ON STABILITY OF TANGENT BUNDLE OF TORIC VARIETIES

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Abstract. Let $X$ be a nonsingular complex projective toric variety. We address the question of semi-stability as well as stability for the tangent bundle $TX$. In particular, a complete answer is given when $X$ is a Fano toric variety of dimension four with Picard number at most two, complementing earlier work of Nakagawa. We also give an infinite set of examples of Fano toric varieties for which $TX$ is unstable; the dimensions of this collection of varieties are unbounded. Our method is based on the equivariant approach initiated by Klyachko and developed further by Perling and Kool.

1. Introduction

Let $X$ be a smooth complex projective variety. If the canonical line bundle $K_X$ is ample, then from a theorem of Yau, [23], and Aubin, [2], it follows that the tangent bundle $TX$ is semistable (in the sense of Mumford and Takemoto) with respect to the polarization $K_X$. The variety $X$ is said to be Fano if the anti-canonical line bundle $K_X^{-1}$ is ample. Fano varieties are very basic objects in birational classification of complex algebraic varieties (minimal model program), for example a theorem of Birkar-Cascini-Hacon-McKernan [3] says that every uniruled variety is birational to a variety which has a fibration with a Fano general fiber. Stability of the tangent bundle of a nonsingular Fano variety with respect to polarization $K_X^{-1}$ is a question of interest originated mostly from a differential geometric point of view. The existence of Einstein-Kähler metric on $X$ implies the polystability of the tangent bundle with respect to $K_X^{-1}$ [14,17]. In general converse of this result is not true. The simplest example is the surface $\Sigma_2$ obtained by blowing up the complex plane $\mathbb{P}^2$ at two points. In this paper we are interested in studying semi-stability as well as stability of the tangent bundle $TX$ when $X$ is a toric variety and in particular when $X$ is a Fano toric variety.

Let $X$ be a nonsingular complex projective toric variety of dimension $n$, equipped with an action of the $n$–dimensional complex torus $T$. A coherent torsion-free sheaf $\mathcal{E}$ on $X$ is said to be $T$–equivariant (or $T$–linearized) if it admits a lift of the $T$–action on $X$, which is linear on the stalks of $\mathcal{E}$. Fix a polarization $H$ of $X$, where $H$ is a $T$–equivariant very ample line bundle (equivalently, $T$–invariant very ample divisor) of $X$.

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A $T$–equivariant coherent torsion-free sheaf $\mathcal{E}$ on $X$ is said to be equivariantly stable (respectively, equivariantly semistable) if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$) for every proper $T$–equivariant proper subsheaf $\mathcal{F} \subset \mathcal{E}$ (see Section 2). From the uniqueness of the Harder-Narasimhan filtration it follows easily that the notions of semi-stability and equivariant semi-stability of an equivariant torsion-free sheaf on a nonsingular toric variety are equivalent. Further, in case of equivariant torsion-free sheaves, the notions of equivariant stability and stability coincide (see Theorem 2.1 or [16, Proposition 4.13]). Using this equivariant approach, we investigate the stability and semi-stability of the tangent bundle of a nonsingular toric variety.

Our main results are as follows.

1. Determination of the stability (or otherwise) of the tangent bundle of Hirzebruch surfaces for an arbitrary polarization; see Theorem 6.2 and Corollary 6.3.

2. A very simple proof of the well-known result that $T\mathbb{P}^n$ is stable with respect to the anti-canonical polarization (Theorem 7.1).

3. We identify all nonsingular Fano toric 4-folds with Picard number at most two that have semi-stable tangent bundle (Theorem 9.3). In particular, we get an example of a Fano toric 4-fold (namely, $B_3$) which has a strictly semi-stable tangent bundle, but does not admit Einstein-Kähler metric (cf. [19]).

4. Construction of an infinite family of Fano toric varieties with unstable tangent bundle, consisting of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(m))$ for all $n \geq 2$ and $m \leq n$ (Theorem 8.1). The case $n = 2$ was settled earlier by Steffens in [22].

The general strategy here is as follows. We use the isotypical decomposition of an equivariant sheaf to describe it in terms of certain combinatorial data, following Perling [21] and Kool [15]. (Of course, both draw inspiration from the seminal work of Klyachko [13].) We prove a formula in Lemma 4.3 that calculates the rank of an equivariant torsion–free coherent sheaf on $X$ from the combinatorial data. With a little bit of fine-tuning, this specializes to a very useful rank formula, see (5.6), for an arbitrary equivariant coherent subsheaf of the tangent bundle. We obtain a similar type of formula for the degree of such a subsheaf, see (5.9), using a formula of Kool for the first Chern class. Using these formulas, we can identify the combinatorial data that may be associated to a subsheaf of a given rank whose slope exceeds that of the tangent bundle; see Lemma 6.1 and Lemma 9.1. We then examine if a subsheaf with the given rank and corresponding to such combinatorial data really exists by studying the transition maps associated to the combinatorial data.

2. FROM EQUIVARIANT STABILITY TO STABILITY

Given a coherent torsion-free sheaf $\mathcal{E}$ on a projective variety $X$ of dimension $n$, the slope $\mu(\mathcal{E})$ with respect to a polarization $H$ on $X$ is defined as the ratio

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\text{rank} \mathcal{E}},$$

where the degree of $\mathcal{E}$ is defined as the intersection product $\deg \mathcal{E} := c_1(\mathcal{E}) \cdot H^{n-1}$. A subsheaf $\mathcal{F}$ of $\mathcal{E}$ is said to be a proper subsheaf if $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$. A
torsion-free sheaf \( \mathcal{E} \) is said to be \( \mu \)-stable (respectively, \( \mu \)-semistable) if \( \mu(\mathcal{F}) < \mu(\mathcal{E}) \) (respectively, \( \mu(\mathcal{F}) \leq \mu(\mathcal{E}) \)) for every proper subsheaf \( \mathcal{F} \subset \mathcal{E} \). The notion of \( \mu \)-stability (semistability) was first introduced by Mumford and Takemoto. In this article, (semi)stability we will always mean \( \mu \)-(semi)stability, unless otherwise specified. Also, a sheaf \( \mathcal{E} \) will be called unstable if \( \mathcal{E} \) is not semistable.

Stable and semistable sheaves play an important role in the structure theory of coherent sheaves (cf. [10]). Every torsion-free coherent sheaf admits the Harder-Narasimhan filtration such that each successive quotient is semistable. A semistable sheaf, in turn, admits a Jordan-Holder filtration such that each successive quotient is stable of same slope.

In this section we give a proof of the crucial fact that for an equivariant torsion-free sheaf on a nonsingular toric variety, equivariant stability is equivalent to usual stability. This result is also proved by Kool [16, Proposition 4.13] for reflexive sheaves. The proof given here is different.

**Theorem 2.1.** Let \( \mathcal{E} \) be an equivariant torsion-free sheaf on a projective toric variety \( X \). Then \( \mathcal{E} \) is equivariantly stable if and only if \( \mathcal{E} \) is stable.

**Proof.** If \( \mathcal{E} \) is stable, then it is evidently equivariantly stable. We will prove that \( \mathcal{E} \) is stable if it is equivariantly stable.

We first note that it is enough to prove this under the extra assumption that \( \mathcal{E} \) is reflexive. Indeed, if \( \mathcal{E} \) is torsion-free and equivariantly stable, then \( \mathcal{E}^{\vee \vee} \) is reflexive and equivariantly stable. On the other hand, if \( \mathcal{E}^{\vee \vee} \) is stable, then clearly \( \mathcal{E} \) is also stable. Indeed, for any coherent subsheaf \( \mathcal{F} \subset \mathcal{E}^{\vee \vee} \), we have \( \deg \mathcal{F} = \deg(\mathcal{F} \cap \mathcal{E}) \).

So assume that \( \mathcal{E} \) is reflexive and equivariantly stable. We will first show that \( \mathcal{E} \) is semistable.

To prove semistability by contradiction, assume that \( \mathcal{E} \) is not semistable. Then there is a unique maximal destabilizing semistable subsheaf

\[
\mathcal{F} \subset \mathcal{E}.
\]

In other words, \( \mathcal{F} \) is the smallest nonzero subsheaf of \( \mathcal{E} \) in the Harder-Narasimhan filtration of \( \mathcal{E} \).

As before, \( T \subset \text{Aut}(X) \) is the torus acting on \( X \). Let

\[
\Phi : T \times \mathcal{E} \longrightarrow \mathcal{E}
\]

be an action of the torus \( T \) on \( \mathcal{E} \) lifting the action of \( T \) on \( X \). For any element \( t \in T \), the homomorphism

\[
\mathcal{E} \longrightarrow \mathcal{E}, \quad v \longmapsto \Phi(t, v)
\]

will be denoted by \( \Phi_t \). Note that \( \Phi_t \) is an automorphism of \( \mathcal{E} \) over the automorphism of \( X \) given by the action of \( t \) on it. For any \( t \in T \), and any coherent subsheaf \( \mathcal{V} \subset \mathcal{E} \), the coherent subsheaf \( \Phi_t(\mathcal{V}) \subset \mathcal{E} \) will be denoted by \( t \cdot \mathcal{V} \). The above automorphism \( \Phi_t \) produces an isomorphism

\[
(t^{-1})^* \mathcal{V} \xrightarrow{\sim} t \cdot \mathcal{V}
\]

over the identity map of \( X \).
Since $\mu(t^*V) = \mu(V)$ for any coherent sheaf $V$ on $X$, from (2.2) it follows that $\mu(t \cdot V) = \mu(V)$ for every coherent subsheaf $V \subset \mathcal{E}$. This and (2.2) together imply that the subsheaf $\mathcal{F}$ in (2.1) has the property that $t \cdot \mathcal{F}$ is also a maximal destabilizing subsheaf of $\mathcal{E}$. Therefore, from the uniqueness of the maximal destabilizing subsheaf it is deduced that

$$t \cdot \mathcal{F} = \mathcal{F} \subset \mathcal{E}.$$ 

Consequently, the action of $T$ on $\mathcal{E}$ preserves the subsheaf $\mathcal{F}$. Therefore, $\mathcal{F}$ is equivariant. This contradicts the given condition that $\mathcal{E}$ is equivariantly semistable. Hence we conclude that $\mathcal{E}$ is semistable.

Let $\mathcal{H}$ be the socle of $\mathcal{E}$; in other words, $\mathcal{H}$ is the maximal polystable subsheaf of $\mathcal{E}$ with the same slope as $\mathcal{E}$ [7, Proposition 3.1], [10, p. 23, Lemma 1.5.5]. From the uniqueness of $\mathcal{H}$, it follows that $t \cdot \mathcal{H} = \mathcal{H}$ for every $t \in T$. Therefore, $\mathcal{H}$ is a $T$–equivariant subsheaf of $\mathcal{E}$. Since $\mathcal{E}$ is equivariantly stable, and the slopes of $\mathcal{H}$ and $\mathcal{E}$ coincide, we must have $\mathcal{H} = \mathcal{E}$. This implies that $\mathcal{E}$ is polystable.

Since $\mathcal{E}$ is polystable, it suffices to show that $\mathcal{E}$ is indecomposable. Note that $\mathcal{E}$ is indecomposable if the dimension of a maximal torus in the algebraic group $\text{Aut}(\mathcal{E})$ is one [1, p. 201, Proposition 16]; the automorphism group $\text{Aut}(\mathcal{E})$ is a Zariski open subset of the affine space $H^0(X, \text{End}(\mathcal{E}))$.

The action of $T$ on $\mathcal{E}$ produces an action of $T$ on the group $\text{Aut}(\mathcal{E})$:

$$(t \cdot A)(v) = t \cdot A(t^{-1} \cdot v), \quad \forall \ A \in \text{Aut}(\mathcal{E}), \ v \in \mathcal{E},$$

for every $t \in T$. We will show that there is a maximal torus $\widetilde{T} \subset \text{Aut}(\mathcal{E})$ on which $T$ acts trivially. For this, first consider the semi-direct product $\text{Aut}(\mathcal{E}) \rtimes T$ for this action of $T$ on $\text{Aut}(\mathcal{E})$. Let $\widetilde{T}' \subset \text{Aut}(\mathcal{E}) \rtimes T$ be a maximal torus containing the subgroup $T$ of $\text{Aut}(\mathcal{E}) \rtimes T$. Then

$$\widetilde{T} := \widetilde{T}' \cap \text{Aut}(\mathcal{E}) \subset \text{Aut}(\mathcal{E})$$

is a maximal torus of $\text{Aut}(\mathcal{E})$ on which $T$ acts trivially. Now $\widetilde{T}$ produces an eigenspace decomposition of $\mathcal{E}$ for the characters of $\widetilde{T}$

$$\mathcal{E} = \bigoplus_{\chi \in \widetilde{T}^*} \mathcal{E}_\chi;$$

any $t \in \widetilde{T}$ acts on $\mathcal{E}_\chi$ as multiplication by $\chi(t)$. The direct summands in this decomposition are preserved by the action of $T$ on $\mathcal{E}$, because $T$ acts trivially on $\widetilde{T}$ (see [6] p. 55, Proposition 1.2) for a general result). But $\mathcal{E}$ is equivariantly stable, so it does not admit any nontrivial $T$–equivariant decomposition. This implies that $\dim \widetilde{T} = 1$, because that action of $\widetilde{T}$ on $\mathcal{E}$ is faithful. As noted before, this implies that $\mathcal{E}$ is indecomposable. \hfill \Box

3. Equivariant coherent sheaves on $X$

We briefly review the classification of equivariant coherent sheaves on a nonsingular toric variety following Perling [21]. The notation established in this section will be used extensively in the rest of the paper.
Let \( X \) be a nonsingular complex projective toric variety of dimension \( n \), equipped with the action of an \( n \)-dimensional torus \( T \). Let \( M \) and \( N \) denote the group of characters of \( T \) and the group of one-parameter subgroups of \( T \) respectively. Then both \( M \) and \( N \) are free \( \mathbb{Z} \)-modules of rank \( n \) that are naturally dual to one another. Let \( \Delta \) denote the fan of \( X \). It is a collection of rational cones in the real vector space \( N \otimes \mathbb{R} \) closed under the operations of taking faces, and performing intersections. Denote the set of \( d \)-dimensional cones of the fan \( \Delta \) of \( X \) by \( \Delta(d) \). For any one-dimensional cone (ray) \( \alpha \in \Delta(1) \), its primitive co-character generator is also denoted by \( \alpha \); this is for notational convenience. We refer the reader to \cite{9,20} for details on toric varieties.

Let \( E \) be a \( T \)-equivariant coherent sheaf over \( X \) of rank \( r \). Let \( X_\sigma \) be any affine toric subvariety of \( X \) corresponding to a cone \( \sigma \). Denote by \( S_\sigma \) the semigroup \( \{ m \in M \mid \langle m, \alpha \rangle \geq 0 \ \forall \ \alpha \in \sigma \} \subseteq M \).

Let \( k[S_\sigma] \) be the finitely generated semigroup algebra which is the coordinate ring of \( X_\sigma \). Let \( E_\sigma \) denote the \( k[S_\sigma] \)-module \( \Gamma(X_\sigma, E) \) consisting of sections of \( E \) over \( X_\sigma \).

Consider the isotypical decomposition

\[
E_\sigma = \bigoplus_{m \in M} E_\sigma^m. \tag{3.1}
\]

For any \( m' \in M \) with \( \chi(m') \in k[S_\sigma] \), there is a natural multiplication map

\[
\chi(m') : E_\sigma^m \rightarrow E_\sigma^{m+m'}. \tag{3.2}
\]

This homomorphism is injective if \( E \) is torsion-free.

Let

\[
S_{\sigma^\perp} = \{ m \in M \mid \langle m, \alpha \rangle = 0 \ \forall \ \alpha \in \sigma \}. \tag{3.3}
\]

Define

\[
M_\sigma = M/S_{\sigma^\perp}. \tag{3.4}
\]

Denote by \([ m ]\) the equivalence class of \( m \) in \( M_\sigma \). Note that \( M_\sigma \) may be identified with the character group of an appropriate subtorus \( T_\sigma \) of \( T \), namely the maximal subtorus of \( T \) that has a fixed point in \( X_\sigma \). Let \( A_\sigma \) be the subvariety of \( X_\sigma \) defined by the ideal generated by the set \( \{ \chi(m) - 1 \mid m \in S_{\sigma^\perp} \} \). Then \( A_\sigma \) has a dense \( T_\sigma \)-orbit. So elements of \( M_\sigma \) generate the field of rational functions on \( A_\sigma \).

If \( m' \in S_{\sigma^\perp} \), then \( \chi(m') \) in \( (3.2) \) is an isomorphism. Denote the isomorphism class of \( E_{\sigma m'} \), for \( m' \in m + S_{\sigma^\perp} \), by \( E_{[m]}^\sigma \). The space \( E_{[m]}^\sigma \) may be identified with the space of sections of the \( T_\sigma \)-equivariant bundle \( E|_{A_\sigma} \) of weight \([ m ]\in M_\sigma \). We have, in fact, an isotypical decomposition

\[
\Gamma(A_\sigma, E) = \bigoplus_{[m] \in M_\sigma} E_{[m]}^\sigma. \tag{3.5}
\]

Moreover, for \([ m ]\in M_\sigma \) and any \( m' \in M \) such that \( \chi(m') \in k[S_\sigma] \), the map \( \chi(m') \) in \( (3.2) \) induces a map

\[
\chi^\sigma([m']) : E_{[m]}^\sigma \rightarrow E_{[m+m']}^\sigma. \tag{3.6}
\]

Here, we may naturally identify \( \chi^\sigma([m']) \) with a character of \( T_\sigma \).
Definition 3.1. Define an equivalence relation $\leq_\sigma$ on $M$ by setting $m \leq_\sigma m'$ if and only if $m' - m \in S_\sigma$. This yields a directed pre-order on $M$ which is a partial order when $\sigma$ is of top dimension.

If $m \leq_\sigma m'$, but $m' \not\leq_\sigma m$ does not hold, we say that $m <_\sigma m'$.

Definition 3.2. Let $\{E_m^\sigma \mid m \in M\}$ be a family of $k$–vector spaces. For each relation $m \leq_\sigma m'$, let there be given a $k$–linear map

$$\chi^\sigma(m, m') : E_m^\sigma \to E_m'^\sigma,$$

such that $\chi^\sigma(m, m) = 1$ for all $m \in M$, and also

$$\chi^\sigma(m, m'') = \chi^\sigma(m', m'') \circ \chi^\sigma(m, m')$$

for all triples $m \leq_\sigma m' \leq_\sigma m''$. We refer to the $\chi^\sigma(m, m')$’s as multiplication maps.

Denote such data by $\hat{E}\sigma$ and call it a $\sigma$–family. A morphism $\phi^{\sigma} : \hat{E}\sigma \to \hat{E}\sigma'$ of $\sigma$–families is given by a collection of linear maps $\{\phi_m^\sigma : E_m^\sigma \to E_m'^\sigma \mid m \in M\}$ respecting the multiplication maps.

Definition 3.3. A $\sigma$–family $\hat{E}\sigma$ is called finite if

1. all the $k$–vector spaces $E_m^\sigma$ are finite dimensional,
2. for each chain $\ldots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \ldots$ of elements of $M$, there exists an $i_0 \in \mathbb{Z}$ such that $E_{m_i}^\sigma = 0$ for all $i < i_0$, and
3. there are only finitely many vector spaces $E_m^\sigma$ such that the map

$$\bigoplus_{m' <_\sigma m} E_{m'}^\sigma \to E_m^\sigma$$

defined by the summation of $\chi^\sigma(m', m)$’s is not surjective.

Theorem 3.4 ([21]). The category of $T$–equivariant coherent sheaves on $X_\sigma$ is equivalent to the category of finite $\sigma$–families $\{\hat{E}\sigma\}$.

Let $\tau \leq_\sigma \sigma$ be a subcone. Let $i_{\tau,\sigma} : X_\tau \to X_\sigma$ be the corresponding inclusion map. Define

$$i_{\tau,\sigma}^\ast(E^\sigma) = E^\sigma \otimes_{k[S_\tau]} k[S_\tau].$$

Note that $i_{\tau,\sigma}^\ast(E^\sigma)$ has a natural $M$–grading.

Definition 3.5. Let $\Delta$ be a fan. A collection $\{\hat{E}\sigma \mid \sigma \in \Delta\}$ of finite $\sigma$–families is called a finite $\Delta$–family, denoted $\hat{E}\Delta$, if for every pair $\tau < \sigma$, there is an isomorphism $\eta_{\tau,\sigma} : i_{\tau,\sigma}^\ast \hat{E}\sigma \to \hat{E}\tau$, such that for every triple $\rho < \tau < \sigma$, the following holds:

$$\eta_{\rho,\sigma} = \eta_{\rho,\tau} \circ i_{\rho,\tau}^\ast \eta_{\tau,\sigma}.$$ 

A morphism of finite $\Delta$–families is a collection of morphisms

$$\{\phi^\sigma : \hat{E}\sigma \to \hat{E}\sigma' \mid \sigma \in \Delta\}$$
of finite $\sigma$–families such that for all $\tau < \sigma$ the following diagram commutes:

$$
\begin{array}{ccc}
\hat{E}_\tau^\sigma & \xrightarrow{i_\tau^*(\phi^\sigma)} & \hat{E}_\tau^\sigma \\
\eta_{\tau\sigma} \downarrow & & \eta_{\tau\sigma} \downarrow \\
\hat{E}_\tau & \xrightarrow{\phi^\sigma} & \hat{E}_\tau
\end{array}
$$

Since $S_{\sigma^\perp} \subseteq S_{\tau^\perp}$, there is a surjective group homomorphism

$$
M/S_{\sigma^\perp} \longrightarrow M/S_{\tau^\perp}.
$$

Then $\eta_{\tau\sigma}$ induces an isomorphism $\eta_{\tau\sigma} : (i_\tau^*(E_\sigma^m)) \longrightarrow E_\tau^m$.

**Theorem 3.6** ([21]). The category of finite $\Delta$–families is equivalent to the category of coherent $T$–equivariant sheaves over $X$.

For a $T$–equivariant subsheaf $F$ of $E$, one has $F_\sigma^m \subseteq E_\sigma^m$ for every $\sigma \in \Delta$ and $m \in M$.

**4. Rank of an equivariant torsion–free coherent sheaf**

In this section, we derive a formula for the rank of an equivariant torsion–free coherent sheaf on $X$.

**Definition 4.1.** For an equivariant torsion–free coherent sheaf $F$ and an $n$–dimensional cone $\sigma$, define

$$
Gen(\hat{F}_\sigma) = \left\{ m' \in M \mid \dim F_\sigma^m < \dim F_\sigma^{m'} \quad \forall \ m <_\sigma m' \right\}.
$$

Since $F$ is a coherent sheaf, it follows that $Gen(\hat{F}_\sigma)$ is finite for every $\sigma$. Note that the finite collection of graded vector spaces

$$
\left\{ F_\sigma^m \mid m \in Gen(\hat{F}_\sigma), \ \sigma \in \Delta(n) \right\},
$$

and the isomorphisms $\eta_{\tau\sigma}$ of the previous section, together determine the $\Delta$–family $\hat{F}_\Delta$.

A coherent sheaf is locally free on some open subset and its rank equals the rank of its restriction to such an open subset. By equivariance, the coherent sheaf $F$ must be locally free on the dense torus orbit $X_{\{0\}}$, where $\{0\}$ denotes the trivial cone. By localizing to the dense torus orbit, we find that $\rank F = \dim F_\sigma^{m(0)}$ for all $m \in M$. Now it is straight-forward to check that

$$
\rank F = \dim F_\sigma^m, \text{ where } m' <_\sigma m \text{ for all } m' \in Gen(\hat{F}_\sigma), \quad (4.1)
$$

for any $\sigma \in \Delta(n)$.

Let $\alpha$ be any one dimensional cone. Note that the spaces $F_\alpha^m$ and $F_\alpha^{m'}$ are isomorphic if $m - m' \in S_{\alpha^\perp}$, or in other words if $\langle \alpha, m \rangle = \langle \alpha, m' \rangle$.

**Definition 4.2.** Let $F$ be a $T$–equivariant torsion–free coherent sheaf on $X$. For a subcone $\alpha \in \Delta(1)$ and $\lambda \in \mathbb{Z}$, define

$$
d(F, \alpha, \lambda) = \dim F_\alpha^m, \quad \text{where} \quad \lambda = \langle \alpha, m \rangle.
$$
Define
\[ e(\mathcal{F}, \alpha, \lambda) = d(\mathcal{F}, \alpha, \lambda) - d(\mathcal{F}, \alpha, \lambda - 1). \]

We remark that \( d(\mathcal{F}, \alpha, \lambda) = \dim F^\alpha_{[m]} \), where \([m]\) denotes the equivalence class of \( m \) in \( M_\alpha \).

**Lemma 4.3.** For an equivariant torsion–free coherent sheaf \( \mathcal{F} \) on \( X \), the equality
\[ \text{rank}(\mathcal{F}) = \sum_{\lambda \in \mathbb{Z}} e(\mathcal{F}, \alpha, \lambda) \]
holds for all \( \alpha \in \Delta(1) \).

**Proof.** Fix \( \alpha \in \Delta(1) \). We may identify \( M_\alpha \) with \( \mathbb{Z} \) via the association \( [m] \mapsto \langle \alpha, m \rangle \). Note that \( F^\alpha_j \) has a natural inclusion in \( F^\alpha_{j+1} \) under the multiplication by the character \( \chi^\alpha(1) \) of the torus \( T_\alpha \); see (3.6).

By the finiteness of \( \text{Gen}(\hat{\mathcal{F}}^\sigma) \) for all \( \sigma \), the set \( S := \{ \lambda \in \mathbb{Z} | e(\mathcal{F}, \alpha, \lambda) \neq 0 \} \) is finite. Suppose \( \lambda_1 < \cdots < \lambda_m \) are the elements of \( S \). By (4.1), we have
\[ \text{rank} \mathcal{F} = \dim(F^\alpha_j), \quad \text{for } j \geq \lambda_m. \] (4.2)

Recall the affine subvariety \( A_\alpha \) of \( X_\alpha \) defined by the ideal generated by \( \{ \chi^\alpha(p) - 1 | m \in S_\alpha^+ \} \). This is, in fact, a nonsingular affine curve. Therefore, as \( \mathcal{F} \) is torsion–free, \( \Gamma(A_\alpha, \mathcal{F}) \) is a free \( k[A_\alpha] \)-module. Let \( \{ f_i \in F^\alpha_{\lambda_i} | 1 \leq i \leq m \} \) be any collection such that \( f_i \) is not in the image of \( F^\alpha_{\lambda_{i-1}} \) under the multiplication by \( \chi^\alpha(1) \). To prove the lemma, in view (4.2) of it is enough to show that \( f_1, \cdots, f_m \) are \( k[A_\alpha] \)-linearly independent.

Assume the contrary, namely, \( f_1, \cdots, f_m \) are not \( k[A_\alpha] \)-linearly independent. Note that \( k[A_\alpha] \) is generated as a \( k \)-vector space by \( \{ \chi^\alpha(p) | p \in \mathbb{Z}_{\geq 0} \} \). Therefore, we have
\[ \sum_{i=1}^{m} c_i \chi^\alpha(d_i) f_i = 0 \] (4.3)
for some nontrivial \( c_i \in k \), and some nonnegative integers \( d_i \). We may assume without loss of generality that \( c_m \neq 0 \). Moreover, by considering the direct sum decomposition (3.3), we may assume without loss of generality that each summand \( \chi^\alpha(d_i) f_i \) belongs to the same graded component \( F^\alpha_d \), where \( d \geq m \). Then, dividing (4.3) by \( c_m \chi^\alpha(d - m) \), we obtain that \( f_m \) belongs to the image of \( F^\alpha_{\lambda_m - 1} \) which is a contradiction. This concludes the proof. \( \square \)

**5. Equivariant coherent subsheaves of \( TX \)**

Let \( \sigma \) be an \( n \)-dimensional cone in \( \Delta \). Let \( \alpha_1^\sigma, \cdots, \alpha_n^\sigma \) be the primitive integral generators of the one-dimensional faces of \( \sigma \). Since \( X_\sigma \) is nonsingular, the vectors \( \alpha_1^\sigma, \cdots, \alpha_n^\sigma \) form a basis of the \( \mathbb{Z} \)-module \( N \). Let
\[ \sigma^\vee = \{ m \in M \otimes \mathbb{R} | \langle m, v \rangle \geq 0 \ \forall \ v \in \sigma \} \]
be the dual cone of $\sigma$. Define $m_i^\sigma \in \sigma^\vee \cap M$ by

$$\langle m_i^\sigma, \alpha_j \rangle = \delta_{ij},$$

where $\delta_{ij}$ denotes the Kronecker delta. Note that $m_1^\sigma, \ldots, m_n^\sigma$ form a $\mathbb{Z}$–basis of $M$.

Set $E = TX$. Then $E^\sigma$ is a free $\mathcal{O}_{X,\sigma}$–module of rank $n$, with generators having $T$–weights $-m_i^\sigma, 1 \leq i \leq n$. To be precise, let

$$z_i = \chi(m_i^\sigma) \quad \text{and} \quad \partial_{z_i} = \partial_{\partial z_i} \quad \text{for } 1 \leq i \leq n.$$ 

Then $E^\sigma$ is freely generated over $\mathcal{O}_{X,\sigma}$ by $\{\partial_{z_1}, \ldots, \partial_{z_n}\}$. In the convention followed by Perling [21], $T$ acts on $\chi(m)$ with weight $m$. Hence, the section $\partial_{z_i}$ has $T$–weight $-m_i^\sigma$. Note that $\dim E^\sigma_m = 1$ for $1 \leq i \leq n$. Consequently, the $\sigma$–family $\hat{E}^\sigma$ has the following properties:

(a) $\dim E^\sigma_m = |\{m_i^\sigma \mid -m_i^\sigma \leq \sigma m\}|$.

(b) For every $m \leq_{\sigma} m'$, the multiplication map $\chi^\sigma(m, m')$ is injective.

For a subcone $\tau < \sigma$, the $\tau$–family $\hat{E}^\tau$ satisfies the following:

(a) $\dim E^\tau_m = |\{m_i^\tau \mid -m_i^\tau \leq \tau m\}|$.

(b) For every $m \leq_{\tau} m'$, the multiplication map $\chi^\tau(m, m')$ is injective.

By (5.2),

$$e(TX, \alpha, \lambda) = \begin{cases} 1 & \text{if } \lambda = -1, \\ n - 1 & \text{if } \lambda = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for every $\alpha \in \Delta(1)$.

Let $\mathcal{F}$ be any equivariant coherent subsheaf of $TX$. Then, $\mathcal{F}$ is torsion–free. Therefore, if $e(\mathcal{F}, \alpha, \lambda) \neq 0$, then $d(\mathcal{F}, \alpha, \lambda) \neq 0$, and consequently $E^\alpha_m \neq 0$, where $\langle \alpha, m \rangle = \lambda$ and $E^\alpha = \Gamma(TX_\alpha)$. As a result, we have

$$e(\mathcal{F}, \alpha, \lambda) \neq 0 \implies \lambda \geq -1. \quad (5.5)$$

Hence, for an equivariant coherent subsheaf $\mathcal{F}$ of $TX$, the rank formula of Lemma 4.3 takes the form,

$$\text{rank}(\mathcal{F}) = \sum_{\lambda \in \mathbb{Z}_{\geq -1}} e(\mathcal{F}, \alpha, \lambda) \quad (5.6)$$

where $\alpha$ is an arbitrary element of $\Delta(1)$.

Let $D_\alpha$ denote the torus invariant Weil divisor of $X$ corresponding to $\alpha \in [\Delta(1)]$.

**Theorem 5.1** ([15 Corollary 1.2.18]). The first Chern class of $\mathcal{F}$ has the expression

$$c_1(\mathcal{F}) = - \sum_{\alpha \in \Delta(1), \lambda \in \mathbb{Z}} \lambda e(\mathcal{F}, \alpha, \lambda) D_\alpha.$$ 

Let $H = \sum a_\alpha D_\alpha$ be a polarization of $X$; in other words, $H$ is an ample Cartier divisor on $X$. Let $P$ be the polytope of $X$ in $M \otimes \mathbb{R}$ associated to $H$. The convex polytope $P$ basically encodes the linear system of $H$. It has a facet $P_\alpha$ corresponding
to each $\alpha$. The facet $P_\alpha$ lies in the hyperplane $\{v \in M \otimes \mathbb{R} \mid \langle v, \alpha \rangle = -a_\alpha\}$. Now $P$ is easily determined from these supporting hyperplanes, and has the explicit formula

$$P = \{v \in M \otimes \mathbb{R} \mid \langle v, \alpha \rangle \geq -a_\alpha \ \forall \ \alpha \in [\Delta(1)]\}.$$  

By [9, Corollary on p. 112], the intersection product

$$D_\alpha \cdot H^{n-1} = (n-1)! \text{Vol}(P_\alpha),$$  

(5.7)

where the volume is measured with respect to the lattice $S_\alpha^-$ (see (3.3)).

Using Theorem 5.1, (5.7), (5.4) and (5.5), we now have

$$\text{deg} \ TX = (n-1)! \sum_{\alpha \in \Delta(1)} \text{Vol}(P_\alpha),$$  

(5.8)

and

$$\text{deg} \ F = -(n-1)! \sum_{\alpha \in \Delta(1), \lambda \in \mathbb{Z}_{\geq -1}} \lambda e(F, \alpha, \lambda) \text{Vol}(P_\alpha).$$  

(5.9)

Lemma 5.2. An upper bound for the slope of an equivariant coherent subsheaf $F$ of $TX$ of rank $r$ is given by

$$\mu(F) \leq \frac{(n-1)!}{r} \sum_{\alpha \in \Delta(1)} \text{Vol}(P_\alpha).$$

Proof. Fix an $\alpha \in \Delta(1)$ and an $n$-dimensional cone $\sigma$ containing $\alpha$. Consider the summand

$$-(n-1)! \text{Vol}(P_\alpha) \sum_{\lambda \in \mathbb{Z}_{\geq -1}} \lambda e(F, \alpha, \lambda)$$

of (5.9) corresponding to $\alpha$. As the sum is decreasing in the $\lambda$’s, the optimal choice would be $\lambda = -1$. However if $e(F, \alpha, -1) \neq 0$, then there is an element of the form $g(z)\partial z_\alpha$ in $F^\sigma$ with $g(z) \in k[S_\alpha]$. Moreover all such elements are multiples of one another when restricted to the dense torus $X_{(0)}$. Therefore, we have $e(F, \alpha, -1) \leq 1$. Hence,

$$-(n-1)! \text{Vol}(P_\alpha) \sum_{\lambda \in \mathbb{Z}_{\geq -1}} \lambda e(F, \alpha, \lambda) \leq (n-1)! \text{Vol}(P_\alpha).$$

This completes the proof of the lemma.  

The above upper bound is not very sharp. We will use finer estimates in what follows.

Lemma 5.3. Suppose $F, G$ are equivariant coherent subsheaves of $TX$ having the same rank. If $F$ is a proper subsheaf of $G$, then $\mu(F) < \mu(G)$.

Proof. For subsheaves $F \subseteq G$ of $TX$, we have $F^\sigma_m \subseteq G^\sigma_m$. If, in addition, rank($F$) = rank($G$), then Lemma 4.3 implies that generators of $F$ are associated with bigger values of $\lambda$. Therefore, we have

$$\sum_{\alpha \in \Delta(1), \lambda \in \mathbb{Z}} \lambda e(G, \alpha, \lambda) \text{Vol}(P_\alpha) \leq \sum_{\rho \in \Delta(1), \lambda \in \mathbb{Z}} \lambda e(F, \alpha, \lambda) \text{Vol}(P_\alpha).$$
Thus, when \( F \subseteq G \subseteq TX \), and \( \text{rank}(F) = \text{rank}(G) \), by \((5.9)\), we have
\[
\deg F \leq \deg G.
\] (5.10)

Now, suppose \( F \) is a proper subsheaf of \( G \). Then there exists an \( n \)-dimensional cone \( \sigma \) and an \( m \in \text{Gen}(\hat{G}^\sigma) \) such that \( F_m^\sigma \subseteq G_m^\sigma \). Then there exist at least one \( \alpha_i \in \sigma \cap \Delta(1) \) such that \( e(F, \alpha_i, \langle \alpha_i, m \rangle) < e(G, \alpha_i, \langle \alpha_i, m \rangle) \). The lemma follows. \( \square \)

6. Hirzebruch surfaces

In this section we study semi-stability of tangent bundle for smooth projective toric surface. Following lemma is very crucial in computing degree of rank 1 subsheaves of the tangent bundle.

**Lemma 6.1.** To any equivariant rank one coherent subsheaf \( F \) of \( TX \) on a complete nonsingular toric surface \( X \), one can associate an integral vector
\[
\vec{\lambda} := (\lambda_1, \cdots, \lambda_p)
\] (6.1)
where

1. \( p = |\Delta(1)| \), and \( \Delta(1) = \{\alpha_1, \ldots, \alpha_p\} \)
2. \( e(F, \alpha_i, \lambda_i) = 1 \) for each \( i \),
3. each \( \lambda_i \in \mathbb{Z}_{\geq -1} \), and
4. \( (\lambda_i, \lambda_j) \neq (-1, -1) \) if \( \alpha_i, \alpha_j \) form a cone in \( \Delta \).

**Proof.** Given a rank one subsheaf \( F \) of \( TX \), for each \( \alpha_i \in \Delta(1) \), by \((5.6)\) and \((5.5)\) there exists a unique \( \lambda_i \in \mathbb{Z}_{\geq -1} \) such that \( e(F, \alpha_i, \lambda_i) = 1 \).

Now, suppose \( \alpha_i, \alpha_j \) generate a cone \( \sigma \). Denote the characters \( \chi(m_i^\sigma) \), \( \chi(m_j^\sigma) \) by \( z_i \) and \( z_j \) respectively. Then \( \lambda_i = -1 \) implies that \( F_m^\sigma \neq 0 \) for some \( m = -m_i^\sigma + km_j^\sigma \) where \( k \geq 0 \). Therefore, \( F^\sigma \) has a generator of the form \( z_i^k \partial_{z_j} \). Similarly, \( F^\sigma \) has a generator of the form \( z_i^k \partial_{z_j} \) if \( \lambda_j = -1 \). Thus \( \text{rank}(F) \) would exceed one if \( (\lambda_i, \lambda_j) = (-1, -1) \). \( \square \)

However, not every vector of type \((6.1)\) corresponds to an equivariant rank one coherent subsheaf of \( TX \). To illustrate this, we consider the example of a Hirzebruch surface. The Hirzebruch surface \( X = \mathbb{F}_m \) is a projective toric surface which may be obtained by the projectivization of the bundle \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m) \) on \( \mathbb{P}^1 \). \( \mathbb{F}_m \) has fan \( \Delta \) with \( \Delta(1) = \{\alpha_1, \cdots, \alpha_4\} \) where
\[
[\alpha_1 \alpha_2 \alpha_3 \alpha_4] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & m & -1 \end{bmatrix}.
\]
We denote the 2-dimensional cone generated by \( \alpha_i \) and \( \alpha_{i+1 \text{mod } 4} \) by \( \sigma_i \).

Assume that \( m \geq 1 \). Consider a collection \((6.1)\) such that \( \lambda_1 = \lambda_3 = -1 \). Denote
\[
t_1 = \chi(m_1^\sigma) = \chi((1, 0)), \quad t_2 = \chi(m_2^\sigma) = \chi((0, 1)).
\]
As \( \lambda_1 = -1 \), there is an element \( s_1 \) of the form \( t_2^k \partial_{s_1} \) in \( F^{\sigma_1} \). Note that \( m_2^\sigma = (m, 1) \) and \( m_3^\sigma = (-1, 0) \). (Here, it is convenient to continue to use the notation introduced in the proof of Lemma \( 6.1 \) as opposed to the last section.) Denote
\[
z_2 = \chi(m_2^\sigma) = t_2^m t_2, \quad z_3 = \chi(m_3^\sigma) = t_1^{-1}.
\]
This implies that
\[ t_1 = z_3^{-1}, \quad t_2 = z_2 z_3^m. \]
Therefore, we have the Jacobian
\[
\frac{\partial (t_1, t_2)}{\partial (z_2, z_3)} = \begin{bmatrix} 0 & -z_3^{-2} \\ z_3^m & m z_2 z_3^{m-1} \end{bmatrix} = \begin{bmatrix} 0 & -t_1^2 \\ t_1^{-m} & m t_1 t_2 \end{bmatrix}.
\]
Thus we have
\[ \partial z_2 = t_1^{-m} \partial t_2, \quad \partial z_3 = -t_2^2 \partial t_1 + m t_1 t_2 \partial t_2. \]
Again, as \( \lambda_3 = -1 \), there must be an element \( s_2 \) in \( F_{\sigma_2} \) of the form \( z_2 \partial z_3 \). The restrictions of \( s_1 \) and \( s_2 \) to the dense torus \( X_{(0)} \) are independent. Therefore, \( F \) must have rank 2, which is a contradiction.

However, it can be verified that the vector
\[ \vec{\lambda} = (0, -1, 0, -1) \quad (6.2) \]
corresponds to a rank one coherent subsheaf of \( T X \) on every \( F_m \). An analogous, but more general example is worked out in Theorem 8.1.

**Theorem 6.2.** The tangent bundle of the Hirzebruch surface \( F_m \) is unstable with respect to every polarization if \( m \geq 2 \).

**Proof.** Let \( D_i \) denote the divisor of \( X = F_m \) corresponding to \( \alpha_i, 1 \leq i \leq 4 \). The \( D_i \)’s generate the Picard group of \( X \). It is easy to check that \( H = \sum a_i D_i \) is ample if and only if \( a := a_1 + a_3 - ma_2 > 0 \) and \( b := a_2 + a_4 > 0 \). The polytope \( P \) associated to \( H \) has vertices \( A = (a_1, -a_2), \quad B = (a_3 - ma_2, -a_2), \quad C = (ma_4 + a_3, a_4) \) and \( D = (-a_1, a_4) \). The facets (edges) are \( P_{\alpha_1} = AD, \quad P_{\alpha_2} = AB, \quad P_{\alpha_3} = BC \) and \( P_{\alpha_4} = CD \). Their volumes are \( b, \ a, \ b \) and \( a + mb \) respectively.

The slope of a rank one coherent subsheaf \( F \) associated to the collection \( 6.1 \) is
\[ \mu(F) = \deg(F) = - \sum_{\alpha_i \in \Delta(1)} \lambda_i \text{Vol}(P_{\alpha_i}). \quad (6.3) \]
It follows that the slope is a decreasing function of each \( \lambda_i \). Since there is no rank one equivariant coherent subsheaves of \( T X \) with \( \lambda_1 = -1 = \lambda_3 \), the rank one equivariant coherent subsheaf with maximum slope corresponds to the collection \( 6.2 \). The slope of this subsheaf is
\[ \mu(F) = \text{Vol}(P_{\alpha_2}) + \text{Vol}(P_{\alpha_4}) = 2a + mb. \]
Note that the rank one subsheaves with \( \vec{\lambda} = (-1, 0, 0, 0), \ (0, -1, 0, 0), \ (0, 0, -1, 0) \) and \( (0, 0, 0, -1) \), if they exist, have slopes \( b, \ a, \ b \) and \( a + mb \) respectively; and all of them are less than \( 2a + mb \).

On the other hand, by \( 5.8 \), the slope of \( T X \) is
\[ \mu(T X) = \frac{1}{2} \sum_{i=1}^{4} \text{Vol}(P_{\alpha_i}) = a + \frac{m + 2}{2} b. \]
Since \( a \) and \( b \) are positive, the theorem follows. \( \square \)
It is known that $TF_0$ and $TF_1$ are semistable with respect to the anti-canonical polarization, namely when each $a_i = 1$. We can deduce more about $F_1$ from the above calculations.

**Corollary 6.3.** For a polarization with $2a < b$, for example when $0 < 2a_1 < a_4$ and $a_2 = a_3 = 0$, the tangent bundle $TF_1$ is stable. On the other hand, if $2a > b$, for example when $0 < a_4 < 2a_1$ and $a_2 = a_3 = 0$, then $TF_1$ is unstable.

### 7. Projective spaces

The method of the previous section can be adapted to give an alternative proof of the well-known result that the tangent bundle of the complex projective space $\mathbb{P}^n$ is stable with respect to the anti-canonical polarization (cf. [10, Section 1.4]).

**Theorem 7.1.** $T\mathbb{P}^n$ is stable with respect to the anti-canonical polarization for every $n$.

**Proof.** Let $\Delta$ be the fan of $\mathbb{P}^n$. Then $\Delta(1) = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$ where

$$\{\alpha_1 = (1, 0, \ldots, 0, 0), \ldots, \alpha_n = (0, 0, \ldots, 0, 1)\}$$

is the standard basis of $\mathbb{R}^n$ and

$$\alpha_{n+1} = -\sum_i \alpha_i = (-1, -1, \ldots, -1).$$

$\Delta$ has $n+1$ cones of dimension $n$ which may be enumerated as

$$\sigma_i = \langle \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{n+1} \rangle, \quad 1 \leq i \leq n+1.$$

Let $P$ denote the polytope of $\mathbb{P}^n$ with respect to the anti-canonical polarization. For any pair $(i, j)$ there is an automorphism $A_{ij}$ of the lattice $N$ that interchanges $\alpha_i$ and $\alpha_j$ keeping the other $\alpha$’s fixed. This implies that every facet $P_{\alpha_i}$ of $P$ has the same volume, say $V_n$. Hence, by (5.9), the slope of an equivariant rank $r$ coherent subsheaf $F$ of the tangent bundle is

$$\mu(F) = -(n-1)! \frac{V_n}{r} \sum_{\alpha \in \Delta(1), \lambda \in \mathbb{Z}_{\geq -1}} \lambda e(F, \alpha, \lambda). \quad (7.1)$$

We claim that, if $r < n$, then there can be at most $r$ many $\alpha_i$’s with $e(F, \alpha_i, -1) = 1$. This would imply that $\mu(F) \leq (n-1)! V_n$. However, by (5.8)

$$\mu(T\mathbb{P}^n) = (n-1)! \frac{(n+1)V_n}{n} > (n-1)! V_n.$$

Thus $\mu(F) < \mu(T\mathbb{P}^n)$ for every proper sub sheaf $F$ of $T\mathbb{P}^n$.

To prove the claim, assume that there are at least $(r+1)$ many $\alpha_i$’s with $e(F, \alpha_i, -1) = 1$. Since $r+1 \leq n$, there exists an $n$-dimensional cone $\sigma$ containing $(r+1)$ of these $\alpha_i$’s. Note that the corresponding $(r+1)$ many $\partial_{z_i}$’s (up to multiplication by characters) are all generators of $F^\sigma$, contradicting the fact that rank$(F) = r$. \qed
8. Unstable Fano examples in higher dimensions

If $X$ is a Fano toric variety, then the polytope $P$ corresponding to the anti-canonical polarization is a reflexive polytope (cf. [4]). This implies that $P$ has integral vertices and the origin is the unique integral point which is in the interior of $P$. When $X$ is a surface, it is easy to tabulate such polytopes. One easily checks that the tangent bundle of a nonsingular Fano toric surface is semistable with respect to the anti-canonical polarization. However, it is known that the tangent bundle of any nonsingular Fano toric surface is semistable with respect to the anti-canonical polarization (cf. [8]).

The product of two Fano varieties $X_1 \times X_2$ is Fano. If $TX_1$ and $TX_2$ are both semistable with respect to their respective anti-canonical polarizations, then the same holds for $T(X_1 \times X_2)$ (cf. [22]). This yields more examples of semistable Fano toric varieties. However there are a lot of Fano toric varieties with unstable tangent bundle in higher dimensions, as the following result shows. For $n = 3$, the result below had appeared in [22].

**Theorem 8.1.** Suppose $X$ is the Fano toric variety $\mathbb{P}(O_{\mathbb{P}^{n-1}} \oplus O_{\mathbb{P}^{n-1}}(m))$ where $0 < m \leq n - 1$ and $n \geq 3$. Then $TX$ is unstable with respect to the anti-canonical polarization.

**Proof.** Let $\Delta$ be the fan of $X$. Then $\Delta(1) = \{\alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}\}$, where $\{\alpha_1 = (1, 0, \cdots, 0, 0), \cdots, \alpha_n = (0, 0, \cdots, 0, 1)\}$ is the standard basis of $\mathbb{R}^n$ and

$$
\alpha_{n+1} = -\alpha_n, \quad \alpha_{n+2} = -\sum_{j=1}^{n} \alpha_j + (m+1)\alpha_n = (-1, -1, \cdots, -1, m).
$$

$\Delta$ has $2n$ cones of dimension $n$. They may be divided into two groups, upper and lower. The upper cones contain $\alpha_n$ but not $\alpha_{n+1}$, and vice versa for the lower cones. There are $n$ upper cones, each missing one of the remaining $\alpha_i$'s, and similarly for lower cones.

We verify that there is an equivariant subsheaf $F$ of $TX$ of rank 1 associated to the vector $\vec{\lambda} = (0, \cdots, 0, -1, -1)$, i.e. $\lambda_n = \lambda_{n+1} = -1, \lambda_j = 0$ for other $j$ (see Lemma 6.1). Note that for this $\vec{\lambda}$, in every $n$–dimensional cone $\sigma$ there is exactly one $\alpha_i$, say $\alpha_i$, such that $e(F, \alpha_i, -1) \neq 0$. The number $i_0$ is $n$ for upper cones, and $n + 1$ for lower cones.

Consider the upper cone $\sigma$ generated by $\alpha_1, \cdots, \alpha_n$. Denote $\chi(m_\sigma^\alpha)$ by $t_i$ for $1 \leq i \leq n$. Then $\lambda_n = -1$ implies that $F^\sigma$ is generated by an element of the form $g(t_1, \cdots, t_{n-1})\partial_{i_0}$. Set $g(t_1, \cdots, t_{n-1}) = 1$.

Now consider any other upper cone $\tau$. Assume $\tau$ is missing the ray $\alpha_j$, where $j \leq n$. Let $z_i^\tau := \chi(m_i^\tau)$ where $1 \leq i \leq n + 2$ and $i \neq j, n + 1$. Then $F^\tau$ is generated
by an element of the form \( f(z) \partial z^i \). We have

\[
z_i^\tau = \begin{cases} 
t_i t_j^{-1} & \text{if } i \neq n, n + 2, \\
t_i^m t_n & \text{if } i = n, \\
t_j^{-1} & \text{if } i = n + 2.
\end{cases}
\]

It follows easily that \( \partial t_n = t_j^m \partial z_n = (z_{n+2})^{-m} \partial z_n^\tau \). Note that \( \lambda_n = -1 \) means \( F^\tau \) should be generated by an element of the form \( h(z_1^\tau, \ldots, z_n^\tau, \ldots, z_{n+2}) \partial z_n^\tau \). Set \( h = (z_{n+2})^{-m} \).

The calculations for the lower cones are quite analogous. Finally, it is enough to consider the transition between the upper cone \( \sigma \) and the lower cone \( \delta \) generated by \( \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n+1} \). The coordinates on \( X_\delta \) are \( t_1, \ldots, t_{n-1} \) and \( w = \chi(m^\delta) = t_n^{-1} \). As \( \lambda_{n+1} = -1 \), we assume that \( F^\delta \) is generated by \( \partial_w \). Note that \( \partial_w = -t_n^m \partial t_n \) and \( t_n \) is invertible on \( X_\sigma \cap X_\delta \). This confirms the existence of the desired rank one sheaf \( F \).

Consider the anti–canonical divisor \( H \) on \( X \). The associated polytope is combinatorially equivalent to a prism \( S^{n-1} \times I \) where \( S^{n-1} \), \( I \) denote a simplex of dimension \( n - 1 \) and an interval respectively. The top and bottom facets, \( P_{\alpha_{n+1}} \) and \( P_{\alpha_n} \), are \( (n - 1) \)-dimensional simplices and lie on the hyperplanes \( x_n = 1 \) and \( x_n = -1 \) respectively. The top facet \( P_{\alpha_{n+1}} \) has the vertices

\[
( -1, -1, \ldots, -1, 1), \ (n + m - 1, -1, \ldots, -1, 1), \ (-1, n + m - 1, \ldots, -1, 1),
\]

\[
\cdots, ( -1, -1, \ldots, n + m - 1, 1).
\]

The bottom facet \( P_{\alpha_n} \) has the vertices

\[
( -1, -1, \ldots, -1, -1), \ (n - m - 1, -1, \ldots, -1, -1), \ (-1, n - m - 1, \ldots, -1, -1),
\]

\[
\cdots, ( -1, -1, \ldots, n - m - 1).
\]

We have

\[
\text{Vol}(P_{\alpha_{n+1}}) = \frac{(n + m)^{n-1}}{(n - 1)!} \quad \text{and} \quad \text{Vol}(P_{\alpha_n}) = \frac{(n - m)^{n-1}}{(n - 1)!}.
\]

Consequently, using \([7,9]\), we have,

\[
\mu(F) = (n + m)^{n-1} + (n - m)^{n-1}.
\]

The prism \( P \) has \( n \) side facets corresponding to the rays \( \alpha_1, \ldots, \alpha_{n-1}, \) and \( \alpha_{n+2} \). Each side facet, in turn, is a prism of height 2 bounded by a facet of \( P_{\alpha_{n+1}} \) and \( P_{\alpha_n} \) on top and bottom respectively. Each of these side facets have volume

\[
\frac{(n + m)^{n-2} + (n - m)^{n-2}}{(n - 2)!}.
\]

Therefore, using \([5,8]\), we have,

\[
\mu(TX) = \frac{(n + m)^{n-1} + (n - m)^{n-1} + n(n - 1)(n + m)^{n-2} + (n - m)^{n-2}}{n}.
\]

It is now easy to verify \( \mu(F) > \mu(TX) \) using \((a^{n-2} - b^{n-2})(a - b) > 0\) where \( a = n + m \) and \( b = n - m \). Here, we have used \( n \geq 3 \). The theorem follows. \( \square \)
9. Fano toric 4–folds with small Picard number

Steffens has studied the stability of the tangent bundle of all Fano 3–folds in [22]. Moreover, Nakagawa [18,19] has identified all Fano toric 4–folds that admit Einstein-Kähler metrics. We study which Fano toric 4–folds with Picard number \( \leq 2 \) have semi-stable tangent bundle.

A list of Fano toric 4–folds is given by Batyrev in [5], see also earlier work of Kleinschmidt [12]. There are 10 classes of these varieties with Picard number \( \leq 2 \), which are:

\[
\begin{align*}
(1) & \quad \mathbb{P}^4 \\
(2) & \quad B_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(3)) \\
(3) & \quad B_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2)) \\
(4) & \quad B_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)) \\
(5) & \quad B_4 = \mathbb{P}^1 \times \mathbb{P}^3 \\
(6) & \quad B_5 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \\
(7) & \quad C_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \\
(8) & \quad C_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \\
(9) & \quad C_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \\
(10) & \quad C_4 = \mathbb{P}^2 \times \mathbb{P}^2
\end{align*}
\]

However, we will first need to fine-tune our tools to deal with subsheaves of higher rank. We will describe a generalization of Lemma [6,1] for rank \( r \) subsheaves.

**Lemma 9.1.** To any equivariant rank \( r \) coherent subsheaf \( \mathcal{F} \) of \( TX \) on a complete nonsingular toric variety \( X \), one can associate a unique \( r \times p \) matrix of integers \( \Lambda_{r \times p} := (\lambda_{ij}) \), where

\[
\begin{align*}
(1) & \quad p = |\Delta(1)|, \\
(2) & \quad e(\mathcal{F}, \alpha_j, \lambda_{ij}) \neq 0 \text{ for each } 1 \leq j \leq p, \\
(3) & \quad -1 \leq \lambda_{ij} \leq \lambda_{2j} \leq \cdots \leq \lambda_{rj} \text{ for every } j, \\
(4) & \quad \sum e(\mathcal{F}, \alpha_j, \lambda_{ij}) = r \text{ for every } j, \text{ where the sum is over any maximal set of row indices } i \text{ such that corresponding } \lambda_{ij} \text{'s in the column } j \text{ are distinct.} \\
(5) & \quad (\lambda_{ij_1}, \cdots, \lambda_{ij_{r+1}}) \neq (-1, \cdots, -1) \text{ if } \alpha_{j_1}, \cdots, \alpha_{j_{r+1}} \text{ form a cone in } \Delta, \text{ and} \\
(6) & \quad (\lambda_{ij}, \lambda_{2j}) \neq (-1, -1) \text{ for any } j.
\end{align*}
\]

**Proof.** We are tabulating which \( e(\mathcal{F}, \alpha_j, \lambda) \neq 0 \) such that each entry \( \lambda_{ij} \) in the \( j \)-th column contributes 1 to \( e(\mathcal{F}, \alpha_j, \lambda_{ij}) \). This implies that if \( e(\mathcal{F}, \alpha_j, \lambda) = k \), then \( k \) entries of the \( j \)-th column have entry \( \lambda \).

Then the proof is almost immediate. Condition (3) is a choice of order made for the sake of uniqueness. Condition (4) follows from Theorem [4.3] Condition (5) ensures that rank of \( \mathcal{F} \) does not exceed \( r \). Condition (6) follows from the dependence of relevant generators over \( k[S_{(0)}] \). It is equivalent to saying \( e(\mathcal{F}, \alpha_j, -1) \leq 1 \) for each \( j \). \( \square \)

We use the associated matrix \( \Lambda \) below to give a proof of the semi-stability of the toric Fano 4–fold \( B_5 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \) from the above list.

**Theorem 9.2.** Suppose \( X \) is the Fano toric 4–fold \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \). Then \( TX \) is strictly semistable with respect to the anti-canonical polarization.

**Proof.** Let \( \Delta \) be the fan of \( X \). Then \( \Delta(1) = \{ \alpha_1 = (1, 0, 0, 0), \alpha_2 = (0, 1, 0, 0), \alpha_3 = (0, 0, 1, 0), \alpha_4 = (-1, -1, -1, 0), \alpha_5 = (0, 0, 0, 1), \alpha_6 = (1, 0, 0, -1) \} \) (see [5]).
We have $\text{Vol}(P_{\alpha_2}) = 8$, $\text{Vol}(P_{\alpha_2}) = \frac{56}{3}$, $\text{Vol}(P_{\alpha_3}) = \frac{56}{3}$, $\text{Vol}(P_{\alpha_4}) = \frac{56}{3}$, and $\text{Vol}(P_{\alpha_5}) = \frac{32}{3}$ with respect to anti-canonical divisor. Then

$$\mu(TX) = \frac{3!}{4} \sum_{i=1}^{6} \text{Vol}(P_{\alpha_i}) = 128$$

We denote the 4-dimensional cones by $\sigma_i$ and we have 8 of them. They are

$$
\begin{align*}
\sigma_1 &= <\alpha_2, \alpha_3, \alpha_4, \alpha_6> \\
\sigma_2 &= <\alpha_1, \alpha_2, \alpha_3, \alpha_5> \\
\sigma_3 &= <\alpha_1, \alpha_2, \alpha_3, \alpha_6> \\
\sigma_4 &= <\alpha_1, \alpha_2, \alpha_4, \alpha_5> \\
\sigma_5 &= <\alpha_1, \alpha_2, \alpha_4, \alpha_6> \\
\sigma_6 &= <\alpha_1, \alpha_3, \alpha_4, \alpha_5> \\
\sigma_7 &= <\alpha_1, \alpha_3, \alpha_4, \alpha_6> \\
\sigma_8 &= <\alpha_2, \alpha_3, \alpha_4, \alpha_5>
\end{align*}
$$

Denote

$$t_1 = \chi((1, 0, 0, 0)), \quad t_2 = \chi((0, 1, 0, 0)), \quad t_3 = \chi((0, 0, 1, 0)), \quad t_4 = \chi((0, 0, 0, 1)).$$

First consider rank 1 equivariant subsheaves of $TX$. There is a possible such subsheaf $\mathcal{F}$ associated to the vector $\lambda = (0, 0, 0, -1, -1)$.

The other choices for $\lambda$ that satisfy the conditions of Lemma 6.1 have at most one $-1$ entry. Then it is easy to check using Lemma 5.3 that $\mathcal{F}$ has the possible maximum slope among rank 1 subsheaves. The slope of $\mathcal{F}$ is

$$\mu(\mathcal{F}) = \frac{3!}{1}(\text{Vol}(P_{\alpha_5}) + \text{Vol}(P_{\alpha_6})) = 128.$$

As this equals $\mu(TX)$, we need to check the existence of $\mathcal{F}$.

First consider the 4-dimensional cones containing $\alpha_5$, these are $\sigma_2, \sigma_4, \sigma_6, \sigma_8$. Define

$$x_i := \chi(m_i^{\sigma_2}), \quad y_i := \chi(m_i^{\sigma_4}), \quad z_i := \chi(m_i^{\sigma_6}) \quad \text{and} \quad w_i := \chi(m_i^{\sigma_8})$$

where $1 \leq i \leq 4$.

Then

$$x_1 = t_1, \quad x_2 = t_2, \quad x_3 = t_3, \quad x_4 = t_4, \quad y_1 = t_1t_3^{-1}, \quad y_2 = t_2t_3^{-1}, \quad y_3 = t_3^{-1}, \quad y_4 = t_4, \quad z_1 = t_1t_3^{-1}, \quad z_2 = t_2t_3^{-1}, \quad z_3 = t_2^{-1}, \quad z_4 = t_4, \quad w_1 = t_1^{-1}t_2, \quad w_2 = t_1^{-1}t_3, \quad w_3 = t_1^{-1}, \quad w_4 = t_4. \quad (9.1)$$

As $\lambda_5 = -1$, there is an element generated by (meaning, a multiple of) $\partial_{x_4}$ in $F^{\sigma_2}$. Similarly, there are elements generated by $\partial_{y_4}$, $\partial_{z_4}$ and $\partial_{w_4}$ in $F^{\sigma_4}$, $F^{\sigma_6}$ and $F^{\sigma_8}$, respectively. But, using the Jacobians of the transformations between the $t$ and other coordinates, we have

$$\partial_{x_4} = \partial_{y_4} = \partial_{z_4} = \partial_{w_4} = \partial_{t_4}.$$

Thus the generators agree on the dense torus.
Secondly, consider cones containing $\alpha_6$ as a generator, which are $\sigma_1, \sigma_2, \sigma_5, \sigma_7$. Call now $p_i := \chi(m_i^{\sigma_1}), q_i := \chi(m_i^{\sigma_2}), r_i := \chi(m_i^{\sigma_5})$ and $s_i := \chi(m_i^{\sigma_7})$ where $1 \leq i \leq 4$. Similar to above computations, find that

$$
p_1 = t_1^{-1}t_2t_4^{-1}, \quad p_2 = t_1^{-1}t_3t_4^{-1}, \quad p_3 = t_1^{-1}t_4^{-1}, \quad p_4 = t_4^{-1},
q_1 = t_1t_4, \quad q_2 = t_2, \quad q_3 = t_3, \quad q_4 = t_4^{-1},
r_1 = t_1r_4, \quad r_2 = t_2^{-1}t_4, \quad r_3 = t_3, \quad r_4 = t_4^{-1},
s_1 = t_1t_2^{-1}t_4, \quad s_2 = t_2^{-1}t_3, \quad s_3 = t_2^{-1}, \quad s_4 = t_4^{-1}.
$$

(9.2)

As $\lambda_6 = -1$, there are elements generated by $\partial_{p_4}, \partial_{q_4}, \partial_{r_4}$ and $\partial_{s_4}$ in $F^{\sigma_1}, F^{\sigma_2}, F^{\sigma_5}$ and $F^{\sigma_7}$ respectively. Using various Jacobians, we have

$$\partial_{p_4} = \partial_{q_4} = \partial_{r_4} = \partial_{s_4} = t_1t_4 \partial_{t_1} - t_2^2 \partial_{t_4}.\]$$

However, on the dense torus, the generators $\partial_{t_4}$ and $t_1t_4 \partial_{t_1} - t_2^2 \partial_{t_4}$ are linearly independent. Hence the rank of $\mathcal{F}$ must be at least 2, leading to a contradiction. We conclude that $\mu(\mathcal{F}) < 128 = \mu(TX)$ for every rank 1 equivariant subsheaf of $TX$.

Next we consider rank 2 equivariant subsheaves of $TX$ with maximum possible slope. By condition (5) of Lemma 9.1 a subsheaf $\mathcal{F}$ with $\lambda_{ij} = -1$ for 3 values of $j$ is only possible when 2 of the $j$’s are 5 and 6. Then, using condition (6) of Lemma 9.1 the slope of $\mathcal{F}$ has the following bound,

$$\mu(\mathcal{F}) \leq \frac{3!}{2}(\text{Vol}(P_{\alpha_1}) + \text{Vol}(P_{\alpha_5}) + \max_{1 \leq j \leq 4} \text{Vol}(P_{\alpha_j})) = 120.$$

Thus there is no destabilizing subsheaf of rank 2.

There is a rank 3 equivariant subsheaf $\mathcal{F}$ of $TX$ with associated matrix $A_{3 \times 6} = (\lambda_{ij})$ such that $\lambda_{ij} = -1$ for $1 \leq j \leq 4$, and all other $\lambda_{ij} = 0$. For every 4-dimensional cone $\sigma_i, F^{\sigma_i}$ is generated over $k[S_{\sigma_i}]$ by $\partial_{\chi(m_i^{\sigma_i})}, 1 \leq i \leq 3$. The generators for different cones $\sigma_1$ and $\sigma_m$ are related by Jacobians of corresponding transition maps which are naturally regular on $X_{\sigma_1 \cup \sigma_m}$. Moreover, as $\chi(m_i^{\sigma_i})$ is either $t_4$ or $t_4^{-1}$, it follows easily that $\partial_{\chi(m_i^{\sigma_i})}, 1 \leq i \leq 3$, is always a combination of $\partial_{t_1}, \partial_{t_2}$ and $\partial_{t_3}$. Thus the subsheaf $\mathcal{F}$ indeed exists. It has the maximum slope among rank 3 subsheaves, and the slope is

$$\mu(\mathcal{F}) = \frac{3!}{3}(\text{Vol}(P_{\alpha_1}) + \text{Vol}(P_{\alpha_5}) + \text{Vol}(P_{\alpha_3}) + \text{Vol}(P_{\alpha_4})) = 128 = \mu(TX).$$

This concludes the proof. \hfill \Box

Here is the classification of all nonsingular Fano toric 4-folds with Picard number $\leq 2$ according to stability with respect to the anticanonical polarization.

**Theorem 9.3.** Suppose $X$ is one of the Fano toric 4-folds with Picard number $\leq 2$. Then

1. $TP^4$ is stable,
2. $TB_1$ and $TC_4$ are polystable,
3. $TB_5$ is strictly semistable,
4. $TB_1$, $TB_2$ and $TB_3$ are unstable,
5. $TC_1$, $TC_2$ and $TC_3$ are unstable.
Proof. (1) This is well-known. See Theorem 7.1 for an alternative proof.
(2) $B_4$ and $C_4$ admit Einstein-Kähler metrics (see [19, Theorem 3.4]). Hence, these are polystable.
(3) See Theorem 9.2.
(4) These are special cases of Theorem 8.1.
(5) $TC_1$ and $TC_2$ are destabilized by rank 2 subsheaves, and $TC_3$ is destabilized by a rank 1 subsheaf. These may be verified by following a similar approach as in the proof of Theorem 9.2.

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