A Blockchain-Based Approach for Saving and Tracking Differential-Privacy Cost

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Abstract—An increasing amount of users’ sensitive information is now being collected for analytics purposes. To protect users’ privacy, differential privacy has been widely studied in the literature. Specifically, a differentially private algorithm adds noise to the true answer of a query to generate a noisy response. As a result, the information about the dataset leaked by the noisy output is bounded by the privacy parameter. Oftentimes, a dataset needs to be used for answering multiple queries (e.g., for multiple analytics tasks), so the level of privacy protection may degrade as more queries are answered. Thus, it is crucial to keep track of the privacy spending which should not exceed the given privacy budget. Moreover, if a query has been answered before and is asked again on the same dataset, we may reuse the previous noisy response for the current query to save the privacy cost.

In view of the above, we design and implement a blockchain-based system for tracking and saving differential-privacy cost. Blockchain provides a distributed immutable ledger that records each query’s type, the noisy response used to answer each query, the associated noise level added to the true query result, and the remaining privacy budget in our system. As a result, the owner of the dataset will have full knowledge about how the dataset has been used and be confident that no new privacy cost will be incurred for answering queries once the specified privacy budget is exhausted. Furthermore, since the blockchain records the noisy response used to answer each query, we also design an algorithm to reuse previous noisy response if the same query is asked repeatedly. Specifically, considering that different requests of the same query may have different privacy requirements, our algorithm (via a rigorous proof) is able to set the optimal reuse fraction of the old noisy response and add new noise (if necessary) to minimize the accumulated privacy cost. Experimental results show that the proposed algorithm can reduce the privacy cost significantly without compromising data accuracy.

We envision our system to be useful in practical applications handling sensitive information (e.g., the Apple iOS system which adopts differential privacy to collect user’s emoji behavior), and to motivate more usage of blockchains.

Index Terms—Blockchain, differential privacy, data analytics, Gaussian mechanism.

I. INTRODUCTION

Massive volumes of users’ sensitive information are being collected for data analytics and machine learning. To protect personal privacy, many countries have strict policies about how technology companies collect and process users’ data. However, the companies need to analyze users’ data for service quality improvement. To preserve privacy while revealing useful information about datasets, differential privacy (DP) has been proposed [1,2]. Intuitively, by incorporating some noise, the output of an algorithm under DP will not change significantly due to the presence or absence of one user’s information in the dataset. Due to its introduction [1,2], DP has attracted much interest from both academia [3–7] and industry [8–10]. For example, Apple has incorporated DP into its mobile operating system iOS [8]; Google has implemented a DP tool called RAPPOR in the Chrome browser to collect information [9].

Roughly speaking, a randomized mechanism achieving $(\epsilon, \delta)$-DP [1] means that except with a (typically small) probability $\delta$, altering a record in a database cannot change the probability that an output is seen by more than a multiplicative factor $\epsilon^*$. Thus, the information about the dataset leaked by the noisy output of an $(\epsilon, \delta)$-DP algorithm is bounded by the privacy parameters $\epsilon$ and $\delta$. Smaller $\epsilon$ and $\delta$ mean stronger privacy protection and less information leakage. Note that non-zero information leakage is necessary to achieve non-zero utility. Usually, a dataset may be used for answering multiple queries (e.g., for multiple analytics tasks), thus accumulating the information leakage and degrading the privacy protection level, which can be intuitively understood as the increase of privacy spending. Therefore, it is necessary to record the privacy cost to prevent it from exceeding the privacy budget. Besides, we reduce privacy cost by reusing old noisy response to answer the current query if the query was answered before.

Traditionally, the privacy cost incurred by answering queries on a dataset is claimed by the dataset holder. Users whose information is in the dataset are not clear about the usage. It is possible that privacy consumption has exceeded the privacy budget. To solve this problem, the emerging blockchain technology provides a new solution to manage the privacy cost. Blockchain is a chain of blocks storing cryptographic and tamper-resistant transaction records without using a centralized server [11,12]. With blockchain recording how the dataset is used for answering queries, users have full knowledge of how their information is analyzed. Users can easily access the blockchain to check the consumption of the privacy budget. The dataset holder has the motivation to adopt our blockchain-based approach to provide the following accountability guarantee to users whose information is in the dataset: if the dataset holder uses the dataset more than the set of queries recorded by the blockchain, measures can be taken to catch the dataset holder with cheating because transactions written into the blockchain are tamper-resistant.

In view of the above, we propose a blockchain-based system to track and manage differential-privacy cost, which uses blockchain to make the privacy spending transparent to the data owner. Consequently, the data owner can track how dataset used by checking blockchain transactions’ information, including each query’s type, the noisy response used to answer each
First, to the best of our knowledge, we are the first to use the blockchain ensure that no new privacy cost will be incurred, and this can be verified. Furthermore, since the blockchain stores the noisy response used to answer each query, we also design an algorithm to minimize the accumulated privacy cost by reusing previous noisy response if the same query is asked again. Our algorithm (via a rigorous proof) is able to set the optimal reuse fraction of the old noisy response and add new noise (if necessary) considering different requests of the same query may be sent with different privacy requirements. In our blockchain-based system, reusing noisy responses not only saves privacy cost, but also reduces communication overhead when the noisy response is generated without contacting the server hosting the dataset.

The proposed system is useful for practical applications which contain sensitive information and motivates more usage of blockchains in the domain of privacy preservation. Furthermore, as more laws related to privacy get launched, more people and companies will pay attention to privacy issues. Therefore, if a company uses our proposed blockchain system to support privacy cost and queries history tracking, it will be more reliable and attract more users and partners.

**Contributions.** The major contributions of this paper are summarized as follows:

- First, to the best of our knowledge, we are the first to use the Ethereum platform to track and reuse noise and the differential privacy cost. Unlike traditional usage of blockchain, we design a system that enables the smart contract to access the external server to fetch real-world data safely without breaking the deterministic state of the blockchain.
- Second, a novel privacy-preserving algorithm with a rigorous mathematical proof is designed to minimize accumulated privacy cost by reusing previous noisy responses if the same query is received. Thus a dataset can satisfy more queries while preventing the privacy leakage, which is essential for the datasets with frequent queries, e.g., medical record datasets.
- Third, by combining our designed system with the algorithm, a data owner can host datasets locally while opening access to others in a privacy-preserving mode. Data owners can set a privacy budget and multiple query types, and then the blockchain smart contract will record every request, the associated privacy cost, and the noisy response. Unlike calling the data hosting server every time in naive solutions, our approach reduces the number of times to request the server significantly by taking advantage of recorded noisy results.
- Fourth, we implement the proposed system and algorithm according to a detailed sequence diagram and conduct experiments by using a real-world dataset. Numerical results demonstrate that our proposed system and algorithm are effective in saving the privacy cost while keeping accuracy.

**Organization.** The rest of the paper is organized as follows. Section 1 introduces preliminaries about differential privacy and blockchains. Section 2 presents system design including our proposed noise reuse algorithm. Section 3 describes challenges in implementing our system. In Section 4 we discuss experimental results to validate the effectiveness of our system. Section 5 surveys related work. Section 6 concludes this paper and identifies future directions.

**Notation.** Throughout the paper, \( \mathbb{P}[\cdot] \) denotes the probability, and \( \mathcal{F}[\cdot] \) stands for the probability density function. The notation \( \mathcal{N}(0, A) \) denotes a Gaussian random variable with zero mean and variance \( A \), and means a fresh Gaussian noise when it is used to generate a noisy query response.

## II. Preliminaries

### A. Differential Privacy

Differential privacy intuitively means that the adversary cannot determine with high confidence whether the randomized output comes from a dataset \( D \) or its neighboring dataset \( D' \) which differs from \( D \) by one record. The formal definition of \((\epsilon, \delta)\)-differential privacy is given in Definition 1 and the notion of neighboring datasets is discussed in Remark 2.

**Definition 1 \((\epsilon, \delta)\)-Differential privacy** \([13]\). A randomized mechanism \( Y \), which generates a randomized output given a dataset as the input, achieves \((\epsilon, \delta)\)-differential privacy if

\[
\mathbb{P}[y(D) \in \mathcal{Y}] \leq e^{\epsilon} \mathbb{P}[y(D') \in \mathcal{Y}] + \delta,
\]

for \( D \) and \( D' \) iterating through all pairs of neighboring datasets, and for \( \mathcal{Y} \) iterating through all subsets of the output range,

where \( \mathbb{P}[\cdot] \) denotes the probability, and the probability space is over the coin flips of the randomized mechanism \( Y \).

**Remark 1.** The notion of \((\epsilon, \delta)\)-differential privacy under \( \delta = 0 \) becomes \( \epsilon \)-differential privacy, \( \epsilon \)-Differential privacy and \((\epsilon, \delta)\)-differential privacy is also referred to as the pure and approximate differential privacy, respectively, in many studies \([3]-[5]\).

**Remark 2 (Notion of neighboring datasets).** Two datasets \( D \) and \( D' \) are called neighboring if they differ only in one tuple. There are still variants about this. In the first case, the sizes of \( D \) and \( D' \) differ by one so that \( D' \) is obtained by adding one record to \( D \) or deleting one record from \( D \). In the second case, \( D \) and \( D' \) have the same size (say \( n \)), and have different records at only one of the \( n \) positions. Finally, the notion of neighboring datasets can also be defined to include both the cases above. Our results in this paper apply to all of the above cases.

Among various mechanisms to achieve DP, the Gaussian mechanism for real-valued queries proposed in \([1]\) has received much attention. The improved result given by \([13]\) is Lemma 1.

**Lemma 1 (Theorem A.1 by Dwork and Roth \([13]\)).** To answer a query \( Q \) with \( \ell_2 \)-sensitivity \( \Delta_Q \), adding a zero-mean Gaussian noise with standard deviation

\[
\sqrt{2 \ln \frac{1.25}{\delta}} \times \frac{\Delta_Q}{\epsilon} \quad \text{(denoted by Gaussian}(\Delta_Q, \epsilon, \delta) \text{ hereafter in this paper)}
\]

to each dimension of the true query result achieves \((\epsilon, \delta)\)-differential privacy. The above \( \ell_2 \)-sensitivity \( \Delta_Q \) of a query \( Q \) is defined as the maximal \( \ell_2 \) distance between the true query results for any
two neighboring datasets $D$ and $D'$ that differ in one record; i.e., $\Delta Q = \max_{\text{neighboring } D, D'} \|Q(D) - Q(D')\|_2$.

More discussions on the $\ell_2$-sensitivity of a query are given in Section III-C on Page 8.

B. Blockchain, Ethereum and Smart Contracts

Blockchain. The blockchain technology is popularly used in systems requiring high security and transparency, such as Bitcoin and Ethereum [13]. The blockchain can be effectively used to solve the double-spending problem in Bitcoin transaction by using a peer-to-peer network. The solution is to hash transaction information in a chain of hash-based Proof-of-Work (PoW, used by Bitcoin) which is the consensus mechanism algorithm used to confirm transactions and produce new blocks to the chain. Once the record is formed, it cannot be changed except redoing the Proof-of-Work.

Besides, the blockchain is constantly growing with appending 'completed' blocks. Blocks consisting of the most recent transactions are added to the chain in chronological order [13]. Each blockchain node can have a copy of the blockchain. The blockchain allows participants to track their transactions without centralized control.

Ethereum. Ethereum is a blockchain platform which allows users to create decentralized end-to-end applications [16]. The miners in the Ethereum use the Proof-of-Work consensus algorithm to complete transaction verification and synchronization. Besides, Ethereum can run smart contracts elaborated below.

Smart Contract. The smart contract was first proposed by Nick Szabo as a computerized transaction protocol that can execute terms of a contract automatically [17]. It intends to make a contract digitally, and allows to maintain credible transactions without a third party. With the development of blockchains, such as the Ethereum, smart contracts are stored in the blockchain as scripts. A blockchain with a Turing-complete programming language allows everyone to customize smart contract scripts for transactions [18]. Smart contracts are triggered when transactions are created or generated on the blockchain to finish specific tasks or services.

III. System Description

Our blockchain-based system provides differentially private responses to queries while minimizing the privacy cost via noise reuse. We design a web application to implement our Algorithm 1 which generates noisy responses to queries with the minimal privacy cost by setting the optimal reuse fraction of the old noisy response and adding new noise (if necessary). For clarity, we defer Algorithm 1 to Page 5 and its discussion to Section III-C on Page 8. Our blockchain-based system is illustrated in Fig. 1 and we discuss the detailed design in the following.

A. System Architecture

Our system includes the client, the blockchain, the server, and smart contract followed by more details as below.

Client. The primary function of the client is to transfer users’ queries to the blockchain smart contract. The client computes the required parameter standard deviation for the server to generate the Gaussian noise using the privacy parameters $\epsilon$ and $\delta$ and forwards the query to the blockchain. Also, the client can display the query result to the analyst after getting the noisy response to the query.

Blockchain Smart Contract. The blockchain serves as a middleware between the client and the server. It decides which query should be submitted to the server. The blockchain records the remaining privacy budget, query type, the noisy response to answer the query, the privacy parameters, and the amount of corresponding noise. If the remaining privacy budget is enough, the smart contract will execute the query match function with the recorded history. Otherwise, the smart contract will reject this query. If the current query does not match with any query in the history, the smart contract will call the server to calculate the result. If the query has been received before, the blockchain smart contract will not call the server if the noisy response can be completely generated by old noisy answers and will call the server if access to the dataset is still needed to generate the noisy response.

Server. The server provides APIs to answer analysts’ queries. When the API is called, the server will query the dataset to calculate the respective answer. After the true value $Q(D)$ is calculated, the server will add noise to perturb the answer. Then the server returns the noisy answer to the blockchain.

In the rest of the paper, we use Blockchain, Client, and Server to denote the blockchain, client, and server, respectively.

B. System Functionality

Match query with query history and generate noisy response: Blockchain compares the current query type with saved query types to retrieve previous query results. If it is the first time for Blockchain to see the query, Blockchain will forward the query to the server, and Server will return the perturbed result which satisfies differential privacy to Blockchain. If the current query type matches previous answers’ query type, Blockchain will compare the computed amount of noise with all previously saved amounts of noise under the same query type. Based on the comparison
result, Blockchain will completely reuse old responses or call Server.

Manage privacy budget: Blockchain updating the privacy budget as queries are answered and the Blockchain ensures no new privacy cost will be incurred for answering queries once the specified privacy budget is exhausted.

C. Our Algorithm[1] based on Reusing Noise

We present our solution for reusing noise in Algorithm [1] on Page 5. We consider real-valued queries so that the Gaussian mechanism can be used. Extensions to non-real-valued queries can be regarded as the future work, where we can apply the exponential mechanism of [6].

To clarify notation use, we note that $Q_i$ means the $i$-th query (ordered chronologically) and is answered by a randomized algorithm $Q_i$. A type $t$-query means that the query’s type is $t$. Queries asked at different time can have the same query type. This is the reason that we reuse noise in Algorithm[1].

Suppose a dataset $D$ has been used to answer $m-1$ queries $Q_1, Q_2, \ldots, Q_{m-1}$, where the $i$-th query $Q_i$ for $i = 1, 2, \ldots, m-1$ is answered under $(\epsilon_i, \delta_i)$-differential privacy (by reusing noise, or generating fresh noise, or combining both). For $i = 1, 2, \ldots, m$, we define $\sigma_i := \text{Gaussian}(\Delta Q_i, \epsilon_i, \delta_i)$, where $\Delta Q_i$ denotes the $\ell_2$-sensitivity of $Q_i$, where we defer the discussion of $\Delta Q_i$ to Section III-G on Page 8. As presented in Algorithm[1] we have several cases discussed below. For better understanding of these cases, we later discuss an example given in Table I on Page 5.

Case 1): If $Q_m$ is seen for the first time, we obtain the noisy response $Q_m(D)$ by adding a zero-mean Gaussian noise with standard deviation $\text{Gaussian}(\Delta Q_m, \epsilon_m, \delta_m)$ independently to each dimension of the true result $Q_m(D)$ (if the privacy budget allows), as given by Line 7 of Algorithm [1] where $\text{Gaussian}(\Delta Q_m, \epsilon_m, \delta_m) := \sqrt{2 \ln \frac{1}{\delta_m} \times \Delta Q_m}$ from Lemma [1].

Case 2): If $Q_m$ has been received before, suppose $Q_m$ is a type $t$-query, and among the previous $m-1$ queries $Q_1, Q_2, \ldots, Q_{m-1}$, let $\Sigma_t$ consist of the corresponding noise amounts for previous instances of type $t$-query; i.e., $\Sigma_t := \{\sigma_j : \sigma_j$ has been recorded in Blockchain and $Q_j$ is a type $t$-query$\}$. Blockchain compares $\sigma_m$ and the values in $\Sigma_t$, resulting in the following subcases.

Case 2A): If there exists $\sigma_j \in \Sigma_t$ such that $\sigma_m = \sigma_j$, then $Q_m(D) = Q_j(D)$ is set as $Q_m(D)$.

Case 2B): This case considers that $\sigma_m$ is less than $\min(\Sigma_t)$ which denotes the minimum in $\Sigma_t$. Let $\bar{Q}_{t, \text{min}}(D)$ denote the noisy response (kept in Blockchain) corresponding to $\min(\Sigma_t)$; specifically, if $\min(\Sigma_t) = \sigma_j$ for some $j$, then $\bar{Q}_{t, \text{min}}(D) = Q_j(D)$. Under $\sigma_m < \min(\Sigma_t)$, to minimize the privacy cost, we reuse $\sigma_m$ fraction of noise in $\bar{Q}_{t, \text{min}}(D)$ to generate $Q_m(D)$ (if the privacy budget allows). This will be obtained by Theorem[1]s Result (ii) to be presented on Page 6. Specifically, under $\min(\Sigma_t) > \sigma_m$, as given by Line 22 of Algorithm[1], $Q_m(D)$ is set by $Q_m(D) \leftarrow Q_m(D) + \frac{\sigma_m^2}{\min(\Sigma_t)^2} \times [\bar{Q}_{t, \text{min}}(D) - Q_m(D)] + N(0, 1) \times \sqrt{\frac{\sigma_m^2}{\min(\Sigma_t)^2}}$. Note that if $Q_m$ is multidimensional, independent Gaussian noise will be added to each dimension according to the above formula. This also applies to other places of this paper.

Case 2C): This case considers that $\sigma_m$ is greater than $\min(\Sigma_t)$ and $\sigma_m$ is different from all values in $\Sigma_t$. Let $\sigma_i$ be the maximal possible value in $\Sigma_t$ that is also smaller than $\sigma_m$; i.e., $\sigma_i = \max\{\sigma_j : \sigma_j \in \Sigma_t$ and $\sigma_i < \sigma_m\}$. Then $Q_m(D)$ is set as $\bar{Q}_i(D) + N(0, 1) \times \sqrt{\sigma_m^2 - \sigma_i^2}$. This will become clear by Theorem[1]s Result (ii) to be presented on Page 6.

An example to explain Algorithm[1] Table I provides an example for better understanding of Algorithm[1]. We consider three types of queries. In particular, $Q_1, Q_4, Q_6, Q_{10}, Q_{12}$ are type 1-queries; $Q_2, Q_5, Q_8, Q_9, Q_{11}$ are type 2-queries, and $Q_3, Q_7, Q_{13}$ are type 3-queries.

D. Explaining the Noise Reuse Rules of Algorithm[1]

Our noise-reuse rules of Algorithm[1] are designed to minimize the accumulated privacy cost. To explain this, inspired by [7], we define the privacy loss to quantify privacy cost. We analyze the privacy loss to characterize how privacy degrades in a fine-grained manner, instead of using the composition theorem by Kairouz et al. [19]. Although [19] gives the state-of-the-art results for the composition of differentially private algorithms, the results do not assume the underlying mechanisms to achieve differential privacy. In our analysis, by analyzing the privacy loss of Gaussian mechanisms specifically, we can obtain smaller privacy cost.

For a randomized algorithm $Y$, neighboring datasets $D$ and $D'$, and output $y$, the privacy loss $L_Y(D, D'; y)$ represents the multiplicative difference between the probabilities that the same output $y$ is observed when the randomized algorithm $Y$ is applied to $D$ and $D'$. Specifically, we define

$$L_Y(D, D'; y) := \ln \frac{\mathbb{P}[Y(D) = y]}{\mathbb{P}[Y(D') = y]} \quad (2)$$

where $\mathbb{P}[\cdot]$ denotes the probability density function.

For simplicity, we use probability density function $\mathbb{P}[\cdot]$ in Eq. (2) above by assuming that the randomized algorithm $Y$ has the continuous output. If $Y$ has the discrete output, we replace $\mathbb{P}[\cdot]$ by probability mass function $\mathbb{P}$. [19]

When $y$ follows the probability distribution of random variable $Y(D)$, $L_Y(D, D'; y)$ follows the probability distribution of random variable $L_Y(D, D'; Y(D))$, which we write as $L_Y(D, D')$ for simplicity.

We denote the composition of some randomized mechanisms $Y_1, Y_2, \ldots, Y_m$ for a positive integer $m$ by $Y_1 || Y_2 || \ldots || Y_m$. For the composition, the privacy loss with respect to neighboring datasets $D$ and $D'$ when the outputs of randomized mechanisms $Y_1, Y_2, \ldots, Y_m$ are $y_1, y_2, \ldots, y_m$ is defined by

$$L_{Y_1 || Y_2 || \ldots || Y_m}(D, D'; y_1, y_2, \ldots, y_m) := \ln \frac{\mathbb{P}[Y_1, Y_2, \ldots, Y_m = y_1, y_2, \ldots, y_m]}{\mathbb{P}[Y_1, Y_2, \ldots, Y_m = y_1]}$$

When $y_i$ follows the probability distribution of random variable $Y_i(D)$ for each $i \in \{1, 2, \ldots, m\}$.
Algorithm 1: Our proposed algorithm to answer the $m$-th query and adjust remaining privacy cost.

**Input:** $D$: dataset; $Q_m$: the $m$-th query; $(\epsilon_m, \delta_m)$: requested privacy parameters for query $Q_m$; $(\sqrt{\epsilon_{\text{squared \_remaining \_budget}}}, \delta_{\text{budget}})$: remaining privacy budget (at the beginning, it is $(\sqrt{\epsilon_{\text{squared \_budget}}}, \delta_{\text{budget}})$ for $\epsilon_{\text{squared \_budget}} = \epsilon_{\text{budget}}^2$; $\Delta Q_m$: $\ell_2$ sensitivity of privacy budget $Q_m$;

**Output:** $Q_m(D)$: noisy query response for query $Q_m$ on dataset $D$ under $(\epsilon_m, \delta_m)$-differential privacy;

1. $\sigma_m \leftarrow \text{Gaussian}(\Delta Q_m, \epsilon_m, \delta_m)$; /*Comment: From Lemma 2 it holds that $\text{Gaussian}(\Delta Q_m, \epsilon_m, \delta_m) := \sqrt{2 \ln \frac{1.25}{\delta_m} \times \frac{\Delta Q_m}{\epsilon_m}}$. */

2. if the query $Q_m$ is seen for the first time then
   3. Client computes $\epsilon_{\text{squared \_cost}}$ such that $\text{Gaussian}(\Delta Q_m, \sqrt{\epsilon_{\text{squared \_cost}}}, \delta_{\text{budget}}) = \sigma_m$;

4. /*Comment: This means $\sqrt{2 \ln \frac{1.25}{\delta_{\text{budget}}} \times \frac{\Delta Q_m}{\sqrt{\epsilon_{\text{squared \_cost}}}}}$ = $\sigma_m$, where $\sigma_m$ as Gaussian($\Delta Q_m, \epsilon_m, \delta_m$) is $\sqrt{2 \ln \frac{1.25}{\delta_m} \times \frac{\Delta Q_m}{\epsilon_m}}$. */

5. Client computes $\epsilon_{\text{squared \_remaining \_budget}} := \epsilon_{\text{squared \_remaining \_budget}} - \epsilon_{\text{squared \_cost}}$;

6. if $\epsilon_{\text{squared \_remaining \_budget}} \geq 0$ then

7. return $Q_m(D) = Q_m(D) + N(0, 1) \times \sigma_m$; /*Comment: We refer to this Case 1) on Page 4 */

8. Blockchain records $(Q_m$’s query type, $\epsilon_m, \delta_m, Q_m(D))$; /*Comment: This information will be kept together with a cryptographic hash of the dataset $D$, which Blockchain stores so it knows which records are for the same dataset $D$. */

9. else
10. return an error of insufficient privacy budget;
11. end if
12. else
13. Suppose $Q_m$ is a type $t$-query. Blockchain compares $\sigma_m$ with values in $\Sigma_t := \{\sigma_j : \sigma_j$ has been recorded in Blockchain and $Q_j$ is a type $t$-query$\}$ (i.e., $\Sigma_t$ consists of the corresponding noise amounts for previous instances of type $t$-query), resulting in the following subcases.
14. if there exists $\sigma_j \in \Sigma_t$ such that $\sigma_m = \sigma_j$ then
15. Blockchain returns $Q_m(D) \leftarrow Q_j(D)$; /*Comment: We refer to this Case 2A) on Page 4 */
16. else if $\sigma_m < \min(\Sigma_t)$ then
17. /*Comment: The case of partially reusing an old noise: */
18. Client computes $\epsilon_{\text{squared \_cost}}$ such that $[\text{Gaussian}(\Delta Q_m, \sqrt{\epsilon_{\text{squared \_cost}}}, \delta_{\text{budget}})]^{-2} = \sigma_m^{-2} - [\min(\Sigma_t)]^{-2}$;
19. Client computes $\epsilon_{\text{squared \_remaining \_budget}} := \epsilon_{\text{squared \_remaining \_budget}} - \epsilon_{\text{squared \_cost}}$;
20. if $\epsilon_{\text{squared \_remaining \_budget}} \geq 0$ then
21. Blockchain computes NoiseReuseRatio := $\frac{\sigma_m^2}{[\min(\Sigma_t)]^2}$ and AdditionalNoise := $N(0, 1) \times \sqrt{\sigma_m^{-2} - \frac{\sigma_m^2}{[\min(\Sigma_t)]^2}}$;
22. Blockchain contacts Server to compute $Q_m(D) = Q_m(D) + \text{NoiseReuseRatio} \times [Q_{\min}(D) - Q_m(D)] + \text{AdditionalNoise}$, where $Q_{\min}(D)$ denotes the noisy response (kept in Blockchain) corresponding to $\min(\Sigma_t)$; /*Comment: We refer to this Case 2B) on Page 4 */
23. Blockchain records $(Q_m$’s query type, $\epsilon_m, \delta_m, Q_m(D))$;
24. else
25. return an error of insufficient privacy budget;
26. end if
27. else
28. /*Comment: The case of fully reusing an old noise: */
29. With $\sigma_t$ denoting the maximal possible value in $\Sigma_t$ that is also smaller than $\sigma_m$. Blockchain reuses $Q_t(D)$, which denotes the noisy response (kept in Blockchain) corresponding to $\sigma_t$;
30. Blockchain computes $Q_t(D) \leftarrow Q_t(D) + N(0, 1) \times \sqrt{\sigma_m^{-2} - \sigma_t^{-2}}$; /*Comment: We refer to this Case 2C) on Page 4 */
31. Blockchain records $(Q_m$’s query type, $\epsilon_m, \delta_m, Q_m(D))$;
32. end if
33. end if

Table 1: An example to explain Algorithm 1

| Query Type | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q_7$ | $q_8$ | $q_9$ | $q_{10}$ | $q_{11}$ | $q_{12}$ | $q_{13}$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|
| $\sigma_m$ (computed by line 2) | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| $\sigma_t$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| Case involved in Alg 1 | $Q_1 \leftarrow Q_1 + N(0, 1) \times \sigma_1$ with accessing $D$ | $Q_2 \leftarrow Q_2 + N(0, 1) \times \sigma_2$ with accessing $D$ | $Q_3 \leftarrow Q_3 + N(0, 1) \times \sigma_3$ with accessing $D$ | $Q_4 \leftarrow Q_3$ without accessing $D$ | $Q_5 \leftarrow Q_2$ without accessing $D$ | $Q_6 \leftarrow Q_1$ without accessing $D$ | $Q_7 \leftarrow Q_3$ without accessing $D$ | $Q_8 \leftarrow Q_1$ without accessing $D$ | $Q_9 \leftarrow Q_3$ without accessing $D$ | $Q_{10} \leftarrow Q_1$ without accessing $D$ | $Q_{11} \leftarrow Q_1$ without accessing $D$ | $Q_{12} \leftarrow Q_1$ without accessing $D$ | $Q_{13} \leftarrow Q_1$ without accessing $D$ |
We clearly require the probability distribution of random variable

\[ L_{Y_1}\|Y_2\|\ldots\|Y_m}(D, D'; y_1, y_2, \ldots, y_m) \]

which we write as \( L_{Y_1}\|Y_2\|\ldots\|Y_m}(D, D') for simplicity.

With the privacy loss defined above, we now analyze how to reuse noise when a series of queries are answered under differential privacy. To this end, we present Theorem 1 which presents the optimal ratio of reusing noise to minimize privacy cost.

**Theorem 1 (Optimal ratio of reusing noise to minimize privacy cost).** Suppose that before answering query \( Q_m \) and after answering \( Q_1, Q_2, \ldots, Q_{m-1} \), the privacy loss \( L_{\tilde{Q}_1\|\tilde{Q}_2\|\ldots\|\tilde{Q}_{m-1}}(D, D') \) is given by \( N(\frac{A(D, D')}{2}, A(D, D')) \) for some \( A(D, D') \). For the \( m \)-th query \( Q_m \), suppose that \( Q_m \) is the same as \( Q_j \) for some \( j \in \{1, 2, \ldots, m-1\} \) and we reuse \( r \) fraction of noise in \( \tilde{Q}_j(D) \) to generate \( Q_m(D) \) for \( 0 \leq r \leq 1 \) satisfying \( \sigma_r^2 = r^2\sigma_j^2 > 0 \), where \( r \) is a constant to be decided.

If \( \tilde{Q}_j(D) - Q_j(D) \) follows a Gaussian probability distribution with mean 0 and standard deviation \( \sigma_m \), we generate the noisy response \( \tilde{Q}_m(D) \) to answer query \( Q_m \) as follows:

\[
\tilde{Q}_m(D) = Q_m(D) + r[\tilde{Q}_j(D) - Q_j(D)] + N(0, \sigma_m^2 - r^2\sigma_j^2),
\]

so that \( \tilde{Q}_m(D) - Q_m(D) \) follows a Gaussian probability distribution with mean 0 and standard deviation \( \sigma_m \).

Note that \( \Delta_{Q_m} \) and \( \Delta_{Q_j} \) are the same since \( Q_m \) and \( Q_j \) are the same. Then we have the following results.

(i) After answering the \( m \) queries \( Q_1, Q_2, \ldots, Q_m \), the privacy loss \( L_{\tilde{Q}_1\|\tilde{Q}_2\|\ldots\|\tilde{Q}_m}(D, D') \) will be \( N(\frac{B_r(D, D')}{2}, B_r(D, D')) \) for \( B_r(D, D') := A(D, D') + \|Q_m(D) - Q_m(D')\|_2^2(1 - r)^2 \).

(ii) We clearly require \( r \geq 0 \) and \( \sigma_m^2 - r^2\sigma_j^2 \geq 0 \) in (i) above (note that \( N(0, 0) \equiv 0 \)). To minimize the total privacy cost (which is equivalent to minimize \( B_r(D, D') \) above), the optimal \( r \) is given by

\[
r_{optimal} = \begin{cases} 
1, & \text{if } \sigma_m \geq \sigma_j, \\
(\frac{\sigma_m}{\sigma_j})^2, & \text{if } \sigma_m < \sigma_j,
\end{cases}
\]

so that substituting Eq. (4) into the expression of \( B_r(D, D') \) gives

\[
B_{r_{optimal}}(D, D') = \begin{cases} 
A(D, D'), & \text{if } \sigma_m \geq \sigma_j; \\
A(D, D') + \|Q_m(D) - Q_m(D')\|_2^2 \left( \frac{1}{\sigma_m^2} - \frac{1}{\sigma_j^2} \right), & \text{if } \sigma_m < \sigma_j.
\end{cases}
\]

Note that if \( \sigma_m = \sigma_j \) for some \( j \in \{1, 2, \ldots, m-1\} \), we have \( r_{optimal} = 1 \) and just set \( \tilde{Q}_m(D) = Q_j(D) \).

**Proof.** The proof is in Appendix A of the online full version [10] (i.e., this paper). \[ \square \]

Eq. (4) of Theorem 1 clearly indicates the noise use ratio \( \frac{\sigma_m^2}{\min(\sigma_j^2)} \) of Case 2B) in Algorithm 1 (see Line 22 of Algorithm 1, and the noise use ratio 1 of Cases 2A) and 2C) in Algorithm 1 (see Lines 15 and 20 of Algorithm 1).

By considering \( r = 0 \) in Result (i) of Theorem 1 we obtain Corollary 1 which presents the classical result on the privacy loss of a single run of the Gaussian mechanism.

**Corollary 1.** By considering \( m = 1 \) in Result (ii) of Theorem 1 we have that for a randomized algorithm \( Q \) which adds Gaussian noise amount \( \sigma \) to a query \( Q \), the privacy loss with respect to neighboring datasets \( D \) and \( D' \) is given by

\[
N(\frac{A(D, D')}{2}, A(D, D')) \text{ for } A(D, D') := \|Q(D) - Q(D')\|_2^2.
\]

Corollary 1 has been shown in many prior studies [3][1][13] on the Gaussian mechanism for differential privacy.

**Corollary 2 (Privacy loss of the naive algorithm where each query is answered independently).** Suppose a dataset has been used to answer \( n \) queries \( Q_1, Q_2, \ldots, Q_n \) under differential privacy. Specifically, for \( i = 1, 2, \ldots, n \), to answer the \( i \)-th query \( Q_i \) under \( (\epsilon_i, \delta_i) \)-differential privacy, a noisy response \( \tilde{Q}_i(Q_i) \) is generated by adding independent Gaussian noise \( \sigma_i \) := Gaussian(\( \Delta_{Q_i}, \epsilon_i, \delta_i \)) to the true query result \( Q_i \), where \( \Delta_{Q_i} \) is the \( L_2 \)-sensitivity of \( Q_i \). Then after answering \( n \) queries \( Q_1, Q_2, \ldots, Q_n \) independently as above, the privacy loss with respect to neighboring datasets \( D \) and \( D' \) is given by

\[
N(\frac{F(D, D')}{2}, F(D, D')) \text{ for } F(D, D') := \sum_{i=1}^{n} \|Q_i(D) - Q_i(D')\|_2^2.
\]

E. Explaining Privacy Cost Update in Algorithm 7

Among the above cases, Cases 2A) and 2C) do not incur additional privacy cost since they just use previous noisy results and generate fresh Gaussian noise, without accessing to the dataset \( D \). In contrast, Cases 1) and 2B) incur additional privacy cost since they need to access the dataset \( D \) to compute the true query result \( Q_m(D) \). Hence, in Algorithm 1 the privacy cost is updated in Cases 1) and 2B), but not in Cases 2A) and 2C).

In this section, we explain the reason that the privacy cost is updated in Algorithm 1 (according to Lines 5 and 6 for Case 1), and Lines 15 and 17 for Case 2B).

When our Algorithm 1 is used, we let the above randomized mechanism \( Y_i \) be our noisy response function \( Q_i \). When \( Q_1, Q_2, \ldots, Q_{i-1} \) on dataset \( D \) are instantiated as \( y_1, y_2, \ldots, y_{i-1} \), if the generation of \( Q_i \) on dataset \( D \) uses \( Q_j \) for some \( j < i \), then the auxiliary information \( a_{ij} \) in the input to \( Q_i \) contains \( y_j \) (aux1 is \( \emptyset \)). For the consecutive use of our Algorithm 1 it will become clear that the privacy loss, defined by

\[
\frac{\sum_{i=1}^{n} \|\tilde{Q}_i(D) = y_i\|_2^2}{\min(\text{neighboring datasets } D, D')},
\]

follows a Gaussian probability distribution with mean \( \frac{1}{2} \) and variance \( V \) for some \( V \), denoted by \( N\left(\frac{1}{2}, V\right) \). For such a reason that form of privacy loss, the corresponding differential-privacy level is given by the following lemma.
Lemma 2. If the privacy loss of a randomized mechanism $Y$ with respect to neighboring datasets $D$ and $D'$ is given by $N(\frac{N(D, D')}{2}, V(D, D'))$ for some $V(D, D')$, then $Y$ achieves $(\epsilon, \delta)$-differential privacy for $\epsilon$ and $\delta$ satisfying $\max_{\text{neighboring datasets } D, D'} V(D, D') = \text{Gaussian}(1, \epsilon, \delta)^2$. 

Proof. The proof details are in Appendix B of the online full version [20] (i.e., this paper).

Based on the privacy loss defined above, we have the following theorem which explains the rules to update the privacy cost in our Algorithm [1].

Theorem 2. We consider the consecutive use of Algorithm [1] here. Suppose that after answering query $Q_1, Q_2, \ldots, Q_{m-1}$ and before answering query $Q_m$, the privacy loss with respect to neighboring datasets $D$ and $D'$ is given by $N(\frac{A(D, D')}{2}, A(D, D'))$ for some $A(D, D')$, and the corresponding privacy level can be given by $(\epsilon_{old}, \delta_{budget})$-differential privacy. Then in Algorithm [7] after answering all $m$ queries $Q_1, Q_2, \ldots, Q_{m-1}, Q_m$, we have:

- **the privacy loss with respect to neighboring datasets $D$ and $D'$**
  
  \[ \text{Lemma 2.} \] will still be $N(\frac{A(D, D')}{2}, A(D, D'))$ in Cases 2A and 2C),

- **will be $N(\frac{B(D, D')}{2}, B(D, D'))$ in Case 1 for $B(D, D') : = A(D, D') + \|Q_m(D) - Q_m(D')\|_2^2$.**

- **will be $N(\frac{C(D, D')}{2}, C(D, D'))$ in Case 2B) for $C(D, D') : = A(D, D') + \|Q_m(D) - Q_m(D')\|_2^2 \times \frac{1}{\min(\Sigma_1)}].**

- **the corresponding privacy level can be given by $(\epsilon_{old}, \delta_{budget})$-differential privacy with the following $\epsilon_{new}$:**

- $\epsilon_{new} = \epsilon_{old}$ in Cases 2A and 2C),

- $\epsilon_{new}^2 = \epsilon_{old}^2 + \epsilon_{squared \text{-} cost}$ in Case 1 for $\epsilon_{squared \text{-} cost}$

- $\epsilon_{new}^2 = \epsilon_{old}^2 + \epsilon_{squared \text{-} cost}$ in Case 2B) for $\epsilon_{squared \text{-} cost}$

- $\epsilon_{new}^2 = \epsilon_{old}^2 + \epsilon_{squared \text{-} cost}$ satisfying Gaussian($\Delta Q_m, \epsilon_{squared \text{-} cost}, \delta_{budget}) = \sigma_m$.

Theorem 2 explains the rules to update the privacy cost in Algorithm [1]. Specifically, Result 5 gives Lines 3 and 5 for Case 1), and Result 6 gives Lines 15 and 19 for Case 2B).

Proof. The proof is in Appendix C of the online full version [20] (i.e., this paper).

F. Analyzing the Total Privacy Cost

Based on Theorem 2 we now analyze the total privacy cost when our system calls Algorithm [1] consecutively.

At the beginning when no query has been answered, we have $V = 0$ (note that $\mathcal{N}(0, 0) \equiv 0$). Then by induction via Corollary 1 and Theorem 2 for the consecutive use of Algorithm [1] the privacy loss is always in the form of $N(\frac{1}{2}, V)$ for some $V$. In our Algorithm [1] the privacy loss changes only when the query being answered belongs to Cases 1) and 2B). More formally, we have the following theorem.

Theorem 3. Among queries $Q_1, Q_2, \ldots, Q_m$, let $N_1, N_2A, N_2B$, and $N_2C$ be the set of $i \in \{1, 2, \ldots, m\}$ such that $Q_i$ is in Cases 1), 2A), 2B), and 2C), respectively. For queries in Case 2B), let $T_{2B}$ be the set of query types. In Case 2B), for query type $t \in T_{2B}$, suppose the number of type-$t$ queries $m_t$, and let these type-$t$ queries be $Q_{j_1, t}, Q_{j_2, t}, \ldots, Q_{j_{m_t}, t}$ for indices $j_1, j_2, \ldots, j_{m_t}$ (ordered chronologically) all belonging to $N_{2B}$. From Case 2B) of Algorithm [7] we have $\sigma_{j_1, t} > \sigma_{j_2, t} > \ldots > \sigma_{j_{m_t}, t}$, and for $k \in \{2, 3, \ldots, m_t\}$, $\tilde{Q}_{j_{k-1}, t}$ is answered by reusing $\sigma_{j_{k-1}, t}^2$ fraction of old noise in $\tilde{Q}_{j_{k-1}, t}$, more specifically, $\tilde{Q}_{j_{k-1}, t} = Q_{j_{k-1}, t} + \frac{\sigma_{j_{k-1}, t}^2}{\sigma_{j_{k-1}, t}^2} \tilde{Q}_{j_{k-1}, t} - Q_{j_{k-1}, t} + \mathcal{N}(0, \sigma_{j_{k-1}, t}^2 - \sigma_{j_{k-1}, t}^2)$ from Line 22 of Algorithm [7] for Case 2B). We also consider that $Q_{j_1, t}$ is answered by reusing $\sigma_{j_1, t}^2$ fraction of old noise in $\tilde{Q}_{j_1, t}$. Let the $\ell_2$-sensitivity of a type-$t$ query be $\Delta$.

In the example provided in Table 7 [20] we have $N_1 = \{1, 2, 3\}$, $N_2A = \{7\}$, $N_2B = \{6, 9, 10, 11, 13\}$, and $N_2C = \{8, 12\}$. $T_{2B} = \{type-1, type-2, type-3\}$. In Case 2B), the number of type-1 queries is $m_1 = 2$, and these type-1 queries are $Q_6$ and $Q_{10}$ so $j_{1, 1} = 6$ and $j_{1, 2} = 10$ (also $j_{1, 1} = j_{1, 2} = 1$ since $Q_6$ reuses $Q_1$); the number of type-2 queries is $m_2 = 3$, and these type-2 queries are $Q_5, Q_9$, and $Q_{11}$ so $j_{2, 1} = 5$ and $j_{2, 2} = 9$. $j_{2, 3} = 11$ (also $j_{2, 1} = 2$ since $Q_5$ reuses $Q_3$); the number of type-3 queries is $m_3 = 1$, and this type-3 query is $Q_{13}$ so $j_{3, 1} = 13$ (also $j_{3, 0} = 3$ since $Q_{13}$ reuses $Q_3$).

Then after Algorithm [7] is used to answer all $m$ queries with query $Q_i$ being answered under $(\epsilon_i, \delta_i)$-differential privacy, we have:

- The total privacy loss with respect to neighboring datasets $D$ and $D'$ is given by $N(\frac{G(D, D')}{2}, G(D, D'))$, where

\[
G(D, D') : = \sum_{t \in N_1} \frac{\|Q_{j_{m_t}, t} - Q_{j_{m_t}, t}'\|_2^2}{\sigma_{j_{m_t}, t}^2} + \sum_{t \in T_{2B}} \left[ \frac{\|Q_{j_{1, t}}(D) - Q_{j_{1, t}}(D')\|_2^2}{\sigma_{j_{1, t}}^2} - \frac{\|Q_{j_{1, t}}(D) - Q_{j_{1, t}}(D')\|_2^2}{\sigma_{j_{1, t}}^2} \right],
\]

(6)

and the first summation is the contribution from queries in Case 1), and the second summation is the contribution from queries in Case 2B). When $D$ and $D'$ iterate the space of neighboring datasets, the maximum of $\|Q_{j_{1, t}}(D) - Q_{j_{1, t}}(D')\|$ is $Q_i$’s $\ell_2$-sensitivity $\Delta Q_i$, and the maximum of both $\|Q_{j_{m_t}, t} - Q_{j_{m_t}, t}'\|_2$ and $\|Q_{j_{1, t}}(D) - Q_{j_{1, t}}(D')\|_2$ are $\Delta$ (type-4) since $Q_{j_{m_t}, t}$ and $Q_{j_{1, t}}$ are both type-$t$ queries, we obtain

\[
\max_{\text{neighboring datasets } D, D'} G(D, D') = \sum_{t \in N_1} \frac{\Delta Q_{j_{1, t}}^2}{\sigma_{j_{1, t}}^2} + \sum_{t \in T_{2B}} \left[ \frac{\Delta Q_{j_{m_t}, t}^2}{\sigma_{j_{m_t}, t}^2} - \frac{\Delta Q_{j_{m_t}, t}^2}{\sigma_{j_{m_t}, t}^2} \right].
\]

(7)

In the example provided in Table 4 on Page 5 max $\max_{\text{neighboring datasets } D, D'} G(D, D')$ is given by

\[
\frac{\Delta Q_6^2}{\sigma_6^2} + \frac{\Delta Q_{10}^2}{\sigma_{10}^2} + \frac{\Delta Q_5^2}{\sigma_5^2} + \left[ \frac{\Delta Q_5^2}{\sigma_5^2} - \frac{\Delta Q_5^2}{\sigma_5^2} \right] + \left[ \frac{\Delta Q_9^2}{\sigma_9^2} - \frac{\Delta Q_9^2}{\sigma_9^2} \right] + \left[ \frac{\Delta Q_{11}^2}{\sigma_{11}^2} - \frac{\Delta Q_{11}^2}{\sigma_{11}^2} \right] + \left[ \frac{\Delta Q_{11}^2}{\sigma_{11}^2} - \frac{\Delta Q_{11}^2}{\sigma_{11}^2} \right].
\]


\[
\frac{[\Delta(\text{type-2})]^2}{\sigma_1^2} - \frac{[\Delta(\text{type-2})]^2}{\sigma_2^2} + \frac{[\Delta(\text{type-3})]^2}{\sigma_1^2} - \frac{[\Delta(\text{type-3})]^2}{\sigma_2^2} = \frac{[\Delta(\text{type-1})]^2}{\sigma_{11}^2} + \frac{[\Delta(\text{type-2})]^2}{\sigma_{12}^2} + \frac{[\Delta(\text{type-3})]^2}{\sigma_{12}^2}.
\]

- From Lemma 2 the total privacy cost of our Algorithm 7 can be given by \((\epsilon, \delta, \delta_{\text{budget}})\)-differential privacy for \(\epsilon_{\text{ours}}\) satisfying
\[
[\text{Gaussian}(1, \epsilon_{\text{ours}}, \delta_{\text{budget}})]^{-2} = \max_{\text{neighboring datasets } D, D'} G(D, D'),
\]

or \((\epsilon, \delta)\)-differential privacy for any \(\epsilon\) and \(\delta\) satisfying
\[
[\text{Gaussian}(1, \epsilon, \delta)]^{-2} = \max_{\text{neighboring datasets } D, D'} G(D, D').
\]

**Proof.** The proof is in Appendix D of the online full version \([\text{20}]\) (i.e., this paper).

**Remark 3.** Theorem 3 can be used to understand that our Algorithm 7 incurs less privacy cost than that of the naive algorithm where \(n\) queries are answered independently. As given in Corollary 2, the privacy loss with respect to neighboring datasets \(D\) and \(D'\) is given by \(F(D) - F(D') = \sum_{i=1}^{n} \left| Q_i(D) - Q_i(D') \right|^2 \) for \(F(D) = \sum_{i=1}^{n} \left| Q_i(D) - Q_i(D') \right|^2 \), where the expression of \(G(D, D')\) can be given by Eq. (8) above. From Lemma 2 the privacy cost of the naive algorithm can be given by \((\epsilon_{\text{naive}}, \delta_{\text{budget}})\)-differential privacy for \(\epsilon_{\text{naive}}\) satisfying
\[
[\text{Gaussian}(1, \epsilon_{\text{naive}}, \delta_{\text{budget}})]^{-2} = \max_{\text{neighboring datasets } D, D'} F(D, D'),
\]

which with Eq. (8) in Theorem 3 and the expression of \(G(D, D')\) in Lemma 7 implies \(\epsilon_{\text{ours}} = \sqrt{\frac{\max_{\text{neighboring datasets } D, D'} G(D, D')}{\max_{\text{neighboring datasets } D, D'} F(D, D')}} \leq 1\), where the equal sign is taken only when all \(n\) queries are different so no noise reuse is incurred in our Algorithm 7.

G. **Computing the \(\ell_2\)-sensitivity of A Query**

The \(\ell_2\)-sensitivity of a query \(Q\) is defined as the maximal \(\ell_2\) distance between the (true) query results for any neighboring datasets \(D\) and \(D'\) that differ in one record: 
\[
\Delta_Q = \max_{\text{neighboring datasets } D, D'} ||Q(D) - Q(D')||_2.
\]

For one-dimensional real-valued query \(Q\), \(\Delta_Q\) is simply the maximal absolute difference between \(Q(D)\) and \(Q(D')\) for any neighboring datasets \(D\) and \(D'\). In Section VIII for performance evaluation, we define neighboring datasets by considering modifying an entry. Then if the dataset has \(n\) users’ information, and the domain of each user’s income is within the interval \([\text{min\_income}, \text{max\_income}]\), \(\Delta_Q\) for query \(Q\) being the average income of all users is \(\frac{\text{max\_income} - \text{min\_income}}{n}\) since this is the maximal variation in the output when a user’s record changes. Similarly, \(\Delta_Q\) for query \(Q\) being the percentage of female users is \(\frac{n}{\text{max\_users} - \text{min\_users}}\).

**IV. IMPLEMENTATION CHALLENGES OF OUR BLOCKCHAIN-BASED SYSTEM**

We now discuss challenges and countermeasures during the design and implementation of our blockchain-based system.

**Smart Contract fetches external data.** Ethereum blockchain applications, such as Bitcoin scripts and smart contracts are unable to access and fetch directly the external data they need. However, in our application, Blockchain needs to fetch data from Server then returns them to Client. This requires smart contract to send the HTTP POST request. Hence, we use the Provable, a service integrated with a number of blockchain protocols and can be accessed by non-blockchain applications as well. It guarantees that data fetched from the original data-source is genuine and unaltered.

By using the Provable, smart contracts can directly access data from web sites or APIs. In our case, Blockchain can send HTTP requests to Server with parameters, and then process and store data after Server responds successfully.

**Mathematical operations with Solidity.** Blockchain is written using soliity language which is designed to target the Ethereum Virtual Machine. However, current soliity language does not have inherent functions for complex mathematical operations, such as taking the square root or logarithm. We write a function to implement the square root operation. To avoid using Lemma 1 to compute logarithm in Blockchain, we generate Gaussian noise in Client, and pass the value to Blockchain as one of the parameters in function Query-Match. Besides, current Solidity version cannot operate float or double type data. To keep the precision, we scale up the noise amount during calculation, and then scale down the value before returning the noisy data to analysts.

**V. EXPERIMENTS**

In this section, we perform experiments to validate the proposed system and algorithm are effective in saving privacy cost according to the system flow shown in Fig. 2. More specifically, a user sends a query through the UI, and then Client receives the query and forwards it to Blockchain smart contract. After the smart contract checks with stored data, it will decide whether to return the noisy response to Client directly or forward the request to Server. If Server receives the request, it will generate and return a noisy response to the smart contract.

**A. Experiment Setup**

We prototype a web application based on the system description in Section III. We use the Javascript language to write Client, whereas the Solidity language is for Blockchain.
smart contract. Besides, Web3 is used as the Javascript API to exchange information between Client and Blockchain smart contract, and then Node.js and express web framework are leveraged to set up Server. In addition, MongoDB is used as the database to host the real-world dataset. Our designed smart contracts are deployed on the Ropsten testnet with the MetaMask extension of the Chrome browser. The Ropsten testnet is a testing blockchain environment maintained by Ethereum, and it implements the same Proof-of-Work protocol as the main Ethereum network.

We evaluate the performance of the proposed differential privacy mechanism based on a real-world dataset containing American community survey samples extracted from the Integrated Public Use Microdata Series at https://www.ipums.org. There are 5000 records in the dataset. Each record includes the following numerical attributes: “Total personal income”, “Total family income”, “Age”, and categorical attributes: “Race”, “Citizenship status”. We consider five types of queries: “average family income”, “Age”, and categorical attributes: “Race”, “Citizenship status”. We set the privacy budget as 8, 10, 0.2, 0.2, and 0.2, respectively. More details about the sensitivity computation of a query are given in Section III-G.

B. Experimental Results

The benchmark of our experiment is a naive scheme which does not contain the Algorithm 1 in the smart contract. That is, every query will be forwarded by the smart contract to Server to get the noisy response. Hence, no differential privacy cost can be reused in the naive scheme.

First, we use an experiment to validate that our proposed Algorithm 1 is effective in saving privacy cost. Thus, we design a performance comparison experiment by tracking privacy cost using our Algorithm 1 and the naive scheme, respectively. Specifically, we deploy two smart contracts implementing our Algorithm 1 and the naive scheme on the Ropsten testnet, respectively. Then, we send 150 requests randomly selected in five query types from Client of the web application, and record the privacy cost of each query. As shown in Fig. 3 compared with the naive scheme, the proposed algorithm saves significant privacy cost. When the number of the queries is 150, the differential-privacy cost of Algorithm 1 is about 52% less than that of the naive algorithm. We also observe that the privacy cost in the proposed scheme increases slowly when the number of queries increases, even trending to converge to a specific value. The reason is that, in Algorithm 1 for each query type, we can always partially or fully reuse previous noisy answers when the query type is asked for a second time or more. Therefore, in our scheme, many queries are answered without incurring additional privacy cost if noisy responses fully reuse previous noisy answers.

Second, to prove that the proposed Algorithm 1 retains the accuracy of the dataset, we design another experiment to compare the sum of relative errors. We use the same smart contracts as those in the last experiment. We accumulate relative errors incurred in each query. Fig. 4 shows that the sum of relative errors of the Algorithm 1 is comparable with that of the naive scheme. Since relative errors are similar between two schemes, our results demonstrate that the proposed Algorithm 1 keeps the accuracy.

As a summary, Fig. 3 and Fig. 4 together demonstrate that our Algorithm 1 can save privacy cost significantly without sacrificing the accuracy of the dataset.

VI. RELATED WORK

In this section, we first compare our paper and a closely related study [22], and then discuss other related work.

A. Comparison with Yang et al. [22]

Note that Yang et al. [22] utilized blockchain and differential privacy technologies to achieve the security and privacy protection during data sharing. Compared with [22], we summarize the differences between our work and [22] as follows.

• Although Algorithm 1 of [22] claims to satisfy $\epsilon$-differential privacy, it does not since the noisy output’s domain (i.e., the set of all possible values) depends on the input. The explanation is as follows. In [22], for two neighboring datasets $D$ and $D'$, there exists a subset $Y$ of outputs such that $\Pr[\tilde{Q}(D) \in Y] > 0$ but $\Pr[\tilde{Q}(D') \in Y] = 0$. This means...
• \([22]\) assumes the same privacy parameter for all queries.
• \([22]\) does not discuss how to choose the small additional privacy parameter in its Algorithm \([1]\)
• In \([22]\), when a query is asked for the first time, the Laplace mechanism of \([2]\) for \(\epsilon\)-differential privacy is used to add Laplace noise to the true query result. Afterwards, \([22]\) adds new Laplacian noise on previous noisy output, which makes the new noisy response no longer follow Laplace distribution since the sum of independent Laplace random variables does not follow a Laplace distribution. Hence, the analysis in \([22]\) is not effective.

We consider \((\epsilon, \delta)\)-differential privacy by using the Gaussian noise. The advantage of Gaussian noise over Laplace noise lies in the easier privacy analysis for the composition of different privacy-preserving algorithms, since the sum of independent Gaussian random variables still follows the Gaussian distribution, while the sum of independent Laplace random variables does not obey a Laplace distribution.

\section*{B. Other Related Work}

Differential privacy, a strong mathematical model to guarantee the database’s privacy, has attracted much attention in recent years. Blockchain is a fast-growing technology to provide security and privacy in a decentralized manner \([12,23]\). Feng \textit{et al.} \([27]\) summarized prior studies about privacy protection in blockchain system, including methodology for identity and transaction privacy preservation. In the following, we will introduce more recent studies utilizing blockchain or privacy techniques to provide privacy or security protection in identity, data, and transactions.

\textbf{Leveraging Blockchains for Identity Privacy/Security Protection.} A few studies have focused on leveraging the blockchain to guarantee privacy/security in access control management or identity protection. For example, Zyskind \textit{et al.} \([28]\) and Xia \textit{et al.} \([29]\) both used blockchain in access control management. Zyskind \textit{et al.} \([28]\) created a decentralized personal data management system to address users’ concerns about privacy when using third-party mobile platforms. Xia \textit{et al.} \([29]\) proposed a permissioned blockchain-based data sharing framework to allow only verified users to access the cloud data. Lu \textit{et al.} \([30]\) developed a private and anonymous decentralized crowdsourcing system ZebraLancer, which overcame data leakage and identity breach in traditional decentralized crowdsourcing. The above studies focused on identity privacy because the Blockchain is anonymous, whereas they did not consider the privacy protection for the database.

\textbf{Leveraging Blockchains for Data Privacy/Security Protection.} In addition to the identity privacy preservation, Hu \textit{et al.} \([31]\) replaced the central server with a smart contract and constructed a decentralized privacy-preserving search scheme for computing encrypted data while ensuring the privacy of data to prevent from misbehaviors of a malicious centralized server. Luongo \textit{et al.} \([32]\) used secure multi-party computation to design a privacy primitive named Keep which allows contracts to manage and use private data without exposing the data to the public blockchain for protecting smart contracts on public blockchains. Alternatively, we use the differential privacy standard to guarantee privacy. Moreover, blockchains are popular to be used for security protection of data sharing in IoT scenarios \([23,24,29]\).

\textbf{Leveraging Blockchains for Transaction Privacy/Security Protection.} Moreover, some previous studies used blockchain to guarantee security and privacy in transactions. For example, Henry \textit{et al.} \([11]\) proposed that the blockchain should use mechanisms that piggyback on the overlay network, which was ready for announcing transactions to de-link users’ network-level information instead of using an external service such as Tor to protect users’ privacy. Gervais \([33]\) proposed a quantitative framework to analyze the security of Proof-of-Work in blockchains, where the framework’s inputs included security, consensus, and network parameters. Herrera-Joancomartí and Pérez-Solà \([34]\) focused on privacy in bitcoin transactions. Sani \textit{et al.} \([35]\) proposed a new blockchain Xyreum with high-performance and scalability to secure transactions in the Industrial Internet of Things.

\textbf{Reusing Noisy Answers in Differentially Private Algorithms.} Some differential privacy algorithms reusing noisy answers were proposed to provide privacy protection. Xiao \textit{et al.} \([36]\) proposed an algorithm to correlate the Laplace noise added to different queries to improve the overall accuracy. Given a series of counting queries, the mechanism proposed by Li and Miklau \([37]\) selected a subset of queries to answer privately and used their noisy answers to derive answers for the remaining queries. For a set of non-overlapping counting queries, Kellaris and Papadopoulos \textit{et al.} \([38]\) pre-processed the counts by elaborate grouping and smoothing them via averaging to reduce the sensitivity and thus the amount of injected noise. Given a workload of queries, Yaroslavtsev \textit{et al.} \([39]\) introduced a solution to balance accuracy and efficiency by answering some queries more accurately than others.

\section*{VII. CONCLUSION AND FUTURE DIRECTIONS}

In this paper, we use a blockchain-based approach for tracking and saving differential-privacy cost. In our design, we propose an algorithm that reuses noise fully or partially for different instances of the same query type to minimize the accumulated privacy cost. The algorithm is proved the efficiency via a rigorous mathematical proof. Moreover, we design a blockchain-based system for conducting real-world experiments to confirm the effectiveness of the proposed approach.

A future direction is that for different but correlated queries, we will investigate how to reuse answers between them. In addition, our system is currently implemented based on Ethereum. To enable more extensions, we will implement our system using the Hyperledger Fabric whose smart contract code supports more sophisticated operations compared with Ethereum. Moreover, Ethereum 2.0 is expected to launch in 2020, which will equip with the Proof-of-Stake consensus mechanism \([40]\). Then we will upgrade our algorithm and system design to be compatible with new features.
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A. Proof of Theorem [1]

Proof. (i) \( \tilde{Q}_j \) is zero-mean Gaussian random variables with variance \( \sigma_j^2 \), privacy loss satisfies \( N(0, \sigma_j^2) \). After answering \( Q_1, Q_2, \ldots, Q_{m-1} \), the privacy loss \( L_{Q_1\|Q_2\|\ldots\|Q_{m-1}}(D, D') \) is given by \( N\left( \frac{A(D, D')}{2}, A(D, D') \right) \) as stated in Theorem [1].

When \( \tilde{Q}_j(D) \) and \( \tilde{Q}_m(D) \) take \( y_j \) and \( y_m \) respectively, \( \tilde{Q}_j(D) - Q_j(D) \) and \( \tilde{Q}_m(D) - Q_m(D) - r[\tilde{Q}_j(D) - Q_j(D)] \) take the following defined \( g_j \) and \( g_m \) respectively:

\[
\begin{align*}
g_j &:= y_j - Q_j(D), \quad g_m := y_m - Q_m(D) - r[y_j - Q_j(D)].
\end{align*}
\]

(9) For \( D' \) being a neighboring dataset of \( D \), we further define

\[
\begin{align*}
h_j &:= Q_j(D) - Q_j(D'), \quad h_m := Q_m(D) - Q_m(D'),
\end{align*}
\]

(10) so that

\[
\begin{align*}
g_j + h_j &= y_j - Q_j(D'), \quad g_m + h_m - rh_j = y_m - Q_m(D') - r[y_j - Q_j(D')].
\end{align*}
\]

(13) Note that \( h_j \) and \( h_m \) are the same since \( Q_j \) and \( Q_m \) are the same. From the above analysis, we obtain:

\[
\begin{align*}
&\mathbb{E}\left[ \tilde{Q}_m(D) = y_m \mid \tilde{Q}_j(D) = y_j \right] \\
&= \mathbb{E}\left[ Q_m(D) - Q_m(D) - r[\tilde{Q}_j(D) - Q_j(D)] = g_m \mid \tilde{Q}_j(D) = y_j \right] \\
&= \frac{1}{\sqrt{2\pi(\sigma_m^2 - r^2\sigma_j^2)}} e^{-\frac{(y_m - g_m)^2}{2(\sigma_m^2 - r^2\sigma_j^2)}},
\end{align*}
\]

(15)

where step (b) follows since where \( \tilde{Q}_j(D) - Q_j(D) \) is a zero-mean Gaussian random variable with variance \( \sigma_j^2 \) and \( \tilde{Q}_m(D) - Q_m(D) - r[\tilde{Q}_j(D) - Q_j(D)] \) is a zero-mean Gaussian random variable with variance \( \sigma_m^2 - r^2\sigma_j^2 \).

Similarly, for dataset \( D' \), we have:

\[
\begin{align*}
&\mathbb{E}\left[ \tilde{Q}_m(D') = y_m \mid \tilde{Q}_j(D') = y_j \right] \\
&= \mathbb{E}\left[ Q_m(D') - Q_m(D') - r[\tilde{Q}_j(D') - Q_j(D') \mid \tilde{Q}_j(D') = y_j \right] \\
&= \frac{1}{\sqrt{2\pi(\sigma_m^2 - r^2\sigma_j^2)}} e^{-\frac{(y_m + h_m - rh_j)^2}{2(\sigma_m^2 - r^2\sigma_j^2)}},
\end{align*}
\]

(16)

where step (b) follows since where \( \tilde{Q}_j(D') - Q_j(D') \) is a Gaussian random variable with variance \( \sigma_j^2 \) and \( \tilde{Q}_m(D') - Q_m(D') - r[\tilde{Q}_j(D') - Q_j(D')] \) is a zero-mean Gaussian random variable with variance \( \sigma_m^2 - r^2\sigma_j^2 \).

Then,

\[
\begin{align*}
\ln \mathbb{P}\left[ \tilde{Q}_m(D) = y_m \mid \tilde{Q}_j(D) = y_j \right] \\
&= \ln \mathbb{P}\left[ \tilde{Q}_m(D') = y_m \mid \tilde{Q}_j(D') = y_j \right] \\
&= \ln \frac{1}{\sqrt{2\pi(\sigma_m^2 - r^2\sigma_j^2)}} e^{-\frac{(y_m - g_m)^2}{2(\sigma_m^2 - r^2\sigma_j^2)}} \\
&= \frac{1}{\sqrt{2\pi(\sigma_m^2 - r^2\sigma_j^2)}} e^{-\frac{(y_m - h_m + rh_j)^2}{2(\sigma_m^2 - r^2\sigma_j^2)}} \\
&= \frac{g_m(h_m - rh_j)^2 - g_m^2}{2(\sigma_m^2 - r^2\sigma_j^2)} \\
&= \frac{g_m(h_m - rh_j)^2 + (h_m - rh_j)^2}{2(\sigma_m^2 - r^2\sigma_j^2)}. \quad (17)
\end{align*}
\]

Since \( g_m \) follows a zero-mean Gaussian distribution with variance \( \sigma_m^2 - r^2\sigma_j^2 \), clearly \( \frac{g_m(h_m - rh_j)^2}{(\sigma_m^2 - r^2\sigma_j^2)} \) follows a zero-mean Gaussian distribution with variance given by

\[
\begin{align*}
&\frac{(h_m - rh_j)^2}{\sigma_m^2 - r^2\sigma_j^2} \\
&= \frac{h_m - rh_j)^2}{\sigma_m^2 - r^2\sigma_j^2}. \quad (18)
\end{align*}
\]

Since \( Q_m(D) \) and \( Q_j(D) \) are the same, we obtain from Eq. (11) and Eq. (12) that \( h_j = h_m = Q_m(D) - Q_m(D') \), which we use to write Eq. (18) as

\[
\begin{align*}
&\frac{\|Q_m(D) - Q_m(D')\|_2^2(1 - r)^2}{\sigma_m^2 - r^2\sigma_j^2}.
\end{align*}
\]

(19)

Summarizing the above, privacy loss is

\[
B_r(D, D') := A(D, D') + \frac{\|Q_m(D) - Q_m(D')\|_2^2(1 - r)^2}{\sigma_m^2 - r^2\sigma_j^2}. \quad (20)
\]

(ii) The optimal \( r \) is obtained by minimizing \( B_r(D, D') \) and hence minimizing \( \frac{(1-r)^2}{\sigma_m^2 - r^2\sigma_j^2} \). Analyzing the monotonicity of this expression, we derive the optimal \( r \) as in Eq. (4). The first-order derivative of \( B_r(D, D') \) to \( r \) is:

\[
\begin{align*}
B_r(D, D')' &= \frac{-2r(\sigma_j^2 - \sigma_m^2)(r - 1)}{r^2\sigma_j^2 - \sigma_m^2}. \quad (21)
\end{align*}
\]

- Case 1: if \( \sigma_m \geq \sigma_j \), \( B_r(D, D') \geq 0 \) when \( r \in [1, \frac{\sigma_m}{\sigma_j}] \), and \( B_r(D, D')' < 0 \) when \( r \in (-\infty, 1) \cup (\frac{\sigma_m}{\sigma_j}, \infty) \). Hence, the optimal \( r \) to minimize \( B_r(D, D') \) is at \( r = 1 \).

- Case 2: if \( \sigma_m < \sigma_j \), \( B_r(D, D') \geq 0 \) when \( r \in [\frac{\sigma_m}{\sigma_j}, 1] \), and \( B_r(D, D')' < 0 \) when \( r \in (-\infty, \frac{\sigma_m}{\sigma_j}) \cup (1, \infty) \). Hence, the optimal \( r \) to minimize \( B_r(D, D') \) at \( r = \frac{\sigma_m}{\sigma_j} \).

Thus, we obtain optimal values of \( r \) as Eq. (4).

□

B. Proof of Lemma [2]

Proof. Consider a query \( R \) with \( \ell_2 \)-sensitivity being 1. Let \( \tilde{R} \) be the mechanism of adding Gaussian noise amount \( \mu := \sqrt{\max_{D, D' \in V(D, D')} (D, D')} \) to \( R \). From Corollary [1] the privacy loss of randomized mechanism \( R \) with respect to neighboring datasets \( D \) and
$D'$ is given by $\mathcal{N}(U(D, D'), U(D, D'))$ for $U(D, D') := \frac{(R(D) - R(D'))}{\|R(D) - R(D')\|_2^2}$. By considering the $\ell_2$-sensitivity of $R$ (i.e., $\|R(D) - R(D')\|_2$) as 1, $\max_{\text{neighboring datasets } D, D'} V(D, D')$ and $\max_{\text{neighboring datasets } D, D'} U(D, D')$ are the same. In addition, from Theorem 5 of [41], letting $Y$ (resp., $\tilde{R}$) satisfy $(\epsilon, \delta)$-differential privacy can be converted to a condition on $\max_{\text{neighboring datasets } D, D'} V(D, D')$ (resp., $\max_{\text{neighboring datasets } D, D'} U(D, D')$). Then letting $Y$ satisfy $(\epsilon, \delta)$-differential privacy is the same as letting $\tilde{R}$ satisfy $(\epsilon, \delta)$-differential privacy. From Lemma 1, $\tilde{R}$ achieves $(\epsilon, \delta)$-differential privacy with $\mu = \text{Gaussian}(1, \epsilon, \delta)$; i.e., if $\max_{\text{neighboring datasets } D, D'} V(D, D') = \text{Gaussian}(1, \epsilon, \delta)$. Summarizing the above, we complete proving Lemma 2.

C. Proof of Theorem 2

Proof. We use Theorem 1 to show Results 1, 2, and 3 of Theorem 2. Proof of 1: In Case 2A) and Case 2C), $Q_m$ can reuse previous noise. Hence, the privacy loss will still be $\mathcal{N}(\mu, \sigma_m)$, where $\mu$ is the overall contribution of queries in Cases 1) to $G(D, D')$ and $\sigma_m$ is given by Eq. (6). From Theorem 2, among all $Q_m$ with query $Q_i$ being answered under $(\epsilon, \delta)^\prime$-differential privacy, the total privacy loss with respect to neighboring datasets $D$ and $D'$ is given by $\mathcal{N}(\sigma^2_{Q_{2m}} G(D, D'))$ for some $G(D, D')$.

Next, we use Theorem 2 to further show that the expression of $G(D, D')$ is given by Eq. (6). From Theorem 2, among all queries, only queries belonging to Cases 1) and 2B) contribute to $G(D, D')$. Below we discuss the contributions respectively.

With $N_1$ denoting the set of $i \in \{1, 2, \ldots, n\}$ such that $Q_i$ is in Cases 1), we know from Result 2 of Theorem 2 that the contributions of queries in Cases 1) to $G(D, D')$ is given by

$$\sum_{i \in N_1} \|Q_i - \tilde{Q}_i\|^2_{\sigma^2_i}.$$

Below we use Result 3 of Theorem 2 to compute the contributions of queries in Case 2B) to $G(D, D')$. For $T_{2B}$ being the set of query types in Case 2B), we discuss each query type $t \in T_{2B}$ respectively.

From Result 3 of Theorem 2, the contribution to $G(D, D')$ by answering $Q_{jt, 1}$ under differential privacy is

$$\|Q_{jt, 1} - \tilde{Q}_{jt, 1}\|^2_{\sigma^2_{jt, 1}}.$$ 

Similarly, the contribution to $G(D, D')$ by answering $Q_{jt, 2}$ under differential privacy is

$$\|Q_{jt, 2} - \tilde{Q}_{jt, 2}\|^2_{\sigma^2_{jt, 2}}.$$ 

Similar analyses are repeated for additional type-$t$ queries in Case 2B). In particular, for each $s \in \{1, 2, \ldots, m_t\}$, the contribution to $G(D, D')$ by answering $Q_{jt, s}$ under differential privacy is

$$\sum_{s \in \{1, 2, \ldots, m_t\}} \|Q_{jt, s} - \tilde{Q}_{jt, s}\|^2_{\sigma^2_{jt, s}}.$$ 

Summing all contributions for each query type $t \in T_{2B}$, the contributions to $G(D, D')$ by answering $Q_{jt, 1}, Q_{jt, 2}, \ldots, Q_{jt, m_t}$ under differential privacy is

$$\sum_{s \in \{1, 2, \ldots, m_t\}} \|Q_{jt, s} - \tilde{Q}_{jt, s}\|^2_{\sigma^2_{jt, s}}.$$
s ∈ \{1, 2, \ldots, m_t\}. Hence, we write (24) as

\[
\sum_{s \in \{1, 2, \ldots, m_t\}} \left\{ \left[ \frac{\|Q_{j_{t,s}}(D) - Q_{j_{t,s}}(D')\|_2^2}{\sigma_{j_{t,s}}^2} \right] - \left[ \frac{\|Q_{j_{t,s-1}}(D) - Q_{j_{t,s-1}}(D')\|_2^2}{\sigma_{j_{t,s-1}}^2} \right] \right\} = \left[ \frac{\|Q_{j_{t,m_t}}(D) - Q_{j_{t,m_t}}(D')\|_2^2}{\sigma_{j_{t,m_t}}^2} \right] - \left[ \frac{\|Q_{j_{t,0}}(D) - Q_{j_{t,0}}(D')\|_2^2}{\sigma_{j_{t,0}}^2} \right].
\] (25)

Summing all (25) for \( t \in T_{2B} \), the contributions to \( G(D, D') \) by answering all queries in Case 2B) is

\[
\sum_{t \in T_{2B}} \left\{ \left[ \frac{\|Q_{j_{t,m_t}}(D) - Q_{j_{t,m_t}}(D')\|_2^2}{\sigma_{j_{t,m_t}}^2} \right] - \left[ \frac{\|Q_{j_{t,0}}(D) - Q_{j_{t,0}}(D')\|_2^2}{\sigma_{j_{t,0}}^2} \right] \right\}. \] (26)

Then \( G(D, D') \) as the sum of (22) and (26) is given by Eq. (6).

Summarizing the above, we have proved that after Algorithm 1 is used to answer all \( n \) queries under differential privacy, the total privacy loss with respect to neighboring datasets \( D \) and \( D' \) is given by \( \mathcal{N}(\frac{G(D, D')}{2}, G(D, D')) \) for \( G(D, D') \) in Eq. (6). Furthermore, under

\[
\max_{\text{neighboring datasets } D, D'} \|Q_i(D) - Q_i(D')\|_2 = \Delta Q_i
\]

and

\[
\max_{\text{neighboring datasets } D, D'} \|Q_{j_{t,m_t}}(D) - Q_{j_{t,m_t}}(D')\|_2 = \max_{\text{neighboring datasets } D, D'} \|Q_{j_{t,0}}(D) - Q_{j_{t,0}}(D')\|_2 = \Delta(\text{type-t}),
\]

we use Eq. (6) to have \( \max_{\text{neighboring datasets } D, D'} G(D, D') \) given by Eq. (7).

Finally, from Lemma 2, the total privacy cost of our Algorithm 1 can be given by \((\epsilon_{\text{ours}}, \delta_{\text{budget}})\)-differential privacy for \( \epsilon_{\text{ours}} \) satisfying

\[
[\text{Gaussian}(1, \epsilon_{\text{ours}}, \delta_{\text{budget}})]^{-2} = \max_{\text{neighboring datasets } D, D'} G(D, D').
\]