RECENT DEVELOPMENTS ON ISING AND CHIRAL POTT'S MODEL

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After briefly reviewing selected Ising and chiral Potts model results, we discuss a number of properties of cyclic hypergeometric functions which appear naturally in the description of the integrable chiral Potts model and its three-dimensional generalizations.

1. Ising Model and Integrable Chiral Potts Model

1.1. Z-Invariant Ising Model

Baxter’s Z-invariant Ising model is the prototype integrable lattice model in statistical mechanics. It is “exactly solvable” for two reasons, namely because of a complete parametrization in terms of Yang–Baxter rapidities but also because of reformulations in terms of free fermions. This does not mean that the calculation of its pair-correlation or its susceptibility is a straightforward exercise. A more detailed description of the singularity structure of the zero-field susceptibility of the square-lattice Ising model has been obtained only recently.¹

Both integrability features were exploited in our recent studies of the pair-correlation function and the wavevector-dependent susceptibility of Ising models with quasiperiodic coupling constants²,³ and of the pentagrid Ising model⁴ of Korepin.

1.2. Integrable Chiral Potts Model

An N-state generalization of the Ising model with fermions replaced by cyclic parafermions is given by the integrable chiral Potts model⁵,⁶,⁷ One version of this model is given in terms of a square lattice of horizontal and vertical rapidity lines with rapidities q and p, respectively pointing left and
up. After black-and-white checkerboard coloring of the faces, Potts spins are placed on the black faces. Boltzmann weights \( W(a-b) \) and \( \overline{W}(a-b) \) are assigned to each nearest-neighbor pair of spins in states \( \omega^a \) and \( \omega^b \),

\[
\omega \equiv e^{2\pi i/N},
\]

\((a, b = 1, \ldots, N)\), as in Fig. 1. Here the difference \( a - b \) is to be taken mod \( N \). The Boltzmann weights \( W \) and \( \overline{W} \) can be parametrized as

\[
\frac{W_{pq}(n)}{W_{pq}(0)} = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - x_p \omega^j}{y_p - x_q \omega^j}, \quad \frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \left(\frac{\mu_p \mu_q}{\mu_p \mu_q}\right)^n \prod_{j=1}^{n} \frac{\omega x_p - x_q \omega^j}{y_q - y_p \omega^j}. \tag{2}
\]

The rapidities \( p \) and \( q \) lie on a higher genus curve with moduli \( k, k' \), with \( k^2 + k'^2 = 1 \). The \( p \)-curve is parametrized by \((x_p, y_p, \mu_p)\) satisfying the algebraic equations

\[
y_p^N = (1 - k' \lambda_p)/k, \quad x_p^N = (1 - k'/\lambda_p)/k, \quad \mu_p^N = \lambda_p, \tag{3}
\]

\[
\lambda_p + \lambda_p^{-1} = (1 + k'^2 - k^2 t_p^N)/k', \quad t_p = x_p y_p, \tag{4}
\]

which follow from the two mod \( N \) conditions \( W_{pq}(N) = W_{pq}(0) \) and \( \overline{W}_{pq}(N) = \overline{W}_{pq}(0) \). Given a value of \( t_p \) one can choose \( |\lambda_p| > 1 \) or \( |\lambda_p| < 1 \). Then \( x_p, y_p, \mu_p \) are given by (3) up to powers of \( \omega \).

### 1.3. Chiral Potts Free Energy and Order Parameters

Baxter has derived several exact results for the free energy of the chiral Potts model. Most of his work is based on a set of functional equations for the transfer matrices.\(^8\) An account with results for all four regimes, with each of \( |\lambda_p| \) and \( |\lambda_q| \) > 1 or < 1, can be found in Ref. 9. Baxter also...
obtained results for the interfacial tension, which can be much simplified in the symmetric case.\textsuperscript{10}

For the order parameters of the integrable chiral Potts model we have

\[ \langle \sigma_0^n \rangle = (1 - k^2)^{\beta_n}, \quad \beta_n = \frac{n(N - n)}{2N^2}, \quad (1 \leq n \leq N - 1, \quad \sigma_0^N = 1), \quad (5) \]

which was conjectured\textsuperscript{11} early in 1988 and proved only very recently by Baxter.\textsuperscript{12,13}

2. Cyclic Hypergeometric Functions

2.1. Basic Hypergeometric Series at Root of Unity

The basic hypergeometric hypergeometric series is defined as

\[ \Phi_p^{\alpha_1, \cdots, \alpha_{p+1}; \beta_1, \cdots, \beta_p; z} = \sum_{l=0}^{\infty} \frac{(\alpha_1; q)_l \cdots (\alpha_{p+1}; q)_l}{(\beta_1; q)_l \cdots (\beta_p; q)_l q^l} z^l, \quad (6) \]

where

\[ (x; q)_l \equiv \prod_{j=1}^{l} (1 - xq_j), \quad l \geq 0. \quad (7) \]

Setting first \( \alpha_{p+1} = q^{-N} \) and then \( q \to \omega = e^{2\pi i/N} \), we get

\[ \Phi_p^{\omega, \alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_p; z} = \sum_{l=0}^{N-1} \frac{(\alpha_1; \omega)_l \cdots (\alpha_p; \omega)_l}{(\beta_1; \omega)_l \cdots (\beta_p; \omega)_l} \omega^l. \quad (8) \]

We note

\[ (x; \omega)^l = (1 - x^N)(x; \omega)_l \quad \text{and} \quad (\omega; \omega)_l = 0, \quad l \geq N. \quad (9) \]

Requiring

\[ z^N = \prod_{j=1}^{p} \gamma_j^N, \quad \gamma_j^N = \frac{1 - \beta_j^N}{1 - \alpha_j^N}, \quad (10) \]

we obtain from (8) the “cyclic hypergeometric function” with summand periodic mod \( N \). Of special importance is the Saalschütz case, defined by

\[ z = q = \frac{\beta_1 \cdots \beta_p}{\alpha_1 \cdots \alpha_{p+1}} \quad \text{or} \quad \omega^2 \alpha_1 \alpha_2 \cdots \alpha_p = \beta_1 \beta_2 \cdots \beta_p, \quad \omega = \omega. \quad (11) \]

The theory of cyclic hypergeometric series is intimately related with the theory of the integrable chiral Potts model and its generalizations in three dimensions. We note that our notations differ from those of Bazhanov and Baxter\textsuperscript{14,15} and of others,\textsuperscript{16,17,18,19} who have an upside-down version of the \( q \)-Pochhammer symbol \( (x; q)_l \).
2.2. Integrable Chiral Potts Model Weights

The weights $W$ and $\overline{W}$ of the integrable chiral Potts model can be written in product form:

$$\frac{W(n)}{W(0)} = \gamma_n (\alpha; \omega)_n, \quad \frac{\gamma^N}{1 - \alpha^N}. \quad (12)$$

This is periodic with period $N$.

The dual weights are given by Fourier transform, i.e.

$$\hat{W}(k) = \sum_{n=0}^{N-1} \omega^{nk} W(n) = 2 \Phi_1 \left[ \frac{\omega, \alpha}{\beta, \omega} ; \gamma \omega^k \right] W(0). \quad (13)$$

They have the same structure as the original weights:

$$\frac{\hat{W}(n)}{W(0)} = \hat{\gamma}_n (\hat{\alpha}; \omega)_n, \quad \text{with} \quad \hat{\alpha} = \gamma, \quad \hat{\beta} = \frac{\omega \alpha \gamma}{\beta}, \quad \hat{\gamma} = \frac{\omega}{\beta}. \quad (14)$$

2.3. Summation Formula for $2\Phi_1$

The $2\Phi_1$ is exactly summable as a product. More precisely, we introduce the functions

$$\Delta(z) \equiv (1 - z^n)^{1/N}, \quad p(z) \equiv \prod_{j=1}^{N-1} (1 - \omega^j z)^{1/N}, \quad (15)$$

$$p_0(z) \equiv \frac{p(z)}{\Delta(z)^{(N-1)/2}} = \prod_{j=1}^{N-1} \left[ \frac{(1 - \omega^j z)}{\Delta(z)} \right]^{1/(N-1)}, \quad (16)$$

with all have cuts for $z^N \geq 1$ real, with the exception that $p(z)$ is regular on the positive real $z$-axis, where $p(1) = \sqrt{N} \Phi_0$, $\Phi_0 \equiv \omega^{(N-1)(N-2)/24}$.

With these definitions,

$$2\Phi_1 \left[ \frac{\omega, \alpha}{\beta}; \gamma \right] = F_\alpha \omega^{-\frac{1}{2}k(k+1)-mk} \frac{N}{\gamma^{\frac{1}{2}(N-1)}} \frac{p(\beta)p(\gamma)p(\varepsilon)}{p(\alpha)p(1)p(\delta)}. \quad (17)$$

where

$$m \equiv \left\lfloor \frac{N}{2\pi} \text{ arg } \alpha \right\rfloor, \quad n \equiv \left\lfloor \frac{N}{2\pi} \text{ arg } \beta \right\rfloor, \quad \gamma \equiv \omega^k \frac{\Delta(\beta)}{\Delta(\alpha)}, \quad \delta \equiv \frac{\beta}{\alpha}, \quad \varepsilon \equiv \frac{\beta}{\alpha \gamma}. \quad (18)$$

with $\lfloor x \rfloor$ the floor of $x$ and
The phase factor $F_*$ can take several values. If we keep $\alpha$ fixed and move $\beta$ in the complex plane, we encounter the cuts in Fig. 2. From a detailed analysis at each cut we find

$$ F_I = 1, \quad F_{II} = \omega^k, \quad F_{III} = \omega^{m-n+k}, \quad \text{if } \text{Im}\alpha_N > 0, $$

$$ F_I = 1, \quad F_{II}' = \omega^{-k}, \quad F_{III}' = \omega^{n-m-k}, \quad \text{if } \text{Im}\alpha_N < 0. $$

(20)

Noting

$$(z; \omega)_n \equiv \prod_{j=1}^{n} (1 - \omega^{-1} z), \quad z = 0, \ldots, N - 1. $$

(21)

we see that

$$ \frac{p(\omega^n z)}{p(z)} = \frac{p_0(\omega^n z)}{p_0(z)} = \frac{(z; \omega)_n}{\Delta(z)^n} \equiv \langle (z; \omega) \rangle_n, $$

(22)

which is a “cyclic Pochhammer symbol” $\langle (z; \omega) \rangle_{n+N} = \langle (z; \omega) \rangle_n$. On the principal sector $0 < \arg z < 2\pi/N$, we find

$$ p_0(z)p_0(\omega/z) = \omega^{(N^2-1)/12} = \Phi_0^2, \quad \frac{\Delta(\omega/z)}{\Delta(z)} = \frac{\omega^{n+\frac{1}{2}}}{z}. $$

(23)

2.4. $\mathbb{Z}_4$ Symmetry of $2\Phi_1$

The Fourier transform (14) defines a transformation $\mu$,

$$ \mu : \begin{cases} 
\alpha \rightarrow \gamma \rightarrow \omega \beta \rightarrow \frac{\beta}{\alpha\gamma} \rightarrow \alpha, \\
\beta \rightarrow \frac{\omega\alpha\gamma}{\beta} \rightarrow \omega \rightarrow \frac{\omega}{\gamma} \rightarrow \beta.
\end{cases} $$

(24)

From (13) we may infer

$$ \hat{W}(0) = 2\Phi_1 \left[ \frac{\omega, \alpha}{\beta ; \gamma} \right]. $$

(25)
Using this and applying Fourier transform $\mu$ four times, we find
\[
2 \Phi_1 \left[ \frac{\omega}{\beta}, \frac{\alpha}{\gamma} \right] = \frac{N}{2 \Phi_1 \left[ \frac{\omega}{\beta}, \frac{\alpha}{\gamma} \right]} = 2 \Phi_1 \left[ \frac{\omega}{\beta}, \frac{\alpha}{\gamma} \right],
\]
which is a $\mathbb{Z}_4$ symmetry.

2.5. The $3 \Phi_2$ identities

Using the convolution theorem, we find
\[
3 \Phi_2 \left[ \frac{\omega}{\beta_1}, \beta_1, \frac{\alpha_2}{\gamma_1} \right] = N^{-1} \sum_{k=0}^{N-1} 3 \Phi_2 \left[ \frac{\omega}{\beta_1}, \frac{\alpha_1}{\gamma_1} \right] 3 \Phi_1 \left[ \frac{\omega}{\beta_1}, \frac{\alpha_1}{\gamma_1} \right],
\]
where $\gamma_i = \Delta(\beta_i)/\Delta(\alpha_i)$, $i = 1, 2$. We can use the recurrence relation\textsuperscript{7, 20}
\[
W(n) = \frac{2 \Phi_1 \left[ \frac{\omega}{\beta_1}, \frac{\alpha_1}{\gamma_1} \right]}{2 \Phi_1 \left[ \frac{\omega}{\beta_1}, \frac{\alpha_1}{\gamma_1} \right] = \gamma_n (\beta_1, \omega)_n},
\]
which is a $\mathbb{Z}_4$ symmetry.

More generally, one can generate the symmetry relations of the cube in the Baxter–Bazhanov model under the 48 elements of the symmetry group of the cube, see also the work of Sergeev et al.\textsuperscript{18}

The group is generated by two generators. The first one is $t_\alpha$: $\alpha_1 \leftrightarrow \alpha_2$ resulting in
\[
3 \Phi_2 \left[ \frac{\omega}{\beta_1}, \beta_1, \frac{\alpha_2}{\gamma_1} \right] = 3 \Phi_2 \left[ \frac{\omega}{\beta_1}, \beta_1, \frac{\alpha_2}{\gamma_1} \right] = 3 \Phi_2 \left[ \frac{\omega}{\beta_1}, \beta_1, \frac{\alpha_2}{\gamma_1} \right],
\]
with
\[
A \equiv N^{-1} 2 \Phi_1 \left[ \frac{\omega}{\beta_1}, \frac{\alpha_1}{\gamma_1} \right] 2 \Phi_1 \left[ \frac{\omega}{\beta_1}, \frac{\alpha_1}{\gamma_1} \right].
\]
with \( \gamma_3 = \Delta(\beta_1)/\Delta(\alpha_2) \) and \( \gamma_4 = \Delta(\beta_2)/\Delta(\alpha_1) \), so that \( \gamma_3 \gamma_4 = \gamma_1 \gamma_2 \). The second generator is \( M = \mu^{-1} \otimes \mu \), which results in

\[
3\Phi_2 \left[ \omega, \alpha_1, \alpha_2 ; \beta_1, \beta_2 ; \gamma_1 \gamma_2 \right] = \frac{N \Phi_2 \left[ \omega, \tilde{\alpha}_2, \tilde{\alpha}_1 ; \tilde{\beta}_1, \tilde{\beta}_2 ; \tilde{\gamma}_1 \tilde{\gamma}_2 \right]}{\Phi_1 \left[ \omega, \tilde{\alpha}_1 ; \tilde{\gamma}_1 \right] \Phi_1 \left[ \omega, \tilde{\alpha}_2 ; \tilde{\gamma}_2 \right]}, \tag{32}
\]

with

\[
\tilde{\alpha}_1 = \frac{\beta_1}{\alpha_1 \gamma_1}, \quad \tilde{\beta}_1 = \frac{\omega}{\gamma_1}, \quad \tilde{\gamma}_1 = \alpha_1, \\
\tilde{\alpha}_2 = \gamma_2, \quad \tilde{\beta}_2 = \frac{\omega \alpha_2 \gamma_2}{\beta_2}, \quad \tilde{\gamma}_2 = \frac{\omega}{\beta_2}. \tag{33}
\]

This is the inverse of (29). We can use (17) to evaluate the \( 2\Phi_1 \)'s, but this will lead to a phase factor depending on the positions of the \( \alpha \)'s and \( \beta \)'s with respect to the cuts defined by Fig 2. Eqs. (29), (31) and (33) are valid in general, independent of choices of Riemann sheets or branch cuts.

### 2.6. Connection with Sergeev, Mangazeev and Stroganov

In several of the Russian works\(^{16,17,18,19} \) one uses points, \( p, p', \text{ etc.} \), from the Fermat curve \( \Gamma \) in homogeneous notation, i.e.,

\[
p \in \Gamma \iff p = (x, y, z) \quad \text{with} \quad x^N + y^N = z^N. \tag{34}
\]

In our affine notation, \( p \leftrightarrow \alpha, p' \leftrightarrow \beta, \text{ etc.} \), we would identify

\[
\alpha \equiv \frac{\omega x}{z}, \quad \Delta(\alpha) \equiv \frac{y}{z} = (1 - \alpha^N)^{1/N}. \tag{35}
\]

The assignment of Riemann sheets and branch cuts is more subtle in their homogeneous notation. They deal with that by breaking up the curve \( \Gamma \) in parts \( \Gamma^m \),

\[
p \in \Gamma_{l,m} \equiv \Gamma^m_{l} \iff \alpha \equiv \frac{\omega^{m+1} x}{z}, \quad \Delta(\alpha) \equiv \frac{\omega^{-l} y}{z},
\]

\[
-\frac{\pi}{N} + l \frac{2\pi}{N} < \arg \frac{y}{z} < + \frac{\pi}{N} + l \frac{2\pi}{N}, \quad 0 < \arg \frac{x}{z} < \frac{2\pi}{N}, \tag{36}
\]

and by using the notation \( (p, m) \) for points in \( \Gamma^m_0 \). How their notations translate into ours is also indicated in (36).

The \( \omega \)-Pochhammer symbol \( (x; \omega)_l \) is defined upside-down and is not even unique in the various Russian papers. It is to be translated as

\[
w(x, y, z|l) \equiv \prod_{s=1}^{l} \frac{y}{z - x \omega^s} = \left( \frac{y}{z} \right)^l \frac{1}{(\omega x/z; \omega)_l} \tag{37}
\]
in Ref. 16. However, for the work of Sergeev et al.\(^{18}\) one must identify

\[
w(p|m + \sigma) = \frac{1}{p_0(\omega^\sigma \alpha)}, \quad w(p|0) = \frac{1}{p_0(\omega^\sigma \beta)}, \quad (38)
\]

\[
\alpha = \omega^{m+1} \frac{x'}{z'}, \quad \Delta(\alpha) = y', \quad \beta = \omega^{m+1} \frac{x}{z}, \quad \Delta(\beta) = \frac{y}{z}, \quad (39)
\]

with \(p_0(z)\) defined in (16), as they normalize \(\prod w(x|l) = 1\), not \(w(p|0) = 1\).

Therefore, for the appendix of Ref. 18 we have to make the translation

\[
r \Psi_r \left( \begin{array}{c} (p_1, m_1), \ldots, (p_r, m_r) \\ (p'_1, m'_1), \ldots, (p'_r, m'_r) \end{array} \right) \propto C r_{+1} \Phi_r \left[ \begin{array}{c} \omega, \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_r \end{array} ; z \right], \quad (40)
\]

\[
C \equiv \frac{1}{\sqrt{N}} \frac{p_0(\alpha_1) \cdots p_0(\alpha_r)}{p_0(\beta_1) \cdots p_0(\beta_r)}, \quad z \equiv \omega^n \frac{\Delta(\beta_1) \cdots \Delta(\beta_r)}{\Delta(\alpha_1) \cdots \Delta(\alpha_r)}. \quad (41)
\]

### 2.7. Other Identities for Cyclic Hypergeometric Functions

One can derive many other identities for the cyclic hypergeometric function (8), (10). Without giving explicit expressions, we list some of the types of identities in Table 1.

| Conditions | \(p_{+1} \Phi_p = \prod / \prod\) | \(p_{+1} \Phi_p \propto p_{+1} \Phi_p\) |
|------------|-----------------|-----------------|
| None       | 2 \(\Phi_1\)     | 3 \(\Phi_2\)     |
| \(z = \omega\) | 3 \(\Phi_2\)     | 4 \(\Phi_3\)     |
| Saalschütz| 4 \(\Phi_3\)     | 5 \(\Phi_4\)     |

One type of identity is the evaluation of \(p_{+1} \Phi_p\) in terms of a ratio of products. This is shown in the middle column of Table 1. Another type of identity is the proportionality of two \(p_{+1} \Phi_p\)'s where the proportionality factor can be expressed in terms of \(2 \Phi_1\)'s or, equivalently, products. This is shown in the last column of Table 1. The conditions under which such identities can be found are listed in the first column.

The two cases where there are no further conditions have been discussed in previous subsections. Other cases requiring the conditions \(z = \omega\) and the more restrictive Saalschütz condition (11) have also been discussed in Ref. 21. The star–triangle equation of the integrable chiral Potts model is a special case of the Saalschützian \(4 \Phi_3\) identities.\(^{20,21}\)
It must be noted that identities of all six types in Table 1 have been derived by Sergeev, Mangazeev and Stroganov in the appendix of Ref. 18. However, one needs the translation (40) to see the connections with more standard basic hypergeometric notations and with the Saalschütz condition.

Many other identities can be derived. For example, Watson’s analogue of Whipple’s theorem for $8\Phi_7$ reduces to $7\Phi_6 \propto 4\Phi_3$. Moreover, new identities can be found in the $N \to \infty$ limit.\textsuperscript{22}

3. Final Remarks

We have presented several results on the deep connection of the integrable chiral Potts model with the theory of cyclic hypergeometric functions. Eq. (17) with $F_*$ as specified in Sec. 2.3 is new and is easier to use than a formulation with multiple Riemann sheets, especially when doing numerical computations with it. Finally, translation (40) is also new and may make the results of Sergeev et al.\textsuperscript{18} more accessible to a wider audience familiar with basic hypergeometric series.

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