The Stellar Distribution Function and Local Vertical Potential from Gaia DR2

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ABSTRACT
We develop a novel method to simultaneously determine the vertical potential, force and stellar $z - v_z$ phase space distribution function (DF) in our local patch of the Galaxy. We assume that the Solar Neighborhood can be treated as a one-dimensional system in dynamical equilibrium and directly fit the number density in the $z - v_z$ plane to what we call the Rational Linear DF (RLDF) model. This model can be regarded as a continuous sum of isothermal DFs though it has only one more parameter than the isothermal model. We apply our method to a sample of giant stars from Gaia Data Release 2 and show that the RLDF provides an excellent fit to the data. The well-known phase space spiral emerges in the residual map of the $z - v_z$ plane. We use the best-fit potential to plot the residuals in terms of the frequency and angle of vertical oscillations and show that the spiral maps into a straight line. From its slope, we estimate that the phase spirals were generated by a perturbation $\sim 540$ Myr years ago. We also determine the differential surface density as a function of vertical velocity dispersion, a.k.a. the vertical temperature distribution. The result is qualitatively similar to what was previously found for SDSS/SEGUE G dwarfs. Finally, we address parameter degeneracies and the validity of the 1D approximation. Particularly, the mid-plane density derived from a cold subsample, where the 1D approximation is more secure, is closer to literature values than that derived from the sample as a whole.

Key words: Galaxy: kinematics and dynamics - Galaxy: Solar Neighborhood - Galaxy: disc - Galaxy: structure - Galaxy: evolution

1 INTRODUCTION
At the Sun’s position in the Galaxy, the vertical force, that is, the gravitational force perpendicular to the Galactic plane, is dominated by baryons near the plane and dark matter above 1 kpc. It therefore provides a powerful constraint on mass models of the Galaxy and helps break the disc-halo degeneracy. Moreover, when combined with measurements of the rotation curve near the Sun and the local baryon density, the vertical force yields an estimate for the local density of dark matter (see e.g. Read 2014; de Salas & Widmark 2020).

Stellar dynamics has been used to probe the vertical force since the pioneering work of Kapteyn (1922), Jeans (1922), and Oort (1932). These studies lay the groundwork for present-day analyses by introducing three key assumptions for the vicinity of the Sun: (1) Stars behave like an incompressible fluid in six-dimensional phase space as described by the Collisionless Boltzmann Equation (CBE). (2) Near the mid-plane, the vertical force can be approximated by a function of $z$ while the stellar distribution function (DF) can approximated by a function of $z$ and $v_z$. Here, $z$ is the position relative to the mid-plane and $v_z$ is the velocity component perpendicular to the mid-plane. (3) Stars are in dynamical equilibrium with the gravitational potential. These assumptions together imply that the DF $f_z(z, v_z)$ depends on $z$ and $v_z$ only through the vertical energy $E_z$, a result that follows from the Jeans Theorem. Alternatively, one can derive the vertical Jeans Equation, that is, the $v_z$ moment of the CBE, under the assumption of planar symmetry. The Jeans Theorem and the vertical Jeans Equation lead to various relations among the number density, velocity dispersion, and DF that can be used to estimate the vertical force and potential from kinematic data (Flynn & Fuchs 1994; Holmberg & Flynn 2000, 2004; Zhang et al. 2013; Sivertsson et al. 2018; Buch et al. 2019; Guo et al. 2020).

The mean field assumption is justified from the fact that the two-body relaxation time of stars in the disc is much greater than the age of the Galaxy (Binney & Tremaine 2008). As for the second assumption, which we refer to as the 1D approximation, radial variations become important as one moves further from the mid-plane with corrections reaching 10-20% for $z > 1$ kpc. They are also more important for stellar populations with higher radial velocity dispersion. (See Read 2014 for a review and Bovy & Rix 2013 for a scheme to infer the vertical potential and force as a function of Galactocentric radius.) Finally, there is the assumption that the disc is in dynamical equilibrium. Oort looked for, but did not
find, systematic motions in the direction perpendicular to the Galactic plane, and emphasized that this result lends “support to the assumption that in the z-direction the stars are thoroughly mixed” (Oort 1932). However, we now know that the the disc has motions perpendicular to the plane, the most prominent being those associated with warping of the disc at large Galactocentric radii (see reviews by Binney 1992 and Kalberla & Kerp 2009). Furthermore, there is evidence that the warp extends into the Solar Neighbourhood (Schönrich & Dehnen 2018). In addition, vertical bulk motions in disc stars and asymmetric variations in the number counts about the Galactic mid-plane have been identified by several groups using different surveys (Widrow et al. 2012; Yanny & Gardner 2013; Williams et al. 2013; Carlin et al. 2013; Bennett & Bovy 2019; Salomon et al. 2020). Perhaps the most striking indication of vertical disequilibrium comes from the phase spirals uncovered by Antoja et al. (2018). They can be seen in number counts as well as mean azimuthal and radial velocities across the $z - v_z$ plane and provide compelling evidence that the disc in the Solar Neighbourhood is not fully mixed (Antoja et al. 2018; Binney & Schönrich 2018; Darling & Widrow 2019a,b; Bland-Hawthorn et al. 2019; Laporte et al. 2019; Li & Shen 2020). Taken together, these observations bring into question the equilibrium assumption that is often made in attempts to measure the vertical force and local dark matter density (Banik et al. 2017; Salomon et al. 2020) and call for new fitting methods that explicitly include disequilibrium features.

In this paper, we outline a method that represents the first step towards this goal and apply it to a sample of giants from Gaia’s Second Data Release (GDR2). For this first step, we retain the three assumptions described above while simultaneously modelling the potential and DF directly in the full $z - v_z$ phase space. Specifically, we compare star counts in the $z - v_z$ plane, $N(z, v_z)$, with predictions from a model in which the DF and potential are described by simple parametric functions of the phase space variables. The best-fit potential is the one in which contours of constant $E_2$ come closest to contours of constant $N$. The method has the advantage of working directly with the DF and the potential. Furthermore, evidence of disequilibrium, such as the phase spirals, emerge in the residuals of the equilibrium model. By contrast, $n(z)$ (the z distribution of stars) and $f_z(E_2)$ can hide manifestations of disequilibrium since they are constructed by integrating out one of the phase space coordinates ($v_z$ for $n(z)$ and an angle variable in the $z - v_z$ plane for $f_z$). As an extension of the model, we map the residuals into frequency-angle space via the best-fit potential and use it to infer the time when the disc was perturbed. In a subsequent paper, we will explore models where the phase spiral, underlying equilibrium DF, and potential are fit simultaneously.

The use of number count contours in the $z - v_z$ plane was considered by Kuijken & Gilmore (1989a) who dismissed it for two reasons. First, the limited number of stars with full kinematic measurements available at that time would have necessitated fairly coarse bins in the $z - v_z$ plane. Second, measurement errors were difficult to handle and model uncertainties were difficult to estimate. Presently, we have GDR2, which includes over 6 million stars with complete position and velocity measurements (Gaia Collaboration et al. 2018a). By Gaia’s Third and Fourth Data Releases, the number of stars with 6D phase space measurements will swell by over two orders of magnitude. Thus, we have the opportunity to fit $z - v_z$ contours with a sufficiently fine grid. Moreover, Markov Chain Monte Carlo methods allow us to efficiently estimate uncertainties via Bayesian statistics. We are therefore well-positioned to address the obstacles described in Kuijken & Gilmore (1989a).

The outline of the paper is as follows: We present our fitting algorithm in Section 2 and test it on mock data in Section 3. In Section 4, we describe the selection criteria for our sample of GDR2 giants. We then present the results derived from this sample in Section 5 and discuss them in Section 6. We conclude in Section 7 with a summary and thoughts on future directions for this line of research.

2 THE FITTING ALGORITHM

2.1 likelihood function

Consider a sample of stars with fully determined positions and velocities that are selected according to their intrinsic properties and locations within the Galaxy. In what follows, we assume that the sample is complete. That is, all stars inside the region of the sample and with the chosen set of stellar properties are assumed to be included in the sample. Generally, the number density of stars in the $z - v_z$ plane predicted by a model DF $f(x, v)$ is given by:

$$n(z, v_z) = \int f(x, v) S(x) dx dy dv_R dv_\phi$$

where the geometric selection function $S(x)$ is unity inside the sample volume $V$ and zero outside. In this paper, we further assume that the system is in dynamical equilibrium, axisymmetric, and symmetric about the mid-plane of the Galaxy. By Jeans theorem, the DF depends on the positions and velocities through the integrals of motion. For an axisymmetric system, the component of the angular momentum along the symmetry axis $L_z$ and the total energy $E$ are both exact integrals, while the vertical structure of the disc is determined by the dependence of the DF on the vertical energy

$$E_z = \frac{1}{2}v_z^2 + \psi(R, z) - \Psi(R, 0),$$

which is only approximately conserved.

As mentioned in Section 1, we make two additional assumptions for the Solar Neighborhood: (1) the stellar distribution function (DF) can be approximated by some $f_z(z, v_z)$, and (2) the vertical force is only a function of $z$. In this way, we can define the vertical potential as

$$\psi(z) \equiv \Psi(R, z) - \Psi(R, 0).$$

The vertical energy is then

$$E_z = \frac{1}{2}v_z^2 + \psi(z),$$

which is an integral of motion. Therefore, Equation 1 implies that:

$$n(z, v_z) \propto G(z) f_z(z, v_z) = G(z) f_z(E_2)$$

where

$$G(z) = \int S(x) dx dy$$

is essentially the area of the intersection of the sample volume and a horizontal plane at height $z$. 

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We sort stars into $z - v_z$ bins and define $O_i$ to be the observed star count in the $i$‘th bin, centred on the point $(z_i, v_{z,i})$. Let $A_i$ be the area of the $i$-th bin in $z - v_z$ phase space so that it has dimensions of distance $\times$ velocity. The prediction for the star counts in the $i$-th bin is found by multiplying Equation 5 by $A_i$:

$$M_i \propto A_i G(z_i) f_z(E_{z,i})$$

with the normalization condition

$$\sum_i M_i = N$$

where $N$ is the total number of stars used in the analysis. Since there are always a few stars falling outside our $z - v_z$ grid, $N$ is generally a bit less than the total number of stars in the sample. The probability of a star falling into the $i$-th bin is given by $p_i = M_i/N$. Our likelihood function is then the product of the binomial distribution over all bins:

$$L = \prod_i \frac{N!}{O_i!(N - O_i)!} p_i^{O_i} (1 - p_i)^{N - O_i}$$

### 2.2 gravitational potential

In this work, we adopt a reparameterization of the potential introduced by Kuijken & Gilmore (1989a,b):

$$\psi(z) = \omega_1^2 D \left( \frac{z^2}{\sqrt{\omega_2^2 + D^2}} - D \right) + \frac{1}{2} \omega_2^2 z^2.$$ (10)

They identify the first term, which is quadratic near the mid-plane and linear for $z \gg D$, with the disc. In this interpretation, the disc has thickness $D$ and surface density $\Sigma_d = \omega_1^2 D/(2\pi G)$. Likewise, they identify the second term with an effective halo having a constant density $\rho_{halo} = \omega_2^2/(4\pi G)$. The term “effective halo” is used since the bulge and disc also contribute quadratic components to the potential. However for our purposes Equation 10 is simply a convenient fitting formula for the potential.

For an axisymmetric system, Poisson’s equation is given by

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \psi}{\partial R} \right) = 4\pi G \rho(R, z)$$

where $\rho$ is the total mass density. If we assume that the radial contribution is independent of $z$, then the left-hand side of Equation 11 becomes

$$4\pi G \rho = \frac{\omega_1^2}{(1 + z^2/D^2)^{3/2}} + \omega_2^2 + 2 \left( B^2 - A^2 \right)$$

where $A$ and $B$ are the Oort constants. The integrated surface density $\Sigma$ within a distance $z$ ($z > 0$) from the mid-plane is given by

$$2\pi G \Sigma(z) = F_z(z) + 2 \left( B^2 - A^2 \right) z$$

where

$$F_z(z) = \left( \frac{\omega_1^2}{\sqrt{1 + z^2/D^2}} + \omega_2^2 \right) z$$

is the magnitude of the vertical force at a distance $z$ from the mid-plane. In the limit $z \rightarrow 0$ we have

$$4\pi G \rho_0 = \omega_1^2 + \omega_2^2 + 2 \left( B^2 - A^2 \right)$$

where $\rho_0$ is the total density at the mid-plane.

### 2.3 distribution function

In the 1D approximation, Jeans Theorem implies that the DF is a function of $E_z$. A particularly simple choice is the well-known isothermal DF (Spitzer 1942; Camm 1950),

$$f_z(E_z) \propto e^{-E_z/\sigma_z^2}.$$ (16)

The assumption of a vertically isothermal disc has been embedded in a number of three-integral DFs. For example, Kuijken & Dubinski (1995) introduced the following disc DF for their numerical disc-bulge-halo models:

$$f(E_p, L_z, E_z) = \tilde{F}(R_c) \exp \left[ -\frac{E_p - E_{p0}(R_c)}{\sigma_{p0}(R_c)} - \frac{E_z}{\sigma_z^2(R_c)} \right],$$ (17)

where $R_c$ is the radius of a circular orbit for a star with angular momentum $L_z$. Equation 17 is an extension of the planar DF found in Shu (1969) and Kuijken & Tremaine (1991). It neatly factors into isothermal terms for in-plane and vertical motions. However, since $R_c$ is a function of $L_z$, it is not strictly separable into the form $f(x, v) = f_\parallel(x, y, v_R, v_\parallel) f_z(z, v_\parallel)$. A second example is the quasi-isothermal DF introduced by Binney (2010) and Binney & McMillan (2011), which is approximately isothermal in the vertical action $J_z$ rather than $E_z$.

It is common in studies of the vertical force to model the stellar distribution as the sum of multiple isothermal components (Bahcall 1984b; Holmberg & Flynn 2000, 2004; Flynn et al. 2006). The idea is at the heart of the Bovy & Rix (2013) measurement of the vertical force as a function of $R$. That analysis was built on earlier work by Bovy et al. (2012a) and Bovy et al. (2012b) where stars were divided into mono-abundance sub-populations that were approximately isothermal with a vertical velocity dispersion that varied smoothly with elemental abundance.

To accommodate these ideas, we introduce the Rational Linear Distribution Function (RLDF)

$$f_z(E_z) = f_0 \left( 1 + \frac{E_z}{\alpha \sigma_z^2} \right)^{-\alpha}$$ (18)

where $f_z d^2x dv_z$ is the number of stars in a volume element $d^2x$ and velocity element $dv_z$. The number density of stars as a function of $z$ is then given by

$$\nu(z) = \int_{-\infty}^{\infty} f(E_z) dv_z = \nu_0 \left( 1 + \frac{\psi(z)}{\alpha \sigma_z^2} \right)^{1/2-\alpha}$$ (19)

where

$$\nu_0 = \sqrt{2\pi \alpha} \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha)} \sigma_z \nu_0.$$ (20)

Likewise, the velocity dispersion is

$$\langle v_z^2 \rangle^{1/2} = \sigma_z \sqrt{\frac{\alpha}{\alpha - 3/2} \left( 1 + \frac{\psi(z)}{\alpha \sigma_z^2} \right)}$$ (21)

Note that in the limit $\alpha \rightarrow \infty$, these expressions reduce to those for the isothermal DF, namely Equation 16 for the DF and

$$\nu(z) \rightarrow \nu_0 e^{-\psi(z)/\sigma_z^2} \text{ and } \langle v_z^2 \rangle^{1/2} \rightarrow \sigma_z$$ (22)

for the stellar density and velocity dispersion. Also note that we require $\alpha > 3/2$ for the system to have a finite velocity dispersion.
The bin size to be \( \Delta f \) by \( \omega \) fitting procedure mixed. stars from populations with velocity dispersion between \( \mu \) and \( \omega \) priors for parameters given the data via Bayes theorem assuming linear gives the probability of the data given the model. We calcul-

\[
f_\xi(E_z) = f_0 \int_0^\infty \frac{d\mu_z}{\mu_z} g(\sigma_z/\mu_z; \alpha) e^{-E_z/\mu_z^2} \tag{23}\]

where

\[
g(x; \alpha) = \frac{2\alpha^\alpha}{\Gamma(\alpha)} x^{2\alpha} e^{-\alpha x^2} \tag{24}\]

The differential DF is then

\[
\frac{df}{d\mu_z} = f_0 g(\sigma_z/\mu_z; \alpha) e^{-E_z/\mu_z^2}. \tag{25}\]

We integrate this expression over \( z \) and \( v_z \) to obtain

\[
\frac{d\Sigma}{d\mu_z} = g(\sigma_z/\mu_z; \alpha) \int_0^\infty dz e^{-\psi(z)/\mu_z^2} \tag{26}\]

where \( \frac{d\Sigma}{d\mu_z} \Delta \mu_z \) is the contribution to the surface density for stars from populations with velocity dispersion between \( \mu_z \) and \( \mu_z + \Delta \mu_z \) assuming that stellar populations are well mixed.

### 2.4 fitting procedure

To summarize, the potential is defined by three parameters \( \omega_1, \omega_2, \) and \( D \) through Equation 10, while the DF is defined by \( \sigma_z \) and \( \alpha \) through Equation 18. The normalization factor \( f_0 \) is calculated from other parameters via Equation 8. We fix the bin size to be \( (\Delta z, \Delta v_z) = (40 \text{ pc}, 2 \text{ km/s}) \). Equation 9 gives the probability of the data given the model. We calculate the probability distribution function (PDF) of the model parameters given the data via Bayes theorem assuming linear priors for \( \omega_1, D, \omega_2, \sigma_z \) and \( \ln \alpha \) (instead of \( \alpha \)) as listed in Table 1. To do so, we using the Markov chain Monte Carlo (MCMC) sampler emcee (Foreman-Mackey et al. 2013).

### 3 MOCK DATA TESTS

#### 3.1 Data generation

In this section, we test our method on two mock datasets. The first is drawn from an equilibrium model that is constructed using the GALACTICS code (Kuijken & Dubinski 1995; Widrow & Dubinski 2005; Deg et al. 2019). It comprises 200k stars in an annulus centered at the Solar radius \( R_0 = 8.3 \text{ kpc} \) with a half-width of 500 pc. We assume a sample volume that does not depend on \( z \) so that \( G(z) = 1 \). The particular GALACTICS model is described in Darling & Widrow (2019a) and designed to match the basic properties of the Milky Way. It comprises an exponential disc, a Sérsic bulge and an NFW halo (Navarro et al. 1996) with properties as follows:

- The disc has a radial scale length \( R_d = 2.8 \text{ kpc} \) and a mass of \( M_d \simeq 3.8 \times 10^{10} M_\odot \);
- The bulge has a Sérsic index of 2, a scale length of 700 pc and a mass of \( M_b \simeq 1.3 \times 10^{10} M_\odot \);
- The circular speed at the solar circle is \( \simeq 255 \text{ km/s} \);
- At the Solar Circle, the thickness of the disc is \( (z^2)^{1/2} = 320 \text{ pc} \), and the vertical velocity dispersion is \( \sigma_z = 17 \text{ km/s} \);
- The disc has a Toomre Q-parameter \( Q = 1.5 \) at \( R = 2.2 R_d \).

Our second partially perturbed dataset includes a phase spiral, which is qualitatively similar to the one discovered by Antoja et al. (2018). To generate the phase spiral, we give \( \pm 10\% \) of the particles from the equilibrium dataset a \( v_z \) perturbation of \( +20 \text{ km/s} \). We then evolve the system for 500 Myr in the fixed potential of the equilibrium galaxy. Thus, the phase spiral is purely kinematic. (For a discussion of the importance of self-gravity for the development of phase spirals, see Darling & Widrow 2019a.) For each of the two datasets, we only keep stars within \( |z| < 1.48 \text{ kpc} \) and \( |v_z| < 80 \text{ km/s} \) for fitting, as few stars reach beyond this region of \( z - v_z \) phase space.

#### 3.2 results

In Figure 1, we show the results for \( \psi(z) \), \( F_z = d\psi/dz \) and \( d^2\psi/dz^2 \) along with the true profiles as determined by the GALACTICS code. In the case of \( d^2\psi/dz^2 \), the true profile is assumed to be given by \( 4\pi G \rho - 2(B^2 - A^2) \). With the equilibrium dataset, the model does an excellent job of recovering the potential and force as can be seen in the left-hand panels of Figure 1. The model overestimates \( d^2\psi/dz^2 \) near the mid-plane by \( \sim 5\% \) and underestimates for \( z \gtrsim 600 \text{ pc} \) by \( \sim 20\% \). The relatively poor fit at large \( z \) is not surprising given the small number of tracers at these large distances above the mid-plane and the fact that one is attempting to extract a second derivative of the potential.

As one might expect, the potential and force are not recovered as well when the mock data includes a phase spiral. The model underestimates the potential and the force and overestimating \( d^2\psi/dz^2 \).

In Figure 2, we show the actual and model-predicted number densities in the \( z - v_z \) plane for both mock datasets as well as the residuals. For the equilibrium dataset, the model captures the structure of the number density map extremely well. In particular, the residuals are dominated by shot noise. On the other hand, the spirals dominate the residuals in the perturbed mock dataset.

#### 3.3 residuals in frequency-angle space

The phase spirals seen in Figure 2 for the perturbed mock dataset are the result of phase mixing after the initial perturbation in an anharmonic vertical potential. A star with vertical energy \( E_z \) orbits the \( z - v_z \) plane with an angular frequency \( \Omega(E_z) = 2\pi/T(E_z) \)

\[
T(E_z) = 2 \int_{-z_m}^{z_m} \frac{dz}{|v|} = 2\sqrt{2} \int_0^{z_m} \frac{dz'}{\sqrt{E_z - \psi(z')}} \tag{27}\]
is the orbital period and \( z_m = z_m(E_z) \) is the maximum vertical excursion of a particle with energy \( E_z \). Note that for \( E_z = 0 \), \( \Omega(0) \equiv \Omega_0 = (\omega_z^2 + \omega_t^2)^{1/2} \). The usual angle variable of angle-action coordinates \( \theta \)\(^1\) is then given by

\[
\theta(z, v_z) = -\text{sgn}(v_z) \cdot 2\pi \cdot \tau / T(E_z)
\]

where \( \text{sgn} \) is the sign function, \( E_z \) follows Equation 4 and \( \tau \) is the time it takes for a star at \( z \) with energy \( E_z \) to travel upwards to its maximum excursion.

The initial perturbation in our second mock dataset amounts to a displacement of the peak in the \( z - v_z \) DF along the \( v_z \) direction. Over time \( t \), this peak is sheared due to variations in \( \Omega(E_z) \) with \( E_z \). So long as the particles evolve kinetically, the peak becomes a straight ridge in the \( \Omega(E_z) - \theta \) plane as defined by the linear equation

\[
\theta = t \cdot \Omega(E_z) + \theta_0.
\]

\( \text{Antoja et al. 2018; Binney & Sch"onrich 2018; Darling & Widrow 2019a} \). In this equation, \( \theta_0 \) is the angle of the initial displacement. For our perturbed dataset, the true values are \( t = 500 \) Myr and \( \theta_0 = -\pi/2 \).

In Figure 3, we map the residuals shown in Figure 2 to the \( \Omega(E_z) - \theta \) plane using the best-fit potential from our maximum likelihood analysis. The spirals do indeed become straight lines with the obvious wrap-around effect due to periodicity in \( \theta \). To infer \( t \) and \( \theta_0 \), we model the residual as a Fourier series:

\[
R_F(\theta_i, \Omega_i; t, \theta_0) = \sum_{n=1}^{n_{\text{max}}} A_n \cos(\theta_i - \Omega_i t - \theta_0)
\]

where the subscript \( i \) refers to the \( i \)th bin of our \( \Omega - \theta \) grid. We then take the log-likelihood function to be

\[
\ln L(t, \theta_0, \lambda) = -N_b \ln (2\pi \lambda) - \frac{1}{2\lambda^2} \sum_{i=1}^{N_b} [R_F(\theta_i, \Omega_i; t, \theta_0) - R(\theta_i, \Omega_i)]^2
\]

where \( N_b \) is the number of \( \Omega - \theta \) bins, \( R(\theta_i, \Omega_i) \) is the actual density residual in the \( i \)th bin and \( \lambda \) is a parameter that characterizes the uncertainties in the model. We fit the \( \Omega - \theta \) space residuals over the range \( \Omega_{\text{min}} = 45 \) km/s/kpc and \( \Omega_{\text{max}} = \Omega_0 \) and \( -\pi < \theta < \pi \). The Fourier coefficients \( A_n \) are given by

\[
A_n = \frac{1}{\pi (\Omega_{\text{max}} - \Omega_{\text{min}})} \int_{\Omega_{\text{min}}}^{\Omega_{\text{max}}} d\Omega \int_{-\pi}^{\pi} R(\theta, \Omega) d\theta .
\]

An emcee calculation of the likelihood function yields the best-fit parameters \( t = 556 \pm 3 \) Myr and \( \theta_0 = 2.2 \pm 0.2 \) rad for \( n_{\text{max}} = 4 \), though the results are virtually the same for \( n_{\text{max}} = 1, 2, \) or 3. Recall that for the perturbed mock data, the true values are \( t = 500 \) Myr and \( \theta_0 = -\pi/2 \) rad. Thus, we recover \( t \) to about 10%, but don’t recover \( \theta_0 \). Given that \( \Omega \sim 55 \) km/s/kpc for our dataset, this is not surprising as it only takes an error of \( \Delta t \simeq 50 \) Myr, that is, a fractional error of \( \geq 10\% \) for \( t \), to scramble the value of \( \theta_0 \) by \( \pi \).
4 GIANT GDR2 SAMPLE

In this section, we describe the steps needed to apply our fitting algorithm to GDR2 data. These include sample selection, a completeness check, and a method for handling measurement uncertainties.

4.1 Star catalog and quality cuts

Historically, the choice of stellar populations to study the local vertical force has been guided by the availability of accurate position and velocity measurements as well as the notion that old populations are well-mixed since they will have made many oscillations through the Galactic plane. Common choices include main sequence stars, such as F, G, and K Dwarfs (Hill et al. 1979; Bahcall 1984a; Kuijken & Gilmore 1989b; Zhang et al. 2013; Xia et al. 2016; Guo et al. 2020), and K Giants (Bahcall 1984c; Kuijken & Gilmore 1989c; Holmberg & Flynn 2004). More recently, Bienaymé et al. (2014); Hagen & Helmi (2018) and Salomon et al. (2020) considered red clump stars which have the advantage that their distances can be accurately determined from photometry since they are good standard candles (Groenewegen 2008; Girardi 2016; Hawkins et al. 2017; Ruiz-Dern et al. 2018).

The Gaia mission aims to determine the positions and velocities for ~1.2 billion stars. Already, the radial velocity sample from GDR2 provides complete measurements for the phase space components for ~7 million stars. In this study, we draw our sample of giants from the GaiaIVdelpecqspelspj3 catalog constructed by Schönhrich et al. (2019). The authors show that GDR2 parallaxes are systematically biased and propose corrected parallaxes given by \( \pi_\odot = \pi_G + \sqrt{(0.043\text{mas})^2 + \overline{\epsilon^2}} \) where \( \pi_G \) and \( \overline{\epsilon^2} \) are GDR2 parallaxes and their uncertainties. We use their distance expectation value, \( r_{\text{dist}} \), as a distance estimate. We note that differences between \( r_{\text{dist}} \) and \( 1/\pi_\odot \) are typically less than 1%. We take distance uncertainties to be

\[
\epsilon_r = \sqrt{(r^2 - \langle r \rangle^2) / \langle r \rangle^2}
\]

where \( \langle r \rangle \) is given by \( E_{\text{dist}} \) and \( r^2 \) is given by the second moment of the distance probability distribution, \( \text{distm2} \). We calculate the \( z - v_z \) coordinates using the astropy.coordinates Python package\(^3\) where we assume the Sun's distance to the Galactic center as \( R_\odot = 8.3 \text{kpc} \) (Gillessen et al. 2009), the Sun's vertical displacement from the mid-plane as \( z_\odot = 20.3 \text{pc} \) (Bennett & Bovy 2019), and the Sun’s vertical motion with respect to the local standard of rest as \( v_z,\odot = 7.24 \text{km/s} \) (Schönrich et al. 2010).

We implement the following quality cuts as recommended by Schönhrich et al. (2019) to insure better parallax precision:

- \( 0 < G < 14.5, G_{\text{RP}} > 0, G_{\text{BP}} > 0 \) where \( G_{\text{BP}}, G \) and \( G_{\text{RP}} \) are apparent magnitudes in Gaia’s three broad colour bands.
- \( \epsilon_{v_{\text{rad}}}/\sigma < 0.1 \) mas and \( \pi/\sigma > 5 \) where \( \sigma \) is the uncertainties of the parallax measurement. The second of these cuts is in accord with other papers that have analyzed Gaia DR2 data (Antoja et al. 2018; Bennett & Bovy 2019; Guo et al. 2020; Li & Shen 2020).
- \( \text{visibility period} n_{\text{vis}} > 5 \)
- \( 1.172 < \text{bp rp excess factor} < 1.3 \)
- \( d > 80 \text{pc} \) where \( d \) is the distance from the Sun. This cut reduces systematic distance errors to < 4% (Schönhrich et al. 2019).

In addition to these cuts, we remove stars with Galactocentric speed \( |v| > 550 \text{km/s} \) since these stars have speeds close to or exceeding the escape speed of the Galaxy at the Solar circle (Williams et al. 2017; Monari et al. 2018; Marchetti et al. 2019). We also exclude stars identified by Boubert et al. (2019) as potentially having large radial velocity errors due to contamination of their spectra by a star in close alignment. We exclude stars within 15° of the Galactic mid-plane (i.e. \( |b| < 15° \)) to avoid losing stars due to obscuration (Katz et al. 2019). Finally, we remove stars with \( G < 3 \) as the whole GDR2 is only complete for \( 3 \lesssim G \lesssim 17 \) (Bennett & Bovy 2019). The final cut on apparent magnitude is then \( 3 < G < 14.5 \).

4.2 Sample volume and CMD region

In their discovery paper on phase spirals, Antoja et al. (2018) selected stars from an annular wedge with Galactocentric radius \( 8.24 \text{kpc} < R < 8.44 \text{kpc} \) and Galactic azimuthal angle \( \varphi \) within 4° of the Sun. Thus, their volume has an extent in the azimuthal direction more than five times larger than the extent in the radial direction. They use all stars from the GDR2 radial velocity survey, which has a magnitude limit at the faint end of \( G = 17 \). However, for all stellar populations combined, the GDR2 radial velocity survey is only complete for \( 4 \lesssim G \lesssim 12.5 \) (Katz et al. 2019). Thus, their \( z - v_z \) number count map has a \( z \)-dependent selection function which isn’t accounted for.

For our analysis, we carve out a region with the same shape

\(^2\) See https://zenodo.org/record/2557803 for their data

\(^3\) See https://docs.astropy.org/en/stable/coordinates/index.html for documentation.

\(^4\) See https://arxiv.org/src/1901.10460v1/anc/ for a catalog of these stars.
Figure 5. In the upper panel, we show the volume number density profile with $80 \text{ pc} < |z - z_0| < 1.48 \text{ kpc}$ since Figure 4 is only for illustrative purposes. In this way, the extent of the sample volume in the azimuthal and radial directions are similar. As for $z$ and $v_z$, we use stars with $80 \text{ pc} < |z - z_0| < 1.48 \text{ kpc}$ (instead of $|z| < 1.48 \text{ kpc}$ for our mock data) and $|v_z| < 80 \text{ km/s}$ for our fitting. We modify the $z$ selection criteria for two reasons: (1) the sample volume is Sun-centered, and (2) there is little volume left to analyze within $|z - z_0| < 80 \text{ pc}$ compared to the whole sample volume since we have removed the region within $80 \text{ pc}$ from the Sun and also within $15^\circ$ of the mid-plane.

By considering a sample volume that is larger than that of Antoja et al. (2018), we are able to choose stars from a region of the CMD that guarantees completeness for $3 < G < 14.5$ while maintaining an adequate sample size. Since giant stars are more likely to have available RVS due to their intrinsic brightness, we select a sample of giants with $G_{BP} - G_{RP} > 1$ and $M_G < 2$ on the CMD. For our sample volume, the distance to the Sun ranges between $r_{\text{min}} = 80 \text{ pc}$ and $r_{\text{max}} = 1.67 \text{ kpc}$. Given our apparent magnitude cut $3 < G < 14.5$, we apply an absolute magnitude cut of

$$3 - 5\log_{10} \frac{r_{\text{min}}}{10 \text{ pc}} = -1.52 < M_G < 14.5 - 5\log_{10} \frac{r_{\text{min}}}{10 \text{ pc}} = 3.83$$

(34)


to avoid Malmquist Bias. Combined with the cut $M_G < 2$ for giants, we arrive at our absolute magnitude cut of $-1.52 < M_G < 2$, which concludes all cuts we apply on our data. Our final sample comprises 108,852 stars.

In Figure 4, we show the region of the CMD used in this work as well as other studies of stellar kinematics in the vicinity of the Sun. For example, Gaia Collaboration et al. (2018b) constructed a giant star catalog from GDR2 that was selected by having $G$-band absolute magnitudes of $M_G < 3.9$ and intrinsic colors of $G_{BP} - G_{RP} > 0.95$. This sample of over three million sources was then used to produce mean velocity and velocity dispersion maps in a volume extending to $\sim 4 \text{ kpc}$ from the Sun. Since the authors are computing moments of the intrinsic colors of $G$ by having constructed a giant star catalog from GDR2 that was selected for stars with $G_{BP} - G_{RP} > 1$ and $M_G < 2$ on the CMD, they use in their Jeans equation analysis of Red Clump stars.

4.3 vertical number density profile and sample completeness

We examine the distribution of stars as a function of $z$ in Figure 5. In the upper panel, we show the volume number density profile $n(z) \equiv n(z)/S(z)$ for both our sample and a sample that includes stars that do not have radial velocity measurements. For stars that are not in the gaiaRVdelpqspdelpsf3catalog catalogue, we estimate distances following the procedure outlined in Schöning et al. (2019), that is, $d = 1/\varpi_S$ where $\varpi_S = \varpi_G + \sqrt{(0.043 \text{mas})^2 + \varepsilon^2}$. In addition, we highlight differences between the number counts north and south of the mid-plane by plotting $\nu(z)$ separately for $z > 0$ and $z < 0$. The middle panel shows the ratio of $\nu(z)$ from the radial velocity sample and the full sample. The fact that the ratio is close to unity implies that the radial velocity sample is as complete as the full GDR2 survey. That is, for our giant sample, the restriction to stars with radial velocity measurements doesn’t introduce any new selection effects.

In the lower panel, we show our results for the North-South asymmetry, $A(z) = [\nu(z) - \nu(-z)] / [\nu(z) + \nu(-z)]$, which is consistent with what has been found in Widrow et al. (2012); Bennett & Bovy (2019); Salomon et al. (2020). This asymmetry has been interpreted as evidence for disequilibrium in the stellar disk.

4.4 measurement uncertainties

Statistical uncertainties in stellar distances, proper motions and radial velocities imply uncertainties in $z$ and $v_z$. Thus, the $z - v_z$ bin assigned to a given star is also uncertain. In addition, the true position of a star can lie outside the sample volume even its measured position lies inside it. By calculating the uncertainties in $z$ and $v_z$ (see formulae in Johnson & Soderblom (1987)) we estimate that $\sim 15\%$ of stars have a true bin different from on the one implied by the measured kinematics.

To account for these uncertainties, we use a bootstrap method to generate 100 datasets where astrometric quantities are sampled from measured quantities under the assumption the uncertainties are Gaussian. We then convert each dataset to $x$, $y$, $z$ and $v_x$ using the astropy.coordinates Python package and conduct data selection and fitting as previously discussed. Finally, the MCMC chains from each of the datasets are combined to yield a PDF for the model param-

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5 See https://gea.esac.esa.int/archive/documentation/GDR2/. We use Johnson-Cousins B-V values as a proxy for Hipparcos B-V since Figure 4 is only for illustrative purposes.
eters. This PDF can be sampled to determine best-fit values and uncertainties for the vertical potential, the vertical force, the DF as well as the parameters themselves.

5 RESULTS

In Table 2, we present best-fit values and 1σ errors for all parameters as well as ψ(z) and Fz at z = 0.5 kpc, 1.0 kpc and 1.5 kpc. We also present Σ(z) at these heights, which are derived from Equation 13. To do so, we assume A = 15.45 ± 0.34 km/s/kpc and B = −12.27 ± 0.40 km/s/kpc, which are averages of recent measurements by Bovy (2017); Vityazev et al. (2017); Bobylev & Bajkova (2018); Nouh & Elsanhoury (2020); Krisanova et al. (2020).

In Figure 6, we show one- and two-dimensional projections of the likelihood function via the so-called corner plot (Foreman-Mackey 2016). We see that the peak of the likelihood function is well inside the region defined by the prior probabilities for the parameters given in Table 1. The likelihood function appears to be reasonably well behaved, though there is a tail of outliers, which will be discussed in detail in Section 6.1. Along the tail, D is large and therefore the two terms in Equation 10 can both be regarded as quadratic. Hence, ω1 and ω2 are approximately degenerate and therefore anti-correlated. In addition, a positive correlation exists between α and σz. To see this, consider Equation 21 in the limit z = 0. We then have ω(z) = ασz(1 − ασz)−1. Thus, for fixed mid-plane dispersion, an increase in α requires an increase in σz.

In Figure 7, we show profiles of ψ, Fz, and d2ψ/dz2, which are calculated using 1000 samples of the model parameters from the MCMC chain. In Figure 8, we compare our ψ(z) and Fz profiles with those obtained by Holmberg & Flynn (2000, 2004); Bovy & Rix (2013); Zhang et al. (2013); Bienaymé et al. (2014); Xia et al. (2016); Hagen & Helmi (2018) and Guo et al. (2020). Our results for ψ(z) are similar to those from the literature. As expected, differentiation amplifies the differences as can be seen in the lower panel of the Figure where we show Fz(z). Our estimate for the force is generally lower than the literature values for |z| ≲ 700 pc but is consistent with the literature values for z ≃ 800 pc, which is also where the scatter in the models is at a minimum. At larger values of z the scatter increases due to the

| Quantity | Best-fit value and 1σ error |
|----------|-----------------------------|
| ω1      | 52.8±1.4 km/s/kpc           |
| D        | 0.36±0.07 kpc              |
| ω2      | 31.4±3.4 km/s/kpc          |
| σz      | 12.8±0.1 km/s              |
| ln α    | 0.97±0.01                  |
| ψ(0.5 kpc) | 376.1±4.6 (km/s)²       |
| ψ(1.0 kpc) | 1187±9 (km/s)²        |
| ψ(1.5 kpc) | 2276±39 (km/s)²        |
| f2(0.5 kpc) | 1294±16 (km/s²)/kpc       |
| f2(1.0 kpc) | 1914±35 (km/s²)/kpc       |
| f2(1.5 kpc) | 2438±102 (km/s²)/kpc       |
| Σ(0.5 kpc) | 44.6±0.8 M⊙/pc⁻²         |
| Σ(1.0 kpc) | 64.3±1.7 M⊙/pc⁻²         |
| Σ(1.5 kpc) | 80.4±4.1 M⊙/pc⁻²         |

Table 2. Best-fit values and 1σ errors of all parameters as well as ψ(z), Fz and Σ(z) at z = 0.5 kpc, 1.0 kpc and 1.5 kpc.
lack of data. Based on these results we expect that our estimate for the matter density near the mid-plane will be below the values quoted in the literature. Indeed, we obtain $\rho_0 = 0.067 \pm 0.004 M_{\odot} pc^{-3}$ from equation 15 with a correction of $-0.003 M_{\odot} pc^{-3}$ from the radial term $2(B^2 - A^2)$. Our value is roughly 30% lower than typical literature values which cluster around $0.09 M_{\odot} pc^{-3}$ (Kuijken & Gilmore 1989c; Holmberg & Flynn 2000, 2004; Bovy & Rix 2013; Zhang et al. 2013; Bienaymé et al. 2014; Bovy et al. 2012b; Yanny et al. 2009). They arrive at their estimate by first dividing the sample into mono-abundance sub-populations as defined by $[\alpha/Fe]$ and $[Fe/H]$. These sub-populations are found to have a velocity dispersion that is approximately constant in $z$, which implies that the sub-populations are approximately isothermal. A scatter plot of the surface density for each sub-population as a function of dispersion and $[\alpha/Fe]$ is shown in their Figure 8 and reproduced here in Figure 10. Also shown is the histogram derived by binning the sub-populations in $\mu_z^2$. We see that the distributions indicated by our curve and their histogram are qualitatively similar though they are derived for different populations of stars.

In Figure 11 we plot the number densities in the $z - v_z$ plane.

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23. Apart from a sharp rise with $\mu_z$ near $\mu_z = 0$, $d\Sigma/d\mu_z^2$ is a decreasing function of $\mu_z$. Bovy et al. (2012b) approximate this function, which they call the vertical temperature distribution, for a sample of G-dwarf from SDSS/SEGUE (Abazajian et al. 2009; Yanny et al. 2009). They arrive at their estimate by first dividing the sample into mono-abundance sub-populations as defined by $[\alpha/Fe]$ and $[Fe/H]$. These sub-populations are found to have a velocity dispersion that is approximately constant in $z$, which implies that the sub-populations are approximately isothermal. A scatter plot of the surface density for each sub-population as a function of dispersion and $[\alpha/Fe]$ is shown in their Figure 8 and reproduced here in Figure 10. Also shown is the histogram derived by binning the sub-populations in $\mu_z^2$. We see that the distributions indicated by our curve and their histogram are qualitatively similar though they are derived for different populations of stars.
Figure 9. Vertical distribution function $f_z(E_z)$ for the data and model. The solid red curve is the DF for the stars in our sample where the vertical energy is calculated using best-fit parameters for the potential from Table 2. The dotted blue curve shows our RLDF model (Equation 18) with $\alpha = 2.64$ and $\sigma_z = 12.8 \, \text{km s}^{-1}$.

Figure 10. Vertical temperature distribution in the Solar Neighbourhood from an analysis of SDSS/SEGUE G-dwarf data by Bovy et al. (2012b) and from our analysis of GDR2 giants. The points and histogram are the same as in Figure 8 in Bovy et al. (2012b). The points show the surface density for mono-abundance populations as a function of their vertical velocity dispersion squared, $\mu_z^2$. Colors indicate the $[Fe/\alpha]$ abundance. The histogram gives the contribution to the surface density, $\Delta \Sigma$ for bins of width $\Delta \mu_z^2 = 260 \, \text{km}^2 \, \text{s}^{-2}$. The solid blue curve is our model prediction for $\Delta \Sigma = (\Delta \mu_z^2 / 2 \mu_z) (d\Sigma / d\mu_z)$ where $d\Sigma / d\mu_z$ is given by Equation 26 with $\sigma_z = 12.8 \, \text{km s}^{-1}$ and $\alpha = 2.64$. We normalize our curve so that the area under our curve equals the area under the histogram.

Figure 11. Density in $z-v_z$ plane of our GDR2 sample (upper panel) and our best-fit model (middle panel) along with the fitting residual (lower panel) in unit of $\text{kpc}^{-3} \cdot \text{km/s}^{-1}$. The overdensity in the residual panel forms a clear spiral pattern.

The perturbation age is in agreement with Antoja et al. (2018) who estimate that the perturbing event started $\sim 500$ Myr ago with a likely range of $300 \sim 900$ Myr.
Diatom values of potential is approximately quadratic for small and intermediate values of $D$. The bridge and island stretch to higher values of plane, we conclude that it is related to the potential and not the DF. The bridge and island stretch to higher values of $D$. In this region of parameter space, the first term in the potential is approximately quadratic for small and intermediate values of $\omega_2$. We therefore expect $\omega_1$ and $\omega_2$ to be anti-correlated for this region and indeed this is what is seen in the $\omega_1$ projection of the PDF. We explore this degeneracy further in Figure 13 where we plot $\theta$ and $F_\theta$ for parameters characteristic of the island, namely $\omega_1 = 57$ km/s/kpc, $D = 0.65$ kpc and $\omega_2 = 8$ km/s/kpc. The potential and force are nearly the same as those obtained from our best-fit values from Table 2 for $z \lesssim 900$ pc but strongly diverge at larger $z$. Also shown are the results from an MCMC analysis where $\omega_2$ is fixed to be zero. As with the “island” model, the potential and force are nearly the same as those obtained with the three-parameter potential out to $z \sim 900$ pc.

These results are symptomatic of the well-known fact that it is difficult to disentangle the disc, bulge, and halo contributions from the potential despite many attempts (see, for example, Zhang et al. 2013; Xia et al. 2016; Sivertsson et al. 2018; Guo et al. 2020). As mentioned in Section 2, Kuijken & Gilmore (1989a,b) identify the first and second terms in Equation 10 with the disc and (effective) halo respectively. But it is clear that the potential within the first few scale heights of the disc is adequately fit by a simple two-parameter model. Note that while the potentials are similar, the inferred values for the mid-plane density, which is derived from the second derivative at $z = 0$ can vary considerably. We can see this already in our models which show an increase in the scatter of $d^2\psi/dz^2$ as $z \to 0$. In fact, the estimates for $\omega_1^2 + \omega_2^2$ for our three-parameter and island models differ by 13%. The baryon-dark matter degeneracy was illustrated in Guo et al. (2020) who illustrated that their inferred values for the dark matter density in the Solar Neighbourhood were sensitive to the choice of priors for the stellar surface and volume densities (see their Table 2). However, as seen in Figure 8, the potential and force are relatively insensitive to priors on the stellar component.

### 6.2 separability assumption and mid-plane density

The analysis presented in this work is based on the 1D approximation wherein the vertical dynamics of Solar Neighbourhood stars is separable from their dynamics in the Galactic plane and that the vertical force is independent of $R$. In reality, in-plane and vertical motions are coupled since the full potential $\Psi(R,z)$ is not separable. The effects of a non-separable potential are greatest for stars with large radial and azimuthal velocity dispersion since these stars make the largest excursions from the Solar Neighbourhood. Following Li & Shen (2020) we reanalyze our data by first dividing the sample into cold and hot sub-samples. As we will see, the results may shed light on the discrepancy between our inferred value for $\rho_0$ and the values found in the literature.

We define the effective radial energy of a star as

$$E_{R,\text{eff}}(R,v_R) \simeq \frac{v_R^2}{2} + \chi_{\text{eff}}(R)$$

$$= \frac{v_R^2}{2} + \left[\Psi(R,0) + \frac{L_z^2}{2R^2}\right]$$

where $\chi_{\text{eff}}(R) \equiv \Psi(R,0) + \frac{L_z^2}{2R^2}$ is the effective radial potential and $L_z$ is the vertical angular momentum. We assume that near the Sun, the potential in the mid-plane has the form $\Psi(R,0) = v_c^2 \ln R$ with $v_c = 220$ km s$^{-1}$ such that the rotation curve of the Solar Neighborhood is flat with a rotation speed

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**Figure 12.** The residual of phase-space density plotted in $\Omega - \theta$ space. Parallel straight lines corresponding to $t = 543$ Myr and $\theta_0 = -1.2$ rad are overplotted in black. The unit of the residual is kpc$^{-1} \cdot \text{s} \cdot \text{km}^{-1} \cdot \text{rad}^{-1}$.

**Figure 13.** Comparison of $\psi(z)$ and $F_z$ from 2-parameter and 3-parameter potential fitting and the degenerate case of 3-parameter potential.
of $v_c$. A star with angular momentum $L_z$ and $v_R = 0$ follows a circular orbit with $R = \frac{L_z}{v_c}$. The radial energy, which we define as

$$\Delta E_{R,\text{eff}} \equiv E_{R,\text{eff}}(R, v_R) - E_{R,\text{eff}} \left( \frac{L_z}{v_c}, 0 \right)$$

(36)

indicates the extent of the radial motion. We sort stars in our sample by increasing $\Delta E_{R,\text{eff}}$ and take first and the second halves to be the cold and hot sub-samples, respectively. The hot sample has larger extent of radial motion and is therefore more affected by the coupling of in-plane and vertical motion. In Figure 14, we randomly select 10 thousand cold and hot stars respectively from our sample and plot them on the $v_R - v_\phi$ plane. The figure shows a clear elliptical boundary between the two sub-samples and is in good agreement with Figure 7 of Li & Shen (2020) where the radial action $J_R$ is used to separate hot and cold components.

Results for our analysis for the two sub-samples are presented in Table 3 as well as $\psi(z)$ and $F_z$ shown as Figure 15. While the inferred $\psi(z)$ almost coincide the underlying parameter values, the corresponding $F_z$ clearly differ from each other. For the mid-plane density, we find that $\rho_{0,\text{cold}} = 0.091^{+0.014}_{-0.0092} M_\odot \text{kpc}^{-3}$ and $\rho_{0,\text{hot}} = 0.0552^{+0.0020}_{-0.0022} M_\odot \text{kpc}^{-3}$ (radial correction included) from the parameter sampling. That is to say, $\rho_{0,\text{cold}}$ is in good agreement with literature value while $\rho_{0,\text{hot}}$ is even worse than what we got for the whole sample. Therefore, our low $\rho_0$ is likely caused by the non-separable nature of the Galactic potential.

### Table 3. Best-fit parameter values and 1σ errors for the cold and hot sub-sample.

| Parameter | the cold sub-sample | the hot sub-sample |
|-----------|---------------------|-------------------|
| $\omega_1$ | $62.0^{+4.8}_{-3.4}$ km/s/kpc | $53.6^{+1.9}_{-3.0}$ km/s/kpc |
| $D$       | $0.17 \pm 0.04$ kpc       | $0.69^{+0.09}_{-0.11}$ kpc |
| $\omega_2$ | $35.8^{+1.5}_{-1.9}$ km/s/kpc | $16.6^{+8.8}_{-9.2}$ km/s/kpc |
| $\sigma_\alpha$ | $12.0 \pm 0.1$ km/s | $15.2 \pm 0.2$ km/s |
| ln $\alpha$ | $1.23 \pm 0.02$ | $0.91 \pm 0.02$ |

7 CONCLUSIONS

In this work, we introduce a method for inferring the local vertical potential and stellar DF from kinematic measurements of stars in the vicinity of the Sun. The method is based on Jeans Theorem under the 1D approximation where $f_z$ is a function of $z$ and $v_z$ through $E_z$ and $\psi$ is a function of $z$. The likelihood function, which drives the method, compares the stellar number density in the $z - v_z$ plane to the model prediction. The best-fit potential is the one in which contours of constant $E_z$ coincide with contours of constant number density. The method has several advantages over other approaches. First, in contrast with methods based on the Jeans Equations, it works directly with the $z - v_z$ DF rather than its moments. Second, the DF and potential are inferred simultaneously rather than sequentially as in the approaches of Kuijken & Gilmore 1989a,b; Holmberg & Flynn 2000, 2004. Finally, evidence for disequilibrium such as the phase spiral emerge from the residuals of the model.

We also use this work to introduce the RLDF as a parametric model for $f_z$. In a sense, the RLDF serves as an alternative to models with thin and thick disc components. In fact, it has one fewer parameter than a model with two isothermal components. Since the RLDF can be written as the integral sum of isothermal components, it provides a simple mathemati-
cal model for the continuous mono-abundance sub-population proposal of Bovy et al. (2012a,b); Bovy & Rix (2013).

From our analysis of a sample of GDR2 giants, we inferred the vertical potential, force, and density for $|z| < 1.5$ kpc. Our results were in general agreement with those found in the literature though our inferred value for the total mid-plane density was below the published values. A reanalysis using stars with relatively low radial energy (i.e., the ”cold” population) yielded a value closer to the ones from the literature. We also calculated the vertical temperature distribution (differential surface density as a function of vertical velocity dispersion) and found that the form was similar to the corresponding distribution for SDSS/SEGUE G dwarfs from Bovy et al. (2012b). Finally, we viewed the residuals of the DF in the frequency-angle plane and found that the phase spiral mapped to a straight line whose slope yielded an estimate of $\sim 540$ Myr for the time since the event that perturbed the disc.

We conclude by mentioning two ways in which the model can be improved. The first is to tackle its most significant shortcoming, namely the use of the 1D approximation. To do so, one can model the DF by a three-integral DF from Kuijken & Dubinski (1995) (see our Equation 17) or the quasi-isothermal model from Binney (2010); Binney & McMillan (2011). We note that in either case, we can replace the isothermal factor with an RLDF one. The second improvement is to incorporate disequilibrium features such as the phase spiral into the model. So long these features are incorporated disequilibrium features such as the phase spiral to a straight diagonal line. 

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**DATA AVAILABILITY**

The Gaia Data Release 2 is available at the following website: https://gea.esac.esa.int/archive/. All other data used for our work is available through the links posted in the footnotes where necessary.

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