A Bombieri – Vinogradov type result for exponential sums over primes

S. I. Dimitrov

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Abstract

In this paper we establish a new Bombieri – Vinogradov type result for exponential sums over primes. This result is a major application of the large sieve method.

Keywords: Bombieri – Vinogradov theorem · Dirichlet character · Exponential sum · Primes.

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1 Introduction and statement of the result

Siegel-Walfisz and Bombieri – Vinogradov theorems are extremely important results in analytic number theory and have various applications.

Siegel-Walfisz theorem is a refinement both of the prime number theorem and of Dirichlet’s theorem on primes in arithmetic progressions. It states that for any fixed $A > 0$ there exists a positive constant $c$ depending only on $A$ such that

$$\sum_{p \leq x} \log p = \frac{x}{\varphi(d)} + O\left(\frac{x}{e^{c\sqrt{\log x}}}\right),$$

whenever $x \geq 2$, $(a, d) = 1$, $d \leq (\log x)^A$ and $\varphi(n)$ is Euler’s function.

The celebrated Bombieri – Vinogradov theorem concerns the distribution of primes in arithmetic progressions, averaged over a range of moduli and states the following. Let $A > 0$ be fixed. Then

$$\sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \max_{y \leq X} \max_{(a, d) = 1} \left| \sum_{p \leq y} \frac{\log p - \frac{y}{\varphi(d)}}{\varphi(d)} \right| \ll \frac{X}{\log^A X}.$$
In 2017 Tolev established a Siegel-Walfisz type result for exponential sums over primes. It states the following. Let $\delta$, $\xi$ and $\mu$ be positive real numbers depending on $c > 1$, such that

$$\xi + 3\delta < \frac{12}{25}, \quad \mu < 1.$$ 

Let $D = X^\delta$ and $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \text{ if } 2|d \text{ or } \mu(d) = 0.$$ 

If

$$L(t, X) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X} \frac{e(tp^c) \log p}{p + 2 \equiv 0 (d)},$$

then for $|t| < X^{\xi - 1}$ the asymptotic formula

$$L(t, X) = \left( \int_{\mu X}^{X} e(ty^c) dy \right) \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} + O\left( \frac{X}{\log^A X} \right),$$

holds. Here $\mu(n)$ is Möbius’ function and $A > 0$ is an arbitrary large constant.

Motivated by these results in this paper we prove a Bombieri – Vinogradov type theorem for exponential sums over primes. More precisely we establish the following upper bound.

**Theorem 1.** Let $c > 1$ and $A > 0$ be fixed. Then for $|t| < X^{1/4 - c}$ the inequality

$$\sum_{d \leq \sqrt{X/(\log X)^A + \delta}} \max_{y \leq X} \max_{(a, d) = 1} \left| \sum_{\mu X < p \leq y \atop p \equiv a \mod (d)} e(tp^c) \log p - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c) dx \right| \ll \frac{X}{\log^A X}$$

holds.

**2 Notations**

Assume that $X$ is a sufficiently large positive number. The letter $p$ with or without subscript will always denote prime numbers. We write $e(t) = \exp(2\pi it)$. As usual $\varphi(n)$ is Euler’s function, $\Lambda(n)$ is von Mangoldt’s function, $\mu(n)$ is Möbius’ function and $\tau(n)$ denotes the number of positive divisors of $n$. We shall use the convention that a congruence, $m \equiv n \pmod{d}$ will be written as $m \equiv n (d)$. The letter $\chi$ denotes an Dirichlet character to given modulo. The sums $\sum_{\chi(d)}$ and $\sum_{*\chi(d)}$ denotes respectively summation over all
characters and all primitive characters to modulo $d$. Let $\mu$, $c$ be fixed with $0 < \mu < 1$, $c > 1$ and $|t| < X^{1/4-c}$.

Denote

$$S(t) = \sum_{\mu X < p \leq X} e(tp^c) \log p; \quad (1)$$

$$I(t) = \int_{\mu X} e(ty^c) \, dy; \quad (2)$$

$$\Psi(y, \chi, t) = \sum_{\mu y < n \leq y} \Lambda(n) \chi(n)e(tn^c); \quad (3)$$

$$E(y, t, d, a) = \sum_{\mu y < n \leq y \atop n \equiv a \pmod{d}} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)} \int_{\mu y}^y e(tx^c) \, dx. \quad (4)$$

### 3 Preliminary lemmas

**Lemma 1.** Let $|t| < X^{1/4-c}$. Then for the exponential sum denoted by (1) and the exponential integral denoted by (2) the asymptotic formula

$$S(t) = I(t) + O\left(\frac{X}{e(\log X)^{1/5}}\right)$$

holds.

**Proof.** See ([2], Lemma 14).

**Lemma 2.** Let $\delta$, $\xi$ and $\mu$ be positive real numbers depending on $c > 1$, such that

$$\xi + 3\delta < \frac{12}{25}, \quad \mu < 1.$$

Let $D = X^\delta$ and $B > 0$ is an arbitrarily large constant. Then for $|t| < X^{\xi-c}$ the upper bound

$$\sum_{1 < d \leq D} \frac{1}{\varphi(d)} \max_{\chi(d)} \left| \Psi(y, \chi, t) \right| \ll \frac{X}{\log^B X}$$

holds. Here $\Psi(y, \chi, t)$ is denoted by (3).

**Proof.** See ([3], Lemma 10).
Lemma 3. (Pólya – Vinogradov inequality) Suppose that $M, N$ are positive integers and $\chi$ is a non-principal character modulo $q$. Then

$$\left| \sum_{M < n \leq M+N} \chi(n) \right| \leq 6\sqrt{q} \log q .$$

Proof. See (1, Theorem 12.5)

Lemma 4. For any complex numbers $a_n$ and positive integers $M, N, Q$ we have

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \left| \sum_{\chi(q)} a_n \chi(n) \right|^2 \ll (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2$$

Proof. See (1, Theorem 7.13).

Lemma 5. We have

$$\int_{\mu \mathcal{X}} \, \, y^{\beta-1+i\gamma} e(ty^\epsilon) \, dy \ll \begin{cases} \frac{X^\beta}{\sqrt{|t|X^\epsilon}} & \text{for } |\gamma| < 4\pi c |t| X^\epsilon, \\ \frac{X^\beta}{|t|^\epsilon} & \text{for } |\gamma| \geq 4\pi c |t| X^\epsilon. \end{cases}$$

Proof. See (3, Lemma 10).

4 Proof of the Theorem

In order to prove our theorem we will use the formula

$$\sum_{\mu y < n \leq y \atop p \equiv a \pmod{d}} e(tp^\epsilon) \log p = \sum_{\mu y < n \leq y \atop n \equiv a \pmod{d}} \Lambda(n) e(tn^\epsilon) + \mathcal{O} \left( X^{\frac{1}{2} + \epsilon} \right). \quad (5)$$

Define

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is principal}, \\ 0 & \text{otherwise}. \end{cases} \quad (6)$$

By the orthogonality of characters we have

$$\sum_{\mu y < n \leq y \atop n \equiv a \pmod{d}} \Lambda(n) e(tn^\epsilon) - \frac{1}{\varphi(d)} \int_{\mu y} e(tx^\epsilon) \, dx$$

$$= \sum_{\mu y < n \leq y \atop n \equiv a \pmod{d}} \Lambda(n) e(tn^\epsilon) \frac{1}{\varphi(d)} \sum_{\chi(d)} \chi(n) \overline{\chi(a)} - \frac{1}{\varphi(d)} \int_{\mu y} e(tx^\epsilon) \, dx$$

$$= \frac{1}{\varphi(d)} \sum_{\chi(d)} \left( \overline{\chi(a)} \sum_{\mu y < n \leq y \atop \chi(n) = 1} \Lambda(n) e(tn^\epsilon) - \delta(\chi) \int_{\mu y} e(tx^\epsilon) \, dx \right)$$
and therefore

\[
\max_{(a, d) = 1} \left| \sum_{\mu y < n \leq y} \Lambda(n)e(tn^c) \right| - \frac{1}{\varphi(d)} \int_{\mu y}^{u} e(tx^c) \, dx \right| \\
\leq \frac{1}{\varphi(d)} \sum_{\chi(d)} \left| \Psi(y, \chi, t) - \delta(\chi) \sum_{\mu y} e(tx^c) \right| ,
\]

where \(\Psi(y, \chi, t)\) is defined by (3). Denote

\[
\Sigma = \sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(d)} \max_{y \leq X} \left| E(y, t, d, a) \right| .
\]

From (4), (6), (7) and (8) we obtain

\[
\Sigma \leq \Sigma' + \Sigma'',
\]

where

\[
\Sigma' = \sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(d)} \max_{y \leq X} \left| \sum_{\mu y < n \leq y} \Lambda(n)e(tn^c) \right| - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c) \, dx + O \left( \log^2 y \right) \right| ,
\]

\[
\Sigma'' = \sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(d)} \sum_{\chi(d)} \left| \Psi(y, \chi, t) \right| .
\]

By (5), (10) and Lemma 1 we find

\[
\Sigma' \ll \frac{1}{e(\log X)^{1/8}} \sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(d)} \ll \frac{X}{(\log X)^{A+3}} ,
\]

Next we consider \(\Sigma''\). Moving to primitive characters from (11) we deduce

\[
\Sigma'' \ll \sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(d)} \sum_{r \mid d} \sum_{\chi(r)} \max_{y \leq X} \left| \Psi(y, \chi, t) \right| + \frac{\sqrt{X}}{(\log X)^{A+3}}
\]

\[
\ll \sum_{1 < r \leq \sqrt{X}/(\log X)^{A+5}} \left( \sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(d)} \sum_{\chi(r)} \max_{y \leq X} \left| \Psi(y, \chi, t) \right| + \frac{\sqrt{X}}{(\log X)^{A+3}} \right)
\]

\[
\ll (\log X) \sum_{1 < r \leq \sqrt{X}/(\log X)^{A+5}} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} \left| \Psi(y, \chi, t) \right| + \frac{\sqrt{X}}{(\log X)^{A+3}}
\]

\[
= (\Omega_1 + \Omega_2) \log X + \frac{\sqrt{X}}{(\log X)^{A+3}} ,
\]

(13)
where
\[
\Omega_1 = \sum_{r \leq R_0} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{\ast} \max_{y \leq X} |\Psi(y, \chi, t)|,
\] (14)
\[
\Omega_2 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{\ast} \max_{y \leq X} |\Psi(y, \chi, t)|,
\] (15)
\[
R_0 = (\log X)^{A+5}, \quad R = \frac{\sqrt{X}}{\log X^{A+34+3}}.
\] (16)

Taking into account (14), (16) and Lemma 2 with \(B = 2A + 6\) we obtain
\[
\Omega_1 \ll \sum_{r \leq R_0} \max_{y \leq X} \frac{y}{\log B} \ll \frac{X}{(\log X)^{A+1}}.
\] (17)

Next we consider \(\Omega_2\). From (3) and (15) it follows
\[
\Omega_2 \ll \Omega_3 + \Omega_4,
\] (18)
where
\[
\Omega_3 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{\ast} \max_{y \leq X} |\Psi_1(y, \chi, t)|,
\] (19)
\[
\Omega_4 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{\ast} \max_{y \leq X} |\Psi_2(y, \chi, t)|,
\] (20)
and where
\[
\Psi_1(y, \chi, t) = \sum_{u < n \leq y} \Lambda(n) \chi(n) e(tn^c),
\] (21)
\[
\Psi_2(y, \chi, t) = \sum_{u < n \leq \mu y} \Lambda(n) \chi(n) e(tn^c),
\] (22)
\[
u \leq (\log X)^{2A+12}.
\] (23)

The choice of the parameter \(\nu\) will be made later.

We shall estimate only the sum \(\Omega_3\). The sum \(\Omega_4\) can be estimated likewise. Using (21) and Vaughan's identity (see [4]) we get
\[
\Psi_1(y, \chi, t) = U_1(y, \chi, t) - U_2(y, \chi, t) - U_3(y, \chi, t) - U_4(y, \chi, t),
\] (25)
where

\[ U_1(y, \chi, t) = \sum_{d \leq u} \mu(d) \sum_{u < dl \leq y} \chi(dl)e(tdlc) \log l, \]

(26)

\[ U_2(y, \chi, t) = \sum_{d \leq u} c(d) \sum_{u < dl \leq y} \chi(dl)e(tdlc), \]

(27)

\[ U_3(y, \chi, t) = \sum_{u < dl \leq y} \chi(dl)e(tdlc), \]

(28)

\[ U_4(y, \chi, t) = \sum_{d > u, l > u} a(d) \Lambda(l) \chi(dl)e(tdlc), \]

(29)

and where

\[ |c(d)| \leq \log d, \quad |a(d)| \leq \tau(d). \]

(30)

Now (19), (25) – (29) give us

\[ \Omega_3 \ll \Omega_3^{(1)} + \Omega_3^{(2)} + \Omega_3^{(3)} + \Omega_3^{(4)}, \]

(31)

where

\[ \Omega_3^{(j)} = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{y \leq X} \max_{\chi(r)} \left| U_j(y, \chi, t) \right|, \quad j = 1, 2, 3, 4. \]

(32)

**Estimation of \( \Omega_3^{(1)} \) and \( \Omega_3^{(2)} \)**

From (16), (26), (32), Abel’s summation formula and Lemma it follows

\[ \Omega_3^{(1)} \ll \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{y \leq X} \max_{\chi(r)} \left| \sum_{d \leq u} \chi(dl)e(tdlc) \log l \right| \]

\[ \ll X^{1/4}(\log X) \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \sum_{y \leq X} \max_{d \leq u} \max_{u/d < x/d} \left| \sum_{u/d < l \leq x} \chi(l) \right| \]

\[ \ll X^{1/4}(\log X) \sum_{R_0 < r \leq R} r^{1/2} \log r \]

\[ \ll X^{1/4}uR^{3/2} \log^2 X. \]

(33)

Working similarly to \( \Omega_3^{(1)} \) we deduce

\[ \Omega_3^{(2)} \ll X^{1/4}uR^{3/2} \log^2 X. \]

(34)
Estimation of $\Omega_3^{(3)}$ and $\Omega_3^{(4)}$

We split the range of $l$ of the exponential sum (29) into dyadic subintervals of the form $L < l \leq 2L$, where $u < L \leq y/2d$. Further we use (30), Abel’s summation formula, Perron’s formula with parameters

\[ \varkappa = 1 + \frac{1}{\log X}, \quad T = X^2, \]  

(35)

Lemma 4 and partial integration to find

\[ U_3(y, \chi, t) \ll (\log X) \left| \sum_{l \sim L} \sum_{u < d \leq y/l} \Lambda(l) a(d) \chi(dl) e(td^\varepsilon l^c) \right| \]

\[ = (\log X) \left| e(ty^c) \sum_{l \sim L} \sum_{u < d \leq y/l} \Lambda(l) a(d) \chi(dl) \right| \]

\[ - \int_{u^2/l}^{y/l} \left( \sum_{l \sim L} \sum_{u < d \leq x} \Lambda(l) a(d) \chi(dl) \right) \left| \frac{de(tx^c l^c)}{x^\varkappa (1 + T \log \frac{4X}{dl})} \right| dx \]

\[ = (\log X) \left| e(ty^c) \left( \sum_{l \sim L} \sum_{u < d \leq x} \Lambda(l) a(d) \chi(dl) \right) \frac{dy s}{s} ds \right| \]

\[ + \mathcal{O} \left( \sum_{l \sim L} \sum_{u < d \leq x} \frac{x^\varkappa \Lambda(l) \tau(d)}{(dl)^\varkappa (1 + T \log \frac{4X}{dl})} \right) \]

\[ - \int_{u^2/l}^{y/l} \left( \sum_{l \sim L} \sum_{u < d \leq x} \Lambda(l) a(d) \chi(dl) \frac{dy s}{s} ds \right) \]

\[ + \mathcal{O} \left( \sum_{l \sim L} \sum_{u < d \leq x} \frac{x^\varkappa \Lambda(l) \tau(d)}{(dl)^\varkappa (1 + T \log \frac{4X}{dl})} \right) \]

\[ \ll \left| \sum_{l \sim L} \Lambda(l) \chi(l) \right| \left| \sum_{u < d \leq x} \frac{a(d) \chi(d)}{dx^{1+\gamma_0}} \right| \log^2 X + X^{1/4}u^{-2} \log^2 X + \log^2 X \]

\[ + \int_0^T \sum_{l \sim L} \sum_{u < d \leq x} \frac{\Lambda(l) \chi(dl) a(d)}{(dl)^{x-1+\gamma_0}} \left( \int_{u^2/l}^{y/l} x^{x-2+\gamma_0} e(tl^c x^c) dx \right) d\gamma \left| \log X \right| \]

(36)

for some $|\gamma_0| \leq T$. Now (32) and (36) imply

\[ \Omega_3^{(4)} \ll (\Xi_1 + \Xi_2 + RX^{1/4}u^{-2} + R) \log^2 X, \]  

(37)
where

\[
\Xi_1 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^* \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{x-1+i\gamma_0}} \right| \sum_{u < d \leq X/L} a(d) \chi(d) \left| \frac{d^{x-1+i\gamma_0}}{d^{x-1+i\gamma_0}} \right|, \tag{38}
\]

\[
\Xi_2 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^* \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{x-1+i\gamma_0}} \right| \sum_{u < d \leq X/L} a(d) \chi(d) \left| \frac{d^{x-1+i\gamma_0}}{d^{x-1+i\gamma_0}} \right|.
\tag{39}
\]

First we estimate \(\Xi_1\). By (16), (30), (35), Cauchy’s inequality and Lemma 4 we obtain

\[
\sum_{R_0 < r \leq R} \frac{r}{\varphi(r)} \sum_{\chi(r)}^* \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{x-1+i\gamma_0}} \right| \sum_{u < d \leq X/L} a(d) \chi(d) \left| \frac{d^{x-1+i\gamma_0}}{d^{x-1+i\gamma_0}} \right|
\]

\[
\ll \left( \sum_{R_0 < r \leq R} \frac{r}{\varphi(r)} \sum_{\chi(r)}^* \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{x-1+i\gamma_0}} \right|^2 \right)^{1/2}
\times \left( \sum_{R_0 < r \leq R} \frac{r}{\varphi(r)} \sum_{\chi(r)}^* \left| \sum_{u < d \leq X/L} a(d) \chi(d) \frac{d^{x-1+i\gamma_0}}{d^{x-1+i\gamma_0}} \right|^2 \right)^{1/2}
\ll \left( L + R^2 \right)^{1/2} \left( \frac{X}{L} + R^2 \right)^{1/2} \left( \sum_{l \sim L} \Lambda^2(l) \right)^{1/2} \left( \sum_{u < d \leq X/L} \tau^2(d) \right)^{1/2}
\ll \left( X + X R u^{-1/2} + X^{1/2} R^2 \right) \log^2 X.
\tag{40}
\]

From (16), (38), (40) and Abel’s summation formula it follows

\[
\Xi_1 \ll \left( X R_0^{-1} + X u^{-1/2} \log X + X^{1/2} R \right) \log^2 X.
\tag{41}
\]

Next we consider \(\Xi_2\). Put

\[
\Gamma : z = f(\gamma) = \frac{\int_0^y x^{x-2+i\gamma e(t x^e)} \log x}{u^2} dx, \quad 0 \leq \gamma \leq T.
\tag{42}
\]

By (39) and (42) we get

\[
\Xi_2 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^* \left| \frac{1}{\Gamma} \int_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(d) \chi(a)}{(dl^2)x-1+i\gamma_0} dz \right|
\ll \left( |f(0)| + |f(T)| \right) \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^* \left| \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(d) \chi(a)}{(dl^2)x-1+i\gamma_0} \right|.
\tag{43}
\]
for some \( z_0 \in \Gamma \). Bearing in mind (35), (42), (43) and proceeding as in \( \Xi_1 \) we find

\[
\Xi_2 \ll (XR_0^{-1} + Xu^{-1/2} \log X + X^{1/2} \log X)^2 \int_{u^2}^{y} \frac{x^2}{\log x} \, dx
\ll (XR_0^{-1} + Xu^{-1/2} \log X + X^{1/2} \log X)^2 \log^2 X.
\]  

(44)

Now (37), (41) and (44) give us

\[
\Omega_3^{(4)} \ll (XR_0^{-1} + Xu^{-1/2} \log X + X^{1/2} \log X)^2 \log^4 X.
\]  

(45)

Working similarly to \( \Omega_3^{(4)} \) we deduce

\[
\Omega_3^{(3)} \ll (XR_0^{-1} + Xu^{-1/2} + X^{1/2} \log X)^2 \log^4 X.
\]  

(46)

From (31), (33), (34), (45) and (46) we obtain

\[
\Omega_3 \ll (X^{1/4} uR^{3/2} + XR_0^{-1} + Xu^{-1/2} \log X + X^{1/2} \log X)^2 \log^4 X.
\]  

(47)

Using (13), (16), (17), (18) and (47) we get

\[
\Sigma' \ll (X^{1/4} uR^{3/2} + XR_0^{-1} + Xu^{-1/2} \log X + X^{1/2} \log X)^2 \log^5 X.
\]  

(48)

Summarizing (9), (12), (16), (48) and choosing

\[
u = (\log X)^{2A+12}
\]

we find

\[
\Sigma \ll \frac{X}{\log^A X}.
\]  

(49)

Bearing in mind (4), (5), (8) and (49) we establish Theorem 1.

**References**

[1] H. Iwaniec, E. Kowalski, *Analytic number theory*, Colloquium Publications, 53, Amer. Math. Soc., (2004).

[2] D. Tolev, *On a diophantine inequality involving prime numbers*, Acta Arith., 61, (1992), 289 – 306.

[3] D. Tolev, *On a diophantine inequality with prime numbers of a special type*, Proceedings of the Steklov Institute of Mathematics, 299, (2017), 261 – 282.
[4] R. C. Vaughan, *An elementary method in prime number theory*, Acta Arithmetica, **37**, (1980), 111 – 115.

S. I. Dimitrov
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, St.Kliment Ohridski Blvd.
1756 Sofia, BULGARIA
e-mail: sdimitrov@tu-sofia.bg