A VARIATIONAL APPROACH TO THE RIEMANN PROBLEM FOR HYPERBOLIC CONSERVATION LAWS

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Abstract. Within the framework of strictly hyperbolic systems of conservation laws endowed with a convex entropy, it is shown that the admissible solution to the Riemann problem is obtained by minimizing the entropy production over all wave fans with fixed end-states.

1. Introduction. We consider strictly hyperbolic systems of $n$ conservation laws in one spatial dimension:

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0.$$  \hfill (1.1)

The state vector $U$ takes values in $\mathbb{R}^n$. The flux $F$ is a given smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and for any $U \in \mathbb{R}^n$ the Jacobian matrix $DF(U)$ has real distinct eigenvalues

$$\lambda_1(U) < \cdots < \lambda_n(U),$$  \hfill (1.2)

which are the characteristic speeds.

The books [4][8][10][15] provide detailed expositions of the current state of the theory of such systems. When the flux is nonlinear, solutions starting out from even smooth initial values eventually develop jump discontinuities that propagate on as shock waves. Thus, the Cauchy problem in the large must be set in the realm of weak solutions. Furthermore, in order to secure well-posedness, one has to impose admissibility conditions, which are usually motivated by physics.

The subject of this paper is the Riemann Problem, namely the Cauchy problem for (1.1) with initial data of the form

$$U(x,0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0, \end{cases}$$  \hfill (1.3)

where $U_L$ and $U_R$ are constant states. Since both (1.1) and (1.3) are invariant under uniform stretching of the spatial and temporal coordinates, this problem possesses self-similar solutions $U(x,t) = V(x/t)$.

Self-similar solutions play a central role, as they govern both the local structure and the large time behavior of general weak solutions, that is, figuratively speaking, they depict how solutions look under the microscope or through a telescope. Moreover, solutions to Riemann problems serve as the building blocks for constructing $BV$ solutions to the Cauchy problem by the random choice method [9] or by the front tracking algorithm [4].

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Because of strict hyperbolicity (1.2), any self-similar solution to the Riemann problem (1.1), (1.3) is a wave fan consisting of \( n+1 \) constant states \( U_L = U_0, U_1, \ldots, U_n = U_R \) joined by \( n \) waves, one for each characteristic family. In turn, the \( i \)-wave, associated with the \( i \)-th characteristic family and joining the constant states \( U_{i-1} \) and \( U_i \), is generally composed of a finite or infinite number of \( i \)-rarefaction waves and \( i \)-shocks propagating at or near the characteristic speed \( \lambda_i \). The wave fan is admissible when all of its shocks satisfy proper admissibility conditions. Riemann [14] introduced the standard method of synthesizing the wave fan from its elementary wave components with the help of wave curves. This approach, which has the advantage of constructing solutions to the Riemann problem explicitly, is particularly efficient when the system is genuinely nonlinear [12], in which case any admissible \( i \)-wave is either a single \( i \)-rarefaction wave or a single compressive \( i \)-shock. The method of wave curves is still applicable, albeit considerably more complicated, when the system is merely piecewise genuinely nonlinear, as in that case admissible \( i \)-waves may contain an at most finite number of \( i \)-rarefaction waves and/or \( i \)-shocks [13], but it becomes quite cumbersome for more general systems, where the number of elementary waves contained in some admissible \( i \)-wave may be infinite [11]. It has also been established that the admissible solution constructed by the method of wave curves is unique.

An alternative, implicit construction of solutions to the Riemann problem is induced by the vanishing viscosity method. This approach is particularly well adapted to the task at hand when the viscosity is allowed to vary linearly with time [6][16], in which case the resulting equation
\[
\partial_t U(x,t) + \partial_x F(U(x,t)) = \epsilon \partial_x^2 U(x,t) \tag{1.4}
\]
inherits from (1.1) the invariance under uniform stretching of the spatial and temporal coordinates, and thereby admits self-similar solutions \( U_\epsilon(x,t) = V_\epsilon(x/t) \). Starting out from such solutions, with boundary conditions \( V_\epsilon(-\infty) = U_L, V_\epsilon(\infty) = U_R \), and passing to the limit \( \epsilon \downarrow 0 \), one obtains admissible wave fan solutions to (1.1), (1.3). In fact, it is through this approach that the existence of admissible solutions to the Riemann problem was first established, for general, not necessarily piecewise genuine nonlinear, strictly hyperbolic systems of conservation laws [17]. More recently, it was shown [2][3] that even the standard vanishing viscosity method, with constant, time independent, viscosity coefficient, may be employed for the same purpose. The connection between the methods of wave curves and vanishing viscosity is discussed in [2][8].

In this paper, the Riemann problem will be solved by still another method, through variational techniques. This approach is applicable to systems that are endowed with an entropy-entropy flux pair \( (\eta, q) \), where \( \eta(U) \) is uniformly convex. This class contains, in particular, virtually all hyperbolic systems of conservation laws, in one spatial dimension, that arise in continuum physics.

The entropy-entropy flux pair satisfies the compatibility condition
\[
Dq(U) = D\eta(U)DF(U), \tag{1.5}
\]
and admissible solutions of (1.1) must satisfy the inequality
\[
\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) \leq 0, \tag{1.6}
\]
in the sense of distributions. In particular, any classical solution is admissible, since it satisfies (1.6), as an equality, by virtue of (1.5).
The variational principle that will be employed in the construction of solutions to the Riemann Problem, is induced by the entropy rate admissibility condition\[5\][7][8], according to which admissible solutions must minimize the rate of entropy production, within the class of all (weak) solutions that satisfy the initial condition (1.3). It will be shown that when \(|UR - UL|\) is sufficiently small there exists a self-similar solution of (1.1), (1.3) with small total variation, which satisfies this admissibility criterion. The approach will then be validated by showing that, when the system is piecewise genuinely nonlinear, the solution constructed by the variational principle coincides with the unique solution synthesized by the classical method of wave curves.

The results derived in this paper have already been announced in [8], with a sketch of (an alternative) proof. However, detailed proofs are presented here for the first time.

2. Wave fans. Consider the strictly hyperbolic system (1.1). For \(U \in \mathbb{R}^n, DF(U)\) possesses real distinct eigenvalues (1.2) and thereby associated, linearly independent, right (column) eigenvectors \(R_1(U), \ldots, R_n(U)\) and left (row) eigenvectors \(L_1(U), \ldots, L_n(U)\), normalized by

\[L_i(U)R_j(U) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}\] (2.1)

The aim is to study the structure of self-similar BV solutions of (1.1), with small oscillation. We fix a reference state \(\bar{U}\) and set \(\lambda_i(\bar{U}) = \bar{\lambda}_i, R_i(\bar{U}) = \bar{R}_i, L_i(\bar{U}) = \bar{L}_i\). We also fix a positive constant \(\delta\) that is small in comparison to the gaps \(\bar{\lambda}_{j+1} - \bar{\lambda}_j\) between characteristic speeds, and then, if necessary, restrict its size even further so that, for \(i = 1, \ldots, n\), the oscillation of \(\lambda_i, R_i\) and \(L_i\) over the ball \(B\) of radius \(\delta\), centered at \(\bar{U}\), is sufficiently small.

A function \(V(\xi)\), of bounded variation on \((−∞, ∞)\), taking values in \(B\), induces a self-similar, wave fan solution \(U(x, t) = V(x/t)\) to the Riemann problem (1.1), (1.3) if it satisfies the ordinary differential equation

\[\dot{F}(V(\xi)) - \xi V(\xi) = 0,\] (2.2)

in the sense of distributions on \((−∞, ∞)\), together with the boundary conditions

\[V(−∞) = U_L, \quad V(∞) = U_R.\] (2.3)

Here and throughout an overdot denotes differentiation with respect to \(\xi\). Notice that (2.2) may be rewritten as

\[\left[F(V(\xi)) - \xi V(\xi)\right]^+ + V(\xi) = 0,\] (2.4)

which implies in particular that \(F(V(\xi)) - \xi V(\xi)\) is Lipschitz on \((−∞, ∞)\). Furthermore, by virtue of the chain rule of Volpert [18], for functions of bounded variation, (2.2) may also be written as

\[\widetilde{DF}(V) - \xi I]\dot{V} = 0,\] (2.5)

in the sense of measures, where

\[\widetilde{DF}(V)(\xi) = \int_0^1 DF(\tau V(\xi−) + (1 - \tau)V(\xi+))d\tau.\] (2.6)

Finally, as a function of bounded variation, \(V\) is differentiable almost everywhere and (2.5) will be satisfied pointwise at any point \(\xi\) of continuity where \(\dot{V}(\xi)\) exists.
With the wave fan $V$ is associated a partition of $(-\infty, \infty)$ into three pairwise disjoint subsets $\mathcal{C}, \mathcal{S}$ and $\mathcal{W}$, as follows:

(a) $\mathcal{C}$ is the maximal open subset of $(-\infty, \infty)$ on which the measure $\dot{V}$ vanishes, i.e., it is the complement of the support of $\dot{V}$. Thus $\mathcal{C}$ is the (at most) countable union of disjoint open intervals, on each of which $V$ is constant.

(b) $\mathcal{S}$ is the (at most) countable set of points of jump discontinuity of $V$. Since $F(V(\xi)) - \xi V(\xi)$ is Lipschitz on $(-\infty, \infty)$, the Rankine-Hugoniot jump condition

$$ F(V(\xi^+)) - F(V(\xi^-)) = \xi [V(\xi^+) - V(\xi^-)] \quad (2.7) $$

must hold for any $\xi \in \mathcal{S}$. In particular, for $\delta$ sufficiently small, any $\xi \in \mathcal{S}$ must satisfy $|\xi - \lambda_i| = O(\delta)$, for some $i = 1, \ldots, n$, in which case it is dubbed an $i$-shock. The (possibly empty) set of $i$-shocks will be denoted by $\mathcal{S}_i$.

(c) $\mathcal{W}$ is the (possibly empty) set of points of continuity of $V$ that lie in the support of the measure $\dot{V}$. When $\xi \in \mathcal{W}$,

$$ \lambda_i(V(\xi)) = \xi, \quad (2.8) $$

for some $i \in \{1, \ldots, n\}$. Indeed, in the case $\xi$ is the limit of a sequence of $i$-shocks $\{\xi_m\} \subset \mathcal{S}_i$, then $V(\xi_m^+) - V(\xi_m^-) \to 0$, as $m \to \infty$, and (2.8) follows from the Rankine-Hugoniot jump condition (2.7). On the other hand, if $\xi$ lies in the interior of the set of points of continuity of $V$, then (2.8) fails for all $i = 1, \ldots, n$, then $\lambda_i(V(\zeta)) \neq \zeta$ for $\zeta \in (\xi - \varepsilon, \xi + \varepsilon)$, $\varepsilon > 0$, in which case, by virtue of (2.5), $\dot{V}$ would vanish on $(\xi - \varepsilon, \xi + \varepsilon)$, in contradiction to the assumption that $\xi \in \text{spt}\dot{V}$. The set of $\xi \in \mathcal{W}$ satisfying (2.8) for a particular $i$ will be denoted by $\mathcal{W}_i$. Thus $\mathcal{W}_i$ represents the $i$-rarefaction wave of the wave fan $V$. When $\xi \in \mathcal{W}_i$ is also a point of differentiability of $V$, then (2.5), (2.6) imply that $\dot{V}(\xi)$ is colinear to $R_i(V(\xi))$. Moreover, by (2.8), $\mathcal{D}\lambda_i(V(\xi)) \dot{V}(\xi) = 1$ so that $\dot{V}(\xi)$ is necessarily a point of genuine nonlinearity of the $i$-th characteristic family.

The smallest closed interval $[\zeta_i, \zeta_i']$ that contains $\mathcal{S}_i \cup \mathcal{W}_i$ represents the $i$-wave of the wave fan $V$. At this level of generality, in the absence of any admissibility restriction on waves, $[\zeta_i, \zeta_i']$ may also contain open intervals of points of $\mathcal{C}$. Thus $i$-waves generally consist of $i$-shocks, $i$-rarefaction waves and constant states.

The complement of the union of the $i$-waves consists of the open intervals $(-\infty, \zeta_1), (\zeta_1, \zeta_2), \ldots, (\zeta_i, \zeta_{i+1}), \ldots, (\zeta_n, \infty)$, on each of which $V$ is respectively the constant state $U_L = U_0, U_1, \ldots, U_i, \ldots, U_n = U_R$.

Henceforth, we focus attention to wave fans with monotone $i$-waves, characterized by the property that $L_i[V(\zeta_2^+) - V(\zeta_1^-)]$ does not change sign for all $\zeta_i \leq \xi_1 \leq \zeta_2 \leq \zeta_i'$. In the forthcoming third edition of [8] it is shown that when $\delta$ is sufficiently small,

$$ |L_i[V(\zeta_2^+) - V(\zeta_1^-)]| \leq O(\delta)|L_i[V(\zeta_2^+) - V(\zeta_1^-)]| \quad (2.9) $$

holds for all $j \neq i$ and any $\zeta_i \leq \xi_1 \leq \zeta_2 \leq \zeta_i'$.

So long as $|U_R - U_L|$ is sufficiently small, wave fans $V$ with monotone $i$-waves and end-states (2.3) always exist. In the simplest example of such a $V$, each $i$-wave is a (not necessarily admissible) single $i$-shock. For future reference, we outline the standard method of constructing this particular wave fan, with the help of shock curves.

As is well-known (e.g. [8,12,15]), the Hugoniot locus of states $U$ that may be joined to some fixed state $\bar{U}$ by an $i$-shock of small strength is a smooth (so called
i-shock) curve \( U = W_i(\tau; \tilde{U}) \). The speed of the shock is \( s_i(\tau; \tilde{U}) \). We select the parameterization of the i-shock curve so that
\[ L_{i}[W_{i}(\tau; \tilde{U}) - \tilde{U}] = \tau. \] (2.10)
Moreover, \( W_i(0; \tilde{U}) = \tilde{U}, \) \( s_i(0; \tilde{U}) = \lambda_i(\tilde{U}) \), and after normalizing the eigenvectors of \( DF(\tilde{U}) \), \( W_i(0; \tilde{U}) = R_i(\tilde{U}) \).

To construct the wave fan \( V \) with end-states (2.3), we seek numbers \( \tau_i, \cdots, \tau_n \) and states \( U_L = U_0, U_1, \cdots, U_n = U_R \) such that \( U_i = W_i(\tau_i; U_{i-1}), i = 1, \cdots, n \). Equivalently, we consider the function
\[ \Omega(\tau_1, \cdots, \tau_n; U_L) = W_n(\tau_n; W_{n-1}(\tau_{n-1}; \cdots W_1(\tau_1; U_L) \cdots)) \] (2.11)
and seek \( (\tau_1, \cdots, \tau_n) \) such that \( \Omega(\tau_1, \cdots, \tau_n; U_L) = U_R \). Observing that
\[ \Omega(\tau_1, \cdots, \tau_n; U_L) = U_L + \sum_{i=1}^{n} \tau_i R_i(U_L) + \Phi(\tau_1, \cdots, \tau_n; U_L), \] (2.12)
where \( \Phi \) and its first derivatives vanish at \( (0, \cdots, 0) \), we conclude that a unique solution exists when \( |U_R - U_L| \) is sufficiently small.

If \( V \) is a wave fan with monotone i-waves, (2.9) implies
\[ TV_{[\zeta_i, \tilde{\zeta}_i]} V(\xi) \leq c|\tilde{L}_i[V(\tilde{\zeta}_i^+) - V(\tilde{\zeta}_i^-)]|. \] (2.13)
Since
\[ U_R - U_L = \sum_{i=1}^{n} (U_i - U_{i-1}) = \sum_{i=1}^{n} [V(\zeta_i^+) - V(\zeta_i^-)], \] (2.14)
(2.9) and (2.13) imply that the total variation of any wave fan \( V \) with monotone i-waves, \( i = 1, \cdots, n \), taking values in \( B \), with \( \delta \) sufficiently small, and joining the state \( U_L \), on the left, to the state \( U_R \), on the right, is bounded:
\[ TV_{(-\infty, \infty)} V(\xi) \leq c|U_R - U_L|. \] (2.15)

3. **Minimizing the entropy production.** We assume that the system (1.1) is endowed with an entropy-entropy flux pair \((\eta(U), q(U))\), where \( D^2\eta(U) \) is positive definite on \( B \). In particular, (1.5) holds. For self-similar solutions \( U(x, t) = V(x/t) \), the entropy admissibility condition (1.6) reduces to
\[ \dot{q}(V(\xi)) - \xi \dot{q}(V(\xi)) \leq 0, \] (3.1)
in the sense of measures. By virtue of the chain rule of Volpert [18], for functions of bounded variation, (3.1) may be written as
\[ [D\tilde{q}(V) - \xi D\tilde{\eta}(V)] \dot{V} \leq 0, \] (3.2)
in the sense of measures, where
\[ \left\{ \begin{array}{l}
\tilde{\eta}(V)(\xi) = \int_{0}^{1} D\eta(\tau V(\xi^-) + (1 - \tau) V(\xi^+)) d\tau \\
\tilde{\eta}(V)(\xi) = \int_{0}^{1} D\eta(\tau V(\xi^-) + (1 - \tau) V(\xi^+)) d\tau.
\end{array} \right. \] (3.3)

From (3.2), (1.5) and (2.5) we infer that the measure \( \dot{q}(V) - \xi \dot{q}(V) \) is concentrated in the set of points of jump discontinuity of \( V \). Thus the entropy admissibility condition (3.1) reduces to
\[ q(V(\xi^+)) - q(V(\xi^-)) - \xi [\dot{q}(V(\xi^+)) - \eta(V(\xi^-))] \leq 0, \] (3.4)
for all shocks \( \xi \in \mathcal{S} \), and the total entropy production, i.e., the total mass of the measure \( \dot{q}(V) - \xi \dot{\eta}(V) \), is

\[
\mathcal{P}_V = \sum_{\xi \in \mathcal{S}} (q(V(\xi^+)) - q(V(\xi^-)) - \xi[\eta(V(\xi^+)) - \eta(V(\xi^-))]).
\] (3.5)

The entropy-entropy flux pair \((\eta(U), q(U))\) induces a whole family \((\eta(U) + AU + a, q(U) + AF(U))\) of entropy-entropy flux pairs, for arbitrary \(A \in \mathbb{M}^{1 \times n}\) and \(a \in \mathbb{R}\). Because of the Rankine-Hugoniot jump condition (2.7), all of these entropy-entropy flux pairs incur the same entropy production. We select the particular member in the family, which we rename \((\eta(U), q(U))\), that satisfies the normalization condition

\[
\eta(U_L) = \eta(U_R) = 0,
\] (3.6)
in which case the total entropy \( \int \eta(U) dx \) of solutions with initial values (1.3) is finite.

The rate of change of the total entropy is

\[
\mathcal{H}_V = \frac{d}{dt} \int_{-\infty}^{\infty} \eta(U(x, t)) dx = \frac{d}{dt} \int_{-\infty}^{\infty} \eta(V(\frac{x}{t})) dx = \int_{-\infty}^{\infty} \eta(V(\xi)) d\xi.
\] (3.7)

On account of the identity

\[
\eta(V(\xi)) = [\eta(V(\xi)) - q(V(\xi))] + \dot{\eta}(V(\xi)) - \xi \dot{\eta}(V(\xi)),
\] (3.8)
which holds in the sense of measures, one immediately deduces that \(\mathcal{H}_V\) and \(\mathcal{P}_V\) are related by

\[
\mathcal{H}_V = \mathcal{P}_V + q(U_L) - q(U_R).
\] (3.9)

We now invoke the entropy rate principle [5][7][8] according to which admissible solutions of (1.1) must minimize the entropy production. In the present setting, a wave fan \(V\) with monotone \(i\)-waves and end-states (2.3) will satisfy this requirement if \(\mathcal{P}_V \leq \mathcal{P}_V\) where \(V(\xi)\) is any other wave fan with monotone \(i\)-waves and the same end-states (2.3). It should be noted that the restriction to wave fans with monotone \(i\)-waves is only made for simplicity, as it can be shown that the \(i\)-waves of the wave fan that minimizes the entropy production \(\mathcal{P}\) are necessarily monotone.

By virtue of (3.9), minimizing the entropy production \(\mathcal{P}_V\) is equivalent to minimizing the entropy rate

\[
\mathcal{H}_V = \int_{-\infty}^{\infty} \eta(V(\xi)) d\xi.
\] (3.10)

As shown in Section 2, the set of wave fans with monotone \(i\)-waves is nonempty. Furthermore, (2.15) implies that if \(U_L \in \mathcal{B}\) and \(|U_R - U_L|\) is sufficiently small, then any wave fan with monotone \(i\)-waves and end-states (2.3) takes values in \(\mathcal{B}\) and has uniformly bounded total variation over \((-\infty, \infty)\). Therefore, by virtue of Helly’s theorem, any minimizing sequence of (3.10) must contain a subsequence that converges to a wave fan \(V\) that satisfies (2.2), (2.3) and minimizes the entropy production. We thus have established the following

**Theorem 3.1** If \(|U_R - U_L|\) is sufficiently small, there exists a wave fan \(V\), with monotone \(i\)-waves and end-states (2.3), which satisfies the entropy rate principle and thus induces a solution to the Riemann problem (1.1), (1.3).

In the following section it will be shown that the solution to the Riemann problem derived above coincides with the solution constructed by the standard method of wave curves and thus, in particular, it is unique.
4. Admissibility and uniqueness of solutions. Suppose that a certain state $U_-$, on the left, is joined to some state $U_+$, on the right, by an $i$-shock of speed $s$. Thus, as pointed out in Section 2, $U_+$ must lie on the $i$-shock curve emanating from $U_- : U_+ = W_i(\tau_+; U_-)$. The shock is said to satisfy the Liu $E$-condition if

$$s = s_i(\tau_+; U_-) \leq s_i(\tau; U_-), \text{ for all } \tau \text{ between } 0 \text{ and } \tau_+. \quad (4.1)$$

A basic theorem, established by Liu [13], states that if the system is piecewise genuinely nonlinear and $|U_R - U_L|$ is sufficiently small, then there exists a unique solution to the Riemann problem (1.1), (1.3), with shocks satisfying the Liu $E$-condition. In the standard proof of this theorem, the solution is constructed explicitly with the help of wave curves. It will be shown here that the shocks of the solution obtained in Section 3 by minimizing entropy production, likewise satisfy the Liu $E$-condition, and hence both methods yield the same, unique solution to the Riemann problem.

**Theorem 4.1** The shocks of any wave fan that minimizes the functional $\mathcal{H}_V$ over all wave fans with monotone $i$-waves and end-states (2.3), satisfy the Liu $E$-condition.

**Proof.** We will establish the assertion of the theorem by contradiction: Given any wave fan containing some shock that violates the Liu $E$-condition, we will construct a wave fan with lower entropy production. In order to convey the essence of the argument with minimal technicalities, we will first discuss, in full detail, the special case where the given wave fan consists of a single shock. Then we will describe briefly how to handle the general case.

Accordingly, assume that $U_L$ and $U_R$ are joined by an $i$-shock of speed $s$,

$$F(U_R) - F(U_L) = s|U_R - U_L|, \quad (4.2)$$

that violates the Liu $E$-condition. Under the normalization assumption (3.6), the entropy rate $\mathcal{H}$ for this shock is zero. The aim is to construct a wave fan $V$, with the same end-states (2.3) and $\mathcal{H}_V < 0$.

Let $U_R = W_i(\tau_R; U_L)$. For definiteness, assume $\bar{L}_i|U_R - U_L| > 0$ so that $\tau_R > 0$. Since the Liu $E$-condition is violated, the set of $\tau \in (0, \tau_R)$ with $s_i(\tau; U_L) < s$ is nonempty. Let $\tau_L$ be the infimum of that set, and assume $\tau_L > 0$, as the case $\tau_L = 0$ is simpler. In particular, $s_i(\tau_L; U_L) = s$, and $s_i(\tau; U_L)$ is decreasing at $\tau_L$.

We consider the generic case $s_i(\tau_L; U_L) < 0$.

Upon setting $U_M = W_i(\tau_L; U_L)$,

$$F(U_M) - F(U_L) = s|U_M - U_L|. \quad (4.3)$$

Combining (4.2) and (4.3),

$$F(U_R) - F(U_M) = s|U_R - U_M|, \quad (4.4)$$

which shows that $U_M$ also lies on the $i$-shock curve emanating from $U_R$, say $U_M = W_i(\tau_R; U_R)$ and $s_i(\tau_R; U_R) = s$.

One may thus regard the $i$-shock joining $U_L$ with $U_R$ as the superposition of two $i$-shocks, one that joins $U_L$ with $U_M$ and one that joins $U_M$ with $U_R$, both propagating with the same speed $s$. The aim is to perform a perturbation that splits the original shock into two shocks, one with speed slightly lower than $s$ and the other with speed slightly higher than $s$, and then show that the resulting wave fan has negative entropy rate.

We begin by fixing a small positive number $\epsilon$. The wave fan $V$ will consist of $(n + 2)$ constant states $U_L = U_0, \cdots, U_{i-1}, U_i, U_{i+1}, \cdots, U_n = U_R$, joined by shocks.
For $j = 1, \ldots, i - 1$, $U_j$ is joined to $U_{j-1}$ by a $j$-shock, $U_j = W_j(\tau_j; U_{j-1})$, of speed $s_j$:  
\[ F(U_j) - F(U_{j-1}) = s_j[U_j - U_{j-1}], \quad j = 1, \ldots, i - 1. \tag{4.5} \]

$\tilde{U}$ is joined to $U_{i-1}$ by an $i$-shock, $\tilde{U} = W_i(\tau_L + \varepsilon; U_{i-1})$ of speed $s_-$:  
\[ F(\tilde{U}) - F(U_{i-1}) = s_[-\tilde{U} - U_{i-1}]. \tag{4.6} \]

$\tilde{U}$ is also joined to $U_i$ by an $i$-shock, $\tilde{U} = W_i(\tau_R + \tau_i; U_i)$ of speed $s_+$:  
\[ F(\tilde{U}) - F(U_i) = s_+[\tilde{U} - U_i]. \tag{4.7} \]

Finally, for $j = i + 1, \ldots, n$, $U_{j-1}$ is joined to $U_j$ by a $j$-shock, $U_{j-1} = W_j(\tau_j; U_j)$ of speed $s_j$:  
\[ F(U_{j-1}) - F(U_j) = s_j[U_{j-1} - U_j], \quad j = i + 1, \ldots, n. \tag{4.8} \]

The amplitudes $(\tau_1, \ldots, \tau_n)$ of the waves are computed from the equation  
\[ \Omega(\tau_1, \ldots, \tau_n; \varepsilon) = 0, \tag{4.9} \]

where  
\[ \Omega(\tau_1, \ldots, \tau_n; \varepsilon) = W_i(\tau_R + \tau_i; W_{i+1}(\tau_{i+1}; \ldots W_n(\tau_n; U_R) \ldots)) \tag{4.10} \]

\[ -W_i(\tau_L + \varepsilon; W_{i-1}(\tau_{i-1}; \ldots W_1(\tau_1; U_L) \ldots)). \]

Observing that  
\[ \Omega(\tau_1, \ldots, \tau_n; \varepsilon) = -\sum_{j=1}^{i-1} \tau_j R_j(U_L) + \sum_{j=i+1}^n \tau_j R_j(U_R) \tag{4.11} \]

\[ -\varepsilon W_i(\tau_L; U_L) + \tau_i W_i(\tau_R; U_R) \]

\[ + \Phi(\tau_1, \ldots, \tau_n; \varepsilon), \]

where $\Phi$ and its first derivatives vanish at $(0, \ldots, 0, 0)$, we conclude that, for $\varepsilon$ sufficiently small, (4.9) has a unique solution. With an eye to treating the general case, it should be pointed out here that differentiability of $\Phi$ is not necessary. It would suffice to have $\Phi$ Lipschitz continuous, with small Lipschitz constant at the origin. One still needs to verify that $s_- = s_i(\tau_L + \varepsilon; U_{i-1})$ is smaller than $s_+ = s_i(\tau_R + \tau_i; U_i)$ and this will be done below.

By combining (4.2), (4.5), (4.6), (4.7) and (4.8), we deduce  
\[ (s - s_-)[\tilde{U} - U_L] + (s - s_+)[U_R - \tilde{U}] \tag{4.12} \]

\[ = \sum_{j=1}^{i-1} (s_j - s_-)[U_j - U_{j-1}] + \sum_{j=i+1}^n (s_j - s_+)[U_j - U_{j-1}]. \]

For $j = 1, \ldots, i - 1, i + 1, \ldots, n$,  
\[ U_j - U_{j-1} = a_j \tilde{R}_j + V_j, \quad |V_j| = O(\delta)|a_j|. \tag{4.13} \]

Thus, upon setting  
\[ \ell = |(s - s_-)\tilde{L}_i[\tilde{U} - U_L]| \tag{4.14} \]

we deduce, from (4.12):  
\[ |a_j| = O(\delta)\ell, \quad j = 1, \ldots, i - 1, i + 1, \ldots, n, \tag{4.15} \]

\[ (s - s_-)[\tilde{U} - U_L] + (s - s_+)[U_R - \tilde{U}] = O(\delta^2)\ell, \tag{4.16} \]

which implies, in particular, that $(s - s_-)$ and $(s - s_+)$ have opposite signs. Recalling that $s_i(\tau_L; U_L) < 0$, we conclude that $s_- < s < s_+$. 


We now compute $\mathcal{H}_V$ from (3.10):

$$\mathcal{H}_V = \sum_{j=1}^{i-2} (s_{j+1} - s_j)\eta(U_j) + (s_- - s_{i-1})\eta(U_{i-1}) + (s_+ - s_-)\eta(\bar{U})$$

(4.17)

$$+ (s_{i+1} - s_+)(U_i) + \sum_{j=i+1}^{n-1} (s_{j+1} - s_j)\eta(U_j).$$

Upon rearranging the terms in the above summations:

$$\mathcal{H}_V = \sum_{j=1}^{i-1} (s_- - s_j)[\eta(U_j) - \eta(U_{j-1})] + (s_+ - s_-)\eta(\bar{U})$$

(4.18)

$$+ \sum_{j=i+1}^{n} (s_+ - s_j)[\eta(U_j) - \eta(U_{j-1})].$$

By adding and subtracting terms, the above may be rewritten as

$$\mathcal{H}_V = (s_- - s_-)[\eta(\bar{U}) - D\eta(\bar{U})]\bar{U} - U_L] - \eta(U_L)]$$

(4.19)

$$+ (s_+ - s_-)[\eta(\bar{U}) - D\eta(\bar{U})]\bar{U} - U_R] - \eta(U_R)]$$

$$+ \sum_{j=1}^{i-1} (s_- - s_j)[\eta(U_j) - \eta(U_{j-1}) - D\eta(\bar{U})][U_j - U_{j-1}]$$

$$+ \sum_{j=i+1}^{n} (s_+ - s_j)[\eta(U_j) - \eta(U_{j-1}) - D\eta(\bar{U})][U_j - U_{j-1}].$$

Since $\eta$ is strictly convex, and recalling (4.14) and (4.16),

$$\begin{cases}
-s_-\{\eta(\bar{U}) - D\eta(\bar{U})]\bar{U} - U_L] - \eta(U_L]) \geq \alpha \ell [\bar{U} - U_L] \\
-s_+\{\eta(\bar{U}) - D\eta(\bar{U})]\bar{U} - U_R] - \eta(U_R]) \geq \alpha \ell [\bar{U} - U_R],
\end{cases}$$

(4.20)

with $\alpha > 0$. On the other hand, by virtue of (4.15),

$$\begin{cases}
|\eta(U_j) - \eta(U_{j-1}) - D\eta(\bar{U})]U_j - U_{j-1}| \leq O(\delta)\ell |\bar{U} - U_L|, j = 1, \ldots, i - 1 \\
|\eta(U_j) - \eta(U_{j-1}) - D\eta(\bar{U})]U_j - U_{j-1}| \leq O(\delta)\ell |\bar{U} - U_R|, j = i + 1, \ldots, n.
\end{cases}$$

(4.21)

Hence, $\mathcal{H}_V < 0$, which establishes the assertion of the theorem in the special case where the wave fan consists of a single shock violating the Liu $E$-condition.

We now turn to the general case, where a wave fan $\bar{V}$ is given, with monotone $i$-waves, end-states (2.3) and at least one shock violating the Liu $E$-condition, and the objective is to construct another wave fan $V$, with the same end-states, such that $\mathcal{H}_V < \mathcal{H}_\bar{V}$. The $i$-waves of $V$ will be perturbations of the $i$-waves of $\bar{V}$. We discuss briefly how such perturbations may be realized.

With any $i$-wave $V(\xi)$, defined on an interval $[\zeta, \bar{\zeta}]$ and joining states $\bar{U}$ and $\bar{\bar{U}}$, we associate the Lipschitz continuous track curve $W(\tau)$ of the wave, in state space, which is constructed by the following process. We set $\phi(\xi) = \bar{L}_i[V(\xi) - \bar{U}]$ for any point $\xi \in [\zeta, \bar{\zeta}]$ of continuity of $V$. If $\tau = \phi(\xi)$, we set $W(\tau) = V(\xi)$. On the other hand, if $\tau \in [\phi(\xi-), \phi(\xi+)]$, we define $W(\tau) = W_i(\tau - \phi(\xi-); V(\xi-))$. Thus $W(\tau)$
joins $\hat{U}$ and $\tilde{U}$ by a (finite or infinite) sequence of $i$-shock curves and/or $i$-rarefaction curves. Conversely, from the track curve one may recover the wave. The track curve provides the most efficient way for performing perturbations of waves. Simple but tedious analysis yields the following conclusion.

Assume that the state $\hat{U}$, on the left, is connected to a state $\tilde{U}$, on the right, by an $i$-wave. Then there exists a Lipschitz function $\Phi_i(U, \tau)$, defined for $U$ in a small neighborhood of $\hat{U}$ and for $\tau$ in a small interval $(-\varepsilon, \varepsilon)$, and taking values in the state space, with the following properties:

(a) $\Phi_i(\hat{U}, 0) = \tilde{U}$.

(b) The state $\Phi_i(U, \tau)$ is joined to the state $U$ by an $i$-wave.

(c) If $U = \hat{U} + \sum_{j=1}^{n} \alpha_j \bar{R}_j$, then

$$\Phi_i(U, \tau) = \tilde{U} + \sum_{j \neq i} \alpha_j \bar{R}_j + \tau \bar{R}_i + \Psi_i(U, \tau), \quad (4.22)$$

where $\Psi_i$ is a Lipschitz function with small Lipschitz constant:

$$|\Phi_i(U_1, \tau_1) - \Phi_i(U_2, \tau_2)| \leq O(\delta)||U_1 - U_2|| + |\tau_1 - \tau_2|. \quad (4.23)$$

(d) When the given $i$-wave contains an $i$-shock with left state $U_-$, right state $U_+$ and speed $s$, then there exist Lipschitz functions $P_-(U, \tau), P_+(U, \tau)$ and $\sigma(U, \tau)$, with $P_-(\hat{U}, 0) = U_-, P_+(\hat{U}, 0) = U_+$ and $\sigma(\hat{U}, 0) = s$, such that the $i$-wave that joins $U$ with $\Phi_i(U, \tau)$ contains an $i$-shock with left state $P_-(U, \tau)$, right state $P_+(U, \tau)$ and speed $\sigma(U, \tau)$.

Armed with the above information it is easy to adapt the proof presented earlier for the case of the single shock to the present situation of a general wave fan. This task shall be left to the reader.

REFERENCES

[1] L. Ambrosio, N. Fusco and D. Pallara, “Functions of Bounded Variation and Free Discontinuity Problems,” Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] S. Bianchini, On the Riemann problem for non-conservative hyperbolic systems, Arch. Rational Mech. Anal., 166 (2003), 1–26.

[3] S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. of Math. (2), 161 (2005), 223–342.

[4] A. Bressan, “Hyperbolic Systems of Conservation Laws. The One-dimensional Cauchy Problem,” Oxford Lecture Series in Mathematics and its Applications, 20, Oxford University Press, Oxford, 2000.

[5] C. M. Dafermos, The entropy rate admissibility criterion for solutions of hyperbolic conservation laws, J. Diff. Eqs., 14 (1973), 202–212.

[6] C. M. Dafermos, Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method, Arch. Rational Mech. Anal., 52 (1973), 1–9.

[7] C. M. Dafermos, Admissible wave fans in nonlinear hyperbolic systems, Arch. Rational Mech. Anal., 106 (1989), 243–260.

[8] C. M. Dafermos, “Hyperbolic Conservation Laws in Continuum Physics,” 2nd Edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 325, Springer-Verlag, Berlin, 2005.

[9] James Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math., 18 (1965), 697–715.
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[10] H. Holden and N. H. Risebro, “Front Tracking for Hyperbolic Conservation Laws,” Applied Mathematical Sciences, 152, New York: Springer-Verlag, 2002.

[11] T. Iguchi and P. G. LeFloch, Existence theory for hyperbolic systems of conservation laws with general flux functions, Arch. Rational Mech. Anal., 168 (2003), 165–244.

[12] P. D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math., 10 (1957), 537–566.

[13] T.-P. Liu, Admissible solutions of hyperbolic conservation laws, Memoirs AMS, 30 (1981).

[14] B. Riemann, Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite, Gött. Abh. Math. Cl., 8 (1860), 43–65.

[15] D. Serre, “Systems of Conservation Laws,” Vols. 1-2, Cambridge: Cambridge University Press, 1999.

[16] V. A. Tupciev, On the method for introducing viscosity in the study of problems involving the decay of a discontinuity, Soviet Math., 14 (1973), 978–982.

[17] A. E. Tzavaras, Materials with internal variables and relaxation to conservation laws, Arch. Rational Mech. Anal., 146 (1999), 129–155.

[18] A. I. Volpert, The spaces BV and quasilinear equations, Math. USSR Sbornik, 2 (1967), 225–267.

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