CONSTRUCTION OF SYMMETRIC CUBIC SURFACES

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Abstract. We consider the action of the group PGL₄(𝐾) on the smooth cubic surfaces of ℙ³(K) (𝐾 an algebraically closed field of characteristic zero).
We classify, in an explicit way, all the smooth cubic surfaces with non trivial stabilizer, the corresponding stabilizers and obtain a geometric description of each group in terms of permutations of the Eckardt points, of the 27 lines or of the 45 tritangent planes.

1. Introduction

Traditionally, as it is claimed in the Segre’s book [12], “the study of the general cubic surface dates from 1849, in which year the 27 lines were found by Cayley and Salmon”. Nevertheless, as pointed out by [5], cubic surfaces were considered for the first time in a work of Plücker, which dates of 1829. Certainly, the subject is very old and classic. A wide historical overview on the theme can be found in [5] (see also Nguyen’s Thesis [9]).

Nevertheless, the beauty and richness of the properties of these surfaces inspired and till inspire new researches on the subject. Consider that, even in very recent years, an entire issue of the Journal “Le Matematiche” has been devoted to the cubic surfaces. It is hard to draw up complete references on the topic, we point out however the ample bibliography in the paper [11] (where an interesting list of open problems is given) and in the book [5], where an entire chapter is devoted to give a modern view to the cubic surfaces.

In addition, in the last years several authors have considered the problem of classifying cubic surfaces over finite fields (see, for instance, [6], [3] and the references given there).

Concerning Segre’s investigation, in [12] he described, in particular, the groups of symmetries of the smooth cubic surfaces and gave the list of them. He also realized that non-trivial symmetries are connected to the existence of Eckardt points (also known in the literature as star points), i.e. points which are the intersection of three coplanar lines of the surface.

In more recent years, many authors studied both these topics. In the paper [8], Hosoh reconsidered the problem of possible automorphisms of cubic surfaces and, starting from their description as the blow-ups of 6 generic points of the plane, obtained all the groups. In particular, he pointed out some mistakes in the book of Segre and found a further surface (whose symmetric group is C₈), that was missed in [12]. In 2012 Dolgachev, in Chapter 9 of his book [5], gave another, complete description, of the automorphisms groups and the possible types of the corresponding surfaces.

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Another problem that has received the attention of many authors is the determination of the moduli space of cubic surfaces (see [2], [5] Section 9.4.5) and also [10], where the description of the moduli space is connected to the presence of Eckardt points.

In the present paper, we consider one more time the classification of the automorphisms of smooth cubic surfaces in the three-dimensional projective space.

We follow an approach which allows us to obtain a four dimensional family of cubic surfaces which, from one hand, parametrizes (up to a projectivity) all the smooth cubic surfaces and, from the other hand, allows us to get the explicit equations of the 27 lines for each cubic surface of the family. From the knowledge of the lines then we can easily determine the 45 tritangent planes and the possible Eckardt points. Therefore we stratify the initial four-dimensional family into subfamilies which parametrize cubic surfaces with all the possible configurations of Eckardt points.

Consequently, we study the stabilizer (w.r.t. the action of $\text{PGL}_4(K)$) of every surface of the obtained families, considering the corresponding permutations of the Eckardt points, of the 27 lines or of the tritangent planes.

The paper is organized as follows: in Section 2 we introduce several tools in order to get the equation of a suitable family of cubic surfaces which parametrizes (up to a projectivity) all the smooth cubic surfaces of $\mathbb{P}^3$ and for which the 27 lines are explicitly determined.

As a consequence, in Section 3 we are able to re-obtain many known results on the configuration of the possible Eckardt points which allow us to subdivide the surfaces into several subfamilies.

The knowledge of the lines is the basic point for the construction given in Section 4 (where we compute the projectivities of $\mathbb{P}^3$ stabilizing the considered surfaces) and for the study given in Section 5 (where we complete the determination of the stabilizers, introducing further subfamilies of surfaces). In this way, we get the list of the stabilizers which is clearly the same list obtained by Segre in [12] (with the above mentioned exceptions), by Hosoh in [8], by Dolghacev in [5]. Our approach (which is very elementary) gives, in addition, a uniform way to understand the classification and, furthermore, all the object we manipulate (surfaces, Eckardt points, lines, tritangent planes, projectivities, . . .) are totally explicit.

The final Section, making use of the previous constructions, presents a geometric interpretation of the automorphisms groups in terms of permutations of lines, tritangent planes and Eckardt points.

While we are aware that it is certainly not easy to obtain truly new results after so many years of research on the subject, as far as we know, the approach we propose is not present in the literature and provides some new tools to treat the subject.

In order to obtain these results, we have intensively used packages of symbolic computation (see [1] and [13]) and, in particular, we have implemented some specific Sage software, available at the repository: https://github.com/FedericoPolli/Simmetries_Of_Cubic_Surfaces. The interested reader can download from this site all the procedures we used and also (in the directory computations) several Sage sections which contain the construction of all the possible families of surfaces according to their Eckardt points.
and, for each family, the elements of the automorphisms group, as used in the present paper.

2. Preparatory results

Let us set the following notation: \( K \) is an algebraically closed field of characteristic zero, \( \mathbb{P}^3_K \) is the projective space on \( K \), whose homogeneous coordinates are \([x, y, z, t] \), and \( \text{PGL}_4(K) \) is the projective general linear group (4 \times 4 invertible matrices, up to a scalar) acting in the canonical way on \( \mathbb{P}^3_K \). We refer to it as the group of projectivities of \( \mathbb{P}^3_K \).

Finally, let \( K[x, y, z, t] \) be the ring of polynomials in four variables over \( K \).

If \( F \in K[x, y, z, t] \) is a homogeneous polynomial of degree three, the set of zeroes of \( F \), denoted by \( S = V(F) \), is a cubic surface in \( \mathbb{P}^3_K \).

It is well known that, if \( S \) is smooth, it contains 27 lines having a precise configuration (see [7], chap. V, 4) that we briefly recall. They can be labelled by:

\[
E_i, G_j \quad \text{(for } i = 1, \ldots, 6), \quad F_{ij} \quad \text{(for } 1 \leq i < j \leq 6)
\]

and they intersect according to the rules: \( E_i \) intersects \( G_j \) if and only if \( i \neq j \), \( E_i \) or \( G_i \) intersects \( F_{hk} \) if and only if \( i \in \{ h, k \} \), \( F_{ij} \) intersects \( F_{hk} \) if and only if \( i, j, h, k \) are all distinct.

Cubic surfaces are parametrized by \( \mathbb{P}^{19}_K \), hence a space describing them, up to projectivities, is four-dimensional. In the literature one can find many different ways to introduce it. The construction of the four dimensional family we present here has the advantage to explicitly give all the lines of the cubic surfaces.

In order to do so, we summarize the general approach in [4] even if, in the present paper, we restrict ourselves to the smooth case.

Definition 2.1. An \( L \)-set is a quintuple \((l_1, l_2, l_3, l_4, l_5)\) of lines of \( \mathbb{P}^3_K \) such that \( l_2 \) intersects \( l_1, l_3 \) and \( l_5 \), while \( l_4 \) intersects only \( l_1 \) and \( l_3 \) and there are no further intersections.

Lemma 2.2. If \( L_1 = (l_1, \ldots, l_5) \) and \( L_2 = (l'_1, \ldots, l'_5) \) are two \( L \)-sets, then there exists a unique projectivity of \( \mathbb{P}^3_K \) which sends \( l_i \) to \( l'_i \), for \( i = 1, \ldots, 5 \).

Notation 2.3. We denote such a projectivity by \( M(L_1, L_2) \in \text{PGL}_4(K) \).

It is showed that every smooth cubic surface contains an \( L \)-set (and precisely 25,920). As a consequence, if we choose a specific \( L \)-set then the family of cubic surfaces containing it represents all the smooth cubic surfaces, up to a projectivity.

In the sequel, we will choose a specific \( L \)-set as follows.

Definition 2.4. We call basic \( L \)-set, and denote it by \( L_b \), the following quintuple

\[
(2.2) \quad L_b = (l_1, l_2, l_3, l_4, l_5) = ((y, z), (x, y), (x, t), (x - z, y - z), (x - y, z + t)).
\]

The family of all cubic surfaces passing through \( L_b \) is the four dimensional linear system given by

\[
(2.3) \quad a(2x^2y - 2xy^2 + xz^2 - xzt - yt^2 + yzt) + b(x - t)(xz + yt) + c(z + t)(yt - xz) + d(y - z)(xz + yt) + g(x - y)(yt - xz) = 0.
\]
The parameters $a, b, c, d, g$ give singular surfaces if and only if satisfy $\sigma = 0$, where

\[
\sigma = (a + b - c)(2a + b - d)(a - c - d)(a + c + g) - (a + c - g)(4ac - g^2)(a^2 + ac - 2ad + ag + d^2 - dg) - (a^2 + 2ab + ac - ag + b^2 - bg) - (4a^2 + 3ab - 4ac - 3ad - bc - 2bd + bg + cd + dg).
\]

(2.4)

and we briefly say that it is the singular locus inside the family $\{2, 3\}$.

**Definition 2.5.** A plane $\pi$ is called tritangent to a smooth cubic surface $S$ if $\pi \cap S$ consists of three (distinct) lines.

If $r$ and $s$ are two meeting lines of a smooth cubic surface $S$, then the plane containing them is tritangent $S$ and the third line will be denoted by $\text{res}(r, s)$ and called the residue line of $r$ and $s$.

If $r$ and $s$ are as above, the tritangent plane containing them will be denoted by the triplet of lines $(r, s, \text{res}(r, s))$ or simply by $(r, s)$.

The properties of incidence of the 27 lines (see (2.1)) allow us to realize that there are 45 tritangent planes to a smooth cubic surface $S$ (see Table 1).

| $\tau_1$ : $(E_1, G_2)$ | $\tau_2$ : $(E_1, G_3)$ | $\tau_3$ : $(E_1, G_4)$ | $\tau_4$ : $(E_1, G_5)$ | $\tau_5$ : $(E_1, G_6)$ |
|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| $\tau_6$ : $(E_2, G_1)$ | $\tau_7$ : $(E_2, G_3)$ | $\tau_8$ : $(E_2, G_4)$ | $\tau_9$ : $(E_2, G_5)$ | $\tau_{10}$ : $(E_2, G_6)$ |
| $\tau_{11}$ : $(E_3, G_1)$ | $\tau_{12}$ : $(E_3, G_2)$ | $\tau_{13}$ : $(E_3, G_4)$ | $\tau_{14}$ : $(E_3, G_5)$ | $\tau_{15}$ : $(E_3, G_6)$ |
| $\tau_{16}$ : $(E_4, G_1)$ | $\tau_{17}$ : $(E_4, G_2)$ | $\tau_{18}$ : $(E_4, G_3)$ | $\tau_{19}$ : $(E_4, G_4)$ | $\tau_{20}$ : $(E_4, G_5)$ |
| $\tau_{21}$ : $(E_5, G_1)$ | $\tau_{22}$ : $(E_5, G_2)$ | $\tau_{23}$ : $(E_5, G_3)$ | $\tau_{24}$ : $(E_5, G_4)$ | $\tau_{25}$ : $(E_5, G_5)$ |
| $\tau_{26}$ : $(E_6, G_1)$ | $\tau_{27}$ : $(E_6, G_2)$ | $\tau_{28}$ : $(E_6, G_3)$ | $\tau_{29}$ : $(E_6, G_4)$ | $\tau_{30}$ : $(E_6, G_5)$ |
| $\tau_{31}$ : $(F_{12}, F_{34})$ | $\tau_{32}$ : $(F_{12}, F_{35})$ | $\tau_{33}$ : $(F_{12}, F_{36})$ | $\tau_{34}$ : $(F_{13}, F_{24})$ | $\tau_{35}$ : $(F_{13}, F_{25})$ |
| $\tau_{36}$ : $(F_{13}, F_{26})$ | $\tau_{37}$ : $(F_{14}, F_{23})$ | $\tau_{38}$ : $(F_{14}, F_{25})$ | $\tau_{39}$ : $(F_{14}, F_{26})$ | $\tau_{40}$ : $(F_{15}, F_{23})$ |
| $\tau_{41}$ : $(F_{15}, F_{24})$ | $\tau_{42}$ : $(F_{15}, F_{26})$ | $\tau_{43}$ : $(F_{16}, F_{23})$ | $\tau_{44}$ : $(F_{16}, F_{24})$ | $\tau_{45}$ : $(F_{16}, F_{25})$ |

Here we list a few simple properties about the 27 lines of a smooth cubic surface $S$.

**Remark 2.6.** Let $r, s, t$ be three lines on $S$. If $r$ and $s$ are incident and $t$ is skew with $r$ and $s$, then $t$ meets $\text{res}(r, s)$.

**Lemma 2.7.** Let $\{s_1, s_2\}$ and $\{r_1, r_2, r_3\}$ two sets of skew lines on a smooth cubic surface such that $s_i \cap s_j \neq \emptyset$, for all $i, j$. Then the line $t = \text{res}(\text{res}(r_1, s_1), \text{res}(r_2, s_2))$ intersects $r_3$.

**Proof.** The lines $\text{res}(r_1, s_1)$ and $\text{res}(r_2, s_2)$ are skew with $r_3$, so, by Remark 2.6, $t$ intersects $r_3$. □

Again from the incidence properties (see (2.1)), a simple argument shows that, given two skew line $s_1$ and $s_2$ on $S$, there are five lines of $S$ intersecting $s_1$ and $s_2$. The next result is more precise and leads to determine by residuality all the lines of $S$. 
Proposition 2.8. Let \( \{s_1, s_2\} \) and \( \{r_1, r_2, r_3, r_4\} \) two sets of skew lines on a smooth cubic surface \( S \) such that \( s_i \cap r_j \neq \emptyset \), for all \( i, j \). Then all the 27 lines of \( S \) can be determined by residuality.

Proof. The line \( t = \text{res}(\text{res}(r_1, s_1), \text{res}(r_2, s_2)) \) intersects \( r_3 \) and \( r_4 \) from Lemma 2.7. Consider now the line \( u = \text{res}(\text{res}(r_3, s_2), \text{res}(r_4, t)) \). Then, again from Lemma 2.7, \( u \) intersects \( s_1 \). Since \( s_1 \) and \( u \) are skew with \( s_2 \), we get from Remark 2.6 that the line \( r_5 = \text{res}(s_1, u) \) intersects \( s_2 \), and clearly also \( r_1 \).

Now it is straightforward to see that \( \{s_1, s_2\} \) and \( \{r_1, r_2, r_3, r_4, r_5\} \) give rise to the remaining 20 lines of \( S \) in the following way:

\[
\begin{align*}
\text{res}(r_i, s_j) & \text{ for } i = 1, \ldots, 5, \quad j = 1, 2 \\
\text{res}(\text{res}(r_i, s_1), \text{res}(r_j, s_2)) & \text{ for } i = 1, \ldots, 4, \quad j = i + 1, \ldots, 5.
\end{align*}
\]

\[\square\]

This result leads us to introduce the notion of another useful sextuple of lines.

Definition 2.9. A 6-uple of lines \( (l_1, \ldots, l_6) \) of a smooth cubic surface \( S \) such that \( L = (l_1, \ldots, l_5) \) is an L-set and \( l_6 \) intersects \( l_2 \) and \( l_4 \) and is skew with the other lines of \( L \) is called an extended L-set of \( S \).

Proposition 2.10. Given an extended L-set \( (l_1, \ldots, l_6) \) of a smooth cubic surface \( S \), all the other lines of \( S \) can be determined by residuality, in a unique way.

Proof. Indeed, \( l_1, l_3, \text{res}(l_2, l_5), l_6 \) are four skew lines meeting both \( l_2 \) and \( l_4 \). From Proposition 2.8 we can find all the lines of \( S \). \[\square\]

Corollary 2.11. As soon as the labels of the lines of an extended L-set are chosen, the labels of all the 27 lines are uniquely determined.

This gives immediately the following result.

Corollary 2.12. There is a one to one correspondence between extended L-sets and the permutations of the 27 lines preserving the incidence relations.

The above facts (Proposition 2.10 and Corollary 2.12) do not hold concerning L-sets, as the following result shows.

Proposition 2.13. For any L-set \( L = (l_1, \ldots, l_5) \) of \( S \), there exist exactly two lines \( l_6 \) and \( l_6' \) of \( S \) such that \( L_e = (l_1, \ldots, l_5, l_6) \) and \( L'_e = (l_1, \ldots, l_5, l'_6) \) are two extended L-sets.

Proof. There are exactly five (mutually skew) lines meeting both \( l_2 \) and \( l_4 \). Clearly, \( l_1, l_3, \text{res}(l_2, l_5) \) are three among them. Setting \( l_6 \) and \( l'_6 \) the two remaining lines, it follows that \( L_e \) and \( L'_e \) are two extended L-sets. \[\square\]

Observe that both \( l_6 \) and \( l'_6 \) have the same incidence relations with respect the lines of \( L \); in other words, an L-set cannot distinguish the two extended L-sets which contain it.

Remark 2.14. Since the number of L-sets on a smooth cubic surface is 25,920, there are 51,840 extended L-sets and therefore, by Corollary 2.12, the group of permutations of the 27 lines of a cubic surface has order 51,840, in accordance to the order of the Weyl group \( E_6 \) (see, for instance, [1]).
Notation 2.15. For this reason, from now on, we denote by $E_6$ the group of the permutations of the 27 lines. Clearly, each of such permutations preserves the incidence relations among all the lines.

From Corollary 2.11 all the extended $L$-sets are equivalent. For this reason, we assume that the basic $L$-set is

$$L_b = (l_1, l_2, l_3, l_4, l_5) = (E_1, G_4, E_2, G_3, E_3)$$

and the basic extended $L$-set is

$$L_{bc} = (l_1, l_2, l_3, l_4, l_5, l_6) = (E_1, G_4, E_2, G_3, E_3, E_5).$$

All the smooth cubic surfaces containing the basic $L$-set $L_b$ (whose lines have the equations given in (2.2)) are described by the equation (2.3), where $\sigma(a, b, c, d, g) \neq 0$. However, it is convenient to make the substitution

$$g = e + f, \quad a = ef/c$$

in (2.3) and (2.4) and this is possible since $c = 0$ gives a singular cubic surface. We then obtain the equation (representing a four-dimensional quadric in $\mathbb{P}_K^3$) of the family parametrizing, up to a projectivity, all the smooth cubic surfaces of $\mathbb{P}_K^3$:

$$bc(t - x)(xz + yt) + c^2(z + t)(xz - yt) - cd(y - z)(xz + yt) + c(e + f)(x - y)(xz - yt) - ef(2x^2y - 2xy^2 + x^2z - xzt + yzt - yt^2) = 0$$

(2.5)

and the corresponding singular locus

(2.6)

$$\Sigma_0 = c(e - f)(-c + e)(e + f)(e + e)(-e + f)(-cd + cf + ef)(-cd + ce + ef) \cdot$$

$$(-c^2 - cd + ef)(bc - cf + ef)(bc - ce + ef)(bc - cd + 2ef)(bc - c^2 + ef) \cdot$$

$$(bc^2 + c^2d + bc^2f - 2c^2f - cdf + 2ef^2)(bc^2 + c^2d + bc - 2c^2e - cde + 2ef^2) \cdot$$

$$(-bc^3 - 2bc^2d + c^3d + bc^2e + c^2de + bc^2f + c^2df + 3bcef - 4c^2ef$$

$$- 3cdef + 4e^2f^2).$$

Following the proof of Proposition 2.13, we can determine exactly two lines $l_6$ and $l_6'$ that complete $L_b$ to an extended $L$-set. We choose as $l_6$ (or, equivalently, $E_5$) the line having Plücker coordinates:

$$[0, (f - c)(cd - cf - ef)(bc - cf + ef),$$

$$c - f)(cd - cf - ef)^2, (c + f)(bc - cf + ef)^2,$$

$$c + f)(cd - cf - ef)(cf - ef - bc), 2f(cd - cf - ef)(cf - ef - bc)].$$

One can see that $l_6'$ has the Plücker coordinates obtained from (2.7) by exchanging $e$ and $f$ and this is consistent with the fact that (2.5) and (2.6) are symmetric in $e$ and $f$. After this choice, all the remaining lines can be explicitly obtained from the extended $L$-set $L_{bc}$ as in Proposition 2.10.

The knowledge of the equations of the 27 lines for the generic cubic of the family (2.5) is the starting point for the constructions that we are going to introduce.
3. Eckardt Points and Eckardt Families

**Definition 3.1.** A point on a smooth cubic surface $S$ is called an *Eckardt point* (see [12] or [5], Section 9.1.4) if it is the intersection of three (necessarily coplanar) lines of $S$. The tritangent plane containing these three lines is said an *Eckardt plane*.

**Notation 3.2.** If $P = r \cap s \cap t$ is an Eckardt point of $S$, it uniquely determines the Eckardt plane $\pi = (r, s, t)$. Therefore, we shall use the same notation to denote both, i.e., we shall write $P = (r, s, t)$ (or simply $P = (r, s)$).

In this way, Table 3 lists either the tritangent planes and the *possible* Eckardt points (planes).

There are many ways to determine conditions to impose the existence of Eckardt points, but, since we know the equations of all the lines of $S$, as soon as a tritangent plane is given, we can easily obtain the conditions on the coefficients $b, c, d, e, f$ of (2.5) forcing the lines of a certain tritangent plane to meet in a common point.

**Example 3.3.** Consider the tritangent plane $\tau_3 = (E_1, G_4, F_{14})$. It has an Eckardt point if and only if the three coplanar lines (here given by their Plücker coordinates) $E_1 = [0, 0, 1, 0, 0, 0], G_4 = [0, 0, 0, 0, 0, 1], F_{14} = [0, bc + c^2 + ef, -c^2 - cd + ef, 0, 0, c(-b + e + f)]$ have a common point. It turns out the corresponding condition is $bc + c^2 + ef = 0$. Similarly, $\tau_8 = (E_2, G_4, F_{24})$ has an Eckardt point if and only if $c^2 - cd + ef = 0$.

In the same way, for each tritangent plane $\tau_i$ we can determine a polynomial $P_i \in K[b, c, d, e, f]$ such that $\tau_i$ is an Eckardt plane if and only if $P_i(b, c, d, e, f) = 0$.

We collect all the conditions and get a list $Q$ of the 45 polynomials $P_i(b, c, d, e, f)$, $i = 1, \ldots, 45$, here omitted for shortness.

**Remark 3.4.** It is not difficult to see that every cubic surface with at least one Eckardt point is projectively equivalent to a cubic surface containing the basic $L$-set (2.2) and such that the Eckardt point is $A = \tau_3 = (E_1, G_4)$. The corresponding condition is $\Lambda_1 = \{b = -(c^2 + ef)/c\}$ and this substitution in equation (2.5) yields to a family, say $S_1$, representing all the cubic surfaces with at least one Eckardt point up to a projectivity and whose singular locus is given by $\Sigma_1$, obtained from (2.6) together with the condition $\Lambda_1$.

The above substitution annihilates precisely one of the 45 polynomials of $Q$ (clearly the polynomial $P_3$), hence the general element of $S_1$ is a smooth cubic surface with exactly one Eckardt point.

The same kind of argument can be used to detect families of surfaces with a larger number of Eckardt points.

**Remark 3.5.** A cubic surface $S$ containing two Eckardt points (on a line of the surface) is projectively equivalent to a surface containing the basic $L$-set $L_b$ (2.2) and such that the Eckardt points are $A = E_1 \cap G_4$ and $D = E_2 \cap G_4$. A direct computation shows that this family, denoted by $S_2$ (see Table 3 and Table 4), and its singular locus $\Sigma_2$, can be obtained by the substitutions

$$\Lambda_2 = \{b = -(c^2 + ef)/c, \quad d = (c^2 + ef)/c\}$$

into (2.5) and (2.6).

Since, in the above list $Q$, no other polynomial but $P_3$ and $P_8$ vanishes, we obtain
that the general element of $\mathcal{S}_2$ is a smooth cubic surface with exactly two Eckardt points (contained in a line of the surface).

**Remark 3.6.** Suppose now that a smooth cubic surface has two Eckardt points not contained in one of its lines. Again, we can assume, up to a projectivity, that the surface passes through the $L$-set $L_b$ and that $A = E_1 \cap G_4 = \tau_3$ and $C = E_2 \cap G_3 = \tau_7$ are Eckardt points. The corresponding set of conditions is

$$A_3 = \{b = -(c^2 + ef)/c, d = (3ef - c^2 + cf + ce)/(2e)\}.$$

When we make this substitution in $Q$, we obtain that vanish not only $P_3$ and $P_7$, but also $P_{34}$. This means that $\tau_{34} = (F_{13}, F_{24}, F_{36})$ is an Eckardt point, that turns out to be collinear with $A$ and $C$.

Since no other polynomial of $Q$ vanishes, we get a family $\mathcal{S}_3$ which contains (up to a projectivity) all the smooth cubic surfaces with three collinear Eckardt points.

Collecting the three Remarks above, we have the following result (see also [12] Ch. IV, Sect. XIV and [5], Proposition 9.1.26).

**Proposition 3.7.** The following facts hold.

1. There exist cubic surfaces with precisely one Eckardt point.
2. If a cubic surface has two Eckardt points, then the line joining them either is contained in the surface (and in this case it cannot contain other Eckardt points) or is not a line of the surface (and in this case it intersects the surface in a third point which is another Eckardt point).
3. There exist cubic surfaces with precisely two Eckardt points.
4. There exist cubic surfaces with precisely three Eckardt points. In this case the three points are collinear.

**Proof.** (2) We have only to see that it is not possible to have three collinear Eckardt points contained in a line of the surface. This is a consequence of the fact that if we impose that $(E_1, G_4)$, $(E_2, G_4)$ and $(E_3, G_4)$ are Eckardt points, we obtain that the corresponding cubic surfaces are singular.

(4) Suppose that a cubic surface has precisely three Eckardt points $A_1$, $A_2$ and $A_3$ which are not collinear. Then, from part (2), the lines $r_1 = A_1 + A_2$, $r_2 = A_1 + A_3$ and $r_3 = A_2 + A_3$ must be contained in the surface, so the tritangent plane $A_1 + A_2 + A_3$ contains $r_1, r_2, r_3$ and also the lines $\text{res}(r_1, r_2), \text{res}(r_1, r_3), \text{res}(r_2, r_3)$ and this is impossible. □

**Corollary 3.8.** If a smooth cubic surface contains at least three Eckardt points then, up to a projectivity, it is a cubic surface of the family $\mathcal{S}_3$.

**Proof.** If the three points $A_1$, $A_2$ and $A_3$ are collinear, the situation is described in Remarks 3.5 and 3.6, in this case the surface is projectively equivalent to an element of $\mathcal{S}_3$. Otherwise, with the same argument of Proposition 3.7 - (4), at least one of the three lines connecting $A_1$, $A_2$ and $A_3$ cannot be contained in the surface. Therefore it contains a further Eckardt point and, so, the surface contains three collinear Eckardt points. □

The two results above show that, in order to describe all the possible configurations of $n$ Eckardt points of a smooth cubic surface (with $n \geq 3$), we have to study the subfamilies of $\mathcal{S}_3$. 
Therefore, from now on in this Section, we consider the family $\mathcal{S}_{e_3}$ given by (2.6) under the conditions $\Lambda_3$. In this setting, the list $Q$ specializes in the following way: clearly $P_3$, $P_7$ and $P_{34}$ vanish since they correspond to the Eckardt planes $\tau_3 = (E_1, G_4, F_{14})$, $\tau_7 = (E_2, G_3, F_{23})$ and $\tau_{34} = (F_{13}, F_{24}, F_{56})$, as described in Remark 3.6. Moreover, from Proposition 3.7 - (2), to impose a fourth Eckardt point could force the surface to have some further Eckardt point. In this case, several polynomials of $Q$ coincide. The remaining distinct and non zero polynomials are 14. After renaming them as $Q_{11}, \ldots, Q_{14}$, we obtain:

\[
\begin{align*}
Q_1 &= 5c^2 - ce - cf + ef; & Q_2 &= 3c^2 + ce + cf - ef; \\
Q_3 &= c^2 + 3ce - cf + ef; & Q_4 &= c^2 - ce + 3cf + ef; \\
Q_5 &= 5c^2 + ce + cf + ef; & Q_6 &= 3c^2 - ce - cf - ef; \\
Q_7 &= c^2 + ce - 3cf + ef; & Q_8 &= c^2 - 3ce + cf + ef; \\
Q_9 &= c^2 + ef; & Q_{10} &= 3c^2 + e^2; \\
Q_{11} &= 3c^2 + f^2; & Q_{12} &= 2c - e + f; \\
Q_{13} &= 2c + e - f; & Q_{14} &= e + f
\end{align*}
\]

and they are associated to the Eckardt points accordingly to Table 2.

**Table 2.** List of Eckardt planes associated to $Q_i = 0$.

| $Q_1$ | $\tau_{11}, \tau_{18}$ | $Q_2$ | $\tau_2, \tau_{36}$ |
|-------|------------------------|-------|---------------------|
| $Q_3$ | $\tau_{11}, \tau_{28}, \tau_{35}$ | $Q_4$ | $\tau_5, \tau_{23}, \tau_{36}$ |
| $Q_5$ | $\tau_6, \tau_{13}, \tau_{17}$ | $Q_6$ | $\tau_8$ |
| $Q_7$ | $\tau_9, \tau_{29}, \tau_{41}$ | $Q_8$ | $\tau_{10}, \tau_{24}, \tau_{14}$ |
| $Q_9$ | $\tau_{12}, \tau_{16}, \tau_{31}$ | $Q_{12}$ | $\tau_{25}, \tau_{39}, \tau_{13}$ |
| $Q_{13}$ | $\tau_{30}, \tau_{38}, \tau_{40}$ | $Q_{14}$ | $\tau_{37}$ |
| $Q_{10}$ | $\tau_{14}, \tau_{20}, \tau_{22}, \tau_{26}, \tau_{33}, \tau_{42}$ | $Q_{11}$ | $\tau_{15}, \tau_{19}, \tau_{21}, \tau_{27}, \tau_{32}, \tau_{45}$ |

**Remark 3.9.** The polynomials $Q_4, Q_8, Q_{11}$ and $Q_{13}$ can be ignored since they are obtained, respectively, from $Q_3, Q_7, Q_{10}$ and $Q_{12}$ exchanging $e$ and $f$ (and we already observed that $Q_{23}$ is invariant with respect to this exchange).

We can see, from Table 2, that there are cubic surfaces with 4 Eckardt points (when $Q_2$ or $Q_6$ or $Q_{14}$ are zero) or cubic surfaces with 6 Eckardt points (when one of the polynomial $Q_1, Q_3, Q_4, Q_5, Q_7, Q_8, Q_9, Q_{12}, Q_{13}$ is zero) or cubic surfaces with 9 Eckardt points (when $Q_{10}$ is zero).

**Lemma 3.10.** Let $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$ be the three subfamilies of $\mathcal{S}_{e_3}$ given by the conditions $Q_2 = 0, Q_6 = 0, Q_{14} = 0$, respectively.

Then, for every smooth surface $T \in \mathcal{T}_1$ (respectively, $T_2$) there exists $S \in \mathcal{T}_0$ such that $S$ and $T$ are projectively equivalent, and conversely.

**Proof.** The cubic surfaces of $\mathcal{S}_{e_3}$ have $\tau_3 = (E_1, G_4)$, $\tau_7 = (E_2, G_3)$, $\tau_{34} = (F_{13}, F_{24})$ as Eckardt points. Moreover, those of $\mathcal{T}_0$ also $\tau_2 = (E_1, G_3)$ and those of $\mathcal{T}_1$ also $\tau_6 = (E_2, G_4)$.

Therefore, on one hand, $S \in \mathcal{T}_0$ if and only if it has the following Eckardt points

$$(l_1, l_2), (l_1, l_4), (l_3, l_4), (\text{res}(l_1, l_4), \text{res}(l_2, l_3))$$
with respect to the basic $L$-set
$$L_0 = (l_1, l_2, l_3, l_4, l_5) = (E_1, G_4, E_2, G_3, E_3).$$

On the other hand, $T \in \mathcal{T}_1$ if and only if it has the following Eckardt points
$$(r_1, r_2), (r_1, r_4), (r_3, r_4), (\text{res}(r_1, r_4), \text{res}(r_2, r_3)),$$
with respect to the $L$-set
$$L = (r_1, r_2, r_3, r_4, r_5) = (G_4, E_1, G_3, E_2, G_2).$$

So, if we put $M = M(L, L_0)$, then for each $T \in \mathcal{T}_1$ the cubic surface $M(T)$ belongs to the family $\mathcal{T}_0$. The converse holds by exchanging the roles of $\mathcal{T}_0$ and $\mathcal{T}_1$.

The same argument runs concerning $\mathcal{T}_2$. □

Since the above three subfamilies of $Se_3$ are projectively equivalent, we can choose one of them as the space parametrizing cubic surfaces having four Eckardt points. So, we set $Se_4 = \mathcal{T}_0$.

The conditions and the equation defining $Se_4$ are contained in Table 3 and in Table 4 (the denominator $c + e$ can be assumed non zero, since it appears as a factor in the polynomial (2.6) defining the singular locus).

Concerning cubic surfaces with 6 Eckardt points, one can see from Table 2 that there are 9 families of this type. With an argument quite similar to the proof of Lemma 3.10, we can show that these 9 families are all projectively equivalent and, so, all the cubic surfaces with 6 Eckardt points are parametrized by the subfamily of $Se_3$ defined, for instance, by the further condition $Q_5 = 0$. We denote such a family by $Se_6$.

Finally, let us consider the case of 9 Eckardt points. From Table 2 and Remark 3.9 it is enough to consider the only condition $Q_{10} = 3c^2 + e^2 = 0$. This gives rise to two families $\mathcal{T}_1$ and $\mathcal{T}_2$ of cubic surfaces:
$$\mathcal{T}_1 : \Lambda_3 \cup \{e = \sqrt{-3}c\}, \quad \mathcal{T}_2 : \Lambda_3 \cup \{e = -\sqrt{-3}c\}$$

**Lemma 3.11.** For every $T_1(c, f)$ in $\mathcal{T}_1$ there exist $c', f'$ such that the corresponding surface $T_2(c', f') \in \mathcal{T}_2$ is projectively equivalent to $T_1$ and conversely.

**Proof.** Consider the $L$-set of $T_1$ given by $L = (F_{46}, G_6, F_{26}, F_{15}, E_3)$ and the unique matrix $M = M(L, L_0)$ (see Lemma 2.2). It can be easily checked that $M^{-1}(T_1) = T_2(f - c, -c + 2\sqrt{-3}c - f)$. □

The families $Se_4 = Se_4(c, e), Se_6 = Se_6(c, e)$ and $Se_9 = Se_9(c, f)$ depend on two parameters, so are one dimensional families. We specialize the polynomials of Table 2 with conditions $\Lambda_4, \Lambda_6$ and $\Lambda_9$ respectively, in order to see if these families contain surfaces with more Eckardt points. We obtain several cubic surfaces with 10 Eckardt points and several others with 18 Eckardt points. Similar arguments to those discussed above show that, up to projectivities, there is only one cubic surface $Se_{10}$ with 10 Eckardt points obtained specializing (2.5) with the conditions $\Lambda_{10} = \Lambda_4 \cup \{e = (2 - \sqrt{5})c\}$ (whose singular locus is $\Sigma_{10} = \emptyset$, since $Se_{10}$ does not depend on parameters) and only one cubic surface $Se_{18}$ with 18 Eckardt points obtained by the conditions $\Lambda_{18} = \Lambda_4 \cup \{e = \sqrt{-3}c\}$ (again, the singular locus is $\Sigma_{18} = \emptyset$).

We conclude this section summarizing the above results (see also [12], Ch. IV, Sect. xiv).
Theorem 3.12. A smooth cubic surface can have only 0, 1, 2, 3, 4, 6, 9, 10 or 18 Eckardt points. If \( S \) is a smooth cubic surface with \( n \) Eckardt points, then it is projectively equivalent to a cubic surface of the family \( S_{eq} \) (here, for completeness, by \( S_{eq} \) we denote the family \( 2.5 \)) whose singular locus is given by \( \Sigma_n \).

**Definition 3.13.** The families of cubic surfaces \( S_{eq} \), where \( n = 0, 1, 2, 3, 4, 6, 9, 10, 18 \), will be called **Eckardt families**.

In Table 3 we list the conditions to impose to \( 2.5 \) in order to obtain the families \( S_{eq} \). In the third column, we list the Eckardt points, labelled accordingly to Table 1. In the last column we report the dimension of the Eckardt families. The families \( S_{eq}', S_{eq}'' \) and \( S_{eq}''' \) will be introduced in Section 5.

**Table 3.** Eckardt families and Eckardt points

| Family | Conditions | Eckardt points | Dim |
|--------|------------|----------------|-----|
| \( S_{e1} \) | \( A_1 : \{ b = -(c^2 + ef)/c \} \) | \( \tau_3 \) | 3 |
| \( S_{e1}' \) | \( A_1 + (5.1) \) | '' | 1 |
| \( S_{e1}'' \) | \( A_1 + (5.1) + (5.2) \) | '' | 0 |
| \( S_{e2} \) | \( A_2 : A_1 \cup \{ d = (c^2 + ef)/c \} \) | \( \tau_3, \tau_8 \) | 2 |
| \( S_{e3} \) | \( A_3 : A_1 \cup \{ d = (3ef - c^2 + ef + ce)/(2e) \} \) | \( \tau_3, \tau_7, \tau_34 \) | 2 |
| \( S_{e4} \) | \( A_4 : A_3 \cup \{ f = c(3e - c)/c \} \) | \( \tau_3, \tau_7, \tau_8, \tau_34 \) | 1 |
| \( S_{e6} \) | \( A_6 : A_3 \cup \{ f = c(5c + e)/(c + e) \} \) | \( \tau_3, \tau_6, \tau_7, \tau_13, \tau_17, \tau_34 \) | 1 |
| \( S_{e9} \) | \( A_9 : A_3 \cup \{ e = \sqrt{-3}c \} \) | \( \tau_3, \tau_7, \tau_14, \tau_20, \tau_22, \tau_26, \tau_33, \tau_34, \tau_42 \) | 1 |
| \( S_{e9}' \) | \( A_9 + (5.3) \) | '' | 0 |
| \( S_{e10} \) | \( A_{10} : A_4 \cup \{ e = (2 - \sqrt{5})c \} \) | \( \tau_1, \tau_3, \tau_7, \tau_8, \tau_{11}, \tau_{12}, \tau_{16}, \tau_{18}, \tau_{31}, \tau_{34} \) | 0 |
| \( S_{e18} \) | \( A_{18} : A_4 \cup \{ e = \sqrt{-3}c \} \) | \( \tau_2, \tau_3, \tau_7, \tau_8, \tau_{14}, \tau_{15}, \tau_{19}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{26}, \tau_{27}, \tau_{32}, \tau_{33}, \tau_{34}, \tau_{37}, \tau_{32}, \tau_{45} \) | 0 |
For sake of shortness, Stab($S$) for a generic $S \in \mathcal{S}_e$ will be denoted by Stab($\mathcal{S}_e$).

Assume now that $S$ is a cubic surface containing the basic $L$-set $L_b$ (see (2.2)) and take $M \in$ Stab($S$). Then $M$ sends $L_b$ to another $L$-set of $S$. Hence, in order to find Stab($S$), it is enough to compute the matrices $M = M(L_b, L)$ where $L$ varies in the set of 25,920 $L$-sets and check whether $M(S) = S$. 

---

### Table 4. Equations of the Eckardt families

| Family | Equation |
|--------|----------|
| $\mathcal{S}_0$ | (2) |
| $\mathcal{S}_1$ | $-x^2z - xz^2 - yxt + yzt + 2yt^2)c^2 + (xy - x^2 - y^t + zt)c - z(x^2 - xyt - yzt)c(e + f) + (2x^2y - 2xy^2 - x^2z + xyt + yzt)c^2f$ |
| $\mathcal{S}_1'$ | $(x^2z - (w + 1)x^2 - xz^2 - (w + 2)yt + 2yt^2 + wyzt + (w + 1)yt^2)c^2 + \(- (w + 3)x^2y + (w + 3)xy + 2x^2z + (w - 2)xy + xz^2 + 2(w + 1)yt - (w + 2)yt^2 - wyzt - (w + 1)yt^2)c^2f + ((w - 1)x^2y - (w - 1)xy + x^2z - xyt - wxy + wy^2t)^2f$ |
| $\mathcal{S}_2$ | $18x^2y - 18xy^2 - 6x^2z - 3x^2(4 - 4w - 1)xyz + (w^3 - 5w^2 + 6)xz^2 + (w^3 - 3w^2 + 7w - 9)yt - (w^3 + w - 6)yt^2 + (w^3 - 2w - 1)yt^2$ |
| $\mathcal{S}_3$ | $18x^2y - 18xy^2 - 6x^2z - 3x^2(4 - 4w - 1)xyz + (w^3 - 5w^2 + 6)xz^2 + (w^3 - 3w^2 + 7w - 9)yt - (w^3 + w - 6)yt^2 + (w^3 - 2w - 1)yt^2$ |
| $\mathcal{S}_4$ | $(2x^2z + xz^2 + 3x^2y - y^t + yzt + 2yt^2)c^2 + \(2x^2z - 3x^2y - x^2z - 2xy + y^2t + yzt)c(e + f) + (- 2x^2y + 4x^2y + 2x^2z - 3xy + xz^2 + 2x^2y - 3y^2t + yzt)c^2f$ |
| $\mathcal{S}_5$ | $(- 2x^2z + 2xy + x^2z + xyt - y^t + 2yt^2)c^2 + \(3x^2y + 3x^2y - 2x^2z + 2xy - x^2z - 2xy + 2yt + y^2t)c(e + f) + (- x^2y + x^2y + x^2y - y^2t)c^2f$ |
| $\mathcal{S}_6$ | $(- 2x^2z - 1x^2z + x^2z + x^2z - xyt + y^t + 2yt^2)c^2 + \(- 5x^2 + 5x^2y + 2x^2z - 4x^2y + x^2z + 2x^2yt - 4y^2t + 2yt^2 + y^2t^2)c(e + f) + (- x^2y + x^2y + x^2y + y^2t)c^2f$ |
| $\mathcal{S}_7$ | $312x^2y - 312x^2y + 2(5w^4 - 20w^2 + 8w - 32)x^2z - (w^3 - 5w^2 + 64w - 152)x^2y + (w^3 - 20w^2 + 8w - 32)x^2z + 4(w^3 + 9w^2 - 14w - 48)x^2y + (w^3 - 20w^2 + 8w + 288)y^2t - (9w^3 + 16w^2 - 48w + 88)y^2t - 2(7w^3 - 20w^2 - 20w + 28)y^2t$ |
| $\mathcal{S}_{10}$ | $x^2y - x^2y + 2x^2z - 2xy + x^2z - 2x^2y + 2y^t + y^t^2$ |
| $\mathcal{S}_{18}$ | $3x^2y - 3x^2y + 2x^2z - 2xy + x^2z + 2x^2y + 2y^t + y^t^2$ |

### 4. Stabilizers of Cubic Surfaces of $\mathcal{S}_e$

By Stab($S$) we denote the stabilizer of a smooth cubic surface $S$ with respect to the action of the group of projectivities on $\mathbb{P}^3_K$, i.e. it is the subgroup of PGL$_4(K)$ defined by

$$\text{Stab}(S) = \{ M \in \text{PGL}_4(K) \mid M(S) = S \}.$$ 

Obviously, the equality $M(S) = S$ means that the polynomial defining $M(S)$ is a multiple of that defining $S$.

This section is devoted to describe Stab($S$) as far as $S \in \mathcal{S}_e$, for all possible $n$.

By Stab($S$) for a generic $S \in \mathcal{S}_e$, will be denoted by Stab($\mathcal{S}_e$).
The computation for the generic cubic (2.5), although feasible, is lengthy and complex, but if we specialize the parameters \( b, c, d, e, f \) to random numeric values, it becomes reasonably fast and suffices to prove that the stabilizer of the general smooth cubic surface is trivial.

It is then interesting to detect the cases in which the stabilizer is not trivial.

If \( l \subset S \) is a line and \( M \in \text{Stab}(S) \), then \( M(l) \) is a line of \( S \), so \( M \) induces a permutation of the 27 lines (preserving the incidence relations), naturally defined as

\[
\pi_M = \begin{pmatrix}
    l_1 & l_2 & \ldots & l_{27} \\
    M(l_1) & M(l_2) & \ldots & M(l_{27})
\end{pmatrix} \in \mathbb{E}_6.
\]

In particular, if \( L = (l_1, \ldots, l_5) \) is an \( L \)-set, with \( \pi_M(L) \) we denote the quintuple \((\pi_M(l_1), \ldots, \pi_M(l_5))\) that is clearly still an \( L \)-set.

**Proposition 4.1.** If \( S \) is any smooth cubic surface, then there is a natural group monomorphism

\[
\phi : \text{Stab}(S) \to \mathbb{E}_6 \quad \text{defined by } M \mapsto \pi_M.
\]

Consequently, \(|\text{Stab}(S)|\) divides \( 2^7 \cdot 3^4 \cdot 5 \).

**Proof.** Obviously, \( \pi_{MN} = \pi_M \circ \pi_N \), for all \( M \) and \( N \) in \( \text{Stab}(S) \). Moreover, \( \phi \) is injective. Namely, if \( \pi_M \) is the identity permutation, then, in particular, \( M(L_b) = L_b \) and therefore \( M \) is the identity matrix by Lemma 2.2.

The last claim follows from \(|\mathbb{E}_6| = 51,840 = 2^7 \cdot 3^4 \cdot 5\). \(\square\)

From the above result and Lemma 2.2 we have immediately the following fact.

**Corollary 4.2.** For all \( M \in \text{Stab}(S) \) it holds \( M = M(L_b, \pi_M(L_b)) \).

The following result describes a relationship between Eckardt points and symmetries of cubic surfaces.

**Lemma 4.3.** Let \( S \) be a smooth cubic surface and \( M \in \text{Stab}(S) \). The following facts hold:

1. if \( \text{ord}(M) = 2 \), then \( S \) has at least one Eckardt point;
2. if \( \text{ord}(M) = 3 \), then \( S \) has at least three Eckardt points;
3. if \( \text{ord}(M) = 5 \), then \( S \) has at least ten Eckardt points.

**Proof.** (1) Since the number of lines of \( S \) is odd, at least one of them (say \( l_2 \)) is fixed by \( M \). As recalled in Section 2, there are 5 tritangent planes containing \( l_2 \): one is certainly fixed, the other 4 are either exchanged in couples or fixed. We cover all the possible cases by considering three tritangent planes \( \pi, \alpha, \beta \) such that \( M(\pi) = \pi, M(\alpha) = \alpha \) and \( M(\beta) = \beta \) or \( M(\pi) = \pi, M(\alpha) = \beta \) and \( M(\beta) = \alpha \). Consider now the first possibility. Let \( \{l_1, \text{res}(l_1, l_2)\}, \{l_3, \text{res}(l_2, l_3)\}, \{l_5, \text{res}(l_2, l_3)\} \) be the lines of \( S \cap \pi, S \cap \alpha \) and \( S \cap \beta \) respectively (different from \( l_2 \)). There are some cases to consider, since \( M \) can fix the lines of these sets or exchange them. In each case, consider the skew lines \( l_1, l_3, l_5 \) and the line \( l_2 \) which intersects them. We can complete these four lines with a line \( l_4 \) such that \( (l_1, l_2, l_3, l_4, l_5) \) is an \( L \)-set. So, up to projectivities, we can assume that this \( L \)-set is \( L_b \). (In particular, we can assume that \( S \) has an equation given by (2.5).) If \( M \) fixes the three couples of lines above, then \( M \) sends \( L_b \) to either \( L_b \) itself or to \((E_1, G_4, E_2, F_{12}, E_3)\). If \( M \) permutes only the two lines of \( \beta \) \((E_3 \text{ and } F_{34})\) and fixes the lines of \( \alpha \), then \( M \) sends \( L_b \) to either \((E_1, G_4, E_2, G_5, F_{34})\) or \((E_1, G_4, E_2, G_6, F_{34})\), and so on. In this way we collect all
the matrices $M_1, \ldots, M_{12}$ which could stabilize $S$. For $i = 1, \ldots, 12$, we impose $M_i(S) = S$, obtaining the corresponding set of conditions on the parameters of $S$. It turns out that the corresponding surfaces, when not singular, have at least one Eckardt point.

(2) If $M$ has order 3, we consider three cases: either one of the lines of $S$ is fixed or there are three lines $r_1, r_2$ and $r_3$ such that $r_1 \mapsto r_2 \mapsto r_3 \mapsto r_1$ and are coplanar or are skew. Again, in the first case we consider the five tritangent planes passing through the fixed lines. At least two of them are fixed, so the lines on these planes are also fixed. In particular, we can find three lines $l_1$, $l_2$, $l_3$ on $S$ fixed by $M$ and such that $l_1$ and $l_2$ are skew and $l_2$ meets $l_1$ and $l_3$. Moreover, there are 5 lines intersecting $l_1$ and $l_3$; one is $l_2$ and of the 4 remaining at least one, say $l_4$, must be fixed. We complete these four lines to an $L$ set and we can assume it is $L_6$. Again, we study where $L_b$ can be sent by $M$ and we collect the possible values for $M$. Similar considerations allow us to obtain other matrices in the cases there are no fixed lines for $M$. As in the previous case, we see that the condition that $M(S) = S$ translates into conditions on the parameters of $S$ which give either singular cubic surfaces or smooth cubic surfaces with at least three Eckardt points.

(3) The case in which $\text{ord}(M) = 5$ can be solved in a similar way. 

Consider now the Eckardt families $S_{e_i}$, for $i \in \{1, 2, 3, 4, 6, 9, 10, 18\}$, introduced in the previous section.

**Definition 4.4.** A permutation $\pi \in E_6$ of the 27 lines of a surface $S \in S_{e_n}$ is called $n$-admissible if it maps three coplanar lines passing through an Eckardt point to three coplanar lines passing through an Eckardt point.

The set of $n$-admissible permutations will be denoted by $A_n$ and is a subgroup of the Weyl group $E_6$.

Notice that, since the Eckardt points of each surface $S \in S_{e_n}$ are in a precise configuration, representable by $n$ formal line triplets, the above definition does not depend on the particular surface $S$ of $S_{e_n}$.

Using Proposition 4.1 and Corollary 4.2 we obtain the following property.

**Proposition 4.5.** For any $n \in \{1, 2, 3, 4, 6, 9, 10, 18\}$, let $S \in S_{e_n}$. Then the group monomorphism $\phi$ defined in Proposition 4.1 restricts to

$$\phi : \text{Stab}(S) \rightarrow A_n.$$ 

The inverse map (when defined) is given by $\pi \mapsto M(L_b, \pi(L_b))$. In particular, $\text{Stab}(S)$ is contained in the set of matrices

$$M_n = \{M(L_b, \pi(L_b)) \mid \pi \in A_n\}.$$

**Remark 4.6.** As observed in Section 2, an $L$-set is not enough to determine all the labels of the 27 lines (see Proposition 2.13), but it is necessary to consider the extended $L$-sets (see Corollary 2.12). This holds for all $S \in S_{e_n}$ with $n \neq 9$. Namely, if $S \in S_9$, the Eckardt point $\tau_{22} = (E_5, G_2)$ allows us to uniquely determine the line $E_5$ as the line which intersects $G_2$ in an Eckardt point. Indeed, $\tau_{27} = (E_6, G_2)$ is not an Eckardt plane.

As a consequence of this observation,

$$|A_n| = 2|M_n|, \text{ for } n \neq 9, \quad \text{and} \quad |A_9| = |M_9|.$$
The computation of the groups \( \mathcal{A}_n \) is fast, since we have only to select, among the elements of \( E_6 \) (permutations of the 27 symbols \( E_1, \ldots, F_{56} \) preserving the incidence relations), those that preserve also the Eckardt points. From the elements of \( \mathcal{A}_n \) we can choose the elements of \( M_n \) (and represent them by symbolic \( L \)-sets). It turns out that the order of the sets \( M_n \) are the following:

\[
\begin{align*}
|M_1| &= 576 & |M_2| &= 96 & |M_3| &= 108 & |M_4| &= 36 \\
|M_5| &= 48 & |M_6| &= 1296 & |M_{10}| &= 120 & |M_{18}| &= 648.
\end{align*}
\]

The next step is to explicitly determine \( \text{Stab}(S_{e_n}) \): for all \( L \)-sets \( L \in M_n \), we compute the matrix \( M = M(L_b, L) \). Then \( \text{Stab}(S_{e_n}) \) is just the collection of all the above matrices \( M \) such that \( M(S) = S \) (where \( S \) is the generic surface of \( S_{e_n} \)). We get in this way the stabilizers and their orders:

\[
\begin{align*}
|\text{Stab}(S_{e_1})| &= 2 & |\text{Stab}(S_{e_2})| &= 4 & |\text{Stab}(S_{e_3})| &= 6 & |\text{Stab}(S_{e_4})| &= 12 \\
|\text{Stab}(S_{e_5})| &= 24 & |\text{Stab}(S_{e_6})| &= 54 & |\text{Stab}(S_{e_{10}})| &= 120 & |\text{Stab}(S_{e_{18}})| &= 648
\end{align*}
\]

In particular, Lemma 4.3 and the above computations prove the following relevant fact (see also [12], Ch. IV, Sect. XIV).

**Theorem 4.7.** The stabilizer of a smooth cubic surface \( S \) is non trivial if and only if \( S \) has some Eckardt points.

Let us sketch the computation required in a particular case.

**Example 4.8.** The family \( S_{e_6} \) consists of cubic surfaces having six Eckardt points given by

\[
(E_1, G_4), (E_2, G_1), (E_2, G_3), (E_3, G_4), (E_4, G_2), (F_{13}, F_{24})
\]

(see Table 3 and Table 4). Its equation is given in Table 4 and the corresponding singular locus is \( \Sigma_6 = c(c - e)(3c + e)(c + e)(5c^2 + 2ce + e^2) \).

A direct computation shows that the group \( \mathcal{A}_6 \) has 96 elements, representable by extended \( L \)-sets, like

\[
(E_1, G_4, E_2, G_3, E_3, E_5) = L_{be}, (E_1, G_4, E_2, G_3, E_3, E_6), (E_1, G_4, E_2, F_{12}, E_3, F_{15}),
\]

\[
(E_1, G_4, E_2, F_{12}, E_3, F_{16}), (E_1, G_4, F_{24}, G_2, F_{34}, E_3), (E_1, G_4, F_{24}, G_2, F_{34}, E_6), \ldots
\]

By deleting the last element of the extended \( L \)-sets above, we obtain 48 distinct \( L \)-sets which represent the matrices of \( M_6 \). For instance, \( L = (E_1, G_4, F_{24}, F_{13}, F_{34}) \) represents the matrix \( M = M(L_b, L) \in M_6 \), where

\[
M = \begin{pmatrix}
  c(c + e) & 0 & 0 & (e - c)(3c + e) \\
  0 & c(c + e) & 0 & c^2 - 4ce - e^2 \\
  0 & 0 & c(c + e) & 2c(c + e) \\
  0 & 0 & 0 & c(c + e)
\end{pmatrix}
\]

and it is easy to verify that this matrix belongs to \( \text{Stab}(S_{e_6}) \). By repeating the same check for all the 48 elements of \( M_6 \), we see that only 24 of these matrices stabilize \( S \in S_{e_6} \), so they are the elements of the group \( \text{Stab}(S_{e_6}) \).

5. **Subfamilies with larger stabilizer**

So far, we have computed the stabilizers \( \text{Stab}(S) \) for the generic cubic surface \( S \in S_{e_n} \). In this section, we check whether, for some \( n \), there are smooth surfaces in \( S_{e_n} \), with \( n \) Eckardt points and a larger stabilizer.

The case of \( S_{e_7} \) requires some attention in order to avoid hard computations, while subfamilies of the families \( S_{e_n} \) \( (n \geq 2) \) with larger stabilizer can be detected
in a simpler way. In these cases, indeed, it is not too difficult to understand if in the group $A_n$ there are further $L$-sets that stabilize cubic surfaces under some conditions on the parameters.

We obtain that in the families $S_{e_i}$, for $i \in \{2, 4, 6\}$, there are no cubic surfaces having the same number of Eckardt points and a larger stabilizer.

5.1. The subfamily $S_{e_1}'$. If $S \in S_{e_1}$ contains only one Eckardt point, then, by Lemma 4.3 the order of $\text{Stab}(S)$ is $2^r$, where $1 \leq r \leq 7$.

Following the proof of Lemma 4.3 we again select the 12 $L$-sets which give rise to the 12 matrices $M_i$. One of them, say $M_1$, comes from the $L$-set $(E_1, G_4, F_{14}, F_{13}, F_{35})$. One can see that $\text{ord}(M_1) = 2$ and that $M_1$ stabilizes $S_{e_1}$, i.e. $M_1$ generates the cyclic group $\text{Stab}(S)$, for the general $S$.

At this point we check whether, for some values of the parameters, we can find surfaces stabilized by other matrices of order 2. It turns out that the conditions we get give only singular cubic surfaces, so in $\text{Stab}(S)$, for all possible $S \in S_{e_1}$, there is only one element of order 2.

Therefore, if $\text{Stab}(S)$ contains other elements, at least one of them, say $M_2$, must have order 4. In this case, $M_2$ fixes the plane $\tau_3 = (E_1, G_4, F_{14})$ and also, consequently, the three lines of $\tau_3$. Since $E_2$ meets $G_4$, then $M_2(E_2)$ is another line which intersects $G_4$; it is easy to see that it can be assumed to be $E_3$.

Hence we collect all the $L$-sets of the form $(E_1, G_4, E_3, *, *)$ and check if they give matrices which, under suitable conditions on the parameters, stabilize the corresponding cubic surface. In this way, we find that the matrix given by the $L$-set $(E_1, G_4, E_3, G_6, F_{24})$ is in the stabilizer of a cubic surface of $S_{e_1}$ as long as the conditions:

\[
\begin{align*}
    e &= \frac{(2\sqrt{-1} - 1)c(5c - (4\sqrt{-1} - 3)f)}{5(c-f)} \\
    d &= \frac{(2\sqrt{-1} + 1)(5c^2 - (4\sqrt{-1} + 8)cf - 5f^2)}{5(f-c)}
\end{align*}
\]

are satisfied (note that the denominator is not zero, since the polynomial $c - f$ is a factor of $\Sigma_4$).

The conditions (5.1) plus condition $\Lambda_1$ of Table 3 will be denoted by $\Lambda_1'$ and give a family $S_{e_1}'$ depending on two parameters ($c$ and $f$) such that $\text{Stab}(S_{e_1}')$ is (in general) of order 4 (and has only one Eckardt point). Other $L$-sets give other families of cubic surfaces with a matrix of order 4 in the stabilizer, but are all projectively equivalent to $S_{e_1}'$.

5.2. The subfamily $S_{e_1}''$. Now we impose to the generic cubic of $S_{e_1}'$ that the matrix corresponding to the $L$-set $(G_4, E_1, F_{15}, E_5, F_{12})$ is in the stabilizer, we get the conditions

\[
\begin{align*}
    c &= \sqrt{-1} - \sqrt{2} \\
    f &= 3
\end{align*}
\]

which gives a cubic surface $S_{e_1}''$ with one Eckardt point and stabilizer of order 8. Also the matrices obtained from other $L$-sets, like

\[(E_1, G_4, E_2, F_{12}, E_3), \quad (E_1, G_4, F_{46}, G_6, F_{45}), \quad (E_1, G_4, E_6, G_5, E_3)\]

and several others, give cubic surfaces with stabilizer of order 8, but these surfaces are all projectively equivalent to $S_{e_1}''$. Therefore the condition $\Lambda_1''$ given by $\Lambda_1'$ plus
conditions 5.2 (when substitute into the polynomial \((2.5)\)), gives the unique surface with one Eckardt point and stabilizer of order 8.

**Remark 5.1.** In the list of possible stabilizers of cubic surfaces given in [12] this case is missing, as also remarked in [8].

5.3. **The subfamily** \(S_{e}'9\). Finally, the family \(S_{e}9\) contains cubic surfaces with larger stabilizer. One of them is obtained from \(S_{e}9\) by imposing the conditions:

\[
\begin{align*}
  f &= 1/4(\sqrt{-1} - \sqrt{3})^3 - (\sqrt{-1} - \sqrt{3}) - 1 \\
  c &= 1
\end{align*}
\]

on its parameters. Hence the conditions \(\Lambda_{9}'\), obtained by adding (5.3) to \(\Lambda_{9}\), give the unique (up to projectivities) cubic surface, say \(S_{e}'9\), still having 9 Eckardt points, but stabilizer of order 108.

6. **Structure of the automorphisms groups**

The knowledge of the explicit equations of the cubic surfaces of each of the Eckardt families \(S_{e_{i}}\), their lines and their stabilizers allows us to determine the structure of these groups and also a graphic representation of them.

Here we first describe the stabilizers giving (in a rough way) for each group, the generators in terms of permutations of the numbers 1, \ldots, 27 which represent the 27 lines, according to the following correspondence:

\[
\begin{align*}
  E_1 &\rightarrow F_{12}, \quad E_2 \rightarrow F_{13}, \quad E_3 \rightarrow F_{14}, \quad E_4 \rightarrow F_{15}, \quad E_5 \rightarrow F_{16}, \quad E_6 \rightarrow F_{17}, \\
  G_1 &\rightarrow F_{18}, \quad G_2 \rightarrow F_{19}, \quad G_3 \rightarrow F_{20}, \quad G_4 \rightarrow F_{21}, \quad G_5 \rightarrow F_{22}, \quad G_6 \rightarrow F_{23}
\end{align*}
\]

\[
\begin{align*}
  F_{24} &\rightarrow F_{25}, \quad F_{26} \rightarrow F_{27}
\end{align*}
\]

| \(S\)   | type of \(\text{Stab}(S)\)   | generators | \([\text{Stab}(S)]\) |
|--------|-----------------------------|------------|-------------------|
| \(S_{e_{1}}\) | \(C_2\)         | \(g_1\)    | 2                 |
| \(S_{e_{1}'}\) | \(C_4\)         | \(g'_1\)   | 4                 |
| \(S_{e_{1}''}\) | \(C_8\)        | \(g''_1\)  | 8                 |
| \(S_{e_{2}}\) | \(C_2 \times C_2\) | \(g_2, h_2\) | 4                 |
| \(S_{e_{3}}\) | \(S_3\)         | \(g_3, h_3\) | 6                 |
| \(S_{e_{4}}\) | \(C_2 \times S_3\) | \(g_4, h_4\) | 12                |
| \(S_{e_{6}}\) | \(S_4\)         | \(g_6, h_6\) | 24                |
| \(S_{e_{9}}\) | \(((C_3 \times C_3) \times C_3) \times C_2\) | \(g_9, h_9, k_9\) | 54               |
| \(S_{e_{9}'}\) | \(((C_3 \times C_3) \times C_3) \times C_4\) | \(g'_9, h'_9\) | 108              |
| \(S_{e_{10}}\) | \(S_5\)         | \(g_{10}, h_{10}\) | 120              |
| \(S_{e_{18}}\) | \((C_3 \times C_3 \times C_3) \times S_4\) | \(g_{18}, h_{18}\) | 648              |

**Table 5.** Stabilizers of Eckardt families and their generators in terms of permutations of the lines.
where:
\[
g_1 = (2,19)(3,22)(4,7)(5,25)(6,26)(8,13)(9,14)(11,16)(12,17)(18,27)(20,24)(21,23)
\]
\[
g'_1 = (2,3,19,22)(4,21,7,23)(5,26,25,6)(8,16,13,11)(9,12,14,17)(18,24,27,20)
\]
\[
g_1'' = (1,10)(2,11,3,8,19,16,22,13)(4,24,21,27,7,20,23,18)(5,17,26,9,25,12,6,14)
\]
\[
g_2 = (1,15)(3,22)(4,8)(5,25)(6,7,13)(9,18)(11,20)(12,21)(14,27)(16,24)(17,23)
\]
\[
h_2 = (2,19)(3,22)(4,7)(5,25)(6,26)(8,13)(9,14)(11,16)(12,17)(18,27)(20,24)(21,23)
\]
\[
g_3 = (2,19)(3,22)(4,7)(5,25)(6,26)(8,13)(9,14)(11,16)(12,17)(18,27)(20,24)(21,23)
\]
\[
h_3 = (1,9,14)(2,19,10)(3,13,4)(5,21,17)(6,20,16)(7,8,22)(11,24,26)(12,23,25)(15,18,27)
\]
\[
g_4 = (2,19)(3,22)(4,7)(5,25)(6,26)(8,13)(9,14)(11,16)(12,17)(18,27)(20,24)(21,23)
\]
\[
h_4 = (1,18,14,15,9,27)(2,19,10)(3,7,4,22,13,8)(5,12,17,25,21,23)(6,11,16,26,20,24)
\]
\[
g_6 = (1,13)(2,10)(3,18)(5,20)(6,21)(7,15)(9,22)(11,25)(12,26)(14,27)(16,24)(17,23)
\]
\[
h_6 = (1,3,15,22)(2,19)(4,13,14,18)(5,6)(7,27,9,8)(11,23,21,16)(12,24,20,17)(25,26)
\]
\[
g_9 = (2,19)(3,22)(4,7)(5,25)(6,26)(8,13)(9,14)(11,16)(12,17)(18,27)(20,24)(21,23)
\]
\[
h_9 = (1,9,14)(2,19,10)(3,13,4)(5,21,17)(6,20,16)(7,8,22)(11,24,26)(12,23,25)(15,18,27)
\]
\[
k_9 = (1,21,3)(2,7,13)(4,27,20)(5,26,19)(6,24,9)(8,12,14)(10,16,23)(11,15,22)(17,25,18)
\]
\[
g_9' = (2,3,19,22)(4,20,7,24)(5,6,25,26)(8,17,13,12)(9,11,14,16)(18,23,27,21)
\]
\[
h_9' = (1,4,6)(2,3,5)(7,10,12)(8,9,11)(13,22,27)(14,25,21)(15,26,17)(16,19,24)(18,23,20)
\]
\[
g_{10} = (1,13)(2,9)(4,19)(5,20)(6,21)(7,14)(10,22)(11,23)(12,24)(15,27)(16,26)(17,25)
\]
\[
g_{10} = (1,3,15,8,7)(2,9,22,27,19)(4,13,14,10,18)(5,17,11,24,21)(6,16,12,23,20)
\]
\[
g_{18} = (1,17)(2,5)(3,4,24,26)(6,14,22,15)(7,9,20,10)(8,19,16,18)(11,25,13,23)(21,27)
\]
\[
h_{18} = (1,26,7)(2,11,20)(3,8,18)(4,17,10)(5,9,23)(6,15,12)(13,16,14)(19,25,22)(21,27,24)
\]

**Remark 6.1.** The content of the above Table clearly coincides with Table 9.6 in [8] and with the description of the groups given in [8]. Our approach is, however, totally explicit and allows us to immediately obtain the projectivities (i.e. the matrices) which stabilize the surfaces, accordingly to the procedure in Example 4.8.

There are however other ways to interpret the stabilizers which show hidden symmetries involving Eckardt points, tritangent planes, Sylvester pentahedron, ...
Figure 1. Representation of the stabilizers of $S e_1$, $S e'_1$ and $S e''_1$. Two vertices are connected by a line if the corresponding lines are incident. The groups Stab($S$) for $S$ in $S e_1, S e'_1, S e''_1$ are the rotations around the center of the figure by $180^\circ$, by $90^\circ$ and by $45^\circ$, respectively.

Figure 1 allows us to visualize these three stabilizers. The 8 vertices $E_2$, $G_2$, $F_{34}, \ldots$ represent 8 lines of $S$ and two vertices are connected if and only if the corresponding lines are coplanar (and hence determine a tritangent plane). For instance, the vertices $E_2$ and $G_5$ are connected, since the corresponding lines meet on the tritangent plane $\tau_9 = (E_2, G_5, F_{25})$.

In the stabilizer of a generic element $S \in S e_1$ we have the 2-cycles $(E_2, F_{24})$, $(G_2, F_{12})$, $(F_{34}, E_3)$ and $(G_5, F_{15})$, which correspond to the rotation of $180^\circ$ of Figure 1 around its center. Hence Stab($S$) can be seen as a rotation group.

The action of Stab($S$) on the tritangent planes fixes the planes $\tau_8 = (E_2, F_{24})$, $\tau_1, \tau_{13}, \tau_4$ and exchanges the planes of the couples $(\tau_6, \tau_{17}), (\tau_{12}, \tau_{31}), (\tau_{42}, \tau_{14}), (\tau_9, \tau_{41})$ and these permutations are also coherent with the rotation of the Figure 1.

Similarly, the stabilizer of a generic cubic of $S e'_1$ is isomorphic to the group generated by the clockwise rotation of the figure around its center by $90^\circ$. Also here, a rotation of the figure can be identified either with a permutation of lines or of tritangent planes.

Finally, the stabilizer of $S e''_1$ is isomorphic to the group generated by the clockwise rotation of the figure around its center of $45^\circ$.

6.2. Stabilizer of $S e_2$. The generic smooth surface $S \in S e_2$ has two Eckardt points $(E_1, G_4), (E_2, G_4)$ (see Remark 3.5). The only line fixed by Stab($S$) is $G_4$. The five tritangent planes through $G_4$ are fixed as well and, so, also the two Eckardt points.

If $\{e, g_1, g_2, g_3\}$ are the four elements of the stabilizer and $\{E_1, F_{14}, E_2, F_{24}\}$ are all the lines on the two Eckardt planes, out of $G_4$, then, up to renaming the elements, $g_1$ is the 2-cycle $(E_1, F_{14})$, $g_2$ is the 2-cycle $(E_2, F_{24})$ and $g_3$ exchanges $E_1$ with $F_{14}$ and $E_2$ with $F_{24}$, i.e. $g_3 = g_1 \circ g_2$. This is another description of Stab($S$) as $C_2 \times C_2$. Moreover, concerning the action of Stab($S$) on the tritangent planes, it acts (for instance) on the set of four planes $\{\tau_1, \tau_{16}, \tau_6, \tau_{17}\}$ exchanging the first two or the last two or both couples.

6.3. Stabilizer of $S e_3$. For a generic $S \in S e_3$, the group Stab($S$) $\cong S_3$ does not fix lines and permutes, in all the possible ways, the three lines $E_1, G_3, F_{13}$ on the plane $\tau_2$. In addition, also the lines on the planes $\tau_8$ and $\tau_{37}$ are permuted in all the possible ways.
The action of \( \text{Stab}(S) \) on the tritangent planes is even more explicit, since corresponds to all the permutations of the planes \( \tau_3, \tau_7, \tau_{34} \), i.e. to all the permutations of the three Eckardt points.

6.4. \textbf{Stabilizer of } \( S_{e4} \). The generic cubic surface \( S \in S_{e4} \) has four Eckardt points: \( \tau_8 = (E_2, G_4, F_{24}), \tau_3 = (E_1, G_4, F_{14}), \tau_7 = (E_2, G_3, F_{23}) \) and \( \tau_{34} = (F_{13}, F_{24}, F_{56}) \). They belong to the Eckardt plane \( \tau_8 \) and three of them are collinear: \( \tau_3, \tau_7, \tau_{34} \).

Clearly, each of these three points belongs to one of the three lines through \( \tau_8 \) and this plane is fixed by \( G = \text{Stab}(S) \).

The action of \( G \) on the lines gives rise to the following five orbits:

\[
\{E_1, F_{23}, F_{13}, F_{14}, G_3, F_{56}\}, \quad \{E_2, F_{24}, G_4\}, \quad \{E_3, F_{34}, G_1, F_{12}, G_2, E_4\},
\{E_5, F_{45}, G_6, F_{26}, F_{35}, F_{16}\}, \quad \{E_6, F_{46}, G_5, F_{25}, F_{36}, F_{15}\}
\]

The action of \( G \) on the three collinear Eckardt points \( \tau_3, \tau_7, \tau_{34} \) gives all their permutations, meanwhile \( G \) exchanges the planes \( \tau_2 \) and \( \tau_{37} \). This is another way to see that \( G \) is isomorphic to the direct product \( C_2 \times S_3 \).

The group \( G \) can be also represented by considering an hexagon whose vertices are the six lines of one of the orbits above. Rearranging the labels of the vertices in a suitable way, \( G \) turns out to be isomorphic to the dihedral group \( D_6 \).

6.5. \textbf{Stabilizer of } \( S_{e6} \). The group \( G = \text{Stab}(S) \) for a generic \( S \in S_{e6} \) is \( S_4 \). The six Eckardt points are \( P_1 = \tau_3, P_2 = \tau_6, P_3 = \tau_7, P_4 = \tau_{13}, P_5 = \tau_{17}, P_6 = \tau_{34} \) and are all contained in the plane \( \tau_8 \) of equation \( x = 0 \).

The action of \( G \) on the tritangent planes has seven orbits, two of them are:

\[
\{\tau_1, \tau_{11}, \tau_{18}, \tau_{37}\}, \quad \{\tau_2, \tau_{12}, \tau_{16}, \tau_{31}\}.
\]

The action of \( G \), restricted on each of these two orbits, gives all the possible permutations of the four planes, confirming that \( G \cong S_4 \).

Moreover it is straightforward to see that

\[
S \cap \tau_2 \cap \tau_{37} = \{P_1, P_3, P_5\}, \quad S \cap \tau_{11} \cap \tau_{31} = \{P_2, P_4, P_6\},
S \cap \tau_{12} \cap \tau_{18} = \{P_3, P_4, P_3\}, \quad S \cap \tau_1 \cap \tau_{16} = \{P_1, P_2, P_3\}.
\]

Hence \( G \) can also be seen as the group of permutations of the dotted lines in Figure 2, they are not lines of \( S \) and each of them contains three Eckardt points.
6.6. Stabilizer of $S_9$. For a generic $S \in S_9$, the nine Eckardt points ($\tau_5$, $\tau_7$, $\tau_{14}$, $\tau_{20}$, $\tau_{22}$, $\tau_{26}$, $\tau_{33}$, $\tau_{34}$, $\tau_{42}$) are coplanar, belonging to the plane $\pi : (1 - \sqrt{3})x - y + z = 0$. Moreover, they are the nine inflection points of the cubic curve $\pi \cap S$ and are therefore in the Hesse configuration.

The group $\text{Stab}(S)$ contains the following permutations:

- $g_1$, given by the product of the disjoint cycles
  $$(E_1, F_{14}, G_3), (E_2, G_3, F_{23}), (E_3, G_5, F_{35}), (E_4, F_{46}, G_6), (E_5, G_2, F_{25}), (E_6, F_{16}, G_1), (F_{12}, F_{36}, F_{45}), (F_{13}, F_{56}, F_{24}), (F_{15}, F_{26}, F_{34}),$$
  fixes the 9 Eckardt points and rotates clockwise around their baricenters the small triangles of Figure $3$ of $120^\circ$;

- $g_2$, given by the product of the disjoint cycles
  $$(E_1, G_3, F_{13}), (E_2, F_{24}, G_4), (E_3, F_{12}, E_4), (E_5, F_{26}, F_{16}), (E_6, F_{25}, F_{15}), (G_1, G_2, F_{31}), (G_5, F_{36}, F_{46}), (G_6, F_{35}, F_{45}), (F_{14}, F_{23}, F_{56}),$$
  rotates clockwise around their baricenters the three medium triangles of Figure $3$ of $120^\circ$;

- $g_3$, given by the product of disjoint cycles
  $$(E_1, G_3), (E_2, G_4), (E_3, G_1), (E_4, G_2), (E_5, G_6), (E_6, G_5), (F_{12}, F_{34}), (F_{14}, F_{23}), (F_{15}, F_{36}), (F_{16}, F_{35}), (F_{25}, F_{46}), (F_{26}, F_{45}),$$
  rotates clockwise around its baricenter the large triangle of Figure $3$ and, simultaneously, rotates clockwise the three small triangles $(E_1, F_{14}, G_4)$, $(G_2, F_{25}, E_5)$ and $(F_{12}, F_{36}, F_{45})$ of $240^\circ$ and the tree small triangles $(E_2, G_3, F_{23})$, $(F_{15}, F_{26}, F_{34})$, $(G_6, E_4, F_{46})$ of $120^\circ$; therefore also $g_3$ has order 3.

- $g_4$, given by the product of disjoint cycles
  $$(E_1, G_3), (E_2, G_4), (E_3, G_1), (E_4, G_2), (E_5, G_6), (E_6, G_5), (F_{12}, F_{34}), (F_{14}, F_{23}), (F_{15}, F_{36}), (F_{16}, F_{35}), (F_{25}, F_{46}), (F_{26}, F_{45}),$$
  is the reflection of the large triangle of Figure $3$ along the dotted line.

This shows that the group $\text{Stab}(S)$ is isomorphic to

$$\langle (g_1) \times (g_2) \rangle \times (g_3) \times (g_4) \cong (C_3 \times C_3) \times C_3 \times C_2.$$ 

6.7. Stabilizer of $S_9'$. Since $S_9' \in S_9$, then $\text{Stab}(S_9')$ is a subgroup of $\text{Stab}(S_9)$ (see Sections $4$ and $5$).

Obviously, the nine Eckardt points are again coplanar and are the nine inflection points of the cubic curve obtained as intersection of their plane and the surface.

Nevertheless, the group structure of $\text{Stab}(S_9')$ is different. Keeping the notation for $g_1, g_2, g_3, g_4$, defined in the case $S_9$, let us set $g_5$ to be the permutation of the lines given by the product of the following disjoint cycles:

$$(E_1, F_{25}, G_3, F_{46}), (E_2, E_4, G_4, G_2), (E_3, F_{26}, G_1, F_{45}), (E_5, F_{21}, G_6, F_{14}), (E_6, F_{12}, G_5, F_{34}), (F_{15}, F_{16}, F_{35}, F_{36})$$

It is not difficult to see that $\text{ord}(g_3) = 4$ and $g_3^2 = g_4$. Moreover, $g_5$ acts on the small triangles of Figure $3$ (i.e. on the nine Eckardt points) as follows:

$$\tau_3 \mapsto \tau_{22} \mapsto \tau_7 \mapsto \tau_{20} \mapsto \tau_3, \quad \tau_{14} \mapsto \tau_{42} \mapsto \tau_{26} \mapsto \tau_{33} \mapsto \tau_{14}, \quad \tau_{34} \mapsto \tau_{34}$$

Therefore the group structure of the stabilizer is

$$\text{Stab}(S_9') = \langle (g_1) \times (g_2) \rangle \times (g_3) \times (g_5) \cong (C_3 \times C_3) \times C_3 \times C_4.$$
6.8. Stabilizer of $S_{e10}$. The cubic surface $S_{e10}$ has the 10 Eckardt points: $\tau_1, \tau_3, \tau_7, \tau_8, \tau_{11}, \tau_{12}, \tau_{16}, \tau_{18}, \tau_{31}, \tau_{34}$.

It turns out that the action of $\text{Stab}(S_{e10})$ on the 45 tritangent planes gives rise to an orbit consisting of five planes, $\pi_2, \pi_6, \pi_{13}, \pi_{17}, \pi_{37}$ (none of them is an Eckardt plane) and that $\text{Stab}(S_{e10})$ permutes them in all possible ways. In this way, we obtain an explicit representation of $\text{Stab}(S_{e10})$ as $S_5$.

The polynomials defining the above planes are:

\[
\begin{align*}
\pi_2 : & \quad y - z, \\
\pi_6 : & \quad x - y, \\
\pi_{13} : & \quad x + t, \\
\pi_{17} : & \quad x - 2y + t, \\
\pi_{37} : & \quad 2x - y + z
\end{align*}
\]

and it is easy to verify that $S_{e10}$ is the sum of the cubes of these five linear polynomials. Therefore the five planes above compose the Sylvester pentahedron of $S_{e10}$ (see [12], Ch. IV, §84). Figure 4 shows the five planes and the 10 Eckardt points (there labelled by 1, 2, ..., 10).

6.9. Stabilizer of $S_{e18}$. The cubic surface $S_{e18}$ has the 18 Eckardt points:

$\tau_2, \tau_3, \tau_7, \tau_8, \tau_{14}, \tau_{15}, \tau_{19}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{26}, \tau_{27}, \tau_{32}, \tau_{33}, \tau_{34}, \tau_{37}, \tau_{42}, \tau_{45}$

and the group $G = \text{Stab}(S_{e18})$ has order 648.

Also in this case, the action of $G$ on the 45 tritangent planes has a particular orbit but, here, it consists of the 18 Eckardt planes listed above.

A more detailed analysis shows that the 18 Eckardt points are contained in four planes, represented in Figure 5 as the faces of a tetrahedron:

\[
\begin{align*}
\pi_1 = & \quad A + B + C, \\
\pi_2 = & \quad B + C + D, \\
\pi_3 = & \quad A + B + D, \\
\pi_4 = & \quad A + C + D
\end{align*}
\]

and whose equations are:

\[
\begin{align*}
\pi_1 : & \quad x - (1 + \sqrt{-3})y - t, \\
\pi_2 : & \quad (1 + \sqrt{-3})(x - t) - 4y, \\
\pi_3 : & \quad (1 - \sqrt{-3})x - y + z, \\
\pi_4 : & \quad 2(\sqrt{-3} - 1)x - (1 + \sqrt{-3})(y - z).
\end{align*}
\]

Each plane $\pi_i$ contains 9 Eckardt points, as shown in Figure 5 and these are precisely the inflection points of the cubic curve $\pi_i \cap S_{e18}$.
The action of $G$ on the 18 Eckardt points gives a group $H$ of permutations of these points and $|H| = 648$, so $H \cong G$. It contains the three permutations:

$$g_1 = (\tau_3, \tau_{34}, \tau_7)(\tau_{15}, \tau_{32}, \tau_{19})(\tau_{22}, \tau_{42}, \tau_{26})$$
$$g_2 = (\tau_3, \tau_{34}, \tau_7)(\tau_{14}, \tau_{31}, \tau_{20})(\tau_{21}, \tau_{45}, \tau_{27})$$
$$g_3 = (\tau_2, \tau_8, \tau_{37})(\tau_{15}, \tau_{32}, \tau_{19})(\tau_{21}, \tau_{45}, \tau_{27})$$

Each of them acts cyclically on the triplets of collinear Eckardt points, as can be seen in Figure 5, therefore, setting $N = \langle g_1, g_2, g_3 \rangle$, it is clear that $N \cong C_3 \times C_3 \times C_3$. Moreover $N$ is a normal subgroup of $H$ and the quotient group $H/N$ is generated by the two classes $[g_4]$ and $[g_5]$, where:

$$g_4 = (\tau_2, \tau_2)(\tau_3, \tau_{42})(\tau_7, \tau_{22})(\tau_8, \tau_{45})(\tau_{20}, \tau_{33})(\tau_{26}, \tau_{34})(\tau_{27}, \tau_{37})$$
$$g_5 = (\tau_2, \tau_7)(\tau_3, \tau_{37}, \tau_{34}, \tau_8)(\tau_{14}, \tau_{22}, \tau_{32}, \tau_{21})(\tau_{15}, \tau_{45}, \tau_{33}, \tau_{42})(\tau_{19}, \tau_{27}, \tau_{29}, \tau_{28})$$

Concerning the action of these two elements on the four planes of the tetrahedron, note that

$$g_4 : \pi_2 \leftrightarrow \pi_4, \quad g_5 : \pi_1 \leftrightarrow \pi_3 \leftrightarrow \pi_2 \leftrightarrow \pi_4 \leftrightarrow \pi_1$$

Therefore $\text{ord}(g_4) = 2$, $\text{ord}(g_5) = 4$ and $H/N \cong S_4$. Setting $K = \langle g_4, g_5 \rangle$ the corresponding subgroup of $H$, we get that $H = N \rtimes K$, hence $\text{Stab}(S_{e18})$ is now described by

$$\langle (g_1) \times (g_2) \times (g_3) \rangle \rtimes \langle g_4, g_5 \rangle \cong (C_3 \times C_3 \times C_3) \rtimes S_4.$$
Figure 5. A representation of the group Stab(S_{18}).

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