On $\delta$-deformations of polygonal dendrites.

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Abstract

We find the conditions under which the attractor $K(S')$ of a deformation $S'$ of a contractible polygonal system $S$ is a dendrite.

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This is a very convenient though rather restrictive way to define post-critically finite self-similar dendrites in the plane using contractible P-polygonal systems. This approach was discussed in [7],[8],[9]. It turned out that well-known examples of self-similar dendrites are obtained using such systems. Nevertheless, if we move slightly the vertices of the main polygon $P$ and of polygons $P_i$, defining the polygonal system $S$, and change the system $S$ accordingly, we often obtain a system $S'$ of a more general type whose attractor $K'$ is a dendrite too. We call such systems generalized polygonal systems (Definition 8) and in the case when polygons $P_i'$ differ from the polygons $P_i$ less than by $\delta$, we call such systems $\delta$-deformations (Definition 12) of the polygonal system $S$. In this paper we begin initial study of generalized polygonal systems and $\delta$-deformations of contractible polygonal systems.

In Theorem 9 we formulate sufficient conditions under which the attractor $K$ of a generalized polygonal system $S$ is a dendrite. These conditions are expressed in terms of intersections $K_i \cap K_j$ of the pieces of the attractor $K$. In Theorem 14 we show that a $\delta$-deformation $S'$ of a contractible polygonal system $S$ defines a continuous map $f : K \to K'$ of respective attractors of these systems which agrees with the action of $S$ and $S'$ and give conditions under which $f$ is a homeomorphism. In Theorem 20 we show that Parameter Matching Condition is a necessary condition for a generalized polygonal system to generate a dendrite. In Theorem 27 we show that if $\delta$ is sufficiently small and the system $S'$ is $\delta$-deformation of a contractible P-polygonal system $S$, which satisfies Parameter Matching Condition, then the attractor $K(S')$ is a dendrite, homeomorphic to $K(S)$.

1 Preliminaries

1.1 Self-similar sets

Definition 1. Let $S = \{S_1, S_2, \ldots, S_m\}$ be a system of (injective) contraction maps on the complete metric space $(X,d)$. A nonempty compact set $K \subset X$ is called the attractor of the system $S$, if $K = \bigcup_{i=1}^{m} S_i(K)$.  

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The system $S$ defines its Hutchinson operator $T$ by $T(A) = \bigcup_{i=1}^{m} S_i(A)$. By Hutchinson’s Theorem, the attractor $K$ is unique for $S$ and for any compact set $A \subset X$ the sequence $T^n(A)$ converges to $K$. We also call the subset $K \subset X$ self-similar with respect to $S$.

Throughout the whole paper, the maps $S_i \in S$ are supposed to be similarities and the set $X$ to be $\mathbb{R}^2$. We will use complex notation for the point on the plane so each similarity will be written as $S_j(z) = q_j e^{i\alpha_j}(z - z_j) + z_j$, where $q_j = \text{Lip} S_j$ and $z_j = \text{fix}(S_j)$. For a system $S$, let $q_{\text{min}} = \min\{q_j, j \in I\}$ and $q_{\text{max}} = \max\{q_j, j \in I\}$.

Here $I = \{1, 2, \ldots, m\}$ is the set of indices, while $I^* = \bigcup_{n=1}^{\infty} I^n$ is the set of all finite $I$-tuples, or multiindices $j = j_1 j_2 \ldots j_n$. The length $n$ of the multiindex $j = j_1 \ldots j_n$ is denoted by $|j|$ and $ij$ denote the concatenation of the corresponding multiindices. We say $i \sqsubset j$, if $j = il$ for some $l \in I^*$; if $i \not\sqsubset j$ and $j \not\sqsubset i$, $i$ and $j$ are incomparable.

For a multiindex $j \in I^*$ we write $S_j = S_{j_1 j_2 \ldots j_n} = S_{j_1} S_{j_2} \ldots S_{j_n}$ and for the set $A \subset X$ we denote $S_j(A)$ by $A_j$; we also denote by $G_S = \{S_j, j \in I^*\}$ the semigroup, generated by $S$;

$I^\infty = \{\alpha = \alpha_1 \alpha_2 \ldots, \quad \alpha_i \in I\}$ denotes the index space; and $\pi : I^\infty \rightarrow K$ is the index map, which sends $\alpha$ to the point $\bigcap_{n=1}^{\infty} K_{\alpha_1 \ldots \alpha_n}$.

Along with a system $S$ we will consider its $n$-th refinement $S^{(n)} = \{S_j, j \in I^n\}$, whose Hutchinson’s operator is equal to $T^n$.

**Definition 2.** The system $S$ satisfies the open set condition (OSC) if there exists a non-empty open set $O \subset X$ such that $S_i(O), \{1 \leq i \leq m\}$ are pairwise disjoint and all contained in $O$.

Let $\mathcal{C}$ be the union of all $S_i(K) \cap S_j(K)$, $i, j \in I, i \neq j$. The post-critical set $\mathcal{P}$ of the system $S$ is the set of all $\alpha \in I^\infty$ such that for some $j \in I^*$, $S_j(\alpha) \in \mathcal{C}$. In other words, $\mathcal{P} = \{\sigma^k(\alpha) : \alpha \in \mathcal{C}, k \in \mathbb{N}\}$, where the map $\sigma^k : I^\infty \rightarrow I^\infty$ is defined by $\sigma^k(\alpha_1 \alpha_2 \ldots) = \alpha_{k+1} \alpha_{k+2} \ldots$. A system $S$ is called post-critically finite (PCF) if its post-critical set $\mathcal{P}$ is finite. Thus, if the system $S$ is postcritically finite then there is a finite set $\mathcal{V} = \pi(\mathcal{P})$ such that for any non-comparable $i, j \in I^*$, $K_i \cap K_j = S_i(\mathcal{V}) \cap S_j(\mathcal{V})$.

1.2 **Dendrites**

A **dendrite** is a locally connected continuum containing no simple closed curve.

The order $\text{Ord}(p, X)$ of the point $p$ with respect to a dendrite $X$ is the number of components of the set $X \setminus \{p\}$. Points of order 1 in a dendrite $X$ are called end points of $X$; a point $p \in X$ is called a cut point of $X$ if $X \setminus \{p\}$ is disconnected; points of order at least 3 are called ramification points of $X$.

A continuum $X$ is a dendrite iff $X$ is locally connected and uniquely arcwise connected.

1.3 **Contractible polygonal systems**

Let $P \subset \mathbb{R}^2$ be a finite polygon homeomorphic to a disk, $\mathcal{V}_P = \{A_1, \ldots, A_{n_P}\}$ be the set of its vertices. Let also $\Omega(P, A)$ denote the angle with vertex $A$ in the polygon $P$. We consider a system of similarities $S = \{S_1, \ldots, S_m\}$ in $\mathbb{R}^2$ such that:

(D1) for any $i \in I$ set $P_i = S_i(P) \subset P$;

(D2) for any $i \neq j$, $i, j \in I, P_i \cap P_j = \mathcal{V}_{P_i} \cap \mathcal{V}_{P_j}$ and $\#(\mathcal{V}_{P_i} \cap \mathcal{V}_{P_j}) < 2$;

(D3) $\mathcal{V}_P \subset \bigcup_{i \in I} S_i(\mathcal{V}_P)$;

(D4) the set $\tilde{P} = \bigcup_{i=1}^{m} P_i$ is contractible.
Definition 3. The system $S$ satisfying the conditions (D1 – D4), is called a contractible $P$-polygonal system of similarities.

This theorem was proved by the authors in [8, Theorem 4.(g)](or [10, Theorem 10.(g)]):

Theorem 4. Let $S$ be a contractible $P$-polygonal system of similarities.
(a) The system $S$ satisfies (OSC).
(b) $P_j \subset P_i$ if $j \supset i$.
(c) If $i \subset j$, then $S_i(V_P) \cap P_j \subset S_j(V_P)$.
(d) For any incomparable $i, j \in I^*$, $|(P_i \cap P_j) \cap 1$ and $P_i \cap P_j = S_i(V_P) \cap S_j(V_P)$.
(e) The set $G_S(V_P)$ of vertices of polyhedra $P_j$ is contained in $K$.
(f) If $x \in K \setminus G_S(V_P)$, then $\# \pi^{-1}(x) = 1$.
(g) For any $x \in G_S(V_P)$ there is $\varepsilon > 0$ and a finite system $\{\Omega_1, ..., \Omega_n\}$, where $n = \# \pi^{-1}(x)$, of disjoint angles with vertex $x$, such that if $x \in P_j$ and diam $P_j < \varepsilon$, then for some $k \leq n$, $\Omega(P_j, x) = \Omega_k$. Conversely, for any $\Omega_k$ there is such $j \in I^*$, that $\Omega(P_j, x) = \Omega_k$.

Polygonal system and its attractor

Local structure of $K$ near the vertex B.(rotated)

Definition 5. Let $S$ be a contractible $P$-polygonal system of similarities. The vertex $A \subset V_P$ is called a cyclic vertex, if there is such multiindex $i = i_1 i_2 ... i_k$, that $S_i(A) = A$. The least number $k = |i|$ among all $i$ for which $S_i(A) = A$ is called the order of the cyclic vertex $A$.

Definition 6. A point $B \in \bigcup_{i=1}^m V_P$ is subordinate to a cyclic vertex $A$, if for certain multiindex $i, S_i(A) = B$.

Proposition 7. Let $S$ be a contractible $P$-polygonal system of similarities. Then:
(1) Each vertex $B \in V_P$ is subordinate to some cyclic vertex.
(2) There is such $n$, that in the system $S^{(n)} = \{S_i, i \in I^n\}$ all the cyclic vertices have order 1.

Proof. Notice that if $A \in V_P$ is a cyclic vertex, then there is such $j \in I^*$ that $S_j(A) = A$. Therefore if for some $j \in I^*$, $A \in P_j$, then for some $n$, $S_j^n(P) \subset P_j \subset P, A$ being a vertex of each of these polygons. Since $\Omega(S_j^n(P), A) = \Omega(P, A)$, for any $j \in I^*$, for which $A \in P_j$, $\Omega(P_j, A) = \Omega(P, A)$. This implies that $\# \pi^{-1}(A) = 1$ and for any $n$ there is unique $j \in I^n$ such that $A \in P_j$.

Conversely if for any $i \in I^*$, for which $A \in P_i$, $\Omega(P_i, A) = \Omega(P, A)$ then $\# \pi^{-1}(A) = 1$ and $A$ is a cyclic vertex of the system $S$.

Then, by Theorem 4 for any vertex $B \in G_S(V_P)$ there is a finite set $\{i_1, ..., i_n\}$ of incomparable multiindices such that for any $l, l^* \in P_i \cap P_{i^*} = \{B\}$, the set $\bigcup_{l=1}^k K_i$ is a neighbourhood of the point $B$ in $K$ and for any $l = 1, ..., k$, the point $S_{i_l}^{-1}(B) = A_l$ is a cyclic vertex. This proves (1).

Let now $A_1, ..., A_k$ be the full set of cyclic vertices in $V_P$ and $p_1, ..., p_k$ be their respective orders. Let $N$ be the least common multiple of $p_1, ..., p_k$. Then $S^{(n)}$ is the desired $P$-polygonal system.
1.4 Main parameters of a contractible polygonal system

For any set $X \subset \mathbb{R}^2$ or point $A$ by $V_\varepsilon(X)$ (resp. $V_\varepsilon(A)$) we denote $\varepsilon$-neighbourhood of the set $X$ (resp. of the point $A$) in the plane.

$\rho_0$: Take such $\rho_0 > 0$ that:
(i) for any vertex $A \in V P, V_\rho(A) \cap P_k \neq \emptyset \Rightarrow A \in P_k$;
(ii) for any $x, y \in P$ such that there are $P_k, P_l: x \in P_k, y \in P_l$ and $P_k \cap P_l = \emptyset, d(x, y) \geq \rho_0$.

Choosing the parameters $\alpha_0, \rho_1$ and $\rho_2$ for a polygonal system.

$\rho_1, \rho_2$: As it follows from Theorem 4 for any vertex $B \in V P$ there is a finite set of cyclic vertices $A_i, \ldots, A_k \in V P$, and multiindices $j_1, \ldots, j_k$ such that for any $l = 1, \ldots, k, S_{j_l}(A_l) = B$ and $S_{j_i}(A_l) = A_i$ and the set $\bigcup_{l=1}^{k} S_{j_i}^{n}(K)$ is a neighborhood of the point $B$ in $K$ for any $n \geq 0$.

Let $\rho_1$ and $\rho_2$ be such positive numbers that for any vertex $B \in V P$

$$(V_\rho_1(B) \cap K) \subset \bigcup_{l=1}^{k} S_{j_l}(P_i) \quad \text{and} \quad \bigcup_{l=1}^{k} P_{j_i} \subset V_\rho_2(B).$$ (1)

$\alpha_0$: Let $\alpha_0$ denote the minimal angle between those sides of polygons $P_i, P_j, i, j \in I$, which have common vertex.

Arrangement of maps fixing cyclic vertices. Let $S$ be a contractible $P$-polygonal system all of whose cyclic vertices have order 1. In this case we can arrange the indices in $I$ and enumerate the vertices in $V P$ in such way that each cyclic vertex $A_l$ will be the fixed point of $S_l \in S$. Notice that $S_l$ is a homothety $S_l(z) = q_l(z - A_l) + A_l$ and the polygon $P$ lies inside the angle $\Omega(P, A_l)$ and $K \setminus \{A_l\} = \bigsqcup_{n=0}^{\infty} S_l^n(K \setminus K_l)$. The number of points in $K \setminus S_l(K_l) \cap S_l(K_l)$ is finite and is equal to the ramification order of $A_l$ in $K$.

2 Generalized polygonal systems.

If we omit the condition (D1) in the definition of contractible $P$-polygonal system $S$, we get the definition of a generalized $P$-polygonal system:

Definition 8. A system $S = \{S_1, \ldots, S_m\}$, satisfying the conditions D2-D4, is called a generalized $P$-polygonal system of similarities.
Theorem 9. Let $S$ be a generalized $P$-polygonal system. If for any $i,j \in I$

\[ S_i(K) \cap S_j(K) = P_i \cap P_j, \tag{2} \]

then the attractor $K$ of the system $S$ is a dendrite.

Proof: Let $i,i' \in I$. By a (simple) chain of indices, connecting $i$ and $i'$, we mean a sequence $i = i_1, i_2, \ldots, i_l = i'$ of pairwise different indices such that $P_{i_k} \cap P_{i_{k+1}} = \emptyset$ if $|k' - k| > 1$, and that for any $k = 1, \ldots, l - 1$, $P_{i_k} \cap P_{i_{k+1}} = \{x_k\}$, where $x_k$ denotes a common vertex of the polygons $P_{i_k}$ and $P_{i_{k+1}}$. The last condition also means, that $K_{i_k} \cap K_{i_{k+1}} \ni x_k$ for any $k \in I$.

Since in a generalized polygonal system for any two indices $i, i'$ there is a chain of indices $i = i_1, i_2, \ldots, i_l = i'$ connecting them, then by [5, Theorem 1.6.2], the attractor $K$ is connected, locally connected and arcwise connected. Thus, any two points of $K$ can be connected by some Jordan arc in $K$.

Notice also that if the condition (2) holds, and the indices $i, i' \in I$ can be connected by a chain $i = i_1, i_2, \ldots, i_l = i'$, then for any points $x \in K_i, y \in K_{i'}$ there is some Jordan arc $\gamma_{xy} \subset K$, consisting of subarcs

\[ \gamma_{xx_1} \subset K_{i_1}, \ldots, \gamma_{x_{k-1}x_k} \subset K_{i_k}, \ldots, \gamma_{x_{l-1}y} \subset K_{i_l} \]  

(3)

with disjoint interiors.

At the same time, if the condition (2) holds, and a Jordan arc $\gamma_{xy} \subset K$ with endpoints in $x$ and $y$, meets sequentially the pieces $K_{i_1}, \ldots, K_{i_l}$, then it passes sequentially through the points $x_k$, where $\{x_k\} = K_{i_{k-1}} \cap K_{i_k}$ and consists of subarcs of the form (3) with disjoint interiors.

And vice versa, if the condition (2) holds, then for any Jordan arc $\gamma_{xy}$ in $K$ there is unique chain of indices $i_1, \ldots, i_l$, such that $\gamma_{xy}$ consists of subarcs of the form (3).

We need a small Lemma to continue the proof:

Lemma 10. Let $j \in I^*$ and $x, y \in K_j$. If the condition (2) holds, then for any two Jordan arcs $\lambda_1, \lambda_2$ with endpoints $x, y$, the distance $d_H(\lambda_1, \lambda_2) \leq q_{\text{max}} \text{diam } K_j$.

Proof: Indeed, consider the Jordan arcs $\lambda_1 = S_j^{-1}(\lambda_1)$ and $\lambda_2 = S_j^{-1}(\lambda_2)$, connecting $x' = S_j^{-1}(x)$ and $y' = S_j^{-1}(y)$ in $K$. Let $x' \in K_i$ and $y' \in K_{i'}$, and let $i_1, i_2, \ldots, i_l$ be the chain, connecting $i$ and $i'$. Then each of the arcs $\lambda_1', \lambda_2'$ consists of subarcs, connecting sequentially the pairs of points $x_k, x_{k+1}$ in the sequence $x', x_1, \ldots, x_{l-1}, y'$, and lying in respective pieces $K_{i_k}$. 

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Since the diameters of these sets are not greater than \( q_{\text{max}} \text{diam} K \), \( d_H(\lambda_1, \lambda_2) \leq q_{\text{max}} \text{diam} K \). Then \( d_H(\lambda_1, \lambda_2) \leq q_{\text{max}} \text{diam} K_j \leq \text{diam} K^{[j]+1} \). \[\blacksquare\]

Now we can finish the proof of the Theorem. Let \( \lambda \) and \( \lambda' \) be Jordan arcs in \( K \) with endpoints at \( x \) and \( y \). Applying the Lemma 10 by induction to the subarcs of which the arcs \( \lambda \) and \( \lambda' \) consist, we get that for any \( n > |j| \), \( d_H(\lambda_1, \lambda_2) \leq q_{\text{max}} \text{diam} K_j \leq |j| + 1 \). Taking a limit with \( n \to \infty \), we obtain that a Jordan arc, connecting the points \( x \) and \( y \) is unique. Therefore \( K \) is a dendrite. \[\blacksquare\]

Remark 1. It is possible for a generalized \( P \)-polygonal system \( S \) not to satisfy the condition (2) and to have the attractor \( K \) which is a dendrite. The attractor \( K \) of a generalized polygonal system \( S \) on the picture below is a dendrite, but \( P_7 \cap P_9 = \emptyset \), whereas \( K_7 \cap K_9 \) is a line segment.

Corollary 11. Let \( S \) be a generalized \( P \)-polygonal system, satisfying the condition (2). For any subarc \( \gamma_{xy} \subset K \) and for any \( n \), there is unique chain of pairwise different multiindices \( i_1, i_2, ..., i_l \in I^n \), which divides \( \gamma_{xy} \) to sequential arcs \( \gamma_{xx_1} \subset K_{i_1}, ..., \gamma_{xx_{k-1}x_k} \subset K_{i_k}, ..., \gamma_{x_{l-1}y} \subset K_{i_l} \).

3 \( \delta \)-deformations of contractible polygonal systems.

Definition 12. Let \( \delta > 0 \). A generalized \( P' \)-polygonal system \( S' = \{S'_1, ..., S'_m\} \) is called a \( \delta \)-deformation of a \( P \)-polygonal system \( S = \{S_1, ..., S_m\} \), if there is a bijection \( f : \bigcup_{k=1}^m V_{P_k} \to \bigcup_{k=1}^m V_{P'_k} \), such that

a) \( f|_{V_{P_r}} \) extends to a homeomorphism \( \tilde{f} : P \to P' \);

b) \( |f(x) - x| < \delta \) for any \( x \in \bigcup_{k=1}^m V_{P_k} \)

c) \( f(S_k(x)) = S'_k(f(x)) \) for any \( k \in I \) and \( x \in V_P \).

A polygonal system \( S \) and its \( \delta \)-deformation \( S' \)

Notice that by the Definition 12 if \( z_1, z_2 \in V_P, i, j \in I \) and \( S_i(z_1) = S_j(z_2) \), then \( S'_i(f(z_1)) = S'_j(f(z_2)) \). Moreover, we have the following
Lemma 13. If $A_1, A_2 \in \mathcal{V}_P$, $i, j \in I^*$ and $S_i(A_1) = S_j(A_2)$, then $S'_i(f(A_1)) = S'_j(f(A_2))$.

**Proof:** Suppose $S_i(A) = B \in \mathcal{V}_P$ for some $A \in \mathcal{V}_P$ and let $i = i_1 i_2 \ldots i_n$. Denote $S_{i_{k+1} \ldots i_n}(A)$ by $A_k$.

Then we have a finite sequence of relations between $B \in \mathcal{V}_P$, the vertices $A_k \in \mathcal{V}_P$:

$$B = S_{i_1}(A_1); \quad A_1 = S_{i_2}(A_2); \quad \ldots A_{n-1} = S_{i_n}(A)$$  \hspace{1cm} (4)

Since, by c), $f(S_k(A_k)) = S'_k(A'_k)$, $A'_{k-1} = f(A_{k-1}) = f(S_k(A_k)) = S'_k(A'_k)$, therefore the map $f$ transforms the relations to

$$B' = S'_1(A'_1); \quad A'_1 = S'_2(A'_2); \quad \ldots A'_{n-1} = S'_n(A')$$  \hspace{1cm} (5)

which implies $S'_i(A') = B'$.

Therefore if $S_i(A_1) = S_j(A_2) \in \mathcal{V}_P$, then $S'_i(f(A_1)) = S'_j(f(A_2))$.

Now suppose $S_i(A_1) = S_j(A_2)$ and $i = i'$, $j = j'$ and $S_i(A_1) = S_i(A_2) = S_i(B)$ for some $B \in \mathcal{V}_P$. Then $S'_i(A_1) = S'_j(A_2) = B$, therefore $S'_i(f(A_1)) = S'_j(f(A_2)) = f(B)$ and $S'_i(f(A_1)) = S'_j(f(A_2)) = S'_i(f(B))$. \[\blacksquare\]

**Theorem 14.** Let $K$ and $K'$ be the attractors of a contractible P-polygonal system $S$ and of its $\delta$-deformation $S'$ respectively and $\pi: I^\infty \to K$, $\pi': I^\infty \to K'$ be respective address maps.

(i) There is unique continuous map $\hat{f} : K \to K'$ such that $\hat{f} \circ \pi = \pi'$.

(ii) If $S'$ satisfies condition 3, then $\hat{f}$ is a homeomorphism.

**Remark 2.** Equivalent formulation of the statement (i) of the Theorem is:

There is unique continuous map $\hat{f} : K \to K'$ such that for any $z \in K$ and $i \in I^*$,

$$\hat{f}(S_i(z)) = S'_i(\hat{f}(z)).$$  \hspace{1cm} (6)

**Proof:** The proof is similar to (cf.[II, Lemma 1.]). First, we define the function $\hat{f}$ which is a surjection of the dense subset $G_S(\mathcal{V}_P) \subset K$ to the dense subset $G_{S'}(\mathcal{V}_{P'}) \subset K'$. Second, we show that it is Hölder continuous on $G_S(\mathcal{V}_P)$, and therefore has unique continuous extension to a surjection of $K$ to $K'$, which we denote by the same symbol $\hat{f}$. Third, we show that the condition 3 implies that $\hat{f}$ is injective and therefore is a homeomorphism.

1. Define a map $\hat{f}(z) : G_S(\mathcal{V}_P) \to G_{S'}(\mathcal{V}_{P'})$ by:

$$\hat{f}(z) = S'_i(f(S^{-1}_i(z))) \text{ if } z \in S_i(\mathcal{V}_P)$$  \hspace{1cm} (7)

As it follows from Lemma 13 if $S_i(A_1) = S_j(A_2) = z$ then $S'_i(f(S^{-1}_i(z))) = S'_j(f(S^{-1}_j(z)))$, so the map $\hat{f}$ is well-defined.

Obviously, $\hat{f}(G_S(\mathcal{V}_P)) = G_{S'}(\mathcal{V}_{P'})$ because if $A' \in \mathcal{V}_{P'}$ and $z' = S'_i(A')$, then there is a vertex $A = f^{-1}(A') \in \mathcal{V}_P$, therefore $z' = \hat{f}(S_i(A))$.

Moreover, for any $z \in G_S(\mathcal{V}_P)$ and $i \in I^*$, $\hat{f}(S_i(z)) = S'_i(\hat{f}(z))$ and if $z_1, z_2 \in G_S(\mathcal{V}_P)$, $i, j \in I^*$ and $S_i(z_1) = S_j(z_2)$, then $S'_i(\hat{f}(z_1)) = S'_j(\hat{f}(z_2))$.

2. Let $q_k = \text{Lip } S_k$, $q'_k = \text{Lip } S'_k$, $\beta = \min_{k \in I} \frac{\log q'_k}{\log q_k}$.
Then, following the proof of [7, Theorem 27, step 4], in which for our estimates we use $K'$ instead of $P'$, we see that for any $z_1, z_2 \in G_S(V_P)$,
\[
|z'_1 - z'_2| \leq \frac{2|K'|}{(\rho_0 \cdot \sin(\alpha_0/2))\beta} |z_1 - z_2|^{\beta}.
\]

Therefore the map $\hat{f}$ can be extended to a Hölder continuous surjective map of $K$ to $K'$. Since for any $z \in K$ and any $k \in I$, $\hat{f}(S_k(z)) = S'_{k}(f(z))$, $\hat{f} \circ \pi = \pi'$.

3. Now, suppose the system $S'$ satisfies the condition $[2]$. Suppose for some $\sigma = i_1i_2\ldots \in I^\infty$ and $\tau = j_1j_2\ldots \in I^\infty$, $\hat{f} \circ \pi(\sigma) = \hat{f} \circ \pi(\tau)$. Then, if $i_1 \neq j_1$, then, by condition $[2]$ $P'_i \cap P'_{j_1} \neq \emptyset$, therefore $P_{i_1} \cap P_{j_1} = \{B\}$ for some $B \in V_{\hat{p}}$ and $\pi(\sigma) = \pi(\tau) = B$.

Suppose now $\sigma = l\sigma'$ and $\tau = l\tau'$ and $\hat{f} \circ \pi(\sigma') = \hat{f} \circ \pi(\tau')$, so if first indices in $\sigma'$ and $\tau'$ are different, then $\pi(\sigma) = \pi(\tau) = S_1(B)$ for some $B \in V_{\hat{p}}$.

This implies injectivity of the map $\hat{f}$. So $\hat{f}$ is a homeomorphism of compact sets $K$ and $K'$.

4 Parameter matching theorem.

The Definition 5 of cyclic vertices can be applied to generalized polygonal systems. In this case, if $A$ is a cyclic vertex of a generalized P-polygonal system $S$, the map $S_i$ for which $S_i(A) = A$, need not be a homothety and we have to define the rotation parameter for such map. Though the rotation angle $\alpha_i$ of the map $S_i$ is defined up to $2n\pi$, the number $n$ is unique defined by the set $\hat{P}$.

**Definition 15.** Let $A$ be a cyclic vertex and $S_i(z) = re^{ia}(z - A) + A$, then the parameter $\lambda_A$ of the cyclic vertex $A$ is a number $\frac{\alpha}{\ln r}$, where the angle $\alpha$ is defined by the geometrical configuration of the system.

**Remark 3.** The following picture shows how the angle $\alpha$ depends on the geometric configuration of the system $S$.

![Diagram showing angle dependence](image)

**Definition 16.** Generalized P-polygonal system $S$ of similarities satisfies the parameter matching condition, if for any $B \in \bigcup_{i=1}^{m} V_P$ and any cyclic vertices $A, A'$ such that for some $i, j \in I^*$, $S_i(A) = S_j(A') = B$, the equality $\lambda_A = \lambda_{A'}$ holds.

**Lemma 17.** Let $S$ be a generalized P-polygonal system, satisfying the condition $[2]$. For any vertices $A, B \in V_P$ there are $A', B' \in V_P$ and a map $S_i \in S$ such that $S_i(A') = A$ and $S_i(\gamma_{A'B'}) \subset \gamma_{AB}$.

**Proof:** Consider the unique arc $\gamma_{AB}$, connecting $A$ and $B$.

For the arc $\gamma_{AB}$ we consider the chain $i_1, i_2, \ldots, i_l$, which partitions it to subarcs $\gamma_{A_{i_1}} \subset K_{i_1}$, $\ldots, \gamma_{x_{k-1}x_k} \subset K_{i_k} \ldots, \gamma_{x_{l-1}x_1} \subset K_{i_1}$ (possibly to the only arc $\gamma_{AB}$ if $\gamma_{AB} \subset K_{i_1}$). Put $A' = S_{i_1}^{-1}(A)$, $B' = S_{i_1}^{-1}(x_1)$, and $\gamma(A'B') = S_{i_1}^{-1}(\gamma_{AxB})$. ■
Proposition 18. Let \( S \) be a generalized \( P \)-polygonal system satisfying the condition \([3]\) and let \( A \) be a cyclic vertex of the polygon \( P \). Then there is such vertex \( B \in V_P \) and a multiindex \( \mathbf{i} \in I^* \), that \( S_1(A) = A \) and the Jordan arc \( \gamma_{AB} \subset K \) satisfies the inclusion \( S_1(\gamma_{AB}) \subset \gamma_{AB} \).

**Proof:** Notice that if \( S \) is a contractible \( P \)-polygonal system then for any cyclic vertex \( A \) and for any \( n \) there is unique multiindex \( \mathbf{i} \in I^n \), and unique vertex \( B \in V_P \), such that \( S_1(B) = A \). Therefore, if \( S_1(A) = A \), the piece \( S_1(K) \) separates the point \( A \) from the other part of the attractor \( K \) of the system \( S \), i.e. \( A \notin K \setminus S_1(K) \) and each Jordan arc \( \gamma_{AB} \) where \( B \in V_P \setminus \{A\} \), contains a point \( B' \in S_1(V_P \setminus \{A\}) \).

In the case when \( S \) is a generalized polygonal system, the situation is more complicated. Since the attractor \( K \) is a dendrite in the plane which has one-point intersection property, it follows from \([3]\) that the system \( S \) satisfies OSC and each vertex \( A' \in V_P \) has finite ramification order. Let \( U_1, ..., U_s \) be the components of \( K \setminus \{A\} \). Since \( S_1 \) fixes \( A \), there is a permutation \( \sigma \) of the set \( \{1, ..., s\} \), such that for any \( l \in \{1, ..., s\} \), \( S_1(U_l) \subset U_{\sigma(l)} \). Therefore there is such \( N \) that \( \sigma^N(N) = \text{Id} \) and \( S_1 = S_1^N \) sends each \( U_l \) to a subset of \( U_l \). Each of those components \( U_l \) which have non-empty intersection with \( V_P \setminus \{A\} \) has also non-empty intersection with \( S_j(V_P \setminus \{A\}) \), therefore each arc \( \gamma_{AB}, B \in V_P \) contains a point \( B' \in S_j(V_P) \).

Let us enumerate the vertices of \( P \) so that \( A = A_1 \) and other vertices are \( A_2, ..., A_{n_p} \). For each vertex \( A_k, k \geq 2 \) there is unique vertex \( A_{k'} \) such that \( \gamma_{A_1A_k} \cap S_1(V_P) = S_1(A_{k'}) \). The formula \( \phi(k) = k' \) defines a map \( \phi \) of \( \{2, 3, ..., n_p\} \) to itself. There is some \( N' \) such that \( \phi^{N'} \) has a fixed point \( k \). Therefore \( S_1^{N'}(\gamma_{A_1A_k}) \subset \gamma_{A_1A_k} \). ■

**Definition 19.** The arc \( \gamma_{AB} \) is called an invariant arc of the cyclic vertex \( A \).

From Propositions \([7]\) and \([18]\) and V.V.Aseev’s Lemma about disjoint periodic arcs \([1]\) Lemma 3.1] we come to the following Parameter Matching Theorem:

**Theorem 20.** Let the generalized \( P' \)-polygonal system \( S' \) be a \( \delta \)-deformation of a contractible \( P \)-polygonal system \( S \) and the attractor \( K' \) of the system \( S' \) be a dendrite. Then the system \( S' \) satisfies parameter matching condition.

**Proof:** Let \( S \) be a generalized polygonal system whose attractor \( K \) is a dendrite. Let \( C \in \cup_{i=1}^{n_1} V_P \) and \( A, A' \in V_P \) be such cyclic vertices that for some \( i, j \in I \), \( S_1(A) = S_j(A') = C \). Denote the images \( S_1(K) \) and \( S_j(K) \) by \( K_i, K_j \) respectively. Without loss of generality we can suppose that the point \( C \) has coordinate \( 0 \) in \( C \). Since for some \( i, j \in I^* \), \( S_1(A) = A \) and \( S_j(A') = A' \), the maps \( S_i = S_iS_i^{-1} \) and \( S_{i2} = S_jS_j^{-1} \) have \( C \) as their fixed point and \( S_{i1}(K_i) \subset K_i \) and \( S_{i2}(K_j) \subset K_j \). Let \( S_{i1}(z) = q_i e^{i\alpha_i}z \) and \( S_{i2}(z) = q_i e^{i\alpha_i}z \). So the parameters of the vertices \( A \) and \( A' \) will be \( \lambda_1 = \frac{\alpha_1}{\log q_1} \) and \( \lambda_2 = \frac{\alpha_1}{\log q_2} \). Let \( \gamma_{AB} \subset K \) and \( \gamma'A'B' \subset K \) be invariant arcs for the vertices \( A \) and \( A' \). Let also \( \gamma_1 = S_1(\gamma_{AB}) \) and \( \gamma_2 = S_j(\gamma_{A'B'}) \). Then \( S_1(\gamma_1) \subset \gamma_1 \) and \( S_j(\gamma_2) \subset \gamma_2 \). By V.V.Aseev’s Lemma on disjoint periodic arcs \([1]\) Lemma 3.1] it follows that if \( \gamma_1 \cap \gamma_2 = \{C\} \), then \( \lambda_1 = \lambda_2 \). ■

5 Main theorem.

**Some assumptions.** From now on we will use the following convention: \( S = \{S_1, ..., S_m\} \) will denote a contractible \( P \)-polygonal system and \( S' = \{S'_1, ..., S'_m\} \) will be a \( P' \)-polygonal system which is a \( \delta \)-deformation of \( S \) defined by a map \( f \).

For any \( k \in I \), \( S_k(z) = q_k e^{i\alpha_k}(z - z_k) + z_k \) and \( S'_k(z) = q'_k e^{i\alpha'_k}(z - z'_k) + z'_k \), where \( z_k = \text{fix}(S_k) \).

We also suppose by default that \( \text{diam} P = 1 \). We suppose that

\[
\delta < q_{\min}/8 \quad \text{and} \quad \delta < (1 - q_{\max})/8 \tag{8}
\]
Lemma 21. Let $S' = \{S'_1, \ldots, S'_m\}$ be a $\delta$-deformation of a contractible $P$-polygonal system $S$. For sufficiently small $\delta$, and for any $k \in I$, 

$$\frac{q_k - 2\delta}{1 + 2\delta} \leq q'_k \leq \frac{q_k + 2\delta}{1 - 2\delta} \quad \text{and} \quad |\alpha'_k - \alpha_k| \leq \arcsin 2\delta + \arcsin \frac{2\delta}{q_k}. \quad (9)$$

**Proof:** Let $A, B$ be such vertices of $P$ that $|B - A| = 1$. Let us write $S'_k(A) = A_k$ and $f(A) = A'$ and use the similar notation for all vertices so by definition, $S'_k(A') = A'_k = f(A_k)$. Notice that $\frac{B_k - A_k}{B - A} = q_k e^{i\alpha_k}$ and $\frac{B'_k - A'_k}{B' - A'} = q'_k e^{i\alpha'_k}$.

Since the map $f$ moves $A, B, A_k, B_k$ to a distance at most $\leq \delta$, so $|(B - A) - (B' - A')| \leq 2\delta$ and $|(B_k - A_k) - (B'_k - A'_k)| \leq 2\delta$. Therefore $|(B_k - A_k)| - 2\delta \leq |(B'_k - A'_k)| \leq |(B_k - A_k)| + 2\delta$ and

$$\alpha'_i - \alpha_i = \arg \frac{B'_i - A'_i}{B' - A'} \subset \frac{B_k - A_k}{B_k - A_k} = \arg \frac{B'_k - A'_k}{B_k - A_k} - \arg \frac{B' - A'}{B - A} \quad (10)$$

This implies the inequalities (9). \[\square\]

Under the Assumptions (8), $3q_{\min}/5 < \frac{q_{\min} - 2\delta}{1 + 2\delta} < q'_k < \frac{q_{\max} + 2\delta}{1 - 2\delta} < 1 + 3q_{\max}/3 + q_{\max}$; taking into account that $q_k < 1$ and $1 - 2\delta > 3/4$, and that if $0 < x < .5$, then $\arcsin x < 1.05x$, we have

$$\Delta q_k = |q'_k - q_k| < \frac{2\delta(1 + q_k)}{1 - 2\delta} < 6\delta \quad \text{and} \quad \Delta \alpha_k = |\alpha'_k - \alpha_k| < C_\delta \quad (11)$$

where $C_\alpha = 2.1(1 + 1/q_{\min})$.

Let $V_\delta(P)$ denote $\delta$-neighborhood of the polygon $P$.

Lemma 22. Let $S' = \{S'_1, \ldots, S'_m\}$ be a $\delta$-deformation of a contractible $P$-polygonal system $S$. The set $U = V_{\delta_1}(P)$, where $\delta_1 = \frac{8\delta}{1 + 3q_{\max}}$, satisfies the condition

for any $k \in I$, $S_k(U) \subset U$ and $S'_k(U) \subset U \quad (12)$

**Proof:** By Definition 12, $V_\delta(P_k) \supset P'_k$, $V_\delta(P'_k) \supset P_k$ and since vertices of $P$ are also moved at distance less than $\delta$, $V_\delta(P) \supset P'$ and $V_\delta(P') \supset P$.

So we can write $S'_k(P') \subset V_\delta(P) \subset V_{2\delta}(P)$ from which it follows that $S'_k(P') \subset V_{2\delta}(P_k) \subset V_{2\delta}(P)$. For any positive $\rho$ we have the inclusion $S'_k(V_{\rho}(P)) \subset V_{2\delta + q_{\max} \rho}(P)$. In the case when $\rho = 2\delta + q_k \rho$ this implies $S'_k(V_{\rho}(P)) \subset V_{\rho}(P)$ where $\rho = \frac{2\delta}{1 - q_k}$. Since $q'_k \leq q_k + 2\delta$, $q'_{\max} \leq q_{\max} + 2\delta < 3q_{\max} + 1/4$, we come to inclusions (12). \[\square\]

Lemma 23. For any $z \in V_{\delta_1}(P)$, $|S'_k(z) - S_k(z)| < C_\Delta \delta$, where $C_\Delta = 14 + 2C_\alpha$.

**Proof:** Take $z \in V_{\delta_1}(P)$ and consider the difference $S'_k(z) - S_k(z)$. It can be represented in the form $S'_k(A) - S_k(A) + (q_k e^{i\alpha_k} - q_k e^{i\alpha_k})(z - A)$. So

$$|S'_k(z) - S_k(z)| < |S'_k(A) - S_k(A)| + (|q_k - q_k| + |q_k e^{i\alpha_k} - e^{i\alpha_k}|)|z - A|. \quad (13)$$

Since $|z - A| < 1 + \delta_1 < 2$ and $|S'_k(A) - S_k(A)| < 2\delta$, the right hand side of (13) is no greater than $2\delta + 2(6\delta + C_\alpha \delta)$. \[\square\]
Proposition 24. Let $\pi : I^\infty \to K$ and $\pi' : I^\infty \to K'$ be the address maps for the systems $S$ and $S'$ respectively.

1. Under the assumptions (8), for any $\sigma \in I^\infty$,

$$|\pi'(\sigma) - \pi(\sigma)| < C_K \delta \text{ where } C_K = \frac{2C_\Delta}{1 - q_{\text{max}}}$$

(14)

2. For any $n$, if the system $S''^{(n)}$ is a generalized polygonal system, then it is $C_K \delta$-deformation of the system $S^{(n)}$.

Remark 4. Let $S' = \{S'_1, ..., S'_m\}$ be a $\delta$-deformation of a contractible $P$-polygonal system $S$. Let $A \in S_j(V_P)$ for some $j \in I$. Let $g(z) = z - A + A'$ and $S''_k = g \circ S'_k \circ g^{-1}$. Then $S'' = \{S''_1, ..., S''_m\}$ is a $2\delta$-deformation of the system $S$, for which $A'' = A$, $K'' = g(K')$, $P''_j = g(P_j)$. Since $g$ is a translation, the estimates (9) and (11) for $S''$ remain the same with the same $\delta$, while $|\pi''(\sigma) - \pi(\sigma)| < (C_K + 1)\delta$. Thus we will denote $\delta_2 = (C_K + 1)\delta$.

Taking into account the Propositions 7 and 24, it is sufficient to prove the Theorem for the case when all cyclic vertices of the system $S$ have order 1.

Proposition 25. Let $P'$-polygonal system $S'$ be a $\delta$-deformation of a contractible $P$-polygonal system $S$. Let $A \in V_P$ be a cyclic vertex (of order 1) and $S_k(z) = q_k e^{\alpha_k}(z - A) + A$. Then the rotation angle $\alpha_k$ of the map $S_k'$ does not exceed $\arcsin 2\delta + \arcsin \frac{2\delta}{q_k}$ and the parameter $\lambda_k$ of the map $S_k'$ satisfies the inequality

$$|\lambda_k| \leq \frac{\arcsin 2\delta + \arcsin \frac{2\delta}{q_k}}{\log(q_k + 2\delta) - \log(1 - 2\delta)}$$

(15)

Proof: The formula (15) follows directly from Lemma 21.

Under the assumptions (8),

$$|\lambda_k| < C_\lambda \delta,$$

where $C_\lambda = \frac{2.1(1 + 1/q_{\text{max}})}{\log(3 + q_{\text{max}}) - \log(3q_{\text{max}} + 1)}$.

(16)

Lemma 26. Let $S$ be a contractible $P$-polygonal system whose cyclic vertices have order 1 and $S'$ be its $\delta$-deformation. Then if

$$2.1 \frac{\delta_2}{\rho_1} + \lambda \log \frac{\rho_2 + \delta_2}{\rho_1 - \delta_2} < \alpha_0 \text{ and } 2\delta_2 < \rho_0,$$

(17)

then the system $S'$ satisfies the Condition (3).

Proof. Take a vertex $B \in V_{P'}$. We may suppose for convenience that $B = 0$ and, following Remark 4, we can suppose that the mapping $f$ fixes the vertex $B = 0$, so $B' = B = 0$. Let $W_l = S_l(K \setminus K_{\eta})$. The maps $\tilde{S}_l = S_l, S_{-1}, S_{-2}$ are homotheties with a fixed point $B$ such that

$$K_{h \setminus \{B\}} = \bigcup_{n=0}^{\infty} \tilde{S}_l^n(W_l)$$

(18)

Similarly, let $W_l' = \tilde{f}(W_l)$ and $\tilde{S}_l = S_l'. S_{-1} S_{-2}^{-1}$. Then

$$K_{h \setminus \{B\}} = \bigcup_{n=0}^{\infty} \tilde{S}_l'^n(W_l')$$

(19)

11
Notice that for any \( l \), \( \tilde{S}_l(z) = q_{li}z \) and \( \tilde{S}'_l(z) = q'_{li} e^{\alpha_{li}} z \), and due to parameter matching condition, there is such \( \lambda \), that for any \( l \), \( \alpha_{li} = \lambda \log q'_{li} \).

Consider the map \( z = \exp(w) \) of the plane \( (w = \varrho + i\varphi) \) as universal cover of the punctured plane \( \mathbb{C} \setminus \{0\} \).

Consider polygons \( P_j \), and choose their liftings in the plane \( (w = \varrho + i\varphi) \). We may suppose these liftings lie in respective horizontal strips \( \theta^l_+ - \varphi \leq \theta^l \leq \theta^l_+ \), where \( 0 < \theta^l_+ < \theta^l_+ < 2\pi \) and \( \theta^l_+ + \alpha_0 < \theta^l_+ + 1 \) for any \( l < k \) and \( \theta^l_+ + \alpha_0 < \theta^l_+ + 1 \). We also consider liftings of \( K_j, W_l, K'_j \), and \( W'_l \). We denote these liftings by \( \mathcal{K}_j, W_l, \mathcal{K}'_j \), and \( W'_l \). It follows from the equations \( 18 \) and \( 19 \) that

\[
\mathcal{K}_j = \bigcup_{n=0}^{\infty} \tilde{T}_l^n(W_l) \quad \text{and} \quad \mathcal{K}'_j = \bigcup_{n=0}^{\infty} \tilde{T}'_l^n(W'_l) \tag{20}
\]

where \( T_l(w) = w + \log q_l \) and \( T'_l(w) = w + (1 + i\lambda) \log q'_l \) are parallel translations for which \( T_l(\mathcal{K}_l) \subset \mathcal{K}_l \) and \( T'_l(\mathcal{K}'_l) \subset \mathcal{K}'_l \).

The images of the set \( K' \) under the map \( w = \log(z - O) \) and the map \( w = \log(z - B) \).

The sets \( \mathcal{K}_l \) lie in the half-strips \( \varrho \leq \log \rho_2, \theta^l_+ - \varphi \leq \theta^l \), while the sets \( W_l \) are contained in rectangles \( R_l = \{ \log \rho_1 \leq \varrho \leq \log \rho_2, \theta^l_+ - \varphi \leq \theta^l \} \).

Then the sets \( W'_l \) lie in a rectangle

\[
R'_l = \left\{ \log(\rho_1 - \delta_2) \leq \varrho \leq \log(\rho_2 + \delta_2), \theta^l_+ - 1.05 \frac{\delta_2}{\rho_1} \leq \varphi \leq \theta^l_+ + 1.05 \frac{\delta_2}{\rho_1} \right\}
\]

Each union \( \bigcup_{n=0}^{\infty} T'^n_l(R'_l) \) lies in a half strip

\[
\left\{ \begin{array}{l}
\varrho \leq \log(\rho_2 + \delta_2) \\
\theta^l_+ - 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_2 + \delta_2) \leq \varphi - \lambda \varrho \leq \theta^l_+ + 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_1 - \delta_2)
\end{array} \right.
\]

Therefore the set \( \mathcal{K}'_j \) also lies in this half-strip. So, if

\[
\theta^l_{i-1} + 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_1 - \delta_2) < \theta^l_i - 1.05 \frac{\delta_2}{\rho_1} - \lambda \log(\rho_2 + \delta_2) \tag{21}
\]

then \( \mathcal{K}'_{j-1} \cap \mathcal{K}'_j = \emptyset \).

We can guarantee that such inequality holds for any \( l \) if \( 2.1 \frac{\delta_2}{\rho_1} + \lambda \log \frac{\rho_2 + \delta_2}{\rho_1 - \delta_2} < \alpha_0 \).

If, moreover, \( 2\delta_2 < \rho_0 \), then for any \( i_1, i_2 \in I \) such that \( P_{i_1} \cap P_{i_2} = \emptyset \), \( P'_{i_1} \cap P'_{i_2} = \emptyset \) and \( K'_{i_1} \cap K'_{i_2} = \emptyset \) which implies the condition \( [2] \).

**Theorem 27.** Let \( S \) be a contractible \( P \)-polygonal system. There is such \( \delta > 0 \) that for any \( \delta \)-deformation \( S' \) of the system \( S \), satisfying parameter matching condition, the attractor \( K(S') \) is a dendrite, homeomorphic to \( K(S) \).
Proof: Let all the cyclic vertices of the $P$-polygonal system $S$ have order 1. If we suppose that $\delta_2 < \rho_1/4, \text{and } \delta_2 < (1 - \rho_2)/4$ and combine the inequalities $\frac{\rho_0}{2(C_K + 1)}$, we see that if the following inequalities hold:

1. $\delta < \frac{q_{\min}}{8}$;  
2. $\delta < \frac{1}{q_{\max}} - \frac{1}{8}$;  
3. $\delta < \frac{\rho_0}{2(C_K + 1)}$;  
4. $\delta < \frac{\rho_1}{4(C_K + 1)}$;
5. $\delta < \frac{1 - \rho_2}{4(C_K + 1)}$;  
6. $\delta < \frac{2.1(C_K + 1)}{\rho_1} + \frac{\alpha_0}{3 \rho_1} + C_\lambda \log \left(1 + 3 \rho_2\right)$,

then the attractor $K'$ of $\delta$-deformation $S'$ of the system $S$ satisfies the condition [2]. Therefore $K'$ is a dendrite. By Theorem 14, the map $f : K \to K'$ is a bijection and therefore it is a homeomorphism.

Suppose now that $S$ has cyclic vertices of order greater than 1 and let $M = 12 + 4.2 \left(1 + \frac{1}{q_{\min}}\right)$. There is such $n$, that the system $S^{(n)}$ has cyclic vertices of order 1. Suppose any $\delta$-deformation of the system $S^{(n)}$ generates a dendrite. Then for any $\delta/M$-deformation deformation $S'$ of the system $S$, the system $S^{(n)}$ is a $\delta$-deformation of the system $S^{(n)}$.

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