In this paper, we investigate the microlensing effects of blackhole-like wormholes. We evaluate the deflection angle upon the second order under weak field approximation with Gauss-Bonnet theorem. We elaborate on the deflection angle of the Ellis-Bronnikov wormhole as an example. Following the same procedure, we study the magnification of three typical wormholes (WH): Schwarzschild WH, Kerr-like WH, and RN WH, as well as their blackhole correspondence. We find that the prograde case of Kerr-like metric will lead to the multi-peaks of magnification as the mass part is compatible with the charge part. Moreover, the first two gentle peaks of Kerr blackhole are larger than the wormhole case by one order of magnitude, while the main peak of Kerr blackholes and wormholes are of the same order. For other cases, the behavior of magnification wormholes and their corresponding blackholes is similar. Our result may shed new light on exploring compact objects through the microlensing effect.

I. INTRODUCTION

Wormhole (WH) [1–5] is a hypothetical geometric structure connecting two otherwise remote regions. Wormholes may permit faster-than-light travel and time travel [6]. Furthermore, in the framework of General Relativity, the construction of traversable wormholes requires the violation of Null Energy Condition (NEC) [7, 8], and exotic matters beyond our current scope are necessary. Hence, the existence of wormholes may improve our understanding of new physics. Thus, It is important to study wormhole physics.

Gravitational lensing is a promising approach to search wormholes [9–11]. In the literature, the lensing effect of a wormhole is extensively studied [12–34]. Conventionally, wormholes are treated as a dark compact object [35]. The resulting lensing effect is relevant to the asymptotic behavior of the wormholes, while their geometric and topological structures do not play an important role. In this sense, a large variety of the wormholes mimics the black holes (BHs), and it would be difficult to distinguish them by astrophysical observations, such as lensing, accretion and quasi-normal-mode ringing [36, 37]. Thus, people are motivated to distinguish wormholes from other compact objects with various techniques such as shadow [38, 39], especially for which compares with various WHs and BHs in light of the shadow of Sgr A* [40], accretion [41, 42], deflection angle of massive particles [43] and quasi-normal ringing [44].

In this paper, we proceed with a slightly different approach. We wish to study the lensing effect when the light ray is close to the wormhole throat/blackhole horizon. The lensing effect of the simplest Ellis wormhole in the strong-field limit is well-studied [45, 46]. However, the magnification of more generic wormholes might be hard to evaluate. Hence, we still work in the weak field approximation. That is, the impact parameter $b_I$ is much larger than the intrinsic parameters of the wormhole/blackhole, such as the mass of a Schwarzschild metric. However, we evaluate the deflection angle upon second order, and numerically study the magnification of each case. For convenience, we adopt the technique introduced in [47–49], where the deflection angle is evaluated through the Gaussian-Bonnet theorem (GBT). The GBT formalism is wildly applied to study the deflection angle of various wormhole models [50–57].

We organize the paper as follows. In section I, we briefly introduce wormhole physics, gravitational lensing physics, and how to evaluate the deflection angle using GBT formalism. We explicitly show how GBT formalism works by using Ellis wormhole as an example in section II. We then study the magnification of Schwarzschild WH/BH in section III, Kerr WH/BH in section IV, and RN WH/BH in section V. We find that it is possible to distinguish Kerr WH and BH through the difference of magnitudes between their gentle peaks and main peaks. We conclude in section VI.

I. BASIC FORMALISM

In this section, we firstly review the basics of wormhole physics. After that, we discuss the gravitational lensing, and show how to use the GBT formalism to study the lensing physics.

Wormhole physics

For simplicity, we shall consider static spherically symmetric wormholes only. We start with the Morris-Throne
wormhole [1]. The metric is given by

\[ ds^2 = -e^{2\Lambda(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 d\Omega_2^2, \]  

(1)

where \( d\Omega_2^2 \) is the metric of a unit 2-sphere. The function \( \Lambda(r) \) and \( b(r) \) are the redshift function and shape function, respectively. The wormhole structure is characterized by its throat that connects two regions of spacetime. We illustrate a typical wormhole structure in figure 1, in which we only consider the microlensing effects that occurred on one side (spacetime 2 or 1). We impose the flare-out condition, which states that the geometry is the minimality of the wormhole throat in the embedded spacetime. For metric \( [1] \), the condition is given by

\[ b(r_0) = r_0, \quad \frac{b(r) - rb'(r)}{2b(r)^2} > 0, \]  

(2)

with \( b'(r) \equiv db(r)/dr \). \( r = r_0 \) labels the location of the throat. We see that the structure of the wormhole is solely determined by the shape function \( b(r) \). We also impose the asymptotic flatness, which sets \( \lim_{r \to \infty} b(r)/r = 0 \).

Finally, a traversable wormhole should have no horizon, i.e. \( g_{tt} \neq 0 \) everywhere. In the metric \( [1] \) this translates into \( \Lambda(r) \) is finite everywhere.

Gravitational lensing

A typical gravitational lensing geometry is illustrated in figure 2. For an infinitesimal source, the images observed will be magnified or demagnified due to the change of cross-section of a bundle of rays. The magnification is determined by the ratio between the solid angles

\[ |\mu| = \frac{d\omega_i}{d\omega_s} = \left| \frac{\beta}{\theta} \frac{d\beta}{d\theta} \right|^{-1}. \]  

(3)

The lensing geometry in figure 2 gives the lens equation

\[ \beta = \theta - \frac{D_{ls}}{D_s} \alpha. \]  

(4)

Hence, if we work out the deflection angle \( \alpha \) as a function of \( \theta \), we can use (4) to get \( \beta(\theta) \). Then, with the help of (3) we have the magnification \( |\mu| \), which is an important observable in astrophysics.

Finally, the lens equation (4) may admit more than one solution of \( \beta(\theta) \), corresponding to multiple images. For simplicity, we shall consider the microlensing case, where the separation of images is too small to be resolved by existing telescopes. In this case, we observe the combined light intensity, i.e. the observed magnification should be the summation of magnifications of each image:

\[ |\mu_{total}| = \sum_i |\mu_i|. \]  

(5)

Formalism with Gauss-Bonnet Theorem

For scenarios with relatively strong gravity and potentially non-trivial geometry, GBT is useful to calculate the deflection angle since it is a pure geometric description of gravitation. In GBT formalism, the deflection angle is given by

\[ \alpha = -\int \int_{D_\infty} K d\sigma, \]  

(6)

where \( D_\infty \) is a domain outside the light ray, the \( K \) stands for the Gaussian optical curvature and \( d\sigma \) is the elementary surface area of the optical geometry. In the lensing geometry as illustrated in figure 3, the formula (6) be-
and after a coordinate transformation \( \rho \) wormhole is given by plest traversable wormhole models. The metric of an Ellis integrals in equation (9) and get the deflection angle. With the wormhole geometry, we can then evaluate the boundary, \( \gamma_{2} \) is the sum of the exterior angle when taking the domain as a polygon. The function \( \kappa \) is the curvature of the geodesics. Note that, the external angles and interior angles are connected by \( \gamma_{2}. \) Hence, we further have

\[
\theta_{S} + \theta_{O} = 2\pi + \oint_{\gamma_{L}} \kappa \, d\sigma + \oint_{A_{1}} \kappa \, d\sigma,
\]

Finally, in the limit \( r \to \infty \), we have \( \theta_{S} + \theta_{O} = \pi \). With the wormhole geometry, we can then evaluate the integrals in equation (10) and get the deflection angle.

II. ELLIS WORMHOLE AS AN EXAMPLE

In this section, we illustrate how the GBT formalism works by applying it to the Ellis wormhole, the simplest traversable wormhole models. The metric of an Ellis wormhole is given by

\[
ds^{2} = -dt^{2} + dr^{2} + (r^{2} + r_{0}^{2}) \, d\Omega_{2}^{2},
\]

and after a coordinate transformation \( \rho = \sqrt{r^{2} + r_{0}^{2}} \), the metric returns to a Morris-Throne type

\[
ds^{2} = -dt^{2} + \frac{d\rho^{2}}{1 - r_{0}^{2}/\rho^{2}} + \rho^{2} \, d\Omega_{2}^{2},
\]

and it’s easy to see that the throat radius is \( r_{0} \).

For a photon, the geodesic equation is \( ds^{2} = 0 \). Also, we may simplify the problem by working in the equatorial plane with \( \theta = \pi/2 \). The geodesics of a photon is then described by

\[
dt^{2} = dr^{2} + (r^{2} + r_{0}^{2}) d\varphi^{2}.
\]

Now we define auxiliary functions \( du = dr, \, \zeta(u) = \sqrt{r^{2} + r_{0}^{2}} \), such that equation (12) becomes

\[
dt^{2} = h_{ab} \, d\lambda^{a} \, d\lambda^{b} = du^{2} + \zeta^{2}(u) \, d\varphi^{2}.
\]

The Gaussian optical curvature is then

\[
\mathcal{K} = -\frac{1}{\zeta^{2}(u)} \left[ \frac{1}{\zeta(u)} \left( \frac{d\zeta}{du} \left( \frac{d\zeta}{du} \right) \left( \frac{d\zeta}{du} \right) + \left( \frac{d\zeta}{du} \right)^{2} \right) \right].
\]

Moreover, we have \( \kappa \, dt = d\varphi \), then equation (10) becomes

\[
\int_{0}^{\pi} \frac{d\varphi}{\sin \varphi} + \int_{0}^{\pi} \frac{d\varphi}{\sin \varphi} \mathcal{K} \, \sqrt{\det h_{ab} \, dr \, d\varphi} = \pi, \tag{15}
\]

and the deflection angle is

\[
\alpha = -\int_{0}^{\pi} \frac{d\varphi}{\sin \varphi} \mathcal{K} \, \sqrt{\det h_{ab} \, dr \, d\varphi}. \tag{16}
\]

Here, the \( r \) integral ranges from the source to the observation. Using the lens geometry in figure 2 and with the help of equations (13) and (14), we finally get

\[
\alpha = \left( \varphi - \sqrt{\frac{r_{0}^{2} + 2b_{1}^{2} - r_{0}^{2} \cos[2\varphi]}{2(r_{0}^{2} + b_{1}^{2}) \text{Csc}[\varphi] E[\varphi, -r_{0}^{2}/b_{1}^{2}]}} \right) \frac{3\pi}{32}, \tag{17}
\]

where \( E[\varphi, -r_{0}^{2}/b_{1}^{2}] \) is the Elliptic funcnt. In weak field approximation, the impact parameter \( b_{1} \) would be much greater than the throat radius \( r_{0} \). We expand our result in Taylor series with respect to \( r_{0}/b_{1} \), and gets

\[
\alpha = \frac{\pi}{4} \left( \frac{r_{0}}{b_{1}} \right)^{2} + \frac{3\pi}{32} \left( \frac{r_{0}}{b_{1}} \right)^{4} + \mathcal{O} \left( \frac{r_{0}}{b_{1}} \right)^{6}, \tag{18}
\]

where the first term is agreed with [58]. In this section, we only implement GBT to calculate its deflection angle which is consistent with the geodesic line method. Since the ADM mass of Ellis wormhole is zero, thus one cannot find its corresponding blackhole. In the following examples, we will investigate the microlensing effects of Schwarzschild wormhole, Kerr-like wormhole and RN-like wormhole, especially for calculating their magnification effects under GBT as well as the corresponding BHs.

III. SCHWARZSCHILD WORMHOLE (BLACKHOLE)

In this section, we will investigate the microlensing effects of the Schwarzschild wormhole and its corresponding blackhole.
The metric of Schwarzschild wormhole can be written as follows [39],

\[ ds^2 = -\left(1 - \frac{2M}{r} + \lambda^2\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2, \quad (19) \]

where \( \lambda \) is a parameter, the Schwarzschild blackhole is restored as \( \lambda = 0 \) and \( 8\pi G = 1 \). The key case is that \( \lambda \) is nonzero which will lead to the geometry of the wormhole structure with \( r_0 = 2M \). However, we will use it as a free parameter to simulate the total magnification. Using GBT formula into Gaussian curvature \((14)\), one could obtain its second order as follows [53],

\[ K = \frac{6(\lambda^2 + 1) - 7(\lambda^2 + 1)rM^2 + r^2(\lambda^2 + 2)M}{(-r + 2M)r^4}. \quad (20) \]

Noting that this Gaussian curvature is an exact formula without using weak field approximation. In order to capture more information via the lensing effects, we calculate its corresponding deflection angle up to the second order of \( \frac{M}{b_l} \) using the weak field approximation,

\[ \alpha \approx \frac{4M}{b_l} + \frac{2\lambda r^2}{b_l} + \frac{7M^2\pi}{4b_l^2} + \frac{7M^2\pi\lambda^2}{4b_l^2}. \quad (21) \]

Its first order is consistent with Ref. [53]. The deflection angle \((21)\) of Schwarzschild wormhole contains the information of Schwarzschild blackhole as \( \lambda = 0 \). According to the weak field approximation, we could also obtain that \( \lambda \ll b_l \). Thus, the deflection angle \((21)\) is sufficient for investigating the microlensing effects for Schwarzschild wormhole and Schwarzschild blackhole.

### Magnification I

In this part, we will study the microlensing effects of the Schwarzschild wormhole and its corresponding blackhole via magnification. By implementing Eqs. \((1)\) and \((3)\), we will obtain its magnification as showing in figure 4, in which we numerically simulate the magnification as a function \( \mu = \mu(r_0, \lambda, b_l) \). To describe the impact of throat structure on the lensing effects, we change the variable \( r_0 = 2M \) \((G = 1)\) in figure 4. The scale that we consider is within the galaxy whose radius is around \( 5 - 100 \) kpc, thus we set \( D_t = 10 \) kpc as a quite reasonable input, meanwhile, we also set \( b_l = \frac{1}{2} \) for simplicity. Figure 4 indicates that the magnification is a function of \( b_l \). In the upper panel, it shows the peak of magnification of the Schwarzschild wormhole will be enhanced as improving the value of \( r_0 \) with a fixed \( \lambda = 0.1 \). The lower panel shows that \( \mu \) will be enhanced by increasing the value of \( \lambda \), in which the blue line corresponds to the Schwarzschild blackhole. Thus, we may conclude that the magnification of Schwarzschild’s blackhole is minimal in determining the mass. From the perspective of observations, we could distinguish these two objects via their peaks since one can determine the mass of a condensed object without any charge, and meanwhile, we could fix the distance \( D_{ls}, D_s \), and \( D_l \).

### IV. KERR WORMHOLE (BLACKHOLE)

In this section, we will investigate the microlensing effects of Kerr-like wormhole and its corresponding blackhole.

#### Metric II

The Kerr-like wormhole was discovered by [59], its corresponding metric reads as follows,

\[ ds^2 = -\left(1 - \frac{2Mr}{\Sigma}ight)dt^2 - \frac{4Mar\sin^2(\theta)}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2\sin^2(\theta)}{\Sigma}\right)\sin^2 \theta d\phi^2, \quad (22) \]
with
\[ \Sigma = r^2 + a^2 \cos^2(\theta) \]
\[ \hat{\Delta} = r^2 - 2M(1 + \lambda^2)r + a^2 \] (23)
where \( a \equiv \frac{J}{MC} \) (\( c = 1 \)) with the angular momentum \( J \).
\( \lambda = 0 \) will become the geometry of Kerr blackhole. As for the non-trivial \( \lambda \), the topology will change dramatically. The radius of throat will be given by \( \Delta = 0 \) whose formula is \( r_+ = M(1 + \lambda^2) + \sqrt{M^2(1 + \lambda^2) - a^2} \). Observing that \( r < r_+ \) will not allow the points exist. We cannot easily define the \( r_0 = r_+ \) in metric (22). Thus, we will retain the \( M \) and \( a \) to simulate the magnification.

**Deflection angle II**

The first essential quantity is the deflection angle. Ref. [53] already has evaluated the deflection angle to the first order. Noticing that their first order of deflection angle \( \alpha \approx \frac{2M(\lambda^2 + 2)}{b_I^2} \pm \frac{4Ma}{b_I^2} \), in which the “−” (minus) sign denotes the prograde light ray, will change the structure of magnification since the first term and the second term will contribute the opposite part. In light of this simple observation, its higher-order of deflection angle is also essential.

We will follow the method of [60] to calculate the deflection angle of metric (22). First, we need to calculate its corresponding \( d\sigma = \sqrt{\det h_{ab}} dr d\phi \) as follows,
\[ d\sigma = \sqrt{\frac{\Sigma^2}{\Delta(\Sigma - 2mr)}} \left( r^2 + a^2 + \frac{2a^2mr}{\Sigma - 2mr} \right) \frac{\Sigma}{(\Sigma - 2mr)} dr d\phi \] (24)
where \( \hat{\Delta} \) and \( \Sigma \) have defined in eq. (23). Since Kerr-like wormhole (blackhole) will rotate, the geodesic curvature will not vanish. Hence we obtain
\[ \kappa \approx -\frac{2aM}{r^3} + \frac{2M^2a\lambda^2}{r^4} - \frac{2aM^2}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) \] (25)
where the first term is consistent with eq. (36) of [60]. We also make the same approximation as [60] implemented as \( b_I \approx r/\cos \theta \) and \( l \approx b_I \tan \theta \) (\( 0 < \theta < 2\pi \)), in which \( \theta \) could be approximated to \( 2\pi \) as observer and light source is very remote with each other (approximated to be infinity for simplicity). Meanwhile, when we transform into the variable \( b \), we already have used the weak field approximation \( M, a \ll b_I, r \). Here, we should emphasize that this calculation for \( \kappa \) is the prograde case (\( dl > 0 \)). As for the retrograde case, the calculation is the same but the sign is the opposite, thus one can write down the geodesic curvature as follows,
\[ \kappa \approx \pm \left( -\frac{2aM}{r^3} + \frac{2M^2a\lambda^2}{r^4} - \frac{2aM^2}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) \right) \] (26)
Secondly, we will consider the Gaussian curvature, we will explicitly implement the result of [53] since \( \sigma \) already has been approximated to the second order,
\[ \kappa \approx \frac{(\lambda^2 + 2)M}{r^3}. \] (27)

Meanwhile, we will adopt the linear approximation of \( b_I \), then we will obtain the deflection angle from the Gauss curvature and geodesic curvature,
\[ \alpha = -\int_{\phi_s}^{\phi_R} K \sqrt{\gamma} dr d\phi + \int_{l_S}^{l_R} k dl, \] (28)
where we will set that \( l_R \) and \( l_S \) are infinity. Subsequently, we first get the prograde case,
\[ \alpha_{pro} = \frac{2M(\lambda^2 + 2)}{b_I} - \frac{4Ma}{b_I^2} + \frac{3M^2\pi(\lambda^2 + 2)}{2b_I^2} - \frac{aM^2\pi(\lambda^2 + 5)}{b_I^2}. \] (29)
The retrograde case is as follows,
\[ \alpha_{ret} = \frac{2M(\lambda^2 + 2)}{b_I} + \frac{4Ma}{b_I^2} + \frac{3M^2\pi(\lambda^2 + 2)}{2b_I^2} - \frac{3aM^2\pi(\lambda^2 + 1)}{b_I^2}. \] (30)
The first two terms of eq. (29) and eq. (30) are in the accordance with [53]. Before investigating the magnification of Kerr-like wormhole (blackhole), we will first compare the deflection angle between the prograde case and retrograde case. Figure 5 indicates the deflection angle for the prograde (29) and retrograde case (30). In the upper panel, we find that the difference will be apparent as setting \( \lambda = 0.01 \) and \( a = 1 \) where \( b_I \) is smaller than 2 kpc. In the lower panel, our numerical results show that these two cases almost cannot be distinguished as setting \( \lambda = 0.1, a = 0.1 \). Through figure 5 we find that the large value of momentum \( a \) will lead to the apparent difference of these two cases, which means that the fast rotation of Kerr-like wormhole (blackhole) will lead to the distinctive microlensing effects.

**Magnification II**

In this subsection, we will analyze the magnification of a Kerr-like wormhole (blackhole). As mentioned in the previous discussions, we will analyze the magnifications of Kerr-like wormhole (blackhole) in various cases: (a). The contribution of the mass part is much larger than the angular momentum part; (b). The contribution of the angular momentum part is much larger than the mass part; (c). These two parts are comparable.

\[ M \gg a \]

In this case, a Kerr-like wormhole will behave like a quasi-static object. Figure 6 shows the magnification of Kerr Kerr wormhole (blackhole) metric (22), in which it indicates that trend of magnification is almost the same
FIG. 5: This plot shows the deflection angle for case (29) and (30). For both cases, we have set $M = 0.01$ (weak field approximation). In the upper panel, we have set $\lambda = 0.01$, $a = 1$. The blue solid line corresponds to the prograde case (29) and the yellow solid line corresponds to the case (30). For the lower panel, we have set $\lambda = 0.1$, $a = 0.1$. For the prograde case and retrograde case, the maximal order of this magnification is around $10^3$. The Kerr blackhole corresponds to the blue solid line in figure 6. Our numerical results show that the peak will be appeared as enhancing the value of $b_I$. Thus, one can find the difference between the prograde case and retrograde case via the occurrence of the peak at different scales for $b_I$. Here, we should emphasize that there is only one peak for the Kerr wormhole or Kerr blackhole in the weak field region. If someone wants to distinguish the Kerr-like wormhole and Kerr blackhole, the value of $\lambda$ should be better larger than 0.1. Another special place is that the peak occurs from 5 kpc to 6 kpc, which means that the impact parameter is quite large compared with the radius of the throat. As for the other scale of $b_I$, it is trivial that has no magnification effects.

$M \ll a$

In this case, we will investigate the microlensing effects as $M \ll a$, which means the Kerr wormhole (blackhole) will have a large angular momentum. Comparing with figure 6, it indicates that the peak will appear at the smaller scale of $b_I$, whose order is also smaller (around 300) compared with $M \gg a$. In this case, it is difficult to distinguish the Kerr-like wormhole and Kerr blackhole since these three curves are almost overlapping with each other, for which the blue solid line corresponds to the Kerr blackhole. However, we can still distinguish $M \gg a$ and $M \ll a$ since the order of peak of $\mu$ is different.

$M \approx a$

In this subsection, we will investigate the microlensing effects of metric (22) in which the contribution from the mass is comparable with the angular momentum part. First, we observe the second and third terms of deflection angle (29) that will contribute compatible if we take the value of $a$ is ten times larger than $M$ at most.

In figure 8, it shows that the magnification will be clearly distinguished by these two cases. The most essential difference comes via the multi-peaks for the prograde case in the upper panel of figure 8. Especially for the Kerr blackhole, it will be three peaks in the prograde case, in which the first one is less than unity and the second one will be around 10. As for the Kerr-like wormhole
FIG. 7: This plot shows the magnification of Kerr wormhole (blackhole) metric \( \mu \equiv \mu(M, b_I, \lambda, a) \), including the prograde case and retrograde case. We have defined \( \mu \equiv \mu(M, b_I, \lambda, a) \). The upper panel shows the magnification of the prograde case, and the lower panel shows the retrograde case. All of these units have unified and the unit of 0.1 kpc \(< b_I < 3\) kpc ensures the weak field approximation. The maximal order of the magnification is around \(10^2\).

(yellow and green solid line), the first peak of them is too tiny to be detected, in which the second peak of yellow and green lines will be smaller one order compared with the Kerr blackhole case. To detect its validity, we need more accurate observations. As for the yellow and green solid lines, the first peak will be suppressed by enhancing the value of \(M\), but it is not so relevant with \(\lambda\).

The lower panel shows the retrograde case, in which one cannot find any multi-peaks. The difference between the Kerr blackhole (blue solid line) and Kerr-like wormhole is that the peak will be improved as changing the value of \(\lambda\), which does not highly depend on \(M\). As for the maximal values of these two cases, both of them will reach around 200 – 250. Thus, if one detected the magnification had multi-peaks as shown in figure 7, one possibility is the prograde case of Kerr-like wormhole (blackhole).

V. RN WORMHOLE (BLACKHOLE)

In this section, we will investigate the microlensing effects of RN wormhole (blackhole).

FIG. 8: This plot shows the magnification of Kerr wormhole (blackhole) metric \( \mu \equiv \mu(M, b_I, \lambda, a) \), including the prograde case and retrograde case. We have defined \( \mu \equiv \mu(M, b_I, \lambda, a) \). The upper panel shows the magnification of the prograde case, and the lower panel shows the retrograde case. All of these units have unified and the unit of 0.1 kpc \(< b_I < 2\) kpc ensures the weak field approximation. The maximal order of the magnification is around \(10^2\).

RN wormhole

The metric of RN wormhole combines the typically spherically symmetric structure and RN blackhole’s structure, its metric is found by [61].

\[
ds^2 = -\left(1 + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{r_0^2}{r^2} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2,
\]

where \(Q\) is the charge and \(r_0\) is the radius of the throat and we only consider the special case of RN spacetime, namely \(\frac{b(r)}{r} = \frac{r_0}{r}\). \((b(r)\) is the shape function). As \(Q = 0\), it will become the Ellis wormhole. And \(b(r) = r_0^2/r = 0\) will nicely recover the geometry of the RN blackhole. Thus, \(r_0\) is non-vanishing which denotes the geometry will change dramatically (from blackhole to wormhole). Here, we will utilize \(r_0\) and \(Q^2\) as two essential parameters to simulate the magnification. Being armed with GBT, we obtain the Gaussian curvature up the second order,

\[
\mathcal{K} = \frac{3Q^2 - r_0^2}{r^4} - \frac{4r_0^2Q^2}{r^6} + \mathcal{O}(Q^4/r_0^4),
\]

(32)
Then, we can obtain its corresponding deflection angle as follows,

\[ \alpha = \frac{r_0^2 \pi}{4b_I^2} - \frac{3\pi Q^2}{4b_I^2} + \frac{3r_0^2 \pi Q^2}{8b_I^2}. \] 

(33)

Once obtained the second order of deflection angle, then we can simulate the magnification of RN wormhole (blackhole).

Deflection angle III

In the following investigation, we will also study three cases: (a). The contribution of \( r_0 \) part is much larger than the part of \( Q \) part. (b). The contribution of \( Q \) part is much larger than \( r_0 \) part. (c). The part of \( r_0 \) is compatible with the part of \( Q \). The magnification is parametrized by \( \mu \equiv \mu(r_0, b_I, Q) \).

\[ r_0 \ll Q \]

As \( r_0 \gg Q \), the geometry will almost recover RN blackhole. Figure 9 shows the magnification of metric (31), in which it tells that there are no magnification effects as \( b_I > 1.5 \) kpc. Being different from the usual case, the image of RN wormhole (blackhole) will be shrunk as the \( b_I \) is less than 1 kpc. As for the pure blackhole case (the blue solid line), the shrinking effects will be milder compared to the RN wormhole case (green and yellow solid line).

\[ \mu(0, b_I, 0.01) \]

\[ \mu(10^{-7}, b_I, 0.05) \]

\[ \mu(10^{-7}, b_I, 0.1) \]

FIG. 9: This plot shows the magnification of metric (31). The range of \( b_I \) is from 0.1 kpc to 5.8 kpc keeping the weak field approximation. The blue line corresponds to the RN blackhole. The yellow and green solid line corresponds to the case of \( Q = 0.05 \) and \( Q = 0.01 \), respectively.

\[ r_0 \gg Q \]

In this case, it corresponds to the Ellis wormhole. In figure 10, it indicates that the magnification is dramatically different as \( r_0 \gg Q \) since there are peak values in some specific scales (various value of \( b_I \)). In our numerical simulations, we find that the peak of magnification of RN wormhole will be larger and larger as enhancing the value of \( r_0 \) with a fixed \( Q \). The precise physical meaning is that the magnification effects will be enhanced by improving the radius of the throat as \( r_0 \gg Q \).

\[ Q \approx r_0 \]

In this case, we will investigate the magnification effects of RN wormhole (blackhole) when the contribution of charge part is compatible with the radius of throat \( r_0 \).

\[ \mu(0.1, b_I, 0.1) \]

\[ \mu(0.01, b_I, 0.01) \]

\[ \mu(0.001, b_I, 0.001) \]

FIG. 10: This plot shows the magnification of metric (31). The range of \( b_I \) is from 0.1 kpc to 2.5 kpc keeping the weak field approximation. We have set \( Q = 10^{-7} \). We gradually reduce the values of \( r_0 \) from 0.1 to 0.02 corresponding to the blue, yellow, and green lines, respectively. The peak of these cases is from 80 to 230.

\[ \mu(0.1, b_I, 10^{-7}) \]

\[ \mu(0.05, b_I, 10^{-7}) \]

\[ \mu(0.02, b_I, 10^{-7}) \]

FIG. 11: This plot shows the magnification of metric (31). The range of \( b_I \) is from 0.1 kpc to 2.5 kpc keeping the weak field approximation. We have set \( Q = 10^{-7} \). We gradually reduce the values of \( r_0 \) from 0.1 to 0.02 corresponding to the blue, yellow, and green lines, respectively.
Figure 11 we can clearly see that the magnification will be approaching unity as $b_f > 1.5$ kpc. Our numerical results also indicate that the shrinking effects will be enhanced by improving the values of $r_0$ and $Q$, in which the geometry of spacetime will be more and more trivial as decreasing the contribution of matter.

VI. CONCLUSION

We explore the microlensing effects of a wormhole with a relatively strong presence of gravity. Although we still work in the weak field approximation, we investigate the deflection angle in the second order. The three typical wormholes (blackholes), the Schwarzschild WH/BH, Kerr-like WH/BH, and RN WH/BH, are investigated.

We find that it is possible to distinguish the Kerr-like wormhole and its corresponding blackhole. More specifically, in the prograde case, there will be multi-peaks when the contributions from mass and angular momentum are comparable. As shown in figure 5 the magnitude of the main peak for a Kerr BH is two orders of magnitude higher than the first gentle peak. While, for the wormhole case, the main peak is three orders of magnitude higher than the first gentle peak. The other cases are hard to distinguish, since the behavior of magnifications of wormholes and corresponding blackholes are similar, and the only difference is the magnitude of magnification.

Our work is a preliminary check on this topic. There are many interesting ideas to be studied in the future. Firstly, to fully address the issue, we need to go to the strong field limit. Then, in the strong field case, it is possible that the non-trivial topology influence not only the deflection angle, but also the lens equations. Hence, we need to improve the techniques in this paper. Secondly, we only studied selected models, and we need more examples to strengthen our conclusion. For example, the no-hair theorem [62, 63] tells that blackholes are uniquely determined by their mass, charge, and angular momentum. It is interesting to study wormholes and blackholes sharing these three same parameters. Finally, GBT may fail for certain modified gravity theories, such as the gravitational theory with a modification to the Gauss-Bonnet term [64], can we improve the GBT formalism in this situation?

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