A STRING PROJECT IN MULTICOLOUR QCD

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Abstract

Some old and new evidence for the existence of the string (planar random surfaces) representation of multicolour QCD are reviewed. They concern the random surface representation of the strong coupling expansion in lattice multicolour gauge theory in any dimension.

Our old idea of modified strong coupling expansion in terms of planar random surfaces, valid for the physical weak coupling phase of the four-dimensional QCD, is explained in detail. Some checks of the validity of this expansion are proposed. (The lectures given in the Trieste Spring School and Workshop-1993 on String Theory)

1. Introduction

The principal model considered in this communication will be the Wilson lattice $U(N)$ gauge theory without quarks, particularly its large N (multicolour) limit.
The action of the theory is defined on the hypercubic D-dimensional lattice with the vertices labeled by the D-dimensional vectors \( x \) and the vectors of the links as \( \mu, \nu, \text{etc.} \) (so that the plaquette is denoted as \((x, \mu, \nu)\)).

\[
S[U] = N\beta \text{tr} \sum_{x,\mu,\nu} U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^+ U_{x,\nu}^+ + \text{compl. conj.} \tag{1.1}
\]

The partition function \( Z \), the free energy \( F \) and the Wilson average \( W(C) \) along a closed loop \( C \) on the lattice are now expressed as:

\[
Z = e^{N^2F} = \int \prod_{x,\mu} [dU_{x,\mu}]_H \exp N\beta S[U] \tag{1.2}
\]

\[
W(C) = \langle \text{tr} \prod_{(x,\mu)\in C} U_{x,\mu} \rangle \tag{1.3}
\]

with the obvious definition of the average \( \langle ... \rangle \).

The Haar measure \([dU]_H\) for the U(N) integrals is characterised by the unitarity condition

\[
U^+U = UU^+ = I \tag{1.4}
\]

The famous paper of K.Wilson \[1\], which explains the confinement of quarks in terms of the strong coupling (SC) expansion in lattice QCD, left a wide area in which to improve upon; this lattice “superconfinement” resulted in a string tension \( K(\beta) \) which was wrongly scaled to describe the weak coupling (WC) phase. Namely, from the Wilson calculations for the planar loop \( C \) of sufficiently large size one obtains in the (SC) limit the area law asymptotics:

\[
W(C) \sim \exp[-K(\beta)A_{\text{min}}(C)] \tag{1.5}
\]

where \( A_{\text{min}}(C) \) is the minimal area of the surface spent on \( C \), with

\[
K(\beta) \sim -\log(\beta) \tag{1.6}
\]

instead of the correct physical asymptotics

\[
K(\beta) \sim \Lambda^2 \exp[-\frac{48\pi^2}{11}\beta] \tag{1.7}
\]
dictated by the dimensional transmutation mechanism of asymptotically free theory.

It is widely believed, after years of computer simulations of the lattice SU(N) gauge theory, that in the case \( N=2 \) and \( N=3 \) gauge groups, if we move from small
\( \beta \), where (1.6) is valid, to larger \( \beta \), we meet a sharp crossover: an abrupt change in behaviour in physical quantities as functions of \( \beta \), before we reach the physical confinement with the string tension (1.7). Nobody knows whether we can describe at least in principle the physical phase by summing up the SC series. Even if the answer is positive, practically, it turns out to be impossible.

The situation becomes even worse if \( N \) increases. The crossover turns into a real first order phase transition, toppling any hope that the SC expansion can be extended into the physical phase. This phase transition corresponds to the breakdown of the \( Z(N) \) symmetry of the centre of the gauge group. It has indeed to be broken in the physical phase of the theory, since we know very well that each gauge matrix \( U \) on the lattice has to fluctuate close to some singled out value on the group, which leaves no space for the discrete \( Z(N) \) degrees of freedom. This phase transition is clearly seen even in rather naive mean field calculations.

In the limit of large \( N \), which is our only hope for any free string representation (if not exact solution) of QCD, a new complication arises, namely, the Gross-Witten (GW) phase transition [2]. This is also characterised by a narrow distribution of any particular lattice gauge variable around some singled out point on the group space, once one increases \( \beta \) beyond the GW transition point. If we consider the eigenvalues of a gauge variable (which are the gauge invariants) we can say that their distribution in the WC phase does not take the whole unit circle, as in the SC phase, but is squeezed into a smaller interval. The \( Z(N) \) symmetry of the group centre will be broken anyway. Note that although the neighbouring gauge variables should not be far away from each other on the group space, on the scale of the correlation radius of the theory (in other words, physical mass, or confinement scale), different \( U \)-variables cover the whole group space. This property itself can be viewed as a definition of the correlation radius.

All this looks like very bad news for the SC expansion in the Wilson lattice theory. Even the hope of an exact string representation of QCD would be significantly reduced by this failure, since one of the few indications of its existence is the representation of the SC expansion in terms of the sum over non-interacting random surfaces on the hypercubic \( D \)-dimensional lattice, found in [3, 4] and advanced in [5, 6, 7] (we will present below a short discription of this construction).

Nevertheless, Wilson’s original idea of confinement on the lattice seems to be too nice to discard so easily. A long time ago, the author proposed a possible way out of the impasse [4]. Namely, it was suggested to modify the SC expansion by the introduction of new weights on the links of random surfaces, which have a non-trivial dependence on \( \beta \) and must be calculated selfconsistently from the unitarity condition (1.4) for the gauge variables. These weights can be expressed in terms of averages of traces of products of the Lagrange multipliers for the constraints (1.4) on each link. The weights do not fluctuate in the large \( N \) limit due to the factorization property, and are uniform in the physical space in light of the translational and rotational invariance. So, they can be considered as the effective ”world sheet coupling constants”,

\[ \text{weights} \]
though this notion is a bit vague on the lattice.

The whole construction is very close to the Stanley solution of the N-vector field theory (on the lattice) in the large N limit (though our construction is as yet far from a solution). Let us recall these simple and nice arguments of Stanley, to compare with ours.

The partition function of the N-vector field $n_a(x)$ normalized as

$$n_b^2(x) = 1 \quad (1.8)$$

is defined as follows:

$$Z = \int \prod_x Dn(x) \delta(n_b^2(x) - 1) \exp\left[ N \frac{\beta}{2} \sum_{x,\mu} n_b(x)n_b(x + \mu) \right] \quad (1.9)$$

We can try to investigate this model in terms of the SC expansion by expanding (1.9) in powers of $\beta$ and integrating order by order with respect to the compact measure in $n(x)$. The strong coupling diagrams will look like sums over paths with the weights $\beta^{\text{length}}$, which might have nontrivial couplings in the points of self-intersections. Let us demonstrate, say, the picture corresponding to the constraint (1.8):

$$< n_b^2(x) > = 1 \quad (1.10)$$

The typical strong coupling diagram is shown in fig.1. In the large N limit the diagrams will simply be trees of loops attached to each other at single points in physical space.

Let us remind ourselves how to sum up these trees.

We introduce the lagrange multiplier field $\alpha(x)$ to impose the constraint (1.4) on the integration measure, and then integrate out the $n$-field to obtain the following functional integral:

$$Z = \int \prod_x D\alpha(x) \exp\left[ N \sum_x \alpha(x) - \frac{N}{2} \text{Tr} \log \left( -\beta (\delta_{x,x+\mu} + \delta_{x,x-\mu} + \alpha(x)) \right) \right] \quad (1.11)$$

In the limit of large N we can solve this problem by looking for the translationary invariant saddle point for $\alpha$,

$$\frac{\delta S_{\text{eff}}[\alpha]}{\delta \alpha(x)} = 0 \quad (1.12)$$
which gives the equation:

\[
1 = \frac{1}{(2\pi)^2} \int_0^{2\pi} d^2p \frac{1}{2\beta \sum \cos(p_\mu) + \alpha}
\]  

(1.13)

In the limit \( \beta \to \infty \) we shift the variable \( \alpha \)

\[
\alpha = \beta(4 - m^2)
\]

(1.14)

expand the \( \cos p \) in (1.13) (since we are going to get the continuous limit) and obtain

\[
\beta = \frac{1}{(2\pi)^2} \int d^2p \frac{1}{p^2 + m^2} = \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2}
\]

(1.15)

which gives the well-known dimensional transmutation for the physical mass scale:

\[
m^2 = \Lambda^2 e^{-4\pi \beta}
\]

(1.16)

where \( \Lambda \) is the ultraviolet cutoff.

The end result is a multiplet of \( N \) scalar noninteracting particles with the same mass (1.16). This result is exact in the large \( N \) limit. One can easily see that the saddle point condition (1.13) corresponds to the normalisation condition (1.10).

If we expand the formula (1.13) in powers of \( \beta \) we get:

\[
1 = \sum_{\text{paths} P_{xx}} \left( \frac{\beta}{<\alpha>} \right)^{\text{Length}(P_{xx})}
\]

(1.17)

where \( P_{xx} \) is a path starting and finishing at the same point \( x \) on the lattice. Instead of summing over the whole tree of the loops we sum over only one free closed path with the renormalised weight:

\[
\left( \frac{\beta}{<\alpha>} \right)^{\text{Length}(P_{xx})} = (4 - m^2)^{\text{Length}(P_{xx})}
\]

(1.18)

We see that this sum over random paths diverges at \( m = 0 \), as it should do, in order to have an appropriate continuous limit.

This is an example of a modified SC (MSC) expansion which works in the physical SC phase.
It will be useful for the future comparison with QCD to show the relationship between the lagrange multiplier field and a vector field condensate (I thank A.Polyakov for his comments on this subject). We have the following equation for the correlation function of the vector field in the background of the lagrange multiplier:

\[ \beta \sum_{\mu = \pm 1, \pm 2} n_b(x + \mu)n_b(y) + \alpha n_b(x)n_b(y) + \delta_{x,y} = 0 \tag{1.19} \]

For \( x = y \) we obtain in the continuous limit:

\[ <\alpha> = 4\beta - \beta \Lambda^{-2}\pi^2 <n_b(x)\Delta n_b(x)> -1 \tag{1.20} \]

\[ = 4\beta + \frac{1}{4\Lambda^2} \int \frac{d^2 pp}{p^2 + m^2} - 1 \tag{1.21} \]

\[ = 4\beta - \frac{\pi}{4\Lambda^2} m^2 \log \frac{\Lambda}{m} \tag{1.22} \]

This is the relation between the modified weight in the sum over paths and the vector field condensate. Of course, it agrees with (1.16).

We presented here this well-known solution of Stenley model since it will be our guideline for the construction of the MSC expansion in lattice QCD, which looks to be generalizable to the physical WC phase.

2. Free random surface representation of the strong coupling expansion in D-dimensional lattice gauge theory

In this section we are going to reformulate the standard SC expansion of Wilson gauge theory in terms of FREE random surfaces on the D-dimensional hypercubic lattice. This formal representation will be exact order by order in \( \beta \). The result will consist of a description of the elementary geometrical objects (plaquettes, ”saddles” et.c) from which we construct this surface.

To demonstrate the idea let us start from a simpler model of random surfaces on the lattice: the Weingarten model. It has the same action as Wilson gauge theory, but instead of the unitary Haar measure one takes a gaussian measure:

\[ [dU]_W = d^{2N^2}U \exp \left[-N\text{tr}U^+U\right] \tag{2.1} \]

It can be represented in terms of lattice random surfaces by the standard SC expansion. We expand the exponent in (1.2) in powers of \( \beta \) and perform gaussian integrals using Wick’s theorem:

\[ \int [d^{2N^2}U]_W U_{ij}^+ U_{kl} = \frac{1}{N} \delta_{il}\delta_{jk} \tag{2.2} \]
\[ \int [d^{2N^2}] U U_{i,j} U_{k,l} U^{+} = \frac{1}{N^2} \delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2} \]  
and so on.

Geometrically this means that we can glue plaquettes together by means of these integrations over the common link variables, as is shown in fig.2 for (2.2) for two plaquettes, and in fig.3 for (2.3) for four plaquettes. Continuing this process we arrive at closed planar surfaces built on the hypercubic lattice. Due to the matrix structure of the theory, every connected piece of the surface will be weighed with the weight \(N^{2-2G}\), where \(G\) is its genus. This factor is explained in the standard way, for matrix models. In the Weingarten model the 'tHooft limit of large \(N\) leads precisely to planar surfaces. Due to the equal weights of different terms in the formulae (2.2)-(2.3), there are no extra weights at the intersections of surfaces which means the surfaces are the world sheets of a "free string".

Finally, the SC expansion for the free energy in the Weingarten model can be represented in the large \(N\) limit as a sum over planar surfaces \(\sigma_W\) on the hypercubic lattice:

\[ F(\beta) = \sum_{\sigma_W} \beta^{\mathcal{A}(\sigma_W)} \]  

where \(\mathcal{A}(\sigma_W)\) is the area of the surface (the number of plaquettes from which it is built).

It is known that for any dimension higher than 1 this model has a pathological behaviour and degenerates into tree like configurations at the critical point [8].

What will change if we take instead of (2.1) the U(N) Haar measure of the Wilson gauge theory? Nothing will be different for the gluing of two plaquettes, since the integral (2.2) will be the same. However, for four plaquettes, we will have instead of (2.3):

\[ \int [d^{2N^2}] U U_{i,j} U_{k,l} U^{+} = \frac{1}{N^2} [\delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2} + \delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2}] \]  

\[ - \frac{1}{N^2} [\delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2} - \delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2}] \]  

\[ + \frac{1}{N^2} \delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2} - \frac{1}{N^2} \delta_{i_1i_2} \delta_{j_1j_2} \delta_{k_1k_2} \]  

The first line in the r.h.s. of (2.8) represents the original Wick contractions of (2.3), whereas the second corresponds to new, cyclic contractions of indices (and plaquettes), etc.

In the large \(N\) limit, we have to expand the \(N\)-dependent coefficients in (2.8) in powers of \(1/N\) and interpret each term geometrically in terms of pieces of a free planar random surface on the D-dimensional hypercubic lattice. The most natural
interpretation is presented graphically in fig.4. The first term represents one of Wick contractions of indices (the rest of the gauge variables on these four plaquettes is contracted into two separate traces at the boundaries of two disconnected couples of plaquettes).

The second term corresponds to the cyclic coupling of indices (the rest of the gauge variables are contracted in a single trace around a single boundary). This term looks like a saddle (though quite a singular one, with zero radius of curvature). Topologically it is a disc which is glued into the random surface. It has the an extra $1/N$ power with respect to the previous term, corresponding to smaller Euler characteristics $\kappa$ (it is equal to 1 for each disc). In this way the topological expansion of 'tHooft attaches to our surfaces the factor:

$$N^{2-2G} = N^\kappa$$

(2.9)

The saddles seem to play an important role in the whole construction. Note that the sign is negative in front of this term.

The last term in fig.3 represent the next term of the $1/N$ expansion of the coefficient in the first term in the l.h.s. of (2.8) and can be described as a tube connecting two couples of plaquettes. It again has the topology of a disc.

All these terms can in principle contribute in the large $N$ limit. Higher order terms in the $1/N$ expansion in (2.8) describe higher order topologies.

This procedure can be continued for higher $n$-correlators of gauge matrices of the type

$$\int [d^{2N^2}]_H U_{i_1j_1}^+ U_{k_1l_1}^+ \ldots U_{i_nj_n}^+ U_{k_nl_n}$$

(2.10)

These correlators, being calculated in the same manner, give all possible connected or disconnected objects like multiple "saddles", drawn in fig.5, and give rise to a cyclic contraction of indices in (2.11), like

$$\delta_{i_1l_1} \delta_{k_1j_2} \delta_{i_2l_3} \ldots \delta_{k_nj_1}$$

(2.11)

as well as tubes and their mixtures, which can be build from $2n$ plaquettes. Every connected part of such a configuration is accompanied by some numerical coefficient which corresponds to appropriate index contractions in the integral (2.11).

This interpretation of the strong coupling expansion in terms of random lattice surfaces was proposed in [3, 4]. The factors corresponding to the $n$-saddles, consisting from cycling gluings of $2n$ plaquettes, were found there to be the Catalan numbers $f_n$ (see next section for their calculation):

$$f_n = -(-1)^n \frac{(2n)!}{2(2n-1)(n!)^2}$$

(2.12)
so that
\[ f_1 = 1, \quad f_2 = -1, \quad f_3 = 2, \quad f_4 = -5, \quad f_5 = 14 \] (2.13)

The whole variety of coefficients corresponding to this zoo of objects was calculated in [5]. Let us comment also that the rules for the sum over surfaces found here for the free energy, are directly generalisable to the Wilson average \( W(C) \). One can consider the gauge variables forming the loop factor in (1.3) as the edge of a surface, to which the plaquettes can be attached either by Weingarten type contractions, or by saddles, tubes, et c. The Wilson average can be viewed as an open string amplitude.

Fortunately, as was shown by Kostov [6, 7], with the exception of the multiple saddles, these complicated objects can be ignored and the sum over surfaces can be reduced to a sum over one-link-irreducible surfaces with multiple saddles. It was shown by Kostov that any surfaces which can be cut into two pieces by cutting along a single link (one-link-reducible surfaces) cancel each other due to sign changing factors.

This theorem of Kostov is a direct consequence of the XSunitarity of the gauge variables (1.4). Let us sketch the proof of it (the details can be found in [7]).

One can use the "backtracking" condition for an arbitrary Wilson average (1.3): if we cut the contour \( C \) in a point and glue in this cut a path consisting from two links \( l \ast l^{-1} \) going there and back in the same direction \( \mu \) on the lattice, the Wilson average will not change (see fig.5):
\[
W(C \ast l \ast l^{-1}) = W(C) \] (2.14)

Now consider dressed saddles: by definition these correspond to the sum of all surfaces attached to the link on which the saddle is situated. Namely, it is the bare saddle, considered above, plus all surfaces connected to it by "tubes" (also considered above). We can have in the sum over surfaces in the l.h.s. of (2.14) three situations: a dressed link vertex of the n-th order can be attached to both links \( l, l^{-1} \), or two link vertices of the orders (n-k) and k can be attached separately to each link, or there will be no plaquettes attached to these links at all (which corresponds precisely to \( W(C) \) without back tracking). So, introducing the weights \( F_k \) of dressed saddles, we obtain from (2.14):
\[
\sum_n W_n(C) [F_n + \sum_{k=0}^n F_{n-k}F_k] + W(C) = W(C) \] (2.15)

where \( W_n(C) \) is the sum over surfaces spanned on the contour \( C \) with dressed n-saddle attached to two backtracking links.

Since \( C \) can be arbitrary here, we conclude that
\[
F_n + \sum_{k=0}^n F_{n-k}F_k = 0 \] (2.16)
which defines the catalan numbers (2.12). From here we conclude that the weights of the dressed saddles are also equal to the Catalan numbers: $F_n = f_n$. Hence, we can throw away the one-link- irreducible surface, together with the objects more complicated then multiple saddles, from the sum over surfaces.

As usual, the partition function corresponds to the sum of various disconnected random surfaces, each having the topology of a sphere. Since these surfaces are non-selfinteracting (there is no excluded volume problem for them) we can be sure that the free energy corresponds to the sum over only connected random surfaces.

Finally, the random surface representation of the SC expansion in the Wilson gauge theory looks quite simple:

$$F(\beta) = \sum_{\sigma} \beta^{\text{Area}(\sigma)} \prod_{s\sigma} f_{n_s}$$

where $\sigma$ are the one-link-irreducible planar surfaces built from plaquettes glued together in saddles denoted by $s$, on the D-dimensional hypercubic lattice. $f_{n_s}$ are the Catalan factors attached to these saddles (we consider normal contraction of two plaquettes as a saddle of order 1).

The analogous representation for $W(C)$ can be given in terms of a sum over surfaces spanned on a loop $C$. Very few things are known about the critical behaviour of this SC expansion in multicolour Wilson gauge theory. The no-go theorem of [8] is not directly applicable here, since we have sign changing terms. It might or might not have a nontrivial critical behaviour, but its direct continuation to the physical WC phase seems quite unprobable in virtue of one of the two phase transitions mentioned in the introduction.

Can we modify the random surface representation (2.17) in order to describe the WC phase of QCD as well? We will propose a possible modification in the next section, and then we will give some arguments in favour of it.

3. Modified strong coupling expansion for the physical phase in $QCD_4$

To explain the idea of the modified SC expansion, proposed in [4] and tested on some examples in [3], we consider two similar, but technically slightly different approaches.

The first one, originally proposed in [4], leads to a simpler geometrical picture for random surfaces, whereas the second based on the representation introduced in [7] leads to a more promising quantitative scheme.
3.1 First construction for modified SC expansion

In the first case we introduce for each link gauge variable $U_{\mu}$ a hermitean lagrange multiplier matrix $\alpha_{\mu}$, parametrizing the Haar measure in one of two following ways:

$$[dU]_H = d^{2N^2}U \delta^{N^2}(U + U - I) =$$
$$\int d^{N^2} \alpha e^{Ntrα(I - U + U)} =$$
$$\int d^{N^2} \alpha e^{Ntrα(I - UU^+)}$$

(3.1) (3.2) (3.3)

It is our choice whether to take the second or the third line as a definition of $\alpha$ for any link variable. Let us choose it in the most symmetric way: we classify all the vertices of the D dimensional lattice in chess order, as even and odd, and we take the definition of $\alpha_{\mu}$ according to the second line of (3.3), if the corresponding link goes from an even to an odd vertex (in the positive direction of the coordinate axes $\mu$), and the line three of (3.3), if otherwise.

Now we have a double functional integral for (1.2): in hermitean matrices $\alpha$ and in complex matrices $U$. Let us perform first the integral over $U$’s by means of strong coupling expansion in $\beta$. In each order the integrals are now purely gaussian, with the matrix propagators equal to

$$\int d^{2N^2} e^{NtrαU + U} U_{ij}^+ U_{kl} = \frac{1}{N}(\alpha^{-1})_{il} \delta_{jk}$$

(3.4)

for the ”even” links, and

$$\int d^{2N^2} e^{NtrαUU^+} U_{ij}^+ U_{kl} = \frac{1}{N} \delta_{il} (\alpha^{-1})_{kj}$$

(3.5)

for the ”odd” links.

The integration over $U$’s can be performed according to the same rules as for the standard Weingarten model with the measure (2.1). We will get the sum over exactly the same hypercubic surfaces $\sigma_W$ as in Waingarten model, but since the propagators are now modified, new $\alpha$-depending weights should be attached to every even vertex $es$ of the surface:

$$F(\beta) = \sum_{\sigma_W} \beta^{Area(\sigma_W)} \prod_{es \in W, \sigma} g(\mu_{es})(\beta)$$

(3.6)

where

$$g(\mu) = g_{\mu_1 \mu_2 ... \mu_n}(\beta) = \frac{1}{N} tr[\alpha_{\mu_1} \alpha_{\mu_2} ... \alpha_{\mu_n}]^{-1} >$$

(3.7)

where $\mu_1, \mu_2, ..., \mu_n$ are the directions, either positiv or negativ, of links around a given even vertex on the surface).
Instead of further integration over $\alpha$’s, we have already substituted in (3.7) the emerging traces of products of $\alpha^{-1}_\mu$’s by their averages, according to the large $N$ factorization theorem [10, 11]. We see from this construction, that we can formally represent any physical quantity in multicolour lattice QCD as a sum over random hypercubic surfaces (Weingarten type surfaces in this case) with special factors attached to the vertices of surfaces. In virtue of translational and rotational invariances of any physical averages, the $g$-factors should not depend on the position of the vertex of the surface in the $D$-dimensional lattice, but only on the sequence of the links $\mu_1, ..., \mu_n$ around a given vertex of the surface (up to the obvious rotations). So $g_\{\mu\}$ play the role of ”string coupling constants”, though this notion is quite vague for the lattice strings.

These constants are in fact non-trivial functions of $\beta$ and should be calculated separately. One can propose the following formal method for it.

In the large $N$ limit, we can in principle define all $g$-factors from the chain of obvious equations, following from the unitarity of gauge matrices:

$$< \frac{1}{N} \text{tr} U_{\mu_1}^+ U_{\mu_1} ... U_{\mu_n}^+ U_{\mu_n} > = 1 \quad (3.8)$$

where $\mu_1, ..., \mu_n$ is any sequence of directions of links around an even vertex on the lattice. The l.h.s. of (3.8) is just a wilson average for a loop $C\{\mu\}$ consisting of only backtracking along the links surrounding a given vertex. If we re-express these Wilson averages in terms of the same sums over random surfaces spanned on these contours, as it was done for the free energy (3.6), we obtain the following conditions on $g$’s:

$$\sum_{\sigma_W, \partial \sigma_W = C\{\mu\}} \beta^{\text{Area} (\sigma_W)} \prod_{es \in \sigma_W} g_{\{e\}} (\beta) = 1 \quad (3.9)$$

where the sum is taken over all the surfaces having as a boundary the abovementioned backtracking contour $C\{\mu\}$.

This chain of nonlinear equations can serve at least in principle for the calculation of the string couplings $g_{\{\nu\}} (\beta)$. Note the similarity (not accidental) with the equation (1.17) for the $N$-vector field: there the sum over paths pinned to a point in the physical space, with the renormalized hopping parameter $\beta/\alpha$ was equal to $1$, as a consequence of the normalization of the vector field, where as here the sum over random surfaces pinned to a sequence of links around a vertex on the lattice, with renormalized string couplings, is equal to $1$, as the consequence of unitarity.

Of course, technically all this looks too complicated. There are too many factors $g$ to calculate. Therefore we use this approach only to demonstrate that the sum over random surfaces is quite a natural representation of the Wilson multicolour QCD.

Why should this modified strong coupling expansion work in physical WC phase of the theory? Our hope in this approach is that the equations (3.8) can have two
different branches of solution: the SC branch, which corresponds to the standard $\beta$ expansion considered in the previous section and valid up to some critical $\beta_c$, and the physical WC branch valid beyond $\beta_c$. It is obvious that in the SC phase the g-factors will be just $\beta$-independent numbers, whereas in the WC phase they should be nontrivial functions of $\beta$. The traces of $\alpha$-matrices serve here as an order parameter for the corresponding phase transition (of Gross-Witten or $Z_N$ breaking type). We will show in the next section that this modified SC expansion works indeed in both phases in the simplest example: the one plaquette model.

This construction is hardly useful for practical calculations, but the existence of a simple random surface representation of Weingarten type suggests that the search for the continuous QCD-string is not in vain.

### 3.2 Second construction for modified SC expansion

Let us describe the second construction for the modified SC expansion. The corresponding surfaces in this construction will be more complicated, including now the $n$-saddles. However technically it will be much more tractable since we will effectively obtain a sequence of weights $g_n\beta$ (traces of only one $\alpha$-matrix), attached to $n$-saddles labeled by only one integer $n$.

According to the trick proposed in ([7]), let us double every gauge variable on each link $l$:

$$U_l \rightarrow U_lV_l^+$$

We can integrate now in each of the matrices separately. This is the same Wilson gauge theory, since the matrix $V$ can be easily absorbed into $U$, in virtue of the invariance of the Haar measure.

Now the Haar measure on every link can be parametrized as:

$$[dU]_H[dV]_H = \int d^{N^2}\alpha_1 \int d^{N^2}\alpha_2 \int d^{2N^2}U \int d^{2N^2}V e^{N\text{tr}(I-U^+U)}e^{N\text{tr}(I-V^+V)}$$

We are ready now to integrate out the $U, V$ variables by means of the formal $\beta$-expansion with fixed $\alpha$’s. Since each of the link variables is doubled we have to apply the Wick theorem separately to $U$-half-link and $V$-half-link independently. Let us consider two examples of one link integrals over $U$ and $V$ with $\alpha$ fixed: with two link-variables:

$$\langle (UV^+)_{ij}(VU^+)_{kl}\rangle_{U,V}^{(0)} = \frac{1}{N} \delta_{il} \delta_{jk} \frac{\text{tr}}{N} \alpha^{-2}$$

and with four link-variables:

$$\langle (VU^+)_{i_1j_1}(UV^+)_{k_1l_1}(VU^+)_{i_2j_2}(UV^+)_{k_2l_2}\rangle_{U,V}^{(0)} = \frac{1}{N^2} (\frac{\text{tr}}{N} \alpha^{-2})^2 \delta_{i_1l_1} \delta_{j_1k_1} \delta_{i_2l_2} \delta_{j_2k_2}$$

(3.11)
\[
\frac{1}{N^4} \text{tr} \alpha^{-4} \delta_{i_1 l_2} \delta_{k_2 j_2} \delta_{i_2 l_1} \delta_{k_1 j_1} + (i_1, j_1 \leftrightarrow i_2, j_2) \tag{3.15}
\]

where we introduced the hermitean matrix

\[
\alpha^2 = \alpha_1^2 \alpha_2 \alpha_1^2 \tag{3.17}
\]

So the cyclic contraction of indices corresponding to a saddle, appears here in a natural manner, even on the phase of gaussian integrations. To every n-saddle a factor

\[
g_n(\beta) = \frac{\text{tr}}{N} \alpha^{-2n} \tag{3.18}
\]

should be attached. As in the previous construction, in virtue of translational and rotational invariance, and the large N factorization theorem, we can already take the average for each factor \(g_n\), which will not depend on the position or orientation of the link. So \(g_n\)'s are nontrivial functions of \(\beta\), labeled by only one integer \(n\). The corresponding sum over surfaces \(\sigma^*\) will consist from surfaces built from plaquettes glued together by saddle-like configurations \(s\):

\[
F(\beta) = \sum_{\sigma^*} \beta^{\text{Area}(\sigma^*)} \prod_s g_{n_s} \tag{3.19}
\]

Note that unlike the surfaces \(\sigma\) in (2.17), the surfaces \(\sigma^*\) emerging here can be one link reducible.

The random surface picture here is more complicated than in the previous construction, and a good question is whether we can describe the emerging saddles as special vertex operators in a continuous string theory. We will discuss this possibility in the concluding section.

Now we have to calculate the weights \(g_n\). The best way to do it is to write down the effective action for two matrix variables \(\alpha_1, \alpha_2\) (as in the previous construction, they will not depend either on the space coordinates, or on orientations in the large N limit) defining these factors through eq. (3.18):

\[
S_{\text{eff}}(\alpha_1, \alpha_2) = N \text{tr}[\alpha_1 + \alpha_2 - \log[\alpha_1 \alpha_2]] + N^2 F[\alpha^4 \alpha_2] \tag{3.20}
\]

where \(F[\alpha_1 \alpha_2]\) is the same as \(F\) in (3.19), but with unaveraged traces of \(\alpha\)'s instead of \(g_n\)'s.

The partition function of QCD can be written as a two matrix problem:

\[
Z(\beta) = \int dN^2 \alpha_1 \int dN^2 \alpha_2 e^{S_{\text{eff}}(\alpha_1, \alpha_2)} \tag{3.21}
\]
We propose here to reduce the double integral in $\alpha_1, \alpha_2$ to an integral over the only matrix variable (3.17). For this purpose one has to integrate out the extra degrees of freedom. Introducing the variable (3.17) and

$$\gamma = \alpha_1$$

we rewrite (3.21) as

$$Z(\beta) = \int dN^2 \alpha_{2} \int dN^2 \gamma \exp \left( N \text{tr}[\gamma + \gamma^{-1} \alpha^2 - \log \gamma] + N^2 F[\alpha^2] \right)$$

(3.23)

Let us compare it with the well-known integral over the unitary matrix $U$ in an external field $\alpha^{\frac{1}{2}}$:

$$e^{N^2 B[\alpha]} = \int [dU]_H e^{N \text{tr} \left( U + U^+ \right)}$$

(3.24)

By introducing the parametrization

$$[dU]_H = d^{2N^2} U \delta^{N^2} (U^+ U - I) = \int dN^2 \gamma e^{N \text{tr} \gamma (I - U^+ U)}$$

(3.25)

and integrating out $U$ we can see that the integral over $\gamma$ in (3.23) is precisely given by (3.24):

$$Z(\beta) = \int dN^2 \alpha^2 \exp N^2 \left( B[\alpha] + F[\alpha^2] \right)$$

(3.26)

One can find in [12] the expression for both the SC and WC phases of this integral. We need only the WC branch of this solution, found in [13]:

$$B[\alpha] = 2/N \sum_{k=1}^{N} \alpha_k - \frac{1}{2N^2} \sum_{k>j} \log[\alpha_k + \alpha_j] - \frac{3}{4}$$

(3.27)

where $\alpha_1, \ldots, \alpha_N$ are the eigenvalues of the $\alpha$-matrix.

One can easily obtain this result by using the representation $U = e^{iA} = 1 + iA - \frac{1}{2} A^2 + \ldots$ of the unitary matrix in (3.24) and keeping only guassian terms. This approximate calculation gives nevertheless the exact result in the WC phase, as is the case for the matrix integral in [14].

Finally, we can define the momenta (3.7) as the saddle point condition on the density of eigenvalues $\rho(\alpha)$ in the one-matrix integral (3.26):

$$-1 + \sum_{n=1}^{\infty} \alpha^{-2n-1} F_n[g_1, g_2, \ldots] = P \int d\mu \rho(\mu) \left( \frac{1}{\alpha - \mu} + \frac{1}{\frac{1}{2} \alpha + \mu} \right)$$

(3.28)
where \( F_n \) are the sums over the surfaces built according the same rules as (3.19), but attached to a fixed saddle of the order \( n \). So to compute \( g_n \)'s we have first to calculate these sums over surfaces (the most nontrivial part of the problem) then find \( \rho(\alpha) \) from (3.28) as a function of \( g_n \)'s and then to find them from the selfconsistency condition:

\[
\int d\alpha \rho(\alpha) \alpha^{-2n} = g_n(\beta)
\]

(3.29)

One can recognize the geometrical similarity of the eq. (3.28) with the eqs. (1.17), (3.9).

The idea of the modified SC here is similar to the previous construction: we hope that the factors \( g_n(\beta) \) will have two different branches as functions of \( \beta \). In the SC phase they are just \( \beta \)-independent numbers, where as in the physical WC phase their behaviour changes and they turn out to be nontrivial functions of \( \beta \). We will find these two branches for the simple one-plaquette model in the next section, and then we will demonstrate their existence in the 4D Wilson theory.

4. Simple examples: one-plaquette model

In this section we will check the idea of the modified SC expansion on a simple example: one-plaquette model, which was defined and solved in [2]. Its partition function is:

\[
Z_P(\beta) = \int [dU] e^{N \text{tr}[U+U^+]} \]

(4.1)

We will use a method for its solution explained in [3] which recalls our first construction. Namely we parametrize the Haar measure in (4.1) by the lagrange multiplier \( \alpha \), as in (3.3), and after the integration over the complex matrix \( U \) we obtain the following 1-matrix model:

\[
Z_P(\beta) = \int dN^2 \alpha e^{N \text{tr}[\alpha - \log \alpha + \beta^2 \alpha^{-1}]} \]

(4.2)

By standard methods [15] we obtain the integral equation for the density of eigenvalues \( \rho(\alpha) \):

\[
-1 + \frac{1}{\alpha} + \beta^2 \alpha^{-2} = P \int d\mu \rho(\mu) \frac{2}{\alpha - \mu}
\]

(4.3)

If we introduce the analytic (outside the cut) function

\[
F(\alpha) = P \int d\mu \rho(\mu) \frac{2}{\alpha - \mu}
\]

(4.4)
which satisfies the condition:

\[ F(\alpha) \to_{\alpha \to \infty} 2/\alpha \]  

(4.5)

and has no singularities at \( \alpha = 0 \), with its imaginary part being equal to \( \pi \rho(\alpha) \), we obtain two solutions for it, separated by the Gross-Witten phase transition at \( \beta_c = 1/2 \). The SC solution is valid only for \( \beta < 1/2 \) and corresponds to the density

\[ \rho(\alpha) = \frac{1}{2\pi i} \left( \frac{1}{1 + \frac{1}{\alpha}} \right) \]  

(4.6)

The cut is collapsed to a point here, and the calculation of momenta of \( \alpha \) should be understood as a contour integral around the origin. So we obtain:

\[ \frac{1}{N} \alpha^{-n} = \oint d\alpha \rho(\alpha) \alpha^{-n} = 1, \quad \text{if } n = 0, 1 \]  

(4.7)

\[ = 0, \quad \text{for other } n \]  

(4.8)

It easy to obtain the following expression for the plaquette averages:

\[ W_n(\beta) = \langle \frac{\text{tr}}{N} (U^n + U^{+n}) \rangle = 2\beta^n < \frac{\text{tr}}{N} \alpha^{-n} > \]  

(4.9)

We conclude from the two last formulae that:

\[ W_1(\beta) = \beta \]  

(4.10)

and all other \( W_n \) are equal to zero in the SC phase, as it should be.

In the WC phase \( (\beta > 1/2) \) we find the solution for \( F(\alpha) \) in the form

\[ F(\alpha) = -1 + \frac{1}{\alpha} + \frac{\beta^2}{\alpha^2} + \left( \frac{1}{\alpha} + \frac{\beta}{\alpha^2} \right) \sqrt{-(\alpha^2 + 2(1 - \beta)\alpha + \beta^2)} \]  

(4.11)

Comparing it with the expansion

\[ F(\alpha) = \sum_{k=1}^{\infty} \alpha^k < \frac{\text{tr}}{N} \alpha^{-k-1} > \]  

(4.12)

we can obtain:

\[ W_1 = 1 - \frac{1}{4\beta} \]  

(4.13)

\[ W_2 = (1 - \frac{1}{2\beta})^2, \text{ etc.} \]  

(4.14)
We see that unlike the SC phase, all \( W_n \) are functions of \( \beta \) tending to 1 as \( \beta \to \infty \), as it should be.

It is easy to see that in the SC phase the large \( \beta \) asymptotics of the momenta of \( \alpha \) are always:

\[
< \frac{\tr}{N} \alpha^{-n} > \rightarrow_{\beta \to \infty} \beta^{-n}
\]  
(4.15)

which is similar to the asymptotics of a single lagrange multiplier of the vector field, as one sees from (1.14).

It is curious to see how the unitarity relations \((U^*U)^n = I\) are satisfied here. Say, from the integral over \( U \) with fixed \( \alpha \) we have

\[
< \frac{\tr}{N} (U^*U) >= < \frac{\tr}{N} \alpha^{-1} > + \beta^2 < \frac{\tr}{N} \alpha^{-2} >= 1
\]  
(4.16)

One can observe that the WC solution and the SC solution satisfy (4.16) in very different ways.

Of course the example considered here is the simplest possible one: the modified SC expansion consists here only from one term: in (4.2) this is the last term in the exponent, where the SC expansion parameter \( \beta^2 \) is modified by the weight \( < \frac{\tr}{N} \alpha^{-1} > \).

In the SC phase this weight is just one, and \( \beta^2 \) corresponds to the area of the "surface" consisting from two plaquettes. In the WC phase the weight is already nontrivial, as you see from (4.14). The same is true for the "Wilson loops" \( W_n(\beta) \).

One might think that the situation considered in this example is too simple to test the validity of the modification of SC, since we have only one surface in the sum over lattice surfaces here. Let us try another formulation of this model which already has an infinite sum over the "surfaces" (built on one single plaquette from an arbitrary number of copies of this plaquette). Namely let us test our second construction of the previous section. The doubling of the matrice \( U \) gives instead of (4.1) the two unitary matrix integrals:

\[
Z_P(\beta) = \int [dU]_H \int [dV]_H e^{N(\beta[U V^* + V U^*])}
\]  
(4.17)

Introducing the lagrange multipliers as in (3.11) and integrating over \( U, V \) we obtain the sum over surfaces with modified weights in (3.26) in the form:

\[
F(\alpha) = - \frac{1}{N} \log(\alpha^2 - \beta^2) = - \log \alpha^2 + \sum_{k=1}^{\infty} \frac{\beta^2 k}{k} \frac{\tr}{N} \alpha^{-2k}
\]  
(4.18)

The saddle point equation (3.28) now reads:

\[
-1 + \frac{\alpha}{\alpha^2 - \beta^2} = P \int d\mu \rho(\mu) \left( \frac{1}{\alpha - \mu} + \frac{1}{\alpha + \mu} \right)
\]  
(4.19)
This equation seems to be rather nontrivial, but nevertheless it can be easily solved in the relevant WC phase. Let us shift the matrix variable $\alpha$, defining a new variable $\epsilon$:

$$\alpha = \beta + \epsilon \quad (4.20)$$

The eq.(4.19) now reads:

$$-1 + \frac{\epsilon + \beta}{2\epsilon \beta + \epsilon^2} = P \int d\epsilon' \rho(\epsilon') \left( \frac{1}{\epsilon - \epsilon'} + \frac{1}{2 \beta + \epsilon + \epsilon'} \right) \quad (4.21)$$

We can try now to solve this equation by expanding the second term in the l.h.s. and the second term in the r.h.s. in powers of $\beta^{-1}$. In the r.h.s. we will obtain in this way the momenta $<\epsilon>$ as the coefficients of expansion. It happens to be selfconsistent to set all of them to zero.

$$<\epsilon^n> = 0, \text{ for } n = 1, 2, 3, \ldots \quad (4.22)$$

Then the equation (4.21) turns out to be

$$-1 + \frac{1}{2\epsilon} = P \int d\epsilon' \rho(\epsilon') \frac{1}{\epsilon - \epsilon'} \quad (4.23)$$

with the solution:

$$\rho(\epsilon) = \frac{1}{2\pi i} (1 + \frac{1}{2\alpha}) \quad (4.24)$$

As in the eq.(4.6) the corresponding integrals should be understood as contour integrals around the origin.

It seems to be strange that the solution (4.24) does not depend on $\beta$ and is very similar to the SC one (refscr). Nevertheless it reproduces the correct WC behaviour of Wilson averages, say,

$$W_1(\beta) = \langle \frac{\text{tr}}{N}(UV^* + VU^*) \rangle = \langle \frac{\beta}{N \alpha^2 - \beta^2} \rangle = \int d\epsilon \rho'(\epsilon) \frac{1}{2\epsilon + \epsilon^2} = 1 - \frac{1}{4\beta} \quad (4.25)$$

We see from the last formula, that the modified SC expansion contains here an infinite number of terms:

$$W_1(\beta) = \beta \sum_{k=0}^{\infty} \beta^2 k < \frac{\text{tr}}{N} \alpha^{-2k-2} > \quad (4.26)$$

At $\beta = \infty$ this series diverge since $\alpha \sim \beta$. This divergence is situated at $\epsilon = 0$, therefore it gives a necessary contribution to the integral in (4.23). For every
particular term in the expansion (4.26) we would loose this singularity, therefore to get the correct weights we have to regularise the singularity in the density of eigenvalues (4.24).

One of the regularizations would be the above mentioned expansion of (4.21) in $1/\beta$ up to a finite number $M$ of terms and keeping the corresponding momenta (4.22) nonzero for a while. Then we can solve the resulting equation by standard methods and define the inverse momenta $<\frac{N}{k}\alpha^{-k}>$ (the weights of the "surface") from the self-consistency condition. The "sum over surfaces" with these weights will reproduce the correct result (4.25) for sufficiently big $M$. This method was checked numerically in [9] on a similar construction and the convergency with $M$ turned out to be very fast.

Anyway, this example shows that the modified SC expansion works well in the WC phase of the one-plaquette model and obeys the following general features which should hold in more realistic models:

1. There exist two different branches (SC and WC) of the solution for the weights of the modified SC expansion: they are numbers in the SC phase, and non-trivial functions of $\beta$ in the WC phase.

2. The matrix lagrange multiplier has the asymptotic behaviour:

$$\alpha \sim \beta, \quad \text{for} \quad \beta \to \infty$$

(4.27)

tending to cancel the big factor $\beta^{Area}$ in the sum over surfaces.

3. The corresponding sum over surfaces diverges at the point $\beta = \infty$ and needs to be regularized.

5. The existence for the modified weights of random surface in physical phase of $\hat{QCD}_4$

The problem of calculating the sum over surfaces in a physical four dimensional situation in the formulae (3.19) is non-trivial. Even its reformulation in terms of a continuous functional integral of some string theory with a definite two dimensional world sheet action is far from being achieved.

Our task here is rather to demonstrate the possibility of the existence of such a representation in principle, then to get a well elaborated quantitative approach to the multicolour QCD.

Namely, we will show the relationship between gluon condensates and the momenta of the lagrange multiplier matrix.
Let us vary the action (1.1) with the Haar measure parametrized as in (3.11) with respect to the fields $U^+ \delta U^+, V \delta V$. We get the following matrix operatorial equations of motion:

$$\alpha_{1x,\mu} = \beta \sum_{\nu} (UV^+)_{x,\nu}(U(VU^+))_{x+\nu,\mu}(VU^+)_{x,\mu}$$  \hspace{1cm} (5.1)$$

and

$$\alpha_{2x,\mu} = \beta \sum_{\nu} (UV^+)_{x,\mu}(U(VU^+))_{x+\nu,\mu}(VU^+)_{x,\nu}$$  \hspace{1cm} (5.2)$$

Since we know that any physical gauge invariant quantity depends only on the combination $\alpha_1 \alpha_2$ of these matrices, we find from last equations:

$$\alpha = \alpha_1 \alpha_2$$  \hspace{1cm} (5.3)$$

$$\beta \sum_{\nu_1,\nu_2} (UV^+)_{x,\nu_1}(U(VU^+))_{x+\nu_1,\mu}(VU^+)_{x+\nu_2,\mu}(VU^+)_{x,\nu_2}$$  \hspace{1cm} (5.4)$$

The product of matrices in the r.h.s. runs along the 6 link boundary of two plaquettes attached to each other along a fixed link $\mu$. This “chair-like” configuration of plaquettes was used extensively for various improvements of the Wilson action in lattice computer simulations of QCD.

So any momentum of the matrix $\alpha$ which acquires the space and direction independent average in the large $N$ limit, can be represented as an average of the power of the combination in the r.h.s. of (5.3). Any positive power looks like a saddle configuration of the n-th order (built from 2n plaquettes) presented in fig.4:

$$< \frac{\text{tr}}{N} \alpha^n > = \beta^{2n} < \frac{\text{tr}}{N} \left( \sum_{\nu_1,\nu_2} U_{x,\nu_1} U_{x+\nu_1,\mu} U^+_{x+\mu,\nu_1} U_{x+\mu,\nu_2} U^+_{x+\nu_2,\mu} U^+_{x,\nu_2} \right)^n > \hspace{1cm} (5.6)$$

$$+ \text{ contact terms}$$  \hspace{1cm} (5.7)$$

If we represented the n’th power in the r.h.s. as a sum of individual terms it would be expressed through Wilson loops running around the boundaries of saddles of the n-th order (we restored the standard Wilson gauge variables $U$, without the doubling on each link).

For negative $n$’s (5.4) gives a local expression of the weights of the modified SC expansion in (3.19). Again, the situation is very similar here to the calculation of the lagrange multiplier through the local condensate of the vector field, give in the Introduction.

Let us discuss a possible continuous limit for the expression (5.7). it is clear that for $\beta \to \infty$ any wilson loop tends to 1, so, $\alpha \to 4(D-1)^2 \beta$. As usually, we make a shift:

$$\alpha = 4(D-1)^2 \beta + \epsilon$$  \hspace{1cm} (5.8)$$
It already shows that the big weight $\beta^{\text{Area}}$ will be cancelled in modified SC expansion (3.19). Going to the local limit in (5.7) we obtain

$$< \alpha_{-n} >$$

$$\simeq \left( [2D - 2]^{-2n} \frac{1}{(D-1)} \right) < \epsilon > + ...$$

(5.9)

$$\simeq \beta \left< \frac{4}{N} F_{\mu \nu}^2(x) >_{QCD} > + \text{subtractions} \right.$$ (5.10)

(5.11)

(5.12)

where $F_{\mu \nu} = A_{\nu, \mu} + A_{\mu, \nu} + [A_{\mu}, A_{\nu}]$ is the gauge field strength.

We see that the weights of the modified SC expansion are defined through the local condensates of the gluon field $A_{\mu}(x)$ in QCD and can in principle be calculated perturbatively, from the Feynman perturbation technique. All terms proportional to the powers of cut-off in (5.12) should be subtracted, since they should cancel with the contact terms, as happened for the vector-field.

However, the perturbative calculation cannot give us the most interesting exponentially small terms in the gluon condensates, which should in principle define the renormalized string tension. This problem cannot be solved if we are unable to calculate the corresponding sum over surfaces (3.19) and get the effective action for the $\alpha$ variables. The only purpose of the relations found here, between the gluon condensates and the momenta of the lagrange multiplier, was to show the existence of the new nontrivial weights in the modified SC expansion leading to a lattice string representation for the multicolour gauge theory.

6. Conclusions

Presenting this lattice construction of the random surface representation in QCD, we did not want to create an impression that a real chromodynamical string theory is around the corner. All of this can serve only as an intuitive picture from which one can probe different continuum string world sheet lagrangians. We tried here to demonstrate in principle the existence of the modified strong coupling expansion with the weights of the expansion being defined selfconsistently in the physical weak coupling phase. We managed to demonstrate the existence of these weights in a toy one-plaquette model, and then we gave some hints as to how to calculate them through gluon condensates.

What are the main ingredients of this world sheet lagrangean?

As it was noted in papers [18] and [20], the fact that the Wilson loop average should obey the Makeenko-Migdal equations implies the existence (finiteness) of the
area variational derivative of the Wilson loop \([19]\). This means that the cosmological constant term should be absent, and the action contains the term:

$$\int d^2\xi T_{\mu\nu}\epsilon_{ab}\partial_a X_\mu \partial_b X_\nu$$  \hspace{1cm} (6.1)$$

where \(X_\mu(\xi)\) is the world sheet coordinate in the physical space, and \(T_{\mu\nu}\) is either the independent antisymmetric field on the world sheet (in \([18]\)), or an antisymmetric function of \(X_\mu\) (in \([20]\)).

This might be a good starting point but it is far from the final action. According to our construction of this paper, a probable object to be added with a new coupling constant to this action would be a saddle like configuration. This saddle is quite singular: for example, for the saddle of the second order shown in fig.3, we have a deficit of the angle (curvature) equal to \(4\pi\) (since it consists from 4 plaquettes) and concentrated in one point on the world sheet. The radii of curvature at these points are equal to zero. Nevertheless, this singular object is localized on the world sheet and can in principle be described by a local operator in the action, depending on the extrinsic geometry. The problem is to guess the continuous form of these terms for saddles of any orders. Their coupling constants should be defined from the unitarity condition, say in the form of the back-tracking independence of the Wilson average (see section 3).

One possibility to describe the external geometry is to introduce the external curvature-dependent terms, like

$$\int d^2\xi \sqrt{g}[\Delta(g)X_\mu]^2$$  \hspace{1cm} (6.2)$$

as was proposed by Polyakov. It is not clear whether this simple introduction of the extrinsic geometry dependence is sufficient to describe the same effects as saddles do for QCD.

The saddles seem to be relevant factors for the whole problem. In 2 dimensions, they define the so-called branch points of the world sheet \([7, 13, 18, 17]\), like on the Riemann surface of the function \(y = \sqrt{x}\). This branch point can have a negative weight, as seen from the expression for the Wilson average with the loop C encircling twice some two dimensional domain of the area \(A\) \([21]\):

$$W(A) = e^{-\beta^{-1}A}(1 - \beta^{-1}A)$$  \hspace{1cm} (6.3)$$

The last term in the pre-exponent emerges from the entropy of the branch point on the minimal area surface covering this contour. The weight of the branch point is negative, and for \(A > \beta\) the Wilson loop becomes negative. One might wonder whether the same phenomenon could happen in the four dimensional QCD in the deep
confinement regime, when only the minimal area gives the main contribution. For
the same contour the minimal surface should not look different as in two dimensions,
with the same branch point.

The only way to check it is by a Monte-Carlo simulation for the lattice gauge
theory. A good indication of the relevance of special saddle points on the world sheet
of the QCD string would be the negative sign of this Wilson average. It would be a
completely non-perturbative result since in perturbation theory the Wilson average
is always close to one. Work on this calculation is in progress on the APE in Rome
[22].

Of course, one can easily imagine a much more pessimistic scenario. The saddles
could be quite big and densely distributed on a typical world sheet for the physically
relevant values of $\beta$. Such a condensation of saddles might completely cripple the
surface, and the whole surface description might be irrelevant in this situation. This
looks to be a probable scenario for big $\beta$, as we have already shown in the one
plaquette model, where big saddles dominated in the simple sum over "surfaces". However, one might hope that these saddles are sufficiently rare on the surface in the
deep confinement regime. If the worst scenario turns out to be true, it might mean
that our hopes for a string representation of QCD are hopeless.

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Figure Captions

Fig.1. The typical strong coupling diagram for the vector field.

Fig.2. Glueing together of two plaquettes after a link variable integration.

Fig.3. Formation of saddles, tubes et.c. from four plaquettes after a link variable integration.

Fig.4 Multiple saddle.

Fig.5 Backtracking condition for the wilson loop, following from the unitarity.