Conference Key Agreement and Quantum Sharing of Classical Secrets with Noisy GHZ States

Kai Chen and Hoi-Kwong Lo
Center for Quantum Information and Quantum Control (CQIQC)
Department of Electrical and Computer Engineering (ECE) and
Department of Physics, University of Toronto, Toronto, ON M5S 3G4, Canada
Email: hkle@comm.utoronto.ca

Abstract—We propose a wide class of distillation schemes for multi-partite entangled states that are CSS-states. Our proposal provides not only superior efficiency, but also new insights on the connection between CSS-states and bipartite graph states. We then consider the applications of our distillation schemes for two cryptographic tasks—namely, (a) conference key agreement and (b) quantum sharing of classical secrets. In particular, we construct “prepare-and-measure” protocols. Also we study the yield of those protocols and the threshold value of the fidelity above which the protocols can function securely. Surprisingly, our protocols will function securely even when the initial state does not violate the standard Bell-inequalities for GHZ states. Experimental realization involving only bi-partite entanglement is also suggested.

I. INTRODUCTION

Entanglement is the hallmark of quantum mechanics and has become the most important resources for various quantum information tasks nowadays. For practical aims, one may at first sight ask, how many standard states (e.g., Einstein-Podolsky-Rosen (EPR) states and Greenberger-Horne-Zeilinger (GHZ) states) can be distilled through a noisy channel [1]. While this problem is really hard to work out, except for the case of bipartite pure state, because one has to consider all the possible strategy for entanglement distillation (for a recent review, see [2]). A practical way is to study specific strategies for entanglement distillation. While the classification of bipartite pure-state entanglement has been solved, currently the classification of multi-partite entanglement and mixed state bipartite entanglement is an important open problem [2]. Our focus on this paper is to study the distillation of multi-partite entanglement and its applications to multi-party quantum cryptography. Our motivations is threefold: theoretically for better understanding and quantifying multi-partite entanglement; practically to propose new applications of quantum cryptography in the multi-party setting particularly in the presence of noises; and finally to provide a bridge between theory and practice.

For convenience, we use the standard stabilizer formulation, particularly specialize to the case where the output state is a so-called Calderbank-Shor Steane (CSS) state [3], [4], which is a simultaneous eigenstate of a complete set of (commuting) stabilizer generators each of either X-type or Z-type. We note that CSS states are equivalent to bipartite graph codes, which have previously been analyzed by Dür, Aschauer and Briegel [5]. We remark that one part of this equivalence—that CSS states are bipartite graph states—was due to Eric Rains [6]. Our observation puts the earlier work of [5] in the more systematic setting of CSS formulation. Moreover, we apply the idea of CSS-state distillation to the construction of multi-party quantum cryptographic protocols.

We specifically consider two multi-party cryptographic tasks, namely (a) conference key agreement where three parties, Alice, Bob, and Charlie, would like to obtain a common random string of number, known as the conference key, \( k \), and to ensure that \( k \) is secure from any eavesdropper, Eve; and (b) quantum sharing of classical secrets [7] where Alice would like to divide up her secret password between two parties, B' and C', in such a way that neither B' nor C' alone knows anything about the password and yet when B' and C' come together, they can re-generate the password. To secure the classical communications between the parties in both protocols, we assume that each pair of the three parties are authenticated by standard unconditionally secure method of authentication.

For practical interest, we will construct “prepare-and-measure” type protocols for both quantum cryptography tasks, for which the participants do not need to have full-blown quantum computers to implement them. In such protocols, there is a preparer who prepares some number, say N, copies of a standard entangled multi-partite state and distributes them to other participants. Each of other participants just performs some local individual measurement on his/her share of each copy of the state. The participants then perform classical post-processing (i.e., classical computations and classical communications (CCCs)) that can be performed by strictly classical devices). Our study is a natural generalization of the security proofs by Shor-Preskill [8] and Gottesman-Lo [9] of the BB84 [10] quantum key distribution protocol to the multi-partite case. [The first proof of security of BB84 was by Mayers [11].]

Our paper is organized as follows. In Section 2, we study the entanglement distillation of the GHZ state and present an improved hashing protocol. It can distill prefect GHZ state successfully whenever the fidelity \( F \geq 0.7554 \) comparing the result of \( F \geq 0.8075 \) in [12] for a tripartite Werner-like state. In Section 3, we generalize our results from the GHZ state to a general so-called CSS state and show that the
various subroutines that we have studied, in fact, apply to a general CSS state. Also we show the equivalence of CSS states and bipartite graph states. In Sections 4 & 5, we apply our formulation to study the three-party conference key agreement and secret sharing problem, and show that for tripartite Werner state, conference key agreement is possible whenever the fidelity $F \geq 0.3976$, while for secret sharing whenever fidelity $F \geq 0.5372$. We remarked that our protocols, will work even when the initial GHZ-state does not violate standard Bell inequalities.

## II. Distillation of the GHZ State

Suppose three distant parties, Alice, Bob, and Charlie, share a GHZ state $|\Psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle |0\rangle + |1\rangle |1\rangle |1\rangle)$, which is the +1 eigenstate of commuting observables (stabilizer generators): $S_0 = X \otimes X \otimes X, S_1 = Z \otimes Z \otimes I, S_2 = Z \otimes I \otimes Z$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let us denote a GHZ basis as following:

$$|\Psi_{p,i_1,i_2}\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle |i_1\rangle |i_2\rangle + (-1)^p |1\rangle |\bar{i}_1\rangle |\bar{i}_2\rangle), \quad (1)$$

where $p$ and the $i$’s are zero or one and a bar over a bit value indicates its logical negation. Here, $(p, i_1, i_2)$ correspond to the eigenvalues of the 3 stabilizer generators $S_0, S_1, S_2$ by correspondence relation: eigenvalue $1 \rightarrow$ label 0, and eigenvalue $-1 \rightarrow$ label 1. Thus, a density matrix which is diagonal in the GHZ-basis would be:

$$\rho_{ABC} = \text{diagal}\{p_{000}, p_{100}, p_{011}, p_{111}, p_{010}, p_{101}, p_{001}, p_{101}\}.$$  

We can think of an GHZ-state to be in one of the eight possible basis states.

### A. Error rate estimation and derivation of density matrix (GHZ-basis diagonal)

Suppose three parties share a general tri-partite density matrix that is not necessarily diagonal in the GHZ-basis. But according to [13], [14], one can depolarize a general 3-party density matrix by applying with the operator $XXX$, followed by $ZZZ$ and finally, $ZZZ$ with a probability $1/2$ separately. The overall operation makes $\rho$ diagonal in the basis Eq. (1) without changing the diagonal coefficients. For this reason, we will focus only on the diagonal elements of a density matrix in the GHZ-basis.

In this paper, we often assume the three parties share $n$ trios of qubits. Notice that there is no need to assume an i.i.d. (independent identical distribution) for the $n$ trios. Instead, we consider the most general setting where those $n$ trios can be fully entangled among themselves and perhaps also with some additional ancillas. As noted in the last paragraph, we consider only the $n$-GHZ-basis for the $n$ trios. Each basis vector can be denoted by a sequence of $n$ objects each of which takes one of the $2^3 = 8$ possible values. In fact, following Gottesman-Lo [9], we argue that the protocols presented in this paper will work well (with high probability) so long as the “type” of sequence is known. Recall that, to encode the information about the type, we need only consider the relative frequency of the eight basis states for the $n$ trios. We can easily summarize this information by the marginal density matrix:

$$\rho_{ABC} = \text{diagal}\{p_{000}, p_{100}, p_{011}, p_{111}, p_{010}, p_{101}, p_{001}, p_{101}\}.$$  

Now the three parties can estimate the eight matrix elements of the aforementioned marginal density matrix reliably by using local operations and classical communications (LOCCs) only. They just measure along $X, Y, Z$ basis and compare the results of their local measurements. In fact, they will find that the error pattern for the seven non-trivial group elements of $XXX, ZZI, ZIZ, -YYX, IZZ, -YZY, -YY, -XYY$. For example, the error rate for $XXX$ is $p_{100} + p_{101} + p_{110} + p_{111}$, the similar error pattern for other group elements. Since these error rates are linearly dependent on $p_{ijk}$ and can be determined by local operations and classical communications (LOCCs), the above equations relate the diagonal matrix elements of the density matrix, $\rho_{ABC}$ to experimental observables.

It should be clear that the above error estimation procedure will give accurate information on the type of the sequence. Therefore, in what follows, when one studies multi-partite entanglement distillation protocol, one can reduce the problem of a general initial state of $n$ trios of qubits to the case where the $n$ trios are treated as i.i.d. This reduction idea via the method of the type is a generalization of the bipartite case studied in [9]. Such a reduction technique greatly simplifies the problem.

### B. Improved multi-party hashing for distillation of GHZ states

Suppose $M$ (> 2) parties share an ensemble of $n$ identical mixed multi-partite states and they would like to distill out almost perfect (generalized) GHZ states. Maneva and Smolin [12] found an efficient multi-partite entanglement distillation protocol—multi-party hashing method. They used multilateral quantum XOR gates (MXOR) as shown in the Fig.4 of [12] where every party imposes identical Control-NOT operations on some of their own particles from one single source particle or to one single target particle. If we write an unknown $N$-qubit state as an $N$-bit string $b_0, b_1, b_2, \ldots b_{N-1}$ where $b_0$ corresponds to the eigenvalue of the operator $XXX \cdots X$ and $b_i$ ($i > 0$) corresponds to the eigenvalue of the operator $Z_iZ_{i+1}$. (See Eq. (3) of [12]). Note that $b_0$ denotes the phase error pattern and $b_i$ ($i > 0$) denotes the bit-flip error pattern. Maneva and Smolin’s protocol involves applying first a number of rounds of random hashing in the amplitude bits followed by performing a number of rounds of random hashing in the phase bit(s). They showed that its yield (per input mixed state) was, in the asymptotic large $n$ limit, given by $D_h = 1 - \max_{x \in \{H(b_0), H(b_2)|b_1\}} H(x)$, where $H(x)$ is the standard Shannon entropy in classical information theory [15]. We will argue that, in fact, the yield can be increased to:

$$D_h = 1 - \max\{H(b_1), H(b_2|b_1)\} - H(b_0) + I(b_0; b_1, b_2), \quad (2)$$

for tri-partite case where $I(X; Y)$ is the standard mutual information between $X$ and $Y$.

The key point is that there may be some correlations between $b_0$ and $(b_1, b_2)$ and also between $b_1$ and $b_2$ i.e., $I(b_0; b_1, b_2) \geq 0, I(b_1; b_2) \geq 0$ If these quantities are non-
zero, one can improve Maneva-Smolin’s protocol by consider the following strategy:

1. The three parties perform $nH(b_1)$ rounds of hashing to work out the value of $b_1$’s completely (Suppose here $H(b_1) \leq H(b_2)$). Afterwards, the uncertainty in the variables $b_2$’s is reduced to $nH(b_2|b_1)$. Therefore, only $nH(b_2|b_1)$ rounds of hashing is needed to work out the value of $b_2$’s. Note that the hashing of $b_1$ and $b_2$ can be simultaneously executed. Therefore, in total, we still only need $n\max\{H(b_1),H(b_2|b_1)\}$ rounds of random hashing in amplitude bits.

2. They use the information on the pattern of $b_1,b_2$ (the amplitude bits) to reduce their ignorance on the pattern of $b_0$ (the phase bit) from $nH(b_0)$ to $nH(b_0|b_1,b_2) = n[H(b_0) - I(b_0|b_2)] = n[H(b_0, b_1, b_2) - H(b_1, b_2)]$.

3. They apply a random hashing by using multilateral quantum XOR gates with regard to one single source particle of every party, shown in detail in Fig.4b of [12] to identify the pattern of $b_0$. Now only (slightly more than) $n[H(b_0, b_1, b_2) - H(b_1, b_2)]$ rounds of random hashing is needed.

Thus the yield of our method gives Eq. 1.

Suppose, one has prepared a class of Werner-like states $\rho_W = \alpha \langle \Phi^+ \rangle |\Phi^+\rangle + \frac{1-\alpha}{2} I, \quad 0 \leq \alpha \leq 1$, where $|\Phi^+\rangle$ denotes the so-called cat state and $I$ is the identity matrix.

We remark that in the GHZ-basis, the state can be rewritten into $\rho_W = F|0,0,0,0,0,0,0,0\rangle \langle 0,0,0,0,0,0,0,0| + \frac{1-F}{2} (|0,0,0,0,0,0,0,0\rangle \langle 0,0,0,0,0,0,0,0|)$.

Using the random hashing method of Maneva and Smolin, for the tri-partite case, we can obtain perfect GHZ states with nonzero yield whenever $F > 0.8075$. With our improved random hashing method, we still get with nonzero yield whenever $F \geq 0.7554$. This is a substantial improvement of the original method.

III. CSS states

In this section, we generalize our results on the GHZ state to a general CSS state. We first define a CSS state and show that, just like the GHZ state, a CSS state can be distilled by the hashing protocol and a recurrence method developed by Murao et al in [16]. We derive the yield for the hashing protocol and show that a CSS state is equivalent to a bipartite (i.e., two-colorable) graph state, a subject of recent attention.

A. Distillation of CSS-states

A CSS-state is basically a CSS-code where the number of encoded qubit is zero. For instance, an encoded $|0\rangle$ state of a CSS code is a CSS-state. More formally, we have the following definition.

**CSS states:** A CSS-state is a +1 eigenstate of a complete set of (commuting) stabilizer generators such that each stabilizer element is of X-type or Z-type only. For example, a GHZ-state is a CSS-state with stabilizer generators: $XXX$, $ZZI$ and $ZIZ$.

**Claim 1:** Suppose we label its simultaneous eigenstate for a CSS-state by its simultaneous eigenvalues $(\hat{b}, \hat{p})$ where $\hat{b} = \{b_1, b_2, \cdots , b_m\}$ is a vector that denotes the tuples of Z-type eigenvalues and $\hat{p} = \{p_1, p_2, \cdots , p_n\}$ is a vector that denotes the tuples of X-type eigenvalues and $\tilde{p} = \{p_1, p_2, \cdots , p_n\}$ is a vector that denotes the tuples of Y-type eigenvalues and $\tilde{p} = \{p_1, p_2, \cdots , p_n\}$ is a vector that denotes the tuples of Z-type eigenvalues.

**Claim 2:** Bipartite (i.e., two-colorable) graph states are equivalent to CSS-states.

**Proof:** For details, see our preprint [17] (one part of the equivalence: CSS implies bipartite, is due to Eric Rains).

**Claim 2** establishes the equivalence of two different mathematical formulations: CSS-states and bipartite graph states. Thus much of what we have learnt about the distillation of bipartite/two-colorable graph states through the work of [5] can be interpreted in the more systematic language of CSS-states. In particular, it is natural to consider the bit-flip and phase error patterns separately and consider their propagations in quantum computational circuit. From Claim 2, we learn that Claim 1, which originally refers to CSS states, can be applied directly to any bipartite graph states. Moreover, the improvement that we have found in the last section, in fact, applies to bipartite graph states.
IV. 3-PARTY CONFERENCE KEY AGREEMENT WITH NOISY GHZ STATES

In this section, we consider a prepare-and-measure conference key agreement scheme, where we allow the participants to perform only local individual quantum measurements and local classical computations and classical communications (CCCCs) to obtain the same random string of secret, known as a key, which can be useful for securing a conference call against eavesdropping attacks.

How do we prove the security of our protocol against an eavesdropper? We start with a CSS-based GHZ state distillation protocol and try to convert it to a “prepare and measure” protocol.

The protocol involves two subprotocols. For bit-flip error detection (B step), we use the P2 step of Murao et al. [16] as shown in Fig. 1a. We remark that the P1 step used in [16], is, in fact, a phase error detection step (more precisely, subsequent measurement operators applied are conditional to phase error syndromes found in earlier measurements) and is strictly forbidden in a “prepare-and-measure” protocol [9].

Thus for phase error correction, we design a procedure borrowing the idea of Gottesman and Lo [9] as shown in Fig. 1b, but apply it to the multi-partite case.

Our multi-partite B step: Using the language of stabilizer, this step can be reformulated by the following transformation of the three GHZ-like states

\[
\begin{align*}
(p, i_1, i_2), (q, j_1, j_2), (r, k_1, k_2) \\
&\rightarrow (p \oplus q, i_1, i_2), (q \oplus i_1 \oplus j_1, i_2 \oplus j_2),
\end{align*}
\]

where \((p, i_1, i_2), (q, j_1, j_2)\) denote the phase bit and amplitude bits for the first and second GHZ-like states. If \(i_1 \oplus j_1 = i_2 \oplus j_2 = 0 \mod 2\), we keep the first GHZ state, otherwise discard all the two states. This corresponds to the prescription that we keep the first trio if \(M_A = M_B\) and \(M_A = M_C\) (i.e., Alice, Bob and Charlie get the same measurement outcome). This step just changes the 8 elements of diagonal entries accordingly and can be obtained by straight calculation.

Our multi-partite P step: This step (shown in Fig. 1b) can be reformulated by the following transformation of the three GHZ-like states

\[
\begin{align*}
&\left[(p, i_1, i_2), (q, j_1, j_2), (r, k_1, k_2)\right] \\
&\rightarrow \left[(p \oplus q, j_1, j_2), (p \oplus q, j_1, j_2), (r, k_1, k_2)\right]
\end{align*}
\]

If \(p \oplus q = p \oplus r = 1 \mod 2\), we apply \(p \rightarrow p + 1 \mod 2\), otherwise keep the first GHZ-like state invariant. Note that P can also be performed locally by each party, which regards the circuit as implementing a 3-qubit phase error correction code.

We now argue that the aforementioned entanglement distillation subprotocol (P step) can be converted to a prepare-and-measure protocol. This is in the spirit of [9]. In conference key agreement, Alice, Bob and Charlie do not need to perform phase error correction. They only need to prove that phase error correction would have been successful, if they had performed it. Thus in the second part of Fig. 1b, each of Alice, Bob and Charlie simply takes the parity \((Z_A + Z_B + Z_C) \mod 2\) of their own three particles. No classical communication is needed. Moreover, we can apply the same conversion idea to any concatenated protocols involving B steps, P steps and following by a hashing protocol. This conversion result means that we can obtain secure protocols by considering the convergence of GHZ distillation protocols involving those operations.

By direct numerical calculation, we can verify that our scheme can distill GHZ state with nonzero yield whenever \(F \geq 0.3976\) by some state-dependent sequence of B and P steps, and then change to our random hashing method if it works. [Notice that the yield of a recurrence protocol asymptotically goes to zero (see [1]). Therefore, it is advantageous to switch to a hashing protocol at some point.] We find that a sequence of B and P steps BBBB for \(F = 0.3976\), which is optimal for any sequence with at most 5 steps. our new protocol gives dramatic improvement by using 2-way classical communications compared with the random hashing method of Maneva and Smolin which works only when \(F \geq 0.8075\).

V. QUANTUM SHARING OF CLASSICAL SECRETS IN A NOISY CHANNEL

Quantum sharing of classical secrets—has also been introduced [7]. Here, the goal is to use quantum states to share a classical secret between multiple parties and to ensure that no eavesdropper can learn useful information by passive eavesdropping. Whereas [7] considers only the perfect case, here we consider the case where the initial state is imperfect, following [18], [19].

Phase One: The key point is that the GHZ state is a +1 eigenstate of \(XXX\), and satisfy a classical constraint \(X_A + X_B + X_C = 0 \mod 2\). Note that individually, each of Bob and Charlie has no information on \(X_A\). But, by getting together,
Bob and Charlie can obtain $X_B + X_C \mod 2$ and, therefore, obtain $X_A$.

**Phase two:** Suppose Alice now would like to share a bit value, $b$, with Bob and Charlie. She can broadcast $X_A + b$.

Note that, since the final measurement is now done along the X-axis, the bit-flip measurements now correspond to measurements along X. Therefore, for the bit-flip error detection code ($B'$ step), we use the same procedure as the P1 step of Murao et al [16] as shown in Fig. 2a. As for the phase error correction ($P'$ step), we design a procedure similar to the idea of Gottesman and Lo [9] as shown in Fig. 2b. Details of our protocol can be found in our preprint [17].

Similar argument to conference key agreement case holds for secret sharing. The three parties only need to ensure that, phase error correction ($P'$ step) would have been successful, if they had performed it. We verify that our scheme used in secret sharing in a noisy channel can distill GHZ state with nonzero yield whenever $F \geq 0.5372$ by some state-dependent sequence of $B$ and $P$ steps and then change to our random hashing method if it works. For the optimal sequences within 5 steps, we find it is $(B' B' B' B' B')$ that just gives $F \geq 0.5372$ followed by hashing method.

**Remark 1:** Note that, if one of the parties, say Alice, is actually the preparer of the multi-partite state, in a prepare-and-measure protocol, she is allowed to pre-measure her sub-system. By doing so, she projects an $N$-partite entangled state into one of the various $(N - 1)$-partite entangled state. Therefore, conference key agreement and secret sharing protocols can be implemented with only $(N - 1)$-party entanglement.

**Remark 2:** It was claimed in Refs. [18], [19] that a violation of Bell inequalities is a criterion for security of secret sharing schemes [7] under the assumption of individual attacks by Eve and one-way classical post-processing protocols by Alice and Bob. Violation of Bell inequalities for $N$ particles Werner-like state is shown in Ref. [20] to be $\alpha > 1/\sqrt{2}(N - 1)$. For a tripartite system, this gives $\alpha > 1/2$ and thus $F > 9/16 \approx 0.5625$. This is clearly a higher requirement for the initial fidelity of a Werner-like state than that for our two-way prepare and measure secret sharing scheme which only requires $F \geq 0.5372$. Thus our two-way protocols are secure even when Bell inequalities are not violated.

**Experimental Implementations.** From Remark 1, for the three-party case, our protocols can be done with only bi-partite entangled states and can be experimentally implemented with, for example, parametric down conversion sources. More concretely, imagine that Alice prepares a perfect GHZ state and measures her qubit along the $X$, $Y$, $Z$-axis. After her measurement, Bob and Charlie’s state will be $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ or $\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$; $\frac{1}{\sqrt{2}}(|00\rangle + |i1\rangle)$ or $\frac{1}{\sqrt{2}}(|00\rangle - |i1\rangle)$ (with equal probabilities) separately. Thus Alice could implement the two protocols by simply preparing one of the six states, and sending the two qubits to Bob and Charlie respectively through some quantum channels.

We remark that our protocols are not proven to be optimal. In future, it will be interesting to search for protocols with better yields and higher threshold error rates. Moreover, one may try to generalize our results to quantum sharing of classical secrets for more general access structures. Our results will shed some light on the fundamental questions of the classification of multi-party entanglement, the power and limitations of multi-party quantum cryptography. The details of our work can be found in [17].

We thank helpful discussions with many colleagues and financial support from a number of funding agencies.

**References**

[1] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, *Phys. Rev.*, vol. A54, p. 3824 (1996).

[2] M. Horodecki, R. Horodecki, *Quantum Inf. & Comp.*, vol. 1, p. 45 (2001).

[3] A.R. Calderbank and P.W. Shor, *Phys. Rev.*, vol. A54, p. 1098 (1996).

[4] A.M. Steane, *Phys. Rev. Lett.*, vol. 77, p. 793 (1996).

[5] W. Dür, H. Aschauer, and H.-J. Briegel, *Phys. Rev. Lett.*, vol. 92, p. 107903 (2003).

[6] Eric Rains, private communications.

[7] M. Hillery, V. Bužek, and A. Berthiaume, *Phys. Rev.*, vol. A59, p. 1829 (1999).

[8] P.W. Shor and J. Preskill, *Phys. Rev. Lett.*, vol. 83, p. 441 (2000).

[9] D. Gottesman, and H.-K. Lo, *IEEE Trans. Inf. Theory*, vol. 49, p. 457 (2003).

[10] C. H. Bennett, & G. Brassard, *Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing*, IEEE, 1984, pp. 175-179.

[11] D. Mayers, *Journal of ACM*, vol. 48, Issue 3, p. 351-406. A preliminary version is D. Mayers, “Quantum key distribution and string oblivious transfer in noisy channels”, *Advances in Cryptology-Proceedings of Crypto* 96 (Springer-Verlag, New York, 1996), p. 343.

[12] E.N. Maneva, and J.A. Smolin, “Improved two-party and multi-party purification protocols”, in: *AMS Contemporary Mathematics Series*, vol. 305, S.J. Lomonaco, and H.E. Brandt (Eds.), AMS, (2002), pp. 203-212.

[13] T. Cover & J. Thomas, Elements of Information Theory, Wiley, New York, 1991.

[14] M. Murao, M.B. Plenio, S. Popescu, V. Vedral, and P.L. Knight, *Phys. Rev.*, vol. A57, p. R4075 (1998).

[15] K. Chen and H.-K. Lo, preprint available at http://arxiv.org/abs/quant-ph/0404133

[16] V. Scarani and N. Gisin, *Phys. Rev. Lett.*, vol. 87, p. 117901 (2001).

[17] V. Scarani and N. Gisin, *Phys. Rev.*, vol. A65, p. 012311 (2001).

[18] M. Zukowski and C. Brukner, *Phys. Rev. Lett.*, vol. 88, p. 210401 (2002).