TWISTED B-SPLINES IN THE COMPLEX PLANE

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ABSTRACT. In this paper, we introduce the new class of twisted B-splines and study some properties of these B-splines. We also investigate the system of twisted translates and the wavelets corresponding to these twisted B-splines.

1. Introduction

Let \( X := \{x_0 < x_1 < \cdots < x_k < x_{k+1}\} \) be a set of knots on the real line \( \mathbb{R} \). A spline function, or for short a spline, of order \( n \) on \([x_0, x_{k+1}]\) with knot set \( X \) is a function \( f : [x_0, x_{k+1}] \to \mathbb{R} \) such that

(i) on each interval \((x_{i-1}, x_i)\), \( i = 0, 1, \ldots, k + 1 \), \( f \) is a polynomial of order at most \( n \);

(ii) \( f \in C^{n-2}[x_0, x_{k+1}] \). In the case \( n = 1 \), we define \( C^{-1}[x_0, x_{k+1}] \) to be the space of piecewise constant functions on \([x_0, x_{k+1}]\).

The function \( f \) is called a cardinal spline if the knot set \( X \) is a contiguous subset of \( \mathbb{Z}^n \). As a spline of order \( n \) contains \( n(k + 1) \) free parameters on \([x_0, x_{k+1}]\) and has to satisfy \( n-1 \) differentiability conditions at the \( k \) interior knots \( x_1, \ldots, x_k \), the set \( S_{X,n} \) of all spline functions \( f \) of order \( n \) over the knot set \( X \) forms a real vector space of dimension \( n + k \). As the space of cardinal splines of order \( n \) over a finite knot set \( X \) is finite-dimensional, a convenient and powerful basis for \( S_{X,n} \) is given by the family of polynomial cardinal B-splines introduced by Curry & Schoenberg [4]. For a detailed study of splines and wavelets we refer to [3] and for splines and fractal functions we refer to [9].

The theory of B-splines can be extended to \( \mathbb{C} = \mathbb{R}^2 \) by using the tensor product of functions, namely, \( \mathbb{B}_n(x, y) := B_n(x)B_n(y), \ x, y \in \mathbb{R} \). The B-splines considered in this paper are totally different. In order to define these new twisted B-splines, we make use of the concept of twisted convolution. This is a non-standard convolution which arises while transferring the convolution from the Heisenberg group to the complex plane.

The Heisenberg group \( \mathbb{H} \) is a nilpotent Lie group whose underlying manifold is \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) endowed with a group operation defined by

\[
(x, y, t)(x', y', t') := (x + x', y + y', t + t' + \frac{1}{2}(x'y - y'x)),
\]

and where the Haar measure is Lebesgue measure \( dx \, dy \, dt \) on \( \mathbb{R}^3 \). By the Stone–von Neumann theorem, every infinite dimensional irreducible unitary representation on \( \mathbb{H} \) is unitarily equivalent to the representation \( \pi_\lambda \) given by

\[
\pi_\lambda(x, y, t)\phi(\xi) = e^{2\pi i \lambda t}e^{2\pi i \lambda (\frac{1}{2}xy)}\phi(\xi + y),
\]

for \( \phi \in L^2(\mathbb{R}) \) and \( \lambda \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\} \). This representation \( \pi_\lambda \) is called the Schrödinger representation of the Heisenberg group. For \( f, g \in L^1(\mathbb{H}) \), the group convolution of \( f \) and \( g \) is defined by

\[
f \ast g(x, y, t) := \int_{\mathbb{H}} f((x, y, t)(u, v, s)^{-1})g(u, v, s) \, du \, dv \, ds.
\]
This can be reduced to $\mathbb{R}^2$ as a twisted convolution in the following way. For $f \in L^1(\mathbb{H})$, define

$$f^\lambda(x, y) := \int_{\mathbb{R}} e^{2\pi i \lambda t} f(x, y, t) \, dt.$$ 

Now for $F, G \in L^1(\mathbb{R}^2)$, define

$$F *^\lambda G(x, y) := \int_{\mathbb{R}^2} F(x - u, y - v)G(u, v) \, e^{\pi i \lambda (uy - vx)} \, du \, dv.$$ 

The above operations are called $\lambda$-twisted convolutions on $\mathbb{R}^2$. One can show that for $f, g \in L^1(\mathbb{H})$,

$$(f * g)^\lambda = f^\lambda *^\lambda g.$$ 

In particular, when $\lambda := 1$, the $\lambda$-twisted convolution is simply called the twisted convolution and denoted by $F \times G$ for $F, G \in L^1(\mathbb{R}^2)$.

Now let $\pi(x, y) := \pi_1(x, y, 0)$. Then for $f \in L^1(\mathbb{R}^2)$, the Weyl transform of $f$ is defined to be

$$W(f) := \int_{\mathbb{R}^2} f(x, y)\pi(x, y) \, dx \, dy,$$

with $\pi(x, y)\phi(\xi) = e^{2\pi i (x\xi + \frac{1}{2}xy)}\phi(\xi + y), \phi \in L^2(\mathbb{R})$. In order to study problems concerned with the group Fourier transform on $\mathbb{H}$, an important technique is to take the partial Fourier transform in the $t$-variable and reduce the study to the case $\mathbb{R}^2$. Then the analysis on the Heisenberg group can be studied by first looking into an analysis based on the Weyl transform and twisted convolution. As we intend to study in the future $B$-splines on the Heisenberg group, we lay the ground work for it in this paper by studying $B$-splines on $\mathbb{C}$ using twisted convolution. It is important to mention here that $L^1(\mathbb{R}^2)$ turns out to be a non-commutative Banach algebra under twisted convolution, unlike ordinary convolution on $\mathbb{R}^2$. For a study of the Heisenberg group, we refer to [6] and [15].

In [12], Radha and Adhikari studied the problem of characterizing the system of twisted translates to be a frame sequence or a Riesz sequence. This study was later extended to the Heisenberg group in [14]. In [13], Radha and Adhikari studied some properties of twisted translates of a square-integrable function on $\mathbb{C}$ and later extended those results to the Heisenberg group. Recently, in [1], the orthonormality of a wavelet system associated with twisted translates and left translates on the Heisenberg group were investigated.

We organize the paper as follows. In Section 2, we provide the necessary background for the remainder of this article. In Section 3, we define twisted $B$-splines and study some of their properties. In Section 4, we investigate the system of twisted translates of the twisted $B$-splines and prove that this collection is a Riesz basis for the second order $B$-spline. In Section 5, we introduce the notion of multiresolution analysis on $L^2(\mathbb{R}^2)$ using twisted translations and standard dilations. We also prove that by taking the first order twisted $B$-spline as a scaling function, the corresponding mother wavelet, which we call a twisted Haar wavelet, leads to an orthonormal basis for $L^2(\mathbb{R}^2)$.

It is worth mentioning here that the presence of the “twist” term in the definition of a twisted $B$-spline complicates the computations and induces an entirely different structure from that of the classical $B$-splines on the real line.

2. Notation and Background

Let $0 \neq \mathcal{H}$ be a separable Hilbert space.
Definition 2.1. A sequence \( \{f_k : k \in \mathbb{N}\} \) of elements in \( \mathcal{H} \) is said to be a Bessel sequence for \( \mathcal{H} \) if there exists a constant \( B > 0 \) satisfying
\[
\sum_{k \in \mathbb{N}} | \langle f, f_k \rangle |^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H}.
\]

Definition 2.2. A sequence of the form \( \{U e_k : k \in \mathbb{N}\} \), where \( \{e_k : k \in \mathbb{N}\} \) is an orthonormal basis of \( \mathcal{H} \) and \( U \) is a bounded invertible operator on \( \mathcal{H} \), is called a Riesz basis. If \( \{f_k : k \in \mathbb{N}\} \) is a Riesz basis for \( \text{span}\{f_k : k \in \mathbb{N}\} \), then it is called a Riesz sequence.

Equivalently, \( \{f_k : k \in \mathbb{N}\} \) is said to be a Riesz sequence if there exist constants \( A, B > 0 \) such that
\[
A \|\{c_k\}\|_{\ell^2(\mathbb{N})}^2 \leq \sum_{k \in \mathbb{N}} |c_k f_k|^2 \leq B \|\{c_k\}\|_{\ell^2(\mathbb{N})}^2,
\]
for all finite sequences \( \{c_k\} \subset \ell^2(\mathbb{N}) \).

Definition 2.3. The Gramian \( G \) associated with a Bessel sequence \( \{f_k : k \in \mathbb{N}\} \) is a bounded operator on \( \ell^2(\mathbb{N}) \) defined by
\[
G\{c_k\} := \left\{ \sum_{k \in \mathbb{N}} \langle f_k, f_j \rangle c_k \right\}_{j \in \mathbb{N}}.
\]

It is well known that \( \{f_k : k \in \mathbb{N}\} \) is a Riesz sequence iff
\[
A \|\{c_k\}\|_{\ell^2(\mathbb{N})}^2 \leq \langle G\{c_k\}, \{c_k\} \rangle \leq B \|\{c_k\}\|_{\ell^2(\mathbb{N})}^2,
\]
for constants \( A, B > 0 \).

Definition 2.4. A closed subspace \( V \subset L^2(\mathbb{R}) \) is called a shift-invariant space if \( f \in V \Rightarrow T_k f \in V \) for any \( k \in \mathbb{Z} \), where \( T_k \) denotes the translation operator \( T_k f(y) := f(y - k) \). In particular, if \( \phi \in L^2(\mathbb{R}) \) then \( V(\phi) = \text{span}\{T_k \phi : k \in \mathbb{Z}\} \) is called a principal shift-invariant space.

For a detailed study of Riesz bases on \( \mathcal{H} \) and shift-invariant spaces on \( L^2(\mathbb{R}) \), we refer to [2].

Definition 2.5. Let \( \psi \in L^2(\mathbb{R}) \) and \( j, k \in \mathbb{Z} \). Define \( \psi_{j,k}(x) := 2^{j/2} \psi(2^j x - k) \), \( x \in \mathbb{R} \). The latter can also be written as
\[
\psi_{j,k} = D_{2^j} T_k \psi \quad j, k \in \mathbb{Z},
\]
where \( D_{2^j} f(x) := 2^{j/2} f(2^j x) \) and \( T_k f(x) := f(x - k), f \in L^2(\mathbb{R}) \), are unitary operators on \( L^2(\mathbb{R}) \). If \( \{\psi_{j,k} : j, k \in \mathbb{Z}\} \) is an orthonormal basis for \( L^2(\mathbb{R}) \) then the function \( \psi \) is called a wavelet or mother wavelet and \( \{\psi_{j,k} : j, k \in \mathbb{Z}\} \) is called a wavelet basis for \( L^2(\mathbb{R}) \).

The following definition is useful in constructing wavelets for \( L^2(\mathbb{R}) \). This notion was first introduced by Meyer in [11] and independently studied and developed by Mallat in [8].

Definition 2.6. A sequence \( \{V_j\}_{j \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}) \) is said to form a multiresolution analysis for \( L^2(\mathbb{R}) \) if the following conditions hold:

(i) \( V_j \subset V_{j+1}, \forall j \in \mathbb{Z} \).
(ii) \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \) and \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \).
(iii) \( f \in V_j \iff f(2 \cdot) \in V_{j+1}, \forall j \in \mathbb{Z} \).
(iv) There exists \( \phi \in V_0 \) such that \( \{T_k \phi : k \in \mathbb{Z}\} \) is an orthonormal basis for \( V_0 \).

The function \( \phi \) is called a scaling function of the given multiresolution analysis.
Example 2.7. Let $\phi := \chi_{[0,1]}$, where $\chi_{[0,1]}$ is the characteristic function of $[0,1]$. Let $V_0 = \text{span} \{T_k\phi : k \in \mathbb{Z} \}$. Define

$$\psi(x) := \begin{cases} 1, & 0 \leq x < \frac{1}{2}; \\ -1, & \frac{1}{2} \leq x < 1; \\ 0, & \text{otherwise}. \end{cases}$$

Then $\psi$ is called the Haar wavelet.

For a study of wavelets we refer to, e.g., [5].

Definition 2.8. Let $\chi_{[0,1]}$ denote the characteristic function of $[0,1]$. For $n \in \mathbb{N}$, set

$$B_1 : [0,1] \to [0,1], \quad x \mapsto \chi_{[0,1]}(x);$$

$$B_n := B_{n-1} * B_1, \quad 1 < n \in \mathbb{N}. \quad (2.1)$$

An element of the discrete family $\{B_n\}_{n \in \mathbb{N}}$ is called a (cardinal) polynomial B-spline of order $n$.

Remark 2.9. The adjective “cardinal” refers to the fact that $B_n$ is defined over the set of knots $\{0,1,\ldots,n\}$.

Using this definition, it can be shown that $B_n$ has a closed representation of the form

$$B_n(x) = \frac{1}{\Gamma(n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n-1},$$

where $\Gamma$ denotes the Euler Gamma function and $x^p_+ := \max\{0,x\}^p$ a truncated power function.

Now (2.1) implies that the Fourier representation of $B_n$ is given by

$$\hat{B}_n(\omega) := \mathcal{F}(B_n)(\omega) := \int_{\mathbb{R}} B_n(x)e^{-i\omega x} dx = \left(1 - e^{-i\omega}\right)^n.$$

Next, we summarize some of the properties of the cardinal B-splines $B_n$ that are relevant for the remainder of this paper.

(i) $\text{supp } B_n = [0,n]$.

(ii) $B_n > 0$ on $(0,n)$.

(iii) Partition of Unity: $\sum_{k \in \mathbb{Z}^n} B_n(x-k) = 1$, for all $x \in \mathbb{R}$.

(iv) The collection $\{B_n : n \in \mathbb{N}\}$ constitutes a family of piecewise polynomial functions with $B_n \in C^{n-2}[0,n]$, $n \in \mathbb{N}$. In the case $n = 1$, we define $C^{-1}[0,1]$ as the family of piecewise constant functions on $[0,1]$.

A particular property of B-splines which we consider in the setting of the present paper is the following.

Theorem 2.10. [2, Theorem 9.2.6] For each $n \in \mathbb{N}$, the sequence $\{T_k B_n\}_{k \in \mathbb{Z}}$ is a Riesz sequence.

In addition, polynomial (cardinal) B-splines satisfy a two-scale refinement equation of the form

$$B_n(x) = \sum_{k=0}^{n} \frac{1}{2^{m-1}} \binom{n}{k} B_n(2x-k),$$

and can be used to define certain classes of wavelets. We observe that when $B_1 = \chi_{[0,1]}$, the corresponding wavelet is the Haar wavelet. For more details, we refer to, for instance, [3].
The Weyl transform, defined in the Introduction, maps \( L^1(\mathbb{R}^2) \) into the space of bounded operators on \( L^2(\mathbb{R}) \), denoted by \( \mathcal{B}(L^2(\mathbb{R})) \). Moreover, the Weyl transform \( W(f) \) is an integral operator whose kernel \( K_f(\xi, \eta) \) is given by

\[
K_f(\xi, \eta) = \int_{\mathbb{R}} f(x, \eta - x) e^{\pi i x (\xi + \eta)} \, dx.
\]

The map \( W \) can be uniquely extended to a bijection from the class of tempered distributions \( S'(\mathbb{R}^2) \) onto the space of continuous linear maps from \( S(\mathbb{R}) \) into \( S'(\mathbb{R}) \). Here, \( S(\mathbb{R}) \) denotes the Schwarz space of real-valued functions on \( \mathbb{R} \).

When \( f \in L^2(\mathbb{R}^2) \), \( W(f) \) turns out to be a Hilbert-Schmidt operator on \( L^2(\mathbb{R}) \). Furthermore, \( W(f) \) enjoys the properties reminiscent of those of the Fourier transform on \( \mathbb{R} \). For instance, we have the Plancherel formula

\[
\|W(f)\|_{\mathcal{B}_2} = \|f\|_{L^2(\mathbb{R}^2)},
\]

where \( \mathcal{B}_2 := \mathcal{B}_2(L^2(\mathbb{R})) \) is the space of Hilbert-Schmidt operators on \( L^2(\mathbb{R}) \). Moreover,

\[
\langle W(f), W(g) \rangle_{\mathcal{B}_2} = \langle f, g \rangle_{L^2(\mathbb{R}^2)} = \langle K_f, K_g \rangle_{L^2(\mathbb{R}^2)}.
\]

The inversion formula is given by

\[
f(x, y) = \text{tr}(\pi(x, y)^*W(f)),
\]

where \( \text{tr} \) denotes the trace, and, in addition,

\[
W(f \times g) = W(f)W(g), \quad \text{for } f, g \in L^2(\mathbb{R}^2).
\]

**Definition 2.11.** Let \( \phi \in L^2(\mathbb{R}^2) \). For \((k, l) \in \mathbb{Z}^2\), the twisted translation of \( \phi \), denoted by \( T^t_{(k,l)} \phi \), is defined to be

\[
T^t_{(k,l)} \phi(x, y) := e^{\pi i l(y-k)} \phi(x-k, y-l), \quad (x, y) \in \mathbb{R}^2.
\]

**Definition 2.12.** The twisted shift-invariant space of \( \phi \), denoted by \( V^t(\phi) \), and defined by

\[
V^t(\phi) := \text{span}\{T^t_{(k,l)} \phi : k, l \in \mathbb{Z}\}
\]

is a closed subspace of \( L^2(\mathbb{R}^2) \).

In order to give a reader a feeling for twisted translation, we provide some properties which were proved in [12].

1. The adjoint \( (T^t_{(k,l)})^* \) of \( T^t_{(k,l)} \) is \( T^t_{(-k,-l)} \).
2. \( T^t_{(k_1,l_1)} T^t_{(k_2,l_2)} = e^{-\pi i (k_1 l_2 - l_1 k_2)} T^t_{(k_1+k_2, l_1+l_2)} \).
3. \( T^t_{(k,l)} \) is an unitary operator on \( L^2(\mathbb{R}^2) \), for each \((k, l) \in \mathbb{Z}^2\).
4. The Weyl transform of \( T^t_{(k,l)} \phi \) is given by \( W(T^t_{(k,l)} \phi) = \pi(k, l)W(\phi) \).

3. Definition and Various Properties of Twisted B-splines

**Definition 3.1.** Let \( \phi_1(x, y) := \chi_{[0,1)}(x)\chi_{[0,1)}(y) \). Define

\[
\phi_n = \prod_{i=1}^{n-1} \phi_i := \phi_1 \times \phi_1 \times \cdots \times \phi_1, \quad ((n-1)\text{-fold twisted convolution}).
\]

Then \( \phi_n \) is called an \( n \)-th order twisted B-spline.
Using this definition, we can derive an explicit formula for \( \phi_2(x, y) \).

\[
\phi_2(x, y) = \phi_1 \times \phi_1(x, y)
\]

\[
= \iint_{\mathbb{R}^2} \phi_1(x - u, y - v)\phi_1(u, v) e^{\pi i(xu - vy)} \, du \, dv
\]

\[
= \iint_{[0,1] \times [0,1]} \chi_{[0,1]}(x - u)\chi_{[0,1]}(y - v) e^{\pi i(xu - vy)} \, du \, dv
\]

\[
= \left( \int_0^1 \chi_{[0,1]}(x - u) e^{\pi iuy} \, du \right) \left( \int_0^1 \chi_{[0,1]}(y - v) e^{\pi ivx} \, dv \right)
\]

\[
= \left( \int_{u \in [0,1] \cap (x-1,x]} e^{\pi iuy} \, du \right) \left( \int_{v \in [0,1] \cap (y-1,y]} e^{\pi ivx} \, dv \right)
\]

Then a straightforward computation leads to the following:

\[
\phi_2(x, y) = \begin{cases}
\frac{1}{\pi xy} (e^{\pi ixy} - 1)(e^{-\pi ixy} - 1), & (x, y) \in (0,1] \times (0,1]; \\
\frac{1}{\pi xy} (e^{\pi ixy} - 1)(e^{-\pi ixy} - e^{-\pi ixy(y-1)}), & (x, y) \in (0,1] \times (1,2]; \\
\frac{1}{\pi xy} (e^{\pi ixy} - e^{\pi i(x-1)y})(e^{-\pi ixy} - 1), & (x, y) \in (1,2] \times (0,1]; \\
\frac{1}{\pi xy} (e^{\pi ixy} - e^{\pi i(x-1)y})(e^{-\pi ixy} - e^{-\pi ixy(y-1)}), & (x, y) \in (1,2] \times (1,2]; \\
0, & \text{otherwise}.
\end{cases}
\]

Using trigonometric identities, \( \phi_2 \) can be written in the form

\[
\phi_2(x, y) = \begin{cases}
\frac{2}{\pi xy} (1 - \cos(\pi xy)), & (x, y) \in (0,1] \times (0,1]; \\
\frac{2}{\pi xy} (\cos(\pi x(y - 1)) - \cos(\pi x)), & (x, y) \in (0,1] \times (1,2]; \\
\frac{2}{\pi xy} (\cos(\pi (x - 1)y) - \cos(\pi y)), & (x, y) \in (1,2] \times (0,1]; \\
\frac{2}{\pi xy} (\cos(\pi (x - y)) - \cos(\pi (x + y - xy))), & (x, y) \in (1,2] \times (1,2]; \\
0, & \text{otherwise}.
\end{cases}
\]

Figure 1 displays the graphs of the classical second order tensor product B-spline \( B_2 = B_2 \otimes B_2 \) and the second order twisted B-spline \( \phi_2 \).

![Figure 1](image_url)

**Figure 1.** The classical B-spline \( B_2 \) (left) and twisted B-spline \( \phi_2 \) (right).

We note that

\[
\phi_{n+1}(x, y) = \phi_n \times \phi_1(x, y)
\]

\[
= \iint_{\mathbb{R}^2} \phi_n(x - u, y - v)\chi_{[0,1]}(u)\chi_{[0,1]}(v) e^{\pi i(xy - vx)} \, du \, dv.
\]
Thus
\begin{equation}
\phi_{n+1}(x, y) = \int_0^1 \int_0^1 \phi_n(x - u, y - v)e^{\pi i (uy - vx)}
\end{equation}
\tag{3.2}

**Proposition 3.2.** \( \text{supp} \phi_n = [0, n] \times [0, n], \forall n \in \mathbb{N} \).

**Proof.** By (3.1), we observe that \( \text{supp} \phi_2 = [0, 2] \times [0, 2] \). Assume that \( \text{supp} \phi_n = [0, n] \times [0, n] \). Now,
\[
\phi_{n+1}(x, y) = \int_{u \in [x-1, x] \cap [0, n]} \int_{v \in [y-1, y] \cap [0, n]} \phi_n(u, v) e^{\pi i (vx - uy)}
\]
which means \( 0 < x < n + 1, \ 0 < y < n + 1 \). Thus if \((x, y) \notin (0, n + 1) \times (0, n + 1)\), then it follows that \( \phi_{n+1}(x, y) = 0 \) showing that \( \text{supp} \phi_{n+1} = [0, n+1] \times [0, n+1] \).

**Proposition 3.3.** We have
\[
\iint_{\mathbb{R}^2} \phi_1(x, y) \ dx \ dy = 1
\]
and
\[
\iint_{\mathbb{R}^2} \phi_2(x, y) \ dx \ dy = \frac{1}{\pi^2} \left( -\gamma - \log \pi + \text{Ci}(\pi) + i \text{Si}(\pi) \right) \left( -\gamma - \log \pi + \text{Ci}(\pi) - i \text{Si}(\pi) \right),
\]
where \( \gamma \) denotes the Euler–Mascheroni constant, and
\[
\text{Ci}(z) := \int_0^z \frac{\sin t}{t} \ dt, \ \text{Si}(z) := \int_0^z \frac{\cos t}{t} \ dt, \ |\arg z| < \pi,
\]
the cosine and sine integrals, respectively [7].

For \( n \geq 1 \),
\[
\iint_{\mathbb{R}^2} \phi_{n+1}(x, y) \ dx \ dy = \iint_{Q} L_n(u, v) \ du \ dv,
\]
where \( Q = [0, 1] \times [0, 1] \) and \( L_n \) is evaluated recursively as follows:
\[
L_{n+1}(u, v) = \iint_{Q} L_n(p + u, q + v) e^{\pi i (uq - vp)} \ dp \ dq,
\]
with
\[
L_1(u, v) = \frac{1}{\pi^2 uv} (e^{\pi i u} - 1)(e^{-\pi i v} - 1).
\]

**Proof.** For \( n = 1 \), we obtain
\[
\iint_{\mathbb{R}^2} \phi_1(x, y) \ dx \ dy = \iint_{\mathbb{R}^2} \chi_{[0,1]}(x)\chi_{[0,1]}(y) \ dx \ dy = 1.
\]
For \( n = 2 \), making use of (3.2) and Fubini’s theorem, we get
\[
\iint_{\mathbb{R}^2} \phi_2(x, y) \ dx \ dy = \iint_{\mathbb{R}^2} \left( \int_0^1 \int_0^1 \phi_1(x - u, y - v) e^{\pi i (uy - vx)} \ dv \ du \right) \ dx \ dy
\]
\[
= \iint_{Q} \left( \iint_{\mathbb{R}^2} \phi_1(x - u, y - v) e^{\pi i (uy - vx)} \ dx \ dy \right) \ du \ dv
\]
\[
= \iint_{Q} \left( \iint_{Q} e^{\pi i (us - vs)} \ dr \ ds \right) du \ dv,
\]
where we applied the change of variables \( x - u = r \) and \( y - v = s \). However,
\[
\iint_{Q} e^{\pi i (us - vs)} \ dr \ ds = \frac{1}{\pi^2 uv} (e^{\pi i u} - 1)(e^{-\pi i v} - 1).
\]
Thus
\[ \iint_{\mathbb{R}^2} \phi_2(x, y) \, dx \, dy = \frac{1}{\pi^2} \left( \int_0^1 \frac{e^{\pi i u} - 1}{u} \, du \right) \left( \int_0^1 \frac{e^{-\pi i v} - 1}{v} \, dv \right) \]
\[ = \frac{1}{\pi^2} \left( -\gamma - \log \pi + \text{Ci}(\pi) + i \text{Si}(\pi) \right) \left( -\gamma - \log \pi + \text{Ci}(\pi) - i \text{Si}(\pi) \right) . \]

Now, assume that the result is true for \( n \) (3.4).

Using (3.2), Fubini’s theorem, and then applying a change of variables, yields
\[ \iint_{\mathbb{R}^2} \phi_{n+1}(x, y) \, dx \, dy = \iint_{\mathbb{R}^2} \left( \int_0^1 \phi_n(x - u, y - v) e^{\pi i (uy-vx)} \, dv \, du \right) \, dx \, dy \]
\[ = \iint_Q \left( \iint_{\mathbb{R}^2} \phi_n(x - u, y - v) e^{\pi i (uy-vx)} \, dx \, dy \right) \, du \, dv \]
\[ = \iint_Q \left( \iint_{\mathbb{R}^2} \phi_n(r, s) e^{\pi i (us-vr)} \, dr \, ds \right) \, du \, dv . \]

(3.3)

Thus, we need to compute
\[ \iint_{\mathbb{R}^2} \phi_n(r, s) e^{\pi i (us-vr)} \, dr \, ds, \quad \forall \ n \in \mathbb{N} . \]

We shall use induction on \( n \). First, for \( n = 1 \), we set
\[ \iint_{\mathbb{R}^2} \phi_1(r, s) e^{\pi i (us-vr)} \, dr \, ds = \left( \int_0^1 e^{-\pi ivr} \, dr \right) \left( \int_0^1 e^{\pi ius} \, ds \right) \]
\[ = \frac{1}{\pi^2} \left( \frac{e^{\pi i u} - 1}{u} \right) \left( \frac{e^{-\pi iv} - 1}{v} \right) \]
\[ =: L_1(u, v) . \]

(3.4)

Now, assume that the result is true for \( n \geq 1 \), that is,
\[ \iint_{\mathbb{R}^2} \phi_n(r, s) e^{\pi i (us-vr)} \, dr \, ds = L_n(u, v) . \]

Using (3.2), Fubini’s theorem, and then applying a change of variables, yields
\[ \iint_{\mathbb{R}^2} \phi_{n+1}(r, s) e^{\pi i (us-vr)} \, dr \, ds = \iint_Q \left( \iint_{\mathbb{R}^2} \phi_n(r, s) e^{\pi i (p+u)(q+v)r} \, dr \, ds \right) e^{\pi i (pq-vp)} \, dp \, dq \]
\[ = \iint_Q L_n(p + u, q + v) e^{\pi i (pq-vp)} \, dp \, dq . \]

Hence, if we denote the left-hand side of the above equation by \( L_{n+1}(u, v) \), we obtain
\[ L_{n+1}(u, v) = \iint_Q L_n(p + u, q + v) e^{\pi i (pq-vp)} \, dp \, dq . \]

Therefore, by (3.4), we get
\[ L_2(u, v) = \iint_Q L_1(p + u, q + v) e^{\pi i (pq-vp)} \, dp \, dq \]
\[ = \frac{1}{\pi^2} \left( \int_0^1 \frac{e^{\pi i (p+u)} - 1}{p+u} e^{-\pi ivp} \, dp \right) \left( \int_0^1 \frac{e^{-\pi i(q+v)} - 1}{q+v} e^{\pi iuq} \, dq \right) \]
\[ = \frac{1}{\pi^2} \left( \text{Ei}(-\pi iuv) - \text{Ei}(-\pi iu(v-1)) - \text{Ei}(-\pi i(u+1)v) + \text{Ei}(-\pi i(u+1)(v-1)) \right) \times \]
\[ \left( \text{Ei}(\pi iuv) - \text{Ei}(\pi i(u+1)v) - \text{Ei}(\pi i(u+1)v) + \text{Ei}(\pi i(u+1)(v+1)) \right) , \]
with \( \text{Ei} \) denoting the exponential integral
\[
\text{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} \, dt, \quad |\arg z| < \pi,
\]
where the principal value of the integral is taken and the branch cut ranges from \(-\infty\) to 0. (Cf. [7].) Now, using (3.5) in (3.3) and induction on \( n \), we can obtain the values for
\[
\iint_{\mathbb{R}^2} \phi_{n+1}(x, y) \, dx \, dy
\]
for all \( 2 \leq n \in \mathbb{N} \).

\[\Box\]

**Proposition 3.4.** The kernel of the Weyl transform of \( \phi_1 \) is given by
\[
K_{\phi_1}(\xi, \eta) = e^{\pi i (\xi + \eta)} \sin \left( \frac{\xi + \eta}{2} \right) \chi_{[0,1]}(\eta - \xi), \quad \xi, \eta \in \mathbb{R}.
\]
For \( n \geq 2 \), the kernel of the Weyl transform of \( \phi_n \) can be evaluated recursively as follows:
\[
K_{\phi_n}(\xi, \eta) = e^{\pi i \eta} \int_0^1 e^{-\pi i y_{n-1}} \sin \left( \frac{2\eta - y_{n-1}}{2} \right) K_{\phi_{n-1}}(\xi, \eta - y_{n-1}) \, dy_{n-1}.
\]

**Proof.** Recall that \( \phi_1(x, y) = \chi_{[0,1]}(x)\chi_{[0,1]}(y) \) and that the Weyl transform of \( \phi_1 \) is an integral operator whose kernel \( K_{\phi_1} \) is given by
\[
K_{\phi_1}(\xi, \eta) = \int_{\mathbb{R}} \phi_1(x, \eta - \xi) e^{\pi ix(\xi + \eta)} \, dx
\]
\[
= \chi_{[0,1]}(\eta - \xi) \int_0^1 e^{\pi ix(\xi + \eta)} \, dx
\]
\[
= \frac{1}{\pi i (\xi + \eta)} \left( e^{\pi i (\xi + \eta)} - 1 \right) \chi_{[0,1]}(\eta - \xi)
\]
\[
= e^{\pi i (\xi + \eta)} \frac{\sin \left( \frac{\pi}{2} (\xi + \eta) \right)}{\frac{\pi}{2} (\xi + \eta)} \chi_{[0,1]}(\eta - \xi), \quad \xi + \eta \neq 0.
\]
If \( \xi + \eta = 0 \), then
\[
K_{\phi_1}(\xi, \eta) = \chi_{[0,1]}(\eta - \xi).
\]
Thus, using the sinc function, we express the kernel in the form
\[
K_{\phi_1}(\xi, \eta) = e^{\pi i (\xi + \eta)} \sin \left( \frac{\xi + \eta}{2} \right) \chi_{[0,1]}(\eta - \xi).
\]
As, \( \phi_m = \bigotimes_{i=1}^{m-1} \phi_i \), \( W(\phi_m) = W(\phi_1)^m \) since \( W(\phi \times \psi) = W(\phi)W(\psi) \). It is known that if \( T \) is an integral operator with kernel \( K(\xi, \eta) \), then \( T^2 \) is an integral operator with kernel
\[
\int K(\xi, y)K(y, \eta) \, dy.
\]
Thus, in general, \( T^n \) is an integral operator with kernel
\[
\int \cdots \int K(\xi, y_1)K(y_1, y_2) \cdots K(y_{n-2}, y_{n-1})K(y_{n-1}, \eta) \, dy_{n-1} \cdots dy_1.
\]
Hence, the kernel of Weyl transform of $\phi_n$ is given by

$$K_{\phi_n}(\xi, \eta) = e^{\frac{2\pi i}{2}((\xi+\eta))} \int_{y_1 \in \mathbb{R}} \cdots \int_{y_{n-1} \in \mathbb{R}} e^{\pi i (y_1 + \cdots + y_{n-1})} \frac{\chi_{[0,1)}(y_1 - \xi)}{2} \frac{\chi_{[0,1)}(y_1 + \eta)}{2} \chi_{[0,1)}(y_2 - y_1) \cdots \frac{\chi_{[0,1)}(y_{n-1} - y_{n-2})}{2} \chi_{[0,1)}(y_{n-1} + \eta) \times$$

$$\text{sinc} \left( \frac{y_1 + y_2}{2} \right) \chi_{[0,1)}(y_2 - y_1) \cdots \frac{\chi_{[0,1)}(y_{n-2} + y_{n-1})}{2} \chi_{[0,1)}(y_{n-1} - y_{n-2}) \times$$

$$\text{sinc} \left( \frac{y_{n-1} + \eta}{2} \right) \chi_{[0,1)}(\eta - y_{n-1}) \; dy_1 \cdots dy_{n-1}.$$ 

First, we recursively express the kernel $K_{\phi_n}$ in terms of $K_{\phi_{n-1}}$. Observe that

$$K_{\phi_2}(\xi, \eta) = e^{\frac{2\pi i}{2}((\xi+\eta))} \int_{y_1 \in \mathbb{R}} e^{\pi iy_1} \frac{\chi_{[0,1)}(y_1 - \xi)}{2} \frac{\chi_{[0,1)}(\eta - y_1)}{2} \chi_{[0,1)}(y_1 - \xi) \chi_{[0,1)}(\eta - y_1) \; dy_1$$

$$= e^{\frac{2\pi i}{2}((\xi+\eta))} \int_{y_1 \in \mathbb{R}} e^{\pi iy_1} \frac{\chi_{[0,1)}(y_1 - \xi)}{2} \frac{\chi_{[0,1)}(\eta - y_1)}{2} \; dy_1.$$ 

The above integral can be rewritten as

$$K_{\phi_2}(\xi, \eta) = e^{\frac{2\pi i}{2}((\xi+\eta))} e^{\pi iy_1} \int_0^1 e^{-\pi iy_1} \frac{\chi_{[0,1)}(y_1 - \xi)}{2} \frac{\chi_{[0,1)}(\eta - y_1)}{2} \chi_{(\eta-\xi-1,\eta-2)}(y_1) \; dy_1.$$ 

Therefore,

$$K_{\phi_n}(\xi, \eta) = e^{\frac{2\pi i}{2}((\xi+\eta))} \int_{y_{n-1} \in \mathbb{R}} \left( \int_{y_1 \in \mathbb{R}} \cdots \int_{y_{n-2} \in \mathbb{R}} e^{\pi i (y_1 + \cdots + y_{n-2})} \frac{\chi_{[0,1)}(y_1 - \xi)}{2} \cdot \frac{\chi_{[0,1)}(y_1 + \eta)}{2} \chi_{[0,1)}(y_2 - y_1) \cdots \frac{\chi_{[0,1)}(y_{n-2} + y_{n-1})}{2} \chi_{[0,1)}(y_{n-1} - y_{n-2}) \times$$

$$\text{sinc} \left( \frac{y_1 + y_2}{2} \right) \chi_{[0,1)}(y_2 - y_1) \cdots \frac{\chi_{[0,1)}(y_{n-2} + y_{n-1})}{2} \chi_{[0,1)}(y_{n-1} - y_{n-2}) \times$$

$$\text{sinc} \left( \frac{y_{n-1} + \eta}{2} \right) \chi_{[0,1)}(\eta - y_{n-1}) \; dy_{n-2} \cdots dy_1 \right)$$

$$= e^{\frac{2\pi i}{2}((\xi+\eta))} \int_{y_{n-1} \in \mathbb{R}} e^{-\frac{2\pi i}{2}(\xi+y_{n-1})} K_{\phi_{n-1}}(\xi, y_{n-1}) e^{\pi iy_{n-1}} \frac{\chi_{[0,1)}(y_{n-1} - \xi)}{2} \frac{\chi_{[0,1)}(\eta - y_{n-1})}{2} \chi_{[0,1)}(y_{n-1} - y_{n-2}) \; dy_{n-1}.$$ 

Thus,

$$K_{\phi_n}(\xi, \eta) = e^{\pi iy_1} \int_0^1 e^{-\frac{2\pi i}{2}y_{n-1}} \frac{\chi_{[0,1)}(y_{n-1} - \xi)}{2} \frac{\chi_{[0,1)}(\eta - y_{n-1})}{2} K_{\phi_{n-1}}(\xi, \eta - y_{n-1}) \; dy_{n-1}.$$ 

\[\square\]

**Proposition 3.5.** Given $f \in L^1(\mathbb{R}^2)$, we have the following identities:

$$\int\int_{\mathbb{R}^2} f(x,y)\phi_1(x,y) \; dx \; dy = \int\int_{Q} f(x_1, y_1) \; dx_1 \; dy_1,$$

$$\int\int_{\mathbb{R}^2} f(x,y)\phi_2(x,y) \; dx \; dy = \int\int_{Q^2} e^{\pi i (x_2 y_1 - y_2 x_1)} f(x_1 + x_2, y_1 + y_2) \; dx_1 \; dy_1 \; dx_2 \; dy_2$$

and

$$\int\int_{\mathbb{R}^2} f(x,y)\phi_3(x,y) \; dx \; dy = \int\int_{Q^3} e^{\pi i (x_2 x_3 - x_1 y_1 + y_2 - y_3)} f(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \; dx_1 \; dx_2 \; dy_1 \; dx_3 \; dy_3.$$
For \( n \geq 4 \),
\[
\int_{\mathbb{R}^2} f(x, y)\phi_n(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\pi i[(x+\cdots+x_n)(y+\cdots+y_n)-(x_1+\cdots+x_n)(y_1+\cdots+y_n)]} \times \\
e^{\pi i[(y_2(x_3+\cdots+x_{n-1})-x_2)(y_3+\cdots+y_{n-1})]+(y_3(x_4+\cdots+x_{n-1})-x_3)(y_4+\cdots+y_{n-1})} \cdots (y_{n-3}(x_{n-2}+x_{n-1})-x_{n-3}(y_{n-2}+y_{n-1}))} \times \\
e^{\pi i(y_{n-2}x_{n-1}-x_{n-2}y_{n-1})} f(x_1 + x_2 + \cdots + x_n, y_1 + y_2 + \cdots + y_n) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \cdots \, dx_n \, dy_n.
\]

**Proof.** We have
\[
\int_{\mathbb{R}^2} f(x, y)\phi_1(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} f(x, y) \, dx \, dy = \int_{Q} f(x_1, y_1) \, dx_1 \, dy_1.
\]
Using (3.2) and then applying Fubini’s theorem yields
\[
\int_{\mathbb{R}^2} f(x, y)\phi_1(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y)\left(\int_{u=0}^{1} \int_{v=0}^{1} \phi_1(x-u, y-v) e^{\pi i(uy-vx)} \, dv \, du\right) \, dx \, dy
\]
\[
= \int_{Q} \left(\int_{\mathbb{R}^2} f(x, y)\phi_1(x-u, y-v) e^{\pi i(uy-vx)} \, dx \, dy\right) \, du \, dv.
\]
Applying the change of variables \( x-u = r, y-v = s \), we obtain
\[
\int_{\mathbb{R}^2} f(x, y)\phi_2(x, y) \, dx \, dy = \int_{Q} \left(\int_{\mathbb{R}^2} f(r+u, s+v) \phi_1(r, s) e^{\pi i(su-rv)} \, dr \, ds\right) \, du \, dv
\]
\[
= \int_{Q} \int_{Q} f(r+u, s+v) e^{\pi i(su-rv)} \, dr \, ds \, du \, dv
\]
\[
= \int_{Q^2} e^{\pi i(y_2x_1-y_1x_2)} f(x_1 + x_2, y_1 + y_2) \, dx_1 \, dy_1 \, dx_2 \, dy_2.
\]
Further, for \( n = 3 \)
\[
\int_{\mathbb{R}^2} f(x, y)\phi_3(x, y) \, dx \, dy = \int_{\mathbb{R}^2} e^{\pi i((x_2+x_3)(y_1+y_2)-(x_1+x_2)(y_2+y_3))} f(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \, dx_3 \, dy_3.
\]
Proceeding in this way, we get for \( n \geq 4 \),
\[
\int_{\mathbb{R}^2} f(x, y)\phi_n(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y)\left(\int_{x_n=0}^{1} \int_{y_n=0}^{1} \phi_{n-1}(x-x_n, y-y_n) e^{\pi i(x_ny-y_nx)} \, dy_n \, dx_n\right) \, dx \, dy
\]
\[
= \int_{Q} \left(\int_{\mathbb{R}^2} f(x, y)\phi_{n-1}(x-x_n, y-y_n) e^{\pi i(x_ny-y_nx)} \, dx \, dy\right) \, dx_n \, dy_n
\]
\[
(3.6)
\]
\[
= \int_{Q} \left(\int_{\mathbb{R}^2} T_{(-x_n,-y_n)}^t f(x, y)\phi_{n-1}(x, y) \, dx \, dy\right) \, dx_n \, dy_n.
\]
As (3.6) holds also for \( T_{(-x_n,-y_n)}^t f \), we obtain
\[
\int_{\mathbb{R}^2} T_{(-x_n,-y_n)}^t f(x, y)\phi_{n-1}(x, y) \, dx \, dy = \int_{Q} \left(\int_{\mathbb{R}^2} T_{(-x_n,-y_n)}^t \left(\int_{\mathbb{R}^2} T_{(-x_n,-y_n)}^t f(x, y)\phi_{n-2}(x, y) \, dx \, dy\right) \, dx \, dy\right) \, dx_n \, dy_n-1
\]
Hence, (3.6) becomes
\[
\int_{\mathbb{R}^2} f(x, y)\phi_n(x, y) \, dx \, dy = \int_{Q^2} \left(\int_{\mathbb{R}^2} T_{(-x_n,-y_n)}^t \left(\int_{\mathbb{R}^2} T_{(-x_n,-y_n)}^t f(x, y)\phi_{n-2}(x, y) \, dx \, dy\right) \, dx \, dy\right) \, dx_n \, dy_n \, dx_n \, dy_n-1.
\]
Proceeding in this way, produces
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y) \phi_n(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} T^t_{(x_2, y_2)} T^t_{(x_3, y_3)} \cdots T^t_{(x_n, y_n)} f(x, y) \phi_1(x, y) \, dx \, dy \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} T^t_{(x_2, y_2)} T^t_{(x_3, y_3)} \cdots T^t_{(x_n, y_n)} f(x_1, y_1) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \cdots dx_n \, dy_n
\]
Now, composing all twisted translation operators, yields
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y) \phi_n(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\pi i [(x_3 y_2 - y_3 x_2) + \{x_4 (y_2 + y_3) - y_4 (x_2 + x_3)\} + \cdots + \{x_n (y_2 + \cdots + y_{n-1}) - y_n (x_2 + \cdots + x_{n-1})\} + \cdots + \{x_1 (y_2 + \cdots + y_{n-1}) - y_1 (x_2 + \cdots + x_{n-1})\}]} \\
\times T^t_{(x_2, \cdots, x_{n-1}, y_2 - y_3 \cdots - y_n)} f(x_1, y_1) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \cdots dx_n \, dy_n \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\pi i [(y_2 x_1 - x_1 y_1) + \{y_3 (x_1 + y_2) - x_3 (y_1 + x_2)\} + \cdots + \{y_n (x_1 + \cdots + y_{n-1}) - x_n (y_1 + \cdots + x_{n-1})\} + \cdots + \{y_1 (x_1 + \cdots + y_{n-1}) - x_1 (y_1 + \cdots + x_{n-1})\}]} \\
\times f(x_1 + x_2 + \cdots + x_n, y_1 + y_2 + \cdots + y_n) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \cdots dx_n \, dy_n.
\]
Finally, we can rewrite (3.7) in the form
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y) \phi_n(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\pi i [(x_2 + \cdots + x_n) (y_1 + \cdots + y_{n-1}) - (x_1 + \cdots + x_{n-1}) (y_2 + \cdots + y_n)]} \\
\times e^{\pi i [(y_2 x_1 - x_1 y_1) + \{y_3 (x_1 + y_2) - x_3 (y_1 + x_2)\} + \cdots + \{y_n (x_1 + \cdots + y_{n-1}) - x_n (y_1 + \cdots + x_{n-1})\} + \cdots + \{y_1 (x_1 + \cdots + y_{n-1}) - x_1 (y_1 + \cdots + x_{n-1})\}]} \\
\times f(x_1 + x_2 + \cdots + x_n, y_1 + y_2 + \cdots + y_n) \, dx_1 \, dy_1 \, dx_2 \, dy_2 \cdots dx_n \, dy_n.
\]

**Proposition 3.6.** The twisted B-splines \( \phi_1 \) and \( \phi_2 \) satisfy the following partition-of-unity-line property:
\[
\int_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T^t_{(k,l)} \phi_1(x, y) \, dx \, dy = 1
\]
and
\[
\int_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T^t_{(k,l)} \phi_2(x, y) \, dx \, dy = C_{\phi_2},
\]
where the constant \( C_{\phi_2} \approx 0.000160507 / \pi^2 \). Moreover, the functions \( L_n \) defined by
\[
\int_{\mathbb{R}^2} L_n(u, v) \, du \, dv := \int_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T^t_{(k,l)} \phi_{n+1}(x, y) \, dx \, dy,
\]
can be recursively computed in terms of their lower orders via
\[
L_{n+1}(u, v) = \int_{\mathbb{R}^2} e^{\pi i (u - v)} L_n(u + s, v + t) \, ds \, dt.
\]

**Proof.** Recall that \( \phi_1(x, y) = \chi_{(0,1)}(x) \chi_{(0,1)}(y) \). Then
\[
T^t_{(k,l)} \phi_1(x, y) = e^{\pi i (lx - ky)} \chi_{[k,k+1)}(x) \chi_{[l,l+1)}(y).
\]
For each \( x \in \mathbb{R} \), there exists a unique \( p_x \in \mathbb{Z} \) such that \( p_x \leq x < p_x + 1 \). In fact, \( p_x = \lfloor x \rfloor \). Thus
\[
\sum_{k,l \in \mathbb{Z}} T^t_{(k,l)} \phi_1(x, y) = e^{\pi i (|y - \lfloor x \rfloor |)}.
Hence,
\[
\int_\mathbb{R}^2 \sum_{k,l \in \mathbb{Z}} T_{(k,l)}^t \Phi_1(x, y) \, dx \, dy = \int_\mathbb{R}^2 e^{\pi i (|y| - [x] y)} \, dx \, dy
\]
\[
= \lim_{M \to \infty} \int_{[-M,M]^2} e^{\pi i (|y| - [x] y)} \, dx \, dy
\]
\[
= \lim_{M \to \infty} \int_{x=-M}^M \int_{y=-M}^M e^{\pi i (|y| - [x] y)} \, dy \, dx,
\]
(3.8)
where we used the Fubini-Tonelli theorem to obtain the third line. But
\[
\int_{x=-M}^M \int_{y=-M}^M e^{\pi i (|y| - [x] y)} \, dx \, dy = \sum_{k,l=-M}^{M-1} \int_{x=k}^{k+1} \int_{y=l}^{l+1} e^{\pi i (lx - ky)} \, dy \, dx
\]
(3.9)
say, where

\[
A = \sum_{k,l=-M}^{M-1} \int_{x=k}^{k+1} \int_{y=l}^{l+1} e^{\pi i (lx - ky)} \, dy \, dx,
\]

\[
B = \sum_{k=-M}^{M-1} \int_{x=k}^{k+1} \int_{y=0}^{1} e^{-\pi ik y} \, dy \, dx,
\]

\[
C = \sum_{l=-M}^{M-1} \int_{x=0}^{1} \int_{y=l}^{l+1} e^{\pi il x} \, dy \, dx.
\]

Integrating $A$, we obtain

\[
A = \frac{1}{\pi^2} \sum_{k,l=-M}^{M-1} \frac{1}{kl} (e^{\pi il} - 1)(e^{-\pi ik} - 1)
\]
\[
= \frac{4}{\pi^2} \sum_{k,l=-M}^{M-1} \frac{1}{kl} = \frac{4}{\pi^2} L_M^2,
\]
where

\[
L_M = \sum_{k=-M}^{M-1} \frac{1}{k} = \begin{cases} 0, & M \text{ is even;} \\ -\frac{1}{M}, & M \text{ is odd.} \end{cases}
\]
Since \( L_M \to 0 \) as \( M \to \infty \), we have \( A \to 0 \) as \( M \to \infty \). On the other hand, integrating \( B \) and \( C \), we get
\[
B + C = -\sum_{k=-M \atop k \neq 0}^{M-1} \frac{1}{\pi ik}(e^{-\pi ik} - 1) - \sum_{l=-M \atop l \neq 0}^{M-1} \frac{1}{\pi il}(e^{\pi il} - 1)
\]
\[
= -\sum_{k=-M \atop k \neq 0}^{M-1} \frac{1}{\pi ik}((-1)^k - 1) - \sum_{l=-M \atop l \neq 0}^{M-1} \frac{1}{\pi il}((-1)^l - 1)
\]
\[
= 0.
\]

Making use of (3.9) and (3.10) in (3.8) yields
\[
\iint_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T^i_{(k,l)} \phi_1(x,y) \, dx \, dy = 1 + \lim_{M \to \infty} A
\]
\[
= 1.
\]

Now, consider
\[
\iint_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T^i_{(k,l)} \phi_2(x,y) \, dx \, dy = \frac{1}{\pi^2} \int_0^1 \int_0^1 \sum_{p,q \in \mathbb{Z}} e^{\pi i(uq-vp)}(e^{\pi i(u-p)} - 1)(e^{-\pi i(v-q)} - 1) \frac{1}{(u-p)(v-q)} \, du \, dv.
\]

First, we define the integrals
\[
\mathcal{I}(p, q) := \int_0^1 \int_0^1 e^{\pi i(uq-vp)}(e^{\pi i(u-p)} - 1)(e^{-\pi i(v-q)} - 1) \frac{1}{(u-p)(v-q)} \, du \, dv,
\]

where \( p, q \in \mathbb{Z} \). In case that \( |p|, |q| \notin \{0, 1\} \), the above integral evaluates to
\[
[Ei(i\pi p(q-1)) - Ei(i\pi pq) + Ei(i\pi (p+1)q) - Ei(i\pi (p+1)(q-1))] \times
\]
\[
[Ei(-i\pi (p-1)q) - Ei(-i\pi pq) + Ei(-i\pi p(q+1) - Ei(-i\pi (p-1)(q+1)))]
\]

Setting
\[
F_1(p, q) := Ei(i\pi p(q-1)) - Ei(i\pi pq) + Ei(i\pi (p+1)q) - Ei(i\pi (p+1)(q-1)),
\]
\[
F_2(p, q) := Ei(-i\pi (p-1)q) - Ei(-i\pi pq) + Ei(-i\pi p(q+1) - Ei(-i\pi (p-1)(q+1)),
\]

we note that \( F_2(p, q) = \overline{F_1(q,p)} \). Thus,
\[
\mathcal{I}(p, q) := F_1(p, q)F_1(q,p), \quad |p|, |q| \notin \{0, 1\}.
\]

Employing the identity \( Ei(-ix) = \overline{Ei(ix)} \), for \( x > 0 \), we obtain the following relations: \( \mathcal{I}(-p, -q) = \overline{\mathcal{I}(p, q)} \), \( \mathcal{I}(-p, q) = \overline{\mathcal{I}(q, -p)} \), and \( \mathcal{I}(p, -q) = \overline{\mathcal{I}(-q, p)} \).
For $|p|, |q| \in \{0, 1\}$, the integrals $\mathcal{I}(p,q)$ evaluate to

\[
\begin{align*}
\mathcal{I}(0, 0) &= (\gamma - \text{Ci}(x) + \log \pi)^2 + \text{Si}^2(x); \\
\mathcal{I}(1, 0) &= (\text{Ei}(-2i\pi) + \text{Ei}(-i\pi) + \log 2)(\text{Ci}(\pi) - i\text{Si}(\pi) - \gamma - \log \pi); \\
\mathcal{I}(0, 1) &= \overline{\mathcal{I}(1, 0)}; \\
\mathcal{I}(1, 1) &= (\text{Ei}(i\pi) - \text{Ei}(2i\pi) + \log 2)(\text{Ei}(i\pi) - \text{Ei}(2i\pi) + \log 2); \\
\mathcal{I}(0, -1) &= \mathcal{I}(1, 0); \\
\mathcal{I}(-1, 0) &= \mathcal{I}(0, 1) = \overline{\mathcal{I}(1, 0)}; \\
\mathcal{I}(-1, -1) &= \mathcal{I}(1, 1); \\
\mathcal{I}(1, -1) &= (\text{Ei}(-i\pi) - 2\text{Ei}(-2i\pi) + \text{Ei}(-4i\pi))(\text{Ci}(\pi) + i\text{Si}(\pi) - \gamma - \log \pi); \\
\mathcal{I}(-1, 1) &= \overline{\mathcal{I}(1, -1)},
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{I}(p, 0) &= (\text{Ei}(-i\pi(p - 1)) - \text{Ei}(-i\pi p) + \log \left(\frac{p}{p - 1}\right)) \left(-\text{Ei}(-i\pi p) + \text{Ei}(-i\pi(p + 1)) + \log \left(\frac{p}{p + 1}\right)\right); \\
\mathcal{I}(p, 1) &= (\text{Ei}(-i\pi(p - 1)) + \text{Ei}(-2i\pi(p - 1)) + \text{Ei}(-i\pi p) - \text{Ei}(-2i\pi p)) \times \left(\text{Ei}(i\pi p) - \text{Ei}(i\pi(p + 1)) + \log \left(\frac{1 + p}{p}\right)\right); \\
\mathcal{I}(p, -1) &= (\text{Ei}(-i\pi p) - \text{Ei}(-2i\pi p) - \text{Ei}(-i\pi(p + 1)) + \text{Ei}(-2i\pi(p + 1)) \times \left(-\text{Ei}(i\pi(p - 1)) + \text{Ei}(i\pi p) + \log \left(\frac{p - 1}{p}\right)\right); \\
\mathcal{I}(0, q) &= \overline{\mathcal{I}(q, 0)}; \\
\mathcal{I}(1, q) &= \overline{\mathcal{I}(q, 1)}; \\
\mathcal{I}(-1, q) &= \overline{\mathcal{I}(q, -1)}.
\end{align*}
\]

Finally, for $p \geq 2$ and $q \leq -2$,

\[
\begin{align*}
\mathcal{I}(p, q) &= (\text{Ei}(i\pi(p - 1)(q - 1)) - \text{Ei}(i\pi p(q - 1)) - \text{Ei}(i\pi(p - 1)q) + \text{Ei}(i\pi pq) \times (\text{Ei}(i\pi pq) - \text{Ei}(-i\pi(p + 1)q) - \text{Ei}(-i\pi(p + 1)q) + \text{Ei}(-i\pi(p + 1)(q + 1))).
\end{align*}
\]

Next, we will show that $\sum_{p,q \in \mathbb{Z}} \mathcal{I}(p,q)$ converges as an iterated sum which allows the interchange of integrals and sums in (3.11).

For this purpose, we need to write $\sum_{p,q \in \mathbb{Z}} \mathcal{I}(p,q)$ as follows.

\[
\begin{align*}
\sum_{p,q \in \mathbb{Z}} \mathcal{I}(p,q) &= \sum_{|p|, |q| \leq 1} \mathcal{I}(p,q) + \sum_{|p| \geq 2} \mathcal{I}(p, 0) + \sum_{|p| \geq 2} \mathcal{I}(p, 1) + \sum_{|p| \geq 2} \mathcal{I}(p, -1) \\
&\quad + \sum_{|q| \geq 2} \mathcal{I}(0, q) + \sum_{|q| \geq 2} \mathcal{I}(1, q) + \sum_{|q| \geq 2} \mathcal{I}(-1, q) \\
&\quad + \sum_{p \geq 2, q \leq -2} \mathcal{I}(p, q) + \sum_{p \leq 2, q \geq 2} \mathcal{I}(p, q) + \sum_{|p|, |q| \geq 2} \mathcal{I}(p,q).
\end{align*}
\]
Using the afore-mentioned identities between the integrals $I(p, q)$, the above sums reduce to

$$\sum_{p,q \in \mathbb{Z}} I(p, q) = \sum_{|p|,|q| \leq 1} I(p, q) + 2 \sum_{p \geq 2} 2 \text{Re} I(p, 0) + 2 \sum_{p \geq 2} 2 \text{Re} I(p, 1) + 2 \sum_{p \geq 2} 2 \text{Re} I(p, -1) + \sum_{p \geq 2, q \leq -2} 2 \text{Re} I(p, q) + \sum_{p,q \geq 2} 2 \text{Re} I(p, q).$$

Recalling that $\text{Ei}(ix) = \text{Ci}(x) + i(\text{Si}(x) - \frac{x}{2})$ [4], we express $I(p, 0)$, $I(p, 1)$, $I(p, -1)$, and $I(p, q)$ ($p \geq 2$, $q \leq -2$) in terms of $\text{Ci}$ and $\text{Si}$, and take the real part. This yields,

$$\text{Re} I(p, 0) = \left( \text{Ci}(\pi(p - 1)) - \text{Ci}(\pi p) + \log \left( \frac{p}{p - 1} \right) \right) \left( \text{Ci}(\pi(p + 1)) - \text{Ci}(\pi p) + \log \left( \frac{p}{p + 1} \right) \right) - \left( \text{Si}(\pi p) - \text{Si}(\pi(p - 1)) \right) \left( \text{Si}(\pi p) - \text{Si}(\pi(p + 1)) \right);$$

$$\text{Re} I(p, 1) = \left( \text{Ci}(\pi p) - \text{Ci}(\pi(p - 1)) + \text{Ci}(2\pi(p - 1)) - \text{Ci}(2\pi p) \right) \left( \text{Ci}(\pi(p + 1)) - \text{Ci}(\pi p) + \log \left( \frac{p + 1}{p} \right) \right) - \left( \text{Si}(\pi p) - \text{Si}(\pi(p - 1)) + \text{Si}(2\pi(p - 1)) - \text{Si}(2\pi p) \right) \left( \text{Si}(\pi p) - \text{Si}(\pi(p + 1)) \right);$$

$$\text{Re} I(p, -1) = \left( \text{Ci}(\pi p) - \text{Ci}(2\pi p) + \text{Ci}(2\pi(p + 1)) - \text{Ci}(\pi(p + 1)) \right) \left( \text{Ci}(\pi(p - 1)) - \text{Ci}(\pi p) + \log \left( \frac{p - 1}{p} \right) \right) - \left( \text{Si}(\pi p) - \text{Si}(2\pi p) + \text{Si}(2\pi(p + 1)) - \text{Si}(\pi(p + 1)) \right) \left( \text{Si}(\pi p) - \text{Si}(\pi(p - 1)) \right);$$

$$\text{Re} I(p, q) = \left( \text{Ci}(\pi(p - 1)(q - 1)) - \text{Ci}(\pi q(q - 1)) + \text{Ci}(\pi pq) - \text{Ci}(\pi(p - 1)q) \right) \times \left( \text{Ci}(\pi(p + 1)(q + 1)) - \text{Ci}(\pi(p + 1)q) + \text{Ci}(\pi pq) - \text{Ci}(\pi(p + 1)q + 1) \right) + \left( \text{Si}(\pi(p - 1)(q - 1)) - \text{Si}(\pi q(q - 1)) + \text{Si}(\pi pq) - \text{Si}(\pi(p - 1)q) \right) \times \left( \text{Si}(\pi(p + 1)(q + 1)) - \text{Si}(\pi(p + 1)q) + \text{Si}(\pi pq) - \text{Si}(\pi(p + 1)q + 1) \right).$$

Similarly, we write $F_1(p, q)$ in terms of $\text{Ci}$ and $\text{Si}$ and obtain

$$F_1(p, q) = [\text{Ci}(\pi p(q - 1)) - \text{Ci}(\pi pq) + \text{Ci}(\pi(p + 1)q) - \text{Ci}(\pi(p + 1)(q - 1))] + i [\text{Si}(\pi p(q - 1)) - \text{Si}(\pi pq) + \text{Si}(\pi(p + 1)q) - \text{Si}(\pi(p + 1)(q - 1))]
= u(p, q) + iv(p, q).$$

The sum $\sum_{p,q \geq 2} 2 \text{Re} I(p, q)$ has then the form

$$\sum_{p,q \geq 2} 2 \text{Re} I(p, q) = \sum_{p,q \geq 2} 2 [u(p, q)u(q, p) + v(p, q)v(q, p)].$$

The main idea to show convergence of the infinite sums in (3.12) is to estimate the $\text{Ci}$ and $\text{Si}$ terms. To this end, we use the following asymptotic representations of these two functions found
in [7].

\[ (3.13a) \quad \text{Ci}(x) = \frac{\sin x}{x} P(x) - \frac{\cos x}{x} Q(x), \]

\[ (3.13b) \quad \frac{\pi}{2} - \text{Si}(x) = \frac{\cos x}{x} P(x) + \frac{\sin x}{x} Q(x), \]

where

\[ (3.14a) \quad P(x) = \sum_{k=0}^{n} \frac{(-1)^k (2k)!}{x^{2k}} + O(|x|^{-2n-2}), \]

\[ (3.14b) \quad Q(x) = \sum_{k=0}^{n} \frac{(-1)^k (2k + 1)!}{x^{2k+1}} + O(|x|^{-2n-3}). \]

In addition, we need estimates on the logarithmic terms appearing in the above formulas for the real parts of \( I(p, 0), I(p, 1), \) and \( I(p, -1). \) We summarize these estimates in the following lemma.

**Lemma 3.7.** Let \( p \geq 2. \) Then

(a) \( \log \left( \frac{p}{p-1} \right) \leq \frac{1}{p-1} \leq \frac{2}{p}; \)

(b) \( \left| \log \left( \frac{p}{p+1} \right) \right| \leq \frac{1}{p-1}; \)

(c) \( \left| \log \left( \frac{p}{p-1} \right) \log \left( \frac{p}{p+1} \right) \right| \leq \frac{1}{(p-1)^2}. \)

**Proof.** These statements follow immediately from rewriting the inequalities in terms of the exponential function and using its power series expansion. \( \square \)

Using (3.13a) and (3.13b), we rewrite \( \text{Re} I(p, 0) \) in the following way.

\[
\text{Re} I(p, 0) = \left[ \frac{(-1)^p}{\pi(p-1)} Q(\pi(p-1)) - \frac{(-1)^{p+1}}{\pi p} Q(\pi p) + \log \left( \frac{p}{p-1} \right) \right] \times \\
\left[ \frac{(-1)^p}{\pi(p+1)} Q(\pi(p+1)) - \frac{(-1)^{p+1}}{\pi p} Q(\pi p) + \log \left( \frac{p}{p+1} \right) \right] \\
- \left[ \frac{(-1)^{p-1}}{\pi(p-1)} P(\pi(p-1)) - \frac{(-1)^p}{\pi p} P(\pi p) \right] \left[ \frac{(-1)^{p+1}}{\pi(p+1)} P(\pi(p+1)) - \frac{(-1)^p}{\pi p} P(\pi p) \right]
\]

Therefore,

\[
|\text{Re} I(p, 0)| \leq \left[ \frac{|Q(\pi(p-1))|}{\pi(p-1)} + \frac{|Q(\pi p)|}{\pi p} + \log \left( \frac{p}{p-1} \right) \right] \left[ \frac{|Q(\pi(p+1))|}{\pi(p+1)} + \frac{|Q(\pi p)|}{\pi p} + \log \left( \frac{p}{p+1} \right) \right] \\
+ \left[ \frac{|P(\pi(p-1))|}{\pi(p-1)} + \frac{|P(\pi p)|}{\pi p} \right] \left[ \frac{|P(\pi(p+1))|}{\pi(p+1)} + \frac{|P(\pi p)|}{\pi p} \right]
\]

By (3.14a) and (3.14b), there exist positive constants \( c_1, \ldots, c_4 \) such that

\[
|\text{Re} I(p, 0)| \leq \left[ \frac{c_1}{p-1} + \log \left( \frac{p}{p-1} \right) \right] \left[ \frac{c_2}{p} + \log \left( \frac{p}{p+1} \right) \right] + \left[ \frac{c_3}{p-1} \right] \left[ \frac{c_4}{p} \right] \\
\leq \frac{c_5}{p(p-1)} + \frac{c_6}{p^2} + \frac{c_7}{(p-1)^2} \leq \frac{c_8}{(p-1)^2}.
\]

Here, we used the statements in the lemma to obtain the last line.
In a similar fashion, one shows that $|\text{Re} \mathcal{I}(p, \pm 1)|$ is bounded above by $\frac{c\pi}{(p-1)^2}$, for positive constants $c_\pm$. Therefore, the single infinite sums in (3.12) are all bounded above by a convergent series of the form $c \sum_{p=2}^{\infty} (p-1)^{-2}$, $c > 0$, and are thus themselves convergent.

It remains to establish the convergence of the double infinite sums $\text{Re} \mathcal{I}(p, q)$ ($p \geq 2, q \leq -2$) and $\text{Re} \mathcal{I}(p, q)$ ($p, q \geq 2$). This, however, can be done by applying essentially the same arguments as above for the single infinite sums. One thus obtains

$$|\text{Re} \mathcal{I}(p, q)| \leq \frac{c}{(p-1)^2(q-1)^2}, \quad c > 0,$$

for either range of parameters $p$ and $q$. Hence, the double infinite series in (3.12) are also convergent.

A numerical evaluation of the first 1,000 terms in $\sum_{p,q \in \mathbb{Z}} \mathcal{I}(p, q)$ gives a value of approximately $0.000160507$. Hence,

$$\int \int_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T_{(k,l)}(x, y) \, dx \, dy = C_{\phi_2},$$

where $C_{\phi_2} \approx 0.000160507/\pi^2$.

To verify the recursive formulas for the functions $L_n$, note that for $n \geq 1$, we have that

$$\sum_{k,l \in \mathbb{Z}} T_{(k,l)}(x, y) \phi_{n+1}(x, y) = \sum_{k,l \in \mathbb{Z}} e^{\pi i (lx - ky)} \phi_{n+1}(x - k, y - l)$$

$$= \sum_{k,l \in \mathbb{Z}} e^{\pi i (lx - ky)} \int_{u=0}^{1} \int_{v=0}^{1} \phi_n(x - k - u, y - l - v) e^{\pi i (u(y-l)-v(x-k))} \, dv \, du$$

$$= \int \int_{Q} e^{\pi i (uy-vx)} \sum_{k,l \in \mathbb{Z}} e^{\pi i ((x-u)-k(y-v))} \phi_n(x - u - k, y - v - l) \, du \, dv$$

$$= \int \int_{Q} e^{\pi i (uy-vx)} \sum_{k,l \in \mathbb{Z}} T_{(k,l)}(x - u, y - v) \, du \, dv,$$

and, therefore,

$$\int \int_{\mathbb{R}^2} \sum_{k,l \in \mathbb{Z}} T_{(k,l)}(x, y) \phi_{n+1}(x, y) \, dx \, dy = \int \int_{Q} \left( \int \int_{\mathbb{R}^2} e^{\pi i (uy-vx)} \sum_{k,l \in \mathbb{Z}} T_{(k,l)}(x - u, y - v) \, dx \, dy \right) \, du \, dv$$

$$= \int \int_{Q} L_n(u, v) \, du \, dv.$$

Thus, for $n \geq 1$,

$$L_{n+1}(u, v) = \int \int_{\mathbb{R}^2} e^{\pi i (uy-vx)} \sum_{k,l \in \mathbb{Z}} T_{(k,l)}(x - u, y - v) \, dx \, dy$$

$$= \int \int_{\mathbb{R}^2} e^{\pi i (uy-vx)} \sum_{k,l \in \mathbb{Z}} e^{\pi i ((x-u)-k(y-v))} \phi_{n+1}(x - u - k, y - v - l) \, dx \, dy$$

$$= \int \int_{\mathbb{R}^2} e^{\pi i (uy-vx)} \sum_{k,l \in \mathbb{Z}} e^{\pi i ((x-u)-k(y-v))} \left( \int \int_{Q} \phi_n(x - u - k - s, y - v - l - t) \times \right.$$

$$\left. e^{\pi i (s(y-v-l)-t(x-u-k))} \, ds \, dt \right) \, dx \, dy.$$
Now, applying Fubini’s theorem yields

\[ L_{n+1}(u, v) = \int_Q \sum_{k,l \in \mathbb{Z}} \left( \int_{\mathbb{R}^2} e^{2\pi i (uy-vx)} e^{2\pi i(s(y-v)-(t(x-u))} e^{2\pi i(t(x-u)-k(y-v)-t)} \right) dsdt \]

\[ \phi_n(x - u - k - s, y - v - l - t) \ dx \ dy \]

\[ = \int_Q \sum_{k,l \in \mathbb{Z}} \left( \int_{\mathbb{R}^2} e^{2\pi i (uy-vx)} e^{2\pi i(s(y-v)-(t(x-u))} T_{(k,l)}^t \phi_n(x - u - s, y - v - t) \ dx \ dy \right) dsdt \]

\[ = \int_Q e^{2\pi i (ut-ve)} \sum_{k,l \in \mathbb{Z}} T_{(k,l)}^t \phi_n(x - u - s, y - v - t) \ dx \ dy \ ds \ dt \]

\[ = \int_Q e^{2\pi i (ut-ve)} L_n(u + s, v + t) \ ds \ dt. \]

\[ \square \]

4. System of Twisted Translates of \( \phi_n \)

First, we prove that \( \{ T_{(k,l)}^t \phi_1 : k, l \in \mathbb{Z} \} \) forms an orthonormal system in \( L^2(\mathbb{R}^2) \). Consider

\[ \langle T_{(k,l)}^t \phi_1, T_{(k',l')}^t \phi_1 \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} T_{(k,l)}^t \phi_1(x, y) T_{(k',l')}^t \phi_1(x, y) \ dx \ dy \]

\[ = \int_{\mathbb{R}^2} e^{2\pi i(x(l-1)-y(k-k'))} \phi_1(x - k, y - l) \ \phi_1(x - k', y - l') \ dx \ dy. \]

Applying a change of variables, we get

\[ \langle T_{(k,l)}^t \phi_1, T_{(k',l')}^t \phi_1 \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{2\pi i(k'-l'k)} e^{2\pi i(x(l-1)-y(k-k'))} \phi_1(x, y) \ \phi_1(x + k - k', y + l - l') \ dx \ dy \]

\[ = (-1)^{k'-l'k} \int_0^1 \int_0^1 e^{2\pi i(x(l-1)-y(k-k'))} \chi_{[0,1]}(x + k - k') \times \]

\[ \chi_{[0,1]}(y + l - l') \ dy \ dx \]

\[ = (-1)^{k'-l'k} \int_0^1 \int_0^1 e^{2\pi i(x(l-1)-y(k-k'))} \chi_{[k'-k,k+k+1]}(x) \times \]

\[ \chi_{[l'-l,l'+l+1]}(y) \ dy \ dx \]

proving that \( \{ T_{(k,l)}^t \phi_1 : k, l \in \mathbb{Z} \} \) forms an orthonormal system.

**Theorem 4.1.** Let \( \phi_2 \) denote the second order twisted B-spline. Then \( \{ T_{(k,l)}^t \phi_2 : k, l \in \mathbb{Z} \} \) is a Riesz basis for the twisted shift-invariant space \( V^t(\phi_2) \).

**Proof.** Recall that \( V^t(\phi_2) = \text{span} \{ T_{(k,l)}^t \phi_2 : k, l \in \mathbb{Z} \} \). Hence is enough to prove that the collection \( \{ T_{(k,l)}^t \phi_2 : k, l \in \mathbb{Z} \} \) is a Riesz sequence. Let \( G \) denote the Gramian associated with the system \( \{ T_{(k,l)}^t \phi_2 : k, l \in \mathbb{Z} \} \) and let \( \{ c_{k,l} \} \in \ell^2(\mathbb{Z}^2) \). We need to prove that there exist \( A,B > 0 \) such that

\[ A \| \{ c_{k,l} \} \|_{\ell^2(\mathbb{Z}^2)}^2 \leq \langle G \{ c_{k,l} \}, \{ c_{k,l} \} \rangle_{\ell^2(\mathbb{Z}^2)} \leq B \| \{ c_{k,l} \} \|_{\ell^2(\mathbb{Z}^2)}^2. \]
Consider
\[ \langle G\{c_k,l\}, \{c_k,l\} \rangle_{\ell^2(\mathbb{Z}^2)} = \sum_{k,k',l,l' \in \mathbb{Z}} c_{k,l} \overline{c_{k',l'}} \left\langle T_{(k,l)}^t \phi_2, T_{(k',l')}^t \phi_2 \right\rangle_{L^2(\mathbb{R}^2)} \]
\[ = \sum_{k,k',l,l' \in \mathbb{Z}} c_{k,l} \overline{c_{k',l'}} \int_{\mathbb{R}^2} e^{\pi i(x(-k'l') - y(k-k'))} \phi_2(x - k, y - l) \overline{\phi_2(x - k', y - l')} \, dx \, dy \]
\[ = \sum_{k,k',l,l' \in \mathbb{Z}} c_{k,l} \overline{c_{k',l'}} (-1)^{k'l'-l'k} \int_{\mathbb{R}^2} e^{\pi i(x(-l') - y(k-k'))} \phi_2(x + k' - k, y + l' - l) \times \phi_2(x, y) \, dx \, dy, \]

after applying a change of variables. As the support of \( \phi_2 = [0, 2] \times [0, 2] \), it can be easily shown that in the sum for \( k, l \in \mathbb{Z} \) only three values of \( k \) and \( l \) survive. Namely,
\[ \begin{align*}
    k &= k' - 1, k', k' + 1, \\
    l &= l' - 1, l', l' + 1.
\end{align*} \]

Thus, we can write
\[ \langle G\{c_k,l\}, \{c_k,l\} \rangle_{\ell^2(\mathbb{Z}^2)} = S_1 + S_2 + \cdots + S_9, \]

depending upon the above choices of \( k \) and \( l \). Now
\[ S_1 = \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'-1} \overline{c_{k',l'}} \int_{x=0}^2 \int_{y=0}^2 e^{\pi i(y-x)} \phi_2(x + 1, y + 1) \phi_2(x, y) \, dy \, dx. \]

In the integral limits, only \((x, y) \in (0, 1) \times (0, 1)\) contributes and \((x, y) \in (0, 1) \times (1, 2)\), \((x, y) \in (1, 2) \times (0, 1)\) and \((x, y) \in (1, 2) \times (1, 2)\) cannot contribute using the supports of intersection of \( \phi_2(x+1, \cdot +1) \) and \( \phi_2(\cdot, \cdot) \). Thus, from [3.1], we obtain
\[ S_1 = \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'-1} \overline{c_{k',l'}} \frac{4}{\pi^4} \int_{x=0}^1 \int_{y=0}^1 e^{\pi i(y-x)} \frac{\cos(\pi(y-x)) - \cos(\pi(1-xy))}{xy(x+1)(y+1)} \times (1 - \cos(\pi xy)) \, dy \, dx. \]

In a similar way, we get that
\[ S_2 = \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'} \overline{c_{k',l'}} \int_{x=0}^1 \int_{y=0}^1 e^{-\pi i(y-x)} \phi_2(x + 1, y + 1) \phi_2(x, y) \, dy \, dx \]
\[ = S_1, \]

\[ S_3 = \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'} \overline{c_{k',l'}} \phi_2(x + 1, y) \phi_2(x, y) \, dx \, dy \]
\[ = \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'} \overline{c_{k',l'}} \frac{4}{\pi^4} \left( \int_{x=0}^1 \int_{y=0}^1 \frac{e^{\pi iy}}{x(x+1)y^2} \left( \cos(\pi xy) - \cos(\pi y) \right)(1 - \cos(\pi xy)) \, dy \, dx + \right. \]
\[ \left. \int_{x=0}^1 \int_{y=1}^2 \frac{e^{\pi iy}}{x(x+1)y^2} \left[ \cos(\pi(x+y+1)) - \cos(\pi x + y + 1) \right] \cos(\pi x(y-1)) \cos(\pi x) \, dy \, dx \right), \]
Notice that the integrand in $S_5$ is same as the integrand in $\overline{S_3}$ by interchanging the roles of $x$ and $y$.

\[
S_6 = \sum_{k', l' \in \mathbb{Z}} c_{k', l'} e^{\pi i (k + l')} \int_{\mathbb{R}^2} e^{-\pi i (x + y)} \phi_2(x + 1, y - 1) \overline{\phi_2(x, y)} \, dx \, dy
\]

Notice that the integrand in $S_6$ is same as the integrand in $S_3$ with $x$ and $y$ interchanged.

\[
S_7 = \sum_{k', l' \in \mathbb{Z}} c_{k', l' + 1} \overline{c_{k', l'}} e^{\pi i (k + l')} \int_{\mathbb{R}^2} e^{-\pi i (x + y)} \phi_2(x + 1, y - 1) \overline{\phi_2(x, y)} \, dx \, dy
\]

\[
S_8 = \sum_{k', l' \in \mathbb{Z}} c_{k', l' + 1} \overline{c_{k', l'}} e^{-\pi i (k + l')} \int_{\mathbb{R}^2} e^{-\pi i (x + y)} \phi_2(x - 1, y + 1) \overline{\phi_2(x, y)} \, dx \, dy
\]

Notice that the integrand in $S_8$ is same as the integrand in $S_7$ with $x$ and $y$ interchanged.
Thus,
\[
S_9 = \sum_{k',l' \in \mathbb{Z}} |c_{k',l'}|^2 \int_{\mathbb{R}^2} |\phi_2(x,y)|^2 \, dx \, dy
\]
\[
= \sum_{k',l' \in \mathbb{Z}} |c_{k',l'}|^2 \frac{4}{\pi^4} \left( \int_{x=0}^{1} \int_{y=0}^{1} \frac{1}{x^2 y^2} [1 - \cos(\pi x)]^2 \, dy \, dx + \int_{x=1}^{2} \int_{y=1}^{2} \frac{1}{x^2 y^2} [\cos(\pi x(y - 1)) - \cos(\pi x)]^2 \, dy \, dx \right.
\]
\[
\quad \left. + \int_{x=1}^{2} \int_{y=1}^{2} \frac{1}{x^2 y^2} [\cos(\pi(x - y)) - \cos(\pi(x + y - xy))]^2 \, dy \, dx \right) .
\]

Thus,
\[
\langle G\{c_{k,l}\}, \{c_{k,l}\} \rangle_{\ell^2(\mathbb{Z})} = \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'-1} \overline{c_{k',l'}} \, e^{\pi i (l'-k')} \frac{1}{\pi^4} I_1 + \overline{S_1} + \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'} \overline{c_{k',l'}} \, e^{\pi i l'} \frac{1}{\pi^4} I_3 + \overline{S_3} + \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'-1} \overline{c_{k',l'}} \, e^{-\pi i k'} \frac{1}{\pi^4} I_5 + \overline{S_5} + \sum_{k',l' \in \mathbb{Z}} c_{k'-1,l'+1} \overline{c_{k',l'}} \, e^{\pi i (k'+l')} \frac{1}{\pi^4} I_7 + \overline{S_7} + \sum_{k',l' \in \mathbb{Z}} |c_{k',l'}|^2 \frac{1}{\pi^4} I_9.
\]

As the values of the integral \( I_5 \) and \( I_6 \) are same as the integral in \( S_3 \) and \( I_3 \) respectively, it is enough if one evaluates the integrals \( I_1, I_3, I_7 \) and \( I_9 \). These integrals are numerically computed and the values are the following:
\[
I_1 = 0.531003 - i \, 0.467628
\]
\[
I_3 = -1.97877 + i \, 0.56791
\]
\[
I_7 = 0.906616 - i \, 0.390131
\]
\[
I_9 = 14.3661.
\]

Hence \( |I_1| = 0.707559, |I_3| = 2.05865 \) and \( |I_7| = 0.986993 \). Now
\[
\langle G\{c_{k,l}\}, \{c_{k,l}\} \rangle_{\ell^2(\mathbb{Z})} \leq \frac{1}{\pi^4} \|\{c_{k,l}\}\|_{\ell^2(\mathbb{Z})}^2 (2|I_1| + 4|I_3| + 2|I_7| + |I_9|)
\]
\[
= \frac{1}{\pi^4} \|\{c_{k,l}\}\|_{\ell^2(\mathbb{Z})}^2 (11.6237 + 14.3661)
\]
\[
= \frac{25.9898}{\pi^4} \|\{c_{k,l}\}\|_{\ell^2(\mathbb{Z})}^2 .
\]

On the other hand
\[
\langle G\{c_{k,l}\}, \{c_{k,l}\} \rangle_{\ell^2(\mathbb{Z})} \geq \frac{1}{\pi^4} \|\{c_{k,l}\}\|_{\ell^2(\mathbb{Z})}^2 \{I_9 - (2|I_1| + 4|I_3| + 2|I_7|)\}
\]
\[
= \frac{1}{\pi^4} \|\{c_{k,l}\}\|_{\ell^2(\mathbb{Z})}^2 (14.3661 - 11.6237)
\]
\[
= \frac{2.7424}{\pi^4} \|\{c_{k,l}\}\|_{\ell^2(\mathbb{Z})}^2 .
\]

Thus \( \{T_{(k,l)}^l \phi_2 : k, l \in \mathbb{Z}\} \) is a Riesz sequence. \( \square \)
We conjecture that \( \{ T_{(k,l)}^t \phi_n : k, l \in \mathbb{Z} \} \) is also a Riesz sequence for any twisted \( B \)-spline of order \( n \). In fact, we show that there exists a constant \( B > 0 \) such that

\[
\left\| \sum_{k,l \in \mathbb{Z}} c_{k,l} T_{(k,l)}^t \phi_n \right\|_{L^2(\mathbb{R}^2)}^2 \leq B \left\| \{ c_{k,l} \} \right\|_{l^2(\mathbb{Z}^2)}^2.
\]

The proof follows by induction on \( n \). By Theorem 4.1, the result is true for \( n = 2 \). Assume that the result is true for \( n \). We need to show the result for \( n + 1 \). Consider a finite sequence \( \{ c_{k,l} \} \) in \( l^2(\mathbb{Z}^2) \) and compute

\[
\left\| \sum_{k,l \in \mathbb{Z}} c_{k,l} T_{(k,l)}^t \phi_{n+1} \right\|_{L^2(\mathbb{R}^2)}^2 = \left\| \sum_{k,l \in \mathbb{Z}} c_{k,l} W(T_{(k,l)}^t \phi_{n+1}) \right\|_{B_2(L^2(\mathbb{R}))}^2
\]

\[
= \left\| \sum_{k,l \in \mathbb{Z}} c_{k,l} \pi(k,l) W(\phi_{n+1}) \right\|_{B_2(L^2(\mathbb{R}))}^2
\]

\[
= \left\| \left( \sum_{k,l \in \mathbb{Z}} c_{k,l} \pi(k,l) W(\phi_n) \right) W(\phi_1) \right\|_{B_2(L^2(\mathbb{R}))}^2
\]

\[
\leq \left\| \sum_{k,l \in \mathbb{Z}} c_{k,l} \pi(k,l) W(\phi_n) \right\|_{B_2(L^2(\mathbb{R}))}^2 \left\| W(\phi_1) \right\|_{B_2(L^2(\mathbb{R}))}^2
\]

\[
= \left\| \sum_{k,l \in \mathbb{Z}} c_{k,l} W(T_{(k,l)}^t \phi_n) \right\|_{B_2(L^2(\mathbb{R}))}^2 \left\| \phi_1 \right\|_{L^2(\mathbb{C})}^2
\]

\[
\leq B \left\| \{ c_{k,l} \} \right\|_{l^2(\mathbb{Z}^2)}^2.
\]

So we obtain an upper bound.

The conjecture is mainly for obtaining the lower bound. In the classical case of classical splines \( B_n \) on \( \mathbb{R} \), the result follows using the fact that \( \{ T_k B_n : k \in \mathbb{Z} \} \) is a Riesz sequence iff there exist constants \( A, B > 0 \) such that

\[
A \leq \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + k)|^2 \leq B, \quad \text{for a.e. } \xi \in \mathbb{R}.
\]

In the case of twisted shift-invariant spaces, we have a similar result (see Theorem 5.1 in [12]) but under the requirement that “\( \phi \) satisfies condition \( C \)””. However, the splines \( \phi_n \) do not satisfy this condition for \( n \geq 2 \). This can be easily checked for \( \phi_2 \). Thus, we leave the conjecture to the interested reader.

5. Twisted Haar Wavelet

Recall that \( \phi_1(x, y) = \chi_{[0,1]}(x) \chi_{[0,1]}(y) \) and \( \{ T_{(k,l)}^t \phi_1 : k, l \in \mathbb{Z} \} \) is an orthonormal system. Our aim is to obtain a wavelet \( \psi_1 \) by treating \( \phi_1 \) as a scaling function generating a multiresolution analysis on \( \mathbb{C} = \mathbb{R}^2 \). We shall call the resulting wavelet \( \psi_1 \) a twisted Haar wavelet. Using the twisted translation and dilation on \( \mathbb{R}^2 \), we shall define a multiresolution analysis in \( L^2(\mathbb{R}^2) \). In fact, a
multiresolution analysis has been studied for the Heisenberg group \( \mathbb{H} \) in [10]. Keeping in mind this multiresolution analysis for \( L^2(\mathbb{R}) \) and \( L^2(\mathbb{H}) \), we provide the following definition for \( L^2(\mathbb{R}^2) \).

We recall the definition of twisted translation of a function \( \phi \in L^2(\mathbb{R}^2) \) from (2.2). The dilation is defined to be the unitary operator on \( L^2(\mathbb{R}^2) \) given as follows. For \( a > 0 \),

\[
D_a f(x, y) := a f(ax, ay),
\]

for \( x, y \in \mathbb{R} \).

**Definition 5.1** (MRA). We say that a sequence of closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) of \( L^2(\mathbb{R}^2) \) forms a twisted multiresolution analysis for \( L^2(\mathbb{R}^2) \) if the following conditions are satisfied.

(i) \( V_j \subset V_{j+1}, \forall j \in \mathbb{Z} \).

(ii) \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^2) \) and \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \).

(iii) \( f \in V_j \iff D_2 f \in V_{j+1}, \forall j \in \mathbb{Z} \).

(iv) There exists a function \( \phi \in V_0 \) such that \( \{T_{(k,l)}^t \phi : k, l \in \mathbb{Z}\} \) forms an orthonormal basis for \( V_0 \). The function \( \phi \) is called the scaling function associated with the twisted multiresolution analysis.

Let \( W_0 \) be the orthogonal complement of \( V_0 \) in \( V_1 \). As in the classical case, if we can find a \( \psi \in W_0 \) such that \( \{T_{(k,l)}^t \psi : k, l \in \mathbb{Z}\} \) forms an orthonormal basis for \( W_0 \) then by defining \( V_j \) to be the orthogonal complement of \( V_j \) in \( V_{j+1} \), one can show that \( L^2(\mathbb{R}^2) = \bigoplus_{j \in \mathbb{Z}} W_j \).

**Definition 5.2.** Let \( \psi \in L^2(\mathbb{R}^2) \). If \( \{D_2 T_{(k,l)}^t \psi : j, k, l \in \mathbb{Z}\} \) is an orthonormal basis for \( L^2(\mathbb{R}^2) \), then \( \psi \) is called a twisted wavelet for \( L^2(\mathbb{R}^2) \).

Now, our aim is to find \( \psi \) explicitly for the scaling function \( \phi_1 \) using the twisted multiresolution analysis. Towards this end, we prove the following result.

**Theorem 5.3.** Let \( \phi \in L^2(\mathbb{R}^2) \) such that

(i) \( \text{supp } \phi \subset [0, 1] \times [0, 1] \).

(ii) \( \{T_{(k,l)}^t \phi : k, l \in \mathbb{Z}\} \) forms an orthonormal system.

(iii) \( \{V_j, \phi : j \in \mathbb{Z}\} \) generates a twisted multiresolution analysis for \( L^2(\mathbb{R}^2) \).

Let \( c_{k,l} := \langle D_{1/2} \phi, T_{(k,l)}^t \phi \rangle \). Define

\[
\psi := \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} c_{k,l} D_2 T_{(-k+1,l)}^t \phi.
\]

Then \( \psi \) is a twisted wavelet for \( L^2(\mathbb{R}^2) \). In other words \( \{D_2 T_{(k,l)}^t \psi : j, k, l \in \mathbb{Z}\} \) is an orthonormal basis for \( L^2(\mathbb{R}^2) \).

**Proof.** Let \( V_0 := \text{span}\{T_{(k,l)}^t \phi : k, l \in \mathbb{Z}\} \) and \( V_j := \text{span}\{D_2 T_{(k,l)}^t \phi : k, l \in \mathbb{Z}\} \) for \( j \in \mathbb{Z} \). As \( \phi \in V_0 \), \( \phi \in V_1 \). Then \( D_{1/2} \phi \in V_0 \) and therefore

\[
D_{1/2} \phi = \sum_{k,l \in \mathbb{Z}} c_{k,l} T_{(k,l)}^t \phi.
\]

Taking the Weyl transform on both sides, we obtain

\[
W(D_{1/2} \phi) = \left( \sum_{k,l \in \mathbb{Z}} c_{k,l} \pi(k, l) \right) W(\phi) =: m_0 W(\phi).
\]
Define
\[
m_1 := \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} \overline{c_{k,l}} \pi(-k+1,l)
\]
and take \( W(D_{1/2} \psi) = m_1 W(\phi) \). Then,
\[
W(D_{1/2} \psi) = \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} \overline{c_{k,l}} W(T_{(-k+1,l)}^t \phi).
\]
Hence,
\[
D_{1/2} \psi = \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} \overline{c_{k,l}} T_{(-k+1,l)}^t \phi,
\]
which in turn implies that
\[
\psi = \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} \overline{c_{k,l}} D_2 T_{(-k+1,l)}^t \phi.
\]
Clearly \( \psi \in V_1 \). We need to show that \( \phi \perp \psi \) and \( \{ T_{(k,l)}^t \phi : k, l \in \mathbb{Z} \} \perp \{ T_{(k,l)}^t \psi : k, l \in \mathbb{Z} \} \). Now,
\[
c_{k,l} = \int_{\mathbb{R}^2} D_{1/2} \phi(x,y) \overline{T_{(k,l)}^t \phi(x,y)} \, dx \, dy
= \frac{1}{2} \int_{\mathbb{R}^2} \phi(x/2,y/2) e^{-\pi i (lx-ky)} \overline{\phi(x-k,y-l)} \, dx \, dy
= 2 \int_{\mathbb{R}^2} \phi(x,y) e^{-2\pi i (lx-ky)} \overline{\phi(2x-k,2y-l)} \, dx \, dy.
\]
As \( \text{supp} \, \phi \subset [0,1] \times [0,1] \), we obtain
\[
c_{k,l} = 2 \int_{x=0}^1 \int_{y=0}^1 \phi(x,y) \overline{\phi(2x-k,2y-l)} e^{-2\pi i (lx-ky)} \, dy \, dx
= \frac{1}{2} \int_{x=-k}^{2-k} \int_{y=-l}^{2-l} \phi\left(\frac{x+k}{2}, \frac{y+l}{2}\right) \overline{\phi(x,y)} e^{-\pi i (lx-ky)} \, dy \, dx,
\]
by a change of variables. This shows that \( c_{k,l} \) can only contribute for \( k = 0,1 \) and \( l = 0,1 \). Thus, it follows that \( \text{supp} \, \psi \subset [0,1] \times [0,1] \). As \( \text{supp} \, \phi \subset [0,1] \times [0,1] \), \( \text{supp} \, T_{(k,l)}^t \phi \subset [k,k+1] \times [l,l+1] \). Hence,
\[
\langle \psi, T_{(k,l)}^t \phi \rangle = 0, \quad \forall \, (k,l) \neq (0,0).
\]
But,
\[
(5.1) \quad \langle \phi, \psi \rangle = \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} c_{k,l} \langle D_{1/2} \phi, T_{(-k+1,l)}^t \phi \rangle.
\]
As \( D_{1/2} \phi \in V_0, \) \( D_{1/2} \phi = \sum_{k',l' \in \mathbb{Z}} c_{k',l'} T_{(k',l')} \phi \). Substitution into (5.1) yields
\[
\langle \phi, \psi \rangle = \sum_{k,l,k',l' \in \mathbb{Z}} (-1)^{k+l} c_{k,l} c_{k',l'} \langle T_{(k',l')}^t \phi, T_{(-k+1,l)}^t \phi \rangle
= \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} c_{k,l} c_{-k+1,l}.
\]
Now it is straight-forward to show that the right-hand side of the above equation is zero. Hence it follows that the system \( \{ T_{(k,l)}^t \phi : k, l \in \mathbb{Z} \} \) is orthogonal to the system \( \{ T_{(k,l)}^t \psi : k, l \in \mathbb{Z} \} \).

Let \( W_0 \) be the orthogonal complement of \( V_0 \) in \( V_1 \). Then \( \psi \in W_0 \). Further, one can verify that \( \{ T_{(k,l)}^t \psi : k, l \in \mathbb{Z} \} \) is an orthonormal basis for \( W_0 \). Then it follows from the definition of *twisted*
multiresolution analysis that the system \( \{ D_2 T_{(k,l)}^t \psi : j, k, l \in \mathbb{Z} \} \) forms an orthonormal basis for \( L^2(\mathbb{R}^2) \).

\[ \square \]

**Example 5.4.** Let \( \phi := \phi_1 \). We have already shown that \( \{ T_{(k,l)}^t \phi_1 : k, l \in \mathbb{Z} \} \) is an orthonormal system. Let \( V_0 := \text{span}\{ T_{(k,l)}^t \phi_1 : k, l \in \mathbb{Z} \} \). As the support of \( T_{(k,l)}^t \phi_1 \) is same as the support of the ordinary translates \( T_{(k,l)} \phi_1 \), we can prove as in the classical case that \( \{ V_j, \phi_1 \} \) generates a twisted multiresolution analysis. In this case, the \( c_{k,l} \) can be obtained as follows:

\[
\begin{align*}
c_{0,0} &= \frac{1}{2}, & c_{0,1} &= \frac{1}{\pi i}, \\
c_{1,0} &= -\frac{1}{\pi i}, & c_{1,1} &= \frac{2}{\pi^2}.
\end{align*}
\]

The resulting wavelet, which we call a twisted Haar wavelet is given by

\[
\psi = \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} \overline{c_{k,l}} \ D_2 T_{(-k+1,l)}^t \phi_1.
\]

\[
= \frac{1}{2} D_2 T_{(1,0)}^t \phi_1 + \frac{1}{\pi i} D_2 T_{(1,1)}^t \phi_1 - \frac{1}{\pi i} D_2 T_{(0,0)}^t \phi_1 + \frac{2}{\pi^2} D_2 T_{(0,1)}^t \phi_1.
\]

The real part, imaginary part and magnitude of \( \psi \) are shown in the Figure 3. We can compare this with the classical Haar wavelet on \( \mathbb{R}^2 \), given by \( \psi(x,y) = \psi_H(x)\psi_H(y) \) which is shown in Figure 2.

\[ \text{Figure 2. A classical tensor product Haar wavelet.} \]

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