Singularities of integrable systems and nodal curves

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Abstract

The relation between integrable systems and algebraic geometry is known since the XIXth century. The modern approach is to represent an integrable system as a Lax equation with spectral parameter. In this approach, the integrals of the system turn out to be the coefficients of the characteristic polynomial $\chi$ of the Lax matrix, and the solutions are expressed in terms of theta functions related to the curve $\chi = 0$.

The aim of the present paper is to show that the possibility to write an integrable system in the Lax form, as well as the algebro-geometric technique related to this possibility, may also be applied to study qualitative features of the system, in particular its singularities.

Introduction

It is well known that the majority of finite dimensional integrable systems can be written in the form

$$\frac{d}{dt} L(\lambda) = [L(\lambda), A(\lambda)]$$

(1)

where $L$ and $A$ are matrices depending on the time $t$ and additional parameter $\lambda$. The parameter $\lambda$ is called a spectral parameter, and equation (1) is called a Lax equation with spectral parameter.

The possibility to write a system in the Lax form allows us to solve it explicitly by means of algebro-geometric technique. The algebro-geometric scheme of solving Lax equations can be briefly described as follows. Let us assume that the dependence on $\lambda$ is polynomial. Then, with each matrix polynomial $L$, there is an associated algebraic curve $C(L) \subset \mathbb{C}^2$ called the spectral curve. The Lax equation implies that this curve does not depend on time. Consider the set $\mathcal{S}_C$ of matrix polynomials having the same spectral curve $C$. For each $L \in \mathcal{S}_C$, there is an associated linear bundle over $C$. This bundle is obtained by considering for each point $(\lambda, \mu) \in C$ the kernel of the operator $L(\lambda) - \mu E$, where $E$ stands for the identity matrix. In this way, we obtain a map from $\mathcal{S}_C$ to the Jacobian variety of the spectral curve. The classical result is that this map linearizes the Lax flow. For details, see e.g. the reviews [3–5], as well as references therein and Section 1.1 of the present paper.

The aim of our paper is to show that the possibility to write an integrable system in the Lax form, as well as the algebro-geometric technique related to this possibility, may also be applied to study qualitative features of the system.

In the last 30 years, there has been considerable interest in topology of singular Lagrangian fibrations associated to integrable systems [6–16]. The generic structure of such fibrations is described by the classical Arnold-Liouville theorem which asserts that the phase space of an integrable system is almost everywhere foliated into invariant tori. This description breaks down on the singular set, that is the subset of the phase space where the first integrals become dependent. Though the set of such points is of measure zero, these are singularities

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1 More generally, any equation of the form $\dot{L} = [L, A]$ where $L, A$ are operators is called a Lax equation. Equations of such kind were first considered by Lax [1] in connection with the KdV equation. Lax equations with spectral parameter were first considered in [2].

2 In this paper, we mostly deal with integrable systems given by holomorphic functions on complex manifolds, so, formally speaking, there are no tori. However, the systems we deal with have a property of being algebraically completely integrable, so that each of their regular invariant sets can be identified with an open subset in a certain Abelian variety, which is already a torus.
which mainly determine the global topology of the system. Furthermore, the most remarkable solutions, such as fixed points, stable periodic trajectories, and heteroclinic connections, belong to the singular set. Apart from this, singularities also arise in problems of quantization \[17, 18\], nearly-integrable systems \[19, 20\], and mirror symmetry \[21\].

Although singularities have been studied for a long time, their relation to algebro-geometric description of integrable systems seems to be not well understood. We note that singularities of integrable systems can be, in principle, described by means of straightforward computations using explicit formulas for commuting Hamiltonians. However, firstly, these computations are rather tedious even for low-dimensional systems and, secondly, they do not allow us to see the relation between singularities and algebraic geometry related to the problem. Since first integrals of most of the known integrable systems arise as coefficients of an algebraic curve equation, and the solutions of these systems are expressed in terms of theta functions related to that curve, it seems to be inconsistent to ignore algebraic geometry when studying singularities. In this paper, we show that singularities naturally fit it the classical algebro-geometric scheme of solving Lax equations.

Let us get down to the details. Let \( m, n \in \mathbb{N}^* \) be positive integers, and let \( J \in \mathfrak{gl}(n, \mathbb{C}) \) be a fixed matrix with distinct eigenvalues. Consider the space

\[
\mathcal{L}^m_J(\mathfrak{gl}(n, \mathbb{C})) = \left\{ \sum_{i=0}^{m} L_i \lambda^i \mid L_i \in \mathfrak{gl}(n, \mathbb{C}), L_m = J \right\}
\]

of matrix-valued polynomials of degree \( m \) with a fixed leading term \( J \). It is well known that this space has a structure of a Poisson manifold. The Poisson structure on \( \mathcal{L}^m_J(\mathfrak{gl}(n, \mathbb{C})) \) is related to the decomposition of the loop algebra \( \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) into a sum of two subalgebras \[22, 23\]. The Poisson bracket turns the space of holomorphic functions on \( \mathcal{L}^m_J(\mathfrak{gl}(n, \mathbb{C})) \) into a Lie algebra. This Lie algebra has a natural large commutative subalgebra. Namely, let \( \psi \in \mathbb{C}[\mu, \lambda^{-1}] \) be a polynomial in \( \mu \) and \( \lambda^{-1} \). Define a holomorphic function \( H_{\psi} : \mathcal{L}^m_J(\mathfrak{gl}(n, \mathbb{C})) \to \mathbb{C} \) by the following formula:

\[
H_{\psi}(L) = \text{Res}_{\lambda=0} \lambda^{-1} \text{Tr} \psi(L(\lambda), \lambda^{-1}).
\]

Then for each \( \psi_1, \psi_2 \in \mathbb{C}[\mu, \lambda^{-1}] \) we have \( \{ H_{\psi_1}, H_{\psi_2} \} = 0 \), so that the space

\[
\mathcal{F} = \{ H_{\psi} \mid \psi \in \mathbb{C}[\mu, \lambda^{-1}] \}
\]

is a Poisson-commutative subalgebra of the space of holomorphic functions on \( \mathcal{L}^m_J(\mathfrak{gl}(n, \mathbb{C})) \). Moreover, \( \mathcal{F} \) is an integrable system, which means that the space

\[
d\mathcal{F}(L) = \{ dH(L) \mid H \in \mathcal{F} \}
\]

is maximal isotropic at almost every point \( L \in \mathcal{L}^m_J(\mathfrak{gl}(n, \mathbb{C})) \), see \[24\] for the \( m = 1 \) case, in which this construction coincides with the so-called argument shift method, and \[28, 29, 31\] for the general case.

In order to relate the above definition of integrability with the classical one, let us consider the functions

\[
H_{jk}(L) = \text{Res}_{\lambda=0} \lambda^{-1} \text{Tr} \lambda^{-j} L(\lambda)^k
\]

where \( 1 \leq k \leq n \) and \( 0 \leq j < mk \). It is claimed that the functions \( H_{jk} \) Poisson commute and are independent almost everywhere. Among the Hamiltonians \( H_{jk} \), there are \( mn \) Casimir functions, and the number of the remaining functions equals \( \frac{1}{2}mn(n-1) \), that is exactly one half of the dimension of a generic symplectic leaf. Therefore, Hamiltonian flows generated by each of the functions \( H_{jk} \) are completely integrable in the Liouville sense.\(^3\) Note that the functions \( H_{jk} \) are “generators” of the family \( \mathcal{F} \), which means that each function \( H_{\psi} \in \mathcal{F} \) is a function of \( H_{jk} \)’s.

For each \( H_{\psi} \in \mathcal{F} \), the Hamiltonian flow corresponding to \( H_{\psi} \) have the Lax form

\[
\frac{d}{dt} L(\lambda) = [L(\lambda), \phi(L(\lambda), \lambda^{-1})_+]
\]

where \( \phi = \partial \psi / \partial \mu \) and \( (\ldots)_+ \) denotes the sum of the terms of positive degree.

\(^3\)We should mention that there also exists another approach to singularities of integrable systems based on the notion of a bi-Hamiltonian structure. See \[22, 24\].

\(^4\)We note that topology of integrable systems, with no relation to singularities, was studied from the algebro-geometric point of view by Audin and her collaborators \[25, 26\]. See also our work \[27\] where we discuss the relation between algebraic geometry and stability of solutions of integrable systems.

\(^5\)We note that some definitions of Liouville integrability include the requirement of completeness of Hamiltonian flows. In our case, this requirement is not satisfied.
As it is mentioned above, with each matrix polynomial \( L \in \mathcal{L}_m \), we can associate the curve \( \phi \) called the spectral curve. For each fixed curve \( C \), the isospectral set

\[
\mathcal{S}_C = \{ L \in \mathcal{L}_m(\mathfrak{gl}(n, \mathbb{C})) \mid C(L) = C \}
\]

is preserved by each of the flows \( \psi \). As it is easy to see, coefficients of the spectral curve equation are linear combinations of \( H_{jk} \)’s and vice versa, so \( \mathcal{S}_C \) coincides with a common level set of Hamiltonians \( H_{jk} \). The fibration \( \mathcal{L}_m = \bigsqcup \mathcal{S}_C \), where \( C \) varies in the set of affine algebraic curves, is a \textit{singular Lagrangian fibration}. A fiber \( \mathcal{S}_C \) is called \textit{regular} if each point \( L \in \mathcal{S}_C \) is non-singular for the integrable system \( \mathcal{F} \), i.e. if the Hamiltonians \( H_{jk} \) are independent at each point \( L \in \mathcal{S}_C \) (or, which is the same, the space \( d\mathcal{F}(L) \) is maximal isotropic). Each regular fiber \( \mathcal{S}_C \) is smooth, moreover it is a Lagrangian submanifold in the ambient symplectic leaf of \( \mathcal{L}_m \). Fibers which are not regular are called \textit{singular}.

As is well known, if the curve \( C \) is non-singular, then the fiber \( \mathcal{S}_C \) is non-singular as well. Furthermore, in this case \( \mathcal{S}_C \) can be explicitly described as an open dense subset in the total space of a principal \((\mathbb{C}^*)^{n-1}\)-bundle over the Jacobian of \( C \), and the mapping \( \mathcal{S}_C \to \text{Jac}(C) \) linearizes each of the flows \( \psi \).

If the curve \( C \) is singular, then some points \( L \in \mathcal{S}_C \) may become singular, which means that the differentials of the Hamiltonians \( H_{jk} \) become dependent. The goal of this paper is to describe singularities arising on \( \mathcal{S}_C \) when the curve \( C \) is nodal.

The first part of the paper (Section 1) is devoted to the description of the set \( \mathcal{S}_C \) itself. Namely, we show that if \( C \) is a nodal, possibly reducible curve, then \( \mathcal{S}_C \) is subdivided into natural smooth strata indexed by partial normalizations of \( C \) and integer points in a certain convex polytope. For each stratum, there is a map to the generalized Jacobian of the corresponding partial normalization, and the image of each of the flows \( \psi \) under this map is a linear flow. Main result of this part of the paper is Theorem 1 (see Section 1.1).

In the second part of the paper (Section 2), we prove that if the spectral curve \( C \) is nodal, then all singular points on \( \mathcal{S}_C \) are non-degenerate. Non-degenerate singularities of integrable systems are in a sense analogous to Morse singular points of smooth functions. In the complex case, all non-degenerate singularities of the same rank are locally symplectomorphic to each other. In the real case, each non-degenerate singularity can be represented as a product of three basic singularities: elliptic, hyperbolic, and focus-focus. In the same way, there are three kinds of nodal points of real algebraic curves: acnodes (isolated points in the real part of the curve), crunodes (double points in the real part), and nodes which do not belong to the real part. In the case when the system under consideration is real, we show that acnodes, crunodes, and complex nodes in the spectral curve correspond to elliptic, hyperbolic, and focus-focus singularities respectively. Main results of this part are Theorems 4 and 5 (see Section 2.2).

Let us make one important remark. The Hamiltonians \( \{ \psi \} \) and equations \( \{ \psi \} \) may seem to be of a rather special form. Nevertheless, it turns out that almost all known finite-dimensional integrable systems can be written in this form (see [3] and references therein), so the construction discussed is quite universal. However, in order to obtain physically interesting examples, we need to pass to a certain subspace \( \mathcal{L}' \subset \mathcal{L}_m(\mathfrak{gl}(n, \mathbb{C})) \) which are Poisson manifolds, and if we pick those flows \( \psi \) which leave the subspace \( \mathcal{L}' \) invariant, we obtain an integrable hierarchy\(^6\) on \( \mathcal{L}' \). In this paper, we only consider the hierarchy \( \mathcal{F} \) either on the whole space \( \mathcal{L}_m(\mathfrak{gl}(n, \mathbb{C})) \), or on its real counterpart \( \mathcal{L}_m(\mathfrak{so}(n, \mathbb{R})) \). However, almost all of our results, in particular Theorems 4 and 5 can be extended to restricted systems in a more or less straightforward way. In particular, we claim that singularities of such classical integrable systems as Euler, Lagrange and Kovalevskaya tops, spherical pendulum, geodesic flow on ellipsoid etc., as well as their multidimensional generalizations, can be described using our approach.

\(^6\)We note that the real version of the integrable system \( \mathcal{F} \) discussed above is constructed in exactly the same way. Its phase space is \( \mathcal{L}_m(\mathfrak{so}(n, \mathbb{R})) = \{ \sum_{i=1}^m L_i \lambda^i \mid L_i \in \mathfrak{gl}(n, \mathbb{R}), L_m = J \} \). The Hamiltonians are of the same form \( \{ \psi \} \) where the polynomial \( \lambda \) is real, and the corresponding Hamiltonian flows have the form \( \psi \) where \( \phi \) is also real. For each \( L \in \mathcal{L}_m(\mathfrak{gl}(n, \mathbb{R})) \), the associated spectral curve \( C(L) \) is a \textit{real algebraic curve}, i.e. an affine algebraic curve over \( \mathbb{C} \) endowed with an antiholomorphic involution \( (\lambda, \mu) \mapsto (\lambda, \mu) \).

\(^7\)For example, consider the subspace \( \mathcal{L}' = \mathcal{L}_m(\mathfrak{so}(n, \mathbb{R})) \subset \mathcal{L}_m(\mathfrak{gl}(n, \mathbb{R})) \) which consists of those polynomials \( L(\lambda) \) which satisfy \( L(\lambda)^2 = -L(\lambda) \). As it is easy to see, the space \( \mathcal{L}_m(\mathfrak{so}(n, \mathbb{R})) \) is invariant with respect to the flow \( \psi \) if and only if the polynomial \( \phi \) is real and odd in the variable \( \mu \). Considering the flows \( \phi \) for all such polynomials \( \phi \), we obtain a completely integrable system on \( \mathcal{L}_m(\mathfrak{so}(n, \mathbb{R})) \). In particular, taking \( m = 2 \) and \( n = 3 \), we obtain the Lagrange top.\(^8\)
1 Singular spectral curves, generalized Jacobians, and convex polytopes

1.1 Description of the set $S_C$

In this section, we assume that $C$ is a nodal curve and describe the set $S_C$ defined by (4). We note that the simplicity of the spectrum of the leading term $J$ implies that the spectral curve $C$ is necessarily reduced, i.e. its defining polynomial has no multiple factors.

It is clear that the curve $C$ should satisfy some additional assumptions in order for the set $S_C$ to be non-empty. Namely, let $C_{spec}$ be the set of plane affine algebraic curves in with defining polynomial $\chi(\lambda, \mu)$ satisfying

$$\lim_{z \to 0} \left( \chi \left( \frac{1}{z}, \frac{w}{z^m} \right) z^n \right) = \det(J - wE).$$

Clearly, for $S_C$ to be non-empty, we should have $C \in C_{spec}$. So, in what follows, we consider the set $S_C$ only for $C \in C_{spec}$.

First, assume that the spectral curve $C$ is non-singular. The description of the set $S_C$ in this case is well-known. Namely, consider the Riemann surface $\Omega$ which is obtained from the spectral curve $C$ by adding points at infinity. Let

$$\text{PGL}(\mathbb{C}, J) = \{ R \in \text{PGL}(n, \mathbb{C}) | RJ = JR \}.$$ 

The set $S_C$ carries the natural action of $\text{PGL}(\mathbb{C}, J)$ by conjugation. This action is free and preserves each of the flows (4). Let $S_C = S_C/\text{PGL}(\mathbb{C}, J)$. Then, as shown in [33–35], there exists a biholomorphic map

$$\hat{\Phi} : S_C \to \text{Pic}_{g+1}(X) \setminus (\Theta_{g-1} + [D_\infty])$$

where $g$ is the genus of $X$, $\Theta_{g-1} \subset \text{Pic}_{g-1}(X)$ is the theta divisor, and $D_\infty$ is the pole divisor of $\lambda$. Furthermore, the image of the flow (4) under the map $\hat{\Phi}$ is a linear flow given by

$$\omega \left( \frac{d\xi}{dt} \right) = \sum_{\pi : \lambda(\pi) = \infty} \text{Res}_\pi \phi \omega$$

(6)

where $\xi \in \text{Pic}_{g+1}(X)$, $\omega \in \Omega^1(X)$, and the cotangent space to $\text{Pic}_{g+1}(X)$ is identified with the space $\Omega^1(X)$ of holomorphic differentials on $X$.

The set $S_C$ itself has a structure of a holomorphic principal $\text{PGL}(\mathbb{C}, J)$-bundle over $\hat{S}_C$. The structure of this bundle is described in [38]. Let $\infty_1, \ldots, \infty_n$ be the poles of $\lambda$, and let $X'$ be the curve obtained from $X$ by identifying $\infty_1 \sim \cdots \sim \infty_n$. Then there exists a biholomorphic map

$$\Phi : S_C \to \text{Pic}_{g+1}(X') \setminus \pi^{-1}(\Theta_{g-1} + [D_\infty])$$

where $\pi$ is the natural projection $\text{Pic}_{g+1}(X') \to \text{Pic}_{g+1}(X)$. The projection $\pi$ defines a principal bundle structure on $\text{Pic}_{g+1}(X')$, and the following diagram commutes:

$$
\begin{array}{ccc}
S_C & \xrightarrow{\Phi} & \text{Pic}_{g+1}(X') \\
\downarrow & & \downarrow \pi \\
\hat{S}_C & \xrightarrow{\hat{\Phi}} & \text{Pic}_{g+1}(X)
\end{array}
$$

The image of the flow (4) under the map $\hat{\Phi}$ is given by the same formula (6) where $\omega$ is now not necessarily holomorphic, but may have poles of the first order at points $\infty_1, \ldots, \infty_n$.

The goal of this part of the paper is to extend these results to the case when the spectral curve is nodal and possibly reducible. Recall that a singular point $(\lambda, \mu)$ of a plane affine algebraic curve $\{ \lambda, \mu \in \mathbb{C}^2 | p(\lambda, \mu) = 0 \}$ is called a node, or an ordinary double point, if the Hessian $d^2p(\lambda, \mu)$ is non-degenerate. A plane algebraic curve is called nodal if all its singular points are nodes. See Section 1.2 for details on nodal curves.

Let $C$ be a nodal, possibly reducible curve, and let $\text{Sing} C$ be the set of its nodes. Let $L \in S_C$. As is well known (see [57], Chapter 5.2), for all points $(\lambda, \mu) \in C \setminus \text{Sing} C$, we have

$$\dim \text{Ker} (L(\lambda) - \mu E) = 1.$$ 

For any $(\lambda, \mu) \in \text{Sing} C$, there are two possibilities:

$$\dim \text{Ker} (L(\lambda) - \mu E) = 1 \quad \text{or} \quad \dim \text{Ker} (L(\lambda) - \mu E) = 2.$$
This dichotomy gives rise to a natural stratification of $\mathcal{S}_C$. For $L \in \mathcal{S}_C$, let

$$K(L) = \{(\lambda, \mu) \in \text{Sing} C(L) \mid \dim \text{Ker} (L(\lambda) - \mu E) = 2\}.$$

Let $K \subset \text{Sing} C$, and let

$$\mathcal{S}_C^K = \{L \in \mathcal{S}_C \mid K(L) = K\}.$$

The set $\mathcal{S}_C^K$ is a quasi-affine variety. Clearly, we have

$$\mathcal{S}_C = \bigsqcup_{K \subset \text{Sing} C} \mathcal{S}_C^K$$

where the union is disjoint in set-theoretical, not in topological sense. Stratification (7) is preserved by each of the flows (4), that is all these flows leave $\mathcal{S}_C^K$ invariant for each $K$. Below, we give a geometric characterization of the sets $\mathcal{S}_C^K$.

Let $X_K$ be the curve which is obtained from $C$ by adding points at infinity and blowing up at the points of $K$, and let $X'_K$ be the curve obtained from $X_K$ by identifying $\infty_1 \sim \cdots \sim \infty_n$. Our first result is that $\mathcal{S}_C^K$ is biholomorphic to an open subset in the disjoint union of $r$ copies of the generalized Jacobian of $X'_K$ where $r$ is the number of integer points in a certain polytope constructed from the curve $X'_K$. Let us describe the construction of this polytope.

Let $Y$ be a curve, and let $Y_1, \ldots, Y_k$ be its irreducible components. A multidegree on $Y$ is a mapping $d: \{Y_1, \ldots, Y_k\} \to \mathbb{Z}$. The total degree of a multidegree $d$ is the number $|d| = \sum_{i=1}^k d(Y_i)$. For each $I \subset \{1, \ldots, k\}$, define the subcurve $Y_I \subset Y$ as the union $\bigcup_{i \in I} Y_i$. For each subcurve $Y_I \subset Y$ we can restrict $d$ on $Y_I$ and get a multidegree $d_I$ on $Y_I$ (see also Section 1.2).

Let $d$ a multidegree on $Y$ of total degree $g(Y)$ where $g(Y)$ is the arithmetic genus of $Y$ (see Section 1.2). We say that $d$ is uniform if for each subcurve $Y_I \subset Y$ we have

$$|d_I| \geq g(Y_I).$$

The set of uniform multidegrees is non-empty and coincides with the set of integer points in the convex polytope

$$P = \left\{ x \in \mathbb{R}^k : \sum_{i=1}^k x_i = g(Y); \sum_{i \in I} x_i \geq g(Y_I) \forall I \subset \{1, \ldots, k\} \right\}.$$ (8)

Let $\Delta_K$ be the set of uniform multidegrees on $X'_K$.

**Theorem 1.** Assume that $C \subset \mathcal{C}_{\text{spec}}$ is a nodal curve, and let $K \subset \text{Sing} C$. Then:

1. The set $\mathcal{S}_C^K$ is a complex analytic manifold of dimension

$$\dim \mathcal{S}_C^K = g(X'_K) = \frac{mn(n-1)}{2} - |K|. $$

2. There exists a biholomorphic map

$$\Phi: \mathcal{S}_C^K \to \bigsqcup_{d \in \Delta_K} (\text{Pic}_d(X'_K) \setminus \Upsilon_d)$$

where $\Delta_K$ is the set of uniform multidegrees on $X'_K$, and $\Upsilon_d \subset \text{Pic}_d(X'_K)$ is a subset of positive codimension.

3. The image of the flow (4) under the mapping $\Phi$ is given by the formula (4) where $\omega$ is any regular differential on $X'_K$.

4. The flows (4) span the tangent space to $\mathcal{S}_C^K$ at every point.

**Remark 1.1.** We note that Theorem 1 remains true over the reals. Namely, the set

$$\text{Res} \mathcal{S}_C^K = \mathcal{S}_C^K \cap \mathcal{L}_m^f(\text{gl}(n, \mathbb{R}))$$

is a smooth manifold diffeomorphic to an open subset in the real part of $\bigcup_{d \in \Delta_K} \text{Pic}_d(X'_K)$.

**Remark 1.2.** Regular differentials on $X'_K$ can be viewed as meromorphic differentials on its non-singular compact model $X$ with special properties. Namely, a meromorphic differential $\omega$ on $X$ is regular on $X'_K$ if all poles of $\omega$ are simple, and for each $Q \in X'_K$, we have

$$\sum_{P \in \pi(P) = Q} \text{Res}_P \omega = 0$$

where $\pi: X \to X'_K$ is the normalization map. See Section 1.2 for details on regular differentials on curves.
Corollary 1.1. Assume that $C \in \mathcal{C}_{\text{spec}}$ is a nodal curve. Then:

1. The dimension of $S_C$ is equal to $\frac{1}{2}mn(n-1)$.

2. The number of irreducible components of $S_C$ is equal to the number of uniform multidegrees on the curve $X$.

Proof. Assertion 1 is obvious. Assertion 2 is proved in Section 1.2.2.

The proof of Theorem 1 is given in Section 1.3. In Section 1.4 we consider an example. Namely, we take $m = 1$ and discuss the set $S_C$ when the spectral curve $C$ is $n$ straight lines in general position. In this case, the polytope $\mathcal{P}$ turns out to be the permutohedron $P_n$. It turns out that solutions of (4) corresponding to $d$ are vertices of $P_n$. Then $\text{Pic}(X)_{d}$ is completely contained in the image of $\Phi$, i.e. the exceptional set $\Upsilon_d$ is empty. For integer points in the interior of $P_n$, this is no longer so, and the solutions turn out to be rational functions of exponents.

Let us also define $\hat{S}_C$ as the quotient of $S_C$ by the $\mathbb{P}\text{GL}(C,J)$ action. We note that this action is no longer free, however there always exists a subgroup $H \subset \mathbb{P}\text{GL}(C,J)$ such that $\mathbb{P}\text{GL}(C,J)/H$ acts freely, and $H$ acts trivially.

The following statement follows from Theorem 1.

Corollary 1.2. Assume that $C \in \mathcal{C}_{\text{spec}}$ is a nodal curve, and that $K \subset \text{Sing} C$. Then:

1. The set $\hat{S}_C$ is a complex analytic manifold of dimension

$$\dim \hat{S}_C = g(X) = \frac{mn(n-1)}{2} - n - |K| + \dim H_0(X_K).$$

2. There exists a biholomorphic map

$$\hat{\Phi}: \hat{S}_C \rightarrow \bigsqcup_{d \in \Delta_K} \left( \text{Pic}_d(X_K) \setminus \hat{\Upsilon}_d \right)$$

where $\Delta_K$ is the set of uniform multidegrees on $X_K$ (not $X_K$!), and $\hat{\Upsilon}_d \subset \text{Pic}_d(X_K)$ is a subset of positive codimension.

3. The image of the flow (11) under the mapping $\hat{\Phi}$ is given by the formula (10) where $\omega$ is any regular differential on $X_K$.

4. The flows (11) span the tangent space to $\hat{S}_C$ at every point.

5. Let $\pi: \text{Pic}(X_K) \rightarrow \text{Pic}(X_K)$ be the natural projection. Then the following diagram commutes:

$$\begin{array}{ccc}
S_C & \xrightarrow{\Phi} & \text{Pic}(X_K) \\
\downarrow & & \downarrow \pi \\
\hat{S}_C & \xrightarrow{\Phi} & \text{Pic}(X_K)
\end{array}$$

Remark 1.3. We note that it is also possible to describe the set $\Delta_K$ in terms of the curve $X_K$ itself.

Let $Y$ be a nodal curve, and let $d$ a multidegree on $Y$ such that $|d| = g(Y) - \dim H_0(Y)$. Then $d$ is called semistable if for each subcurve $Y_i \subset Y$ we have

$$|d| \geq g(Y_i) - \dim H_0(Y_i).$$

A condition equivalent to (9) first appeared in [39]. The term semistable multidegree is suggested in [39].

As it is to see (see Proposition 1.2.2), we have

$$\Delta_K = \{ d \in \mathbb{Z}^n \text{ such that } d - \deg D_\infty \text{ is semistable} \}.$$ 

We note that the multidegree $\deg D_\infty$ has a transparent description. If the defining polynomial of $C$ is $\chi = \chi_1 \cdots \chi_k$, then $\deg D_\infty = (\deg \chi_1, \ldots, \deg \chi_k)$.

1.2 Nodal curves and generalized Jacobians

The theory of generalized Jacobians is due to Rosenlicht [40, 41], see also Serre [42]. In this section, we present an elementary exposition of this theory for nodal, possibly reducible, curves. We note that all the presented results are well-known, at least in the irreducible case. As for the reducible case, we were not able to find some statements in the literature, in particular, Proposition 1.1.1 concerning effective Weil divisors on reducible curves.
assumption that the orientation is fixed. However, we do not need it.

Of course, if we rename $P_i$ to $P_i^-$ and vice versa, the orientation will be reversed. However, it is convenient to assume that the orientation is fixed.

### 1.2.1 Nodal curves and arithmetic genus.

**Definition 1.** A plane affine algebraic curve $\{\lambda, \mu \in \mathbb{C}^2 \mid p(z, w) = 0\}$ is called a *plane nodal curve* if $\det d^2 p \neq 0$ at all points where $dp = 0$.

Below, we give a more abstract definition of nodal curves.

Let $X = X_1 \sqcup \ldots \sqcup X_r$, where $r \geq 1$ be a disjoint union of connected Riemann surfaces, and let $\Sigma = \{P_1, \ldots, P_n\}$ be a finite set of pairwise disjoint 2-element subsets of $X$. Assume that $\mathcal{P}_i = \{P_i^+, P_i^\sigma\}$, and consider the topological space $X/\Sigma$ obtained from $X$ by identifying $P_i^+$ with $P_i^\sigma$ for each $1 \leq i \leq \sigma$.

Let $\pi: X \to X/\Sigma$ be the natural projection, and let $\text{supp}(\Sigma) = \bigcup_{i=1}^\sigma \mathcal{P}_i$. A function $f: X/\Sigma \to \mathbb{C}$ is called meromorphic on $X/\Sigma$ if $\pi^*(f)$ is a meromorphic function on $X$ which does not have poles at the points of $\text{supp}(\Sigma)$.

**Definition 2.** The space $X/\Sigma$ endowed with the described ring of meromorphic functions is called a *nodal curve*.

Obviously, a plane nodal curve completed at infinity is a nodal curve. The converse is of course not true, i.e. not any nodal curve can be obtained in this way.

In what follows, we prefer to work with the non-singular curve $X/\Sigma$ rather than with the singular curve $X/\Sigma$. Our approach to singular curves is rather close to the original approach of Rosenlicht.

**Definition 3.** We say that a function on $X$ is *$\Sigma$-regular* if it is meromorphic and takes same finite values at $P_i^\pm$.

Obviously, $\Sigma$-regular functions on $X$ are in one-to-one correspondence with meromorphic functions on $X/\Sigma$. The ring of $\Sigma$-regular functions will be denoted by $M(X, \Sigma)$:

$$M(X, \Sigma) = \{f \in M(X) : f(P_i^+) = f(P_i^-) \neq \infty \forall 1 \leq i \leq |\Sigma|\},$$

where $M(X)$ is the ring of functions meromorphic on $X$.

**Definition 4.** A meromorphic differential $\omega$ on $X$ is *$\Sigma$-regular* if it is meromorphic and takes same finite values at $P_i^\pm$.

$$\text{res}_{P_i^+} \omega + \text{res}_{P_i^-} \omega = 0 \quad \forall 1 \leq i \leq |\Sigma|. \quad (10)$$

We denote the space of $\Sigma$-regular differentials by $\Omega^1(X, \Sigma)$.

**Definition 5.** The number $\dim \Omega^1(X, \Sigma)$ is called the *arithmetic genus* of $X/\Sigma$.

Let us denote the arithmetic genus by $g(X, \Sigma)$. We shall also use the notation $g(X)$ for the geometric genus of $X$ (that is the dimension of the space $\Omega^2(X)$ of holomorphic differentials on $X$), $c(X)$ for the number of irreducible components of $X$, and $c(X, \Sigma)$ for the number of connected components of $X/\Sigma$. These notations are summarized in Table 1.

| Notation | Meaning |
|----------|---------|
| $g(X)$   | $\dim \Omega^1(X) = \frac{1}{2} \dim H_1(X, \mathbb{C})$, genus of $X$ |
| $c(X)$   | $\dim H_0(X, \mathbb{C})$, number of irreducible components of $X/\Sigma$ |
| $g(X, \Sigma)$ | $\dim \Omega^1(X, \Sigma)$, arithmetic genus of $X/\Sigma$ |
| $c(X, \Sigma)$ | $\dim H_0(X/\Sigma, \mathbb{C})$, number of connected components of $X/\Sigma$ |

**Table 1: Notations**

---

*Formally speaking, to turn $X/\Sigma$ into a complex analytic space, we should have described its structure sheaf. However, we do not need it.*

*Of course, if we rename $P_i^-$ to $P_i^+$ and vice versa, the orientation will be reversed. However, it is convenient to assume that the orientation is fixed.*
We denote the dual graph of \( X/\Sigma \) by \( \Gamma(X, \Sigma) \) or, when it does not cause confusion, just \( \Gamma \). We note that
\[
\dim H_0(\Gamma, \mathbb{C}) = c(X, \Sigma) = \dim H_0(X/\Sigma, \mathbb{C}).
\]
With each \( \omega \in \Omega^1(X, \Sigma) \), we associate a 1-chain in the dual graph:
\[
Z(\omega) = \sum_{i=1}^{2g} \left( \text{Res}_{r_i^+} \omega \right) e_i.
\]

**Proposition 1.1.** For each \( \omega \in \Omega^1(X, \Sigma) \), the chain \( Z(\omega) \) is a cycle.

**Proof.** This follows from condition (10).

The following is simple.

**Proposition 1.2.**
1. The mapping \( Z: \Omega^1(X, \Sigma)/\Omega^1(X) \to H_1(\Gamma(X, \Sigma), \mathbb{C}) \) is an isomorphism.
2. The arithmetic genus of a nodal curve is given by
\[
g(X, \Sigma) = g(X) + \dim H_1(\Gamma, \mathbb{C}) = g(X) + |\Sigma| + c(X, \Sigma) - c(X).
\]

### 1.2.2 Generalized Jacobian of a nodal curve.

Let us define the generalized Jacobian of a nodal curve. As in the non-singular case, there is a natural mapping
\[
\mathcal{I}: H_1(X \setminus \text{supp}(\Sigma), \mathbb{Z}) \to \Omega^1(X, \Sigma)^*
\]
given by
\[
(\mathcal{I}(\gamma), \omega) = \oint_{\gamma} \omega.
\]

The image of this mapping is a lattice \( L(X, \Sigma) \subset \Omega^1(X, \Sigma)^* \) called the period lattice.

**Definition 6.** The quotient
\[
\text{Jac}(X, \Sigma) = \Omega^1(X, \Sigma)^*/L(X, \Sigma)
\]
is called the generalized Jacobian of \( X/\Sigma \).

Let us describe the structure of the generalized Jacobian. By Proposition 1.2 we have an exact sequence
\[
0 \longrightarrow H^1(\Gamma, \mathbb{C}) \overset{Z^*}{\longrightarrow} \Omega^1(X, \Sigma)^* \overset{\pi^*}{\longrightarrow} \Omega^1(X)^* \longrightarrow 0
\]
where \( \pi^* \) is the restriction map. Obviously, \( \pi^* \) maps \( L(X, \Sigma) \) onto \( L(X) \) where \( L(X) \) is the usual period lattice of \( X \). The kernel of the mapping \( \pi^* : L(X, \Sigma) \to L(X) \) consists of integrals over cycles contractible in \( X \), i.e. functionals \( \xi \) of the form
\[
(\xi, \omega) = 2\pi i \sum_{i=1}^{2g} k_i \text{Res}_{r_i^+} \omega, \quad k_i \in \mathbb{Z}.
\]

We get another exact sequence
\[
0 \longrightarrow H^1(\Gamma, 2\pi i \mathbb{Z}) \overset{Z^*}{\longrightarrow} L(X, \Sigma) \overset{\pi^*}{\longrightarrow} L(X) \longrightarrow 0.
\]
Combining these two exact sequences, we get the following commutative diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\Gamma, 2\pi i \mathbb{Z}) \overset{Z^*}{\longrightarrow} L(X, \Sigma) \overset{\pi^*}{\longrightarrow} L(X) \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\Gamma, \mathbb{C}) \overset{Z^*}{\longrightarrow} \Omega^1(X, \Sigma)^* \overset{\pi^*}{\longrightarrow} \Omega^1(X)^* \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\Gamma, \mathbb{C}/2\pi i \mathbb{Z}) \overset{Z^*}{\longrightarrow} \text{Jac}(X, \Sigma) \overset{\pi^*}{\longrightarrow} \text{Jac}(X) \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0 \\
\end{array}
\]
(11)

The columns and the two top rows of this diagram are exact, so the bottom row is exact as well. We conclude that the generalized Jacobian \( \text{Jac}(X, \Sigma) \) is the extension of the usual Jacobian \( \text{Jac}(X) \) by the group \( H^1(\Gamma, \mathbb{C}/2\pi i \mathbb{Z}) \cong (\mathbb{C}^*)^m \) where
\[
m = \dim H_1(\Gamma, \mathbb{C}) = |\Sigma| - c(X) + c(X, \Sigma).
1.2.3 Abel map. Now, let us construct the Abel map for nodal curves. For each Weil divisor $D$ on $X$, we define its multidegree $\deg D = (d_1, \ldots, d_r) \in \mathbb{Z}^r$ where $r = c(X)$, $d_i = \deg (D |_{x_i})$. The total degree of $D$ is the number

$$|\deg D| = \sum_{i=1}^r d_i.$$  

We denote the set of divisors of multidegree $d$ by $\text{Div}_d(X)$.

Definition 7. A divisor $D$ on $X$ is called $\Sigma$-regular, if its support does not intersect $\text{supp}(\Sigma)$.

The set of $\Sigma$-regular divisors of multidegree $d$ will be denoted by $\text{Div}_d(X, \Sigma)$. The set of all $\Sigma$-regular divisors

$$\text{Div}(X, \Sigma) = \bigcup_{d \in \mathbb{Z}^k} \text{Div}_d(X, \Sigma)$$

is a $\mathbb{Z}^k$-graded Abelian group.

Let $M^*(X, \Sigma)$ be the set of invertible elements in $M(X, \Sigma)$:

$$M^*(X, \Sigma) = \{f \in M(X, \Sigma) : f(P) \neq 0 \ \forall \ P \in \text{supp}(\Sigma), \ f |_{x_j} \neq 0 \ \forall \ 1 \leq j \leq c(X)\}.$$  

For each $f \in M^*(X, \Sigma)$, its divisor $(f)$ is a $\Sigma$-regular divisor.

Definition 8. Divisors of the form $(f)$ where $f \in M^*(X, \Sigma)$ will be called $\Sigma$-principal. Two $\Sigma$-regular divisors are $\Sigma$-linearly equivalent if their difference is a $\Sigma$-principal divisor.

We denote the space of $\Sigma$-principal divisors by $\text{PDiv}(X, \Sigma)$. For two $\Sigma$-equivalent divisors, we write:

$$D_1 \sim \Sigma D_2.$$

Let $D$ be a $\Sigma$-regular divisor of multidegree 0. Then $D$ can be written as

$$D = \sum_{i=1}^{c(X)} (D_i^+ - D_i^-)$$

where $D_i^+$ are effective divisors on $X_i$, and $\deg D_i^+ = \deg D_i^-$. For a $\Sigma$-regular differential $\omega$, we set

$$\int_D \omega = \sum_{i=1}^{c(X)} \int_{D_i^+} \omega.$$  

This integral is defined up to periods of $\omega$, hence we obtain a map

$$A_\Sigma : \text{Div}_0(X, \Sigma) \rightarrow \text{Jac}(X, \Sigma)$$

which is the analogue of the usual Abel map.

Proposition 1.3. The following diagram is commutative:

$$\begin{array}{ccc}
\text{Div}_0(X, \Sigma) & \longrightarrow & \text{Div}_0(X) \\
\downarrow A_\Sigma & & \downarrow A \\
\text{Jac}(X, \Sigma) & \longrightarrow & \text{Jac}(X)
\end{array}$$

where the upper arrow is the natural inclusion, and $A$ is the usual Abel map.

Proof. Obvious.

1.2.4 Abel theorem and generalized Picard group. Let $\text{PDiv}(X)$ be the group of principal divisors on $X$, and let $D \in \text{PDiv}(X)$. Then we can find a meromorphic function $f$ such that $D = (f)$. This function is defined up to a factor which is constant on each component of $X$. To each edge $e_i$ of the dual graph $\Gamma(X, \Sigma)$ we assign a number

$$r_i = f(P_i^+) / f(P_i^-) \in \mathbb{C}^*.$$  

The numbers $\{r_i\}$ define a 1-cocycle on the dual graph. Since $f$ is defined up to a locally constant factor, this cocycle is defined up to a coboundary. Therefore, to each principal divisor we can assign a cohomology class. Denote this class by $R(D)$. We have a mapping

$$R : \text{PDiv}(X) \cap \text{Div}(X, \Sigma) \rightarrow H^1(\Gamma(X, \Sigma), \mathbb{C}^*).$$

and an exact sequence

$$0 \longrightarrow \text{PDiv}(X, \Sigma) \longrightarrow \text{PDiv}(X) \cap \text{Div}(X, \Sigma) \overset{R}{\longrightarrow} H^1(\Gamma, \mathbb{C}^*) \longrightarrow 0. \quad (12)$$
Proposition 1.4. The following diagram commutes:

\[
\begin{array}{c}
P\text{Div}(X) \cap \text{Div}(X, \Sigma) \xrightarrow{\mathcal{A}_\Sigma} \text{Jac}(X, \Sigma) \\
R \downarrow \quad Z^* \downarrow \\
H^1(\Gamma, C^*) \xrightarrow{\ln_*} H^1(\Gamma, C/2\pi i\mathbb{Z})
\end{array}
\]

where \(\ln_*\) is the map induced by \(\ln: C^* \to C/2\pi i\mathbb{Z}\).

Proof. Let \(D = (f) \in P\text{Div}(X) \cap \text{Div}(X, \Sigma)\). Choose a path \(\gamma\) joining \(0\) and \(\infty\) on the Riemann sphere such that \(f^{-1}(\gamma)\) does not intersect \(\text{supp}(\Sigma)\). Let \(\omega\) be a \(\Sigma\)-regular differential. Then

\[
\langle \mathcal{A}_\Sigma(D), \omega \rangle = \int_{f^{-1}(\gamma)} \omega = \int_{0}^{\infty} \sum_{i=1}^{[\Sigma]} \left( \frac{1}{z-P_i^+} - \frac{1}{z-P_i^-} \right) \text{Res}_{P_i} f \, dz =
\]

\[
= \sum_{i=1}^{[\Sigma]} \ln \frac{f(P_i^+)}{f(P_i^-)} \, \text{Res}_{P_i}^+ \omega = \langle \ln_* R(D), Z(\omega) \rangle = (Z^* (\ln_* R(D)), \omega).
\]

\[\square\]

Proposition 1.5 (Abel theorem for nodal curves). A \(\Sigma\)-regular divisor \(D\) of multidegree zero is \(\Sigma\)-principal if and only if \(\mathcal{A}_\Sigma(D) = 0\).

Proof. Let \(D\) be a \(\Sigma\)-principal divisor. Then \(R(D) = 0\), so \(\mathcal{A}_\Sigma(D) = Z^* (\ln_* R(D)) = 0\). Vice versa, let \(\mathcal{A}_\Sigma(D) = 0\). Applying Proposition 1.4, we conclude that \(A(D) = 0\). So, by the standard Abel theorem, \(D\) is principal. Since \(\mathcal{A}_\Sigma(D) = 0\), and \(Z^*\) is injective, we conclude that \(R(D) = 0\), so the divisor \(D\) is \(\Sigma\)-principal.

\[\square\]

Using diagrams (11) and (12), the diagram (13) is reduced to

\[
\begin{array}{c}
P\text{Div}(X) \cap \text{Div}(X, \Sigma)/P\text{Div}(X, \Sigma) \xrightarrow{\mathcal{A}_\Sigma} \text{Ker} \pi^* \\
R \downarrow \quad Z^* \downarrow \\
H^1(\Gamma, C^*) \xrightarrow{\ln_*} H^1(\Gamma, C/2\pi i\mathbb{Z})
\end{array}
\]

where \(R\), \(\ln_*\), and \(Z^*\) are isomorphisms. Therefore, \(\mathcal{A}_\Sigma\) is also an isomorphism.

Definition 9. The generalized Picard group is

\[\text{Pic}_0(X, \Sigma) = \text{Div}(X, \Sigma)/P\text{Div}(X, \Sigma)\]

The group \(\text{Pic}(X, \Sigma)\) is a \(\mathbb{Z}^2\)-graded Abelian group:

\[\text{Pic}(X, \Sigma) = \bigsqcup_{d \in \mathbb{Z}^2(X)} \text{Pic}_d(X, \Sigma)\]

where the multidegree \(d\) graded Picard variety \(\text{Pic}_d(X, \Sigma)\) is the set of \(\Sigma\)-regular divisors of multidegree \(d\) modulo linear equivalence. We denote the \(\Sigma\)-linear equivalence class of a divisor \(D\) by \([D]_\Sigma\), or just \([D]\).

Proposition 1.6 (Abel-Jacobi theorem for nodal curves). The Abel map is an isomorphism between \(\text{Pic}_0(X, \Sigma)\) and \(\text{Jac}(X, \Sigma)\).

Proof. By Proposition 1.5 the Abel map \(\mathcal{A}_\Sigma: \text{Pic}_0(X, \Sigma) \to \text{Jac}(X, \Sigma)\) is injective. Let us prove that it is surjective. Take \(x \in \text{Jac}(X, \Sigma)\). Then \(\pi^*(x) \in \text{Jac}(X)\), and by the classical Abel-Jacobi theorem, there exists \(D \in \text{Div}_U(X)\) such that \(\mathcal{A}(D) = \pi^*(x)\). As it is easy to see, \(D\) may be chosen to be \(\Sigma\)-regular. Then \(x - \mathcal{A}_\Sigma(D) \in \text{Ker} \pi^*\). As it is proved above, the Abel map is an isomorphism between \((\text{PDiv}(X) \cap \text{Div}(X, \Sigma))/\text{PDiv}(X)\) and \(\text{Ker} \pi^*\), so there exists \(D' \in \text{Div}_U(X, \Sigma)\) such that \(\mathcal{A}_\Sigma(D') = x - \mathcal{A}_\Sigma(D)\), i.e. \(x = \mathcal{A}_\Sigma(D' + D)\), q.e.d.

The variety \(\text{Pic}_d(X, \Sigma)\) is thus a principal homogeneous space of the group

\[\text{Pic}_0(X, \Sigma) \simeq \text{Jac}(X, \Sigma)\]

for each multidegree \(d\). In particular, \(\text{Pic}_d(X, \Sigma)\) has a canonical affine structure and its tangent space at each point can be naturally identified with \(\Omega^1(X, \Sigma)^*\).

1.2.5 Partial normalizations and subcurves. Let \(\Lambda \subset \Sigma\). Then \(X/\Lambda\) is a partial normalization of the curve \(X/\Sigma\). Each \(\Sigma\)-regular divisor is also a \(\Lambda\)-regular divisor, hence there is a natural inclusion map \(\text{Div}(X, \Sigma) \to \text{Div}(X, \Lambda)\).
**Proposition 1.7.** There exists a unique graded epimorphism $\pi^*_\Lambda$ which makes the following diagram commutative:

$$
\begin{array}{c}
\text{Div}(X, \Sigma) & \longrightarrow & \text{Div}(X, \Lambda) \\
\downarrow & & \downarrow \\
\text{Pic}(X, \Sigma) & \overset{\pi^*_\Lambda}{\longrightarrow} & \text{Pic}(X, \Lambda)
\end{array}
$$

where the upper arrow is the natural inclusion, and vertical arrows are natural projections.

**Proof.** Obvious.

The notation $\pi^*_\Lambda$ reflects the fact that this map is backward with respect to the partial normalization map $\pi_\Lambda: X/\Lambda \to X/\Sigma$.

Now, let us discuss subcurves. Let $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, c(X)\}$, and let

$$X_I = \bigsqcup_{i \in I} X_i.$$

Let also

$$\Sigma_I = \{\{P^+_i, P^-_i\} \in \Sigma | P^+_i \in X_I, P^-_i \in X_I\}.$$

Then $X_I/\Sigma_I$ is a subcurve of $X/\Sigma$. Subcurves of $X/\Sigma$ correspond to complete subgraphs of its dual graph. If $X_I/\Sigma_I$ is a sub-curve of $X/\Sigma$, then for each multidegree $d = (d_1, \ldots, d_{c(X)})$ on $X$ there is a natural restriction map $\text{Div}_d(X, \Sigma) \to \text{Div}_{d_I}(X_I, \Sigma_I)$ where $d_I = (d_{i_1}, \ldots, d_{i_p})$.

**Proposition 1.8.** There exists a unique epimorphism $i^*_I$ which makes the following diagram commutative:

$$
\begin{array}{c}
\text{Div}_d(X, \Sigma) & \longrightarrow & \text{Div}_{d_I}(X_I, \Sigma_I) \\
\downarrow & & \downarrow \\
\text{Pic}_d(X, \Sigma) & \overset{i^*_I}{\longrightarrow} & \text{Pic}_{d_I}(X_I, \Sigma_I)
\end{array}
$$

where the upper arrow is the natural restriction map, and vertical arrows are natural projections.

**Proof.** Obvious.

The notation $i^*_I$ reflects the fact that this map is backward with respect to the natural inclusion map $i_I: X_I/\Sigma_I \to X/\Sigma$.

### 1.2.6 Riemann’s inequality and effective divisors

Let $D \in \text{Div}(X, \Sigma)$, and let

$$L(D, \Sigma) = \{f \in M(X, \Sigma) \mid \text{ord}_P f \geq -D(P) \quad \forall \ P \in X\}$$

where $\text{ord}_P f$ is the order of $f$ at the point $P$, and we set $\text{ord}_P f = \infty$ if $P \in X_i$ and $f \mid_{X_i} \equiv 0$. Obviously, the set $L(D, \Sigma)$ is a vector space.

**Proposition 1.9** (Riemann’s inequality for nodal curves). For each $D \in \text{Div}(X, \Sigma)$, the following inequality holds

$$\dim L(D, \Sigma) \geq |\deg D| - g(X, \Sigma) + c(X, \Sigma).$$

**Proof.** Let $L(D) = L(D, \emptyset)$. We have

$$\dim L(D) = \sum_{i=1}^{c(X)} \dim L(D \mid x_i) \geq \sum_{i=1}^{c(X)} (\deg D \mid x_i - g(x_i) + 1) = |\deg D| + c(X, \Sigma) - g(X, \Sigma).$$

Consider a map $\delta: L(D) \to \mathbb{C}^{[\Sigma]}$ given by $\delta(f) = (f(P^+_1), f(P^-_1), \ldots)$. We have $L(D, \Sigma) = \text{Ker} \delta$, so

$$\dim L(D, \Sigma) = \dim L(D) - \dim \text{Im} \delta \geq \dim L(D) - |\Sigma| = |\deg D| - g(X, \Sigma) + c(X, \Sigma).$$

Let $D$ and $D'$ be $\Sigma$-linearly equivalent divisors. Then it easy to see that there exists an isomorphism $\phi: L(D, \Sigma) \to L(D', \Sigma)$. This allows us to define the set

$$W_d(X, \Sigma) = \{[D] \in \text{Pic}_d(X, \Sigma) : L(D, \Sigma) \neq 0\}.$$
Definition 10. Let \( d \) be a multidegree of total degree \( g(X, \Sigma) - c(X, \Sigma) \). Then \( d \) is called semistable if for each subcurve \( X_I / \Sigma_I \subset X / \Sigma \) we have
\[
|d_I| \geq g(X_I, \Sigma_I) - c(X_I, \Sigma_I). \tag{15}
\]

Proposition 1.10. Let \( d \) be a multidegree of total degree \( g(X, \Sigma) - c(X, \Sigma) \). Then
a) if \( d \) is semistable, then \( W_d(X, \Sigma) \) has a positive codimension in \( \text{Pic}_d(X, \Sigma) \);
b) if \( d \) is not semistable, then \( W_d(X, \Sigma) = \text{Pic}_d(X, \Sigma) \).

This result is due to Beauville [25] and Alexeev [24]. We will get a proof of this statement as a by-product of our further considerations.

Similarly to \( W_d(X, \Sigma) \), we define \( E_d(X, \Sigma) \subset \text{Pic}_d(X, \Sigma) \) as the set of those classes of divisors which are representable by effective divisors. Obviously, we have \( E_d(X, \Sigma) \subset W_d(X, \Sigma) \).

Moreover, these two sets are equal for non-singular connected curves.

If we define \( \bar{X}_i = X_i \setminus (X_i \cap \text{supp}(\Sigma)) \), then the set \( E_d(X, \Sigma) \) can be described as the image of the map
\[
(\bar{X}_1)^{d_1} \times \cdots \times (\bar{X}_k)^{d_k} \to \text{Pic}_d(X, \Sigma)
\]
which maps a collection of points to the corresponding effective divisor. This description makes it obvious that \( E_d(X, \Sigma) \) has a positive codimension in \( \text{Pic}_d(X, \Sigma) \) if \( |d| < g(X, \Sigma) \). However, for reducible curves, the set \( E_d(X, \Sigma) \) can have a positive codimension even if \( |d| \geq g(X, \Sigma) \).

This motivates us to give the following definition.

Definition 11. Let \( d \) be a multidegree of total degree \( g(X, \Sigma) \). We say that \( d \) is uniform if for each subcurve \( X_I / \Sigma_I \subset X / \Sigma \) we have
\[
|d_I| \geq g(X_I, \Sigma_I).
\]

Proposition 1.11. Let \( d \) be a multidegree of total degree \( g(X, \Sigma) \).

a) if \( d \) is uniform, then \( E_d(X, \Sigma) \) is dense in \( \text{Pic}_d(X, \Sigma) \);
b) if \( d \) is not uniform, then \( E_d(X, \Sigma) \) has positive codimension in \( \text{Pic}_d(X, \Sigma) \).

We shall postpone the proof of this result until we have obtained several preliminary statements. Let us consider the following decomposition of the space \( L(D, \Sigma) \):
\[
L(D, \Sigma) = L^{(1)}(D, \Sigma) \cup L^{(2)}(D, \Sigma) \cup L^{(n)}(D, \Sigma)
\]
where
\[
L^{(1)}(D, \Sigma) = L(D, \Sigma) \cap \mathcal{M}^+(X, \Sigma), \\
L^{(2)}(D, \Sigma) = \{ f \in L(D, \Sigma) : f |_{X_i} \equiv 0 \text{ for some } 1 \leq i \leq k \}, \\
L^{(n)}(D, \Sigma) = \{ f \in L(D, \Sigma) \setminus L^{(1)}(D, \Sigma) : f(P_i^+) = 0 \text{ for some } 1 \leq i \leq |\Sigma| \}.
\]

Proposition 1.12. Let \( D \in \text{Div}_d(X, \Sigma) \). Then \( D \in E_d(X, \Sigma) \) if and only if \( L^{(1)}(D, \Sigma) \neq \emptyset \).

Proof. Obvious. \( \square \)

It is easy to see that if \( D \) and \( D' \) are \( \Sigma \)-linearly equivalent, then the aforementioned isomorphism \( \phi : L(D, \Sigma) \to L(D', \Sigma) \) maps \( L^{(1)}(D, \Sigma) \) to \( L^{(1)}(D', \Sigma) \) and \( L^{(n)}(D, \Sigma) \) to \( L^{(n)}(D', \Sigma) \). Therefore, for each multi-degree \( d \), we can define the sets
\[
R_d(X, \Sigma) = \{ [D] \in \text{Pic}_d(X, \Sigma) : L^{(1)}(D, \Sigma) \neq 0 \}, \\
N_d(X, \Sigma) = \{ [D] \in \text{Pic}_d(X, \Sigma) : L^{(n)}(D, \Sigma) \neq \emptyset \}.
\]

We have \( W_d(X, \Sigma) = E_d(X, \Sigma) \cup R_d(X, \Sigma) \cup N_d(X, \Sigma) \). We note that \( R_d \) is empty for irreducible curves, while \( N_d \) is empty for non-singular curves.

Proposition 1.13. Let \( d \) be a multidegree of total degree \( |d| \leq g(X, \Sigma) \). Then \( N_d(X, \Sigma) \) has positive codimension in \( \text{Pic}_d(X, \Sigma) \).

Proof. Let \( \Lambda \subset \Sigma \), \( S = \text{supp}(\Sigma \setminus \Lambda) \) and let \( D_\Lambda = \sum_{P \in S} P \). Let also \( e(\Lambda) = d - \deg D_\Lambda \). We claim that
\[
N_d(X, \Sigma) \subset \bigcup_{\Lambda \subset \Sigma} (\pi_\Lambda)^{-1}(E_{e(\Lambda)}(X, \Lambda) + [D_\Lambda]).
\]

Indeed, let \([D] \in N_d(X, \Sigma) \). Then there exists \( f \in L^{(1)}(X, \Sigma) \). Assume that \( f \) vanishes at points \( P_1^+ \), \( P_{-1}^+ \), \( P_2^+ \), \( P_{-2}^+ \), and let
\[
\Lambda = \Sigma \setminus \{ (P_1^+, P_{-1}^-), \ldots , (P_{-2}^-, P_{-2}^+) \}.
\]

We have \( f \in L^{(1)}(D - D_\Lambda, \Lambda) \), which implies that \([D] \in (\pi_\Lambda)^{-1}(E_{e(\Lambda)}(X, \Lambda) + [D_\Lambda]) \), q.e.d.
Further, we have
\[ |e(\Lambda)| = |d| - 2p < |d| - p \leq g(X, \Sigma) - p \leq g(X, \Lambda), \]
so \( E_{\omega(\Lambda)}(X, \Lambda) \) has positive codimension in \( \text{Pic}_{\omega(\Lambda)}(X, \Lambda) \), which implies the proposition. \( \square \)

Let \( X_I / \Sigma_I \subset X / \Sigma \) be a subcurve. Taking \( I' = \{ 1, \ldots, c(X) \} \setminus I \), we obtain the complimentary subcurve \( X_{I'} / \Sigma_{I'} \). Let us define the number \( \kappa(I) = |\Sigma| - |\Sigma_I| - |\Sigma_{I'}| \) which is equal to the geometric number of points in the intersection \( X_I / \Sigma_I \cap X_{I'} / \Sigma_{I'} \).

**Definition 12.** Let \( d \) be a multidegree. We say that \( d \) is R-semistable if for each proper non-empty subcurve \( X_I / \Sigma_I \subset X / \Sigma \) we have
\[ |d_I| \leq g(X_I, \Sigma_I) - c(X_I, \Sigma_I) + \kappa(I). \quad (16) \]

**Proposition 1.14.** Let \( d \) be a multidegree. Then
\begin{itemize}
  \item[a)] if \( d \) is R-semistable, then \( R_d(X, \Sigma) \) has positive codimension in \( \text{Pic}_d(X, \Sigma) \);
  \item[b)] if \( d \) is not R-semistable, then \( R_d(X, \Sigma) = \text{Pic}_d(X, \Sigma) \).
\end{itemize}

**Proof.** Assume that \( d \) is R-semistable. Let \( X_I / \Sigma_I \subset X / \Sigma \) be a subcurve, and let \( X_{I'} / \Sigma_{I'} \) be the complimentary subcurve. We have the following decomposition of the dual graph:
\[ \Gamma(X, \Sigma) = \Gamma(X_I, \Sigma_I) \cup \Gamma(X_{I'}, \Sigma_{I'}) \cup \{ \epsilon_{j_1}, \ldots, \epsilon_{j_n} \} \]
where \( \kappa = \kappa(I) \). Without loss of generality, we assume that \( P_{j_1}^+, \ldots, P_{j_n}^+ \in X_I \), and \( P_{j_1}^-, \ldots, P_{j_n}^- \in X_{I'} \). Let \( D_I = P_{j_1}^+ + \cdots + P_{j_n}^+ \in \text{Div}(X_I, \Sigma_I) \), and let \( e(I) = d_I - \deg D_I \). We claim that
\[ R_d(X, \Sigma) \subset \bigcup_{\ell \subset J} (\ell_i)^{-1} \left( (E_{\omega(I)}(X_I, \Sigma_I) \cup N_{\omega(I)}(X_I, \Sigma_I)) + [D_I] \right) \]
where \( J = \{ 1, \ldots, c(X) \} \). Indeed, let \( [D] \in R_d(X, \Sigma) \). Then there exists \( f \in L^{(\ell)}(D, \Sigma) \), \( f \neq 0 \). Assume that \( f \equiv 0 \) on irreducible components \( X_{i_1}, \ldots, X_{i_p} \). Let \( I' = \{ i_1, \ldots, i_p \} \), and let \( I = J \setminus I' \). Let also \( f_I = f |_{X_I} \). Then
\[ f_I \in L^{(\ell)}(D |_{X_I} - D_I, \Sigma_I) \cup L^{(\ell)}(D |_{X_I} - D_I, \Sigma_I), \]
therefore \( [D] \in (\ell_i)^{-1} \left( (E_{\omega(I)}(X_I, \Sigma_I) \cup N_{\omega(I)}(X_I, \Sigma_I)) + [D_I] \right) \), q.e.d.

Further, we have
\[ |e(I)| = |d_I| - \deg D_I = |d_I| - \kappa(I) < g(X_I, \Sigma_I), \]
so \( E_{\omega(I)}(X_I, \Sigma_I) \) and \( N_{\omega(I)}(X_I, \Sigma_I) \) have positive codimension in \( \text{Pic}_{\omega(I)}(X_I, \Lambda_I) \), which implies that \( R_d(X, \Sigma) \) has positive codimension in \( \text{Pic}_d(X, \Sigma) \).

Now, assume that \( d \) is not R-semistable. Then there exists a subcurve such that
\[ |d_I| \geq g(X_I, \Sigma_I) - c(X_I, \Sigma_I) + \kappa(I) + 1. \]
Let \( [D] \in \text{Pic}_d(X, \Sigma) \). We shall prove that \( L^{(\ell)}(D, \Sigma) \neq 0 \). We have
\[ |\deg(D |_{X_I} - D_I)| \geq g(X_I, \Sigma_I) - c(X_I, \Sigma_I) + 1, \]
so by Riemann’s inequality there exists \( f \neq 0 \) such that \( f \in L(D |_{X_I} - D_I, \Sigma_I) \). Take \( \tilde{f} = f \) for \( P \in X_I \), and \( \tilde{f} = 0 \) for \( P \notin X_I \). Then \( f \in L^{(\ell)}(D, \Sigma) \), q.e.d.

**Proof of Proposition** [1.10] Using the obvious formula
\[ g(X, \Sigma) = g(X_I, \Sigma_I) + g(X_{I'}, \Sigma_{I'}) + \kappa(I) + c(X, \Sigma) - c(X_I, \Sigma_I) - c(X_{I'}, \Sigma_{I'}), \quad (17) \]
we show that if \( |d| = g(X, \Sigma) - c(X, \Sigma) \), then inequality [15] for a subcurve \( X_I / \Sigma_I \) is equivalent to inequality [10] for the complimentary subcurve \( X_{I'} / \Sigma_{I'} \), and vice versa. So, for \( |d| = g(X, \Sigma) - c(X, \Sigma) \), the notions “semistable” and “R-semistable” coincide (cf. [13], Remark 1.3.3). Therefore, if the multidegree \( d \) is semistable, then it is R-semistable, and the set \( R_d(X, \Sigma) \) has positive codimension in \( \text{Pic}_d(X, \Sigma) \). At the same time, since \( |d| < g(X, \Sigma) \), the sets \( E_d(X, \Sigma) \) and \( N_d(X, \Sigma) \) also have positive codimension in \( \text{Pic}_d(X, \Sigma) \), thus the same is true for \( W_d(X, \Sigma) \).

Vice versa, if \( d \) is not semistable, then it is not R-semistable, therefore \( R_d(X, \Sigma) = \text{Pic}_d(X, \Sigma) \), and \( W_d(X, \Sigma) = \text{Pic}_d(X, \Sigma) \). \( \square \)
Proof of Proposition \[\text{[17]}\]. Assume \(d\) uniform. If the curve \(X/\Sigma\) can be represented as the disjoint union of two proper subcurves \(X_I/\Sigma_I\) and \(X_{I'}/\Sigma_{I'}\), then the multidegrees \(d_I = d |_{X_I}\) and \(d_{I'} = d |_{X_{I'}}\) are also uniform. Furthermore, we have
\[
\text{Pic}_d(X, \Sigma) = \text{Pic}_{d_I}(X_I, \Sigma_I) \times \text{Pic}_{d_{I'}}(X_{I'}, \Sigma_{I'}), \quad E_d(X, \Sigma) = E_{d_I}(X_I, \Sigma_I) \times E_{d_{I'}}(X_{I'}, \Sigma_{I'}).
\]
Therefore, without loss of generality, we may assume that \(X/\Sigma\) is connected. Let us show that uniform multidegrees on connected curves are \(R\)-semistable. Using \([17]\), we get
\[
g(X, \Sigma) \leq g(X_I, \Sigma_I) + g(X_{I'}, \Sigma_{I'}) + \kappa(I) - c(X_I, \Sigma_I).
\]
Using uniformity condition, we have
\[
|d_I| = |d| - |d_{I'}| = g(X, \Sigma) - |d_{I'}| \leq g(X, \Sigma) - g(X_{I'}, \Sigma_{I'}) \leq g(X_I, \Sigma_I) + \kappa(I) - c(X_I, \Sigma_I).
\]
We conclude that \(R_d(X, \Sigma)\) has positive codimension in \(\text{Pic}_d(X, \Sigma)\). Since \(N_d(X, \Sigma)\) also has positive codimension, and
\[
\text{Pic}_d(X, \Sigma) = W_d(X, \Sigma) = E_d(X, \Sigma) \cup R_d(X, \Sigma) \cup N_d(X, \Sigma),
\]
we conclude that \(E_d(X, \Sigma)\) is dense in \(\text{Pic}_d(X, \Sigma)\), q.e.d.

Now, assume that \(d\) is not uniform. Then there exists a subcurve \(X_I/\Sigma_I\) such that \(|d_I| < g(X_I, \Sigma_I)|\), so that the set \(E_{d_I}(X_I, \Sigma_I)\) has positive codimension in \(\text{Pic}_{d_I}(X_I, \Sigma_I)\). At the same time, we have an inclusion \(\iota_d^{-1}(E_{d_I}(X, \Sigma)) \subseteq E_{d_I}(X_I, \Sigma_I)\), which proves that \(E_{d_I}(X, \Sigma)\) has positive codimension in \(\text{Pic}_{d_I}(X, \Sigma)\), and thus is not dense.

1.2.7 On more general curves. We can generalize the discussion of this section to a slightly more general class of curves which we call \textit{generalized nodal curves}. Let \(X = X_1 \sqcup \ldots \sqcup X_\sigma\) be a disjoint union of connected Riemann surfaces, and let \(\Sigma = \{\mathcal{P}_1, \ldots,\mathcal{P}_\sigma\}\) be a set of pairwise disjoint finite subsets of \(X\). A generalized nodal curve \(X/\Sigma\) is obtained from \(X\) by gluing points within each \(\mathcal{P}_i\) to a single point. The ring of meromorphic functions is defined as
\[
\mathcal{M}(X, \Sigma) = \{f \in \mathcal{M}(X) \mid P, Q \in \mathcal{P}_i \Rightarrow f(P) = f(Q) \neq \infty \forall i = 1, \ldots, \sigma\},
\]
and \(\Sigma\)-regular differentials are those which are holomorphic outside \(\text{supp}(\Sigma) = \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_\sigma\) and satisfy
\[
\sum_{P \in \mathcal{P}_i} \text{Res}_P \omega = 0
\]
for each \(i\). The study of such curves can be easily reduced to nodal curves. Assume that \(\mathcal{P}_i = \{P_{i,1}, \ldots, P_{i,s}\}\). Consider a Riemann sphere \(Y \cong \mathbb{CP}^1\) with \(s\) marked points \(Q_{1,i}, \ldots, Q_{s,i}\). Let \(X' = X \sqcup Y\), and let
\[
\Sigma' = \Sigma \cup \{\{P_{i,1}, Q_{1,i}\}, \ldots, \{P_{i,s}, Q_{s,i}\}\} \setminus \{\mathcal{P}_i\}.
\]
The curve \(X'/\Sigma'\) is “equivalent” to \(X/\Sigma\) in the following sense: there are natural identifications
\[
\mathcal{M}(X, \Sigma) \simeq \{f \in \mathcal{M}(X', \Sigma') \mid f \mid_{Y = \text{const}}\}, \quad \Omega^1(X, \Sigma) \simeq \Omega^1(X', \Sigma').
\]
These identifications allow to reduce the study of \(X/\Sigma\) to \(X'/\Sigma'\). Repeating this operation for each \(i\) such that \(|\mathcal{P}_i| > 2\), we obtain a nodal curve, which shows that all results of this section are true for generalized nodal curves as well.

Proposition 1.15. The \textit{arithmetic genus} of a generalized nodal curve is given by
\[
g(X, \Sigma) = g(X) + |\text{supp}(\Sigma)| - |\Sigma| + c(X, \Sigma) - c(X).
\]

1.3 Proof of Theorem 1

1.3.1 Preliminaries. Let \(X\) be the non-singular compact model of \(C\). Then \(\lambda\) and \(\mu\) are meromorphic functions on \(X\). Using the simplicity of the spectrum of \(J\), we conclude that \(\lambda\) has exactly \(n\) simple poles on \(X\). We denote these poles by \(\infty_1, \ldots, \infty_n\). Without loss of generality, we may assume that \(J = \text{diag}(j_1, \ldots, j_n)\), and that \(\mu \lambda^{-m}\) takes value \(j_i\) at the point \(\infty_i\). We also define
\[
D_\infty = \sum_{i=1}^n \infty_i \in \text{Div}_n(X),
\]
and \(X_\infty = \text{supp}(D_\infty)\). Let us also consider the projection \(\pi: X \setminus X_\infty \to C\) given by \(P \mapsto (\lambda(P), \mu(P))\), and let \(X_S = \pi^{-1}(\text{Sing} C)\). Let also
\[
D_s = \sum_{P \in X_S} P \in \text{Div}(X).
\]
Let us find the genus of $X$. Let $(\lambda)_R = \sum_{P \in X}(\text{mult}_P \lambda - 1)$ be the ramification divisor of $\lambda: X \to \mathbb{C}P^1$. The following is clear.

**Proposition 1.16.** We have $(\lambda)_R = (\partial_X/\partial \mu)|_0 - D$, where $\chi$ is the defining polynomial of the curve $C$, and $(\ldots)_0$ denotes the divisor of zeros.

We conclude that
\[
\deg(\lambda)_R = \deg \left( \frac{\partial \chi}{\partial \mu} \right)_0 - \deg D = \deg \left( \frac{\partial \chi}{\partial \mu} \right)_\infty - \deg D
\]
where $(\ldots)_\infty$ is the pole divisor. Further, from the condition $C \in \mathcal{C}_{\text{spec}}$, we easily conclude that $|\deg (\partial \chi/\partial \mu)_\infty| = mn(n-1)$, so, by Riemann-Hurwitz formula, we have
\[
g(X) = c(X) - n + \frac{1}{2}|\deg(\lambda)_R| = c(X) - n + \frac{mn(n-1)}{2} - |\text{Sing} C|.
\]

We shall work with two singularizations of $X$, namely the curves $X_K$ and $X'_K$. In terms of Section 1.2, they are described as follows.

Let $K = \text{Sing } C \setminus K$, and assume that $K = \{P_1, \ldots, P_s\}$. Then $\pi^{-1}(P_i)$ consists of two points $P^-_i, P^+_i$. Let us define
\[
\Sigma = \{\{P^-_1, P^+_1\}, \ldots, \{P^-_s, P^+_s\}\}.
\]
Then $X/\Sigma = X_K$. Similarly, we define $\Sigma' = \Sigma \cup \{X_\infty\}$, so that $X/\Sigma' = X'_K$.

**Proposition 1.17.** We have
\[
g(X, \Sigma) = \frac{mn(n-1)}{2} - |K| + c(X, \Sigma) - n, \quad g(X, \Sigma') = \frac{mn(n-1)}{2} - |K|.
\]

**Proof.** Use formula (18) and Proposition 1.16. \qed

### 1.3.2 Construction of the map $\Phi$

Let us construct a map $\Phi: S^N_X \to \text{Pic}(X, \Sigma')$. Let $L \in S^N_X$, and let $X_0 = X \setminus (\pi^{-1}(K) \cup X_\infty)$. Then $X_0 \in X$. Then the matrix $L(\lambda(P)) - \mu(P)E$ is finite and has one-dimensional kernel. In this way, we obtain a holomorphic mapping $\psi: X_0 \to \mathbb{C}P^{n-1}$ which maps $P$ to $\text{Ker } L(\lambda(P)) - \mu(P)E$. As it is easy to see, the mapping $\psi$ can be uniquely extended to a holomorphic mapping defined on the whole $X$ (see e.g. [27, Proposition 8.2]). Obviously, we have $\psi(P) \in \text{Ker } L(\lambda(P)) - \mu(P)E$ for each $P \in X \setminus X_\infty$, and $\psi(\infty)^i = \delta_i^\ell$ where $\delta_i^\ell$ is the Kronecker delta.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^*)^n$, and let
\[
D_\alpha = \left( \sum_{i=1}^n \alpha_i \psi^i \right)_0.
\]
In other words, we define
\[
h_\alpha = \psi \left( \sum_{i=1}^n \alpha_i \psi^i \right)^{-1},
\]
and set $D_\alpha = (h_\alpha)_\infty$.

We would like to fix $\alpha$, and set $\Phi(L) = [D_\alpha]_{\Sigma'}$ for each $L \in S^N_X$. However, this is not possible, since the divisor $D_\alpha$ is not necessarily $\Sigma'$-regular. Nevertheless, for each $L \in S^N_X$, we can find $\alpha$ such that $D_\alpha$ is $\Sigma'$-regular. The problem is that if we take distinct $\alpha, \beta \in (\mathbb{C}^*)^n$, then $D_\alpha$ and $D_\beta$ are $\Sigma'$-linearly equivalent, but not $\Sigma'$-linearly equivalent. Let us show how to overcome this difficulty.

Let $\alpha \in (\mathbb{C}^*)^n$, and choose $f_\alpha \in \mathcal{M}^\Sigma(X, \Sigma)$ such that $f_\alpha(\infty) = \alpha$. Then $(f_\alpha)$ is a $\Sigma'$-regular divisor, and its $\Sigma'$-linear equivalence class does not depend on the choice of $f_\alpha$.

**Proposition 1.18.** Assume that $D_\alpha$ and $D_\beta$ are $\Sigma'$-regular. Then $D_\alpha - (f_\alpha) \sim D_\beta - (f_\beta)$.

**Proof.** Let
\[
f = \left( \sum \alpha_i \psi^i \right) f_\alpha^{-1} \left( \sum \beta_i \psi^i \right)^{-1} f_\beta.
\]
Then $D_\alpha - (f_\alpha) = D_\beta + (f_\beta) = (f)$, and $f \in \mathcal{M}^\Sigma(X, \Sigma')$. \qed

Let $U_\alpha = \{L \in S^N_X: D_\alpha \in \text{Div}(X, \Sigma')\}$. Define $\Phi_\alpha: U_\alpha \to \text{Pic}(X, \Sigma')$ by setting
\[
\Phi_\alpha(L) = [D_\alpha - (f_\alpha)]_{\Sigma'}.
\]
Then $S^N_X = \bigcup_{\alpha} U_\alpha$, and for each $L \in U_\alpha \cap U_\beta$, we have $\Phi_\alpha(L) = \Phi_\beta(L)$. In this way, we obtain a mapping $\Phi: S^N_X \to \text{Pic}(X, \Sigma')$ given by $\Phi(L) = \Phi_\alpha(L)$ for each $L \in U_\alpha$. 

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1.3.3 Multidegree count. We have \( \deg \Phi(L) = \deg D_a \). To find the total degree of \( D_a \), we use a standard trick (see [25], Chapter 5.2). Let \( a \in \mathbb{C} \) be a regular value of the function \( \lambda \), and let \( \lambda^{-1}(a) = \{ P_1, \ldots, P_n \} \). Set

\[
r(a) = \det^2(h_a(P_1), \ldots, h_a(P_n)).
\]

Then it is easy to see that \( r \) can be extended to a meromorphic function on the whole \( \mathbb{C} \). Clearly, we have

\[
\deg (r)_\infty = 2|\deg (h_a)_\infty| = 2|\deg D_a|.
\]

To count the poles of \( r \), we count its zeros. As it is easy to see, it is possible to choose such \( \alpha \) that \( L(\lambda) \in U_a \) and the divisor \( \lambda_a(D_a) \) does not intersect the divisor \( \lambda_\ast((\partial \chi/\partial \mu)_a) \) where \( \chi \) is the defining polynomial of \( C \). Further, for the sake of simplicity, we shall assume that the curve \( C \) satisfies the following genericity assumption: for each \( a \in \mathbb{C} \), we have

\[
|C \cap \{ \lambda = a \}| \geq n - 1. \tag{19}
\]

This means that each line \( \lambda = a \) can contain either at most one node, or at most one simple ramification point. Furthermore, for all nodes of \( C \), both tangents are non-vertical. It follows that the divisor \( (\lambda_a = (\lambda - a)_a) \) can be of one of the following types:

1. \( (\lambda)_a = \sum_{i=1}^n P_i \).
2. \( (\lambda)_a = 2P_{n-1} + \sum_{i=1}^{n-2} P_i \).
3. \( (\lambda)_a = P^+ + P^- + \sum_{i=1}^{n-2} P_i, \pi(P^\pm) \in K \).
4. \( (\lambda)_a = P^+ + P^- + \sum_{i=1}^{n-2} P_i, \pi(P^\pm) \in \hat{K} \).

In all cases, the points \( P_1, \ldots, P_{n-1}, P^+, P^- \) are pairwise distinct, and \( \pi(P) \notin \text{Sing}(C) \).

**Proposition 1.19.** Let \( (\lambda)_a = \sum_{i=1}^n P_i \). Then \( r(a) \neq 0 \).

**Proof.** Obvious. \( \square \)

**Proposition 1.20.** Let \( (\lambda)_a = 2P_{n-1} + \sum_{i=1}^{n-2} P_i \). Then

a) the matrix \( L(a) \) has a 2 \times 2 Jordan block with eigenvalue \( \mu(P_{n-1}) \), eigenvector \( h_a(P_{n-1}) \), and generalized eigenvector \( (h_a)^\ast(P_{n-1}) \) where \( z \) is any local coordinate near \( P_{n-1} \);

b) \( r \) has a simple zero at \( a \).

**Proof.** Take a local coordinate \( z \) such that \( \lambda = a + z^2 \). Differentiating the equation

\[
(L(\lambda(z)) - \mu(z)E)h_a(z) = 0
\]

with respect to \( z \), we prove item a). To prove item b), let \( b \to a \), then

\[
r(b) = 4(\lambda - a) \left( \det^2 \left( h_a(P_1), \ldots, h_a(P_{n-2}), h_a(P_{n-1}), \frac{d h_a}{d z}(P_{n-1}) \right) + o(1) \right). \]

\( \square \)

**Proposition 1.21.** Let \( (\lambda)_a = P^+ + P^- + \sum_{i=1}^{n-2} P_i \) where \( \pi(P^\pm) \in K \). Then

a) the vectors \( h_a(P^+) \) and \( h_a(P^-) \) are linearly independent;

b) \( r(a) \neq 0 \).

**Proof.** If we assume that \( h_a(P^+) \) and \( h_a(P^-) \) are linearly dependent, then we necessarily have \( h_a(P^+) = h_a(P^-) \). Take \( \lambda \) as a local coordinate near \( P^+ \) and \( P^- \). Differentiating

\[
(L(\lambda) - \mu E)h_a(\lambda) = 0
\]

with respect to \( \lambda \) at \( P^+ \) and \( P^- \) and subtracting the obtained equations, we get

\[
(L(a) - \mu(P^\pm)E) \left( \frac{d h_a}{d \lambda}(P_+) - \frac{d h_a}{d \lambda}(P_-) \right) = \left( \frac{d \mu}{d \lambda}(P_+) - \frac{d \mu}{d \lambda}(P_-) \right) h_a(P^\pm). \tag{20}
\]

Since the singular point \( \pi(P^\pm) \) is nodal, we have \( d\mu/d\lambda(P_+) \neq d\mu/d\lambda(P_-) \), so \( L_a \) has a Jordan block, which contradicts the definition of the set \( K \). This proves item a). Item b) obviously follows. \( \square \)

**Proposition 1.22.** Let \( (\lambda)_a = P^+ + P^- + \sum_{i=1}^{n-2} P_i \) where \( \pi(P^\pm) \in \hat{K} \). Then

a) the matrix \( L(a) \) has a 2 \times 2 Jordan block with eigenvalue \( \mu(P^+) = \mu(P^-) \), eigenvector \( h_a(P^+) = h_a(P^-) \), and generalized eigenvector \( (h_a)^\ast(P_+) - (h_a)^\ast(P_-) \);

b) \( r \) has a double zero at \( a \).
Proof. Since the geometric multiplicity of \( \mu(P^\pm) \) is equal to 1, we have \( h_a(P^+) = h_a(P^-) \), which implies equation (19) and hence item a). To prove item b), let \( b \to a \), then

\[
q.e.d.
\]

We have

\[
\text{Proof.}
\]

\[
\text{Let}
\]

\[
D_\Sigma = \sum_{i=1}^{2|\Sigma|} (P_i^+ + P_i^-).
\]

Considering Propositions 1.20, 1.21 we conclude with the following:

**Proposition 1.23.** We have \( (r)_a = \lambda_a((\lambda)_R + D_\Sigma) \).

Using (15), we conclude that

\[
\deg D_\alpha = \frac{1}{2} \deg (r)_\infty = \frac{1}{2} \deg (r)_a = \frac{mn(a-1)}{2} - |K| = g(X, \Sigma').
\]

Now, let us prove that \( \deg D_\alpha \) is uniform. Let \( X_I/X_I' \) be a subcurve of \( X/\Sigma' \), and let \( q = \deg \lambda | x_I \). Then \( |X_I \cap X_\infty| = q \). Without loss of generality, we may assume that \( \infty_1, \ldots, \infty_q \in X_I \). Let \( pr: \mathbb{C}^n \to \mathbb{C}^q \) be a map given by \( pr(x^1, \ldots, x^n) = (x^1, \ldots, x^q) \). Let also \( a \in \mathbb{C} \) be a regular value of the function \( \lambda \). Then \( \lambda^{-1}(a) \cap X_I \) consists of \( q \) points \( P_1, \ldots, P_q \). Set

\[
r_I(a) = \det^2 (pr(h_a(P_1)), \ldots, pr(h_a(P_q))).
\]

We have \( r_I(\infty) \neq 0 \), so \( r_I \neq 0 \). Repeating the above arguments, we get

\[
|\deg D_\alpha | x_I | \geq \frac{1}{2} \deg (r)_\infty = \frac{1}{2} \deg (r)_a \geq g(X_I, \Sigma_I'),
\]

q.e.d.

1.3.4 Injectivity.

**Proposition 1.24.** Let \( L \in U_\alpha \). Then \( \dim L(D_\alpha - D_\infty, \Sigma) = 0 \).

**Proof.** Let \( \hat{\lambda}: L(D_\alpha - D_\infty, \Sigma) \to L(D_\alpha, \Sigma) \) and \( A: L(D_\alpha, \Sigma) \to L(D_\alpha, \Sigma - D_\infty) \) be linear maps given by

\[
\hat{\lambda}f = \lambda f, \quad Af = f - \sum_{i=1}^{n} \alpha_i f(\infty_i) h^i_\alpha.
\]

Assume that \( \dim L(D_\alpha - D_\infty, \Sigma) > 0 \). Then the operator

\[
A\hat{\lambda}: L(D_\alpha - D_\infty, \Sigma) \to L(D_\alpha - D_\infty, \Sigma)
\]

must have an eigenvector \( g_\alpha \in L(D_\alpha - D_\infty, \Sigma) \). Denote the corresponding eigenvalue by \( a \). We have

\[
(\lambda - a) g_\alpha = \sum_{i=1}^{n} c_i h^i_\alpha \tag{21}
\]

where \( c_1, \ldots, c_n \in \mathbb{C} \). As it is easy to see, the spectrum of \( p\hat{\lambda} \) does not depend on the choice of \( \alpha \), so we can assume that \( \text{supp}(D_\alpha) \cap \lambda^{-1}(a) \) is empty. Then (21) implies that

\[
\sum_{i=1}^{n} c_i h^i_\alpha(P) = 0 \quad \forall \ P \in \lambda^{-1}(a). \tag{22}
\]

Let us assume that genericity assumption (19) is satisfied, and hence \( (\lambda)_a \) belongs to one of the four aforementioned types. Let us consider each of these types and show that (22) cannot hold. The proof in the general case is analogous.

1. Let \( (\lambda)_a = \sum_{i=1}^{n} P_i \). Then (22) implies that \( h_\alpha(P_1), \ldots, h_\alpha(P_n) \) are linearly dependent, which is not possible.

2. Let \( (\lambda)_a = 2P_{n-1} + \sum_{i=2}^{n-2} P_i \). Let \( z \) be a local coordinate near \( P_{n-1} \). We have \( \lambda_z(P_{n-1}) = 0 \), so by (21) we have

\[
\sum_{i=1}^{n} c_i \frac{dh^i_\alpha}{dz}(P_{n-1}) = 0
\]

and, using (22), we conclude that \( h_\alpha(P_1), \ldots, h_\alpha(P_{n-2}), h_\alpha(P_{n-1}), (h_\alpha)_z'(P_{n-1}) \) are linearly dependent. By item a) of Proposition 1.20 this is not possible.
3. Let \((\lambda) = P^+ + P^- + \sum_{i=1}^{n-2} P_i\) where \(\pi(P^\pm) \in K\). In view of Proposition 1.21, this case is analogous to Case 1.

4. Let \((\lambda) = P^+ + P^- + \sum_{i=1}^{n-2} P_i\) where \(\pi(P^\pm) \in K\). By (21), we have

\[
g_\alpha(P^+) - g_\alpha(P^-) = \sum_{i=1}^{n} c_i \left( \frac{d h_i}{d X}(P^+) - \frac{d h_i}{d X}(P^-) \right).
\]

(23)

Since \(g_\alpha \in M(X, \Sigma)\), we should have \(g_\alpha(P^+) = g_\alpha(P^-)\), so (22) and (23) imply that \(h_\alpha(P_1), \ldots, h_\alpha(P_{n-2}), h_\alpha(P^+), (h_\alpha)'_\lambda(P_1) - (h_\alpha)'_\lambda(P_-)\) are linearly dependent. This is impossible by item a) of Proposition 1.22.

\[\Box\]

Proposition 1.25. Let \(L \in U_a\). Then \(\dim L(D_\alpha - D_{\infty} + \infty_i, \Sigma) = 1\).

Proof. Let \(f, g \in L(D_\alpha - D_{\infty} + \infty_i, \Sigma)\). Then \(f(\infty_i) - g(\infty_i) \in L(D_\alpha - D_{\infty}, \Sigma)\). So, by Proposition 1.24 we have \(f(\infty_i) - g(\infty_i) = 0\), and \(\dim L(D_\alpha - D_{\infty} + \infty_i, \Sigma) \leq 1\). On the other hand, \(h_\alpha \in L(D_\alpha - D_{\infty} + \infty_i, \Sigma)\), so \(\dim L(D_\alpha - D_{\infty} + \infty_i, \Sigma) = 1\).

\[\Box\]

Proposition 1.26. Let \(L \in U_a\). Then \(\dim L(D_\alpha, \Sigma') = 1\).

Proof. Consider the linear map \(A : L(D_\alpha, \Sigma') \to L(D_\alpha - D_{\infty}, \Sigma)\) given by \(A(f) = f - f(\infty_i)\). We have

\[\dim L(D_\alpha, \Sigma') \leq \dim L(D_\alpha - D_{\infty}, \Sigma) + \dim \text{Ker} A = 1\]

On the other hand, we have \(1 \in L(D_\alpha, \Sigma')\), so \(\dim L(D_\alpha, \Sigma') = 1\).

\[\Box\]

Now, let us prove that \(\Phi\) is injective. Assume that \(L^{(1)} \neq L^{(2)}\), and that \(\Phi(L^{(1)}) = \Phi(L^{(2)})\). As it is easy to see, there exists \(\alpha\) such that \(L^{(1)}, L^{(2)} \in U_a\). We have

\[|D^{(1)}_{\alpha}|_{\Sigma'} = |D^{(2)}_{\alpha}|_{\Sigma'},\]

so \(D^{(2)}_\alpha - D^{(1)}_\alpha = (f)\) where \(f \in L(D^{(1)}_{\alpha}, \Sigma')\). By Proposition 1.26 we have \(f = \text{const}\), therefore \(D^{(2)}_\alpha = D^{(1)}_\alpha\).

Further, let us show that \(L^{(1)} = L^{(2)}\). We have

\[(h^{(1)}_\alpha)i^t, (h^{(2)}_\alpha)i^t \in L(D^{(1)}_\alpha - D_{\infty} + \infty_i, \Sigma),\]

and using Proposition 1.25 we conclude that \((h^{(1)}_\alpha)i^t\) and \((h^{(2)}_\alpha)i^t\) are proportional. At the same time, we have \(h^{(1)}(\infty_i) = h^{(2)}(\infty_i) = (\infty_i)^{-1}\), so \((h^{(1)}_\alpha)i^t = h^{(2)}_\alpha i^t\), and \(h^{(1)}_\alpha = h^{(2)}_\alpha\). Consequently, for each \(a \in C\), the matrices \(L^{(1)}(a)\) and \(L^{(2)}(a)\) have same eigenvalues and eigenvectors, and must coincide.

1.3.5 Denseness of the image.

Proposition 1.27. Let \(d\) be a multidegree of total degree \(g(X, \Sigma')\). Then \(d\) is uniform on \(X/\Sigma'\) if and only if \(d - \deg D_{\infty}\) is semistable on \(X/\Sigma\).

Proof. This follows from the obvious formula

\[g(X_i, \Sigma_i) - |\deg D_{\infty}| X_{i} = g(X_i, \Sigma_i) - c(X_i, \Sigma_i),\]

satisfied for any \(I \subset \{1, \ldots, c(X)\}\).

\[\Box\]

Let \(d\) be a uniform degree on \(X/\Sigma'\). By Proposition 1.10 the set \(E_d(X, \Sigma')\) is dense in \(\text{Pic}_d(X, \Sigma')\). Further, let \(d_r = d - \deg D_{\infty}\). Then, by Proposition 1.10 the set \(W_{d_r}(X, \Sigma)\) has positive codimension in \(\text{Pic}_{d_r}(X, \Sigma)\). Let

\[E_{d_r}(X, \Sigma') = E_d(X, \Sigma) \cap (\Sigma')^{-1}(\text{Pic}_{d_r}(X, \Sigma) \setminus W_{d_r}(X, \Sigma) + [D_{\infty}]).\]

The set \(E_{d_r}(X, \Sigma')\) is dense in \(\text{Pic}_{d_r}(X, \Sigma')\). Let us show that \(\text{Im} \Phi \supset \text{Pic}_{d_r}(X, \Sigma')\), so that \(\text{Im} \Phi\) is also dense. Let \(\xi \in \text{Pic}_{d_r}(X, \Sigma')\). Then we can find a \(\Sigma'\)-regular effective divisor \(D\) such that \([D] = \xi\). By Riemann’s inequality, we have

\[\dim L(D - D_{\infty} + \infty_i, \Sigma) \geq 1,\]

Let \(h^i \in L(D - D_{\infty} + \infty_i, \Sigma) \setminus \{0\}\). By the construction of the set \(E_{d_r}(X, \Sigma')\), we have

\[\dim L(D - D_{\infty}, \Sigma) = 0,\]

so \(h^i(\infty_i) \neq 0\), and we can normalize \(h^i\) by \(h^i(\infty_i) = 1\). Define \(h = (h^1, \ldots, h^n)\). We need to
show that there exists \( L \in S^S_k \) such that
\[
(L(\lambda(P)) - \mu(P)E)h(P) = 0.
\]

Let \( a \in \mathbb{C} \) be a regular value of \( \lambda \), and let \( \lambda^{-1}(a) = \{P_1, \ldots, P_n\} \). Let
\[
r(a) = \det^2(h(P_1), \ldots, h(P_n)).
\]

**Proposition 1.28.** Proposition 1.14 item b) of Proposition 1.20 item b) of Proposition 1.21 and item b) of Proposition 1.22 hold for \( r(a) \).

**Proof.** Arguments similar to that of Section 1.3.3 show that \( r(a) \) does not have poles except for the pole at infinity, and that \( L(a) - bE \) has two-dimensional kernel if and only if \( (a,b) \) lies outside \( C \), which proves the proposition. q.e.d.

Then \( L(a) \) is meromorphic in \( a \) and satisfies (24). Local analysis using Proposition 1.23 shows that \( L(a) \) does not have poles except for the pole at infinity, and that \( L(a) - bE \) has two-dimensional kernel if and only if \( (a,b) \) lies outside \( C \), which proves the proposition. q.e.d.

### Linearization of flows

Let us consider the solution curve of (4) and show that its

1.3.6 Linearization of flows.

Let us consider a bilinear pairing
\[
\langle \cdot, \cdot \rangle : \mathbb{C}[\mu, \lambda^{-1}] \times \Omega^1(X, \Sigma') \to \mathbb{C}
\]
given by
\[
\langle \phi, \omega \rangle_\Sigma = \sum_{i=1}^n \text{Res}_{\Sigma_i} \phi \omega.
\]

We need to show that the mapping \( \mathbb{C}[\mu, \lambda^{-1}] \to \Omega^1(X, \Sigma')^* \) given by \( \phi \mapsto \langle \phi, \cdot \rangle_\Sigma \) is surjective,
or, which is the same, that the right radical of the form \( \langle , \rangle_\infty \) is trivial. Let \( s = \max, \text{ord}_\infty, \omega \). Then
\[
\text{ord}_\infty, \mu^j \lambda^{-k} \omega \geq k - mj + s,
\]
and if \( k - mj + s = -1 \), then
\[
\text{Res}_\infty, \mu^j \lambda^{-k} \omega = j^j \text{Res}_\infty, \lambda^{s+1} \omega,
\]
and
\[
\langle \mu^j \lambda^{-k}, \omega \rangle_\infty = \sum_{i=1}^{n} j^i \text{Res}_\infty, \lambda^{s+1} \omega.
\]
Assume that
\[
\sum_{i=1}^{n} j^i \text{Res}_\infty, \lambda^{s+1} \omega = 0 \forall j, k \geq 0 : k - mj + s = -1. \tag{27}
\]

Consider two cases.

1. If \( j_i \neq 0 \) for each value of \( i \), then (27) implies that \( \text{Res}_\infty, \lambda^{s+1} \omega = 0 \) for each \( i \), which contradicts the choice of \( s \).

2. If, say, \( j_1 = 0 \), then (27) implies that \( \text{Res}_\infty, \lambda^{s+1} \omega = 0 \) for \( 2 \leq i \leq n \). Therefore, according to our choice of \( s \), we have \( \text{Res}_\infty, \lambda^{s+1} \omega \neq 0 \). At the same time, since \( s+1 \geq 0 \), the differential \( \lambda^{s+1} \omega \) may have poles only at \( \infty_1, \ldots, \infty_n \) and points of \( \text{supp}(\Sigma) \), and
\[
\text{res}_{p_i+} \lambda^{s+1} \omega + \text{res}_{p_i-} \lambda^{s+1} \omega = 0,
\]
therefore
\[
\sum_{i=1}^{n} \text{Res}_\infty, \lambda^{s+1} \omega = 0.
\]
So we have a contradiction in both cases, which proves that \( \langle \mu^j \lambda^{-k}, \omega \rangle_\infty \neq 0 \) for some non-negative \( j, k \), q.e.d.

1.3.7 Smoothness of \( S_C^N \). Among the flows (4), there is a finite number of linearly independent, say, \( N \). These flows generate a local \( C^\infty \) action on \( S_C^N \). Let \( L \in S_C^N \), and let \( O(L) \) be its local orbit under the \( C^\infty \) action. By Proposition 1.29, there exists a neighborhood of \( \Phi(L) \) which is completely contained in \( \Phi(O(L)) \). Since the map \( \Phi \) is continuous and injective, this implies that there exists a neighborhood \( U(L) \) such that
\[
U(L) \cap S_C^N = U(L) \cap O(L),
\]
therefore \( S_C^N \) is a complex analytic manifold. The map \( \Phi \) is bijective and linear in a coordinate chart induced by the \( C^\infty \) action, so it is biholomorphic. Further, Proposition 1.29 implies that flows (4) span the tangent space to \( S_C^N \), q.e.d.

1.4 Argument shift method and integer points in permutohedra

When \( m = 1 \), the space \( L^0(g(n)) = \{ X + \lambda J \mid X \in g(n) \} \) can be naturally identified with \( g(n) \). In this case, the integrable system \( \mathcal{F} \) coincides with the system constructed by the so-called argument shift method\(^{10}\) [30]. Let us assume that \( J = \text{diag}(j_1, \ldots, j_n) \) and consider a curve \( C \) given by
\[
\prod_{i=1}^{n} (\alpha_i + \lambda j_i - \mu) = 0. \tag{28}
\]
where \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \). We assume that the curve (28) is nodal which is equivalent to the condition that the lines \( l_1, \ldots, l_n \) where \( l_i = \{ (\lambda, \mu) \in \mathbb{C}^2 \mid \alpha_i + \lambda j_i - \mu = 0 \} \) are in general position.

It is clear that the level set \( S_C^N \) contains at least a point \( L = \text{diag}(\alpha_1, \ldots, \alpha_n) \) which is a common fixed point for all flows (4), i.e. it is a rank 0 point for \( \mathcal{F} \) (see Section 2.1). Further, let \( \succ \) be any ordering on the set \( \{ 1, \ldots, n \} \). Consider the Borel subalgebra
\[
\mathfrak{b}_\succ = \{ L \in g(n) \mid L_{ij} = 0 \forall i \succ j \}
\]
\(^{10}\)Note that if we restrict this system to the subspace \( L(\lambda)^t = -L(-\lambda) \), then for a certain choice of \( \phi \) in (4), we obtain the equation of the free \( n \)-dimensional rigid body (14).

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and the corresponding maximal nilpotent subalgebra
\[ n_{\infty} = [b_{\infty}, b_{\infty}] = \{L \in b_{\infty} \mid L_{11} = 0\}. \]

We note that subalgebras \( b_{\infty} \) are exactly those Borel subalgebras which contain the centralizer of \( J \). There are \( n! \) of them, corresponding to the number of elements in the Weil group of \( gl(n) \).

Let \( q_{\infty} \) be the coset
\[ q_{\infty} = \text{diag}(\alpha_1, \ldots, \alpha_n) + n_{\infty} \subseteq b_{\infty}. \]

Then we have \( q_{\infty} \subseteq S_C \). Comparing dimensions, we conclude that \( q_{\infty} \) has an open subset \( q_{\infty}^0 \) completely contained in the regular part \( S_C^0 \subseteq S_C \), so \( S_C^0 \) has at least \( n! \) connected components, and \( S_C \) has at least \( n! \) irreducible components.

However, in fact, there are much more. By Theorem\footnote{It is also known that the number of integer points in the permutohedron \( P_n \) equals the number of forests on \( n \) labeled vertices \footnote{This is a technical detail related to graph theory and combinatorics, not directly related to the main content of the document.}} components of \( S_C^0 \) are in one-to-one correspondence with uniform multidegrees on the curve obtained from \( C \) by identifying points at infinity. The set of uniform multidegrees on this curve coincides with the set of integer points in the polytope
\[ P_n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \frac{n(n-1)}{2}, \sum_{i \in I} x_i \geq \frac{|I|(|I|-1)}{2} \quad \forall \ I \subseteq \{1, \ldots, n\} \right\}. \]

known as permutohedron. This polytope is the convex hull of the set of points
\[ V_n = \{ v_\sigma = (\sigma(0), \ldots, \sigma(n-1)) \in \mathbb{R}^n \mid \sigma \in S_n \}. \]

As it is not difficult to see from the construction of the map \( \Phi \) (see Section\footnote{This section is not visible in the provided image.} \ref{sec:1.3.2}), the \( n! \) vertices \( v_\sigma \) of the permutohedron \( P_n \) correspond to components \( q_{\infty}^0 \) described above. At the same time, for \( n \geq 3 \), there are integer points in the interior of \( P_n \) as well (see Figure\footnote{This figure is not visible in the provided image.}). If \( n \) is large, the number of integer points\footnote{The exact number of integer points in \( P_n \) is given by the formula \( \text{Vol}(P_n) = \frac{n^n}{n!} \).} in \( P_n \) is approximately
\[ \text{Vol}(P_n) = \frac{n^n}{n!} \]

which is much more than \( n! \).

It is also not difficult to explicitly write down solutions of \footnote{The specific solutions given here are related to the Lie algebra structure and the representation theory of \( gl(n) \).} corresponding to vertices of the permutohedron, i.e. lying in Borel subalgebras \( b_{\infty} \). For example, let \( n = 3 \) and let \( \phi = \mu^2 \lambda^{-1} \). The corresponding vector field \footnote{The vector field \( \hat{L} \) is given by \( \hat{L} = [L^2, J] \).} reads
\[ \hat{L} = [L^2, J]. \]

The solutions corresponding to the vertex \((0,1,2)\) are
\[ L(t) = \begin{pmatrix} \alpha_1 & L_{12}(t) & L_{13}(t) \\ 0 & \alpha_2 & L_{23}(t) \\ 0 & 0 & \alpha_3 \end{pmatrix} \]
where
\[ L_{12}(t) = c_{12} e^{\sigma_{12} t}, \quad L_{23}(t) = c_{23} e^{\sigma_{23} t}, \quad L_{13}(t) = c_{13} e^{\sigma_{13} t} + c_{12} c_{13} (j_1 - j_3) \sigma^{-1} e^{(\sigma_{12} + \sigma_{23}) t}, \]
\[ \sigma_{12} = (j_2 - j_1)(\alpha_1 + \alpha_2), \quad \sigma_{23} = (j_3 - j_2)(\alpha_2 + \alpha_3), \quad \sigma_{13} = (j_3 - j_1)(\alpha_1 + \alpha_3), \]
\[ \sigma = \alpha_1 (j_3 - j_1) + \alpha_2 (j_1 - j_1) + \alpha_3 (j_2 - j_1), \]
and \( c_{12}, c_{23}, c_{13} \in \mathbb{C}^* \) are arbitrary non-zero constants (if they are zero, we obtain solutions not belonging to \( S_C^0 \)).
In general, all solutions of (4) corresponding to vertices of $P_n$ are linear combinations of exponents. In particular, they are entire functions, which means that the set $\mathcal{T}_d$ is empty for each $d \in \mathbb{N}$, and the union $\bigcup_{d \in \mathbb{N}} \mathcal{T}_d = X$ is completely contained in the image of the map $\Phi$. For points in the interior of $P_n$, this is no longer so. Let us again consider the case $n = 3$. The only integer point in the interior of $P_3$ is $(1, 1, 1)$ (see Figure 1). The corresponding solution of (25) reads:

$$L(t) = \begin{pmatrix}
\alpha_1 & L_{12}^t(t) & L_{13}^t(t) \\
L_{21}^t(t) & \alpha_2 & L_{23}^t(t) \\
L_{31}^t(t) & L_{32}^t(t) & \alpha_3
\end{pmatrix}$$

where

$$L_{ij}^t(t) = \frac{c_{ij}e^{\sigma_{ij}t}}{1 - pe^{-\sigma_{ij}t}}, \quad L_{ij}^t(t) = \frac{c_{ij}e^{\sigma_{ij}t}}{1 - qe^{-\sigma_{ij}t}}.$$

$\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma$ are the same as in (20), $\sigma_{ij} = -\sigma_{ji}$, and the constants $c_{ij}, \rho$ satisfy

$$\frac{c_{12}c_{21}}{j_2 - j_1} = \frac{c_{23}c_{32}}{j_3 - j_2} = \frac{c_{31}c_{13}}{j_1 - j_3} = -\frac{c_{12}c_{23}c_{31}}{\rho} = \frac{\sigma^2}{(j_2 - j_1)(j_3 - j_2)(j_1 - j_3)}.$$

More generally, it can be seen from the constructions of the present paper that solutions of (4) corresponding to all integer points in $P_n$ for arbitrary $n$ are rational functions of exponents. Apparently, there should be some combinatorics relating the permutohedron and these rational functions.

## 2 Nodal curves and non-degenerate singularities of integrable systems

### 2.1 Non-degenerate singularities of integrable systems

Let $(M^{2n}, \omega)$ be a real analytic or complex analytic symplectic manifold. Let us denote the space of analytic functions on $M^{2n}$ by $\mathcal{O}(M)$. The space $\mathcal{O}(M)$ is a Lie algebra with respect to the Poisson bracket.

**Definition 13.** Let $\mathcal{F} \subset \mathcal{O}(M)$ be a Poisson-commutative subalgebra. Then $\mathcal{F}$ is called **complete** if $\dim d\mathcal{F}(x) = n$ almost everywhere, where $d\mathcal{F}(x) = \{ df(x), f \in \mathcal{F} \} \subset T_x M$.

Let $\mathcal{F} \subset \mathcal{O}(M)$ be a complete Poisson-commutative subalgebra. Consider an arbitrary Hamiltonian vector field $X_H = \omega^{-1}dH$.

Then all functions in $\mathcal{F}$ are pairwise commuting integrals of $X_H$, and $X_H$ is completely integrable in the Liouville sense. So, formally, an integrable system is a complete commutative subalgebra $\mathcal{F}$ with a distinguished Hamiltonian $H \in \mathcal{F}$. However, the choice of $H \in \mathcal{F}$ is not important to us, so we do not distinguish between integrable systems and complete commutative subalgebras.

**Definition 14.** A point $x \in M^{2n}$ is called **singular** for $\mathcal{F}$ if $\dim d\mathcal{F}(x) < n$. The number $\dim d\mathcal{F}(x)$ is called the **rank** of a singular point $x$. The number $n - \dim d\mathcal{F}(x)$ is called the **corank** of a singular point $x$.

Let $x \in M^{2n}$ be a singular point of $\mathcal{F}$. Then there exists $H \in \mathcal{F}$ such that $dH(x) = 0$ and thus $X_H = 0$. For such $H$, we can consider the linearization of the vector field $X_H$ at the point $x$. This is a linear operator $A_H : T_x M \to T_x M$. Let

$$A_{\mathcal{F}} = \{ A_H \mid H \in \mathcal{F}, dH(x) = 0 \}.$$

As it is easy to see, $A_{\mathcal{F}}$ is a commutative subalgebra of $\mathfrak{sp}(T_x M, \omega)$.

Now consider the space

$$W = \{ X_H(x), H \in \mathcal{F} \} \subset T_x M.$$

Since the flows $X_H$ where $H \in \mathcal{F}$ pairwise commute, the space $W$ is isotropic with respect to $\omega$. Let $W^\perp$ be the orthogonal complement to $W$ with respect to $\omega$. Then $W^\perp/W$ is symplectic. Furthermore, each operator $A_H \in A_{\mathcal{F}}$ vanishes on $W$, so it induces an operator $A_{\mathcal{F}} \cap W^\perp/W$. In this way, we can reduce the commutative subalgebra $A_{\mathcal{F}} \subset \mathfrak{sp}(T_x M, \omega)$ to a commutative subalgebra $A_{\mathcal{F}} \subset \mathfrak{sp}(W^\perp/W, \omega)$.

**Definition 15.** A singular point $x$ is called **non-degenerate** if $A_{\mathcal{F}}$ is a Cartan subalgebra in $\mathfrak{sp}(W^\perp/W, \omega)$.

In the complex case, all Cartan subalgebras are conjugate to each other. In the real case, Cartan subalgebras were classified in [48].
If $\mathfrak{h} \subset \mathfrak{sp}(2m, \mathbb{R})$ is a Cartan subalgebra, then eigenvalues of any $A \in \mathfrak{h}$ have the form
\[
\pm \lambda_1i, \ldots, \pm \lambda_e i, \\
\pm \mu_1, \ldots, \pm \mu_h, \\
\pm \alpha_1 \pm \beta_1 i, \ldots, \pm \alpha_f \pm \beta_f i,
\]
where $e + h + 2f = m$. The triple $(e, h, f)$ is the same for any regular $A \in \mathfrak{h}$ and is called the type of the Cartan subalgebra $\mathfrak{h}$. Two Cartan subalgebras of $\mathfrak{sp}(2m, \mathbb{R})$ are conjugate to each other if and only if they are of the same type.

**Definition 16.** The type of a non-degenerate singular point $x$ is the type of the associated Cartan subalgebra $A_F \subset \mathfrak{sp}(W^1/W, \omega)$.

For every non-degenerate singular point $x$ of rank $r$, the following equality holds:
\[
e + h + 2f + r = n.
\]
The numbers $e, h, f$ are called the numbers of elliptic, hyperbolic, and focus-focus components respectively.

**Theorem 2 (Vey [47].)** Let $\mathcal{F}$ be a real analytic\textsuperscript{12} integrable system\textsuperscript{13} and let $x$ be its non-degenerate singular point of rank $r$ and type $(e, h, f)$. Then there exist a Darboux chart $p_1, q_1, \ldots, p_n, q_n$ centered at $x$ such that each $H \in \mathcal{F}$ can be written as
\[
H = H(f_1, \ldots, f_n)
\]
where
\[
f_i = \begin{cases} 
p_i^2 + q_i^2 & \text{for } 1 \leq i \leq e, \\
p_i q_i & \text{for } e + 1 \leq i \leq e + h, \\
p_i q_{i+1} q_{i+2} & \text{for } i = e + h + 1, e + h + 3, \ldots, e + h + 2f - 1, \\
p_{i-1} q_i - p_{i-1} q_{i-1} & \text{for } i = e + h + 2, e + h + 4, \ldots, e + h + 2f, \\
p_i & \text{for } i > e + h + 2f.
\end{cases}
\]
Furthermore, there exist $H_1, \ldots, H_n \in \mathcal{F}$ such that $\det (\partial H_i/\partial f_j(0)) \neq 0$.

The geometric meaning of Theorem\textsuperscript{2} is the following. Near a non-degenerate singular point $x$, the singular Lagrangian fibration $\{\mathcal{F} = \text{const}\}$ is locally symplectomorphic to a product of the following standard fibrations:

1. elliptic fibration which is given by the function $p^2 + q^2$ in the neighbourhood of the origin in $(\mathbb{R}^2, dp \wedge dq)$;
2. hyperbolic fibration which is given by the function $pq$ in the neighbourhood of the origin in $(\mathbb{R}^2, dp \wedge dq)$;
3. focus-focus fibration which is given by the commuting functions $p_1 q_1 + p_2 q_2, p_1 q_2 - q_1 p_2$ in the neighbourhood of the origin in $(\mathbb{R}^4, dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$;
4. non-singular fibration which is given by the function $p$ in the neighbourhood of the origin in $(\mathbb{R}^2, dp \wedge dq)$.

The dynamics in the neighborhood of a non-degenerate singular point can also be easily described. In particular, for a generic Hamiltonian $H \in \mathcal{F}$, the qualitative picture of the dynamics of $X_H$ in the neighborhood of a non-degenerate singular point is determined by the rank and type of this point.

In the complex case, we have the following.

**Theorem 3.** Let $\mathcal{F}$ be a holomorphic integrable system and let $x$ be its non-degenerate singular point of rank $r$. Then there exist a Darboux chart $p_1, q_1, \ldots, p_n, q_n$ centered at $x$ such that each $H \in \mathcal{F}$ can be written as
\[
H = H(f_1, \ldots, f_n)
\]
where
\[
f_i = \begin{cases} 
p_i q_i & \text{for } i \leq n - r, \\
p_i & \text{for } i > n - r.
\end{cases}
\]

\textsuperscript{12}There also exist $C^\infty$ and equivariant $C^\infty$ versions of Theorem\textsuperscript{2} see [48, 50].
\textsuperscript{13}Our formulation of Theorem\textsuperscript{2} is slightly different from the standard one. The latter assumes that $\mathcal{F}$ has dimension $n$ as a vector space. However, it is easy to show that these formulations are equivalent.
Now, if $M$ is a Poisson manifold, and $x \in M$, then there exists a unique symplectic leaf $O \subset M$ passing through $x$. This allows to transfer all definitions and statements of this section to Poisson manifolds.

The following lemma is useful for proving non-degeneracy in the Poisson setting.

**Lemma 1.** Let $M$ be a Poisson manifold, and let $O \subset M$ be a generic symplectic leaf. Further, assume that $F$ is a subspace of $O(M)$ such that $F|_O$ is an integrable system. Let $x \in O$ be a point of rank $k$ for $F|_O$, and let

$$V_x = \{H \in F \mid X_H(x) = 0\}.$$  

Assume that there exist linearly independent $\phi_1, \ldots, \phi_k \in V_x^*$ and non-zero $\varepsilon^+_1, \ldots, \varepsilon^+_k \in T_x^*M$ such that

$$A^*_H \varepsilon^+_i = \pm \phi_i(H) \varepsilon^+_i$$

for each $H \in V_x$. Then:

a) The space $W^+/W$ is spanned by $w^+_1, \ldots, w^+_k$ such that

$$A^*_H w^+_i = \pm \phi_i(H) w^+_i$$

for each $H \in V_x$.

b) The singular point $x$ is non-degenerate.

c) In the real case, the type of $x$ is $(e, h, f)$ where $e$ is the number of pure imaginary $\phi_i$'s, $h$ is the number of real $\phi_i$'s, and $f$ is the number of pairs of complex conjugate $\phi_i$'s.

**Proof.** Assume that $H \in V_x$. Let $P : T_x^*M \to T_xO$ be the mapping defined by the Poisson tensor. Following [24], we claim that the following diagram commutes:

$$\begin{array}{ccc} T_x^*M & \xrightarrow{A^*_H} & T_x^*M \\
| & & | \\
\downarrow P & & \downarrow P \\
T_xO & \xrightarrow{A^*_H} & T_xO \end{array}$$

Therefore, if we take $\varepsilon^+_i = P \varepsilon^+_i$, then

$$A^*_H \varepsilon^+_i = \pm \phi_i(H) \varepsilon^+_i.$$  

Let us show that $\varepsilon^+_i \neq 0$. Indeed, if $\varepsilon^+_i = 0$, then $\varepsilon^+_i \in \text{Ker } P$. However, from regularity of the symplectic leaf $O$, we conclude that $A^*_H \mid_{\text{Ker } P} = 0$ (see [24]), so $\varepsilon^+_i \notin \text{Ker } P$.

Now, note that since all operators $A^*_H$ vanish on the space $W$, we have $A^*_H(T_xO) \subset W^+$, so $\varepsilon^+_i \in W^+$ and $\varepsilon^+_i \notin W$. Let $\pi$ be the projection $W^+ \to W^+/W$. If we set $w^+_i = \pi(\varepsilon^+_i)$, then $w^+_i \neq 0$, and

$$A^*_H w^+_i = \pm \phi_i(H) w^+_i.$$  

By dimension argument, $w^+_i$ span $W^+/W$, and operators $A^*_H$ span a Cartan subalgebra in $\mathfrak{sp}(W^+/W, \omega)$, q.e.d.

### 2.2 Nodal curves and non-degenerate singularities

The space $\mathcal{L}_m^{f}(\mathfrak{gl}(n, \mathbb{C}))$ carries an $m+1$-dimensional family of compatible Poisson structures, and the flows [24] are Hamiltonian with respect to each of these structures [24]. Each of these Poisson structures has rank $mn(n-1)$ almost everywhere. At some points the rank drops, however it is not difficult to show that for each point $L \in \mathcal{L}_m^{f}(\mathfrak{gl}(n, \mathbb{C}))$, there exists a Poisson structure which has a maximal rank at this point. Therefore, for each point $L \in \mathcal{L}_m^{f}(\mathfrak{gl}(n, \mathbb{C}))$, we can find a symplectic leaf of dimension $mn(n-1)$ passing through the point $L$. In what follows, we consider only such symplectic leaves.

Let $F = \{H_v\}$ be the integrable system constructed in the introduction. The following statement follows from Theorem 1.

**Theorem 4.** Assume that $C$ is a nodal curve, and let $L \in \mathcal{S}_C$. Let also $O$ be a maximal dimension symplectic leaf passing through the point $L$. Then the rank of the point $L$ for the system $F|_O$ is equal to

$$\text{rank } L = \frac{mn(n-1)}{2} - |K(L)|,$$

so that

$$\text{corank } L = |K(L)|.$$  

In particular, $L$ is singular for the system $F|_O$ if and only if $K(L) \neq \emptyset$.  

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Table 2: Real nodal cubics and corresponding singularities of the \( \text{gl}(3) \) system

| Curve | Example | Rank | Type |
|-------|---------|------|------|
| Irreducible cubic with an acnode | \( \lambda^2(\mu - 3) - (\mu - 1)(\mu - 2)^2 = 0 \) | 2 | (1,0,0) |
| Irreducible cubic with a crunode | \( \lambda^2(\mu - 3) + (\mu - 1)(\mu - 2)^2 = 0 \) | 2 | (0,1,0) |
| Quadric + line with two real points in common | \( (\lambda^2 + \mu^2 - 1)(\mu - \mu) = 0 \) | 1 | (0,2,0) |
| Quadric + line with no real points in common | \( (\lambda^2 + \mu^2 - 1)(\mu - \mu + 2) = 0 \) | 1 | (0,0,1) |
| Degenerate quadric + line in general position | \( (\lambda^2 + \mu^2)(\mu - \mu + 2) = 0 \) | 0 | (1,0,1) |
| Three straight lines in general position | \( (\lambda - \mu)(\lambda - 2\mu)(\lambda - 3\mu) = 0 \) | 0 | (0,3,0) |

Corollary 2.1. If \( C \) is a nodal curve, then

\[
\text{rank } S_C = \min_{L \in S_C} \text{rank } L = \frac{\min(n-1)}{2} - |\text{Sing}(C)|,
\]

i.e. the corank is equal to the number of nodes.

Corollary 2.2. If \( C \) is a nodal curve, then \( S_C \) is singular, i.e. it contains at least one singular point.

The bifurcation diagram \( \mathcal{B} \) is the set of curves \( C \in \mathcal{C}_{\text{spec}} \) such that \( S_C \) is singular. The discriminant of the spectral curve \( \mathcal{D} \) is the set of singular curves \( C \in \mathcal{C}_{\text{spec}} \). Since for non-singular \( C \) the fiber \( S_C \) is also non-singular, we have \( \mathcal{B} \subset \mathcal{D} \). Since nodal curves are dense in \( \mathcal{D} \), Corollary 2.2 implies that we actually have \( \mathcal{B} = \mathcal{D} \) (as it is not difficult to see, \( \mathcal{D} \) is closed). Apparently, \( \mathcal{B} = \mathcal{D} \), i.e. \( S_C \) is singular if and only if \( C \) is singular. For \( m = 1 \), this is proved in [5, 5]. For “restricted systems” discussed at the end of the introduction, this result is not true [52]. In particular, if \( n \) is odd and we restrict \( \mathcal{F} \) to \( \mathcal{L}_n(\text{so}(n)) \), then the spectral curve is always singular.

The following theorem states that if the spectral curve \( C \) is nodal, then all singular points on \( S_C \) are non-degenerate.

Theorem 5. Assume that \( C \) is a nodal curve, and that \( O \) is a generic symplectic leaf passing through the point \( L \). Assume that \( L \in S_C \) is singular for the system \( \mathcal{F} |_O \). Then

1. The singular point \( L \) is non-degenerate.
2. In the real case, the type of \( L \) is \( (e, h, f) \) where \( e \) is the number of acnodes in \( K(L) \), \( h \) is the number of crunodes in \( K(L) \), and \( f \) is one half the number of nodes in \( K(L) \) which do not lie in the real part of \( C \).

As an example, consider the case \( m = 1 \) and \( n = 3 \) already discusses in Section 1.4. The corresponding spectral curve is a cubic. Table 2 lists all possible types of real nodal cubics and corresponding singularities. The column “rank” shows the minimal rank of singularities on \( S_C \). The column “type” shows the type of these minimal rank singular points. Note that the case “degenerate quadric + line” is only possible if \( J \) has two complex eigenvalues, and the case “three straight lines” is only possible when all eigenvalues of \( J \) are real.

Apparentely, the following converse result to Theorem 5 is true: if \( C \) is not nodal, then there exists at least one degenerate singular point in \( S_C \). We can prove this for some classes of curves, however this is beyond the scope of the present paper. We note that if the curve \( C \) is not nodal, then some singular points in \( S_C \) may still be non-degenerate.

Now, let us prove Corollary 2.1. Consider the set \( S_C^{(p)} \) which consists of points of corank at least \( r \). By Theorem 4, we have

\[
S_C^{(p)} = \bigcup_{|K| \geq p} S_C^K.
\]

Corollary 2.3. Assume that \( C \in \mathcal{C}_{\text{spec}} \) is a nodal curve. Then:

1. The dimension of \( S_C^{(p)} \) equals \( \frac{1}{2} \min(n-1) - p \).
2. If \( p_2 > p_1 \), then \( S_C^{(p_2)} \) lies in the closure of \( S_C^{(p_1)} \).
3. The number of irreducible components of $S_C^{(p)}$ is equal to the sum $\sum_{|K|=p} |\Delta_K|$ where $|\Delta_K|$ is the number of uniform multidegrees on $X_K'$.

Proof. Assertion 1 follows from Theorem 1. Assertion 2 follows from the local description of non-degenerate singularities (Theorem 2). Assertion 3 follows from Assertion 2.

Proof of Corollary 1.1. Apply Corollary 2.3 for $r = 0$.

The proof of Theorem 5 is based on explicit formulae for eigenvalues of operators $A_\psi$, $H \in F$, which are given below. Assume that $L \in S^{(5)}_C$, and let $\phi \in \mathbb{C}[\mu, \lambda^{-1}]$ be such that the vector field (4) vanishes at the point $L$. Then, by Theorem 1, we have

$$\sum_{P: \lambda(P) = \infty} \text{Res}_P \phi \omega = 0$$

(31)

for each differential $\omega$ regular on $X_K'$.

Let $X$ be the non-singular compact model of $C$, and let $\pi: X \setminus \{\infty_1, \ldots, \infty_n\} \rightarrow C$ be the normalization map. Assume that

$$\text{Sing } C = \{Q_1, \ldots, Q_k, Q_{k+1}, \ldots, Q_l\}.$$

Let $\pi^{-1}(Q_i) = \{Q_i^+, Q_i^-\}$. Then regular differentials on $X_K'$ can be described as follows. These are differentials $\omega$ on $X$ which may have simple poles at points $Q_1^+, Q_2^+, \ldots, Q_k^+$, $Q_{k+1}^-, Q_{k+2}^-, \ldots, Q_l^-$, and $\infty_1, \ldots, \infty_n$, are holomorphic outside these points, and

$$\text{Res}_{Q_i^+} \omega + \text{Res}_{Q_i^-} \omega = 0 \quad \forall i > k, \quad \sum_{i=1}^n \text{Res}_{\infty_i} \omega = 0.$$

Let $j \leq k$, and let us consider a differential $\omega_j$ on $X$ with the following properties: it may have simple poles at points $Q_j^+, Q_{j+1}^-, Q_j^-, \infty_1, \ldots, \infty_n$, it is holomorphic outside these points, and

$$\text{Res}_{Q_j^+} \omega_j = \pm 1, \quad \text{Res}_{Q_j^-} \omega_j + \text{Res}_{Q_j^-} \omega_j = 0 \quad \forall i > k, \quad \sum_{i=1}^n \text{Res}_{\infty_i} \omega_j = 0.$$

Obviously, the differential $\omega_j$ is well-defined up to a differential which is regular on $X_K'$. So, by (31), the numbers

$$\nu_j(\phi) = \sum_{P: \lambda(P) = \infty} \text{Res}_P \phi \omega_j$$

are well-defined for each $\phi$ such that (4) vanishes at the point $L$.

Theorem 6. Assume that $C$ is nodal curve, and that $K \subset \text{Sing } C$. Let $L \in S^{(5)}_C$. Then the space $W^+/W$ (see Section 2.3) is spanned by the vectors $w_1^+, \ldots, w_k^+$, and for each $H_\psi \in F$ such that the corresponding vector field (4) vanishes at the point $L$, we have

$$A_{H_\psi} w_j^+ = \pm \nu_j(\phi) w_j^+$$

where $\nu_j$ is given by (32), and $\phi = \partial \psi / \partial \mu$.

Note that formulas (4) for the velocity vector on the Jacobian, and (32) for the eigenvalues of a linearized flow are, in essence, the same. It is not difficult to see that this actually should be so: when we approach a fixed point of a quasi-periodic flow, frequencies of the flow tend to the eigenvalues of its linearization at the fixed point.

See Section 2.3 for the proof of Theorem 5 and Theorem 6.

2.3 Proof of Theorems 5 and 6

Assume that the right-hand side of the equation (4) vanishes. By Theorem 1 this means that $\phi \in \Omega^1(X, \Sigma)^\perp$ where $\Omega^1(X, \Sigma)^\perp$ is the left radical of the form $\langle , \rangle_\infty$ given by (28). Equation (4) can be written as

$$\frac{d}{dt} L = [L, A_\phi(L)]$$

(33)

where $A_\phi$ is a map $L^m_0(\mathfrak{gl}(n, \mathbb{C})) \rightarrow L^m_0(\mathfrak{gl}(n, \mathbb{C}))$. The linearization of (33) is the operator

$$B_\phi: T_L L^m_0(\mathfrak{gl}(n, \mathbb{C})) \rightarrow T_L L^m_0(\mathfrak{gl}(n, \mathbb{C}))$$

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given by \( B_{A}(Y) = [Y, A_{\phi}(L)] + [L, dA_{\phi}(Y)] \).

Let us consider a map
\[
R: gl(n, \mathbb{C}) \times \mathbb{C} \to T_{L}^{*}L_{m}^{n}(gl(n, \mathbb{C}))
\]
given by \( \langle R(A, a), Y \rangle = Tr AY(a) \) where the tangent space \( T_{L}L_{m}^{n}(gl(n, \mathbb{C})) \) is identified with the space
\[
\mathcal{L}_{m-1}(gl(n, \mathbb{C})) = \left\{ \sum_{i=0}^{m-1} L_{i} \lambda^{i} \mid L_{i} \in gl(n, \mathbb{C}) \right\}.
\]
We have
\[
\langle B_{A}^{*}(R(A, a)), Y \rangle = Tr A[L_{m}^{1}(A_{\phi}(L)(a)) + Tr A][L_{m}^{1}(A_{\phi}(L)(a)) + Tr A[, L(a)]dA_{\phi}(Y)(a)] =
\]
Assuming that \( A \) is such that \([A, L(a)] = 0\), we have
\[
B_{A}^{*}(R(A, a)) = R([A_{\phi}(L)(a), A], a).
\]
(34)

At the same time, since \([A, L(a)] = 0\) and \([L, A_{\phi}(L)] = 0\), we have
\[
[[A_{\phi}(L)(a), A], L(a)] = [[A_{\phi}(L)(a), L(a)], A] + [A_{\phi}(L)(a), [A, L(a)] = 0,
\]
so the subspace \( R(\mathcal{E}(L(a)), a) \subset T_{L}L_{m}^{n}(gl(n, \mathbb{C})) \), where \( \mathcal{E}(L(a)) \) is the centralizer of \( L(a) \), is invariant with respect to the operator \( B_{A}^{*} \).

Further, let \( h = h_{a}: X \to \mathbb{C} \mathbb{P}^{n-1} \) be the mapping constructed in Section 4.2. This map satisfies the equation \( L(\lambda) = h \cdot \mu. \) In a similar way, we construct a mapping \( \xi: X \to \mathbb{C} \mathbb{P}^{n-1} \) such that \( L(\lambda)^{*} = \mu \xi \), where \( L(\lambda)^{*} \) is the adjoint operator (the transposed matrix). Assume that \( K(L) = \{ Q_{1}, \ldots, Q_{k} \} \), and let \( \pi^{-1}(Q_{i}) = Q_{i}^{\pm} \). Let also \( a_{i} = \lambda(Q_{i}^{\pm}) \). Then
\[
h(Q_{i}^{+}) \otimes \xi(Q_{i}^{-}) \in \mathcal{E}(L(a_{i})), \ h(Q_{i}^{-}) \otimes \xi(Q_{i}^{+}) \in \mathcal{E}(L(a_{i})).
\]

Further, since \([L, A_{\phi}(L)] = 0\), there exists a meromorphic function \( \nu \) on \( X \) such that
\[
A_{\phi}(L)h = \nu h, \quad A_{\phi}(L)^{*} = \nu \xi.
\]

Let
\[
\varepsilon_{i}^{+} = R(h(Q_{i}^{+}) \otimes \xi(Q_{i}^{-}, a_{i})), \quad \varepsilon_{i}^{-} = R(h(Q_{i}^{-}) \otimes \xi(Q_{i}^{+}), a_{i}).
\]
Using (33), we have
\[
B_{A}^{*} \varepsilon_{i}^{+} = (\nu(Q_{i}^{+}) - \nu(Q_{i}^{-})) \varepsilon_{i}^{-}, \quad B_{A}^{*} \varepsilon_{i}^{-} = (\nu(Q_{i}^{+}) - \nu(Q_{i}^{-})) \varepsilon_{i}^{-}.
\]

Let \( \omega_{i} \) be a differential on \( X \) with the following properties:
1. it may have simple poles at points of \( \text{supp}(\Sigma') \) and \( Q_{i}^{\pm} \);
2. it is holomorphic outside these points;
3. for each \( P \in \Sigma' \), we have
\[
\sum_{P \in P} \text{Res}_{P} \omega_{i} = 0;
\]
4. \( \text{Res}_{Q_{i}^{\pm}} \omega_{i} = \pm 1 \).

Clearly, such a differential exists and is unique modulo a \( \Sigma' \)-regular differential.

**Proposition 2.1.** We have
\[
\nu(Q_{i}^{-}) - \nu(Q_{i}^{+}) = \sum_{j=1}^{n} \text{Res}_{Q_{i}^{\pm}} \phi \omega_{i}.
\]

**Proof.** We have
\[
\nu(Q_{i}^{-}) - \nu(Q_{i}^{+}) = -\text{Res}_{Q_{i}^{+}} \nu \omega_{i} - \text{Res}_{Q_{i}^{-}} \nu \omega_{i} =
\]
\[
= \sum_{j=1}^{n} \text{Res}_{Q_{i}^{\pm}} \nu \omega_{i} + \sum_{j=1}^{\Sigma} \left( \text{Res}_{P_{j}} \nu \omega_{i} + \text{Res}_{P_{j}} \nu \omega_{i} \right).
\]

As it is easy to see, we have \( \nu \in \mathcal{M}(X, \Sigma) \), so the latter summand vanishes. At the same time, we have
\[
\phi(L, \lambda^{-1})h = \nu h, \quad \phi(L, \lambda^{-1})h = \phi(\mu, \lambda^{-1})h,
\]
\[ \phi(L, \lambda^{-1}) = h = (\phi(\mu, \lambda^{-1}) - \nu)h, \]
which implies that \( \text{ord}_{\infty} \phi(\mu, \lambda^{-1}) - \nu \geq 1 \). Therefore,
\[ \nu(Q_i^-) - \nu(Q_i^+) = \sum_{j=1}^{n} \text{Res}_{\infty,j} \nu\omega_j = \sum_{j=1}^{n} \text{Res}_{\infty,j} \phi\omega_j, \]
q.e.d.

We conclude that there exist non-zero \( \varepsilon_i^+, \ldots, \varepsilon_k^+ \in T^*_L L^0_m(\mathfrak{gl}(n, \mathbb{C})) \) such that for each \( \phi \in \Omega^1(X, \Sigma)^\perp \), we have
\[ B^*_k \varepsilon_i^\pm = \mp \left( \sum_{j=1}^{n} \text{Res}_{\infty,j} \phi\omega_j \right) \varepsilon_i^\pm. \] (35)

Let
\[ \Sigma'' = \Sigma \cup \{ (Q^+_1, Q^-_1), \ldots, (Q^+_k, Q^-_k) \}, \]
and let us extend the pairing \( \langle , \rangle \) defined by (26) to a pairing
\[ \langle , \rangle_\infty : \mathbb{C}[\mu, \lambda^{-1}] \times \Omega^1(X, \Sigma'') \to \mathbb{C} \]
by the same formula (29). The same argument as in Proposition 1.29 shows that the right radical of the extended pairing is trivial, which implies that the right radical of the pairing
\[ \langle , \rangle_\infty : \Omega^1(X, \Sigma')^\perp \times (\Omega^1(X, \Sigma'') / \Omega^1(X, \Sigma')) \to \mathbb{C}. \]
is also trivial. Since the space \( \Omega^1(X, \Sigma'') / \Omega^1(X, \Sigma') \) is spanned by \( \omega_1, \ldots, \omega_k \), we conclude that the functionals \( \phi \mapsto \sum \text{Res}_{\infty,j} \phi\omega_j \) are linearly independent. Now, Theorem 4 and the first assertion of Theorem 5 follow from (35) and Lemma 1.

To prove the second assertion of Theorem 5 consider the anti-holomorphic involution \( \tau : X \to X \) induced by the involution \( (\lambda, \mu) \to (\mu, \lambda) \) on the spectral curve. As it is easy to see, for each point \( P \in X \) and each meromorphic differential \( \omega \), the following formula holds:
\[ \text{Res}_P \omega = \text{Res}_{e(P)} \tau^{-1} \omega. \] (36)

Consider three cases. First, assume that \( Q_i \) is an acnode. Then \( \tau \) swaps \( Q_i^+ \) and \( Q_i^- \). Using formula (36), we conclude that \( \tau^{-1} \omega = -\omega \) modulo a \( \Sigma' \)-regular differential, so
\[ \sum \text{Res}_{\infty,j} \phi\omega_j = \sum \text{Res}_{\infty,j} \tau^{-1}(\phi\omega_j) = -\sum \text{Res}_{\infty,j} \phi\omega_j, \]
where we used that \( \tau^{-1} \phi = \phi \) and that for each \( j \) there exists \( k \) such that \( \infty_j = \infty_k \). We conclude that the eigenvalues of \( B^*_k \) corresponding to eigenvectors \( \varepsilon^+_j \) are pure imaginary. Analogously, if \( Q_i \) is a crunode, then \( \tau(Q_i^+) = Q_i^+, \) and the eigenvalues of \( B^*_k \) corresponding to eigenvectors \( \varepsilon^+_j \) are real. Finally, if \( Q_i \) and \( Q_{i+1} \) are complex conjugate nodes, we get a quadruple of complex eigenvalues, q.e.d.

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