On the Evolution of Compressible and Incompressible Viscous Fluids with a Sharp Interface

Takayuki Kubo 1,* and Yoshihiro Shibata 2

1 Faculty of Research Natural Science Division, Ochanomizu University, 2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610, Japan
2 Department of Mathematics, Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan; yshibata@waseda.jp
* Correspondence: kubo.takayuki@ocha.ac.jp; Tel.: +81-3-5978-5300

Abstract: In this paper, we consider some two phase problems of compressible and incompressible viscous fluids' flow without surface tension under the assumption that the initial domain is a uniform \( W_{q}^{2,1/4} \) domain in \( \mathbb{R}^N \) (\( N \geq 2 \)). We prove the local in the time unique existence theorem for our problem in the \( L_p \) and \( L_q \) in space framework with \( 2 < p < \infty \) and \( N < q < \infty \) under our assumption. In our proof, we first transform an unknown time-dependent domain into the initial domain by using the Lagrangian transformation. Secondly, we solve the problem by the contraction mapping theorem with the maximal \( L_p-L_q \) regularity of the generalized Stokes operator for the compressible and incompressible viscous fluids' flow with the free boundary condition. The key step of our proof is to prove the existence of an \( R \)-bounded solution operator to resolve the corresponding linearized problem. The Weis operator-valued Fourier multiplier theorem with \( R \)-boundedness implies the generation of a continuous analytic semigroup and the maximal \( L_p-L_q \) regularity theorem.

Keywords: Navier–Stokes equations; two phase problem; local in time unique existence theorem; \( R \)-bounded operator

1. Introduction

It is an important mathematical problem to consider the unsteady motion of a bubble in an incompressible viscous fluid or that of a drop in a compressible viscous one. The problem is, in general, formulated mathematically by the Navier–Stokes equations in a time-dependent domain separated by an interface, where one part of the domain is occupied by a compressible viscous fluid and another part by an incompressible viscous fluid. More precisely, we consider two fluids that fill a region \( \Omega \subset \mathbb{R}^N \) (\( N \geq 2 \)). Let \( \Gamma \subset \Omega \) be a given surface that bounds the region \( \Omega_+ \) occupied by a compressible barotropic viscous fluid and the region \( \Omega_- \) occupied by an incompressible viscous one. We assume that the boundary of \( \Omega_\pm \) consists of two parts, \( \Gamma_\pm \), where \( \partial\Omega_{\pm} = \Gamma_\pm \cap \Gamma = \emptyset \), and \( \Omega = \Omega_+ \cup \Omega_- \). Let \( \Gamma_t, \Gamma_{t+}, \Omega_{t+}, \) and \( \Omega_{t-} \) be the time evolution of \( \Gamma, \Gamma_+ \), and \( \Omega_+ \), respectively. We assume that the two fluids are immiscible, so that \( \Omega_{t+} \cap \Omega_{t-} = \emptyset \) for any \( t \geq 0 \). Moreover, we assume that no phase transitions occur, and we do not consider the surface tension at the interface \( \Gamma_t \) and the boundary \( \Gamma_{t-} \). Thus, in this paper, we consider that the motion of the fluids is governed by the following system of equations:

\[
\begin{align*}
\partial_t \rho_+ + \text{div} (\rho_+ \mathbf{v}_+) &= 0 \quad \text{in } \Omega_{t+}, \\
\rho_+ (\partial_t \mathbf{v}_+ + \mathbf{v}_+ \cdot \nabla \mathbf{v}_+) - \text{Div} \mathbf{S}_+ (\mathbf{v}_+) + \nabla p(\rho_+) &= 0 \quad \text{in } \Omega_{t+}, \\
\text{div} \mathbf{v}_- &= 0 \quad \text{in } \Omega_{t-}, \\
\rho_0_- (\partial_t \mathbf{v}_- + \mathbf{v}_- \cdot \nabla \mathbf{v}_-) - \text{Div} \mathbf{S}_- (\mathbf{v}_-) + \nabla \pi_- &= 0, \quad \text{in } \Omega_{t-},
\end{align*}
\]

(1)
subject to the interface condition:

\[
\begin{align*}
\left(\mathbf{S}_+ (\mathbf{v}_+ ) - p(\rho_+ ) I \right) \mathbf{n}_+ |_{t=0} & = - p(\rho_0 ) \mathbf{n}_0 |_{t=0}, \\
\mathbf{v}_+ |_{t=0} & = \mathbf{v}_- |_{t=0} = 0,
\end{align*}
\]

on \( \Gamma_t \), boundary conditions:

\[
\mathbf{v}_+ |_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{v}_-) - \pi_- I) \mathbf{n}_- |_{\Gamma_-} = 0,
\]

kinematic conditions:

\[
V_\nu = \mathbf{u}_- \cdot \mathbf{n}_- \quad \text{on} \quad \Gamma_t, \quad V_\nu = \mathbf{u}_- \cdot \mathbf{n}_- \quad \text{on} \quad \Gamma_-,
\]

for any \( t > 0 \), and initial conditions:

\[
(\rho_+, \mathbf{v}_+) |_{t=0} = (\rho_0 + \theta_0 + v_0^+, \mathbf{v}_0^+) \quad \text{in} \quad \Omega_+, \quad \mathbf{v}_- |_{t=0} = \mathbf{v}_0^- \quad \text{in} \quad \Omega_-.
\]

Here, \( \mathbf{v}_\pm = (v_{1\pm}, \ldots, v_{N\pm}) \) are the unknown velocity fields of the fluids, \( \rho_{0\pm} \) positive numbers describing the mass densities of \( \Omega_{\pm} \), \( \rho_+ \) the unknown mass density of \( \Omega_+ \), \( \pi_- \) the unknown pressure, \( \theta_0 + v_0^+ \) the prescribed initial data, \( p(s) \) the prescribed pressure, which is a \( C^\infty \) function defined on an open interval \( (\rho_{0+}/2, 2\rho_{0+}) \) satisfying the condition: \( p'(s) \geq 0 \) on \( (\rho_{0+}/2, 2\rho_{0+}) \), \( \mathbf{n}_+ \) the outward normal to \( \Gamma_t \), pointing from \( \Omega_- \) to \( \Omega_+ \), \( \mathbf{n}_- \) the unit outward normal to \( \Gamma_- \), \( V_\nu \) the evolution speed of \( \Gamma_t \) along \( \mathbf{n}_- \) and \( V_\nu \) the evolution speed of \( \Gamma_- \) along \( \mathbf{n}_- \).

Moreover, for any point \( x_0 \in \Gamma_t \), \( f |_{\Gamma_t \pm}(x_0, t) \) is defined by:

\[
f |_{\Gamma_t \pm}(x_0, t) = \lim_{x \to x_0 \pm \Gamma_t \pm \Omega_\pm} f(x, t),
\]

and the stress tensors \( \mathbf{S}_\pm \) are defined by:

\[
\mathbf{S}_+(\mathbf{v}_+) = \mu_+ \mathbf{D}(\mathbf{v}_+) + (\mathbf{v}_+ - \mu_+ \div \mathbf{v}_+) I, \quad \mathbf{S}_-(\mathbf{v}_-) = \mu_- \mathbf{D}(\mathbf{v}_-)
\]

with viscosity coefficients \( \mu_\pm \) and \( v_+ \), which are positive constants in this paper, where \( \mathbf{D}(\mathbf{v}) \) denotes the deformation tensor whose \((j,k)\) components are \( D_{jk}(\mathbf{v}) = \partial v_k / \partial x_j + \partial v_j / \partial x_k \) and \( I \) is the \( N \times N \) identity matrix. Finally, for an \( N \times N \) matrix function \( \mathbf{K} = (K_{ij}) \), \( \div \mathbf{K} \) is an \( N \)-vector whose \( i \)th components are \( \sum_{j=1}^{N} K_{ij} \), and also, for any vector functions \( \mathbf{v} = (v_1, \ldots, v_N) \) and \( \mathbf{v} : \nabla \mathbf{v} = (\sum_{i=1}^{N} v_i \partial v_i, \ldots, \sum_{i=1}^{N} v_j \partial v_N) \), for any functions \( f \) defined on \( \Omega_\pm \), \( f \) denotes a function defined by \( f = f \pm \Omega_\pm \).

Aside from the dynamical system (1) subject to (2), (3), and (5), a kinematic condition (4) for \( \Gamma_t \) and \( \Gamma_- \) gives:

\[
\Gamma_t = \{ x = x(\xi, t) \mid \xi \in \Gamma \}, \quad \Gamma_- = \{ x = x(\xi, t) \mid \xi \in \Gamma_- \},
\]

where \( x(\xi, t) \) is the solution of the Cauchy problem:

\[
\frac{dx}{dt} = \mathbf{v}(x, t) = \begin{cases} 
\mathbf{v}_+ & \text{in} \ \Omega_+, \\
\mathbf{v}_- & \text{in} \ \Omega_-,
\end{cases} \quad x |_{t=0} = \xi \in \Omega.
\]

This expresses the fact that the interface \( \Gamma_t \) and the free surface \( \Gamma_- \) consist for all \( t > 0 \) of the same fluid particles, which do not leave them and are not incident on them from inside \( \Omega_t \). It is clear that \( \Omega_t \) is given by:

\[
\Omega_t = \{ x = x(\xi, t) \mid \xi \in \Omega_t \}.
\]
Problem (1) with (2)–(5) can therefore be written as an initial boundary value problem with interface \( \Gamma \) in the given domain \( \Omega \) if we go over the Euler coordinates \( x \in \Omega_{\pm} \) to Lagrange coordinates \( \xi \in \Omega_{\pm} \) with \( x \) by (7). If velocity vector fields \( u_{\pm}(\xi, t) \) defined on \( \Omega_{\pm} \) are known as functions of the Lagrange coordinates \( \xi \in \Omega_{\pm} \), then this connection can be written in the form:

\[
x = \xi + \int_0^t u_{\pm}(\xi, s) \, ds \equiv X_{u_{\pm}}(\xi, t)
\]

and \( u_{\pm}(\xi, t) = v(X_{u_{\pm}}(\xi, t), t) \). Let \( A_{\pm} \) be the Jacobi matrix of the transformation (9) with element \( a_{ij}^{\pm} = \delta_{ij} + \int_0^t (\partial_{\xi_j} u_{\pm,j})(\xi, s) \, ds \) with \( \delta_{ij} \) being the Kronecker delta symbols. There exists a small number \( \sigma > 0 \) such that \( A_{\pm} \) is invertible, that is det \( A_{\pm} \neq 0 \), whenever:

\[
\max_{ij = 1, \ldots, N} \sup_{\xi \in \Omega_{\pm}} \left| \int_0^t (\partial_{\xi_i} u_{\pm,j})(\xi, s) \, ds \right| < \sigma \quad (t > 0),
\]

while \( \det A_{-} = 1 \) in \( \Omega_{-} \), because of the incompressibility. Whenever (10) is valid, we have:

\[
\nabla x = A_{\pm}^{-1} \nabla \xi = \left( I + V_0 \left( \int_0^t \nabla u_{\pm}(\xi, s) \, ds \right) \right) \nabla \xi,
\]

with \( \nabla x = \left( \partial_{\xi_1}, \ldots, \partial_{\xi_N} \right)^T M \) denotes the transposed \( M \) and \( \nabla \xi = \left( \partial_{\xi_1}, \ldots, \partial_{\xi_N} \right)^T \), where \( V_0 = V_0(w) \) is the \( N \times N \) matrix of \( C^\infty \) functions with respect to \( w = (w_1, \ldots, w_N) \) defined on \( |w| < \sigma \) and \( V_0(0) = 0 \). Let \( n \) and \( n_- \) be unit outward normals to \( \Gamma \) and \( \Gamma_{-} \), respectively, and then, by (8), we have:

\[
\n = A_{\pm}^{-1} n, \quad n_- = A_{\pm}^{-1} n_-.
\]

Setting \( \rho_{\pm}(X_{u_{\pm}}(\xi, t), t) = \rho_{0+} + \theta_{\pm}(\xi) + \theta_{\pm}(\xi, t) \) and \( p_{-} = \pi_{\pm}(X_{u_{\pm}}(\xi, t), t) \) and using the facts that \( \rho_{\pm}(X_{u_{\pm}}(\xi, t), t) = J_{\pm}(\xi, t)^{-1}(\rho_{0+} + \theta_{\pm}(\xi)) \) with \( J_{\pm} = \det A_{\pm} \) and \( \text{div}_x v_{\pm} = J_{\pm}^{-1} \text{div}_x J_{\pm} A_{\pm}^{-1} v_{\pm} \) with \( v_{\pm}(\xi, t) = v_{\pm}(X_{u_{\pm}}(\xi, t), t) \), we can write Equations (1)–(5) with Lagrange coordinates in the form:

\[
\begin{vmatrix}
\frac{d}{dt} \theta_{+} + (\rho_{0+} + \theta_{+}) \text{div} u_{+} = F_{+} & \text{in } \Omega_{+}, \\
(\rho_{0+} + \theta_{+}) \partial_{\xi_i} u_{+} - \text{Div} S_{+}(u_{+}) + \nabla (p_{\rho_{0+} + \theta_{+}}(\theta_{+})) = g_{+} + G_{+} & \text{in } \Omega_{+}, \\
(\rho_{0-} + \theta_{+}) \partial_{\xi_i} u_{-} - \text{Div} S_{-}(u_{-}) + \nabla p_{-} = G_{-} & \text{in } \Omega_{-}, \\
\text{div} u_{-} = F_{-} & \text{in } \Omega_{-}, \\
(S_{+}(u_{+}) - p_{\rho_{0+} + \theta_{+}}(\theta_{+}))n|_{\Gamma_{+} = 0} - (S_{-}(u_{-}) - p_{-})n|_{\Gamma_{-} = 0} = h + H_{+}, \\
u_{+}|_{\Gamma_{+} = 0} - u_{-}|_{\Gamma_{-} = 0} = 0, \quad u_{+}|_{\Gamma_{+} = 0} = 0, \quad (S_{-}(u_{-}) - p_{-})n|_{\Gamma_{-} = 0} = H_{-}
\end{vmatrix}
\]

for \( t > 0 \) subject to the initial condition:

\[
(\theta_{+}, v_{+})|_{t=0} = (0, v_{0+}) \text{ in } \Omega_{+}, \quad u_{-}|_{t=0} = v_{0-} \text{ in } \Omega_{-}.
\]

Here, \( g_{+} = -p'_{\rho_{0+} + \theta_{+}}(\theta_{+}) = h = -(p_{\rho_{0+} + \theta_{+}} - p(\rho_{0+}))n \), and \( F_{\pm}, G_{\pm}, H_{\pm} \) are nonlinear functions with respect to \( \theta_{\pm}, u_{\pm}, w_{\pm} = \int_0^t \nabla u_{\pm}(\xi, s) \, ds \) of the forms:

\[
F_{+} = -\{ \theta_{+} \text{div} u_{+} + (\rho_{0+} + \theta_{+}) \text{tr} (V_{0+} \nabla u_{+}) \}, \\
G_{+} = -\theta_{+} \partial_{\xi_i} u_{+} + \text{Div} (\mu_{\rho_{0+} \theta_{+}} V_{0+} u_{+} + (\nu_{\rho_{0+} \theta_{+}} + \mu_{\rho_{0+} \theta_{+}}) \text{tr} (V_{0+} \nabla u_{+}) I) + V_{0+} \nabla (\mu_{\rho_{0+} \theta_{+}} D(u_{+}) + V_{D+} \nabla u_{+}) + (\nu_{\rho_{0+} \theta_{+}} - \mu_{\rho_{0+} \theta_{+}}) (\text{div} u_{+} + \text{tr} (V_{0+} \nabla u_{+})) I) \\
- \left( \int_0^t p''_{\rho_{0+} + \theta_{+}}(\theta_{+}) + \tau_{\theta_{+}}(1 - \tau) \, d\tau \right) \nabla (\theta_{0+} + \theta_{+}), \\
G_{-} = -\rho_{0-} V_{-} \partial_{\xi_i} u_{-} - \mu_{\rho_{0-} \theta_{-}} (\text{Div} (V_{0-} \nabla u_{-}) + V_{-} \text{Div} (D(u_{-}) + V_{D-} \nabla u_{-})), \\
F_{-} = (1 - J_{-}) \text{div} u - \text{tr} (V_{0-} \nabla u_{-}) = \text{div} ((1 - J_{-}) u_{-} - T_{0-} J_{-} u_{-}), \\
H = -\mu_{\rho_{0-} \theta_{-}} (V_{D-} \nabla u_{-} + V_{-} (D(u_{-}) + V_{D-} \nabla u_{-})) n
\]

Here, \( \tau_{\theta_{+}} = (1 - \tau) \theta_{+} + \theta_{+}(1 - \tau) \).
\[-(v_+ - \mu_+) [\text{tr}(\nabla v_+ + \nabla u_+)] n + \left[ \int_0^1 (1 - \tau)p''(\rho_{0+} + \theta_{0+} + \tau\theta_+^2) \, d\tau \right] n \]
\[+ \mu_+ [\nabla \nabla u_+ + V \cdot D(u_+)] + V \cdot D(u_+) + (1 + V \cdot D(u_+)] + V \cdot D(u_+) \nabla u_+)] n \]
\[H_+ = -\mu_+ [\nabla \nabla u_+ + V \cdot D(u_+)] + V \cdot D(u_+) + (1 + V \cdot D(u_+)] + V \cdot D(u_+) \nabla u_+)] n \]

with \(V_{B+} = V_0(w_{\pm}), V_{D+} = V_D(w_{\pm}), \) and \(J_+ = \det(\nabla X_{u+}). \) In the formula (13), \(V_{-1} = (1 + V_0)^{-1} - I, \) \(\text{tr} B \) means the trace of \(N \times N \) matrix \(B, \) and \(V_D(w) \) is a matrix of the \(C^\infty \) function with respect to \(w \) defined on \(|w| < \sigma, \) which satisfies \(V_D(0) = 0 \) and relations: \(D(v) = D(\dot{v}) + V_D(\int_0^t \nabla \dot{v} \, ds) \nabla \dot{v} \) with \(\dot{v} = v(X(u), t), t). \)

Since the pioneering work [1] on the well-posedness of the Navier–Stokes equations around a free surface, there have been many studies on the free boundary problem. Here, we introduce the known results concerning compressible and incompressible viscous two-phase fluids.

Denisova [2,3] proved the local well-posedness theorem and the global well-posedness theorem for Equations (1)–(3) and (5) in the \(L_2 \) framework. The purpose of this paper is to prove the local well-posedness for Equations (1)–(3) and (5) in the \(L_p \) in time and \(L_q \) in space framework with \(2 < p < \infty \) and \(N < q < \infty \) under the physically reasonable assumption on the viscosity coefficients, that is \(\mu \geq 0 \) and \(v_+ > 0. \) The regularity of solutions in our result is optimal in the sense of the maximal regularity, while the \(L_2 \) framework used by Denisova [2,3] loses regularity from the point of view of Sobolev’s imbedding theorem.

Moreover, we consider the problem with full generality about the domain. Namely, we consider the problem in a uniform \(W_{2+}^{1/q} \) domain, the conditions of which are satisfied by bounded domains, exterior domains, half-spaces, perturbed half-spaces, and layer domains (cf. Shibata [4]).

Symbols 1. To state our theorem on the local in time unique existence of solutions to Equations (1)–(3) and (5), we introduce some functional spaces and the definition of the uniform \(W_{2+}^{1/r} \) domain. For the differentiations of scalar functions \(f \) and \(N\)-vector functions \(g, \) we use the following symbols:

\[
\nabla f = (\partial_1 f, \ldots, \partial_N f), \quad \nabla^2 f = (\partial_i \partial_j f \mid i, j = 1, \ldots, N), \quad \nabla^2 g = (\partial_i \partial_j g_k \mid i, j, k = 1, \ldots, N),
\]

where \(\partial_i = \partial/\partial x_i. \) For any domain \(D \) and \(1 \leq q \leq \infty, \) \(L_q(D), W_q^m(D), \) and \(B_{q,p}^m(D) \) denote the standard Lebesgue space, Sobolev space, and Besov space, while \(\| \cdot \|_{L_q(D)}, \| \cdot \|_{W_q^m(D)}, \) and \(\| \cdot \|_{B_{q,p}^m(D)} \) denote their norms. We set \(W_q^{1,q}(D) = L_q(D) \) and \(W_q^0(D) = B_{q,q}^0(D). \) In addition, \((a, b)_D \) denotes the inner product on \(D \) defined by \((a, b)_D = \int_D a(x)b(x) \, dx. \)

Let \(X \) be any Banach space with norm \(\| \cdot \|_X. \) We set \(X^d = \{ f = (f_1, \ldots, f_d) \mid f_i \in X (i = 1, \ldots, d), \} \), while its norm is denoted by \(\| \cdot \|_{X^d} \) instead of \(\| \cdot \|_{X^d} \) for short. Let \(W_{1,q}^0(D) \) and \(W_{1,0}^0(D) \) be homogeneous spaces defined by \(W_{1,q}^0(D) = \{ v \in L_{aq}(D) \mid \nabla v \in L_q(D)^N \} \) and \(W_{1,0}^1(D) = \{ v \in \tilde{W}_{2}^1(D) \mid \| v \|_{\tilde{W}_{2}^1(D)} = 0 \}, \) respectively, where \(\tilde{W}_{2}^1(D) \) is the boundary of \(D. \)

Moreover, we set \(W_{1,0}^0(D) = \{ v \in W_{1,0}^0(D) \mid \| v \|_{\tilde{W}_{2}^1(D)} = 0 \}. \) For \(1 \leq p \leq \infty, \) \(L_p((a, b), X) \) and \(W_p^m((a, b), X) \) denote the usual Lebesgue space and Sobolev space of \(X\)-valued functions defined on an interval \((a, b), \) while \(\| \cdot \|_{L_p((a, b), X)} \) and \(\| \cdot \|_{W_p^m((a, b), X)} \) denote their norms, respectively. For any \(N\)-vector \(w = (w_1, \ldots, w_N) \) and \(z = (z_1, \ldots, z_N) \), we define \(\langle w, z \rangle, \) \(T_\alpha[w], \) and \(N_\alpha[w] \) by:

\[
\langle w, z \rangle = \sum_{j=1}^N w_j z_j, \quad T_\alpha[w] = w - \langle w, z \rangle z, \quad N_\alpha[w] = \langle w, z \rangle z,
\]

respectively. Here, \(T_\alpha[w] \) denotes the tangential part of \(w \) with respect to \(z. \) For \(1 < q < \infty, \) \(q' \) denotes the dual exponent defined by \(q' = q/(q - 1). \) We use the letter \(C \) to denote
generic constants, and \( C_{a,b} \) denotes that the constant \( C_{a,b} \) essentially depends on the quantities \( a, b, \cdots \). Constants \( C, C_{a,b}, \cdots \) may change from line to line.

In this paper, let \( f_q(\Omega_-) \) be a solenoidal space defined by setting:

\[
J_q(D) = \{ u_\cdot \in L_q(D) \mid (u_\cdot, \nabla \varphi)_D = 0 \quad \text{for any } \varphi \in \dot{W}^1_{q,0}(D) \}. \tag{15}
\]

We write \( \operatorname{div}u = f = \operatorname{div}f \) in \( D \) for \( f \in W^1_q(D), f \in L_q(D)^N, \) and \( u \in W^1_q(D) \), if:

\[
(f, \varphi)_D = -(f, \nabla \varphi)_D \quad \text{for any } \varphi \in W^1_{q,0}(D), \quad \operatorname{div}u = f \quad \text{in } D, \quad \text{and } u - f \in J_q(D). \tag{16}
\]

We now introduce a few definitions.

**Definition 1.** Let \( 1 < r < \infty \), and let \( D \) be a domain in \( \mathbb{R}^N \) with boundary \( \partial D \). We say that \( D \) is a uniform \( W^{2-1/r}_r \) domain, if there exist positive constants \( a, b, \) and \( K \) such that for any \( x_0 = (x_{01}, \ldots, x_{0N}) \in \partial D \), there exist a coordinate number \( j \) and a \( W^{2-1/r}_r \) function \( h(x) \) (\( \hat{x} = (x_1, \ldots, \hat{x}_j, \ldots, x_N) \)) defined on \( B^1_a(x_0) \) with \( \hat{x}_0 = (x_{01}, \ldots, \hat{x}_{0j}, \ldots, x_{0N}) \) and \( \| h \|_{W^{2-1/r}_r(B^1_a(x_0))} \leq K \) such that:

\[
D \cap B^1_b(x_0) = \{ x \in \mathbb{R}^N \mid x_j > h(x') \quad (x' \in B^1_a(x_0')) \} \cap B_b(x_0), \\
\partial D \cap B^1_b(x_0) = \{ x \in \mathbb{R}^N \mid x_j = h(x') \quad (x' \in B^1_a(x_0')) \} \cap B_b(x_0). \tag{17}
\]

Here, \( (x_1, \ldots, \hat{x}_j, \ldots, x_N) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \), \( B^1_a(x_0') = \{ x' \in \mathbb{R}^{N-1} \mid \| x' - \hat{x}_0' \| < a \} \), and \( B^1_b(x_0) = \{ x \in \mathbb{R}^N \mid \| x - x_0 \| < b \} \).

Second, we introduce the assumption of the solvability of the weak Dirichlet problem, which is needed to treat the divergence condition for the incompressible part.

**Definition 2.** Let \( 1 < q < \infty \). We say that the weak Dirichlet problem is uniquely solvable on \( \dot{W}^1_{q,0}(\Omega_-) \) with exponents \( q \), if for any \( f \in L_q(\Omega_-)^N \), there exists a unique solution \( \theta \in \dot{W}^1_{q,0}(\Omega_-) \) of the variational problem:

\[
(\nabla \theta, \nabla \varphi)_{\Omega_-} = (f, \nabla \varphi)_{\Omega_-} \quad \text{for any } \varphi \in \dot{W}^1_{q,0}(\Omega_-). \tag{18}
\]

**Remark 1.** (1) Since \( \partial \Omega_- = \Gamma \cup \Gamma_- \) with \( \Gamma \cap \Gamma_- = \emptyset \), \( \dot{W}^1_{q,0}(\Omega_-) = \{ v \in \dot{W}^1_{q,0}(\Omega_-) \mid v|_\Gamma = v|_{\Gamma_-} = 0 \} \). (2) When \( q = 2 \), the weak Dirichlet problem is uniquely solvable on \( \Omega_- \) without any restriction, but for \( q \in (1, \infty) \setminus \{2\} \), we do not know the unique solvability in general. For example, we know the unique solvability of the weak Dirichlet problem in bounded domains, exterior domains, half-space, layer, and tube domains. (cf. Galdi [5], as well as Shibata [4,6]).

**Remark 2.** Let \( K \) be a linear operator defined by \( K(f) = \theta \). Then, combining the unique solvability with Banach’s closed range theorem implies the estimate:

\[
\| \nabla K(f) \|_{L_q(\Omega_-)} \leq C(\| f \|_{L_q(\Omega_-)}). \tag{19}
\]

Moreover, for any \( f \in L_q(\Omega_-)^N \) and \( g \in W^1_{q}(\Omega_-) \), \( v = g + K(f - \nabla g) \in W^1_{q}(\Omega_-) + \dot{W}^1_{q,0}(\Omega_-) \) satisfies the variational equation: \( (\nabla v, \nabla \varphi)_{\Omega_-} = (f, \nabla \varphi)_{\Omega_-} \) for any \( \varphi \in \dot{W}^1_{q,0}(\Omega_-) \), subject to \( v = g \) on \( \Gamma \) and \( \Gamma_- \). Here, we set \( W^1_{q}(\Omega_-) + \dot{W}^1_{q,0}(\Omega_-) = \{ p_1 + p_2 \mid p_1 \in W^1_{q}(\Omega_-), \ p_2 \in \dot{W}^1_{q,0}(\Omega_-) \} \), which is the space for the pressure term \( p_- \) in the incompressible part.

The following theorem is our main result about local in time unique existence of solutions to Equations (11) with (12).
Theorem 1. Let $2 < p < \infty$, $N < q < \infty$, $2/p + N/q < 1$, and $R > 0$. Let $\rho_{0+}$ be positive constants describing the reference mass density on $\Omega_\pm$, and let $p(s)$ be a $C^\infty$ function defined on $(\rho_{0+}/2, 2\rho_{0+})$ such that $0 \leq p(s) \leq \rho_{1+}$ with some positive constant $\rho_{1+}$ for any $\rho \in (\rho_{0+}/2, 2\rho_{0+})$. Let $\Omega_\pm$ be uniform $W^{2-1/q}_q(\Omega_\pm)$ domains in $\mathbb{R}^N$ ($N \geq 2$). Assume that the weak Dirichlet problem is uniquely solvable on $W^{1,0}_q(\Omega_\pm)$ with exponents $q$ and $q'$. Let $\theta_{0+} \in W^1_q(\Omega_+)$ and $\nu_{0+} \in B^{2(1-1/p)}(\Omega_\pm)^N$ be initial data with:

$$\|\theta_{0+}\|_{W^1_q(\Omega_+)} + \|\nu_{0+}\|_{B^{2(1-1/p)}(\Omega_+)} + \|\nu_{0-}\|_{B^{2(1-1/p)}(\Omega_-)} \leq R,$$

which satisfy the compatibility condition:

$$\mathcal{T}_n[S_+(\nu_{0+})n]|_{\Gamma_0} - \mathcal{T}_n[S_-(\nu_{0-})n]|_{\Gamma_0} = 0, \quad \nu_{0+}|_{\Gamma_+} = 0, \quad \nu_{0-}|_{\Gamma_-} = 0, \quad \text{div} \nu_{0-} \in L_q(\Omega_-),$$

and the range condition:

$$\frac{3}{4}\rho_{0+} < \rho_{0+} + \theta_{0+}(x) < \frac{7}{4}\rho_{0+} \quad (x \in \Omega_+). \quad (21)$$

Then, there exists a $T > 0$ depending on $R$ such that the system of Equations (11) with (12) admits a unique solution $(\theta_+, \nu_\pm)$ with:

$$\theta_+ \in W^1_p((0, T), W^1_q(\Omega_+)), \quad \nu_\pm \in W^1_p((0, T), L_q(\Omega_\pm)^N) \cap L_p((0, T), W^2_q(\Omega_\pm)^N)$$

satisfying (10) and the estimate:

$$\|\theta_+\|_{W^1_p(0, T), W^1_q(\Omega_+)} + \sum_{\ell = +, -} \left(\|\nu_\ell\|_{L_p((0, T), W^2_q(\Omega_\pm))} + \|\partial_t \nu_\ell\|_{L_p((0, T), L_q(\Omega_\pm))}\right) \leq C_R$$

with some constant $C_R$ depending on $R$, $\rho_{0\pm}$, $p$, and $q$.

Using the argument due to Ströhmer [7], we can show the injectivity of the map $x = X_{a+}(x, t)$, so that we have the following local in time unique existence theorem for (1)–(5).

Theorem 2. Let $N < q < \infty$, $2 < p < \infty$, $2/p + N/q < 1$, and $R > 0$. Assume that $\Omega_\pm$ are uniform $W^{2-1/q}_q$ domains. Assume that the weak Dirichlet problem is uniquely solvable on $W^{1,0}_q(\Omega_\pm)$ with exponents $q$ and $q'$. Let $\theta_{0+} \in W^1_q(\Omega_+)$ and $\nu_{0\pm} \in B^{2(1-1/p)}(\Omega_\pm)^N$ be initial data that satisfy the compatibility condition (20), range condition (21), and:

$$\|\theta_{0+}\|_{W^1_q(\Omega_+)} + \|\nu_{0+}\|_{B^{2(1-1/p)}(\Omega_+)} + \|\nu_{0-}\|_{B^{2(1-1/p)}(\Omega_-)} \leq R.$$

Then, there exists a $T > 0$ depending on $R$ such that Equation (1) subject to the interface condition (2), boundary condition (3), kinematic condition (4), and initial condition (5) admits a unique solution $(\rho_+, \nu_\pm)$ with:

$$\rho - \rho_{0+} \in W^1_p((0, T), L_q(\Omega_+)) \cap L_p((0, T), W^1_q(\Omega_+)), \quad \nu_\pm \in W^1_p((0, T), L_q(\Omega_\pm)^N) \cap L_p((0, T), W^2_q(\Omega_\pm)^N).$$

Remark 3. Here, $f \in W^{m,p}_p((0, T), W^{m,p}_q(\Omega_\pm))$ denotes that for almost all $t \in (0, T)$, $\partial_t^m f(\cdot, t) \in W^{m,p}_q(\Omega_\pm)$ and:

$$\|f\|_{W^{m,p}_p((0, T), W^{m,p}_q(\Omega_\pm))} := \sum_{k=0}^m \left(\int_0^T \|\partial_t^k f(\cdot, t)\|_{W^{m,p}_q(\Omega_\pm)}^p \, dt\right)^{1/p} < \infty.$$
Theorem 1 is proven by using a standard fixed point argument based on the maximal $L^p$-$L^q$ regularity for solutions to the linear problem:

\[
\begin{cases}
\partial_t \theta_+ + \gamma_2 \div \mathbf{u}_+ = f_+ & \text{in } \Omega_+, \\
\gamma_0 \partial_t \mathbf{u}_+ - \Div S^+(\mathbf{u}_+) + \nabla (\gamma_1 \theta_+) = \mathbf{g}_+ & \text{in } \Omega_+,
\end{cases}
\]
\[
\begin{cases}
\rho_0 \partial_t \mathbf{u}_- - \Div S^-(\mathbf{u}_-) + \nabla p_- = \mathbf{g}_- & \text{in } \Omega_-,
\end{cases}
\]
\[
\begin{cases}
\mathbf{u}_+ = \mathbf{f}_- = \div \mathbf{f}_- & \text{in } \Omega_+,
\end{cases}
\]
\[
\begin{cases}
(S^+(\mathbf{u}_+) - \gamma_1 \theta_+ \mathbf{n})|_{\Gamma_1} + (S^-(\mathbf{u}_-) - p_- \mathbf{n})|_{\Gamma_1} = \mathbf{h} & \text{for } t > 0,
\mathbf{u}_+|_{\Gamma_1} = \mathbf{u}_-|_{\Gamma_1} = 0 & \text{for } t > 0,
\mathbf{u}_+|_{\Gamma_2} = 0, \quad (S_-(\mathbf{u}_-) - p_- \mathbf{n})|_{\Gamma_2} = \mathbf{h}_- & \text{for } t > 0,
(\theta_+, \mathbf{u}_+)|_{t=0} = (\theta_{0+}, \mathbf{u}_{0+}) & \text{in } \Omega_+, \quad \mathbf{u}_-|_{t=0} = \mathbf{u}_{0-} & \text{in } \Omega_-.
\end{cases}
\]

(22)

Here, $\gamma_i = \gamma_i(x) \ (i = 0, 1, 2)$ are uniformly continuous functions defined on $\overline{\Omega}_+$ such that:

\[
\frac{1}{2} \rho_0 + \gamma_0(x) \leq \gamma_0(x) \leq 2 \rho_0, \quad 0 \leq \gamma_k(x) \leq \rho_2^+ \ (x \in \Omega), \quad \|\nabla \gamma_+\|_{L^p(\Omega_+)} \leq \rho_2^+ \quad (23)
\]

for $k = 1, 2$ and $\ell = 0, 1, 2$ with some positive constant $\rho_2^+$ and $N < r < \infty$. We may consider the case where $\gamma_1 = 0$, which corresponds to the Lamé system.

Symbols 2. To state our main result for linear Equation (22), we introduce more symbols and functional spaces used throughout this paper. Set:

\[W^m_{p,\text{loc}}(a, b, X) = \{f(t) \mid f(t) \in W^m_p((c, d), X) \text{ for any } c, d \text{ with } a < c < d < b\}\]

and $W^0_{p,\text{loc}}(\mathbb{R}, X) = L^p_{p,\text{loc}}(\mathbb{R}, X)$. Moreover, we set:

\[W^m_{p,\gamma}(\mathbb{R}, X) = \{f(t) \in L^p_{p,\text{loc}}(\mathbb{R}, X) \mid e^{-\gamma t} \partial_t^j f(t) \in L^p_p(\mathbb{R}, X) \ (j = 0, 1, \ldots, m)\}\]

with $\partial_t^0 f(t) = f(t)$ and $W^0_{p,\gamma}(\mathbb{R}, X) = L^p_{p,\gamma}(\mathbb{R}, X)$ and $W^0_{p,\gamma,0}(\mathbb{R}, X) = L^p_{p,\gamma,0}(\mathbb{R}, X)$.

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform defined by:

\[\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) \, dt, \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\tau) \, d\tau\]

with $\lambda = \gamma + i \tau \in \mathbb{C}$, respectively. Given $s \in \mathbb{R}$ and $X$-valued function $f(t)$, we set:

\[A^s_\gamma f(t) = \mathcal{L}^{-1}[\lambda^s \mathcal{L}[f](\lambda)](t)\]

We introduce a Bessel potential space of $X$-valued functions of order $s > 0$ as follows:

\[H^s_{p,\gamma}(\mathbb{R}, X) = \{f \in L^p_p(\mathbb{R}, X) \mid e^{-\gamma\tau} A^s_\gamma f(t) \in L^p_p(\mathbb{R}, X) \text{ for any } \gamma' \geq \gamma\}\]

We have the following theorem.

**Theorem 3.** Let $1 < p, q < \infty$, $N < r < \infty$, $2/p + N/q \neq 1$, and $2/p + N/q \neq 2$. Assume that $r \geq \max(q, q')$, that $\Omega_\pm$ are uniformly $W^{2-1/q}_q$ domains, and that the weak Dirichlet problem is uniquely solvable on $\Omega_-$ with exponents $q$ and $q'$. Then, there exists a positive number $\gamma_0$ such that the following three assertions are valid.

**Existence** For any initial data $\theta_{0+} \in W^1_q(\Omega_+)$ and $\mathbf{u}_{0+} \in B^{2(1-1/p)}_{p,p}(\Omega_\pm)$, and any right members $f_+, f_- = \div f_-, \mathbf{g}_+, \mathbf{h}_+$ and $\mathbf{h}_-$ with:
\[ f_+ \in L_{p,\gamma_0}(\mathbb{R}, W_0^1(\Omega_+)), \quad g_\pm \in L_{p,\gamma_0}(\mathbb{R}, L_q(\Omega_\pm)^N), \quad h \in L_{p,\gamma_0}(\mathbb{R}, W_0^1(\Omega))^N \cap H_+^{1/2}(\mathbb{R}, L_q(\Omega)^N), \]
\[ f_- \in L_{p,\gamma_0}(\mathbb{R}, W_0^1(\Omega_-)) \cap H_{-}^{1/2}(\mathbb{R}, L_q(\Omega_-)), \quad f_- \in W_0^1(\mathbb{R}, L_q(\Omega_-)), \]
\[ h_- \in L_{p,\gamma_0}(\mathbb{R}, W_0^1(\Omega_-)^N) \cap H_{-}^{1/2}(\mathbb{R}, L_q(\Omega_-)^N), \]

satisfying the compatibility conditions:
\[ T_n[S_+(u_0) n]|_{t=0} - T_n[S_-(u_0) n]|_{t=0} = T_n[h]|_{t=0}, \quad u_0|_{t=0} - u_0 - |_{t=0} = 0, \]
\[ u_0|_{t=0} = 0, \quad T_n[S_-(u_0) n]|_{t=0} = T_n[h_-]|_{t=0}. \]
\[ \text{div} u_0 = f_-|_{t=0} \text{ in } \Omega_- \quad (u_0 - \nabla \varphi)_{\Omega_-} = (f_- - \nabla \varphi)_{\Omega_-} \text{ for any } \varphi \in W_0^1_{q,0}(\Omega_-). \]

Equation (22) admits solutions \( \theta_+ \) and \( u_\pm \) with:
\[ \theta_+ \in W_0^{1, \gamma_0}(\mathbb{R}, W_0^1(\Omega_+)), \quad u_\pm \in L_{p,\gamma_0}(\mathbb{R}, W_0^2(\Omega_\pm)^N) \cap W_0^1(\mathbb{R}, L_q(\Omega_\pm)^N) \]

possessing the estimate:
\[ \|e^{-\gamma t}(\partial_t \theta_+ + \gamma \theta_+)\|_{L_p(\mathbb{R}, W_0^1(\Omega_+))} + \sum_{\ell=+,-} \|e^{-\gamma t}(\partial_t u_\ell, \gamma u_\ell, \Lambda_{1/2} \nabla u_\ell, \nabla^2 u_\ell)\|_{L_p(\mathbb{R}, L_q(\Omega_\ell))} \leq C \left( \|\theta_0\|_{W_0^1(\Omega_+)} + \sum_{\ell=+,-} \|u_0\|_{W_0^{3,1/p}(\Omega_\ell)} \right) \]
\[ + \|e^{-\gamma t} f_+\|_{L_p(\mathbb{R}, W_0^1(\Omega_+))} + \|e^{-\gamma t}(\Lambda_{1/2} f_- - \nabla f_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \]
\[ + \|e^{-\gamma t}\partial_t f_+\|_{L_p(\mathbb{R}, L_q(\Omega_+))} + \|e^{-\gamma t} g_+\|_{L_p(\mathbb{R}, L_q(\Omega_+))} + \|e^{-\gamma t} g_-\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \]
\[ + \|e^{-\gamma t}(\Lambda_{1/2} h, \nabla h)\|_{L_p(\mathbb{R}, L_q(\Omega_+))} + \|e^{-\gamma t}(\Lambda_{1/2} h_-, \nabla h_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \]

for any \( \gamma \geq \gamma_0 \), where \( C \) is a constant independent of \( \gamma \).

Uniqueness Let \( \theta_+ \) and \( u_\pm \) satisfy (26) and Equation (22) with \( \theta_{0+} = 0, u_{0\pm} = 0, f_\pm = 0, \)
\[ g_\pm = 0, f_- = 0, \text{ and } h = 0, \text{ then } \theta_+ = 0 \text{ and } u_\pm = 0. \]

To prove Theorem 3, Problem (22) is divided into two parts: One is the case where the right side in (22) is considered for all \( t \in \mathbb{R} \), while the initial conditions are not taken into account. The other case is non-homogeneous initial conditions and a zero right side in (22). In the first case, solutions are represented by the Laplace inverse transform of solution formulas represented by using \( k \)-bounded solution operators for the generalized resolvent problem corresponding to (22). Combining the \( k \)-boundedness and Weis’s operator-valued Fourier multiplier theorem yields the maximal \( L_p-L_q \) estimate of solutions to Equation (22) with zero initial conditions. Moreover, the \( k \)-bounded solution operators yield the generator of the continuous analytic semigroup associated with Equation (22), which, combined with some real interpolation technique, yields the \( L_p-L_q \) maximal regularity for the initial problem for Equation (22). Combining these two results gives Theorem 3. To prove the generation of the continuous analytic semigroup, we have to eliminate the pressure term \( p_- \) in Equation (22), and so, using the assumption of the unique existence of the weak Dirichlet problem, we define the reduced generalized resolvent problem (RGRP) (cf. (41) in Section 2 below) according to Grubb and Solonnikov [8], which is the equivalent system to the generalized resolvent problem (GRP) corresponding to (22).

The paper is organized as follows. In Section 2, we introduce (GRP) and state main results for (GRP). Secondly, we drive (RGRP) and discuss some equivalence between (GRP) and (RGRP). Thirdly, we state the main results for (RGRP), which implies the results for (GRP) according to the equivalence between (GRP) and (RGRP). In Section 3, we discuss the model problems in \( \mathbb{R}^N \). In Section 4, we discuss the bent half space problems for (RGRP). In Section 5, we prove the main result for (GRP) and also Theorem 3. In Section 6, we prove Theorem 1 by the Banach fixed point argument based on Theorem 3.
2. $R$-Bounded Solution Operators

To prove the generation of the continuous analytic semigroup and the maximal $L_p^rL_q^s$ regularity for the linear problem (22), we show the existence of $R$-bounded solution operators to the following generalized resolvent problem (GRP) corresponding to: (22):

\[
\begin{align*}
\lambda \theta_+ + \gamma_2+ & \nabla u_+ = f_+ & \text{in } \Omega_+, \\
\lambda u_+ - \gamma_0^{-1}(\nabla S_+(u_+) - \nabla (\gamma_1+\theta_+)) &= g_+ & \text{in } \Omega_+, \\
\lambda u_- - \rho_0^{-1}(\nabla S_-(u_-) - \nabla p_-) &= g_- & \text{in } \Omega_-, \\
\text{div } u_- &= f_- = \text{div } f_- & \text{in } \Omega_-, \\
(S_+(u_+) - \gamma_1+\theta_+)n\big|_{\Gamma^0} - (S_-(u_-) - p_-)n\big|_{\Gamma^-} &= h & \text{in } \Gamma, \\
(S_-(u_-) - p_-)n_-\big|_{\Gamma^-} &= h_- & \text{in } \Gamma^-,
\end{align*}
\]

(28)

When $\lambda \neq 0$, setting $\theta_+ = \lambda^{-1}(f_+ - \gamma_2+ \text{div } u_+)$, we transfer the second equation and the fifth equation in (28) to:

\[
\begin{align*}
\lambda u_+ - \gamma_0^{-1}(\text{div } S_+(u_+)) + \lambda^{-1}\nabla (\gamma_1+\gamma_2+\text{div } u_+) &= g_+ - \lambda^{-1}\gamma_0^{-1}\nabla (\gamma_1+f_+) & \text{in } \Omega_+, \\
(S_+(u_+) + \lambda^{-1}\gamma_1+\gamma_2+\text{div } u_+ I n - (S_-(u_-) - p_- I)n\big|_{\Gamma^-} &= h + \lambda^{-1}\gamma_1+f_+n\big|_{\Gamma^0},
\end{align*}
\]

respectively. Thus, $g_+ - \lambda^{-1}\gamma_0^{-1}\nabla (\gamma_1+f_+)$ and $h + \lambda^{-1}\gamma_1+f_+n\big|_{\Gamma^0}$, being renamed $g_+$ and $h$, respectively, and setting $\gamma_1+\gamma_2+ = \gamma_3+$, from now on, we consider the following problem:

\[
\begin{align*}
\lambda u_+ - \gamma_0^{-1}(\text{div } S_+(u_+)) + \delta \nabla (\gamma_3+\text{div } u_+) &= g_+ & \text{in } \Omega_+, \\
\lambda u_- - \rho_0^{-1}(\nabla S_-(u_-) - \nabla p_-) &= g_- & \text{in } \Omega_-, \\
\text{div } u_- &= f_- = \text{div } f_- & \text{in } \Omega_-, \\
(S_+(u_+) + \delta \gamma_3+\text{div } u_+ I n\big|_{\Gamma^0} - (S_-(u_-) - p_- I)n\big|_{\Gamma^-} &= h & \text{in } \Gamma, \\
(S_-(u_-) - p_- I)n_-\big|_{\Gamma^-} &= h_- & \text{in } \Gamma^-,
\end{align*}
\]

(29)

Here, $\delta$ and $\lambda$ satisfy one of the following three conditions:

(C1) $\delta = \lambda^{-1}$, $\lambda \in \Lambda_{\varepsilon,\lambda_0} = K_{\varepsilon} \cap \Sigma_{\varepsilon,\lambda_0}$,

(C2) $\delta = \delta_0 \in \Sigma_{\varepsilon}$ with $\text{Re } \delta_0 < 0$, $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \lambda_0$ and $\text{Re } \lambda \geq |\text{Im } \lambda| |\text{Re } \delta_0| |\text{Im } \lambda|$

(C3) $\delta = \delta_0$ with $\text{Re } \delta_0 \geq 0$, $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \lambda_0$,

where we set $\Sigma_{\varepsilon} = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\text{arg } \lambda| \leq \pi - \varepsilon \}$ with $0 < \varepsilon < \pi/2, \Sigma_{\varepsilon,\lambda_0} = \{ \lambda \in \Sigma_{\varepsilon} \mid |\lambda| \geq \lambda_0 \}$, and:

\[
K_{\varepsilon} = \{ \lambda \in \mathbb{C} \mid (\text{Re } \lambda + \rho_3\nu^{-1} + \varepsilon)^2 + (\text{Im } \lambda)^2 \geq (\rho_3\nu^{-1} + \varepsilon)^2 \}
\]

(30)

with $\rho_3 = \sup_{x \in \Omega_1} \gamma_1+\gamma_2+ |(x)\leq \rho_3^2)$. We may include the case where $\gamma_1 = 0$, which corresponds to the Lamé system. The former case (C1) is used to prove the existence of $R$-bounded solution operators to (28), and the latter cases (C2) and (C3) enable the application of a homotopic argument for proving the exponential stability of the analytic semigroup in bounded domains. For the sake of simplicity, we introduce the set $\Gamma_{\varepsilon,\lambda_0}$ defined by:

\[
\Gamma_{\varepsilon,\lambda_0} = \begin{cases} 
\Lambda_{\varepsilon,\lambda_0} & \text{when } \delta = \lambda^{-1}, \\
\{ \lambda \in \mathbb{C} \mid \text{Re } \lambda \geq \lambda_0, \text{Re } \lambda \geq |\text{Re } \delta_0| |\text{Im } \lambda| \} & \text{when } \delta = \delta_0 \in \Sigma_{\varepsilon}, \text{Re } \delta_0 < 0, \\
\{ \lambda \in \mathbb{C} \mid \text{Re } \lambda \geq \lambda_0 \} & \text{when } \delta = \delta_0 \text{ with } \text{Re } \delta_0 \geq 0.
\end{cases}
\]

(31)

Note that $|\delta| \leq \max(|\delta_0|, \lambda_0^{-1})$. 

Before stating our main results for the linear problem, we introduce a few symbols and the definition of the \( R \)-bounded operator family and the operator-valued Fourier multiplier theorem due to Weis [9].

Symbols 3. For any two Banach spaces \( X \) and \( Y \), \( \mathcal{L}(X, Y) \) denotes the set of all bounded linear operators from \( X \) to \( Y \), and we write \( \mathcal{L}(X) = \mathcal{L}(X, X) \) for short. \( \text{Hol}(\mathbb{U}, X) \) denotes the set of all \( X \)-valued holomorphic functions defined on a complex domain \( \mathbb{U} \). Let \( \mathcal{D}(\mathbb{R}, X) \) and \( \mathcal{S}(\mathbb{R}, X) \) be the set of all \( X \)-valued \( C^\infty \)-functions having compact support and the Schwartz space of rapidly decreasing \( X \)-valued functions, respectively, while \( \mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X) \). Given \( M \in L_{1,\text{loc}}(\mathbb{R} \setminus \{0\}, X) \), we define the operator \( T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \rightarrow \mathcal{S}'(\mathbb{R}, Y) \) by:

\[
T_M \varphi = \mathcal{F}^{-1}[M \mathcal{F}[\varphi]] \quad (\mathcal{F}[\varphi] \in \mathcal{D}(\mathbb{R}, X)).
\]

Here, \( \mathcal{F}_x \) and \( \mathcal{F}_x^{-1} \) denote the Fourier transform and its inversion defined by:

\[
\mathcal{F}_x[u](\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} u(x)dx, \quad \mathcal{F}_x^{-1}[v](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix \cdot \xi} \hat{v}(\xi)d\xi,
\]
respectively.

**Definition 3.** Let \( X \) and \( Y \) be Banach spaces. A family of operators \( \mathcal{T} \subset \mathcal{L}(X, Y) \) is called \( \mathcal{R} \)-bounded on \( \mathcal{L}(X, Y) \), if there exist constants \( C > 0 \) and \( p \in [1, \infty) \) such that for any \( n \in \mathbb{N} \), \( \{T_j\}_{j=1}^n \subset \mathcal{T}, \{x_j\}_{j=1}^n \subset X \), and sequences \( \{r_j(u)\}_{j=1}^n \) of independent, symmetric, \( \{-1,1\} \)-valued random variables on \([0,1]\), there holds the inequality:

\[
\int_0^1 \| \sum_{j=1}^n r_j(u)T_jx_j \|^p_Y du \leq C \int_0^1 \| \sum_{j=1}^n r_j(u)x_j \|^p_X du.
\]

The smallest such \( C \) is called the \( \mathcal{R} \)-bound of \( \mathcal{T} \), which is denoted by \( \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) \).

The following theorem was obtained by Weis [9].

**Theorem 4.** Let \( X \) and \( Y \) be two UMD spaces and \( 1 < p < \infty \). Let \( M \) be a function in \( C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y)) \) such that:

\[
\mathcal{R}_{\mathcal{L}(X,Y)}\left( \left\{ \left( \frac{d}{d\tau} \right)^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\} \right\} \right) \leq \kappa < \infty \quad (\ell = 0, 1)
\]

with some constant \( \kappa \). Then, the operator \( T_M \) defined in (32) may uniquely be extended to a bounded linear operator from \( L_p(\mathbb{R}, X) \) to \( L_p(\mathbb{R}, Y) \). Moreover, denoting this extension by \( T_M \), we have:

\[
\| T_M \|_{\mathcal{L}(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))} \leq C\kappa
\]

for some positive constant \( C \) depending on \( p, X, \) and \( Y \).

**Remark 4.** For the definition of the UMD space, we refer to the monograph by Amann [10]. For \( 1 < q < \infty \) and \( m \in \mathbb{N} \), Lebesgue spaces \( L_q(\Omega) \) and Sobolev spaces \( W^m_q(\Omega) \) are UMD spaces.

For the calculation of the \( \mathcal{R} \)-norm, we use the following lemmas.

**Lemma 1.** (1) Let \( X \) and \( Y \) be Banach spaces, and let \( \mathcal{T} \) and \( \mathcal{S} \) be \( \mathcal{R} \)-bounded families in \( \mathcal{L}(X, Y) \). Then, \( \mathcal{T} + \mathcal{S} = \{ T + S \mid T \in \mathcal{T}, S \in \mathcal{S} \} \) is also an \( \mathcal{R} \)-bounded family in \( \mathcal{L}(X, Y) \) and:

\[
\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).
\]
Theorem 5. Let \( m(\lambda) \) be a bounded function defined on a subset \( \Lambda \subset \mathbb{C} \), and let \( M_m(\lambda) \) be a multiplication operator with \( m(\lambda) \) defined by \( M_m(\lambda)f = m(\lambda)f \) for any \( f \in L_q(D) \). Then,

\[
\mathcal{R}_{L(X,Z)}(S T) \leq \mathcal{R}_{L(X,Y)}(T) \mathcal{R}_{L(Y,Z)}(S).
\]

Lemma 2. Let \( 1 < p, q < \infty \), and let \( D \) be a domain in \( \mathbb{R}^N \).

(1) Let \( m(\lambda) \) be a bounded function defined on a subset \( \Lambda \subset \mathbb{C} \), and let \( M_m(\lambda) \) be a multiplication operator with \( m(\lambda) \) defined by \( M_m(\lambda)f = m(\lambda)f \) for any \( f \in L_q(D) \). Then,

\[
\mathcal{R}_{L(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \leq C_{N,q,D} \|m\|_{L_\infty(\Lambda)}.
\]

(2) Let \( n(\tau) \) be a \( C^1 \) function defined on \( \mathbb{R} \setminus \{0\} \) that satisfies the conditions: \( |n(\tau)| \leq \gamma \) and \( |\tau n'(\tau)| \leq \gamma \) with some constant \( \gamma > 0 \) for any \( \tau \in \mathbb{R} \setminus \{0\} \). Let \( T_n \) be an operator-valued Fourier multiplier defined by \( T_nf = F^{-1}[nF[f]] \) for any \( f \) with \( F[\phi] \in D(\mathbb{R}, X) \). Then, \( T_n \) is extended to a bounded linear operator from \( L_p(\mathbb{R}, L_q(D)) \) into itself. Moreover, denoting this extension also by \( T_n \), we have:

\[
\|T_n\|_{L(L_p(\mathbb{R}, L_q(D)))} \leq C_{p,q,D} \gamma.
\]

Remark 5. For the proofs of Lemma 1 and Lemma 2, we refer to [11], p.28, 3.4. Proposition and p.27, 3.2. Remarks (4) (cf. also Bourgain [12]), respectively.

2.1. Existence of \( \mathcal{R} \)-Bounded Solution Operators for Problems (28) and (29)

We state two theorems about the existence of \( \mathcal{R} \)-bounded solution operators to Problems (28) and (29).

Theorem 5. Let \( 1 < q < \infty \), \( 0 < \epsilon < \pi/2 \) and \( N < r < \infty \). Assume that \( r \geq \max(q, q') \), that \( \Omega_\pm \) are uniform \( W^2_{\infty} \) domains, and that the weak Dirichlet problem is uniquely solvable on \( \Omega_- \) with exponents \( q \) and \( q' \). Let \( X^0_q(\Omega) \) and \( \Lambda^0_q(\Omega) \) be the spaces defined by:

\[
X^0_q(\Omega) = \{G^0 = (g_+, g_-, h_0, f_+, f_-) \mid g_+ \in L_q(\Omega_+)^N, h_0 \in W^1_q(\Omega)^N, h_- \in W^1_q(\Omega_-)^N, f_+ = f_0(\Omega_+)^N, f_- = \text{div } f_-\},
\]

\[
\Lambda^0_q(\Omega) = \{F^0 = (F_1, \ldots, F_7) \mid F_1 \in L_q(\Omega_+)^N, F_2, F_3, F_4, F_5, F_6, F_7 \in L_q(\Omega_-)^N, F_7 \in L_q(\Omega_-), F_8 \in W^1_q(\Omega_-)\}.
\]

Then, there exist a constant \( \lambda_0 > 0 \) and operator families \( A^0_\pm(\lambda) \) and \( B^0_\pm(\lambda) \) with:

\[
A^0_\pm(\lambda) \in \text{Hol}(\Gamma_{e,\lambda_0}, L(X^0_q(\Omega), W^1_q(\Omega_\pm)^N)), B^0_\pm(\lambda) \in \text{Hol}(\Gamma_{e,\lambda_0}, L(\Lambda^0_q(\Omega), W^1_q(\Omega_-) + W^1_{q,0}(\Omega_-)))))
\]

such that \( u_\pm = A^0_\pm(\lambda)F^0G^0 \) and \( p_- = B^0_\pm(\lambda)F^0_\pmG^0 \) are unique solutions to Problem (29) for any \( \lambda \in \Gamma_{e,\lambda_0} \) and \( G^0 = (g_+, g_-, h_0, f_+, f_-) \in X^0_q(\Omega) \), and:

\[
\mathcal{R}_{L(X^0_q(\Omega), W^1_q(\Omega_-)^N)}((\tau \partial_\tau)^\ell (A^0_\pm(\lambda)) \mid \lambda = \gamma + i\tau \in \Gamma_{e,\lambda_0}) \leq C,
\]

\[
\mathcal{R}_{L(\Lambda^0_q(\Omega), L_q(\Omega_-)^N)}((\tau \partial_\tau)^\ell (B^0_\pm(\lambda)) \mid \lambda = \gamma + i\tau \in \Gamma_{e,\lambda_0}) \leq C
\]

for \( j = 0, 1, 2 \) and \( \ell = 0, 1 \). Here, we set \( F^0_\lambda G^0 = (g_+, g_-, \lambda^{1/2}h_0, h_-, \lambda^{1/2}f_-, f_-, \lambda^{1/2}f_-) \).

Remark 6. (i) The constants depend on \( \epsilon, q, r, \rho_0, \rho_2, \mu_\pm, \nu_+ \), and \( \delta_0 \), but we do not mention this dependence.

(ii) The variables \( F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \) and \( F_9 \) correspond to \( g_+, g_-, \lambda^{1/2}h_0, h_-, \lambda^{1/2}f_-, f_-, \lambda^{1/2}f_-) \).
Let $\theta_+ = \lambda^{-1}(f_+ - \gamma_2 + \text{div}\ u_+)$ in (28), the following theorem follows immediately from Theorem 5 and Lemma 1.

**Theorem 6.** Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, and $N < r < \infty$. Assume that $r \geq \max(q,q')$, that $\Omega_\pm$ are uniform $W_{\varepsilon}^{r-1/q'}$ domains, and that the weak Dirichlet problem is uniquely solvable on $\Omega_-$ with exponents $q$ and $q'$. Let $X^1_q(\Omega)$ and $X^1_q(\Omega)$ be the sets defined by:

$$
X^1_q(\Omega) = \{ G^1 = (G^0, f_+) \mid G^0 \in X^0_q(\Omega), \ f_+ \in W^1_q(\Omega_+) \},
$$

$$
X^1_q(\Omega) = \{ F^1 = (F^0, F_{10}) \mid F^0 \in X^0_q(\Omega), \ F_{10} \in W^1_q(\Omega+) \}.
$$

Then, there exist a constant $\lambda_0 > 0$ and operator families $A^1_\pm(\lambda)$ and $B^1_\pm(\lambda)$ with:

$$
A^1_\pm(\lambda) = \text{Hol}(\Gamma_{e,\lambda_0}, \mathcal{L}(X^1_q(\Omega), W^1_q(\Omega_+))) \quad \text{and} \quad B^1_\pm(\lambda) = \text{Hol}(\Gamma_{e,\lambda_0}, \mathcal{L}(X^1_q(\Omega), W^1_q(\Omega_+))),
$$

such that $u_\pm = A^1_\pm(\lambda)F^1_\pm G^1$, $\theta_+ = B^1_\pm(\lambda)F^1_\pm G^1$, and $p_- = B^1_\pm(\lambda)F^1_\pm G^1$ are unique solutions to Problem (28) for any $\lambda \in \Gamma_{e,\lambda_0}$ and $G^1 = (g_+, g_-, h_+, f_-, f_+) \in X^1_q(\Omega)$, and:

$$
\mathcal{R}_\mathcal{L}(\mathcal{X}^1_q(\Omega), W^1_q(\Omega_+))\{(\tau\partial_\tau)^f(\lambda^j A^1_\pm(\lambda)) \mid \lambda = \gamma + i\varepsilon \in \Gamma_{e,\lambda_0} \} \leq C,
$$

$$
\mathcal{R}_\mathcal{L}(\mathcal{X}^1_q(\Omega), W^1_q(\Omega_+))\{(\tau\partial_\tau)^f(\lambda^j B^1_\pm(\lambda)) \mid \lambda = \gamma + i\varepsilon \in \Gamma_{e,\lambda_0} \} \leq C,
$$

$$
\mathcal{R}_\mathcal{L}(\mathcal{X}^1_q(\Omega), L^1_q(\Omega_-))\{(\tau\partial_\tau)^f(-\nabla B^1_\pm(\lambda)) \mid \lambda = \gamma + i\varepsilon \in \Gamma_{e,\lambda_0} \} \leq C
$$

for $j = 0, 1, 2$, $k = 0, 1$, and $\ell = 0, 1$. Here, we set $F^1_\pm G^1 = (F^0 \pm G^0, f_+)$.

**Remark 7.** The variable $F_{10}$ corresponds to $f_+$, and we set:

$$
\| G^1 \|_{\mathcal{X}^1_q(\Omega)} = \| G^0 \|_{\mathcal{X}^0_q(\Omega_+)} + \| f_+ \|_{W^1_q(\Omega_+)} \quad \| F^1 \|_{\mathcal{X}^1_q(\Omega)} = \| F^0 \|_{\mathcal{X}^0_q(\Omega)} + \| F_{10} \|_{W^1_q(\Omega_+)}.
$$

2.2. Reduced Generalized Resolvent Problem

Since the pressure term $p_-$ has no time evolution in (22), we eliminate $p_-$ from (29) and derive a reduced problem. Before this discussion, we consider the resolvent problem for the Laplace operator with non-homogeneous Dirichlet condition of the form:

$$
\lambda(w, \varphi)_\Omega + \rho_0^{-1}(\nabla w, \nabla \varphi)_\Omega = -(f, \nabla \varphi)_\Omega_- \quad \text{for any } \varphi \in W^1_q(\Omega_-)
$$

subject to $w|_\Gamma = g_1$ and $w|_\Gamma = g_2$. Here and in the following, we write $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ for short. Note that:

$$
(w, \nabla \varphi)_\Omega = -(\lambda^{-1}(f + \rho_0^{-1}\nabla w), \nabla \varphi)_\Omega_- \quad \text{for any } \varphi \in W^1_q(\Omega_-).
$$

We can show the following theorem by using the method in Shibata [13].
Theorem 7. Let $1 < q < \infty$, $N < r < \infty$, and $0 < \epsilon < \pi/2$. Assume that $r \geq \max(q, q')$ and that $\Omega_-$ is a uniform $W^{1-1/r}_r$ domain. Set:

\[ X^2_q(\Omega_-) = \{ G^2 = (f, g_1, g_2) \mid f \in L_q(\Omega_-)^N, g_1 \in W^1_q(\Omega_-), g_2 \in W^1_q(\Omega_-) \}, \]

\[ A^2_q(\Omega_-) = \{ F^2 = (f_2, f_{11}, \ldots, f_{14}) \mid f_2, f_{11}, f_{13} \in L_q(\Omega_-)^N, f_{12}, f_{14} \in W^1_q(\Omega_-) \}. \]

Then, there exists a $\lambda_0 > 0$ and an operator family $\vartheta(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, L(X^2_q(\Omega_-), W^1_q(\Omega_-)))$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $G^2 = (f, g_1, g_2) \in X^2_q(\Omega_-)$, $w = \vartheta(\lambda) F^2 G^2$ is a unique solution to (35), and:

\[ \mathcal{R}_{L(\chi^2_q(\Omega_-), W^1_q(\Omega_-))}(\{q_{\partial_w} (\lambda^{k/2} \vartheta(\lambda)) \mid \lambda = \gamma + ir \in \Sigma_{\epsilon, \lambda_0} \}) \leq C \quad (\ell = 0,1, k = 0,1). \]  

(35)

Here, we set $F^2 G^2 = (f, \lambda^{1/2} g_1, \lambda^{1/2} g_2, g_2)$.

Remark 8. (i) $F_{11}, F_{12}, F_{13}$, and $F_{14}$ are the corresponding variables to $\lambda^{1/2} g_1, g_1, \lambda^{1/2} g_2$ and $g_2$.

(ii) Since $\mathcal{R}$-boundedness implies the usual boundedness, by (34) and (35) we have:

\[ \|\lambda w\|_{W^1_q(\Omega_-)} + \|\lambda^{1/2} w\|_{L_q(\Omega_-)} + \|w\|_{W^1_q(\Omega_-)} \leq C\|F(\lambda^{1/2} g_1, \lambda^{1/2} g_2)\|_{L_q(\Omega_-)} \]

with $w = \vartheta(\lambda) F^2 (f, g_1, g_2)$. Here, $W^1_{q,0}(\Omega_-)$ is the dual space of $W^1_{q'}(\Omega_-)$.

We start our main discussion in this subsection. Given $w_+ \in W^1_{q'}(\Omega_+)$, let $\text{Ext}^{-} [w_+]$ denote an extension of $w_+$ to $\Omega_-$ such that $\text{Ext}^{-} [w_+]|_{\Gamma^-} = w_+|_{\Gamma^0}$ and $\|\text{Ext}^{-} [w_+]|_{W^1_q(\Omega_-)} \leq C\|w_+\|_{W^1_{q'}(\Omega_+)}$. Since we can choose some uniform covering of $\Omega_\pm$ (cf. Proposition 4 in Section 5 below), $\text{Ext}^{-} [w_+]$ is defined by the even extension of $w_+$ in each local chart. For $u_\pm \in W^2_q(\Omega_\pm)^N$, we define an operator $K(u_+, u_-)$ by $K(u_+, u_-) = \tilde{K}(f, g_1, g_2)$, where we set $	ilde{K}(f, g_1, g_2) = \vartheta(\lambda)|_{(0, g_1, g_2)} + K(f - \nabla \vartheta(\lambda)|_{(0, g_1, g_2)})$.

\[ f = f(u_-) = \text{Div} S(-u_-) - \nabla \text{div} u_- \]

and $\tilde{K}$ is the operator defined in Remark 2. Note that $K(u_+, u_-) \in W^1_q(\Omega_-) + \tilde{W}^1_{q,0}(\Omega_-)$ and satisfies the variational equation:

\[ (\nabla K(u_+, u_-), \nabla \varphi)_{\Omega_-} = (\rho_{\lambda}^{-1} (\text{Div} S(-u_-) - \nabla \text{div} u_-), \nabla \varphi)_{\Omega_-} \]

for any $\varphi \in W^1_q(\Omega_-)$

subject to:

\[ K(u_+, u_-)|_{\Gamma^-} = (S(-u_-) n, n > -\text{div} u_-) - (S(u_+) + \delta \gamma_3, \text{div} u_+ I) n, n > |_{\Gamma^-}, \]

\[ K(u_+, u_-)|_{\Gamma^+} = (S(-u_-) n, n > -\text{div} u_-) |_{\Gamma^+}, \]

and the estimate:

\[ \|\nabla K(u_+, u_-)\|_{L_q(\Omega_-)} \leq C(\|\nabla u_+\|_{W^1_q(\Omega_+)} + \|\nabla u_-\|_{W^1_q(\Omega_+)}) . \]

(40)

The reduced generalized resolvent problem (RGRP) is the following:

\[ \begin{cases} 
\lambda u_+ - \gamma_0^{-1} (\text{Div} S(u_+) + \delta \nabla (\gamma_3, \text{div} u_+)) = g_+ & \text{in } \Omega_+,
\lambda u_- - \rho_0^{-1} (\text{Div} S(-u_-) - \text{Div} K(u_+, u_-)) = g_- & \text{in } \Omega_-,
(S(u_+) + \delta \gamma_3, \text{div} u_+ I) n|_{\Gamma^+} = (S(-u_-)|_{\Gamma^-} - (S(u_-) - K(u_+, u_-)) n)|_{\Gamma^-} = h|_{\Gamma^-},
(S(u_+) - K(u_+, u_-)) n|_{\Gamma^+} = h|_{\Gamma^+},
\end{cases} \]

(41)
Using $T_z[w]$ defined in (14), we can write the interface condition and free boundary condition in (41) as follows:

\[
\begin{align*}
h|_r &= T_n[S_+(u_+)|n]|r=0 - T_n[S_-(u_-)|n]|r=0 - (\nabla u_-|r=0)\mathbf{n}, \\
h_+|r_- &= T_n[S_-(u_-)|n]|r_-=0 + (\nabla u_-|r_-)\mathbf{n}.
\end{align*}
\]

(42)

We say that $(u_+, u_-)$ is a solution to (41) with $(g_+, g_-, h, h_-)$ if $u_{\pm} \in W^2_0(\Omega_{\pm})$ and $u_{\pm}$ satisfies Equation (41). Furthermore, we say that $(u_+, u_-, p_-)$ is a solution to (29) with $(g_+, g_-, f_- = \text{div } f_-, h, h_-)$ if $u_{\pm} \in W^2_0(\Omega_{\pm})$, $p_- \in W^1_0(\Omega) + W^1_0(\Omega)$ and $u_{\pm}$ and $p_-$ satisfy Equation (29). In this subsection, we show the equivalence of the solutions between (29) and (41).

**Assertion 1.** If (29) is solvable, then so is (41).

In fact, we define $f_- \in W^1_0(\Omega_-)$ by $f_- = \nabla(\lambda)F^2_3(g_-, h, h_-)$ with $g_- \in L_q(\Omega_-)$, $h = \langle h, n \rangle > 0$ and $h_- = \langle h, n_+ \rangle > 0$. Notice that $f_- = \text{div } f_-$ with $f_-$ = $\lambda^{-1}(g_- + \rho_0^{-1}\nabla f_-)$. Let $(u_+, u_-, p_-)$ be a solution to (29) with $(g_+, g_-, f_- = \text{div } f_-, h, h_-)$. In particular, $\text{div } u_- = f_- = \nabla(\lambda^{-1}(g_- + \rho_0^{-1}\nabla f_-))$, namely $\text{div } u = f_-$ and $Au - \nabla h = -\rho_0^{-1}\nabla f_-$ is $\in L_q(\Omega_-)$. From the second equation of (29), it follows that for any $\varphi \in W^1_{q',0}(\Omega_-)$:

\[
(g_-, \nabla \varphi)_{\Omega_-} = \lambda(u_-, \nabla \varphi) - (\rho_0^{-1}\text{div } u_-, \nabla \varphi)_{\Omega_-} - (\rho_0^{-1}(\text{Div } S_-(u_-) - \nabla \text{div } u_-), \nabla \varphi)_{\Omega_-} + (\rho_0^{-1}\nabla p_-, \nabla \varphi)_{\Omega_-} = (g_- + \rho_0^{-1}\nabla f_-, \nabla \varphi)_{\Omega_-} - (\rho_0^{-1}\nabla f_-, \nabla \varphi)_{\Omega_-} + (\rho_0^{-1}(-\text{K}(u_+, u_-)), \nabla \varphi)_{\Omega_-},
\]

which yields that $(\rho_0^{-1}\nabla(p_- - \text{K}(u_+, u_-)), \nabla \varphi)_{\Omega_-} = 0$ for any $\varphi \in W^1_{q',0}(\Omega_-)$. Moreover,

\[
\begin{align*}
p_- - \text{K}(u_+, u_-) &= <h, n> + \text{div } u_- = <h, n> + f_- = 0 \\
p_- - \text{K}(u_+, u_-) &= <h, n_- > + \text{div } u_- = <h, n_- > + f_- = 0
\end{align*}
\]

Thus, the uniqueness yields that $p_- = \text{K}(u_+, u_-)$, and so, $(u_+, u_-)$ is a solution to (41) with $(g_+, g_-, h, h_-)$.  

**Assertion 2.** If (41) is solvable, then so is (29).

In fact, given $g_- \in L_q(\Omega_-)$, $h \in W^1_0(\Omega)$, and $h_- \in W^1_0(\Omega_-)$, we define $\hat{\rho}_1$ by $\hat{\rho}_1 = f_2 + K(g_- - \nabla \hat{p}_2)$ with $\hat{p}_2 = \nabla(\lambda)F^2_3(0, h, -h_-)$. Next, given $f_- = \text{div } f_-$, we define $\hat{\rho}_3$ by:

\[
\hat{\rho}_3 = \nabla(\lambda)F^2_3(0, f_-, f_-) + K(-\nabla(\lambda)F^2_3(0, f_-, f_-))
\]

with $F = \lambda f_- - \rho_0^{-1}\nabla f_-$. Let $(u_+, u_-)$ be a solution to equations:

\[
\begin{align*}
\lambda u_+ - \gamma_0^{-1}(\text{Div } S_+(u_+) + \delta(\gamma_3, \text{div } u_+)) &= g_+ \quad \text{in } \Omega_+,
\lambda u_- - \rho_0^{-1}(\text{Div } S_-(u_-) - \nabla K(u_+, u_-)) &= g_- - \rho_0^{-1}\nabla(\hat{\rho}_1 + \hat{\rho}_3) \quad \text{in } \Omega_-,
(S_+(u_+) + \delta \gamma_3 \text{div } u_+ | n|_{r_0} - (S_-(u_-) - K(u_+, u_-) I | n|_{r_0}) = (T_n[h] - f_- n)|_{r_0} = 0,
(S_-(u_-) - K(u_+, u_-) I | n|_{r_-} = (T_n[h_-] + f_- n_-)|_{r_-} = 0,
\end{align*}
\]

(43)

Setting $p_- = \text{K}(u_+, u_-) + \hat{\rho}_1 + \hat{\rho}_3$, we see that $(u_+, u_-, p_-)$ is a solution to (29) with $(g_+, g_-, f_- = \text{div } f_-, h, h_-)$. In our task is to prove that $\text{div } u_- = f_- = \text{div } f_-$, Notice that $\rho_0^{-1}(\nabla \hat{p}_1, \nabla \varphi)_{\Omega_-} = (g_- - \nabla \varphi)_{\Omega_-}$ and $\rho_0^{-1}(\nabla \hat{p}_3, \nabla \varphi)_{\Omega_-} = -(\lambda f_- - \rho_0^{-1}\nabla f_-)_{\Omega_-}$.

Thus, by (38):

\[
\begin{align*}
(\lambda f_- - \rho_0^{-1}\nabla f_-)_{\Omega_-} &= (g_- - \rho_0^{-1}\nabla(\hat{\rho}_1 + \hat{\rho}_3), \nabla \varphi)_{\Omega_-} = \lambda(u_-, \nabla \varphi)_{\Omega_-} - \rho_0^{-1}(\nabla \text{div } u_-, \nabla \varphi)_{\Omega_-}
\end{align*}
\]

Therefore, $(u_+, u_-)$ is also a solution to (29).
for any \( \varphi \in \tilde{W}^1_q(\Omega_-) \), which yields that:

\[
\lambda(u - f_-, \nabla \varphi)_{\Omega_-} - \rho_0^{-1}(\nabla (\text{div } u_-- f_-), \nabla \varphi)_{\Omega_-} = 0 \quad \text{for any } \varphi \in \tilde{W}^1_q(\Omega_-) . \tag{44}
\]

Taking \( \varphi \in W^1_q(\Omega_-) \subset \tilde{W}^1_q(\Omega_-) \) in (44), using the divergence theorem of Gauss, and noticing that \( \text{div } f_- = f_- \) give that:

\[
\lambda(\text{div } u_- - f_-)_{\Omega_-} + \rho_0^{-1}(\nabla (\text{div } u_- - f_-), \nabla \varphi)_{\Omega_-} = 0 \quad \text{for any } \varphi \in W^1_q(\Omega_-) .
\]

Moreover, from the third equation in (42) and (39), it follows that:

\[
\begin{align*}
(1) & \quad f_-|_\Gamma = < T_n[h], n > - \{ < S_+(u_+) + \delta_{\gamma_3}, \text{div } u_+ > I_n|_{\Gamma \cap \partial} n > - < S_-(u_-)n|_{\Gamma \cap \partial} n n > \} - k(u_+, u_-)|_\Gamma \\
& = \text{div } u_-|_\Gamma, \\
(2) & \quad \text{div } f_+|_\Gamma = < T_n[h], n > + < S_-(u_-) - k(u_+, u_-)I_n > N_{\Gamma \cap \partial} n > = \text{div } u_-|_\Gamma .
\end{align*}
\]

Thus, the uniqueness yields that \( \text{div } u_- = f_- \) in \( \Omega_- \). Inserting this fact into (44) and using the fact that \( \lambda \neq 0 \), we have \( u - f_- \in L_q(\Omega) \), which shows that \( \text{div } u = \text{div } f_- \) on \( \Gamma_- \).

Noting that \( \tilde{\rho}_1 = h = < h, n > \) and \( \tilde{\rho}_3 = f_- \) on \( \Gamma \) and that \( \tilde{\rho}_1 = h_- = < h, n_+ > \) and \( \tilde{\rho}_3 = -f_- \) on \( \Gamma_- \), we have:

\[
\begin{align*}
(1) & \quad (S_+(u_+) + \delta_{\gamma_3}, \text{div } u_+ > I_n|_{\Gamma \cap \partial} n > - < S_-(u_-)n|_{\Gamma \cap \partial} n n > ) = T_n[h] - f_- n + < h, n > + f_- n = h \quad \text{on } \Gamma , \\
(2) & \quad (S_-(u_-) - (K(u_+, u_-) + \tilde{\rho}_1 + \tilde{\rho}_3)I_n n = T_n[h] + f_- n - < h, n > > n + f_- n = h \quad \text{on } \Gamma .
\end{align*}
\]

Thus, \( (u_+, u_-, p_-) \) is a solution of Equation (29) with \( (g_+, g_-, f_-) = (\text{div } f_-, h, h_-) \).

### 2.3. Existence of \( R \)-Bounded Solution Operators for Problem (41)

The following theorem is concerned with the existence of \( R \)-bounded solution operators to Problem (41).

**Theorem 8.** Let \( 1 < q < \infty, 0 < e < \pi/2 \) and \( N < r < \infty \). Assume that \( r \geq \max(q,q') \), that \( \Omega_{\pm} \) are uniform \( W^2_\gamma /r \) domains, and the weak Dirichlet problem is uniquely solvable in \( \Omega_- \) with exponents \( q \) and \( q' \). Let \( X_q(\Omega) \) and \( X_q(\Omega) \) be the sets defined by:

\[
\begin{align*}
X_q(\Omega) &= \{ G = (g_+, g_-, h, h_-) \mid g_+ \in L_q(\Omega_{\pm}), h \in W^1_q(\Omega)^N, h_- \in W^1_q(\Omega_-)^N \}, \\
X_q(\Omega) &= \{ F = (f_1, \ldots, f_6) \mid f_1 \in L_q(\Omega_{\pm})^N, F_2, F_3 \in L_q(\Omega_-)^N, F_4 \in W^1_q(\Omega)^N, F_5 \in W^1_q(\Omega_-)^N \}.
\end{align*}
\]

Then, there exist a constant \( C_0 > 0 \) and operator families \( S_{\pm}(\lambda) \in \text{Hol}(\Gamma_{e_{\lambda}}, L(\Omega), W^2_\gamma(\Omega) \odot N) \) such that for any \( \lambda \in \Gamma_{e_{\lambda}} \) and \( G = (g_+, g_-, h, h_-) \) in \( X_q(\Omega) \), \( u_{\pm} = S_{\pm}(\lambda) F \) is a unique solution to (41) and:

\[
R_{\ell}(\gamma_{\ell}, W^2_\gamma(\Omega) \odot N) \{ \{ (\tau_\ell)_{\lambda_0}^{\ell/2}(S_{\pm}(\lambda)) \mid \lambda \in \Gamma_{e_{\lambda}} \} \} \leq C,
\]

for \( \ell = 0, 1 \) and \( j = 0, 1, 2 \), where we set \( F_3 G = (g_+, g_-, \lambda^{1/2} h, \nabla h, \lambda^{1/2} h-, \nabla h_-) \) and \( G_{\lambda} u = (\lambda u, \gamma_{\ell} u, \lambda^{1/2} \nabla u, \nabla^2 u) \).

**Remark 9.** For any subdomain \( \Omega \subset \Omega \), we set:

\[
||G||_{X_q(\Omega)} = ||g_+||_{L_q(\Omega_{\pm})} + ||g_-||_{L_q(\Omega_{\pm})} + ||h||_{W^1_q(\Omega)} + ||h_-||_{W^1_q(\Omega_-)},
\]

\[
||F||_{I_q(\Omega)} = ||F_1||_{L_q(\Omega_{\pm})} + ||F_2||_{L_q(\Omega_-)} + ||F_3||_{W^1_q(\Omega)} + ||F_4||_{L_q(\Omega_-)} + ||F_5||_{W^1_q(\Omega_-)}.
\]

Obviously, according to Assertion 2 in Section 2.2, by Theorem 8, Lemma 1, and Lemma 2 we have Theorem 8. Thus, we shall prove Theorem 8 only.
2.4. The Uniqueness of Solutions to Problem (41)

Assuming the existence of solutions to Problem (41) with exponent \( q' \), we prove the uniqueness of solutions to (41). Namely, we prove the following lemma.

**Lemma 3.** Let \( 1 < q < \infty \) and \( N < r < \infty \). Assume that \( r \geq \max(q, q') \), that \( \Omega_{\pm} \) are uniform \( W_{q-1/r}^{2} \) domains, and that the weak Dirichlet problem is uniquely solvable on \( \Omega_{-} \) with exponents \( q \) and \( q' \). If there exists a \( \lambda_0 > 0 \) such that Problem (41) is solvable with exponent \( q' \) for any \( \lambda \in \Gamma_{2}\lambda_{0} \), then the uniqueness for (41) with exponent \( q \) is valid for any \( \lambda \in \Gamma_{2}\lambda_{0} \).

**Remark 10.** (i) The reason why we assume that \( r \geq \max(q, q') \) is that we use the existence of solutions to the dual problem to prove the uniqueness.

(ii) The uniqueness means that if \( (u_{+}, u_{-}) \) is a solution to (41) with \((0,0,0)\), then \( u_{\pm} = 0 \).

Before proving Lemma 3, we first prove that if \((u_{+}, u_{-})\) is a solution to (41) with \((g_{+}, g_{-}, 0, 0)\) and if \( g_{-} \in L_{q}(\Omega_{-}) \), then \( u_{-} \in L_{q}(\Omega_{-}) \), as well. In fact, for any \( \varphi \in W_{q,r}^{1}(\Omega_{-}) \), we have:

\[
0 = (g_{-}, \nabla \varphi)_{\Omega_{-}} = (A u_{-} - \rho_{0}^{-1}(\text{Div} S_{-}(u_{-}) - \nabla K(u_{+}, u_{-})), \nabla \varphi)_{\Omega_{-}}
\]

Choosing \( \lambda \in W_{q,r}^{1}(\Omega_{-}) \), we have:

\[
\lambda(\text{div} u_{-} \varphi)_{\Omega_{-}} + \rho_{0}^{-1}(\nabla \text{div} u_{-}, \nabla \varphi)_{\Omega_{-}} = 0 \quad \text{for any } \varphi \in W_{q,r}^{1}(\Omega_{-}).
\]

In addition, we have:

\[
0 = (S_{+}(u_{+}) + \delta \gamma_{33} \text{Div} u_{+}) n|_{\Gamma_{+}}, n > = - (S_{-}(u_{-}) - K(u_{+}, u_{-})) n|_{\Gamma_{-}}, n > = \text{div} u_{-} \quad \text{on } \Gamma,
\]

Thus, the uniqueness guaranteed by Theorem 7 implies that \( \text{div} u_{-} = 0 \), which inserted into (45) yields that \( u_{-} \in L_{q}(\Omega_{-}) \). Secondly, for any \( u_{\pm} \in W_{q}^{2}(\Omega_{\pm}) \) and \( v_{\pm} \in W_{q}^{2}(\Omega_{\pm}) \) with \( u_{-} \in L_{q}(\Omega_{-}) \) and \( v_{-} \in L_{q}(\Omega_{-}) \):

\[
(-\text{Div} S_{+}(u_{+}) + \delta \gamma_{33} \text{Div} u_{+}), (\nabla \varphi)_{\Omega_{+}} + (-\text{Div} S_{-}(u_{-}) - \nabla K(u_{+}, u_{-})), \varphi)_{\Omega_{-}}
\]

provided that \( w_{+}|_{\Gamma_{+}} = w_{-}|_{\Gamma_{-}} \) with \( w = u \) and \( v \), where for \( G = \Gamma \) and \( \Gamma_{-} \), we set \((a, b)_{G} = \int_{G} a(x)b(x) dx \) being the surface element on \( G \). In fact, setting \( K(u_{+}, u_{-}) = p_{1} + p_{2} \in W_{q}^{1}(\Omega_{-}) + \hat{W}_{q,r}^{1}(\Omega_{-}) \), by the divergence theorem of Gauss, we have:

\[
(-\text{Div} S_{+}(u_{+}) + \delta \gamma_{33} \text{Div} u_{+}), (\nabla \varphi)_{\Omega_{+}} + (-\text{Div} S_{-}(u_{-}) - \nabla K(u_{+}, u_{-})), \varphi)_{\Omega_{-}}
\]

with:

\[
C = \sum_{\ell=+,-} \frac{h_{\ell}}{2}(D(u_{\ell}), D(v_{\ell}))_{\Omega_{\ell}} - (\delta \gamma_{33} \text{Div} u_{+}, \text{Div} v_{+})_{\Omega_{+}} + (\nu_{+} - \mu_{+}) (\text{Div} u_{+}, \text{Div} v_{+})_{\Omega_{+}},
\]
because \( K(u_+, u_-) \big|_{\Gamma^0} = p_1 \big|_{\Gamma^0} \) as follows from \( p_2 \big|_{\Gamma^0} = 0 \). Analogously, we have:

\[
(u_+, -(\text{Div} \ S_+ (v_+) + \delta \nabla (\gamma_3 \text{div} \ v_+)))_{\Omega^+} + (u_-, -(\text{Div} \ S_- (v_-) - \nabla K(v_+, v_-)))_{\Omega^-} = B + (u_-, \nabla p_2^2)_{\Omega^-} + C
\]

with \( K(v_+, v_-) = p_1^* + p_2^* \in W^1_q (\Omega^-) + W^1_q (\Omega^-) \). Since \( u_- \in I_q (\Omega^-) \) and \( v_- \in \tilde{I}_q (\Omega^-) \), we have \((\nabla p_2, v_-)_{\Omega^-} = (u_-, \nabla p_2^2)_{\Omega^-} = 0\), so that we have (47).

**Proof of Lemma 3.** Let \((u_+, u_-)\) satisfy (41) with \((0, 0, 0, 0)\), that is let \((u_+, u_-)\) satisfy the homogeneous equation. In particular, \(u_- \in I_q (\Omega^-)\). Let \(f_+\) and \(f_-\) be any vectors of functions in \( C^0 (\Omega_+) \). We define \( \psi \) by \( \psi = K(f_-) \in \tilde{W}^{1, q}_{q, \beta} (\Omega_-) \), and then, \( f_- - \nabla \psi \in I_q (\Omega_-) \). Let \((v_-, v_-)\) be a solution to (41) with \((f_+, f_- - \nabla \psi, 0, 0)\). Since \( f_- - \nabla \psi \in I_q (\Omega_-), v_- \in \tilde{I}_q (\Omega_-) \), so that by (47) and the fact that \((u_-, \nabla \psi)_{\Omega^-} = 0\), we have:

\[
0 = (\gamma_0^+, u_+, f_+)_{\Omega^+} + (\rho_0^-, u_-, f_-)_{\Omega^-}.
\]

Since \( f_+ \) are chosen arbitrarily, we have \( u_+ = 0 \), which completes the proof of Lemma 3. \( \square \)

### 3. Model Problems

In this section, we consider a model problem for the incompressible-compressible viscous fluid in \( \mathbb{R}^N \). In what follows, we set:

\[
\mathbb{R}^N_+ = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0 \}, \quad \mathbb{R}^N_0 = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0 \},
\]

and \( n_0 = (0, \ldots, 0, 1) \). Before stating the main results of this section, we notice that the following two variational problems are uniquely solvable:

\[
p_0^{-1} (\nabla v, \nabla \varphi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in \tilde{W}^{1, q}_{q, \beta} (\mathbb{R}^N), \tag{48}
\]

\[
\lambda (w, \nabla \varphi) + p_0^{-1} (\nabla w, \nabla \varphi)_{\mathbb{R}^N} = (g, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in \tilde{W}^{1, q}_{q, \beta} (\mathbb{R}^N) \tag{49}
\]

subject to \( w |_{\mathbb{R}^N_0} = g \). More precisely, let \( 1 < q < \infty \). As is well known, for any \( f \in L_q (\mathbb{R}^N)^N \), Problem (48) admits a unique solution \( \varphi \in \tilde{W}^{1, q}_{q, \beta} (\mathbb{R}^N) \) possessing the estimate: \( \| \nabla \varphi \|_{L^q (\mathbb{R}^N)} \leq C \| f \|_{L^q (\mathbb{R}^N)} \). We define an operator \( P \) acting on \( f \) by setting \( v = Pf \).

Moreover, for any \( g \in L_q (\mathbb{R}^N)^N, \ g \in H^1_q (\mathbb{R}^N), \) and \( \lambda \in \Sigma_{\gamma} \). Problem (49) admits a unique solution \( w \in \tilde{W}^{1, q}_{q, \beta} (\mathbb{R}^N) \) possessing the estimate: \( |\lambda|^{1/2} \| \nabla w \|_{L^q (\mathbb{R}^N)} + \| \nabla w \|_{L^q (\mathbb{R}^N)} \leq C \| g \|_{L^q (\mathbb{R}^N)} \), where \( C \) is independent of \( \lambda \). This assertion is also known (cf. [13]). In particular, we have \( w = -\text{div } \lambda^{-1} \left( g - p_0^{-1} \nabla w \right) \).

In this section, assuming that \( \gamma_0 + \gamma_3 + \gamma_3 + \gamma_0^{-1} \) are positive constants such that:

\[
p_0 + 2 \leq \gamma_0 + \gamma_3 + \gamma_3 + \gamma_0^{-1} \leq 2 \rho_0^+, \quad 0 \leq \gamma_3 + \gamma_3 + (p_2 + 2)^2,
\]

we consider the following interface problem in \( \mathbb{R}^N \):

\[
\lambda u_+ - \gamma_0^{-1} \text{Div } T_+ (u_+) = g_+ \quad \text{in } \mathbb{R}^N_+, \quad \lambda u_- - \rho_0^{-1} \text{Div } T_- (u_-, K_0^2 (u_+, u_-)) = g_- \quad \text{in } \mathbb{R}^N_-, \tag{50}
\]

\[
T_+ (u_+) n_0 |_{x_N = 0^+} - T_- (u_-, K_0^2 (u_+, u_-)) n_0 |_{x_N = 0^-} = h |_{x_N = 0}, \quad u_+ |_{x_N = 0^+} = u_- |_{x_N = 0^-}.
\]

Here, \( g_+ \in L_q (\mathbb{R}^N_+) \) and \( h \in W^1_q (\mathbb{R}^N) \) are prescribed functions, and for notational simplicity, we set:

\[
T_+ (u_+) = S_+ (u_+) + \delta \gamma_3 + \text{div } u_+ I, \quad T_- (u_-, p) = S_- (u_-) - p I. \tag{51}
\]

Moreover, \( v = K_0^2 (u_+, u_-) \) is a unique solution to the variational problem:

\[
(\nabla v, \nabla \varphi)_{\mathbb{R}^N} = (\text{Div } S_- (u_-) - \nabla \text{div } u_-) \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in \tilde{W}^{1, q}_{q, \beta} (\mathbb{R}^N) \tag{52}
\]
subject to \( v = g_1 \) on \( \mathbb{R}^N_0 = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0 \} \) with:

\[
g_1 = \langle S_-(u_-)n_0, n_0 \rangle_{|x_N=0} - \nabla u_--T_+(u_+)n_0, n_0 \rangle_{|x_N=0+}.
\]

We prove the following theorem.

**Theorem 9.** Let \( 1 < q < \infty, 0 < \epsilon < \pi/2, \lambda_0 > 0 \). Let \( X_3^q(\mathbb{R}^N) \) and \( \Lambda_3^q(\mathbb{R}^N) \) be the sets defined by:

\[
X_3^q(\mathbb{R}^N) = \{ G^3 = (g_+, g_-, h) \mid g_\pm \in L_q(\mathbb{R}^N_{\pm}), \ h \in W^1_q(\mathbb{R}^N) \}.
\]

\[
\Lambda_3^q(\mathbb{R}^N) = \{ F^3 = (F_1, F_2, F_3, F_4) \mid \ h \in L_q(\mathbb{R}^N_{\pm}), \ F_2 \in L_q(\mathbb{R}^N_{+}), \ F_3 \in L_q(\mathbb{R}^N_{+}), \ F_4 \in L_q(\mathbb{R}^N_{+}) \}.
\]

Then, there exist operator families \( L_+^q(\lambda) \in \text{Hol}(\Gamma_{e_\lambda}, \mathcal{L}(X_3^q(\mathbb{R}^N), W^2_q(\mathbb{R}^N_{\pm}))) \) such that for any \( \lambda \in \Gamma_{e_\lambda} \) and \( G^3 = (g_+, g_-, h) \in X_3^q(\mathbb{R}^N), \ u_\pm = E_\pm(\lambda)F_3^qG^3 \) is a unique solution to (50), and:

\[
R_{\mathcal{L}(X_3^q(\mathbb{R}^N), W^2_q(\mathbb{R}^N_{\pm}))}(\{ (\tau \partial \tau)^{\ell}(\lambda^{1/2}E_{\pm}(\lambda)) \mid \lambda \in \Gamma_{e_\lambda} \} \leq r_b
\]

for \( \ell = 0, 1 \) and \( j = 0, 1, 2 \) with some constant \( r_b \) depending on \( e, q, \lambda_0, \delta_0, \mu_\pm, \nu_\pm, \rho_\pm, \rho_\pm, \) and \( N \). Here, we set \( \lambda F_3^qG^3 = (g_+, g_-, \lambda^{1/2}h, \nabla h) \).

**Remark 11.** We set:

\[
\| G^3 \|_{X_3^q(\mathbb{R}^N)} = \| g_+ \|_{L_q(\mathbb{R}^N_+)} + \| g_- \|_{L_q(\mathbb{R}^N_-)} + \| h \|_{W^1_q(\mathbb{R}^N)},
\]

\[
\| F^3 \|_{X_3^q(\mathbb{R}^N)} = \| F_1 \|_{L_q(\mathbb{R}^N_+)} + \| F_2 \|_{L_q(\mathbb{R}^N_-)} + \| (F_3, F_4) \|_{L_q(\mathbb{R}^N)}. \tag{53}
\]

According to Assertion 1 in Section 2.2, we consider the following system of equations:

\[
\lambda v_+ - \gamma_0^{-1} \text{Div} T_+(v_+) = g_+ \quad \text{in} \ \mathbb{R}^N_+,
\]

\[
\lambda v_- - \mu_0^{-1} \text{Div} T_-(v_-, h) = g_- \quad \text{in} \ \mathbb{R}^N_-,
\]

\[
div v_- = f_- = \text{div} f_- \quad \text{in} \ \mathbb{R}^N_-.
\]

\[
T_+(v_+)n_0|_{x_N=0} - T_-(v_-, h)n_0|_{x_N=0} = h,
\]

\[
v_+|_{x_N=0} = v_-|_{x_N=0}.
\]

Then, Theorem 9 follows from the following theorem, because (49) is uniquely solvable.

**Theorem 10.** Let \( 1 < q < \infty, 0 < \epsilon < \pi/2, \lambda_0 > 0 \). Let \( Y^0_q(\mathbb{R}^N) \) and \( Y^0_q(\mathbb{R}^N) \) be the sets defined by:

\[
Y^0_q(\mathbb{R}^N) = \{ G^0 = (g_+, g_-, h, f_-, f_-) \mid g_\pm \in L_q(\mathbb{R}^N_{\pm}), \ h \in W^1_q(\mathbb{R}^N), \ f_- \in W^1_q(\mathbb{R}^N), \ f_- \in L_q(\mathbb{R}^N_{\pm}), \ f_- = \text{div} f_- \},
\]

\[
Y^0_q(\mathbb{R}^N) = \{ F^0 = (F_1, F_2, F_3, F_4, F_7, F_8, F_9) \mid F_1 \in L_q(\mathbb{R}^N_{\pm}), \ F_2 \in L_q(\mathbb{R}^N_{\pm}), \ F_3 \in L_q(\mathbb{R}^N_{\pm}), \ F_4 \in W^1_q(\mathbb{R}^N), \ F_7 \in L_q(\mathbb{R}^N), \ F_9 \in W^1_q(\mathbb{R}^N) \}.
\]

Then, there exist operator families \( A^1_\pm(\lambda) \) and \( B^1_\pm(\lambda) \) with:

\[
A^1_\pm(\lambda) \in \text{Hol}(\Gamma_{e_\lambda}, \mathcal{L}(Y^0_q(\mathbb{R}^N), W^2_q(\mathbb{R}^N_{\pm}))), \quad B^1_\pm(\lambda) \in \text{Hol}(\Gamma_{e_\lambda}, \mathcal{L}(Y^0_q(\mathbb{R}^N), W^1_q(\mathbb{R}^N) + W^1_q(\mathbb{R}^N))).
\]
such that for any \( \lambda \in \Gamma_{e,\lambda_0} \) and \( G^0 = (g_+, g_-, h, f_-) \in \hat{Y}_0^0(\mathbb{R}^N) \), \( v_\pm = A_\pm^1(\lambda) F_\lambda^\pm G^0 \) and \( p_- = B^0(\lambda) F_\lambda^0 G^0 \) are unique solutions to (54), and:

\[
\mathcal{R}_{L(\lambda)}(\hat{Y}_0^0, \mathcal{W}_{2-1}^0) \bigl( \{(\partial_\tau)^f(\lambda)^{1/2} A^1_\pm(\lambda) \mid \lambda \in \Gamma_{e,\lambda_0} \} \bigr) \leq r_0,
\]

\[
\mathcal{R}_{L(\lambda)}(\hat{Y}_0^0, \mathcal{L}_d^0) \bigl( \{(\partial_\tau)^f(\lambda) B^1_\pm(\lambda) \mid \lambda \in \Gamma_{e,\lambda_0} \} \bigr) \leq r_0
\]

for \( \ell = 0, 1 \) and \( j = 0, 1, 2 \) with some constant \( r_0 \) depending on \( \delta, \rho_\pm, \varepsilon, \lambda_0, \) and \( q \). Here, we set \( F_\lambda^0 G^0 = (g_+, g_-, \lambda^{1/2} h, h, \lambda^{1/2} f_- f_- f_-, f_- f_- f_-) \).

**Remark 12.** We set:

\[
\|G^0\|_{\hat{X}_q(\mathbb{R}^N)} = \|g_+\|_{\mathcal{L}_d(\mathbb{R}^N)} + \|g_-\|_{\mathcal{L}_d(\mathbb{R}^N)} + \|h\|_{\mathcal{W}_q(\mathbb{R}^N)} + \|f_-\|_{\mathcal{W}_q(\mathbb{R}^N)} + \|f_+\|_{\mathcal{L}_d(\mathbb{R}^N)};
\]

\[
\|F_0\|_{\hat{X}_q(\mathbb{R}^N)} = \|F_1\|_{\mathcal{L}_d(\mathbb{R}^N)} + \|F_2, F_0\|_{\mathcal{L}_d(\mathbb{R}^N)} + \|F_3\|_{\mathcal{L}_d(\mathbb{R}^N)} + \|F_4\|_{\mathcal{W}_q(\mathbb{R}^N)} \quad (55)
\]

To prove Theorem 10, we first reduce the problem to the case where \( f_- = \text{div} f_- = 0 \) and \( g_+ = 0 \). Concerning the incompressible part, we consider the following equations:

\[
\lambda w_- - \rho_0^{-1} \text{Div} (\mathbf{S}_- (w_-) - p_- I) = g_-, \quad \text{in } \mathbb{R}^N,
\]

\[
\text{div } w_- = f_- = \text{div } f_-, \quad \text{in } \mathbb{R}^N,
\]

\[
p_-|_{\mathbb{R}^N} = 0, \quad w_0|_{\mathbb{R}^N} = 0, \quad \mathbf{D}_N w|_{\mathbb{R}^N} = f_-,
\]

where \( w_- = \sum_{j=1}^{n-1} w_j \). We start with proving that for any \( w_- \in \mathcal{W}_{q,0}^1(\mathbb{R}^N)^N \) and \( \varphi \in \hat{W}_{q,0}^1(\mathbb{R}^N)^N \):

\[
(\Delta w_-, \nabla \varphi)_{\mathbb{R}^N} = (\nabla \text{div } w_-, \nabla \varphi)_{\mathbb{R}^N}.
\]

Since \( C_0^\infty(\mathbb{R}^N) \) is not dense in \( \hat{W}_{q,0}^1(\mathbb{R}^N) \) in general (cf. Shibata [6]), we give a proof below. To prove (57), we use an inequality:

\[
\|x_N^{-1} \varphi\|_{\mathcal{L}_d(\mathbb{R}^N)} \leq C\|\nabla \varphi\|_{\mathcal{L}_q(\mathbb{R}^N)} \quad (58)
\]

for any \( \varphi \in \hat{W}_{q,0}^1(\mathbb{R}^N) \) and \( 1 < q < \infty \). In fact, representing \( \varphi(x', x_N) = -\int_{-x_N}^{0} (\mathbf{D}_N \varphi)(x', s) \, ds \) with \( x' = (x_1, \ldots, x_{n-1}) \) and using the Hardy inequality, we have:

\[
\|x_N^{-1} \varphi\|^q_{\mathcal{L}_d(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \int_{0}^{\infty} \chi_N^{-1} \int_{0}^{x_N} |\varphi(x', s)| \, ds \, dx_N \, dx' \leq C_q \int_{\mathbb{R}^N} \int_{0}^{\infty} \|\mathbf{D}_N \varphi(x', s)| \, ds \, dx',
\]

which yields (58). To prove (57), we take \( \psi(x_N) \in C_0^\infty(\mathbb{R}) \), which equals one for \( |x_N| < 1 \) and zero for \( |x_N| > 2 \), and set \( \psi_R(x_N) = \psi(x_N/R) \). For any \( v \in \mathcal{L}_d(\mathbb{R}^N)^N \) and \( \varphi \in \hat{W}_{q,0}^1(\mathbb{R}^N) \),

\[
(v, \nabla \varphi)_{\mathbb{R}^N} = \lim_{R \to \infty} (v, \nabla (\psi_R \varphi))_{\mathbb{R}^N}.
\]

In fact, by (58):

\[
|(v, (\mathbf{D}_N \varphi)(x_N)_{\mathbb{R}^N})| \leq C\|v\|_{\mathcal{L}_d(\mathbb{R}^N) \times [-2 R \leq x_N \leq - R]} \|\nabla \varphi\|_{\mathcal{L}_q(\mathbb{R}^N)} \to 0
\]
as $R \to \infty$, which yields (59). We now prove (57). Notice that $\psi R \varphi \in W^1_{\varphi, 0}(\mathbb{R}^N)$. Since $C^0_{ad}(\mathbb{R}^N)$ is dense in $W^1_{\varphi, 0}(\mathbb{R}^N)$, we take a sequence $\{\omega_j\}_{j=1}^\infty$ of $C^0_{ad}(\mathbb{R}^N)$ such that $\|\omega_j - \psi R \varphi\|_{W^1_{\varphi, 0}(\mathbb{R}^N)} \to 0$ as $j \to \infty$. Then, by (59),

$$(\Delta w_-, \nabla \varphi)_{\mathbb{R}^N} = \lim_{k \to \infty} \lim_{j \to \infty} (\Delta w_-, \nabla \omega_j)_{\mathbb{R}^N}$$

$$= \lim_{k \to \infty} \lim_{j \to \infty} (\nabla \text{div} w_-, \nabla \omega_j)_{\mathbb{R}^N} = (\nabla \text{div} w_-, \nabla \varphi)_{\mathbb{R}^N},$$

which shows (57).

We now consider equations:

$$\lambda w_- - \rho_0^{-1} \text{div} (S_- (w_-) - p_- I) = g_-, \quad \text{div} w_- = f_- = \text{div} f_- \quad \text{in} \, \mathbb{R}^N,$$

$$w_j|_{\mathbb{R}^N_0} = 0, \quad D_N w_N|_{\mathbb{R}^N_0} = f_-$$

for $j = 1, \ldots, N - 1$, where $w_- = \tau (w_1, \ldots, w_N)$. Noticing that $\text{Div} S_- (w_-) = \Delta w_- + \nabla \text{div} w_-$ and using (57), for any $\varphi \in W^1_{\varphi, 0}(\mathbb{R}^N)$, we have:

$$(g_-, \nabla \varphi)_{\mathbb{R}^N} = (\lambda (w_-, \nabla \varphi)_{\mathbb{R}^N} - 2 \rho_0^{-1} (\nabla \text{div} w_-, \nabla \varphi)_{\mathbb{R}^N} + \rho_0^{-1} (\nabla p_-, \nabla \varphi)_{\mathbb{R}^N}$$

$$= (\lambda (f_-, \nabla \varphi)_{\mathbb{R}^N} - 2 \rho_0^{-1} (\nabla f_-, \nabla \varphi)_{\mathbb{R}^N} + \rho_0^{-1} (\nabla p_-, \nabla \varphi)_{\mathbb{R}^N}.$$

Hence, we have $p_- = \rho_0^{-1} \text{P} (g_- - \lambda f_- + 2 \rho_0^{-1} \nabla f_-)$, and so, the first equation in (60) is reduced to equations:

$$\lambda w_- - \rho_0^{-1} \Delta w_- = g_- - \nabla \text{P} (g_- - \lambda f_- + 2 \rho_0^{-1} \nabla f_-) + \rho_0^{-1} \nabla f_- \quad \text{in} \, \mathbb{R}^N,$$

$$\text{div} w_- = f_- = \text{div} f_- \quad \text{in} \, \mathbb{R}^N,$$

$$w_j|_{\mathbb{R}^N_0} = 0, \quad D_N w_N|_{\mathbb{R}^N_0} = f_-$$

for $j = 1, \ldots, N - 1$. The first equations and third equations in (61) become the following equations:

$$\lambda w_j - \rho_0^{-1} \Delta w_j = g_j, \quad w_j|_{\mathbb{R}^N_0} = 0$$

(62)

$$\lambda w_N - \rho_0^{-1} \Delta w_N = g_N \quad \text{in} \, \mathbb{R}^N, \quad D_N w_N|_{\mathbb{R}^N_0} = f_-$$

(63)

where $g_j$ denotes the $j$th component of $N$-vector $\hat{g} := g_- - \nabla \text{P} (g_- - \lambda f_- + 2 \rho_0^{-1} \nabla f_-) + \rho_0^{-1} \nabla f_-$. We use the following theorem, which was proven in [13].

**Proposition 1.** Let $1 < q < \infty$, $0 < e < \pi/2$ and $\lambda_0 > 0$. Then, the following two assertions hold: (1) There exists an operator family $S_q(\lambda) \in \text{Hol} (L_{\varphi}, L_q(\mathbb{R}^N), W^2_{\varphi}(\mathbb{R}^N))$ such that for any $\lambda \in \Sigma_{e, \lambda_0}$ and $g_j \in L_q(\mathbb{R}^N)$, $w_j = S_q(\lambda) g_j$ ($j = 1, \ldots, N - 1$) are unique solutions of Equation (62), and:

$$R_{L_q(\mathbb{R}^N), W^2_{\varphi}(\mathbb{R}^N)} \{((\tau \partial_\tau)^l (\lambda^{1/2} S_q(\lambda)) \mid \lambda \in \Sigma_{e, \lambda_0})\} \leq r_0$$

for $l = 0, 1$, and $j = 0, 1, 2$ with some constant $r_0$ depending on $\lambda_0$.

(2) Let

$$Y^1_q(\mathbb{R}^N) = \{(g_- - f_-) \mid g \in L_q(\mathbb{R}^N)^N, \quad f_- \in W^1_q(\mathbb{R}^N)\},$$

$$Y^1_q(\mathbb{R}^N) = \{(F_2, F_7, F_8) \mid F_2 \in L_q(\mathbb{R}^N)^N, F_7 \in L_q(N^N), \quad F_8 \in W^1_q(\mathbb{R}^N)\}. $$
Then, there exists an operator family $S_\lambda(\lambda) \in \text{Hol}(\Sigma_{c,\lambda_0}, \mathcal{L}(Y_q^1(\mathbb{R}^N), W_\xi^2(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{c,\lambda_0}$ and $(\hat{g}_N, f_-) \in Y_q^1(\mathbb{R}^N)$, $w_N = S_\lambda(\lambda)(\hat{g}_N, \lambda^{1/2} f_-, f_-)$ is a unique solution of Equation (63), and:

$$\mathcal{R}_\mathcal{L}(Y_q^1(\mathbb{R}^N), W_\xi^2(\mathbb{R}^N))\left(\{(\tau \partial_\tau)^f(\lambda^{1/2} S_\lambda(\lambda)) \mid \lambda \in \Sigma_{c,\lambda_0}\}\right) \leq r_b$$

for $\ell = 0, 1$, and $j = 0, 1, 2$ with some constant $r_b$ depending on $\lambda_0$.

Finally, we prove that $\text{div } w_-= f_-$ with $w_-= T(w_1, \ldots, w_N)$. In fact, for any $\varphi \in \bar{W}_q^{1,0}(\mathbb{R}^N)$, by (57), (62), and (63) we have:

$$(g_- - \nabla \mathcal{P}(g_- - \lambda f_- + 2\rho_0^{-1}\nabla f_-) + \rho_0^{-1}\nabla f_-, \nabla \varphi)_{R^N} = (\lambda w_- - \rho_0^{-1}\Delta_w, \nabla \varphi)_{R^N}$$

which yields that:

$$\lambda(w_-, \nabla \varphi)_{R^N} - \rho_0^{-1}(\nabla (\text{div } w_-, f_-), \nabla \varphi)_{R^N} = 0$$

for any $\varphi \in \bar{W}_q^{1,0}(\mathbb{R}^N)$. By the divergence theorem of Gauss and the assumption that $f_- = \text{div } f_-$, we have:

$$\lambda(\text{div } w_-, f_-)_{R^N} + \rho_0^{-1}(\nabla (\text{div } w_-, f_-), \nabla \varphi)_{R^N} = 0$$

for any $\varphi \in W_0^1(\mathbb{R}^N)$. Since $\text{div } w_-, \varphi = D_N w_N \mid_{x_N=0} = f_-$, therefore the uniqueness yields that $\text{div } w_-= f_-$. Thus, by (64), we have $w_- - f_- \in I_q(\mathbb{R}^N)$, which shows that $w_-$ and $\rho$ satisfy (56).

Summing up, we proved the following proposition.

**Proposition 2.** Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\lambda_0 > 0$. Let:

$$Y_q^1(\mathbb{R}^N) = \{(g_-, f_-, f_-) \mid g_-, f_- \in L_q(\mathbb{R}^N), f_- = \text{div } f_-, \}$$

$$Y_q^1(\mathbb{R}^N) = \{(F_2, F_7, F_8, F_9) \mid F_2 \in L_q(\mathbb{R}^N), F_7 \in L_q(\mathbb{R}^N), F_8 \in W_\xi^2(\mathbb{R}^N)\}$$

Then, there exists an operator family $T^1_{\lambda}(\lambda) \in \text{Hol}(\Sigma_{c,\lambda_0}, \mathcal{L}(Y_q^1(\mathbb{R}^N), W_\xi^2(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{c,\lambda_0}$ and $(\hat{g}_N, f_-) \in Y_q^1(\mathbb{R}^N)$, $w_- = T^1_{\lambda}(\lambda)(g_-, \lambda^{1/2} f_-, f_-)$ is a unique solution of Equation (56), and:

$$\mathcal{R}_\mathcal{L}(Y_q^1(\mathbb{R}^N), W_\xi^2(\mathbb{R}^N))\left(\{(\tau \partial_\tau)^f(\lambda^{1/2} T^1_{\lambda}(\lambda)) \mid \lambda \in \Sigma_{c,\lambda_0}\}\right) \leq r_b$$

for $\ell = 0, 1$, and $j = 0, 1, 2$ with some constant $r_b$ depending on $\lambda_0$.

Concerning the compressible part, we consider the equations:

$$\lambda w_+ - \gamma_0^{-1}\text{Div } T_+(w_+) = g_+ \quad \text{in } \mathbb{R}^N_+, \quad \gamma_0^{-1}\text{Div } T_+(w_+)|_{\partial \mathbb{R}^N} = 0.$$  

(65)

We know the following theorem, which was proven by Götz and Shibata [14].

**Proposition 3.** Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, $\delta_0 > 0$, and $\lambda_0 > 0$. Then, there exists an operator family $T^1_{\lambda}(\lambda) \in \text{Hol}(\Gamma_{c,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N), W_\xi^2(\mathbb{R}^N)))$ such that for any $g_+$ and $\lambda \in \Gamma_{c,\lambda_0}$, $w_+ = T^1_{\lambda}(\lambda) g_+$ is a unique solution to Problem (65), and:

$$\mathcal{R}_\mathcal{L}(Y_q^1(\mathbb{R}^N), W_\xi^2(\mathbb{R}^N))\left(\{(\tau \partial_\tau)^f(\lambda^{1/2} T^1_{\lambda}(\lambda)) \mid \lambda \in \Gamma_{c,\lambda_0}\}\right) \leq r_b$$

for $\ell = 0, 1$, and $j = 0, 1, 2$ with some constant $r_b$ depending on $\lambda_0$. 


for $\ell = 0, 1, j = 0, 1, 2$ with some constant $r_0$ depending on $\epsilon, \lambda_0, \rho, \chi, \sigma, q,$ and $N$.

We now set $v_\pm = w_\pm + u_\pm$ and $q_- = p_- + \theta$ in Equation (54), and then, the equations for $u_\pm$ and $\theta$ are the following:

$$
\lambda u_+ - \gamma_0^{-1} \text{Div} T_+(u_+) = 0 \quad \text{in } \mathbb{R}_+^N, \\
\lambda u_- - \rho_0^{-1} \text{Div} T_-(u_- + \theta) = 0, \quad \text{div } u_- = 0 \quad \text{in } \mathbb{R}_-^N,
$$

(66)

Concerning Equation (66), we know the following theorem, which was proven by Kubo, Shibata, and Soga [15].

**Theorem 11.** Let $1 < q < \infty$, $0 < \epsilon < \pi / 2$, and $\lambda_0 > 0$. Let:

$$
Y^0_\lambda(\mathbb{R}^N) = \{(h, k) \mid h \in W_0^1(\mathbb{R}^N), \ k \in W_0^1(\mathbb{R}^N) \},
$$

$$
Y^1_\lambda(\mathbb{R}^N) = \{(F_3, F_4) \mid F_3, F_4 \in L_q(\mathbb{R}_+^N), \ F_3, F_4 \in W_0^1(\mathbb{R}_+^N) \}.
$$

Then, there exist operator families $T^1_\pm(\lambda)$ and $Q_- (\lambda)$ with:

$$
T^1_\pm(\lambda) \in \text{Hol}(\Gamma_{p,\phi}, \mathcal{L}(Y^0_\lambda(\mathbb{R}^N), W^1_q(\mathbb{R}_+^N))), \quad Q^0_\lambda (\lambda) \in \text{Hol}(\Gamma_{p,\phi}, \mathcal{L}(Y^0_\lambda(\mathbb{R}^N), W^1_q(\mathbb{R}_+^N) + W^1_q(\mathbb{R}_-^N))),
$$

such that for any $\lambda \in \Gamma_{p,\phi}$ and $G^3 = (h, k) \in Y^0_\lambda(\mathbb{R}^N)$, $u_\pm = T^1_\pm(\lambda) G^3$ and $\theta = Q_-(\lambda) F^3_\lambda G^3$ are unique solutions to (66), where $F^3_\lambda G^3 = (\lambda^{1/2} h, \lambda^{1/2} k, \lambda^{1/2} k, k)$,

$$
\mathcal{R} \mathcal{E}(Y^0_\lambda(\mathbb{R}^N), W^1_q(\mathbb{R}_+^N)) \{ (\tau \partial_\gamma)^{\ell} (\lambda^{1/2} T^1_\pm(\lambda)) \mid \lambda \in \Gamma_{p,\phi} \} \leq r_0,
$$

$$
\mathcal{R} \mathcal{E}(Y^0_\lambda(\mathbb{R}^N), L_q(\mathbb{R}_+^N)) \{ (\tau \partial_\gamma)^{\ell} (\nabla Q_- (\lambda)) \mid \lambda \in \Gamma_{p,\phi} \} \leq r_0
$$

for $\ell = 0, 1$ and $j = 0, 1, 2$ with some constant $r_0$ depending on $\epsilon, \lambda_0, \rho, \chi, \sigma, q,$ and $N$.

**Remark 13.** $F_3, F_4, F_{10}, F_{11},$ and $F_{12}$ are the corresponding variables to $\lambda^{1/2} h$, $\lambda^{1/2} k$, and $k$. We set:

$$
\| (F_3, F_4, F_{10}, F_{11}, F_{12}) \| Y^0_\lambda(\mathbb{R}^N) = \| (F_3, F_{10}) \| L_q(\mathbb{R}^N) + \| (F_4, F_{11}) \| W^1_q(\mathbb{R}^N) + \| F_{12} \| W^1_q(\mathbb{R}^N).
$$

Combining Proposition 2 and Proposition 3 with Lemma 1 and Lemma 2, we have Theorem 10. This completes the proof of Theorem 9.

### 4. Several Problems in Bent Spaces

Let $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a bijection of the $C^1$ class, and let $\Phi^{-1}$ be its inverse map. Writing $\nabla \Phi = A + B(x)$ and $\nabla \Phi^{-1} = A_- + B_-(x)$, we assume that $A$ and $A_-$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_-(x)$ are matrices of functions in $W^1_{1,0}(\mathbb{R}^N)$ with $N < r < \infty$ such that:

$$
\| (B, B_-) \| L_{\infty}(\mathbb{R}^N) \leq M_1, \quad \| \nabla (B, B_-) \| L_{r}(\mathbb{R}^N) \leq M_2.
$$

(67)

We will choose $M_1$ small enough eventually, and so we may assume that $0 < M_1 < 1 \leq M_2$. We set $D_+ = \Phi(\mathbb{R}^N)$ and $S = \Phi(\mathbb{R}_-^N)$, and we denote the unit outward normal to $S$ pointing from $D_- \to D_+$ by $n_\pm$. Since $S$ is represented by $\Phi_{-1,N}(y) = 0$ with $\Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})$, we have:

$$
n_\pm = \frac{\nabla \Phi_{-1,N}}{\| \nabla \Phi_{-1,N} \|} = \frac{(A_{N1} + B_{N1}, \ldots, A_{NN} + B_{NN})}{(\sum_{i=1}^N (A_{Ni} + B_{Ni})^2)^{1/2}},
$$

(68)
where we set $A_{-1} = (A_{ij})$ and $B_{-1} = (B_{ij})$. Notice that $n_+$ is defined on the whole $\mathbb{R}^N$. By (67) with small $M_1$,  
$$\left\{ \sum_{i=1}^{N} (A_{Ni} + B_{Ni})^2 \right\}^{-1/2} = 1 + b_0$$  
with $b_0 \in W_{r,\text{loc}}^1(\mathbb{R}^N)$ possessing the estimate: $\|b_0\|_{L^r(\mathbb{R}^N)} \leq C_N M_1$ and $\|\nabla b_0\|_{L^r(\mathbb{R}^N)} \leq C M_2$. Let $\gamma_{0+}(x)$ and $\gamma_{3+}(x)$ be real-valued functions defined on $\mathbb{R}^N$ satisfying the following conditions:  
$$\rho_{0+}/2 \leq \gamma_{0+} \leq 2\rho_{0+}, \quad 0 \leq \gamma_{3+} \leq (\rho_{2+})^2 \quad (x \in D_+),$$  
$$\|\gamma_{\ell+} - \hat{\gamma}_{\ell+}\|_{L^r(D_+)} \leq M_1, \quad \|\nabla \gamma_{\ell+}\|_{L^r(D_+)} \leq C M_2$$  
for $\ell = 0$ and 3, where $\hat{\gamma}_{\ell+} (\ell = 0, 3)$ are some constants with $\rho_{0+}/2 < \hat{\gamma}_{0+} < 2\rho_{0+}$ and $0 \leq \hat{\gamma}_{3+} \leq (\rho_{2+})^2$.

First, we consider the following problem:  
$$\lambda u_+ - \gamma_{0+}^{-1} \text{Div} T_+ (u_+) = g_+ \quad \text{in } D_+,$$  
$$\lambda u_- - \rho_{0-}^{-1} \text{Div} T_- (u_-, K_+(u_+, u_-)) = g_- \quad \text{in } D_-,$$  
$$T_+(u_+) n_+ |_{S_0 - 0} - T_- (u_-, K_+(u_+, u_-)) n_+ |_{S_0 - 0} = h |_{S_0}, \quad u_+ |_{S_0 + 0} = u_- |_{S_0 - 0}.$$  
Moreover, $v = K_+(u_+, u_-)$ is a solution to the weak Dirichlet problem:  
$$(\nabla v, \nabla \varphi)_{D_-} = (\text{Div } S_- (u_-) - \nabla \text{Div } u_-, \nabla \varphi)_{D_-}$$  
for any $\varphi \in W^1_{q,0}(D_-)$ (72)

subject to $v = \langle S_- (u_-) n_+, n_+ > |_{S_0 - 0} - \text{div } u_- |_{S_0 - 0} - \langle S_+ (u_+) + \delta \gamma_{3+} \text{div } u_+, 1)n_+, n_+ > |_{S_0 + 0}$.

We have the following theorem.

**Theorem 12.** Let $1 < q < \infty$, $0 < e < \pi/2$, and $r \geq \max (q, q')$. Let $X^3(D)$ and $X^3(D)$ be sets defined by replacing $\mathbb{R}^N$ and $\mathbb{R}^N_D$ by $D$ and $D_\pm$, respectively, in Theorem 9. Then, there exist constants $M_1 \in (0, 1)$, $\lambda_0 = \lambda_{M_2} \geq 1$ and operator families $E^1_\lambda (\lambda) \in \text{Hol}(\Gamma_{e_0}, L(X^3_0(D), W^2_{q}(D_\pm)))$ such that for any $\lambda \in \Gamma_{e_0 A_0}$ and $G^3 = (g_+, g_-, h) \in X^3_0(D)$, $u_\pm = E^1_\lambda (\lambda) F^3 G_3$ is a unique solution to (71), and:

$$\mathcal{R}_{L_\lambda (X^3_0(D), W^2_{q}(-D_\pm))} \left( \{ (\tau \partial_1)^{(\lambda/2) E^1_\lambda (\lambda)} | \lambda \in \Gamma_{e_0 A_0} \} \right) \leq C M_2 \quad (\ell = 0, 1, j = 0, 1, 2).$$

**Remark 14.** Here and in the following, $M_1$ depends on $e, q, \mu_\pm, \nu_+, \rho_{0\pm}, \rho_{2+}$, but is independent of $M_2$. In addition, constants denoted by $\lambda_{M_2}$ and $C_{M_2}$ depend on $M_2, e, q, \mu_\pm, \nu_+, \rho_{0\pm}, \rho_{2+}$, and $N$, but we mention only dependence on $M_2$.

**Proof.** The idea of the proof here follows Shibata [16] and von Below, Enomoto, and Shibata [17]. Using the change of variable: $x = \Phi^{-1}(y)$ with $y \in D$ and $x \in \mathbb{R}^N$ and the change of unknown functions: $v_\pm = A_{-1} u_\pm \circ \Phi$, writing $\gamma_{0-}^{-1} = \gamma_{0-}^{-1} + (\gamma_{0-}^{-1} - \gamma_{0+}^{-1})$ and $\gamma_{3+} = \hat{\gamma}_{3+} + (\gamma_{3+} - \hat{\gamma}_{3+})$, and setting $p = K_+(u_+, u_-)$, we see that Problem (71) is transferred to the following equivalent problem:

$$\lambda v_+ - \gamma_{0+}^{-1} |\text{Div } S_+ (v_+) + \delta \gamma_{3+} \text{div } v_+| - F^1_+ (v_+) = G_+ \quad \text{in } \mathbb{R}^N,$$

$$\lambda v_- - \rho_{0-}^{-1} |\text{Div } S_- (v_-) - \nabla p| - F^1_- (v_-) + p^1 \nabla p = G_- \quad \text{in } \mathbb{R}^N$$  
subject to the interface condition: $v_+ |_{S_0 = 0} = v_- |_{S_0 = 0}$ and:

$$\{ S_+ (v_+) + \delta \gamma_{3+} \text{div } v_+ 1)n_0 + F^2_+(v_+) \}|_{S_0 = 0} - \{ S_- (v_-) - p \} n_0 + F^2_-(v_-) \}|_{S_0 = 0} = H.$$  
$p$ satisfies the following variational equation:

$$(\nabla p, \nabla \varphi) + (p^2 \nabla p, \nabla \varphi) = (\text{Div } S_- (v_-) - \nabla \text{Div } v_- + F^1_- (v_-), \nabla \varphi)$$  
for any $\varphi \in W^1_{q,0}(\mathbb{R}^N)$ (74)
subject to:

\[ p|_{x_N=0^-} = \langle \mathbf{S}_-(v_-) \mathbf{n}_0, \mathbf{n}_0 \rangle - \text{div}v_- + \mathcal{F}_+^1(v_-)\|_{x_N=0^-} - \langle \mathbf{S}_+(v_+) + \delta \gamma_3, \text{div}v_+ \mathbf{I} \mathbf{n}_0, \mathbf{n}_0 \rangle + \mathcal{F}_-^3(v_+)\|_{x_N=0^+}. \]

Here, we write \((\cdot, \cdot) = (\cdot, \cdot)_{\mathbb{R}^N}\) for short, and \(G_+ = A_-\mathbf{g}_+ \circ \Phi, H = A_-\mathbf{h} \circ \Phi\), and \(\mathcal{F}_\pm^i(v_\pm)\) are the vector functions of the forms:

\[ \mathcal{F}_\pm^i(v_\pm) = R_\pm^1 \nabla^2 v_\pm + S_\pm \nabla v_\pm, \quad \mathcal{F}_+^i(v_+) = R_+^i \nabla v_+, \quad \mathcal{F}_-^i(v_-) = R_+^i \nabla v_- \quad (75) \]

for \(i = 2, 3\) and \(j = 2, 3, 4\). In view of (67)-(70), we can assume that \(R_\pm^i, S_\pm\), and \(\mathcal{P}_i\) possesses the following estimate:

\[ \|(R_\pm^i, R_\pm^i, \mathcal{P}_i)\|_{L_2(\mathbb{R}^N)} \leq CM_1, \quad \|(\nabla R_\pm^i, \nabla R_\pm^i, \nabla \mathcal{P}_i, S_\pm)\|_{L_2(\mathbb{R}^N)} \leq CM_2 \quad (76) \]

for \(i = 1, 2, 3\), \(j = 1, 2, 3, 4\), and \(k = 1, 2\). Following Shibata ([16] Section 4), we treat the \(\mathbb{R}^N\) side as follows: Let \(\nu = K_0^0(v_+, v_-)\) be a function defined in (52), which satisfies the estimate:

\[ \|\nabla K_0^0(v_+, v_-)\|_{L_4(\mathbb{R}^N)} \leq C(\|\nabla v_+\|_{W_4^1(\mathbb{R}^N)} + \|\nabla v_-\|_{W_4^1(\mathbb{R}^N)}). \quad (77) \]

Setting \(p = K_0^0(v_+, v_-) + p_1\), we see that \(p_1\) satisfies the variational equation:

\[ (\nabla p_1, \nabla \varphi) + (\mathcal{P}_2^2 \nabla p_1, \nabla \varphi) = (\mathcal{P}_2^3(v_-) - \mathcal{P}_2^3(v_+), \nabla \varphi) \quad \text{for any } \varphi \in W_4^1(\mathbb{R}^N) \quad (78) \]

subject to:

\[ p_1|_{x_N=0^-} = \mathcal{F}_+^1(v_-)\|_{x_N=0^-} - \mathcal{F}_+^3(v_+)\|_{x_N=0^+}. \quad (79) \]

Since \(\|\mathcal{P}_2^2\|_{L_2(\mathbb{R}^N)}\) is small enough, we can show the following lemma by the small perturbation from the weak Dirichlet problem in \(\mathbb{R}^N\).

**Lemma 4.** Let \(1 < q < \infty\). Then, there exist a constant \(M_1 \in (0, 1)\) and an operator \(\Psi\) with:

\[ \Psi \in \mathcal{L}(L_q(\mathbb{R}^N)^{2N}, W_4^1(\mathbb{R}^N) + \tilde{W}_4^1(\mathbb{R}^N)) \]

such that for any \(f \in L_q(\mathbb{R}^N)^N\) and \(g \in \tilde{W}_4^1(\mathbb{R}^N), \theta = \Psi(f, \nabla g)\) is a unique solution to the variational problem:

\[ (\nabla \theta, \nabla \varphi) + (\mathcal{P}_2^2 \nabla \theta, \nabla \varphi) = (f, \nabla \varphi) \quad \text{for any } \varphi \in \tilde{W}_4^1(\mathbb{R}^N) \quad (80) \]

subject to \(\theta|_{x_N=0^-} = g|_{x_N=0^-}\).

By Lemma 4, \(p_1 = p_1(v_+, v_-) = \Psi(f, \nabla g)\) with \(f = \mathcal{F}_+^3(v_-) - \mathcal{P}_2^2 \nabla K_0^0(v_+, v_-)\) and \(g = \mathcal{F}_-^3(v_-) - \mathcal{F}_+^3(v_+)\). Inserting \(p = K_0^0(v_+, v_-) + p_1(v_+, v_-)\) into (73), we have:

\[ \lambda v_- - \tilde{\gamma}_0^{-1}[\text{Div} \mathbf{S}_+(v_+) - \delta \gamma_3 \text{div} \nabla v_+] - \mathcal{F}_+^1(v_+) = G_+ \quad \text{in } \mathbb{R}^N, \]

\[ \lambda v_- - p_0^{-1}[\text{Div} \mathbf{S}_-(v_-) - \nabla K_0^0(v_+, v_-)] - \mathcal{F}_+^3(v_-) \]

\[ + p_0^1[\mathcal{P}_1 \nabla K_0^0(v_+, v_-) + (I + \mathcal{P}_1) \nabla p_1(v_+, v_-)] = G_- \quad \text{in } \mathbb{R}^N \quad (81) \]

subject to the interface conditions \(v_+|_{x_N=0^+} = v_-|_{x_N=0^-}\) and:

\[ \{ \langle \mathbf{S}_+(v_+) + \delta \gamma_3 \text{div} v_+ \mathbf{I} \mathbf{n}_0, \mathbf{n}_0 \rangle + \mathcal{F}_+^2(v_+) + \mathcal{F}_+^3(v_+) \mathbf{n}_0 \|_{x_N=0^+} - \langle \mathbf{S}_-(v_-) - (K_0^0(v_+, v_-) + \mathcal{F}_+^1(v_-) \mathbf{I}) \mathbf{n}_0 + \mathcal{F}_-^2(v_-) \|_{x_N=0^-} = \mathbf{H}. \]

Here, we used (78).
To solve (81) for any right members $G^3 = (g_+, g_-, h) \in X^3_N(\mathbb{R}^N)$, we set $v_\pm = E_\pm(\lambda)F^{3}_\pm G^3$ in (81), where $E_\pm(\lambda)$ are operators given in Theorem 9, and then, we have:

\[
\begin{align*}
\lambda v_+ - \gamma^{-1}_3 (\text{Div } s_+(v_+) - \delta_3 \nabla \text{Div } v_+) - \mathcal{F}_+^1(v_+) &= G_+ - F^1_+(\lambda)G^3 \quad \text{in } \mathbb{R}^N_+,
\lambda v_- - \gamma^{-1}_3 (\text{Div } s_-(v_-) - \delta_3 \nabla^0 k_0^3(v_+,v_-)) - \mathcal{F}_-^1(v_-) + \rho_0^{-1} [\mathcal{D}^1 \nabla k_0^3(v_+ ,v_-) + (I + \mathcal{D}^1) \nabla p_1(v_+,v_-)] &= G_- - F^1_-(\lambda)G^3 \quad \text{in } \mathbb{R}^N_-
\end{align*}
\]

subject to the interface conditions $v_+|_{x_N = 0^+} = v_-|_{x_N = 0^-}$ and:

\[
\begin{align*}
\{ (s_+(v_+) + \delta_3 \text{Div } v_+ I)n_0 + \mathcal{F}_+^2(v_+ + \mathcal{F}_+^3(v_+)n_0) \}|_{x_N = 0^+} - \{ (s_-(v_-) - (k_0^3(v_+,v_-) + \mathcal{F}_-^4(v_-)I)n_0 + \mathcal{F}_-^2(v_-)n_0) \}|_{x_N = 0^-} &= H - F^2(\lambda)G^3.
\end{align*}
\]

Here, we set:

\[
\begin{align*}
F^1_+(\lambda)G^3 &= \mathcal{F}_-^1(\mathcal{E}_+(\lambda)F^3_+ G^3),
F^1_-(\lambda)G^3 &= \mathcal{F}_-^1(\mathcal{E}_-(\lambda)F^3_+ G^3) - \rho_0^{-1} [\mathcal{D}^1 \nabla k_0^3(\mathcal{E}_+(\lambda)F^3_+ ,\mathcal{E}_-(\lambda)F^3_+ G^3) + (I + \mathcal{D}^1) \nabla p_1(\mathcal{E}_+(\lambda)F^3_+ G^3, \mathcal{E}_-(\lambda)F^3_+ G^3)],
F^2(\lambda)G^3 &= -\text{Ext}^+ [\mathcal{F}_-^2(\mathcal{E}_+(\lambda)F^3_+ G^3) + \mathcal{F}_-^1(\mathcal{E}_-(\lambda)F^3_+ G^3)]n_0 + \text{Ext}^+ [\mathcal{F}_-^2(\mathcal{E}_-(\lambda)F^3_+ G^3)n_0 - \mathcal{F}_-^2(\mathcal{E}_-(\lambda)F^3_+)G^3],
\end{align*}
\]

and $\text{Ext}^+ [f_\pm]$ denote the even extension of functions $f_\pm$ defined on $\mathbb{R}^N_\pm$ to $\mathbb{R}^N$. Note that:

\[
\| \nabla^\ell \text{Ext}^+ [f_\pm] \|_{L_q(\mathbb{R}^N)} \leq 2\| f_\pm \|_{L_q(\mathbb{R}^N)} \quad (\ell = 0,1)
\]

with $\nabla^0 f_\pm = f_\pm$. Let us define the corresponding $\mathcal{R}$- bounded operators $\mathcal{R}^1_\pm(\lambda)$ and $\mathcal{R}^2(\lambda)$ by:

\[
\begin{align*}
\mathcal{R}^1_+(\lambda)F^3 &= \mathcal{F}_+^1(\mathcal{E}_+(\lambda)F^3),
\mathcal{R}^1_-(\lambda)F^3 &= \mathcal{F}_-^1(\mathcal{E}_-(\lambda)F^3) - \rho_0^{-1} [\mathcal{D}^1 \nabla k_0^3(\mathcal{E}_+(\lambda)F^3, \mathcal{E}_-(\lambda)F^3) + (I + \mathcal{D}^1) \nabla p_1(\mathcal{E}_+(\lambda)F^3, \mathcal{E}_-(\lambda)F^3)],
\mathcal{R}^2(\lambda)F^3 &= -\text{Ext}^+ [\mathcal{F}_-^2(\mathcal{E}_+(\lambda)F^3) + \mathcal{F}_-^1(\mathcal{E}_-(\lambda)F^3)]n_0 - \text{Ext}^+ [\mathcal{F}_-^2(\mathcal{E}_-(\lambda)F^3)n_0 - \mathcal{F}_-^2(\mathcal{E}_-(\lambda)F^3)].
\end{align*}
\]

Set $\mathcal{R}(\lambda)G^3 = (\mathcal{F}_+^1(\lambda)G^3, \mathcal{F}_-^1(\lambda)G^3, \mathcal{F}_-^2(\lambda)G^3)$ and $\mathcal{R}(\lambda)F^3 = (\mathcal{R}^1_+(\lambda)F^3, \mathcal{R}^1_-(\lambda)F^3, \mathcal{R}^2(\lambda)F^3).$ Obviously:

\[
\mathcal{R}(\lambda)G^3 = \mathcal{R}(\lambda)F^3 G^3.
\]

To obtain:

\[
\mathcal{R}(\mathcal{E}_+(\lambda)G^3) \{ \{ (\tau \partial_q)^\ell \mathcal{R}(\lambda) | \lambda \in \Gamma_{\epsilon_A,0} \} \} \leq C(\sigma + M_1 + C_{\sigma}M_2 \lambda_0^{-1/2}) \quad (\ell = 0,1),
\]

we use the following lemma (cf. Shibata ([4] Lemma 2.4)).

Lemma 5. Let $D = \mathbb{R}^N$ or $\mathbb{R}^N_\pm$. Let $1 < q \leq r < \infty$ and $N < r < \infty$. Then, there exists a constant $C_{N,q,r}$ such that for any $\sigma > 0$, $a \in L_q(D)$ and $b \in W^1_q(D)$, it holds that:

\[
\| ab \|_{L_q(D)} \leq \sigma \| \nabla b \|_{L_q(D)} + C_{N,q,r} \sigma^{-N} \| a \|_{L_q(D)} \| b \|_{L_q(D)}.
\]
To prove (85), for example, we treat $\text{Ext}^{-}[F^2_\lambda (E_+(\lambda)F^3)]$. Recalling (75) and using (83), (76), Lemma 5, Lemma 2, Theorem 9, and (55), we have:

$$
\int_0^1 \left\| \sum_{k=1}^n r_k(u) \text{Ext}^{-}[F^2_\lambda (E_+(\lambda)F^3)] \right\|_{L^q_\mathbb{R}(\mathbb{R}^N)} du = \int_0^1 \left\| \nabla \text{Ext}^{-} \left[ \frac{n}{\lambda^2} r_k(u) F^2_\lambda (E_+(\lambda)F^3) \right] \right\|_{L^q_{\mathbb{R}^{2N}}} du
$$

$$
\leq 2^l \int_0^1 \left\| F^2_\lambda (E_+(\lambda)F^3) \right\|_{L^q_{\mathbb{R}^N}} du
$$

$$
\leq C_\rho \int_0^1 \left\{ \rho \left\| \sum_{k=1}^n r_k(u) \nabla^2 E_+(\lambda)F^3 \right\|_{L^q_{\mathbb{R}^N}} + C_{\rho,M_2} \left\| \sum_{k=1}^n r_k(u) (\lambda \nabla E_+(\lambda)F^3) \right\|_{L^q_{\mathbb{R}^N}} \right\}
$$

$$
\leq C_\rho \sigma \int_0^1 \left\{ \sum_{k=1}^n r_k(u) \nabla^2 E_+(\lambda)F^3 \right\}_{L^q_{\mathbb{R}^N}} du
$$

$$
+ C_\rho (C_{\rho,M_2} \lambda^{-1/2}) \int_0^1 \left\| \sum_{k=1}^n r_k(u) \nabla E_+(\lambda)F^3 \right\|_{L^q_{\mathbb{R}^N}} du
$$

$$
\leq C_\rho (\sigma + C_{\rho,M_2} \lambda^{-1/2}) \int_0^1 \left\| \sum_{k=1}^n r_k(u) F^3 \right\|_{L^q_{\mathbb{R}^N}} du.
$$

Analogously, we can estimate the $R$-bound of any other terms, and therefore, we have (85).

Recalling (55) and $F^3_\lambda G^3 = (g_+, g_-, \lambda^{1/2}, h, \nabla h)$, we see that $\|F^3_\lambda G^3\|_{X^q_{\mathbb{R}^N}}$ gives equivalent norms of $X^q_{\mathbb{R}^N}$. By (84) and (85), we have:

$$
\| R(\lambda) G^3 \|_{X^q_{\mathbb{R}^N}} \leq C (\sigma + M_1 + C_{\rho,M_2} \lambda_0^{-1/2}) \|F^3_\lambda G^3\|_{X^q_{\mathbb{R}^N}}
$$

for any $G^3 = (g_+, g_-, h) \in X^q_{\mathbb{R}^N}$. Thus, choosing $\sigma$ so small and $M_1$ so large and $\lambda_0$ so large that $C (\sigma + M_1 + C_{\rho,M_2} \lambda_0^{-1/2}) \leq 1/2$, we have:

$$
\| R(\lambda) \|_{L(X^q_{\mathbb{R}^N})} \leq 1/2
$$

for any $\lambda \in \Gamma_{\epsilon, \lambda_0}$, and therefore, $(I - R(\lambda))^{-1}$ exists in $L(X^q_{\mathbb{R}^N})$. If we set $v_\pm = E_\pm(\lambda) F^3_\lambda (I - R(\lambda))^{-1} G^3$, with $G = (G_+, G_-, H)$, then in view of (82), $v_\pm$ solve (81). Moreover, using (84), we have $F^3_\lambda (I - R(\lambda))^{-1} = (I - F^3_\lambda R(\lambda))^{-1} F^3_\lambda$, and so, defining operators $\hat{E}_\pm(\lambda) = E_\pm(\lambda) (I - F^3_\lambda R(\lambda))^{-1}$ and using (85) and Theorem 9, we see that $v_\pm = \hat{E}_\pm(\lambda) F^3_\lambda G^3$ with $G^3 = (G_+, G_-, h)$ is a unique solution to (81), and:

$$
\mathcal{R}_{L(X^q_{\mathbb{R}^N}), L^q_{\mathbb{R}^N}} \{ (\tau \partial r)^\ell G_\lambda \hat{E}_\pm(\lambda) \mid \lambda \in \Gamma_{\epsilon, \lambda_0} \} \leq C
$$

(\ell = 0, 1).

Since $u_\pm = (A^{-1})_{-1} [v_\pm \circ \Phi^{-1}]$ is a unique solution to (71), we have Theorem 12 by the pullback.

Next, for the compressible part, we consider the following two problems:

$$
\lambda v_+ - \gamma_0^{-1} (\text{Div } S_+(v_+) + \delta \nabla (\gamma_3 \text{div } v_+)) = g_+ \quad \text{in } D_+, \quad v_+|_{\partial D} = 0; \quad (86)
$$

$$
\lambda v_+ - \gamma_0^{-1} (\text{Div } S_+(v_+) + \delta \nabla (\gamma_3 \text{div } v_+)) = g_+ \quad \text{in } \mathbb{R}^N. \quad (87)
$$

Since we know the existence of $\mathcal{R}$-bounded solution operators in $\mathbb{R}^N_+$ and $\mathbb{R}^N$ (cf. Enomoto and Shibata [18]), in a similar fashion to the proof of Theorem 9, we can prove the following theorem (cf. von Below, Enomoto and Shibata [17]).

**Theorem 13.** Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $r \geq \max(q, q')$. Then, there exist constants $M_1 \in (0, 1)$ and $\lambda_0 = \lambda M_2 \geq 1$ such that the following two assertions hold:
There exists an operator family $E_{D+}(λ) ∈ Hol(Γ_{e,λ,0}, L_q(D^-)N, W^2_q(D^-)N)$ such that for any $λ ∈ Γ_{e,λ,0}$ and $g_+ ∈ L_q(D^-)N$, $v_+ = E_{D+}(λ)g_+$ is a unique solution to (86), and:

$$R_{L_q(D^-)N, W^2_q(D^-)N}\left\{ ((τ∂_τ)^f(λ/2E_{D+}(λ)) | λ ∈ Γ_{e,λ,0}) ≤ C \ (ℓ = 0, 1, j = 0, 1, 2) \right.$$  (2)

There exists an operator family $E_{0+}(λ) ∈ Hol(Γ_{e,λ,0}, L_q(ℝ^N)N, W^2_q(ℝ^N)N)$ such that for any $λ ∈ Γ_{e,λ,0}$ and $g_+ ∈ L_q(ℝ^N)N$, $v_+ = E_{0+}(λ)g_+$ is a unique solution to (87), and:

$$R_{L_q(ℝ^N)N, W^2_q(ℝ^N)N}\left\{ ((τ∂_τ)^fE_{0+}(λ) | λ ∈ Γ_{e,λ,0}) ≤ C \ (ℓ = 0, 1, j = 0, 1, 2) \right.$$  (2)

Finally, for the incompressible part, we consider the following two problems:

$$λv_--p_{0,-}^{-1}(\\text{Div} S_-(v_-) - \nabla K_F(v_-)) = g_- \text{ in } D_-, \quad (S_-(v_-) - K_F(v_-))n_+ |_S = h_- |_S, \quad (88)$$

$$λv_--p_{0,-}^{-1}(\\text{Div} S_-(v_-) - \nabla K_0(v_-)) = g_-, \text{ in } ℝ^N, \quad (89)$$

where $K_F(v_-)$ and $K_0(v_-)$ are unique solutions to the following variational problems:

$$((\nabla K_F(v_-), \nabla \varphi)_{D_+} = (S_-(v_-) - \nabla \text{div} v_-, \nabla \varphi)_{D_+} \text{ for any } \varphi ∈ \hat{W}^1_q(0)(D_-)$$

subject to $K_F(v_-) = < S_-(v_-)n_+, n_+ > - \text{div} v_-$ on $S$, and:

$$((\nabla K_0(v_-), \nabla \varphi)_{ℝ^N} = (S_-(v_-) - \nabla \text{div} v_-, \nabla \varphi)_{ℝ^N} \text{ for any } \varphi ∈ \hat{W}^1_q(0)(ℝ^N), \quad (88)$$

respectively. Since we know the existence of $R$-bounded solution operators in $ℝ^N_+$ and $ℝ^N$ (cf. Shibata and Shimizu [19]), in a similar fashion to the proof of Theorem 9, we can prove the following theorem (cf. Shibata [16]).

**Theorem 14.** Let $1 < q < ∞$ and $0 < e < π/2$. Then, there exist constants $M_1 ∈ (0, 1)$ and $λ_0 = λ_M ≥ 1$ such that the following two assertions hold:

1. Let $X^5_q(D_-)$ and $X^5_q(D^-)$ be sets defined by:

$$X^5_q(D_-) = \{(g_-, h_-) | g_- ∈ L_q(D_-)N, \ h_- ∈ W^1_q(D_-)N\},$$

$$X^5_q(D^-) = \{F^5 = (F_2, F_5, F_6) | F_2, F_5 ∈ L_q(D_-)N, \ F_6 ∈ L_q(D^-)N\}.$$

Then, there exists an operator family $E_{D-}(λ) ∈ Hol(Γ_{e,λ,0}, L_q(X^5_q(D_-), W^2_q(D^-)N))$ such that for any $λ ∈ Γ_{e,λ,0}$ and $G^5 = (g_-, h_-) ∈ X^5_q(D_-)$, $v_- = E_{D-}(λ)F^5G^5$ is a unique solution to (88), and:

$$R_{L_q(X^5_q(D_-)N, W^2_q(D^-)N)}\left\{ ((τ∂_τ)^f(λ/2E_{D-}(λ)) | λ ∈ Γ_{e,λ,0}) ≤ C \ (ℓ = 0, 1, 1, j = 0, 1, 2) \right.$$  (2)

Here, $F^5G^5 = (g_-, λ^{1/2}h_-, ϕh_-)$.  

2. There exists an operator family $E_{0-}(λ) ∈ Hol(Γ_{e,λ,0}, L_q(ℝ^N)N, W^2_q(ℝ^N)N)$ such that for any $λ ∈ Γ_{e,λ,0}$ and $g_- ∈ L_q(ℝ^N)N$, $v_- = E_{0-}(λ)g_-$ is a unique solution to (89), and:

$$R_{L_q(ℝ^N)N, W^2_q(ℝ^N)N}\left\{ ((τ∂_τ)^fE_{0-}(λ) | λ ∈ Γ_{e,λ,0}) ≤ C \ (ℓ = 0, 1, j = 0, 1, 2) \right.$$  (2)

5. A Proof of Theorem 8

5.1. Some Preparations for the Proof of Theorem 8

We first give several properties of the uniform $W^2_2^{-1/r}$ domain in the following proposition.
Proposition 4. Let \( N < r < \infty \), and let \( \Omega_{\pm} \) be uniform \( W^{2-1/r}_r \) domains in \( \mathbb{R}^N \). Let \( M_1 \) be the number given in (67). Then, there exist constants \( M_2 > 0 \), \( 0 < d^i < 1 \) \( (i = 1, \ldots, 5) \), at most countably many \( N \)-vectors of functions \( \Phi_j^i \in W^2_r(\mathbb{R}^N)^N \) \( (i = 1, \ldots, 5, \ j \in \mathbb{N}) \), and points \( x^j_i \in \Gamma, \ x^j_0 \in \Gamma_+ \) \( x^j_1 \in \Omega \), \( x^j_2 \in \Gamma_+ \) \( x^j_3 \in \Omega_+ \), and \( x^j_4 \in \Omega_- \), such that the following assertions hold:

(i) The maps: \( \mathbb{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbb{R}^N \) \( (i = 1, 2, 3, \ j \in \mathbb{N}) \) are bijective such that
\[
\nabla \Phi_j^i = A_{j_1}^i + B_{j_2}^i, \ \nabla (\Phi_j^i)^{-1} = A_{j_1}^i - 1 + B_{j_2}^i, \ \text{ where } A_{j_1}^i \text{ and } A_{j_1}^i - 1 \text{ are } N \times N \text{ constant orthonormal matrices, and } B_{j_2}^i \text{ and } B_{j_2}^i^{-1} \text{ are } N \times N \text{ matrices of } W^2_r(\mathbb{R}^N) \text{ functions that satisfy the conditions: } \| (B_{j_2}^i, B_{j_2}^{-1}) \|_{L_0(\mathbb{R}^N)} \leq M_1 \text{ and } \| \nabla (B_{j_2}^i, B_{j_2}^{-1}) \|_{L_0(\mathbb{R}^N)} \leq M_2.
\]

(ii) \( \Gamma = \left\{ \bigcup_{i=1,2,3} \bigcup_{j=1}^{\infty} (\Phi_j^i(H_i) \cap B^0_\delta(x_i^j)) \right\} \cup \left\{ \bigcup_{i=4,5} \bigcup_{j=1}^{\infty} B^0_\delta(x_i^j) \right\} \) with \( H_1 = \mathbb{R}^N \), \( H_2 = \mathbb{R}_{+}^N \) and \( H_3 = \mathbb{R}_{-}^N \). \( \Phi_j^1(\mathbb{R}^N) \cap B_\delta(x_i^j) = \Omega \cap B_\delta(x_i^j), \ \Phi_j^2(\mathbb{R}^N) \cap B_\delta(x_i^j) = \Omega_+ \cap B_\delta(x_i^j), \ \Phi_j^3(\mathbb{R}^N) \cap B_\delta(x_i^j) = \Omega_- \cap B_\delta(x_i^j), \ B_\delta(x_i^j) \subset \mathbb{R}^N \), \( \Phi_j^2(\mathbb{R}^N) \cap B_\delta(x_i^j) = \Gamma_1 \cap B_\delta(x_i^j) \) \( (i = 1, 2, 3) \). Here and in the following, we set \( \Gamma_1 = \Gamma, \ \Gamma_2 = \Gamma_+, \text{ and } \Gamma_3 = \Gamma_- \) for notational convenience.

(iii) There exist \( C^\infty \) functions \( \zeta_j^i \) \( (i = 1, \ldots, 5, \ j \in \mathbb{N}) \) such that \( \| (\zeta_j^i, \zeta_j^\bar{i}) \|_{W^{1,\infty}_0(\mathbb{R}^N)} \leq c_0, \ 0 \leq \zeta_j^i \leq 1, \ \supp \zeta_j^i \subset \Gamma_1 \cap B_\delta(x_i^j), \ \zeta_j^i = 1 \) \( \text{on } \Gamma_3 \cap \sum_{j=1}^{\infty} \zeta_j^i = 1 \) \( \text{on } \Omega \cap \sum_{j=1}^{\infty} \zeta_j^i = 1 \) \( \text{on } \Gamma_1 \cap (i = 1, 2, 3) \).

(iv) There exists a natural number \( L \geq 2 \) such that any \( L + 1 \text{ distinct sets of } \{ B^0_\delta(x_i^j) \mid i = 1, \ldots, 5, \ j \in \mathbb{N} \} \) have an empty intersection.

Proof. For a detailed proof, we refer to Enomoto and Shibata ([18] Appendix). \( \square \)

In the following, choosing \( M_2 \) larger if necessary, we may assume that
\[
\| \nabla \gamma_k + \|_{L_1(B^1_\delta(x_i^j) \cap \Omega_+)} \leq M_2 \quad (k = 0, 3, i = 1, 3, 5, j \in \mathbb{N}),
\]
which is a weaker assumption than the last condition in (23). Since functions in \( W^1_2 \) are Hölder continuous of order \( \alpha \) with \( 0 < \alpha < 1 - N/r \), as follows from Sobolev’s imbedding theorem, we have
\[
| \gamma_k(x) - \gamma_k(x_i^j) | \leq C \| \nabla \gamma_k \|_{W^{1,\infty}_0(B^0_\delta(x_i^j))} | x - x_i^j |^\alpha
\]
for any \( x \in B^1_\delta(x_i^j) \) \( (k = 0, 3, i = 1, 3, 5, j \in \mathbb{N}) \) with some constant \( C \) independent of \( j \), and so choosing \( d^i > 0 \) smaller and more points \( x_i^j \) suitably, we may assume that \( | \gamma_k(x) - \gamma_k(x_i^j) | \leq M_1 \) for \( x \in B^2_\delta(x_i^j) \) \( (k = 0, 3, i = 1, 3, 5, j \in \mathbb{N}) \).

Here and in the following, constants denoted by \( C \) are independent of \( j \in \mathbb{N} \). In addition, in view of (68), we may assume that each unit outward normal \( n_i^j \) to \( \Phi_j^1(\mathbb{R}^N) \) \( (i = 1, 3, 4, j \in \mathbb{N}) \) is defined on \( \mathbb{R}^N \) and satisfies the conditions: \( \| n_i^j \|_{L_\infty(\mathbb{R}^N)} = 1 \) and \( \| \nabla n_i^j \|_{L_1(\mathbb{R}^N)} \leq CM_2 \).

Note that \( n = n_i^1 \) on \( B^2_\delta(x_i^j) \cap \Gamma_+ \text{ and } n_- = n_i^5 \) on \( B^2_\delta(x_i^j) \cap \Gamma_- \).

Summing up, from now on, we may assume that:
\[
\| \gamma_k + \gamma_k(x_i^j) \|_{L_\infty(\mathbb{R}^N)} \leq CM_1, \ \| \nabla \gamma_k + \|_{L_1(B^1_\delta(x_i^j))} \leq M_2 \quad (k = 0, 3),
\]
\[
\| \nabla n \|_{L_1(B^1_\delta(x_i^j) \cap \Omega)} \leq M_2, \ \| \nabla n_- \|_{L_1(B^1_\delta(x_i^j) \cap \Omega_-)} \leq M_2,
\]
and that both \( n \) and \( n_- \) are defined on \( \mathbb{R}^N \) with \( \| n \|_{L_\infty(\Omega)} = 1 \) and \( \| n_- \|_{L_\infty(\Omega_-)} = 1 \), respectively.

Next, we prepare two lemmas used to construct a parametrix.

Lemma 6. Let \( X \) be a Banach space and \( X^* \) its dual space, while \( \| \cdot \|_X, \| \cdot \|_{X^*} \) and \( \langle \cdot, \cdot \rangle > \) are the norm of \( X \), the norm of \( X^* \), and the duality of \( X \) and \( X^* \), respectively. Let \( n \in \mathbb{N} \), and for \( i = 1, \ldots, n \), let \( a_i \in \mathbb{C} \), let \( \{ s_j^{(i)} \}_{j=1}^{\infty} \) be sequences in \( X^* \). Let \( \{ s_j^{(i)} \}_{j=1}^{\infty} \) and \( \{ h_j^{(i)} \}_{j=1}^{\infty} \) be sequences of positive numbers. Assume that there exist maps \( N_j^i : \mathbb{R} \to [0, \infty) \) such that:
\[
| \langle f_j^{(i)} \phi \rangle | \leq M_3 s_j^{(i)} N_j^i (\phi) \quad (i = 1, \ldots, n), \quad | \langle \sum_{i=1}^{n} a_i s_j^{(i)} \phi \rangle | \leq M_3 h_j^{(i)} N_j^i (\phi)
\]
for any $\varphi \in L_{q'}(D)$ with some constant $M_3$ independent of $j \in \mathbb{N}$. If:

$$\sum_{j=1}^{\infty} (g_j^{(i)})^q < \infty, \quad \sum_{j=1}^{\infty} (h_j)^q < \infty, \quad \sum_{j=1}^{\infty} N_j(\varphi)^{q'} \leq M_4^{q'} \|\varphi\|_{X'}^q,$$

then the infinite sum $f^{(i)} = \sum_{j=1}^{\infty} f_j^{(i)}$ exists in the strong topology of $X'$ and:

$$\|f^{(i)}\|_{X'} \leq M_3 M_4 \left( \sum_{j=1}^{\infty} (g_j^{(i)})^q \right)^{1/q}, \quad \|\sum_{i=1}^{n} a_i f^{(i)}\|_{X'} \leq M_3 M_4 \left( \sum_{j=1}^{\infty} (h_j)^q \right)^{1/q}.$$

**Lemma 7.** Let $D$ be a domain in $\mathbb{R}^N$, and assume that there exists at most countably many covering \(\{B_j\}_{j=1}^{\infty}\) such that $D \subset \bigcup_{j=1}^{\infty} B_j$ and \(\{B_j\}_{j=1}^{\infty}\) has a finite intersection property of order $L$, that is any $L + 1$ distinct sets of \(\{B_j\}_{j=1}^{\infty}\) have an empty intersection. Let $1 < q < \infty$. Then, the following assertions hold.

(i) There exists a constant $C_{q,L}$ such that:

$$\left( \sum_{j=1}^{\infty} \|f\|_{L_q(D \cap B_j)}^q \right)^{1/q} \leq C_{q,L} \|f\|_{L_q(D)} \quad \text{for any } f \in L_q(D).$$

(ii) Let $m \in \mathbb{N}_0$. Let \(\{f_j\}_{j=1}^{\infty}\) be a sequence in $W^m_q(D)$, and let \(\{g_j^{(\ell)}\}_{j=1}^{\infty} (\ell = 0, 1, \ldots, m)\) be sequences of positive numbers. Assume that:

$$\sum_{j=1}^{\infty} (g_j^{(\ell)})^q < \infty, \quad \|D^\ell f_j, \varphi\|_{L_q(D \cap B_j)} \leq M_3 g_j^{(\ell)} \|\varphi\|_{L_q(D \cap B_j)} \quad \text{for any } \varphi \in L_q(D) \text{ and } \ell = 0, 1, \ldots, m$$

with some constant $M_3$ independent of $j \in \mathbb{N}$. Then, $f = \sum_{j=1}^{\infty} f_j$ exists in the strong topology of $W^m_q(D)$ and:

$$\|D^\ell f\|_{L_q(D)} \leq C_{q,L} M_3 \left( \sum_{j=1}^{\infty} (g_j^{(\ell)})^q \right)^{1/q}.$$

**Remark 15.** To prove Lemma 6, we consider the difference of finite sum $\sum_{j=1}^{N} f_j^{(i)}$ and use the Hölder inequality for the sequence. The assertion (i) of Lemma 7 follows immediately from the property of the Lebesgue measure and suitable decomposition of covering sets \(\{B_j\}_{j=1}^{\infty}\), and the assertion (ii) of Lemma 7 follows from Lemma 6 and Lemma 7 (i).

### 5.2. Local Solutions

In the following, we write $B_{\rho_i}(x_i^j) = B_{\rho_i}(x_i) (i = 1, \ldots, 5)$, $H_i^1 = \Phi_i^1(\mathbb{R}^N)$, $H_i^{1/2} = \Phi_i^{1/2}(\mathbb{R}^N)$, $H_i^2 = \Phi_i^2(\mathbb{R}^N)$, $H_i^3 = \Phi_i^3(\mathbb{R}^N)$, $H_i^4 = H_i^5 = \mathbb{R}^N$, $\Gamma_i^{1/2} = \Phi_i^{1/2}(\mathbb{R}^N)$, $\Gamma_i^2 = \Phi_i^2(\mathbb{R}^N)$, and $\Gamma_i^3 = \Phi_i^3(\mathbb{R}^N)$ for short. $n_i$ denote the unit outward normals to $\Gamma_i^{1/2}$ pointing from $H_i^{1/2}$ to $H_i^{1/2}$ and $n_i^3$ denote the unit outward normals to $\Gamma_i^3$ for $j \in \mathbb{N}$. In view of (90), we define the functions $\gamma^j_{ik}$ by:

$$\gamma^j_{ik}(x) = (\gamma_{ik} + \gamma_{ik}(x_i^j))(\tilde{\gamma}_i^j(x) + \gamma_{ik}(x_i^j))$$

for $k = 0, 3, i = 1, 2, 4$, and $j \in \mathbb{N}$. Noting that $0 \leq \tilde{\gamma}_{i}^j \leq 1$ and $\|\nabla \tilde{\gamma}_{i}^j\|_{L_{\infty}(\mathbb{R}^N)} \leq c_0$, by (90) and (23):

$$\rho_0/2 \leq \gamma^j_0(x) \leq 2\rho_0, \quad 0 \leq \gamma^j_3(x) \leq (\rho_2)^2 (x \in \{H_i^{1/2}, H_i^2, H_i^4\}),$$

$$\|\gamma^j_{ik}(\cdot) - \gamma^j_{ik}(x_i^j)\|_{L_{\infty}(H_i^{1/2})} \leq CM_1, \quad \|\nabla \gamma^j_{ik}\|_{L_{\infty}(H_i^{1/2})} \leq M_2$$

(91)
for $k = 0, 3, i = 1, 2, 4,$ and $j \in \mathbb{N}.$ In addition, we have:
\[
\gamma^i_k(x) = \gamma_k(x) \quad (x \in \text{supp } \zeta^i_j, k = 0, 3, i = 1, 2, 4, j \in \mathbb{N}),
\]
(92)
because $\zeta^i_j = 1$ on $\text{supp } \zeta^i_j.$ For $G = (g_+ g_-, h, h_-) \in X_q(\Omega),$ we consider the equations:
\[
\begin{align*}
\lambda v^j_{+1} - (\gamma^j_0)^{-1} & \{ \text{Div } S_+ (v^j_{+1}) + \delta \nabla (\gamma^j_0 \text{div } v^j_{+1}) \} = \xi^j_0 g_+ \quad \text{in } H^1_{+j}, \\
\lambda v^j_{-1} - \rho_0^{-1} & \{ \text{Div } S_- (v^j_{-1}) - \nabla K^j_1 (v^j_{+1}) \} = \xi^j_0 g_- \quad \text{in } H^1_{-j}, \\
(S_+ (v^j_{+1}) + \delta \gamma^j_0 \text{div } v^j_{+1} I) n^j_1 |_{r_1} = (S_- (v^j_{-1}) - K^j_1 (v^j_{+1}) I) n^j_1 |_{r_1} = \zeta^j_1 h |_{r_1}, \\
v^j_{+1} |_{r_1} = v^j_{-1} |_{r_1} = 0.
\end{align*}
\]
(93)
Here, $v = K^j_1 (v^j_{+1}, v^j_{-1})$ is a unique solution to the variational problem:
\[
(\nabla v, \nabla \varphi)_{H^1_{+j}} = \rho_0^{-1} \{ \text{Div } S_- (v^j_{-1}) - \nabla \text{div } v^j_{-1}, \nabla \varphi \}_{H^1_{-j}} \quad \text{for any } \varphi \in \tilde{W}^1, \theta(\mathcal{H}^1_{-j})
\]
(94)
such that $v |_{r_1} = (S_- (v^j_{-1}) n^j_1 n^j_1 > -\text{div } v^j_{-1} |_{r_1} = (S_+ (v^j_{+1}) - \delta \gamma^j_0 \text{div } v^j_{+1} I) n^j_1, n^j_1 |_{r_1} > n_{1,4}.$ Here and in the following, $X_q(\Omega)$ and $\mathcal{X}_q(\Omega)$ denote the spaces defined in Theorem 8 in Section 2.3.

Moreover, we consider the following four problems:
\[
\begin{align*}
\lambda v_j^2 - (\gamma^j_0)^{-1} & \{ \text{Div } S_+ (v_j^2) + \delta \nabla (\gamma^j_0 \text{div } v_j^2) \} = \xi^j_0 g_+ \quad \text{in } H^2_{j}, \\
\lambda v_j^3 - (\gamma^j_0)^{-1} & \{ \text{Div } S_+ (v_j^3) + \delta \nabla (\gamma^j_0 \text{div } v_j^3) \} = \xi^j_0 g_+ \quad \text{in } H^3_{j}, \\
\lambda v_j^3 - \rho_0^{-1} & \{ \text{Div } S_- (v_j^3) - \nabla K^j_0 (v_j^3) \} = \xi^j_0 g_- \quad \text{in } H^3_{j}, \\
\lambda v_j^5 - \rho_0^{-1} & \{ \text{Div } S_- (v_j^5) - \nabla K^j_1 (v_j^5) \} = \xi^j_0 g_- \quad \text{in } H^5_{j}.
\end{align*}
\]
(95)
Here, $K^j_0 (v_j^3)$ and $K^j_1 (v_j^5)$ are unique solutions to the variational problem:
\[
(\nabla K^j_0 (v_j^3), \nabla \varphi)_{H^2_{j}} = \rho_0^{-1} \{ \text{Div } S_- (v_j^3) - \nabla \text{div } v_j^3, \nabla \varphi \}_{H^2_{j}} \quad \text{for any } \varphi \in W^1, \theta(\mathcal{H}^2_{j})
\]
(96)
such that $K^j_0 (v_j^3) |_{r_1} = (S_- (v_j^3) n_j^3 n_j^3 > -v_j^3 |_{r_1} = 0,$ and the variational problem:
\[
(\nabla K^j_1 (v_j^5), \nabla \varphi)_{H^5_{j}} = \rho_0^{-1} \{ \text{Div } S_- (v_j^5) - \nabla \text{div } v_j^5, \nabla \varphi \}_{H^5_{j}} \quad \text{for any } \varphi \in W^1, \theta(\mathcal{H}^5_{j}).
\]
(97)
Here and in the following, $\lambda, \theta$ and general constants denoted by $C$ are independent of $i = 1, \ldots, 5$ and $j \in \mathbb{N}.$ By Theorem 12 in Section 4, there exist operator families $T^j_{\ell j}(\lambda) \in \text{Hol}(\mathcal{G}_q, L(\mathcal{X}_q, (H^2_{+j}, W^2_{-j}(H^1_{+j})^N)))$ such that for any $\lambda \in \mathcal{G}(\mathcal{X}_q, \mathcal{X}_q)$ and $v_{\ell j} = T^j_{\ell j}(\lambda) F^j_{\ell j} G^j_{\ell j}$ are unique solutions to the problem in (93), and:
\[
\mathcal{R}_{L,j}(\mathcal{X}_q, (H^1_{+j}, W^2_{-j}(H^1_{+j}))^N) \supseteq \{(\tau \partial_\tau)^{\ell} (\lambda^{m/2} T^j_{\ell j}(\lambda)) | \lambda \in \mathcal{G}(\mathcal{X}_q, \mathcal{X}_q)\} \leq C
\]
(98)
for $\ell = 0, 1$ and $m = 0, 1, 2.$ Moreover, by Theorem 13 and Theorem 14 in Section 4, there exist operator families $T_{j,k}^j(\lambda) \in \text{Hol}(\mathcal{G}_q, L(\mathcal{X}_q, (H^2_{+j}, W^2_{-j}(H^1_{+j})^N)))$ such that for any $\lambda \in \mathcal{G}(\mathcal{X}_q, \mathcal{X}_q)$ and $v_{\ell j} = T_{j,k}^j(\lambda) F_{\ell j}(H_{\ell j})^2$ are unique solutions of the problems in (95) ($k = 2, 3, 4, 5,$) and:
\[
\mathcal{R}_{L,j}(\mathcal{X}_q, (H^1_{+j}, W^2_{-j}(H^1_{+j}))^N) \supseteq \{(\tau \partial_\tau)^{\ell} (\lambda^{m/2} T_{j,k}^j(\lambda)) | \lambda \in \mathcal{G}(\mathcal{X}_q, \mathcal{X}_q)\} \leq C
\]
(99)
for $\ell = 0, 1$ and $m = 0, 1, 2.$ Here and in the following, we set $X_2^j (H^1_{+j}) = X_5^j (H^1_{+j}) = L_2^j (H^2_{+j})^N, X_3^j (H^2_{+j}) = X_4^j (H^2_{+j}), X_3^j (H^1_{+j}) = X_5^j (H^1_{+j})$ (cf. $X_3^j$ and $X_5^j$ were given in Theorem 14), $X_4^j (H^1_{+j}) = X_4^j (H^1_{+j}) = L_2^j (H^2_{+j})^N, H_3^j (H^1_{+j}) = H_3^j (H^5_{+j}) = L_2^j (H^5_{+j})^N.$ Moreover, we set
\( F_1^j \mathcal{G}_j = \xi_j^1(\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2} \mathbf{h}, \mathbf{h}) \), \( F_2^j \mathcal{G}_j = \xi_j^2 \mathbf{g}_+, \) \( F_3^j \mathcal{G}_j = \xi_j^3(\mathbf{g}_-, \lambda^{1/2} \mathbf{h}_-, \mathbf{h}_-) \), \( F_4^j \mathcal{G}_j = \xi_j^4 \mathbf{g}_+ \), and \( F_5^j \mathcal{G}_j = \xi_j^5 \mathbf{g}_- \). Since \( \mathcal{R} \)-boundedness implies the usual boundedness, by \((98)\) and \((99)\),

\[
\sum_{\pm}(\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|^{1/2}\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|^{1/2}\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|^{1/2}\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)})
\leq C(\sum_{\pm}|\xi_j^1 \mathbf{g}_+|_{L^1(\mathcal{R}^N)} + |\lambda|^{1/2}|\xi_j^2 \mathbf{h}|_{L^1(\mathcal{R}^N)} + |\lambda|^{1/2}|\xi_j^3 \mathbf{g}_+|_{L^1(\mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{1j}\|_{L^1(\mathcal{R}^N)}).
\]

for any \( \lambda \in \Gamma_{C_{i0}}, \) because \( |\lambda| \geq C_{i0} \geq 1 \).

### 5.3. Construction of Parametrices

For \( \mathbf{G} = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}_-, \mathbf{h}_-) \in X_q(\Omega) \), we define parametrices \( \mathbf{U}_\pm(\lambda) \) by:

\[
\mathbf{U}_+(\lambda) \mathbf{G} = \sum_{j=1}^{\infty} \xi_j \mathbf{v}_{+j} + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \tilde{\xi}_i \mathbf{v}_{+j}^i, \quad \mathbf{U}_-(\lambda) \mathbf{G} = \sum_{j=1}^{\infty} \xi_j \mathbf{v}_{-j} + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \tilde{\xi}_i \mathbf{v}_{-j}^i
\]

Set \( G_\lambda \mathbf{v} = (\lambda \mathbf{v}, \lambda^{1/2} \nabla \mathbf{v}, \nabla^2 \mathbf{v}) \), where \( \nabla \mathbf{v} = (\nabla \mathbf{v}, \nabla \mathbf{v}, \mathbf{v}) \) and \( \nabla^2 \mathbf{v} = (\nabla^2 \mathbf{v}, \nabla \mathbf{v}, \mathbf{v}) \), and \( B_j^i = B_\delta(x_j^i) \) for notational simplicity. By \((100)\),

\[
\|G_\lambda(\xi_j \mathbf{v}_{+j}), \varphi_\pm\|_{\mathcal{A}_1} \leq C(\|\mathbf{g}_+\|_{L^1(\Omega \cap \mathcal{R}^N)} + |\lambda|^{1/2}\|\mathbf{h}_+\|_{L^1(\Omega \cap \mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{+j}\|_{L^1(\mathcal{R}^N)} + |\lambda|\|\mathbf{v}_{+j}\|_{L^1(\mathcal{R}^N)})
\]

for any \( \varphi \in L_q'((\Omega \pm)^N), \) and so, by Lemma 6 and Lemma 7, the infinite sums in \((101)\) exist in the strong topology of \( W_{q}^2((\Omega \pm)^N) \) and \( \mathbf{U}_\pm(\lambda) \in \text{Hol}(\Gamma_{C_{i0}}, \mathcal{L}(X_q(\Omega), W_{q}^2((\Omega \pm)^N))). \) By \((37), (42), (92), (93), (95)\)–\((97), \text{and} (101)\), setting \( \mathbf{w}_\pm = \mathbf{U}_\pm(\lambda) \mathbf{G} \), we have:

\[
\lambda \mathbf{w}_+ - \gamma_3^{-1}(\text{Div} \mathbf{S}_+(\mathbf{w}_+) + \delta \nabla(\gamma_3 + \text{div} \mathbf{w}_+)) = \mathbf{g}_+ - R_1^1(\lambda) \mathbf{G} \quad \text{in} \ \Omega_+, \\
\lambda \mathbf{w}_- - \rho_1^{-1}(\text{Div} \mathbf{S}_-(\mathbf{w}_-) - \delta \nabla(\mathbf{w}_+)) = \mathbf{g}_- - (R_1^1(\lambda) \mathbf{G} - L(\lambda) \mathbf{G}) \quad \text{in} \ \Omega_+, \\
(\mathbf{S}_+(\mathbf{w}_+) + \delta \gamma_3 + \text{div} \mathbf{w}_+) \mathbf{n}_{\Gamma_0} - (\mathbf{S}_-(\mathbf{w}_-) - \delta \mathbf{w}_+) \mathbf{n}_{\Gamma_0} = \mathbf{h} - R_3^1(\lambda) \mathbf{G}, \\
\mathbf{w}_+|_{\Gamma_0} = \mathbf{w}_-|_{\Gamma_0} = \mathbf{h}_- - R_3^1(\lambda) \mathbf{G}, \quad \text{where we set:}
\]

\[\mathbf{w}_+ = \mathbf{w}_- = \mathbf{h}_- = R_3^1(\lambda) \mathbf{G},\]
\[ R_+^1(\lambda)G = \sum_{j=1}^{\infty} \gamma_{0,j}^{-1} [\nabla S_+(\zeta_j^3v_{1,j}^-) - \zeta_j^3\nabla S_+(v_{1,j}^-) + \delta\{\nabla(\gamma_j^3 \nabla S_+(v_{1,j}^-)) - \zeta_j^3\nabla(\gamma_j^3 \nabla v_{1,j}^-)\}] \]
\[ + \sum_{i=2,4,j=1}^{\infty} \gamma_{0,j}^{-1} [\nabla S_-(\zeta_j^3v_{1,j}^-) - \zeta_j^3\nabla S_+(v_{1,j}^-) + \delta\{\nabla(\gamma_j^3 \nabla S_+(v_{1,j}^-)) - \zeta_j^3\nabla(\gamma_j^3 \nabla v_{1,j}^-)\}] \]
\[ R_+^3(\lambda)G = \sum_{j=1}^{\infty} \gamma_{0,j}^{-1} [\nabla S_+(\zeta_j^3v_{1,j}^-) - \zeta_j^3\nabla S_+(v_{1,j}^-)] + \sum_{i=3,5,j=1}^{\infty} \rho_{0,j}^{-1} [\nabla S_-(\zeta_j^3v_{1,j}^-) - \zeta_j^3\nabla S_-(v_{1,j}^-)] \]
\[ L(\lambda)G = \nabla K(U_+(\lambda)G, U_-(\lambda)G) - \sum_{j=1}^{\infty} \{\zeta_j^3\nabla K_j^1(v_{1,j}^-) - \zeta_j^3\nabla K_j(v_{1,j}^-) + \zeta_j^3\nabla K_j(v_{1,j}^-)\} \] (103)
\[ R_+^2(\lambda)G = -\sum_{j=1}^{\infty} \{\nabla S_+(\zeta_j^3v_{1,j}^-) - \zeta_j^3S_+(v_{1,j}^-)\n_j\} - T_{n_j}^{-1}[\nabla S_-(\zeta_j^3v_{1,j}^-) - \zeta_j^3S_-(v_{1,j}^-)\n_j]\]
\[ R_+^3(\lambda)G = -\sum_{j=1}^{\infty} T_{n_j}^{-1}[\nabla S_-(\zeta_j^3v_{1,j}^-) - \zeta_j^3S_-(v_{3,j}^-)\n_j] + \sum_{j=1}^{\infty} \{\nabla(\zeta_j^3v_{1,j}^-) - \zeta_j^3\nabla v_{1,j}^-\}. \]

Finally, we construct \( R \)-bounded solution operators that represent \( U_\pm(\lambda)G \). For \( F = (F_1, \ldots, F_6) \in X_{\lambda}(\Omega) \), we set:
\[ U_+^1(\lambda)F = T_{n_j}^1(\lambda)\xi_j^1(F_1, F_2, F_3, F_4), \quad U_+^3(\lambda)F = T_{n_j}^3(\lambda)\xi_j^3(F_2, F_5, F_6), \quad U_+^2(\lambda)F = T_{n_j}^2(\lambda)\xi_j^2(F_1, F_2, F_3, F_4) \] (104)

Obviously,
\[ v_j = U_+^j(\lambda)F_\lambda G \quad (G = (g_+, g_-, h, h_-) \in X_{\lambda}(\Omega)), \] (105)

where \( F_\lambda G = (g_+, g_-, \lambda^{1/2}h, h, \lambda^{1/2}h_-, h_-) \). By (98), we have:
\[ \int_0^1 \| \sum_{k=1}^{n} r_k(u)G_k U_+^j(\lambda_k)F_k \|^q_{L_q(\Omega)} du \leq C \int_0^1 \| \sum_{k=1}^{n} r_k(u)F_k \|^q_{L_q(\Omega \cap B_j)} du \] (106)

for any \( n \in \mathbb{N} \), \( \{\lambda_k\}_{k=1}^{n} \subset \Gamma_{\varepsilon, \lambda_0} \) and \( \{F_k\}_{k=1}^{n} \subset X_{\lambda}(\Omega) \), where \( \{r_k(u)\}_{k=1}^{n} \) are the same as in Definition 3. By (98), Lemma 6, and Lemma 7, \( U_+^j(\lambda)F = \sum_{j=1}^{\infty} \zeta_j^jU_+^j(\lambda)F \) exist in the strong topology of \( W^1_q(\Omega_{\lambda})^N \), \( U_0^j(\lambda)F = \sum_{j=1}^{\infty} \zeta_j^jU_0^j(\lambda)F \) exist in the strong topology of \( W^1_q(\Omega_{\lambda})^N \) with \( \Omega_i = \Omega_{\lambda} \) for \( i = 2, 4 \) and \( \Omega_i = \Omega_\lambda \) for \( i = 3, 5 \), and:
\[ \| \sum_{k=1}^{n} a_k G_k U_+^j(\lambda_k)F_k \|^q_{L_q(\Omega_{\lambda})} \leq C_{q, j, L} \sum_{k=1}^{\infty} \| a_k F_k \|^q_{L_q(\Omega_{\lambda})} \] (106)

for any complex numbers \( a_k, \lambda_k \in \Gamma_{\varepsilon, \lambda_0} \) and \( F_k \in X_{\lambda}(\Omega) \) \( (k = 1, \ldots, n, n \in \mathbb{N}) \). Setting \( U_+^j(\lambda)F = U_+^j(\lambda)F_\lambda G + \sum_{j=3,5} U_0^j(\lambda)F_\lambda G \) and \( U_0^j(\lambda)F = U_0^j(\lambda)F_\lambda G + \sum_{j=2,4} U_0^j(\lambda)F_\lambda G \), by the facts that \( v_j = U_+^j(\lambda)F_\lambda G \) and (106), we have:
\[ U_0^j(\lambda)G \in \text{Hol}(\Gamma_{\varepsilon, \lambda_0} L(\lambda, W^1_q(\Omega_{\lambda})^N)), \quad U_\pm(\lambda)F_\lambda G = U_\pm(\lambda)G \quad (G \in X_{\lambda}(\Omega)), \]
\[ R_{\lambda}(X_{\lambda}(\Omega), W^1_q(\Omega_{\lambda})) \subseteq \{ (\tau a_j)^{(\lambda^{1/2}U_\pm(\lambda))} | \lambda \in \Gamma_{\varepsilon, \lambda_0} \} \subseteq C \quad (\ell = 0, 1, j = 0, 1, 2). \] (107)
5.4. Estimates of the Remainder Terms

We introduce the operators that represent $\mathcal{R}^1_+(\lambda)$, $\mathcal{R}^1_-(\lambda)$, $\mathcal{R}^2(\lambda)$, and $\mathcal{R}^3(\lambda)$ as follows:

$$\mathcal{R}^1_+(\lambda) \mathbf{F} = \sum_{j=1}^{\infty} \gamma_0^{-1} \{ \text{Div} \, \mathbf{S}_+ (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \text{Div} \, \mathbf{S}_+ (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) \}$$

$$+ \delta \{ \nabla (\gamma_{3+} \text{div} \, (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F})) \} - \zeta_j^1 \nabla (\gamma_{3+} \text{div} \, (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F})) \} \}$$

$$+ \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \gamma_0^{-1} \{ \text{Div} \, \mathbf{S}_+ (\zeta_j^1 \mathcal{U}^1_i(\lambda) \mathbf{F}) - \zeta_j^1 \text{Div} \, \mathbf{S}_+ (\mathcal{U}^1_i(\lambda) \mathbf{F}) \}$$

$$+ \delta \{ \nabla (\gamma_{3+} \text{div} \, (\zeta_j^1 \mathcal{U}^1_i(\lambda) \mathbf{F})) \} - \zeta_j^1 \nabla (\gamma_{3+} \text{div} \, (\mathcal{U}^1_i(\lambda) \mathbf{F})) \} \} ;$$

$$\mathcal{R}^1_-(\lambda) \mathbf{F} = \sum_{j=1}^{\infty} \rho_0^{-1} \{ \text{Div} \, \mathbf{S}_- (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \text{Div} \, \mathbf{S}_- (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) \}$$

$$+ \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \rho_0^{-1} \{ \text{Div} \, \mathbf{S}_- (\zeta_j^1 \mathcal{U}^1_i(\lambda) \mathbf{F}) - \zeta_j^1 \text{Div} \, \mathbf{S}_- (\mathcal{U}^1_i(\lambda) \mathbf{F}) \}$$

$$+ \mathcal{L}(\lambda) \mathbf{F} = \nabla \mathbf{K} (\mathcal{U}^1_+ (\lambda) \mathbf{F}, \mathcal{U}^- (\lambda) \mathbf{F}) - \sum_{j=1}^{\infty} \zeta_j^1 \nabla \mathbf{K} (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}, \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F})$$

$$- \sum_{j=1}^{\infty} \zeta_j^3 \nabla \mathbf{K} (\mathcal{U}^1_i (\lambda) \mathbf{F}) - \sum_{j=1}^{\infty} \zeta_j^3 \nabla \mathbf{K} (\mathcal{U}^1_i (\lambda) \mathbf{F}) ;$$

$$\mathcal{R}^2(\lambda) \mathbf{F} = - \sum_{j=1}^{\infty} [\mathcal{T}^1_{\beta_j}] [\text{Ext}^- (\mathbf{S}_+ (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \mathbf{S}_+ (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F})) \mathbf{n}_j^1]$$

$$- [\mathcal{T}^1_{\beta_j}] [\text{Ext}^+ (\mathbf{S}_- (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \mathbf{S}_- (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F})) \mathbf{n}_j^1] - \{ \text{div} (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \text{div} (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) \} ,$$

$$\mathcal{R}^3(\lambda) \mathbf{F} = - \sum_{j=1}^{\infty} [\mathcal{T}^1_{\beta_j}] (\mathbf{S}_- (\zeta_j^3 \mathcal{U}^1_1(\lambda) \mathbf{F}) - \zeta_j^3 \mathbf{S}_- (\mathcal{U}^1_1(\lambda) \mathbf{F})) \mathbf{n}_j^3] + \{ \text{div} (\zeta_j^3 \mathcal{U}^1_1(\lambda) \mathbf{F}) - \zeta_j^3 \text{div} (\mathcal{U}^1_1(\lambda) \mathbf{F}) \}$$

(108)

for any $\mathbf{F} \in \mathcal{X}_0(\Omega)$. Setting:

$$\mathcal{R}(\lambda) \mathbf{F} = (\mathcal{R}^1_+(\lambda) \mathbf{F}, \mathcal{R}^1_-(\lambda) \mathbf{F} + \mathcal{L}(\lambda) \mathbf{F}, \mathcal{R}^2(\lambda) \mathbf{F}, \mathcal{R}^3(\lambda) \mathbf{F}),$$

$$\mathbf{R}(\lambda) \mathbf{G} = (\mathcal{R}^1_+(\lambda) \mathbf{G}, \mathcal{R}^1_-(\lambda) \mathbf{G} + \mathcal{L}(\lambda) \mathbf{G}, \mathcal{R}^2(\lambda) \mathbf{G}, \mathcal{R}^3(\lambda) \mathbf{G}),$$

we have:

$$F_{\ell} \mathbf{R}(\lambda) F_{\ell} \mathbf{G} = F_{\ell} \mathbf{R}(\lambda) \mathbf{G},$$

(109)

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_0(\Omega))} \{(\tau \partial_\tau)^{\ell} \mathcal{F}_\lambda \mathbf{R}(\lambda) \left| \mathcal{A} \mathbf{A}_{\mathcal{R}(\lambda) = 0} \right. \} \leq C \sigma + C_\sigma \lambda_0^{-1/2} \quad (\ell = 0, 1)$$

(110)

for any $\sigma > 0$ with some constant $C_\sigma$ depending on $\sigma$. If we prove (110), then, choosing $\sigma > 0$ so small and $\lambda_0 \geq 1$ so large that $C_\sigma + C_\sigma \lambda_0^{-1/2} \leq 1/2$, we see that $I - F_{\ell} \mathbf{R}(\lambda)$ exists in $\text{Hol}(\Gamma_{\mathcal{R}(\lambda)} \mathcal{L}(\mathcal{X}_0(\Omega)))$. Thus, in view of (107), (109), (110), and (102), we see easily that $S_{\pm}(\lambda) = \mathcal{U}_{\pm}(\lambda)(I - F_{\ell} \mathbf{R}(\lambda))^{-1}$ has a required $\mathbb{R}$-bounded solution operator to (41), which completes the proof of Theorem 8.

Thus, we prove (110) in the following. By direct use of Lemma 6, Lemma 7, Lemma 1, Lemma 2, Remark 9, (98), and (104), we can estimate $\mathcal{R}^1_+(\lambda)$, $\mathcal{R}^2(\lambda)$, and $\mathcal{R}^3(\lambda)$, except for $\mathcal{L}(\lambda)$. In fact, for example,

$$|\text{Div} \, \mathbf{S}_\pm (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \text{Div} \, \mathbf{S}_\pm (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) | \leq C \| \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F} \|_{W^2_0(H^{1/2}_{\mathcal{X}_0(\Omega)})} \| \varphi \|_{L^q(\Omega_{\pm} \cap \mathcal{B}^\varepsilon)}$$

for any $\varphi \in L^q(\Omega_{\pm})$, and so, there exists an operator family $\mathcal{R}^1_{\pm}(\lambda) \in \text{Hol}(\Gamma_{\mathcal{R}(\lambda)} \mathcal{L}(\mathcal{X}_0(\Omega), L_q(\Omega_{\pm})))$ such that $\mathcal{R}^1_{\pm}(\lambda) \mathbf{F} = \sum_{j=1}^{\infty} \{ \text{Div} \, \mathbf{S}_\pm (\zeta_j^1 \mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) - \zeta_j^1 \text{Div} \, \mathbf{S}_\pm (\mathcal{U}^1_{j,\pm}(\lambda) \mathbf{F}) \}$ exists in
the strong topology of $L_q(\Omega \pm)$ and $\|R_{1\pm}^1(\lambda)F\|_{L_q(\Omega \pm)}^q \leq C \sum_{j=1}^\infty \|U_{1\pm}^j(\lambda)F\|_{W_{q}^1(\mathcal{H}_{\lambda}^j)}^q$ By (98) and the monotone convergence theorem, for any $n \in \mathbb{N}$, $\{\lambda_k\}_{k=1}^n \subset \Gamma_{c,\lambda_0}$, and $\{F_k\}_{k=1}^n \subset X_q(\Omega)$:

$$\int_0^1 \sum_{k=1}^n r_k(u)R_{1\pm}^1(\lambda_k)F_k\|_{L_q(\Omega \pm)}^q du \leq C \sum_{j=1}^\infty \int_0^1 \sum_{k=1}^n r_k(u)U_{1\pm}^j(\lambda_k)F_k\|_{L_q(\mathcal{H}_{\lambda}^j)}^q du$$

$$\leq C \sum_{j=1}^\infty \left\{ \lambda_0^{-2} \int_0^1 \sum_{k=1}^n r_k(u)\lambda_k^2 \nabla U_{1\pm}^j(\lambda_k)F_k\|_{L_q(\mathcal{H}_{\lambda}^j)}^q du + \lambda_0^{-q} \int_0^1 \sum_{k=1}^n r_k(u)\lambda_k U_{1\pm}^1(\lambda_k)F_k\|_{L_q(\mathcal{H}_{\lambda}^j)}^q du \right\}$$

$$\leq C\lambda_0^{-2} \sum_{j=1}^\infty \int_0^1 \sum_{k=1}^n r_k(u)F_k\|_{X_q(\Omega \cap B_j^c)}^q du \leq C\lambda_0^{-2} \int_0^1 \sum_{k=1}^n r_k(u)F_k\|_{X_q(\Omega \cap B_j^c)}^q du,$$

from which it follows that $R_{L}(X_q(\Omega))(\{R_{1\pm}^1(\lambda) | \lambda \in \Gamma_{c,\lambda_0}\}) \leq C\lambda_0^{-1/2}$. The other terms except for $L(\lambda)$ can be estimated in the same manner. Namely, we have:

$$R_{L}(X_q(\Omega),L_q(\Omega))^{\infty}(\{(\tau \partial_t)^i R_{1\pm}^1(\lambda) | \lambda \in \Gamma_{c,\lambda_0}\}) \leq C\lambda_0^{-1/2},$$

$$R_{L}(X_q(\Omega),L_q(\Omega))^1(\{(\tau \partial_t)^i (\lambda^{1/2}R^2(\lambda)) | \lambda \in \Gamma_{c,\lambda_0}\}) \leq C\lambda_0^{-1/2},$$

$$R_{L}(X_q(\Omega),W_{q}^1(\Omega))^{\infty}(\{(\tau \partial_t)^i R^2(\lambda) | \lambda \in \Gamma_{c,\lambda_0}\}) \leq C\lambda_0^{-1/2},$$

$$R_{L}(X_q(\Omega),L_q(\Omega))^{1}(\{(\tau \partial_t)^i (\lambda^{1/2}R^3(\lambda)) | \lambda \in \Gamma_{c,\lambda_0}\}) \leq C\lambda_0^{-1/2},$$

$$R_{L}(X_q(\Omega),W_{q}^1(\Omega))^{1}(\{(\tau \partial_t)^i R^3(\lambda) | \lambda \in \Gamma_{c,\lambda_0}\}) \leq C\lambda_0^{-1/2}.$$

Next, we estimate $L(\lambda)$. We use the following two lemmas due to Shibata [4].

**Lemma 8.** Let $1 < q < \infty$. Then, there exists a constant $c$ independent of $j \in \mathbb{N}$ such that:

$$\|\psi\|_{W_{q}^1(\Omega \cap B_j^c)} \leq c \|\nabla \psi\|_{L_q(\Omega \cap B_j^c)}$$

for any $\psi \in \dot{W}_{q,0}^1(\Omega \cap B_j^c)$, $i = 1, 3$,

$$\|\psi - c_j(\psi)\|_{W_{q}^1(\Omega \cap B_j^c)} \leq c \|\nabla \psi\|_{L_q(\Omega \cap B_j^c)}$$

for any $\psi \in \dot{W}_{q,0}^1(\Omega \cap B_j^c)$, $i = 1, 3$,

where $c_j(\psi)$ are suitable constants depending on $\psi$.

**Lemma 9.** Let $1 < q < \infty$. Then, there exists a constant $c$ independent of $j \in \mathbb{N}$ such that:

$$\|K^1_j(u_+, u_-)\|_{L_q(\mathcal{H}_j^1)} \leq c \sum_{\pm} (\|\nabla u_{\pm}\|_{L_q(\mathcal{H}_j^1)}^\pm + \|\nabla u_{\pm}\|_{L_q(\mathcal{H}_j^1)}^{1-1/q} \|\nabla^2 u_{\pm}\|_{L_q(\mathcal{H}_j^1)}^{1/q})$$

$$\|K^1_j(v)\|_{L_q(\mathcal{H}_j^1)} \leq c (\|\nabla^2 v\|_{L_q(\mathcal{H}_j^1)} + \delta_1 \|\nabla v\|_{L_q(\mathcal{H}_j^1)} \|\nabla^2 v\|_{L_q(\mathcal{H}_j^1)}^{1/q}$$

for any $u_{\pm} \in W_{q}^2(\mathcal{H}_j^1)$ and $v \in W_{q}^2(\mathcal{H}_j^1)$, $i = 3, 5$, where $\delta_1$ are symbols defined by $\delta_1 = 1$ and $\delta_5 = 0$.

**Lemma 10.** Let $1 < q < \infty$. Then,

$$\|v\|_{L_q(\Omega_j)} \leq C_q (\|\nabla v\|_{L_q(\Omega_j)} + \|\nabla v\|_{L_q(\Omega_j)} \|\nabla v\|_{L_q(\Omega_j)}^{1-1/q}$$

for $i = 1, 3$ and $j \in \mathbb{N}$, where $C_q$ is a constant independent of $j \in \mathbb{N}$. 
To estimate $\mathcal{L}(\lambda)$, we write $\mathcal{L}(\lambda)\mathbf{F} = \nabla \mathcal{L}^1(\lambda)\mathbf{F} + \mathcal{L}^2(\lambda)\mathbf{F}$ with:

$$\mathcal{L}^1(\lambda)\mathbf{F} = K(U_+ (\lambda)\mathbf{F}, U_- (\lambda)\mathbf{F}) - \sum_{j=1}^{\infty} \zeta_j^3 K_j^1(U_{+j}(\lambda)\mathbf{F}, U_{-j}(\lambda)\mathbf{F})$$

$$- \sum_{j=1}^{\infty} \zeta_j^3 K_j^3(U_{+j}(\lambda)\mathbf{F}) - \sum_{j=1}^{\infty} \zeta_j^3 K_j^2(U_{-j}(\lambda)\mathbf{F}),$$

$$\mathcal{L}^2(\lambda)\mathbf{F} = \sum_{j=1}^{\infty} (\nabla \zeta_j^1) K_j^1(U_{+j}(\lambda)\mathbf{F}, U_{-j}(\lambda)\mathbf{F}) + \sum_{j=1}^{\infty} (\nabla \zeta_j^2) K_j^2(U_{+j}(\lambda)\mathbf{F}) + \sum_{j=1}^{\infty} (\nabla \zeta_j^5) K_j^2(U_{-j}(\lambda)\mathbf{F}).$$

By (98), (99), Lemma 9, Lemma 6, and Lemma 7, we have:

$$\mathcal{R}_{\mathcal{L}(\chi_q(\Omega), \lambda_0(\lambda))}((\tau \partial_\tau) \mathcal{L}^2(\lambda) | \lambda \in \Gamma_{e, \lambda_0}) \leq C \sigma + C_0 \lambda_0^{-1/2}.$$
5.5. A Proof of Theorem 3

By (98), (99), Lemma 8, Lemma 9, and Lemma 10, we have:

\[
\|L(\lambda)F\|_{W^{1,q}_{\rho,0}(\Omega_+)^*} \leq C_s \left( \sum_{j=1}^{\infty} \|U_{F_j}(\lambda)F\|_{W_{q,0}^2(\Omega_+ \cap B_j^j)}^q + \sum_{j=1}^{\infty} \|U_{F_j}(\lambda)F\|_{W_{q,0}^2(\Omega_+ \cap B_j^j)}^q \right) + C_r \left( \sum_{j=1}^{\infty} \|U_{F_j}(\lambda)F\|_{W_{q,0}^2(\Omega_+ \cap B_j^j)}^q + \sum_{j=1}^{\infty} \|U_{F_j}(\lambda)F\|_{W_{q,0}^2(\Omega_+ \cap B_j^j)}^q \right),
\]

which, combined with (98) and (99), yields that:

\[
\mathcal{R}_{L(\lambda_0^q(\Omega),W_{q,0}^1(\Omega_+)^*)}(\{(\tau_\partial v)^t L(\lambda) \mid \lambda \in \Gamma_+\}) \leq C_s + C_r \lambda_0^{-1/2}.
\]

Identifying \(\hat{W}_{1,q,0}(\Omega_-) = \{\nabla \varphi \mid \varphi \in \hat{W}_{1,q,0}(\Omega_-) \subset L_q(\Omega_-)^N\}\), by the Hahn–Banach theorem, there exists an operator family \(\mathbf{M}(\lambda) \in \text{Hol}(\Gamma_\rho, \mathcal{L}(\mathcal{X}_q(\Omega_-), L_q(\Omega_-)^N))\) such that \((\mathbf{M}(\lambda)F, \nabla \varphi)_{\Omega_-} = < L(\lambda)F, \varphi >\), and:

\[
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_{1,q}^1(\Omega_-))}(\{(\tau_\partial v)^t \mathbf{M}(\lambda) \mid \lambda \in \Gamma_+\}) \leq C_s + C_r \lambda_0^{-1/2}.
\]

Moreover, by (39), (94), and (96), there exists an operator family \(\mathbf{m}(\lambda) \in \text{Hol}(\Gamma_\rho, \mathcal{L}(\mathcal{X}_q(\Omega), W_{1,q}^1(\Omega_-)))\) such that \((\mathbf{m}(\lambda)F, \nabla \varphi)_{\Omega_-} = < L(\lambda)F, \varphi >\), and:

\[
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_{1,q}^1(\Omega_-))}(\{(\tau_\partial v)^t \mathbf{m}(\lambda) \mid \lambda \in \Gamma_+\}) \leq C_s + C_r \lambda_0^{-1/2}.
\]

Since the weak Dirichlet problem is assumed to be uniquely solvable, we have \(L^1(\lambda)F = m(\lambda)F + K(\mathbf{M}(\lambda)F - \nabla \mathbf{m}(\lambda)F)\), which yields that:

\[
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_{1,q}^1(\Omega_-))}(\{(\tau_\partial v)^t (\nabla L^1(\lambda)) \mid \lambda \in \Gamma_+\}) \leq C_s + C_r \lambda_0^{-1/2}.
\]

Therefore, we have (110), and so, the proof of Theorem 8 is complete.

5.5. A Proof of Theorem 3

Instead of Problem (22), we consider:

\[
\begin{cases}
\gamma_{1+} \partial_t u_+ - \text{Div} (S_+(u_+)) - \delta \nabla (\gamma_{2+} u_+) = g_+ & \text{in } \Omega_+ \times (0, \infty), \\
\rho_{1-} \partial_t u_- - \text{Div} S_-(u_-) + \nabla p_- = g_- & \text{in } \Omega_- \times (0, \infty), \\
\text{div } u_- = f_- = \text{div } f_- & \text{in } \Omega_- \times (0, \infty), \\
(S_+(u_+) + \delta \gamma_{2+} \text{div } S(u_+))n|_{\Gamma_+0} - (S_-(u_-) - p_-)n|_{\Gamma_-0} = h & \text{for } t > 0, \\
\frac{u_+}{|_{\Gamma_+0}} - \frac{u_-}{|_{\Gamma_-0}} = 0 & \text{for } t > 0, \\
\frac{u_+}{|_{t=0}} = \frac{u_0+}{\Omega_+} & \text{in } \Omega_+, & \frac{u_-}{|_{t=0}} = \frac{u_0-}{\Omega_-} & \text{in } \Omega_-.
\end{cases}
\]

We first consider the generation of the \(C^0\) analytic semigroup associated with the following equations:

\[
\begin{cases}
\partial_t p_+ + \gamma_{2+} \text{div } w_+ = 0 & \text{in } \Omega_+ \times (0, \infty), \\
\gamma_{1+} \partial_t w_+ - \text{Div} (S_+(w_+)) + \nabla (\gamma_{1+} p_+) = 0 & \text{in } \Omega_+ \times (0, \infty), \\
\rho_{1-} \partial_t w_- - \text{Div} S_-(w_-) + \nabla K(w_-, w_-) = 0 & \text{in } \Omega_- \times (0, \infty), \\
(S_+(w_+) - \gamma_{1+} p_+ n|_{\Gamma_+0} - (S_-(w_-) - K(w_-, w_-))n|_{\Gamma_-0} = 0 & \text{for } t > 0, \\
w_+|_{\Gamma_+0} - w_-|_{\Gamma_-0} = 0 & \text{for } t > 0, \\
w_+|_{t=0} = (p_{0+}, w_{0+}) & \text{in } \Omega_+, & w_-|_{t=0} = w_{0-} & \text{in } \Omega_-. \end{cases}
\]
In view of Theorem 8, let $u_\pm = S_\pm(\lambda)(g_+, g_-, 0, 0, 0, 0)$, and set $\theta_+ = \lambda^{-1}(f_+ - \gamma_2 + \operatorname{div} u_+)$, then $u_\pm$ and $\theta_+$ are unique solutions of Equation (28) with $p_- = K(u_+, u_-)$ and $f_- = h = h_- = 0$ and possess the estimates:

$$|\lambda|\theta_+^i_{W_1^q(\Omega_+)} + \sum_{\pm}(|\lambda| u_\pm^i_{L_1^q(\Omega_\pm)} + |u_\pm|^i_{W_2^q(\Omega_\pm)}) \leq C(|f_+|^i_{W_1^q(\Omega_+) + |g_+|^i_{L_1^q(\Omega_+) + |g_-|^i_{L_1^q(\Omega_-)})} \tag{114}$$

for any $\lambda \in \Gamma_{\lambda_0}$. Here, using the same argument as in Assertion 2 in Sect.2, we see that $u_- \in I_q(\Omega_-)$, and so, $\operatorname{div} u_- = 0$ in (28).

Set:

$$X_q(\Omega) = \{(\theta_+, u_+, u_-) | \theta_+ \in W_1^q(\Omega_+), \ u_+ \in L_q(\Omega_+), \ u_- \in I_q(\Omega_-)\},$$

$$Y_q(\Omega) = \{\theta_+, u_+, u_- \} \times X_q(\Omega) \setminus (S_+(u_+) - \gamma_1 + \theta_+)n|\Gamma_0 - (S_-(u_-) - K(u_+, u_-)I)n|\Gamma_0 = 0, \quad S_-(u_-) - K(u_+, u_-)I)n|\Gamma_0 = 0, \quad u_+|\Gamma_0 = u_-|\Gamma_0, \quad u_+|\Gamma_0 = 0\}.$$

Then, Problem (113) generates a $C^0$ analytic semigroup on $X_q(\Omega)$. Let $D_q(\Omega) = (X_q, Y_q)_{-1/p,p'}$, where $\langle \cdot, \cdot \rangle_{-1/p,p'}$ denotes a real interpolation functor. By a standard real interpolation method (trace method), we see that Problem (113) admits unique solutions $p_+$ and $w_\pm$ with:

$$p_+ \in W_1^q((0, \infty), W_1^q(\Omega_+)), \quad w_\pm \in W_1^q((0, \infty), L_q(\Omega_\pm) N) \cap L_p((0, \infty), W_1^q(\Omega_\pm) N), \tag{115}$$

possessing the estimate:

$$\|e^{-\gamma t}p_+\|_{L_p((0,\infty),W_1^q(\Omega_+))} + \sum_{\pm}\|e^{-\gamma t}w_\pm\|_{L_p((0,\infty),L_q(\Omega_\pm))} + \|e^{-\gamma t}\nabla K(w_+, w_-)\|_{L_p((0,\infty),L_q(\Omega_\pm))} \leq C(\|p_0\|_{L_p(\Omega, W_1^q(\Omega_+))} + \sum_{\pm}\|w_0\|_{L_p(\Omega, W_1^q(\Omega_\pm))}^{1-1/p}) \tag{116}$$

for any $\gamma > \lambda_0$. Moreover, $w_- \in I_q(\Omega_-)$ for any $t \in (0, \infty)$. The uniqueness follows from Duhamel’s principle.

We now consider equations:

$$\begin{cases}
\partial_t q_+ + \gamma_2 + \operatorname{div} v_+ = f_+ & \text{in } \Omega_+ \times \mathbb{R}, \\
\gamma_0 \partial_t v_+ - \operatorname{Div} S_+(v_+) + \nabla (\gamma_1 + q_+) = g_+ & \text{in } \Omega_+ \times \mathbb{R}, \\
\rho_0 \partial_t v_- - \operatorname{Div} S_-(v_-) + \nabla q_- = g_- & \text{in } \Omega_- \times \mathbb{R}, \\
\operatorname{div} v_- = f_- & \text{in } \Omega_- \times \mathbb{R}, \\
(S_+(v_+) - \gamma_1 + q_+ + 1)n|\Gamma_0 = 0, \quad (S_-(v_-) - q_+ - 1)n|\Gamma_0 = 0 & \text{for } t \in \mathbb{R}, \\
v_+|\Gamma_0 - v_-|\Gamma_0 = 0 & \text{for } t \in \mathbb{R}, \\
v_+|\Gamma_0 = 0, \quad (S_-(v_-) - q_+ - 1)n_-|\Gamma_0 = h_- & \text{for } t \in \mathbb{R}.
\end{cases} \tag{117}$$

Applying the Laplace transform to (117) and setting $\varphi_\pm = \mathcal{L}[v_\pm](\lambda)$ and $\varphi_\pm = \mathcal{L}[q_\pm](\lambda)$, we have:

$$\begin{cases}
\lambda \varphi_+ + \gamma_2 + \operatorname{div} \varphi_+ = \hat{f}_+ & \text{in } \Omega_+,

\lambda \varphi_- - \gamma_0^{-1}(\operatorname{Div} S_+(\varphi_+) - \nabla (\gamma_1 + \hat{q}_+)) = \hat{g}_+ & \text{in } \Omega_+,

\lambda \varphi_- - \rho_0^{-1}(\operatorname{Div} S_-(\varphi_-) - \nabla \hat{q}_-) = \hat{g}_- & \text{in } \Omega_-,

\operatorname{div} \varphi_- = \hat{f}_- = \text{div } \hat{f}_- & \text{in } \Omega_-,

(S_+(\varphi_+) - \gamma_1 + \hat{q}_+ + 1)n|\Gamma_0 = 0, \quad (S_-(\varphi_-) - \hat{q}_+ - 1)n|\Gamma_0 = \hat{h}_|\Gamma_0,

\varphi_+|\Gamma_0 - \varphi_-|\Gamma_0 = 0 & \text{for } t \in \mathbb{R}, \\
\varphi_+|\Gamma_0 = 0, \quad (S_-(\varphi_-) - \hat{q}_+ - 1)n_-|\Gamma_0 = \hat{h}_- & \text{for } t \in \mathbb{R}. \tag{118}
\end{cases}$$

Applying Theorem 5 yields that $\varphi_\pm = A_0^0(\lambda)^{-1}f_+^0G^0$, $\varphi_\pm = \lambda^{-1}(f_+ - \gamma_2 + \operatorname{div} \varphi_+)$, and $\hat{q}_+ = B_0^0(\lambda)^{-1}f_+^0G^0$ satisfy Equation (118), and so, $v_\pm = L^{-1}[\varphi_\pm]$ and $q_\pm = L^{-1}[\hat{q}_\pm]$ satisfy Equation (118). Moreover, applying Theorem 4 yields that:
\[ \| e^{-\gamma t} q_+ \|_{W^1_q(\Omega_+)} + \sum_{\pm} \bigl( \| e^{-\gamma t} \partial_t v_\pm \|_{L_p(\Omega_\pm)} + \| e^{-\gamma t} v_\pm \|_{L_p(\Omega_\pm)} \bigr) + \| \nabla q_- \|_{L_p(\Omega_-)} \]
\[
\leq C \biggl( \sum_{\pm} \| e^{-\gamma t} g_\pm \|_{L_p(\Omega_\pm)} + \| e^{-\gamma t} \Lambda_\gamma^{1/2} (f_-, h_-) \|_{L_p(\Omega_-)} + \| e^{-\gamma t} (f_-, h_-) \|_{L_p(\Omega_-)} \biggr)
\]
\[
+ \| e^{-\gamma t} \Lambda_\gamma^{1/2} h \|_{L_p(\Omega)} + \| e^{-\gamma t} h \|_{L_p(\Omega)} \biggr) \}
\]
\[ \text{for any } \gamma > \lambda_0. \]  
(119)

Here, we may assume that \( \lambda_0 \geq 1. \)

To prove Theorem 3, setting \( \theta_+ = q_+ + p_+, u_\pm = v_\pm + w_\pm \) and \( p_- = q_- + \theta_- \) in (22), we see that \( p_+, w_\pm, \) and \( \theta_- = K(u_+, u_-) \) satisfy Equation (113) with \( p_0+ = \theta_0+ - q_+ |_{t=0} \) and \( w_0\pm = u_0\pm - v_\pm |_{t=0}. \) By compatibility conditions (25) and the assumption that \( 2/p + N/q \neq 1, 2, \) we see that \( (\theta_0+ - q_+ |_{t=0}, u_0+, v_+ |_{t=0}, u_0-, v_- |_{t=0}) \in \mathbb{R}^N \) and so, Problem (113) admits unique solutions \( p_+ \) and \( w_\pm \) satisfying (115) and (116). By the real interpolation theorem, we have:
\[
\| p_+ |_{t=0} \|_{W^1_q(\Omega_+)} \leq C \| e^{-\gamma t} p_+ \|_{W^1_q(\Omega_+)} \biggr),
\]
\[
\| v_\pm |_{t=0} \|_{B^{2(1-1/p)}_{q,p}(\Omega_\pm)} \leq C \bigl( \| e^{-\gamma t} \partial_t v_\pm \|_{L_p(\Omega_\pm)} \bigr),
\]
which, combined with (119), yields the existence part of Theorem 3, because \( w_- \in L^q(\Omega_-) \)

6. A Proof of Theorem 1

In what follows, we assume that \( 2 < p < \infty, N < q < \infty, 2/p + N/q < 1, \) that \( \Omega_\pm \) are uniform \( W^{1-1/q}_q \) domains in \( \mathbb{R}^N \) (\( N \geq 2 \)), and that the weak Dirichlet problem is uniquely solvable in \( \Omega_- \). By Sobolev’s imbedding theorem, we have:
\[
W^1_q(\Omega_\pm) \subset L^\infty(\Omega_\pm), \quad \| \prod_{j=1}^m f_j \|_{W^1_q(\Omega_\pm)} \leq C \prod_{j=1}^m \| f_j \|_{W^1_q(\Omega_\pm)},
\]  
(120)

Let \( \theta_0+ \in W^1_q(\Omega_+) \) and \( u_\pm \in B^{2(1-1/p)}_{q,p}(\Omega_\pm) \) be initial data satisfying the compatibility condition (20), range condition (21), and \( \| \theta_0+ \|_{W^1_q(\Omega)} + \| v_0+ \|_{B^{2(1-1/p)}_{q,p}(\Omega_+)} + \| v_0- \|_{B^{2(1-1/p)}_{q,p}(\Omega_-)} \leq R_1. \)

To prove Theorem 1, we follow the argument due to Shibata and Shimizu ([21] Section 2). Let \( \Pi_+ \) and \( Z_\pm \) be solutions to linear problem:
\[
\left\{ \begin{array}{l}
\partial_t \Pi_+ + (\rho_{0+} + \theta_{0+}) \text{div } Z_+ = 0 \quad \text{in } \Omega_+,

(\rho_{0+} + \theta_{0+}) \partial_t Z_+ - \text{Div } S_+(Z_+) + \nabla (\rho'_{0+} + \theta_{0+}) \Pi_+ = g_+ \quad \text{in } \Omega_+,

\rho_{0-} \partial_t Z_- - \text{Div } S_-(Z_-) + \nabla p_- = 0 \quad \text{in } \Omega_-,

div Z_- = 0 \quad \text{in } \Omega_-,

(S_+(Z_+) - (\rho_{0+} + \theta_{0+}) \Pi_+ I) n |_{\Gamma_0} = 0, \quad (S_-(Z_-) - p_- I) n |_{\Gamma_-} = 0,

Z_+ |_{\Gamma_0} = 0, \quad Z_- |_{\Gamma_-} = 0,
\end{array} \right.
\]  
(121)

for any \( t > 0 \) subject to the initial condition: \( (\Pi_+, Z_+ |_{t=0}) = (0, v_0+) \) in \( \Omega_+ \) and \( Z_- |_{t=0} = v_0- \) in \( \Omega_- \) with some pressure term \( p_- \), where \( g_+ = -p'_{0+} \text{div } \theta_+ - p_{0+} \text{div } n \) and \( h = -p(\rho_{0+} + \theta_{0+}) n. \) Since \( v_0\pm \) satisfy the compatibility condition (20), by Theorem 3, we know the unique existence of \( \Pi_+ \) and \( Z_\pm \) with:
\[
\Pi_+ \in W^1_{p,\gamma_0}(\mathbb{R}_+, W^1_q(\Omega_+)), \quad Z_\pm \in W^1_{p,\gamma_0}(\mathbb{R}_+, W^2_q(\Omega_\pm)) \cap L_p(\mathbb{R}_+, W^2_q(\Omega_\pm)) \]  
(122)

with large \( \gamma_0 \) depending on \( R_1 \) possessing the estimate:
\[
\| e^{-\gamma t} \Pi_+^0 \|_{W^1_2((\mathbb{R}, W^1_2(\Omega)))} + \sum_{\ell = +,-} \left\{ \| e^{-\gamma t} Z_\ell \|_{W^1_2(\mathbb{R}, L^q(\Omega)))} + \| e^{-\gamma t} Z_\ell \|_{L^p(\mathbb{R}, W^2_2(\Omega)))} \right\} \leq C_{R_1} \sum_{\ell = +,-} \| v_0 \|_{B^{(1-\rho)/2}(\Omega))} \leq C_{R_1} R_1
\]

(123)

for any \( \gamma \geq \gamma_0 \). In the following, \( \gamma \) is fixed such as \( \gamma \geq \gamma_0 \). Let \( \Pi_+^0 \) be the zero extension of \( \Pi_+ \) to \( t < 0 \) and \( Z^\pm_\ell \) be the even extension of \( Z^\pm_\ell \) to \( t < 0 \), then:

\[
\Pi_+^0(x,t) = \begin{cases} 
\Pi_+(x,t) & (t \geq 0), \\
0 & (t < 0), 
\end{cases} \quad Z^\pm_\ell(x,t) = \begin{cases} 
Z^\pm(x,t) & (t \geq 0), \\
Z^\pm(x,-t) & (t < 0), 
\end{cases}
\]

Let \( \psi(t) \) be a function in \( C^\infty(\mathbb{R}) \) such that \( \psi(t) = 1 \) for \( t > -1/2 \) and \( \psi(t) = 0 \) for \( t < -1 \), and set \( Z^\pm_\ell = \psi Z^\pm_\ell \). By (123):

\[
\| e^{-\gamma t} \Pi_+^0 \|_{W^1_2((\mathbb{R}, W^1_2(\Omega)))} + \sum_{\ell = +,-} \left\{ \| e^{-\gamma t} Z^\ell_\ell \|_{W^1_2((\mathbb{R}, L^q(\Omega)))} + \| e^{-\gamma t} Z^\ell_\ell \|_{L^p((\mathbb{R}, W^2_2(\Omega)))} \right\} \leq C_{R_1} R_1.
\]

(124)

We look for a solution to (11) of the form: \( \theta_+ = \Pi_+^0 + \rho_+ \) and \( u_\pm = Z^\pm_\ell + v_\pm \), so that \( \rho_+ \) and \( v_\pm \) enjoy the equations:

\[
\partial_t \rho_+ + (\rho_0 + \theta_0) \partial_t v_+ - \text{Div} S_+ (v_+) + \nabla (p'(\rho_0 + \theta_0) \rho_+) = G_+ (\Pi_+^0 + \rho_+, Z^\ell_\ell + v_+) \quad \text{in } \Omega_+,
\]

\[
\rho_0 \partial_t v_- - \text{Div} S_- (v_-) + \nabla \rho_- = G_- (Z^\ell_\ell + v_-) \quad \text{in } \Omega_-,
\]

\[
(S_+ (v_+) - (p'(\rho_0 + \theta_0) \rho_+) I) n|_{\Gamma^+} = (S_- (v_-) - p_- I) n|_{\Gamma^-} = H(\Pi_+^0 + \rho_+, Z^\ell_\ell + v_+) |_{\Gamma},
\]

\[
v_+|_{\Gamma^+} = v_-|_{\Gamma^-}, \quad v_+|_{\Gamma^+} = 0, \quad (S_- (v_-) - p_- I) n_-|_{\Gamma^-} = H_- (Z^\ell_\ell + v_-) |_{\Gamma^-}.
\]

(125)

for \( 0 < t < T \) subject to the initial condition: \( (\rho_+, v_+)|_{t=0} = (0,0) \) in \( \Omega_+ \) and \( v_-|_{t=0} = 0 \) in \( \Omega_- \) with some pressure term \( p_- \). We solve (125) by the contraction mapping principle. For this purpose, we introduce an underlying space \( \mathcal{I}_{R,T} \) defined by:

\[
\mathcal{I}_{R,T} = \{ (\rho_+, v_+, v_-) | \ (\rho_+, v_+)|_{t=0} = (0,0) \text{ in } \Omega_+, \ v_-|_{t=0} = 0 \text{ in } \Omega_- ,
\]

\[
\rho_+ \in W^2_1((0,T), W^2_1(\Omega_+)), v_+ \in W^1_2((0,T), L^q(\Omega_+)^N) \cap L^p((0,T), W^2_2(\Omega_+)^N), \]

\[
\| \rho_+ \|_{W^2_1((0,T), W^1_2(\Omega_+))} + \sum_{\ell = +,-} \left\{ \| v_+ \|_{W^1_2((0,T), L^q(\Omega_+))} + \| v_- \|_{L^p((0,T), W^2_2(\Omega_+)^N)} \right\} \leq R.
\]

(126)

We choose \( T > 0 \) so small eventually that we may assume that \( 0 < T < 1 \). We choose \( R > 0 \) large enough that \( C_\ell R_1 \leq R \) in (124) in such a way that:

\[
\| e^{-\gamma t} \Pi_+^0 \|_{W^1_2((\mathbb{R}, W^1_2(\Omega)))} + \sum_{\ell = +,-} \left\{ \| e^{-\gamma t} Z^\ell_\ell \|_{W^1_2((\mathbb{R}, L^q(\Omega)))} + \| e^{-\gamma t} Z^\ell_\ell \|_{L^p((\mathbb{R}, W^2_2(\Omega)))} \right\} \leq R.
\]

(127)

In the following, \( C \) denotes a generic constant depending on \( R_1 \), but we do not mention this dependence. For any function \( f \) defined on \( (0,T) \) with \( f(x,0) = 0 \), \( f^0 \) denotes the zero extension of \( f \) to \( t < 0 \), and we define \( E[f](x,t) = \int_0^x f(s,t) ds \) for \( t \leq T \) and \( E[f](x,t) = f^0(x,2T-t) \) for \( t > T \). Note that \( E[f] = 0 \) for \( t \not\in [0,2T] \) and that \( \partial_t E[f] = \partial_t f \) for \( 0 < t < T \), \( \partial_t E[f](\cdot,t) = - (\partial_t f)(\cdot,2T-t) \) for \( T < t < 2T \), and \( \partial_t E[f] = 0 \) for \( t \not\in [0,2T] \). For \((\kappa_+, w_+, w_-) \in \mathcal{I}_{R,T} \), we set:

\[
F_1[\kappa_+] = \Pi_+^0 + E[\kappa_+], \quad F_{2\pm}[w_\pm] = Z^\ell_\pm + E[w_\pm], \quad F_{3\pm}[w_\pm] = E \left[ \int_0^T \nabla F_{2\pm}[w_\pm](\cdot,s) ds \right].
\]

Note that:
\[ F_1[\kappa_+] = \Pi_+^0 + \kappa_+, \quad F_2[\mathbf{w} \pm] = Z_\pm + \mathbf{w}_\pm, \quad F_3[\mathbf{w} \pm] = \int_0^t \nabla(Z_\pm + \mathbf{w}_\pm) \, ds \quad \text{when } t \in (0, T). \tag{128} \]

Employing the same argument due to Shibata and Shimizu ([21] Section 2) and using (126) and (127), we have:

\[ \| F_1[\kappa_+] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega))} \leq C R^{1/p'}, \quad \| F_3[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega))} \leq C R^{1/p'}, \tag{129} \]

where we used the fact that \[ F_1[\kappa_+](\cdot, t) = \int_0^t (\partial_s F_1[\kappa_+])(\cdot, s) \, ds. \] In addition, by (126) and (127):

\[ \sum_{\ell = +, -} \left\{ \| e^{-\ell t} F_{2\ell}[\mathbf{w} \pm] \|_{L^1_0(\mathbb{R}, L^q_0(\Omega_\pm))} + \| e^{-\ell t} F_{2\ell}[\mathbf{w}] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \right\} \leq CR. \tag{130} \]

Moreover, we have:

\[ \| e^{-\ell t} \Lambda^{1/2} F_{2\pm}[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq CR, \quad \| e^{-\ell t} F_{2\pm}[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq CR, \tag{131} \]

\[ \| \partial_1 F_{3\pm}[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, L^q_0(\Omega_\pm))} \leq CR, \quad \| \partial_1 F_{3\pm}[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq CR. \tag{132} \]

In fact, as was seen in Shibata and Shimizu ([22], \( L_{p, \gamma}(\mathbb{R}, W^2_q(\Omega_\pm)) \) \( \cap \) \( W^1_{p, \gamma}(\mathbb{R}, L_q(\Omega_\pm)) \)) is continuously imbedded into \( H^{1/2}_p(\mathbb{R}, W^1_q(\Omega_\pm)) \), and so, we have the first estimate in (131) by (130). Replacing the Fourier multiplier theorem of the Mihlin type [23] by that of Bourgain [12] (cf. Lemma 2) in the paper due to Calderón [24] about the Bessel potential space (cf. Amann [25]), we see that \( H^{1/2}_p(\mathbb{R}, L_q(\Omega_\pm)) \) is continuously imbedded into the space \( \{ v \mid e^{-\ell t} v \in L^0(\mathbb{R}, L_q(\Omega_\pm)) \} \) if \( p > 2 \). Thus, we have:

\[ \| e^{-\ell t} F_{2\pm}[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq C \| e^{-\ell t} \Lambda^{1/2} F_{2\pm}[\mathbf{w}_\pm] \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))}, \]

and therefore, the second estimate in (131) follows from the first one. Since \( \partial_1 F_{3\pm}[\mathbf{w}_\pm] = \nabla F_{2\pm}[\mathbf{w}_\pm] \) for \( 0 \leq t \leq T, \partial_1 F_{3\pm}[\mathbf{w}_\pm] = -\nabla F_{2\pm}[\mathbf{w}_\pm](\cdot, T-t) \) for \( T \leq t \leq 2T, \) and \( \partial_1 F_{3\pm}[\mathbf{w}_\pm] = 0 \) for \( t \notin [0, 2T], \) (132) follows from (131) and (130).

We choose \( T \in (0, 1) \) so small that:

\[ CRT^{1/p'} < \rho_0/4, \quad CRT^{1/p'} < \sigma/2, \tag{133} \]

and therefore, we can define \( p(\rho_{0+} + \theta_{0+} + \tau F_1[\kappa_+]) \) (\( 0 \leq \tau \leq 1 \)) and \( V_\ell(F_{3\pm}[\mathbf{w}_\pm]) \) (\( \ell = 0, D, -1 \)). Since \( V_\ell(0) = 0 \) (\( \ell = 0, D, -1 \)), by (129) and (132):

\[ \| V_\ell(F_{3\pm}[\mathbf{w}_\pm]) \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq C R^{1/p'}, \quad \| \partial_1 V_\ell(F_{3\pm}[\mathbf{w}_\pm]) \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq CR, \tag{134} \]

\[ \| \partial_1 V_\ell(F_{3\pm}[\mathbf{w}_\pm]) \|_{L^1_0(\mathbb{R}, L^q_0(\Omega_\pm))} \leq CR, \quad \| \partial_1 V_\ell(F_{3\pm}[\mathbf{w}_\pm]) \|_{L^1_0(\mathbb{R}, W^1_q(\Omega_\pm))} \leq CR \]

for \( \ell = 0, D, -1 \).

We define \( f_+(x_+, w_+), g_+(x_+, w_+), g_-(w_+), f_-(w_-) = \text{div } f_-(w_-), h(x_+, w_\pm), \) and \( h_-(w_-) \) by:

\[ f_+(x_+, w_+) = -\{ F_1[\kappa_+] \text{div } F_2[w_+] + (\rho_{0+} + \theta_{0+} + F_1[\kappa_+]) \text{tr}(V_0(F_3[\mathbf{w}_+]) \nabla F_2[w_+]) \}, \]

\[ g_+(x_+, g_+) = -F_1[\kappa_+] \nabla F_2[w_+] + \text{Div } \{ \mu_+ V_D(F_3[\mathbf{w}_+]) \nabla F_2[w_+] + (\nu_+ - \mu_+) \text{tr}(V_0(F_3[\mathbf{w}_+]) \nabla F_2[w_+] \mathbf{I}) \} \]

\[ + V_0(F_3[\mathbf{w}_+]) \nabla \{ \mu_+ (D(F_2[w_+]) + V_D(F_3[\mathbf{w}_+])) \nabla F_2[w_+] \} \]

\[ + (\nu_+ - \mu_+) \text{div } F_2[w_+] + \text{tr}(V_0(F_3[\mathbf{w}_+]) \nabla F_2[w_+] \mathbf{I}) \]

\[ - \nabla \left( \int_0^1 p''(\rho_{0+} + \theta_{0+} + F_1[\kappa_+])(1 - \tau) \, d\tau \, (F_1[\kappa_+])^2 \right) \]

\[ - V_0(F_3[\mathbf{w}_+]) p'(\rho_{0+} + \theta_{0+} + F_1[\kappa_+]) \nabla (\theta_{0+} + F_1[\kappa_+]), \]
\[ g_-(w_-) = -\rho_0 + \nu_1(F_3[w_-]) \partial_t F_2 + \mu_1 \text{Div} (V_D(F_3[w_-]) \nabla F_2[w_-]) + V_1(F_3[w_-]) \text{Div} \{ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \}, \]
\[ f_-(w_-) = -\nu(\nabla V_0(F_3[w_-]) \nabla F_2[w_-]), \]
\[ f_{-1}(w_-) = T V_0(F_3[w_-]) F_2[w_-], \]
\[ h(x_+, w_{+\pm}) = h^1(w_{\pm}) + h^2(x_+), \]
\[ h_-(w_-) = -\nu(1 - \nu) \text{Ext}^+ \left[ V_D(F_3[w_-]) \nabla F_2[w_-] \right] + V_1(F_3[w_-]) \text{Div} \{ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \} + (I + \nu(1 - \nu)) \text{Div} \{ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \} V_0(F_3[w_-]), \]

where we set:
\[ h^1(w_{\pm}) = -\nu(1 - \nu) \text{Ext}^+ \left[ V_D(F_3[w_-]) \nabla F_2[w_-] \right] - (\nu - \nu) \text{Ext}^+ \left[ \nu(1 - \nu) \text{Ext} \left[ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \right] \right] n_+ \]
\[ -\nu(1 - \nu) \text{Ext}^+ \left[ V_1(F_3[w_-]) \text{Ext} \left[ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \right] \right] n_+ \]
\[ -\nu(1 + \nu) \text{Ext}^+ \left[ V_1(F_3[w_-]) \text{Ext} \left[ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \right] \right] n_+ \]
\[ + \nu(1 + \nu) \text{Ext}^+ \left[ V_1(F_3[w_-]) \nabla F_2[w_-] + V_1(F_3[w_-]) (D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-]) + (1 + \nu(1 - \nu)) \text{Div} \{ D(F_2[w_-]) + V_D(F_3[w_-]) \nabla F_2[w_-] \} V_0(F_3[w_-]) \right] n_+ \]

and:
\[ h^2(x_+) = \text{Ext}^+ \left[ \int_0^1 (1 - \tau) \nu(\rho_0 + \theta_0 + \nu F_1(x_+)) d\tau \right] (F_1[x_+])^2 n. \]

By (128), we have:
\[ f_+(x_+, w_+) = F_+(\Pi^0_+ + x_+, Z^\nu_+ + w_+), \quad g_+(x_+, w_+) = G_+(\Pi^0_+ + x_+, Z^\nu_+ + w_+), \quad g_-(w_-) = G_-(Z^\nu_- + w_-), \quad f_-(w_-) = F_-(Z^\nu_- + w_-), \]
\[ h(x_+, w_{\pm})|_{\Gamma} = H(\Pi^0_+ + x_+, Z^\nu_\pm + w_\pm), \quad h_-(w_-)|_{\Gamma} = H_-(Z^\nu_- + w_-) \]
for \( 0 < t < T \). By (120), (129), (130), and (134), we have:
\[ \| e^{-\gamma t} f_+(x_+, w_+) \|_{L^p(\mathbb{R}; L^q(\Omega))} + \| e^{-\gamma t} g_+(x_+, w_+) \|_{L^p(\mathbb{R}; L^q(\Omega))} + \| e^{-\gamma t} g_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} \]
\[ + \| e^{-\gamma t} \nabla f_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} + \| e^{-\gamma t} \nabla h_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} + \| e^{-\gamma t} \nabla h_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} \]
\[ \leq C_\xi T^{1/p'} \]

with some constant \( C_\xi \) depending on \( R \). Since \( \partial_t F_3[w_-] = 0 \) for \( t \notin [0, T] \), we have:
\[ \| e^{-\gamma t} \partial_t f_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} \leq \| V_0(F_3[w_-]) \|_{L^1(\mathbb{R}; L^q(\Omega))} \| e^{-\gamma t} F_2[w_-] \|_{L^1(\mathbb{R}; L^q(\Omega))} + T^{1/p} \| \partial_t V_0(F_3[w_-]) \|_{L^1(\mathbb{R}; L^q(\Omega))} \| e^{-\gamma t} F_2[w_-] \|_{L^1(\mathbb{R}; L^q(\Omega))}, \]

and so, by (134), (131), and (130), we have:
\[ \| e^{-\gamma t} \partial_t f_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} \leq C_\xi T^{1/p}. \]

To estimate \( \| e^{-\gamma t} \Lambda^{1/2}_y f_-(w_-) \|_{L^p(\mathbb{R}; L^q(\Omega))} \), we use the following lemma due to Shibata and Shimizu ([21] Lemma 2.6).
Lemma 11. Let $2 < p < \infty$, $N < q < \infty$, and $0 < T \leq 1$. Let $f \in L_\omega(\mathbb{R}, W^1_q(\Omega_+)) \cap W^2_0(\mathbb{R}, L_q(\Omega_+))$ and $g \in H^{1/2}(\mathbb{R}, L_q(\Omega_-)) \cap L_{p,\gamma}(\mathbb{R}, W^1_q(\Omega_-))$. If $\partial_t f \in L_p(\mathbb{R}, W^1_q(\Omega_-))$ and $f(\cdot, t) = 0$ for $t \not\in [0, 2T]$, then we have:

\[
\|e^{-\gamma t}A^{1/2}_\gamma (fg)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
\leq C \{ \|f\|_{L_\omega(\mathbb{R}, W^1_q(\Omega_-))} + T^{(q-N)/(pq)} \|\partial_t f\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))} \|\partial_t f\|_{L_p(\mathbb{R}, L^1_q(\Omega_-))} \} \\
x (\|e^{-\gamma t}A^{1/2}_\gamma\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))} + \|\partial_t f\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))}).
\]

Applying Lemma 11 to $f_-(w_-)$, we have:

\[
\|e^{-\gamma t}A^{1/2}_\gamma f_-(w_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
\leq C \{ \|V_0(F_3 \cdot w_-)\|_{L_\omega(\mathbb{R}, W^1_q(\Omega_-))} + T^{(q-N)/(pq)} \|\partial_t V_0(F_3 \cdot w_-)\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))} \} \\
x (\|e^{-\gamma t}A^{1/2}_\gamma \nabla F_2 - w_-\|_{L_p(\mathbb{R}, L_q(\Omega_-))} + \|e^{-\gamma t}A^{1/2}_\gamma \nabla F_2 - w_-\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))}),
\]

and so, by (130) and (134):

\[
\|e^{-\gamma t}A^{1/2}_\gamma f_-(w_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \leq C_R(T^{1/p'} + T^{(q-N)/(pq)}).
\] (138)

Analogously, we have:

\[
\|e^{-\gamma t}A^{1/2}_\gamma h_1(w_\pm)\|_{L_p(\mathbb{R}, L_q(\Omega_\pm))} + \|e^{-\gamma t}A^{1/2}_\gamma h_2(w_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \leq C_R(T^{1/p'} + T^{(q-N)/(pq)}).
\] (139)

Since \[\|e^{-\gamma t}A^{1/2}_\gamma h^2(\kappa_+)\|_{L_p(\mathbb{R}, L_q(\Omega_+))} \leq C\|e^{-\gamma t}\partial_\kappa h^2(\kappa_+)\|_{L_p(\mathbb{R}, L_q(\Omega_+))},\] by (120), (129), and (130), we have:

\[
\|e^{-\gamma t}A^{1/2}_\gamma h^2(\kappa_+)\|_{L_p(\mathbb{R}, L_q(\Omega_+))} \leq C_R T^{1/p'}.
\] (140)

Let $\rho_+$ and $v_\pm$ be solutions to equations:

\[
\partial_t \rho_+ + (\rho_0 + \theta_0)\text{div } v_+ = f_+(\kappa_+, w_+) \quad \text{in } \Omega_+,
\]

\[
(\rho_0 + \theta_0)\partial_\kappa v_+ - \text{div } S_+(v_+) + \nabla (p_+\rho_0 + \theta_0) = g(v_+, w_+) \quad \text{in } \Omega_+,
\]

\[
\rho_0 \partial_\kappa v_- - \text{div } S_-(v_-) + \nabla p_- = g(v_-, w_-) \quad \text{in } \Omega_-,
\]

\[
\text{div } v_- = f_- - (w_-) \quad \text{in } \Omega_-,
\]

\[
(S_+(v_+) - (p_+(\rho_0 + \theta_0)\rho_0 + \rho_0, 1)n|_{r=0} - (S_-(v_-) - p_-)n|_{r=0} = h(\kappa_+, w_\pm)|_{r},
\]

\[
v_+|_{r=0} = v_-|_{r=0}, \quad v_+|_{r}= 0, \quad (S_-(v_-) - p_-)n|_{r}= h_-(w_-)|_{r}.
\] (141)

for $0 < t < T$ subject to the initial condition: $(\rho_+, v_+)|_{t=0} = (0, 0)$ in $\Omega_+$ and $v_-|_{t=0} = 0$ in $\Omega_-$ with some pressure term $p_-$. By Theorem 3 and the estimates (136), (137), (139), and (140), we have:

\[
\rho_+ \in W^{1, 2}(\mathbb{R}, W^1_q(\Omega_+)) \cap W^{1, 0}_0(\mathbb{R}, L_q(\Omega_\pm)) \cap L_{p, \gamma, 0}(\mathbb{R}, W^2_q(\Omega_\pm)),
\]

\[
\|e^{-\gamma t}p\|_{L_p(\mathbb{R}, W^1_q(\Omega_+))} + \sum_{\ell = 1}^{\omega} \|e^{-\gamma t}(\partial_\kappa v_\ell, A^{1/2}_\gamma \nabla v_\ell, \nabla^2 v_\ell)\|_{L_p(BR_0, L_q(\Omega_\pm))} \leq C_R T^\omega,
\] (142)

with some constant $C_R$ depending on $R$ and $\omega = \min(1/p', (q - N)/(pq))$. By (135), $\rho_+$ and $v_\pm$ satisfy equations:

\[
\partial_t \rho_+ + (\rho_0 + \theta_0)\text{div } v_+ = F_+ (\Pi_+ + \kappa_+, Z^\prime_+ + w_+) \quad \text{in } \Omega_+,
\]

\[
(\rho_0 + \theta_0)\partial_\kappa v_+ - \text{div } S_+(v_+) + \nabla (p_+(\rho_0 + \theta_0)\rho_0) = G_+ (\Pi_+ + \kappa_+, Z^\prime_+ + w_+) \quad \text{in } \Omega_+,
\]

\[
\rho_0 \partial_\kappa v_- - \text{div } S_-(v_-) + \nabla p_- = G_-(Z^\prime_- + w_-) \quad \text{in } \Omega_-,
\]

\[
\text{div } v_- = \cdots.
\]
\[
\begin{align*}
\text{div } v_- &= f_-(Z_-^t + w_-) \quad \text{in } \Omega_-,
(S_+(v_+)) - (p'(\rho_0 + \theta_0)\rho_0 I) n|_{\Gamma^+} - (S_-(v_-) - p_- I) n|_{\Gamma^-} = H(I(1\mathbb{I}_p^0 + \kappa_+, Z_+^t + w_+),

v_+|_{\Gamma^+} = v_-|_{\Gamma^-}, \quad v_+|_{\Gamma_+} = 0, \quad (S_-(v_-) - p_- I) n_-|_{\Gamma_+} = H_-(Z_-^t + w_-)
\end{align*}
\]

for \(0 < t < T\) subject to the initial condition: \((\rho_+, v_+)|_{t=0} = (0, 0)\) in \(\Omega_+\) and \(v_-|_{t=0} = 0\) in \(\Omega_-\) with some pressure term \(p_-\).

Let \(\Phi\) be a map defined by \(\Phi(v_+, w_+) = \), the restriction of \((\rho_+, v_+)\) to the time interval \((0, T)\). Since:

\[\|\rho\|_{L^1((0,T),W^1_2(\Omega_+))} + \sum_{\ell=+,-} \|v_\ell\|_{L^p((0,T),W^{1,2}_2(\Omega_\ell))} \in C_T^\epsilon C_R T^\omega\]

as follows from (143), choosing \(T > 0\) so small that \(C_T^\epsilon C_R T^\omega \leq R\), we see that \(\Phi\) is the map from \(I_{R,T}\) into itself. Choosing \(T > 0\) smaller if necessary, we can show that \(\Phi\) is a contraction map on \(I_{R,T}\), and so by the Banach fixed point theorem \(\Phi\) has a unique fixed point \((\rho_+, v_+)\) that solves Equation (125) uniquely. This completes the proof of Theorem 1.

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