Fluctuating loops and glassy dynamics of a pinned line in two dimensions

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Abstract

We represent the slow, glassy equilibrium dynamics of a line in a two-dimensional random potential landscape as driven by an array of asymptotically independent two-state systems, or loops, fluctuating on all length scales. The assumption of independence enables a fairly complete analytic description. We obtain good agreement with Monte Carlo simulations when the free energy barriers separating the two sides of a loop of size $L$ are drawn from a distribution whose width and mean scale as $L^{1/3}$, in agreement with recent results for scaling of such barriers.

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Slow dynamics is perhaps the most significant characteristic of the glassy state of matter, affecting essentially all experimental measurements. An intuitively appealing picture, which explains this remarkable slowing down, is that the configuration space of a glass consists of many nearly degenerate free energy minima, separated by high potential barriers \[ E_B \gg k_B T \]. The dynamics is dominated by transitions between configurations whose free energy difference \( \Delta E \) is less than the thermal energy \( k_B T \), and which are separated by free energy barriers \( E_B \gg k_B T \). In a genuine glass such degeneracies occur on all length scales \( L \). The dependence of \( \Delta E \) and \( E_B \), or more precisely, of their probability distributions, on the linear extent \( L \) has become the focus of great theoretical interest \[ 1–8 \]. If transitions mainly occur between pairs of low-energy configurations which can be regarded asymptotically independent for large \( L \), the simple model of a gas of fluctuating two-level systems \[ 9 \] allows for a fairly complete description of the dynamics. In general, however, a much more complicated hierarchical interdependence of transitions on different length scales may take place \[ 1 \].

The subject of this Letter is an elastic line (henceforth called an interface) in a two-dimensional random potential landscape \[ 3,7,8 \]. This system combines a fair degree of realism e.g. as a model for a domain boundary \[ 5 \] or a magnetic flux line trapped between two copper-oxide planes in a dirty high-temperature superconductor \[ 6,8 \], with a simplicity that has allowed a substantial body of knowledge to accumulate over the past decade \[ 10,11 \]. An almost degenerate two-level system in this case is simply a *loop*: a segment of the line between two points, which can flip between two low energy paths (valleys in the potential landscape) separated by a barrier (a mountain in the landscape), this is illustrated in the inset of Fig. 1. It has been well established \[ 3,4,7 \] that the transverse size of such a loop scales as \( \Delta h \propto L^{\zeta} \) with \( \zeta = 2/3 \), while the free energy differences between the two valley configurations, \( \Delta E \) are distributed with mean zero and variance \( \langle \Delta E^2 \rangle \propto L^{2\theta} \), where \( \theta = 1/3 \). Recent work \[ 8 \] provides evidence that the barriers between such configurations are distributed with mean \( \langle E_B(L) \rangle \propto L^\rho \) and the variance \( \langle E_B^2 \rangle \propto L^{2\theta} \) with likely logarithmic corrections.

These results, when combined, provide all necessary ingredients for developing the dynamic description of the line under the assumption of asymptotic independence of the transi-
tions within large loops. In this Letter we outline such a description, and show that it agrees well with dynamical Monte Carlo simulations. We first numerically confirm that nearly degenerate paths form loops of various sizes, and at first stage we analyze the dynamics of loops of a fixed length. The dynamics of a single loop can be described as a sequence of flips where the interface position moves from one arm of the loop to another. By studying loops of different lengths we determine how the flipping-rate distribution depends on the size of the loop. At the second stage we study the fluctuations of a low-energy interface. Using numerical simulations we determine the time-dependent fluctuations of the interface position, which we compare with an analytic model that assumes that interface fluctuations are due to flipping loops. Since parameters describing loops are determined at the first stage, there are no adjustable parameters left at this point. We find that the prediction of the loop model is in good agreement with the simulations, supporting the conjecture that interface dynamics is due to fluctuating loops.

The numerical model: We study a model that excludes overhangs so that the interface height \( h(x) \) is at all times a single-valued function of the spatial coordinate \( x \). For the numerics we use a lattice model where the interface is discretized to have a unit slope between the lattice points, \(|h(x) − h(x+1)| = 1\), and use fixed boundary conditions \( h(0) = h(L_0) = 0 \), where \( L_0 \) is the interface length. The Hamiltonian of the lattice model for a particular realization of the random medium is

\[
H_\mu[h] = \sum_{x=1}^{L_0} \mu(h(x), x),
\]

where \( h(x), x = 1, \ldots, L_0 \) represents the interface, and \( \mu \) is the potential landscape. The random potential is uncorrelated and uniformly distributed over the range \( 0 \leq \mu(h, x) \leq 1 \) [12]. This Hamiltonian belongs to the universality class of a directed polymer in a random medium (DPRM) [11,13], and in this context the requirement of unit slope between lattice points effectively adds an implicit line tension [11]. The dynamics is implemented in a spirit similar to that of previously introduced mappings onto spin chains [14,15]; the dynamics is described by the master equation based on the transition probability \( P[h(x) \rightarrow h'(x)] = \)}
exp{\(\beta(H_{\mu}[h] - H_{\mu}[h'])/2\)}dt, where \(\beta\) is the inverse temperature, \(dt\) an infinitesimal time interval, and \(h(x)\) and \(h'(x)\) are two interface configurations that differ in only one position. Numerically the dynamics of the master equation is exactly modeled by an algorithm that uses time steps sampled from a Poisson distribution \[16\]. We have used \(\beta = 2\) in the simulations that are presented in this Letter. In general the computation time grows very rapidly with increased \(\beta\).

**Single loop statistics:** In order to explore the ideas of loop-based dynamics we use the free energy landscape to define loops. The free energy of a point \((x, h)\) is defined as
\[ F(x, h) = -\beta^{-1} \ln(P(x, h)), \]
where \(P(x, h)\) is the probability that an interface with fixed ends \(((x, h) = (0, 0)\) and \((L_0, 0))\) crosses this point. The probability \(P(x, h)\) is easily calculated using transfer matrix methods \[11\]. A lattice point is defined to be part of an *island* if it is not part of any interface that includes only points with \(P(x, h) > 0.1\). The level 0.1 is chosen to obtain well defined islands, but the specific value does not greatly affect the final results of the analysis. Interface segments encircling an island form a loop, and we measure the loop size in terms of its length \(L = x_r - x_l + 1\), where \((x_r, h_r)\) and \((x_l, h_l)\) are the right and left ends of the island, respectively. The center height of the island, \(h_{is}\), is defined as the \(h\)-component of its center of mass, where every lattice point in the island is regarded as a mass point. We say that the loop surrounding the island changes its state (flips) when the interface height, averaged over the island length, crosses \(h_{is}\). This definition is convenient although it sometimes catches events that would not intuitively be considered as flips.

In order to study the relationship between static and dynamic scaling, and to unambiguously determine the numerical values of the amplitudes to be later used in the fit, we collect statistics of the dynamics of individual loops. A loop is inscribed in a bounding box which sets the boundaries within which the interface is free to move (showed with dashed lines in Fig. [1]). The interface is constrained to pass through the left and right corners of the bounding box, \((x_l - 2, h_l)\), and \((x_r + 2, h_r)\). We collect statistics of the time between consecutive flips of the loop. In Fig. [4] we plot a typical example of the measured probability that a loop stays in the upper (lower) state for at least time \(t\). The decay of this probability
is exponential for sufficiently large \( t \), which is consistent with the two-state model discussed below (Eq. (2)). The deviation for small \( t \) is probably due to our simplified criterion for flipping the loop. Before making a least-square fit to an exponential decay we exclude the part of the short-time data that strongly deviates from the expected form. We also exclude the largest times (0.1% of the data) since the statistics of these very rare events is poor. From this we obtain the characteristic decay times \( \tau_{+-} \) and \( \tau_{-+} \) for flipping the loop from its upper to lower state and vice versa.

For an individual loop we collect data from 10,000 flips, and make the fit described above. The rate constant, \( \Gamma \), characteristic for the loop, is calculated by \( \Gamma = \tau_{+-}^{-1} + \tau_{-+}^{-1} \) (we assume that the free energy difference between the arms of the loop is unimportant). By collecting statistics from 1000 loops of the same size \( L \) we find that \( \Gamma \) is log-normally distributed. This is consistent with a simple activated behavior, \( \Gamma = \overline{\Gamma} e^{-\beta \Delta} \), where \( \Delta \) is a Gaussian-distributed energy barrier separating the two sides of the loop, and \( \overline{\Gamma} \) is a constant setting the unit of time. In Fig. 2 we show fits to the log-normal distribution by plotting \( Q^{-1}(P(\Gamma)) \) against \( \ln(\Gamma) \), where \( Q \) is the complement of the cumulative normal distribution, and \( P(\Gamma) \) is the measured cumulative probability of the rate constant \( \Gamma \) (hence, a straight line would correspond to a log-normal distribution). The distribution is characterized by the average barrier height \( \Delta(L) \) and the standard deviation \( \sigma(L) \). The values for \( \beta \sigma(L) \) are obtained as the standard deviations of \( \ln(\Gamma) \), which also give the inverse slopes of the lines in Fig. 2. Similarly, the crossings between the lines and the zero axis represent the averages of \( \ln(\Gamma) \), which give values for \( \ln(\overline{\Gamma}) - \beta \Delta(L) \). The average barrier height \( \Delta(L) \) and the standard deviation \( \sigma(L) \) scale with loop size as \( L^{1/3} \). This allows us to determine the time scale \( \overline{\Gamma} \) and the multiplicative constants in \( \Delta(L) \) and \( \sigma(L) \). The mean and standard deviation of \( \ln(\Gamma) \) are plotted as functions of \( L^{1/3} \) in Fig. 3 confirming the \( L^{1/3} \) scaling, and giving for \( \beta = 2 \) the parameter values \( \ln(\overline{\Gamma}) = 2.2, \beta \Delta(L) = 3.0 L^{1/3} \), and \( \beta \sigma(L) = 0.28 L^{1/3} \).

*Full interface numerics:* The same numerical algorithm is used for studying the dynamics of the full interface but keeping only the ends of the interface fixed. We start the interface from a random initial state (chosen from the equilibrium ensemble), measure how
the height \( h_c(t) \) of the center point of the interface varies with time, and collect statistics of \( \delta h^2(t) = [h_c(t) - h_c(0)]^2 \). The equilibrium ensemble is used to normalize \( \delta h^2(t) \) with respect to \( \delta h^2(\infty) \). The data points in Fig. 4 show the results of simulating interfaces of lengths 20, 40, 60, and 128 for \( \beta = 2 \), where \( \delta h^2(t) \) has been averaged over 20,000 realizations of the random medium.

**The analytic model:** We model the loop dynamics using a two-state model, where the probability of a given loop being in the upper or lower state, respectively, is given by \( P_+(t) \) and \( P_-(t) \). The time development of these probabilities is governed by the coupled differential equations

\[
\frac{dP_+}{dt} = -\Gamma_{+-}P_+(t) + \Gamma_{++}P_-(t),
\]

\[
\frac{dP_-}{dt} = -\Gamma_{--}P_+(t) + \Gamma_{-+}P_-(t).
\] (2)

The fluctuation in the height of the center position of a loop of width \( wL^{2/3} \) is given in the two-state model by

\[
\delta h_{\Gamma,L}^2(t) = \langle [h_c(t) - h_c(0)]^2 \rangle = \frac{w^2}{2} L^{4/3} (1 - e^{-\Gamma t}),
\] (3)

where we assumed for simplicity \( \Gamma_{+-} = \Gamma_{--} = \Gamma/2 \), i.e. that the two arms of the loop are exactly degenerate.

We consider an interface of length \( L_0 \), and denote the average barrier height of the largest loops (i.e. loops of size \( L_0 \)) by \( \Delta_0 \), and its standard deviation by \( \sigma_0 \). We assume that the number of loops of a given size scales as \( L^{-1} \), and that the fluctuations of different loops are independent and additive. We find that the total fluctuation of the interface, as implied by loop dynamics, is given by

\[
\frac{\delta h^2(t)}{\delta h^2(\infty)} = 1 - \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \int_{0}^{1} du A u^3 e^{-\frac{1}{2}u^2} e^{-\gamma t} \exp[u\beta(y\sigma_0 - \Delta_0)].
\] (4)

The first integral corresponds to integrating over loops of fixed length but variable rate constants, and the second integral adds up the contributions of loops of different lengths. The infinite time fluctuations \( \delta h^2(\infty) \) are given by the equilibrium result, and scale as \( L_0^{4/3} \).
The two-state model (Eq. (4)) implies that the natural time scale in the problem is given by $t = (\Gamma)^{-1} \ln(2) \exp(\beta \Delta_0 / 2^{1/4})$ in the sense that at time $t = T$ the fluctuations have reached approximately 50% of the equilibrium value; more precisely $0.5 \leq \frac{\delta h^2(T)}{\delta h^2(\infty)} \leq 0.58$ for all $\beta$, $\Delta_0$, and $\sigma_0$. Hence, at low temperatures interface dynamics are exponentially slow as is typical for glassy systems [17].

The solid lines in Fig. 4 are the results of the two-state model using the parameters determined by studying individual loops. The time scale $\Gamma$ determines the position of the curves, the average barrier height, $\Delta_0$, affects both the position and the slope of the curves, and the standard deviation of the barrier heights, $\sigma_0$, has only a minor effect on the results for small $\sigma_0 / \Delta_0$ (taking the limit $\sigma_0 / \Delta_0 \to 0$ would not significantly change the results). Considering that there are no adjustable parameters, the agreement with the results of the numerical simulations in Fig. 4 is quite good. This suggests that the hypothesis that interface dynamics is due to fluctuating loops is indeed valid. The deviations in the short time behavior are, we believe, due in part to more complicated loops on small length scales, which cannot be described as independent simple loops. Another contribution to the short time deviations is that in the lattice model the arms of a loop can have a non-zero width (i.e. two adjacent interface positions are not separated by a barrier), which naturally enhances fluctuations in short time scales. The interpretation that small time differences are due to discretization and finite size corrections is further supported by the fact that the deviations are smaller for larger system sizes. Another possible source of deviations is logarithmic corrections to the scaling forms of $\Delta(L)$ and $\sigma(L)$, however, our data on individual loops is insufficient to determine these corrections.

In conclusion, we have studied the dynamics of a one-dimensional interface in a two-dimensional random medium. We have shown that the dynamics can be understood quantitatively in terms of loops formed by nearly degenerate interface paths. At low temperatures the dynamics is exponentially slow, which is typical for glassy systems [12,17].
REFERENCES

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[1] M. Mezard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific, Singapore (1987).

[2] D. S. Fisher and D. A. Huse, Phys. Rev. B 38, 386 (1988); 38, 373 (1988).

[3] D. A. Huse and C. L. Henley, Phys. Rev. Lett. 54, 2708 (1985).

[4] D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. 55, 2924 (1985).

[5] L. Ioffe and V. M. Vinokur, J. Phys. C 20, 6149 (1987).

[6] D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B 43, 130 (1991).

[7] D. S. Fisher and D. A. Huse, Phys. Rev. B 43, 10728 (1991).

[8] L. V. Mikheev, B. Drossel, and M. Kardar, Phys. Rev. Lett. 75, 1170 (1995).

[9] P. W. Anderson, B. I. Halperin, and C. M. Varma, Philos. Mag. 25, 1 (1972).

[10] M. Kardar, in *Les Houches 1994, Session LXII, fluctuating geometries in statistical mechanics and field theory*, edited by F. David, P. Ginsparg, and J. Zinn-Justin (http://xxx.lanl.gov/lh94/e-book) or cond-mat/9411022).

[11] T. Halpin-Healy and Y. C. Zhang, Phys. Rep. 254, 215 (1995).

[12] A Gaussian distribution with the same mean and standard deviation gives results that marginally differ from those of the uniform distribution.

[13] H. Yoshino, unpublished (cond-mat/9510024).

[14] L.-H. Gwa and H. Spohn, Phys. Rev. A 46, 844 (1992).

[15] H. C. Fogedby, A. B. Eriksson, and L. V. Mikheev, Phys. Rev. Lett. 75, 1883 (1995).
[16] K. Binder, Sec. 1 in *Monte Carlo Methods in Statistical Physics*, edited by K. Binder (Springer-Verlag, Berlin 1979).

[17] K. H. Fisher and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1991).
FIG. 1. Typical time dependence of the probability that a loop stays in the upper (lower) state for at least a time $t$. The data is based on 10,000 crossings in each direction for an island of size $L = 10$. Note that the time scales are normalized to make the right-end value 0.1% of the initial value. The inset schematically shows a loop formed by two nearly degenerate alternative paths of the interface.

FIG. 2. The cumulative distribution, $P(\ln(\Gamma))$, and the cumulative normal distribution $Q$ are used to illustrate the fit to a log-normal distribution of the $\Gamma$ data (solid lines) for the loop sizes $L = 1, 2, 4, 8, 12, 16, 20$ (from right to left). The statistical data is based on 1000 loops of each size.
FIG. 3. The parameters $\Delta(L)$, $\sigma(L)$, and $\Gamma$ are determined by fitting $\beta \Delta(L) - \ln(\Gamma)$ to the mean, and $\beta \sigma(L)$ to the standard deviation of the ln($\Gamma$) data. The error bars show the 95% confidence interval of the statistical error.

FIG. 4. The points show the time dependence of $\delta h^2(t)$ for interfaces of lengths 20, 40, 60, and 128 averaged over 20,000 samples each. The solid lines show the result of the loop model (Eq. (4)) where the parameters of the single-island analysis have been used.