AdS$_5 \times S^5$ Untwisted

M.J. Duff$^1$, H. Lü$^\ddagger$ and C.N. Pope$^\ddagger$

$^\ddagger$Laboratoire de Physique Théorique de l’École Normale Supérieure
24 Rue Lhomond - 75231 Paris CEDEX 05

$^\ddagger$Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843

ABSTRACT

Noting that $T$-duality untwists $S^5$ to $CP^2 \times S^1$, we construct the duality chain: $n = 4$ super Yang-Mills $\rightarrow$ Type IIB superstring on AdS$_5 \times S^5$ $\rightarrow$ Type IIA superstring on AdS$_5 \times CP^2 \times S^1$ $\rightarrow$ M-theory on AdS$_5 \times CP^2 \times T^2$. This provides another example of supersymmetry without supersymmetry: on AdS$_5 \times CP^2 \times S^1$, Type IIA supergravity has $SU(3) \times U(1) \times U(1) \times U(1)$ and $N = 0$ supersymmetry but Type IIA string theory has $SO(6)$ and $N = 8$. The missing superpartners are provided by stringy winding modes. We also discuss IIB compactifications to AdS$_5$ with $N = 4$, $N = 2$ and $N = 0$.

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1 Introduction

There is now persuasive evidence that certain conformal field theories in various dimensions are dual to M/string theory compactified to anti-de Sitter space \([1]\). In particular \((n = 4, d = 4)\) super Yang-Mills theory is dual to type IIB string theory on \(\text{AdS}_5 \times S^5\). In this paper we note that \(S^5\) may be regarded as a \(U(1)\) bundle over \(CP^2\) and hence that the theory admits a \(T\)-dual type IIA description. In fact \(T\)-duality untwists \(S^5\) to \(CP^2 \times S^1\). Thus we can construct the duality chain: \(n = 4\) super Yang-Mills \(\rightarrow\) type IIB superstring on \(\text{AdS}_5 \times S^5\) \(\rightarrow\) type IIA superstring on \(\text{AdS}_5 \times CP^2 \times S^1\) \(\rightarrow\) \(M\)-theory on \(\text{AdS}_5 \times CP^2 \times T^2\). This provides another example of the phenomenon of \textit{supersymmetry without supersymmetry} \([2]\), but this time without involving Dirichlet 0-branes. On \(\text{AdS}_5 \times CP^2 \times S^1\) type IIA \textit{supergravity} has \(SU(3) \times U(1) \times U(1) \times U(1)\) and \(N = 0\) supersymmetry. Indeed, since \(CP^2\) does not admit a spin structure, its spectrum contains no fermions at all! Nevertheless, type IIA \textit{string theory} has \(SO(6)\) and \(N = 8\) supersymmetry. The missing superpartners (and indeed all the fermions) are provided by stringy winding modes. These winding modes also enhance \(SU(3) \times U(1)\) to \(SO(6)\), while the gauge bosons of the remaining \(U(1) \times U(1)\) belong to massive multiplets.

We also describe the way in which \(p\)-brane solutions of type IIA and type IIB are related when the compactification takes the form of a \(U(1)\) fibration rather than an \(S^1\), which is the case, for example, for all odd-dimensional spheres.

In addition to the \(S^5\) compactification of type IIB, which has the maximal \(N = 8\) supersymmetry in \(D = 5\), we also exhibit new compactifications on the spaces \(Q(n_1, n_2)\), which are \(U(1)\) bundles over \(S^2 \times S^2\), with winding numbers \(n_1\) and \(n_2\) respectively. Generically, these have \(N = 0\) supersymmetry, but the special case \(Q(1, 1)\) has \(N = 4\) supersymmetry. Reversing the orientation of \(Q(1, 1)\), we get a solution with \(N = 0\) supersymmetry. However, if these are dual to \((n = 2, d = 4)\) or \((n = 0, d = 4)\) super Yang-Mills theories with vanishing \(\beta\) function, they would have to be unusual ones with non-integer conformal dimensions. We also present compactifications that give \(N = 2\) and \(N = 0\) supersymmetries, but with integer conformal dimensions, in which \(S^5\) is replaced by a lens space. (Reducing the supersymmetries by using lens spaces was discussed in \([3]\).)

The organisation of this paper is as follows. Since supergravity and supermembranes on \(\text{AdS}\) and their relation to singletons and superconformal theories was an active area of research some years ago, we begin with a brief review in section 2. Then, in section 3, we introduce some ideas and notation that will be relevant also in the later parts of the paper. Specifically, we consider a third kind of dimensional reduction for \(p\)-branes, in addition to
the usual double-dimensional reduction and vertical dimensional reduction schemes, which is applicable to cases where the transverse space is even dimensional. Viewing the transverse space as a foliation of spheres, we note that in these cases the spheres are odd dimensional, and hence can be viewed as $U(1)$ bundles over complex projective spaces. We can perform a dimensional reduction on the $U(1)$ direction; we shall refer to this as “Hopf reduction.” Combined with a T-duality transformation, this reduction allows the “untwisting” of the $U(1)$ fibres. In section 4, we consider the near-horizon limit of this kind of reduction, in the case of the self-dual 3-brane of the type IIB theory \[4, 5\]. This corresponds to the AdS$_5 \times S^5$ solution, and we show how it may be mapped by T-duality into an AdS$_5 \times CP^2 \times S^1$ solution of the type IIA theory, and hence to an AdS$_5 \times CP^2 \times T^2$ solution in M-theory. We then discuss the spectrum of states in the dual description, and show in particular how the fermions, including the gravitini, are present only in the string theory or M-theory spectrum, but not in the supergravity spectrum. Many other solutions of the form AdS$_5 \times M_5$ exist, where $M_5$ is any Einstein space (which need not even be homogeneous). In section 5 we consider an infinite family of examples, where $M_5$ is the space $Q(n_1, n_2)$ described by the $U(1)$ bundle over $S^2 \times S^2$ that has winding numbers $n_1$ and $n_2$ over the two 2-sphere factors in the base. We show that the space $Q(1, 1)$ gives an $N = 4$ supersymmetric solution, and that with an orientation reversal, it gives $N = 0$. These type IIB solutions can be mapped via T-duality to type IIA solutions of the form AdS$_5 \times S^2 \times S^2 \times S^1$. We also obtain solutions on the products of AdS$_5$ and lens spaces, with $N = 2$ and $N = 0$. In section 6, we give further examples of solutions of the form AdS$_5 \times$ spheres, including some that arise as the near-horizon limit of multiply-charged intersecting $p$-brane solutions. Finally, after our conclusions, we give the detailed form of the T-duality mapping between the type IIA and type IIB theories in an appendix.

2 Anti-de-Sitter space, branes, singletons, superconformal field theories and all that

In the early 80’s there was great interest in $N$-extended supergravities for which the global $SO(N)$ is promoted to a gauge symmetry \[6\], in particular the maximal $N = 8$, $SO(8)$ theory \[7\]. In these theories the underlying symmetry is described by the $D = 4$ anti-de Sitter (AdS$_4$) supersymmetry algebra, and the Lagrangian has a non-vanishing cosmological constant proportional to the square of the gauge coupling constant. This suggested that there might be a Kaluza-Klein interpretation, and indeed this maximal theory was seen
to correspond to the massless sector of $D = 11$ supergravity compactified on an $S^7$ whose metric admits an $SO(8)$ isometry [8]. An important ingredient in these developments that had been insufficiently emphasized in earlier work on Kaluza-Klein theory was that the $AdS_4 \times S^7$ geometry was not fed in by hand but resulted from a spontaneous compactification, i.e. the vacuum state was obtained by finding a stable solution of the higher-dimensional field equations [1]. The mechanism of spontaneous compactification appropriate to the $AdS_4 \times S^7$ solution eleven-dimensional supergravity was provided by the Freund-Rubin mechanism [11] in which the 4-form field strength in spacetime $F_{\mu\nu\rho\sigma}$ ($\mu = 0, 1, 2, 3$) is proportional to the alternating symbol $\epsilon_{\mu\nu\rho\sigma}$ [11]. By applying a similar mechanism to the 7-form dual of this field strength one could also find compactifications on $AdS_7 \times S^4$ [12] whose massless sector describes gauged maximal $N = 4, SO(5)$ supergravity in $D = 7$ [13]. A summary of these AdS compactifications of $D = 11$ supergravity may be found in [13]. Type IIB supergravity in $D = 10$, with its self-dual 5-form field strength, also admits a Freund-Rubin compactification on $AdS_5 \times S^5$ [16, 17] whose massless sector describes gauged maximal $N = 8$ supergravity in $D = 5$ [18, 19].

In the three cases given above, the symmetry of the vacuum is described by the supergroups $OSp(4/8), SU(2,2/4)$ and $OSp(6,2/4)$ for the $S^7$, $S^5$ and $S^4$ compactifications respectively. Each of these groups is known to admit the so-called singleton or doubleton representations [20, 21]. Curiously, although they appeared in the Kaluza-Klein harmonic expansions [22, 23], they could be gauged away. In fact they reside not in the bulk of AdS but on the boundary [20] where the above supergroups correspond to the superconformal groups [24]. In the case of $S^7$, one finds an $(n = 8, d = 3)$ supermultiplet with 8 scalars and 8 spinors; in the case of $S^5$ one finds a $(n = 4, d = 4)$ supermultiplet with 1 vector, 4 spinors and 6 scalars, and in the case of $S^4$ one finds a $((n_+, n_-) = (2, 0), d = 6)$ supermultiplet with 1 2-form with self-dual field strength, 8 spinors and 5 scalars.

With the discovery of the eleven-dimensional supermembrane [25], it was noted that the physical degrees of freedom on the worldvolume of the membrane also correspond to the $(n = 8, d = 3)$ supermultiplet with 8 scalars and 8 spinors and it was conjectured that these may in fact admit the interpretation of singletons [27]. There followed a good deal of activity relating super $p$-branes on $AdS_{p+2} \times S^{D-p-2}$, singletons and superconformal field theories [28, 29, 30, 31, 32, 33, 24, 34, 35, 36, 37]. In particular, it was possible to find solutions of the combined supergravity supermembrane equations describing a membrane occupying the $S^1 \times S^2$ boundary of $AdS_4$: the “membrane at the end of the universe” [28, 29]. The action and transformation rules for the $OSp(4/8)$ singleton conformal field
theory were presented in [31] and a general correspondence between super $p$-branes and the superconformal field theories in Nahm’s classification [24] was discussed in [31, 35, 33].

These early works focussed on scalar supermultiplets because these were the only $p$-branes known at the time [41]. However, with the discovery of type $II$ $p$-brane solitons [42, 43, 4, 5, 44], vector and tensor multiplets were also seen to play a role. In particular, the worldvolume fields of the self-dual type IIB superthreebrane were shown to be described by a $(n = 4, d = 4)$ gauge theory [5]. The subsequent realisation that this theory admitted the interpretation of a Dirichlet 3-brane [45], and the observation that the superposition of $N$ such branes yields an $SU(N)$ gauge theory [46] are, of course, crucial to the duality with type IIB theory [1]. For earlier related work on coincident threebranes and $n = 4$ super Yang Mills, see [47, 48, 49, 50].

More recently, AdS has emerged in the near-horizon geometry of black $p$-brane solutions [37, 38, 39, 10] in $D$ dimensions. The dual brane, with worldvolume dimension $\tilde{\rho} + 1 = D - p - 3$, interpolates between $D$-dimensional Minkowski space and $AdS_{\tilde{\rho}+2} \times S^{p+2}$ (or $M_{\tilde{\rho}+2} \times S^3$ if $p = 1$). It is the subset of those solutions with constant or zero dilaton, the non-dilatonic $p$-branes, that have been conjectured to be dual to conformal theories with vanishing beta function [1]. Generically, however, the gradient of the dilaton plays the role of a conformal Killing vector on AdS [38] and these may be related to theories with non-vanishing beta function [1]. Moreover, new super $p$-branes with fewer supersymmetries may also be constructed which interpolate between $D$-dimensional flat space and $AdS \times M$ where $M$ is any Einstein space [52, 53], not necessarily a round sphere. For example $N = 1$ for the squashed $S^7$ [54, 55]. Note that the space at large distance, although asymptotically locally flat, is not asymptotic to Minkowski spacetime. Rather, it approaches a flat metric on a generalised cone [52].

Since in gauged supergravity the gauge coupling is related to the AdS cosmological constant, the beta function is determined by the renormalisation of the cosmological constant [50] which in turn is fixed by the Weyl anomaly [54, 58]. The Weyl anomaly at one-loop (and indeed all odd-loop orders) vanishes trivially in the case of $(N = 4, D = 7)$ and $(N = 8, D = 5)$ supergravities, since the spacetime has odd dimension [57]. One might naively have expected to find a non-vanishing beta function for gauged $(N = 8, D = 4)$ supergravity, but remarkably it vanishes [59]. This result continues to hold when the massive Kaluza-Klein multiplets are included [81, 11], by virtue of the $N > 4$ spin-moment sum rules [62, 63]. Moreover, the argument has been extended to all orders [64].

This vanishing of the beta function in the $AdS_4 \times S^7$ compactification is an answer
that has been looking for a question for eighteen years [54]. Now, however, we see that this is entirely consistent with the recent conjectured duality between supergravity on AdS and certain superconformal field theories whose coupling constant in given by the AdS cosmological constant [1], a duality that forms the subject of the present paper.

Following Maldacena’s conjecture [1], a number of papers appeared reviving the old singleton-AdS-membrane-superconformal field theory connections [55, 56, 61, 64, 71, 72, 73, 74, 75, 76, 77, 78] and applying them to this new duality context. In particular, the there is seen to be a correspondence between the Kaluza-Klein mass spectrum in the bulk and the conformal dimension of operators on the boundary [70, 72]. The philosophy is that supergravity is a good approximation for large $N$ and that stringy excitations correspond to operators whose dimensions diverge for $N \to \infty$. Under the IIB/IIA $T$-duality discussed in the present paper, however, the Kaluza-Klein and certain stringy excitations trade places and so the type IIA (or $D = 11$) supergravity picture may throw some light on the finite $N$ regime.

### 3 Hopf fibrations of M-theory and string theory

There are two kinds of dimensional reduction of $p$-brane solitons that are commonly considered. The simpler is double dimensional reduction, where a Kaluza-Klein reduction on a $p$-brane worldvolume coordinate is performed. This is always possible, since the $p$-brane solitons have translational isometries on their worldvolumes. The effect is to reduce a $p$-brane in $(D + 1)$ dimensions to a $(p - 1)$-brane in $D$ dimensions. The second kind of dimensional reduction, known as vertical reduction, involves performing the Kaluza-Klein reduction on a transverse-space direction instead. In order to do this, it is necessary first to construct a $p$-brane solution in $(D + 1)$-dimensions with an isometry along a transverse-space direction. This can be done by considering a multi-soliton configuration, with a uniform continuum of $p$-branes along the chosen reduction axis. In this reduction scheme, a $p$-brane in $(D + 1)$ dimensions is reduced to a $p$-brane in $D$ dimensions.

In this section, we shall consider various examples of a third kind of dimensional reduction scheme, which is possible whenever the transverse space has dimension $2n$ that is even. This means that in hyperspherical polar coordinates, the usual flat metric $dy^i \, dy^i$ can be written in the form $dr^2 + r^2 \, d\Omega^2_{2n-1}$, where $d\Omega^2_{2n-1}$ is the metric on the unit $(2n-1)$-sphere. Now the odd-dimensional sphere $S^{2n-1}$ can be written as a $U(1)$ bundle over $CP^{n-1}$, and we may then perform a Kaluza-Klein reduction on the $U(1)$ direction of the Hopf fibres,
since this corresponds to an isometry of the sphere, and hence of the transverse space. This “Hopf” reduction scheme is more akin to the vertical reduction described above, in that it involves a reduction in the transverse space, while the Poincaré symmetry on the $p$-brane worldvolume survives unscathed. However, unlike normal vertical reduction, it is not necessary first to construct a uniform distribution of $p$-branes; a single $p$-brane solution in $D$ dimensions already has the necessary $U(1)$ isometry.

In the following subsections, we shall consider examples where we perform Hopf reductions on $p$-branes in M-theory, the type IIA string, and the type IIB string.

### 3.1 Hopf reductions in M-theory

We shall take as our starting point the BPS-saturated membrane solution of $D = 11$ supergravity. The bosonic Lagrangian of 11-dimensional supergravity contains the metric and a 4-form field strength $F_4 = dA_3$, and is given by

$$L = e^R - \frac{1}{35} F_4^2 + \frac{1}{(12)^4} \epsilon^{M_1 \cdots M_{11}} F_{M_1 \cdots M_4} F_{M_5 \cdots M_8} A_{M_9 \cdots M_{11}} .$$

It admits the extremal membrane solution

$$ds^2_{11} = H^{-2/3} dx^\mu dx^{\nu} \eta_{\mu\nu} + H^{1/3} (dr^2 + r^2 d\Omega_7^2)$$

$$F_4 = d^3x \wedge dH^{-1} = Q \ast \Omega_7 ,$$

where $H = c + \frac{1}{6} Q r^{-6}$ is harmonic in the 8-dimensional transverse space whose flat metric is described in terms of the radial coordinate $r$ and the metric $d\Omega_7^2$ on the unit 7-sphere.

We are using $\Omega_7$ to denote the volume form on the unit 7-sphere, $\ast$ denotes the Hodge dual, and $Q$ is the electric charge carried by the membrane. Conventionally, the constant $c$ is chosen to be unity, so that the metric tends to the standard eleven-dimensional Minkowski metric as $r$ tends to infinity.

The odd-dimensional spheres can be viewed as Hopf fibrations over complex projective spaces, and specifically, $S^7$ is a $U(1)$ bundle over $CP^3$. Indeed the standard unit-radius metric $d\Omega_7^2$ on $S^7$ can be written as:

$$d\Omega_7^2 = d\Sigma_6^2 + (dz + \bar{A})^2 ,$$

where $d\Sigma_6^2$ is the standard Fubini-Study metric on $CP^3$, and the Kaluza-Klein vector potential $\bar{A}$ has field strength $\bar{F}$ given by $\bar{F} = 2J$, where $J$ is the Kähler form on $CP^3$. The Fubini-Study metric $\bar{g}_{ij}$ on $CP^3$ is Einstein, and we choose a normalisation where its Ricci tensor satisfies $\bar{R}_{ij} = 8\bar{g}_{ij}$. The potential is given in terms of its components by
\( \tilde{A} = \tilde{A}_i dy^i \), where \( y^i \) are the coordinates on \( CP^3 \) and the Kähler form satisfies \( \nabla_i J_{jk} = 0 \) and \( J_{ij} J_{jk} = -\delta_i^k \). The coordinate \( z \) has period \( 4\pi \).

Using (3.3), the membrane metric (3.2) can be written as

\[
\begin{align*}
    ds_{11}^2 &= H^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + H^{1/3} (dr^2 + r^2 d\Sigma_6^2) + H^{1/3} r^2 (dz + \tilde{A})^2 .
\end{align*}
\]

We may now compactify the solution on the circle parameterised by the U(1) fibre coordinate \( z \). In general, the Kaluza-Klein reduction of a metric from \( D + 1 \) to \( D \) dimensions takes the form

\[
    ds_{D+1}^2 = e^{-2\alpha \varphi} ds_D^2 + e^{2(D-2)\alpha \varphi} (dz + \tilde{A})^2 ,
\]

where \( \alpha = ((2(D - 1)(D - 2))^{-1/2} \), and the parameterisation is such that the lower-dimensional metric is in the Einstein frame (i.e. \( \sqrt{g_{D+1}} R_{D+1} \) reduces to \( \sqrt{g_D} R_D \)). Applying this to (3.4), we see that the dimensionally-reduced ten-dimensional solution is given by

\[
\begin{align*}
    ds_{10}^2 &= H^{-5/8} r^{1/4} dx^\mu dx^\nu \eta_{\mu\nu} + H^{3/8} r^{1/4} (dr^2 + r^2 d\Sigma_6^2) ,
    e^{4\varphi} &= H r^6 ,
    F_4 &= d^3 x \wedge dH^{-1} = Q e^{-\frac{1}{2} \varphi} + \Sigma_6 ,
    \mathcal{F} &= 2J ,
\end{align*}
\]

where \( \Sigma_6 \) is the volume form of \( CP^3 \). Note from (3.8) that it is related to the volume form of the 7-sphere by \( \Omega_7 = (dz + \tilde{A}) \wedge \Sigma_6 = dz \wedge \Sigma_6 \).

Unlike the usual membrane solution in \( D = 10 \) type IIA string, which could be obtained by vertical dimensional reduction of the membrane in \( D = 11 \), the field strength \( \mathcal{F} \) of the Kaluza-Klein vector \( A \), associated with the Kähler form on the \( CP^3 \), also acquires a charge, in addition to the charge carried by the 4-form field strength. Nevertheless, since the \( U(1) \) compactification of M-theory gives rise to the IIA string, it follows that the above configuration solves the equations of motion of IIA massless supergravity in \( D = 10 \) [81].

Up to the conformal factor \( r^{1/4} \), its metric is reminiscent of the usual membrane solution in \( D = 10 \), except for the fact that it is the \( CP^3 \) metric \( d\Sigma_6^2 \), rather than the unit 6-sphere metric \( d\Omega_6^2 \), that appears in the part describing the transverse space. Although it came by dimensional reduction from a BPS solution of \( D = 11 \) supergravity preserving 16 of the 32 components of supersymmetry, it does not itself preserve 16 as a solution of type IIA supergravity. This is because some of the Killing spinors are described by Dirichlet 0-branes that are absent in the type IIA supergravity picture, and indeed they are absent in perturbative type IIA string theory. In fact the solution will preserve either 12 or 0 components of supersymmetry, depending on the orientation of the \( CP^3 \) [3]. Of course, all 16 will be present in the non-perturbative type IIA string theory. The moral is that neither type IIA supergravity nor perturbative type IIA string theory is always a reliable guide to the number of supersymmetries preserved in M-theory.
The ten-dimensional solution (3.5) can be further compactified on $CP^3$, giving rise to the four-dimensional metric
\[ ds_4^2 = H^{-1/2} r \, dx^\mu dx^\nu \eta_{\mu\nu} + H^{1/2} r \, dr^2. \] (3.6)

Let us now return for a moment to the eleven-dimensional membrane solution (3.2). There is an horizon at $r = 0$, and in this neighbourhood the metric $ds_{11}^2$ is of the form $AdS_4 \times S^7$, where $AdS_4$ is four-dimensional anti-de Sitter spacetime. This may be seen by noting that at sufficiently small $r$ the constant $c$ in the harmonic function $H = c + \frac{1}{6} Q r^{-6}$ becomes negligible, and indeed if we temporarily set $c = 0$ the metric $ds_{11}^2$ becomes
\[ ds_{11}^2 = k_3 \left( \frac{1}{k} e^{4 \rho} \, dx^\mu dx^\nu \eta_{\mu\nu} + d\rho^2 \right) + k_3 \, d\Omega_7^2, \] (3.7)
where $\rho = \log r$ and $k = \frac{1}{6} Q$. The terms inside the first bracket describe the metric on $AdS_4$ written in horospherical [38, 39, 82] coordinates $x^\mu$ and $\rho$. Thus the membrane solution (3.2) with $c = 1$ can be viewed as interpolating between $AdS_4 \times S^7$ at the horizon and eleven-dimensional Minkowski spacetime at radial infinity [37, 38, 39].

The $AdS_4 \times S^7$ solution (3.7) can be reinterpreted as an $AdS_4$ solution of gauged $N = 8$ supergravity in four dimensions, since this latter theory is obtained by Kaluza-Klein reduction of eleven-dimensional supergravity on the 7-sphere [8, 15]. On the other hand, it can also be viewed first of all as an $AdS_4 \times CP^3$ solution of the type IIA string, and then, by compactifying on the $CP^3$, we again obtain a theory on $AdS_4$ in four dimensions. This can be seen from the discussion in this section, by considering the near-horizon structure of the dimensionally-reduced solution (3.5). It was first studied in the supergravity context in [31], and more recently an M-theory discussion was given in [2].

This completes our discussion of the Hopf reduction of the membrane solution in M-theory. Note that we cannot carry out an analogous Hopf reduction of the 5-brane, since now the transverse space has dimension 5, and is a foliation of 4-spheres, which cannot be described as Hopf bundles.

### 3.2 Hopf reductions of type II $p$-branes

We now turn to a consideration of the $p$-branes of the type IIA and type IIB strings in ten-dimensions, and show how in certain cases they may be viewed as $U(1)$ fibrations from a nine-dimensional point of view. In these cases we may establish a correspondence between a $p$-brane solution of the type IIA or type IIB theory and a different kind of solution of the type IIB or type IIA theory, respectively, by making use of the T-duality between the type
IIA and type IIB strings compactified on a circle. Specifically, these correspondences may be implemented whenever the transverse space in the \( p \)-brane solution is even-dimensional, implying that it is described in hyperspherical coordinates in terms of a foliation of odd-dimensional spheres. Then, in a manner analogous to the 7-sphere discussion in the previous section, we can compactify on the \( U(1) \) fibre coordinate of the sphere.

The \( p \)-brane solitons in the type IIA theory arise for \( p = \{0_D, 1, 2, 4, 5, 6_D, 8\} \), corresponding to foliations of the spheres \( \{S^8, S^7, S^6, S^4, S^3, S^2, S^0\} \) respectively. (The subscripts in the list of \( p \)-branes indicate the ones that are D-branes. The D8-brane arises only in the massive IIA supergravity.) For the type IIB string, the \( p \)-branes are \( \{-1_D, 1, 1_D, 3, 5, 5_D, 7\} \), associated with the spheres \( \{S^9, S^7, S^7, S^5, S^3, S^1\} \) respectively. Note that in the type IIB case all the \( p \)-branes are associated with odd-dimensional spheres, and so they may all be compactified on a \( U(1) \) fibre coordinate. On the other hand in the type IIA case only the string and the 5-brane are associated with odd-dimensional spheres, whilst all the type IIA D-branes are associated with even-dimensional spheres.

As a preliminary, we shall show how to construct the odd-dimensional unit spheres \( \mathbb{S}^{2n+1} \) as \( U(1) \) bundles over \( \mathbb{C}P^n \). The construction, which generalises the 7-sphere example that we used in the previous section, involves writing the metric \( d\Omega^2_{2n+1} \) on the unit \((2n+1)\)-sphere in terms of the Fubini-Study metric \( d\Sigma^2_{2n} \) on \( \mathbb{C}P^n \) as

\[
d\Omega^2_{2n+1} = d\Sigma^2_{2n} + (dz + \bar{A})^2. \tag{3.8}
\]

In fact we may give general results for any metric of the form

\[
ds^2 = c^2 (dz + \bar{A})^2 + ds^2 \tag{3.9}
\]

on a \( U(1) \) bundle over a base manifold with metric \( ds^2 \), where \( c \) is a constant. Choosing the vielbein basis \( e^z = c (dz + \bar{A}) \), \( e^i = \bar{e}^i \), one finds that the Riemann tensor for \( ds^2 \) has non-vanishing vielbein components given by

\[
R_{ijkl} = R_{ijkl} - \frac{1}{4} c^2 (\bar{F}_{ik} \bar{F}_{jl} - \bar{F}_{il} \bar{F}_{jk} + 2 \bar{F}_{ij} \bar{F}_{kl}) , \\
R_{zizj} = \frac{1}{4} c^2 \bar{F}_{ik} \bar{F}_{jk} , \quad R_{zizk} = \frac{1}{2} c \nabla_j \bar{F}_{ij} . \tag{3.10}
\]

In all the cases we shall consider, the components \( R_{zizk} \) will be zero, since \( \bar{F} = d\bar{A} \) will be proportional to covariantly-constant tensors, such as Kähler forms. The Ricci tensor for \( ds^2 \) has the vielbein components

\[
R_{zz} = \frac{1}{4} c^2 \bar{F}_{ij} \bar{F}_{ij} , \quad R_{iz} = R_{ij} - \frac{1}{2} c^2 \bar{F}_{ik} \bar{F}_{jk} , \quad R_{zi} = -\frac{1}{2} c \nabla_j \bar{F}_{ij} , \quad R_{ij} = \bar{R}_{ij} . \tag{3.11}
\]

\(^{1}\)The idea of generating new solutions from old by means of T-duality transformations has been considered in various contexts. See, for example, \([83, 84]\).
Applied to our present case, where the unit $(2n + 1)$-sphere should have a Ricci tensor satisfying $R_{ab} = 2n \delta_{ab}$, we see that this is achieved by taking the field strength to be given by $\tilde{F}_{ij} = 2J_{ij}$, where $J_{ij}$ is the covariantly-constant Kähler form on $CP^n$. Furthermore, the Fubini-Study Einstein metric on $CP^n$ should be scaled such that its Ricci tensor satisfies $\tilde{R}_{ij} = 2(n + 1) \delta_{ij}$. The volume form $\Omega_{2n+1}$ on the unit $(2n + 1)$-sphere is related to the volume form $\Sigma_{2n}$ on $CP^n$ by $\Omega_{2n+1} = dz \wedge \Sigma_{2n}$. Note also that the volume form on $CP^n$ is related to the Kähler form by $\Sigma_{2n} = \frac{1}{n!} J^n$.

\begin{equation}
\Sigma_{2n} = \frac{1}{n!} J^n .
\end{equation}

### 3.3 Type IIA p-branes

Let us begin by constructing the new solutions of the type IIB theory that are related by T-duality in the Hopf-fibred nine-dimensional background to the string and the 5-brane of the type IIA theory. The type IIA string solution is given by

\begin{align}
\text{ds}^2 &= H^{-3/4} dx^\mu dx_\mu + H^{1/4} (dr^2 + r^2 d\Omega_7^2) ,
\nonumber
e^{-2\phi_1} &= H ,
F_3^{(1)} &= d^2 x \wedge dH^{-1} = Q e^{\phi_1} \ast \Omega_7 , \tag{3.13}
\end{align}

where $H$ is an harmonic function on the transverse space, of the form $H = 1 + \frac{1}{6} Q r^{-6}$. Using the expression (3.3), we may reduce this solution to nine dimensions in the same manner as we previously reduced the membrane of eleven-dimensional supergravity to $D = 10$. Thus we obtain the nine-dimensional solution

\begin{align}
\text{ds}_9^2 &= r^{2/7} \left[ H^{-5/7} dx^\mu dx_\mu + H^{2/7} (dr^2 + r^2 d\Sigma_6^2) \right] ,
\nonumber
e^{-2\phi_1} &= H ,
F_3^{(1)} &= d^2 x \wedge dH^{-1} = Q e^{\phi_1 - \frac{1}{\sqrt{7}} \phi_2} \ast \Sigma_6 ,
F_2^{(2)} &= 2J , \tag{3.14}
\end{align}

where $d\Sigma_6^2$ is the metric on $CP^3$, with Kähler form $J$ and volume form $\Sigma_6$, and $\phi_2$ is the Kaluza-Klein scalar arising in the reduction $ds_{10}^2 = e^{-2\alpha\phi_2} ds_9^2 + e^{4\alpha\phi_2} (dz_2 + A_1^{(2)})^2$, with $\alpha = 1/(4\sqrt{7})$. Note that the exponential factor appearing in the expression for $F_3^{(1)}$ in terms of $\ast \Sigma_6$ is precisely the inverse of the exponential dilaton prefactor of the kinetic term for $F_3^{(1)}$ in the nine-dimensional Lagrangian. This is an example of the general rule that a field strength with kinetic term $e^{a\phi} F^2$ has the form $F = Q e^{-a\phi} \ast \Sigma$ when it carries an electric charge.

Using the T-duality transformation to the type IIB variables given in the appendix, we find that the dilatonic scalars become $e^{-\phi} = H^{1/2} r$ and $e^{2\sqrt{7}\phi} = H r^6$, implying that upon
oxidising to $D = 10$ according to $ds_{10}^2 = e^{-2\alpha \phi} \, ds_9^2 + e^{14\alpha \phi} (dz_2 + A_1)^2$, we obtain the type IIB solution

$$
\begin{align*}
 ds_{10}^2 &= r^{1/2} \left[ H^{-3/4} \, dx^\mu \, dx_\mu + H^{1/4} \, (dt^2 + r^2 \, d\Sigma_6^2 + r^{-2} \, dz_2^2) \right] , \\
 e^{-2\phi} &= H \, r^2 , \\
 F_3^{(\text{NS})} &= Q \, e^{\phi} \star (dz_2 \wedge \Sigma_6) + 2 \, J \wedge dz_2 \quad (3.15)
\end{align*}
$$

Since this is related by T-duality to the string solution of the type IIA theory, it follows that this is a solution of the type IIB theory. However, as a type IIB supergravity solution, it will not preserve the same number of supersymmetries as it did as a solution of type IIA supergravity. In fact, it will preserve either 12 or 0 of the 32 components of supersymmetry, rather than the 16 when it is a type IIA solution. This discrepancy is because some of the Killing spinors are described by stringy winding modes in the type IIB picture. (The counting is the same as in the Hopf reduction of the M2-brane discussed in section 3.1, since in both cases the transverse space is foliated by 7-spheres.) The moral to be drawn from this is that supergravity is not always a reliable guide to the number of supersymmetries preserved in string theory.

The solution (3.15) can be viewed as a string solution of the type IIB theory, since it has a 2-dimensional Poincaré symmetry on the worldsheet. However, it is a string of a rather unusual kind, with an unconventional geometry in the transverse space, and an unconventional configuration for the 3-form field strength. However, since by construction it is related by T-duality to the standard string solution of the type IIA theory, it gives an equivalent description of these degrees of freedom. It can be contrasted with two other procedures for using T-duality to relate the type IIA string to solutions of the type IIB theory. One of these involves wrapping the type IIA string around a circle, giving a particle in $D = 9$ which, after transforming to type IIB variables can be oxidised to a pp-wave solution of the type IIB theory in $D = 10$:

$$
\begin{align*}
 ds^2 &= -H^{-1} \, dt^2 + dr^2 + r^2 \, d\Omega_7^2 + H \, (dz - H^{-1} \, dt)^2 , \\
 \phi &= 0 . \quad (3.16)
\end{align*}
$$

The other procedure involves first constructing a line of type IIA strings $D = 10$, by taking $H$ to be independent of one of the Cartesian coordinates of the transverse space, and then compactifying along this direction. After transforming this $D = 9$ solution into type IIB variables, its oxidation back to $D = 10$ will give a standard solution describing a line of NS-NS strings in the type IIB theory.
We now turn to the type IIA 5-brane solution, which is given by
\[
\begin{align*}
    ds^2 &= H^{-1/4} d\alpha^\mu d\alpha^\nu + H^{3/4} (dr^2 + r^2 d\Omega_3^2), \\
    e^{2\phi_1} &= H, \quad F_3^{(1)} = Q \Omega_3,
\end{align*}
\]
where $\Omega_3$ is the volume form on the unit 3-sphere, and $Q$ is the magnetic charge carried by the 3-form field strength. Using (3.8), we may write the 3-sphere metric as the $U(1)$ fibration over $CP^1 \sim S^2$. Dimensionally reducing on the $U(1)$ fibres gives the nine-dimensional solution
\[
\begin{align*}
    ds^2_9 &= r^{2/7} \left[ H^{-1/7} d\alpha^\mu d\alpha^\nu + H^{6/7} (dr^2 + r^2 d\Sigma_2^2) \right], \\
    e^{2\phi_1} &= H, \quad e^{2\sqrt{7}\phi_2} = H^3 r^8, \\
    F_2^{(12)} &= Q \Sigma_2, \quad F_2^{(2)} = 2\Sigma_2,
\end{align*}
\]
where $\Sigma_2$ is the volume form of the $CP^1$ metric, which is nothing but the unit 2-sphere. Note that in this particular case the Kähler form $J$ on $CP^1$ is identical to the volume form $\Sigma_2$. Converting to type IIB variables, and oxidising to $D = 10$, we obtain the type IIB solution
\[
\begin{align*}
    ds^2 &= r^{1/2} \left[ d\alpha^\mu d\alpha^\nu + H, (dr^2 + r^2 d\Sigma_2^2) + H^{-1} r^{-2} (dz_2 + A_1)^2 \right], \\
    e^{-\phi} &= r, \quad F_2 = Q \Sigma_2, \quad F_3^{(NS)} = 2(dz_2 + A_1) \wedge \Sigma_2.
\end{align*}
\]
This configuration, being related by T-duality in $D = 9$ to the standard 5-brane of the type IIA theory, is a solution of type IIB supergravity. As in the previous case of the string solution, the full set of 16 out of the 32 components of supersymmetry will be preserved only once the winding modes of the type IIB string theory are included.

Again, as in the case of the string solution that we discussed previously, the type IIB solution (3.19) can be contrasted with two other type IIB solutions that can be obtained from the type IIA 5-brane by T-duality. One of these involves a diagonal dimensional reduction of the type IIA 5-brane, giving a 4-brane in $D = 9$. After transforming to type IIB variables, this can be oxidised to $D = 10$ where it describes the standard NS-NS 5-brane of the type IIB theory. The other type IIB solution is obtained by vertical dimensional reduction of the type IIA 5-brane to a 5-brane in $D = 9$, given by
\[
\begin{align*}
    ds_9^2 &= H^{-1/7} d\alpha^\mu d\alpha^\nu + H^{6/7} (dr^2 + r^2 d\Omega_3^2), \\
    e^{2\phi_1} &= H, \quad e^{2\sqrt{7}\phi_2} = H^3, \\
    F_2^{(12)} &= Q \Sigma_2.
\end{align*}
\]
After transforming to type IIB variables, this oxidises to give the NUT solution
\[
\begin{align*}
 ds^2 &= dx^\mu dx_\mu + H (dr^2 + r^2 d\Omega_2^2) + H^{-1} (dz_2 + A_1)^2, \\
 \phi &= 0, \\
 F_2 &= Q \Sigma_2
\end{align*}
\] (3.22)
of the type IIB theory in ten dimensions.

### 3.4 Type IIB p-branes

The \(p\)-brane solutions in the type IIB string are given by
\[
\begin{align*}
 ds^2 &= H^{-\tilde{d}/8} dx^\mu dx_\mu + H^{d/8} (dr^2 + r^2 d\Omega_{\tilde{d}+1}^2), \\
 e^{\pm \phi} &= H^{\tilde{d}/4-1},
\end{align*}
\] (3.23)
where the plus sign in the expression for the dilaton corresponds to D-branes, and the minus sign to NS-NS branes. The \(p\)-brane has world-volume dimension \(d = p + 1\), and \(\tilde{d} = 8 - d\). The rank \(n\) of the field strength \(F_n\) that supports the solution is given by \(n = \min(d+1, \tilde{d}+1)\). When \(d < \tilde{d}\) the field strength carries an electric charge, whilst when \(d > \tilde{d}\) it carries a magnetic charge:
\[
\begin{align*}
 n = d + 1: & \quad F_n = d^d x \wedge dH^{-1} = Q H^{dd/8-1} \ast \Omega_{\tilde{d}+1}, \\
 n = \tilde{d} + 1: & \quad F_n = Q \Omega_{\tilde{d}+1}.
\end{align*}
\] (3.24)

In the case \(d = \tilde{d} = 4\), the 3-brane is supported by the self-dual 5-form
\[
F_5 = Q(\Omega_5 + \ast \Omega_5).
\] (3.25)
Since \(\tilde{d}\) is even for all the \(p\)-branes, the \((\tilde{d}+1)\)-sphere is always odd dimensional, and hence can be written in the \(U(1)\)-fibred form \([3,8]\).

First, let us consider the Dp-brane solutions of the type IIB theory, which exist for \(p = \{-1, 1, 3, 5, 7\}\). After dimensionally reducing on the \(U(1)\) fibre coordinate, we obtain the nine-dimensional solutions
\[
\begin{align*}
 ds_9^2 &= r^{2/7} \left[ H^{-\tilde{d}/7} dx^\mu dx_\mu + H^{d/7} (dr^2 + r^2 d\Sigma_{\tilde{d}}^2) \right], \\
 e^\phi &= H^{\tilde{d}/4-1}, \\
 e^{\sqrt{7} \phi} &= H^{d/4} r^4
\end{align*}
\] (3.26)
If the field strength \(F_n\) supporting the type IIB \(p\)-brane carries an electric charge, then it remains unchanged in nine dimensions, whereas if it carries a magnetic charge, it is reduced to \(F_{n-1}\) given by \(F_n \to F_{n-1} \wedge (dz_2 + A_1)\), where \(F_2 = dA_1 = 2J\) and \(J\) is the Kähler form
on $\mathbb{C}P^{\tilde{d}/2}$. After transforming to type IIA variables, we may then oxidise the solutions to the ten-dimensional type IIA theory, where we find

$$
\begin{align*}
 ds^2 &= r^{1/2} \left[ H^{-(\tilde{d}-1)/8} (dx^\mu dx_\mu + r^{-2} dz_2^2) + H^{(d+1)/8} (dr^2 + r^2 d\Sigma_{\tilde{d}}^2) \right], \\
 e^{-\phi_1} &= H^{(d-3)/4} r.
\end{align*}
$$

(3.27)

The type IIA solutions are all supported by two field strengths, one of which is universal, namely the N-NS 3-form

$$
F_3^{(1)} = 2Q dz_2 \wedge J.
$$

(3.28)

The other non-vanishing field strength is the R-R field strength of the type IIA theory whose rank $n = 0, 2$ or 4 is given by $n = \min(d+2, \tilde{d})$. Note that the “0-form field strength” $F_0$ is the cosmological term of the massive IIA theory. The various cases are summarised in the table below, where the $\hat{*}$ symbol denotes the appropriately scaled Hodge dual incorporating the necessary dilaton-dependent factor $\hat{*} = e^{-a\phi} \ast$, where the field strength has a dilaton prefactor $e^{a\phi}$ in its kinetic term.

| $p$ | IIB | $D = 9$ | IIA |
|-----|-----|--------|-----|
| -1  | $d\chi = Q \hat{\Omega}_9$ | $d\chi = Q \hat{\Sigma}_8$ | $F_2^{(1)} = Q \hat{\Sigma}_8$ |
| 1   | $F_3^R = Q \hat{\Omega}_7$ | $F_3^R = Q \hat{\Sigma}_6$ | $F_4 = Q \hat{\Sigma}_6$ |
| 3   | $F_5 = Q (\Omega_5 + *\Omega_5)$ | $F_4 = Q \Sigma_4$ | $F_4 = Q \Sigma_4$ |
| 5   | $F_5^R = Q \Omega_3$ | $F_2^R = Q \Sigma_2$ | $F_2^{(1)} = Q \Sigma_2$ |
| 7   | $d\chi = Q \Omega_1$ | $F_0 = Q$ | $F_0 = Q$ |

Table 1: Type IIB $D_p$-branes and their T duals

The configurations discussed here will be solutions of type IIA supergravity, but the full set of 16 out of 32 components of supersymmetry will only be found once the type IIA string winding modes are included. For example, in the case of the type IIB 3-brane, the corresponding solution that we obtain by Hopf dualising on the fibres of the foliating 5-spheres in the transverse space will, as a solution of type IIA supergravity, preserve no supersymmetry at all. This example emphasises the moral that just because a solution is non-BPS in supergravity, it does not necessarily follow that it is non-BPS in string theory. (The apparent disappearance of Killing spinors under T-duality has also been discussed in [83, 85].)
The solutions of the type IIA theory obtained above can be further oxidised to $D = 11$. From (3.27), we obtain the eleven-dimensional metrics

$$ds_{11}^2 = r^{2/3} \left[ H^{d/6 - 1} (dx^\mu dx_\mu + r^{-2} dz_2^2) + r^{-2} H^{1-d/3} (dz_1 + A^{(1)}_1)^2 + H^{d/6} (dr^2 + r^2 d\Sigma_4^2) \right].$$

(3.29)

The field strength $F_4$ in $D = 11$ will have a universal term of the form $2Q (dz_1 + A^{(1)}_1) \wedge dz_2 \wedge J$, together with an extra term of the form given in Table 1, in the cases $p = 1$ or $p = 3$. The Kaluza-Klein vector $A^{(1)}_1$ will be zero when $p = 1$ or 3, while $dA^{(1)}_1$ will give the field strengths $F^{(1)}_2$ listed in Table 1 when $p = -1$ or $p = 5$. The $p = 7$ case is a solution of the massive IIA theory, and presumably cannot be oxidised to $D = 11$.

Note that in the case of $p = 3$, we find from (3.29) that the Hopf dualisation of the type IIB self-dual 3-brane gives the eleven-dimensional solution

$$ds_{11}^2 = r^{2/3} \left[ H^{-1/3} (dx^\mu dx_\mu + r^{-2} (dz_1^2 + dz_2^2)) + H^{2/3} (dr^2 + r^2 d\Sigma_4^2) \right],$$

$$F_4 = 2Q dz_1 \wedge dz_2 \wedge J + Q \Sigma_4.$$

(3.30)

In addition to the D-branes discussed above, the type IIB theory also has NS-NS string and 5-brane solutions. The T-duality transformation of these into solutions of the type IIA theory proceeds, mutatis mutandis, identically to the discussion of the transformation to type IIB of the type IIA string and 5-brane in the previous subsection. This follows from the fact that the NS-NS sector of the two type II theories are identical, and invariant under the T-duality.

4 \ AdS$_5 \times$ S$^5$ compactification of type IIB

As we discussed in section 3, certain extremal $p$-brane solutions, namely those where the dilaton is finite on the horizon, have a spacetime structure that approaches $\text{AdS}_5 \times S^n$ as the horizon of the $p$-brane is approached. In this section, we shall consider a particular such example, namely the $\text{AdS}_5 \times S^5$ solution of the type IIB theory, which may be viewed as the near-horizon limit of the self-dual 3-brane. For the present purposes, however, we shall find it more convenient not to obtain the solution by this limiting process, but rather, to work directly with the $\text{AdS}_5 \times S^5$ solution.

This solution involves just the metric tensor and the self-dual 5-form field strength $H_{(5)}$ of the type IIB theory, whose relevant equations of motion can be written simply as

$$R_{MN} = \frac{1}{36} H_{MPQRS} H^N_{PQRS},$$

$$H_{(5)} = * H_{(5)}.$$  

(4.1)
where, in the absence of the other fields of the theory, we have simply $H_{(3)} = dB_{(4)}$. We may find a solution on $\text{AdS}_5 \times S^5$ of the form

$$
\begin{align*}
    ds^2 &= ds^2(\text{AdS}_5) + ds^2(S^5), \\
    H_{(5)} &= 4m\Omega_{\text{AdS}_5} + 4m\Omega_{S^5},
\end{align*}
$$

(4.2)

where $\Omega_{\text{AdS}_5}$ and $\Omega_{S^5}$ are the volume forms on $\text{AdS}_5$ and $S^5$ respectively, $m$ is a constant, and the metrics on $\text{AdS}_5$ and $S^5$ satisfy

$$
R_{\mu\nu} = -4m^2 g_{\mu\nu}, \quad R_{mn} = 4m^2 g_{mn}
$$

(4.3)

respectively. Since the unit 5-sphere has metric $d\Omega_5^2$ with Ricci tensor $\bar{R}_{mn} = 4\bar{g}_{mn}$, it follows that we can write

$$
ds^2(S^5) = \frac{1}{m^2} d\Omega_5^2.
$$

(4.4)

From (3.8), it follows that we can write this as

$$
ds^2(S^5) = \frac{1}{m^2} d\Sigma^2_4 + \frac{1}{m^2} (dz + \bar{A})^2,
$$

(4.5)

where $d\Sigma^2_4$ is the metric on the “unit” $\text{CP}^2$, and $d\bar{A} = 2J$, where $J$ is the Kähler form on $\text{CP}^2$.

We may now perform a dimensional reduction of this solution to $D = 9$, by compactifying on the circle of the $U(1)$ fibres, parameterised by $z$. Comparing with the general Kaluza-Klein prescription, for which

$$
\begin{align*}
    ds^2_{10} &= ds^2_{9} + (dz_2 + A)^2, \\
    H_{(5)} &= H_{(5)} + H_{(4)} \wedge (dz_2 + A),
\end{align*}
$$

(4.6)

we see, from the fact that the $S^5$ and $\text{CP}^2$ volume forms are related by $\Omega_5 = (dz + A) \wedge \Sigma_4$, that the solution will take the 9-dimensional form

$$
\begin{align*}
    ds^2_9 &= ds^2(\text{AdS}_5) + \frac{1}{m^2} d\Sigma^2_4, \\
    F_{(4)} &= \frac{1}{m^2} \Sigma_4, \quad F_{(2)} = \frac{2}{m} J.
\end{align*}
$$

(4.7)

(Note that in the dimensional reduction of the 5-form of the type IIB theory, its self-duality translates into the statement that the fields $H_{(5)}$ and $H_{(4)}$ in $D = 9$ must satisfy $H_{(4)} = \ast H_{(5)} = F_{(4)}$.)

We now perform the T-duality transformation to the fields of the $D = 9$ reduction of the type IIA theory. The relation between the IIB and the IIA fields is given in the appendix.
Thus in the IIA notation, we have the nine-dimensional configuration

\[ ds_9^2 = ds^2(AdS_5) + \frac{1}{m^2} d\Sigma_4^2 , \]
\[ F_{(4)} = \frac{4}{m^2} \Sigma_4 , \quad F_{(12)}^{(12)} = \frac{2}{m} J . \]  

(4.8)

The crucial point is that the 2-form field strength \( F_{(12)}^{(12)} \) of the IIA variables is no longer a Kaluza-Klein field coming from the metric; rather, it comes from the dimensional reduction of the 3-form field strength in \( D = 10 \). Indeed, if we trace the solution (4.8) back to \( D = 10 \), we have the type IIA configuration

\[ ds_{10}^2 = ds^2(AdS_5) + \frac{1}{m^2} d\Sigma_4^2 + dz_2^2 , \]
\[ F_{(4)} = \frac{4}{m^2} \Sigma_4 , \quad F_{(1)}^{(1)} = \frac{2}{m} J \wedge dz_2 . \]  

(4.9)

The solution has the topology \( AdS_5 \times CP^2 \times S^1 \). This should be contrasted with the topology \( AdS_5 \times S^5 \) for the original \( D = 10 \) solution in the type IIB framework. Thus the T-duality transformation in \( D = 9 \) has “unravelled” the twisting of the \( U(1) \) fibre bundle over \( CP^2 \), leaving us with a direct product \( CP^2 \times S^1 \) compactifying manifold in the type IIA description.

A further oxidation to \( D = 11 \) can now be performed. Upon doing so, we obtain the configuration

\[ ds_{11}^2 = ds^2(AdS_5) + \frac{1}{m^2} d\Sigma_4^2 + dz_1^2 + dz_2^2 , \]
\[ F_{(4)} = \frac{1}{m^2} \Sigma_4 - \frac{2}{m} J \wedge dz_1 \wedge dz_2 . \]  

(4.10)

which is just the near-horizon limit of (3.31). The topology of this solution is \( AdS_5 \times CP^2 \times T^2 \).

At first sight, the T-duality transformation that we have performed has a somewhat surprising implication. We began with a solution on \( AdS_5 \times S^5 \), which admits a spin structure, and mapped it via T-duality to a solution on \( AdS_5 \times CP^2 \times S^1 \), which does not admit a spin structure (because \( CP^2 \) does not admit a spin structure). In particular, this means that the spectrum of Kaluza-Klein excitations in the \( CP^2 \times S^1 \) compactification of type IIA supergravity contains no fermions at all!

To understand this, we should first look at the situation in the type IIB language, after having performed the reduction on the circle of the \( U(1) \) fibres, but before we make the T-duality transformation to the type IIA fields. Here too, we have a compactification involving \( CP^2 \), namely the \( AdS_5 \times CP^2 \) solution of \( D = 9 \) supergravity. However, at this stage we have done nothing but re-write the \( AdS_5 \times S^5 \) solution in terms of reduced \( D = 9 \) fields.
crucial point is that in this description, given in \((4.7)\), the Kaluza-Klein potential \(A\) has a topologically non-trivial form, with its field strength being proportional to the Kähler form of \(CP^2\). All the fermions in the Kaluza-Klein expansion of the \(D = 10\) type IIB fermions will be charged with respect to this Kaluza-Klein potential. As discussed in \([86]\), \(CP^2\) does admit a spin\(^c\) structure, or generalised spin structure, in which the spinors are charged under the gauge potential whose field strength is the Kähler form \(J\). In fact the non-existence of a standard spin structure is caused by the fact that spinors transported around a family of closed curves spanning the non-trivial 2-cycle in \(CP^2\) differ in phase by a factor of \(-1\), implying an inconsistency. This inconsistency is removed by considering instead charged spinors, minimally coupled to the gauge potential for \(J\), whose charges \(q\) are chosen to be precisely intermediate between the values that would normally be required by the Dirac quantisation condition in the presence of the magnetic charge \(\int J\); in other words of the form \(q = n + \frac{1}{2}\). Normally, this would give a minus sign inconsistency in the fermion phases, but here it precisely cancels the previously-discussed minus-sign inconsistency, allowing the existence of the charged spinors. Under the dimensional reduction on the \(U(1)\) fibres of \(S^5\), all the fermions automatically acquire proper charges that are consistent with the generalised spin structure. (This is discussed in some detail in \([87]\).) Thus in the type IIB description, there is a complete consistency between the \(D = 10\) and \(D = 9\) pictures.

Now, let us perform the \(T\)-duality transformation to the nine-dimensional type IIA field variables. In particular, this means that the non-trivial 1-form potential will no longer be a Kaluza-Klein vector potential, but instead it is the winding-mode potential \(A_1^{(12)}\) coming from the dimensional reduction of the NS-NS 2-form \(A_2^{(1)}\) in \(D = 10\) type IIA. The Kaluza-Klein modes in \(D = 9\) type IIA do not carry charges with respect to this potential. Consequently, Kaluza-Klein fermions in \(D = 9\) could not evade the sign-inconsistency problem that precludes the existence of a spin structure in \(CP^2\). In other words, there can be no fermions at all in the spectrum of Kaluza-Klein excitations of the \(D = 10\) type IIA theory compactified on \(CP^2 \times S^1\). In the string theory there are, however, also winding modes to be considered. These modes are charged with respect to the winding mode potential \(A_1^{(12)}\), and it is fermions in this sector of the complete \(D = 9\) spectrum that will carry the necessary charges that allow them to exist consistently on \(CP^2\).

We are now in a position to consider in more detail the relation between the spectrum of states in the \(AdS_5 \times S^5\) compactification of the type IIB theory, and the \(AdS_5 \times CP^2 \times S^1\) compactification of the type IIA theory. The easiest way to describe this is by looking first at the states in type IIB, which carry representations of the \(SO(6) = SU(4)\) isometry.
group of $S^5$, and decompose them with respect to the $SU(3) \times U(1)$ subgroup which is the isometry group of $CP^2$ times the $U(1)$ gauge symmetry of the Kaluza-Klein gauge potential. In particular, the 8 gravitini and the 15 gauge bosons of $SO(6)$ decompose as

$$15 \rightarrow 8_0 + 1_0 + 3_{-2} + \bar{3}_2 ,$$
$$4 + \bar{4} \rightarrow 3_{-1/2} + 1_{3/2} + \bar{3}_{1/2} + 1_{-3/2} .$$

(4.11)

The subscripts on the $SU(3)$ representations denote their charges with respect to $U(1)$. These are the Kaluza-Klein charges $q$, associated with the dependence $e^{iqz}$ on the compactifying coordinate $z$. In particular, we see that the gravitini have non-zero charges, and so all are truncated out in a dimensional reduction to the $z$-independent sector. Furthermore, as we mentioned previously, they have half-integer charges, precisely as is needed for consistency in the $CP^2$ manifold.

Turning to the gauge bosons, we see that the $8 + 1$ gauge bosons of the $SU(3)$ isometry of $CP^2$, together with the $U(1)$ Kaluza-Klein gauge potential, survive in a truncation to the zero-mode sector. The rest of the $SO(6)$ gauge symmetry of the 5-sphere is recovered only if the charged Kaluza-Klein modes are retained.²

After dualising to the $D = 9$ type IIA picture, the $U(1)$ charges in (4.11) will be carried instead by the gauge potential $A^{(12)}_1$ coming from the dimensional reduction of the 2-form potential $A_2^{(1)}$ of the type IIA theory. Thus the charged representations in (4.11) will not be seen at all in the Kaluza-Klein spectrum of the compactification of type IIA supergravity on $AdS_5 \times CP^2 \times S^1$. It is only by including the winding modes of the type IIA string that the charged representations will be recovered. (Subtleties involving the interpretation of T-duality as the interchange of Kaluza-Klein modes and winding modes in cases such as we are considering, where there is a $U(1)$ isometry but no non-contractible loop, are discussed in [83]. We shall continue to use the term “winding mode,” even though it is perhaps not wholly appropriate in this context.)

Note that although the type IIA Kaluza-Klein spectrum has states of maximum spin 2, these states are not BPS, and indeed will belong to long supermultiplets of the type IIA string when the winding modes are included.

Finally, we address the puzzle that on the type IIA side the gauge symmetry is $SU(3) \times U(1) \times U(1) \times U(1)$, but only an $SU(3) \times U(1)$ sits inside the $SO(6)$ of the type IIB side. (Two of the $U(1)$ factors are the obvious Kaluza-Klein $U(1)$’s from the $T^2$ compactification

²Analogous considerations of the Hopf fibration of $S^7$ in the context of $AdS_4 \times S^7$ compactifications of $D = 11$ supergravity and M-theory were explored in [81, 2].
of M-theory, while the third is associated with the gauge potential $A_1^{(12)}$; see Table 4 in the Appendix.) The R-R and NS-NS vectors $A_i^{(1)}$ which form an $SL(2, Z)$ doublet of $D = 9$ type IIA supergravity survive, after compactification on $CP^2$, but in the type IIB interpretation belong to “massive” multiplets of $N = 8$, $D = 5$ supersymmetry. (That is to say, they are not in the massless supergravity multiplet [L].)

5 Non-maximally-supersymmetric compactifications

The AdS$_5 \times S^5$ solution of the previous section can be generalised to any other configuration of the form AdS$_5 \times M_5$, where $M_5$ is any five-dimensional compact Einstein space with positive Ricci tensor. A particularly interesting class of such solutions is provided by taking $M_5$ to be a $U(1)$ bundle over $S^2 \times S^2$. We may denote these spaces by $Q(n_1, n_2)$, where the integers $n_1$ and $n_2$ are the winding numbers of the fibres over the two $S^2$ factors in the base manifold. Natural metrics on these spaces are given by [88]

$$ds^2 = c^2 (dz + \vec{A})^2 + \frac{1}{\Lambda_1} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{\Lambda_2} (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) ,$$

(5.1)

where $c$ is a constant, $\Lambda_1$ and $\Lambda_2$ are the “cosmological constants” of the two 2-spheres in the base manifold, and $z$ has period $2\pi$. The potential $\vec{A}$ can be taken to be

$$\vec{A} = -n_1 \cos \theta_1 d\phi_1 - n_2 \cos \theta_2 d\phi_2 ,$$

(5.2)

giving a field strength with the vielbein components

$$\bar{F}_{i_1j_1} = n_1 \Lambda_1 \epsilon_{i_1j_1} , \quad \bar{F}_{i_2j_2} = n_2 \Lambda_2 \epsilon_{i_2j_2} ,$$

(5.3)

where $\epsilon_{i_1j_1}$ and $\epsilon_{i_2j_2}$ are the Levi-Civita tensors on the two 2-spheres. Substituting into (3.11), we see find that the Ricci tensor for $ds^2$ has the vielbein components

$$R_{i_1j_1} = (\Lambda_1 - \frac{1}{2} n_1^2 c^2 \Lambda_1^2) \delta_{i_1j_1} , \quad R_{i_2j_2} = (\Lambda_2 - \frac{1}{2} n_2^2 c^2 \Lambda_2^2) \delta_{i_2j_2} ,$$

$$R_{zz} = \frac{1}{2} c^2 (n_1^2 \Lambda_1^2 + n_2^2 \Lambda_2^2) ,$$

(5.4)

It is easy to see that for each choice of integers $n_1$ and $n_2$, there are uniquely determined quantities $x_1$ and $x_2$ such that the metric $ds^2$ is Einstein, with $\Lambda_1 = x_1 c^{-2}$ and $\Lambda_2 = x_2 c^{-2}$ [88]. In fact $x_1$ and $x_2$ are the real roots of the cubic polynomials

$$9n_1^4 n_2^2 x_3^3 - 24n_1^2 n_2^2 x_1^2 + 8(n_1^2 + 2n_2^2) x_1 - 8 = 0 ,$$

$$9n_2^4 n_1^2 x_3^3 - 24n_2^2 n_1^2 x_2^2 + 8(n_2^2 + 2n_1^2) x_2 - 8 = 0 .$$

(5.5)
Thus we have an Einstein metric for each $Q(n_1,n_2)$ space. We may take $n_1$ and $n_2$ to be relatively prime, since if they had a common divisor, it would simply imply that the period for $z$ could have been taken to be $2\pi$ times $\gcd(n_1,n_2)$, rather than simply $2\pi$. Thus if we always take $z$ actually to have the period $2\pi$, the $Q(n_1,n_2)$ spaces where $n_1$ and $n_2$ have a common divisor are “lens spaces,” where the fibres have been identified under the freely-acting group $Z_p$, where $p = \gcd(n_1,n_2)$. In fact, we have in general that $Q(n_1,n_2) = Q(n_1/p,n_2/p)/Z_p$.

The situation of greatest interest to us is when the Einstein space $M_5$ admits Killing spinors, implying that the $\text{AdS}_5 \times M_5$ solution will preserve some supersymmetries. There will be an unbroken supersymmetry for each solution $\eta$ of the Killing-spinor equation on $M_5$. In this AdS context the Killing-spinor equation is

$$D_a \eta \equiv D_a \eta - \frac{1}{2} m \Gamma_a \eta = 0 ,$$  \hspace{1cm} (5.6)$$

where we assume that the Einstein metric on $M_5$ is such that $R_{ab} = 4m^2 \delta_{ab}$ (in vielbein components). The integrability condition for the existence of solutions to (5.6), obtained by taking the commutator of the $D_a$ derivatives, is

$$[D_a,D_b] \eta = \frac{4}{3} R_{abcd} \Gamma^{cd} \eta - \frac{1}{2} m^2 \Gamma_{ab} \eta = 0 .$$  \hspace{1cm} (5.7)$$

We find that the Einstein metrics on $Q(n_1,n_2)$ admit Killing spinors only if $n_1 = n_2 = 1$. In this case, solving the Einstein conditions following from (5.4), we obtain an Einstein metric on $Q(1,1)$ for which $R_{ab} = 4m^2 \delta_{ab}$, provided that the parameters in (5.1) are given by

$$\Lambda_1 = \Lambda_2 = 6m^2 , \quad c = \frac{1}{3m} .$$  \hspace{1cm} (5.8)$$

It is straightforward now to substitute the resulting expressions for the Riemann tensor, given by (3.10), into the integrability condition (5.7). We find that Killing spinors exist provided that they satisfy the condition

$$\Gamma_{1234} \eta = \eta ,$$  \hspace{1cm} (5.9)$$

where 1 and 2 are the vielbein indices for the first 2-sphere, and 3 and 4 are the indices for the second 2-sphere. Thus there are half the number of Killing spinors on $Q(1,1)$ as there are on $S^5$, and so we have a solution $\text{AdS}_5 \times Q(1,1)$ with $N = 4$ spacetime supersymmetry. The isometry group of $Q(1,1)$ is $SO(4) \times U(1) = SU(2) \times SU(2) \times U(1)$.

Curiously enough, the topology of the space $Q(1,1)$ is in fact $S^2 \times S^3$, although the Einstein metric is

\[\]
to one of the gauged $N = 4$, $D = 5$ theories described in \cite{10}. The massive Kaluza-Klein modes will in general have irrational AdS$_5$ energies in this compactification, implying that we would need to take the covering space of AdS$_5$ where the time coordinate is non-compact. Alternatively, if the AdS$_5$ spacetime is still required to have its usual periodic time coordinate then the the AdS energies will necessarily be integer or half-integer, implying that only a subset of the Kaluza-Klein modes will survive.

Whenever an Einstein space that is not a round sphere has Killing spinors, its orientation reversal gives a space with no Killing spinors \cite{13}. (Except for round spheres, the equation (5.6) admits solutions only for one choice of sign of $m$, and by reversing the orientation of the manifold, (5.6) admits solutions for the “wrong sign,” implying that there are no spacetime supersymmetries.) By this means we can also obtain a compactification on AdS$_5 \times Q(1,1)$ that has no supersymmetry.

We may now follow steps analogous to those described for $S^5$ in the previous section, and reduce the AdS$_5 \times Q(1,1)$ solution of the type IIB theory to $D = 9$, and perform a T-duality transformation. Upon oxidation back to the $D = 10$ type IIA theory, we have a solution on AdS$_5 \times S^2 \times S^2 \times S^1$. This can be oxidised further to $D = 11$, giving a solution on AdS$_5 \times S^2 \times S^2 \times T^2$.

There are also other ways in which we may obtain AdS$_5 \times M_5$ compactifications with less than maximal supersymmetry. In particular, we may take the standard AdS$_5 \times S^5$ solution, and simply replace $S^5$ by the cyclic lens space of order $k$, obtained by identifying the fibre coordinate of the $U(1)$ bundle over $CP^2$ with a period which is $1/k$ times the period in the $S^5$ case. We may denote these lens spaces by $S^5/Z_k$. (This mechanism for reducing the supersymmetries was proposed in \cite{3}.) It is evident that the mode functions on $S^5/Z_k$ will be the subset of mode functions on $S^5$ whose $U(1)$ charges $q$ are of the form

$$q = \frac{1}{2} k n,$$

(5.10)

where $n$ is any integer. Thus the spectrum of Kaluza-Klein states in AdS$_5$ will now be given by this subset of states of the AdS$_5 \times S^5$ compactification. All that is necessary in order to determine what survives is to take the decompositions of all the $SO(6)$ representations under the $SU(3) \times U(1)$ subgroup, and retain only those whose $U(1)$ charges satisfy (5.10). For not the standard direct-product metric on $S^2 \times S^3$ \cite{89}. The space $Q(1,1)$ is the coset $SO(4)/SO(2)$, which is the $S^2$ bundle of unit tangent vectors over $S^3$. This bundle is trivial, since $S^3$ is parallelisable, which explains why the topology of $Q(1,1)$ is just the direct product $S^2 \times S^3$. The spaces $Q(0,1)$ and $Q(1,0)$ also have topology $S^2 \times S^3$, but give rise to the “standard” direct-product Einstein metric for this topology; in these cases there will be no Killing spinors.)
example, we can see from (4.11) that if we consider the lens space $S^5/Z_3$, only the two $SU(3)$ singlet gravitini will survive, and so the $N = 8$ supersymmetry of the $S^5$ compactification will be broken to $N = 2$. At the same time, only the gauge bosons of $SU(3) \times U(1)$ will survive. The $U(1)$ gauge boson is in the $N = 2$ supergravity multiplet, and the $SU(3)$ gauge bosons will be in matter multiplets. Another possibility is to consider the lens space $S^5/Z_k$ for any other values of $k$ apart from $k = 1$ or $k = 3$. Now, we see from (4.11) that none of the gravitini (nor indeed any fermions at all if $k$ is even) will survive, and so we obtain $N = 0$ solutions. These are different from the $N = 0$ theory obtained in [73]. Note that in all these example, since the surviving states are a subset of the original states on $S^5$, their AdS energies, and hence the conformal weights of the associated operators in the Yang-Mills theory, will all be of standard integer form. This is quite different from the situation for the $Q(1,1)$ compactification that we described previously (since generic Kaluza-Klein compactifications will give rise to fractional, and indeed irrational, mass eigenvalues [15]).

6 Further examples

Non-dilatonic $p$-branes are defined to be either $p$-branes where there is no dilaton coupling, or those where the dilatons are regular on the horizon. Consequently, these $p$-branes have the common feature that their metrics are regular on the horizon. In particular, the metrics have the form of $AdS_{p+2} \times S^{D-p-2}$. Thus they can be viewed as interpolating between $AdS_{p+2} \times S^{D-p-2}$ on the horizon and $D$-dimensional Minkowski spacetime asymptotically at infinity. The M-theory membrane and 5-brane in $D = 11$, and the self-dual 3-brane in the type IIB theory, are the three examples, which we discussed earlier. These $p$-branes are supported by a single field strength which carries either a single electric or magnetic or self-dual charge, and the solutions involve a single harmonic function. Non-dilatonic $p$-branes also exist in lower dimensions, for example the dyonic strings in $D = 6$ [92], three-charge black holes and strings in $D = 5$ [93], and four-charge black holes in $D = 4$ [94, 95]. Upon oxidising these solutions to $D = 11$ or $D = 10$, they become intersections of $p$-branes, waves and NUTs.

We shall study the horizons of these non-dilatonic $p$-branes as solutions in their own

\footnote{This example is the same as one discussed in [73, 78], which was obtained by describing $S^3$ as the unit sphere in $C^3$, and then identifying the three complex coordinates $z^i$ under $z^i \to e^{i\alpha} z^i$, with $\alpha = 2\pi/3$. In fact, from the construction of the Fubini-Study metric on $CP^2$ as the Hopf fibration of $S^3$, one can see that making such an identification for any $\alpha = 2\pi/k$ will give rise to the cyclic lens space $S^5/Z_k$. (See, for example, [71].)}
right. Since the dilatonic scalars decouple (i.e. they are constants), it follows that the oxidation of these metrics to \( D = 11 \) must be described by \( \text{AdS}_{p+2} \times S^{D-p-2} \times T^{11-D} \), where \( T^{11-D} \) is an \((11-D)\)-dimensional torus.\(^{[1]}\) Oxidising instead to \( D = 10 \), the metric must be of the form \( \text{AdS}_{p+2} \times S^{D-p-2} \times T^{10-D} \), independent on whether it is oxidised to the type IIA or the type IIB theory. (Here we consider lower-dimensional solutions that are supported by field strengths come from the dimensional reduction of antisymmetric tensors from \( D = 11 \) or \( D = 10 \). We comment on the cases later where the solutions are supported by field strengths that come from the dimensional reduction of the metric.) Thus to summarise, M-theory, type IIA strings and type IIB strings have the following solutions that are of AdS structure, namely

| M-theory       | type IIA       | type IIB       |
|----------------|----------------|----------------|
| \( \text{AdS}_2 \times S^2 \times T^7 \)  | \( \text{AdS}_2 \times S^2 \times T^6 \)  | \( \text{AdS}_2 \times S^2 \times T^6 \)  |
| \( \text{AdS}_3 \times S^2 \times T^6 \)  | \( \text{AdS}_3 \times S^2 \times T^5 \)  | \( \text{AdS}_3 \times S^2 \times T^5 \)  |
| \( \text{AdS}_2 \times S^3 \times T^6 \)  | \( \text{AdS}_2 \times S^3 \times T^5 \)  | \( \text{AdS}_2 \times S^3 \times T^5 \)  |
| \( \text{AdS}_3 \times S^3 \times T^5 \)  | \( \text{AdS}_3 \times S^3 \times T^4 \)  | \( \text{AdS}_3 \times S^3 \times T^4 \)  |
| \( \text{AdS}_4 \times S^7 \)  | \( \text{AdS}_3 \times S^3 \times T^4 \)  | \( \text{AdS}_5 \times S^5 \)  |
| \( \text{AdS}_7 \times S^4 \)  | \( \text{AdS}_3 \times S^3 \times T^4 \)  | \( \text{AdS}_5 \times S^5 \)  |

Table 2: AdS and sphere structures in type IIA, IIB and M-theory

Note that the solutions that can be viewed as the horizons of M-branes or self-dual 3-branes that are supported by either the 4-form field strength or the self-dual 5-form, in a uniquely determined way; the 4-form field strength in \( D = 11 \) is given by the volume forms of the \( \text{AdS}_4 \) or \( S^4 \) respectively and the self-dual 5-form in \( D = 10 \) is given by the sum of the volume forms of the \( \text{AdS}_5 \) and \( S^5 \). The other solutions, which can be viewed as the horizons of intersecting branes, can be supported by different field strengths. To see this, let us consider some examples. In \( D = 6 \), a dyonic string can be supported by a field strength \( F_3^{(i)} \), which carries both electric and magnetic charges. Its horizon is \( \text{AdS}_3 \times S^3 \). Oxidising this back to \( D = 11 \), we obtain the metric \( \text{AdS}_3 \times S^3 \times T^5 \), which is the horizon of the intersection of a membrane and a 5-brane, and the solution is unique. If we oxidise the solution to \( D = 10 \) type IIA, different situations can arise depending on the value of the

\(^{[1]}\)Alternatively, some or all of the torus directions could be taken to be non-compact, so that \( T^{10-D} \) would be replaced by \( T^{m_1} \times E^{m_2} \), where \( m_1 + m_2 = 10 - D \). Note also that there are solutions where \( T^{10-D} \) is replaced by any Ricci-flat \((10-D)\)-dimensional space.

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internal index $i$. If $i = 1$, then the solution will become the horizon of the intersection of a string and a 5-brane, supported by the NS-NS 3-form; if $i = 2, 3, 4$ or 5, then the solution will become the horizon of the intersection of a D2-brane and a D4-brane, supported by the R-R 4-form. Now the situation is more complicated if we oxidise the solution to type IIB. If $i = 1$, the result is the same as in type IIA, namely it becomes the horizon of the NS-NS string and 5-brane; if $i = 2, 3, 4$ or 5, then it becomes intersection of two D3-branes. This example shows that although the metrics for all these solution have the same form, namely $\text{AdS}_3 \times \mathbb{S}^3 \times T^5$, they can be supported by quite different types of field strengths. Another way of obtaining the geometries in Table 2 is by performing duality transformations on intersecting brane solutions [96, 97, 98, 99].

The situation is analogous, but more complicated for three-charge and four-charge non-dilatonic $p$-branes. The possible field strengths that can be used to construct such a solution are given by [100]

\begin{eqnarray}
D = 5 & : & \{ F_2^{(ij)}, F_2^{(kk)}, F_2^{(mn)} \}_{15}, \quad \{ *F_3^{(i)}, F_2^{(j)}, F_2^{(ij)} \}_{30}, \\
D = 4 & : & \{ F_2^{(ij)}, F_2^{(kk)}, F_2^{(mn)}, *F_2^{(p)} \}_{105+105}, \quad \{ F_2^{(ij)}, *F_2^{(ik)}, F_2^{(j)}, F_2^{(k)} \}_{210}, \\
& & \{ F_2^{(ij)}, F_2^{(kk)}, *F_2^{(ik)}, *F_2^{(jl)} \}_{210}, \\
\end{eqnarray}

(6.1)

where the indices $(i, j, \ldots)$ are all different, and run over all the internal dimensions. The field strengths with $*$ and without $*$ carry electric and magnetic charges respectively, or else magnetic and electric charges. The subscripts denote the multiplicities of the solutions. These solutions form 45- and 630-dimensional representations of the Weyl group of $E_6$ and $E_7$ [101]. Although different solutions have the same metric configuration, they are supported by quite different field strengths. In particular, it implies that the AdS$_2$ or AdS$_3$ solutions in type IIA, type IIB or M-theory can have very different field-strength configurations. For example, the black hole solution supported by the field strengths $\{ F_2^{(13)}, F_2^{(24)}, F_2^{(56)} \}$ becomes the intersection of three membranes in $D = 11$, or one string and two membranes in type IIA, or one NS-NS string, one D-string and one D3-brane in type IIB. The classification of such correspondences between lower-dimensional solutions and higher-dimensional intersections can be found in [100]. The various field strengths appearing in (6.1) and (6.2) divide between NS-NS and R-R as follows:

\begin{eqnarray}
\text{NS-NS} & : & F_3^{(1)}, \quad F_2^{(1\alpha)}, \quad F_2^{(\alpha)}, \\
\text{R-R} & : & F_3^{(\alpha)}, \quad F_2^{(\alpha\beta)}, \quad F_2^{(1)}.
\end{eqnarray}

(6.3)
There is one more example that is worth mentioning. In $D = 4$, one can construct dyonic black holes if a single 2-form field strength carries both electric and magnetic charges. The solution is non-supersymmetric and describes a bound state with negative binding energy [102]. The horizon of this solution is $\text{AdS}_2 \times S^2$, and hence the supersymmetry is fully restored at the horizon. This provides further field configurations that support AdS structures in supergravities.

In the above discussion, we have considered lower dimensional solutions that are supported by the field strengths that come from the dimensional reduction of antisymmetric tensors in $D = 11$ or $D = 10$. In these cases, upon oxidation, the metric is still diagonal, and the AdS structure is manifest. As we have seen in (6.1) and (6.2), there can also be lower-dimensional solutions that are supported by Kaluza-Klein 2-form field strengths, coming from the dimensional reduction of the metric. They can be used to construct Reissner-Nordstrøm black holes in $D = 5$ and $D = 4$, which approach $\text{AdS}_2$ near the horizon. Upon oxidation, the metric acquires off-diagonal components, and describes a gravitational wave. The form of the metric near the horizon is not $\text{AdS}_2 \times S^2 \times T^n$, as it would be for those examples in the first list in (6.1) and the third list in (6.2). In fact the near-horizon form of the higher-dimensional metric in this case approaches a metric which is locally $\text{AdS}_3 \times \text{sphere} \times \text{torus}$ [103, 104, 105].

In the above AdS and sphere solutions of the type IIA, type IIB and M theories, there are further examples where the sphere has odd dimension, namely $S^3$, which can be viewed as a $U(1)$ bundle over $CP^1 = S^2$. Thus for such a type IIA or type IIB solution with $S^3$ in $D$ dimensions, we can perform a Hopf T-duality transformation on the $U(1)$ coordinate, upon reducing to $(D - 1)$ dimensions. If the solution is supported by R-R fields, then this transformation will have the effect of untwisting the $S^3$ to give a solution on $CP^1 \times S^1$ in $D$ dimensions. Note that the T-duality in question here is part of the T-duality symmetry of the $(D - 1)$-dimensional theory. If these metrics are in the type IIA theory, they can then be further mapped to M-theory with $T^2 \times CP^1$. Thus we can summarise the AdS and $CP^n$ structures as follows:

| M-theory               | type IIA               | type IIB               |
|------------------------|------------------------|------------------------|
| $\text{AdS}_2 \times CP^1 \times T^7$ | $\text{AdS}_2 \times CP^1 \times T^6$ | $\text{AdS}_2 \times CP^1 \times T^6$ |
| $\text{AdS}_3 \times CP^1 \times T^6$ | $\text{AdS}_3 \times CP^1 \times T^5$ | $\text{AdS}_3 \times CP^1 \times T^5$ |
| $\text{AdS}_5 \times CP^2 \times T^2$ |                         |                        |

Table 3: AdS and $CP^n$ structures in type IIA, IIB and M-theory
So far we have looked at the AdS structures in maximal supergravities. Such structures also exist in non-maximal supergravities. For example, in the AdS solutions listed in Table 2 and 3, the $T^n$ torus can be replaced by any Ricci flat space of the same dimension. For instance, $K3$ can replace $T^4$, and any Calabi-Yau 6-manifold can replace $T^6$. These lead to lower-dimensional theories with less supersymmetry. Let us consider a specific example, namely the compactification of type IIA and type IIB on the K3 manifold. The resulting six-dimensional theories are also related by a T-duality in $D = 5$, after a compactification on $S^1$. Both six-dimensional theories admit self-dual \cite{106}, and more generally, dyonic \cite{92}, string solutions whose horizons have the metric form $\text{AdS}_3 \times S^3$ \cite{38}. In the case of the type IIB six-dimensional theory, there exist R-R 3-forms which can support the dyonic string solutions. Applying the Hopf T-duality on the $U(1)$ fibre coordinate of $S^3$, we obtain the structure $\text{AdS}_3 \times CP^1 \times S^1$. If it is oxidised to $D = 10$, we then have a solution of the form $\text{AdS}_3 \times CP^1 \times S^3$, if $K3$. If it is further oxidised to $D = 11$, it becomes $\text{AdS}_3 \times CP^1 \times T^2 \times K3$.

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A T-duality of type IIA and type IIB

The Lagrangian of $D = 9$, $N = 2$ supergravity as the low-energy limit of type IIA string compactified on a circle can be obtained from the dimensional reduction of type IIA supergravity in $D = 10$, which itself can be obtained from dimensional reduction of eleven-dimensional supergravity. Using the notation adopted in [107], the bosonic sector of the theory contains the vielbein, a dilaton $\phi$ together with a second dilatonic scalar $\varphi$, (which measures the size of the compactifying circle,) one 4-form field strength $\tilde{F}_4 = dA_3$, two 3-forms $\tilde{F}^{(i)}_3 = dA^{(i)}_2$, three 2-forms $\tilde{F}^{(12)}_2 = dA^{(12)}_1$ and $\tilde{F}^{(i)}_2 = dA^{(i)}_1$ and one 1-form $\tilde{F}^{(12)}_1 = dA^{(12)}_0$. The full bosonic Lagrangian is given by [107, 101]

\[ e^{-1}L_{\text{IIA}} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\mathcal{F}^{(12)}_1)^2 e^{\frac{3}{2}\sqrt{2}\varphi} + \frac{5}{2}\varphi \]

\[ - \frac{1}{48}(F_4)^2 e^{\frac{1}{2}\phi + \frac{3}{2\sqrt{2}}\varphi} - \frac{1}{12}(F^{(1)}_3)^2 e^{-\phi + \frac{1}{2\sqrt{2}}\varphi} - \frac{1}{12}(F^{(2)}_3)^2 e^{\frac{1}{2}\phi - \frac{5}{2\sqrt{2}}\varphi} \quad (A.1) \]

\[ - \frac{1}{4}(F^{(12)}_2)^2 e^{-\phi - \frac{3}{2\sqrt{2}}\varphi} - \frac{1}{4}(F^{(1)}_2)^2 e^{\frac{3}{2}\phi + \frac{1}{2\sqrt{2}}\varphi} - \frac{1}{4}(F^{(2)}_2)^2 e^{\frac{3}{2}\sqrt{2}\varphi} \]

\[ - \frac{1}{2e} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A^{(12)}_1 - \frac{1}{2} \tilde{F}^{(1)}_3 \wedge \tilde{F}^{(2)}_3 \wedge A_3 . \]

Here we are using the notation that field strengths without tildes include the various Chern-Simons modifications, whilst field strengths written with tildes are unmodified. Thus we have

\[ F_4 = \tilde{F}_4 - \tilde{F}^{(1)}_3 \wedge A^{(1)}_1 - \tilde{F}^{(2)}_3 \wedge A^{(2)}_1 - \frac{1}{2} \tilde{F}^{(12)}_2 \wedge A^{(1)}_1 \wedge A^{(2)}_1, \]

\[ F^{(1)}_3 = \tilde{F}^{(1)}_3 - \tilde{F}^{(12)}_2 \wedge A^{(1)}_1, \]

\[ F^{(2)}_3 = \tilde{F}^{(2)}_3 + F^{(12)}_2 \wedge A^{(1)}_1 - A^{(12)}_0 (\tilde{F}^{(1)}_3 - \tilde{F}^{(12)}_2 \wedge A^{(12)}_1), \]

\[ F^{(12)}_2 = \tilde{F}^{(12)}_2, \quad \mathcal{F}^{(1)}_2 = \mathcal{F}^{(1)}_2 + A^{(12)}_0 \mathcal{F}^{(2)}_1, \quad \mathcal{F}^{(2)}_2 = \mathcal{F}^{(2)}_2, \quad \mathcal{F}^{(12)}_1 = \mathcal{F}^{(12)}_1. \quad (A.2) \]

The Lagrangian of the nine-dimensional supergravity as the low-energy limit of the type IIB string compactified on a circle can be obtained from the dimensional reduction of type IIB supergravity in $D = 10$. It is given by

\[ e^{-1}L_{\text{IIB}} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 \]

\[ - \frac{1}{48}e^{-\frac{1}{2\sqrt{2}}\varphi} F_4^2 - \frac{1}{12}e^{-\phi + \frac{1}{2\sqrt{2}}\varphi} (F^{(NS)}_3)^2 - \frac{1}{2}e^{\frac{3}{2}\sqrt{2}\varphi} (F^{(R)}_3)^2 \]

\[ - \frac{1}{4}e^{\frac{3}{2}\sqrt{2}\varphi} (F^{(2)}_2)^2 - \frac{1}{4}e^{\phi - \frac{3}{2\sqrt{2}}\varphi} (F^{(NS)}_2)^2 \]

\[ - \frac{1}{2e} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1 - \frac{1}{2} \tilde{F}^{(NS)}_3 \wedge \tilde{F}^{(R)}_3 \wedge A_3 . \]

Note that in $D = 10$ there are two 2-form potentials, one of which is the NS-NS field $A^{(NS)}_2$, and the other is the R-R field $A^{(R)}_2$. The dimensional reduction of these two potentials gives
rise to two 2-form potentials and also two vector potentials in $D = 9$, denoted by $A^{(\text{NS})}_1$ and $A^{(\text{R})}_1$ respectively.

The $D = 10$ IIA string and IIB string are related by a perturbative T-duality, in that type IIA string compactified on a circle with radius $R$ is equivalent to type IIB string compactified on a circle with radius $1/R$. At the level of their low-energy effective actions, this implies that there is only one $D = 9$, $N = 2$ supergravity. The Lagrangians (A.1) and (A.3) are related to each other by local field redefinitions. The relations between the gauge potentials of these two nine-dimensional theories (including the axions) are summarised in Table 4

|                  | IIA          | D = 10 | D = 9 | T-duality | IIB          | D = 9 | D = 10 |
|------------------|--------------|--------|-------|-----------|--------------|-------|--------|
| R-R fields       | $A_3$        | $A_3$  | $\leftrightarrow$ | $A_3$    | $B_4$        |
|                  | $A_2^{(2)}$  | $\leftrightarrow$ | $A_2^{R}$ | $A_2^{R}$  |
|                  | $A_1^{(1)}$  | $\leftrightarrow$ | $A_1^{R}$  |             |
|                  | $A_0^{(12)}$ | $\leftrightarrow$ | $\chi$    | $\chi$    |
| NS-NS fields     | $G_{\mu\nu}$| $A_1^{(2)}$ | $\leftrightarrow$ | $A_1^{NS}$ | $A_2^{NS}$ |
|                  | $A_2^{(1)}$  | $\leftrightarrow$ | $A_2^{NS}$ |             |
|                  | $A_1^{(12)}$ | $\leftrightarrow$ | $A_1$     | $G_{\mu\nu}$ |

Table 4: Gauge potentials of type II theories in $D = 10$ and $D = 9$

The relation between the dilatonic scalars of the two nine-dimensional theories is given by

$$
\begin{pmatrix}
\phi \\
\varphi
\end{pmatrix}_{\text{IIA}} =
\begin{pmatrix}
3/4 \\
-\sqrt{7}/4
\end{pmatrix}
\begin{pmatrix}
\phi \\
\varphi
\end{pmatrix}_{\text{IIB}}.
$$

(A.4)

The dimensional reduction of the ten-dimensional string metric to $D = 9$ is given by

$$
ds_{\text{str}}^2 = e^{\frac{1}{2} \phi} ds_{10}^2
= e^{\frac{1}{2} \phi} (e^{-\varphi/(2\sqrt{7})} ds_9^2 + e^{\sqrt{7} \varphi/2} (dz_2 + A)^2),
$$

(A.5)

where $ds_{10}^2$ and $ds_9^2$ are the Einstein-frame metrics in $D = 10$ and $D = 9$. The radius of the compactifying circle, measured using the ten-dimensional string metric, is therefore given by $R = e^{\frac{1}{4} \phi + \sqrt{7} \varphi/4}$. It follows from (A.4) that the radii $R_{\text{IIA}}$ and $R_{\text{IIB}}$ of the compactifying circles, measured using their respective ten-dimensional string metrics, are related by $R_{\text{IIA}} = 1/R_{\text{IIB}}$. 

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References

[1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, hep-th/9711200.

[2] M.J. Duff, H. Lü and C.N. Pope, *Supersymmetry without supersymmetry*, Phys. Lett. B409 (1997) 136, hep-th/9704186.

[3] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Compactification of D = 11 supergravity on K3×T^3*, Phys. Lett. B129 (1983) 39.

[4] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, Nucl. Phys. B360 (1991) 197.

[5] M.J. Duff and J.X. Lu, *The self-dual Type IIB superthreebrane*, Phys. Lett. B273 (1991) 409.

[6] D.Z. Freedman and A. Das, *Gauge internal symmetry in extended supergravity*, Phys. Lett. B74 (1977) 333.

[7] B. De Wit and H. Nicolai, *N = 8 supergravity*, Nucl. Phys. B208 (1982) 323.

[8] M.J. Duff and C.N. Pope, *Kaluza-Klein supergravity and the seven sphere*, in Supersymmetry and Supergravity 82, proceedings of the Trieste conference.

[9] E. Cremmer and J. Scherk, *Spontaneous compactification of extra space dimensions*, Nucl. Phys. B118 (1977) 61.

[10] P.G O. Freund and M. A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. B97 (1980) 233.

[11] M.J. Duff and P. van Nieuwenhuizen, *Quantum inequivalence of different field representations*, Phys. Lett. B94 (1980) 179.

[12] K. Pilch, P.K. Townsend and P. van Nieuwenhuizen, *Compactification of d = 11 supergravity on S^4 (or 11 = 7 + 4, too)*, Nucl. Phys. B242 (1984) 377.

[13] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged maximally extended supergravity in seven dimensions*, Phys. Lett. B143 (1984) 103.

[14] P.K. Townsend and P. van Nieuwenhuizen, *Gauged seven-dimensional supergravity*, Phys. Lett. B125 (1983) 41-6.
[15] M.J. Duff, B.E.W. Nilsson and C.N. Pope, Kaluza-Klein supergravity, Phys. Rep. 130 (1986) 1.

[16] M. Gunaydin and N. Marcus, The spectrum of the $S^5$ compactification of the $N = 2$, $D = 10$ supergravity and the unitary supermultiplet, Class. Quant. Grav. 2 (1985) L11.

[17] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, Mass spectrum of chiral ten-dimensional $N = 2$ supergravity on $S^5$, Phys. Rev. D32 (1985) 389.

[18] M. Pernici, K. Pilch and P. van Nieuwenhuizen, Gauged $N = 8$, $d = 5$ supergravity, Nucl. Phys. B259 (1985) 460.

[19] M. Gunaydin, L.J. Romans and N.P. Warner, Gauged $N = 8$ supergravity in five dimensions, Phys. Lett. B154 (1985) 268.

[20] C. Fronsdal, Dirac supermultiplet, Phys. Rev. D26 (1982) 1988.

[21] M. Gunaydin, in Supermembranes and Physics in 2 + 1 Dimensions, proceedings of the Trieste conference (eds M.J. Duff, C.N. Pope and E. Sezgin, World Scientific, 1990).

[22] E. Sezgin, The spectrum of the eleven dimensional supergravity compactified on the round seven sphere, Trieste preprint, 1983, in Supergravity in Diverse Dimensions, vol. 2, 1367, (eds A. Salam and E. Sezgin World Scientific, 1989); Fortschr. Phys. 34 (1986) 217.

[23] M. Gunaydin, L.J. Romans and N.P. Warner, Spectrum generating algebras in Kaluza-Klein theories, Phys. Lett. B146 (1984) 401.

[24] W. Nahm, Supersymmetries and their representations, Nucl. Phys. B135. (1978) 149.

[25] E. Bergshoeff, E. Sezgin and P.K. Townsend, Supermembranes and eleven-dimensional supergravity, Phys. Lett. B189 (1987) 75.

[26] E. Bergshoeff, E. Sezgin and Y. Tanii. Stress tensor commutators and Schwinger terms in singleton theories Int. J. Mod. Phys. A5 (1990) 3599.

[27] M.J. Duff, Supermembranes: The first fifteen weeks, Class. Quant. Grav. 5, 189 (1988).

[28] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, Supersymmetric supermembrane vacua and singletons, Phys. Lett. B199, 69 (1988).
[29] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, *Compactifications of the eleven-dimensional supermembrane*, Phys. Lett. B224, 71 (1989).

[30] H. Nicolai and E. Sezgin, *Singleton representations of OSp(N,4)* Phys. Lett. B143 (1984) 389.

[31] M.P. Blencowe and M.J. Duff, *Supersingletons*, Phys. Lett. B203, 229 (1988).

[32] M.J. Duff, *Classical and quantum supermembranes*, Class. Quant. Grav. 6, 1577 (1989).

[33] M.P. Blencowe and M.J. Duff, *Supermembranes and the signature of spacetime*, Nucl. Phys. B310 387 (1988).

[34] E. Bergshoeff, A. Salam, E. Sezgin, Y. Tanii, *N = 8 supersingleton quantum field theory*, Nucl.Phys. B305 (1988) 497.

[35] H. Nicolai, E. Sezgin and Y. Tanii, *Conformally invariant supersymmetric field theories on S^p x S^1 and super p-branes*, Nucl. Phys. B305 (1988) 483.

[36] M.J. Duff, C.N. Pope and E. Sezgin, *A stable supermembrane vacuum with a discrete spectrum*, Phys. Lett. B225, 319 (1989).

[37] G.W. Gibbons and P.K. Townsend, *Vacuum interpolation in supergravity via super p-branes* Phys. Rev. Lett. 71 (1993) 3754.

[38] M.J. Duff, G.W. Gibbons and P.K. Townsend, *Macroscopic superstrings as interpolating solitons*, Phys. Lett. B. 332 (1994) 321.

[39] G.W. Gibbons, G.T. Horowitz and P.K. Townsend, *Higher-dimensional resolution of dilatonic black hole singularities*, Class. Quant. Grav. 12 (1995) 297.

[40] M.J. Duff, R.R. Khuri and J.X. Lu, *String Solitons*, Phys. Rep. 259 (1995) 213.

[41] A. Achucarro, J. Evans, P.K. Townsend and D. Wiltshire, *Super p-branes*, Phys. Lett. B198 (1987) 441.

[42] C.G. Callan, J.A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons*, Nucl. Phys. B359 (1991) 611.

[43] C.G. Callan, J.A. Harvey and A. Strominger, *Worldbrane actions for string solitons*, Nucl. Phys. B367 (1991) 60.
[44] M.J. Duff and J.X. Lu, *Type II p-branes: the brane scan revisited*, Nucl. Phys. B390 (1993) 276.

[45] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, Phys. Rev. Lett. 75 (1995) 4724.

[46] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. B460 (1996) 335, [hep-th/9510135](https://arxiv.org/abs/hep-th/9510135).

[47] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, Phys. Rev. D54 (1996) 3915, [hep-th/9602135](https://arxiv.org/abs/hep-th/9602135).

[48] I.R. Klebanov, *World volume approach to absorption by non-dilatonic branes*, Nucl. Phys. B496 231, [hep-th/9702076](https://arxiv.org/abs/hep-th/9702076).

[49] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *String theory and classical absorption by three-branes*, Nucl. Phys. B499 (1997) 217, [hep-th/9703040](https://arxiv.org/abs/hep-th/9703040).

[50] S.S. Gubser and I.R. Klebanov, *Absorption by branes and Schwinger terms in the world volume theory*, Phys. Lett. B413 (1997) 41, [hep-th/9708003](https://arxiv.org/abs/hep-th/9708003).

[51] N. Itzhak, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, [hep-th/9802042](https://arxiv.org/abs/hep-th/9802042).

[52] M.J. Duff, H. Lü, C.N. Pope and E. Sezgin, *Supermembranes with fewer supersymmetries*, Phys. Lett. B371 (1996) 206, [hep-th/9511162](https://arxiv.org/abs/hep-th/9511162).

[53] L. Castellani, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and M. Trigiante, *G/H M-branes and AdS_{p+2} geometries*, [hep-th/9803039](https://arxiv.org/abs/hep-th/9803039).

[54] M.A. Awada, M.J. Duff and C.N. Pope, *N = 8 supergravity breaks down to N = 1*, Phys. Rev. Lett. 50 (1983) 294.

[55] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Spontaneous supersymmetry breaking via the squashed seven sphere*, Phys. Rev. Lett. 50 (1983) 2043.

[56] S.M. Christensen and M.J. Duff, *Quantizing Gravity with a Cosmological Constant*, Nucl. Phys. B170, 480 (1980).

[57] M.J. Duff, *Twenty Years of the Weyl Anomaly*, Class. Quantum. Grav. 11 (1994) 1387.

33
[58] S. Deser, M.J. Duff and C.J. Isham, *Non-Local Conformal Anomalies*, Nucl. Phys. **B111**, 45 (1976).

[59] S.M. Christensen, M.J. Duff, G.W. Gibbons and M. Rocek, *Vanishing One-Loop $\beta$-Function in Gauged $N > 4$ Supergravity*, Phys. Rev. Lett. **45**, 161 (1980).

[60] G.W. Gibbons and H. Nicolai, *One loop effects on the round seven-sphere*, Phys. Lett. **B143** (1984) 108.

[61] T. Inami and K. Yamagishi, *Vanishing quantum vacuum energy in 11-dimensional supergravity on the round seven-sphere*, Phys. Lett. **143** (1984) 115.

[62] T. Curtwright, *Charge renormalization and high spin fields*, Phys. Lett. **B102** (1981) 17.

[63] M.J. Duff, *Ultraviolet divergences in extended supergravity*, in “Supergravity 81” (eds. Ferarra and Taylor, C. U. P. 1982)

[64] K.S. Stelle, P.K. Townsend, *Vanishing beta functions in extended supergravities*, Phys. Lett. **B113** (1982) 25.

[65] S. Ferrara and C. Fronsdal, *Conformal Maxwell theory as a singleton field theory on AdS$_5$, IIB three branes and duality*, hep-th/9712239.

[66] R. Kallosh, J. Kumar and A. Rajaraman, *Special conformal symmetry of worldvolume actions*, hep-th/9712073.

[67] H.J. Boonstra, B. Peters, K. Skenderis, *Branes and anti de Sitter space-time*, hep-th/9801076.

[68] P. Claus, R. Kallosh, J. Kumar, P.K. Townsend and A. Van Proeyen, *Supergravity and the large $N$ limit of theories with sixteen supercharges*, hep-th/9801206.

[69] M. Gunaydin and D. Minic, *Singletons, doubletons and $M$-theory*, hep-th/9802047.

[70] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, hep-th/9802109.

[71] G.T. Horowitz and H. Ooguri, *Spectrum of large $N$ gauge theory from supergravity*, hep-th/9802116.

[72] E. Witten, *Anti de Sitter space and holography*, hep-th/9802150.
[73] S. Kachru and E. Silverstein, 4d conformal field theories and strings on orbifolds, hep-th/9802183.

[74] M. Berkooz, A supergravity dual of a (1,0) field theory in six dimensions, hep-th/9802193.

[75] S. Ferrara and C. Fronsdal, On $N = 8$ supergravity on $AdS_5$ and $N = 4$ superconformal Yang-Mills theory, hep-th/9802203.

[76] Soo-Jong Rey and Jungtay Lee, Macroscopic strings as heavy quarks in large $N$ gauge theory and anti de Sitter supergravity, hep-th/9803001.

[77] J. Maldacena, Wilson loops in large $N$ field theories, hep-th/9803002.

[78] A. Lawrence, N. Nekrasov and C. Vafa, On conformal field theories in four dimensions, hep-th/9803015.

[79] E. Cremmer, B. Julia and J. Scherk, Supergravity theory in eleven-dimensions, Phys. Lett. B76 (1978) 409.

[80] M.J. Duff and K.S. Stelle, Multi-membrane solutions of $D = 11$ supergravity, Phys. Lett. B253 (1991) 113.

[81] B.E.W. Nilsson and C.N. Pope, Hopf fibration of $D = 11$ supergravity, Class. Quantum Grav. 1 (1984) 499.

[82] H. Lü, C.N. Pope and P.K. Townsend, Domain walls from anti-de Sitter spacetime, Phys. Lett. B391 (1997) 39, hep-th/9607164.

[83] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, An introduction to T-duality in string theory, Nucl. Phys. Proc. Suppl. 41 (1995) 1, hep-th/9410237.

[84] E. Bergshoeff, C.M. Hull and T. Ortin, Duality in the type II superstring effective action, Nucl. Phys. B451 (1995) 547, hep-th/9504081.

[85] I. Bakas and K. Sfetsos, T-duality and world-sheet supersymmetry, Phys. Lett. B349 (1995) 448, hep-th/9502063.

[86] S.W. Hawking and C.N. Pope, Generalised spin structures in quantum gravity, Phys. Lett. B73 (1978) 42.

[87] C.N. Pope, Eigenfunctions and spin$^c$ structures in $CP^2$, Phys. Lett. B97 (1980) 417.
[88] D.N. Page and C.N. Pope, *Which compactifications of D = 11 supergravity are stable?* Phys. Lett. **B144** (1984) 346.

[89] W. Ziller, *Homogeneous Einstein metrics*, in: Global Riemannian Geometry, eds T.J. Willmore and N. Hitchin, (Wiley, New York, 1984).

[90] L.J. Romans, *Gauged N = 4 supergravities in five dimensions and their magnetovac backgrounds*, Nucl. Phys. **B267** (1986) 443.

[91] G.W. Gibbons and C.N. Pope, *CP² as a gravitational instanton*, Commun. Math. Phys. **61** (1978) 239.

[92] M.J. Duff, S. Ferrara, R. Khuri and J. Rahmfeld, *Supersymmetry and dual string solitons*, Phys. Lett. **B356** (1995) 479.

[93] A.A. Tseytlin, *Extreme dyonic black holes in string theory*, Mod. Phys. Lett. **A11** (1996) 689, [hep-th/9601177](https://arxiv.org/abs/hep-th/9601177).

[94] M. Cvetic and A.A. Tseytlin, *General class of BPS saturated black holes as exact superstring solutions*, Phys. Lett. **B366** (1996) 95, [hep-th/9510097](https://arxiv.org/abs/hep-th/9510097).

[95] M. Cvetic and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, Phys. Rev. **D53** (1996) 5619, [hep-th/9512031](https://arxiv.org/abs/hep-th/9512031).

[96] S. Hyun, *U-duality between three and higher dimensional black holes*, [hep-th/9704055](https://arxiv.org/abs/hep-th/9704055).

[97] H.J. Boonstra, B. Peeters and K. Skenderis, *Duality and asymptotic geometries*, Phys. Lett. **B411** (1997) 59, [hep-th/9706192](https://arxiv.org/abs/hep-th/9706192).

[98] E. Bergshoeff and K. Behrndt, *D-instantons and asymptotic geometries*, [hep-th/9803090](https://arxiv.org/abs/hep-th/9803090).

[99] E. Cremmer, I.V. Lavrinenko, H. Lü, C.N. Pope, K.S. Stelle and T.A. Tran, *Euclidean signature supergravities, dualities and instantons*, [hep-th/9803255](https://arxiv.org/abs/hep-th/9803255).

[100] H. Lü, C.N. Pope, T.A. Tran and K.-W. Xu, *Classification of p-branes, NUTs, Waves and Intersections*, to appear in Nucl. Phys. **B**, [hep-th/9708053](https://arxiv.org/abs/hep-th/9708053).

[101] H. Lü, C.N. Pope and K.S. Stelle, *Weyl group invariance and p-brane multiplets*, Nucl. Phys. **B476** (1996) 89, [hep-th/9602140](https://arxiv.org/abs/hep-th/9602140).
[102] G.W. Gibbons and R.E. Kallosh, Topology, entropy and Witten index of dilaton black holes, Phys. Rev. D51 (1995) 2839, hep-th/9407118.

[103] K. Sfetsos and K. Skenderis, Microscopic derivation of the Bekenstein-Hawking entropy formula for non-extremal black holes, hep-th/9711138.

[104] D. Birmingham, I. Sachs and S. Sen, Entropy of three-dimensional black holes in string theory, hep-th/9801019.

[105] V. Balasubramanian and F. Larsen, Near horizon geometry and black holes in four dimensions, hep-th/9802198.

[106] M.J. Duff and J.X. Lu, Black and super p-branes in diverse dimensions, Nucl. Phys. B416 (1994) 301.

[107] H. Lü and C.N. Pope, p-brane solitons in maximal supergravities, Nucl. Phys. B465 (1996) 127, hep-th/9512012.