HAUSDORFF DIMENSION OF ASYMPTOTIC SELF-SIMILAR SETS

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Abstract. In this paper, we introduce the notion of asymptotic self-similar sets on general doubling metric spaces by extending the notion of self-similar sets, and determine their Hausdorff dimensions, which gives an extension of Balogh and Rohner’s result. This is carried out by introducing the notions of almost similarity maps and asymptotic similarity systems. These notions have an advantage of making geometric constructions possible. Actually, as an application, we determined the Hausdorff dimension of general Sierpinski gaskets on complete surfaces constructed by a geometric way in a natural manner.

1. Introduction

The notion of self-similar sets or general Cantor sets have played significant roles in fractal geometry. These sets are usually defined by means of iterated function systems \( \{ f_1, \cdots, f_k \} \) consisting of contracting similarity maps on a complete metric space as the unique nonempty compact set \( K \), called an attractor or an invariant set, satisfying \( K = \bigcup_{i=1}^{n} f_i(K) \). Hutchinson [10] (cf. Kigami [12], Schief [18]) introduced the notion of the open set condition and determined the Hausdorff dimension of self-similar sets in Euclidean space \( \mathbb{R}^n \) satisfying the open set condition. Balogh and Rohner extended Hutchinson’s result to doubling metric spaces [2]. However, it is difficult to construct a similarity map in general metric spaces. Actually, similarity maps do not always exist on curved metric spaces. To overcome this difficulty, in the previous work [22], the first named author introduced the notion of \((\lambda, c, \nu)\)-almost similarity maps extending that of \(\lambda\)-similarity maps in order to construct generalized Cantor sets in general metric measure spaces, and determined the Hausdorff dimension of such a generalized Cantor set. However the basic subsets considered in [22]

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are assumed to be disjoint each other, and therefore generalized Cantor sets like Sierpinski gaskets are excluded in the results of [22].

In the present paper, we extend both Balogh and Rohner’s result and our previous result to the case when basic subsets may have intersections with their boundary by introducing a generalized open set condition. As an application, we determine the Hausdorff dimension of Sierpinski gaskets on complete surfaces defined via geometric way.

Let $X$ be a proper complete metric space. We assume that $X$ is doubling in the sense of [2] (see Section 2 for the precise definition). Complete Riemannian manifolds with Ricci curvature bounded from below are typical examples of doubling metric spaces (cf. [8]). Doubling metric spaces also appear in metric measure spaces satisfying a doubling condition. Nowadays, geometric analysis on doubling metric measure spaces has been very active (see for instance Assouad [1], Gromov [8], Heinonen [9], Villani [20]), and therefore it is quite natural to study self-similarity sets in such doubling metric spaces.

Let $\bar{U} \supset \bar{V}$ be bounded domains in $X$ homeomorphic to each other, where $\bar{U}$ and $\bar{V}$ denote the closures of the open subsets $U$ and $V$. Fix constants $0 < \lambda < 1$, $0 < \nu < 1$ and a continuous increasing function $\varphi : (0, \infty) \to (0, \infty)$ with $\lim_{x \to +0} \varphi(x) = 0$. We call a homeomorphism $f : \bar{U} \to \bar{V}$ a $(\lambda, \varphi(|U|), \nu)$-almost similarity map if for every $x, y \in \bar{U}$,

\begin{align}
&\left| \frac{d(f(x), f(y))}{d(x, y)} - \lambda \right| \leq \lambda \varphi(|U|), \\
&|V| \leq \nu|U|.
\end{align}

where $|U|$ is the diameter of $U$. Then the set $\bar{V}$ is called a $(\lambda, \varphi(|\bar{U}|), \nu)$-almost similar set of $\bar{U}$.

In this paper, we assume the following conditions for $\varphi$:

\begin{align}
&\varphi : (0, \infty) \to (0, \infty) \text{ is increasing with } \lim_{x \to +0} \varphi(x) = 0; \\
&\int_1^\infty \varphi(au^x) \, dx < \infty \text{ for some constants } a > 0 \text{ and } 0 < \nu < 1.
\end{align}

Note that the second condition (2) above does not depend on the choice of $a > 0$ and $0 < \nu < 1$, and that for any $\alpha > 0$ and any positive integer $n$, the following functions satisfy the above conditions:

$\varphi(y) = y^\alpha$, $\varphi(y) = -(\log y)^{-1-\frac{2}{2\nu+1}}$.

For a fixed positive integer $k$, we let $I = \{1, 2, \ldots, k\}$. We denote by $I^*$ the set of all ordered multi-indices $I = i_1 \cdots i_n$ with $n \geq 1$, $i_j \in I$ for every $1 \leq j \leq n$. We set $|I| = |i_1 \cdots i_n| = n$ and call it the length of $I$. Let $I^n$ denote the set of all $I \in I$ of length $n$.

In the present paper, we investigate an asymptotic self-similar set in $X$, which is defined under the following hypothesis: For $0 < \nu < 1$ and
a > 0, let \( \varphi : (0, \infty) \to (0, \infty) \) be a continuous function satisfying the above conditions \([13]\).

**Definition 1.1.** Suppose that ratio coefficients \( 0 < \lambda_i < 1, (i = 1, \ldots, k) \) together with a non-empty open subset \( V \subset X \) are given for which we have

1. For each \( i \in \mathcal{I} \), a \((\lambda_i, \varphi(|\bar{V}|), \nu)\)-almost similarity map \( f_i : \bar{V} \to \bar{V}_i \subset \bar{V} \)

is given in such a way that \( V_i \cap V_j = \emptyset \) for every \( i \neq j \in \mathcal{I} \), where \( V_i := f_i(V) \);

2. For each \( ij \in \mathcal{I}^2 \), a \((\lambda_j, \varphi(|\bar{V}_i|), \nu)\)-almost similarity map \( f_{ij} : \bar{V}_i \to \bar{V}_{ij} \subset \bar{V}_i \)

is given in such a way that \( V_{ij} \cap V_{ij'} = \emptyset \) for every \( j \neq j' \in \mathcal{I} \), where \( V_{ij} := f_{ij}(V_i) \);

3. For each \( I' \in \mathcal{I}^{n-1} \) and \( i_n \in \mathcal{I} \) with \( I := I'i_n \), a \((\lambda_{i_n}, \varphi(|\bar{V}_{I'}|), \nu)\)-almost similarity map \( f_I : \bar{V}_{I'} \to \bar{V}_I \subset \bar{V}_{I'} \)

is defined in such a way that \( V_{I'i} \cap V_{I'j} = \emptyset \) for every \( i \neq j \in \mathcal{I} \), where \( V_I := f_I(V_{I'}) \).

We call \( \{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}} \) an \((\{\lambda_i\}_{i=1}^k, \varphi, \nu)\)-asymptotic similarity system.

Then the set \( K \) defined as

\[
K = \bigcap_{n=1}^{\infty} \left( \bigcup_{I \in \mathcal{I}^n} \bar{V}_I \right),
\]

is called an **asymptotic self-similar set** in \( X \).

Let us consider the case of iterated function system of contracting similarity maps \( \{f_1, \ldots, f_k\} \) with open set condition

1. \( V \supset f_1(V) \cup \cdots \cup f_k(V) \);
2. \( f_i(V) \cap f_j(V) \neq \emptyset \) for every \( i \neq j \);

for some non-empty open set \( V \subset X \). In this case, for each \( I = i_1 \cdots i_n \in \mathcal{I}^n \), let

\[
V_I := f_{i_n} \circ \cdots \circ f_{i_1}(V), \quad f_I := f_{i_n} : \bar{V}_{I'} \to \bar{V}_I.
\]

Then this gives a \((\{\lambda_i\}_{i=1}^k, \varphi = 0, \lambda_{\text{max}})\)-asymptotic similarity system \( \{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}} \), where \( \lambda_{\text{max}} = \max \lambda_i \). Thus the notion of \((\{\lambda_i\}_{i=1}^k, \varphi, \nu)\)-asymptotic similarity system is an extension of iterated function system of contracting similarity maps with open set condition.

Our main result in the present paper is stated as follows.
Theorem 1.2. Let $X$ be a complete doubling metric space and let $K$ be the asymptotic self-similar set associated with a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$. Then the Hausdorff and the box dimensions of $K$ are given as

$$\dim_H K = \dim_B K = s,$$

where $s$ is a unique number satisfying $\sum_{i=1}^k \lambda_i^s = 1$.

In [2], Balogh and Rohner suggested a problem. They considered an iterated function system of contracting asymptotically similarity maps in the sense that for all $I = i_1 \cdots i_n \in \mathcal{I}$

$$c_1 \lambda_I \leq \frac{|f_I(x), f_I(y)|}{|x, y|} \leq c_2 \lambda_I,$$

where $f_I = f_{i_n} \circ \cdots \circ f_{i_1}$, $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_n}$ and $c_1, c_2$ are uniform positive constants. They posed a problem: What happens if an iterated function system of contracting similarity maps is replaced by one of contracting asymptotically similarity maps? Rajala and Vilppolainen completely solved the above problem in Theorem 4.9 of [16] by introducing a more general notion of a semiconformal iterated function system. A $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$ is closely related with Balogh and Rohner’s iterated function system of contracting asymptotically similarity maps and Rajala and Vilppolainen’s semiconformal iterated function system under the open set condition. Actually our notion of asymptotic similarity system provides a controlled Moran construction in the sense of Rajala and Vilppolainen ([16]) (see Lemma 3.12). However an asymptotic self-similar set introduced in the present paper is constructed by means of a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system, which consists of infinite series of almost similarity maps. Therefore in general, it is not simply defined by a finite iterated function system. For example, a generalized Sierpinski gasket on a general complete surfaces constructed in this paper is an asymptotic self-similar set. It would be an interesting question to determine whether a generalized Sierpinski gasket on a general complete surface can be defined by means of a finite iterated function system due to Balogh-Rohner or Rajala-Vilppolainen (see Section 4). Anyway the notion of asymptotic self-similar sets introduce in this paper has an advantage of making geometric constructions in general curved spaces much easier.

As indicated above, we consider a Sierpinski gasket on a complete surface $M$ as an application of Theorem 1.2 which is naturally defined in a geometric way as follows.

Now let $\mathcal{I} = \{1, 2, 3\}$, and let $\Delta$ be a closed domain contained in a convex domain of $M$ bounded by a geodesic triangle. By joining the midpoints of the edges of $\Delta$ by minimal geodesics, we divide $\Delta$ into
four triangles, and remove the center triangle to get three geodesic triangles $\Delta_1$, $\Delta_2$ and $\Delta_3$. Repeating this procedure for each $\Delta_i$ infinitely many times, we obtain a system of geodesic triangles $\{\Delta_i\}_{i \in \mathcal{I}}$. The \textit{generalized Sierpinski gasket} $K_\Delta$ on $M$ associated with $\Delta$ is defined as

$$K_\Delta = \bigcap_{n=1}^{\infty} \left( \bigcup_{i \in \mathcal{I}_n} \Delta_i \right),$$

We say that $\Delta$ is \textit{asymptotically non-degenerate} if all the divided small triangles $\Delta_i$ are $\delta$-non-degenerate for some constant $\delta > 0$. (See Section 4 for the precise definition). For example, every geodesic triangle region $\Delta$ of perimeter less than $2\pi$ on a unit sphere is asymptotically non-degenerate (see Example 4.3). We show that a small geodesic triangle region on a surface is asymptotically non-degenerate (see Lemma 4.9).

\textbf{Theorem 1.3.} If a geodesic triangle domain $\Delta$ in a convex domain on a complete surface is asymptotically non-degenerate, then

1. for some $0 < \nu < 1$ there exists a $\{(1/2, 1/2, 1/2), \varphi, \nu\}$-asymptotic similarity system $\{ (\Delta_i, f_i) \}_{i \in \mathcal{I}}$- associated with $\Delta$, where $\varphi(x) = cx^2$ for some constant $c > 0$;

2. the Hausdorff and box dimensions of the generalized Sierpinski gasket $K_\Delta$ associated with $\Delta$ are given by

$$\dim_H K_\Delta = \dim_B K_\Delta = \frac{\log 3}{\log 2}.$$

The following result gives a condition for $\Delta$ to be asymptotically non-degenerate.

\textbf{Corollary 1.4.} A geodesic triangle domain $\Delta$ in a convex domain on a complete surface is asymptotically non-degenerate if and only if for some $0 < \nu < 1$ there exists a $\{(1/2, 1/2, 1/2), \varphi, \nu\}$-asymptotic similarity system $\{ (\Delta_i, f_i) \}_{i \in \mathcal{I}}$- associated with $\Delta$, where $\varphi(x) = cx^2$ for some constant $c > 0$.

The organization of the present paper is as follows: In Section 2, we discuss some basic notions needed in the proof of the above results. In Section 3, we prove Theorem 1.2. In Section 4, we discuss generalized Sierpinski gaskets on complete surfaces, and prove Theorem 1.3 and Corollary 1.4.

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2. PRELIMINARIES

The distance between points $x, y$ in a metric space will be denoted as $d(x, y)$. For $r > 0$, $B(x, r)$ denotes the open ball of radius $r$ around $x$. 
Definition 2.1. A metric space $X$ is said to be doubling if there exists a positive integer $C$ such that for any $x \in X$ and any $r > 0$, there exist $\{x_i\}_{i=1}^C \subset X$ such that

$$B(x, r) \subset \bigcup_{i=1}^C B(x_i, r/2)$$

Note that $C$, called the doubling constant of $X$, does not depend on the choices of $x$ or $r$.

For the proof of the following lemma, see Lemma 3.3 of [2].

Lemma 2.2. Let $X$ be a doubling metric space with doubling constant $C$. For any $0 < \delta < 1$, there exists a constant $C(\delta)$ such that the number of mutually disjoint balls $B(x_i, \delta r)$ in a ball $B(x, r)$ of $X$ is bounded by $C(\delta)$.

Definition 2.3. Let $X$ be a metric space, $A \subset X$ and $\alpha$ be a non-negative real number. An $\epsilon$-cover $\{U_i\}$ of $A$ is a finite or countable collection of sets $U_i$ covering $A$ with $|U_i| \leq \epsilon$. Define $H_\alpha^\epsilon(A)$ by

$$H_\alpha^\epsilon(A) = \inf \left\{ \sum_{i=1}^\infty |U_i|^\alpha \mid \{U_i\} : \epsilon\text{-cover of } A \right\}.$$ 

The $\alpha$-dimensional Hausdorff measure of $A$ is defined by

$$H_\alpha(A) = \lim_{\epsilon \to 0} H_\alpha^\epsilon(A),$$

and the Hausdorff dimension $\dim_H A$ of $A$ is defined as

$$\dim_H A := \sup \{ \alpha \geq 0 \mid H_\alpha(A) = \infty \} = \inf \{ \alpha \geq 0 \mid H_\alpha(A) = 0 \}.$$ 

Let $A$ be a bounded subset of a metric space $X$. Let $N_\epsilon(A)$ denote the minimal number of subsets of diameter $\leq \epsilon$ needed to cover $A$. The lower box dimension and the upper box dimension of $A$ are defined respectively as

$$\underline{\dim_B} A = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}, \quad \overline{\dim_B} A = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$ 

When both the lower and the upper box dimensions are equal, the common value

$$\dim_B A = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}$$

is called the box dimension of $A$.

The following is a standard fact (see [7] for instance):

(2.5) \[ \dim_H A \leq \underline{\dim_B} A \leq \overline{\dim_B} A. \]

Next we discuss self-similarity measures. In the rest of this section, we always assume that $Y$ is a compact metric space unless otherwise stated.
Let $\mathcal{M}(Y)$ be the set of all Borel probability measures on $Y$. Consider the Kantrovich-Rubinshtein metric $d_M$ and the modified Kantrovich-Rubinshtein metric $d'_M$ on $\mathcal{M}(Y)$ defined by

$$d_M(\mu_1, \mu_2) = \sup \left\{ \left| \int_Y \phi \, d\mu_1 - \int_Y \phi \, d\mu_2 \right| : \phi \in \text{Lip}_1(Y), \sup_{x \in Y} |\phi(x)| \leq 1 \right\},$$

$$d'_M(\mu_1, \mu_2) = \sup \left\{ \left| \int_Y \phi \, d\mu_1 - \int_Y \phi \, d\mu_2 \right| : \phi \in \text{Lip}_1(Y) \right\},$$

where $\text{Lip}_1(Y)$ denotes the set of all Lipschitz functions on $Y$ with Lipschitz constant $\leq 1$.

It is well known that $(\mathcal{M}(Y), d_M)$ is complete (see Theorem 8.10.43 of [3]). Further, we have from the definition

$$d_M(\mu_1, \mu_2) \leq d'_M(\mu_1, \mu_2) \leq \max\{|Y|, 1\} d_M(\mu_1, \mu_2).$$

In particular, $(\mathcal{M}(Y), d'_M)$ is also complete.

Let $\{f_i\}_{i=1}^m$ be a family of contracting maps in a compact metric space $Y$. Namely, there are some constants $0 < \lambda_i < 1$ such that

$$\frac{d(f_i(x), f_i(y))}{d(x, y)} \leq \lambda_i < 1,$$

for every $x \neq y \in Y$ and $1 \leq i \leq m$.

**Lemma 2.4.** (cf. [11]) Let $Y$ and $\{f_i\}_{i=1}^m$ be as above. Then for any positive numbers $a_i, 1 \leq i \leq m$, with $\sum_{i=1}^m a_i = 1$, there exists a unique Borel probability measure $\mu_0$ such that

$$\mu_0(A) = a_1 \mu_0(f_1^{-1}(A)) + \cdots + a_m \mu_0(f_m^{-1}(A))$$

for every measurable subset $A \subset Y$. In other words,

$$\mu_0 = \sum_{i=1}^m a_i (f_i)_*(\mu_0),$$

where $(f_i)_*(\mu_0)$ is the push-forward measure of $\mu_0$ by $f_i$.

**Proof.** Define the map $F^*(a_1, \ldots, a_m) : (\mathcal{M}(Y), d'_M) \to (\mathcal{M}(Y), d'_M)$ by

$$F^*(a_1, \ldots, a_m)(\mu) = \sum_{i=1}^m a_i (f_i)_*(\mu).$$

If $\phi \in \text{Lip}_1(Y)$, $\phi \circ f_i$ has Lipschitz constant $\leq \lambda_{\text{max}}$, where $\lambda_{\text{max}} = \max\{\lambda_1, \ldots, \lambda_m\}$. This implies that $F^*(a_1, \ldots, a_m)$ is $\lambda_{\text{max}}$-contracting. Since $(\mathcal{M}(Y), d'_M)$ is complete, it has a fixed point $\mu_0$ in $\mathcal{M}(K)$ by the contraction mapping theorem. This completes the proof. \qed
3. Proof of Theorem 1.2

Let \( K \) be the asymptotic self-similar set in a complete doubling metric space \( X \) associated with a \((\{\lambda_i\}_i^{k}, \varphi, \nu)\)-asymptotic similarity system \( \{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}} \). For each \( I = i_1 \cdots i_n \in \mathcal{I}^* \), we set
\[
g_I := f_I \circ \cdots \circ f_{i_1 i_2} \circ f_{i_1} : \bar{V} \to \bar{V}, \quad \bar{V}_I := g_I(\bar{V}) \subset \bar{V}.
\]

Note that
\[
|V_I| \leq \nu^{|I|}|V|.
\]

Let \( s \) be a unique solution of \( \sum_{i=1}^{k} \lambda_i^s = 1 \)

**Lemma 3.1.** Let \( \varphi : (0, \infty) \to [0, \infty) \) be a continuous function satisfying the conditions (1.3). Then
\[
\prod_{i=0}^{\infty} (1 + \varphi(\nu^{|I|}|V|)) < \infty, \quad \prod_{i=0}^{\infty} (1 - \varphi(\nu^{|I|}|V|)) > 0.
\]

**Proof.** By the condition on \( \varphi \), we have
\[
\sum_{i=0}^{\infty} \log(1 + \varphi(\nu^{|I|}|V|)) \leq \sum_{i=0}^{\infty} \varphi(\nu^{|I|}|V|) < \infty.
\]

Similarly we have
\[
\sum_{i=0}^{\infty} \log(1 - \varphi(\nu^{|I|}|V|)) \geq -2 \sum_{i=0}^{\infty} \varphi(\nu^{|I|}|V|) > -\infty.
\]

These complete the proof. \( \square \)

Let \( I = i_1 \cdots i_{m-1} i_m \in \mathcal{I}^* \). We use the notation
\[I_- = i_1 \cdots i_{m-1},\]
and write naturally like \( I = I_- i_m \) as before.

**Lemma 3.2.** \( \dim_H K \leq s \)

**Proof.** By the construction, we have \(|V_{i_1 \cdots i_m}| \leq |V_{i_1 \cdots i_{m-1}}| \nu \). For any \( \epsilon > 0 \) take a sufficiently large \( n \) such that \( \mathcal{U}_n := \{ V_I \mid I \in \mathcal{I}^n \} \) is an \( \epsilon \)-cover of \( K \). From the definition of \((\lambda_i, \varphi, \nu)\)-almost similarity map \( f_I : V_I \to V_I, I = I' i_n \), we have
\[
|V_I| \leq \lambda_n(1 + \varphi(|V_{I'}|)|V_{I'}|).
\]
It follows from Lemma 3.1 that
\[
\mathcal{H}^s(K) \leq \sum_{I \in \mathcal{I}^n} |V_I|^s \\
= \sum_{I' \in \mathcal{I}^{n-1}} (|V_{I'1}|^s + \cdots + |V_{I'k}|^s) \\
\leq \sum_{I' \in \mathcal{I}^{n-1}} (1 + \varphi(|V_I|))|V_I|^s(\lambda_1^s + \cdots + \lambda_k^s) \\
\leq (1 + \varphi(n^{-1}|V|))^s \sum_{I' \in \mathcal{I}^{n-1}} |V_I|^s \\
\leq \cdots < \prod_{i=0}^{\infty} (1 + \varphi(n^i |V|))^s|V| < C|V|,
\]
where \( C \) is a constant, and therefore \( \dim_H K \leq s \).

\[ \square \]

**Lemma 3.3.** Let \( X \) be as in Theorem 1.2, and let \( \mathcal{V} = \{V_i\} \) be a collection of disjoint open sets of \( X \) such that each \( V_i \) contains a closed ball of radius \( c_1 \rho \) and is included in a closed ball of radius \( c_2 \rho \) for some positive constants \( c_1 < c_2 \) and \( \rho \). Then every closed \( \rho \)-ball \( \bar{B}(x, \rho) \) in \( X \) intersects at most \( \mathcal{C}(\delta) \) elements of \( \mathcal{V} = \{\bar{V}_i\} \), where \( \delta = \frac{c_1}{c_1 + 2c_2 + 2} \) and \( \mathcal{C}(\delta) \) is a constant given in Lemma 2.2.

**Proof.** Take \( x_1, x_2 \in X \) satisfying \( \bar{B}(x_1, c_1 \rho) \subset V_1 \subset \bar{B}(x_2, c_2 \rho) \). Let \( \bar{V}_1, \ldots, \bar{V}_N \) intersect \( \bar{B}(x, \rho) \).

Taking any point \( z \in \bar{V}_i \cap \bar{B}(x, \rho) \), we have
\[
d(x_1, x) \leq d(x_1, z) + d(z, x) \leq (2c_2 + 1)\rho.
\]
Furthermore, for any \( y \in B(x_1, c_1 \rho) \), we have
\[
d(y, x) \leq d(y, x_1) + d(x_1, x) < (c_1 + 2c_2 + 1)\rho.
\]
Thus we get
\[
\bigcup_{i=1}^{N} B(x_1, c_1 \rho) \subset B(x, (c_1 + 2c_2 + 1)\rho).
\]
Since \( B(x_1, c_1 \rho) \) are mutually disjoint, from Lemma 2.2 we obtain the conclusion of the lemma. This completes the proof. \[ \square \]

The rest of this section is mainly devoted to prove the following.

**Lemma 3.4.** \( \dim_H K \geq s \).

We set
\[
\bar{V}^n := \bigcup_{I \in \mathcal{I}^n} \bar{V}_I.
\]

Note that
\[
K = \bigcap_{n=1}^{\infty} \bar{V}^n.
\]
For a large \( n_0 \), fix an arbitrary \( I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}^{n_0} \), and consider
\[
\bar{V}_{I_0} := g_{I_0}(V) = f_{I_0} \circ \cdots \circ f_{i_1} \circ f_i(\bar{V}), \quad K_{I_0} := K \cap \bar{V}_{I_0}.
\]
It suffices to prove that \( \dim_H K_{I_0} \geq s \). Therefore we start with
\[
W := V_{I_0},
\]
instead of \( V \).

For every \( 1 \leq i \leq k \), put
\[
h_i := f_{I_0} : W \to \bar{W}_i,
\]
where
\[
\bar{W}_i := h_i(W) \subset \bar{W}.
\]
Recall from the definition
\[
\left| \frac{d(h_i(x), h_i(y))}{d(x, y)} - \lambda_i \right| < o(n_0),
\]
for every \( x \neq y \in \bar{W} \), where
\[
o(n_0) = \lambda_{\max} \varphi(\nu^{n_0}|V|),
\]
and therefore \( \lim_{n_0 \to \infty} o(n_0) = 0 \). For \( J = j_1 \cdots j_m \in \mathcal{I}^s \) and every \( 1 \leq \ell \leq m \), we use the notation
\[
h_{j_1 \cdots j_{\ell}} := f_{I_{j_1 \cdots j_{\ell}}} : \bar{W}_{j_1 \cdots j_{\ell-1}} \to \bar{W}_{j_1 \cdots j_{\ell}},
\]
as before, and define \( g_J : \bar{W} \to \bar{W}_J \) by
\[
g_J := h_{j_1} \circ \cdots \circ h_{j_{j_2}} \circ h_{j_1}.
\]

**Lemma 3.5.** For every \( x \neq y \in \bar{W} \), we have
\[
\left| \frac{d(g_J(x), g_J(y))}{d(x, y)} - \lambda_J \right| < o(n_0)\lambda_J,
\]
where \( \lambda_J = \lambda_{j_1} \cdots \lambda_{j_m} \).

**Proof.** Put \( J_{\ell} := j_1 \cdots j_{\ell} \) for each \( 1 \leq \ell \leq m \). From Lemma 3.4, we obtain
\[
\frac{d(g_J(x), g_J(y))}{d(x, y)} = \frac{d(g_{J_1}(x), g_{J_1}(y))}{d(g_{J_m-1}(x), g_{J_m-1}(y))} \cdots \frac{d(g_{J_{\ell}}(x), g_{J_{\ell}}(y))}{d(g_{J_1}(x), g_{J_1}(y))} \frac{d(g_J(x), g_J(y))}{d(x, y)}
\leq \lambda_J \prod_{\ell=0}^{\infty} (1 + \varphi(\nu^{n_0+\ell}|V|))
= \lambda_J (1 + o(n_0)).
\]
An estimate from below is similar, and hence omitted. \( \square \)

For a small \( \epsilon > 0 \) compared with \( |W| \), let \( \{U_i\} \) be any \( \epsilon \)-covering of
\[
\bar{K} := K_{I_0}.
\]
Replacing \( U_i \) by balls \( B_i \) of radius \( 2|U_i| \), we have a covering \( \{B_i\} \) of \( \bar{K} \). Thus
\[
\sum |U_i|^s \geq 2^{-s} \sum |B_i|.
\]
Fix $B_i$ and take $c_1 > 0$ and $c_2 > 0$ such that $W$ contains a ball of radius $c_1|W|$ and is contained in a ball of radius $c_2|W|$.

**Definition 3.6.** We denote by $\mathcal{I}^\infty$ the set of all infinite sequences $J = j_1j_2 \cdots$ with $j_\ell \in \mathcal{I}$ for all $\ell \geq 1$. We call a finite subset $\mathcal{S}$ of $\mathcal{I}^\ast$ a **simple family** if for each $J = j_1j_2 \cdots \in \mathcal{I}^\infty$, there is a unique $m$ such that $J_m = j_1j_2 \cdots j_m \in \mathcal{S}$.

For instance, $\mathcal{I}_m$ is a simple family for every $m \geq 1$.

**Lemma 3.7.** For every simple family $\mathcal{S}$, we have

$$\sum_{I \in \mathcal{S}} \lambda_I^s = 1.$$ 

**Proof.** Let $m := \max_{I \in \mathcal{S}} |I|$. We prove the lemma by the reverse induction on $m$. Take $I \in \mathcal{S}$ with $|I| = m$, and let $I = i_1 \cdots i_m$. Recall $I_\ell = i_1 \cdots i_{m-1}$ and note that $I_\ell j \in \mathcal{S}$ for all $j \in \mathcal{I}$. It follows that

$$\sum_{j=1}^{k} \lambda_{I_\ell j}^s = \lambda_{I_\ell}^s.$$ 

Set

$$\mathcal{S}_m := \mathcal{S} \cap \mathcal{I}_m, \quad \mathcal{S}' := (\mathcal{S} \setminus \mathcal{S}_m) \cup \{I_\ell \mid I \in \mathcal{S}_m\}.$$ 

Since $\mathcal{S}'$ is a simple family, it follows from the inductive hypothesis that

$$\sum_{I \in \mathcal{S}} \lambda_I^s = \sum_{I \in \mathcal{S}'} \lambda_I^s = 1.$$ 

□

**Assertion 3.8.** For each $i$, there is a simple family $\mathcal{S}_i$ consisting of $J$ satisfying that $W_J$ is contained in a ball of radius $c_2|B_i|$ and contains a ball of radius $\lambda_{\min}c_1c_2|B_i|$ for some uniform constant $0 < \lambda_{\min} \leq \lambda_{\min}^\infty$.

**Proof.** For each $J = j_1j_2 \cdots \in \mathcal{I}^\infty$, there is a unique $m$ such that

(3.8) 

$$|W_{j_1 \cdots j_m} - 1| > c_2|B_i|, \quad |W_{j_1 \cdots j_m}| \leq c_2|B_i|.$$ 

Set $J_m := j_1 \cdots j_m$. Obviously, $W_{J_m}$ is contained in a ball of radius $c_2|B_i|$. Since $W$ contains a ball of radius $c_1|W|$ and since $W_{J_m}$ is open, $W_{J_m}$ contains a ball of radius $(1 - o(n_0))\lambda_{J_m}c_1|W|$. From the choice of $J_m$,

$$(1 - o(n_0))\lambda_{J_m}c_1|W| \geq (1 - o(n_0))^2\lambda_{J_m}c_1c_2|B_i|.$$ 

Let $\mathcal{S}$ be the set of all $J_m \in \mathcal{I}^\ast$ when $J$ runs over $\mathcal{I}^\infty$. (3.16) implies that $\nu^{m-1} \geq c_2|B_i||W|$, and therefore $\mathcal{S}$ is finite. This completes the proof. □

Applying Lemma 2.4 to the contracting maps $g_I : \bar{W} \to \bar{W}$, $I \in \mathcal{S}$, we have
Assertion 3.9. Let $S = S_i$ be as in Assertion 3.8. Then there is a unique Borel probability measure $\mu = \mu_S$ in $\mathcal{M}(\bar{W})$ such that
$$\mu = \sum_{I \in S} \lambda_I^*(g_I)_*(\mu),$$
where $\lambda_I^* = (\lambda_I)^*$. Since $\bar{W} \supset \tilde{K}$, it follows from Lemma 3.5 and the property of $S$ that for any $J \in S$,
$$2^s c_2^s |B_i|^s \geq |W_J|^s \geq (1 - o(n_0)) \lambda_J^* |\tilde{K}|^s.$$
By Lemma 3.3, the number of $\bar{W}_J$ with $J \in S$ meeting $B_i$ is uniformly bounded by some constant $C = C(\delta)$, where $\delta = \delta(c_1, c_2, \lambda_{\min})$. Let $\mu$ be the measure constructed in Assertion 3.9. Then we have
$$\mu(B_i) = \sum_{I \in S} \lambda_I^*(g_I)_*(\mu)(B_i) = \sum_{I \in S} \lambda_I^*(g_I)_*(\mu)(B_i \cap \bar{W}_I) \leq C(\delta) \max_{I \in S, W_I \cap B_i \neq \emptyset} \lambda_I^s.$$
It follows from (3.9) and (3.10) that
$$c_2^s |B_i|^s \geq (1 - o(n_0)) C(\delta)^{-1} |K|^s \mu(B_i).$$
Since
$$\sum_{|J| = m} \lambda_J^s = 1,$$
for each $m \geq 1$, applying Lemma 2.4 to the contracting maps $g_J : \bar{W} \to \bar{W}, J \in \mathcal{T}^m$, we have a unique measure $\mu_m \in \mathcal{M}(\bar{W})$ such that
$$\mu_m = \sum_{|J| = m} \lambda_J^s(g_J)_*(\mu_m).$$
Assertion 3.10. For $m > \max_{I \in S} |I|$, we have $\mu = \mu_m$.

Proof. For each $J \in \mathcal{T}^m$, there are unique $I \in S$ and $J_\alpha \in \mathcal{T}^s$ such that $J = IJ_\alpha$. Let $A_I$ be the set of all the indices $\alpha$ with $J = IJ_\alpha$ for some $J \in \mathcal{T}^m$. We can write as
$$\mu_m = \sum_{I \in S, \alpha \in A_I} \lambda_{IJ_\alpha}^s(g_{IJ_\alpha})_*(\mu_m).$$
By iterating $\ell$-times, we have
$$\mu_m = \sum_{J_1, \ldots, J_\ell \in \mathcal{T}^m} \lambda_{J_1}^s \cdots \lambda_{J_\ell}^s(g_{J_1} \circ \cdots \circ g_{J_\ell})_*(\mu_m) = \sum_{I_i \in S, \alpha_i \in A_{I_i}} \lambda_{I_1J_{\alpha_1}}^s \cdots \lambda_{I_{\ell}J_{\alpha_\ell}}^s(g_{J_1} \circ \cdots \circ g_{J_\ell})_*(\mu_m).$$
Since $A_I = \mathcal{I}^{m-|I|}$, similarly to (3.12) we see
\begin{equation}
\sum_{\alpha \in A_I} \lambda_{J_\alpha}^s = 1.
\end{equation}

It follows that
\[
\mu = \sum_{I \in S} \lambda_I^s(g_I)_\ast(\mu) = \sum_{I \in S, \alpha \in A_I} \lambda_{I,\alpha}^s(g_I)_\ast(\mu).
\]

By iterating $\ell$-times, we obtain
\[
\mu = \sum_{I_\ell, \alpha_\ell \in A_{I_\ell}} \lambda_{I_\ell,\alpha_\ell}^s \cdots \lambda_{I_1,\alpha_1}^s(g_{I_\ell} \circ \cdots \circ g_{I_1})_\ast(\mu).
\]

It follows that
\[
d^*_M(\mu, \mu_m) \leq \sum_{I_\ell, \alpha_\ell \in A_{I_\ell}} \lambda_{I_\ell,\alpha_\ell}^s \cdots \lambda_{I_1,\alpha_1}^s.
\]

Here,
\[
\left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu - \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m \right| \\
\leq \left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu - \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m \right| \\
+ \left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m - \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m \right|.
\]

For a constant $\tilde{\lambda}$ with $\lambda_{\text{max}} < \tilde{\lambda} < 1$, choose a large $n_0$ such that
\[(1 + o(n_0))\lambda_{\text{max}} < \tilde{\lambda} < 1.\]
Then the Lipschitz constant of $g_{I_\ell} \circ \cdots \circ g_{I_1}$ satisfies
\[
L(g_{I_\ell} \circ \cdots \circ g_{I_1}) \leq (1 + o(n_0))^\ell \lambda_{I_\ell} \cdots \lambda_{I_1} < \tilde{\lambda}_{I_\ell \cdots I_1},
\]
where we put $\tilde{\lambda}_{I_\ell \cdots I_1} := (\tilde{\lambda})^{|I_\ell| + \cdots + |I_1|}$. Therefore we obtain
\[
\left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu - \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m \right| \\
\leq \tilde{\lambda}_{I_\ell \cdots I_1} d^*_M(\mu, \mu_m).
\]

On the other hand, from the inclusion
\[g_{I_\ell} \circ \cdots \circ g_{I_1}(\tilde{W}) \supset g_{I_\ell} \circ \cdots \circ g_{I_1}(W),\]
we have
\[
\sup_{x \in \tilde{W}} |\phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1}(x) - \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1}(x)| \\
\leq |g_{I_\ell} \circ \cdots \circ g_{I_1}(\tilde{W})| \\
\leq (1 + o(n_0))^\ell \lambda_{I_\ell} \cdots \lambda_{I_1} < \tilde{\lambda}_{I_\ell \cdots I_1}.
Thus letting $n = \min_{I \in S} |I|$ together with (3.13), we have
\[
d^*_M(\mu, \mu_m) \leq \sum_{I_1, \ldots, I_\ell, \alpha_1, \ldots, \alpha_\ell} \lambda^*_I \cdots \lambda^*_I \tilde{\lambda}^{I_1 \cdots I_\ell} (d^*_M(\mu, \mu_m) + 1)
\]
\[
\leq \tilde{\lambda}^{n\ell} \sum_{I_1, \ldots, I_\ell, \alpha_1, \ldots, \alpha_\ell} \lambda^*_I \cdots \lambda^*_I (d^*_M(\mu, \mu_m) + 1)
\]
\[
= \tilde{\lambda}^{n\ell} \sum_{I_1, \ldots, I_\ell} \lambda^*_I \cdots \lambda^*_I (d^*_M(\mu, \mu_m) + 1)
\]
\[
= \tilde{\lambda}^{n\ell} (d^*_M(\mu, \mu_m) + 1),
\]
which yields
\[
d^*_M(\mu, \mu_m) \leq \frac{1}{1 - \tilde{\lambda}^{n\ell}}.
\]
Letting $\ell \to \infty$, we conclude that $\mu = \mu_m$. \qed

Proof of Lemma 3.1. From the last assertion, we have
\[
supp(\mu) \subset \bigcap_{m=1}^{\infty} \left( \bigcup_{|J|=m} g_J(\bar{W}) \right) = \tilde{K}.
\]
It follows from (3.11) that
\[
\sum 2^{-s} |B_i|^s \geq (1 - o(n_0)) 4^{-s} c^* C(\delta)^{-1} |\tilde{K}| \sum \mu(B_i)
\]
\[
\geq (1 - o(n_0)) 4^{-s} C(\delta)^{-1} |\tilde{K}|.
\]
This shows that $\dim_H \tilde{K} \geq s$. We have completed the proof of lemma 3.1. \qed

Finally we show that

**Lemma 3.11.** $\dim_B K \leq s$.

**Proof.** For every $\epsilon > 0$ and $J_\infty = j_1 j_2 \cdots \in \mathcal{I}^\infty$, take a minimal $m$ satisfying $|W_J| \leq \epsilon$ for $J := J_m = j_1 \cdots j_m$. Note that
\[
|W_J| \geq \lambda_{\min} / 2 |W_J| \geq \epsilon \lambda_{\min} / 2.
\]
Thus we have a simple family $S = \{J \mid J_\infty \in \mathcal{I}^\infty \}$. By Lemma 3.7, we have
\[
\sum_{J \in S} \lambda^*_J = 1.
\]
By Lemma 3.5, we have
\[
\frac{|W_J|}{|W|} - \lambda_J < \lambda J o(n_0).
\]
It follows from (3.14) and (3.16) that
\[
(\epsilon \lambda_{\min} / 2)^s \leq 2^s \lambda^*_J |W|^s.
\]
Using (3.15), we obtain
\[ \sum_{J \in \mathcal{S}} (\epsilon \lambda_{\min}/2)^s \leq 2^s |W|^s. \]
Since \( \{W_J \mid J \in \mathcal{S}\} \) is disjoint, we conclude that
\[ N_{\epsilon}(\tilde{K}) \leq 2^s |W|^s (\epsilon \lambda_{\min}/2)^{-s}. \]
This shows that \( \dim_B \tilde{K} \leq s \), and the conclusion of the lemma follows.

It follows from Lemmas 3.4, 3.11 and (2.5) that \( \dim_H K = \dim_B K = s \). This completes the proof of Theorem 1.2.

Finally we point out that our notion of asymptotic similarity system provides a controlled Moran construction defined in Rajala and Vilppolainen [16]:

**Lemma 3.12.** Let \( \{(\tilde{V}_I, f_I)\}_{I \in \mathcal{I}} \) be a \( (\{\lambda_i\}_{i=1}^k, \varphi, \nu) \)-asymptotic similarity system. Then \( \{\tilde{V}_I\}_{I \in \mathcal{I}} \) is a controlled Moran construction defined in Rajala and Vilppolainen [16]. Namely, there exists a constant \( D \geq 1 \) such that for every \( I, J \in \mathcal{I}^* \)

1. \( \tilde{V}_I \subset \tilde{V}_J \);
2. there exists a positive integer \( n \) such that
   \[ \max_{i \in \mathcal{I}^n} |\tilde{V}_I| < D^{-1}; \]
3. \( D^{-1} \leq \frac{|\tilde{V}_{IJ}|}{|V_I||V_J|} \leq D. \)

**Proof.** (1) is clear. In view of (3.6), (2) is obvious. To show (3), we go back to the situation of Lemma 3.5. Let \( o(n_0) \) be as in (3.7). For a large \( n_0 \), fix an arbitrary \( I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}^{n_0} \), and consider \( W = V_{I_0} \). If we take \( n_0 \) with \( o(n_0) < 1/2 \), we have from Lemma 3.5
\[ \frac{1}{2} \lambda_J |\tilde{W}| < |\tilde{W}_J| < 2\lambda_J |\tilde{W}|, \quad \frac{1}{2} \lambda_J |\tilde{W}| < |\tilde{W}_J| < 2\lambda_J |\tilde{W}|, \]
which imply
\[ \frac{1}{4|W||W_J||W_I|} < |\tilde{W}_{IJ}| < \frac{4}{|W||W_J||W_I|}. \]
Now (3) is immediate, since we have only finitely many choices for \( I_0 \).
4. Sierpinski gaskets on surfaces

Let $D$ be a domain in a complete surface $M$. We assume that $D$ is convex in the sense that for every two points of $D$ there exists a unique minimal geodesic joining them and it is contained in $D$. For simplicity, we assume that the absolute value of the Gaussian curvature of $M$ is at most 1 on $D$. Let $\Delta$ be a domain in $D$ bounded by a geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$. We call $\Delta$ a geodesic triangle region. The set of lengths $\{L(\gamma_i)\}_{i=1}^3$ is called the side-length of $\Delta$.

**Definition 4.1.** We say that $\Delta$ is $\delta$-non-degenerate if each angle $\tilde{\alpha}$ of a comparison triangle $\tilde{\Delta}$ of $\Delta$ in $\mathbb{R}^2$ satisfies $\delta < \tilde{\alpha} < \pi - \delta$, where a comparison triangle means that $\tilde{\Delta}$ has the same side-length as $\Delta$.

In this section, we let $I = \{1, 2, 3\}$. Let $\{\Delta_I\}_{I \in I^*}$ be the system of geodesic triangles obtained by dividing $\Delta$ into smaller triangles $\Delta_I$ consecutively, as stated in Introduction.

**Definition 4.2.** We say that the system $\{\Delta_I\}_{I \in I^*}$ is non-degenerate if there is a $\delta > 0$ such that $\Delta_I$ is $\delta$-non-degenerate for every $I \in I^*$. In this case, we also say that $\Delta$ is asymptotically non-degenerate.

**Example 4.3.** Let $S^2$ denote the unit sphere around the origin in $\mathbb{R}^3$, and let $\Delta$ be a geodesic triangle domain on $S^2$ of perimeter less than $2\pi$. Joining the vertexes $p_1, p_2, p_3$ of $\Delta$ by shortest segments in $\mathbb{R}^3$, we have a geodesic triangle region $\hat{\Delta}$ on the plane through $p_1, p_2, p_3$. By the projection along the rays from the origin of $\mathbb{R}^3$, we have a canonical map

$$\pi: \Delta \to \hat{\Delta},$$

which is a bi-Lipschitz homeomorphism. From a system of geodesic triangles $\{\Delta_I\}_{I \in I^*}$ of $\Delta$, setting $\hat{\Delta}_I := \pi(\Delta_I)$, we have the system of geodesic triangles $\{\hat{\Delta}_I\}_{I \in I^*}$ of $\hat{\Delta}$. Note that each $\hat{\Delta}_I$ is $2^{-|I|}$-similar to $\hat{\Delta}$ in the usual sense. Since $\Delta_I$ is bi-Lipschitz homeomorphic to $\hat{\Delta}_I$,

$$\text{Area}(\Delta_I) \geq L^{-2}\text{Area}(\hat{\Delta}_I),$$

where $L$ is the bi-Lipschitz constant of $\pi$. It follows that $\Delta$ is asymptotically non-degenerate. Now we have the formula (1.4) for the Sierpinski gasket $K_\Delta$ associated with $\Delta$ by two reasons. One is by Theorem 1.3 and the other one is due to the well-known formula for $K_\hat{\Delta}$.

Example 4.3 is the special case. For a geodesic triangle region on a general complete surface, it seems impossible to reduce the problem to a triangle region in $\mathbb{R}^2$.

The main purpose of this section is to prove the following result.

**Theorem 4.4.** For every $\delta > 0$ there exists an $r > 0$ such that

1. every geodesic triangle region $\Delta$ on $D$ with $|\Delta| \leq r$ is asymptotically non-degenerate;
(2) the Hausdorff and box dimensions of the Sierpinski gasket $K_\Delta$ associated with $\Delta$ are given by (1.4).

If $\Delta$ be asymptotically non-degenerate as in Theorem 1.3, we can apply Theorem 4.4 to $\Delta_I$ for each $I \in \mathcal{I}$ with large enough $|I|$. Therefore Theorem 4.4 yields Theorem 1.3.

The following lemma is a consequence of law of cosine, and hence is omitted.

**Lemma 4.5.** For any $\delta > 0$ there exists an $\epsilon > 0$ such that if a geodesic triangle $\Delta$ of side length $(a_1, a_2, a_3)$ is $\delta$-non-degenerate, and if the side length $(a_1', a_2', a_3')$ of a geodesic triangle $\Delta'$ satisfies

$$(4.17) \quad (1 - \epsilon) \frac{a_j}{a_i} < \frac{a_j'}{a_i'} < (1 + \epsilon) \frac{a_j}{a_i},$$

for any $i \neq j$, then $\Delta'$ is $\delta/2$-non-degenerate.

**Proof.** We may assume that $\Delta$ and $\Delta'$ are triangles in $\mathbb{R}^2$. Set $(a, b, c) := (a_1, a_2, a_3)$ and $(a', b', c') := (a_1', a_2', a_3')$ for simplicity. Rescaling $\Delta'$, we may assume that $c = c'$. It suffices to show that if $\Delta'$ has side-length $(a', b', c') = (a', b, c)$ satisfying (4.17), then the angles $\alpha, \beta$ (resp. $\alpha', \beta'$) opposite to the edges of length $a$ and $b$ in $\Delta$ (resp $a'$ and $b$ in $\Delta'$) satisfy that $|\alpha' - \alpha| < \delta/4$ and $|\beta' - \beta| < \delta/4$ for a suitable $\epsilon = \epsilon(\delta) > 0$.

**Sublemma 4.6.** If a geodesic triangle $\Delta$ of side lengths $(a_1, a_2, a_3)$ is $\delta$-non-degenerate, then there exists a constant $C(\delta)$ such that

$$C(\delta)^{-1} < \frac{a_j}{a_i} < C(\delta),$$

for every $1 \leq i, j \leq 3$.

**Proof.** This is an immediate consequence of the law of sines. One can take $C(\delta) = 1/ \sin \delta$. \hfill $\Box$

By trigonometry, we have

$$\sin^2 \alpha/2 = (a + c)(a + b)/bc, \quad \sin^2 \alpha'/2 = (a' + c)(a' + b)/bc.$$

It follows from the assumption and Sublemma 4.6 with $|a' - a| < \epsilon a$ that

$$(4.18) \quad |\sin^2 \alpha'/2 - \sin^2 \alpha/2| \leq a(a + a'b + c)\epsilon/bc \leq 5C(\delta)^2 \epsilon.$$

Since $\sin \alpha'/2 + \sin \alpha/2 > \sin(\delta/2)$, we obtain

$$|\sin \alpha'/2 - \sin \alpha/2| \leq 5C(\delta)^2 \epsilon/ \sin(\delta/2).$$

From $\alpha < \pi - 2\delta$, we have $\cos \frac{\alpha + \alpha}{4} > \sin(\delta/4)$. It follows that

$$(4.19) \quad |\alpha' - \alpha| \leq 8 \left| \sin \frac{\alpha' - \alpha}{4} \right| < 5C(\delta)^2 \epsilon/ \sin^2(\delta/4).$$
Similarly we have
\[
|\sin^2\beta'\frac{1}{2} - \sin^2\beta\frac{1}{2}| = |a - a'|b(b + c)/aa'c \leq b(b + c)e/ca' \\
\leq \frac{\epsilon}{1 - \epsilon}\frac{b(b + c)}{a} \leq \frac{\epsilon}{1 - \epsilon}2C(\delta)^2,
\]
which implies
\[
(4.20) \quad |\beta' - \beta| < 8\epsilon \left(\frac{C(\delta)}{\sin(\delta/2)}\right)^2.
\]
Thus from (4.19), (4.20), we obtain
\[
|\alpha' - \alpha| < \delta/4 \quad \text{and} \quad |\beta' - \beta| < \delta/4
\]
for a suitable \(\epsilon \leq \epsilon(\delta)\). This completes the proof. \(\square\)

Let \(\Delta\) be a geodesic triangle region on \(D\) bounded by a geodesic triangle \((\gamma_1, \gamma_2, \gamma_3)\) with vertices \(p_1, p_2, p_3\). By the convexity of \(D\), we have
\[
|\Delta| = \max_{1 \leq i \leq 3} a_i,
\]
where we put \(a_i := L(\gamma_i)\). Fix a vertex \(p_1\) and let \(\gamma_i\) be parametrized on \([0, 1]\) in such a way that \(\gamma_2(0) = \gamma_3(0) = p_1\). Let \(\varphi: [0, 1] \times [0, 1] \rightarrow \Delta\) be a parametrization of \(\Delta\) such that \(t \rightarrow \varphi(t, s), 0 \leq t \leq 1\), is the geodesic, denoted by \(\sigma_s\), from \(\gamma_2(s)\) to \(\gamma_3(s)\) for each \(s \in [0, 1]\). Namely \(\varphi(t, s) = \sigma_s(t)\). We set
\[
a_1(s) := L(\sigma_s).
\]
Now define the map \(f_1: \Delta \rightarrow \Delta\) by
\[
f_1(\varphi(t, s)) = \varphi(t, s/2).
\]
Note that the image \(\Delta_1\) of \(f_1\) is the geodesic triangle region bounded by \((\gamma_2|_{[0, 1/2]}, \gamma_3|_{[0, 1/2]}, \sigma_{1/2})\) and that \(\Delta_1\) has side-length \((a_1(1/2), a_2/2, a_3/2)\). We put
\[
r := |\Delta|.
\]

**Lemma 4.7.** For any \(s \in (0, 1)\), we have
\[
1 - r^2 < \frac{a_1(s)}{sa_1} < 1 + r^2.
\]
In particular, \(|\Delta_1| \leq \frac{1}{2}(1 + r^2)|\Delta|\).

**Proof.** Let \(\tilde{\gamma}_i(s) := \exp_{p_1}^{-1}(\gamma_i(s)), i = 2, 3\). The Rauch comparison theorem (see [4]) implies
\[
(4.21) \quad \frac{\sin r}{r} < \frac{a_1}{d(\tilde{\gamma}_2(1), \tilde{\gamma}_3(1))} < \frac{\sinh r}{r}
\]
\[
(4.22) \quad \frac{\sin r}{r} < \frac{a_1(s)}{d(\tilde{\gamma}_2(s), \tilde{\gamma}_3(s))} < \frac{\sinh r}{r}.
\]
Since \(d(\tilde{\gamma}_2(s), \tilde{\gamma}_3(s)) = sd(\tilde{\gamma}_2(1), \tilde{\gamma}_3(1))\), the conclusion follows. \(\square\)
Let us denote by \((a_{1,1}, a_{1,2}, a_{1,3})\) the side length \((a_1(1/2), a_2/2, a_3/2)\)
of \(\Delta_1\). Lemma 4.7 implies that
\[
(1 - r^2)\frac{a_i}{a_j} < \frac{a_{1,i}}{a_{1,j}} < (1 + r^2)\frac{a_i}{a_j},
\]
for every \(1 \leq i, j \leq 3\).

In a similar way, we construct a map \(f_{i_1}: \Delta \to \Delta_{i_1} \subset \Delta\) for each \(1 \leq i_1 \leq 3\). Repeating this procedure for each \(\Delta_i\) inductively, for each multi-index \(I = i_1 \cdots i_{n-1}i_n\), we have a geodesic triangle region \(\Delta_I\) and a map \(f_I: \Delta_I \to \Delta_I\), where \(I' = i_1 \cdots i_{n-1}\). The side-length \((a_{I,1}, a_{I,2}, a_{I,3})\) of \(\Delta_I\) is also suitably defined inductively. Take \(r < 1\) and set
\[
\nu := 1/2(1 + r^2) < 1.
\]

**Lemma 4.8.** There exists an \(L(r) > 1\) such that for every \(I\) and \(1 \leq i, j \leq 3\)
\[
L(r)^{-1}\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < L(r)\frac{a_i}{a_j},
\]

**Proof.** Repeating use of (4.23) and Lemma 4.7 applied to \(s = 1/2\) implies that for each \(I = i_1 \cdots i_m\),
\[
(1 - r_m^2) \cdots (1 - r_1^2)(1 - r^2)\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < (1 + r_m^2) \cdots (1 + r_1^2)(1 + r^2)\frac{a_i}{a_j}
\]
for every \(1 \leq i, j \leq 3\), where \(r_k := |\Delta_{i_1 \cdots i_k}|, 1 \leq k \leq m\). Since \(r_k \leq 1/2(1 + r_{k-1}^2)r_{k-1} < \nu r_{k-1} < \cdots < \nu^k r\).

it follows that
\[
\prod_{m=0}^{\infty} \left(1 - \nu^{2m-2}\right)\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < \prod_{m=1}^{\infty} \left(1 + \nu^{2m-2}\right)\frac{a_i}{a_j}.
\]
This completes the proof. \(\square\)

From (4.24), one can take \(L(r)\) as
\[
L(r) := e^{2r^2/2r^2}.
\]

For every \(s \in (0,1]\) we denote by \(\Delta(1 : s)\) the geodesic triangle \((\gamma_2|_{\{0,s\}}, \gamma_3|_{\{0,s\}}, \sigma_s)\). Similarly, \(\Delta(i : s)\) and \(\Delta_I(i : s)\) are defined for every \(1 \leq i \leq 3\) and every multi-index \(I \in \mathcal{I}^*\).

Lemmas 4.5, 4.7 and 4.8 imply

**Lemma 4.9.** For every \(\delta > 0\), there exists a positive number \(r\) such that if \(\Delta\) is \(\delta\)-non-degenerate and the diameter \(|\Delta|\) of \(\Delta\) is less than \(r\), then \(\Delta_I\) as well as \(\Delta_I(i : s)\) is \(\delta/2\)-non-degenerate for every multi-index \(I, 1 \leq i \leq 3\) and \(s \in (0,1]\).
By Lemma 4.9, we get the conclusion (1) of Theorem 4.4. In view of Theorem 1.2, to prove the conclusion (2) of Theorem 4.4, it suffices to prove the following.

**Theorem 4.10.** There is a positive number \( c = c(\delta) \) such that \( \{(\Delta_I, f_I)\}_{I \in \mathcal{I}} \) gives a \( (1/2, \varphi_c, \nu) \)-asymptotic similarity system, where \( \varphi_c(x) = cx^2 \).

**Proof.** In view of Lemma 4.9, it suffices to prove that the map \( f := f_1 : \Delta \to \Delta_1 \subset \Delta \) is a \( (1/2, \varphi_c, \nu) \)-almost similarity map for a uniform positive constant \( c = c(\delta) \). Note that \( J_s(t) := \frac{\partial \varphi_c}{\partial s}(t, s) = \sigma_s(t) \). Observe that

\[
\tag{4.25}
df(T_s(t)) = T_{s/2}(t), \quad df(J_s(t)) = \frac{1}{2} J_{s/2}(t).
\]

Lemma 4.7 shows that

\[
\left| \frac{L(\sigma_{s/2})}{L(\sigma_s)} - \frac{1}{2} \right| < 3r^2,
\]

which implies that

\[
\tag{4.26}
\left| \frac{|df(T_s)|}{|T_s|} - \frac{1}{2} \right| < 3r^2.
\]

Next we show

**Lemma 4.11.** For every \( s, u \in (0, 1] \) and \( t \in [0, 1] \), we have

\[
\tag{4.27}
\left| \frac{|J_u(t)|}{|J_s(t)|} - 1 \right| < C(\delta)r^2.
\]

From now on, we shall use the general symbols \( C(\delta) \) or \( c(\delta) \) to denote constants depending only on \( \delta \) unless otherwise stated.

**Proof.** For any fixed \( s \), take unique Jacobi fields \( Y_1 \) and \( Y_2 \) along \( \sigma_s \) and the reverse geodesic \( \sigma_s^-(t) := \sigma(1 - t) \) respectively such that

\[
Y_1(0) = 0, \quad Y_1(1) = J_s(1), \quad Y_2(1) = J_s(0), \quad Y_2(0) = 0,
\]

to have

\[
J_s(t) = Y_1(t) + Y_2(1 - t).
\]

We denote by \( S^2 \) and \( H^2 \) the sphere and the hyperbolic plane of constant curvature 1 and \(-1\) respectively.

Recall that \( \Delta \) is a \( \delta \)-non-degenerate geodesic triangle region of side lengths \( (a_1, a_2, a_3) \) in \( D \) whose diameter is denoted by \( r \).

**Lemma 4.12.** Let \( \alpha_{i+} \) and \( \alpha_{i-} \) be the angles of comparison triangles \( \Delta_+ \) and \( \Delta_- \) of \( \Delta \) in \( S^2 \) and \( H^2 \) respectively at the vertices opposite to the edge of length \( a_i \). Then we have

\[
\left| \alpha_{i+} - \alpha_{i-} \right| < C(\delta)r^2.
\]
Proof. Put $(a, b, c) := (a_1, a_2, a_3)$, and let $\alpha_+, \alpha_-$ and $\alpha$ be the angles of comparison triangles of $\Delta$ in $S^2$, $H^2$ and $\mathbb{R}^2$ respectively at the vertices opposite to the edge of length $a$. By the laws of cosines, we have
\[
\sin b \sin c \cos \alpha_+ = \cos a - \cos b \cos c \\
\sinh b \sinh c \cos \alpha_- = \cosh b \cosh c - \cosh a \\
2bc \cos \alpha = b^2 + c^2 - a^2,
\]
which imply
\[
2bc \cos \alpha_+ = 2bc \cos \alpha + O(b^3 c) + O(bc^3) + O(a^4) \\
2bc \cos \alpha_- = 2bc \cos \alpha + O(b^3 c) + O(bc^3) + O(a^4).
\]

It follows from Sublemma 4.6 that

\[
| \cos \alpha_+ - \cos \alpha | \leq O(b^2) + O(c^2) + O(a^4/bc) \\
\leq C(\delta) r^2.
\]

Since $\delta < \alpha < \pi - \delta$, we obtain $|\alpha_+ - \alpha| \leq C(\delta) r^2$. Similarly we get $|\alpha_- - \alpha| \leq C(\delta) r^2$, and hence $|\alpha_+ - \alpha_-| \leq C(\delta) r^2$. \qed

Let $\alpha_s$ and $\beta_s$ be the angle of the geodesic triangle $\Delta(1 : s) = (\gamma_2|_{s}, \gamma_3|_{s}, \sigma_s)$ at $\gamma_2(s)$ and $\gamma_3(s)$ respectively.

**Lemma 4.13.**

\[
|\alpha_s - \alpha_t| < c(\delta) r^2, \quad |\beta_s - \beta_t| < c(\delta) r^2,
\]

for every $s, t \in (0, 1]$.

**Proof.** Let $\alpha_s^+, \alpha_s^-$, $\alpha_t^0$ denote the angles of comparison triangles in $S^2$, $H^2$, and $\mathbb{R}^2$ respectively at the vertices corresponding $\gamma_2(s)$. By Toponogov’s theorem (cf. [5]), we have

\[
(4.28) \quad \alpha_s^- \leq \alpha_s, \quad \alpha_t^0 \leq \alpha_s^+.
\]

By the law of cosines, we have
\[
\cos \alpha_s^0 = \frac{a_2^2 + (a_1(s)/s)^2 - a_3^2}{2a_2(a_1(s)/s)} \\
\cos \alpha_t^0 = \frac{a_2^2 + (a_1(t)/t)^2 - a_3^2}{2a_2(a_1(t)/t)}.
\]
By Lemma 4.9, we have
\[ \cos \alpha_s^0 - \cos \alpha_t^0 \]
which imply with Lemma 4.7
\[ \frac{a_1^2 + a_2^2(1 + r^2) - a_3^2}{2a_2a_1(1 - r^2)} - \frac{a_1^2 + a_2^2(1 - r^2) - a_3^2}{2a_2a_1(1 + r^2)} = \frac{r^2(2a_1^2 + a_2^2 - a_3^2)}{a_1a_2(1 - r^2)(1 + r^2)} \]
\[ = \frac{r^2}{1 - r^2} \left( \frac{2a_1}{a_2} + \frac{a_2}{a_1} - \frac{a_3^2}{a_1a_2} \right) \leq C(\delta)r^2. \]

Reversing the role of $s$ and $t$, we have
\[ |\cos \alpha_s^0 - \cos \alpha_t^0| \leq C(\delta)r^2. \]

By Lemma 4.13, we have $\delta/2 < (\alpha_s^0 + \alpha_t^0)/2 < \pi - \delta/2$, which implies $\sin \frac{\alpha_s^0 + \alpha_t^0}{2} > \sin(\delta/2)$. Therefore we conclude that
\[ |\alpha_s^0 - \alpha_t^0| \leq 4 \left| \sin \left( \frac{\alpha_s^0 - \alpha_t^0}{2} \right) \right| \leq C_1(\delta)r^2, \]
where $C_1(\delta) := \frac{2C(\delta)}{\sin(\delta/2)}$. Using (4.28) and Lemma 4.12, we see
\[ \alpha_s \leq \alpha_s^0 + C(\delta)r^2 \leq \alpha_t^0 + C(\delta)r^2 + C_1(\delta)r^2 \leq \alpha_t + 2C(\delta)r^2 + C_1(\delta)r^2. \]

Reversing the role of $s$ and $t$ completes the proof. \qed

Next we analyze the behavior of the norm of Jacobi field $J_s$. For a fixed $s \in (0, 1]$, let $Y_i(t) = Y_i^N(t) + Y_i^T(t)$, $i = 1, 2$, be the orthogonal decompositions of $Y_i$ to the normal and tangential components to $\hat{\sigma}_s$. We can write $Y_1(t)$ and $Y_1(t)^N$ as
\[ Y_1(t) = d \exp_{\gamma_2(s)}(t(V_1)_{\hat{\sigma}_s(0)}), \quad Y_2(t) = d \exp_{\gamma_3(s)}(t(V_2)_{\hat{\sigma}_s(0)}), \]
(4.29)
\[ Y_1^N(t) = d \exp_{\gamma_2(s)}(t(V_1^N)_{\hat{\sigma}_s(0)}), \quad Y_2^N(t) = d \exp_{\gamma_3(s)}(t(V_2^N)_{\hat{\sigma}_s(0)}), \]
(4.30)
where $V_1$ and $V_2$ are some parallel vector fields on the tangent spaces satisfying
\[ d \exp_{\gamma_2(s)}((V_1)_{\hat{\sigma}_s(0)}) = \gamma_3(s), \quad d \exp_{\gamma_3(s)}((V_2)_{\hat{\sigma}_s(0)}) = \gamma_2(s). \]
The Rauch comparison theorem shows that
\[ |Y_1^N(t)| \equiv t|V_1^N| \equiv t|\gamma_3(t)^N|, \quad |Y_2^N(1-t)| \equiv (1-t)|V_2^N| \equiv (1-t)|\gamma_2(t)^N|. \]
Here and hereafter we use the symbol \( a \equiv b \) whenever \( |a - b| < C(\delta)r^2 \).

It follows from \( \dim M = 2 \) that
\[
|J_N^s(t)| = |Y_1^N(t)| + |Y_2^N(1 - t)|
\]
\[
\equiv t|\dot{\gamma}_3(t)| + (1 - t)|\dot{\gamma}_2(t)|
\]
\[
= t \sin \beta_s a_3 + (1 - t) \sin \alpha_s a_2,
\]
where we recall \( a_i = L(\gamma_i) = |\dot{\gamma}_i(t)| \).

Similarly we have
\[
|J_u^N(t)| \equiv t \sin \beta_u a_3 + (1 - t) \sin \alpha_u a_2.
\]

It follows from that
\[
|J_N^s(t)| \equiv |J_u^N(t)|.
\]

Next we show that
\[
|J_T^s(t)| \equiv |J_T^u(t)|.
\]

We use the expression (4.29) with Gauss’s lemma to obtain
\[
\langle Y_1(t), T_s(t) \rangle = ta_3|T_s| \cos \beta_s,
\]
\[
\langle Y_2(t), T_s(t) \rangle = -(1 - t)a_2|T_s| \cos \alpha_s.
\]

Thus we get
\[
|J_T^s(t)| = |ta_3 \cos \beta_s - (1 - t)a_2 \cos \alpha_s|.
\]

From an inequality for \( |J_T^s(t)| \) similar to the above and Lemma 4.13 we have (4.35). Now (4.27) follows from (4.34), (4.35). Thus we have completed the proof of Lemma 4.11.

The expression (4.29) also yields
\[
|Y_1(t)| \equiv t|V_1| \equiv ta_3, \quad |Y_2(1 - t)| \equiv (1 - t)|V_2| \equiv (1 - t)a_2.
\]

In particular we have
\[
|J_s(t)| \leq 2r.
\]

Since \( |J_N^N(t)| \geq c(\delta)r \) from (4.33), (4.36) implies that the angle \( \theta_s(t) := \angle(J_s(t), T_s(t)) \) has definite lower and upper bounds:
\[
0 < c(\delta) \leq \theta_s(t) \leq \pi - c(\delta).
\]

(4.25), (4.26), (4.27) and (4.37) yield that
\[
\left| \frac{|df(v)|}{|v|} - \frac{1}{2} \right| < C(\delta)r^2,
\]
for every tangent vector \( v \). Thus we conclude that \( f : \Delta \to \Delta_1 \) is a \((1/2, \varphi_{C(\delta)}, \nu)\)-almost similarity map, with \( \varphi_{C(\delta)}(x) = C(\delta)x^2 \). This completes the proof of Theorem (2) 4.10.
Proof of Corollary 1.4. In view of Theorem 1.2, it suffices to show that for a geodesic triangle region $\Delta$ on a convex domain of a complete surface, if the collection $\{ (\Delta_I, f_I) \}_{I \in I^*}$ gives a $\{1/2, 1/2, 1/2\}, \varphi_C, \nu$-asymptotic similarity system with $\varphi_C(x) = Cx^2$ and $0 < \nu < 1$, then $\Delta$ is asymptotically non-degenerate.

For a large $n_0$, fix an arbitrary $I_0 = i_1 \cdots i_{n_0} \in I_{n_0}$, and set

$$W := \Delta_{I_0} = g_{I_0}(\Delta) = f_{i_0} \circ \cdots \circ f_{i_{n_0}}(\Delta).$$

For every $1 \leq i \leq k$, put $h_i := f_{i_{i_0}} : W \to W_i = h_i(W) \subset W$,

and recall from the definition

$$\left| \frac{d(h_i(x), h_i(y))}{d(x, y)} - \lambda_i \right| < o(n_0),$$

where $o(n_0) = \lambda_i \varphi(\nu^{n_0} |\Delta|)$ and therefore $\lim_{n_0 \to \infty} o(n_0) = 0$. For $J = j_1 \cdots j_m$, define $g_J : W \to W_J$ by

$$g_J := h_{j_1} \circ \cdots \circ h_{j_{j_2}} \circ h_{j_1},$$

where we use the notation $h_{j_1 \cdots j_k} := f_{j_{j_1} \cdots j_k} : W_{j_1 \cdots j_k-1} \to W_{j_1 \cdots j_k}$, as before. By Lemma 3.5 we have

$$\left| \frac{d(g_J(x), g_J(y))}{d(x, y)} - \lambda_J \right| < o(n_0) \lambda_J,$$

for every $x, y \in W$. We denote by $\text{inrad}(W)$, the inradius of $W$, the largest $r > 0$ such that an $r$-ball is contained in $W$. It follows that

$$\frac{|W_J|}{\text{inrad}(W_J)} \leq \frac{1 + o(n_0)}{1 - o(n_0)} \frac{|W|}{\text{inrad}(W)},$$

for every $J \in I^*$. This implies that there exists a $\delta > 0$ such that $\Delta_I$ is $\delta$-nondegenerate for every $I \in I^*$. \hfill $\square$

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