Abstract. The linear quadratic regulator is the fundamental problem of optimal control. Its state feedback version was set and solved in the early 1960s. However static output feedback problem has no explicit-form solution. It is suggested to look at both of them from another point of view as a matrix optimization problem, where the variable is a feedback matrix gain. The properties of such a function are investigated, it turns out to be smooth, but not convex, with possible non-connected domain. Nevertheless, the gradient method for it converges to the optimal solution in state feedback case and to a stationary point in output feedback case. The results can be extended for the general framework of reduced gradient method for optimization with equality-type constraints.

Key words. linear quadratic regulators, nonconvex minimization, state and output control

AMS subject classifications. Primary, 49N10; Secondary, 49M37, 90C26, 90C52

1. Introduction. The linear quadratic regulator (LQR) problem is formulated as an optimization problem of minimizing a quadratic integral cost with respect to control function. It has been extensively analysed in the last century since the seminal works of Kalman in 1960 [17, 18]. The main result claims that for infinite-horizon LTI system the optimal control can be expressed as linear static state feedback. The optimal gain can be found by solving algebraic matrix Riccati equation (ARE). The results became classical and were immediately included in textbooks on control [4, 5, 21]. New approaches to the problem were based on the techniques of semidefinite programming — reduction to convex optimization with Linear Matrix Inequalities (LMIs) as constraints [10, 16, 6, 20]. Linear static feedback is very natural and simple form of control for engineers, thus there were many attempts to extend the technique for other control problems.

The nearest relative of LQR is output feedback — the same LTI system with quadratic performance in the case when full state is not measured but some output (linear function of state) is available. The attempts to apply static output feedback (SOF) met numerous difficulties. The problem was first addressed by Levine and Athans [22], but it was discovered that such stabilizing control may be lacking and there are no simple optimality certificaties if it does exist. Serious theoretical efforts were directed on formulation of existence conditions, see [34, 9], but the problem still remains open. If a system is stabilizable via a static output controller, there are just necessary conditions for optimality, moreover these conditions are formulated as a system of nonlinear matrix equations [22]. Thus design of optimal SOF implies application of numerical methods. The first one was proposed in [22], but it requires to solve nonlinear matrix equations on each iteration. The method suggested by Anderson and Moore [4] is based on solution of linear matrix equations only, but its convergence is not obvious. Since then, numerous iterative schemes have been proposed, see [35, 25, 29, 26, 12, 32, 16] and references therein. However rigorous validation is lacking for many of them, while some others include hard nonlinear problems to be solved at each iteration. To sum up, optimization of SOF remains a challenging problem.

A promising tool for solving both state and output feedback control is the direct gradient method. Matrix gain $K$ for state $u(t) = Kx(t)$ or output $u(t) = Ky(t)$ control is considered as variable for optimization of the objective function which is expressed as $f(K)$. This function is well-defined for the set of stabilizing controllers $\mathcal{S}$ (otherwise the quadratic integral performance index is not defined). The set $\mathcal{S}$ is open and the minimum of $f(K)$ is achieved at the interior point. Thus a simple gradient method for unconstrained minimization of $f(K)$ can be applied

$$K_{j+1} = K_j - \gamma_j \nabla f(K_j)$$

provided that the initial stabilizing controller $K_0$ is known. Gradient $\nabla f(K)$ for state feedback case has been found in the pioneering paper of Kalman [17], for output feedback it was obtained by Levine and Athans.
Recently there was a breakthrough in this field. First there appeared papers devoted to discrete-time version of state-feedback LQR [11, 13]. \( f(K) \), despite being non-convex is shown to satisfy the so called Lezanski-Polyak-Lojasiewicz (LPL) condition. This condition was proposed in the works [23, 30, 24] back in 1960s and still remains a powerful tool in non-convex optimization [19]. Based on LPL condition it was possible to prove convergence of the gradient method to optimal controller. Important works [27, 28] overcome nonconvexity obstacle for classical continuous-time LQR. It was proved that LPL condition holds for this case and gradient method converges.

Situation is more complicated for output control. As we mentioned above, the domain \( S \) can be nonconnected, and values of local minima at different connected components are different. Moreover several local minima points can exist in a single component. Thus it is hard to expect something better than convergence to a stationary point.

**Contributions of the paper** We consider gradient method in the unified setup, common for state and output control. We analyse the properties of the objective \( f(K) \) and its domain \( S \). Simple examples exhibit their properties, and some of them are new — e.g. the existence of local minima and saddle points for output case. The sublevel set \( S_0 = \{ K \in S : f(K) \leq f(K_0) \} \) is proved to be bounded and minimum \( f(K) \) on \( S_0 \) exists at \( K_0 \) with \( \nabla f(K_0) = 0 \). The important result is that \( f(K) \) on \( S_0 \) is L-smooth. Moreover LPL condition holds for state control case, this allows to prove convergence of the gradient method to the unique minimizer with linear rate. The result is close to the one obtained in [27], but the technique of the proof is completely different. In [27] the problem was converted into convex optimization by change of variables. However such transformation is possible for state control only, while we use the technique which fits for both state and output cases. This allows to prove convergence to a stationary point for output feedback. Finally we exhibit that the results can be considered in the framework of reduced gradient method for abstract optimization problems with equality-type constraints.

**Organization of the paper** In section 2 we formulate the LQR as matrix optimization problem with nonlinear equality constraints. Then it is reduced to matrix unconstrained minimization with objective \( f(K) \) and its domain \( S \). Section 3 discusses the properties of this function defined on a generally non-convex set. The most important are L-smoothness property; for state feedback case LPL condition holds. In section 4 the gradient flow on this set is showed to be exponentially stable and the discrete gradient method is introduced. The convergence guaranties are presented. Section 5 illustrates the numerical experiments for the proposed method. In section 6 we address the links between the proposed method and general notion of reduced gradient. Finally section 7 we discuss directions for future research. The proofs of the results are relegated to Appendix.

**2. Problem Statement.** We use standard notation: \( \| \cdot \| \) — spectral norm of a matrix; \( \| \cdot \|_F \) — its Frobenius norm; \( S_n \) — the set of symmetric matrices; \( I \) is the identity matrix; \( A \succ B \ (A \succeq B) \) means that the matrix \( A - B \) is positive (semi-)definite; the eigenvalues \( \lambda_i(A) \) of a matrix \( A \in \mathbb{R}^{n \times n} \) are indexed in an increasing order with respect to their real parts, i.e., \( \Re(\lambda_1(A)) \leq \ldots \leq \Re(\lambda_n(A)) \).

Consider linear time-invariant system

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]
\[
y(t) = Cx(t),
\]
\[
Ex(0)x(0)\top = \Sigma,
\]
where matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{r \times n} \). The infinite-horizon LQR performance criterion is given by

\[
\mathbb{E} \int_0^\infty [x(t)\top Q x(t) + u(t)\top R u(t)] \, dt,
\]

\[
(2.1)
\]

\[
(2.2)
\]
where the expectation is taken over the distribution of an initial condition \( x(0) \) with zero mean and covariance matrix \( \Sigma \), and the quadratic cost is parameterized by \( 0 \prec Q \in S_n \), and \( 0 \prec R \in S_m \).

The static feedback control is \( u(t) = -Ky(t) \), where \( K \in \mathbb{R}^{m \times r} \), is a constant matrix. Then the closed loop system is given by

\[
\dot{x}(t) = A_K x(t), \quad A_K = (A - BKC)
\]

and objective function becomes

\[
f(K) = \mathbb{E} \int_0^\infty \left[ x(t)^\top (Q + C^\top K^\top RKC)x(t) \right] dt.
\]

We use notation \( f(K) \) to underline that the performance index depends on gain only; all other ingredients of system description are known. Thus our optimization problem is

\[
f(K) \rightarrow \min_{K \in \mathcal{S}},
\]

here \( \mathcal{S} \) is the set of stabilizing feedback gains,

\[
\mathcal{S} = \{ K \in \mathbb{R}^{m \times n} : \Re \lambda_i(A - BKC) < 0, \forall i \in 1, n \}.
\]

Indeed, \( f(K) \) is defined for stabilizing controllers \( K \in \mathcal{S} \) only.

The problem of existence of stable output feedback is hard, see e.g. [9, 34]. However we are not interested in this, our main assumption is that a stabilizing controller exists and is available:

\( K_0 \in \mathcal{S} \) is known.

For instance if \( A \) is Hurwitz then we can take \( K_0 = 0 \). This controller will be taken as the initial approximation for iterative methods. Thus our goal is to improve the performance of the known regulator. Denote \( S_0 \) – the sublevel set

\[
S_0 = \{ K \in \mathcal{S} : f(K) \leq f(K_0) \}.
\]

We suppose the following Assumptions hold:

- \( K_0 \in \mathcal{S} \) exists;
- \( Q, R, \Sigma \succ 0 \);
- \( \text{rank}(C) = r \).

Notice that there are no assumptions on controllability/observability, existence of \( K_0 \in \mathcal{S} \) suffices. Also we assume \( B \neq 0 \), otherwise the problem is trivial. Condition \( Q \succ 0 \) in many cases can be relaxed to \( Q \succeq 0 \), but we do not focus on this.

We distinguish two main versions of the problem:

1. \textbf{SLQR} - state LQR - if \( C = I \), that is the state \( x(t) \) is available as control input. If it is needed to specify the performance index \( f(K) \) for this case, we denote it as \( f_S(K) \).
2. \textbf{OLQR} - output LQR - if \( C \neq I \), when output \( y(t) \) is the only information available. We use notation \( f_O(K) \) to specify this case, while \( f(K) \) is used in general situation.

Let us formulate the problem as matrix constrained optimization one. To avoid calculation of integrals Bellman lemma [7] is instrumental.

**Lemma 2.1.** Given \( W \succ 0 \), and a Hurwitz matrix \( A \). Then on the solution of the LTI system

\[
\dot{x}(t) = Ax(t), \quad x(0) = x_0
\]

it holds that

\[
\int_0^\infty x^\top(t)Wx(t)dt = x_0^\top X x_0,
\]

where \( X \) is the solution of the Lyapunov matrix equation

\[
A^\top X + XA = -W.
\]

Applying this result we rewrite 2.5 in the final form
Problem 2.2.

\begin{equation}
(2.6) \quad f(K) := \text{Tr}(X\Sigma) \rightarrow \min_{K}
\end{equation}

\begin{equation}
(2.7) \quad (A - BKC)^\top X + X(A - BKC) + C^\top K^\top RKC + Q = 0, \quad X > 0.
\end{equation}

This is optimization problem with matrix variables $K, X$ and nonlinear equality-type constraint (2.7). For $K \in \mathcal{S}$ the solution $X > 0$ of this equation exists (Lyapunov theorem), we denote it as $X(K)$. Thus the problem is rewritten in the form (2.5) with $f(K) = \text{Tr}(X(K)\Sigma)$.

In the next section we analyse the properties of the function $f(K)$, its domain $\mathcal{S}$ and level set $\mathcal{S}_0$.

3. Properties of $f(K)$.

3.1. Examples. We start with few simple examples to exhibit the variety of situations.

**Example 3.1.** Let us consider 1D example with parameters $A = 0 \in \mathbb{R}, Q = R = 2B = 1 \in \mathbb{R}, K = k \in \mathbb{R}$. The function

\[ f(k) = k + \frac{1}{k} \]

can be written explicitly.

Here $\mathcal{S}$ is convex and unbounded, $\mathcal{S}_0$ – bounded, $f(K)$ is convex and unbounded on $\mathcal{S}$ (see Figure 1).

**Example 3.2.** Let $n = 2, m = 2$, set $A, B$ and $C$ to be identity matrices. Then

\[ \mathcal{S} = \{ K \in \mathbb{R}^{2 \times 2} : k_{11} + k_{22} < 1 + k_{11}k_{22} + k_{12}k_{21}, k_{11} + k_{22} < 2 \} . \]

We see that $\mathcal{S}_0$ is not convex. This can be verified if one takes a cut $x = k_{11} = k_{12}, y = k_{22} = k_{21}$ (see Figure 2).

**Example 3.3.** Let $n = 3, m = 1$ consider the matrices $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $C = I$. Then

\[ \mathcal{S} = \{ K \in \mathbb{R}^{1 \times 3} : k_1 > 0, k_2k_3 > k_1 \} . \]
Again $S_0$ is not convex. For instance, the cut $x = k_1, y = k_2 = k_3$ (see Figure 3) is not convex.

Previous examples related to SLQR (state feedback). Now we proceed to OLQR (output feedback).

**Example 3.4.** Consider an example with a scalar control and $Q = I_3, R = 1, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -\alpha \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $C = (5 \quad 2 \quad 1)$.

Then

$$S = \{k \in \mathbb{R} : k + \alpha > 0, (k + \alpha)(2k + 1) > 5k + 1 > 0\}.$$

If $\alpha = -1$ this set is non-connected, it has two connectivity components. The function is illustrated on Figure 4. It has a single minima at each of the components. If $\alpha = -1.4$ this set is connected, it is a ray $k > -0.2$. The function is illustrated on Figure 5. It has two local minima located in the same connected component.

**Example 3.5.**

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -\alpha \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, Q = I_3, R = 1.$$

The set

$$S = \{K \in \mathbb{R}^{1 \times 2} : \alpha + k_2 > 0, 1 + k_1 + k_2 > 0, (1 + k_2)(\alpha + k_1 - k_2) > 1 + k_1 + k_2\}$$

is connected with two local minima and a saddle point $K = (1.95, 0.38)$ for $\alpha = 1.2$ Figure 6. If $\alpha$ is set to 0.9 there are two connectivity components with a single local minimum in each component Figure 7.

We conclude that domain $S$ of $f(K)$ can be nonconvex even for SLQR, and disconnected for OLQR. Function $f(K)$ can be unbounded on its domain but it looks smooth. We shall validate these properties below.
3.2. Connectedness of $S$. It was known that $S$ in state feedback case is connected [27], and the same is true for $S_0$.

**Lemma 3.6 (Connectedness of $S_0$).** Let $C = I$. The sets $S, S_0$ are connected for every $K_0 \in S$.

**Proof.** For $C = I$ equation (2.7) becomes $(A - BK)^\top X + X(A - BK) + K^\top RK + Q = 0$. It is proved in [20] that equality here can be replaced with inequality and after change of variables $P = X^{-1}$ definition of stabilizing controllers becomes

$$S = \{K = R^{-1}B^TP^{-1}, AP + PA^T - BR^{-1}B^T + PQP \preceq 0, P > 0.\}$$

The inequality for $P$ can be rewritten as block LMI and defines a convex set. Its image given by the continuous map $K = R^{-1}B^TP^{-1}$ is connected. Similarly the set $S_0$ is defined by the same map for the same set of $P$ with extra constraint $TrP^{-1} \Sigma \leq f(K_0)$ which is convex (again it can be written as LMI in $P$), this implies connectedness of $S_0$.

We provided the proof to demonstrate well known technique of variable change [10] which allows to transform the original problem to a convex one. This line of research was developed in [27, 28] to validate the gradient method. Unfortunately this trick does not work for output feedback — there exist no convex reparametrization in this case.

As we have seen in Examples, the set $S$ can be non-connected. Upper estimates for the number $N
of connected elements for particular cases may be found in [15]. For instance, if \( m = r = 1 \) (single-input single-output system) then \( N \leq n + 1 \). For more general problems with additional condition \( K \in L, L \) being a linear subspace in the set of matrices (so-called decentralised control) the number of components can grow exponentially, see [14], where numerous examples can be found.

3.3. \( f(K) \) is coercive and \( S_0 \) is bounded. The Examples exhibit that function \( f(K) \) is unbounded on its domain. Below we analyse its behavior in more details.

**Definition 3.7.** A continuous function \( f : K \mapsto f(K) \in \mathbb{R} \) defined on the set \( S \) is called coercive if for any sequence \( \{K_j\}_{j=1}^\infty \subseteq S \)

\[
f(K_j) \to +\infty
\]

if \( \|K_j\| \to +\infty \) or \( K_j \to K \in \partial S \).

**Lemma 3.8.** The function \( f(K) = \text{Tr} (X(K)\Sigma) \) is coercive and the following estimates hold

\[
(3.1) \quad f(K) \geq \lambda_1(\Sigma)\lambda_1(Q) - \Re \lambda_n(A_K),
\]

\[
(3.2) \quad f(K) \geq \frac{\lambda_1(\Sigma)\lambda_1(R)||K||_F^2\lambda_1(CC^T)}{2||A|| + 2||K||_F||B||||C||}.
\]

The proof of the Lemma and further results can be found in Appendices B and C. From estimate (3.2) we immediately get

**Corollary 3.9.** For any \( K_0 \in S \) the set \( S_0 \) is bounded.

On the other hand a minimum point of \( f(K) \) on \( S_0 \) exists (continuous function on a compact set) but \( S_0 \) has no common points with boundary of \( S \) due to (3.1). Hence

**Corollary 3.10.** If a stabilizing \( K_0 \) exists, then there exists a minimum point \( K_* \in S \).

This reasoning can be seen as an alternative proof of lemma 2.1 in [35].

3.4. **Gradient of \( f(K) \).** Differentiability of \( f(K) \) is well known fact, proved in the pioneering papers by Kalman [17] for SLQR and by Levine and Athans [22] for OLQR. We provide it for completeness.

**Lemma 3.11.** For all \( K \in S \) the gradient of (2.6) is

\[
(3.3) \quad \nabla f(K) = 2(RKC - B^TX)YC^T,
\]

where \( Y \) is the solution to the Lyapunov matrix equation

\[
(3.4) \quad A_KY + YA_K^T + \Sigma = 0.
\]

**Proof.** Consider the increment of the Lyapunov equation (2.7)

\[
A_K^TDX + DXA_K + dA_K^TX + XdA_K + C^tdK^TRKC + C^TK^TRdKC = 0,
\]

\[
A_K^TDX + DXA_K + C^tdK^T(RKC - B^TX) + (C^TK^TR - XB)dKC = 0.
\]

Denote \( M := RKC - B^TX \) then

\[
df(K) = \text{Tr} (\Sigma dX) = 2 \text{Tr} (YC^T dK^T M) = \langle 2MYC^T, dK \rangle,
\]

where \( Y \) is the solution to (3.4).

The necessary condition for the minimizer of \( f(K) \) is \( \nabla f(K_*) = 0 \) (because \( K_* \) exists and belongs to the open set \( S \)). This condition implies the set of three nonlinear matrix equations for \( K_* : \nabla f(K_*) = 0, \) (3.4), (2.7). In general they can not be solved explicitly and numerical methods are required.
However there is the famous case of state feedback control $C = I$ when explicit form of the solution (going back to Kalman [17]) can be obtained. Then by setting the gradient calculated in Lemma 3.11 to zero and noting that $Y_* > 0, C = I$ we get

$$K_* = R^{-1}B^TX_*.$$  

Further, substituting the control matrix in (2.7) by the expression for $K_*$ we obtain the well known Riccati equation for $X_*

$$A^TX_* - X_*BR^{-1}B^TX_* + X_*A - X_*BR^{-1}B^TX_* + X_*BR^{-1}RR^{-1}B^TX_* + Q = 0,

$$A^TX_* + X_*A - X_*BR^{-1}B^TX_* + Q = 0.

Of course this is not completely explicit solution because Riccati equation should be solved numerically, but the methods for this purpose are well developed [3, 8].

3.5. Second derivative of $f(K)$. The performance index $f(K)$ is twice differentiable. To avoid tensors, we restrict analysis with the action of the Hessian $\nabla^2 f(K)[E,E]$ on a matrix $E \in \mathbb{R}^{m \times n}$. It is given by the expression

$$\frac{1}{2} \nabla^2 f(K)[E,E] = \langle (REC - B^TX'(K)[E]) YC^T, E \rangle + \langle MY'(K)[E]C^T, E \rangle,$$

where $X' := X'(K)[E]$ and $Y' := Y'(K)[E]$ are the solutions to equations

$$A_K^T X' + X'^T A_K + (-BEC)^T X + X(-BEC) + C^T E^T RKC + C^T K^T REC = 0,

A_K Y' + Y'^T A_K + (-BEC) Y + Y(-BEC)^T = 0,

which can be equivalently rewritten as

$$A_K^T X' + X'^T A_K + M^T EC + (M^T EC)^T = 0,

A_K Y' + Y'^T A_K - (BECY + (BECY)^T) = 0.$$

These Lyapunov equations have solutions

$$X' = \int_0^\infty e^{tA_K} \left( M^T EC + (M^T EC)^T \right) e^{tA_K} dt,$$

$$Y' = -\int_0^\infty e^{tA_K} \left( BECY + (BECY)^T \right) e^{tA_K^T} dt,$$

Then substituting $Y'$ with $X'$ in the last term of (3.5) we obtain

**Lemma 3.12.** For all $K \in \mathcal{S}$ the gradient of $f(\cdot)$ is differentiable and the action of the Hessian of $f(\cdot)$ on any $E \in \mathbb{R}^{m \times n}$ satisfies

$$\frac{1}{2} \nabla^2 f(K)[E,E] = \langle REYC^T, E \rangle - 2 \langle B^T X'YC^T, E \rangle.$$

As Examples show, $f(K)$ is in general nonconvex. However for state feedback case we can guarantee local strong convexity in the neighborhood of the minimum point $K_*$.  

**Corollary 3.13.** Let $C = I$ and $K_* \in \mathcal{S}$. Then $f(\cdot)$ is strongly convex in the neighborhood of $K_*$.  

**Proof.** Note that when $K = K_*$ the second term in (3.6) turns to zero. If we recall that $R, Y > 0$ it is straightforward to show that

$$\langle REY, E \rangle = \text{Tr} \left( (R^2 E) Y (R^2 E)^T \right) > 0,$$

Then the Hessian is positive at $K_*$ and there is a neighbourhood of $K_*$ where the function $f(\cdot)$ is strongly convex. \hfill \square
Moreover quadratic lower bound is available for this case.

**Lemma 3.14.** Let \( C = I \) and \( K_\ast \in \mathcal{S} \) be the optimal feedback gain and \( K \in \mathcal{S} \). Then

\[
    f(K) - f(K_\ast) = \operatorname{Tr} \left( R^2 (K - K_\ast) Y (K - K_\ast)^\top R^2 \right) = \| R^2 (K - K_\ast) Y \|_F^2,
\]

where \( Y > 0 \) is the solution to the Lyapunov matrix equation

\[
    A_K Y + Y A_K^\top + \Sigma = 0
\]

Proof. Subtracting the Lyapunov equations at points \( K_1, K_2 \in \mathcal{S} \) it is possible to obtain similarly to (C.6) the identity

\[
    A_K^\top (X_2 - X_1) + (X_2 - X_1) A_K + (K_2 - K_1) M_2 + M_2^\top (K_2 - K_1) - (K_2 - K_1)^\top R (K_2 - K_1) = 0.
\]

Putting \( K_1 = K_2 = K_\ast \), where \( K_\ast \) is a stationary point and noting that \( \nabla f(K_\ast) = 2 M Y_\ast = 0 \) we obtain

\[
    A_K^\top (X - X_\ast) + (X - X_\ast) A_K + (K - K_\ast) R (K - K_\ast) = 0.
\]

The result follows directly from this expression.

The upper bound is also available.

**Lemma 3.15.** On the set \( \mathcal{S} \) the action of the Hessian \( \nabla f(K) \) on a matrix \( E \in \mathbb{R}^{m \times n}, \|E\|_F = 1 \) can be bounded as

\[
    \frac{1}{2} \nabla^2 f(K)[E, E] \leq \lambda_n(R)\lambda_n(CYC^\top) + \|X'\|_F \|B\| \|C\| \|Y\|,
\]

where \( X' \) and \( Y \) are solutions to the Lyapunov matrix equations

\[
    A_K^\top X' + X' A_K + M^\top E C + (M^\top E C)^\top = 0,
\]

\[
    A_K Y + Y A_K^\top + \Sigma = 0.
\]

Proof. It follows from (3.6) that

\[
    \frac{1}{2} \sup_{\|E\|_F = 1} \| \nabla^2 f(K)[E, E] \| \leq \sup_{\|E\|_F = 1} \left( | \langle REYC^\top, E \rangle | + 2 | \langle B^\top X' Y, E \rangle | \right)
\]

Now we estimate both terms in this expression assuming \( \|E\|_F = 1 \).

\[
    \langle REYC^\top, E \rangle = \operatorname{Tr} \left( REYC^\top E^\top \right) \leq \lambda_n(R)\lambda_n(CYC^\top).
\]

By Cauchy-Schwarz inequality

\[
    | \langle B^\top X' Y C^\top, E \rangle | = | \langle X', BECY \rangle | \leq \|X'\|_F \| BECY \|_F.
\]

It suffices to bound \( \| BECY \|_F \) when \( \|E\|_F = 1 \)

\[
    \| BECY \|_F = \sqrt{\operatorname{Tr}(BECCY^\top E^\top B^\top)} \leq \|B\| \|C\| \|Y\|.
\]

**3.6.** \( f(K) \) is \( L \)-smooth on \( S_0 \). A function is called \( L \)-smooth, if its gradient satisfies Lipschitz condition with constant \( L \). Function \( f(K) \) fails to be \( L \)-smooth on \( S \), however it has this property on sublevel set \( S_0 \).

**Theorem 3.16.** On the set \( S_0 \) the function \( f(K) \) is \( L \)-smooth with constant

\[
    L = \frac{2f(K_0)}{\lambda_1(Q)} \left( \frac{\lambda_n(R)\|C\| + n\|B\| \|C\| \|f(K_0)\xi\|_F}{\lambda_1(\Sigma)} \right),
\]

where \( \xi = \frac{1}{\lambda_1(\Sigma)} \left( \frac{f(K_0)\|B\|}{\lambda_1(\Sigma)\lambda_1(Q)} + \sqrt{\left( \frac{f(K_0)\|B\|}{\lambda_1(\Sigma)\lambda_1(Q)} \right)^2 + \lambda_n(R)} \right) \).
For the proof see Appendix B.

**Corollary 3.17.** The following inequality holds for $K \in S_0$:

$$\left| \nabla^2 f(K)[E,E] \right| \leq L \|E\|^2_F$$

where $L$ is given in (3.8).

Indeed for twice differentiable functions Lipschitz constant $L$ for gradients equals to the upper bound for the norm of second derivatives.

### 3.7. Gradient domination property.

As we have seen, $f(K)$ can be nonconvex even for state feedback case (SLQR). However there is a useful property which replaces convexity in validation of minimization methods. This property is referred to in the optimization literature as gradient domination or Lezanski-Polyak-Lojasiewicz (LPL) condition \[30, 23, 24, 19\].

**Theorem 3.18.** Consider the state feedback control (i.e. $C = I$). The function $f(K)$ defined in (2.6) satisfies the LPL condition on the set $S_0$

$$\frac{1}{2} \|\nabla f(K)\|^2_F \geq \mu (f(K) - f(K_*))$$

where $\mu > 0$ is given by

$$\mu = \frac{\lambda_1(R) \lambda_1^2(\Sigma) \lambda_1(Q)}{8 f(K_*) \left( \|A\| + \frac{\|B\|^2 f(K_0)}{\lambda_1(\Sigma) \lambda_1(R)} \right)^2}$$

Constant $\mu$ in the LPL condition depends on $K_0$ and tends to zero when $f(K)$ tends to infinity. The condition is false for the entire set $S$, as can be seen for Example 3.1. The condition can not be applied for output feedback - for instance, in Example 3.4 there are two disconnected components with different values of minima. Moreover in Example 3.5 there are two local minima in the connected domain.

### 4. Methods.

Now we proceed to versions of gradient method for minimization of $f(K)$. This is not a standard task, because function $f(K)$ is defined not on the entire space of matrices, it is unbounded on its domain and can be nonconvex. However the properties of the function obtained in Section 3 allow to get convergence results. In all cases gradient methods behave monotonically. For SLQR global convergence to the single minimum point with linear rate can be validated. For OLQR global convergence to a stationary point holds. In all versions of the method the known stabilizing controller $K_0$ serves as the initial point.

#### 4.1. Continuous Method.

First we consider the gradient flow defined by the system of ordinary differential equations

$$\begin{cases}
\dot{K}(t) = -\nabla f(K), \\
K(0) = K_0 \in S.
\end{cases}$$

**Theorem 4.1.** The solution of the above system $K_t = K(t) \in S_0$ exists for all $t \geq 0$, $f(K_t)$ is monotone decreasing and

$$\nabla f(K_t) \xrightarrow{t \to \infty} 0, \quad \min_{0 \leq t \leq T} \|\nabla f(K_t)\|^2 \leq \frac{f(K_0)}{T}.$$ 

If $C = I$ then $K_t$ converges to the global minimum point $K_*$ exponentially:

$$\|K_t - K_*\|_F \leq \sqrt{\frac{2L(f(K_0) - f(K_*))}{\mu}} e^{-\mu t},$$

where $\mu$ and $L$ are determined in Theorems 3.16 and 3.18.

The main idea of the proof is the equality $\frac{d}{dt} f(K) = -\|\nabla f(K)\|^2$, the details are in Appendix D.
4.2. Discrete Method. Consider the gradient method in general form

\[ K_{j+1} = K_j - \gamma_j \nabla f(K_j). \]  

The properties obtained in Theorems 3.16 and 3.18 allow to establish convergence guaranties for the above method.

**Theorem 4.2.** For arbitrary \( 0 < \gamma_j < \frac{2}{L} \) method (4.4) generates nonincreasing sequence \( f(K_j) \):

\[ f(K_{j+1}) \leq f(K_j) - \gamma_j \frac{L \gamma_j}{2} \|
abla f(K_j)\|^2. \]

Moreover if \( 0 < \varepsilon_1 \leq \gamma_j \leq \frac{2}{L} - \varepsilon_2, \varepsilon_2 > 0 \) then

\[ \nabla f(K_j) \to 0, \min_{0 \leq j \leq k} \|\nabla f(K_j)\|^2 \leq \frac{f(K_0)}{k}. \]

and for \( C = I \) the method converges to the global minimum \( K_* \) with linear rate

\[ \|K_j - K_*\| \leq cq^j, \ 0 \leq q < 1 \]

The simplest choice is \( \gamma_j = 1/L \), then in the last inequality constants \( c, q \) can be written explicitly. The proof in Appendix D is the elementary replica of the standard ones in [30].

4.3. Algorithm. The method above is just a “conceptual” one, we do not know constant \( L \) and it is hard to estimate it. Thus an implementable version of the algorithm is needed. It can be constructed as follows. Inequality (4.5) provides the opportunity to apply Armijo-like rule: step-size \( \gamma \) satisfies this rule if

\[ f(K - \gamma \nabla f(K)) \leq f(K) - \alpha \gamma \|\nabla f(K)\|^2 \]

for some \( 0 < \alpha < 1 \). We can achieve this inequality by subsequent reduction of the initial guess for \( \gamma \) due to (4.5). This initial guess can be taken as follows. Consider a univariate function

\[ \varphi(t) = f(K - t\nabla f(K)), \]

One iteration of Newton method for minimization of \( \varphi(t) \) starting from \( t_0 = 0 \) implies

\[ t_1 = \frac{\varphi'(0)}{\varphi''(0)} \]

Calculating derivatives we get

\[ t_1 = \frac{\|\nabla f(K)\|^2}{\nabla^2 f(K) \nabla f(K)} \cdot \frac{2}{\nabla^2 f(K) \nabla f(K)} \cdot \frac{2}{\nabla^2 f(K) \nabla f(K)}. \]

But expressions for these quantities were obtained in section 3 (see (3.3) and (3.6)). Notice that \( t_1 \geq 1/L \) due to (3.9), thus such step-size is bounded below. Taking \( \gamma_j \) with some \( T_1 > 0 \) (such upper bound is needed to restrict the step-size) for \( K = K_j \) in gradient method we arrive to the basic algorithm below.

**Theorem 4.3.** For Algorithm 4.1 the number of step reductions is bounded uniformly for all iterations and convergence results of Theorem 4.2 hold true.

The proof follows the same lines as for Theorem 4.2 and is given in the Appendix.

There are different ways to choose constants \( T_1, \alpha \) in the Algorithm. We do not discuss them here, because there are various implementations of the Algorithm and they deserve separate consideration. Moreover simulation (see below) confirmed that neither upper bound nor step-reduction is needed in practical calculation.

It is also possible to consider a different approach for a stepsize choice. For instance, it can be chosen in such a way that guaranties that a new iterate remains stabilizing. Then there is no need to check if \( K \in S \) on every iteration. Consider the Lyapunov equation

\[ (A - BKC)Y + (A - BKC)^T + I = 0. \]
Algorithm 4.1 Gradient method

1: Return: $K$
2: Initialization: $K_0 \in S$, $\epsilon > 0$, $\alpha \in (0, 1)$, $T_1 > 0$.
3: While $\|\nabla f(K)\|_F \geq \epsilon$ do
4:   Solve for $X$: $A_K^TX + XA_K + Q + K^TRK = 0$.
5:   Solve for $Y$: $A_KY + YA_K^T + \Sigma = 0$.
6:   $M \leftarrow RK - B^TX, \nabla f(K) \leftarrow 2MY$.
7:   Solve for $X'$: $A_K^TX' + X'A_K + M^T\nabla f(K) + \nabla f(K)^TM = 0$.
8:   $\nabla^2 f(K)[\nabla f(K), \nabla f(K)] \leftarrow 2(R\nabla f(K)Y, \nabla f(K)) - 4(B^TX'Y, \nabla f(K))$.
9: $t \leftarrow \min\{T_1, \frac{\|\nabla f(K)\|_F^2}{\|\nabla^2 f(K)[\nabla f(K), \nabla f(K)]\|_F^2}\}$, $K_{prev} \leftarrow K$.
10: Gradient step: $K \leftarrow K - t\nabla f(K)$.
11: if $K \in S$ or $f(K) \geq f(K_{prev}) - \alpha t\|\nabla f(K_{prev})\|_F^2$ then
12:   $t \leftarrow \alpha t$,
13: end if
14: return the gradient step.
15: end while

Denote $K_t = K - t\nabla f(K)$ and $G = (B\nabla f(K)C)Y + Y(B\nabla f(K)C)^T$.

$AY + YA^T - [(BK_tC)Y + Y(BK_tC)^T] + I - tG = 0,$

$AK_tY + YA_K^T + I - tG = 0.$

The function $V(x) = x^TY^{-1}x$ remains the quadratic Lyapunov function for a new $A_{K_t}$ when $I - tG > 0$. If $\lambda_{max}(G) \leq 0$, then $K_t \in S, \forall t > 0$. Otherwise, $K_t \in S$ if $0 < t < \frac{1}{\lambda_{max}(G)}$.

5. Simulation. We have started with the comparison of various versions of the step-size choice for low-dimensional tests, such as Examples 3.1 to 3.5. In all cases Algorithm 4.1 was superior, thus all other simulations were performed with this Algorithm only. In all cases it converged to global or local minimizers with high accuracy for 10–20 iterations.

For large-scale simulation we generated matrices with dimensions $n = 100, m = 10$ for SLQR problem:

$C = I, A = \frac{1}{4}rand(n, n) - I, B = ones(n, m) + \frac{1}{2}rand(n, m),$

where $ones(n, m)$ is a $n \times m$ matrix with all entries equal to one and $rand(n, m)$ is a $n \times m$ matrix with every entry generated from a uniform distribution between 0 and 1. Three different starting points $K_0$ are determined by pole placement procedure. The half of poles ($\frac{1}{2}$) are chosen equidistantly with real parts from $-3$ to $-1$. These poles are then determined to have a complex part $+i$ with a probability $\frac{1}{2}$. If the pole has a complex part, then it also has its complex conjugate pair. The convergence for three different starting points is illustrated in Figure 8. It is interesting to notice that from different points the algorithm converged to three different controllers with very close values of the performance index.

Of course these calculations are preliminar, much more should be done to develop reliable and efficient gradient-based algorithms for state feedback which can win in competition with classical algorithms based on Riccati-equation techniques.

6. Links with Reduced Gradient method. Gradient method for feedback minimization can be considered in general setup of abstract optimization problem with equality-type constraints

$$\min_{x,y} f(x, y),$$

$$s.t. g(x, y) = 0.$$  

Suppose that the solution $x(y)$ of the equality $g(x, y) = 0$ for fixed $y \in S$ can be found either explicitly or with minor computational efforts. Define $F(y) := f(x(y), y)$. Thus problem is converted to unconstrained
Figure 8. Algorithm 4.1 for \( n = 100, m = 10 \) and \( C = I \).

Optimization

\[
\min_y F(y), \quad y \in S.
\]

Gradient of \( F(y) \) can be written with no problems

\[
\nabla F(y) = - (\nabla_y g(x, y))^\top ((\nabla_x g(x, y))^{-1})^\top \nabla_x f(x, y)^\top + \nabla_y f(x, y)^\top
\]

and gradient method with \( y_0 \in S \) becomes so called reduced gradient method:

\[
y_{j+1} = y_j - \gamma_j \nabla F(y_j), \quad x_j = x(y_j).
\]

The method has been proposed by Ph.Wolfe [36] and implemented in numerous algorithms, see e.g. [1]. The standard setup was assumption \( S = \mathbb{R}^n \). However the method for nonlinear equality constraints had just local theoretical validation (see e.g. Theorem 8, Chapter 8.2 in [31]), while the main interest is its global convergence. In the setup of the present paper \( x \) corresponds to \( Y \), \( y \) to \( K \). The main tool for proving convergence in general case is to obtain the conditions which are the analogs of our results on \( L \)-smoothness and LPL-condition (Theorems 3.16 and 3.18). If such results hold, the proof is a replica of our considerations.

7. Conclusion. The results can be extended in several directions. First, more efficient computational schemes are of interest. Gradient method is the simplest method for unconstrained smooth optimization. Accelerated algorithms - such as conjugate gradient, heavy ball, Nesterov acceleration - are developed for
strongly convex functions. But we have proved (Corollary 3.13) that \( f_\delta(K) \) is strongly convex in the neighborhood of the optimal solution \( K^* \). Thus such methods are applicable to accelerate local convergence. Second, more research should be devoted to output minimization. For instance, how common is the effect of multiple minima in one connectivity component (as in Example 3.5)? Does the method converge to a local minima only or it can be a saddle point? Third, the gradient method can be easily extended to decentralised control (this additional condition \( K \in L, L \) being a linear subspace in the space of matrices), see e.g. [14]. However its validation remains open question.

**Appendix A. Basic Facts.** The following lemmas are helpful throughout the paper.

**Lemma A.1.** Let \( X \) and \( Y \) be the solutions to the dual Lyapunov equations with Hurwitz matrix \( A \)

\[
A^T X + X A + W = 0,
\]
\[
AY + YA^T + V = 0.
\]

Then \( \text{Tr}(XY) = \text{Tr}(YW) \).

**Lemma A.2.** Let \( W_1 \succ W_2 \succeq 0 \) and \( X_1, X_2 \) be the solutions to Lyapunov equations with Hurwitz matrix \( A \)

\[
A^T X_1 + X_1 A + W_1 = 0,
\]
\[
A^T X_2 + X_2 A + W_2 = 0.
\]

Then \( X_1 \succ X_2 \).

**Appendix B. Analysis of the OLQR.**

**B.1. Proof of Lemma 3.8.**

Proof. Let us first consider the sequence \( \{K_j\}_{j=1}^\infty \subseteq S: K_j \to K \in \partial S \), where \( \partial S \) is the imaginary axis. The stability degree of a matrix \( \sigma(A) := \max_\lambda |\lambda| \) is a continuous map, i.e. \( \sigma(A - BK_j C) \to \sigma(A - BKC) \). Therefore, \( \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} \) such that

\[
|\sigma(A - BK_j C) - \sigma(A - BKC)| = \sigma(A - BKC) < \varepsilon,
\]

\( \forall j \geq N \). Let \( x_j \) be the solution to the corresponding Lyapunov equation (2.7) associated with \( K_j \). Note that for any eigenvalue \( \lambda \) and the corresponding eigenvector \( v \) of the matrix \( A_K \)

\[
e^{\text{Re}(\lambda)t} \|v\|_2 \leq \|e^{\lambda t}v\|_2 = \|e^{\lambda t}v\|_2 \leq \|e^{\lambda t}\|_2 \leq \|e^{\lambda t}\|_2.
\]

Therefore, this lower bound on the norm of the matrix exponential \( \|e^{\lambda t}\|_f \geq |e^{-\sigma(A_K)t}| \) allows to observe that

\[
f(K_j) = \text{Tr}(X_j) = \text{Tr}(Y_j(Q + C^T K_j^T R K_j C)) \geq \text{Tr}(Y_j Q) \geq \lambda_1(S) \lambda_1(Q) \int_0^\infty \|e^{\lambda t}\|_f^2 dt
\]

\[
\geq \frac{\lambda_1(S) \lambda_1(Q)}{2\sigma(A_K)} \geq \frac{\lambda_1(S) \lambda_1(Q)}{2\varepsilon} \to +\infty, \text{ if } \varepsilon \to 0.
\]

On the other hand, suppose that the sequence \( \{K_j\}_{j=1}^\infty \subseteq S: \|K_j\| \to +\infty. \)

\[
f(K_j) \geq \text{Tr}(Y_j C^T K_j^T R K_j C) \geq \text{Tr}(C^T K_j^T R K_j C) \lambda_1(Y_j) \geq \lambda_1(R)\|K_j\|_f^2 \lambda_1(Y_j) \lambda_1(C C^T),
\]

where \( Y_j \) is the solution to the Lyapunov equation

\[
A_K Y_j + Y_j A_K^T + \Sigma = 0,
\]

for which we have the lower bound (1.16) in [33]

\[
\lambda_1(Y_j) \geq \frac{\lambda_1(S)}{2\|A_K\|} > 0.
\]
Further we note that
\[ \|A_K\| \leq \|A - BK_jC\| \leq \|A\| + \|B\|\|C\|\|K_j\|_F. \]
Therefore,
\[ (B.3) \quad f(K_j) \geq \lambda_1(R)\|K_j\|_F^2\lambda_1(Y_j)\lambda_1(C\Sigma^T) \geq \frac{\lambda_1(\Sigma)\lambda_1(R)\|K_j\|_F^2\lambda_1(C\Sigma^T)}{2\|A\| + 2\|K_j\|_F\|B\|\|C\|} \to +\infty, \]
if \( \|K_j\|_F \to +\infty \) \hspace{1cm} \( \square \)

\textbf{B.2. Proof of Theorem 3.16.}

\textit{Proof.} Note that in Lemma 3.15 \( Y \) depends on \( K \) and \( X' \) on \( K \) and \( E \). We must obtain a uniform estimate that depends only on the problem parameters and \( K_0 \). The first term in Lemma 3.15 can be upper bounded as
\[ (B.4) \quad \lambda_n(R)\lambda_n(CY^T) \leq \frac{\lambda_n(R)}{\lambda_1(Q)} \|f(K_0)\|. \]
For the second we have
\[ (B.5) \quad \|B\|\|C\|_F\|Y\| \leq \frac{\sqrt{n}\|B\|\|C\|_F}{\lambda_1(Q)} \|f(K_0)\|. \]
Further it suffices to bound \( \|X\|_F \). We first show that \( X' \leq \alpha X \) with some constant \( \alpha \). Recall that \( X' \) and \( X \) are solutions to the corresponding Lyapunov equations
\[ A^T_KX' + X'A_K + C^T K^T REC + C^T E^T RK - (XBE + (XBE)^T) = 0 \]
\[ (B.6) \quad A^T_KX + XA_K + Q + C^T K^T RK = 0 \]
For any \( \alpha, \beta > 0 \) we obtain that \( X' \leq \tilde{X}' \), where \( \tilde{X}' \) is the solution to
\[ A^T_K\tilde{X}' + \tilde{X}'A_K + \alpha C^T K^T RK + \frac{1}{\alpha} C^T E^T REC + (\beta X^2 + \frac{1}{\beta}(BEC)^T BEC) = 0. \]
Further we choose the constants \( \alpha \) and \( \beta \) such that \( \tilde{X}' \leq \alpha X \).
\[ (B.7) \quad A^T_K \left( \frac{\tilde{X}'}{\alpha} \right) + \left( \frac{\tilde{X}'}{\alpha} \right) A_K + C^T K^T RK + \frac{1}{\alpha^2} C^T E^T REC + \frac{1}{\alpha}(\beta X^2 + \frac{1}{\beta}(BEC)^T BEC) = 0. \]
Consider the matrix function of two variables
\[ F(\alpha, \beta) := C^T E^T \left( \frac{1}{\alpha} R + \frac{1}{\beta} B^T B \right) EC + \beta X^2 - \alpha Q. \]
To obtain an upper bound on \( \tilde{X}' \) we solve the two dimensional minimization problem on \((\alpha, \beta)\) with the relaxed matrix inequality constraint \( F_1(\alpha, \beta) \leq F(\alpha, \beta) \leq 0 \)
\[ \alpha \to \min_{\alpha, \beta > 0}, \quad F_1(\alpha, \beta) \leq 0, \]
where
\[ F_1(\alpha, \beta) := \left( \frac{1}{\alpha} \lambda_n(R) + \frac{1}{\beta} \|B\|^2 + \beta \|X\|^2 - \alpha \lambda_1(Q) \right) I. \]
The solution is the pair
\[ \alpha_* = \frac{\|X\|\|B\| + \sqrt{\|X\|^2\|B\|^2 + \lambda_1(Q)\lambda_n(R)}}{\lambda_1(Q)} \quad \beta_* = \frac{\|B\|}{\|X\|}. \]
Note that it trivially follows from $X \leq \frac{f(K_0)}{\lambda_1(\Sigma)} I$ that $X^2 \leq \frac{f^2(K_0)}{\lambda_1(\Sigma)} I$. Therefore,

$$X' \leq \alpha_* X \leq \frac{\alpha_*}{\lambda_1(\Sigma)} f(K_0) I.$$ \hfill{(C.1)}

Let us denote $\gamma := \frac{\alpha_*}{\lambda_1(\Sigma)} f(K_0)$ and note that as $X'$ and $\gamma I$ commute we obtain the bound on the Frobenius norm

$$\|X'\|_F \leq \sqrt{\gamma} \leq \frac{\sqrt{n} f(K_0)}{\lambda_1(\Sigma)} \left( \frac{f(K_0)\|B\|}{\lambda_1(\Sigma)} + \sqrt{\left( \frac{f(K_0)\|B\|}{\lambda_1(\Sigma)} \right)^2 + \lambda_n(R)} \right). \hfill{(B.8)}$$

The result (3.8) follows directly from Lemma 3.15 if we apply the obtained bounds (B.4), (B.5), and (B.8). \hfill{□}

**Appendix C. Analysis of SLQR.**

**C.1. Technical Lemmas.**

**Lemma C.1.** Consider the state feedback control (i.e. $C = I$). Let $K_* \in S$ be the optimal feedback gain and $K_0 \in S$. Then for $K \in S_0$

$$(C.1) \quad f(K) - f(K_*) \leq \frac{(\|A\| + \|K\|\|B\|)^2 \lambda_n(Y_*)}{\lambda_1(R) \lambda_1(\Sigma)} \|\nabla f(K)\|_F^2,$$

where $Y_*$ is the solution to the Lyapunov matrix equation

$$(C.2) \quad A_{K_*} Y_* + Y_* A_{K_*}^T + \Sigma = 0.$$ \hfill{Proof.}

The Lyapunov equations for an arbitrary $X = X(K)$ and $X_* = X(K_*)$ are

$$A_{K_*}^T X + X A_{K_*} + K^T R K + Q = 0,$$

$$(C.3) \quad A_{K_*}^T X_* + X_* A_{K_*} + K_*^T R K_* + Q = 0.$$ \hfill{(C.4)}

Substituting (C.3) from (C.4) gives

$$A_{K_*}^T X - A_{K_*}^T X_* + X A_{K_*} - X_* A_{K_*} + K^T R K - K_*^T R K_* = 0,$$

which is equivalent to

$$A_{K_*}^T (X - X_*) + (X - X_*) A_{K_*} + (K - K_*)^T M + M^T (K - K_*) - (K - K_*)^T R (K - K_*) = 0,$$

where $M = R K - B^T X$.

For any $\alpha > 0$

$$(K - K_*)^T M + M^T (K - K_*) \leq \frac{1}{\alpha} (K - K_*^T) (K - K_*) + \alpha M^T M.$$ \hfill{(C.5)}

Therefore, picking $\alpha = \frac{1}{\lambda_1(R)}$ we obtain

$$\leq \frac{1}{\alpha} (K - K_*)^T M + \frac{1}{\alpha} (\frac{1}{\lambda_1(R)} I - R) (K - K_*^T R (K - K_*) = 0,$$

where $M = R K - B^T X$.

Let $Z$ be the solution to

$$A_{K_*}^T Z + Z A_{K_*} + \frac{1}{\lambda_1(R)} M^T M = 0.$$ \hfill{(C.6)}
Then \((X - X_*) \leq Z\). Further,

\[
f(K) - f(K_*) = \text{Tr}((X - X_*)\Sigma) \leq \text{Tr}(Z\Sigma) = \frac{1}{\lambda_1(R)} \text{Tr} \left( (M^\top MY) \right) \leq \frac{\lambda_n(Y_*)}{\lambda_1(R)} \text{Tr} \left( (M^\top M) \right)
\]


\[
\leq \frac{\lambda_n(Y_*)}{\lambda_1(R)\lambda_1^2(Y)} \text{Tr} \left( (Y^\top M^\top MY) \right) = \frac{\lambda_n(Y_*)}{4\lambda_1(R)\lambda_1^2(Y)} \|\nabla f(K)\|^2_F,
\]

where \(Y\) satisfies

\[
A_K Y + YA_K^\top + \Sigma = 0.
\]

It follows from (B.2) that

\[
\lambda_1(Y) \geq \frac{\lambda_1(\Sigma)}{2\|A_K\|} \geq \frac{\lambda_1(\Sigma)}{2(\|A\| + \|B\|\|K\|_F)} > 0.
\]

Therefore,

\[
f(K) - f(K_*) \leq \frac{\|A\| + \|B\|\|K\|_F)^2 \lambda_n(Y_*)}{\lambda_1(R)\lambda_1^2(Y)} \|\nabla f(K)\|^2_F.
\]

**Lemma C.2.** For \(K \in S\) and the solution to the Lyapunov matrix equation

\[
A_K Y + YA_K^\top + \Sigma = 0
\]

it holds that

\[
(C.7) \quad \lambda_n(Y) \leq \frac{f(K)}{\lambda_1(Q + C^\top K^\top RKC)}.
\]

**Proof.**

\[
\lambda_1(Q + C^\top K^\top RKC) \text{Tr} (Y) \leq \text{Tr} \left( \frac{\lambda_n(Y_*)}{\lambda_1(\Sigma)} \|\nabla f(K)\|^2_F \right).
\]

**Lemma C.3.** For \(K \in S\) the norm \(\|K\|_F\) is bounded for \(f(K)\) bounded and

\[
(C.8) \quad \|K\|_F \leq \frac{2\|B\|f(K)}{\lambda_1(\Sigma)\lambda_1(R)} + \frac{\|A\|}{\|B\|}.
\]

**Proof.** Indeed, consider (3.2) as a quadratic equation with respect to \(\|K\|_F\). Bounding its largest root we obtain an explicit expression

\[
\|K\|_F \leq \frac{2\|B\|f(K) + 2\|B\|f(K) \sqrt{1 + \frac{2\|A\|\lambda_1(\Sigma)\lambda_1(R)}{\|B\|f(K)}}}{2\lambda_1(\Sigma)\lambda_1(R)} \leq \frac{2\|B\|f(K)}{\lambda_1(\Sigma)\lambda_1(R)} + \frac{\|A\|}{\|B\|}.
\]

**C.2. Proof of Theorem 3.18.**

**Proof.** In view of Lemma C.1 it suffices to plug (C.7) and (C.8) into (C.1).

**Appendix D. Analysis of the Methods.**

**D.1. Proof of Theorem 4.1.**

**Proof.** The proof is the direct replica of Theorems 8, 9 in [30]. The only difference is that in [30] the objective function was defined on the entire space while here it is defined on \(S\) and is \(L\)-smooth on \(S_0 \in S\). But differentiating \(f(K_t)\) as function of \(t\) we get \(\frac{d}{dt} f(K_t) = -\|\nabla f(K_t)\|^2\), thus \(f(K_t)\) is monotone and \(K(t)\) remains in \(S_0\) for all \(t \geq 0\). The estimate 4.2 follows from

\[
f(K_0) \geq f(K_0) - f(K_T) = \int_0^T \|\nabla f(K_t)\|^2 dt \geq T \min_{0 \leq t \leq T} \|\nabla f(K_t)\|^2.
\]
D.2. Proof of Theorem 4.2.

Proof. Again the proof is the same as for Theorems 3, 4 in [30], the only detail is to establish that \( K_j \in S_0 \) for all \( j \). But it follows from the inequality (4.5). \( \square \)

D.3. Proof of Theorem 4.3. The proof can be easily reconstructed via the comments which led to the formulation of the algorithm.

REFERENCES

[1] J. Abadie and J. Carpentier, Generalization of the Wolfe reduced gradient method to the case of nonlinear constraints., in Optimization, R. Fletcher, ed., Academic Press, London, 1969, pp. 37–47.

[2] J. Ackermann, Parameter space design of robust control systems, IEEE Transactions on Automatic Control, 25 (1980), pp. 1058–1072, https://doi.org/10.1109/TAC.1980.1102505, http://ieeexplore.ieee.org/document/1102505/.

[3] H. M. Amman and H. Neudecker, Numerical solutions of the algebraic matrix riccati equation, Journal of Economic Dynamics and Control, 21 (1997), pp. 363 – 369, https://doi.org/https://doi.org/10.1016/S0165-1889(96)00936-0, http://www.sciencedirect.com/science/article/pii/S0165188996009360.

[4] B. D. Anderson and J. B. Moore, Linear Optimal Control, Prentice Hall, Englewood Cliffs, N.J., 1971.

[5] M. Athans and P. Falb, Optimal Control, McGraw-Hill, 1 ed., 1966.

[6] D. V. Balandin and M. M. Kogan, Synthesis of linear quadratic control laws on basis of linear matrix inequalities, Automation and Remote Control, 68 (2007), pp. 371–385, https://doi.org/10.1134/S0005117907030010, http://link.springer.com/10.1134/S0005117907030010.

[7] R. Bellman, Notes on matrix theory. X. A problem in control, Quarterly of Applied Mathematics, 14 (1957), pp. 417–419, https://doi.org/10.1090/qam/82592, http://www.ams.org/qam/1957-14-04/S0033-569X-1957-82592-1/.

[8] P. Benner, J.-R. Li, and T. Penzl, Numerical solution of large-scale Lyapunov equations, Riccati equations, and linear-quadratic optimal control problems, Numerical Linear Algebra with Applications, 15 (2008), pp. 755–777, https://doi.org/10.1002/nla.622, https://hal.inria.fr/hal-00781119.

[9] V. Blondel and J. N. Tsitsiklis, NP-Hardness of Some Linear Control Design Problems, SIAM Journal on Control and Optimization, 35 (1997), pp. 2118–2127, https://doi.org/10.1137/S0036141094272630.

[10] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, Linear matrix inequalities in system and control theory, SIAM studies in applied mathematics, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[11] J. Bu, A. Mesbahi, M. Fazel, and M. Mesbahi, LQR through the lens of first order methods: Discrete-time case, 2019, https://arxiv.org/abs/1907.08921.

[12] J. Eilbrecht, M. Jilg, and O. Stürbig, Distributed H 2 -optimized Output Feedback Controller Design using the ADMM, IFAC-PapersOnLine, 50 (2017), pp. 10389–10394, https://doi.org/10.1016/j.ifacol.2017.08.1706, https://linkinghub.elsevier.com/retrieve/pii/S2405896317321337.

[13] M. Fazel, R. Ge, S. Kakade, and M. Mesbahi, Global convergence of policy gradient methods for the linear quadratic regulator, in Proceedings of the 35th International Conference on Machine Learning, J. Dy and A. Krause, eds., vol. 80 of Proceedings of Machine Learning Research, Stockholmsmssan, Stockholm Sweden, Jul 2018, PMLR, pp. 1467–1476, https://proceedings.mlr.press/v80/fazel18a.html.

[14] J. He, L. Ghaoui, and N. Bertsimas, On the exponential number of connected components for the feasible set of optimal decentralized control problems, in 2019 American Control Conference (ACC), Philadelphia, PA, USA, July 2019, IEEE, pp. 1430–1437, https://doi.org/10.23919/ACC.2019.8814952, https://ieeexplore.ieee.org/document/8814952/.

[15] E. N. Gryazina, B. T. Polyak, and A. A. Tremba, D-decomposition technique state-of-the-art, Automation and Remote Control, 69 (2008), pp. 1991–2026, https://doi.org/10.1134/S0005117908120011, http://link.springer.com/10.1134/S0005117908120011.

[16] T. Iwasaki, R. Skelton, and J. Geromel, On the exponential number of connected components for the feasible set of optimal decentralized control systems, IEEE Transactions on Automatic Control, 69 (2015), pp. 2143–2155, https://doi.org/10.1109/TAC.1970.1099363.

[17] J. Kalman, Contributions to the theory of optimal control, Boletín de la Sociedad Matemática Mexicana, (1960), pp. 102–119, https://springer.com/us/en/book/978-3-662-48855-9.

[18] R. Kalman, On the general theory of control systems, Proc. First Internat. Congress Automat. Contr., (1960), pp. 481–491.

[19] H. Karimi, J. Nutini, and M. Schmidt, Linear convergence of gradient and proximal-gradient methods under the Polyak-ojasiewicz condition, vol. 9851, 09 2016, pp. 795–811.

[20] H. Krawkiewicz, P. S. Sichrovskii, and V. N. Chestnov, Linear-quadratic regulator. I. a new solution, Automation and Remote Control, 26 (1967), pp. 371–385, https://doi.org/10.1134/S0005117915200018, http://link.springer.com/10.1134/S0005117915200018.

[21] H. Kwakernaak and R. Sivan, Linear Optimal Control, Wiley, 1 ed., 1972, https://doi.org/10.1115/1.3426828.

[22] W. Levine and M. Athans, On the determination of the optimal constant output feedback gains for linear multivariable systems, IEEE Transactions on Automatic Control, 15 (1970), pp. 44–48, https://doi.org/10.1109/TAC.1970.1099363.

[23] T. Leask, ber die Methode des "schnellsten Falles" fr das Minimumproblem von Funktionalen in Hilbertschen Rumen, Studia Mathematica, 28 (1967), pp. 183–192, https://doi.org/10.4064/sm-28-2-183-192, http://www.impan.pl/get/doi/10.4064/sm-28-2-183-192.

[24] S. Lojasiewicz, Une propriet e topologique des sous-ensembles analytiques reels, Les Equations aux Derivees Partielles, Paris: CNRS, (1963), pp. 87–89.
[25] P. Makila and H. Toivonen, Computational methods for parametric LQ problems—A survey, IEEE Transactions on Automatic Control, 32 (1987), pp. 658–671, https://doi.org/10.1109/TAC.1987.1104686, http://ieeexplore.ieee.org/document/11040686/.

[26] D. Moerder and A. Calise, Convergence of a numerical algorithm for calculating optimal output feedback gains, IEEE Transactions on Automatic Control, 30 (1985), pp. 900–903, https://doi.org/10.1109/TAC.1985.1104073, http://ieeexplore.ieee.org/document/1104073/.

[27] H. Mohammadi, A. Zare, M. Soltanolkotabi, and M. Jovanovic, Convergence and sample complexity of gradient methods for the model-free linear quadratic regulator problem, 12 2019, https://arxiv.org/abs/1912.11899.

[28] H. Mohammadi, A. Zare, M. Soltanolkotabi, and M. R. Jovanovi, Global exponential convergence of gradient methods over the nonconvex landscape of the linear quadratic regulator, in 2019 IEEE 58th Conference on Decision and Control (CDC), 2019, pp. 7474–7479, https://doi.org/10.1109/CDC40024.2019.9029985.

[29] E.-S. M. Mostafa, A Conjugate Gradient Method for Discrete–Time Output Feedback Control Design, Journal of Computational Mathematics, 30 (2012), pp. 279–297, https://doi.org/10.4208/jcm.1109-m3364.

[30] B. Polyak, Gradient methods for the minimisation of functionals, USSR Computational Mathematics and Mathematical Physics, 3 (1963), pp. 864–878, https://doi.org/10.1016/0041-5553(63)90382-3, https://linkinghub.elsevier.com/retrieve/pii/0041555363903823.

[31] B. T. Polyak, Introduction to optimization, Optimization Software, New York, 1987.

[32] T. Rautert and E. W. Sachs, Computational Design of Optimal Output Feedback Controllers, SIAM Journal on Optimization, 7 (1997), pp. 837–852, https://doi.org/10.1137/S1052623495290441, http://epubs.siam.org/doi/10.1137/S1052623495290441.

[33] S. Savov, Solution bounds for algebraic equations in control theory, Prof. Marin Drinov Academic Publishing House, 2014, https://doi.org/10.2478/cait-2014-0027.

[34] V. Syrmos, C. Abdallah, and P. Dorato, Static output feedback: a survey, in Proceedings of 1994 33rd IEEE Conference on Decision and Control, vol. 1, Lake Buena Vista, FL, USA, 1994, IEEE, pp. 837–842, https://doi.org/10.1109/CDC.1994.410963, http://ieeexplore.ieee.org/document/410963/.

[35] H. T. Toivonen, A globally convergent algorithm for the optimal constant output feedback problem, International Journal of Control, 41 (1985), pp. 1589–1599, https://doi.org/10.1080/0020718508961217, http://www.tandfonline.com/doi/abs/10.1080/0020718508961217.

[36] P. Wolfe, Methods of nonlinear programming, in J. Abadie, ed., Nonlinear programming (Wiley, New York) ch. 6, 1967, pp. 97–131.