Exact free energy distribution function of a randomly forced directed polymer

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We study the elastic (1+1)-dimensional string subject to a random gaussian potential on scales smaller than the correlation radius of the disorder potential (Larkin problem). We present an exact calculation of the probability function \( \mathcal{P}(F, (u, L)) \) for the free energy \( F \) of a string starting at \((0, 0)\) and ending at \((u, L)\). The function \( \mathcal{P}(F) \) is strongly asymmetric, with the left tail decaying exponentially \( \ln \mathcal{P}(F \to -\infty) \propto F \) and the right tail vanishing as \( \ln \mathcal{P}(F \to +\infty) \propto -F^3 \). Our analysis defines a strategy for future attacks on this class of problems.

Over the recent years, the interest in static and dynamic aspects of random manifold problems \([1] \) has increased significantly, with numerous applications in the field of random magnets \([2] \), vortex matter in type II superconductors \([3] \), dislocations in metals \([4] \), charge-density waves in solids \([5] \), and many more. Also, the physics of random elastic manifolds is related to other fascinating topics, e.g., Burgers turbulence \([6] \), stochastic growth \([7] \), or systems of interacting bosons \([8] \), all of which have attracted considerable interest recently. In this context, the Larkin model \([9] \) has become a generic model system whose distribution function \( F \) is characterized by the tails \( F \to -\infty \) and \( F \to +\infty \), respectively.

Due to the random nature of the potential \( U \), the free energy \( F \) is a stochastic variable and we are interested in its distribution function \( \mathcal{P}(F) \), see Fig. 1.

The imaginary time Schrödinger equation \( T \partial_z Z = (T^2/2\epsilon)\partial^2_u Z - UZ, z \in [0, L] \), satisfied by the partition function \( Z \), directly maps to the Kardar-Parisi-Zhang equation \( \partial_z F = (T^2/2\epsilon)\Delta F - (\nabla F)^2/2\epsilon + U \) describing the stochastic growth of the ‘surface’ \( F(u, z) \), and differentiating the KPZ-equation with respect to \( u \) and going over to the gradient field \( v = \nabla F \) leads to the Burgers equation \([10] \).
role of Planck’s constant and the internal coordinate z of the string mapping on to the imaginary time coordinate of the boson system. In the new variables $t = \sqrt{\alpha/2T} \, z$ and $x_i = (\alpha \epsilon_i / 2T^3)^{1/4} \, u_i$, Eq. (5) takes the simple form $\partial_t G = (1/2) \sum_i \partial_i^2 G + \sum_{i,j} x_i \epsilon_{ij} G$. The specific form of the Hamiltonian motivates an Ansatz $G = g(t) \exp[[1/2] \sum_{i,j} a_{ij}(t) x_i x_j]$ for the Green function, reducing the problem to a system of ordinary differential equations $\dot{g}/g = (1/2) \sum_k a_{kk} + \dot{a}_{ij} = \sum_k a_{ik} a_{jk} + 2$. We assume the matrix $A = (a_{ij})$ to be Hermitian, i.e., $a_{ij} = a$ and $a_{i,j} = b$ (this assumption implies no restriction, as the present problem has a unique solution).

The three resulting differential equations $\dot{g}/g = na/2$, $b = (n-2)b^2 + 2ab + 2$, and $a = (n-1)b^2 + a^2 + 2$ are easily solved: concentrating first on the difference $a - b$ we find, using the initial condition at $t = 0$, the result $a - b = -1/t$. Using the Ansatz $c(t) = \exp[−n \int^t dt' \dot{b}(t')]$ in the equation for $b$, we can reduce the problem to the Bessel equation and find the solution $b(t) = 1/nt - 2t \cot(\sqrt{2nt})/\sqrt{2nt}$, where we have again made use of the initial condition. Finally, the result for $g(t)$ reads $g(t) = (2n)^{1/4} \epsilon^T \exp[-(1/4) \ln \sin^2(\sqrt{2nt})]$ and assembling the various elements, we arrive at the following final expression for the moments $P(n) = (Z^n(x,t)/Z_{th}(x,t))$ of the replicated partition function (the arguments in $P$ are $x \leftrightarrow u$ and $t \leftrightarrow L$; we subtract the free energy $F_{th}$ of the thermal model ($U = 0$) as we are only interested in the difference $\Delta F = F - F_{th}$ with and without disorder; in the following we drop the symbol ‘$\Delta$’),

$$P(n) = \exp \left[ \frac{n x^2}{2t} - \frac{1}{4} \ln \sin^2(\sqrt{2nt}) - n^2 x^2 t \cot(\sqrt{2nt}) / \sqrt{2nt} \right].$$

(6)

The moments (5) give us access to all characteristic quantities of the random directed polymer on short scales. In particular, the $n \to 0$ limit of (5) determines the free energy average $\langle F \rangle$ [6,13] and the probability function $P(u,L)$ for the displacement field $u$ [14]. The large $n$ limit determines the ‘tails’ ($F \to \pm \infty$) of the distribution function $P(F)$; the ‘left tails’ $P(F \to -\infty)$ for the random directed polymer problem have been studied on short [14,15] and long scales [16,17]. However, in order to determine the ‘body’ of the function $P(F)$ one has to calculate all the moments (5) (in the boson language the determination of the ‘tails’ and the ‘body’ of the distribution function $P(F)$ amount to calculating the ground state of the $n$-boson problem and the full spectrum, respectively). In the following, we present a brief derivation of these quantities as they follow from (6).

The $n \to 0$ limit of (6) determines the disorder averaged free energy [19] $\langle F \rangle = -T \langle \ln[Z/Z_0] \rangle = -T \lim_{n \to 0} [(Z/Z_0)^n - 1]/n = -\alpha L^2 / 12c$. Similarly, this limit determines the correlation functions for the displacement field $u$: These follow from the probability function $P(u,L) = \langle Z(u,L)/ \int_{-\infty}^{+\infty} du \, Z(u,L) \rangle = \sqrt{\gamma / \pi} \exp(-\gamma u^2)$, where
\[
\gamma^{-1} = 2(TL/\epsilon + \alpha L^3/3c^3).
\]
The mean squared displacement \(\langle u^2(L) \rangle = \langle (u(L) - u(0))^2 \rangle\) on scale \(L\) then is given by \(\langle u^2(L) \rangle = (T/c)L + (\alpha/3c^2)L^3\) and crosses over from diffusive thermal- to random wandering at a length scale
\(L \sim \sqrt{cT/\alpha}.)

We turn to the calculation of the probability distribution function \(\mathcal{P}(F)\). The moments \(\mathcal{P}(n)\), see (1), are nothing but the Mellin transform of \(\mathcal{P}(F)\) (2).

\[
\mathcal{P}(n) = \left\langle \frac{Z^n}{Z_{th}^n} \right\rangle = \int_{-\infty}^{+\infty} dF \mathcal{P}(F) \exp(-nF/T),
\]
and we have reduced the problem of calculating \(\mathcal{P}(F)\) to that of inverting the Mellin transform. The latter depends only on the combinations \(\lambda \equiv 2t^2 n = (\alpha L^2/cT) n\) and \(\beta \equiv x^2/2t^3 = \epsilon^2u^2/\alpha L^3\), i.e., we can rewrite (1) in the form \(F_0 \equiv 2tL^2 = \alpha L^2/\epsilon\)

\[
f(\beta, \lambda(n)) = \int_{-\infty}^{+\infty} dF \mathcal{P}(F) \exp(-\lambda F/F_0) \quad (= \mathcal{P}(n))
\]

\[= \exp \left[ \frac{\beta}{2} \lambda - \frac{1}{4} \sin^2 \left( \frac{\sqrt{\lambda}}{\lambda} \right) - \frac{\beta}{2} \lambda^2 \cot \left( \frac{\sqrt{\lambda}}{\lambda} \right) \right].
\]

The function \(f(\beta, \lambda)\) is analytic in the halfplane \(\text{Re}(\lambda) < \pi^2\) with \(f(\beta, \lambda \to \pi^2 - 0) \to \infty\). We then can invert the Mellin transform and write the function \(\mathcal{P}(F)\) as an integral of \(f\) over the imaginary axis, once we have found the proper analytical continuation of \(f(\beta, \lambda)\) (see below),

\[
\mathcal{P}(F) = \frac{1}{2\pi F_0} \int_{-\infty}^{+\infty} d\lambda f(\beta, \lambda) \exp(i\lambda F/F_0).
\]

Note that the condition for the existence of the inverse Mellin transform, \(f d\lambda |f(\beta, i\lambda)| < \infty\) is satisfied: for \(\beta = 0\) we have \(|f(0, i\lambda) \to \pm \infty\rangle \sim \exp(-\sqrt{\lambda}/2\lambda)|\) while for \(\beta \neq 0\), \(|f(\beta, i\lambda) \to \pm \infty\rangle \sim \exp(-\sqrt{\lambda}/2\lambda)|\) remains finite at finite \(|\lambda|\); we conclude that \(\mathcal{P}(F)\) decays faster than exponential. We can invert the Mellin transform asymptotically via the method of steepest descents and obtain

\[
\mathcal{P}(F \to \infty) \sim \exp(-16F^3/27\beta^2F_0^3) \quad (\beta \neq 0).
\]

The case \(\beta = 0\) has to be treated separately: With \(f(\beta = 0, \lambda \to \infty) \sim \exp(-\sqrt{\lambda}/2\lambda)|\), we find \(\mathcal{P}(F > 0) = 0\). Using the Ansatz \(\mathcal{P}(F) \propto \exp(-A/F\alpha)\) in combination with the method of steepest descents produces the result

\[
\mathcal{P}(F \to -0) \sim \exp(-F_0/16F) \quad (\beta = 0).
\]

**Body:** We have to find the proper analytic continuation of the Mellin transform (3) to imaginary \(e\lambda\) values (imaginary boson number), see (4). Inspiration how to carry out this analytic continuation can be obtained via the alternative route defining the distribution function in terms of a path integral over the stochastic field \(f\),

\[
\mathcal{P}(F) = \int D[f(z)] \delta(F - \mathcal{H}[u_0(f)]) + cu^2/2L,
\]

where \(u_0(z)\) is the solution of the equation \(\epsilon u_0''(z) = -f(z)\) with the appropriate boundary conditions \(u_0(0) = 0\) and \(u_0(L) = u\) and \(\mathcal{H}[u_0(z)]\) is the associated energy \((x, y)\) denotes the usual scalar product,

\[
\mathcal{H}[u_0(z)] = \frac{cu^2}{2L} + \frac{1}{2\epsilon}(f, 1/f) - \frac{u}{L}(z, f).
\]

We determine the spectrum of the Hermitian operator \(\hat{A}\)
\((A_n = -L^2/\pi^2n^2)\) and after integration over the stochastic field \(f\), we arrive at the distribution function in the form

\[
\mathcal{P}(F) = \int_{-\infty}^{+\infty} (d\mu/2\pi) \exp[i\mu F - g(\mu)],
\]

\[
g(\mu) = \sum_{n=1}^{\infty} \ln \left[ 1 - \frac{i\alpha L^2\mu}{\epsilon \pi^2 n^2} \right] + \frac{\alpha L^2\mu^2}{\pi^2} \left[ n^2 - \frac{i\alpha L^2\mu}{\epsilon \pi^2} \right]^{-1}
\]

(defining \(\mu = i\lambda/F_0\), a few algebraic manipulations transform this result back to (8). The above result allows for a straightforward inversion of the Mellin transform via numerical integration and thus to reconstruct the full distribution function \(\mathcal{P}(F)\), see Fig. 1. Furthermore, it implies that \(\mathcal{P}(F > 0) = 0\) for \(u = 0\): We integrate over the contour in the \(\mu\)-plane made from the real axis and the upper \((F > 0)\) semi-circle. The integral over the arc vanishes asymptotically. With all branching points situated in the lower half-plane, the contour integral vanishes and \(\mathcal{P}(F > 0) \equiv 0\). This property of \(\mathcal{P}(F)\) can be understood in terms of a minimization of the functional \(\mathcal{H}[u(z)]\): With the endpoint \((0, L)\), the test function \(u(z)\) satisfies the boundary conditions and we have \(\mathcal{H}[u_0(z)] \leq \mathcal{H}[u_0(z)] = 0\), hence \(\mathcal{P}(F > 0) = 0\). This analysis cannot be applied to other endpoints with \(u \neq 0\).

The distribution function \(\mathcal{P}(F)\) for the Larkin model, particularly its ‘tails’, have been studied before and a comparative discussion is instructive. Expanding the correlator \(\langle K(u) \rangle\) in a Taylor series, \(\langle (u, z)U(u', z') \rangle \approx [K(0) - \alpha(u - u')]d(z - z')\), the authors of Refs. (8) arrive at the asymptotic expression \((Z^n) \sim \exp[-C\sqrt{n(n - 1)L}]\), which cannot be identified with the Mellin transform of some distribution function \(\mathcal{P}(F)\), however. Keeping only the second term \(\exp(C\sqrt{n(n - 1)L})\), as suggested in Ref. (8) is questionable: though the left \(\sim \exp(-BL^2/|F|)\) reproduces the desired answers.
for the fluctuations of the free energy $\delta F \propto L^2$ and the line wandering $u \propto L^{3/2}$, this tail is unacceptable for a properly normalizable distribution function. The origin of the problem seems to lie in the correlator expansion itself: determining the exact Mellin transform using the expanded correlator we find the result

$$P_{\text{exp}}(n) = \exp \left[ \frac{n-1}{4} \ln \frac{2n^2}{\sinh^2 (\sqrt{2n})} + \frac{\hat{K}}{2} n^2 \right],$$

where $\hat{K} = K(0)L/T^2$. While the leading term has the correct sign, the fact that $P_{\text{exp}}(n = -\pi^2/2t) = 0$ implies that $P_{\text{exp}}(F)$ assumes negative values, in conflict with the required positivity of $P$. The findings of Parisi [11] show that the expansion provides correct results for the mean value of the free energy and the displacement correlation function, however, higher moments emphasizing larger distances $u - u'$, where the expanded correlator becomes negative, turn out wrong and so does the function $P(F)$. Carrying the expansion of the potential further, $U(u, z) \approx U(0, z) - f(z)u + g(z)u^2/2$, the calculation of the distribution function $P(F)$ can be done along the lines described above (see (13)) and we recover the result for the Larkin model for $\beta = 0$, see Fig. 1: as expected, the answer for the translation invariant random polymer problem [1] and [2] with a homogeneously correlated $K(u - u')$ is independent of the coordinate $u$ of the polymer’s endpoint.

Next, we address the large scale behavior of the random directed polymer, which can be solved via the Bethe Ansatz technique [14]. The ground state energy $E_0(n) \propto -n^2(n-1)$ produces the left tail $\ln P(F) \propto -|F|^{3/2}/L^{1/2}$ and provides the scaling $\delta F \propto L^{1/3}$ for the fluctuations in $F$ and the correct exponent $\zeta = 2/3$. Still, the generalization of this procedure seems not straightforward. E.g., for the $(1 + 2)$-problem the ground state energy increases faster than any power of $n$, implying that $\delta F$ behaves logarithmically and $\zeta = 1/2$, in contradiction with the expected superdiffusive ($\zeta > 1/2$) behavior of the $(1 + 2)$-dimensional string. It then seems important to know the complete Mellin transform $P(n)$ for an accurate calculation of the wandering exponent $\zeta$.

An interesting remark concerns the wandering exponent $\zeta = 3/2$ for the Larkin problem and its relation with the Kolmogorov scaling for the small-distance velocity fluctuations in the Burgers turbulence problem, $(\langle |v(u, L = \infty) - v(-u, L = \infty)|^2 \rangle \propto u^{2\zeta}$, with $\nu = 1/3$ (see also [1]). The consistency between the two exponents $\zeta$ and $\nu$ is easily checked: The fields in the two problems are related via $v = \partial_t F \sim F/u$, hence $F \propto u^{1+\nu}$. In the Larkin model $F \propto L^2 \propto u^{2/\zeta}$ and we obtain $2/\zeta = 1 + \nu$, which is indeed satisfied. However, this result does not carry a deep physical meaning but is simply a consequence of scaling. Physically, the two problems are very different: In the Larkin model we study the free energy $F$ both as a function of $u$ and (finite) $L$ with only one pinning valley relevant, whereas for the case of Burgers turbulence the stationary correlation functions $m_k(u) = \langle |v(u, L = \infty) - v(-u, L = \infty)|^k \rangle$ are studied, where many disorder-induced valleys are relevant (in the language of the directed polymer problem).

In the end, the calculation of the probability distribution $P[F(u, L)]$ turns out to be quite a non-trivial problem: a successful attack on the problem requires the precise knowledge of all the moments $P(n)$. Under these circumstances, we then can proceed with the analytical continuation to imaginary boson number and invert the Mellin transform to obtain $P(F)$. The successful application of this strategy to the Larkin model uncovers the limitations of previous attacks on the problem and may serve as a guideline for future studies.

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