On the quantum Geroch group

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Abstract. The Geroch group is an infinite dimensional transitive group of symmetries of cylindrically symmetric gravitational waves which acts by non-canonical transformations on the phase space of these waves. The unique Poisson bracket on the Geroch group which makes this action Lie-Poisson is obtained. A quantization of the Geroch group is proposed, at a formal level, that is very similar to an $\mathfrak{sl}_2$ Yangian, and a certain action of this quantum Geroch group on gravitational observables is shown to preserve the commutation relations of Korotkin and Samtleben’s quantization of cylindrically symmetric gravitational waves. The action also preserves three of the four additional conditions that define their quantization. It is conjectured that the action preserves the remaining condition as well and is, in fact, a symmetry of their model.

1. Introduction

Cylindrically symmetric vacuum general relativity is an important truncation of full general relativity (GR) which has infinitely many degrees of freedom, yet is tractable because it is integrable \cite{BZ78,Mai78}. It is useful as a toy model for studying both classical and quantum gravitational phenomena, and as a testbed for methods of quantizing full vacuum GR \cite{Ash96,Var00,GP97,DT99,AM00,FMV03,KN96,NS00,Nie03,CMN03,HS89,Hus96}. It is, for instance, highly relevant to the quantization of null initial data for vacuum GR, since the Poisson algebra for these in the general case is almost identical to that of the cylindrically symmetric case \cite{FR17}.

In 1971 Kuchar \cite{Kuc71,All87,AP96} quantized cylindrically symmetric vacuum GR subject to a further restriction on the polarization of gravitational waves, and in 1997 Korotkin and Samtleben \cite{KS98a,KS98b} obtained an almost complete quantization without this restriction. Specifically, Korotkin and Samtleben obtained a quantization of the Poisson algebra of classical observables (phase space functions) of full cylindrically symmetric vacuum GR, but not a representation of the resulting algebra of quantum observables on a Hilbert space that respects the reality conditions.

Cylindrically symmetric vacuum GR possesses a transitive group of symmetries called the Geroch group \cite{Ger72,Kin77,KC77,KC78a,KC78b,Jul85}. These symmetry transformations were initially used as a means to generate new solutions to the field equations from old, but this is not the focus of our interest here. We are rather exploring the possibility that this symmetry is a deep structural feature of the model, which is preserved in the quantum theory. Since the Geroch group acts transitively on the phase space \cite{HE01}, that is, any solution can be reached...
from any other via an action of the group, one expects that the Geroch group will act as a spectrum generating symmetry of the quantum theory \cite{Maj11,MM92}. Our result supports this expectation for the quantum theory of Korotkin and Samtleben: We obtain a quantization of the Geroch group and of its action on the cylindrically symmetric gravitational field which appears to be a symmetry of the algebra of quantum observables of this field. We verify that it preserves the commutators and all other relations defining this algebra, save one which we are unable to check.

This result is not entirely surprising since the quantization of \cite{KS98b} is based on the integrability of the model, which in turn is closely related to the symmetry under the Geroch group. Indeed, Korotkin and Samtleben explored the action of the Geroch group in their formalism. They showed in \cite{KS98b} how the non-canonical Geroch group transformations of the classical phase space are generated by a family of phase space functions as generalized Hamiltonian flows, and they noted that the Poisson brackets between these generators indicated that the quantization of the Geroch group should be an $\mathfrak{sl}_2$ Yangian, a well known quantum group introduced by Drinfel’d in \cite{D83}. Samtleben \cite{Sam98} even makes a partial proposal for the action of the generators on observables in the quantum theory.

Our results are complementary to theirs. We determine the Poisson bracket on the Geroch group from the requirement that the action of the group on the phase space of cylindrically symmetric GR be a Lie-Poisson map, which is to say, that it satisfies (1). (See \cite{BR00,STS85}.) With this bracket the group has a natural quantization, a quantum group very similar to an $\mathfrak{sl}_2$ Yangian. We then make an ansatz for the form of the action of the quantum Geroch group on the gravitational observables, a natural two parameter generalization of the classical action, and fix the parameters by requiring the action to be a symmetry. Our approach does not rely on finding the generators of the Geroch group in the algebra of observables.

Finding the generators is, however, a crucial step for turning Geroch group into a spectrum generating symmetry of the quantum theory. Although we will not carry out this program here let us outline it to put our result in context. A spectrum generating group is a group $G$ of symmetries together with a homomorphism $\pi$ of this group into unitary elements of the (complexified) algebra $Q$ of quantum observables. Thus, $\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$. Now suppose $\rho : Q \to \text{End}(\mathcal{H})$ is a homomorphism defining a $*$-representation of $Q$ by linear operators on a Hilbert space $\mathcal{H}$, that is a representation such that $*$ is mapped to adjoint, $\dagger$. Then $U = \rho \circ \pi$ defines a representation of $G$ by operators on the Hilbert space $\mathcal{H}$. Acting with $g \in G$ on states maps the matrix element $\langle a | X b \rangle$ of an operator $X$ between states $a, b \in \mathcal{H}$ to $\langle a | U(g)^\dagger X U(g) b \rangle$. This is equivalent to the map

$$X \mapsto U(g)^\dagger X U(g)$$

\[ \dagger \] The algebraic relations that define our quantum Geroch group are the same as those of the Yangian in the RTT presentation (see \cite{FRT90} and \cite{Mol03}). But while the Yangian is obtained by imposing these relations on formal power series in the inverse spectral parameter, in the present case the quantum Geroch group is obtained by imposing them on sums of Fourier modes in the spectral parameter, over a continuous range of frequencies. This difference arises because the classical Geroch group is a “line group”, consisting of $SL(2)$ valued functions on the real axis, while the Yangian quantizes a loop group, consisting of $SL(2)$ valued functions on the unit circle. These are not equivalent because the Poisson bracket, which takes the same form for both groups in the complex spectral parameter plane containing the real axis and unit circle, is invariant under translations in the complex plane, but not under the Moebius transformation that takes the real axis to the unit circle. Although we outline how such Fourier sums might be defined in terms of distributions, we do not work out the details. Rather, we proceed formally in the quantum theory, deducing our conclusions from the algebraic properties of the quantum Geroch group, which we assume exists.
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Note that it is only because $\pi$, and thus also the representation $U$, is unitary that this map preserves algebraic relations between observables.

The utility of a spectrum generating group lies in the fact that any $*$-representation of $Q$ is an extension of a representation of the spectrum generating group, so finding a suitable $*$-representation of the algebra of observables (which is an open problem in the case of Korotkin and Samtleben’s quantization of cylindrically symmetric gravity) reduces to an examination of the representations of the group and their possible extensions. In particular, if the group acts transitively on the classical phase space, as the Geroch group does in the case of cylindrically symmetric GR, it is reasonable to expect it also to act transitively on the Hilbert space of states in the quantum theory, and thus that the representation $U$ of the group is irreducible. If this is the case then the representation of the group and $\pi$ give detailed information about the matrix elements of the observables in $Q$ in the representation $\rho$. (The requirement that the action of the group be irreducible in $H$ is usually part of the definition of a spectrum generation group [Maj11].)

Because

$$U^\dagger [A, B] U = [U^\dagger AU, U^\dagger BU]$$

(2)

for any pair of observables $A$ and $B$, the transformation (1) preserves commutation relations of observables in the quantum theory, and thus the Poisson algebra of phase space functions in the classical limit. But the action of the Geroch group on the phase space of cylindrically symmetric GR does not do this; Geroch group actions are not canonical transformations. So the Geroch group does not fit into the framework of spectrum generating groups we have outlined.

However, the Geroch group does fit if the framework is extended by allowing $G$ to become a quantum group in the quantum theory. In this case $G$ is still a true group in the classical limit, but it acquires a non-trivial Poisson bracket which turns it into a phase space, and it is quantized along with the phase space $\Gamma$ of the physical system in the quantum theory. The matrix elements $\langle a | U b \rangle$ of the symmetry transformations in the Hilbert space $H$ of the physical system become operators: If $G$ were not quantized they would be functions of the group manifold taking a complex number value at each point $g \in G$, but when $G$ is quantized they become operators on the Hilbert space of the quantum group, having an expectation value for each state of this group.

The notion of a spectrum generating symmetry generalizes directly to quantum groups. See [MM92] and also [Timm08] Chapter 3. In particular, $U$ is still unitary, but in the sense of quantum group actions:

$$\langle U a | U b \rangle = \langle a | b \rangle \otimes 1$$

(3)

where $1$ is the unit operator on the group Hilbert space. Such a transformation preserves inner products in $H$ in the sense that the expectation value of $\langle U a | U b \rangle$ in any state of the group is equal to the untransformed inner product $\langle a | b \rangle$. It also preserves the commutation relations of observables, because once again equation (2) holds. But note that now the commutator $[U^\dagger AU, U^\dagger BU]$ receives a contribution from the non-trivial commutators between the matrix elements of $U$. As a consequence the Poisson bracket is still preserved in the classical limit, but only when a contribution to the transformed bracket from the Poisson bracket on $G$ is included:

$$\{ A, B \}_\Gamma(g \triangleright \xi) \equiv \{ A, B \}_\Gamma^G(g, \xi) = \{ A^G(g, \cdot), B^G(g, \cdot) \}_\Gamma(\xi) + \{ A^G(\cdot, \xi), B^G(\cdot, \xi) \}_G(g)$$

(4)

$$= \{ A^G, B^G \}_{G \times \Gamma}(g, \xi)$$

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Here \( g \triangleright \xi \) denotes the action of a group element \( g \in G \) on a phase space point \( \xi \in \Gamma \), and \( f^G(g, \xi) = f(g \triangleright \xi) \) is the corresponding action of \( G \) on a function \( f \) of \( \Gamma \). The right side is simply the Poisson bracket on \( \Gamma \) of the functions \( A^G \) and \( B^G \) with \( g \in G \) held fixed, plus the bracket on \( G \) of these two functions but with \( \xi \) held fixed. This is just the standard Poisson bracket of \( A^G \) and \( B^G \) on the phase space \( G \times \Gamma \), the phase space of a pair of subsystems that are independent in the sense that their phase coordinates Poisson commute. Following [BR00] a Lie group action satisfying (4) will be called a Lie-Poisson action.

As already said, we will not work out the homomorphism \( \pi \), which turns the Geroch group into a spectrum generating (quantum) group. A large part, perhaps most, of this task has already been completed in [KS97], [KS98b], and [Sam98]. Rather, we will choose a parametrization of the phase space of cylindrically symmetric vacuum GR which makes the classical action of the Geroch group particularly simple, by the so called “deformed 2-metric” \( \mathcal{M} \), which is a one parameter family of \( 2 \times 2 \) matrices. Then we will use the Poisson brackets of \( \mathcal{M} \), which are also quite nice, to determine the unique Poisson bracket on the Geroch group that ensures that (4) is satisfied. The Geroch group with this Poisson bracket has a natural quantization very similar to the \( \mathfrak{sl}_2 \) Yangian. Finally, we obtain the action (111) of the quantized Geroch group on \( \mathcal{M} \), a small modification of the classical action, which appears to define an automorphism of the quantum algebra of observables (which is generated by the \( \mathcal{M} \)s) in the quantization of cylindrically symmetric vacuum GR of Korotkin and Samtleben [KS98b]. Specifically, we show that this action preserves the exchange relation of \( \mathcal{M} \), as well as its symmetry, reality and positive semi-definiteness. We have not been able to show that the action preserves the Korotkin and Samtleben’s quantization of the condition \( \det \mathcal{M} = 1 \), but suspect that it does.

The roles of our results and those of Korotkin and Samtleben in developing the Geroch group as a spectrum generating symmetry of cylindrically symmetric GR can be illustrated by a simple analogy. Consider the rotation group \( SO(3) \) as a symmetry in the non-relativistic classical and quantum mechanics of a spinless particle in three dimensional space. Depending on the potential energy function of the particle, rotations may or may not be symmetries of the Hamiltonian, but they certainly preserve Poisson brackets, so we can use them as a spectrum generating group to organize the Hilbert space. (Actually, since \( SO(3) \) does not act irreducibly on the Hilbert space it does not quite satisfy the usual definition of a spectrum generating group [Maj11], but it is close enough for the purpose of our analogy.) Classically a rotation by a vectorial angle \( \theta \) transforms the Cartesian coordinate position \( q \) and its conjugate momentum \( p \), or indeed any vector \( x \), according to

\[
x \mapsto e^{\theta \times x}.
\]

where \( \times \) denotes the 3-vector cross product operation. Quantum mechanically the rotation of any observable is obtained by conjugation with \( \pi(e^{\theta \times}) = e^{\pi \theta \cdot J} \) where \( J = q \times p \). For vector observables

\[
e^{-\frac{i}{\hbar} \theta \cdot J} x e^{\frac{i}{\hbar} \theta \cdot J} = e^{\theta \times} x.
\]

The work of Korotkin and Samtleben to obtain the generators of the Geroch group, [KS97] [KS98b] [Sam98], corresponds in this analog to identifying the angular momentum operator \( J \) as the generator of rotations. Our result refers more to the right side of (7). The Cartesian position \( q \) and momentum \( p \) play the role of the deformed metric \( \mathcal{M} \) as the phase space coordinates that transform in a simple way under the group, and the transformation of the quantum \( q \) and \( p \) vectors on the right side of (7) corresponds to our transformation law (111).
of the quantum deformed metric $\mathcal{M}$. Of course, in the case of the particle the transformations of $q$ and $p$ under rotations in the quantum theory is identical to their classical transformations, while in the case of the Geroch group the transformation of $\mathcal{M}$ is changed in non-trivial ways in the quantum case: The group itself is quantized, and the form of the transformation law has to be adjusted a little.

The paper is organized as follows: In Section 2 the definition of cylindrically symmetric vacuum GR is recalled, as well as the structures associated with its integrability, in particular the auxiliary linear problem and the deformed metric $\mathcal{M}_{ab}$. This leads to a precise description of the space of solutions, in terms of which the classical Geroch group is defined in a very simple way. (This section is based on ideas of [BZ78, HE01, BM87, Nic91, KS98b, Fuchs] and others. Although the underlying ideas are not new the specific development given here, which is both mathematically rigorous and short, is new as far as the authors know.) In Section 3 the Poisson brackets on the space of solutions is recalled, and from this the Poisson bracket on the Geroch group that makes the action of this group Lie-Poisson is obtained. Section 4 is dedicated to the quantum theory. A quantization of the Geroch group as a quantum group closely analogous to an $\mathfrak{sl}_2$ Yangian double is proposed, and an action of this quantum group on the observables of Korotkin and Samtleben’s quantization of cylindrically symmetric vacuum GR which preserves the commutators (exchange relations) and three of four further conditions defining the algebra of these observables is found. The paper closes with some concluding remarks.

2. Cylindrically symmetric vacuum gravity and the Geroch group

We will consider smooth ($C^\infty$) solutions to the vacuum Einstein field equations with cylindrical symmetry. Cylindrically symmetric spacetime geometries have two commuting spacelike Killing fields that generate cylindrical symmetry orbits. Furthermore, the Killing orbits should be orthogonal to a family of 2-surfaces, a requirement called ”orthogonal transitivity”. The line element then takes the form

$$\text{d}s^2 = \Omega^2 (-\text{d}t^2 + \text{d}\rho^2) + \rho h_{ab} \text{d}\theta^a \text{d}\theta^b,$$

where $\theta^a = (\phi, z)$ are coordinates on the symmetry cylinders such that $\partial_z$ and $\partial_\phi$ are Killing vectors, with $\phi \in (0, 2\pi)$ the angle around the the constant $z$ sections, which are circles. $h_{ab} = \rho e_{ab}$ is the induced metric on the symmetry orbits in these coordinates, with $\rho = \sqrt{\det h}$ the corresponding area density. The “conformal metric” of the orbits, $e_{ab}$, is thus a unit determinant, $2 \times 2$, symmetric matrix. Cylindrical symmetry requires that neither $\rho$, nor $e$, nor the conformal factor $\Omega$ that appears in (8), depend on the coordinates $\theta^a$. They are all functions only of the reduced spacetime $\mathcal{S}$, the quotient of the full spacetime by the symmetry orbits, which is coordinatized by $\rho$ and $t$.

Note that the time coordinate $t$ is determined up to an additive constant by the requirement that $t \pm \rho$ be null coordinates of the metric. In fact, since the field equations require that $\Box \rho = 0$ on the reduced spacetime [NKS97]

$$\rho = \frac{1}{2} (\rho^+ + \rho^-),$$

with $\rho^\pm$ constant on left/right moving null geodesics, making them null coordinates, and the time coordinate is

$$t = \frac{1}{2} (\rho^+ - \rho^-).$$
The coordinates \((t, \rho)\) are good on all of the reduced spacetime provided \(d\rho\) is spacelike throughout \(S\).

Orthogonal transitivity need not be included in the concept of cylindrical symmetry \cite{Carot99} but traditionally it has been assumed in formulating the cylindrically symmetric reduction of GR, and it is not known whether the model is still integrable if this assumption is dropped. It is not as stringent a condition as it might appear, since it is actually enforced by the vacuum field equations provided only two numbers, the so called twist constants, vanish. See \cite{Wald84} Theorem 7.1.1. and \cite{Chr90}.

The model quantized by Korotkin and Samtleben is subject to two further restrictions: regularity of the spacetime geometry at the symmetry axis, and asymptotic flatness at spatial infinity, in a sense to be described below. Regularity at the axis ensures that the twist constants vanish, and it eliminates \(\Omega\) as an independent field: Absence of a conical singularity at the axis requires that \(\Omega^2 = \lim_{\rho \to 0} e_{\phi\phi}/\rho\), and the field equations then determine \(\Omega\) as a functional of \(e_{ab}\) on the rest of the spacetime \cite{NKS97}. The field \(e_{ab}\) thus contains all the degrees of freedom of the model.

Korotkin and Samtleben do not work with these regular, asymptotically flat spacetimes directly, but rather with their Kramer-Neugebauer duals. The Kramer-Neugebauer transformation \cite{KN68} \cite{BM87} is an invertible map from solutions to solutions of cylindrically symmetric GR which takes 4-metrics that are regular on the axis to geometries for which \(e_{ab}\) is regular there. Moreover it maps flat spacetime to a solution in which \(e_{ab}\) is constant everywhere. This suggests part of the definition of asymptotic flatness we will adopt: As spatial infinity is approached, that is in the limit \(\rho \to \infty\), \(e_{ab}\) of the Kramer-Neugebauer dual is required to tend to a constant matrix \(e_{\infty\infty}\) (in the basis \(\partial_\phi, \partial_z\) of Killing vectors). We shall return to the issue of asymptotic flatness in Subsection 2.3 on the classical Geroch group, where it will be defined in terms of the deformed metric \(M\). However, asymptotic flatness will not play an important role in our results.

Thus reduced, vacuum general relativity becomes a non-linear sigma model coupled to a fixed (background) dilaton \(\rho\). Evaluating the Hilbert action on the class of cylindrically symmetric field configurations under consideration one obtains the action \cite{BM87} \cite{Nic91} \cite{KS98b} (see also \cite{HPS08})

\[
I = -\frac{1}{8} \int_S \rho (\partial_t e^{ab} \partial_t e_{ab} - \partial_\rho e^{ab} \partial_\rho e_{ab}) d\rho dt = \frac{1}{2} \int_S \rho \text{tr} [P_t^2 - P_\rho^2] d\rho dt,
\]

where \(e_{ab}\) is the inverse of \(e_{ab}\).\footnote{The Hilbert action usually includes a normalization factor depending on Newton’s constant. This factor is important for the quantum theory. It scales the action in the Feynmann path integral, and thus the size of quantum effects. In the absence of measurements of quantum gravity effects the factor is determined by the measured strength of the gravitational fields of matter fields coupled to gravity, together with the measured size of quantum effects in the matter fields, such as the energy of a light quantum of given frequency. We have not included such a factor in the action for cylindrically symmetric fields because it is not unambiguously determined by the action of the physical theory, without cylindrical symmetry: Since the symmetry orbits are non-compact the action of a complete cylindrically symmetric spacetime is, in general, infinite. One can obtain a finite action by integrating over only a finite range of values of the coordinates \((\phi, z)\) parametrizing the symmetry orbits, and the result is proportional to the action above, but the factor of proportionality depends on the range of \((\phi, z)\) chosen. Ultimately this ambiguity is not so strange, since quantum gravity probably has no states in which the field fluctuations respect exactly cylindrical symmetry, meaning that the cylindrically symmetric theory is not a sector of the full theory at the quantum level.}

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In the second form the action is a functional, via $P_\mu$, of $\mathcal{V}$, a real, positively oriented, zweibein for $e$:

$$\mathcal{V}^i a b c j = \epsilon_{a b c} e_{a b} \epsilon_{i j} \mathcal{V}^i a b c j = 1.$$ (12)

$P_\mu$ is an $\mathfrak{sl}_2$ valued 1-form on the reduced spacetime $S$, defined as the symmetric component of the flat $\mathfrak{sl}_2$ connection

$$J_\mu i j = V^{-1} a b \partial_\mu V^a b.$$ (13)

That is, $P_\mu = \frac{1}{2} (J_\mu + J_t \mu)\), where the superscript $t$ indicates transposition, or, more explicitly, $P^\mu i j = \frac{1}{2} (J_\mu i j + \delta_{il} J_\mu k J_\mu j k).$

Indices from the beginning of the latin alphabet, $a, b, c, ..., \) correspond to the tangent spaces of the cylindrical symmetry orbits, while letters $i, j, ..., \) from the middle of the alphabet denote internal indices, which label the elements of the zweibein viewed as a basis of the space $F$ of constant 1-forms of density weight $-\frac{1}{2}$ on the symmetry orbits. $\epsilon_{a b}$ and $\epsilon_{i j}$ are antisymmetric symbols, with $\epsilon_{a b} = 1 = \epsilon_{1 2}$; and $\delta_{i j}$ is the Kronecker delta. $\mathcal{V}$ may also be viewed as a linear map from an internal vector space to $F$. Then $\delta$ is a Euclidean metric on the internal space, and the internal indices $i, j, ... \) refer to an orthonormal basis in this space. Finally, Greek indices $\mu, \nu, ... \) label the reduced spacetime coordinates $x^0 \equiv t$ and $x^1 \equiv \rho$.

Notice that the conformal 2-metric $e$ determines $\mathcal{V}$ only up to internal rotations at each point, which form the group $SO(2)$. The action is also invariant under these rotations, so they constitute a gauge invariance of the model.

The reduced spacetime field equation for $e$ is the Ernst equation \[Ern68\], \[KS98b\]:

$$\partial^\mu [\rho e_{a b} \partial_\mu e_{b c}] = 0.$$ (14)

Equivalently

$$\partial_\mu (\rho P^\mu) - [Q_\mu, \rho P^\mu] = 0.$$ (15)

Here spacetime indices are raised and lowered with the flat metric $\eta = -dt \otimes dt + d\rho \otimes d\rho$, and $Q_\mu$ is the antisymmetric component of $J_\mu$. $Q$ transforms as a connection under rotations of the zweibein $\mathcal{V}$ and the presence of the comutator term makes (15) invariant under these gauge transformations.

Note that if a real, unit determinant reference zweibein $Z$ is chosen, then $\mathcal{V}^i a b c j$ may be parametrized by an $SL(2, \mathbb{R})$ group valued scalar field $\mathcal{V}^i a b c j = Z^i a b c j$. The choice of a reference zweibein is not necessary for any of our constructions, but it allows us to describe them in the language of Lie groups. Since the conformal 2-metric $e$ determines $\mathcal{V}$ only up to local internal $SO(2)$ rotations the space of possible $e$s at each point is the coset $SL(2, \mathbb{R})/SO(2)$. The action (11) shows that the model is in fact an $SL(2, \mathbb{R})/SO(2)$ coset sigma model with a fixed, non-dynamical, dilaton $\rho$.

The cylindrically symmetric truncations of several theories of gravity coupled to matter, such as GR with electromagnetism, and supergravity, can be accomodated in the same framework of coset sigma models with dilaton, by replacing $SL(2)$ by another semi-simple Lie group $G$, and $SO(2)$ by the maximal compact subgroup $Q$ of $G$. The quantization scheme of Korotkin and Samtleben extends to these models \[KS98b\] \[Sam98\] \[KNS99\], and we expect that our results on the Geroch group do also.
2.1. Integrability, the auxiliary linear problem, and the deformed metric

The integrability of the model manifests itself through the existence of an auxiliary linear problem: From the symmetric \((P)\) and antisymmetric \((Q)\) components of the flat connection \(J\) a new \(\mathfrak{sl}_2\) connection, the Lax connection \(\hat{J}\), is constructed which is flat if and only if the field equation \((15)\) on \(\mathcal{V}\) holds. The Lax connection will be defined as \([BM87]\ [Nic91] [NKS97]\)

\[
\hat{J}_\mu = Q_\mu + \frac{1 + \gamma^2}{1 - \gamma^2} P_\mu - \frac{2\gamma}{1 - \gamma^2} \varepsilon_{\mu\nu} P^\nu, \tag{16}
\]

with

\[
\gamma = \frac{\sqrt{w + \rho^+} - \sqrt{w - \rho^-}}{\sqrt{w + \rho^+} + \sqrt{w - \rho^-}}, \tag{17}
\]

and \(w \in \mathbb{C}\) a spacetime independent parameter called the constant spectral parameter. \((\gamma\) is called the variable spectral parameter.) \(\varepsilon\) is the antisymmetric symbol of the reduced spacetime, with \(\varepsilon_{t\rho} = 1\).

Note that the square roots in the definition \((17)\) of \(\gamma\) are principal roots, defined by \(\sqrt{re^{i\varphi}} = \sqrt{r}e^{i\varphi/2}\) for all \(r \in [0, \infty), \varphi \in (-\pi, \pi]\). From this, and the fact that \(\Im(w + \rho^+) = \Im(w - \rho^-)\), it follows that \(|\gamma| \leq 1\), and if \(|\gamma| = 1\) then \(\Im(\gamma) \leq 0\).

The Lax connection may also be expressed in terms of the null coordinates \((\rho^+, \rho^-)\):

\[
\hat{J} = Q - \frac{\gamma - 1}{\gamma + 1} P_+ d\rho^+ - \frac{\gamma + 1}{\gamma - 1} P_- d\rho^- = Q + \frac{\sqrt{w - \rho^-}}{\sqrt{w + \rho^+} + \sqrt{w - \rho^-}} P_+ d\rho^+ + \frac{\sqrt{w + \rho^+}}{\sqrt{w + \rho^+} + \sqrt{w - \rho^-}} P_- d\rho^- \tag{18}
\]

A direct calculation yields the curvature of the Lax connection. Taking into account that \(J\) is flat, that is \([\partial_\mu - J_\mu, \partial_\nu - J_\nu] = 0\), one obtains for the curvature of \(\hat{J}\)

\[
[\partial_\mu - \hat{J}_\mu, \partial_\nu - \hat{J}_\nu] = -\frac{1}{\sqrt{w + \rho^+} + \sqrt{w - \rho^-}} \varepsilon_{\mu\nu} (\partial_\sigma (\rho P^\sigma) - [Q_\sigma, \rho P^\sigma]), \tag{19}
\]

which vanishes iff \((15)\) holds.

Since \(\hat{J}\) is flat on solutions there exists on these a zweibein \(\hat{\mathcal{V}}\), the “deformed zweibein”, that is covariantly constant with respect to \(\hat{J}\):

\[
d\hat{\mathcal{V}} = \hat{\mathcal{V}} \hat{J}, \tag{20}
\]

or equivalently \(\hat{J} = \hat{\mathcal{V}}^{-1} d\hat{\mathcal{V}}\). That is, \(\hat{\mathcal{V}}\) bears the same relationship to \(\hat{J}\) as \(\mathcal{V}\) does to \(J\).

Equation \((20)\) is the auxiliary linear problem. A particularly useful solution, \(\hat{\mathcal{V}}_0\), is obtained by setting \(\hat{\mathcal{V}} = \mathcal{V}\) at a reference point \(0\) on the worldline in \(\mathcal{S}\) of the symmetry axis. Then

\[
\hat{\mathcal{V}}_0(x; w) = \mathcal{V}(0) \mathcal{P} e^{\int_0^x \hat{J}(\cdot; w)} \quad \forall x \in \mathcal{S}, \tag{21}
\]

with \(\mathcal{P}\) indicating a path ordered exponential. Notice that \(\hat{J} = J\) when \(\gamma = 0\). This is the case on the axis, where \(\rho^+ = -\rho^-\), so the location of \(0\) within the axis doesn’t matter, since \(\hat{\mathcal{V}}_0 = \mathcal{V}\) on the whole axis. It also occurs on the whole reduced spacetime in the limit that \(w \to \infty\). \(\hat{\mathcal{V}}_0\) therefore equals \(\mathcal{V}\) everywhere in this limit. It follows that if \(\hat{\mathcal{V}}_0\) at a given point \(x \in \mathcal{S}\) is expressed as a function of \(\gamma\), via

\[
w = \frac{\rho}{2}(\gamma + 1/\gamma) - t = \frac{1}{4\gamma}[(1 - \gamma)^2 \rho^+ + (1 + \gamma)^2 \rho^+], \tag{22}
\]

then \(\mathcal{V}(x)\) is just \(\hat{\mathcal{V}}_0(x; \gamma)\) evaluated at \(\gamma = 0\).

A solution \(\mathcal{V}\) is thus easily recovered from the deformed zweibein \(\hat{\mathcal{V}}_0\) it defines. This provides a means to solve the field equation because \(\hat{\mathcal{V}}_0\) can be obtained as the solution of a
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The unit circle in the $\gamma$ plane, shown in the figure, corresponds to the branch cut of $\gamma(w)$ along the real segment $[-\rho^+, \rho^-]$ in the $w$ plane, with $\gamma = 1$ and $\gamma = -1$ at the branch points $w = \rho^-$ and $w = -\rho^+$ respectively. The domain $D$, consisting of the open disk and the lower unit semi-circle (including 1 and $-1$), is the range of the function $\gamma(w)$ defined by (17), and the $\gamma$ domain of $\hat{V}_0(x; \gamma)$.

Figure 1.

"factorization problem". Just as the zweibein $V$ determines the conformal metric $e = V V^d$ the deformed zweibein $\hat{V}$ defines the deformed (conformal) metric mentioned in the Introduction:

$$M(x; \gamma) = \hat{V}(x; \gamma) \hat{V}^d(x; \frac{1}{\gamma}).$$

(23)

As we shall see shortly, $\hat{V}_0$ can be recovered from $M$, and $M$, in turn, can be computed by integration from initial data. In fact, $M$ is altogether a useful parametrization of the solutions of the theory and plays a central role in the present paper.

But before explaining this we must clarify the definition (23) of $M$. For $\gamma$ in the set $D$ consisting of the open unit disk, $|\gamma| < 1$, and the lower half of the unit circle, $|\gamma| = 1$, $\Re \gamma \leq 0$, $\hat{V}(x; \gamma)$ will be set to $\hat{V}_0(x; \gamma)$. See Fig. 1. One could equate $\hat{V}$ with $\hat{V}_0$ also in the complement of $D$, using the fact that $w(\gamma) = w(1/\gamma)$ according to (22), but this would in general lead to a $\hat{V}$ which is discontinuous on the unit circle. It is more useful to define $\hat{V}$ on the unit circle to be the limit of $\hat{V}_0$ inside the unit disk.

Note that $D$ is precisely the range of $\gamma(w)$ as defined by (17), and that $\gamma(w)$ is holomorphic in $w$ on all of the Riemann sphere except at a branch cut on the segment $-\rho^+ \leq w \leq \rho^-$ of the real axis. On the branch cut $\gamma(w)$ is continuous from above, that is $\gamma(w) = \lim_{\epsilon \to 0, \epsilon > 0} \gamma(w + i \epsilon)$.

From the definition (20, 21) it follows that $\hat{V}_0(x; w)$ shares these analyticity and continuity properties in $w$: The left side of (20) is holomorphic in $\hat{V}$ and $w$ away from the lines $\rho^- = w$ and $\rho^+ = -w$ in spacetime where $\gamma = \pm 1$, implying (by [Lef77] proposition 10.3, Chapter II, section 5) that $\hat{V}_0$ is holomorphic in $w$ provided the path of integration in (21) may be chosen

\[ M \] is sometimes called a “scattering matrix” [BMS7], although that word is also used for the quantum $R$ matrix on occasion, or a “monodromy matrix” in [BMS7, NKS97] and [KS98b], but that word is more often used for another object, the holonomy of the Lax connection along the entire length of a one dimensional space [FT87, BR00].
to avoid these lines, which is the case for all $w \in \mathbb{C}$ except the branch cut $-\rho^+ \leq w \leq \rho^-$. Since the singularities of $\hat{J}$ at the singular lines are integrable a dominated convergence argument adapted to path ordered exponentials (see prop. 4 of appendix A of [FR17]) shows that $\hat{V}_0$ is continuous from above in $w$ at the branch cut. At any point $x$ of the reduced spacetime $\gamma(w)$ maps the complement of the branch cut to the interior of the unit disk, so $\mathcal{V}_0(x; \gamma)$ is analytic in $\gamma$ in $|\gamma| < 1$, and approaching the branch cut from above in $w$ corresponds to approaching the lower half of the $\gamma$ unit circle $\{|\gamma| = 1, \Im \gamma \leq 0\}$ from the inside of the unit disk, so $\hat{V}_0(x; \gamma)$ on the lower half of the unit circle is the limit of $\hat{V}_0(x; \gamma)$ inside the unit disk.

It remains to analyze the implications of setting $\hat{V}(x; \gamma)$ on the upper half of the $\gamma$ unit circle equal to the limit of $\hat{V}_0(x; \gamma)$ inside the unit disk. Off the branch cut $\hat{V}_0(x; w) = \hat{V}_0(x; \bar{w})$ and thus $\mathcal{V}_0(x; \gamma) = \hat{V}_0(x; \bar{\gamma})$ for $|\gamma| < 1$, which by continuity holds for $\hat{V}(x; \gamma)$ on the entire closed unit disk, including the circle $|\gamma| = 1$. Notice that $\gamma(w)$ can be extended to a double valued analytic function of $w$, namely the inverse of $w(\gamma)$ defined by (22) which takes values $\gamma(w)$ and $1/\gamma(w)$ at each $w \in \mathbb{C}$. To define $\hat{V}(x; \gamma)$ on the closed $\gamma$ unit disk we have made also $\hat{V}(x; w)$ double valued on the branch cut $-\rho^+ \leq w \leq \rho^-$, taking the values $\mathcal{V}_0(x; w) = \hat{V}_0(x; w)$ and $\mathcal{V}(x; w) = \hat{V}_0(x; \bar{w})$ corresponding to $\gamma(w)$ and $\gamma(w) = 1/\gamma(w)$ respectively. $\hat{V}(x; \gamma)$ satisfies the auxiliary linear problem with connection $\hat{J}(2) = \hat{V}(2) - 1_j \hat{V}(2) = \hat{V}(1) - 1_j \hat{V}(1) = \hat{J}(1)$. This is just $\hat{J}$ evaluated using $\gamma(w) = 1/\gamma(w)$ in place of $\gamma(w)$, which by (16) yields $-\hat{J}(1)^t$. Thus

$$\hat{J}(2) = -\hat{J}(1)^t. \quad (24)$$

Having defined $\hat{V}(x; \gamma)$ on the closed unit disk we have defined $\mathcal{M}(x; \gamma)$ only on the unit circle, since for any $\gamma$ that lies in the interior of the unit disk $1/\gamma$ lies outside the closed unit disk. But this will suffice for our purposes. Equation (23) provides a factorization of $\mathcal{M}(x; \gamma)$ on the $\gamma$ unit circle into a product of an $SL(2, \mathbb{C})$ valued function $\hat{V}(x; \gamma)$ holomorphic inside the unit circle having a continuous limit on the circle itself, and an $SL(2, \mathbb{C})$ valued function $\hat{V}(x; 1/\gamma)$ holomorphic outside the unit circle in the Riemann sphere, having a continuous limit on the circle. This factorization is essentially unique, for suppose $\hat{V}'(\gamma)$ defines another such factorization, then $\hat{V}'(\gamma)\hat{V}(1/\gamma) = \hat{V}(\gamma)\hat{V}'(1/\gamma)$ on $|\gamma| = 1$, which implies that $f_+(\gamma) = \hat{V}^{-1}(\gamma)\hat{V}'(\gamma)$ and $f_-(\gamma) = \hat{V}'(1/\gamma)\hat{V}^{-1}(1/\gamma) = f_+^{-1}(1/\gamma)$ are functions that are holomorphic inside and outside the unit circle respectively and match on the circle. Together they therefore define a continuous function on the Riemann sphere which is holomorphic off the unit circle. By Morera’s theorem this function is actually holomorphic on the whole Riemann sphere and thus constant. Indeed it must be a constant orthogonal matrix since it must equal $f_+(1) = f_+^{-1}(1)$. $\mathcal{M}$ therefore determines $\hat{V}(x; \gamma)$, and thus also $\mathcal{V}(x)$, up to an $SO(2)$ gauge transformation.

Since we will reuse this holomorphy argument several times let us enshrine it in a lemma:

**Lemma 1** Suppose a continuous function $f$ on the Riemann sphere is holomorphic everywhere except possibly on the unit circle $|z| = 1$, then it is a constant.

**Proof:** By Cauchy’s integral theorem the integral of $f$ around a closed, rectifiable, piecewise $C^1$ curve entirely inside or entirely outside the unit circle vanishes. Because $f$ is continuous a uniform convergence argument shows that the integral also vanishes on curves that contain segments of the unit circle but do not cross this circle. This shows that the integral around any closed, rectifiable, piecewise $C^1$ curve at all vanishes. By Morera’s theorem $f$ is therefore holomorphic on the entire Riemann sphere, and thus, by Liouville’s theorem, constant. \[\blacksquare\]
Figure 2. The reduced spacetime $S$ is indicated by the shaded area. It is bounded by the worldline of the cylindrical symmetry axis, shown as a vertical line on the left. For a given value of $w$ the functions $\gamma(x;w)$ and $V_0(x;w)$ are singular (non-holomorphic) on the diagonal lines $\rho^- = w$ and $\rho^+ = -w$. The deformed metric $M(w)$ is equal to the conformal metric $e$ on the symmetry axis at the time $t = -w$ where the two singular lines meet.

A solution may therefore be recovered, up to gauge, from the deformed metric $M$ it defines. We will now show a remarkable property of $M$ expressed as a function of $x$ and $w$ instead of $x$ and $\gamma$. Namely, that $M(x;w)_{ab}$ corresponding to a solution of the field equation depends only on $w$, and in fact is equal to the conformal metric $e_{ab}$ on the axis worldline at time $t = -w$. In the next subsection we will see that any $C^\infty$ function $M(w)$ sharing the algebraic properties of a conformal metric (that it be a real, positive definite, symmetric matrix of unit determinant) determines a solution uniquely up to gauge, so $M(w)$ provides a natural parametrization of the space of solutions.

Let us explore the properties of $M(x;w)$. To each $w$ lying on the branch cut $[-\rho^+; \rho^-]$ there corresponds a point $\gamma(w)$ on the lower half of the $\gamma$ unit circle, and another point $1/\gamma(w) = \overline{\gamma(w)}$ lying on the upper half of the unit circle. A priori these define two values of $M$,

$$M^{(1)}(x;w) \equiv M(x;\gamma(w)) = \hat{V}^{(1)}(x;\gamma(w))\hat{V}^{(2)t}(x;w) = \hat{V}_0(x;w)\overline{V}_0(x;w)$$

and

$$M^{(2)}(x;w) \equiv M(x;\overline{\gamma(w)}) = \hat{V}^{(2)}(x;\overline{\gamma(w)})\hat{V}^{(1)t}(x;w) = \overline{M^{(1)}(x;w)}$$

respectively. Now notice that by (23) the spacetime gradient of $M^{(1)}$ at constant $w$ vanishes!

$$dM^{(1)} = \hat{V}^{(1)}\hat{j}^{(1)}\hat{V}^{(2)t} + \hat{V}^{(1)}\hat{j}^{(2)t}\hat{V}^{(2)t} = 0.$$

If $w$ is real it lies in the branch cut of the function $\gamma(x;w)$ whenever the spacetime point $x$ is spacelike separated from the point $t = -w$ on the symmetry axis worldline. These spacetime points form a wedge bounded to the past by the null line $\rho^+ = -w$ and to the future by the null line $\rho^- = w$, lines at which $w$ lies on the edge of the branch cut and $\gamma = -1$ and $\gamma = 1$ respectively. See Fig. 2. $M^{(1)}(x;w)$ is clearly independent of position on the interior of this wedge. On the boundary of the wedge $\hat{j}^{(1)}(x;w)$ is singular in $x$, but the singularity is
integrable so \( \tilde{V}_0(x; w) \) defined by the path ordered exponential (21) is continuous in \( x \) throughout \( S \). \( M^{(1)} \) thus has the same constant value on the boundaries of the wedge as in its interior. It may therefore be evaluated at the vertex of the wedge, the point \( t = -w \) on the symmetry axis worldline, where \( \tilde{V}_0 = \mathcal{V} \) and hence \( M^{(1)} = \mathcal{V}' \mathcal{V} = e \). Since this is real it also equals \( M^{(2)} \):  

\[
M(x; w) \equiv M^{(1)}(x; w) = M^{(2)}(x; w) = e(\rho = 0, t = -w).
\]

(28)

One may conclude that \( M_{ab}(x; w) \) is real, positive definite, of unit determinant, symmetric in its indices, and independent of \( x \), being a smooth function of \( w \) only. As a consequence of the last property

\[
M(x; \bar{\gamma}) = M(x; \gamma)
\]

(29)

on the \( \gamma \) unit circle. (Note that while the \( x \) independence of \( M(x; w) \) has not been demonstrated for all \( x \) in spacetime it has been demonstrated for all \( x \) such that \( |\gamma(x; w)| = 1 \), which is the subset of spacetime on which \( M(\cdot; w) \) has been defined.)

Equation (28) provides an interpretation of \( M \) as the conformal metric on the axis worldline in the reduced spacetime. Given \( e \) on a segment \( a \leq t \leq b \) of the axis worldline this determines \( M(w) \) for \( a \leq -w \leq b \), which in turn determines \( M(x; \gamma) \) on the whole \( \gamma \) unit circle at all points \( x \) within the “causal diamond”, \( \Delta(a, b) \), of \( a \leq t \leq b \), that is, within the intersection of the past light cone of the point \( t = b \) and the future light cone of the point \( t = a \) on the axis. See Fig. 3. As we have seen this allows the solution \( \mathcal{V} \) to be recovered within the causal diamond up to \( SO(2) \) gauge. \( M \) may also be calculated from \( \mathcal{V} \) and its normal derivative along any Cauchy surface of this diamond by first calculating \( \tilde{V}_0 \) on this surface (really a line in \( S \)). For instance, it can be calculated from \( \mathcal{V} \) on one of the null boundaries of the diamond, as discussed in detail in [FR17].
2.2. The deformed metric and the space of smooth solutions to the field equation

We have seen that the deformed metric $\mathcal{M}$, or equivalently the conformal metric on the axis, determines the solution uniquely up to $SO(2)$ gauge. In this subsection it will be shown that any smooth function $\mathcal{M}$ from a real interval $a \leq -w \leq b$ to real, symmetric, positive definite, unit determinant $2 \times 2$ matrices defines a smooth solution $\mathcal{V}(\rho, t)$ on the causal diamond $\Delta(a, b)$ such that $e_{cd}(t = -w) = \mathcal{M}_{cd}(w)$ on the axis worldline.

The space $\mathcal{D}_n$ of smooth functions from a given interval in $\mathbb{R}$ to the group $GL(n, \mathbb{C})$ of invertible $n \times n$ complex matrices naturally has the structure of a group, under pointwise multiplication, and of an infinite dimensional manifold modeled on the topological vector space $E$ formed by smooth functions taking values in the set of all (not necessarily invertible) $n \times n$ complex matrices with the topology of uniform convergence of the functions and all their partial derivatives on compacta. (See [PrSe86] Chapter 3.) It follows that the space of solutions in the causal diamond of the axis worldline segment $a \leq t \leq b$ is precisely the submanifold of $\mathcal{D}_2$ for the interval $a \leq -w \leq b$ in which the matrices are real, symmetric, positive definite, and of unit determinant.

The demonstration of the existence of solutions corresponding to all $\mathcal{M}$ relies on the existence of factorizations of “loops”, which are smooth functions from the unit circle to $GL(n, \mathbb{C})$. The set $\mathcal{L}_n$ of all such functions admits a natural group structure and manifold structure exactly analogous to that of $\mathcal{D}_n$, and is called the “$GL(n, \mathbb{C})$ loop group” [PrSe86].

The loop in our proof will be the deformed metric $\mathcal{M}$ “expressed as function of $\gamma$ at a given reduced spacetime point $x$”. More precisely, at each point $x$ in the causal diamond $\Delta(a, b)$ the given function $\mathcal{M}(w)$ on $a \leq -w \leq b$ defines the function

$$\mathcal{M}_C(x; \gamma) \equiv \mathcal{M} \circ w(x; \gamma) = \mathcal{M}(\frac{\rho}{2}(\gamma + 1/\gamma) - t)$$

for all $\gamma$ on the unit circle in $\mathbb{C}$. This is “$\mathcal{M}$ as a function of $x$ and $\gamma$”, which we denote $\mathcal{M}_C$ here to avoid confusion because it is, after all, distinct as a function from the function $\mathcal{M}$ of $w$. (The subscript $C$ is meant to suggest “circle” or “$\gamma$”.) $\mathcal{M}_C$, like $\mathcal{M}$, takes values in the real, symmetric, positive definite, unit determinant $2 \times 2$ matrices. The definition (30) implies, furthermore, that it is a smooth function of $\gamma$ (and $\rho$ and $t$), and that

$$\overline{\mathcal{M}_C(\gamma)} = \mathcal{M}_C(\bar{\gamma})$$

(31)

on the $\gamma$ unit circle, since $\bar{\gamma} = 1/\gamma$ there. (The $x$ dependence of the two sides of this equation has been left implicit, as it will be in other equations in which it plays no role.)

We shall see that these properties of $\mathcal{M}_C$, imply that $\mathcal{M}_C$ admits a factorization of the form

$$\mathcal{M}_C(\gamma) = \hat{\nu}(\gamma) \hat{\nu}^t(\gamma^{-1})$$

(32)

which defines a deformed zweibein $\hat{\nu}$ and a zweibein $\nu(x) = \hat{\nu}(x; \gamma = 0)$. This zweibein will be the solution defined by $\mathcal{M}$.

The main theorem underlying this result is ([PrSe86] Theorem 8.1.2):

**Theorem 2 (Birkhoff factorization theorem)** Let $M$ be a smooth function from the unit circle $|z| = 1$ in the complex plane to the group $GL(n, \mathbb{C})$ of invertible $n \times n$ complex matrices, that is $M \in \mathcal{L}_n$, then there exist smooth functions $A$ and $B$ from the unit disk $|z| \leq 1$ to $GL(n, \mathbb{C})$, which are holomorphic on the interior $|z| < 1$ of the disk, such that

$$M(z) = A(z)D(z)B(z^{-1})$$

(33)
on $|z| = 1$, where $D(z) = \text{diag}[z^{k_1}, z^{k_2}, \ldots, z^{k_n}]$ with $k_1 \geq k_2 \geq \ldots \geq k_n$ integers called partial indices.

The partial indices are uniquely determined by $M$. For a dense open subset of $\mathcal{L}_n$ all the partial indices vanish, and then $A$ and $B$ are also uniquely determined by $M$, provided the value of $B(0)$ is fixed. Indeed on this open subset of $\mathcal{L}_n$, once $B(0)$ is fixed, the map $\mathcal{L}_n \to \mathcal{L}_n \times \mathcal{L}_n$ defined by $M \mapsto (A, B)$ is a diffeomorphism.

(In the statement of the theorem in [PrSe86] $A$ and $B$ are smooth functions on the circle $|z| = 1$ which are the boundary values of functions holomorphic in $|z| < 1$. But it is easy to show that this implies that these functions are smooth on $|z| \leq 1$: The smoothness of such a function along the boundary $|z| = 1$ implies that the power series in $z$ for the function and for each of its derivatives converge uniformly on $|z| \leq 1$, ensuring that each derivative is continuous in this domain. Seeley’s theorem [See64] then implies that the function has a $C^\infty$ extension to the whole $z$ plane.)

We will also use Theorem 1.13 of [GKS03]:

**Theorem 3** If $M \in \mathcal{L}_n$ is Hermitian and positive definite, then all partial indices of $M$ vanish.

Now we are ready to use the properties of $\mathcal{M}_C$ to demonstrate (32):

**Proposition 4** Suppose $\mathcal{M}_C \in \mathcal{L}_2$ is real, symmetric, positive definite and of unit determinant, and that $\mathcal{M}_C(\gamma) = \mathcal{M}_C(\bar{\gamma})$. Then there exists a smooth, $2 \times 2$ matrix valued function $\hat{\nu}$ on $|\gamma| \leq 1$ which is holomorphic on $|\gamma| < 1$ and satisfies

$$\mathcal{M}_C(\gamma) = \hat{\nu}(\gamma)\hat{\nu}(\gamma) = \hat{\nu}(\gamma)\hat{\nu}(\gamma^{-1})$$

for all $|\gamma| = 1$, \det $\hat{\nu} = 1$,

$$\mathcal{M}_C(\gamma) = \hat{\nu}(\gamma)\hat{\nu}(\gamma)$$

and

$$\overline{\hat{\nu}(\gamma)} = \hat{\nu}(\gamma).$$

These conditions determine $\hat{\nu}$ up to multiplication by a constant real $SO(2)$ matrix $K$: $\hat{\nu}' = \hat{\nu}K$ also satisfies the conditions.

**Proof:** By Theorem 2 there exist smooth functions $A_0$ and $B_0$ from the closed $\gamma$ unit disk to $GL(2, \mathbb{C})$, holomorphic on $|\gamma| < 1$ such that

$$\mathcal{M}_C(\gamma) = A_0(\gamma)B_0(\gamma^{-1})$$

with $B_0(0) = 1$. The diagonal factor in the factorization is absent by Theorem 3. Define $A_1(\gamma) = A_0(\gamma)B_0(1)$ and $B_1(\gamma) = B_0^{-1}(1)B(\gamma)$, so that

$$\mathcal{M}_C(\gamma) = A_1(\gamma)B_1(\gamma^{-1})$$

on $|\gamma| = 1$, with $B_1(1) = 1$ and $A_1(1) = \mathcal{M}_C(1)$.

Because $\mathcal{M}_C$ is Hermitian

$$A_1(\gamma)B_1(\gamma) = \mathcal{M}_C(\gamma) = \mathcal{M}_C^\dagger(\gamma) = B_1^\dagger(\gamma)A_1^\dagger(\gamma)$$

so

$$A_1^{-1}(\gamma)B_1^\dagger(\gamma) = B_1(\bar{\gamma})A_1^{-1}(\gamma) = [A_1^{-1}(\gamma)B_1^\dagger(\gamma)]^\dagger$$
on $|\gamma| = 1$. Note that the left side is holomorphic on $|\gamma| < 1$ while the right side is holomorphic on $|\gamma| > 1$, and both sides are continuous at $|\gamma| = 1$. Thus by Lemma both sides are equal to a constant Hermitian matrix $K_1$:

$$A^{-1}_1(\gamma)B^\dagger_1(\gamma) = K_1 = K^\dagger_1.$$

(41)

Evaluating at $\gamma = 1$ shows that $K_1 = \mathcal{M}^{-1}_C(1)$.

Since $\mathcal{M}_C(1)$ is real, symmetric, and positive definite there exists a real zweibein $C$ for $\mathcal{M}_C(1)$ such that $\mathcal{M}_C(1) = C C^\dagger$. This zweibein is made unique by requiring it to be an upper triangular matrix with positive diagonal entries (so that it defines a Cholesky decomposition of $\mathcal{M}_C(1)$). Then $B_1(\gamma) = C^{-1}C^{-1}A^\dagger_1(\gamma)$, so

$$\mathcal{M}_C(\gamma) = \hat{\nu}(\gamma)\hat{\nu}(\gamma)$$

(42)

on $|\gamma| = 1$, with $\hat{\nu}(\gamma) = A_1(\gamma)C^{-1}$. As a consequence $\hat{\nu}(1) = C$.

Equation (42) and the condition $\mathcal{M}_C(\gamma) = \mathcal{M}_C(\gamma)$ implies that $\hat{\nu}(\gamma)\hat{\nu}(\gamma) = \hat{\nu}(\gamma)\hat{\nu}(\gamma)$, and thus

$$\hat{\nu}(\gamma)\hat{\nu}(\gamma) = \hat{\nu}(\gamma)\hat{\nu}(\gamma) = [\hat{\nu}(\gamma)\hat{\nu}(\gamma)]^{-1}$$

(43)

on $|\gamma| = 1$. Once again Lemma shows that the two sides must be equal to a constant matrix. Evaluating at $\gamma = 1$ shows that this constant is $C^t C^{-1} = 1$, since $C$ is real. Thus

$$\hat{\nu}(\gamma) = \hat{\nu}(\gamma).$$

(44)

The fact that $\det \hat{\nu} = 1$ is demonstrated similarly: $\det \mathcal{M}_C = 1$ implies that $\det \hat{\nu}(\gamma) = 1/\det \hat{\nu}(\gamma)$ on $|\gamma| = 1$, so Lemma shows that $\det \hat{\nu}(\gamma)$ is independent of $\gamma$. Thus

$$\det \hat{\nu}(\gamma) = \det \hat{\nu}(1) = \det C = 1.$$  

(45)

Here we have taken into account that $1 = \det \mathcal{M}_C(1) = (\det C)^2$ and $\det C > 0$ because $C$ is upper triangular with positive diagonal elements.

The function $\hat{\nu}$ which has been found satisfies conditions (34), (35), and (36), and $\hat{\nu}(1) = C$ is upper triangular with positive diagonal elements.

It is evident the zweibein $\hat{\nu}^t = \hat{\nu}^t K$ also satisfies (34), (35), and (36) provided $K$ is a constant real $SO(2)$ matrix. This is so only if $K$ is a constant real $SO(2)$ matrix because if (33) holds for both $\hat{\nu}$ and $\hat{\nu}^t$ then $\hat{\nu}^{-1}(\gamma)\hat{\nu}(\gamma) = \hat{\nu}^t(1/\gamma)\hat{\nu}^{-1}(1/\gamma)$ on $|\gamma| = 1$. Lemma then implies that $K = \hat{\nu}^{-1}(\gamma)\hat{\nu}(\gamma)$ is an orthogonal matrix independent of $\gamma$.

An important issue is how $\hat{\nu}(x; \gamma)$ depends on the spacetime position $x$.

**Proposition 5** If the $SO(2)$ freedom in $\hat{\nu}$ is fixed by requiring $\hat{\nu}(1)$ to be upper triangular with positive diagonal elements then $\hat{\nu}$ is a smooth function of $\gamma$ and spacetime position $x$ on $|\gamma| = 1$.

**Proof:** Fix an arbitrary smooth curve, parametrized by $\lambda \in \mathbb{R}$, in the reduced spacetime. It defines a curve $\mathcal{M}_C(x(\lambda), \gamma)$ in $\mathcal{L}_2$. Our first task will be to show that this curve is $C^\infty$ in $\mathcal{L}_2$. Then, adopting the notation of (37) we know that $\mathcal{M}_C(x; \gamma) = A_0(x; \gamma)B_0(x; \gamma^{-1})$ at each spacetime point, with $A_0$ and $B_0$ smooth on $|\gamma| \leq 1$ and holomorphic on $|\gamma| < 1$, and $B_0(x; 0) = 1$. Theorem 2 tells us, in addition, that $A_0(x(\lambda), \gamma)$ and $B_0(x(\lambda), \gamma)$ viewed as curves in $\mathcal{L}_2$ parametrized by $\lambda$ are smooth. The second step of the proof consists in demonstrating that this implies that $A_0(x(\lambda), \gamma)$ and $B_0(x(\lambda), \gamma)$ are $C^\infty$ functions of $\lambda$ and $\gamma$ on $|\gamma| = 1$. From this the claim of the proposition is then easily deduced.
Let us turn to the first task, showing that \( \mathcal{M}_C(x(\lambda), \gamma) \) is a \( C^\infty \) curve in \( \mathcal{L}_2 \). Let \( e^{i\theta} = \gamma \) on \(|\gamma|=1\), then
\[
\mathcal{M}_C(x(\lambda), e^{i\theta}) = \mathcal{M}(\rho(\lambda) \cos(\theta) - t(\lambda)).
\] (46)

We will work with the coordinates of this curve in the atlas that Pressley and Segal [PrSe86] use to define the manifold structure of \( \mathcal{L}_2 \). This atlas is composed of right translates (under \( \theta \)-pointwise multiplication) of a chart \( \Phi_1 \) on a neighborhood of the identity in \( \mathcal{L}_2 \), the \( \Phi_1 \) coordinates of a point \( X \in \mathcal{L}_2 \) in this neighborhood being the matrix elements \([\ln X]_{ab}(\theta)\) of the logarithm (inverse exponential map) of \( X \). Note that since \( GL(2, \mathbb{C}) \) is a finite dimensional Lie group the logarithm is a diffeomorphism in a neighborhood of the identity in this group.

Let \( \lambda_0 \) be the value of \( \lambda \) at some particular point of the curve (46), then \( F(\lambda, \theta) = \mathcal{M}_C(x(\lambda), \gamma) \mathcal{M}_C^{-1}(x(\lambda_0), \gamma) \) is a right translate of this curve that passes through the identity of \( \mathcal{L}_2 \) at \( \lambda = \lambda_0 \), and \( \ln F(\lambda, \theta) \) is the matrix of coordinates of this curve in the chart \( \Phi_1 \). From the definition (46), and the fact that \( \mathcal{M} \) is \( C^\infty \), it follows that \( \ln F \) is \( C^\infty \) in its arguments, and vanishes at \( \lambda = \lambda_0 \). As a consequence \( \ln F \) and all its \( \theta \) derivatives converge to 0 uniformly in \( \theta \) at \( \lambda = \lambda_0 \), that is, they converge uniformly to \( \ln F(\lambda_0, \theta) \) and its \( \theta \) derivatives. \( F \) is thus a continuous curve in \( \mathcal{L}_2 \) at \( \lambda = \lambda_0 \), and so is \( \mathcal{M}_C(x(\lambda), e^{i\theta}) \).

Now let us consider the first derivative in \( \lambda \). The difference
\[
[\ln F(\lambda, \theta) - \ln F(\lambda_0, \theta)]/(\lambda - \lambda_0) - \partial_\lambda \ln F(\lambda_0, \theta)
\] (47)
is \( C^\infty \) in its arguments and vanishes at \( \lambda = \lambda_0 \). (Here \( \partial_\lambda \) is the partial derivative in \( \lambda \) at constant \( \theta \).) Thus the ratio \([\ln F(\lambda, \theta) - \ln F(\lambda_0, \theta)]/(\lambda - \lambda_0)\) converges to \( \partial_\lambda \ln F(\lambda_0, \theta) \) uniformly in \( \theta \), and all \( \theta \) derivatives of the ratio converge uniformly to the corresponding \( \theta \) derivatives of \( \partial_\lambda \ln F(\lambda_0, \theta) \). Thus the first derivative in \( \lambda \) of the curve \( F \) in \( \mathcal{L}_2 \) exists, and its components in the chart \( \Phi_1 \) are simply the matrix elements of \( \partial_\lambda \ln F \). This has been established for \( \lambda = \lambda_0 \) but, because right translations act as diffeomorphisms, it clearly holds for all \( \lambda \) such that \( F(\lambda, \cdot) \) lies in the domain of \( \Phi_1 \). Now the preceding argument can be applied to \( \partial_\lambda \ln F \), instead of \( \ln F \), to establish that the second derivative also exists, and finally, that the curve \( F \), and also \( \mathcal{M}_C(x(\lambda), e^{i\theta}) \), is \( C^\infty \) in \( \mathcal{L}_2 \).

From this result it follows, by Theorem 2 that the factors \( A_0 \) and \( B_0 \) in the factorization \( \mathcal{M}_C(x; \gamma) = A_0(x; \gamma)B_0(x; \gamma^{-1}) \) also define smooth curves, \( A_0(x(\lambda), \cdot) \) and \( B_0(x(\lambda), \cdot) \), in \( \mathcal{L}_2 \). Our aim is now to show that this implies that \( A_0(x(\lambda), e^{i\theta}) \) and \( B_0(x(\lambda), e^{i\theta}) \) are \( C^\infty \) functions of \( \lambda \) and \( \theta \).

It is sufficient to consider \( A_0 \). The coordinates in the chart \( \Phi_1 \) of the right translate of \( A_0(x(\lambda), \cdot) \) that passes through the origin at \( \lambda = \lambda_0 \) are the matrix elements of \( \ln H \) where \( H(\lambda, \theta) = \ln(A_0(x(\lambda), e^{i\theta})A_0^{-1}(x(\lambda_0), e^{i\theta})) \). The definition of \( \mathcal{L}_2 \) implies that \( A_0(x(\lambda), e^{i\theta}) \), and thus also \( \ln H \), is \( C^\infty \) in \( \theta \).

Since \( A_0(x(\lambda), \cdot) \) is a differentiable curve in \( \mathcal{L}_2 \) the ratio \([\ln H(\lambda) - \ln H(\lambda_0)]/(\lambda - \lambda_0)\) and all its \( \theta \) derivatives converge uniformly in \( \theta \) as \( \lambda \to \lambda_0 \). Therefore \( \partial_\lambda \ln H \), as well as \( \partial_\lambda \partial_\theta^n \ln H \) exist at \( \lambda = \lambda_0 \) for all integers \( n > 0 \), and, because the convergence is uniform, the multi derivatives obtained by changing the order of differentiation in any one of the above expressions are defined and equal to the original expression. This argument is equally valid at any other value of \( \lambda \) such that \( \ln H(\lambda) \) lies in the domain of \( \Phi_1 \), so the derivatives are defined at these values of \( \lambda \) as well.

Of course \( A_0(x(\lambda), \cdot) \) is actually a \( C^\infty \) curve, so the argument can be applied to \( \partial_\lambda \ln F \) in place of \( \ln F \) to demonstrate the existence of the second \( \lambda \) derivatives, and, iterating, the
\[ \lambda \text{ derivatives of all orders. In } H \text{ is therefore a } C^\infty \text{ function of } \lambda \text{ and } \theta. \] It follows that 
\[ H = A_0(x(\lambda), e^{i\theta})A_0^{-1}(x(\lambda), e^{i\theta}) \] is also. Thus, since \( A_0(x(\lambda_0), e^{i\theta}) \) is a \( C^\infty \) function of \( \theta \), \( A_0(x(\lambda), e^{i\theta}) \) is \( C^\infty \) in \( \lambda \) and \( \theta \). Because \( x(\lambda) \) is an arbitrary smooth curve in the reduced spacetime, it follows that \( A_0(x, e^{i\theta}) \) is a \( C^\infty \) function of \( x \) and \( \theta \).

The definitions in the proof of Theorem 3 imply that \( \dot{\nu}(x; e^{i\theta}) = A_0(x; e^{i\theta})B_0(1)C(x)^{-1t} = A_0(x; e^{i\theta})A_0^{-1}(x; 1)C(x) \) where \( C(x) \) is the upper triangular matrix with positive definite diagonal elements appearing in the Cholesky factorization \( M_C(x; 1) = C(x)C(x)^t \). It follows that \( \dot{\nu}(x; e^{i\theta}) \) is \( C^\infty \) in its arguments, because \( A_0 \) and \( M_C \) are \( C^\infty \) in their arguments, with \( \det A_0 = 1 \), and \( C(x) \) is holomorphic in the components of \( M_C(x; 1) \).

Now consider the limit of \( \dot{\nu}(x; \gamma) \) as \( x \) approaches the worldline of the symmetry axis. \( \mathcal{M}_C(x; \gamma) = \mathcal{M}(\frac{\xi}{2}((\gamma + 1/\gamma) - t) \) approaches the constant, that is \( \gamma \) independent, value \( \mathcal{M}(w = -t) \) in the topology of the manifold \( L_2 \). \( \dot{\nu} \) must therefore approach a factorization of this constant. Since a factorization by constant matrices exists, and the factorization is unique up to a constant \( SO(2) \) transformation, \( \dot{\nu} \) must be \( \gamma \) independent at the symmetry axis. It follows that \( \dot{\nu}(x; \gamma) = \dot{\nu}(x; 0) = \nu(x) \) there.

So far we have shown that any smooth function \( \mathcal{M}(w) \) from the interval \( a \leq -w \leq b \) to \( 2 \times 2 \) matrices that are real, symmetric, positive definite, and of unit determinant defines a zweibein \( \nu(x) \equiv \dot{\nu}(x; \gamma = 0) \) at each point \( x \) in the causal diamond \( \Delta(a, b) \) of the segment \( a \leq t \leq b \) of the axis worldline in the reduced spacetime. But, is this zweibein a solution to the field equations? Does its deformed metric reproduce the function \( \mathcal{M} \) we started with? We will now show that the answer to both questions is “yes”.

The key point is that \( \mathcal{M} \) depends only on \( w \). Thus the spacetime differential of \( \mathcal{M}_C(x; \gamma) = \mathcal{M}(w(x; \gamma)) \) at constant \( w \) vanishes:

\[ 0 = [d\mathcal{M}_C]_w. \tag{48} \]

This forces \( \dot{j} \equiv \dot{\nu}^{-1}[d\nu]_w \) to be exactly the Lax connection (116) constructed from the zweibein \( \nu \). As a consequence, since \( \dot{\nu} = \nu \) on the axis, \( \dot{\nu} \) is the deformed zweibein \( \dot{\nu}_0 \) constructed from \( \nu \) via the integral (21), and the given function \( \mathcal{M} \) is the corresponding deformed metric. On the other hand the equation \( \dot{j} = \dot{\nu}^{-1}[d\dot{\nu}]_w \) also implies that \( \dot{j} \) is flat. Since the Lax connection is flat iff the zweibein satisfies the field equation \( \nu \) is a solution, which moreover satisfies the boundary condition \( e(t) = \mathcal{M}(w = -t) \) on the symmetry axis worldline.

Let us carry out this argument in detail. Specifically, let us demonstrate the key claim that \( \dot{j} \) equals the Lax connection defined by \( \nu \): \( [d\mathcal{M}_C]_w(\gamma) = [d\dot{\nu}(\gamma)]_w \dot{\nu}^t(\gamma^{-1}) + (\dot{\nu}(\gamma))[d\dot{\nu}^t(\gamma^{-1})]_w \), so

\[ 0 = \dot{\nu}^{-1}(\gamma)[d\mathcal{M}_C]_w(\gamma)\dot{\nu}^{-1}(\gamma^{-1}) = \dot{\nu}^{-1}(\gamma)[d\dot{\nu}(\gamma)]_w + [\dot{\nu}^{-1}(\gamma^{-1})[d(\dot{\nu}(\gamma^{-1})]_w]^t. \tag{49} \]

The differential of \( \dot{\nu} \) at constant \( w \) is

\[ [d\dot{\nu}]_w = [d\dot{\nu}]_\gamma + \frac{\partial \dot{\nu}}{\partial \gamma}[d\gamma]_w, \tag{50} \]

where

\[ [d\gamma]_w = -\frac{\gamma}{2\rho} \left[ \frac{\gamma - 1}{\gamma + 1} d\rho^+ + \frac{\gamma + 1}{\gamma - 1} d\rho^- \right], \tag{51} \]

as follows from the differential of the expression (22) for \( w \) as a function of \( \gamma, \rho^+ \), and \( \rho^- \),

\[ dw = \frac{1}{4\gamma} [(1 - \gamma)^2 d\rho^+ + (1 + \gamma)^2 d\rho^-] + \frac{\rho}{2}(1 - \frac{1}{\gamma^2}) d\gamma. \tag{52} \]
On the quantum Geroch group

The variable spectral parameter \( \gamma \) is of course double valued as a function of \( w \) (at a given spacetime position). This does not introduce any ambiguity in \([d\gamma]_w\) expressed as a function \( \gamma \) since the value of \( \gamma \) identifies the branch. The fact that \( \gamma \) and \( \gamma^{-1} \) correspond to the same \( w \) does lead to the identity

\[
[d\gamma^{-1}]_w(g) = -\gamma^{-2}[d\gamma]_w(g) = [d\gamma]_w(g^{-1}),
\]

which may easily be verified directly from (51). Here the argument of each expression is the value of \( \gamma \) at which it is evaluated. It follows that

\[
[d\hat{\nu}(\gamma^{-1})]_w(g) = [d\hat{\nu}](g^{-1}) + \frac{\partial \hat{\nu}(\gamma^{-1})}{\partial \gamma^{-1}}(g)[d\gamma^{-1}]_w(g)
\]

\[
= [d\hat{\nu}](g^{-1}) + \frac{\partial \hat{\nu}(\gamma)}{\partial \gamma}(g^{-1})[d\gamma]_w(g^{-1}) = [d\hat{\nu}]_w(g^{-1})
\]

Defining

\[
\hat{j} \equiv \hat{\nu}^{-1}[d\hat{\nu}]_w = \hat{\nu}^{-1}[d\hat{\nu}] - \hat{\nu}^{-1}\frac{\partial \hat{\nu}}{\partial \gamma} \frac{\gamma - 1}{\gamma + 1} \frac{dp^+}{2\rho} + \frac{\gamma + 1}{\gamma - 1} \frac{dp^-}{2\rho}
\]

the condition (59) may be expressed as

\[
\hat{j}(\gamma) = -\hat{j}^{-1}(\gamma^{-1}) \quad \text{on } |\gamma| = 1.
\]

The connection \( \hat{j} \) is holomorphic inside the \( \gamma \) unit disk. This follows from (56), the definition of \( \hat{\nu} \) and the fact that \([d\hat{\nu}]_\gamma \) is holomorphic in this domain. The latter follows from Theorem 5. The holomorphicity of \( \hat{\nu} \) inside the unit \( \gamma \) disk implies that it can expressed in terms of its boundary values on the unit circle via Cauchy’s integral formula. The spacetime partial derivative at constant \( \gamma \), \( \partial_\mu \), may be taken inside this integral because by Theorem 5 \([d\hat{\nu}]_\gamma \) is jointly continuous in spacetime position and \( \gamma \) on \( |\gamma| = 1 \). Therefore

\[
\partial_\mu \hat{\nu}(\gamma) = \frac{1}{2\pi i} \int_{|\gamma| = 1} \frac{\partial \hat{\nu}(\gamma')}{\gamma' - \gamma} \frac{d\gamma'}{\gamma'}
\]

at all \( \gamma \) inside the unit disk, which implies that \([d\hat{\nu}]_\gamma \) is indeed holomorphic there.

Equation (57) is thus yet another condition that requires a function holomorphic inside the unit circle and a function that is holomorphic outside this circle to match on the circle itself. But this time the limiting functions on the circle are not bounded. In fact

\[
\hat{j} = q - \frac{\gamma - 1}{\gamma + 1} p_+ dp^+ - \frac{\gamma + 1}{\gamma - 1} p_- dp^-,
\]

where \( q \) is holomorphic inside the unit disk and continuous on its boundary, and \( p_\pm = \mp \left[ \frac{\hat{\nu}^{-1} \frac{\partial \hat{\nu}}{\partial \gamma}}{\gamma''} \right]_{\gamma = \pm 1} \frac{1}{dp} \). This should match

\[
-q'(\gamma^{-1}) = -q'(\gamma^{-1}) - \frac{\gamma - 1}{\gamma + 1} p_{\pm} dp^+ - \frac{\gamma + 1}{\gamma - 1} p_{\pm} dp^-.
\]

(Note that \( \gamma^{-1} = \frac{\gamma}{\gamma + 1} \)). Equality requires firstly that the coefficients of the singular terms match:

\[
p_\pm = p_{\pm}^l,
\]

and, secondly, that the regular remainder also matches, which requires that

\[
q(x; \gamma) = -q'(x; \gamma^{-1})
\]
on $|\gamma| = 1$. Because $q$ is continuous on $|\gamma| = 1$ Lemma 11 applies, so $q$ is in fact independent of $\gamma$, though it can still depend on $x$, and it is antisymmetric: $q(x) = -q^t(x)$.

Now evaluate $\hat{j}(x; \gamma)$ at $\gamma = 0$. From (59) it follows that

$$\hat{j}(x; 0) = q + p_+ d\rho^+ + p_- d\rho^-,$$

while from (56) it follows that $\hat{j}(x) = \nu^{-1} d\nu(x)$, because the second term in this expression, proportional to $\gamma$, vanishes at $\gamma = 0$. $p = p_+ d\rho^+ + p_- d\rho^-$ and $q$ are thus the symmetric and antisymmetric components, respectively, of $j$, and $\hat{j}$ is the Lax connection calculated from the zweibein field $\nu$.

2.3. The classical Geroch group for cylindrically symmetric gravity

The Geroch group is a group of mappings of the solution space to itself. We saw in the last subsection that the space of solutions on the causal diamond $\Delta(a, b) \subset S$ of a segment $a \leq t \leq b$ of the axis worldline can be identified with the set of smooth conformal metrics $e_{cd}(t)$ on this segment. Equivalently, it consists of the smooth deformed metrics $\nu$ calculated from the zweibein field $e$ or $M$. The classical Geroch group for cylindrically symmetric gravity consists of the smooth functions $s : [-b, -a] \to SL(2, \mathbb{R})$ of $w$, and acts via

$$\mathcal{M}(w) \mapsto s(w)\mathcal{M}(w)s^t(w).$$

The action is a symmetry in the sense that it maps solutions to solutions: The only conditions that $\mathcal{M}$ must satisfy is that it be smooth in $w$ and that $\mathcal{M}(w)$ be real, symmetric positive definite, and of unit determinant. (64) clearly preserves all these conditions.

The action is also transitive. The properties of $\mathcal{M}$ guarantee that there exists a Cholesky decomposition $\mathcal{M}(w) = s(w)s^t(w)$ with $s$ smooth in $w$ and $s(w) \in SL(2, \mathbb{R})$ (and upper triangular with positive diagonal elements). Any solution can therefore be reached via a Geroch group action from the solution corresponding to $\mathcal{M} = \mathbf{1}$ (the Kramer-Neugebauer dual of flat spacetime), and of course by applying the inverse element $s^{-1}$ of the Geroch group one returns from the given solution to $\mathcal{M} = \mathbf{1}$. It follows that any solution $\mathcal{M}'$ can be reached via a Geroch group action from any other $\mathcal{M}$ by mapping $\mathcal{M} \mapsto \mathbf{1} \mapsto \mathcal{M}'$, so the Geroch group acts transitively on the space of solutions.

Now let us consider the (Kramer-Neugebauer duals of) infinite asymptotically flat solutions treated in Kortkin and Samtleben’s model of cylindrically symmetric gravitational waves. In this case the space of solutions corresponds to smooth conformal metrics $e_{cd}$ on the whole real $t$ axis, or equivalently smooth $\mathcal{M}$ on the whole real $w$ axis. Asymptotic flatness of the Kramer-Neugebauer dual requires that $e_{cd}(x)$ tends to an asymptotic value $e_{\infty cd}$ at spacelike infinity, that is, as $\rho$ tends to $\infty$ while $t$ is held constant. (Of course $e_{\infty cd} = \delta_{cd}$ in a suitable basis.) Thus at spacelike infinity the solution is the Kramer-Neugebauer dual of Minkowski space.

The following theorem shows that this is ensured by requiring that $e$ tends to $e_\infty$ as $t \to \pm \infty$ along the symmetry axis worldline. Equivalently, one requires that $\mathcal{M}(w)$ tends to $e_\infty$ as $w \to \mp \infty$.

**Proposition 6** If $\mathcal{M}(w) \to e_\infty$ as $w \to \pm \infty$ then $e(x) \to e_\infty$ as $\rho \to \infty$ with $t$ held constant.
Proof: Let us consider the limit \( \rho \to \infty \) with \( t \) held constant. In this limit \( \mathcal{M}_C(x; e^{i\theta}) = \mathcal{M}(\rho \cos \theta - t) \) tends to \( e^{-i\theta} \) for all \( \theta \) except \( \pm \pi/2 \), while it also remains bounded since \( \mathcal{M} \) is bounded: \( ||\mathcal{M}_C(x, e^{i\theta})|| \equiv \max_{ab}(\mathcal{M}_{C, ab}(x, e^{i\theta})) \leq \sup_w ||\mathcal{M}(w)|| < \infty \). It follows that
\[
\langle \mathcal{M}_C \rangle \to e_{\infty}
\] (65)
where
\[
\langle \mathcal{M}_C \rangle(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{M}_C(x; e^{i\theta}) d\theta.
\] (66)
Now recall that \( \mathcal{M}_C(e^{i\theta}) = \hat{\nu}(e^{i\theta}) \hat{\nu}^\dagger(e^{-i\theta}) = \hat{\nu}(e^{i\theta}) \hat{\nu}^\dagger(e^{i\theta}) \) with
\[
\hat{\nu}(e^{i\theta}) = \nu + \sum_{n=1}^{\infty} \alpha_n e^{i n \theta},
\] (67)
since \( \hat{\nu}(\gamma) \) is holomorphic in \( |\gamma| < 1 \) and \( \hat{\nu}(0) = \nu \). Thus
\[
\langle \mathcal{M}_C \rangle = e + \sum_{n=1}^{\infty} \alpha_n \alpha_n^\dagger.
\] (68)
It turns out that this and (65) implies that \( e \to e_{\infty} \) as \( \rho \to \infty \). Each of the matrices \( e \) and \( \alpha_n \alpha_n^\dagger \) are Hermitian and positive semi-definite. (\( e \) of course also has unit determinant and is real by Theorem 4.) Hermitian \( 2 \times 2 \) matrices are the real span of the unit matrix \( 1 \) and the Pauli matrices. A Hermitian matrix \( A \) may therefore be represented by a real 4-vector \( a \) defined by
\[
A = \begin{bmatrix}
a^0 + a^3 & a^1 - ia^2 \\
a^1 + ia^2 & a^0 - a^3
\end{bmatrix} = a^0 \mathbf{1} + a^1 \sigma_1.
\] (69)
The determinant of \( A \) is the Lorentzian norm squared \( [a^0]^2 - [a^1]^2 - [a^2]^2 - [a^3]^2 \). The unit determinant Hermitian matrices correspond to the unit norm shell, and the positive semi-definite ones to the closed future light cone. It follows that \( e(x) \) lies on the future unit norm shell, and, by (68), in the closed past lightcone of \( \langle \mathcal{M}_C \rangle(x) \). But \( e_{\infty} \), being the limit of \( \mathcal{M} \), must be real, symmetric, positive definite and have unit determinant, just like \( e(x) \), so it too lies on the future unit norm shell. It follows that if \( \langle \mathcal{M}_C \rangle \to e_{\infty} \) then \( e \to e_{\infty} \) as well.

How fast should \( \mathcal{M} \) approach \( e_{\infty} \)? This is really a question of convenience. Asymptotic flatness is a mathematical fiction constructed to treat isolated systems without having to deal with the rest of the Universe, and not a model of the real world, so we may choose the falloff rates as we wish. Note that we need not worry that overly strong falloff conditions limit the local dynamics, because the asymptotic behaviour of \( \mathcal{M} \) has no effect at all on the solutions within a given finite causal diamond \( \Delta(a, b) \), which depends only on \( \mathcal{M}(w) \) in the interval \( a \leq -w \leq b \).

A reasonable prescription might be to require \( \mathcal{M} \) to be a Schwartz function, a smooth function which falls off more rapidly than any inverse power of \( w \). Then the Geroch group would also consist of Schwarz functions \( s : \mathbb{R} \to SL(2, \mathbb{R}) \) and would act transitively. In the present work it will not be necessary to commit ourselves to a particular choice of falloff condition.

The Geroch group as defined here is not a loop group because it maps the real line, instead of a circle, into the group \( SL(2, \mathbb{R}) \). It is tempting to turn the real line into a unit circle via the Moebius transformation \( w \mapsto z = \frac{1+iw}{1-w} \). Smooth functions on the unit circle \( |z| = 1 \) then correspond to smooth functions of \( w \) which are also smooth functions of \( 1/w \) on \( w \neq 0 \). These necessarily fall off to a constant limiting value as \( w \to \pm \infty \), but this falloff can be as slow as \( 1/w \). The Geroch group for asymptotically flat spacetimes with a suitable falloff condition on
\[ \mathcal{M} \] is therefore equivalent to a loop group. However, for our purposes there is a serious problem with this way of presenting the group: The Poisson bracket \( \{ q_1, q_2 \} \) depends only on the difference \( w_1 - w_2 \) but is not similarly translation invariant when expressed in terms of the Moebius transformed spectral parameter \( z \). The lack of translation invariance complicates the quantization because the close connection to the \( \mathfrak{sl}_2 \) Yangian is lost. We therefore choose to continue with the Geroch group in its “almost loop group form” in terms of functions on the real \( w \) line. It might aptly be called a “line group”.

3. Poisson brackets on the space of solutions and on the Geroch group

The deformed metric \( \mathcal{M} \) given as a function of \( w \) characterizes solutions completely, and any smooth, real, symmetric, unit determinant function \( \mathcal{M} \) defines a solution. The function \( \mathcal{M} \) can thus be taken as a coordinate system on the space of solutions, that is, the phase space \( \Gamma \) of asymptotically flat cylindrically symmetric gravitational waves. Any observable is therefore a functional of \( \mathcal{M} \). To specify the Poisson bracket on \( \Gamma \) space it is thus sufficient to give the Poisson brackets between the matrix elements \( \mathcal{M}_{ab}(v) \) and \( \mathcal{M}_{cd}(w) \).

These Poisson brackets have been evaluated from the action (11) by Korotkin and Samtleben [KS95], and in a different way in [FR17]. The result is

\[
\{ \mathcal{M}(v), \mathcal{M}(w) \} = \text{p.v.} \left( \frac{1}{v-w} \right) \left[ \mathcal{M}(v) \mathcal{M}(w) + \mathcal{M}(v) \Omega \mathcal{M}(w) + \mathcal{M}(v) \Omega^t \mathcal{M}(w) + \mathcal{M}(v) \Omega \mathcal{M}(w) + \mathcal{M}(v) \Omega^t \mathcal{M}(w) \right].
\]

(70)

Here “tensor notation” has been used, in which a tensor product \( A \otimes B \) of a tensor \( A \) acting on space 1 and a tensor \( B \) acting on space 2 is denoted \( \overline{AB} \). \( \Omega \) denotes the Casimir element of \( \mathfrak{sl}_2 \): In terms of Pauli’s sigma matrices \( \Omega = \frac{1}{2} \left( \sigma_x \sigma_x + \sigma_y \sigma_y + \sigma_z \sigma_z \right) \). In terms of Kronecker deltas

\[
\Omega_{a \ c}^{\ b \ d} = \delta_{a \ d} \delta_{c \ b} - \frac{1}{2} \delta_{a \ b} \delta_{c \ d},
\]

(71)

with indices \( a, b \) corresponding to space 1 and indices \( c, d \) corresponding to space 2. \( \Omega^t \) denotes \( \Omega \) transposed in the indices corresponding to space 2, that is, \( \Omega_{a \ c}^{\ b \ d} = \Omega_{a \ b}^{\ c \ d} \). \( \Omega^t \Omega \) is \( \Omega \) similarly transposed in the indices corresponding to space 1, and \( \Omega \Omega^t \) is obtained by transposing in both spaces. Finally, \( \text{p.v.} \left( \frac{1}{u} \right) \) denotes the Cauchy principal value of \( \frac{1}{u} \), a distribution defined by

\[
\text{p.v.} \left( \frac{1}{u} \right) = \lim_{\epsilon \to 0^+} \frac{1}{2} \left( \frac{1}{u + i\epsilon} + \frac{1}{u - i\epsilon} \right),
\]

(72)

where the limit is taken after integration against the test function.

The Poisson bracket (70) is really just a symmetrization, in both the indices associated with space 1 and in those associated with space 2, of a simpler expression:

\[
\{ \mathcal{M}(v), \mathcal{M}(w) \} = \text{p.v.} \left( \frac{1}{v-w} \right) \text{Sym}_1 \text{Sym}_2 \left[ \mathcal{M}(v) \mathcal{M}(w) \right],
\]

(73)

where \( \text{Sym}X = X + X^t \) for any matrix \( X \).

Recall that the action of a Geroch group element \( g \) on the deformed metric \( \mathcal{M} \) maps \( \mathcal{M} \) to

\[
\mathcal{M}^G(w)(g, \xi) \equiv \mathcal{M}(w)(g \triangleright \xi) = s(w)\mathcal{M}(w)s^t(w)
\]

(74)
at all points \( \xi \) in the phase space \( \Gamma \), which defines the action on any observables since these are functionals of \( \mathcal{M} \).

We will now show that there exists a unique Poisson bracket on the Geroch group such that this action is a Poisson map \( G \times \Gamma \rightarrow \Gamma \). That is, that it preserves the Poisson algebra of observables: For any pair, \( A \) and \( B \) of observables \( \{ A, B \} \equiv \{ A, B \} \Gamma(g \triangleright \xi) \) equals \( \{ A^G, B^G \} \Gamma(g, \xi) = \{ A^G(g, \cdot), B^G(g, \cdot) \Gamma(\xi) + \{ A^G(\cdot, \xi), B^G(\cdot, \xi) \} G(g) \).

**Theorem 7** The action of the Geroch group \( G \) on the space of solutions \( \Gamma \) is a Poisson map \( G \times \Gamma \rightarrow \Gamma \) if and only if the Poisson bracket on \( G \) is

\[
\{ \hat{s}(v), \frac{2}{3} \hat{s}(w) \} = p.v. \left( \frac{1}{v-w} \right) \left[ \Omega, \frac{1}{3} s(v) \hat{s}(w) \right] \tag{75}
\]

*Proof:* Since all observables are functionals of \( \mathcal{M} \) it is necessary and sufficient to verify that the action \((74)\) preserves equation \((70)\), that is, that

\[
\{ \frac{1}{3} \hat{s} M^2, \frac{2}{3} \hat{s} M^2 \} \Gamma(g, \xi) = \{ \hat{s}, \hat{M} \} \Gamma(g \triangleright \xi). \tag{76}
\]

When the argument is not given, tensors in space 1 are evaluated at spectral parameter \( v \), and those in space 2 at \( w \).

The right side of equation \((76)\) is equal to

\[
\{ \frac{1}{3} \hat{s} M^2, \frac{2}{3} \hat{s} M^2 \} G \times \Gamma = \frac{1}{12} \left[ \Omega, \frac{1}{2} \hat{\mathcal{M}} M + \frac{1}{2} \hat{\mathcal{M}} M + \frac{1}{2} \hat{\mathcal{M}} M + \frac{1}{2} \hat{\mathcal{M}} M \right] \frac{1}{l^2} s \tag{77}
\]

Substituting the Poisson bracket \((70)\) between deformed metrics into this equation, and setting \( \mathcal{Y} \equiv \{ s, s \} G s^{-1} s^{-1} \), one obtains

\[
\{ \frac{1}{3} \hat{s} M^2, \frac{2}{3} \hat{s} M^2 \} G \times \Gamma = \frac{1}{12} s \ p.v. \left( \frac{1}{v-w} \right) \left[ \Omega, \frac{1}{2} \hat{\mathcal{M}} M + \frac{1}{2} \hat{\mathcal{M}} M + \frac{1}{2} \hat{\mathcal{M}} M + \frac{1}{2} \hat{\mathcal{M}} M \right] \frac{1}{l^2} s \tag{78}
\]

The left side of \((76)\) is just the Poisson bracket \((70)\) evaluated with \( \mathcal{M} G \) in place of \( \mathcal{M} \):

\[
\text{Sym}_1 \text{Sym}_2 \left[ p.v. \left( \frac{1}{v-w} \right) \Omega \frac{1}{2} \hat{\mathcal{M}} M \right]. \tag{80}
\]

The action of the Geroch group maps the solution space to itself, so in order to satisfy \((70)\) on all of \( G \times \Gamma \) it is necessary and sufficient that

\[
\text{Sym}_1 \text{Sym}_2 \left[ p.v. \left( \frac{1}{v-w} \right) s \Omega \frac{1}{2} \hat{\mathcal{M}} M + \Omega \frac{1}{2} \hat{\mathcal{M}} M \right] = \text{Sym}_1 \text{Sym}_2 \left[ p.v. \left( \frac{1}{v-w} \right) \Omega \frac{1}{2} \hat{\mathcal{M}} M \right] \tag{81}
\]

for all \( \mathcal{M} \) corresponding to solutions. This equation may be written as

\[
\text{Sym}_1 \text{Sym}_2 \frac{1}{2} \left[ Z \mathcal{M} \mathcal{M} \right] = 0 \tag{82}
\]

with

\[
\frac{1}{2} Z = p.v. \left( \frac{1}{v-w} \right) \left[ \frac{1}{2} s \Omega \frac{1}{2} \hat{\mathcal{M}} M + \Omega \frac{1}{2} \hat{\mathcal{M}} M \right]. \tag{83}
\]
Clearly, setting $Z = 0$ is sufficient to ensure that (82) is satisfied for all $\mathcal{M}$, and thus that the action (74) preserves the Poisson bracket (70) on the solution space. $Z = 0$ implies that

$$\{s, s\}_G = \mathbb{Y}^{12}_{s s} = p.v. \left( \frac{1}{v - w} \right) \left[ \Omega^{12}_{s s} - \frac{1}{s s \Omega} \right], \tag{84}$$

which is precisely the claimed result (75).

In fact $Z = 0$ is also necessary. That is, the validity of (82) for all $\mathcal{M}$ corresponding to solutions implies $Z = 0$ (and thus of course (75)). To show this we must simplify (82). The space of solutions corresponds to smooth functions $\mathcal{M}(w)$ that are real, symmetric, positive definite, and of unit determinant. But if (82) hold for $\mathcal{M}$ it also holds for a multiple of $\mathcal{M}$ by a scalar function of $w$, so if (82) holds for all $\mathcal{M}$ corresponding to solutions it must still hold if the unit determinant requirement is dropped. In other words, it must hold for all smooth matrix valued functions which are real, symmetric, and positive definite. In particular, if $A$ and $B$ are two such functions, it must hold for $A + B$ (because the sum of positive definite matrices is positive definite). This implies that

$$0 = \text{Sym}_1 \text{Sym}_2 [Z(1 + B)(1 + B)] = \text{Sym}_1 \text{Sym}_2 [Z(\hat{A} \hat{B} + B \hat{A})] \tag{85}$$

Now note that as a distribution $\hat{Z}(v, w)$ is symmetric under the simultaneous interchange of the spaces 1 and 2, and the arguments $v$ and $w$. That is, $Z_{abc}(v, w) = Z_{abcd}(w, v)$ where $a, b$ are the indices corresponding to space 1, and $c, d$ are those corresponding to space 2. It therefore follows from (85) that

$$0 = \text{Sym}_1 \text{Sym}_2 [Z AB]. \tag{86}$$

Finally, note that real symmetric positive definite matrices span all real symmetric matrices, so (86) must hold for all smooth, real symmetric matrix functions $A$ and $B$.

The following Lemma is useful for extracting the consequences of this condition

**Lemma 8** If $X$ is a matrix such that $0 = \text{Sym}[XM]_{ab} = 2X_{(a} v_{b)}c$ for all real symmetric matrices $M$, then $X = 0$.

*Proof:* Let $M_{ab} = v_a v_b$ for an arbitrary vector $v$, then $X_{(a} v_{b)}v_{c} = 0$. If $w_a = X_{a} v_{c}$ this last condition becomes $w_{(a}v_{b)} = 0$. Contracting this relation with any dual vector $\psi^a$ one obtains $w_a v_b \psi^b + v_a w_b \psi^b = 0$, so $w$ and $v$ are linearly dependent, implying that $w_{[a} v_{b]} = 0$, and thus that $0 = w_a = X_a v_{c}$. Since this relation holds for all vectors $v$ it follows that $X = 0$. \hfill \Box

Applying the Lemma to space 1 one finds that if (86) holds for all $A$ then requires that $0 = \text{Sym}_2 [Z AB]$. Applying the Lemma again shows that if (86) holds for all $B$ as well then $Z = 0$. \hfill \Box

The bracket (75) is the well known Sklyanin bracket [Skl79] [BR00] for the $SL(2, \mathbb{R})$ line group - the Geroch group.

Note that (75) may also be written as

$$\{s(v), s(w)\}_G = \lim_{\epsilon \to 0^+} \frac{1}{v - w + i\epsilon} \left[ \Omega, \frac{1}{s} \left( s(v) \right) \right], \tag{87}$$

where the limit is taken after integration against a test function. This expression, with either sign before the $\epsilon$ term, reproduces (75), as does any linear combination of these with coefficients adding up to 1. The reason is that the difference

$$\lim_{\epsilon \to 0^+} \left( \frac{1}{u + i\epsilon} - \frac{1}{u - i\epsilon} \right) = -2i\pi \delta(u), \tag{88}$$
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and

$$\delta(v - w)[\Omega, \hat{s}(v)\hat{s}(w)] = \delta(v - w)[\Omega, \hat{s}(v)\hat{s}(v)] = 0,$$

(89)

since $\Omega$ is invariant under simultaneous equal $SL(2, \mathbb{C})$ transformations in both space 1 and space 2.

It is straightforward to verify the classical Yang-Baxter equation

$$[\hat{r}_{12}^{13}, \hat{r}_{13}^{23}] + [\hat{r}_{12}^{13}, \hat{r}_{13}^{23}] + [\hat{r}_{12}^{13}, \hat{r}_{13}^{23}] = 0$$

(90)

for either of the two alternative classical $r$ matrices $\hat{r}_{13}^{ij} \equiv \lim_{\epsilon \to 0^+} \left( \frac{1}{\epsilon w_1 - w_2 + i\epsilon v} \right)_{ij} \Omega$ appearing in the form \[87\] of the bracket. This guarantees the Jacobi relation for the bracket, so it really is a Poisson bracket.

It is also straightforward to check that the action of the Geroch group on itself via the multiplication map, $(g_1, g_2) \mapsto g_1g_2$, that takes two group elements to their product is a Poisson map:

$$\{ \hat{s}(g_1) \hat{s}(g_2) \} = \{ \hat{s}(g_1) \hat{\nu}(\cdot) \hat{s}(g_2) \} = \{ \hat{s}(\cdot) \hat{\nu}(\cdot) \} = \{ \hat{s}(\cdot) \hat{s}(\cdot) \}$$

(91)

The Geroch group equipped with the bracket \[75\] is thus a Poisson-Lie group, and its action \[74\] on the phase space $\Gamma$ is a Lie-Poisson action in the terminology of [BR00].

4. Quantization of the Geroch group

The Geroch group with the Sklyanin bracket \[75\] admits a natural quantization, defined by the exchange relation

$$R(v - w)\hat{s}(v)\hat{s}(w) = \hat{s}(w)\hat{s}(v)R(v - w),$$

(92)

with $R(u) = (u - i\hbar/2)I - i\hbar\Omega$, or more explicitly

$$R(u)^{bc}_{\phantom{bc}de} = wu^b s^d_c - i\hbar\hat{s}^d_s s^b_c.$$

(93)

($I = 1_{12}$ is the product of the identity matrices in spaces 1 and 2.)

The connection between the exchange relation \[92\] and the Sklyanin bracket \[75\] can be made manifest by writing \[92\] as

$$[\hat{s}(v), \hat{s}(w)] = \frac{i\hbar}{v - w - i\hbar/2} \left( \Omega \hat{s}(v)\hat{s}(w) - \hat{s}(w)\hat{s}(v) \Omega \right),$$

(94)

which is clearly a quantization of \[87\] with $\hbar/2$ seemingly in the role of $\epsilon$, and a minus before the $\epsilon$ term. In fact the quantization does not prefer one of the forms of \[87\] over the other. By multiplying \[92\] by $R(v - w)$ from both the left and the right one obtains the equivalent exchange relation

$$R(v - w)\hat{s}(w)\hat{s}(v) = \hat{s}(v)\hat{s}(w)R(v - w),$$

(95)

which may be written as

$$[\hat{s}(v), \hat{s}(w)] = \frac{i\hbar}{v - w + i\hbar/2} \left( \Omega \hat{s}(w)\hat{s}(v) - \hat{s}(v)\hat{s}(w) \Omega \right).$$

(96)

The quantum operators $s_{ab}^c(w)$ also satisfy the quantization of the unit determinant condition:

$$q\det s(w) \equiv s_1^1(w + i\hbar/2)s_2^2(w - i\hbar/2) - s_2^1(w + i\hbar/2)s_1^2(w - i\hbar/2) = 1,$$

(97)
and the reality condition

\[ [s^a{}_b(w)]^* = s^b{}_a(\bar{w}). \] (98)

Classically the functions \( s \) are defined only on real values of the spectral parameters \( w \), and are real matrix valued by the reality conditions. But to formulate the quantum theory they need to be defined also for \( w \) with non-zero imaginary components, at least for imaginary components \( \pm i\hbar/2 \). \( s(w) \) at such complex values of \( w \) will be taken to be the analytic continuation of \( s \) on the real axis. This analytic continuation exists for each individual Fourier mode \( e^{-i kw}\tilde{s}(k)^a{}_b \), and \( s(w) \) will be defined as the sum of these modes, the latter satisfying the reality condition

\[ \tilde{s}(-k)^a{}_b = [\tilde{s}(k)^a{}_b]^*. \] (99)

This is of course not a detailed definition of \( s(w) \), and such a definition will not be given here. Rather, conclusions will be drawn from the defining relations (92), (97), and (98) or (99) assuming an object satisfying these conditions exists. Nevertheless, let us outline one way \( s(w) \) might be defined:

In this approach, \( \tilde{s}(k)^a{}_b \) is a tempered distribution on \( \mathbb{R} \), valued in a unital, associative \(*\)-algebra and \( s(w)^a{}_b \) its Fourier transform. Contracted with a test function and a state on the Geroch group it yields a \( \mathbb{C} \) number. Note that the integral of \( s(w) \) against a test function along a contour from \(-\infty \) to \( \infty \) in the complex plane is defined for test functions which are entire and restrict to Schwarz functions on the real line. A state is a linear function \( \varphi \) on the \(*\)-algebra which returns the expectation value of the \(*\)-algebra element it acts on. States are required to be positive and normalized - \( \varphi(AA^*) \geq 0 \) for all elements \( A \) of the \(*\)-algebra and \( \varphi(1) = 1 \). States correspond essentially to density matrices in the standard Hilbert space formulation of quantum mechanics: Via the Gelfand-Naimark-Segal construction (extended to \(*\)-algebras) a Hilbert space can be found for any given state such that that state, and a subset of the other states on the \(*\)-algebra, are represented by density matrices on the Hilbert space. See [KM15] for a recent review.

The tempered distributions valued in a \(*\)-algebra form a vector space and may be multiplied, provided the arguments \( w \) of the factors are all independent. Together with the \(*\)-algebra unit, they form a unital associative algebra. The quotient of this algebra by the exchange relation is also a unital, associative algebra, which satisfies the exchange relations as distributional equations. (The reality condition (99) is of course built in from the start in the tempered distributions admitted.) This is not quite enough. To justify some important applications of the exchange relations it is necessary to limit the singularity of some products, by restricting the states the Geroch group is allowed to have. For instance products of the form \( \frac{1}{i}\tilde{s}(w + i\hbar/2)\tilde{s}(w - i\hbar/2) \) with \( w \in \mathbb{R} \) must be defined. The quantum determinant is a sum of terms of this form, and the action of the quantum Geroch group on the gravitational field involves this product.

Let us assume from here on that these issues, essentially issues of the convergence of Fourier transforms, can be adequately resolved. There remains the question as to whether the conditions (92), (97), and (98) are algebraically compatible, or whether, on the contrary, they make the algebra generated by \( s(w) \) trivial, or restrict it overly, so that the classical limit does not recover the full Geroch group. In fact, the conditions are compatible: Taking the \(*\) conjugate of the exchange relation (92) and applying (98) one obtains the exchange relation (95), which is equivalent to (92).
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where to merit further exploration. Not to have been studied in detail before, although it was used implicitly in [KS98b], and seems

\[ s_{2}^{2}(v + i\hbar/2)s_{1}^{1}(v - i\hbar/2) - s_{1}^{2}(v + i\hbar/2)s_{2}^{1}(v - i\hbar/2) = 1, \quad (100) \]

for all \( v \) since (97) holds for all \( w \). This turns out to be equivalent to (97) because of the exchange relation. The \( R \) matrix (93) evaluated at spectral parameter difference \( u = i\hbar \) is \( R(i\hbar)_{a}^{b} = i\hbar\varepsilon_{ac}\varepsilon^{bd} \), so (92) implies that

\[ \varepsilon_{ac}\varepsilon^{bd}s(v + i\hbar/2)_{b}^{e}s(v - i\hbar/2)_{d}^{f} = s(v - i\hbar/2)_{c}^{d}s(v + i\hbar/2)_{a}^{b}\varepsilon_{bd}\varepsilon^{ef}. \quad (101) \]

Therefore \( \varepsilon_{bd}s(v + i\hbar/2)_{e}s(v - i\hbar/2)_{d}^{f} \) is antisymmetric in the indices \( e, f \), and thus equal to \( q\text{det}(s(v))_{e}^{f} \). The left side of (100) is therefore \( -q\text{det}sc_{21}^{2} = q\text{det}s \).

The quantum determinant condition (97) is also compatible with the exchange relation (92) because this exchange relation implies that the quantum determinant \( q\text{det}(s(w)) \) commutes with all matrix elements \( s(v)_{a}^{b} \). This is demonstrated in [Sam98] section 5.1. See also [Mol03] sections 2.6 and 2.7. It can therefore be set to 1 without implying additional relations.

Finally, the exchange relation is compatible with itself, that is consistent, because the matrix \( R \) satisfies the quantum Yang-Baxter equation

\[ R(w_{1} - w_{2})R(w_{1} - w_{3})R(w_{2} - w_{3}) = \frac{1}{2}R(w_{2} - w_{3})R(w_{1} - w_{3})R(w_{1} - w_{2}). \quad (102) \]

(The overset numbers indicate the pair of spaces on which \( R \) acts in each case.) This implies that the result of reordering the factors in a product of three or more \( s \) matrix elements by any sequence of exchanges depends only on the final order of the factors. Were it not so, and different sequences of exchanges lead to different expressions in the same final order, then the equality of these these expressions would place constraints on the products of the \( s \) matrix elements. That is, linear combinations of components of a product \( \frac{1}{2}(w_{1})\frac{1}{2}(w_{2})\frac{1}{2}(w_{3}) \) would have to vanish.

The quantum Geroch group generated by \( s(w) \) is very similar to the \( \mathfrak{sl}(2, \mathbb{R}) \) Yangian. It is defined by the same algebraic conditions (92), (97), and (98), and therefore forms a Hopf algebra with co-product \( \Delta(s(w)_{a}^{b}) = s(w)_{a}^{c}s(w)_{c}^{b} \), antipode \( S(s(w)_{a}^{b}) = s^{-1}(w)_{a}^{b} = \varepsilon_{ac}\varepsilon^{bd}s(w + i\hbar)_{d}^{c} \) and co-unit \( \epsilon(s(w)_{a}^{b}) = \delta_{a}^{b} \). (The formula for the inverse of \( s \) follows from (101).) However, unlike the generators of the Yangian, the generators \( s(w) \) of the quantized Geroch group are not formal series of negative powers of the spectral parameter \( w \), but rather a sum of Fourier modes \( e^{-ikw} \) with \( k \) taking all real values. It can be thought of as an analog of the Yangian which quantizes an \( SL(2, \mathbb{R}) \) line group instead of an \( SL(2, \mathbb{R}) \) loop group. This structure seems not to have been studied in detail before, although it was used implicitly in [KS98], and seems to merit further exploration.

We turn now to the question of whether and how this quantum Geroch group is a symmetry of quantum cylindrically symmetric cylindrically symmetric vacuum general relativity. In [KS98] Korotkin and Samtleben proposed a quantization of the phase space \( \Gamma \) of this system. In their quantization the exchange relation for the basic data \( \mathcal{M}(w) \) is

\[ R(v - w)\mathcal{M}(v)R'(v - w + 2i\hbar)\mathcal{M}(w) = \mathcal{M}(w)R'(v - w + 2i\hbar)\mathcal{M}(v)R(v - w) \quad (103) \]

where \( R \) is defined as in (92) and \( R' \) is the “twisted R matrix”, defined by

\[ R'(u) = (u - i\hbar/2)I + i\hbar\Omega, \quad (104) \]

or equivalently

\[ R'(u)_{a}^{b} = (u - i\hbar)\delta_{a}^{b}\delta_{c}^{d} + i\hbar\delta_{a}^{c}\delta_{b}^{d}. \quad (105) \]
It is easy to check that this exchange relation quantizes the Poisson bracket \([70]\).

In addition to the exchange relation \(M\) also satisfies quantum versions of all the conditions that define the set of classical deformed metrics corresponding to classical solutions, namely reality, symmetry, positive definiteness and unit determinant. The reality and symmetry conditions are straightforward: \([M(w)_{ab}]^* = M(w)_{ab}\) and \(M_{ab} = M_{ba}\), just as for the classical \(M\). Note that \(M\) is defined only on real \(w\), even in the quantum theory. The remaining conditions are expressed in terms of a factorization of \(M\):

\[
M(w) = T_+(w)T_+^\dagger(w)
\]

with \(T_-(w) = [T_+(w)]^*\) and \(T_+\) holomorphic in the upper half \(w\) plane (with at most power a power law divergence at infinity) or, equivalently, a sum of purely negative frequency modes, \(e^{ikw} k \geq 0\), in \(w\). The positivity condition, which seems to be just positive semi-definiteness in the quantum case, is a consequence of the existence of this factorization:

\[
M_{ab}v^av^b = \sum_c v^aT(w)_+ + \varepsilon^av^bT(w)_+ + \varepsilon^bT(w)_+ \]

for any real 2-vector \(v\), which, because of the positivity of states, has expectation value \(\varphi(M_{ab}v^av^b) \geq 0\). The unit determinant condition is quantized by requiring that \(q\det T_\pm = 1\).

The question arises as to whether the action \(M \mapsto M^G = sMS^d\) of the classical Geroch group can be quantized in such a way that it is an automorphism of the quantum theory of cylindrically symmetric gravitational waves of Korotkin and Samtleben. Here we will show that the classical action with a small “quantum” modification \([111]\) does indeed preserve the exchange relation \([103]\) and the symmetry, reality, and positive semi-definiteness of \(M\). We will not be able to check whether the unit determinant conditions is preserved, because we have not found an expression for an action on \(T_\pm\) corresponding to the action \([111]\) on \(M\).

For this reason, and also because \(M\) is quite naturally the central object of the classical theory, it would be interesting to define the quantum theory of cylindrically symmetric gravitational waves entirely of this family of operators. But this project will be left to future investigations.

We will make the ansatz that the action of the quantum Geroch group on the quantum deformed metric \(M\) takes the form

\[
M(w)_{ab} \mapsto M^G(w)_{ab} = s(w + c_1)_a^cM(w)_{cd}s(w + c_2)_b^d,
\]

where \(c_1\) and \(c_2\) are complex constants to be determined. Since the matrix elements of \(s\) Poisson commute classically with those of \(M\) it will be assumed that they commute in the quantum theory.

We will find below, in Proposition \([10]\) that this action preserves the exchange relation \([103]\) if \(c_1 - c_2 = i\hbar\). The condition \(c_1 - c_2 = i\hbar\) also implies that the action \([107]\) preserves the symmetry of \(M\). Recall that the exchange relations \([92]\) reduces to \([101]\) when the two spectral parameters differ by \(i\hbar\), and that this implies that \(\varepsilon_c^eM(w)_{ab}s(v+i\hbar/2)_a^eM(w)_{cd}s(v-i\hbar/2)_d^f\) is antisymmetric in the indices \(e, f\), for any \(v\). This is true in particular if \(v = w + c\) with \(c = (c_1 + c_2)/2\), so if \(M(w)_{ab}\) is symmetric in \(a, b\) then \(M^G(w)_{ab}\) is also, because \(s(v+i\hbar/2)_a^cM(w)_{cd}s(v-i\hbar/2)_b^d\) vanishes.

The action of the Geroch group should also preserve the reality of \(M(w)\) on real \(w\). The reality of \(M^G(w)_{ab}\) requires that

\[
s(v+i\hbar/2)_a^cM(w)_{cd}s(v-i\hbar/2)_b^d = [s(v-i\hbar/2)_b^d]^*[M(w)_{cd}]^*[s(v+i\hbar/2)_a^c]^*.
\]

By the reality conditions on \(M\) and \(s\), and the symmetry of both \(M\) and \(M^G\) this reduces to

\[
s(v+i\hbar/2)_a^cM(w)_{cd}s(v-i\hbar/2)_b^d = s(\bar{v} + i\hbar/2)_a^cM(w)_{cd}s(\bar{v} - i\hbar/2)_b^d.
\]
which holds identically if \( c \) is real, since then \( v = w + c \) is real. If, on the contrary, \( c \) is not real \((109)\) is a highly non-trivial condition on \( s \) which would have to be implemented by restrictions on the states of the Geroch group. This would seem to prevent many classical Geroch group elements from being realized as classical limit states of the quantum Geroch group, but this last claim has not been proved.

Finally, \( c \in \mathbb{R} \) also ensures that the positive semidefiniteness of \( M \) is preserved, because then

\[
M^G(w)_{ab} v^a v^b = \sum_d v^a s(w + c + i\hbar/2)_a c^d [v^b s(w + c + i\hbar/2)_b c^d]^*, \tag{110}
\]

for any real 2-vector \( v \), which has positive semi-definite expectation value on any state. Perhaps this ensures that \( M^G \) also has a factorization of the form \( M^G(w) = T^G_+ (w) T^G_- (w) \). The conditions \( \text{qdet} T_+ = 1 \) would then be preserved by the quantum Geroch group action provided the factorization can be chosen such that \( \text{qdet} T^G_+ = 1 \), which does not seem implausible since \( \text{qdet} s = 1 \). This question will be left to future investigations. (Note, that \( T^G_+ (w) \) is not \( s(w + c + i\hbar/2) T(w) \), because this latter expression is not purely negative frequency in \( w \). That is, it is not holomorphic in the upper half complex plane with at most a power law divergence at infinity.)

When \( c \) is real it can actually be set to zero, because one may replace \( s(w) \) by \( s(w + c) \) altering only the action of the Geroch group on \( M \) and not any of the conditions that define the quantum Geroch group (that is, the exchange relations, reality conditions and unit determinant condition). This freedom exists already at the classical level: One may represent a given group element \( g \) by the shifted matrix function \( s_\Delta (w) \equiv s(w - c) \), in terms of which the action of the Geroch group becomes \((g \triangleright M)(w) = s_\Delta (w + c) M(w) s^\dagger_\Delta (w + c)\).

If \( c_1 - c_2 = i\hbar \) and \( c = (c_1 + c_2)/2 = 0 \) then \( c_1 = i\hbar/2 \) and \( c_2 = -i\hbar \), so the action of the Geroch group becomes

\[
M(w) \mapsto M^G(w) = s(w + i\hbar/2) M(w) s^\dagger (w - i\hbar/2). \tag{111}
\]

Clearly this reduces to the classical Geroch group action \((74)\) in the classical limit.

Our result is that this action preserves the exchange relations of \( M \) as well as the symmetry and reality of \( M \). That it preserves symmetry and reality has already been proved. We turn now to the proof of the claimed invariance of the exchange relation \((103)\) under the action \((111)\), indeed under the action \((107)\) provided only that \( c_1 - c_2 = i\hbar \). It will not be proved that this last condition is necessary, only that it is sufficient. But to give an idea of why \( c_1 - c_2 = i\hbar \) is special \( c_1 \) and \( c_2 \) will be kept arbitrary until the end of the proof.

The following lemma, which restates the two equivalent forms \((92)\) and \((93)\) of the exchange relations for the \( s \) in terms of the twisted \( R \) matrix \(R'\), will be useful. Note that in the following a presuperscript \( t \), that is \( t^t X \), will once more denote transposition in space 1, while a postsuperscript \( t \) will denote transposition in space 2.

**Lemma 9** For any \( u, v, w \in \mathbb{C} \)

\[
\frac{1}{i} s(v)^t R'(u) \frac{2}{i} s(w) = \frac{2}{i} s(w) R'(u)^t s(v) + (u + v - w - i\hbar)[\frac{1}{i} s(v)^t, \frac{2}{i} s(w)] \tag{112}
\]

and

\[
\frac{1}{i} s(v)^t R''(u) \frac{2}{i} s(w) = \frac{2}{i} s(w) R''(u)^t s(v) + (u + v - w - i\hbar)[\frac{1}{i} s(v), \frac{2}{i} s(w)]. \tag{113}
\]
Proposition 10

Proof: By (92)

\[ R(v-w)\frac{1}{2}s(v)s(w) - \frac{2}{2}\frac{1}{2}s(w)s(v)R(v-w) = 0, \]  

with \( R(v-w) = (v-w - i\hbar/2)I - i\hbar\Omega. \) Notice that \( R(z) = R(v-w) + (z-v+w)I, \) so

\[ R(z)\frac{1}{2}s(v)s(w) - \frac{2}{2}\frac{1}{2}s(w)s(v)R(z) = (z-v+w)[\frac{1}{2}s(v), \frac{2}{2}s(w)]. \]  

(115)

But, by (104) \( R'(u) = -iR(-u + i\hbar). \) Thus, setting \( z = -u + i\hbar \) and taking the transpose of (115) in space 1 one obtains equation (112).

To demonstrate (113) recall that the exchange relation (92) is equivalent to equation (95) obtained from (92) by exchanging spaces 1 and 2, and the spectral parameters \( v \) and \( w. \) Thus

\[ \frac{1}{2}s(v)s(w)R(z) - R(z)\frac{1}{2}s(w)s(v) = (z-v+w)[\frac{1}{2}s(v), \frac{2}{2}s(w)]. \]  

(116)

Equation (113) is obtained from this equation by setting \( z = -u + i\hbar \) and transposing in space 2. Note that \(-R'(-u + i\hbar) = R'(u). \)

We are now ready to prove the invariance of (103) under the action of the quantum Geroch group.

Proposition 10 If \( c_1 - c_2 = i\hbar, \) then \( \mathcal{M}^G \) as defined by (107) satisfies the same exchange relation (103) as \( \mathcal{M} \) does.

Proof: We wish to evaluate the effect of substituting \( \mathcal{M}^G \) for \( \mathcal{M} \) in the exchange relation (103). To this end the left side, \( L, \) of (103) with \( \mathcal{M}^G \) in place of \( \mathcal{M} \) will be rearranged until it takes the form of the right side of (103) with \( \mathcal{M}^G \) again in place of \( \mathcal{M}, \) plus remainder terms, which will be seen to vanish when \( c_1 - c_2 = i\hbar. \)

According to (107)

\[ L = R(v-w)\frac{1}{2}\mathcal{M}^G(v)R'(w-v + 2i\hbar)\frac{2}{2}\mathcal{M}^G(w) \]  

(117)

\[ = R(v-w)\frac{1}{2}s(v + c_1)\frac{1}{2}\mathcal{M}(v)\frac{1}{2}s(v + c_2)R'(w-v + 2i\hbar)\frac{2}{2}s(w + c_1)\frac{2}{2}\mathcal{M}(w)\frac{2}{2}s(w + c_2). \]  

(118)

As a first step the order of the factors in the middle of the expression is reversed using Lemma 9 putting \( v + c_2, \) \( w + c_1, \) and \( w - v + 2i\hbar \) in place of \( v, w, \) and \( u \) respectively in equation (112):

\[ L = R(v-w)\frac{1}{2}s(v + c_1)\frac{1}{2}\mathcal{M}(v)\frac{2}{2}s(w + c_2)R'(w-v + 2i\hbar)\frac{2}{2}s(v + c_2)\frac{1}{2}\mathcal{M}(w)\frac{2}{2}s(w + c_2) \]

+ \( (c_2 - c_1 + i\hbar)R(v-w)\frac{1}{2}s(v + c_1)\frac{1}{2}\mathcal{M}(v)\frac{1}{2}s(v + c_2)\frac{1}{2}\mathcal{M}(w)\frac{2}{2}s(w + c_2). \)  

(119)

Recall that the matrix elements of \( s \) commute with those of \( \mathcal{M}. \) By exchanging such matrix elements the first line of the preceding expression can be written as

\[ R(v-w)\frac{1}{2}s(v + c_1)\frac{2}{2}s(w + c_1)\frac{1}{2}\mathcal{M}(v)\frac{1}{2}s(v + c_2)\frac{2}{2}s(w + c_2) \]  

(120)

Now the exchange relation (92) may be used to reorder the first three factors, and then (103) to reorder the four middle factors, yielding

\[ \frac{2}{2}s(w + c_1)\frac{1}{2}\mathcal{M}(w)\frac{1}{2}s(v + c_1)\frac{1}{2}\mathcal{M}(v)\frac{2}{2}s(v + c_2)\frac{2}{2}s(w + c_2)R'(v-w + 2i\hbar)\frac{2}{2}s(v + c_2)\frac{2}{2}s(w + c_2). \]  

(121)

Next, the order of the last three factors is reversed using equation (95) transposed in both space 1 and space 2, and factors of \( s \) and \( \mathcal{M} \) are interchanged, with the result

\[ \frac{2}{2}s(w + c_1)\frac{2}{2}\mathcal{M}(w)\frac{1}{2}s(v + c_1)\frac{1}{2}\mathcal{M}(v)\frac{2}{2}s(v + c_2)\frac{2}{2}s(w + c_2)R'(v-w)\frac{2}{2}s(v + c_2)\frac{2}{2}\mathcal{M}(w)\frac{2}{2}s(w + c_2). \]  

(122)
Finally, $L$ is used to reverse the order of the middle factors again, producing also another commutator term:

$$L = \frac{2}{s(w + c_1)} \mathcal{M}(w) \frac{2}{s(w + c_2)} R^i(v - w + 2i\hbar) \frac{1}{s(v + c_1)} \mathcal{M}(v) \frac{1}{s(v + c_2)} R^i(v - w - 2i\hbar) \frac{v - w - 2i\hbar}{v - w + 2i\hbar} + (c_2 - c_1 + i\hbar) \left\{ R(v - w) \frac{1}{s(v + c_1)} \mathcal{M}(v) [\frac{1}{s(v + c_2)} \frac{2}{s(w + c_1)} \mathcal{M}(w) \frac{2}{s(v + c_2)} \mathcal{M}(v) \frac{1}{s(v + c_2)} R^i(v - w) \right\}. \ (123)$$

If $c_2 - c_1 + i\hbar = 0$, then only the first line remains, so

$$L = \frac{2}{s(w + c_1)} \mathcal{M}(w) \frac{2}{s(w + c_2)} R^i(v - w + 2i\hbar) \frac{1}{s(v + c_1)} \mathcal{M}(v) \frac{1}{s(v + c_2)} R^i(v - w - 2i\hbar) \frac{v - w - 2i\hbar}{v - w + 2i\hbar}. \ (124)$$

which is precisely the right side of $L$ with $\mathcal{M}^G$ in place of $\mathcal{M}$, proving the claim of the proposition.

5. Conclusions

In the present work the Geroch group of symmetries of the space of solutions classical cylindrically symmetric vacuum GR was obtained in a rigorous and concise manner using the inverse scattering method (following ideas of [BZ78, HE01, BM87, Nie91, KS98b, Fuchs] and others); The Poisson bracket on the Geroch group was obtained directly from the requirement that its action be Lie-Poisson; The Geroch was quantized using this bracket; And an action of the quantum Geroch group was found which seems to be a symmetry of quantum cylindrically symmetric vacuum GR as formulated by Korotkin and Samtleben. It was possible to verify the invariance of all but one of the conditions that define the algebra of observables of this theory under this action.

One direction for future work is to extend our results to cylindrically symmetry reductions of theories of gravity coupled to various forms of matter (or other 2 commuting Killing field reductions of these models). As already mentioned Korotkin and Samtleben’s quantization of gravity extends to many such models, in particular to maximal supergravity [KNS99]. We expect that our results could also be extended to this context.

Another direction is to define the quantum Geroch group rigorously. Here we have defined it only formally, as a Hopf algebra that is very similar to a Yangian, but not quite the same. Recall that the generators $s(w)$ of the quantum Geroch group are sums of Fourier modes $e^{ikw}$ for all real $k$, while the analogous generators of the $sl_2$ Yangian are formal power series in $1/w$. This difference seems to be deeply rooted. It arises from the fact that the Geroch group is a line group, consisting of $SL(2)$ valued functions on the real axis in the complex spectral parameter plane, while the Yangian quantizes a loop group, consisting of $SL(2)$ valued functions on the unit circle in this plane, and the Poisson bracket, which is of the same form in terms of the spectral parameter for both groups, is not invariant under the Moebius transformation that takes the real axis to the unit circle. The present work, and also the quantization of cylindrically symmetric vacuum gravity by Korotkin and Samtleben, is based on ignoring this difference as much as possible. To go further it would probably be useful to face the issue directly and build a theory of “line Yangians” that quantize line groups. Exploring this new structure could also be fruitful mathematically.

Finally, one might try to extend the model to which the Geroch group, and it’s quantization, applies, by relaxing the conditions imposed at the symmetry axis that eliminated
the conformal factor $\Omega$ in the reduced spacetime metric as a degree of freedom. This would be especially interesting for the program of [FR17], in which the quantization of cylindrically symmetric gravity is used to guide the quantization of null initial data without cylindrical symmetry, because a degree of freedom analogous to $\Omega$ at the axis is present in general null initial data.

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