No relativistic Newton–Wigner probability current for any spin

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Abstract
We investigate whether the Newton–Wigner position probability density, extended from spinless particles to electrons/positrons and particles of higher spin, can be incorporated as the zero component of a four-component probability current density that transforms locally as a four-vector function of the spacetime coordinates. We find that this is not possible, in all cases.

Keywords: relativistic quantum mechanics, quantum theory, Lorentz transformations

1. Introduction
Throughout this paper, we use Heaviside–Lorentz units, in which \( \hbar = c = \varepsilon_0 = \mu_0 = 1 \).

Perhaps the most important result of Dirac’s theory of the electron and positron [1] is the construction of the four-component charge current density operator (as a single particle operator)

\[
J_0^\mu(x) = q \frac{m_e}{(2\pi)^3} \int \frac{d^3p_a}{\omega_a} \sum_{m_a=\pm\frac{1}{2}} \int \frac{d^3p_b}{\omega_b} \sum_{m_b=\pm\frac{1}{2}} \langle p_a, m_a | \gamma^\mu u(p_a, m_a) \gamma^0 u(p_b, m_b) | p_b, m_b \rangle \epsilon^{(n_a-n_b) \cdot x} \bar{u}(p_a, m_a) \gamma^\mu u(p_b, m_b) \langle p_b, m_b | p_a, m_a \rangle,
\]

(1)

with the spinors and gamma matrices as in [2]. It transforms locally as a four-vector function of the spacetime coordinates for all Lorentz transformations, \( L \),

\[
U^\dagger(L) J_0^\mu(x) U(L) = L^\mu_{\nu} J_0^\nu(L^{-1}x).
\]

(2)

It translates in spacetime as

\[
J_0^\mu(x + a) = U(T(a)) J_0^\mu(x) U^\dagger(T(a)).
\]

(3)
It is locally conserved, 
\[ \partial_\mu J_\mu^0(x) = 0, \]  
and as a consequence the charge density is globally conserved in all frames: 
\[ \int d^3 x J_0^0(x) = q. \]  
It contains both momentum- and spin-dependent contributions, as can be seen using the Gordon decomposition [2]:

\[ \pi(p_a, m_a) \gamma^\mu u(p_a, m_a) = \frac{1}{2m_e} (p_a + p_b)^\mu \pi(p_a, m_a) u(p_b, m_b) + \frac{i}{2m_e} \pi(p_a, m_a) \sigma^{\mu\nu} u(p_b, m_b) (p_a - p_b)_\nu. \]

Furthermore, the charge density, the zero component of the four-vector, is negative (positive) semidefinite for the electron (positron).

Note that this result can be used for electrons, by taking \( q = -e \), or positrons, by taking \( q = +e \).

There is a way, consistent with special relativity, of defining a spatial probability density for an electron or positron. In this paper we investigate whether this Newton–Wigner [3] probability operator can be taken as the zero component of a four-component probability current that transforms like the Dirac current and is conserved. We will find that this is not possible for the electron or positron, for a spinless particle or, in fact, for any spin.

If such a probability current were possible, we suppose that a charge current could be obtained simply by multiplying by the charge. If this were possible we might have a second way to construct a quantum electrodynamics, and would have to compare with the standard results, and with experiment. The impossibility of this construction gives us confidence in the Dirac current density as the correct representation of the physical current.

The spinless case was treated by Rosenstein and Horwitz [4], where they claim to have constructed such a relativistic probability current. We find that they are in error.

The organization of this paper is as follows. First, in section 2, we review results on relativistic probability amplitudes and the construction of a total position probability density (summed over spin z-components) for the electron or positron. In section 3 we derive the fundamental commutation relations between the boost generators and a four-vector. We also find the representation of the boost generators as differential operators acting on momentum-spin component wavefunctions. In section 4 we construct the spatial components of what is possibly a four-vector probability current density for the electron/positron. Then we show that the second set of commutators from section 2 is not satisfied for these four components. In section 5 we briefly consider the spinless case and show, again, that a four-vector probability current density cannot be constructed. This requires examination of the result of Rosenstein and Horwitz to show where they are in error. The case of general spin is then considered, with the same conclusion of no possible four-vector probability current density. Conclusions follow in section 6.

2. Relativistic probability amplitudes and the position probability density

We begin with the improper basis vectors, \( |p, m\rangle \ (m = \pm 1/2) \), that carry the unitary, irreducible representations of the Poincaré group [5] for free electrons and positrons. They are eigenvectors of four-momentum with eigenvalue components \( p^\mu = (\omega, p)^\mu \). We deal only with positive energies \( p^0 = \omega = \sqrt{p^2 + m_e^2} \). The spin label, \( m \), carries the representation
of rotations in the rest frame. A charge label is to be understood, the only action necessary to distinguish positrons from electrons. We choose to give these basis vectors the covariant normalization

\[ \langle p_1, m_1 | p_2, m_2 \rangle = \omega_1 \delta^3(p_1 - p_2). \] (6)

If we construct a wavepacket state vector, a superposition of the basis vectors, normalized to unity, as

\[ |\psi\rangle = \int \frac{d^3p}{\sqrt{\omega}} \sum_{m=\pm \frac{1}{2}} |p, m\rangle \Psi_m(p), \] (7)

then \( \Psi_m(p) \) can be interpreted as a momentum-spin component probability amplitude just as in the nonrelativistic theory. The normalization condition is

\[ \int d^3p \sum_{m=\pm \frac{1}{2}} |\Psi_m(p)|^2 = 1. \] (8)

The expectations of the four-momentum operator and the rest-frame spin \( z \)-component operator are, respectively,

\[ \langle \psi | p^\mu | \psi \rangle = \int d^3p \sum_{m=\pm \frac{1}{2}} |\Psi_m(p)|^2 p^\mu \quad \text{and} \quad \langle \psi | s^z | \psi \rangle = \int d^3p \sum_{m=\pm \frac{1}{2}} |\Psi_m(p)|^2 m. \] (9)

This is a covariant, but not manifestly covariant theory. That means that the Lorentz transformation properties of the expressions we write require a nontrivial derivation. In contrast, for a manifestly covariant formalism, the transformation properties of expressions are usually immediately obvious to the reader as those expressions are written in terms of tensors and objects with simple, known, transformation properties. The postulates of special relativity do not require that all quantities of physical interest transform as scalars, four-vectors and tensors. In combination with the rules of quantum mechanics, they merely require that such transformations be well-defined and unitary (or antiunitary in the case of time reversal) and depend only on the translation, rotation or boost parameters.

The unitary transformations of the \( \Psi_m(p) \) can be easily derived [6]. The technique is to apply the unitary (or antiunitary) transformation to \( |\psi\rangle \) and thus to the basis vectors, then manipulate the expression into the form

\[ U/A |\psi\rangle = \int \frac{d^3p}{\sqrt{\omega}} \sum_{m=\pm \frac{1}{2}} |p, s, m\rangle \Psi'_m(p), \] (10)

then extract the \( \Psi'_m(p) \) by orthonormality.

The transformation rules for the Poincaré transformations are:

Spacetime translations : \( \Psi'_m(p) = \Psi_m(p) e^{+ip\cdot a} \),

Rotations : \( \Psi'_m(p) = \sum_{m'=-\frac{1}{2}}^{\frac{1}{2}} D^{(\frac{1}{2})}_{mm'}(R) \Psi_{m'}(R^{-1}p) \),

Boosts : \( \Psi'_m(p) = \sqrt{\gamma_0(1-\beta_0 \cdot \beta)} \sum_{m'=\pm \frac{1}{2}} \Psi^{(\frac{1}{2})}_{mm'}(p \leftarrow \Lambda^{-1}p) \Psi_{m'}(\Lambda^{-1}p) \). (11)
where $\beta_0$ is the boost velocity, $\gamma_0 = 1/\sqrt{1 - \beta_0^2}$ and $\beta = p/\omega$ is the velocity of the particle. For the inversions, we have

$$\text{Space inversion : } \Psi_m'(\omega, p) = \eta \Psi_m(\omega, -p),$$

$$\text{Time reversal : } \Psi_m'(\omega, p) = (-1)^{\frac{1}{2} + m} \Psi_{-m}(\omega, -p).$$

In these expressions $W$ is a matrix element of a Wigner rotation, which can be evaluated by

$$W(\Lambda p \leftarrow p) = \Lambda^{-1}[\Lambda p] \Lambda [p]$$

where

$$\Lambda[p] \equiv \Lambda\left(\frac{p}{\omega}\right)$$

and $\Lambda(\beta)$ is a function of the boost velocity, $\beta$. Explicit forms of the Wigner rotations are given in [7]. Two successive, noncollinear, boosts (from the rest momentum to $p$ and then from $p$ to $\Lambda p$) produce a boost (from the rest momentum to $\Lambda p$) preceded by a rotation in the rest frame. This is the physics behind the Thomas precession [8].

These transformations are all unitary (antiunitary for time reversal) in that they preserve the modulus-squared of the scalar product

$$|\langle \varphi | \psi \rangle|^2 = \left| \int d^3p \sum_{m=\pm \frac{1}{2}} \varphi^*(p,m)\psi(p,m) \right|^2.$$  

The improper state vector for an electron or positron localized at position $x$ at time $t$ with spin $z$-component $m$ is given by Newton and Wigner [3] (with $x^\mu = (t, x)^\mu$) as

$$|x, m\rangle = \int d^3p \sqrt{\omega} \Psi_m(p) \frac{e^{ip \cdot x}}{(2\pi)^{\frac{3}{2}}}.$$  

At equal times, these satisfy the orthogonality (and normalization) condition

$$\langle t, x_1, m_1 | t, x_2, m_2 \rangle = \delta_{m_1 m_2} \delta^3(x_1 - x_2).$$  

Note that the spin component in each rest frame becomes the spin component at a position and time.

The amplitudes for our state vector on this basis are then

$$\psi_m(x) \equiv \langle x, m | \psi \rangle = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \Psi_m(p) e^{-ip \cdot x},$$

a Fourier transform of the momentum-spin component amplitudes as in the nonrelativistic theory, with the form of the energy changed. The unitary/antiunitary transformations of these position-spin component amplitudes are given in [6]. We merely note that a boost transformation involves a nonlocal transformation of the amplitudes.

Now the total position probability density operator, summed over spin components, is

$$\rho(t, x) = \sum_{m=\pm \frac{1}{2}} \langle t, x, m | t, x, m \rangle,$$

a time-dependent operator in the Heisenberg picture. We want to investigate whether this operator can be the zero component of a four-component probability current density that
transforms locally as a four-vector function of the spacetime coordinates. A possible obstacle
to this is the fact that $\rho(x)$, unitarily transformed on its own, transforms nonlocally.

For comparison, the Dirac charge density with a factor of the charge removed can be written

$$\rho_D(x) = \frac{m_e}{(2\pi)^3} \sum_{a=1}^{4} |x, a\rangle \langle x, a|,$$

with

$$|x, a\rangle = \int \frac{d^3p}{\omega} \sum_{m=\pm \frac{1}{2}} |p, m\rangle u_{ma}(p) e^{i\omega x}$$

and

$$u_{ma}(p) = u_{am}(p)$$

and

$$\begin{pmatrix} u_{1m}(p) \\ u_{2m}(p) \\ u_{3m}(p) \\ u_{4m}(p) \end{pmatrix} = (p, m).$$

The $|x, a\rangle$ boost locally.

3. The commutators between the boost generators and the components of a four-vector

Under the Lorentz transformations as defined, the spacetime origin is invariant, so we need
only attempt to construct $J^\mu(0)$, which is then required to transform as a four-vector. Then we
may translate it to general $x$ using equation (3).

For a boost by infinitesimal rapidity, $\zeta$, equation (2) becomes

$$(1 + i\zeta \cdot K) J^0(0) (1 - i\zeta \cdot K) = J^0(0) + \zeta \cdot J(0),$$

$$(1 + i\zeta \cdot K) J(0) (1 - i\zeta \cdot K) = J(0) + J^0(0) \zeta.$$  

(24)

This gives the two sets of commutators

$$i[K, J^0(0)] = J(0)$$

(25)

and

$$i[K, J_j(0)] = \delta_j^0 J^0(0).$$

(26)

We need the representation of the boost generators, $K$, acting on the $\Psi_m(p)$. We take

$$(1 - i\zeta \cdot K) |\psi\rangle = \int \frac{d^3p}{\sqrt{\omega}} \sum_{m=\pm \frac{1}{2}} |p, x, m\rangle \Psi_m(p),$$

(27)

with $\Psi'_m(p)$ given by the third equation of the set equation (11), approximated to first order in
$\zeta$. We need to approximate the explicit formula for the Wigner rotation. We find
where the $\sigma$ are the usual Pauli matrices. We note that this representation of the operator is explicitly Hermitian, as it must be since the form of the scalar product is simply given by

$$\langle \phi | \psi \rangle = \int d^3p \sum_{m=\pm \frac{1}{2}} |p, s, m\rangle \langle p, s, m|.$$

The anticommutator of two operators is defined by $\{A, B\} = AB + BA$.

### 4. Construction and test of a possible four-vector probability current density

First we write $J^0(0) = \rho(0)$ in momentum space using equations (16) and (19). This gives

$$J^0(0) = \frac{1}{(2\pi)^3} \int \frac{d^3p_a}{\sqrt{\omega_a}} \int \frac{d^3p_b}{\sqrt{\omega_b}} \sum_{m=\pm \frac{1}{2}} |p_a, m\rangle \langle p_b, m|.$$

From equation (25), using equation (28), this gives the spatial part of the possible four-vector as

$$J(0) = \frac{1}{(2\pi)^3} \int \frac{d^3p_a}{\sqrt{\omega_a}} \int \frac{d^3p_b}{\sqrt{\omega_b}} \sum_{m=\pm \frac{1}{2}} |p_a, m\rangle \{3 - \frac{1}{4} |\beta_a - \beta_b|^2 - \frac{1}{2} |p_a - m | \langle p_b, m |}.$$

where $\beta_{a/b} = p_{a/b}/\omega_{a/b}$. We note the explicit Hermiticity of this expression. It is easily verified that $J^0(0)$ is rotationally invariant and that $J(0)$ rotates as a three-vector, as required.

Now we must test the other commutators, equation (26). To simplify the calculation, we test

$$i \sum_{i=1}^3 [K_i, J_i(0)] = 3 J^0(0),$$

which is a necessary but not sufficient condition for covariance. After some calculation, we find

$$i \sum_{i=1}^3 [K_i, J_i(0)] = \frac{1}{(2\pi)^3} \int \frac{d^3p_a}{\sqrt{\omega_a}} \int \frac{d^3p_b}{\sqrt{\omega_b}} \sum_{m=\pm \frac{1}{2}} |p_a, m\rangle \{3 - \frac{1}{4} |\beta_a - \beta_b|^2 - \frac{1}{2} |p_a - m | \langle p_b, m |}.$$

Hermitian and rotationally invariant, as expected. Clearly the condition in equation (26) is not satisfied, so it is not possible to construct a four-vector probability current for the electron or positron.

### 5. The spinless case, comparison with other work and the case of general spin

It is a simple matter to eliminate spin from these equations and arrive at the candidate four components for the spinless case:
\[ J_0^0(0) = \frac{1}{(2\pi)^3} \int \frac{d^3 p_a}{\sqrt{\omega_a}} \int \frac{d^3 p_b}{\sqrt{\omega_b}} |p_a\rangle \langle p_b| \]  
\begin{equation}
(34)
\end{equation}

and

\[ J_0(0) = \frac{1}{(2\pi)^3} \int \frac{d^3 p_a}{\sqrt{\omega_a}} \int \frac{d^3 p_b}{\sqrt{\omega_b}} |p_a\rangle \frac{1}{2} (\beta_a + \beta_b) \langle p_b| . \]  
\begin{equation}
(35)
\end{equation}

This was the form obtained by Rosenstein and Horwitz [4]. The commutator of equation (32) comes out in this case to be

\[ i \sum_{i=1}^{3} [K_i, J_0^0(0)] = \frac{1}{(2\pi)^3} \int \frac{d^3 p_a}{\sqrt{\omega_a}} \int \frac{d^3 p_b}{\sqrt{\omega_b}} |p_a\rangle \{ 3 - \frac{1}{4} |\beta_a - \beta_b|^2 \} \langle p_b| , \]  
\begin{equation}
(36)
\end{equation}

not equal to \(3J_0^0(0)\). So this proposed current is not covariant.

It is informative to try to write the candidate spinless four-current in manifestly covariant form, using

\[ \frac{1}{2} (1, \beta_{a/b})^\mu = \frac{1}{2} \frac{p^\mu_{a/b}}{\omega_{a/b}} . \]  
\begin{equation}
(37)
\end{equation}

This gives

\[ J_0^\mu(0) = \frac{1}{(2\pi)^3} \int \frac{d^3 p_a}{\omega_a} \int \frac{d^3 p_b}{\omega_b} |p_a\rangle \frac{1}{2} \left\{ \frac{\omega_b}{\omega_a} p^\mu_a + \frac{\omega_a}{\omega_b} p^\mu_b \right\} \langle p_b| , \]  
\begin{equation}
(38)
\end{equation}

clearly not covariant.

We note that Kowalski and Rembieliński [9] derive a four-component object, with the position probability density

\[ \rho_0(x) = |x\rangle \langle x| \]  
\begin{equation}
(39)
\end{equation}

as the zero component, that is locally conserved in one frame. This is done by solving the local conservation equation. Their object clearly does not have the required transformation properties.

For general spin \(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\), we start with the basis vectors \(|p, s, m\rangle\) (\(m = -s, -s+1, \ldots, s-1, s\)). Much of the calculation is similar to the two particular cases just considered. The Pauli matrices are replaced by

\[ \frac{1}{2} [\sigma]_{m, m_0} \rightarrow \langle s, m_a | s, m_b \rangle , \]  
\begin{equation}
(40)
\end{equation}

and the sums over spin \(z\)-components take values from \(-s\) to \(+s\). Here \(s\) is the spin angular momentum operator, with well-known matrix elements [10]. It is clear that the term \(-\frac{1}{4} |\beta_a - \beta_b|^2\) will always appear within the representations of the commutator equation (32), along with rotationally invariant terms contributed by the spin, with the same negative sign and thus unable to cancel, destroying any possibility of covariance.

6. Conclusions

We have seen that the relativistic density operator as in equation (19) for an electron/positron cannot be incorporated as the zero component of a four-vector probability current density. The same is true for any spin.
The time evolution of an expectation value of the position probability density operator can be calculated by taking an expectation of equation (19) or (30). The result is relativistic since we know exactly how any boosted frame would view this evolution, and the integral over all space is invariant. We have found that this density does not ‘flow’ like the density of a relativistic fluid with a local conservation law. This result adds to the discussion around the results of Hegerfeldt [11–13] regarding violation of a classical notion of causality in relativistic quantum mechanics. Since the probability density does not flow like the density of a fluid, it certainly does not flow like the density of a fluid with a field of velocities that are constrained to be less than the speed of light in magnitude.

The result for spin-1/2 obtained in this paper strongly supports the conclusion that the Dirac current is the unique choice for the electromagnetic current of an electron or positron.

Another aim of this paper is to promote the use of relativistic probability amplitudes. It would make little sense to second quantize these amplitudes to form nonlocally transforming field operators. Yet these amplitudes have a definite role to play in quantum field theory. If one is to construct a realistic description of a scattering experiment, one needs to describe the essentially free initial and final states with wavepackets, with the probability distributions in momentum, position and spin specified. To do this it is essential to use relativistic probability amplitudes.

In addition, it has been shown that the use of wavepackets eliminates some divergences in scattering calculations [14].

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