A comparison of different methods
in the study of dynamical fluctuations
in high energy $e^+e^-$collisions

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Different methods in the study of anomalous scaling of factorial moments in high energy $e^+e^-$collisions are examined in some detail. It is shown that the horizontal and vertical factorial moments are equivalent only when they are used in combination with the cumulant variables. The influence of different reference frames and that of phase space restrictions is also discussed.

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I. INTRODUCTION

Since the observation of exotic multiparticle events in Cosmic Ray experiments and especially the discovery of unexpectedly large local fluctuations recorded by the JACEE collaboration, the investigation of nonlinear phenomena (NLP) in high energy collisions has attracted much attention. One of the signals of these NLP is the phenomena (NLP) in high energy collisions.

The influence of different reference frames and that of phase space restrictions is also discussed.

One of the signals of these NLP is the diminishing of phase space scale: $C_m = 1/M \sum_{m=1}^{M} (p_{m})^q$, (1)

where a certain phase space region $\Delta$ is divided in a proper way (isotropically for a self-similar fractal while anisotropically for a self-affine fractal) into $M$ sub-cells, $p_m$ is the probability for a particle to fall into the $m$th sub-cell, $\langle \cdot \cdot \cdot \rangle$ denotes the average over an event sample. If the $C_q(M)$’s have a power law behaviour with the diminishing of phase space scale:

$C_q(M) \sim M^{\phi_q} \quad (M \to \infty), \quad (2)$

then it is said to be anomalous scaling and the system is a fractal.

In real experiments the probability $p_m$ is unobservable and the corresponding moments $C_q$ is unaccessible. This problem has been solved by Bialas and Peschanski, who are able to show that the factorial moments $F_q(M)$ are equal to the probability moments $C_q$ provided the statistical fluctuations are Poissonian. Thus the scaling property of factorial moments, sometimes called intermittency, becomes a central problem in the study of nonlinear phenomena in high energy collisions.

Various methods have been developed in this study.

Firstly people noticed that in the definition Eq.(3) of factorial moments the average over event sample (vertical average) is carried out first and then is the average over the $M$ sub-cells (horizontal average). It was proposed to exchange the order of these two averages and define the horizontal factorial moments (HFM) as

$$F_q^{(H)}(M) = \frac{\langle M^{-1} \sum_{m=1}^{M} (n_m(n_m-1)\cdots(n_m-q+1)) \rangle}{\langle M^{-1} \sum_{m=1}^{M} n_m \rangle^{q}}, \quad (4)$$

Accordingly, the $F_q$ defined in Eq.(3) is called vertical factorial moments (VFM).

Note that the equality of factorial moments $F_q$ and probability moments $C_q$ has been proved only for the VFM. Therefore, in the study of the nonlinear phenomena — fractal property of multiparticle system, the HFM is appropriate only when it is equal to VFM. We will see in the following that this equality holds in some cases but does not hold in some other cases.

Secondly, various methods have been proposed to correct for the unflatness of the phase-space variable distributions. One is to divide the factorial moments by a factor $R_q$,

$$F_q^{C} = \frac{F_q}{R_q}, \quad R_q = \frac{M^{-1} \sum_{m=1}^{M} \langle n_m \rangle^q}{\langle M^{-1} \sum_{m=1}^{M} n_m \rangle^q}. \quad (5)$$

Another one is to change the phase space variable $x$ into the corresponding cumulant variable $x_c$ before calculating the factorial moments. The cumulant variable is defined as

$$x_c = \frac{\int_{x_{\min}}^{x} \rho(x)dx}{\int_{x_{\min}}^{x_{\max}} \rho(x)dx}. \quad (6)$$

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Another problem arises while carrying on this kind of study in $e^+e^-$ collisions. In these collisions the thrust (or sphericity) axis is chosen as the $z$ axis (longitudinal axis) to define the phase space variables: rapidity $y$, transverse momentum $p_t$ and azimuthal angle $\phi$. Different frames could be used to define the azimuthal angle $\phi$. The first one is to choose the minor axis of thrust (or sphericity) analysis as the $x$ axis, and use it as the starting point for counting the azimuthal angle $\phi$. The second one is to put the $z$ axis still on the major thrust axis, but turn the coordinate system around it and let the new $x$ axis lie on the $z_0-z$ plane, where $x_0, y_0, z_0$ denote the axes of the lab system and $x, y, z$ those of the turned system, as shown in Fig.1. In the following this frame will be referred to as the “rotated frame”. The third method is to rotate the frame in each event for a random angle around the $z$ axis. This is called the random frame.

All these methods have been used in the literature for studying the anomalous scaling of factorial moments, making the results hard to be compared. In the present paper we will examine these methods in some detail and discuss their applicability in physical problems. We will take $e^+e^-$ collisions at the $Z^0$ energy $\sqrt{s} = 91.2$ GeV as example and use JETSET7.4 [12] Monte Carlo code to generate 500 000 multihadron events as the event sample.

### II. AVERAGE DISTRIBUTIONS OF PHASE SPACE VARIABLES

In Fig. 2 (a), (b) and (d) are shown the average distributions of $y$, $p_t$ and $\phi$, where the rapidity is defined as $y = 0.5 \ln[(E+p_z)/(E-p_z)]$ with $z$ along the thrust axis; the azimuthal angle $\phi$ is defined in the plane perpendicular to the thrust axis, calculated with respect to the minor axis.

It can be seen from the figures that all the distributions are unflat. Especially, the distribution of $p_t$ is exponential and is highly concentrated in low $p_t$. A simple variable transformation $p_t \rightarrow \ln p_t$ [13] can make it a litter flatter as shown in Fig.2 (c).

### III. THE CORRECTION FACTOR AND CUMULANT VARIABLES

The unflat average distribution will cause additional variation of factorial moments with the diminishing of phase space scale and make the scaling property of factorial moments unequal to that of the probability moments even for the VFM. This effect has to be corrected.

Fialkowski proposed a factor [8], cf. Eq.(5), to correct for this effect. This method works good when the distribution of the phase space variable is not far from flat, e.g. the distribution of rapidity $y$ in a restricted central region $|y| < Y_c$ with $Y_c = 2$ as shown in Fig.2 (a), and is not good when the distribution is far from flat. This is especially the case for the distribution of $p_t$, cf. Fig.2
FIG. 4: VFM (solid circles) and HFM (solid curves) using correction factor method (restricted phase space). In (b) the upward triangles are for $q = 5$, downward ones for $q = 4$.

(b). Therefore, people sometimes transform $p_t \to \ln p_t$ first [13] and calculate $F_q(\ln p_t)$ instead of $F_q(p_t)$, and then correct the result by the factor $R_q$ given in Eq.(5). Note that the highly asymmetric region $0.1 \leq p_t \leq 2$ is transformed to the region $-2.3 \leq \ln p_t \leq 0.69$, which is distributed symmetrically around the pick of distribution, cf. the two arrows in Fig's.2 (b) and (c).

Since a transformation to a flatter distribution is necessary before calculating factorial moments, it is evident that the best way is to transform all the phase space variables to a flat distribution first. This could be established through the transformation to cumulant variables [9], cf. Eq.(6). The corresponding distributions are shown in Fig.3.

IV. VERTICAL AND HORIZONTAL FACTORIAL MOMENTS

Now let us turn to the comparison of vertical and horizontal factorial moments (VFM and HFM).

As noticed in the Introduction, our aim is to explore the anomalous scaling of probability moments as shown in Eq.(2) but the equivalence of factorial and probability
moments has been proved only for the VFM. So, the HFM is appropriate only when it is equal to VFM.

In Figures 4 and 5 (a), (b), (d), (e), (f) are shown the 1-D and 3-D VFM (solid circles) and HFM (solid curves) for the moment orders \( q = 2, 3, 4, 5 \) calculated using the correction factor, Eq.(5), method and the cumulant variables Eq.(6), respectively.

It can be seen from the figures that the VFM and HFM are equal only when using together with the cumulant variables and are unequal, especially for 1-D \( F_q(\phi) \) and \( F_q(p_t) \), when using the correction factor method.

The results from VFM + Correction Factor method are about the same as that from VFM + Cumulant Variables for 1-D \( F_q(y) \) and 3-D \( F_q(y, p_t, \phi) \) but are not the case for 1-D \( F_q(\phi) \) and \( F_q(p_t) \), cf. Fig’s. 4 and 5.

V. THE INFLUENCE OF PHASE SPACE REDUCTION

In hadron-hadron and nucleus-nucleus collisions the central rapidity regions with the rapidity \( y \) restricted to \( |y| \leq Y_c \) for some value of \( Y_c \) is commonly used. This is physically meaningful, because in these collisions the final state particles are mainly produced in the central region, while the particles in the regions \( |y| > Y_c \) are mainly come from the fragmentation of incident particles (leading particle effect).

On the contrary, in \( e^+e^- \) collisions, the multihadron final state is produced from a point source — virtual photon or \( Z^0 \), and no leading particle effect is present. The rapidity \( y \) is usually defined with respect to the thrust or sphericity axis. In this case, to carry on the study in a "central rapidity region" \( |y| \leq Y_c \) is physically doubtful. This is especially evident when the collision energy is so high that 3, 4 or even more jets can be produced. In a 2-jet event the restriction \( |y| \leq Y_c \) will cut out the most energetic particles from the two opposite jets symmetrically, but in a 3-jet event the same cut will cut out the most energetic particles only from one jet while retain almost all the particles in the other two jets. This asymmetric cut will results in unexpected phenomena, and the physical meaning may be difficult to interpret. Therefore, the study of multiparticle dynamics in \( e^+e^- \)-collisions is better to be carried out in the full phase space. However, the central rapidity region is sometimes also used in the literature for the study of \( e^+e^- \)-collisions \[13\]. Therefore, to investigate the influence of rapidity cut is worthwhile.

In Fig.6 are shown the results of VFM and HFM in nearly the full phase space — \(-5 < y < 5, 0.1 < p_t < 3 \text{ GeV}\) to be compared with the results shown in Fig.5 for a restricted phase space — \(-2 < y < 2, 0 < \phi < 2\pi, 0.1 < p_t < 2 \text{ GeV}\).

It can be seen from Fig.6 that the first point in 3-D \( F_q(y, p_t, \phi) \) and 1-D \( F_q(y) \) and the first 3 points in 1-D \( F_q(\phi) \) do not lie on a scaling curve together with the other points. This is due to the momentum conservation effect \[14\]. In the anomalous-scaling study, these points should be omitted.

The momentum conservation effect will also be reduced in a restricted phase space region, which was first pointed out in Ref.\[13\] and has been proposed as a second method for eliminating the influence of this effect. This explains the reason why the first points in Fig’s.5 lie on the scaling curves.
VI. THE FLUCTUATIONS IN AZIMUTHAL ANGLE

The fluctuations in azimuthal angle are worthwhile special investigation. It is commonly expected that there should have cylindrical symmetry around the \( z \) axis. If that is the case, then the fluctuations should have no correlation with the \( x \) axis chosen for counting the azimuthal angle \( \varphi \). In Fig’s. 4, 5, 6 (b), are shown the \( F_q(\varphi) \) with \( x \) axis along the minor of thrust analysis, while in the corresponding figures (c) are shown the results after rotating the \( x \) axis to let it lie on the \( z-z_0 \) plane \([10]\), cf. Fig.1. It can be seen that in the rotated frame, Fig’s. 4, 5, 6 (c), the \( F_q(\varphi) \) increases much faster as the diminishing of phase space scale than that in the thrust-minor frame, Fig’s. 4, 5, 6 (b). As discussed in Ref. \([10]\) this is because the thrust-minor axis is basically determined by the first hard gluon emission and taking this axis as \( x \) axis to count the azimuthal angle \( \varphi \) will highly reduce the fluctuation of the direction of first hard gluon emission. After rotation, the correlation between \( x \) axis and the direction of first hard gluon emission is relaxed and the full dynamical fluctuations are exhibited.

We could also rotate the \( x \) axis around \( z \) for a random angle in each event \([14]\). The resulting \( F_q(\varphi) \) turn out to be the same as those in the rotated frame with \( x \) on the \( z-z_0 \) plane shown in Fig’s.4, 5, 6 (c). This confirms the cylindrical symmetry of the fluctuation in \( \varphi \) after the correlation with the thrust-minor is relaxed.

VII. CONCLUSIONS

The following conclusions could be drawn from the above investigation:

1) The horizontal factorial moments (HFM) are equivalent to the vertical ones (VFM) only after the cumulant-variable transformation. Therefore, in the study of non-linear phenomena (intermittency or fractal) in high energy collisions the HFM could be used only in combination with the cumulant variables. On the other hand, the HFM is in its own right useful in single-event analysis. It can be seen from Eq. (4) that HFM is the average of the so called single-event factorial moments \( f_q^{(e)} \) \([13]\)

\[
F_q^{(H)}(M) = \langle f_q^{(e)} \rangle / \langle f_1^{(e)} \rangle^q, \tag{7}
\]

\[
f_q^{(e)}(M) = \frac{1}{M} \sum_{m=1}^{M} n_m^{(e)}(n_m^{(e)} - 1) \cdots (n_m^{(e)} - q + 1) \tag{8}
\]

where \( n_m^{(e)} \) is the number of particles of a single event in the \( m \)th sub-cell. The fluctuation of the single-event factorial moments around its average (HFM) is a characteristic of single-event fluctuations \([13]\).

2) The scaling properties of factorial moments in transverse directions (\( \varphi, p_t \)) are very sensitive to the correction method used. They are unstable when using the correction factor method, Eq.(5), even after the transformation \( p_t \to \ln p_t \) has been made. Using this method, the VFM in \( \varphi \) and the HFM in \( p_t \) fall down instead of increase with the diminishing of phase space scale, while at the same time the HFM in \( \varphi \) and the VFM in \( p_t \) do increase with the diminishing of scale, cf. Fig’s. 4 (b) and (d).

3) In the full phase space, the first few points of factorial moments do not lie on the scaling curve with the other points, due to the momentum conservation effect. This effect can be eliminated either through neglecting these points or through a cut in phase space.

4) The thrust (or sphericity) major-minor frame is inappropriate for the study of the scaling property of the azimuthal angle \( \varphi \), because this frame is strongly correlated with the direction of first hard gluon emission. Rotate the \( x \) axis to let it lie on the \( z-z_0 \) plane or rotate it randomly for each event can relax this correlation and exhibit the full dynamical fluctuations in \( \varphi \).

Therefore, the cumulant variables together with a frame rotated around the thrust (or sphericity) axis is the best for the investigation of the nonlinear phenomena (anomalous scaling of probability moments) in \( e^+e^- \) collisions. The VFM and HFM are equivalent in this case.

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