LEVI-CIVITA CONNECTION FOR $SU_q(2)$

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Abstract. We prove that the $4D_\pm$ calculi on the quantum group $SU_q(2)$ satisfy a metric-independent sufficient condition for the existence of a unique bicovariant Levi-Civita connection corresponding to every bi-invariant pseudo-Riemannian metric.

1. Introduction

The quantum group $SU_q(2)$ introduced in [7] and the notion of bicovariant differential calculi on Hopf algebras was introduced in [8] by Woronowicz. The question of bicovariant Levi-Civita connections on bicovariant differential calculi of compact quantum groups have been investigated by Heckenberger and Schmüdgen in [5] for the quantum groups $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$. On the other hand, Beggs, Majid and their collaborators have studied Levi-Civita connections on quantum groups and homogeneous spaces in various articles, and a comprehensive account can be found in [1].

More recently, in [3], bicovariant connections on arbitrary bicovariant differential calculi of compact quantum groups and the notion of their metric compatibility with respect to arbitrary bi-invariant pseudo-Riemannian metrics was studied. In that article, the construction of a canonical bicovariant torsionless connection on a calculus was presented (Theorem 5.3 of [3]), provided that Woronowicz’s braiding map $\sigma$ for the calculus satisfies a diagonalisability condition. Also, a metric-independent sufficient condition for the existence of a unique bicovariant Levi-Civita connection (in the sense of Definition 6.3 of [3]) was provided in Theorem 7.9 of [3].

In this article, we will investigate the theory of [3] in the context of the $4D_\pm$ calculi of the compact quantum group $SU_q(2)$ which were explicitly described in [7] and then [6]. In Section 2, we recall the notion of covariant Levi-Civita connections on bicovariant differential calculi as formulated in [3]. In Section 3, the $4D_\pm$ calculi on $SU_q(2)$ are recalled and we show that Woronowicz’s braiding map for the $4D_\pm$ calculi satisfy the diagonalisability condition mentioned above. In Section 4, we construct a bicovariant torsionless connection. In Section 5, we will show that the sufficiency condition of Theorem 7.9 of [3] is satisfied by both calculi, except for at most finitely many values of $q$, and hence we can conclude the existence of a unique bicovariant Levi-Civita connection, corresponding to a bi-invariant pseudo-Riemannian metric.

2. Levi-Civita connections on bicovariant differential calculi

In this section, we recall the notion of Levi-Civita connections on bicovariant differential calculi as formulated in [3].

We say that $(\mathcal{E}, \Delta_{\mathcal{E}}, \varepsilon\Delta)$ is a bicovariant bimodule over a Hopf algebra $A$ if $\mathcal{E}$ is a bimodule over $A$, $(\mathcal{E}, \Delta_{\mathcal{E}})$ is a left $A$-comodule, $(\mathcal{E}, \varepsilon\Delta)$ is a right $A$-comodule, subject to the following compatibility conditions:

$$\Delta_{\mathcal{E}}(ap) = \Delta(a)\Delta_{\mathcal{E}}(p), \quad \Delta_{\mathcal{E}}(\rho a) = \Delta_{\mathcal{E}}(\rho)\Delta(a)$$

$$\varepsilon\Delta(ap) = \Delta(a)\varepsilon\Delta(p), \quad \varepsilon\Delta(\rho a) = \varepsilon\Delta(\rho)\Delta(a),$$

where $\rho$ is an arbitrary element of $\mathcal{E}$ and $a$ is an arbitrary element of $A$. If $(\mathcal{E}, \Delta_{\mathcal{E}}, \varepsilon\Delta)$ is a bicovariant bimodule over $A$, we say that an element $e$ in $\mathcal{E}$ is left (respectively, right) invariant if $\Delta_{\mathcal{E}}(e) = 1 \otimes e$ (respectively, $\varepsilon\Delta(e) = e \otimes 1$). In this article, we will denote the vector space of elements of $\mathcal{E}$ invariant under the left coaction of $A$ by $a\mathcal{E}$, and that of elements invariant under
the right coaction of \( \mathcal{A} \) by \( \mathcal{E}_0 \). If \( \mathcal{E} \) and \( \mathcal{F} \) are two bicovariant bimodules over \( \mathcal{A} \), a \( \mathbb{C} \)-linear map \( T : \mathcal{E} \rightarrow \mathcal{F} \) is said to be left covariant if \( \Delta x \circ T = (\text{id} \otimes \mathbb{C} T) \circ \Delta x \). \( T \) is said to be right covariant if \( x \Delta \circ T = (T \otimes \mathbb{C} \text{id}) \circ x \Delta \). \( T \) is called bicovariant if it is both left-covariant and right-covariant.

A (first order) differential calculus \((\mathcal{E}, d)\) over a Hopf algebra \( \mathcal{A} \) is called a bicovariant differential calculus if the following conditions are satisfied:

(i) For any \( a_k, b_k \) in \( \mathcal{A}, k = 1, \ldots, K \),
\[
(\sum_k a_k db_k = 0) \text{ implies that } (\sum_k \Delta(a_k)(\text{id} \otimes \mathbb{C} d)\Delta(b_k) = 0),
\]

(ii) For any \( a_k, b_k \) in \( \mathcal{A}, k = 1, \ldots, K \),
\[
(\sum_k a_k db_k = 0) \text{ implies that } (\sum_k \Delta(a_k)(d \otimes \text{id})\Delta(b_k) = 0).
\]

Woronowicz ([8]) proved that a bicovariant differential calculus is endowed with canonical left and right-comodule coactions of \( \mathcal{A} \), making it into a bicovariant bimodule \((\mathcal{E}, \Delta_{\mathcal{E}}, \varepsilon_{\mathcal{E}})\). Moreover, the map \( d : \mathcal{A} \rightarrow \mathcal{E} \) is a bicovariant map.

Next, we state the construction of the associated space of two-forms, \( \Omega^2(\mathcal{A}) \) for a bicovariant differential calculus of an arbitrary unital Hopf algebra \( \mathcal{A} \) as in [8]. To do so, we need to recall the braiding map \( \sigma \) for bicovariant bimodules.

**Proposition 2.1.** (Proposition 3.1 of [8]) Given a bicovariant bimodule \( \mathcal{E} \) on a Hopf algebra \( \mathcal{A} \), there exists a unique bimodule homomorphism
\[
\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}
\]
such that \( \sigma(\omega \otimes \mathcal{A} \eta) = \eta \otimes \mathcal{A} \omega \) \((1)\)

for any left-invariant element \( \omega \) and right-invariant element \( \eta \) in \( \mathcal{E} \), under the coactions of \( \mathcal{A} \). \( \sigma \) is an invertible bicovariant \( \mathcal{A} \)-bimodule map from \( \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \) to itself. Moreover, \( \sigma \) satisfies the following braid equation on \( \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \):
\[
(\text{id} \otimes \mathcal{A} \sigma)(\sigma \otimes \mathcal{A} \text{id})(\text{id} \otimes \mathcal{A} \sigma) = (\sigma \otimes \mathcal{A} \text{id})(\text{id} \otimes \mathcal{A} \sigma)(\sigma \otimes \mathcal{A} \text{id}).
\]

The symbol \( \wedge \) denotes the quotient map, which is a bicovariant bimodule map,
\[
\wedge : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \Omega^2(\mathcal{A}).
\]
The map \( d : \mathcal{A} \rightarrow \mathcal{E} \) extends to a unique exterior derivative map (to be denoted again by \( d \)),
\[
d : \mathcal{E} \rightarrow \Omega^2(\mathcal{A}),
\]
such that, for all \( a \) in \( \mathcal{A} \) and \( \rho \) in \( \mathcal{E} \),

(i) \( d(a \rho) = da \wedge \rho + ad(\rho) \),

(ii) \( d(\rho a) = d(\rho) a - \rho \wedge da \),

(iii) \( d \) is bicovariant.

Let us, from now on, denote the subspace of left-invariant elements of an arbitrary bicovariant bimodule \( \mathcal{E} \) by the symbol \( \mathcal{E}_0 \). By Proposition 2.5 of [2], the vector space \( \mathcal{E}_0 \otimes \mathcal{E}_0 \mathcal{E} \) can be identified with the space \( \mathcal{E}_0(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \) of left-invariant elements of \( \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \). The isomorphism \( \mathcal{E}_0 \otimes \mathcal{E}_0 \mathcal{E} \rightarrow \mathcal{E}_0(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \) is given by
\[
\omega_i \otimes \mathbb{C} \omega_j \mapsto \omega_i \otimes_{\mathcal{A}} \omega_j \quad (2)
\]
where \( \{\omega_i\}_i \) is a vector space basis of \( \mathcal{E}_0 \).

Moreover, by the bicovariance of the map \( \sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \), we get the restriction (see Equation 18 of [3]):
\[
\sigma_{\mathcal{E}} := \sigma|_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} : \mathcal{E}_0 \otimes \mathbb{C} \mathcal{E} \rightarrow \mathcal{E}_0 \otimes \mathbb{C} \mathcal{E}.
\]

From now on, we are going to work under the assumption that \( \sigma_{\mathcal{E}} \) is a diagonalisable map between finite dimensional vector spaces. In [3], this assumption was crucially used to set up a framework for the existence of a unique bicovariant Levi-Civita connection on a bicovariant differential calculus satisfying the assumption.
Let us introduce some notations and definitions so that we can recall the framework mentioned above.

**Definition 2.2.** Suppose the map \( \sigma \) is diagonalisable. The eigenspace decomposition of \( \mathcal{E} \otimes \mathbb{C} \mathcal{E} \) will be denoted by \( \mathcal{E} \otimes \mathbb{C} \mathcal{E} = \bigoplus_{\lambda \in \Lambda} V_{\lambda} \), where \( \Lambda \) is the set of distinct eigenvalues of \( \sigma \) and \( V_{\lambda} \) is the eigenspace of \( \sigma \) corresponding to the eigenvalue \( \lambda \). Thus, for example, \( V_1 \) will denote the eigenspace of \( \sigma \) for the eigenvalue \( \lambda = 1 \).

Moreover, we define \( \mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \) to be the eigenspace of \( \sigma \) with eigenvalue 1, i.e.,

\[
\mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} := V_1.
\]

We also define \( \mathcal{F} := \bigoplus_{\lambda \in \Lambda \setminus \{1\}} V_{\lambda} \). Finally, we will denote by \( \rho(\text{sym}) \) the idempotent element in \( \text{Hom}(\mathcal{E} \otimes \mathbb{C} \mathcal{E}, \mathcal{E}) \) with range \( \mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \) and kernel \( \mathcal{F} \).

Let us introduce the notion of bi-invariant pseudo-Riemannian metric on a bicovariant \( \mathcal{A} \)-bimodule \( \mathcal{E} \).

**Definition 2.3.** \([5], \text{Definition } 4.1 \text{ of } [3]\) Suppose \( \mathcal{E} \) is a bicovariant \( \mathcal{A} \) bimodule and \( \sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \) be the map as in Proposition 2.1. A bi-invariant pseudo-Riemannian metric for the pair \((\mathcal{E}, \sigma)\) is a right \( \mathcal{A} \)-linear map \( g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{A} \) such that the following conditions hold:

(i) \( g \circ \sigma = g \).

(ii) If \( g(\rho \otimes_{\mathcal{A}} \nu) = 0 \) for all \( \nu \in \mathcal{E} \), then \( \rho = 0 \).

(iii) The map \( g \) is bi-invariant, i.e. for all \( \rho, \nu \in \mathcal{E} \),

\[
\begin{align*}
(id \otimes C \nu)(\Delta(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) (\rho \otimes_{\mathcal{A}} \nu)) &= g(\rho \otimes_{\mathcal{A}} \nu), \\
(e \otimes id)(\Delta(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) (\rho \otimes_{\mathcal{A}} \nu)) &= g(\rho \otimes_{\mathcal{A}} \nu).
\end{align*}
\]

Now we can define the torsion of a connection and the compatibility of a left-covariant connection with a bi-invariant pseudo-Riemannian metric.

**Definition 2.4.** \([5]\) Let \( (\mathcal{E}, d) \) be a bicovariant differential calculus on \( \mathcal{A} \). A (right) connection on \( \mathcal{E} \) is a \( C \)-linear map \( \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \) such that, for all \( a \) in \( \mathcal{A} \) and \( \rho \) in \( \mathcal{E} \), the following equation holds:

\[
\nabla(\rho a) = \nabla(\rho)a + \rho \otimes_{\mathcal{A}} da.
\]

The map \( \nabla \) is said to be a left-covariant, right-covariant or bicovariant connection if it is a left-covariant, right-covariant or bicovariant map, respectively. The torsion of a connection \( \nabla \) on \( \mathcal{E} \) is the right \( \mathcal{A} \)-linear map

\[
T_{\nabla} := \Lambda \circ \nabla + d : \mathcal{E} \rightarrow \Omega^2(\mathcal{A}).
\]

The connection \( \nabla \) is said to be torsionless if \( T_{\nabla} = 0 \).

Our notion of torsion is the same as that of \([5]\), with the only difference being that they work with left connections.

**Definition 2.5.** \((\text{Definitions } 6.1 \text{ and } 6.3 \text{ of } [3])\) Let \( \nabla \) be a left-covariant connection on a bicovariant calculus \( (\mathcal{E}, d) \) such that the map \( \sigma \) is diagonalisable, and \( g \) a bi-invariant pseudo-Riemannian metric. Then we define

\[
\Pi_{\nabla}^0(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \text{ by the following formula :}
\]

\[
\Pi_{\nabla}^0(\nabla)(\omega_1 \otimes \omega_2) = 2(id \otimes g)(\sigma \otimes id)(\nabla \otimes id)0(\text{sym})(\omega_1 \otimes \omega_2).
\]

Next, for all \( \omega_1, \omega_2 \) in \( \mathcal{E} \) and \( a \) in \( \mathcal{A} \), we define \( \Pi_{\nabla} \) \( : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \) by

\[
\Pi_{\nabla}(\nabla)(\omega_1 \otimes \omega_2) = \Pi_{\nabla}^0(\nabla)(\omega_1 \otimes \omega_2)a + g(\omega_1 \otimes A \omega_2)da.
\]

Finally, \( \nabla \) is said to be compatible with \( g \), if, as maps from \( \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \) to \( \mathcal{E} \),

\[
\Pi_{\nabla}(\nabla) = dg.
\]

This allows us to give the definition of a Levi-Civita connection.

**Definition 2.6.** Let \( (\mathcal{E}, d) \) be a bicovariant differential calculus such that the map \( \sigma \) is diagonalisable and \( g \) a pseudo-Riemannian bi-invariant metric on \( \mathcal{E} \). A left-covariant connection \( \nabla \) on \( \mathcal{E} \) is called a Levi-Civita connection for the triple \( (\mathcal{E}, d, g) \) if it is torsionless and compatible with \( g \).
In [3], it was shown that this suitably generalises the notion of Levi-Civita connections for bicovariant differential calculi on Hopf algebras.

Then, we have the following metric-independent sufficient condition for the existence of a unique bicovariant Levi-Civita connection.

\textbf{Theorem 2.7.} (Theorem 7.9 of [3]) Suppose \((\mathcal{E},d)\) is a bicovariant differential calculus over a cosemisimple Hopf algebra \(\mathcal{A}\) such that the map \(\sigma\) is diagonalisable and \(\omega\) be a bi-invariant pseudo-Riemannian metric. If the map

\[(\sigma(P_{\text{sym}}))_{ij} : (\mathcal{E} \otimes_{\mathbb{C}} \mathcal{E}) \otimes \mathcal{E} \to \mathcal{E} \otimes_{\mathbb{C}} (\mathcal{E} \otimes_{\mathbb{C}} \mathcal{E})\]

is an isomorphism, then the triple \((\mathcal{E},d,\omega)\) admits a unique bicovariant Levi-Civita connection.

3. The 4\(D_{\pm}\) Calculi on \(SU_q(2)\) and the Braiding Map

In this section we recall briefly the definition quantum group \(SU_q(2)\) and the 4\(D_{\pm}\) calculus on \(SU_q(2)\). Then we show that the map \(\sigma : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}\) is actually diagonalisable. Our main reference for the details is [6].

For \(q \in [-1,1]\), \(SU_q(2)\) is the \(C^*\)-algebra generated by the two elements \(\alpha\) and \(\gamma\), and their adjoints, satisfying the following relations:

\[
\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,
\]

\[
\gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.
\]

The comultiplication map \(\Delta\) is given by

\[
\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.
\]

This makes \(SU_q(2)\) into a compact quantum group. We will denote the Hopf *-algebra generated by the elements \(\alpha, \gamma\) by the symbol \(\mathcal{A}\).

In [6], it is explicitly proven that there does not exist any three-dimensional bicovariant differential calculus and exactly two inequivalent four-dimensional calculi for \(SU_q(2)\). We use the description of the two bicovariant calculi, \(4D_+\) and \(4D_-\), as given in [6]. We will rephrase some of the notation to fit our formalism. For \(q \in \{-1,1\}\), the first order differential calculi \(\mathcal{E}\) of both the \(4D_+\) and \(4D_-\) calculi are bicovariant \(\mathcal{A}\)-bimodules such that the space \(\mathcal{E}\) of one-forms invariant under the left coaction of \(\mathcal{A}\) is a 4-dimensional vector space. We will denote a preferred basis of \(\mathcal{E}\) by \(\{\omega_i\}_{i=1,2,3,4}\). Here we have replaced the notation in [6] with \(\omega_i = \Omega_i\).

The following is the explicit description of the exterior derivative \(d\) on \(\mathcal{E}\) for the preferred basis \(\{\omega_i\}_{i=1}^4\) mentioned above.

\textbf{Proposition 3.1.} (Equation (5.2) of [6]) Let \(d : \mathcal{E} \to \Omega^2(\mathcal{A})\) be the exterior derivative of the 4\(D_{\pm}\) calculus.

\[
d(\omega_1) = \pm \sqrt{r} \omega_1 \wedge \omega_3, \quad d(\omega_2) = \mp \sqrt{r} \omega_2 \wedge \omega_3,
\]

\[
d(\omega_3) = \pm \sqrt{q} \omega_1 \wedge \omega_2, \quad d(\omega_4) = 0,
\]

where the upper sign stand for \(4D_+\) and the lower for \(4D_-\), and \(r = 1 + q^2\).

Now we show that the map \(\sigma\) for \(SU_q(2)\) satisfies the diagonalisability condition by giving explicit bases for eigenspaces of \(\sigma\). We will use the explicit action of \(\sigma\) on elements \(\omega_i \otimes_{\mathcal{A}} \omega_j\), \(i, j = 1, 2, 3, 4\) as given in Equation (4.1) of [6]

\textbf{Proposition 3.2.} For \(SU_q(2)\), the map \(\sigma\) is diagonalisable and has the minimal polynomial equation

\[(\sigma - 1)(\sigma + q^2)(\sigma + q^{-2}) = 0.
\]

\textit{Proof.} The proof of this result is by explicit listing of eigenvectors of \(\sigma\) for eigenvalues \(1, q^2, q^{-2}\) and by a dimension argument. Throughout we make use of the canonical identification \(\omega_i \otimes_{\mathcal{A}} \omega_j \mapsto \omega_i \otimes \omega_j\) as stated in (2).
Either by directly applying $\sigma_0$ on the following linearly independent two-tensors or from Equation (4.2) of [6], we get that the following are in the eigenspace corresponding to eigenvalue 1:

$$\omega_1 \otimes \omega_1, \omega_2 \otimes \omega_2, \omega_3 \otimes \omega_3 + t \omega_1 \otimes \omega_2, \omega_4 \otimes \omega_4,$$

$$\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1, \omega_3 \otimes \omega_3 + q^2 \omega_3 \otimes \omega_2,$$

$$q^2 \omega_1 \otimes \omega_3 + \omega_3 \otimes \omega_1, \frac{tk}{q^2} \omega_2 \otimes \omega_3 - \omega_2 \otimes \omega_4 - \omega_4 \otimes \omega_2,$$

$$\frac{tk}{q^2} \omega_1 \otimes \omega_3 + \omega_3 \otimes \omega_1,$$ 

By explicit computation, the following linearly independent two-tensors are in the eigenspace corresponding to the eigenvalue $q$:

$$\frac{tk}{q^2} \omega_3, \omega_1 \otimes \omega_3 - q^2 \omega_2 \otimes \omega_4 - \frac{tk}{q^2} \omega_4 \otimes \omega_2, \omega_4 \otimes \omega_2,$$

$$-\frac{tk}{q^2} \omega_3 \otimes \omega_4 + \frac{tk}{q^2} \omega_1 \otimes \omega_1 + \frac{tk}{q^2} \omega_1 \otimes \omega_4 + \omega_4 \otimes \omega_1,$$

$$-\frac{tk}{q^2} \omega_1 \otimes \omega_3 + \frac{tk}{q^2} \omega_3 \otimes \omega_4 - q^2 \omega_3 \otimes \omega_4 + q^2 \omega_4 \otimes \omega_3.$$

By explicit computation, the following linearly independent two-tensors are in the eigenspace corresponding to the eigenvalue $q^{-2}$:

$$\frac{tk}{q^2} \omega_2 \otimes \omega_3 - q^2 \omega_2 \otimes \omega_4 - \frac{tk}{q^2} \omega_4 \otimes \omega_2, \omega_4 \otimes \omega_2,$$

$$-\frac{tk}{q^2} \omega_3 \otimes \omega_4 + \frac{tk}{q^2} \omega_1 \otimes \omega_1 + \frac{tk}{q^2} \omega_1 \otimes \omega_4 - q^2 \omega_4 \otimes \omega_2,$$

$$-\frac{tk}{q^2} \omega_1 \otimes \omega_3 + \frac{tk}{q^2} \omega_3 \otimes \omega_4 - q^2 \omega_3 \otimes \omega_4 + q^2 \omega_4 \otimes \omega_3.$$

We have thus accounted for 16 linearly independent elements of $\sigma E \otimes_c \rho E$. Since $\rho E$ has dimension 4, $\sigma E \otimes_c \rho E$ has dimension 16. Hence we have a basis, and in particular bases for the eigenspace decomposition, of $\sigma E \otimes_c \rho E$. Moreover, $\sigma_0$ satisfies the minimal polynomial

$$(\sigma_0 - 1)(\sigma_0 + q^2)(\sigma_0 + q^{-2}) = 0.$$ 

\[\square\]

4. A BICOVARIANT TORSIONLESS CONNECTION

In this section, using the fact that $\sigma_0$ is diagonalizable and $\sigma E \otimes_c \rho E$ admits an eigenspace decomposition, we construct a bicovariant torsionless connection on the $4D_{\pm}$ calculus.

**Remark 4.1.** Note that since any element $\rho$ in $E$ can be uniquely expressed as $\rho = \sum \omega_i a_i$ for some $a_i$ in $A$ (Theorem 2.1 of [8]), a connection on $E$ is determined by its action on the basis $\{\omega_i\}$. 

By Proposition 3.2, we have the eigenspace decomposition

$$\sigma E \otimes_c \rho E = \text{Ker}(\sigma_0 - \text{id}) \oplus \text{Ker}(\sigma_0 + q^2) \oplus \text{Ker}(\sigma_0 + q^{-2}).$$

Since $\text{Ker}(\wedge) = \text{Ker}(\sigma_0 - \text{id})$, we have that

$$\text{Ker}(\sigma_0 + q^2) \oplus \text{Ker}(\sigma_0 + q^{-2}) \cong \Omega^2(A),$$

with the isomorphism being given by $\wedge|_{\text{Ker}(\sigma_0 + q^2) \oplus \text{Ker}(\sigma_0 + q^{-2})}$. Let us denote $\text{Ker}(\sigma_0 + q^2) \oplus \text{Ker}(\sigma_0 + q^{-2})$ by $\rho F$ from now on. This is consistent with the notation adopted in Definition 2.2.

**Theorem 4.2.** Let $\{\omega_i\}_{1}$ be the preferred basis for the $4D_\pm$ calculus on $SU_q(2)$. For $i = 1, 2, 3, 4$, we define

$$\nabla_0(\omega_i) = -(\wedge|_{\rho F})^{-1} \circ d(\omega_i) \in \sigma E \otimes_c \rho E.$$
Then, $\nabla$ extends to a bicovariant torsionless connection on $\mathcal{E}$. More explicitly,

\[
\nabla_0(\omega_1) = \pm \frac{r}{k^2(q^2 + 1)^2} \left( \frac{2t k}{\sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 + t q \omega_1 \otimes \mathcal{C} \omega_4 - \frac{2t k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 + t q \omega_4 \otimes \mathcal{C} \omega_1 \right)
\]

\[
\nabla_0(\omega_2) = \pm \frac{r}{k^2(q^2 + 1)^2} \left( \frac{2t k}{\sqrt{r}} \omega_2 \otimes \mathcal{C} \omega_3 - t q \omega_2 \otimes \mathcal{C} \omega_4 - \frac{2t k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_2 - t q \omega_4 \otimes \mathcal{C} \omega_2 \right)
\]

\[
\nabla_0(\omega_3) = \pm \frac{q r}{k^2(q^2 + 1)^2} \left( \frac{2t k}{\sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_2 - \frac{2t k}{\sqrt{r}} \omega_2 \otimes \mathcal{C} \omega_1 - \frac{t^2 k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_3 + t q \omega_3 \otimes \mathcal{C} \omega_4 + t q \omega_4 \otimes \mathcal{C} \omega_3 \right)
\]

\[
\nabla_0(\omega_4) = 0
\]

**Proof.** By the definition of $\nabla_0$,

\[
\land \circ \nabla_0(\omega_i) = - \land \circ (\land_{\alpha})^{-1} \circ d(\omega_i) = -d(\omega_i).
\]

Therefore, for any element $\rho = \sum_i \omega_i a_i$ in $\mathcal{E}$,

\[
\land \circ \nabla_0(\sum_i \omega_i a_i) = \land \circ \sum_i (\nabla_0(\omega_i) a_i + \omega_i \wedge a_i)
\]

\[
= - \sum_i (\land \circ (\land_{\alpha})^{-1} \circ d(\omega_i) a_i + \omega_i \wedge a_i)
\]

\[
= - \sum_i (d(\omega_i) a_i + \omega_i \wedge a_i) = - \sum_i d(\omega_i a_i).
\]

Hence $\nabla_0$ is a torsionless connection. The construction of $\nabla_0$ is the same as that in Theorem 5.3 of [3]. Hence, by that theorem, our connection $\nabla_0$ is bicovariant.

Now we derive $\nabla_0$ explicitly on each $\omega_i$ using the formulas for $d(\omega_i)$ in Proposition 3.1.

We have that $d(\omega_1) = \pm \sqrt{r} \omega_1 \wedge \omega_3$. The decomposition of $\omega_1 \otimes \mathcal{C} \omega_3$ as a linear combination of the basis eigenvectors listed in Proposition 3.2 is given by

\[
\omega_1 \otimes \mathcal{C} \omega_3 = \frac{2q^2}{(q^2 + 1)^2} (q^2 \omega_1 \otimes \mathcal{C} \omega_3 + \omega_3 \otimes \mathcal{C} \omega_1)
\]

\[
- \frac{q^2 \sqrt{r}}{k(q^2 + 1)^2} \left( \frac{t^2 k}{\sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 + \omega_1 \otimes \mathcal{C} \omega_4 + \omega_4 \otimes \mathcal{C} \omega_1 \right)
\]

\[
- \frac{\sqrt{r}}{t^2 k(q^2 + 1)^2} \left( - \frac{t k}{q \sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 - q^2 \omega_1 \otimes \mathcal{C} \omega_4 + \frac{t q k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 + \omega_4 \otimes \mathcal{C} \omega_1 \right)
\]

\[
- \frac{\sqrt{r}}{t^2 k(q^2 + 1)^2} \left( \frac{t k}{q \sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 + \omega_1 \otimes \mathcal{C} \omega_4 + \frac{t q k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 - q^2 \omega_4 \otimes \mathcal{C} \omega_1 \right).
\]

Since the first two terms in the above decomposition are elements of $\text{Ker}(\sigma - \text{id}) = \text{Ker}(\land)$, applying $\land$ on both sides, we have

\[
\omega_1 \wedge \omega_3 = - \frac{\sqrt{r}}{t^2 k(q^2 + 1)^2} \left( - \frac{t k}{q \sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 - q^2 \omega_1 \otimes \mathcal{C} \omega_4 + \frac{t q k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 + \omega_4 \otimes \mathcal{C} \omega_1 \right)
\]

\[
- \frac{\sqrt{r}}{t^2 k(q^2 + 1)^2} \left( \frac{t k}{q \sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 + \omega_1 \otimes \mathcal{C} \omega_4 + \frac{t q k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 - q^2 \omega_4 \otimes \mathcal{C} \omega_1 \right),
\]

and since the last two terms in the decomposition are from $\mathcal{F}$,

\[
(\land_{\alpha})^{-1}(\omega_1 \wedge \omega_3) = - \frac{\sqrt{r}}{t^2 k(q^2 + 1)^2} \left( - \frac{t k}{q \sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 - q^2 \omega_1 \otimes \mathcal{C} \omega_4 + \frac{t q k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 + \omega_4 \otimes \mathcal{C} \omega_1 \right)
\]

\[
- \frac{\sqrt{r}}{t^2 k(q^2 + 1)^2} \left( \frac{t k}{q \sqrt{r}} \omega_1 \otimes \mathcal{C} \omega_3 + \omega_1 \otimes \mathcal{C} \omega_4 + \frac{t q k}{\sqrt{r}} \omega_3 \otimes \mathcal{C} \omega_1 - q^2 \omega_4 \otimes \mathcal{C} \omega_1 \right).
\]
Thus, by the construction of $\nabla_0$, we have
\[
\nabla_0(\omega_1) = \pm \left( -\frac{r}{t^2k(q^2 + 1)^2} \left( -\frac{tk}{\sqrt{r}} \omega_1 \odot \omega_3 - q^2 \omega_1 \odot \omega_4 + \frac{tk}{\sqrt{r}} \omega_3 \odot \omega_1 + \omega_4 \odot \omega_1 \right) \right)
- \frac{r}{t^2k(q^2 + 1)^2} \left( -\frac{tk}{\sqrt{r}} \omega_1 \odot \omega_3 + \omega_1 \odot \omega_4 + \frac{tk}{\sqrt{r}} \omega_3 \odot \omega_1 - q^2 \omega_4 \odot \omega_1 \right))
= \pm \frac{r}{t^2k(q^2 + 1)^2} \left( \frac{2tk}{\sqrt{r}} \omega_1 \odot \omega_3 + t\omega_1 \odot \omega_4 - \frac{2tk}{\sqrt{r}} \omega_3 \odot \omega_1 + t\omega_4 \odot \omega_1 \right)
\]

Proposition 3.1 also gives that $d(\omega_2) = \pm \frac{kr}{q} \omega_2 \wedge \omega_3, d(\omega_3) = \pm \frac{kr}{q} \omega_1 \wedge \omega_2$ and $d(\omega_4) = 0$. So, similarly, we have
\[
\omega_2 \odot \omega_3 = \frac{2}{(q^2 + 1)^2} \left( \omega_2 \odot \omega_3 + q^2 \omega_1 \odot \omega_2 \right) - \frac{q^4 r}{k(q^2 + 1)^2} \left( \frac{t^2k}{\sqrt{r}} \omega_2 \odot \omega_3 \odot \omega_1 \odot \omega_4 - \omega_2 \odot \omega_4 \omega_1 \odot \omega_2 \right)
+ \frac{q^2 r}{t^2k(q^2 + 1)^2} \left( \frac{tk}{\sqrt{r}} \omega_2 \odot \omega_3 \odot \omega_1 \odot \omega_4 - \frac{tk}{\sqrt{r}} \omega_1 \odot \omega_2 \odot \omega_4 \odot \omega_2 \right)
+ \frac{q^2 r}{t^2k(q^2 + 1)^2} \left( \frac{tk}{\sqrt{r}} \omega_2 \odot \omega_3 \odot \omega_1 \odot \omega_4 - \frac{tk}{\sqrt{r}} \omega_2 \odot \omega_4 \odot \omega_1 \odot \omega_2 \right),
\]
and hence,
\[
\nabla_0(\omega_2) = \pm \frac{r}{t^2k(q^2 + 1)^2} \left( \frac{2tk}{\sqrt{r}} \omega_2 \odot \omega_3 \omega_4 - t\omega_2 \odot \omega_4 \omega_1 - \frac{2tk}{\sqrt{r}} \omega_3 \odot \omega_2 \omega_4 - t\omega_4 \odot \omega_1 \right).
\]

Moreover,
\[
\omega_1 \odot \omega_2 = \frac{2tq^2}{(q^2 + 1)^2} \left( \omega_1 \odot \omega_2 + \omega_2 \odot \omega_1 \right) + \frac{2q^2}{(q^2 + 1)^2} \left( \omega_3 \odot \omega_3 + t\omega_1 \odot \omega_2 \right)
- \frac{q^4 r}{k(q^2 + 1)^2} \left( \frac{t^2k}{\sqrt{r}} \omega_1 \odot \omega_2 \odot \omega_3 \odot \omega_4 + \omega_1 \odot \omega_4 \odot \omega_3 \odot \omega_2 \right)
- \frac{q^2 r}{tk(q^2 + 1)^2} \left( \frac{tk}{\sqrt{r}} \omega_1 \odot \omega_2 \odot \omega_3 \odot \omega_4 - \frac{tk}{\sqrt{r}} \omega_2 \odot \omega_3 \odot \omega_4 \odot \omega_1 \right)
- \frac{q^2 r}{tk(q^2 + 1)^2} \left( \frac{tk}{\sqrt{r}} \omega_1 \odot \omega_2 \odot \omega_3 \odot \omega_4 - \frac{tk}{\sqrt{r}} \omega_2 \odot \omega_4 \odot \omega_1 \odot \omega_2 \right),
\]
and hence,
\[
\nabla_0(\omega_3) = \pm \frac{qr}{tk(q^2 + 1)^2} \left( \frac{2tk}{\sqrt{r}} \omega_1 \odot \omega_2 \omega_4 - \frac{2tk}{\sqrt{r}} \omega_2 \odot \omega_4 \omega_1 - \frac{2tk}{\sqrt{r}} \omega_3 \odot \omega_1 \odot \omega_4 + t\omega_4 \odot \omega_1 \odot \omega_2 \right)
\]
Lastly, since $d(\omega_4) = 0$, $\nabla_0(\omega_4) = 0$

Thus, we are done with our proof. \(\square\)

5. Existence of a unique b covariant Levi-Civita connection

In this section, we prove that for every $q \in \mathbb{C}$, the $4D_\pm$ calculi admit a unique b covariant Levi-Civita connection for every bi-invariant pseudo-Riemannian metric (as defined in Definition 2.3) on $\mathcal{E}$. We achieve this by verifying the hypotheses of Theorem 2.7.

Recall that for the $4D_\pm$ calculi, we had the decomposition
\[
o \mathcal{E} \otimes \mathcal{E} = \text{Ker}(o\sigma - \text{id}) \oplus \text{Ker}(o\sigma + q^2) \oplus \text{Ker}(o\sigma + q^{-2}).\]

We have already fixed the symbol $o\mathcal{F}$ for $\text{Ker}(o\sigma + q^2) \oplus \text{Ker}(o\sigma + q^{-2})$. Let us now denote $\text{Ker}(o\sigma - \text{id})$ by $o \mathcal{E} \otimes \mathcal{E}^\text{sym} \mathcal{E}$. Moreover, as in Definition 2.2, we define the $\mathbb{C}$-linear map
\[
o(P_{\text{sym}}) : o \mathcal{E} \otimes \mathcal{E} \rightarrow o \mathcal{E} \otimes \mathcal{E}
\]
to be the idempotent with range $o \mathcal{E} \otimes \mathcal{E}^\text{sym} \mathcal{E}$ and kernel $o \mathcal{F}$. Since, $o(P_{\text{sym}})$ is the idempotent onto the eigenspace of $o\sigma$ with eigenvalue one, and with kernel the eigenspaces with eigenvalues $q^2$ and $q^{-2}$, it is of the form (see (22) of [3])
\[
o(P_{\text{sym}}) = \frac{o\sigma + q^2}{1 + q^2} \frac{o\sigma + q^{-2}}{1 + q^{-2}}.
\]
By Proposition 3.2, the set \( \{ \nu_i \}_{i=1}^{10} \) forms a basis of \( \mathfrak{a} \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E} \), where \( \nu_i \) are given as follows:

\[
\begin{align*}
\nu_1 &= \omega_1 \otimes \omega_1, \\
\nu_2 &= \omega_2 \otimes \omega_2, \\
\nu_3 &= \omega_3 \otimes \omega_3 + i\omega_1 \otimes \omega_2, \\
\nu_4 &= \omega_4 \otimes \omega_4, \\
\nu_5 &= \omega_2 \otimes \omega_1 + \omega_1 \otimes \omega_2, \\
\nu_6 &= \omega_3 \otimes \omega_2 + \frac{1}{q^2} \omega_2 \otimes \omega_3, \\
\nu_7 &= \omega_3 \otimes \omega_1 + q^2 \omega_1 \otimes \omega_3, \\
\nu_8 &= \omega_4 \otimes \omega_2 + \omega_2 \otimes \omega_4 - \frac{t^2 k}{q^2 \sqrt{r}} \omega_2 \otimes \omega_1, \\
\nu_9 &= \omega_4 \otimes \omega_1 + \omega_1 \otimes \omega_4 + \frac{t^2 k}{q^2 \sqrt{r}} \omega_1 \otimes \omega_3, \\
\nu_{10} &= \omega_4 \otimes \omega_3 + \omega_3 \otimes \omega_4 + \frac{t^2 k}{q^2 \sqrt{r}} \omega_1 \otimes \omega_2.
\end{align*}
\]

Thus, an arbitrary element of \( (\mathfrak{a} \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E}) \otimes \mathfrak{C} \mathfrak{E} \) is given by \( X = \sum_{i,j} A_{ij} \nu_i \otimes \omega_j \), for some complex numbers \( A_{ij} \). Hence, if we show that \( (a(P_{\mathfrak{sym}}))_{23}(\sum_{i,j} A_{ij} \nu_i \otimes \omega_j) = 0 \) implies that \( A_{ij} = 0 \) for all \( i, j \), then \( (a(P_{\mathfrak{sym}}))_{23} \) is a one-one map from \( (\mathfrak{a} \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E}) \otimes \mathfrak{C} \mathfrak{E} \) to \( \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E} \). However, \( \dim((\mathfrak{a} \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E}) \otimes \mathfrak{C} \mathfrak{E}) = \dim(\mathfrak{E} \otimes \mathfrak{C} \mathfrak{E})(a \mathfrak{E} \mathfrak{C} \mathfrak{E}) \), so that \( (a(P_{\mathfrak{sym}}))_{23} \) is a linear isomorphism from \( (\mathfrak{a} \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E}) \otimes \mathfrak{C} \mathfrak{E} \) to \( \mathfrak{a} \mathfrak{E} \otimes \mathfrak{C} \mathfrak{E} \). Suppose \( \{ A_{ij} \}_{ij} \) are complex numbers such that \( (a(P_{\mathfrak{sym}}))_{23}(\sum_{i,j} A_{ij} \nu_i \otimes \omega_j) = 0 \). Then, by (5), we have

\[
((q^2(a(\sigma))_{23} + 1)((a(\sigma))_{23} + q^2))(\sum_{ij} A_{ij} \nu_i \otimes \omega_j) = 0.
\]

We want to show that except for finitely many values of \( q \), the above equation implies that all the \( A_{ij} \) are equal to \( 0 \). This involves a long computation, including a series of preparatory lemmas. We will be using the explicit form of \( a(\sigma)(\omega_i \otimes \omega_j) \) as given in Equation (4.1) of [6] as well as (6) to express the left hand side of (7) as a linear combination of basis elements \( \omega_i \otimes \omega_j \otimes \omega_k \). Then we compare coefficients to derive relations among the \( A_{ij} \). We do not provide the details of the computation. However, for the purposes of book-keeping, each equation is indexed by a triplet \( (i, j, k) \) meaning that it is obtained by collecting coefficients of the basis element \( \omega_i \otimes \omega_j \otimes \omega_k \) in the expansion of \( ((q^2(a(\sigma))_{23} + 1)((a(\sigma))_{23} + q^2))(\sum_{mn} A_{mn} \nu_m \otimes \omega_n) \).

**Lemma 5.1.** We have the following equations:

\[
\begin{align*}
A_{11} &= 0 \\
A_{12}(q^4 + 2) + (tA_{31} + A_{51} + A_{101} \frac{t^2 k}{q^2 \sqrt{r}})2q^2 + (A_{73}q^2 + A_{93} \frac{t^2 k}{q^2 \sqrt{r}})2q(q^2 - 1) &= 0 \\
A_{13}(q^4 + 2q^2 - 1) + A_{14}(\frac{k}{\sqrt{r}}(q^2 - 2 + q^{-2})) \\
+ (A_{71}q^2 + A_{91} \frac{t^2 k}{q^2 \sqrt{r}})2q^2 + A_{91}(\frac{k}{\sqrt{r}}q^{-2}(q^2 - 1)) &= 0 \\
A_{13}(-2q^2 \frac{t^2 k}{k} + A_{14}(q^4 + 1) + (A_{71}q^2 + A_{91} \frac{t^2 k}{q^2 \sqrt{r}})q^4 + A_{91}(q^4 + 1) &= 0
\end{align*}
\]

**Proof.** The above equations are derived by comparing the coefficients of \( \omega_1 \otimes \omega_1 \otimes \omega_1 \), \( \omega_1 \otimes \omega_2 \), \( \omega_1 \otimes \omega_1 \otimes \omega_3 \) and \( \omega_1 \otimes \omega_1 \otimes \omega_4 \) in (7).
Lemma 5.2. We have the following equations:

\[
A_{12}(2q^2 - 1) + (tA_{31} + A_{51} + A_{10,1} \frac{t^2k}{\sqrt{r}})(q^4 + 1) + (A_{73}q^2 + A_{94} \frac{t^2k}{\sqrt{r}})(-2q(q^2 - 1)) \\
+ A_{93}(\frac{k}{\sqrt{r}}(q^2 - 1)^2) + (A_{74}q^2 + A_{94})(-\frac{k}{\sqrt{r}}(q^2 - 1)^2) = 0 \\
\text{(1,2,1)}
\]

\[
tA_{32} + A_{52} + A_{10,2} \frac{t^2k}{\sqrt{r}} = 0 \\
\text{(1,2,2)}
\]

\[
(tA_{34} + A_{54} + A_{10,4} \frac{t^2k}{\sqrt{r}})(-\frac{k}{\sqrt{r}}(q^2 - 1)^2) + (tA_{34} + A_{53} + A_{10,3} \frac{t^2k}{\sqrt{r}})(-(q^2 - 1)^2) \\
+ (A_{72}q^2 + A_{92} \frac{t^2k}{\sqrt{r}})2q^2 + A_{92}(\frac{k}{\sqrt{r}}(q^2 - 1)^2) = 0 \\
\text{(1,2,3)}
\]

\[
(tA_{33} + A_{53} + A_{10,3} \frac{t^2k}{\sqrt{r}})q^4 + (tA_{34} + A_{54} + A_{10,4} \frac{t^2k}{\sqrt{r}})(q^4 + 1) \\
+ (A_{72}q^2 + A_{92} \frac{t^2k}{\sqrt{r}})(-q^2) + A_{92}(q^4 + 1) = 0 \\
\text{(1,2,4)}
\]

Proof. The above equations are derived by comparing the coefficients of $\omega_1 \otimes \omega_2 \otimes \omega_1$, $\omega_1 \otimes \omega_2 \otimes \omega_1$, $\omega_1 \otimes \omega_2 \otimes \omega_1$ and $\omega_1 \otimes \omega_2 \otimes \omega_1$ in (7). \hfill \Box

Lemma 5.3. We have the following equations:

\[
A_{12}q^2 + A_{14} \frac{k}{\sqrt{r}}(-q^2(q - q^{-1})^2) \\
+ (A_{71}q^2 + A_{94} \frac{t^2k}{\sqrt{r}})(-q^4 + 2q^2 + 1) + A_{91} \frac{k}{\sqrt{r}}(-q^2 - 1)^2 = 0 \\
\text{(1,3,1)}
\]

\[
(tA_{33} + A_{53} + A_{10,3} \frac{t^2k}{\sqrt{r}})2q^2 + (tA_{34} + A_{54} + A_{10,4} \frac{t^2k}{\sqrt{r}})(q^2 - 2q + q^{-2}) \\
+ (A_{72}q^2 + A_{92} \frac{t^2k}{\sqrt{r}})(q^4 + 2q^2 - 1) + A_{92} \frac{k}{\sqrt{r}}q^{-2}(q^2 - 1)^2 = 0 \\
\text{(1,3,2)}
\]

\[
(tA_{31} + A_{51} + A_{10,1} \frac{t^2k}{\sqrt{r}})(-2q^3 + 2q) + A_{12}2q(q^2 - 1) + A_{93}(\frac{k}{\sqrt{r}}q^{-2}(q^2 - 1)^3) \\
+ (A_{73}q^2 + A_{93} \frac{t^2k}{\sqrt{r}})(-q^4 + 6q^2 - 1) + (A_{74}q^2 + A_{94} \frac{t^2k}{\sqrt{r}})(-\frac{k}{\sqrt{r}}q^{-2}(q^2 - 1)^3) = 0 \\
\text{(1,3,3)}
\]

\[
(tA_{31} + A_{51} + A_{10,1} \frac{t^2k}{\sqrt{r}})(-2q^3 + 2q) + (A_{73}q^2 + A_{93} \frac{t^2k}{\sqrt{r}})(q^4) \\
+ A_{93}(3(q^2 - 1)^2 + 2q^2) + (A_{74}q^2 + A_{94} \frac{t^2k}{\sqrt{r}})(q^4 + 1) = 0 \\
\text{(1,3,4)}
\]

Proof. The above equations are derived by comparing the coefficients of $\omega_1 \otimes \omega_3 \otimes \omega_1$, $\omega_1 \otimes \omega_3 \otimes \omega_1$, $\omega_3 \otimes \omega_2$, $\omega_1 \otimes \omega_3 \otimes \omega_3$ and $\omega_1 \otimes \omega_3 \otimes \omega_1$ in (7). \hfill \Box
The above equations are derived by comparing the coefficients of $\frac{\omega}{C}$.

\[
\begin{align*}
A_{13}(\frac{q^2}{k}) + A_{14}(q^4 + 1) + (A_{71}q^2 + A_{91}\frac{\omega^2k}{q\sqrt{r}})q^4\sqrt{r} + A_{91}(q^4 + 1) &= 0 \\
(tA_{33} + A_{35} + A_{103}\frac{\omega}{C}) + (tA_{34} + A_{54} + A_{104}\frac{\omega}{C})q^4 + 1) &= 0 \\
+A_{72}q^2 + A_{92}\frac{\omega^2k}{\sqrt{r}} + A_{92}(q^4 + 1) &= 0 \\
A_{12}(\frac{r}{k}) + (tA_{31} + A_{51} + A_{101}\frac{\omega^2k}{q\sqrt{r}})\sqrt{r} + A_{93}(q^4 - 1) &= 0 \\
+A_{73}q^2 + A_{93}\frac{\omega^2k}{\sqrt{r}}(q^2 - 1) + (A_{74}q^2 + A_{94}\frac{\omega^2k}{\sqrt{r}})(q^4 + 1) &= 0 \\
A_{94} &= 0
\end{align*}
\]

\textit{Proof.} The above equations are derived by comparing the coefficients of $\omega_1 \otimes \omega_4 \otimes \omega_1$, $\omega_1 \otimes \omega_4 \otimes \omega_1$ and $\omega_1 \otimes \omega_4 \otimes \omega_1$ in (7).

\[
\begin{align*}
A_{51} &= 0 \\
A_{52}(q^4 + 2) + A_{21}(2q^2) + (A_{63}q^{-2} + A_{81}\frac{\omega^2k}{q^2\sqrt{r}})2q(q^2 - 1) &= 0 \\
+A_{83}(\frac{k}{\sqrt{r}}(q^2 - 1) - 1) + (A_{64}q^{-2} + A_{84}\frac{\omega^2k}{q^2\sqrt{r}})\sqrt{r}q(q^2 - 2 + q^2) &= 0 \\
A_{53}(q^4 + 2q^2 - 1) + A_{54}\frac{k}{\sqrt{r}}(q^2 - 2 + q^2) &= 0 \\
+(A_{64}q^{-2} + A_{84}\frac{\omega^2k}{q^2\sqrt{r}})2q^2 + A_{81}\frac{k}{\sqrt{r}}q^{-2}(q^2 - 1) &= 0 \\
A_{53}(\frac{k}{\sqrt{r}}q^2) + A_{54}(q^4 + 1) + (A_{61}q^{-2} + A_{81}\frac{\omega^2k}{q^2\sqrt{r}})\sqrt{r}k + A_{81}(q^4 + 1) &= 0
\end{align*}
\]

\textit{Proof.} The above equations are derived by comparing the coefficients of $\omega_2 \otimes \omega_1 \otimes \omega_1$, $\omega_2 \otimes \omega_1 \otimes \omega_2$, $\omega_2 \otimes \omega_1 \otimes \omega_1$ and $\omega_2 \otimes \omega_1 \otimes \omega_1$ in (7).

\[
\begin{align*}
A_{52}(2q^2 - 1) + A_{21}(q^4 + 1) + (A_{63}q^{-2} + A_{83}\frac{\omega^2k}{q^2\sqrt{r}})(-2q(q^2 - 1)) &= 0 \\
+A_{83}(\frac{k}{\sqrt{r}}(q^2 - 1) - 2) + (A_{64}q^{-2} + A_{84}\frac{\omega^2k}{q^2\sqrt{r}})(\frac{k}{\sqrt{r}}q(q^2 - 2 + q^2)) &= 0 \\
A_{22} &= 0 \\
A_{23}(-q^4 + 2q^2 - 1) + A_{24}(\frac{k}{\sqrt{r}}(q^4 - 2q^2 + 1)) &= 0 \\
+(A_{62}q^{-2} + A_{82}\frac{\omega^2k}{q^2\sqrt{r}})2q^2 + A_{82}(\frac{k}{\sqrt{r}}q^2 - 1) &= 0 \\
A_{23}q^{-2} + A_{24}(q^4 + 1) + (A_{62}q^{-2} + A_{82}\frac{\omega^2k}{q^2\sqrt{r}})\sqrt{r}k &+ A_{82}(q^4 + 1) &= 0
\end{align*}
\]

\textit{Proof.} The above equations are derived by comparing the coefficients of $\omega_2 \otimes \omega_2 \otimes \omega_1$, $\omega_2 \otimes \omega_2 \otimes \omega_1$ and $\omega_2 \otimes \omega_2 \otimes \omega_1$ in (7).
Lemma 5.7. We have the following equations:

\[
A_{53}(2q^2) + A_{54}(q^4 + 1) + (A_{61}q^{-2} + A_{81}\frac{t^k}{q^2\sqrt{r}})(-q^4 + 2q^2 + 1) + A_{81}\frac{k}{\sqrt{r}}(-q^2 - 1)^2 = 0 \quad (2,3,1)
\]

\[
A_{52}2q(-q^2 - 1) + A_{21}(-2q^3 + 2q) + (A_{63}q^{-2} + A_{83}\frac{t^2k}{q^2\sqrt{r}})(-q^4 + 6q^2 - 1) + A_{83}\frac{k}{\sqrt{r}}(-q^2 - 1)^3 = 0 \quad (2,3,2)
\]

\[
A_{21}\sqrt{q}(-q^2 + 2 + q^2) + (A_{64}q^{-2} + A_{84}\frac{t^2k}{q^2\sqrt{r}})\sqrt{q}q^4 + A_{83}(q^2 - 1)^2 + 2q^2
\]

\[
+ (A_{64}q^{-2} + A_{84}\frac{t^2k}{q^2\sqrt{r}})(q^4 + 1) = 0 \quad (2,3,4)
\]

Proof. The above equations are derived by comparing the coefficients of \(\omega_2 \otimes\omega_3 \otimes\omega_1, \omega_2 \otimes\omega_3 \otimes\omega_2, \omega_3 \otimes\omega_2 \otimes\omega_3 \otimes\omega_3 \) and \(\omega_2 \otimes\omega_3 \otimes\omega_3 \otimes\omega_4 \) in (7).

Lemma 5.8. We have the following equations:

\[
A_{53}\frac{k}{\sqrt{r}}(-q^2) + A_{54}(q^4 + 1) + (A_{61}q^{-2} + A_{81}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{q}}{k}q^4 + A_{81}(q^4 + 1) = 0 \quad (2,4,1)
\]

\[
A_{52}\frac{k}{\sqrt{r}}q^4 + A_{24}(q^4 + 1) + (A_{62}q^{-2} + A_{82}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{q}}{k}(-q^2) + A_{82}(q^4 + 1) = 0 \quad (2,4,2)
\]

\[
A_{21}\sqrt{q}(-q^3) + A_{21}\frac{k}{\sqrt{r}}q^3 + (A_{31}q^{-2} + A_{83}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{q}}{k}q^2(q^2 - 1) + A_{83}(q^4 - 1) + (A_{64}q^{-2} + A_{84}\frac{t^2k}{q^2\sqrt{r}})(q^4 + 1) = 0
\]

\[
A_{84} = 0 \quad (2,4,4)
\]

Proof. The above equations are derived by comparing the coefficients of \(\omega_2 \otimes\omega_4 \otimes\omega_1, \omega_2 \otimes\omega_3 \otimes\omega_4 \otimes\omega_3 \) and \(\omega_2 \otimes\omega_3 \otimes\omega_4 \otimes\omega_4 \) in (7).

Lemma 5.9. We have the following equations:

\[
A_{71} = 0 \quad (3,1,1)
\]

\[
A_{72}(q^4 + 2) + A_{61}2q^2 + A_{83}2q(q^2 - 1)
+ A_{10,3}\frac{k}{\sqrt{r}}q^{-1}(-q^2 - 1)^2 + A_{34}\frac{k}{\sqrt{r}}q^2(q^2 - 2 + q^2) = 0 \quad (3,1,2)
\]

\[
A_{73}(q^4 + 2q^2 - 1) + A_{74}\frac{k}{\sqrt{r}}q^2 - 2 + q^2 + A_{31}2q^2 + A_{10,1}\frac{k}{\sqrt{r}}q^2(-q^2 - 1) = 0 \quad (3,1,3)
\]

\[
A_{73}\frac{k}{\sqrt{r}}(-q^2) + A_{74}(q^4 + 1) + A_{31}\frac{\sqrt{q}}{k}q^4 + A_{10,1}(q^4 + 1) = 0 \quad (3,1,4)
\]

Proof. The above equations are derived by comparing the coefficients of \(\omega_3 \otimes\omega_1 \otimes\omega_1, \omega_3 \otimes\omega_3 \otimes\omega_2 \otimes\omega_3 \) and \(\omega_3 \otimes\omega_1 \otimes\omega_1 \otimes\omega_4 \) in (7).
Lemma 5.10. We have the following equations:

\[ A_{72}(2q^2 - 1) + A_{61}(q^4 + 1) + A_{33}2q(-q^2 - 1) \]
\[ + A_{10,3}\frac{k}{\sqrt{r}}(-q^{-1}(q^2 - 1)^2) + A_{34}\frac{k}{\sqrt{r}}(-q(q^2 - 2 + q^{-2})) = 0 \] (3,2,1)
\[ A_{62} = 0 \] (3,2,2)
\[ A_{63}(-q^4 + 2q^2 + 1) + A_{64}\frac{k}{\sqrt{r}}(-(q^4 - 2q^2 + 1)) + A_{33}2q^2 + A_{10,2}\frac{k}{\sqrt{r}}(-q^2 - 1)^2) = 0 \] (3,2,3)
\[ A_{63}\frac{\sqrt{r}}{k}q^4 + A_{64}(q^4 + 1) + A_{32}\frac{\sqrt{r}}{k}(-q^2) + A_{10,2}(q^4 + 1) = 0 \] (3,2,4)

Proof. The above equations are derived by comparing the coefficients of \( \omega_3 \otimes \omega_2 \otimes \omega_1, \omega_3 \otimes \omega_2 \otimes \omega_3, \omega_3 \otimes \omega_2 \otimes \omega_3 \), and \( \omega_3 \otimes \omega_2 \otimes \omega_4 \) in (7).

Lemma 5.11. We have the following equations:

\[ A_{72}3q^2 + A_{74}\frac{k}{\sqrt{r}}(-(q^2 - 1)^2) + A_{31}(-q^4 + 2q^2 + 1) + A_{10,1}\frac{k}{\sqrt{r}}(-(q^2 - 1)^2) = 0 \] (3,3,1)
\[ A_{63}2q^2 + A_{64}\frac{k}{\sqrt{r}}(q^2 - 2 + q^{-2}) + A_{32}q^4 + 2q^2 - 1) + A_{10,2}\frac{k}{\sqrt{r}}q^{-2}(q^2 - 1)^2 = 0 \] (3,3,2)
\[ A_{61}(-2q^3 + 2q) + A_{33}(-q^4 + 6q^2 - 1) \]
\[ + A_{10,3}\frac{k}{\sqrt{r}}(-q^{-2}(q^2 - 1)^3) + A_{34}\frac{k}{\sqrt{r}}(-q(q - 1)^3) = 0 \] (3,3,3)
\[ A_{64}\frac{\sqrt{r}}{k}q^3 + A_{33}\frac{\sqrt{r}}{k}q^4 + A_{10,3}(3(q^2 - 1)^2 + 2q^2) + A_{34}(q^4 + 1) = 0 \] (3,3,4)

Proof. The above equations are derived by comparing the coefficients of \( \omega_3 \otimes \omega_3 \otimes \omega_1, \omega_3 \otimes \omega_3 \otimes \omega_3 \), \( \omega_3 \otimes \omega_3 \otimes \omega_3 \), and \( \omega_3 \otimes \omega_3 \otimes \omega_3 \) in (7).

Lemma 5.12. We have the following equations:

\[ A_{73}\frac{\sqrt{r}}{k}(-q^2) + A_{74}(q^4 + 1) + A_{31}\frac{\sqrt{r}}{k}q^4 + A_{10,1}(q^4 + 1) = 0 \] (3,4,1)
\[ A_{63}\frac{\sqrt{r}}{k}q^3 + A_{64}(q^4 + 1) + A_{32}\frac{\sqrt{r}}{k}(-q^2) + A_{10,2}(q^4 + 1) = 0 \] (3,4,2)
\[ A_{72}\frac{\sqrt{r}}{k}(-q^3) + A_{64}\frac{\sqrt{r}}{k}q^3 = 0 \] (3,4,3)
\[ A_{10,4} = 0 \] (3,4,4)

Proof. The above equations are derived by comparing the coefficients of \( \omega_3 \otimes \omega_4 \otimes \omega_1, \omega_3 \otimes \omega_4 \otimes \omega_3 \), \( \omega_4 \otimes \omega_2, \omega_3 \otimes \omega_4 \otimes \omega_3 \), and \( \omega_4 \otimes \omega_4 \otimes \omega_4 \) in (7).

Lemma 5.13. We have the following equations:

\[ A_{91} = 0 \] (4,1,1)
\[ A_{92}(q^4 + 2) + A_{81}2q^2 + A_{10,3}2q(q^2 - 1) \]
\[ + A_{43}\frac{k}{\sqrt{r}}q^{-1}(q^2 - 1)^2 + A_{10,4}\frac{k}{\sqrt{r}}q(q^2 - 2 + q^{-2}) = 0 \] (4,1,2)
\[ A_{93}(q^4 + 2q^2 - 1) + A_{94}\frac{k}{\sqrt{r}}(q^2 - 2 + q^{-2}) + A_{10,1}2q^2 + A_{41}\frac{k}{\sqrt{r}}q^{-2}(q^2 - 1) = 0 \] (4,1,3)
\[ A_{93}\frac{\sqrt{r}}{k}(-q^2) + A_{94}(q^4 + 1) + A_{10,3}\frac{\sqrt{r}}{k}q^4 + A_{41}(q^4 + 1) = 0 \] (4,1,4)

Proof. The above equations are derived by comparing the coefficients of \( \omega_4 \otimes \omega_1 \otimes \omega_1, \omega_4 \otimes \omega_3 \otimes \omega_2, \omega_4 \otimes \omega_1 \otimes \omega_3\), and \( \omega_4 \otimes \omega_1 \otimes \omega_4 \) in (7).
Lemma 5.14. We have the following equations:

\[ A_{92}(q^2 - 1) + A_{81}(q^4 + 1) + A_{10,3}2q(q^2 - 1) + A_{43}\frac{k}{\sqrt{r}}(-q^{-1}(q^2 - 1)^2) \]

\[ + A_{10,4}\frac{k}{\sqrt{r}}(q^2 - 2 + q^{-2}) = 0 \quad (4,2,1) \]

\[ A_{82} = 0 \quad (4,2,2) \]

\[ A_{83}(-q^4 + 2q^2 + 1) + A_{84}\frac{k}{\sqrt{r}}(-q^4 + 2q^2 - 1) + A_{10,2}2q^2 + A_{42}\frac{k}{\sqrt{r}}(q^2 - 1)^2 = 0 \quad (4,2,3) \]

\[ A_{83}\frac{\sqrt{r}}{k}q^4 + A_{84}(q^4 + 1) + A_{10,2}\frac{\sqrt{r}}{k}(-q^2) + A_{42}(q^4 + 1) = 0 \quad (4,2,4) \]

**Proof.** The above equations are derived by comparing the coefficients of \( \omega \otimes \omega \otimes \omega_1 \), \( \omega_4 \otimes \omega_2 \otimes \omega_3 \), and \( \omega_4 \otimes \omega_2 \otimes \omega_4 \) in (7).

Lemma 5.15. We have the following equations:

\[ A_{93}(q^4 + 2q^2 - 1) + A_{94}\frac{k}{\sqrt{r}}(q^2 - 2 + q^{-2}) + A_{10,1}(-q^4 + 2q^2 + 1) \]

\[ + A_{41}\frac{k}{\sqrt{r}}(-q^{-1}(q^2 - 1)^2) = 0 \quad (4,3,1) \]

\[ A_{83}2q^2 + A_{84}\frac{k}{\sqrt{r}}(q^2 - 2 + q^{-2}) + A_{10,2}(q^4 + 2q^2 - 1) + A_{42}\frac{k}{\sqrt{r}}(-q^2 - 1)^2 = 0 \quad (4,3,2) \]

\[ A_{92}2q(q^2 - 1) + A_{81}(-2q^3 + 2q) + A_{10,3}(-q^4 + 6q^2 - 1) + A_{43}\frac{k}{\sqrt{r}}(-q^{-2}(q^2 - 1)^3) \]

\[ + A_{10,4}\frac{k}{\sqrt{r}}(-q^{-1}(q^2 - 1)^3) = 0 \quad (4,3,3) \]

\[ A_{83}\frac{\sqrt{r}}{k}q^3 + A_{10,3}\frac{\sqrt{r}}{k}q^4 + A_{43}(3(q^2 - 1)^2 + 2q^2) + A_{10,4}q^4 + 1 = 0 \quad (4,3,4) \]

**Proof.** The above equations are derived by comparing the coefficients of \( \omega_4 \otimes \omega_3 \otimes \omega_1 \), \( \omega_4 \otimes \omega_2 \otimes \omega_3 \), and \( \omega_4 \otimes \omega_2 \otimes \omega_4 \) in (7).

Lemma 5.16. We have the following equations:

\[ A_{93}\frac{\sqrt{r}}{k}(-q^2) + A_{94}(q^4 + 1) + A_{10,1}\frac{\sqrt{r}}{k}q^4 + A_{41}(q^4 + 1) = 0 \quad (4,4,1) \]

\[ A_{83}\frac{\sqrt{r}}{k}q^4 + A_{83}(q^4 + 1) + A_{10,2}\frac{\sqrt{r}}{k}(-q^2) + A_{42}(q^4 + 1) = 0 \quad (4,4,2) \]

\[ A_{92}\frac{\sqrt{r}}{k}(-q^3) + A_{81}\frac{\sqrt{r}}{k}q^3 + A_{10,3}\frac{\sqrt{r}}{k}q^4(2q^2 - 1) + A_{43}(q^4 - 1) + A_{10,4}q^4 + 1 = 0 \quad (4,4,3) \]

\[ A_{44} = 0 \quad (4,4,4) \]

**Proof.** The above equations are derived by comparing the coefficients of \( \omega_4 \otimes \omega_4 \otimes \omega_1 \), \( \omega_4 \otimes \omega_2 \otimes \omega_4 \) and \( \omega_4 \otimes \omega_4 \otimes \omega_4 \) in (7).

Theorem 5.17. For the 4D\(_\pm\) calculi, the map

\[ ((0(P_{\text{sym}}))_{23}) : (\mathcal{E} \otimes_{\mathcal{E}} \mathcal{E}) \otimes \mathcal{E} \rightarrow \mathcal{E} \otimes (\mathcal{E} \otimes_{\mathcal{E}} \mathcal{E}) \]

is an isomorphism except for, possibly, finitely many values of \( q \in (-1, 1) \backslash \{0\} \). Hence, for each bi-invariant pseudo-Riemannian metric \( g \), there exists a unique bicovariant Levi-Civita connection for each calculus.

**Proof.** By the discussion preceding the above series of preparatory lemmas, we need to show that the system of equations given above admit only the trivial solution for \( A_{ij} \), \( i = 1, \ldots, 10 \), \( j = 1, \ldots, 4 \). We then proceed to solve these equations for all \( A_{ij} \). Note that the following variables are all identically zero in the above over-determined system:

\( A_{11} \) (by (1,1,1)), \( A_{94} \) (by (1,4,4)), \( A_{51} \) (by (2,1,1)), \( A_{22} \) (by (2,2,2)), \( A_{84} \) (by (2,4,4)), \( A_{71} \) (by (3,1,1)), \( A_{62} \) (by (3,2,2)), \( A_{10,4} \) (by (3,4,4)), \( A_{91} \) (by (4,1,1)), \( A_{82} \) (by (4,2,2)) and \( A_{44} \) (by (4,4,4)).
This reduces the equations (1,3,1) and (1,4,1) to the following exact system of linear equations in the variables $A_{13}$ and $A_{14}$, with the associated matrix having determinant $q^2(q^2 + 1)^2$:

$$A_{13}2q^2 + A_{14}\frac{k}{q}(q^2(q - q^{-1})^2) = 0$$

$$A_{13}(\frac{q}{k} + 2) + A_{14}(q^4 + 1) = 0$$

Hence the solution for the variables $A_{13}$ and $A_{14}$ is zero.

We repeat this process for the rest of the $A_{ij}$, identifying a subset of equations which has been reduced to an exact one due to the previously solved $q$ in the current set are also solved to be 0 except for at most finitely many value of $q \in (-1,1) \setminus \{0\}$. (2,2,3) and (2,2,4) reduce to the following system of linear equations in $A_{23}$ and $A_{24}$ with determinant $(q^2 + 1)^2$:

$$A_{23}(-q^4 + 2q^2 + 1) + A_{24}(-\frac{k}{q}(q^4 - 2q^2 + 1)) = 0$$

$$A_{23}(\frac{q}{k} + 2) + A_{24}(q^4 + 1) = 0$$

(4,1,3), (4,1,4) and (4,3,1) reduce to the following system of linear equations in $A_{41}$, $A_{93}$, $A_{10,1}$ with determinant $2q^{10} - 2q^4 - 2q^2 + 2$:

$$A_{93}(q^4 + 2q^2 - 1) + A_{10,1}(2q^2 + A_{41}\frac{k}{q}q^{-2}(q^2 - 1)) = 0$$

$$A_{93}(\frac{q}{k} - q^2) + A_{10,1}(q^4 + 1) = 0$$

$$A_{93}(q^4 + 2q^2 - 1) + A_{10,1}(q^4 - 2q^2 + 1) + A_{41}\frac{k}{q}(q^2 - 1)^2 = 0$$

(4,1,2), (4,2,1), (4,3,3) and (4,4,3) reduce to the following system of linear equations in $A_{43}$, $A_{81}$, $A_{92}$, $A_{10,3}$ with determinant $4q^{14} + 10q^{12} - 10q^{10} - 8q^8 + 26q^4 - 26q^2 + 4$:

$$A_{92}(q^4 + 2) + A_{43}(2q^2 + A_{10,3}(-2q^2 - 1) + A_{41}\frac{k}{q}(q^2 - 1)^2) = 0$$

$$A_{92}(2q^2 - 1) + A_{43}(q^4 + 1) + A_{10,3}(2q^2 - 1) + A_{41}\frac{k}{q}(q^2 - 1)^2 = 0$$

$$A_{92}(2q^2 - 1) + A_{43}(-2q^3 + 2q) + A_{10,3}(-q^4 + 6q^2 - 1) + A_{41}\frac{k}{q}(q^2 - 1)^3 = 0$$

$$A_{92}(\frac{q}{k} - q^3) + A_{43}(\frac{q}{k} - q^3) + A_{10,3}\frac{q}{k}q^2(q^2 - 1) + A_{43}(q^4 - 1) = 0$$

(3,4,3), (3,1,2), (3,2,1) and (3,3,3) reduce to the following system of linear equations in $A_{33}$, $A_{34}$, $A_{61}$, $A_{72}$ with determinant $-2q^2(q - 1)^2(q + 1)^2(q^2 + 1)^2$:

$$A_{72}\frac{q}{k}(q^3) + A_{61}\frac{q}{k}q^3 = 0$$

$$A_{72}(q^4 + 2) + A_{61}(2q^2 + A_{33}(-2q^2 - 1) + A_{34}\frac{k}{q}(q^2 - 2 + q^{-2}) = 0$$

$$A_{72}(2q^2 - 1) + A_{61}(q^4 + 1) + A_{33}(-2q^2 - 1) + A_{34}\frac{k}{q}(q^2 - 2 + q^{-2}) = 0$$

$$A_{61}(-2q^3 + 2q) + A_{33}(-q^4 + 6q^2 - 1) + A_{34}\frac{k}{q}(q^4 - q^{-1})^3 = 0$$

(2,1,3) and (2,1,4) reduce to the following system of equations in $A_{53}$ and $A_{54}$ with determinant $q^4(q^2 + 1)^2$:

$$A_{53}(q^4 + 2q^2 - 1) + A_{54}\frac{k}{q}(q^2 - 2 + q^{-2}) = 0$$

$$A_{53}(\frac{q}{k} - q^2) + A_{54}(q^4 + 1) = 0$$

(1,1,2), (1,2,1), (1,3,3) and (1,3,4) reduce to a system of equations in $A_{12}$, $A_{31}$, $A_{73}$, $A_{74}$ with determinant a non-zero polynomial in $q$: 
By Theorem\(2.12)\), \((2.2.1), (2.3.3), (2.3.4)\) and \((2.4.3)\) reduce to a system of equations in \(A_{21}, A_{52}, A_{63}, A_{64}, A_{83}\) with determinant a non-zero polynomial in \(q\):

\[
A_{52}(q^4 + 2) + A_{21}(2q^2) + (A_{63}q^2 + A_{64}\frac{t^2k}{q^2}\sqrt{r})2q(q^2 - 1)
+ A_{63}\left(k\frac{1}{q^2}\sqrt{r}(q^2 - 1)^2\right) + (A_{64}q^2 + A_{84}\frac{t^2k}{q^2}\sqrt{r})(q^2 - 2 + q^2) = 0
\]

\[
A_{52}(2q^2 - 1) + A_{21}(q^2 + 1) + (A_{63}q^2 + A_{83}\frac{t^2k}{q^2}\sqrt{r})(-2q(q^2 - 1))
+ A_{83}\left(k\frac{1}{q^2}\sqrt{r}(q^2 - 1)^2\right) + (A_{64}q^2 + A_{84}\frac{t^2k}{q^2}\sqrt{r})(-q^2 + 2 + q^2) = 0
\]

\[
A_{52}2q^2(q^2 - 1) + A_{21}(-2q^3 + 2q) + (A_{63}q^2 + A_{83}\frac{t^2k}{q^2}\sqrt{r})(-q^4 + 6q^2 - 1)
+ A_{83}\left(k\frac{1}{q^2}\sqrt{r}(-q^2 - q - 1)^3\right) + (A_{64}q^2 + A_{84}\frac{t^2k}{q^2}\sqrt{r})(q^2 + 2q^2)
\]

\[
A_{21}\left(k\frac{1}{q^2}\sqrt{r}q^3 + (A_{63}q^2 + A_{83}\frac{t^2k}{q^2}\sqrt{r})q^4 + A_{83}(3q^2 - 1) + 2q^2)\right)
+ (A_{64}q^2 + A_{84}\frac{t^2k}{q^2}\sqrt{r})(q^4 + 1) = 0
\]

\[
A_{52}\left(k\frac{1}{q^2}\sqrt{r}(-q^3) + A_{21}\left(k\frac{1}{q^2}\sqrt{r}q^3 + (A_{63}q^2 + A_{83}\frac{t^2k}{q^2}\sqrt{r})q^2\right)\right)q^2 = 0
\]

\[
A_{83}(q^4 - 1) + (A_{64}q^2 + A_{84}\frac{t^2k}{q^2}\sqrt{r})(q^4 + 1) = 0
\]

\((3.3.2)\) and \((3.4.2)\) reduce to a system of equations in \(A_{32}, A_{10,2}\) with determinant \(q^4(q^2 + 1)^2\):

\[
A_{32}(q^4 + 2q^2 - 1) + A_{10,2}\frac{t^2k}{q^2}\sqrt{r}q^2(q^2 - 1)^2 = 0
\]

\[
A_{32}\left(k\frac{1}{q^2}\sqrt{r}q^2(q^2 - 1)^2\right) + A_{10,2}(q^4 + 1) = 0
\]

Finally, \((4.2.3)\) reduces identically to \(A_{42} = 0\).

Hence we have shown that all \(A_{ij}\) are identically equal to zero except for almost finitely many values of \(q \in (-1, 1)\). Therefore, \((\sigma(P_{\text{Sym}}))_{21}|_{o(\mathcal{O})_{(2, 0)\mathcal{O} \mathcal{C} \mathcal{E}}}\) is an isomorphism if \(q\) does not belong to this finite subset.

Since \(SU_q(2)\) is a cosemisimple Hopf algebra, and we have shown that the map \(\sigma\) is diagonalisable, by Theorem 2.7, for each bi-invariant pseudo-Riemannian metric \(g\), each of the \(4D\_\pm\) calculi admits a unique bicovariant Levi-Civita connection for all but finitely many \(q\).

The proof of Theorem 2.7, as given in [3], involves explicitly constructing a Levi-Civita connection for each triple \((\mathcal{E}, d, g)\), subject to the accompanying hypothesis. In Theorem 5.17, we have shown that the hypothesis holds for the \(4D\_\pm\) calculi and for any bi-invariant pseudo-Riemannian metric. In this subsection, we provide the explicit construction of the Levi-Civita connection for a fixed arbitrary bi-invariant pseudo-Riemannian metric \(g\). For this we will need to recall some definitions and results from [3].
Definition 5.18. Let $E$ and $g$ be as above. We define a map

$$V_g : oE \to (oE)^*, \quad V_g(e)(f) = g(e \otimes_A f).$$

Definition 5.19. Let $g$ be as above. We define a map

$$g^{(2)} : (oE \otimes_C oE) \otimes (oE \otimes_C oE) \to C$$

by the formula

$$g^{(2)}((e_1 \otimes_C e_2) \otimes (e_3 \otimes_C e_4)) = g(e_1 \otimes_A g(e_2 \otimes_A e_3) \otimes_A e_4)$$

for all $e_1, e_2, e_3, e_4$ in $oE$.

We also define a map $V_{g^{(2)}} : (oE \otimes_C oE) \to (oE \otimes_C oE)^* := \text{Hom}_C(oE \otimes_C oE, C)$ by the formula

$$V_{g^{(2)}}(e_1 \otimes_C e_2)(e_3 \otimes_C e_4) = g^{(2)}((e_1 \otimes_A e_2) \otimes_A (e_3 \otimes_A e_4)).$$

Proposition 5.20. (Lemma 4.4 and 4.9 of [3]) The map $V_g$ is one-one and hence a vector space isomorphism from $oE$ to $(oE)^*$. Moreover, the map $V_{g^{(2)}}$ is a vector space isomorphism from $oE \otimes_C oE$ onto $(oE \otimes_C oE)^*$.

Definition 5.21. Let $V$ and $W$ be finite dimensional complex vector spaces. The canonical vector space isomorphism from $V \otimes_A W^*$ to $\text{Hom}_C(W, V)$ will be denoted by the symbol $\zeta_{V,W}$. It is defined by the formula:

$$\zeta_{V,W} \left( \sum v_i \otimes \phi_i(w) \right) = \sum v_i \phi_i(w). \quad (8)$$

Lemma 5.22. (Lemma 3.12 of [3]) The following maps are vector space isomorphisms:

$$\zeta_{oE \otimes_C oE} : (oE \otimes_C oE) \otimes (oE)^* \to \text{Hom}_C(oE \otimes_C oE, oE)$$

$$\zeta_{oE \otimes_C oE} : oE \otimes (oE \otimes_C oE)^* \to \text{Hom}_C(oE \otimes_C oE, oE)$$

Definition 5.23. Given the maps $\zeta_{oE \otimes_C oE}, \zeta_{oE \otimes_C oE}, V_g, V_{g^{(2)}}$ and $0(P_{sym})_{23}$, the map

$$\tilde{\Phi}_g : \text{Hom}_C(oE \otimes_C oE, oE) \to \text{Hom}_C(oE \otimes_C oE, oE)$$

is defined such that the following diagram commutes:

$$\xymatrix{ \text{Hom}_C(oE \otimes_C oE, oE) & (oE \otimes_C oE)^* \ar[l]_{\zeta_{oE \otimes_C oE}} \ar[r]^{(0P_{sym})_{23}} & (oE \otimes_C oE, oE) \ar[l]_{\tilde{\Phi}_g} }$$

Remark 5.24. In Theorem 5.17, we proved that the map $0(P_{sym})_{23} : (oE \otimes_C oE) \otimes (oE \otimes_C oE) \to (oE \otimes_C oE, oE)$ is an isomorphism. By Proposition 5.20 and Lemma 5.22, the remaining legs of the above commutative diagram are isomorphisms. Hence, $\tilde{\Phi}_g : \text{Hom}_C(oE \otimes_C oE, oE) \to \text{Hom}_C(oE \otimes_C oE, oE)$ is also an isomorphism.

Theorem 5.25. For a fixed bi-invariant pseudo-Riemannian metric $g$, the bicovariant Levi-Civita connection $\nabla$ is defined on elements of $oE$ by

$$\nabla = \nabla_0 + \tilde{\Phi}_g^{-1}(dg - \bar{\Pi}_0^g(\nabla_0)), \quad (9)$$

where $\nabla_0$ is the bicovariant torsionless connection constructed in Theorem 4.2. Here, $\nabla_0$ and $g$ are considered as restrictions on $oE$ and $oE \otimes_C oE$ respectively.

Proof. Let us recall from Remark 4.1, that it is sufficient to define a connection on $oE$ to define it on the whole of $E$. Next, by (4), $\bar{\Pi}_0^g(\nabla_0)$ is a well-defined map in $\text{Hom}_C(oE \otimes_C oE, oE)$. The map $dg$ is a well-defined map in $\text{Hom}_C(oE \otimes_C oE, oE)$. (Indeed it is the zero-map, since $g$ maps $oE \otimes_C oE$ to $C$, and $d$ maps $C$ to 0. That we write it at all in the formula of $\nabla$ is because of how it appears in the proof of Proposition 7.3 of [3].) We have already remarked that $\tilde{\Phi}_g$ is a well-defined isomorphism from $\text{Hom}_C(oE \otimes_C oE, oE)$ to $\text{Hom}_C(oE \otimes_C oE, oE)$. Hence, the right-hand side of (9) is a well-defined map in $\text{Hom}_C(oE \otimes_C oE, oE)$. That it defines the unique bicovariant Levi-Civita connection on $E$ follows from the proofs of Proposition 7.3 and Theorem 7.8 of [3], and we leave out the details.
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