New inequalities of Steffensen’s type for s-convex functions

Mohammad W. Alomari

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Abstract In this work, new inequalities connected with the Steffensen’s integral inequality for s-convex functions are proved.

Keywords Steffensen’s inequality · Hayashi’s inequality · s-Convex function

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1 Introduction

In order to study certain inequalities between mean values, Steffensen [11] has proved the following inequality (see also [9, p. 311]):

**Theorem 1** Let \( f \) and \( g \) be two integrable functions defined on \((a, b)\), \( f \) is decreasing and for each \( t \in (a, b) \), \( 0 \leq g(t) \leq 1 \). Then, the following inequality

\[
\left\{ \begin{array}{c}
\int_{b-\lambda}^{b} f(t) \, dt \\
\int_{a}^{b} f(t) g(t) \, dt \leq \\
\int_{a}^{a+\lambda} f(t) \, dt
\end{array} \right. \quad (1.1)
\]

holds, where \( \lambda = \int_{a}^{b} g(t) \, dt \).

Some minor generalization of Steffensen’s inequality (1.1) was considered by Hayashi [5], using the substituting \( g(t)/A \) for \( g(t) \), where \( A \) is positive constant. For other result involving Steffensen’s type inequality, see [3,5,8–11].

In the recent work [1], Alomari et al. proved the following result:

**Theorem 2** Let \( f, g : [a, b] \to \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \), for all \( t \in [a, b] \) such that \( \int_{a}^{b} g(t) \, dt \) exists. If \( f \) is absolutely continuous on \([a, b]\) with \( f' \in L_p[a, b] \), \( 1 \leq p \leq \infty \), then we have

M. W. Alomari (✉)
Department of Mathematics, Faculty of Science, Jerash University, 26150 Jerash, Jordan
e-mail: mwomat@gmail.com
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inequalities of Hermite–Hadamard type see [2] and [7].

Let

Lemma 1 Let us start with the following lemma due to Mitrinović et al. [9]:

2

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functions in the second sense. 123

A function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), where \( \mathbb{R}^+ = [0, \infty) \), is said to be \( s \)-convex in the second sense if

\[
\alpha x + \beta y \leq \alpha^s f(x) + \beta^s f(y)
\]

for all \( x, y \in [0, \infty) \), \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \) and for some fixed \( s \in (0, 1] \). This class of \( s \)-convex functions is usually denoted by \( K_f^2 \), (see [6]). It can be easily seen that for \( s = 1 \), \( s \)-convexity reduces to the ordinary convexity of functions defined on \([0, \infty)\).

In [4], Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for \( s \)-convex functions in the second sense:

\[
2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s+1}.
\]

2 The results

Let us start with the following lemma due to Mitrinović et al. [9]:

Lemma 1 Let \( f, g : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \), for all \( t \in [a, b] \) and \( \int_a^b g(t) \, dt \) exists. Then we have the following representation

\[
\begin{align*}
\left[\int_a^b f(t) \, dt - \int_a^b f(t) g(t) \, dt\right]^{\frac{a+\lambda}{2}} &
\leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s+1}.
\end{align*}
\]

The constant \( k = \frac{1}{s+1} \) is the best possible in the second inequality in (1.4). For another inequalities of Hermite–Hadamard type see [2] and [7].

The aim of this paper is to establish new inequalities of Steffensen’s type for \( s \)-convex functions in the second sense.
and

\[ \int_{a}^{b} f(t) g(t) \, dt - \int_{b-\lambda}^{b} f(t) \, dt = - \int_{a}^{b} \left( \int_{a}^{x} g(t) \, dt \right) f'(x) \, dx - \int_{b-\lambda}^{b} \left( \int_{x}^{b} (1 - g(x)) \, dt \right) f'(x) \, dx, \quad (2.2) \]

where \( \lambda := \int_{a}^{b} g(t) \, dt. \)

**Proof**

Integrating by parts

\[- \int_{a}^{a+\lambda} \left( \int_{a}^{x} (1 - g(t)) \, dt \right) f'(x) \, dx - \int_{a+\lambda}^{b} \left( \int_{x}^{b} g(t) \, dt \right) f'(x) \, dx \]

\[= - \left( \int_{a}^{a+\lambda} (1 - g(t)) \, dt \right) f(a+\lambda) + \int_{a}^{a+\lambda} f(x) \left( \int_{x}^{b} (1 - g(t)) \, dt \right) \]

\[+ \int_{a+\lambda}^{b} g(t) \, dt \, f(a+\lambda) + \int_{a+\lambda}^{b} f(x) \left( \int_{a}^{x} g(t) \, dt \right) \]

\[= - \left( \int_{a}^{a+\lambda} (1 - g(t)) \, dt \right) f(a+\lambda) + \int_{a}^{a+\lambda} f(x) \, dx \]

\[= - \lambda f(a+\lambda) + f(a+\lambda) \int_{a}^{a+\lambda} g(t) \, dt + \int_{a}^{a+\lambda} f(x) \, dx \]

\[= - \lambda f(a+\lambda) + f(a+\lambda) \int_{a}^{a+\lambda} g(t) \, dt + f(a+\lambda) \int_{a}^{a+\lambda} f(x) \, dx \]

\[= - \lambda f(a+\lambda) + f(a+\lambda) \int_{a}^{a+\lambda} g(t) \, dt + f(a+\lambda) \int_{a+\lambda}^{b} g(t) \, dt \]

\[+ \int_{a}^{a+\lambda} f(x) \, dx - \int_{a}^{a+\lambda} g(x) \, dx \]

\[= \int_{a}^{a+\lambda} f(x) \, dx - \int_{a}^{b} f(x) g(x) \, dx, \]

which gives the desired representation (2.1). The identity (2.2) can be also proved in a similar way, we shall omit the details. \(\square\)
2.1 Inequalities involving $s$-convexity

In the following, inequalities for absolutely continuous functions whose first derivatives are $s$-convex ($s$-concave) are given:

**Theorem 3** Let $f, g : [a, b] \subset \mathbb{R}^+ \to \mathbb{R}$ be integrable such that $0 \leq g(t) \leq 1$, for all $t \in [a, b]$ such that $\int_a^b g(t) f'(t) \, dt$ exists. If $f$ is absolutely continuous on $[a, b]$ such that $|f'|$ is $s$-convex on $[a, b]$, for some fixed $s \in (0, 1]$ then we have

$$\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^{b} f(t) \, g(t) \, dt \right| \leq \frac{1}{(s + 1)(s + 2)} \left[ \lambda^2 \left| f'(a) \right| + (b - a - \lambda)^2 \left| f'(b) \right| \right]$$

and

$$\left| \int_a^{b} f(t) \, g(t) \, dt - \int_a^{b} f(t) \, dt \right| \leq \frac{1}{(s + 1)(s + 2)} \left[ \lambda^2 \left| f'(b) \right| + (b - a - \lambda)^2 \left| f'(a) \right| \right]$$

where $\lambda := \int_a^b g(t) \, dt$.

**Proof** Utilizing the triangle inequality on (2.1), and since $|f'|$ is $s$-convex, we have

$$\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^{b} f(t) \, g(t) \, dt \right|$$

$$\leq \left| \int_a^{a+\lambda} f(t) \, dt \right| + \left| \int_a^{b} f(t) \, g(t) \, dt \right|$$

$$\leq \int_a^{a+\lambda} f(t) \, dt + \int_a^{b} f(t) \, g(t) \, dt$$

$$\leq \int_a^{a+\lambda} f(t) \, dt + \int_a^{b} f(t) \, g(t) \, dt$$

$$\leq \int_a^{a+\lambda} f(t) \, dt + \int_a^{b} f(t) \, g(t) \, dt$$

$$\leq \int_a^{a+\lambda} f(t) \, dt + \int_a^{b} f(t) \, g(t) \, dt$$

$$\leq \frac{f'(a+\lambda)}{\lambda^s} \int_a^b \left( \int_a^x |1 - g(t)| \, dt \right) (x - a)^s \, dx$$

$$+ \frac{f'(a)}{\lambda^s} \int_a^b \left( \int_a^x |1 - g(t)| \, dt \right) (a + \lambda - x)^s \, dx$$
New inequalities of Steffensen’s type

\[ + \frac{|f'(b)|}{(b-a-\lambda)^s} \int_{a+\lambda}^{b} \left( \int_{x}^{b} |g(t)| \, dt \right) (x-a-\lambda)^s \, dx \]

\[ + \frac{|f'(a+\lambda)|}{(b-a-\lambda)^s} \int_{a+\lambda}^{b} \left( \int_{x}^{b} |g(t)| \, dt \right) (b-x)^s \, dx \]

\[ \leq \frac{|f'(a+\lambda)|}{\lambda^s} \int_{a}^{a+\lambda} (x-a)^{s+1} \, dx + \frac{|f'(a)|}{\lambda^s} \int_{a}^{a+\lambda} (x-a) (a+\lambda-x)^s \, dx \]

\[ + \frac{|f'(b)|}{(b-a-\lambda)^s} \int_{a+\lambda}^{b} (b-x) (x-a-\lambda)^s \, dx + \frac{|f'(a+\lambda)|}{(b-a-\lambda)^s} \int_{a+\lambda}^{b} (b-x)^{s+1} \, dx \]

\[ = \frac{1}{(s+1)(s+2)} \left[ \lambda^2 |f'(a)| + (b-a-\lambda)^2 |f'(b)| \right] \]

\[ + \frac{1}{s+2} \left[ \lambda^2 + (b-a-\lambda)^2 \right] |f'(a+\lambda)| \]

which proves the first inequality in (2.3). In similar way and using (2.2) we may deduce the desired inequality (2.4), and we shall omit the details. \( \square \)

**Corollary 1** In (2.3) if one chooses \( s = 1 \) then

\[ \left| \int_{a}^{b} f(t) \, dt - \int_{a}^{a+\lambda} f(t) \, dt \right| \]

\[ \leq \frac{1}{6} \lambda^2 |f'(a)| + \frac{1}{3} [\lambda^2 + (b-a-\lambda)^2] |f'(a+\lambda)| + \frac{1}{6} (b-a-\lambda)^2 |f'(b)| \]

(2.5)

also, in (2.4) if \( s = 1 \), then

\[ \left| \int_{a}^{b} f(t) g(t) \, dt - \int_{b-\lambda}^{b} f(t) \, dt \right| \]

\[ \leq \frac{1}{6} \lambda^2 |f'(b)| + \frac{1}{3} [\lambda^2 + (b-a-\lambda)^2] |f'(b-\lambda)| + \frac{1}{6} (b-a-\lambda)^2 |f'(a)| \]

(2.6)

**Remark 1** In the inequalities (2.3) and (2.4), choose \( \lambda = 0 \), then we have

\[ \left| \int_{a}^{b} f(t) g(t) \, dt \right| \]

\[ \leq \frac{(b-a)^2}{(s+1)(s+2)} \min \left\{ (s+1) |f'(a)| + |f'(b)|, |f'(a)| + (s+1) |f'(b)| \right\} . \]

(2.7)
Another approach leads to the following result:

**Theorem 4** Let \( f, g : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \), for all \( t \in [a, b] \) such that \( \int_a^b g(t) f'(t) \, dt \) exists. If \( f \) is absolutely continuous on \([a, b]\) with \( |f'| \) is \( s \)-convex on \([a, b]\), for some fixed \( s \in (0, 1) \) then we have

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right| \\
\leq \frac{1}{s + 1} \left[ \int_a^b g(t) \, dt \right] \cdot \left[ \lambda \int f'(a) \, dt + (b-a) \int f'(a+\lambda) \, dt + (b-a-\lambda) \int f'(b) \, dt \right]
\]

and

\[
\left| \int_a^b f(t) g(t) \, dt - \int_a^{b-\lambda} f(t) \, dt \right| \\
\leq \frac{1}{s + 1} \left[ \int_{b-\lambda}^b g(t) \, dt \right] \cdot \left[ (b-a-\lambda) \int f'(a) \, dt + (b-a) \int f'(b-\lambda) \, dt + \lambda \int f'(b) \, dt \right]
\]

where \( \lambda := \int_a^b g(t) \, dt \).

**Proof** From Lemma 1, we may write

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right| \\
\leq \sup_{x \in [a, a+\lambda]} \left[ \int_a^x (1-g(t)) \, dt \right] \cdot \int_a^{a+\lambda} f'(x) \, dx + \sup_{x \in [a+\lambda, b]} \left[ \int_a^x g(t) \, dt \right] \cdot \int_{a+\lambda}^b f'(x) \, dx.
\]

Since \( |f'| \) is \( s \)-convex on \([a, b]\), then by (1.4) we have

\[
\int_a^{a+\lambda} f'(x) \, dx \leq \lambda \cdot \frac{|f'(a)| + |f'(a+\lambda)|}{s + 1},
\]

and

\[
\int_{a+\lambda}^b f'(x) \, dx \leq (b-a-\lambda) \cdot \frac{|f'(a+\lambda)| + |f'(b)|}{s + 1}.
\]
Therefore, we have

$$\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right|$$

$$\leq \lambda \cdot \frac{|f'(a)| + |f'(a + \lambda)|}{s + 1} \cdot \int_a^{a+\lambda} (1 - g(t)) \, dt$$

$$+ (b - a - \lambda) \cdot \frac{|f'(a + \lambda)| + |f'(b)|}{s + 1} \cdot \int_a^b g(t) \, dt$$

$$\leq \max \left\{ \int_a^{a+\lambda} (1 - g(t)) \, dt, \int_a^b g(t) \, dt \right\} \cdot \left[ \lambda \cdot \frac{|f'(a)| + |f'(a + \lambda)|}{s + 1} \right.$$

$$\left. + (b - a - \lambda) \cdot \frac{|f'(a + \lambda)| + |f'(b)|}{s + 1} \right]$$

$$= \frac{1}{s + 1} \left[ \int_a^b g(t) \, dt \right] \cdot \left[ \lambda \cdot |f'(a)| + (b - a) \cdot |f'(a + \lambda)| + (b - a - \lambda) \cdot |f'(b)| \right],$$

which proves the first inequality in (2.8). The second inequality in (2.8) follows directly, since $0 \leq g(t) \leq 1$ for all $t \in [a, b]$, then

$$0 \leq \int_a^b g(t) \, dt \leq (b - a - \lambda).$$

The inequalities in (2.9) may be proved in the same way using the identity (2.2), we shall omit the details. \hfill \Box

2.2 Inequalities involving $s$-concavity

**Theorem 5** Let $f, g : [a, b] \subset \mathbb{R}^+ \to \mathbb{R}$ be integrable such that $0 \leq g(t) \leq 1$, for all $t \in [a, b]$ such that $\int_a^b g(t) \, f'(t) \, dt$ exists. If $f$ is absolutely continuous on $[a, b]$ with $|f'|$ is $s$-concave on $[a, b]$, for some fixed $s \in (0, 1]$ then we have

$$\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right|$$

$$\leq 2^{s-1} \left[ \int_a^b g(t) \, dt \right] \cdot \left[ \lambda \cdot |f'(a + \lambda)| + (b - a - \lambda) \cdot \left| f'\left(\frac{a + b + \lambda}{2}\right)\right| \right]$$

$$\leq 2^{s-1} (b - a - \lambda) \cdot \left[ \lambda \cdot \left| f'(a + \lambda) \right| + (b - a - \lambda) \cdot \left| f'\left(\frac{a + b + \lambda}{2}\right)\right| \right]$$

(2.10)
and

\[
\left| \int_{a}^{b} f(t) g(t) \, dt - \int_{a}^{b} f(t) \, dt \right| \\
\leq 2^{s-1} \left[ \int_{b}^{b} g(t) \, dt \right] \cdot \left( b - a - \lambda \right) \left| f' \left( \frac{a + b - \lambda}{2} \right) \right| + \lambda \left| f' \left( b - \frac{\lambda}{2} \right) \right| \\
\leq \lambda 2^{s-1} \left[ (b - a - \lambda) \left| f' \left( \frac{a + b - \lambda}{2} \right) \right| + \lambda \left| f' \left( b - \frac{\lambda}{2} \right) \right| \right] \tag{2.11}
\]

where \( \lambda := \int_{a}^{b} g(t) \, dt \).

\textbf{Proof} Utilizing the triangle inequality on (2.1), and since \( |f'| \) is s-concave on \([a, b]\) then by (1.4) we may state

\[
\left| \int_{a}^{a+\lambda} f(t) \, dt - \int_{a}^{b} f(t) g(t) \, dt \right| \\
\leq \sup_{x \in [a, a+\lambda]} \left[ \int_{a}^{x} (1 - g(t)) \, dt \right] \cdot \left| f' \left( \frac{a + \lambda}{2} \right) \right| \cdot \left( b - a - \lambda \right) \left| f' \left( \frac{a + b + \lambda}{2} \right) \right| \\
\leq 2^{s-1} \lambda \left| f' \left( \frac{a + \lambda}{2} \right) \right| \cdot \left[ \left( b - a - \lambda \right) \left| f' \left( \frac{a + b + \lambda}{2} \right) \right| \right.
\]

which proves the first inequality in (2.10). The second inequality in (2.10) follows directly, since \( 0 \leq g(t) \leq 1 \) for all \( t \in [a, b] \), then

\[
0 \leq \int_{a+\lambda}^{b} g(t) \, dt \leq (b - a - \lambda).
\]

The inequalities in (2.11) may be proved in the same way using the identity (2.2), we shall omit the details. \( \square \)

Another result is incorporated in the following theorem:

\textbf{Theorem 6} Let \( f, g : [a, b] \subset \mathbb{R}^+ \to \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \), for all \( t \in [a, b] \) such that \( \int_{a}^{b} g(t) f'(t) \, dt \) exists. If \( f \) is absolutely continuous on \([a, b]\) with \( |f'|^q \) is s-concave on \([a, b]\), for some fixed \( s \in (0, 1] \) and \( q > 1 \), then we have

\[
\left| \int_{a}^{a+\lambda} f(t) \, dt - \int_{a}^{b} f(t) g(t) \, dt \right| \\
\leq \frac{2^{(s-1)/q}}{(p + 1)^{1/p}} \left[ \lambda^2 \left| f' \left( \frac{a + \lambda}{2} \right) \right| + (b - a - \lambda)^2 \left| f' \left( \frac{a + b + \lambda}{2} \right) \right| \right], \tag{2.12}
\]

\( \square \) Springer
and

\[ \left| \int_{a}^{b} f(t) \, g(t) \, dt - \int_{b-\lambda}^{b} f(t) \, dt \right| \leq \frac{2^{(s-1)/q}}{(p+1)^{1/p}} \left[ (b-a)^2 \left| f'(b - \frac{\lambda}{2}) \right| + \lambda^2 \left| f'(\frac{a+b-\lambda}{2}) \right| \right], \quad (2.13) \]

where \( \lambda := \int_{a}^{b} g(t) \, dt \).

**Proof** From Lemma 1 and using the Hölder inequality for \( q > 1 \), and \( p = \frac{q}{q-1} \), we obtain

\[
\left| \int_{a}^{a+\lambda} f(t) \, dt - \int_{a}^{b} f(t) \, g(t) \, dt \right| \\
\leq \int_{a}^{a+\lambda} \left| \int_{a}^{x} (1 - g(t)) \, dt \right| \left| f'(x) \right| \, dx + \int_{a}^{b} \left| \int_{x}^{b} g(t) \, dt \right| \left| f'(x) \right| \, dx \\
\leq \left( \int_{a}^{a+\lambda} \left| \int_{a}^{x} (1 - g(t)) \, dt \right|^{p} \, dx \right)^{1/p} \left( \int_{a}^{a+\lambda} \left| f'(x) \right|^{q} \, dx \right)^{1/q} \\
+ \left( \int_{a}^{b} \left| \int_{x}^{b} g(t) \, dt \right|^{p} \, dx \right)^{1/p} \left( \int_{a+\lambda}^{b} \left| f'(x) \right|^{q} \, dx \right)^{1/q} := M, \quad (2.14)
\]

where \( p \) is the conjugate of \( q \).

By the inequality (1.4), we have

\[ \int_{a}^{a+\lambda} \left| f'(x) \right|^{q} \, dx \leq 2^{s-1} \lambda \left| f'(a + \frac{1}{2} \lambda) \right|^{q}, \]

and

\[ \int_{a+\lambda}^{b} \left| f'(x) \right|^{q} \, dx \leq 2^{s-1} (b - a - \lambda) \left| f'(\frac{a+b+\lambda}{2}) \right|^{q}, \]

which gives by (2.14)

\[ M \leq 2^{(s-1)/q} \lambda^{1/q} \left| f'(a + \frac{\lambda}{2}) \right| \left( \int_{a}^{a+\lambda} (x-a)^p \, dx \right)^{1/p} \]

\[ + 2^{(s-1)/q} (b - a - \lambda)^{1/q} \left| f'(\frac{a+b+\lambda}{2}) \right| \left( \int_{a+\lambda}^{b} (b-x)^p \, dx \right)^{1/p} \]
\[
\begin{align*}
\frac{2^{(s-1)/q}}{(p + 1)^{1/p}} \left[ \lambda^{1/p + 1/q} \left| f'(a + \frac{\lambda}{2}) \right| + (b - a - \lambda)^{1/p + 1/q} \left| f'(\frac{a + b + \lambda}{2}) \right| \right] \\
= \frac{2^{(s-1)/q}}{(p + 1)^{1/p}} \left[ \lambda^2 \left| f'(a + \frac{\lambda}{2}) \right| + (b - a - \lambda)^2 \left| f'(\frac{a + b + \lambda}{2}) \right| \right],
\end{align*}
\]
giving the inequality (2.12).

The inequality (2.13) may be proved in the same way using the identity (2.2), we shall omit the details. \(\square\)

**Remark 2** The interested reader may obtain several inequalities for log-convex, quasi-convex, \(r\)-convex and \(h\)-convex functions by replacing the condition on \( |f'| \).

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