Photon-added one-photon and two-photon nonlinear coherent states

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Abstract

From the photon-added one-photon nonlinear coherent states $a^\dagger m |\alpha, f\rangle$, we introduce a new type of nonlinear coherent states with negative values of $m$. The nonlinear coherent states corresponding to the positive and negative values of $m$ are shown to be the result of nonunitarily deforming the number states $|m\rangle$ and $|0\rangle$, respectively. As an example, we study the sub-Poissonian statistics and squeezing effects of the photon-added geometric states with negative values of $m$ in detail. Finally we investigate the photon-added two-photon nonlinear coherent states and find they are still the two-photon nonlinear coherent states with certain nonlinear functions.

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1. Introduction

Recently there has much interest in the study of nonlinear coherent states (NLCSs) [1,2], which are right-hand eigenstates of the product of the boson annihilation operator $a$ and a nonlinear function $f(N)$ of the number operator $N$,

$$f(N)a|\alpha, f\rangle = \alpha|\alpha, f\rangle.$$  \hspace{1cm} (1)

Here $\alpha$ is a complex eigenvalue. It has been shown that a class of NLCSs may appear as stationary states of the centre-of-mass motion of a trapped ion [1]. These nonlinear coherent states exhibit nonclassical features like squeezing and self-splitting.

Another type of interesting nonclassical states consists of the photon-added states [3,4]

$$|m, \psi\rangle = \frac{a^m|\psi\rangle}{\langle\psi|a^m a^m|\psi\rangle},$$ \hspace{1cm} (2)

where $|\psi\rangle$ may be an arbitrary quantum state, $a^\dagger$ is the boson creation operator, $m$ is a non-negative integer-the number of added quanta. For the first time these states were introduced by Agarwal and Tara [3] as photon-added coherent states. The photon-added squeezed states [5], even(odd) photon-added states [6] and photon-added thermal state [7] were also introduced and studied. The photon-added states can be produced in the interaction of a two-level atom with a cavity field initially prepared in the state $|\psi\rangle$ [3].

Sivakumar showed that the photon-added coherent states are nonlinear coherent states [8]. As a generalization we showed a general result that photon-added NLCSs (PANLCSs) are still NLCSs with different nonlinear functions [9]. The PANLCSs are defined as

$$|m, \alpha, f\rangle = \frac{a^m|\alpha, f\rangle}{\langle\alpha, f|a^m a^m|\alpha, f\rangle}.$$ \hspace{1cm} (3)

They satisfy [9]

$$f(N - m)[1 - m/(N + 1)]a|m, \alpha, f\rangle = \alpha|m, \alpha, f\rangle.$$ \hspace{1cm} (4)

As seen from Eq.(4), the PANLCS is an NLCS with the nonlinear function $f(N - m)[1 - m/(N + 1)]$. Naturally Eq.(4) reduces to Eq.(1) when $m = 0$. The well-known geometric
states (GSs) [10] and negative binomial states (NBSs) [11] are NLC Ss [9]. Therefore, the photon-added GSs [12,13] and photon-added NBSs [14] are still NLC Ss and are special cases of the PANLCSs.

In the present paper we show that the PANLCSs are the result of nonunitarily deforming the number state $|m\rangle$. We introduce the PANLCS with negative values of $m$, which are the result of nonunitarily deforming the vacuum state $|0\rangle$. As an example, we study the sub-Poissonian statistics and squeezing effects of the photon-added geometric states with negative values of $m$ in detail. We also investigate the photon-added two-photon nonlinear coherent states.

2. The PANLCS as deformed number state $|m\rangle$

In this section we show that the PANLCS can be written as a nonunitarily deformed number state. This is achieved by the method given by Shanta et al [15]. Here we give a brief review of the method.

Consider an annihilation operator $A$ which annihilates a set of number states $|n_i\rangle, i = 1, 2, \ldots k$. Then we can construct a sector $S_i$ by repeatedly applying $A\dagger$, the adjoint of $A$, on the number state $|n_i\rangle$. Thus we have $k$ sectors corresponding to the states that are annihilated by $A$. A given sector may turn out to be either of finite or infinite dimension. If a sector, say $S_j$, is of infinite dimension then we can construct an operator $G\dagger_j$ such that $[A, G\dagger_j] = 1$ holds in that sector. Then the eigenstates of $A$ can be written as $\exp(\alpha G\dagger_j)|n_j\rangle$. If an operator $A$ is of the form $f(N) a^p$, where $p$ is non-negative integer, such that it annihilates the number state $|j\rangle$ then $G\dagger_j$ is constructed as [15]

$$G\dagger_j = \frac{1}{p} A\dagger \frac{1}{AA\dagger}(a\dagger a + p - j).$$

(5)

It is interesting that the operator $f(N-m)[1 - m/(N + 1)] a$ in Eq. (4) annihilates both the vacuum state $|0\rangle$ and Fock state $|m\rangle$. The states between the vacuum state and Fock state $|m\rangle$ are not annihilated. To discuss the case of the PANLCS $|m, \alpha, f\rangle$ let

$$A = f(N-m)[1 - m/(N + 1)] a, A\dagger = a\dagger f(N-m)[1 - m/(N + 1)]$$

(6)
We construct sector \( S_0 \) by repeated applying \( A^\dagger \) on the vacuum state. \( S_0 \) is the set \(|i\rangle, i = 0, 1, 2, ..., m - 1 \) and it is of finite dimension. The sector \( S_m \), built by the repeated application of \( A^\dagger \) on \(|m\rangle\), is the set \(|i\rangle, i = m, m + 1, ... \) and it is of infinite dimension. Hence we can construct an operator \( G^\dagger \) such that \([A, G^\dagger] = 1\) holds in \( S_m \). To construct \( G^\dagger \), we set \( p = 1 \) and \( j = m \) in Eq.(5) and this yields

\[
G^\dagger = a^\dagger \frac{1}{f(N - m)}
\]

In fact, by direct verification, we have

\[
[f(N - m)[1 - m/(N + 1)]a, a^\dagger \frac{1}{f(N - m)}] = 1.
\]

Therefore the PANLCS can be written as

\[
|m, \alpha, f\rangle = \exp(G^\dagger)|m\rangle = \exp[a^\dagger \frac{1}{f(N - m)}]|m\rangle
\]

up to a normalization constant. From the above equation it is shown that the PANLCS can be viewed as nonunitarily deformed Fock(number) state \(|m\rangle\).

3. The PANLCS with negative \( m \)

The form of \( A \), given by Eq.(6), suggests that it is a well-defined operator-valued function also for negative values of \( m \) on the Fock space. In this section the PANLCS with negative \( m \) is constructed. Denoting the the PANLCS with negative \( m \) by \(|-m, \alpha, f\rangle\), the equation to determine them are

\[
f(N + m)[1 + m/(N + 1)]a|-m, \alpha, f\rangle = \alpha|-m, \alpha, f\rangle.
\]

The operator \( A = f(N + m)[1 + m/(N + 1)]a \) only annihilates the vacuum state. When \( f(N) \equiv 1 \), the state \(|-m, \alpha, f\rangle\) reduces to that studied in Ref. [8]. The sector \( S_0 \), built by the repeated application of \( A^\dagger = a^\dagger f(N + m)[1 + m/(N + 1)] \) on \(|0\rangle\), is the set \(|i\rangle, i = 0, 1, ... \).
and it is just the infinite dimensional Fock space. To construct $G^\dagger$, corresponding to the operator $A = f(N + m)[1 + m/(N + 1)]a$, we set $p = 1$ and $j = 0$ in Eq.(5) and this yields

\[ G^\dagger = a^\dagger \frac{N + 1}{f(N + m)(N + m + 1)} \]  

(11)

Thus the PANLCS with negative $m$ can be written as

\[ | - m, \alpha, f \rangle = \exp(G^\dagger)|0\rangle = \exp[a^\dagger \frac{N + 1}{f(N + m)(N + m + 1)}]|0\rangle \]  

(12)

up to a normalization constant. The state $| - m, \alpha, f \rangle$ is obtained by nonunitarily deforming the vacuum state $|0\rangle$ while the state $|m, \alpha, f \rangle$ is obtained by nonunitarily deforming the Fock state $|m\rangle$.

The PANLCS is obtained by the action of $a^{1m}$ on the NLCS $|\alpha, f \rangle$. The state $| - m, \alpha, f \rangle$ can be written in a similar form using the inverse operators $a^{-1}$ and $a^{\dagger -1}$. These operators are defined in terms of their actions on the number state $|n\rangle$ as follows

\[ a^{-1}|n\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle, \]  

\[ a^{\dagger -1}|n\rangle = \frac{1}{\sqrt{n}}|n-1\rangle, \]  

\[ a^{\dagger -1}|0\rangle = 0. \]  

(13)

Using these inverse operators the state $| - m, \alpha, f \rangle$ can be rewritten as $| - m, \alpha, f \rangle = a^{\dagger -m}a^{-m}|\alpha, f'\rangle$ up to a normalization constant. Here $|\alpha, f'\rangle$ is the NLCS with the nonlinear function $f'(N) = f(N + m)$. The state $| - m, \alpha, f \rangle$ is obtained by the action of the operator $a^{\dagger -m}a^{-m}$ on the NLCS $|\alpha, f'\rangle$ while the state $|m, \alpha, f \rangle$ is obtained by the action of the operator $a^{\dagger m}$ on the NLCS $|\alpha, f \rangle$.

From Eq.(12) the number state expansion of the PANLCS with negative $m$ can be easily obtained as

\[ | - m, \alpha, f \rangle = \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{n!}}{f(n + m - 1)...f(0)(n + m)!} |n\rangle \]  

(14)

up to a normalization constant. The expansion is useful in the following discussions.
4. Photon-added geometric state with negative $m$

In this section we consider a special example of the PANLCS with negative $m$, the photon-added geometric state with negative $m$.

The geometric state is defined as [10]

$$|\eta\rangle = \eta^{1/2} \sum_{n=0}^{\infty} (1 - \eta)^{n/2} |n\rangle, \quad 0 < \eta < 1,$$

It satisfies

$$\frac{1}{\sqrt{N+1}} a|\eta\rangle = \sqrt{1 - \eta}|\eta\rangle. \quad (16)$$

In comparison with Eq.(1), we see that the geometric state is an NLCS with the nonlinear function $1/\sqrt{N + 1}$. The photon-added geometric state is defined as

$$|m, \eta\rangle = \frac{a^m |\eta\rangle}{\langle \eta|a^m a^\dagger m|\eta\rangle} \quad (17)$$

$$= \eta^{(m+1)/2} \sum_{n=0}^{\infty} \binom{m+n}{n} (1 - \eta)^{n/2} |n\rangle, \quad (18)$$

which is just the negative binomial state introduced by Barnett [12]. We have studied the statistical properties and algebraic characteristics of the photon-added geometric state in detail [13]. From Eqs.(4) and (16) we get

$$\frac{\sqrt{N-m+1}}{N+1} a|m, \eta\rangle = \sqrt{1 - \eta}|m, \eta\rangle. \quad (19)$$

The state $|m, \eta\rangle$ is an NLCS with the nonlinear function $f(N) = \sqrt{N-m+1/(N+1)}$. When $m = 0$, Eq.(19) reduces to Eq.(16) as we expected.

We would like to study the state $|-m, \eta\rangle$, the photon-added geometric state with negative values of $m$, which satisfies

$$\frac{\sqrt{N+m+1}}{N+1} a|-m, \eta\rangle = \sqrt{1 - \eta}|-m, \eta\rangle. \quad (20)$$
From Eq.(14), the number state expansion of the state $| - m, \eta \rangle$ is given by

$$| - m, \eta \rangle = \sqrt{\frac{m!}{2F_1(1, 1; m + 1; 1 - \eta)}} \sum_{n=0}^{\infty} \frac{(1 - \eta)^n}{n!} \frac{n!}{(n + m)!} |n\rangle,$$

(21)

where $2F_1(1, 1; m + 1; 1 - \eta)$ is the hypergeometric function.

The photon statistics of a quantum state can be conveniently studied by Mandel’s $Q$-parameter [17]

$$Q = \frac{\langle N^2 \rangle - \langle N \rangle^2 - \langle N \rangle}{\langle N \rangle}$$

(22)

A negative $Q$ indicates that the photon number distribution is sub-Poissonian and it is a nonclassical feature. A positive $Q$ indicates the super-Poissonian distribution and $Q=0$ indicates Poissonian distribution. The photon-added geometric state $|m, \eta\rangle$ can be sub-Poissonian depending on the parameter $\eta$ [13]. For the state $| - m, \eta \rangle$(Eq.(21)), the mean value of $N^k$ is easily obtained as

$$\langle N^k \rangle = \frac{m!}{2F_1(1, 1; m + 1; 1 - \eta)} \sum_{n=0}^{\infty} \frac{n^k(1 - \eta)^n}{n!} \frac{n!}{(n + m)!}.$$  

(23)

In Fig.1 the $Q$-parameter, calculated using Eqs.(22) and (23), for the state $| - m, \eta \rangle$ is shown as a function of $\eta$. The $Q$-parameter is always greater than zero indicating that they are super-Poissonian. For larger values of $\eta$, the $Q$-parameter is close to zero since the state $| - m, \eta \rangle$ reduces to $|0\rangle$ in the limit $\eta \to 1$.

5. Squeezing in $| - m, \eta \rangle$

Define the quadrature operators $X$ (coordinate) and $Y$ (momentum) by

$$X = \frac{1}{2}(a + a^\dagger), Y = \frac{1}{2i}(a - a^\dagger).$$

(24)

Then their variances

$$Var(X) = \langle X^2 \rangle - \langle X \rangle^2, Var(Y) = \langle Y^2 \rangle - \langle Y \rangle^2$$

(25)

obey the Heisenberg’s uncertainty relation

$$Var(X)Var(Y) \geq \frac{1}{16}.$$ 

(26)

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If one of the variances is less than 1/4, the squeezing occurs. In the present case, $\langle a \rangle$ and $\langle a^2 \rangle$ are real. Thus, the variances of $X$ and $Y$ can be written as

$$\text{Var}(X) = \frac{1}{4} + \frac{1}{2}(\langle a^\dagger a \rangle + \langle a^2 \rangle - 2\langle a \rangle^2),$$

(27)

$$\text{Var}(Y) = \frac{1}{4} + \frac{1}{2}(\langle a^\dagger a \rangle - \langle a^2 \rangle).$$

(28)

From Eq.(21), the expectation value $\langle -m, \eta | a^k | -m, \eta \rangle$ is directly obtained as

$$\langle -m, \eta | a^k | -m, \eta \rangle = \frac{m!}{2\text{F}_1(1, 1; m + 1; 1 - \eta)} \sum_{n=0}^{\infty} \frac{(n + k)! (1 - \eta)^{n+k/2}}{(n + m)! (n + m + k)!}$$

(29)

The variances of $X$ and $Y$ can be calculated from Eqs.(27), (28) and (29). In Fig.2 we show the variances of the quadrature operators $X$ and $Y$ as a function of $\eta$ for different values of $m$. The squeezing exists in the quadrature $Y$. For the quadrature $Y$, the degree of the squeezing becomes deep with the increase of the parameter $m$. In the limit $\eta \to 1$, the variances are all equal to 1/4. This is because the state $| -m, \eta \rangle$ reduces the vacuum state $|0\rangle$ in this limit.

5. Photon-added two-photon nonlinear coherent states

In this section, we investigate the photon-added two-photon nonlinear coherent states. The two-photon nonlinear coherent state is defined as [18]

$$F(N)a^2|\alpha, F\rangle = \alpha|\alpha, F\rangle,$$

(30)

and the corresponding photon-added two-photon nonlinear coherent state is

$$|\alpha, F, m\rangle = a^m|\alpha, F\rangle$$

(31)

up to a normalization constant.

Acting the operator $a^2a^m$ on Eq.(31) from the left, we obtain

$$F(N - m + 2)a^2a^m|\alpha, F\rangle = \alpha(N + 1)(N + 2)a^{1(m-2)}|\alpha, F\rangle.$$  

(32)
Since
\[ a^2 a^\dagger m a^2 = (N + 4 - m)(N + 3 - m)a^2 a^\dagger(m-2), \] (33)
we obtain
\[ F(N-m+2)(N+4-m)(N+3-m)a^2 a^\dagger(m-2)|\alpha, F\rangle = \alpha(N+1)(N+2)a^\dagger(m-2)|\alpha, F\rangle. \] (34)
Let \( m - 2 \to m \) in the above equation and note that the operator \((N + 1)(N + 2)\) is positive in the whole Fock space, we get
\[ F(N - m)(1 - \frac{m}{N + 2})(1 - \frac{m}{N + 1})a^2|\alpha, F, m\rangle = \alpha|\alpha, F, m\rangle. \] (35)
This shows that the photon-added nonlinear coherent states \(|\alpha, F, m\rangle\) are still nonlinear coherent states with the nonlinear function
\[ F(N - m)(1 - \frac{m}{N + 2})(1 - \frac{m}{N + 1}). \] (36)

Since the squeezed vacuum state and squeezed first Fock state are two-photon nonlinear coherent state [18], we conclude that the photon-added squeezed vacuum state and photon-added squeezed first Fock state are also two-photon nonlinear coherent states as discussed in Ref. [19]. We can also introduce the photon-added two-photon nonlinear coherent states with negative \( m \) and make a similar discussion as one-photon case. We will not explicitly present them here.

6. Conclusions

In conclusion, we have studied a special NLCSs, the PANLCSs. From the PANLCS we introduce a new type of quantum state, the PANLCS with negative values of \( m \). The states corresponding to the positive and negative values of \( m \) are shown to be the result of nonunitarily deforming the number states \(|m\rangle\) and \(|0\rangle\), respectively. As an example, we study the sub-Poissonian statistics and squeezing effects in the photon-added geometric state with negative values of \( m \) in detail. The results shows that photon-added geometric state with negative values of \( m \) are always super-Poissonian and the state can be squeezed in
the quadrature $Y$. We also consider the photon-added two-photon nonlinear coherent states and find a similar conclusion as one-photon case, i.e, the photon-added two-photon nonlinear coherent states are still two-photon nonlinear coherent states with certain nonlinear functions.

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**Figure Captions:**

Figure 1, Mandel’s Q parameter as a function of $\eta$ for different values of $m$.

Figure 2, Variances of the quadrature operators $X$ and $Y$ as a function of $\eta$ for different values of $m$. 
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Fig. 2