear in $c_{lk}$ and $c_{lk}^\dagger$. Thus we obtain $I = e \int_{-\infty}^{\infty} dt \langle [\hat{X}^\dagger X^\dagger - \hat{X} X] [X^\dagger(t) \hat{X}(t) + X(t) \hat{X}(t)] \rangle + \text{c.c.}$ Two terms $XXX\hat{X}$ and $\hat{X}^\dagger X^\dagger X^\dagger \hat{X}^\dagger$ describe tunneling of two electrons from the same lead, and therefore they do not contribute to the normal current. We then combine all other terms to extend the integral to $+\infty$,

$$I = e \int_{-\infty}^{\infty} dt \langle \hat{X}^\dagger(t) X^\dagger(t) X \bar{X} - \bar{X} X X^\dagger(t) \hat{X}^\dagger(t) \rangle$$

Finally, we use $\int_{-\infty}^{\infty} dt X(t) \bar{X}(t) = -\int_{-\infty}^{\infty} dt \bar{X}(t) X(t)$ (since $\bar{X}(\infty) = 0$) to get Eq. 3 with $A = X \bar{X}$. Here, again, we drop terms $D_1 \hat{D}_1^\dagger$ and $D_2 \hat{D}_2$ responsible for tunneling of two electrons from the same lead, and obtain $A$ as in Eq. 7.

Next, we derive Eq. 8 for the noise power. At small frequencies $\omega \ll \Delta_{\pm}$ fluctuations of $I_s$ are suppressed because of charge conservation (see below), and we can replace $\hat{I}_d$ in the correlator Eq. 8 with $\hat{I}_s$. We expand $S(\omega)$ up to fourth order in $T_{lk}p$, use $\int_{-\infty}^{\infty} dt \hat{I}_d(t) e^{\pm i\omega t} = 0$, and repeat the steps leading to Eq. 14. Doing this we obtain,

$$S(\omega) = -\int_{-\infty}^{\infty} dt \cos(\omega t) \langle [\bar{V}(t), \hat{I}_d(t)] [\bar{V}, \hat{I}_d] \rangle .$$

Then, we replace $V$ and $\hat{I}_d$ with $X$ and $X^\dagger$. We again keep only terms relevant for cotunneling, and in addition we neglect terms of order $\omega^2 / \Delta_{\pm}$ (applying same arguments as before, see Eq. 20). We then arrive at Eq. 8 with the operator $A$ given by Eq. 7.

Finally, in order to show that fluctuations of $I_s$ are suppressed, we replace $\hat{I}_d$ in Eq. 14 with $\hat{I}_s$, and then use the operators $Y$ and $Y^\dagger$ instead of $X$ and $X^\dagger$. In contrast to Eq. 8 terms such as $Y^\dagger Y^\dagger YY$ do not contribute, because they contain integrals of the form $\int_{-\infty}^{\infty} dt \cos(\omega t) D_1(t) \hat{D}_1(t) = 0$. The only nonzero contribution can be written as

$$S_{ss}(\omega) = \frac{e^2}{4} \omega^2 \int_{-\infty}^{\infty} dt \cos(\omega t) \langle [\bar{Y}(t), \hat{Y}(t)] [\bar{Y}^\dagger, \hat{Y}] \rangle ,$$

where we have used integration by parts and the property $\bar{Y}(\infty) = 0$. Compared to Eq. 8 this expression contains an additional integration over $t$, and thereby it is of order $(\omega / \Delta_{\pm})^2$.

References

1. For a recent review on shot noise, see: Ya. M. Blanter and M. Büttiker, Shot Noise in Mesoscopic Conductors, Phys. Rep. 336, 1 (2000).
2. D. V. Averin and Yu. V. Nazarov, in Single Charge Tunneling, eds. H. Grabert and M. H. Devoret, NATO ASI Series B: Physics Vol. 294, (Plenum Press, New York, 1992).
3. D. C. Glattli et al., Z. Phys. B 85, 375 (1991).
4. For an review, see D. V. Averin and K. K. Likharev, in Mesoscopic Phenomena in Solids, edited by B. L. Al'tshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991).
5. E. V. Sukhorukov, G. Burkard, and D. Loss, Phys. Rev. B 63, 125315 (2001).
6. D. Loss and E. V. Sukhorukov, Phys. Rev. Lett. 84, 1035 (2000).
7. Such a non-equilibrium FDT was derived for single barrier junctions long ago by D. Rogovin, and D. J. Scalapino, Ann. Phys. (N. Y.) 86, 1 (1974).
8. G. D. Mahan, Many Particle Physics, 2nd Ed. (Plenum, New York, 1993).
9. M. H. Devoret, and R. J. Schoelkopf, Nature 406, 1039 (2000).