Representation of One as the Sum of Unit Fractions

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Abstract

One is expressed as the sum of the reciprocals of a certain set of integers. We give an elegant proof to the fact applying the polynomial theorem and basic calculus.

1 Introduction

Let us consider the representation of one as the sum of unit fractions. For examples, we can take 2, 3 and 6 for

\[ \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1, \]

and 3, 4, 4, 8 and 24 for

\[ \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{24} = 1. \]

It is well known that any positive rational number can be written as the sum of unit fractions. In the paper, we give a part of solutions to the Diophantine equation

\[ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1, \]

where the \( x_j \) are not necessarily distinct integers for \( j = 1, 2, \ldots, n \).

To explain our result for \( n = 6 \), we find all possible combinations of

\[ \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in \mathbb{N}^6 \]
### Table 1: Possible combinations of $\alpha$ for $n = 6$

| $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | denominator |
|------------|------------|------------|------------|------------|------------|-------------|
| 6          | 0          | 0          | 0          | 0          | 0          | $6! = 720$  |
| 4          | 1          | 0          | 0          | 0          | 0          | $4! \cdot 2 = 48$ |
| 3          | 0          | 1          | 0          | 0          | 0          | $3! \cdot 3 = 18$ |
| 2          | 2          | 0          | 0          | 0          | 0          | $2! \cdot 2! \cdot 2^2 = 16$ |
| 2          | 0          | 0          | 1          | 0          | 0          | $2! \cdot 4 = 8$ |
| 1          | 1          | 1          | 0          | 0          | 0          | $2 \cdot 3 = 6$ |
| 1          | 0          | 0          | 0          | 1          | 0          | $5$ |
| 0          | 3          | 0          | 0          | 0          | 0          | $3! \cdot 2^3 = 48$ |
| 0          | 1          | 0          | 1          | 0          | 0          | $2 \cdot 4 = 8$ |
| 0          | 0          | 2          | 0          | 0          | 0          | $2! \cdot 3^2 = 18$ |
| 0          | 0          | 0          | 0          | 0          | 1          | $6$ |

such that

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 = 6,$$

(5)

where $\alpha_j \in \mathbb{N} = \{0, 1, 2, \ldots\}$ for $j = 1$ to 6. Next, we take the quantities

$$\prod_{j=1}^{6} \alpha_j! j^{\alpha_j} = \alpha_1!^{\alpha_1} \cdot \alpha_2!^{2\alpha_2} \cdot \ldots \cdot \alpha_6!^{6\alpha_6}$$

(6)

for each possible $\alpha$. Then, we can calculate the sum of reciprocals of the above quantities

$$\frac{1}{720} + \frac{1}{48} + \frac{1}{18} + \frac{1}{16} + \frac{1}{8} + \frac{1}{6} + \frac{1}{5} + \frac{1}{48} + \frac{1}{8} + \frac{1}{18} + \frac{1}{6}$$

(7)

which is equal to 1.

The example is generalized to our main result:

**Theorem.** For any positive integer $n$,

$$\sum_{\alpha \in S_n} \prod_{j=1}^{n} \frac{1}{\alpha_j! j^{\alpha_j}} = 1,$$

(8)

where the summation over $S_n$ runs through all possible $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{N}^n$ such that

$$\sum_{j=1}^{n} j^{\alpha_j} = n.$$

(9)
2 Preliminaries

Lemma 1 (Polynomial theorem). Let $n$ and $m$ be positive integers. For any $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$,

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \quad (10)$$

where $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{N}^n$ and the summation runs through all possible $\alpha$ in $\mathbb{N}^n$ such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = m$.

Proof. Each coefficient of

$$x^\alpha = \prod_{j=1}^n x_j^{\alpha_j} \quad (11)$$

in the right-hand side for some $\alpha$ in $\mathbb{N}^n$ with $|\alpha| = m$ is equal to the number of combinations of the products among $x_1, x_2, \ldots, x_n$. \qed

Lemma 2. Given a polynomial

$$f(x) = \sum_{j=0}^n a_j x^j. \quad (12)$$

Then, the $j$th coefficient of $f(x)$ can be expressed by

$$a_j = \frac{1}{j!} f^{(j)}(0), \quad (13)$$

where $f^{(j)}$ stands for the $j$th derivative.

Proof. $f$ is infinitely differentiable, since the $j$-times differential function of $x^k$ is

$$\left( \frac{d}{dx} \right)^j x^k = \frac{k!}{(k-j)!} x^{k-j} \quad (14)$$

if $j \leq k$, and

$$\left( \frac{d}{dx} \right)^j x^k = 0 \quad (15)$$

if $j > k$. We have

$$f^{(j)}(x) = \sum_{k=0}^n a_k \left( \frac{d}{dx} \right)^j x^k = \sum_{k=j}^n \frac{k!}{(k-j)!} a_k x^{k-j} \quad (16)$$
for any $j$ between 0 and $n$, then

$$f^{(j)}(0) = j!a_j,$$

which implies the conclusion of the lemma

$$a_j = \frac{1}{j!}f^{(j)}(0).$$

\[\Box\]

**Lemma 3.** Let $n$ be a positive integer. We put

$$g(x) = \left(x + \frac{1}{2}x^2 + \cdots + \frac{1}{n}x^n\right)^n,$$

then $g^{(n)}(0) = n!$.

**Proof.** Put

$$g(x) = x^n \left(1 + \frac{1}{2}x + \cdots + \frac{1}{n}x^{n-1}\right)^n = x^n h(x),$$

where

$$h(x) = \left(1 + \frac{1}{2}x + \cdots + \frac{1}{n}x^{n-1}\right)^n.$$

Leibniz rule implies

$$g^{(n)}(x) = \sum_{j=0}^{n} \frac{n!}{(n-j)!j!} \left(\frac{d}{dx}\right)^{n-j} x^n \cdot \left(\frac{d}{dx}\right)^j h(x)$$

$$= \sum_{j=0}^{n} \frac{n!n!}{(n-j)!j!j!} x^j \cdot \left(\frac{d}{dx}\right)^j h(x).$$

Therefore, we obtain

$$g^{(n)}(0) = n!h(0) = n!.$$
3 Proof of the main result

We begin with the relation

\[
\left(x_1 + \frac{1}{2}x_2 + \cdots + \frac{1}{n}x_n\right)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x_1^{\alpha_1} \left(\frac{1}{2}x_2^{\alpha_2}\right) \cdots \left(\frac{1}{n}x_n^{\alpha_n}\right)
\]

by Lemma [1] Putting \(m = n\) and \(x_j = t\) for \(j = 1, 2, \ldots, n\) implies

\[
\left(t + \frac{1}{2}t^2 + \cdots + \frac{1}{n}t^n\right)^n = n! \sum_{|\alpha|=n} \frac{1}{\alpha!} t^{\sum_j \alpha_j}. \tag{25}
\]

Compare to the coefficients of \(t^n\) in both side of the identity. We obtain \(n!\) from the left-hand side of the identity by Lemma [2] and Lemma [3]. On the other hand, the coefficient of \(t^n\) in the right-hand side is the sum of all terms with \(t^n\), which is written by

\[
\sum_{\alpha \in S_n} \prod_{j=1}^n \frac{1}{\alpha_j! j^{\alpha_j}}, \tag{26}
\]

where the summation runs through all possible \(\alpha\) in \(S_n\) defined by

\[
S_n = \{\alpha \in \mathbb{N}^n; \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n\}. \tag{27}
\]

Hence, we obtain

\[
\sum_{\alpha \in S_n} \prod_{j=1}^n \frac{1}{\alpha_j! j^{\alpha_j}} = 1 \tag{28}
\]

which completes the proof of our main result.

4 Concluding Remarks

In this paper, a part of reciprocal bases of one is investigated from analytic point of view. In particular, the polynomial theorem and the multi-index analysis play important role in the proof. Although these are not all of solutions to the Diophantine equation, one is presented as the sum of the reciprocal numbers of a certain set of integers.
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