Lebesgue constants on compact manifolds

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Abstract

Sharp asymptotic for norms of Fourier projections on compact homogeneous manifolds $M^d$ (for example, the real spheres $S^d$, the real, complex and quaternionic projective spaces $P^d(\mathbb{R})$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$ and the Cayley elliptic plain $P^{16}(\text{Cay})$) are established. These results extend sharp asymptotic estimates found by Fejer [4] in the case of $S^1$ in 1910 and then by Gronwall [6] in 1914 in the case of $S^2$. As an application of these results we give solution of the problem of Kolmogorov on sharp asymptotic for the rate of convergence of Fourier sums on a wide range of sets of multiplier operators.

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1 Introduction

Let $M^d$ be a compact globally symmetric space of rank 1 (two-point homogeneous space), $\nu$ its normalized volume element, $\Delta$ its Laplace-Beltrami operator. It is well-known that the eigenvalues $\theta_k$, $k \geq 0$ of $\Delta$ are discrete, nonnegative and form an increasing sequence $0 \leq \theta_0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \cdots$ with $+\infty$ the only accumulation point. Corresponding eigenspaces $H_k$, $k \geq 0$ are finite dimensional, $d_k = \dim H_k < \infty$, $k \geq 0$, orthogonal and $L_2(M^d, \nu) = \bigoplus_{k=0}^{\infty} H_k$. Let $\{Y_j^k\}_{k=1}^{d_k}$ be an orthonormal basis of $H_k$. Assume that $\phi$ is a continuous function $\phi \in C(M^d)$ with the formal Fourier expansion

$$\phi \sim \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} c_k,j(\phi)Y_j^k, \quad c_{k,j}(\phi) = \int_{M^d} \phi \overline{Y_j^k} d\nu.$$
Consider the sequence of Fourier sums
\[ S_n(\phi) = \sum_{k=0}^{n} \sum_{j=1}^{d_k} c_{k,j}(\phi) Y^k_j. \]

The main aim of this article is to establish sharp asymptotic for the sequence \( \|S_n\|_{C(M^d)} \to C(M^d) \) as \( n \to \infty \). As a consequence we give the solution of the problem of Kolmogorov on sharp asymptotic for the rate of convergence of Fourier sums on sets generated by pseudo-differential operators (multiplier operators) on compact manifolds.

Observe that this set of problems is closely related to the problem of uniform convergence of Fourier series on \( M^d \). Indeed, let
\[ E_n(\phi) = \inf \{ \|\phi - t_n\|_{C(M^d)} \mid t_n \in T_n \} \]
be the best approximation of a function \( \phi \in C(M^d) \) by the subspace \( T_n \) of polynomials of order \( \leq n \), \( T_n = \bigoplus_{k=0}^{n} H_k \). Then, by the Lebesgue inequality [10] we get
\[ \|\phi - S_n(\phi)\|_{C(M^d)} \leq \left( 1 + \|S_n\|_{C(M^d)} \to C(M^d) \right) E_n(\phi), \]
where \( \|S_n\|_{C(M^d)} \to C(M^d) = \sup\{\|S_n(\phi)\|_{C(M^d)} \mid \|\phi\|_{C(M^d)} \leq 1 \} \). It means that \( S_n(\phi, x) \) converges uniformly to \( \phi \) if
\[ \lim_{n \to \infty} E_n(\phi) \|S_n\|_{C(M^d)} \to C(M^d) = 0. \]

In the case of the circle, \( S^1 \), the following result has been found by Fejer in 1910 [4],
\[ \|S_n\|_{C(S^1)} \to C(S^1) = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)| dt = \frac{4}{\pi^2} \ln n + O(1), \]
where \( D_n(t) = 1/2 + \sum_{k=1}^{n} \cos kt \) is the Dirichlet kernel. In the case of \( S^2 \), the two-dimensional unit sphere in \( \mathbb{R}^3 \), the estimates of \( \|S_n\|_{C(S^2)} \to C(S^2) \) as \( n \to \infty \), have been established by Gronwall [6]. Namely, it was shown that
\[ \|S_n\|_{C(S^2)} \to C(S^2) = n^{1/2} \frac{2}{\pi^{3/2}} \int_{0}^{\pi} \sqrt{\cot \left( \frac{\eta}{2} \right)} \, d\eta + O(1), \]
\[ = n^{1/2} \frac{2^{3/2}}{\pi^{3/2}} \pi^{-1/2} + O(1). \]

2 Harmonic Analysis on Compact Manifolds

We shall be mostly interested here in compact globally symmetric spaces of rank 1 (two-point homogeneous spaces) and the complex sphere \( S^d_c \). Such manifolds of dimension \( d \) will be denoted by \( M^d \). In particular, each \( M^d \) can be considered as the orbit space of some compact subgroup \( H \) of the orthogonal group \( G \),
that is $M^d = G/H$. Let $\pi : G \to G/H$ be the natural mapping and $e$ be the identity of $G$. The point $o = \pi(e)$ which is invariant under all motions of $H$ is called the pole (or the north pole) of $M^d$. On any such manifold there is an invariant Riemannian metric $d(\cdot, \cdot)$, and an invariant Haar measure $d\nu$. Two-point homogeneous spaces admit essentially only one invariant second order differential operator, the Laplace-Beltrami operator $\Delta$. A function $Z(\cdot) : M^d \to \mathbb{R}$ is called zonal if $Z(h^{-1} \cdot) = Z(\cdot)$ for any $h \in H$. A complete classification of the two-point homogeneous spaces was given by Wang [12]. For information on this classification see, e.g., Cartan [2], Gangolli [5], and Helgason [7, 8]. The geometry of these spaces is in many respects similar. All geodesics in a given one of these spaces are closed and have the same length $2L$. Here $L$ is the diameter of $G/H$, i.e., the maximum distance between any two points. A function $Z(\cdot) : M^d \to \mathbb{R}$ is called invariant under the left action of $H$ on $G/H$ if and only if it depends only on the distance of its argument from $o = e_H$. Since the distance of any point of $G/H$ from $e_H$ is at most $L$, it follows that a $H$-spherical function $Z$ on $G/H$ can be identified with a function $\tilde{Z}$ on $[0, L]$. Let $\theta$ be the distance of a point from $e_H$. We may choose a geodesic polar coordinate system $(\theta, u)$ where $u$ is an angular parameter. In this coordinate system the radial part $\Delta_\theta$ of the Laplace-Beltrami operator $\Delta$ has the expression

$$\Delta_\theta = (A(\theta))^{-1} \frac{d}{d\theta} \left( A(\theta) \frac{d}{d\theta} \right),$$

where $A(\theta)$ is the area of the sphere of radius $\theta$ in $G/H$ which can be computed in terms of the structure of the Lie algebras of $G$ and $H$ (see Helgason [8, p.251], [7, p.168] for the details). It can be shown that

$$A(\theta) = \omega_{\sigma+\rho+1} \lambda^{-\sigma} (2\lambda)^{-\rho} (\sin \lambda \theta)^{\sigma} (\sin 2\lambda \theta)^{\rho},$$

where $\omega_d$ is the area of the unit sphere in $\mathbb{R}^d$ and

$$\begin{align*}
S^d : \quad & \sigma = 0, \rho = d - 1, \lambda = \pi/2L, d = 1, 2, 3, \ldots; \\
P^d(\mathbb{R}) : \quad & \sigma = 0, \rho = d - 1, \lambda = \pi/4L, d = 2, 3, 4, \ldots; \\
P^d(\mathbb{C}) : \quad & \sigma = d - 2, \rho = 1, \lambda = \pi/2L, d = 4, 6, 8, \ldots; \\
P^d(\mathbb{H}) : \quad & \sigma = d - 4, \rho = 3, \lambda = \pi/2L, d = 8, 12, \ldots; \\
P^{16}(\text{Cay}) : \quad & \sigma = 8, \rho = 7, \lambda = \pi/2L.
\end{align*}$$

Applying (1) and (2) we can write the operator $\Delta_\theta$ (up to some numerical constant) in the form

$$\Delta_\theta = \frac{1}{(\sin \lambda \theta)^{\sigma} (\sin 2\lambda \theta)^{\rho}} \frac{d}{d\theta} (\sin \lambda \theta)^{\sigma} (\sin 2\lambda \theta)^{\rho} \frac{d}{d\theta},$$

Using a simple change of variables $t = \cos 2\lambda \theta$, this operator takes the form (up to a positive multiple),

$$\Delta_t = (1 - t)^{-\alpha}(1 + t)^{-\beta} \frac{d}{dt} (1 - t)^{1+\alpha}(1 + t)^{1+\beta} \frac{d}{dt},$$

(3)
where
\[ \alpha = \frac{\sigma + \rho - 1}{2}, \quad \beta = \frac{\rho - 1}{2}. \] (4)

For all manifolds considered here \( \alpha = (d - 2)/2 \). We will need the following statement Szegö [10, p.60]:

**Proposition 1.** The Jacobi polynomials \( y = P_k^{(\alpha, \beta)} \) satisfy the following linear homogeneous differential equation of the second order:

\[ (1 - t^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)t)y' + k(k + \alpha + \beta + 1)y = 0, \]

or
\[ ((1 - t)^{\alpha + 1}(1 - t)^{\beta + 1}y')' + k(k + \alpha + \beta + 1)(1 - t)^\alpha (1 + t)^\beta y = 0. \]

It follows from the above proposition that the eigenfunctions of the operator \( \Delta \), defined in (3) are well-known Jacobi polynomials \( P_k^{(\alpha, \beta)} \) and the corresponding eigenvalues are \( \theta_k = -k(k + \alpha + \beta + 1) \). In this way zonal \( \mathcal{H} \)-invariant functions on \( \mathbb{M}^d = \mathbb{G}/\mathcal{H} \) can be easily identified in each of the five cases above since the elementary zonal functions are eigenfunctions of the Laplace-Beltrami operator.

We shall call them \( Z_k, k \in \mathbb{N} \cup \{0\} \), with \( Z_0 \equiv 1 \). Let \( \tilde{Z}_k \) be the corresponding functions induced on \([0, L] \) by \( Z_k \). Then

\[ \tilde{Z}_k(\theta) = C_k(\mathbb{M}^d)P_k^{(\alpha, \beta)}(\cos 2\lambda\theta), \quad k \in \mathbb{N} \cup \{0\}, \] (5)

where \( \alpha \) and \( \beta \) have been specified above. If \( \mathbb{M}^d = \mathbb{P}^d(\mathbb{R}) \), then only the polynomials of even degree appear because, due to the identification of antipodal points on \( \mathbb{S}^d \), only the even order polynomials \( P_k^{(\alpha, \alpha)} \), \( k = 2m, m \in \mathbb{N} \cup \{0\} \), can be lifted to be functions on \( \mathbb{P}^d(\mathbb{R}) \).

In the case of \( \mathbb{S}^d \) we have \( \sigma = 0, \rho = d - 1 \), so that, \( \alpha = \beta = (d - 2)/2 \) and the polynomials \( P_k^{(\alpha, \beta)} \) reduce to \( P_k^{(d-2)/2,(d-2)/2)} \) which is a multiple of the Gegenbauer polynomial \( P_k^{(d-1)/2} \). A detailed treatment of the Jacobi polynomials can be found in Szegö [11]. We remark that the Jacobi polynomials \( P_k^{(\alpha, \beta)}(t), \alpha > -1, \beta > -1 \) are orthogonal with respect to \( \omega^{\alpha,\beta}(t) = c^{-1}(1 - t)^\alpha (1 + t)^\beta \) on \((-1, 1)\). The above constant \( c \) can be found using the normalization condition \( \int_{\mathbb{S}^d} d\nu = 1 \) for the invariant measure \( d\nu \) on \( \mathbb{M}^d \) and a well-known formula for the Euler integral of the first kind

\[ B(p, q) = \int_0^1 \xi^{p-1}(1 - \xi)^{q-1}d\xi = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0, \quad q > 0. \] (6)

Applying (6) and a simple change of variables we get

\[ 1 = \int_{\mathbb{M}^d} d\nu = \int_{-1}^1 \omega^{\alpha,\beta}(t)dt = c^{-1}\int_{-1}^1 (1 - t)^\alpha (1 + t)^\beta dt, \]

so that,

\[ c = \int_{-1}^1 (1 - t)^\alpha (1 + t)^\beta dt = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}. \] (7)
We normalize the Jacobi polynomials as follows:

\[ P_k^{(\alpha, \beta)}(1) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k + 1)}. \]

This way of normalization is coming from the definition of Jacobi polynomials using the generating function Szegö [11, p.69].

Let \( L_p(\mathbb{M}^d) \) be the set of functions of finite norm given by

\[ \| \varphi \|_p = \| \varphi \|_{L_p(\mathbb{M}^d)} = \left\{ \left( \int_{\mathbb{M}^d} |\varphi(x)|^p \, d\nu(x) \right)^{1/p} \right\}, \quad 1 \leq p < \infty, \quad \text{ess sup} |\varphi|, \quad p = \infty. \]

Further, let \( U_p = \{ \varphi \mid \varphi \in L_p(\mathbb{M}^d), \quad \| \varphi \|_p \leq 1 \} \) be the unit ball of the space \( L_p(\mathbb{M}^d) \). The Hilbert space \( L_2(\mathbb{M}^d) \) with usual scalar product \( \langle f, g \rangle = \int_{\mathbb{M}^d} f(x)g(x) \, d\nu(x) \) has the decomposition \( L_2(\mathbb{M}^d) = \bigoplus_{k=0}^{\infty} H_k \), where \( H_k \) is the eigenspace of the Laplace-Beltrami operator corresponding to the eigenvalue \( \theta_k = -k(k + \alpha + \beta + 1) \). Let \( \{ Y_j^k \}_{j=1}^{d_k} \) be an orthonormal basis of \( H_k \). The following addition formula is known Koornwinder [9]

\[ \sum_{j=1}^{d_k} Y_j^k(x)Y_j^k(y) = \tilde{Z}_k(\cos 2\lambda \theta), \quad (8) \]

where \( \theta = d(x, y) \). Comparing (8) with (5) we get

\[ \sum_{j=1}^{d_k} Y_j^k(x)Y_j^k(y) = \tilde{Z}_k(\cos \theta) = C_k(\mathbb{M}^d) P_k^{(\alpha, \beta)}(\cos 2\lambda \theta). \quad (9) \]

### 3 Sets of smooth functions and multiplier operators on \( \mathbb{M}^d \)

Using multiplier operators we introduce a wide range of smooth functions on \( \mathbb{M}^d \). Let \( \varphi \in L_p(\mathbb{M}^d), \quad 1 \leq p \leq \infty \), with the formal Fourier expansion

\[ \varphi \sim \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k, \quad c_{k,j}(\phi) = \int_{\mathbb{M}^d} \phi Y_j^k \, d\nu. \]

Let \( \Lambda = \{ \lambda_k \}_{k \in \mathbb{N}} \) be a sequence of real (complex) numbers. If for any \( \phi \in L_p(\mathbb{M}^d) \) there is a function \( f = \Lambda \phi \in L_q(\mathbb{M}^d) \) such that

\[ f \sim \sum_{k=0}^{\infty} \lambda_k \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k, \]

then we shall say that the multiplier operator \( \Lambda \) is of \( (p, q) \)-type with norm \( \| \Lambda \|_{p,q} = \sup_{\phi \in U_p} \| \Lambda \phi \|_q \). We shall say that the function \( f \) is in \( \Lambda U_p \oplus \mathbb{R} \) if

\[ \Lambda \phi = f \sim C + \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k, \]

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where \( C \in \mathbb{R} \) and \( \varphi \in U_p \). In particular, the \( \gamma \)-th fractional integral (\( \gamma > 0 \)) of a function \( \varphi \in L_1(M_d^d) \) is defined by the sequence \( \lambda_k = (k(k + \alpha + \beta + 1))^{-\gamma/2} \). Sobolev’s classes \( W^\gamma_p(M_d) \) on \( M_d^d \) are defined as sets of functions with formal Fourier expansions

\[
C + \sum_{k=1}^{\infty} (k(k + \alpha + \beta + 1))^{-\gamma/2} \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k,
\]

where \( C \in \mathbb{R} \) and \( \| \phi \|_p \leq 1 \). Let \( Z \) be a zonal integrable function on \( M_d^d \). For any integrable function \( g \) we can define convolution \( h \) on \( M_d^d \) as the following

\[
h(\cdot) = (Z * g)(\cdot) = \int_{M_d^d} Z(\cos(2\lambda d(\cdot, x)) g(x) d\nu(x).
\]

For the convolution on \( M_d^d \) we have Young’s inequality \( \| (z * g) \|_q \leq \|z\|_p \|g\|_r \), where \( 1/q = 1/p + 1/r - 1 \) and \( 1 \leq p, q, r \leq \infty \). It is possible to show that for any \( \gamma > 0 \) the function \( G_{\gamma} = G_{\gamma, \eta} = \sum_{k=1}^{\infty} (k(k + \alpha + \beta + 1))^{-\gamma/2} Z_k^\eta \) is integrable on \( M_d^d \) and for any function \( g \in W^\gamma_p(M_d^d) \) we have an integral representation \( g = C + G_{\gamma} * \phi \), where \( C \in \mathbb{R} \) and \( \phi \in U_p \).

### 4 The Orthogonal Projection

The main result of this article is the following statement.

**Theorem 1.** Let \( M_d^d = S^d, P^d(C), P^d(H), P^{16}(Cay), d \geq 2 \), then

\[
\| S_n \|_{C(M_d^d) \to C(M_d^d)} = K(M_d^d) n^{(d-1)/2} + O \left\{ \begin{array}{ll}
1, & d = 2, 3 \\
\frac{1}{n^{(d-3)/2}}, & d \geq 4
\end{array} \right.,
\]

where

\[
K(M_d^d) = \frac{4}{\pi^{5/2} \Gamma(d/2)} \int_0^{\pi/2} (\sin \eta)^{(d-3)/2} (\cos \eta) \chi(M_d^d) d\eta,
\]

and

\[
\chi(M_d^d) = \left\{ \begin{array}{ll}
(d - 1)/2, & M_d^d = S^d, d = 2, 3, 4, \ldots, \\
1/2, & M_d^d = P^d(C), d = 4, 6, 8, \ldots, \\
2, & M_d^d = P^d(H), d = 8, 12, 16, \ldots, \\
7/2, & M_d^d = P^{16}(Cay).
\end{array} \right.
\]

If \( M_d^d = P^d(\mathbb{R}), d = 2, 3, \ldots, \) then

\[
\| S_{2n} \|_{C(P^d(\mathbb{R})) \to C(P^d(\mathbb{R}))}
= \frac{4 n^{(d-1)/2}}{\pi^{5/2} \Gamma(d/2)} \int_0^{\pi/2} (\sin \eta)^{(d-3)/2} d\eta + O \left\{ \begin{array}{ll}
1, & d = 2, 3 \\
\frac{1}{n^{(d-3)/2}}, & d \geq 4
\end{array} \right.,
\]

**Proof.** Consider the case \( M_d^d = S^d, P^d(C), P^d(H), P^{16}(Cay) \) first. We will need an explicit representation for the constant \( C_k(M_d^d) \) defined in [9] for our
applications. Putting $y = x$ in (9) and then integrating both sides with respect to $d\nu(x)$ we get

$$d_k = \dim H_k = \sum_{j=1}^{d_k} |Y_j^k(x)|^2 d\nu(x) = C_k(M^d)P_k^{(\alpha, \beta)}(1). \quad (10)$$

Taking the square of both sides of (9) and then integrating with respect to $d\nu(x)$ we find

$$\sum_{j=1}^{d_k} |Y_j^k(y)|^2 = C_k^2(M^d) \int_{M^d} \left( P_k^{(\alpha, \beta)}(\cos(2\lambda d(x, y))) \right)^2 d\nu(x). \quad (11)$$

Since $d\nu$ is shift invariant then

$$\int_{M^d} \left( P_k^{(\alpha, \beta)}(\cos(2\lambda d(x, y))) \right)^2 d\nu(x) = c^{-1} \| P_k^{(\alpha, \beta)} \|_2^2,$$

where the constant $c$ is defined by (14) and (see [11, p.68])

$$\| P_k^{(\alpha, \beta)} \|_2^2 = \int_{-1}^{1} \left( P_k^{(\alpha, \beta)}(t) \right)^2 (1 - t)^\alpha (1 + t)^\beta dt$$

$$= \frac{2^{\alpha+\beta+1}}{2k + \alpha + \beta + 1} \frac{\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}{\Gamma(k + 1)\Gamma(k + \alpha + \beta + 1)}.
$$

So that, (11) can be written in the form

$$\sum_{j=1}^{d_k} |Y_j^k(y)|^2 = c^{-1} C_k^2(M^d) \| P_k^{(\alpha, \beta)} \|_2^2.$$

Integrating the last line with respect to $d\nu(y)$ we obtain

$$d_k = c^{-1} C_k^2(M^d) \| P_k^{(\alpha, \beta)} \|_2^2.$$

It is sufficient to compare this with (10) to obtain

$$C_k(M^d) = \frac{cP_k^{(\alpha, \beta)}(1)}{\| P_k^{(\alpha, \beta)} \|_2^2}. \quad (12)$$

We get now an integral representation for the Fourier sums $S_n(\phi, x)$ of a function $\phi \in L_1(M^d)$,

$$S_n(\phi, x) = \sum_{k=0}^{n} \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k(x)$$

$$= \sum_{k=0}^{n} \sum_{j=1}^{d_k} \left( \int_{M^d} \phi(y) Y_j^k(y) d\mu(y) \right) Y_j^k(x)$$

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\begin{align*}
&= \int_{M^d} \sum_{k=0}^{n} \left( \sum_{j=1}^{d_k} Y_{j}^{k}(y) Y_{j}^{k}(x) \right) \phi(y) d\nu(y) \\
&= \int_{M^d} \sum_{k=0}^{n} Z_{k}^{n}(y) \phi(y) d\nu(y) \\
&= \int_{M^d} K_{n}(x, y) \phi(y) d\nu(y), \\
\end{align*}
\begin{equation}
\text{where} \quad K_{n}(x, y) = \sum_{k=0}^{n} Z_{k}^{n}(y). 
\end{equation}

By (5) and (12) we have
\begin{equation}
K_{n}(x, y) = c \sum_{k=0}^{n} \frac{P_{k}^{(\alpha, \beta)}(1)}{\|P_{k}^{(\alpha, \beta)}\|_{2}^{2}} P_{k}^{(\alpha, \beta)}(\cos 2\lambda d(x, y)).
\end{equation}

Put
\begin{equation}
G_{n}(\gamma, \delta) = \sum_{k=0}^{n} \frac{P_{k}^{(\alpha, \beta)}(\gamma) P_{k}^{(\alpha, \beta)}(\delta)}{\|P_{k}^{(\alpha, \beta)}\|_{2}^{2}},
\end{equation}
then Szegö \[ p.71, \]
\begin{equation}
G_{n}(\gamma, 1) = \sum_{k=0}^{n} \frac{P_{k}^{(\alpha, \beta)}(\gamma) P_{k}^{(\alpha, \beta)}(1)}{\|P_{k}^{(\alpha, \beta)}\|_{2}^{2}} = 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_{n}^{(\alpha+1, \beta)}(\gamma).
\end{equation}

It means that the kernel function (14) in the integral representation (13) can be written in the form
\begin{equation}
K_{n}(x, y) = c 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_{n}^{(\alpha+1, \beta)}(\cos 2\lambda d(x, y)).
\end{equation}

Let \( o \) be the north pole of \( M^d \), then since \( K_{n} \) is a zonal function and \( d\nu \) is shift invariant,
\begin{align*}
\|S_{n}\|_{C(M^d) \rightarrow C(M^d)} &= \sup_{\|\phi\|_{C(M^d)} \leq 1} \|S_{n}(\phi, x)\|_{C(M^d)} \\
&= \sup \left\{ \int_{M^d} |K_{n}(x, y)| d\nu(y) \mid x \in M^d \right\} \\
&= \int_{M^d} |K_{n}(o, y)| d\nu(y) \\
&= c 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \int_{M^d} |P_{n}^{(\alpha+1, \beta)}(\cos 2\lambda d(o, y))| d\nu(y)
\end{align*}
where

\[ \text{It is known Szegö [11, p.196] that for } 0 < \theta < \pi \]

\[ \text{Comparing (16) - (18), applying a simple Taylor series arguments and elementary estimates of the derivative of the function} \]

\[ \sigma(\eta) = \left( \sin \frac{\eta}{2} \right)^{\alpha-1/2} \left( \cos \frac{\eta}{2} \right)^{\beta+1/2} \]

we get

\[ I_n = \pi^{-1/2} n^{-1/2} \int_0^\pi \left( \sin \frac{\eta}{2} \right)^{\alpha-1/2} \left( \cos \frac{\eta}{2} \right)^{\beta+1/2} \]

\[ \times \cos \left( \left( n + \frac{\alpha + \beta + 2}{2} \right) \eta - \frac{d+1}{4} \pi \right) \]

\[ = 2\pi^{-3/2} n^{-1/2} \int_0^\pi \left( \sin \frac{\eta}{2} \right)^{\alpha-1/2} \left( \cos \frac{\eta}{2} \right)^{\beta+1/2} \]

\[ + n^{-1/2} O \left\{ \begin{array}{ll} n^{-1/2}, & \alpha = 0 \\ n^{-1}, & \alpha \geq 1/2 \end{array} \right\}, \ n \to \infty. \] (19)

Remind that \( \alpha = (d-2)/2, d \geq 2 \) for any manifold \( M^d \) under consideration.

Put \( \chi(M^d) = \beta + 1/2 \), then from (16) and (19) it follows that

\[ \|S_n\|_{C(M^d) \to C(M^d)} = \mathcal{K}(M^d)n^{\alpha+1/2} + O(n^{\alpha-1/2}), \]

where

\[ \mathcal{K}(M^d) = \frac{2}{\pi^{3/2} \Gamma(\alpha + 1)} \int_0^\pi \left( \sin \frac{\eta}{2} \right)^{\alpha-1/2} \left( \cos \frac{\eta}{2} \right)^{\beta+1/2} d\eta \]

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\[
\frac{4}{\pi^{3/2} \Gamma(d/2)} \int_0^{\pi/2} (\sin \eta)^{(d-3)/2} (\cos \eta)^{\chi(M^d)} d\eta,
\]

since \(\alpha = (d-2)/2\). Hence,
\[
\|S_n\|_{C(M^d) \to C(M^d)} = K(M^d)n^{(d-1)/2} + O \left\{ \begin{array}{l}
  n^{(d-2)/2}, \quad d = 2 \\
  n^{(d-3)/2}, \quad d \geq 3
\end{array} \right\}.
\]

Finally, the value of \(\chi(M^d)\), where \(M^d = S^d, P^d(\mathbb{C}), P^d(\mathbb{H}), P^{16}(\text{Cay})\), can be easily calculated using (2) and (4).

The case of \(P^d(\mathbb{R})\) needs a special treatment. In this case \(\alpha = \beta = (d-2)/2\), \(\lambda = \pi/(4L)\) and the kernel function \(K^*_2(x, y)\) in the integral representation for the Fourier sums,
\[
S_{2n}(\phi, x) = \int_{P^d(\mathbb{R})} K^*_2(x, y)\phi(y) d\nu(y)
\]

has the form
\[
K^*_2(x, y) = \sum_{k=0}^{n} Z^2_k(y) = \sum_{k=0}^{n} C_{2k}(P^d(\mathbb{R})) P^{(l,\alpha)}_{2k} \cos(2\lambda d(x, y))
\]

\[
= \sum_{k=0}^{n} C_{2k}(P^d(\mathbb{R})) P^{(d-2)/2,2/2} \left( \cos \left( \frac{\pi}{2L} d(x, y) \right) \right).
\]

Let the constant \(c^*\) be such that
\[
1 = \int_{P^d(\mathbb{R})} d\nu = \int_0^1 \omega^{(d-2)/2,2/2}(t) dt = (c^*)^{-1} \int_0^1 (1-t^2)^{(d-2)/2} dt,
\]

then \(c^* = c/2\) and
\[
C_{2k}(P^d(\mathbb{R})) = \frac{c^* P^{(d-2)/2,2/2}(1)}{\|P^{(d-2)/2,2/2}\|_{2,*}^2} = \frac{c P^{(d-2)/2,2/2}(1)}{\|P^{(d-2)/2,2/2}\|_{2}^2},
\]

where
\[
\|P^{(d-2)/2,2/2}\|_{2,*}^2 = \int_0^1 \left( P^{(d-2)/2,2/2}(t) \right)^2 (1-t^2)^{(d-2)/2} dt
\]

\[
= 2^{-1} \|P^{(d-2)/2,2/2}\|_{2}^2.
\]

Let \(O\) be the north pole of \(P^d(\mathbb{R})\), then since \(K_2^*\) is a zonal function and \(d\nu\) is shift invariant,
\[
\|S_{2n}\|_{C(P^d(\mathbb{R})) \to C(P^d(\mathbb{R}))} = \sup_{\|\phi\|_{C(P^d(\mathbb{R}))} \leq 1} \|S_{2n}(\phi)\|_{C(P^d(\mathbb{R}))}
\]

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Consider the function

\[ G_{2n}^*(\gamma, 1) = \sum_{k=0}^{n} \frac{P_k^{(\alpha, \alpha)}(\gamma)}{\|P_k^{(\alpha, \alpha)}\|^2} P_k^{(\alpha, \alpha)}(1). \]

Since \( P_k^{(\alpha, \beta)}(\gamma) = (-1)^k P_k^{(\beta, \alpha)}(-\gamma) \), Szegö [11, p.59], then

\[ G_{2n}^*(\gamma, 1) = \frac{G_{2n}(\gamma, 1) + G_{2n}(-\gamma, 1)}{2} \]

\[ = \frac{2^{-d+1}\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} \left( \frac{P_{2n}^{(d/2,(d-2)/2)}(\gamma) + P_{2n}^{(d/2,(d-2)/2)}(-\gamma)}{2} \right) \]

\[ = \frac{2^{-d}\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} \left( P_{2n}^{(d/2,(d-2)/2)}(\gamma) + P_{2n}^{((d-2)/2,d/2)}(\gamma) \right) \]

where \( G_{2n}(\gamma, 1) \) is defined in [15]. Consequently, (20) takes the form

\[ \|S_{2n}\|_{C(P^n(\mathbb{R})) \rightarrow C(P^{d}(\mathbb{R}))} = \frac{c 2^{-d} \Gamma(2n+d)}{c^{d} \Gamma(d/2)\Gamma(2n+d/2)} \]

\[ \times \int_{P^{d}(\mathbb{R})} \left\| P_{2n}^{(d/2,(d-2)/2)}(\cos(\pi d(o, y)/(4L))) + P_{2n}^{((d-2)/2,d/2)}(\cos(\pi d(o, y)/(4L))) \right\| d\nu(y) \]

\[ = \frac{c 2^{-d} \Gamma(2n+d)}{c^{d} \Gamma(d/2)\Gamma(2n+d/2)} \int_{0}^{1} \left\| P_{2n}^{(d/2,(d-2)/2)}(t) + P_{2n}^{((d-2)/2,d/2)}(t) \right\| (1-t^2)^{(d-2)/2} dt \]

\[ = \frac{2^{-d} \Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} I'_n, \quad (21) \]

where

\[ I'_n = \int_{0}^{1} \left\| P_{2n}^{(d/2,(d-2)/2)}(t) + P_{2n}^{((d-2)/2,d/2)}(t) \right\| (1-t^2)^{(d-2)/2} dt \]

\[ = \int_{0}^{\pi/2} \left\| P_{2n}^{(d/2,(d-2)/2)}(\cos \eta) + P_{2n}^{((d-2)/2,d/2)}(\cos \eta) \right\| (\sin \eta)^{d-1} d\eta \]

Applying [15] we get

\[ I'_n = \frac{1}{\pi^{1/2}2^{1/2}n^{1/2}} \int_{0}^{\pi/2} d\eta (\sin \eta)^{d-1} \times \left\| \left( \sin \frac{\eta}{2} \right)^{-d/2-1/2} \left( \cos \frac{\eta}{2} \right)^{-(d-2)/2-1/2} \right\| \]

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converges uniformly if
\[
\lim_{n \to \infty} \left| (2n + \frac{d}{2} + (d - 2)/2 + 1) \eta - \frac{(d/2 + 1/2)\pi}{2} \right| + O(n^{-3/2})
\]
\[
= \frac{1}{\pi^{1/2}2^{1/2}n^{1/2}} \left( \int_0^{\pi/2} (\sin \eta)^{d-1} \left( \frac{\sin \eta}{2} \right)^{-d/2-1/2} \left( \frac{\cos \eta}{2} \right)^{-d/2-1/2} \right) \left\| \cos \left( \frac{2n + d}{2} \right) \eta - \frac{(d + 1)\pi}{4} \right\| d\eta + O(n^{-3/2})
\]

Comparing (21) with the last line we get
\[
\| S_{2n} |_{C^1 \rightarrow C^1} = K(P^d) n^{(d-1)/2} + O \left\{ \begin{array}{l}
n^{(d-2)/2}, \quad d = 2 \\
n^{(d-3)/2}, \quad d \geq 3
\end{array} \right\},
\]
where
\[
K(P^d) = \frac{4}{\pi^{3/2} \Gamma(d/2)} \int_0^{\pi/2} (\sin \eta)^{(d-3)/2} d\eta.
\]

\[\square\]

**Remark 1.** Let \( M^d = S^d, P^d, P^d(C), P^d(\mathbb{R}), P^d(\mathbb{H}), P^{16}(\text{Cay}) \). It is known [1] that for any \( \gamma > 0 \),
\[
E_n(W^\gamma(M^d)) = \sup \{ E_n(f) \mid f \in W^\gamma(M^d) \} \propto n^{-\gamma}.
\]

From the Theorem 1 and the Lebesgue inequality it follows that the Fourier series of a function \( f \in W^\infty(M^d) \) converges uniformly if \( \gamma > (d-1)/2 \). In general, let \( \Delta^0 \lambda_k = \lambda_k, \Delta^1 \lambda_k = \lambda_k - \lambda_{k+1}, \Delta^{s+1} \lambda_k = \Delta^s \lambda_k - \Delta^s \lambda_{k+1}, \ k, s \in \mathbb{N} \) and
\[
M := \left\{ \begin{array}{l}
(d + 1)/2, \quad d = 3, 5, \ldots
\end{array} \right\}
\]
Let \( \Lambda = \{ \lambda_k \}_{k \in \mathbb{N}} \) be a multiplier operator, \( \Lambda : L^\infty(M^d) \to L^\infty(M^d) \) and \( \Lambda U^\infty(M^d) \) be the respective set of smooth functions, then from the Theorem 2, [1] p.317 it follows that the Fourier series of a function \( f \in \Lambda U^\infty(M^d) \) converges uniformly if
\[
\lim_{n \to \infty} n^{(d-1)/2} \sum_{k=n+1}^{\infty} |\Delta^{M+1} \lambda_k| k^M = 0,
\]
since \( E_n(\Lambda U_\infty(M^d)) \ll \sum_{k=n+1}^{\infty} |\Delta^{M+1}\lambda_k| k^M \). In particular, let
\[
\Lambda = \{\lambda_k\}_{k \in \mathbb{N}}, \quad \lambda_k = k^{-(d-1)/2} (\ln k)^{-\alpha},
\]
where \( \alpha > 0 \), then the Fourier series of any function \( f \in \Lambda U_\infty(M^d) \) converges uniformly. A similar result is valid for \( P^d(\mathbb{R}) \).

Remark 2. In terms of gamma function we have the following representations for the constant \( \mathcal{K}(M^d) \):
\[
\mathcal{K}(\mathbb{S}^d) = \frac{2 \Gamma \left( \frac{d-1}{4} \right) \Gamma \left( \frac{d+1}{4} \right)}{\pi^{3/2} \left( \Gamma \left( \frac{d}{2} \right) \right)^2}, \quad d = 2, 3, 4, \ldots
\]
\[
\mathcal{K}(P^d(\mathbb{R})) = \frac{2 \Gamma \left( \frac{d-1}{4} \right)}{\pi \Gamma \left( \frac{d}{4} \right) \Gamma \left( \frac{d+1}{4} \right)}, \quad d = 2, 3, 4, \ldots
\]
\[
\mathcal{K}(P^d(\mathbb{C})) = \frac{2 \Gamma \left( \frac{d-1}{4} \right) \Gamma \left( \frac{3}{4} \right)}{\pi^{5/2} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d+2}{4} \right)}, \quad d = 4, 6, 8, \ldots
\]
\[
\mathcal{K}(P^d(\mathbb{H})) = \frac{\Gamma \left( \frac{d-1}{4} \right)}{\pi \Gamma \left( \frac{d}{4} \right) \Gamma \left( \frac{d+2}{4} \right)}, \quad d = 8, 12, 16, \ldots
\]
\[
\mathcal{K}(P^{16}(\text{Cay})) = \frac{11 \cdot 2^{1/2}}{2949120 \cdot \pi^{1/2}}.
\]

Here we present the solution of the problem of Kolmogorov on sharp asymptotic of the rate of convergence of Fourier series on sets of smooth functions on manifolds. First we will need some definitions.

For a given multiplier sequence \( \Lambda = \{\lambda_k\} \) let
\[
C_n^\delta = \frac{\Gamma(n + \delta + 1)}{\Gamma(\delta + 1) \Gamma(n + 1)} \asymp n^\delta
\]
and
\[
S_n^\delta = \frac{1}{C_n^\delta} \sum_{m=0}^{n} C_n^{\delta-m} Z_m.
\]
It is known [1] that
\[
\|S_n^\delta\|_1 \ll \begin{cases} 1, & \delta > (d-1)/2, \\ \log n, & \delta = (d-1)/2, \\ n^{(d-1)/2-\delta}, & 0 \leq \delta < (d-1)/2. \end{cases}
\]  \( \tag{22} \)

Definition 1. Let \( \mathcal{T} \) be the set of multipliers \( \Lambda = \{\lambda_k\}_{k \in \mathbb{N}} \) such that
\[
\lim_{m \to \infty} \left\| \sum_{s=0}^{d-1} \Delta^s \lambda_{m-s} C_{m-s}^n S_{m-s}^n \right\|_1 = 0,
\]
and
\[
\left\| \sum_{k=n+1}^{\infty} \Delta^d \lambda_k C_{d-1} S_{k-1} + \sum_{s=1}^{d} \Delta^s \lambda_{n+1} C_n^s S_n^s \right\|_1 = o(\lambda_{n+1} ||S_n^0||_1).
\]

As a simple consequence of Theorem 1 we get

**Theorem 2.** Let $\Lambda \in \mathcal{T}$ then

\[
\sup_{f \in \Lambda \cup P(d^d)} \|f - S_n(f)\|_p = K(M^d)|\lambda_{n+1}|n^{(d-1)/2}(1 + o(1)),
\]

where $p = 1, \infty$, $\Lambda \in \mathcal{T}$, $M^d = S^d, P^d(\mathbb{C}), P^d(\mathbb{H}), P^d(Cay)$ and

\[
\sup_{f \in \Lambda \cup P(d^d)} \|f - S_{2n}(f)\|_p = K(P^d(\mathbb{R}))|\lambda_{n+1}|n^{(d-1)/2}(1 + o(1)), \quad p = 1, \infty.
\]

**Proof** Since $\Lambda \in \mathcal{T}$ then

\[
\left\| \sum_{k=n+1}^{\infty} \lambda_k Z_k \right\|_1 = |\lambda_{n+1}||S_n^0||_1(1 + o(1)).
\]

If $M^d = P^d(\mathbb{R})$ and $\{\lambda_k\}_{k \in \mathbb{N}} \in \mathcal{T}$ then

\[
\left\| \sum_{k=n+1}^{\infty} \lambda_k Z_{2k} \right\|_1 = |\lambda_{n+1}||S_n^0||_1(1 + o(1)),
\]

where

\[
S_n^\delta = \frac{1}{\mathcal{C}_n^\delta} \sum_{m=0}^{n} C_{n-m}^\delta Z_{2m}, \quad \delta > 0.
\]

Hence, applying Theorem 1 we get (23) and (24).

**Remark 3.** In particular, let $\lambda_k = (k(k+\alpha+\beta+1))^{-\gamma/2}$, $\gamma > 0$, then using (23) it is easy to show that for any fixed $\gamma > 0$, $\Lambda = \{\lambda_k\}_{n \in \mathbb{N}} \in \mathcal{T}$ and

\[
\sup_{f \in W^p_\gamma(M^d)} \|f - S_n(f)\|_p = K(M^d)n^{-\gamma+(d-1)/2} + O \left( n^{-\gamma} \begin{cases} 1, & d = 2 \\ \ln n, & d = 3 \\ n^{(d-3)/2}, & d \geq 4 \end{cases} \right),
\]

$p = 1, \infty$, $\Lambda \in \mathcal{T}$, $M^d = S^d, P^d(\mathbb{C}), P^d(\mathbb{H}), P^d(Cay)$,

\[
\sup_{f \in W^p_\gamma(P^d(\mathbb{R}))} \|f - S_{2n}(f)\|_p = K(P^d(\mathbb{R}))n^{-\gamma+(d-1)/2} + O \left( n^{-\gamma} \begin{cases} 1, & d = 2 \\ \ln n, & d = 3 \\ n^{(d-3)/2}, & d \geq 4 \end{cases} \right).
\]

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The case $M^d = S^d$ was considered by C. Xirong, D. Feng, W. Kunyang (Estimations of the remainder of spherical harmonic series, Math. Proc. Cambridge Philos. Soc. 145 (2008), no. 1, 243-255). Namely, it was shown that for any $\gamma > 0$ and $d \geq 2$, 

$$\sup_{f \in W^\infty_{\gamma}(S^d)} \| f - S_n(f) \|_\infty = K_d n^{-\gamma + \frac{d+1}{2}} + H(n, \gamma, d)$$

where

$$K_d = \frac{\pi^{3/2}}{(d-1)} \frac{\left( \Gamma \left( \frac{d+1}{2} \right) \right)^2}{(d-1)! \Gamma \left( \frac{d}{2} \right)}$$

$$= \frac{8}{2^{(d+1)/2}} \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d}{2} \right)}{\omega_d}$$

$$= \frac{8}{2^{(d+1)/2}} \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d}{2} \right)}{\omega_d}$$

$$\omega_{d+1} = \int_{S^d} d\sigma(x), \text{ and } H(n, \gamma, d) \text{ satisfies the condition}$$

$$|H(n, \gamma, d)| \leq \begin{cases} 
C n^{-\gamma}, & \text{if } d = 2; \\
C n^{-\gamma} \log n, & \text{if } d = 3; \\
C n^{-\gamma + \frac{d+1}{2}}, & \text{if } d \geq 4.
\end{cases}$$

Unfortunately this result is incorrect. In particular, for any $d \geq 2$, 

$$\frac{\pi^{3/2}}{(d-1)} \frac{\left( \Gamma \left( \frac{d+1}{2} \right) \right)^2}{(d-1)! \Gamma \left( \frac{d}{2} \right)} \neq \frac{8}{2^{(d+1)/2}} \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d}{2} \right)}{\omega_d}.$$

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