Infinite-message Distributed Source Coding for Two-terminal Interactive Computing

Nan Ma and Prakash Ishwar

Abstract

A two-terminal interactive function computation problem with alternating messages is studied within the framework of distributed block source coding theory. For any arbitrary fixed number of messages, a single-letter characterization of the sum-rate-distortion function was provided in previous work using traditional information-theoretic techniques. This, however, does not directly lead to a satisfactory characterization of the infinite-message limit, which is a new, unexplored dimension for asymptotic-analysis in distributed block source coding involving potentially infinitesimal-rate messages. This paper introduces a new convex-geometric approach to provide a blocklength-free single-letter characterization of the infinite-message sum-rate-distortion function as a functional of the joint source pmf and distortion levels. This characterization is not obtained by taking a limit as the number of messages goes to infinity. Instead, it is in terms of the least element of a family of partially-ordered marginal-perturbations-concave functionals defined by the coupled per-sample distortion criteria. For computing the Boolean AND function of two independent Bernoulli sources at one and both terminals, with zero Hamming distortion, the respective infinite-message minimum sum-rates are characterized in closed analytic form. These sum-rates are achievable using infinitely many infinitesimal-rate messages. The convex-geometric functional viewpoint also suggests an iterative algorithm for evaluating any sum-rate-distortion function, including, as a special case, the Wyner-Ziv rate-distortion function.

I. Introduction

In this paper we study a two-terminal interactive function computation problem with alternating messages (Fig. 1) within a distributed block source coding framework. Here, \( n \) samples of one component of a discrete memoryless multi-source

\[
\begin{align*}
X & := X^n := (X(1), \ldots, X(n)) \in X^n \\
Y & \in Y^n \text{ are available at terminal } A \text{ and } n \text{ samples of another component of the multi-source } Y \in Y^n \text{ are available at a different terminal } B. 
\end{align*}
\]

The two component sources of the multi-source are statistically dependent. Terminal \( A \) is required to produce a sequence \( \tilde{Z}_A \in Z^n_A \) such that \( d_A^n(X, Y, \tilde{Z}_A) \leq D_A \) where \( d_A^n \) is a distortion function of \( 3n \) variables. Similarly, terminal \( B \) is required to produce a sequence \( \tilde{Z}_B \in Z^n_B \) such that \( d_B^n(X, Y, \tilde{Z}_B) \leq D_B \). All alphabets and distortion functions are assumed to be finite. To achieve the desired objective, \( t \) coded messages, \( M_1, \ldots, M_t \), of respective

\[\text{Fig. 1. Interactive distributed source coding with } t \text{ alternating messages.}\]

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The authors are with the Department of Electrical and Computer Engineering, Boston University, Boston, MA 02215, USA {nanma, pi}@bu.edu
bit rates (bits per source sample), $R_1, \ldots, R_t$, are sent alternately from the two terminals starting with some terminal. The message sent from a terminal can depend on the source samples at that terminal and on all the previous messages (which are available to both terminals). There is enough memory at both terminals to store all the source samples and messages. After $t$ messages, terminal $A$ produces a sequence $\tilde{Z}_A \in \mathbb{Z}^n$ and terminal $B$ produces a sequence $\tilde{Z}_B \in \mathbb{Z}^n$. The sum-rate-distortion function $R_{\text{sum}}(D_A, D_B)$ is the infimum of the sum of all rates $\sum_{i=1}^{t} R_i$ for which the following criteria hold: $P(d_A^n(x, y, \tilde{Z}_A) > D_A)$ and $P(d_B^n(x, y, \tilde{Z}_B) > D_B) \to 0$ as $n \to \infty$.

When the criterion is of the form of vanishing probability of block-error (Sec. [II-A]), or of the form of expected per-sample distortion (Sec. [VI-A]), for any fixed number $t$, a single-letter characterization of the set of all feasible coding rate-distortion tuples (the rate-distortion region) and the sum-rate-distortion function $R_{\text{sum}}(D_A, D_B)$, was provided in our previous work [1], [2] using traditional information-theoretic techniques. This characterization, however, does not directly lead to a satisfactory characterization of the infinite-message limit $R_{\text{sum},\infty}(D_A, D_B) := \lim_{t \to \infty} R_{\text{sum}}(D_A, D_B)$. The objective of this paper is to provide a characterization of $R_{\text{sum},\infty}(D_A, D_B)$ which is not obtained by taking a limit as the number of messages goes to infinity and also an iterative algorithm to evaluate it. Understanding the sum-rate-distortion function in the limit where potentially an infinite number of alternating messages are allowed to be exchanged will shed light on the fundamental benefit of cooperative interaction in two-terminal problems. While asymptotics involving blocklength, rate, quantizer step-size, and network size have been explored in the distributed block source coding literature, asymptotics involving an infinite number of messages, each with potentially infinitesimal rate, has not been studied. The number of messages is a relatively unexplored resource and a new dimension for asymptotic analysis.

This paper introduces a new convex-geometric approach to provide a blocklength-free single-letter characterization of the infinite-message sum-rate-distortion function as a functional of the joint source distribution and distortion levels. This characterization is not obtained by taking a limit as the number of messages goes to infinity. Instead, it is in terms of the least element of a family of partially-ordered, marginal-perturbations-concave functionals defined by the coupled per-sample distortion criteria. For computing the Boolean AND function of two independent Bernoulli sources at one/both terminals with zero Hamming distortion, the respective infinite-message minimum sum-rates are characterized in closed analytic form and shown to be achievable using infinitely many infinitesimal-rate messages. The functional viewpoint also leads to an iterative algorithm for evaluating any finite-message sum-rate-distortion functions.

Related interactive computation problems have been studied extensively in the area of communication complexity [3], [4] where the main focus is on exact zero error computation, without regard for the statistical dependencies in samples across terminals, and where computing efficiency is gauged in terms of the order-of-magnitude of the total number of bits exchanged; not bit-rate (notable exceptions to this main focus are [5], [6]). Two-way distributed block source coding where the goal is to reproduce the sources with a non-zero per-sample distortion, as opposed to computing functions, was studied by Kaspi [7] who characterized the $t$-message sum-rate-distortion function in each direction. Orlitsky and Roche [8] studied two-terminal samplewise function computation with a vanishing block-error probability and characterized the feasible rates and the minimum sum-rate for two alternating messages ($t = 2$). A more detailed account of related work appears in [2].

If we choose the per-sample distortion function to be the Hamming distortion with respect to a function of $X$ and $Y$ and set the distortion level to zero, the characterization of the sum-rate-distortion function essentially reduces to the characterization of the minimum sum-rate of the problem with the criterion of vanishing probability of block-error [2, Proposition 3]. In this sense, the per-sample distortion criterion is more general than the vanishing probability of error criterion. For clarity of exposition, however, we first focus on the vanishing probability of block-error criterion in Sec. [III] After formulating the problem in Sec. [II-A] in Sec. [II-B] we recap results from [1], [2] that are needed for the subsequent development. We present the characterization of the infinite-message minimum sum-rate in Sec. [III] We provide an iterative algorithm for evaluating any finite-message minimum sum-rate in Sec. [IV] We evaluate the infinite-message minimum sum-rates for two examples in Sec. [V] In Sec. [VI] we show how these results extend to the general rate-distortion problem.

Notation: Vectors are denoted by boldface letters; the dimension will be clear from the context. The acronym ‘iid’ stands
for independent and identically distributed and ‘pmf’ stands for probability mass function. With the exception of the symbols $R, D, N, A,$ and $B$, random quantities are denoted in upper case and their specific instantiations in lower case. For integers $i, j$, with $i \leq j$, $V_i^j$ denotes the sequence of random variables $V_i, \ldots, V_j$. For $i \geq 1$, $V_i$ is abbreviated to $V^i$. If $j < i$ then “$V^j_i$" denotes the null expression “". More generally, if $\{Q_i\}_{i \in S}$ is a set of quantities $Q_i$ indexed by a subset $S$ of integers then for all integers $i$ not in $S$, “$Q_i$” = “". For a set $S$, $S^c$ denotes the n-fold Cartesian product $S \times \ldots \times S$. The support-set of a pmf $p$ is the set over which it is strictly positive and is denoted by $\text{supp}(p)$. If $\text{supp}(q) \subseteq \text{supp}(p)$ then we write $q \ll p$. The set of all pmfs on alphabet $\mathcal{A}$, i.e., the probability simplex in $\mathbb{R}^{(|\mathcal{A}|)}$, is denoted by $\Delta(\mathcal{A})$. $X \sim \text{Ber}(p)$ means $p_X(1) = 1 - p_X(0) = p$, and $h_2(p)$ denotes its entropy. $X \perp Y$ means $X$ and $Y$ are independent. The indicator function of set $S$ which is equal to one if $x \in S$ and is zero otherwise, is denoted by $1_S(x)$. Symbols $\land$ and superscript $c$ represent Boolean AND and complement respectively.

II. Interactive function computation problem

A. Problem formulation

We consider two statistically dependent discrete memoryless stationary sources taking values in finite alphabets. For $i = 1, \ldots, n$, let $(X(i), Y(i)) \sim \text{iid } p_{XY}(x, y), x \in X, y \in Y, |X| < \infty, |Y| < \infty$. Here, $p_{XY}$ is a joint pmf which describes the statistical dependencies among the samples observed at the two terminals at each time instant $i$. Let $f_A : X \times Y \rightarrow Z^A$ and $f_B : X \times Y \rightarrow Z^B$ be functions of interest at terminals $A$ and $B$ respectively, where $Z^A$ and $Z^B$ are finite alphabets. The desired outputs at terminals $A$ and $B$ are $Z^A$ and $Z^B$ respectively, where for $i = 1, \ldots, n$, $Z_A(i) := f_A(X(i), Y(i))$ and $Z_B(i) := f_B(X(i), Y(i))$.

**Definition 1:** A two-terminal interactive distributed source code (for function computation) with initial terminal $A$ and parameters $(t, n, |M_i|, \ldots, |M_i|)$ is the tuple $(e_1, \ldots, e_t, g_A, g_B)$ of $t$ block encoding functions $e_1, \ldots, e_t$ and two block decoding functions $g_A, g_B$, of blocklength $n$, where for $j = 1, \ldots, t$,

(Enc.$j$) $e_j : \begin{cases} (X^n \times \bigotimes_{i=1}^{j-1} M_i) \rightarrow M_j, & \text{if } j \text{ is odd} \\ (Y^n \times \bigotimes_{i=1}^{j-1} M_i) \rightarrow M_j, & \text{if } j \text{ is even} \end{cases}$

(Dec.$A$) $g_A : X^n \bigotimes_{j=1}^{t} M_j \rightarrow Z^A_n$.

(Dec.$B$) $g_B : Y^n \bigotimes_{j=1}^{t} M_j \rightarrow Z^B_n$.

The output of $e_j$, denoted by $M_j$, is called the $j$-th message, and $t$ is the number of messages. The outputs of $g_A$ and $g_B$ are denoted by $\hat{Z}_A$ and $\hat{Z}_B$ respectively. For each $j$, $(1/n) \log_2 |M_j|$ is called the $j$-th block-coding rate (in bits per sample). The sum of all the individual rates $(1/n) \sum_{j=1}^{t} \log_2 |M_j|$ is called the sum-rate.

**Definition 2:** A rate tuple $R = (R_1, \ldots, R_t)$ is admissible for $t$-message interactive function computation with initial terminal $A$ if, $\forall \epsilon > 0, \exists N(\epsilon, t)$ such that $\forall n > N(\epsilon, t)$, there exists an interactive distributed source code with initial terminal $A$ and parameters $(t, n, |M_i|, \ldots, |M_i|)$ satisfying

\[ \frac{1}{n} \log_2 |M_j| \leq R_j + \epsilon, \quad j = 1, \ldots, t, \]
\[ \mathbb{P}(Z_A \neq \hat{Z}_A) \leq \epsilon, \quad \mathbb{P}(Z_B \neq \hat{Z}_B) \leq \epsilon. \]

Note that of interest here are the probabilities of block error $\mathbb{P}(Z_A \neq \hat{Z}_A)$ and $\mathbb{P}(Z_B \neq \hat{Z}_B)$ which are multi-letter distortion functions. The set of all admissible rate tuples, denoted by $R^A_t$, is called the operational rate region for $t$-message interactive function computation with initial terminal $A$. The rate region is closed and convex due to the way it has been defined. The minimum sum-rate $R_{sum}^A$ is given by $\min\left(\sum_{j=1}^{t} R_j\right)$ where the minimization is over $R \in R^A_t$. For initial terminal $B$, the rate region and the minimum sum-rate are denoted by $R^B_t$ and $R_{sum}^B$ respectively. The focus of this paper is on the minimum sum-rate rather than the rate region.
We allow the number of messages $t$ to be equal to 0. When $t = 0$, there is no message transfer and the initial terminal is irrelevant. Thus for $t = 0$, in the notation for the minimum sum-rate, we omit the superscript and denote the minimum sum-rate as $R_{\text{sum},0}$.

For a given initial terminal, for $t = 0$ and $t = 1$, function computation may not be feasible for general $p_{XY}$, $f_A$, $f_B$. If the computation is infeasible, $R^A_t$ is empty and we set $R^A_{\text{sum},t} = +\infty$. If for some specific $p_{XY}$, $f_A$, $f_B$, the computation is feasible, then $R^A_{\text{sum},t}$ will be finite. Note that for $t \geq 2$, the computation is always feasible and $R^A_{\text{sum},t}$ is finite.

For all $j \leq t$, null messages, i.e., messages for which $|A_j| = 1$, are permitted by Definition 1. Hence, a $(t-1)$-message interactive code is a special case of a $t$-message interactive code. Thus, $R^A_{\text{sum},(t-1)} \geq R^A_{\text{sum},t}$ and $R^A_{\text{sum},(t-1)} \geq R^B_{\text{sum},t}$ (see [1, Proposition 1] for a detailed discussion). Therefore, $\lim_{t \to \infty} R^A_{\text{sum},t} = \lim_{t \to \infty} R^B_{\text{sum},t} =: R_{\text{sum},\infty}$. The limit $R_{\text{sum},\infty}$ is the infinite-message minimum sum-rate.

Depending on the specific joint pmf $p_{XY}$ and functions $f_A$ and $f_B$, it may be possible to reach the infinite-message limit $R_{\text{sum},\infty}$ with finite $t$ (see end of Sec. V-B for examples).

For all finite $t$, a single-letter characterization of the operational rate region $R^A_t$ and the minimum sum-rate $R^A_{\text{sum},t}$ were respectively provided in Theorem 1 and Corollary 1 of [1]. As discussed in Sec. II-B this does not, in general, directly lead to a satisfactory characterization of the infinite-message limit $R_{\text{sum},\infty}$ which is a new, unexplored dimension for asymptotic-analysis in distributed block source coding involving potentially an infinite number of infinitesimal-message rates. The main goal and contribution of this paper is the development of a general convex-geometric blocklength-free characterization of this infinite-message limit.

**B. Characterization of $R^A_{\text{sum},t}$ for finite $t$**

Let $U_1, \ldots, U_t$ be finite alphabets whose cardinalities are bounded as follows

$$|U_i| \leq \begin{cases} |X| |U_{i-1}| + t - j + 3, & \text{j odd,} \\ |Y| |U_{i-1}| + t - j + 3, & \text{j even.} \end{cases}$$

Note that these bounds are independent of blocklength $n$. For $j = 1, \ldots, t$, $j$ odd, let $p_{U_j|XU_{j-1}}$ denote a conditional pmf where for each $(x, u_{j-1}) \in X \times U_1 \times \ldots \times U_{j-1}$, $p_{U_j|XU_{j-1}}(\cdot|x, u_{j-1}) \in \Delta(U_j)$. Similarly, for $j = 1, \ldots, t$, $j$ even, let $p_{U_j|YU_{j-1}}$ denote a conditional pmf where for each $(y, u_{j-1}) \in Y \times U_1 \times \ldots \times U_{j-1}$, $p_{U_j|YU_{j-1}}(\cdot|y, u_{j-1}) \in \Delta(U_j)$. Let $X, Y, U_1, \ldots, U_t$ denote random variables taking values in $X \times Y, U_1, \ldots, U_t$ respectively with joint pmf $p_{XYU^t} = p_{XY}p_{U^t|XY}$ where for all $(x, y) \in X \times Y$ and all $u^t \in \bigotimes_{i=1}^t U_i$,

$$p_{U^t|XY}(u^t|x, y) = p_{U_1|X}(u_1|x) \cdot p_{U_2|YU_1}(u_2|y, u_1) \cdot p_{U_3|YU^2}(u_3|x, u^2) \ldots .$$

Here, $X$ and $Y$ are referred to as the source random variables and $U^t$ as the auxiliary random variables. Note that $p_{U^t|XY}$ is a conditional pmf where for each $(x, y) \in X \times Y$, $p_{U^t|XY}(\cdot|x, y) \in \Delta(U_1 \times \ldots \times U_t)$. The factorization of $p_{U^t|XY}(u^t|x, y)$ in (2) is equivalent to the following Markov chain conditions involving $X, Y, U^t$: for $i = 1, \ldots, t$, if $i$ is odd, $U_i - (X, U^{i-1}) - Y$ forms a Markov chain, otherwise $U_i - (Y, U^{i-1}) - X$ forms a Markov chain. Let

$$\mathcal{P}^A_{\text{mc},t} := \{ \text{all conditional pmfs } p_{U^t|XY} \text{ of the form (2)} \} .$$

Thus, $\mathcal{P}^A_{\text{mc},t}$ is a family of conditional pmfs parameterized (continuously) by the conditional pmfs $p_{U_1|X}$, $p_{U_2|YU_1}$, $\ldots$. For finite $t$, $\mathcal{P}^A_{\text{mc},t}$ is a compact subset of a finite-dimensional Euclidean space. Let

$$\mathcal{P}_{\text{ent},t}(p_{XY}, f_A, f_B) := \{ p_{U^t|XY} : H(f_A(X, Y), X^t, U^t) = H(f_B(X, Y), Y^t, U^t) = 0 \} .$$

Note that for all $t \geq 2$, the set $\mathcal{P}_{\text{ent},t}$ is not empty because one can choose $U_1$ and $U_2$ such that $H(X|U_1) = H(Y|U_2) = 0$: take $U_1$ (respectively $U_2$) to be a deterministic one-to-one mapping from $X$ to $U_1$ (respectively $Y$ to $U_2$) (note that $|X| \leq |U_1|$ and $|Y| \leq |U_2|$). Also note that $H(f_A(X, Y)|X, U^t)$ and $H(f_B(X, Y)|Y, U^t)$ are continuous functionals of the joint pmf $p_{XYU^t}$; and for each fixed $p_{XY}$, they are continuous functionals of $p_{U^t|XY}$. Thus, for finite $t$, $\mathcal{P}_{\text{ent},t}(p_{XY}, f_A, f_B)$ is a compact subset
of a finite-dimensional Euclidean space. Therefore, \( P_t^A(p_{XY}, f_A, f_B) := P_{mc,t}^A \cap P_{ent,t}^A(p_{XY}, f_A, f_B) \) is a compact subset of a finite-dimensional Euclidean space. Generally speaking, \( P_t^A \) is determined by \( p_{XY}, f_A, \) and \( f_B \). In the rest of this paper, however, \( f_A \) and \( f_B \) are fixed (but have general form) and \( p_{XY} \) is variable. Therefore, we drop \( f_A \) and \( f_B \) from the notation and speak of the family of conditional pmfs \( P_t^A(p_{XY}) \) associated with \( p_{XY} \). For initial terminal \( B \), the corresponding set is denoted by \( P_B^B(p_{XY}) \). We are now ready to state the characterization of \( R_{sum,t}^A(p_{XY}) \) developed in [1].

**Fact 1:** (Characterization of \( R_{sum,t}^A(p_{XY}) \) [1, Corollary 1])

\[
R_{sum,t}^A = \min_{P_{U|XY}} \{I(X; U|Y) + I(Y; U|X)\}. \tag{5}
\]

Note that the conditional mutual information quantities in (5) are continuous functionals of the joint pmf \( p_{XYU} \). In the minimization in (5), \( p_{XY}, f_A, \) and \( f_B \) are fixed. Since we are minimizing a continuous functional over a compact set, a minimizer exists in \( P_t^A(p_{XY}) \). Since the arguments live in a finite dimensional Euclidean space, the minimization in (5) is a finite dimensional optimization problem.

The characterization of \( R_{sum,t}^A(p_{XY}) \) in (5) does not directly inform us how quickly \( R_{sum,t}^A(p_{XY}) \) converges to \( R_{sum,\infty}^A \), i.e., bounds on the rate of convergence are unavailable for general \( p_{XY}, f_A, \) and \( f_B \). In the absence of such bounds, one pragmatic approach to estimate \( R_{sum,\infty}^A \) is to compute \( R_{sum,t}^A(p_{XY}) \) by numerically solving (with some machine precision) the finite-dimensional optimization problem in (5) for increasing values of \( t \) until the difference between \( R_{sum,t-1}^A(p_{XY}) \) and \( R_{sum,t}^A(p_{XY}) \) is smaller than some small number. Although (5) provides a single-letter characterization for \( R_{sum,t}^A(p_{XY}) \) for each finite \( t \), as \( t \) increases, an increasing number of auxiliary random variables \( U \) are involved in the optimization problem. In fact, due to (1), the upper bounds for \( |U| \) increase exponentially with respect to \( t \). Therefore, the dimension of the optimization problem in (5) explodes as \( t \) increases. Each iteration is computationally much more demanding than the previous one. To make matters worse, there appears to be no obvious way of re-using the computations done for evaluating \( R_{sum,t-1}^A(p_{XY}) \) when evaluating \( R_{sum,t}^A(p_{XY}) \), i.e., every time \( t \) is increased, a new optimization problem needs to be solved all over again. Finally, if we need to estimate \( R_{sum,\infty}^A \) for a different joint pmf \( p_{XY} \) (but for the same functions \( f_A \) and \( f_B \)), we would need to repeat this entire process for the new \( p_{XY} \).

In Sec. III, we take a new fundamentally different approach. We first develop a general convex-geometric blocklength-free characterization of \( R_{sum,\infty}^A(p_{XY}) \) which does not involve taking a limit as \( t \to \infty \). Furthermore, instead of developing the characterization of \( R_{sum,\infty}^A(p_{XY}) \) for a fixed joint pmf \( p_{XY} \) – which is a single nonnegative real number – we characterize the entire infinite-message minimum sum-rate surface \( R_{sum,\infty}^A(p_{XY}) \) – which is a functional of the joint pmf \( p_{XY} \) – in a single concise description. This leads to a simple test for checking if a given achievable sum-rate functional of \( p_{XY} \) coincides with \( R_{sum,\infty}^A(p_{XY}) \). It also provides a whole new family of lower bounds for \( R_{sum,\infty}^A(p_{XY}) \). In Sec. IV we use the new characterization to develop an iterative algorithm for computing the surfaces \( R_{sum,\infty}^A(p_{XY}) \) and \( R_{sum,t}^A(p_{XY}) \) (for any finite \( t \)) in which, crudely speaking, the complexity of computation in each iteration does not grow with iteration number and results from the previous iteration are re-used in the following one. In Sec. V we use the new characterization to evaluate \( R_{sum,\infty}^A(p_{XY}) \) exactly, in closed analytic form, for two specific examples. For one of the examples (Sec. VI-A), in an earlier work we had derived an upper bound for \( R_{sum,\infty}^A(p_{XY}) \) using an achievable distributed source coding strategy that uses infinitely many infinitesimal-rate messages, but had been unable to establish the optimality of that strategy. The new characterization, however, shows this to be optimal. In Sec. VI we show how these results extend to the general rate-distortion problem.

### III. Characterization of \( R_{sum,\infty}^A(p_{XY}) \)

#### A. The rate reduction functional \( \rho_t^A(p_{XY}) \)

If the goal is to losslessly reproduce the sources \( f_A(x,y) = y, f_B(x,y) = x \), the minimum sum-rate is equal to \( H(X|Y) + H(Y|X) \) and this can be achieved by Slepian-Wolf coding. The sum-rate needed for computing functions can only be smaller than that needed for reproducing sources losslessly. The reduction in the minimum sum-rate for function computation in
comparison to source reproduction is given by
\[\rho^A_t := H(X|Y) + H(Y|X) - R^A_{\text{sum},t} = \max_{p_u|Y,U \in P_{XY}} [H(X|Y,U') + H(Y|X,U')].\]  

For interactive distributed source codes with initial terminal B, the minimum sum-rate and rate reduction are denoted by \(R^B_{\text{sum}}\) and \(p^B_t\) respectively. A quantity which plays a key role in the characterization of \(R^B_{\text{sum,nc}}\) is \(\rho^A_t\) corresponding to the “rate reduction” for zero messages (there are no auxiliary random variables in this case). Since the initial terminal has no significance when \(t = 0\), \(\rho^A_0 = \rho^B_0 = \rho_0\). Let
\[\mathcal{P}_{f_A,f_B} := \{p_{XY} \in \Delta(X \times Y) : H(f_A(X),Y|X) = H(f_B(X),Y|Y) = 0\}.
\]

Error-free computations can be performed without any message transfers if, and only if, \(p_{XY} \in \mathcal{P}_{f_A,f_B}\). Thus,
\[R_{\text{sum,0}} = \begin{cases} 0, & \text{if } p_{XY} \in \mathcal{P}_{f_A,f_B}, \\ +\infty, & \text{otherwise}, \end{cases}\]
\[\rho_0 = \begin{cases} H(X|Y) + H(Y|X), & \text{if } p_{XY} \in \mathcal{P}_{f_A,f_B}, \\ -\infty, & \text{otherwise}. \end{cases}\]

**Remark 1:** If \(f_A(x,y)\) is not a function of \(x\) alone and \(f_B(x,y)\) is not a function of \(y\) alone, then for all \(p_{XY} \in \mathcal{P}_{f_A,f_B}\), we have \(\text{supp}(p_{XY}) \neq X \times Y\). Such \(p_{XY}\) can only lie on the boundary of the probability simplex \(\Delta(X \times Y)\).

Evaluating \(R^A_{\text{sum}}\) is equivalent to evaluating the rate reduction \(\rho^A_t\). Notice, however, that in (6), all the auxiliary random variables appear only as conditioned random variables whereas this is not the case in (5). As discussed in Sec. III-C, this difference is critical as it enables us to characterize \(\rho_0 := \lim_{t \to \infty} \rho^A_t = \lim_{t \to \infty} \rho^B_t\) which then gives us a characterization of \(R^B_{\text{sum,nc}}\) as \(R^B_{\text{sum,nc}} = H(X|Y) + H(Y|X) - \rho_0\). The rate reduction functional is the key to the characterization.

**B. Marginal-perturbations-closed family of joint pmfs \(\mathcal{P}_{XY}\)**

Generally speaking, \(R^A_{\text{sum}}, \rho^A_t, R_{\text{sum,0}}\) and \(\rho_0\) are functionals of \(p_{XY}, p_A,\) and \(f_B\). We will view \(R^A_{\text{sum}}(p_{XY}), \rho^A_t(p_{XY}), R_{\text{sum,nc}}(p_{XY})\) and \(\rho_0(p_{XY})\) as functionals of \(p_{XY}\) with \(f_A\) and \(f_B\) fixed to emphasize the dependence of \(p_{XY}\). Instead of evaluating \(\rho_0(p_{XY})\) for one particular \(p_{XY}\) as it is done in the numerical evaluation of single-terminal and Wyner-Ziv rate-distortion functions, our approach is to evaluate \(\rho_0(p_{XY})\) for all \(p_{XY}\) belonging to \(\mathcal{P}_{XY}\) – a collection of joint pmfs of interest which is closed in the sense of Definition 4. We will develop a characterization of \(\rho_0(p_{XY})\) for the entire pmf-collection \(\mathcal{P}_{XY}\); not just for one particular \(p_{XY}\). Central to the definition of \(\mathcal{P}_{XY}\) is the idea of a marginal perturbation set which is discussed next.

**Definition 3:** *(X-marginal and Y-marginal perturbation sets \(\mathcal{P}_{XY}(p_{XY})\) and \(\mathcal{P}_{X|Y}(p_{XY})\))*
The set of X-marginal perturbations of a pmf \(p_{XY} \in \Delta(X \times Y)\) is defined as
\[\mathcal{P}_{X|Y}(p_{XY}) := \{p'_{XY} \in \Delta(X \times Y) : p'_{XY} \ll p_{XY}, p'_{XY} | p_X = p_{XY} | p_X\}\]
where \(p_X\) and \(p'_{X}\) denote the X-marginals of \(p_{XY}\) and \(p'_{XY}\) respectively. Similarly, let
\[\mathcal{P}_{Y|X}(p_{XY}) := \{p'_{XY} \in \Delta(X \times Y) : p'_{XY} \ll p_{XY}, p'_{XY} | p_Y = p_{XY} | p_Y\}\]
declare the set of Y-marginal perturbations of \(p_{XY}\) where \(p_Y\) and \(p'_{Y}\) denote the Y-marginals of \(p_{XY}\) and \(p'_{XY}\) respectively.

The sets \(\mathcal{P}_{X|Y}(p_{XY})\) and \(\mathcal{P}_{Y|X}(p_{XY})\) are nonempty as they contain \(p_{XY}\). Notice that a pmf \(p'_{XY} \in \mathcal{P}_{X|Y}(p_{XY})\) iff \(p'_{X} \ll p_X\) and \(\forall (x,y) \in \text{supp}(p'_{X}) \times Y, p'_{X}(y|x) = p_{XY}(y|x), \) where \(p'_{X}, p'_{X}(y|x)\) and \(p_X, p_{XY}(y|x)\) are X-marginal and conditional pmfs of \(p'_{XY}\) and \(p_{XY}\) respectively. Essentially, \(\mathcal{P}_{X|Y}(p_{XY})\) is the collection of all joint pmfs \(p'_{XY}\) which have the same conditional pmf \(p_{Y|X}\) or \(p'_{XY} = p_{Y|X} \cdot p'_{X}\) on \(\text{supp}(p'_{X})\). The subtlety is that the conditional pmf \(p_{Y|X}\) of the joint pmf \(p'_{XY}\) is well-defined only on \(\text{supp}(p'_{X})\). Corresponding statements can be made for \(\mathcal{P}_{Y|X}(p_{XY})\). Marginal perturbation sets can be viewed as neighborhoods of \(p_{XY}\).
Remark 2: For all $p_{XY}$: (i) $\mathcal{P}_{XY}(p_{XY})$ and $\mathcal{P}_{XY}(p_{XY})$ are convex sets of joint pmfs; (ii) if $p_{XY}' \in \mathcal{P}_{XY}(p_{XY})$ then $\mathcal{P}_{XY}(p_{XY}') \subseteq \mathcal{P}_{XY}(p_{XY})$; and (iii) if $p_{XY}' \in \mathcal{P}_{XY}(p_{XY})$ then $\mathcal{P}_{XY}(p_{XY}') \subseteq \mathcal{P}_{XY}(p_{XY})$.

We will develop a characterization of $\rho_{\infty}(p_{XY})$ for all $p_{XY}$ belonging to any family of joint pmfs $\mathcal{P}_{XY}$ which is closed with respect to $X$-marginal and $Y$-marginal perturbations.

Definition 4: (Marginal-perturbations-closed family of joint pmfs $\mathcal{P}_{XY}$) A family of joint pmfs $\mathcal{P}_{XY} \subseteq \Delta(\mathcal{X} \times \mathcal{Y})$ will be called marginal-perturbations-closed if for all $p_{XY} \in \mathcal{P}_{XY}$, $\mathcal{P}_{XY}(p_{XY}) \cup \mathcal{P}_{XY}(p_{XY}) \subseteq \mathcal{P}_{XY}$.

Examples of such marginal-perturbations-closed families of joint pmfs include (i) the set of all joint pmfs with supports contained in a specified subset of $\mathcal{X} \times \mathcal{Y}$, i.e., $\mathcal{P}_{XY} = \Delta(S)$ where $S \subseteq \mathcal{X} \times \mathcal{Y}$ and (ii) the set of all joint pmfs of all independent sources: $\mathcal{P}_{XY} = \{p_{XY}(p_{X}p_{Y}) \mid p_{X} \in \Delta(\mathcal{X}), p_{Y} \in \Delta(\mathcal{Y})\}$ (see Sec. [?]). In fact, if $q_{XY}$ belongs to any marginal-perturbations-closed family with supp$(q_{X}) = \mathcal{X}$ and supp$(q_{Y}) = \mathcal{Y}$, then the family contains $\Delta(\mathcal{X}) \times \Delta(\mathcal{Y})$, that is, all product pmfs on $\mathcal{X} \times \mathcal{Y}$.

C. Main result

To describe the characterization of the functional $R_{\text{sum,}\infty}(p_{XY})$, it is convenient to define the following family of functionals associated with computing $f_{A}$ and $f_{B}$.

Definition 5: (Marginal-perturbations-concave, $\rho_{0}$-majorizing family of functionals $\mathcal{F}(\mathcal{P}_{XY})$) Let $\mathcal{P}_{XY}$ be any marginal-perturbations-closed family of joint pmfs on $\Delta(\mathcal{X} \times \mathcal{Y})$. The set of marginal-perturbations-concave, $\rho_{0}$-majorizing family of functionals $\mathcal{F}(\mathcal{P}_{XY})$ is the set of all the functionals $\rho: \mathcal{P}_{XY} \to \mathbb{R}$ satisfying the following three conditions:

1) $\rho_{0}$-majorization: $\forall p_{XY} \in \mathcal{P}_{XY}$, $\rho(p_{XY}) \geq \rho_{0}(p_{XY})$.
2) Concavity with respect to $X$-marginal perturbations: $\forall p_{XY} \in \mathcal{P}_{XY}$, $\rho$ is concave on $\mathcal{P}_{XY}(p_{XY})$.
3) Concavity with respect to $Y$-marginal perturbations: $\forall p_{XY} \in \mathcal{P}_{XY}$, $\rho$ is concave on $\mathcal{P}_{XY}(p_{XY})$.

Remark 3: Since $\rho(\rho_{0}(p_{XY})) = -\infty$ for all $p_{XY} \notin \mathcal{P}_{f_{A}f_{B}}$, condition 1) of Definition 5 is trivially satisfied for all $p_{XY} \in \mathcal{P}_{XY} \setminus \mathcal{P}_{f_{A}f_{B}}$ (we use the convention that $\forall a \in \mathbb{R}$, $a \geq -\infty$). Thus the statement that $\rho$ majorizes $\rho_{0}$ on the set $\mathcal{P}_{XY}$ is equivalent to the statement that $\rho$ majorizes $H(X|Y) + H(Y|X)$ on the set $\mathcal{P}_{f_{A}f_{B}} \cap \mathcal{P}_{XY}$.

Remark 4: Conditions 2) and 3) do not imply that $\rho$ is concave on $\mathcal{P}_{XY}$. In fact, $\mathcal{P}_{XY}$ itself may not be convex. For example, the set $\mathcal{P}_{XY} = \{p_{XY}(p_{X}p_{Y}) \mid p_{X} \in \Delta(\lambda), p_{Y} \in \Delta(\nu)\}$ is not convex.

We now state and prove the main result of this paper.

Theorem 1: (i) $\rho_{\infty} \in \mathcal{F}(\mathcal{P}_{XY})$. (ii) For all $\rho \in \mathcal{F}(\mathcal{P}_{XY})$, and all $p_{XY} \in \mathcal{P}_{XY}$, we have $\rho_{\infty}(p_{XY}) \leq \rho(p_{XY})$.

The set $\mathcal{F}(\mathcal{P}_{XY})$ is partially ordered with respect to majorization. The theorem says that $\mathcal{F}(\mathcal{P}_{XY})$ has a least element and that $\rho_{\infty}$ is the least element. Note that there is no parameter $t$ which needs to be sent to infinity in this characterization of $\rho_{\infty}$.

To prove Theorem 1 we will establish a connection between the $t$-message interactive coding problem and a $(t-1)$-message interactive coding subproblem. Intuitively, to construct a $t$-message interactive code with initial terminal $A$, we need to begin by choosing the first message. This corresponds to choosing the auxiliary random variable $U_{1}$. Then for each realization $U_{1} = u_{1}$, constructing the remaining part of the code becomes a $(t-1)$-message subproblem with initial terminal $B$ with the same desired functions, but with a different source pmf $p_{XY|U_{1}}(\cdot, |u_{1}) \in \mathcal{P}_{XY}(p_{XY})$. We can repeat this procedure recursively to construct a $(t-1)$-message interactive code with initial terminal $B$. After $t$ steps of recursion, we will be left with the trivial 0-message problem.

Proof: (i) We need to verify that $\rho_{\infty}$ satisfies all three conditions in Definition 5.

1) Since $\forall p_{XY} \in \mathcal{P}_{XY}$, $R_{\text{sum,}\infty}(p_{XY}) \leq R_{\text{sum,0}}(p_{XY})$, we have $\rho_{\infty}(p_{XY}) \geq \rho_{0}(p_{XY})$. Thus $\rho_{\infty}$ is $\rho_{0}$-majorizing.

2) For an arbitrary $q_{XY} \in \mathcal{P}_{XY}$, consider two arbitrary joint pmfs $p_{XY,1}, p_{XY,0} \in \mathcal{P}_{XY}(q_{XY})$. For every $\lambda \in \{0, 1\}$, let $p_{XY,\lambda} := \lambda p_{XY,1} + (1 - \lambda)p_{XY,0}$. Let $p_{X}(x), p_{XY,0}(y|x)$ and $p_{X,1}(x), p_{XY,1}(y|x)$ and $p_{X,\lambda}, p_{XY,\lambda}(y|x)$ denote the $X$-marginal and conditional pmfs of $p_{XY,0}$ and $p_{XY,1}$ respectively. Due to Remark 3), $p_{XY,\lambda} \in \mathcal{P}_{XY}(q_{XY})$. We need to show that $\rho_{\infty}(p_{XY,\lambda}) \geq \lambda \rho_{\infty}(p_{XY,1}) + (1 - \lambda)\rho_{\infty}(p_{XY,0})$. 


Let $(X, Y)$ be a pair of source random variables with joint pmf $p_{XY}$. Consider an auxiliary random variable $U^*_t$ taking values in $\mathcal{U}_t := \{0, 1\}$ such that $(X, Y, U^*_t) \sim p_{XYU}p_{U|X}$ where $\forall x \in \text{supp}(p_{X|x}), p_{U|X}(1|x) = \lambda p_{X|x}(x)$ and $p_{U|X}(0|x) = (1 - \lambda) p_{X|x}(x)/p_{X|x}(x)$.

It follows that the marginal pmf of $U^*_t$ is $\text{Ber}(\lambda)$ and $Y - X - U^*_t$ is a Markov chain. Consequently, $\mathcal{V}(x, u_1) \in \text{supp}(p_{X|x})$ for all joint pmfs in $p_{XYU}$ such that $(x, y, u_1) \in \text{supp}(p_{XYU})$. Let $(x, y, u_1) \in \mathcal{V}(x, u_1) \times \mathcal{U}_t$, $p_{XYU} (x, y, u_1) = p_{XYU} (x, y)$, and in the last but one equality we used the crucial property that all joint pmfs in $\mathcal{P}_{XY}(q_{XY})$ have the same conditional pmf.

Now, for all $t \in \mathbb{Z}^+$ we have,

$$
\rho^t_1(p_{XY}) = \max_{p_{U|X} \in \mathcal{P}_{U|X}} \{ H(X|Y, U^t) + H(Y|X, U^t) \}
$$

$$
= \max_{p_{U|X} \in \mathcal{P}_{U|X}} \left( \max_{p_{U|XU^t} \in \mathcal{P}_{U|XU^t}} \left\{ H(X|Y, U^t) + H(Y|X, U^t) \right\} \right)
$$

$$
\geq \max_{p_{U|XU^t} \in \mathcal{P}_{U|XU^t}} \left\{ H(X|Y, U^t_1, U^*_t, U^t_2, U^t_1) + H(Y|X, U^t_2, U^* = 1) \right\}
$$

$$
+ (1 - \lambda) \cdot \max_{p_{U|XU^t} \in \mathcal{P}_{U|XU^t}} \left\{ H(X|Y, U^t_2, U^t_1 = 0) + H(Y|X, U^t_2, U^t_1 = 0) \right\}
$$

$$
= \lambda \rho^1_1(p_{XY}) + (1 - \lambda) \rho^0_{t-1}(p_{XY}). \tag{8}
$$

In step (a) we replaced $p_{U|X}$ with the particular $p_{U|X}$ defined above. Step (b) follows from the “law of total conditional entropy” with the additional observations that conditioned on $U^*_t = u_t$, $p_{XYU^*_t}(x, y|u_t) = p_{XYU}(x, y)$ and $(H(X|Y, U^*_t, U^t_1 = u_t) + H(Y|X, U^t_2, U^*_t = u_t))$ only depends on $p_{U|XU^t}$. Conditioned on $U^*_t = u_t$, (i) $p_{U|XU^t} \in \mathcal{P}_{U|XU^t}$ if $p_{U|XU^t} \in \mathcal{P}_{U|XU^t}$ and (ii) $p_{U|XU^t} \in \mathcal{P}_{XYU^t}$ if $p_{U|XU^t} \in \mathcal{P}_{XYU^t}$. Therefore, $\rho^t_1(p_{XY}) \geq \lambda \rho^1_1(p_{XY}) + (1 - \lambda) \rho^0_{t-1}(p_{XY})$. Therefore, $\rho^t_1(p_{XY})$ satisfies condition 2) in Definition 5.

(3) In a similar manner, by reversing the roles of terminals $A$ and $B$ in the above proof, it can be shown that $\rho^t_1$ also satisfies condition 3) in Definition 5. Thus, $\rho^t \in \mathcal{F}(p_{XY})$.

(ii) It is sufficient to show that: $\forall \rho \in \mathcal{F}(p_{XY})$, $\forall p_{XY} \in \mathcal{P}_{XY}$, $\forall t \in \mathbb{Z}^+$, $\rho(p_{XY}) \leq \rho(p_{XY})$ and $\rho^t_1(p_{XY}) \leq \rho(p_{XY})$. We prove this by induction on $t$. For $t = 0$, the result is true by condition 1) in Definition 5. $\rho^0_0(p_{XY}) = \rho^0_0(p_{XY}) = \rho_0(p_{XY}) \leq \rho(p_{XY})$. Now assume that for an arbitrary $t \in \mathbb{Z}^+$, $\rho^t_1(p_{XY}) \leq \rho(p_{XY})$ and $\rho^t_1(p_{XY}) \leq \rho(p_{XY})$ hold. We will show that
\[ \rho^A(p_{XY}) \leq \rho(p_{XY}) \] and \[ \rho^B(p_{XY}) \leq \rho(p_{XY}) \] hold.

\[ \rho^A(p_{XY}) = \max_{p_{U|Y} \in \mathcal{P}^A(p_{XY})} \left\{ H(X|Y, U^t) + H(Y|X, U^t) \right\} \]

\[ = \max_{p_{U|Y}} \left\{ \max_{p_{U|XU^t}: p_{U|XU^t} \in \mathcal{P}^A(p_{XY})} \left\{ H(X|Y, U^t) + H(Y|X, U^t) \right\} \right\} \]

\[ \begin{aligned}
&\overset{(d)}{=} \max_{p_{U|Y}} \left\{ \sum_{u \in \text{supp}(p_{U|Y})} p_U(u_1) \max_{p_{U|XU^t}: p_{U|XU^t} \in \mathcal{P}^A(p_{XY})} \left\{ H(X|Y, U_1^t, U_1 = 1) + H(Y|X, U_1^t, U_1 = 1) \right\} \right\} \\
&\overset{(e)}{=} \max_{p_{U|Y}} \left\{ \sum_{u \in \text{supp}(p_{U|Y})} p_U(u_1) \rho_{XU^t}(p_{XY}|U_1, \cdot, |u_1) \right\} \\
&\overset{(f)}{=} \max_{p_{U|Y}} \left\{ \sum_{u \in \text{supp}(p_{U|Y})} p_U(u_1) \rho(p_{XY}|U_1, \cdot, |u_1) \right\} \\
&\overset{(g)}{=} \max_{p_{U|Y}} \left\{ \rho \left( \sum_{u \in \text{supp}(p_{U|Y})} p_U(u_1) p_{XY|U_1, \cdot, |u_1} \right) \right\} \\
&= \rho(p_{XY}).
\] (9)

The reasoning for steps (d) and (e) are similar to those for steps (b) and (c) respectively in the proof of part (i) (see equation array (3)) but for step (e) we need to also confirm that \( p_{XY|U_1}(\cdot, |u_1) \in \mathcal{P}_{XY}(p_{XY}) \) for all \( u_1 \in \text{supp}(p_{U_1}) \). This is confirmed by noting that since \( Y - X - U_1 \) is a Markov chain, \( \forall u_1 \in \text{supp}(p_{U_1}) \) and \( \forall x \in \text{supp}(p_X) \), we have \( p_{Y|XU_1}(y|x, u_1) = p_{Y|X}(y|x) \) (see para after Definition 3). Step (f) is due to the inductive hypothesis \( \rho^B_{U^t}(p_{XY}) \leq \rho(p_{XY}) \). Step (g) is Jensen’s inequality applied to \( \rho(p_{XY}) \) which is concave on \( \mathcal{P}_{XY}(p_{XY}) \). Using similar steps as above, we can also show that \( \rho^B(p_{XY}) \leq \rho(p_{XY}) \).

Remark 5: In the proof of Theorem 1 there are only two places where the marginal-perturbations-closed property of \( \mathcal{P}_{XY} \) is used. It is first used in part (i) to show that \( p_{XY|U_1}(x, y|u_1) = p_{XY|U_1}(x, y) \). It is used in part (ii) to show that \( p_{XY|U_1}(\cdot, |u_1) \in \mathcal{P}_{XY}(p_{XY}) \).

Remark 6: It can be verified that the functional \( (H(X|Y) + H(Y|X)) \) belongs to \( \mathcal{F}(\Delta(X|Y)) \). Whereas both \( (H(X|Y) + H(Y|X)) \) and \( \rho_{XU}(p_{XY}) \) are concave on \( X \)-marginal and \( Y \)-marginal perturbation sets of \( p_{XY} \), it cannot be claimed that \( R_{\text{sum}, \rho_{XU}}(p_{XY}) = (H(X|Y) + H(Y|X)) - \rho_{XU}(p_{XY}) \) will be convex on the marginal perturbation sets of \( p_{XY} \). For each \( t \), \( \rho^A_t \) is the maximum of \( (H(X|Y, U^t) + H(Y|X, U^t)) \), where \( U^t \) appear only as conditioned random variables. This enables us to use the “law of total conditional entropy” (which corresponds to convexification) and arrive at (3) and (2). Notice, however, that \( R_{\text{sum}, \rho_{XU}} \) is the minimum value of \( (H(X; U_1|Y) + I(Y; U_1|X)) \) over all \( U^t \) where \( U^t \) are not conditioned. Therefore, \( R^A_{\text{sum}, \rho_{XU}} \) cannot be expressed as a convex combination of \( R^B_{\text{sum}, \rho_{XU}, t-1} \). Due to these reasons, although evaluating \( \rho_{XU} \) is equivalent to evaluating \( R_{\text{sum}, \rho_{XU}} \), the rate reduction functional is the key to the characterization as remarked in Sec. III-A.

Since every \( \rho \in \mathcal{F}(\mathcal{P}_{XY}) \) gives an upper bound for \( \rho_{XU} \), \( (H(X|Y) + H(Y|X) - \rho) \) gives a lower bound for \( R_{\text{sum}, \rho_{XU}} \). This fact provides a way testing if an achievable sum-rate functional is optimal. If \( R^t \) is a sum-rate functional which is achievable then \( \forall p_{XY} \in \mathcal{P}_{XY}, R^t(p_{XY}) \geq R_{\text{sum}, \rho_{XU}}(p_{XY}) \). If it can be verified that \( \rho^* := (H(X|Y) + H(Y|X) - R^t) \) belongs to \( \mathcal{F}(\mathcal{P}_{XY}) \), then by Theorem 1 \( R^* = R_{\text{sum}, \rho_{XU}} \). The nontrivial part of the test is to verify if \( R^* \) is concave on \( X \)-marginal and \( Y \)-marginal perturbation sets. We will demonstrate this test on two examples in Sec. VI.

IV. ITERATIVE ALGORITHM FOR COMPUTING \( R^A_{\text{sum}, \rho_{XU}}(\cdot) \) AND \( R_{\text{sum}, \rho_{XU}}(\cdot) \)

Although Theorem 1 provides a characterization of \( \rho_{XU} \) and \( R_{\text{sum}, \rho_{XU}} \), that is not obtained by taking a limit, it does not directly provide an algorithm to evaluate \( R_{\text{sum}, \rho_{XU}} \). To efficiently represent and search for the least element of \( \mathcal{F}(\mathcal{P}_{XY}) \) is nontrivial.
because each element is a functional; not a scalar. The proof of Theorem 1 however, inspires an iterative algorithm for evaluating $R^A_{\text{sum},t}$ and $R^B_{\text{sum},\infty}$.

Equation (9) states that $\rho^A_t(p_{XY})$ is the maximum value of $\rho \in \mathbb{R}$ such that $(p_{XY}, \rho)$ is a finite convex combination of $\left\{ (p_{XY}|U_i, \cdot, \cdot | U_1), \rho^B_{t-1}(p_{XY}|U_i, \cdot, \cdot | U_1) \right\}_{U_1 \in \text{supp}(p_{U_1})}$, where $p_{XY}|U_i, \cdot, \cdot | U_1$ belongs to $\mathcal{P}_{XY}(p_{XY})$ for all $U_1$ in $\text{supp}(p_{U_1}) \subseteq \mathcal{U}_1$. Consider the hypograph of $\rho^B_{t-1}(\cdot) \colon \mathcal{P}_{XY}(p_{XY}) : \text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^B_{t-1} := \left\{(p_{XY}, \rho) : p_{XY} \in \mathcal{P}_{XY}(p_{XY}), \rho \leq \rho^B_{t-1}(p_{XY})\right\}$. Due to (9), the convex hull of $\text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^B_{t-1}$ is $\text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^A_t$. This enables us to evaluate $\rho^A_t$ from $\rho^B_{t-1}$ on the set $\mathcal{P}_{XY}(p_{XY})$: $\rho^A_t$ is the least concave functional on $\mathcal{P}_{XY}(p_{XY})$ that majorizes $\rho^B_{t-1}$. In the convex optimization literature, $(-\rho^A_t)$ is called the double Legendre-Fenchel transform or convex biconjugate of $(-\rho^B_{t-1})$ [9]. Thus $\rho^A_t$ can be determined through a convex biconjugation operation (taking a convex hull of a hypograph) on any given $X$-marginal perturbation set. To determine $\rho^A_t(p_{XY})$ for all $p_{XY} \in \mathcal{P}_{XY}$, we can, in principle, first choose a cover for $\mathcal{P}_{XY}$ made up of $X$-marginal perturbation sets, say $\{\mathcal{P}_{XY}(p_{XY})\}_{p_{XY} \in \mathcal{A}}$, where $\mathcal{A} \subseteq \mathcal{P}_{XY}$, and then perform the convex biconjugation operation in every $X$-perturbation set in the cover. This relationship between $\rho^A_t$ and $\rho^B_{t-1}$ leads to the following iterative algorithm.

**Algorithm to evaluate $R^A_{\text{sum},t}$ and $R^B_{\text{sum},t}$**

- **Initialization:** Choose a marginal-perturbations-closed family $\mathcal{P}_{XY}$ containing all source joint pmfs of interest. Define $\rho^A_0(p_{XY}) = \rho^B_0(p_{XY}) = \rho_0(p_{XY})$ by equation (7) in the domain $\mathcal{P}_{XY}$. Choose a cover for $\mathcal{P}_{XY}$ made up of $X$-marginal perturbation sets, denoted by $\{\mathcal{P}_{XY}(p_{XY})\}_{p_{XY} \in \mathcal{A}}$, where $\mathcal{A} \subseteq \mathcal{P}_{XY}$. Also choose a cover for $\mathcal{P}_{XY}$ made up of $Y$-marginal perturbation sets, denoted by $\{\mathcal{P}_{XY}(p_{XY})\}_{p_{XY} \in \mathcal{B}}$, where $\mathcal{B} \subseteq \mathcal{P}_{XY}$.

- **Loop:** For $t = 1$ through $t$ do the following.
  
  For every $p_{XY} \in \mathcal{A}$ do the following in the set $\mathcal{P}_{XY}(p_{XY})$.
  - Construct $\text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^B_t$.
  - Let $\rho^A_t$ be the upper boundary of the convex hull of $\text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^B_t$.

  For every $p_{XY} \in \mathcal{B}$ do the following in the set $\mathcal{P}_{XY}(p_{XY})$.
  - Construct $\text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^B_t$.
  - Let $\rho^B_t$ be the upper boundary of the convex hull of $\text{hyp}_{\mathcal{P}_{XY}(p_{XY})}^B_t$.

- **Output:** $R^A_{\text{sum},t}(p_{XY}) = H(X|Y) + H(Y|X) - \rho^A_t(p_{XY})$ and $R^B_{\text{sum},t}(p_{XY}) = H(X|Y) + H(Y|X) - \rho^B_t(p_{XY})$.

To make numerical computation feasible, $\mathcal{P}_{XY}$ has to be discretized. Once discretized, however, in each iteration, the amount of computation is the same and is fixed by the discretization step-size. Also note that results from each iteration are re-used in the following one. Therefore, for large $t$, the complexity to compute $R^A_{\text{sum},t}$ grows linearly with respect to $t$.

$R_{\text{sum},\infty}$ can also be evaluated to any precision, in principle, by running this iterative algorithm for $t = 1, 2, \ldots$ until some stopping criterion is met, e.g., the maximum difference between $\rho^A_{t-1}$ and $\rho^A_t$ on $\mathcal{P}_{XY}$ falls below some threshold. Developing stopping criteria with precision guarantees requires some knowledge of the rate of convergence which is not established in this paper; the rate may, however, be empirically estimated. For the example presented in Sec. 3.B, the process of convergence and the impact of the discretization step-size to the iterative evaluation is discussed. When the objective is to evaluate $R_{\text{sum},\infty}(p_{XY})$ for all pmfs in $\mathcal{P}_{XY}$, this iterative algorithm is much more efficient than using (5) to solve for $R^A_{\text{sum},t}$ for each $p_{XY}$ for $t = 1, 2, \ldots$, an approach which follows the definition of $R_{\text{sum},\infty}$ literally as the limit of $R^A_{\text{sum},t}$ as $t \to \infty$. Our iterative algorithm is based on Theorem 1 which is a characterization of $R_{\text{sum},\infty}$ without taking a limit involving $t$.

Since $-\rho^A_t$ is the convex biconjugate of $-\rho^B_{t-1}$ on all $X$-marginal perturbation sets and $-\rho^B_t$ is the convex biconjugate of $-\rho^A_{t-1}$ on all $Y$-marginal perturbation sets, it follows that for all $t > 0$, $\rho^A_t$ satisfies conditions 1) and 2) in Definition 5 (\(\rho_0\)-majorization and concavity with respect to $X$-marginal perturbations), and $\rho^B_t$ satisfies satisfies conditions 1) and 3) (\(\rho_0\)-majorization and concavity with respect to $Y$-marginal perturbations). By Theorem 1 $\rho_{\infty}$ satisfies all three conditions of Definition 5 and is not larger than any $\rho$ which satisfies all three conditions. Also for all $t$, by definition, $\rho^A_t \leq \rho_{\infty}$ and $\rho^B_t \leq \rho_{\infty}$. Hence, if for some $t$, $\rho^A_t$ satisfies 3) then $\rho^A_t = \rho_{\infty}$. Similarly, if for some $t$, $\rho^B_t$ satisfies 2) then $\rho^B_t = \rho_{\infty}$. Thus, $\rho^A_t$ and $\rho^B_t$ equal $\rho_{\infty}$ if they satisfy all three conditions. If all three conditions are not satisfied (two are always satisfied), it is
beneficial to increase the number of messages. Specifically, if $\rho^i_A$ is not concave on a $Y$-marginal perturbation set, then for some $p_{XY}$, $\rho^i_A(p_{XY}) < \rho^{i+1}_A(p_{XY}) \leq \rho^i_{r+2}(p_{XY})$. Note that if for some $t > 0$, $\rho^i_t = \rho^i_{r+1}$, then this functional must satisfy all three conditions, therefore $\rho^i_A = \rho_\infty$. That is, if it is not beneficial to add one message in the beginning of the communication, it is never beneficial to add arbitrarily many messages.

V. Examples

A. $R_{sum,\infty}$ for independent binary sources and Boolean AND function computed at both terminals

In [1, Sec. IV.F], we studied the samplewise computation of the Boolean AND function at both terminals for independent Bernoulli sources, i.e., $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, $X \perp Y$, $X \sim \text{Ber}(p)$, $Y \sim \text{Ber}(q)$, and $f_A(x, y) = f_B(x, y) = x \land y$. An interesting interactive coding scheme was described in [1] where the individual rate for each message vanished as the number of messages went to infinity. The (achievable) infinite-message sum-rate of this scheme, denoted by $R^*$, was evaluated in closed form as

$$R^*(p, q) = \begin{cases} h_2(p) + ph_2(q) + p \log_2 q + p(1 - q) \log_2 e, & \text{if } 0 \leq p \leq q \leq 1, \\ R^*(q, p), & \text{if } 0 \leq q \leq p \leq 1. \end{cases}$$

(10)

This expression was derived in [1, Sec. IV.F] for the situation $0 < p < q < 1$. The situation $0 < q \leq p < 1$ follows by symmetry. The remaining situations $pq = 0$ and $(1 - p)(1 - q) = 0$ easily follow using zero or one message. Since $R^*(p, q)$ is an achievable sum-rate, $R^* \geq R_{sum,\infty}$. Using Theorem 1, we shall now prove that $R^*$ is, in fact, equal to $R_{sum,\infty}$. We will verify that $\rho^* := H(X|Y) + H(Y|X) - R^*$ belongs to $\mathcal{F}(\mathcal{P}_{XY})$ for the product pmf family $\mathcal{P}_{XY}$, which will imply, by Theorem 1(ii), that $\rho^* \geq \rho_\infty$, i.e., $R^* \leq R_{sum,\infty}$. Note that $R_{sum,\infty}$ is not evaluated using Theorem 1. Only part (ii) of Theorem 1 is used as a converse proof to show that the achievable sum-rate $R^*$ is $R_{sum,\infty}$.

Since the sources are independent, we take the marginal-perturbations-closed family to be $\mathcal{P}_{XY} = \{p_{XY}|p_X \in \mathcal{A}(\mathcal{X}), p_Y \in \mathcal{A}(\mathcal{Y})\}$. For each product pmf $p_{XY}$, the $X$-marginal and $Y$-marginal perturbation sets are $\mathcal{P}_{XY}(p_{XY}) = \{p'_X : p'_X \ll p_X\}$ and $\mathcal{P}_{XY}(p_{XY}) = \{p'_Y : p'_Y \ll p_Y\}$ respectively. Since $p_X$ and $p_Y$ are parameterized by $p$ and $q$ respectively, each product pmf $p_{XY}$ can be represented by a point $(p, q) \in [0, 1]^2$. For all pmfs $(p, q) \in (0, 1)^2$, the $X$-marginal and $Y$-marginal perturbation sets are line segments $[0, 1] \times [q]$ and $[p] \times [0, 1]$ respectively. For all pmfs $(0, q)$, where $q \in (0, 1)$, the $X$-marginal and $Y$-marginal perturbation sets are $(0, q)$ and $(0) \times (0, 1]$ respectively. For the pmfs $(0, 0)$, both the $X$-marginal and $Y$-marginal perturbation sets are $(0, 0)$. The marginal perturbation sets of remaining pmfs on the boundary of $[0, 1]^2$ can be derived using symmetry (swap $p$ and $q$; then swap symbols 0 and 1).

It is easy to see that

$$R_{sum,\infty}(p, q) = \begin{cases} 0, & \text{if } (p, q) \in \mathcal{P}_{f_a,f_b}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\mathcal{P}_{f_a,f_b} = \{(p, q) : p = 0 \text{ or } q = 0 \text{ or } p = q = 1\}$. It is also easy to verify that for all $(p, q)$, $R^*(p, q) \leq R_{sum,\infty}(p, q) = 0$, or equivalently, $\rho^*(p, q) \geq \rho_0(p, q)$. By taking the first and second-order partial derivatives of $\rho^*(p, q) = h_2(p) + h_2(q) - R^*(p, q)$ with respect to $p$ and $q$, we can verify that for any fixed $q$, $\rho^*(p, q)$ is concave with respect to $p$, and for any fixed $p$, $\rho^*(p, q)$ is concave with respect to $q$. Therefore, $\rho^*(p, q)$ is concave in every $X$-marginal and $Y$-marginal perturbation set. Therefore, $\rho^*(p, q) \in \mathcal{F}(\mathcal{P}_{XY})$, which implies that $R_{sum,\infty}(p, q) \geq R^*(p, q)$ due to Theorem 1(ii). Since $R^*(p, q)$ is both an upper bound and a lower bound of $R_{sum,\infty}(p, q)$, we have $R_{sum,\infty}(p, q) = R^*(p, q)$.

B. $R_{sum,\infty}$ for independent binary sources and Boolean AND function computed at only terminal $B$

We change the problem in Sec. V.A to the problem of computing the Boolean AND function at only terminal $B$, i.e., $f_A(x, y) = 0$ and $f_B(x, y) = x \land y$. The source statistics are unchanged: $X \perp Y$, $X \sim \text{Ber}(p)$, $Y \sim \text{Ber}(q)$. An achievable
sum-rate $R^*$ presented below can be derived using the same technique presented in [1, Sec. IV.F].

$$R'(p, q) = \begin{cases} h_2(p) + ph_2(q) + p \log_2 q + p(1 - 2q) \log_2 e, & \text{if } 0 \leq p \leq q \leq 1/2, \\ R'(q, p), & \text{if } 0 \leq q \leq p \leq 1/2, \\ R'(1 - p, q), & \text{if } 0 \leq q \leq 1/2 \leq p \leq 1, \\ h_2(p), & \text{if } 1/2 \leq q \leq 1. \end{cases}$$ (11)

The proof of the achievability of $R^*$ is given in Appendix IV. Following the method in Sec. IV.A, it can be verified that $\mathcal{P}_{f,fb} = \{(p, q) : p = 0$ or $q = 0$ or $p = 1\}$ and $R'(p, q) = (h_2(p) + h_2(q) - R'(p, q))$ belongs to $\mathcal{F} (\mathcal{P}_{XY})$, where $\mathcal{P}_{XY} = \{p_X p_Y | p_X \in \Delta(X), p_Y \in \Delta(Y)\}$ is the same marginal-perturbations-closed family used in Sec. IV.A. Therefore, $R^* = R_{sum,co}$.

Iterative algorithm: Now we use this example to demonstrate the numerical implementation of the iterative algorithm discussed in Sec. IV.

- **Initialization:** Choose $\mathcal{P}_{XY} = [0, 1]^2$. Choose $\mathcal{A} = \{(1/2, q)\}_{q \in [0, 1]}$, which leads to a cover for $\mathcal{P}_{XY}$ made up of $X$-marginal perturbation sets $\{(0, 1) \times (q)\}_{q \in [0, 1]}$. Similarly, choose $\mathcal{B} = \{(p, 1/2)\}_{p \in [0, 1]}$, which leads to a cover made up of $Y$-marginal perturbation sets $\{p \times (0, 1)\}_{p \in [0, 1]}$. In order to perform numerical computation, we discretize $\mathcal{P}_{XY}$ into an $N \times N$ grid $\mathcal{P}^N_{XY} := \{(\frac{i}{N}, \frac{j}{N})\}_{i=0}^{N-1} \times j \in [0, 1]$. The two covers are correspondingly discretized into the collection of the columns and the collection of the rows of $\mathcal{P}^N_{XY}$. Compute $\rho_0^N(p, q) = \rho_0^N(p, q) = \rho_0(p, q)$ according to (7) for all $(p, q) \in \mathcal{P}^N_{XY}$ as follows.

$$\rho_0(p, q) = \begin{cases} h(p) + h(q), & \text{if } p = 0 \text{ or } q = 0 \text{ or } p = 1, \\ -\infty, & \text{otherwise.} \end{cases}$$

- **Loop:** For $\tau = 1$ to $d$ do the following.

  - For every $j \in \{0, \ldots, N - 1\}$, do the following. Let $\mathcal{H}(\rho_{\tau-1}^B)$ to be the set of points $\left(\frac{i}{N}, \frac{j}{N}, \frac{j}{N}\right)$ for all $i \in \{0, \ldots, N - 1\}$ such that $\rho_{\tau-1}^B \left(\frac{i}{N}, \frac{j}{N}, \frac{j}{N}\right) \neq -\infty$. For $i \in \{0, \ldots, N - 1\}$, define $\mathcal{H}(\rho_{\tau-1}^A) \left(\frac{i}{N}, \frac{j}{N}, \frac{j}{N}\right)$ in such a way that $\left(\frac{i}{N}, \frac{j}{N}, \frac{j}{N}\right)$ is on the upper boundary of the convex hull of $\mathcal{H}(\rho_{\tau-1}^A)$. For some $i$, there is no point in the convex hull with the first coordinate $\frac{j}{N}$, then set $\mathcal{H}(\rho_{\tau-1}^A) \left(\frac{i}{N}, \frac{j}{N}, \frac{j}{N}\right) = -\infty$. With this definition, $-\rho_{\tau}^B$ is the convex biconjugate of $-\rho_{\tau-1}^B$ on the $X$-marginal perturbation set taking the symbol $-\infty$ into consideration.

  - For every $i \in \{0, \ldots, N - 1\}$, do the following. Let $\mathcal{H}(\rho_{\tau-1}^A)$ to be the set of points $\left(\frac{i}{N}, \frac{i}{N}, \frac{j}{N}\right)$ for all $j \in \{0, \ldots, N - 1\}$ such that $\rho_{\tau-1}^A \left(\frac{i}{N}, \frac{i}{N}, \frac{j}{N}\right) \neq -\infty$. For $j \in \{0, \ldots, N - 1\}$, define $\rho_{\tau}^B \left(\frac{i}{N}, \frac{i}{N}, \frac{j}{N}\right)$ in such a way that $\left(\frac{i}{N}, \frac{i}{N}, \frac{j}{N}\right)$ is on the upper boundary of the convex hull of $\mathcal{H}(\rho_{\tau-1}^A)$. For some $j$, there is no point in the convex hull with the first coordinate $\frac{i}{N}$, then set $\rho_{\tau}^B \left(\frac{i}{N}, \frac{i}{N}, \frac{j}{N}\right) = -\infty$.

- **Output:** $R_{sum,t}^N(p, q) = h(p) + h(q) - \rho_{\tau}^B(p, q)$ for all $(p, q)$ on the grid.

Fig. 2 shows some plots for the rate reduction functions with different $t$. Since for $t \geq 2$, $\rho_1$ and $\rho_\infty$ are hardly distinguishable, we use the brightness to show $\rho_\infty - \rho_1 = (R_{sum,t} - R_{sum,co})$ in Fig. 2 to highlight the difference. Depending on specific joint pmf, the limit $R_{sum,co}$ could possibly be reached by $R_{sum,t}$ with a finite $t$. Specifically, for all $(p, q) \in \mathcal{P}_{f,fb}$, $R_{sum,0} = 0 = R_{sum,co}$ and no message needs to be sent. For all $(p, q) \in (0, 1) \times \{1/2, 1\}$, $R_{sum,co} = h_2(p)$ and this sum-rate can be achieved with $t = 1$ message from $A$ to $B$, thus $R_{sum,1} = R_{sum,co}$. Note that $R_{sum,0} = \infty$ because $(p, q) \notin \mathcal{P}_{f,fb}$ and $R_{sum,1} = \infty$. For $(p, q) \in \{1/2\} \times (0, 1/2)$, $R_{sum,co} = h_2(q)$. In [8, Sec. V.C] it was shown that this sum-rate can be achieved with $t = 2$ messages, the first from $B$ to $A$ and the second from $A$ to $B$. Thus $R_{sum,2} = R_{sum,co}$. Note that $R_{sum,1} = \infty$ and in [1, Sec. IV.C] we showed that $R_{sum,1} = \log_2 2 = 1$. For these distributions $(p, q)$ discussed above, $R_{sum,co}$ can be reached by $t = 0$, 1 or 2. However, we can see from Fig. 3 that when $(p, q)$ is close to the line segments $p = q < 1/2$ and $1 - p = q < 1/2$, we do need a large $t$ to get a $R_{sum,t}$ close to $R_{sum,co}$.

When the iterative algorithm is numerically implemented to approximate $\rho_\infty$, the accuracy depends on the number of iterations and the discretization step-size. Fig. 4 shows the dependency of the maximum error $\max_{p,q} (\rho_\infty(p, q) - \rho_1(p, q))$ with respect to $t$ and $N$. For each $N$, the maximum error decreases as $t$ increases until an error floor is reached. For a finer
Fig. 2. The rate reduction functions with different $t$ for $x \wedge y$ computed only at terminal B (Sec. V-B). $\rho_\infty$ is generated by the closed form expression. The other plots are generated by the iterative algorithm.

Fig. 3. Difference between $\rho_\infty$ and $\rho_t$ for $x \wedge y$ computed only at terminal B (Sec. V-B). The brightness represents the scaled logarithm of $(\rho_\infty - \rho_t)$. The white color means a large $(\rho_\infty - \rho_t)$ and the black color means $(\rho_\infty - \rho_t) < 10^{-4}$.

discretization with a larger $N$, the error floor is lower. Fig. 4(b) shows the relation between the error floor level and the computation time needed to reach the error floor for different $N$. Roughly speaking, when $N$ is doubled, the error floor level is halved and the computation time to reach the error floor is approximately multiplied by four.
the expected distortion conditions in Definition 6 by the conditions are denoted by way it has been defined. The sum-rate-distortion function message interactive function computation with initial terminal desired functions

\[ (a) \]

\[ \text{Dependency of the maximum error, } \max_{p,q} \rho_A(p,q) - \rho(p,q), \text{ with respect to } t \text{ and } N \text{ for Sec. (a)} \]

(b) relation between the error floor level and the computation time needed to reach the error floor for different \( N \).

\[ (b) \]

VI. EXTENSION TO INTERACTIVE RATE-DISTORTION PROBLEM

A. Problem formulation

In [2] we studied the interactive coding problem with per-sample distortion criteria. Let \( d_A : X \times Y \times Z_A \rightarrow D \) and \( d_B : X \times Y \times Z_B \rightarrow D \) be bounded single-letter distortion functions, where \( D = [0, d_{\text{max}}] \). The fidelity of function computation can be measured by the per-sample average distortion

\[ d^{(n)}_A(x, y, \hat{z}_A) := \frac{1}{n} \sum_{i=1}^{n} d_A(x(i), y(i), \hat{z}_A(i)), \quad d^{(n)}_B(x, y, \hat{z}_B) := \frac{1}{n} \sum_{i=1}^{n} d_B(x(i), y(i), \hat{z}_B(i)). \]

Of interest here are the expected per-sample distortions \( E[d^{(n)}_A(X, Y, \hat{Z}_A)] \) and \( E[d^{(n)}_B(X, Y, \hat{Z}_B)] \). Note that although the desired functions \( f_A \) and \( f_B \) do not explicitly appear in these fidelity criteria, they are subsumed by \( d_A \) and \( d_B \) because they accommodate general relationships between the sources and the outputs of the decoding functions. The performance of \( t \)-message interactive coding for function computation is measured as follows.

**Definition 6:** A rate-distortion tuple \( (R, D) = (R_1, \ldots, R_t, D_A, D_B) \) is admissible for \( t \)-message interactive function computation with initial terminal \( A \) if, \( \forall \epsilon > 0, \exists (N(e), t) \) such that \( \forall n > N(e), t \), there exists an interactive distributed source code with initial terminal \( A \) and parameters \( (i, n, |M_1|, \ldots, |M_t|) \) satisfying

\[ \frac{1}{n} \log_2 |M_j| \leq R_j + \epsilon, \quad j = 1, \ldots, t, \]

\[ E[d_A^{(n)}(X, Y, \hat{Z}_A)] \leq D_A + \epsilon, \quad E[d_B^{(n)}(X, Y, \hat{Z}_B)] \leq D_B + \epsilon. \]

The set of all admissible rate-distortion tuples, denoted by \( \mathcal{RD}_A^t \), is called the operational rate-distortion region for \( t \)-message interactive function computation with initial terminal \( A \). The rate-distortion region is closed and convex due to the way it has been defined. The sum-rate-distortion function \( R_{\text{sum},A}(D) \) is given by \( \min \left( \sum_{j=1}^{t} R_j \right) \) where the minimization is over all \( R \) such that \( (R, D) \in \mathcal{RD}_A^t \). For initial terminal \( B \), the rate-distortion region and the minimum sum-rate-distortion function are denoted by \( \mathcal{RD}_B^t \) and \( R_{\text{sum},B}(D) \) respectively. For any fixed \( D \), we define \( R_{\text{sum},A}(D) := \lim_{t \to \infty} R_{\text{sum},A}(D) = \lim_{t \to \infty} R_{\text{sum},B}(D) \).

The admissibility of a rate-distortion tuple can also be defined in terms of the probability of excess distortion by replacing the expected distortion conditions in Definition 6 by the conditions \( \mathbb{P}(d_A^{(n)}(X, Y, \hat{Z}_A) > D_A) \leq \epsilon \) and \( \mathbb{P}(d_B^{(n)}(X, Y, \hat{Z}_B) > D_B) \leq \epsilon \).
Although these conditions appear to be more stringent, it can be shown that they lead to the same operational rate-distortion region. For simplicity, we focus on the expected distortion conditions as in Definition 6.

B. Characterization of $R_{\text{sum,0}}^A(p_{XY}, D)$ and $\rho_{0}^A(p_{XY}, D)$ for finite $t$ [2]

The single-letter characterization of $R_{\text{sum,0}}^A(p_{XY}, D)$ is given by

$$R_{\text{sum,0}}^A(p_{XY}, D) = \min_{(p_{U'|Y}, \hat{g}_A, \hat{g}_B) \in \mathcal{P}_{A}^\delta(p_{XY}, D)} [I(X; U'|Y) + I(Y; U'|X)],$$

where $P_{A}^\delta(p_{XY}, D) := \{(p_{U'|Y}, \hat{g}_A, \hat{g}_B) : p_{U'|Y} \in \mathcal{P}_{\text{sec, } A}$, deterministic functions $\hat{g}_A, \hat{g}_B$ satisfying $E[d_A(X, Y, \hat{g}_A(U', X)) \leq D_A, E [d_B(X, Y, \hat{g}_B(Y))] \leq D_B\}$. Compared with (5), the expected distortion constraints replace the conditional entropy constraints. The rate reduction functional is defined as follows.

$$\rho_{0}^A(p_{XY}, D) := H(X|Y) + H(Y|X) - R_{\text{sum,0}}^A(D) = \max_{(p_{U'|Y}, \hat{g}_A, \hat{g}_B) \in \mathcal{P}_{A}^\delta(p_{XY}, D)} [H(X|Y, U') + H(Y|X, U')].$$

For $t = 0$, let $P_{f_{A,Y}D} := \{p_{XY} : \exists \hat{g}_A, \hat{g}_B, s.t. E[d_A(X, Y, \hat{g}_A(X))] \leq D_A, E [d_B(X, Y, \hat{g}_B(Y))] \leq D_B\}$. Then we have

$$R_{\text{sum,0}}(p_{XY}, D) = \begin{cases} 0, & \text{if } (p_{XY}, D) \in P_{f_{A,Y}D}, \\ +\infty, & \text{otherwise}. \end{cases}$$

$$\rho_{0}(p_{XY}, D) = \begin{cases} H(X|Y) + H(Y|X), & \text{if } (p_{XY}, D) \in P_{f_{A,Y}D}, \\ -\infty, & \text{otherwise}. \end{cases}$$

C. Characterization of $R_{\text{sum,oo}}^A(p_{XY}, D)$

We can use the same technique as in Sec. III to characterize the functional $\rho_{\infty}(p_{XY}, D)$.

Definition 7: (Marginal-perturbations-distortion-concave, $\rho_{0}$-majorizing family of functionals $\mathcal{F}_{D}(\mathcal{P}_{XY})$) Let $\mathcal{P}_{XY}$ be any marginal-perturbations-closed family of joint pmfs on $\Delta(X \times Y)$. The set of marginal-perturbations-distortion-concave, $\rho_{0}$-majorizing family of functionals $\mathcal{F}_{D}(\mathcal{P}_{XY})$ is the set of all the functionals $\rho : \mathcal{P}_{XY} \times \mathcal{D}^2 \rightarrow \mathbb{R}$ satisfying the following conditions:

1) $\rho_{0}$-majorization: $\forall p_{XY} \in \mathcal{P}_{XY}$ and $\forall D \in \mathcal{D}^2$, $\rho(p_{XY}) \geq \rho_{0}(p_{XY})$.
2) Concavity with respect to $X$-marginal perturbations and distortion vector: $\forall p_{XY} \in \mathcal{P}_{XY}$, $\rho$ is concave on $\mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2$.
3) Conavity with respect to $Y$-marginal perturbations and distortion vector: $\forall p_{XY} \in \mathcal{P}_{XY}$, $\rho$ is concave on $\mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2$.

The following characterization of $\rho_{\infty}(p_{XY}, D)$ is the generalization of Theorem 1 to the rate-distortion problem.

Theorem 2: (i) $\rho_{\infty}(p_{XY}, D) \in \mathcal{F}_{D}(\mathcal{P}_{XY})$. (ii) For all $\rho \in \mathcal{F}_{D}(\mathcal{P}_{XY})$ and $\forall (p_{XY}, D) \in \mathcal{P}_{XY} \times \mathcal{D}^2$, we have $\rho_{\infty}(p_{XY}, D) \leq \rho(p_{XY}, D)$.

The proof of Theorem 2 is parallel to that of Theorem 1.

Proof: (i) We need to verify that $\rho_{\infty}$ satisfies all three conditions in Definition 7.

1) Since $R_{\text{sum,oo}}^A(p_{XY}, D) \leq R_{\text{sum,0}}^A(p_{XY}, D)$, we have $\rho_{\infty}(p_{XY}, D) \geq \rho_{0}(p_{XY}, D)$.

2) For an arbitrary $q_{XY} \in \mathcal{P}_{XY}$, consider two tuples $(p_{XY,1}, D_1), (p_{XY,0}, D_0) \in \mathcal{P}_{XY}(q_{XY}) \times \mathcal{D}^2$. For every $\lambda \in (0, 1)$, let $(p_{XY,\lambda}, D_\lambda) := \lambda(p_{XY,1}, D_1) + (1 - \lambda)(p_{XY,0}, D_0)$. We need to show that $\rho_{\infty}(p_{XY,\lambda}, D_\lambda) \geq \lambda \rho_{\infty}(p_{XY,1}, D_1) + (1 - \lambda) \rho_{\infty}(p_{XY,0}, D_0)$. Using the same method as in the proof of Theorem 1 part (i.2), we construct the joint pmf $p_{XY,\lambda}$ with the following properties (P1) $U_1^\prime$ is Ber($\lambda$), (P2) $\forall (x, y, u_1) \in \text{supp}(p_{XY,1}) \times U_1^\prime, p_{XY,\lambda}(x, y|u_1) = p_{XY,\lambda}(x, y, u_1)$, and (P3) $Y - X + U_1^\prime$ is a Markov.
For every $t \in \mathbb{Z}^+$, we have
\[
\rho^A_t(p_{XY:t}, D_A) = \max_{(p_{U:t|X}, \hat{g}_A, \hat{g}_B) \in \mathcal{P}^A(p_{XY:t}, D_A)} \{H(X|Y, U^t) + H(Y|X, U^t)\}
\]
\[
= \max_{p_{U:1}} \max_{(p_{U:2|XY:1}, \hat{g}_A, \hat{g}_B) \in \mathcal{P}^A(p_{XY:1}, D_A)} \{H(X|Y, U^t) + H(Y|X, U^t)\}
\]
\[
\geq \max_{(p_{U:1}|X)} \{H(X|Y, U^t, U_1^t) + H(Y|X, U_2^t, U_1^t)\}
\]
\[
= \lambda \cdot \max_{(p_{U:1}|X)} \{H(X|Y, U_1^t, U_2^t, U_1^t) = 1) + H(Y|X, U_2^t, U_1^t = 1)\}
\]
\[
+ (1 - \lambda) \cdot \max_{(p_{U:1}|X)} \{H(X|Y, U_1^t, U_2^t, U_1^t = 0) + H(Y|X, U_2^t, U_1^t = 0)\}
\]
\[
\rho^{B}_{t-1}(p_{XY:1}, D_1) + (1 - \lambda) \rho^{B}_{t-1}(p_{XY:0}, D_0).
\]

In step (a) we replaced $p_{U:1|X}$ with the particular $p_{U:1|X}$ defined above, and replaced the overall distortion constraints $E[d_A(X, Y, \hat{g}_A(U_1^t, X))] \leq D_{A,t}$ and $E[d_B(X, Y, \hat{g}_A(U_2^t, Y))] \leq D_{B,t}$ by the stronger individual distortion constraints $E[d_A(X, Y, \hat{g}_A(U_1^t, U_2^t, X))]U_1^t = u_1] \leq D_{A,u_1}$ and $E[d_B(X, Y, \hat{g}_A(U_2^t, Y))]\{U_1^t = u_1 = 0 or 1. Step (b) follows from the “law of total conditional entropy” with the additional observations that conditioned on $U_1^t = u_1$, $(H(X|Y, U_1^t, U_2^t, U_1^t = u_1) + H(Y|X, U_2^t, U_1^t = u_1)$ only depends on $p_{U:2|XY:1}(\cdot, u_1)$, $\hat{g}_A(u_1, \ldots)$, and $\hat{g}_B(u_1, \ldots)$. Step (c) is due to the observation that for a fixed $p_{U:1|X}$, conditioned on $U_1^t = u_1$, $(p_{U:1|X}p_{U:2|XY:1}^t, \hat{g}_A, \hat{g}_B) \in \mathcal{P}^B_t(p_{XY,u_1}, D_{u_1})$ (iff $p_{U:2|XY:1}^t, \hat{g}_A, \hat{g}_B) \in \mathcal{P}^B_t(p_{XY,u_1}, D_{u_1})$. Now send $t$ to infinity in both the left and right sides of (13. We have $\rho_{n}\geq \lambda \rho_o(p_{XY,1}, D_1) + (1 - \lambda) \rho_o(p_{XY,0}, D_0)$. Therefore, $\rho_{o}$ satisfies condition 2) in the Definition 7. Similarly, it also satisfies condition 3).

(ii) It is sufficient to show that $\forall \rho \in \mathcal{T}_{D}(\mathcal{P}_{XY})$, for every tuple $(p_{XY}, D) \in \mathcal{P}_{XY} \times \mathcal{D}_{2}^{*}$, for every $t \in \mathbb{Z} \cup \{0\}$, $\rho^{A}_t(p_{XY}, D) \leq \rho(p_{XY}, D)$ and $\rho^{B}_t(p_{XY}, D) \leq \rho(p_{XY}, D)$ hold. We show this argument by induction on $t$. For $t = 0$, the statement is true by condition 1) in Definition 7. Then we assume that for an arbitrary $t \in \mathbb{Z}^+$, $\rho^{A}_{t-1}(p_{XY}, D) \leq \rho(p_{XY}, D)$ and $\rho^{B}_{t-1}(p_{XY}, D) \leq \rho(p_{XY}, D)$.
\(\rho(p_{XY}, D)\) hold. We will show that \(\rho^A(p_{XY}, D) \leq \rho(p_{XY}, D)\) and \(\rho^B(p_{XY}, D) \leq \rho(p_{XY}, D)\) hold.

\[
\rho^A(p_{XY}, D) = \max_{(p_{XY}, \Delta X, \Delta Y) \in \mathcal{P}_A(p_{XY}, D)} \left\{ H(X|Y, U^A) + H(Y|X, U^A) \right\}
\]

\[
\rho^B(p_{XY}, D) = \max_{(p_{XY}, \Delta X, \Delta Y) \in \mathcal{P}_B(p_{XY}, D)} \left\{ H(X|Y, U^B) + H(Y|X, U^B) \right\}
\]

The reasoning for steps (d), (e) and (f) are similar to that for steps (a), (b) and (c) in the proof of part (i). In step (d) when the overall distortion constraints are replaced by the individual distortion constraints, the maximum is not changed, because we go through all the possibilities for the individual distortion levels \(D_{ui}\) satisfying \(E[D_{ui}] = \sum_{i} D_{ui} p_{ui}(u_i) = D\). In step (f) we need to confirm that \(p_{XY|U_1}(\cdot, \cdot | u_1) \in \mathcal{P}_{XY}(p_{XY})\). The reasoning is as same as in step (e) in the proof of Theorem 1. Step (g) is due to the inductive hypothesis \(\rho^B_{r-1}(p_{XY}, D) \leq \rho(p_{XY}, D)\). Step (h) is due to the Jensen’s inequality applied to the concave functional \(\rho\). Using similar steps as above, we can also show \(\rho^B(p_{XY}, D) \leq \rho(p_{XY}, D)\).

Theorem 2 conveys the same intuition discussed in Sec. III. The main difference is that for each realization \(U_1 = u_1\), the distortion vector \(D_{ui}\) in the \((t-1)\)-message subproblem could be different from the original distortion vector \(D\), as long as \(E[D_{ui}] = D\). Therefore, we need to convexify over the distortion vector.

D. Iterative algorithm for computing \(R^A_{sum}(p_{XY}, D)\) and \(R^B_{sum}(p_{XY}, D)\)

The iterative algorithm presented in Sec. [X] can also be extended to the rate-distortion problem as follows. Equation (16) states that \(\rho^A(p_{XY}, D)\) is the maximum value of \(\rho E D^2\) such that \((p_{XY}, D, \rho)\) is a convex combination of \(\{(p_{XY|U_1}(\cdot, \cdot | u_1), D_{ui})\}_{ui \in \text{supp}(p_{ui})}\). Consider the hypograph of \(\rho^B_{r-1}(\cdot)\) on \(\mathcal{P}_{XY}(p_{XY}) \times D^2\): \(\text{hypo}_{\mathcal{P}_{XY}(p_{XY}) \times D^2} \rho^B_{r-1} := \{(p_{XY}, D, \rho) : (p_{XY}, D) \in \mathcal{P}_{XY}(p_{XY}) \times D^2, \rho \leq \rho^B_{r-1}(p_{XY}, D)\}\). Due to (16), the convex hull of \(\text{hypo}_{\mathcal{P}_{XY}(p_{XY}) \times D^2} \rho^B_{r-1}\) is \(\text{hypo}_{\mathcal{P}_{XY}(p_{XY}) \times D^2} \rho^A\). Therefore, we have the following algorithm which is similar to the one presented in Sec. III.

Algorithm to evaluate \(R^A_{sum}(D)\) and \(R^B_{sum}(D)\)

- **Initialization**: Choose a marginal-perturbations-closed family \(\mathcal{P}_{XY}\) containing all source joint pmfs of interest. Define \(\rho^A_0(p_{XY}, D) = \rho^B_0(p_{XY}, D) = \rho_0(p_{XY}, D)\) by equation (13) in the domain \(\mathcal{P}_{XY} \times D^2\). Choose a cover for \(\mathcal{P}_{XY}\) made up of \(X\)-marginal perturbation sets, denoted by \(\{P_{XY}(p_{XY})\}_{p_{XY} \in \mathcal{A}}\), where \(\mathcal{A} \subseteq \mathcal{P}_{XY}\). Also choose a cover for \(\mathcal{P}_{XY}\) made up of \(Y\)-marginal perturbation sets, denoted by \(\{P_{XY}(p_{XY})\}_{p_{XY} \in \mathcal{B}}\), where \(\mathcal{B} \subseteq \mathcal{P}_{XY}\).
• Loop: For \( \tau = 1 \) through \( t \) do the following.
  
  For every \( p_{XY} \in \mathcal{A} \), do the following in the set \( \mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2 \).
  
  - Construct hyp \( \mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2 \rho^B_{t-1} \).
  - Let \( \rho^A_t \) be the upper boundary of the convex hull of hyp \( \mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2 \rho^B_{t-1} \).
  
  For every \( p_{XY} \in \mathcal{B} \), do the following in the set \( \mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2 \).
  
  - Construct hyp \( \mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2 \rho^A_{t-1} \).
  - Let \( \rho^B_t \) be the upper boundary of the convex hull of hyp \( \mathcal{P}_{XY}(p_{XY}) \times \mathcal{D}^2 \rho^A_{t-1} \).

• Output: \( R^A_{\text{sum},j}(p_{XY}, \mathcal{D}) = H(X|Y) + H(Y|X) - \rho^A_t(p_{XY}, \mathcal{D}) \), and \( R^B_{\text{sum},j}(p_{XY}, \mathcal{D}) = H(X|Y) + H(Y|X) - \rho^B_t(p_{XY}, \mathcal{D}) \).

Here we need to discretize the set \( \mathcal{P}_{XY} \times \mathcal{D}^2 \). \( R_{\text{sum},n}(p_{XY}, \mathcal{D}) \) can also be evaluated to any precision by running this algorithm to a large enough value of \( t \), until the change between \( \rho^A_t(p_{XY}, \mathcal{D}) \) and \( \rho^A_t(p_{XY}, \mathcal{D}) \) is below a certain threshold.

In the special case \( t = 1 \), and \( d_1 \equiv 0 \), the interactive problem reduces to the Wyner-Ziv problem. If we further assume that \( |\mathcal{Y}| = 1 \), the Wyner-Ziv problem reduces to the single-terminal rate-distortion problem. Therefore, the algorithm described above can be used to evaluate the single-terminal and Wyner-Ziv rate-distortion functions as special cases.

VII. Concluding remarks

In this work, we studied a two-terminal interactive function computation problem with alternating messages within the framework of distributed block source coding theory. We introduced a new convex-geometric approach to provide a blocklength-free single-letter characterization of the infinite-message sum-rate-distortion function as a functional of the joint source pmf and distortion levels. This characterization is not obtained by taking a limit as the number of messages goes to infinity. Instead, it is in terms of the least element of a family of partially-ordered, marginal-perturbations-concave functionals defined by the coupled per-sample distortion criteria. An interesting direction would be to find an efficient algorithm to search for the least element in the family of functionals.

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Appendix I

Achievability of \( R^* \) in Sec. V-B

The achievability proof of \( R^*(p, q) \) for \( (p, q) \in (0, 1/2]^2 \) uses the same technique presented in [1, Sec. IV.F]. Once we have established \( R^*(p, q) \) for \( (p, q) \in (0, 1/2]^2 \), \( R^*(1 - p, q) \) is an achievable sum-rate for \( (p, q) \in [1/2, 1) \times (0, 1/2] \). The reason is that when \( p \geq 1/2 \), \( X \sim \text{Ber}(1 - p) \) where \( (1 - p) \leq 1/2 \). Using the achievable scheme for \( (p, q) \in (0, 1/2]^2 \), \( B \) can compute \( X^c \wedge Y \). Then \( X \wedge Y = (X^c \wedge Y)^c \wedge Y \) can be computed locally at \( B \). For \( 1/2 \leq q \leq 1 \), the rate \( H(X) = h_2(p) \) is achieved using one message from \( A \) to \( B \) sending \( X \). Now we will show the achievability of \( R^*(p, q) \) for \( (p, q) \in (0, 1/2]^2 \).

Define real auxiliary random variables \( (V_s, V_y) \sim \text{Uniform}([0, 1]^2) \). If \( X := I_{[1-p, 1]}(V_s) \) and \( Y := I_{[1-q, 1]}(V_y) \), then \( (X, Y) \) has the correct joint pmf, i.e., \( p_X(1) = 1 - p_X(0) = p, p_Y(1) = 1 - p_Y(0) = q \), and \( X \not\perp Y \). We will interpret 0 and 1 as real zero and real one respectively as needed. This interpretation will allow us to express Boolean arithmetic in terms of real arithmetic. Thus \( X \wedge Y \) (Boolean AND) = \( XY \) (real multiplication). Define a rate-allocation curve \( \Gamma \) parametrically by \( \Gamma := \{(\alpha(s), \beta(s)) \in [0, 1] \times [0, 1] \mid 0 \leq s \leq 1 \} \) where \( \alpha \) and \( \beta \) are real, nondecreasing, absolutely continuous functions with \( \alpha(0) = \beta(0) = 0 \), \( \alpha(1) = (1 - p) \), and \( \beta(1) \in [0, 1 - q] \). Note that in [1, Sec. IV.F] where the AND function is computed at both terminals rather than ony terminal \( B \), \( \Gamma \) need to satisfy a different condition \( \beta(1) = (1 - q) \). The significance of \( \Gamma \) will become clear later.

Now choose a partition of \( [0, 1] \), \( 0 = s_0 < s_1 < \ldots < s_{t-1} < s_{t/2} = 1 \), such that \( \max_{i=1,\ldots,t/2} (s_i - s_{i-1}) < \Delta_t \). For \( i = 1, \ldots, t/2 \), define \( t \) auxiliary random variables as follows,

\[
U_{2i-1} := I_{[\alpha(s_i), 1)] \cap [\beta(s_{i-1}), 1]}(V_s, V_y), \quad U_{2i} := I_{[\alpha(s_i), 1)] \cap [\beta(s_{i-1}), 1]}(V_s, V_y).
\]
In Fig. 5(a), \((V_x, V_y)\) is uniformly distributed on the unit square and \(U^t\) are defined to be 1 in rectangular regions which are nested. The following properties can be verified:

**P1:** \(U_1 \geq U_2 \geq \ldots \geq U_i.\)

**P2:** \(p_{U|XY} \in \mathcal{P}_{\text{ent,f}},\) or equivalently, \(H(X \land Y | Y, U^t) = 0;\) since \(U_t = 1_{[1-p,1]}(V_x, V_y)\) and \(Y = 1_{[1-q,1]}(V_y).\) Therefore \(U_t \land Y = 1_{[1-p,1]}(V_x, V_y) \land Y = X \land Y.\)

**P3:** \(p_{U|XY} \in \mathcal{P}_{\text{mac,f}},\) which can be equivalently written as Markov chain conditions: for example, consider \(U_{2i-1} - (Y, U_{2i-1}) - X.\)

\(U_{2i-1} = 0 \Rightarrow U_{2i} = 0\) and the Markov chain holds. \(U_{2i-1} = Y = 1 \Rightarrow (V_x, V_y) \in [\alpha(s_i), 1] \times [1-q, 1] \Rightarrow U_{2i} = 1\) and the Markov chain holds. Given \(U_{2i-1} = 1, Y = 0, (V_x, V_y) \sim \text{Uniform}([\alpha(s_i), 1] \times [\beta(s_{i-1}), 1-q]) \Rightarrow V_x\) and \(V_y\) are conditionally independent. Thus \(X \equiv U_{2i}(U_{2i-1} = 1, Y = 0)\) because \(X\) is a function of only \(V_x\) and \(U_{2i}\) is a function of only \(V_y\) upon conditioning. So the Markov chain \(U_{2i} - (Y, U_{2i-1}) - X\) holds in all situations.

**P4:** \((Y, U_{2i}) \equiv X|U_{2i-1} = 1:\) this can be proved by the same method as in **P3.**

**P2** and **P3** show that \(p_{U|XY} \in \mathcal{P}_{\alpha}(p_{XY}).\)
For $i = 1, \ldots, t/2$, the $(2i)$-th rate is given by

$$I(Y; U_2|x, U_2^{2i-1}) =$$

$$p_1 = I(Y; U_2|x, U_{2i-1} = 1)p_U(1)$$

$$p_4 = I(Y; U_2|U_2 = 1)q_U(1)$$

$$= H(Y|U_2 = 1)q_U(1) - H(Y|U_2, U_{2i-1} = 1)p_U(1)$$

$$= (1 - \alpha(s_i)) \left(1 - \beta(s_i)\right)h_2\left(q \left| \frac{1}{1-\beta(s_i)} \right. \right)$$

$$- (1 - \beta(s_i))h_2\left(q \left| \frac{1}{1-\beta(s_i)} \right. \right)$$

$$= \int_{\{0\leq x \leq 1\}} w_y(v_y, q)dv_y$$

where step (a) is due to property P4 and because $(U_{2i-1}, U_2) = (1, 0) \Rightarrow Y = 0$, hence $H(Y|U_2, U_{2i-1} = 1)p_U(1) = H(Y|U_2 = 1, U_{2i-1} = 1)p_U(1)$, and step (b) is because

$$\frac{\partial}{\partial v_y} \left(-(1 - v_y)h_2\left(q \left| \frac{1}{1-v_y} \right. \right) \right) = \log_2\left(\frac{1-v_y}{1-q-v_y} \right) = w_y(v_y, q).$$

The $2i$-th rate can thus be expressed as a 2-D integral of a weight function $w_y$ over the rectangular region $\mathcal{R}_{\text{Reg}(2i)} := [\alpha(s_i), 1] \times [\beta(s_i), 1]$ (a horizontal bar in Fig. 5a). Therefore, the sum of rates of all messages sent from terminal $B$ to terminal $A$ is the integral of $w_y$ over the union of all the corresponding horizontal bars in Fig. 5a. Similarly, the sum of rates of all messages sent from terminal $A$ to terminal $B$ can be expressed as the integral of another weight function $w_x(v_x, p) := \log_2((1-v_x)/(1-p-v_x))$ over the union of all the vertical bars in Fig. 5a.

Now let $t \to \infty$ such that $\Delta_t \to 0$. Since $\alpha$ and $\beta$ are absolutely continuous, $(\alpha(s_i) - \alpha(s_i-1)) \to 0$ and $(\beta(s_i) - \beta(s_i-1)) \to 0$. The union of the horizontal (resp. vertical bars) in Fig. 5a tends to the region $\mathcal{W}_y$ (resp. $\mathcal{W}_x$) in Fig. 5b. Hence an achievable infinite-message sum-rate given by

$$\int_{\mathcal{W}_x} w_x(v_x, p)dv_x + \int_{\mathcal{W}_y} w_y(v_y, q)dv_y$$

depends on only the rate-allocation curve $\Gamma$ which coordinates the progress of source descriptions at $A$ and $B$. When $0 < p < q \leq 1/2$, choose $\Gamma = \Gamma_1$ to be the piecewise linear curve connecting $(0,0), (1-p/q, 0), (1-2p, 1-2q), (1-p, 1-2q)$ in that order (see Fig. 5c). When $0 < q \leq p \leq 1/2$, choose $\Gamma = \Gamma_2$ to be the piecewise linear curve connecting $(0,0), (0, 1-q/p), (1-2p, 1-2q), (1-p, 1-2q)$ in that order (see Fig. 5d). For these two choices of the rate-allocation curve, (1.1) can be evaluated in closed form and is given by the expressions in the first two cases of (1.1), which completes the proof.

**Remark 7:** The two curves $\Gamma_1$ and $\Gamma_2$ were specifically chosen to minimize the value of (1.1). Although the minimization steps are nontrivial, they are omitted because the achievability of $R^*$ does not rely on them.

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