AN INTEGRO-PDE MODEL
FOR EVOLUTION OF RANDOM DISPERAL

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ABSTRACT. We consider an integro-PDE model for a population structured by the spatial variables and a trait variable which is the diffusion rate. Competition for resource is local in spatial variables, but nonlocal in the trait variable. We focus on the asymptotic profile of positive steady state solutions. Our result shows that in the limit of small mutation rate, the solution remains regular in the spatial variables and yet concentrates in the trait variable and forms a Dirac mass supported at the lowest diffusion rate. Hastings and Dockery et al. showed that for two competing species in spatially heterogeneous but temporally constant environment, the slower diffuser always prevails, if all other things are held equal [13, 15]. Our result suggests that their findings may well hold for arbitrarily many or even a continuum of traits.

1. INTRODUCTION

In this paper, we focus on the concentration phenomena in a mutation-selection model for the evolution of random dispersal in a bounded, spatially heterogeneous and temporally constant environment. This model concerns a population structured simultaneously by a spatial variable $x \in D$ and the motility trait $\alpha \in \mathcal{A}$ of the species. Here $D$ is a bounded open domain in $\mathbb{R}^N$, and $\mathcal{A} = [\underline{\alpha}, \overline{\alpha}]$, with $\overline{\alpha} > \underline{\alpha} > 0$, denotes a bounded set of phenotypic traits. We assume that the spatial diffusion rate is parameterized by the variable $\alpha$, while mutation is modeled by a diffusion process with constant rate $\epsilon^2 > 0$. Each individual is in competition for resources with all other individuals at the same spatial location. Denoting by $u(t, x, \alpha)$ the population density of the species with trait $\alpha \in \mathcal{A}$ at location $x \in D$ and time $t > 0$, the model is given as

\begin{align}
\begin{cases}
    u_t &= \alpha \Delta u + \left[ m(x) - \hat{u}(x, t) \right] u + \epsilon^2 u_{\alpha\alpha}, \quad x \in D, \alpha \in (\underline{\alpha}, \overline{\alpha}), t > 0, \\
    \frac{\partial u}{\partial n} &= 0, \quad x \in \partial D, \alpha \in (\underline{\alpha}, \overline{\alpha}), t > 0, \\
    u_\alpha &= 0, \quad x \in D, \alpha \in \{\underline{\alpha}, \overline{\alpha}\}, t > 0, \\
    u(0, x, \alpha) &= u_0(x, \alpha), \quad x \in D, \alpha \in (\underline{\alpha}, \overline{\alpha}).
\end{cases}
\end{align}

Here $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator in the spatial variables,

$$\hat{u}(x, t) := \int_{A} u(t, x, \alpha) \, d\alpha,$$

$n$ denotes the outward unit normal vector on the boundary $\partial D$ of the spatial domain $D$, and $\frac{\partial}{\partial n} = n \cdot \nabla$. The function $m(x)$ represents the quality of the habitat, which
is assumed to be non-constant in \(x\) to reflect that the environment is spatially heterogeneous but temporally constant.

The model \((1.1)\) can be viewed as a continuum (in trait) version of the following mutation-selection model considered by Dockery et al. \([13]\), concerning the competition of \(k\) species with different dispersal rates but otherwise identical:

\[
\begin{aligned}
\frac{\partial}{\partial t} u_i &= \alpha_i \Delta u_i + \left[ m(x) - \sum_{j=1}^{k} u_j \right] u_i + \epsilon^2 \sum_{j=1}^{k} M_{ij} u_j \\
\frac{\partial m}{\partial n} u_i &= 0 \\
u_i(x, 0) &= u_{i,0}(x)
\end{aligned}
\tag{1.2}
\]

where \(0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k\) are constants, \(m(x) \in C^2(\overline{D})\) is non-constant, \(M_{ij}\) is an irreducible real \(k \times k\) matrix that models the mutation process so that \(M_{ii} < 0\) for all \(i\), and \(M_{ij} \geq 0\) for \(i \neq j\) and \(\epsilon^2 \geq 0\) is the mutation rate.

Model \((1.2)\) was introduced to address the question of evolution of random dispersal. In the case when there is no mutation, i.e. \(\epsilon = 0\), this question was considered in \([15]\), where it was shown that in a competition model of two species with different diffusion rates but otherwise identical, a rare competitor can invade the resident species if and only if the rare species is the slower diffuser. Dockery et al. \([13]\) generalized the work of Hastings \([15]\) to \(k\) species situation, and proved that no two species can coexist at equilibrium, i.e. the set of non-trivial, non-negative steady states of the system \((1.2)\) is given by

\[
\{(\theta_{\alpha_1}, 0, \ldots, 0), (0, \theta_{\alpha_2}, 0, \ldots, 0), \ldots, (0, \ldots, \theta_{\alpha_k})\},
\]

where \(\theta_{\alpha}\) is the unique positive solution of

\[
\alpha \Delta \theta + \theta(m - \theta) = 0 \quad \text{in } D, \quad \frac{\partial m}{\partial n} = 0 \quad \text{on } \partial D.
\]

Moreover, among the non-trivial steady states, only \((\theta_{\alpha_1}, \ldots, 0)\), the steady state where the slowest diffuser survives, is stable and the rest of the steady states are all unstable. Furthermore, when \(k = 2\), the steady state \((\theta_{\alpha_1}, 0)\) is globally asymptotically stable among all non-negative, non-trivial solutions. Whether such a result holds for three or more species remains an interesting and important open question.

Dockery et al. \([13]\) further inquired the effect of small mutation. More precisely, when \(0 < \epsilon \ll 1\), it is shown that \((1.2)\) has a unique steady state \(\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)\) in the space of non-trivial, non-negative functions, such that \(\tilde{u}_i > 0\) for all \(i\), and \(\tilde{U} \to (\theta_{\alpha_1}, 0, \ldots, 0)\) as \(\epsilon \to 0\); i.e. the system \((1.2)\) equilibrates only when the slowest species is dominant and all other species remain at low densities.

It is natural, then, to inquire if the situation in the discrete (in trait) framework carries over to the continuum framework. The aim of this paper is to study the asymptotic behavior of steady state(s) of \((1.1)\). Let \(u_\epsilon\) be any positive steady state of \((1.1)\), we will show that, as \(\epsilon \to 0\),

\[
u_\epsilon(x, \alpha) \to \delta(\alpha - \alpha) \theta_{\alpha}(x),
\]

i.e. \(u_\epsilon\) converges to a Dirac mass supported at the lowest possible trait value \(\alpha\). See Theorem 2.3 for precise descriptions of our main results.

Mutation-selection models for a continuum of trait values have been studied extensively, when the phenotypic trait is associated only with growth advantages \([4, 8, 9, 12, 17, 19, 21]\). See also \([16]\) for a pure selection model. The consideration of a spatial trait is more recent \([1, 2, 7, 20]\).
System (1.1) is also considered in an unbounded spatial domain $x \in \mathbb{R}$. A formal argument concerning the existence of an “accelerating wave” is presented in [6], which provides a theoretical explanation of the accelerating invasion front of cane toads in Australia [23]. Rigorous results are obtained when $\alpha \in \mathcal{A} = [\underline{\alpha}, \bar{\alpha}]$ more recently in [5, 24]. It can be summarized that the highest diffusion rate is selected when the underlying spatial domain is unbounded, which stands in contrast to the case of bounded spatial domains we consider in this paper, where the lowest possible diffusion rate is selected.

The rest of the paper is organized as follows: The main results are stated in Section 2. Section 3 concerns various estimates on steady states of (1.1). In Section 4 we introduce an auxiliary eigenvalue problem and a transformed problem of (2.1). The limit of $\hat{u}_\epsilon$ is determined in Section 5. In Section 6, we analyze the qualitative properties of solutions to the transformed problem. The proof of our main result is given in Section 7. Finally, the Appendices A to C establish the existence results, the smooth dependence of principal eigenvalue on coefficients as well as a Liouville-type results concerning positive harmonic functions on cylinder domains.

2. Main Results

In this paper, we consider the asymptotic behavior of positive steady states of (1.1), denoted by $u_\epsilon$. That is, $u_\epsilon$ satisfies the following mutation-selection equation of a randomly diffusing population:

$$
\begin{align*}
\alpha \Delta u_\epsilon + \epsilon^2 (u_\epsilon)_{\alpha \alpha} + [m(x) - \hat{u}_\epsilon(x)] u_\epsilon &= 0 \quad \text{in } \Omega := D \times (\underline{\alpha}, \bar{\alpha}), \\
\frac{\partial u_\epsilon}{\partial n} &= 0 \quad \text{on } \partial D \times (\underline{\alpha}, \bar{\alpha}), \\
(u_\epsilon)_\alpha &= 0 \quad \text{in } D \times \{\underline{\alpha}, \bar{\alpha}\},
\end{align*}
$$

where

$$
\hat{u}_\epsilon(x) = \int_\alpha^{\bar{\alpha}} u_\epsilon(x, \alpha) \, d\alpha.
$$

Throughout this paper, we assume

(A) $m(x)$ is a non-constant function in $C(\bar{D})$ such that $\int_D m(x) \, dx > 0$.

The existence of positive solutions to (2.1) can be stated as follows:

**Theorem 2.1.** Suppose (A) holds, then (2.1) has at least one positive solution for all $\epsilon > 0$.

We postpone the proof of Theorem 2.1 to Appendix A. For the rest of the paper we will focus on the asymptotic behavior of positive solutions of (2.1) as $\epsilon \to 0$. To this end, we define the following quantities:

**Definition 2.2.**
(i) Let $\theta_\underline{\alpha}(x)$ be the unique positive solution of

$$
\begin{align*}
\alpha \Delta \theta + \theta (m(x) - \theta) &= 0 \quad \text{in } D, \\
\frac{\partial \theta}{\partial n} &= 0 \quad \text{on } \partial D.
\end{align*}
$$

(ii) For each $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, we denote the principal eigenvalue and principal positive eigenfunction of the following problem by $\sigma^*(\alpha)$ and $\psi^*(x, \alpha)$, respectively:

$$
\begin{align*}
\alpha \Delta \psi + (m(x) - \theta_\underline{\alpha}(x)) \psi + \sigma \psi &= 0 \quad \text{in } D, \\
\frac{\partial \psi}{\partial n} &= 0 \quad \text{on } \partial D, \quad \text{and} \quad \int_D \psi^2 \, dx = \int_D \theta_\underline{\alpha}^2 \, dx.
\end{align*}
$$
(Note that by (i), \( \theta_\alpha(x) \) is a positive eigenfunction for \( \frac{\partial^2\psi}{\partial \alpha^2} + \frac{1}{\cosh^2\alpha} \cdot \psi \) when \( \alpha = \alpha \). By uniqueness of the (normalized) principal eigenfunction, we have \( \sigma^*(\alpha) = 0 \), and \( \psi^*(x,\alpha) = \theta_\alpha(x) \) for \( x \in D \).

(iii) Denote by \( \eta^*(s) \) the unique positive solution to \( \frac{\partial^2\eta}{\partial s^2} + (a_0 - a_1 s)\eta = 0 \) for \( s > 0 \),
\[
\begin{cases}
\eta'(0) = 0 = \eta(+\infty) \\
i_0 = 1 = \int_0^{\infty} \eta(s) \, ds = 1,
\end{cases}
\]
where \( a_0, a_1 \) are positive constants determined by \( a_1 = \frac{a_0}{a_0}, \) and \( a_0 = (a_1)^{2/3} A_0 \), where \( A_0 \) is the absolute value of the first negative zero of the derivative of the Airy function.

When \( m(x) \equiv 1 \), one can easily show that \( u_\epsilon \equiv 1/(\alpha - \alpha) \), i.e. there is no selection in the trait variable. Our main result shows that the outcome changes drastically when \( m(x) \) is non-constant. In fact, \( u_\epsilon \) concentrates at the lowest value in the trait variable, as \( \epsilon \to 0 \). This phenomenon is also known as spatial sorting.

**Theorem 2.3.** Let \( u_\epsilon \) be any positive solution of \( \frac{\partial^2\psi}{\partial \alpha^2} + \frac{1}{\cosh^2\alpha} \cdot \psi \). Then for all \( \beta > 0 \), there exists \( C > 0 \) independent of \( \epsilon > 0 \) such that
\[
(2.6) \quad u_\epsilon(x, \alpha) \leq C \epsilon^{-2/3} \exp\left(-\beta(\alpha - \alpha)\epsilon^{-2/3}\right)
\]
in \( \Omega = D \times (\alpha, \alpha) \). Moreover, as \( \epsilon \to 0 \)
\[
(2.7) \quad \left\| \epsilon^{2/3} u_\epsilon(x, \alpha) - \theta_\alpha(x) \eta^*(\frac{\alpha - \alpha}{\epsilon^{2/3}}) \right\|_{L^\infty(\Omega)} \to 0
\]
where \( \theta_\alpha(x) \) and \( \eta^*(s) \) are given as above. In particular, we have
\[
(2.8) \quad \hat{u}_\epsilon(x) = \int_\Omega u_\epsilon(x, \alpha) \, d\alpha \to \theta_\alpha(x) \quad \text{as } \epsilon \to 0.
\]

As the proof of Theorem 2.3 is fairly technical, we briefly outline the main ingredients for readers. Our idea is to establish the “separation of variables” formula \( (2.7) \) for \( u_\epsilon \), by introducing the scaling \( s = (\alpha - \alpha)/\epsilon^{2/3} \) and writing \( u_\epsilon(x, \alpha) = \psi_\epsilon(x, \alpha) w_\epsilon(x, s) \), where \( \psi_\epsilon \) is the principal eigenfunction of \( -\alpha^2 \Delta \psi + (\tilde{\alpha} - m) \psi = \sigma \psi \), subject to the zero Neumann boundary condition and the integral constraint \( \int_D \psi_\epsilon^2 = \int_D \theta_\alpha \). The main body of our paper is devoted to the proof of following two things: (i) As \( \epsilon \to 0 \), \( \tilde{\alpha} \to \alpha \) uniformly, which implies that \( \psi_\epsilon(x, \alpha, \epsilon) \to \psi_\alpha \). The concentration phenomenon of \( u_\epsilon \) on the subset \( D \times \{ \alpha \} \) of \( \Omega \), i.e. \( (2.8) \), is established in Section 5 with the help of several key estimates proved in Sections 3 and 4, such as the \( L^\infty \) estimate of \( \tilde{u}_\epsilon \), as well as the limit \( \lim_{\epsilon \to 0}(\tilde{u}_\epsilon - m) \) being non-constant; (ii) As \( \epsilon \to 0 \), \( \epsilon^{-2/3} w_\epsilon(x, s) \) converges to \( \eta(s) \) uniformly for some function \( \eta \) which is independent of the spatial variable \( x \). This is done in Section 6 along with some key estimates established in the earlier sections.

**Remark 2.4.** After this work is completed, the authors learned that a closely related result, under a slightly different formulation, is independently proved by B. Perthame and P.E. Souganidis under a different approach, where an intermediate trait attains the minimum diffusion rate and an interior Dirac mass is found when the mutation rate tends to zero. Apart from the distinction in our approaches, we note the following distinct features of our work: (i) A boundary concentration is found in our set-up, instead of an interior concentration in \( [22] \) which predicts different scalings in powers of \( \epsilon \); (ii) Our method does not assume the convexity
of spatial domain $D$; (iii) Various detailed $L^\infty$ estimates and asymptotic limits are obtained (Theorem 2.3) which paves the way to the proof of asymptotic stability and uniqueness of $u_\epsilon$ in a future paper; (iv) The key estimate of the limit $h_0(x) = \lim_{\epsilon \to 0} \hat{u}_\epsilon(x) - m(x)$ being non-constant (Lemma 3.4) reflects the effect of spatial heterogeneity, the underlying mathematical reason for the selection of small diffusion rate. See also Proposition 3.7 which makes the connection to [22, Lemma 4.3].

3. Properties of $\hat{u}_\epsilon$

In this section we establish various properties of $\hat{u}_\epsilon$. Recall that $\hat{u}_\epsilon$ is defined in (2.2).

**Lemma 3.1.** There exists some positive constant $\delta_1 = \delta_1(\alpha, \bar{\alpha}, m)$ independent of $\epsilon$ such that

$$\delta_1 \leq \hat{u}_\epsilon(x) \leq 1/\delta_1$$

in $D$ for all $\epsilon > 0$. In particular,

$$h_\epsilon(x) := \hat{u}_\epsilon(x) - m(x)$$

is bounded uniformly in $L^\infty(D)$.

**Proof.** The idea of the upper bound follows from [24]. Define

$$v_\epsilon(x) = \int_\alpha^\bar{\alpha} \alpha u_\epsilon(x, \alpha) \, d\alpha.$$

Then we have

$$\alpha \hat{u}_\epsilon(x) \leq v_\epsilon(x) \leq \bar{\alpha} \hat{u}_\epsilon(x)$$

in $D$.

Integrating (2.1) over $\alpha$ gives

$$\begin{cases}
\Delta v_\epsilon(x) + (m(x) - \hat{u}_\epsilon(x)) \hat{u}_\epsilon(x) = 0 & \text{in } D, \\
\frac{\partial v_\epsilon}{\partial n} = 0 & \text{on } \partial D.
\end{cases}$$

Let $\max_D v_\epsilon = v_\epsilon(x_0)$, then $\hat{u}_\epsilon(x_0) \leq m(x_0) \leq \max_D m$ (see [18, Proposition 2.2]), and by (3.3),

$$\alpha \max_D \hat{u}_\epsilon \leq \max_D v_\epsilon = v_\epsilon(x_0) \leq \bar{\alpha} \hat{u}_\epsilon(x_0) \leq \bar{\alpha} \max_D m.$$

Hence we deduce that $\|\hat{u}_\epsilon\|_{L^\infty(D)}$ and $\|h_\epsilon\|_{L^\infty(D)}$ are bounded uniformly in $\epsilon$, where $h_\epsilon(x) = \hat{u}_\epsilon(x) - m(x)$ is given in (3.4).

Next, we show the lower bound of $\hat{u}_\epsilon$. By (3.3), we deduce that

$$\hat{u}_\epsilon(x) = k_\epsilon(x)v_\epsilon(x)$$

for some $k_\epsilon(x) \in L^\infty(D)$ such that $\bar{\alpha}^{-1} \leq k_\epsilon(x) \leq \alpha^{-1}$. So that $v_\epsilon$ is a positive solution of

$$-\Delta v_\epsilon + h_\epsilon(x)k_\epsilon(x)v_\epsilon = 0 \quad \text{in } D,$$

and

$$\frac{\partial v_\epsilon}{\partial n} = 0 \quad \text{on } \partial D,$$

where we have already shown that $h_\epsilon = \hat{u}_\epsilon - m$ is uniformly bounded (in $L^\infty(D)$) in $\epsilon$. Therefore, the Harnack inequality applies so that

$$\max_D v_\epsilon \leq C' \min_D v_\epsilon$$

for some positive constant $C'$ independent of $\epsilon$. Thus, we conclude that $\hat{u}_\epsilon$ is uniformly bounded from below and above by the constants $\delta_1$ and $\delta_1^{-1}$, respectively.
for some constant $C' > 1$ independent of $\epsilon$. Combining with (3.3), we have

\[(3.6) \quad \alpha \max_D \hat{u}_\epsilon \leq \max_D v_\epsilon \leq C' \min_D v_\epsilon \leq C' \alpha \min_D \hat{u}_\epsilon.\]

Now, if we divide (2.1) by $u_\epsilon$ and integrate by parts over $\Omega = D \times (\alpha, \overline{\alpha})$, we obtain

\[(3.7) \quad (\overline{\alpha} - \alpha) \int_D (\hat{u}_\epsilon - m) \, dx = \int_\Omega h_\epsilon(x) \, d\alpha \, dx = \int_\Omega \frac{\alpha |\nabla_x u_\epsilon|^2 + \epsilon^2 |(u_\epsilon)_\alpha|^2}{u_\epsilon^2} \, dx > 0.\]

We deduce by (3.6) and (3.7) that

\[\frac{C' \overline{\alpha}}{\alpha} \min_D \hat{u}_\epsilon \geq \max_D \hat{u}_\epsilon \geq \frac{1}{|D|} \int_D \hat{u}_\epsilon(x) \, dx \geq \frac{1}{|D|} \int_D m(x) \, dx > 0.\]

This establishes the uniform lower bound of $\hat{u}_\epsilon$. \hfill \Box

**Remark 3.2.** Since $\|\hat{u}_\epsilon\|_{L^\infty(D)}$ and $\|v_\epsilon\|_{L^\infty(D)}$ are bounded uniformly in $\epsilon$, applying elliptic $L^p$ estimate to (3.1) implies that $\|v_\epsilon\|_{W^{2,p}(D)}$ is bounded uniformly in $\epsilon$. In particular, there exists sequence $\epsilon_k \to 0$ such that $v_{\epsilon_k}$ converges uniformly on $D$.

**Lemma 3.3.** There exists a constant $C > 0$ such that for any positive solution $u_\epsilon$ of (2.1),

\[\sup_{D \times (\alpha, \overline{\alpha})} u_\epsilon \leq C \epsilon^{-1}.\]

**Proof.** Choose $x_\epsilon$ and $\alpha_\epsilon$ such that the supremum of $u_\epsilon$ is attained at $(x_\epsilon, \alpha_\epsilon)$. Next, let $U_\epsilon(x, \tau) = u_\epsilon(x, \alpha_\epsilon + \epsilon \tau)$, then (extending $u_\epsilon$ to $D \times [\alpha - \epsilon_0, \overline{\alpha} + \epsilon_0]$ by reflection across the boundary portions $D \times \{\alpha, \overline{\alpha}\}$ if necessary) one may observe that $U_\epsilon$ satisfies a uniformly elliptic equation with uniformly bounded (in $L^\infty$) coefficients

\[\begin{cases} \alpha \Delta_x U_\epsilon + U_\epsilon_{\tau\tau} + h_\epsilon(x)U_\epsilon = 0 & \text{in } D \times [-2,2], \\
\frac{\partial U_\epsilon}{\partial \eta} = 0 & \text{on } \partial D \times [-2,2], \end{cases}\]

where $\alpha = \alpha_\epsilon + \epsilon \tau$ is always bounded between $[\alpha - \epsilon_0, \overline{\alpha} + \epsilon_0] \subset (0, +\infty)$. Hence, we may apply the Harnack’s inequality to yield a positive constant $C$ independent of $\epsilon$ such that

\[u_\epsilon(x, \alpha_\epsilon + \epsilon \tau) \geq C u_\epsilon(x_\epsilon, \alpha_\epsilon) = C \sup_{D \times (\alpha, \overline{\alpha})} u_\epsilon\]

for all $x \in D$, $\tau \in [-1,1]$. Hence,

\[\hat{u}_\epsilon(x) = \int_\alpha^{\overline{\alpha}} u_\epsilon(x, \alpha) \, d\alpha \geq C \epsilon \sup_{D \times (\alpha, \overline{\alpha})} u_\epsilon\]

for all $\epsilon$ sufficiently small. By Lemma 3.1 we deduce that

\[\sup_{D \times (\alpha, \overline{\alpha})} u_\epsilon \leq C' \epsilon^{-1}\]

for some positive constant $C'$. \hfill \Box

By Lemma 3.1 $h_\epsilon$ is bounded in $L^\infty(D)$ uniformly in $\epsilon$. Therefore, up to subsequences $\epsilon_j \to 0$, $h_{\epsilon_j}$ converges weakly in $L^p(D)$ for all $p > 1$. We first prove an important property of any subsequential limit $h_0$.

**Lemma 3.4.** Let $h_0$ be a weak (subsequential) limit of $h_\epsilon(x)$ in $L^p(D)$ ($p > 1$) as $\epsilon \to 0$, then $h_0(x)$ is non-constant in $D$.

**Proof.** Suppose to the contrary that for some $c \in \mathbb{R}$, $h_\epsilon(x) \to c$ weakly in $L^p(D)$ for all $p > 1$.\hfill \Box
Claim 3.5. \( c = 0 \).

By taking \( \epsilon \to 0 \) in \((3.7)\), we deduce that \( c \geq 0 \); i.e. for some \( c \geq 0 \), \( \hat{u}_\epsilon = h_\epsilon + m(x) \to c + m(x) \) in \( L^p \) for all \( p > 1 \).

Next, integrating \((2.3)\) with respect to \( \alpha \) and then \( x \), we obtain
\[
\int_D \hat{u}_\epsilon(x)(m(x) - \hat{u}_\epsilon(x)) = 0,
\]
so that
\[
\int_D m(x)(m(x) + c) = \lim_{\epsilon \to 0} \int_D m(x) \hat{u}_\epsilon \geq \liminf_{\epsilon \to 0} \int_D (\hat{u}_\epsilon)^2 \geq \int_D (m(x) + c)^2,
\]
where the first inequality follows from \((3.8)\), and the second inequality from expanding \( \int_D [\hat{u}_\epsilon(x) - (m(x) + c)]^2 \geq 0 \) as
\[
\int_D \hat{u}_\epsilon^2 \geq 2 \int_D \hat{u}_\epsilon(m + c) - \int_D (m + c)^2 \to \int_D (m + c)^2.
\]
Hence, by \((3.9)\), we have \( c = 0 \); i.e. \( h_\epsilon(x) \to 0 \) weakly in \( L^p(D) \) for all \( p > 1 \).

Note that we are done if \( m < 0 \) somewhere, since then \( h_\epsilon(x) \geq -m(x) > 0 \) in some open subset of \( D \) independent of \( \epsilon \), which contradicts \( h_\epsilon \to 0 \). For the general case of \( m(x) \) being possibly non-negative, we continue via a blow-up argument. Let
\[
C_\epsilon := \sup_{\alpha \in (\underline{\alpha}, \overline{\alpha})} \left( \frac{\sup_{x \in D} u_\epsilon(x, \alpha)}{\inf_{x \in D} u_\epsilon(x, \alpha)} \right).
\]

It is enough to show that

Claim 3.6. \( C_\epsilon \searrow 1 \) as \( \epsilon \to 0 \).

Assuming Claim 3.6, then by definition of \( C_\epsilon \),
\[

u_\epsilon(x, \alpha) \leq C_\epsilon u_\epsilon(y, \alpha) \quad \text{for all } x, y \in D \text{ and } \underline{\alpha} < \alpha < \overline{\alpha}.
\]

This gives, upon integrating over \( \alpha \in (\underline{\alpha}, \overline{\alpha}) \),
\[
\sup_{x \in D} \hat{u}_\epsilon(x) \leq C_\epsilon \inf_{y \in D} \hat{u}_\epsilon(y).
\]

Hence \( \hat{u}_\epsilon(x) \) converges to a constant. But this also means that \( h_\epsilon = \hat{u}_\epsilon - m(x) \) converges to a non-constant function, as \( m(x) \) is non-constant. This is a contradiction.

It remains to prove Claim 3.6. Assume to the contrary that there exist some constant \( c_0 > 1 \), and sequences \( \epsilon_k \to 0 \), \( \alpha_k \to \alpha_0 \), \( x_k, y_k \in D \) such that
\[
(3.10) \quad u_{\epsilon_k}(x_k, \alpha_k) \geq c_0 u_{\epsilon_k}(y_k, \alpha_k).
\]

Extend \( u_\epsilon \) to \( D \times [\underline{\alpha} - \epsilon_0, \overline{\alpha} + \epsilon_0] \) for some fixed \( \epsilon_0 \) small by reflection on the boundary \( D \times \{\underline{\alpha}, \overline{\alpha}\} \), and define
\[
U_k(x, s) := \frac{u_{\epsilon_k}(x, \alpha_k + \epsilon_k s/\sqrt{\alpha_k})}{\sup_{x \in D} u_{\epsilon_k}(x, \alpha_k)} \quad \text{in } D \times \left(\frac{\alpha - \epsilon_0 - \alpha_k}{\epsilon_k}, \frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon_k}\right).
\]

Then \((3.10)\) says that for some \( c_0 > 1 \) independent of \( k \)
\[
(3.11) \quad \inf_{x \in D} U_k(x, 0) \leq \frac{1}{c_0} \quad \text{for all } k.
\]

Moreover, \( U_k \) satisfies
\[
\begin{cases}
\Delta_x U_k + \frac{\alpha_k}{\alpha_k + \epsilon_k s/\sqrt{\alpha_k}} U_{k,ss} + \frac{h_\epsilon(x)}{\alpha_k + \epsilon_k s/\sqrt{\alpha_k}} U_k = 0 & \text{in } D \times \left(\frac{\alpha - \epsilon_0 - \alpha_k}{\epsilon_k}, \frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon_k}\right), \\
U_k(x, s) > 0 & \text{in } D \times \left(\frac{\alpha - \epsilon_0 - \alpha_k}{\epsilon_k}, \frac{\overline{\alpha} + \epsilon_0 - \alpha_k}{\epsilon_k}\right), \quad \sup_{x \in D} U_k(x, 0) = 1.
\end{cases}
\]
Since $\alpha_k \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$, $\epsilon_k \to 0$, the domain of $U_k$ converges to $D \times \mathbb{R}$ as $k \to \infty$. Moreover, by the uniform boundedness of $\|h_{\epsilon_k}\|_{L^\infty(D)}$ in $k$ (Lemma 3.1), we have for each $M > 0$ the coefficients of the equation of $U_k(x, s)$ are bounded in $L^\infty(\tilde{D} \times [-M, M])$ uniformly in $k$. Since $\sup_{x \in D} U_k(x, 0) = 1$, together with (3.11) we may apply Harnack inequality to obtain a constant $C = C(M)$ independent of $k$ such that

$$C^{-1} \leq U_k(x, s) \leq C \quad \text{for } x \in D \text{ and } |s| \leq M.$$ 

By $L^p$ estimates (applied to $\overline{D} \times [-M, M]$ for each $M$), there is a subsequence $U_{k_i}$ that converges uniformly in compact subsets of $\overline{D} \times \mathbb{R}$ to a positive solution of $\Delta x U_0 + (U_0)_{ss} = 0$ on $D \times \mathbb{R}$. (The limiting domain is $D \times \mathbb{R}$ as $\frac{\alpha - \alpha_0}{\epsilon} \to -\infty$ and $\frac{\alpha + \alpha_0}{\epsilon} \to \infty$.) Now, we apply Proposition C.1 for positive harmonic functions on a cylinder domain, so that $U_0 \equiv c_1$ for some positive constant $c_1$. Since $\sup_{x \in D} U_k(x, 0) = 1$ for all $k$, we have $c_1 = 1$. In particular, we set $s = 0$ and find a subsequence $U_{k_i}(x, 0)$ converges to 1 uniformly for $x \in D$. This is in contradiction to (3.11) and proves Claim 3.6. This completes the proof. \hfill \(\blacksquare\)

The following result generalizes a key estimate of [22], proved wherein via Bernstein’s method under the additional assumption that $D$ is convex. Although not needed for the rest of the paper, Proposition 3.7 enables one to follow the elegant Hamilton-Jacobi approach as in [22] to show the concentration phenomenon.

**Proposition 3.7.** Let $u_\epsilon$ be a positive solution of (2.1). Then there exists a constant $C > 0$ independent of $\epsilon$ such that

$$\epsilon \left\| \frac{u_{\epsilon, \alpha}}{u_\epsilon} \right\|_{L^\infty(\Omega)} + \left\| \frac{\nabla x u_{\epsilon}}{u_\epsilon} \right\|_{L^\infty(\Omega)} \leq C.$$

**Proof.** Extend the definition of $u_\epsilon$ to $D \times (2\overline{\alpha} - \alpha, 2\overline{\alpha} - \underline{\alpha})$ by reflecting along the boundary portions $D \times \{\overline{\alpha}, \underline{\alpha}\}$. For each $\alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$, define

$$U_\epsilon(x, \tau) := u_\epsilon(x, \alpha_0 + \epsilon \tau).$$

Then $U_\epsilon$ is a positive solution to

$$A(\tau, \epsilon) \Delta x U_\epsilon + U_{\epsilon, \tau \tau} - h_\epsilon(x) U_\epsilon = 0$$

in $D \times (\epsilon^{-1}(2\overline{\alpha} - \alpha_0, \epsilon^{-1}(2\overline{\alpha} - \alpha_0))$ and satisfies the Neumann boundary condition on $\partial D \times \epsilon^{-1}(2\overline{\alpha} - \alpha_0, \epsilon^{-1}(2\overline{\alpha} - \alpha_0))$. Here $A(\tau, \epsilon)$ is a continuous function such that $\underline{\alpha} \leq A(\tau) \leq \overline{\alpha}$. This, together with the boundedness of $\|h_\epsilon\|_{L^\infty(D)}$ (Lemma 3.1), one may apply the Harnack inequality to $D \times (-1, 1)$ and deduce the following.

**Claim 3.8.** There exists $C > 0$ independent of $\alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$ and $\epsilon$ such that

$$\sup_{D \times (-1, 1)} U_\epsilon(x, \tau) \leq C \inf_{D \times (-1, 1)} U_\epsilon(x, \tau).$$

Next, we apply elliptic $L^p$ estimates to $U_\epsilon$ in $D \times (-1, 1)$, so that

(3.12) \[ \sup_{x \in D} \|U_{\epsilon, \tau}(x, 0)\| + \left\| \nabla x U_\epsilon(x, 0) \right\| \leq C \|U_\epsilon\|_{L^p(D \times (-1, 1))} \leq C \sup_{D \times (-1, 1)} U_\epsilon(x, \tau). \]

In view of Claim 3.8 we deduce for any $x \in D$,

$$|U_{\epsilon, \tau}(x, 0)| + \left| \nabla x U_\epsilon(x, 0) \right| \leq C \inf_{D \times (-1, 1)} U_\epsilon \leq C U_\epsilon(x, 0).$$
i.e. \( |u_\varepsilon(x,\alpha_0)| + |\nabla_x u_\varepsilon(x,\alpha_0)| \leq C u_\varepsilon(x,\alpha_0) \) for all \( x \in D \). Since \( C \) is independent of \( x, \alpha_0 \) and \( \varepsilon \), this proves Proposition 3.7. \( \square \)

4. Two Eigenvalue Problems

4.1. An Auxiliary Eigenvalue Problem. Consider, for each \( \alpha > 0 \) and \( \varepsilon > 0 \) the eigenvalue problem (recall \( h_\varepsilon(x) = \hat{u}_\varepsilon(x) - m(x) \))

\[
\begin{aligned}
-\alpha \Delta \psi + h_\varepsilon \psi &= \sigma \psi & \text{in } D, \\
\frac{\partial \psi}{\partial n} &= 0 & \text{on } \partial D \quad \text{and} \quad \int_D \psi^2 \, dx = \int_D \theta^2 \, dx,
\end{aligned}
\]

and denote the principal eigenvalue and positive eigenfunction by \( \sigma_\varepsilon(\alpha) \) and \( \psi_\varepsilon(x,\alpha) \), respectively. At this point, we have not shown how the two eigenvalue problems (4.1) and (2.4) are related yet.

For each \( \varepsilon > 0 \), \( \sigma_\varepsilon(\alpha) \) is a smooth function of \( \alpha > 0 \) (Proposition B.5(ii)), and it has a Taylor expansion at \( \alpha = \overline{\alpha} \):

\[
\sigma_\varepsilon(\alpha) = \sigma_{0,\varepsilon} + \sigma_{1,\varepsilon}(\alpha - \overline{\alpha}) + \sigma_{2,\varepsilon}(\alpha - \overline{\alpha})^2 + O((\alpha - \overline{\alpha})^3),
\]

where \( \sigma_{0,\varepsilon} = \sigma_\varepsilon(\overline{\alpha}) \) and \( \sigma_{1,\varepsilon} = \frac{\partial}{\partial \alpha} \sigma_\varepsilon(\overline{\alpha}) \).

Lemma 4.1. Let \( \sigma_\varepsilon \) and \( \psi_\varepsilon \) be given as above.

(i) For each \( k \geq 0 \), \( \frac{\partial}{\partial \alpha} \sigma_\varepsilon(\alpha) \) is bounded uniformly in \( \varepsilon > 0 \) and \( \alpha \in [\underline{\alpha}, \overline{\alpha}] \).

(ii) For each \( k \geq 0 \) and \( p > 1 \), \( \frac{\partial}{\partial \alpha^k} \psi_\varepsilon(\cdot, \alpha) \) is bounded in \( W^{2,p}(D) \) (and hence \( C(D) \)) uniformly in \( \varepsilon > 0 \) and \( \alpha \in [\underline{\alpha}, \overline{\alpha}] \).

(iii) There exists \( c_0 > 0 \) such that

\[
\liminf_{\varepsilon \to 0} \frac{\partial \sigma_\varepsilon}{\partial \alpha}(\alpha) \geq c_0 > 0 \quad \text{for all } \alpha \in [\underline{\alpha}, \overline{\alpha}].
\]

In particular, \( \liminf_{\varepsilon \to 0} \sigma_{1,\varepsilon} = \liminf_{\varepsilon \to 0} \frac{\partial \sigma_\varepsilon}{\partial \alpha}(\overline{\alpha}) > 0 \).

(iv) There exist positive constants \( c_1, c_2 \) such that for all \( \varepsilon > 0 \),

\[
c_1 \leq \psi_\varepsilon(x,\alpha) \leq c_2 \quad \text{for all } x \in D \text{ and } \alpha \in [\underline{\alpha}, \overline{\alpha}].
\]

Corollary 4.2. There exists \( C > 0 \) independent of \( \varepsilon \) such that

\[
\left\| \frac{\psi_{\varepsilon,\alpha}}{\psi_\varepsilon} \right\|_{L^\infty(D \times [\underline{\alpha}, \overline{\alpha}])} + \left\| \frac{\psi_{\varepsilon,\alpha}}{\psi_\varepsilon} \right\|_{L^\infty(D \times [\underline{\alpha}, \overline{\alpha}])} \leq C.
\]

Proof of Lemma 4.1. By the uniform boundedness of \( \|h_\varepsilon\|_{L^\infty(D)} \) in \( \varepsilon \) (Lemma 3.7), assertions (i) and (ii) follow from Proposition B.5(i). To show (iii), it suffices to show, given any sequence \( \varepsilon_j \to 0 \), and \( \alpha_j \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}] \), \( \liminf_{j \to \infty} \frac{\partial}{\partial \alpha} \sigma_\varepsilon(\alpha_j) > 0 \). By Lemma 3.7, we may assume without loss of generality that for some \( h_0 \in L^\infty(D) \), \( h_{\varepsilon_j} \to h_0 \) weakly in \( L^p(D) \) for all \( p > 1 \). Then, in the notation of Appendix B Proposition B.5(ii) implies that

\[
\frac{\partial}{\partial \alpha} \sigma_\varepsilon(\alpha_j) = \frac{\partial}{\partial \alpha} \lambda_1(\alpha_j, h_{\varepsilon_j}) \to \frac{\partial}{\partial \alpha} \lambda_1(\alpha_0, h_0).
\]

Since \( h_0 \) is non-constant (Lemma 3.4), Proposition B.1 implies that the last expression is positive. This proves (iii).

For (iv), suppose that along a sequence \( \varepsilon_j \to 0 \) and \( \alpha_j \to \alpha_0 > 0 \), either \( \inf_x \psi_{\varepsilon_j}(x,\alpha_j) \to 0 \) or \( \sup_x \psi_{\varepsilon_j}(x,\alpha_j) \to \infty \). By the uniform boundedness of \( \|h_\varepsilon\|_{L^\infty(D)} \) (Lemma 3.7), we may assume without loss that \( h_{\varepsilon_j} \) converges weakly in \( L^p(D) \) for all \( p > 1 \). Hence by Proposition B.5(ii), \( \psi_{\varepsilon_j} = \varphi_1(\cdot, \alpha_j, h_{\varepsilon_j}) \) converges to
\( \varphi_1(\cdot; \alpha_0, h_0) \) uniformly in \( D \), and the latter is a strictly positive function in \( C(\overline{D}) \). This is a contradiction, and proves (iv).

4.2. A Transformed Problem. By the fact that \( u_\epsilon(x, \alpha) \) is the principal eigenfunction, with zero as the corresponding principal eigenvalue, of the problem

\[
\begin{aligned}
- \alpha \Delta \phi - \epsilon^2 \phi_{\alpha\alpha} + h_\epsilon(x) \phi &= 0 & & \text{in } \Omega = D \times (\overline{\alpha}, \overline{\alpha}), \\
\frac{\partial \phi}{\partial n} &= 0 & & \text{on } \partial D \times (\overline{\alpha}, \overline{\alpha}), \quad \text{and} \quad \phi_\alpha = 0 & & \text{on } D \times \{ \overline{\alpha}, \overline{\alpha} \},
\end{aligned}
\]

we have the following variational characterization

\[
\begin{aligned}
0 &= \inf_{\phi \in H^1(\Omega)} J_\epsilon[\phi],
\end{aligned}
\]

where

\[
J_\epsilon[\phi] = \int_\Omega \left[ \alpha |\nabla x \phi|^2 + \epsilon^2 |\phi_\alpha|^2 + h_\epsilon \phi^2 \right].
\]

Define

\[
s_\epsilon = (\overline{\alpha} - \alpha)/\epsilon^{2/3},
\]

and

\[
w_\epsilon(x, s) := u_\epsilon(x, \alpha + \epsilon^{2/3} s)/\psi_\epsilon(x, \alpha + \epsilon^{2/3} s) \quad \text{for } x \in D, \; 0 \leq s \leq s_\epsilon,
\]

where \( \psi_\epsilon \) is given by (4.1). Then \( w_\epsilon \) satisfies

\[
\begin{aligned}
- \alpha \nabla_x \cdot (\psi_\epsilon^{-2} \nabla_x w_\epsilon) - \epsilon^2/3 (\psi_\epsilon^2 w_{\epsilon, s})_s + \psi_\epsilon^2 \left[ \sigma_\epsilon(\alpha) - \epsilon^2 \psi_{\epsilon, \alpha} \right] w_\epsilon &= 0 & & \text{in } D \times (0, s_\epsilon), \\
\frac{\partial w_\epsilon}{\partial n} &= 0 & & \text{on } \partial D \times (0, s_\epsilon), \\
w_{\epsilon, s} &= -\epsilon^{2/3} \psi_{\epsilon, \alpha} \psi_\epsilon w_\epsilon & & \text{on } D \times \{0, s_\epsilon\}.
\end{aligned}
\]

The corresponding variational characterization can be written as

\[
- \sigma_{0, \epsilon} = -\sigma_\epsilon(\alpha) = \inf_{\phi \in H^1(D \times (0, s_\epsilon)) \setminus \{0\}} \tilde{J}_\epsilon[\phi]
\]

where

\[
\tilde{J}_\epsilon[\phi] = \int_D \int_0^{s_\epsilon} \psi_\epsilon \left[ \alpha |\nabla_x \phi|^2 + \epsilon^2 |\phi_\alpha|^2 + \left( \sigma_\epsilon(\alpha) - \epsilon^2 \psi_{\epsilon, \alpha} \right) \phi^2 \right] ds dx + \epsilon^{4/3} \int_D \psi_{\epsilon, \alpha} \phi^2 ds = 0.
\]

Lemma 4.3. \( 0 \leq -\sigma_{0, \epsilon} \leq O(\epsilon^{2/3}) \).

Proof. First, \( \sigma_\epsilon(\alpha) \) is the principal eigenvalue (with principal eigenfunction \( \tilde{\phi}(x, \alpha) = \psi_\epsilon(x, \alpha) \)) of

\[
- \alpha \Delta \tilde{\phi} - \epsilon^2 \tilde{\phi}_{\alpha\alpha} + h_\epsilon(x) \tilde{\phi} = \sigma \tilde{\phi} \quad \text{in } \Omega = D \times (\overline{\alpha}, \overline{\alpha})
\]

subject to homogeneous Neumann boundary condition, with a variation characterization analogous to (4.4) and (4.5). Since the integrand in (4.5) is monotone increasing in \( \overline{\alpha} \leq \alpha \leq \overline{\alpha} \), \( \sigma_\epsilon(\alpha) \) is necessarily less than the principal eigenvalue of (4.3), which is zero. This proves \( \sigma_{0, \epsilon} = \sigma_\epsilon(\alpha) \leq 0 \).

For the upper estimate, we use a test function \( \phi(x, s) = \eta(s) \) for (4.9), where \( \eta : [0, \infty) \to [0, 1] \) satisfies

\[
\eta(s) > 0 \quad \text{for } 0 \leq s \leq 1, \quad \text{and} \quad \eta(s) = 0 \quad \text{for } s \geq 2.
\]
Then upon using $\nabla_x \eta = 0$, (4.2), and
\[
\int_D \left[ \psi \psi_{e,\alpha} \right]_{\alpha=0}^{s} \, dx = \frac{\eta^2(0)}{2} \frac{\partial}{\partial \alpha} \left[ \int_D \psi^2 \, dx \right]_{\alpha=0} = 0,
\]
(since $\eta(s_\epsilon) = 0$ for $\epsilon$ small, and by normalization we have $\int_D \psi^2(x, \alpha) \, dx \equiv 1$ for all $\alpha$), we obtain from (4.9) that
\[
-\sigma_0, \epsilon \leq \int_0^2 \psi^2 \left[ \frac{\epsilon^2}{3} \left| \eta' \right|^2 + (\sigma_1, \epsilon \epsilon^2/3 + \sigma_2, \epsilon \epsilon^3/3 + O(\epsilon^2)) \right] \, ds \, dx.
\]
The conclusion follows from Lemma 4.1(iv).

5. UNIFORM LIMIT OF $\hat{u}_\epsilon$.

In this section, we show that $\hat{u}_\epsilon$ converges to $\theta_\alpha$ in $C(\bar{D})$. In particular, $h_\epsilon \to \theta_{\alpha - m}$ in $C(\bar{D})$.

Recall that $w_\epsilon$ is defined in (4.7).

**Lemma 5.1.** For all $\beta > 0$, there exists $C > 0$ independent of $\epsilon$, such that
\[
w_\epsilon(x, s) \leq C \epsilon^{-1} e^{-\beta s}
\]
for all $x \in D$ and $0 \leq s \leq s_\epsilon$, where $s_\epsilon = (\alpha - \alpha)/\epsilon^2/3$.

**Proof.** First, we derive a rough upper bound of $w_\epsilon$ from Lemma 3.3.

**Claim 5.2.** There exists $C > 0$ such that
\[
\sup_{D \times (\underline{\alpha}, \overline{\alpha})} w_\epsilon \leq C \epsilon^{-1}.
\]
By definition, $\sup w_\epsilon \leq (\sup u_\epsilon)/(\inf \psi)$, and the claim follows from the upper bound of $u_\epsilon$ (Lemma 3.3) and Lemma 4.1(iv).

Next, we construct a supersolution to prove the exponential decay.

**Claim 5.3.** For each $\beta > 0$, there exists $s_0 > 0$ independent of $\epsilon$ such that
\[
\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\alpha)}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\alpha)}{\epsilon^{2/3}} - \frac{\epsilon^{2/3} \phi_{e,\alpha}(x, \alpha)}{\psi_e(x, \alpha)} \geq 2\beta^2
\]
for all $\alpha \in [\underline{\alpha} + \epsilon^{2/3} s_0, \overline{\alpha}]$.

To see the claim, we note that since $\sigma_\epsilon$ is monotone increasing in $\alpha$ (Proposition B.1(iii)), for $\alpha = \underline{\alpha} + \epsilon^{2/3} s$ and $s \geq s_0$,
\[
\frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\alpha)}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\alpha)}{\epsilon^{2/3}} \geq \frac{\sigma_\epsilon(\alpha + \epsilon^{2/3} s_0) - \sigma_\epsilon(\alpha)}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\alpha)}{\epsilon^{2/3}}.
\]
By (4.2) and Lemma 4.1(i),
\[
\liminf_{\epsilon \to 0} \frac{\sigma_\epsilon(\alpha + \epsilon^{2/3} s_0) - \sigma_\epsilon(\alpha)}{\epsilon^{2/3}} \geq (\liminf_{\epsilon \to 0} \sigma_{1, \epsilon}) s_0.
\]
Taking also Lemma 4.3 and Corollary 4.2 into account, we conclude that for $\alpha = \underline{\alpha} + \epsilon^{2/3} s$ and $s \geq s_0$,
\[
\liminf_{\epsilon \to 0} \left[ \frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\alpha)}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\alpha)}{\epsilon^{2/3}} - \frac{\epsilon^{2/3} \phi_{e,\alpha}(x, \alpha)}{\psi_e(x, \alpha)} \right] \geq (\liminf_{\epsilon \to 0} \sigma_{1, \epsilon}) s_0 - C.
\]
Since $\liminf_{\epsilon \to 0} \sigma_{1, \epsilon} > 0$ by Lemma 4.1(iii), Claim 5.3 holds by choosing $s_0$ large.
Claim 5.4. For each \( \beta > 0 \), there exists \( s_0 > 0 \) independent of \( \epsilon \) and a supersolution

\[
W(x,s) := \left( \sup_{x \in D, \ 0 \leq s \leq s_0} w_\epsilon \right) \left[ \exp(-\beta(s-s_0)) + \exp(\beta(s-(3/2)s_0)) \right],
\]

deﬁned on \( D \times (s_0, s_\epsilon) \) such that

\[
\begin{align*}
&\frac{\partial W}{\partial s} = 0 \quad \text{on } \partial D \times (s_0, s_\epsilon), \\
&W(x,s_0) \geq w_\epsilon(x,s_0) \quad \text{for } x \in D, \\
&W_s(x,s_\epsilon) \geq -\epsilon^{2/3} \psi_{s\alpha}(x,s) W(x,s_\epsilon) \quad \text{for } x \in D,
\end{align*}
\]

where \( s_\epsilon = (\alpha - \alpha) / \epsilon^{2/3} \).

To show the differential inequality, note that the term involving derivatives in \( x \) vanishes, and that by Claim 5.3 and Corollary 4.2

\[
\frac{1}{\psi_\epsilon} \left( \psi_\epsilon^2 W_s \right)_s + \left( \frac{\sigma_\epsilon(\alpha) - \sigma_\epsilon(\alpha)}{\epsilon^{2/3}} + \frac{\sigma_\epsilon(\alpha)}{\epsilon^{2/3}} - \epsilon^{4/3} \psi_{s\alpha} \psi_\epsilon \right) W \geq 0
\]

It remains to check the boundary condition on \( D \times \{s_\epsilon\} \), as the rest follows by deﬁnition. Note that \( s_\epsilon \to \infty \) as \( \epsilon \to \infty \), so that \( \exp(-\beta(s_\epsilon-s_0)) \ll \exp(-\beta s_\epsilon/2) \).

We therefore have

\[
\frac{W_s(x,s_\epsilon)}{W(x,s_\epsilon)} + \epsilon^{2/3} \psi_{s\alpha}(x,s) \psi_\epsilon(x,s) = \frac{-\beta \exp(-\beta(s_\epsilon-s_0)) + \beta \exp(-\beta s_\epsilon/2)}{\exp(-\beta(s_\epsilon-s_0)) + \exp(-\beta s_\epsilon/2)} + O(\epsilon^{2/3})
\]

which converges to a positive constant \( \beta \) uniformly for \( x \in D \). This proves the claim.

Now, we claim that

\[
w_\epsilon \leq W \quad \text{in } D \times (0, s_\epsilon).
\]

By deﬁnition, it is easy to see that \( w_\epsilon \leq W \) in \( D \times [0, s_0] \). That the inequality holds in \( D \times (s_0, s_\epsilon) \) is due to the fact that \( W \) is a strict positive supersolution of the linear problem (5.1) with homogeneous Dirichlet boundary condition on \( D \times \{s_0\} \), and Neumann condition on the remaining boundary portions. Standard maximum principle applies and shows that the quotient \( \frac{W_s}{\epsilon W} \) is non-negative. (See, e.g. [4] p. 48.)

Finally, we obtain Lemma 5.1 by combining Claim 5.2 and (5.2). \( \square \)

Lemma 5.5. Let \( v_\epsilon \) be given by (3.2), then

\[
\sup_{x \in D} |v_\epsilon(x) - \alpha u_\epsilon(x)| \to 0.
\]
Proof. Given \( \epsilon \), take \( \delta = \sqrt{\epsilon\delta_1} \), where \( \delta_1 \) is given by Lemma 3.1.

\[
|v_\epsilon(x) - \frac{\alpha}{\epsilon} \hat{u}_\epsilon(x)| = \left| \int_{\alpha}^{\bar{\alpha}} (\alpha - \alpha) u_\epsilon(x, \alpha) \, d\alpha \right| \\
\leq \delta \left| \int_{\alpha}^{\alpha + \delta} u_\epsilon(x, \alpha) \, d\alpha \right| + (\bar{\alpha} - \alpha) \left| \int_{\alpha}^{\bar{\alpha}} u_\epsilon(x, \alpha) \, d\alpha \right| \\
\leq \delta \hat{u}_\epsilon(x) + C_1 \int_{\alpha}^{\bar{\alpha}} \epsilon^{-1} \exp(-\beta(\alpha - \bar{\alpha})/\epsilon^{2/3}) \, d\alpha \\
\leq \sqrt{\epsilon} + o(1)
\]

where we have used Lemma 4.1(iv) and Lemma 5.1 in the second last inequality. This proves the lemma. \( \square \)

Proposition 5.6. \( \hat{u}_\epsilon \to \theta_\alpha \) in \( C(\overline{\Omega}) \), where \( \theta_\alpha \) is the unique positive solution of (2.3). In particular, \( h_\epsilon \to \theta_\alpha - m \) in \( C(\overline{\Omega}) \).

Proof. By Remark 3.2 and Lemma 5.5, we deduce that up to a subsequence \( \epsilon_j \to 0 \), both \( u_\epsilon \) and \( v_\epsilon/\hat{u}_\epsilon \) converges uniformly in \( D \) to some \( \hat{u}_0 \in W^{2,p}(D) \). We claim that \( \hat{u}_0 \) is a (strong and therefore classical) solution of (2.3), i.e. for each \( z(x) \in C^\infty(\overline{\Omega}) \),

\[
(5.3) \quad \alpha \int_D [\hat{u}_0 \Delta z + \hat{u}_0 (m - \hat{u}_0) z] \, dx - \alpha \int_{\partial D} \hat{u}_0 \frac{\partial z}{\partial n} \, dx = 0.
\]

To show (5.3), multiply (3.4) by a test function \( z(x) \) and integrate by parts, using the Neumann boundary condition of \( v_\epsilon \), we obtain

\[
\int_D [v_\epsilon \Delta z + \hat{u}_\epsilon (m - \hat{u}_\epsilon) z] \, dx - \int_{\partial D} v_\epsilon \frac{\partial z}{\partial n} \, dx = 0.
\]

Then one can pass to the limit to obtain (5.3) by invoking Lemma 5.1. By the lower estimate in Lemma 3.1 there exists \( \delta_1 > 0 \) such that \( \hat{u}_0(x) \geq \delta_1 \) for all \( x \in D \). Hence \( \hat{u}_0 \) is the unique positive solution of (2.3), i.e. \( \hat{u}_\epsilon \to \theta_\alpha \) in \( C(\overline{\Omega}) \). Since the limit is independent of subsequences, we deduce that that \( \hat{u}_\epsilon \to \theta_\alpha \) as \( \epsilon \to 0 \) (not just along subsequences \( \epsilon_j \to 0 \)). This proves the proposition. \( \square \)

Corollary 5.7. Let \( \sigma_\epsilon(\alpha) \) and \( \psi_\epsilon(x, \alpha) \) be the principal eigenvalue and eigenfunction of (4.1), and let \( \sigma^*(\alpha) \) and \( \psi^*(x, \alpha) \) be those of (2.3). Then as \( \epsilon \to 0 \), \( \sigma_\epsilon \to \sigma^* \) in \( C^k([\alpha, \bar{\alpha}]) \) for all \( k \), and \( \psi_\epsilon(., \alpha) \to \psi^*(., \alpha) \) in \( C^k([\alpha, \bar{\alpha}]; W^{2,p}(D)) \). In particular,

\[
(5.4) \quad \sigma_{0, \epsilon} \to \sigma_0^* := \sigma^*(\alpha) \quad \text{and} \quad \sigma_{1, \epsilon} \to \sigma_1^* := \frac{\partial \sigma^*}{\partial \alpha}(\alpha) > 0.
\]

Proof. Since now \( h_\epsilon \to h_0 = \theta_\alpha - m \) in \( L^\infty(D) \), the corollary follows from Proposition 3.1(ii). Since \( h_0 = \theta_\alpha - m \) is non-constant (Lemma 5.1, Proposition 3.1(iv) asserts that \( \sigma_1^* > 0 \). \( \square \)

6. Convergence of \( w_\epsilon \)

Let \( w_\epsilon \) be given by (4.7), define the normalized version \( \tilde{w}_\epsilon = \tilde{w}_\epsilon(x, s) \) on \( D \times [0, s_\epsilon] \) by

\[
\tilde{w}_\epsilon(x, s) := \left( \int_D \frac{\theta_\alpha^2}{\psi_\epsilon^2} \, dx \right)^{1/2} w_\epsilon(x, s)
\]
Corollary 6.2. \[ \text{fact proved the following.} \]

(6.1) \[
\int_D \int_0^s \psi^2_r(x, \omega + \varepsilon^{2/3} s) \tilde{w}_r^2(x, s) \, ds \, dx = \int_D \theta^2_\omega \, dx > 0.
\]

Proposition 6.1.

(i) \( \|\tilde{w}_\epsilon\|_{H^1(D \times (0, \epsilon))} \) is bounded uniformly in \( \epsilon \).

(ii) For any \( \beta > 0 \), there exists \( C_1 > 0 \) such that for all \( \epsilon \) sufficiently small, \( \tilde{w}_\epsilon(x, s) \leq C_1 \epsilon^{-\beta s} \) for all \( 0 \leq s \leq s_\epsilon \).

(iii) As \( \epsilon \to 0 \), \( \tilde{w}_\epsilon(x, s) \) converges locally uniformly in \( \tilde{D} \times [0, +\infty) \) to the unique positive solution of the problem

(6.2) \[
\begin{aligned}
\eta_{ss} + (a_0 - \sigma_1^* s) \eta &= 0 \quad \text{for } s \geq 0, \\
\eta_s(0) = 0 &= \tilde{\eta}(+\infty), \\
\int_0^\infty \eta^2 \, ds &= 1,
\end{aligned}
\]

where \( a_0 = (\sigma_1^*)^{2/3} A_0 \), with \( \sigma_1^* \) given by (5.3) and \( A_0 \) being the absolute value of the first negative root of the derivative of the Airy function.

Since \( \tilde{\eta}(+\infty) = 0 \) and \( \tilde{w}_\epsilon(x, s) \to 0 \) as \( s \to +\infty \) uniformly in \( x \in \tilde{D} \), we have in fact proved the following.

Corollary 6.2. \( \|\tilde{w}_\epsilon(x, s) - \tilde{\eta}(s)\|_{L^\infty(D \times (0, s_\epsilon))} \to 0 \) as \( \epsilon \to 0 \).

Proof. By Lemma 4.3 and the fact that \( \tilde{w}_\epsilon \) is a minimizer of (4.9), we obtain

\[
\int_D \int_0^s \psi_r^2 \left[ \alpha \epsilon^{-2/3} |\nabla_x \tilde{w}_r|^2 + |(\tilde{w}_r)_s|^2 + \left( \frac{\sigma_r(\alpha) - \sigma_r(0)}{\epsilon^{2/3}} + O(\epsilon^{4/3}) \right) \tilde{w}_r^2 \right] \, ds \, dx \\
+ \epsilon^{2/3} \int_D [\psi_{\epsilon, \alpha} \psi_r \tilde{w}_r^2]_{s=0}^s \, dx \\[ \leq C \left( \int_D \int_0^s \psi_r^2 \tilde{w}_r^2 \, ds \, dx \right) = C.
\]

Claim 6.3. \( \left| \int_D [\psi_{\epsilon, \alpha} \psi_r \tilde{w}_r^2]_{s=0}^s \, dx \right| \leq C \int_D \int_0^s \psi_r^2 \left( |\nabla_x \tilde{w}_r|^2 + |\tilde{w}_{r, \alpha}|^2 + \tilde{w}_r^2 \right) \, ds \, dx. \)

To prove Claim 6.3, we apply the Trace Theorem, so that there is \( C > 0 \) independent of \( \epsilon \) such that

\[
\left| \int_D [\psi_{\epsilon, \alpha} \psi_r \tilde{w}_r^2]_{s=0}^s \, dx \right| \leq C \int_D \int_0^s \left( |\nabla_x \tilde{w}_r|^2 + |\tilde{w}_{r, \alpha}|^2 + \tilde{w}_r^2 \right) \, ds \, dx \\
\leq C \int_D \int_0^s \psi_r^2 \left( |\nabla_x \tilde{w}_r|^2 + |\tilde{w}_{r, \alpha}|^2 + \tilde{w}_r^2 \right) \, ds \, dx,
\]

where we have used \( \|\psi_{\epsilon, \alpha} \psi_r\|_{L^\infty(D \times (0, s_\epsilon))} \leq C \) (Corollary 5.7) for the first inequality, and Lemma 4.4(4) for the second inequality.

From Claim 6.3, the normalization (6.1), the estimate in the beginning of the proof, and the monotonicity of \( \sigma_r(\alpha) \) in \( \alpha \) (Proposition 3.1(4)), we have

(1-C \epsilon^{2/3}) \int_D \int_0^s \psi_r^2 \left[ \alpha \epsilon^{-2/3} |\nabla_x \tilde{w}_r|^2 + |(\tilde{w}_r)_s|^2 + \tilde{w}_r^2 \right] \leq C \int_D \int_0^s \psi_r^2 \tilde{w}_r^2 \, ds \, dx = C.

By Lemma 4.4(4), we deduce

(6.3) \[
\int_D \int_0^s \left[ \epsilon^{-2/3} |\nabla_x \tilde{w}_r|^2 + |(\tilde{w}_r)_s|^2 + \tilde{w}_r^2 \right] \leq C,
\]

which implies our assertion (i). Passing to a subsequence, \( \tilde{w}_\epsilon(x, s) \) converges weakly in \( H^1_{loc}(D \times [0, \infty)) \) to some function \( \tilde{\eta} \) of \( x \). Moreover, as \( \int_D \int_0^s |\nabla_x \tilde{w}_r|^2 \, ds \, dx \leq C \epsilon^{2/3} \), it follows that \( \nabla_x \tilde{\eta} = 0 \) a.e..
We outline the rest of the proof of Proposition 6.1. First we will show (iii) except the normalization condition

\[(6.4) \quad \int_0^\infty \tilde{\eta}^2 \, ds = 1.\]

Second, we will show the estimate (ii). Finally we will use (ii) to derive (6.4) from (6.5).

We claim that \(\tilde{\eta}\) must satisfy the equation in (6.2). To see this claim, note that the equation for \(\tilde{w}_\varepsilon\) is

\[(6.5) \quad 0 = -\frac{\alpha}{c_\varepsilon^{2/3}} \nabla_x \cdot \left( \psi^2(x, \tilde{w}_\varepsilon) \right) + \left( \frac{\sigma_\varepsilon(\alpha) - \sigma_\varepsilon(s)}{c_\varepsilon^{2/3}} + \frac{\sigma_\varepsilon(\alpha)}{c_\varepsilon^{2/3}} - \varepsilon^{4/3} \frac{\psi_{x,\alpha\alpha}}{\psi_x} \right) \psi^2 \tilde{w}_\varepsilon.

We argue via the weak formulation.

**Claim 6.4.** There exists a constant \(\bar{a}_0\) such that for each test function \(z(s)\) that is compactly supported in \([0, \infty)\),

\[\int_0^\infty \left[-zs\tilde{\eta} + (\bar{a}_0 - \sigma_\varepsilon^2)z\tilde{\eta}\right] \, ds = 0.\]

In particular, \(\tilde{\eta}\) satisfies the equation \(\tilde{\eta}_{ss} + (\bar{a}_0 - \sigma_\varepsilon^2)\tilde{\eta} = 0\) on \((0, \infty)\) and \(\tilde{\eta}(0) = 0\).

Multiplying (6.5) by a test function \(z = z(s)\), and integrating over \(x \in D\), we see that the term involving derivatives in \(x\) vanishes (by the Neumann boundary condition \(\frac{\partial \tilde{w}_\varepsilon}{\partial n} = 0\)), and obtain

\[(6.6) \quad 0 = -z \int_D \psi^2(\tilde{w}_\varepsilon) \, dx + z \int_D \left[ \frac{\sigma_\varepsilon(\alpha) - \sigma_\varepsilon(s)}{c_\varepsilon^{2/3}} + \frac{\sigma_\varepsilon(\alpha)}{c_\varepsilon^{2/3}} - \varepsilon^{4/3} \frac{\psi_{x,\alpha\alpha}}{\psi_x} \right] \psi^2 \tilde{w}_\varepsilon \, dx.

Next, integrate the first term of (6.6) over \(s \in [0, s_e]\), we see that

\[
\begin{align*}
- \int_0^{s_e} z \int_D \psi^2(\tilde{w}_\varepsilon) \, dx \, ds & = \int_0^{s_e} \int_D z \psi^2(\tilde{w}_\varepsilon) \, dx \, ds - \left[ z \int_D \psi^2(\tilde{w}_\varepsilon) \, dx \right]_{s=0}^{s_e} \\
& = \int_0^{s_e} \int_D z \psi^2(\tilde{w}_\varepsilon) \, dx \, ds + \varepsilon^{2/3} \left[ z \int_D \psi_x \psi_{x,\alpha} \tilde{w}_\varepsilon \, dx \right]_{s=0}^{s_e},
\end{align*}
\]

where we have used the boundary condition \((\tilde{w}_\varepsilon)_s = -\varepsilon^{2/3} \psi_{x,\alpha} \tilde{w}_\varepsilon / \psi_x\) on \(D \times \{0, s_e\}\).

Since \(z(s)\) has compact support in \([0, \infty)\), the boundary term evaluated at \(s = s_e\) vanishes, and the remaining boundary term is of order \(O(\varepsilon^{2/3})\) (since \(\tilde{w}_\varepsilon\) is bounded in \(H^1(D \times (0, s_e))\) by assertion (i), and hence bounded in \(L^2(D \times \{0\})\) by the Trace Theorem). Hence, we have

\[(6.7) \quad - \int_0^{s_e} z \int_D \psi^2(\tilde{w}_\varepsilon) \, dx \, ds = \int_0^{s_e} \int_D z \psi^2(\tilde{w}_\varepsilon) \, dx \, ds + o(1).
\]

Also, in the support of \(z(s)\), \((\psi^2(x, \alpha + \varepsilon^{2/3} s)) \to (\psi^+)^2(x, \alpha)\) uniformly, so we may use (6.7) to integrate (6.6) over \(s \in [0, s_e]\) and pass to the limit to get

\[(6.8) \quad 0 = \left( \int_D (\psi^+)^2(x, \alpha) \, dx \right) \left[ \int_0^\infty z \tilde{\eta} \, ds + \int_0^\infty (\sigma_\varepsilon^2 s - \bar{a}_0) z \tilde{\eta} \, ds \right].
\]
where we have used Corollary 5.7 and that \(a_0\) is a subsequential limit of \(-\sigma_\epsilon(\alpha)/\epsilon^{2/3}\) (see also Lemma 4.3). This proves Claim 6.4. Next, we claim that

\[
(6.10) \quad \eta_{ss} + (\alpha_0 - \sigma_1^+) \eta = 0, \quad \eta \geq 0 \text{ on } [0, +\infty), \quad \text{and} \quad \eta_\epsilon(0) = 0.
\]

It remains to show that \(\eta(+\infty) = 0\), and that the subsequential limit \(\alpha_0\) must be determined by \(\tilde{a}_0\) of the proposition. By (6.10) and \(\eta > 0, \eta_{ss} \geq 0\) for all \(s\) sufficiently large. Hence \(\eta(+\infty)\) exists in \([0, +\infty]\). By (6.9), \(\eta(+\infty) = 0\).

Hence, \(\eta\) is a constant multiple of Airy\((\sigma_1^+)^{1/3}A_0\), where \(A_0\) is the absolute value of the first negative root of the derivative of the Airy function \(\text{Airy}(x)\). In particular the subsequential limit \(\tilde{a}_0\) given in (6.8) is uniquely determined by \(\tilde{a}_0 = (\sigma_1^+)^{2/3}A_0\) (i.e. the full limit \(\lim_{\epsilon \rightarrow 0} \sigma_\epsilon(\alpha)/\epsilon^{2/3}\) exists). This shows Claim 6.5. To finish the proof of (iii) except for (6.4), it remains to establish the following.

**Claim 6.6.** \(\tilde{w}_\epsilon(x, s) \rightarrow \tilde{\eta}(s) \text{ locally uniformly in } D \times [0, \infty).\) In particular, for each \(M > 0\), \(\|\tilde{w}_\epsilon\|_{L^\infty(D\times[0,M])} \text{ is bounded uniformly in } \epsilon.\)

It is enough to show that for each \(M > 0\),

\[
(6.11) \quad \sup_{s \in [0, M]} \frac{\sup_{x \in D} \tilde{w}_\epsilon(x, s)}{\inf_{x \in D} \tilde{w}_\epsilon(x, s)} \rightarrow 1 \quad \text{as } \epsilon \searrow 0.
\]

For, assuming (6.11), one can write

\[
(6.12) \quad \tilde{w}_\epsilon(x, s) = \tilde{w}_\epsilon(x_0, s)(1 + \delta_\epsilon(x, s)) \quad \text{for some } x_0 \in D,
\]

where \(\delta_\epsilon(x, s) \rightarrow 0 \text{ in } L^\infty_{\text{loc}}(D \times [0, +\infty)).\) Now, if we integrate (6.12) over \(x \in D\), then

\[
\tilde{W}_\epsilon(s) := \frac{1}{|D|} \int_D \tilde{w}_\epsilon(x, s) \, dx = \tilde{w}_\epsilon(x_0, s)(1 + \hat{\delta}_\epsilon(s)),
\]

where \(\hat{\delta}_\epsilon(s) \rightarrow 0 \text{ in } L^\infty_{\text{loc}}([0, \infty)).\) Since \(\tilde{w}_\epsilon\) is bounded in \(H^1(D \times (0, s_0))\), one can easily deduce that \(\tilde{W}_\epsilon(s) \in H^1_{\text{loc}}((0, +\infty)) \subset C^{1/2}_{\text{loc}}([0, +\infty)).\) Therefore, by Arzelà-Ascoli Theorem, \(\tilde{W}_\epsilon(s)\) and hence \(\tilde{w}_\epsilon(x_0, s)\) converges to \(\tilde{\eta}(s)\) in \(C^{1/2}_{\text{loc}}([0, \infty)).\) Finally, (6.12) implies that \(\tilde{w}_\epsilon(x, s) \rightarrow \tilde{\eta}(s)\) locally uniformly in \(D \times [0, +\infty).\)

It remains to show (6.11) in a similar fashion as in Claim 6.6. Assume to the contrary that there exists some constant \(c_0 > 1, \epsilon = \epsilon_k \rightarrow 0, s_k \rightarrow s_0 < +\infty,\) such that

\[
(6.13) \quad \sup_{x \in D} \tilde{w}_\epsilon(x, s_k) \geq c_0 \inf_{x \in D} \tilde{w}_\epsilon(x, s_k).
\]
Similarly as before, we extend \( \tilde{w}_\epsilon \) by reflection on \( D \times \{0\} \) so that \( \tilde{w}_\epsilon \) is defined on \( D \times (-s_\epsilon, s_\epsilon) \), and define

\[
W_k(x, \tau) := \frac{\tilde{w}_\epsilon(x, s) \epsilon^{1/3}/\sqrt{\alpha}}{\sup_{x' \in D} \tilde{w}_\epsilon(x', s)} \quad \text{in } D \times \left(- (s_\epsilon + s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}}, (s_\epsilon - s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}} \right).
\]

Recall that \( s_\epsilon \) is defined in (6.6). By the equation (6.5) satisfied by \( \tilde{w}_\epsilon \), \( W_k \) satisfies

\[
\begin{cases}
-\frac{d}{dt} \nabla \cdot (\psi^2 \nabla W_k) = - (\psi^2 W_{k, \tau}) + \frac{1}{\alpha} \left( \sigma_\epsilon \left( \alpha + \epsilon \sqrt{\frac{\alpha}{\epsilon^{1/3}}} s_\epsilon + \epsilon \frac{\tau}{\sqrt{\alpha}} \right) - \epsilon^2 \psi_{\epsilon, \alpha} \psi \right) \psi^2 W_k = 0, \\
W_{k, \tau} = - \frac{\epsilon}{\sqrt{\alpha}} \psi_{\epsilon, \alpha} W_k 
\end{cases}
\]

in \( D \times \left(- (s_\epsilon + s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}}, (s_\epsilon - s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}} \right) \), where

\[
\alpha = \alpha(\tau) = \alpha + \left| \epsilon \sqrt{\frac{\alpha}{\epsilon^{1/3}}} s_\epsilon + \epsilon \frac{\tau}{\sqrt{\alpha}} \right|
\]

and the boundary conditions

\[
\begin{cases}
\frac{\partial}{\partial n} W_k = 0 \quad \text{on } \partial D \times \left(- (s_\epsilon + s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}}, (s_\epsilon - s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}} \right) \\
W_{k, \tau} = - \frac{\epsilon}{\sqrt{\alpha}} \psi_{\epsilon, \alpha} W_k \quad \text{on } D \times \left(- (s_\epsilon + s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}}, (s_\epsilon - s) \sqrt{\frac{\alpha}{\epsilon^{1/3}}} \right).
\end{cases}
\]

Since \( s_\epsilon \to +\infty \) as \( \epsilon \to 0 \) and that \( s_k \) remains bounded, we see in particular that the domain of \( W_k \) tends to \( D \times \mathbb{R} \) as \( k \to \infty \).

**Claim 6.7.** For each \( M > 0 \), \( \sigma_\epsilon \left( \alpha + s_\epsilon \epsilon \sqrt{\frac{\alpha}{\epsilon^{1/3}}} + \epsilon \frac{\tau}{\sqrt{\alpha}} \right) \to 0 \) as \( \epsilon \to 0 \), uniformly for \( \tau \in [-M, M] \).

By Lemma 4.1(i), \( \sigma_\epsilon \) are bounded in \( C^1([\underline{\alpha}, \overline{\alpha}]) \) uniformly in \( \epsilon \). Hence we may write

\[
\left| \sigma_\epsilon \left( \alpha + s_\epsilon \epsilon \sqrt{\frac{\alpha}{\epsilon^{1/3}}} + \epsilon \frac{\tau}{\sqrt{\alpha}} \right) \right| \leq |\sigma_\epsilon(\alpha)| + C \left| \epsilon \sqrt{\frac{\alpha}{\epsilon^{1/3}}} s_\epsilon + \epsilon \frac{\tau}{\sqrt{\alpha}} \right|
\]

and conclude that \( \sigma_\epsilon \left( \alpha + s_\epsilon \epsilon \sqrt{\frac{\alpha}{\epsilon^{1/3}}} + \epsilon \frac{\tau}{\sqrt{\alpha}} \right) \) goes to zero by Lemma 4.3 and boundedness of \( s_k, \tau \). This proves Claim 6.7.

Since the coefficients of (6.14) are bounded in \( L^p_{\text{loc}}(\overline{D} \times \mathbb{R}) \) uniformly in \( k \), Harnack inequality, and the normalization condition \( \sup_{x \in D} W_k(x, 0) = 1 \) implies that \( W_k \) are bounded in \( L^p_{\text{loc}}(\overline{D} \times \mathbb{R}) \) uniformly in \( k \). Hence we may apply elliptic \( L^p \) estimates similarly as in Claim 3.6 to conclude that a subsequence of \( W_k \) converges in \( L^p_{\text{loc}}(\overline{D} \times \mathbb{R}) \) to a positive solution of \( (\psi_0(x, \alpha))^{-2} \nabla_x \cdot (\psi_0^2(x, \alpha) \nabla_x W) + W_{\tau} = 0 \) in \( D \times \mathbb{R} \). (Here we used Claim 6.7.) Now, we apply the following Liouville Theorem, whose proof is exactly analogous to Proposition C.1 and is skipped.

**Proposition 6.8.** Let \( \psi(x) \) be a smooth positive function defined in \( \tilde{D} \), then every positive solution \( W \) to \( \psi^{-2} \nabla_x \cdot (\psi^2 \nabla_x W) + W_{\tau} = 0 \) in \( D \times \mathbb{R} \), subject to Neumann boundary condition on \( \partial D \times \mathbb{R} \), is necessarily a constant.

So that by normalization \( \sup_{x \in D} W_k(x, 0) = 1 \), \( W_k(x, 0) \to 1 \) uniformly in \( D \). This contradicts (6.13) and proves (6.11). This establishes Claim 6.6. Claims 6.6 and 6.7 establish assertion (iii) except for condition (6.4).

Next, we proceed to show the estimate in (ii). By the preceding argument in the proof of Lemma 5.1, specifically the construction of supersolution \( \overline{W} \) in Claim 5.4,
we can show that for all $\beta > 0$, there exists $s_0 > 0$ such that
\[
\tilde{w}(x, s) \leq \left( \sup_{\alpha \in D} \tilde{u}_\epsilon \right) \left[ \exp(-\beta(s - s_0)) + \exp(\beta(s - (3/2)s_\epsilon)) \right]
\]
for $x \in D$ and $s \in [s_0, s_\epsilon)$. Then (ii) follows from Claim 6.6 as the expression inside paranthesis is bounded uniformly in $\epsilon$. We do not repeat the details.

For (iii), it remains to show (7.4). By assertion (ii), and that
\[
\psi_\epsilon(x, \alpha + \epsilon^{2/3}s) \to \psi^*(x, \alpha) \quad \text{and} \quad \tilde{w}_\epsilon(x, s) \to \tilde{\eta}(s)
\]
in $L^\infty_{loc}(D \times [0, \infty))$ (by Lemma 4.1(iv) and Claim 6.6 resp.), we may pass to the limit in (6.1) to obtain
\[
\int_D \theta^2_\alpha \, dx = \int_D \int_0^{s_\epsilon} \psi^2_\epsilon(x, \alpha + \epsilon^{2/3}s) \tilde{w}^2_\epsilon(x, s) \, ds \, dx \to \int_D (\psi^*)^2(x, \alpha) \, dx \int_0^\infty \tilde{\eta}^2 \, ds
\]
Upon noting that (see Definition 2.2(ii))
\[
\psi^*(x, \alpha) = \theta_\alpha(x) \quad \text{in} \quad D,
\]
the proof is completed. \hfill \Box

7. PROOF OF THEOREM 2.3

Proof of Theorem 2.3. First, we note that by Proposition 5.6,
\[
e^{2/3} \int_D \int_0^{s_\epsilon} \psi_\epsilon w_\epsilon \, ds \, dx = \int_D \int_0^{s_\epsilon} u_\epsilon(x, \alpha) \, d\alpha \, dx = \int_D \tilde{u}_\epsilon \, dx \to \int_D \theta_\alpha \, dx
\]
as $\epsilon \to 0$. Furthermore, by (6.15), (6.16) and the estimate of Proposition 6.1(ii),
\[
\int_D \int_0^{s_\epsilon} \psi_\epsilon \tilde{w}_\epsilon \, ds \, dx \to \int_D \psi^*(x, \alpha) \, dx \int_0^\infty \tilde{\eta}(s) \, ds = \int_D \theta_\alpha(x) \, dx \int_0^\infty \tilde{\eta}(s) \, ds.
\]
By the definition of $w_\epsilon$ and $\tilde{w}_\epsilon$, there is a function $\Gamma(\epsilon)$ such that
\[
w_\epsilon(x, s) = \Gamma(\epsilon) \tilde{w}_\epsilon(x, s).
\]
By (7.1) and (7.2), we have
\[
\lim_{\epsilon \to 0} e^{2/3} \Gamma(\epsilon) = \left( \int_0^\infty \tilde{\eta} \, ds \right)^{-1}.
\]
Hence, by 7.3 and Corollary 6.2,
\[
\left\| e^{2/3} w_\epsilon(x, (\alpha - \alpha)/\epsilon^{2/3}) - \left( \int_0^\infty \tilde{\eta} \, ds \right)^{-1} \tilde{\eta} \left( \frac{\alpha - \alpha}{\epsilon^{2/3}} \right) \right\|_{L^\infty(\Omega)} \to 0.
\]
By the fact that $\left( \int_0^\infty \tilde{\eta} \, ds \right)^{-1} \tilde{\eta}(s) = \eta^*(s)$ where $\eta^*$ is given in Definition 2.2(iii), we also have
\[
\left\| e^{2/3} w_\epsilon(x, (\alpha - \alpha)/\epsilon^{2/3}) - \eta^* \left( \frac{\alpha - \alpha}{\epsilon^{2/3}} \right) \right\|_{L^\infty(\Omega)} \to 0.
\]
Using Lemma 4.1(iv), we have
\[
\left\| e^{2/3} u_\epsilon(x, \alpha) - \psi_\epsilon(x, \alpha) \eta^* \left( \frac{\alpha - \alpha}{\epsilon^{2/3}} \right) \right\|_{L^\infty(\Omega)} \to 0.
\]
By the fact that $\eta^*(s) \leq Ce^{-\beta s}$ for some $C, \beta > 0$, (6.15) and (6.16), we have
\[(7.6) \quad \left\| \psi(\alpha, \alpha_\sigma) - \theta_\sigma(x) \eta^*(\frac{\alpha - \sigma}{\epsilon^{2/3}}) \right\|_{L^\infty(\Omega)} \to 0.
\]
And (2.7) follows from (7.5) and (7.6).

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Appendix A. Existence Results

In this section we show the existence of positive solution to (2.1). For this purpose, we fix positive parameters $\epsilon$ and $\tau > 0$, and denote (in this section only) the principal eigenvalue and eigenfunction of the following problem by $\mu_1$ and $\phi_1$.

\[(A.1) \quad \begin{cases} \alpha \Delta \phi + \epsilon^2 \phi + m(x)\phi + \mu \phi = 0 & \text{in } \Omega := D \times (\alpha, \overline{\alpha}), \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \partial D \times (\alpha, \overline{\alpha}), \\ \phi_\sigma = 0 & \text{on } D \times \{\alpha, \overline{\alpha}\}.
\end{cases}
\]

**Theorem A.1.** If $\mu_1 \geq 0$, then the equation (2.1) has no positive steady-state. If $\mu_1 < 0$, then the equation (2.1) has at least one positive steady-state.

**Proof.** First, we prove the non-existence result. Suppose $\mu_1 \geq 0$ and let $u$ be a non-negative solution of (2.1). Multiply (2.1) by the principal eigenfunction $\phi_1$ of (A.1), and integrate by parts, we obtain
\[\mu_1 \int_D \int_\alpha^{\overline{\alpha}} \phi_1^2 \, d\alpha \, dx + \int_D \int_\alpha^{\overline{\alpha}} u\phi_1 \, d\alpha \, dx = 0.
\]
Since $\mu_1 \geq 0$, both terms are non-negative, and both must be identically zero, i.e. $u \equiv 0$.

For the existence result, we consider, for $\tau \in [0, 1]$, the positive solutions of
\[(A.2) \quad \begin{cases} \alpha \Delta u + \epsilon^2 (u)_{\sigma\sigma} + (m(x) - \tau \hat{u} - (1 - \tau)u)u = 0 & \text{in } D \times (\alpha, \overline{\alpha}), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (\alpha, \overline{\alpha}), \\ (u)_{\sigma} = 0 & \text{in } D \times \{\alpha, \overline{\alpha}\}.
\end{cases}
\]
Here we recall that $\hat{u} = \int_\alpha^{\overline{\alpha}} u(x, \alpha) \, d\alpha$. It remains to show the following claim, from which existence of positive solution to (2.1) follows by a standard topological degree argument, as the existence of a unique, linearly stable positive solution to (A.2) when $\tau = 0$ is standard.

**Claim A.2.** For some $\delta > 0$ independent of $\tau \in [0, 1]$, any positive solution $u$ of (A.2) satisfies
\[\delta < \|u\|_{L^1(\Omega)} < 1/\delta.
\]
For the upper bound, one can integrate (A.2) over \( \Omega \) to get
\[
\int_D \hat{u} m \, dx = \tau \int_D \hat{u}^2 \, dx + (1 - \tau) \int_D \int_\alpha u^2 \, d\alpha \, dx
\]
\[
\geq \left( \tau + \frac{1 - \tau}{\alpha - \alpha} \right) \int_D \hat{u}^2 \, dx
\]
\[
\geq c_0 \int_D \hat{u}^2 \, dx
\]
\[
\geq \frac{c_0}{|D|} \left( \int_D \hat{u} \, d\alpha \right)^2 = \frac{c_0}{|D|} \| u \|^2_{L^1(\Omega)},
\]
from which the upper bound follows.

For the lower bound, let \( u = v \phi_1 \), where \( \phi_1 > 0 \) is the principal eigenfunction corresponding to the principal eigenvalue \( \mu_1 < 0 \) of (A.1). Moreover, if we normalize \( \phi_1 \) by \( \int_\Omega \phi_1^2 = 1 \), then \( \sup_\Omega \phi_1 \) and \( \inf_\Omega \phi_1 \) are fixed positive constants independent of \( \tau \), as (A.1) is independent of \( \tau \). The equation for \( v \) can be written as
\[
\begin{cases}
\alpha \nabla_x \cdot (\phi_1^2 \nabla_x v) + \epsilon^2 (\phi_1^2 v_\alpha) + \phi_1^2 v (\mu_1 - \tau \hat{u} - (1 - \tau) u) = 0 & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial D \times (\alpha, \alpha), \\
v_\alpha = 0 & \text{on } D \times \{\alpha, \alpha\}.
\end{cases}
\]
Hence, if we divide by \( v \) and integrate by parts, we have
\[
\int_\Omega \phi_1^2 (\mu_1 + \tau \hat{u} + (1 - \tau) u) \, d\alpha \, dx = \int_\Omega \phi_1^2 \frac{\alpha^2 |\nabla_x v|^2 + \epsilon^2 |v_\alpha|^2}{v^2} \, d\alpha \, dx > 0.
\]
Hence we have
\[
\left( \sup_\Omega \phi_1 \right)^2 \left[ \tau (\alpha - \alpha) + (1 - \tau) \right] \| u \|^2_{L^1(\Omega)} = -\mu_1 \int_\Omega \phi_1^2 \, d\alpha \, dx = -\mu_1 > 0.
\]
Since \( \mu_1 \) and \( \sup_\Omega \phi_1 \) are independent of \( \tau \), we have
\[
\| u \|^2_{L^1(\Omega)} \geq \frac{-\mu_1}{(\sup_\Omega \phi_1)^2 \left[ \tau (\alpha - \alpha) + (1 - \tau) \right]}.
\]
Since the latter term is bounded from below uniformly in \( \tau \in [0, 1] \), the claim is proved.

**Corollary A.3.** If \( \int_D m(x) \, dx > 0 \), then for any \( \epsilon > 0 \), (2.1) has at least one positive solution.

**Proof.** Divide the equation (A.1) by the principal eigenfunction \( \phi_1 \) and integrate by parts over \( \Omega \), we get
\[
\int_D \int_\Omega \frac{\alpha |\nabla_x \phi_1|^2 + \epsilon^2 |\phi_1,\alpha|^2}{\phi_1^2} \, d\alpha \, dx + \int_D \int_\Omega (m(x) + \mu_1) \, d\alpha \, dx = 0.
\]
Hence for all \( \epsilon > 0 \),
\[
\mu_1 \leq -\frac{1}{|D|} \int_D m(x) \, dx < 0,
\]
and the existence of positive solution of (2.1) follows from Theorem A.1. \( \square \)
APPENDIX B. EIGENVALUE PROBLEMS WITH DIFFUSION PARAMETER $\alpha$ AND WEIGHT FUNCTION $h(x)$

For each $\alpha > 0$ and $h \in L^\infty(D)$, let $\lambda_1 = \lambda_1(\alpha, h) \in \mathbb{R}$ and $\varphi(x) = \varphi_1(x; \alpha, h)$ be the normalized principal eigenvalue and principal eigenfunction of the following problem.

\begin{equation}
\begin{aligned}
-\alpha \Delta_x \varphi + h \varphi &= \lambda \varphi & \text{in } D, \\
\frac{\partial \varphi}{\partial n} &= 0 & \text{on } \partial D, \\
\int_D \varphi^2 \, dx &= 1.
\end{aligned}
\end{equation}

We shall state and prove a number of properties of $\lambda_1$ and $\varphi_1$, and its dependence on the parameters $\alpha$ and $h$, some of which is folklore among specialists.

**Proposition B.1.**

(i) For each $\alpha > 0$ and $h \in L^\infty(D)$, the problem \[(B.1)\] has a principal eigenvalue $\lambda_1$ which is simple, and the corresponding eigenfunction $\varphi_1$ can be chosen positive and uniquely determined by the constraint \[
\int_D \varphi_1^2 \, dx = 1.
\]

(ii) For each $p > 1$, the mapping $(\alpha, h) \mapsto (\lambda_1, \varphi_1(\cdot))$ is smooth from $\mathbb{R}_+ \times L^\infty(D)$ to $\mathbb{R} \times W^{2,p}_N(D)$, where $W^{2,p}_N(D) = \{ \phi \in W^{2,p}(D) : \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D \}$.

(iii) If $h \in L^\infty(D)$ is non-constant, then $\frac{\partial \lambda_1}{\partial \alpha}(\alpha, h) > 0$ for all $\alpha > 0$.

**Proof.** Part (i) is well-known. See, e.g. [14, Section 8.12]. In particular, the principal eigenvalue is given by the variational characterization

\begin{equation}
\lambda_1(\alpha, h) = \inf_{\varphi \in C^1(D) \setminus \{0\}} \frac{\int_D (\alpha|\nabla_x \varphi|^2 + h \varphi^2) \, dx}{\int_D \varphi^2 \, dx}.
\end{equation}

Fix $p > N$ ($N$ being the dimension of $D$). Consider the following mapping $F : W^{2,p}_N(D) \times \mathbb{R} \times \mathbb{R}_+ \times L^\infty(D) \to L^p(D) \times \mathbb{R}$, given by

\begin{equation}
F(\varphi, \lambda, \alpha, h) = (\alpha \Delta_x \varphi - h \varphi + \lambda \varphi, \int_D \varphi^2 \, dx - 1),
\end{equation}

Then for each $\alpha > 0$ and $h \in L^\infty(D)$, the principal eigenpair $(\varphi_1(\cdot; \alpha, h), \lambda_1(\alpha, h))$ of \[(B.1)\] satisfies

\begin{equation}
F(\varphi_1(\cdot; \alpha, h), \lambda_1(\alpha, h), \alpha, h) = (0, 0).
\end{equation}

Assertion (ii) follows from the following claim, in view of the Implicit Function Theorem and the smooth dependence of the operator $F$ on $\alpha$ and $h$.

**Claim B.2.** For each fixed $\alpha > 0$ and $h \in L^\infty(D)$,

\begin{equation}
D_{(\varphi, \lambda)} F(\varphi_1(\cdot; \alpha, h), \lambda_1(\alpha, h), \alpha, h) : W^{2,p}_N \times \mathbb{R} \to L^p(D) \times \mathbb{R}
\end{equation}

is a bijection.

We shall follow the arguments in the proof of [10, Lemma 2.1]. Suppose for some $(\Phi, t) \in W^{2,p}_N \times \mathbb{R}$, $D_{(\varphi, \lambda)} F(\varphi_1(\cdot; \alpha, h), \lambda_1(\alpha, h), \alpha, h)[\Phi, t] = (0, 0)$, i.e.

\begin{equation}
\alpha \Delta_x \Phi - h \Phi + \lambda_1 \Phi + t \varphi_1 = 0 & \text{in } D, \quad \frac{\partial \Phi}{\partial n} = 0 & \text{on } \partial D,
\end{equation}

and

\begin{equation}
2 \int_D \Phi \varphi_1 \, dx = 0,
\end{equation}

where $\lambda_1 = \lambda(\alpha, h)$ and $\varphi_1 = \varphi(x; \alpha, h)$. The Fredholm alternative implies that $\int_D t \varphi_1^2 \, dx = 0$, i.e. $t = 0$. Hence $\Phi = c \varphi_1$ for some constant $c$ (as $\lambda_1 = \lambda_1(\alpha, h)$
is a simple eigenvalue). But then (B.3) implies that $c = 0$. This shows that the kernel of $D(\varphi, \lambda)F(\varphi(\cdot; \alpha, h), \lambda_1(\alpha, h), \alpha, h)$ is trivial. Now let $(f, q) \in L^p(D) \times \mathbb{R}$ be given, we need to solve for $(\Phi, t)$ in

(B.6) \[ \alpha \Delta_x \Phi - h \Phi + \lambda_1 \Phi + t \varphi_1 = f \quad \text{in } D, \quad \text{and } \quad \frac{\partial \Phi}{\partial n} = 0 \quad \text{on } \partial D, \]

and

(B.7) \[ 2 \int_D \Phi \varphi_1 \, dx = q. \]

Set $t = (\int_D f \varphi_1 \, dx) / (\int_D \varphi_1^2 \, dx)$, then $\int_D (f - t \varphi_1) \varphi_1 \, dx = 0$, so (B.6) has solution of the form $\Phi = s \varphi_1 + \Phi^*$ where $\Phi^* \in W_1^{2,p}(D)$ is unique and satisfies $\int_D \Phi^* \varphi_1 \, dx = 0$. Finally, if we set $s = q / (2 \int_D \varphi_1^2 \, dx)$ then $(\Phi, t)$ solves (B.6) and (B.7). This proves Claim [B.2] which implies assertion (ii).

For (iii), we differentiate (B.1) with respect to $\alpha$,

(B.8) \[ \begin{cases} \frac{-\alpha \Delta_x \varphi_1}{\partial \alpha} + \frac{h \frac{\partial \varphi_1}{\partial \alpha}}{\partial \alpha} - \lambda_1 \frac{\partial \psi_1}{\partial \alpha} = \Delta_x \varphi_1 + \frac{\partial \lambda_1}{\partial \alpha} \varphi_1 & \text{in } D, \\ \frac{\partial \Phi}{\partial n} = 0 & \text{on } \partial D, \quad \text{and } \quad \int_D \frac{\partial \psi_1}{\partial \alpha} \varphi_1 \, dx = 0. \end{cases} \]

Multiply (B.8) by $\varphi_1$ and integrate by parts, we have $\frac{\partial \lambda_1}{\partial \alpha} \int_D \varphi_1^2 \, dx = \int_D |\nabla_x \varphi_1|^2 \, dx$. Since $h(x)$ is non-constant in $x$, $\varphi_1 = \varphi_1(\cdot; \alpha, h)$ is non-constant in $x$ and this implies that $\frac{\partial \lambda_1}{\partial \alpha} > 0$. This proves (iii).

First, we show that $\lambda_1$ and $\varphi_1$ are continuous with respect to the weak topology of $\mathbb{R}_+ \times \cap_{p>1} L^p(D)$.

**Lemma B.3.** Let $\lambda_1(\alpha, h)$ and $\varphi_1(\cdot; \alpha, h)$ be the principal eigenpair of (B.1),

(i) For each $p > 1$, there exists $C_0' = C_0'(p, M, \Omega, \overline{\Omega}, \partial D)$ such that

[\lambda_1(\alpha, h)] + \|\varphi_1(\cdot; \alpha, h)\|_{W_1^{2,p}(D)} \leq C_0'.

provided $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $\|h\|_{L_\infty(D)} \leq M$.

(ii) If $\alpha_j \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$, $\sup_{j \geq 0} \|h_j\|_{L_\infty(D)} < +\infty$ and $h_j \to h_0$ in $L^p(D)$ for all $p > 1$, then as $j \to \infty$, $\lambda_1(\alpha_j, h_j) \to \lambda_1(\alpha_0, h_0)$ and $\varphi_1(\cdot; \alpha_j, h_j) \to \varphi_1(\cdot; \alpha_0, h_0)$ weakly in $W_1^{2,p}(D)$ for all $p > 1$.

**Proof.** By [B.2], $\lambda_1 := \lambda_1(\alpha, h)$ forms a bounded sequence in $[-\|h\|_{L_\infty(D)}, \|h\|_{L_\infty(D)}]$. The $L^p$ estimate (for $p > N$) applied to (B.1) and interpolation inequality together imply

\[ \|\varphi_1\|_{W_1^{2,p}(D)} \leq C \|\varphi_1\|_{L^p(D)} \leq \frac{1}{2} \|\varphi_1\|_{W_1^{2,p}(D)} + C \|\psi_1\|_{L_2(D)}, \]

where $C$ is a generic constant, depending on $\|h\|_{L_\infty(D)}, \Omega, \overline{\Omega}$ and the domain $D$ that changes from line to line. This proves (i).

For (ii), let $\alpha_j \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$ and $h_j$ be a uniformly bounded sequence in $L_\infty(D)$ and $h_j \to h_0$ weakly in $L^p(D)$. Denote $\lambda_{1,j} = \lambda_1(\alpha_j, h_j)$ and $\varphi_{1,j} = \varphi_1(\cdot; \alpha_j, h_j)$. By assertion (i), there are subsequences $\lambda_{1,j'}$ and $\varphi_{1,j'}$ such that $\lambda_{1,j} \to \lambda$ and $\varphi_{1,j} \to \tilde{\varphi}$ weakly in $W_1^{2,p}(D)$, for some $\lambda \in \mathbb{R}$ and $\tilde{\varphi} \in W_1^{2,p}(D)$. Take $\alpha = \alpha_j'$, $h = h_j$ in (B.1), and pass to the weak limit $j' \to \infty$, we deduce

\[ \begin{cases} -\alpha_0 \Delta_x \tilde{\varphi} + h_0 \tilde{\varphi} = \lambda \tilde{\varphi} & \text{in } D, \\ \frac{\partial \tilde{\varphi}}{\partial n} = 0 & \text{on } \partial D, \quad \int_D \tilde{\varphi}^2 \, dx = 1. \end{cases} \]

Hence $(\tilde{\varphi}, \lambda)$ is an eigenpair of (B.1) when $\alpha = \alpha_0$, $h = h_0$ and such that $\tilde{\varphi} \geq 0$. Moreover, $\tilde{\varphi}$ is non-trivial, as $\int_D \tilde{\varphi}^2 \, dx = 1$. By uniqueness of principal eigenpair,
it follows that \( \tilde{\lambda} = \lambda_1(\alpha_0, h_0) \) and \( \tilde{\varphi} = \varphi_1(\cdot; \alpha_0, h_0) \). Since the limit is independent of subsequence, we deduce that the full sequence \( \lambda_1(\alpha_j, h_j) \to \lambda(\alpha_0, h_0) \) and \( \varphi_1(\cdot; \alpha_j, h_j) \to \varphi_1(\cdot; \alpha_0, h_0) \) weakly in \( W^{2,p}(D) \). This proves the assertion (ii). \( \square \)

Next, we show the following uniform estimate of \((D_{(\varphi, \lambda)})^{-1}\).

**Lemma B.4.** There exists \( C_2 = C_2(M, \underline{\alpha}, \underline{\alpha}, D) \) such that for any \( \alpha \in [\underline{\alpha}, \overline{\alpha}] \) and \( \|h\|_{L^\infty(D)} \leq M \), if

\[
D_{(\varphi, \lambda)} F(\varphi_1(\cdot; \alpha, h), \lambda_1(\alpha, h), \alpha, h)\Phi, t] = (f, q),
\]

i.e. (B.6) and (B.7) hold with \( \lambda_1 = \lambda(\alpha, h) \) and \( \varphi_1 = \varphi_1(x; \alpha, h) \), then

\[
|t| + \|\Phi\|_{W^{2,p}(D)} \leq C_2(|q| + \|f\|_{L^p(D)}).
\]

**Proof.** Let \( M > 0 \) be given. Suppose to the contrary that there are \( \alpha_j \in [\underline{\alpha}, \overline{\alpha}] \), \( h_j \), \( \Phi_j \), \( t_j \), \( q_j \), \( f_j \) such that

\[
\sup_j \|h_j\|_{L^\infty(D)} \leq M, \quad |t_j| + \|\Phi_j\|_{W^{2,p}(D)} \to \infty, \quad |q_j| + \|f_j\|_{L^p(D)} \leq 1.
\]

Without loss of generality, we may assume \( \alpha_j \to \alpha_0 \in [\underline{\alpha}, \overline{\alpha}] \) and for some \( h_0 \in L^\infty(D) \), \( h_j \to h_0 \) weakly in \( L^p(D) \). Denote

\[
\lambda_{1,j} = \lambda_1(\alpha_j, h_j), \quad \text{and} \quad \varphi_{1,j} = \varphi_1(\cdot; \alpha_j, h_j) \quad \text{for} \quad j \in \mathbb{N} \cup \{0\}.
\]

The above arguments ensure that

\[
\Phi_j = \Phi_j^+ + q_j/(2 \int_D \varphi_{1,j}^2 \, dx)\varphi_{1,j}, \quad \text{and} \quad t_j = \int_D f_j \varphi_{1,j} \, dx / \int_D \varphi_{1,j}^2 \, dx
\]

where \( \Phi_j^+ \) is the unique solution of (B.6) subject to the constraint \( \int_D \Phi_j^+ \varphi_{1,j} \, dx = 0 \).

By the normalization \( \int_D \varphi_{1,j}^2 \, dx = 1 \), we have

\[
\Phi_j = \Phi_j^+ + \frac{q_j}{2} \varphi_{1,j} \quad \text{and} \quad t_j = \int_D f_j \varphi_{1,j} \, dx.
\]

Since we have shown that \( |\lambda_{1,j}| \) and \( \|\varphi_{1,j}\|_{W^{2,p}(D)} \) remain bounded uniformly in \( j \), (B.10) and (B.11) imply that \( \|\Phi_j^+\|_{W^{2,p}(D)} \to \infty \). Apply \( L^p \) estimate to the equation of \( \Phi_j^+ \), which is

\[
\begin{cases}
\alpha_j \Delta_x \Phi_j^+ - h_j \Phi_j^+ + \lambda_{1,j} \Phi_j^+ = f_j - (\int_D f_j \varphi_{1,j} \, dx)\varphi_{1,j} & \text{in } D, \\
\frac{\partial \Phi_j^+}{\partial n} = 0 & \text{on } \partial D, \quad \text{and} \quad \int_D \Phi_j^+ \varphi_{1,j} \, dx = 0.
\end{cases}
\]

Using the boundedness of \( \varphi_{1,j} \) in \( W^{2,p}(D) \) and hence in \( L^\infty(D) \), we have

\[
\|\Phi_j^+\|_{W^{2,p}(D)} \leq C \left[ \|\Phi_j^+\|_{L^p(D)} + \|f_j - (\int_D f_j \varphi_{1,j} \, dx)\varphi_{1,j}\|_{L^p(D)} \right]
\]

\[
\leq C(\|\Phi_j^+\|_{L^\infty(D)} + \|f_j\|_{L^p(D)}).
\]

Hence we must have \( \|\Phi_j^+\|_{L^\infty(D)} \to \infty \) as \( j \to \infty \). Define \( \tilde{\Phi}_j := \Phi_j^+ / \|\Phi_j^+\|_{L^\infty(D)} \), then \( \tilde{\Phi}_j \) satisfies

\[
\begin{cases}
\alpha_j \Delta_x \tilde{\Phi}_j - h_j \tilde{\Phi}_j + \lambda_{1,j} \tilde{\Phi}_j = \tilde{f}_j & \text{in } D, \\
\frac{\partial \tilde{\Phi}_j}{\partial n} = 0 & \text{on } \partial D, \quad \int_D \tilde{\Phi}_j \varphi_{1,j} \, dx = 0, \quad \text{and} \quad \sup \tilde{\Phi}_j = 1,
\end{cases}
\]

where \( \tilde{f}_j = |f_j - (\int_D f_j \varphi_{1,j} \, dx)\varphi_{1,j}| / \|\Phi_j^+\|_{L^\infty(D)} \) converges to zero in \( L^p(D) \) as \( j \to \infty \). By \( L^p \) estimates, \( \tilde{\Phi}_j \) is bounded uniformly in \( W^{2,p}(D) \). Hence, there is a subsequence \( \tilde{\Phi}_{j'} \) that converges, weakly in \( W^{2,p}(D) \) and strongly in \( C^1(D) \), to
some function $\Phi_0$. By normalization $\sup_D |\Phi_0| = \lim_{j \to \infty} \left( \sup_D \Phi_0 \right) = 1$. Moreover, $\Phi_0$ satisfies (using Lemma [B.3] ii))

$$\begin{align*}
\alpha_0 \Delta_x \Phi_0 - h_0 \Phi_0 + \lambda_1(\alpha_0, h_0) \Phi_0 &= 0 \quad \text{in } D, \\
\frac{\partial \Phi_0}{\partial n} &= 0 \quad \text{on } \partial D, \quad \text{and } \int_D \Phi_0 \varphi_1 \, dx = 0.
\end{align*}$$

Since $\Phi_0$ is non-negative, Proposition [B.1] i) implies $\Phi_0 = c \varphi_1(\cdot; \alpha_0, h_0) = c \varphi_{1,0}$ but the integral constraint implies that $c = 0$. i.e. $\Phi_0 = 0$. This is a contradiction to $\sup_D \Phi_0 = 1$. This proves [B.9].

**Proposition B.5.** Let $\lambda_1(\alpha, h)$ and $\varphi_1(\cdot; \alpha, h)$ be the principal eigenpair of [B.1].

(i) For each $k$, there exists $C_k' = C_k'(M, \alpha, \tau, D)$ such that

$$\sum_{j=0}^{k} \left| \frac{\partial^j}{\partial \alpha^j} \lambda_1(\alpha, h) \right| + \left\| \frac{\partial^j}{\partial \alpha^j} \varphi_1(\cdot; \alpha, h) \right\|_{W^{2,p}(D)} \leq C_k' \quad \text{provided } \alpha \in [\alpha, \overline{\alpha}] \text{ and } \| h \|_{L^\infty(D)} \leq M.$$

(ii) If $\sup_{j \geq 0} \| h \|_{L^\infty(D)} < +\infty$ and $h_j \to h_0$ in $L^p(D)$ for all $p > 1$, then for each $k \geq 0$,

$$\frac{\partial^k}{\partial \alpha^k} \lambda_1(\cdot, h_j) \to \frac{\partial^k}{\partial \alpha^k} \lambda_1(\cdot, h_0) \quad \text{in } C_{loc}([0, \infty)) \quad \text{as } j \to \infty.$$

Moreover, given $k \geq 0$, $p > 1$, and sequence $\alpha_j \to \alpha_0 > 0$,

$$\frac{\partial^k}{\partial \alpha^k} \varphi_1(\cdot; \alpha_j, h_j) \to \frac{\partial^k}{\partial \alpha^k} \varphi_1(\cdot; \alpha_0, h_0) \quad \text{weakly in } W^{2,p}(D) \quad \text{as } j \to \infty.$$

**Proof.** Assertions (i) and (ii) for the case $k = 0$ are exactly Lemma [B.3]. We first prove assertion (i) for $k = 1$, differentiate the relation [B.3] with respect to $\alpha$,

$$D_{(\varphi, \lambda)} F \left[ \frac{\partial}{\partial \alpha} \varphi_1(\cdot; \alpha, h), \frac{\partial}{\partial \alpha} \lambda_1(\alpha, h) \right] = -D_\alpha F,$$

where the partial derivatives of $F$ are evaluated at $(\varphi_1(\cdot; \alpha, h), \lambda_1(\alpha, h), \alpha, h)$. By [B.2], we may write

$$\lambda_1' = (D_{(\varphi, \lambda)} F)^{-1} [-D_\alpha F] = (D_{(\varphi, \lambda)} F)^{-1} \left( -\Delta_x \varphi_1, 0 \right)$$

and deduce by Lemma [B.3] that

$$|\lambda_1'| + \| \varphi_1' \|_{W^{2,p}(D)} \leq C_2 \| \Delta_x \varphi_1 \|_{L^p(D)} \leq C_2 \| \varphi_1 \|_{W^{2,p}(D)} \leq C.$$

i.e. assertion (i) holds for $k = 1$. We argue inductively for $k > 1$. Suppose (i) holds for $k = K - 1$. We can write

$$D_{(\varphi, \lambda)} F \left[ \frac{\partial^K}{\partial \alpha^K} \varphi_1(\cdot; \alpha, h), \frac{\partial^K}{\partial \alpha^K} \lambda_1(\alpha, h) \right] = \mathcal{F}_K(\alpha, h)$$

where

$$\mathcal{F}_K(\alpha, h) := \left( -K \frac{\partial^{K-1}}{\partial \alpha^{K-1}} (-\Delta_x \varphi_1) - \sum_{k=1}^{K-1} \binom{K}{k} \frac{\partial^k}{\partial \alpha^k} \lambda_1 \frac{\partial^{K-k}}{\partial \alpha^{K-k}} \varphi_1, 0 \right).$$

By the form of $\mathcal{F}_K$, we can deduce the following result.

**Claim B.6.** $\| \mathcal{F}_K(\alpha, h) \|_{L^\infty(D)} \leq C \sum_{k=0}^{K-1} \left( \left| \frac{\partial^k}{\partial \alpha^k} \lambda_1 \right| + \left\| \frac{\partial^k}{\partial \alpha^k} \varphi_1 \right\|_{W^{2,p}(D)} \right).$
By the induction assumption (i.e. (i) holds for \( k = K - 1 \)) we have \( \|\tilde{\mathcal{F}}_K\|_{L^p(D)} \leq C(M, \alpha, \pi, D) \). Hence we may apply Claim B.4 to (B.15) to conclude the assertion (i) for the case \( K \). This induction argument proves (i).

By Lemma B.3(ii), it remains to prove assertion (ii) for case \( k \geq 1 \). Let \( \alpha_j \to \alpha_0 \in [\alpha, \pi] \) and \( h_j \) be a uniformly bounded sequence in \( L^\infty(D) \) and \( h_j \to h_0 \) weakly in \( L^p(D) \). Denote \( \lambda_{1,j} = \lambda_1(\alpha_j, h_j) \) and \( \varphi_{1,j} = \varphi_1(\cdot; \alpha_j, h_j) \). By assertion (i), there are subsequences \( \lambda_{1,j'} \) and \( \varphi_{1,j'} \) such that for all \( k \geq 0 \), \( \frac{\partial^k}{\partial \alpha^k} \lambda_1(\alpha_{j'}, h_{j'}) \to \hat{\lambda}_k \) and \( \frac{\partial^k}{\partial \alpha^k} \varphi_1(\cdot; \alpha_{j'}, h_{j'}) \to \hat{\varphi}_k \) weakly in \( W^{2,p}(D) \), for some \( \hat{\lambda}_k \in \mathbb{R} \) and \( \hat{\varphi}_k \in W^{2,p}(D) \). Passing to the limit in (B.14), we deduce that

\[
(B.17) \quad D_{(\varphi, \lambda)}F [\hat{\varphi}_1, \hat{\lambda}_1] = -D_{\alpha}F,
\]

where the partial derivatives of \( F \) are evaluated at \( (\varphi_1(\cdot; \alpha_0, h_0), \lambda_1(\alpha_0, h_0), \alpha_0, h_0) \).

Since we also have

\[
(B.18) \quad D_{(\varphi, \lambda)}F \left[ \frac{\partial}{\partial \alpha} \varphi_1(\cdot; \alpha_0, h_0), \frac{\partial}{\partial \alpha} \lambda_1(\alpha_0, h_0) \right] = -D_{\alpha}F,
\]

where the partial derivatives of \( F \) are evaluated at \( (\varphi_1(\cdot; \alpha_0, h_0), \lambda_1(\alpha_0, h_0), \alpha_0, h_0) \), we may invert \( D_{(\varphi, \lambda)}F \) in both (B.17) and (B.18), and conclude that

\[
\hat{\varphi}_1 = \frac{\partial}{\partial \alpha} \varphi_1(\cdot; \alpha_0, h_0) \quad \text{and} \quad \hat{\lambda}_1 = \frac{\partial}{\partial \alpha} \lambda_1(\alpha_0, h_0).
\]

Since the limit is determined independent of the subsequence, we conclude assertion (ii) for the case \( k = 1 \).

Again, we may argue inductively for \( k > 1 \). Suppose (ii) is proved for \( k = 1, \ldots, K - 1 \). The following can be easily observed from (B.16).

Claim B.7. If assertion (ii) holds for \( k = 1, \ldots, K - 1 \), then

\[
\tilde{\mathcal{F}}_K(\alpha_j, h_j) \to \tilde{\mathcal{F}}_K(\alpha_0, h_0)
\]

weakly in \( L^p(D) \), where \( \tilde{\mathcal{F}}_K \) is defined in (B.16).

Based on Claim B.7 and the assertion (ii) for the cases \( k = 1, \ldots, K - 1 \), we may pass to the limit in (B.15). Together with the uniform boundedness of \( [D_{(\varphi, \lambda)}F]^{-1} : L^p(D) \to W^{2,p}(D) \) (Lemma B.4), this implies \( \frac{\partial^k}{\partial \alpha^k} \lambda_1(\alpha_j, h_j) \to \frac{\partial^k}{\partial \alpha^k} \lambda_1(\alpha_0, h_0) \) and

\[
\frac{\partial^k}{\partial \alpha^k} \varphi_1(\cdot; \alpha_j, h_j) \to \varphi_1(\cdot; \alpha_0, h_0) \text{ in } W^{2,p}(D).
\]

Thus assertion (ii) follows by induction on \( k \).

\[\square\]

Appendix C. Liouville Theorem for Positive Harmonic Functions in Cylinder Domain

We give a proof of the Liouville-type theorem for positive harmonic functions in cylinder domains, since we cannot locate a proper reference for this result.

Proposition C.1. Let \( k \in \mathbb{N} \), \( D \) be a bounded smooth domain in \( \mathbb{R}^N \) and \( u \) be a non-negative harmonic function on \( \Omega := D \times \mathbb{R}^k \subset \mathbb{R}^{N+k} \), so that \( \frac{\partial u}{\partial n} = 0 \) on \( \partial D \times \mathbb{R}^k \). Then \( u \) is necessarily a constant.
Proof. Let $x \in D$, $y \in \mathbb{R}^k$ and let $u(x, y)$ be a non-negative harmonic function on $\Omega = D \times \mathbb{R}^k$, subject to Neumann boundary condition on $\partial D \times \mathbb{R}^k$. By subtracting a positive constant from $u$, we may assume that $\inf_{\Omega} u = 0$.

Harnack inequality says that there is a constant $C > 1$ such that for all $y' \in \mathbb{R}^k$, we have

$$\sup_{x \in D, |y-y'|<2} u \leq C \inf_{x \in D, |y-y'|<2} u.$$  

Define $v(y) = \frac{1}{|D|} \int_D u(x', y) \, dy$, then $v$ is a harmonic function on $\mathbb{R}^k$ and must be equal to a non-negative constant $v_0$. Hence for each $y' \in \mathbb{R}^k$, there exists $x' \in \bar{D}$ such that $u(x', y') = v_0$. It follows that for each $y' \in \mathbb{R}^k$,

$$v_0 \leq C \inf_{x \in D, |y-y'|<2} u(x, y).$$

Taking infimum in $y' \in \mathbb{R}^k$, it follows that from $\inf_{\Omega} u = 0$ that $v_0 = 0$. Hence,

$$\frac{1}{|D|} \int_D u(x, y) \, dx = v(y) = v_0 = 0$$

for all $y \in \mathbb{R}^k$, i.e. $u \equiv 0$ in $\Omega$. \qed

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