Diffeomorphism type via aperiodicity in Reeb dynamics

Myeonggi Kwon, Kevin Wiegand and Kai Zehmisch

Abstract. We characterise boundary-shaped disc-like neighbourhoods of certain isotropic submanifolds in terms of aperiodicity of Reeb flows. We prove uniqueness of homotopy and diffeomorphism type of such contact manifolds assuming non-existence of short periodic Reeb orbits.

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1. Introduction

In their seminal work, Gromov [1] and Eliashberg [2] observed that foliations by holomorphic curves can be used to prove uniqueness of the diffeomorphism (in fact symplectomorphism) type of minimal symplectic fillings of the standard contact 3-sphere, i.e., all such fillings are diffeomorphic to the 4-ball $D^4$. The method they used, the so-called filling by holomorphic curves method, is obstructed by bubbling off of holomorphic spheres. Related classification results in dimension 4 can be found in [3–8].

On the other hand, Hofer [9] discovered a fundamental property of holomorphic curves in symplectisations; non-compactness properties of holomorphic curves of finite energy are strongly related to the existence of periodic Reeb orbits. Combining the method of filling by holomorphic curves with the theory of finite energy planes Eliashberg–Hofer [10] determined the diffeomorphism (in fact contactomorphism) type of certain contact manifolds with boundary $S^2$: any compact contact manifold with boundary $S^2 = \partial D^3$ is diffeomorphic to $D^3$ provided there exists a contact form that is equal to

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the standard contact form on $D^3$ near the boundary $S^2$ such that the corresponding Reeb vector field does not admit a periodic orbit with period less than $\pi$. A similar characterisation of $D^2 \times S^1$ in terms of Reeb dynamics was obtained by Kegel–Schneider–Zehmisch [11].

In higher dimensions, the diffeomorphism type of symplectically aspherical fillings of the standard contact sphere was determined by Eliashberg–Floer–McDuff [12, Theorem 1.5]: any such filling is diffeomorphic to the ball $D^{2n}$. The proof they used was refined to the so-called \textit{degree method} (see Sect. 3.2 for an explanation) by Barth–Geiges–Zehmisch [13] allowing a much wider class of contact type boundaries, see also [14–16].

The contact theoretic counterpart in higher dimensions was not clear for a while. It was conjectured by Bramham–Hofer [17] that the existence of trapped Reeb orbits on a compact contact manifold, whose boundary neighbourhoods look like neighbourhoods of $S^{2n} = \partial D^{2n+1}$ in $D^{2n+1}$, implies the existence of periodic Reeb orbits. A counterexample to that conjecture was given by Geiges–Röttgen–Zehmisch [18]. It suggests that the diffeomorphism type in higher dimensional contact geometry should be determined via a method not based on non-existence of trapped orbits as done in Eliashberg–Hofer [10].

In fact, using the degree method, Geiges–Zehmisch [19] proved that any compact strict contact manifold that has an aperiodic Reeb flow is diffeomorphic to $D^{2n+1}$ provided that the following condition is satisfied: A neighbourhood of the boundary admits a strict contact embedding into the standard $D^{2n+1}$ mapping the boundary to $S^{2n} = \partial D^{2n+1}$. This was generalised by Barth–Schneider–Zehmisch [20] to situations in which $D^{2n+1}$ is replaced by the disc bundle of $\mathbb{R} \times T^*Q \times \mathbb{C} \times \mathbb{C}^{n-1-d}$ whenever $n-1 \geq d$.

The aim of this work is to replace the torus $T^d$ by more general $d$-dimensional manifolds, see Theorem 2.1 below. Again the argument will be based on the construction of a proper degree 1 evaluation map on the moduli space of 1-marked holomorphic discs with varying Lagrangian boundary conditions. The restriction to $T^d$ in [20] was caused by the choice of the boundary conditions set up for the holomorphic discs. This led to trivialising the cotangent bundle of $T^d$ in a Stein holomorphic fashion. To replace $D^{2n+1}$ by the disc bundle of $\mathbb{R} \times T^*Q \times \mathbb{C} \times \mathbb{C}^{n-1-d}$ for a wider class of manifolds $Q$ we choose different boundary conditions for the holomorphic discs. Instead of taking a foliation of $T^*Q$ by sections we consider the foliation $T^*Q$ given by the cotangent fibres. This will result in a more advanced analysis for the holomorphic discs. The essential point here will be a target rescaling argument in Sect. 7, which was invented by Bae–Wiegand–Zehmisch [21] in the context of virtually contact structures, to ensure $C^0$-bounds on holomorphic discs in the situation of general manifolds $Q$. Furthermore, to obtain $C^0$-bounds of holomorphic discs along their boundaries in $T^*Q$-direction, we develop an integrated maximum principle in Sects. 5 and 6.5.
2. Aperiodicity and boundary shape

Strict contact manifolds \((M, \alpha)\) are naturally equipped with a nowhere vanishing vector field, namely the Reeb vector field of \(\alpha\). Assuming \(\alpha\) to be aperiodic, i.e., assuming that the Reeb vector field does not admit any periodic solution, the diffeomorphism type of \(M\) can be determined in many situations. Here we are interested in comparing compact manifolds with boundary \(M\) with neighbourhoods of isotropic submanifolds of the sort \(D(T^*Q \oplus \mathbb{R}^{2n+1-2d})\). This requires boundary conditions for the Reeb vector field as we will explain in the following:

2.1. A model

Let \(Q\) be a closed, connected Riemannian manifold of dimension \(d\) and let \(n \in \mathbb{N}\) such that \(n-1 \geq d\). Define a strict contact manifold \((C, \alpha_0)\) by setting

\[
C := \mathbb{R} \times T^*Q \times \mathbb{C} \times \mathbb{C}^{n-1-d}
\]

and

\[
\alpha_0 := db + \lambda + \frac{1}{2}(x_0dy_0 - y_0dx_0) - \sum_{j=1}^{n-1-d} y_j dx_j,
\]

where \(b \in \mathbb{R}\), \(\lambda\) is the Liouville 1-form of \(T^*Q\), \(x_0 + iy_0\) and \(x_j + iy_j\) are coordinates on \(\mathbb{C}\) and \(\mathbb{C}^{n-1-d}\), resp. Throughout the text, we will use vector notation \(x\) and \(y\) for the coordinate tuples \((x_1, \ldots, x_{n-1-d})\) and \((y_1, \ldots, y_{n-1-d})\), resp., so that we can abbreviate

\[
-ydx = - \sum_{j=1}^{n-1-d} y_j dx_j.
\]

The Reeb vector field of \(\alpha\) is given by \(\partial_b\), which is tangent to the real lines \(\mathbb{R} \times \{\ast\}\).

By [22, Theorem 6.2.2] \((C, \alpha_0)\) is the model neighbourhood of an isotropic submanifold \(Q\) of a strict contact manifold provided that \(Q\) has trivial symplectic normal bundle and the dimension \(d\) of \(Q\) is smaller than \(n\). Observe, that \((C, \alpha_0)\) is the contactisation of the Liouville manifold

\[
\left(T^*Q \times \mathbb{C} \times \mathbb{C}^{n-1-d}, \lambda + \frac{1}{2}(x_0dy_0 - y_0dx_0) - ydx\right).
\]

The statements about the model neighbourhood situation and contactisation of course hold in the critical case \(d = n\) also. Simply ignore the Euclidean factors in the formulations.

2.2. Fibrewise shaped

The space \(C\) itself is the total space of the stabilised cotangent bundle \(T^*Q \oplus \mathbb{R}^{2n+1-2d}\). Let \(S \subset C\) be a hypersurface diffeomorphic to the unit sphere bundle \(S(T^*Q \oplus \mathbb{R}^{2n+1-2d})\) such that
(1) $S$ intersects each fibre transversely in a sphere

$$S_q := S \cap (T^*_q Q \oplus \mathbb{R}^{2n+1-2d}), \quad q \in Q,$$

of dimension $2n - d$, and

(2) each $S_q$ intersects the flow lines of $\partial_b$ in at most two points. We require transverse intersections if such a flow line intersects $S_q$ in two points. Points of tangency, i.e. points that correspond to single intersections, form a submanifold diffeomorphic to a $(2n - d - 1)$-sphere.

In view of condition (1), we remark that the hypersurface $S$ bounds a bounded domain $D$ inside $C$, whose closure is diffeomorphic to the closed unit disc bundle $D(T^*Q \oplus \mathbb{R}^{2n+1-2d})$. Condition (2) will play an important role in Sect. 3.1. We call $S$ a shape.

2.3. Standard near the boundary

Let $(M, \alpha)$ be a strict contact manifold of dimension $2n + 1$ that is standard near the boundary, i.e.

(1) connected, compact with boundary $\partial M$ diffeomorphic to

$$\partial M \cong S(T^*Q \oplus \mathbb{R}^{2n+1-2d}),$$

(2) such that there exist an open collar neighbourhood $U \subset M$ of $\partial M$ and an embedding $\varphi : (U, \partial U = \partial M) \to (D, S)$ such that $\varphi^* \alpha_0 = \alpha$ on $U$.

If $\varphi$ is given we will call $S$ the shape of $M$.

To quantify aperiodicity of $(M, \alpha)$ we denote by $\inf_0(\alpha) > 0$ the minimal action of all contractible closed Reeb orbits w.r.t. $\alpha$. By Darboux’s theorem, $\inf_0(\alpha)$ is indeed positive. For aperiodic $\alpha$, we set $\inf_0(\alpha)$ to be $\infty$.

A second ingredient for quantisation comes with the subset

$$Z := \mathbb{R} \times T^*Q \times D \times C^{n-1-d}$$

of $C$ denoting the closed unit disc in $\mathbb{C}$ by $D$. We may assume that $S \subset \text{Int } Z$ by scaling radially via $(t^2b, t^2w, tz_0, tz)$, $t \in (0, 1)$, if necessary. The contact form $\alpha$ on $M$ will be replaced by $t^2\alpha$ accordingly.

2.4. Main theorem

We compare the homology, homotopy and diffeomorphism type of $M$ with the one of $D(T^*Q \oplus \mathbb{R}^{2n+1-2d})$. This will be done in terms of embeddings

$$D(T^*Q \oplus \mathbb{R}^{2n+1-2d}) \to M$$

determined by a small neighbourhood of a section $Q \to S$ as constructed, e.g. at the beginning of Sect. 9. We denote the image of such an embedding by

$$M_0 := D(T^*Q \oplus \mathbb{R}^{2n+1-2d}).$$

**Theorem 2.1.** Let $Q$ be an oriented, closed, connected Riemannian manifold of dimension $d$. Let $n \in \mathbb{N}$ such that $n - 1 \geq d$. Let $(M, \alpha)$ be a strict contact manifold that is standard near the boundary as described in Sect. 2.3. Assume that the shape $S \cong \partial M$ of $M$ is contained in the interior of $(Z, \alpha_0)$. If $\inf_0(\alpha) \geq \pi$, then the following is true:
(i) Any embedding $Q \to M$ given by a section $Q \to S$ induces isomorphisms of homology and surjections of fundamental groups. If in addition $\pi_1 Q$ is abelian, then the surjections are injective.

(ii) Assume that $\pi_1 Q$ is abelian and that at least one of the following conditions is satisfied:
   
   (a) $\pi_1 Q$ is finite.
   
   (b) $Q$ is aspherical.
   
   (c) $Q$ is simple and $S \to Q$ a trivial sphere bundle, or, more generally, $S$ is a simple space.

   Then, $M$ is homotopy equivalent to $M_0$.

(iii) If in addition to the assumptions in (ii) (including choices of one of the conditions (a)–(c)) we have that $2n + 1 \geq 7$ and that the Whitehead group of $\pi_1 Q$ is trivial, then $M$ is diffeomorphic to $M_0$.

2.5. Comments on Theorem 2.1

In view of the contact connected sum, the bound $\pi$ in the theorem is optimal, cf. [19, Remark 1.3.(1)]. The shape boundary condition can be isotoped to a round shape through shaped hypersurfaces. Hence, we recover the results from [19,20] and obtain independence of the choice of metric.

The orientation of $Q$ will not be used in the compactness argument below. But will be needed for an orientation of the moduli space. Without orientation we only can talk about the mod-2 degree of the evaluation map. Hence, if $Q$ is not orientable, only part (i) of the theorem remains true replacing homology by homology with $\mathbb{Z}_2$-coefficients.

Similarly, the boundary of $M$ is necessarily connected, cf. [19, Remark 1.3.(4)]. Indeed, suppose $\partial M$ has several components that have individually a shape embedding into potentially different stabilised cotangent bundles. Here, different $Q$s with varying dimensions are allowed. $M$ itself satisfies the remaining stated properties from Theorem 2.1. In this situation one can set up the moduli space of holomorphic discs with respect to one distinguished boundary component; the other components will come with the maximum principle for holomorphic curves. In other words, the holomorphic disc analysis will be un effected and the evaluation map on the moduli space will be of degree one. This contradicts the fact that no holomorphic disc can exceed one of the additional boundary components due to the maximum principle.

Example 2.2. In view of the Hadamard–Cartan and the Farrell–Jones theorems, the assumptions of Theorem 2.1 part (b) in (ii) and (iii) are satisfied for all Riemannian manifolds $Q$ with abelian fundamental group and non-positive sectional curvature. Hence, we recover $Q = T^d$ from [20].

Example 2.3. A particular class of manifolds $Q$ that satisfy the assumptions of Theorem 2.1 part (c) in (ii) and (iii) are products of unitary groups and spheres of any dimensions. Indeed, such $Q$ always have stably trivial tangent bundle, are simple with fundamental group free abelian so that in particular the Whitehead group of those is trivial.
Remark 2.4. If we know more about the handle body structure of $M$ conditions on the topology of $Q$ can be relaxed. For example if $M$ has the homotopy type of a CW complex of codimension 2 so that the inclusion $\partial M \subset M$ is $\pi_1$-injective the assumption $\pi_1 Q$ abelian in Theorem 2.1 can be dropped everywhere.

If $M$ admits a handle body structure with all handles of index at most $\ell$ and if $d+\max(d, \ell) \leq 2n-1$, then $M$ and $M_0$ are homotopy equivalent without any further conditions. This follows with the argument from [13, Theorem 7.2] using the diagram in Sect. 9.2. In particular, the CW-dimension of $M$ must be equal to $d$. In fact, one can conclude with the diffeomorphism type as in [13, Theorem 9.4], cf. [13, Example 9.5].

3. The degree method

We will explain the main idea of the proof of Theorem 2.1, which will be given in Sects. 4–9.

3.1. Completion via gluing

Assuming $S \subset \mathrm{Int} Z$, we define smooth manifolds

$$\hat{C} := (C \setminus \mathrm{Int} D) \cup_\varphi M, \quad \hat{Z} := (Z \setminus \mathrm{Int} D) \cup_\varphi M$$

by gluing via $\varphi$ and equip both with the contact form

$$\hat{\alpha} := \alpha_0 \cup_\varphi \alpha$$

that coincides with $\alpha$ on $M$ and with $\alpha_0$ on $C \setminus \mathrm{Int} D$. Because of the contact embedding $\varphi$ of $U \supset \partial M$ into $(Z, \alpha_0)$ this is well defined. According to the second shape condition in Sect. 2.2, the gluing does not create additional periodic Reeb orbits inside $(\hat{C}, \hat{\alpha})$ so that $\inf_0(\alpha)$ and $\inf_0(\hat{\alpha})$ coincide.

3.2. Filling by holomorphic discs

To prove Theorem 2.1, we will argue as in [19,20]: The Liouville manifold

$$\left( T^* Q \times \mathbb{D} \times \mathbb{C}^{n-1-d}, \lambda + \frac{1}{2} \left( x_0 dy_0 - y_0 dx_0 \right) - yd\mathbf{x} \right)$$

is foliated by holomorphic discs $\{ w \} \times \mathbb{D} \times \{ s + it \}$. Using the Niederkrüger transformation from Sect. 4.3 these discs can be lifted to holomorphic discs in the symplectisation of the contactisation $(Z, \alpha_0)$ and are called standard discs. After gluing some of the standard discs will survive, namely those which correspond to the end of $(\hat{Z}, \hat{\alpha})$ in the symplectisation $(W, \omega)$ of $(\hat{Z}, \hat{\alpha})$. We will study the corresponding moduli space $W$ of holomorphic discs

$$u = (a, f) : \mathbb{D} \longrightarrow W$$

subject to varying Lagrangian boundary conditions, which will differ substantially from those used in [19,20]. This requires a different argument to obtain $C^0$-bounds for holomorphic discs, which at the end allows a wider class of base manifolds $Q$. 
It will turn out that the evaluation map
\[ \text{ev}: \mathcal{W} \times \mathbb{D} \longrightarrow \hat{Z} \]
\[ ((a, f), z) \mapsto f(z) \]
either is proper of degree one or there will be breaking off of finite energy planes. The first alternative allows conclusions on the diffeomorphism type of \( M \) with the \( s \)-cobordism theorem as in [13]. The second results in the existence of a short contractible periodic Reeb orbit of \( \alpha \) on \( M \) by a result of Hofer [9]. Short here means that the action of the Reeb orbit is bounded by the area of \( \mathbb{D} \).

The condition \( \inf_0(\alpha) \geq \pi \) will exclude breaking of holomorphic discs along periodic Reeb orbits of action less than \( \pi \). But in fact, under the assumptions of Theorem 2.1 the shape \( S \) of \( M \) actually is contained in \( \mathbb{R} \times T^*Q \times B_r(0) \times \mathbb{C}^{n-1-d} \) for \( r \in (0, 1) \). Working out the proof of Theorem 2.1 with that slightly smaller radius \( r \) we will see that requiring non-existence of short periodic Reeb orbits with period bounded by \( \pi r^2 \) will be sufficient. In other words, we can assume that \( \inf_0(\alpha) > \pi r^2 \) to prove properness of the evaluation map \( \text{ev} \). To simplify notation, we will assume \( r = 1 \), i.e. from now on we assume \( \inf_0(\alpha) > \pi \).

4. Standard holomorphic discs

In this section, we construct standard holomorphic discs. We will follow [19, Section 2] and [20, Section 2] adding adjustments to the current situation.

4.1. The contactisation

We consider the Liouville manifold
\[ (V, \lambda_V) := \left( T^*Q \times \mathbb{D} \times \mathbb{C}^{n-1-d}, \lambda + \frac{1}{2} \left( x_0 dy_0 - y_0 dx_0 \right) - y dx \right), \]
whose contactisation \( (\mathbb{R} \times V, db + \lambda_V) \) is \( (Z, \alpha_0) \). The induced contact structure \( \xi_0 = \ker \alpha_0 \) on \( Z \) is spanned by tangent vectors of the form \( v - \lambda_V(v) \partial_b, \)
\( v \in TV \).

4.2. Liouville manifold and potential

Denote by \( J_{T^*Q} \) the almost complex structure on \( T^*Q \) that is compatible with \( d\lambda \) and satisfies \( \lambda = -dF \circ J_{T^*Q} \). Here \( F \) is a strictly plurisubharmonic potential in the sense of [23, Section 3.1] that coincides with the kinetic energy function near the zero section of \( T^*Q \) and interpolates to the length function on the complement of a certain disc bundle in \( T^*Q \), see [24, Section 3.1]. In Sect. 5, we will present a construction of \( (F, J_{T^*Q}) \).

Define an almost complex structure on the Liouville manifold \( (V, \lambda_V) \) by setting
\[ J_V := J_{T^*Q} \oplus i \oplus i. \]
\( J_V \) is compatible with the symplectic form \( d\lambda_V \) and satisfies \( \lambda_V = -d\psi \circ J_V \), where \( \psi \) is the strictly plurisubharmonic potential
\[ \psi(w, z_0, z) := F(w) + \frac{1}{4}|z_0|^2 + \frac{1}{2}|y|^2 \]
denoting by $w \in T^*Q$ a co-vector of $Q$, $z_0 \in \mathbb{D}$ and using complex coordinates $z_j = x_j + iy_j$, $j = 1, \ldots, n-1-d$ on $\mathbb{C}^{n-1-d}$. Again the tuple $(z_1, \ldots, z_{n-1-d})$ is abbreviated by $z$ so that $\frac{1}{2}|y|^2$ reads as

$$\frac{1}{2} \sum_{j=1}^{n-1-d} y_j^2.$$ 

In particular, $(V, J_V)$ is foliated by holomorphic discs $\{w\} \times \mathbb{D} \times \{s + it\}$.

### 4.3. The symplectisation

Let $\tau \equiv \tau(a)$ be a strictly increasing smooth function $\mathbb{R} \to (0, \infty)$. We consider the symplectisation $(\mathbb{R} \times Z, d(\tau \alpha_0))$ of $(Z, \alpha_0)$. Define a compatible, translation invariant almost complex structure $J$ that preserves the contact hyperplanes $\xi_0$ on all slices $\{a\} \times Z$ by requiring that $J(\partial_a) = \partial_b$ and that

$$J(v - \lambda_V(v)\partial_b) = J_V v - \lambda_V(J_V v)\partial_b$$

for all $v \in TV$. The Niederkrüger map is the biholomorphism

$$\Phi: (\mathbb{R} \times \mathbb{R} \times V, J) \longrightarrow (\mathbb{C} \times V, i \oplus J_V)$$

$$\begin{array}{cccc}
(a, b, z) & \longrightarrow & (a - \psi(z) + ib, z)
\end{array}$$

recalling that $Z = \mathbb{R} \times V$, see [25, Proposition 5] and [19, Proposition 2.1].

### 4.4. The Niederkrüger transform

Using the inverse of $\Phi$, we lift the holomorphic discs

$$\{a + ib\} \times \{w\} \times \mathbb{D} \times \{s + it\}$$

from $(\mathbb{C} \times V, i \oplus J_V)$ to the symplectisation $(\mathbb{R} \times \mathbb{R} \times V, J)$ of $(Z, \alpha_0)$. For fixed $b \in \mathbb{R}$, $w \in T^*Q$, and $s, t \in \mathbb{R}^{n-1-d}$, the resulting standard holomorphic discs

$$\mathbb{D} \longrightarrow \mathbb{R} \times \mathbb{R} \times T^*Q \times \mathbb{D} \times \mathbb{C}^{n-1-d}$$

are parametrised by

$$u_{s,b}^t(w, z) = \left(\frac{1}{4}(|z|^2 - 1), b, w, z, s + it\right),$$

cf. [19, Section 2.2].

To set boundary conditions for the standard discs we define a $(n-1)$-dimensional family of cylinders

$$L_q^t := \{0\} \times \mathbb{R} \times T_q^*Q \times \partial \mathbb{D} \times \mathbb{R}^{n-1-d} \times \{t\},$$

where $t \in \mathbb{R}^{n-1-d}$ and $q \in Q$ are the parameters. Observe, that the $L_q^t$ foliate $\{0\} \times \partial Z$. Furthermore the restriction of $d(\tau \alpha_0)$ to the tangent bundle of $\{0\} \times Z$ equals $\tau(0)d\alpha_0$, which is a positive multiple of

$$d\lambda + dx_0 \wedge dy_0 + dx \wedge dy.$$

Therefore, $L_q^t$ is a Lagrangian cylinder because the dimension of $L_q^t$ is $n + 1$. 

4.5. Class independence

Preparing the definition of the moduli space $\mathcal{W}$ we consider the space $\mathbb{R} \times T_q^*Q \times \mathbb{R}^{n-1-d}$ of tuples $(b, w, s)$. Assuming $n \geq 2$ this space is at least 2-dimensional, so that the complement of any ball in $\mathbb{R} \times T_q^*Q \times \mathbb{R}^{n-1-d}$ is path-connected. Therefore, we find $R > 0$ such that

1. the shape $S$ is contained in the closed disc bundle $D_R(T^*Q \oplus \mathbb{R}^{2n+1-2d})$
2. all standard discs $u_{s,b}^{t,w}$ of level $(q, t), w \in T_q^*Q$, that are contained in $\mathbb{R} \times (Z \setminus D_R(T^*Q \oplus \mathbb{R}^{2n+1-2d}))$

are homotopic therein relative $L_q^b$ via a homotopy inside $\{0\} \times \mathbb{R} \times T_q^*Q \times D \times \mathbb{R}^{n-1-d} \times \{t\}$.

5. Symplectic potentials on cotangent bundles

We prepare the proof of geometric bounds on holomorphic discs that belong to the moduli space $\mathcal{W}$. The aim of this section is to construct an almost complex structure on $T^*Q$.

The almost complex structure on $T^*Q$ that belongs to the Levi-Civita connection of $Q$ is the one that is induced by the kinetic energy function. The one coming from symplectising the unit cotangent bundle in contrast belongs to the length functional and does not extend over the zero section. Here we want to interpolate the two in order to obtain $C^0$-bounds on holomorphic curves in the complement of the unit codisc bundle that we after all can identify with the positive symplectisation also holomorphically.

5.1. Dual connection

We denote the covariant derivative of the Levi-Civita connection of $Q$ by $\nabla$. The corresponding covariant derivative $\nabla^*$ of the dual connection is defined via chain rule by

$$(\nabla^* \beta)(X, Y) := (\nabla^*_X \beta)(Y) := X(\beta(Y)) - \beta(\nabla_X Y)$$

for 1-forms $\beta$ and vector fields $X, Y$ on $Q$, cf. [21, Section 4]. Denoting the Christoffel symbols of $\nabla$ by $\Gamma^k_{ij}$ the Christoffel symbols $(\Gamma^*)^k_{ij}$ of $\nabla^*$ can be expressed by $(\Gamma^*)^k_{ij} = -\Gamma^l_{ik}$. The connection map of the dual connection $K: TT^*Q \to T^*Q$ and the tangent functor $T$ are related via $K \circ T = \nabla^*$ and defines a splitting of $TT^*Q = \mathcal{H} \oplus \mathcal{V}$ into horizontal

$$\mathcal{H} := \ker (K: TT^*Q \to T^*Q)$$

and vertical distribution

$$\mathcal{V} = \ker (T \tau: TT^*Q \to TQ),$$
where $T\tau$ is the linearisation of the cotangent map $\tau: T^*Q \to Q$. Observe that $T\tau$ defines a bundle isomorphism from $\mathcal{H}$ onto $\tau^*TQ$ and that $\mathcal{V}$ can be identified with $\tau^*T^*Q$ canonically.

5.2. Orthogonal splitting
Denoting the metric of $Q$ by $g$, contraction defines a bundle isomorphism

$$G: TQ \longrightarrow T^*Q$$

$$v \longmapsto i_v g.$$ 

The dual metric $g^\flat$ is defined by

$$g^\flat(\alpha, \beta) = g(G^{-1}(\alpha), G^{-1}(\beta))$$

for co-vectors $\alpha, \beta \in T^*Q$ on $Q$, so that the dual norm $\alpha \mapsto |\alpha|_\flat$ defines the length function on $T^*Q$. The kinetic energy function reads as

$$k(\beta) = \frac{1}{2} |\beta|_\flat^2.$$ 

For a smooth, strictly increasing function $\chi: \mathbb{R} \to \mathbb{R}$ with $\chi(0) = 0$ we define

$$F = \chi \circ k: T^*Q \to [0, \infty).$$

This leads to a Riemannian metric $h$ on $T^*Q$ defined by

$$h(v \oplus \alpha, w \oplus \beta) := \frac{1}{\chi' \circ k} \cdot g(T\tau(v), T\tau(w)) + (\chi' \circ k) \cdot g^\flat(\alpha, \beta),$$

where $v, w \in \mathcal{H}$ and $\alpha, \beta \in \mathcal{V}$. The metric $h$ turns $TT^*Q = \mathcal{H} \oplus \mathcal{V}$ into an orthogonal splitting.

5.3. Taming structure
The Liouville form $\lambda$ on $T^*Q$ is given by $\lambda_w = w \circ T\tau$ for $w \in T^*Q$ and defines a symplectic form via $d\lambda$. Observe that for $v, w \in \mathcal{H}$ and $\alpha, \beta \in \mathcal{V}$

$$\lambda_v(v \oplus \alpha) = w(T\tau(v))$$

and

$$d\lambda(v \oplus \alpha, w \oplus \beta) = \alpha(T\tau(w)) - \beta(T\tau(v)).$$

In view of the splitting $TT^*Q = \mathcal{H} \oplus \mathcal{V}$, we define the almost complex structure $J_{T^*Q}$ by setting

$$J_{T^*Q}(v \oplus \alpha) := (\chi' \circ k) \cdot G^{-1}(\alpha) \oplus \frac{-1}{\chi' \circ k} \cdot G(v)$$

for $v \in \mathcal{H}$ and $\alpha \in \mathcal{V}$. This yields

$$h = d\lambda(\cdot, J_{T^*Q} \cdot),$$

i.e. $J_{T^*Q}$ is compatible with the symplectic form $d\lambda$. Non-degeneracy of the metric $h$ and the symplectic form $d\lambda$ shows that the almost complex structure $J_{T^*Q}$ is uniquely determined.
5.4. Potentials

We claim that the function $F$ is a symplectic potential on the tame symplectic manifold $(T^*Q, d\lambda, J_{T^*Q})$ in the sense that

$$\lambda = -dF \circ J_{T^*Q}.$$ 

Indeed, in local $(q, p)$-coordinates on $T^*Q$ induced by Riemann coordinates on $Q$ about $q \equiv 0$ we have

$$H(0, p) = \{ (0, p, \dot{q}, 0) \mid \dot{q} \in \mathbb{R}^d \}, \quad \mathcal{V}(0, p) = \{ (0, p, 0, \dot{p}) \mid \dot{p} \in \mathbb{R}^d \},$$

as well as

$$\lambda(0, p) = p \, dq, \quad d\lambda(0, p) = dp \wedge dq,$$

and

$$(J_{T^*Q}(0, p)) = \begin{pmatrix} 0 & \chi'\left(\frac{1}{2}p^j p^i\right)^{-1} \\ -\left(\chi'\left(\frac{1}{2}p^j p^i\right)\right)^{-1} & 0 \end{pmatrix}$$

using block matrix notation and writing e.g. $\chi'\left(\frac{1}{2}p^j p^i\right)$ instead of $\chi'\left(\frac{1}{2}p^j p^i\right) \mathbf{1}$. Because of

$$dF|_{(0, p)} = \chi'\left(\frac{1}{2}p^j p^i\right) \cdot p^j dp^i$$

we get, therefore,

$$-dF \circ J_{T^*Q}|_{(0, p)} = p^j dq^j|_{(0, p)} = \lambda(0, p)$$

as claimed.

5.5. Interpolating geodesic and normalised geodesic flow

We choose the strictly increasing function $\chi : \mathbb{R} \to \mathbb{R}$ from Sect. 5.2 to satisfy $\chi(t) = t$ for $t \leq 1/4$ and $\chi(t) = \sqrt{2}t$ for $t \geq 1/2$ to interpolate the kinetic energy with the length function.

We would like to understand the interpolation given by $\chi$ in terms of symplectisation. For that, we consider the diffeomorphism

$$\Phi : (\mathbb{R} \times ST^*Q, e^{a} \alpha) \longrightarrow (T^*Q \setminus Q, \lambda) \quad (a, w) \longmapsto e^{a} w$$

of Liouville manifolds, where $\alpha := \lambda|_{ST^*Q}$. Observe, that

$$\Phi^*F(a, w) = \chi \circ k(e^a w) = \chi\left(\frac{1}{2} e^{2a}\right)$$

equals $e^a$ for $a \geq 0$. Since $\Phi$ is a symplectomorphism $I := \Phi^*J_{T^*Q}$ is a compatible almost complex structure on the symplectisation $(\mathbb{R} \times ST^*Q, d(e^a \alpha))$. Moreover, on the positive part $\{a > 0\}$ of the symplectisation, where $\Phi^*F = e^a$, we obtain $\Phi^*dF = e^a da$. Therefore,

$$e^a \alpha = \Phi^* \lambda = \Phi^* \left( -dF \circ J_{T^*Q} \right) = -e^a da \circ I,$$

which implies

$$\alpha = -da \circ I.$$ 

Consequently, $I$ preserves the contact structure $\xi = \ker \alpha \cap \ker(da)$ induced by $\alpha$ on all slices. Moreover, denoting the Reeb vector field of $\alpha$ by $R$ we get

$$1 = \alpha(R) = -da(IR).$$
Hence,
\[ I \partial_a = R. \]

We remark that \( \partial_a \) is the Liouville vector field of \((\mathbb{R} \times ST^*Q, e^\alpha)\). Therefore, \( \Phi_* \partial_a = Y \), where \( Y \) is the Liouville vector field on \( T^*Q \) determined by \( \lambda = i_Y d\lambda \).

We claim that the almost complex structure \( I \) is invariant under translation in \( \mathbb{R} \)-direction along \( \mathbb{R}^+ \times ST^*Q \). Indeed, using local Riemann coordinates as in Sect. 5.4 the restriction of \( J_{T^*Q} \) to \( \{|p|_\beta > 1\} \) is given by
\[
(J_{T^*Q})_{(0,p)} = \begin{pmatrix} 0 & 1/p \\ -|p| & 0 \end{pmatrix}
\]
abbreviating, e.g. \( |p| = |p|_\beta \mathbb{1} \). As the flow of \( Y \) scales by \( e^t \) in \( p \)-direction the pullback of \( J_{T^*Q} \) with respect to the flow of \( Y = p \partial_p \) at \((0,p)\) equals
\[
\begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & e^t/p \\ -e^t|p| & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} 0 & 1/p \\ -|p| & 0 \end{pmatrix}.
\]
This shows that the Lie derivative \( L_Y J_{T^*Q} \) vanishes. Hence, \( \Phi_* \partial_a = Y \) implies \( L_{\partial_a} I = 0 \), i.e. \( I_{(a,p)} = I_{(a+t,p)} \) for all \( a, a + t > 0 \).

In other words, \( I \) is a compatible almost complex structure on the positive part of the symplectisation \((\mathbb{R}^+ \times ST^*Q, d(e^\alpha))\). \( I \) is translation invariant, preserves the contact structure \( \xi = \ker \alpha \), and sends the Liouville vector field \( \partial_a \) to the Reeb vector field \( R \) of \( \alpha \).

### 6. A boundary value problem

Following [19, Section 3] and [20, Section 3] we introduce the moduli space \( W \) of holomorphic discs to understand the topology of the manifold \( M \). We consider the glued strict contact manifold \((\hat{Z}, \hat{\alpha})\) introduced in Sect. 3.1 and form its symplectisation \((W, \omega)\), i.e. we set
\[
(W, \omega) := \left( \mathbb{R} \times \hat{Z}, d(\tau \hat{\alpha}) \right)
\]
for a positive, strictly increasing smooth function \( \tau \) defined on \( \mathbb{R} \) such that \( \tau(a) = e^a \) for all \( a \geq 0 \). Compared to the constructions in [19,20], there will be a substantial difference in setting up the boundary conditions for the holomorphic discs.

#### 6.1. An almost complex structure

We denote by \( \hat{\xi} \) the contact structure defined by \( \hat{\alpha} \). On the symplectisation \((W, \omega)\) we choose a compatible almost complex structure \( J \) that is \( \mathbb{R} \)-invariant, sends \( \partial_a \) to the Reeb vector field of \( \hat{\alpha} \), and restricts to a complex bundle structure on \(( \hat{\xi}, d\hat{\alpha})\).

To incorporate standard holomorphic discs we define the box \( B \) by
\[
B := [-b_0, b_0] \times DR T^*Q \times D^2_r \times D^2_{R^2}.
\]
where $0 < b_0, r \in (0, 1), 1 \leq R$ are real numbers chosen such that $S \subset \text{Int} B$. Here, $D^2_{\rho} \subset \mathbb{C}^2$ denotes the closed 2-disc of radius $\rho$ and $D_{\rho}T^*Q$ is the closed $\rho$-disc subbundle of $T^*Q$. Set

$$\hat{B} := (B \setminus \text{Int} D) \cup \varphi M.$$  

We require the almost complex structure $J$ to be the one defined in Sect. 4 on the complement of $\mathbb{R} \times \text{Int}(\hat{B})$ in $\mathbb{R} \times \hat{Z}$. On $\mathbb{R} \times \text{Int}(\hat{B})$ we will choose $J$ generically, see Sect. 8.

6.2. The moduli space

The moduli space $\mathcal{W}$ is the set of all holomorphic discs

$$u = (a, f): \mathbb{D} \longrightarrow (W, J)$$

that satisfy the following conditions:

(w_1) There exists a level $(q, t) \in Q \times \mathbb{R}^{n-1-d}$ such that

$$u(\partial \mathbb{D}) \subset L^t_q.$$  

(w_2) There exist $b \in \mathbb{R}, w \in T^*_qQ, s \in \mathbb{R}^{n-1-d}$ such that

$$[u] = [u^{t,w}_{s,b}] \in H_2(W, L^t_q),$$

where $(q, t)$ is the level of $u$.

(w_3) $u$ maps the marked points 1, i, −1 to the characteristic leaves $L^t_q \cap \{z_0 = 1\}$, $L^t_q \cap \{z_0 = i\}$, and $L^t_q \cap \{z_0 = -1\}$, resp., i.e. for $k = 0, 1, 2$ we have

$$f(i^k) \in \mathbb{R} \times T^*_qQ \times \{i^k\} \times \mathbb{R}^{n-1-d} \times \{t\}.$$  

The parameters $b, w, s$ in condition (w_2) are assumed to be sufficiently large so that the standard disc $u^{t,w}_{s,b}$ defines a holomorphic disc in $(W, J)$. With Sect. 4.5 the relative homology class of $u^{t,w}_{s,b}$ is independent of the choice of $b, w, s$.

6.3. Uniform energy bounds

The symplectic energy $\int_{\mathbb{D}} u^*\omega$ is bounded by $\pi$ for all $u = (a, f) \in \mathcal{W}$. Indeed, by Stokes theorem, the symplectic energy of $u$ is equal to the action $\int_{\partial \mathbb{D}} f^*\tilde{\alpha}$ of the boundary circle. This also holds for any standard disc homologous to $u$. The claim follows as the symplectic energy is the same for all holomorphic discs of the same level and as the action of the boundary circle of standard discs equals $\pi$.

By a similar argument, we obtain that the symplectic energy of any non-constant holomorphic disc that takes boundary values in some Lagrangian cylinder $L^t_q$ is a positive multiple of $\pi$.  

6.4. Maximum principle
Let \( u = (a, f) \in \mathcal{W} \) be a holomorphic disc of level \((q, t)\). By [19, Lemma 3.6.(i)], the function \( a \) is subharmonic and, hence, \( a < 0 \) on \( \partial \mathbb{D} \).

The set \( G := f^{-1}(\mathring{Z} \setminus \mathring{B}) \) is an open subset of \( \mathbb{D} \) that contains a neighbourhood of \( \partial \mathbb{D} \) in \( \mathbb{D} \). Restricting \( f \) to \( G \), we can write
\[
 f = (b, w, h_0, h)
\]
w.r.t. coordinate functions on \( \mathbb{R} \times T^*Q \times \mathbb{D} \times \mathbb{C}^{n-1-d} \). As the Niederkrüger map is biholomorphic, the function \( b \) is harmonic and the maps \( w, h_0, h \) are holomorphic, see Sect. 4.3.

In particular, if \( G = \mathbb{D} \), then \( u \) is one of the discs \( u_{s,b} \). This follows as in [19, Lemma 3.7]. Simply use the fact that a holomorphic map \( w : \mathbb{D} \to T^*Q \) with boundary on \( T^*_q Q \) is constant by Stokes theorem and \( w^* \lambda = 0 \) on \( \partial \mathbb{D} \).

Motivated by this, we introduce the notion of standard holomorphic discs to the glued manifold \( \mathcal{W} \):

**Definition 6.1.** A holomorphic disc \( u = (a, f) \in \mathcal{W} \) is called a **standard disc** if \( f(\mathring{D}) \subset \mathring{Z} \setminus \mathrm{Int} \mathring{B} \). Holomorphic discs \( u = (a, f) \in \mathcal{W} \) with \( f(\mathring{D}) \cap \mathrm{Int} \mathring{B} \neq \emptyset \) are called **non-standard**.

Applying the strong maximum principle and the boundary lemma by E. Hopf to \( h_0 \) we obtain as in [19, Lemma 3.6.(ii)] and on [19, p. 669 and p. 671]:
\[
\begin{align*}
(1) & \ f(\mathrm{Int} \mathbb{D}) \subset \mathrm{Int} \mathring{Z} . \\
(2) & \ u|_{\partial \mathbb{D}} \text{ is an embedding.}
\end{align*}
\]

**Remark 6.2.** In the situation, \( u \) is a non-constant holomorphic disc \((W, J)\) that satisfies just the boundary condition \( u(\partial \mathbb{D}) \subset L^t_q \) the conclusions from this section that rely on the maximum principle continue to hold. The corresponding replacement of the statement in (2) which does not use the homological assumption is the following: \( h_0 \) restricts to an immersion on \( \partial \mathbb{D} \) so that \( u(\partial \mathbb{D}) \) is positively transverse to each of the characteristic leaves \( L^t_q \cap \{ z_0 = e^{i\theta} \}, \ \theta \in [0, 2\pi) \).

**Remark 6.3.** The monotonicity argument used in [19, Lemma 3.9] implies that there exists a compact ball \( K \subset \mathbb{C}^{n-1-d} \) such that \( h(G) \subset K \) for all non-standard disc \( u \in \mathcal{W} \), i.e. with \( u = (a, f) \) we have
\[
 f^{-1}\left( \mathbb{R} \times T^*Q \times \mathbb{D} \times (\mathbb{C}^{n-1-d} \setminus K) \right) = \emptyset .
\]

6.5. Integrated maximum principle
Let \( u = (a, f) \in \mathcal{W} \) be a holomorphic disc of level \((q, t)\). As in Sect. 6.4 we consider \( G := f^{-1}(\mathring{Z} \setminus \mathring{B}) \) so that we can write \( f = (b, w, h_0, h) \) on \( G \). In Sect. 6.4 we obtained uniform \( C^0 \)-bounds on \( h_0 \) and \( h \) relying on the maximum principle from [19,20]. As the boundary conditions in \( T^*Q \)-direction are considerably different form the one used in [20] uniform \( C^0 \)-bounds on \( w \) require a new argument.

First of all we remark that by Stokes theorem, the symplectic energy of \( u \) (which we computed in Sect. 6.3 to be equal to \( \pi \)) is equal to the area \( \int_\mathbb{D} f^* d\omega \)
of \( f \). Because \( f^*d\hat{\alpha} \) is an area density by our compatibility assumptions we obtain

\[
\int_G w^*d\lambda \leq \int_G f^*d\alpha_0 \leq \pi.
\]

Recall the diffeomorphism \( \Phi: (\mathbb{R} \times ST^*Q, e^\alpha \alpha) \to (T^*Q \setminus Q, \lambda) \) of Liouville manifolds from Sect. 5.5, which pulls \( J_{T^*Q} \) back to \( I \). Define \( v := \Phi^{-1} \circ w \) and replace \( G \) by the subset \((|w|)^{-1}((R, \infty))\), \( R \geq 1 \) appearing in the definition of the box in Sect. 6.1, so that

\[ v = (c, k): G \longrightarrow (\ln R, \infty) \times ST^*Q \]

is an \( I \)-holomorphic map subject to the following boundary conditions:

\[ c(\partial G \setminus \partial \mathbb{D}) = \{\ln R\}, \quad k(\partial \mathbb{D} \cap G) \subset ST^*_qQ. \]

Further we have

\[
\int_G v^*d(e^\alpha \alpha) \leq \pi
\]

for the symplectic energy of \( v \).

We consider the subdomain

\[ G_t := c^{-1}((t, \infty)) \]

of \( G \) for \( t \geq \ln R \). Note that \( G_{\ln R} = G \). In order to allow partial integration we denote by \( \mathcal{R} \) the set of all regular values \( t \in (\ln R, \infty) \) of the functions \( c \) and \( c|_{\partial \mathbb{D} \cap G} \). By Sard’s theorem \( \mathcal{R} \) has full measure. Therefore, the open set \( \mathcal{R} \) is dense in \((\ln R, \infty)\).

For \( t \in \mathcal{R} \) the domain \( G_t \) has piecewise smooth boundary

\[ \partial G_t = \partial \mathbb{D} \cap G_t + \partial G_t \setminus \partial \mathbb{D}, \]

which we equip with the boundary orientation. Up to a null set the interior boundary \( \partial G_t \setminus \partial \mathbb{D} \) is given by \( c^{-1}(t) \). Observe that \( ST^*_qQ \) is a Legendrian sphere in the unit cotangent bundle so that the restrictions of \( k^*\alpha \) to the tangent spaces of \( \partial \mathbb{D} \cap G_t \) vanish. Stokes theorem applied twice implies

\[
\int_{G_t} v^*d(e^\alpha \alpha) = e^t \int_{c^{-1}(t)} k^*\alpha = e^t \int_{G_t} k^*d\alpha,
\]

where we used \( v^*d(e^\alpha \alpha) = d(e^ck^*\alpha) \).

On the other hand, using Leibniz rule, we have a decomposition

\[ v^*d(e^\alpha \alpha) = e^c dc \wedge k^*\alpha + e^ck^*d\alpha \]

into energy densities. Define the \( \alpha\)-energy functional by

\[ e(t) := \int_{G_t} e^c dc \wedge k^*\alpha \geq 0. \]

Therefore,

\[
\int_{G_t} v^*d(e^\alpha \alpha) = e(t) + \int_{G_t} e^ck^*d\alpha \geq e(t) + e^t \int_{G_t} k^*d\alpha
\]

using \( e^c \geq e^t \) on \( G_t \).
Combining these expressions for the symplectic energy, we get \( e(t) \leq 0 \). Hence, \( e(t) = 0 \) for all \( t \in \mathcal{R} \), i.e. the \( \alpha \)-energy functional \( e = e(t) \) vanishes identically. Because of

\[
dc \wedge f^* \alpha = (c_x^2 + c_y^2) \, dx \wedge dy,
\]

we deduce that \( c|_{G_t} = \text{const} \) and, since \( k^* \alpha = -dc \circ 1 \), that \( k^* \alpha|_{G_t} = 0 \) as well as \( k^* da|_{G_t} = 0 \). We conclude that \( v|_{G_t} = \text{const} \) for all \( t \in (\ln R, \infty) \). An open and closed argument for \( G = (|w|)^{-1}(R, \infty) \) implies that either \( G = \emptyset \) or \( v = \text{const} \) on all of \( G = \mathbb{D} \), which in turn implies that \( u \in \mathcal{W} \) was a standard disc. This shows uniform \( C^0 \)-bounds in \( T^*Q \)-direction for all non-standard discs \( u \in \mathcal{W} \):

**Proposition 6.4.** If \( u = (a, f) \in \mathcal{W} \) is a non-standard holomorphic discs, then

\[
f^{-1}\left(\mathbb{R} \times (T^*Q \setminus D_R T^*Q) \times \mathbb{D} \times \mathbb{C}^{n-1-d}\right) = \emptyset.
\]

### 7. Compactness

Consider a non-standard disc \( u = (a, f) \in \mathcal{W} \) of level \((q, t)\). On the preimage \( G := f^{-1}(\hat{Z} \setminus \hat{B}) \), we write

\[
f = (b, w, h_0, h)
\]

w.r.t. to the decomposition \( \mathbb{R} \times T^*Q \times \mathbb{D} \times \mathbb{C}^{n-1-d} \). In Sects. 6.4 and 6.5, we obtained uniform bounds on

(i) \( a \) from above by 0,

(ii) \( h_0 \) in the sense \( |h_0| \leq 1 \),

(iii) \( w \) and \( h \) in the sense that \( |w|_b \) and \( |h| \), resp., are bounded by a geometric constant.

The coordinate function \( b \) completes to a holomorphic function

\[
a - F(w) - \frac{1}{4}|h_0|^2 - \frac{1}{2}|\text{Im } h|^2 + ib
\]

on \( G \), where the restriction of the real part to \( \partial \mathbb{D} \) equals \( F(w)|_{\partial \mathbb{D}} \) up to a constant. In [19, Lemma 3.8], where no \( T^*Q \)-component appears, we used Schwarz reflection and the maximum principle to establish uniform bounds on \( |b| \). In our situation this would require real analyticity of \( F(w)|_{\partial \mathbb{D}} \), which in general does not hold.

We will work around this utilising a bubbling off analysis that uses target rescaling along the Reeb vector field \( \partial_b \) on \( \hat{Z} \setminus \hat{B} \). This will require ideas from [21]. In fact, by the elliptic nature of the holomorphic curves equation the bubbling off analysis directly yields compactness properties of holomorphic curves. Therefore, we will combine the target rescaling in \( b \)-direction with the usual target rescaling along the Liouville vector field \( \partial_b \):

By the maximum principle \( |b| \) attains its maximum on \( \partial G \). Observe that because of \( f(\partial \mathbb{D}) \subset \hat{Z} \setminus \hat{B} \) the boundary of \( G \) decomposes

\[
\partial G = \partial \mathbb{D} \sqcup f^{-1}(\partial \hat{B}).
\]
Assuming $|b| \not\leq b_0$ we get therefore that $|b|$ attains its maximum on $\partial \mathbb{D}$.

Suppose there exist sequences $\zeta_\nu \in \mathbb{D}$ and $u_\nu = (a_\nu, f_\nu) \in \mathcal{W}$ of non-standard such that

$$|b_\nu(\zeta_\nu)| \to \infty$$

writing $f_\nu = (b_\nu, w_\nu, h_\nu^0, h_\nu)$. We may assume that $\zeta_\nu \in \partial \mathbb{D}$ for all $\nu$ and that $\zeta_\nu \to \zeta_0$ in $\partial \mathbb{D}$. By the mean value theorem, we find a sequence $z_\nu$ in $\mathbb{D}$ such that $|\nabla u_\nu(z_\nu)| \to \infty$. This implies that uniform gradient bounds for non-standard holomorphic discs in $\mathcal{W}$ result in uniform bounds on $b$.

**Proposition 7.1.** Under the assumptions of Theorem 2.1 each sequence of non-standard discs $u_\nu \in \mathcal{W}$ has a $C^\infty$-converging subsequence.

**Proof.** Consider a sequence of non-standard discs $u_\nu = (a_\nu, f_\nu) \in \mathcal{W}$ of level $(q_\nu, t_\nu)$ such that $|\nabla u_\nu(z_\nu)| \to \infty$ for a sequence $z_\nu \to z_0$ in $\mathbb{D}$. By compactness of $Q$ and Remark 6.3 we can assume that $(q_\nu, t_\nu) \to (q_0, t_0)$.

Observe that modifications as made in [19, Section 4.1] that fix the varying boundary conditions we will mention in Sect. 8.3 are not necessary for the following compactness argument.

Up to a choice of a subsequence, we distinguish two cases:

1. $f_\nu(z_\nu) \in \hat{Z} \setminus \hat{B}$ for all $\nu$, and
2. $f_\nu(z_\nu) \in \hat{B}$ for all $\nu$.

In the first case, additionally, we can assume that the sequences $w_\nu(z_\nu)$, $h_\nu^0(z_\nu)$, and $h_\nu(z_\nu)$ converge and that either

1.1 $b_\nu(z_\nu) \to \pm \infty$, or
1.2 $b_\nu(z_\nu) \to b_\infty \in \mathbb{R}$.

In case (1.1), we use bubbling off analysis as in [26, Section 6], but this time applied to the holomorphic maps

$$(a_\nu - a_\nu(z_\nu), b_\nu - b_\nu(z_\nu), w_\nu, h_\nu^0, h_\nu)$$

defined on $G_\nu := f_\nu^{-1}(\hat{Z} \setminus \hat{B})$ for interior bubbling; for bubbling along the boundary perform the shift w.r.t. the real parts $x_\nu$ of the $z_\nu$. For both observe that shift in $b$-direction is a strict contactomorphism of $(Z, \alpha_0)$ and does not effect the Hofer energy. To have enough space inside $G_\nu$ during the domain rescaling use the trick in [26, Case 1.2.b] explained on [26, p. 547]; this time make use of the stretching of the holomorphic discs $u_\nu$ in $b$-direction instead of the $a$-direction. In the cases (2) and (1.2) apply the usual bubbling off analysis as in [9,27–29], cf. [26, Cases 1.1, 1.2.a, 2 in Section 6].

Finally, in all cases, we can argue as in [19, Section 4]. By the aperiodicity assumption $\inf_0(\alpha) \geq \pi$, which with Sect. 3.1 implies $\inf_0(\dot{\alpha}) \geq \pi$, there is no bubbling off of finite energy planes. This is because finite energy planes asymptotically converge to contractible periodic Reeb orbits. The asymptotic analysis of the finite energy planes possibly requires a bubbling off analysis that involves target rescaling in $b$-direction as explained above, cf. [21, Section 5.2].

Because there are no bubble spheres by exactness of $(W, \omega)$ we are left with bubbling off of holomorphic discs, cf. [21, Section 5.3]. This will lead us
to a contradiction as in [19, Section 4.2]. Indeed, the Hofer energy of a bubble discs is a positive multiple of $\pi$, see Sect. 6.3. As the Hofer energy of all $u_\nu$ equals $\pi$ by Sect. 6.3 there is at most one bubble discs. Hence, we can assume that $u_\nu$ converge in $C^\infty_{\text{loc}}$ on $\mathbb{D} \setminus \{z_0\}$ for some $z_0 \in \partial \mathbb{D}$. By our assumption on the 3 fixed marked points in the definition of $W$ after removing the singularity $z_0$ the limiting holomorphic disc will be non-constant; and, therefore, will also have energy equal to a positive multiple of $\pi$. But the sum of energies of the bubble disc and the limiting disc can not exceed $\pi$. This contradiction shows uniform gradient bounds for any sequence $u_\nu$ of holomorphic discs in $W$. \hfill $\square$

8. Transversality

In Sect. 7, we established properness of the evaluation map

$$\text{ev}: \quad W \times \mathbb{D} \longrightarrow \hat{Z} \quad (u = (a, f), z) \longmapsto f(z).$$

The aim of this section is to show that ev has degree 1. We will follow the considerations from [19, Section 5] and [20, Section 3.5] and just indicate the adaptations to the present situation.

8.1. Maslov index

For all $u \in W$ the Maslov index of the bundle pair

$$(u^*TW, (u|_{\partial \mathbb{D}})^*TL^t_q)$$

equals 2, where $(t, q)$ is the level of $u$. Indeed, following [19, Lemma 3.1], by homotopy invariance it suffices to show the claim for standard discs

$$u(z) = u_{s, b}^t, w(z) = \left(\frac{1}{2}(|z|^2 - 1), b, w, z, s + it\right),$$

$w \in T_q^2$, assuming $W = \mathbb{R} \times \mathbb{R} \times T^*Q \times \mathbb{D} \times \mathbb{C}^{n-1-d}$. In particular, $u^*TW \cong \mathbb{C}^{n+1}$. Moreover, $(u|_{\partial \mathbb{D}})^*TL^t_q$ is isomorphic to $i\mathbb{R} \oplus i\mathbb{R}^d \oplus e^{i\theta}\mathbb{R} \oplus \mathbb{R}^{n-1-d}$ over $e^{i\theta} \in \partial \mathbb{D}$. Hence, the Maslov index equals 2 by normalisation.

8.2. Simplicity

First of all, we remark that the classes $[u] \in H_2(W, L^t_q), u \in W$, are $J$-indecomposable. Otherwise, we would find a decomposition

$$[u] = \sum_{j=1}^N m_j [v_j]$$
in $H_2(W, L^t_q)$, for simple holomorphic discs $v_j$ with boundary on $L^t_q$ and multiplicities $m_j \geq 1$. Writing $v_j = (a_j, f_j)$ we get for the energy

$$\pi = \sum_{j=1}^N m_j \int_{\partial \mathbb{D}} f_j^* \alpha_0.$$
Writing \((b_j, w_j, h^j_0, x_j + it_j)\) for the restriction of \(f_j|_\partial \mathbb{D}\) the left hand side reads as
\[
\sum_{j=1}^{N} m_j \int_{\partial \mathbb{D}} \left[ b^*_j db + w^*_j \lambda + (h^j_0)^* \frac{1}{2} (x_0 dy_0 - y_0 dx_0) - (x_j + it_j)^* (y dx) \right].
\]

The first and last summand vanish by exactness of the form we pull back to the circle \(\partial \mathbb{D}\); the second vanishes because \(w_j(\partial \mathbb{D}) \subset T^*_0 Q\). Hence, writing \(r_j\) for the winding number of \(h^j_0|_{\partial \mathbb{D}}\), which is positive for non-constant \(h^j_0\) by the argument principle, we get
\[
\pi = \pi \cdot \sum_{j=1}^{N} m_j r_j \geq N \cdot \pi.
\]

We conclude that \(N = 1, m_1 = 1\), i.e. \([u]\) is \(J\)-indecomposable.

Consulting [19, Lemma 3.4] we see that \(u\) must be simple. Because \(u|_{\partial \mathbb{D}}\) is an embedding, see Sect. 6.4, we obtain as in [19, Lemma 3.5] that the set of \(f\)-injective points is open and dense in \(\mathbb{D}\).

### 8.3. Variable boundary conditions

There is a natural way to identify the boundary conditions

\[
L^t_q = \{0\} \times \mathbb{R} \times T^*_0 Q \times \partial \mathbb{D} \times \mathbb{R}^{n-1-d} \times \{t\}
\]

for the holomorphic discs in \(W\). Observe, that the union of \(L^t_q\) over all parameters \(t \in \mathbb{R}^{n-1-d}\) and \(q \in Q\) equals

\[
\{0\} \times \partial \hat{\mathbb{D}} = \{0\} \times \mathbb{R} \times T^*_0 Q \times \partial \mathbb{D} \times \mathbb{C}^{n-1-d}
\]

so that flows induced by tangent vectors \(v \in T_q \mathbb{R}^{n-1-d}\) and \(v \in T_q Q\) can be taken for the identifications. Consider a chart \((\mathbb{R}^d, 0) \to (Q, q)\) of \(Q\) about \(q\) and extend \(v\) to a vector field on \(\mathbb{R}^d\) that has compact support and is constant near 0. The induced flow on \(Q\) naturally lifts to a fibre and Liouville form preserving flow on \(T^*_0 Q\), see [30, p. 92]. Similarly, extend \(v \in T_q \mathbb{R}^{n-1-d}\) to a compactly supported vector field on \(\mathbb{R}^{n-1-d}\) that is constant near \(t \in \mathbb{R}^{n-1-d}\).

We regard \((v, v)\) as a vector field on \(\mathbb{R} \times \mathbb{R} \times T^*_0 Q \times \partial \mathbb{D} \times \mathbb{C}^{n-1-d}\) cutting off \((v, v)\) with a bump function that has support on a small neighbourhood of \(\{0\} \times [-b_0, b_0] \times T^*_0 Q \times \partial \mathbb{D} \times \mathbb{C}\) and equals 1 on a smaller neighbourhood. We denote the corresponding flow on \(W\) by \(\psi^{(v, v)}_t\). Given a level \((q_0, t_0)\) we find a neighbourhood \(U\) of \((q_0, t_0)\) \(Q \times \mathbb{R}^{n-1-d}\) and a vector field \((v, v)\) as above such that the time-1 map \(\psi^{(v, v)}_1\) sends \(L^t_{q_0}\) to \(\psi^{(v, v)}_1(L^t_{q_0}) = L^t_q\) for all \((q, t) \in U\). Simply define \((v, v)\) to be \((q - q_0, t - t_0)\) on \(U\).

### 8.4. Admissible functions

Denote by \(\mathcal{B}\) the separable Banach manifold consisting of all continuous maps \(u : (\mathbb{D}, \partial \mathbb{D}) \to (W, \{0\} \times \hat{C})\) of Sobolev class \(W^{1,p}\), \(p > 2\), that satisfy the conditions \((w_1)-(w_3)\) in the definition of the moduli space \(W\), see Sect. 6.2.
The Banach manifold structure is given as follows: The subset $B_q^t \subset B$ of all $u$ of level $(q, t)$ is a separable Banach manifold whose tangent spaces are

$$T_u B_q^t = W^{1,p}(u^*TW, (u|_{\partial D})^*TL_q^t).$$

Consider the level projection map $B \rightarrow Q \times \mathbb{R}^{n-1-d}$ that assigns to all $u \in B$ the corresponding level $(q, t)$. Using the identifying maps the $\psi^{(v,v)}_1$ from Sect. 8.3 these defines a locally trivial fibration on the Banach manifold $B$ with fibres $B_q^t$.

### 8.5. Linearised Cauchy–Riemann operator

In particular,

$$T_u B = T_u B_q^t \oplus (T_q Q \oplus \mathbb{R}^{n-1-d})$$

so that the linearised Cauchy–Riemann operator at $u \in B$ of level $(q, t)$ splits as

$$D_u = D_u^{(q,t)} \oplus K_u,$$

where $D_u^{(q,t)} := D_u|_{T_u B_q^t}$ is the linearised Cauchy–Riemann operator in fibre direction and $K_u : T_q Q \oplus \mathbb{R}^{n-1-d} \rightarrow L^p(u^*TW)$ is a compact perturbation, see [19, Section 5.1]. The index of $D_u^{(q,t)}$ equals $n$, as the Maslov index of the problem with fixed boundary level was 2 (see Sect. 8.1), so that the total index equals $\text{ind } D_u = 2n - 1$.

If $Q$ is oriented, we can orient $D_u$ via the determinant bundle

$$\det D_u = \det D_u^{(q,t)} \otimes \det (T_q Q \oplus \mathbb{R}^{n-1-d})$$

as follows: The line bundle $\det D_u^{(q,t)}$ is oriented by the construction in [31, Section 8.1] via the trivial bundle $TL_q^t \cong T_q^*Q \oplus \mathbb{R}^{n+1-d}$ and the orientation of $T_q^*Q \cong \mathbb{R}^d$ so that the bundle pair

$$(u^*TW, (u|_{\partial D})^*TL_q^t)$$

admits a natural trivialisation. The line bundle $\det (T_q Q \oplus \mathbb{R}^{n-1-d})$ is oriented via the orientation of $Q \times \mathbb{R}^{n-1-d}$.

### 8.6. Lifting topology

As in [19, Section 5.2], we choose $J$ to be regular by perturbing the induced complex structure on $\hat{\xi}$ over $\hat{B}$. Regularity of $J$ along standard discs is obvious. Hence, the moduli space $\mathcal{W}$ is a smooth oriented manifold of dimension $2n - 1$ whose end is made out of standard holomorphic discs. Therefore, the evaluation map $\text{ev}$, which is proper, has degree 1. With [19, Section 6] and [13, Section 2] we see that $\text{ev}$ induces surjections of homology groups and of $\pi_1$.

Identify $Q$ with the subset

$$Q \equiv \{0\} \times Q \times \{1\} \times \{0\}$$

of

$$\mathbb{R} \times T^*Q \times \{1\} \times \mathbb{C}^{n-1-d} \subset \partial \hat{Z}.$$
Observe that $M$ is a strong deformation retract of $\hat{Z}$. We choose a deformation retraction such that the inclusion $Q \subset \hat{Z}$ is isotoped to an embedding $Q \to M$. Combining this with the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{W} \times D & \xrightarrow{\text{ev}} & \hat{Z} \\
\uparrow \subset & & \uparrow \subset \\
\mathcal{W} \times \{1\} & \xrightarrow{\text{ev}} & \mathbb{R} \times T^*Q \times \{1\} \times \mathbb{C}^{n-1-d}
\end{array}
$$

yields:

**Proposition 8.1.** Under the assumptions of Theorem 2.1 the isotoped inclusion $Q \to M$ induces a surjection of homology and fundamental groups.

**Proof.** This follows with the homology epimorphism argument from [13, Section 2.3] and the covering argument from [13, Section 2.5]. □

9. The homotopy type

We compute the homotopy type of $M$ in terms of $D(T^*Q \oplus \mathbb{R}^{2n+1-2d}).$ For that we assume that, up to fibre preserving isotopy, the shape $S$ is equal to the shape given by the unit sphere bundle in $T^*Q \oplus \mathbb{R}^{2n+1-2d}.$ This results into the same construction for $\hat{Z}$ as in Sect. 3.1 up to ambient diffeotopy.

We identify $Q$ with the section of the sphere bundle $\partial M = S(T^*Q \oplus \mathbb{R}^{2n+1-2d})$ given by

$$Q \equiv \{0\} \times Q \times \{1\} \times \{0\}$$

in

$$\mathbb{R} \times T^*Q \times D \times \mathbb{C}^{n-1-d}.$$

Observe that this defines a natural embedding of $D(T^*Q \oplus \mathbb{R}^{2n+1-2d})$ into $M$ via a small disc bundle about

$$\{0\} \times Q \times \{(1 - \varepsilon)\} \times \{0\},$$

$\varepsilon > 0$ small. Indeed, simply shift a small disc bundle in $\mathbb{R} \times T^*Q \times \mathbb{C}^{n-1-d}$ in direction of $\{0\} \times Q \times \{(1 - \varepsilon)\} \times \{0\}.$ The image is denoted by $M_0.$

9.1. Homology type and fundamental group

Proposition 8.1 implies that the inclusion $Q \subset M$ is surjective in homology and $\pi_1.$ Based on that we show:

**Proposition 9.1.** Under the assumptions of Theorem 2.1, the inclusion $M_0 \subset M$ induces isomorphisms of homology groups.

**Proof.** The arguments are similar to [20, p. 42] and [13, Section 2.4]. Recall the general assumption $n - 1 \geq d.$

From Proposition 8.1, we immediately obtain $H_k M = 0$ for $k \geq d + 1$ so that the homology isomorphism property of the inclusion $M_0 \subset M$ is automatic in all degrees $k \geq d + 1.$
By general position, any section $Q \to \partial M$ of the sphere bundle induces an isomorphism in homology in degree $k \leq 2n-1-d$. Therefore, the inclusion of the sphere bundle into the disc bundle of $T^*Q \oplus \mathbb{R}^{2n+1-2d}$ is isomorphic in homology of degree $k \leq 2n-1-d$. We claim that the inclusion $\partial M \to M$ shares the same property. With $d+1 \leq 2n-1-d$, the proposition will be immediate.

By Poincaré duality and the universal coefficient theorem, we have $H_k(M, \partial M) \cong H_{2n+1-k}M \cong FH_{2n+1-k}M \oplus TH_{2n-k}M$, where $FH_*$ and $TH_*$ denote the free and the torsion part of $H_*$, respectively. By the above $H_k(M, \partial M) = 0$ for $k \leq 2n-d-1$. The long exact sequence of the pair $(M, \partial M)$ implies that $\partial M \to M$ is isomorphic in degree $k \leq 2n-2-d$ and epimorphic in degree $k = 2n-1-d$. Because the homology of the sphere bundle $\partial M$ vanishes in degree $k = 2n-1-d$ the epimorphism is in fact injective.

\begin{corollary}
Under the assumptions of Theorem 2.1 the inclusion $M_0 \subset M$ induces an epimorphism on fundamental groups. If in addition $\pi_1 Q$ is abelian, then the inclusion $M_0 \subset M$ will be $\pi_1$-isomorphic.

Proof. Using the $\pi_1$-isomorphism $M_0 \simeq Q \subset \partial M$, the claim follows from Proposition 8.1 and 9.1 as in [13, Section 2.5].
\end{corollary}

Proof of Theorem 2.1 (i). The claim directly follows from Proposition 9.1 and Corollary 9.2. Simply observe that the specific choice of section into the sphere bundle is irrelevant here.

9.2. A cobordism

Implementing the construction from [20, Section 4.2] in the situation at hand we define a cobordism $X := M \setminus \text{Int} M_0$.

The construction comes with the following diagram

\begin{center}
\begin{tikzpicture}
  \node (M) at (0,0) {$M_0$};
  \node (M0) at (0,-2) {$\partial M_0$};
  \node (Q) at (2,-2) {$Q$};
  \node (Q0) at (2,0) {$Q_0$};
  \node (X) at (4,0) {$X$};
  \node (M1) at (4,-2) {$\partial M$};

  \draw[->, dashed] (M) -- (M0);
  \draw[->] (M) -- (Q);
  \draw[->] (M) -- (X);
  \draw[->] (M0) -- (Q0);
  \draw[->] (M0) -- (Q);
  \draw[->] (X) -- (M);

  \node at (1.5,-0.5) {$\cong$};
  \node at (1.5,-1.5) {$\cong$};
  \node at (3.5,-0.5) {$\cong$};

  \node at (-0.2,-0.5) {time-1 map of isotopy};
  \node at (0.8,-2.5) {gen. pos.};
  \node at (1.5,-1.25) {time-1 map of former isotopy};
  \node at (2.1,-0.5) {of isotopy};

  \node at (-0.2,0.5) {time-1 map of isotopy};
  \node at (0.8,-2.5) {gen. pos.};
  \node at (2.6,0.5) {gen. pos.};
  \node at (3.5,-0.5) {time-1 map of isotopy};

  \node at (4.2,0.5) {time-1 map of isotopy};
  \node at (4.2,-2.5) {gen. pos.};

  \node at (0.2,0.5) {gen. pos.};
  \node at (0.2,-2.5) {gen. pos.};
  \node at (2.6,0.5) {gen. pos.};
  \node at (2.6,-2.5) {gen. pos.};

\end{tikzpicture}
\end{center}

that commutes up to homotopy. We explain the diagram: Set

$Q_0 \equiv \{0\} \times Q \times \{(1-\varepsilon')\} \times \{0\}$,
where \( \varepsilon' \in (0, \varepsilon) \) is chosen such that \( Q_0 \subset \partial M_0 \). All arrows are given by inclusion except those whose label refers to an isotopy. The mentioned isotopy is an isotopy of \( Q_0 \) inside \( M \) that is the restriction of a diffeotopy on \( \mathbb{R} \times T^* Q \times \mathbb{D} \times \mathbb{C}^{n-1-d} \) obtained by shifting and rescaling that brings \( Q_0 \) to \( Q \) and \( \partial M_0 \) to \( \partial M \). The arrow \( M_0 \to M \) is obtained from an extension of the isotopy of \( Q_0 \subset M \) to \( M_0 \).

**Proposition 9.3.** Under the assumptions of Theorem 2.1 the inclusion maps \( \partial M_0, \partial M \subset X \) induce isomorphisms of homology groups. If in addition \( \pi_1 Q \) is abelian (or more generally the inclusion \( Q \subset M \) is \( \pi_1 \)-injective) then the inclusions \( \partial M_0, \partial M \subset X \) will be \( \pi_1 \)-isomorphic.

**Proof.** The argumentation is the one given at the end of [20, Section 4.2]: For low degrees \( k \leq 2n - d - 1 \) use general position arguments as indicated in the diagram and the results from Sect. 9.1. In higher degrees \( k \geq d + 1 \) essentially this is Poincaré duality and excision. \( \square \)

**Proof of Theorem 2.1 part (a) in (ii) and (iii).** We have to establish homotopy equivalence, resp., a diffeomorphism between \( M \) and \( M_0 \). With Proposition 9.3 this essentially follows from the relative Hurewicz and the \( s \)-cobordism theorem. The arguments are precisely as in the proof of [13, Theorem 1.5] for \( Q \) simply connected and [13, Theorem 5.3] via finite coverings in the non-simply connected case. \( \square \)

### 9.3. Infinite coverings

We assume the inclusion map \( \partial M \subset M \) to be \( \pi_1 \)-injective. This will be satisfied if \( \pi_1 Q \) is abelian for example. If \( Q \) is simply connected vanishing in relative homology of the cobordism \( \{ \partial M_0, X, \partial M \} \), which will be simply connected too, implies triviality of relative homotopy groups. If \( Q \) is not simply connected, one way to work around this is to lift along the universal covering of \( X \). For \( \pi_1 Q \) finite the universal covering space \( \tilde{X} \) will be compact so that we are in the situation of the previous sections. This was used in the proof of Theorem 2.1 part (a) in (ii) and (iii) in Sect. 9.2.

If \( \pi_1 Q \) is infinite, we reset the moduli space: The \( \pi_1 \)-isomorphism \( \partial M \subset M \) ensures that the universal cover of \( \tilde{Z} \) is obtained by gluing similarly to Sect. 3.1; this time we glue the universal covers of the involved objects along a lift of \( \varphi \). This makes it possible to consider the moduli space \( \mathcal{W}' \) of holomorphic discs in \( \tilde{W} \) defined as in Sect. 6.2; just replace \( Q \) with \( \tilde{Q} \) in the definition of the Lagrangian boundary cylinders. This places us into the situation of [20, Section 4.4]. The change of the boundary condition is inessential and the special choice \( Q = T^d \) is not really used. Hence, we obtain a covering \( \mathcal{W}' \to \mathcal{W} \) together with a proper degree 1 evaluation map

\[
\text{ev}: \quad \mathcal{W}' \times \mathbb{D} \longrightarrow \tilde{Z} \\
(u = (a, f), z) \longmapsto f(z),
\]

see [13, Lemma 6.1]. Similar to Proposition 8.1 and [13, Proposition 6.2 and Lemma 6.3] we obtain:
Proposition 9.4. Under the assumptions of Theorem 2.1 the inclusion $\tilde{Q} \rightarrow \tilde{M}$ of universal covers induces a surjection of homology and fundamental groups. Further, the inclusion $\partial\tilde{M}_0 \rightarrow \tilde{X}$ is homology surjective.

Because the universal cover $\tilde{X}$ is not compact for $\pi_1Q$ infinite Poincaré duality delivers no information about relative homology groups in contrary to our argument in Proposition 9.1. But we can say the following:

Theorem 2.1 part (b) in (ii) and (iii). Because the universal cover of $Q$ is contractible so is $\tilde{M}$ by Proposition 9.4. Hence, the inclusion $\tilde{M}_0 \subset \tilde{M}$ is a homotopy equivalence. This follows with the arguments from the proof of [13, Theorem 7.2]. With the proof of [13, Theorem 9.1], which in our situation is particularly easy because of the extra codimension, it follows that the boundary inclusions of $\tilde{X}$ are homotopy equivalences. Hence, $X$ is in fact an $h$-cobordism. For the diffeomorphism type, then apply the $s$-cobordism theorem. □

If $\partial M$ is a simple space, which for example is satisfied whenever $Q$ is a simple space and $\partial M \rightarrow Q$ a trivial sphere bundle, then vanishing of relative homology of $(X, \partial M_0)$ and $(X, \partial M)$, resp., implies homotopy equivalence of each of the boundary inclusions of the cobordism $\partial M_0, X, \partial M$. The basic idea here is that the kernel of the Hurewicz homomorphism is made out of the action of the fundamental group, which we now assume to be trivial, see [13, Section 8]:

Theorem 2.1 part (c) in (ii) and (iii). Follows with the same arguments as in [13, Theorem 1.7 and Example 9.3 (b)]. □

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Myeonggi Kwon
Department of Mathematics Education
Sunchon National University
Suncheon 57922
Korea
e-mail: mkwon@scnu.ac.kr

Kevin Wiegand
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
Germany
e-mail: kwiegand@mathi.uni-heidelberg.de

Kai Zehmisch
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
44801 Bochum
Germany
e-mail: kai.zehmisch@rub.de

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