SHARP VARIATIONAL CHARACTERIZATION AND A SCHRÖDINGER EQUATION WITH HARTREE TYPE NONLINEARITY

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Abstract. In this paper, we first give a sharp variational characterization to the smallest positive constant $C_{VGN}$ in the following Variant Gagliardo-Nirenberg interpolation inequality:

$$
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha} \, dx \, dy \leq C_{VGN} \|\nabla u\|_{L^2}^{N(p-2)+\alpha} \|u\|_{L^2}^{2p-(N(p-2)+\alpha)},
$$

where $u \in W^{1,2}(-\mathbb{R}^N)$ and $N \geq 1$. Then we use this characterization to determine the sharp threshold of $\|\varphi_0\|_{L^2}$ such that the solution of $i\varphi_t = -\Delta \varphi - \varphi |\varphi|^{p-2}(|x|^{-\alpha} * |\varphi|^p)$ with initial condition $\varphi(0, x) = \varphi_0$ exists globally or blows up in a finite time. We also outline some results on the applications of $C_{VGN}$ to the Cauchy problem of $i\varphi_t = -\Delta \varphi - \varphi |\varphi|^{p-2}(|x|^{-\alpha} * |\varphi|^p)$.

1. Introduction. In this note, we study the following semilinear Schrödinger equation with a Hartree type nonlinearity

$$
i\varphi_t = -\Delta \varphi - \varphi |\varphi|^{p-2}(|x|^{-\alpha} * |\varphi|^p), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N
$$

as well as Schrödinger equation with harmonic potential and Hartree type nonlinearity

$$
i\varphi_t = -\Delta \varphi + |x|^2 \varphi - \varphi |\varphi|^{p-2}(|x|^{-\alpha} * |\varphi|^p), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N,
$$

where $N \geq 1$, $0 < \alpha < \min\{N, 4\}$, $\varphi := \varphi(t, x) : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{C}$ is a complex-valued function and $\ast$ is the standard convolution in $\mathbb{R}^N$.

Problems of this kind arise naturally from various physical situations. For example, Eq.(1) can be considered as the classical limit of the field equations describing a quantum mechanical nonrelativistic many boson system [10]. Eq.(2) can be used to describe average pulse propagation in dispersion-managed fibers, see e. g. [13].

When $\alpha = 1$, $p = 2$ and $N = 3$, Eq.(2) is also equivalent to the Schrödinger-Poisson system with harmonic potential,

$$
\begin{cases}
  i\varphi_t + \Delta \varphi = |x|^2 \varphi - V(x) \varphi, & x \in \mathbb{R}^N \\
  \Delta V = |\varphi|^2.
\end{cases}
$$

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This equation arises typically in the mean field approximation of the many body effects, modeled by the Poisson equation with a confinement modeled by the quadratic potential of the harmonic oscillator. For other variant of Schrödinger equation, we refer the interested readers to [2] and the references therein.

There are many studies on the Eq.(1) and Eq.(2) respectively. But some problems are still unsolved. For example when proposing an initial condition in [7] and we denote the potential of the harmonic oscillator. For other variant of Schrödinger equation, we have effects, modeled by the Poisson equation with a confinement modeled by the quadratic equation arises typically in the mean field approximation of the many body effects.

\[ \varphi(t, x)|_{t=0} = \varphi_0(x), \]  

(3)

the existence and asymptotical behavior of solutions of Eqs.(1)+(3) and the solutions of Eqs.(2)+(3) have been important and interesting topics. For every $\varphi_0 \in W^{1,2}(\mathbb{R}^N)$, we know from [3] that Eqs.(1)+(3) has a local solution for $2 \leq p < 2^*(\alpha)$ (where $2^*(\alpha) = (2N - \alpha)/(N - 2)$ for $N \geq 3$ and $2^*(\alpha) = +\infty$ for $N = 1, 2$); when $2 \leq p < 2 + (2 - \alpha)/N$, the solutions of Eqs.(1)+(3) exist globally for any $\varphi_0 \in W^{1,2}(\mathbb{R}^N)$. While for $2 + (2 - \alpha)/N \leq p < 2^*(\alpha)$, the solutions of Eqs.(1)+(3) exist globally for $\|\varphi_0\|_{L^2(\mathbb{R}^N)}$ small enough and the solutions may blow up in a finite time for some $\|\varphi_0\|_{L^2(\mathbb{R}^N)}$. So a natural and interesting question is what is the optimal value of $\|\varphi_0\|_{L^2(\mathbb{R}^N)}$?

For the Cauchy problem of Eq.(2), there are similar problems unknown. Denote $\Sigma = \{u \in W^{1,2}(\mathbb{R}^N) : |x|u \in L^2(\mathbb{R}^N)\}$. Then for any $N \geq 1$ and every $\varphi_0 \in \Sigma$, we know from [7] that Eqs.(2)+(3) has a local solution for $2 \leq p < 2^*(\alpha)$; Eqs.(2)+(3) has a global solution provided that $2 + (2 - \alpha)/N \leq p < 2^*(\alpha)$ and $\|\varphi_0\|_{L^2(\mathbb{R}^N)}$ is small; while the solutions of Eqs.(2)+(3) may blow up in a finite time for $2 + (2 - \alpha)/N \leq p < 2^*(\alpha)$ and suitable bigger $\|\varphi_0\|_{L^2}$. So the determination of the sharp threshold of $\|\varphi_0\|_{L^2}$ is an interesting problem.

Partial results for radially symmetric functions in $W^{1,2}(\mathbb{R}^N)$ has been obtained in [7] (where we assume $N \geq 2$). But general results for $N \geq 1$ are still unknown. In the present paper, we have two goals. One is to give a sharp variational characterization of $C_{VGN}$ related to the following Variant Gagliardo-Nirenberg interpolation inequality: there is a positive constant $C$ such that for all $u \in W^{1,2}(\mathbb{R}^N)$,

\[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x - y|^\alpha} \, dx \, dy \leq C \left( \int |\nabla u|^2 \, dx \right)^A \left( \int |u|^2 \, dx \right)^B, \]  

(4)

where $A = \frac{N(p-2) + \alpha}{2}$ and $B = p - A$. Note that the inequality (4) has been shown in [7] and we denote

\[ C_{VGN} := \min\{C > 0 : C \text{ is such that } (4) \text{ holds}\}. \]

The other is to use the characterization of $C_{VGN}$ to solve the unknown problems mentioned above. We will study Eq.(2) in details but only outline some counterpart for Eq.(1).

To explain our ideas and strategies of solving these problems, we recall first that for the semilinear Schrödinger equation $i \psi_t + \Delta \psi + |\psi|^{q-1}\psi = 0$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Weinstein [14] has originally solved similar problems mentioned above by studying the following Gagliardo-Nirenberg inequality

\[ \|u\|_{L^{q+1}}^{q+1} \leq C_{GN} \|\nabla u\|_{L^2}^{\frac{q(q+1)}{2}} \|u\|_{L^2}^{\frac{2(q+1) - N(q-1)}{2}}, \quad u \in W^{1,2}(\mathbb{R}^N). \]

Weinstein [14] got a sharp estimate on the smallest positive constant $C_{GN}$ by solving the minimization problem

\[ C_{GN}^{-1} = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{N(q-1)}{4}} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{2(q+1) - N(q-1)}{4}}}{\int_{\mathbb{R}^N} |u|^{q+1} \, dx}. \]
Weinstein [14] managed to do this by restricting the spatial dimension $N \geq 2$ and using the fact that the embedding of

$$W^{1,2}_r(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N)$$

is compact for $1 < q < 2^* - 1$, \hspace{1cm} (5)

where $W^{1,2}_r(\mathbb{R}^N) = \{ u \in W^{1,2}(\mathbb{R}^N) : u(x) = u(|x|) \}$, $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = +\infty$ for $N = 2$.

Noticing that in the case of $N = 1$, we do not know if the embedding (5) is compact or not. It seems that the methods of Weinstein [14] can not be used directly to study this for $N = 1$. In Section 2, we develop a scaling method and characterize $C_{VGN}$ for all spatial dimension $N \geq 1$. The main result is contained in Theorem 2.4. We emphasize here that the characterization of this $C_{VGN}$ not only has independent interests, but also can be applied to study some other properties of solutions of Eqs.(1)+(3) and Eqs.(2)+(3). The variational characterization of $C_{VGN}$ is the main contribution of the present paper. We also point out that this scaling method can be used to study a class of interpolation inequality involving $p$-Laplace operator, which will be published elsewhere. In Section 3, we summarize some local existence results on the Eqs.(2)+(3). In Section 4, we determined the threshold of $\|\varphi_0\|_{L^2}$ such that the solutions of Eqs.(2)+(3) exist globally or blow up in a finite time. The main results are Theorem 4.1 and Theorem 4.2. In Section 5, we study the existence of global solutions of Eqs.(2)+(3) for initial data whose $L^2(\mathbb{R}^N)$ norm can be as large as you want. In Section 6, we outline some counterpart results for the Eqs.(1)+(3).

Notations. Throughout this paper, all integrals are taken over $\mathbb{R}^N$ unless stated otherwise. The $W^{1,2}(\mathbb{R}^N)$ is the standard Sobolev space with the norm $\|u\|^2 = \int (\nabla u^2 + |u|^2)dx$. The norm in $L^q(\mathbb{R}^N)$ ($1 \leq q \leq \infty$) is denoted by $\|\cdot\|_{L^q}$. $M_j$ ($j \in \mathbb{N}$) denote various positive constants whose exact value are not important.

2. Variational characterization of $C_{VGN}$. This section is devoted to the characterization of $C_{VGN}$. We start with the following Variant Gagliardo-Nirenberg interpolation inequality.

Proposition 1. [7] Let $0 < \alpha < N$ and $(2N - \alpha)/N < p < 2^*(\alpha)$. Then there is a positive constant $C(p,\alpha,N)$ depending on $p$, $\alpha$ and $N$ such that for any $u \in W^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x - y|^\alpha}dxdy \leq C(p,\alpha,N) \left( \int |\nabla u|^2dx \right)^A \left( \int |u|^2dx \right)^B,$$

where $A = \frac{N(p-2)+\alpha}{2}$ and $B = p - A$. The $C_{VGN}$ denotes the infimum of all possible $C(p,\alpha,N)$.

The characterization of $C_{VGN}$ will be divided into several steps. In the first place, we consider the following auxiliary maximization problem with constraint. For any $\lambda > 0$, we denote $S_\lambda = \{ u \in W^{1,2}(\mathbb{R}^N) : \|u\|^2 = \lambda \}$ and set

$$m_\lambda = \sup \{ g(u) : u \in S_\lambda \}, \quad \text{where} \quad g(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x - y|^\alpha}dxdy.$$

We will adapt the method of Lions [11] to prove that for any given $\lambda > 0$, the $m_\lambda$ is achieved. Note that $m_\lambda > 0$ for any $\lambda > 0$. Moreover we have the following lemma.
Lemma 2.1. Let \((u_n)_{n \in \mathbb{N}}\) be a maximizing sequence of the \(m_\lambda\). Then for any \(\varepsilon > 0\), there is \(0 < R_0 < +\infty\), a sequence \(z_k \in \mathbb{R}^N\) \((k \in \mathbb{N})\), and a subsequence \((u_{n_k})_{k \in \mathbb{N}}\) such that

\[
\int_{z_k + B_{R_0}} \left( |\nabla u_{n_k}|^2 + |u_{n_k}|^2 \right) \, dx \geq \lambda - \varepsilon.
\]

Proof. According to Lions’ concentration compactness lemma [11], we divide the proof into two steps. Denote \(\rho_n(x) = |\nabla u_n(x)|^2 + |u_n(x)|^2\). In the first step, we claim that for any given \(R > 0\),

\[
\liminf_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{z + B_R} \rho_n(x) \, dx > 0.
\]

We prove this claim indirectly. Suppose that there is a subsequence of \((\rho_n)_{n \in \mathbb{N}}\), still denoted by \((\rho_n)_{n \in \mathbb{N}}\), such that

\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{z + B_R} \rho_n(x) \, dx = 0 \quad \text{for some} \quad R > 0.
\]

Then from an inequality of [12], see also [7, Lemma 2.2], we have that

\[
g(u_n) \leq M_1 \left( \int |u_n|^{\frac{2Np}{Np-\alpha}} \, dx \right)^{\frac{Np}{Np-\alpha}}.
\]

By the assumption of \(p\), one deduces from Sobolev inequality that

\[
\int_{z + B_R} |u_n|^{\frac{2Np}{Np-\alpha}} \, dx \leq M_2 \left( \int_{z + B_R} \rho_n(x) \, dx \right)^{\frac{Np}{Np-\alpha}},
\]

where \(M_2\) is independent of \(z\), \(R\) and \(u_n\). Now covering \(\mathbb{R}^N\) by balls of radius \(R\), in such a way that each point of \(\mathbb{R}^N\) is contained in at most \(N + 1\) balls, we hence obtain that

\[
\int |u_n|^{\frac{2Np}{Np-\alpha}} \, dx \leq M_2(N + 1) \left( \sup_{z \in \mathbb{R}^N} \int_{z + B_R} \rho_n(x) \, dx \right)^{\frac{Np}{Np-\alpha} - 1} \|u_n\|^2
\]

\[
= \lambda M_2(N + 1) \left( \sup_{z \in \mathbb{R}^N} \int_{z + B_R} \rho_n(x) \, dx \right)^{\frac{Np}{Np-\alpha} - 1} \to 0 \quad \text{as} \quad n \to \infty.
\]

Therefore \(g(u_n) \to 0\) as \(n \to \infty\), which is a contradiction since \(u_n\) satisfies \(\lim_{n \to \infty} g(u_n) = m_\lambda > 0\).

In the second step, we prove that the case of

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{z + B_R} \rho_n(x) \, dx = \beta \in (0, \lambda)
\]

does not occur either. Arguing by a contradiction, we may assume that (7) holds for some \(\beta \in (0, \lambda)\). Then for any given \(\varepsilon > 0\) and \(\varepsilon\) small enough, there is \(R_0 > 0\) such that for all \(R \geq R_0\), there are \(n_0 \in \mathbb{N}\) and a sequence \(z_n \in \mathbb{R}^N\) \((n \in \mathbb{N})\) such that

\[
\beta - 2\varepsilon \leq \int_{z_n + B_{R_0}} \rho_n(x) \, dx \leq \beta + 2\varepsilon.
\]

Moreover, we also have that for any \(R_1 > R_0\),

\[
\beta - 2\varepsilon \leq \sup_{z \in \mathbb{R}^N} \int_{z + B_{R_1}} \rho_n(x) \, dx \leq \beta + 2\varepsilon.
\]
Hölder inequality and an elementary calculation that \( \text{Supp} \) Lemma 2.1 holds. The proof is complete.

Therefore, we have that
\[
0 \leq \int_{B_{(z_n,R_1)\setminus B(z_n,R_0)}} \rho_n(x) \, dx \leq 4 \varepsilon
\]  
and
\[
(\lambda - \beta) + 2\varepsilon \geq \int_{\mathbb{R}^N \setminus B(z_n,R_1)} \rho_n(x) \, dx \geq (\lambda - \beta) - 2\varepsilon.
\]

Now we define a cut-off function \( \eta \) on \([0, \infty)\) such that \( \eta_{R_0}(s) = 1 \) for \( s \leq R_0 \) and \( \eta_{R_0}(s) = 0 \) for \( s \geq 2R_0 \) and \( |\eta''_{R_0}(s)| \leq \frac{2}{R_0^2} \). Let \( \eta_{R_0}''(s) = 1 - \eta_{R_0}(s) \). Set
\[
v_n(x) = \eta_{R_0}(|x|)u_n(x + z_n) \quad \text{and} \quad w_n(x) = \eta_{R_0}''(|x|)u_n(x + z_n).
\]

From a direct calculation and using the fact of
\[
\int |\nabla \eta_{R_0} u_n|^2 \, dx \leq \frac{4}{R_0^2} \int_{R_0 \leq |x - z_n| \leq 2R_0} |u_n|^2 \, dx \leq \frac{M_3}{R_0^2}
\]
and
\[
\int |\eta_{R_0} \nabla \eta_{R_0} u_n \nabla u_n| \, dx \leq \frac{M_4}{R_0},
\]
we obtain that
\[
\beta - 2\varepsilon - \frac{M_5}{R_0} \leq \|v_n\|^2 \leq \beta + 2\varepsilon + \frac{M_5}{R_0} + 4\varepsilon
\]
Similarly, we can deduce that
\[
(\lambda - \beta) - 2\varepsilon - \frac{M_6}{R_0} \leq \|w_n\|^2 \leq (\lambda - \beta) + 2\varepsilon + \frac{M_6}{R_0} + 4\varepsilon.
\]

Note that for any \( a, b \in \mathbb{R} \) and \( q > 1 \), there holds
\[
||a + b|^q - |a|^q - |b|^q| \leq M (|a|^{q-1}|b| + |a||b|^{q-1}).
\]
Since \( \text{Supp}u_n \cap \text{Supp}w_n = \{x \in \mathbb{R}^N : R_0 \leq |x - z_n| \leq 2R_0, u_n \geq 0\} \), we obtain from (8), Hölder inequality and an elementary calculation that
\[
|g(u_n) - g(v_n) - g(w_n)| \leq M_7 \varepsilon + \frac{M_8}{R_0}.
\]

Therefore,
\[
m_\lambda + o(1) = g(u_n)
\leq \|v_n\|^{2p} + \frac{|\lambda v_n|^p}{\lambda^p} g(\sqrt{\lambda v_n}) + \|w_n\|^{2p} + \frac{|\lambda w_n|^p}{\lambda^p} g(\sqrt{\lambda w_n}) + M_9 \varepsilon + \frac{M_{10}}{R_0}
\leq \frac{m_\lambda}{\lambda^p} ( (\beta - \varepsilon)^p + (\lambda - \beta - \varepsilon)^p) + M_9 \varepsilon + \frac{M_{11}}{R_0}
< \frac{m_\lambda}{\lambda^p} (\text{since } p > 1)
\]
by letting \( n \to \infty \) and then \( \varepsilon \to 0 \) (since \( R_0 \to \infty \) as \( \varepsilon \to 0 \)). Which is a contradiction.

It is now deduced from Lions’ Concentration Compactness lemma \([11]\) that Lemma 2.1 holds. The proof is complete. \( \square \)

**Proposition 2.** Let \( N \geq 1 \) and \( \frac{2N-\alpha}{N} < p < 2^*(\alpha) \). Then for any \( \lambda > 0 \), the \( m_\lambda \) is achieved by some \( u \in W^{1,2}(\mathbb{R}^N) \).
Proposition 3. Let \((v_n)_{n \in \mathbb{N}}\) be a maximizing sequence of \(m_\lambda\), i.e., \(\|v_n\|^2 = \lambda\) and \(g(v_n) \to m_\lambda\). Then by Lemma 2.1, we have that for any \(\varepsilon > 0\), there is \(R_0 < +\infty\) and a sequence \(z_k \in \mathbb{R}^N\) and a subsequence \((u_{n_k})_{k \in \mathbb{N}} \subset (u_n)_{n \in \mathbb{N}}\) such that
\[
\int_{z_k + B_{R_0}} (|\nabla u_{n_k}|^2 + |u_{n_k}|^2) \, dx \geq \lambda - \varepsilon.
\]
This and Proposition 1 imply that \(g(u_{n_k}) \to g(u)\) and \(u\) is a maximizer of the \(m_\lambda\).

In the second place, we consider another auxiliary problem. On \(W^{1,2}(\mathbb{R}^N)\), we define the functionals
\[
L(u) = \frac{1}{2} \int (|\nabla u|^2 + |u|^2) \, dx - \frac{1}{2p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} \, dx \, dy
\]
and
\[
I(u) = \int (|\nabla u|^2 + |u|^2) \, dx - \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} \, dx \, dy.
\]
Set \(\mathcal{N} = \{ u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\} : I(u) = 0 \}\) and define
\[
d = \min \{ L(u) : u \in \mathcal{N} \}.
\]
To proceed, we point out first that for any \(\lambda > 0\), \(m_\lambda = \lambda \mu_1\). From the Proposition 2 and Lagrange multiplier rule, we know that for \(\lambda = 1\), there is \(V_1 \in W^{1,2}(\mathbb{R}^N)\setminus \{0\}\) and a \(\theta_1 \in \mathbb{R}\) such that
\[
-\Delta V_1 + V_1 = \theta_1 V_1 |V_1|^{p-2} \int_{\mathbb{R}^N} \frac{|V_1(y)|^p}{|x-y|^\alpha} \, dy.
\]
A simple scaling argument implies that \(\theta_1^{\frac{1}{1-p^*}} V_1(x)\) is a solution of
\[
-\Delta u + u = u |u|^{p-2} \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\alpha} \, dy, \quad u \in W^{1,2}(\mathbb{R}^N).
\] (13)
Moreover, we have \(\|\theta_1^{\frac{1}{1-p^*}} V_1\|^2 = g(\theta_1^{\frac{1}{1-p^*}} V_1)\) and the following proposition.

**Proposition 3.** Let \(N \geq 1\), \(\frac{2N-\alpha}{N} < p < 2^*(\alpha)\) and \(\Lambda = \|\theta_1^{\frac{1}{1-p^*}} V_1\|^2\). Then a maximizer \(U\) of the \(m_\Lambda\) is a minimal action solution of Eq.(13). That is to say, \(U\) is a solution of Eq.(13) and for any \(v \in \mathcal{N}\), one has that \(L(U) \leq L(v)\).

**Proof.** Firstly we prove that \(\|U\|^2 = g(U)\). Indeed, since \(U\) is a maximizer of the \(m_\Lambda\), there is \(\theta_\Lambda\) such that
\[
-\Delta U + U = \theta_\Lambda U |U|^{p-2} \left(|x|^{-\alpha} * |U|^p \right).
\]
Hence \(\|U\|^2 = \theta_\Lambda g(U)\). Note that \(\|U\|^2 = \Lambda = \|\theta_1^{\frac{1}{1-p^*}} V_1\|^2\) and
\[
g(U) = m_\Lambda = \Lambda^p m_1 = \Lambda^p g(V_1) = \theta_1^{\frac{1}{1-p^*}} g(V_1) = g(\theta_1^{\frac{1}{1-p^*}} V_1) = \|\theta_1^{\frac{1}{1-p^*}} V_1\|^2.
\]
We obtain that \(\|U\|^2 = g(U)\). Therefore \(\theta_\Lambda = 1\).

Secondly, we prove that \(U\) is a minimal action solution of Eq. (13). For any \(v \in \mathcal{N}\), we have that \(\frac{\sqrt{\Lambda v}}{\|v\|} \in S_\Lambda\). Since \(U\) is a maximizer of the \(m_\Lambda\),
\[
m_\Lambda = g(U) \geq g(\frac{\sqrt{\Lambda v}}{\|v\|}) = \frac{\Lambda^p}{\|v\|^{2p}} g(v).
\]
It is now deduced from $\|U\|^2 = g(U)$ and $I(v) = 0$ that $\|U\|^2 \leq \|v\|^2$. Therefore,

$$L(U) = \left(\frac{1}{2} - \frac{1}{2p}\right) \|U\|^2 \leq \left(\frac{1}{2} - \frac{1}{2p}\right) \|v\|^2 = L(v).$$

This implies that $U$ is a minimizer of the minimum $d$. We obtain from [15, Theorem 4.3] that $U$ is a minimal action solution of Eq. (13).

In the third place, we give another characterization to a minimal action solution of Eq. (13), which will play an important role in the study of $C_{VGN}$.

**Proposition 4.** Let $N \geq 1$ and $\frac{2N-\alpha}{N} < p < 2^*(\alpha)$. If $\phi$ is a minimal action solution of Eq. (13), then $\phi$ is a maximizer of $m_r = \|\phi\|^2$.

**Proof.** Clearly, from the definition we know that $m_r \geq g(\phi)$. It remains to prove that $m_r \leq g(\phi)$. For any $u \in W^{1,2}(\mathbb{R}^N)$ satisfying $\|u\|^2 = \|\phi\|^2$, there is $s_0 = \frac{1}{\|u\|^{\frac{2p}{p-1}} g(u)^{\frac{2}{p-1}}} \|u\|^2$ such that $I(s_0 u) = 0$, which implies that $\|s_0 u\|^2 = s_0^2 g(u)$. Since $\phi$ is a minimal action solution of Eq. (13), we obtain that

$$L(\phi) = \left(\frac{1}{2} - \frac{1}{2p}\right) \|\phi\|^2 \leq L(s_0 u) = \left(\frac{1}{2} - \frac{1}{2p}\right) \|s_0 u\|^2 = \left(\frac{1}{2} - \frac{1}{2p}\right) \|u\|^2 = \frac{2p}{p-1} g(u)\|u\|^\frac{1}{p-1}.$$

According to the fact of $g(\phi) = \|\phi\|^2 = \|u\|^2$, we deduce that

$$g(u) \leq \|u\|^2 = \|\phi\|^2.$$

Since $u$ is chosen arbitrarily, we get that $m_r \leq \|\phi\|^2 = g(\phi)$. This completes the proof.

We are now in a position to give a variational characterization of $C_{VGN}$. To simplify the notations, we denote $K := N(p - 2) + \alpha$ in the rest of this section. Define the following functional

$$Q(u) = \int |\nabla u|^2 \, dx - \frac{K}{2p} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} \, dx \, dy, \quad u \in W^{1,2}(\mathbb{R}^N).$$

**Lemma 2.2.** Let $\frac{2N-\alpha}{N} < p < 2^*(\alpha)$ and $\phi$ be a minimal action solution of Eq. (13). Then $Q(\phi) = 0$.

**Proof.** The proof is the same as [6, lemma 4.2].

**Lemma 2.3.** Let $\frac{2N-\alpha}{N} < p < 2^*(\alpha)$ and $\phi$ be a minimal action solution of Eq. (13). Then

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} \, dx \, dy = \frac{2p}{2p-K} \int |\phi|^2 \, dx$$

and

$$\int |\nabla \phi|^2 \, dx = \frac{K}{2p-K} \int |\phi|^2 \, dx.$$

**Proof.** Since $\phi$ is a minimal action solution of Eq. (13), one has that $I(\phi) = 0$. Combining this with Lemma 2.2, one can get the conclusion immediately by direct calculation.
Theorem 2.4. Let $N \geq 1$ and $(2N - \alpha)/N < p < 2^\ast(\alpha)$. Then the smallest positive constant $C_{VGN}$ can be characterized as

$$C_{VGN} = \frac{2p}{K} \left( \frac{K}{2p - K} \right)^{1 - \frac{p}{2}} \left( \frac{2p - K}{p - 1} \right)^{1 - p} = \frac{2p}{K} \left( \frac{K}{2p - K} \right)^{1 - \frac{p}{2}} \|\phi\|_{L^2}^{2(1 - p)},$$

where $\phi$ is a minimal action solution of Eq. (13).

Proof. We prove this theorem by studying the following minimization problem

$$C_{VGN}^{-1} = \inf \left\{ J(u) : u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\} \right\},$$

where

$$J(u) = \left( \int |\nabla u|^2 \, dx \right)^A \left( \int |u|^2 \, dx \right)^B \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)|^p |u(y)|^p \, |x-y|^\alpha \, dxdy \right)^{-1}.$$

In the first step, we claim that

$$C_{VGN}^{-1} \geq \frac{2p - K}{2p} \left( \frac{K}{2p - K} \right)^A \left( \int |\phi|^2 \, dx \right)^{p-1}.$$

Indeed from $\phi \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}$ and Lemma 2.3, one deduces that

$$J(\phi) = \left( \int |\nabla \phi|^2 \, dx \right)^A \left( \int |\phi|^2 \, dx \right)^B \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} |\phi(x)|^p |\phi(y)|^p \, |x-y|^\alpha \, dxdy \right)^{-1}$$

$$= \frac{K}{2p - K} \int |\phi|^2 \, dx \left( \int |\phi|^2 \, dx \right)^{A - 1} \left( \frac{2p}{2p - K} \int |\phi|^2 \, dx \right)^{-1}$$

$$= \frac{2p - K}{2p} \left( \frac{K}{2p - K} \right)^A \left( \int |\phi|^2 \, dx \right)^{p-1}.$$

Therefore (15) holds since $\phi$ is chosen arbitrarily.

In the second step, we prove that

$$C_{VGN}^{-1} \leq \frac{2p - K}{2p} \left( \frac{K}{2p - K} \right)^A \left( \int |\phi|^2 \, dx \right)^{p-1}.$$

To prove this, for any $u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}$, we define $w(x) := \xi u(\mu x)$, where $\xi$ and $\mu$ are positive parameters which will be determined later. By direct computation, one obtains that

$$\int |\nabla w(x)|^2 \, dx = \xi^2 \mu^{-N} \int |\nabla u(x)|^2 \, dx,$$

$$\int |w(x)|^2 \, dx = \xi^2 \mu^{-N} \int |u(x)|^2 \, dx$$

and

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |w(x)|^p |w(y)|^p \, |x-y|^\alpha \, dxdy = \xi^{2p} \mu^{-2N} \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)|^p |u(y)|^p \, |x-y|^\alpha \, dxdy.$$

Now we determine $\xi$ and $\mu$ by the following equations

$$\xi^2 \mu^{-N} \int |u(x)|^2 \, dx := \int |\phi|^2 \, dx$$

and

$$\xi^2 \mu^{2-N} \int |\nabla u(x)|^2 \, dx := \int |\nabla \phi|^2 \, dx.$$
Noting that from Lemma 2.3 we know that
\[ \int |\nabla \phi|^2 dx = \frac{K}{2p-K} \int |\phi|^2 dx. \]
It is deduced from Eq.(19) and Eq.(20) that
\[ \mu = \left( \frac{K}{2p-K} \right)^{\frac{1}{2}} \left( \frac{\int |u|^2 dx}{\int |\nabla u|^2 dx} \right)^{\frac{1}{2}} \left( \int |\nabla u|^2 dx \right)^{-\frac{1}{2}} \] (21)
and
\[ \xi^2 = \left( \frac{K}{2p-K} \right)^{\frac{N}{2}} \int |\psi|^2 dx \left( \int |u|^2 dx \right)^{1+\frac{N}{2}} \left( \int |\nabla u|^2 dx \right)^{-\frac{N}{2}} . \] (22)
From the choice of \( w \), we have that \( \| w \|^2 = \| \phi \|^2 \). Since \( \phi \) is a minimal action solution of Eq.(13), we obtain from Proposition 4 that
\[ g(w) \leq g(\phi). \]
Therefore
\[ \xi^{2p} \mu^{a-2N} g(u) \leq g(\phi) = \frac{2p}{2p-K} \int |\phi|^2 dx. \]
Combining this with Eq.(21) and Eq.(22), we deduce that
\[ \left( \frac{\int |\nabla u|^2 dx}{\int |u|^2 dx} \right)^A \left( \frac{\int |u|^2 dx}{\int |\nabla u|^2 dx} \right)^B (g(u))^{-1} \geq \frac{2p-K}{2p} \left( \frac{K}{2p-K} \right)^{Np+q-2N} \left( \int |\phi|^2 dx \right)^{p-1}. \]
The expression of \( J(u) \) implies that
\[ J(u) \geq \frac{2p-K}{2p} \left( \frac{K}{2p-K} \right)^{Np+q-2N} \left( \int |\phi|^2 dx \right)^{p-1}. \]
Since \( u \) is chosen arbitrarily, we obtain that
\[ C_{VGN}^{-1} \geq \frac{2p-K}{2p} \left( \frac{K}{2p-K} \right)^{Np+q-2N} \left( \int |\phi|^2 dx \right)^{p-1}. \]
From the above two steps, we get that
\[ C_{VGN} = \frac{2p}{K} \left( \frac{K}{2p-K} \right)^{1-\frac{K}{2}} \| \phi \|^2_{L^2}. \]
Finally, since \( d = L(\phi) = \left( \frac{1}{q} - \frac{1}{2p} \right) \| \phi \|^2 \), we deduce from Lemma 2.3 that \( \int |\phi|^2 dx = \frac{2p-K}{p-1} d \). Therefore
\[ C_{VGN} = \frac{2p}{K} \left( \frac{K}{2p-K} \right)^{1-\frac{K}{2}} \left( \frac{2p-K}{p-1} d \right)^{1-p}. \]
The proof is complete. \( \square \)
3. Cauchy problem of Eq. (2). In this section, we sketch some results on the existence of a local or global solution of Eqs.(2)+(3). Define $\Sigma = \{ u \in H^1(\mathbb{R}^N) : \int |x|^2 |u|^2 dx < +\infty \}$. Then $\Sigma$ is a Hilbert space under the inner product
\[
(u, v)_\Sigma = \text{Re} \int (\nabla u \nabla \bar{v} + |x|^2 u \bar{v} + u \bar{v}) dx.
\]
The norm on $\Sigma$ is denoted by $\| u \|^2_\Sigma = \int (|\nabla u|^2 + |x|^2 |u|^2 + |u|^2) dx$. Then we have the following Proposition 5 and Theorem 3.1.

**Proposition 5.** [7] Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$ and $2 \leq p < 2^*(\alpha)$. For any $\varphi_0 \in \Sigma$, there is a $T = T(\|\varphi_0\|_\Sigma) > 0$ and a unique solution $\varphi$ of Eq. (2) with $\varphi \in C([0, T), \Sigma)$ and $\varphi(0) = \varphi_0$. Moreover, for all $t \in [0, T)$, we have that
\[
\int |\varphi|^2 dx \equiv \int |\varphi_0|^2 dx
\]
and
\[
E(\varphi) = \frac{1}{2} \int (|\nabla \varphi|^2 + |x|^2 |\varphi|^2) dx - \frac{1}{2p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\alpha} dx dy \equiv E(\varphi_0),
\]
where either $T = +\infty$ or $T < +\infty$ and $\lim_{t \to T^-} \|\varphi\|_\Sigma = +\infty$.

**Theorem 3.1.** [7] Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$ and $2 \leq p < 2^*(\alpha)$.

- If $2 \leq p < 2 + (2 - \alpha)/N$, then for any $\varphi_0 \in \Sigma$ the solution $\varphi(x, t)$ of Eq. (2) exists globally in time.
- If $2 \leq p = 2 + (2 - \alpha)/N$, then the solution $\varphi(x, t)$ of Eq. (2) exists globally in time provided the initial data $\|\varphi_0\|_{L^2}$ sufficiently small.

4. Critical mass for critical nonlinearity. In this section, we will use “our estimate” of $C_{VGN}$ to give a sharp condition on the solution of Eqs.(2)+(3) which exists globally in time or blows up in a finite time. The new aspect is in the case of $N = 1$, since for $N \geq 2$, we have used the radial symmetric properties of $W^{1,2}_r(\mathbb{R}^N)$ to study similar phenomena in [7]. Here with the help of Theorem 2.4, we can deal with Eq. (2) generally for $N \geq 1$. We give an answer on the question: how small an initial data can ensure the existence of global solution of Eqs.(2)+(3) in the case of $p = 2 + (2 - \alpha)/N$. The answer is simple as we see below.

**Theorem 4.1.** Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$ and $p = 2 + (2 - \alpha)/N$. If $\varphi_0 \in \Sigma$ and
\[
\|\varphi_0\|_{L^2} < \|w\|_{L^2},
\]
where $w$ is a minimal action solution of Eq. (13), then Eqs.(2)+(3) has a global solution $\varphi(x, t) \in C(\mathbb{R}^+, \Sigma)$.

**Proof.** Let $\varphi_0 \in \Sigma$ and $\varphi(x, t) \in C([0, T), \Sigma)$ be a solution of Eqs.(2)+(3) in the case of $p = 2 + (2 - \alpha)/N$. Using the inequality (4) and Theorem 2.4, one deduces that
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\alpha} dx dy \leq \frac{2N + 2 - \alpha}{N} \left( \int \frac{|\varphi|^2 dx}{|w|^2 dx} \right)^{N+\alpha-2} \int |\nabla \varphi|^2 dx.
\]
On the other hand, since the function $w$ is the solution of Eqs. (2)+(3). If $\phi$ satisfies the condition (25) is sharp in the sense of the following theorem.

**Remark 1.** We point out that for Schrödinger equation with harmonic potential, the condition (25) is sharp in the sense of the following theorem.

**Theorem 4.2.** Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$ and $p = 2 + (2 - \alpha)/N$. If $\phi_0 \in \Sigma$ and

$$\phi_0(x) = c \lambda^{\frac{N}{2}} w(\lambda x),$$

where $\lambda > 0$, $w$ is a minimal action solution of Eq. (13) and $c$ is a complex number with $|c| \geq 1$, then

$$\|\phi_0\|_{L^2} \geq \|w\|_{L^2}.$$  \hspace{1cm} (28)

Moreover the solution $\phi(x, t)$ of Eqs. (2)+(3) must blow up in a finite time.

In order to prove Theorem 4.2, we need several lemmas. Firstly, we have the following virial identity which originated from Glassey [9].

**Proposition 6.** Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$, $\phi_0 \in \Sigma$ and $\phi \in C([0, T], \Sigma)$ be the solution of Eqs. (2)+(3). If $h(t) = \frac{1}{2} \int |x|^2 |\phi|^2 dx$ and $p = 2 + (2 - \alpha)/N$, then one has that

$$h''(t) = 8E(\phi_0) - 16h(t).$$  \hspace{1cm} (29)

**Proof.** For $N \geq 2$, this proposition has been proven in [7]. While for $N \geq 1$, the proof is the same and we omit the details here.

**Lemma 4.3.** Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$ and $p = 2 + (2 - \alpha)/N$. If $\phi_0 \in \Sigma \setminus \{0\}$ and $\phi_0$ satisfies that

$$h(0) = \frac{1}{2} \int |x|^2 |\phi_0|^2 dx \geq E(\phi_0),$$

then the solution $\phi$ of Eqs. (2)+(3) blows up in a finite time.

**Proof.** For $N \geq 2$, this lemma has been proven in [7]. While for $N \geq 1$, the proof is the same and we omit the details here.

**Proof of Theorem 4.2.** For any positive constant $\lambda$ and complex number $c$ with $|c| \geq 1$, a direct computation yields that

$$\int |\phi_0|^2 dx = |c|^2 \int |\lambda \frac{N}{2} w(\lambda x)|^2 dx = |c|^2 \int |w|^2 dx \geq \int |w|^2 dx.$$

On the other hand, since the function $w(x)$ makes the inequality (4) with constant $C_{VGN}$ into equality, one obtains that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(x)|^p |w(y)|^p}{|x-y|^\alpha} dxdy = \frac{2N + 2 - \alpha}{N} \int |\nabla w|^2 dx.$$
Therefore
\[
E(\varphi_0) = \frac{1}{2} \int |\nabla \varphi|^2 dx - \frac{N}{2(2N + 2 - \alpha)} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi_0(x)|^p|\varphi_0(y)|^p}{|x - y|^\alpha} dxdy + h(0) \\
= \frac{1}{2}(1 - |c|^{\frac{2N + 4 - 2\alpha}{N}})\lambda^2 c^2 \int |\nabla w|^2 dx + h(0) \\
\leq h(0).
\]

It follows from Lemma 4.3 that \( \varphi(x,t) \) blows up in a finite time. The proof of Theorem 4.2 is complete. \( \square \)

**Remark 2.** From Theorem 4.1 and Theorem 4.2, we see that when \( p = 2 + (2 - \alpha)/N \), \( \|w\|_{L^2} \) is the critical mass for the solutions of Eqs.(2)+(3) which exists globally in time or blows up in a finite time. The prescribed initial data in Theorem 4.2 also implies that the existence of blow-up solutions of Eqs.(2)+(3) not only depends on the mass of the initial data but also on the profile of the initial data. So it is very reasonable to **conjecture** that for some class of initial data \( \varphi_0 \) with \( \|\varphi_0\|_{L^2} \geq \|w\|_{L^2} \), the solutions of Eqs.(2)+(3) exist globally in time. In fact, this conjecture is true in the case of \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \). Furthermore, we can prove that when \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \), the solutions of Eqs.(2)+(3) exist globally in time for a large class of initial data whose norm can be taken as large as one wants.

5. **Global solutions for supercritical nonlinearity.** After developing the critical mass for the existence of global solutions and the blow-up solutions of Eqs.(2)+(3) in the critical nonlinearity \( p = 2 + (2 - \alpha)/N \), attentions are now focused on the existence of global solutions of Eqs.(2)+(3) in the case of supercritical nonlinearity \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \). An interesting aspect is that we can get the global solutions for arbitrarily large data. We emphasize that the use of Theorem 2.4 is essential. We also point out that for \( N \geq 2 \), we have studied these in [7]. Here we give very general results for \( N \geq 1 \). The following lemma 5.1 will be useful in what follows.

**Lemma 5.1.** [1, 5] Let \( I \subset \mathbb{R} \) be an open interval, \( s_0 \in I, \theta > 1, a > 0, b > 0 \) and \( \Phi(s) \in C(I, \mathbb{R}^+) \). Set \( f(y) = a - y + by^\theta \) for any \( y \geq 0 \), \( y_s = (b\theta)^{-\frac{1}{\theta - 1}} \) and \( b_s = \frac{a - 1}{b} y_s \). Assume that \( \Phi(s_0) < y_s \), \( a \leq b_s \) and \( f \circ \Phi > 0 \). Then \( \Phi(s) < y_s \) for any \( s \in I \).

Next, we define a real valued function \( V(\lambda) \) as follows,
\[
V(\lambda) = \left( \frac{A - 1}{B} \right)^{\frac{2\lambda - 1}{2\lambda - 2}} \|w\|_{L^2}^{\frac{p-1}{2}} \lambda^{-\frac{4\lambda - 1}{4\lambda - 2}}, \quad \lambda > 0,
\]
where as before \( A = \frac{N(p-2)+\alpha}{2} \) and \( B = \frac{2p-(N(p-2)+\alpha)}{2} \). Denote
\[
\mathcal{S} = \{ u \in \Sigma; \|u\|_{L^2} \leq V(\|\nabla u\|_{L^2}^2 + \|uxu\|_{L^2}^2) \}.
\]

**Lemma 5.2.** Let \( N \geq 1, 0 < \alpha < \min\{N, 4\} \) and \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \). \( \mathcal{S} \) is an unbounded subset of \( \Sigma \).

**Proof.** The proof is the same as [7, lemma 5.2]. \( \square \)
Theorem 5.3. Let $N \geq 1$, $0 < \alpha < \min\{N, 4\}$ and $2 + (2 - \alpha)/N < p < 2^*(\alpha)$. If $\varphi_0 \in S$, then the solutions $\varphi(x, t)$ of Eqs. (2)+(3) exists globally in $t \in [0, +\infty)$. Moreover for any $t \in (0, T)$ we have that
\[
\|\varphi(t)\|_{L^2}^{2\alpha} (\|\nabla \varphi(t)\|_{L^2}^{2} + \|x \varphi(t)\|_{L^2}^{2}) < \frac{A}{B} \|w\|_{L^2}^{2(p-1)}
\]
and
\[
\|\varphi(t)\|^2 \leq \frac{2N(p-2) + 2\alpha}{N(p-2) + \alpha - 2} E(\varphi_0) + \|\varphi_0\|^2_{L^2}.
\]

Proof. The proof is the same as [7, Theorem 5.3]. We write here for the readers’ convenience. For any $t \in [0, T)$, applying Theorem 2.4 to $\varphi(t, x)$ and using the choice of $A$ and $B$, we get that
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\alpha} dxdy \leq C_{VGN} \|\varphi_0\|^2_{L^2} \|\nabla \varphi\|^2_{L^2}.
\]
Denote $a = \int (|\nabla \varphi_0|^2 + |x|^2 |\varphi_0|^2) > 0$. It is deduced from the energy identity and (31) that
\[
\int (|\nabla \varphi|^2 + |x|^2 |\varphi|^2) dx = 2E(\varphi) + \frac{1}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\alpha} dxdy
\]
\[
= 2E(\varphi_0) + \frac{1}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\alpha} dxdy
\]
\[
< a + \frac{1}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\alpha} dxdy
\]
\[
\leq a + \frac{C_{VGN}}{p} \|\varphi_0\|^2_{L^2} \|\nabla \varphi\|^2_{L^2}.
\]
Let
\[
b := \frac{C_{VGN}}{p} \|\varphi_0\|^2_{L^2} = \frac{1}{B} \left( \frac{B}{A} \right)^A \|w\|^2_{L^2} \|\varphi_0\|^2_{L^2},
\]
\[
\theta = A = \frac{N(p-2) + \alpha}{2} > 1 \quad \text{and}
\]
\[
\Phi(t) = \int (|\nabla \varphi(t)|^2 + |x|^2 |\varphi(t)|^2) dx.
\]
Obviously $\Phi(0) = a$. At the same time, we define $f(y) = a - y + by^\theta$. Then (31) implies that
\[
0 < a - y + by^\theta, \quad \text{where} \quad y = \Phi(t).
\]
Denote
\[
ya = (b\theta)^{-\frac{1}{\theta-1}}, \quad \fb = \frac{\theta-1}{\theta} \nya.
\]
Then $\fb < \nya$. By direct computation and the exact value of $C_{VGN}$ we have that
\[
y_\ast = \frac{A}{B} \|w\|_{L^2}^{2(p-1)} \|\varphi_0\|_{L^2}^{-\frac{2\alpha}{p-1}}
\]
and
\[
\fb = \frac{A - 1}{B} \|w\|_{L^2}^{2(p-1)} \|\varphi_0\|_{L^2}^{-\frac{2\alpha}{p-1}}.
\]
Moreover, using \( \|\varphi_0\|_{L^2} \leq V(\|\nabla \varphi_0\|_{L^2}^2 + \|x\varphi_0\|_{L^2}^2) = V(a) \), we know that

\[
\|\varphi_0\|_{L^2} \leq \left( \frac{A - 1}{B} \right)^{\frac{1}{1-2p}} \|w\|_{L^2}^{\frac{p-1}{1-2p}} a^{-\frac{1}{1-2p}},
\]

which implies that

\[
a \leq \frac{A - 1}{B} \|w\|_{L^2}^{\frac{2(p-1)}{1-2p}} \|\varphi_0\|_{L^2}^{-\frac{2p}{1-2p}} \quad \text{(34)}
\]

because of \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \).

Now using (33), (34) and Lemma 5.1, we get that \( \Phi(t) < y_* \) for any \( t \in [0, T] \).

It follows from \( \int |\varphi|^2 \equiv \int |\varphi_0|^2 \) that \( \|\varphi(t)\|_{L^2}^2 \) is bounded from above uniformly with respect to \( t \in [0, T] \). In other words, the solutions of Eqs. (2)-(3) with \( \varphi_0 \) satisfying (30) exist globally in \( t \in [0, +\infty) \).

Since \( \Phi(t) < y_* \) for any \( t \in [0, T] \), we get that

\[
\|\varphi_0\|_{L^2} \leq \left( \frac{2p}{1-2p} \right) \left( \|\nabla \varphi(t)\|_{L^2}^2 + \|x\varphi(t)\|_{L^2}^2 \right) < \frac{A}{B} \|w\|_{L^2}^{\frac{2(p-1)}{1-2p}}.
\]

Next, for the solutions obtained in the above, we give an explicit upper bound on the \( \|\varphi(t)\|_{L^2}^2 \). Firstly we have that

\[
E(\varphi_0) = E(\varphi) = \frac{1}{2} \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) - \frac{1}{2p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x - y|^\alpha} dx dy
\]

\[
\geq \frac{1}{2} \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) - \frac{C_{VGN}}{2p} \|\varphi\|_{L^2}^2 \|\nabla \varphi\|_{L^2}^2
\]

\[
\geq \frac{1}{2} \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) - \frac{C_{VGN}}{2p} \|\varphi\|_{L^2}^2 \left( \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) \right)^A
\]

\[
= \frac{1}{2} \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) \left[ 1 - \frac{1}{p} \left( \frac{AC_{VGN}}{p} \|\varphi_0\|_{L^2}^2 \right)^{-1} \right]
\]

\[
\times \left( \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) \right)^{-1} \left[ 1 - \frac{1}{p} \left( \frac{AC_{VGN}}{p} \|\varphi_0\|_{L^2}^2 \right)^{-1} \right]
\]

Since \( \Phi(t) < y_* \), which is that

\[
\left( \frac{AC_{VGN}}{p} \|\varphi_0\|_{L^2}^2 \right)^{-\frac{1}{p - 1}} \left( \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) \right)^{-1} > 1
\]

by using the exact value of \( C_{VGN} \) obtained in Theorem 2.4. Therefore

\[
\left[ \left( \frac{AC_{VGN}}{p} \|\varphi_0\|_{L^2}^2 \right)^{-\frac{1}{p - 1}} \left( \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) \right)^{-1} \right]^{1 - A} < 1
\]

because of \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \). It follows that

\[
E(\varphi_0) \geq \frac{A - 1}{2A} \int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2),
\]

which yields that

\[
\int ((\nabla \varphi)^2 + |x|^2 |\varphi|^2) dx \leq \frac{2N(p - 2) + 2\alpha}{N(p - 2) + \alpha - 2} E(\varphi_0).
\]
Therefore
\[ \|\varphi(t)\|_{L^2}^2 \leq \frac{2N(p-2) + 2\alpha}{N(p-2) + \alpha - 2} E(\varphi_0) + \|\varphi_0\|_{L^2}^2. \]

The proof is complete. \( \Box \)

**Remark 3.** By Lemma 5.2, we get that Eqs. (2)+(3) possesses global solutions for a large class of initial data whose norm can be as large as we want. On the other hand, from the definition of \( V(\lambda) \) and Theorem 5.3, we know that \( V(\lambda) \rightarrow \|w\|_{L^2}^{2} \) as \( p \rightarrow 2 + (2 - \alpha)/N \). So we obtain the sharp condition for global existence in the case of initial data \( \|\varphi_0\|_{L^2} < \|w\|_{L^2} \), which coincides with Theorem 4.1. In the case of critical nonlinearity \( p = 2 + (2 - \alpha)/N \), the condition (25) is sharp. However, we do not know whether or not the condition (30) is sharp in the case of super critical nonlinearity \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \).

6. **Remarks on the Cauchy problem of Eq. (1).** In this section, we outline some applications of \( C_{V,G,N} \) to the Cauchy problem of Eq. (1).

**Theorem 6.1.** Let \( N \geq 1, 0 < \alpha < \min\{N,4\} \) and \( p = 2 + (2 - \alpha)/N \). If \( \varphi_0 \in W^{1,2}(\mathbb{R}^N) \) and
\[ \|\varphi_0\|_{L^2} < \|w\|_{L^2}, \]
where \( w \) is a minimal action solution of Eq. (13), then Eqs. (1)+(3) has a global solution \( \varphi(x,t) \in C(\mathbb{R}_{+}, W^{1,2}(\mathbb{R})). \)

**Theorem 6.2.** Let \( N \geq 1, 0 < \alpha < \min\{N,4\} \) and \( p = 2 + (2 - \alpha)/N \). If \( \varphi_0 \in W^{1,2}(\mathbb{R}^N) \) with the form
\[ \varphi_0(x) = c\lambda^{N/2} w(\lambda x), \]
where \( \lambda > 0, w \) is a minimal action solution of Eq. (13) and \( c \) is a complex number with \( |c| \geq 1 \), then \( \|\varphi_0\|_{L^2} \geq \|w\|_{L^2} \). Moreover the solution \( \varphi(x,t) \) of Eqs. (1)+(3) must blow up in a finite time.

**Remark 4.** We point out that when \( p = 2 + 2\alpha/N \) and \( w \) is a minimal action solution of Eq. (13), then the following
\[ \varphi(x,t) = (T - t)^{-N/2} e^{-\frac{1}{4}t|x|^2} w\left(\frac{x}{T - t}\right) e^{\frac{\alpha}{2} - \frac{\alpha}{4} t} \]
is a solution of \( i\varphi_t + \Delta \varphi + |\varphi|^p - 2(|x|^{-\alpha} * |\varphi|^p) = 0 \) and \( \varphi \) blows up at finite time. But for the general \( p \), the blow-up derived by self-similarity is still open and we can not solve it at this moment.

Next, if we use the function \( V(\lambda) \) defined as before and set \( S_0 = \{ u \in W^{1,2}(\mathbb{R}^N) : \|u\|_{L^2} \leq V(\|\nabla u\|_{L^2}^2) \} \), then we know from a proof similar to Lemma 5.2 that \( S_0 \) is an unbounded set in \( W^{1,2}(\mathbb{R}^N) \). Moreover, similar to Theorem 5.3, we have the following theorem.

**Theorem 6.3.** Let \( N \geq 1, 0 < \alpha < \min\{N,4\} \) and \( 2 + (2 - \alpha)/N < p < 2^*(\alpha) \). If \( \varphi_0 \in S_0 \), then the solutions \( \varphi(x,t) \) of Eqs. (1)+(3) exists globally in \( t \in [0, +\infty) \). Moreover for any \( t \in (0, T) \) we have that
\[ \|\varphi_0\|_{L^2}^2 \|\nabla \varphi(t)\|_{L^2}^2 < \frac{A}{B} \|w\|_{L^2}^{2p-2} \]
and
\[ \|\nabla \varphi(t)\|_{L^2}^2 \leq \frac{2N(p-2) + 2\alpha}{N(p-2) + \alpha - 2} E(\varphi_0). \]
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