A two dimensional natural convective flow of nanofluid over a vertical plate in the presence of time-exponential temperature has been studied analytically. We have incorporated the effects of thermophoresis and Brownian motion in the considered nanofluid model. Moreover on the boundary of plate, it is assumed that the volumetric concentration of nanoparticles is passively controlled. The non-linear coupled governing equations have been solved with the help of the modified simple function equation method. Exact expressions for the concentration, temperature and velocity distribution profile have been established. Numerical computations are carried out in order to see the influence of different physical parameters like Lewis number, Prandtl number, buoyancy ratio parameter, thermophoresis parameter and Brownian motion parameter on the volumetric concentration of nanoparticles, temperature and velocity distribution profile and results are illustrated graphically and with tables. It is found that for decreasing values of the Brownian diffusion coefficient, therefore for increasing values of the Lewis number, the values of temperature and of particles concentration decrease. The temperature decreases with Prandtl number; while the nanoparticles concentration increases with Prandtl number (the thermal diffusivity has different influence on temperature and concentration).

1. Introduction
High performance thermal transport fluid models are the subject of various investigations in recent decades. Owing to high thermal conductivities behavior of metals, the thermal conduct of suspensions of particulate solid in liquids has been studied a lot in the past. The main purpose of this study is to utilize the high thermal conductivity obtained from the solid particles. In comparison with the conventional thermal transport like water, the thermal conductivity obtained from the solid particles is thousands and thousands times high (Chon, Kihm, Lee, & Choi, 2005). Earlier the study of particulate solids in liquids was done to suspension width millimeters or micro sized particles (Frijns, van den Akker, Hilbers, Stephan, & Van Steenhoven, 2010). Such suspensions suffer from rheological stability problems; even they do in fact show the coveted increment in thermal conductivities. In particular in the mini and micro-channels, the particles have a tendency to rapidly settle out of the suspension which results in terrible clogging (Hardt, Schilder, Tiemann, Kolb, Hessel, & Stephan, 2007; Xuan & Li, 2003). Recent development in nanotechnology has permitted another classification of fluids named nanofluids (Ibrahem, Rabbo, Gambaryan-Roisman, & Stephan, 2010). The suspensions of particles in such fluids are significantly smaller than 100 nanometers. This shoots up the thermal conductivity of the resulting fluid higher than the base fluid. Choi was the first who termed such fluids as ‘nanofluids’ (Choi & Estman, 1995). The main characteristic features of nanofluids include excellent stability and thermal conductivities enhancement than that anticipated by the microscopic models currently available.

There are quite a number of publications which describe the present and future applications of nanofluids (Wong & De Leon, 2010; Kaka & Pramanjaroenkij, 2009; Wang & Mujumdar, 2008a,b; Das, Choi, Yu, & Pradeep, 2007; Saleh, Alali, & Ebaid, 2017). Owing to numerous applications of nanofluids, it has become quite an interesting area of study for researchers and scientists. For example, Rashidi et al. have proposed the volume of fluid model to study the potential of Al2O3 water nanofluid for the improvement in the production of a single slope solar still (Rashidi, Akar, Bovand, & Ellahi, 2018). They have observed that the productivity enhanced by 25% with the increase in the solid volume fraction from 0 to 5%. Zeeshan et al. have
studied the activation of energy in the radioactive MHD Couette-Poiseuille nanofluid flow along horizontal channel in the presence of convective boundary conditions (Zeeshan, Shehzad, & Ellahi, 2018). They have observed that the chemical reaction in the presence of energy activation is directly proportional to the volumetric concentration of nanoparticles. Many interesting results on the current study for the nanofluid flow can be found in Ijaz, Zeeshan, Bhatti, and Ellahi (2018), Ellahi, Zeeshan, Shehzad, and Alamri, (2018), Akbarzadeh, Rashidi, Karimi, and Ellahi, (2018), Ellahi, Tariq, Hassan, and Vafai, (2017), Srinivasacharya and Shafeeuarraman (2017), and Mahanthesh, Gireesha, and Gorla, (2017).

The investigation for the natural convection flows has been a matter of interest from the last many years (Yang, 1988; Catton, 1978). It has many applications in engineering and geophysical sciences. These include cooling of electronic equipment, solar energy collectors, cooling of nuclear reactors, cooling, geothermal systems, space technology and lubricants etc. For instance, in designing a good nuclear wastes disposal canister, the study of the forecast in the characteristics of the natural convection thermal transport due to heated body embedded in a porous medium is a key point.

The flow of fluid and thermal transport through an infinite vertical plate has a lot of applications in science and technology. These include packed-bed storage tanks and catalytic reactors, thermal insulation, geothermal reservoirs, grain storage, gas production and petroleum resources, etc. (Lai & Kulacki, 1991). The thermal convection fluid flows with ramped boundary temperature of the wall in the presence of slip boundary conditions are investigated by Vieru, Imran, and Rauf (2015). The thermal radiation effects on the vertical infinite plate have been studied by Hossain, Khanafer, and Vafai (2001).

Recently, nanoscale flows have been widely used to nanofiber fabrication especially in electrospinning and bubble electrosprning. Interesting results in this direction have been obtained by (Liu, Zhao, & He, 2018; Zhao, Liu, & He, 2017). In this paper we have considered unsteady two-dimensional nanofluid flows over a vertical plate in the presence of time-exponential temperature on the boundary.

2. Mathematical formulation of the problem

Consider a Cartesian coordinate system \((x', y')\), where \(x'\) is taken to be vertical and \(y'\) is orthogonal to it. We considered a vertical infinite plate parallel to \(x'\)-axis. The origin point \((x', y') = (0, 0)\) lies on the plate. We considered a two-dimensional natural convection multi-phase nanofluid flow along this plate. The influence of the Brownian motion and the thermophoresis are incorporated in the considered nanofluid model. At time \(t' = 0\), there is no fluid flow along with the ambient volumetric nanoparticle concentration is \(\phi_{\infty}'\), and temperature is \(T_{\infty}'\). At time \(t' = 0^+\), the surface of the plate is conducted with a time-exponential temperature given by boundary condition (15). Furthermore, it is assumed that the there is no normal flux of the volumetric nanoparticle concentration at the surface of the plate. At \(y' \to \infty\), (i.e. when we are far away from the plate) the ambient values of the volumetric nanoparticle concentration and the temperature are obtained. We further assume that the nanoparticle concentration is dilute.

With the assumption of standard boundary layer and by employing Oberbeck-Boussinesq approximation for nanofluid model, the governing equations for the considered problem can be written as (Buongiorno, 2006; Nield & Kuznetsov, 2009; Kuznetsov & Nield, 2010),

\[
\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = 0, \quad (1)
\]

\[
\begin{align*}
\rho_l (\frac{\partial u'}{\partial t'} + v \frac{\partial u'}{\partial y'} + u \frac{\partial u'}{\partial x'}) + g(\rho_p - \rho_l) (\phi' - \phi_{\infty}')
& = \mu (\frac{\partial^2 u'}{\partial y'^2} + \rho_l g \beta (1 - \phi_{\infty}') (T' - T_{\infty})),
\end{align*}
\]

\[
(\rho_c_p) \frac{\partial T'}{\partial t'} + v \frac{\partial T'}{\partial y'} + u \frac{\partial T'}{\partial x'} = (\rho_c_p) \left( \frac{D_T}{T_{\infty}} \right) \left( \frac{\partial^2 T'}{\partial y'^2} \right) + k'(\frac{\partial^2 T'}{\partial y'^2})^2,
\]

\[
\frac{\partial \phi'}{\partial t'} + v \frac{\partial \phi'}{\partial y'} + u \frac{\partial \phi'}{\partial x'} = \frac{D_p}{T_{\infty}} \frac{\partial \phi_{\infty}'}{\partial y'^2} + \frac{D_p}{\alpha} \frac{\partial \phi_{\infty}'}{\partial y'^2},
\]

where \((u', v')\), \((\rho_{c,p})_p, (\rho_{c,p})_l, \rho_p, \rho_l, \nu, \phi', T', K', g, D_T\) and \(D_p\) are the \((x', y')\)-components of the velocity of the fluid, the effective heat capacity of the nanoparticle material, the heat capacity of the fluid, the nanoparticles density, the base fluid density, the kinematic viscosity, the nanoparticle volume fraction, the temperature of the nanofluid, the thermal conductivity, the gravity acceleration, the thermophoretic diffusion coefficient and the Brownian diffusion coefficient, respectively. We introduced the following non-dimensional identities in Equations (1-4)

\[
\begin{align*}
t & = \frac{Gr^{1/2} t' \nu}{L^2}, \quad \nu = \frac{Gr^{1/4} y'}{L}, \quad \frac{X}{L} = \frac{Gr^{-1/4} v' L}{v} ,
\end{align*}
\]

\[
\begin{align*}
U & = \frac{u}{v}, \quad \frac{Y}{L} = \frac{r}{(1 - \phi_{\infty}')} / \phi_{\infty}',
\end{align*}
\]

\[
\begin{align*}
\frac{\beta_1}{\alpha} T & = \frac{Gr^{1/2}}{v} \frac{T_{\infty}}{T_{\infty}'}, \quad \alpha = \frac{k'}{(\rho_{c,p})_p}, \quad Pr = \frac{\nu}{\alpha},
\end{align*}
\]

\[
\begin{align*}
L_e & = \frac{\alpha}{D_p}, \quad G_r = \frac{g L^3 \beta_1}{v^2} (1 - \phi_{\infty}'),
\end{align*}
\]

\[
\begin{align*}
N_r = \frac{(\rho_p - \rho_l) \phi_{\infty}'}{\rho_l \beta T_{\infty}' (1 - \phi_{\infty}')}, \quad \frac{N_r}{T_{\infty}'} = \frac{D_T \epsilon t'}{L^2}, \quad \epsilon = \frac{(\rho_{c,p})_p}{(\rho_{c,p})_l}, \quad N_r = \frac{D_p \phi_{\infty}'}{\alpha},
\end{align*}
\]

where \(T_1 = \frac{2D_T t'}{L^2}\), \(L\) is the characteristic plate length, \(\mu\) is the viscosity coefficient, \(P_r\) is the Prandtl
number, \( N_i \) is the thermophoresis parameter, \( N_t \) is the buoyancy ratio parameter, \( N_b \) is the Brownian motion parameter, \( L_e \) is the Lewis number and \( G_i \) is the thermal Grashof number. Equations (1–4) reduced to

\[
\frac{\partial V}{\partial Y} + \frac{\partial U}{\partial X} = 0, \tag{5}
\]

\[
\frac{\partial U}{\partial T} + V \frac{\partial U}{\partial Y} + U \frac{\partial U}{\partial X} = \frac{\partial^2 U}{\partial Y^2} - N_t \phi + T, \tag{6}
\]

\[
\frac{\partial T}{\partial T} + V \frac{\partial T}{\partial Y} + U \frac{\partial T}{\partial X} = \frac{1}{P_r} \left[ N_t \left( \frac{\partial^2 T}{\partial Y^2} \right)^{\frac{2}{3}} + \frac{\partial^2 T}{\partial Y^2} + N_b \frac{\partial \phi}{\partial T} \right] \tag{7}
\]

\[
\frac{\partial \phi}{\partial T} + V \frac{\partial \phi}{\partial Y} + U \frac{\partial \phi}{\partial X} = \frac{1}{P_r L_e} \left[ \left( \frac{\partial^2 \phi}{\partial Y^2} \right)^{\frac{2}{3}} + \left( \frac{N_t}{N_b} \right) \frac{\partial^2 T}{\partial Y^2} \right]. \tag{8}
\]

Introducing \( U = \frac{\psi}{\phi} \), \( V = -\frac{\partial \psi}{\partial X} \); where \( \psi := \psi(X, Y, t) \). Equation (5) is satisfied and Equations (6)–(8) reduced to

\[
\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} + \frac{\partial \psi}{\partial X} \frac{\partial \psi}{\partial Y} = \frac{\partial^3 \psi}{\partial Y^3} - N_t \phi + T, \tag{9}
\]

\[
\frac{\partial T}{\partial T} + \frac{\partial \psi}{\partial X} \frac{\partial T}{\partial Y} + \frac{\partial \psi}{\partial Y} \frac{\partial T}{\partial X} \frac{1}{P_r} \left[ N_t \left( \frac{\partial^2 T}{\partial Y^2} \right)^{\frac{2}{3}} + \frac{\partial^2 T}{\partial Y^2} + N_b \frac{\partial \phi}{\partial T} \right] \tag{10}
\]

\[
\frac{\partial \phi}{\partial T} + \frac{\partial \psi}{\partial X} \frac{\partial \phi}{\partial Y} + \frac{\partial \psi}{\partial Y} \frac{\partial \phi}{\partial X} = \frac{1}{P_r L_e} \left[ \left( \frac{\partial^2 \phi}{\partial Y^2} \right)^{\frac{2}{3}} + \left( \frac{N_t}{N_b} \right) \frac{\partial^2 T}{\partial Y^2} \right]. \tag{11}
\]

Along with the Equations (9)–(11), we consider the following initial-boundary conditions

\[
\psi(X, Y, t) = 0, \psi(X, Y, 0) = \frac{a_0}{c_0}, \tag{12}
\]

\[
\psi''(X, Y, t) = \frac{a_0}{c_0} \text{ at } (X, Y, t) = (0, 0, 0), \tag{13}
\]

\[
V = 0 = \frac{\partial \psi}{\partial X} = 0, U = \frac{\partial \psi}{\partial X} = 0, \text{ at } t = 0, Y \geq 0, X \geq 0, \tag{14}
\]

\[
\phi(X, Y, 0) = \frac{1}{L_e N_t} \left[ \frac{\alpha_1 + \left( 1 - 2 \sqrt{L_e(1-L_e)} \right) e^{-\sqrt{\theta} \left( X + c_0 Y \right)}}{\alpha_1 + e^{-\sqrt{\theta} \left( X + c_0 Y \right)}}, \tag{15}
\]

\[
\frac{\partial \phi(X, Y, t)}{\partial T} = \frac{2}{N_t} \ln \left( \frac{\alpha_1 + e^{-\sqrt{\theta} \left( X + c_0 Y \right)}}{\alpha_1 + e^{-\sqrt{\theta} \left( X + c_0 Y \right)}} \right) \tag{16}
\]

where \( \alpha_0, \alpha_1, b, c \in (0, \infty) \), \( a \in (-\infty, 0) \) are constants and \( L_e \in (0, 1), b \leq 1 \).

### 2.1. Solutions for the concentration and temperature

Now we make the transformations \( \xi = a t + b X + c Y \), \( \psi = A(\xi), \ T = B(\xi), \ \phi = C(\xi) \) in above system (9–11). (Notice; \( \xi^\prime = a \frac{\xi}{c}, \ \frac{\xi}{c} = b \frac{\xi}{c}, \ \frac{\xi}{c} = c \frac{\xi}{c} \))

\[
C^\prime = c^2 \phi = ac^2 + B - N_t C = 0, \tag{19}
\]

\[
aP_b B' = c^2 B + N_t c^2 B' + N_t c^2 (B')^2, \tag{20}
\]

\[
aP_c L_c C' = c^2 C' + \frac{N_t}{N_b} c^2 B'^2. \tag{21}
\]

We will solve Equations (19–21) with the help of modified simple function method. From Equations (19–21) and neglecting the constant of integration we can write

\[
C^\prime = a_1 C' + a_2 C^2 + a_3 C^3 = 0, \tag{22}
\]

where \( c \neq 0 \), \( a_1 = \frac{aP(1+L_e)}{c}, \ a_2 = \frac{aP^2 L_e}{c^2}, \ a_3 = \frac{aP^3 L_e^2 N_t}{c^3} \). Let

\[
C(\xi) = \sum_{k=0}^{m} \theta(\xi)^k, \tag{23}
\]

where the arbitrary constants \( \theta(\xi) \) \( (k = 0, 1, 2, \ldots, m) \) and the unknown function \( \theta(\xi) \) are to be determined such that \( \alpha_m \neq 0 \). Balancing \( C^\prime(\xi) \) and \( C^2(\xi) \) gives \( m = 2 \). This means that

\[
C(\xi) = a_0 + a_1 \left( \frac{\theta(\xi)}{\theta(\xi)^2} \right) + a_2 \left( \frac{\theta(\xi)}{\theta(\xi)^2} \right). \tag{24}
\]

By substitution (24) into (22), we obtain

\[
n a_0 + a_1 \left( \frac{\theta(\xi)}{\theta(\xi)^2} \right) + a_2 \left( \frac{\theta(\xi)}{\theta(\xi)^2} \right) = 0. \tag{25}
\]
Comparing the coefficients of \( \theta^0, \theta^{-1}, \theta^{-2}, \theta^{-3}, \theta^{-4}, \theta^{-5} \) to zero, we get the corresponding equations respectively:

\[
\begin{align*}
    n_2a_0 - n_3a_2^2 &= 0, \\
    a_1\theta^0 - n_1a_0\theta^0 + n_4a_0a_1\theta^0 + n_2a_1\theta^0 - 2n_2a_0a_1\theta^0 &= 0, \\
    (n_0a_2^2 - 3a_1 - 2n_2a_2 + 2n_4a_0a_2)\theta^0\theta^0 + 2n_3(\theta^2 + \theta^0\theta^0) &= 0, \\
    + (n_2a_2 + n_1a_1 - n_3a_2^2 - 2n_2a_0a_2 - 2n_4a_1a_0)\theta^2 &= 0, \\
    2a_1\theta^3 - 10a_2a_0\theta^3 + 2n_1a_1\theta^3 - 2n_2a_2\theta^3 &= 0, \\
    + (n_0a_2a_0\theta^3 - n_3a_2^2\theta^3 + 3n_4a_1a_0\theta^2a_0 + 2n_2a_1a_0\theta^2) &= 0, \\
    6a_2\theta^4 - 3a_1^2\theta^4 - 3n_4a_1a_0a_2\theta^4 + 2n_2a_0\theta^4 &= 0, \\
    2n_0a_2^2\theta^5 &= 0.
\end{align*}
\]

Solving above system of equations simultaneously, we get

\[
\begin{align*}
a_0 &= \frac{D_{a_1}}{L_{a_0}}, \\
a_1 &= \frac{2}{n_3}, \\
a_2 &= 0 \\
\theta(\zeta) &= A_1 + A_2e^{\sqrt{\gamma}} + A_3e^{-\sqrt{\gamma}},
\end{align*}
\]

where \( \gamma = n_2(1-L_2) \). Using these values of \( a_0, a_1, a_2 \) and \( \theta(\zeta) \) in Equation (24), we get

\[
C(\zeta) = \frac{1}{L_{a_0}N_0} + \frac{2\sqrt{\gamma}}{n_3}A_2e^{\sqrt{\gamma}} - A_3e^{-\sqrt{\gamma}}.
\]

Using the conditions (14), (16) into (33) we find that \( A_1 = \alpha_1, \quad A_2 = 0 \quad \text{and} \quad A_3 = 1 \), therefore \( C(\zeta) \) is given by

\[
C(\zeta) = \frac{1}{L_{a_0}N_0}a_1 + \beta_1e^{-\sqrt{\gamma}}.
\]

where \( \beta_1 = 1-2\sqrt{L_2(1-L_2)} \). Using Equations (24) and (32) with \( A_1 = \alpha_1, \quad A_2 = 0 \quad \text{and} \quad A_3 = 1 \), we have

\[
\int C(\zeta)d\zeta = a_0 \zeta + a_1 \ln(\alpha_1 + e^{-\sqrt{\gamma}}) + D_0.
\]

Equation (21) leads to

\[
B(\zeta) = \frac{aP_lL_{a_0}}{C_0^2 N_0} \int C(\zeta)d\zeta - \frac{N_0}{N_t}C(\zeta) + D_1 + D_2.
\]

Using (35) in (36) we get

\[
B(\zeta) = \frac{2}{N_t} \ln(\alpha_1 + e^{-\sqrt{\gamma}}) + \left( \frac{aP_l}{C_0^2 N_0} + D_1 \right) \xi
\]

\[
+ D_3 \frac{1}{N_{t_1}} \frac{1}{\alpha_1 + e^{-\sqrt{\gamma}}},
\]

where \( D_3 = \frac{aP_lL_{a_0}}{C_0^2 N_0}D_0 + D_2 \). In order to have \( \lim_{\zeta \to \infty} B(\zeta) < \infty \) we must put \( D_1 = - \frac{\alpha_0}{N_{t_1}} \). Using the boundary condition (15) we get \( D_3 = 0 \), so

\[
B(\zeta) = \frac{2}{N_t} \ln(\alpha_1 + e^{-\sqrt{\gamma}}) - \frac{1}{L_{a_0}N_0} \frac{1}{\alpha_1 + e^{-\sqrt{\gamma}}} - \frac{N_0}{N_t}C(\zeta).
\]

In order to obtain \( \lim_{\zeta \to \infty} B(\zeta) = 0 \), we must consider for \( \alpha_1 \) the value \( \alpha_1 = \exp\left[\frac{1}{2\sqrt{\gamma}}\right] \).

2.2. Calculations for the stream function and velocity components

Using Equations (38) and (34) in Equation (19) we can write

\[
c^3A''(\zeta) - acA''(\zeta) = h(\zeta),
\]

where
The system of fundamental solutions for the homogeneous equations obtained to Equation (39) are 1, \( \xi \), \( \exp(\frac{a}{c^2} \xi) \). The general solution of Equation (39) is

\[
A(\xi) = K_1(\xi) + K_2(\xi)\xi + K_3(\xi) \exp\left(\frac{a}{c^2} \xi\right). \tag{41}
\]

Applying the method of variations of constants, we have the system

\[
K_1'(\xi) + K_2'(\xi)\xi + K_3'(\xi) \exp\left(\frac{a}{c^2} \xi\right) = 0, \tag{42}
\]

\[
K_2'(\xi) + \frac{a}{c^2} K_2'(\xi) \exp\left(\frac{a}{c^2} \xi\right) = 0, \tag{43}
\]

\[
\frac{a^2}{c^4} K_3'(\xi) \exp\left(\frac{a}{c^2} \xi\right) = h(\xi). \tag{44}
\]

Solving above system of Equations (42)–(44) simultaneously we get

\[
K_1(\xi) = E_1 + \int_0^\xi \left[ \frac{c^2}{a} \frac{z}{h(z)} - \frac{c^4}{a^2} h(z) \right] dz, \tag{45}
\]

\[
K_2(\xi) = E_2 + \frac{c^2}{a} \int_0^\xi h(z) dz; \quad K_3(\xi) = E_3 + \frac{c^4}{a^2} \int_0^\xi \frac{z}{h(z)} e^{-\frac{a}{c^2} h(z)} dz. \tag{46}
\]

where \( E_1, E_2 \) and \( E_3 \) are constants. Using Equations (45) and (46) into (41) we obtain

\[
A(\xi) = E_1 + \int_0^\xi \left[ \frac{c^2}{a} \frac{z}{h(z)} - \frac{c^4}{a^2} h(z) \right] dz + E_2 \xi + \frac{c^4}{a^2} \int_0^\xi h(z) e^{-\frac{a}{c^2} h(z)} dz + E_3 \exp\left(\frac{a}{c^2} \xi\right) \tag{47}
\]

Using initial conditions (12) in Equation (47) we get \( E_1 = -\alpha_0, E_2 = 0 \) and \( E_3 = \alpha_0 \). Differentiating (47) with respect to \( \xi \) we have

\[
A'(\xi) = -\frac{c^2}{a} \int_0^\xi h(z) dz + \frac{c^2}{a} \exp\left(\frac{a}{c^2} \xi\right) \int_0^\xi e^{-\frac{a}{c^2} h(z)} dz
\]

\[\quad + \frac{\alpha_0}{c^2} \exp\left(\frac{a}{c^2} \xi\right). \tag{48}\]

Hence the require velocity components for the fluid flow are

\[
U(X, Y, t) = \frac{\partial \psi}{\partial Y} = \frac{\partial A}{\partial Y} = cA'(\xi), \tag{49}
\]

\[
V(X, Y, t) = -\frac{\partial \psi}{\partial X} = -bA'(\xi), \tag{49}
\]

where \( \xi = at + bx + cy \).

3. Numerical validation of results

In order to have a validation of the solutions given by Equations (34) and (47), we have used a numerical subroutine from the software Mathcad, namely the subroutine ‘Odesolve’ for the numerical integration of the ordinary differential equations. It is observed from Equations (19)–(22) that, if the concentration \( C(\xi) \) it is known, then the temperature \( B(\xi) \) and the stream function \( A(\xi) \) are determined. From this reason we have given the numerical solution only for Equation (22). By introducing notations

\[
y_1(\xi) = C(\xi), \quad y_2(\xi) = C'(\xi), \quad y(\xi) = \left( \frac{y_1(\xi)}{y_2(\xi)} \right), \tag{49}
\]

Equation (22) can be written as a system of two differential equations of order one, namely

\[
y'(\xi) = \left( \frac{n_1 y_2(\xi) - n_2 y_1(\xi) + n_3 y_1(\xi) y_2(\xi) - n_4 y_1(\xi) y_2(\xi)}{y_2(\xi)} \right). \tag{50}
\]

Numerical solutions of the above system have been obtained by using the subroutine Odesolve(Y, \( \xi, t, 10^7 \)). The results given by the numerical method are compared with the numerical results given by the analytical expression (34). Results are in good agreement and are presented in the Table 1 and Figure 1.

4. Result and discussion

A two dimensional natural convective nanofluid flow over a vertical plate in the presence of time-exponential temperature has been analyzed analytically. Both the thermophoresis and Brownian motion effects are incorporated in the considered nanofluid model. We have used the modified simple function method to solve the corresponding governing equations. Exact expression for temperature, volumetric nanoparticle concentration and velocity distribution have been recovered. In this section numerical computations have been established in order to see the influence of different non-dimensional physical parameters like Lewis number, Pradtl number, thermophoresis parameter, buoyancy ratio parameter and Brownian motion parameter on the volumetric concentration.
concentration of nanoparticles, temperature and velocity distribution profile and results are illustrated graphically and with tables.

The values for the non-dimensional parameters which we have used in our numerical computations are \(a = -0.02\), \(b = 0.1\), \(c_1 = 0.05\), \(N_t = 0.75\) and \(D_f = 5\). Figure 2 illustrates the volumetric nanoparticles concentration increases with the passage of time and a constant value is attained when the fluid is quit away from the plate. This is in resemblance with the applied boundary condition that the ambient value of temperature is attained far away from the plate. Figure 3 depicts that the thermal transport distribution profile decreases with the increase in time. The thermal transport distribution profile is very high near the plate as compared to the nanofluid temperate being away from the plate. Figure 3 also shows that temperature profile decay exponentially as a function of spatial coordinate \(y\) and an ambient value of temperature is attained far away from vertical plate. Figures 4 and 5 demonstrates the influence of Prandtl number \(Pr\) and Lewis number \(Le\) on the volumetric nanoparticles concentration \(\phi\). The effect of the Prandtl number \(Pr\) on the volumetric nanoparticles concentration is depicted in the Figure 4. It has been observed that the volumetric nanoparticles concentration increases with the increase in the values of Prandtl number \(Pr\). Figure 5 elucidates the effects of Lewis number \(Le\) on the volumetric nanoparticles concentration. It is clear that a decrease in the volumetric nanoparticles concentration \(\phi\) has been observed with the increase in the values of the physical number \(Le\). Figures 6–9 describes the effects of physical parameters Prandtl number \(Pr\), Lewis number \(Le\), the Brownian motion parameter \(N_b\) and the thermophoresis parameter \(N_t\) on the thermal transport profile distribution. In each case it is observed that the thermal transport profile decreases with the increase in one of the physical parameters \(Pr\), \(N_b\), \(N_t\) and \(Le\). It is observed from the temperature profiles that, for increasing values of the Prandtl number, values of the temperature are decreasing. This behavior is because values of Prandtl number are increasing for lower values of the thermal diffusivity coefficient. As a
consequence, fluid velocity decreases and the volumetric nanoparticle concentration increases. Also, it must be highlighted that the temperature is decreasing with the Brownian motion parameter $N_b$ and with the thermophoresis parameter $N_t$ because the increasing values of these parameters are due to the low fluid thermal conductivity. As it is observed from Table 2, the velocity $U(\zeta)$ is increasing with the thermophoresis parameter $N_t$. As it is observed from Table 2, the velocity $U(\zeta)$ is increasing with the thermophoresis parameter $N_t$.

5. Conclusions

Unsteady two-dimensional convective nanofluid flows over a vertical heated plate have been analytically studied. To have a validation of the obtained results, the subroutine Odesolve from Mathcad software has been employed in order to obtain numerical solutions. By comparing both the analytical and numerical results, a good agreement has been found.

From numerical studies and graphical representations, some interesting conclusions have been found on the behavior and movement of nanofluid.

- The nanoparticle concentration increases for fluids with low thermal conductivity while the fluid temperature is decreasing.
- The temperature of nanofluid is decreasing function with the Brownian motion parameter and with the thermophoretic parameter.
The nanoparticle concentration decreases with the Lewis number. Increasing values of the Lewis parameter means a smaller Brownian diffusion coefficient.

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