Fisher informations and local asymptotic normality for continuous-time quantum Markov processes

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Abstract
We consider the problem of estimating an arbitrary dynamical parameter of an open quantum system in the input–output formalism. For irreducible Markov processes, we show that in the limit of large times the system-output state can be approximated by a quantum Gaussian state whose mean is proportional to the unknown parameter. This approximation holds locally in a neighbourhood of size $t^{-1/2}$ in the parameter space, and provides an explicit expression of the asymptotic quantum Fisher information in terms of the Markov generator. Furthermore we show that additive statistics of the counting and homodyne measurements also satisfy local asymptotic normality and we compute the corresponding classical Fisher informations. The general theory is illustrated with the examples of a two-level system and the atom maser. Our results contribute towards a better understanding of the statistical and probabilistic properties of the output process, with relevance for quantum control engineering, and the theory of non-equilibrium quantum open systems.

Keywords: quantum open systems, system identification, quantum Markov processes, quantum Fisher information, local asymptotic normality, continuous time measurements

(Some figures may appear in colour only in the online journal)

1. Introduction

The last decades have witnessed rapid progress in the development of quantum technologies [1, 2]. These successes rely on the ability to create and control specific target states which are
used as resources in quantum communication [3], quantum computing [4] or quantum metrology [5], and other applications. Effective quantum control is a challenging experimental task, partly because it requires a good understanding of the system’s Hamiltonian and its interaction with the environment. Therefore, the estimation of dynamical parameters becomes an essential enabling tool for quantum technology.

In this paper, the system identification problem refers to the estimation of dynamical parameters of an open system in the input–output formalism [6], which is routinely used in quantum optics [7] and quantum control theory [8]. As illustrated in figure 1, the system is indirectly monitored by performing continuous-time measurements in the output channels [9, 10]. The stochastic measurement trajectory is then used for the estimation of an unknown parameter [11–13], e.g. the coupling constant between the system and the field. Similar problems have been investigated in other system identification scenarios such as quantum channel tomography [14], the estimation of the Hamiltonian of a closed quantum system [15, 16], or the estimation of the Lindblad generator of an open system in the Markov approximation [17].

Our study focuses on two distinct aspects of the system identification problem. Firstly we look at the joint system-output state in the limit of large times. We show that this state can be approximated by a quantum Gaussian state whose mean is proportional to the unknown parameter, for a range of parameters localized in a region of the size of the statistical uncertainty $t^{-1/2}$. From a statistical perspective, the quantum statistical model becomes equivalent to a Gaussian one, which allows us to compute the asymptotic quantum Fisher information, providing the absolute upper bound on the estimation precision. An alternative method to compute the quantum Fisher information is described in [13], see also [18] and [19].

The second result concerns the statistical properties of the counting and homodyne continuous-time measurements performed on the output. We show that the total counts statistics and the integrated homodyne currents, also satisfy local asymptotic normality (LAN), in the sense of convergence in distribution to one-dimensional Gaussian models with unknown mean and fixed variance. Furthermore we provide explicit expressions for their classical Fisher informations. In general, such statistics are less informative, but computationally much cheaper, than standard estimation methods such as the maximum likelihood estimator. It is therefore useful to better understand the statistical power of different output statistics, as it has been shown in recent indepth studies of the atom maser [20, 21].

LAN for quantum systems has been previously investigated for systems of independent qubits [22, 23] or independent finite dimensional systems [24, 25]. Our work is a generalization for continuous time models of the theory developed for finitely correlated systems in [18, 26]. We also point out that the LAN of classical Markov processes has been derived in [27].

Figure 1. The input–output formalism. The input fields interact with a system characterized by a Hamiltonian $H^\alpha$ and Lindblad operators $L_j^\alpha$ and evolve into the output fields. The output is continuously monitored and the measurement outcomes are used to infer the unknown parameter $\alpha$. 
The paper is structured as follows. In the beginning of section 2 we introduce the model, an open quantum system whose markovian dynamics depends on an unknown parameter, and the tools of quantum stochastic calculus needed to prove the main result. Using the Trotter–Kato theorem, we then establish a general result concerning the convergence of ergodic one-parameter semigroups. In section 3.1 we use this convergence theorem to show that the joint system-output model converges to a Gaussian model in the limit of large times. Furthermore, in section 4 we prove that additive functionals of continuous output measurement processes such as counting and homodyne converge to a Gaussian model in the asymptotic regime. We illustrate the theoretical results with two examples: a two-level system (section 5.1) and the atom maser (section 5.2).

2. Quantum open systems background

In this section we briefly review the physical setup and mathematical formalism needed to derive the main result of this paper. We consider a quantum system coupled to an environment through $k$ interaction channels. We assume the environment is memory-less such that the dynamics of the open system is Markovian, i.e. the time evolution of the system is described by the master equation which integrates to a one-parameter semigroup of completely positive operators. The joint dynamics of the system and environment is described by a unitary operator which is the solution of a quantum stochastic differential equation driven by bosonic quantum noises representing the environment degrees of freedom [28]. The picture is completed by the input–output formalism of Gardiner and Zoller [6] depicted in figure 1, which describes the evolution of the input fields (initially in the vacuum state) into the output fields; the system can be monitored indirectly through continuous-time measurements in the output (e.g. photon counting or homodyne) to extract information about its state or about the dynamics.

In the following paragraphs we introduce the formalism of quantum stochastic calculus of Hudson and Parthasarathy [28, 29]. This is used to derive the equations for the dynamics of the model. The time evolution of the states and operators in the model is given in terms of certain one-parameter semigroups. Using the Trotter–Kato theorem we derive a convergence property for these semigroups similar to the results found in [30]. This derivation is essential for proving the main result of this paper, the LAN of quantum Markov processes.

2.1. Quantum stochastic calculus

Let $\mathcal{H}$ be the Hilbert space of the system which we assume to be finite dimensional. The Hilbert space of $k$ independent bosonic fields is $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^+; \mathbb{C}^k))$, which is the symmetric Fock space over the one particle space $\mathbb{C}^k \otimes L^2(\mathbb{R}^+; \mathbb{C}^k)$. Thus

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{m=1}^{\infty} L^2(\mathbb{R}^+; \mathbb{C}^k)^{\otimes m}.$$ 

We define the coherent vector associated to wave functions $f \in L^2(\mathbb{R}^+; \mathbb{C}^k)$ by

$$e(f) = e^{-\frac{1}{2} |f|^2} \left( 1 \oplus \bigoplus_{m=1}^{\infty} \frac{f^{\otimes m}}{\sqrt{m!}} \right).$$ (1)

The vacuum state is given by $e(0)$ and the inner product of two coherent vector is defined as

$$\langle e(f), e(g) \rangle = \exp \{-\frac{1}{2} ||f||^2 - \frac{1}{2} ||g||^2 + \langle f, g \rangle \}.$$ The coherent vectors are linearly independent and their linear span $\mathcal{D}$ is dense in $\mathcal{F}$. 

Let \( f_j \) be the \( j \)th component (with \( j = 1, \ldots, k \)) of \( f \in L^2(\mathbb{R}^+; \mathcal{C}^k) \) according to the standard basis in \( \mathcal{C}^k \). On \( \mathcal{D} \) we define the creation process \( A_{j,t}^\epsilon \), annihilation process \( A_{j,t} \), and counting process \( A_{j,t} \) defined by

\[
A_{j,t} e(f) = \langle \chi_{[0,t]} f_j \rangle e(f) = \int_0^t f_j(s) ds e(f),
\]

\[
\{ \langle e(g), A_{j,t}^\epsilon e(f) \rangle \} = \langle g_0 \chi_{[0,t]} \rangle \{ \langle e(g), e(f) \rangle \} = \int_0^t g(s) f_j(s) ds \{ \langle e(g), e(f) \rangle \},
\]

\[
\{ \langle e(g), A_{j,t} e(f) \rangle \} = \langle g_0 \chi_{[0,t]} f_j \rangle \{ \langle e(g), e(f) \rangle \} = \int_0^t g(s) f_j(s) ds \{ \langle e(g), e(f) \rangle \}. \tag{2}
\]

For \( 0 < s < t \) we can write \( L^2(\mathbb{R}^+; \mathcal{C}^k) = L^2((0, s); \mathcal{C}^k) \oplus L^2((s, t); \mathcal{C}^k) \oplus L^2((t, \infty); \mathcal{C}^k) \) which combined with the factorization property of the Fock space gives rise to the following tensor product structure

\[
\mathcal{F}(L^2(\mathbb{R}^+; \mathcal{C}^k)) = \mathcal{F}(L^2((0, s); \mathcal{C}^k)) \otimes \mathcal{F}(L^2((s, t); \mathcal{C}^k)) \otimes \mathcal{F}(L^2((t, \infty); \mathcal{C}^k)).
\]

This in turn, allows for the identification of the coherent vector with the product

\[
e(f) \equiv e(f_0) \otimes \ldots \otimes e(f_l) \otimes \ldots \otimes e(f_{l+t}) \quad \text{where} \quad f_{l+t} \equiv f_{X_{l+t}} \quad \text{and} \quad f_l \equiv f_{X_{l}}.
\]

Let \( M^{(k)}_t \) be one of the three processes defined in (2). Then \( M^{(k)}_t \) acts only on the ‘past’ and present Fock space i.e. given the factorization property of the Fock space we can write \( M^{(k)}_t = M^{(k)}_t \otimes \mathbb{1}_{(t, \infty)} \) and we say \( M^{(k)}_t \) is adapted with respect to this factorization. This property is used to define the stochastic increment

\[
dM^{(k)}_t e(f) \equiv \left( M^{(k)}_t \right)_d e(f) = e(f_0) \otimes \left( M^{(k)}_t \right)_d e(f_{l+t} \otimes \ldots \otimes e(f_{l+t})), \tag{3}
\]

Let \( X_{1,t} \) and \( X_{2,t} \) be two stochastic processes of the type (2), or more generally, processes defined by quantum stochastic differential equations [29]

\[
dX_{i,t} = \sum_k \alpha^{(k)}_{i,t} dM^{(k)}_t,
\]

where \( \alpha^{(k)}_{i,t} \) are adapted operator valued coefficients. Then the process \( X_{1,t}, X_{2,t} \) is adapted and its increment satisfies the quantum Ito rule

\[
d(X_{1,t}, X_{2,t}) = X_{1,t} dX_{2,t} + X_{2,t} dX_{1,t} + dX_{1,t} dX_{2,t}. \tag{4}
\]

The rules of multiplication of stochastic increments defined at the same time \( t \) are given in the following table

| \[ \frac{dA_{j,t}^\epsilon \ dA_{k,t} \ dt \ d\Lambda_{i,t} }{ } \] |
|---|---|---|---|---|
| \[ dA_{j,t}^\epsilon \] | 0 | 0 | 0 | 0 |
| \[ dA_{j,t} \] | \[ \delta_{ij} dt \] | 0 | 0 | \[ \delta_{ij} dA_{j,t} \] |
| \[ dt \] | 0 | 0 | 0 | 0 |
| \[ d\Lambda_{j,t} \] | \[ \delta_{ij} dA_{j,t}^\epsilon \] | 0 | 0 | \[ \delta_{ij} d\Lambda_{i,t} \] |

### 2.2. The Markov semigroup of the reduced system evolution

We consider a system with a finite dimensional Hilbert space \( \mathcal{H}_t \) and denote by \( H \) and \( L_j, j = 1, \ldots, k \) the system Hamiltonian and coupling with the \( k \)-bosonic fields representing the environment. The joint unitary dynamics is given by the unique solution of the quantum stochastic differential equation [28]
\[ dU_t = \left\{ \sum_{j=1}^{k} \left( L_j dA_j^* - L_j^* dA_j - \frac{1}{2} L_j L_j^* dt \right) - iH dt \right\} U_t, \]

where \( L_j dA_j^* \) stands for \( L_j \otimes dA_j^* \), and \( U_0 = 1 \) and e.g.

The joint state of the system and fields at time \( t \) is given by

\[ \rho_t = U_t (\rho_0 \otimes \omega) U_t^\dagger, \]

where \( \rho_0 \) is initial state of the system and \( \omega = \langle \Omega | \Omega \rangle \) is the joint vacuum state of the bosonic fields. Using the Ito rules and the fact that the expectation value of the stochastic increments vanishes in the vacuum, one can show that the reduced state of the system is \( \rho_t = \text{Tr}_\omega \{ \rho(t) \} \) can be written in terms of a one-parameter semigroup. Indeed by taking time differentials we obtain the following form of the master equation

\[ \begin{align*}
    \frac{d\rho_t}{dt} &= \left\{ \langle \Omega | \{ dU_t \rho_0 U_t + U_t \rho_0 dU_t + dU_t \rho_0 dU_t \} | \Omega \rangle \right\} \\
    &= \left\{ -i[H, \rho_t] + \sum_{j=1}^{k} \left( L_j \rho_t L_j^* - \frac{1}{2} \{ L_j^* L_j, \rho_t \} \right) \right\} dt \\
    &= \mathcal{L}_s(\rho_t) dt,
\end{align*} \]

where \( \mathcal{L}_s \) is the Lindblad generator in the Schrödinger picture. In its integral form, the reduced evolution of the system is given in terms of a semigroup of trace preserving completely positive maps, characteristic of Markov dynamics

\[ \rho_t = e^{t \mathcal{L}_s} [\rho_0] \equiv T_{t}[\rho_0]. \]

In the dual, or Heisenberg picture the Lindblad generator is \( \mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \)

\[ \mathcal{L}(X) = -i[H, X] + \sum_{j=1}^{k} \left( L_j^* X L_j - \frac{1}{2} \{ L_j^* L_j, X \} \right) \]

and the following duality holds for generators as well as for the semigroups

\[ \text{Tr}(\rho \mathcal{L}(X)) = \text{Tr}(\mathcal{L}_s(\rho)X), \quad \text{Tr}(\rho T_t(X)) = \text{Tr}(T_t(\rho)X). \]

The Markov dynamics has at least one stationary state, i.e. \( T_{\infty}[\rho_0] = \rho_\infty \) or equivalently \( \mathcal{L}_s \rho_\infty = 0 \). Throughout the paper we will restrict our attention to irreducible semigroups which are characterized by the fact that the stationary state is unique and full rank, and any initial state converges in the long run to this stationary state i.e.

\[ \lim_{t \to \infty} T_{t}[\rho_0] = \rho_\infty. \]

An important property of irreducible semigroups which will be used in the paper is the existence of a spectral gap: the Lindblad generator has a non-degenerate eigenvalue equal to zero (corresponding to the stationary state) and all other eigenvalues have strictly negative real part, see theorem 5.4 in [31].

2.3. Output processes

We now turn our attention to the evolution of observables, in particular field observables which carry information about the dynamics, and can be measured continuously in time. This is described by the input–output formalism [6], in which the ‘input’ fields are perturbed by the interaction with the system and propagate out as ‘output’ fields. Let \( M_t \) be one of the fundamental stochastic process of the type (2), which can be seen as an ‘input’ process \( M_t^m \equiv M_t \); the corresponding ‘output’ is obtained by evolving the input with the unitary \( U_t \).
The observed stochastic processes correspond to physical measurements in the environment. We consider two such processes here corresponding to particle counting and homodyne measurements.

### 2.3.1. Counting measurements

We first consider the counting process obtained by detecting photons in the \( i \)-th output channel. The associated quantum stochastic process is

\[
\Lambda_{\text{out}}^{i} = U_{i}^{*} \Lambda_{i} U_{i},
\]

where \( X_{i} = U_{i}^{*} (X \otimes 1) U_{i} \) denotes the evolved system observable \( X \). This implies that in the stationary regime the average counts rate per unit of time is \( \langle L_{i} \rangle_{\text{st}} = \text{Tr}(\rho_{\text{in}} L_{i}^{*} L_{i}) \).

For later use, we introduce a contractions semigroup on \( B(H_{i}) \), which can be used to compute the characteristic function of \( \Lambda_{\text{out}}^{i} \), and therefore encodes the distribution of the counting operators. Similarly to the derivation of the master equation, one can show that \( S_{\text{st}}^{i} : B(H_{i}) \to B(H_{i}) \) defined by

\[
S_{\text{st}}^{i}(X) = \left\langle \Omega \left| U_{i}^{*} (X \otimes e^{i\Lambda_{i}^{\text{out}}}) U_{i} \right| \Omega \right\rangle.
\]

is a contractions semigroup with generator

\[
\mathcal{L}^{i}(X) = \mathcal{L}(X) + (e^{i\theta} - 1)L_{i}^{*}XL_{i}.
\]

In particular, the characteristic function of \( \Lambda_{\text{out}}^{i} \) for an initial state \( \rho_{\text{in}} \) is

\[
\phi_{\Lambda}^{i}(s) = \mathbb{E}(e^{s\Lambda_{\text{out}}^{i}}) = \text{Tr}\left(\rho_{\text{in}} S_{\text{st}}^{i}(1)\right).
\]

### 2.3.2. Homodyne measurements

We consider now measurements of a given quadrature of the \( i \)-th output field. Let \( Z_{i} = e^{i\phi} \Lambda_{i}^{*} + e^{-i\phi} \Lambda_{i}^{\text{out}} \) be the corresponding stochastic process in the environment with \( \phi \) defining the measured quadrature. We have that

\[
dZ_{i}^{\text{out}} = e^{-i\phi} d\Lambda_{i} + e^{i\phi} d\Lambda_{i}^{*} + e^{-i\phi} L_{i}^{*} d\theta + e^{i\phi} L_{i,\theta}^{*} d\theta,
\]

and therefore

\[
\left[Z_{i}^{\text{out}}\right]_{ss} = t \text{Tr}\left(\rho_{\text{in}} \left(e^{-i\phi} L_{i}^{*} + e^{i\phi} L_{i,\theta}^{*}\right)\right).
\]

As in the case of counting, we define the contractions semigroup \( T_{i}^{(\phi)} : B(H_{i}) \to B(H_{i}) \)

\[
T_{i}^{(\phi)}(X) = \left\langle \Omega \right| U_{i}^{*} (X \otimes e^{i\phi Z}) U_{i} \left| \Omega \right\rangle.
\]

whose generator can be computed by applying the quantum Ito rules (4):

\[
\mathcal{L}^{(\phi)}(X) = \mathcal{L}(X) + i\phi \left(e^{-i\phi} L_{i}^{*} X + X e^{i\phi} L \right) - \frac{\phi^{2}}{2} X.
\]

Then the characteristic function of \( Z_{i} \) for an initial state \( \rho_{\text{in}} \) is given by

\[
\phi_{Z}^{i}(p) = \mathbb{E}(e^{pZ}) = \text{Tr}\left(\rho_{\text{in}} T_{i}^{(\phi)}(1)\right).
\]
2.4. Convergence of one-parameter semigroups

In this section we discuss a general semigroup convergence result which will be used as a technical tool in the LAN results. We start with the following Trotter–Kato theorem, see [30] (theorem 3.17).

**Theorem 2.1.** Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}_0$ be a closed subspace of $\mathcal{B}$. For each $n \geq 0$, let $S_t^{(n)}$ be a strongly continuous one-parameter contraction semigroup on $\mathcal{B}$ with generator $\mathcal{L}^{(n)}$. Moreover, let $S_t$ be a strongly continuous one-parameter contraction semigroup on $\mathcal{B}_0$ with generator $\mathcal{L}$. Let $\mathcal{D}$ be a core for $\mathcal{L}$. The following conditions are equivalent:

1. For all $X \in \mathcal{D}$ there exist $X_1, X_2$ such that
   \[ \lim_{n \to \infty} X_n = X, \quad \lim_{n \to \infty} \mathcal{L}^{(n)}(X_n) = \mathcal{L}(X). \]

2. For all $0 \leq T < \infty$ and all $X \in \mathcal{B}_0$
   \[ \lim_{n \to \infty} \sup_{0 \leq t \leq T} \left\| S_t^{(n)}(X) - S_t(X) \right\| = 0. \]

We will apply the Trotter–Kato theorem to the following scenario. Let us assume that the generator $\mathcal{L}^{(n)}$ can be expanded as
\[
\mathcal{L}^{(n)}(X) = n\mathcal{L}_0(X) + \sqrt{n}\mathcal{L}_1(X) + \mathcal{L}_2(X) + \mathcal{O}(n^{-1/2}).
\]
Moreover we assume that $(\ker(\mathcal{L}_0) + \text{ran}(\mathcal{L}_0))$ is dense in $\mathcal{B}(\mathcal{H})$. In this case [30, theorem 5.1] there exists a projection $P : \mathcal{B} \to \mathcal{B}$ such that $\ker(P) = \text{ran}(\mathcal{L}_0)$ and $\ker(P) = \ker(\mathcal{L}_0)$. With $Q = I - P$ we have $P\mathcal{L}_0 P = Q\mathcal{L}_0 P = P\mathcal{L}_0 Q = 0$, but $Q\mathcal{L}_0 Q \neq 0$. Furthermore, we assume there exists a map $\hat{\mathcal{L}} : \mathcal{B} \to \mathcal{B}$ such that $\hat{\mathcal{L}}\mathcal{L}_0 = \mathcal{L}_0\hat{\mathcal{L}} = Q$, and that $\mathcal{L}_1(X) \in \text{ran}(\mathcal{L}_0)$ for all $X \in P\mathcal{B}(\mathcal{H})$.

**Theorem 2.2.** Let $S_t^{(n)}$ be a sequence of semigroups on a Banach space $\mathcal{B}$ with generators $\mathcal{L}^{(n)}$ satisfying the above assumptions. Suppose that
\[
\mathcal{L} = -P\mathcal{L}_1\mathcal{L}_1 + P\mathcal{L}_2 P,
\]
generates a one parameter contraction semigroup on $\mathcal{B}_0 := PB$. Then
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left\| S_t^{(n)}(X) - \exp(t\mathcal{L}(X)) \right\| = 0,
\]
for all $X \in \mathcal{B}_0$ and $0 \leq T < \infty$.

**Proof.** For any $X \in \mathcal{B}_0$, we will construct an expansion $X^{(n)} = X + \frac{1}{\sqrt{n}}X_1 + \frac{1}{n}X_2$. Since $\lim_{n \to \infty} X^{(n)} = X$, if we find a suitable choice for $X_1$ and $X_2$ such that $\lim_{n \to \infty} \mathcal{L}^{(n)}(X^{(n)}) = \mathcal{L}(X)$ then our conclusion follows from the Trotter–Kato theorem. We find that...
\[
\lim_{n \to \infty} \mathcal{L}^n(X^{(n)}) = \lim_{n \to \infty} (n \mathcal{L}_0 X + \sqrt{n} \mathcal{L}_0 X_1 + \mathcal{L}_0 X_2 + \sqrt{n} \mathcal{L}_1 X + \mathcal{L}_1 X_1 + \mathcal{L}_2 X).
\]

Note that \( \mathcal{L}_0 X = 0 \) for \( X \in \mathcal{B}_0 \). Moreover, if we choose \( X_1 = -\mathcal{L}_1 X \) then \( \mathcal{L}_0 X_1 + \mathcal{L}_1 X = 0 \). This leads to
\[
\lim_{n \to \infty} \mathcal{L}^n(X^{(n)}) = \mathcal{L}_0 X_2 - \mathcal{L}_1 \mathcal{L}_1 X + \mathcal{L}_2 X
= \mathcal{L}_0 X_2 - Q \mathcal{L}_1 \mathcal{L}_1 X + Q \mathcal{L}_2 X - P \mathcal{L}_1 \mathcal{L}_1 X + P \mathcal{L}_2 X.
\]

We now choose \( X_2 := \mathcal{L} Q \mathcal{L}_1 \mathcal{L}_1 X - \mathcal{L} Q \mathcal{L}_2 X \in Q \mathcal{B} \) and find
\[
\lim_{n \to \infty} \mathcal{L}^n(X^{(n)}) = -P \mathcal{L}_1 \mathcal{L}_1 X + P \mathcal{L}_2 X = \mathcal{L}(X).
\]

The convergence follows from Trotter–Kato theorem. □

Similar convergence results for the asymptotic behavior of one parameter semigroup with different properties have been derived in [30].

In this paper we will use a rather special case of theorem 2.2. We consider contraction semigroups on \( \mathcal{B}(\mathcal{H}_s) \), with \( \mathcal{H}_s \) a finite dimensional Hilbert space, and such that the first term \( \mathcal{L}_0 \) in the expansion (15) is the generator of a irreducible Markov semigroup. This means that the Schrödinger picture generator \( \mathcal{L}_0 \) has a unique stationary state \( \rho^s_\theta \) which has full rank, while the Heisenberg picture generator \( \mathcal{L}_0 \) has \( 1 \) as the unique zero eigenvector. In this case \( \rho^s_\theta \) leaves \( \mathcal{B}_1 \) invariant and its restriction to this space is invertible. Indeed, if \( X \in \mathcal{B}_1 \) then
\[
\text{Tr}(\rho^s_\theta \mathcal{L}_1 X) = \text{Tr}(\mathcal{L}_0 \rho^s_\theta (\rho^s_\theta X)) = 0
\]
so \( \mathcal{L}_0 (X) \in \mathcal{B}_1 \). Moreover, let \( Y \in \mathcal{B}_1 \) be such that \( Y \) is orthogonal onto the range of \( \mathcal{L}_0 \) in the sense that \( \text{Tr}(Y \mathcal{L}_0 (X)) = 0 \) for all \( X \). Then, by using the duality property (7) we find \( \text{Tr}(\mathcal{L}_0 (Y) X) = 0 \) for all \( X \), which implies that \( \mathcal{L}_0 \rho^s_\theta (Y) = 0 \) so that \( Y = c \rho^s_\theta \). But since \( Y \in \mathcal{B}_1 \), we have \( \text{Tr}(\rho^s_\theta Y) = 0 \), which implies \( Y = 0 \). Therefore the range of \( \mathcal{L}_0 \) is \( \mathcal{B}_1 \) and the inverse \( \tilde{\mathcal{L}} : \mathcal{B}_1 \to \mathcal{B}_1 \) is well defined.

Besides irreducibility, the only additional condition which will need to be verified when applying theorem 2.2 is then
\[
\text{Tr}(\rho^s_\theta \mathcal{L}_1 (1)) = 0.
\]

### 3. LAN for the output state

We return now to the Markov model introduced in the previous section, and assume that the interaction between the quantum system and the environment depends on an unknown parameter \( \theta \in \mathbb{R} \). The goal is to find how well we can estimate \( \theta \) when we are allowed to perform arbitrary measurements on the output. This question can be approached by invoking the quantum Cramér–Rao bound [33], and computing the quantum Fisher information of the output state [34]. However since we are dealing with a time-correlated state, it is not obvious that the quantum Cramér–Rao bound is achievable in a ‘single shot’ measurement even in the large time limit. Instead we will take a more fundamental approach aimed at characterizing the asymptotic ‘shape’ of the quantum statistical model, which provides both the quantum Fisher information and its asymptotic achievability, together with the Gaussian distribution of the optimal estimator. The relevant statistical concept is that of LAN [35]. We will first briefly
review its meaning in the case of quantum statistical models consisting of ensembles of identically prepared systems. After this we formulate the extension to quantum Markov processes, which is one of the main results of the paper.

3.1. LAN for ensembles of identically prepared systems

We illustrate the idea of quantum LAN through the simplest example of a one parameter quantum statistical model [18]. Let \( |\psi\rangle \in \mathbb{C}^d \) be a pure quantum state and define a family of states

\[
|\psi_q\rangle = e^{-i\theta J}|\psi\rangle
\]

indexed by an unknown parameter \( \theta \in \mathbb{R} \). The generator \( J \) is a self-adjoint operator and we assume that \( \langle \psi | J | \psi \rangle = 0 \). The quantum Cramér–Rao bound [33] asserts that for any measurement and any unbiased estimator \( \hat{\theta} \) (i.e. \( \mathbb{E}(\hat{\theta}) = \theta \)), the mean square error (MSE) is lower bounded as

\[
\mathbb{E}\left[(\hat{\theta} - \theta)^2\right] \geq F_q^{-1},
\]

where \( F_q \) is the QFI which is determined by the variance of the generator \( J \). If we are given \( n \) identical copies of \( |\psi_q\rangle \), then the corresponding Fisher information is \( nF_q \), and therefore \( \theta \) can be estimated with error rate scaling as \( n^{-1/2} \).

The underlying idea of LAN is that for large \( n \) the parameter can be localized in a region of size \( n^{-1/2+\epsilon} \) with high probability, e.g. by using a proportion \( n^{1-\epsilon} \) of the system to produce a rough estimator \( \hat{\theta}_0 \). Therefore, in asymptotics it suffices to understand the local properties of the model, and it is natural to work with the equivalent parametrization \( \theta = \theta_0 + \frac{u}{\sqrt{n}} \) where \( \theta_0 \) is fixed and known, and \( u \) is the ‘local parameter’ to be estimated. Let us denote the joint state of the ensemble by \( |\psi_{n,u}\rangle := |\psi_{\theta_0 + u/\sqrt{n}}\rangle^\otimes n \), and notice that since we are dealing with pure states, all properties of the statistical model are encoded in the inner products [26]. The following calculation shows that in the limit of large \( n \), the local statistical model converges to a limit:

\[
\lim_{n \to \infty} \langle \psi_{n,u} | \psi_{n,v} \rangle = \lim_{n \to \infty} \langle \psi | e^{i(u-v)J/\sqrt{n}} | \psi \rangle^n = \lim_{n \to \infty} \left(1 - \frac{(u-v)^2 F}{8n} + o(n^{-1})\right)^n = e^{-\frac{(u-v)^2 F}{8}} = \langle \sqrt{F/2} u | \sqrt{F/2} v \rangle.
\]

Above, \( u, v \) are arbitrary local parameters, \( F = F_{\theta_0} \), and the vector \( \sqrt{F/2} u \) denotes a one parameter model consisting of a coherent state of a one-mode continuous variable system with means \( \langle Q \rangle = \sqrt{F/2} u \) and \( \langle P \rangle = 0 \). The convergence (17) is an example of LAN for pure states quantum models. Its statistical interpretation is that for large \( n \), the task of estimating \( u \) in the original model becomes equivalent to that of estimating \( u \) in the limit Gaussian model. In the case of the latter, measuring \( Q/\sqrt{F/2} \) produces an unbiased, normally distributed estimator \( \hat{u} \) with MSE \( \mathbb{E}[(\hat{u} - u)^2] = F^{-1} \). The ‘weak’ convergence defined above can be strengthened to an operational notion formulated in terms of quantum channels implementing the convergence [22, 26], which can be applied to general models with mixed states and arbitrary number of parameters. This provides a rigorous framework for studying asymptotically optimal estimation procedures and establishing the asymptotic normality of the estimator [36]. In this paper we limit ourselves to proving the weak form of
LAN (in terms of inner products for system and output states) and we refer to [26] on how this can be extended to strong convergence of the output state model.

3.2. QLAN for the Markov model

We assume that the Markov dynamics described in section 2.2 depends on an unknown one-dimensional parameter $\theta$, more precisely $H = H_0$ and $L_i = L_i(\theta)$ and the dependence is smooth with respect to $\theta$. Moreover, we assume that the Markov semigroup is irreducible for any $\theta$. We consider that initially the system is in the pure state $|\chi_0\rangle$, so that the joint initial state of system and environment is $|\Psi(0)\rangle = |\chi_0\rangle \otimes |\Omega\rangle \in \mathcal{H}_s \otimes \mathcal{F}(L^2(\mathbb{R}^2; C^4))$, where $|\Omega\rangle$ is the joint vacuum state of the bosonic fields.

As in the case of identically prepared systems, we expect that by measuring the (stationary) output for a time $t$, allows us to localize $\theta$ within a neighbourhood of size $t^{-1/2}$. Therefore, we write $\theta = \theta_0 + \frac{u}{\sqrt{t}}$ with $\theta_0$ fixed and $u \in \mathbb{R}$ the unknown local parameter. The evolution of the joint initial state gives rise to the family of pure states

$$|\Psi^u_t\rangle := U_t^{\theta_0 + \frac{u}{\sqrt{t}}} |\Psi(0)\rangle,$$

(18)

Since the vector state is only defined up to a complex phase, we make the following choice which allows to establish the convergence of the inner products. Let

$$|\tilde{\Psi}^u_t\rangle = e^{i \tilde{A} u t} |\Psi^u_t\rangle,$$

(19)

where

$$\tilde{A} = \text{Tr} \left( H + \text{Im} \sum_{i=1}^d \frac{L_i^* L_i}{i} \right),$$

and $H, L_i$ denote the derivative of $H_0$ and $L_i(\theta)$ with respect to $\theta$, at $\theta = \theta_0$. The following theorem establishes the LAN of the joint system and output state. The notations used in the Theorem and its proof are those introduced in section 2.4. In particular, $\tilde{L}$ is the inverse of $\mathcal{L}_0 = \mathcal{L}_{\theta_0}$ on the orthogonal complement of the identity.

**Theorem 3.1.** Consider an open system with space $\mathcal{H}_s$ characterized by its Hamiltonian $H_0$ and the jump operators $L_{i,\theta}$, all of which depend smoothly on an unknown parameter $\theta \in \mathbb{R}$. We assume that the dynamics is irreducible for $\theta = \theta_0$. Let $\theta = \theta_0 + \frac{u}{\sqrt{t}}$ be the local parametrization around $\theta_0$ and let $|\tilde{\Psi}^u_t\rangle$ be the joint system-output state at time $t$, as defined in (19).

Then the quantum statistical model $\{|\tilde{\Psi}^u_t\rangle, u \in \mathbb{R}\}$ converges weakly to the coherent states model $\{|\sqrt{F}u\rangle : u \in \mathbb{R}\}$, i.e. for $u, v \in \mathbb{R}$

$$\lim_{t \to \infty} \langle \tilde{\Psi}^u_t | \tilde{\Psi}^v_t \rangle = \left( \sqrt{\frac{F}{2}} u \right) \left( \sqrt{\frac{F}{2}} v \right).$$

(20)
with limiting quantum Fisher information

\[
F = 8 \left( \frac{1}{2} \sum_i L_i^* L_i - \text{Re}(\mathcal{H}\tilde{B}) - \text{Im}\left( \sum_i \frac{L_i^* \tilde{B} L_i}{2} \right) + \frac{1}{2} \text{Im}\left[ \sum_i \left( \frac{L_i^* L_i + L_i^* L_i}{2} \right) \tilde{B} \right] \right)_{ss}
\]

\[
\tilde{B} = \hat{L} \left( \hat{H} + \text{Im}\sum_i L_i^* L_i \right) = \left( \hat{H} + \text{Im}\sum_i L_i^* L_i \right) \mathbf{1},
\]

where \( \langle \cdot \rangle_{ss} = \text{Tr}(\cdot \rho_{ss}) \) denotes the expectation with respect to the stationary state at \( \theta_0 \).

**Proof.** For a fixed triple \((t, u, v)\) we let \( \theta = \theta_0 + \frac{u}{\sqrt{\tau}}, \theta' = \theta_0 + \frac{v}{\sqrt{\tau}} \) and define the one parameter contractions semigroup with parameter \( \tau \)

\[
T^{(t,u,v)}_\tau : \mathcal{B}(\mathcal{H}_s) \rightarrow \mathcal{B}(\mathcal{H}_s)
\]

\[
T^{(t,u,v)}_\tau : X \mapsto e^{-i\tau(\theta - \theta')}\mathcal{A}\left\{ \Omega \left[ U^{\theta_0 + u/\sqrt{\tau}}_{t} U^{\theta_0 + v/\sqrt{\tau}}_{t} \right] (X \otimes \mathbf{1}) U^{\theta_0}_{t} \right\} \Omega.
\]

The fact that \( T^{(t,u,v)}_\tau \) is a semigroup can be shown by differentiation and by using the quantum Itô rules. Its generator \( \mathcal{L}^{(t,u,v)} \) is

\[
\mathcal{L}^{(t,u,v)}(X) = -\frac{1}{2} \left( L^*_i L_i X + XL^*_i L_i \right) - i(\theta - \theta')\mathcal{A}.
\]

The inner products can then be computed as

\[
\langle \Psi_t | \Psi_t' \rangle = e^{-i\tau(\theta - \theta')}\mathcal{A}\left\{ \chi_0 \otimes \Omega | U^{\theta_0 + u/\sqrt{\tau}}_{t} U^{\theta_0 + v/\sqrt{\tau}}_{t} \right\} | \chi_0 \otimes \Omega \rangle = \langle \chi_0 | T^{(t,u,v)}_\tau(\mathbf{1}) | \chi_0 \rangle.
\]

By applying theorem 2.2 we obtain the limit

\[
\lim_{\tau \rightarrow \infty} \langle \chi_0 | T^{(t,u,v)}_\tau(\mathbf{1}) | \chi_0 \rangle = e^{i(v^2 - u^2)G} e^{-(u - v)^2} e^G \mathbf{1} = e^{i(v_2 - u_2)G} \mathbf{1},
\]

where \( G \in \mathbb{R} \) is a constant, \( \left\langle \frac{F}{\sqrt{2}} v \right\rangle \) is the one mode coherent state with mean \( \langle Q \rangle = \left\langle \frac{F}{\sqrt{2}} v \right\rangle, \langle P \rangle = 0 \). The details of this calculation are found in the appendix A.1.

Since the complex phase pre-factor can be absorbed in the definition of the coherent state, we conclude that the system-output model converges weakly to the one parameter coherent state limit model.

We have shown that asymptotically the joint state of the system and environment are locally statistically equivalent to the Gaussian model of coherent states. The coefficient \( F \) in (21) is the quantum Fisher information per unit of time of the local states (19). However the result does not tell us how the quantum Fisher information can be achieved in a realistic setting, so in the next section we focus on the statistical properties of simple measurements such as counting and homodyne.
4. LAN for measurements on the output

In this section we prove that additive statistics of continuous measurements on the environment satisfy (the classical version of) LAN. More precisely, let $X_t$ be a real-valued random variable indexed by $t \in \mathbb{R}$ (a summary statistic at time $t$), and that its distribution depends on an unknown parameter $\theta \in \mathbb{R}$; we suppose that the ‘amount of information’ about $\theta$ grows linearly with $t$. As before, we write $\theta = \theta_0 + u \sqrt{t}$ and we say that the process satisfies LAN if the following convergence in distribution holds (under $\theta$) as $t \to \infty$

$$\frac{1}{\sqrt{t}}(X_t - \mathbb{E}_{\theta_0}(X_t)) \overset{d}{\to} \mathcal{N}(\mu u, \sigma^2).$$

(25)

The limit is the normal distribution with mean $\mu u$ and variance $\sigma^2$. Its classical Fisher information is the rescaled limiting Fisher information of $X_t$ and is given by the signal to noise ratio $I = \mu^2 / \sigma^2$. As a consequence of (25), we find that the estimator

$$\hat{\theta} = \theta_0 + \hat{u} / \sqrt{t} := \theta_0 + \left( X_t - \mathbb{E}_{\theta_0}(X_t) \right) / (t\mu)$$

is asymptotically normal and its MSE satisfies

$$t\mathbb{E}_{\theta_0} \left( (\hat{\theta} - \theta)^2 \right) \Bigg\to I^{-1}.$$

To prove (25) it suffices to show the convergence of the characteristic functions

$$\lim_{t \to \infty} \mathbb{E}_{\theta_0 + u \sqrt{t}} \left( e^{i\omega X_t} \right) = e^{i\omega \mu u - \frac{1}{2} \sigma^2 \omega^2}. \quad (26)$$

Below, we apply this recipe to the total counts and integrated homodyne current statistics. We stress that these results are for summary statistics, i.e. they do not take into account time correlations and typically have smaller Fisher information than the whole stochastic measurement process.

Before describing the two cases, let us make a few remarks on the problem of estimation based on more general statistics. A class of models for which the Fisher information of the full counting process can be derived analytically, is that of dissipative processes where the jump operator is of rank one (e.g. a two level atom decaying to the ground state). In this case the counting process is of renewal type, and the Fisher information can be computed from the waiting time distribution of a single jump [40]. However, we do not expect that a general closed form expression of the Fisher information of the full process can be derived for generic Lindblad dynamics. To go beyond the total counts statistics one could consider time averages of functions of the detection record over a given time window. Since the output process has a finite correlation time, the optimal size of the time window is expected to be of the order of the correlation time. A central limit theory for such statistics has been developed in [37] for the case of discrete time quantum Markov chains. Its appropriate extension to the continuous time setting, perhaps using the ‘jump by jump’ description of the output process introduced in [41], would provide the asymptotic normality result (25) (and the Fisher information) for the time averaged function. This could then be optimized to find the best choice of multi-photon statistic. A small step in this direction has been made in [21], for the problem of estimating of Rabi angle of the one-atom maser.

4.1. Counting process

We return to the counting process introduced in section 2.3.1 and consider for simplicity that the system is coupled with a single bosonic field. The multi-channel case can be treated
similarly. We assume that the dynamics depends on the unknown one-dimensional parameter \( \theta \in \mathbb{R} \), so that \( H = H_\theta, L = L_\theta \). Recall that \( \Lambda^\text{out}_t \) is the counting process resulting from detecting output excitations. We define the ‘compensated’ counting process

\[
Y_t = \Lambda^\text{out}_t - t \langle L^*_0 L_0 \rangle_s,
\]

where \( \langle L^*_0 L_0 \rangle_s \) is a known quantity equal to the stationary counting rate when the parameter \( \theta \) takes the value \( \theta_0 \). We will show that the rescaled process \( Y_t/\sqrt{t} \) satisfies LAN, and can be used to construct an asymptotically normal estimator of \( \theta \) whose MSE can be calculated explicitly.

**Theorem 4.1.** Consider an open system with space \( \mathcal{H}_s \) characterized by its Hamiltonian \( H_\theta \) and a jump operator \( L_\theta \), both of which depend smoothly on an unknown parameter \( \theta \in \mathbb{R} \). We assume that the dynamics is irreducible for \( \theta = \theta_0 \). Let \( \theta = \theta_0 + u/\sqrt{t} \) be the local parametrization around \( \theta_0 \) and let \( Y_t \) be the counting process defined in (27). Then \( Y_t \) satisfies LAN, i.e. the following convergence in distribution holds as \( t \to \infty \), with \( \theta = \theta_0 + u/\sqrt{t} \)

\[
\frac{1}{\sqrt{t}} Y_t \overset{D}{\to} N(\mu_t, V_t).
\]

The limit is the normal distribution with mean \( \mu_t \) and variance \( V_t \), both of which can be computed explicitly (see end of proof). In particular, the asymptotic rescaled classical Fisher information of \( Y_t \) is given by

\[
I_t = \frac{\mu_t^2}{V_t} \leq F
\]

and the estimator \( \hat{\theta}_t = \theta_0 + Y_t/(\mu_t) \) is asymptotically normal and satisfies

\[
\lim_{t \to \infty} t \mathbb{E} \left[ (\hat{\theta}_t - \theta)^2 \right] = I_t^{-1}.
\]

**Proof.** To prove (28) it suffices to prove the convergence of the corresponding characteristic functions

\[
\lim_{t \to \infty} \mathbb{E} \left( e^{iuY_t/\sqrt{t}} \right) = e^{iu\mu_t - \frac{1}{2}V_t u^2}.
\]

To establish this, we introduce a family of contractions semigroups \( S_{t,u,s}^{(u,s)} : \mathcal{B}(\mathcal{H}_s) \to \mathcal{B}(\mathcal{H}_s) \), where \( u, s \) are considered fixed and \( t \) is an index playing the role of \( n \) in theorem 2.2. The semigroups are given by

\[
S_{t,u,s}^{(u,s)}(X) = \left\{ \Omega \left| X \otimes e^{-iuK_{u,t}/\sqrt{t}} U^{\theta_0 + u/\sqrt{t}}_{t\tau} \right| \Omega \right\}.
\]

Using theorem 2.2 we will show that

\[
\lim_{t \to \infty} S_{t,u,s}^{(u,s)}(1) = e^{iu\mu_t - \frac{1}{2}V_t u^2} 1,
\]

where \( \mu_t \) and \( V_t \) are constants whose explicit expression is given at the end of the proof.

The limit (29) follows from (31) by setting \( \tau = 1 \) and taking expectation with respect to the system’s initial state on both sides. The asymptotic rescaled Fisher information of \( Y_t \) is the Fisher information of the Gaussian shift model \( \{ N(\mu_t, V_t) : u \in \mathbb{R} \} \) which is equal to \( I_t = \mu_t^2/V_t \).
The proof of the limit (31) can be found in appendix A.2.

4.2. Homodyne measurement

In the same setup as the previous section, we consider the homodyne measurement with quadrature angle $\phi$, described by the quantum output process (integrated homodyne current) $Z_t = e^{-i\phi t}A_{t^+}^{\text{out}}(t) + e^{i\phi t}A_{t^-}^{\text{out}}(t)$. As before, we define the random variable which is centred for $\theta = \theta_0$

$$W_t = Z_t - t \left( e^{-i\phi t}L_{\theta_0}^* + e^{i\phi t}L_{\theta_0} \right).$$

We will show that $W_t$ satisfies LAN, as $t \to \infty$.

**Theorem 4.2.** Consider an open system with space $\mathcal{H}_s$ characterized by its Hamiltonian $H_0$ and a jump operator $L_q$, both of which depend smoothly on an unknown parameter $\theta \in \mathbb{R}$. We assume that the dynamics is irreducible for $\theta = \theta_0$. Let $\theta = \theta_0 + u/\sqrt{t}$ be the local parametrization around $\theta_0$ and let $W_t$ be the integrated homodyne current defined in (32). Then $W_t$ satisfies LAN, i.e. the following convergence in distribution holds as $t \to \infty$, for $\theta = \theta_0 + u/\sqrt{t}$

$$\frac{1}{\sqrt{t}} W_t \xrightarrow{D} N \left( \mu_h u, V_h \right).$$

The limit is the normal distribution with mean $\mu_h u$ and variance $V_h$, both of which can be computed explicitly (see end of proof). In particular, the asymptotic rescaled classical Fisher information of $W_t$ is given by

$$I_h = \frac{\mu_h^2}{V_h} \leq F$$

and the estimator $\hat{\theta}_t = \theta_0 + \frac{W_t}{(t\mu_h)}$ is asymptotically normal and satisfies

$$\lim_{t \to \infty} t \mathbb{E} \left[ \left( \hat{\theta}_t - \theta \right)^2 \right] = I_h^{-1}.$$  

**Proof.** To prove (28) it suffices to show the convergence of characteristic functions

$$\lim_{t \to \infty} \mathbb{E} \left[ e^{i p \hat{w}_t} \right] = e^{i p \mu_h - \frac{1}{2} p^2 V_h}.$$  

We define a family of one-parameter contractions semigroups $T^{(t, u, p)}_t : \mathcal{B}(\mathcal{H}_s) \to \mathcal{B}(\mathcal{H}_s)$ indexed by $(t, u, p)$, with $u, p$ fixed and $t$ playing the role of $n$ in theorem 2.2. The semigroups are given by

$$T^{(t, u, p)}_t(X) = \{ \Omega \left[ U_{t^+}^{n \mu_p + u \sqrt{\tau}/\sqrt{t}} \left( X \otimes e^{i p_{\hat{w}_t}} \right) U_{t^-}^{n \mu_p - u \sqrt{\tau}/\sqrt{t}} \right] \Omega \},$$

Using theorem 2.2 we will show that

$$\lim_{t \to \infty} T^{(t, u, p)}_t(1) = e^{i p \mu_h - \frac{p^2}{\tau} V_h 1},$$

where $\mu_h$ and $V_h$ are constants whose explicit expression is given at the end of the proof. The limit (34) follows from (36) by setting $\tau = 1$ and taking expectation with respect to the system’s initial state on both sides. The asymptotic rescaled Fisher information of $W_t$ is
the Fisher information of the Gaussian shift model \( \{ N(\mu_\theta u, V_\theta) : u \in \mathbb{R} \} \) which is equal to 

\[ I_\theta = \frac{\mu_\theta^2}{V_\theta}. \]

The proof of the limit (36) can be found in appendix A.3.

\[ \square \]

5. Examples

In this section we apply the general results to two examples, a two level system and the atom maser.

5.1. Two-level system

In this example we consider a two level open system with Hilbert space \( \mathcal{H}_s = \mathbb{C}^2 \). The system has zero Hamiltonian and its open dynamics is given by spontaneous decay with jump operator \( L = \theta \sigma_x \), where \( \sigma_x = \frac{1}{2} (\sigma_x - i \sigma_y) \) with \( \sigma_{x,y} \) Pauli matrices. The corresponding Lindblad operator is

\[ \tilde{L}(X) = L^*XL - \frac{1}{2} \{ L^*L, X \}. \]  \hfill (37)

where \( \theta \in \mathbb{R} \) will play the role of the unknown parameter.

If the input channel is in the vacuum state, then the system decays to the ground state, after which the output is in the vacuum and no further information about \( \theta \) can be obtained. The situation is different if we consider that the atom is driven by a laser with constant amplitude \( z \). This can be modelled by the unitary dynamics \( U_t \) defined in (5) but with the vacuum input state replaced by the coherent state (see (1)) \( e(z\chi_{0,t}) = W(z\chi_{0,t})\Omega \), where \( z = |z| e^{i\phi} \) is the laser amplitude and \( W(z\chi_{0,t}) \) is the unitary Weyl operator defined by the quantum stochastic differential equation

\[ dW(z\chi_{0,t}) = \left\{ zdA^x(t) - \dot{z}dA(t) - \frac{1}{2} |z|^2 dt \right\} W(z\chi_{0,t}). \]

Therefore, we can equivalently describe the dynamics by the unitary \( U_t = U_t \cdot W(z\chi_{0,t}) \) while the input is in the vacuum state. By differentiation we obtain [42]

\[ U_t^z = \left\{ \left( \theta \sigma_x + z \right)dA^x(t) - \left( \theta \sigma_x + z \right)dA(t) - \frac{1}{2} \left( \theta^2 \sigma_x \sigma_x + |z|^2 + 2z\theta \sigma_x \right) dt \right\} U_t^z \]

and the corresponding Lindblad operator is

\[ \mathcal{L}_\theta(X) = i[H_\theta, X] + L^a_\theta X L_\theta - \frac{1}{2} \left\{ L_\theta^a L_\theta, X \right\}, \] \hfill (38)

where \( L_\theta = \theta \sigma_x + z \) and \( H_\theta = \frac{i}{2} \theta (\bar{z} \sigma_x - z \sigma_x) \) for \( \sigma_\pm = \frac{1}{2} (\sigma_i \pm i \sigma_i) \). In this picture the Hamiltonian and jump operator depend on the laser amplitude \( z \), and in the stationary regime, the counting rate in the output channel is equal to the intensity \( |z|^2 \) of the input coherent state. In a more realistic model one could consider that the driving laser is carried by a forward channel while an additional side channel can be used to monitor fluorescence photons emitted by the atom [42]. However, since our main purpose is to illustrate the different notions of Fisher information, we restrict ourselves to the simple model of a single channel. The Master dynamics has a unique stationary state given by
Our goal is to compute the quantum Fisher information of the output, and the classical Fisher informations for counting and homodyne, as described in theorems 3.1, 4.1, 4.2 for the parameter $\theta$.

Following the method of sections 3 and 4, we localize the unknown parameter as $u = 0$. We remark that in both sections the generators of the corresponding one-parameter semigroups have a common lowest order perturbation term given by

\[ L_0(X) = i\left[ H_{\theta_0}, X \right] + L_{\theta_0}^*XL_{\theta_0} - \frac{1}{2}\left\{ L_{\theta_0}^*L_{\theta_0}, X \right\}, \]

The reduced dynamics of the atom has a unique stationary state as shown above. Therefore the kernel of $L_0$ is spanned by the identity operator and the projection $P$ can be defined by tracing with the stationary state $i.e.$ $PX = \text{Tr}(\rho_0 X)I$. The following vectors form a basis in $M(\mathbb{C}^2)$

\[ e_1 = \sigma_z + aI, \quad e_2 = \sigma_z + bI, \quad e_3 = \sigma_x + cI, \quad e_4 = I, \]

where $a = 1 - 2c$, $b = -c$, $c = -b$; moreover the vectors $e_1, e_2, e_3$ are chosen in such a way that they span the orthogonal complement of $P$, i.e. $\text{Tr}(e_i \rho_0) = 0$, for $i = 1, 2, 3$.

With respect to the basis $(e_1, e_2, e_3)$ any operator acting on $\mathbb{C}^2$ can be represented as a vector and the map $L_0$ defined in (40) is given by

\[ L_0 = \begin{pmatrix}
-\theta_0^2 & z\theta_0 & z\theta_0 & 0 \\
-2\xi\theta_0 & \frac{\theta_0^2}{2} & 0 & 0 \\
-2\xi\theta_0 & 0 & \frac{\theta_0^2}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \]

Following the assumptions in section 2.4 we define the map $\tilde{L}$ as the inverse of $L_0$ on the subspace spanned by the vectors $e_1, e_2, e_3$. With respect to the basis defined above this map is given by

\[ \tilde{L} = \frac{1}{\theta_0^2(\theta_0^2 + 8|z|^2)} \begin{pmatrix}
-\theta_0^2 & -2\xi\theta_0 & -2\xi\theta_0 & 0 \\
4\xi\theta_0 & 8\xi^2 - \theta_0^2 & 0 & 0 \\
0 & 0 & 8\xi^2 - \theta_0^2 & 0
\end{pmatrix}. \]

5.1.1. LAN for output states. We first apply the theory of section 3 to this particular example. For two neighbouring parameters $\theta = \theta_0 + \frac{\theta'}{\sqrt{t}}$ and $\theta' = \theta_0 + \frac{\theta''}{\sqrt{t}}$ the generator in (22) reads

\[ L^{(\theta, \theta')}_{\theta}(X) = i\left[ \left( H_0 X - X H_{\theta'} \right) + L_{\theta}^*XL_{\theta'} - \frac{1}{2}\left( L_{\theta}^*L_{\theta}X + XL_{\theta}^*L_{\theta'} \right) \right] \]

The assumptions in section 2.4 are satisfied, so that theorem 3.1 holds. The details of the analytical calculations are shown in appendix A.4. The quantum Fisher information is found to be
5.1.2. Classical measurements. If we consider the counting process we remark that the average number of photons in the output channel is

\[ \langle \tilde{L}_\phi \tilde{L}_\theta \rangle_{ss} = \frac{\theta_0^2}{2} \left\{ \sigma_z \right\}_{ss} + \theta_0 \left( \tilde{z}\sigma_- + \tilde{z}\sigma_+ \right)_{ss} + \frac{\theta_0^2}{2} |\tilde{z}|^2 \]

\[ = -\frac{\theta_0^2}{2} \tilde{a} - \theta_0 \left( \tilde{z}\tilde{b} + \tilde{z}\tilde{c} \right) + \frac{\theta_0^2}{2} |\tilde{z}|^2 = |\tilde{z}|^2. \]  

This equals the intensity of the driving laser, result which is expected due to conservation of energy. Therefore no information on the unknown parameter is contained in the total number of counts statistic.

We compute the Fisher information for homodyne measurements in the manner of section 4.2. The details of the calculations are shown in appendix A.5. The Fisher information is given by

\[ I_{\theta_0} = \frac{4}{\theta_0^2} \left( \begin{array}{c} \bar{F} \left( \theta_0 \right) \\ 0 \end{array} \right), \]

where

\[ \bar{F} \left( \theta_0 \right) = \frac{128 |r|^4}{\left( 8 |r|^2 + 1 \right)} \quad \text{and} \quad r := \frac{\tilde{z}}{\theta_0}. \]  

Figure 2. Fisher information for homodyne measurements: left—dependence on \( \phi \) for \(|\tilde{r}| = |\tilde{z}|/\theta_0 = 0.66 \) with \( \tilde{b} = 0 \) (blue line) and \( \tilde{b} = \pi/3 \) (red dotted line); right—dependence on \( \theta_0 \) for \( \tilde{b} = \phi = 0 \) and different laser strengths.
and
\[ B_\phi = 1 + \frac{2}{(1 + 8 |r|^2)^2} \left( 4 \text{Im}^2(e^{i\phi}r) - 16 |r|^2 + 192 |r|^4 + 512 |r|^4 \text{Im}^2(e^{i\phi}r) \right). \]

In figure 2 we plot the Fisher information for homodyne detection as a function of \( \phi \) for all other parameters fixed (left) and as a function of \( \theta_0 \) for fixed \(|z|, \delta \) and \( \phi \) (right). The Fisher information is a function of the ratio \(|r| = |z|/\theta_0\) and of \( \cos^2(\phi + \delta) \) therefore it is maximized whenever the laser and the detector are aligned in such a way that \( \cos(\phi + \delta) = \pm 1 \).

5.2. The atom maser

A one-atom maser [38] consists of a beam of excited two level atoms interacting resonantly with a single mode of an electromagnetic field enclosed in a dissipative cavity. The ‘system’ is the field in the cavity, and in a certain time coarse graining approximation, the interaction with a Poissonian beam of atoms, and that with a positive temperature thermal bath can be modelled as a coupling to four bosonic channels, one for each possible interaction. Two channels correspond to the two possible outcomes of the excited atom passing through the cavity and the other two channels are the photon exchange channels between the cavity and a thermal bath of constant temperature. The Lindblad generator is
\[
\mathcal{L}(X) = \sum_{i=1}^{4} L_{i,\phi}^* X L_{i,\phi} - \frac{1}{2} \{ L_{i,\phi}^* L_{i,\phi}, X \}
\]
with the four jump operators defined as
\[
L_{1,\phi} = \sqrt{N_{\text{ex}}} a^* \frac{\sin(\sqrt{\nu} a^*)}{\sqrt{\nu} a^*}, \quad L_{2,\phi} = \sqrt{N_{\text{ex}}} \cos(\sqrt{\nu} a^*),
\]
\[
L_{3,\phi} = \sqrt{\nu + 1} a, \quad L_{4,\phi} = \sqrt{\nu} a^*,
\]
where \( N_{\text{ex}} \) is the rate of the incoming atoms, \( \nu \) is the average number of photons in the bath, \( a \) and \( a^* \) creation and annihilation operators for the field, and \( \phi \) is the accumulated Rabi angle which is proportional to the interaction strength.

The atom maser has a unique stationary state which is diagonal in the number basis, and its coefficients are
\[
\rho_s(n) = \rho_s(0) \prod_{i=1}^{n} \left[ \frac{\nu}{\nu + 1} + \frac{N_{\text{ex}} \sin^2(\sqrt{\nu})}{\nu + 1} \right].
\]

The large deviations theory and the central limit theorem for counting measurements has been studied in [39], while the problem of estimating the parameter \( \phi \) has been investigated in detail in [20] and [21].

In [20] it was shown that the quantum Fisher information described in theorem 3.1 is
\[
F = 4 \text{Tr} \left( \rho_s \sum_{i=1}^{4} L_{i}^* L_{i} \right) = 4N_{\text{ex}} \text{Tr} \left( \rho_s a a^\dagger \right) = 4N_{\text{ex}} \sum_{k} (k + 1) \rho_s(k).
\]

This is plotted in the left panel of figure 3 for \( N_{\text{ex}} = 16 \) and \( \nu = 0.1 \). For comparison, the classical Fisher informations associated to total counts of ground state atoms, excited state atoms and ground and excited state jointly, is plotted in the right panel. We note that all informations are equal to zero at a particular value \( \phi_0 \) where the mean photon number in the
cavity, and the rate of ground state atoms are at a maximum, and therefore the derivative with respect to $\phi$ is zero. One can show however [21], that the Fisher information of the full detection record is strictly larger than zero for all $\phi$. This shows the importance of extending the theory developed here for total counts and time integrated statistics, to more general statistics depending on time correlations.

Besides counting, one could in theory consider that a homodyne measurement is performed on the photon loss channel. However the Fisher information of the integrated homodyne current is equal to zero as the homodyne current has mean zero. Interestingly, if the homodyne measurement is performed jointly with the counting measurement on the atomic channels, then the joint classical Fisher information of the total counts and integrated current is slightly larger than that of the total counts alone. This is due to small non-diagonal terms in the covariance matrix of the two statistics. Note that although the integrated current has zero Fisher information, the full homodyne process may contain information about $\phi$, but our results are not sufficiently general to analyse its statistical properties.

6. Conclusion

In this paper we have extended the discrete-time results of [18, 26] to the domain of continuous-time quantum Markov processes, in the input–output formalism. For an irreducible system whose dynamics depends on an unknown parameter, we have shown that for large time, the output state can be approximated by a quantum Gaussian state (asymptotic normality) and found the explicit expression of the asymptotic quantum Fisher information of this quantum statistical model. This provides an absolute bound on the estimation precision of any measurement procedure.

We have then analysed the statistical properties of the counting and homodyne continuous-time measurements. We showed that the total counts and the integrated homodyne current also satisfy asymptotic normality and computed the general expression of the corresponding classical Fisher informations. We then considered two examples (two level system and the atom maser), in which the performance of these measurements is compared with that prescribed by the quantum Fisher information. Finding the optimal measurement and estimation scheme is an important open problem which goes beyond the scope of this paper. Another remaining problem is to extend the present results to a multi-dimensional parameter

![Figure 3. The quantum Fisher information of the atom maser as a function of the Rabi angle $\phi$ (left panel). Classical Fisher information corresponding to the total counts of ground state atoms, excited state atoms, and joint statistics (right panel).](image)
set-up, and to derive the general quantum central limit theorem which underpins the asymptotic normality results [43]. We note that the ergodicity assumption is crucial for LAN, as it has been shown [19, 44] that non-ergodic systems (i.e. systems with several stationary states) can exhibit Heisenberg precision scaling when the initial state is a coherent superposition of vectors from different stationary phases.

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Appendix A

A.1. Details of the proof of theorem 3.1

We apply theorem 2.2 for the family of semigroups $T^{(u,v)}_t$ where $t$ plays the role of index (instead of the discrete index $n$). The generator in equation (15) can be expanded as

$$\mathcal{L}^{(t, \theta, \theta')} (X) = t\mathcal{L}_0(X) + \sqrt{t} \mathcal{L}_1(X) + \mathcal{L}_2(X) + O\left( t^{-1/2} \right),$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ depend on $(u, v)$, and the terms are given by

$$\mathcal{L}_0(X) = i[H, X] + \sum_{i=1}^{d} \left( L_i^* L_i - \frac{1}{2} \left( L_i^* L_i, X \right) \right),$$

$$\mathcal{L}_1(X) = i \left( u H X - v X H \right) + \sum_{i=1}^{d} \left( u L_i^* X L_i + v L_i^* X L_i \right) - \frac{1}{2} \left( u \left( L_i^* L_i + L_i^* L_i \right) X + v X \left( L_i^* L_i + L_i^* L_i \right) \right)$$

$$- i(u - v) \left( H + \text{Im} \sum_{i=1}^{d} L_i^* L_i \right) X,$$

$$\mathcal{L}_2(X) = \frac{1}{2} \left( u^2 H X - v^2 X H \right) + \sum_{i=1}^{d} \left( \frac{u^2}{2} L_i^* X L_i + \frac{v^2}{2} L_i^* X L_i + u v L_i^* X L_i \right)$$

$$- \frac{1}{2} \sum_{i=1}^{d} \left( \frac{u^2}{2} \left( L_i^* L_i + L_i^* L_i + 2 L_i^* L_i \right) X + \frac{v^2}{2} X \left( L_i^* L_i + L_i^* L_i + 2 L_i^* L_i \right) \right).$$

Since $\text{Ker}(\mathcal{L}_0) = \mathbb{C}1$ we choose the projection $P(X) = \text{Tr} (\rho_{ss} X) 1$ where $\rho_{ss}$ is the stationary state of the system. Then, since $P$ is a one-dimensional projection, the limit generator in theorem 2.2 is of the form

$$\mathcal{L}^{(u,v)} := -P \mathcal{L}_1 \hat{L}_1 + P \mathcal{L}_2 \hat{P} = f(u, v) P,$$

where $f(u, v)$ is a (complex valued) function. The limit of the inner product in (23) is therefore given by

$$\lim_{t \to \infty} \langle \Psi_t | \tilde{\Psi}_t \rangle = \lim_{t \to \infty} \left( \chi_0 \left| T^{(u,v)}_t(\mathbf{1}) \right| \chi_0 \right) = \left( \chi_0 \left| \Psi^{(u,v)}_1 \right| \chi_0 \right) = e^{f(u,v)}.$$
We will now calculate the function $f(u, v)$. For the second term in (53) we have

$$
\mathcal{L}_2(1) = \left( u^2 - v^2 \right) \left( \frac{i}{2} \mathcal{H} + \frac{1}{4} \sum_i \left( \hat{L}_i^* \hat{L}_i - \hat{L}_i^* \hat{L}_i \right) \right) - \frac{1}{2} (u - v)^2 \sum_i \hat{L}_i^* \hat{L}_i
$$

$$
= \frac{i}{2} \left( u^2 - v^2 \right) \left( \hat{H} + \text{Im} \sum_i \hat{L}_i^* \hat{L}_i \right) - \frac{1}{2} (u - v)^2 \sum_i \hat{L}_i^* \hat{L}_i
$$

and

$$
\mathcal{L}_1(1) = (u - v) \left( i \hat{H} + \frac{1}{2} \sum_i \left( \hat{L}_i^* \hat{L}_i - \hat{L}_i^* \hat{L}_i \right) \right) - i(u - v) \left( \hat{H} + \text{Im} \sum_{i=1}^n \hat{L}_i^* \hat{L}_i \right) \mathbf{1}
$$

$$
= i(u - v) \left( \hat{H} + \text{Im} \sum_i \hat{L}_i^* \hat{L}_i - \left( \hat{H} + \text{Im} \sum_{i=1}^n \hat{L}_i^* \hat{L}_i \right) \mathbf{1} \right) \equiv i(u - v) \hat{B}.
$$

Note that the condition $\text{Tr}(\mathcal{L}_1(1))_{\text{ss}} = 0$ is satisfied. For simplicity we denote $\hat{B} = \tilde{\mathcal{L}}(B)$, and note that $(\hat{B})_{\text{ss}} = 0$ since $\tilde{\mathcal{L}}$ leaves the set of zero expectation observables invariant. Rearranging the terms in a suitable way we find that

$$
\mathcal{L}_1 \tilde{\mathcal{L}}_1(1) = i \left( u^2 - v^2 \right) \left\{ -\text{Im}(\hat{H} \hat{B}) + \text{Re} \left[ \sum_i \hat{L}_i^* \hat{B} \hat{L}_i \right] - \frac{1}{2} \text{Re} \left[ \sum_i \left( \hat{L}_i^* \hat{L}_i + \hat{L}_i^* \hat{B} \hat{L}_i \right) \hat{B} \right] \right\}
$$

$$
- (u - v)^2 \left\{ \text{Re}(\hat{H} \hat{B}) + \text{Im} \left[ \sum_i \hat{L}_i^* \hat{B} \hat{L}_i \right] - \frac{1}{2} \text{Im} \left[ \sum_i \left( \hat{L}_i^* \hat{L}_i + \hat{L}_i^* \hat{B} \hat{L}_i \right) \hat{B} \right] \right\}
$$

$$
- \left( \hat{H} + \text{Im} \sum_{i=1}^n \hat{L}_i^* \hat{L}_i \right) \hat{B} \right\}.
$$

Therefore we find

$$
f(u, v) = \exp \left( i \left( u^2 - v^2 \right) \langle X_2 \rangle_{\text{ss}} - (u - v)^2 \langle X_1 \rangle_{\text{ss}} \right),
$$

where $X_1$ and $X_2$ are the selfadjoint operators

$$
X_1 = \frac{1}{2} \sum_i \hat{L}_i^* \hat{L}_i - \text{Re}(\hat{H} \hat{B}) - \text{Im} \left[ \sum_i \hat{L}_i^* \hat{B} \hat{L}_i \right] + \frac{1}{2} \text{Im} \left[ \sum_i \left( \hat{L}_i^* \hat{L}_i + \hat{L}_i^* \hat{B} \hat{L}_i \right) \hat{B} \right],
$$

$$
X_2 = \frac{1}{2} \left( \hat{H} + \text{Im} \sum_i \hat{L}_i^* \hat{L}_i \right) + \text{Im}(\hat{H} \hat{B}) - \text{Re} \left[ \sum_i \hat{L}_i^* \hat{B} \hat{L}_i \right]
$$

$$
+ \frac{1}{2} \text{Re} \left[ \sum_i \left( \hat{L}_i^* \hat{L}_i + \hat{L}_i^* \hat{B} \hat{L}_i \right) \hat{B} \right].
$$

We conclude that the limit overlap can be expressed in terms of the overlap of two one-mode coherent states, with a certain choice of phase which does not have a physical significance

$$
\lim_{t \to \infty} \langle \Psi_t^u \vert \Psi_t^v \rangle = e^{-i(u-v)\sqrt{F/2}u} e^{i(u-v)\sqrt{F/2}v} e^{i(u-v)^2/2} = e^{i(u-v)^2/2} = \left\langle \sqrt{F/2} u \right| \sqrt{F/2} v \right\rangle.
$$

The constant $F = 8 \langle X_1 \rangle_{\text{ss}}$ is the quantum Fisher information of the limiting coherent state model $\{ \vert \sqrt{F/2} u \rangle : u \in \mathbb{R} \}$. This completes the proof of the quantum LAN theorem 3.1.
A.2. Details of the proof of theorem 4.1

In the following we drop the subscript when \( \theta = \theta_0 \) and denote \( L = L_{\theta_0}, H = H_{\theta_0} \). By differentiating (30) and using the quantum Ito rules it can be checked that \( S_{(u,s)}^{(l,u,s)} \) is a semi-group with generator

\[
\mathcal{L}_{(l,u,s)}^{(l,u,s)}(X) = \frac{t}{2} \left[ \mathcal{L}_0(X) + \left( e^{i\alpha/\sqrt{t}} - 1 \right) L_{\theta_0}^* XL_{\theta_0} - \frac{i s}{\sqrt{t}} \langle L^* L \rangle_{\alpha} X \right]
\]

\[
= t \mathcal{L}_0(X) + \sqrt{t} \mathcal{L}_1(X) + \mathcal{L}_2(X) + O\left( t^{-1/2} \right).
\]

where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) depend on \((u, s)\), and the terms are given by

\[
\mathcal{L}_0(X) = i[H, X] + L^* XL - \frac{1}{2} \{ L^* L, X \},
\]

\[
\mathcal{L}_1(X) = \frac{s^2}{2} (\dot{L}^* XL + L^* X \dot{L}) + \frac{u^2}{2} \left( L^* XL + L^* \dot{X}L \right) - \frac{u^2}{4} \left( \dot{L}^* LX + L^* XL \right) - \frac{u^2}{4} \left( L^* LX + L^* LX + 2L^* XL + 2XL^* L + 2XL^* L \right).
\]

Therefore we have

\[
\mathcal{L}_1(1) = i\left( L^* L - \langle L^* L \rangle_{ss} 1 \right), \quad \mathcal{L}_2(1) = -\frac{s^2}{2} L^* L + i s u \left( \dot{L}^* L + L^* \dot{L} \right).
\]

and in particular \( \text{Tr}(\beta_\theta \mathcal{L}_1(1)) = 0 \). Since the dynamics is irreducible at \( \theta_0 \), we have \( \text{Ker}(\mathcal{L}_0) = \mathbb{C} \mathbf{1} \) and the conditions of theorem 2.2 are fulfilled with \( P \) being the projection \( P(X) = \text{Tr}(\beta_\theta X) \mathbf{1} \). Therefore

\[
\lim_{t \to -\infty} S_{(l,u,s)}^{(l,u,s)}(1) = S_{(l,u,s)}^{(u,s)}(1) = \exp\left( \tau \mathcal{L}_1(s, u) \right) \mathbf{1},
\]

where \( S_{(l,u,s)}^{(u,s)} \) is a semigroup on the one-dimensional space \( \mathbb{C} \mathbf{1} \) with generator

\[
\mathcal{L}^{(u,s)} := -P \mathcal{L}_1 \mathcal{L}_1 + P \mathcal{L}_2 P = f_\epsilon(s, u) P.
\]

It now remains to compute the function \( f_\epsilon(s, u) \). With the above expressions for \( \mathcal{L}_1(1) \) and \( \mathcal{L}_2(1) \) we get

\[
\left[ -P \mathcal{L}_1 \mathcal{L}_1 + P \mathcal{L}_2 P \right](1) = \left( i u s \mu_\epsilon - \frac{s^2}{2} V_\epsilon \right) \mathbf{1},
\]

where

\[
\mu_\epsilon := \left\{ i \left[ \dot{H}, A \right] + L^* AL + L^* AL + 2 \text{Re} \left( \dot{L}^* L \right) - \left( \text{Re} \left( \dot{L}^* L \right) A + A \text{Re} \left( \dot{L}^* L \right) \right) \right\}_{ss},
\]

\[
V_\epsilon := \langle L^* L + 2L^* AL \rangle_{ss}.
\]
With $A$ given by $A := -\hat{L}(L^a L)_{\alpha\beta}$. In the expression of final expression of $V_c$ we used the fact that $\langle A \rangle_{ss} = 0$ since $\hat{L}$ leaves the space of zero mean observables invariant.

A.3. Details of the proof of theorem 4.2

In the following we drop the subscript when $\theta = \theta_0$ and denote $L = L_{\theta_0}$, $H = H_{\theta_0}$. The generator of the semigroup $T_{t}(u,p)$ is

$$
L_{(u,p)}(X) = i \left[ H_{\theta_0}, X \right] + L^a_{\theta_0} X L^a_{\theta_0} - \frac{1}{2} \left( \frac{L^a_{\theta_0} L^a_{\theta_0} X + XL^a_{\theta_0} L^a_{\theta_0}}{2} \right)
$$

$$
+ \frac{i p}{\sqrt{t}} \left( e^{-itL^a_{\theta_0} X} + X e^{itL^a_{\theta_0}} \right) - \frac{p^2}{2t} X - \frac{i p}{\sqrt{t}} \left( e^{-itL^a_{\theta_0}} + e^{itL^a_{\theta_0}} \right)_{ss} X
$$

$$
= t \mathcal{L}_1(X) + \sqrt{t} \mathcal{L}_1(X) + \mathcal{L}_2(X) + O(t^{-1/2}),
$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ depend on $(u, p)$ and the terms are given by

$$
\mathcal{L}_0(X) = i[H, X] + L^a XL^a - \frac{1}{2} \left( L^a X, L \right),
$$

$$
\mathcal{L}_1(X) = i u \left[ H, X \right] + u \left( L^a XL^a + L^a XL \right) - \frac{u}{2} \left( L^a L + L^a L \right) \left( L^a + L^a L \right)
$$

$$
+ \frac{i p}{\sqrt{t}} \left( e^{-itL^a X} + e^{itXL} \right) - \frac{i p}{\sqrt{t}} \left( e^{-itL^a} + e^{itXL} \right)_{ss} X
$$

$$
\mathcal{L}_2(X) = \frac{u^2}{2} \left[ H, X \right] + i u p \left( e^{-itL^a X} + e^{itXL} \right) + \frac{u^2}{2} \left( L^a XL + L^a + L^a XL_2 \right)
$$

$$
- \frac{u^2}{4} \left( L^a LX + L^a + L^a LX + 2L^a L^a + XL^a L + XL^a L + 2XL^a L \right) - \frac{p^2}{2} X.
$$

Then

$$
\mathcal{L}_1(1) = i p \left( e^{-itL^a} + e^{itL} \right) - i p \left( e^{-itL^a} + e^{itL} \right)_{ss} 1,
$$

$$
\mathcal{L}_2(1) = i u p \left( e^{-itL^a} + e^{itL} \right) - \frac{p^2}{2} 1.
$$

and in particular $\text{Tr}(\rho_a \mathcal{L}_1(1)) = 0$. Since the dynamics is irreducible, we have $\text{Ker}(\mathcal{L}_0) = \mathbb{C}1$ and the conditions of theorem 2.2 are fulfilled with $P$ being the projection $P(X) = Tr(\rho_a X) 1$. Therefore

$$
\lim_{t \to \infty} T_{(u,p)}(1) = T_{(u,p)}(1) = \exp(\tau f_{\theta_0}(u, p)) 1,
$$

where $T_{(u,p)}$ is a semigroup on the one-dimensional space $\mathbb{C}1$ with generator

$$
\mathcal{L}^{(u,p)} := -PL_1 \hat{L} L_1 + P L_2 P = f_{\theta_0}(u, p) P.
$$

It now remains to compute the function $f_{\theta_0}(u, p)$. With the above expressions for $\mathcal{L}_1(1)$ and $\mathcal{L}_2(1)$ and with the notation $i p B := -\hat{L} \mathcal{L}_1(1)$ such that $\text{Tr}(\rho_a B) = 0$, we have

$$
e^{-P \mathcal{L}_1 + P L_2 P}(1) = e^{i p B - \frac{p^2}{2}} 1.
where
\[ \mu_\hbar = \left(i[H, B] + L^a L^b + L^b L^a + e^{-i\phi} L^a + e^{i\phi} L^b \right) \]
\[ - \frac{1}{2} \left( (L^a L^b + L^b L^a)B + B (L^a L^b + L^b L^a) \right)_{ss}, \]
\[ V_\hbar = 1 + 2 \langle e^{-i\phi} L^a B + e^{i\phi} BL \rangle_{ss}. \]

**A.4. Calculation of the QFI for the two-level atom**

The generator in (44) can be expanded in a perturbation series in powers of \( t \) with coefficients
\[
\mathcal{L}_0(X) = \theta_0 \left[ z \sigma_+ - z \sigma_- , X \right] + \theta_0^2 \left( \sigma_+ \sigma_- \frac{1}{2} \{ \sigma_+ , \sigma_- , X \} \right),
\]
\[
\mathcal{L}_1(X) = \theta_0 \left( (u + v) \sigma_+ X \sigma_- - (u \sigma_+ X + v X \sigma_-) \right)
+ \frac{u + v}{2} \left[ z \sigma_+ - z \sigma_- , X \right] + (u - v) (z \sigma_+ X - X \sigma_-),
\]
\[
\mathcal{L}_2(X) = u v \sigma_+ \sigma_- \frac{1}{2} (v^2 \sigma_+ X + u^2 X \sigma_-).
\] (55)

Our immediate aim is to write explicitly the form of the limit generator
\[
\mathcal{L}^{(u,v)} \mathcal{L} \mathcal{L} = - P \mathcal{L}_1 \mathcal{L}_1 X + P \mathcal{L}_2 \mathcal{P} X
\] (56)
for \( X \in \text{Ker}(\mathcal{L}_0) \). For simplicity we take \( X = I \).

A first observation is that \( \sigma_+ \sigma_- = \frac{1}{2} z + \frac{1 - \theta_0}{2} e_4 \) and therefore we can compute
\[
P \mathcal{L}_2(I) = - \frac{(u - v)^2}{2} P (\sigma_+ \sigma_-) = - \frac{(u - v)^2}{2} \frac{1 - \theta_0}{2} I = - \frac{(u - v)^2}{2} a I.
\] (57)

We proceed with the calculation of the first term in the generator. Note that
\[
\mathcal{L}_1(I) = (u - v) (z \sigma_+ - z \sigma_-) = (u - v) (ze_2 - \bar{z}e_3).
\] (58)

Therefore
\[
\hat{\mathcal{L}} \mathcal{L}_1(I) = \frac{2(u - v)}{\theta_0^2} (e_3 - e_2).
\] (59)

This leads to
\[
\mathcal{L}_1 \hat{\mathcal{L}} \mathcal{L}_1(I) = - \frac{u - v}{\theta_0^2} (v \theta_0 \bar{z} \sigma_- - u \theta_0 z \sigma_+ - 2(u - v) |z|^2 \sigma_+ \sigma_-).
\] (60)

Thus we get
\[
-P \mathcal{L}_1 \hat{\mathcal{L}} \mathcal{L}_1(I) = - \frac{2 \left( \frac{8}{8} \frac{|z|^2 - \theta_0^2}{|z|^2 + \theta_0^2} \right) |z|^2}{(8 |z|^2 + \theta_0^2) \theta_0^2} (u - v)^2 I.
\] (61)

Putting everything together we find the desired expression
\[
-P \mathcal{L}_1 \hat{\mathcal{L}} \mathcal{L}_1(I) + P \mathcal{L}_2(I) = - \frac{16}{(8 |z|^2 + \theta_0^2) \theta_0^2} (u - v)^2 I.
\] (62)
The corresponding quantum Fisher information follows

\[ F(\theta_0) = \frac{128 \, |z|^4}{8 \, |z|^2 + \theta_0^2} \theta_0^2. \]  

(A.5. Calculation of the Fisher information for homodyne measurements for the two-level atom)

The generator of dynamics can be expanded in the usual perturbation series with coefficients

\[ \mathcal{L}_0(X) = \theta_0 \left[ \sigma_+ - \bar{\sigma}_-, X \right] + \theta_0^2 \left[ 2 \sigma_+ X + \frac{1}{2} \left\{ \sigma_-, \sigma_+ X \right\} \right], \]

\[ \mathcal{L}_1(X) = u \left[ \sigma_+ - \bar{\sigma}_-, X \right] + 2u \theta_0 \sigma_+ X - u \theta_0 \left( \sigma_+ X + X \sigma_- \right) \]

\[ + \, i \theta_0 \left( e^{i\theta_0} \bar{\zeta} + e^{i\theta_0} \zeta \right) X + X e^{i\theta_0} \left( \theta_0 \sigma_- + \zeta \right) \]

\[ - \, i \theta_0 \left( e^{-i\theta_0} \bar{\zeta} + e^{-i\theta_0} \zeta \right) \}

\[ \mathcal{L}_2(X) = iu \left( e^{-i\theta_0} \sigma_+ + e^{i\theta_0} \sigma_- X + u^2 \sigma_+ X \sigma_- - \frac{u^2}{2} \left( \sigma_+ X + X \sigma_- \right) \right) - \frac{B_0^2}{2} X. \]  

The numerator in the Fisher information is defined in terms of the mean value

\[ \left\{ e^{-i\theta_0} \left( \theta_0 \sigma_+ + \bar{\zeta} \right) + e^{i\theta_0} \left( \theta_0 \sigma_- + z \right) \right\}_{ss} = \theta_0 \left\{ e^{-i\theta_0} \sigma_+ + e^{i\theta_0} \sigma_- \right\}_{ss} + 2 \text{Re} \left( ze^{i\theta} \right) \]

\[ = - \theta_0 \left( e^{-i\theta_0} \bar{\zeta} + e^{i\theta_0} \zeta \right) + 2 \text{Re} \left( ze^{i\theta} \right) \]

\[ = 2 \text{Re} \left( ze^{i\theta} \right) - 4 \text{Re} \left( ze^{i\theta} \bar{a} \right). \]  

The denominator is the coefficient of \(-p^2/2\) in the expression \(-P \mathcal{L}_1 \mathcal{L}_1(I) + P \mathcal{L}_2(I)\). Remark that the coefficient of \(p^2/2\) in \(P \mathcal{L}_2(I)\) is simply \(-1\).

Starting from

\[ \mathcal{L}_1(I) = i \theta_0 \left( e^{-i\theta_0} \sigma_+ + e^{i\theta_0} \sigma_- \right) \]

\[ = i \theta_0 \left( e^{-i\theta_0} \sigma_+ + e^{i\theta_0} \sigma_- \right) \]  

We find that

\[ \tilde{\mathcal{L}} \mathcal{L}_1(I) = \frac{i \theta_0}{\theta_0^2 - 8 \, |z|^2} \left( -4 \text{Re} \left( \left( e^{i\theta_0} \sigma_+ e_1 + \left( 2 \theta_0^2 - 8 \, |z|^2 \right) e^{i\theta_0} + 8 \theta_0^2 e^{i\theta_0} \right) e_2 + \left( 2 \theta_0^2 - 8 \, |z|^2 \right) e^{i\theta_0} + 8 \theta_0^2 e^{-i\theta_0} \right) e_3 \right). \]  

Then the denominator in the Fisher information reads

\[ B_0 = 1 + \frac{2}{\theta_0^2 - 8 \, |z|^2} \left( \theta_0^4 \left( 4 \text{Im}^2 \left( e^{i\theta_0} \sigma_+ e_1 \right) - 16 \, |z|^2 \right) + 192 \theta_0^5 \, |z|^4 + 512 \, |z|^4 \text{Im}^2 \left( e^{i\theta_0} \sigma_+ e_1 \right) \right). \]
And the Fisher information is given by $I_h = \frac{A_h^2}{B_h}$, with

$$A_h = 64\theta_0 \left| c \right|^2 Re\left( e^{i\theta_0} \right) \left( \frac{\theta_0}{\theta_0^2 + 8 \left| c \right|^2} \right)^2.$$  \hspace{1cm} (69)

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