Removal of the Resolvent-like Dependence on the Spectral Parameter from Perturbations

The spectral problem \((A + V(z))\psi = z\psi\) is considered with \(A\), a self-adjoint operator. The perturbation \(V(z)\) is assumed to depend on the spectral parameter \(z\) as resolvent of another self-adjoint operator \(A' : V(z) = -B(A' - z)^{-1}B^*\). It is supposed that the operator \(B\) has a finite Hilbert-Schmidt norm and spectra of the operators \(A\) and \(A'\) are separated. Conditions are formulated when the perturbation \(V(z)\) may be replaced with a “potential” \(W\) independent of \(z\) and such that the operator \(H = A + W\) has the same spectrum and the same eigenfunctions (more precisely, a part of spectrum and a respective part of eigenfunctions system) as the initial spectral problem. The operator \(H\) is constructed as a solution of the non-linear operator equation \(H = A + V(H)\) with a specially chosen operator-valued function \(V(H)\). In the case if the initial spectral problem corresponds to a two-channel variant of the Friedrichs model, a basis property of the eigenfunction system of the operator \(H\) is proved. A scattering theory is developed for \(H\) in the case where the operator \(A\) has continuous spectrum.

1. Introduction

Perturbations, depending on the spectral parameter (usually energy of system) arise in a lot of quantum-mechanical problems typically (see e.g. Ref. [1]) as a result of dividing the Hilbert space \(H\) of physical system in two subspaces, \(H = H_1 \oplus H_2\). The first one, say \(H_1\), is interpreted as a space of some “external” degrees of freedom. The second one, \(H_2\), is associated with an “internal” structure of the system. The Hamiltonian \(H\) of the system looks as a matrix,

\[
H = \begin{bmatrix}
A_1 & B_{12} \\
B_{21} & A_2
\end{bmatrix}
\]

(1)

with \(A_\alpha, \alpha = 1, 2\), the channel Hamiltonians (self-adjoint operators in \(H_\alpha\)) and \(B_{12}, B_{21} = B_{21}^*\), the coupling operators. Reducing the spectral problem \(HU = zU\), \(U = \{u_1, u_2\}\) to the channel \(\alpha\) only one gets the spectral problem

\[
[A_\alpha + V_\alpha(z)]u_\alpha = zu_\alpha, \quad \alpha = 1, 2,
\]

(2)

where the perturbation

\[
V_\alpha(z) = -B_{\alpha\beta}(A_\beta - zI_\beta)^{-1}B_{\beta\alpha}, \quad \beta \neq \alpha,
\]

(3)

depends on the spectral parameter \(z\) as the resolvent \((A_\beta - zI_\beta)^{-1}\) of the Hamiltonian \(A_\beta\). Here, by \(I_\beta\) we understand the identity operator in \(H_\beta\).

The present paper is a summary of the author’s works [2]—[4] considering a possibility to “remove” the energy dependence from perturbations of the type (3). Namely, in [2]—[4] we search for such a new perturbation (“potential”) \(W_\alpha\) not depending on \(z\) that spectrum of the Hamiltonian \(H_\alpha = A_\alpha + W_\alpha\) is (a part of) the spectrum of the problem (3). At the same time, the respective eigenvectors of \(H_\alpha\) become also those for (3). An interest to the problem of such a removal of dependence on the spectral parameter from perturbations is stimulated in particular by a rather conceptual question (see for instance Ref. [5]) concerning a use of the two-body energy-dependent potentials in few-body nonrelativistic scattering problems. Since the energies of pair subsystems are not fixed in the N-body (N > 3) system, a direct embedding of such potentials into the few-body Hamiltonian is impossible. Thus, the replacements of the type (3) energy-dependent potentials with the respective new potentials \(W_\alpha\) could be considered as a way to overcome this difficulty (see Ref. [4] for discussion).

The Hamiltonians \(H_\alpha\) are found in [2]—[4] as solutions of the non-linear operator equations (first appeared in
The operator-value function $V_\alpha(Y)$ of the operator variable $Y$, $Y : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$, is defined by us in such a way [see formula (1)] that eigenfunctions $\psi$ of $Y$, $Y\psi = z\psi$, become automatically those for $V_\alpha(Y)$ and $V_\alpha(Y)\psi = V_\alpha(z)\psi$. We have proved a solvability of Eq. (2) in the case where the Hilbert–Schmidt norm $\|B_{\alpha\beta}\|_2$ of the operators $B_{\alpha\beta}$ satisfies the condition $\|B_{\alpha\beta}\|_2 < \frac{1}{4}\text{dist}(\sigma(A_1), \sigma(A_2))$ in supposition that spectra $\sigma(A_\alpha)$ of the operators $A_\alpha$ are separated, $\text{dist}(\sigma(A_1), \sigma(A_2)) > 0$ (see Theorem 1).

In Ref. [2], the problem of the removal of energy dependence from the type (2) perturbations was considered in details when one of the operators $A_\alpha$ is the Schrödinger operator in $L^2(\mathbb{R}^n)$ and another one has a discrete spectrum only. The report [3] announces the results concerning the equations (4) and properties of their solutions $H_\alpha$ in a rather more general situation where the Hamiltonian $H$ may be rewritten in terms of a two-channel variant of the Friedrichs model investigated by O.A.LADYZHENSKAYA and L.D.FADDEEV [7]. In particular in [2] and [3] a scattering problem is studied for $H_\alpha$ in the case if $A_\alpha$ has continuous spectrum and the basis property of the eigenfunction system of the operator $H_\alpha$ is shown.

In the paper [4], we specify the assertions from [3] and give proofs for them. Also, we pay attention to an important circumstance disclosing a nature of solutions of the basic equations (4). Thing is that the operators $W_\alpha = V_\alpha(H_\alpha)$ may be present in the form $W_\alpha = B_{\alpha\beta}Q_{\beta\alpha}$ with $Q_{\beta\alpha}$ satisfying the stationary Riccati equations (3). Exactly the same equations always arise if one makes a block diagonalization of the type (2) operator matrices in the way described below in Lemma 1, so that the solutions $H_\alpha$, $\alpha = 1, 2$, of Eqs. (2) determine in fact parts of the operator $H$ in respective invariant subspaces. The idea of such a diagonalization was applied already by S.OKUBO [8] to some quantum–mechanical Hamiltonians. It was used later by V.A.MALYSHEV and R.A.MINLOS [9] in a method of construction of invariant subspaces for a class of self–adjoint operators in statistical physics. This idea was used also in the recent paper [10] by V.M.ADMIAN and H.LANGER who studied spectral properties of a class of the type (2) spectral problems and in particular, a possibility to choose among their solutions a Riesz basis in $\mathcal{H}_\alpha$.

2. Construction of the operators $H_\alpha$

We study the spectral problem (2) with perturbation $V_\alpha(z)$ given by (3). We suppose that $B_{\beta\alpha}$ is a linear operator from $\mathcal{H}_\alpha$ to $\mathcal{H}_\beta$ with a finite Hilbert–Schmidt norm $\|B_{\beta\alpha}\|_2$, $\|B_{\beta\alpha}\|_2 < \infty$. A goal of the work is a construction of such an operator $H_\alpha$ that its each eigenfunction $u_\alpha$, $H_\alpha u_\alpha = z u_\alpha$, together with eigenvalue $z$, satisfies Eq. (2). The operator $H_\alpha$ is searched for as a solution of the non-linear operator equation (4). To obtain this equation we introduce the following operator-value function $V_\alpha(Y)$ of the operator variable $Y$:

$$V_\alpha(Y) = B_{\alpha\beta} \int_{\sigma_\beta} E_\beta(d\mu) B_{\beta\alpha}(Y - \mu I_\alpha)^{-1},$$

(5)

$Y : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$. Here, $\sigma_\beta$ is the spectrum and $E_\beta$, the spectral measure of the operator $A_\beta$. The integral over $E_\beta$ in (5) for $Y$ such that $\sup \|Y - \mu I_\alpha\|_\infty < \infty$ may be constructed in the same way as the usual integrals of scalar functions over spectral measure. For $\|B_{\beta\alpha}\|_2 < \infty$, the existence of this integral as a bounded operator from $\mathcal{H}_\alpha$ to $\mathcal{H}_\beta$ is proved in [4]. We notice that if $\phi$ is an eigenfunction of $Y$, $Y\phi = z\phi$, then automatically $V_\alpha(Y)\phi = B_{\alpha\beta} \int_{\sigma_\beta} E_\beta(d\mu) B_{\beta\alpha}(z - \mu)^{-1}\phi = B_{\alpha\beta}(z - A_\beta)^{-1}B_{\beta\alpha}\phi = V_\alpha(z)\phi$. This means that $H_\alpha$ satisfies with its eigenfunctions $\psi_\alpha$ the relation $H_\alpha \psi_\alpha = (A_\alpha + V_\alpha(H_\alpha))\psi_\alpha$ and one can spread this relation over all the linear combinations of the eigenfunctions. Supposing that the eigenfunctions system of $H_\alpha$ is dense in $\mathcal{H}_\alpha$, we can use Eq. (4) means that the construction of the operator $H_\alpha$ is reduced to the searching for the operator $Q_{\beta\alpha} = \int_{\sigma_\beta} E_\beta(d\mu) B_{\beta\alpha}(A_\alpha + B_{\alpha\beta}Q_{\beta\alpha} - \mu I_\alpha)^{-1}$. Since $H_\alpha = A_\alpha + B_{\alpha\beta}Q_{\beta\alpha}$, we have

$$Q_{\beta\alpha} = \int_{\sigma_\beta} E_\beta(d\mu) B_{\beta\alpha}(A_\alpha + B_{\alpha\beta}Q_{\beta\alpha} - \mu I_\alpha)^{-1}, \quad \beta \neq \alpha.$$

(6)

We restrict ourselves to a study of Eq. (3) solvability only in the case where spectra $\sigma_1$ and $\sigma_2$ are separated, $d_0 = \text{dist}(\sigma_1, \sigma_2) > 0$. Applying to Eq. (3) the contracting mapping theorem, one comes to the following:

**Theorem 1.** Let $M_{\beta\alpha}(\delta)$ be a set of bounded operators $X$, $X : \mathcal{H}_\alpha \to \mathcal{H}_\beta$, satisfying the inequality $\|X\| \leq \delta$
with $\delta > 0$. If this $\delta$ and the norm $\|B_{\alpha\beta}\|_2$ satisfy the condition $\|B_{\alpha\beta}\|_2 < d_0 \min\{\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\}$, then Eq. (4) is uniquely solvable in $M_{\beta\alpha}(\delta)$. In particular the equation (3) is uniquely solvable in the unit ball $M_{\beta\alpha}(1)$ for any $B_{\alpha\beta}$ such that $\|B_{\alpha\beta}\|_2 < \frac{1}{2}d_0$.

Eq. (3) can be rewritten (see [3], [4]) also in symmetric form as a stationary Riccati equation,

$$Q_{\beta\alpha}A_{\alpha} - A_{\beta}Q_{\beta\alpha} + Q_{\beta\alpha}B_{\alpha\beta}Q_{\beta\alpha} = B_{\beta\alpha}. \quad (7)$$

One finds immediately from Eqs. (3), $\alpha = 1, 2$, that if $Q_{\beta\alpha}$ gives a solution $H_{\alpha} = A_{\alpha} + B_{\alpha\beta}Q_{\beta\alpha}$ of the problem (4) in the channel $\alpha$ then $Q_{\alpha\beta}^* = -Q_{\beta\alpha}^* = -\int (H_{\alpha}^* - \mu I_{\alpha})^{-1}B_{\alpha\beta}E_{\beta}(d\mu)$ gives an analogous solution $H_{\beta} = A_{\beta} + B_{\beta\alpha}Q_{\alpha\beta}$ in the channel $\beta$.

**Lemma 1.** Let $Q_{\beta\alpha}$ and $Q_{\alpha\beta} = -Q_{\beta\alpha}^*$ be solutions of Eqs. (3). Then the transform $H' = Q^{-1}HQ$ with $Q = \begin{bmatrix} I_1 & Q_{12} \\ Q_{21} & I_2 \end{bmatrix}$ reduces the operator $H$ to the block-diagonal form, $H' = \text{diag}\{H_1, H_2\}$ with $H_{\alpha} = A_{\alpha} + B_{\alpha\beta}Q_{\beta\alpha}$.

One can find assertions analogous to Lemma 1 in Refs. [9] and [10]. A solvability (for sufficiently small $\|B_{\alpha\beta}\|$) of the equation (3) was proved in [9], [10] for different situations and by rather different methods but also in the supposition $\text{dist}\{\sigma(A_{\alpha}), \sigma(A_{\beta})\} > 0$.

**Remark 1.** Let $X_{\alpha} = I_{\alpha} - Q_{\beta\alpha}Q_{\beta\alpha} = I_{\alpha} + Q_{\beta\alpha}Q_{\alpha\beta}$, and $H_{\alpha} = A_{\alpha} + B_{\alpha\beta}X_{\alpha}$.

Since $H_{\alpha} = \text{diag}\{H_{\alpha}^0, H_{\alpha}^0\}$ with $H_{\alpha}^0 = X_{\alpha}^{-1}H_{\alpha}X_{\alpha}^{-1}$, the operators $H_{\alpha}^0$, $\alpha = 1, 2$, are self-adjoint on $D(A_{\alpha})$ in $H_{\alpha}$. Moreover, the operators $H'_{\alpha} = Q^\dagger \cdot \text{diag}\{H_{\alpha}^0, 0\} \cdot Q^\dagger = Q \cdot \text{diag}\{H_{\alpha}, 0\} = Q^{-1}$ represent parts of the Hamiltonian $H$ in the corresponding invariant subspaces $H_{\alpha}^0 = \{f : f = \{f_{\alpha}, f_{\beta}\} \in H, f_{\alpha} \in H_{\alpha}, f_{\beta} = Q_{\beta\alpha}f_{\alpha}\}$ (see also Refs. [9], [10]).

### 3. Spectra of the Hamiltonians $H_{\alpha}$ and basis properties of their eigenfunctions

Let us suppose that $Q_{\beta\alpha}$ and $Q_{\alpha\beta} = -Q_{\beta\alpha}^*$ are solutions of Eqs. (3) and (7) which are spoken about in Theorem 1. Since we take $B_{\alpha\beta}$ with $\|B_{\alpha\beta}\|_2 < \|B_{\alpha\beta}\|_2 < d_0/2$ and $\|Q_{\beta\alpha}\| < 1$, the spectra $\sigma(H_1)$ and $\sigma(H_2)$ do not intersect (actually, when these spectra are discussed in Refs. [3], [4], a more general case is also considered where not necessary $\|Q_{\beta\alpha}\| < 1$). By Lemma 1, the operator $H' = \text{diag}\{H_1, H_2\}$ is connected with the (self-adjoint) operator $H$ by a similarity transform. Thus, the spectra $\sigma(H_1)$ and $\sigma(H_2)$ of the operators $H_{\alpha}$, $\alpha = 1, 2$, are real and $\sigma(H_1) \cup \sigma(H_2) = \sigma(H)$. Continuous spectrum $\sigma_c(H_{\alpha})$ of every $H_{\alpha}$ coincides with that, $\sigma_c^0$, of the operator $A_{\alpha}$, $\sigma_c(H_{\alpha}) = \sigma_c^0$, since due to $\|B_{\alpha\beta}\| < +\infty$, the potential $W_{\alpha} = B_{\alpha\beta}Q_{\beta\alpha}$ is a compact operator.

For more concrete statements concerning the spectra of the operators $H_{\alpha}$ we adopt some presuppositions restricting us as regards $H$ to the case of a two-channel variant of the Friedrichs model in the form [7] reproducing often encountered quantum-mechanical situations. At first, we assume that the operator $H$ is defined in that representation where the operators $A_{\alpha}$, $\alpha = 1, 2$, are diagonal. We suppose that the continuous spectra $\sigma_{\alpha}^0$ are absolutely continuous and consist of a finite number of finite (and may be one or two infinite) intervals. At second, we suppose that discrete spectra $\sigma_{\alpha}^0$ of the operators $A_{\alpha}$, $\alpha = 1, 2$, do not intersect with $\sigma_{\alpha}^0$, $\sigma_{\alpha}^0 \bigcap \sigma_{\alpha}^0 = \emptyset$, and consist of a finite number of points with finite multiplicity. The coupling operators $B_{\alpha\beta}$ are supposed to be the integral ones with sufficiently quickly decreasing (in the case of unbounded $\sigma_{\alpha}^0$) kernels being smooth in the Hölder sense (see Refs. [3], [4] for details).

With these presuppositions the continuous spectrum $\sigma_{\alpha}(H) = \sigma_{\alpha}^0 \bigcup \sigma_{\alpha}^0$ of the operator $H$ is absolutely continuous and its part $H^c$ acting in respective invariant subspace, is unitary equivalent to the operator $H_0 = A_{\alpha}^{(0)} \oplus A_{\beta}^{(0)}$ with $A_{\alpha}^{(0)}$, $\alpha = 1, 2$, the part of $A_{\alpha}$ acting in the invariant subspace $H_{\alpha}^c$ corresponding to $\sigma_{\alpha}^0$. Namely, there exist the wave operators $U^{(+)}$ and $U^{(-)}$, $U^{(\pm)} = \begin{pmatrix} \pm u_{11}^{(\pm)} & u_{12}^{(\pm)} \\ u_{21}^{(\pm)} & \pm u_{22}^{(\pm)} \end{pmatrix}$, $s = \lim_{t \to \pm \infty} e^{Ht}e^{-Ht^*}$ with the properties:

$H^{(\pm)} = U^{(\pm)}H_0, U^{(\pm)}H^{(\pm)*} = I, U^{(\pm)}H^{(\pm)*} = I - P_\delta$. Here, by $P_\delta$ we understand the orthogonal projector on the subspace corresponding to the discrete spectrum $\sigma_d(H)$ of $H$. The kernel $u_{\alpha\alpha}^{(\pm)}(\lambda, \lambda')$ of the component $u_{\alpha\alpha}^{(\pm)}$, $\alpha = 1, 2$, represents a (generalized) eigenfunction of continuous spectrum of the problem (3) for $z = \lambda' \pm i0, \lambda' \in \sigma_{\alpha}^0$. At the same time $u_{\alpha\alpha}^{(\pm)}(\lambda, \lambda')$ is the problem (3) eigenfunction corresponding to $\lambda' \in \sigma_{\beta}^0$.

By $U_j$, $j = 1, 2, \ldots$, we denote eigenvectors, $U_j = \{u_1^{(j)}, u_2^{(j)}\}$, $\|U_j\| = 1$, and by $z_j, z_j \in \mathbb{R}$, the respective eigenvalues of $\sigma_d(H)$. We assume that in the case of multiple discrete eigenvalues, certain $z_j$ may be repeated in the
numeration. The component \( u^{(j)}_\alpha \) of the vector \( U_j \) is a solution of Eq. (3) for \( z = z_j \).

Let us return, with the presuppositions above, to the operators \( H_\alpha \). First, let us assume \( \sigma_d(H_\alpha) \neq \emptyset \). Then, it follows from the construction of the function (3) that if \( z \in \sigma_d(H_\alpha) \) then this \( z \) becomes automatically a point of the discrete spectrum of the initial spectral problem (3). At the same time \( \psi_\alpha \) becomes its eigenfunction. We shall denote the eigenfunctions of the \( H_\alpha \) by \( \psi_\alpha^{(j)} \), \( \psi_\alpha^{(j)} = u^{(j)}_\alpha \), keeping for them the same numeration as for the eigenvectors \( U_j, U_j = \{ u^{(j)}_\alpha, u^{(j)}_\beta \} \), of the Hamiltonian \( \mathbf{H}, \mathbf{H}U_j = z_j U_j, z_j \in \sigma_d(\mathbf{H}) \). Respectively eigenvectors of the adjoint operator \( H^*_\alpha, H^*_\alpha = A_\alpha + Q^*_\beta \alpha B^*_{\beta \alpha}, \) are \( \tilde{\psi}^{(j)}_\alpha = \psi_\alpha^{(j)} - Q_{\alpha \beta}u^{(j)}_\beta \). Due to Lemma 1, \( \sigma_d(\mathbf{H}) = \sigma_d(H_1) \cup \sigma_d(H_2). \) Since in conditions of Theorem 1 \( \sigma(H_1) \cap \sigma(H_2) = \emptyset \), we have also \( \sigma_d(H_1) \cap \sigma_d(H_2) = \emptyset \).

Dealing with the continuous spectrum of \( H_\alpha \) we take into account the fact that the solutions \( Q_{\beta \alpha} \) and \( Q_{\alpha \beta} = -Q^*_\beta \alpha \) of Eqs. (3) and (4) corresponding to the operators \( B_{\alpha \beta} \) with Hölder kernels, have the Hölder kernels themselves. The same is true as well for \( W_{\alpha \beta} = B_{\alpha \beta}Q_{\beta \alpha} \). Then we can prove [3], [4] that the operators \( \tilde{\Psi}^{(\pm)}_\alpha = u^{(\pm)}_\alpha \) turn out to be the wave operators between \( H_\alpha \) and \( A^{(0)}_\alpha: \tilde{\Psi}^{(\pm)}_\alpha = \lim_{t \to \pm \infty} \exp(iH_\alpha t) \exp(-iA^{(0)}_\alpha t) \) and \( H_\alpha \tilde{\Psi}^{(\pm)}_\alpha = \tilde{\Psi}^{(\pm)}_\alpha A^{(0)}_\alpha \). At the same time the operators \( \tilde{\Psi}^{(\pm)}_\alpha = \tilde{\Psi}^{(\pm)}_\alpha - Q_{\alpha \beta}u^{(\pm)}_{\beta \alpha} \) become those ones for \( H^*_\alpha \).

**Theorem 2.** The following orthogonality relations take place: \( \langle \psi_\alpha^{(j)}, \psi_\alpha^{(k)} \rangle = \delta_{jk}, \psi^{(\pm)*}_\alpha \tilde{\Psi}^{(\pm)}_\alpha = I_\alpha|_{H^*_\alpha}, \tilde{\Psi}^{(\pm)}_\alpha = \psi^{(\pm)}_\alpha \). For all this \( S^{(\alpha)} = \tilde{\Psi}^{(-)*}A^{(0)}_\alpha \) represents a scattering operator for a system described by the Hamiltonian \( H_\alpha \). In fact, this operator coincides with the component \( S_{\alpha \alpha} \) of the scattering operator \( S, S = U^{(-)*}U^{(+)} \), for a system described by the two–channel Hamiltonian \( \mathbf{H} \).

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