Critical effects of overlapping of connectivity and dependence links on percolation of networks

Ming Li\textsuperscript{1}, Run-Ran Liu\textsuperscript{2,4}, Chun-Xiao Jia\textsuperscript{2} and Bing-Hong Wang\textsuperscript{1,3,4}

\textsuperscript{1} Department of Modern Physics, University of Science and Technology of China, Hefei 230026, People’s Republic of China
\textsuperscript{2} Institute of Information Economy, Hangzhou Normal University, Hangzhou 310036, People’s Republic of China
\textsuperscript{3} Complex System Research Center, University of Shanghai for Science and Technology and Shanghai Academy of System Science, Shanghai 200093, People’s Republic of China
E-mail: runranliu@gmail.com and bhwang@ustc.edu.cn

New Journal of Physics 15 (2013) 093013 (9pp)
Received 27 April 2013
Published 10 September 2013
Online at http://www.njp.org/
doi:10.1088/1367-2630/15/9/093013

Abstract. In a recent work Parshani \textit{et al} (2011 \textit{Proc. Natl Acad. Sci. USA} \textbf{108} 1007), dependence links have been introduced to the percolation model and used to study the robustness of the networks with such links, which shows that the networks are more vulnerable than the classical networks with only connectivity links. This model usually demonstrates a first order transition, rather than the second order transition found in classical network percolation. In this paper, considering the real situation that the interdependent nodes are usually connected, we study the cascading dynamics of networks when dependence links partially overlap with connectivity links. We find that the percolation transitions are not always sharpened by making nodes interdependent. For a high fraction of overlapping, the network is robust for random failures, and the percolation transition is second order, while for a low fraction of overlapping, the percolation process shows a first order transition. This work demonstrates that the crossover between two types of transitions does not only depend on the...
density of dependence links but also on the overlapping fraction of connectivity
and dependence links. Using generating function techniques, we present exact
solutions for the size of the giant component and the critical point, which are in
good agreement with the simulations.

Contents

1. Introduction 2
2. Analytical solution 3
3. Simulation results and discussion 7
4. Conclusions 8
Acknowledgments 8
References 9

1. Introduction

Percolation theory has been extensively studied in statistical physics and mathematics with
applications in diverse fields [1–4]. In the last decade, percolation theory was particularly useful
in studying the robustness of the networks against random failure and intentional attack [5–9].
Generally speaking, for network science, percolation theory explains the existence of the giant
component of a network after a fraction $1 - p$ of nodes are removed, which usually exhibits a
second order phase transition [5, 6]. For some special networks, such as a random network and
its extensions, the critical point $p_c$ can be solved exactly using the typical method in random
graph theory [10] or the popular generating function techniques [11–14].

Various models have been proposed to study failure propagation among nodes in a realistic
network. These studies mainly focus on two aspects, one addresses on the local dependence
failures [15–18], the other addresses on the global overload failures [19–21]. All these works
demonstrate the fragility of the real networks in theory. For example, Watts’s work shows that
even the failure of one node could also lead to a global cascade [16]. Furthermore, cascades
in percolation are also studied as an avalanche [22–24], branching process [25] or sandpile
model [26]. In these models, the topological structure of the network confines the propagation
of failures, so a few stable nodes could limit the spread of failures on the network to make it
more robust [5]. At the same time, however, if the stable nodes fail (targeted attack), the network
will be very fragile [5, 8, 9].

Recently, based on the motivation that modern infrastructures are becoming significantly
more dependent on each other, percolation on interdependent networks has been studied
[27, 28]. To model interdependent networks, the dependence link has been proposed to represent
the dependence of nodes between different networks. Specifically, a dependence link between
node $i$ and node $j$ means that if node $i$ fails node $j$ also fails, and vice versa. Buldyrev et al
[27] showed that the interdependent networks are more fragile than a single network, and
demonstrate a first order phase transition. On the other hand, as shown by Son et al [29], it
depends on the topology of the network whether dependencies lead to first order transitions or
not. For networks embedded in low dimensional space, the transition is still second order.

Following the work of Buldyrev et al [27], interdependent networks with different
topological properties and attack strategies have been studied extensively in the past 3 years,
such as inter-similarity [30], multiple support-dependence relations [31], targeted attack [32], network of networks [33] and assortativity [34]. Besides, dependence links have been introduced into the study of the cascading failures on a single network [35]. In this model, due to the iterative process of cascading failures caused by dependence links, the network is more fragile than the network without dependence links. In addition, for a high density of dependence links, the model demonstrates a first order phase transition, which is different from the network composed by connectivity links only. Furthermore, with a view that more than two nodes depend on each other, a dependence group is used to replace the dependence link and the model has been developed as a general one with arbitrary size distributions of dependence groups [36]. In [37], the general formalism for analyzing random and random-regular networks with dependence groups has been summarized, which gives a clear framework for further study.

The studies of the percolation on a single network with two types of links are all based on the assumption that the dependence and connectivity links are independent of each other. However, in reality, two nodes probably have a dependence link and a connectivity link simultaneously. That is to say, there can be some overlapping between dependence links and connectivity links in real networks. For example, considering a financial network, the trading and sales connections enable the companies to interact with each other, which can be understood as connectivity links [35]. But for two companies with the supply–demand relationship, the trading and sales connections between them also represents the dependence relationship (dependence links). In this case, if one company fails, the other one will also fail due to the loss of customers (or raw materials).

In this paper, we will study the cascading dynamics of networks with the overlapping of dependence and connection links. Following the model proposed by Parshani et al [35], we consider a network connected by connectivity links with a degree distribution $P_k$ and each node has exactly one dependence partner (i.e. the case of $q = 1$ in [35]). As the setting used in [35], if node $i$ depends on node $j$, then node $j$ depends on node $i$. To describe the overlapping of the two types of links, we assume that a fraction $\beta$ of the dependence links overlaps with the connectivity links. That means a fraction $\beta$ of nodes are adjacent to their dependence partners. These partners are of local dependence compared to the randomly chosen one. When $\beta \to 0$, our model will reduce to the case of the dependence link density $q = 1$ in [35].

This paper is organized as follows. In the next section, we will give the analytical results using generating function techniques. In section 3, the simulation results will be presented to test the analysis results. In the last section, we will summarize our findings in this paper.

2. Analytical solution

The cascading process begins by randomly removing a fraction, $1 - p$, of nodes of the network and their links. The failure of a node will lead to the failure of its dependence partner, even though it still connects to the network by connectivity links. The failures of connectivity links may cause the other nodes to disconnect from the network. These two cascading processes are called the dependence process and the percolation process, respectively [35]. Thus, the two cascading processes will occur alternately after the initial removal of nodes until no further splitting and link removal can occur.

We solve this model by considering the final state after the cascades as the method used in [29, 38]. Let $S$ be the order parameter usually used in percolation theory, that is the probability that a randomly chosen node belongs to the giant component of the final network, i.e. the size...
Figure 1. Schematic representation of the giant component after the cascades. The solid and dashed lines represent the connectivity and dependence links, respectively. In the giant component, there are generally two kind of nodes represented by (a) and (b), respectively. (a) The two dependence nodes are not connected (fraction $1 - \beta$). For this case, both of the nodes must connect to the giant component. (b) The two dependence nodes are connected (fraction $\beta$). For this case, only one of the nodes is needed to connect to the giant component.

of the giant component of the final network. In addition, let $R$ be the probability that the node, arriving by following a randomly chosen link (the connectivity links excluding the overlapping links), belongs to the giant component of the final network. Then, in the steady state, $S$ is equal to the probability that a randomly chosen node and its dependence node belong to the giant component simultaneously:

$$S = p^2 f(R).$$    (1)

Here, $f(R)$ is the probability that the two nodes belong to the giant component simultaneously, and $p^2$ expresses that they are preserved after the initial removal.

Next, we check the conditions of a node belong to the giant component to write down the expression for $f(R)$. In the giant component, the dependence partners not linked by connectivity links (with a fraction $1 - \beta$) must connect to two other nodes in the giant component, respectively. For the remaining nodes (with a fraction $\beta$) in the giant component, in each dependence pair at least one of the two nodes should connect to the giant component, since they are linked by the overlapping links. At the same time, the effective degree of the latter type of nodes is reduced by one. These two cases are schematically shown in figure 1. Thus, $f(R)$ can be written as

$$f(R) = (1 - \beta) \left[ 1 - G_0(1 - R) \right]^2 + \beta \left[ 1 - [G_1(1 - R)]^2 \right],$$    (2)

where $G_0(x)$ is the corresponding generating function of the degree distribution $G_0(x) = \sum_k P_k x^k$, and $G_1(x)$ is the corresponding generating function of the underlying branching processes $G_1(x) = \sum_k P_k k x^{k-1} / \langle k \rangle = G_0'(x) / G_0'(1)$. Here, the brackets $\langle \cdots \rangle$ denote an average over the degree distribution $P_k$. 

New Journal of Physics 15 (2013) 093013 (http://www.njp.org/)
Similarly, one can write down the equation for $R$:

$$R = p^2 g(R). \quad (3)$$

Here, $g(R)$ is

$$g(R) = (1 - \beta) [1 - G_1 (1 - R)] [1 - G_0 (1 - R)] + \beta [1 - G_2 (1 - R) G_1 (1 - R)], \quad (4)$$

where $G_2(x)$ is the generating function $G_2(x) = \sum_k P_k k (k-1) x^{k-2} / \langle k (k-1) \rangle = G'_0(x) / G''_0(1)$.

For the arbitrary degree distribution $P_k$ and the fraction of initial removal $1 - p$, we can solve (3) to obtain $R$, and then insert it into (1) to obtain the order parameter $S$. In particular, we consider a random network with degree distribution [10]

$$P_k = \frac{e^{-\langle k \rangle } \langle k \rangle ^k}{k!}. \quad (5)$$

Then, the generating functions can be written as

$$G_0(x) = G_1(x) = G_2(x) = e^{-(1-x)} e^{-\langle k \rangle (1-x)}. \quad (6)$$

Hence, $f(R) = g(R)$, and (1) is equivalent to (3). Consequently, for a random network, we have a simple self-consistent equation for the order parameter $S$:

$$S = p^2 \left\{ (1 - \beta) \left[ 1 - e^{-(\langle k \rangle S)} \right]^2 + \beta \left[ 1 - e^{-2(\langle k \rangle S)} \right] \right\}. \quad (7)$$

Substituting $\beta = 0$ into (7), one obtains $S = p^2 \left[ 1 - e^{-(\langle k \rangle S)} \right]^2$. This coincides with the exact result found in [36, 37]. Equation (7) has a trivial solution at $S = 0$, which means that the network is completely fragmented. The nontrivial solution of $S$ can be presented by the crossing points of the curve $W(S) = S - p^2 f(S)$ and $S$-axis as shown in figure 2. One can find that the system shows two different types of phase transition. For the case of $\beta = 0.3$, the percolation transition

**Figure 2.** Graphical solutions for (7) with $\langle k \rangle = 16$. (a) $\beta = 0.3$, the percolation transition is first order with the critical point $p_c \approx 0.3202$. The inset shows the discontinuous transition of the order parameter $S$ as $p$ increases. (b) $\beta = 0.4$, the percolation transition is second order with the critical point $p_c \approx 0.2795$. The inset shows the continuous transition of the order parameter $S$ as $p$ increases.
Figure 3. The size of the giant component $S$ versus $p$, the fraction of nodes that have been left after random removal, for random network with $\langle k \rangle = 16$ and 10,000 nodes. The symbols represent simulation results, and the solid lines show the theoretical predictions by (7). Note that, when there are too many small degree nodes in the network and each node has only one dependence link, it becomes almost impossible to choose dependence partners randomly for a large $\beta$ in the simulation. This is one of the reasons that there is a small disagreement between simulations and the exact solution for larger $\beta$.

is discontinuous, while for the case of $\beta = 0.4$, the percolation transition is continuous. The exact value of the tricritical point $\beta_c$, at which the type of percolation transition changes, will be given later.

Next, we analyze (7) to obtain the first order transition point $p^I_c$ and the second order transition point $p^{II}_c$. The second order transition point $p^{II}_c$ corresponds to the solution of (7), in the condition of $S \rightarrow 0$. Thus,

$$p^{II}_c = \frac{1}{\sqrt{2\beta \langle k \rangle}}, \quad \beta > \beta_c.$$  

The first order transition point $p^I_c$ corresponds to the tangential intersection of $W(S)$ and $S$-axis, meaning that the derivative of $W(S)$ with respect to $S$ is zero. This yields

$$p^I_c = \left[ 2 \langle k \rangle e^{-\langle k \rangle S} \left( 1 - \beta - (1 - 2\beta) e^{-\langle k \rangle S} \right) \right]^{-1/2}, \quad \beta < \beta_c,$$  

where $S$ is the solution of (7). Equation (7) has no closed-form solution (other than the trivial solution $S = 0$), so we do not have a closed-form $p^I_c$. But it is easy to obtain the numerical solution of (7) and (9). When the conditions for the first and second order transitions are satisfied simultaneously, we will obtain the tricritical point $p^*_c$ and $\beta_c$. In addition, according the graphical solution of (7) shown in figure 2, the tricritical points also satisfy the equation $d^2 W(S)/dS^2 = 0$. This yields

$$e^{-\langle k \rangle S} = \frac{1 - \beta}{2(1 - 2\beta)}.$$  

New Journal of Physics 15 (2013) 093013 (http://www.njp.org/)
Figure 4. The critical point $p_c$ for different $\beta$. The average degrees of the networks are 14 (black rectangle), 16 (red circle) and 18 (green triangle). The solid and dashed lines are the theoretical prediction of (9) and (8), respectively. The dotted line separates the first order and second order regions obtained using (11) and (12).

Together with (8) and (9), we obtain

$$\beta_c = \frac{1}{3}. \tag{11}$$

$$p^*_c = \sqrt{\frac{3}{2\langle k \rangle}}. \tag{12}$$

Equation (11) means that the type of the percolation transition is independent of the average degree $\langle k \rangle$. A network with $\beta > \beta_c$ undergoes a second order phase transition, while for $\beta < \beta_c$ the network undergoes a first order transition.

3. Simulation results and discussion

To validate the analytical results presented in the last section, we plot the size of the giant component in the steady state as a function of the parameter $p$ by both simulation and analysis in figure 3. One can find that the analytical results are in agreement with the simulation results well. When a node’s dependence node is in its neighborhood, the dependence link can only lead to a local failure. Conversely, if a node’s dependence node is a randomly chosen one, its failure may trigger a series of failures in a larger scale range. In other words, the randomly chosen dependence links are easier to propagate the failures over the network than the local dependence links. Therefore, for a larger parameter value of $\beta$, the network is more robust as shown in figure 3. On the other hand, one can find that for a small fraction of initial
removal $1 - p$, the overlapping has no obvious impact on the size of the giant component, and the giant components are almost the same for different $\beta$.

Next, in order to validate the critical points obtained by (9) and (8), we use the simulation method developed by Parshani et al [35]. For the first order phase transition, the number of iterative failures sharply increases with approaching the critical point $p_c^I$. In addition, the size of the second largest component reaches its maximum values at the second order transition point $p_c^{II}$, which provides a useful method for identifying $p_c^{II}$ at the second order region. Using these methods, we plot the simulation results of the phase transition point $p_c$ shown in figure 4, from which one can find that the simulation results are in agreement with the theoretical results well. As the theory predicts, the phenomenon of crossover in phase transition can be found in figure 4, and the critical point $p_c$ decreases with the increasing of $\beta$. For a large $\beta$, although the dependence link density is 1, the network is still robust and the percolation transition is second order, which is different to the results in [35–37]. This indicates that the robustness of the networks with connectivity and dependence links relies not only on the number of the dependence links [35] but also on the correlation between the two types of links.

4. Conclusions

In this paper, we have studied the effects of the overlapping of connectivity and dependence links on network percolation. Although the detailed analysis is only presented for random networks, it is easily applied to other network topologies. Both simulation and analytical results reveal the existence of the crossover between the first and second order phase transitions. Our results demonstrate that a high density of dependence links does not always make the percolation transition sharpened. For a small fraction of overlapping, the dependence links could easily propagate the failures to the whole network as the results shown in [35–37] and the network is vulnerable. Thus, the percolation transition is first order. For a large fraction of overlapping, most nodes are adjacent to their dependence partners, so the cascading failures caused by dependence links are confined to some small areas. Therefore, the networks are more robust than that with a small fraction of overlapping, and the system demonstrates a second order phase transition. On the other hand, our model illustrates the significance of the correlation of dependence partners to the robustness of networks. Comparing with the randomly chosen dependence partners, the local dependence partners have less impact on the robustness of the networks. To investigate the robustness of networks with dependence links, the correlation of dependence partners needs to be taken into account in the future work. Moreover, if the degree correlation in real networks is considered, the local and randomly chosen dependence partners may act on the robustness of the networks in a different way, which needs to be revealed by further studies.

Acknowledgments

We acknowledge two anonymous referees for very valuable comments and suggestions. This work is funded by the National Natural Science Foundation of China (grant numbers 11275186 and 91024026). RRL and CXJ acknowledge the support of the research start-up fund of Hangzhou Normal University (grant numbers 2011QDL29 and 2011QDL31). RRL is also supported by the Zhejiang Provincial Natural Science Foundation of China under grant number...

New Journal of Physics 15 (2013) 093013 (http://www.njp.org/)
LY12A05003 and the National Natural Science Foundation of China under grant number 11305042.

References

[1] Broadbent S R and Hammersley J M 1957 Proc. Camb. Phil. Soc. 53 629
[2] Kirkpatrick S 1973 Rev. Mod. Phys. 45 574
[3] Kesten H 1982 Percolation Theory for Mathematicians (Boston, MA: Birkhäuser)
[4] Stauffer D and Aharony A 1992 Introduction to Percolation Theory (London: Taylor and Francis)
[5] Albert R, Jeong H and Barabási A L 2000 Nature 406 378
[6] Cohen R, Erez K, ben-Avraham D and Havlin S 2000 Phys. Rev. Lett. 85 4626
[7] Callaway D S, Newman M E J, Strogatz S H and Watts D J 2000 Phys. Rev. Lett. 85 5468
[8] Cohen R, Erez K, ben-Avraham D and Havlin S 2001 Phys. Rev. Lett. 86 3682
[9] Gallos L K, Cohen R, Argyrakis P, Bunde A and Havlin S 2005 Phys. Rev. Lett. 94 188701
[10] Bollobás B 1985 Random Graphs (London: Academic)
[11] Wiff H 1994 Generatingfunctionology (London: Academic)
[12] Newman M E J, Strogatz S H and Watts D J 2001 Phys. Rev. E 64 026118
[13] Newman M E J 2002 Phys. Rev. E 66 016128
[14] Newman M E J 2009 Phys. Rev. Lett. 103 058701
[15] Sachs, M L, Carreras B A and Lynch V E 2000 Phys. Rev. E 61 4877
[16] Watts D J 2002 Proc. Natl Acad. Sci. USA 99 5766
[17] Gleson J P and Cahalan D J 2007 Phys. Rev. E 75 056103
[18] Liu R-R, Wang W-X, Lai Y-C and Wang B-H 2012 Phys. Rev. E 85 026110
[19] Motter A E and Lai Y-C 2002 Phys. Rev. E 66 065102
[20] Crucitti P, Latora V and Marchiori M 2004 Phys. Rev. E 69 045104
[21] Motter A E 2004 Phys. Rev. Lett. 93 098701
[22] Samuelsson B and Socolar J E S 2006 Phys. Rev. E 74 036113
[23] Larremore D B, Carpenter M Y, Ott E and Restrepo J G 2012 Phys. Rev. E 85 066131
[24] Bakke J Ø H, Hansen A and Kertész J 2006 Europhys. Lett. 76 717
[25] Athreya K B and Ney P E 1972 Branching Processes (Berlin: Springer)
[26] Goh K-I, Lee D-S, Kahng B and Kim D 2003 Phys. Rev. Lett. 91 148701
[27] Buldyrev S V, Parshani R, Paul G, Stanley H E and Havlin S 2010 Nature 464 1025
[28] Parshani R, Buldyrev S V and Havlin S 2010 Phys. Rev. Lett. 105 048701
[29] Son S-W, Grassberger P and Paczuski M 2011 Phys. Rev. Lett. 107 195702
[30] Parshani R, Rozenblat C, Ietri D, Ducruet C and Havlin S 2010 Europhys. Lett. 92 68002
[31] Shao J, Buldyrev S V, Havlin S and Stanley H E 2011 Phys. Rev. E 83 036116
[32] Huang X, Gao J, Buldyrev S V, Havlin S and Stanley H E 2011 Phys. Rev. E 83 065101
[33] Gao J, Buldyrev S V, Havlin S and Stanley H E 2011 Phys. Rev. Lett. 107 195701
[34] Zhou D, Stanley H E, D’Agostino G and Scala A 2012 Phys. Rev. E 86 066103
[35] Parshani R, Buldyrev S V and Havlin S 2011 Proc. Natl Acad. Sci. USA 108 1007
[36] Bashan A, Parshani R and Havlin S 2011 Phys. Rev. E 83 051127
[37] Bashan A and Havlin S 2011 J. Stat. Phys. 145 686
[38] Son S-W, Bizhani G, Christensen C, Grassberger P and Paczuski M 2012 Europhys. Lett. 97 16006

New Journal of Physics 15 (2013) 093013 (http://www.njp.org/)