Abstract

We state some inequalities for m-divisible and infinite divisible characteristic functions. Basing on them we propose statistical test for a distribution to be infinite divisible.

Key words: infinite divisible distributions; statistical tests.

1 Inequalities for characteristic functions. Estimates from below

Let $f(t)$ be characteristic function of a distribution on real line. We say that $f(t)$ is $m$-divisible ($m$ is positive integer) if $f^{1/m}(t)$ is a characteristic function as well. The function $f(t)$ is infinitely divisible if it is $m$-divisible for all positive integers $m$. Properties of infinitely divisible characteristic functions were well-studied (see, for example, [3]). In particular, any probability distribution with a compact support is not infinitely divisible. However, it is not true for $m$-divisible distributions. Really, let us take arbitrary distribution with a compact support. Denote its characteristic function by $g(t)$ and define $f(t) = g^m(t)$. Clearly, $f(t)$ is $m$-divisible characteristic function of a distribution with compact support.

Let us start with the case of arbitrary symmetric distribution having compact support.

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**Theorem 1.1.** Let $g(t)$ be a characteristic function of an even non-degenerate distribution function having its support in interval $[-A, A]$, $A > 0$. Then the inequality

$$ \cos(\sigma \cdot t) \leq g(t) \quad (1.1) $$

holds for all $t \in (-4.49/A, 4.49/A)$. Here $\sigma$ is standard deviation of the distribution with characteristic function $g(t)$.

**Proof.** Denote by $G(x)$ a distribution function corresponding to the characteristic function $g(t)$. We have

$$ g(t) = \int_{-A}^{A} \cos(tx)dG(x) = \int_{0}^{A} \cos(tx)d\tilde{G}(x) = \int_{0}^{A^2} \cos(t\sqrt{y})dH(y), \quad (1.2) $$

where $\tilde{G}(x) = 2(G(x) - 1/2)$ and $H(y) = \tilde{G}(\sqrt{y})$. In view of the fact that $\cos(t)$ is an even function we can consider positive values of $t$ only.

It is not difficult to see that the function $\cos(t\sqrt{y})$ is convex in $y$ for the case when $0 \leq t\sqrt{y} \leq z_0$, where $z_0$ is the first positive root of the equation $\sin z - z \cos z = 0$. Numerical calculations show that $z_0 > 4.49$. For the case of $|t| \leq 4.49/A$ let us apply Jensen inequality to (1.2). We obtain

$$ g(t) = \int_{0}^{A^2} \cos(t\sqrt{y})dH(y) \geq \cos \left( t\sqrt{\int_{0}^{A^2} ydH(y)} \right) = \cos(\sigma t). $$

The condition of support compactness may be changed by the restriction of absolute fifth moment existence. Namely, the following result holds.

**Theorem 1.2.** Let $G(x)$ be a symmetric probability distribution function. Denote by $a_j$ its absolute moments and suppose that $a_{10}$ is finite. Denote by $g(t)$ corresponding characteristic function and $\sigma^2 = a_2$. Then we have

$$ \cos(\sigma t) \leq g(t) \quad (1.3) $$

for all $|t| \leq 5(a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5a_5 + a_{10})^{1/2}$.

**Proof.** Suppose that $g(t)$ is not identical to $\cos(\sigma t)$. Define

$$ \varphi(t) = g(t) - \cos(\sigma t). $$
It is easy to verify that \( \varphi(t) \) and its derivatives at the point \( t = 0 \) satisfy
\[
\varphi^{(k)}(0) = 0, \quad \text{for } k = 0, 1, 2, 3,
\]
and \( \varphi^{(4)}(0) = a_4 - a_2^2 > 0 \). Therefore, \( \varphi^{(4)}(t) \) is positive in some neighborhood of the point \( t = 0 \) and, consequently, \( \varphi(t) \) is non-negative at least in some neighborhood obtained by means of forth times integration. Our aim now is to estimate the length of the interval for \( \varphi^{(4)}(t) \) positiveness. For this consider derivative of \( \varphi^{(4)}(t) \) that is \( \varphi^{(5)}(t) \). We have
\[
|\varphi^{(5)}(t)| = \left| \int_{-\infty}^{\infty} (\sigma^5 \sin(\sigma t) - \sin(t x) x^5) dG(x) \right| \leq \left( \int_{-\infty}^{\infty} (\sigma^5 \sin(\sigma t) - \sin(t x) x^5)^2 dG(x) \right)^{1/2} \leq \left( \sigma^{10} + 2\sigma^5 a_5 + a_{10} \right)^{1/2}.
\]
In view of the facts that \( \varphi^{(4)}(0) = a_4 - \sigma^4 > 0 \) we see that \( \varphi^{(4)}(t) \geq 0 \) on the interval \([0, (a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}]\), and, because of symmetry, on interval \([- (a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}, (a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}]\). This guarantees that
\[
\varphi(t) \geq \frac{t^4}{24} \left( (a_4 - \sigma^4) - (\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2} t/5 \right).
\]
From this and the symmetry of \( \varphi(t) \) follows the result. \( \square \)

Let us turn to the of \( m \)-divisible distribution with compact support.

**Theorem 1.3.** Let \( f(t) \) be a characteristic function of \( m \)-divisible symmetric distribution having compact support in \([-A, A]\), \( A > 0 \) and standard deviation \( \sigma \). Then
\[
\cos^m(\sigma t/\sqrt{m}) \leq f(t)
\]
for \( |t| \leq \min(4.49m/A, \pi \sqrt{m}/(2\sigma)) \).

**Proof.** Denote \( g(t) = f^{1/m}(t) \). It is clear that:

1. \( g(t) \) is a symmetric characteristic function;
2. \( m \sigma^2(g) = \sigma^2(f) = \sigma^2 \) (because variance of sum of independent random variables equals to the sum of their variances);
3. distribution with characteristic function \( g \) has compact support in 
\([-A/m, A/m]\) (see, for example, [2], Theorem 3.2.1).

Applying Theorem 1.1 to the function \( g(t) \) we find
\[
\cos(\sigma t/\sqrt{m}) \leq g(t) = f^{1/m}(t)
\]
for \(|t| \leq 4.49m/A\). However, for \(|t| \leq \min(4.49m/A, \pi \sqrt{m}/(2\sigma))\) the left
hand side of previous inequality is non-negative and we come to the conclusions of Theorem 1.3.

Note that \( \cos(\sigma t/\sqrt{m}) \) is monotone increasing in \( m \) for \(|t| \leq \pi \sqrt{m}/(2\sigma)\)
and, therefore the estimator (1.5) is more precise than (1.1).

Let us give a little bit different result.

**Theorem 1.4.** Let \( f(t) \) be a characteristic function of \( m \)-divisible symmetric
distribution having finite tenth moment \( a_{10} \). Then
\[
\cos^{m}(\sigma t/\sqrt{m}) \leq f(t)
\]
for \(|t| \leq \min(C \sqrt{m}, \pi \sqrt{m}/(2\sigma))\), where positive \( C \) depends on absolute mo-
ments \( a_{k} \), \( (k = 1, \ldots, 10) \) only.

**Proof.** Denote \( g(t) = f^{1/m}(t) \). It is clear that:

1. \( g(t) \) is a symmetric characteristic function;

2. \( m\sigma^{2}(g) = \sigma^{2}(f) = \sigma^{2} \) (because variance of sum of independent random
   variables equals to the sum of their variances);

3. distribution with characteristic function \( g \) has finite absolute moments
   up to tenth order (see, for example, [2]).

It is not difficult to verify that
\[
a_{k,m} \sim a_{k}/m \quad \text{as} \quad m \to \infty,
\]
where \( a_{k,m} \) is \( k \)th absolute moment of \( g \). To finish the proof it is enough to
apply Theorem 1.2 and the relation (1.6). 

Consider now the case of infinitely divisible distribution.
Theorem 1.5. Let \( f(t) \) be a symmetric infinite divisible characteristic function with finite second moment \( \sigma^2 \). Then
\[
\exp\{-\sigma^2 t^2/2\} \leq f(t)
\] (1.7)
for all \( t \in \mathbb{R}^1 \).

Proof. If \( f(t) \) has finite tenth moment it is sufficient pass to limit in (1.5) as \( m \to \infty \). In general case one can approximate \( f \) by infinitely divisible characteristic functions with finite tenth moment. \( \square \)

Another proof. Kolmogorov representation formula (see, for example \[3\]) for \( f(t) \) allows us to rewrite (1.7) in the form
\[
\exp\{-\sigma^2 t^2/2\} \leq \exp\{-\int_{-\infty}^{\infty} (1 - \cos(tx))/x^2 dK(x)\}
\]
or, equivalently,
\[
\int_{-\infty}^{\infty} (1 - \cos(tx))/x^2 dK(x) \leq \sigma^2 t^2/2. \tag{1.8}
\]
However,
\[
2(1 - \cos(tx))/x^2 \leq 4 \sin^2(t x/2)/x^2 \leq t^2.
\]
This leads to (1.8) with
\[
\sigma^2 = \int_{-\infty}^{\infty} dK(x) = -f''(0).
\]
\( \square \)

From the last proof it follows that if the equality in (1.7) attends in a point \( t_o \neq 0 \) then it holds for all \( t \in \mathbb{R}^1 \).

Theorem 1.5 shows an extreme property of Gaussian distribution among the class of infinite divisible distributions with finite second moment. Another extreme property without any moment conditions was given in [1]. Let us give this result here.

Theorem 1.6. Let \( f(t) \) be a symmetric infinite divisible characteristic function. Then
\[
f(t) \geq f^4(t/2)
\] (1.9)
for all \( t \in \mathbb{R}^1 \). If the equality in (1.9) attends in a point \( t_o \neq 0 \) then it holds for all \( t \in \mathbb{R}^1 \).
Proof. From Lévy-Khinchin representation we have
\[ f(t) = \exp \left\{ - \int_{-\infty}^{\infty} \left( 1 - \cos(tx) \right) \frac{1 + x^2}{x^2} d\Theta(x) \right\}, \]
\[ f^4(t/2) = \exp \left\{ -4 \int_{-\infty}^{\infty} \left( 1 - \cos(tx/2) \right) \frac{1 + x^2}{x^2} d\Theta(x) \right\}. \]
However,
\[ 1 - \cos(tx) = 2 \sin^2(tx/2) = 8 \sin^2(tx/4) \cos^2(tx/4) \leq \]
\[ \leq 8 \sin^2(tx/4) = 4 (1 - \cos(tx/2)) \]
\[ \square \]
Let us note that Theorem 1.5 may be obtained from Theorem 1.6. Really, assuming the existence of finite second moment we have
\[ f(t) \geq f^4(t/2) \geq f^{4^2}(t/2^2) \geq \ldots \geq f^{4^k}(t/2^k) \rightarrow \exp \{-\sigma^2 t^2/2\} \]
as \( k \rightarrow \infty \). This proves the inequality (1.7).

2 Inequalities for characteristic functions. Estimates from above

Our aim here is to proof the following result.

**Theorem 2.1.** Let \( g(t) \) be a characteristic function of an even non-degenerate function having its support in interval \([-A, A] \), \( A > 0 \). Then the inequality
\[ g(t) \leq \cos(a_{1/\gamma} \cdot t) \] (2.10)
holds for all \( t \in (-\pi/(2A^\gamma), \pi/(2A^\gamma)) \). Here \( a_{1/\gamma} \) is absolute moment of the order \( 1/\gamma \) of the distribution with characteristic function \( g(t) \), and \( \gamma > 1 \).

**Proof.** Denote by \( G(x) \) probability distribution function corresponding to characteristic function \( g(t) \). Set \( \tilde{G}(x) = 2(G(xk) - 1/2) \), \( H(y) = \tilde{G}(y^\gamma) \). We have
\[ g(t) = \int_{-A}^{A} \cos(tx)dG(x) = \int_{0}^{A} \cos(tx)d\tilde{G}(x) = \]
\[ = \int_{0}^{A} \cos(ty^\gamma)dH(y) \leq \cos(a_{1/\gamma} \cdot t) \]
for all \( |t| \leq \pi/(2A^\gamma) \). Here we used the fact that the function \( \cos(ty^\gamma) \) is concave in \( y \) if \( 0 \leq y|t| \leq \pi/2 \) and applied Jensen inequality. \( \square \)
3 Inequalities for some moments of infinitely divisible distributions

Let us give some inequalities comparing the moments of infinitely divisible distributions with corresponding characteristics of Gaussian distribution. 

**Theorem 3.1.** Let $X$ be a random variable having symmetric infinitely divisible distribution with finite second moment. Suppose that $0 < r < 2$. Then

$$\mathbb{E}|X|^r \leq \mathbb{E}|Y|^r = \frac{2^{r/2}\sigma^r\Gamma\left((1 + r)/2\right)}{\sqrt{\pi}}$$

(3.1)

where random variable $Y$ has symmetric Gaussian distribution with the same second moment $\sigma^2$ as $X$. The equality in (3.1) attends if and only if $X$ has Gaussian distribution.

**Proof.** Recall that if $Z$ is a random variable with characteristic function $h(t)$ then

$$\mathbb{E}|Z|^r = C_r \int_0^\infty \frac{1 - \text{Re}(h(t))}{t^{r+1}} dt,$$

where $C_r$ depends on $r$ only ($0 < r < 2$). From (1.7) it follows that

$$1 - \exp\{-\sigma^2 t^2/2\} \geq 1 - f(t)$$

and

$$C_r \int_0^\infty \frac{1 - \exp\{-\sigma^2 t^2/2\}}{t^{r+1}} dt \geq C_r \int_0^\infty \frac{1 - f(t)}{t^{r+1}} dt.$$

\[\Box\]

**Acknowledgement**

The study was partially supported by grant GAČR 19-04412S (Lev Klebanov) and by grant SGS18/065/OHK4/1T/13 Czech Technical University in Prague (Irina Volchenkova).
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