Probability Representation in Quantum Field Theory

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Abstract

The recently proposed probability representation of quantum mechanics is generalized
to quantum field theory. We introduce a probability distribution functional for field con-
figurations and find an evolution equation for such a distribution. The connection to the
time-dependent generating functional of Green’s functions is elucidated and the classical
limit is discussed.

The use of statistical methods for describing quantum physics gives the opportunity to
describe classical and quantum phenomena in a unified approach. Since the beginning of
quantum mechanics there have been attempts to understand its nature in a classical-like con-
text, namely to describe quantum states in terms of a classical distribution of probability.
It is this philosophy which inspired the so-called quasi-probability distribution functions of
Wigner, Husimi, Glauber and Sudarshan [1]. The original goal was not completely achieved
(the above distribution functions are not always positive defined or they do not describe mea-
surable variables) until Cahill and Glauber in [2] introduced a class of distribution functions,
known as marginal distribution functions (MDF), which enjoyed all the properties of a density
of probability. Nevertheless, it was realized only recently [3] that quantum mechanics could be
described entirely in terms of a distribution of such a family, suitably defined for a random vari-
able, which we will specify below. In [3] a consistent scheme has been proposed, the so-called
probability representation, which has been shown to be completely equivalent to the ordinary
formulation. Quantum states are described by a distribution of probability, the MDF, and the
time evolution by an integro-differential equation for the MDF. Invertible relations have been
established between the MDF and the density matrix [3, 4] and between the Green’s functions
of the related evolution equations [5].

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In this framework classical and quantum phenomena, both statistically described, only differ by the evolution equations of the distributions of probabilities for the relevant observables. The quantum evolution equation, of Fokker-Plank type, is seen to reduce to Boltzmann equation for the classical distribution of probability when the classical limit is considered.

In this letter we generalize the probability representation first to the case of \( N \) interacting particles, then to non relativistic quantum field theory. We consider a system of interacting oscillators which describe, when the limit to the continuum is performed, a self-interacting scalar field theory with generic self-interaction potential. We introduce the notion of MDF for the quantum state of such system and derive the evolution equation both for the discrete and continuous cases. The MDF is seen to be a distribution of probability with the same arguments used for 1-d quantum mechanics.

Interestingly similar ideas have been developed by Wetterich in \(^6\) in connection with the approach to equilibrium in non-equilibrium quantum field theories. There an evolution equation is found for a suitably defined time-dependent generating functional \(^7\). We establish the connection between our MDF and Wetterich’s generating functional in terms of a (functional-) Fourier transform.

In section \(^1\) we briefly review the probability representation of quantum mechanics for the simple case of a one-dimensional quadratic Hamiltonian. The derivation of the evolution equation for the MDF is performed in detail in a slightly different manner from its original derivation \(^3\), but more suitable for generalizations. In section \(^2\) we consider a quadratic Hamiltonian describing \( N \) interacting particles. We define the MDF and show that this is a well defined probability distribution. We then find the evolution equation. In section \(^3\) we consider a scalar field theory with self-interacting potential, which may be seen as the limit to the continuum of the previous model. We define the MDF which is now a probability distribution functional, and derive its exact evolution equation. In section \(^4\) we derive the evolution equations for the above mentioned systems, directly in terms of the Fourier transform of the MDF, the quantum characteristic function. We show then how our results may be connected to those found in \(^6\).

## 1 The Probability Representation of Quantum Mechanics

The MDF of a random variable \( X \) was introduced in \(^2\) as the Fourier transform of the quantum characteristic function \( \chi(k) = \langle e^{ik\hat{X}} \rangle \), to be

\[
w(X, t) = \frac{1}{2\pi} \int dk \ e^{-ikX} \langle e^{ik\hat{X}} \rangle ,
\]  

(1.1)

where \( \hat{X} \) is the operator associated to \( X \), \( \langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A}) \), and \( \hat{\rho} \) is the time-dependent density operator. It is shown in \(^2\) that \( w(X, t) \) is positive and normalized to unity, provided \( \hat{X} \) is an observable. This theorem may be easily proven taking for simplicity \( \hat{\rho} \) to be the density operator for a pure state. Then, evaluating the trace in (1.1) on eigenstates of the operator
\[ X, \text{ it can be verified that (1.1)} \]
\[ \text{yields } w(X, t) = \rho(X, X, t) \text{ which is positive and normalized to unity.} \]

We recall that the quantum characteristic function is, up to factors of \( i \), the generating function of the momenta of any order, for the probability distribution of the operator \( \hat{X} \). Hence it plays in quantum statistical mechanics the same rôle as the generating functionals for the Green’s functions in quantum field theory. In ref. \[ 4 \] \( X \) is taken to be a variable of the form
\[
X = \mu q + \nu p, \tag{1.2}
\]
where \( \mu, \nu \) are real parameters labelling different reference frames in the phase space. \( \mu \) is dimensionless, while \( [\nu] = [m^{-1}][t] \). Thus, \( X \) represents the position coordinate taking values in an ensemble of reference frames. For such a choice of \( X \) it was shown that there exists an invertible relation among the MDF and the density matrix, respectively in \[ 4 \] for the 1-d case, and in \[ 8 \] for the 2-d case. This relation was originally understood through the Wigner function: the MDF was expressed in terms of the Wigner function which is in turn related to the density matrix and viceversa. The evolution equation of the MDF was then found starting from an evolution equation for the Wigner function established by Moyal in \[ 9 \]. This intermediate step in terms of the Wigner function is not necessary. We can directly invert (1.1), when the variable \( X \) and the associated operator are given by (1.2). The evolution equation is then obtained (in the Schrödinger representation) by means of the Liouville equation for the density operator, in coordinate representation. In view of the subsequent generalization to \( N \) degrees of freedom and to field theory, let us derive these results in some detail for a one dimensional system. Equation (1.1) is explicitly written as
\[
w(X, \mu, \nu, t) = \frac{1}{2\pi} \int dk \int dZ e^{-ikX} < Z | \hat{\rho} e^{ik\hat{X}} | Z > = \frac{1}{2\pi} \int dk \int dZ \rho(Z, Z - k\nu \hbar) e^{-ik[X - \mu(Z - k\nu \hbar)/2]}. \tag{1.3}
\]
The MDF so defined is normalized with respect to the \( X \) variable: \( \int dX w(X, \mu, \nu, t) = 1 \).
Performing the change of variables \( Z' = Z, Z'' = Z - k\nu \hbar \) we may reexpress the MDF in the more convenient form:
\[
w(X, \mu, \nu, t) = \frac{1}{2\pi |\nu| \hbar} \int \rho(Z', Z'', t) \exp \left[ -i \frac{Z' - Z''}{\nu \hbar} \left( X - \mu \frac{Z + Z'}{2} \right) \right] dZ' dZ'' \tag{1.4}
\]
which can be inverted to
\[
\rho(X, X', t) = |\alpha| \int w(Y, \mu, \frac{X - X'}{h\alpha}) \exp \left[ i\alpha \left( Y - \mu \frac{X + X'}{2} \right) \right] d\mu dY \tag{1.5}
\]
where \( \alpha \) is a parameter with dimension of an inverse length. The density matrix is independent of \( \alpha \). In facts, using the homogeneity of the MDF, \( w(\alpha X, \alpha \mu, \alpha \nu) = |\alpha|^{-1} w(X, \mu, \nu) \), which is evident from the definition, (1.5) may be written as
\[
\rho(X, X', t) = \int w(Y, \mu, X - X') \exp \left[ \frac{i}{\hbar} \left( Y - \mu \frac{X + X'}{2} \right) \right] d\mu dY \tag{1.6}
\]
where the variables $Y, \mu$ have been rescaled by $\alpha$. It is important to note that, for (1.4) to be invertible, it is necessary that $X$ be a coordinate variable taking values in an ensemble of phase spaces; in other words, the specific choices $\mu = 1, \nu = 0$ or any other fixing of the parameters $\mu$ and $\nu$ would not allow to reconstruct the density matrix. Hence, the MDF contains the same amount of information on a quantum state as the density matrix, only if Eq. (1.2) is assumed.

We now address the problem of finding the evolution equation for the MDF, for Hamiltonians of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}). \quad (1.7)$$

Using the Liouville equation

$$\frac{\partial \hat{\rho}}{\partial t} + i\hbar [\hat{H}, \hat{\rho}] = 0 \quad (1.8)$$

and substituting into Eq. (1.4), we have

$$\dot{w}(X, \mu, \nu, t) = -\frac{i}{2\pi} \left| \nu \right| \hbar \int \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial Z^2} - \frac{\partial^2}{\partial Z'^2} \right) (V(Z) - V(Z')) \right] \rho(Z, Z', t)$$

$$\times \exp \left[ -i \frac{Z - Z'}{\nu} \left( X - \mu \frac{Z + Z'}{2} \right) \right] dZ dZ'. \quad (1.9)$$

Integrating by parts and assuming the density matrix to be zero at infinity, we finally have

$$\dot{w}(X, \mu, \nu, t) = \left\{ \frac{1}{m} \mu \frac{\partial}{\partial \nu} + i \hbar \left[ V \left( -\left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} - \frac{i \nu \hbar}{2} \frac{\partial}{\partial X} \right) \right] - V \left( -\left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + \frac{i \nu \hbar}{2} \frac{\partial}{\partial X} \right) \right\} w(X, \mu, \nu, t), \quad (1.10)$$

where the operator $\left( \frac{\partial}{\partial X} \right)^{-1}$ is so defined

$$(\frac{\partial}{\partial X})^{-1} \int f(Z) e^{g(Z)X} dZ = \int \frac{f(Z)}{g(Z)} e^{g(Z)X} dZ. \quad (1.11)$$

This equation, which plays the rôle of the Schrödinger equation in the alternative scheme just outlined, has been studied and solved for some quantum mechanical systems [10],[11]. The classical limit of (1.10) is easily seen to be

$$\dot{w}(X, \mu, \nu, t) = \left\{ \frac{\mu}{m} \frac{\partial}{\partial \nu} + \nu V' \left( -\left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \right) \frac{\partial}{\partial X} \right\} w(X, \mu, \nu, t), \quad (1.12)$$

where $V'$ is the derivative of the potential with respect to the argument. Equation (1.12) may be checked to be equivalent to Boltzmann equation for a classical distribution of probability $f(q, p, t)$,

$$\frac{\partial f}{\partial t} + \frac{p \partial f}{m \partial q} - \frac{\partial V}{\partial q} \frac{\partial f}{\partial p} = 0, \quad (1.13)$$

after performing the change of variables

$$w(X, \mu, \nu, t) = \frac{1}{2\pi} \int f(q, p, t) e^{ik(X-\mu q-\nu p)} dk \ dq \ dp \ ; \quad (1.14)$$

Hence, the classical and quantum evolution equations only differ by terms of higher order in $\hbar$. Moreover, for potentials quadratic in $\hat{q}$, higher order terms cancel out and the quantum
evolution equation coincides with the classical one. This leads to the remarkable result that there is no difference between the evolution of the distributions of probability for quantum and classical observables, when the system is described by a Hamiltonian quadratic in positions and momenta. For this kind of systems, the propagator is the same \[5\]. Of course, what makes the difference is the initial condition.

2 Generalization to \(N\) degrees of freedom

We consider now a system of \(N\) interacting particles sitting on the sites of a lattice (we choose it to be one-dimensional for simplicity). The Hamiltonian of the system is

\[
\hat{H} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m_i} + V(\hat{q}).
\]

(2.15)

We assume the masses to be equal to unity. We take the potential to be of the form:

\[
V(\hat{q}) = \sum_{i=1}^{N} \left( \frac{1}{2a^2}(\hat{q}_{i+1} - \hat{q}_i)^2 + U(\hat{q}_i) \right)
\]

(2.16)

where \(a\) is the lattice spacing and \(U(\hat{q}_i)\) is the part of the potential which depends only on the position of the \(i\)-th particle. This specification is not essential for the purposes of this section, but it will become necessary for understanding the limit to the continuum, which will be considered in next section. The quantum characteristic function for the \(N\) dimensional system may be defined as

\[
\chi(k_1, \ldots k_N) = \langle e^{i \sum_i k_i \hat{X}_i} \rangle,
\]

(2.17)

where \(\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A})\), and \(\hat{\rho}\) is the density operator of the system. (In case there is no interaction between different sites of the lattice the density operator may be factorized and the characteristic function is just the product \(\chi(k_1, \ldots k_N) = \prod_{i=1}^{N} \chi(k_i)\).)

Performing the Fourier transform of (2.17) the MDF is then given by

\[
w(X_\sigma, \mu_\sigma, \nu_\sigma, t) = \frac{1}{(2\pi)^N} \int dk_1 \ldots dk_N \ e^{-i \sum_i k_i X_i} < e^{i \sum_i k_i \hat{X}_i} > ;
\]

(2.18)

where \(\sigma\) is a collective index. It may be shown that this is a probability distribution, namely that it is positive definite and normalized, provided \(\hat{X}_i\) are observables. The proof goes along with the one-dimensional case. We first suppose that there is no interaction between different sites at some initial time \(t_0\) and we assume for simplicity that the system be in a pure state \(|\psi > = |\psi_1 > \otimes \ldots \otimes |\psi_N >\). Using the factorization property of the quantum characteristic function Eq. (2.18) may be seen to reduce to the product \(w(X_\sigma, \mu_\sigma, \nu_\sigma, t_0) = \prod_i \rho_i(X_i, X_i, t_0) = \rho(X, X, t_0)\), which is positive and normalized. Then Liouville equation guarantees that this result stays valid when the interaction is switched on. In analogy with the one-dimensional case we now introduce the variables

\[
X_i = \mu_i \hat{q}_i + \nu_i \hat{p}_i
\]

(2.19)
with $\hat{X}_i$ accordingly defined.

Introducing the notation $|Z_\sigma \rangle \equiv |Z_1 > \otimes \ldots \otimes |Z_N >$ we rewrite (2.18) as

$$w(X_\sigma, \mu_\sigma, \nu_\sigma, t) = \frac{1}{(2\pi\hbar)^N} \int \prod_{i=1}^{N}dk_i \sum_{i} e^{-i\sum_{j=1}^{N} k_j X_j} \langle Z_\sigma | \hat{\rho} e^{i\sum_{i=1}^{N} k_i \hat{X}_i} | Z_\sigma \rangle$$

$$= \frac{1}{(2\pi\hbar)^N} \int \prod_{i=1}^{N}dk_i \sum_{\rho} \rho(Z_\sigma, Z_\sigma - k_i \nu_i \hbar) e^{-i\sum_{j=1}^{N} \nu_j k_j \langle X_j - \mu_j(Z_j - k_j \nu_j \hbar/2) \rangle}, \tag{2.20}$$

where $\rho(Z_\sigma, Z'_\sigma) = \langle Z_\sigma | \hat{\rho} | Z'_\sigma >$. Performing the change of variables

$$Z''_i = Z_i \quad Z''_i = Z_i - k_i \nu_i \hbar \tag{2.21}$$

we have

$$w(X_\sigma, \mu_\sigma, \nu_\sigma, t) = \frac{1}{(2\pi\hbar)^N} \int \prod_{i=1}^{N}d\mu_i dY_i w(Y_\sigma, \mu_\sigma, X_\sigma - X'_\sigma) \exp \left[ -i \sum_{j=1}^{N} \frac{Z''_j - Z'_j}{\nu_j \hbar} \left( X_j - \mu_j \frac{Z_j + Z'_j}{2} \right) \right]. \tag{2.22}$$

which can be inverted to

$$\rho(X_\sigma, X'_\sigma, t) = \frac{1}{(2\pi\hbar)^N} \int \prod_{i=1}^{N}d\mu_i dY_i w(Y_\sigma, \mu_\sigma, X_\sigma - X'_\sigma) \exp \left[ -i \sum_{j=1}^{N} \frac{Z_i - Z'_i}{\nu_i \hbar} \left( X_i - \mu_i \frac{Z_i + Z'_i}{2} \right) \right]. \tag{2.23}$$

Once again, we recall that the inversion of (2.22) is made possible by choosing the variables $X_i$ as in (2.19). We now use the Liouville equation (1.8) to get

$$\dot{w}(X_\sigma, \mu_\sigma, \nu_\sigma, t) = -\frac{i}{(2\pi\hbar)^N} \int \prod_{i=1}^{N} \left( \frac{1}{\nu_i} dZ_i \sum_{\rho} \right) \left[ \sum_{j=1}^{N} -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial Z_j^2} - \frac{\partial^2}{\partial Z'_j^2} \right) \right]$$

$$+ \left( V(Z_\sigma) - V(Z'_\sigma) \right) \rho(Z_\sigma, X'_\sigma, t) \exp \left[ -i \sum_{i=1}^{N} \frac{Z_i - Z'_i}{\nu_i \hbar} \left( X_i - \mu_i \frac{Z_i + Z'_i}{2} \right) \right]. \tag{2.24}$$

Integrating by parts and assuming the density matrix to be zero at infinity, we finally have

$$\dot{w}(X_\sigma, \mu_\sigma, \nu_\sigma, t) = \left\{ \sum_{i=1}^{N} \mu_i \frac{\partial}{\partial \nu_i} + \frac{i}{\hbar} \left[ V \left( \left( \frac{\partial}{\partial X_\sigma} \right)^{-1} + \frac{i \nu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right) \right] \right\} w(X_\sigma, \mu_\sigma, \nu_\sigma, t), \tag{2.25}$$

where the inverse derivative is defined as in (1.13). We report for future convenience the term containing the potential when explicating the interaction between neighbours:

$$\left[ V \left( \left( \frac{\partial}{\partial X_\sigma} \right)^{-1} + \frac{i \nu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right) - V \left( \left( \frac{\partial}{\partial X_\sigma} \right)^{-1} + \frac{i \nu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right) \right]$$

$$= \left[ U \left( \left( \frac{\partial}{\partial X_\sigma} \right)^{-1} + \frac{i \nu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right) - U \left( \left( \frac{\partial}{\partial X_\sigma} \right)^{-1} + \frac{i \nu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right) \right]$$

$$- \frac{i \hbar}{\alpha^2} \sum_{i=1}^{N} \frac{\partial}{\partial \nu_i} \left[ \left( \frac{\partial}{\partial X_i} \right)^{-1} - 2 \frac{\partial}{\partial \mu_i} + \frac{\partial}{\partial \mu_{i-1}} \left( \frac{\partial}{\partial X_{i-1}} \right)^{-1} \frac{\partial}{\partial \mu_{i-1}} \right]. \tag{2.26}$$
When considering the classical limit we have

\[ w(X, \mu, \nu, t) = \left\{ \sum_{i=1}^{N} \mu_i \frac{\partial}{\partial \nu_i} + \nu \frac{1}{a^2} \left[ \frac{\partial^2}{\partial X_{i+1}} \left( -\frac{\partial}{\partial \mu_{i+1}} \right) \right] - 2 \frac{\partial}{\partial \mu_i} + \frac{\partial}{\partial X_i} \left( \frac{\partial}{\partial X_{i-1}} \right)^{-1} \frac{\partial}{\partial \mu_{i-1}} \right\} w(X, \mu, \nu, t), \tag{2.27} \]

where \( U_i \) is the derivative of the self-interaction potential with respect to the \( i \)-th variable. Equation (2.27) may be seen to be equivalent to the Boltzmann equation as in the one-dimensional case. Moreover, Hamiltonians which are quadratic in positions and momenta yield the same evolution equations for classical and quantum probability distributions.

### 3 Generalization to Field Theory

We now consider a scalar quantum field theory described by the Hamiltonian

\[ \hat{H} = \int d^d x \left[ \frac{1}{2} \hat{\pi}^2(x) + \frac{1}{2} \sum_{b=1}^{d} (\partial_b \hat{\phi}(x))^2 + U(\hat{\phi}(x)) \right]. \tag{3.28} \]

\( d \) is the spatial dimension, while \( U(\phi(x)) \) is the self-interacting potential, polynomial in the field \( \hat{\phi} \). The Hamiltonian (3.28) is easily seen to be obtained by the discrete Hamiltonian (2.15) by taking the limit to the continuum \((a \to 0)\) with the following rules:

\[
\begin{align*}
&a^{d/2} \hat{q}_i \to \hat{\phi}(x) \\
&\quad a^{-(d/2+1)}(\hat{q}_{i+1} - \hat{q}_i) \to \frac{\partial \hat{\phi}(x)}{\partial x_b}.
\end{align*}
\]

(3.29)

In analogy with the discrete case we introduce the field

\[ \hat{\Phi}(x) = \mu(x) \hat{\phi}(x) + \nu(x) \hat{\pi}(x) \tag{3.30} \]

where \( \mu(x) = \lim_{a \to 0} a^{-d/2} \mu_i \), and \( \nu(x) = \lim_{a \to 0} a^{-d/2} \nu_i \).

The quantum characteristic functional, which now will play the rôle of generating functional for correlation functions of the fields, may be defined as

\[ \chi(k(x)) = \langle e^{i \int d^d x \ k(x) \hat{\Phi}(x)} \rangle = \text{Tr} \left( \hat{\rho}(t) e^{i \int d^d x \ k(x) \hat{\Phi}(x)} \right). \tag{3.31} \]

The functional Fourier transform of \( \chi(k(x)) \), what we will call the marginal distribution functional (MDF), still defines a probability distribution. This can be understood by recognizing that it is the limit of the MDF for the discrete \( N \)-dimensional system considered in the previous section:

\[ w(\Phi(x), \mu(x), \nu(x), t) = \int Dk \ e^{-i \int k(x) \Phi(x) dx} \chi(k) = \lim_{a \to 0} w(X, \mu, \nu, t), \tag{3.32} \]
\[ \prod_{i} \frac{dk_i}{(2\pi)^N} \rightarrow \int \mathcal{D}k. \] Also, the density matrix functional may be defined as the limit of Eq. (2.23) to be
\[ \rho(\Phi, \Phi', t) = \int \mathcal{D}\mu \mathcal{D}\Psi \ w(\Psi, \mu, \Phi - \Phi') \exp \left\{ \frac{i}{\hbar} \int dy \left[ \Psi(y) - \mu(y) \left( \frac{\Phi(y) - \Phi'(y)}{2} \right) \right] \right\}. \quad (3.33) \]

Then, the evolution equation for the probability distribution functional is easily obtained by taking the limit of (2.25):
\[ \dot{w}(\Phi(x), \mu(x), \nu(x), t) = \left\{ \int d^dx \left[ \mu(x) \frac{\delta}{\delta \nu(x)} + 2\nu(x) \frac{\delta}{\delta \Phi(x)} \frac{\delta}{\delta \mu(x)} \right] \right\} w(\Phi(x), \mu(x), \nu(x), t). \quad (3.34) \]

The inverse functional derivative \( \left( \frac{\delta}{\delta \Phi(x)} \right)^{-1} \) is so defined:
\[ \left( \frac{\delta}{\delta \Phi(x)} \right)^{-1} \int \mathcal{D}k \ e^{-i \int k(y) \Phi(y) dy} = \int \mathcal{D}k \ \frac{i}{k(x)} e^{-i \int k(y) \Phi(y) dy}, \quad (3.35) \]
while the notation \( \Delta[f(x)] \) stands for \( f(x + \Delta x) - f(x) \). Performing an expansion in powers of \( \hbar \) the classical limit may be obtained as in the previous sections.

4 The Quantum Characteristic Function as a Generating Function

In this section we discuss the connection between the probability representation described above both for quantum mechanics and quantum field theory and a slightly different point of view developed in [6], where evolution equations are found for a suitably defined euclidean partition function. There are two main ingredients in our approach: one is the probabilistic interpretation for the distribution describing the observables, the other is the equivalence between the description based on the MDF and the conventional description based on the density matrix. The first aspect is guaranteed by the Glauber theorem which states that the Fourier transform of the quantum characteristic function associated to observables is a probability distribution. The second aspect, namely the invertibility of the MDF in terms of the density matrix, is achieved by introducing configuration space variables which take value in an ensemble of reference frames in phase space, each labelled by the two parameters, \( \mu, \nu \). Thus, the evolution equations which we have found ((1.10), (2.25), (3.34)), together with suitable initial conditions, completely characterize the state of the given quantum system. These equations assume a simpler form when their Fourier transform is performed. We have
\[ \chi(k, \mu, \nu, t) = \int dX e^{ikX} w(X, \mu, \nu, t), \quad (4.36) \]
with obvious generalizations to the $N$ dimensional case and to field theory. For the one-dimensional quantum systems considered in section 1, the Fourier transform of Eq.(1.10) yields an evolution equation for the quantum characteristic function itself:

$$\dot{\chi}(k, \mu, \nu, t) = \left\{ \frac{1}{m} \frac{\partial}{\partial \nu} + \frac{i}{\hbar} \left[ V \left( \frac{1}{ik} \frac{\partial}{\partial \mu} - k \hbar \right) - V \left( \frac{1}{ik} \frac{\partial}{\partial \mu} + k \hbar \right) \right] \right\} \chi(k, \mu, \nu, t). \quad (4.37)$$

For the $N$-dimensional quantum systems considered in section 2, the Fourier transform of Eq.(2.25) yields

$$\dot{\chi}(k_\sigma, \mu_\sigma, \nu_\sigma, t) = \left\{ \frac{1}{m_N} \sum_{i=1}^{N} \frac{\mu_i}{\partial \nu_i} + \frac{i}{\hbar} \left[ V \left( \frac{1}{ik_\sigma} \frac{\partial}{\partial \mu_\sigma} - \nu_\sigma k_\sigma \hbar \right) - V \left( \frac{1}{ik_\sigma} \frac{\partial}{\partial \mu_\sigma} + \nu_\sigma k_\sigma \hbar \right) \right] \right\} \chi(k_\sigma, \mu_\sigma, \nu_\sigma, t), \quad (4.38)$$

while Fourier transforming Eq.(3.34) we have, for quantum field theory,

$$\dot{\chi}(k(x), \mu(x), \nu(x), t) = \left\{ \int d^d x \left[ \frac{1}{m} \frac{\delta}{\delta \nu(x)} - 2i \hbar k(x) \nu(x) \Delta \left[ \frac{1}{ik(x)} \frac{\delta}{\delta \mu(x)} \right] \right] + \frac{i}{\hbar} \left[ U \left( \frac{1}{ik(x)} \frac{\delta}{\delta \mu(x)} - k(x) \nu(x) \hbar \right) \right] - \frac{i}{\hbar} \left[ U \left( \frac{1}{ik(x)} \frac{\delta}{\delta \mu(x)} + k(x) \nu(x) \hbar \right) \right] \right\} \chi(k(x), \mu(x), \nu(x), t). \quad (4.39)$$

Now the comparison with the results of [6] may be easily understood. Let us stick to quantum field theory for definiteness. The quantum characteristic functional which is a generating functional for correlation functions of the $\Phi$ field coincides with the generating functional considered in [6]

$$Z(\mu', \nu', t) = \text{Tr} \left( \hat{\rho}(t) \exp \left\{ \int d^d x \ \mu'(x) \hat{\phi}(x) + \nu'(x) \hat{\pi}(x) \right\} \right), \quad (4.40)$$

after rescaling the parameters $\mu$ and $\nu$ to $\mu' = ik \mu$ and $\nu' = ik \nu$ (of course, the same holds for quantum mechanics). Consequently, the evolution equations for the characteristic functional may be seen to equal to those found in [6] for the generating functional (4.40) provided the parameters $\mu'$ and $\nu'$ are rescaled as specified.

Going back to the initial remark of this section, we may conclude that, the quantum characteristic function and its evolution equation (or the generating functional in (4.40)) are more interesting from an operative point of view as they determine the correlation functions and their time evolution. On the other hand the introduction of the MDF is both relevant and necessary from a theoretical point of view. In facts it allows a unified description of classical and quantum phenomena in terms of probability distributions obeying different evolution equations. Also it justifies the introduction of the $X$ variable, as a variable taking values in an ensemble of reference frames (1.2), in view of the invertibility of the MDF in terms of the density matrix. This seems to us the profound motivation for introducing such a combination of phase space variables in the quantum characteristic functional and in the generating functional (4.40). We stress once again that $X$ and its field analogue $\Phi$ are, for each couple $(\mu, \nu)$, configuration space variables in the transformed reference frame labelled by $(\mu, \nu)$. 9
5 Conclusions

In this letter we have presented an extension of the probabilistic representation of quantum mechanics to quantum field theory. In this framework classical and quantum phenomena, both statistically described, only differ by the evolution equations of the distributions of probabilities for the relevant observables. Quantum observables are described by a distribution of probability, the MDF, and the time evolution by an integro-differential equation for the MDF. We recently addressed the problem of finding the Green’s function for the time-evolution equation of the MDF [10]. The problem was solved for quadratic Hamiltonians, and a characterization of such a propagator in terms of the time–dependent invariants of the system was found. This propagator represents the transition probability of the system from a quantum state to another. Thus, a generalization to quantum field theory would be interesting in our opinion, and is presently under consideration. Another promising application of the probabilistic point of view is suggested in [6] where it is used to study the approach to equilibrium of non equilibrium quantum field theories. An extension to relativistic quantum field theory would be also interesting, though it poses problems of interpretation which are not understood at the moment.

REFERENCES

[1] E. Wigner, Phys. Rev. 40, 749 (1932); K. Husimi, Proc. Phys. Math. Soc. Jpn. 23, 264 (1940); R. J. Glauber, Phys. Rev. Lett. 10, 84 (1963); E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963).

[2] K. Cahill and R. Glauber, Phys. Rev. 177, 1882 (1969).

[3] S. Mancini, V. I. Man’ko, and P. Tombesi, Phys. Lett. A213, 1 (1996); Found. Phys. 27, 801 (1997).

[4] S. Mancini, V. I. Man’ko, and P. Tombesi, Quantum Semiclass. Opt. 7, 615 (1995).

[5] O. V. Man’ko and V. I. Man’ko J. Russ. Laser Research 18, 407 (1997).

[6] C. Wetterich Phys. Rev. E56, 2687 (1997).

[7] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford U. P., Oxford 1989).

[8] G. D’Ariano, S. Mancini, V. I Man’ko, and P. Tombesi, Quantum Semiclass. Opt. 8, 1017 (1996).

[9] J. E. Moyal, Proc. Cambridge Phylos. Soc. 45, 99 (1949).

[10] V. I. Man’ko, L. Rosa, and P. Vitale, Phys. Rev. A 57, 3291 (1998).

[11] V. I. Man’ko, in Symmetry in Science IX, pag. 215, edited by B. Gruber and M. Ramek (Plenum, New York, 1997).