ABSTRACT
Sparse signal recovery has been dominated by the basis pursuit denoise (BPDN) problem formulation for over a decade. In this paper, we propose an algorithm that outperforms BPDN in finding sparse solutions to underdetermined linear systems of equations at no additional computational cost. Our algorithm, called WSPGL1, is a modification of the spectral projected gradient for ℓ1 minimization (SPGL1) algorithm in which the sequence of LASSO subproblems are replaced by a sequence of weighted LASSO subproblems with constant weights applied to a support estimate. The support estimate is derived from the data and is updated at every iteration. The algorithm also modifies the Pareto curve at every iteration to reflect the new weighted ℓ1 minimization problem that is being solved. We demonstrate through extensive simulations that the sparse recovery performance of our algorithm is superior to that of ℓ1 minimization and approaches the recovery performance of iterative re-weighted ℓ1 (IRWL1) minimization of Candès, Wakin, and Boyd, although it does not match it in general. Moreover, our algorithm has the computational cost of a single BPDN problem.

Index Terms— Sparse recovery, compressed sensing, iterative algorithms, weighted ℓ1 minimization, partial support recovery

1. INTRODUCTION
The problem of recovering a sparse signal from an underdetermined system of linear equations is prevalent in many engineering applications. In fact, this problem has given rise to the field of compressed sensing which presents a new paradigm for acquiring signals that admit sparse or nearly sparse representations using fewer linear measurements than their ambient dimension [1][2].

Consider an arbitrary signal $x \in \mathbb{R}^N$ and let $y \in \mathbb{R}^n$ be a set of measurements given by $y = Ax + e$, where $A$ is a known $n \times N$ measurement matrix, and $e$ denotes additive noise that satisfies $\|e\|_2 \leq \epsilon$ for some known $\epsilon \geq 0$. Compressed sensing theory states that it is possible to recover $x$ from $y$ (given $A$) even when $n < N$, that is, using very few measurements. When $x$ is strictly sparse—i.e., when there are only $k < n$ nonzero entries in $x$—and when $e = 0$, one may recover an estimate $\hat{x}$ of the signal $x$ by solving the constrained ℓ0 minimization problem

$$\min_{u \in \mathbb{R}^N} \|u\|_0 \text{ subject to } Au = y.$$ (1)

However, ℓ0 minimization is a combinatorial problem and quickly becomes intractable as the dimensions increase. Instead, the convex relaxation given by the ℓ1 minimization problem

$$\min_{u \in \mathbb{R}^N} \|u\|_1 \text{ subject to } \|Au - y\|_2 \leq \epsilon$$ (BPDN)

also known as basis pursuit denoise (BPDN) [3], can be used to recover an estimate $\hat{x}$. Candès, Romberg and Tao [2] and Donoho [1] show that it is possible to recover a stable and robust approximation of $x$ by solving (BPDN) instead of (1) at the cost of increasing the number of measurements taken.

Several works in the literature have proposed alternate algorithms that attempt to bridge the gap between ℓ0 and ℓ1 minimization. These include using ℓp minimization with $0 < p < 1$ which has been shown to be stable and robust under weaker conditions than those of ℓ1 minimization, see [4][5][6]. Weighted ℓ1 minimization is another alternative if there is prior information regarding the support of the signal to-be-recovered as it incorporates such information into the recovery by weighted basis pursuit denoise (w-BPDN)

$$\min_{u} \|u\|_{1,w} \text{ subject to } \|Au - y\|_2 \leq \epsilon,$$ (w-BPDN)

where $w \in (0, 1]^N$ and $\|u\|_{1,w} := \sum_i w_i |u_i|$ is the weighted ℓ1 norm (see [7][8][9]).

When no prior information is available, the iterative reweighted ℓ1 minimization (IRWL1) algorithm, proposed by Candès, Wakin, and Boyd [10] and studied by Needell [11], solves a sequence of weighted ℓ1 minimization problems with the weights $w^{(t)}_i \approx 1/|x^{(t-1)}_i|$, where $x^{(t-1)}_i$ is the solution of the $(t - 1)$th iteration and $w^{(0)}_i = 1$ for all $i \in \{1 \ldots N\}$. More recently, Mansour and Yilmaz [12] proposed a support driven iterative reweighted ℓ1 minimization (SDRL1) algorithm that also solves a sequence of weighted ℓ1 minimization problems with constant weights $w^{(t)}_i = \omega \in [0, 1]$ when $i$ belongs to support estimates $\Lambda^{(t)}$ that are updated in every iteration. The performance of SDRL1 is shown to match that of IRWL1.
Motivated by the performance of constant weighting in the SDRL1 algorithm, we present in this paper an iterative algorithm called WSPGL1 that converges to the solution of a weighted \( \ell_1 \) problem (wBPDN) with a two set weight vector \( w_A = \omega \) and \( w_{A^c} = 1 \), where \( \omega \in [0, 1] \) and \( \Lambda \) is a support estimate. The set \( \Lambda \) to which the algorithm converges is not known a priori but is derived and updated at every iteration. Our algorithm is a modification of the spectral projected gradient for \( \ell_1 \) minimization (SPGL1) algorithm [13] which solves a sequence of LASSO [14] subproblems to arrive at the solution of the BPDN problem. We give an overview of the SPGL1 algorithm in section 2. In contrast, our algorithm solves a sequence of weighted LASSO subproblems that converge to the solution of the wBPDN problem with weights \( \omega \) applied to a support estimate \( \Lambda \). We discuss the details of this algorithm in section 2 and present preliminary recovery results in section 3 demonstrating its superior performance in recovering sparse signals from incomplete measurements compared with \( \ell_1 \) minimization. We limit the scope of this paper to discussing the algorithm and presenting sparse recovery results and leave the analysis of the algorithm for future work.

**Notation:** For a vector \( x \in \mathbb{R}^N \), an index set \( \Lambda \subset \{1 \ldots N\} \) and its complement \( \Lambda^c \), let \( x_k \) and \( x|_k \) refer to the largest \( k \) entries of \( x \), \( x(k) \) is the \( k \)th largest entry of \( x \), \( x_\Lambda \) refers to the entries of \( x \) indexed by \( \Lambda \), and \( x^{(t)} \) is the vector \( x \) at iteration \( t \).

## 2. THE SPGL1 ALGORITHM

In this section, we give an overview of the SPGL1 algorithm, developed by van den Berg and Friedlander [13], that finds the solution to the BPDN problem.

### 2.1. General overview

The SPGL1 algorithm finds the solution of the BPDN problem by efficiently solving a sequence of LASSO subproblems

\[
\min_{u \in \mathbb{R}^N} \|Au - y\|_2 \text{ subject to } \|u\|_1 \leq \tau \quad (\text{LS}_\tau)
\]

using a spectral projected-gradient algorithm. The single parameter \( \tau \) determines a Pareto curve \( \phi(\tau) = \|r\|_2 \), where \( r = y - Ax \) and \( x^* \) is the solution of (LS\(_\tau\)). The Pareto curve traces the optimal trade-off between the least-squares fit and the one-norm of the solution.

The SPGL1 algorithm is initialized at a point \( x^{(0)} \) which gives an initial \( \tau_0 = \|x^{(0)}\|_1 \). The parameter \( \tau \) is then updated according to the following rule

\[
\tau_{t+1} = \tau_t + \frac{\|x^{(t)}\|_2 - \epsilon}{\|AHx^{(t)}\|_\infty / \|r^{(t)}\|_2}, \quad (2)
\]

where superscript \( H \) indicates Hermitian transpose, and \( \epsilon = \|e\|_2 = \|y - Ax\|_2 \). Consequently, the next iterate \( x^{(t+1)} \) is given by the solution of (LS\(_{\tau_{t+1}}\)) and the algorithm proceeds until convergence.

### 2.2. Probing the Pareto curve

One of the main contributions of [13] lies in recognizing and proving that the Pareto curve is convex and continuously differentiable over all solutions of (LS\(_\tau\)). This gives rise to the update rule for \( \tau \) shown in (2) and guarantees the convergence of SPGL1 to the solution of BPDN.

The update rule (2) is in fact a Newton-based root-finding method that solves \( \phi'(\tau) = \epsilon \). The update rule generates a sequence of parameters \( \tau_t \) according to the Newton iteration

\[
\tau_{t+1} = \tau_t + \frac{\epsilon - \phi(\tau_t)}{\phi'(\tau_t)}
\]

where \( \phi'(\tau) \) is the derivative of \( \phi \) at \( \tau \). It is then shown that the \( \phi'(\tau) \) is equal to the negative of the dual variable \( \lambda \) of (LS\(_\tau\)) resulting in the expression \( \phi'(\tau) = -\lambda = -\frac{\|AHx\|_\infty}{\|r\|_2} \). Figure 1 illustrates an example of a Pareto curve and the root finding method used in SPGL1.

## 3. THE PROPOSED WSPGL1 ALGORITHM

In this section, we describe the proposed WSPGL1 algorithm for sparse signal recovery as a variation of the SPGL1 algorithm. The WSPGL1 algorithm solves a sequence of weighted LASSO subproblems to arrive at the solution to a weighted BPDN problem with weights \( \omega \in [0, 1] \) applied to a support set \( \Lambda \). The set \( \Lambda \) is derived and updated from the solutions of the weighted LASSO subproblems (LS\(_\tau\), \( w \)).

### 3.1. Algorithm description

The two algorithms SPGL1 and WSPGL1 follow exactly the same initial steps until the solution \( x^{(t)} \) of the first LASSO
The WSPGL1 algorithm converges to the solution of a weighted BPDN problem with weights \( \omega \in [0, 1] \) applied to a support set \( \Lambda \). When the sparse signal is recovered exactly, the set \( \Lambda \) coincides with the true support of the sparse signal \( x \). Figure 2(a) illustrates the solution path of WSPGL1 which follows the Pareto curve of the BPDN problem until the first (LS) is solved. The algorithm then uses the support information from \( x^{\tau_1} \) to switch to the Pareto curve of the wBPDN problem. Figure 2(b) compares the solution paths of WSPGL1, SPGL1, and weighted SPGL1 with oracle support information. Both WSPGL1 and the oracle weighted SPGL1 use \( \omega = 0.3 \).

It is still not clear under what conditions the WSPGL1 algorithm achieves exact recovery. What is clear is that WSPGL1 can exactly recover signals with far more nonzero coefficients than what BPDN can recover. The WSPGL1 algorithm is motivated by the work in \cite{9} and \cite{12}, which show that weighted \( \ell_1 \) minimization can recover less sparse signals than BPDN when the weights are applied to a support...
estimate that is at least 50% accurate. Moreover, it is possible to draw a support estimate from the solution of BPDN and improve that support estimate by solving wBPDN using the initial support estimate. Based on these results, we conjectured that the solution of every LASSO subproblem in SPGL1 allows us to find a support estimate that is accurate enough to improve the recovery conditions of the corresponding wBPDN problem. A full analysis of this algorithm will be the subject of future work.

4. NUMERICAL RESULTS

We tested the WSPGL1 algorithm by comparing its performance with SDRL1 [12], IRWL1 [10] and standard $\ell_1$ minimization using SPGL1 [13]. The signals have an ambient dimension $N = 2000$ and the sparsity and number of measurements are varied. The results are averaged over 100 experiments.

![Fig. 3: Comparison of the percentage of exact recovery of sparse signals between the proposed WSPGL1, SDRL1 [12], IRL1 [10], and standard $\ell_1$ minimization using SPGL1 [13]. The signals have an ambient dimension $N = 2000$ and the sparsity and number of measurements are varied. The results are averaged over 100 experiments.](image)

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