Quantum Virasoro algebra with central charge $c = 1$
on the horizon of a $2D$-Rindler spacetime.

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Abstract. Using the holographic machinery built up in a previous work, we show that the hidden $SL(2, \mathbb{R})$ symmetry of a scalar quantum field propagating in a Rindler spacetime admits an enlargement in terms of a unitary positive-energy representation of Virasoro algebra defined in the Fock representation. That representation has central charge $c = 1$. The Virasoro algebra of operators gets a manifest geometrical meaning if referring to the holographically associated QFT on the horizon: It is nothing but a representation of the algebra of vector fields defined on the horizon equipped with a point at infinity. All that happens provided the Virasoro ground energy $h := \mu^2/2$ vanishes and, in that case, the Rindler Hamiltonian is associated with a certain Virasoro generator. If a suitable regularization procedure is employed, for $h = 1/2$, the ground state of that generator corresponds to thermal states when examined in the Rindler wedge, taking the expectation value with respect to Rindler time. This state has inverse temperature $1/(2\beta)$, where $\beta$ is the parameter used to define the initial $SL(2, \mathbb{R})$ unitary representation. (As a consequence the restriction of Minkowski vacuum to Rindler wedge is obtained by fixing $h = 1/2$ and $2\beta = \beta_U$, the latter being Unruh’s inverse temperature). Finally, under Wick rotation in Rindler time, the pair of QF theories which are built up on the future and past horizon defines a proper two-dimensional conformal quantum field theory on a cylinder.
1 Introduction and summary of previously obtained results.

1.1 Introduction. A number of papers has been concerned with the issue of the statistical origin of black-hole entropy. Holographic principle [1, 2, 3] arose by the idea that gravity near the horizon should be described by a low dimensional theory with a higher dimensional group of symmetry. Maldacena and Witten [4, 5] showed that there is a correspondence between quantum field theory in an asymptotically AdS spacetime, the “bulk”, and a conformal theory on its “boundary” at spacelike infinity. Rehren proved rigorously some holographic results for free quantum fields in a AdS background, establishing a correspondence between bulk observables and boundary observables without employing string machinery [6, 7]. Dealing with QFT in 2D-Rindler spacetime, we have proved in a recent work [8], that it is possible to define a free quantum theory on the horizon of a two dimensional Rindler space. That theory enjoys holographic interplay with the analogous theory defined in the bulk. More precisely, there are two holographic theorems. The former shows that there is a *-algebra injective homomorphism from the algebra of the bulk observables associated with the Rindler free field to the algebra of the horizon observables associated with the horizon free field. The latter identifies the observables of the theories form the point of view of unitary equivalences whenever the theory is represented in suitable Fock spaces (in that case also the vacuum states are in correspondence thorough the unitary operator which realizes holography). An interesting consequences is that the “hidden” SL(2, R) symmetry of free quantum field theory in the bulk found in [9] becomes manifest when transposed on the Killing horizon by means of unitary holography. In fact, due to the spectrum of the Hamiltonian operator, QFT theory in the bulk turns out to be invariant under a unitary representation of SL(2, R) but such a quantum symmetry cannot be induced by the geometric background because the isometries of Rindler space has a Lie algebra different form that of SL(2, R) [9]. Nevertheless, the unitary representation of SL(2, R) which realizes that bulk symmetry becomes manifests, i.e. it reveals a clear geometric meaning, if it is examined on the horizon by means of the holographic machinery. All that is summarized within subsection 1.2 in some details.

Overlap with ideas and results of [8] is present in the literature especially due to Schroer [10], Schroer and Wiesbrock [11], Schroer and Fassarella [12]. In those papers an approach to holography similar to ours is implemented in the framework of LightFront Holography developed at algebraic level using nets of local observable algebras. From a very elementary point of view, a relevant difference with our machinery is the fact that the quantization of the bulk field used by Schroer and collaborators is that referred to Minkowski vacuum and Minkowski time instead of Rindler vacuum and time. From a pure physical point of view, perhaps, the quantization with respect Rindler frame is more interesting if one tries to use our machinery as a starting point to investigate QFT near the bifurcate Killing horizon of a black hole: Rindler quantization corresponds to quantization in a reference frame that gives rise to Minkowski coordinates far from the black hole and the associated particles should be those things are made of. However the interplay of Schroer and collaborators’ ideas and achievements and procedures and results presented in our paper deserves further investigation. Another relevant paper which merits particular quotation is that by Guido, Longo, Roberts and Verch [13]. Overlap with some results
arising by our approach are present in section 4 of [13]. In that section, in the very general context of QFT in curved spacetime in terms of nets of local $C^*$ algebras (and Von Neumann representations) and making use of very general theorems by Wiesbrock on local quantum field theory defined on $S^1$ and covariant with respect to \( PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\pm I \), it is proven the existence of a local quantum field theory (covariant with respect to \( PSL(2,\mathbb{R})/\pm I \)) defined on the bifurcate Killing horizon. This is done by considering a net of Von Neumann algebras in the representation of a state which is, in restriction to the subnet of observables which are localized at the horizon, a KMS state at Hawking temperature for the Killing flow. In [8] we found some clues for the existence of a whole unitary representation of Virasoro algebra which extends the \( SL(2,\mathbb{R}) \) unitary representation on the horizon. In this paper we prove the very existence of a full unitary representation of Virasoro algebra with central charge \( c = 1 \) for quantum field theory defined on the horizon. That fact is interesting for several reasons in relation with the problem of the statistical interpretation of black hole entropy. In fact, there are several attempts to give a statistical explanation to black hole entropy by counting microstates in terms of the degeneracy of an eigenspace of a certain Virasoro generator in a suitable irreducible unitary representations of Virasoro algebra [14]. This is done by means of the so-called “Cardy’s formula”. These approaches are, in fact, based on the existence of a Virasoro algebra (with central charge \( c \neq 0 \)) in terms of generators of diffeomorphisms of the black hole manifolds considering the horizon as a boundary. The algebra of the associated generators in the Hamiltonian ADM formulation of gravity gets a non-vanishing central charge. Under the supposition that a quantum version of that Virasoro representation exists, that the value of central charge is not affected by the quantization procedure and that the actual value of the black hole mass is a eigenvalue of the Virasoro generator \( L_0 \), it is possible to compute the degeneracy of that eigenspace by means of Cardy’s formula because of the presence of a central charge. The logarithm of the degeneracy gives the very black-hole entropy law barring logarithmic corrections.

The main problem of all of those approaches is that the Virasoro algebra representation with non vanishing central charge is proven to exists at classical level only in the Hamiltonian formulation. All derivations of black hole entropy by that way are based on the found classical formulas and on the supposition that there is a quantum version of the found Hamiltonian structure (in order to use Cardy’s formula).

To make contact with the content of this paper where a quantum scalar field propagating in a 2D-Rindler space is considered, we notice that in the approaches outlined above, the only near-horizon structure is sufficient to use the Virasoro-Cardy machinery [15]. Moreover, for a Schwarzschild black-hole manifold, the relevant algebra of diffeomorphisms is that of diffeomorphisms in the plane \( r,t \) which preserve the horizon structure. Hence it seems that 2D-Rindler models are relevant to this context. On the other hand, a scalar field arises naturally in these 2D-Rindler space approaches by dimensional reduction form the gravitational theory in 4D in the presence of spherical symmetry. That field supports information of part of 4D-dimensional gravity in the 2D model. Concerning the problem of the existence of a Virasoro representation at quantum level we stress that, in this paper, we prove that a very positive-energy unitary representation of Virasoro algebra does exist at quantum level for the quantum field defined on
the horizon. That algebra of operators can be defined also for the scalar field propagating in the bulk via unitary holography.

In fact, Section 2 of this paper is devoted to show that the bulk hidden \( SL(2, \mathbb{R}) \) symmetry admits an enlargement in terms of a unitary representation of Virasoro algebra with central charge \( c = 1 \) defined in Fock representation. The Virasoro algebra of operators gets a manifest geometrical meaning if referring to the holographically associated QFT on the horizon: It gives rise to a unitary representation of a group of automorphisms of the \( \ast \)-algebra generated by field operators. This representation is induced by group of the diffeomorphisms of the horizon compactified by adding a point at infinity. Moreover, a sub-representation which is generated by three certain Virasoro generators reduces to the \( SL(2, \mathbb{R}) \) representation previously found.

Under Wick rotation with respect to Rindler time, the pair of QF theories which are built up on the future and past horizon defines a proper two-dimensional conformal quantum field theory. That CFT can be realized on the Riemann surface given by a two-dimensional cylinder. In Section 3 we prove that, with a suitable choice of the weight of the found Virasoro algebra of operators, a certain generator which generalizes Rindler Hamiltonian, admits a ground state \( \Psi \) which enjoys notable thermodynamic properties: When that state is examined in the bulk via holography, and a suitable mean is computed with respect to Rindler-time evolution, \( \Psi \) reveals itself as a thermal state whose inverse temperature is \( 2/\beta \). \( \beta \) being the parameter initially used to build up the unitary \( SL(2, \mathbb{R}) \) representation in the bulk.

1.2. QFT on a Killing horizon, unitary and algebraic holographic theorems and \( SL(2, \mathbb{R}) \) symmetry. We summarize part of the content of [8] relevant for this work within the following five steps.

[a] Consider the globally-hyperbolic spacetime \( \mathbb{R} \), called two-dimensional Rindler wedge with metric \( ds_R^2 = -\kappa^2 y^2 dt^2 + dy^2 \), that can be obtained by a suitable near horizon approximation of a general Schwarzschild-like metric also dropping the angular coordinates \( \mathbb{S} \), above \( t \in \mathbb{R}, \quad y \in (0, +\infty) \) are global coordinates. A free Klein-Gordon scalar field \( \phi \) in \( \mathbb{R} \) satisfies the equation of motion \( -\partial_t^2 \phi + \kappa^2 (y \partial_y \phi - y^2 m^2) \phi = 0 \). In Rindler quantization, the one-particle Hilbert space \( H \) consists of the space of complex linear combinations of the positive frequency parts of smooth real solutions \( \psi \) of the K-G equation with compact Cauchy data. The natural symplectic form on that space is \( \Omega(\psi, \psi') := \int_\Lambda (\psi'^\mu \nabla^\nu \psi - \psi \nabla^\nu \psi') n_\mu \, d\sigma \), \( \Lambda \) being any Cauchy surface, \( d\sigma \) the induced measure and \( n \) a unit future-oriented normal vector. Every \( \psi \) decomposes into \( \partial_t \)-stationary modes as

\[
\psi(t, y) = \psi_+(t, y) + c.c. = \int_0^{+\infty} e^{-iEt} \sum_\alpha \Phi_E^{(\alpha)}(y) \tilde{\psi}_+^{(\alpha)}(E) \, dE + c.c. \tag{1}
\]

The index \( \alpha \) distinguishes between two cases: if \( m > 0 \) there is a single mode \( \Phi_E^{(\alpha)} = \Phi_E \). If \( m = 0 \) there are two values of \( \alpha \), corresponding to ingoing and outgoing modes, \( \Phi_E^{(\text{in})/(\text{out})} = e^{\pm iE \ln(\kappa y)/\kappa/\sqrt{4\pi E}} \). In that case both ingoing and outgoing components in (1) must have Cauchy compact support. The one-particle Hilbert space of wavefunctions is obtained by taking the completion of the space of complex linear combinations of positive frequency wavefunctions
(obtained from Cauchy support compactly real wavefunctions) with respect to the Hermitian scalar product $-i\Omega(\psi^\prime_+,\psi_+)$. \(\partial_t\) evolution of a wavefunction \(\psi\) is equivalent to the action of the one-parameter subgroup generated by an Hamiltonian \(H\) on the associated \(\psi_+ \in \mathcal{H}\). \(\sigma(H) = [0, +\infty)\) for \(m \geq 0\). If \(m > 0\) there is no energy degeneration and the one-particle Hilbert space \(\mathcal{H}\) is isomorphic to \(L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^+, dE)\). Let us pass to the bosonic Fock space, \(\mathfrak{F}(\mathcal{H})\), associated with \(\mathcal{H}\). The quantum field \(\Omega(\cdot, \phi)\) of our theory is the map

\[
\psi \mapsto \Omega(\psi, \phi) := ia(\psi_+) - ia^\dagger(\psi_+),
\]

where \(\psi\) is any real compactly supported wavefunction and \(a(\psi_+)\) and \(a^\dagger(\psi_+)\) respectively denote the annihilation and construction operator associated with the one-particle states \(\psi_+\) and \(\psi_+\) respectively, defined in \(\mathfrak{F}(\mathcal{H})\) and referred to the Rindler vacuum \(|0\rangle\) (that is \(|0\rangle_m \otimes |0\rangle\) out if \(m = 0\)). \(\Omega(\psi, \phi)\) is essentially self-adjoint in the dense invariant subspace spanned by all states containing a finite arbitrarily large number of particles with states given by positive-frequency wavefunctions. Every wavefunction \(\psi, \phi\) in (1) can be obtained as \(\psi = E(f)\) where \(f\) is an associated compactly supported smooth function in \(\mathbb{R}\) and \(E\) is the causal propagator (the “advanced-minus-retarded” two point function) of Klein-Gordon operator. Moreover

\[
\int_{\mathbb{R}} \psi f \, d\mu_g = \Omega(Ef, \psi) \quad \text{and} \quad \int_{\mathbb{R}} h(x)(Ef)(x) \, d\mu_g(x) = \Omega(Ef, Eh),
\]

\(\mu_g\) being the measure induced by the metric in \(\mathbb{R}\). \([18]\) suggests to define a quantum-field operator smeared with compactly-supported complex-valued functions \(f\), as the linear map

\[
f \mapsto \hat{\phi}(f) := \Omega(Ef, \hat{\phi}),
\]

which is formally equivalent to the non-rigorous but popular definition

\[
\hat{\phi}(t, y) = \int_0^\infty \sum_{\alpha} e^{-iEt} \Phi^{(a)}_E(y)a_{E\alpha} + e^{iEt} \Phi^{(a)}_E(y)a_{E\alpha}^\dagger dE.
\]

The rigorous version of the formal identity \([\hat{\phi}(x), \hat{\phi}(x')] = -iE(x, x')\) is

\[
[\hat{\phi}(f), \hat{\phi}(h)] = -iE(f, h) := -i \int_{\mathbb{R}} h(x)(Ef)(x) \, d\mu_g(x).
\]

[b] In \([9, 8]\) we have established that, if \(m > 0\), \(\mathcal{H}\) is irreducible under a (uniquely determined) strongly-continuous unitary representation of \(SL(2, \mathbb{R})\) whose Lie algebra is given by the (uniquely determined) self-adjoint extension of the real linear combinations of operators \(H_0, D, C\):

\[
H_0 := E, \quad D := -i \left( \frac{1}{2} + E \frac{d}{dE} \right), \quad C := - \frac{d}{dE} E \frac{d}{dE} + \frac{(k - \frac{1}{2})^2}{E}.
\]
$k$ can arbitrarily be fixed in \{1/2, 1, 3/2, \ldots \}. $iH_0, iC, iD$ enjoy the commutation relations of the Lie algebra of $SL(2, \mathbb{R})$ in a suitable dense and invariant domain $\mathcal{D}_k$ where they, and their real linear combinations, are essentially self-adjoint [8] and $\mathcal{H}_0 = H$. $\mathcal{D}_k$ is the subspace spanned by the eigenvectors of the operator

$$K_\beta := \frac{1}{2} \left( \beta H_0 + \frac{1}{\beta} C \right),$$

(8)

$\beta$ being a constant with the dimensions of an inverse energy. The unitary representation does not depend on the value of $\beta$. The spectrum of the self-adjoint operator $K_\beta$ (initially defined on $\mathcal{D}_k$) is a pure point spectrum without degeneracy, it does not depend on $\beta$ itself and is $\sigma(K_\beta) = \{ \lambda_n \mid \lambda_n = n \, , \, n = k, k + 1, k + 2, \ldots \}$. If $L_\beta^{(\alpha)}$ are modified Laguerre’s polynomials [17], the associated normalized eigenvectors (which are the same as those of $K_\beta$) are

$$Z_n^{(k)}(E) := \eta_n \sqrt{\frac{\Gamma(n - k + 1)}{E \Gamma(n + k)}} e^{-\beta E} (2\beta E)^k L_{n-k}^{(2k-1)}(2\beta E), \quad n = k, k + 1, \ldots ,$$

(9)

$\eta_n$ being a pure phase which can be arbitrarily fixed (in [9] we used $\eta_n = 1$). As noticed in [9], if $\beta$ is interpreted as an inverse temperature, the exponential $e^{-\beta E}$ suggests an interpretation in terms of a canonical ensemble of the energetic content of these states. In this paper we examine in depth this possibility finding out very interesting results.

If $m = 0$ and so $\mathcal{H} \cong L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^+, dE)$, an analogue representation exists in each space $L^2(\mathbb{R}^+, dE)$. Making use of Heisenberg representation it is simply proven that the algebra generated by $H, D, C$, with depending-on-time coefficients, is made of constant of motions [8]. Thus $SL(2, \mathbb{R})$ is a symmetry of the one-particle system. That can straightforwardly be extended to the free quantum field in Fock space. The crucial point is that the found symmetry is hidden: It cannot be induced by the background geometry since the Killing fields of Rindler spacetime enjoy a different Lie algebra from that of $H_0, D, C$ and no representation of $SL(2, \mathbb{R})$ exists in terms of isometries of $\mathbb{R}$ (see [8] for definitions and details). The picture changes dramatically when the found $SL(2, \mathbb{R})$ symmetry is examined on the horizon as said in [c] below.

[c] The space $\mathbb{R}$ is naturally embedded in a Minkowski spacetime which contains the horizon associated with the rindler metric. Rindler light coordinates $u = t - \log(ky)/\kappa$, $v = t + \log(ky)/\kappa$ (where $u, v \in \mathbb{R}$) cover the (open) Rindler space $\mathbb{R}$. Separately, $v$ is well defined on the future horizon $\mathcal{F}$, $u \to +\infty$, and $u$ is well defined on the past horizon $\mathcal{P}$, $v \to -\infty$ (see figure). A wavefunction in (10) admits well-defined limits toward the future horizon $u \to +\infty$:

$$\psi(v) = \int \frac{e^{-iEv}}{\sqrt{4\pi E}} e^{i\rho_{m, \kappa}(E)} \psi_+(E) \, dE + c.c.$$ 

(10)

$e^{i\rho_{m, \kappa}(E)}$ being a pure phase [8]. In coordinate $u \in \mathbb{R}$, the restriction of $\psi$ to $\mathcal{P}$ is similar with $v$ replaced for $u$ and $\rho_{m, \kappa}(E)$ replaced by $-\rho_{m, \kappa}(E)$. If $m = 0$ restrictions to $\mathcal{F}$ and $\mathcal{P}$ are similar to (10) with the difference that $e^{i\rho_{m, \kappa}(E)}$ is replaced by 1, only ingoing components survive in the limit toward $\mathcal{F}$ and only outgoing components survive in the limit toward $\mathcal{F}$ ($v$ must be
replaced for \( u \) in that case). Discarding the phase it is possible to consider the following real “wavefunction on the (future) horizon \( \mathbf{F} \)":

\[
\varphi(v) = \int_{\mathbb{R}^+} \frac{e^{-iEv}}{\sqrt{4\pi E}} \tilde{\varphi}_+(E) \, dE + \int_{\mathbb{R}^+} \frac{e^{+iEv}}{\sqrt{4\pi E}} \bar{\tilde{\varphi}}_+(E) \, dE,
\]

where \( \varphi \) is any real function in Schwartz’ space on \( \mathbb{R} \equiv \mathbf{F} \), as the basic object in defining a quantum field theory on the future horizon. The same is doable concerning \( \eta \) in Schwartz’ space. More precisely, field operators and exact 1-forms \( \eta \). Thus, if

\[
\langle \varphi'_{+}, \varphi_{+} \rangle_{\mathbf{F}} := -i \Omega_{\mathbf{F}}(\varphi_{+}, \varphi_{+}).
\]

The one-particle Hilbert space \( \mathcal{H}_{\mathbf{F}} \) is the completion with respect to that scalar product of the space of complex combinations of positive frequency parts \( \tilde{\varphi}_{+}(E) \), of horizon wavefunctions \( \varphi \). As \( \langle \varphi'_{+}, \varphi_{+} \rangle_{\mathbf{F}} = \int_{\mathbb{R}^+} \tilde{\varphi}^\dagger_{+}(E)\tilde{\varphi}_{+}(E) \, dE, \mathcal{H}_{\mathbf{F}} \) turns out to be isomorphic to \( L^2(\mathbb{R}^+, dE) \) once again. The field operator is defined in the symmetrized Fock space \( \mathfrak{S}(\mathcal{H}_{\mathbf{F}}) \), with vacuum state \( |0\rangle_{\mathbf{F}} \), with rigorous symplectic definition given by

\[
\varphi \mapsto \Omega_{\mathbf{F}}(\varphi, \hat{\varphi}_{\mathbf{F}}) := i a(\varphi_{+}) - ia^\dagger(\varphi_{+}),
\]

where \( \varphi \) is any horizon wavefunction in the space specified above. With these definitions, in spite of the absence of any equation of motion the essential features of free quantum field theory are preserved by that definition [5]. Degeneracy of the metric on the horizon prevents from smearing field operators by functions due to the ill-definiteness of the induced volume measure. However, employing the symplectic approach [8], a well-defined smearing-procedure is that of

\[
\eta(v) \mapsto \hat{\varphi}_{\mathbf{F}}(\eta) := \Omega_{\mathbf{F}}(E_{\mathbf{F}}\eta, \hat{\varphi}_{\mathbf{F}}),
\]

which is the rigorous meaning of

\[
\hat{\varphi}_{\mathbf{F}}(\eta) = \int_0^{\infty} \frac{dE}{\sqrt{4\pi E}} \left( \int_{\mathbb{R}} e^{-iEv} \eta(v) \right) a_E + \left( \int_{\mathbb{R}} e^{iEv} \eta(v) \right) a_E^\dagger.
\]

Horizon wavefunctions \( \varphi \) and 1-forms \( \eta, \eta' \) in the spaces said above enjoy the same properties as in the bulk. More precisely one has

\[
\int_{\mathbf{F}} \varphi \eta = \Omega_{\mathbf{F}}(E_{\mathbf{F}}\eta, \varphi) \quad \text{and} \quad \int_{\mathbb{R}} (E_{\mathbf{F}}\eta) \eta' = \Omega_{\mathbf{F}}(E_{\mathbf{F}}\eta, E_{\mathbf{F}}\eta'),
\]

\[
[\hat{\varphi}_{\mathbf{F}}(\eta), \hat{\varphi}_{\mathbf{F}}(\eta')] = -i E_{\mathbf{F}}(\eta, \eta') = \int_{\mathbf{F}} \psi_{\eta'}d\psi_{\eta} - \psi_{\eta}d\psi_{\eta'}.
\]
The latter is nothing but the rigorous meaning of the formal equation \([\hat{\phi}(v), \hat{\phi}(v')] = -iE_{\mathcal{F}}(v, v')\). Finally a “locality property” holds true:

\[
[\hat{\phi}_{\mathcal{F}}(\eta), \hat{\phi}_{\mathcal{F}}(\eta')] = 0 \quad \text{if} \quad \text{supp}(\eta) \cap \text{supp}(\eta') = \emptyset.
\]

Everything we have stated for \(\mathcal{F}\) can analogously be stated for \(\mathcal{P}\).

[\(d\)] It is possible to prove the existence of a unitary equivalence between the theory in the bulk and that on the horizon in the sense we are going to describe.

**Theorem 1.1.** If \(f\) is any real smooth compactly supported function \(f\) used to smear the bulk field, define \(\eta_f := 2d(E(f) |_{\mathcal{F}})\), and \(\omega_f := 2d(E(f) |_{\mathcal{P}})\), \(E(f) |_{\mathcal{F}/\mathcal{P}}\) being the limit toward \(\mathcal{F}\), respectively \(\mathcal{P}\), of \(E(f)\) (see figure).

(a) If \(m > 0\), there is a unique injective unital \(*\)-algebras homomorphism \(U_{\mathcal{F}} : \mathfrak{F}(\mathcal{H}) \to \mathfrak{F}(\mathcal{H}_{\mathcal{F}})\) such that

\[
U_{\mathcal{F}}|_0 = |0\rangle_{\mathcal{F}}, \quad \text{and} \quad U_{\mathcal{F}}^{-1}\hat{\phi}_{\mathcal{F}}(\eta_f)U_{\mathcal{F}} = \hat{\phi}(f).
\]

(b) If \(m = 0\) two unitary operators arise \(V_{\mathcal{F}/\mathcal{P}} : \mathfrak{F}(\mathcal{H}_{\text{in/out}}) \to \mathfrak{F}(\mathcal{H}_{\mathcal{F}/\mathcal{P}})\) such that

\[
V_{\mathcal{F}/\mathcal{P}}|_0|_{\text{in/out}} = |0\rangle_{\mathcal{F}/\mathcal{P}}
\]

and

\[
V_{\mathcal{F}/\mathcal{P}}^{-1}\hat{\phi}_{\mathcal{F}}(\eta_f)V_{\mathcal{F}} = \hat{\phi}_{\text{in}}(f), \quad \text{and} \quad V_{\mathcal{F}/\mathcal{P}}^{-1}\hat{\phi}_{\mathcal{P}}(\omega_f)V_{\mathcal{P}} = \hat{\phi}_{\text{out}}(f).
\]

\(\mathcal{H}_{\text{in/out}}\) is the bulk Hilbert space associated with the ingoing/outgoing modes and \(\hat{\phi}_{\text{in/out}}(f)\) is the part of bulk field operator built up using only ingoing/outgoing modes.

Details on the construction of \(U_{\mathcal{F}}, V_{\mathcal{F}}, V_{\mathcal{P}}\) are supplied in [8]. Similarly to the extent in the bulk case, one focuses on the algebra \(A_{\mathcal{F}}\) of linear combinations of product of field operators \(\hat{\phi}_{\mathcal{F}}(\omega)\) varying \(\omega\) in the space of allowed complex 1-forms. We assume that \(A_{\mathcal{F}}\) also contains the unit operator \(I\). The Hermitean elements of \(A_{\mathcal{F}}\) are the natural observables associated with the horizon field. From an abstract point of view the found algebra is a unital \(*\)-algebra of formal operators \(\phi_{\mathcal{F}}(\eta)\) with the additional properties \([\phi_{\mathcal{F}}(\eta), \phi_{\mathcal{F}}(\eta')] = -iE_{\mathcal{F}}(\eta, \eta')\), \(\phi_{\mathcal{F}}(\eta)^* = \phi_{\mathcal{F}}(\overline{\eta})\) and linearity in the form \(\eta\). (The analogous algebra of operators in the bulk fulfil the further requirement \(\phi(f) = 0\) if (and only if) \(f = Kg\), \(K\) being the Klein-Gordon operator. No analogous requirement makes sense for \(A_{\mathcal{F}}\) since there is no equation of motion on the horizon.) \(A_{\mathcal{F}}\) can be studied no matter any operator representation in any Fock space. Operator representations are obtained via GNS theorem once an algebraic state has been fixed [1]. \(A_{\mathcal{P}}\) can analogously be defined. Below \(A_{\mathcal{R}}\) denotes the unital \(*\)-algebra associated with the bulk field operator. If \(m = 0\), \(A_{\mathcal{R}}\) naturally decomposes as \(A_{\text{in}} \otimes A_{\text{out}}\) (see [8]) with obvious notation. We have the following result which is independent from any choice of vacuum state and Fock representation. The proof can be found in [8].

**Theorem 1.2.** Assume the same notation as in Theorem 1.1 concerning \(\eta_f\) and \(\omega_f\).

(a) If \(m > 0\), there is a unique injective unital \(*\)-algebras homomorphism \(\chi_{\mathcal{F}} : A_{\mathcal{R}} \to A_{\mathcal{F}}\) such
that \( \chi_F(\phi(f)) = \phi_F(\eta_f) \). Moreover in GNS representations in the respectively associated Fock spaces \( \mathcal{F}(\mathcal{H}_{in/out}) \) built up over \( |0\rangle_{in/out} \) and \( |0\rangle_{F/P} \) respectively, \( \chi_F \) has a unitary implementation naturally induced by \( U_F \) (e.g. \( \chi_F(\phi(f)) = U_F \phi(f) U_F^{-1} \)).

(b) If \( m = 0 \), there are two injective unital *-algebras homomorphisms \( \Pi_F/P : A_{in/out} \to A_{F/P} \) such that \( \Pi_F(\phi(f)) = \phi_F(\eta_f) \) and \( \Pi_P(\phi(f)) = \phi_P(\omega_f) \). Moreover in GNS representations in the respectively associated Fock spaces \( \mathcal{F}(\mathcal{H}_{in/out}) \), \( \mathcal{F}(\mathcal{H}_{F/P}) \) built up over \( |0\rangle_{in/out} \) and \( |0\rangle_{F/P} \) respectively, \( \Pi_F \) and \( \Pi_P \) have unitary implementations and reduce to \( V_F \) and \( V_P \) respectively.

Notice that, in particular \( \chi_F \) preserves the causal propagator, in the sense that it must be 
\[-iE(f,g) = [\phi(f),\phi(g)] = [\phi_F(f),\phi_P(g)] = -iE_F(\eta_f,\eta_g).\]

[e] Consider quantum field theory on \( F \), but the same result holds concerning \( P \). In \( \mathcal{H}_F \cong L^2(\mathbb{R}^+,dE) \) define operators \( H_{F0}, D_F, C_F \) as the right-hand side of the equation that respectively defines \( H_0, D, C \) in (7). They and their real linear combinations are essentially self-adjoint if restricted to the invariant dense domain \( D_k(F) \) defined with the same definition as \( D_k \) in [b]. Exactly as in the bulk case, operators \( iH_F = (iH_{F0},iD_F,iC_F \) generate a strongly-continuous unitary \( SL(2, \mathbb{R}) \) representation \( \{U_g(F) \} \) \( g \in SL(2, \mathbb{R}) \). Hence, varying \( g \in SL(2, \mathbb{R}) \), the unitary operators obtained by unitary holomorphy \( (U_g(F) |_{\mathcal{C}})^{-1} U_g(F) \) define a representation of \( SL(2, \mathbb{R}) \) for the system in the bulk. By construction \( (U_F |_{\mathcal{C}})^{-1} H_F U_F |_{\mathcal{C}} = H \). As a consequence every \( U_g(F) \) gives rise to a \( SL(2, \mathbb{R}) \) symmetry of the bulk field and the group of these symmetries is unitary equivalent to that generated by \( iH, iD, iC \). In particular the one-parameter group associated with \( H_F \) generates \( v \)-displacements of horizon wavefunctions which are equivalent, under unitary holomorphy, to \( t \)-displacements of bulk wavefunctions. Now, it make sense to investigate the geometrical nature of the \( SL(2, \mathbb{R}) \) representation \( \{U_g(F) \} \) that, as we said, induces, up to unitary equivalences, the original \( SL(2, \mathbb{R}) \) symmetry in the bulk, but now can be examined on the horizon. In fact, the symmetry has a geometrical meaning: The action of every \( U_g(F) \) on a state \( \tilde{\varphi}_+ = \tilde{\varphi}_+(E) \) is essentially equivalent to the action of a corresponding \( F \)-diffeomorphism on the associated (by \( \Pi_F \)) horizon wavefunction \( \varphi \). More precisely [3]:

**Theorem 1.3.** Assume \( k = 1 \) in (7), take a matrix \( g \in SL(2, \mathbb{R}) \). Let \( \varphi = \varphi(v) \) be a real Schwartz’ horizon wavefunction with positive frequency part \( \varphi_+ = \varphi_+(E) \) and such that \( \varphi(0) = 0 \) and \( v \mapsto \varphi(1/v) \) belongs to Schwartz’ space too.

The wavefunction \( \varphi_g(v) \) associated with \( U_g(F) \tilde{\varphi}_+ \) reads

\[
\varphi_g(v) = \varphi \left( \frac{av + b}{cv + d} \right) - \varphi \left( \frac{b}{d} \right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g^{-1}.
\]

Moreover one has the particular cases:

(a) The unitary one-parameter group generated by \( iH_F \) is associated to the one-parameter group of \( F \)-diffeomorphisms generated by \( \partial_v \).

In other words, for every \( t \in \mathbb{R} \) and positive-frequency part wavefunction \( \tilde{\varphi}_+ \), the positive-
frequency part wavefunction $e^{iH_F \hat{\varphi}^+}$ is associated with the horizon wavefunction $\varphi_0$, such that

$$\frac{\partial \varphi_0}{\partial t}\bigg|_{t=0} = -\partial_v \varphi.$$  

(b) With the same terminology as in the case (a), the unitary one-parameter group generated by $i\hat{D}_F$ is associated to the one-parameter group of $\mathbf{F}$-diffeomorphisms generated by $v\partial_v$.

(c) With the same terminology as in the case (a), the unitary one-parameter group generated by $i\hat{C}_F$ is associated to the one-group of $\mathbf{F}$-diffeomorphisms generated by $v^2\partial_v$.

The term $-\varphi(b/d)$ in (17) assures that $\varphi$ vanishes as $v \to \pm \infty$. Notice that the added term disappears when referring to $d\varphi$ rather than $\varphi$. The group of diffeomorphisms of $\mathbf{F} \equiv \mathbb{R}$ used above,

$$v \mapsto av + b \quad \frac{cv + d}{(a \ b) \in SL(2, \mathbb{R})}$$

in fact gives a representation of $SL(2, \mathbb{R})$. It can be obtained by finite composition of one-parameter subgroups associated with the following three vector fields on $\mathbf{F}$: $\partial_v$, $v\partial_v$, $v^2\partial_v$. It is simply proven that the Lie brackets of $-\partial_v$, $-v\partial_v$, $-v^2\partial_v$ produce the same algebra as the Lie algebra of $SL(2, \mathbb{R})$. We conclude that the bulk $SL(2, \mathbb{R})$-symmetry is manifest when examined on the horizon, in the sense that it is induced by the geometry.

2 From the line to the circle: The full Virasoro Algebra

Noli tangere circulos meos.

(Archipedes’ last words.)

The algebra of vector fields $\partial_v, v\partial_v, v^2\partial_v$ can be extended to include the class of fields defined on the horizon $v^{n+1}\partial_v$ with $n \in \mathbb{Z}$. It is interesting to notice that these fields (more precisely the fields $-v^{n+1}\partial_v$) enjoy Virasoro commutation relations without central charge. In fact there is a central representation of Virasoro algebra which presents a central charge and is directly defined in terms of operators acting in the Fock space of the horizon particles. The representation can be introduced after one has given a convenient definition of quantum field operator on the circle $\mathbf{F} = \mathbf{F} \cup \{\infty\}$.

2.1. QFT on the circle $\mathbf{F} = \mathbf{F} \cup \{\infty\}$. Consider the vector field on $\mathbf{F}$,

$$\mathcal{K} := \frac{1}{2} \left( \beta \partial_v + \frac{1}{\beta} v^2 \partial_v \right).$$

That field is associated with the essentially self-adjoint operator that is defined on $\mathcal{D}^{(\mathbf{F})}_1$

$$K_{\beta} := \frac{1}{2} \left( \beta H_{\mathbf{F}} + \frac{1}{\beta} C_{\mathbf{F}} \right),$$

10
in the Lie algebra of the unitary representation of $SL(2, \mathbb{R})$ because of of Theorem 1.4. It is simply proven that the integral line of $K$ with origin in $v = 0$ is $v = \beta \tan(\theta/2)$ with $\theta \in (-\pi, \pi)$ and $v = 0$ corresponding to $\theta = 0$. One can use $\theta$ as a new coordinate on $\mathcal{F}$ with the advantage that this new coordinate gets finite values in the whole compactified manifold $\mathcal{F} \cup \{\infty\} \equiv \mathcal{F}$ (in the sense of Alexandrov’s procedure), the added point $\infty$ corresponding to $\theta = \pi \equiv -\pi$ in the circle. As a consequence of our definitions, it turns out that

$$\mathcal{K} = \partial_{\theta}.$$  \hfill (21)

In fact this formula smoothly extends the left-hand side on the whole circle $\mathcal{F}$. By construction, there is the natural submanifold embedding $\mathcal{F} \subset \mathcal{F}$. We want to show that such an inclusion can be extended to free quantum field theory if a suitable definition of QFT on $\mathcal{F}$ is given. We follow a procedure very similar to that used for the horizon. As a final result we show that more strongly, the “inclusion” of QFT on $\mathcal{F}$ into QFT on $\mathcal{F}$ is actually a unitary equivalence as well as a $*$-algebras inclusion. The observable $K_{\mathcal{F}, \beta}$ plays a central role in that identification. The associated quantum field theory on $\mathcal{F}$ will be proved to support a nice unitary Virasoro’s algebra representation with an explicit geometric meaning that extends the unitary representation of $SL(2, \mathbb{R})$.

Consider the space of real $C^\infty$ functions $\rho$ on $\mathcal{F}$, $C^\infty(\mathcal{F}; \mathbb{R})$, and define a subsequent real vector space $S(\mathcal{F})$ by taking the quotient with respect to the equivalence relation, for $\rho, \rho' \in C^\infty(\mathcal{F}; \mathbb{R})$

$$\rho \sim \rho' \text{ iff } d(\rho - \rho') = 0.$$  \hfill (22)

From now on, the elements of $S(\mathcal{F})$ are called circle wavefunctions. The following symplectic form on $S(\mathcal{F})$ is well-defined and nondegenerate (the latter is not true on $C^\infty(\mathcal{F}; \mathbb{R})$):

$$\Omega_{\mathcal{F}}(\rho, \rho') := \int_{\mathcal{F}} \rho' d\rho - \rho d\rho'.$$  \hfill (23)

The elements of $C^\infty(\mathcal{F}; \mathbb{C})$ can be expanded in Fourier series. If $\rho \in C^\infty(\mathcal{F}; \mathbb{R})$, with a re-arrangement of the Fourier coefficients it holds either in $L^2(\mathcal{F}, d\theta)$ and in the uniform sense

$$\rho(\theta) = \rho_0 + \sum_{n=1}^{\infty} \frac{e^{-in\theta} \tilde{\rho}_+(n)}{\sqrt{4\pi n}} + \frac{e^{in\theta} \tilde{\rho}_-(n)}{\sqrt{4\pi n}} + c.c.$$.

As $\tilde{\rho}_+(n)$ is proportional to $\int_{\mathcal{F}} e^{-in\theta} \rho(\theta)d\theta$ and $\int_{\mathcal{F}} e^{-in\theta} d\theta = 0$, if $n > 0$, the coefficients $\tilde{\rho}(n)$ are not affected if $\rho$ is replaced by $\rho'$ with $\rho - \rho' = \text{constant}$, and thus the coefficients $\tilde{\rho}_+(n)$, for $n > 0$, are well-associated with an element of $S(\mathcal{F})$. In the following we indicate the elements of $S(\mathcal{F})$ simply by $\rho$ instead of $[\rho]$. In the sense clarified above, if $\rho \in S(\mathcal{F})$ we have the expansion

$$\rho(\theta) = \sum_{n=1}^{\infty} \frac{e^{-in\theta} \tilde{\rho}_+(n)}{\sqrt{4\pi n}} + c.c. = \rho_+(\theta) + c.c.$$  \hfill (24)
To define the one-particle Hilbert space, define the Hermitean scalar product
\[
\langle \rho^\prime_+, \rho_+ \rangle_F := -i \Omega_F(\overline{\rho^\prime_+}, \rho_+) .
\]
The one-particle Hilbert space \( \mathcal{H}_F \) is the completion with respect to that scalar product of the space of complex combinations of positive frequency parts \( \{ \tilde{\rho}^+_+(n) \} \), of circle wavefunctions \( \tilde{\rho}^+_+ \). It is simply proven that, if \( \rho \in C^\infty(F; \mathbb{C}) \) with Fourier coefficients \( \{ C_n \}_{n \in \mathbb{Z}} \), for every \( p = 0, 1, \ldots \) there is a real \( K_p \) such that \( |n|^p |C_n| \leq K_p \) for all \( n \in \mathbb{Z} \). As a consequence, \( \sum_{n \in \mathbb{Z}} n|C_n|^2 < \infty \).

We conclude that, if \( \rho \in S(F) \), the sequence of complex numbers \( \{ \tilde{\rho}^+_+(n) = \sqrt{2} n C_n \}_{n=1}^{\infty} \) is an element of \( \ell^2(\mathbb{C}) \). A direct computation shows that \( \mathcal{H}_F \) turns out to be isomorphic to \( \ell^2(\mathbb{C}) \) because
\[
\langle \rho^\prime_+, \rho_+ \rangle_F = \sum_{n=1}^{\infty} \overline{\tilde{\rho}^+_+(n)} \tilde{\rho}^+_+(n)
\]
Using the Hilbert base of \( \mathcal{H}_F \) given by the eigenvectors of the operator \( K_\beta_F \), \( \{ Z_n^{(1)} \}_{n=1,2,\ldots} \) (where the phase of \( Z_n^{(1)} \) in (9) is fixed to be \( \eta_n = (-1)^{n+1} \)), the unitary map \( M : \mathcal{H}_F \to \mathcal{H}_F \) can be defined such that
\[
M : \varphi \mapsto \{ \langle Z_n^{(1)}, \varphi \rangle \}_{n=1,2,\ldots} .
\]
That isomorphism has a natural geometric interpretation stated in the former part of the theorem below.

**Theorem 2.1.** Let \( \varphi = \varphi(v) \) be a real horizon wavefunction (which belongs to Schwartz’ space on \( \mathbb{R} \equiv F \)) associated with a quantum state \( \tilde{\varphi}^+_+ \). If \( \rho \) is the circle wavefunction associated with \( \varphi \) by means of the unitary transformation (25), that is \( \tilde{\rho}^+_+ := M(\tilde{\varphi}^+_+) \), one has
\[
\rho(\theta) = \varphi(v(\theta)) ,
\]
where \( v(\theta) = \beta \tan(\theta/2) \), \( \theta \in (-\pi, \pi] \). In other words
\[
\varphi(v(\theta)) = \sum_{n=1}^{\infty} \frac{\langle Z_n^{(1)}, \tilde{\varphi}^+_+ \rangle}{\sqrt{4\pi n}} e^{-in\theta} + \text{c.c.} + \text{constant} .
\]
The linear map \( \varphi \mapsto \rho \) defined in (26) is injective and preserves the symplectic forms of the respective spaces, that is, if \( \rho' \) is associated with \( \varphi' \) by the map (25) itself,
\[
\Omega_F(\rho, \rho') = \Omega_F(\varphi, \varphi') .
\]

**Proof.** Notice that, if the real horizon wavefunction \( \varphi = \varphi(v) \) is in the Schwartz’ space, the function \( (-\pi, \pi] \ni \theta \mapsto \varphi(v(\theta)) \) is well-defined and belongs to \( C^\infty(F; \mathbb{R}) \) with \( \varphi(v(\pm \pi)) = 0 \) with all its derivatives of any order. So the thesis makes sense. The second part can straightforwardly be proven by using the given definitions, so we focus on the former only. If the real horizon
wavefunction \( \varphi = \varphi(v) \) is in the Schwartz’ space, the associated positive frequency part \( \tilde{\varphi}_+(E) \) is such that \( \tilde{\varphi}_+(E)/\sqrt{4\pi E} \) is the restriction to \( \mathbb{R}^+ \) of a Schwartz’ function. As a consequence \( \varphi_+(v) = \int_0^{+\infty} dE \, e^{-iEv} \tilde{\varphi}_+(E)/\sqrt{4\pi E} \) is smooth and

\[
\varphi_+(v(\theta)) \sim \text{const.} \frac{\tilde{\varphi}_+(E)}{\sqrt{E}} |_{E=0} (\theta \mp \pi)^2
\]
as \( \theta \to \pm \pi \). So the Fourier expansion of \( \varphi_+ \) makes sense and each coefficient of the Fourier expansion of \( \varphi \) is the sum of the corresponding coefficients of the Fourier expansion of \( \varphi_+ \) and \( \overline{\varphi}_+ \), it being \( \varphi = \varphi_+ + \overline{\varphi}_+ \). We want to evaluate the Fourier coefficients of \( \varphi_+ \). First consider the Fourier coefficients with \( n > 0 \). By direct computation \[17\] one finds

\[
\int_0^{+\infty} \frac{e^{-iE\beta \tan(\theta/2)} Z_n^{(1)}(E)}{\sqrt{4\pi E}} dE = \frac{1}{\sqrt{4\pi n}} \left( (-1)^{n+1} + e^{-in\theta} \right),
\]
with \( \theta \in (-\pi, \pi] \) (notice that the dependence form \( \beta \) cancels out due the shape \[19\] of functions \( Z_n^{(1)} \) by passing to the new variable of integration \( E \beta \) in the integral). As a consequence, defining \( Z_n^{(1)}(E) = 0 \) if \( E < 0 \), inverting the Fourier(-Plancherel) transform (and changing the integration variable \( v \to -v \),

\[
\frac{Z_n^{(1)}(E)}{\sqrt{2E}} = \lim_{L \to +\infty} \int_{-L}^{L} dv \, e^{-iEv} \frac{(-1)^{n+1} + e^{2in\tan^{-1}(v/\beta)}}{\sqrt{4\pi n}},
\]
the limit being computed in the sense of \( L^2(\mathbb{R}, dE) \). Since \( E \mapsto \tilde{\varphi}_+(E)/\sqrt{E} \) is the restriction to \( \mathbb{R}^+ \) of a Schwartz function, \( E \mapsto \psi(E) = \sqrt{2E} \tilde{\varphi}_+(E) \) (\( \psi(E) := 0 \) for \( E < 0 \)) is a function in \( L^1(\mathbb{R}, dE) \cap L^2(\mathbb{R}, dE) \). The functions \( E \mapsto Z_n^{(1)}(E) \) and \( E \mapsto Z_n^{(1)}(E)/\sqrt{2E} \) (assumed to vanish for \( E < 0 \)) are real and belong to \( L^1(\mathbb{R}, dE) \cap L^2(\mathbb{R}, dE) \). It holds

\[
\langle Z_n^{(1)}, \tilde{\varphi}_+ \rangle = \int_{-\infty}^{\infty} Z_n^{(1)}(E) \tilde{\varphi}_+(E) dE = \int_{-\infty}^{\infty} \frac{Z_n^{(1)}(E)}{\sqrt{2E}} \psi(E) dE.
\]

Using \[29\] and taking the \( L^2 \)-continuity of the scalar product into account, one gets

\[
\langle Z_n^{(1)}, \tilde{\varphi}_+ \rangle = \lim_{L \to +\infty} \int_{-\infty}^{\infty} dE \, \psi(E) \int_{-L}^{L} dv \, e^{-iEv} \frac{(-1)^{n+1} + e^{2in\tan^{-1}(v/\beta)}}{\sqrt{4\pi n}},
\]
that is

\[
\langle Z_n^{(1)}, \tilde{\varphi}_+ \rangle = \lim_{L \to +\infty} \int_{0}^{\infty} dE \tilde{\varphi}_+(E) \int_{-L}^{L} dv \, 2E e^{-iEv} \frac{(-1)^{n+1} + e^{in\theta(v)}}{\sqrt{4\pi n}},
\]
Using \( E e^{-iEv} = i \frac{\partial}{\partial \theta} e^{-iEv} \) and integrating by parts it arises

\[
\langle Z_n^{(1)}, \tilde{\varphi}_+ \rangle = C(L) + \lim_{L \to +\infty} \sqrt{2n} \int_{0}^{\infty} dE \tilde{\varphi}_+(E) \int_{-L}^{L} dv \, e^{-iEv} \frac{e^{in\theta(v)}}{\sqrt{4\pi E}} \frac{d\theta}{\sqrt{2\pi}},
\]
where $C(L)$ is a boundary term which vanishes in the limit $L \to \infty$ by Riemann-Lebesgue’s lemma. Interchanging the integration symbols and taking the limit as $L \to \infty$ we finally get the $n$-th Fourier coefficient of $\varphi_+$ and the $(−n)$-th Fourier coefficient of $\bar{\varphi}_+$

$$
\int_{−\pi}^{\pi} \varphi_+(v(\theta)) \frac{e^{in\theta}}{\sqrt{2\pi}} d\theta = \frac{\langle Z_n^{(1)}, \varphi_+ \rangle}{\sqrt{2n}}, \quad \int_{−\pi}^{\pi} \varphi_+(v(\theta)) \frac{e^{-in\theta}}{\sqrt{2\pi}} d\theta = \frac{\langle Z_n^{(1)}, \bar{\varphi}_+ \rangle}{\sqrt{2n}}.
$$

Now we pass to consider the remaining Fourier coefficients. Since in (29) $Z_n^{(1)}(E)$ is defined to vanish for $E < 0$, one has that, if $n > 0$

$$
\lim_{L \to +\infty} \int_{−\infty}^{0} dE \int_{−L}^{L} dv e^{−ivE} \frac{(−1)^{n+1} + e^{2im\tan^{-1}(v/\beta)}}{\sqrt{4\pi n}} = 0,
$$

which, after complex conjugation and change of variables $E \to −E$, is equivalent to

$$
\lim_{L \to +\infty} \int_{0}^{+\infty} dE \int_{−L}^{L} dv e^{−ivE} \frac{(−1)^{m+1} + e^{2im\tan^{-1}(v/\beta)}}{\sqrt{4\pi|m|}} = 0, \quad (30)
$$

where $m = −n < 0$ and $g \in L^2(\mathbb{R}, dE)$. Using $g(E) = \sqrt{2E} \hat{\varphi}_+(E)$ for $E \geq 0$ and $g(E) = 0$ otherwise and following the same procedure as for the case $n > 0$, (30) implies that, if $n < 0$

$$
\int_{−\pi}^{\pi} \varphi_+(v(\theta)) \frac{e^{in\theta}}{\sqrt{2\pi}} d\theta = 0.
$$

As a consequence,

$$
\int_{−\pi}^{\pi} \varphi_+(v(\theta)) \frac{e^{-in\theta}}{\sqrt{2\pi}} d\theta = 0.
$$

Putting all together we get

$$
\varphi(v(\theta)) = \varphi_+(v(\theta)) + \varphi_+(v(\theta)) = \text{constant} + \sum_{n=1}^{\infty} \frac{\langle Z_n^{(1)}, \varphi_+ \rangle}{\sqrt{4\pi n}} e^{-in\theta} + \sum_{n=1}^{\infty} \frac{\langle Z_n^{(1)}, \bar{\varphi}_+ \rangle}{\sqrt{4\pi n}} e^{in\theta}
$$

which concludes the proof. □

The result stated in Theorem 2.1 suggests to define a quantum field on the circle $F := \mathbb{F} \cup \{\infty\}$ whose Hilbert space is the symmetrized Fock space $\mathcal{F}(\mathcal{H}_F) \cong \mathcal{F}(\mathcal{H}_F)$, where the isomorphism is that naturally induced by $M$ of eq. (25) and the vacuum $|0\rangle_F$ is associated with $|0\rangle_F$ by the isomorphism itself. Formally the quantum field operator on $\mathcal{F}$ reads

$$
\hat{\varphi}(\theta) = \sum_{n=1}^{\infty} \frac{e^{-in\theta} \alpha_n}{\sqrt{4\pi n}} + \frac{e^{in\theta} \alpha_n^\dagger}{\sqrt{4\pi n}}, \quad (31)
$$

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where $\alpha_n$ and $\alpha_n^\dagger$ are the annihilator and constructor operator of modes $Z_n^{(1)}$.

The field operator is defined in the symmetrized Fock space $\mathcal{F}(\mathcal{H}_F)$, with rigorous symplectic definition given by

$$\rho \mapsto \Omega_F(\rho, \hat{\phi}_F) := i\alpha(\rho_-) - i\alpha^\dagger(\rho_+), \quad (32)$$

where $\rho$ is any circle wavefunction and respectively $\alpha(\rho_-), \alpha^\dagger(\rho_+)$ annihilates and creates the states $\rho_-$ and $\rho_+$. Once again a well-defined smearing-procedure is that of field operators and exact 1-forms $\eta = d\rho$ where $\rho \in \mathcal{S}(\mathcal{F})$. Notice that $d\rho$ does not depend on the chosen element of the class of equivalence associated with $\rho$. More precisely, we introduce the “causal propagator” on $\mathcal{F}$

$$\eta(\theta) \mapsto E_F(\eta) = \frac{1}{4} \int_{\mathcal{F}} \left[ \text{sign}(\theta - \theta') + (\theta' - \theta)/\pi \right] \eta(\theta'),$$

where it is understood that one has to take the quotient with respect the equivalence relation defining $\mathcal{S}(\mathcal{F})$ after the action of $E_F$. $E_F$ gives rise to a bijective linear map from the space of real exact $C^\infty$ one-forms on $\mathcal{F}$ (which will be denoted by $\mathcal{D}(\mathcal{F})$) and $\mathcal{S}(\mathcal{F})$ itself. Indeed, it results that if $\rho \in \mathcal{S}(\mathcal{F})$

$$E_F(\omega) = \rho \quad \text{if and only if} \quad \omega = 2d\rho. \quad (33)$$

We can define the field operator smeared by elements of $\mathcal{D}(\mathcal{F}; \mathbb{R})$ as

$$\eta \mapsto \hat{\phi}_F(\eta) := \Omega_F(E_F\eta, \hat{\phi}_F), \quad (34)$$

which is the rigorous meaning of

$$\hat{\phi}_F(\eta) = \sum_{n=1}^{\infty} \left( \int_{\mathcal{F}} e^{-in\theta} \eta(\theta) \right) \frac{\alpha_n}{\sqrt{4\pi n}} + \left( \int_{\mathcal{F}} e^{in\theta} \eta(\theta) \right) \frac{\alpha_n^\dagger}{\sqrt{4\pi n}}. \quad (35)$$

Circle wavefunctions $\rho$ and 1-forms $\eta, \eta'$ in the spaces said above enjoy the same properties as in the bulk. More precisely one has

$$\int_{\mathcal{F}} \rho\eta = \Omega_F(E_F\rho, \eta) \quad \text{and} \quad \int_{\mathcal{F}} (E_F\eta)\eta' = \Omega_F(E_F\eta, E_F\eta'), \quad (36)$$

$$[\hat{\phi}_F(\eta), \hat{\phi}_F(\eta')] = -iE_F(\eta, \eta'). \quad (37)$$

The latter is nothing but the rigorous meaning of the formal equation $[\hat{\phi}_F(\theta), \hat{\phi}_F(\theta')] = -iE_F(\theta, \theta')$. Notice that as a consequence of (33), (36), (37), a “locality property” holds

$$[\hat{\phi}_F(\eta), \hat{\phi}_F(\eta')] = 0 \quad \text{if} \quad \text{supp}(\eta) \cap \text{supp}(\eta') = \emptyset.$$

Everything we said about the future circle $\mathcal{F} = \mathcal{F} \cup \{\infty\}$ can be restated, with obvious changes of notation, for the past circle $\mathcal{P} := \mathcal{P} \cup \{\infty\}$.

Theorem 2.1 together with Theorems 1.1 and 1.2 has two straightforward consequences.
Theorem 2.2. If $f$ is any real smooth compactly supported function $f$ used to smear the bulk field, extend on $F$ and $P$ the forms $\eta_f$ and $\omega_f$ defined in Theorem 1.1 by putting $\eta_f(\infty) := 0$ and $\omega_f(\infty) := 0$ and consider these forms as elements of $D(F)$ and $D(P)$ respectively. 

(a) If $m > 0$, there is a unitary map $U_F : \hat{F} (\mathcal{H}) \rightarrow \hat{F} (\mathcal{H}_F)$ such that

$$U_F |0\rangle_F = |0\rangle_F, \quad \text{and} \quad U_F^{-1}\hat{\phi}_F (\eta_f) U_F = \hat{\phi}(f).$$

(b) If $m = 0$, two unitary operators arise $V_{F/P} : \hat{F} (\mathcal{H}_{\text{in/out}}) \rightarrow \hat{F} (\mathcal{H}_{F/P})$ such that

$$V_{F/P} |0\rangle_{\text{in/out}} = |0\rangle_{F/P}$$

and

$$V^{-1}_{F/P} \hat{\phi}_F (\eta_f) V_F = \hat{\phi}_{\text{in}} (f), \quad \text{and} \quad V^{-1}_{F/P} \hat{\phi}_P (\omega_f) V_P = \hat{\phi}_{\text{out}} (f).$$

$\mathcal{H}_{\text{in/out}}$ is the bulk Hilbert space associated with the ingoing/outgoing modes and $\hat{\phi}_{\text{in/out}} (f)$ is the part of bulk field operator built up using only ingoing/outgoing modes.

Sketch of proof. The unitary operator $U_F$ in Theorem 1.1 is obtained (see [8]) as the unitary operator that fulfils the following pair of conditions. (1) $U_F |0\rangle_F = |0\rangle_F$; (2) for every natural $n$, consider the subspace of $\hat{F} (\mathcal{H})$, $\mathcal{H}^n \otimes$, spanned by (symmetrized) states with $n$ particles; on every $\mathcal{H}^{(n)}$, $U_F$ reduces to the tensor product of $n$ copies of the unitary operator $U_F : \mathcal{H} \rightarrow \mathcal{H}_F$, where, under the identifications (working in the energy representations) $\mathcal{H} \cong L^2 (\mathbb{R}^+, dE)$, $\mathcal{H}_F \cong L^2 (\mathbb{R}^+, dE)$, $U_F$ is nothing but the identity operator. Now consider the composite unitary operator $U_F := M \circ U_F : \mathcal{H} \rightarrow \mathcal{H}_F$, where $M$ is as in eq. (20), and define $U_F$ such that $U_F |0\rangle_F = |0\rangle_F$ and the restriction of $U_F$ to every $\mathcal{H}^{(n)}$ coincides with to the tensor product of $n$ copies of the unitary operator $U_F$. Theorems 1.1 and 2.1 and the definition of $\hat{\phi}_F$ immediately imply the validity of the thesis. The case of $m = 0$ can be proven by the same way. ☐

Similarly to the extent on the horizon case, one can focus on the algebra $A_F$ of linear combinations of product of field operators $\phi_F (\omega)$, varying $\omega$ in the space $D(F; \mathbb{C}) := D(F) + i D(F)$ and defining $\hat{\phi}_F (\omega + i \omega') := \hat{\phi}_F (\omega) + i \hat{\phi}_F (\omega')$. We assume that $A_F$ also contains the unit operator $I$. The Hermitean elements of $A_F$ are the natural observables associated with the horizon field. From an abstract point of view the found algebra is a unital $*$-algebra of formal operators $\hat{\phi}_F (\eta)$ with the additional properties $[\hat{\phi}_F (\eta), \hat{\phi}_F (\eta')] = - i E_F (\eta, \eta')$, $\hat{\phi}_F (\eta)^* = \hat{\phi}_F (\bar{\eta})$ and linearity in the form $\eta$. $A_F$ can be studied no matter any operator representation in any Fock space. Operator representations are obtained via GNS theorem once an algebraic state has been fixed [15]. Everything we said can be extended to the analogous *-algebra defined on $P$, $A_P$. We have a second result.

Theorem 2.3. Assume the same notation as in Theorem 2.2 concerning $\eta_f$ and $\omega_f$.

(a) If $m > 0$, there is a unique injective unital *-algebras homomorphism $\chi_F : A_R \rightarrow A_F$ such that $\chi_F (\phi (f)) = \phi_F (\eta_f)$. Consider the GNS representations in the Fock spaces $\hat{F} (\mathcal{H})$, $\hat{F} (\mathcal{H}_F)$ built up over $|0\rangle$ and $|0\rangle_F$ respectively associated with $A_R$ and $A_F$, in these representations $\chi_F$
has a unitary implementation naturally induced by $U_F$ (e.g. $\chi_F(\hat{\phi}(f)) = U_F \hat{\phi}(f) U_F^{-1}$).

(b) If $m = 0$, there are two injective unital $\ast$-algebras homomorphisms $\Pi_{\infty/F}: A_{\infty/F} \to A_{\infty/F}$ such that $\Pi_{\infty}(\hat{\phi}(f)) = \phi_{\infty}(\eta_f)$ and $\Pi_{F}(\hat{\phi}(f)) = \phi_{F}(\omega_f)$. Moreover, considering the GNS representations in the Fock spaces $\mathcal{F}(\mathcal{H}_{in/out})$, $\mathcal{F}(\mathcal{H}_{F/P})$ built up over $|0\rangle_{in/out}$ and $|0\rangle_{F/P}$ respectively associated with $A_{\infty/F}$ and $A_{F/P}$, $\Pi_F$ and $\Pi_P$ have unitary implementations and reduce to $V_F$ and $V_P$ respectively.

Sketch of proof. Consider the map $\chi_F': \hat{\phi}_F(\eta) \mapsto \hat{\phi}_F(\eta)$ where $\eta = d\varphi$, $\varphi$ being any real Schwartz function on $F \equiv \mathbb{R}$. In $\hat{\phi}_F(\eta)$, $\eta$ is supposed extended to the whole $F$ by means of $\eta(\infty) := 0$ so that $\eta \in D(F;\mathbb{R})$. Using the fact that it holds $[\hat{\phi}_F(\eta), \hat{\phi}_F(\eta')] = -i\Omega_F(E_F(\eta'), E_F(\eta)) = -i\Omega_F(E_F(\eta'), E_F(\eta)) = [\hat{\phi}_F(\eta), \hat{\phi}_F(\eta')]$, one proves that $\chi_F'$ uniquely extends into a injective $\ast$-algebra homomorphism from $A_F$ to $A_F$. The injective $\ast$-algebra homomorphism $\chi_F$ is nothing but $\chi_F' \circ \chi_F$. The remaining properties are straightforward consequences of the properties of $\chi_F$ stated in Theorem 1.2. The case $m = 0$ is analogous. □

2.2. Virasoro algebra with $c = 1$ in the Fock space of the circle. The unitary map $M: \mathcal{H}_F \to \mathcal{H}_F$ associates the essentially self-adjoint operators $H_{F_0}, D_F, C_F$ defined on $D_1(F) \subset \mathcal{H}_F$ with analogous essentially self-adjoint operators acting on one-particle circle states $\mathcal{H}_F$, respectively $H_{F_0}, D_F, C_F$. More precisely, the real linear combinations of these operators are essentially self-adjoint in the dense invariant domain $D_1(F) = M(D_1(F))$ spanned by the eigenvectors of $K_{\beta F}$ associated with the analogous operator $K_{\beta F}$.

The Lie algebra spanned by the operators above in $F$ gives rises to a strongly-continuous unitary $SL(2,\mathbb{R})$ representation $\{U^{(F)}_g\}_{g \in SL(2,\mathbb{R})}$ on the Hilbert space of the circle that is related, by means of $M$, with the analogous unitary representation $\{U^{(F)}_g\}_{g \in SL(2,\mathbb{R})}$ found on the horizon $F$ discussed in [e] of 1.2. And thus, in turn, it induces just the bulk symmetry induced by $\{U^{(F)}_g\}_{g \in SL(2,\mathbb{R})}$ (see [e] of 1.2) by means of unitary holography. In particular $H_F := H_{F_0}$ turns out to be associated with the generator of Rindler-time displacements $H$.

$H_{F_0}, D_F, C_F$ are a basis of the Lie algebra of $SL(2,\mathbb{R})$. An equivalent, but more useful in the following, basis of the Lie algebra of $SL(2,\mathbb{R})$ made of essentially self-adjoint operators in $D_1(F)$ is that of the operators $K_{\beta F}, D_F, S_F$ with

$$S_F := \frac{1}{2} \left( \beta H_{F_0} - \frac{1}{\beta} C_F \right).$$

Now, it make sense to investigate the geometrical nature of the $SL(2,\mathbb{R})$ representation $\{U^{(F)}_g\}$ on the circle $F$ instead of the horizon $F$. First of all one has to notice that the vector fields $\frac{\partial}{\partial \theta}, v \frac{\partial}{\partial \theta}, v^2 \frac{\partial}{\partial \theta}$, which give rise to the geometric interpretation of $\{U^{(F)}_g\}$ when working on $F$, span in $F$ the same space as that spanned by the three smooth vector fields defined on the whole circle $F$,

$$\partial_{\theta}, \cos(\theta)\partial_{\theta}, \sin(\theta)\partial_{\theta},$$
The proof is straightforward using the relation \( v = \beta \tan(\theta/2) \) only. Then consider the transformations (18) of the line \( F \), translate them in the variable \( \theta = 2 \arctan(v/\beta) \) extended to the domain \((-\pi, \pi]\) so to include \( \infty \). The new transformations so obtained define a representation of \( SL(2, \mathbb{R}) \) in terms of orientation-preserving diffeomorphisms \( d_g \) of the circle \( F \):

\[
d_g : \theta \mapsto 2 \arctan \left( \frac{a\beta \tan(\theta/2) + b}{c\beta^2 \tan(\theta/2) + bd} \right), \quad g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).
\] (39)

There is another, more elegant, way to write the elements of same diffeomorphism group:

\[
\delta_h : e^{i\theta} \mapsto \frac{\zeta e^{i\theta} + \eta}{\eta e^{i\theta} + \zeta}, \quad h := \begin{pmatrix} \zeta & \eta \\ \eta & \zeta \end{pmatrix} \in SU(1,1),
\] (40)

where we have used the group isomorphism \( SL(2, \mathbb{R}) \ni g \mapsto h \in SU(1,1) \) with:

\[
\zeta := \frac{\beta a + \beta d + i(b - \beta^2 c)}{2}, \quad \eta := \frac{\beta d - \beta a - i(b + \beta^2 c)}{2}.
\]

The condition \( h \in SU(1,1) \), when \( h \) has the form in (40), can equivalently be written

\[
|\zeta|^2 - |\eta|^2 = 1.
\] (41)

**Remark.** Notice that the transformation \( \rho \mapsto \rho g \) does not mix Fourier components with positive frequency and Fourier components with negative frequency and vise versa. (This fact allows one to look for unitary representations of the considered group in the one-particle Hilbert space which is constructed by using positive frequency part of wavefunctions.) Indeed, using (40), \( e^{-in\theta} \) is mapped into \( \left( \frac{\zeta e^{i\theta} + \eta}{\eta e^{i\theta} + \zeta} \right)^{-n} \). Fourier coefficients with strictly “negative frequency” \(-m\) are proportional to the integrals, where \( m, n \geq 1 \) are integer,

\[
\int_{-\pi}^{\pi} \left( \frac{\zeta e^{i\theta} + \eta}{\eta e^{i\theta} + \zeta} \right)^{-n} e^{-im\theta} d\theta = \oint_{+S^1} \frac{-i}{z^{m+1}} \left( \frac{\eta z + \zeta}{\zeta z + \eta} \right)^{n} dz = \oint_{+S^1} \frac{-i}{z^{m+1}} \left( \frac{\eta}{\zeta} + \frac{1/\zeta}{\zeta z + \eta} \right)^{n} dz,
\]

where \( +S^1 \) is the circle \(|z| = 1\) with positive orientation. Expanding the expression under the last integration symbol using the binomial formula one reduces to a linear combination of contributions of integrals with form

\[
\oint_{+S^1} \frac{dz}{z^{m+1}(z - z_0)^p}
\]

with \( p = 0, 1, \ldots n \geq 1, m \geq 1 \) and where \( z_0 = -\eta/\zeta \) with \( \zeta \neq 0 \) (due to (39)). If \( p > 0 \), using the condition (41) one sees that, whatever the values of \( \eta \) and \( \zeta \) and barring the pole at \( z = 0 \) with order \( m + 1 \), there is another pole of order \( p \) inside the region with boundary \( S^1 \), at \( z = -\eta/\zeta \). Cauchy formula for \( p > 0 \) proves that the contribution of the two residues in each integral cancel out each other and the final result is zero. The case \( p = 0 \) gives the same result automatically.
So it makes sense to look for a unitary representation of the group in the one-particle space.

Exactly as the case of Theorem 1.3, if $\rho$ is a real circle wavefunction, with associated one-particle quantum state $\hat{\rho}_+ = \hat{\rho}_+(n)$, the action of every $U_g^{(F)}$ on $\hat{\rho}_+$ is equivalent to the action of a corresponding $F$-diffeomorphism, $d_g$, on the horizon wavefunction $\rho$ itself.

**Theorem 2.4.** Assume $k = 1$ in the definition of $D_k^{(F)}$, that is, in (40). If $g \in SL(2, \mathbb{R})$ and $\hat{\rho}_+ = \hat{\rho}_+(n)$ is the positive frequency part of $\rho \in S(F)$, the state $U_g^{(F)} \hat{\rho}_+$ can be associated with the wavefunction $\rho_g \in S(F)$ with

$$\rho_g(\theta) = \rho(d_g^{-\theta}) \ , \ \text{for all } \theta \in (-\pi, \pi]. \quad (42)$$

In particular (with the same terminology as that used in (a) of Theorem 1.3):

(a) The unitary one-parameter group generated by $iK_{\beta}^{(F)}$ is associated with the one-parameter group of $F$-diffeomorphisms generated by $\partial_{\beta}$;

(b) the unitary one-parameter group generated by $iD_{\theta}^{(F)}$ is associated to the one-parameter group of $F$-diffeomorphisms generated by $\sin \theta \partial_{\theta}$;

(c) the unitary one-parameter group generated by $iS_{\theta}^{(F)}$ is associated to the one-parameter group of $F$-diffeomorphisms generated by $\cos \theta \partial_{\theta}$.

The Lie algebra spanned by fields $\partial_{\theta}$, $\cos \theta \partial_{\theta}$, $\sin \theta \partial_{\theta}$ is a realization of the Lie algebra of $SL(2, \mathbb{R})$.

**Proof.** The first part can be proven as follows. Take $\rho \in C^\infty(F; \mathbb{R})$. As $H_F = \ell^2(\mathbb{C})$, the associated positive frequency part in “frequency picture” $\hat{\rho}_+ \in \mathcal{F}_F$ is a sequence $\{C_n\}_{n=1,2,\ldots}$. The associated positive frequency part in “$\theta$ picture” can be written

$$\rho_+^{(\theta)} = \mathcal{F}(W \hat{\rho}_+)$$

where $\mathcal{F} : \ell^2(\mathbb{C}) \to L^2((-\pi, \pi), d\theta)$ and $W : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ are respectively the continuous linear operators,

$$\mathcal{F} : \{C_n\}_{n=1,2,\ldots} \mapsto \{C_n \sqrt{2\pi} e^{-in\theta} \}_{n=1,2,\ldots} \quad (43)$$

$$W : \{C_n\}_{n=1,2,\ldots} \mapsto \{\frac{C_n}{\sqrt{2\pi}}\}_{n=1,2,\ldots} \quad (44)$$

On the other hand, in the sense of the topology of $\ell^2(\mathbb{C})$,

$$\hat{\rho}_+ = \sum_{m=1}^{\infty} C_m \Psi_m$$

where $\{\Psi_m\}_{m=1,2,\ldots}$ is the Hilbert base $\Psi_m = \{\delta_{mn}\}_{n=1,2,\ldots}$. By linearity, continuity and the absence of positive-negative frequency mixing,

$$(\rho_g)_+ = \mathcal{F}(W U_g^{(F)} \hat{\rho}_+) = \mathcal{F}\left(W U_g^{(F)} \sum_n C_n \Psi_n\right) = \sum_n \mathcal{F}\left(W U_g^{(F)} C_n \Psi_n\right). \quad (45)$$
Now one notice that Theorem 1.3 holds true also for the horizon wavefunction $\varphi^{(n)}$ with positive frequency part, in “frequency picture” given by $Z^{(1)}_n(E)$ (with and $k = 1, 2, \ldots$) as the associated positive frequency wavefunction. (The proof of this fact is, in fact, the same proof as that of Theorem 1.3, that is Theorem 4.7 in $\mathbb{S}$, with trivial adaptations which make sense in the considered case. In particular Proposition 4.2 in $\mathbb{S}$ can directly be proven by using (28). It is useful to notice that $\Theta Z^{(1)}_n = (-1)^n Z^{(1)}_n$.) By (28),

$$
\varphi^{(n)}(v(\theta)) = \frac{1}{\sqrt{4\pi n}} \left\{ (-1)^{n+1} (C_n + \overline{C_n}) + C_n e^{-in\theta} + \overline{C_n} e^{in\theta} \right\},
$$

and thus, Theorem 1.3 says that

$$
\varphi^{(n)}_g(v(\theta)) = \frac{1}{\sqrt{4\pi n}} \left[ C_n e^{-ind_g^{-1}\theta} - e^{-ind_g^{-1}\pi} C_n \right] + c.c. .
$$

Since, by Theorem 2.1, $Z_n$ in $\mathbb{H}_F$ is transformed into $\Psi_n$ of $\mathbb{H}_F$ and the same transformation associates $U^{(F)}_g$ with $U^{(F)}_g$, (17) can be re-written in the space $\mathbb{H}_F$ and in the circle $F$:

$$
\mathbb{F} \left( WU^{(F)}_g C_n \Psi_n \right) (\theta) + c.c. = C_n \varphi^{(n)}_g(v(\theta)) + c.c. = \frac{1}{\sqrt{4\pi n}} \left[ C_n e^{-ind_g^{-1}\theta} - C_n e^{-ind_g^{-1}\pi} \right] + c.c. .
$$

Inserting it in (45), one concludes that

$$
\rho_g(\theta) = \sum_{n=1}^{+\infty} C_n \frac{e^{-ind_g^{-1}\theta}}{\sqrt{4\pi n}} - C_n \frac{e^{-ind_g^{-1}\pi}}{\sqrt{4\pi n}} + c.c. .
$$

The convergence must be understood in the sense of $L^2(F, d\theta)$. However since $\rho \in C^\infty(F)$ and thus $\rho \circ d_g^{-1} \in C^\infty(F)$, the latter admit a uniformly convergent Fourier series and the series $\sum_n |C_n|/\sqrt{4\pi n}$ converges too. By the uniqueness property of Fourier series it must hold,

$$
\rho_g(\theta) = \rho(d_g^{-1}\theta) + constant, \text{ for all } \theta \in F.
$$

This concludes the proof if the functions are considered as elements of $\mathbb{S}(F)$. Let us pass to prove the statement in (b), the remaining cases can be proven following a strongly analogous proof (which is much more simple in the case (a)). If $t \mapsto g_t$ is the one-parameter subgroup of $SL(2, \mathbb{R})$ whose associated one-parameter group of diffeomorphisms is that generated by the field $\sin \theta \partial_\theta$, consider the transformed wavefunction $\rho_{g(t)}(\theta) := \rho(d_t^{-1}(\theta)) = \rho(d_{-t}(\theta))$ and the associated positive frequency part $\rho_{g(t)}^+ \sim \rho_{g(t)}^+$ in “frequency representation”. Since $SL(2, \mathbb{R})$ acts on positive frequency wavefunctions by means of a strongly continuous unitary representation, there must be some self-adjoint generator $A$ (not depending on $\rho_+$) such that $\rho_{g(t)}^+ = e^{itA} \rho_+$. Our thesis states that $A = D_F$. With the formalism introduced above, the statement turns out to be proved if it holds for all the states $\rho^{(k)}_+ = \Psi_k = \{\delta_{nk}\}_{n=1,2,\ldots} \in \ell^2(\mathbb{R})$, $k = 1, 2, \ldots$. Hence
we want to show that for \( k = 1, 2, \ldots, \), \( \rho_{g(t)}^{(k)} = e^{i D_{F} \Psi_k} \). To this end it is sufficient to show that in the topology of \( \mathfrak{H}_F \cong \ell^2(\mathbb{C}) \),

\[
\frac{d}{dt} \rho_{g(t)}^{(k)} \bigg|_{t=0} = i D_{F} \Psi_k .
\]

Indeed, by Stone’s theorem the derivative in the left-hand side is \( iA \Psi_k \), on the other hand, since \( D_{F} \) is essentially self-adjoint in the linear space finitely spanned by the vectors \( \Psi_k \), it must be \( A = D_{F} \). Let us prove \((50)\). From now on \( \rho_{g(t)}^{(k)} = \{ C_n(t) \}_{n=1,2,\ldots} \). Defining \( \theta_t(\theta) := d_{-t}(\theta) \), one has

\[
C_n(t) = \sqrt{\frac{n}{k}} \int_{-\pi}^{\pi} \frac{e^{-i(k\theta(t) - n\theta)}}{2\pi} d\theta ,
\]

because of \((47)\) where, by construction, \( \rho_{g(t)}^{(k)}(\theta) = \varphi_{g(t)}^{(k)}(v(\theta)) \) and using the fact that there is no mixing of positive end negative frequencies under the action of the group. By direct computation one finds that \( C_n(0) = 0 \) and \( dC_n(t)/dt|_{t=0} = 0 \) if \( n \neq k, k \pm 1 \). So if the derivative in the left-hand side of \((50)\) is computed term by term, \((50)\) can be re-written

\[
\frac{\delta_{n,k-1}\sqrt{k(k-1)} - \delta_{n,k+1}\sqrt{k(k+1)}}{2} = \langle \Psi_n, i D_{F} \Psi_k \rangle_{\mathbb{F}}
\]

where we have computed the derivatives of \( dC_n(t)/dt|_{t=0} \) using \((51)\). However, it also holds

\[
\langle \Psi_n, i D_{F} \Psi_k \rangle_{\mathbb{F}} = \langle Z_{n}^{(1)}, i D_{F} Z_{k}^{(1)} \rangle_{\mathbb{F}} ,
\]

and the right-hand side can be computed trivially (for instance by employing the formalism in p. 137 of [9]) and it turns out to coincide with the left-hand side of \((52)\), so \((52)\) holds true. To conclude the proof, it is sufficient to show that the derivative in left-hand side of \((50)\), which is computed with respect to the topology of \( \ell^2(\mathbb{C}) \), can equivalently be computed deriving term by term the sequence of complex which defines \( \rho_{g(t)}^{(k)} \). Expanding the term under the integral symbol in \((51)\) by means of Taylor formula in the variable \( t \) about \( t = 0 \), using the Lagrange formula for the remnant and, finally, using integration by parts and the fact that the integrated functions are smooth and periodic on \( S^1 \), one proves that for some constant \( A \), for all \( t \) in a neighborhood of 0 and for all \( n \neq k, k \pm 1 \):

\[
\left| \frac{C_n(t)}{t} \right|^2 \leq \frac{A}{n^2} .
\]

Thus

\[
\sum_{n=1}^{\infty} \left| \frac{C_n(t) - C_n(0)}{t} - \frac{dC_n(t)}{dt} \right|_{t=0}^2 = \sum_{n\neq k, k \pm 1} \left| \frac{C_n(t)}{t} \right|^2 + \sum_{n=k, k \pm 1} \left| \frac{C_n(t) - C_n(0)}{t} - \frac{dC_n(t)}{dt} \right|_{t=0}^2 .
\]
\( \frac{C_n(t)}{t} \to 0 \) in our hypotheses for \( 0 < n \neq k, k \pm 1 \) and so the sum of the corresponding series above vanishes too due to Lebesgue’s dominated convergence theorem with respect to the Dirac measure with support on the points \( n \neq k, k \pm 1 \) as a consequence of [53]. We finally gets

\[
\lim_{t \to 0} \sum_{n=1}^{\infty} \left| \frac{C_n(t) - C_n(0)}{t} - \frac{dC_n(t)}{dt} \right|_{t=0} = \lim_{t \to 0} \sum_{n=k,k \pm 1} \left| \frac{C_n(t) - C_n(0)}{t} - \frac{dC_n(t)}{dt} \right|_{t=0} = 0 .
\]

We conclude that the derivative in the left-hand side of (50) computed with respect to the topology of \( \ell(\mathbb{C}) \) coincides with that computed term by term. This concludes the proof because the last statement can straightforwardly be proven by direct inspection. \( \square \)

The theorem states that, in fact, the bulk \( SL(2, \mathbb{R}) \) symmetry becomes manifest when examined on the circle \( \mathbb{F} = \mathbb{F} \cup \{ \infty \} \). However that is not the whole story because the found circle unitary \( SL(2, \mathbb{R}) \) representation is just a little part of a larger unitary representation with geometrical meaning. We can, in fact, consider the Lie algebra \( Vect(S^1) \) of the infinite dimensional Lie group [19] of orientation-preserving diffeomorphisms of the circle \( Diff^+(S^1) \), where \( S^1 = \mathbb{F} \) in our case. To make contact with Virasoro algebra we have to consider an associated complex Lie algebra [20]. Consider the complex Lie algebra \( d(\mathbb{F}) := Vect(\mathbb{F}) \oplus iVect(\mathbb{F}) \) equipped with usual Lie brackets \( \{ \cdot, \cdot \} \) and the involution \( \omega : X \mapsto -\bar{X} \) for \( X \in d(\mathbb{F}) \), so that \( \omega(\{X,Y\}) = \{\omega(Y),\omega(X)\} \). An algebraic basis of that algebra is made of the complex smooth fields on \( \mathbb{F} \):

\[
\mathcal{F}_n := i e^{int} \partial_\theta , \quad \text{with} \ n \in \mathbb{Z} .
\]  
(54)

The vector fields \( \mathcal{F}_n \) enjoy the celebrated Virasoro commutation rules with central charge \( c = 0 \):

\[
\{ \mathcal{F}_n, \mathcal{F}_m \} = (n-m) \mathcal{F}_{n+m} ,
\]  
(55)

and the Hermiticity condition

\[
\omega(\mathcal{F}_n) = \mathcal{F}_{-n} .
\]  
(56)

In the presented picture \( Vect(\mathbb{F}) \) is nothing but the sub-algebra of \( d(\mathbb{F}) \) containing all of the vectors fixed under \( -\omega \). An algebraic basis of \( Vect(\mathbb{F}) \) is that made of the fields

\[
\mathcal{F}_n^{(+)} := \frac{\omega(\mathcal{F}_n) + \mathcal{F}_n}{2i} = \cos(n\theta) \partial_\theta , \quad \mathcal{F}_n^{(-)} := \frac{\omega(\mathcal{F}_n) - \mathcal{F}_n}{2} = \sin(n\theta) \partial_\theta ,
\]  
(57)

where \( n = 0, 1, \ldots \) while \( m = 1, 2, \ldots \). Conversely, the base of \( d(\mathbb{F}) \), \( \{ \mathcal{F}_n \}_{n \in \mathbb{Z}} \) can be obtained from the base above as, where \( n = 1, 2, \ldots \),

\[
\mathcal{F}_0 := i \mathcal{F}_0^{(+)} , \quad \mathcal{F}_n := \mathcal{F}_n^{(+)} + i \mathcal{F}_n^{(-)} , \quad \mathcal{F}_{-n} := \mathcal{F}_{-n}^{(+)} - i \mathcal{F}_{-n}^{(-)} .
\]  
(58)

Notice that the three fields \( \mathcal{F}_0^{(+)}, \mathcal{F}_1^{(+)}, \mathcal{F}_1^{(-)} \) are in fact generators of a finite-dimensional sub algebra of \( Vect(\mathbb{F}) \), namely the representation of the Lie algebra of \( SL(2, \mathbb{R}) \) found above, which
is equivalently generated by the three fields $v^{n+1}\partial_v$ with $n = -1, 0, 1$. However for $|n| > 1$, the algebras spanned by generators $v^{n+1}\partial_v$ and $F_n^{(\pm)}$ are different and we focus attention on the latter ones.

By direct inspection one proves that each of the fields $F_n^{(\pm)} \in Vect(F)$ generate a global one-parameter group of $F$ orientation-preserving diffeomorphisms (this fact does not hold for fields $v^{n+1}\partial_v$ in $F$). Global means here that the additive parameter which labels the group ranges over the entire real line $\mathbb{R}$. In turn, that group of diffeomorphisms generates a group of automorphisms of the algebra of the quantum field $A_F$. Let us explain how it happens. If $d_{\lambda}^{(F_n^{(\pm)})} : F \rightarrow F$ is an element of the one-parameter (orientation-preserving) diffeomorphism group generated by $F_n^{(\pm)}$, $\lambda \in \mathbb{R}$ being the additive parameter, and $\rho \in S(F)$, as usual we define the associated group of wavefunction transformations: $(\alpha^{(F_n^{(\pm)})}_\lambda \rho)(\theta) := \rho(d_{-\lambda}^{(F_n^{(\pm)})}(\theta))$. Notice that the transformations $\alpha^{(F_n^{(\pm)})}_\lambda$ are, in fact, automorphisms of the real vector space $S(F)$ equipped with the symplectic form $\Omega_F$ because the latter is orientation-preserving diffeomorphism invariant.

Remark. By direct inspection one realizes that, if $n > 1$ the transformation $\alpha^{(F_n^{(\pm)})}_\lambda$ does not admit the space of positive frequency wavefunctions as invariant space. As a consequence it is not possible to represent $\alpha^{(F_n^{(\pm)})}_\lambda$ unitarily in the one-particle space $F_n$. To implement the transformation $\alpha^{(F_n^{(\pm)})}_\lambda$ at quantum level, the entire Fock space is necessary if $n > 1$.

As we want to deal with quantum fields smeared by exact 1-forms of $D(F)$, we define a natural action of the diffeomorphisms $d_{\lambda}^{(F_n^{(\pm)})}$ also on these forms by using [33]: If $\omega \in D(F)$ and $\rho := E_F(\omega)$, we define the one-parameter group of transformations of 1-forms $\{\beta^{(F_n^{(\pm)})}_{\lambda}\}_{\lambda \in \mathbb{R}}$, such that

$$\beta^{(F_n^{(\pm)})}_{\lambda}(\omega) := 2d\alpha^{(F_n^{(\pm)})}_{\lambda}^{-1}(E_F(\omega)).$$

Finally we can define the action of quantum fields by means of

$$\gamma^{(F_n^{(\pm)})}_{\lambda}(\phi_F(\omega)) := \phi_F\left(\beta^{(F_n^{(\pm)})}_{-\lambda}(\omega)\right).$$

Using the given definitions, the fact that $\alpha^{(F_n^{(\pm)})}_\lambda$ preserves $\Omega_F$ as well as [36] and [37], one finds

$$\left[\gamma^{(F_n^{(\pm)})}_{\lambda}(\phi_F(\omega)), \gamma^{(F_n^{(\pm)})}_{\lambda}(\phi_F(\omega'))\right] = [\phi_F(\omega), \phi_F(\omega')].$$

It is possible to prove by that identity that $\gamma^{(F_n^{(\pm)})}_{\lambda}$ naturally extends into a $*$-algebra automorphism of the algebra $A_F$. The procedure to do it is very similar to those used in [3] to extend transformations of field operators into $*$-algebra homomorphisms. So, in fact, every field $F_n^{(\pm)}$
gives rise to a one-parameter group of automorphisms of the algebra $A_F$ that we indicate by
\[
\left\{ \gamma^{(F_n)}_{\lambda} \right\}_{\lambda \in \mathbb{R}}
\] once again. A natural question arises:

Is there a representation of the (infinite dimensional complex) Lie algebra $d(F)$, in terms of
operators defined in $\mathfrak{g}(H_F)$ such that the fields $F_n^{(\pm)}$ are mapped into (essentially) anti-self-adjoint
operators $-iF_n^{(\pm)}$, whose associated unitary one-parameter groups implement the respective one-
parameter group of $A_F$-automorphisms
\[
\left\{ \gamma^{(F_n^{(\pm)})}_{\lambda} \right\}_{\lambda \in \mathbb{R}}
\]
(at least at the first order)? That is
\[
e^{-i\lambda F_n^{(\pm)}} \phi_F(\omega) e^{i\lambda F_n^{(\pm)}} = \gamma^{(F_n^{(\pm)})}_{\lambda}(\hat{\phi}_F(\omega)),
\]
(62)
or some other formally related, but perhaps weaker, identity holds.

The answer is yes provided one uses an operator algebra which represents a central extensions
of the algebra $d(F)$. In other words one has to permit to change, at quantum level, the relation
(55) by adding in the right hand side a further term which commutes with the elements of the
representation itself. The obtained algebra is properly called Virasoro’s algebra.

More precisely, the quantum representation on a hand is a straightforward extension of that
previously found for the group $SL(2, \mathbb{R})$, on the other hand is, in fact, a positive-energy and
unitary representation of Virasoro algebra with central charge $c = 1$ [20]. To built up such a
representation the entire Fock space, and not only the one-particle Hilbert space, is necessary. A
relevant point is that the found unitary representation of the Virasoro algebra can be exported
in the bulk via unitary holography.

In the circle Fock space $\mathfrak{g}(H_F)$, consider the basis obtained by taking all the symmetrized tensor
products of one-particle states $Z_n^{(1)}$, namely the eigenvectors of the operator $K_{F,\beta}$. Henceforth $\alpha_n$
and $\alpha_n^\dagger$ are respectively the creation and annihilation operator associated with the one-particle
state $Z_n^{(1)}$ with $n = 1, 2, \ldots$. As a consequence
\[
[\alpha_n, \alpha_n^\dagger] = \delta_{n,m} I, \quad [\alpha_n, \alpha_m] = [\alpha_n^\dagger, \alpha_m^\dagger] = 0.
\]
(63)

Now, fix $\mu \in \mathbb{R}$ and introduce the operators, $a_n$, with $n \in \mathbb{Z}$ such that
\[
a_n = \begin{cases} 
\mu I & \text{if } n = 0, \\
\sqrt{n} \alpha_n & \text{if } n > 0, \\
-i \sqrt{-n} \alpha_n^\dagger & \text{if } n < 0
\end{cases}.
\]
(64)

By (63) these operators satisfy the oscillator algebra commutation relations [20]
\[
[a_m, a_n] = m \delta_{m,-n} I,
\]
(65)
and the so called Hermiticity conditions
\[
a_n^\dagger = a_{-n}
\]
(66)
(actually in the left-hand side is considered only the restriction of $a_n^\dagger$ to the domain of $a_{-n}$). With these definitions, the formal expression for $\hat{\phi}_F$ takes the form

$$\hat{\phi}_F(\theta) = \frac{1}{i\sqrt{4\pi}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{-in\theta}}{n} a_n,$$

(67)

moreover, formally but also with a rigorous meaning in terms of a field operator smeared by an exact 1-form (34), it holds

$$a_n = \frac{1}{\sqrt{\pi}} \int_F \hat{\phi}_F(\theta) e^{in\theta}, \quad \text{if } n \in \mathbb{Z} \setminus \{0\}.$$

(68)

Finally, define the operators (denoted by $L_k$ in [20])

$$F_k := \frac{\epsilon_k}{2} a_k^2/2 + \sum_{n > -k/2} a_{-n} a_{n+k}, \quad k \in \mathbb{Z}$$

(69)

where $\epsilon_k = 0$ if $k$ is odd, $\epsilon_k = 1$ if $k$ is even (including $k = 0$). The various sums are, in fact, finite when acting on a vector, since we adopt as a common domain of those operators, the dense subspace $D_1^{(F)} \subset \mathcal{F}(\mathcal{F})$ made of the finite linear combination of vectors containing any finite number of particles in states $Z^{(1)}_n$.

**Theorem 2.5.** The operators $F_k$, $k \in \mathbb{Z}$ enjoy the following properties on their domain $D_1^{(F)}$.

(a) The complex Lie algebra finitely spanned by operators $F_k$ (equipped with the usual operator commutator and Hermitian conjugation) is a positive-energy unitary Virasoro algebra representation $\text{Vir}(\mathcal{F})$ with central charge $c = 1$. Indeed it holds

$$[F_m, F_n] = (m - n)F_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} I,$$

(70)

Hermiticity relations are fulfilled

$$F_m^\dagger \Psi = F_{-m} \Psi, \quad \text{for every } \Psi \in D_1^{(F)},$$

(71)

$F_0$ is essentially self-adjoint and $\overline{F_0}$ is positive defined with discrete spectrum

$$\sigma(\overline{F_0}) = \left\{ \frac{\mu^2}{2} + N \middle| N = 0, 1, \ldots \right\}.$$

(72)

(b) For $n = 1, 2, \ldots$, the operators

$$F_0^{(+)} := F_0, \quad F_n^{(+)} := \frac{F_{-n} + F_n}{2} \quad \text{and} \quad F_n^{(-)} := \frac{i(F_{-n} - F_n)}{2}$$

(73)

are essentially self-adjoint in $D_1^{(F)}$. (It is worth stressing that the interplay of fields $\mathcal{F}_n^{(\pm)}$ and $\mathcal{F}_n$ is the same as that operators $-iF_n^{(\pm)}$ and $F_n$ and not $F_n^{(\pm)}$ and $F_n$, this is because the operator involution $\dagger$ corresponds to the field involution $\omega$ instead of the simpler complex conjugation.)
Proof. Barring the statements on $F_0$, the properties in (a) are proven in [20] (see sections 2.1, 2.2, 2.3 and 3.1) as consequences of (65), (66) and (69). The operators $F_0$, $F_m + F_{-m}$ and $i(F_m - F_{-m})$ are symmetric by construction and one can prove by direct inspection that the element of $\mathcal{D}_1^{(F)}$ are analytic vectors for these operators. Thus they are essentially self-adjoint. This proves (b) and the essential self-adjointness of $F_0$. By direct inspection and using (64), one finds

$$F_0 = \frac{\mu^2}{2} I + \sum_{n=1}^{\infty} n \alpha_n^\dagger \alpha_n.$$  

(74)

The Hilbert basis of $\mathcal{F}(\mathcal{H}_F), \{|L\rangle\}_{L \in \mathbb{N}}$ made of the vectors (labeled with an arbitrary order by the index $L$) containing any finite number of states $Z^{(1)}_m$ is a basis of eigenvectors of $F_0$. The eigenvalue associated with $|L\rangle$ is of the form $\mu^2 / 2 + N_L$ where $N_L$ ranges everywhere in $\mathbb{N}$. This fact suggests to consider the self-adjoint operator

$$F'_0 := \sum_{L=0}^{\infty} N_L |L\rangle \langle L|$$

where now the sum is interpreted in the strong operator topology in the domain $\mathcal{D}(F'_0)$ containing the vectors $|\Psi\rangle \in \mathcal{F}(\mathcal{H}_F)$ with

$$\sum_{L=0}^{\infty} N_L^2 |\langle L|\Psi\rangle|^2 < \infty.$$  

By construction, $F'_0$ has the spectrum (72). On the other hand, since $F_0 \subset F'_0$ by construction, uniqueness of self-adjoint extensions of $F_0$ implies $F_0 = F'_0$. $\square$

Finally we show that (1) the whole Virasoro representation extends the circle $SL(2,\mathbb{R})$ unitary representation and (2) it has the geometric meaning [62].

**Theorem 2.6.** Referring to the Virasoro representation of Theorem 2.5,

(a) if (and only if) $\mu = 0$, the operators $F_0^{(+)}, F_1^{(+)}, F_1^{(-)}$ admit $\mathcal{D}_1^{(F)} \subset \mathcal{H}_F$ as invariant space and

$$F_0^{(+)} |\gamma_F\rangle = K_{\beta_F},$$

(75)

$$F_1^{(+)} |\gamma_F\rangle = S_F,$$  

(76)

$$F_1^{(-)} |\gamma_F\rangle = D_F,$$  

(77)

and so these operators generate the $SL(2,\mathbb{R})$ representation $\{U_{\theta}^{(F)}\}_{\theta \in SL(2,\mathbb{R})}$.

(b) If (and only if) $\mu = 0$, for every $n \in \mathbb{N}$ ($n > 0$ in the case $(-)$), (62) holds true at the first order at least,

$$\left[ F_n^{(\pm)}, \hat{\phi}_F(\omega) \right] = i \frac{d}{d\lambda} \left. \gamma_{\lambda}^{(3\gamma^{(\pm)}_n)} (\hat{\phi}_F(\omega)) \right|_{\lambda = 0}$$

(78)

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for every \( \omega = \eta + i\eta' \in \mathcal{D}(\mathbb{F}; \mathbb{C}) \) such that the real wavefunctions \( E_{\mathbb{F}}\eta \) and \( E_{\mathbb{F}}\eta' \) are associated with states in \( \mathcal{D}^{(F)} \) and the derivative is computed in the strong operator topology in \( \mathcal{D}_1^{(F)} \).

**Proof.** (a) The proof of the first case is a trivial consequence of (74). Concerning the second and third cases, we notice that using operators \( \alpha \) and \( \beta \),

\[
F_{-1} = -i\mu\alpha_1^\dagger + \sum_{n=1}^{\infty} \sqrt{n(n+1)}\alpha_{n+1}^\dagger \alpha_n \quad (79)
\]

\[
F_1 = i\mu\alpha_1 + \sum_{n=1}^{\infty} \sqrt{n(n+1)}\alpha_n^\dagger \alpha_{n+1} . \quad (80)
\]

It is obvious that, because of the terms containing \( \mu(\alpha_1^\dagger \pm \alpha_1) \), the operators above admit \( \mathcal{D}_1^{(F)} \) as an invariant space if and only if \( \mu = 0 \). In that case, the restrictions to \( \mathcal{D}_1^{(F)} \) coincide respectively with the operators \( A_1 \) and \( A_\sum \) defined in [21] or (23) of [9] (where the coefficient \( \beta \) is indicated by \( \lambda/\kappa \) and \( k = 1 \)). With our notations

\[
A_\sum = \frac{1}{2} \left( \beta H_{\mathbb{F}0} - \frac{1}{\beta} C_{\mathbb{F}} \right) + iD_{\mathbb{F}} , \quad (81)
\]

so that \( A_\sum Z_1^{(1)} = 0 \) and \( A_\sum Z_n^{(1)} = \sqrt{n(n+1)}Z_n^{(1)} \). (81) implies (76) and (77) straightforwardly taking (78) into account.

Let us come to the last part. It is simply proven that every exact 1-form \( \omega = \eta + i\eta' \), where the real exact 1-forms \( \eta, \eta' \) determine circle wavefunctions with positive frequency in \( \mathcal{D}_1^{(F)} \), is a finite complex linear combination of forms \( \omega_m(\theta) := de^{im\theta} \) with \( m \in \mathbb{Z} \setminus \{0\} \). Hence it is sufficient to prove (78) for every \( \phi(\omega_m) \) with \( m \in \mathbb{Z} \setminus \{0\} \) and \( k \in \mathbb{Z} \). By direct computation and using (68) and (2.12) in [20], one finds that, for every \( \Psi \in \mathcal{D}_1^{(F)} \),

\[
[L_k^{(\pm)}, \hat{\phi}(\omega_m)]\Psi = -\frac{m\sqrt{\pi}}{2} (i)_j (a_{m-k} \pm a_{m+k})\Psi \quad (82)
\]

where \((i)_j := 1 \) if \( j = + \), \((i)_j := i \) if \( j = - \). The identity above holds provided \( a_{m-k} \) and \( a_{m+k} \) are interpreted as the multiplicative operator \( \mu I \) [20]. On the other hand,

\[
\gamma_{\lambda}^{(\pm)} \left( \hat{\phi}_{\mathbb{F}}(\omega_m) \right) = i\alpha(\phi_{m\lambda}) - i\alpha^\dagger(\psi_{m\lambda}) ,
\]

where the vectors \( \psi_{m\lambda} \) and \( \phi_{m\lambda} \) are defined by

\[
\psi_{m\lambda} := \left\{ \sqrt{n} \int_{-\pi}^{\pi} e^{im\theta} e^{im\theta}(\theta) d\theta \right\}_{n=1,2,...}, \quad \phi_{m\lambda} := \left\{ \sqrt{n} \int_{-\pi}^{\pi} e^{-im\theta} e^{im\theta}(\theta) d\theta \right\}_{n=1,2,...} \quad (83)
\]

and \( \lambda \mapsto \theta_{\lambda}(\theta) \) is the integral curve of \( \gamma_{\lambda}^{(\pm)} \) starting from \( \theta \). Notice that the linear maps \( \psi \mapsto \alpha(\psi)\Psi \) and \( \psi \mapsto \alpha^\dagger(\psi)\Psi \) are continuous for every fixed vector \( \Psi \in \mathcal{D}_1^{(F)} \), so that

\[
\frac{d}{d\lambda}|_{\lambda=0} \gamma_{\lambda}^{(\pm)} \left( \hat{\phi}_{\mathbb{F}}(\omega_m) \right) \Psi = i\alpha \left( \frac{d}{d\lambda}|_{\lambda=0} \phi_{m\lambda} \right) \Psi - i\alpha^\dagger \left( \frac{d}{d\lambda}|_{\lambda=0} \psi_{m\lambda} \right) \Psi .
\]
In turn, using a procedure very similar to that used in the proof of (b) in Theorem 2.4, one sees that the derivatives \( \frac{d}{d\lambda}|_{\lambda=0}\phi_{m\lambda} \) and \( \frac{d}{d\lambda}|_{\lambda=0}\psi_{m\lambda} \) evaluated by using the topology of \( l^2(\mathbb{C}) \) coincide with the analogous derivatives computed term-by-term for the sequences of \( l^2(\mathbb{C}) \) which define \( \phi_{m\lambda} \) and \( \psi_{m\lambda} \). These derivatives can be computed straightforwardly and give rise to

\[
\frac{d}{d\lambda}|_{\lambda=0}\gamma(\xi_{\pm}^2) \left( \hat{\phi}_F(\omega_m) \right) \Psi = -\frac{im\sqrt{\pi}}{2}(i)^\pm(a_{m-k} \pm a_{m+k})\Psi,
\]

where, in the right-hand side, \( a_{m-k} \) and \( a_{m+k} \) must be interpreted as the null operator. By comparison with (82) we find that (78) holds true provided \( \mu = 0 \).

Remarks. (1) A natural question concerns whether or not \( \mathfrak{F}(\mathcal{H}_F) \) is irreducible with respect to the found Virasoro representation. The answer depend on the value of \( \mu \). If and only if \( \mu \in \mathbb{Z} \) (and in particular if \( \mu = 0 \)) the answer is negative because of several results by Kac, Segal and Wakimoto-Yamada (see Theorem 6.2 in [20] where the parameter \( \mu \) used below is indicated by \( \mu \) which differs from the parameter \( \mu \) used herein). If \( \mu = -m \in \mathbb{Z} \), one has the orthogonal decomposition

\[
\mathfrak{F}(\mathcal{H}_F) = \bigoplus_{k \in \mathbb{Z}^+, k \geq -m} V(1, (m + 2k)^2/4),
\]

where \( V(c, h) \) is the up-to-isomorphism unique highest-weight unitary Virasoro representation (which is irreducible by consequence) with central charge \( c \) and weight \( h \). We recall the reader that if \( c = 1 \) and \( h = l^2/4 \) with \( l \in \mathbb{Z} \), \( V(c, h) \) is not a Verma representation. In other words the system of generators of \( V(c, h) \) built up over the singular vector of \( V(c, h) \) by application of products of Virasoro generators contains linearly dependent vectors. Conversely, if \( c = 1 \) and \( h \neq l^2/4 \) with \( l \in \mathbb{Z} \), \( V(c, h) \) is a Verma representation.

(2) It is possible to build up a free scalar standard 2D-CFT by using \( \hat{\phi}_F \) and the analogous field \( \hat{\phi}_p \) defined on \( P := \mathbb{P} \cup \{\infty\} \). In fact, consider the Wick rotation in Rindler coordinates \( t \mapsto it \). Under that continuation, light-Rindler coordinates transforms into \( v \mapsto it + log(ky)/k \), \( u \mapsto it - log(ky)/k \) and so \( \theta = 2\arctan(v/\beta) \mapsto z \), \( \theta' = 2\arctan(u/\beta) \mapsto \bar{z} \) which is the coordinate on \( P \) which is defined analogously to \( \theta \). With the given definitions, \( z \) turns out to be defined on a cylinder \( C \) obtained by taking \( Im(z) \in \mathbb{R} \) and \( Re(z) \in (-\pi, \pi] \) with the identification \( -\pi \equiv \pi \). By this way the fields \( \hat{\phi}_F \) and \( \hat{\phi}_p \) become respectively the Euclidean holomorphic and anti holomorphic fields in \( \mathfrak{F}(\mathcal{H}_F) \otimes \mathfrak{F}(\mathcal{H}_p) \):

\[
\hat{\phi}(z) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{z^n}{n} a_n, \quad \hat{\phi}(\bar{z}) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\bar{z}^n}{n} b_n
\]

where the operators \( b_n \) are defined on \( P \) similarly to operators \( a_n \) and \( [a_n, b_m] = 0 \). The operators \( F_n \) and the analogues \( P_n \) defined in \( P \) are those usually denoted by \( L_n \) and \( T_n \) respectively.

(3) The bulk evolution is generated by the Hamiltonian \( H \) which is the quantum generator associated with the bulk killing vector \( \partial_t \). Consider the operator \( H_F \) associated with \( H \) by holography and naturally extended it in the whole Fock space \( \mathfrak{F}(\mathcal{H}_F) \) by assuming to work with
massive non interacting particles in the bulk. The obtained operator $H_F^\otimes$ coincides with the self-adjoint Virasoro generator

$$H_F^\otimes := \frac{1}{\beta} \left( 2F_1^{(+)} + F_0 \right)$$

(84)

provided $\mu = 0$. Under this hypothesis, $F_0^{(+)}$, $F_1^{(+)}$ span a finite-dimensional Lie algebra and $H_F^\otimes$ is the closure of an element of the algebra. (That is noting but the Lie algebra of a unitary representation of $SL(2, \mathbb{R})$.) As a consequence it is possible to define time-depending observables $F_0(t), F_1^{(\pm)}(t)$ which are constant of motion in Heisenberg picture. These are finite linear combinations of generators $F_0(t), F_1^{(\pm)}(t)$. The proof of that fact is essentially the same as that of Theorem 2.1 – item (b) in particular – in [8]. We conclude that $F_0(t), F_1^{(\pm)}(t)$ generate a symmetry of the system when they are realized, by unitary holography, as operators acting in the bulk. Conversely, this result does not apply as it stands for $F_n^{(\pm)}$ if $n > 1$. This is because there is no finite-dimensional Lie algebra containing both $F_n^{(\pm)}$ and $H_F^\otimes$. However if one assumes that $F_0^{(+)}$ (which is associated with $K_{\beta F}$ in the bulk) is the Hamiltonian of the theory on $F$, the observables $F_n^{(\pm)}$ with $n > 1$ can be considered as symmetries of the system. This is because, for every fixed integer $n > 0$ and also if $\mu \neq 0$, $F_0^{(+)}$, $F_n^{(\pm)}$, $F_n^{(-)}$ span a finite-dimensional Lie algebra (which is, in fact, a representation of the Lie algebra of $SL(2, \mathbb{R})$).

### 3 Appearance of thermal states from Virasoro generators.

Let us focus on the class of “Hamiltonian operators” defined for the theory on the circle $F$,

$$H_F^\otimes := \frac{1}{\beta} \left( 2F_1^{(+)} + F_0 \right),$$

(85)

where, differently from (84), now $\mu \in \mathbb{R}$ and thus, barring the value $\mu = 0$, $H_F^\otimes$ cannot be associated with the Rindler Hamiltonian in the bulk by means of holography. In the following we study some properties of these Hamiltonians and associated ground states which can be considered as operators and states of the theory in the bulk. We shall not give rigorous proofs since the treatment of the issue involves a singular Bogoliubov transformation as well as a regularization procedure. We have formally

$$H_F^\otimes = \frac{\mu^2}{2\beta} I + H_F^\otimes + i \frac{\mu}{2\beta} \left( \alpha_1 - \alpha_1^\dagger \right).$$

(86)

We look for a, formally unitary, transformation $U_\mu$ such that

$$H_F^\otimes = U_\mu H_F^\otimes U_\mu^\dagger.$$

It is convenient to work in the Fock space $\mathcal{F}(\mathcal{H}_F)$ which is isomorphic to $\mathcal{F}(\mathcal{H}_F)$ by means of the isomorphism $M : \mathcal{H}_F \rightarrow \mathcal{H}_F$ used in Theorem 2.1. In this representation

$$H_F^\otimes = \frac{\mu^2}{2\beta} I + \int_{\mathbb{R}^+} dE \ a_E^\dagger a_E + i \frac{\mu}{2\beta} \int_{\mathbb{R}^+} dE \ Z_1^{(1)}(E) \left( a_E - a_E^\dagger \right),$$

(87)
where $a_E$ and $a_E^\dagger$ are as in eq. (14). By that way, it turns out that formally

\[
U_\mu = \exp \left( -i \int_0^\infty Z_1^{(1)}(E) \frac{\mu}{2\beta} (a_E + a_E^\dagger) \frac{dE}{E} \right). \tag{88}
\]

Notice that when $\mu$ is equal to zero the unitary transformation becomes the identity and $H_{F_0}^\otimes = H_F^\otimes$ as is due. For completeness we say that it is possible to rewrite $U_\mu$ in terms the operators $\alpha_n$ as follow:

\[
U_\mu = \exp \left( -i \sum_{n>0} (-1)^{(n+1)} \frac{\mu}{\sqrt{n}} (\alpha_n + \alpha_n^\dagger) \right). \tag{89}
\]

The ground state of $H_{F_0}^\otimes$, can be obtained as

\[
\Psi_\mu := U_\mu^\dagger |0\rangle_F . \tag{90}
\]

$\Psi_\mu$ is not invariant under Rindler evolution generated by $H_F^\otimes$ but it enjoys interesting thermal properties when one considers expectation values of observables also averaged during a long period of Rindler time $T \to \infty$. Consider the expectation value of the operator $A$:

\[
\langle A \rangle_\mu := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \langle \Psi_\mu(t), A \Psi_\mu(t) \rangle_F dt , \tag{91}
\]

where $\Psi_\mu(t) := \exp\{-iH_F^\otimes\}\Psi_\mu$. The direct computation is affected by mathematical problems which can be made harmless by making discrete the energy spectrum and taking the limit toward the continuous case after the evaluation of the expectation value. The discrete spectrum can be obtained by reducing to a known regularization procedure, consisting into a suitable version of the so-called “box quantization”. Start from noticing that it can be re-written, by using the adimensional parameter $\lambda = \log(E/E^*)$, $E^*$ being an arbitrarily fixed energy scale,

\[
U_\mu = \exp \left( -i \int_{-\infty}^{+\infty} Z_1^{(1)}(E(\lambda)) \frac{\mu}{2\beta} (a_{E(\lambda)} + a_{E(\lambda)}^\dagger) d\lambda \right). \tag{92}
\]

Now, differently from $E$, $\lambda$ ranges over the whole real line and box-quantization can be used as follow. First of all define the operators $c_\lambda = \sqrt{E'(\lambda)}a_{E(\lambda)}$ and $c_\lambda^\dagger = \sqrt{E'(\lambda)}a_{E(\lambda)}^\dagger$ so that bosonic commutation relations of $a_E$ and $a_E^\dagger$ turns out to be equivalent to

\[
[c_\lambda, c_{\lambda'}^\dagger] = \delta(\lambda - \lambda') , \quad [c_\lambda, c_{\lambda'}] = 0 , \quad [c_\lambda^\dagger, c_{\lambda'}^\dagger] = 0 . \tag{93}
\]

Finally, to get the discrete spectrum in $\lambda$, assume that values $\lambda$ describe the spectrum of a “momentum operator”. These values can be made discrete by working a $1D$ box with length $L$ with periodic boundary conditions, the continuous spectrum being restored in the limit $L \to \infty$. 

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Within this framework, if \( \lambda_n = \frac{2\pi n}{L} \) with \( n \in \mathbb{Z} \), the operators \( c_j := c_{\lambda_j} \) enjoy the commutation relations
\[
\begin{align*}
[c_i, c_j^\dagger] &= \delta_{ij}, \\
[c_i, c_j] &= 0, \\
[c_i^\dagger, c_j^\dagger] &= 0.
\end{align*}
\] (94)

With that regularization procedure, the Hamiltonian \( H^\oplus_F \) can be re-written as
\[
H^\oplus_F = \int_{\mathbb{R}^+} E a_E^\dagger a_E dE = \int_{\mathbb{R}} E(\lambda) c_\lambda^\dagger c_\lambda d\lambda \to \sum_j E_j c_j^\dagger c_j, \] (95)

where \( E_j := E(\lambda_j) \). Similarly, using \( E = E^* e^\lambda \) and (9) for \( k = n = 1 \), the regularized unitary transformation \( U_\mu \) reads:
\[
U_\mu = \prod_j \exp \left( -i\mu e^{-\beta E_j}(c_j + c_j^\dagger) \right). \] (96)

and so the state \( \Psi_\mu \) can be expanded as
\[
\Psi_\mu = \prod_j \exp \left( \frac{\mu^2}{2} e^{-2\beta E_j} \right) \sum_n \mu^n e^{-\beta E_j n} \frac{c_j^\dagger n}{n!}|0\rangle_F. \] (97)

We are now ready to compute \( \langle A \rangle_\mu \). Using (97) in (98) one gets straightforwardly
\[
\langle A \rangle_\mu = Z^{-1}_\beta \sum_{\{n_j\}} e^{-2\beta \sum_j E_j n_j} \mu^2 \sum_j n_j \langle \{n_j\} | A | \{n_j\} \rangle, \] (98)

with
\[
Z_\beta = \sum_{\{n_j\}} e^{-2\beta \sum_j E_j n_j} \mu^2 \sum_j n_j, \] (99)

and the final limit \( L \to \infty \) is understood. Let us consider all the developed machinery as referred to the theory in the bulk making use of the holographic theorem (Theorem 2.2) for a massive field. In that way, \( A \) must be considered as an observable for an observer in the Rindler wedge and \( \Psi_\mu \) is a state for a quantum field propagating in the Rindler wedge. If \( \mu = 1 \), (98) states that the time-averaged state \( \Psi_{\mu=1} \) viewed by an observer in the bulk who uses Rindler time-evolution, is a thermal state with inverse temperature \( 1/(2\beta) \). In particular, if we also choose \( \beta = \beta_U/2 \), where \( \beta_U \) is the inverse Unruh temperature, we get formally and in the sense of the pointed regularization-procedure out,
\[
\langle A \rangle_{\mu=1} = tr(\rho_{\beta_U} A), \] (100)

where
\[
\rho_{\beta_U} := \frac{e^{-\beta_U H^\oplus_F}}{tr e^{-\beta_U H^\oplus_F}}.
\]
is the density matrix of a thermal state, which coincides with the restriction of Minkowski vacuum to Rindler wedge because of celebrated results of QFT (Bisognano-Wichmann-Sewell theorems, see [22] for a general discussion). In the case $\mu \neq 1$, (98) suggests to interpret
\[
\frac{\log \mu^2}{2\beta}
\]
as a chemical potential and the associated state can be seen as a grand canonical ensemble state.

4 Overview and open problems

In this paper we have shown that quantum field theory for free fields propagating in a 2D-Rindler background is unitary equivalent to the analogue defined on the compactified Killing horizons. The same equivalence can be implemented at algebraic level. The key point of this holographic description is the hidden $SL(2,\mathbb{R})$ symmetry found in [9] for the fields propagating in the bulk. Indeed that hidden representation of $SL(2,\mathbb{R})$ becomes geometrically manifest when the theory is represented on the Killing horizon. Preserving a clear geometric meaning, the representation can be enlarged up to include a positive-energy unitary representation of Virasoro algebra with central charge $c = 1$. Notice that the Virasoro algebra is realized in the many particles description of the fields, namely it describes a representation in the Fock space. The appearance of the pair of Virasoro algebras in the future and past horizon leads naturally to an (Euclidean) 2D conformal field theory on a cylinder which is holographically associated with QFT in the bulk. In the last section we have proposed the idea that, for a particular choice of the parameter $\beta$ and the ground energy $h = \mu^2/2$ of the Virasoro Hamiltonian $F_0$, the ground state $\Psi_\mu(t)$ of another Virasoro generator which generalizes Rindler Hamiltonian has thermal properties. $\Psi_\mu(t)$, seen in Fock space built up over the Rindler vacuum $|0\rangle$, reveals to be an infinite particle state in thermal equilibrium temperature $1/(2\beta)$. It can be useful to describe the Hawking effect. These thermal properties are shown here without rigorous proof because of the use of a necessary regularization procedure in computing the mean value of the state with respect to Rindler time. Further investigation in that direction, perhaps based on KMS condition, is necessary.

Another issue which deserves investigation is the existence of any relation between the results of this paper and the attempts to give a statistical explanation to black hole entropy by counting microstates of irreducible unitary representations of Virasoro algebra [15]. This is done by means of the so-called Cardy’s formula after a suitable dimensional regularization which gives rise to a scalar field (supporting part of information of 4D gravity) propagating in a 2D spacetime. The main problem of those approaches is that they must assume the existence of a quantum Virasoro representation. The existence of such a representation has been established in this paper: It is worthwhile to investigate about the possible interplay between the quantum Virasoro representation found here and that necessary in those approaches.
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