ON THE MULTIPLICITY OF THE HYPERELLIPTIC INTEGRALS

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Abstract. Let \( I(t) = \oint_{\delta(t)} \omega \) be an Abelian integral, where \( H = y^2 - x^{n+1} + P(x) \) is a hyperelliptic polynomial of Morse type, \( \delta(t) \) a horizontal family of cycles in the curves \( \{ H = t \} \), and \( \omega \) a polynomial 1-form in the variables \( x \) and \( y \). We provide an upper bound on the multiplicity of \( I(t) \), away from the critical values of \( H \). Namely: \( \text{ord } I(t) \leq n - 1 + \frac{\deg \omega}{2} \) if \( \deg \omega < \deg H = n + 1 \). The reasoning goes as follows: we consider the analytic curve parameterized by the integrals along \( \delta(t) \) of the \( n \) “Petrov” forms of \( H \) (polynomial 1-forms that freely generate the module of relative cohomology of \( H \)), and interpret the multiplicity of \( I(t) \) as the order of contact of \( \gamma(t) \) and a linear hyperplane of \( \mathbb{C}^n \). Using the Picard-Fuchs system satisfied by \( \gamma(t) \), we establish an algebraic identity involving the wronskian determinant of the integrals of the original form \( \omega \) along a basis of the homology of the generic fiber of \( H \). The latter wronskian is analyzed through this identity, which yields the estimate on the multiplicity of \( I(t) \). Still, in some cases, related to the geometry at infinity of the curves \( \{ H = t \} \subseteq \mathbb{C}^2 \), the wronskian occurs to be zero identically. In this alternative we show how to adapt the argument to a system of smaller rank, and get a nontrivial wronskian. For a form \( \omega \) of arbitrary degree, we are led to estimating the order of contact between \( \gamma(t) \) and a suitable algebraic hypersurface in \( \mathbb{C}^{n+1} \). We observe that \( \text{ord } I(t) \) grows like an affine function with respect to \( \deg \omega \).

1. Introduction

Consider a complex bivariate polynomial \( H(x, y) \in \mathbb{C}[x, y] \). It is well known that the polynomial mapping \( H : \mathbb{C}^2 \to \mathbb{C} \) defines a locally trivial differentiable fibration over the complement of a finite subset of
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Under some restrictions on the principal part of $H$, this set limits to $\text{crit}(H)$, the set of critical values of $H$ (see [3], [5]). One can then consider the homology bundle $\cup_t(H_1\{H = t\}, \mathbb{Z}) \to \mathbb{C} \setminus \text{crit}(H)$, equipped with the Gauss-Manin connection. Take a class $\delta(t)$ in the homology group $H_1\{H = t\}, \mathbb{Z}$ of a generic fiber of $H$. As the base of the homology bundle is 1-dimensional, the connection is flat, and the parallel transport of $\delta$ depends only on the homotopy class of the path in $\mathbb{C} \setminus \text{crit}(H)$. The transport of the homology class $\delta(t)$ along a loop that encircles a critical value of $H$ results in a nontrivial outcome, due to the action of the monodromy on $\delta$. Let $\omega$ be a polynomial 1-form in the variables $x$ and $y$. Its restriction on any fiber of $H$ is a closed form, therefore the integral of $\omega$ on a cycle lying in a regular level curve of $H$ depends only on the homology class of this cycle. Consider the complete Abelian integral $I(t) = \oint_{\delta(t)} \omega$. This function admits an analytic extension in the complement of $\text{crit}(H)$. We refer to [1], [13] for a detailed survey.

Let $t_0 \in \mathbb{C}$ be a regular value of $H$. One can ask for an estimate on the multiplicity of $I(t)$ at $t_0$. The result is expected to depend on two parameters, namely the degree of $H$ and the degree of $\omega$. The number of parameters can be reduced by looking first at the case $\deg \omega < \deg H$. This case is the most interesting regarding the connection with the infinitesimal Hilbert 16th problem, that takes place in the real setting. Assume $H$ has real coefficients. Consider the Hamiltonian distribution $dH$, and the one-parameter perturbation $dH + \epsilon \omega$ by an arbitrary real polynomial 1-form $\omega$. The first order term in the Taylor expansion at $\epsilon = 0$ of the corresponding displacement function $d(t, \epsilon)$ is an integral $I(t)$ of $\omega$ along an oval in a level curve of $H$. Under the assumption $\deg \omega < \deg H$, it is proved by Yu. Ilyashenko (cf with [8]) that: $I(t) \equiv 0$ if and only if $\omega$ is exact, so that the perturbation is still a Hamiltonian distribution. In the case when $I(t) \not\equiv 0$, the multiplicity of $I(t)$ at a point $t_0$ provides an upper bound for the cyclicity of the displacement function $d(t, \epsilon)$ at $(t_0, 0)$. Thus, the vanishing of the Abelian integral is relevant to the number of limit cycles born by small perturbation of the Hamiltonian distribution $dH$.

For polynomials $H$ with generic principal part ($H$ regular at infinity), an answer on the order of $I(t)$ is given by P. Mardesic in [9]: a step towards the multiplicity of the Abelian integrals consists in measuring the multiplicity of their Wronskian determinant, which is a globally
univalued function on \( \mathbb{C}P^1 \), hence rational, with poles at the critical values of \( H \) and possibly at infinity.

We focus on the Abelian integrals performed on level curves of hyperelliptic polynomials. We establish a relation between the Wronskian and a polynomial that we build up from the Picard-Fuchs system. As the Picard-Fuchs system reflects the topology of the level sets of \( H \), our approach depends whether \( \text{deg} \, H \) is even or odd, still our estimate on the multiplicity of \( I(t) \) is always quadratic with respect to \( \text{deg} \, H \).

We finally show that, for a fixed hyperelliptic Hamiltonian, the growth of the multiplicity of \( I(t) \) is linear with respect to \( \text{deg} \, \omega \).

2. Preliminary observations

We begin by recalling a result about flatness of solutions of a linear differential system. Consider a system \( dx = \Omega x \) of order \( n \), whose coefficient matrix \( \Omega \) is meromorphic on \( \mathbb{C}P^1 \). Denote by \( t_1, \ldots, t_s \) the poles of \( \Omega \). Fix a point \( t_0 \), distinct from the poles, and consider a solution \( \gamma(t) \subseteq \mathbb{C}^n \), analytic in a neighbourhood of \( t_0 \). Take a linear hyperplane \( \{ h = \sum_{i=1}^n c_ix_i = 0 \} \subseteq \mathbb{C}^n \). If this hyperplane does not contain the solution \( \gamma \), then (cf [10]):

**Theorem 1.**

\[
\text{ord}_{t=t_0} (h \circ \gamma)(t) \leq n - 1 + \frac{n(n-1)}{2} \left( \sum_{i=1}^s (-\text{ord}_{t_i} \Omega) - 2 \right)
\]

where \( \text{ord}_{t_i} \Omega \) is the minimum order of the pole \( t_i \) over the entries of \( \Omega \).

We give here a simplified algorithm of the proof: write the system in the affine chart \( t \) in the form \( \dot{x} = A(t)x \), for a polynomial matrix \( A \) and scalar polynomial \( P \). Replace the derivation \( \frac{\partial}{\partial t} \) by \( D = P(t) \frac{\partial}{\partial t} \). Then the curve \( \gamma \) satisfies: \( D\gamma = A\gamma \). Due to the linearity, we can write \( y(t) = (h \circ \gamma)(t) \) as the product of the row matrix \( q_0 = (c_1, \ldots, c_n) \) by the column matrix \( \gamma \): \( y(t) = q_0 \cdot \gamma(t) \). The successive derivatives of \( y \) with respect to \( D \) can be written in a similar way: \( D^k y(t) = q_k(t) \cdot \gamma(t) \), where the row vectors \( q_k \) have polynomial coefficients and are constructed inductively by: \( q_{k+1} = Dq_k + q_k A \). We observe that the sequence of \( \mathbb{C}(t) \)-vector spaces \( V_k \subseteq \mathbb{C}(t)^n \) spanned by the vectors \( q_0, q_1, \ldots, q_k \), is strictly increasing (before stabilizing), hence we may extract from the matrix \( \Sigma \) with rows \( q_0, \ldots, q_{n-1} \), a nondegenerate minor \( \Delta \) of rank...
$l \leq n$, such that any vector $q_k$ decomposes according to the Cramer rule:

$$q_k(t) = \sum_{i=0}^{l-1} \frac{p_{ik}(t)}{\Delta(t)} q_i(t), \quad k \geq l$$

with polynomial coefficients $p_{ik}(t)$. This shows that the function $y$ is a solution of an infinite sequence of linear differential equations of the form:

$$(1) \quad \Delta \cdot D^k y = \sum_{i=0}^{l-1} p_{ik}(t) D^i y, \quad k \geq l$$

Then, by deriving in an appropriate way each of these relations, one arrives at the key-assertion:

$$\text{ord}_{t_0} y \leq l - 1 + \text{ord}_{t_0} \Delta$$

Thus, the flatness of a particular solution is correlated to the multiplicity of a polynomial constructed from the system.

One can derive an analytic version of this assertion by complementing the solution $\gamma$ by $n-1$ vector-solutions $\Gamma_2, \ldots, \Gamma_{n-1}$, so as to obtain a fundamental matrix in a simply connected domain around $t_0$. In particular: $\det(\gamma, \Gamma_2, \ldots, \Gamma_n)(t)$ does not vanish in this domain.

We restrict these solutions on the hyperplane $\{h = 0\}$ and set: $y_1 = y = (h \circ \gamma), y_i = (h \circ \Gamma_i), \quad i = 2, \ldots, n$. Let $l \leq n$ be the maximum number of independent functions among $y_1, \ldots, y_n$. Their Wronskian determinant $W = W(y_1, \ldots, y_l)$ is analytic and does not vanish identically around $t_0$. Expand $W$ with respect to any of its columns, it follows that:

$$\text{ord}_{t_0} W \geq \min_{k=0, \ldots, l-1} \{\text{ord}_{t_0} y_i^{(k)}\} + \text{ord}_{t_0} D_k$$

where $D_k$ is the minor corresponding to the element $y_i^{(k)}$. Hence: $\text{ord}_{t_0} W \geq \text{ord}_{t_0} y_i - (n - 1)$, for any $i = 1, \ldots, n$. So:

$$\text{ord}_{t_0} (h \circ \gamma) \leq n - 1 + \text{ord}_{t_0} W$$

Naturally, the order of vanishing of $W$ does not depend on the particular choice of fundamental system. Besides, one arrives at the same conclusion by forming the Wronskian determinant $W_D(y_1, \ldots, y_l)$ with respect to the derivation $D = P(t) \frac{\partial}{\partial t}$, since $W_D = P^{(l-1)/2} \cdot W$, and $D$ is not singular at $t_0$ (meaning that $P(t_0) \neq 0$).
Note that, like in the algebraic situation, one can interpret $W_D$ as the principal coefficient of a linear differential equation of order $l \leq n$ satisfied by $y_1, \ldots, y_l$:

$$W_D(y_1, \ldots, y_l)D^l y + a_{l-1}(t)D^{l-1} y + \ldots + a_0(t)y = 0$$

with coefficients $a_i(t)$ analytic in a neighbourhood of $t_0$. The method leading to such an equation is standard. For any linear combination $y$ of $y_1, \ldots, y_l$, the following Wronskian determinant of size $l + 1$ is zero identically:

$$\begin{vmatrix}
    y_1 & \ldots & y_l & y \\
    Dy_1 & \ldots & D^{l}y_l & D^{l}y \\
    \vdots & \vdots & \vdots & \vdots \\
    D^{l-1}y_1 & \ldots & D^{l-1}y_l & D^{l-1}y \\
    D^{l}y_1 & \ldots & D^{l}y_l & D^{l}y \\
\end{vmatrix}$$

Expanding this determinant with respect to its last column gives the equation. This is the analytic analogue of the $l$th order equation in the sequence (1). Both of them admit the same solutions.

Suppose that $\Sigma$ is a nondegenerate matrix, that is, its determinant $\Delta(t)$ is not the null polynomial. Let $P$ be a fundamental matrix of solutions of the system $Dx = Ax$. From the construction, we obtain immediately the following matrix relation:

**Lemma 1.** $\Sigma \cdot P = W_D(y_1, \ldots, y_n)$, where $W_D(y_1, \ldots, y_n)$ is the Wronski matrix of $y_1, \ldots, y_n$, computed with the derivation $D$.

**Remark 1.** The matrix $\Sigma$ defines a meromorphic gauge equivalence between the original system and the companion system of the equation (2).

This yields the relation between determinants:

$$\det \Sigma \cdot \det P = W_D(y_1, \ldots, y_n) = P(t)^{\frac{n(n-1)}{2}} \cdot W(y_1, \ldots, y_n)$$

$W(y_1, \ldots, y_n)$ being the usual Wronskian $W_{\frac{\partial}{\partial t}}(y_1, \ldots, y_n)$. 
Now, at the non-singular point \( t_0 \) of the system, both \( P \) and \( \det \mathcal{P} \) are nonzero, so that the order at \( t_0 \) of the analytic function \( W(y_1, \ldots, y_n) \) is exactly the order at \( t_0 \) of the polynomial determinant \( \det \Sigma \).

3. Multiplicity of the integrals

3.1. Petrov forms and Picard-Fuchs system. Consider a bi-variate hyperelliptic polynomial \( H \in C[x, y] \), \( H = y^2 - x^{n+1} + \mathcal{P}(x) \), with \( \deg H = n - 1 \). Hyperelliptic polynomials are examples of semi quasi-homogeneous polynomials. Recall that a polynomial \( H \) is said to be semi quasi-homogeneous if the following holds: the variables \( x \) and \( y \) being endowed with weights \( w_x \) and \( w_y \) (so that a monomial \( x^\alpha y^\beta \) has weighted degree \( \alpha w_x + \beta w_y \)), \( H \) decomposes as a sum \( H^* + \mathcal{P} \), and the highest weighted-degree part \( H^* \) possesses an isolated singularity at the origin. Moreover, a semi quasi-homogeneous polynomial has only isolated singularities. In the sequel, the notation “\( \deg \)” will stand for the weighted degree. For \( H \) hyperelliptic, \( \deg H = n + 1 \), with \( w_x = 1 \), \( w_y = (n + 1)/2 \). The weighted degree extends to polynomial 1-forms: for \( \omega = P(x, y)dx + Q(x, y)dy \), \( \deg \omega \) is the maximum \( \max(\deg P + w_x, \deg Q + w_y) \). The symbol \( \Lambda^k \) will designate the \( C[x, y] \)-module of polynomial \( k \)-forms on \( C^2 \).

Consider the quotient \( \mathcal{P}_H = \frac{\Lambda^1}{\mathcal{P} + dx + dy} \). It is a module over the ring of polynomials in one indeterminate. Note that the integral of a 1-form in \( \Lambda^1 \) depends only on its class in \( \mathcal{P}_H \). In addition, working in the Petrov module of \( H \) enables to exhibit a finite number of privileged 1-forms, that we will call the Petrov forms: indeed, \( \mathcal{P}_H \) is freely generated by the monomial 1-forms \( \omega_1 = ydx \), \( \omega_2 = xydx \), \ldots, \( \omega_n = x^{n-1}ydx \). Moreover, the class of any 1-form in \( \mathcal{P}_H \) decomposes as a sum: \( p_1(t)\omega_1 + \ldots + p_n(t)\omega_n \), with the following estimates on the degrees of the polynomials \( p_i \):

\[
\deg p_i \leq \frac{\deg \omega - \deg \omega_i}{\deg H}
\]

These assertions belong to a general theorem due to L. Gavrilov (6), where the Petrov module of any semi quasi-homogeneous polynomial \( H \) is described. The number of Petrov forms is the global Milnor number of \( H \).

Consider a hyperelliptic integral \( \oint_{\delta(t)} \omega \), in a neighbourhood of a regular value \( t_0 \) of the Hamiltonian \( H \). It becomes natural to consider the
germ of analytic curve $\gamma(t) = (\oint_{\delta} \omega_1, \oint_{\delta} \omega_2, \ldots, \oint_{\delta} \omega_n)$ parameterized by the integrals of the Petrov forms. We shall start with forms of small degree, that is $\deg \omega \leq n$, and study the behaviour of the multiplicity of the integral with respect to $n$. This restriction on the degree implies that $\omega$ is a linear combination of the Petrov forms, with constant coefficients: $\omega = \sum_{i=1}^{n} c_i \omega_i$, $c_i \in \mathbb{C}$. Therefore, the question amounts to estimating the order of contact at the point $t = t_0$ of the curve $\gamma(t)$ and of the linear hyperplane $\{\sum_{i=0}^{n} c_i x_i = 0\}$ whose coefficients are prescribed by the decomposition of $\omega$.

In order to apply the argument presented in Section 2, we have to interpret $\gamma$ as a solution of a linear differential system. We recall the procedure described by S. Yakovenko in [13, Lecture 2]. For any $i = 1, \ldots, n$, divide the 2-forms $Hd\omega_i$ by $dH$. Then apply the Gelfand-Leray formula and decompose the Gelfand-Leray residue in the Petrov module of $H$. For a hyperelliptic Hamiltonian, it is clear that the $\mathbb{C}$-vector space of relative 2-forms $\Lambda^2 \Lambda^1$ is spanned by the differentials of the Petrov forms $d\omega_1, \ldots, d\omega_n$. Whence the decomposition:

\begin{equation}
H \cdot d\omega_i = dH \wedge \eta_i + \sum_{j=1}^{n} a_{ij} d\omega_j, \ a_{ij} \in \mathbb{C}, \ \eta_i \in \Lambda^1.
\end{equation}

Now, by the Gelfand-Leray formula,

\begin{equation}
t \frac{d}{dt} \oint_{\delta} \omega_i - \sum_{j=1}^{n} a_{ij} \frac{d}{dt} \oint_{\delta} \omega_j = -\oint_{\delta} \eta_i,
\end{equation}

and this relation does not depend on the cycle of integration.

In order to obtain the system, one decomposes the residues $\eta_i$ in $\mathcal{P}_H$. So, an estimate on the degree of $\eta_i$ is required, which can be quite cumbersome when starting from a general Hamiltonian. Yet, in the hyperelliptic case, (4) is completely explicit (cf with [12]), and one sees immediately that: for any $i$, $\eta_i = \sum_{j=1}^{n} b_{ij} \omega_j$, $b_{ij} \in \mathbb{C}$. Then (5) appears as the expanded form of the linear system:

\begin{equation}
(tE - A)\dot{x} = Bx
\end{equation}

where $A = (a_{ij})$ and $B = (b_{ij})$ are constant matrices, and $E$ is the ($n \times n$) Identity matrix.

We can write it as well as a system with rational coefficients: $\dot{x} = C(t)F(t)x$, where the polynomial matrix $C(t)$, obtained as $C(t) = \text{Ad}(tE -$
A) \cdot B, has degree $n - 1$, and the scalar polynomial $P(t) = \det(tE - A)$ has degree $n$. The polynomial $P$ can be explicited: evaluation of the relation (4) at a critical point $(x_*, y_*)$ of $H$ shows that the corresponding critical value $t_* = H(x_*, y_*)$ is an eigenvalue of $A$. If the critical values of $H$ are assumed pairwise distinct, then: $P(t) = (t - t_1) \ldots (t - t_n)$. Thus, the singular points of the system are the critical values of $H$ and the point at infinity. All of them are Fuchsian.

We can apply Theorem 1 and get the following estimate on the multiplicity at a zero of the integral: $\text{ord}_{t_0} \oint_{\delta} \omega \leq n - 1 + \frac{n(n - 1)^2}{2}$. We are going to show how to improve this bound, applying (3).

3.2. Main result. We now formulate the theorem. We impose an additional requirement on the hyperelliptic Hamiltonian: $H$ has to be of Morse type, that is, with nondegenerate critical points as well as distinct critical values.

Theorem 2. Let $H$ be a hyperelliptic polynomial $H = y^2 - x^{n+1} + \overline{H}(x)$, where $\overline{H}$ is a polynomial of degree $n - 1$, of Morse type. Let \{\omega_1, \ldots, \omega_n\} be the set of monomial Petrov forms associated to $H$, and let $\omega = \sum_{i=1}^{n} c_i \omega_i$, $c_i \in \mathbb{C}$ be an arbitrary linear combination. Consider the Abelian integral of $\omega$ along a horizontal section $\delta(t)$ of the homology bundle. Let $t_0$ be a regular value of $H$. If $\oint_{\delta(t)} \omega \neq 0$, then:

$$\text{ord}_{t_0} \left( \oint_{\delta(t)} \omega \right) \leq n - 1 + \frac{n(n - 1)}{2}$$

The rest of this section is devoted to the proof of Theorem 2.

Recall that a fundamental matrix of the Picard-Fuchs system is obtained by integrating some $n$ suitable polynomial 1-forms $\Omega_1, \ldots, \Omega_n$ over any basis of the homology groups $H_1(\{H = t\}, \mathbb{Z})$:

$$\mathcal{P} = \begin{pmatrix} \oint_{\delta} \Omega_1 & \cdots & \oint_{\delta} \Omega_n \\ \vdots & \vdots & \vdots \\ \oint_{\delta} \Omega_n & \cdots & \oint_{\delta} \Omega_n \end{pmatrix}$$

As explained in [13, Lecture 2], the determinant of a fundamental system of solutions has to be a polynomial, divisible by $(t - t_1) \ldots (t - t_n)$. Its actual degree depends on the choice of the integrands. It is shown by L. Gavrilov in [6] and D. Novikov in [11] that one can plug
the Petrov forms into the period matrix $\mathcal{P}$ and get $\det \mathcal{P} = c \cdot (t - t_1) \ldots (t - t_n) = P(t)$, with a nonzero constant $c$.

Next, we form the vectors $q_0, \ldots, q_{n-1}$, $q_0 = (c_1, \ldots, c_n)$, $q_{k+1} = Dq_k + q_k C(t)$, $D = P(t) \frac{\partial}{\partial t}$, and collect them in a matrix $\Sigma$. Lemma 1 says:

$$\Sigma \cdot \mathcal{P} = W_D \left( \oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega \right)$$

which gives:

$$(6) \quad \Delta \cdot (t - t_1) \ldots (t - t_n) = P^{n(n-1)/2} \cdot W \left( \oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega \right)$$

The disadvantage of this formula is that it does not resist possible degeneracy of the matrix $\Sigma$: the determinant $\Delta = \det \Sigma$ is a polynomial in the variable $t$, whose coefficients are homogeneous polynomials with respect to the components of $q_0$.

$$\Delta_{q_0}(t) = P_0(q_0) + P_1(q_0) \cdot t + \ldots + P_D(q_0) \cdot t^D$$

where $D$ is the maximum possible degree for $\Delta$, achieved for generic $q_0$. One cannot a priori guarantee that the algebraic subset

$$S = \{ q_0 \in \mathbb{C}^n : P_i(q_0) = 0, i = 0, \ldots, D \}$$

is reduced to zero.

We are now going to analyze on what conditions the equality (6) makes sense. It involves the geometry at infinity of the hyperelliptic affine curves $\{ H = t \} \subseteq \mathbb{C}^2$. Suppose that for the form $\omega = \sum c_i \omega_i$, the Wronskian $W(\oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega)$ vanishes identically as a function of $t$. This means that one can find a cycle $\sigma$, complex combination of $\delta_1, \ldots, \delta_n$, such that the integral $\oint_\sigma \omega$ is identically zero.

**Lemma 2.** If an Abelian integral $\oint_{\sigma(t)} \omega$ is zero identically, then $\sigma(t)$ belongs to the kernel of the intersection form on $H_1(\{ H = t \}, \mathcal{C})$.

**Proof.** Assume on the contrary, that $\sigma$ has a nonzero intersection number with a cycle from $H_1(\{ H = t \}, \mathcal{C})$, while $\oint_\sigma \omega$ vanishes identically. The semi quasi-homogeneity property implies that $H$ defines a trivial fibration at infinity, and the homology of a regular fiber can be generated by a basis of $n$ vanishing cycles $v_i$ ($v_i$ contracts to a point
when $t$ approaches $t_i$). Necessarily, $\sigma$ intersects one of the vanishing cycles - say $v_1$: $(\sigma, v_1) \neq 0$. Continue analytically the integral $\oint_{v_1} \omega$ along a loop around $t_1$: the Picard-Lefschetz formula states that the monodromy changes $\sigma$ into $\sigma - (\sigma, v_1)v_1$. On the level of the integral: $0 = 0 - (\sigma, v_1)\oint_{v_1} \omega$, hence $\oint_{v_1} \omega$ is zero.

Moreover, the assumptions of hyperellipticity and Morse type imply that one can produce a basis $\{v_1, \ldots, v_n\}$ of vanishing cycles in which any two consecutive cycles intersect: $(v_i, v_{i+1}) = \pm 1$ (cf [7]). One proceeds inductively with the rest of the critical values $t_2, \ldots, t_n$: every integral of $\omega$ along $v_2, \ldots, v_n$ vanishes identically as well. This means that the restrictions of $\omega$ on any generic fiber of $H$ are exact. We can now apply L. Gavrilov’s result [6, Theorem 1.2] and deduce the global statement: the form $\omega$ has to be exact, hence zero in the Petrov module. But this is clearly impossible since $\omega$ is a combination of $\mathbb{C}[t]$-independent forms.

Consequently, if the integral of $\omega$ vanishes along $\sigma$, this means that $\sigma$ becomes homologous to zero when the affine fiber is embedded in its normalization. That is, $\sigma$ lies in the kernel of the morphism $i_*: H_1(\Gamma, \mathbb{Z}) \to H_1(\overline{\Gamma}, \mathbb{Z})$, denoting the affine curve $\{H = t\} \subset \mathbb{C}^2$ by $\Gamma$ and its normalized curve in $\mathbb{C}P^2$ by $\overline{\Gamma}$.

**Lemma 3.** If $n$ is even, then $S$ is limited to $\{0\}$.

**Proof.** The projection $\Pi : \overline{\Gamma} \to \mathbb{C}P^1$, $(x, y) \mapsto x$ is a double ramified covering of $\mathbb{C}P^1$. From the affine equation $y^2 = \Pi_{i=1}^{n+1}(x - x_i(t))$, one finds $n + 1$ ramification points of $\Pi$ in the complex plane, hence the total number of ramification points of $\Pi$ is $n + 1$ or $n + 2$. On the other hand, using the Riemann-Hurwitz formula, the number of ramification points of $\Pi$ is $2g_{\overline{\Gamma}} + 2$. This shows that the genus $g_{\overline{\Gamma}}$ is equal to $[n/2]$. Therefore, if $n$ is even, the homology group of $\overline{\Gamma}$ has rank $2g_{\overline{\Gamma}} = n$, and $i_*$ is an isomorphism. In this case, there is a single point at infinity on $\overline{\Gamma}$ above $\infty \in \mathbb{C}P^1$ and $\sigma$ is zero in the homology of the affine level curve $\{H = t\}$. This means that no relation can occur between the integrals $\oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega$, unless $\omega$ is zero. \(\square\)

For even $n$, we can carry out the analysis further. The Wronskian $W = W(\oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega)$ is a rational function, so the sum of its orders at all points of $\mathbb{C}P^1$ equals 0. As a consequence, the order at one of
its zeros can be deduced from the order at its poles and at the point at infinity:

$$\text{ord}_{t_0} W \leq -\text{ord}_{\infty} W - \sum_{i=1}^{n} \text{ord}_{t_i} W$$

From (6), we get:

$$\text{ord}_{\infty} W = \text{ord}_{\infty} \Delta - n + \frac{n^2(n-1)}{2}$$

One gets easily an upper bound on $\deg \Delta$: from the inductive construction the degree of each component of a vector $q_k$ is no larger than $k(n-1)$, this yields: $\deg \Delta \leq \frac{n(n-1)^2}{2}$. In the right hand side of (7), the cubic terms cancel out each other, so that we get the estimate:

$$\text{ord}_{\infty} W \geq \frac{n^2-3n}{2}$$. This shows in particular that $\infty$ is a zero of $W$.

As for the order of the $W$ on the finite singularities, we reproduce an argument due to P. Mardesic ([9]): the critical points of $H$ are Morse, which allows to fix the Jordan structure of the monodromy matrix at $t_i$, by choosing an adapted basis of cycles. This imposes the structure of the integrals in a neighbourhood of $t_i$: they are all analytic at $t_i$, except one of them that undergoes ramification. The pole of the Wronskian at $t_i$ may only result from the derivation of this integral. The estimate follows automatically: $\text{ord}_{t_i} W \geq 2 - n$. The contribution of the poles is $\sum_{i=1}^{n} \text{ord}_{t_i} W \geq 2n - n^2$. Therefore, we have obtained the following upper bound:

$$\text{ord}_{t_0} W \leq \sum_{t_0 \in \mathbb{CP}^1, t_0 \neq \infty, t_0 \neq t_i} \text{ord}_{t_0} W \leq \frac{n(n-1)}{2}$$

which proves Theorem 2 for even $n$.

We now return to the case of odd $n$. The homology group $H_1(\Gamma, \mathbb{Z})$ has rank $2g_{\Gamma} = n - 1$, and $\ker \ i_*$ is generated by one cycle that we will call $\delta_\infty$. The forms $\omega$ that annihilate the Wronskian $W$ are those with zero integral along the cycle $\delta_\infty$. In order to describe this subspace of forms, we prove a dependence relation among Abelian integrals:

**Lemma 4.** The complex vector space generated by the residues of the Petrov forms at infinity has dimension 2, that is:

$$\dim_{\mathbb{C}} \left( \oint_{\delta_\infty} \omega_1, \ldots, \oint_{\delta_\infty} \omega_n \right) = 2$$
Proof. In order to estimate the residues $\rho(\omega_i)$ of the forms $\omega_i = x_i^{-1} y dx$, $i = 1, \ldots, n - 1$, at the point at infinity on the curve $y^2 - x^{n+1} + H(x) = t = 0$, ($[0 : 1 : 0] \in \mathbb{CP}^2$), we pass to the chart $u = 1/x$ and, expressing $y$ as a function of $x$: $y = \pm(x^{n+1} + H(x) - t)^{1/2}$, we get meromorphic 1-forms at $u = 0$:

$$\omega_i = \left(\frac{1}{u}\right)^{i-1} \cdot \left(\frac{1}{u}\right)^2 \cdot \left(\frac{1}{u}\right)^{(n+1)/2} \cdot (1 + R(u) + tu^{n+1})^{1/2} du$$

where $R$ is a polynomial, $\deg R = n+1$, $R(0) = 0$. We have to compute the coefficient of $1/u$. The Taylor expansion of the square root gives: for $i < \frac{n+1}{2}$, $\rho(\omega_i)$ is a constant with respect to $t$, and for $i \geq \frac{n+1}{2}$, $\rho(\omega_i)$ is a polynomial of degree 1. This proves that the space of residues has dimension 2 over $\mathbb{C}$, and is generated by any pair $\{\rho(\omega_i), \rho(\omega_{i+(n+1)/2})\}$, $i < \frac{n+1}{2}$. □

Remark 2. In the case of a Hamiltonian of even degree (that is, for odd $n$), we detect solutions of the Picard-Fuchs system that are included in a hyperplane. Indeed, whenever a form $\omega = \sum_{i=1}^{n} c_i \omega_i$ has a zero residue at infinity, then its coefficients define the equation of a hyperplane $\{ h = \sum_{i=1}^{n} c_i x_i = 0 \}$ that contains the integral curve $\Gamma_1(t) = (\oint_{\delta_{\infty}} \omega_1, \ldots, \oint_{\delta_{\infty}} \omega_n)$. This implies that the global monodromy of the Picard-Fuchs system is reducible: extend $\Gamma_1$ to a fundamental system by adjoining solutions $\Gamma_2, \ldots, \Gamma_n$. Then, the $\mathbb{C}$-space spanned by the solutions $\Gamma_i$ such that $h \circ \Gamma_i(t) \equiv 0$ is invariant by the monodromy (see [2, Lemma 1.3.4]).

It follows from Lemma 4 that the set $S$ coincides with the set of relations between the residues of the Petrov forms at infinity. It is thus a codimension 2 linear subspace of $\mathbb{C}^n$. Therefore, we arrive at the conclusion of Theorem 2, for odd $n$ and $\omega \notin S$. In the remaining cases, that is for $\omega \in S$, the identity (6) is useless, since both sides are 0.

We aim at reconstructing the identity (6), initiating the reasoning from a linear differential system of size smaller than $n$. First, we know the exact number of independent integrals among $\oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega$.

Lemma 5. If $\omega$ belongs to $S$, then: $\dim_{\mathbb{C}} \left( \oint_{\delta_1} \omega, \ldots, \oint_{\delta_n} \omega \right) = n - 1$. 

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Proof. The relations between these integrals constitute the space
\[ \left\{ (d_1, \ldots, d_n) \in \mathbb{C}^n : d_1 \oint_{\delta_1} \omega + \ldots + d_n \oint_{\delta_n} \omega \equiv 0 \right\} \]
From Lemma 2, any relation \((d_1, \ldots, d_n)\) must verify: \(d_1\delta_1 + \ldots + d_n\delta_n\) is a multiple of \(\delta_{\infty}\). This defines a 1-dimensional vector space. □

From now on, we work with an adapted basis of cycles in \(H_1(\{H = t\}, \mathbb{Z})\), that includes \(\delta_{\infty}\), and that we denote by \((\delta_1, \ldots, \delta_n, \delta_{\infty})\). Then it is clear that \(W(\oint_{\delta_1} \omega, \ldots, \oint_{\delta_{n-1}} \omega) \neq 0\). We also make several changes in the Petrov frame: recall that the matrix \(A = (a_{ij})\) in (4) describes the vectors \(Hd\omega_i\) via the correspondence that associates the \(i\)-th canonical vector \(e_i \in \mathbb{C}^n\) to the monomial \(x^i\). We already noticed that the matrix \(A\) was diagonalizable. We assume \(A\) diagonal (which corresponds to combining linearly the forms \(\omega_i\)). Moreover, we know that among the forms \(\omega_i\) associated to an eigenbasis of \(A\), two of them will have independent residues at infinity. Up to permutation, these are \(\omega_{n-1}\) and \(\omega_n\). Now, after adding a scalar multiple of \(\omega_n\) to \(\omega_{n-1}\), we may assume that \(\oint_{\delta_{\infty}} \omega_{n-1}\) is a constant, while the residue of the form \(\omega_n\) is a polynomial of degree 1 in the variable \(t\). With respect to such a basis \((\omega_1, \ldots, \omega_n)\), a nonzero off-diagonal entry, \(a_{n,n-1}\), may appear in \(A\). Thus, with our choice of basis of \(\mathbb{C}^n\), \(A\) has the form:

\[
A = \begin{pmatrix}
a_{1,1} & & \\
& \ddots & \\
& & a_{n-1,n-1}
\end{pmatrix}
\]

Now, a form \(\omega = \sum_{i=1}^{n} c_i \omega_i\) belongs to \(S\) if and only if
\[
c_1 \oint_{\delta_{\infty}} \omega_1 + \ldots + c_n \oint_{\delta_{\infty}} \omega_n \equiv 0
\]
So that after a linear change of coordinates in \(\mathbb{C}^n\), we may write \(\omega\) as:
\[
\omega = c_1 \widetilde{\omega}_1 + \ldots + c_{n-2} \widetilde{\omega}_{n-2}, \text{ with } \oint_{\delta_{\infty}} \tilde{\omega}_i = 0, \ i = 1, \ldots, n-2.
\]
We set:
\[
\tilde{\omega}_{n-1} = \omega_{n-1} \text{ and } \tilde{\omega}_n = \omega_n.
\]
Thus, the integral \(\oint_{\delta_{\infty}} \omega\) reads:
\[
q_0 : \overline{\gamma}(t), \ \ \ q_0 \in \mathbb{C}^{n-2}, \ \overline{\gamma}(t) = (\oint_{\delta_{\infty}} \tilde{\omega}_1, \ldots, \oint_{\delta_{\infty}} \tilde{\omega}_{n-2}).
\]
When passing to the Petrov frame \((\tilde{\omega}_i), \ i = 1, \ldots, n\), the matrix \(A\) is changed into \(A' = P^{-1}AP\), and part of the structure of the matrix \(P = (p_{i,j})\) is known: \(p_{n-1,n-1} = p_{n,n} = 1; \ p_{i,n-1} = 0 \text{ for } i \neq n - 1\,\text{, and} \)

\[
p_{i,j} = \begin{cases}
0 & \text{if } i = n - 1, j = n, \\
1 & \text{if } i = n, j = n-1, \\
\text{otherwise} & \text{for } i, j < n-1.
\end{cases}
\]
Since the residues of $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}$ are constants, these equalities entail: $\oint_{\delta_\infty} \tilde{\eta}_i = 0$, $i = 1, \ldots, n - 1$. Hence the Gelfand-Leray forms $\tilde{\eta}_i$ admit a decomposition with respect to $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-2}$ only: $\tilde{\eta}_i = \sum_{j=1}^{n-2} b'_{i,j} \tilde{\omega}_j$, $i = 1, \ldots, n - 1$, $b'_{i,j} \in \mathbb{C}$. Besides, from the $n$-th equality
\[ \frac{t}{dt} \oint_{\delta_\infty} \tilde{\omega}_n - a'_{n,n} \frac{d}{dt} \oint_{\delta_\infty} \tilde{\omega}_n = \oint_{\delta_\infty} \tilde{\eta}_n \]
it follows that $\tilde{\eta}_n$ has a nonzero component along $\tilde{\omega}_n$.

This provides information on the matrix $B'$ related to the new frame $(\tilde{\omega}_i)$: $b'_{i,n-1} = b'_{i,n} = 0$ for $i = 1, \ldots, n - 1$.

The curve $\gamma(t) = (\oint_{\delta_1} \tilde{\omega}_1, \ldots, \oint_{\delta_1} \tilde{\omega}_n)$ is a solution of the linear system $\det(tE - A') \cdot \dot{x} = \text{Ad}(tE - A')B' \cdot x = C \cdot x$ with polynomial matrix $C = (C_{i,j})$. From the structure of $A'$ and $B'$, most of the entries in the last two columns of $C$ are zeros, in particular: $C_{i,j} = 0$, for $i = 1, \ldots, n - 2$ and $j = n - 1, n$. This means that the truncated curve $\overline{\gamma}(t) = (\oint_{\delta_1} \tilde{\omega}_1, \ldots, \oint_{\delta_1} \tilde{\omega}_{n-2})$ satisfies the linear system whose matrix $C$ is the $(n - 2) \times (n - 2)$ upper-left corner of $C$.

Starting from $q_0 = (c_1, \ldots, c_{n-2}) \in \mathbb{C}^{n-2}$, we derive the vectors $q_1, \ldots, q_{n-3} \in \mathbb{C}[t]^{n-2}$ by: $q_{k+1} = Dq_k + q_k \cdot C$, with the same $D$ as before: $D = \det(tE - A') = (t-t_1) \cdots (t-t_n)$. They satisfy: $D^k(q_0 \cdot \overline{\gamma}) = q_k \cdot \overline{\gamma}$. Let $\Delta$ be the wedge product of $q_0, \ldots, q_{n-3}$.

On the other hand, consider the matrix
\[
\tilde{P} = \begin{pmatrix}
\oint_{\delta_1} \tilde{\omega}_1 & \cdots & \oint_{\delta_{n-2}} \tilde{\omega}_1 \\
\vdots & \ddots & \vdots \\
\oint_{\delta_1} \tilde{\omega}_{n-2} & \cdots & \oint_{\delta_{n-2}} \tilde{\omega}_{n-2}
\end{pmatrix}
\]

We obtain:

\[ \Delta \cdot \det \tilde{P} = P^{\nu} \cdot \overline{\nu} \]
with \( P(t) = (t - t_1) \ldots (t - t_n), \nu = (n - 3)(n - 2)/2, \) and \( \overline{W} = W(f_{\delta_1} \omega, \ldots, f_{\delta_{n-2}} \omega). \)

As \( \overline{W} \) is nonzero (by the choice of the basis of the homology), both determinants \( \Delta \) and \( \det \tilde{P} \) are non identically vanishing. Notice that \( \deg \Delta \leq \frac{(n-1)(n-2)(n-3)}{2} \). A closer look at \( \det \tilde{P} \) shows that it is polynomial, of degree:

**Lemma 6.** \( \deg(\det \tilde{P}) \leq n. \)

**Proof.** Note that the matrix \( \tilde{P} \) is the product \( R \cdot \overline{P} \) of a \((n - 2) \times n\) constant matrix \( R = (r_{i,j}), 1 \leq i \leq n - 2, 1 \leq j \leq n, \) of rank \( n - 2 \) by \( \overline{P} \), obtained by removing the last two columns in the standard period matrix \( \mathcal{P} \). There is no restriction in supposing the first \( n - 2 \) columns of \( R \) independent (this amounts to permuting the cycles in \( \overline{P} \)), and consider \( \overline{R} \) the corresponding square matrix of rank \( n - 2 \). Form the product \( \overline{R}^{-1} \cdot \tilde{P} \). Its determinant is the same as \( \det \tilde{P} \), up to a nonzero constant. On the other hand, this matrix has the expression:

\[
\overline{R}^{-1} \cdot \tilde{P} = \begin{pmatrix}
\delta f_{\delta_1} (\omega_1 + \Omega_1) & \cdots & \delta f_{\delta_{n-2}} (\omega_1 + \Omega_1) \\
\vdots & \ddots & \vdots \\
\delta f_{\delta_1} (\omega_{n-2} + \Omega_{n-2}) & \cdots & \delta f_{\delta_{n-2}} (\omega_{n-2} + \Omega_{n-2})
\end{pmatrix}
\]

where \( \Omega_1, \ldots, \Omega_{n-2} \) belong to the span \( \mathbb{C}(\omega_{n-1}, \omega_n) \). Expanding \( \det(\overline{R}^{-1} \cdot \tilde{P}) \), it turns out that the term that brings the highest degree (with respect to \( t \)) is the determinant:

\[
\begin{vmatrix}
\delta f_{\delta_1} \Omega_1 & \cdots & \delta f_{\delta_{n-2}} \Omega_1 \\
\delta f_{\delta_1} \omega_2 & \cdots & \delta f_{\delta_{n-2}} \omega_2 \\
\vdots & \ddots & \vdots \\
\delta f_{\delta_1} \omega_{n-2} & \cdots & \delta f_{\delta_{n-2}} \omega_{n-2}
\end{vmatrix}
\]

Setting \( x = t^{1/(n+1)} x', \ y = t^{1/2} y' \), it follows that the leading term of this determinant has degree \( \frac{D}{n+1} \), where \( D \) is the weighted degree \( \deg \omega_n + \deg \omega_2 + \ldots + \deg \omega_{n-2} = n + \frac{n+1}{2} + \sum_{i=2}^{n-2} (i + \frac{n+1}{2}) \), hence \( \frac{D}{n+1} \leq n \). The leading coefficient appears as the determinant of the integrals of the forms \( \omega_n(x', y'), \omega_2(x', y'), \ldots, \omega_{n-2}(x', y') \) over cycles in the level sets of the principal quasi-homogeneous part of \( H, y^2 - x^{n+1} \) (cf. [G]). The latter determinant is guaranteed to be nonzero since the differentials of the forms involved are independent in the quotient ring of \( \mathbb{C}[x, y] \) by the Jacobian ideal \((H_x, H_y)\). \( \square \)
Again, we observe that the identity (8) returns a quadratic lower estimate: $\text{ord}_\infty W \geq \frac{n^2 - 7n + 6}{2}$. On every finite pole: $\text{ord}_t W \geq n - 4\ell + 3n$, and at a zero $t_0$ of $W$: $\text{ord}_{t_0} W \leq n - 3 + \frac{n^2 - n - 6}{2}$. Finally: $\text{ord}_t \oint \delta \omega \leq n - 3 + \frac{n^2 - n - 6}{2}$. The proof of Theorem 2 is completed.

3.3. Intersection with an algebraic hypersurface. We consider the asymptotic behaviour of the integral with respect to the degree $d$ of the form $\omega$.

**Theorem 3.** Under the same assumptions on the Hamiltonian, consider the Abelian integral of a 1-form $\omega$ of degree $d$. If $\oint \delta \omega \neq 0$, then:

$$\text{ord}_{t_0} \left( \oint \delta \omega \right) \leq A(n) + d \cdot B(n)$$

An idea could be first to decompose $\omega$ in the Petrov module of $H$:

$$\omega = p_1(t)\omega_1 + \ldots + p_n(t)\omega_n,$$

which implies: $\oint \delta \omega = p_1(t) \oint \delta \omega_1 + \ldots + p_n(t) \oint \delta \omega_n$, with $\deg p_i \leq \frac{d}{n+1}$, $i = 1, \ldots, n$. Then apply the above reasoning, noticing that the vector

$$(\oint \delta \omega_1, \ldots, \oint \delta \omega_n, t \oint \delta \omega_1, \ldots, t \oint \delta \omega_n, \ldots, t^{[d/(n+1)]} \oint \delta \omega_1, \ldots, t^{[d/(n+1)]} \oint \delta \omega_n)$$

is a solution of a hypergeometric Picard-Fuchs system of size $n \cdot \left([d/(n+1)] + 1\right)$. This would give a bound that is quadratic with respect to the degree $d$ of the form. Yet, one should expect linear growth, since, for a fixed Hamiltonian, even the growth of the number of zeros of the integrals was proven by A. Khovanski to be linear in the degree of the form.

**Proof.** We define the curve $t \mapsto \Gamma(t) = (t, \oint \delta \omega_1, \ldots, \oint \delta \omega_n) \subseteq \mathbb{C}^{n+1}$, together with the algebraic hypersurface $\{(t, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}: h(t, x_1, \ldots, x_n) = 0\}$, setting $h(t, x_1, \ldots, x_n) = p_1(t)x_1 + \ldots + p_n(t)x_n$, in view of the above Petrov decomposition. Thus, $\oint \delta \omega$ is the composition $(h \circ \Gamma)(t)$. This reads also as the matrix product of the row vector $Q_0 = (p_1(t), \ldots, p_n(t)) \in \mathbb{C}[t]^n$, by the column vector $\gamma(t) = (\oint \delta \omega_1, \ldots, \oint \delta \omega_n)$. The construction of vectors $Q_k \in \mathbb{C}[t]^n$ can be performed likewise. Their degrees, as well as the degree of their
exterior product, have affine growth with respect to $d$. The lower estimate on the order of the Wronskian $W(f_{\delta_1} \omega, \ldots, f_{\delta_n} \omega)$ at its finite poles is not affected by $d$.

□

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