On Closed Space Curves in Minkowski Space–Time $E^n_v$

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Abstract A criterion for a 3-space curve to be closed has been presented in “Chung: Proc. Am. Math. Soc. 83, 357–361 (1981)”. In this paper, following same procedure, we give generalization of the criterion for a 3-space curve to an $n$-space curve to be closed in Minkowski space–time $E^n_v$. Furthermore, we apply this criterion for a curve lying on an oriented surface in the Minkowski 3-space $E^3_1$ as an application.

Keywords Minkowski space–time · Closed curves · Frenet–Serret frame

Mathematics Subject Classification 31A35 · 53C50 · 53B30

Introduction

In 1978, Professor Shiing-Shen Chern raised the following question at the Institute of Mathematics of Academia Sinica:

“What is the necessary and sufficient condition to be satisfied by the curvature and torsion, so that space curve to be a closed one?”

The answer of the question was given by Chung in 1981. By considering the Frenet frame of a space curve in Euclidean 3-space $E^3$ and using well-known method of successive approximation due to E. Picard, Chung presented a criterion for a 3-space curve to be closed [5]. He used a system of linear equations which are derivatives of Frenet vectors. By using this criterion, he found that the curve determined by curvature $\kappa(s) = \text{const.} > 0$ and torsion $\sigma(s) = \text{const.}$ is periodic of period $\omega$, if and only if $\sigma = 0$ and $\kappa = 2\pi/\omega$. Köse et al. [8]
generalized the method of Chung to the space curves in n-dimensional Euclidean space $E^n$. Some applications of this method can be obtained in references [10, 12, 13].

Moreover, the Minkowski space is more interesting than the Euclidean space. In this space, curves and surfaces have different characters such as timelike, spacelike and null (lightlike). In [1], Altun has given the Frenet–Serret frame and harmonic curvatures of non-null curves in Minkowski space–time $E^n$. Furthermore, the geometry of the curves lying on an oriented surface has been given in Minkowski 3-space $E^3$ in [7, 16–18].

Moreover, the 4D Minkowski space has an important role in the theory of relativity, especially the closed timelike curves [3, 6, 15]. But what is the condition for a space curve to be closed in Minkowski space–time? In this paper, our goal is to answer this question. We give a criterion for closed space curves in Minkowski space–time $E^n$. In addition to this, as a special case we considered a curve lying on a surface in Minkowski 3-space $E^3$ and by regarding the Darboux frame of this curve we obtained a necessary and sufficient condition to be satisfied by the geodesic curvature, normal curvature and geodesic torsion.

**Preliminaries**

Minkowski space–time $E^n_v$ is an Euclidean space $E^n$ provided with the standard flat metric given by

$$\langle , \rangle = - \sum_{i=1}^{v} \mathrm{d}x^2_i + \sum_{i=v+1}^{n} \mathrm{d}x^2_i$$

where $(x_1, x_2, \ldots, x_n)$ is a rectangular coordinate system in $E^n_v$. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $\vec{v} \in E^n_v$ can have one of three causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Let $\bar{x}(s) : I \subset \mathbb{R} \rightarrow E^n_v$ be a regular curve in $E^n_v$. Analogues to the vectors, the curve $\bar{x}(s)$ can locally be spacelike, timelike or null (lightlike) in $E^n_v$ if all of its velocity vectors $\bar{x}'(s)$ are spacelike, timelike or null (lightlike), respectively [9]. A timelike (resp. spacelike) curve $\bar{x}(s)$ is said to be parameterized by a pseudo-arclength parameter $s$, if $\langle \bar{x}'(s), \bar{x}'(s) \rangle = -1$ (resp. $\langle \bar{x}'(s), \bar{x}'(s) \rangle = 1$). Also, recall that the pseudo-norm of an arbitrary vector $\vec{v} \in E^n_v$ is given by $\| \vec{v} \| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. Therefore $\vec{v}$ is a unit vector iff $\langle \vec{v}, \vec{v} \rangle = \pm 1$. The velocity of the curve $\bar{x}(s)$ is given by $\| \bar{x}'(s) \|$. Next, vectors $\vec{v}$, $\vec{w}$ in $E^n_v$ are said to be orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$.

Let $\{e_1, e_2, \ldots, e_n\}$ denote an orthonormal basis in the Minkowski space–time $E^n_v$. Then the position vector of a curve $\bar{x}(s)$ can be given by $\bar{x}(s) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$. Let $\omega$ be the total length of the curve $\bar{x}(s)$. When $\bar{x}(s)$ is a closed curve, the position vector $\bar{x}(s)$ must be a periodic function of period $\omega$ [8].

Let $\bar{x}(s)$ be a regular curve in $E^n_v$ and $\varphi = \{\bar{x}'(s), \bar{x}''(s), \ldots, \bar{x}^{(n)}(s)\}$ a maximal linear independent and non-null set. The orthonormal system $\{\bar{V}_1(s), \bar{V}_2(s), \ldots, \bar{V}_n(s)\}$ can be obtained from $\varphi$. This system is called a moving Frenet–Serret frame along the curve $x(s)$ in the space $E^n_v$ [1]. In this paper, we consider the curve $\bar{x}(s)$ whose derivatives $\bar{x}^{(l)}(s)$, $(1 \leq l \leq n)$ are non-null.

**Definition 2.1** Let $\bar{x}(s)$ be a regular curve in $E^n_v$ and $\{\bar{V}_1(s), \bar{V}_2(s), \ldots, \bar{V}_n(s)\}$ denote the non-null Frenet frame of $\bar{x}(s)$. The functions $k_i : I \rightarrow \mathbb{R}$ defined by

$$k_i(s) = \varepsilon_i \varepsilon_{i-1} \left( \bar{V}_i(s), \bar{V}_{i+1}(s) \right), \quad 1 \leq i \leq n - 1,$$
are called curvature functions of $\bar{x}(s)$. Here $\varepsilon_i = \left< \tilde{V}_i(s), \tilde{V}_i'(s) \right> = \pm 1$. Furthermore, the real number $k_i(s)$ is called the $i$-th curvature on $x$ at the point $\bar{x}(s)$ [1].

**Theorem 2.2** Let $\bar{x}(s)$ be a unit speed curve in $E_1^n$ and let the set $\{\tilde{V}_1(s), \tilde{V}_2(s), \ldots, \tilde{V}_n(s)\}$ denote the non-null Frenet–Serret frame at the point $\bar{x}(s)$. Then, the derivative formulae are given as follows

$$
\begin{align*}
\tilde{V}_1' &= \varepsilon_1 k_1(s) \tilde{V}_2 \\
\tilde{V}_i' &= -\varepsilon_i k_{i-1}(s) \tilde{V}_{i-1}(s) + \varepsilon_i k_i(s) \tilde{V}_{i+1}(s), \ (1 < i < n) \\
\tilde{V}_n' &= -\varepsilon_n k_{n-1}(s) \tilde{V}_{n-1}(s)
\end{align*}
$$

or in the matrix form

$$
\begin{bmatrix}
\tilde{V}_1' \\
\tilde{V}_2' \\
\tilde{V}_3' \\
\vdots \\
\tilde{V}_{n-1}' \\
\tilde{V}_n'
\end{bmatrix} =
\begin{bmatrix}
0 & \varepsilon_1 k_1 & 0 & 0 & \cdots & 0 & 0 \\
-\varepsilon_2 k_1 & 0 & \varepsilon_2 k_2 & 0 & \cdots & 0 & 0 \\
0 & -\varepsilon_3 k_2 & 0 & \varepsilon_3 k_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\varepsilon_{n-1} k_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -\varepsilon_n k_{n-1}
\end{bmatrix}
\begin{bmatrix}
\tilde{V}_1 \\
\tilde{V}_2 \\
\tilde{V}_3 \\
\vdots \\
\tilde{V}_{n-1} \\
\tilde{V}_n
\end{bmatrix}
$$

(See [1]). Here to avoid of singularities it is assumed that the result is fulfilled in generic points and when assumed that the curve does not necessarily lie in lower dimensions it is supposed that $\det(x', x'', \ldots, x^{(n)}) \neq 0$.

In the special case $v = 1, n = 3$ we have Minkowski 3-space $E_1^3$ and we can give the followings for this space.

**Definition 2.3** A surface in the Minkowski 3-space $E_1^3$ is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [2].

Let $S$ be an oriented surface in 3D Minkowski space $E_1^3$ and let consider a non-null curve $\bar{x}(s)$ lying on $S$ fully. Since the curve $\bar{x}(s)$ lies on the surface $S$ there exists another frame of the curve $\bar{x}(s)$ which is called Darboux frame and denoted by $\{\tilde{T}, \tilde{g}, \tilde{n}\}$. In this frame $\tilde{T}$ is unit tangent of the curve, $\tilde{n}$ is unit normal of the surface $S$ and $\tilde{g}$ is a unit vector defined by $\tilde{g} = \tilde{n} \times \tilde{T}$.

According to the Lorentzian causal characters of the surface $S$ and the curve $\bar{x}(s)$ lying on $S$, the derivative formulae of the Darboux frame can be changed as follows:

(i) If the surface $S$ is a timelike surface, then the curve $\bar{x}(s)$ lying on $S$ can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\bar{x}(s)$ is given by

$$
\begin{bmatrix}
\tilde{T}' \\
\tilde{g}' \\
\tilde{n}'
\end{bmatrix} =
\begin{bmatrix}
0 & k_g & -\varepsilon k_n \\
k_g & 0 & \varepsilon \tau_g \\
k_n & \tau_g & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{T} \\
\tilde{g} \\
\tilde{n}
\end{bmatrix}, \ \langle \tilde{T}, \tilde{T}' \rangle = \varepsilon = \pm 1, \ \langle \tilde{g}, \tilde{g}' \rangle = -\varepsilon, \ \langle \tilde{n}, \tilde{n}' \rangle = 1. \quad (2)
$$
If the surface $S$ is a spacelike surface, then the curve $\vec{x}(s)$ lying on $S$ is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\vec{x}(s)$ is given by

$$\begin{bmatrix} \frac{\dot{T}}{\hat{n}} \\ \frac{\dot{\phi}}{\hat{n}} \\ \frac{\ddot{\phi}}{\hat{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \ddot{T} \\ \ddot{\hat{g}} \\ \ddot{n} \end{bmatrix}, \quad \{\ddot{T}, \ddot{\hat{g}}\} = 1, \quad \langle \ddot{n}, \dddot{n} \rangle = -1. \quad (3)$$

In these formulae $k_g, k_n$ and $\tau_g$ are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively (for details [7, 16–18]).

Closed Space Curves in Minkowski Space–Time $E^n_v$

In this section, by considering the Frenet–Serret frame of the curve, we give a criterion for closed space curves in Minkowski space–time $E^n_v$. For this purpose, we use the method given in [5, 8].

Firstly, let consider the system of linear differential equations

$$\frac{d\phi_i}{ds} = \sum_{j=1}^{n} a_{ij}(s)\phi_j, \quad (i = 1, 2, \ldots, n), \quad (4)$$

where $a_{ij}(s)$ are assumed to be continuous and periodic of period $\omega$. Let the initial conditions be $\phi_i(0) = \lambda_i$. The matrix form of (4) can be given by

$$\frac{d\phi}{ds} = A(s)\phi, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}, \quad A(s) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$ 

Since the $a_{ij}(s)$ are periodic, $A(s)$ is a continuous periodic matrix function of period $\omega$. Let us write the integral of $A(s)$ as a $n \times n$ matrix as follows

$$\int_{0}^{s} A(s)ds = \begin{bmatrix} \int_{0}^{s} a_{11}(s)ds & \int_{0}^{s} a_{12}(s)ds & \cdots & \int_{0}^{s} a_{1n}(s)ds \\ \int_{0}^{s} a_{21}(s)ds & \int_{0}^{s} a_{22}(s)ds & \cdots & \int_{0}^{s} a_{2n}(s)ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_{0}^{s} a_{n1}(s)ds & \int_{0}^{s} a_{n2}(s)ds & \cdots & \int_{0}^{s} a_{nn}(s)ds \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}.$$ 

$$\phi(s, \lambda) = \begin{bmatrix} \phi_1(s, \lambda) \\ \phi_2(s, \lambda) \\ \vdots \\ \phi_n(s, \lambda) \end{bmatrix}.$$ 

Then the Eq. 4 may be abbreviated to the form

$$\frac{d\phi}{ds} = A(s)\phi, \quad \phi(0) = \lambda. \quad (5)$$

(See [4, 11]). From (5) we get the integral equation $\phi(s) = \lambda + \int_{0}^{s} A(\sigma)\phi(\sigma)d\sigma$ [4, 14]. Then following Lemma can be given:

**Lemma 3.1** The solution $\phi(s, \lambda)$ of the equation $\frac{d\phi}{ds} = A(s)\phi, \quad \phi(0) = \lambda, \quad (A(s+\omega)\equiv A(s))$ is periodic of period $\omega$ if and only if $\int_{0}^{\omega} A(\sigma)\phi(\sigma, \lambda)d\sigma = 0$ [5].
In the same paper, using the well-known method of successive approximation due to E. Picard, the solution \( \phi(s, \lambda) \) is constructed as

\[
\phi(s, \lambda) = \left\{ I + \xi A(s) + \xi^{(2)} A(s) + \cdots + \xi^{(n)} A(s) + \cdots \right\} \lambda
\]

where \( I \) is \( n \times n \) unit matrix. The following theorem is proved by applying Lemma 3.1 to the expression

\[
\int_0^\omega A(s) \phi(s, \lambda) ds = \left\{ \xi A(s) + \xi^{(2)} A(s) + \cdots + \xi^{(n)} A(s) + \cdots \right\} \lambda = M(s) \lambda
\]

where

\[
\xi A(s) = \xi^{(1)} A(s) = \int_0^s A(\sigma) d\sigma, \quad \xi^{(n)} A(s) = \int_0^s A(\sigma) \xi^{(n-1)} A(\sigma) d\sigma, \quad (n > 1)
\]

and

\[
M(s) = \xi A(s) + \xi^{(2)} A(s) + \cdots + \xi^{(n)} A(s) + \cdots
\]

**Theorem 3.2** The equations \( \frac{d\phi}{ds} = A(s) \phi \) have a non-vanishing periodic solution of period \( \omega \) if and only if \( \det (M(\omega)) = 0 \). In particular, the equations \( \frac{d\phi}{ds} = A(s) \phi \) have \( n \)-linearly independent periodic solutions of period \( \omega \) if and only if the matrix \( M(\omega) = 0 \) [5].

Let now consider the Frenet–Serret formulae given in Eq. 1. If we write the Frenet–Serret vectors \( \vec{V}_1, \vec{V}_2, \ldots, \vec{V}_n \) in coordinates form we get

\[
\vec{V}_i = \sum_{j=1}^n v_{ij} \vec{e}_j, \quad (i = 1, 2, \ldots, n).
\]

(6)

From (1) and (6) we obtain the systems of linear differential equations:

\[
\begin{aligned}
&\frac{dv_{1j}}{ds} = \varepsilon_1 k_1(s) v_{2j} \\
&\frac{dv_{2j}}{ds} = -\varepsilon_2 k_1(s) v_{1j} + \varepsilon_2 k_2(s) v_{3j} \\
&\vdots \\
&\frac{dv_{(n-1)j}}{ds} = -\varepsilon_{n-2} k_{n-2}(s) v_{(n-2)j} + \varepsilon_{n-1} k_{n-1}(s) v_{nj} \\
&\frac{dv_{nj}}{ds} = -\varepsilon_n k_{n-1}(s) v_{(n-1)j}, \quad (j = 1, 2, \ldots, n)
\end{aligned}
\]

(7)

Thus, we observe that

\[
(v_{11}, v_{21}, \ldots, v_{n1}) \\
(v_{12}, v_{22}, \ldots, v_{n2}) \\
\vdots \\
(v_{1n}, v_{2n}, \ldots, v_{nn})
\]
are $n$-independent solutions of the following system of differential equations

\[
\begin{align*}
\frac{d\phi_1}{ds} &= \varepsilon_1 k_1(s)\phi_2 \\
\frac{d\phi_2}{ds} &= -\varepsilon_2 k_1(s)\phi_1 + \varepsilon_2 k_2(s)\phi_3 \\
&\vdots \\
\frac{d\phi_{n-1}}{ds} &= -\varepsilon_{n-1} k_{n-2}(s)\phi_{n-2} + \varepsilon_{n-1} k_{n-1}(s)\phi_n \\
\frac{d\phi_n}{ds} &= -\varepsilon_n k_{n-1}(s)\phi_{n-1}
\end{align*}
\]  

(8)

The system in (8) is a special case of the general equations given in (4).

Theorem 3.3 Let $\vec{x}(s)$ be a curve in Minkowski space–time $E_v^n$ with curvatures $k_1(s), k_2(s), \ldots, k_{n-1}(s)$ and let the curvatures $k_1(s), k_2(s), \ldots, k_{n-1}(s)$ be continuous periodic functions of period $\omega$. Then $\vec{x}(s)$ is a periodic curve with period $\omega$ if and only if

(i) the matrix $M(\omega) = 0$,

(ii) $\omega + \int_0^\omega m_{11} ds = \int_0^\omega m_{12} ds = \int_0^\omega m_{13} ds = \cdots = \int_0^\omega m_{1n} ds = 0$

where

\[
M(t) = \xi A(t) + \xi^{(2)} A(t) + \cdots + \xi^{(n)} A(t) + \cdots,
\]

\[
A(t) = \begin{bmatrix}
0 & \varepsilon_1 k_1(t) & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\varepsilon_2 k_1(t) & 0 & \varepsilon_2 k_2(t) & 0 & \cdots & 0 & 0 & 0 \\
0 & -\varepsilon_3 k_2(t) & 0 & \varepsilon_3 k_3(t) & 0 & \cdots & 0 & 0 \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & -\varepsilon_{n-1} k_{n-2}(t) & 0 & \varepsilon_{n-1} k_{n-1}(t) \\
0 & 0 & 0 & 0 & \cdots & 0 & -\varepsilon_n k_{n-1}(t) & 0
\end{bmatrix}
\]

and $m_{ij}(s)$ are the entries of the matrix $M(t)$.

Proof By considering Theorem 3.1 for the system (8) we have

\[
\xi A(s) = \begin{bmatrix}
0 & \int_0^s \varepsilon_1 k_1(t)dt & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\int_0^s \varepsilon_2 k_1(t)dt & 0 & \int_0^s \varepsilon_2 k_2(t)dt & 0 & \cdots & 0 & 0 & 0 \\
0 & -\int_0^s \varepsilon_3 k_2(t)dt & 0 & \int_0^s \varepsilon_3 k_3(t)dt & 0 & \cdots & 0 & 0 \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & -\int_0^s \varepsilon_{n-1} k_{n-2}(t)dt & 0 & \int_0^s \varepsilon_{n-1} k_{n-1}(t)dt \\
0 & 0 & 0 & 0 & \cdots & 0 & -\int_0^s \varepsilon_n k_{n-1}(t)dt & 0
\end{bmatrix}
\]

and so on. When and only when the matrix $M(\omega)$ is a zero matrix, there exist $n$-orthonormal vector functions $\vec{V}_1(s), \vec{V}_2(s), \ldots, \vec{V}_n(s)$ of period $\omega$ such that each set of functions $\{v_{1j}, v_{2j}, \ldots, v_{nj}\}, (j = 1, 2, \ldots, n)$ forms a solution of the equation $\frac{d\phi}{ds} = A(s)\phi$ corresponding to the initial condition $(a_{1j}, a_{2j}, \ldots, a_{nj})$.

The vector function defining the curve $\vec{x}(s) = \int_0^s \vec{V}_1(s)ds$ where $\vec{V}_1(s)$ is given by

\[
\begin{bmatrix}
v_{1j} \\
v_{2j} \\
\vdots \\
v_{nj}
\end{bmatrix} = (I + M(s)) \begin{bmatrix}
a_{1j} \\
a_{2j} \\
\vdots \\
a_{nj}
\end{bmatrix}, \quad (j = 1, 2, \ldots, n)
\]
and \( I \) is \( n \times n \) unit matrix. The curve \( x(s) \) is periodic of period \( \omega \) if and only if
\[
\int_0^{\omega} \tilde{V}_1(s) \, ds = 0.
\]

Let now the initial condition be
\[
v_{1j}(0) = a_{1j}, \ v_{2j}(0) = a_{2j}, \ldots, v_{nj}(0) = a_{nj}, \quad (j = 1, 2, \ldots, n)
\]
where \( (a_{11}, a_{12}, \ldots, a_{1n}), \ (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{n1}, a_{n2}, \ldots, a_{nn}) \) form an orthonormal frame. Then
\[
\begin{bmatrix}
v_{1j} \\
v_{2j} \\
\vdots \\
v_{nj}
\end{bmatrix} = (I + M(t)) \begin{bmatrix}
a_{1j} \\
a_{2j} \\
\vdots \\
a_{nj}
\end{bmatrix}, \quad (j = 1, 2, \ldots, n).
\]

That is, \( v_{1j} = a_{1j} + a_{1j}m_{11} + a_{2j}m_{12} + \cdots + a_{nj}m_{1j} \). So that
\[
\int_0^{\omega} v_{1j}(s) \, ds = a_{1j} + a_{1j} \int_0^{\omega} m_{11} \, ds + \cdots + a_{nj} \int_0^{\omega} m_{1j} \, ds.
\]

Since the determinant
\[
\begin{vmatrix}
a_{11} & a_{21} & \cdots & a_{n1} \\
a_{12} & a_{22} & \cdots & a_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{nn}
\end{vmatrix} \neq 0.
\]

The condition \( \int_0^{\omega} \tilde{V}_1(s) \, ds = 0 \) is equivalent to
\[
\omega + \int_0^{\omega} m_{11} \, ds = \int_0^{\omega} m_{12} \, ds = \int_0^{\omega} m_{13} \, ds = \cdots = \int_0^{\omega} m_{1n} \, ds = 0.
\]

This completes the proof. \( \square \)

In the following example, this criterion is applied to a curve lying on a surface in Minkowski 3-space \( E_1^3 \) as a special case.

**Example 3.4** Let \( S \) be an oriented timelike surface in three-dimensional Minkowski space \( E_1^3 \) and let consider a non-null curve \( \tilde{x}(s) \) lying on \( S \) fully with Darboux frame \( \{ \tilde{T}, \tilde{g}, \tilde{n} \} \).

Assume that \( k_g = \text{const.}, k_n = \text{const.} \) and \( \tau_g = \text{const.} \) for \( \tilde{x}(s) \). By considering (2), from
Theorem 3.3, we get

\[ m_{11}(\omega) = (k_g^2 - \varepsilon k_n^2) \left[ \frac{\omega^2}{2!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^4}{4!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^6}{6!} + \ldots \right] \]

\[ = (k_g^2 - \varepsilon k_n^2) \cosh \left( \frac{(k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^{1/2} \omega}{k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2} \right) - 1 \]

\[ m_{12}(\omega) = k_g \left[ \omega + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^3}{3!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^5}{5!} + \ldots \right] \]

\[ = k_g \left[ \omega + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^3}{3!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^5}{5!} + \ldots \right] \]

\[ - k_n \tau_g \left[ \frac{\omega^3}{3!} - \varepsilon (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^5}{5!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^7}{7!} + \ldots \right] \]

\[ m_{13}(\omega) = -\varepsilon k_n \left[ \omega + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^3}{3!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^5}{5!} + \ldots \right] \]

\[ + \varepsilon k_n \tau_g \left[ \frac{\omega^3}{3!} - \varepsilon (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^5}{5!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^7}{7!} + \ldots \right] \]

\[ = -\varepsilon k_n \left[ \omega + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^3}{3!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^5}{5!} + \ldots \right] \]

\[ + \varepsilon k_n \tau_g \left[ \frac{\omega^3}{3!} - \varepsilon (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^5}{5!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^7}{7!} + \ldots \right] \]

and

\[ \omega + \int_0^\omega m_{11} \, ds = \omega + (k_g^2 - \varepsilon k_n^2) \left[ \frac{\omega^3}{3!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^5}{5!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^7}{7!} + \cdots \right] \]

\[ \int_0^\omega m_{12} \, ds = k_g \left[ \frac{\omega^2}{2!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^4}{4!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^6}{6!} + \cdots \right] \]

\[ - k_n \tau_g \left[ \frac{\omega^3}{3!} - \varepsilon (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^5}{5!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^7}{7!} + \cdots \right] \]

\[ \int_0^\omega m_{13} \, ds = -\varepsilon k_n \left[ \frac{\omega^2}{2!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^4}{4!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^6}{6!} + \cdots \right] \]

\[ + \varepsilon k_n \tau_g \left[ \frac{\omega^3}{3!} - \varepsilon (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2) \frac{\omega^5}{5!} + (k_g^2 - \varepsilon k_n^2 + \varepsilon \tau_g^2)^2 \frac{\omega^7}{7!} + \cdots \right] \]

Hence the conditions (i) and (ii) of Theorem 3.3 give that the curve \( \bar{x}(s) \) lying on \( S \) fully is periodic of period \( \omega \) if and only if \( (k_g^2 - \varepsilon k_n^2)^{1/2} = \frac{2\pi}{i\omega} \), \( \tau_g = 0 \), where \( i \) is imaginary unit with \( i^2 = -1 \).
Conclusion

Closed curves are very important in mathematical physics. Especially, the closed timelike curves have an important role in general relativity. This is the reason why we goal to give a criterion for closed space curves in Minkowski space–time $E^n_1$ in this paper. We introduce the criterion and, as a special case, apply to a curve lying on a surface in Minkowski 3-space $E^3_1$.

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