Linear Sigma Model Linkage with Nonperturbative QCD

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Abstract

Linkage is demonstrated between the quark mass, quark condensate and coupling strength in the infrared limit of QCD and in the quark-level linear sigma model.

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1 Introduction

Although QCD appears to be the best candidate among existing effective-field-theoretical models for a complete picture of the strong interactions, its non-perturbative, low-energy behaviour is less-well-understood than its high energy behaviour. Consequently, it remains necessary to consider models for the strong interactions that, on the one hand, exhibit linkage to low-energy QCD physics, and, on the other hand, facilitate computations of key phenomenological quantities.

Guidance in this search naturally relies upon chiral symmetry, which plays a dominant role in governing low-energy phenomena in the strong interactions. Consequently, it is important to look for the simplest chiral realization of QCD, which we believe is either the Nambu-Jona-Lasinio (NJL) scheme or the Linear Sigma Model (LσM).

In this context, we consider a quark-level LσM where elementary fermions are treated as quarks and are combined with elementary $\pi$ and $\sigma$ mesons in a chiral-invariant way. Recent evidence for the occurrence of a $\sigma$-meson in $\pi\pi$ scattering [1,2] provides additional motivation for examining the linkage between the linear sigma model and QCD. Moreover, some analyses suggest that this scalar meson has a mass at or near twice the dynamical mass anticipated for light quarks [2], consistent with LσM and NJL expectations [3,4].

In what follows, we examine the correspondence between nonperturbative QCD at freeze-out energy scales and the LσM, based upon the equivalence (or near equiv-
ence) of quark masses, quark-antiquark condensates, and coupling strengths.

In Section 2 we work out in detail the relationship between the dynamically generated quark mass and the nonperturbative quark condensate in QCD. Then in Section 3 we compute these nonperturbative quantities in the quark-level linear sigma model (L$\sigma$M). Next in Section 4 we show how the coupling constant strength associated with chiral symmetry breakdown in QCD corresponds to the anticipated meson-quark coupling strength in the L$\sigma$M. Finally, in Section 5 we discuss how a reappraisal of the $\pi^0 \rightarrow \gamma\gamma$ decay process provides insight into effective theories of low-energy QCD, such as the L$\sigma$M, that contain constituent-mass quarks.

2 Dynamical Quark Mass and Condensate in QCD

Within QCD, the relationship of quark-antiquark condensates to observable physics is even more tenuous (or, more indirect) than that of quark masses. Consequently, there is some value in reviewing the relationship between these two order-parameters of chiral noninvariance. The existence of a nonperturbative quark-antiquark vacuum condensate is fundamental to the dynamical generation of a nonperturbative quark mass. To see this, consider the limit in which no Lagrangian quark mass appears. In the Wick-Dyson expansion of the time-ordered product of fermion-antifermion fields,

$$T\psi_i(x)\psi_j(y) = < O_p | T\psi_i(x)\bar{\psi}_j(y) | O_p > + :\psi_i(x)\bar{\psi}_j(y):,$$

(1)
the expectation value of the time-ordered product between purely-perturbative vacuum states \( | O_p \rangle \) is the massless free-fermion propagator

\[
< O_p | T \psi^\alpha_i (x) \bar{\psi}^\beta_j (y) | O_p > = \int \frac{d^4k}{(2\pi)^4} i\delta^{\alpha\beta} \frac{\gamma^\mu_k \gamma^\nu_k}{k^2} e^{-ik \cdot (x-y)},
\]  

(2)

where \( \{ \alpha, \beta \} \) are internal symmetry indices (e.g. colour) and \( \{ i, j \} \) are Dirac spinorial indices. Let us assume, however, that we are calculating in the presence of a vacuum \( | \Omega \rangle \) whose nonperturbative content admits the existence of a fermion-antifermion condensate:

\[
< \bar{\psi} \psi > = - \lim_{x \to y} < \Omega | : \psi^\alpha_i (x) \bar{\psi}^\beta_i (y) : | \Omega > .
\]  

(3)

The vacuum \( | \Omega \rangle \) is clearly nonperturbative in character, since normal-ordered fields necessarily annihilate the purely-perturbative vacuum \( | O_p \rangle \). The full fermion propagator is obtained by taking the expectation value of (1) between nonperturbative vacuum states \( | \Omega \rangle \),

\[
< \Omega | T \psi^\alpha_i (x) \bar{\psi}^\beta_j (y) | \Omega > = < O_p | T \psi^\alpha_i (x) \bar{\psi}^\beta_j (y) | O_p > + < \Omega | : \psi^\alpha_i (x) \bar{\psi}^\beta_j (y) : | \Omega > .
\]  

(4)

It is evident from (4) that the existence of a quark-antiquark condensate, as defined by (3), implies a difference between the full fermion propagator

\[
< \Omega | T \psi^\alpha_i (x) \bar{\psi}^\beta_j (y) | \Omega >
\]

and the massless free fermion propagator (2). In Landau gauge, in which leading-order quark-antiquark-condensate contributions to the spinorial component of the fermion self-energy are seen to vanish [5], this difference can be
ascribed to a dynamical mass function in the full fermion Landau-gauge propagator,

\[
\langle \Omega | T \psi_i^\alpha(x) \bar{\psi}_j^\beta(y) | \Omega \rangle = \int \frac{d^4k}{(2\pi)^4} i\delta^{\alpha\beta} \frac{[\gamma^\mu k_\mu + M(k^2)\delta_{ij}]}{[k^2 - M^2(k^2)]} e^{-ik \cdot (x-y)}
\] (5)

One can rearrange (4) to relate this dynamical mass function directly to the quark-antiquark vacuum condensate [5,6]

\[
- \langle \bar{\psi} \psi \rangle = \lim_{x \to y} Tr < \Omega | : \psi_i^\alpha(x) \bar{\psi}_i^\beta(y) : | \Omega > = Ni \int \frac{d^4k}{(2\pi)^4} Tr \left[ \frac{\gamma \cdot k + M(k^2)}{k^2 - M^2(k^2) + i\epsilon} - \frac{\gamma \cdot k}{k^2 + i\epsilon} \right] = 4Ni \int \frac{d^4k}{(2\pi)^4} \left[ \frac{M(k^2)}{k^2 - M^2(k^2) + i\epsilon} \right],
\] (6)

where \( N \equiv \delta^{\alpha\alpha} \), the sum of internal symmetry indices [e.g. colour number \( N_c \)].

Equation (6) can be understood diagrammatically as the equivalence of the quark-antiquark condensate to a quark propagator-loop over the full fermion propagator [Fig 1]. This equation, however, demonstrates that the existence of a dynamical mass function \( M(k^2) \) is necessarily linked to the nonperturbative content of the vacuum \( | \Omega > \), as parametrised by the quark-antiquark condensate defined in (3).

It is evident from (6) that the dynamical quark mass scale and the scale of the quark-antiquark condensate are tightly linked. We will assume that \( M(k^2) \) is asymptotically even as \( k^2 \) flips from timelike to spacelike values [5].

If \( m_{dyn} \) is identified with the pole [4,7,8] of the fermion propagator in (5),

\[
m_{dyn} = M(m_{dyn}^2),
\] (7)

5
and if $M(k^2)$ is analytic, then the left-hand side of the dispersion relation

$$Im\left(M(m_{\text{dyn}}^2)\right) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dp^2 \frac{Re(M(p^2))}{p^2 - m_{\text{dyn}}^2},$$

must vanish in order to ensure a quark decay-width of zero (the quark is stable). We assume the right-hand side of (8) is dominated by the asymptotic ($p^2 \to \pm \infty$) behaviour of the integrand. If such is the case, a vanishing result is possible only if the integrand of (8) is asymptotically odd in $p^2$, in which case $Re[M(p^2)]$ must be asymptotically even in $p^2$.

One approximate way of relating the dynamical quark mass to the scale of the condensate is to identify the momentum dependence of $M(k^2)$ with the asymptotic momentum dependence anticipated from Bethe-Salpeter and renormalization-group considerations [7]

$$M(k^2) = \mu^2 M(\mu^2) \left(\ln(|k^2|/\Lambda_{QCD}^2) / \ln(\mu^2/\Lambda_{QCD}^2)\right)^{d-1}. \tag{9}$$

In (9), $\mu^2$ is a renormalization scale, and the anomalous-dimension factor $d$ is given by

$$d = 9(N_c^2 - 1)/(2N_c(11N_c - 2N_f)) \tag{10}$$

for an SU($N_c$) internal symmetry group with $N_f$ fermion flavours. Strictly speaking, (9) describes asymptotic behaviour of $M$ for $|k^2| >> \Lambda_{QCD}^2$ in the Euclidean regime. The absolute value signs in (9) are to ensure that $M(k^2)$ is asymptotically even, as discussed above.

\footnote{Controllable low-energy behaviour and the absence of IR-divergences in the mass function $M(p^2)$ is suggested by the leading-order “freezing out” of $M(p^2)$ at $|p^2|$ less than $m_{\text{dyn}}^2$ [8,9].}
We now seek to relate the renormalized condensate \( \langle \overline{\psi} \psi \rangle_\mu \sim [\ln(\mu^2/\Lambda^2)]^d \) to the dynamical mass function (9). To extract the leading-logarithmic [i.e., renormalization-group (RG)] dependence of the condensate, we substitute (9) into (6), with UV cut off \( \mu \) and IR cut off \( \Lambda_{QCD} \), and find that [5]

\[
- \langle \overline{\psi} \psi \rangle_\mu = \frac{4N_c C i}{(2\pi)^4} \int_{\Lambda_{QCD}}^{\mu} d^4k \frac{[\ln[|k|^2/\Lambda_{QCD}^2]]^{d-1}}{|k|^2 (k^2 - M^2(k^2))},
\]

(11)

where the constant \( C \) is given by

\[
C \equiv \mu^2 M(\mu^2) \left[ \ln(\mu^2/\Lambda_{QCD}^2) \right]^{1-d}.
\]

(12)

For \( \mu^2 \) to be sufficiently large to justify neglect of \( M^2(k^2) \) in the integrand denominator, one can perform a Wick rotation \( d^4k = i\pi k_E^2 dk_E^2 \) and evaluate (11) explicitly:

\[
- \langle \overline{\psi} \psi \rangle_\mu = \frac{N_c C}{4\pi^2 d} \left[ \ln \left( \frac{\mu^2}{\Lambda_{QCD}^2} \right) \right]^d.
\]

(13)

One can eliminate the (leading-)logarithm by utilizing the one-loop RG expression for the running QCD coupling constant

\[
\alpha_s(\mu^2) = \pi d'/[\ln(\mu^2/\Lambda_{QCD}^2)],
\]

(14)

\[
d' = 12/(11N_c - 2N_f),
\]

\footnote{Arc contributions of a finite but large radius occurring after Wick rotation have been neglected, as they vanish in the large \( \mu \) limit and, hence, do not contribute to the leading-log expression for \( \langle \overline{\psi} \psi \rangle \).}
to find from (13) that
\[ -\langle \bar{\psi}\psi \rangle_{\mu} = \frac{N_c d' \mu^2 M(\mu^2)}{4\pi d \alpha_s(\mu^2)} = \frac{2N_c^2 \mu^2 M(\mu^2)}{3\pi(N_c^2 - 1)\alpha_s(\mu^2)}. \] (15)

We note that as the scale \( \mu \) is varied, \( \mu^2 M(\mu^2) \sim [\ln(\mu^2/\Lambda_{QCD}^2)]^{d-1} \) and \( \alpha_s(\mu^2) \sim [\ln(\mu^2/\Lambda_{QCD}^2)]^{-1} \). Thus we confirm that \( \langle \bar{\psi}\psi \rangle_{\mu} \sim [\ln(\mu^2/\Lambda_{QCD}^2)]^{d} \), consistent with the RG invariance of \( m\bar{\psi}\psi \) Lagrangian mass terms [the current quark mass runs as \( m(\mu^2) \sim \left( \ln(\mu^2/\Lambda_{QCD}^2) \right)^{-d} \)]. Consequently, (15) must be regarded as a relationship between the RG quantities \( \langle \bar{\psi}\psi \rangle_{\mu} \) and \( M(\mu^2) \), consistent with the large-\( \mu \) assumptions employed in evaluating (11) and (13).

The result (15) is in fact consistent with that obtained directly from the quark-antiquark condensate contribution (Fig. 2) to the Landau-gauge self-energy, as determined [10] using operator-product expansion methods appropriate for spacelike \((k^2 < 0)\) momenta:
\[ M(k^2) = \frac{3(N_c^2 - 1)}{2N_c^2} \frac{\pi \alpha_s}{k^2} \langle \bar{\psi}\psi \rangle. \] (16)

The use of Landau gauge decouples leading-order wave-function renormalization effects from the quark-antiquark condensate [5], and can be motivated to decouple dynamical-mass-function contributions to vertex functions as well [11]. Renormalization-group improvement of (16) leads to
\[ M(k^2) = \frac{3(N_c^2 - 1)}{2N_c^2} \frac{\pi \alpha_s}{k^2} \left| \langle \bar{\psi}\psi \rangle_k \right|, \] (17)
where even behaviour in \( k^2 \) [consistent with the known positivity of \( M(k^2) \) when \( k^2 < 0 \)] has been imposed for reasons given earlier. Upon algebraic rearrangement
of (17) with $k \to \mu$, eqs. (17) and (15) are seen to be equivalent - the loop integral [Fig. 1] over the RG-improved Bethe-Salpeter dynamical mass function and the Fig 2 contribution to the quark self-energy generate equivalent leading-order expressions. If one chooses $k^2 = m_{\text{dyn}}^2$ in (17) and utilizes the initial condition (7), one finds that

$$-\langle \bar{\psi}\psi \rangle_{m_{\text{dyn}}} = \frac{2N_c^2 m_{\text{dyn}}^3}{3(N_c^2 - 1)\pi \alpha_s(m_{\text{dyn}}^2)}. \quad (18)$$

In Section 4 we argue that $\alpha_s(m_{\text{dyn}}^2)$ can be identified with the critical value $\pi/4$ associated with the onset of dynamical mass generation through the Schwinger-Dyson equation for the quark propagator [12,13]. With $N_c = 3$, $\alpha_s = \pi/4$, we see from (18) that

$$-\langle \bar{\psi}\psi \rangle_{m_{\text{dyn}}} = \frac{3m_{\text{dyn}}^3}{\pi^2}, \quad (19)$$

in which case

$$-\langle \bar{\psi}\psi \rangle_{1\text{GeV}} = \frac{3m_{\text{dyn}}^3}{\pi^2} \left[ \ln \left( \frac{1\text{GeV}^2}{m_{\text{dyn}}^2} \right) \right]^{4/9} \quad (20)$$

at the 1 GeV momentum scale characterizing QCD sum-rule estimates. Thus, a dynamical quark mass of 320 MeV corresponds to a 1 GeV quark-antiquark condensate $-\langle \bar{\psi}\psi \rangle_{1\text{GeV}} = (243 \text{ MeV})^3$, consistent with QCD sum-rule estimates [14].

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<sup>3</sup>In (15), we note that $d = d'$ when $N_c = 3$. We have chosen to keep $N_c$ arbitrary in order to demonstrate that the equivalence of the Fig. 1 approach to (15) and the Fig. 2 approach to (17) is not a consequence of this coincidental equality. We are grateful to R. Tarrach for making us aware of this concern some time ago.
3 Condensate and Quark Mass in the Linear Sigma Model

The linear sigma model [15] treated as an effective theory for the interaction of constituent quarks and mesons [16,17] is controlled by the (chiral) quark level Goldberger-Treiman (GT) relation $g = m_q/f_\pi$ ensuring $\partial \cdot \vec{A} = 0$ in the chiral limit (CL). Thus a quark mass of 320 MeV (as in the previous section) and the CL of the pion decay constant, $f_\pi \approx 90$ GeV correspond to a $\pi q \bar{q}$ coupling constant $g \approx 3.6$, a value compatible with the observed [19] pion-nucleon coupling constant $g_{\pi NN} = 3g g_A$ between 13.0 and 13.5. Elsewhere [4,16], the quark-level linear sigma model (LσM) $\pi q \bar{q}$ coupling has been predicted to be $g = 2\pi/\sqrt{3} \approx 3.63$ for $N_c = 3$. This prediction corresponds to a GT quark mass $m_q = 2\pi f_\pi/\sqrt{3} \approx 326$ MeV, consistent with $m_{dyn} \approx M_N/3$ phenomenological expectations.

This “hard” quark mass can be used via (6) to determine the linear-sigma model quark-loop analogue of the quark-antiquark condensate for $N_c = 3$:

$$- \langle \bar{\psi} \psi \rangle_{L\sigma M} = \frac{12i}{(2\pi)^4} \int_0^\Lambda \frac{d^4k}{k^2 - m_q^2} m_q. \tag{21}$$

In the LσM, the ultraviolet cutoff $\Lambda$ is determined by the logarithmically divergent quark loop diagram for $f_\pi$ [Fig. 3]

$$i f_\pi q_\mu = 12g m_q q_\mu \int_0^\Lambda \frac{d^4k}{(2\pi)^4 [k^2 - m_q^2]^2}. \tag{22}$$

One can show that $1 - f^{CL}_\pi/f_\pi = m_\pi^2/8\pi^2 f_\pi^2 \approx 0.03$ or $f^{CL}_\pi \approx 90$ MeV for the physical $f_\pi \approx 93$ MeV [18]. A $\chi^2$-minimization corresponding to $g_{\pi NN} = 13.145 \pm 0.072$ is presented in the most recent paper in ref. [19].
Since \( f_\pi = m_q/g \), eq. (22) leads to the gap equation

\[
1 = -12ig^2 \int_0^\Lambda d^4p (2\pi)^{-4} (p^2 - m_q^2)^{-2} \\
= \frac{3g^2}{4\pi^2} \left[ \ln \frac{\Lambda^2 + m_q^2}{m_q^2} - \frac{\Lambda^2}{\Lambda^2 + m_q^2} \right].
\]

(23)

If \( g = 2\pi/\sqrt{3} \), the square-bracketed expression in (23) is unity for \( \Lambda = 2.30m_q \). With this value of \( \Lambda \), one finds from (21) that

\[
-\langle \bar{\psi}\psi \rangle_{L\sigma M} = \frac{3m_q^3}{4\pi^2} \left[ \frac{\Lambda^2}{m_q^2} - \ln \left( \frac{\Lambda^2}{m_q^2} + 1 \right) \right] \\
\approx (209\text{MeV})^3
\]

(24)

for \( m_q = 326\text{MeV} \).

It should be noted that the corresponding \( \Lambda \) value \( (750\text{ MeV}) \) is both somewhat above the \( L\sigma M \) scalar meson mass \( m_\sigma = 2m_q \) and somewhat below the \( \rho \)-mass, consistent with phenomenological expectations. The \( L\sigma M \) describes an effective theory for low-energy QCD in which the \( \sigma \)-meson is fundamental but the \( \rho \)-meson is composite. The condensate value in (24) can be compared with the corresponding QCD estimate from the previous section at momentum scale \( \Lambda \). We suggest here that it may be more appropriate to associate \( L\sigma M \) parameters with their corresponding QCD parameters in the infrared limit; i. e. at their low energy “freeze-out” values. In the next section, we discuss the freezing out of the QCD coupling constant at the critical value \( \alpha_s = \pi/4 \). As remarked earlier, a similar freezing out of the dynamical quark mass function \( M(k^2) \) at the critical value \( m_{dyn} \) has been demonstrated both by plane wave \[8\] and coordinate space \[9\] methods. It is evident from (15) and (18) that the latter
equation defines $\langle \bar{\psi}\psi \rangle$ in terms of “frozen out” values for $\alpha_s$ and $M(k^2)$. Thus (19) [equivalent to (18) with explicit use of $\alpha_s(m_{dyn}) = \pi/4$] represents the magnitude of $\langle \bar{\psi}\psi \rangle$ at its low-energy “freeze-out”. Comparison of (19) to (24) is suggestive of near-equality between the L$\sigma$M quark mass (326 MeV) and the QCD dynamical mass (311 MeV) that would generate equivalence between the L$\sigma$M condensate $\langle \bar{\psi}\psi \rangle_{L\sigma M}$ and the “freeze-out” QCD condensate $\langle \bar{\psi}\psi \rangle_{m_{dyn}}$. Alternatively, if $m_{dyn} = 326$ MeV (the L$\sigma$M value), we find from (19) that $\langle -\bar{\psi}\psi \rangle_{m_{dyn}} = 3m_{dyn}^3/\pi = (219$ MeV)$^3$, a value quite comparable to (24).

4 Correspondence Between L$\sigma$M and QCD Coupling Strengths

Thus far we have argued that the link between “frozen-out” QCD and the L$\sigma$M is in the near-equivalence of their order-parameters of chiral-noninvariance:

$$m_{dyn} = m_q , \quad \langle \bar{\psi}\psi \rangle_{m_{dyn}} \approx \langle \bar{\psi}\psi \rangle_{L\sigma M} .$$

Correspondence between QCD and L$\sigma$M coupling strengths can be obtained from the quark mass gap $\sigma$-tadpole graph of Fig. 4, which leads to the L$\sigma$M relation [4]

$$m_q = \frac{g_{\sigma q q}^2}{-m_{\sigma}^2} \langle -\bar{\psi}\psi \rangle_{L\sigma M} ,$$

for zero momentum transferred to the vacuum. On the other hand, we see from (18) (with $N_c = 3$) that [5,20]

$$\langle -\bar{\psi}\psi \rangle_{m_{dyn}} = \frac{3}{4\pi\alpha_s}m_{dyn}^3 ,$$

12
where $\alpha_s$ is assumed to be at its infrared “frozen-out” value. We use (25) to justify substitution of (27) into (26) and find that

$$m^2/\sigma^2 \approx 3g^2/4\pi\alpha_s.$$  

(28)

Since one knows that $m_{\sigma} = 2m_q$ in the Nambu Jona-Lasinio model [3] as well as in the quark-level $L\sigma M$ [4,16] in the chiral limit, (28) then implies that

$$g_{\sigma qq}^2/4\pi \approx (4/3)\alpha_s \equiv \alpha_{\text{eff}}^s.$$  

(29)

Eqs. (25) and (29) together constitute the linkage between $L\sigma M$ and infrared QCD parameters.

Now we proceed to examine separately the scales of the $L\sigma M$ and QCD couplings related by (29). The QCD coupling strength $\alpha_s = g_s^2/4\pi$ runs according to (14) in one-loop order. The ALEPH Collaboration [21] value of $\alpha_s(M_Z^2) = 0.122 \pm 0.007$ can be used [22] to obtain $\alpha_s(2.5GeV^2) = 0.375 \pm 0.07$ at the threshold of the three-flavour region. Substituting this latter coupling back into (14) with $d' = 4/9$ for three quark flavours, one finds $\Lambda_{QCD} \approx 246 MeV$. This three-flavour QCD cutoff in turn generates $\alpha_s(1GeV^2) \approx 0.5$, which is the usual QCD coupling one expects at the $\phi(1020)$ scale [23].

We focus here on the value of $\alpha_s$ at the NJL - $L\sigma M$ scalar mass $m_{\sigma} = 2m_q \approx 650 MeV$, which for $\Lambda_{QCD} = 246 MeV$ is $\alpha_s[(650MeV)^2] \approx 0.72$, using (14) again.

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6Our own calculation, using somewhat smaller PDG [1] world average for $\alpha_s(m^2)[0.118 \pm 0.03]$ evolved down to 2.5 $GeV^2$ via the 3-loop $\overline{MS}$ $\beta$-function [1], yields the range $\alpha_s = 0.338^{+0.030}_{-0.027}$, where the uncertainty includes the PDG uncertainty in the five-flavour threshold [4.1 $GeV \leq m_b \leq 4.5 GeV$]. The corresponding range of (1-loop) $\Lambda$ in (14) is between 167 and 237 MeV, consistent with $\alpha_s(1GeV^2)$ between 0.4 and 0.5.
This value is to be compared with Mattingly and Stevenson’s [24] extension of QCD to the infrared region via “minimal sensitivity”, leading to a freezing out of $\alpha_s$ at the value $\alpha_s[(2m_q)^2]/\pi = 0.26$, $[\alpha_s = 0.82]$. The relevant issue here, however, is not so much the location of the freeze-out momentum scale, but rather the near equivalence of their numerical determination of the frozen coupling to the value $\pi/4 (= 0.785)$ anticipated from chiral symmetry breaking. That is, freezeout is seen to occur at or near $\alpha_s = \pi/4$.

Theoretical justification for this value follows from the “supercriticality” [25, 26] occurring when $\alpha_s^{\text{eff}} = (4/3)\alpha_s = \pi/3(\approx 1)$. At this strength, the underlying Dirac equation breaks down, similar to the perturbative breakdown occurring in large-Z atoms when $Z\alpha \to 1$. Such spontaneous breakdown in QCD is presumably characterized by a nonvanishing quark condensate $<\bar qq> \neq 0$. Stated another way, when $\alpha_s^{\text{eff}} = \pi/3$, Bethe-Salpeter dynamics is at a singular point [25] for nonperturbative QCD, or at an “ultraviolet fixed point” for abelian, quenched planar QED [27]. Near this singular point the $0^-$, $\bar qq$ Bethe-Salpeter bound state wave function in Landau gauge has the asymptotic form

$$\chi(p^2) \sim (p^2)^{-(3\pm\sqrt{1-3\alpha_s^{\text{eff}}}/\pi)/2},$$

which of course suggests $\alpha_s^{\text{eff}} \to \pi/3$ at breakdown. Alternatively, the Schwinger-Dyson approach of Higashijima [12] identifies the onset of chiral symmetry breakdown when $\lambda \equiv (3/\pi)C_2(R)\alpha_s > 1$, confirming the criticality at $\pi/3$ of $\alpha_s^{\text{eff}} = C_2(R)\alpha_s$.

\textsuperscript{7}Mattingly and Stevenson in [24] identify $m_q$ with the current quark mass.
A more qualitative understanding of $\alpha_{s}\!^{eff} \sim 1$ at freezeout is based on the uncertainty principle $p \sim 1/r$ for a relativistic tightly bound $\bar{q}q$ pion with $m_\pi = 0$ [12]. The total pion energy for confining QCD potential $V = -\alpha_{s}\!^{eff}/r$ is then

$$E_\pi = KE + V = p - \alpha_{s}\!^{eff}/r \sim p(1 - \alpha_{s}\!^{eff}).$$

(31)

Then $E_\pi > 0$ is physical for $\alpha_{s}\!^{eff} < 1$, with $\alpha_{s}\!^{eff} = 1$ corresponding to $E_\pi$ becoming unphysical - a signal of the quarks condensing. Further confirmation of this relativistic picture of the pion is that this tight binding (with $E_\pi \rightarrow 0$) corresponds to the two (constituent) quarks in the $\bar{q}q$ pion fusing together with net (L$\sigma$M) pion charge radius [28]

$$r_\pi = \sqrt{3}/2\pi f_\pi = 1/m_q \approx 0.61 \text{ fm},$$

(32)
in close agreement with the measured $r_\pi$.

Finally, we note from (29) that the value $\alpha_{s}\!^{eff} = \pi/3$ implies that the L$\sigma$M coupling constant is $g_{\sigma qq} = 2\pi/\sqrt{3}$. This is precisely the value anticipated ($g \equiv g_{\sigma qq}$ in Section 3) from L$\sigma$M phenomenology [4] (and predicted in [16]), providing a startling confirmation of the correspondence between QCD in the infrared limit and the quark-level L$\sigma$M.

5 Discussion: $\pi^0 \rightarrow \gamma\gamma$ and Effective Theories with Constituent Quark Masses

There is presently no direct derivation from the QCD lagrangian of either the linear sigma model or of any phenomenological model for low energy QCD with constituent-
mass quarks. In the absence of such a derivation, the decay $\pi^0 \to \gamma\gamma$ is of genuine value in discussing the viability of low energy approximations to QCD. In this context, it should be remembered that the axial anomaly as well as partial conservation of the axial-vector current (PCAC) were originally developed from arguments based upon the linear sigma model.

In the linear sigma model, one explicitly calculates a "Steinberger" pseudo-scalar-vector-vector (PVV) quark triangle graph in order to obtain the correct $\pi^0 \to \gamma\gamma$ decay amplitude [29,30]. There are no meson-loop contributions to the amplitude because there is no triple pion coupling in the linear sigma model (such couplings are forbidden because of parity, Lorentz invariance, and G-parity considerations). The PVV loop calculation is, of course, easily seen to be driven by the axial anomaly [31] if one utilizes the right hand side of the axial Ward identity using constituent quark masses:

$$2m_u u\gamma_5 u - 2m_d d\gamma_5 d = \partial_\mu J^{3\mu}_5 - \alpha \vec{F} \cdot \vec{F} / 2\pi. \quad (33)$$

When the quark mass is sufficiently large compared to the pion mass, the contribution of $\partial_\mu J^{3\mu}_5$ is zero by virtue of the Sutherland-Veltman theorem [32], and the PVV loop result is well-known to be driven entirely by the anomaly term. In fact, the linear sigma model PVV decay amplitude is remarkably insensitive to the quark mass, remaining within (now small) experimental error for quark masses as small as 220 MeV [33].

These results are entirely consistent with those obtained via PCAC from the QCD
lagrangian, in which only very small (current-) quark masses are expected to appear. Using PCAC, one can transform the \( \pi^0 \to \gamma\gamma \) decay amplitude into the divergence of the axial-vector vertex of a AVV quark loop. One then utilizes the axial Ward identity in the chiral \( m_{u,d} \to 0 \) limit,

\[
[f_\pi m_\pi^2 \pi^0 =] \partial_\mu J_5^\mu = \alpha \tilde{F} \cdot F/2\pi,
\]

(34)
to relate the divergence of the axial-vector vertex directly to the anomaly term [34].

This striking compatibility between the \( \pi^0 \to \gamma\gamma \) decay amplitude appropriate for QCD-lagrangian quark masses [and obtained via PCAC] and the \( \pi^0 \to \gamma\gamma \) decay amplitude for constituent quark masses within a linear sigma model context is no longer evident in the absence of the pion-quark couplings characterizing the linear sigma model. In the absence of such couplings, any phenomenological model involving constituent mass quarks must necessarily rely upon PCAC to express the \( \pi^0 \to \gamma\gamma \) amplitude in terms of the divergence of an AVV triangle. For such models, however, one can no longer disregard quark masses. The contribution to this amplitude of the other side of the Ward-identity

\[
[f_\pi m_\pi^2 \pi^0 =] \partial_\mu J_5^\mu = \alpha \tilde{F} \cdot F/2\pi + 2m_u \gamma_5 u - 2m_d \gamma_5 d
\]

(35)
cancels for sufficiently large quark masses \((m_{u,d}/m_\pi >> 1)\), because of the equivalence of the anomaly-term insertion and the PVV loop in the soft- pion \((i.e. \text{large-fermion-mass})\) limit, thereby confirming the Sutherland-Veltman theorem. The only way to make such a model work is to introduce an ad hoc modification of PCAC itself through
a redefinition of the interpolating pion field:

\[ f_\pi m_\pi^2 \pi^\circ = \partial_\mu J_5^{3\mu} - \alpha \tilde{F} \cdot F/2\pi, \]  

(36)

Such a redefinition of PCAC [30,35] will ensure the usual anomaly-driven result in the (constituent-quark-) limit of large quark masses for which the Sutherland-Veltman theorem applies—essentially by construction. One must therefore alter PCAC in order to obtain the anticipated anomaly-driven result. By performing this alteration, one is allowing the \( \pi^\circ \to \gamma\gamma \) rate to determine PCAC, rather than the other way around.

Of course, it is certainly true that effective theories and the theories from which they are derived must eventually yield the same results. It is certainly possible that the correct low-energy approximation to QCD is described by a constituent-quark model with an appropriately redefined PCAC—to demonstrate this would entail the derivation of such a model from the QCD lagrangian itself, including a careful treatment of the appropriate anomaly functional [36]. In the absence of such a derivation, however, the redefinition (36) of PCAC constitutes a methodological prescription for obtaining the correct answer with constituent-mass quarks, an answer that emerges much more convincingly within a linear sigma model context.

Such an effective theory is necessarily incomplete in the absence of an explicit mechanism for confinement. When comparing a simple model such as the linear sigma model with QCD in the infrared region, it is important to note that QCD is a theory characterizing a number of nonperturbative phenomena, including both

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8The usual operator statement of PCAC, \( \partial^i A_i = f_\pi m_\pi^2 \phi^I_\pi \), follows directly from the linear sigma model lagrangian (e.g. ref [15]).
confinement and chiral symmetry breaking. In using the linear sigma model as an approximation to low-energy QCD, we are necessarily ignoring confinement in order to use chiral symmetry breaking to extract some of the dynamics of light hadrons. As is pointed out in a recent monograph on symmetry breaking \[37\], such an approach hinges upon the assumption that the dynamics responsible for these light hadrons is insensitive to confinement – in particular, that the size of the pion is smaller than the hadronic confinement radius. Implicit in this picture is the idea of a distinct chiral-symmetry breaking momentum scale \(\Lambda_{\chi}(\approx 1\text{GeV})\) substantially larger than the confinement scale \(\Lambda_{QCD}\) \[38\] characterizing the formation of massive hadrons.

6 Summary

In the present paper, we have considered the quark mass, the quark antiquark condensate, and coupling strength \(\alpha_s\) that emerge from the infrared region of QCD, and we have argued the plausibility of identifying these quantities with those characterizing the linear sigma model. In particular, we have suggested that the quark mass and condensate characterizing the linear sigma model be identified with the infrared freeze-out values of the dynamical mass and condensate of QCD. We also compare mass-gap tadpole equations in the linear sigma model and in QCD, respectively, in order to associate \((g_{\sigma qq})^2/4\pi\) with \(\alpha_s^{eff}\), the critical coupling characterizing chiral-symmetry breakdown in QCD. Independent determinations of \(\alpha_s^{eff} = \pi/3\) and of \(g_{\sigma qq} = 2\pi/\sqrt{3}\) appear to corroborate these conclusions.
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References

[1] N. A. Törnqvist and M. Roos, Phys. Rev. Lett. 76 (1996) 1575; M. Svec, Phys. Rev. D53 (1996) 2343; R. M. Barnett et al [Particle Data Group], Phys. Rev. D54 (1996) Part 1, 1

[2] M. Harada, F. Sannino, and J. Schechter, Phys. Rev. D54 (1996) 1991 and 1996 preprint ph/9609428; S. Ishida et al, Prog. Theor. Phys. 95 (1996) 745

[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345

[4] V. Elias and M. D. Scadron, Phys. Rev. Lett. 53 (1984) 1129

[5] V. Elias, T. G. Steele and M. D. Scadron, Phys. Rev. D 38 (1988) 1584

[6] V. Elias, T. G. Steele, and Mong Tong, Zeit. Phys. C66(1995) 107

[7] K.Lane, Phys. Rev. D 10 (1974) 2605; H. D. Politzer, Nucl. Phys. B 117 (1976) 397; V. A. Miransky, Sov. J. Nucl. Phys. 38 (1983) 280; V. A. Miransky: Dynamical Symmetry Breaking in Quantum Field Theories, p 309, Singapore: World Scientific 1993

[8] L. J. Reinders and K. Stam, Phys. Lett. B 180 (1986) 125

[9] B. Bagan, M. R. Ahmady, V. Elias and T. G. Steele, Z. Phys. C 61 (1994) 157

[10] P. Pascual and E. deRafael, Z. Phys. C 12 (1982) 127

[11] M. R. Ahmady, V. Elias and R. R. Mendel, Phys. Rev. D 44 (1991) 203

21
[12] K. Higashijima, Phys. Rev. D 29 (1984) 1228

[13] T. Maskawa and H. Nakajima, Prog. Theor. Phys. 52 (1974) 1326 and 54 (1975) 860

[14] S. J. Eidelman, L. M. Kurdadze and A. I. Vainshtein, Phys. Lett. B 82 (1979) 278; M. Shifman, A. I. Vainshtein and V. Zakharov, Nucl. Phys. B 147 (1979) 385, 448

[15] M. Gell-Mann and M. Lévy, Nuovo Cimento 16 (1960) 705

[16] R. Delbourgo and M. D. Scadron, Mod. Phys. Lett. A 10 (1995) 251

[17] A. Bramon and M. D. Scadron, Phys. Rev. D 40 (1989) 3779 and Phys. Lett. B 234 (1990) 346; R. E. Karlsen, M. D. Scadron and A. Bramon, Mod. Phys. Lett. A 8 (1993) 97; A. S. Deakin, V. Elias, D. G. C. McKeon, M. D. Scadron and A. Bramon, Mod. Phys. Lett. A 9 (1994) 2381

[18] S. A. Coon and M. D. Scadron, Phys. Rev. C 23 (1981) 1150

[19] R. A. Arndt, R. L. Workman and M. M. Pavan, Phys. Rev. C 49 (1994) 2729; G. Höhler, πN Newsletter 3(1991) 66; R. Koch and E. Pietarinen, Nucl. Phys. A 336 (1980) 331; D. Bugg, A. Carter and J. Carter, Phys. Lett. B 44 (1973) 278.

[20] V. Elias and M. D. Scadron, Phys. Rev. D 30 (1984) 647

[21] ALEPH Collaboration, D. Buskulic et al., Phys. Lett. B 307 (1993) 209

[22] J. Ellis and M. Karliner, Phys. Lett. B341 (1995) 397
[23] A. DeRújula, H. Georgi and S. Glashow, Phys. Rev. D 12 (1975) 147

[24] A. C. Mattingly and P. M. Stevenson, Phys. Rev. Lett. 69 (1992) 1320

[25] P. Fomin and V. A. Miransky, Phys. Lett. B 64 (1976) 166; P. Fomin, V. Gusynin, V. A. Miransky and Y. Sitenko, Riv. Nuovo Comento 6 (1983) 1

[26] J. E. Mandula and J. Weyers, Nucl. Phys. B 237 (1984) 59 and references therein.

[27] W. A. Bardeen, C. N. Leung and S. T. Love, Nucl. Phys. B 273 (1986) 649 and B 323 (1989) 493; W. A. Bardeen, S. T. Love and V. A. Miransky, Phys. Rev. D 42 (1990) 3514

[28] R. Tarrach, Z. Phys. C 2 (1979) 221; S. B. Gerasimov, Sov. J. Nucl. Phys. 29 (1979) 259

[29] J. Steinberger, Phys. Rev. 76 (1949) 1180

[30] See, e.g., C. Itzykson and J.-B. Zuber: Quantum Field Theory, pp 551-553, New York: McGraw Hill 1980

[31] J. S. Bell and R. Jackiw, Nuovo Cimento A 60 (1969) 479; S. L. Adler, Phys. Rev. 177 (1969) 2426

[32] D. G. Sutherland, Phys. Lett. B 23 (1966) 384 and Nucl. Phys. B 2 (1967) 433; M. Veltman, Proc. R. Soc. A 301 (1967) 107 and Nucl. Phys. B 21 (1970) 288

[33] A. S. Deakin, V. Elias and M. D. Scadron, Mod. Phys. Lett. A 9 (1994) 955

23
[34] See, e.g., K. Huang: Quarks, Leptons, and Gauge Fields p 274, Singapore: World Scientific 1992

[35] T.-P. Cheng and L.-F. Li: Gauge Theory of Elementary Particle Physics, p 181, Oxford: Science 1986

[36] A. Manohar and G. Moore, Nucl. Phys. B 243 (1984) 55

[37] V. A. Miransky: Dynamical Symmetry Breaking in Quantum Field Theories, pp 319-322, Singapore, 1993: World Scientific.

[38] C. G. Callan, R. F. Dashen and D. J. Gross, Phys. Rev. D 17 (1984) 189; A. Manohar and H. Georgi, Nucl. Phys. B 234 (1984) 189.
Figures

Fig. 1: The quark-antiquark condensate’s equivalence to the trace of the full fermion propagator loop [Eq. (6)] including a dynamical mass function.

Fig. 2: The quark-antiquark condensate’s contribution to the quark self-energy. The exchanged particle is an SU(N) gluon.

Fig. 3: LσM representation of the constituent quark loop origin of the pion decay constant $f_\pi$.

Fig. 4: Contribution of the LσM quark-antiquark condensate to the LσM quark mass.