Abstract

In this paper, we show that coherent sets of gambles can be embedded into the algebraic structure of information algebra. This leads firstly, to a new perspective of the algebraic and logical structure of desirability and secondly, it connects desirability, hence imprecise probabilities, to other formalism in computer science sharing the same underlying structure. Both the domain free and the labeled view of the information algebra of coherent sets of gambles are presented, considering a special case of possibility space.
1 Introduction and Overview

In a recent paper (Miranda & Zaffalon, 2020) some results about compatibility or consistency of coherent sets of gambles or lower previsions have been derived and it was remarked that these results were in fact results of the theory of information or valuation algebras (Kohlas, 2003). This point of view, however, was not worked out in (Miranda & Zaffalon, 2020). In this paper this issue is taken up and it is shown that coherent sets of gambles, strictly desirable sets of gambles, coherent lower and upper previsions indeed form idempotent information algebras. Like in group theory, certain results concerning particular groups follow from general group theory, so many known results about desirable gambles, lower and linear previsions are indeed properties of an information algebra and follow from the corresponding general theory. Some of these results are discussed in this paper, but there are doubtless many other properties which can be derived from the theory of information algebra.

From the point of view of information algebra, sets of gambles or lower previsions are pieces of information about certain groups of variables with their respective sets of possibilities. Such pieces of information can be aggregated or combined, and the information they contain about other groups of variables can be extracted. This leads to an algebraic structure satisfying a number of simple axioms. In fact there are two different versions of information algebras, a domain-free one and a labeled one. They are closely related and each one can be derived or reconstructed from the other one. Grossly, the domain-free version is better suited for theoretical studies, since it is a structure of universal algebra, whereas the labeled version is better adapted to computational purposes, since it provides more efficient storage structures. In fact, labeled information (or valuation algebras, if idempotency is dropped) are the universal algebraic structures for local computation in join or junction trees as originally proposed for probabilistic networks by (Lauritzen & Spiegelhalter, 1988). Based on this work (Shenoy & Shafer, 1990) proposed a first axiomatic scheme which was sufficient to generalize the Lauritzen-Spiegelhalter scheme to a multitude of other uncertainty formalisms, like Dempster-Shafer belief functions, possibility theory and many others. In (Kohlas, 2003) the algebraic theory is systematically developed and studied. In particular, the domain-free and labeled views are presented (based originally on (Shafer, 1991)). In this paper both the domain-free and the labeled view of desirable gambles and lower previsions are presented.
Sets of desirable gambles are like logical statements, which determine a logical theory obtained by consequence operators, which determine all sentences deducible from the original set of statements. In the case of desirable gambles the consequence operator is natural extension. It has been shown in (Kohlas, 2003) that a consequence operator, satisfying some mild conditions induces an idempotent information algebra of logically closed sets. This theory applies to desirable sets of gambles, if they avoid partial loss. The construction of the domain-free information algebra of coherent sets of gambles is based on this theory and presented in Section 3. Based on the domain-free view, in Section 4 the labeled version is derived by a standard construction for information algebras. Alternatively, a second isomorphic version of a labeled algebra of sets of desirable gambles is presented, which corresponds more to the notion of marginals as used with desirable gambles and lower previsions. In Section 7 we show that there is also an information algebra of lower previsions, and one of upper previsions and that these algebras are homomorphic as information algebras. This has been already stated in (Miranda & Zaffalon, 2020), albeit without proof. Moreover the strictly desirable sets of gambles form a subalgebra of the algebra of coherent sets of gambles, and this subalgebra is isomorphic to the algebras of lower and upper previsions.

Any idempotent information algebra induces an information order, a partial order, where one piece of information is less informative than another one, if the combination of the two pieces yields the second one. Then the first piece of information is "contained" in second one. In the case of coherent sets of gambles this order corresponds to set inclusion and in the case of lower previsions to domination. Moreover since natural extension of desirable sets of gambles is a consequence (or closure) operator, the coherent sets form a complete lattice under information order, where meet is set intersection, and join is essentially natural extension. This carries over to coherent lower previsions where meet is the point-wise infimum of lower previsions and join again natural extension (Section 7). Further, it is well know that there are (in this order) maximal coherent sets (De Cooman & Quaeghebeur, 2012). In the language of information algebra such maximal elements are called atoms. Any coherent set of gambles is contained in a maximal set (an atom) and it is the intersection (meet) of all atoms it is contained in. An information algebra with these properties is called atomistic. In fact any intersection of atoms yields a coherent set of gambles. This is discussed in Section 5. In the case of lower previsions, atoms are linear previsions (Section 8) and the fact that any coherent lower prevision is the meet of atoms it is contained in, is the lower envelope theorem.

Atomistic information algebras have the universal property that they are embedded in a set algebra, that is an information algebra whose elements are sets, combination is simply intersection and extraction is the cylindrification or saturation operator. This is an important representation theorem for information algebras, since set algebras are very special kinds of algebras based on the usual set operations. Conversely any such set algebra of subsets of the possibility space is embedded in the algebra of coherent sets of gambles. This has some importance for conditioning, a subject which however is not pursued in this paper. These links between set algebras and algebras of coherent sets of gambles are discussed in Section 6 and in Section 8.
As stated above, information algebras are the generic structures of local computation in join tree. This is used in Section 9 to reconstruct the proof of [Miranda & Zaffalon, 2020] concerning the running intersection property (RIP) as a condition for pairwise consistency of elements (here coherent sets of gambles or coherent lower previsions) to be sufficient for global consistency, that is to be marginals of some element.

2 Desirable Gambles

Consider a set \( \Omega \) of possible worlds. A gamble over this set is a bounded function

\[ f : \Omega \to \mathbb{R}. \]

We denote the set of all gambles on \( \Omega \) by \( \mathcal{L}(\Omega) \), or more simply by \( \mathcal{L} \) when there is no possible ambiguity. We also let \( \mathcal{L}^+(\Omega) = \{ f \in \mathcal{L}(\Omega) : f \geq 0, f \neq 0 \} \), or simply \( \mathcal{L}^+ \), denote the subset of non-vanishing, non-negative gambles. A coherent set of gambles over \( \Omega \) is a subset \( D \) of \( \mathcal{L}(\Omega) \) such that

1. \( \mathcal{L}^+(\Omega) \subseteq D \),
2. \( 0 \not\in D \),
3. \( f, g \in D \) implies \( f + g \in D \),
4. \( f \in D \), and \( \lambda > 0 \) implies \( \lambda \cdot f \in D \).

So, \( D \) is a convex cone.

If \( D' \) is any subset of \( \mathcal{L}(\Omega) \), then

\[ \mathcal{E}(D') = \text{posi}(\mathcal{L}^+(\Omega) \cup D') ,\]

is called the natural extension of the set of gambles \( D' \), where \( \text{posi}(D) \) is defined in the following way:

\[ \text{posi}(D) = \left\{ \sum_{j=1}^{r} \lambda_j f_j : f_j \in D, \lambda_j > 0, r \geq 1 \right\} . \]

The natural extension of a set of gambles \( D \), \( \mathcal{E}(D) \), is coherent if and only if \( 0 \not\in \mathcal{E}(D) \). Coherent sets are closed under intersection, that is they form a topless \( \cap \)-structure. By standard order theory (Davey & Priestley, 2002), they are ordered by inclusion, intersection is meet in this order and a family of coherent sets of gambles \( D_i, i \in I \), where \( I \) is an index set, have a join if they have an upper bound among coherent sets:

\[ \bigvee_{i \in I} D_i = \bigcap \{ D \text{ coherent} : \bigcup_{i \in I} D_i \subseteq D \} . \]
Also, $E(D')$, the smallest coherent set containing $D'$, if $E(D')$ is coherent,

$$E(D') = \bigcap \{D \text{ coherent} : D' \subseteq D\},$$

so that

$$\bigvee_{i \in I} D_i = E\big(\bigcup_{i \in I} D_i\big)$$

if $E(\bigcup_{i \in I} D_i)$ is coherent. Let $C(\Omega)$ be the family of coherent sets of gambles on $\Omega$. In view of the following section, it is convenient to add $L(\Omega)$ to $C(\Omega)$ and let $\Phi = C(\Omega) \cup \{L(\Omega)\}$. The family of sets in $\Phi$ is still a $\cap$-structure, but now a topped one. So, again by standard results of order theory (Davey & Priestley, 2002), $\Phi$ is a complete lattice under inclusion, meet is intersection and join is defined for any family of sets $D_i \in \Phi$, $\forall i \in I$, as

$$\bigvee_{i \in I} D_i = \bigcap \{D \in \Phi : \bigcup_{i \in I} D_i \subseteq D\}.$$ 

Note that, if a family of coherent sets $D_i$ has no upper bound in $C(\Omega)$, then its join is simply $L(\Omega)$. In this topped $\cap$-structure,

$$C(D') = \bigcap \{D \in \Phi : D' \subseteq D\}$$

is a closure (or consequence) operator on the subsets of gambles, (Davey & Priestley, 2002), i.e., given $D, D' \subseteq L(\Omega)$:

1. $D \subseteq C(D)$,
2. $D \subseteq D'$ implies $C(D) \subseteq C(D')$,
3. $C(C(D)) = C(D)$.

Note that $C(D) = E(D)$, if $0 \notin E(D)$, that is if $E(D)$ is coherent. Otherwise we may have $E(D) \neq L(\Omega)$. We refer to (De Cooman, 2010) for a similar order-theoretic view on belief models. These results prepare the way to an information algebra of coherent sets of gambles (see next section). For further reference, note the following well-known result on consequence operators.

**Lemma 1** If $C$ is a consequence operator on sets of gambles then, for any $D_1, D_2 \subseteq L(\Omega)$:

$$C(D_1 \cup D_2) = C(C(D_1) \cup D_2).$$

**Proof.** Obviously, $C(D_1 \cup D_2) \subseteq C(C(D_1) \cup D_2)$. On the other hand $D_1, D_2 \subseteq D_1 \cup D_2$, hence $C(D_1) \subseteq C(D_1 \cup D_2)$ and $D_2 \subseteq C(D_1 \cup D_2)$. This implies $C(C(D_1) \cup D_2) \subseteq C(C(D_1 \cup D_2)) = C(D_1 \cup D_2)$. 

A coherent set of gambles $M$ is called **maximal**, if it is not a proper subset of a coherent set of gambles. These sets play an important role because of the following facts proved in (De Cooman & Quaeghebeur, 2012):
1. Any coherent set of gambles is a subset of a maximal one,
2. Any coherent set of gambles is the intersection of all maximal coherent sets it is contained in.

In addition, maximal coherent sets of gambles are characterized by the following condition, (De Cooman & Quaeghebeur, 2012)

\[ \forall f \in L \setminus \{0\} : f \notin M \Rightarrow -f \in M. \]

For a discussion of the importance of maximal coherent sets of gambles in relation to information algebras, see Section 5.

A further important class of coherent sets of gambles are the strictly desirable sets of gambles. We shall employ the notation \( D^+ \) for strictly desirable sets of gambles, to differentiate them from the general case of coherent sets of gambles.

In addition to the conditions 1. to 4. above for coherence, the following condition is added:

5. \( f \in D^+ \) implies either \( f \geq 0 \) or \( f - \delta \in D^+ \) for some \( \delta > 0 \).

So, strictly desirable gambles are coherent and they form a subfamily of coherent sets of gambles.

Another concept, which plays an important role in Section 7 are almost desirable sets of gambles, that we indicate with the notation \( \bar{D} \), satisfying the following conditions (Walley, 1991)

1. \( f \in \bar{D} \) implies \( \sup f \geq 0 \),
2. \( \inf f > 0 \) implies \( f \in \bar{D} \),
3. \( f, g \in \bar{D} \) implies \( f + g \in \bar{D} \),
4. \( f \in D \) and \( \lambda > 0 \) imply \( \lambda \cdot f \in \bar{D} \),
5. \( f + \delta \in \bar{D} \) for all \( \delta > 0 \) implies \( f \in \bar{D} \).

Such a set is no more coherent since it contains \( f = 0 \).

Let us indicate with \( \bar{C}(\Omega) \) the family of almost desirable sets of gambles. We remark that \( \bar{C}(\Omega) \) again forms a \( \cap \)-system, still topped by \( L(\Omega) \). Therefore if we add, as before, \( L(\Omega) \) to \( \bar{C}(\Omega) \) then \( \Phi = \bar{C}(\Omega) \cup \{L(\Omega)\} \) is a complete lattice too.

So, we may define the operator:

\[ \bar{C}(D') = \bigcap \{ \bar{D} \in \Phi : D' \subseteq \bar{D} \}. \]

that is still a closure operator on sets of gambles.

So far, we have considered sets of gambles in \( L(\Omega) \) relative to a fixed set of possibilities \( \Omega \). Next some more structure among sets of possibilities is introduced related to family of variables and their possible values. This is the base for relating coherent sets of gambles to information algebras.
3 Domain-Free Information Algebra

We may consider a coherent set of gambles as someone’s evaluation of the “chances” of the elements of Ω as possible answers. This can be considered as a piece of information about Ω. Here, in this spirit, coherent sets of gambles are shown to form an information algebra. We consider for this a special form of a domain Ω, namely a multivariate model.

Let I be an index set of variables $X_i, i \in I$. In practice, I is often assumed to be finite or countable. But we need not make this restriction. Any variable $X_i$ has a domain of possible values $\Omega_i$. For any subset $S$ of I let

$$\Omega_S = \times_{i \in S} \Omega_i$$

and $\Omega = \Omega_I$. We consider coherent sets of gambles on $\Omega$, or rather $\Phi = C(\Omega) \cup \{L(\Omega)\}$. The tuples $\omega$ in $\Omega$ can be seen as functions $\omega : I \rightarrow \Omega$, so that $\omega_i \in \Omega_i$, for any $i \in I$. A gamble $f$ on $\Omega$ is called $S$-measurable, if for all $\omega, \omega' \in \Omega$ with $\omega|S = \omega'|S$ we have $f(\omega) = f(\omega')$ (here $\omega|S$ is the restriction of the map $\omega$ to $S$). Let $L_S$ denote the set of all $S$-measurable gambles. Define $L_\emptyset = \mathbb{R}$, the set of constant gambles, and note that $L_I = L(\Omega)$. For further reference we have the following result.

**Lemma 2** For any subsets $S$ and $T$ of $I$,

$$L_{S \cap T} = L_S \cap L_T.$$

**Proof.** If one of the two subsets is empty we immediatly have the result. Otherwise, assume first $f \in L_{S \cap T}$. Consider two elements $\omega, \mu \in \Omega$ so that $\omega|S = \mu|S$. Then we have also $\omega|S \cap T = \mu|S \cap T$ and $f(\omega) = f(\mu)$. So we see that $f \in L_S$ and similarly $f \in L_T$.

Conversely, assume $f \in L_S \cap L_T$ and consider two tuples so that $\omega|S \cap T = \mu|S \cap T$. Define then a tuple $\lambda$ by

$$\lambda_i = \begin{cases} 
  \omega_i = \mu_i, & i \in S \cap T, \\
  \omega_i, & i \in S - T, \\
  \mu_i, & i \in T - S, \\
  0, & i \in (S \cup T)^c.
\end{cases}$$

Then $\lambda|S = \omega|S$ and $\lambda|T = \mu|T$. Since $f$ is both $S$- and $T$-measurable we have $f(\omega) = f(\mu)$. It follows that $f \in L_{S \cap T}$, and this concludes the proof. □

Now we define in $\Phi$ two operations. Let $D, D_1, D_2 \in \Phi$ and $S \subseteq I$:

1. Combination: $D_1 \cdot D_2 = D_1 \lor D_2 = C(D_1 \cup D_2),$
2. Extraction: $\epsilon_S(D) = C(D \cap L_S).$
Let \( \mathcal{E}_S(D) = \mathcal{C}(D) \cap \mathcal{L}_S \), for any set of gambles \( D \). Then, if \( D \) is coherent, we have:

\[
\epsilon_S(D) = \mathcal{C}(\mathcal{E}_S(D)).
\]

If we consider coherent sets of gambles as pieces of information (about someone’s belief on possible answers in \( \Omega \)) then combination represents the aggregation of two pieces of information and extraction the coarsening or extraction of an information to a subset of variables. Note that \( D_1 \cdot D_2 = \mathcal{L}(\Omega), \text{ for some } D_1, D_2 \in \Phi \), means that the two sets \( D_1 \) and \( D_2 \) are not compatible, that is, \( D_1 \cup D_2 \) is not coherent. So, clearly \( \mathcal{L}(\Omega) \) is the null element of combination and represents contradiction or incompatibility. The set \( \mathcal{L}^+(\Omega) \) is the unit element of combination, representing vacuous information. We claim that \( \Phi \) equipped with these two operations satisfies the properties collected in the following theorem.

**Theorem 1**

1. \( (\Phi; \cdot) \) is a commutative semigroup with a null 0 and unit 1,

2. for any subset \( S \subseteq I \) and \( D, D_1, D_2 \in \Phi \):
   
   \( E_1 \)  \( \epsilon_S(0) = 0 \),
   
   \( E_2 \)  \( \epsilon_S(D) \cdot D = D \),
   
   \( E_3 \)  \( \epsilon_S(\epsilon_S(D_1) \cdot D_2) = \epsilon_S(D_1) \cdot \epsilon_S(D_2) \).

3. for any \( S, T \subseteq I \), \( \epsilon_S \circ \epsilon_T = \epsilon_T \circ \epsilon_S = \epsilon_{S \cap T} \).

**Proof.** That \( (\Phi; \cdot) \) is a commutative semigroup follows from \( D_1 \cdot D_2 = D_1 \lor D_2 \), for any \( D_1, D_2 \) in the complete lattice \( \Phi \). As stated above, 0 = \( \mathcal{L}(\Omega) \) is the null element and 1 = \( \mathcal{L}^+(\Omega) \) the unit element of the semigroup (null and unit in a semigroup are always unique). This proves item 1.

For \( E_1 \) we have

\[
\epsilon_S(0) = \epsilon_S(\mathcal{L}(\Omega)) = \mathcal{C}(\mathcal{L}(\Omega) \cap \mathcal{L}_S) = \mathcal{C}(\mathcal{L}_S) = \mathcal{L}(\Omega) = 0.
\]

for any \( S \subseteq I \).

\( E_2 \) follows since \( D \cap \mathcal{L}_S \subseteq D \) and \( \mathcal{C}(D \cap \mathcal{L}_S) \subseteq D \), for any \( D \in \Phi, \; S \subseteq I \). To prove \( E_3 \) define, using Lemma 1

\[
A = \mathcal{C}(\mathcal{C}(D_1 \cap \mathcal{L}_S) \cup D_2) \cap \mathcal{L}_S = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup D_2) \cap \mathcal{L}_S
\]

\[
B = \mathcal{C}(\mathcal{C}(D_1 \cap \mathcal{L}_S) \cup (\mathcal{C}(D_2 \cap \mathcal{L}_S))) = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup (D_2 \cap \mathcal{L}_S)).
\]

Then we have \( B = \epsilon_S(D_1) \cdot \epsilon_S(D_2) \) and \( \mathcal{C}(A) = \epsilon_S(\epsilon_S(D_1) \cdot D_2) \). Note that \( B \subseteq \mathcal{C}(A) \).

We claim first that:

\[
\epsilon_S(D_1) \cdot \epsilon_S(D_2) = 0 \iff \epsilon_S(D_1) \cdot D_2 = 0.
\]

Indeed, \( \epsilon_S(D_1) \cdot \epsilon_S(D_2) = 0 \) implies a fortiori \( \epsilon_S(D_1) \cdot D_2 = 0 \).
Assume therefore that $\epsilon_S(D_1) \cdot D_2 = 0$. This equivalently means that $0 \in \mathcal{C}(\mathcal{C}(D_1 \cap L_S) \cup D_2) = \mathcal{C}((D_1 \cap L_S) \cup D_2)$, by Lemma \[\text{[1]}\] Now, two cases are possible. If $D_1 = \mathcal{L}(\Omega)$ or $D_2 = \mathcal{L}(\Omega)$ we have immediately the result, otherwise we claim that $0 = f + g'$ with $f \in D_1 \cap L_S$ and $g' \in D_2 \cap L_S$. Indeed, from $0 \in \mathcal{C}((D_1 \cap L_S) \cup D_2)$, we know that $0 = f + g + h'$ with $f \in D_1 \cap L_S$, $g \in D_2$, $h' \in \mathcal{L}^+(\Omega) \subseteq D_2$. Then, if we introduce $g' = g + h'$, we have $0 = f + g'$ with $f \in D_1 \cap L_S$, $g' \in D_2$. However, this implies $g' = -f \in L_S$ and then $g' \in D_2 \cap L_S$.

Notice that $\epsilon_S(D_1) \cdot \epsilon_S(D_2) = B = \mathcal{C}((D_1 \cap L_S) \cup (D_2 \cap L_S))$. Therefore, we have the result.

So, we may now assume that both $\epsilon_S(D_1) \cdot D_2$ and $\epsilon_S(D_1) \cdot \epsilon_S(D_2)$ are coherent. Then we have

$$A = \{ f \in L_S : f \geq \lambda g + \mu h, g \in D_1 \cap L_S, h \in D_2, \lambda, \mu \geq 0, f \neq 0 \}.$$ 

Consider $f \in A$. Then $f = \lambda g + \mu h + h'$, where $h' \in \mathcal{L}^+(\Omega)$. Since $f$ and $g$ are $S$-measurable, $\mu h + h'$ must be $S$-measurable, hence $\mu h + h' \in D_2 \cap L_S$. But this says that $f \in B$, hence we have $\mathcal{C}(A) \subseteq \mathcal{C}(B) = B$.

To prove item 3, note first that $\epsilon_S \circ \epsilon_T(D) = 0$ and $\epsilon_{S \cap T}(D) = 0$ if and only if $D = 0$. So assume $D$ to be coherent. Then we have

$$\epsilon_S(\epsilon_T(D)) = \mathcal{C}(\mathcal{C}(D \cap L_T) \cap L_S),$$

$$\epsilon_{S \cap T}(D) = \mathcal{C}(D \cap L_{S \cap T}) = \mathcal{C}(D \cap L_T \cap L_S).$$

Obviously, $\epsilon_{S \cap T}(D) \subseteq \epsilon_s(\epsilon_T(D))$. Consider then $f \in C(D \cap L_T) \cap L_S$, so that $f \in L_S$, $f \geq g, g \in D \cap L_T$.

Define

$$g'(\omega) = \sup_{\lambda|S=\omega|S} g(\lambda).$$

Then we have $f \geq g'$. Clearly $g'$ is $S$-measurable and belongs to $D$, $g' \in D \cap L_S$. We claim that $g'$ is also $T$-measurable. Consider two elements $\omega$ and $\mu$ so that $\omega|S \cap T = \mu|S \cap T$. Note that we may write

$$g'(\omega) = \sup_{\lambda|S=\omega|S} g(\lambda) = \sup_{\lambda'|I-S} g(\omega|S \cap T, \omega|S - T, \lambda|T - S, \lambda|R),$$

where $R = (S \cup T)^c$. Similarly, we have

$$g'(\mu) = \sup_{\lambda'|S=\mu|S} g(\lambda') = \sup_{\lambda'|I-S} g(\omega|S \cap T, \mu|S - T, \lambda'|T - S, \lambda'|R),$$

Since $g$ is $T$-measurable, we have:

$$g'(\mu) = \sup_{\lambda'|S=\mu|S} g(\lambda') = \sup_{\lambda'|I-S} g(\omega|S \cap T, \omega|S - T, \lambda'|T - S, \lambda'|R),$$
that clearly coincides with \( g'(\omega) \).

This shows that \( g' \) is \( S \cap T \)-measurable, therefore both \( S\)- and \( T\)-measurable, by Lemma 2. So we have \( g' \in D \cap L_S \cap L_T \), hence \( f \in C(D \cap L_T \cap L_S) \). This proves item 3. \( \square \)

A system with two operations satisfying the conditions of Theorem 1 is called a domain-free information algebra [Kohlas, 2003]. There is an alternative, equivalent version, a so-called labeled information algebra, which may derived from it, see the next section. In any information algebra, an information order can be introduced. In the case of coherent sets of gambles, this order is defined as \( D_1 \leq D_2 \) if \( D_1 \cdot D_2 = D_2 \). \( D_2 \) is more informative than \( D_1 \) if adding \( D_1 \) gives nothing new; \( D_1 \) is already contained in \( D_2 \). It is easy to verify that it is a partial order and also that \( D_1 \leq D_2 \) if and only if \( D_1 \subseteq D_2 \). Moreover, combination \( D_1 \cdot D_2 \) yields the supremum, \( D_1 \cdot D_2 = \text{sup}\{D_1, D_2\} \). This is also written as the join \( D_1 \cdot D_2 = D_1 \lor D_2 \). The vacuous information \( L_+ (\Omega) \) is the least information in this order and the inconsistency \( L(\Omega) \) is the top element (although strictly speaking it is no more an information). This means that \( (\Phi, \leq) \) is a join semi-lattice under information order; in fact it is a complete lattice, since information order corresponds to set inclusion. Conditions E1 to E3 in Theorem 1 can also be rewritten using this order as the following: for any subset \( S \subseteq I \) and \( D, D_1, D_2 \in \Phi \),

**E1** \( \epsilon_S(0) = 0 \),

**E2** \( \epsilon_S(D) \leq D \),

**E3** \( \epsilon_S(\epsilon_S(D_1) \lor D_2) = \epsilon_S(D_1) \lor \epsilon_S(D_2) \).

In algebraic logic such an operator is also called an existential quantifier[^1].

We further claim that extraction distributes over intersection (or meet in the complete lattice).

**Theorem 2** Let \( D_j, j \in J \) be any family of sets of gambles in \( \Phi \) and \( S \) any subset of variables. Then

\[
\epsilon_S(\bigcap_{j \in J} D_j) = \bigcap_{j \in J} \epsilon_S(D_j). \tag{3.2}
\]

**Proof.** We may assume that \( D_j \in C(\Omega) \ \forall j \in J \), since if some or all \( D_j = L(\Omega) \), then we may restrict the intersection on both sides over the set \( D_j \in C(\Omega) \), or the intersection over both sides equals \( L(\Omega) \). If \( D_j \in C(\Omega) \), \( \forall j \in J \), we have

\[
\epsilon_S(\bigcap_{j \in J} D_j) = \mathcal{E}(\bigcap_{j \in J} D_j \cap L_S) = \text{posi}(L^+(\Omega) \cup (\bigcap_{j \in J} D_j \cap L_S))
\]

\[
\bigcap_{j \in J} \epsilon_S(D_j) = \bigcap_{j \in J} \mathcal{E}(D_j \cap L_S) = \bigcap_{j \in J} \text{posi}(L^+(\Omega) \cup (D_j \cap L_S)).
\]

[^1]: Although usually operators on a Boolean lattice are considered and the order is inverse to the information order.
Consider first a gamble \( f \in \epsilon_S(\bigcap_{j \in J} D_j) \), so that \( f = \lambda g + \mu h \), where \( \lambda, \mu \) nonnegative and not both equal zero and \( g \in (\bigcap_{j} D_j) \cap L_S \) and \( h \in L^+(\Omega) \). But then \( g \in D_j \cap L_S \) for all \( j \), so that \( f \in \bigcap_{j \in J} \epsilon_S(D_j) \).

Conversely, assume \( f \in \bigcap_{j \in J} \epsilon_S(D_j) \). Then \( f \geq g_j \) for some \( g_j \in (D_j \cap L_S), \forall j \). Hence, \( f(\omega) \geq \sup k g_k(\omega) \), for every \( \omega \in \Omega \). However, \( \sup k g_k \in \cap_{j}(D_j) \) because \( \sup_k g_k(\omega) \geq g_j(\omega), \forall j, \forall \omega \). Moreover, \( \sup_k g_k \in L_S \) because, thanks to the fact that every \( g_j \in L_S \), we have the following: if \( \omega|_S = \omega'|_S \), then \( \sup_k g_k(\omega) = \sup_k g_k(\omega') \). Therefore \( f \in \epsilon_S(\bigcap_{j} D_j) \).

\( \square \)

So \((\Phi, \leq)\) is a lattice under information order and satisfies [322]. Such an information algebra is called a lattice information algebra.

The family of strictly desirable sets of gambles \( D^+ \) is also closed under combination (if incompatibility is admitted) and extraction in \( \Phi \). Therefore, if the null element \( L(\Omega) \) is added and we denote with \( \Phi^+ \) the union of all the strictly desirable sets of gambles and \( L(\Omega) \), we have that \( \Phi^+ \) forms a subalgebra of \( \Phi \).

## 4 Labeled Information Algebra

There is a general method to derive a corresponding labeled information algebra from a domain-free information algebra [Kohlas, 2003]. We describe it here in the frame of the information algebra of coherent sets of gambles. In the particular case of coherent sets of desirable gambles, as well as in the case of lower previsions, there is a second, isomorphic version of the labeled algebra, which is nearer to the intuition and which will be introduced after the general construction of labeled algebras.

From a labeled algebra the domain-free information algebra may be reconstructed [Kohlas, 2003]. So the the two views of information algebras are equivalent. The whole theory presented here could also have been developed in the labeled view. It is a question of convenience whether the domain-free or the labeled view is chosen. Usually, for theoretical considerations, the domain-free view is preferable, because it is nearer to universal algebra. For computational purposes, the labeled view is used in general, because it corresponds better to the needs of efficient data representation.

A subset \( S \) of \( I \) is called support of a coherent set of gambles \( D \), if \( \epsilon_S(D) = D \). This means that the information concerns or is focused to the group \( S \) of variables. Here are a few well-known results on support in domain-free information algebras, let \( D, D_1, D_2 \in \Phi, S, T \subseteq I \):

1. Any \( S \) is a support of the null element 0 \((L(\Omega))\) and the unit \((L^+(\Omega))\) \( S \) of variables,
2. \( S \) is a support of \( \epsilon_S(D) \),
3. if \( S \) is a support of \( D \), then \( S \) is a support of \( \epsilon_T(D) \),
4. if \( S \) and \( T \) are supports of \( D \), then so is \( S \cap T \),
5. if $S$ is a support of $D$, then $\epsilon_T(D) = \epsilon_{S \cap T}(D)$,

6. If $S$ is a support of $D$ and $S \subseteq T$, then $T$ is a support of $D$,

7. if $S$ is a support of $D_1$ and $D_2$, then it is also a support of $D_1 \cdot D_2$,

8. if $S$ is a support of $D_1$ and $T$ a support of $D_2$, then $S \cup T$ is a support of $D_1 \cdot D_2$.

For proofs see [Kohlas, 2003; Kohlas, 2017]. Now, we consider sets $\Psi_S$ of pairs $(D, S)$, where $S \subseteq I$ is a support of $D \in \Phi$. That is, we collect together pieces of information concerning the same set of variables. Let

$$\Psi = \bigcup_{S \subseteq I} \Psi_S.$$ 

In $\Psi$ we define the following operations in terms of the operations of the domain-free algebra, let $(D, S), (D_1, S), (S_2, T) \in \Psi$:

1. Labeling: $d(D, S) = S$,

2. Combination: $(D_1, S) \cdot (D_2, T) = (D_1 \cdot D_2, S \cup T)$,

3. Projection (Marginalization): $\pi_T(D, S) = (\epsilon_T(D), T)$ defined for $T \subseteq S \subseteq I$.

It is well-known and easy to verify, that $\Psi$ with these three operations satisfies the properties in the following theorem [Kohlas, 2003].

**Theorem 3**

Semigroup: $(\Psi, \cdot)$ is a commutative semigroup;

Labeling: $d((D_1, S) \cdot (D_2, T)) = d(D_1, S) \cup d(D_2, T)$, for any $(D_1, S), (D_2, T) \in \Psi$ and $d(\pi_T(D, S)) = T$, for any $(D, S) \in \Psi$, $T \subseteq S \subseteq I$;

Null and Unit: $(0, S) \cdot (D, S) = (0, S)$, $(1, S) \cdot (D, S) = (D, S)$ for any $(D, S) \in \Psi$, $\pi_T(D, S) = (0, T)$ if and only if $(D, S) = (0, S)$, $\pi_T(1, S) = (1, T)$, with $T \subseteq S \subseteq I$ and $(1, S) \cdot (1, T) = (1, S \cup T)$, with $S, T \subseteq I$;

Transitivity of Projection: if $R \subseteq S \subseteq T \subseteq I$, then $\pi_R(D, T) = \pi_R(\pi_S(D, T))$, for any $(D, T) \in \Psi$;

Combination: $\pi_S((D_1, S) \cdot (D_2, T)) = (D_1, S) \cdot \pi_S(\pi_T(D_2, T))$, for any $(D_1, S), (D_2, T) \in \Psi$;

Idempotency: $(D, S) \cdot \pi_T(D, S) = (D, S)$, for any $(D, S) \in \Psi$, $T \subseteq S \subseteq I$;

Identity: $\pi_S(D, S) = (D, S)$, for any $(D, S) \in \Psi$. 
An algebraic system like \( \Psi \) satisfying the conditions in Theorem 3 is called a labeled (idempotent) information algebra. In this sense, coherent sets of gambles form both a domain-free and a labeled information algebra.

We may associate to this labeled algebra another, isomorphic one. For a subset \( S \) of variables, let \( \mathcal{C}_S(\Omega) \) be the family of sets of gambles coherent relative to \( \mathcal{L}_S \), that is sets \( D \cap \mathcal{L}_S \), where \( D \in \mathcal{C}(\Omega) \). Further let \( \tilde{\Psi}_S \) be \( \mathcal{C}_S(\Omega) \cup \{ \mathcal{L}_S \} \) and

\[
\tilde{\Psi} = \bigcup_{S \subseteq I} \tilde{\Psi}_S.
\]

Within \( \tilde{\Psi} \) we define the following operations, let \((D \cap \mathcal{L}_S, S), (D_1 \cap \mathcal{L}_S, S), (D_2 \cap \mathcal{L}_T, T) \in \tilde{\Psi} \):

1. \( d(D \cap \mathcal{L}_S, S) = S \),
2. Combination: \((D_1 \cap \mathcal{L}_S, S) \cdot (D_2 \cap \mathcal{L}_T, T) = ((D_1 \cdot D_2) \cap \mathcal{L}_{S \cup T}, S \cup T) \), where \( D_1 \cdot D_2 = \mathcal{C}(D_1 \cup D_2) \) is the domain-free combination,
3. Projection (Marginalization): \( \pi_T(D \cap \mathcal{L}_S, S) = (D \cap \mathcal{L}_T, T) \) defined if \( T \subseteq S \).

Consider now the map \( h : \Psi \to \tilde{\Psi} \) defined by \((D, S) \mapsto (D \cap \mathcal{L}_S, S) \) and recall that here \( D = \epsilon_S(D) \). The map \( h \) will establish an isomorphism between the labeled information algebra \( \Psi \) and \( \tilde{\Psi} \).

**Theorem 4** The map \( h \) has the following properties:

1. It maintains combination, null and unit, and projection. Let \((D, S), (D_1, S), (D_2, T) \in \Psi \):

\[
\begin{align*}
    h((D_1, S) \cdot (D_2, T)) &= h(D_1, S) \cdot h(D_2, T), \\
    h(\mathcal{L}(\Omega), S) &= (\mathcal{L}_S, S), \\
    h(\mathcal{L}^+(\Omega), S) &= (\mathcal{L}^+_S, S), \\
    h(\pi_T(D, S)) &= \pi_T(h(D, S)), \text{ if } T \subseteq S.
\end{align*}
\]

2. \( h \) is bijective.

**Proof.** 1. we have, by definition

\[
h((D_1, S) \cdot (D_2, T)) = h(D_1 \cdot D_2, S \cup T) = ((D_1 \cdot D_2) \cap \mathcal{L}_{S \cup T}, S \cup T)
\]

\[
= (D_1 \cap \mathcal{L}_S, S) \cdot (D_2 \cap \mathcal{L}_T, \mathcal{L}_T) = h(D_1, S) \cdot h(D_2, T).
\]

Obviously, \( (\mathcal{L}(\Omega), S) \) maps to \( (\mathcal{L}_S, S) \) and \( (\mathcal{L}^+(\Omega), S) \) maps to \( (\mathcal{L}^+_S, S) \). And then we have, again by definition

\[
h(\pi_T(D, S)) = h(\epsilon_T(D, T)) = (\epsilon_T(D) \cap \mathcal{L}_T, T).
\]
Now, we claim that \( \epsilon_T(D) \cap L_T = D \cap L_T \). Clearly \( \epsilon_T(D) \cap L_T \subseteq D \cap L_T \) and, on the other hand, \( D \cap L_T \subseteq C(D \cap L_T) \cap L_T = \epsilon_T(D) \cap L_T \). So we obtain
\[
h(\pi_T(D, S)) = (\epsilon_T(D) \cap L_T, T) = (D \cap L_T, T) = \pi_T(D \cap L_S, S) = \pi_T(h(D, S)).
\]
This proves item 1.

2. Suppose \( h(D_1, S) = h(D_2, T) \). Then first we must have \( S = T \) and further, since then \( D_1 \cap L_S = D_2 \cap L_S \), we have \( D_1 = \epsilon_S(D_1) = C(D_1 \cap L_S) = C(D_2 \cap L_S) = \epsilon_S(D_2) = D_2 \). So the map \( h \) is injective. And for any \( (D \cap L_S, S) \) we have \( h(\epsilon_S(D), S) = (\epsilon_S(D) \cap L_S, S) = (D \cap L_S, S) \) by the first part of the proof. So \( h \) is surjective, hence bijective.

\( \square \)

We remark, that in labeled information algebras, as in domain-free ones, an information order can be defined by \( (D_1 \cap L_S, S) \leq (D_2 \cap L_T, T) \) if
\[
(D_1 \cap L_S, S) \cdot (D_2 \cap L_T, T) = (D_2 \cap L_T, T).
\]
By definition of combination this means that \( S \subseteq T \) and \( D_1 \cdot D_2 = D_2 \), hence \( D_1 \subseteq D_2 \).

In a computational application of this second version of the labeled information algebra, one would use the fact that any set \( D \cap L_S \) is determined by the values of the gambles \( f \) on the set of possibilities \( \Omega_S \), which reduce greatly the efficiency of storage. Observations like this one explain why labeled information algebra are better suited for computational purposes.

## 5 Atoms and Maximal Coherent Sets of Gambles

In certain information algebras there are maximally informative elements, called **atoms** [Kohlas, 2003]. This is in particular the case for the information algebra of coherent sets of gambles. Maximal coherent sets of gambles \( M \) (see Section 2) have the property that, in information order,
\[
M \leq D \quad \text{for} \quad D \in \Phi \Rightarrow M = D \text{ or } D = L(\Omega)
\]
Elements in an information algebra with this property are called atoms [Kohlas, 2003]. In certain cases atoms determine the structure of an information algebra fully. We shall show that this is indeed the case for the algebra of coherent sets of gambles, see in particular the next Section 6. The definition of an atom can alternatively according to the definition of information order also be expressed by combination, \( M \) is an atom if
\[
M \cdot D = L(\Omega) \text{ or } M \cdot D = M, \quad \forall D \in \Phi.
\]
Let \( \text{At}(\Omega) \) denote the set of all atoms (maximal coherent sets) on \( \Omega \). For any coherent set of gambles \( D \), let \( \text{At}(D) \) denote the set of atoms (maximal coherent sets) which contain \( D \),
\[
\text{At}(D) = \{ M \in \text{At}(\Omega) : D \leq M \}.
\]
In general such sets may be empty. Not so in the case of coherent sets of gambles. In the case of the information algebra of coherent sets of gambles, we have in fact a number of additional properties concerning atoms (see Section 2).

1. For any coherent set \( D \) of gambles, \( D \in C(\Omega) \), there is a maximal set (an atom) \( M \) so that in information order \( D \leq M \) (i.e. \( D \subseteq M \)). So \( \text{At}(D) \) is never empty. An information algebra with this property is called \textit{atomic}.

2. For all coherent sets of gambles \( D \in C(\Omega) \), we have
   \[
   D = \inf \text{At}(D) = \bigcap \text{At}(D).
   \]
   An information algebra with this property is called \textit{atomic composed} \textit{or} \textit{atomistic} \textbf{(Kohlas, 2003)}.

3. For any, not empty, subset \( A \) of \( \text{At}(\Omega) \) we have that
   \[
   \inf A = \bigcap A
   \]
   is a coherent set of gambles, i.e. an element of \( C(\Omega) \). Such an algebra is called \textit{completely atomistic}.

The first two properties are proved in \textbf{(De Cooman & Quaeghebeur, 2012)}, the third follows since coherent sets form a \( \bigcap \)-structure. Note that, if \( A \) is a set of maximal sets of gambles, \( A \subseteq \text{At}(\bigcap A) \), and in general \( A \) is a proper subset of \( \text{At}(\bigcap A) \). These properties determine the structure of the information algebra of coherent sets in term of so-called set algebras, as we shall discuss in the following Section 6.

A consideration of the labeled version \( \tilde{\Psi} \) of the information algebra of coherent sets of gambles, exhibits some further structure of atoms. In fact, we may have maximally informative elements relative to a domain \( \Omega_S \) for any subset of variables. If \( M \in \text{At}(\Omega) \), then we have by definition of atom:

\[
(D \cap L_S, S) \geq (M \cap L_S, S) \quad \text{for some} \quad D \in \Phi, \quad \text{if and only if} \quad D = M \quad \text{or} \quad D = L(\Omega)
\]

In other words, we have \( (D \cap L_S, S) \geq (M \cap L_S, S) \) if and only if either \( (D \cap L_S, S) = (M \cap L_S, S) \) or \( (D \cap L_S, S) = (L_S, S) \). This means that the elements \( (M \cap L_S, S) \) are maximally informative relative to the subset of variables \( S \). Such elements are called \textit{atoms relative to} \( S \) \textbf{(Kohlas, 2003)}. Such relative atoms have the following properties.

\textbf{Lemma 3} \textit{Assume} \( M \) \textit{and} \( M_1, M_2 \) \textit{to be atoms,} \( D \in \Phi, \ S \subseteq I \). \textit{Then}

1. \( (M \cap L_S, S) \cdot (D \cap L_S, S) = (L_S, S) \quad \text{or} \quad (M \cap L_S, S) \).
2. \textit{If} \( T \subseteq S \), \textit{then} \( \pi_T(M \cap L_S, S) \) \textit{is an atom relative to} \( T \).
3. \textit{Either} \( (D \cap L_S, S) \leq (M \cap L_S, S) \) \textit{or} \( (D \cap L_S, S) \cdot (M \cap L_S, S) = (L_S, S) \).
4. Either \((M_1 \cap \mathcal{L}_S, S) \cdot (M_2 \cap \mathcal{L}_S, S) = (\mathcal{L}_S, S)\) or \((M_1 \cap \mathcal{L}_S, S) = (M_2 \cap \mathcal{L}_S, S)\).

These are purely properties of information algebras, for a proof see [Kohlas, 2003].

The properties of the algebra of being atomic, atomistic and completely atomistic carry over the the labeled version of the algebra of coherent sets.

1. Atomic: For any element \((D \cap \mathcal{L}_S, S) \in \tilde{\Psi}_S, S \subseteq I\) there is a relative atom \((M \cap \mathcal{L}_S, S)\) so that \((D \cap \mathcal{L}_S, S) \leq (M \cap \mathcal{L}_S, S)\).

2. Atomistic: For any element \((D \cap \mathcal{L}_S, S) \in \tilde{\Psi}_S, S \subseteq I\), \((D \cap \mathcal{L}_S, S) = \inf\{(M \cap \mathcal{L}_S, S) : M \in \text{At}(D)\}\).

3. Completely Atomistic: for any, not empty, subset \(A\) of \(\text{At}(\Omega)\), \(\inf\{(M \cap \mathcal{L}_S, S) : M \in A\}\) exists and belongs to \(\tilde{\Psi}_S\), for every \(S \subseteq I\).

As in the domain-free case these properties imply that the atoms, the maximal coherent sets of gambles, determine the structure of the information algebra of coherent gambles. This will be discussed in the next section.

6 Set Algebras

Important instances of information algebras are set algebras. Its elements are subsets of \(\Omega\). The operation of combination is simply set intersection. Extraction is defined in term of cylindrification: if \(A\) is a subset of \(\Omega\), then its cylindrification with respect to a subset \(S\) of variables is defined as

\[
\sigma_S(A) = \{\omega \in \Omega : \exists \omega' \in A \text{ so that } \omega'|S = \omega|S\}.
\]

This is a saturation operator. The family of subsets of \(\Omega\) with intersection as combination and cylindrification as extraction is a domain-free information algebra [Kohlas, 2017]. Saturation operators are more generally defined relative to partitions or equivalence relations of a set. In the present case we have the relations \(\omega \equiv_S \omega'\) with \(\omega, \omega' \in \Omega\), \(S \subseteq I\), if \(\omega|S = \omega'|S\). Below we shall encounter a more general case.

We show now that such a set algebra can be embedded into the information algebra of coherent sets of gambles. Define for any subset \(A\) not empty of \(\Omega\)

\[
D_A = \{f \in \mathcal{L}(\Omega) : \inf_{\omega \in A} f(\omega) > 0\} \cup \mathcal{L}^+(\Omega).
\]

This is obviously a coherent set of gambles. And we define \(D_\emptyset = \mathcal{L}(\Omega)\).

Now we have the following result.

**Theorem 5** For all subsets \(A\) and \(B\) of \(\Omega\) and subsets \(S\) of \(I\)

1. \(D_\emptyset = \mathcal{L}(\Omega), D_\Omega = \mathcal{L}^+(\Omega)\).
2. \( D_A \cdot D_B = D_{A \cap B} \),
3. \( \epsilon_S(D_A) = D_{\sigma_S(A)} \).

**Proof.** 1.) The first one is a definition, the second one is obvious.

2.) If \( D_A = \mathcal{L}^+ \) or \( D_B = \mathcal{L}^+ \) then \( A = \Omega \) or \( B = \Omega \), hence we have the result.

Now, suppose \( D_A, D_B \neq \mathcal{L}^+ \). If \( A \cap B = \emptyset \), then \( D_{A \cap B} = \mathcal{L}(\Omega) \).

If \( A = \emptyset \) or \( B = \emptyset \) then we have the result. Consider instead the case in which \( A, B \neq \emptyset \). By definition we have \( D_A \cdot D_B = \mathcal{L}(D_A \cup D_B) \). Consider \( f \in D_A \setminus \mathcal{L}^+ \) and \( g \in D_B \setminus \mathcal{L}^+ \). Since \( A \) and \( B \) are disjoint, we have \( \tilde{f} \in D_A \) and \( \tilde{g} \in D_B \), where \( \tilde{f}, \tilde{g} \) are defined in the following way:

\[
\tilde{f}(\omega) = \begin{cases} f(\omega) & \text{for } \omega \in A, \\ -g(\omega) & \text{for } \omega \in B, \\ 0 & \text{for } \omega \in (A \cup B)^c \end{cases}
\]

\[
\tilde{g}(\omega) = \begin{cases} -f(\omega) & \text{for } \omega \in A, \\ g(\omega) & \text{for } \omega \in B, \\ 0 & \text{for } \omega \in (A \cup B)^c \end{cases}
\]

But then \( \tilde{f} + \tilde{g} = 0 \in D_A \cdot D_B \), hence \( D_A \cdot D_B = \mathcal{L}(\Omega) = D_{A \cap B} \).

Assume then that both \( A \cap B \neq \emptyset \). Note that \( D_A \cup D_B \subseteq D_{A \cap B} \) so that \( D_A \cdot D_B \) is coherent and \( D_A \cdot D_B \subseteq D_{A \cap B} \). Consider then a gamble \( f \in D_{A \cap B} \). If \( \inf_A f > 0 \), then \( f \in D_A \) and if \( \inf_B f > 0 \), then \( f \in D_B \) and in both case we have \( f \in D_A \cup D_B \), hence in \( D_A \cdot D_B \). Otherwise select a \( \delta > 0 \) and define two functions

\[
f_1(\omega) = \begin{cases} 1/2f(\omega) & \text{for } \omega \in A \cap B, \omega \in (A \cup B)^c, \\ \delta & \text{for } \omega \in A - B, \\ f(\omega) - \delta & \text{for } \omega \in B - A \end{cases}
\]

\[
f_2(\omega) = \begin{cases} 1/2f(\omega) & \text{for } \omega \in A \cap B, \omega \in (A \cup B)^c, \\ f(\omega) - \delta & \text{for } \omega \in A - B, \\ \delta & \text{for } \omega \in B - A \end{cases}
\]

Then, \( f = f_1 + f_2 \) and \( f_1 \in D_A, f_2 \in D_B \). Therefore \( f \in C(D_A \cup D_B) = D_A \cdot D_B \), hence \( D_A \cdot D_B = D_{A \cap B} \).

3.) If \( A \) is empty, then \( \epsilon_S(D_{\emptyset}) = \mathcal{L}(\Omega) \) so that item 3 holds in this case. So assume \( A \neq \emptyset \). Then we have

\[
\epsilon_S(D_A) = \mathcal{L}(D_A \cap \mathcal{L}_S) = pos\mathcal{L}^+(\Omega) \cup (D_A \cap \mathcal{L}_S)).
\]

Consider then a gamble \( f \in D_A \cap \mathcal{L}_S \). Then, we have \( \inf_A f > 0 \) and \( f \) is \( S \)-measurable. So, if \( \omega|S = \omega'|S \) for some \( \omega' \in A \) and \( \omega \in \Omega \), then \( f(\omega) = f(\omega') \).

Therefore \( \inf_{\sigma_S(A)} f = \inf_A f > 0 \), hence \( f \in D_{\sigma_S(A)} \). Conversely, consider a gamble \( f \in D_{\sigma_S(A)} \). \( D_{\sigma_S(A)} \) is a strictly desirable set of gambles, hence, if \( f \in D_{\sigma_S(A)} \),
$f \in L^+(\Omega)$ or there exists $\delta > 0$ such that $f - \delta \in D_{\sigma_S(A)}$. If $f \in L^+(\Omega)$, then $f \in \epsilon_S(D_A)$. Otherwise, let us define for every $\omega \in \Omega$, $g(\omega) = \inf_{\omega' | S=\omega'} f(\omega') - \delta$. If $\omega \in A$, then $g(\omega) > 0$ since $\inf_{\sigma_S(A)}(f - \delta) > 0$. So we have $\inf_A g \geq 0$ and $g$ is $S$-measurable. However, $\inf_A (g + \delta) = \inf_A (g + \delta) > 0$ hence $(g + \delta) \in D_A \cap L_S$ and $f \geq g + \delta$. Therefore $f \in C(D_A \cap L_S)$. This concludes the proof.

This theorem shows that the map $A \mapsto D_A$ is a homomorphism between the set algebra and the information algebra of coherent sets of gambles. Furthermore, the map is one-to-one, hence an embedding of the set algebra into the algebra of coherent sets of gambles. This is a manifestation of the fact that the theory of desirable gambles covers among other things propositional and predicate logic.

Recall that coherent sets of gambles form a lattice (Section 2) where meet is set intersection. This is also the case for subsets of $\Omega$; they form even a Boolean lattice. We need however to stress that in information order, $A \leq B$ iff $B \subseteq A$. That is, information order is the opposite of the usual inclusion order between sets. This means that in information order meet is set union. Given this observation, it turns out that the map $A \mapsto D_A$ is even a lattice homomorphism.

**Theorem 6** For all subsets $A$ and $B$ of $\Omega$, $D_A \cap D_B = D_{A \cup B}$.

**Proof.** A gamble $f$ belongs to $D_A \cap D_B$ if and only if both $\inf_A f$ and $\inf_B f$ are both positive. But then it belongs to $D_{A \cup B}$. □

But there is much more about set algebras and information algebras of coherent sets of gambles. And this depends on the atomisticity of the information algebra of coherent sets of gambles. We stick in our discussion here to the domain-free view. The labeled view of what follows has been described in [Kohlas, 2003]. In the domain-free case the result we prove below states that $\Phi$ is isomorphic to a set algebra.

Consider the set of all atoms, that is all maximal coherent sets $At(\Phi)$ and define equivalence relations $M \equiv_S M'$ if $\epsilon_S(M) = \epsilon_S(M')$ in $At(\Phi)$ for all subsets of variables $S \subseteq I$. Associated with these equivalence relations are saturation operators $\sigma_S$ defined by $\sigma_S(X) = \{M \in At(\Phi) : \exists M' \in X \text{ so that } M \equiv_S M'\}$ for any subset $X$ of $At(\Phi)$ and $S \subseteq I$. Any saturation operator satisfies a number of important properties which are related to information algebras.

**Lemma 4** Let $\sigma_S$ be a saturation operator on $At(\Phi)$ for some $S \subseteq I$, associated with the equivalence relation $\equiv_S$, and $X, Y$ subsets of $At(\Phi)$. Then
1. \( \sigma_S(\emptyset) = \emptyset \),
2. \( X \subseteq \sigma_S(X) \),
3. \( \sigma_S(\sigma_S(X) \cap Y) = \sigma_S(X) \cap \sigma_S(Y) \),
4. \( X \subseteq Y \) implies \( \sigma_S(X) \subseteq \sigma_S(Y) \),
5. \( \sigma_S(\sigma_S(X)) = \sigma_S(X) \),
6. \( X = \sigma_S(X) \) and \( Y = \sigma_S(Y) \) imply \( X \cap Y = \sigma_S(X \cap Y) \).

Proof. Item 1., 2., 4. and 5. are obvious.

For 6. Consider \( M \in \sigma_S(X \cap Y) \). Then there is a \( M' \in X \cap Y \) so that \( M \equiv_S M' \). In particular, \( M' \in X \), hence \( M \in \sigma_S(X) \). At the same time, \( M' \in Y \), hence \( M \in \sigma_S(Y) \). Then \( M \in \sigma_S(X) \cap \sigma_S(Y) = X \cap Y \). By 2. we must then have equality.

For 3. observe that \( \sigma_S(X) \cap Y \subseteq \sigma_S(X) \cap \sigma_S(Y) \), so that \( \sigma_S(\sigma_S(X) \cap Y) \subseteq \sigma_S(\sigma_S(X) \cap \sigma_S(Y)) = \sigma_S(X) \cap \sigma_S(Y) \) by 4. and 6. For the reverse inclusion note that \( M \in \sigma_S(X) \cap \sigma_S(Y) \) means that there are \( M' \in X \), \( M'' \in Y \) so that \( M \equiv_S M' \) and \( M \equiv_S M'' \). By transitivity we have then \( M' \equiv_S M'' \) so that \( M'' \in \sigma_S(X) \). Then \( M \equiv_S M'' \) and \( M'' \in \sigma_S(X) \cap Y \) imply \( M \in \sigma_S(\sigma_S(X) \cap Y) \). This concludes the proof.

The first three items of the theorem correspond to the properties E1 to E3 of an existential quantifier in an information algebra, if combination is intersection. This is a first step to show that the subsets of \( At(\Phi) \) indeed form an information algebra with intersection as combination and saturation operators \( \sigma_S \) for \( S \subseteq I \) as extraction operator. The missing item will be verified below.

The combination of two equivalence relations is defined as

\[
\equiv_S \cdot \equiv_T = \{(M, M') \in At(\Phi) \times At(\Phi) : \exists M'' \in At(\Phi), \text{ so that } M \equiv_S M'' \equiv_T M'\}
\]

In general this is no more an equivalence relation. In our case however it is an equivalence and the relations commute as the following lemma shows.

**Lemma 5** For the equivalence relations \( \equiv_S \) and \( \equiv_T \) in \( At(\Phi) \), \( S, T \subseteq I \) we have

\[
\equiv_S \cdot \equiv_T = \equiv_T \cdot \equiv_S = \equiv_{S \cap T}.
\]

Proof. Let \( (M, M') \in \equiv_S \cdot \equiv_T \) so that there is an \( M'' \) such that \( \epsilon_S(M) = \epsilon_S(M'') \) and \( \epsilon_T(M') = \epsilon_T(M'') \). It follows that \( \epsilon_{S \cap T}(M) = \epsilon_T(\epsilon_S(M)) = \epsilon_T(\epsilon_S(M'')) = \epsilon_{S \cap T}(M'') \). Similarly we obtain \( \epsilon_{S \cap T}(M') = \epsilon_{S \cap T}(M'') \). But this shows that \( (M, M') \in \equiv_{S \cap T} \).

Conversely, suppose \( (M, M') \in \equiv_{S \cap T} \), that is \( \epsilon_{S \cap T}(M) = \epsilon_{S \cap T}(M') \). We claim that \( \epsilon_S(M) \cdot \epsilon_T(M') \neq 0 \). If, on the contrary \( \epsilon_S(M) \cdot \epsilon_T(M') = 0 \), then \( \epsilon_S(\epsilon_S(M) \cdot \epsilon_T(M')) = \epsilon_S(M) \cdot \epsilon_{S \cap T}(M') = 0 \) and further \( \epsilon_{S \cap T}(\epsilon_S(M) \cdot \epsilon_{S \cap T}(M')) = \epsilon_{S \cap T}(M) \).
\[ \epsilon_{S \cap T}(M') = \epsilon_{S \cap T}(M) = 0. \] But since \( M \) is an atom, this is a contradiction. So there is an atom \( M'' \in At(\epsilon_S(M) \cdot \epsilon_T(M')) \) so that \( \epsilon_S(M) \cdot \epsilon_T(M') \leq M'' \). Then \( \epsilon_S(M'') \geq \epsilon_S(M) \cdot \epsilon_{S \cap T}(M') = \epsilon_S(M) \). Therefore \( \epsilon_S(M'') = \epsilon_S(M'') \cdot \epsilon_S(M) = \epsilon_S(\epsilon_S(M'') \cdot M) = \epsilon_S(M) \) since \( M \) is an atom. In the same way, we deduce \( \epsilon_T(M'') = \epsilon_T(M') \). But this means that \( (M, M') \in \equiv_S \cdot \equiv_T \) and we have proved \( \equiv_S \cdot \equiv_T = \equiv_{S \cap T} \). The other equality follows by symmetry. \( \Box \)

As a corollary, it follows that the associated saturation operators commute.

**Lemma 6** For the equivalence relations \( \equiv_S \) and \( \equiv_T \) in \( At(\Phi) \), \( S, T \subseteq I \) we have

\[ \sigma_S \circ \sigma_T = \sigma_T \circ \sigma_S = \sigma_{S \cap T}. \]

**Proof.** For any subset \( X \) of \( At(\Phi) \) we have

\[
\begin{align*}
\sigma_S \circ \sigma_T(X) &= \{ M \in At(\Phi) : \exists M' \in X, \exists M'' \in At(\Phi) \text{ so that } M \equiv_S M'' \equiv_T M' \} \\
&= \{ M \in At(\Phi) : \exists M' \in X \text{ so that } M \equiv_S \equiv_T M' \} \\
&= \{ M \in At(\Phi) : \exists M' \in X \text{ so that } M \equiv_{S \cap T} M' \} = \sigma_{S \cap T}(X).
\end{align*}
\]

This proves that \( \sigma_S \circ \sigma_T = \sigma_{S \cap T} \). The remaining equality follows by symmetry. \( \Box \)

By this result, we have established that the power set of \( At(\Phi) \) with intersection as combination and saturation as extraction satisfies all items of Theorem \( \Box \). This means that the power set \( \mathcal{P}(At(\Phi)) \) is a domain-free information algebra. Since its elements are subsets and combination and extraction are set operations, it is a set algebra. But in addition \( \mathcal{P}(At(\Phi)) \) is also a complete, atomic Boolean lattice under inclusion, which corresponds to information order in \( \mathcal{P}(At(\Phi)) \).

Now in the following theorem we show that the pair of maps \( D \mapsto At(D) \) and \( \epsilon_S \mapsto \sigma_S \) represents an information algebra homomorphism and also maintains arbitrary joins. Recall that in information order in \( \Phi \) we have \( D_1 \leq D_2 \) if and only if \( D_1 \subseteq D_2 \). So information order is inclusion. In \( \mathcal{P}(At(\Phi)) \) combination is intersection so the information order is \( X \leq Y \) if \( X \cap Y = Y \), hence \( Y \subseteq X \). Therefore, information order in \( \mathcal{P}(At(\Phi)) \) is the inverse of inclusion, join is in this order intersection, meet union. Remark that the following theorems are purely results of atomistic information algebras and not specific to the algebra of coherent sets of gambles. Part of the following has been developed in [Kohlas & Schmid, 2020].

**Theorem 7** The map \( D \mapsto At(D) \) and \( \epsilon_S \mapsto \sigma_S \), are respectively injective and bijective. Moreover, if \( D, D_1, D_2 \) belong to \( \Phi \) and \( S \) is a subset of \( I \), the following are valid

1. \( At(D_1 \cdot D_2) = At(D_1) \cap At(D_2) \),
2. \( At(L^+(\Omega)) = At(\Phi) \) and \( At(L(\Omega)) = \emptyset \),
3. \( At(\epsilon_S(D)) = \sigma_S(At(D)) \).
Proof. The map $\epsilon_S \mapsto \sigma_S$ is bijective by construction. The map $D \mapsto At(D)$ is one-to-one since $\Phi$ is atomistic.

For 2, by definition we have $At(\mathcal{L}(\Omega)) = \emptyset$ and since $\mathcal{L}^+(\Omega) \leq M$ for all atoms $M$, $At(\mathcal{L}^+(\Omega)) = At(\Phi)$ (recall $0 = \mathcal{L}(\Omega)$ and $1 = \mathcal{L}^+(\Omega)$). So, if $D_1 \cdot D_2 = 0$, then $At(D_1) \cap At(D_2) = \emptyset$, since otherwise there would be an atom $M$ such that $D_1 \cdot D_2 \leq M$. Assume therefore $D_1 \cdot D_2 \neq 0$. The since $\Phi$ is atomic, there is an atom $M \in At(D_1 \cdot D_2)$ and since $D_1, D_2 \leq D_1 \cdot D_2 \leq M$, we conclude that $M \in At(D_1) \cap At(D_2)$. On the other hand, if $M \in At(D_1) \cap At(D_2)$, then $D_1 \leq M$ and $D_2 \leq M$, hence $D_1 \cdot D_2 = D_1 \lor D_2 \leq M$ and therefore $M \in At(D_1 \cdot D_2)$ This proves item 1.

For 3, if $D = 0$ we have immediately the result. Otherwise, consider first an atom $M \in \sigma_S(At(D))$, assuming $D \neq 0$. Then there is a $M' \in At(D)$ so that $\epsilon_S(M) = \epsilon_S(M')$. Further $D \leq M'$ implies $\epsilon_S(D) \leq \epsilon_S(M') = \epsilon_S(M) \leq M$ so that $M \in At(\epsilon_S(D))$.

Conversely, if $M \in At(\epsilon_S(D))$ then $\epsilon_S(D) \leq M$. We have $D \leq \epsilon_S(M) \cdot D$. We claim that $\epsilon_S(M) \cdot D \neq 0$. Indeed, otherwise we would have $\epsilon_S(M \cdot \epsilon_S(D)) = \epsilon_S(M) \cdot \epsilon_S(D) = \epsilon_S(M) \cdot D = \epsilon_S(0) = 0$ implying $M \cdot \epsilon_S(D) = 0$ which contradicts $M \in At(\epsilon_S(D))$. So there exists an atom $M' \in At(\epsilon_S(M) \cdot D)$ and thus $D \leq \epsilon_S(M) \cdot D \leq M'$. We conclude that $M' \in At(D)$.

Further $\epsilon_S(\epsilon_S(M) \cdot D) = \epsilon_S(M) \cdot \epsilon_S(D) \leq \epsilon_S(M')$, hence $\epsilon_S(M) \cdot \epsilon_S(M') \cdot \epsilon_S(D) = \epsilon_S(M')$. This implies $\epsilon_S(M) \cdot \epsilon_S(M') \neq 0$. Since $\epsilon_S(M) \cdot \epsilon_S(M') = \epsilon_S(M \cdot \epsilon_S(M'))$, we conclude that $M \cdot \epsilon_S(M') \neq 0$, hence $\epsilon_S(M') \leq M$ since $M$ is an atom. It follows that $\epsilon_S(M') \leq \epsilon_S(M)$.

Proceed in the same way from $\epsilon_S(M) \cdot \epsilon_S(M') = \epsilon_S(M) \cdot M'$ in order to obtain $\epsilon_S(M) \leq \epsilon_S(M')$ so that finally $\epsilon_S(M) = \epsilon_S(M')$. But this means that $M \in \sigma_S(At(D))$. So we have proved that $At(\epsilon_S(D)) = \sigma_S(At(D))$. □

This means that $\Phi$ is embedded in the set algebra $\mathcal{P}(At(\Phi))$ by the map $D \mapsto At(D)$, the algebra of coherent sets of gambles is essentially a set algebra in the technical sense used here. Note that this is purely a result of the theory of information algebra for atomistic algebras, and is not particular to the algebra of coherent sets of gambles. Recall however that we have already seen that $\Phi$ under information order is a complete lattice. And in fact the map $D \mapsto At(D)$ preserves arbitrary joins.

**Corollary 1** Let $D_j, j \in J$ be an arbitrary family of coherent sets of gambles dominated by a coherent set of gambles. Then

$$At\left(\bigvee_{j \in J} D_j\right) = \bigcap_{j \in J} At(D_j).$$

Proof. The proof of item 1 of Theorem 7 carries over to this more general case: By the assumption $\bigvee_{j \in J} D_j \neq 0$. So there is an atom $M \in At(\bigvee_{j \in J} D_j)$ and from $D_j \subseteq \bigvee_{j \in J} D_j$ we conclude that $M \in At(D_j)$, hence $M \in \bigcap_{j \in J} At(D_j)$. Conversely,
if \( M \in \bigcap_{j \in J} \text{At}(D_j) \), then \( D_j \subseteq M \) for all \( j \), hence \( \bigvee_{j \in J} D_j \subseteq M \) and therefore \( M \in \text{At}(\bigvee_{j \in J} D_j) \).

The set algebra \( \mathcal{P}(\Omega) \) introduced at the beginning of the section is embedded to \( \Phi \) (as an information algebra and a lattice), hence into \( \mathcal{P}(\text{At}(\phi)) \). Note that singleton sets \( \{\omega\} \) are atoms in the subset algebra \( \mathcal{P}(\Omega) \), and therefore \( D_{\{\omega\}} \) are also atoms in the algebra of coherent sets of gambles, \( D_{\omega} \in \text{At}(\Phi) \). We may ask how the images \( \text{At}(D) \) of coherent sets of gambles are characterized in \( \text{At}(\Phi) \). A (partial) answer is given in Section 8. We remark also that an analogous analysis can be made relative to the labeled view. We refer to (Kohlas, 2003) for this. In this view, the labeled algebras \( \Psi \) or \( \tilde{\Psi} \) are isomorphic to a generalized relational algebra in the sense of relational database theory (Kohlas, 2003). We come back to this in Section 8 with regard to lower previsions.

### 7 Algebras of Lower and Upper Previsions

Associated with a set of gambles \( D \) on \( \mathcal{L}(\Omega) \) is the lower prevision (Troffaes & De Cooman, 2014; Walley, 1991):

\[
\bar{P}(f) = \sup\{\mu \in \mathbb{R} : f - \mu \in D\}.
\]

There is also an associated upper prevision \( \bar{P} \) defined by \( \bar{P}(f) = -\bar{P}(-f) \). Note that \( \bar{P}(f) \) is defined only if the set \( \{\mu : f - \mu \in D\} \) is not empty and bounded from above. This is the case for all the gambles of a set \( D \) contained in a coherent set of gamble. Since \( \bar{P} \) depends on the set of gambles \( D \), we write \( \bar{P} = \sigma(D) \) and denote the set of gambles for which \( \bar{P} \) is defined as \( \text{dom}(\sigma(D)) \).

#### Lemma 7
For a set of gambles \( D \subseteq \mathcal{L}(\Omega) \) we have

1. if \( 0 \notin \mathcal{E}(D) \), then \( D \subseteq \text{dom}(\sigma(D)) \),
2. if \( D \in \mathcal{C}(\Omega) \), then \( \text{dom}(\sigma(D)) = \mathcal{L}(\Omega) \).

**Proof.** 1.) Consider \( f \in D \). Then the set \( \{\mu : f - \mu \in D\} \) is not empty, since it contains at least 0. Assume \( f - \mu \in D \). If \( \mu \geq \sup f \), then \( f(\omega) - \mu \leq 0 \) for all \( \omega \), but then \( 0 \in \mathcal{E}(D) \) which is a contradiction. So, the set \( \{\mu : f - \mu \in D\} \) is not empty and bounded from above for every \( f \in D \), which means that \( D \subseteq \text{dom}(\sigma(D)) \).

2.) If \( D \) is a coherent set of gambles, then \( 0 \notin \mathcal{C}(D) = \mathcal{E}(D) \) so that \( D \subseteq \text{dom}(\sigma(D)) \). Consider therefore \( f \in \mathcal{L}(\Omega) - D \). If there would be a \( \mu \geq 0 \) so that \( f - \mu \in D \), then \( f - \mu \leq f \in D \), which is a contradiction. Now, if \( \mu \leq \inf f < 0 \), then \( f - \mu \in \mathcal{L}^+(\Omega) \in D \), so it follows \( \inf f \leq \sigma(D)(f) < 0 \) and \( \text{dom}(\sigma(D)) = \mathcal{L}(\Omega) \). □

If \( D \) is a coherent set of gambles, then the associated function \( \bar{P} \) on \( \mathcal{L}(\Omega) \) is called a coherent lower prevision. It is characterized by the following properties: for every \( f, g \in \mathcal{L}(\Omega) \), (Walley, 1991).
1. \( P(f) \geq \inf_{\omega \in \Omega} f(\omega) \),
2. \( P(\lambda f) = \lambda P(f) \), \( \forall \lambda > 0 \),
3. \( P(f + g) \geq P(f) + P(g) \).

The conjugate value given by:

\[
\bar{P}(f) = \inf\{ \mu \in \mathbb{R} : \mu - f \in D \} = -P(-f).
\]

is called upper prevision.

It is called coherent, if the associated lower prevision is.

Let \( P(\Omega) \) denote the family of coherent lower previsions. The map \( \sigma \) maps \( \mathcal{C}(\Omega) \) to \( P(\Omega) \). This map is not one-to-one as different coherent sets of gambles may induce the same lower prevision. We may apply the map \( \sigma \) to almost desirable sets of gambles \( \bar{D} \) and its range is still \( P(\Omega) \) and we recall that moreover the map \( \sigma \) restricted to almost desirable sets of gambles is one-to-one (Walley, 1991), and

\[
P(f) = \max\{ \mu : f - \mu \in \bar{D} \}, \forall f \in \mathcal{L}, \quad \bar{D} = \{ f : P(f) \geq 0 \}.
\]  

(7.2)

There is also a one-to-one relation between coherent lower previsions \( P \) and strictly desirable sets of gamble \( D^+ \), so that, if we restrict \( \sigma \) to strictly desirable sets, we have, (Walley, 1991)

\[
P(f) = \sup\{ \mu : f - \mu \in D^+ \}, \forall f \in \mathcal{L}, \quad D^+ = \{ f : P(f) > 0 \} \cup \mathcal{L}^+(\Omega).
\]

Define the maps \( \tau \) and \( \bar{\tau} \) from coherent lower previsions to strictly desirable sets of gambles and almost desirable sets of gambles accordingly by

\[
\tau(P) = \{ f : P(f) > 0 \} \cup \mathcal{L}^+(\Omega), \quad \bar{\tau}(P) = \{ f : P(f) \geq 0 \}.
\]

Then \( \tau \) and \( \bar{\tau} \) are the inverses of the map \( \sigma \) restricted to strictly desirable and almost desirable sets of gambles respectively. The following lemma shows how coherent, strictly desirable and almost desirable sets are linked relative to the coherent lower previsions they induce

---

**Lemma 8** Let \( D \) be a coherent set of gambles. Then

\[
D^+ = \tau(\sigma(D)) \subseteq D \subseteq \bar{\tau}(\sigma(D)) = \bar{D}
\]

and \( \sigma(D^+) = \sigma(D) = \sigma(\bar{D}) \).

---

\[\text{This result follows also from the fact that, in the sup-norm topology of the linear space } \mathcal{L}(\Omega), \text{a strictly desirable set of gambles } D^+ \text{ is the relative interior of a coherent set } D \text{ plus the non-negative, non-zero gambles and any almost desirable set of gambles } \bar{D} \text{ is the relative closure of a coherent set } D \text{ (Walley, 1991).}\]
Proof. Let $P = \sigma(D)$. Then $f \in D^+$ means that $0 < P(f) = \sup \{ \mu : f - \mu \in D \}$, or $f \in \mathcal{L}^+($). If $f \in \mathcal{L}^+($) then $f \in D$. Otherwise, there is a $\delta$ such that $0 < \delta < P(f)$ and $f - \delta \in D$. Therefore $f \in D$ and $D^+ \subseteq D$. Further, consider $f \in D$. Then we must have $P(f) = \sup \{ f : f - \mu \in D \} \geq 0$, hence $f \in D$. The second part follows since $\tau$ and $\bar{\tau}$ are the inverse maps of $\sigma$ on desirable and almost desirable sets of gambles. \qed

Recall that strictly desirable sets of gambles form a subalgebra of the information algebra of coherent sets of gambles. Then this result establishes a map $D \mapsto D^+$ for any coherent set of gambles to a strictly desirable set. We shall see that this map preserves combination and extraction. To prove this theorem we need the following lemma.

Lemma 9 Let $D$ be a coherent set of gambles and $D^+ = \tau(\sigma(D))$, if $f \notin \mathcal{L}^+($) then $f \in D^+$ if and only if there is a $\delta > 0$ so that $f - \delta \in D$.

Proof. One part is by definition of strictly desirable gambles: if $f \in D^+$ and $f \notin \mathcal{L}^+($), then there is a $\delta > 0$ so that $f - \delta \in D^+ \subseteq D$. Conversely, consider $f - \delta \in D$ for some $\delta > 0$ and note that $D^+ = \{ f : \sigma(D)(f) > 0 \} \cup \mathcal{L}^+($). We have $\sigma(D) = \sup \{ \mu : f - \mu \in D \}$. From $f - \delta \in D$ we deduce that $\sigma(D)(f) > 0$, hence $f \in D^+$. \qed

The next theorem establishes that this map is a weak homomorphism, weak, because it does not apply when $D_1$ and $D_2$ are mutually inconsistent, that is if $D_1 \cdot D_2 = 0$.

Before stating this result we need to introduce a partial order relation on lower previsions. Indeed, we define $P \leq Q$ if $\text{dom}(P) \subseteq \text{dom}(Q)$ and $P(f) \leq Q(f)$ for all $f \in \text{dom}(P)$. This is a partial order on lower previsions (Troffaes & De Cooman, 2014).

Note that, if we consider the restriction of this partial order relation on lower previsions constructed from set $D'$ such that $0 \notin E(D')$, $\sigma$, $\tau$, $\bar{\tau}$ preserve order.

Theorem 8 Let $D_1$, $D_2$ and $D$ be coherent sets of gambles and $S \subseteq I$.

1. If $D_1 \cdot D_2 \neq 0$, then $D_1 \cdot D_2 \mapsto (D_1 \cdot D_2)^+ = D_1^+ \cdot D_2^+$,

2. $\epsilon_S(D) \mapsto (\epsilon_s(D))^+ = \epsilon_S(D^+)$.

Proof. For 1. note first that $D_1^+ \subseteq D_1$ and $D_2^+ \subseteq D_2$ so that

$D_1^+ \cdot D_2^+ = \tau(\sigma(D_1^+ \cdot D_2^+)) \subseteq \tau(\sigma(D_1 \cdot D_2)) = (D_1 \cdot D_2)^+$,

Further

$(D_1 \cdot D_2)^+ = \tau(\sigma(D_1 \cdot D_2)) = \{ f : \sigma(D_1 \cdot D_2)(f) > 0 \} \cup \mathcal{L}^+($)

So, if $f \in (D_1 \cdot D_2)^+$, then either $f \in \mathcal{L}^+($) or

$\sigma(D_1 \cdot D_2)(f) = \sup \{ \mu : f - \mu \in C(D_1 \cup D_2) \} > 0. \quad (7.3)$
In the first case obviously $f \in D_1^+ \cdot D_2^+$. In the second case there is a $\delta > 0$ so that $f - \delta \in C(D_1 \cup D_2)$. This means that $f - \delta = h + \lambda_1 f_1 + \lambda_2 f_2$, where $h \in \mathcal{L}^+(\Omega) \cup \{0\}$, $f_1 \in D_1$, $f_2 \in D_2$ and $\lambda_1, \lambda_2 \geq 0$ and not both equal 0. But then

$$f = h + (\lambda_1 f_1 + \delta/2) + (\lambda_2 f_2 + \delta/2).$$

We have $f_1' = \lambda_1 f_1 + \delta/2 \in D_1$ and $f_2' = \lambda_2 f_2 + \delta/2 \in D_2$. But this, together with $\lambda_1 f_1 = f_1' - \delta/2 \in D_1$ if $\lambda_1 > 0$ or otherwise $f_1' \in \mathcal{L}^+(\Omega)$, and $\lambda_2 f_2 = f_2' - \delta/2 \in D_2$ if $\lambda_2 > 0$ or otherwise $f_2' \in \mathcal{L}^+(\Omega)$, shows according to Lemma 9 that $f_1' \in D_1^+$ and $f_2' \in D_2^+$. So, finally, we have $f \in D_1^+ \cdot D_2^+ = C(D_1^+ \cup D_2^+)$. This proves that $(D_1 \cdot D_2)^+ = D_1^+ \cdot D_2^+$.

To prove 2. note that $D^+ \subseteq D$ so that

$$(\epsilon S(D))^+ = \tau(\sigma(\epsilon S(D))) \supseteq \tau(\sigma(\epsilon S(D^+))) = \epsilon S(D^+).$$

This is valid, because $\epsilon S(D^+)$ is also strictly desirable. Consider then

$$(\epsilon S(D))^+ = \tau(\sigma(\epsilon S(D))) = \{f : \sigma(\epsilon S(D)) > 0\} \cup \mathcal{L}^+(\Omega).$$

Here we have

$$\sigma(\epsilon S(D))(f) = \sup\{\mu : f - \mu \in C(D \cap \mathcal{L}S)\}.$$ 

So, if $f \in (\epsilon S(D))^+$, then either $f \in \mathcal{L}^+(\Omega)$ in which case $f \in \epsilon S(D^+)$ or there is a $\delta > 0$ so that $f - \delta \in \mathcal{L}^+(\Omega) \cup \{0\}$ and $g \in D \cap \mathcal{L}S$. Then we have $f = h + (g + \delta)$ and $g' = g + \delta$ is still $S$- measurable and $g' \in D$. But, given the fact that $g = g' - \delta \in D \cap \mathcal{L}S$, from Lemma 9 we have $g' \in D^+ \cap \mathcal{L}S$ and therefore $f \in \epsilon S(D^+)$. Thus we conclude that $(\epsilon S(D))^+ = \epsilon S(D^+)$ which concludes the proof.

Next, we claim that the map $\sigma$ restricted to coherent or almost desirable sets of gambles preserves also infima. Here we define $\inf\{P_j : j \in J\}$ by $\inf\{P_j(f) : j \in J\}$ for all $f \in \mathcal{L}(\Omega)$.

**Lemma 10** Let $D_j$, $j \in J$ be any family of coherent sets. Then we have

$$\sigma(\bigcap_{j \in J} D_j) = \inf_{j \in J}\{\sigma(D_j)\}$$

**Proof.** Note that the intersection of the coherent sets $D_j$ equals a coherent set $D$. We have $\sigma(\bigcap_{j \in J} D_j) = \sigma(D) = P \leq \sigma(D_j)$, $\forall j \in J$. So $P$ is a lower bound of the $\sigma(D_j), j \in J$. Let $Q$ be another lower bound of $\sigma(D_j), j \in J$. Then we have $\tau(Q) \subseteq D_j$ for all $j \in J$, by definition of $\tau$ and the fact that $Q$ is a lower bound of $\sigma(D_j)$, hence $\tau(Q) \subseteq \bigcap_{j \in J} D_j = D$. But this implies $Q \leq \sigma(D) = P$. Thus $P$ is indeed the infimum of the $\sigma(D_j)$ for $j \in J$. 

If $P'$ is a lower prevision which is dominated by a coherent lower prevision, then its natural extension is defined as the infimum of the coherent lower previsions which dominate it (Walley, 1991),

$$E(P') = \inf\{P \text{ coherent } : P' \leq P\}. \quad (7.4)$$
So, $E(P')$ is the minimal coherent lower prevision, which dominates $P'$. Now, we prove the key result, that the map $\sigma$ commutes with natural extension under certain conditions.

Theorem 9 Let $D'$ be a set of gambles which satisfies the following two conditions.

1. $0 \notin E(D')$,
2. for all $f \in D' - \mathcal{L}^+(\Omega)$ there exists a $\delta > 0$ such that $f - \delta \in D'$.

Then we have

$$\sigma(C(D')) = E(\sigma(D')).$$

Proof. If $D' = \mathcal{L}^+(\Omega)$, then $D' \in C(\Omega)$ and $\sigma(C(D')) = E(\sigma(D'))$ because the lower prevision associated with $\mathcal{L}^+(\Omega)$ is already coherent. So, assume that $D' \neq \mathcal{L}^+(\Omega)$. We have then $E(D') = C(D') \in C(\Omega)$, so that

$$C(D') = \bigcap\{D : D \text{ coherent}, D' \subseteq D\}.$$

Let $P' = \sigma(D')$, then $\sigma(C(D')) \supseteq P'$ and moreover $\sigma(C(D'))$ is coherent, hence $\sigma(C(D')) \supseteq E(P')$, where $E(P')$ defined by (7.4).

Now, consider any coherent lower prevision $P$ so that $P' \leq P$, and $\tau(P)$ the associated strictly desirable set of gambles. We claim that $D' \subseteq \tau(P)$. Indeed, if $f \in D'$, then $P'(f) \geq 0$. If $f \in \mathcal{L}^+(\Omega)$, then $f \in \tau(P)$, otherwise, if $f \in D' - \mathcal{L}^+(\Omega)$, then there is by assumption a $\delta > 0$ such that $f - \delta \in D'$, hence $0 < P'(f) \leq P(f)$. But this means that $f \in \tau(P)$. Since a strictly desirable set of gambles is coherent, it follows:

$$\sigma(C(D')) = \sigma\{\bigcap\{D : D \text{ coherent}, D' \subseteq D\} \subseteq \sigma(\bigcap\{\tau(P) : D' \subseteq \tau(P)\})$$

However, thanks to Lemma [10] we have:

$$\sigma(\bigcap\{\tau(P) : D' \subseteq \tau(P)\}) \leq \inf\{P \text{ coherent} : P' \leq P\} = E(P')$$

so that $\sigma(C(D')) = E(\sigma(D'))$, concluding the proof.

From this result on natural extensions in the two formalisms of coherent sets of gambles and coherent lower previions, we can now introduce into $\mathcal{P}(\Omega)$, like in $\Phi$ operations of combination and extraction. As in Section 4 consider a family of variables $X_i$, $i \in I$ with domains $\Omega_i$, and subsets $S \subseteq I$ of variables with domains $\Omega_S$. Let then for two coherent sets of gambles which are not inconsistent, $D_1 \cdot D_2 \neq 0$, with $P_1 = \sigma(D_1)$ and $P_2 = \sigma(D_2)$,

$$P'(f) = \sigma(D_1 \cup D_2)(f) = \sup\{\mu : f - \mu \in D_1 \cup D_2\} = \max\{P_1(f), P_2(f)\}$$

or $P' = \max\{P_1, P_2\}$. We may take the natural extension of $E(P')$ to define combination of two coherent lower previions $P_1$ and $P_2$. For extraction, we may take the natural extension of the marginal $P_S$ of $P$, defined as the restriction of $P$ to $\mathcal{L}_S$. Thus, in summary, we define $P_1 \cdot P_2$ and $e_S(P)$ by
1. Combination: \( P_1 \cdot P_2(f) = E(\max\{P_1, P_2\})(f), \forall f \in \mathcal{L}(\Omega), \) if \( \max\{P_1, P_2\} \) is dominated by a coherent lower prevision, \( P_1 \cdot P_2(f) = \infty \) for all \( f \in \mathcal{L}(\Omega) \) otherwise.

2. Extraction: \( e_S(\mathcal{P})(f) = E(\mathcal{P}_S)(f), \forall f \in \mathcal{L}(\Omega). \)

The following theorem permits to conclude that the set \( \Psi \) of coherent lower previsions \( \mathcal{P}(\Omega) \) augmented by \( \sigma(\mathcal{L}(\Omega)) \), defined by \( \sigma(\mathcal{L}(\Omega))(f) = \infty \) for all \( f \in \mathcal{L}(\Omega) \), forms a domain-free information algebra under these operations.

**Theorem 10** Let \( D_1^+ \), \( D_2^+ \) and \( D^+ \) be strictly desirable set of gambles and \( S \subseteq I \). Then

1. \( \sigma(D_1^+ \cdot D_2^+) = \sigma(D_1^+) \cdot \sigma(D_2^+) \).
2. \( \sigma(\mathcal{L}(\Omega))(f) = \infty, \sigma(\mathcal{L}^+(\Omega))(f) = \inf f \) for all \( f \in \mathcal{L}(\Omega) \),
3. \( \sigma(e_S(D^+)) = e_S(\sigma(D^+)). \)

Proof. Assume first that \( D_1^+ \cdot D_2^+ = 0 \) and let \( P_1 = \sigma(D_1^+), P_2 = \sigma(D_2^+) \). Then there can be no coherent lower prevision \( P \) dominating both \( P_1 \) and \( P_2 \). Indeed, otherwise we would have \( D_1^+ = \tau(P_1) \leq \tau(P) \) and \( D_2^+ = \tau(P_2) \leq \tau(P) \), where \( \tau(P) \) is a coherent set of gambles. But this is a contradiction. Therefore, we have \( \sigma(D_1^+ \cdot D_2^+)(f) = \infty = \sigma(D_1^+) \cdot \sigma(D_2^+) \), for all gambles \( f \).

Let then \( D_1^+ \cdot D_2^+ \neq 0 \). Then \( D_1^+ \cdot D_2^+ \) as well as \( D_1^+ \cup D_2^+ \) satisfy the condition of Theorem 9. Therefore, applying this theorem, we have

\[
\sigma(D_1^+ \cdot D_2^+) = \sigma(C(D_1^+ \cup D_2^+)) = E(\sigma(D_1^+ \cup D_2^+)) \\
= E(\max\{\sigma(D_1^+), \sigma(D_2^+)\}) = \sigma(D_1^+) \cdot \sigma(D_2^+).
\]

This proves item 1.

Item 2 is obvious.

For 3. we remark that \( D^+ \cap \mathcal{L}_S \) satisfies the condition of Theorem 9. Thus we obtain

\[
\sigma(e_S(D^+)) = \sigma(C(D^+ \cap \mathcal{L}_S)) = E(\sigma(D^+ \cap \mathcal{L}_S)).
\]

Now,

\[
\sigma(D^+ \cap \mathcal{L}_S) = \sup\{\mu : f - \mu \in D^+ \cap \mathcal{L}_S\}.
\]

But \( f - \mu \in D^+ \cap \mathcal{L}_S \) is only possible if \( f \) is \( S \)-measurable and \( f - \mu \in D^+ \). Therefore, we conclude that \( \sigma(D^+ \cap \mathcal{L}_S) = \sigma(D^+) \). Thus, we have indeed \( \sigma(e_S(D^+)) = E(\sigma(D^+)S) = e_S(\sigma(D^+)) \). This concludes the proof \( \square \)

The map \( \sigma \), restricted to the information algebra \( \Phi^+ \), is bijective. It follows that the set \( \Psi = \mathcal{P}(\Omega) \cup \{\sigma(\mathcal{L}(\Omega))\} \) is a domain-free information algebra, isomorphic to \( \Phi^+ \).

There is obviously the connected (isomorphic) information algebra of upper previsions. The following corollary, shows furthermore that \( \Phi \) is weakly homomorphic to \( \Psi \).
Corollary 2 Let $D_1$, $D_2$ and $D$ be coherent sets of gambles so that $D_1 \cdot D_2 \neq 0$ and $S \subseteq I$. Then

1. $\sigma(D_1 \cdot D_2) = \sigma(D_1) \cdot \sigma(D_2)$,
2. $\sigma(\epsilon_S(D)) = \epsilon_S(\sigma(D))$.

Proof. These claims are immediate consequences of Theorems 8 and 10.

The homomorphism does not extend to a pair of inconsistent coherent sets of gambles, as the following example shows.

Example C: Consider a set of possibilities $\Omega = \{\omega_1, \omega_2\}$ and let

$$D_1 = \{f : f(\omega_2) > -2f(\omega_1)\} \cup \{f : f(\omega_2) = -2f(\omega_1), f(\omega_2) \geq 0, f \neq 0\},$$

$$D_2 = \{f : f(\omega_2) > -2f(\omega_1)\} \cup \{f : f(\omega_2) = -2f(\omega_1), f(\omega_2) < 0\}.$$

These are coherent sets of gambles, but they are mutually inconsistent, since we have $D_1 \cdot D_2 = \mathcal{L}(\Omega)$ because $0 \in \mathcal{E}(D_1 \cup D_2)$. But on the other hand, $D_1^+ \cdot D_2^+ = D_1^+ \neq \mathcal{L}(\Omega)$. So already the map $D \mapsto D^+$ does not respect inconsistencies, and then $\sigma(D_1 \cdot D_2) \neq \sigma(D_1) \cdot \sigma(D_2)$ in this example.

Finally, it follows from Lemma 10 and Corollary 2 that for any family of coherent sets of gambles $D_j$,

$$\sigma(\epsilon_S(\bigcap_j D_j)) = \epsilon_S(\sigma(\bigcap_j D_j)) = \epsilon_S(\inf\{\sigma(D_j)\})$$

and

$$\sigma(\bigcap_j \epsilon_S(D_j)) = \inf\{\sigma(\epsilon_S(D_j))\} = \inf\{\epsilon_S(\sigma(D_j))\}.$$ 

Therefore it follows from (3.2) also that for any family $P_{\bar{j}}$ of coherent lower previsions, we have

$$\epsilon_S(\inf\{P_{\bar{j}}\}) = \inf\{\epsilon_S(P_{\bar{j}})\}.$$ 

So, in the information algebra of lower prevision extraction distributes still over meet (infimum).

As always, there is also a labeled version of this domain-free algebra of lower prevision, which will be described very briefly. For any subset $S$ of $I$ let

$$\bar{\Psi}_S = \{P_S : P \text{ coherent lower prevision}\},$$

where again $P_S$ is the restriction of $P$ to $\mathcal{L}_S$. Further let

$$\bar{\Psi} = \bigcup_{S \subseteq I} \bar{\Psi}_S.$$ 

Within $\bar{\Psi}$ we define the following operations, given $P_S, P_{1,S}, P_{2,T} \in \bar{\Psi}$:
1. Labeling: \( d(P_S) = S \),

2. Combination: \( P_{1,S} \cdot P_{2,T} = (P_1 \cdot P_2)_{S \cup T} \), where on the right combination of lower previsions is used,

3. Projection (Marginalization): for \( T \subseteq S \), \( \pi_T(P_S) = P_T \).

It is easy to verify that \( \Psi \) with these operations satisfy the axioms of a labeled information algebra as presented in Theorem 3 for coherent sets of desirable gambles.

We remark also that, if we restrict \( \tau \) to \( L_S \), then \( \tau(P_S) = \{ f : P_S(f) > 0 \} \cup L^+_S = (\{ f : P(f) > 0 \} \cup L^+(\Omega)) \cap L_S = \tau(P) \cap L_S \). This relates the labeled algebra of coherent lower previsions to the second version of the labeled version of coherent or rather strictly desirable sets of gambles at the end of Section 4 as isomorphic.

The results of this section shows that the coherent lower (and upper) previsions form an information algebra closely related to the information algebras of coherent or desirable sets of gambles. This relationship carries over to the labeled versions of the algebras involved, a subject we do not pursue here. However we have seen that the algebra of coherent sets of gambles is completely atomistic. In the next section we discuss what this means for the algebra of lower previsions.

8 Linear Previsions

If \( P(f) = -P(-f) \) for all \( f \) in \( L(\Omega) \), that is if lower and upper previsions coincide, \( P \) is called a linear prevision. Then its usual to write \( P = \bar{P} = P \). Linear previsions have an important role in the theory of imprecise probabilities. Therefore, in this section they will be examined from the point of view of information algebras. First of all a linear prevision is a lower (and upper) prevision. So, if \( P(\Omega) \) denote the set of linear previsions on \( L(\Omega) \), we have \( P(\Omega) \subseteq \bar{P}(\Omega) \). Note that from the third coherence property of lower previsions it follows that \( P(f + g) = P(f) + P(g) \), for every \( f, g \in L(\Omega) \).

Let us concentrate ourselves on the strictly desirable set of gambles associated with a linear prevision,

\[
\tau(P) = \{ f : P(f) > 0 \} \cup L^+(\Omega) = \{ f : -P(-f) > 0 \} \cup L^+(\Omega).
\]

We call these sets maximal strictly desirable sets of gambles. It is possible to show that we have \( M^+ = \tau(P) = \tau(\sigma(M)) \), where \( M \) is a maximal set of gambles, see [Walley, 1991]. Since these sets are atoms in the algebra of coherent sets of gambles, we may presume that maximal strictly desirable sets of gambles are atoms in the algebra of strictly desirable sets of gambles and linear previsions are also atoms in the algebra of coherent prevision. And this is indeed the case.

**Lemma 11** Let \( P \) be an element of \( \Psi \) and \( P \) a linear prevision. Then \( P \leq P \) implies either \( P = P \) or \( P(f) = \infty \) for all \( f \in L(\Omega) \).
Proof. Clearly $P(f) = +\infty$ for all $f \in \mathcal{L}$ is a possible solution. Consider instead the case in which $P$ is coherent.

From (Walley, 1991), we know $P(f) \leq \bar{P}(f), \forall f \in \mathcal{L}$. Then, we have:

$$P(f) \leq \bar{P}(f) = -P(-f) \leq -P(-f) = P(f), \forall f \in \mathcal{L}.$$ \hfill (8.1)

Given the fact that, by hypothesis, we have also $P(f) \geq P(f), \forall f \in \mathcal{L}$, we have the result. \hfill $\square$

**Lemma 12** Let $D^+$ be an element of $\Phi^+$ and $M^+$ a maximal strictly desirable set of gambles. Then $M^+ \leq D^+$ implies either $D^+ = M^+$ or $D^+ = \mathcal{L}(\Omega)$.

Proof.

$M^+ \leq D^+ \Rightarrow \sigma(M^+) \leq \sigma(D^+).$ \hfill (8.2)

From Lemma [11], we have $\sigma(D^+)(f) = +\infty, \forall f \in \mathcal{L}(\Omega)$ or $\sigma(D^+) = \sigma(M^+)$. In the first case, we have $D^+ = \mathcal{L}(\Omega)$. In the second one, we have $D^+ = \tau(\sigma(D^+)) = \tau(\sigma(M^+)) = M^+$. \hfill $\square$

From Lemma 11, we may automatically deduce the properties of atoms in an information algebra, so, for example, we have $P \cdot P = P$ or $P \cdot P = 0$, where here 0 is the null element $P(f) = \infty, \forall f \in \mathcal{L}(\Omega)$. The information algebra $\Phi$ of coherent sets is completely atomistic. It is to be expected that the same holds for the algebra $\Psi$ of coherent lower previsions. Let $\text{At}(\Psi) = \mathcal{P}(\Omega)$ be the set of all linear previsions (atoms) and $\text{At}(P)$ the set of all linear previsions (atoms) dominating the coherent lower prevision $P$,

$$\text{At}(P) = \{P \in \text{At}(\Psi) : P \leq P\}.$$ 

Then the following theorem shows that the information algebra $\Psi$ is completely atomistic.

**Theorem 11** In the information algebra of lower previsions $\Psi$, the following holds:

1. If $P$ is a coherent lower prevision, then

$$P = \inf \text{At}(P).$$

2. If $A$ is any non-empty subset of linear prevision in $\text{At}(\Psi)$, then

$$P = \inf A$$

exists and is a coherent lower prevision.

For the proof of this theorem, see (Walley, 1991) Theorem 3.3.3).
According to this theorem, if $A$ is any non-empty family of linear previsions on $L(\Omega)$, then $\inf A$ exists and is a coherent lower prevision $P$. Then we have $A \subseteq \mathcal{A}(P)$ and it follows

$$P = \inf A = \inf \mathcal{A}(P).$$

So, the coherent lower prevision $P$ is the lower envelope of the linear previsions (atoms) which dominate it.

As any atomistic information algebra, $\Psi$ is embedded in the set algebra of subsets of $\mathcal{A}(\Psi)$ by the map $P \mapsto \mathcal{A}(P)$. This rises the question how to characterize the images of $\Psi$ in the algebra of subsets $\mathcal{A}(\Psi)$. The answer is given by the weak* compactness theorem (Walley, 1991): The sets $\mathcal{A}(P)$ for any coherent lower prevision $P$ are exactly the weak* compact convex subsets of $\mathcal{A}(\Psi)$ in the weak* topology on $\mathcal{A}(\Psi)$. There are many other sets of linear previsions $A$ whose lower envelope equals $P$. If $P = \inf A$ and $A \subseteq B \subseteq \mathcal{A}(P)$, then $P = \inf B$. In fact there is a minimal set $E \subseteq \mathcal{A}(P)$ so that $P = \inf E$ and this is the set of extremal points in the convex set $\mathcal{A}(P)$. This follows from the extreme point theorem (Walley, 1991).

We shall come back to the embedding of the algebra of coherent lower previsions in the algebra of subsets of $\mathcal{A}(\Psi)$ below in the labeled view of the algebra.

We now examine linear previsions in the light of extraction or, equivalently, in the labeled view of the information algebra of coherent lower previsions, see Section 7. Recall that for any subset $S$ of $I$, $P_S$ denotes the restriction of the coherent lower prevision $P$ to $L_S$. Similarly, $P_S$ is the restriction of the linear prevision to $L_S$. These element $P_S$ are local atoms in the labeled information algebra $\tilde{\Psi}$, that is, if $P_S \leq P_S$, then either $P_S = P_S$ or $P_S = 0_S$ for any linear prevision $P$ and $P \in \tilde{\Psi}$ and where $0_S$ is the null element for label $S$, that is $0_S(f) = \infty$ for all $f \in L_S$. We call $P_S$ also a locally linear previsions. There follow as usual some elementary properties of linear previsions, as atoms, for instance such as the equivalent of Lemma 3.

**Lemma 13** Assume $P$ and $P_1$, $P_2$ to be linear previsions, that is atoms and further $P \in \tilde{\Psi}$. Consider also $S \subseteq I$. Then

- $P_S \cdot P_S = 0_S$ or $P_S$,
- if $T \subseteq S$, then $\pi_T(P_S)$ is locally linear relative to $T$, that is an atom relative to $T$,
- either $P_S \leq P_S$ or $P_S \cdot P_S = 0_S$,
- $P_{1,S} \cdot P_{2,S} = 0_S$ or $P_{1,S} = P_{2,S}$.

For a proof of this result on atoms in labeled linear algebras we refer to (Kohlas, 2003). Just as the labeled information algebra of coherent sets of gambles is atomic, atomistic and completely atomistic, the same holds for the labeled algebra of coherent
lower previsions. Let \( At(\tilde{\Psi}_S) \) be the set of all locally linear previsions relative to \( S \) and \( At(P_S) \) the set of locally linear previsions, or atoms, dominating the coherent lower prevision \( P_S \).

- **Atomic:** For every coherent lower prevision \( P \) there is a linear prevision \( P \) so that \( P_S \leq P \). That is, \( At(P_S) \) is not empty.

- **Atomistic:** For every coherent lower previsions \( P \) we have \( P_S = \inf At(P_S) \).

- **Completely Atomistic:** For any, not empty, subset \( A \) of \( At(\tilde{\Psi}_S) \), \( \inf A \) exists and belongs to \( \tilde{\Psi}_S \).

It is further well-known that the local atoms of any atomic labeled information algebra satisfy the following conditions, expressed here for locally linear previsions (Kohlas, 2003).

**Lemma 14** Let

\[
At(\tilde{\Psi}) = \bigcup_{S \subseteq I} At(\tilde{\Psi}_S)
\]

and let \( P_S, P_T, P_{1,S}, P_{2,T} \in At(\tilde{\Psi}) \).

Then,

1. if \( T \subseteq d(P_S) \), then \( d(\pi_T(P_S)) = T \),
2. if \( T \subseteq R \subseteq d(P_S) \), then \( \pi_T(\pi_R(P_S)) = \pi_T(P_S) \),
3. \( \pi_S(P_S) = P_S \),
4. if \( \pi_{S \cup T}(P_{1,S}) = \pi_{S \cup T}(P_{2,T}) \), then there is a \( P_{S \cup T} \in At(\tilde{\Psi}) \) so that \( \pi_S(P_{S \cup T}) = P_{1,S} \) and \( \pi_T(P_{S \cup T}) = P_{2,T} \),
5. for an element \( P_S \), if \( S \subseteq T \), there is a \( P_T \in At(\tilde{\Psi}) \) so that \( \pi_S(P_T) = P_S \).

Most of these properties are immediate consequences from the lower previsions being a labeled information algebra. For a proof of item 4 we refer to (Kohlas, 2003). A system like \( At(\tilde{\Psi}) \) with a labeling and a projection operation satisfying the conditions of Lemma 14 is called a tuple system, since it abstracts the properties of concrete tuples as used in relational database systems. And generalized relational algebras can be defined using tuple systems and they turn out to be labeled information algebras (Kohlas, 2003). In the case of the tuple system \( At(\tilde{\Psi}) \) this goes as follows: A subset \( R \) of \( At(\tilde{\Psi}_S) \) is called a (generalized) relation on \( S \). Denote by \( \mathcal{R}_S \) all these relations on \( S \) and let

\[
\mathcal{R} = \bigcup_{S \subseteq I} \mathcal{R}_S.
\]

Then, in \( \mathcal{R} \), we define the following operations.
1. Labeling: \( d(R) = S \) if \( R \in \mathcal{R}_S \).

2. Natural Join: \( R_1 \Join R_2 = \{ P_{S,T} \in At(\bar{\Psi}_{S,T}) : \pi_S(P_{S,T}) \in R_1, \pi_T(P_{S,T}) \in R_2 \} \) if \( d(R_1) = S \) and \( d(R_2) = T \).

3. Projection: \( \pi_T(R) = \{ \pi_T(P) : P \in R \} \), if \( d(R) = S \).

With these operations, natural join as combination, the algebra of relations \( \mathcal{R} \) becomes a labeled information algebra. Note that the null element of natural join is the empty set and the unit with label \( S \) is \( At(\bar{\Psi}_S) \). This depends only on \( At(\bar{\Psi}) \) being a tuple system (Kohlas, 2003). What is more, it turns out that the atomistic labeled algebra of coherent lower previsions is embedded into this generalized relational algebra.

**Theorem 12** Let \( \bar{\Psi} \) be the labeled information algebra of coherent lower previsions. Let further \( P \) and \( Q \) be elements of \( \bar{\Psi} \). Then

1. \( At(P_S \cdot Q_T) = At(P_S) \Join At(Q_T) \),
2. \( At(\pi_T(P_S)) = \pi_T(At(P_S)) \) if \( T \subseteq S \).

Again, this is a theorem of atomistic information algebras and not particular to lower previsions, (Kohlas, 2003). It says that the map \( P_S \mapsto At(P_S) \), complemented by \( 0_S \mapsto \emptyset \), is a homomorphism between the labeled information algebra of coherent lower previsions and the generalized relational algebra of sets of locally linear previsions. Furthermore, since the map is obviously one-to-one, it tells us that \( \bar{\Psi} \) is embedded into this relational algebra. This is the labeled version of the embedding of the domain-free algebra into a set algebra of atoms.

By definition, any linear prevision is also locally linear relative to any subset of variables. Fix any linear prevision \( Q \) on \( \mathcal{L}_S \) and \( S, T \subseteq I \). Then

\[ \mathcal{M}(Q) = \{ P \in \mathcal{P} : P_S = Q \} \]

is called the linear extension of \( Q \) to \( \mathcal{L}(\Omega) \). Now, define \( Q = E(Q) \) to be the natural extension of \( Q \) to \( \mathcal{L}(\Omega) \). Then, by the definition of extraction we have \( e_S(P) = E(P_S) = Q \) whenever \( P_S = Q \) and \( P \) is coherent. In particular, we have \( e_S(P) = Q \) for all \( P \) in the linear extension of \( Q \), then \( P \geq e_S(P) = Q \), hence \( P \in At(Q) \). Conversely, if \( P \in At(Q) \), then \( P_S = Q \), but this is only possible if \( P_S = Q \). It follows that \( \mathcal{M}(Q) = At(Q) \). This is a version of the natural extension theorem (Walley, 1991). Since extraction distributes over meet, we have

\[ Q = e_S(Q) = e_S(\inf At(Q)) = \inf e_S(At(Q)) \]

and also

\[ Q = e_S(Q) = e_S(\inf \mathcal{M}(Q)) = \inf e_S(\mathcal{M}(Q)) \]

If \( P \in \mathcal{M}(Q) = At(Q) \), then \( e_S(P) = Q \), hence \( P \geq Q \) implies \( e_S(P) = Q \).
9 The Marginal Problem

Consistency, inconsistency, compatibility or incompatibility, whatever is exactly meant by these concepts, are general properties of information. Here these notions will be defined and studied with respect to coherent sets of gambles, but in the frame and using results of information algebras. Two pieces of information can be considered as compatible, if they are not contradictory, that is, if their combination is not the null element. Hence a finite family of coherent sets of gambles $D_1, \ldots, D_n$ is compatible, if $0 \not\in D_1 \cdot \ldots \cdot D_n = E(\cup_{i=1}^n D_i)$. This is called “avoiding partial loss” in desirability (Miranda & Zaffalon, 2020). Otherwise the family is called incompatible. There is, however, a more restrictive concept of compatibility. Here a family of coherent sets of gambles $D_1, \ldots, D_n$, where $D_i$ has support $S_i$, $i = 1, \ldots, n$, is called compatible, if there is a coherent set of gambles $D$ such that $\epsilon_{S_i}(D) = D_i$ for $i = 1, \ldots, n$, see (Miranda & Zaffalon, 2020). To decide whether a family of $D_i$ is compatible in this sense is also called the marginal problem, since extractions are (in the labeled view) projections or marginals. In the view of information algebra, we prefer to call this type of compatibility consistency, since the pieces of information $D_i$ come from or are part of the same piece of information $D$.

In (Miranda & Zaffalon, 2020) two coherent sets $D_i$ and $D_j$, where $D_i$ has support $S_i$ and $D_j$ support $S_j$, are called pairwise compatible, or, in our terminology pairwise consistent, if

$$D_i \cap L_{S_j} = D_j \cap L_{S_i}$$

or $E_{S_j}(D_i) = E_{S_i}(D_j)$. In terms of the information algebra this means that

$$\epsilon_{S_j}(D_i) = \mathcal{C}(E_{S_j}(D_i)) = \mathcal{C}(E_{S_i}(D_j)) = \epsilon_{S_i}(D_j)$$

(9.1)

From this it follows that

$$\epsilon_{S_i \cap S_j}(D_i) = \epsilon_{S_i \cap S_j}(\epsilon_{S_j}(D_i)) = \epsilon_{S_i \cap S_j}(\epsilon_{S_i}(D_j)) = \epsilon_{S_i \cap S_j}(D_j).$$

In an information algebra in general we could take this as a definition of pairwise consistency. From this we may recover [9.1], since by item 5 of the list of properties of support (Section 4), if $S_i$ is a support of $D_i$ and $S_j$ of $D_j$, we have $\epsilon_{S_j}(D_i) = \epsilon_{S_i \cap S_j}(D_i)$ and $\epsilon_{S_i}(D_j) = \epsilon_{S_i \cap S_j}(D_j)$.

Now let $D = D_i \cdot D_j$ and $D_i$ and $D_j$ pairwise consistent. Then $\epsilon_{S_i}(D) = D_i \cdot \epsilon_{S_j}(D) = D_j \cdot \epsilon_{S_j}(D_i) = D_i$ and also $\epsilon_{S_j}(D) = D_j$. So, pairwise consistent pieces of information are consistent. And, conversely, if $D_i$ and $D_j$ are consistent, then $\epsilon_{S_i \cap S_j}(D) = \epsilon_{S_i \cap S_j}(\epsilon_{S_i}(D)) = \epsilon_{S_i \cap S_j}(D_i)$ and similarly $\epsilon_{S_i \cap S_j}(D) = \epsilon_{S_i \cap S_j}(D_j)$ and the two elements are pairwise consistent.

It is well-known that pairwise consistency among a family of $D_1, \ldots, D_n$ of pieces of information is not sufficient for the family to be consistent. It is also well-known that a sufficient condition to obtain consistency from pairwise consistency is that the family of supports $S_1, \ldots, S_n$ of the $D_i$ satisfy the running intersection property (RIP, that is form a join tree or a hypertree construction sequence):
RIP For \( i = 1 \) to \( n - 1 \) there is an index \( p(i) \), \( i + 1 \leq p(i) \leq n \) such that

\[
S_i \cap S_{p(i)} = S_i \cap (\bigcup_{j=i+1}^n S_j).
\]

Then we have the following theorem, see prop. 1 and theorem 2 in (Miranda & Zaffalon, 2020), a theorem, which in fact is a theorem of information algebras in general.

**Theorem 13** If \( D_1, \ldots, D_n \), a family of compatible coherent sets of gambles, with supports \( S_1, \ldots, S_n \) which satisfy RIP, are pairwise consistent, then they are consistent and \( \epsilon_{S_i}(D_1 \cdot \ldots \cdot D_n) = D_i \) for \( i = 1, \ldots, n \).

**Proof.** We give a proof in the framework of general information algebras. Let \( Y_i = S_{i+1} \cup \ldots \cup S_n \) for \( i = 1, \ldots, n - 1 \) and \( D = D_1 \cdot \ldots \cdot D_n \). Then by RIP

\[
\epsilon_{Y_1}(D) = \epsilon_{Y_1}(D_1) \cdot D_2 \cdot \ldots \cdot D_n = \epsilon_{Y_1}(\epsilon_{S_1}(D_1)) \cdot D_2 \cdot \ldots \cdot D_n
\]

\[
= \epsilon_{S_1 \cap Y_1}(D_1) \cdot D_2 \cdot \ldots \cdot D_n = \epsilon_{S_1 \cap S_{p(1)}}(D_1) \cdot D_2 \cdot \ldots \cdot D_n.
\]

But by pairwise compatibility \( \epsilon_{S_1 \cap S_{p(1)}}(D_1) = \epsilon_{S_1 \cap S_{p(1)}}(D_{p(1)}) \), hence by idempotency

\[
\epsilon_{Y_1}(D) = D_2 \cdot \ldots \cdot D_n.
\]

By induction on \( i \), one shows exactly in the same way that

\[
\epsilon_{Y_i}(D) = D_{i+1} \cdot \ldots \cdot D_n, \quad \forall i = 1, \ldots, n - 1.
\]

So, we obtain \( \epsilon_{S_n}(D) = \epsilon_{Y_{n-1}}(D) = D_n \). Now, we claim that \( \epsilon_{S_i}(D) = \epsilon_{S_i \cap S_{p(i)}}(D) \cdot D_i \). Since \( S_{p(i)} \subseteq Y_i \), we have by RIP

\[
D_i \cdot \epsilon_{S_i \cap S_{p(i)}}(D) = D_i \cdot \epsilon_{S_i \cap S_{p(i)}}(\epsilon_{Y_1}(D)) = D_i \cdot \epsilon_{S_i \cap Y_1}(\epsilon_{Y_1}(D)) = D_i \cdot \epsilon_{S_i}(\epsilon_{Y_1}(D))
\]

\[
= D_i \cdot \epsilon_{S_i}(D_{i+1} \cdot \ldots \cdot D_n) = \epsilon_{S_i}(D_i) \cdot D_{i+1} \cdot \ldots \cdot D_n = \epsilon_{S_i}(\epsilon_{Y_{i-1}}(D))
\]

\[
= \epsilon_{S_i}(D).
\]

Then, by backward induction, based on the induction assumption \( \epsilon_{S_i}(D) = D_j \) for \( j > i \), and rooted in \( \epsilon_{S_n}(D) = D_n \), for \( i = n-1, \ldots, 1 \), we have by pairwise consistency

\[
\epsilon_{S_i}(D) = \epsilon_{S_i \cap S_{p(i)}}(D) \cdot D_i = \epsilon_{S_i \cap S_{p(i)}}(\epsilon_{S_{p(i)}}(D)) \cdot D_i
\]

\[
= \epsilon_{S_i \cap S_{p(i)}}(D_{p(i)}) \cdot D_i = \epsilon_{S_i \cap S_{p(i)}}(D_i) \cdot D_i = D_i.
\]

This concludes the proof. \( \square \)

Note that the condition \( \epsilon_{S_i}(D_1 \cdot \ldots \cdot D_n) = D_i \) implies that the family \( D_1, \ldots, D_n \) is consistent. So, this is a sufficient condition for consistency. This theorem is a theorem of information algebra, it holds not only for coherent sets of gambles, but for any information algebra, in particular for the algebra of coherent lower previsions for instance.

The definition of consistency and pairwise consistency depend on the supports (hence indirectly on the \( S \)-measurability) of the elements \( D_i \). But \( D_i \) may have different
supports. How does this influence consistency? Assume \( D_i \) and \( D_j \) are pairwise consistent according to their supports \( S_i \) and \( S_j \), that is \( \epsilon_{S_i \cap S_j}(D_i) = \epsilon_{S_i \cap S_j}(D_j) \). It may be that a set \( S_i' \subseteq S_i \) is still a support of \( D_i \) and a subset \( S_j' \subseteq S_j \) a support of \( D_j \). Then

\[
\epsilon_{S_i' \cap S_j'}(D_i) = \epsilon_{S_i' \cap S_j'}(\epsilon_{S_i \cap S_j}(D_i)) = \epsilon_{S_i' \cap S_j'}(\epsilon_{S_i \cap S_j}(D_j)) = \epsilon_{S_i' \cap S_j'}(D_j).
\]

So, \( D_i \) and \( D_j \) are also pairwise consistent relative to the smaller supports \( S_i' \) and \( S_j' \). The finite supports of a coherent set of gambles \( D_i \), have a least support

\[
d_i = \bigcap\{ S : S \text{ support of } D_i \}.
\]

This is called the dimension of \( D_i \), it is itself a support of \( D_i \) (see item 4 on the list of properties of supports). So, if \( D_i \) and \( D_j \) are pairwise consistent relative to two of their respective supports \( S_i \) and \( S_j \) they are pairwise consistent relative to their dimensions \( d_i \) and \( d_j \). This makes pairwise consistency independent of an ad hoc selection of supports.

But what about consistency? Assume that the family \( D_1, \ldots, D_n \) is consistent relative to the supports \( S_i \) of \( D_i \), that is \( \epsilon_{S_i}(D) = D_i \). Then we have

\[
\epsilon_{d_i}(D) = \epsilon_{d_i}(\epsilon_{S_i}(D)) = \epsilon_{d_i}(D_i) = D_i.
\]

So, the family \( D_1, \ldots, D_n \) is also consistent with respect to the system of their dimensions. Again, this makes the definition of consistency independent of a particular selection of supports. We remark that the set \( \{ S : S \text{ support of } D_i \} \) is an upset, that is with any element \( S \) in the set an element \( S' \supseteq S \) belongs also to the set (item 6 on the list of properties of supports). In fact,

\[
\{ S : S \text{ support of } D_i \} = \uparrow d_i
\]

is the set of all supersets of the dimension. Now, assume \( D_1, \ldots, D_n \) pairwise consistent. The dimensions \( d_i \) may not satisfy RIP, but some sets \( S_i \supseteq d_i \) may (forming a covering join tree for the family \( D_1, \ldots, D_n \)). Then by Theorem 13 and this discussion, \( D_1, \ldots, D_n \) are consistent.

From a point of view of information, consistency of pieces \( D_1, \ldots, D_n \) of information is not always desirable. It is a kind of irrelevance or (conditional) independence condition. In fact, if the members of the family \( D_1, \ldots, D_n \) are pairwise consistent, and the supports \( S_i \) satisfy RIP, then \( D_i = \epsilon_{S_i}(D_1 \cdot \ldots \cdot D_n) \) means that, given the information on the separators \( S_i \cap S_j \), the pieces of information \( D_j \) for \( j \neq i \) give no new information relative to variables in \( S_i \). If, on the other hand, the family of pieces of information \( D_1, \ldots, D_n \) is not consistent, but compatible in the sense that \( D = D_1 \cdot \ldots \cdot D_n \neq 0 \), then, if \( S_1 \) to \( S_n \) satisfy RIP, we have that the family \( \epsilon_{S_i}(D) \supseteq D_i \) (in the information order), that is \( D_j \) may furnish additional information on the variables in \( S_i \) for \( i \neq j \). in (Kohlas, 2003), we have

\[
D = \epsilon_{S_1}(D) \cdot \ldots \cdot \epsilon_{S_n}(D)
\]
Obviously the $D_i^+ = \epsilon_{S_i}(D)$ are pairwise consistent and by definition consistent (this is remark 1 in [Miranda & Zaffalon, 2020]).

To conclude, note that most of this discussion of consistency (in particular Theorem 13) depends strongly on idempotency E2 of the information algebra. For instance, the valuation algebra corresponding to Bayesian networks is not idempotent, as well as many other semiring-valuation algebras [Kohlas & Wilson, 2006]. The RIP condition, Theorem 13, does not apply in these cases.

We have remarked that consistency is essentially an issue of information algebra. So, we may expect that the concepts and results on consistency of coherent sets of gambles carry over to coherent lower and upper previsions. First of all we define the concept of support, also for coherent lower previsions. Analogously to coherent sets of gambles, we say that a subset $S$ of $I$ is called a support of a coherent lower prevision $P$, if there exists a lower prevision $Q$ on $\mathcal{L}_S$ such that $P$ is the natural extension of $Q$, that is $P = E(Q)$.

Coherent lower previsions $P_1, \ldots, P_n$ with supports $S_1, \ldots, S_n$ are called consistent, if there is a coherent lower prevision $P$ such that $\epsilon_{S_i}(P) = P_i$ for all $i = 1, \ldots, n$. Recall that if $P_i$ has support $S_i$, there is a lower prevision $Q_i$ on $\mathcal{L}_{S_i}$ such that $P_i$ is the natural extension of $Q_i$, that is $P_i = E(Q_i)$, hence also $P_i|_{S_i} = Q_i$. We may also call the lower previsions $Q_i$ consistent, if there is a lower prevision $P$ such that $\epsilon_{S_i}(P) = E(Q_i)$. If $P_1, \ldots, P_n$ are consistent, then all pairs $P_i$ and $P_j$ are pairwise consistent, that is $\epsilon_{S_i \cap S_j}(P_j) = \epsilon_{S_i \cap S_j}(P_i)$. This implies also $\epsilon_{S_j}(P_i) = \epsilon_{S_j}(P_j)$ or $P_i|_{S_i \cap S_j} = P_j|_{S_i \cap S_j}$.

Theorem 13 carries over, since it is in fact a theorem of information algebras.

**Theorem 14** If $P_1, \ldots, P_n$, compatible coherent lower previsions with supports $S_1, \ldots, S_n$ which satisfy RIP, are pairwise consistent, then they are consistent and $\epsilon_{S_i}(P_1 \cdot \cdot \cdot P_n) = P_i$ for $i = 1, \ldots, n$.

Of course, there are close relations between consistency of coherent sets of gambles and lower previsions. If $D_1, \ldots, D_n$ are consistent coherent sets of gambles with supports $S_1, \ldots, S_n$, then the associated lower previsions $\sigma(D_i)$ are consistent too, since $\epsilon_{S_i}(\sigma(D_i)) = \sigma(\epsilon_{S_i}(D_i)) = \sigma(D_i)$. Conversely, if $P_1, \ldots, P_n$ are consistent lower previsions with support $S_1$ to $S_n$, then there is a family of consistent strictly desirable sets of gambles $D_i^+ = \tau(P_i)$ with $P_i = \sigma(D_i^+)$. More generally, if $P$ is the lower prevision so that $\epsilon_{S_i}(P) = P_i$, and $P = \sigma(D)$, then $[\epsilon_{S_i}(D)]_{\sigma} = [D_i]_{\sigma}$, where $\sigma(D_i) = P_i$ and $[D]_{\sigma}$ is the equivalence class of coherent sets of gambles associated with the homomorphism $\sigma$, that is $D \equiv_\sigma D'$ if $\sigma(D) = \sigma(D')$. We refer to the homomorphism theorem of universal algebra, see for instance [Kohlas, 2003]. So consistency among coherent sets of gambles is a class property.

Let’s turn to coherent locally linear previsions $P_1, \ldots, P_n$ relative to $S_1$ to $S_n$. Recall that then $P_i|_{S_i} = Q_i$ is a linear prevision on $\mathcal{L}_{S_i}$, $\forall i = 1, \ldots, n$. We claim that if the sequence $S_1, \ldots, S_n$ satisfies RIP, and the the elements of $P_1, \ldots, P_n$ are pairwise consistent, then the whole family is consistent. This is expressed in the following theorem.
**Theorem 15** If $P_1, \ldots, P_n$ are compatible locally linear previsions relative to $S_1$ to $S_n$, pairwise consistent and if $S_1, \ldots, S_n$ satisfies RIP, then $P_1, \ldots, P_n$ are consistent and there is a linear prevision $P$ so that $\epsilon_{S_i}(P) = P_i$ for all $i = 1, \ldots, n$.

**Proof.** Let $P = P_1 \cdot \ldots \cdot P_n$. By Theorem 14 this is a coherent lower prevision and $\epsilon_{S_i}(P) = P_i$. Then there exists a linear prevision (an atom) $P \in At(P)$ since the algebra of coherent lower previsions is atomic. We have then $P \leq P_i$, hence $P_i = \epsilon_{S_i}(P) \leq \epsilon_{S_i}(P)$. But $\epsilon_{S_i}(P)$ and $P_i$ are both local atoms (locally linear), therefore we must have $P_i = \epsilon_{S_i}(P)$.

This last theorem is again a theorem of information algebra. Therefore, it applies equally to the atomic algebra of coherent sets of gambles. But linear previsions have a special appeal since they are linked to probabilities, that is why we choose to present the theorem in this frame.

**10 Outlook**

This paper presents a first approach to information algebras related to desirable gambles, lower and upper previsions. There are many aspects and issues which are not addressed here. Foremost is the issue of conditioning. Conditioning of a coherent set of gambles on an event or set $B$ could be seen as combination of the coherent set with $B$. This concept of conditioning however has yet to be developed and compared with the usual approach to conditioning in imprecise probabilities as proposed originally in [Walley, 1991]. Further, in the multivariate model considered in this paper, desirable gambles and previsions are considered relative to linear spaces $\mathcal{L}(\Omega)$ and $\mathcal{L}_S$ of gambles. In the view of information algebra then coherent sets of $S$-measurable gambles represent pieces of information or belief relative to sets $S$ of variables. More general families of spaces of gambles may be considered, for instance families of gambles measurable relative to certain partitions of the set of possibilities $\Omega$ or certain lattices of Borel fields, etc. How can the concept of information algebras be adapted to such more general frames? Both of these subjects would also serve to deepen the issue of conditional independence, which seems to be fundamental for any theory of information.

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