ON CLOSED GRAPHS II

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Abstract. A graph is closed when its vertices have a labeling by $[n]$ with a certain property first discovered in the study of binomial edge ideals. In this article, we explore various aspects of closed graphs, including the number of closed labelings and clustering coefficients.

1. Introduction

Given a simple graph $G$ with vertices $V(G)$ and edges $E(G)$, a labeling of $G$ is a bijection $V(G) \simeq [n] = \{1, \ldots, n\}$. Given a labeling, we assume $V(G) = [n]$.

Definition 1.1. A labeling of $G$ is closed when $\{j, i\}, \{i, k\} \in E(G)$ with $j > i < k$ or $j < i > k$ implies $\{j, k\} \in E(G)$. Then $G$ is closed if it has a closed labeling.

A labeling of $G$ gives a direction to each edge $\{i, j\} \in E(G)$ where the arrow point from $i$ to $j$ when $i < j$, so the arrow points to the bigger label. In this context, closed means that when two edges point away from a vertex or towards a vertex, the remaining vertices are connected by an edge.

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{i} \\
\text{j} \\
\text{k}
\end{array}
\end{array}
\end{equation}

Closed graphs were first encountered in the study of binomial edge ideals defined in [2] and [9]. Properties of these ideals are explored in [3, 10] and their relation to closed graphs features in [2, 4, 5, 6].

It is natural to ask for a characterization of those graphs that have a closed labeling. One solution was given in [2], which characterizes closed graphs using the clique complex of $G$. Another approach, taken in our previous paper [1], shows that a connected graph is closed if and only if is chordal, claw-free, and narrow (see [1, Def. 1.3] for the definition of narrow).

In this paper, we will use tools developed in [1] to study the combinatorial properties of closed graphs. Our main results include:

- Section 4 Theorem 4.3 counts the number of closed labelings of a closed graph.
- Section 5 Theorem 5.4 counts the number of closed graphs with fixed layer structure (see Section 2 for the definition of layer).

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Section 6. Theorem 6.3 gives a sharp lower bound for the clustering coefficient of a closed graph.

To prepare for these results, we will recall some relevant results and definitions in Section 2 and explore when a labeling remains closed after exchanging two labels in Section 3.

2. Notation and Known Results

We recall some notation and results from [1]. The neighborhood of \( v \in V(G) \) is
\[
N_G(v) = \{ w \in V(G) \mid \{v, w\} \in E(G) \}.
\]
When \( G \) is labeled and \( i \in V(G) = [n] \), we have a disjoint union
\[
N_G(i) = N_G^\geq(i) \cup N_G^\leq(i),
\]
where
\[
N_G^\geq(i) = \{ j \in N_G(i) \mid j > i \} \quad \text{and} \quad N_G^\leq(i) = \{ j \in N_G(i) \mid j < i \}.
\]

Also, vertices \( i, j \in [n] \) with \( i \leq j \) give the interval \( [i, j] = \{ k \in [n] \mid i \leq k \leq j \} \).

Here is a characterization of when a labeling of a connected graph is closed.

**Proposition 2.1** ([1, Prop. 2.4]). A labeling on a connected graph \( G \) is closed if and only if for all \( i \in [n] \), \( N_G^\geq(i) \) is complete and is an interval.

When a connected graph \( G \) has a labeling with \( V(G) = [n] \), we can decompose \( G \) into layers as follows. The \( N \)th layer of \( G \) is the set \( L_N \) of all vertices that are distance \( N \) from vertex 1, i.e.,
\[
L_N = \{ i \in [n] \mid i \text{ is distance } N \text{ from } 1 \}.
\]

Since \( G \) is connected, we have a disjoint union
\[
[n] = L_0 \cup L_1 \cup \cdots \cup L_h,
\]
where \( h = \max\{ N \mid L_N \neq \emptyset \} \). Here is a simple property of layers.

**Lemma 2.2** ([1, Lem. 2.6]). Let \( G \) be labeled and connected. If \( i \in L_N \) and \( \{i, j\} \in E(G) \), then \( j \in L_{N-1}, L_N, \) or \( L_{N+1} \).

When \( G \) is closed and connected, the layers are especially nice.

**Proposition 2.3** ([1, Prop. 2.7]). If \( G \) is connected with a closed labeling, then:

1. Each layer \( L_N \) is complete.
2. If \( d = \max\{ L_N \} \), then \( L_{N+1} = N_G^\geq(d) \).

The diameter of \( G \) is denoted \( \text{diam}(G) \), and a longest shortest path of \( G \) is a shortest path of length \( \text{diam}(G) \). These concepts relate to layers as follows.

**Proposition 2.4** ([1, Prop. 2.8]). If \( G \) is connected with a closed labeling, then:

1. \( \text{diam}(G) \) is the integer \( h \) appearing in (2.1).
2. If \( P \) is a longest shortest path of \( G \), then one endpoint of \( P \) is in \( L_0 \) or \( L_1 \) and the other is in \( L_h \), where \( h = \text{diam}(G) \).
3. Exchangeable Vertices

A closed graph with \( \geq 2 \) vertices has at least two closed labelings, since the reversal of a closed labeling is clearly closed. But there may be other closed labelings, as shown by the simple example:

\[
\begin{array}{c}
\text{(3.1)} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \\
\text{3}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
4 \quad 3 \\
\text{2}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

To explore what makes this example work, we need some definitions.

**Definition 3.1.** Let \( G \) be a graph.

1. The full neighborhood of a vertex \( v \in V(G) \) is \( N^*_G(v) = \{v\} \cup N_G(v) \).
2. \( v, w \in V(G) \) are exchangeable, written \( v \sim w \), if \( N^*_G(v) = N^*_G(w) \).

Vertices 1 and 2 are exchangeable in the left-hand graph of (3.1). Switching labels gives the right-hand graph, which is still closed. Here is the general result.

**Proposition 3.2.** Let \( G \) have a closed labeling. If \( i, j \in [n] \), \( i \neq j \), are exchangeable, then the labeling that switches \( i \) and \( j \) is also closed.

**Proof.** Define \( \phi : [n] \to [n] \) by \( \phi(i) = j \), \( \phi(j) = i \), and \( \phi(k) = k \) for \( k \in [n] \setminus \{i, j\} \).

Pick \( u, v, w \in V(G) \) with \( \{u, v\}, \{v, w\} \in E(G) \), \( u \neq w \), and \( \phi(u) > \phi(v) < \phi(w) \) or \( \phi(u) < \phi(v) > \phi(w) \). We need to prove that \( \{u, w\} \in E(G) \).

If \( \{i, j\} \cap \{u, v, w\} = \emptyset \), then \( \{u, w\} \in E(G) \) since the original labeling is closed. Now suppose \( \{i, j\} \cap \{u, v, w\} \neq \emptyset \) and \( \phi(u) > \phi(v) < \phi(w) \). There are several cases to consider. First suppose that \( i = v \). If \( j \in \{u, w\} \), then without loss of generality we may assume \( j = u \). Then

\[
w \in N^*_G(v) = N^*_G(i) = N^*_G(j) = N^*_G(u)
\]

implies \( \{u, w\} \in E(G) \). If \( j \notin \{u, w\} \), then \( \phi(u) > \phi(i) < \phi(w) \) means that \( u > j < w \). Then \( \{u, w\} \in E(G) \) since the original labeling is closed and \( j \sim i = v \).

The proof when \( j = v \) is similar and is omitted. Then two cases remain:

- \( i = u \) and \( j \notin \{v, w\} \). Thus \( \phi(u) > \phi(v) < \phi(w) \) means that \( j > v < w \). Then \( \{j, w\} \in E(G) \) since the original labeling is closed and \( j \sim i = u \).

Using \( j \sim i = u \) again, we conclude that \( \{u, w\} \in E(G) \).

- \( i = u \) and \( j = w \). Then \( \phi(u) > \phi(v) < \phi(w) \) means \( j > v < i \). Then \( \{u, w\} = \{i, j\} \in E(G) \) since the original labeling is closed.

The proof when \( \phi(u) < \phi(v) > \phi(w) \) is similar and is omitted. \( \square \)

Exchangeability \( v \sim w \) is an equivalence relation on \( V(G) \) with equivalence classes

\[
e(v) = \{w \in V(G) \mid v \sim w\} = \{w \in V(G) \mid N^*_G(w) = N^*_G(v)\}.
\]

Equivalence classes are complete, since \( v \sim w \) implies \( v \in N^*_G(v) = N^*_G(w) \), so that \( \{v, w\} \in E(G) \) whenever \( v \neq w \).
Since permutations are generated by transpositions, Proposition 3.2 implies that when \( G \) has a closed labeling, every permutation of an equivalence class yields a new closed labeling.

When \( G \) is connected and closed, equivalence classes have the following structure.

**Proposition 3.3.** If \( G \) is connected with a closed labeling and \( i \in [n] \), then the equivalence class \( e(i) \) is an interval.

**Proof.** It suffices to show that if \( i < j \) are exchangeable and \( i < k < j \), then \( N_G^*(k) = N_G^*(i) \). First note that \( \{i, k\} \in E(G) \) since \( j \in N_G^*(i) \) and \( N_G^*(i) \) is an interval by Proposition 2.1. Then \( \{j, k\} \in E(G) \) since \( i \sim j \).

Now take \( m \in N_G^*(k) \). We need to show \( m \in N_G^*(i) \). If \( m = k \), this follows from the previous paragraph. If \( \{m, k\} \in E(G) \), there are two possibilities:

- If \( m < k \), then \( m < k > i \), so \( \{m, i\} \in E(G) \) since the labeling is closed.
- If \( m > k \), then \( m > k < j \), so either \( m = j \) or \( \{m, j\} \in E(G) \) by closed.

Since \( N_G^*(i) = N_G^*(j) \), both possibilities imply \( m \in N_G^*(i) \).

Conversely, take \( m \in N_G^*(i) \). If \( m = i \), then \( m \in N_G^*(k) \) since \( \{i, k\} \in E(G) \) by the first paragraph of the proof. If \( \{m, i\} \in E(G) \), then \( \{m, j\} \in E(G) \) since \( i \sim j \).

Again, there are two possibilities:

- If \( m < i \), then \( m < i < k < j \), so \( \{m, k\} \in E(G) \) since \( N_G^*(m) \) is an interval.
- If \( m > i \), then \( m > i < k < j \), so either \( m = k \) or \( \{m, k\} \in E(G) \) by closed.

Thus \( m \in N_G^*(k) \) and the proof is complete. \qed

4. Counting Closed Labelings

Some graphs have no nontrivial exchangeable vertices.

**Definition 4.1.** A graph \( G \) is collapsed if it all exchangeable vertices are equal, i.e., \( N_G^*(v) = N_G^*(w) \) implies \( v = w \).

**Proposition 4.2.** Let \( G \) be a closed graph with \( \geq 3 \) vertices. Then the following are equivalent:

1. \( G \) has exactly two closed labelings.
2. \( G \) is connected and collapsed.

**Proof.** The proof of \( (1) \Rightarrow (2) \) is easy. If \( G \) is not connected, then \( G \) is a disjoint union \( G = G_1 \cup G_2 \), where \( G_i \) is closed. We may assume \( G_1 \) has at least two vertices, so \( G_1 \) has at least two labelings. Then we get at least four closed labelings of \( G \): two where the first is in \( G_1 \), and two where the first is in \( G_2 \). Also, if \( G \) is not collapsed, then some equivalence class \( e(i) \) has at least two elements. If \( |e(i)| \geq 3 \), then switching labels within \( e(i) \) gives at least 6 closed labeling, and if \( |e(i)| = 2 \), then \( G \) has at least one more vertex, which makes it easy to see \( G \) has at least four closed labelings.

The proof of \( (2) \Rightarrow (1) \) will take more work. First note that \( G \) has diameter \( \text{diam}(G) = h \geq 2 \). This follows because \( h = 1 \) would imply that \( G \) is complete, which is impossible since \( G \) is collapsed with \( \geq 3 \) vertices, and \( h = 0 \) is impossible since \( G \) is connected with \( \geq 3 \) vertices.

Fix a closed labeling with \( V(G) = [n] \). This gives layers \( L_0 = \{1\}, L_1, \ldots, L_h \) associated with the labeling, and Proposition 2.2 implies that every longest shortest path has one endpoint in \( L_0 \) or \( L_1 \) and the other in \( L_h \).
Let $\phi : [n] \to [n]$ be another closed labeling which we will call the $\phi$-labeling. Pick $1' \in [n]$ such that $\phi(1') = 1$. Then some longest shortest path of $G$ begins at $1'$. By the previous paragraph, $1' \in L_0 \cup L_1$ or $1' \in L_h$. Replacing $\phi$ with its reversal if necessary, we may assume that $1' \in L_0 \cup L_1$. We claim that $\phi$ is the identity function. This will prove the theorem.

We first show that $1' = 1$, i.e., $\phi(1) = 1$. Recall that $L_1 = N_G(1)$ and that $L_1$ is complete by Proposition 2.3(1). It follows that $N_G^*(1) = L_0 \cup L_1$ is also complete.

The same argument implies that $N_G^*(1')$ is complete. Now suppose $1 \neq 1'$ and pick $m \in N_G^*(1')$ different from $1$. Then $\{1, m\} \in E(G)$ since $1 \in N_G^*(1')$ and $N_G^*(1')$ is complete. This implies $m \in L_1 = N_G(1)$, and then the inclusion $N_G^*(1') \subseteq N_G^*(1)$ follows easily. The opposite inclusion follows by interchanging the two labelings. Hence we have proved $N_G^*(1') = N_G^*(1)$. Since we are assuming $1 \neq 1'$, this contradicts collapsed. Hence we must have $1' = 1$, as claimed.

Now suppose that vertices $1, \ldots, u - 1 \in [n]$ have the same $\phi$-label as in the original labeling, i.e., $\phi(j) = j$ for $1 \leq j \leq u - 1$. Then pick $u' \in [n]$ such that $\phi(u') = u$. To prove that $u' = u$, i.e., $\phi(u) = u$, suppose that $u' \neq u$. Since $\phi$ is the identity on $1, \ldots, u - 1$ and $\phi(u') = u$, we have $u' > u$ and $\phi(u') < \phi(u)$.

We first show that $\{u, u'\} \in E(G)$. Since $G$ is connected, Proposition 2.3 implies that every vertex is connected by an edge to its successor in any closed labeling. For the original labeling, this gives $\{u - 1, u\} \in E(G)$, and for the $\phi$-labeling, this gives $\{u - 1, u'\} \in E(G)$ since $\phi(u - 1) = u - 1$ and $\phi(u') = u$. Proposition 2.3 implies that $N_G^*(u - 1)$ (in the original labeling) is complete, and $\{u, u'\} \in E(G)$ follows.

We next prove that $N_G^*(u) \subseteq N_G^*(u')$. Pick $m \in N_G^*(u)$. Then:

- If $m = u$, then $m \in N_G^*(u')$ since $\{u, u'\} \in E(G)$.
- If $m > u$, then either $m = u'$, in which case $m \in N_G^*(u')$ is obvious, or $m \neq u'$, in which case $m \in N_G^*(u')$ since $m > u' \implies \{m, u'\} \in E(G)$ as the original labeling is closed.
- If $m < u$, then $m \in N_G^*(u')$ since $\phi(m) = m < u \implies \phi(u')$ implies $\{m, u'\} \in E(G)$ as the $\phi$-labeling is closed.

This proves $N_G^*(u) \subseteq N_G^*(u')$. By symmetry, we get $N_G^*(u') = N_G^*(u)$, which contradicts $u' \neq u$ since $G$ is collapsed. We conclude that $u' = u$, and then $\phi$ is the identity by induction on $u$. This completes the proof. \qed

Now suppose that $G$ is a connected graph with a closed labeling. Since each equivalence class is an interval by Proposition 3.3, we can order the equivalence classes

\[(1.1) \quad E_1 < E_2 < \cdots < E_r\]

so that if $i \in E_a$ and $j \in E_b$, then $i < j$ if and only if $a < b$. This induces an ordering on $V(G)/\sim = \{E_1, \ldots, E_r\}$. Then define the graph $G/\sim$ with vertices

\[(2.1) \quad V(G/\sim) = V(G)/\sim = \{E_1, \ldots, E_r\}\]

and edges

\[(3.1) \quad E(G/\sim) = \{\{E_a, E_b\} \mid \{i, j\} \in E(G) \text{ for some } i \in E_a, j \in E_b\}.\]

Since $i \sim i'$ and $j \sim j'$ imply that $\{i, j\} \in E(G)$ if and only if $\{i', j'\} \in E(G)$, we can replace “for some” with “for all” in (3.1).
Theorem 4.3. Let $G$ be connected with a closed labeling and exchangeable equivalence classes $E_1, \ldots, E_r$. Then:

1. The quotient graph $G/\sim$ defined in (4.2) and (4.3) is connected, collapsed, and closed with respect to the labeling $(4.1)$.
2. If $r > 1$, then $G$ has precisely $2 \prod_{a=1}^{r} |E_a|!$ closed labelings.

Proof. For (1), we omit the straightforward proof that $G/\sim$ is connected and closed with respect to $(4.1)$. To prove that $G/\sim$ is collapsed, we first observe that for vertices $u, v \in V(G)$,

$$u \in N^*_G(v) \iff e(u) \in N^*_G(e(v)).$$

We leave the simple proof to the reader. Now suppose that equivalence classes $e(v), e(w)$ satisfy $e(v) \sim e(w)$. Then by (4.4), we have

$$u \in N^*_G(v) \iff e(u) \in N^*_G(e(v)) \iff e(u) \in N^*_G(e(w)) \iff u \in N^*_G(w).$$

This proves that $N^*_G(v) = N^*_G(w)$. Then $v \sim w$, which implies $e(v) = e(w)$. It follows that $G/\sim$ is collapsed.

For (2), first note that $r > 1$ implies $r \geq 3$, for if there were only two equivalence classes $E_1$ and $E_2$, then since $G$ is connected there must be $\{v, w\} \in E(G)$ with $v \in E_1$ and $w \in E_2$. The observation following (4.3) implies that $\{s, t\} \in E(G)$ for all $s \in E_1$ and $t \in E_2$. It follows easily that $G$ is complete, which implies $r = 1$, a contradiction. Hence $r \geq 3$.

According to Proposition 4.2, $G/\sim$ has exactly two closed labelings since it has $r \geq 3$ vertices by the previous paragraph and is connected, closed, and collapsed by (1). It follows from (4.3) that any closed labeling of $G$ induces one of these two closed labelings of $G/\sim$. Hence all closed labelings of $G$ arise from the two ways of ordering the equivalence classes, together with how we order elements within each equivalence class. Proposition 5.2 and the remarks following the proposition imply that we can use any of the $|E|!$ orderings of the elements of an equivalence class $E$. Since different equivalence classes can be ordered independently of each other, we get the desired formula for the total number of closed orderings of $G$. $\square$

5. Counting Closed Graphs

In Theorem 4.3 we fixed a connected graph and counted the number of closed labelings. Here we change the point of view, where we fix a labeling and count the number of connected graphs for which the given labeling is closed.

Here is how a layer of a connected closed graph connects to the next layer.

Definition 5.1. Let $G$ be a connected graph with a closed labeling. Let the layers of $G$ be $L_0 = \{1\}, L_1, \ldots, L_h$, $h = \text{diam}(G)$.

1. Let $a_N = |L_N|$ for $N = 0, \ldots, h$. Note that $a_0 = 1$.
2. If $N < h$, write the vertices of $L_N$ in order. For $1 \leq s \leq a_N$, let $b_s$ be the number of edges of $G$ connecting the $s^{th}$ vertex of $L_N$ to a vertex of $L_{N+1}$.
3. The sequence of $L_N$ is the sequence $S_N = (b_1, b_2, \ldots, b_{a_N})$.

Here is some further notation we will need. First, let $m_N = \min\{L_N\}$ be the minimal element of the $L_N$. Propositions 2.1 and 2.3 imply that $L_N$ is complete and is an interval. Thus $L_N = [m_N, m_N + a_N - 1]$, and the $s^{th}$ vertex of $L_N$ is $u_s = m_N + s - 1$. 


We can now show that the sequence $S_N = (b_1, b_2, \ldots, b_{a_N})$ determines precisely how $L_N$ is connected to $L_{N+1}$.

**Proposition 5.2.** Let $G$ be connected with a closed labeling. If $u_s = m_N + s - 1 \in L_N$ is the $s^{th}$ vertex of $L_N$ and $b_s > 0$, then

$$\{v \in L_{N+1} | \{u_s, v\} \in E(G)\} = [m_{N+1}, m_{N+1} + b_s - 1].$$

Thus $b_s$ determines how $u_s$ links to $L_{N+1}$.

**Proof.** Let $A = \{v \in L_{N+1} | \{u_s, v\} \in E(G)\}$. Note that every $v \in A$ satisfies $v > u_s$ by Proposition 2.3(2). It follows easily that

$$A = N_G^>(u_s) \cap L_{N+1}.$$ 

We know that $L_{N+1}$ is an interval, and the same is true for $N_G^>(u_s)$ by Proposition 2.1. Hence $A$ is an interval. However, if $v \in A$ and $v \neq m_{N+1}$, then $m_{N+1} < v > u_s$ and closed imply $\{u_s, m_{N+1}\} \in E(G)$ since $\{m_{N+1}, v\} \in E(G)$ by the completeness of $L_{N+1}$. Hence $m_{N+1} \in A$, and from here, the proposition follows without difficulty. \qed

Here is an important property of the sequence $S_N$.

**Proposition 5.3.** Let $G$ be connected with a closed labeling. If $N < \text{diam}(G)$, then the sequence $S_N = (b_1, b_2, \ldots, b_{a_N})$ of the layer $L_N$ has the following properties:

1. The last element of $S_N$ is $a_{N+1}$, i.e., $b_{a_N} = a_{N+1}$.
2. $S_N$ is increasing, i.e., $b_s \leq b_{s+1}$ for $s = 1, \ldots, a_N - 1$.

**Proof.** For (1), note that the last vertex of $L_N$ connects to every vertex of $L_{N+1}$ by Proposition 2.3(2). It follows that $b_{a_N} = |L_{N+1}| = a_{N+1}$.

For (2), let $u_s$ be the $s^{th}$ vertex of $L_N$, $1 \leq s \leq a_N - 1$. If $b_s = 0$, then $b_s \leq b_{s+1}$ clearly holds. If $b_s > 0$, then $u_s$ connects to $m_{N+1} + b_s - 1$ by Proposition 5.2 and it connects to $u_{s+1}$ since $L_N$ is complete. Then $m_{N+1} + b_s - 1 > u_s < u_{s+1}$ implies that $u_{s+1}$ connects to $m_{N+1} + b_s - 1$ since the labeling is closed. Using Proposition 5.2 again, we obtain

$$m_{N+1} + b_s - 1 \in [m_{N+1}, m_{N+1} + b_{s+1} - 1],$$

and $b_s \leq b_{s+1}$ follows. \qed

We now come to the main result of this section.

**Theorem 5.4.** Fix $n$ and an integer partition $n = a_0 + a_1 + \cdots + a_h$ with $a_0 = 1$ and $a_N \geq 1$ for $N = 1, \ldots, h$. Also set $L_0 = \{1\}$ and

$$(5.1) \quad L_N = [a_0 + \cdots + a_{N-1} + 1, a_0 + \cdots + a_N]$$

for $N = 1, \ldots, h$, so that $|L_N| = a_N$. Then the number of graphs $G$ satisfying the conditions:

1. $V(G) = [n]$,
2. $G$ is connected and closed with respect to the labeling $V(G) = [n]$, and
3. The $N^{th}$ layer of $G$ is $L_N$ for $N = 0, \ldots, h$,

is given by the product

$$\prod_{N=0}^{h-1} \binom{a_{N+1} + a_N - 1}{a_N - 1}.$$
Proof. Let \( G \) satisfy (1), (2) and (3). Each layer of \( G \) is complete, and every edge of \( G \) connects to the same layer or an adjacent layer by Lemma 2.2. Then Proposition 5.2 shows that the edges of \( G \) are uniquely determined by \( S_0, \ldots, S_{h-1} \).

By Proposition 5.3 each \( S_N = (b_1, b_2, \ldots, b_{a_N}) \) is an increasing sequence of nonnegative integers of length \( a_N \) that ends at \( a_{N+1} \). It is well known that the number of such sequences equals the binomial coefficient \( \binom{a_{N+1}+a_N}{a_N-1} \).

It follows that the product in the statement of the proposition is an upper bound for the number of graphs satisfying (1), (2) and (3).

To complete the proof, we need to show that every sequence counted by the product corresponds to a graph \( G \) satisfying (1), (2) and (3). First note that the minimal element of \( \mathcal{L}_N \) is

\[
m_N = a_0 + \cdots + a_{N-1} + 1
\]

when \( N > 0 \). Now suppose we have sequences \( S_0, \ldots, S_{h-1} \), where each \( S_N = (b_1, b_2, \ldots, b_{a_N}) \) is an increasing sequence of nonnegative integers of length \( a_N \) that ends at \( a_{N+1} \). This determines a graph \( G \) with \( V(G) = [n] \) and the following edges:

(A) All possible edges connecting elements in the same level \( \mathcal{L}_N \).

(B) For each \( N = 0, \ldots, h-1 \), all edges \( \{u_s, v\} \), where \( u_s \) is the \( s^{th} \) vertex of \( \mathcal{L}_N \) and \( v \) is any vertex in the interval \( [m_{N+1}, m_{N+1} + b_s - 1] \subseteq \mathcal{L}_{N+1} \) from Proposition 5.2.

Once we prove that \( G \) is closed and connected with \( \mathcal{L}_N \) as its \( N^{th} \) layer, the theorem will be proved.

Since \( b_{a_N} = a_{N+1} \), we see that for \( N = 0, \ldots, h-1 \), the last element of \( \mathcal{L}_N \) connects to all elements of \( \mathcal{L}_{N+1} \). This enables us to construct a path from 1 to any \( u \in \mathcal{L}_N \) for \( N = 1, \ldots, h \). It follows that \( G \) is connected and that all \( u \in \mathcal{L}_N \) have distance at most \( N \) from vertex 1. Since every edge of \( G \) connects elements of \( \mathcal{L}_M \) to \( \mathcal{L}_M \), \( \mathcal{L}_{M+1} \), or \( \mathcal{L}_{M-1} \), any path connecting 1 to \( u \in \mathcal{L}_N \) must have length at least \( N \). It follows that \( \mathcal{L}_N \) is indeed the \( N^{th} \) layer of \( G \).

It remains to show that \( G \) is closed with respect to the natural labeling given by \( V(G) = [n] \). A vertex of \( G \) is the \( s^{th} \) vertex \( u_s \) of \( \mathcal{L}_N \) for some \( s \) and \( N \). We will show that \( N_G(u_s) \) satisfies Proposition 2.1. The formula (5.1) for \( \mathcal{L}_N \) and the description of the edges of \( G \) given in (A) and (B) make it clear that

\[
N_G(u_s) = [u_{s+1}, a_0 + \cdots + a_N] \cup [m_{N+1}, m_{N+1} + b_s - 1]
\]

where the second equality follows from \( m_{N+1} = a_0 + \cdots + a_N + 1 \). To show that \( N_G(u_s) \) is complete, take distinct vertices \( v, w \in N_G(u_s) \). If both lie in \( \mathcal{L}_N \) or \( \mathcal{L}_{N+1} \), then \( \{v, w\} \in V(G) \) by (A). Otherwise, we may assume without loss of generality that \( v = u_t \), \( t \geq s \), and \( w \in [m_{N+1}, m_{N+1} + b_s - 1] \). Note that \( u_t \) links to every vertex in \( [m_{N+1}, m_{N+1} + b_s - 1] \) by (B). We also have \( b_s \leq b_t \) since \( S_N \) is increasing. It follows that \( \{v, w\} = \{u_t, w\} \in E(G) \). Hence \( N_G(u_s) \) is complete, so that \( G \) is closed by Proposition 2.1. □

6. Local Clustering Coefficients

In a social network, one can ask how often a friend of a friend is also a friend. Translated into graph theory, this asks how often a path of length two has an edge connecting the endpoints of the path. The illustration (1.1) from the Introduction indicates that this should be a frequent occurrence in a closed graph.
There are several ways to quantify the “friend of a friend” phenomenon. For our purposes, the most convenient is the local clustering coefficient of vertex \( v \) of a graph \( G \), which is defined by

\[
C_v = \begin{cases} 
\frac{\text{number of pairs of neighbors of } v \text{ connected by an edge}}{\text{number of pairs of neighbors of } v} & \text{deg}(v) \geq 2 \\
0 & \text{deg}(v) \leq 1.
\end{cases}
\]

Local clustering coefficients are discussed in [8, pp. 201–204].

**Proposition 6.1.** Let \( v \) be a vertex of a closed graph \( G \) of degree \( d = \text{deg}(v) \geq 2 \). Then the local clustering coefficient \( C_v \) satisfies the inequality

\[
C_v \geq \frac{1}{2} - \frac{1}{2(d-1)}.
\]

Furthermore, \( d \geq 3 \) implies that \( C_v \geq \frac{1}{3} \).

**Proof.** Pick a closed labeling of \( G \) and let \( a = |N_G^>(v)| \) and \( b = |N_G^<(v)| \). Then \( a + b = |N_G(v)| = \text{deg}(v) = d \). Since the labeling is closed, any pair of vertices in \( N_G^>(v) \) or in \( N_G^<(v) \) is connected by an edge. It follows that at least

\[
\frac{1}{2}a(a-1) + \frac{1}{2}b(b-1)
\]

pairs of neighbors of \( v \) are connected by an edge. Since the total number of such pairs is \( \frac{1}{2}d(d-1) \) and \( d = a + b \), we obtain

\[
C_v \geq \frac{a(a-1) + b(b-1)}{d(d-1)} = \frac{a^2 + b^2 - d}{d(d-1)} \geq \frac{1}{2d(d-1)} = \frac{1}{2} - \frac{1}{2(d-1)}.
\]

where we use \( a^2 + b^2 - \frac{1}{2}d^2 = \frac{1}{2}(a - b)^2 \geq 0 \). When \( d \geq 4 \), this inequality for \( C_v \) easily gives \( C_v \geq \frac{1}{3} \). When \( d = 3 \), then \( a + b = 3 \), implies that \( a^2 + b^2 \geq 5 \), in which case the left half of (6.1) gives \( C_v \geq \frac{5 - 3}{3(3-1)} = \frac{1}{3} \).

A global version of the clustering coefficient defined by Watts and Strogatz is

\[
C_{WS} = \frac{1}{n} \sum_{v \in V(G)} C_v, \quad n = |V(G)|.
\]

(See reference [323] of [8]. A different global clustering coefficient is discussed in [8, pp. 199–204].) To estimate \( C_{WS} \) for a closed graph, we need the following lemma.

**Lemma 6.2.** Let \( G \) be a connected closed graph.

1. Set \( h = \text{diam}(G) \) and let \( c \) be the number of vertices \( v \in G \) with \( \text{deg}(v) = 2 \) and \( C_v = 0 \). Then \( c \leq h - 1 \).

2. \( G \) has at most two leaves.

**Proof.** For (1), fix a closed labeling for \( G \) with \( V(G) = [n] \) and pick \( v \in V(G) \) with \( \text{deg}(v) = 2 \) and \( C_v = 0 \). We claim that \( v \) is in a layer of its own. To see why, let \( v \in L_N \) and suppose there is \( s \in L_N \) with \( s \neq v \). Then \( \{v, s\} \in E(G) \) since layers are complete by Proposition 2.4(1). Furthermore, \( |L_N| \geq 2 \), so \( N > 0 \). Then \( |s, d| \), \( \{v, d\} \in E(G) \) for \( d = \max|L_{N-1}| \) by Proposition 2.3(2). Since \( \text{deg}(v) = 2 \), we must have \( N_G(v) = \{s, d\} \), and then \( \{s, d\} \in E(G) \) contradicts \( C_v = 0 \). Thus \( \{v\} \) is a layer when \( \text{deg}(v) = 2 \) and \( C_v = 0 \).

Note that if \( \{v\} = L_0 \), then the two vertices in \( N_G(v) = L_1 \) would be linked by an edge. The same holds if \( \{v\} = L_h \), for here the two vertices would be in \( L_{h-1} \).
since \( L_h \) is the highest layer by Proposition 2.3(1). It follows that each of the \( c \) vertices with \( \deg(v) = 2 \) and \( C_v = 0 \) lies in a separate layer distinct from \( L_0 \) or \( L_h \). Since there are only \( h - 1 \) intermediate layers, we must have \( c \leq h - 1 \).

For (2), assume \( G \) has leaves \( u, v, w \) and fix a closed labeling of \( G \). We may assume \( u < v < w \), and let \( u', v', w' \) be the unique vertices adjacent to \( u, v, w \) respectively. A shortest path from \( u \) to \( v \) is directed (see [7] or Proposition 2.1 of [1]) and must pass through \( u' \) and \( v' \), hence \( u < u' \leq v' < v \) since \( u < v \). The same argument applied to \( v \) and \( w \) would imply \( v < v' \leq w' < w \). Thus \( v' < v \) and \( v < v' \), so three leaves cannot exist. \( \square \)

We can now estimate the clustering coefficient \( C_{\text{WS}} \) of a closed graph.

**Theorem 6.3.** If \( G \) is connected and closed with \( n > 1 \) vertices and diameter \( h \), then

\[
C_{\text{WS}} \geq \frac{1}{3} - \frac{h + 1}{3n}.
\]

**Proof.** Since \( n > 1 \) and \( G \) is connected, all vertices of \( G \) have degree \( \geq 1 \). Thus we can write \( V(G) \) as the disjoint union

\[
V(G) = A \cup B \cup C \cup D,
\]

where \( A \) consists of vertices of degree \( \geq 3 \), \( B \) consists of vertices of degree 2 with \( C_v = 1 \), \( C \) consists of vertices of degree 2 with \( C_v = 0 \), and \( D \) consists of the leaves (which have \( C_v = 0 \)). Since \( C_v \geq \frac{1}{3} \) for \( v \in A \) by Proposition 6.1 we have

\[
C_{\text{WS}} \geq \frac{1}{n} \left( \frac{1}{3} \cdot |A| + 1 \cdot |B| + 0 \cdot |C| + 0 \cdot |D| \right) \geq \frac{|A| + |B|}{3n} = \frac{n - (|C| + |D|)}{3n}.
\]

Then we are done since \( |C| \leq h - 1 \) and \( |D| \leq 2 \) by Lemma 6.2. \( \square \)

By Proposition 6.3, the clustering coefficient \( C_{\text{WS}} \) is large when the diameter is small compared to the number of vertices. At the other extreme, both sides of the inequality in Proposition 6.3 are zero when \( G \) is a path graph.

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