Comparing Asynchronous $l$-Complete Approximations and Quotient Based Abstractions

Anne-Kathrin Schmuck, Paulo Tabuada, Jörg Raisch

Abstract

This paper is concerned with a detailed comparison of two different abstraction techniques for the construction of finite state symbolic models for controller synthesis of hybrid systems. Namely, we compare quotient based abstractions (QBA), e.g., described in [10, Part II] with different realizations of strongest (asynchronous) $l$-complete approximations (SA/CA) from [3], [8]. Even though the idea behind their construction is very similar, we show that they are generally incomparable both in terms of behavioral inclusion and similarity relations. We therefore derive necessary and sufficient conditions for QBA to coincide with particular realizations of SA/CA. Depending on the original system, either QBA or SA/CA can be a tighter abstraction.

I. Introduction

The increasing interconnection of physical components and digital hardware in today’s technical systems causes challenges that have been focused on both by the control and the computer science community. Although some efforts have been made to bring these parallel advances together, there still are considerable gaps between the concepts in both fields addressing very similar questions. In this paper, we provide a step towards connecting two methods for finite state symbolic abstraction inspired by these two communities.

Systems where digital hardware is connected to physical components usually lead to hybrid system models. Control synthesis for hybrid systems is a difficult problem, and one common approach to this problem is, first, to simplify a given hybrid control problem by generating a symbolic abstraction of the system to be controlled and, second, to design a symbolic controller using existing synthesis techniques.

Inspired by the computer science community, many proposed solutions apply techniques developed for verification of software processes, as e.g. in [1], [11], [9] and summarized in [10, Part II]. In that work a symbolic abstraction is constructed by partitioning the original state space into a finite number of cells, such that this partition allows for a bisimulation relation between the original state space model and its abstraction. The set of equivalence classes of this partition are used both as outputs and as states of the constructed abstraction. This abstraction method is therefore often referred to as quotient based abstraction (QBA), a terminology we adapt in this paper.

However, in many applications the interface between the system and a controller is given by predefined discrete valued actuator and measurement signals. This implies that the set of input and output symbols is predefined and cannot be used to adjust the abstraction accuracy. One method explicitly addressing this issue is the so called strongest $l$-complete approximation (SICA) [3], which was recently generalized to strongest asynchronous $l$-complete approximations (SA/CA) [8]. Here, the accuracy of the abstraction is adjusted by changing the number $l$ of past input and output symbols considered in the construction of the abstract state space.

SICA and SA/CA were formalized using behavioral systems theory [12], a general framework capturing many different types of dynamics. Therefore, SA/CA are given as behavioral models, i.e. by a set of infinite sequences of symbols. Generally this allows for many different state space realizations. However, almost all previous papers on SICA and SA/CA, e.g., [3], [4], [5], [8], make only use of a particular realization naturally arising in the abstraction process. To be able to compare SICA to QBA we introduce a set of new realizations of SA/CA in this paper.

In the special case where QBA are constructed from a predefined partition of the state space and the bisimulation algorithm [2] is used to refine this partition to increase abstraction accuracy, the idea behind the construction of QBA and a realization of SA/CA is very similar. However, we will show that they are generally incomparable. We therefore derive necessary and sufficient conditions for QBA to coincide with particular realizations of SA/CA.

II. Preliminaries

In this section, we first summarize some elements from behavioral systems theory (e.g., [12]) in Sec. II-A and derive a model of the original system in Sec. II-B as a common starting point to apply QBA and SA/CA. To compare the resulting abstractions we introduce the notion of simulation relations in Sec. II-C.
A. Notation

In the behavioral framework, a dynamical system is given by $\Sigma = (T, W, B)$, consisting of the time axis $T$, the signal space $W$ and the behavior of the system, $B \subseteq (W)^T$, where $(W)^T := \{\omega \mid \omega : T \to W\}$ is the set of all signals evolving on $T$ and taking values in $W$. In this paper we only consider dynamical systems evolving on the discrete time axis $T = \mathbb{N}_0$. However, to simplify notation, we extend the time axis of a behavior $B \subseteq (W)^{\mathbb{N}_0}$ from $\mathbb{N}_0$ to $\mathbb{Z}$ by pre-appending each $\omega \in B$ with the special symbol $\circ$, i.e., $\omega = w_0 w_1 w_2 \ldots \in B$ is transformed to $\circ \circ \circ w_0 w_1 w_2 \ldots \subseteq (W)^{\mathbb{Z}}$. Hence, the notation $\Sigma = (\mathbb{N}_0, W, B)$ refers to a system with behavior $B \subseteq (W)^{\mathbb{Z}}$ s.t. $\forall \omega \in B, k < 0 \Rightarrow \omega(k) = \circ$.

For any $l \in \mathbb{N}_0$, $(W)^l := \{\omega \mid \omega : [0, l-1] \to W\}$ denotes the set of strings $\omega$ with length $l$ and elements in $W$. Now let $I = [t_1, t_2]$ be a bounded interval on $\mathbb{Z}$ with length $|I| = t_2 - t_1 + 1$. Then $\omega|_I = \omega(t_1) \ldots \omega(t_2) \in (W)^{|I|}$ is the result of restricting the map $\omega : \mathbb{Z} \to W$ to the domain $I$ and disregarding absolute time information, i.e., $\omega|_I \in W^{|I|}$ instead of $\omega|_I \in W^2$. Similarly, $B|_I$ results from restricting all trajectories in $B$ to $I$ and disregarding absolute time information. For $t_1 < t_2$ we define $\omega|_{[t_1,t_2]} := \lambda$, where $\lambda$ denotes the empty string.

Now let $W = W_1 \times W_2$ be a product space. Then the projection of a signal $\omega \in (W)^T$ to $W_1$ is given by $\pi_{W_1}(\omega) := \{\omega_1 \in (W_1)^T \mid \exists \omega_2 \in (W_2)^T . \omega = (\omega_1, \omega_2)\}$ and $\pi_{W_1}(B)$ denotes the projection of all signals in the behavior to $W_1$. The concatenation of two strings $\omega_1 \in (W)^{t_1}, \omega_2 \in (W)^{t_2}, t_1, t_2 \in \mathbb{N}_0$ is denoted by $\omega_1 \cdot \omega_2$ (meaning that $\omega_2$ is appended to $\omega_1$).

B. Modelling the Original System

The common starting point of methods generating finite state abstractions of a (possibly continuous) dynamical system is the definition of a finite external signal space $W$. In the context of SA/CA, $W = U \times Y$ is assumed to be predefined by the system to be abstracted, where $U$ is a finite set of control symbols and $Y$ a finite set of measurement symbols. In contrast, the work on QBA usually assumes full sensing and actuating capabilities but defines the finite output set $Y$ based on a specification that the subsequently to be designed controller should guarantee. Therefore, the choice of $W = Y$ is already part of the construction of QBA. In both cases, prior to the abstraction process, a state model of the system to be abstracted is required.

**Definition 1:** A state machine is a tuple $Q = (X, U, Y, \delta, X_0)$, where $X$ is the state space, $X_0$ is the set of initial states, $U$ is the set of inputs, $Y$ is the set of outputs and $\delta \subseteq X \times U \times Y \times X$ is a next state relation.

The set of admissible outputs of a state $x \in X$ is defined by

$$H_\delta(x) := \{y \in Y \mid \exists u \in U, x' \in X . (x, u, y, x') \in \delta\}$$

and $Q$ is said to be output deterministic if

$$\forall x \in X . H_\delta(x) \neq \emptyset \Rightarrow |H_\delta(x)| = 1. \tag{1b}$$

Furthermore,

$$F_\delta(x, u) := \{x' \in X \mid \exists y \in H_\delta(x) . (x, u, y, x') \in \delta\}, \tag{2a}$$

$$T_\delta(x) := \{x' \in X \mid \exists u \in U . x' \in F_\delta(x, u)\}, \tag{2b}$$

are the sets of post-states of a state $x \in X$.

If the state evolution and the output generation of a transition $(x, u, y, x') \in \delta$ can be separated in $Q$, i.e.,

$$\forall x \in X, u \in U . (x, u, y, x') \in \delta \Leftrightarrow (x' \in F_\delta(x, u) \land y \in H_\delta(x)), \tag{3}$$

a state machine can be equivalently defined by the six-tuple $(X, X_0, U, F_\delta, Y, H_\delta)$, which usually defines a transition system. As for the construction of QBA, the set $Y$ is chosen as the set of equivalence classes of a partition of the state space $X$, it is usually assumed that an output of $Q$ is directly generated by a state, rather than a transition. This implies that (3) always holds in this case.

Using a state machine $Q$ to model the original system, its full behavior, i.e., the set of infinite input, state and output sequences compatible with its dynamics, is defined as follows.

1Throughout this paper we use the notation “$\forall . . . \Rightarrow$”, meaning that all statements after the dot hold for all variables in front of the dot. “$\exists . . . \Leftrightarrow$” is interpreted analogously.
Definition 2: Let $Q$ be a state machine as in Def. 1. Then the full behavior of $Q$ is defined by

$$B_f(Q) := \left\{ (\mu, \nu, \xi) \in (U \times Y \times X)^{N_0} \mid \begin{aligned}
    \xi(0) &\in X_0 \\
    \& \forall k \in N_0 . \ (\xi(k), \mu(k), \nu(k), \xi(k+1)) &\in \delta
\end{aligned} \right\} \tag{4}$$

Furthermore, if

$$\forall x \in X_0 . \ \exists (\mu, \nu, \xi) \in B_f(Q) . \ \xi(0) = x$$
and

$$\forall x \in X . \ \exists (\mu, \nu, \xi) \in B_f(Q), k \in N_0 . \ \xi(k) = x \tag{5b}$$

$Q$ is called live and reachable, respectively.

Whenever $Q$ is live and reachable, the dynamics of $Q$ can be equivalently described by its full behavior. As SA/CA are typically constructed from $B_f(Q)$ instead of $Q$, we restrict attention to state machines that are live and reachable, giving the following setup considered in this paper.

Given a dynamical system $S$, we assume that its external dynamics can be modeled by a state machine

$$Q = (X, U, Y, \delta, X_0) , \text{ s.t. (2) and (5) holds} \tag{6a}$$

and the external signal space

$$W \in \{ U \times Y \} \text{ is finite.} \tag{6b}$$

In the remainder of this paper we introduce two methods to construct a finite state abstraction of $Q$ in (6), namely asynchronous l-complete approximations (SA/CA) (from [8]) in Sec. III and quotient based abstractions (QBA) (from [10, part II]) in Sec. IV. To provide a formal comparison of the resulting models in Sec. V we first introduce the notion of simulation relations.

C. Simulation Relations

Simulation relations are commonly used to compare system models in a step-by-step fashion. The idea is to investigate, if there exists a relation between the state spaces of two systems which ensures that trajectories visiting only related states at every time point produce the same external trajectory. To incorporate all possible choices of external signal spaces $W$ as in (6), namely asynchronous l-complete approximations (SA/CA) (from [8]) in Sec. III and quotient based abstractions (QBA) (from [10, part II]) in Sec. IV. To provide a formal comparison of the resulting models in Sec. V we first introduce the notion of simulation relations.

Definition 3: Let $Q_i = (X_i, U_i, Y_i, \delta_i, X_{i0}), \ i \in \{1, 2\}$ be state machines and $W$ a set s.t. $\pi_W(U_1 \times Y_1) = \pi_W(U_2 \times Y_2) \neq \emptyset$. Then $R \subseteq X_1 \times X_2$ s.t.

$$\forall x_1 \in X_{10} , \ (\exists x_2 \in X_{20} . \ (x_1, x_2) \in R) \ \text{ and} \tag{7a}$$

$$\forall (x_1, x_2) \in R, u_1 \in U_1, y_1 \in Y_1, x'_1 \in X_1 , \tag{7b}$$

$$\left\{ \begin{aligned}
    (x_1, u_1, y_1, x'_1) &\in \delta_1 \\
    \exists u_2 \in U_2, y_2 \in Y_2, x'_2 \in X_2 .
\end{aligned} \right\} \Rightarrow
\left\{ \begin{aligned}
    (x_2, u_2, y_2, x'_2) &\in \delta_2 \\
    \land (x_1', x'_2) &\in R \\
    \land \pi_W(u_1, y_1) = \pi_W(u_2, y_2)
\end{aligned} \right\}$$

is a simulation relation from $Q_1$ to $Q_2$ w.r.t. $W$, denoted by $R \in \mathcal{R}_W(Q_1, Q_2)$.

Using Def. 3 we can formally define an ordering on the set of state machines in the usual way.

Definition 4: Given the premises of Def. 3 a state machine $Q_1$ is simulated by $Q_2$ w.r.t. $W$, denoted by $Q_1 \preceq_W Q_2$, if there exists a relation $R \in \mathcal{R}_W(Q_1, Q_2)$. Furthermore, $Q_1$ and $Q_2$ are bisimilar w.r.t. $W$, denoted by $Q_1 \cong_W Q_2$, if there exists a relation $R \in \mathcal{R}_W(Q_1, Q_2)$ also satisfying $R^{-1} \in \mathcal{R}_W(Q_2, Q_1)$.

$^2$As usual, $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$. 
III. STRONGEST ASYNCHRONOUS l-COMPLETE APPROXIMATIONS (SAICA)

The idea of SAICA is to exactly mimic the external behavior of \( Q \) in (9) over finite time intervals of length \( l + 1 \). We therefore consider the behavioral system \( \Sigma = (N_0, W, B(Q)) \), where \( B(Q) \) is the extension of \( \pi_W(B_f(Q)) \) to \( \mathbb{Z} \) as discussed in Sec. [III]. All finite strings of external symbols of length \( l + 1 \) which are consistent with the dynamics of \( Q \) are given by

\[
\Pi_l(B(Q)) := \bigcup_{k \in N_0} B(Q)_{[k-l+1, k]}.
\] (8)

Now consider the following gedankenexperiment: assume playing a sophisticated domino game where \( \Pi_{l+1}(B(Q)) \) is the set of dominos. Pick the first domino to be \( B(Q)_{[-l, 0]} \) (i.e., a domino with only diamonds except for the last symbol) and append any domino from the set \( \Pi_{l+1}(B(Q)) \) if the last \( l \) symbols of the first domino are equivalent to the first \( l \) symbols of the second domino (see Figure 1 (left) for an example). Playing the domino game arbitrarily long and with all possible initial conditions and domino combinations results in the largest, in the sense of set inclusion, behavior \( \hat{B}^l \) satisfying

\[
\hat{B}^l_{[-l, 0]} = B(Q)_{[-l, 0]} \text{ and } \Pi_{l+1}(\hat{B}^l) = \Pi_{l+1}(B(Q)).
\] (9a)

(9b)

defining the behavioral system \( \hat{\Sigma}^l = (N_0, W, \hat{B}^l) \). Observe that the smaller \( l \), the less information in the domino game is used, which generates more freedom in constructing signals, implying \( \hat{B}^l \supseteq \hat{B}^{l+1} \supseteq B(Q) \) for all \( l \in N_0 \). This motivates the use of \( \hat{B}^l \) as an over-approximation of the behavior \( B(Q) \). Obviously, equality \( \hat{B}^r = B(Q) \) holds for all \( r \geq l \) if \( B(Q) \) is itself the largest behavior satisfying (9). In [8], a system \( \Sigma = (N_0, W, B(Q)) \) for which the latter is true was called asynchronously l-complete which inspired the name of SAICA. Following [8], \( \hat{\Sigma}^l \) constructed in the outlined domino game is the unique SAICA of \( \Sigma = (N_0, W, B(Q)) \). However, we are usually interested in a state machine realizing its step by step evolution.

Definition 5: Given (6) and (9), the dynamical system \( \hat{\Sigma}^l = (N_0, W, \hat{B}^l) \) is the SAICA of \( \Sigma = (N_0, W, B(Q)) \). Furthermore, a state machine \( \hat{Q} \) is called a realization of \( \hat{\Sigma}^l \) if \( \hat{B}^l = B(Q) \).

In the literature on SAICA the state space \( \hat{X} \) to construct the realization \( \hat{Q} \) of the SAICA \( \hat{\Sigma}^l \) is usually chosen such that the state represents the “recent past” of length \( l \) of the external signal. Recalling the gedankenexperiment, this choice of \( \hat{X} \) is motivated by the fact that the next feasible domino of length \( l + 1 \) is determined by the last \( l \) symbols of the previous domino (see Fig. 1 (right) for an illustration). Using this state space, the standard state machine realization of SAICA, denoted by \( \hat{Q}^l \) in this paper, is defined as follows.

Proposition 1 ([8], Thm.4): Let \( \hat{\Sigma}^l = (N_0, W, \hat{B}^l) \) be the SAICA of \( \Sigma \) and define

\[
\hat{X}^l := \{ \varnothing \}^l \cup \Pi_l(\hat{B}^l),
\] (10a)

\[
\hat{X}^l_0 := \{ \varnothing \}^l, \text{ and } \hat{\delta}^l := \left\{ (\hat{x}, w, (\hat{x} \cdot w)_{[1, l]}) \mid \hat{x} \cdot w \in \Pi_{l+1}(\hat{B}^l) \right\}.
\] (10c)

Then \( \hat{\Sigma}^l \) is realized by \( \hat{Q}^l = (\hat{X}^l, W, \hat{B}^l, \hat{X}^l_0) \).

Summarizing the abstraction procedure outlined above, constructing the finite state abstraction \( \hat{Q}^l \) in Prop. 1 using SAICA only requires knowledge about the set \( \Pi_{l+1}(B(Q)) \). However, if \( Q \) is available, we can construct \( \hat{Q}^l \) from \( Q \) directly without taking a “detour” via constructing \( B(Q) \) first, as shown in the following section.

As before, \( B(\hat{Q}) \) denotes the extension of \( \pi_W(B_f(\hat{Q})) \) to \( \mathbb{Z} \) as discussed in Sec. [III].
To obtain a transition relation immediately where \( k \) corresponds to the third case. Based on (11) the set of compatible states are defined in Def. 6. Using this intuition it is easy to see that (13) are obtained from two trajectories (\( \omega, \xi \), \( \omega', \xi' \)\( B_S(Q) \) passing \( x \) at time \( k \in \mathbb{N}_0 \) and \( k' \in \mathbb{N}_0 \), respectively, i.e., \( \xi(k) = \xi'(k') = x \) using (12). During this restriction of \( \omega \) (resp. \( \omega' \)) to \( \xi \) (resp. \( \xi' \)) absolute time information is disregarded (see Sec. II-A), implying \( |t-m,l-1| = \omega[|k,k+m-1|] \) and \( |t'-m,l-1| = \omega'[|k',k'+m-1|] \). Therefore, \( Q \) is future unique w.r.t. \( \mathcal{I}_m \) if for all states \( x \in \mathcal{X} \) all trajectories passing \( x \) have the same \( m \)-long future of external symbols, i.e. \( \omega[|k,k+m-1|] = \omega'[|k',k'+m-1|] \). Using this intuition it is easy to see that \( Q \) is always future unique w.r.t. \( \mathcal{I}_0 = [-l,-1] \), as this interval has no future.

We now proceed by constructing \( m \) different finite state machines using the outlined correspondence between \( X \) and \( \Pi_l(B(Q)) \).

Definition 7: Given (6) and (11), define

\[
\hat{X}_m := \left\{ \hat{x} \in \mathcal{X} \mid \exists x \in X_0 \cdot \xi \in E^{\mathcal{I}_m} \right\},
\]

(14a)

and

\[
\hat{X}_0 := \left\{ \hat{x} \in X \cdot \xi \in E^{\mathcal{I}_m} \right\},
\]

(14b)

Then \( \hat{Q}^m = (\hat{X}_m, \hat{U}, \hat{X}_0, \hat{X}_0) \) is called the \( \mathcal{I}_m \),abstract state machine of \( Q \).

A. Some State Machine Realizations of SAICA

Recall from Prop. 1 that the set of finite external sequences of length \( l \), given by \( \Pi_l(B) = \Pi_l(B(Q)) \) (from (9b)) is a suitable candidate for the state space \( \hat{X} \) of the finite state abstraction \( Q \) we are seeking, as \( \Pi_l(B(Q)) \) is finite if \( W \) is finite.
The construction of the abstract state machines in Def. 7 can be interpreted as follows. Using $Q^m_l$ instead of $\hat{X}^m_l \doteq \{o\} \cup \Pi_l(B(Q))$ ensures that $\hat{X}^m_l$ is live and reachable, which is purely cosmetic but allows to simplify subsequent proofs. The last line in the conjunction of (14a) simply says that we have a transition in $\hat{X}^m_l$ from $\hat{x}$ to $\hat{x}'$ if there is a transition in $Q$ between any two states compatible with $\hat{x}$ and $\hat{x}'$, respectively. However, the first two lines in the conjunction of (14a) additionally ensure that $\hat{x}$ and $\hat{x}'$ obey the rules of the domino game, i.e.,

$$\hat{x}|_{[1, l-1]} = \hat{x}'|_{[0, l-2]}$$

as depicted in Fig. 1 (left) and the current external symbol $w = \pi_W(u, y)$ is contained in either $\hat{x}$ or $\hat{x}'$ or both, at the position corresponding to the current time point, i.e.,

$$w = \hat{x}'(l - 1) \text{ if } m = 0,$$
$$w = \hat{x}(l - m) = \hat{x}'(l - 1 - m) \text{ if } 0 < m < l \text{ and}$$
$$w = \hat{x}(0) \text{ if } m = l.$$

As we are interested in state machine realizations of SAICA, we now prove that $\hat{X}^m_l$ actually realizes $\hat{X}^l$ for all choices of $l$ and $m$.

**Theorem 1:** Given (6) and (11), let $\hat{X}^m_l$ be defined as in Def. 7 and let $\hat{X}^l = (\mathbb{N}_0, W, B^l)$ be the unique SAICA of $\Sigma = (\mathbb{N}_0, W, B(Q))$. Then $\hat{X}^m_l$ realizes $\hat{X}^l$.

*Proof:* See Appendix I-A

As an intuitive consequence of Thm. 1, choosing $m = 0$ and the full external symbol set $W = U \times Y$ when construction $\hat{X}^m_l$ in Def. 7 yields the standard realization $\hat{X}^0_l$ of SAICA in Prop. 1.

**Theorem 2:** Given (6) and (11) with $W = U \times Y$, let $\hat{X}^l$ and $\hat{X}^m_l$ be defined as in Prop. 1 and Def. 7 respectively. Then $\hat{X}^l = \hat{X}^0_l$.

*Proof:* See Appendix I-B

### B. Ordering $\hat{X}^m_l$ based on Simulation Relations

Before we discuss the ordering between abstract state machines based on changing $l$ and $m$, we show under which conditions the obtained abstraction $\hat{X}^m_l$ simulates the original state machine $Q$ and when both state machines are bisimilar. This investigation is interesting for the comparison to QBA, as the latter always simulates original state machine $Q$. Furthermore, the framework of QBA allows to construct a bisimilar abstraction whenever it exists. Hence, it is interesting to know if the latter is also true for the setting of SAICA.

The investigation of similarity between $\hat{X}^m_l$ and $Q$ requires the construction of a relation between the original state space $X$ and the abstract state space $\hat{X}^m_l$. As $\hat{X}^m_l$ defines a cover for $X$ s.t. each cell is given by all states $x$ corresponding to a string $\zeta \in \hat{X}^m_l$ via $E^m_l$, the latter is a natural choice for a relation between $X$ and $\hat{X}^m_l$.

Recall from Thm. 1 that the behavior of $Q$ and $\hat{X}^m_l$ coincide, if $B(Q)$ is asynchronously $l$-complete. Behavioral equivalence is always necessary for a relation $R$ to be a bisimulation relation, but usually not sufficient. We therefore introduce a stronger condition, called *state-based asynchronous* $l$-completeness, to serve the latter purpose.

**Definition 8:** Given (6), if

$$\forall x \in X, \zeta \in \Pi_{l+1}(B(Q)) : \zeta|_{[0, l-1]} \in E[l-m, m-1](x) \Rightarrow \zeta \in E[l-m, m](x)$$

(15)

$Q$ is *state-based asynchronous* $l$-complete w.r.t. $T^l_m$.

*Remark 1:* Recall from the beginning of this section that the dynamical system $\Sigma = (\mathbb{N}_0, W, B(Q))$ is asynchronously $l$-complete, as defined in [8, Def.6], if $B(Q)$ is the largest behavior satisfying (9) itself. Intuitively, the latter is true if for all $\zeta \in \Pi_{l+1}(B(Q))$ there exists an $x \in X$ s.t. the second part of (15) holds. Therefore, asynchronous $l$-completeness of $\Sigma$ is always implied by (15), but not vice-versa.

**Theorem 3:** Given (6), (11) and $\hat{X}^m_l$ as in Def. 7 let

$$R = \{(x, \hat{x}) \in X \times \hat{X}^m_l \mid \hat{x} \in E^m_l(x)\}.$$  

Then it holds that

(i) $R \in \mathcal{R}_{U \times Y}(Q, \hat{X}^l_m) \Leftrightarrow Q$ is future unique w.r.t. $T^l_m$ and

(ii) $R^{-1} \in \mathcal{R}_W(\hat{X}^m_l, Q) \Leftrightarrow$

$Q$ is state-based async. $l$-complete w.r.t. $T^l_m$
Theorem 5: Interestingly, we will show that increasing Rem. 1).

Remark 2: Theorem 4: Then it holds that \( Q \) \( \Rightarrow \) \( \hat{Q}^{l+} \) can only simulate \( Q \) iff in every state \( x \in X \) all outgoing transitions agree on this \( w \), i.e., \( Q \) is “output deterministic” w.r.t. \( W \). For \( m > 1 \) applying this reasoning iteratively gives the (rather restrictive) condition of future uniqueness of \( Q \).

As the outlined problems are absent for \( m = 0 \) (as \( Q \) is always future unique w.r.t. \( I_0 \)), \( \hat{Q}^{l_0} \), which we know to coincide with the original realization \( Q \) of SAICA for \( W = U \times Y \), always simulates \( Q \).

Corollary 1: Given (6), (11) and \( \hat{Q}^{l_m} \) as in Def. 7, it holds that \( Q \Rightarrow_{U \times Y} \hat{Q}^{l_m} \).

Remark 2: In the context of SAICA a state machine \( \hat{Q}^{l} \) was introduced in [6] whose state at time \( k \) represents the string of external symbols from time \( k-l+1 \) to time \( k \), i.e., from the interval \( k+I_l \). While the state sets of \( \hat{Q}^{l} \) and \( \hat{Q}^{l_0} \) coincide, their transition structure slightly differs. This is a consequence of the fact that \( \hat{Q}^{l} \) was intended to serve as a set-valued observer for the states of \( Q \).

Recalling the domino game, we know that using longer dominos (i.e., increasing \( l \)) gives less freedom in composing them and therefore yields a tighter abstraction. This intuition carries over to the state space realizations of \( \hat{l} \), inducing an ordering in terms of simulation relations.

Theorem 4: Given (6), (11) and \( \hat{Q}^{l_m} \) as in Def. 7 let
\[
\mathcal{R} = \left\{ (\hat{x}_{l+1}, \hat{x}_l) \in \hat{X}^{l+1} \times \hat{X}^l \mid \hat{x}_l = \hat{x}_{l+1}^{l_1} \right\}.
\]
(17)
Then it holds that
(i) \( \mathcal{R} \in \mathcal{R}_W (\hat{Q}^{l_m}, \hat{Q}^{l_0}) \)
(ii) \( \mathcal{R}^{-1} \in \mathcal{R}_W (\hat{Q}^{l_m}, \hat{Q}^{l_0}) \) \( \Rightarrow \) \( \hat{B}^l = \hat{B}^{l+1} \)

Proof: See Appendix [D].

Thm. 4(ii) implies that the accuracy of the abstraction cannot be increased by increasing \( l > r \) if \( B(Q) \) is asynchronously \( r \)-complete and \( m \) is fixed, e.g. \( m = 0 \). Therefore, the standard realization \( \hat{Q}^l \) for SAICA might never result in a bisimilar abstraction of \( Q \), no matter how large \( l \) is chosen, even if \( \Sigma = (\mathbb{N}_0, W, B(Q)) \) is asynchronously \( r \)-complete. This is due to the fact, that state-based asynchronous \( l \)-completeness of \( Q \) is not implied by asynchronous \( l \)-completeness of \( \Sigma \) (see Rem. 1).

Interestingly, we will show that increasing \( m \), i.e., shifting the interval into the future results in a tighter abstraction w.r.t. simulation relations, i.e. allows to increase the precision of \( \hat{Q}^{l_m} \) for \( l \geq r \) even if \( \Sigma \) is \( r \)-complete.

Theorem 5: Given (6), (11) and \( \hat{Q}^{l_m} \) as in Def. 7 with \( m < l \), let
\[
\mathcal{R} = \left\{ (\hat{x}_{m+1}, \hat{x}_m) \in \hat{X}^{l_{m+1}} \times \hat{X}^l \mid \begin{array}{l}
\begin{cases}
(\hat{x}_{m+1}, \hat{x}_m) = \hat{x}_m \in \hat{X}^l \setminus \hat{X}^{l_{m+1}} \\
\exists x \in X, (\hat{x}_{m+1} \in E_x^{l_{m+1}}(x), \hat{x}_m \in E_x^l(x))
\end{cases}
\end{array} \right\}.
\]
(18)
Then it holds that
(i) \( \mathcal{R} \in \mathcal{R}_W (\hat{Q}^{l_{m+1}}, \hat{Q}^{l_m}) \)
(ii) \( \mathcal{R}^{-1} \in \mathcal{R}_W (\hat{Q}^{l_{m+1}}, \hat{Q}^{l_m}) \) \( \Rightarrow \) \( \hat{Q}^{l_{m+1}} \) is future unique w.r.t. \( I_{m+1} \)
\( \hat{Q}^{l_m} \) is state-based async. \( l \)-complete w.r.t. \( I_m \)

Proof: See Appendix [E].

It is important to note that future uniqueness and state-based asynchronous \( l \)-completeness are incomparable properties, i.e., non is implied by the other. Therefore, there exist situations where \( \hat{Q}^{l_m} \) with \( m > 0 \) simulates \( Q \) (i.e. \( Q \) is future unique w.r.t. \( I_{m+1} \)) and \( \hat{Q}^{l_m} \) is tighter than \( \hat{Q}^{l_0} \) in terms of simulation relations. However, if \( Q \) is both future unique and state-based asynchronously \( l \)-complete w.r.t. a particular interval \( I_m \), Thm. 5 implies that increasing \( l > r \) and \( m > n \) will not result in a tighter abstraction. Moreover, this is not necessary anyway, as Thm. 5 implies that in this case \( \hat{Q}^{l_0} \) is bisimilar to \( Q \). Hence, by using the additional parameter \( m \) when realizing SAICA, we are able to obtain a bisimilar abstraction whenever it exists.
IV. QUOTIENT-BASED ABSTRACTIONS (QBA)

The idea of quotient based abstractions (QBA) is to partition the state space $X$ into a finite set of equivalence classes $\hat{Y}$ which are used as discrete outputs of the original system as well as states of the abstraction. The set $\hat{Y}$ is usually constructed iteratively, by choosing an initial partition $\Phi^0$ and using the refinement algorithm in [2] which terminates if the partition allows to construct a quotient state machine $Q^Y$ which is bisimilar to $Q$.

To draw the connection to the setting of SAICA, we assume that the original system is modelled by (6a) with finite, predefined output set $W = Y$, and initialize the re-partitioning algorithm with the partition induced by $H^{-1}_\delta$.

Definition 9: Given (6) and $l \in \mathbb{N}$, then \[ \Phi^l := \{ H^{-1}_\delta(V) | V \in 2^\hat{Y} \} \] (19a)
and \[ \Phi^{l+1} := \circ_{Z \in \Phi^l} \Phi^l \] (19b)
s.t. \[ \Phi^l := \{ Z' \cap T^{-1}_{\delta}(Z) | Z' \in \Phi^{l-1} \} \cup \{ Z' \setminus T^{-1}_{\delta}(Z) | Z' \in \Phi^{l-1} \} . \] (19c)

iteratively defines the $l^{th}$ partition $\Phi^l$ of $X$ w.r.t. $Y$.

Proposition 2: Given (6), $W = Y$ and $\Phi^l$ as in Def. 9 it holds that \[ \Phi^l = \{ (E_{Z}^{l})^{-1}(V) | V \in 2^\hat{Y} \} . \] (20)

Proof: See Appendix II-A.

Observe that Prop. 2 implies that the equivalence classes of $\Phi^l$ are given by all sets $V \in 2^\hat{Y}$ of $l$-long dominos which are consistent with the behavior of $Q$ and the map $E_{Z}^{l}$ is the natural projection map of $\Phi^l$ taking a state $x \in X$ to its (unique) equivalence class.

Constructing quotient state machines $\hat{Q}^{v}$ from every obtained partition $\Phi^l$, results in a chain of abstractions with increasing precision, similar to increasing $l$ when constructing SAICA. Precisely following the construction of QBA one would first construct an output deterministic version of $Q$ with output space $\hat{Y}^l = 2^{\hat{Y}}$ for every $l$ and its QBA $\hat{Q}^{v}$, also having $\hat{Y}^l$ as its output space. However, to formally compare the resulting state machines to the realizations of SAICA using simulation relations or behavioral inclusion requires identical output spaces. We therefore slightly change the definition of QBA to output values in the set $Y$ rather than in $\hat{Y}$.

Definition 10: Given (6) and $\hat{Y}^l = 2^{\hat{Y}}$, define
\[ \hat{X}^{l,v} = \{ \hat{y} \mid \exists x \in X . \hat{y} = E^{l}_{Z}(x) \} \] (21a)
\[ \hat{X}^{0,v} = \{ \hat{y} \mid \exists x \in X_0 . \hat{y} = E^{l}_{Z}(x) \} , \] and
\[ \hat{\delta}^{l,v} = \left\{ (\hat{x}, u, y, \hat{x'}) \mid \exists x, x' \in X . \left\{ \begin{array}{l} \hat{x} = E^{l}_{Z}(x) \\ \hat{x'} = E^{l}_{Z}(x') \\ \delta(x, u, y, x') \rightarrow (\hat{x}, \hat{x'}) \end{array} \right\} \right\} \] (21c)
Then $\hat{Q}^{v} = (\hat{X}^{l,v}, U \times Y, \hat{\delta}^{l,v}, \hat{X}^{0,v})$ is the quotient state machine of $Q$.

Changing the definition of the output space of QBA from $\hat{Y}$ to $Y$, allows us to prove (bi)-similarity of $Q$ and $\hat{Q}^{v}$ using the usual relation as, e.g., in [10], Thm. 4.18.

Theorem 6: Given (6) and $\hat{Q}^{v}$ as in Def. 10 let \[ \mathcal{R} = \{ (x, \hat{x}) \in (X \times \hat{X}^{l,v}) | \hat{x} = E^{l}_{Z}(x) \} \] (22)
be a relation. Then
(i) $\mathcal{R} \in \mathcal{R}_{l \times Y}(Q, \hat{Q}^{v})$ and
(ii) $\mathcal{R}^{-1} \in \mathcal{R}_{Y}(\hat{Q}^{v}, Q) \Rightarrow \Phi^l$ is a fixed-point of (19).

Proof: See Appendix II-B.

It is easy to see, that increasing $l$ gives a tighter abstraction as long as no fixed-point of (19) is reached, giving the following ordering of abstractions $\hat{Q}^{v}$ w.r.t. $l \in \mathbb{N}_0$.

Theorem 7: Given (6) and $\hat{Q}^{v}$ as in Def. 10 let
\[ \mathcal{R} = \{ (\hat{x}_{l+1}, \hat{x}_{l}) \in (\hat{X}^{l+1,v} \times \hat{X}^{l,v}) | \hat{x}_{l} = \hat{x}_{l+1} | [0, l-1] \} \] (23)

\[ 4 \] In (19b) the operator $\circ_{a \in A} f_a$ composes all functions $f_a$ with $a \in A$ in any order.
be a relation. Then
(i) \( R \in \mathcal{R}_Y (\hat{Q}^{i\nu}, \hat{Q}^{i\nu}) \) and
(ii) \( R^{-1} \in \mathcal{R}_Y (\hat{Q}^{j\nu}, \hat{Q}^{j\nu}) \) if \( \Phi^i \) is a fixed-point of \( \{19\} \).

Proof: See Appendix III-C

It is interesting to note that, whenever a fixed-point \( \Phi^r \) exists, the tightest possible abstraction \( \hat{Q}^{r\nu} \) will be bisimilar to \( Q \).

V. COMPARISON

When it comes to comparing QBA and SA/CA there are two interesting questions to be asked.

(i) Does \( \hat{Q}^{i\nu} \) realize the unique SA/CA \( \hat{\Sigma}' = (N_0, Y, B') \) of \( \Sigma = (N_0, Y, B(\hat{Q})) \)?

(ii) Can we order of the realizability \( \hat{Q}^{i\nu} \) for \( W = Y \) if \( \hat{Q}^{j\nu} \) and \( \hat{Q}^{j\nu} \) in terms of simulation relations for \( W = Y \)?

Unfortunately, we will see that non of the above statements is true in general. We will therefore derive necessary and sufficient conditions on the structure of \( Q \) for these statements to hold. We start by giving the only comparing result that holds in general.

**Theorem 8:** Given \( \Theta \) and \( \Theta \) s.t. \( W = Y \) and \( \hat{Q}^{i\nu} \) as in Def. [10] it holds that \( B(\hat{Q}^{i\nu}) \subseteq B \).

Proof: See Appendix III-A

As behavioral inclusion is a necessary condition for the existence of a simulation relation from \( \hat{Q}^{i\nu} \) to \( \hat{Q}^{j\nu} \) (where the latter behavior is given by \( B \) from Thm. [1] the natural next step is to try to find such a relation. However, thinking back to the results in Thm. [5](i) and Thm. [6](i) there is not much hope for success, as the existence of such a relation would imply that we can also find a simulation relation from \( Q \) to \( \hat{Q}^{j\nu} \) without the need for future-uniqueness of \( Q \) w.r.t. \( \hat{T}^l \). Not surprisingly, the latter condition will turn out to be necessary and sufficient for the naturally chosen relation from \( \hat{Q}^{i\nu} \) to \( \hat{Q}^{j\nu} \) to be a simulation relation.

For the inverse relation to be a simulation relation from \( \hat{Q}^{j\nu} \) to \( \hat{Q}^{i\nu} \) the following property will turn out to be necessary and sufficient.

**Definition 11:** Given \( \Theta \) and \( \Theta \) s.t. \( W = Y \), if

\[
\forall \zeta \in \Pi_{l+1}(B(\hat{Q})), \hat{y} \in \hat{X}^l, \quad \zeta_{|[0,l-1]} \in \hat{y} \Rightarrow \exists x \in (E^{l})^{-1}(\hat{y}) \wedge \zeta_{l} \in E^{l}[0,l](x) \tag{24}
\]

\( Q \) is said to be **domino consistent**.

Intuitively, domino consistency of \( Q \) implies that whenever a string \( \zeta \) is part of an abstract state \( \hat{y} \), i.e., \( \zeta \in \hat{y} \), any domino \( \zeta' \in \Pi_{l+1}(B(\hat{Q})) \) that can be attached to \( \zeta \) in the domino game, i.e., \( \zeta'_{|[0,l-1]} = \zeta \), can be attached for this particular abstract state \( \hat{y} \), i.e., there exists a transition from \( \hat{y} \) to \( \hat{y} \) s.t. \( \zeta'_{[1,l]} \in \hat{y} \). As \( \hat{Q}^{j\nu} \) can do all moves of the domino game, it becomes intuitively clear why this condition in Def. [11] is needed to prove that \( \hat{Q}^{i\nu} \) can simulate \( \hat{Q}^{j\nu} \).

**Theorem 9:** Given \( \Theta \) s.t. \( W = Y \) and \( \hat{Q}^{j\nu} \) and \( \hat{Q}^{i\nu} \) as in Def. [7] and Def. [10] respectively, let

\[
R = \left\{ \langle \zeta, \hat{y} \rangle \in \hat{X}^l \times \hat{X}^{1\nu} \mid \zeta \in \hat{y} \right\} \tag{25}
\]

Then
(i) \( R \in \mathcal{R}_Y (\hat{Q}^{j\nu}, \hat{Q}^{i\nu}) \) if \( Q \) is domino consistent and
(ii) \( R^{-1} \in \mathcal{R}_Y (\hat{Q}^{i\nu}, \hat{Q}^{j\nu}) \) if \( Q \) is future unique w.r.t. \( \hat{T}^l \).

Proof: See Appendix III-B

Combining the results from Thm. [8] and Thm. [9](i) we have the following answer to our first question.

**Corollary 2:** Given \( \Theta \) and \( \Theta \) s.t. \( W = Y \) and \( \hat{Q}^{i\nu} \) as in Def. [10], \( \hat{Q}^{i\nu} \) realizes \( \hat{\Sigma}' = (N_0, Y, B') \) if \( Q \) is domino consistent.工作人员

Even though, we have only given a sufficient condition in Thm. [8] it should be noted that this condition is “almost” necessary in the following sense. The only reason for domino consistency to not be necessary for behavioral equivalence is that for any string \( \nu \in B^l \) domino consistency is only required for all cells this string passes through. As in general not every string passes through all cells that contain any of is \( l \)-long pieces, domino consistency is only necessary for the cells which are actually passed, i.e., for “almost all” cells.

To wrap up the comparison, it is interesting to note that future uniqueness of \( Q \) w.r.t. \( \hat{T}^l \) implies domino consistency and therefore also bisimilarity of \( \hat{Q}^{j\nu} \) and \( \hat{Q}^{i\nu} \) (See Lem. [7] in Appendix II for a formal proof of this statement). However, the inverse is not true, i.e., domino consistency is a weaker condition. Hence, \( \hat{Q}^{j\nu} \) might actually be a tighter abstraction than \( \hat{Q}^{i\nu} \) if \( Q \) is not future unique w.r.t. \( \hat{T}^l \). However, recall from Thm. [3] that in this case, \( \hat{Q}^{j\nu} \) does not simulate \( Q \), i.e., might be “too tight” to suitably abstract \( Q \).

However, if \( Q \) is future unique w.r.t. \( \hat{T}^l \), \( \hat{Q}^{j\nu} \) and \( \hat{Q}^{i\nu} \) are actually equivalent up to a trivial renaming of states.
In this paper we have compared finite state machine abstractions resulting from SA of the “feedback refinement relations” provided in [7], we postpone this idea to future work.

It would therefore be interesting to investigate, which conditions on generally require full state information to be able to control the system based on its abstraction. This motivated the introduction of “feedback refinement relations” in [7], allowing for abstraction based control without the latter.

Therefore, we want to conclude our comparison by investigating the connection between of “feedback refinement relations” in [7], allowing for abstraction based control without the latter. This investigation would require a non-trivial extension of the “feedback refinement relations” provided in [7], we postpone this idea to future work.

Corollary 3: Given the premises of Thm. CA s.t. Q is future unique w.r.t. \( I^l \), let

\[
\mathcal{R} = \left\{ (\zeta, \hat{y}) \in \hat{X}^{I^l} \times \hat{X}^{\Sigma^l} \mid \hat{y} = \{ \zeta \} \right\}
\]

be a relation. Then it holds that \( \mathcal{R} \in \mathcal{R}_Y(\hat{Q}^{I^l}, \hat{Q}^{\Sigma^l}) \) and \( \mathcal{R}^{-1} \in \mathcal{R}_Y(\hat{Q}^{\Sigma^l}, \hat{Q}^{I^l}) \).

Corollary 4: Given the premises of Thm. CA s.t. Q is future unique w.r.t. \( I^l \), then \( \hat{Q}^{I^l} \cong_Y \hat{Q}^{I^l} \). Even though future uniqueness of Q is a very strict requirement, it holds whenever Q is output deterministic and \( l = 1 \) is chosen. In particular, taking the viewpoint of QBA and assuming that Y can be arbitrarily chosen implies that we can always run the refinement algorithm in Def. CA first, before applying QBA and SA/CA. In this case, Q is obviously output deterministic and choosing \( l = 1 \) is sufficient, leading to bisimilar state machines \( \hat{Q}^{I^l} \) and \( \hat{Q}^{\Sigma^l} \). However, it should be kept in mind that in this scenario the standard QBA \( Q^{I^l} \) is usually tighter then the standard realization \( \hat{Q}^{I^l} \) of SA/CA in terms of similarity.

Remark 3: It is interesting to note that for all abstractions discussed in this paper, the relations \( \mathcal{R} \) constructed in Thm. 3 and Thm. 4 are alternating simulation relations from the abstraction to Q in the sense of [10, Def.4.19], i.e., all constructed abstractions are suitable for controller synthesis. However, it was recently shown in [7] that alternating simulation relations generally require full state information to be able to control the system based on its abstraction. This motivated the introduction of “feedback refinement relations” in [7], allowing for abstraction based control without the latter.

It would therefore be interesting to investigate, which conditions on Q allow for control based on predefined input and output symbols using the abstractions constructed in this paper. As this investigation would require a non-trivial extension of the “feedback refinement relations” provided in [7], we postpone this idea to future work.

VI. Conclusion

In this paper we have compared finite state machine abstractions resulting from SA/CA and QBA. For this purpose we have introduced a new parameter \( m \in [0, l] \) to realize SA/CA by different state machines. We have shown that the choice \( m = 0 \) corresponds to relating states in the original state machine \( Q \) to their strict \( l \)-long past of external symbols, reproducing the standard realization of SA/CA. On the other hand, choosing \( m = l \) corresponds to relating states in the original state machine \( Q \) to their \( l \)-long future of external symbols. We have shown that this construction of realizations for SA/CA is closely related to the construction of QBA, if the latter is obtained from a partition resulting from \( l \) steps of the usual repartitioning algorithm.

Even if the latter observation renders both methods conceptually similar, we could show that they are generally incomparable. Only in the special case where the original system is future unique both abstractions are identical up to a renaming of states. While the introduction of the parameter \( m \) was initially motivated by the comparison of SA/CA and QBA, it could also be shown that this allows to obtain a bisimilar abstraction of \( Q \) whenever it exists. This was not possible for the standard realization of SA/CA. It remains an open question how this parameter actually influences the usage of the obtained abstractions for control purposes.

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APPENDIX I
PROOFS FOR SAICA

In this appendix we provide detailed proofs for the theorems in Sec. III. These proofs make substantial use of the construction of $\hat{B}$ from the domino-game discussed in the beginning of Sec. III. Therefore, a definition of concatenation of infinite strings is needed.

**Definition 12:** Given two signals $\omega_1, \omega_2 \in (W)^{\mathbb{N}_0}$ and two time instants $t_1, t_2 \in \mathbb{N}_0$, their concatenation $\omega_3 = \omega_1 \land_{t_2} \omega_2$ is defined by

$$\forall t \in \mathbb{N}_0 : \omega_3(t) = \begin{cases} \omega_1(t), & t < t_1 \\ \omega_2(t - t_1 + t_2), & t \geq t_1 \end{cases} \quad (27)$$

This concatenation can be used to state that for the full behavior $B_f(Q)$ of a state machine $Q$ as defined in (4) the state property holds, i.e., the current state is the only past information required to decide on the future evolution of the full behavior.

**Lemma 1 (e.g. [8], Prop.2):** Let $Q$ be a state machine with full behavior $B_f(Q)$. Then

$$\forall (\mu, \nu, \xi), (\mu', \nu', \xi') \in B_f(Q), k, k' \in \mathbb{N}_0 : \xi(k) = \xi'(k') \Rightarrow (\mu, \nu, \xi) \land_{k,k'} (\mu', \nu', \xi') \in B_f(Q).$$

To simplify the subsequent proofs we now translate the conditions for a transition in $\hat{Q}^{\mathbb{Z}_+}$ into conditions of the domino-game.

**Lemma 2:** Let $Q$ and $\hat{Q}^{\mathbb{Z}_+}$ as in Def. 7. Then

$$((\check{x}, u, y, \check{x}') \in \hat{Q}^{\mathbb{Z}_+} \Rightarrow \begin{array}{l} \check{x} \in [0,t-m-1][\pi_W(u,y)]_{t-m-1} \\
(\check{x} \land \check{x'}) = \check{x} \in [0,t-m-1][\pi_W(u,y)]_{t-m-1} \end{array} \quad (29a)$$

$$\Leftrightarrow \exists \zeta \in \Pi_{t+1}(B(Q)) : \begin{array}{l} \zeta[0,t-1] = \check{x} \\
(\land \zeta[1,t] = \check{x}' \\
(\pi_W(u,y) = \zeta(t-m)) \quad (29b) \end{array}$$

**Proof:** "\(\Rightarrow\):

- Observe from (14c) that $((\check{x}, u, y, \check{x}') \in \hat{Q}^{\mathbb{Z}_+}$ iff the first two lines of the conjunction in (14c) are fulfilled and there exist $x, x' \in X$ s.t.

$$\exists (\omega, \xi) \in \hat{Q}, k : \begin{array}{l} \xi(k) = x \\
\land \check{x} = \omega[k+m-l,k+m+1] \end{array} \quad (30a)$$

$$\exists (\omega', \xi') \in \hat{Q}, k' : \begin{array}{l} \xi'(k') = x' \\
\land \check{x}' = \omega'[k'+m-l,k'+m+1] \end{array} \quad (30b)$$

$$\exists (\omega'', \xi'') \in \hat{Q}, k'' : \begin{array}{l} \xi''(k'') = x \\
\land \omega''(k'') = w \\
\land \xi''(k'' + 1) = x' \end{array} \quad (30c)$$

where $w = \pi_W(u,y)$ and $B_S := \pi_W \times X (B_f(Q))$. Here (30a) and (30b) follow from (12) and (30c) follows from (4).

- Now observe from (30) that $\xi(k) = x = \xi'(k')$ and $\xi'(k') = x' = \xi''(k'' + 1)$. Using (28) we therefore obtain

$$(\check{\omega}, \check{\xi}) = (\omega, \xi) \land_{k,k''} (\omega'', \xi'') \land_{k',k''+1} (\omega', \xi') \in \hat{Q} \quad (30d)$$

giving

$$\check{\omega}[k+m-l,k+m] = \omega[k+m-l,k+1] \cdot \omega''[k',k'+m+1] = \check{x}[0,t-m-1] \cdot \check{x}'[t-m,t-1],$$

and therefore, as $\check{\omega}[k+m-l,k+m] \in \hat{Q}(Q)[k+m-l,k+m]$ (using (8)) we have $\check{x}[0,t-m-1] \cdot \pi_W(u,y) \cdot \check{x}'[t-m,t-1] \in \Pi_{t+1}(B(Q)).$

- Now let $\zeta = \check{\omega}[k+m-l,k+m] \in \Pi_{t+1}(B(Q))$ and observe that $w = \zeta(t-l)$. With this choice of $\zeta$ the first two lines of the conjunction in (14c) immediately imply $\zeta[0,t-1] = \check{x}$ and $\zeta[1,t] = \check{x}'$.

"\(\Leftarrow\):
• Pick $\zeta \in \Pi_{t+1}(B(Q))$ and $u, y, \widehat{x}$ and $\widehat{x}'$ s.t. the right side of (29b) holds.

• It is easy to see that the first two lines of the conjunction in (29a) and (14c) hold with this choice and $\zeta = \widehat{x}|_{[0, t - m - 1]} \cdot w \cdot \widehat{x}'|_{[t - m - 1]} \in \Pi_{t+1}(B(Q))$.

• Now using (3) there exist $(\widehat{\omega}, \widehat{\xi})$ and $\widehat{k} \in \mathbb{N}_0$ s.t. $\zeta = \widehat{\omega}|_{[k + m - l, k + m - 1]}$.

• Now we can choose all signals in (30a) and (30b) equivalent to $(\widehat{\omega}, \widehat{\xi})$ and $x = \widehat{\xi}(\widehat{k})$ as well as $x' = \widehat{\xi}(\widehat{k} + 1)$, giving $\widehat{x} \in \mathbb{E}_m^{\mathbb{Z}_m}(x)$, $\widehat{x}' \in \mathbb{E}_m^{\mathbb{Z}_m}(x)$ and $(x, u, y, x') \in \delta$, hence the last line of the conjunction in (14c) holds.

A. Proof of Thm. 7

Using Def. 5 we prove both directions of the behavior inclusion separately:

1.) Show $\mathbb{E}_k \subseteq \pi_W(B_f(\mathbb{Q}^{\mathbb{Z}_m}))$:

Pick $\widehat{\omega} \in \mathbb{E}_k$, $\widehat{\mu}, \widehat{\nu}$ s.t. $\pi_W(\widehat{\mu}, \widehat{\nu}) = \widehat{\omega}|_{[0, \infty)}$ and $\widehat{\zeta}$ s.t. $\forall k \in \mathbb{N}_0 : \widehat{\zeta}(k) = \widehat{\omega}|_{[k - l + m, k + m - 1]}$. Then we prove both lines in (4) separately:

• Show $\widehat{\zeta}(0) \in \mathbb{X}_0^{\mathbb{Z}_m}$:

– The first line of the conjunction in (9) implies $\omega|_{[t - l, -1]} \in \{\diamond\}$ while the second line implies $\widehat{\omega}|_{[t - l, 0]} \in \Pi_{t+1}(B(Q))$, hence $\widehat{\omega}(0) \neq \diamond$, and $\widehat{\omega}|_{[m, m - l]} \in \Pi_{t+1}(B(Q))$.

– As we know that diamonds can only be appended to the left, we know that $\omega|_{[m, l - t]} \in \{\diamond\}^{l-m}$ while $\widehat{\omega}(r) \neq \diamond$ for all $r < m$.

– Using this and $\widehat{\omega}|_{[m, m - l]} \in \Pi_{t+1}(B(Q))$ in (3), there exists $(\omega', \xi') \in \pi_{W \times X}(B_f(Q))$ s.t. $\omega'|_{[m, m - l]} = \widehat{\omega}|_{[m, m - l]}$.

– Using (12) this implies that $\widehat{\omega}|_{[m, m - l]} \in \mathbb{E}_m^{\mathbb{Z}_m}(\xi'(0))$ with $\xi'(0) \in X_0$, hence $\widehat{\xi}(0) \in \mathbb{X}_0^{\mathbb{Z}_m}$ (from (13b)).

• Show $\forall k \in \mathbb{N}_0 : (\widehat{\xi}(k), \widehat{\mu}(k), \widehat{\nu}(k), \widehat{\xi}(k + 1)) \in \mathbb{X}^{\mathbb{Z}_m}$:

– Recall from (9) that $\forall k \in \mathbb{N}_0 : \omega|_{[k - l + m, k + m - 1]} \in \Pi_{t+1}(B(Q))$.

– Using (3) and (4) this implies

$$\exists (\omega', \xi') \in \mathbb{O}^{\mathbb{B}_S, k'}, \omega'|_{[k' + m - l, k + m]} = \omega|_{[k + m - l, k + m]}.$$ 

Now observe from the choice of $\widehat{\xi}$ and (12) that

$$\widehat{\xi}(k) = \omega'|_{[k' + m - l, k + m]} \in \mathbb{E}_m^{\mathbb{Z}_m}(\xi'(k')) \ni \widehat{\xi}(k + 1) = \omega'|_{[k' + m - l, k + m]} \in \mathbb{E}_m^{\mathbb{Z}_m}(\xi'(k' + 1)),$$

and $\pi_W(\widehat{\mu}(k), \widehat{\nu}(k)) = \omega(k) = \omega'(k')$, hence $(\widehat{\xi}(k), \widehat{\mu}(k), \widehat{\nu}(k), \widehat{\xi}(k + 1)) \in \mathbb{X}^{\mathbb{Z}_m}$ (from (14c)).

2.) Show $\pi_W(B_f(\mathbb{Q}^{\mathbb{Z}_m})) \subseteq \mathbb{E}_k$:

Pick $(\widehat{\mu}, \widehat{\nu}, \widehat{\xi}) \in B_f(\mathbb{Q}^{\mathbb{Z}_m})$ and $\widehat{\omega} \in (W \cup \{\diamond\})^{\mathbb{N}_0}$ s.t. $\pi_W(\widehat{\mu}, \widehat{\nu}) = \widehat{\omega}|_{[0, \infty)}$ and $\forall k < 0 : \widehat{\omega}(k) = \diamond$. It remains to show that $\widehat{\omega} \in \mathbb{O}^{\mathbb{B}_k}$, where the first line of (9) holds by construction. We therefore only prove that $\forall k \geq 0 : \widehat{\omega}|_{[m + k, l - m + l + m]} \in \Pi_{t+1}(B(Q))$.

• Observe that $\widehat{\xi}(0) \in X_0$. Using (13b) this implies there exits $x \in X_0$ s.t. $\widehat{\xi}(0) \in \mathbb{E}_m^{\mathbb{Z}_m}(x)$.

• With (12) this implies $\widehat{\xi}(0)|_{[0, l - m - 1]} = \{\diamond\}^{l-m} = \widehat{\omega}|_{[m - l, -1]}$.

• Using Lem. 2 and 4 we know that for all $k \in \mathbb{N}_0$ holds

$$\widehat{\xi}(k + 1)|_{[0, l - m - 1]} = (\widehat{\xi}(k)|_{[0, l - m - 1]} \cdot \widehat{\omega}(k))|_{[1, l - m]} \ni \widehat{\xi}(k + 1)|_{[t - m - l, -2]} = \widehat{\xi}(k)|_{[t - m - l + 1, l - 1]} \ni \widehat{\xi}(k)|_{[0, l - m - 1]} \cdot \widehat{\omega}(k) \cdot \xi(k + 1)|_{[l - m - l - 1]} \in \Pi_{t+1}(B(Q)).$$

Now it can be easily observed that applying these equations iteratively yields $\widehat{\omega}|_{[k + m - l, k + m]} \in \Pi_{t+1}(B(Q))$ as $\widehat{\xi}(0)|_{[0, l - m - 1]} = \widehat{\omega}|_{[m - l, -1]}$ from above.

B. Proof of Thm. 2

• Show $\mathbb{X}^{\mathbb{Z}_m} = \mathbb{X}^{\mathbb{Z}_m}$:

– First observe from the definition of the projection, (12) and (14a) that

$$\mathbb{X}^{\mathbb{Z}_m} = \bigcup_{k \in \mathbb{N}_0} \mathbb{O}^{\mathbb{B}(Q)}|_{[k - l + m, k + m - 1]} \subseteq \Pi_{t}(B(Q))$$
and therefore (using (33)) \( \hat{X}^{T_0} = \Pi_i(B(Q)) \).

- It remains to show that \( \Pi_i(B(Q)) = \bigcup_{k' \in N_0} \circ \hat{B}^i |_{[k-l,k-1]} \). We prove both directions separately:

  "\( \geq \)"

  \[
  \zeta \in \bigcup_{k \in N_0} \circ \hat{B}^i |_{[k-l,k-1]}
  \iff \exists \omega \in \circ \hat{B}^i, k \in N_0 \cdot (\omega |_{[k-l,k-1]} = \zeta)
  \Rightarrow \zeta \in \Pi_i(B(Q))
  \]

  where the last implication follows from (9).

  "\( \leq \)"

  \[
  \zeta \in \Pi_i(B(Q)) \iff \exists \omega \in \circ B(Q), k \in N_0 \cdot (\omega |_{[k-l,k-1]} = \zeta)
  \Rightarrow \exists \omega \in \circ \hat{B}^i, k \in N_0 \cdot (\omega |_{[k-l,k-1]} = \zeta)
  \iff \zeta \in \bigcup_{k \in N_0} \circ \hat{B}^i |_{[k-l,k-1]}
  \]

  (33b)

  where the second last conclusion follows from \( B(Q) \subseteq \hat{B}^i \).

- Show \( \hat{X}^{T_0} = \hat{X}^i_0 \):

  - Observe from (12) that \( \hat{X}^{T_0} = \circ B(Q) |_{[0,-l-1]} \). As \( B(Q) \subseteq \hat{B}^i \) we have \( \hat{X}^{T_0} = \{ \circ \}^i \).

- Show \( \hat{X}^{T_0} = \hat{G}^i \):

  - If follows from Lem. (2) that \( (\hat{x}, u, y, \hat{x}') \in \hat{X}^{T_0} \) iff

    \[
    \hat{x}' = (\hat{x}|_{[0,-l-1]} \cdot w) |_{[1,j]} = (\hat{x} \cdot w) |_{[1,j]} \quad \text{and} \quad \hat{x}|_{[0,-l-1]} \cdot w = \hat{x} \cdot w \in \Pi_{i+1}(B(Q)),
    \]

  - Using the same reasoning as in (33) we obtain \( \Pi_{i+1}(B(Q)) = \bigcup_{k \in N_0} \circ \hat{B}^i |_{[k-l,k]} \), what proves the statement.

C. Proof of Thm. (2)

We show both statements separately.

(i) \( R \in \mathfrak{R}_{U \times Y}(Q, \hat{Q}^{T_0}_m) \leftrightarrow Q \) is future unique w.r.t. \( T_0^m \).

- We first show that (7a) always holds for \( R \):

  - Pick \( x \in X_0 \). Then it follows from (5a) and (12) that there exists \( \zeta \in \Pi_i(B(Q)) \) s.t. \( \zeta \in E^{T_0}_m(x) \).

  - Using (14b) we get \( \zeta \in \hat{X}^{T_0}_m \), which proves the statement.

- Now we need to show that (7b) holds for \( R \) and \( U \times Y \) iff \( Q \) is future unique w.r.t. \( T_0^m \). We show both directions separately:

  "\( \leq \)" : We first assume that \( Q \) is future unique w.r.t. \( T_0^m \) and prove the implication in (7b)

  - Pick \( (x, \hat{x}) \in R^{T_0}_m \), i.e., \( \hat{x} \in E^{T_0}_m(x) \) and \( u, y, x' \) s.t. \( (x, u, y, x') \in \delta \).

  - It follows from (12) and (4) that

    \[
    \exists (\omega, \xi) \in \circ B_S, k \in N_0 \cdot \left( \begin{array}{l}
    \xi(k) = x \\
    \land \hat{x} = \omega |_{[k+m-l,k+m-1]}
    \end{array} \right) \quad \text{and} \quad \exists (\omega', \xi') \in \circ B_S, k' \in N_0 \cdot \left( \begin{array}{l}
    \xi''(k'') = x' \\
    \land \omega''(k'') = \pi_W((u, y))
    \end{array} \right),
    \]

    where \( B_S := \pi_{W \times X}(B_f(Q)) \).

  - As \( \xi(k) = x = \xi''(k'') \), using (28) gives

    \[
    (\hat{\omega}, \hat{\xi}) = (\omega, \xi) \land \xi'_m, (\omega'', \xi'') \in \circ B_S,
    \]

    where

    \[
    \hat{\xi}(k) = x \quad \hat{\xi}(k + 1) = x' \quad \hat{\omega}|_{[k+m-l,k-1]} = \omega |_{[k+m-l,k-1]} \\
    \hat{\omega}(k) = \pi_W(u, y) = w
    \]

    (35b)

    (35c)

    (35d)
Now observe that (7b) holds for \[\hat{x}' \in \hat{\mathbb{R}}^{l_{k+m-l+1,k+m}}\] and observe that (35b) implies \[\hat{x}' \in E^{\hat{\mathbb{R}}_{m}}(x').\]

Furthermore, (35a) implies \[\hat{x}' \in E^{\hat{\mathbb{R}}_{m}}(x)\] and using (13) and (35c) therefore yields \[\hat{x} = \hat{\omega}^{k+m-l,k+m-1\}_{k+m-l,k+m}]\), hence \[\hat{x}|_{[1,l-1]} = \hat{x}'|_{[0,l-2]}\].

With this, using (35d) immediately shows that (14c) holds, hence \((\hat{x}, u, y, \hat{x}') \in \hat{\delta}^{2m}\).

\[\Rightarrow\]

“\(\Rightarrow\)”: Now using (32) in the proof of Thm. 2 shows that the above statement is equivalent to (15), what proves the statement.

Recall that \(Q\) is always future unique w.r.t. \(\hat{T}^{l_{m}}_{m}\) if \(m = 0\). Therefore, we assume \(m > 0\) and show that (13) holds if (36) holds.

Pick \(x \in X\) and \(\zeta, \zeta' \in E^{\hat{\mathbb{R}}_{m}}(x)\). Using (12), there exist \((\omega, \zeta) \in \hat{\mathcal{B}}_{S}\) and \(k\) s.t. \(x = \xi(k)\) and \(\zeta' = \omega^{k+m-l,k+m-1}\).

Now pick \(x' = \xi'(k + 1)\) and \(u, y\) s.t. \(\pi_{W}(u, y) = \omega(k) = \zeta'(l - m)\) and observe that \((x, u, y, x') \in \delta\) (hence we can apply (36) and)
\[\hat{\zeta} = \zeta'|_{1,l-1} \cdot \omega'(k + m) \in E^{\hat{\mathbb{R}}_{m}}(x').\]

First observe that the right side of (36) implies \(\zeta(l - m) = \pi_{W}(u, y)\) (from (14c) and \(m > 0\)), hence \(\zeta(l - m) = \zeta'(l - m)\).

Furthermore, we know that there exists \(\zeta'' \in E^{\hat{\mathbb{R}}_{m}}(x')\) s.t. \((\zeta, u, y, \zeta'') \in \hat{\delta}^{2m}\), hence (from (14c)) \(\zeta'|_{1,l-1} = \zeta''|_{1,l-2}\).

Using (37) and the same reasoning as before (substituting \(x\) by \(x'\) and \(\zeta, \zeta'\) by \(\hat{\zeta}, \zeta''\)) we immediately obtain \(\zeta''(l - m) = \zeta(l - m) = \zeta''(l - m + 1)\).

As \(\zeta'|_{1,l-1} = \zeta''|_{0,l-2}\) we therefore have \(\zeta(l - m + 1) = \zeta'(l - m + 1)\).

Applying this process iteratively therefore yields \(\zeta'|_{m,m-1} = \zeta'|_{m,m-1}\), what proves the statement.

(ii) \(\mathcal{R}^{-1} \in \mathfrak{R}_{Y}(\hat{\mathcal{B}}^{l_{m}}_{m}, Q) \iff Q\) is state-based async. l-compl.

* First observe that (1a) always holds for \(\mathcal{R}^{-1}\), as we can pick \(\hat{x} \in \hat{X}_{0}^{l_{m}}\) and (14b) implies the existence of \(x \in X_{0}\) s.t. \(\zeta \in E^{\hat{\mathbb{R}}_{m}}(x)\).

* Now observe that (7b) holds for \(\mathcal{R}^{-1}\) and \(Y\) iff
\[\forall \hat{x}, x, u, y, \hat{x}' . \left( \hat{x} \in E^{\hat{\mathbb{R}}_{m}}(x) \land (\hat{x}, u, y, \hat{x}') \in \hat{\delta}^{2m} \right) \Rightarrow \exists x', u' . \left( \hat{x}' \in E^{\hat{\mathbb{R}}_{m}}(x') \land (x, u', y, x') \in \delta \right)\] (38)

* Now we can use Lem. 2 to substitute \((\hat{x}, u, y, \hat{x}') \in \hat{\delta}^{2m}\). Furthermore, we can use the same reasoning as in (30) in the proof of Lem. 2 to observe that \(x \in E^{\hat{\mathbb{R}}_{m}}(x)\) implies that the right side of (38) can only be true if
\[\hat{x}|_{[0,l-m-1]} \cdot \pi_{W}(u, y) \cdot \hat{x}'|_{[m-l,m]} \in E^{[m-l,m]}(x)\].

* (38) is therefore equivalent to
\[\forall \hat{x}, x, u, y, \hat{x}' . \left( \hat{x} \in E^{\hat{\mathbb{R}}_{m}}(x) \land \hat{x}'|_{[0,l-m-1]} = \pi_{W}(u, y) \right) \Rightarrow \hat{x} \in E^{\hat{\mathbb{R}}_{m}}(x)\] (39)

* As \(u, y\) and \(\hat{x}'\) are all-quantified, the above statement is equivalent to
\[\forall \hat{x}, x, w . \left( \hat{x} \in E^{\hat{\mathbb{R}}_{m}}(x) \land \hat{x} \cdot w \in \Pi_{l+1}(\mathcal{B}(Q)) \right) \Rightarrow \hat{x} \cdot w \in E^{[m-l,m]}(x)\]

* Now using (32) in the proof of Thm. 2 shows that the above statement is equivalent to (15), what proves the statement.
D. Proof of Thm. 4

To simplify the proof of Thm. 4(ii) we first show that the necessary and sufficient condition in Thm. 4(ii) are equivalent to a particular condition of the domino-game.

Lemma 3: Given (6) let

\[ \forall z \in W^{l+2}, \left( \left( \frac{\zeta_{[0,l]} \in \Pi_{l+1}(B(Q))}{\zeta_{[l+1]} \in \Pi_{l+1}(B(Q))} \Rightarrow z \in \Pi_{l+2}(B(Q)) \right) \right). \tag{39} \]

Then

\[ (39) \iff B^l = B^{l+1}. \]

Proof: We prove both directions separately.

\( \implies \)

• Recall that \( B^{l+1} \subseteq B^l \) always holds. We therefore only prove that (39) implies \( B^l \subseteq B^{l+1} \).

• Pick \( \omega \in \mathcal{O}B^l \) and recall from (9) that

\[ \forall k < 0, \omega(k) = \circ \text{ and } \forall k \in N_0, \omega_{[k-l,k]} \in \Pi_{l+1}(B(Q)). \]

Using (39) implies \( \forall k \in N_0, \omega_{[k-l-1,k]} \subseteq \Pi_{l+2}(B(Q)) \), hence \( \omega \in \mathcal{O}B^{l+1} \) giving \( B^l \subseteq B^{l+1} \).

\( \impliedby \)

• Pick \( \zeta \in W^{l+2} \) s.t. \( \zeta_{[0,l]} \in \Pi_{l+1}(B(Q)) \) and \( \zeta_{[l+1]} \in \Pi_{l+1}(B(Q)) \).

• Using \( \Pi_{l+1}(B(Q)) = \bigcup_{k \in N_0} \mathcal{O}B_{[k-l,k]} \) from the proof of Thm. 4 this implies the existence of \( \omega, \omega' \in \mathcal{O}B^l \) and \( k, k' \in N_0 \) s.t. \( \omega_{[k-l,k]} = \zeta_{[0,l]} \) and \( \omega'_{[k'-l,k']} = \zeta_{[l+1]}. \)

• Picking \( \omega'' = \omega \land \kappa_{k'\rightarrow k} \) it is easily verified that \( \omega'' \in \mathcal{O}B^l \) and \( \omega''_{[k-l,k+1]} = \zeta. \)

• As \( B^l \subseteq B^{l+1} \) we know that \( \omega''_{[k-l,k+1]} \in \Pi_{l+2}(B(Q)) \), what proves the statement \( \square \).

Proof of Thm. 4 We show both statements separately.

(i) Show \( R \in \mathfrak{R}_W(\hat{Q}^{n_1}, \hat{Q}^{n_2}). \)

• Show (7a):

  • Let \( \hat{x}_{l+1} \in \hat{X}^{l+1}_0 \) and pick \( \hat{x}_l = \hat{x}_{l+1}[1,l]. \)

  • It follows from (14b) that there exists an \( x \in X_0 \) s.t. \( \hat{x}_{l+1} \in E^{l+1}_n(x) \).

  • Now it can be easily observed from (12) that \( \hat{x}_0 \in E^n_0(x) \), hence \( \hat{x}_1 \in \hat{X}^1_0 \) (from (14b)) and \( (\hat{x}_{l+1}, \hat{x}_l) \in R \) (from (17)).

• Show (7b):

  • Pick \( (\hat{x}_{l+1}, \hat{x}_l) \in R, u, v, w \) and \( \hat{x}_{l+1}, \hat{x}_l \) s.t. \( \hat{x}_{l+1} \in \hat{X}^{l+1}_{\omega} \) and \( \hat{x}_l \in \hat{X}^l_{\omega'}. \)

  • With this choice it immediately follows that \( (\hat{x}_{l+1}, \hat{x}_l) \in R \) and it remains to show that there exist \( u', v' \) s.t. \( (\hat{x}_0, u', v', \hat{x}_l) \in \hat{\mathcal{G}}^{l+1}, \) and \( w = \pi_W(u', v'). \)

  • Using Lem. 2 we have

\[
\begin{align*}
\hat{x}_{l+1}[0,l-m] &= (\hat{x}_{l+1}[0,l-m], w[1,l+1-m]) \\
\hat{x}_{l+1}[l+1-m,l+1] &= \hat{x}_{l+1}[l+1-m+2,l] \\
\hat{x}_{l+1}[0,l-m] &= w \cdot \hat{x}_{l+1}[1,l+1-m] \\
\hat{x}_{l+1}[l+1-m,l+1] &\in \Pi_{l+2}(B(Q)) \\
\hat{x}_l &= \hat{x}_{l+1}[1,l] \text{ (from above)} \\
\hat{x}_l[0,l-m] &= (\hat{x}_l[0,l-m] \cdot w)[1,l-m] \\
\hat{x}_l[l+1-m,l+1] &= \hat{x}_l[l+1-m+2,l] \\
\hat{x}_l[l+1-m,l+1] &\in \Pi_{l+1}(B(Q)).
\end{align*}
\]

By using Lem. 2 again this proves the statement .

(ii) Show \( R^{-1} \in \mathfrak{R}_W(\hat{Q}_m^{l+2}, \hat{Q}_m^{l+1}) \iff B^l = B^{l+1} \)

• We first show that (7a) always holds for \( R^{-1} = \)

  • Let \( \hat{x}_l \in \hat{X}^{l+1}_0. \)

  • It follows from (14b) that there exists an \( x \in X_0 \) s.t. \( \hat{x}_0 \in E^{l+1}_n(x) \).

• Using (12) there exists \( (\omega, \xi) \in \pi_{\mathcal{W} \times X}(\mathcal{O}B_{[l,l]}(Q)) \) s.t. \( \xi(0) = x \) and \( \hat{x}_l = \omega_{[m-1, m-1]} \).

• Now pick \( \hat{x}_{l+1} = \omega_{[m-1, m-1]} \) and observe that \( \hat{x}_{l+1} \in E^{l+1}_n(x) \), hence \( \hat{x}_{m+1} \in \hat{X}^{l+1}_0 \) (from (14b)) and \( (\hat{x}_l, \hat{x}_{l+1}) \in \)
Now we show future uniqueness w.r.t. $\mathcal{R}^{-1}$.

- Now we can pick $\tilde{x}_{i+1}$. With this choice we obviously have

$$\tilde{x}_i = \tilde{x}_{i+1} w.r.t. \mathcal{R}^{-1}.$$  \hspace{1cm} (39)

- Now recall from the proof of Thm. 2 that $\tilde{x}_{i+1} \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q}))$. Therefore substituting the right side of (29b) in (40) yields

$$\forall \zeta, \zeta' \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q})), w . \hspace{1cm} \begin{cases} \zeta = \zeta' \text{ w.r.t. } \mathcal{R}^{-1} \text{ and w } \\ \exists \zeta'' \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q})), \hat{\zeta} \in \Pi_{i+2}(\mathcal{B}(\mathcal{Q})). \end{cases} \hspace{1cm} (41)$$

and it can be easily verified that (40) $\iff$ (41) by using (29b).

- It remains to show that (41) $\implies$ (39): 

**”$\Rightarrow$”**

* Pick $\alpha \in \mathcal{W}_{i+2}$, $\zeta' = \alpha|_{[0,l]} \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q}))$ and $\zeta = \alpha|_{[1,l+1]} \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q}))$. With this choice we have $\zeta^l_{|0,l-1} = \zeta'|_{0,l}$, i.e., (41) is true.

* Now pick $\zeta'' \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q})), \hat{\zeta} \in \Pi_{i+2}(\mathcal{B}(\mathcal{Q}))$ s.t. (41) holds.

* With this immediately follows $\hat{\zeta}_{|0,l} = \zeta' = \alpha|_{0,l}$.

* As $\zeta_{|0,l-1} = \zeta'|_{0,l}$, we furthermore have $\zeta_{|l} = \zeta^l_{|0,l-1}$ and $\hat{\zeta}_{|2,l+1} = \zeta''_{|1,l} = \zeta_{|1,l}$ (from (41)), hence $\hat{\zeta}_{|1,l} = \zeta = \alpha|_{1,l}$.

* Therefore, obviously $\alpha = \hat{\zeta}$, hence $\alpha \in \Pi_{i+2}(\mathcal{B}(\mathcal{Q}))$.

**”$\Leftarrow$”**

* Pick $\zeta, \zeta' \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q})), w$ s.t. (41) holds.

* Now we can pick $\hat{\zeta} \in \mathcal{W}_{i+2}$ s.t. $\zeta' = \hat{\zeta}_{|0,l} \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q}))$ and $\zeta = \hat{\zeta}_{|1,l+1} \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q}))$ and (39) implies $\hat{\zeta} \in \Pi_{i+2}(\mathcal{B}(\mathcal{Q}))$.

* Furthermore, we pick $\zeta'' = \zeta$.

* With this choice we obviously have $\hat{\zeta}_{|0,l} = \zeta'$ and $\zeta''_{|1,l} = \zeta_{|1,l}$.

* As $\zeta_{|0,l-1} = \zeta'|_{0,l}$ we furthermore have $\zeta'' = \zeta = \hat{\zeta}_{|1,l+1}$ and therefore $w = \hat{\zeta}(l + 1 - m)$ (as $m \leq l$).

**E. Proof of Thm. 5**

To simplify the proof of Thm. 5(ii) we first show that the necessary and sufficient conditions in Thm. 5(ii) are equivalent to a simpler condition of the domino-game.

**Lemma 4:** Given $\mathcal{B}$ let

$$\forall \zeta, \zeta' \in \Pi_{i+1}(\mathcal{B}(\mathcal{Q})). \left( \zeta|_{0,l-1} = \zeta'|_{0,l-1} \Rightarrow \zeta = \zeta' \right). \hspace{1cm} (42)$$

Then

$$\text{Proof: We show both directions separately:} \hspace{1cm} (42) \iff \left( Q \text{ is future unique w.r.t. } \mathcal{T}_{m+1}^l \right) \wedge \left( Q \text{ is state-based async. } l \text{-complete w.r.t. } \mathcal{T}_m^l \right)$$

**”$\Rightarrow$”**

* We first show future uniqueness w.r.t. $\mathcal{T}_{m+1}^l$:

5To simplify notation we denote the left (resp. right) side of the implication in (41) by (41l) (resp. (41r)).
Pick $x \in X$ and $\zeta, \zeta' \in E^{T_{m+1}}(x)$.

It follows from (12) that there exist $(\omega, \xi), (\omega', \xi') \in \pi_W \times (\circ B_f(Q))$ and $k, k' \in \mathbb{N}_0$ s.t.

$$\xi(k) = \xi(k') = x,$$

$$\omega|_{[k-l+m+1, k+m]} = \zeta$$ and

$$\omega'|_{[k'-l+m+1, k'+m]} = \zeta'.$$

Using (28) we can therefore construct $(\omega'', \xi'') = (\omega, \xi) \land_k (\omega', \xi')$.

Now pick $\zeta = \omega|_{[k-l,k]}$ and $\zeta' = \omega''|_{[k-l,k]}$ and observe

$$\zeta|_{[0,l-1]} = \omega''|_{[k-l,k-1]} = \omega|_{[k-l,k-1]} = \zeta|_{[0,l-1]}.$$ (43)

Using 42 we therefore have $\zeta = \zeta'$, hence $\omega(k) = \zeta(l - m + 1) = \zeta'(l - m + 1) = \omega''(k)$.

Now we can pick $\tilde{\zeta} = \omega|_{[k-l+1,k+1]}$ and $\tilde{\zeta}' = \omega''|_{[k-l+1,k+1]}$ and (by reusing the above argument) obtain $\omega(k) = \zeta(l - m + 2) = \zeta'(l - m + 2) = \omega''(k)$.

Iteratively applying the above reasoning therefore yields $\zeta|_{[l-m+1,l]} = \zeta'|_{[l-m+1,l]}$, what proves the statement.

Now we show that $Q$ is state-based asynchronously $l$-complete w.r.t. $T^l_m$:
- Pick $x \in X$, $\zeta \in \Pi^{l+1}(B(Q))$ s.t. $\zeta|_{[0,l-1]} \in E^{T^l_m}(x)$.
- It follows from (12) that there exist $(\omega, \xi) \in \pi_W \times (\circ B_f(Q))$ and $k \in \mathbb{N}_0$ s.t.

$$\xi(k) = \xi(k') = x$$ and

$$\omega|_{[k-l+m,k+m]} = \zeta|_{[0,l-1]}.$$

Now pick $\zeta' = \omega'|_{[k-l+m,k+m]}$ and observe that $\zeta' \in E^{[m-l,m]}(x)$, $\zeta'|_{[0,l-1]} = \zeta'|_{[0,l-1]}$ and $\zeta' \in \Pi^{l+1}(B(Q))$.

Using 42 we have $\zeta = \zeta'$ and therefore $\zeta \in E^{[m-l,m]}(x)$, what proves the statement.

“\[\supseteq\]”

- Pick $\zeta, \zeta' \in \Pi^{l+1}(B(Q))$ s.t. $\zeta|_{[0,l-1]} = \zeta'|_{[0,l-1]}$.
- Observe that this implies $\zeta|_{[0,l-1]} \in \Pi^l(B(Q))$. Hence, there exists $x \in X$ s.t. $\zeta|_{[0,l-1]} \in E^{T^l_m}(x) = E^{[m-l,m-1]}(x)$.
- Using that $Q$ is state-based asynchronously $l$-complete w.r.t. $T^l_m$ we know that $\zeta, \zeta' \in E^{[m-l,m]}(x)$.

Using 12 this implies that $\zeta|_{[1,l]}, \zeta'|_{[1,l]} \in E^{T^l_{m+1}}(x)$.

As $Q$ is future unique w.r.t. $T^l_{m+1}$ this implies $\zeta|_{[1,l]} = \zeta'|_{[1,l]}$, hence $\zeta = \zeta'$, what proves the statement.

\[\blacksquare\]

\textit{Proof of Thm. 5} We show both statements separately.

(i) Show $R \in \mathfrak{R}_W(\hat{Q}^{T^l_{m+1}}, \hat{Q}^{T^l_{m}})$.

- Show (76):

Let $\hat{x}_{m+1} \in \hat{X}^{T^l_{m+1}}_0$ and pick $\hat{x}_m = \circ \cdot \hat{x}_{m+1}|_{[0,l-2]}$, implying $\hat{x}_{m+1}|_{[0,l-2]} = \hat{x}_m|_{[1,l-1]}$.

Furthermore, it follows from (14b) that there exists an $x \in X_0$ s.t. $\hat{x}_{m+1} \in E^{T^l_{m+1}}(x)$.

Now it can be easily observed from (12) that $\hat{x}_m \in E^{T^l_{m}}(x)$, hence $\hat{x}_m \in \hat{X}^{T^l_{m}}_0$ (from (14b) and $(\hat{x}_{m+1}, \hat{x}_m) \in R$ (from 12).

- Show (79):

Pick $(\hat{x}_{m+1}, \hat{x}_m) \in R$, $u, y$ and $\hat{x}'_{m+1}$ s.t. $(\hat{x}_{m+1}, u, y, \hat{x}'_{m+1}) \in \delta^{T^l_{m+1}}$ and $w = \pi_W(u, y)$.

Using (14c) this implies $\hat{x}_{m+1}|_{[1,l-1]} = \hat{x}'_{m+1}|_{[0,l-2]}$ and $\hat{x}_{m+1}|_{[l-m-1]} = w$.

Now pick $\hat{x}'_m = \hat{x}_{s+1}$ and observe that the first line of (13) implies $\hat{x}_m|_{[1,l-1]} = \hat{x}'_m|_{[0,l-2]}$ while the second line implies

$$\exists x \in X \cdot \left( \hat{x}'_m \in E_{T^l_{m+1}}(x) \land \hat{x}_m \in E_{T^l_{m}}(x) \right)$$

- Using (12) this implies

$$\exists (\omega, \xi) \in B, k \cdot \left( \xi(k) = x \land \hat{x}_m = \omega|_{[k-l,m,k+l]} \right)$$ and

$$\exists (\omega', \xi') \in B, k' \cdot \left( \xi'(k') = x \land \hat{x}'_m = \omega'|_{[k'-l,m,k'+l']} \right).$$ (44a)

As $\xi(k) = \xi'(k') = x$ in (44), using (28) gives

$$\tilde{\omega}, \tilde{\xi} = (\omega, \xi) \land_{k'} (\omega', \xi') \in \mathfrak{B}_S$$, where

$$\tilde{\omega}|_{[k-l+1,k+l]} = \omega|_{[k-l+1,k+l]} \cdot \omega'|_{[k'-l+1,k'-l+1]}$$

$$\tilde{\omega}|_{[0,l-m-1]} = \tilde{x}_m|_{[1,l-m-1]} \cdot \tilde{x}_m|_{[l-m-1,l-1]}.$$ (45)
Now we show that (7b) holds for $\tilde{x}_{m+1}(l-m-1) = w$ this implies
\[
\tilde{x}_m|_{[0,l-m-1]} \cdot w \cdot \tilde{x}_m|_{[l-m,l-1]} \in \Pi_{l+1}(B(Q)).
\]

Now Lem. 2 implies the existence of $u' \in U$ and $y' \in Y$ s.t. $\pi_W(u', y') = w$ and $(\tilde{x}_m, u', y', \tilde{x}_m') \in \tilde{\tau}^I_{m}$. It remains to show that $(\tilde{x}_{m+1}, \tilde{x}_m') \in R$.

First observe that $(\tilde{x}_{m+1}, [0,l-2] = \tilde{x}_{m+1}[1,l-1] = \tilde{x}_m'[1,l-1]$ from above.

Since $(\tilde{x}_{m+1}, u, y, \tilde{x}_{m+1}') \in \tilde{\tau}^I_{m+1}$, Lem. 2 implies
\[
\tilde{x}_{m+1}[0,0] \cdot \tilde{x}_{m+1}' \in \Pi_{l+1}(B(Q))
\]

Using (8) and (4) this implies the existence of $(\tilde{w}, \tilde{z}) \in \tilde{\pi}BS$ and $k$ s.t.
\[
\tilde{w}|_{[k+(m+1)-l,k+(m+1)]} = \tilde{x}_{m+1}[0,0] \cdot \tilde{x}_{m+1}'
\]

Now it is easy to see that
\[
\tilde{x}_{m+1}' \in E_{m+1}^I(\tilde{z}(k+1))
\]
\[
\tilde{x}_m = \tilde{x}_{m+1} \in E_{m}^I(\tilde{z}(k+1)),
\]
what proves the statement.

(ii) We first show that (7a) always holds for $R^{-1}$.
- Let $\tilde{x}_m \in X^I_{0}$.
- It follows from (4b) that there exists an $x \in X_0$ s.t. $\tilde{x}_m \in E_{l}^I(x)$.
- Using (12) there exists $(\omega, \zeta) \in \pi_{W_X}(I^Bf(Q))$ s.t. $\pi(0) = x$ and $\tilde{x}_m = \omega|_{[m-1,m-1]}$.
- Now pick $\tilde{x}_{m+1} = \omega|_{[m+1-m]}$ and observe that $\tilde{x}_{m+1} \in E_{m+1}^I(x)$, hence $\tilde{x}_{m+1} \in X_0^I$ (from (4b)) and $(\tilde{x}_m, \tilde{x}_{m+1}) \in R^{-1}$ (from (18)).
- Now we show that (7b) holds for $R^{-1}$ iff (42) in Lem. 4 holds:
- Observe that (7b) holds for $R^{-1}$ and $W$ iff
\[
\forall \tilde{x}_m, \tilde{x}_m', \tilde{x}_{m+1}, u, y
\]
\[
(\tilde{x}_m, \tilde{x}_{m+1}) \in R^{-1} \land (\tilde{x}_m', \tilde{x}_m, u, y) \in \tilde{\tau}^I
\]
\[
\exists \tilde{x}_{m+1}', u', y'. \quad (\tilde{x}_m', \tilde{x}_{m+1}) \in R^{-1} \land (\tilde{x}_{m+1}, u', y', \tilde{x}_m') \in \tilde{\tau}^I
\]

- Recall from the reasoning in (44) and (45) of part (i) of this proof, that
\[
\forall \tilde{x}_m, \tilde{x}_{m+1}
\]
\[
(\tilde{x}_m, \tilde{x}_{m+1}) \in R^{-1} \Rightarrow \exists \zeta \in \Pi_{l+1}(B(Q)). \quad \left(\zeta|_{[0,l-1]} = \tilde{x}_m \land \zeta|_{[1,l]} = \tilde{x}_{m+1}\right)
\]

Recalling (32) from the proof of Thm. 2 and substituting (47) and (29b) in (46) therefore yields
\[
\forall \zeta, \zeta' \in \Pi_{l+1}(B(Q)), w
\]
\[
(\zeta|_{[0,l-1]} = \zeta'|_{[0,l-1]} \land w = \zeta(l-m)) \Rightarrow
\]
\[
\exists \zeta", \zeta''' \in \Pi_{l+1}(B(Q)). \quad \left(\zeta"|_{[0,l-1]} = \zeta'''|_{[0,l-1]} \land \zeta"|_{[1,l]} = \zeta'|_{[1,l]} \land \zeta'''|_{[0,l-1]} = \zeta|_{[0,l-1]} \land w = \zeta'''|_{[l-m,l-m-1]}\right)
\]
and it can be easily verified that (46) $\Leftrightarrow$ (48) by using (47) and (29b).

It remains to show that (48) $\Leftrightarrow$ (42):

"$\Rightarrow$":

* Pick $\zeta, \zeta' \in \Pi_{t+1}(B(Q))$ s.t. $\zeta |_{[0,t-1]} = \zeta' |_{[0,t-1]}$.
* Let $l = 1$ implying $m = 0$ (as $m < l$). Now (48a) gives $w = \zeta |_{[1,1]}$ and (48r) gives $w = \zeta'' |_{[0,0]}$ and therefore
  
  \[ w = \zeta'' |_{[0,0]} = \zeta |_{[1,1]} = \zeta'' |_{[0,0]} = \zeta' |_{[1,1]} , \]

  hence $\zeta = \zeta'$.
* Let $l > 1$. Then (48r) immediately implies
  
  \[ \zeta' |_{[l,l]} = \zeta'' |_{[l-1,l-1]} = \zeta'' |_{[l-1,l-1]} = \zeta' |_{[l,l]}, \]

  hence $\zeta = \zeta'$.

"⇐" : 
* Pick $\zeta, \zeta' \in \Pi_{t+1}(B(Q))$ and $w$ s.t. $\zeta |_{[0,t-1]} = \zeta' |_{[0,t-1]}$ and $w = \zeta |_{(l,m)}$.
* Furthermore, pick $\zeta'', \zeta''' \in \Pi_{t+1}(B(Q))$ s.t. $\zeta'' |_{[l,l]} = \zeta'' |_{[l,l]}$, $\zeta''' |_{[0,l-1]} = \zeta'' |_{[1,l]}$ and $\zeta''' |_{[0,l-1]} = \zeta'' |_{[1,l]}$.
* As $\zeta = \zeta'$ (from (42)) this immediately implies $\zeta'' |_{[1,l]} = \zeta''' |_{[1,l]}$ and $w = \zeta''' |_{[l-1,m-1,l-1]}$ (as $m < l$), what proves the statement.

**APPENDIX II**

**PROOFS FOR SEC. IV**

In this appendix we provide detailed proofs for the claims made in Sec. IV which make use of the following lemma, restating useful conditions of the partitions $\Phi^l$ from [2].

**Lemma 5:** Given (5) and $\Phi^l$ as in Def. 2 it holds that

\[ \forall Z \in \Phi^l, x, x' \in Z : H_{\delta}(x) = H_{\delta}(x') , \] \hspace{1cm} (49a)

\[ \Phi^l = \left\{ Z \in 2^X | \forall Z' \in \Phi^{l-1}, \left( Z \cap T_{\delta}^{-1}(Z') \neq \emptyset \right) \Rightarrow \left( Z \subseteq T_{\delta}^{-1}(Z') \right) \right\} , \] \hspace{1cm} (49b)

and $\Phi^l$ is a fixed point of (19) if

\[ \forall Z, Z' \in \Phi^l, (Z \cap T_{\delta}^{-1}(Z') \neq \emptyset) \Rightarrow (Z \subseteq T_{\delta}^{-1}(Z')) . \] \hspace{1cm} (49c)

**Proof:** 
* Show (49a): It follows from [2], Prop.3.9 (i) that

\[ \forall Z \in \Phi^l \exists Z' \in \Phi^{l-1}, Z \subseteq Z' \] \hspace{1cm} (50)

Now recall from (19a) that for all $Z^0 \in \Phi^0$ there exists $V \in 2^Y$ s.t. $Z^0 = H_{\delta}^{-1}(V)$ and $\Phi^0$ is a partition. Using (50), we obtain $Z \subseteq H_{\delta}^{-1}(V)$, what proves the statement.

* Show (49b): If follows from [2], Prop.3.9(v) that

\[ \forall Z' \in \Phi^{l-1}, Z \in \Phi^l, (Z \cap T_{\delta}^{-1}(Z') = \emptyset) \] \hspace{1cm} (49d)

Using that $A \Rightarrow B$ is logical equivalent to $\neg \ A \lor B$ and rewriting the previous statement into set-notation gives (49b).

* Show (49c): If follows from [2], Prop.3.10 (iii) that $\Phi^l$ is a fixed point of (19) if $\Phi^{l+1} = \Phi^l$. With this (49c) follows immediately from (49b).
A. Proof of Prop. 2

We prove this statement by induction.

• \( l = 1 \): Recall that \( \mathcal{T}_1^l = [0, 0] \). Therefore (4), (12) and \( W = Y \) implies

\[
E_{\mathcal{T}_1^l}(x) = \left\{ y \in (Y \cup \{\emptyset\}) : \exists (\mu, \nu, \xi) \in \mathcal{B}_f(\mathcal{Q}), k \in \mathbb{N}_0 . \right. \\
\left. \left( \xi(k) = x \land y = \nu|_{[k,k]} \right) \right\}
\]

\[
= \left\{ y \in Y : \exists (\mu, \nu, \xi), k \in \mathbb{N}_0 . \right. \\
\left. \left( \xi(k) = x \land \nu(k) = y \land (\xi(k), \mu(k), \nu(k), \xi(k + 1) \in \delta) \right) \right\}
\]

\[
= \{ y \in Y | \exists u \in U, x' \in X . (x, u, y, x') \in \delta \}
\]

= \( H_\delta(x) \) (51)

where the second last line holds as \( \mathcal{Q} \) is reachable and live, as this implies that all transitions in \( \delta \) are part of an infinite derivation. The rest follows from (19a).

• \( (l - 1) \to l \): Given that

\[
\phi^{l-1} = \left\{ (E_{\mathcal{T}_{l-1}^l})^{-1}(V) \right\} \quad V \in 2^{(Y)^{l-1}}
\]

holds with \( \mathcal{T}_{l-1}^l = [0, l - 2] \), we show (20).

– First observe that

\[
\left\{ (E_{\mathcal{T}_l}^l)^{-1}(V) \right\} \quad V \in 2^{(Y)^{l}}
\]

\[
= \left\{ \{ x \in X | V = E_{\mathcal{T}_l}^l(x) \} \right\} \quad V \in 2^{(Y)^{l}}
\]

\[
= \{ Z \in 2^X | \forall x, x' \in Z . E_{\mathcal{T}_l}^l(x) = E_{\mathcal{T}_l}^l(x') \}
\]

(53)

We therefore have to show that the right side of (53) and the right side of (49b) in Lem. 5 coincide for a fixed \( Z \in 2^X \).

– Using (12) and \( W = Y \) we can rewrite \( E_{\mathcal{T}_l}^l \) as follows.

\[
\zeta \in E^{[0,l-1]}(x) \iff \exists (\mu, \nu, \xi) \in \mathcal{B}_f(\mathcal{Q}), k \in \mathbb{N}_0 .
\]

\[
\left( \xi(k) = x \land \zeta = \nu|_{[k,k+l-1]} \right)
\]

\[
\iff \exists (\mu, \nu, \xi) \in \mathcal{B}_f(\mathcal{Q}), k \in \mathbb{N}_0 .
\]

\[
\left( \xi(k) = x \land (\xi(k), \mu(k), \zeta(0), \xi(k + 1) \in \delta) \land \zeta|_{[1,l-1]} = \nu|_{[k+1,k+l-1]} \right)
\]

\[
\iff \exists (\mu, \nu, \xi) \in \mathcal{B}_f(\mathcal{Q}), k \in \mathbb{N}_0 .
\]

\[
\left( \xi(k) = x \land (\xi(k), \mu(k), (\zeta(0), \xi(k + 1) \in \delta) \land \zeta|_{[1,l-1]} \in E^{[0,l-2]}(\xi + 1) \right)
\]

\[
\iff \exists x \in X, u \in U . \left( (x, u, (\zeta(0), x')) \in \delta \land \zeta|_{[1,l-1]} \in E^{[0,l-2]}(x') \right)
\]

where the last line follows from \( \mathcal{Q} \) being reachable and live.

– Using this equivalence, we can rewrite the right side of (53) for fixed \( x, x' \in Z \) as

\[
E^{[0,l-1]}(x) = E^{[0,l-1]}(x') \iff \forall \zeta \in (Y)^l . \left( \zeta \in E^{[0,l-1]}(x) \Rightarrow \zeta \in E^{[0,l-1]}(x') \right)
\]
Now recall from Prop. 2 and (21a) that there exists
\[ (x, u, \zeta(0), \hat{x}) \in \delta \]
\[ \land \zeta_{[1, \zeta(0)]} \in E_{[0,l-2]}(\hat{x}) \]
\[ \Rightarrow \]
\[ \exists \hat{x}' \in X, u' \in U . \quad (x', u', \zeta(0), \hat{x}') \in \delta \]
\[ \land \zeta_{[1, \zeta(0)]} \in E_{[0,l-2]}(\hat{x}') \]
\[ \forall y \in Y, V \in 2^{(y)} \]
\[ \Rightarrow \]
\[ \exists \hat{x} \in X, u \in U . \quad (x, u, y, \hat{x}) \in \delta \]
\[ \land V = E_{[0,l-2]}(\hat{x}) \]
\[ \forall y \in Y, V \in 2^{(y)} \]
\[ \Rightarrow \]
\[ \exists \hat{x}' \in X, u' \in U . \quad (x', u', y, \hat{x}') \in \delta \]
\[ \land V = E_{[0,l-2]}(\hat{x}') \]
\[ \forall y \in Y, V \in 2^{(y)} \]
\[ \Rightarrow \]
\[ \exists \hat{x}' \in \left( E_{[0,l-2]} \right)^{-1}(V), u \in U . \quad (x, u, y, \hat{x}) \in \delta \]
\[ \Rightarrow \]
\[ \exists \hat{x}' \in \left( E_{[0,l-2]} \right)^{-1}(V), u' \in U . \quad (x', u', y, \hat{x}') \in \delta \]
\[ \forall y \in Y, V \in 2^{(y)} \]

Now it follows from (6) and (3) that
\[ E_{[0,l-1]}(x) = E_{[0,l-1]}(x') \]
\[ H_\delta(x) = H_\delta(x') \]
\[ \land \forall Z \in \Phi^{-1} \cdot \{x\} \cap T_\delta^{-1}(Z^{-1}) \neq \emptyset \] (53) yields
\[ \forall x, x' \in Z . \quad E_{[0,l-1]}(x) = E_{[0,l-1]}(x') \]
\[ \forall x, x' \in Z . \quad H_\delta(x) = H_\delta(x') \]
\[ \Rightarrow \]
\[ \left( \forall Z' \in \Phi^{-1} \cdot (Z \cap T_\delta^{-1}(Z') \neq \emptyset) \Rightarrow \right) \]
\[ \forall x, x' \in Z . \quad H_\delta(x) = H_\delta(x') \]
\[ \land \forall Z \in \Phi^{-1} \cdot \{x\} \cap T_\delta^{-1}(Z^{-1}) \neq \emptyset \] (54)

Now it follows from (49a) in Lem. 5 that \( Z \in \Phi^l \) implies \( \forall x, x' \in Z . \ H_\delta(x) = H_\delta(x') \). We therefore obtain that
\[ Z \in \left( \left( E_{[0,l-1]} \right)^{-1}(V) \right) \]
\[ \forall y \in V \in 2^{(y)} \]
\[ \Rightarrow \]
\[ Z \in \Phi^l \]

what proves the statement.

Lemma 6: Let \( Q \) and \( \hat{Q}^{1} \) as in Def. 10. Then
\[ (V, u, y, V') \in \hat{Q}^{1} \Rightarrow \left( \forall y \in V \in 2^{(y)} \right) \]
\[ \land \forall y \in V \in 2^{(y)} \]

Proof: Observe that \( y \in V_{[0,0]} \) follows from 21c.

Now recall from Prop. 2 and (21a) that there exists \( Z, Z' \in \Phi^l \) s.t. \( Z = (E_{[0,l]}^{1}(V)) \) and \( Z' = (E_{[0,l]}^{1}(V')) \).

We can therefore pick \( \hat{Z'} = (E_{[0,l]}^{1}(V')) \) and observe (from Prop. 2) that \( \hat{Z'} \in \Phi^{-1} \) and (from Lem. 5) \( Z' \subseteq \hat{Z'} \) implying \( T_\delta^{-1}(Z') \subseteq T_\delta^{-1}(\hat{Z'}) \).

Observe that \( Z \cap T_\delta^{-1}(\hat{Z'}) \neq \emptyset \) implying \( Z \cap T_\delta^{-1}(\hat{Z'}) \neq \emptyset \) and therefore (using Prop. 2) \( Z \subseteq T_\delta^{-1}(\hat{Z'}) \).

This implies that
\[ \forall x \in Z . \exists x' \in T_\delta(x) . \quad V_{[0,l-2]} = E_{[0,l-2]}(x') \]
\[ \forall x \in Z . \exists x' \in T_\delta(x) . \quad V_{[0,l-2]} = E_{[0,l-2]}(x') \]
\[ \forall x \in Z . \exists x' \in T_\delta(x) . \quad V_{[0,l-2]} = E_{[0,l-2]}(x') \]

Now recall from 12 that
\[ V_{[0,l-2]} = \bigcup_{x \in T_\delta(Z)} E_{[0,l-2]}(x) \]
implying $V'[0, l-2] \subseteq V[1, l-1]$.

\*\*\* B. Proof of Thm. 6 \*\*\*

The proof of part (i) follows the same lines as the proof in [10], Thm. 4.18. and is therefore omitted. For part (ii) first observe that (7a) always holds for $R^{-1}$, as we can pick $\hat{x} \in X_0^\delta$ and obtain from (21b) that there exists $x \in X_0$ s.t. $\hat{x} = E^{Z_l}(x)$. Now we prove that (7b) holds for $R^{-1}$ if $\Phi_l$ is a fixed-point of (19).

\* Observe that (7b) holds for $R^{-1}$ and $Y$ iff

$$\forall \hat{x}, \hat{x}', u, y, x . \left( \hat{x} = E^{Z_l}(x) \wedge (\hat{x}, u, y, \hat{x}') \in \delta^\gamma \Rightarrow \exists x', u'. \left( \hat{x}' = E^{Z_l}(x') \wedge (x, u', y, x') \in \delta \right) \right)$$

As $Q$ has transition structure, using (21c) it can be easily verified that the previous statement is equivalent to

$$\forall \hat{x}, \hat{x}' \in \hat{X}^\delta . \left( \left( E^{Z_l} \right)^{-1}(\hat{x}) \cap T_{\delta}^{-1}(\left( E^{Z_l} \right)^{-1}(\hat{x}')) \neq \emptyset \Rightarrow \right)$$

$$\left( \left( E^{Z_l} \right)^{-1}(\hat{x}) \subseteq T_{\delta}^{-1}(\left( E^{Z_l} \right)^{-1}(\hat{x}')) \right)$$

\* Using Prop. 2 and (49c) in Lem. 5 this proves the statement.

\*\*\* C. Proof of Thm. 7 \*\*\*

(i) Show $R \in R_Y(\hat{Q}^{+\delta}, \hat{Q}^{\delta})$:

\* Show (7a) holds for $R$: Pick $\hat{x}_{l+1} \in \hat{X}_0^{l+1}$ and observe from (21a) that there exists $x \in X_0$ s.t. $\hat{x}_{l+1} = E^{r_{in+l}}(x)$ implying that $\hat{x}_l = \hat{x}_{l+1}|[0, l-1] = E^{Z_l}(x)$, giving $\hat{x}_l \in X_0^\delta$.

\* Show (7b) holds for $R$:

- Pick $(\hat{x}_{l+1}, \hat{x}_l) \in R$, i.e., $\hat{x}_l = \hat{x}_{l+1}|[0, l-1]$ and $u, y, \hat{x}_{l+1}, \hat{x}_l$ s.t. $(\hat{x}_{l+1}, u, y, \hat{x}_l) \in \delta^{l+1}$ and $\hat{x}_l = \hat{x}_{l+1}|[0, l-1]$, implying that $(\hat{x}_l, \hat{x}_l) \in R$

- Using (55) from Lem. 6 for $l + 1$ we know that

\begin{equation}
\forall x \in \left( E^{r_{in+l}} \right)^{-1}(\hat{x}_{l+1}) . \exists x' \in T_{\delta}(x) . \hat{x}' = E^{Z_l}(x')
\end{equation}

- As $x \in \left( E^{r_{in+l}} \right)^{-1}(\hat{x}_{l+1})$ implies $x \in \left( E^{Z_l} \right)^{l+1}(\hat{x}_l)$, we know from (2b) and (21a) that there exists $u'$ s.t. $(\hat{x}_l, u', y, \hat{x}_l) \in \delta^\gamma$, what proves the statement.

(ii) Show $R^{-1} \in R_Y(\hat{Q}^{\delta}, \hat{Q}^{+\delta}) \Leftrightarrow \Phi_l$ fixed-point of (19): We first prove that (7b) holds for $R^{-1}$ if $\Phi_l$ is a fixed-point of (19).

- Observe that (7b) holds for $R^{-1}$ and $Y$ iff

\begin{equation}
\forall \hat{x}_{l+1}, \hat{x}_l, \hat{x}'_l, u, y . \left( \hat{x}_l = \hat{x}_{l+1}|[0, l-1] \wedge (\hat{x}_l, u, y, \hat{x}'_l) \in \delta^\gamma \Rightarrow \exists x'_l, u'. \left( \hat{x}'_l = \hat{x}_{l+1}|[0, l-1] \wedge (\hat{x}_l, u', y, \hat{x}'_l) \in \delta^{l+1} \right) \right)
\end{equation}

- Reusing the reasoning from Lem. 6 (i.e. (55) for $l + 1$) we know that the previous statement is equivalent to

\begin{equation}
\forall \hat{x}_{l+1}, \hat{x}_l, \hat{x}'_l . \left( \left( E^{Z_l} \right)^{-1}(\hat{x}_{l+1}) \wedge (\left( E^{Z_l} \right)^{-1}(\hat{x}_l)) \neq \emptyset \Rightarrow \right)$$

$$\left( E^{r_{in+l}} \right)^{-1}(\hat{x}_{l+1}) \cap T_{\delta}^{-1}(\left( E^{Z_l} \right)^{-1}(\hat{x}_l)) \neq \emptyset$$

\end{equation}
– Recall from Lem. 5 that (50) holds and \( \Phi^l \) is a partition for any \( l \). Therefore, the right side of (56) implies
\[
\left( E^{\Phi^l}_{\hat{x}^{\Phi^l}} \right)^{-1} (\hat{x}_{l+1}) \subseteq T^{-1}_{\delta} \left( \left( E^{\Phi^l}_{\hat{x}^l} \right)^{-1} (\hat{x}^l) \right)
\]
– Now recall from (12) that
\[
\forall x \in X, \zeta \in \Pi_{l+1}(B(Q)) \cdot \zeta \in E^{\Phi^l}_{\hat{x}^{\Phi^l}}(x) \Rightarrow \zeta|_{[0,l-1]} \in E^{\Phi^l}_{x^l}(x),
\]
hence
\[
\left( E^{\Phi^l}_{x^l} \right)^{-1} (\hat{x}^l) \subseteq \bigcup_{\hat{x}_{l+1} \text{ s.t. } \hat{x}_{l+1} = \hat{x}_{l+1}|_{[0,l-1]}} \left( E^{\Phi^l}_{\hat{x}^{\Phi^l}} \right)^{-1} (\hat{x}_{l+1}).
\]
– As \( \hat{x}_{l+1} \) and \( \hat{x}_l \) are all-quantified in (56), it is now easy to see that (56) is equivalent to
\[
\forall \hat{x}_l, \hat{x}_l',
\]
\[
\left\{ \begin{array}{l}
\left( E^{\Phi^l}_{\hat{x}^l} \right)^{-1} (\hat{x}_l) \cap T^{-1}_{\delta} \left( \left( E^{\Phi^l}_{\hat{x}^l} \right)^{-1} (\hat{x}_l) \right) \neq \emptyset \Rightarrow \\
\left( E^{\Phi^l}_{\hat{x}^l} \right)^{-1} (\hat{x}_l) \subseteq T^{-1}_{\delta} \left( \left( E^{\Phi^l}_{\hat{x}^l} \right)^{-1} (\hat{x}_l) \right)
\end{array} \right.
\]
which is equivalent with \( \Phi^l \) being a fixed-point of (19).

* We conclude the proof by showing that (75) holds for \( R^{-1} \) if \( \Phi^l \) is a fixed-point of (19):
  – Pick \( \hat{x}_l \in \hat{X}_0^{I^l} \) and observe from (21a) that there exists \( x \in X_0 \) s.t. \( \hat{x}_l = E^{\Phi^l}_x(x) \).
  – Now pick any \( \hat{x}_{l+1} \) s.t. \( \hat{x}_l = \hat{x}_{l+1}|_{[0,l-1]} \).
  – As \( \Phi^l \) is a fixed-point of (19) we know that
\[
\left( E^{\Phi^l}_{\hat{x}^l} \right)^{-1} (\hat{x}_l) = \left( E^{\Phi^l}_{\hat{x}^{\Phi^l}} \right)^{-1} (\hat{x}_{l+1})
\]
and therefore \( \hat{x}_{l+1} = E^{\Phi^l}_{\hat{x}^{\Phi^l}}(x) \), hence \( \hat{x}_{l+1} \in \hat{X}_0^{I^{l+1}} \).

APPENDIX III
PROOFS FOR SEC. V

In this appendix we provide detailed proofs for the claims made in Sec. V. Before we start, we prove that future uniqueness is a stronger property than domino consistency.

**Lemma 7:** Given (6) s.t. \( W = Y \). Then
\[
Q \text{ is future unique w.r.t. } I^l_1 \Rightarrow Q \text{ is domino consistent.} \tag{57}
\]

**Proof:**
* Using (13), future uniqueness of \( Q \) w.r.t. \( I^l_1 \) implies that for all \( x \in X \) holds
\[
E^{I^l_1}_x(x) \neq \emptyset \Rightarrow |E^{I^l_1}_x(x)| = 1. \tag{58}
\]

Using (21a) this immediately implies \( |\hat{y}| = 1 \) for all \( \hat{y} \in \hat{X}^{I^l_1} \).

* Now pick \( \zeta \in \Pi_{l+1}(B(Q)) \) and \( \hat{y} \in \hat{X}^{I^l_1} \) s.t. \( \zeta' = \zeta|_{[0,l-1]} \in \hat{y} \), implying \( \hat{y} = \{ \zeta' \} \).

* As \( \zeta \in \Pi_{l+1}(B(Q)) \) we know that there exists \( (\omega, \xi) \in B_S \) and \( k \in N_0 \) s.t. \( \zeta = \omega|_{[k,k+l]} \) and therefore \( \zeta \in E^{[0,l]}(\xi(k)) \) and \( \zeta' \in E^{I^l_1}(\xi(k)) \).

* Observe, that this immediately implies \( \xi(k) \in \left( E^{I^l_1}_x \right)^{-1}(\hat{y}) \), what proves the statement. \( \blacksquare \)

**A. Proof of Thm. 8**

* Pick \( \hat{\nu} \in \pi_Y(B(f(\hat{Q}^{I^l_1}))) \).

* Using (4) we know that there exist \( (\hat{\mu}, \hat{\xi}) \) s.t. \( \hat{\xi}(0) \in \hat{X}_0^{I^l_1} \) and \( \forall k \in N_0 . \hat{\xi}(k), \hat{\mu}(k), \hat{\nu}(k), \hat{\xi}(k + 1) \in \hat{\delta}^{I^l_1} \).

* Using Lem. 3 we know that for all \( k \in N_0 \) it holds that
\[
\hat{\xi}(k + 1)|_{[0,l-2]} \subseteq \hat{\xi}(k)|_{[1,l-1]} \text{ and }
\]
\[
\hat{\nu}(k) \in \hat{\xi}(k)|_{[0,0]}
\]
Applying these equations iteratively yields \( \hat{\nu}|_{[k,k+l-1]} \in \hat{\xi}(k) \).

* Furthermore, we can use (21c) and (4) to know that
\[
\exists (\nu', \xi') \in B_S(Q), k'. \left( \begin{array}{l}
\nu'(k') = \hat{\nu}(k) \\
\land \hat{\xi}(k) = E^{I^l_1}(\xi'(k')) \\
\land \hat{\xi}(k + 1) = E^{I^l_1}(\xi'(k' + 1))
\end{array} \right),
\]
Using (28) we now obtain
\[ \exists (\nu'', \xi''), \in B_{S}(Q), k'', \nu'' \wedge k'' + 1 \in \pi_{Y}(B_{f}(Q)) \Rightarrow B(Q) \]

Using (28) we now obtain
\[ \bar{\nu} = \nu'' \wedge k'' + 1 \nu' \wedge k'' + 1 \nu'' \in \pi_{Y}(B_{f}(Q)) = B(Q) \quad (59) \]

where \( \bar{\nu}[k, k + l] = \bar{\nu}[k', k' + l] \) and therefore \( \bar{\nu}[k, k + l] \in \Pi_{i+1}(B(Q)) \).

Now observe from (21b) and (12) that \( \nu(0) \subseteq \bar{\nu}(0) \subseteq \Pi_{i+1}(B(Q)) \).

Combining that \( \forall k \in N_{0} : \bar{\nu}[k, k + l] \in \Pi_{i+1}(B(Q)) \) and \( \bar{\nu}[0, l - 1] \in B(Q)[0, l - 1] \) we obtain \( \bar{\nu} \in B^{l} \) (from [29]).

\[ \text{B. Proof of Thm. 9} \]

We prove both statements separately.

(i) Show \( \forall \in B Y(\bar{Q}_{v}, \bar{Q}_{v}) \Rightarrow \bar{Q}_{v} \) is domino consistent.

We first show that (7a) always holds for \( \forall \):

- Pick \( \zeta \in \bar{X}_{0}^{l} \). Then it follows from (14b) that there exists \( x \in X_{0} \) s.t. \( \zeta \in E^{l}_{x}(x) \).
- Now pick \( \tilde{y} = E^{l}_{x}(x) \) and observe that \( \tilde{y} \in \bar{X}_{0}^{l} \), which proves the statement.

Now we need to show that (7b) holds for \( \forall \) and \( W = Y \) iff \( \bar{Q}_{v} \) is domino consistent. We show both directions separately:

\( \Rightarrow \) – Observe that (7b) holds for \( \forall \) and \( U \times Y \) iff

\[ \forall \zeta, \zeta', y, y', u, y \cdot \left( \zeta \in \tilde{y} \right) \rightarrow (\exists y, u, y') \left( \zeta' \in y' \wedge (\zeta', u, y, y') \in \bar{\delta}_{v} \right) \quad (60) \]

- Now pick \( \bar{\zeta} \in \Pi_{i+1}(B(Q)) \) and \( \tilde{y} \in \bar{Y} \) s.t. \( \bar{\zeta} \in \bar{\delta}_{v}(0) \) and pick \( \zeta' \) \( \bar{\zeta}_{[0, l - 1]} \) as well as \( y = \bar{\zeta}(0) \). Then it follows from (29b) in Lem. 2 with \( m \) that there exists \( u \) s.t. \( (\zeta, u, y, \zeta') \in \bar{\delta}_{v} \), hence the left side of (60) holds.
- Now we know from the right side of (60) and (21c) that there exists \( x \in \left( E^{l}_{x} \right)^{-1}(\tilde{y}) \) s.t. \( (x, u, y, x') \in \delta \) and \( \zeta' \in E^{l}_{x}(x') \).
- Now it follows immediately from (12) that \( y \cdot \zeta' = \zeta \in E^{l}_{x}(x') \), what proves the statement.

\( \Leftarrow \) – Pick \( (\zeta, \tilde{y}) \in \forall \), i.e., \( \zeta \in \tilde{y} \) and \( y, y', \zeta' \) s.t. \( (\zeta, y, y', \zeta') \in \bar{\delta}_{v} \).
- Now it follows from Lem. 2 that \( y \cdot \zeta' \in \Pi_{i+1}(B(Q)) \).
- Using (24) therefore implies the existence of \( x \in \left( E^{l}_{x} \right)^{-1}(\tilde{y}) \) s.t. \( y \cdot \zeta \in E^{l}_{x}(x) \).
- Using (12) this implies that there exists \( (\mu', u', \xi') \in B(Q) \) and \( k' \in N_{0} \) s.t.

\[ \tilde{y} = E^{l}_{x}(\xi'(k')) \] and \( \zeta' \in E^{l}_{x}(\xi'(k' + 1)) \)

- Now pick \( \tilde{y}' = E^{l}_{x}(\xi'(k' + 1)) \) and observe that \( \zeta' \in \tilde{y}' \).
- Moreover, using (21c) with \( x = \xi'(k'), x' = \xi'(k' + 1) \) and \( u' = \mu'(k') \) immediately implies \( (\tilde{y}, u', y, \tilde{y}') \in \bar{\delta}_{v} \), what proves the statement.

(ii) \( R^{-1} \in \forall Y(\bar{Q}_{v}, \bar{Q}_{v}) \Rightarrow \bar{Q} \) is future unique w.r.t. \( I' \).

\( \Rightarrow \) – Let \( R^{l}_{v} \) and \( R^{v} \) be equivalent to the relations in (16) and (22) (with \( m = l \), respectively. Then it follows from Thm. 3 and Thm. 6 that

\[ R^{v} \in \forall Y(Q, \bar{Q}_{v}) \quad \text{and} \]

\[ R^{l}_{v} \in \forall Y(Q, \bar{Q}^{l}_{v}) \Rightarrow \bar{Q} \text{ is future unique w.r.t. } I' \]
Now take $R$ as in (25) observe that
\[
\mathcal{R}^y \circ \mathcal{R}^{-1} = \left\{ (x, \hat{x}) \in X \times \hat{X}^I \middle| \exists \hat{y} \in \hat{X}^{I^y} . (x, \hat{y}) \in \mathcal{R}^y \land (\hat{y}, \hat{x}) \in \mathcal{R}^{-1} \right\} \\
= \left\{ (x, \hat{x}) \in X \times \hat{X}^I \middle| \exists \hat{y} \in \hat{X}^{I^y} . \hat{y} = E^{I^y}(x) \land \hat{x} \in \hat{y} \right\} \\
= \left\{ (x, \hat{x}) \in X \times \hat{X}^I \middle| \hat{x} \in E^{I^y}(x) \right\} \\
= R^{I^y}
\]

Now combining (61) with
\[
\left( \mathcal{R}^y \in \mathcal{R}_Y(Q, \hat{Q}^{I^y}) \land \mathcal{R}^{-1} \in \mathcal{R}_Y(\hat{Q}^{I^y}, \hat{Q}^{I^y}) \right) \Rightarrow \mathcal{R}^y \circ \mathcal{R}^{-1} \in \mathcal{R}_Y(Q, \hat{Q}^{I^y})
\]
gives
\[
\mathcal{R}^{-1} \in \mathcal{R}_Y(\hat{Q}^{I^y}, \hat{Q}^{I^y}) \Rightarrow Q \text{ is future unique w.r.t. } I^y.
\]

We know that $Q$ is future unique w.r.t. $I^y$. Using (13), this implies that for all $x \in X$ holds
\[
E^{I^y}(x) \neq \emptyset \Rightarrow |E^{I^y}(x)| = 1. \tag{62}
\]

Using (21a) this immediately implies $|\hat{y}| = 1$ for all $\hat{y} \in \hat{X}^{I^y}$. Therefore (25) becomes
\[
\mathcal{R} = \left\{ (\zeta, \hat{y}) \in \hat{X}^{I^y} \times \hat{X}^{I^y} \middle| \hat{y} = \{ \zeta \} \right\} \tag{63}
\]

Show that (7a) holds for $\mathcal{R}^{-1}$:
* Pick $\hat{y} \in \hat{X}^{I^y}_0$. Then it follows from (21b) that there exists $x \in X_0$ s.t. $\hat{y} = E^{I^y}(x)$.
* Now pick $\zeta \in \hat{y}$ and observe that $\zeta \in E^{I^y}(x)$, i.e., $\zeta \in \hat{X}^{I^y}_0$ (from (14b)).
* Show that (7b) holds for $\mathcal{R}^{-1}$:
  * Pick $(\hat{y}, \zeta) \in \mathcal{R}^{-1}$, i.e., $\hat{y} = \{ \zeta \}$ and $u, y, y', \zeta'$ s.t. $(\hat{y}, u, y, \hat{y}') \in \delta^{I^y}$ and $\hat{y}' = \{ \zeta' \}$.
  * Using (63) this immediately implies that $(\hat{y}', \zeta') \in \mathcal{R}^{-1}$. * Now (21c) implies the existence of $x, x'$ s.t. $\{ \zeta \} = E^{I^y}(x)$, $\{ \zeta' \} = E^{I^y}(x')$ and $(x, u, y, x') \in \delta$. Using (14c) this immediately implies $(\zeta, u, y, \zeta') \in \delta^{I^y}$, what proves the statement.