1 Introduction

A natural and efficient method for producing numerous examples of interesting schemes is to consider the vanishing locus of the minors of a homogeneous polynomial matrix. If the matrix satisfies certain genericity conditions then the resulting schemes have a number of well described properties. These objects have been studied in both a classical context and a modern context and go by the name of determinantal schemes. Some of the classical schemes that can be constructed in this manner are the Segre varieties, the rational normal scrolls, and the Veronese varieties. In fact, it can be shown (cf. [10]) that any projective variety is isomorphic to a determinantal variety arising from a matrix with linear entries! Due to their important role in algebraic geometry and commutative algebra, determinantal schemes and their associated rings have both merited and received considerable attention in the literature. Groundbreaking work has been carried out by a number of different authors; we direct the reader to the two excellent sources [1] and [8] for background, history, and a list of important papers.
A homogeneous polynomial matrix can be viewed as defining a map between free modules defined over the underlying polynomial ring. Associated to such a map are a number of complexes. The most important of these are the Eagon-Northcott and Buchsbaum-Rim complexes. Under appropriate genericity conditions, these complexes are exact and it is in this special situation where we will focus our attention. Buchsbaum-Rim sheaves are a family of sheaves associated to the sheafified Buchsbaum-Rim complex. In particular, a first Buchsbaum-Rim sheaf is the kernel of a generically surjective map between two direct sums of line bundles, whose cokernel is supported in the correct codimension. This family of sheaves is described and studied in the two papers [15], [14].

A certain aspect of these sheaves was found to bear an interesting relationship to earlier work of the first author. In [13], Kreuzer obtained the following characterization of 0-dimensional complete intersections in $\mathbb{P}^3$:

**Theorem** ([13] Theorem 1.3) A 0-dimensional subscheme $Y \subset \mathbb{P}^3$ is a complete intersection if and only if $Y$ is arithmetically Gorenstein and there exists an arithmetically Cohen-Macaulay, l.c.i. curve $C$ such that $Y$ is the associated subscheme of an effective Cartier divisor on $C$ and $O_C(Y) \cong \omega_C(-a_Y)$ is globally generated.

Complete intersections form a very important subset of the more general class of standard determinantal schemes (i.e the determinantal subschemes of $\mathbb{P}^n$ arising from the maximal minors of a homogeneous matrix of the “right size”). One immediately observes that to every standard determinantal scheme is associated a number of Buchsbaum-Rim sheaves and to every Buchsbaum-Rim sheaf is associated a standard determinantal ideal. We say a standard determinantal scheme is “good” if one can delete a generalized row from its corresponding matrix and have the maximal minors of the resulting submatrix define a scheme of the expected codimension. In particular, complete intersections are good, as are most standard determinantal schemes.

The paper is organized as follows. In Section 2 we provide the necessary background information. The next section is the heart of the paper. Here we give several characterizations of standard and good determinantal subschemes. Some of these results are summarized in the following:

**Theorem** Let $X$ be a subscheme of $\mathbb{P}^n$ with $\text{codim } X \geq 2$. The following are equivalent.

(a) $X$ is a good determinantal scheme of codimension $r + 1$.

(b) $X$ is the zero-locus of a regular section of the dual of a first Buchsbaum-Rim sheaf of rank $r + 1$.

(c) $X$ is standard determinantal and locally a complete intersection outside a subscheme $Y \subset X$ of codimension $r + 2$ in $\mathbb{P}^n$. 
Several of our results in Section 3 involve the cokernel of the map of free modules mentioned above. We do not quote these results here since we need some notation from Section 2. These results are important in Section 4, though, where we give our main generalizations of Kreuzer’s theorem. We mention two of these.

**Corollary** Let $X \subset \mathbb{P}^n$ be a subscheme of codimension $r + 1 \geq 3$. Then $X$ is a complete intersection if and only if $X$ is arithmetically Gorenstein and there is a good determinantal subscheme $S \subset \mathbb{P}^n$ of codimension $r$ and a canonically defined sheaf $\mathcal{M}_S$ on $S$ (in codimension two, $\mathcal{M}_S \cong \omega_S$ up to twist) such that $X \subset S$ is the zero-locus of a regular section $t \in H^0_S(S, \mathcal{M}_S)$. Furthermore, $S$ and $\mathcal{M}_S$ can be chosen so that $\mathcal{M}_S$ is globally generated.

**Corollary** Suppose $X \subset \mathbb{P}^3$ is zero-dimensional. Then the following are equivalent:

(a) $X$ is good determinantal;

(b) $X$ is standard determinantal and a local complete intersection;

(c) There is an arithmetically Cohen-Macaulay curve $S$, which is a local complete intersection, such that $X$ is a subcanonical Cartier divisor on $S$.

Furthermore, $X$ is defined by a $t \times (t + r)$ matrix if and only if the Cohen-Macaulay type of $X$ is $\binom{r+t-1}{r}$ and that of $S$ is $\binom{r+t-1}{r-1}$.

The last sentence of this corollary gives the connection to Kreuzer’s theorem: recall that the only standard determinantal subschemes with Cohen-Macaulay type 1 (i.e. arithmetically Gorenstein) are complete intersections. In a similar way we characterize good determinantal subschemes of $\mathbb{P}^n$ of any codimension, with special, stronger, results in the case of zeroschemes and the case of codimension two subschemes. We close with a number of examples.

### 2 Preliminaries

Let $R = k[x_0, x_1, \ldots, x_n]$ be a polynomial ring with the standard grading, where $k$ is an infinite field and $n \geq 2$. For any sheaf $\mathcal{F}$ on $\mathbb{P}^n$, we define $H^i_*(\mathbb{P}^n, \mathcal{F}) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t))$. For any scheme $V \subset \mathbb{P}^n$, $I_V$ denotes the saturated homogeneous ideal of $V$ and $\mathcal{I}_V$ denotes the ideal sheaf of $V$ (hence $I_V = H^0_*(\mathbb{P}^n, \mathcal{I}_V)$).

**Definition 2.1** If $A$ is a homogeneous matrix, we denote by $I(A)$ the ideal of maximal minors of $A$. A codimension $r + 1$ scheme, $X$, in $\mathbb{P}^n = \text{Proj}(R)$ will be called a **standard determinantal scheme** if $I_X = I(A)$ for some homogeneous $t \times (t + r)$ matrix, $A$. $X$ will be
called a \textit{good determinantal scheme} if additionally, $A$ contains a $(t - 1) \times (t + r)$ submatrix (allowing a change of basis if necessary—see Example 4.10) whose ideal of maximal minors defines a scheme of codimension $r + 2$. In a similar way we define standard and good determinantal ideals. \hfill \Box

\textbf{Example 2.2} The ideal defined by the maximal minors of the matrix
\[
\begin{bmatrix}
x_1 & x_2 & x_3 & 0 \\
0 & x_1 & x_2 & x_3
\end{bmatrix}
\]
is an example of a standard determinantal ideal which is not good. Note that this ideal is the square of the ideal of a point in $\mathbb{P}^3$, and is not a local complete intersection (see Proposition 3.2). \hfill \Box

Note that standard determinantal schemes form an important subclass of the more general notion of determinantal schemes, where smaller minors are allowed (among other generalizations). See for instance [1], [8], [10].

\textbf{Remark 2.3} In the next section we will make a deeper study of good determinantal schemes. For now, though, we observe the following. Let $X$ be a standard determinantal scheme coming from a $t \times (t + r)$ matrix $A$. Then $X$ is good if and only if there is a $(t - 1) \times (t - 1)$ minor of $A$ which does not vanish on any component of $X$ (possibly after making a change of basis). In particular, we formally include the possibility that $t = 1$, and we include the complete intersections among the good determinantal schemes. \hfill \Box

\textbf{Fact 2.4} Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves of ranks $f$ and $g$ respectively on a smooth variety $Y$. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a generically surjective homomorphism. We can associate to $\phi$ an Eagon-Northcott complex
\begin{equation}
0 \to \wedge^f \mathcal{F} \otimes (S^{f-g} \mathcal{G})^\vee \otimes \wedge^g \mathcal{G}^\vee \to \wedge^{f-1} \mathcal{F} \otimes (S^{f-g-1} \mathcal{G})^\vee \otimes \wedge^g \mathcal{G}^\vee \to \ldots \\
\to \wedge^{g+1} \mathcal{F} \otimes \mathcal{G}^\vee \otimes \wedge^g \mathcal{G}^\vee \to \wedge^g \mathcal{F} \otimes \wedge^g \mathcal{G}^\vee \xrightarrow{\wedge^g \phi} \mathcal{O}_Y \to 0
\end{equation}
and a Buchsbaum-Rim complex
\begin{equation}
0 \to \wedge^f \mathcal{F} \otimes S^{f-g-1} \mathcal{G}^\vee \otimes \wedge^g \mathcal{G}^\vee \to \wedge^{f-1} \mathcal{F} \otimes S^{f-g-2} \mathcal{G}^\vee \otimes \wedge^g \mathcal{G}^\vee \to \ldots \\
\to \wedge^{g+2} \mathcal{F} \otimes \mathcal{G}^\vee \otimes \wedge^g \mathcal{G}^\vee \to \wedge^{g+1} \mathcal{F} \otimes \wedge^g \mathcal{G}^\vee \xrightarrow{\wedge^g \phi} \mathcal{F} \xrightarrow{\phi} \mathcal{G} \to 0
\end{equation}
(see [4], [8], [3], [1], [2]). If the support of the cokernel of $\phi$ has the expected codimension $f - g + 1$ then these complexes are acyclic. \hfill \Box
The consequences of this fact will play a crucial role throughout the paper and they lead us to the following definition.

**Definition 2.5** Let $\mathcal{F}$ and $\mathcal{G}$ be two locally free sheaves which split as the sum of line bundles and let $\phi: \mathcal{F} \to \mathcal{G}$ be a generically surjective homomorphism whose cokernel is supported on a scheme with the “expected” codimension $f - g + 1$. As mentioned in the fact above, the Buchsbaum-Rim complex will be exact and provides a free resolution of the cokernel of the map $\phi$. The kernel of the map $\phi$ will be called a *first Buchsbaum-Rim sheaf*. We use the symbol $\mathcal{B}_\phi$ to represent such a sheaf.

More generally, the $i^{th}$ Buchsbaum-Rim sheaf associated to $\phi$ is the $(i + 1)^{st}$ syzygy sheaf in the Buchsbaum-Rim complex. However, in this paper we will use only the first Buchsbaum-Rim sheaves.

**Remark 2.6** In Fact 2.4 and Definition 2.5, we will allow the rank of $\mathcal{G}$ to be zero, and use the convention that even in this case, $\wedge^0 \mathcal{G} = \mathcal{O}_Y$. Moreover, the Buchsbaum-Rim complex becomes $0 \to \mathcal{F} \to \mathcal{F} \xrightarrow{\delta} 0$, and it follows that the sheafification of any free module is a first Buchsbaum-Rim sheaf.

In Fact 2.4 and Definition 2.5, we can also start with free modules $F$ and $G$, and we get Eagon-Northcott and Buchsbaum-Rim complexes of free modules. The corresponding kernel of the map $\phi$ will then be called a *first Buchsbaum-Rim module*. Note that in this context $\phi$ can be represented by a homogeneous matrix $\Phi$, and the image of $\wedge^g \phi$ is precisely $I(\Phi)$.

Note also that since first Buchsbaum-Rim sheaves (resp. modules) are second syzygy sheaves (resp. modules), they are reflexive.

**Fact 2.7** ([8] exer. 20.6 or [4]) Let $\Phi$ be a matrix whose ideal $I(\Phi)$ of maximal minors vanishes in the expected codimension, and so coker $\Phi$ has a corresponding Buchsbaum-Rim resolution. Then the annihilator of coker $\Phi$ is precisely $I(\Phi)$.

In this paper, we will often be interested in going in the opposite direction, starting with a standard determinantal ideal $J$ and considering the possible associated matrices and cokernels. With this in mind, we make the following definition.

**Definition 2.8** Let $X$ be a standard determinantal scheme of codimension $r + 1$ with corresponding ideal $I_X$. Then we set

$$
\mathcal{M}_X := \left\{ M \mid M \text{ is a f.g. graded } R\text{-module with } Ann_R M = I_X \text{ and a minimal presentation of the form } R^{r+\mu} \to R^\mu \to M \to 0 \right\}
$$
\( M_X \) is the set of possible cokernels of homogeneous matrices whose ideals of maximal minors are precisely \( I_X \). In some situations, \( M_X \) consists of just one element (up to isomorphism and twisting). For example, it can be shown that this happens if \( r = 1 \) (i.e. codimension 2, using Hilbert-Burch theory—see Corollary 4.2). \( M_X \) also consists of just one element if \( X \) is a complete intersection. We do not know the precise conditions which guarantee that all the elements of \( M_X \) are isomorphic up to twisting. In any case, we can at least show that the elements of \( M_X \) look very much alike:

**Lemma 2.9** The elements of \( M_X \) all have the same graded Betti numbers, up to twisting, and in particular come from matrices of the same size.

**Proof:** Let \( M_1, M_2 \in M_X \) and assume that \( M_i \) has \( t_i \) minimal generators, \( i = 1, 2 \). We may also assume that \( M_i \) is the cokernel of a \( t_i \times (t_i + r) \) matrix \( \Phi_i \). By \( \text{8 p. 494, } \text{Rad} (I(\Phi)) = \text{Rad} (\text{Ann}_R M_i) = \text{Rad} (I_X) \). Hence \( I(\Phi) \) is a homogeneous matrix defining a subscheme of \( \mathbb{P}^n \) of codimension \( r + 1 \), the expected codimension, and we may apply the Eagon-Northcott complex to get a minimal free resolution for \( I(\Phi) = I_X \). Hence \( I_X \) has \((r + t_1) \) minimal generators, and \( t_1 = t_2 \).

Now let \( M \in M_X \) and assume that it has \( t \) minimal generators. There is a minimal free resolution

\[
\cdots \rightarrow F^0 \rightarrow G \rightarrow M \rightarrow 0
\]

where \( rk F = t + r \) and \( rk G = t \). As above, \( I(\Phi) \) defines a subscheme of codimension \( r + 1 \), and so the Buchsbaum-Rim complex resolves \( M \) and we are done. \( \Box \)

**Proposition 2.10** Let \( \mathcal{F} \) and \( \mathcal{G} \) be locally free sheaves of ranks \( f \) and \( g \) respectively on \( \mathbb{P}^n \). Let \( \phi : \mathcal{F} \rightarrow \mathcal{G} \) be a generically surjective homomorphism. Assume the cokernel of \( \phi \) is supported on a scheme of codimension \( f - g + 1 \). Let \( I_\phi \) denote the homogeneous ideal of the scheme determined by the cokernel of \( \wedge^g \phi \). Let \( I_s \) denote the homogeneous ideal of the zero-locus of a section, \( s \in H^0(\mathbb{P}^n, \mathcal{B}_\phi) \) (where \( \mathcal{B}_\phi \) denotes the local first Buchsbaum-Rim sheaf of \( \phi \)). Let \( I_t \) denote the homogeneous ideal of the zero-locus of a section, \( t \in H^0(\mathbb{P}^n, \mathcal{B}_\phi^*) \) (where \( \mathcal{B}_\phi^* \) denotes the dual of \( \mathcal{B}_\phi \)). Then for any such section, \( I_s \subset I_\phi \) and \( I_t \subset I_\phi \).

**Proof:** Locally, we can represent the map \( \phi \) by an \( f \times g \) matrix, \( A \). In the same local coordinates, the map from \( \wedge^{g+1} \mathcal{F} \otimes \wedge^g \mathcal{G}^\vee \) to \( \mathcal{F} \) (in the Buchsbaum-Rim complex associated to \( \phi \)) can be expressed by a matrix, \( M \). The entries of \( M \) can be written in terms of \( A \) as follows. Let \( I_A \) denote the ideal of maximal minors of the matrix \( A \). \( I_A \) locally describes the scheme defined by \( I_\phi \). Each column in the matrix, \( M \), arises from choosing \( t + 1 \) columns of the matrix \( A \) and considering all \( t \times t \) minors of this submatrix of \( A \). Thus, each entry in the matrix \( M \) is an element of \( I_A \). Locally, sections of the first Buchsbaum-Rim sheaf of \( \phi \) are determined by an element of the column space of \( M \) (considered as a module). An
immediate consequence of this fact is that the vanishing locus of any section of the first Buchsbaum-Rim sheaf of $\phi$ or the dual of the first Buchsbaum-Rim sheaf of $\phi$ will contain the scheme defined by $I_\phi$. □

Remark 2.11 For clarity, and because of its importance, we restrict ourselves to determinantal subschemes of projective space in the body of this paper. However, the reader will observe that many of our arguments hold true for subschemes of a smooth projective variety and some even for determinantal ideals of an arbitrary commutative ring. □

3 Characterizations of Good Determinantal Schemes

In [15] and [14], regular sections of first Buchsbaum-Rim sheaves were considered, and it was shown that they possess many interesting properties. For example, a regular section of a first Buchsbaum-Rim sheaf of odd rank has a zero-locus whose top dimensional part is arithmetically Gorenstein.

In this paper we are primarily concerned with regular sections of the dual of a first Buchsbaum-Rim sheaf. Our first result gives a property which is analogous to the ones mentioned above for the first Buchsbaum-Rim sheaves.

Theorem 3.1 Let $X$ be a subscheme of $\mathbb{P}^n$ with $\text{codim } X \geq 2$. The following are equivalent.

(a) $X$ is a good determinantal scheme of codimension $r + 1$.

(b) $X$ is the zero-locus of a regular section of the dual of a first Buchsbaum-Rim sheaf of rank $r + 1$.

Proof: We first prove (a) $\Rightarrow$ (b). By assumption there is a homomorphism $\Phi$ such that $I_X = I(\Phi)$, and we have an exact sequence

$$0 \to B \to F \xrightarrow{\Phi} G \to \text{coker } \Phi \to 0 \tag{3}$$

where $rk \ G = t$, $rk \ F = t + r$ and $B$ is a first Buchsbaum-Rim module.

If $t = 1$ then $I(\Phi)$ is a complete intersection of height $r + 1$, which can be viewed as a section of (the dual of) a free module of rank $r + 1$. By Remark 2.6, a free module is a first Buchsbaum-Rim module. Hence we can assume from now on that $t \geq 2$.

Since $X$ is a good determinantal scheme, there is a projection $\pi : G \to G'$, where $G'$ has rank $t - 1$, $G'$ is obtained from $G$ by removing one free summand $R(a)$, and such that
\[ ht(I(\pi \circ \Phi)) = r + 2. \] We get a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow R(a) \rightarrow R(a) \rightarrow 0 \\
\downarrow \\
0 \rightarrow B \rightarrow F \xrightarrow{\Phi} G \rightarrow \text{coker} \Phi \rightarrow 0 \\
\| \downarrow \pi \\
0 \rightarrow B' \rightarrow F' \xrightarrow{\Phi'} G' \rightarrow \text{coker} \Phi' \rightarrow 0 \\
\downarrow \\
0 \rightarrow 0
\end{array}
\] (4)

Let \( \alpha \) be the induced injection from \( B \) to \( B' \). Twist everything in (4) by \(-a\) and relabel, so that the Snake Lemma gives that \( I = \text{coker} \alpha \) is an ideal and we have an exact sequence

\[ 0 \rightarrow R/I \rightarrow \text{coker} \Phi \rightarrow \text{coker} \Phi' \rightarrow 0 \] (5)

It follows that \( I_X = I(\Phi) = \text{Ann}(\text{coker} \Phi) \subset I \) (see Fact 2.7), where \( I_X \) is the saturated ideal of \( X \).

On the other hand, it follows from the same exact sequence that

\[ \text{Ann}(\text{coker} \Phi') \cdot I \subset \text{Ann}(\text{coker} \Phi) = I(\Phi) = I_X. \]

But since \( X \) is good determinantal, it follows that \( I(\Phi') = \text{Ann}(\text{coker} \Phi') \) and \( ht(I(\Phi')) > ht(I(\Phi)) \). Hence \( I \subset I(\Phi) \) and so we conclude \( I = I(\Phi) = I_X \). But then we have a short exact sequence

\[ 0 \rightarrow B \rightarrow B' \rightarrow I_X \rightarrow 0 \]

and so by sheafifying, it follows that \( X \) is the zero-locus of a regular section of the dual of the first Buchsbaum-Rim sheaf \( B' \) as claimed. (Note that \( B' \) is reflexive—see Remark 2.6.)

We now prove (b) \( \rightarrow \) (a). Assume that \( X \) is the zero-locus of a regular section of a sheaf \( (B')^* \), where \( B' \) is the sheafification of a first Buchsbaum-Rim module \( B' \) of rank \( r + 1 \). We are thus given exact sequences (after possibly replacing \( B' \) by a suitable twist)

\[ 0 \rightarrow B' \rightarrow F \xrightarrow{\Phi'} G \rightarrow \text{coker} \Phi' \rightarrow 0 \] (6)

and

\[ 0 \rightarrow R \rightarrow (B')^* \rightarrow Q \rightarrow 0 \] (7)

such that \( rk F = t + r, \ rk G = t - 1, \text{Ann}(\text{coker} \Phi') = I(\Phi') \) (which has height \( r + 2 \)) and

\[ 0 \rightarrow Q^* \rightarrow B' \rightarrow I \rightarrow 0 \]
is exact (again, $B'$ is reflexive), where $I$ is an ideal whose saturation is $I_X$. One can check that dualizing (6) provides
\[ 0 \to G^* \to F^* \to (B')^* \to 0. \]
The mapping cone procedure applied to (7) then gives
\[ 0 \to R \oplus G^* \to F^* \to Q \to 0. \]
Dualizing this, we obtain the following commutative diagram:
\[
\begin{array}{c}
0 \\
\downarrow \\
0 \quad 0 \quad R \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \quad \to \quad Q^* \quad \to \quad F \quad \overset{\Phi}{\to} \quad R \oplus G \quad \to \quad \text{coker } \Phi \quad \to \quad 0 \\
\downarrow \\
0 \quad \to \quad B' \quad \to \quad F \quad \overset{\Phi'}{\to} \quad G \quad \to \quad \text{coker } \Phi' \quad \to \quad 0 \\
\downarrow \\
I \\
\downarrow \\
0
\end{array}
\]
The Snake Lemma then gives
\[ 0 \to R/I \to \text{coker } \Phi \to \text{coker } \Phi' \to 0. \]
It follows that
\[ I \cdot \text{Ann}(\text{coker } \Phi') = I \cdot I(\Phi') \subset \text{Ann}(\text{coker } \Phi). \]
Thus $ht(\text{Ann}(\text{coker } \Phi)) \geq r + 1$. Note that the maximal possible height of $\text{Ann}(\text{coker } \Phi)$ is $r + 1$, hence we get $ht(\text{Ann}(\text{coker } \Phi)) = r + 1$ and $Q^*$ is a first Buchsbaum-Rim module. From the Buchsbaum-Rim complex one can then check that $H^1_*(\mathbb{P}^n, Q^*) = 0$, and hence $I = I_X$ is saturated. Then as in the first part we get $I_X = I(\Phi)$, as desired. \qed

We now give a result which characterizes the good determinantal schemes among the standard determinantal schemes. We use the set $\mathcal{M}_X$ introduced in Definition \ref{def:M_X}.

**Proposition 3.2** Suppose that $X$ is a standard determinantal scheme of codimension $r + 1$. Then the following are equivalent.

(a) $X$ is good determinantal;
(b) There is an $M_X \in \mathcal{M}_X$ and an embedding $R/I_X \hookrightarrow M_X$ whose image is a minimal generator of $M_X$ as an $R$-module, and whose cokernel is supported on a subscheme of codimension $\geq r + 2$.

(c) There is an element $M_X \in \mathcal{M}_X$ which is an ideal in $R/I_X$ of positive height.

Furthermore, if any of the above conditions hold then $X$ is a local complete intersection outside a subscheme $Y \subset \mathbb{P}^n$ of codimension $r + 2$.

Remark 3.3 The first two parts of the above proposition do not even require that the field be infinite. □

Proof of 3.2 We begin with (a) ⇒ (b). Assume that $X$ is a good determinantal scheme arising from a homogeneous matrix $\Phi$. As in the proof of Theorem 3.1 (see the diagram (4)), we have (after possibly twisting) a commutative diagram

\[
\begin{array}{ccccccccc}
0 & & 0 & & R & \rightarrow & R/I_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B & \rightarrow & F & \xrightarrow{\Phi} & G & \rightarrow & M_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \pi & & \downarrow \\
0 & \rightarrow & B' & \rightarrow & F' & \xrightarrow{\Phi'} & G' & \rightarrow & M_Y & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
I_X & & 0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

(8)

where $rk F = t + r$, $rk G = t$, $rk G' = t - 1$, $\Phi'$ is obtained by deleting a suitable row of $\Phi$, $Y$ is the codimension $r + 2$ scheme defined by the maximal minors of $\Phi'$, $B$ and $B'$ are the kernels of $\Phi$ and $\Phi'$, respectively, and $M_X$ and $M_Y$ are the respective cokernels. Then all parts of (b) follow immediately.

This diagram also proves the last part of the Proposition, since by Theorem 3.1 $X$ is the zero-locus of a section of $\mathcal{B}'$, the sheafification of $B'$, which is locally free of rank $r + 1$ outside $Y$.  

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We now prove (b) \(\Rightarrow\) (a). The assumptions in (b) imply a commutative diagram

\[
\begin{array}{cccccccc}
0 & 0 \\
\downarrow & \downarrow \\
R & \rightarrow & R/I_X & \rightarrow & 0 \\
\downarrow & \downarrow \\
F & \xrightarrow{\Phi} & G & \rightarrow & M_X & \rightarrow & 0 \\
\downarrow{\alpha} & \downarrow \\
G' & \xrightarrow{\beta} & \text{coker } s & \rightarrow & 0 \\
0 & 0 \\
\end{array}
\]

with \(\text{rk } F = t + r\), \(\text{rk } G = t\), \(\text{rk } G' = t - 1\). Define \(\Phi' = \alpha \circ \Phi\). One can then show that

\[
F \xrightarrow{\Phi'} G' \xrightarrow{\beta} \text{coker } s \rightarrow 0
\]

is exact. (Either use a mapping cone argument, splitting off \(R\), or else use a somewhat tedious diagram chase.)

The assumption on the support of the cokernel of \(s\) implies \(\text{height}(I(\Phi')) = r + 2\), so \(X\) is good, proving (a).

Now we prove (a) \(\Rightarrow\) (c). The assumption that \(X\) is good implies, in particular, that the ideal of \((t - 1) \times (t - 1)\) minors of \(\Phi\) has height \(\geq r + 2\). Hence after possibly making a change of basis, we can apply Remark 2.3 and Theorem A2.14 (p. 600) to obtain \(M_X = \text{coker } \Phi \cong J/I_X\), where \(J \subset R\) is an ideal of height \(\geq r + 2\), proving (c).

Finally we prove (c) \(\Rightarrow\) (b). Since \(M_X\) is an ideal of positive height in \(R/I_X\), we can find \(f \in R\) with \(\bar{f} = f \mod I_X \in M_X\) such that the map \(R/I_X \rightarrow M_X, 1 \mapsto \bar{f}\) is injective. We can even choose \(f\) so that \(\bar{f}\) is a minimal generator of \(M_X\), considered as an \(R\)-module. Then \(\text{coker } s \cong M_X/(\bar{f} \cdot R/I_X)\) shows that \(I_X + (f) \subset \text{Ann}_R(\text{coker } s)\), so \(\text{coker } s\) is supported on a subscheme of height \(\geq r + 2\). \(\square\)

Next, we want to give an intrinsic characterization of good determinantal subschemes.

**Theorem 3.4** Suppose that \(\text{codim } X = r + 1\). Then the following are equivalent:

(a) \(X\) is good determinantal;

(b) \(X\) is standard determinantal and locally a complete intersection outside a subscheme \(Y \subset X\) of codimension \(r + 2\) in \(\mathbb{P}^n\).

**Proof:** In view of Proposition 3.2, we only have to prove (b) \(\Rightarrow\) (a). We again start with the exact sequence

\[
0 \rightarrow B \rightarrow F \xrightarrow{\Phi} G \rightarrow M_X \rightarrow 0
\]
where $F$ and $G$ are free of rank $t + r$ and $t$ respectively.

Now let $P$ be a point of $X$ outside $Y$, with ideal $\mathfrak{p} \subset R$. By assumption, $X$ is a complete intersection at $P$. We first claim that $(M_X)_{\mathfrak{p}} \cong (R/I_X)_{\mathfrak{p}}$. To see this, we first note that localizing $\Phi$ at $\mathfrak{p}$, we can split off, say, $s$ direct summands until the resulting map is minimal. Then the ideal of maximal minors of this matrix has precisely $(r + t - s)$ minimal generators (Eagon-Northcott complex). On the other hand it is a complete intersection, hence $t - s = 1$ and the cokernel $(M_X)_{\mathfrak{p}}$ of $\Phi_{\mathfrak{p}}$ is as claimed.

Using the above isomorphism, we note that $(M_X)_{\mathfrak{p}}$ has exactly one minimal generator as an $R_{\mathfrak{p}}$-module. Then by [1], Proposition 16.3, it follows that the ideal of submaximal minors of $\Phi$ is not contained in $\mathfrak{p}$. Since $P$ was chosen to be any point outside of $Y$ and $\text{codim } Y = r + 2$, it follows that no component of $X$ lies in the ideal of submaximal minors. That is, the ideal of submaximal minors has height greater than that of $I_X$. Hence by [8], p. 600, Theorem A2.14, we can conclude that $M_X$ is an ideal in $R/I_X$ of positive height. Therefore $X$ is good determinantal, by Proposition [3.2], (c). 

**Remark 3.5** Recall that a subscheme of $\mathbb{P}^n$ is said to be a *generic complete intersection* if it is locally a complete intersection at all its components. In particular, every integral subscheme is a generic complete intersection. This notion occurs naturally in the Serre correspondence which relates reflexive sheaves and generic complete intersections of codimension two (cf., for example, [12]).

Since the locus of points at which a subscheme fails to be locally a complete intersection is closed, for a subscheme $X$ of codimension $r + 1$ the conditions being a generic complete intersection and being locally a complete intersection outside a subscheme $Y \subset X$ of codimension $r + 2$ in $\mathbb{P}^n$ are equivalent. Thus we can reformulate the last result as follows:

A subscheme is good determinantal if and only if it is standard determinantal and a generic complete intersection. 

**Lemma 3.6** Let $A$ be a ring and let $a \subset A$ be an ideal containing an $A$-regular element $f$. Let $b := fA :_A a = \text{Ann}_A(a/fA)$. Then $\text{Hom}_A(a, A) \cong b$.

**Proof:** If grade $a \geq 2$ then it is well-known that $\text{Hom}_A(a, A) \cong A$ (up to shift in the graded case). The interesting case is grade $a = 1$. However, we prove it in the general case. Our main application is to the graded case, where we assume that $a$ and $f$ are homogeneous; then we obtain an isomorphism of graded modules $\text{Hom}_A(a, A) \cong b(\deg f)$.

Consider the exact sequence

\[
0 \to A \to a \to a/fA \to 0
\]

\[
1 \leftrightarrow f
\]
Since $f$ is $A$-regular, dualizing provides

$$0 \to \text{Hom}_A(a/f A, A) \to \text{Hom}_A(a, A) \xrightarrow{\beta} \text{Hom}_A(A, A)$$

We first prove that, up to the isomorphism $\text{Hom}_A(A, A) \cong A$, we get $\text{Hom}_A(a, A) \subset b$. Let $\phi \in \text{Hom}_A(a, A)$ and let $\psi = \beta(\phi)$. Let $b := \psi(1) = \phi(f)$. Then for any $a \in A$ we have

$$\psi(a) = \phi(f \cdot a) = a \cdot b.$$

For any $a \in a$ we have

$$f \cdot \phi(a) = \phi(f \cdot a) = \psi(a) = a \cdot b.$$

Hence $b \cdot a \subset f \cdot A$, i.e. $b \in f A : A a = b$. It follows that $\text{Hom}_A(a, A) \cong \text{im} \beta \subset b$.

For the reverse inclusion we can define for any $b \in b$ a homomorphism $\phi \in \text{Hom}_A(a, A)$ as the composition of

$$a \to f A \quad \text{and} \quad f A \xrightarrow{\sim} A$$

Then $\phi(f) = b$. We conclude that $b = \text{im} \beta \cong \text{Hom}_A(a, A)$.  

**Theorem 3.7** Suppose that $r + 1 \geq 3$. Then

(a) $X$ is standard determinantal of codimension $r + 1$ if and only if there is a good determinantal subscheme $S \subset \mathbb{P}^n$ of codimension $r$ such that $X \subset S$ is the zero-locus of a regular section $t \in H^0(S, \tilde{M}_S) = M_S$ for some $M_S \in \mathcal{M}_S$.

(b) $X$ is good determinantal of codimension $r + 1$ if and only if there is a good determinantal subscheme $S \subset \mathbb{P}^n$ of codimension $r$, such that $X \subset S$ is the zero-locus of a regular section $t \in H^0(S, \tilde{M}_S) = M_S$ for some $M_S \in \mathcal{M}_S$, and the cokernel of this section is isomorphic to an ideal sheaf in $\mathcal{O}_X$ of positive height.

**Proof:** We first assume that $X$ is standard determinantal and we let $\Phi$ be a $t \times (t + r)$ homogeneous matrix with $I(\Phi) = I_X$. Adding a general row to $\Phi$ gives a homogeneous $(t + 1) \times (t + r)$ matrix $\Psi$ whose ideal of maximal minors defines a good determinantal scheme $S \supset X$ of codimension $r$. We have the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \ker \Psi & \to & F & \xrightarrow{\Psi} & G & \to & M_S & \to & 0 \\
| & & \downarrow & & \downarrow & & | & & \downarrow & & | \\
0 & \to & \ker \Phi & \to & F & \xrightarrow{\Phi} & G' & \to & M_X & \to & 0 \\
\end{array}
$$
where \( rk \, F = t + r, \, rk \, G' = t \) and \( rk \, G = t + 1 \). As in Theorem 3.1, after possibly twisting we get the exact sequence

\[
0 \to R/I_S(-\deg t) \xrightarrow{t} M_S \to M_X \to 0.
\]  

(9)

Since \( S \) is good by construction, Proposition 3.2 shows that Lemma 3.6 applies, setting \( A := R/I_S \) and \( a = M_S \). This gives

\[
\text{Hom}_A(M_S, A)(-\deg t) \cong \text{Ann}_A(M_X) \cong I_X/I_S.
\]

Now, dualizing (9) we get

\[
0 \xrightarrow{} \text{Hom}_A(M_X, A) \xrightarrow{} \text{Hom}_A(M_S, A) \xrightarrow{t^*} A(\deg t) \xrightarrow{} 0
\]

It follows that \( X \) is the zero-locus of \( t \), proving the direction \( \Rightarrow \) for case (a). In case (b), we are done by applying Proposition 3.2.

We now consider the direction \( \Leftarrow \). Again let \( A = R/I_S \), where \( I_S = I(\Psi) \) for some homogeneous \((t+1) \times (t+r)\) matrix \( \Psi \), and apply the mapping cone construction to the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
R \\
\downarrow \\
F \\
\downarrow coker \, t \\
0 \\
\end{array} \quad \begin{array}{ccc}
\to & A & \to 0 \\
\downarrow t & \downarrow \\
G & \to M_S \\
\downarrow coker \, t \\
0 \\
\end{array}
\]

where \( rk \, G = t + 1 \). This gives the exact sequence

\[
\cdots \to F \oplus R \xrightarrow{\Phi} G \to coker \, t \to 0
\]

Since \( S \) is good, Proposition 3.2 gives us that \( coker \, t \cong M_S/f \cdot A \) for some \( A \)-regular element \( f \in A \) (see the proof of (c) \( \Rightarrow \) (b)). It follows that \( \text{Ann}_R(coker \, t) \) has grade \( \geq 1 + \text{grade} \, I_S = r + 1 \), thus \( \text{grade} \, I(\Phi) = r + 1 \). Let \( Y \) be the subscheme defined by \( I(\Phi) \). Then we get as above that \( Y \) is the zero-locus of \( t \), and so \( X = Y \), and we are done in case (a). For case (b), again an application of Proposition 3.2 completes the argument since \( coker \, t \in M_X \).

\[ \Box \]

Note that Theorem 3.7 does not mention global generation, while Kreuzer’s theorem mentioned in the introduction does. Conjecture 3.8 and Remark 3.9 address this.
Conjecture 3.8 Given \( X \) a standard determinantal scheme as in Theorem 3.7, one can choose \( S \) and \( M_S \in \mathcal{M}_S \) such that \( X \subseteq S \) is the zero-locus of a regular section \( t \in H^0(S, \tilde{M}_S) \) and such that \( \tilde{M}_S \) is globally generated.

Remark 3.9 Consider a free presentation of \( M_X \) as in the proof of Theorem 3.7:

\[
0 \to B \to F \xrightarrow{\Phi} G \to M_X \to 0.
\]

Suppose that \( \tilde{G} \) is globally generated and furthermore that \( \tilde{B}^* \) has a regular section \( s \). Then we can write

\[
0 \to \mathcal{O} \xrightarrow{s} \tilde{B}^* \to Q \to 0.
\]

A mapping cone gives a free resolution

\[
0 \to \mathcal{O} \oplus \tilde{G}^* \to \tilde{F}^* \to Q \to 0.
\]

Dualizing this sequence gives

\[
0 \to Q^* \to \tilde{F}^* \xrightarrow{\Psi} \mathcal{O} \oplus \tilde{G} \to \mathcal{E}xt^1(Q, \mathcal{O}) \to 0.
\]

Since \( s \) is a regular section, \( \mathcal{E}xt^1(Q, \mathcal{O}) \) is supported on a scheme of codimension one less than the codimension of \( X \). We conclude that \( \Psi \) is a Buchsbaum-Rim matrix, and hence \( \tilde{M}_S = \mathcal{E}xt^1(Q, \mathcal{O}) \) for the scheme \( S \) defined by the maximal minors of \( \Psi \). As in the proof of Theorem 3.7, we obtain the exact sequence

\[
0 \to R/I_S \to M_S \to M_X \to 0.
\]

Since \( \mathcal{O} \oplus \tilde{G} \) is globally generated, we see that \( \tilde{M}_S \) is globally generated as an \( \mathcal{O} \)-module (and hence as an \( \mathcal{O}_S \)-module).

We have just shown that Conjecture 3.8 is true whenever we can simultaneously guarantee that \( \tilde{M}_X \) is globally generated and \( \tilde{B}^* \) has a regular section. Note in particular that \( \tilde{B}^* \) will have a regular section if \( \tilde{F}^* \) is globally generated. The latter holds true, for example, if \( X \) is a complete intersection and we choose \( M_X = R/I_X \).

\( \square \)

Remark 3.10 Analyzing the proof of Theorem 3.7 and noting that \( X \) and \( S \) are defined by the maximal minors of a \( t \times (t + r) \) matrix and a \( (t + 1) \times (t + r) \) matrix, respectively, one observes that there is the following relation between the Cohen-Macaulay types of \( X \) and \( S \), respectively:

\( X \) has Cohen-Macaulay type \( \binom{r+t-1}{r} \) \( \Leftrightarrow S \) has Cohen-Macaulay type \( \binom{r+t-1}{r-1} \).

This follows from the corresponding Eagon-Northcott resolutions. \( \square \)
4 Applications and Examples

In this section we draw some consequences of the results we have shown. We begin with a characterization of complete intersections. It is well-known that every complete intersection is arithmetically Gorenstein but the converse fails in general unless the subscheme has codimension two. For subschemes of higher codimension we have:

**Corollary 4.1** Let $X \subset \mathbb{P}^n$ be a subscheme of codimension $r+1 \geq 3$. Then $X$ is a complete intersection if and only if $X$ is arithmetically Gorenstein and there is a good determinantal subscheme $S \subset \mathbb{P}^n$ of codimension $r$ such that $X \subset S$ is the zero-locus of a regular section $t \in H^0(S, \tilde{M}_S) = M_S$ for some $M_S \in \mathcal{M}_S$. Furthermore, $S$ and $M_S$ can be chosen so that $\tilde{M}_S$ is globally generated.

**Proof:** The result follows immediately from Theorem 3.7, Remark 3.9, and Remark 3.10. $\square$

Next, we consider subschemes of low codimension. As remarked after Definition 2.8, in the case of codimension two we know that $\mathcal{M}_X$ consists of precisely one element (up to isomorphism).

**Corollary 4.2** Suppose $X \subset \mathbb{P}^n$ ($n \geq 2$) has codimension two. Then

(a) $X$ is standard determinantal if and only if $X$ is arithmetically Cohen-Macaulay.

(b) The following are equivalent:

(i) $X$ is good determinantal;

(ii) $X$ is arithmetically Cohen-Macaulay and there are an integer $e \in \mathbb{Z}$ and a section $s \in H^0(X, \omega_X(e))$ generating $\omega_X(e)$ outside a subscheme of codimension 3 as an $\mathcal{O}_X$-module and such that $s$ is a minimal generator of $H^0(\omega_X)$;

(iii) $X$ is arithmetically Cohen-Macaulay and a generic complete intersection.

**Proof:** Part (a) is just the Hilbert-Burch theorem. For (b), the fact that the codimension of $X$ is 2 implies that $\tilde{M}_X \cong \omega_X(e)$ for some $e \in \mathbb{Z}$. Then (b) is just a corollary of Proposition 3.2 and Theorem 3.4. $\square$

**Corollary 4.3** Suppose that $X \subset \mathbb{P}^n$ has codimension 3. Then $X$ is good determinantal if and only if there is a good determinantal subscheme $S \subset \mathbb{P}^n$ of codimension 2 such that $X \subset S$ is the zero-locus of a regular section $t \in H^0(S, \omega_S(e))$ (for suitable $e \in \mathbb{Z}$) whose cokernel is supported on a subscheme of codimension $\geq 4$ and isomorphic to an ideal sheaf of $\mathcal{O}_X$. 

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Proof: This is immediate from Theorem 3.7. □

**Remark 4.4** In general, if $X$ is a good determinantal subscheme of codimension $r + 1$ in $\mathbb{P}^n$ then there is a flag of *good* determinantal subschemes $X_i$ of codimension $i$:

$$X = X_{r+1} \subset X_r \subset \cdots \subset X_2 \subset X_1 \subset \mathbb{P}^n.$$  

In the next corollary we will show that we can choose the various $X_i$ in such a way that they have even better properties than guaranteed by the results of the previous section. □

**Corollary 4.5** If $X \subset \mathbb{P}^n$ has codimension $r + 1 \geq 2$ then the following are equivalent:

(a) $X$ is good determinantal;

(b) There is a good determinantal subscheme $S$ of codimension $r$ which is a local complete intersection outside a subscheme of codimension $r+2$, and a section $t \in H^0(S, \tilde{M}_S)$ inducing an exact sequence

$$0 \to \mathcal{O}_S(e) \to \tilde{M}_S \to \tilde{M}_X \to 0$$

for suitable $M_S \in \mathcal{M}_S$ and $M_X \in \mathcal{M}_X$.

Proof: We first prove (a) $\Rightarrow$ (b). The existence of a good determinantal subscheme $S$ and a section $t$ as in the statement follows from Theorem 3.7 and the exact sequence (9) in particular. The only thing remaining to prove is that $S$ can be chosen to be a local complete intersection outside a subscheme of codimension $r + 2$ (rather than codimension $r + 1$, as guaranteed by Proposition 3.2).

Assume that the matrix $\Phi$, whose maximal minors define $X$, is a homogeneous $t \times (t + r)$ matrix. The scheme $S$ is constructed in Theorem 3.7 by adding a “general row” to $\Phi$, producing a $(t + 1) \times (t + r)$ matrix, $\Psi$. One of the points of the proof of Theorem 3.4 is that the locus $Y$ where $S$ fails to be a local complete intersection is a subscheme of the scheme defined by the ideal of submaximal minors of $\Psi$. In particular, $Y$ is a subscheme of $X$. The fact that $S$ can be chosen to be a local complete intersection outside a subscheme of codimension $r + 2$ will then follow once we show that, given a general point $P$ in any component of $X$, there is at least one submaximal minor of $\Psi$ that does not vanish at $P$.

Since $X$ is good, after a change of basis if necessary we may assume that there is a $(t - 1) \times (t + r)$ submatrix $\Phi'$ whose ideal of maximal minors defines a scheme of codimension $r + 2$ which is disjoint from $P$. Hence there is a maximal minor $A$ of $\Phi'$ which does not vanish at $P$. (We make our change of basis, if necessary, before adding a row to construct $\Psi$. Note that we formally include the possibility that $t = 1$, i.e. that $X$ is a complete...
intersection– see Remark 2.3, Remark 2.6 and Theorem 3.1.) Concatenate another column of $\Phi'$ to $A$ (by abuse we denote by $A$ both the submatrix and its determinant), forming a $(t - 1) \times t$ submatrix of $\Phi'$. Now concatenate the corresponding elements of the “general row” to this matrix, forming a $t \times t$ matrix, $B$, whose determinant is a submaximal minor of $\Psi$. Expanding along this latter row and using the fact that its elements were chosen generally and that $A$ does not vanish at $P$, we get that the determinant of $B$ does not vanish at $P$, as desired. This completes the proof that (a) $\Rightarrow$ (b).

The converse follows exactly as in the proof of Theorem 3.7 (b). Note that the condition of being a local complete intersection away from a subscheme of codimension $r + 2$ is irrelevant in this direction. \[\boxend\]

**Remark 4.6** (i) Using the notation of the previous proof we have seen that given a good determinantal subscheme $X$ we can find subschemes $Y, S$ such that $Y \subset X \subset S$ have decreasing codimensions, $X$ is the zero-locus of a section of $H^0_*(S, \tilde{M}_S)$ and $X, S$ are local complete intersections outside $Y$. In this situation we want to call $X$ a Cartier divisor on $S$ outside $Y$. If $Y$ is empty then $X$ is a Cartier divisor on $S$ in the usual sense.

(ii) Let $X$ be a good determinantal subscheme of codimension $r + 1$ in $\mathbb{P}^n$ and let $X_{r+2} \subset X$ be a subscheme of codimension $r + 2$ such that $X$ is a local complete intersection outside $X_{r+2}$. Then Corollary 4.5 implies that there is a flag of good determinantal subschemes $X_i$ of codimension $i$:

$$X = X_{r+1} \subset X_r \subset \cdots \subset X_2 \subset X_1 \subset X_0 = \mathbb{P}^n$$

such that $X_{i+1}$ is a Cartier divisor on $X_i$ outside $X_{i+2}$ for all $i = 0, \ldots, r$. \[\boxend\]

**Corollary 4.7** If $X \subset \mathbb{P}^n$ is zero-dimensional then the following are equivalent:

(a) $X$ is good determinantal;

(b) There is a good determinantal curve $S$ which is a local complete intersection such that $X$ is a Cartier divisor on $S$ associated to a section $t \in H^0_*(S, \tilde{M}_S)$ inducing an exact sequence

$$0 \to \mathcal{O}_S(e) \xrightarrow{t} \tilde{M}_S \to \mathcal{O}_X(f) \to 0.$$

**Proof:** Note that under the hypotheses that $X$ is zero-dimensional and good, we get in the commutative diagram (4) that $\text{coker } \Phi'$ has finite length, and hence its sheafification is zero. Hence by the exact sequence (3), we get that the sheafification of $\text{coker } \Phi$ is just $\mathcal{O}_X$. Then the result follows from Corollary 4.5. \[\boxend\]

**Corollary 4.8** Suppose $X \subset \mathbb{P}^3$ is zero-dimensional. Then the following are equivalent:
(a) $X$ is good determinantal;

(b) There is an arithmetically Cohen-Macaulay curve $S$, which is a local complete intersection, such that $X$ is a subcanonical Cartier divisor on $S$.

Furthermore, $X$ is defined by a $t \times (t+r)$ matrix if and only if the Cohen-Macaulay type of $X$ is $\binom{r+t-1}{r}$ and that of $S$ is $\binom{r+t-1}{r-1}$.

Proof: Since $S$ has codimension two the exact sequence in the previous result specializes to the sequence

$$0 \rightarrow \mathcal{O}_S(e) \rightarrow \omega_S \rightarrow \mathcal{O}_X(f) \rightarrow 0$$

by Corollary 4.2. Since $S$ is a local complete intersection it implies that $X$ is subcanonical. The statement about the Cohen-Macaulay types is just Remark 3.10. $\square$

Remark 4.9 In view of Remark 3.9 and Remark 3.10, Corollaries 4.1, 4.7 and 4.8 are generalizations of Theorem 1.3 of [13]. $\square$

Example 4.10 In view of Theorem 3.4, we give examples of curves in $\mathbb{P}^3$ (both of degree 3) to show that a good determinantal scheme need not be either reduced or a local complete intersection. For the first, consider the curve defined by the matrix

$$\begin{bmatrix}
  x_0 & x_1 & x_2 \\
  0 & x_0 & x_3
\end{bmatrix}$$

For the second, consider the curve defined by the matrix

$$\begin{bmatrix}
  -x_3 & x_2 & 0 \\
  0 & -x_2 & x_1
\end{bmatrix}$$

This is the defining matrix for the “coordinate axes,” which fail to be a complete intersection precisely at the “origin.” (Recall that in the definition of a good determinantal scheme we allowed for the removal of a generalized row.) $\square$

Example 4.11 The point of Corollary 4.7 is that given a zero-scheme $X$, there is so much “room” to choose the curve $S$ containing it, that $S$ can be assumed to be a local complete intersection even at $X$, where one would normally expect it to have problems. One naturally can ask if there is so much room that $S$ can even be taken to be smooth. The answer is no: for example, the zeroscheme in $\mathbb{P}^3$ defined by the complete intersection $(X_1^2, X_2^2, X_3^2)$ lies on no smooth curve. One can ask, though, if there is any matrix condition analogous to the main result of [4] which guarantees that a “general” choice of $S$ will be smooth. $\square$
Example 4.12 Any regular section of any twist of the tangent bundle of $\mathbb{P}^n$ defines a good determinantal zero-scheme in $\mathbb{P}^n$, by Theorem 3.1. In fact, it can be shown that if $\mathcal{E}$ is any rank $n$ vector bundle on $\mathbb{P}^n$ with $H^i(\mathbb{P}^n, \mathcal{E}) = 0$ for $1 \leq i \leq n-2$, then any regular section of $\mathcal{E}$ defines a good determinantal zero-scheme in $\mathbb{P}^n$. \hfill \Box

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