Abstract The octagon abstract domain is a widely used numeric abstract domain expressing relational information between variables whilst being both computationally efficient and simple to implement. Each element of the domain is a system of constraints where each constraint takes the restricted form $\pm x_i \pm x_j \leq c$. A key family of operations for the octagon domain are closure algorithms, which check satisfiability and provide a normal form for octagonal constraint systems. We present new quadratic incremental algorithms for closure, strong closure and integer closure and proofs of their correctness. We highlight the benefits and measure the performance of these new algorithms.

Keywords Abstract Interpretation · Octagons · Incremental Closure

1 Introduction

The view that simplicity is a virtue in competing scientific theories and that, other things being equal, simpler theories should be preferred to more complex ones, is widely advocated by scientists and engineers. Preferences for simpler theories are thought to have played a role in many episodes in science, and the field of abstract domain design is no exception. Abstract domains that have enduring appeal are typically those that are conceptually simple. Of all the weakly relational domains, for example, octagons [18] are arguably the most popular. One might claim that octagons are more elegant than, say, the two variable per inequality (TVPI) domain [27], and certainly they are easier to understand and implement. Yet one useful operation for this popular domain has remained elusive: incrementally updating an octagonal constraint system.

Inequalities in the octagon domain take the restricted form of $\pm x_i \pm x_j \leq c$, where $x_i$ and $x_j$ are variables and $c$ is a numerical constant. Difference bound matrices (DBMs) can be adapted to represent systems of octagonal constraints, for
which a key family of operations is closure. Closure, in its various guises, provides normal forms for DBMs, allowing satisfiability to be observed and equality to be checked. Closure also underpins operations such as join and projection (the forget operator), hence the concept of closure is central to the design of the whole domain. Closure uses shortest path algorithms, such as Floyd-Warshall \[10,30\], to check for satisfiability. However, octagons can encode unary constraints, which require a stronger notion of closure, known as strong closure, to derive a normal form. Moreover, a refinement to strong closure, called integer closure, is required to detect whether an octagon has an integral solution.

A frequent use-case in program analysis is adding a single new octagonal constraint to a closed DBM and then closing the augmented system. This is incremental closure. Incremental closure not only arises when an octagon for one line is adjusted to obtain an octagon for the next: incremental closure also occurs in integer wrapping \[26\] which involves repeatedly partitioning a space into two (by adding a single constraint), closing and then performing translation. Incremental closure also appears to be useful in access-based localisation \[21\], which involves repeatedly partitioning a space into two (by adding a single constraint), closing and then performing translation. Incremental closure also appears to be useful in access-based localisation \[21\], which analyses each procedure using abstractions defined over only those variables it accesses. One way to adapt localisation to octagons \[5\] is to introduce fresh variables, called anchors, that maintain the relationships which hold when a procedure is entered. One anchor is introduced for each variable that is accessed within the procedure. The body of the callee is analysed to capture how a variable changes relative to its anchor, and then this change is propagated into the caller. The abstraction of the callee is amalgamated with that of the caller by replacing the variables in the caller abstraction with their anchors, imposing the constraints from the callee abstraction, and then eliminating the anchors. If there are only a few non-redundant constraints in the callee \[2\] then incremental closure would be attractive for combining caller and callee abstractions.

In SMT solving, difference logic \[20\] is widely supported, suggesting that an incremental solver for the theory of octagons \[24\] would also be useful. Furthermore field in constraint solving, relational and mixed integer-real abstract domains show promise for enhancing constraint solvers \[22\] and octagons have been deployed for solving continuous constraints \[23\]. In this context, a split operator is used to divide the solution space into two sub-spaces by adding opposing constraints such as $x_i - x_j \leq c$ and $x_j - x_i \leq -c$. Splitting is repeatedly applied until a set of octagons is derived that cover the entire solution space, within a given precision tolerance. Propagation is applied after every split, which suggests incremental closure, and a scheme in which incremental closure is applied whenever a propagator updates a variable.

Closing an augmented DBM is less general than closing an arbitrary DBM and therefore one would expect incremental closure to be both efficient and conceptually simple. However the running time of the algorithm originally proposed for incremental closure \[14\] (Section 4.3.4) is cubic in the number of variables (see Section 4.1 for an explanation of the impact of checks). A quadratic algorithm has been proposed \[8\], but the algorithm missed a form of propagation and therefore did not always close the DBM (see Section 4.2 for a discussion).

The algorithms presented in this paper stem from the desire to provide simple correctness proofs. The act of restructuring and simplifying the proofs for \[8\], exposed a degenerate form of propagation and suggested new algorithmic solutions. These new algorithms yield three different points in the design-space continuum:
ranging from a short incremental closure algorithm that performs strengthening as a separate post-processing step; to a longer queue-based algorithm that performs strengthening on-the-fly. All three algorithms significantly outperform the incremental algorithm of Miné [17, Section 4.3.4], whilst entirely recovering closure.

1.1 Contributions

We summarise the contributions of our work as follows:

- Using new insights, we present new incremental algorithms for closure, strong closure and integer closure (Section 4, Section 5 and Section 6 respectively).
- We prove our algorithms correct and show how proofs for existing closure algorithms can be simplified.
- We give detailed proofs for in-place versions of our algorithms (Section 7).
- We implement these new algorithms which show significant performance improvements over existing closure algorithms (Section 8).

The paper is structured as follows: Section 2 contextualises this study and Section 3 provides the necessary preliminaries. Section 4 critiques the incremental algorithm of Miné, introduces a new incremental quadratic algorithm. Section 5 shows how to recover strong closure incrementally and do so, again, in a single DBM pass. Section 6 explains how to extend incrementally to integer closure. Section 7 suggests various optimisations to the incremental algorithms including in-place update. Experimental results are presented in Section 8 and Section 9 concludes.

2 Related Work

Since the thesis of Miné [17] and his subsequent magnum opus [18], algorithms for manipulating octagons, and even their representations, have continued to evolve. Early improvements showed how strengthening, the act of combining pairs of unary octagon constraints to improve binary octagon constraints, need not be applied repeatedly, but instead can be left to a single post-processing step [2]. This result, which was justified by an inventive correctness argument, led to a significant performance improvement of approximately 20% [2]. Showing that integer octagonal constraints admit polynomial satisfiability represented another significant advance [1], especially since dropping either the two variable or unary coefficient property makes the problem NP-complete [15].

Octagonal representations have come under recent scrutiny [14, Chapter 8]. In Coq, it is natural to realise DBMs as map from pairs of indices (represented as bit sequences) to matrix entries. Look-up becomes logarithmic in the dimension of the DBM, but the DBM itself can be sparse. Strengthening, which combine bounds on different variables, can populate a DBM with entries for binary constraints. Dropping strengthening thus improves sparsity, albeit at the cost of sacrificing a canonical representation. Join can be recovered by combining bounds during join itself, in effect, strengthening on-the-fly. Quite independently, sparse representations have recently been developed for differences [11]. Further field, $O(nm)$ decision procedures have been proposed for unit two variable per inequality (UTVPI)
constraints [16] where $m$ and $n$ are the number of constraints and variables respectively. Subsequently an incremental version was proposed for UTVPI [25] with time complexity $O(m+n \log(n)+p)$ where $p$ is the number of constraints tightened by the additional inequality. Certifying algorithms have also been devised for UTVPI constraints [29], supported by a graphical representation of these constraints, which aids the extraction of a certificate for validating unsatisfiability. DBMs, however, offer additional support for other operations that arise in program analysis such as join and projection. Moreover, there is no reason why each DBM entry could be augmented with a pair of row and column coordinates which records how it was updated, allowing a proof for unsatisfiability to be extracted from a negative diagonal entry.

Recent work [28] has proposed factoring octagons into independent sub-systems, which reduces the size of the DBM. Domain operations are applied point-wise to the independent sub-matrices of the DBM, echoing [12]. The work also shows how the regular access patterns of DBMs enable vectorisation, the step beyond which is harnessing general purpose GPUs [3]. Packs [7] have also been proposed as a factoring device in which the set of programs variables is covered by a sets of variables called packs (or clusters). An octagon is computed for each pack to abstract the DBM as a set of low-dimensional DBMs. Recent work has explored how packs can be introduced automatically using preanalysis and machine learning [13].

The alternative to simplifying the DBM representation is to assume that the DBM satisfies some prerequisites so that a domain operation need not be applied in full generality. Miné [17] showed that incremental version of the closure could be derived by observing that a new constraint is independent of the first $c$ variables of the DBM. This paper stems from an earlier work [8] that adopts an incremental algorithm for disjunctive spatial constraints [4] to DBMs. The work was motivated by the desire to augment [8] with conceptually simple correctness proofs, that revealed a deficiency in [8] which prompted a more thorough study of incrementality.

3 Preliminaries

This section serves as a self-contained introduction to the definitions and concepts required in subsequent sections. For more details, we invite the reader to consult both the seminal [17,18] and subsequent [2] works on the octagon abstract domain.

3.1 The Octagon Domain and its Representation

An octagonal constraint is a two variable inequality of the form $x_i \pm x_j \leq d$ where $x_i$ and $x_j$ are variables and $d$ is a constant. An octagon is a set of points satisfying a system of octagonal constraints. The octagon domain is the set of all octagons that can be defined over the variables $x_0, \ldots, x_{n-1}$.

Implementations of the octagon domain reuse the machinery developed for solving difference constraints of the form $x_i - x_j \leq d$. Miné [18] showed how to translate octagonal constraints to difference constraints over an extended set of variables $x'_0, \ldots, x'_{2n-1}$. A single octagonal constraint translates into a conjunction
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of one or more difference constraints as follows:

$$x_i - x_j \leq d \quad \text{and} \quad x_i - x_j \leq \ell \quad \text{and} \quad x_i - x_j \leq d$$

$$x_i + x_j \leq d \quad \text{and} \quad x_i + x_j \leq d \quad \text{and} \quad x_i + x_j \leq d$$

$$-x_i - x_j \leq d \quad \text{and} \quad -x_i - x_j \leq d \quad \text{and} \quad -x_i - x_j \leq d$$

$$x_i \leq d \quad \text{and} \quad x_i \leq d \quad \text{and} \quad x_i \leq d$$

A common representation for difference constraints is a difference bound matrix (DBM) which is a square matrix of dimension $n \times n$, where $n$ is the number of variables in the difference system. The value of the entry $d = m_{i,j}$ represents the constant $d$ of the inequality $x_i - x_j \leq d$ where the indices $i, j \in \{0, \ldots, n - 1\}$. An octagonal constraint system over $n$ variables translates to a difference constraint system over $2n$ variables, hence a DBM representing an octagon has dimension $2n \times 2n$.

Example 1 Figure 1 serves as an example of how an octagon translates to a system of differences. The entries of the upper DBM correspond to the constants in the difference constraints. Note how differences which are (syntactically) absent from the system lead to entries which take a symbolic value of $\infty$. Observe too how that DBM represents an adjacency matrix for the illustrated graph where the weight of a directed edge abuts its arrow.

The interpretation of a DBM representing an octagon is different to a DBM representing difference constraints. Consequently there are two concretisations for DBMs: one for interpreting differences and another for interpreting octagons, although the latter is defined in terms of the former:

Definition 3.1 Concretisation for rational ($\mathbb{Q}^n$) solutions:

$$\gamma_{\text{diff}}(m) = \{ (v_0, \ldots, v_{n-1}) \in \mathbb{Q}^n \mid \forall i,j : v_i - v_j \leq m_{i,j} \}$$

$$\gamma_{\text{oct}}(m) = \{ (v_0, \ldots, v_{n-1}) \in \mathbb{Q}^n \mid (v_0, v_1, \ldots, v_{n-1}) \in \gamma_{\text{diff}}(m) \}$$

where the concretisation for integer ($\mathbb{Z}^n$) solutions can be defined analogously.

Example 2 Since octagonal inequalities are modelled as two related differences, the upper DBM contains duplicated entries, for instance, $m_{1,2} = m_{3,0}$.
Operations on a DBM representing an octagon must maintain equality between the two entries that share the same constant of an octagonal inequality. This requirement leads to the definition of coherence:

**Definition 3.2 (Coherence)** A DBM $m$ is coherent iff $\forall i, j. m_{i,j} = m_{j,i}$ where $i = i + 1$ if $i$ is even and $i - 1$ otherwise.

*Example 3* For the upper DBM observe $m_{0,3} = 6 = m_{2,1} = m_{3,0}$. Coherence holds in a degenerate way for unary inequalities, note $m_{2,3} = 4 = m_{2,3} = m_{3,2}$.

The bar operation can be realised without a branch using $\bar{i} = i \text{xor} 1$ [17, Section 4.2.2]. Care should be taken to preserve coherence when manipulating DBMs, either by carefully designing algorithms or by using a data structure that enforces coherence [17, Section 4.5]. For clarity, we abstract away from the question of how to represent a DBM by presenting all algorithms for square matrices, rather than triangular matrices as introduced in [17, Section 4.5]. One final property is necessary for satisfiability:

**Definition 3.3 (Consistency)** A DBM $m$ is consistent iff $\forall i. m_{i,i} \geq 0$.

### 3.2 Definitions of Closure

Closure properties define canonical representations of DBMs, and can decide satisfiability and support operations such as join and projection. Bellman [6] showed that the satisfiability of a difference system can be decided using shortest path algorithms on a graph representing the differences. If the graph contains a negative cycle (a cycle whose edge weights sum to a negative value) then the difference system is unsatisfiable. The same applies for DBMs representing octagons. Closure propagates all the implicit (entailed) constraints in a system, leaving each entry in the DBM with the sharpest possible constraint entailed between the variables. Closure is formally defined below:

**Definition 3.4 (Closure)** A DBM $m$ is closed iff

- $\forall i. m_{i,i} = 0$
- $\forall i, j, k. m_{i,j} \leq m_{i,k} + m_{k,j}$

*Example 4* The top right DBM in Figure 1 is not closed. By running an all-pairs shortest path algorithm we get the following DBM:

\[
\begin{array}{cccc}
& x'_0 & x'_1 & x'_2 & x'_3 \\
x'_0 & 11 & 6 & 11 & 6 \\
x'_1 & 6 & 11 & 5 & 9 \\
x'_2 & 9 & 6 & 11 & 4 \\
x'_3 & 5 & 11 & 16 & 11 \\
\end{array}
\]

Notice that the diagonal values have non-negative elements implying that the constraint system is satisfiable. Running shortest path closure algorithms propagates all constraints and makes every explicit all constraints implied by the original system. Once satisfiability has been established, we can set the diagonal values to zero to satisfy the definition of closure.
Closure is not enough to provide a canonical form for DBMs representing octagons. Minè defined the notion of strong closure in [17,18] to do so:

**Definition 3.5 (Strong closure)** A DBM $m$ is strongly closed iff
- $m$ is closed
- $\forall i,j. m_{i,j} \leq m_{i,i}/2 + m_{j,j}/2$

The strong closure of DBM $m$ can be computed by $\text{Str}(m)$, the code for which is given in Figure 3. The algorithm propagates the property that if $x'_j - x'_i \leq c_1$ and $x'_i - x'_i \leq c_2$ both hold then $x'_j - x'_j \leq (c_1 + c_2)/2$ also holds. This sharpens the bound on the difference $x'_j - x'_i$ using the two unary constraints encoded by $x'_j - x'_j \leq c_1$ and $x'_i - x'_i \leq c_1$, namely, $2x'_j \leq c_1$ and $-2x'_i \leq c_2$. Note that this constraint propagation is not guaranteed to occur with a shortest path algorithm since there is not necessarily a path from a $m_{i,i}$ and $m_{j,j}$. An example in Figure 2 shows such a situation: the two graphs represent the octagon, but a shortest path algorithm will not propagate constraints on the left graph; hence strengthening is needed to bring the two graphs to the same normal form. Strong closure yields a canonical representation: there is a unique strongly closed DBM for any (non-empty) octagon [18]. Thus any semantically equivalent octagonal constraint systems are represented by the same strongly closed DBM. Strengthening is the act of computing strong closure.

**Example 5** The lower right DBM in Figure 1 gives the strong closure of the upper right DBM. Strengthening is performed after the shortest path algorithm.

For octagonal constraints over integers, the strong closure property may result in non-integer values due to the division by two. The definition of strong closure for integer octagonal constraints thus needs to be refined. If $x_i$ is integral then $x_i \leq c$ tightens to $x_i \leq \lfloor c \rfloor$. Since $x_i \leq c$ translates to the difference $x_{2i} - x_{2i+1} \leq 2c$, tightening the unary constraint is achieved by tightening the difference to $x_{2i} - x_{2i+1} \leq 2\lfloor c / 2 \rfloor$.

**Definition 3.6 (Tight closure)** A DBM $m$ is tightly closed iff
- $m$ is strongly closed
- $\forall i. m_{i,i}$ is even

For the integer case, a tightening step is required before strengthening. Tightening a closed DBM results in a weaker form of closure, called weak closure. Strong closure can be recovered from weak closure by strengthening [1]. Note, however, that we introduce the property for completeness because our formalisation and proofs do not make use of this notion.
Definition 3.7 (Weak closure) A DBM $m$ is weakly closed iff

\[ \forall i, m_{i,i} = 0 \]
\[ \forall i, j, k, m_{i,k} + m_{k,j} \geq \min(m_{i,j}, m_{i,i}/2 + m_{j,j}/2) \]

3.3 High-level Overview

Figure 3 gives a high-level overview of closure calculation. First a closure algorithm is applied to a DBM. Next, consistency is checked by observing the diagonal has non-negative entries indicating the octagon is satisfiable. If satisfiable, then the DBM is strengthened, resulting in a strongly closed DBM. Note that consistency does not need to checked again after strengthening. The dashed lines in the figure show the alternative path taken for integer problems: to ensure that the DBM entries are integral, a tightening step is applied which is then followed by an integer consistency check and strengthening.

Figure 4 shows how this architecture can be instantiated with algorithms for non-incremental strong closure. A Floyd-Warshall all-pairs shortest path algorithm can be applied to a DBM to compute closure, which is cubic in $n$. The check for consistency involves a pass over the matrix diagonal to check for a strictly negative entry, as illustrated in the figure. (Note that CheckConsistent resets a strictly positive diagonal entry to zero as in [21,5], but the incremental algorithms presented in this paper never relax a zero diagonal entry to a strictly positive value. Hence the reset is actually redundant for the incremental algorithms that follow.) The consistency check is linear in $n$. Strong closure can be additionally obtained by following closure with a single call to $Str$, the code for which is also listed in the figure. This is quadratic in $n$. 
4 Incremental Closure

We are interested in the specific use case of adding a new octagonal constraint to an existing octagon. Min`e designed an incremental algorithm for this very task, which can be refactored into computing closure and then separately strengthening, as depicted in Figure 3. His incremental algorithm, and a refinement, are discussed in Section 4.1. Section 4.2 presents our new incremental algorithm with improved performance.

4.1 Classical Incremental Closure

Min`e designed an incremental algorithm based on the observation that a new constraint will not affect all the variables of the octagon [17, Section 4.3.4]. Without loss of generality, suppose the inequality \( x < b \) is supported by putting \( b = \bar{a} \). Adding \( x_a - \bar{x}_b \leq d \) implies that the equivalent constraint \( x_a - \bar{x}_b \leq d \) is added too, and the entries \( m_{a,b} \) and \( m_{\bar{a},\bar{b}} \) are updated to \( d \) to reflect this. Figure 4 presents a version of the incremental algorithm of Min`e, specialised for adding \( x_a - \bar{x}_b \leq d \) to a closed DBM. The algorithm relies on the observation that updating \( m_{a,b} \) and \( m_{\bar{a},\bar{b}} \) will only (initially) mutate the rows and columns for the \( x_a, x_{\bar{a}}, x_b, x_{\bar{b}}, \bar{x}_b, \bar{x}_{\bar{a}} \) variables. Since \( m \) was closed, despite the updates, it still follows that \( m_{i,j} \leq m_{k,i} + m_{k,j} \) if \( \{i,j,k\} \cap \{a,\bar{a},b,\bar{b}\} = \emptyset \). To restore closure it only remains to enforce \( m_{i,j} \leq m_{k,i} + m_{k,j} \) for \( \{i,j,k\} \cap \{a,\bar{a},b,\bar{b}\} \neq \emptyset \).

The incremental algorithm thus applies Floyd-Warshall closure but only updates an entry \( m_{i,j} \) when \( \{i,j,k\} \cap \{a,\bar{a},b,\bar{b}\} \neq \emptyset \) (lines 7 and 8).

Note that the check \( \{i,j,k\} \cap \{a,\bar{a},b,\bar{b}\} \neq \emptyset \) can be decomposed into three separate checks \( i \in \{a,\bar{a},b,\bar{b}\}, j \in \{a,\bar{a},b,\bar{b}\} \) or \( k \in \{a,\bar{a},b,\bar{b}\} \). Then \( k \in \{a,\bar{a},b,\bar{b}\} \) can be hoisted outside the two inner loops, and likewise \( i \in \{a,\bar{a},b,\bar{b}\} \) can be hoisted

\[
\begin{align*}
1: & \text{function } \text{Close}(m) \\
2: & \text{for } k \in \{0, \ldots, 2n - 1\} \text{ do} \\
3: & \quad \text{for } i \in \{0, \ldots, 2n - 1\} \text{ do} \\
4: & \quad \quad \text{for } j \in \{0, \ldots, 2n - 1\} \text{ do} \\
5: & \quad \quad \quad m'_{i,j} \leftarrow \min(m_{i,j}, m_{i,k} + m_{k,j}) \\
6: & \quad \quad \text{end for} \\
7: & \quad \text{end for} \\
8: & \text{return } m' \\
9: & \text{end function}
\end{align*}
\]

\[
\begin{align*}
1: & \text{function } \text{CheckConsistent}(m) \\
2: & \text{for } i \in \{0, \ldots, 2n - 1\} \text{ do} \\
3: & \quad \text{if } m_{i,i} < 0 \text{ then} \\
4: & \quad \quad \text{return false} \\
5: & \quad \text{else} \\
6: & \quad \quad m_{i,i} = 0 \\
7: & \quad \text{end if} \\
8: & \text{end for} \\
9: & \text{return true} \\
10: & \text{end function}
\end{align*}
\]

Fig. 4: Non-incremental closure and strengthening
1: function MinéIncClose(m, x\_a - x\_b \leq d)
2: m\_a,b \leftarrow \min(m\_a,b, d);
3: m\_b,a \leftarrow \min(m\_b,a, d);
4: for k \in \{0, \ldots, 2n - 1\} do
5: for i \in \{0, \ldots, 2n - 1\} do
6: for j \in \{0, \ldots, 2n - 1\} do
7: if \{i, j, k\} \cap \{a, \bar{a}, b, \bar{b}\} \neq \emptyset then
8: m\_i,j \leftarrow \min(m\_i,j, m\_i,k + m\_k,j)
9: end if
10: end for
11: end for
12: return m
13: end function

outside the inner loop. Furthermore, \(i \in \{a, \bar{a}, b, \bar{b}\}\) can be reduced to four constant-time equality checks \((i = a) \lor (i = \bar{a}) \lor (i = b) \lor (i = \bar{b})\). These strength reductions mitigate against overhead of the check \(\{i, j, k\} \cap \{a, \bar{a}, b, \bar{b}\} \neq \emptyset\) itself.

This guard reduces the number of min operations from \(8n^3\) to \(8n^3 - (2n - 4)^3 = 48n^5 - 96n^3 + 64\) (notwithstanding those in Str), but at the overhead of \(8n^3\) checks. Thus the incremental algorithm in quadratic in the number of min operations but cubic in the number of checks (even with code hoisting).

4.2 Improved Incremental Closure

To give the intuition behind our new incremental closure algorithm, consider adding the constraint \(x\_a' - x\_b' \leq d\) and \(x\_b' - x\_a' \leq d\) to the closed DBM \(m\). The four diagrams given in Figure 6 illustrate how the path between variables \(x\_i'\) and \(x\_j'\) can be shortened. The distance between \(x\_i'\) and \(x\_j'\) is \(c\) \((m\_{i,j} = c)\), the distance between \(x\_a'\) and \(x\_b'\) is \(c\_1\) \((m\_{i,a} = c\_1)\), etc. The wavy lines denote the new constraints \(x\_a' - x\_b' \leq d\) and \(x\_b' - x\_a' \leq d\) and the heavy lines indicate short-circuiting paths between \(x\_i'\) and \(x\_j'\). The bottom left path of the figure illustrates how the distance between \(x\_i'\) and \(x\_k\) can be reduced from \(c\_1\) by the \(x\_k' - x\_a' \leq d\) constraint. The same path illustrates how to shorten the distance between \(x\_a'\) and \(x\_j'\) from \(c\_2\) using the \(x\_a' - x\_b' \leq d\) constraint. The bottom right path of the figure gives two symmetric cases in which \(c\_1\) and \(c\_2\) are sharpened by the addition of \(x\_a' - x\_b' \leq d\) and \(x\_b' - x\_a' \leq d\) respectively. Note that we cannot have the two paths from \(x\_i'\) to \(x\_a'\) and from \(x\_b'\) to \(x\_j'\) both shortened; at most one of them can change. The same holds for the two paths from \(x\_i'\) to \(x\_b'\) and \(x\_a'\) to \(x\_j'\). These extra paths lead to the following strategy for updating \(m\_{i,j}':\)

\[
\begin{pmatrix}
m\_{i,j}' \\
m\_{i,a} + d + m\_{i,b,j} \\
m\_{i,a} + d + m\_{i,b,j} \\
m\_{i,b} + d + m\_{i,a,b,j} \\
m\_{i,b} + d + m\_{i,a,b,j} \\
m\_{i,a} + d + m\_{i,b,j} + d + m\_{i,b,j}
\end{pmatrix}
\]
This leads to the incremental closure algorithm listed in top of Figure 7. Quintic min can be realised as four binary min operations, hence the total number of binary min operations required for $\text{IncClose}$ is $16n^2$, which is quadratic in $n$. The listing in the bottom of the figure shows how commonality can be factored out so that each iteration of the inner loop requires a single ternary min to be computed. Factorisation reduces the number of binary min operations to $2n(2 + 4n) = 8n^2 + 4n$ in $\text{IncCloseHoist}$. Moreover, this form of code hoisting is also applicable algorithms that follow (though this optimisation is not elaborated in the sequel). Furthermore, like $\text{IncClose}$, $\text{IncCloseHoist}$ is not sensitive to the specific traversal order of the DBM, hence has potential for parallelisation. In addition, both $\text{IncClose}$ and $\text{IncCloseHoist}$ do not incur any checks.

Example 6 To illustrate how the incremental closure algorithm of [8], from which the above is derived, omits a form of propagation, consider adding $x_0 \leq 0$, or equivalently $x_0 - x_1 \leq 0$, to the system on the left whose DBM $m$ is given on right. The system is illustrated spatially on the left hand side of Figure 8; the right hand side of the same figure shows the effect of

$$
\begin{align*}
x_0 &\leq 7, \\
x_1 &\leq 0, \\
x_0 - x_1 &\leq 7, \\
x_0 + x_1 &\leq 0
\end{align*}
$$

$$
\begin{pmatrix}
x_0' \\
x_1' \\
x_2' \\
x_3'
\end{pmatrix}
= 
\begin{pmatrix}
0 & 14 & 7 & 7 \\
\infty & 0 & \infty & \infty \\
\infty & 7 & 0 & 0 \\
\infty & 7 & \infty & 0
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{pmatrix}
$$

Fig. 6: Four ways to reduce the distance between $x'_i$ and $x'_j$
1: function IncClose(m, \( x_0 - x_1 \leq d \))
2:     for \( i \in \{0, \ldots, 2n - 1\} \) do
3:         for \( j \in \{0, \ldots, 2n - 1\} \) do
4:             \( m'_{i,j} \leftarrow \min \begin{pmatrix}
                 m_{i,j}, & m_{i,a} + d + m_{a,j}, & m_{i,\bar{b}} + d + m_{\bar{b},j}, & m_{i,a} + d + m_{\bar{b},\bar{b}} + d + m_{\bar{b},j} \\
            \end{pmatrix} \)
5:         end for
6:     end for
7:     if CheckConsistent(m') then
8:         return m'
9:     else
10:        return false
11:     end if
12: end function

1: function IncCloseHoist(m, \( x_0 - x_1 \leq d \))
2:     t_1 \leftarrow d + m_{0,a} + d;
3:     t_2 \leftarrow d + m_{0,\bar{b}} + d;
4:     for \( i \in \{0, \ldots, 2n - 1\} \) do
5:         t_3 \leftarrow \min(m_{i,a} + d, m_{i,\bar{b}} + t_1);
6:         t_4 \leftarrow \min(m_{i,\bar{b}} + d, m_{i,a} + t_2);
7:         for \( j \in \{0, \ldots, 2n - 1\} \) do
8:             \( m'_{i,j} \leftarrow \min(m_{i,j}, t_3 + m_{\bar{b},j}, t_4 + m_{\bar{a},j}) \)
9:         end for
10:     if \( m'_{i,i} < 0 \) then
11:        return false
12:     end if
13:     end if
14:     return m'
15: end function

Fig. 7: Incremental Closure (without and with code hoisting)

adding the constraint \( x_0 - x_1 \leq 0 \). Adding \( x_0 - x_1 \leq 0 \) using the incremental closure algorithm from [8] gives the DBM \( m' \); IncClose gives the DBM \( m'' \):

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\infty & 0 & \infty & \infty \\
\infty & 0 & \infty & 0 \\
\infty & 0 & \infty & \infty \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\infty & 0 & \infty & \infty \\
\infty & 0 & \infty & 0 \\
\infty & 0 & \infty & \infty \\
\end{bmatrix}
\]

The DBM \( m' \) represents the constraint \( x \leq \frac{7}{2} \) but \( m'' \) encodes the tighter constraint \( x \leq 0 \). The reason for the discrepancy between entries \( m'_{0,1} \) and \( m''_{0,1} \) is shown by the following calculations:

\[
m'_{0,1} = \min \left( \begin{bmatrix}
0, & 0, & 0, & 0 \\
0, & \infty, & \infty, & \infty \\
0, & \infty, & \infty, & 0 \\
0, & \infty, & \infty, & \infty \\
\end{bmatrix}
\right) = \begin{bmatrix}
14, & 0, & 0, & 7 \\
0, & 0, & 7, & 0 \\
0, & 0, & 7, & 0 \\
0, & 0, & 7, & 0 \\
\end{bmatrix} = 7
\]
The entry at \( m'_{0,1} \) is calculated using \( m_{2,1} \), but this entry will itself reduce to 0; \( m'_{0,1} \) must take into account the change that occurs to \( m_{2,1} \). More generally, when calculating \( m'_{i,j} \), the min expression of \([8]\) overlooks how the added constraint can tighten \( m_{i,a}, m_{i,b}, m_{i,b} \) or \( m_{a,j} \).

The new incremental algorithm is justified by Theorem 4.1 which, in turn, is supported by the following lemma:

**Lemma 4.1** Suppose \( m \) is a closed DBM, \( m' = \text{INCLOSE}(m, o) \) and \( o = (x'_a - x'_b \leq d) \). If \( m' \) is consistent then

- \( m_{b,a} + d \geq 0 \)
- \( m_{a,b} + d \geq 0 \)
- \( m_{a,a} + d + m_{b,b} + d \geq 0 \)

**Proof** Since \( m' \) is consistent \( m'_{a,a} \geq 0 \) hence

\[
\begin{align*}
\min \left( \begin{array}{c}
m_{a,a} + d + m_{b,a}, \\
m_{a,b} + d + m_{b,a}, \\
m_{b,b} + d + m_{a,a} + d + m_{b,a}, \\
m_{a,b} + d + m_{b,b} + d + m_{a,a}
\end{array} \right) = m'_{a,a} \geq 0
\end{align*}
\]

Therefore \( m_{a,b} + d + m_{a,a} \geq 0 \) and \( m_{a,b} + d + m_{b,b} + d + m_{a,a} \geq 0 \). Since \( m \) is closed \( m_{a,a} = 0 \) hence \( m_{a,b} + d \geq 0 \) and \( m_{a,a} + d + m_{b,b} + d \geq 0 \).

Repeating the argument \( m'_{b,b} \geq 0 \) hence

\[
\begin{align*}
\min \left( \begin{array}{c}
m_{b,b}, \\
m_{b,a} + d + m_{b,b}, \\
m_{b,b} + d + m_{a,a} + d + m_{b,b}, \\
m_{b,a} + d + m_{b,b} + d + m_{a,a}
\end{array} \right) = m'_{b,b} \geq 0
\end{align*}
\]

Therefore \( m_{b,a} + d + m_{b,b} \geq 0 \). Since \( m_{b,b} = 0 \) it follows that \( m_{b,a} + d \geq 0 \). \( \square \)
Theorem 4.1 (Correctness of IncClose) Suppose $m$ is a closed DBM, $m' = \text{IncClose}(m, o)$ and $o = (x_a' - x_b' \leq d)$. Then $m'$ is either closed or it is not consistent.

Proof Suppose $m'$ is consistent. Because $m$ is closed $0 = m_{i,i} \geq m'_{i,i} \geq 0$ hence $m'_{i,i} = 0$. It therefore remains to show $\forall i,j,k.m_{i,k} + m'_{k,j} \geq A$ where

$$A = \min \left( \begin{array}{c}
m_{i,j}, \\
m_{i,a} + d + m_{b,j}, \\
m_{i,b} + d + m_{a,j}, \\
m_{i,b} + d + m_{a,a} + d + m_{b,j}, \\
m_{i,a} + d + m_{b,b} + d + m_{a,j} \end{array} \right)$$

There are 5 cases for $m'_{i,k}$ and 5 for $m'_{k,j}$ giving 25 in total:

1-1. Suppose $m'_{i,k} = m_{i,k}$ and $m'_{k,j} = m_{k,j}$. Because $m$ is closed:

$$m'_{i,k} + m'_{k,j} = m_{i,k} + m_{k,j} \geq m_{i,j} \geq A$$

1-2. Suppose $m'_{i,k} = m_{i,k}$ and $m'_{k,j} = m_{k,a} + d + m_{b,j}$. Because $m$ is closed:

$$m'_{i,k} + m'_{k,j} = m_{i,k} + m_{k,a} + d + m_{b,j} \geq m_{i,a} + d + m_{b,j} \geq A$$

1-3. Suppose $m'_{i,k} = m_{i,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,j}$. Because $m$ is closed:

$$m'_{i,k} + m'_{k,j} = m_{i,k} + m_{k,b} + d + m_{a,j} \geq m_{i,b} + d + m_{a,j} \geq A$$

1-4. Suppose $m'_{i,k} = m_{i,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a} + d + m_{b,j}$. Because $m$ is closed:

$$m'_{i,k} + m'_{k,j} = m_{i,k} + m_{k,b} + d + m_{a,a} + d + m_{b,j} \geq m_{i,b} + d + m_{a,b} + d + m_{b,j} \geq A$$

1-5. Suppose $m'_{i,k} = m_{i,k}$ and $m'_{k,j} = m_{k,a} + d + m_{b,b} + d + m_{a,j}$. Because $m$ is closed:

$$m'_{i,k} + m'_{k,j} = m_{i,k} + m_{k,a} + d + m_{b,b} + d + m_{a,j} \geq m_{i,a} + d + m_{b,b} + d + m_{a,j} \geq A$$

2-1. Suppose $m'_{i,k} = m_{i,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,j}$. Symmetric to case 1-2.

2-2. Suppose $m'_{i,k} = m_{i,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,a} + d + m_{b,j}$. Because $m$ is closed and by Lemma 4.1:

$$m'_{i,k} + m'_{k,j} = m_{i,a} + d + m_{b,k} + m_{k,a} + d + m_{b,j} \geq m_{i,a} + d + m_{b,a} + d + m_{b,j} \geq A$$

2-3. Suppose $m'_{i,k} = m_{i,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,j}$. Because $m$ is closed:

$$m'_{i,k} + m'_{k,j} = m_{i,a} + d + m_{b,k} + m_{k,b} + d + m_{a,j} \geq m_{i,a} + d + m_{b,b} + d + m_{a,j} \geq A$$
2-4. Suppose $m'_{i,k} = m_{i,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a} + d + m_{b,j}$. Because $m$ is closed and by Lemma 4.1,

$$m'_{i,k} + m'_{k,j} = m_{i,a} + d + m_{b,k} + m_{k,b} + d + m_{a,a} + d + m_{b,j} \geq m_{i,a} + d + m_{b,k} + d + m_{a,a} + d + m_{b,j} \geq m_{i,a} + d + m_{b,j} \geq A$$

2-5. Suppose $m'_{i,k} = m_{i,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a} + d + m_{b,j}$. Because $m$ is closed and by Lemma 4.1,

$$m'_{i,k} + m'_{k,j} = m_{i,a} + d + m_{b,k} + m_{k,b} + d + m_{a,a} + d + m_{b,j} \geq m_{i,a} + d + m_{b,k} + d + m_{a,a} + d + m_{b,j} \geq m_{i,a} + d + m_{b,j} \geq A$$

3-1. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a}$ and $m'_{k,j} = m_{k,j}$. Symmetric to case 1-3.

3-2. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a}$ and $m'_{k,j} = m_{k,b} + d + m_{a,j}$. Symmetric to case 2-3.

3-3. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a}$.

Because $m$ is closed and by Lemma 4.1,

$$m'_{i,k} + m'_{k,j} = m_{i,b} + d + m_{a,a} + m_{k,b} + d + m_{a,a} \geq m_{i,b} + d + m_{a,a} + d + m_{a,j} \geq m_{i,b} + d + m_{a,j} \geq A$$

3-4. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a} + d + m_{b,j}$. Because $m$ is closed and by Lemma 4.1,

$$m'_{i,k} + m'_{k,j} = m_{i,b} + d + m_{a,a} + m_{k,b} + d + m_{a,a} \geq m_{i,b} + d + m_{a,a} + d + m_{b,j} \geq m_{i,b} + d + m_{a,j} \geq A$$

3-5. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a} + d + m_{b,j}$. Because $m$ is closed and by Lemma 4.1,

$$m'_{i,k} + m'_{k,j} = m_{i,b} + d + m_{a,a} + m_{k,b} + d + m_{a,a} \geq m_{i,b} + d + m_{a,a} + d + m_{b,j} = m_{i,b} + d + m_{a,j} \geq A$$

4-1. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,j}$. Symmetric to case 1-4.

4-2. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,a} + d + m_{b,j}$. Symmetric to case 2-4.

4-3. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,j}$. Symmetric to case 3-4.

4-4. Suppose $m'_{i,k} = m_{i,b} + d + m_{a,a} + d + m_{b,k}$ and $m'_{k,j} = m_{k,b} + d + m_{a,a} + d + m_{b,j}$. Because $m$ is closed and by Lemma 4.1,

$$m'_{i,k} + m'_{k,j} = m_{i,b} + d + m_{a,a} + d + m_{b,k} + m_{k,b} + d + m_{a,a} \geq m_{i,b} + d + m_{a,a} + d + m_{b,k} + d + m_{a,a} \geq m_{i,b} + d + m_{a,a} + d + m_{b,j} \geq m_{i,b} + d + m_{a,a} + d + m_{b,j} \geq A$$
4-5. Suppose $\mathbf{m}'_{i,k} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,k} + d + \mathbf{m}_{a,j}$ and $\mathbf{m}'_{k,j} = \mathbf{m}_{k,a} + d + \mathbf{m}_{b,j} + d + \mathbf{m}_{a,j}$. Because $\mathbf{m}$ is closed and by Lemma 4.1

$$\mathbf{m}'_{i,k} + \mathbf{m}'_{k,j} = \mathbf{m}_{i,k} + d + \mathbf{m}_{a,a} + d + \mathbf{m}_{b,k} + \mathbf{m}_{k,a} + d + \mathbf{m}_{b,j} + d + \mathbf{m}_{a,j}$$
$$\geq \mathbf{m}_{i,k} + d + \mathbf{m}_{a,a} + d + \mathbf{m}_{b,k} + \mathbf{d} + \mathbf{m}_{a,j}$$
$$\geq \mathbf{m}_{i,k} + d + \mathbf{m}_{a,a} + d + \mathbf{m}_{b,j} + \mathbf{d} + \mathbf{m}_{a,j}$$
$$\geq \mathbf{m}_{i,k} + d + \mathbf{m}_{a,j} \geq A$$

5-1. Suppose $\mathbf{m}'_{i,k} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k}$ and $\mathbf{m}'_{k,j} = \mathbf{m}_{k,j}$. Symmetric to case 1-5.

5-2. Suppose $\mathbf{m}'_{i,k} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k}$ and $\mathbf{m}'_{k,j} = \mathbf{m}_{k,a} + d + \mathbf{m}_{b,j}$. Symmetric to case 2-5.

5-3. Suppose $\mathbf{m}'_{i,k} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k}$ and $\mathbf{m}'_{k,j} = \mathbf{m}_{k,b} + d + \mathbf{m}_{a,j}$. Symmetric to case 3-5.

5-4. Suppose $\mathbf{m}'_{i,k} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k}$ and $\mathbf{m}'_{k,j} = \mathbf{m}_{k,a} + d + \mathbf{m}_{b,j} + \mathbf{d} + \mathbf{m}_{a,j}$. Symmetric to case 4-5.

5-5. Suppose $\mathbf{m}'_{i,k} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k}$ and $\mathbf{m}'_{k,j} = \mathbf{m}_{k,a} + d + \mathbf{m}_{b,b} + \mathbf{d} + \mathbf{m}_{a,j}$. Because $\mathbf{m}$ is closed and by Lemma 4.1

$$\mathbf{m}'_{i,k} + \mathbf{m}'_{k,j} = \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k} + \mathbf{m}_{k,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,j}$$
$$\geq \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k} + \mathbf{m}_{k,a} + d + \mathbf{m}_{b,j} + d + \mathbf{m}_{a,j}$$
$$\geq \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,k} + \mathbf{m}_{k,a} + d + \mathbf{m}_{b,j} + d + \mathbf{m}_{a,j}$$
$$\geq \mathbf{m}_{i,a} + d + \mathbf{m}_{b,b} + d + \mathbf{m}_{a,j} \geq A$$

Note that unsatisfiability can be detected without applying any min operations at all, but we omit this in our algorithms. This is justified by the following corollary of Lemma 4.1

**Corollary 4.1** Suppose $\mathbf{m}$ is a closed DBM, $\mathbf{m}' = \text{InNCLOSE}(\mathbf{m}, o)$ and $o = (x'_a - x'_b \leq d)$. If

- $\mathbf{m}_{b,a} + d < 0$ or
- $\mathbf{m}_{b,b} + d < 0$ or
- $\mathbf{m}_{b,b} + d + \mathbf{m}_{a,a} + d < 0$

then $\mathbf{m}'$ is not consistent.

4.3 Properties of Incremental Closure

By design InNCLOSE recovers closure, but it should also natural for the algorithm to preserve and enforce other properties too. These properties are not just interesting within themselves; they provide scaffolding for results that follow.

4.3.1 Idempotence

An important property of InNCLOSE is idempotence: it formalises the idea that an octagon should not change shape if it is repeatedly intersected with the same inequality. If idempotence did not hold then there would exist $\mathbf{m}' = \text{InNCLOSE}(\mathbf{m}, o)$
and \( m'' = \text{IncClose}(m', o) \) for which \( m' \neq m'' \). This would suggest that \text{IncClose} did not properly tighten \( m \) using the inequality \( o \), but overlooked some propagation, which is the form of suboptimal behaviour we are aim to avoid.

**Proposition 4.1** Suppose that \( m \) is a closed DBM, \( m' = \text{IncClose}(m, o) \), \( m'' = \text{IncClose}(m', o) \) and \( o = (x^a_a - x^b_b \leq d) \). Then either \( m' \) is consistent and \( m'' = m' \) or \( m'' \) is not consistent.

**Proof** Suppose \( m' \) is consistent. By Lemma 4.1 it follows that \( m_{a,a} + d \geq 0 \), \( m_{a,b} + d \geq 0 \), \( m_{a,a} + d + m_{b,b} + d \geq 0 \) and \( m_{b,b} + d + m_{a,a} + d \geq 0 \). Therefore

\[
\begin{pmatrix}
    m_{a,a} + d + m_{b,a}
    
    m_{a,b} + d + m_{a,a}
    
    m_{a,a} + d + m_{b,b} + d + m_{a,a}
    
    m_{b,b} + d + m_{a,a} + d + m_{b,a}
\end{pmatrix} = m_{a,a}
\]

Likewise \( m'_{b,b} = m_{b,b} \). Using the same inequalities it follows

\[
\begin{align*}
    m'_{i,a} &= \min \left( m_{i,a}, m_{i,b} + d + m_{a,a} \right)  \\
    m'_{i,b} &= \min \left( m_{i,b}, m_{i,a} + d + m_{b,b} \right)
\end{align*}
\]

Therefore

\[
\begin{align*}
    m'_{i,a} + d + m'_{b,j} &= \min \left( m_{i,a} + d + m_{b,j}, m_{i,a} + d + m_{b,b} + d + m_{a,j}, m_{i,b} + d + m_{a,a} + d + m_{b,b} + d + m_{a,j}, m_{b,b} + d + m_{a,a} + d + m_{b,j} \right) \\
    &\geq \min \left( m_{i,a} + d + m_{b,j}, m_{i,a} + d + m_{b,b} + d + m_{a,j}, m_{i,b} + d + m_{a,a} + d + m_{b,b} + d + m_{a,j}, m_{b,b} + d + m_{a,a} + d + m_{b,j} \right) \\
    &\geq m'_{i,j}
\end{align*}
\]

Likewise \( m'_{i,b} + d + m'_{a,j} \geq m'_{i,j} \). Now consider

\[
\begin{align*}
    m'_{i,a} + d + m'_{b,b} + d + m'_{a,j} &= \min \left( m_{i,a} + d + m_{b,b} + d + m_{a,j}, m_{i,a} + d + m_{b,b} + d + m_{a,a} + d + m_{b,j}, m_{i,b} + d + m_{a,a} + d + m_{b,b} + d + m_{a,j}, m_{i,b} + d + m_{a,a} + d + m_{b,b} + d + m_{a,j} \right) \\
    &\geq \min \left( m_{i,a} + d + m_{b,b} + d + m_{a,j}, m_{i,a} + d + m_{b,b} + d + m_{a,a} + d + m_{b,j}, m_{i,b} + d + m_{a,a} + d + m_{b,b} + d + m_{a,j}, m_{i,b} + d + m_{a,a} + d + m_{b,b} + d + m_{a,j} \right) \\
    &\geq m'_{i,j}
\end{align*}
\]

Likewise \( m'_{i,b} + d + m'_{a,a} + d + m'_{b,j} \geq m'_{i,j} \). Thus \( m''_{i,j} = m'_{i,j} \). Now suppose \( m' \) is not consistent. Since \( m''_{i,j} \leq m'_{i,j} \) then \( m'' \) is not consistent. \( \square \)
4.3.2 Monotonicity

Another key property of our incremental closure is monotonicity under the pointwise ordering, that is, \( m \leq m' \) iff \( m_{i,j} \leq m'_{i,j} \) for all \( 0 \leq i, j \leq 2n \).

**Proposition 4.2** Suppose \( m \leq m' \) (pointwise) and \( o = (x'_a - x'_b \leq d) \).

\[
\text{IncClose}(m, o) \leq \text{IncClose}(m', o)
\]

4.3.3 Coherence

The following proposition shows that IncClose preserves coherence. The force of this rest is that it is not necessary to enforce coherence as a post-processing step, or even on-the-fly as incremental closure is applied.

**Proposition 4.3** Suppose \( m \) is coherent, \( m' = \text{IncClose}(m, o) \) and \( o = (x'_a - x'_b \leq d) \). Then \( m' \) is coherent.

**Proof**

- Suppose \( m'_{i,j} = m_{i,j} \). Because \( m \) is coherent \( m'_{i,j} = m_{j,i} \geq m'_{j,i} \).
- Suppose \( m'_{i,j} = m_{i,a} + d + m_{b,j} \). Because \( m \) is coherent \( m'_{i,j} = m_{j,b} + d + m_{a,i} \geq m'_{j,i} \).
- Suppose \( m'_{i,j} = m_{i,b} + d + m_{a,j} \). Similar to the previous case.
- Suppose \( m'_{i,j} = m_{i,a} + d + m_{b,j} \). Because \( m \) is coherent \( m'_{i,j} = m_{j,b} + d + m_{a,j} \geq m'_{j,i} \).
- Suppose \( m'_{i,j} = m_{i,a} + d + m_{b,j} + d + m_{a,j} \). Similar to the previous case.

Since \( m'_{i,j} \geq m'_{j,i} \) it follows \( m'_{j,i} \geq m'_{i,j} \) hence \( m'_{i,j} = m'_{j,i} \) as required. \( \square \)

5 Incremental Strong Closure

We now turn our attention from recovering closure to recovering strong closure, which generates a canonical representation for any (non-empty) octagon.

5.1 Classical Strong Closure

The classical strong closure by Minè repeatedly invokes STR within the main Floyd-Warshall loop, but it was later shown by Bagnara et al. [2] that this was equivalent to applying STR just once after the main loop. The following theorem [2, Theorem 3] justifies this tactic, though the proofs we present have been revisited and streamlined:

**Theorem 5.1** Suppose \( m \) is a closed, coherent DBM and \( m' = \text{STR}(m) \). Then \( m' \) is a strongly closed DBM.
Proposition 5.2

Suppose \( \mathbf{m}'_{i,j} = \min(m_{i,i}, (m_{i,i} + m_{i,j})/2) = m_{i,i} \) and likewise \( \mathbf{m}'_{j,j} = m_{j,j} \). Therefore \( \mathbf{m}'_{i,j} \leq m_{i,i} + m_{i,j})/2 = (\mathbf{m}'_{i,j} + \mathbf{m}'_{j,j})/2 \).

Because \( \mathbf{m} \) is closed \( 0 = m_{i,j} \leq m_{i,i} + m_{i,j} \) and thus

\[
\mathbf{m}'_{i,j} = \min(m_{i,i}, (m_{i,i} + m_{i,j})/2) = \min(0, (m_{i,i} + m_{i,j})/2) = 0
\]

To show \( \mathbf{m}'_{i,j} \leq \mathbf{m}'_{i,k} + \mathbf{m}'_{k,j} \) we proceed by case analysis:

- Suppose \( \mathbf{m}'_{i,k} = m_{i,k} \) and \( \mathbf{m}'_{k,j} = m_{k,j} \). Because \( \mathbf{m} \) is closed:

\[
\mathbf{m}'_{i,j} \leq m_{i,j} \leq m_{i,k} + m_{k,j} = \mathbf{m}'_{i,k} + \mathbf{m}'_{k,j}
\]

- Suppose \( \mathbf{m}'_{i,k} \neq m_{i,k} \) and \( \mathbf{m}'_{k,j} = m_{k,j} \). Because \( \mathbf{m} \) is closed and coherent:

\[
2\mathbf{m}'_{i,k} + 2\mathbf{m}'_{k,j} = m_{i,k} + m_{k,k} + 2m_{k,j} \geq m_{i,k} + m_{k,j} + m_{k,j} = m_{i,k} + m_{k,j} + m_{j,j} + 2\mathbf{m}'_{i,j}
\]

- Suppose \( \mathbf{m}'_{i,k} = m_{i,k} \) and \( \mathbf{m}'_{k,j} \neq m_{k,j} \). Symmetric to the previous case.

- Suppose \( \mathbf{m}'_{i,k} \neq m_{i,k} \) and \( \mathbf{m}'_{k,j} \neq m_{k,j} \). Because \( \mathbf{m} \) is closed:

\[
2\mathbf{m}'_{i,k} + 2\mathbf{m}'_{k,j} = m_{i,k} + m_{k,k} + m_{k,j} + m_{j,j} \geq m_{i,k} + m_{k,j} + m_{j,j} + 2\mathbf{m}'_{i,j}
\]

\( \Box \)

5.2 Properties of Strong Closure

We establish a number of properties about \( \text{Str} \) which will be useful when we prove in-place versions of our incremental strong (and tight) closure algorithms.

5.2.1 Idempotence

**Proposition 5.1** Suppose \( \mathbf{m} \) be a DBM and \( \mathbf{m}' = \text{Str}(\mathbf{m}) \). Then \( \mathbf{m}' = \text{Str}(\mathbf{m}') \).

**Proof** Let \( \mathbf{m}'' = \text{Str}(\mathbf{m}) \). Observe \( \mathbf{m}'_{i,i} = \min(m_{i,i}, (m_{i,i} + m_{i,i})/2) = m_{i,i} \) and likewise \( \mathbf{m}'_{j,j} = m_{j,j} \). Therefore

\[
\mathbf{m}''_{i,j} = \min(\mathbf{m}'_{i,j}, (\mathbf{m}'_{i,j} + \mathbf{m}'_{j,j})/2)
\]

\[
= \min(m_{i,j}, (m_{i,j} + m_{j,j})/2), (m_{i,i} + m_{j,j})/2)
\]

\[
= \min(m_{i,j}, (m_{i,i} + m_{j,j})/2) = \mathbf{m}'_{i,j}
\]

\( \Box \)

5.2.2 Monotonicity

**Proposition 5.2** Suppose \( \mathbf{m}^1 \leq \mathbf{m}^2 \) (pointwise). Then \( \text{Str}(\mathbf{m}^1) \leq \text{Str}(\mathbf{m}^2) \).

**Proof**

\[
\text{Str}(\mathbf{m}^2_{i,j}) = \min(m^2_{i,j}, \frac{m^2_{i,i} + m^2_{j,j}}{2})
\]

\[
\geq \min(m^1_{i,j}, \frac{m^1_{i,i} + m^1_{j,j}}{2}) = \text{Str}(\mathbf{m}^1_{i,j})
\]

\( \Box \)
1: function IncStrongClose(m, \( x'_a - x'_b \leq d \))
2:     for i \( \in \{0, \ldots, 2n - 1\} \) do
3:         \( m'_{i,i} \leftarrow \min \left( \begin{array}{c}
m_{i,i},
m_{i,a} + d + m_{b,i},
m_{i,b} + d + m_{a,i},
m_{i,a} + d + m_{b,b} + d + m_{a,i}
\end{array} \right) \)
4:     end for
5:     for i \( \in \{0, \ldots, 2n - 1\} \) do
6:         for j \( \in \{0, \ldots, 2n - 1\} \) do
7:             if j \( \neq i \) then
8:                 \( m'_{i,j} \leftarrow \min \left( \begin{array}{c}
m_{i,j},
m_{i,a} + d + m_{b,j},
m_{i,b} + d + m_{a,j},
m_{i,a} + d + m_{b,b} + d + m_{a,j},
(m'_{i,i} + m'_{j,j})/2
\end{array} \right) \)
9:             end if
10:     end for
11:     if \( m'_{i,i} < 0 \) then
12:         return false
13:     end if
14: end for
15: return m'
16: end function

Fig. 9: Incremental Strong Closure

5.2.3 Reductiveness

**Proposition 5.3** Suppose \( m \) is a DBM and \( m' = \text{STR}(m) \). Then \( m' \leq m \).

*Proof* \( m'_{i,j} = \min(m_{i,j}, \frac{m_{i,i} + m_{j,j}}{2}) \leq m_{i,j} \) \qed

5.2.4 Coherence

**Proposition 5.4** Suppose \( m \) is a closed, coherent DBM. Then \( m' = \text{STR}(m) \) is a coherent DBM.

*Proof*

\[
m'_{i,j} = \min(m_{i,j}, \frac{m_{i,i} + m_{j,j}}{2}) = \min(m_{j,i}, \frac{m_{j,j} + m_{i,i}}{2}) = m'_{j,i}
\]

\qed

5.3 Incremental Strong Closure

Theorem 5.1 states that a strongly closed DBM can be obtained by calculating closure and then strengthening. This is realised by calling IncClose, from Figure 7, followed by a call to STR. Although this is conventional wisdom, it incurs two passes over the DBM: one by IncClose and the other by STR. The two passes can be
unified by observing that strengthening $m'$ critically depends on the entries $m'_{i,t}$ where $i \in \{0, \ldots, 2n - 1\}$. Furthermore, these entries, henceforth called key entries, are themselves not changed by strengthening because:

$$\min(m'_{i,t}, (m'_{i,t} + m'_{i,t} + m'_{i,t} + m'_{i,t} + m'_{i,t})) = m'_{i,t}$$

This suggests precomputing the key entries up front and then using them in the main loop of IncClose to strengthen on-the-fly. This insight leads to the algorithm listed in Figure 9. Line 3 generates the key entries which are closed by construction and unchanged by strengthening. Once the key entries are computed, the algorithm iterates over the rest of the DBM, closing and simultaneously strengthening each entry $m_{i,j}$ at line 8.

The total number of binary min operations required for IncStrongClose is $8n + 10n(2n - 1) = 20n^2 - 2n$, which improves on following IncClose by STR, which requires $16n^2 + 4n^2 = 20n^2$. Furthermore, since $m$ is coherent $m_{i,a} + d + m_{b,i} = m_{a,i} + d + m_{b,i}$ so that the quintic min on line 4 becomes quartic, reducing the min count for IncClose to $20n^2 - 4n$. Furthermore, the entry $m_{i,j}$ can be cached in a linear array $a_i$ of dimension $2n$ and the expression $(m'_{i,1} + m'_{i,2})/2$ in line 8 can be replaced with $(a_i + a_j)/2$, thereby avoiding two lookups in a two-dimensional matrix. We omit the algorithm using array caching for space reasons as this is a simple change to Figure 9.

The following theorem justifies the correctness of the new incremental strong closure algorithm:

**Theorem 5.2 (Correctness of IncStrongClose)** Suppose $m$ is a DBM, $m' = \text{IncStrongClose}(m, o)$, $m^\dagger = \text{IncClose}(m)$, $m^* = \text{Str}(m^\dagger)$ and $o = (x' - x'' \leq d)$. Then $m' = m^*$.

**Proof** We prove that $\forall i, j, m'_{i,j} = m^*_{i,j}$. Pick some $i, j$.

- Suppose $j = i$. Then

$$m^*_{i,i} = \min(m^\dagger_{i,i}, m^\dagger_{i,i}/2 + m^\dagger_{i,i}/2) = m^\dagger_{i,i}$$

$$= \min\left(\begin{array}{c}
m_{i,i} + \frac{1}{2} \left(m_{i,i} + d + m_{i,i} + m_{i,i} + m_{i,i} + m_{i,i}\right)
m_{i,a} + d + m_{b,i},
m_{i,b} + d + m_{a,i},
m_{i,a} + d + m_{b,a} + m_{b,i},
m_{i,a} + d + m_{b,a} + m_{b,i} + m_{i,i}\end{array}\right) = m'_{i,i}$$

- Suppose $j \neq i$. Then

$$m^*_{i,j} = \min(m^\dagger_{i,j}, m^\dagger_{i,j}/2 + m^\dagger_{i,j}/2)$$

$$= \min(m^\dagger_{i,j}, m^\dagger_{i,j}/2 + m^\dagger_{i,j}/2)$$

$$= \min\left(\begin{array}{c}
m_{i,j} + \frac{1}{2} \left(m_{i,j} + d + m_{b,j} + m_{b,j} + m_{b,j} + m_{b,j} + m_{b,j}\right)
m_{i,a} + d + m_{b,j},
m_{i,b} + d + m_{a,j},
m_{i,b} + d + m_{a,j} + d + m_{b,j},
m_{i,a} + d + m_{b,a} + m_{b,j},
m_{i,a} + d + m_{b,a} + m_{b,j} + m_{i,j}\end{array}\right) = m'_{i,j}$$

$\Box$
6 Incremental Tight Closure

The strong closure algorithms previously presented have to be modified to support integer octagonal constraints. If \( x_i \) is integral then \( x_i \leq c \) can be tightened to \( x_i \leq [c] \). Since \( x_i \leq c \) is represented as the difference \( x_{2i} - x_{2i+1} \leq 2c \), tightening is achieved by sharpening the difference to \( x_{2i} - x_{2i+1} \leq 2[c/2] \), so that the constant \( 2[c/2] \) is even. This is achieved by applying \( \text{TIGHTEN}(m) \), the code for which is given in Figure 10. As suggested by Figure 3, closure does not need to be reapplied after tightening to check for consistency; it is sufficient to check that \( m_{i,j} + m_{k,i} < 0 \) \([2] \), which is the role of \( \text{CHECKZCONSISTENT}(m) \). One subtlety that is worthy of note is that after running \( \text{TIGHTEN}(m) \) on a closed DBM \( m \), the resulting DBM will not necessarily be closed but will instead satisfy a weaker property, namely weak closure. Strong closure can be recovered from weak closure, however, by strengthening \([2] \). However, we do not use this approach in the sequel; instead we use tightening and strengthening together to avoid having to work with weakly closed DBMs. First we prove that tightening followed by strengthening will return a closed DBM when the resulting system is satisfiable:

**Lemma 6.1** Suppose \( m \) is a closed, coherent integer DBM. Let \( m' \) be defined as follows:

\[
m'_{i,j} = \min(m_{i,j}, \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,i}}{2} \right\rfloor)
\]

Then \( m' \) is either closed or it is not consistent.

**Proof** Suppose \( m' \) is consistent. Because \( m \) is closed \( m'_{i,i} = m_{i,i} = 0 \) and since \( m' \) is consistent \( 0 \leq m'_{i,i} \) hence \( m'_{i,i} = 0 \). Now to show \( m'_{i,k} + m'_{k,j} \geq m'_{i,j} \).

1. Suppose \( m'_{i,k} = m_{i,k} \) and \( m'_{k,j} = m_{k,j} \). Because \( m \) is closed:

\[
m'_{i,k} + m'_{k,j} = m_{i,k} + m_{k,j} \geq m_{i,j} \geq m'_{i,j}
\]

2. Suppose \( m'_{i,k} \neq m_{i,k} \) and \( m'_{k,j} = m_{k,j} \).

   (a) Suppose \( m_{k,k} \) is even. Because \( m \) is closed and coherent:

\[
m'_{i,k} + m'_{k,j} = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{k,k}}{2} \right\rfloor + m_{k,j} = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{k,k} + 2m_{k,j}}{2} \right\rfloor \\
\geq \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{k,k} + m_{k,j}}{2} \right\rfloor = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,i} + m_{j,k} + m_{j,j}}{2} \right\rfloor \\
\geq \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \frac{m_{j,j}}{2} \geq \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor > m'_{i,j}
\]

(b) Suppose \( m_{k,k} \) is odd. Then

\[
m'_{i,k} + m'_{k,j} = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{k,k}}{2} \right\rfloor + m_{k,j} = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{k,k} - 1 + 2m_{k,j}}{2} \right\rfloor
\]

Because \( m \) is closed and coherent:

\[
\frac{(m_{k,k} - 1) + 2m_{k,j}}{2} \geq \frac{m_{k,j} + m_{k,j} - 1}{2} = \frac{m_{j,k} + m_{j,j} - 1}{2} \geq \frac{m_{j,j} - 1}{2}
\]

i. Suppose \( m_{k,k} + 2m_{k,j} = m_{j,j} \). Since \( m_{k,k} \) is odd \( m_{j,j} \) is odd thus

\[
\frac{m_{j,j} - 1}{2} = \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \text{ and } m'_{i,k} + m'_{k,j} > \left\lfloor \frac{m_{j,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \geq m'_{i,j}
\]
Suppose \( m_{i,k} + 2m_{k,j} > m_{j,j} \). Thus \((m_{i,k} - 1) + 2m_{k,j} \geq m_{j,j}\)

\[ m_i' + m_{k,j}' \geq \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor \geq \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \geq m_i' + m_{k,j}' \]

3. Suppose \( m_i'_{i,k} = m_{i,k} \) and \( m_i'_{k,j} \neq m_{k,j} \). Symmetric to the previous case.

4. Suppose \( m_i'_{i,k} \neq m_{i,k} \) and \( m_i'_{k,j} \neq m_{k,j} \). Then

\[ m_i' + m_{k,j}' = \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor + \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor \]

Since \( m \) is closed and \( m' \) is consistent:

\[ 0 \leq m_{i,k}' = \min(m_{i,k}', \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor) = \min(0, \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor) \]

Therefore

\[ \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor \geq 0 \text{ and } m_{i,k}' + m_{k,j}' \geq \left\lfloor \frac{m_{i,k}}{2} \right\rfloor + \left\lfloor \frac{m_{k,j}}{2} \right\rfloor \geq m_i' + m_{k,j}' \]

\( \square \)

Using the proof that tighten and strengthening gives a closed DBM, we can now show that the resulting DBM is also tightly closed:

**Theorem 6.1** ([5] Theorem 4) Suppose \( m \) is a closed, coherent integer DBM. Let \( m' \) be defined as follows:

\[ m_i' = \min(m_{i,j}', \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor) \]

Then \( m' \) is either tightly closed or it is not consistent.

**Proof** Suppose \( m' \) is consistent. By theorem 6.1 we know that \( m' \) is closed.

We will now show that \( m' \) is strongly closed i.e \( \forall i,j : m'_{i,j} \leq m_i'_{i,j} + m_{j,j}'/2. \)

- Suppose \( m_i'_{i,j} = m_{i,j} \) and \( m_j'_{j,j} = m_{j,j}. \) Then

\[ \frac{m_{i,j}}{2} + \frac{m_{j,j}}{2} = \frac{m_{i,j}}{2} + \frac{m_{j,j}}{2} \geq \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \geq m_i'_{i,j} \]

- Suppose \( m_i'_{i,j} \neq m_{i,j} \) and \( m_j'_{j,j} = m_{j,j}. \) Then

\[ \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \geq \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \geq m_i'_{i,j} + m_{j,j}' \]

- Suppose \( m_i'_{i,j} = m_{i,j} \) and \( m_j'_{j,j} \neq m_{j,j}. \) Symmetric to the previous case.

- Suppose \( m_i'_{i,j} \neq m_{i,j} \) and \( m_j'_{j,j} \neq m_{j,j}. \) Then

\[ \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor = \left\lfloor \frac{m_{i,j}}{2} \right\rfloor + \left\lfloor \frac{m_{j,j}}{2} \right\rfloor \geq m_i'_{i,j} \]
1: function Tighten(m)
2: for i ∈ {0, . . . , 2n − 1} do
3:   m_{i,j} ← 2|m_{i,j}/2|
4: end for
5: end function

1: function TightClose(m)
2: ShortestPathClosure(m)
3: if CheckConsistent(m) then
4:   m ← Tighten(m)
5:   if CheckZConsistent(m) then
6:     return Str(m)
7:   else
8:     return false
9: end if
10: else
11: return false
12: end if
13: end function

Fig. 10: Tight Closure

Thus, if m’ is consistent, it is strongly closed. It remains to show that ∀i,m’_{i,i} is even. Observe that:

m’_{i,i} = \min(m_{i,i}, \frac{m_{i,i}}{2} + \frac{m_{i,i}}{2}) = \min(m_{i,i}, 2\frac{m_{i,i}}{2})

– Suppose m_{i,i} is even. Then 2\frac{m_{i,i}}{2} = m_{i,i} = m’_{i,i} which is even.
– Suppose m_{i,i} is odd. Then 2\frac{m_{i,i}}{2} = m_{i,i} - 1 = m’_{i,i} which is even. □

Notice that the proof of tight closure does not use the concept of weak closure as advocated in [2]. The above proof goes directly from a closed DBM to a tightly closed DBM relying only on simple algebra; it is not based on showing that tightening gives a weakly closed (intermediate) DBM which can be subsequently strengthen to give a tightly closed DBM (see Figure 3).

Tight closure requires the key entries, and only these, to be tightened. This suggests tightening the key entries on-the-fly immediately after they have been computed by closure. This leads to the algorithm given in Figure 11 which coincides with IncStrongClose except in one crucial detail: line 4 tightens the key entries as they are computed. Moreover the key entries are strengthened, with the other entries of the DBM, in the main loop in tandem with the closure calculation, thereby ensuring strong closure. Thus tightening can be accommodated, almost effortlessly, within incremental strong closure.

**Theorem 6.2 (Correctness of IncZClose)** Suppose m is an integer DBM and m’ = IncZClose(m,o) where o = x’_a – x’_b ≤ d. Let m^† = IncClose(m), m^† = Tighten(m^†) and m^* = Str(m^†). Then m^* = m’.

**Proof** We prove that ∀i,j,m_{i,j} = m’_{i,j}. Pick some i,j.

– Suppose j = i. Then
m^*_{i,i} = \min(m^†_{i,i}, m^†_{i,i}/2 + m^†_{i,i}/2) = m^†_{i,i} = 2|m^†_{i,i}/2|
1: function IncZClose(m, x' = x' \in \mathbb{R}^n)
2: for i ∈ \{0, \ldots, 2n - 1\} do
3: \( m'_{i,i} \leftarrow 2 \min \begin{pmatrix} m_{i,i} \\ m_{i,a} + d + m_{b,i} \\ m_{i,b} + d + m_{a,i} \\ m_{i,b} + d + m_{a,a} + d + m_{b,i} \\ m_{i,a} + d + m_{b,b} + d + m_{a,i} \end{pmatrix} / 2 \)
4: end for
5: if CheckZConsistent(m') then
6: for i ∈ \{0, \ldots, 2n - 1\} do
7: if j ∈ \{0, \ldots, 2n - 1\} do
8: if j ≠ i then
9: \( m'_{i,j} \leftarrow \min \begin{pmatrix} m_{j,j} \\ m_{a,j} + d + m_{b,j} \\ m_{b,b} + d + m_{a,j} \\ m_{b,b} + d + m_{a,a} + d + m_{b,j} \\ m_{b,a} + d + m_{b,b} + d + m_{a,j} \end{pmatrix} \)
10: end if
11: end for
12: if \( m'_{i,i} < 0 \) then
13: return false
14: end if
15: end for
16: else
17: return false
18: end if
19: return m'
20: end function

Fig. 11: Incremental Tight Closure

\[ = 2 \min \begin{pmatrix} m_{i,i} \\ m_{i,a} + d + m_{b,i} \\ m_{i,b} + d + m_{a,i} \\ m_{i,b} + d + m_{a,a} + d + m_{b,i} \\ m_{i,a} + d + m_{b,b} + d + m_{a,i} \end{pmatrix} / 2 = m'_{i,j} \]

– Suppose \( j \neq i \). Then

\[ m^*_{i,j} = \min(m^+_{i,j}, m^+_{i,i}/2 + m^+_{i,j}/2) = \min(m^+_{i,j}, m'_{i,i}/2 + m'_{i,j}/2) \]

\[ = \min \begin{pmatrix} m_{i,j} \\ m_{i,a} + d + m_{b,j} \\ m_{i,b} + d + m_{a,j} \\ m_{i,b} + d + m_{a,a} + d + m_{b,j} \\ m_{i,a} + d + m_{b,b} + d + m_{a,j} \end{pmatrix} = m'_{i,j} \]

6.1 Properties of Tight Closure

We prove a number of properties about Tighten which will be useful when we justify the in-place versions of our incremental tight closure algorithm.
6.1.1 Idempotence

Proposition 6.1 Suppose $m$ is a DBM and $m' = \text{Tighten}(m)$. Then $m' = \text{Tighten}(m')$.

Proof Let $m'' = \text{Tighten}(m')$.

- Suppose $j \neq \bar{i}$. Then $m''_{i,j} = m'_{i,j}$.
- Suppose $j = \bar{i}$. Then $m''_{i,\bar{i}} = 2 \left\lfloor \frac{m'_{i,\bar{i}}}{2} \right\rfloor = 2 \left\lfloor \frac{m_{i,\bar{i}}}{2} \right\rfloor = m'_{i,\bar{i}}$.

\[ \square \]

6.1.2 Monotonicity

Proposition 6.2 Suppose $m^1 \preceq m^2$ (pointwise). Then $\text{Tighten}(m^1) \preceq \text{Tighten}(m^2)$.

Proof

- Suppose $j \neq \bar{i}$. Then $\text{Tighten}(m^2_{i,j}) = m^2_{i,j} \geq m^1_{i,j} = \text{Tighten}(m^1_{i,j})$.
- Suppose $j = \bar{i}$. Then $\text{Tighten}(m^2_{i,\bar{i}}) = 2 \left\lfloor \frac{m^2_{i,\bar{i}}}{2} \right\rfloor \geq 2 \left\lfloor \frac{m^1_{i,\bar{i}}}{2} \right\rfloor = \text{Tighten}(m^1_{i,\bar{i}})$.

\[ \square \]

6.1.3 Reductiveness

Proposition 6.3 Suppose $m$ is a DBM and $m' = \text{Tighten}(m)$. Then $m' \preceq m$.

Proof

- Suppose $j = \bar{i}$. Then $m'_{i,j} = m'_{i,\bar{i}} \leq 2 \left\lfloor \frac{m_{i,\bar{i}}}{2} \right\rfloor = m_{i,\bar{i}} = m_{i,j}$.
- Suppose $j \neq \bar{i}$. Then $m'_{i,j} = m_{i,j}$.

\[ \square \]

6.1.4 Coherence

Proposition 6.4 Let $m$ be a coherent DBM and $m' = \text{Tighten}(m)$. Then $m'$ is coherent.

Proof

- Suppose $j = \bar{i}$. Then $m'_{j,\bar{i}} = 2 \left\lfloor \frac{m_{j,\bar{i}}}{2} \right\rfloor = 2 \left\lfloor \frac{m_{i,j}}{2} \right\rfloor = m'_{i,j}$.
- Suppose $j \neq \bar{i}$. Then $m'_{j,\bar{i}} = m_{j,\bar{i}} = m_{i,j} = m'_{i,j}$.

\[ \square \]

7 In-place Update

Closure algorithms are traditionally formulated in a way that is simple to reason about mathematically (see [17, Def 3.3.2]), typically using a series of intermediate DBMs and then present the algorithm itself using in-place update (see [17, Def 3.3.3]). The question of equivalence between the mathematical formulation and the practical in-place implementation is arguably not given the space it should.
Miné, in his magnus opus [17], merely states that equivalence can be shown by using an argument for the Floyd-Warshall algorithm [9, Section 26.2]. However that in-place argument is itself informal. Later editions of the book do not help, leaving the proof as an exercise for the reader. But the question of equivalence is more subtle again for incremental closure. Correctness is therefore argued for incremental closure in Section 7.1, incremental strong closure in Section 7.2 and incremental tight closure in Section 7.3, one correctness argument extending another.

7.1 In-place Incremental Closure

Figure 12 gives an in-place version of IncClose algorithm listed in Figure 7. At first glance one might expect that mutating the entries $m_{i,a}$, $m_{b,\bar{b}}$, $m_{i,\bar{b}}$, $m_{\bar{a},a}$ or $m_{\bar{a},\bar{b}}$ could potentially perturb those entries of $m$ which are updated later. The following theorem asserts that this is not so. Correctness follows from Corollary 7.1 which is stated below:

**Corollary 7.1** Suppose that $m$ is a closed DBM, $m' = \text{incClose}(m,o)$, $o = (x'_{a} - x'_{b} \leq d)$ and $m'$ is consistent. Then the following hold:

- $m'_{i,j} \leq m'_{i,a} + d + m'_{i,b,j}$
- $m'_{i,j} \leq m'_{i,b,j} + d + m'_{i,a,j}$
- $m'_{i,j} \leq m'_{i,a} + d + m'_{b,a} + d + m'_{b,j}$
- $m'_{i,j} \leq m'_{i,a} + d + m'_{b,a} + d + m'_{a,j}$

**Proof** By Proposition 4.1 it follows $m' = \text{incClose}(m', o)$. The result then follows from Theorem 7.1.

The following theorem asserts that in-place update does not compromise correctness. It is telling that the correctness argument does not refer to the entries $m_{i,a}$, $m_{b,\bar{b}}$, $m_{i,b}$, $m_{a,i}$, $m_{a,a}$ or $m_{\bar{a},\bar{b}}$ at all. This is because the corollary on which the theorem is founded follows from the high-level property of idempotence. Notice too that the theorem is parameterised by the traversal order over $m$ and therefore is independent of it.
Theorem 7.1 (Correctness of InPlaceIncClose) Suppose \( \rho : \{0, \ldots, 2n-1\}^2 \to \{0, \ldots, 4n^2 - 1\} \) is a bijective map, \( m \) is a closed DBM, \( m' = \text{InCClose}(m, o) \), \( o = (x_a - x_b \leq d) \), \( m^0 = m \) and

\[
\begin{align*}
m_{i,j}^{k+1} &= \begin{cases} 
  m_{i,j}^k, & \text{if } \rho(i, j) \neq k \\
  \min \left( m_{i,a}^k + d + m_{b,j}^k, m_{i,b}^k + d + m_{a,j}^k, m_{i,a}^k + d + m_{b,j}^k + d + m_{a,j}^k, m_{i,b}^k + d + m_{a,a}^k + d + m_{b,j}^k \right), & \text{if } \rho(i, j) = k
\end{cases}
\end{align*}
\]

Then either \( m' \) is consistent and

- \( \forall \ell \leq k. m_{\rho^{-1}(\ell)} = m'_{\rho^{-1}(\ell)} \)
- \( \forall k \leq 4n^2. m_{\rho^{-1}(\ell)} = m_{\rho^{-1}(\ell)} \)

or \( m' \) is inconsistent.

Proof Suppose \( m' \) is consistent.

Let \( k = 0 \). It vacuously follows that \( \forall \ell \leq k. m_{\rho^{-1}(\ell)} = m'_{\rho^{-1}(\ell)} \). Moreover \( \forall k \leq \ell < 4n^2. m_{\rho^{-1}(\ell)} = m_{\rho^{-1}(\ell)} \) since \( m^0 = m \).

Now let \( k > 0 \) and suppose \( \rho(i, j) = k \) and consider

\[
\begin{align*}
m_{i,j}^{k+1} = \min \left( m_{i,a}^k + d + m_{b,j}^k, m_{i,b}^k + d + m_{a,j}^k, m_{i,a}^k + d + m_{b,j}^k + d + m_{a,j}^k, m_{i,b}^k + d + m_{a,a}^k + d + m_{b,j}^k \right)
\end{align*}
\]

If \( \rho^{-1}(i, a) < k \) then \( m_{i,a}^k = m'_{i,a} \) whereas if \( \rho^{-1}(i, a) \geq k \) then \( m_{i,a}^k = m_{i,a} \). Thus \( m_{i,a}^k \geq m_{i,a} \) and likewise \( m_{b,j}^k \geq m_{b,j} \). By Corollary 7.1 it follows

\[
m_{i,a}^k + d + m_{b,j}^k \geq m_{i,a} + d + m_{b,j} \geq m_{i,j}.
\]

By a similar argument \( m_{i,b}^k + d + m_{a,j}^k \geq m_{i,b} + d + m_{a,j} \) and likewise \( m_{i,b} + d + m_{a,j} \) and \( m_{i,a} + d + m_{b,j} \).

Since \( m_{i,j}^{k+1} = m_{i,j} \) it follows \( m_{i,j}^{k+1} \geq m_{i,j} \). But \( m_{i,j} \leq m \) and by Proposition 4.2 \( m_{i,j}^{k+1} \leq m_{i,j}^{k+1} \). Hence it follows \( \forall \ell \leq k + 1. m_{\rho^{-1}(\ell)} = m_{\rho^{-1}(\ell)} \). Moreover \( k + 1 \leq \ell < 4n^2. m_{\rho^{-1}(\ell)} = m_{\rho^{-1}(\ell)} \).

Suppose \( m' \) is inconsistent hence \( m'_{i,i} < 0 \). Put \( k = \rho(i, i) \). But \( m_{i,i} \leq m \) and by Proposition 4.2 \( m_{i,i} \leq m_{i,i} \). But \( m_{i,i} \geq m_{i,i} \) as required. \( \square \)

7.2 In-place Incremental Strong Closure

The in-place version of the incremental strong closure algorithm is presented in Figure 13. The following lemma shows that running incremental closure followed by strengthening refines the entries in the DBM to their tightest possible value with respect to the new octagonal constraint.
Lemma 7.1 Suppose \( m \) is a closed, coherent DBM and \( m' = \text{IncClose}(m, o) \), \( m'' = \text{Str}(m') \), \( m''' = \text{IncClose}(m'', o) \) and \( o = (x'_a - x'_b \leq d) \). Then either \( m' \) is consistent and \( m'' = m''' \) or \( m'' \) is not consistent.

Proof Suppose \( m' \) is consistent. By Proposition 4.3 \( m' \) is coherent.

1. To show \( m''_{i,j} \leq m''_{i,a} + d + m''_{b,j} \):
   - Suppose \( m''_{i,a} = m'_{i,a} \) and \( m''_{b,j} = m'_{b,j} \). Because \( m' \) is consistent by Corollary 7.1 it follows:
     \[
     m''_{i,a} + d + m''_{b,j} = m'_{i,a} + d + m'_{b,j} \geq m'_{i,j} \geq m''_{i,j}
     \]
   - Suppose \( m''_{i,a} = (m'_{i,a} + m'_{a,a})/2 \) and \( m''_{b,j} = m'_{b,j} \). Because \( m' \) is consistent by Corollary 7.1 it follows \( m'_{a,j} \leq m'_{a,a} + d + m'_{b,j} \) and \( m'_{j,j} \leq m'_{j,a} + d + m'_{b,j} \). Hence
     \[
     m''_{i,a} + d + m''_{b,j} = (m'_{i,i} + m'_{a,a} + 2d + 2m'_{b,j})/2 \\
     \geq (m'_{i,i} + m'_{a,a} + d + m'_{b,j})/2 \\
     \geq (m'_{i,i} + m'_{j,j})/2 \geq m''_{i,j}
     \]
   - Suppose \( m''_{i,a} = m'_{i,a} \) and \( m''_{b,j} = (m'_{b,b} + m'_{j,j})/2 \). Symmetric to the previous case.
   - Suppose \( m''_{i,a} = (m'_{i,i} + m'_{a,a})/2 \) and \( m''_{b,j} = (m'_{b,b} + m'_{j,j})/2 \). Because \( m' \) is consistent by Corollary 7.1 it follows \( m'_{a,b} \leq m'_{a,a} + d + m'_{b,b} \) and \( m'_{i,a} \leq m'_{i,i} + d + m'_{b,a} \) thus \( 0 \leq d + m'_{b,a} \). Hence
     \[
     m''_{i,a} + d + m''_{b,j} = (m'_{i,i} + m'_{a,a} + 2d + m'_{b,b} + m'_{j,j})/2
     \]
2. To show $m''_{i,j} \leq m''_{i,b} + d + m''_{a,j}$. Analogous to the previous case.

3. To show $m''_{i,j} \leq m''_{i,b} + d + m''_{a,a} + d + m''_{b,j}$.
   - Suppose $m''_{i,b} = m'_{i,b}$ and $m''_{b,j} = m'_{b,j}$. Since $m''_{a,a} = m'_{a,a}$ and because $m'$ is consistent by Corollary 7.1 it follows
     \[
     m''_{i,b} + d + m''_{a,a} + d + m''_{b,j} = m'_i + d + m'_{a,a} + d + m'_{b,j} \geq m'_{i,j} \geq m''_{i,j}
     \]
   - Suppose $m''_{i,b} = (m_{i,b} + m_{b,j})/2$ and $m''_{b,j} = m'_{b,j}$. Because $m'$ is consistent by Corollary 7.1 it follows $m'_{a,a} \leq m'_{a,a} + d + m'_{b,j}, m'_{b,a} \leq m'_{b,a} + d + m'_{a,a}$ and $0 \leq d + m'_{b,a}$. Therefore
     \[
     m''_{i,b} + d + m''_{a,a} + d + m''_{b,j} = (m'_{i,b} + m'_{b,j} + 2m'_{a,a} + 4d + 2m'_{b,j})/2 \\
     \geq (m'_{i,b} + m'_{b,j} + m'_{a,a} + 3d + m'_{a,a} + m'_{b,j})/2 \\
     \geq (m'_{i,b} + m'_{b,j} + m'_{a,a} + 2d + m'_{b,j})/2 \\
     \geq (m'_{i,b} + m'_{b,j} + d + m'_{f,j})/2 \\
     \geq (m'_{i,b} + m'_{f,j})/2 \geq m''_{i,j}
     \]

4. To show $m''_{i,j} \leq m''_{i,a} + d + m''_{b,b} + d + m''_{a,j}$. Analogous to the previous case.

It therefore follows that $m''' = m''$. Now suppose $m'$ is not consistent. Hence $m''$ is not consistent thus $m'''$ is not consistent. \qed

Now we move onto the theorem showing the correctness of \textsc{InplaceIncStrongClose}. We show that the in-place version of the algorithm produces the same DBM as the non-in-place version of the algorithm. A bijective map used in the proof to process key entries first before processing non-key entries: the condition $\forall 0 \leq i < 2n, \rho(i, i) < 2n$ ensures this property. Note that this is the only caveat on the order produced by the map; the order in which key entries themselves are ordered is irrelevant and similarly for non-key entries.

**Theorem 7.2** (Correctness of \textsc{InplaceIncStrongClose}) Suppose $m$ is a closed, coherent DBM, $m' = \text{IncClose}(m, a), m'' = \text{Str}(m'), a = (x_0', x_0' + d) \in \{0, \ldots, 2n - 1\}^2 \to \{0, \ldots, 4n^2 - 1\}$

\[
\begin{align*}
(m'_{i,b} + m''_{a,a} + d + m''_{b,j})/2 & \geq (m'_{i,b} + m''_{a,a} + d + m''_{b,j})/2 \\
(m'_{i,b} + m''_{a,a} + 2d + m''_{b,j})/2 & \geq (m'_{i,b} + m''_{a,a} + 2d + m''_{b,j})/2 \\
(m'_{i,b} + m''_{a,a} + 3d + m''_{b,j})/2 & \geq (m'_{i,b} + m''_{a,a} + 3d + m''_{b,j})/2 \\
(m'_{i,b} + m''_{a,a} + 4d + 2m''_{b,j})/2 & \geq (m'_{i,b} + m''_{a,a} + 4d + 2m''_{b,j})/2
\end{align*}
\]
is a bijective map with \( \forall 0 \leq i < 2n, \rho(i, \bar{i}) < 2n, \ m^0 = m \) and

\[
m^{k+1}_{i,j} = \begin{cases} 
m^k_{i,j} & \text{if } \rho(i, j) \neq k \\
\min \left( m^k_{i,j}, m^k_{i,a} + d + m^k_{b,j}, \ m^k_{i,b} + d + m^k_{a,i}, \ m^k_{i,b} + d + m^k_{a,a} + d + m^k_{b,i}, \ m^k_{i,a} + d + m^k_{b,b} + d + m^k_{a,i} \right) & \text{if } \rho(i, j) = k \wedge j = \bar{i} \\
\min \left( m^k_{i,j}, m^k_{i,a} + d + m^k_{b,j}, \ m^k_{i,b} + d + m^k_{a,j}, \ m^k_{i,b} + d + m^k_{a,a} + d + m^k_{b,j}, \ (m^k_{i,a} + m^k_{j,j})/2 \right) & \text{if } \rho(i, j) = k \wedge j \neq \bar{i}
\end{cases}
\]

Then either \( m' \) is consistent and

- \( \forall 0 \leq \ell < k, m^{k-1}_{\rho^{-1}(\ell)} = m'^{\rho^{-1}(\ell)} \)
- \( \forall k < 4n^2, m^{k-1}_{\rho^{-1}(\ell)} = m_{\rho^{-1}(\ell)} \)

or \( m^{4n^2} \) is inconsistent.

**Proof** Suppose \( m' \) is consistent.

Let \( k = 0 \). It vacuously follows that \( \forall 0 \leq \ell < k, m^k_{\rho^{-1}(\ell)} = m'^{\rho^{-1}(\ell)} \). Moreover \( \forall k < 4n^2, m^{k-1}(\ell) = m_{\rho^{-1}(\ell)} \) since \( m^0 = m \).

Suppose \( 0 < k \) and \( \rho(i, j) = k \). Now suppose \( j = \bar{i} \). Then

\[
m^{k+1}_{i,j} = \min \left( m^k_{i,j}, m^k_{i,a} + d + m^k_{b,j}, \ m^k_{i,b} + d + m^k_{a,i}, \ m^k_{i,b} + d + m^k_{a,a} + d + m^k_{b,i}, \ m^k_{i,a} + d + m^k_{b,b} + d + m^k_{a,i} \right)
\]

If \( \rho(i, a) < k \) then \( m^k_{i,j} = m''_{i,a} \) otherwise \( \rho(i, a) \geq k \) then \( m^k_{i,a} = m_{a,a} \geq m''_{i,a} \). Which implies \( m^k_{i,a} \geq m''_{i,a} \). Likewise \( m^k_{b,i} \geq m''_{b,i} \). By Lemma 7.1 and Corollary 7.1 it follows \( m^k_{i,a} + d + m^k_{b,j} \geq m''_{i,a} + d + m''_{b,j} \). By a similar argument \( m^k_{i,b} + m^k_{a,j} \geq m''_{i,j} \). By a similar argument \( m^k_{i,b} + m^k_{a,j} \geq m''_{i,j} \) and \( m^k_{i,a} + d + m^k_{b,b} + d + m^k_{a,i} \geq m''_{i,j} \). Thus \( m^{k+1}_{i,j} \geq m''_{i,j} \). Now to show \( m''_{i,j} \geq m^{k+1}_{i,j} \) Observe

\[
m''_{i,j} = m'_{i,j} = \min \left( m_{i,j}, m_{i,a} + d + m_{b,j}, \ m_{i,b} + d + m_{a,j}, \ m_{i,b} + d + m_{a,a} + d + m_{b,j}, \ m_{i,a} + d + m_{b,b} + d + m_{a,j} \right)
\]
Hence $\forall 0 \leq \ell < k. m_{p^{-1}(\ell)}^{k} = m_{p^{-1}(\ell)}^{\prime}$ follows from the inductive hypothesis and the definition of $m_{p^{-1}(\ell)}^{k+1}$.

Now suppose that $j \neq i$. Then $2n < \rho(i, j)$ and consider

$$m_{i,j}^{k+1} = \min \left( \begin{array}{c} m_{i,j}^{k}, \\
 m_{i,a}^{k} + d + m_{b,j}^{k}, \\
 m_{i,b}^{k} + d + m_{a,j}^{k}, \\
 m_{i,a}^{k} + d + m_{b,j}^{k}, \\
 m_{i,b}^{k} + d + m_{a,j}^{k} \end{array} \right) = m_{i,j}^{k+1}.$$ 

Notice that $m_{i,i}^{k} + m_{j,j}^{k}/2 = m_{i,i}^{\prime} + m_{j,j}^{\prime}/2$, since $\rho(i, i) < 2n \leq \rho(i, j) = k$ and $\rho(j, j) < \rho(i, j) = k$. By Lemma 5.1, $m_{i,i}^{\prime} + m_{j,j}^{\prime}/2 \geq m_{i,j}^{\prime}$.

But $m_{i,i}^{\prime} + m_{j,j}^{\prime}/2 \geq m_{i,j}^{\prime}$ follows from the inductive hypothesis and the definition of $m_{p^{-1}(\ell)}^{k+1}$.

Hence it follows $\forall 0 \leq \ell < k+1. m_{p^{-1}(\ell)}^{k+1} = m_{p^{-1}(\ell)}^{\prime}$. Note $\forall k+1 \leq \ell < 4n^2. m_{p^{-1}(\ell)}^{k+1} = m_{p^{-1}(\ell)}^{\prime}$ follows from the inductive hypothesis and the definition of $m_{p^{-1}(\ell)}^{k+1}$.

Suppose $m_{i,i}^{\prime}$ is inconsistent hence $m_{i,i}^{\prime} < 0$. Put $k = \rho(i, i)$. But $m_{i,i}^{k} \leq m_{i,i}^{k}$ and by Proposition 4.2, $m_{i,i}^{4n^2} = m_{i,i}^{k+1}, \forall k < m_{i,i}^{\prime} < 0$ as required.

7.3 In-place Incremental Tight Closure

The in-place version of the incremental tight closure algorithm is presented in Figure 13. The only difference with incremental strong closure is that for key entries...
Lemma 7.2 Suppose \( m \) is a closed, coherent DBM and \( m' = \text{IncClose}(m, o) \), \( m'' = \text{Tighten}(m', o) \), \( m'' = \text{Str}(m'', o) \), \( m^* = \text{IncClose}(m'', o) \) and \( Wm^* = m'' \) or \( m^* \) is inconsistent.

Proof Suppose \( m'' \) is consistent. By Proposition 5.3 \( m'' \leq m'' \) and by Proposition 5.3 \( m'' \leq m' \) thus \( m' \) is consistent. By Theorem 4.1 \( m' \) is closed hence \( m'_{a,b} = m''_{a,b} = m''_{a,b} = m''_{b,b} = 0 \). By Corollary 7.1 it follows that \( m'_{a,b} \leq m''_{a,b} \leq d \) and \( m'_{b,b} \leq d + m''_{a,a} = d \) therefore \( m''_{a,b} \leq d \) and \( m''_{b,a} \leq d \). By Proposition 4.3 \( m' \) is coherent hence \( m'' \) is closed by Lemma 5.1.

- To show \( m''_{a,b} + d + m''_{b,a} \geq m''_{i,j} \). Since \( m'' \) is closed it follows

\[
\begin{align*}
\quad m''_{a,b} + d + m''_{b,a} & \geq m''_{a,b} + m''_{b,a} + m''_{p} \geq m''_{i,b} + m''_{b,j} \geq m''_{i,j} \\
\quad m''_{b,a} + d + m''_{a,b} & \geq m''_{b,a} + m''_{a,b} + m''_{p} \geq m''_{i,a} + m''_{a,j} \geq m''_{i,j}
\end{align*}
\]
The following theorem is analogous to the theorem for in-place strong closure:

\[
\rho \circ m \vdash \rho \circ m \circ \rho \circ m
\]

By Proposition 4.1 it follows that \( m^* = m^m \). □

The following theorem is analogous to the theorem for in-place strong closure:

**Theorem 7.3 (Correctness of INPLACEINCZCLOSE)** Suppose \( m \) is a closed, coherent DBM, \( m' = \text{INCLOSE}(m, o) \), \( m^m = \text{TIGHTEN}(m') \), \( m^m = \text{STR}(m^m) \), \( o = (x'_a - x'_b < d) \), that \( \rho : \{0, \ldots, 2n-1\}^2 \rightarrow \{0, \ldots, 4n^2-1\} \) is a bijective map with \( \forall 0 \leq i < 2n, \rho(i, i) < 2n \), \( m^0 = m \) and

\[
m^{k+1}_{i,j} = \begin{cases} 
\frac{m^k_{i,j}}{2} & \text{if } \rho(i, j) \neq k \\
\min \left( m^k_{i,a} + d + m^k_{b,i}, \\
m^k_{i,b} + d + m^k_{a,i}, \\
m^k_{i, a} + d + m^k_{b, a} + d + m^k_{b, j}, \\
m^k_{i, b} + d + m^k_{a, b} + d + m^k_{a, j} \\
m^k_{i, j} + m^k_{j, j}/2 \\
\end{cases}
\]

Then either \( m' \) is consistent and

- \( \forall 0 \leq k < l, m^{k-1}_l = m^{m-1}_l \)
- \( \forall k \leq l < 4n^2, m^{k-1}_l = m^{m-1}_l \)

or \( m^{4n^2} \) is inconsistent.

**Proof** Suppose \( m' \) is consistent.
Let \( k = 0 \). It vacuously follows that \( \forall 0 \leq \ell < k. m_{\rho^{-1}(\ell)}^k = m_{\rho^{-1}(\ell)}^0 \). Moreover \( \forall k \leq \ell < 4n^2. m_{\rho^{-1}(\ell)}^k = m_{\rho^{-1}(\ell)}^0 \) since \( m^0 = m \). Now let \( k > 0 \) and suppose \( \rho(i, j) = k \). Now suppose that \( j = \bar{i} \). Then

\[
\begin{align*}
m^k_{i,j} &= \min \left( \begin{array}{c}
m^k_{i,i}, \\
m^k_{i,a} + d + m^k_{b,i}, \\
m^k_{i,b} + d + m^k_{a,i}, \\
m^k_{i,b} + d + m^k_{a,a} + d + m^k_{b,b}, \\
m^k_{i,a} + d + m^k_{b,b} + d + m^k_{a,a}
\end{array} \right) / 2
\end{align*}
\]

If \( \rho^{-1}(i, a) < k \) then \( m^k_{i,a} = m^\prime_{i,a} \) whereas if \( \rho^{-1}(i, a) \geq k \) then \( m^k_{i,a} = m^\prime_{i,a} \); this implies that \( m^k_{i,a} = m^\prime_{i,a} \) and likewise \( m^k_{b,j} = m^\prime_{b,j} \). By Lemma \( 7.2 \) and Corollary \( 7.1 \) it follows that \( m^k_{i,a} + d + m^k_{b,j} \geq m^\prime_{i,a} + d + m^\prime_{b,j} \). By a similar argument \( m^k_{i,b} + d + m^k_{a,j} \geq m^\prime_{i,j} \) and likewise \( m^k_{i,b} + d + m^k_{a,a} + d + m^k_{b,b} \geq m^\prime_{i,j} \). Moreover \( (m^\prime_{i,a} + m^\prime_{i,j})/2 \geq \min(m^\prime_{i,j}, (m^\prime_{i,a} + m^\prime_{i,j})/2) = m^\prime_{i,j} \). Thus \( m^k_{i,j} \geq m^\prime_{i,j} \). Now to show \( m^\prime_{i,j} \geq m^k_{i,j} \), we consider the following cases:

\[
\begin{align*}
m^\prime_{i,j} = m^\prime_{i,j} &= 2 \left( m^\prime_{i,j}/2 \right) = 2 \min \left( \begin{array}{c}
m^k_{i,k}, \\
m^k_{i,a} + d + m^k_{b,i}, \\
m^k_{i,b} + d + m^k_{a,i}, \\
m^k_{i,b} + d + m^k_{a,a} + d + m^k_{b,b}, \\
m^k_{i,a} + d + m^k_{b,b} + d + m^k_{a,a}
\end{array} \right) / 2 \]
\]

Hence it follows \( \forall 0 \leq \ell < k + 1. m^k_{\rho^{-1}(\ell)} + 1 = m^\prime_{\rho^{-1}(\ell)} \). Moreover \( \forall k + 1 \leq \ell < 4n^2. m^k_{\rho^{-1}(\ell)} = m^\prime_{\rho^{-1}(\ell)} \) follows from the inductive hypothesis and the definition of \( m^k_{i,j} \).

Now suppose that \( j \neq \bar{i} \) and consider

\[
\begin{align*}
m^k_{i,j} &= \min \left( \begin{array}{c}
m^k_{i,j}, \\
m^k_{i,a} + d + m^k_{b,j}, \\
m^k_{i,b} + d + m^k_{a,j}, \\
m^k_{i,a} + d + m^k_{b,b} + d + m^k_{a,j}, \\
(m^k_{i,j} + m^k_{j,j})/2
\end{array} \right)
\end{align*}
\]

Notice that \( (m^k_{i,j} + m^k_{j,j})/2 \geq \min(m^k_{i,j} + m^k_{j,j})/2 \) since \( \rho(i, \bar{i}) < 2n \leq \rho(i, j) = k \) and similarly \( \rho(j, \bar{i}) < \rho(i, j) = k \). By Lemma \( 7.2 \) \( m^\prime_{i,j} + m^\prime_{j,j}/2 \geq m^\prime_{i,j} \) and
thus \((m^k_{i,t} + m^k_{i,s})/2 \geq m^{m^p}_{i,j}\). Repeating the argument above it follows that
\(m^{k+1}_{i,j} \geq m^{m^p}_{i,j}\). Now to show \(m^{m^p}_{i,j} \geq m^{k+1}_{i,j}\) observe:

\[
m^{m^p}_{i,j} = m^{m^p}_{i,j} = \min \left( \min \left( \frac{m'_{i,j} + m'_{i,j}}{2} \right), \frac{m'_{i,j} + m'_{j,i}}{2} \right) = m^{k+1}_{i,j}
\]

Hence it follows \(\forall 0 \leq \ell < k \Rightarrow m^{k+1}_{p^{-1}(\ell)} = m^{m^p}_{p^{-1}(\ell)}\). Note \(\forall k+1 \leq \ell < 4n^2.m^{k+1}_{p^{-1}(\ell)} = m^{m^p}_{p^{-1}(\ell)}\) follows by inductive hypothesis and definition of \(m^{k+1}_{i,j}\).

Suppose \(m'\) is inconsistent hence \(m'_{i,i} < 0\). Put \(k = \rho(i, i)\). But \(m^k \leq m\) and by Proposition 4.2 \(m^{4n^2}_{i,i} = m^{k+1}_{i,i} \leq m'_{i,i} < 0\) as required. \(\square\)

8 Experiments

To evaluate and compare algorithms we developed OCaml implementations of the non-incremental and incremental closure algorithms, both for strong and tight closure. With an eye towards retaining clarity of the code, we implemented and tuned in-place versions of the following strong closure algorithms:

- NIC: Close followed by CheckConsistent and Str;
- MICH: MINÉnCClose followed by CheckConsistent and Str;
- MIC: MINÉnCClose followed by CheckConsistent and Str;
- ICH: ICCClosHoist (with its own consistency check) followed by Str, with an additional check for rapidly detecting unsatisfiability using Corollary 4.1;
- ISC: IncStrongClose (again which includes a consistency check) augmented with the rapid unsatisfiability check, again with hoisting. Key entries were calculated and looked-up naively, rather than specialised as suggested in Section 5.3 so as to preserve code clarity.

In addition, the above algorithms were extended with tightening to give:

- NITC: Close, CheckConsistent, Tighten, CheckZConsistent and then Str, as outlined in the architecture diagram of Figure 3;
- MICT: MINÉnCClose followed by CheckConsistent, Tighten, CheckZConsistent and then Str;
Fig. 15: Experiments with strong closure algorithms on 20, 30 and 40 variables
- MICH: Min\'eIncClose followed by CheckConsistent, Tighten, CheckZConsistent and then Str but with aggressive loop-invariant code motion;
- ICT: IncCloseHoist, CheckConsistent, Tighten, CheckZConsistent and then Str, augmented with the rapid unsatisfiability check;
- ITC: IncZClose with the rapid unsatisfiability check, again with hoisting.

For the experiments we randomly generated a feasible octagon with a specific number of variables and constraints, added a single randomly generated constraint, and then calculated strong closure or tight closure. Note that the resulting DBM may be infeasible, short-circuiting a call to Str. The DBM entries were IEEE 754 standard precision floats, using integer rounding to simulate integer entries for the tightening experiments. The resulting DBMs were then all checked for equality against Close (using a tolerance threshold to handle the floats). For each problem instance (number of variables and number of constraints), we repeated
Fig. 17: Experiments with tight closure algorithms on 20, 30 and 40 of variables
Fig. 18: Experiments with tight closure algorithms on 100 and 120 of variables

the experiment 5000 times and averaged the timings. The experiments were run
on a 32-core Intel Xeon workstation with 128GB of memory, using the OCaml
forkwork library to run multiple experiments together.

The results of the strong closure experiments are summarised in Figure 15 and
Figure 16. The labels on the horizontal axis give the number of variables $n$ and
the number of constraints $m$ for each experiment, abbreviated to $n - m$ under to
block of 5 columns. The vertical axis gives average time, in seconds, taken for each
experiment. A log scale is used on the vertical so that the timings for the new
incremental algorithms are discernible.

MIC is faster than NIC and thus the additional overhead of checking the guard
at line 7 of Figure 5 does not negate the saving gained in the min operations.
However the key difference between MIC and MICH is that the guard in MICH is
decomposed into three separate checks to permit loop-invariant code motion. This
suggests that the incremental algorithm of Minè is sensitive to how the check at line 7 of Figure 3 is realised, no doubt because it is applied $O(n^3)$ times. ICH is $O(n^2)$ and is uniformly faster than MICH. ISC is faster again, but the speedup is more modest since both ISC and ICH reside in $O(n^2)$. However, even for small $n$, the speedup between MICH and ISC is at least one order of magnitude. Interestingly, all algorithms do not seem to vary very sensitive to $m$. The running time increases with $n$ for small $m$, as the likelihood of writing a DBM entry increases as the DBM becomes more populated. However, once the DBM is densely populated, which happens when $m$ is large, the running times stabilise, demonstrating that the key parameter is $n$ rather than $m$.

The tight closure experiments are summarised in Figure 17 and Figure 18, mirroring the speedups reported for strong closure in Figure 15 and Figure 16 respectively. To summarise, the four figures present a consistent message: that our new incremental closure algorithms for both strong and tight closure are uniformly and significantly faster than the existing algorithms on all sizes of problem.

9 Concluding Discussion

The octagon domain is used for many applications due to its expressiveness and its easy of implementation, relative to other relational abstract domains. Yet the elegance of their domain operations is at odds with the subtlety of the underlying ideas, and the reasoning needed to justify refinements that appear to be straightforward, such as tightening and in-place update.

This paper has presented novel algorithms to incrementally update an octagonal constraint system. More specifically, we have developed new incremental algorithms for closure, strong closure and integer closure, and their in-place variants. Experimental results with a prototype implementation demonstrate significant speedups over existing closure algorithms. We leave as future work the application of our incremental algorithms for modelling machine arithmetic [26] in binary analysis which, incidentally, was the problem that motivated this study.

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