No-Slip Boundary Condition for Vorticity Equation in 2D Exterior Domain

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Abstract. In this article we derive the no-slip boundary condition for a non-stationary vorticity equation. This condition generates the affine invariant manifold and no-slip integral relations on vorticity can be transferred to a Robin-type boundary condition. This condition leads to the appearance of the non-trivial kernel in Laplace operator and as result it causes the presence of the stationary solution of linearised equation. The no-slip condition causes an orthogonality of the flow to this kernel which is consistent with the Stokes Paradox.

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Preliminary

Fluid dynamics equations written in terms of vorticity favourably differ from ones covering velocity evolution. So, for both 2D and 3D Navier–Stokes systems the vorticity dynamics involves a fewer number of equations rather than in the velocity-pressure formulation. For barotropic fluid the curl removes the most problematic term—pressure. If the system doesn’t involve boundary condition (e.g. Cauchy problem), then the curl operator significantly simplifies corresponding initial-value problem. But in the case of initial-boundary-value problem it isn’t true since Dirichlet no-slip boundary turns out to be a sophisticated integral condition. And so, vorticity boundary conditions mostly are deduced in vorticity-stream
formulation [1–4]. In [5] was studied the parametric boundary condition covering vorticity and stream. Some new results on Newman and Dirichlet boundary conditions for vorticity on solid walls were given in [6].

We will investigate the boundary vorticity which corresponds to the no-slip condition. At first sight there is no analogous to no-slip condition only in terms of vorticity without additional functions involved (e.g. stream function, velocity). Biot–Savar law restores the solenoidal velocity field \( \mathbf{v}(\mathbf{x}) \) induced by vorticity \( \omega(\mathbf{x}) \). It expresses vorticity via velocity field by some integral relation. If the flow interacts with solid by no-slip condition, then it turns to zero integral relation in Biot–Savar law which doesn’t admit explicit integration.

In this paper will be established vorticity boundary condition for both linear and nonlinear Helmholtz equations without any stream function involved. For the linear Stokes system in the exterior of the disc such boundary condition along with explicit formula to Stokes problem was obtained by the author in [7]. In this article we show that for nonlinear vorticity equation in the exterior of the disc of radius \( r_0 \) the boundary condition can be defined in terms of vorticity Fourier coefficients \( w_k(t, r) \) as Robin-type boundary problem:

\[
\frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + |k|w_k(t, r_0) = u_k(t).
\]

And the same boundary-value problem will be actual in more general domains with help of Riemann mapping.

For exterior flows it is typical to study solutions with infinite energy. These solutions with finite Dirichlet integral were studied in [8,9]. For Cauchy problem global existence and bounds on velocity as time \( t \to \infty \) were established in [10–12]. In this paper we will research the vorticity equation for infinite energy solutions. The main difficulty is the fact that the space \( L_2 \) is not enough to describe vorticity dynamics. The solutions with finite vorticity energy but nonzero total circulation may have infinite kinetic energy. But in case of zero circulation it’s not getting considerably better. In order to restore velocity via Biot–Savar law and correctly define the nonlinear term (\( \mathbf{v}, \nabla \omega \)) we need to impose additional requirements on the phase space.

The Laplace operator with Robin boundary condition (0.1) possesses the non-trivial kernel and as result it causes the presence of the stationary solution for the corresponding evolution equation. But from the no-slip condition follows an orthogonality relation between the solution and the kernel (see [7] for more details). It is entirely consistent with the Stokes Paradox.

The paper is arranged as follows. In Sect. 1 we study Biot–Savar law and the integral relation for the no-slip condition. Then in Sect. 2 we will derive the precise integral boundary condition for Stokes system and its local approximation for Helmholtz vorticity equation. In Sect. 3 we will prove the local existence of no-slip condition for vorticity. In Appendix the solvability of the vorticity equation with boundary (0.1) will be established.

**Notations:** In the paper we will exploit \( \mathbb{R}^2 \) as real as well as a complex plane depending on the context. For points from \( \mathbb{R}^2 \) we will use different real and complex notations including polar coordinates \( r, \varphi \) such as \( \mathbf{x} = (x_1, x_2), z = x_1 + ix_2 = re^{i\varphi} \). For functions defined on \( E \subset \mathbb{R}^2 \) along with classic spaces \( L_p(E) \) we will use \( L_p(E; r), L_p(E; \lambda) \) of square-integrable functions with infinitesimals \( r \) \( \lambda \) supplied with norms

\[
\|f\|_{L_p(E; r)} = \int_E |f(r)|^p r \, dr,
\]

\[
\|f\|_{L_p(E; \lambda)} = \int_E |f(\lambda)|^p \lambda d\lambda.
\]

Velocity \( \mathbf{v} \) will be used in both Cartesian \((v_1, v_2)\) and polar coordinates \((v_r, v_\varphi)\).

For vector field \( \mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x})) \) the curl operator is defined as

\[
\omega(\mathbf{x}) = \text{curl} \mathbf{v}(\mathbf{x}) = \partial_{x_1} v_2 - \partial_{x_2} v_1.
\]
Fourier coefficients for vorticity $w(x)$ will be referred to as $w_k$ with prefix $k$:
\[
w_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w_k(r, \varphi) e^{-ik\varphi} d\varphi.
\]

For velocity components $(v_1, v_2)$ and $(v_r, v_\varphi)$ their Fourier coefficients will be designated as $(v_{1,k}, v_{2,k})$ and $(v_{r,k}, v_{\varphi,k})$. For Laplace operator
\[
\Delta f(r, \varphi) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2}
\]
it its Fourier expansion will involve $\Delta_k$ defined as
\[
\Delta_k f_k(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{df_k}{dr} \right) - \frac{k^2}{r^2} f_k(r).
\]

1. Biot–Savar Law in Exterior Domains and No-Slip Integral Condition

Under exterior domain we will understand the complement to some bounded simple-connected closed set: $\Omega = \mathbb{R}^2 \setminus \mathbb{C}$. Now we study when the solenoidal velocity field $v(x)$ can be uniquely restored from its vorticity $w(x)$. Consider the following elliptic problem in exterior domain $\Omega$:
\[
\begin{align*}
\text{div} v(x) &= 0, \quad (1.1) \\
\text{curl} v(x) &= w(x), \quad (1.2) \\
v(x) &= 0, \quad x \in \partial \Omega, \quad (1.3) \\
v(x) &\to v_{\infty}, \quad |x| \to \infty, \quad (1.4)
\end{align*}
\]

where $v_{\infty} \in \mathbb{R}^2$ is the fixed stream velocity at infinity.

Exterior domains are not simply connected and Eqs. (1.1), (1.2) supplied with slip condition on the boundary
\[
(v(x), n) = 0, \quad x \in \partial \Omega, \quad (1.5)
\]
and fixed flow at infinity (1.4) have a unique solution only if we fix circulation at infinity ($n$ is an outer normal to the boundary). No-slip condition (1.3) is stronger than (1.5), and so some additional restrictions on $w(x)$ are required. These restrictions can be realized via moment relations for vorticity. In [7] the solvability of the system above was researched in detail for slip and no-slip conditions in the exterior of the disc. Here we extend these results on more general domains and obtain integral no-slip condition.

We will fix circulation at infinity. From a physical point of view it is natural to suppose zero-circulation:
\[
\lim_{R \to \infty} \oint_{|x| = R} v \cdot dl = 0. \quad (1.6)
\]

Then the solution of the above problem if it exists is given by Biot–Savar formula [7,13]
\[
v(x) = \frac{1}{2\pi} \int_{\Omega} \frac{(x - y)^\perp}{|x - y|^2} w(y) dy + v_\infty, \quad (1.7)
\]
where $x^\perp = (-x_2, x_1)$.

We rewrite this formula in polar coordinates. Boundary condition (1.3) obliges the vorticity $w(y)$ to satisfy some additional restrictions.

The relationship between Cartesian and polar coordinate systems for $v_{\infty} = (v_{1,\infty}, v_{2,\infty})$ is given by formulas:
\[
\begin{align*}
v_{r,\infty} &= v_{1,\infty} \cos \varphi + v_{2,\infty} \sin \varphi, \\
v_{\varphi,\infty} &= v_{2,\infty} \cos \varphi - v_{1,\infty} \sin \varphi.
\end{align*}
\]
Then its Fourier coefficients are determined as
\[
v_r^\infty = \frac{\delta_{|k|, 1}}{2} (v_1^\infty - ikv_2^\infty),
\]
\[
v_\varphi^\infty = \frac{\delta_{|k|, 1}}{2} (v_2^\infty + ikv_1^\infty),
\]
and
\[
v_\varphi^\infty = \text{sign}(k)iv_r^\infty.
\]
Here \(\delta_{|k|, 1} = \begin{cases} 1, & |k| = 1, \\ 0, & |k| \neq 1 \end{cases}\) is the Kronecker symbol.

All Fourier coefficients of the external flow equal to zero except \(k = \pm 1\). For horizontal flow \(v_\infty = (v_\infty, 0)\)
\[
v_r^\infty = \frac{\delta_{|k|, 1}}{2} v_\infty,
\]
\[
v_\varphi^\infty = ik \frac{\delta_{|k|, 1}}{2} v_\infty.
\]

1.1. Biot–Savar Law in Exterior of the Disc

In this subsection the domain under investigation will be the exterior of the disc \(B_{r_0} = \{x \in \mathbb{R}^2, |x| > r_0\}, r_0 > 0\). We will derive Biot–Savar law and no-slip boundary condition for Stokes and Navier–Stokes systems in integral form.

In polar coordinates Eqs. (1.1), (1.2) can be written in Fourier coefficients \(v_{r,k}, v_{\varphi,k}\):
\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_{r,k}) + \frac{ik}{r} v_{\varphi,k} = 0,
\]
\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_{\varphi,k}) - \frac{ik}{r} v_{r,k} = w_k.
\]

The basis for solutions of homogeneous system when \(w_k \equiv 0\) consists of two vectors:
\[
\begin{pmatrix} v_{r,k}^1 \\ v_{\varphi,k}^1 \end{pmatrix} = \begin{pmatrix} ir^{k-1} \\ r^{k-1} \end{pmatrix},
\]
\[
\begin{pmatrix} v_{r,k}^2 \\ v_{\varphi,k}^2 \end{pmatrix} = \begin{pmatrix} ir^{-k-1} \\ -r^{-k-1} \end{pmatrix}.
\]

The solution of this system with boundary relations (1.3), (1.4) and zero-circulation condition (1.6) was derived in [7] as Biot–Savar law in exterior of the disc in the following form for \(k \in \mathbb{Z}\):
\[
v_{r,k} = \text{sign}(k) \frac{ir^{-|k|-1}}{2} \int_{r_0}^{r} s^{|k|+1} w_k(s) \, ds + \text{sign}(k) \frac{ir^{|k|-1}}{2} \int_{r}^{\infty} s^{-|k|+1} w_k(s) \, ds + v_{r,k}^\infty,
\]
\[
v_{\varphi,k} = \frac{r^{-|k|-1}}{2} \int_{r_0}^{r} s^{|k|+1} w_k(s) \, ds - \frac{r^{|k|-1}}{2} \int_{r}^{\infty} s^{-|k|+1} w_k(s) \, ds + v_{\varphi,k}^\infty.
\]

No-slip condition (1.3) and (1.10) lead to moment relations for vorticity \((k \in \mathbb{Z})\):
\[
\int_{r_0}^{r} s^{-|k|+1} w_k(s) \, ds = 2ikv_{r,k}^\infty + 2v_{\varphi,k}^\infty.
\]
From (1.8), (1.9) these moments don’t equal to zero only if $|k| = 1$.

The above formulas (1.11), (1.12) represent Fourier coefficients of Biot–Savar formula (1.7).

Since for $p > 1$ $\nabla v$ is obtained from $\omega$ via a singular integral kernel of Calderon– Zygmund type [15, 16], then

$$\| \nabla v(\cdot) \|_{L_p} \leq C \| w \|_{L_p},$$

(1.14)

But the energy estimate of $\| v(\cdot) \|_{L_p}$ with $p = 2$ from the induced vorticity $w$ via Biot–Savar law causes most difficulties. We will prove the following

**Lemma 1.1.** Let $w(\cdot) \in L_2(B_{r_0})$, the Fourier coefficients at $k = -1, 0, 1$ belong to $L_1(r_0, \infty; r)$. $v(\cdot)$ — be the solution of (1.1)–(1.4), (1.6). Then the following estimate holds with some $C > 0$:

$$\text{sup}_{r \in [r_0, \infty]} \| v(r, \cdot) - v_\infty \|_{H^{1/2}(S_r)} \leq C \left( \| w(\cdot) \|_{L_2(B_{r_0})}^2 + \sum_{k = -1, 0, 1} \| w_k(\cdot) \|_{L_1(r_0, \infty; r)}^2 \right),$$

where $S_r = \{ x \in \mathbb{R}^2, |x| = r \}$.

**Proof.** We will estimate one of the terms in (1.11) when others can be processed in a similar way.

For $|k| > 1$:

$$\left| r^{-|k|-1} \int_r^\infty s^{-|k|+1} w_k(s) ds \right|^2 \leq \frac{\| w_k \|_{L_2(r_0, \infty; r)}^2}{2k - 2},$$

(1.15)

For $|k| = 1$:

$$\left| \int_r^\infty w_{\pm 1}(s) ds \right|^2 \leq \left( 1 + 1/r_0 \right) \| w_{\pm 1} \|_{L_1(r_0, \infty; r)}^2,$$

For $k = 0$:

$$\left| r^{-1} \int_r^\infty w_0(s) ds \right|^2 \leq t_0^{-1} \| w_0 \|_{L_1(r_0, \infty; r)}^2,$$

Fractional differentiation of order $\frac{1}{2}$ corresponds to multiplier $\sqrt{k}$ for Fourier coefficients. Summarizing by $k$ we obtain the required estimate.

We rewrite moment relationship in terms of vorticity $w(x)$ for $k \geq 0$:

$$\int_{r_0}^\infty s^{-|k|+1} w_k(s) ds = \frac{1}{2\pi} \int_{r_0}^\infty \int_0^{2\pi} s^{-k+1} w(s, \varphi)e^{-ik\varphi} ds d\varphi$$

$$= \frac{1}{2\pi} \int_{B_{r_0}} \frac{w(x)}{z^k} d x = 2ikv_{r,k}^\infty,$$

where $z = x_1 + ix_2 = se^{i\varphi}$.

For $k < 0$ we have absolutely the same moments relation:

$$\int_{r_0}^\infty s^{-|k|+1} w_k(s) ds = \frac{1}{2\pi} \int_{r_0}^\infty \int_0^{2\pi} s^{k+1} w(s, \varphi)e^{-ik\varphi} ds d\varphi$$

$$= \frac{1}{2\pi} \int_{B_{r_0}} \frac{w(x)}{z^k} d x = 2ikv_{r,k}^\infty, \quad k < 0.$$

The last formula for $k < 0$ is just complex conjugation of the analogous formula for $k \geq 0$ due to equalities

$$v_{r,k}^\infty = v_{r,-k}^\infty, \quad v_{\varphi,k}^\infty = v_{\varphi,-k}^\infty.$$
Then for no-slip condition we have affine subspace $M$ which must be invariant under vorticity flow:

$$
M = \left\{ w(x) \in L_1(B_{r_0}) \mid \int_{B_{r_0}} \frac{w(x)}{(x_1 + ix_2)^2} \, dx = 4\pi ik v_{r,k}^\infty, \ k \in \mathbb{Z}_+ \right\}.
$$

(1.16)

Note that from (1.8), (1.9) the integral relations in the definition of $M$ are non-zero ones only if $k = 1$.

For the Fourier coefficients the invariance of $M$ means that for vorticity flow which is described by the coefficients $w_k(t, \cdot)$ holds

$$
\int_{r_0}^{\infty} s^{-|k|+1} w_k(t, s) \, ds = \text{const}, \ k \in \mathbb{Z}.
$$

In [7] was proved

**Theorem 1.2** (Biot–Savart Law in polar coordinates). If $w(x) \in M$ then there exists the unique solution of (1.1)–(1.4), (1.6) given by (1.11), (1.12) with Fourier coefficients $v_{r,k}$, $v_{r,k} \in L_\infty(r_0, \infty)$.

### 1.2. Biot–Savart Law in the Simply Connected Domains

Here we derive Biot–Savart law for more general domains. Let $\Omega = \mathbb{R}^2 \setminus \overline{G}$, where $G$ is bounded simply-connected domain with smooth boundary and $\Phi$ be a Riemann mapping from $\Omega$ into exterior of the disc $B_{r_0}$ such that

$$
\Phi(p) = p + O \left( \frac{1}{p} \right),
$$

where $p = y_1 + iy_2 \in \mathbb{C}$. Then for $z = x_1 + ix_2 \in \mathbb{C}$ the inverse transform $\Phi^{-1}(z) : B_{r_0} \to \Omega$ satisfies

$$
\Phi^{-1}(z) = z + O \left( \frac{1}{z} \right).
$$

In addition we suppose

$$
(\Phi^{-1})'(z) = 1 + O \left( \frac{1}{z^2} \right).
$$

So, we have two vector variables: $y = (y_1, y_2) \in \Omega$ and $x = (x_1, x_2) \in B_{r_0}$. We change the variable $y$ into $x$ in the system of equations

$$
\begin{align*}
\text{div } v(y) &= 0, \quad (1.17) \\
\text{curl } v(y) &= w(y).
\end{align*}
$$

Let $v(p)$ defines the vector field in $\Omega$ where $p = y_1 + iy_2$. Then $v = v(\Phi^{-1}(z)) = v(\text{Re } \Phi^{-1}(x_1 + ix_2), \text{Im } \Phi^{-1}(x_1 + ix_2))$ defines the vector field in $B_{r_0}$. Here we will not use the tensor form of the vector field as well as divergence operator and consider the vector field $v(y)$ as a set of two scalar functions $v_1(y)$, $v_2(y)$.

Then with help of Cauchy–Riemann relationship

$$
\begin{align*}
\frac{\partial x_1}{\partial y_1} &= \frac{\partial x_2}{\partial y_2}, \\
\frac{\partial x_2}{\partial y_1} &= -\frac{\partial x_1}{\partial y_2}
\end{align*}
$$
we will have
\[
\text{div } v(y) = \frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_2}{\partial y_1} + \frac{\partial v_2}{\partial y_2} \\
= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_2}{\partial y_1} + \frac{\partial v_2}{\partial y_2} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}
\]
\[
- \frac{\partial x_2}{\partial y_1} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) = \frac{\partial x_1}{\partial y_1} \text{div } v(x) - \frac{\partial x_2}{\partial y_1} \text{curl } v(x).
\]

In a similar way we will have
\[
\text{curl } v(y) = \frac{\partial v_2}{\partial y_1} - \frac{\partial v_2}{\partial y_2} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial y_1} - \frac{\partial v_2}{\partial y_2} \\
= \frac{\partial x_1}{\partial y_1} \text{curl } v(x) + \frac{\partial x_2}{\partial y_1} \text{div } v(x).
\]

Finally, the system (1.17), (1.18) reduces to
\[
\frac{\partial x_1}{\partial y_1} \text{div } v(x) - \frac{\partial x_2}{\partial y_1} \text{curl } v(x) = 0,
\]
\[
\frac{\partial x_2}{\partial y_1} \text{div } v(x) + \frac{\partial x_1}{\partial y_1} \text{curl } v(x) = w(x).
\]

Then
\[
\text{div } v(x) = \frac{\partial x_2}{\partial y_1} \left( \frac{\partial x_1}{\partial y_1} + \frac{\partial x_2}{\partial y_2} \right) w(x),
\]
\[
\text{curl } v(x) = \frac{\partial x_1}{\partial y_1} \left( \frac{\partial x_1}{\partial y_1} + \frac{\partial x_2}{\partial y_2} \right) w(x).
\]

But the complex derivative of \( \Phi'(p) \) is equal to
\[
\Phi'(p) = \frac{\partial x_1}{\partial y_1} + i \frac{\partial x_2}{\partial y_1}.
\]

Then the system (1.17), (1.18) turns to relation
\[
\text{curl } v + i \text{div } v = \frac{\Phi'(p)}{|\Phi'(p)|^2} w.
\]

Using
\[
\frac{\Phi'(p)}{|\Phi'(p)|^2} = \frac{1}{\Phi'(p)} = \overline{(\Phi^{-1})'(z)}
\]
the system (1.1)–(1.4) after Riemann mapping in \( B_{r_0} \) takes the form
\[
\text{div } v(x) = \text{Im} \overline{(\Phi^{-1})(z)} w(x) 
\]
\[
\text{curl } v(x) = \text{Re} (\Phi^{-1})'(z) w(x) 
\]
\[
v(x) = 0, \ |x| = r_0 
\]
\[
v(x) \to v_\infty, \ |x| \to \infty.
\]

Let
\[
r_k(r) = |\text{Im} (\Phi^{-1})'(z) w(x)|_k,
\]
\[
q_k(r) = |\text{Re} (\Phi^{-1})'(z) w(x)|_k,
\]
where subscript \( k \) denotes \( k \)th Fourier harmonic and \( z = re^{i\varphi} = x_1 + ix_2. \)
Rewrite (1.20),(1.21) in polar coordinates in terms of Fourier coefficients $v_{r,k}$, $v_{\varphi,k}$:

\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_{r,k}) + \frac{ik}{r} v_{\varphi,k} = r_k(r),
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_{\varphi,k}) - \frac{ik}{r} v_{r,k} = q_k(r).
\]

Under the assumption of zero-circulation (1.6) from Stokes theorem we have

\[
\lim_{R \to \infty} \oint_{|x| = R} \mathbf{v}(x) \cdot dl = \frac{1}{2\pi} \int_{B_{r_0}} w(x) \, dx = \frac{1}{2\pi} \int_{B_{r_0}} \frac{w(x)}{(|\Phi'|^2}_{k=0} s \, ds = 0.
\]

The solution of the above system is unique (with arbitrary circulation at infinity). Existence will be provided by the moment relations below. The solution of this system for $k \in \mathbb{Z}$ is derived by the similar to (1.11), (1.12) formulas:

\[
v_{r,k} = \text{sign}(k) \frac{ir^{-|k|-1}}{2} \int_{r_0}^{r} s^{k+1}(q_k - i \text{sign}(k)r_k) \, ds \\
+ \text{sign}(k) \frac{ir^{-|k|-1}}{2} \int_{r_0}^{r} s^{-k+1}(q_k + i \text{sign}(k)r_k) \, ds + v_{r,k}^\infty,
\]

(1.24)

\[
v_{\varphi,k} = \frac{r^{-|k|-1}}{2} \int_{r_0}^{r} s^{k+1}(q_k - i \text{sign}(k)r_k) \, ds \\
- \frac{r^{-|k|-1}}{2} \int_{r_0}^{r} s^{-k+1}(q_k + i \text{sign}(k)r_k) \, ds + v_{\varphi,k}^\infty.
\]

(1.25)

Formulas (1.24), (1.25) combined with (1.3) lead to the relations on vorticity ($k \in \mathbb{Z}$):

\[
\int_{r_0}^{\infty} s^{-|k|+1}(q_k(s) + i \text{sign}(k)r_k(s)) \, ds = 2ikv_{r,k}^\infty = 2v_{\varphi,k}^\infty.
\]

(1.26)

The above formulas (1.24), (1.25) are the Fourier coefficients of the Biot–Savard law

\[
\mathbf{v}(x) = \frac{1}{2\pi} \int_{B_{r_0}} \left( \frac{x - y}{|x - y|^2} \text{Re} \left( \Phi^{-1}'(z) \right) + \frac{x - y}{|x - y|^2} \text{Im} \left( \Phi^{-1}'(z) \right) \right) w(y) \, dy \\
+ \mathbf{v}_{\infty}.
\]

(1.27)

**Lemma 1.3.** Let $\Phi$ be a Riemann mapping from exterior domain $\Omega$ with smooth boundary into exterior of the disc $B_{r_0}$, $w(\Phi^{-1}(\cdot)) \in L_2(B_{r_0})$, the Fourier coefficients at $k = -1, 0, 1$ belong to $L_1(r_0, \infty; r)$, $\mathbf{v} = \mathbf{v}(\Phi^{-1}(z)) = \mathbf{v}(\text{Re} \Phi^{-1}(x_1 + ix_2), \text{Im} \Phi^{-1}(x_1 + ix_2))$ defines vector field in $B_{r_0}$ for the solution of (1.1)–(1.4), (1.6) with vorticity $w = \text{curl} \mathbf{v}(\Phi^{-1}(z))$. Then the following estimate holds with some $C > 0$:

\[
v_{\text{vraisup}}_{r \in [r_0, \infty]} \|\mathbf{v}(r, \cdot) - \mathbf{v}_{\infty}\|_{H^{1/2}(S_r)}^2 \leq C \left( \|w(\cdot)\|_{L_2(B_{r_0})}^2 + \sum_{k=-1,0,1} \|w_k(\cdot)\|_{L_1(r_0, \infty; r)}^2 \right),
\]

where $S_r = \{x \in \mathbb{R}^2, |x| = r\}$.

Since $|\Phi'(z)|$ is bounded, then the proof of this lemma is the same as for Lemma 1.1 in the previous subsection.

Consider the flow of 2D velocity field $\mathbf{v}(t, x)$ and its vorticity $w(t, x) = \text{curl} \mathbf{v}(t, x)$. The set of relations (1.26) is the integral form of the no-slip condition (1.3).
1.3. Invariant Affine Manifolds for a No-Slip Condition

Following by the same way as in Theorem 1.2 we define affine subspace $M$ via vorticity moments which must be invariant under vorticity flow:

$$M = \left\{ w(x) \in L_1(B_{r_0}) \left| \int_{B_{r_0}} \frac{\Phi^{-1}(x_1 + ix_2)}{(x_1 + ix_2)^k} \, dx = 4\pi ikv_{r,k}^\infty, \ k \in \mathbb{Z}_+ \right. \right\}. \quad (1.28)$$

In view of (1.8), (1.9) all moments in the definition of $M$ must be equal to zero except $k = 1$.

**Proposition 1.4.** (Biot–Savart law in exterior domain) If $w(\Phi^{-1}(z)) \in M$ then there exists the unique solution of (1.1)–(1.4), (1.6) given by (1.24), (1.25) with Fourier coefficients $v_{r,k}$, $v_{\varphi,k} \in L_\infty(r_0, \infty)$.

**Proof.** All we need is to prove the validity of the no-slip condition. Set $z = se^{i\varphi} = x_1 + ix_2$

$$q_k(s) + ir_k(s) = [(\Phi^{-1})'(z)w(x)]_k,$$

then the equalities (1.24), (1.25) can be written in terms of $w(x)$. And, finally,

$$\int_{r_0}^{\infty} s^{-|k|+1} (q_k(s) + ir_k(s)) \, ds = 1 \quad (1.29)$$

Now, we have

$$= 1 \frac{2\pi}{2} \int_{r_0}^{\infty} \int_0^{2\pi} s^{-|k|+1} h_k(\varphi) e^{-ik\varphi} \, d\varphi \quad ds \, d\varphi$$

$$= 1 \frac{2\pi}{2} \int_{B_{r_0}} (\Phi^{-1})'(z) \frac{w(x)}{z^k} \, dx = 2ikv_{r,k}^\infty. \quad \Box$$

**Proposition 1.5.** Given initial datum $v_0(x)$ satisfying no-slip condition (1.3), infinity condition (1.4), zero-circulation (1.6), such that $w_0 = \text{curl} v_0(\Phi^{-1}(z)) \in L_1(B_{r_0})$, and $w(t, \cdot) = \text{curl} v(\Phi^{-1}(z)) \in L_1(B_{r_0})$ be the vorticity flow. Then $w_0 \in M$ and in order to conserve no-slip condition the affine subspace $M$ must be invariant under the flow, e.g. for any time $t > 0 \ w(t, \cdot) \in M$.

**Proof.** Since $v_0(x)$ satisfies the no-slip condition, then from (1.24), (1.25), (1.29) follows $w(t, \cdot) \in M$. \Box

If $v_\infty = 0$ then this affine manifold $M$ passes through zero and becomes the invariant subspace. If we limit infinite set of relations in (1.28) only by $k = 1, \ldots, N$, we obtain affine manifold $M_N$ of finite codimension:

$$M_N = \left\{ w(x) \in L_1(B_{r_0}) \left| \int_{B_{r_0}} \frac{\Phi^{-1}(z)}{w(x)} \frac{w(x)}{z^k} \, dx = 4\pi ikv_{r,k}^\infty, \ k = 0, 1, \ldots, N \right. \right\}. \quad (1.30)$$

The following lemma says that since $M$ corresponds to a no-slip condition, then $M_N$ is the approximation of this boundary condition.

**Lemma 1.6.** Let for any fixed $t \in [0, T]$ $w_N(t, \cdot)$ are uniformly bounded in $H^1(\Omega)$ by $N$. Then the vector field $v_N(t, \cdot)$ given by (1.27) converges weakly in $H^{1/2}(\partial\Omega)$ to zero for $t \in [0, T]$ as $N \to \infty$.

**Proof.** Without loss of generality assume that $\Omega = B_{r_0}$.

$$v_{r,k}(t, r_0) = \text{sign}(k) \frac{r^{k-1}}{2} \int_{r_0}^{\infty} s^{-|k|+1} w_k(t, s) \, ds + v_{r,k}^\infty$$

$$v_{\varphi,k}(t, r_0) = \frac{r^{k-1}}{2} \int_{r_0}^{\infty} s^{-|k|+1} w_k(t, s) \, ds + v_{\varphi,k}^\infty.$$
From \( w(t, \cdot) \in M_N \) \( v_{r,k}(t, r_0) = v_{\varphi,k}(t, r_0) = 0 \) for \( k = -N, \ldots, N \). For \( k > N \)
\[
\int_{r_0}^{\infty} s^{-|k|+1} w_k(t, s) \, ds + v_{r,k}^\infty = \int_{r_0}^{\infty} s^{-|k|+1} w_k(t, s) \, ds.
\]
Then from (1.15)
\[
\|v^N(t, \cdot)\|_{H^{1/2}(\partial\Omega)} \leq C \sum_{k=N+1}^{\infty} \|w^N_k(t, \cdot)\|_{L^2_2(r_0, \infty, r, dr)}.
\]
For \( w_N(t, \cdot) \in M_N \) the Fourier coefficients of \( v_r, v_\varphi \) with index \( k = -N, \ldots, N \) equal to zero which implies
\[
v^N(t, x') \rightharpoonup 0, \ x' \in \Omega, \text{ (weakly)}
\]in \( H^{1/2}(\partial\Omega) \).

Remark. If \( w_N(t, \cdot) \) are uniformly bounded in \( H^2 \) one can prove that \( v(t, x') \) converges weakly in \( H^{3/2}(\partial\Omega) \) to zero. And from the Rellich-Kondrachov Theorem due to the compact embedding follows that for any \( t \in [0, T] \)
\[
v^N(t, x') \rightarrow 0, \ x' \in \partial\Omega \text{ (strongly)}
\]in \( L^2(\partial\Omega) \).

2. No-Slip Integral Condition for Vorticity in Exterior Domains

Consider the initial-boundary-value problem for the Navier–Stokes system defined in exterior domain \( \Omega \subset \mathbb{R}^2 \) modelling flow around solid with given constant flow at infinity \( v_\infty \in \mathbb{R}^2 \):
\[
\begin{align*}
\partial_t v - \Delta v + (v, \nabla)v &= \nabla p, \quad \text{(2.1)} \\
\text{div } v(t, x) &= 0, \quad \text{(2.2)} \\
v(0, x) &= v_0(x), \quad \text{(2.3)} \\
v(t, x) &= 0, \ |x| = r_0, \quad \text{(2.4)} \\
v(t, x) &\to v_\infty, \ |x| \to \infty. \quad \text{(2.5)}
\end{align*}
\]
Here \( v(t, x) = (v_1(t, x), v_2(t, x)) \) is the velocity field and \( p(t, x) \) is the pressure.

Applying the curl operator \( w(t, x) = \text{curl } v(t, x) = \partial_{x_1} v_2 - \partial_{x_2} v_1 \) we get boundary problem for vorticity equation
\[
\begin{align*}
\frac{\partial w(t, x)}{\partial t} - \Delta w + (v, \nabla)w &= 0, \quad \text{(2.6)} \\
w(0, x) &= w_0(x), \quad \text{(2.7)} \\
\text{curl}^{-1} w(t, x)|_{|x|=r_0} &= 0, \quad \text{(2.8)} \\
w(t, x) &\to 0, \ |x| \to \infty \quad \text{(2.9)}
\end{align*}
\]
with initial datum \( w_0(x) = \text{curl } v_0(x) \).

Vector field \( v(t, x) \) can be derived from \( w(t, x) \) using Green function \( G(x, y) \) for Laplace operator \( \Delta \):
\[
v(t, x) = \int_{\Omega} \nabla^\perp_x G(x, y) w(y) \, dy + v_\infty,
\]
where \( \nabla^\perp_x = (-\partial_{x_2}, \partial_{x_1}) \) and
\[
\Delta_x G(x, y) = 0, \ x \neq y. \quad (2.10)
\]
Indeed
\[
\text{div } \mathbf{v}(t, \mathbf{x}) = \int_{\Omega} (\nabla_{x}, \nabla_{x}^\perp) G(\mathbf{x}, \mathbf{y}) w(\mathbf{y}) \, \mathrm{d} \mathbf{y} = 0,
\]
and
\[
\text{curl } \mathbf{v}(t, \mathbf{x}) = \int_{\Omega} \Delta G(\mathbf{x}, \mathbf{y}) w(\mathbf{y}) \, \mathrm{d} \mathbf{y} = w.
\]

Then the no-slip condition gives the following integral expression
\[
\mathbf{v}(t, \mathbf{x}') = \int_{\Omega} \nabla_{x}^\perp G(\mathbf{x}', \mathbf{y}) w(t, \mathbf{y}) \, \mathrm{d} \mathbf{y} = 0, \quad \mathbf{x}' \in \partial \Omega, \forall t > 0.
\]

Velocity field for Stokes system satisfies
\[
\frac{\partial v(t, \mathbf{x})}{\partial t} - \Delta v(t, \mathbf{x}) = \nabla p \quad (2.11)
\]
when vorticity evolution is described by the heat equation
\[
\frac{\partial w(t, \mathbf{x})}{\partial t} - \Delta w(t, \mathbf{x}) = 0. \quad (2.12)
\]

From no-slip condition follows
\[
\frac{d}{dt} \int_{\Omega} \nabla_{x}^\perp G(\mathbf{x}', \mathbf{y}) w(t, \mathbf{y}) \, \mathrm{d} \mathbf{y} = 0, \quad \mathbf{x}' \in \partial \Omega, \forall t > 0,
\]
and so, the relation
\[
\int_{\Omega} \nabla_{x}^\perp G(\mathbf{x}', \mathbf{y}) w(t, \mathbf{y}) \, \mathrm{d} \mathbf{y} = \text{const}, \quad \mathbf{x}' \in \partial \Omega, \forall t > 0
\]
defines the invariant manifold for vorticity.

We multiply (2.12) by \( \nabla_{x}^\perp G(\mathbf{x}', \mathbf{y}) \) and integrate it over exterior domain using Green formula. Then with help of (2.10) we will have integral boundary condition
\[
\int_{\partial \Omega} \left( \nabla_{x}^\perp G(\mathbf{x}', \mathbf{y}) \frac{\partial w(t, \mathbf{y})}{\partial n} - w(t, \mathbf{y}) \frac{\partial}{\partial n} \nabla_{x}^\perp G(\mathbf{x}', \mathbf{y}) \right) \, \mathrm{d} \mathbf{y} = 0, \quad \forall \mathbf{x}' \in \partial \Omega,
\]
where \( n \) is an outer normal to the boundary. It is still an integral condition but only with the surface integral over \( \partial \Omega \) involved.

For cylindrical domains this integral turns into boundary condition on Fourier coefficients. In this section for 2D Stokes and Navier–Stokes system we will derive boundary condition in terms of Fourier harmonics in exterior simply-connected domains.

### 2.1. Linear Vorticity Equation in the Exterior of the Disc

Consider Stokes flow for vorticity (2.12) in \( B_{r_0} \) and supply it with Robin-type boundary condition:
\[
r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + |k| w_k(t, r_0) = 0, \quad k \in \mathbb{Z} \quad (2.13)
\]
and condition at infinity
\[
w(t, \mathbf{x}) \to 0, \quad |\mathbf{x}| \to \infty. \quad (2.14)
\]

Then \( M \) is invariant under the flow \( w(t, \cdot) \). Indeed, fix \( k > 0 \) and divide Eq. (2.12) by \( z^k \) and integrate over the exterior of the disc \( B_{r_0} \). From moment relations (1.16) follows
\[
\frac{d}{dt} \int_{B_{r_0}} \frac{w(t, \mathbf{x})}{z^k} \, \mathrm{d} \mathbf{x} = 0
\]
and thus
\[ \int_{B_{r_0}} \frac{\Delta w}{z^k} \, dx = 0. \]

In other hand
\[
\int_{B_{r_0}} \frac{\Delta w}{z^k} \, dx = \int_{r_0}^{\infty} \int_{\partial B_{s}}^{\infty} \frac{\Delta w}{s^{k+1}} e^{-ik \varphi} \, s ds ds = 2\pi \int_{r_0}^{\infty} s^{-k+1} \Delta_k w_k(t,s) \, ds
\]
\[
= 2\pi \int_{r_0}^{\infty} s^{-|k|} \left( \frac{\partial}{\partial s} \left( s \frac{\partial}{\partial s} w_k(t,s) \right) - \frac{k^2}{s} w_k(t,s) \right) \, ds
\]
\[
= -2\pi r_0^{-|k|} \left( \frac{\partial w_k(t,r)}{\partial r} \bigg|_{r=r_0} - k \frac{\partial w_k(t,s)}{\partial s} \right) + 2\pi \int_{r_0}^{\infty} s^{-|k|-1} \left( k^2 w_k(t,s) - k^2 w_k(t,s) \right) \, ds
\]
\[
= -2\pi r_0^{-k} \left( \frac{\partial w_k(t,r)}{\partial r} \bigg|_{r=r_0} + k w_k(t,r_0) \right) = 0,
\]
where \( \Delta_k w_k(t,r) \) is defined in (0.3).

In a similar way using complex conjugation of moment relations (1.16) we obtain boundary condition for \( k < 0 \):
\[
\int_{B_{r_0}} \Delta w z^k \, dx = \int_{r_0}^{\infty} \int_{0}^{2\pi} \frac{\Delta w}{s^{-k} e^{ik \varphi}} \, s ds ds = 2\pi \int_{r_0}^{\infty} s^{k+1} \Delta_k w_k(s) \, ds
\]
\[
= -2\pi r_0^{-k} \left( \frac{\partial w_k(t,r)}{\partial r} \bigg|_{r=r_0} - k w_k(t,r_0) \right) = 0.
\]

2.2. Helmholtz Equation in the Exterior of the Disc

Now we are ready to derive the boundary condition to the nonlinear vorticity equation. In fact it will be an integral condition. But its first approximation will be the same boundary condition (2.13) as for Stokes flow.

Consider Helmholtz equation for vorticity
\[
\frac{\partial w(t,x)}{\partial t} - \Delta w(t,x) + (\mathbf{v}, \nabla w) = 0,
\]
where \( \mathbf{v} \) is restored from \( w \) via Biot–Savar law (1.7).

Denote
\[
\mathbf{v}^C = v_1 + iv_2, \quad \mathbf{v}^C_\infty = v_1^\infty + iv_2^\infty.
\]

**Theorem 2.1.** Given initial datum \( \mathbf{v}_0(x) \), \( \text{curl} \, \mathbf{v}_0 \in L_1(B_{r_0}) \) satisfying no-slip condition (1.3), infinity condition (1.4), zero-circulation (1.6) and \( \mathbf{v}(t,x) \) be the solution of (2.1)–(2.5) in \( B_{r_0} \). Then \( w(t,x) = \text{curl} \, \mathbf{v}(t,x) \) satisfies
\[
\left. \frac{\partial w_k(t,r)}{\partial r} \right|_{r=r_0} + k |w_k(t,r_0)| = \left\{ \begin{array}{ll}
|k|r_0^{-|k|} & \frac{2\pi}{2\pi} \int_{B_{r_0}} (\mathbf{v}_0^C - \mathbf{v}_\infty^C) z^{-|k|-1} w(t,x) \, dx, \quad k \geq 0, \\
|k|r_0^{-|k|} & \frac{2\pi}{2\pi} \int_{B_{r_0}} (\mathbf{v}_\infty^C - \mathbf{v}_0^C) z^{-|k|-1} w(t,x) \, dx, \quad k < 0.
\end{array} \right.
\]

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Proof. Fix $k > 0$ and divide the Eq. (2.16) by $z^k$ and integrate over the exterior of the disc $B_{r_0}$. From moment relations (1.16) follows

$$\frac{d}{dt} \int_{B_{r_0}} \frac{w(t, x)}{z^k} \, dx = 0.$$ 

Then

$$\int_{B_{r_0}} \frac{(v, \nabla w)}{z^k} \, dx = - \int_{B_{r_0}} (v, \nabla z^{-k}) w(t, x) \, dx = k \int_{B_{r_0}} \frac{v_1 + iv_2}{z^{k+1}} w(t, x) \, dx = k \int_{B_{r_0}} \frac{v_C}{z^{k+1}} w(t, x) \, dx.$$

From (1.16) follows

$$k \int_{B_{r_0}} \frac{v_C}{z^{k+1}} w(t, x) \, dx = 0$$

and

$$\int_{B_{r_0}} \frac{(v, \nabla w)}{z^k} \, dx = k \int_{B_{r_0}} \frac{v_C - v_C^\infty}{z^{k+1}} w(t, x) \, dx.$$

Using (2.15) we get

$$\int_{B_{r_0}} \Delta w \, dx = -2\pi r_0^{-k} \left( r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + kw_k(t, r_0) \right).$$

Then for $k \geq 0$

$$\left( r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + kw_k(t, r_0) \right) = \frac{kr^k_0}{2\pi} \int_{B_{r_0}} \frac{v_C^\infty - v_C^C}{z^{k+1}} w(t, x) \, dx.$$

Since $w_{-k}(t, r) = w_k(t, r)$ then (2.17) holds for $k < 0$. Theorem is proved. \qed

Remark. If $\|v - v_\infty\|$ is small in some integral norm which is typical for a well-streamlined body, then the right side in (2.17) transfers to boundary condition (2.13). So, (2.13) becomes a rather accurate approximation for Navier–Stokes system. From this fact naturally occurs boundary control problem with unknown function $u_k(t)$:

$$r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + |k| w_k(t, r_0) = u_k(t), \quad k \in \mathbb{Z},$$

(2.18)

where feedback controls $u_k(t)$ are treated as a small correction of zero boundary condition which forces the solution to stay on invariant manifold $M$.

2.3. Linear Vorticity Equation in a Simply Connected Domains

Here we find out that Robin-type boundary condition (2.13) works well as the no-slip boundary condition in exterior domains. It keeps moment relations (1.26) under the Stokes flow, and thus $M$ is an invariant affine subspace.

We impose additional requirement on $\Omega$ that $\Phi^{-1}$ can be represented by absolutely convergent series

$$\Phi^{-1}(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

(2.19)
**Theorem 2.2.** Let $\Omega = \mathbb{R}^2 \setminus \overline{G}$, where $G$ is a bounded simple-connected domain with smooth boundary and $\Phi$ be a Riemann mapping from $\Omega$ onto exterior of the disc satisfying (2.19), $\mathbf{v}_0$—is the initial datum, satisfying no-slip condition (1.3), infinity condition (1.4), zero-circulation (1.6), $\text{curl} \mathbf{v}_0 \in L_1$, and $\mathbf{v}(t,x)$ be the solution of (2.11), (2.2)–(2.5) in $\Omega$. Then $w(t,x) = \text{curl} \mathbf{v}(t,\Phi^{-1}(x))$ satisfies (2.13).

**Proof.** Let us change the variables $p = y_1 + iy_2 \in \Omega$ to $z = x_1 + ix_2 \in B_{\rho_0}$ in heat Eq. (2.12) where $z = \Phi(p)$. Then the Laplace operator $\Delta$ transfers to $|\Phi'(p)|^2 \Delta$ and the Eq. (2.12) transforms to

$$
\partial_t w(t,x) - |\Phi'(p)|^2 \Delta w = 0.
$$

Dividing it by $|\Phi'(p)|^2$ we will have the following equation defined in $B_{\rho_0}$:

$$
|\Phi^{-1}(z)|^2 \partial_t w(t,x) - \Delta w = 0.
$$

Fix $k > 0$ and divide this equation by $z^k = (x_1 + ix_2)^k$. Then integrate it over the exterior of the disc $B_{\rho_0}$. From the moment relations (1.28) follows

$$
\frac{d}{dt} \int_{B_{\rho_0}} \frac{|\Phi^{-1}(z)|^2}{z^k} w(t,x) \, dx = 0.
$$

Using $|\Phi^{-1}(z)|^2 = (\Phi^{-1})'(z)\overline{(\Phi^{-1})'(z)}$ and

$$
(\Phi^{-1})'(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^n}
$$

with some coefficients $c_n$ we have

$$
\frac{d}{dt} \int_{B_{\rho_0}} \frac{|(\Phi^{-1})'(z)|^2}{z^k} w(t,x) \, dx = \frac{d}{dt} \int_{B_{\rho_0}} \sum_{n=0}^{\infty} \frac{c_n (\Phi^{-1})'(z)}{z^{n+k}} w(t,x) \, dx = \sum_{n=0}^{\infty} c_n \frac{d}{dt} \int_{B_{\rho_0}} \frac{(\Phi^{-1})'(z)}{z^{n+k}} w(t,x) \, dx = 0,
$$

and thus

$$
\int_{B_{\rho_0}} \frac{\Delta w}{z^k} \, dx = 0.
$$

In other hand

$$
\int_{B_{\rho_0}} \frac{\Delta w}{z^k} \, dx = \int_{r_0}^{\infty} \int_0^{2\pi} \frac{\Delta w}{s^k e^{ik \varphi}} sdsd\varphi = 2\pi \int_{r_0}^{\infty} s^{-k-1} k w_k(s) \, ds = -2\pi r_0^{-k} \left( \frac{\partial w_k(t,r)}{\partial r} \bigg|_{r=r_0} + k w_k(t,r_0) \right) = 0.
$$

Using $w_{-k}(t,r) = w_k(t,r)$ we will have condition (2.13) for $k < 0$. 

\[\square\]

### 2.4. Helmholtz Equation in a Simply Connected Domains

In the previous subsection we have found the boundary condition for the linear vorticity equation. Here we will find it for nonlinear ones. As in the case of linear vorticity equation we will use Riemann mapping from $\Omega$ to $B_{\rho_0}$ and let us write down the Helmholtz equation in exterior of the disc.

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Changing the variables \( p = y_1 + iy_2 \in \Omega \) to \( z = x_1 + ix_2 \in B_{r_0} \) in the nonlinear term \((\mathbf{v}, \nabla w)\) we will have:

\[
(\mathbf{v}, \nabla w) = v_1 \frac{\partial w}{\partial y_1} + v_2 \frac{\partial w}{\partial y_2} = v_1 \left( \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial y_1} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial y_1} \right) + v_2 \left( \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial y_2} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial y_2} \right)
\]

\[
+ v_2 \left( \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial y_2} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial y_2} \right) = v_1 \frac{\partial w}{\partial x_1} \text{Re} \Phi'(p) + v_1 \frac{\partial w}{\partial x_2} \text{Im} \Phi'(p)
\]

\[
- v_2 \frac{\partial w}{\partial x_1} \text{Im} \Phi'(p) + v_2 \frac{\partial w}{\partial x_2} \text{Re} \Phi'(p) = \text{Re} \Phi'(p)(\mathbf{v}, \nabla w) + \text{Im} \Phi'(p)(\mathbf{v}^\perp, \nabla w),
\]

where \( \mathbf{v}^\perp = (-v_2, v_1) \). Here we used the formulas for complex derivative of \( \Phi'(p) \):

\[
\Phi'(p) = \frac{\partial x_1}{\partial y_1} + i \frac{\partial x_2}{\partial y_2} = \frac{\partial x_1}{\partial y_2} - i \frac{\partial x_1}{\partial y_2}.
\]

Then the vorticity Eq. (2.6) corresponding to Navier–Stokes system after Riemann mapping reduces to

\[
\partial_t w(t, x) - \Phi'(p)^2 \Delta w + \text{Re} \Phi'(p)(\mathbf{v}, \nabla w) + \text{Im} \Phi'(p)(\mathbf{v}^\perp, \nabla w) = 0.
\]

Dividing it by \( |\Phi'(p)|^2 \) we will have the Helmholtz equation in \( B_{r_0} \):

\[
|(\Phi^{-1})'(z)|^2 \partial_t w(t, x) - \Delta w + B(v, w) = 0,
\]

where

\[
B(v, w) = \text{Re}(\Phi^{-1})'(\mathbf{v}, \nabla w) - \text{Im}(\Phi^{-1})'(\mathbf{v}^\perp, \nabla w).
\]

Supply this equation with

\[
r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + |k| w_k(t, r_0) =
\]

\[
\left\{
\begin{array}{ll}
\frac{|k|r_0^{|k|}}{2\pi} \int_{B_{r_0}} (\mathbf{v}_0^\infty - \mathbf{v}^\infty) z^{-|k|-1} (\Phi^{-1})'(w(t, x)) \, d\mathbf{x}, & k \geq 0, \\
\frac{|k|r_0^{|k|}}{2\pi} \int_{B_{r_0}} (\mathbf{v}_0^\infty - \mathbf{v}^\infty) z^{-|k|-1} (\Phi^{-1})'(w(t, x)) \, d\mathbf{x}, & k < 0.
\end{array}
\right.
\]

**Theorem 2.3.** Let \( \Omega, \Phi \) be as in Theorem 2.2, \( \mathbf{v}_0(x) \) be a given initial datum satisfying no-slip, infinity, zero-circulation conditions (1.3), (1.4), (1.6), \( \text{curl} \mathbf{v}_0 \in L_1(\Omega) \), and \( \mathbf{v}(t, x) \) be the solution of the problem (2.1)-(2.5) in \( \Omega \). Then the vorticity \( w(t, \cdot) = \text{curl} \mathbf{v}(t, \Phi^{-1}(\cdot)) \) satisfies (2.21).

**Proof.** Apply the same steps as in the previous subsection: for fixed \( k > 0 \) divide this equation by \( z^k \) and integrate it over \( B_{r_0} \). In the proof of Theorem 2.2 we obtained

\[
\frac{d}{dt} \int_{B_{r_0}} \frac{(\Phi^{-1})'(z)}{z^k} w(t, x) \, d\mathbf{x} = 0,
\]

and

\[
\int_{B_{r_0}} \frac{\Delta w}{z^k} \, d\mathbf{x} = -2\pi r_0^{-k} \left( r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + kw_k(t, r_0) \right).
\]

From (1.20), (1.21)

\[
\int_{B_{r_0}} \frac{\text{Re}(\Phi^{-1})'}{z^k} w \, \text{div} \mathbf{v} \, d\mathbf{x} = -\int_{B_{r_0}} \frac{\text{Im}(\Phi^{-1})'}{z^k} w \, \text{curl} \mathbf{v} \, d\mathbf{x}.
\]
Then using Cauchy–Riemann equations we will have
\[
\int_{B_{r_0}} \frac{B(v, w)}{z^k} \, dx = \int_{B_{r_0}} \frac{\Re(\Phi^{-1})'}{z^k} (v, \nabla w) - \frac{\Im(\Phi^{-1})'}{z^k} (v^\perp, \nabla w) \, dx \\
= \int_{B_{r_0}} \left( -\frac{\Re(\Phi^{-1})'}{z^k} w \, \text{div} \, v - \frac{\Im(\Phi^{-1})'}{z^k} w \, \text{curl} \, v \right) \, dx \\
- \int_{B_{r_0}} w \left( \left( v, \nabla \Re(\Phi^{-1})' \right) - \left( v^\perp, \nabla \Im(\Phi^{-1})' \right) \right) \, dx \\
= k \int_{B_{r_0}} \frac{(\Phi^{-1})'}{z^{k+1}} (v_1 + iv_2) w(t, x) \, dx = k \int_{B_{r_0}} \frac{(\Phi^{-1})'}{z^{k+1}} \mathbf{v}^C w(t, x) \, dx.
\]

From (1.28) we have
\[
k \int_{B_{r_0}} \frac{(\Phi^{-1})'}{z^{k+1}} \mathbf{v}_\infty^C w(t, x) \, dx = 0,
\]
and
\[
\int_{B_{r_0}} \frac{B(v, w)}{z^k} \, dx = k \int_{B_{r_0}} \frac{(\Phi^{-1})'}{z^{k+1}} (\mathbf{v}^C - \mathbf{v}_\infty^C) w(t, x) \, dx,
\]
from which follows (2.21) for \( k \geq 0 \).

For \( k < 0 \) (2.21) follows from the identity \( w_{-k}(t, r) = \overline{w_k}(t, r) \). Theorem is proved. \( \square \)

This theorem says, that for Navier–Stokes system if \( \| \mathbf{v}^C - \mathbf{v}_\infty^C \| \) is small, then (2.13) works well. Nevertheless this boundary condition requires some corrections like boundary control (2.18) in order to stay on invariant affine subspace \( M \).

### 3. No-Slip Boundary Condition for Vorticity in Exterior Domains

In this section we will construct the no-slip boundary condition for vorticity without the assumption of the smallness of \( \| \mathbf{v}^C - \mathbf{v}_\infty^C \| \). With help of Riemann mapping we had reduced the Helmholtz equation (2.6) defined in \( \Omega \) to (2.20) defined in \( B_{r_0} \). Formulas (2.21) present integral condition which approximately equal to boundary condition (2.13) for well streamlined obstacle when \( \mathbf{v} \simeq \mathbf{v}_\infty \). In this section using fixed point theorem we will derive boundary condition for vorticity in the form like (0.1) which ensures the solution to satisfy \( w(t, \cdot) \in M_N \) defined in (1.30). Then the velocity \( \mathbf{v}(t, x) \) restored from \( w(t, x) \) via Biot–Savar law (1.7) will be the solution of Navier–Stokes system with approximate no-slip boundary condition satisfying (1.31).

Fix \( N \in \mathbb{N} \). We supply vorticity equation (2.20) with
\[
w(0, x) = w_0(x), \quad (3.1)
\]
and boundary condition
\[
r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + kw_k(t, r_0) = u_k(t), \quad k = 0, 1, \ldots, N, \quad (3.3)
\]
where \( \overline{u}(t) = \{ u_k(t) \}_{k=0}^N \) is the set of unknown boundary functions. Since \( w_{-k}(t, r) = \overline{w_k}(t, r) \) then for \( k = -N, -N+1, \ldots, -1 \) holds:
\[
r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} - kw_k(t, r_0) = \overline{w_k(t)}. \quad (3.4)
\]
For $|k| > N$ set
\begin{equation}
    r_0 \frac{\partial w_k(t, r)}{\partial r} \bigg|_{r=r_0} + k w_k(t, r_0) = 0. \tag{3.5}
\end{equation}

We construct approximate boundary condition for Helmholtz equation under the assumption of unique resolvability of (2.20), (3.1)–(3.5). In the exterior of the disc the solvability theorems are proved in “Appendixes B and C”.

For the Helmholtz problem we should use Sobolev space $H^1(B_{r_0})$ as the phase space for $w(t, \cdot)$. In order to obtain $v(t, \cdot) \in L_\infty(B_{r_0})$ from Lemma 1.3 we need involve $L_1$ into phase space for $w_{-1}, w_0, w_1$. So, we will use space
\begin{equation}
    W = \{ w \in H^1(B_{r_0}), w_{-1}, w_0, w_1 \in L_1(r_0, \infty; r) \}.
\end{equation}

**Theorem 3.1.** Suppose that for a fixed $N > 0$, $T > 0$ there exists the solution $w_N(t, x) \in C([0, T], W)$ of Helmholtz equation (2.20), (3.1)–(3.5) which is locally Lipschitz continuous due to $u_k \in C[0, T]$, $k = 0, \ldots, N$, $w_0(x) \in H^1(B_{r_0})$. Then for some $M > 0$ and any initial datum with $\|w_0(\cdot)\|_{H^1} \leq M$ there exists such $\tilde{u}(t) = \{u_k(t)\}_{k=0}^N$, that $w_N(t, \cdot) \in M_N$ for all $t \in [0, T]$.

**Proof.** The relation (2.21) can be rewritten as
\begin{equation}
    \tilde{u}(t) = F(\tilde{u}) \tag{3.6}
\end{equation}
with the mapping $F(\tilde{u}) : (C[0, T])^{2N+1} \rightarrow (C[0, T])^{2N+1}$:
\begin{align*}
    F[\tilde{u}(\cdot)] &= \begin{cases} 
        \frac{|k|r_0}{2\pi} \int_{B_{r_0}} (v_\infty^C - v_1^C) z^{-|k|-1}(\Phi^{-1})^t w(t, x) \, dx, & k = 0, \ldots, N \\
        \frac{|k|r_0}{2\pi} \int_{B_{r_0}} (v_\infty^C - v_1^C) z^{-|k|-1}(\Phi^{-1})^t w(t, x) \, dx, & k = -N, \ldots, -1.
    \end{cases}
\end{align*}
where $w(t, x)$ is the solution of boundary-value problem (2.20), (3.1)–(3.5) with boundary condition $\tilde{u}(t)$. $F$ is well-defined since $k/z^{|k|+1} \in L_2(B_{r_0})$ and in virtue of Lemma 1.3 from
\begin{equation}
    \big\| k(v_\infty^C - v_1^C) z^{-|k|-1} \big\|_{L_2(B_{r_0})} \leq C\|w(t, x)\|_W
\end{equation}
follows
\begin{equation}
    |F[\tilde{u}(t)]| \leq C\|w(t, \cdot)\|_W^2
\end{equation}
with some new constant $C > 0$. From $w(t, x) \in C([0, T]; H^1(B_{r_0}))$ follows $F[\tilde{u}(t)] \in (C[0, T])^{2N+1}$.

Fix $M > 0$ and take $\tilde{u}_1(\cdot), \tilde{u}_2(\cdot)$ from $(C[0, T])^{2N+1}$ with $\|\tilde{u}_i\|_{L_\infty} < M$, $i = 1, 2$. Let $w_1$, $w_2$—be the corresponding solutions of (2.20), (3.1)–(3.5) with $w_0(x)$ satisfying $\|w_0(x)\| \leq M$.

$F$ is the contraction mapping with respect to $\tilde{u}$. Indeed
\begin{equation}
    |F[\tilde{u}_1(\cdot)] - F[\tilde{u}_2(\cdot)]| \leq \begin{cases} 
        \frac{|k|r_0}{2\pi} \int_{B_{r_0}} |(v_\infty^C - v_1^C) z^{-|k|-1}(\Phi^{-1})^t w_1(t, x)| \, dx \\
        \frac{|k|r_0}{2\pi} \int_{B_{r_0}} |k(v_\infty^C - v_1^C) z^{-|k|-1}(w_1(t, x) - w_2(t, x))| \, dx \\
        \frac{|k|r_0}{2\pi} \int_{B_{r_0}} |k(v_2^C - v_1^C) z^{-|k|-1} w_2(t, x)| \, dx.
    \end{cases}
\end{equation}
From Lemma 1.3 again we have with some $C > 0$
\[
\|k (v_1^C(t, \cdot) - v_2^C(t, \cdot)) z^{-|k|-1}\|_{L_2(B_{r_0})} \leq C \|w_1(t, x) - w_2(t, x)\|_W
\]
and
\[
\|k (v_\infty^C - v_1^C(t, \cdot)) z^{-|k|-1}\|_{L_2(B_{r_0})} \leq C \|w_1(t, x)\|_W.
\]
Note, that from Lipschitz continuity of $w$ due to $\vec{u}$ the norms $\|w_1(t, x)\|_W$, $\|w_2(t, x)\|_W$ can be arbitrary small for small $M > 0$. Then the estimate
\[
|F[\vec{u}_1(\cdot)] - F[\vec{u}_2(\cdot)]| \leq C(M)\|w_1(t, \cdot) - w_2(t, \cdot)\|_W
\]
holds with $C(M) \to 0$ as $M \to 0$.
Since with some $L > 0$
\[
\|w_1 - w_2\|_{C([0, T], W)} \leq L \|u_1 - u_2\|_{C([0, T])^{2N+1}}
\]
then for small $M > 0$ we will finally have
\[
\|F[\vec{u}_1(\cdot)] - F[\vec{u}_2(\cdot)]\|_{C([0, T])^{2N+1}} \leq K\|\vec{u}_1(\cdot) - \vec{u}_2(\cdot)\|_{C([0, T])^{2N+1}}
\]
with $K < 1$, and from the fixed-point theorem there exists the solution of (3.6). The theorem is proved. \(\square\)

**Remark.** If $w_0(x) \in H^2(B_{r_0})$ then the solution will satisfy $w(t, \cdot) \in H^2(B_{r_0})$ and the velocity on the boundary will tend to zero in $L_2(\partial \Omega)$ as $N \to \infty$ according to (1.32). If we further increase the smoothness of the initial datum then we can obtain the uniform convergence of velocity to no-slip condition on the boundary.

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**Appendix A Stokes Semi-Group Estimates**

Here we study semigroup for Stokes flow and prove uniqueness solvability for Oseen and Helmholtz equations using fixed-point argument. The boundary-value problem (2.12), (2.13), (2.14) can be solved using special Weber–Orr transform (see [7] for more details):
\[
W_{k,l}[f](\lambda) = \int_0^\infty J_k(\lambda s)Y_l(\lambda r_0) - Y_k(\lambda s)J_l(\lambda r_0)\frac{f(s)s}{\sqrt{J_l^2(\lambda r_0) + Y_l^2(\lambda r_0)}}ds, \quad k, l \in \mathbb{Z},
\]
where $J_k(r)$, $Y_k(r)$ - are the Bessel functions of the first and second type (see [14]).
The inverse transform is defined by the formula
\[
W_{k,l}^{-1}[\hat{f}](r) = \int_0^\infty \frac{J_k(\lambda r)Y_l(\lambda r_0) - Y_k(\lambda r)J_l(\lambda r_0)}{\sqrt{J^2_l(\lambda r_0) + Y^2_l(\lambda r_0)}} \hat{f}(\lambda)\lambda d\lambda. \tag{A2}
\]

The case of \(W_{|k|,|k|-1}\) is of special interest. In [7] it was proved the following

**Theorem A.1.** Let vector field \(\mathbf{v}_0(x)\) satisfies (1.1), (1.3), (1.4), (1.6), the vorticity \(\text{curl} \mathbf{v}_0(x) \in L_1(B_{r_0})\), and its Fourier series as well as Fourier series for its vorticity \(w_0(x)\) with coefficients \(w_0^k(r)\) converges, \(\mathbf{v}(t,x)\) be the solution of (2.11), (2.2)–(2.5). Then \(w(t,x) = \text{curl} \mathbf{v}(t,x)\) satisfies Eq. (2.12), boundary conditions (2.13), (2.14) and is given via Fourier coefficients:
\[
w_k(t,r) = W_{k,-1}[|k|,|k|-1] \left[e^{-t^2/4}W_{|k|,|k|-1}[w^0(\cdot)](\lambda)\right](t,r). \tag{A3}
\]

This formula gives an explicit form of the solution for Stokes system in the exterior of the disc in terms of vorticity. From Biot–Savar law one can get the velocity field.

Generalised Weber–Orr transform has a non-trivial kernel. So, the transform (A2) is the inverse one to (A1) only for functions, which are orthogonal to the kernel. But from no-slip condition (1.13) follows orthogonality of Fourier coefficients \(w_k(t, \cdot)\) to the kernel of \(W_{|k|,|k|-1}\). The invertibility of the generalised Weber–Orr transform was studied in detail in [7].

So, Weber–Orr transforms satisfy Bessel-type inequality instead of the Plancherel identity:
\[
\|W_{k,l}[f]\|_{L^2(0,\infty;\lambda)}^2 \leq \|f\|_{L^2(\mathbb{R};\infty;r)}^2. \tag{A4}
\]

From differentiation rules for Bessel functions follows
\[
\frac{\partial}{\partial r} W_{k,-1}[f] = \frac{1}{2} \left(W_{k-1,-1}[\lambda f] - W_{k+1,-1}[\lambda f]\right), \tag{A5}
\]
\[
\frac{k}{r} W_{k,-1}[f] = \frac{1}{2} \left(W_{k+1,-1}[\lambda f] + W_{k-1,-1}[\lambda f]\right). \tag{A6}
\]

In polar coordinates with unit vectors \(\mathbf{e}_r, \mathbf{e}_\varphi\) the gradient is defined as \(\nabla = \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \varphi} \mathbf{e}_\varphi\) with polar coordinates \(\nabla_r, \nabla_\varphi\). Multiplier \(\frac{k}{r}\) corresponds to differentiation with respect to \(\varphi\) of function \(f(r)e^{ik\varphi}\).

Formula (A6) involves Weber transform \(W_{k+1,-1}[f]\) which also possesses non-trivial kernel and satisfy Bessel inequality
\[
\|W_{k+1,-1}[f]\|_{L^2(0,\infty;\lambda)}^2 \leq \|f\|_{L^2(\mathbb{R};\infty;r)}^2.
\]

And vice versa, multiplication by \(\lambda\) transfers to differentiation but in more general sense including not only \(\frac{\partial}{\partial r}\) but also angle derivative \(\frac{\partial}{\partial \varphi}\) expressed by multiplier \(\frac{k}{r}\). Indeed, using
\[
\lambda J_k(\lambda r) = \frac{k-1}{r} J_{k-1}(\lambda r) - \lambda J_{k-1}(\lambda r),
\]
\[
\lambda Y_k(\lambda r) = \frac{k-1}{r} Y_{k-1}(\lambda r) - \lambda Y_{k-1}(\lambda r)
\]
we will have
\[
\lambda W_{k,-1}[f] = W_{k-1,-1}[\frac{yk}{r}] + W_{k-1,-1}[f'(\cdot)]. \tag{A7}
\]

Formula (A3) in theorem defines the Stokes semigroup \(S(t)\) which corresponds to problem (2.12), (2.13), (2.14). The estimates of this semi-group are given by the following

**Proposition A.2.** For \(t > 0\) Stokes semigroup \(S(t)\) for vorticity satisfies
\[
\|S(t)w_0\|_{L^2(B_{r_0})} \leq \|w_0\|_{L^2(B_{r_0})},
\]
\[
\|\nabla S(t)w_0\|_{L^2(B_{r_0})} \leq \frac{1}{\sqrt{ct}}\|w_0\|_{L^2(B_{r_0})},
\]
\[
\|S(t)w_0\|_{H^1(B_{r_0})} \leq \sqrt{3}\|w_0\|_{H^1(B_{r_0})},
\]
Proof. From Bessel inequality we have
\[ \|w_k(t,r)\|^2_{L_2(\sigma,\infty,r)} \leq \|e^{-\lambda^2 t} W_{k,k-1} [w_k^0 (\cdot)](\lambda)\|^2_{L_2(0,\infty,\lambda)} \]
\[ \leq \|W_{k,k-1} [w_k^0 (\cdot)](\lambda)\|^2_{L_2(0,\infty,\lambda)} \leq \|w_k^0 (r)\|_{L_2(\sigma,\infty,r)}^2. \]
Summarizing by \( k \) we obtain the first estimate.

Fix \( k \geq 0 \). Then from estimate
\[ |\lambda e^{\lambda^2 t}| \leq \frac{1}{\sqrt{2et}} \]
and Bessel inequality (A4) from (A5) we have
\[ \left\| \frac{\partial}{\partial r} w_k(t,r) \right\|_{L_2(\sigma,\infty,r)} \leq \|\lambda e^{\lambda^2 t} W_{k,k-1} [w_k^0 (\cdot)](\lambda)\|_{L_2(0,\infty,\lambda)} \]
\[ \leq \|w_k^0 (r)\|_{L_2(\sigma,\infty,r)} \frac{1}{\sqrt{2et}}. \]

From (A6) in a similar way follows
\[ \left\| \frac{k}{r} w_k(t,r) \right\|_{L_2(\sigma,\infty,r)} \leq \|w_k^0 (r)\|_{L_2(\sigma,\infty,r)} \frac{1}{\sqrt{2et}}. \]
Then we have
\[ \|\nabla_r S(t)w_0\|_{L_2(\sigma_0)} \leq \frac{1}{\sqrt{2et}} \|w_0\|_{L_2(\sigma_0)} \]
and the second estimate of the proposition is proved.

Now we will prove the last inequality. Using (A5),(A7)
\[ \left\| \frac{\partial}{\partial r} w_k(t,r) \right\|_{L_2(\sigma,\infty,r)} \leq \|\lambda e^{\lambda^2 t} W_{k,k-1} [w_k^0 (\cdot)](\lambda)\|_{L_2(0,\infty,\lambda)} \]
\[ = \left\| e^{\lambda^2 t} W_{k-1,k} [\frac{k}{r} w_k^0 (\cdot) + \frac{\partial}{\partial r} w_k^0 (\cdot)] \right\| \leq \left\| \frac{k}{r} w_k^0 (\cdot) \right\| + \left\| \frac{\partial}{\partial r} w_k^0 (\cdot) \right\| \]
and so
\[ \|\nabla_r S(t)w_0\|_{L_2(\sigma_0)} \leq \sqrt{2} \|\nabla w_0\|_{L_2(\sigma_0)}. \tag{A8} \]
In a similar way from (A6), (A7)
\[ \left\| \frac{k}{r} w_k(t,r) \right\|_{L_2(\sigma,\infty,r)} \leq \|\lambda e^{\lambda^2 t} W_{k,k-1} [w_k^0 (\cdot)](\lambda)\|_{L_2(0,\infty,\lambda)} \]
\[ \leq \left( \left\| \frac{k}{r} w_k^0 (\cdot) \right\| + \left\| \frac{\partial}{\partial r} w_k^0 (\cdot) \right\| \right) \]
and finally
\[ \|\nabla_\varphi S(t)w_0\|_{L_2(\sigma_0)} \leq \sqrt{2} \|\nabla w_0\|_{L_2(\sigma_0)} \]
combined with (A8) and the first estimate of the proposition gives the last estimate. \( \square \)
Appendix B Existence Theorem for Oseen Equation in Exterior of the Disc

For a fixed irrotational velocity field $\tilde{v}(t, x)$ consider Oseen equation

$$\frac{\partial w(t, x)}{\partial t} - \Delta w + (\tilde{v}, \nabla) w = 0, \quad (B1)$$

with initial datum (3.1) and condition at infinity (3.2).

We supply the problem (B1), (3.1), (3.2) with boundary conditions (3.3)–(3.5).

Define for $k, l \in \mathbb{Z}$

$$R_{k,l}(\lambda, r) = \frac{J_k(\lambda r)Y_l(\lambda r_0) - J_k(\lambda r)Y_l(\lambda r_0)}{\sqrt{J^2_k(\lambda r_0) + Y^2_l(\lambda r_0)}}.$$

From properties of Bessel functions

$$J_{k-1}(r) = \frac{k}{r} J_k(r) + J'_k(r),$$

$$Y_{k-1}(r) = \frac{k}{r} Y_k(r) + Y'_k(r)$$

follows

$$kR_{k,k-1}(\lambda, r_0) + \lambda r_0 R'_{k,k-1}(\lambda, r_0) = r_0 R_{k-1,k-1}(\lambda, r_0) = 0.$$

Also holds ([14]):

$$R_{k,k-1}(\lambda, r_0) = \frac{J_k(\lambda r_0)Y_{k-1}(\lambda r_0) - J_{k-1}(\lambda r_0)J_k(\lambda r_0)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}} = \frac{2}{\pi r_0 \lambda}.$$

We apply Weber–Orr transform $W_{k,k-1}$ (A1) to Helmholtz equation for Oseen flow (B1). First we find how it acts on Laplace operator. Using integration by parts we will have

$$W_{k,k-1}[\Delta w_k(t, r)] = -r_0 \frac{\partial u_k(t, r)}{\partial r} \left. \frac{R_{k,k-1}(\lambda, r)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}} \right|_0^\infty$$

$$+ r_0 w_k(t, r) \left. \frac{\lambda R'_{k,k-1}(\lambda, r)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}} \right|_0^\infty - \lambda^2 W_{k,k-1}[w_k(t, r)]$$

$$= (u_k(t) - kw_k(t, r_0)) \frac{R_{k,k-1}(\lambda, r_0)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}}$$

$$- r_0 w_k(t, r_0) \frac{\lambda R'_{k,k-1}(\lambda, r_0)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}} - \lambda^2 \hat{w}_k(t, \lambda)$$

$$= -\frac{w_k(t, r_0)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}} \left( kR_{k,k-1}(\lambda, r_0) + \lambda r_0 R'_{k,k-1}(\lambda, r_0) \right)$$

$$- \lambda^2 \hat{w}_k(t, \lambda) + u_k(t) \frac{R_{k,k-1}(\lambda, r_0)}{\sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}}$$

$$= \frac{2u_k(t)}{\pi r_0 \lambda \sqrt{J^2_{k-1}(\lambda r_0) + Y^2_{k-1}(\lambda r_0)}} - \lambda^2 \hat{w}_k(t, \lambda).$$

Then the Helmholtz equation (B1) in terms of Fourier coefficients $w_k$ after Weber–Orr transform can be written as

$$\partial \hat{w}_k(t, \lambda) + \lambda^2 \hat{w}_k(t, \lambda) + r_k(\lambda) u_k(t) + W_{k,k-1}[(v, \nabla) w]_k = 0$$
where

\[ r_k(\lambda) = \frac{2}{\pi r_0 \sqrt{J_{k-1}^2(\lambda r_0) + Y_{k-1}^2(\lambda r_0)}}. \]

Finally with the help of integral transform we reduced the Helmholtz equation to the following integral relation

\[ \hat{w}_k(t, \lambda) = e^{-\lambda^2 t} \hat{w}_{0,k}(\lambda) + r_k(\lambda) \int_0^t e^{-\lambda^2(t-\tau)} u_k(\tau) d\tau \]

\[ + \int_0^t e^{-\lambda^2(t-\tau)} W_{k,k-1}[\hat{\nabla}, \hat{\nabla}] w_k(\tau, \lambda) d\tau, \]

where \( \hat{w}_k(t, \cdot) = W_{k,k-1} [w_k(t, \cdot)] \), \( [(\hat{\nabla}, \hat{\nabla}) w]_k \) — \( k \) th Fourier coefficient of the term \( (\hat{\nabla}, \hat{\nabla}) w \). Take the inverse transform \( W_{k,k-1} \) and rewrite it in terms of Stokes semigroup \( S(t) \) corresponding to the problem (2.12), (2.13), (2.14):

\[ w(t, x) = S(t) w_0(x) + \sum_{k=-N}^{N} e^{ikx} W_{k,k-1} \left[ r_k(\lambda) \int_0^t e^{-\lambda^2(t-\tau)} u_k(\tau) d\tau \right] \]

\[ + \int_0^t (\nabla S(t-\tau), \hat{\nabla}) w(\tau, x) d\tau. \]  

**Theorem B.1** (Resolvability of Oseen equation). For given \( \hat{\nabla}(t, x) \in C(R^+ \times B_{r_0}) \cap L^\infty_{\text{loc}}(R^+; L^\infty(B_{r_0})) \) satisfying no-slip condition (2.4), \( u_k(t) \in L^\infty_{\text{loc}}(\mathbb{R}^+) \), \( k = 0, \ldots, N \), \( w_0(x) \in L^2(B_{r_0}) \), the problem (B1), (3.1)-(3.5) has a unique global solution \( w(t, x) \in C \left( [0, \infty), L^2(B_{r_0}) \right) \) which is locally Lipschitz mapping due to \( \{u_k\}_{k=1}^{N}, w_0(x) \).

**Proof.** First we need prove the local resolvability of the Eq. (B2). Fix \( T > 0 \). For given \( \hat{\nabla} \in L^\infty(B_{r_0}), u_k \in C[0, T], k = 0, \ldots, N \). Consider the map

\[ F(w(\tau, \cdot)) = S(t) w_0(x) + \sum_{k=-N}^{N} e^{ikx} W_{k,k-1} \left[ r_k(\lambda) \int_0^t e^{-\lambda^2(t-\tau)} u_k(\tau) d\tau \right] \]

\[ - \int_0^t (\nabla S(t-\tau), \hat{\nabla}) w(\tau, x) d\tau. \]

Consider the space \( Q = C \left( [0, T], L^2(B_{r_0}) \right) \). Since asymptotical behaviour of \( r_k(\lambda) \) is \( 1/\sqrt{\lambda} \) then with some \( C > 0 \)

\[ \| r_k(\lambda) \int_0^t e^{-\lambda^2(t-\tau)} u_k(\tau) d\tau \|_{L^2(0, \infty, \lambda d\lambda)} \leq C \| u_k(\tau) \|_{C[0,T]}. \]

Set

\[ M = \| w_0(\cdot) \|_{L^2} + C \sum_{k=-N}^{N} \| u_k(\tau) \|_{C[0,T]}. \]
From the estimates on $S(t)$ in Proposition A.2 we have

$$
\|F (w(t, \cdot))\|_{L^2} \leq \|w_0(\cdot)\|_{L^2} + C \sum_{k=-N}^{N} \| u_k(\tau) \|_{C[0,T]} + \int_0^t \| (\nabla S(t-\tau), \tilde{\nu}) w(\tau, x) \|_{L^2} \, d\tau \\
\leq M + \| \tilde{\nu} \|_{L^\infty(\mathbb{R}_+ \times B_{r_0})} \| w \|_{Q} \int_0^t \frac{d\tau}{\sqrt{e(t-\tau)}} \\
= M + 2 \sqrt{\frac{T}{e}} \| \tilde{\nu} \|_{L^\infty([0,T] \times B_{r_0})} \| w \|_{Q}.
$$

So we deduced that $F : Q \to Q$ is well-defined and maps $Q$ into itself. Now we prove that $F$ is a strict contraction in $Q$.

$$
F (w_1(\tau, \cdot)) - F (w_2(\tau, \cdot)) = \int_0^t (\nabla S(t-\tau), \tilde{\nu}) (w_1(\tau, x) - w_2(\tau, x)) \, d\tau
$$

and

$$
\|F (w_1(\tau, \cdot)) - F (w_2(\tau, \cdot))\|_{L^2} \leq \text{vraisup}_{t \in [0,T]} \int_0^t \| (\nabla S(t-\tau), \tilde{\nu}) (w_1(\tau, x) - w_2(\tau, x)) \|_{L^2} \, d\tau \\
\leq 2 \sqrt{\frac{T}{e}} \| \tilde{\nu} \|_{L^\infty([0,T] \times B_{r_0})} \| w_1 - w_2 \|_{Q}.
$$

Then for small $T$ by the Banach fixed point theorem, the map $F$ has a unique fixed point $w(t, x)$.

Next, we prove, that $\|w(t, \cdot)\|_{L^2(B_{r_0})}$ cannot blow up in finite time and the solution is global. Since $F(w) = w$ then from estimates above

$$
\|w(t, \cdot)\|_{L^2} \leq M + \| \tilde{\nu} \|_{L^\infty([0,t] \times B_{r_0})} \int_0^t \left( \frac{1}{\sqrt{2e(t-\tau)}} \right) \| w(\tau, \cdot) \|_{L^2} \, d\tau
$$

and from Gronwall's Lemma

$$
\|w(t, \cdot)\|_{L^2} \leq Me^{2 \sqrt{\frac{T}{e}} \text{vraisup}_{t \in [0,T]} \| \tilde{\nu}(\tau, \cdot) \|_{L^\infty(B_{r_0})}}
$$

and $\|w(t, \cdot)\|_{L^2}$ stays finite for all time $t > 0$.

No, we prove uniqueness. If $w_1(t, \cdot), w_2(t, \cdot)$ are two solutions of (B1), (3.1)–(3.5), then its difference $w_1 - w_2$ is the solution of the same problem with zero initial and boundary conditions. Then $M$ defined above equals zero and from the same Gronwall's Lemma

$$
\|w_1(t, \cdot) - w_2(t, \cdot)\|_{L^2} \leq 0,
$$

and the theorem is completely proved. \hfill \square

**Appendix C Existence Theorem for Helmholtz Equation in Exterior of the Disc**

We consider the map

$$
F (w(t, \cdot)) = S(t)w_0(x) + \sum_{k=-N}^{N} e^{ik\varphi} W_{k,k-1} \left[ r_k(\lambda) \int_0^t e^{-\lambda^2(t-\tau)} u_k(\tau) \, d\tau \right] \\
+ \int_0^t S(t-\tau)(v, \nabla w) \, d\tau
$$

(C1)
associated with the boundary-value problem for the Helmholtz equation (2.16). In virtue of Lemma 1.3
we have

\[ \text{Lemma 1.3} \]

\[ \text{Associated with the boundary-value problem for the Helmholtz equation (2.16).} \]

\[ \text{Denote cylinder} \ Z_{r_0,T} = [0,T] \times B_{r_0}. \]

**Theorem C.1** (resolvability of Helmholtz equation). For any \( L > 0 \) there exists \( T = T(L) > 0 \) such that
for all \( w_0 \in H^1(B_{r_0}), \ u_k \in C[0,T], \ k = 0, \ldots, N, \ ||w_0||_{H^1} \leq L, \ ||u_k(\cdot)||_{C[0,T]} \leq L \) the problem (2.16),
(3.1)–(3.5) has a unique solution \( w(t,x) \in Q \).

**Proof.** First we derive that \( F(w(t,\cdot)) \in H^1(B_{r_0}) \) for fixed \( t \in [0,T] \). From the Proposition A.2 \( S(t) \) is continuous mapping in \( H^1(B_{r_0}) \).

From the estimate

\[ \|\lambda r_k(\lambda) \int_0^t e^{-\lambda^2(t-\tau)} u_k(\tau)d\tau\|_{L_2(0,\infty,\lambda d\lambda)} \leq C\|u_k(\tau)\|_{C[0,T]} \]

the second term in (C1) belongs to \( H^1(B_{r_0}) \).

With help of the proposition A.2 we have

\[ \|F(w(t,\cdot))\|_{H^1(B_{r_0})} \leq \sqrt{3}\|w_0(\cdot)\|_{H^1(B_{r_0})} + C \sum_{k=-N}^N \|u_k(\tau)\|_{C[0,T]} \]

\[ + \int_0^t \|S(t-\tau)(v,\nabla w(\tau,x))\|_{H^1(B_{r_0})} d\tau \]

\[ \leq M + \int_0^t \left(1 + \frac{1}{\sqrt{e(t-\tau)}}\right) \|(v, \nabla w(\tau,x))\|_{H^1(B_{r_0})} d\tau \]

with

\[ M = \sqrt{3}\|w_0(\cdot)\|_{H^1} + C \sum_{k=-N}^N \|u_k(\tau)\|_{C[0,T]} \cdot \]

We invoke the following Sobolev inequality:

\[ \|v(t,\cdot) - v_\infty\|_{L_\infty(B_{r_0})} \leq C\|v - v_\infty(t,\cdot)\|_{L_4(B_{r_0})}^{\frac{1}{2}}\|v_\infty(t,\cdot)\|_{L_4(B_{r_0})}^{\frac{1}{2}} \]

From (1.14) follows

\[ \|\nabla v(t,\cdot)\|_{L_4(B_{r_0})} \leq C\|w(t,\cdot)\|_{L_4(B_{r_0})} \]

Then in view of

\[ \|w(t,\cdot)\|_{L_4(B_{r_0})} \leq C\|w(t,\cdot)\|_{L_2(B_{r_0})}^{\frac{1}{2}}\|\nabla w(t,\cdot)\|_{L_2(B_{r_0})}^{\frac{1}{2}} \]

we have

\[ \|\nabla v(t,\cdot)\|_{L_4(B_{r_0})} \leq C\|w(t,\cdot)\|_{L_2(B_{r_0})}^{\frac{1}{2}}\|\nabla w(t,\cdot)\|_{L_2(B_{r_0})}^{\frac{1}{2}} \]

For any \( r \) on circle \( S_r = \{x \in \mathbb{R}^2, \ |x| = r\} \), \( r \geq r_0 \) holds:

\[ \text{vraisup}_{r \in [r_0,\infty)} \|v(t,\cdot) - v_\infty\|_{L_4(S_r)} \leq C\|v(t,\cdot) - v_\infty\|_{L_2(S_r)}^{\frac{1}{2}}\|\nabla v(t,\cdot)\|_{L_2(S_r)}^{\frac{1}{2}} \]

Then from Lemma 1.1

\[ \int_{B_{r_0}} |v(t,\cdot) - v_\infty|^4 dx \leq C \left(\|w(t,\cdot)\|_{L_2(B_{r_0})}^{2} + \sum_{k=-1,0,1} \|w_k(t,\cdot)\|_{L_2(r_0,\infty)}^{2}\right)\|\nabla v(t,\cdot)\|_{L_2(B_{r_0})}^{2} \]
Finally in virtue of (1.14)
\[
\|v(t, \cdot) - v_\infty\|_{L_\infty(B_{r_0})} \leq C \left( \|w(t, \cdot)\|_{L_2(B_{r_0})}^2 + \sum_{k=-1,0,1} \|w_k(t, \cdot)\|_{L_1(r_0, \infty)}^2 \right)^{\frac{1}{2}} \times \|w(t, \cdot)\|_{L_2(B_{r_0})}^{\frac{3}{2}} \|
\]
and
\[
\|v\|_{L_\infty(Z_{r_0}, \tau)} \leq C \|w\|_Q. \quad \text{(C2)}
\]

So, with new constant \(C > 0\)
\[
\left\| \int_0^t S(t - \tau)(v, \nabla w) d\tau \right\|_{H_1(B_{r_0})} \leq C \left( T + 2 \sqrt{\frac{T}{e}} \right) \|v(t, \cdot)\|_{L_\infty(Z_{r_0}, \tau)} \|w(\tau, x)\|_{C([0,T], \mathcal{H}^1(B_{r_0}))}
\]
and \(F(w(t, \cdot)) \in H^1(B_{r_0})\).

Now we prove that the first Fourier coefficients for \(k = -1,0,1\) of the function \(F(w(\tau, \cdot))\) belong to \(L_1(r_0, \infty; r)\).

From (2.13) zero Fourier coefficient of the Stokes semi-group \([S(t)]_0\) generates radially symmetrical solution \(w(t, x)\) of the heat equation with Newman boundary condition
\[
\frac{\partial w(t, x')}{\partial n} = 0, \ |x'| = r_0.
\]

Semigroup coefficients \([S(t)]_{\pm 1}\) correspond to solutions of the heat equation with Robin boundary
\[
\frac{\partial w}{\partial n} + w(t, x') = 0, \ |x'| = r_0.
\]

From \(L_p - L_q\) estimates for heat equation [17] with some \(C > 0\)
\[
\| [S(t)]_k f \|_{L_1(r_0, \infty, r)} \leq C \sqrt{t} \|f\|_{L_2(r_0, \infty, r)}
\]
for \(k = -1,0,1\).

So we deduced that \(F : Q \rightarrow Q\) is well-defined and maps \(Q\) into itself. Now we prove that \(F\) is a strict contraction in \(Q\) in some ball \(B = \{w \in Q \ | \|w\|_Q < L\}\).

Take \(w_1, w_2 \in B\) with corresponding velocity fields \(v_1, v_2\):
\[
F(w_1(\tau, \cdot)) - F(w_2(\tau, \cdot)) = \int_0^t S(t - \tau) (v_1, \nabla (w_1(\tau, x) - w_2(\tau, x))) d\tau
\]
\[
+ \int_0^t S(t - \tau) (v_1 - v_2, \nabla w_2(\tau, x)) d\tau.
\]

Then from (C2) with some constants \(C_1, C_2\)
\[
\|F(w_1(\tau, \cdot)) - F(w_2(\tau, \cdot))\|_{H^1} \leq \sup_{t \in [0,T]} \int_0^t \|S(t - \tau) (v_1, \nabla (w_1(\tau, x) - w_2(\tau, x)))\|_{H^1} d\tau
\]
\[
+ \sup_{t \in [0,T]} \int_0^t \|S(t - \tau) (v_1 - v_2, \nabla w_2(\tau, x))\|_{H^1} d\tau
\]
\[
\leq \left( T + \frac{2}{e} \sqrt{t} \right) \left( \|v_1\|_{L_\infty(Z_{r_0}, \tau)} w_1 - w_2\|_Q + \|v_1 - v_2\|_{L_\infty(Z_{r_0}, \tau)} w_2\|_Q \right)
\]
\[
\leq C_1 \left( T + \frac{2}{e} \sqrt{t} \right) \left( \|v_1\|_{L_\infty(Z_{r_0}, \tau)} + \|w_2\|_Q \right) \|w_1 - w_2\|_Q
\]
\[
\leq C_2 \left( T + \frac{2}{e} \sqrt{t} \right) \|w_1 - w_2\|_Q.
\]
Estimates of Fourier coefficients for $k = -1, 0, 1$ can be held in a similar way using $L_p - L_q$ estimates. So, for any $L > 0$ we can find such a small $T > 0$ that $F$ becomes the strict contraction map in $Q$. Then for small $T$ by the Banach fixed point theorem, the map $F$ has a unique fixed point $w(t, x)$. The theorem is completely proved.  

□

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