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CRITICAL $k$-VERY AMPLENESS FOR ABELIAN SURFACES

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Abstract. Let $(S, L)$ be a $(1, d)$-polarized abelian surface with $c_1(L) = l$ and Picard rank 1. We define a function $\phi$ which takes each ample line bundle $L$ into the least integer $k$ such that $L$ is $k$-very ample but not $(k+1)$-very ample. In this paper we use Bridgeland’s stability conditions and Fourier-Mukai techniques to give a closed formula for $\phi(L^n)$ as a function of $n$. As a byproduct, we calculate the walls in the Bridgeland stability space for certain Chern characters.

1. Introduction

The notion of $k$-very ampleness was introduced in the 1980s initially to understand the idea of higher order embeddings. Weaker notions of $k$-spanned (see [BS91] and [BFS89]) and $k$-jet ampleness (see [BS93]) were also considered. The definitions given all relate to asking that, for a variety $V$ and ample line bundle $L$ on $V$, the natural map $\Gamma(L) \rightarrow \Gamma(L/I)$ surjects for certain classes of sheaf $I$ of 0-subschemes of $V$. The notions differ in how fat points are treated. In this paper we only consider the strongest notion of $k$-very ampleness. Since $k$-very ampleness implies $(k-1)$-very ampleness it is natural to consider the critical value of $k$ when a line bundle $L$ is $k$-very ample but not $(k+1)$-very ample. We shall denote this by $\phi(L)$. In the case when the Neron-Severi group is generated by a single element (as we shall be assuming), we can view the function $\phi$ as a function of a positive integer. It is then natural to hope that $\phi(n+1)$ is related to $\phi(n)$ since $L^{n+1} = L^n \otimes L$. Unfortunately, $k$-very ampleness is not very well behaved with respect to tensoring.

One type of variety where most progress has been made is abelian varieties. There are a number of, by now, classical results on very ampleness of line bundles. For example, $L^2$ is always very ample for any ample line bundle $L$. A little more recently, Debarre, Hulek and Spandaw showed that a suitably generic $(1, d)$ polarization on a $g$-dimensional abelian variety is very ample for $d > 2g$. For the case $g = 2$ and Picard rank 1, this was extended by Bauer and Szenberg ([BS97]) to compute $\phi(1)$ (see Proposition 4.1 below for the details).

There is a clear relation between $k$-very ampleness and so called “weak index theorem” conditions arising in Fourier-Mukai Theory for abelian varieties. These ideas have been extended by Popa and Pareschi ([PP03]) who introduce the notion of M-regularity and relate it to $k$-jet ampleness in [PP04].

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This paper is organized as follows. For the rest of the introduction we define the \( \phi \) function and recall some facts already established in the literature. We also recall some facts about Fourier-Mukai transforms and deduce some easy results about \( \phi \). In section 2 we give a brief introduction to Bridgeland’s stability conditions needed prove the main theorem in this paper. In section 3 we recall the notion of walls and show that, for the Chern character \((0,l,\chi)\) there are never any walls. We use this apply the general stability machinery to provide a useful technical lemma needed to prove our main Theorem. In the final section, we show how to use the technical lemma to bound \( \phi \) from above and then prove that the bound is sharp by computing walls in the stability space associated to the Chern character \((1, nl, (n−1)^2d+d+1)\). We then induct on \( n \) to deduce the main theorem making use of our technical lemma again:

**Theorem 4.3.** Let \((S, L)\) be a \((1,d)\)-polarized abelian surface with \(\text{NS}(S) = \langle L \rangle\), then \(\phi(L^n) = 2(n-1)d - 2\).

1.1. \( k \)-Very Ample. Let \( V \) be a complete algebraic variety of dimension \( g \) over an algebraic closed field \( K \), \( V \) a purely 0-dimensional subscheme of \( V \) with \(|X| = d = \dim(H^0(\mathcal{O}_X))\) and \( L \) an invertible sheaf on \( V \).

**Definition 1.1.** For each 0-scheme \( X \) on \( V \) we can consider the restriction map \( \rho_X \) to \( X \) for the space of sections of \( L \), which fits into the exact sequence:

\[
0 \to H^0(V, L \otimes \mathcal{I}_X) \to H^0(V, L) \xrightarrow{\rho_X} H^0(\mathcal{O}_X) \to H^1(V, L \otimes \mathcal{I}_X) \to H^1(V, L) \to 0
\]

\( L \) is called \( k \)-very ample if \( \rho_X \) is surjective for all purely 0-dimensional subscheme \( X \) of length \(|X| \leq k + 1\).

**Remark 1.2.** The following follows easily from the definition

- \( L \) is 0-very ample if and only if \( L \) is generated by global section.
- \( L \) is 1-very ample if and only if it is very ample.
- If \( L \) is \( k \)-very ample then \( L \) is \((k−1)\)-very ample.

Let \( \text{Amp}(S) \) be the ample cone of \( S \). By the properties above there exists an integer \( k \) for all \( L \in \text{Amp}(S) \) such that \( L \) is \( k \)-very ample but not \((k + 1)\)-very ample.

**Definition 1.3.** Define a map

\[
\phi : \text{Amp}(S) \to \mathbb{Z}_{\geq -1}
\]

which takes \( L \) into the least integer \( k \) such that \( L \) is \( k \)-very ample but not \((k + 1)\)-very ample. Define \( \phi_L(n) := \phi(L^n) \), and \( \phi(n) \) if \( L \) is understood.

There is no obvious reason why this should be a good function of \( n \) for any variety and, even for \( \mathbb{P}^2 \), it is hard to compute. Specific values for some varieties are, however, well known:
Example 1.4. Let $(V, L)$ be a principally polarized abelian variety. Then $\phi_L(2) = 0$.

The following lemma, indirectly proved in [BS97] Proposition (3.2), gives the value of $\phi_L(1)$ and we will reprove it in §4 in the spirit of this paper:

**Proposition 4.1.** If $L$ is an ample line bundle of type $(1, d)$, $d \geq 1$ on an abelian surface $X$ with Picard rank 1, then

$$\phi_L(1) = \left\lfloor \frac{d - 3}{2} \right\rfloor$$

Upper and lower bounds for $\phi$ are also known. It is clear that, if $H^1(L) = 0$ (as is the case, for example, for an ample line bundle on an abelian variety) then an upper bound for $\phi(n)$ can be given by $\chi(L^n) - 1$ since $\chi(L^n \otimes I_X) = \chi(L^n) - |X|$. For a $(1, d)$ polarized abelian surface, this is $n^2d - 1$. A non-trivial lower bound is much harder to come by but Reider’s Theorem (for the most useful version, see [AB11, §2]) provides one, at least when certain divisors do not exist, as it says that if $c_1(L^n)^2 > (k + 2)^2$ then $L^n$ is $k$-very ample. If we apply this to the irreducible $(1, d)$ abelian surface case where such divisors do not exist, we see that $\phi(n) \geq \lceil \sqrt{dn} \rceil - 3$. But this is not even sharp for $d = 1$ and $n = 2$.

Reider’s Theorem arose in the situation where $L \otimes I_Z$ is used to construct vector bundles of rank 2. Key in his construction is the Bogomolov Inequality for semi-stable sheaves. We will also use this in various places and recall it here (see [HL10] for a proof).

**Definition 1.5.** A torsion-free sheaf $E$ is $\mu$-stable ($\mu$-semistable) with respect to $l$ if for each subsheaf $F$ we have

$$\mu(F) < \mu(E)(\mu(F) \leq \mu(E))$$

where $\mu(E) = c_1(E).l^{r-1}/r(E)$.

**Proposition 1.6.** Let $V$ be a smooth projective variety of dimension $n$ and $l$ be an ample divisor on $V$. If $E$ is a $\mu$-semistable (with respect to $l$) torsion sheaf of rank $r$ on $V$, then

$$(r - 1)c_1^2(E).l^{n-2} \leq 2rc_2^2(E).l^{n-2}.$$  

For the case of $(1, d)$-polarization of abelian surface:

$$2r(E)\chi(E) \leq c_1^2(E).$$

We will also need to consider a finer stability for sheaves:

**Definition 1.7.** A torsion-free sheaf $E$ is Gieseker stable (respectively Gieseker semistable) with respect to $l$ if for each subsheaf $F$ we have

$$P(F) < P(E)(P(F) \leq P(E))$$

where $P(E) = \frac{\chi(E \otimes L^n)}{r(E)}$, is the reduced Hilbert polynomial.
We let $\mathcal{M}_{ch}^{GS}$ denote the moduli space of Gieseker semistable sheaves on $S$ with Chern character $\text{ch}$ (or more generally, Simpson semistable sheaves when the rank is zero). These are known to be non-empty on an abelian surface whenever the Bogomolov inequality holds for $\text{ch}$.

1.2. Fourier-Mukai transforms. Let $(V, \Phi)$ be a smooth projective variety over $\mathbb{C}$ and $\Phi$ be its dual. Consider flat projections $V \xrightarrow{\tau} \hat{V} \times \hat{V}$. Let $P \in D(S \times \hat{V})$ where $D(V \times \hat{V})$ denotes the derived category of bounded complexes of coherent sheaves on $V \times \hat{V}$. The Fourier-Mukai transform $\Phi$ is the functor

\begin{equation}
\Phi : D(V) \to D(\hat{V})
\end{equation}

which takes $A$ into $R\pi_!(L\hat{\pi}^*A \otimes P)$ (See [Huy06]). Denote its cohomology by $\Phi^i$. In fact, we shall only consider the classical Fourier-Mukai transform where $P$ is the Poincaré bundle on an abelian surface $V = S$. Then $\Phi$ has a quasi-inverse given (up to shift) by the transform

\begin{equation}
\hat{\Phi} : D(\hat{V}) \to D(V)
\end{equation}

with kernel $\hat{P} \in D(S) \times S$, where $\hat{P} = s^*P$ and $s : S \times \hat{S} \to \hat{S} \times S$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

**Definition 1.8.** An object $E$ satisfies WIT$_n$ if $\Phi^i(E) = 0$ for all $i \neq n$.

**Definition 1.9.** An object $E$ satisfies IT$_n$ if $H^i(E \otimes P) = 0$ for all $\hat{x} \in \hat{S}$ the dual of $S$ and $i \neq n$. In which case, $\Phi^n(E)$ is a locally free sheaf.

**Example 1.10.** [Mum74] Any ample line bundle $L$ on an abelian variety is IT$_0$. Any sheaf which is WIT$_0$ is automatically IT$_0$ by the semi-continuity theorem.

**Proposition 1.11.** If $(S, \Phi)$ is an abelian variety and $L$ is IT$_0$, then $L$ is $k$-very ample if and only if $L \otimes \mathcal{I}_X$ is WIT$_0$ (and hence IT$_0$) for all 0-dimensional subschemes $X$ of length $|X| \leq k + 1$.

**Proof.** "⇒" Suppose that $L$ is $k$-very ample and $L \otimes \mathcal{I}_X$ is not WIT$_0$ for some purely 0-dimensional subscheme $X$ of length $|X| \leq k+1$, so there exists $\hat{x} \in \hat{S}$ such that $H^1(LP_\hat{x} \mathcal{I}_X) \neq 0$. Let $x = \psi_L^{-1}(\hat{x})$ where $\psi_L : S \to \hat{S}$ takes $x$ into $\tau_x L \otimes L^{-1}$, then $H^1(\tau^*_x(L) \mathcal{I}_X) \neq 0$ and so $H^1(\tau^*_x(L \otimes \mathcal{I}_{\tau_x X})) \neq 0$. Hence $H^1(L \otimes \mathcal{I}_{\tau_x X}) \neq 0$ where $|\tau_x X| \leq k + 1$ and this contradicts the assumption.

"⇐" Since $L \otimes \mathcal{I}_X$ is WIT$_0$ for all purely 0-dimensional subscheme $X$ of length $|X| \leq k + 1$, then $H^1(L \otimes \mathcal{I}_X) = 0$. Hence $L$ is $k$-very ample by definition. \hfill $\square$

**Proposition 1.12.** Let $(S, L)$ be an irreducible principally polarized abelian surface, then $L^n$ is not $(2n - 3)$-very ample.

**Proof.** Let $X$ be a 0-dimensional subscheme of $D_L$ of length $2(n - 1)$. Then we have a sequence

$$0 \to L^{n-1} \to L^n \otimes \mathcal{I}_X \to Q \to 0$$
Suppose $Q$ is $\text{IT}_0$. The Chern character of $Q$ is $\text{ch}(Q) = (0, l, (2n - 1) - |X|)$. Since the transform of $Q$ has the Chern character $\text{ch}({\hat{Q}}) = ((2n - 1) - |X|, -l, 0) = (1, -l, 0)$, but $\hat{Q}$ is locally-free which is impossible. So $Q$ is not $\text{IT}_0$ and then $L^n \otimes I_X$ is not $\text{IT}_0$. □

Such $X$ we call collinear as $H^0(L \otimes I_X) \neq 0$ so there exists $x \in S$ such that $X \subset \tau_x D_L$, a translation of the polarization divisor.

**Corollary 1.13.** Let $(S, L)$ be an irreducible principally polarized abelian surface, then $\phi(n) \leq 2n - 4$ for $n \geq 2$.

### 2. Bridgeland’s Stability Condition

Now we will give a brief review of Bridgeland’s stability conditions (See [Bri08]. Define for any $s \in \mathbb{R}$ the following

$$F_s = \{ E \in \text{Coh}_S \mid E \text{ is torsion-free and } \mu_+(E) \leq 2ds \},$$

$$T_s = \{ E \in \text{Coh}_S \mid E \text{ is torsion or } \mu_-(E/\text{tors}(E)) > 2ds \},$$

where $\mu_+(E)$ is the slope of the largest slope $\mu$-destabilizing subsheaf of $E$ and $\mu_-(E)$ is the slope of the lowest slope $\mu$-destabilizing quotient of $E$. We set

$$\mathcal{A}_s = \{ A \in D(S) \mid A^i = 0, i \notin \{0, -1\}, H^{-1}(A) \in F_s, H^0(A) \in T_s \}.$$ 

A group homomorphism $Z_{s,t}$ takes the Chern character $\text{ch}(A)$ into

$$Z_{s,t}(A) = \langle e^{(s+ti)t}, \text{ch}(A) \rangle = -\chi + 2dcs + dr(t^2 - s^2) + 2tdi(c - rs).$$

For each $A \in \mathcal{A}_s$ the slope $\mu_{s,t}(A)$ of $A$ is given by:

$$\mu_{s,t}(A) = \frac{-\text{Re}(Z_{s,t}(A))}{\text{Im}(Z_{s,t}(A))}$$

$$= \frac{\chi - 2dcs - dr(t^2 - s^2)}{2tdi(c - rs)}.$$ 

**Definition 2.1.** We say that $E \in \mathcal{A}_s$ is $\sigma_t$-stable (respectively, $\sigma_t$-semistable) if for all proper injections $F \rightarrow E$ in $\mathcal{A}_s$ we have $\mu_{s,t}(F) < \mu_{s,t}(E)$ ($\mu_{s,t}(F) \leq \mu_{s,t}(E)$, respectively). It is well known that these give sensible stability conditions on any smooth surface.

Then the slope of $E = L^n \otimes I_X$ where $\text{ch}(E) = (1, nl, n^2d - |X|)$ is

$$\mu_{s,t}(E) = \frac{n^2d - |X| - 2dns - d(t^2 - s^2)}{2t(n - s)}.$$ 

Note that $n > s$ as $E \in T_s$. 


Remark 2.2. Now suppose $F \in \mathcal{A}_s$ with $\text{ch}(F) = (r, d, \chi)$ destabilizes $L^n \otimes \mathcal{I}_X$. We have a short exact sequence $F \to E \to Q$ in $\mathcal{A}_s$. Taking cohomology we see that $H^{-1}(F) = 0$. Then $F \in T_s$ and so $c > rs$. Notice also that $H^{-1}(Q) \in F_s$ is torsion-free and since $0 \to H^{-1}(Q) \to F \to E$ is exact, $F$ is also torsion-free.

We also have

$$\mu_{s,t}(F) - \mu_{s,t}(E) \geq 0.$$ 

Therefore

$$\frac{\chi - 2dcs + dr(s^2 - t^2)}{2td(c - rs)} - \frac{n^2d - |X| - 2dns - d(t^2 - s^2)}{2td(n - s)} \geq 0. \quad (2.3)$$

Define $f(F, E)$ to be the numerator of (2.3), then

$$f(F, E) = (\chi - 2dcs + dr(s^2 - t^2))(n - s) - (n^2d - |X| - 2dns - d(t^2 - s^2))(c - rs)$$

$$= (n - s)[\chi - c(2ds - n^2d + |X| + 2dns + d(s^2 + t^2)) + r(n^2ds + |X|s - nd(s^2 + t^2))].$$

We shall be most interested in the case when $s = 0$. Then the destabilizing condition becomes

$$f(F, E) = n\chi - c(n^2d - |X| - dt^2) - drnt^2 \geq 0. \quad (2.4)$$

Therefore

$$n\chi - cn^2d + c|X| \geq (nr - c)dt^2,$$

and $c \leq nr$ because $\mu(H^{-1}(Q)) \leq 0$ and $\mu(F) \leq \mu(F/H^{-1}(Q)) \leq \mu(E)$. Hence a necessary condition for the existence of such a destabilizing object is

$$n\chi - cn^2d + c|X| > 0. \quad (2.5)$$

Recall from [Bri08, Prop 14.2] that in the “large volume limit” as $t \to \infty$, the $\sigma_t$-semistable objects $E$ with $\mu(E) > 0$ are exactly the Gieseker semistable sheaves (when $s = 0$). The case when $\mu(E) < 0$ is similar:

Proposition 2.3. For all $t \gg 0$. $F \in \mathcal{A}_0$ with $\mu(F) < 0$ is $\sigma_t$-semistable and $\mu(F) < 0$ if and only if $H^0(F)$ is supported in dimension 0 and $H^{-1}(F)$ is Gieseker semistable vector bundle.

Proof. Proof follows in same way as that of [Bri08, Prop 14.2] by observing that if $E$ is Bridgeland stable for all $t \gg 0$ then $H^0(E)$ must be supported in dim 0, otherwise $\mu_{0,t}(H^0(E))$ is finite and $H^0(E)$ destabilizes $E$ for $t \gg 0$. Moreover $H^{-1}(E)[1]$ is locally free since

$$0 \to \mathcal{O}_Z \to H^{-1}(E)[1] \to H^{-1}(E)^{**}[1] \to 0$$

is short exact sequence in $\mathcal{A}_0$ and then $\mathcal{O}_Z \to E$ would destabilize $E$. The fact that $H^{-1}(E)$ is Gieseker semistable follows in the same way as [Bri08].

\qed
Remark 2.4. An alternative approach can be seen using an observation of Yanagida and Yoshioka who show that the Bridgeland stability is preserved under \( \Delta \circ \) (see [YY12, Prop 2.9]) at least when \( c_1 \cdot \ell \neq 0 \). So if \( F \) is \( \sigma \)-semistable then \( F^\vee \) is \( \sigma \)-semistable and \( \mu(F^\vee) > 0 \). Then Prop. 14.2 in [Bri08] implies that \( F^\vee \) is \( G \)-semistable sheaf. Therefore \( F^{\vee \vee} \cong F \) takes required form. In particular, observe that \( H^0(F) \neq 0 \) exactly when \( F^\vee \) is not locally-free.

Remark 2.5. Huybrechts ([Huy08]) showed that \( \Phi[1] \) preserves \( A_0 \) and it can also be shown (see for example, [MM12]) that \( E \in A_0 \) is \( \sigma \)-stable if and only if \( \Phi(E) \) is \( \sigma_{1/t} \)-stable (and similarly for semistable).

3. Walls and Moduli Spaces

Definition 3.1. We let \( M_{BS,t}^{ch} \) denote the moduli space of \( \sigma \)-semistable objects in \( A_0 \). It is now well known that these exist (when non-empty) (see, for example [MM12]).

For example, in the large volume limit as \( t \to \infty \), \( M_{BS,t}^{ch} = M_{ch}^{GS} \) when \( c_1(ch) \cdot l > 0 \). Equality here means that the points represent exactly the same objects in \( \text{Coh}(S) \cap A_0 \) up to isomorphism.

It may happen for some value of \( t \) that the two moduli spaces are not equal. In fact, there will be a strictly decreasing sequence \( t_0, t_1, \ldots \) of values of \( t \) on either side of which \( M_{BS,t}^{ch} \) differ. We call these walls. (sometimes they are called mini-walls when we fix \( s \)). Our aim will be to identify these walls when \( ch = (0, l, \chi) \) and \( ch = (1, nl, n^2d - \vert x \vert) \). In the first case we show there are no walls. The following generalizes [Mac12, Prop 4.2]:

Lemma 3.2. For \( s = 0 \) there are no walls for \( ch = (0, l, \chi) \) for any \( \chi \in \mathbb{Z} \).

Proof. Let \( F \in T_0 \) with \( ch(E) = (r, c, k) \) and \( c \geq 0 \) destabilize \( E \in M_{ch}^{GS}(0, l, \chi) \). Then we have the following exact sequence in \( \text{Coh}(X) \)

\[
0 \to F' \to F \to E \to Q \to 0,
\]

where \( F' \in F_0 \). Factoring \( F \to E \) via \( K \hookrightarrow E \), we see that \( K \neq 0 \) implies that \( c_1(K) = l \) as \( E \) is pure. So \( c_1(F') = (c - 1)l \). Then \( Q \) is supported in dimension 0. Let \( \chi(Q) = a \geq 0 \) then \( ch(F') = (r, (c - 1)l, k - \chi + a) \). Since \( F' \in F_0 \), then \( c \leq 0 \). Therefore \( c = 1 \). From (2.2), the stability condition is given by \( k > \chi \). On the other hand, \( F' \) is \( G \)-stable by Proposition 2.3, then \( (k - \chi + a)r \leq 0 \) so \( k < \chi \). Hence there is no walls. \( \square \)

Remark 3.3. In particular, by Remark 2.5, \( ch = (\chi, -l, 0) \) and \( ch = (\chi, l, 0) \) also have no walls. Hence, for all \( t > 0 \)

\[
M_{(\chi,l,0)}^{BS,t} = M_{(\chi,l,0)}^{GS}.
\]
Definition 3.4. We shall say that the moduli space $\mathcal{M}_{(r, c_1, \chi)}^{BS_t}$ of Bridgeland stable sheaves of Chern character $(r, c_1, \chi)$ satisfies $IT_0$ (respectively $WIT_0$) if and only if for each $E$ representing an object of $\mathcal{M}_{(r, c_1, \chi)}^{BS_t}$, $E$ satisfies $IT_0$ (respectively $WIT_0$).

For example $\mathcal{M}_{(1, n, 2d-k)}^{BS_t}$ is $IT_0$ for all $t$ if and only if $L^n$ is $(k-1)$-very ample and so $\phi_L(n) \geq k - 1$.

The following technical result will be useful in the next section:

Lemma 3.5. $\mathcal{M}_{(0, l, \chi)}^{GS}$ is $IT_0$ if and only if $\chi \geq d + 1$.

Proof. We use Proposition 2.3, Remarks 2.4 and 3.3, and Lemma 3.2 to give isomorphisms

$$\mathcal{M}^{GS}_{(0, l, \chi)} \cong [\mathcal{M}^{BS}_{(x, l, 0)}] \Delta \cong \mathcal{M}^{GS}_{(x, l, 0)}$$

for all $t > 0$. Then $[E] \in \mathcal{M}^{GS}_{(0, l, \chi)}$ is $IT_0$ if and only of $[\Phi(E)[1]] \in \mathcal{M}^{BS}_{(x, l, 0)} \cap \mathcal{M}^{GS}_{(x, l, 0)}[1]$ which holds if and only if $\Delta \Phi(E) \in \mathcal{M}^{GS}_{(x, l, 0)}$ is locally-free. But, since all representative sheaves of $\mathcal{M}^{GS}_{(x, l, 0)}$ must be $\mu$-stable, we see that there are non-locally-free sheaves in $\mathcal{M}^{GS}_{(x, l, 0)}$ if and only if $\mathcal{M}^{GS}_{(x, l, 1)} \neq \emptyset$. This happens exactly when the Bogomolov inequality fails for the Chern character $(\chi, l, 1)$, in other words when $\chi \leq d$ as required. \hfill \Box

4. $(1, d)$-Polarization Line Bundles

Let $L$ be $(1, d)$-polarization line bundle on an abelian surface $S$ with $c_1(L) = l$ and $l^2 = 2d$. In this section we will prove some lemmas that help us to find the value of $\phi_L(n)$.

Proposition 4.1. If $L$ is an ample line bundle of type $(1, d)$, $d \geq 1$ on an abelian surface $X$, then

$$\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$$

Proof. The Chern character of $E = L \otimes \mathcal{I}_X$ is $ch(L \otimes \mathcal{I}_X) = (1, l, d - |X|)$. Then $ch(\Phi(E)) = (d - |X|, -l, 1)$. Such objects are all locally-free sheaves exactly when there are no stable sheaves with Chern character $(d - |X|, -l, 2)$. These exist exactly when the Bogomolov inequality holds for such a Chern character (see Definition 3.1). This gives us the criterion $2(d - |X|) \leq d$, so $|X| \leq \left\lfloor \frac{d}{2} \right\rfloor = \left\lfloor \frac{d-1}{2} \right\rfloor$. Hence $\mathcal{M}^{GS}_{(1, l, d-|X|)}$ is $IT_0$ if and only if $|X| \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. Then $\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$. \hfill \Box

Proposition 4.2. Let $(S, L)$ be an irreducible $(1, d)$-polarized abelian surface, then $\phi_L(n) \leq 2(n-1)d - 2$ for $n > 1$. 
Proof. By Lemma 3.5, there is \( Q \) with Chern character \( \text{ch}(Q) = (0, l, d) \) which is not IT\(_0\). Since \( \chi(L^{-n+1} \otimes Q) = d(3 - 2n) < 0 \) for \( n > 1 \) so \( \text{Ext}^1(Q, L^{-n}) \neq 0 \). Pick a non trivial extension

\[
0 \to L^{-n+1} \to E \to Q \to 0
\]

and suppose \( T \hookrightarrow E \) is its torsion subsheaf. Then we have the following diagram:

\[
\begin{array}{ccc}
F & \longrightarrow & Q/T \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L^{-n+1} \\
\downarrow & & \downarrow \\
T & \longrightarrow & L^{-n+1} \\
\end{array}
\]

Then \( Q/T \) must be supported in dimension zero. But then \( \text{Ext}^1(Q/T, L^{-n}) = 0 \) and so \( L^{-n+1} \to F \to Q/T \) must split, which is impossible. Hence \( T = 0 \). Then \( E \cong L^n \otimes I_X \) for some \( X \) of length \( |X| = 2d(n - 1) \) and \( E \) is not IT\(_0\). \( \square \)

The following theorem proves that the upper bound of \( \phi_L(n) \) in Proposition 4.2 is sharp.

**Theorem 4.3.** Let \((S, L)\) be a \((1, d)\) polarized abelian surface with \( \text{NS}(S) = \langle L \rangle \), then \( \phi(L^n) = 2d(n - 1) - 2 \).

**Proof.** By Proposition 4.2 we need to show that \( \phi_L(n) \geq 2(n - 1)d - 2 \), and we do this by showing that \( \mathcal{M}^{\text{BST},t}_{(1, n, d^2 - k)} \) is IT\(_0\) for all \( t \) and \( k = 2d(n - 1) - 1 \). Suppose that \( E \cong L^n \otimes I_X \) where \( |X| = 2d(n - 1) - 1 \) is not IT\(_0\) and \( \Phi(E) \) is \( \sigma_t \)-stable for all \( t \gg 0 \). Then \( \hat{E} \) is a two-step complex such that \( H^{-1}(\hat{E}) \) is \( G \)-stable and \( H^0(\hat{E}) \) is in the form \( \mathcal{O}_Z \), by Proposition 2.3. The Chern character \( \text{ch}(H^{-1}(\hat{E})) = ((n - 1)^2d + d + 1, n, 1 + |Z|) \). By Bogomolov

\[
\left((n - 1)^2d + d + 1\right)(1 + |Z|) \leq n^2d.
\]

Therefore \( |Z| \leq \frac{2d(n - 1) - 1}{dn^2 - 2d(n - 1) + 1} \). But \( d(n - 2)^2 + 2 > 0 \) and so

\[
dn^2 - 2d(n - 1) + 1 > 2d(n - 1) - 1.
\]

Hence, \( |Z| < 1 \). Therefore \( H^0(\hat{E}) = 0 \) and so \( E \) is IT\(_0\). If \( E \) is \( \sigma_t \)-stable for all \( t \), then it follows that \( \Phi(E) \) is \( \sigma_t \)-stable for all \( t \) (and so also for \( t \gg 0 \)). This happens when there are no walls. Unfortunately, there are walls in general. To finish the proof we will identify all the walls and show that all \( \sigma_t \)-semistable objects are IT\(_0\) directly.

**Lemma 4.4.** If \( e \in \mathcal{A}_0 \) destabilizes \( L^n \otimes I_X \) with \( |X| = 2d(n - 1) - 1 \), then \( e \) is a rank 1 torsion-free sheaf.
Proof. By Remark 2.2, \( H^{-1}(e) = 0 \) and \( E := H^0(e) \cong e \) is torsion-free. Suppose \( \text{ch}(E) = (r, g, \chi) \) and let \( q = L^n \otimes \mathcal{I}_X/E \) in \( \mathcal{A}_0 \). Then we have a long exact sequence in \( \text{Coh}(S) \):

\[
(4.2) \quad 0 \to H^{-1}(q) \to E \to L^n \otimes \mathcal{I}_X \to H^0(q) \to 0
\]

Since \( H^{-1}(q) \in \mathcal{F}_0 \) and \( E \in \mathcal{A}_0 \), then \( \mu(E) \geq 0 \geq \mu(H^{-1}(Q)) \).

Then there is an integer \( g > 0 \) such that \( c_1(E) = (nr-g)\ell \). Then \( c_1(H^{-1}(Q)) = (nr-g-nm)\ell \leq 0 \) where \( m \geq 0 \). Therefore \( 0 < nr - g \leq n - m \leq n \). Hence, \( c_1(E) \) can be written as \( c \equiv (n-c)\ell \) for some positive integer \( c < n \). Since any destabilizer of a sheaf can be taken as a \( G \)-stable, the Bogomolov inequality allow us to write

\[
\chi(E) = \frac{(n-c)^2d}{r} - k,
\]

for some rational number \( k \geq 0 \). Since \( E \) is a destabilizer of \( L^n \otimes \mathcal{I}_X \), we have \( f(E, L^n \otimes \mathcal{I}_X) > 0 \). Therefore from a condition (2.5), we get

\[
(4.3) \quad \left( (n-c)^2d - kr \right)n - (n^2d - 2dn + 2d + 1)(n-c)r > 0,
\]

Rearrange (4.3), we obtain

\[
(4.4) \quad (n-c)(- (n-1)^2dr - dr - r + dn^2 - cdn) > krn > 0
\]

As \( n-c > 0 \), then we get walls if \( -(n-1)^2dr - dr - r + dn^2 - cdn > 0 \) so

\[
\frac{1}{r} > \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} + \frac{1}{dn^2} \geq \frac{1}{2},
\]

for all \( n \), since \( d > 0 \). Hence \( r = 1 \). \( \square \)

**Remark 4.5.** The previous lemma proved that the Chern character of any destabilizer of \( L^n \otimes \mathcal{I}_X \) is given by \( \text{ch}(E) = (1, n-c, (n-c)^2d - k) \) which means that \( E \) is in the form \( L^{n-c} \otimes \mathcal{P}_x \otimes \mathcal{I}_Y \), for some \( \hat{x} \in \hat{S} \) and \( |Y| = k \).

**Lemma 4.6.** If \( L^{n-m} \otimes \mathcal{P}_x \otimes \mathcal{I}_Y \) destabilizes \( L^n \otimes \mathcal{I}_X \) where \( |X| = 2d(n-1) - 1 \), then \( m = 1 \).

**Proof.** We assume, without loss of generality, that \( \hat{x} = 0 \). Suppose that \( F = L^{n-m} \otimes \mathcal{I}_Y \) with \( |Y| = k \) is destabilizing \( E = L^n \otimes \mathcal{I}_X \), then \( \mu(F) - \mu(E) > 0 \). Therefore from a condition (4.4), we get

\[
(n-m)(- (n-1)^2dr - dr - r + dn^2 - dmn) > krn.
\]

Then we get walls if and only if

\[
(4.5) \quad \left(1 - \frac{m}{n}\right)(2dn - nmdr - 2d) > k \geq 0.
\]

Since \( 1 - \frac{m}{n} \) is positive, this happens if and only if \( 2dn - nmdr - 2d > 0 \) and then

\[
2 \geq 2 - \frac{d^{-1}}{nd} > m > 0.
\]

Hence, \( m = 1 \). \( \square \)
Lemma 4.7. If $L^{n-1} \otimes \mathcal{P}_x \otimes \mathcal{I}_Y$ destabilizes $L^n \otimes \mathcal{I}_X$ for some $X$ where $|X| = 2d(n-1) - 1$, then $|Y| < d(n-2) \leq 2d(n-2) - 1$.

Proof. Without loss of generality we assume $\hat{x} = 0$. Take $F, E$ as Lemma 4.6, then from 4.5 we get:

$$
(1 - \frac{1}{n}) (dn - 2d) > |Y| \geq 0
$$

Since $0 < 1 - \frac{1}{n} < 1$, then $dn - 2d > |Y|$. □

To complete proof of Theorem 4.3, we now induct on $n \geq 2$. If $n = 2$, then $d(n-2) = 0$ and so there are no walls which establishes the result for $n = 2$.

Suppose that the statement is true for $n - 1 \geq 2$. i.e. $L^{n-1} \otimes \mathcal{I}_X$ is IT$_0$ for all $X$ with $|X| = 2d(n-1) - 1$ To prove that $L^n \otimes \mathcal{I}_X$ is IT$_0$ for all $X$ with $|X| = 2d(n-1) - 1$, we know that the only possible walls are given by $L^{n-1} \otimes \mathcal{I}_Y$ where $|Y| < 2d((n-1) - 1) - 1$. Then there is a short exact sequence

$$0 \to L^{n-1} \otimes \mathcal{I}_Y \to L^n \otimes \mathcal{I}_X \to Q \to 0$$

By induction, $L^{n-1} \otimes \mathcal{I}_Y$ is IT$_0$ and by Lemma 3.5, $Q$ is IT$_0$ as well, since

$$
\chi(Q) = -(n-1)^2 d + |Y| + n^2 d - 2d(n-1) + 1 = 1 + d + |Y| \geq d + 1.
$$

Hence $L^n \otimes \mathcal{I}_X$ is IT$_0$ for all $X$ with $|X| = 2d(n-1) - 1$. □

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