On quantum corrections to geodesics in de-Sitter spacetime

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Abstract

We find a coordinate-independent wave-packet solution of the massive Klein-Gordon equation with the conformal coupling to gravity in the de-Sitter universe. This solution can locally be represented through the superposition of positive-frequency plane waves at any space-time point, assuming that the scalar-field mass $M$ is much bigger than the de-Sitter Hubble constant $H$. We study then how this wave packet propagates over cosmological times, depending on the ratio of $M$ and $H$. In doing that, we find that this packet propagates as a point-like particle of the same mass if $M \gg H$, but, if otherwise, the wave packet behaves highly non-classically.
I. INTRODUCTION

Elementary particles in Minkowski spacetime are related to unitary and irreducible representations of the Poincaré group. Their notion is thus unambiguous in all Lorentz frames, as the Poincaré group represents the isometry group of Minkowski spacetime. It might be then tempting to expect that there is no well-defined particle notion in non-flat spacetimes. In fact, Schrödinger argued that particles may be produced in evolving universes [1]. This quantum effect arises from the absence of time-translation symmetry, which requires the re-definition of creation and annihilation operators during time evolution, while quantum states remain unchanged. A no-particle state at earlier times may not then be interpreted as an empty state at later times [2, 3]. In addition, a single-particle state turns into a multi-particle state over time, resembling, thereby, particle decays in interacting quantum-field models.

In spite of the fact that the observable Universe is dynamically changing all the time, we successfully describe high-energy processes by using the Standard Model of particle physics, in which the Poincaré group plays a crucial role [4]. In the Standard Model, a particle decay may occur if compatible with various conservation laws. In particular, we observe on Earth that energy, momentum and angular momentum are conserved in particle scatterings. As an example, the electron neutrino was foreseen in $\beta$-decay from energy-momentum conservation long before its actual detection [5]. These conservation laws are, in turn, related to the spacetime translation and rotational symmetries which are spontaneously broken in nature. Still, these laws must locally hold, according to the equivalence principle, which is in agreement with the up-to-date observations [6].

It is thus an empirical fact that particles in collider physics are well-defined, even though the observable Universe is evolving. This could be readily explained if wave functions, which describe particles, are well-localised in spacetime. Their non-point-like support is still testable in gravity, namely the quantum interference of non-relativistic neutrons was observed in the Earth’s gravitational field [7]. This observation is consistent with the Schrödinger equation with the Newtonian potential. In general, if the Compton wavelength of particles is negligible with respect to a characteristic curvature length, then the quantum interference induced by gravity can be described by the following coordinate-independent phase factor:

$$\exp\left(-iM\int_A^B ds\right) \quad \text{with} \quad ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu,$$

where $M$ is the particle mass, $A$ and $B$ are initial and final positions of the particle, respectively, which moves along a geodesic connecting these points [8].

The main purpose of this article is to generalise this result to the case when the Compton wavelength of a particle may be comparable to the characteristic curvature length. Besides, particle’s propagation time may be as large as a characteristic curvature time. Since it is not obvious if this generalisation is even possible, we shall consider de-Sitter spacetime, in which one should be able to find a non-perturbative result due to de-Sitter symmetries.

Throughout, we use natural units $c = G = \hbar = 1$, unless otherwise stated.
II. ADIABATIC PARTICLES IN DE-SITTER SPACETIME

It was elaborated in 1968 how adiabatic particles may be created in an expanding universe in linear quantum field models [2]. In this section, we briefly review this adiabatic-particle-creation process in the de-Sitter universe in order to introduce concepts and notations which will be used later on.

Considering de-Sitter spacetime with the Hubble parameter $H$ in flat coordinates $(t, \mathbf{x})$, the particle-creation operator can be defined through the adiabatic modes at past and future cosmic infinities [3]. Following the recent references [9, 10], one has

$$
\hat{a}^\dagger(\varphi_k) = \begin{cases} 
\hat{a}^\dagger(\varphi_{k,-\infty}), & t \to -\infty, \\
\hat{a}^\dagger(\varphi_{k,+\infty}), & t \to +\infty,
\end{cases}
$$

where, in case of the scalar field $\Phi(x)$ with the mass $M$ and conformal coupling to gravity,

$$
\varphi_{k,-\infty}(x) = \left(\frac{\pi}{4Ha^3(t)}\right)^{\frac{1}{4}} e^{\pi \mu z} \frac{H_{1\mu}}{H_0(t)} \left( \frac{|k|}{H_0(t)} \right)^2 e^{i k x}, 
$$

$$
\varphi_{k,+\infty}(x) = \left(\frac{1}{2\mu Ha^3(t)}\right)^{\frac{1}{4}} 2^{i\mu} \Gamma(1+i\mu) J_{i\mu} \left( \frac{|k|}{H_0(t)} \right)^{2} e^{i k x},
$$

where $a(t) = e^{Ht}$ is the de-Sitter scale factor, $\Gamma(z)$, $H_{1\mu}(z)$ and $J_{\mu}(z)$ are, respectively, the gamma, Hankel and Bessel functions, and

$$
\mu \equiv \frac{1}{2} \sqrt{4\nu^2 - 1} > 0 \quad \text{with} \quad \nu \equiv M/H.
$$

The solutions $\varphi_{k,-\infty}(x)$ and $\varphi_{k,+\infty}(x)$ match, respectively, the adiabatic modes at past and future infinity. These will be referred to as the past and future adiabatic modes.

The de-Sitter universe turns into Minkowski spacetime in the limit $H \to 0$. It is straightforward to show in this case that both asymptotic adiabatic modes turn into the Minkowski plane-wave solutions up to a phase factor. However, $\varphi_{k,-\infty}(x)$ gives rise to a “preferred” state in de-Sitter spacetime. In fact, this mode defines the Chernikov-Tagirov aka Bunch-Davies state $|dS\rangle$, which we denote by $|dS\rangle$.

This quantum state is a no-adiabatic-particle state at past infinity (in flat de-Sitter space), in the sense that $|dS\rangle$ is annihilated by $\hat{a}(\varphi_{k,-\infty})$, i.e. $\hat{a}(\varphi_{k,-\infty})|dS\rangle = 0$ and

$$
\hat{a}(\varphi_{k,-\infty}) \equiv +i \int_{\Sigma} d\Sigma^\mu(x) \left( \bar{\varphi}_{k,-\infty}(x) \nabla_\mu \dot{\Phi}(x) - \dot{\Phi}(x) \nabla_\mu \bar{\varphi}_{k,-\infty}(x) \right),
$$

where $\Sigma$ is a space-like Cauchy surface and the bar stands for the complex conjugation. A normalisable single-$\varphi$-particle state can be then defined as

$$
|\varphi_{f_p,-\infty}\rangle \equiv \int d^3k \frac{f_p(k)}{(2\pi)^3} \hat{a}^\dagger(\varphi_{k,-\infty})|dS\rangle \equiv \hat{a}^\dagger(\varphi_{f_p,-\infty})|dS\rangle,
$$
where \( f_p(k) \) is a square-integrable function sharply peaked at \( k = p \) and satisfies

\[
\langle \varphi_{fp,-\infty} | \varphi_{fp,-\infty} \rangle = \int \frac{d^3k}{(2\pi)^3} |f_p(k)|^2 \equiv 1.
\]  

(7)

The state \( |\varphi_{fp,-\infty}\rangle \) does not depend on \( t \), in accordance with the Heisenberg picture we have been working in. Therefore, \( \langle dS | \rangle \) is empty with respect to \( \hat{a}(\varphi_{k,-\infty}) \) at all time moments and the de-Sitter particles are related to unitary and irreducible representations of the de-Sitter symmetry group \([13]\). These particles may be dynamical, i.e. \( |\varphi_{fp,-\infty}\rangle \) depends on time, only in interacting field models \([14, 15]\).

From another side, the de-Sitter mode \( \varphi_{k,-\infty}(x) \) turns into a linear superposition of the positive- and negative-frequency adiabatic modes at future infinity:

\[
\varphi_{k,-\infty}(x) = \alpha(\varphi_{+,\infty}, \varphi_{-\infty}) \varphi_{k,+\infty}(x) + \beta(\varphi_{+,\infty}, \varphi_{-\infty}) \bar{\varphi}_{k,+\infty}(x),
\]

(8)

where the Bogolyubov coefficients can be found in \([9, 10]\). This leads to

\[
\hat{a}^\dagger(\varphi_{k,-\infty}) = \alpha(\varphi_{+,\infty}, \varphi_{-\infty}) \hat{a}^\dagger(\varphi_{k,+\infty}) - \beta(\varphi_{+,\infty}, \varphi_{-\infty}) \hat{a}(\varphi_{-k,+\infty}).
\]

(9)

Hence, the adiabatic-particle-number operator \( \hat{N}(\varphi_{fp}) = \hat{a}^\dagger(\varphi_{fp}) \hat{a}(\varphi_{fp}) \) changes with cosmic time. In particular, one has

\[
\langle dS | \hat{N}(\varphi_{fp}) | dS \rangle = \begin{cases} 0, & t \to -\infty, \\ |\beta(\varphi_{+,\infty}, \varphi_{-\infty})|^2, & t \to +\infty. \end{cases}
\]

(10)

This means that \( |dS\rangle \) is a \( N \)-adiabatic-particle state at future infinity, assuming that

\[
N \equiv \text{floor}(\langle \beta(\varphi_{+,\infty}, \varphi_{-\infty}) |^2)
\]

(11)

where \( N \) can be arbitrarily large, as the Pauli principle does not apply to bosons. This effect is known in the literature as the cosmological adiabatic-particle creation \([2, 3]\).

The particle creation is based on the re-definition of the particle notion over time (see \([2]\)). This procedure implies that particles should be unstable. Specifically, if \( |\varphi_{fp,-\infty}\rangle \) describes a single-adiabatic-particle state at past infinity, then this state should be re-interpreted as a multi-adiabatic-particle state at future infinity. In fact, one finds that

\[
\langle \varphi_{fp,-\infty} | \hat{N}(\varphi_{fp,+\infty}) | \varphi_{fp,-\infty} \rangle = 1 + 2|\beta(\varphi_{+,\infty}, \varphi_{-\infty})|^2,
\]

(12a)

\[
\langle \varphi_{fp,-\infty} | \hat{N}(\varphi_{fp,+\infty}) | \varphi_{fp,-\infty} \rangle = 2|\beta(\varphi_{+,\infty}, \varphi_{-\infty})|^2.
\]

(12b)

This phenomenon may be called as the cosmological adiabatic-particle decay.

A particle decay in interacting field models is the process that may take place if it does not violate various conservation laws. For instance, we observe on Earth that energy, momentum and angular momentum are conserved in collider physics. These conservation laws come from space-time translation and rotational symmetries which are local symmetries of the Universe, according to the equivalence principle. In contrast, the cosmological particle decay cannot be a local process: The gravitational field is the only source of energy which is available for this decay, but the gravitational-field energy is non-localisable \([16]\).
III. COVARIANT PARTICLES IN DE-SITTER SPACETIME

A. Motivation

A scattering process in particle physics usually corresponds to unitary evolution of a $N$-particle state defined at past infinity into a $N$-particle state defined at future infinity. It is evident though that it is impossible to carry out a scattering experiment with the asymptotic states, i.e. states defined at $t \to \pm \infty$, in collider physics, bearing in mind that initial states should then have been arranged at the Big Bang. This apparent tension between theoretical constructions and experiments can be eliminated by taking into account that elementary particles are quantum-field excitations localised in spacetime, namely they are described by wave packets with a finite space-time extent. A Wilson cloud chamber is actually designed to visualise a charged-particle trajectory which is localised within the chamber and, hence, in space. Besides, the free neutron decays into a proton, electron and electron antineutrino, with a mean lifetime of around $10^3$ seconds – free neutrons are also localised in time. For these reasons, initial/final $N$-particle states need to be arranged not at past/future infinity, but rather at a fraction of a second before/after the scattering process. This means particles are essentially non-interacting if their wave packets are well-separated. This observation also explains why the Minkowski-spacetime approximation used in theory works well in practice: the observable Universe locally looks as Minkowski spacetime and, consequently, particles can be considered within a local inertial frame, since their support is normally much smaller than the local-frame extent.

The question of our interest is how the asymptotic states of collider physics emerge locally in curved spacetime. These quantum states describe elementary particles which are free of interactions. In the field model under consideration, this means that we need to determine a single-particle state which can describe a scalar particle to move along a geodesic.

One of the fundamental properties of the geodesic equation is its form invariance under general coordinate transformations. For example, geodesics do not depend on the coordinate parametrisation of de-Sitter spacetime. However, in the closed coordinates to cover the entire de-Sitter hyperboloid, adiabatic modes at past time infinity and the de-Sitter modes do not match [9]. The notion of an adiabatic particle is, in general, coordinate-dependent.

Another basic property of geodesics is that they locally reduce to straight lines. That is a free-particle trajectory $x(\tau)$, where $\tau$ is the proper time, is locally of the form $x(0) + \dot{x}(0) \tau$, where $x(0)$ and $\dot{x}(0)$ are the particle position and velocity at $\tau = 0$, respectively. In quantum theory over Minkowski spacetime, a constant-momentum single-particle state is described by the plane-wave-mode superposition. According to the equivalence principle, this description must also hold in a Fermi normal frame related to a particle geodesic in de-Sitter spacetime if $H|\Delta t| \ll 1$ and $H|\Delta x| \ll 1$, where $|\Delta t| = |\Delta x| = 0$ corresponds to that geodesic. In particle physics, we have also to require that $H\lambda_c \ll 1$, where $\lambda_c$ is the Compton wavelength of the elementary particle (see below). Under these premises, quantum field theory over Minkowski
spacetime should adequately describe this particle locally.

As noted above, \( \varphi_{k,-\infty}(x) \) turns into the Minkowski plane-wave solution if \( H \to 0 \). Since the Hubble parameter is dimensionful, one needs instead to consider \( H \ll M \) and \( H|t| \ll 1 \). The first condition is fulfilled by the massive fields of the Standard Model if \( H \) is identified with the present Hubble parameter, \( H_0 \sim 10^{-26} \text{ m}^{-1} \). The second condition cannot hold for all times. It is known by now that the dark-energy-dominated epoch has started at around \( 10^{16} \text{ s} \) after the Big Bang, whereas the universe age \( t_0 \sim 1/H_0 \) is about \( 10^{18} \text{ s} \) \([17]\). Therefore, \( \varphi_{k,-\infty}(x) \) cannot be reduced to the plane-wave mode all the time over the present de-Sitter-like epoch. From another side, plane-wave modes are successfully applied in particle physics to describe high-energy scattering processes which were taking place over the entire semi-classical history of the Universe.

The later circumstance shows that neither \( \varphi_{k,-\infty}(x) \) nor \( \varphi_{k,+\infty}(x) \) are appropriate for our goal. We intend in what follows to derive a covariant wave-packet solution of the scalar-field equation, which can be locally represented via the superposition of positive-frequency plane waves at any space-time point.

### B. Covariant wave packet in Minkowski spacetime

In particle physics in Minkowski spacetime, a particle, which is localised at \( X = (T, \mathbf{X}) \) in position space and at \( P = (P^T, \mathbf{P}) \) in momentum space, is described by the state

\[
|\varphi_{X,P}\rangle \equiv \int \frac{d^4K}{(2\pi)^3} \theta(K^T) \delta(K^2 - M^2) F_P(K) e^{+iK\cdot X} \hat{a}^\dagger(K)|M\rangle,
\]

where \( F_P(K) \) is sharply peaked at \( K = P \) and the state \( |M\rangle \) stands here for the Minkowski quantum vacuum. The particle-creation operator in momentum space is

\[
\hat{a}^\dagger(K) \equiv -i \int d^3x \left( e^{-iK\cdot x} \partial_t \hat{\Phi}(x) - \hat{\Phi}(x) \partial_t e^{-iK\cdot x} \right),
\]

which satisfies the commutation relation \([\hat{a}(K), \hat{a}^\dagger(P)] = 2\sqrt{K^2 + M^2} (2\pi)^3 \delta(K - P)\). This straightforwardly follows from the commutator of the scalar-field operator at different space-time points. The operators \( \hat{a}^\dagger(K) \) and \( \hat{a}(K) \) provide the standard expansion of the quantum field \( \hat{\Phi}(x) \) over the creation and annihilation operators.

The function \( F_P(K) \) is chosen in such a way that the state \( |\varphi_{X,P}\rangle \) is normalised to unity:

\[
\langle \varphi_{X,P}|\varphi_{X,P}\rangle = \frac{1}{2} \int \frac{d^3K}{(2\pi)^3} \frac{|F_P(K)|^2}{\sqrt{K^2 + M^2}} \equiv 1.
\]

We refer to the reference \([18]\) for further details.
1. Gaussian wave packet in Minkowski spacetime

For later applications, however, it proves useful to introduce a wave packet describing the particle state $|\varphi_{X,P}\rangle$. Specifically, this wave packet reads

$$\varphi_{X,P}(x) \equiv \int \frac{d^4K}{(2\pi)^3} \theta(K^T) \delta(K^2 - M^2) F_P(K) e^{-iK(x - X)} ,$$

(16)

giving rise to

$$\hat{a}^\dagger(\varphi_{X,P}) = -i \int_t d^3x \ (\varphi_{X,P}(x) \partial_t \Phi(x) - \Phi(x) \partial_t \varphi_{X,P}(x)) ,$$

(17)

which produces the state $|\varphi_{X,P}\rangle = \hat{a}^\dagger(\varphi_{X,P})|\text{M}\rangle$. The normalisation condition in terms of the wave packet $\varphi_{X,P}(x)$ takes the form

$$-i \int_t d^3x \ (\varphi_{X,P}(x) \partial_t \varphi_{X,P}(x) - \varphi_{X,P}(x) \partial_t \varphi_{X,P}(x)) = 1 .$$

(18)

In the absence of self-interaction or interaction with other quantum fields, $\hat{a}^\dagger(\varphi_{X,P})$ must be time-independent. This is realised if the wave packet vanishes sufficiently fast in the limit $|x - X| \to \infty$. Considering a Lorentz-invariant Gaussian wave packet [19, 20], namely

$$F_P(K) = \mathcal{N} e^{\frac{PK}{2D^2}} \quad \text{with} \quad P^T \equiv \sqrt{P^2 + M^2} ,$$

(19)

where $D > 0$ is the momentum variance and

$$\mathcal{N} \equiv \frac{2\pi}{D \sqrt{K_1(M^2/2^2)}} ,$$

(20)

where $K_1(z)$ stands for the modified Bessel function of the second kind, we obtain

$$\varphi_{X,P}(x) = \mathcal{N} M^2 K_1 \left( \frac{M^2}{2^2} \left( \frac{1}{4} + i \frac{P^2}{2M^2} P.(x - X) - \frac{P^4}{2M^2} (x - X)^2 \right) \right) .$$

(21)

It follows from $\varphi_{X,P}(x) \propto |\Delta x|^{-3}$ for $|x| \gg |X|$ and $(\Box + M^2)\varphi_{X,P}(x) = 0$ that the creation operator $\hat{a}^\dagger(\varphi_{X,P})$ is time-independent in the linear quantum field theory [21].

2. Wave-packet position in Minkowski spacetime

The trajectory of a freely-moving particle in Minkowski spacetime is a straight line. The same result holds for the trajectory of the wave packet $\varphi_{X,P}(x)$:

$$\langle x(t) \rangle = -i \int_t d^3x \ x \ (\varphi_{X,P}(x) \partial_t \varphi_{X,P}(x) - \varphi_{X,P}(x) \partial_t \varphi_{X,P}(x)) = X + \langle V \rangle (t - T) ,$$

(22)

where

$$\langle V \rangle \equiv \frac{1}{2} \int \frac{d^4K}{(2\pi)^3} \frac{|F_P(K)|^2}{\sqrt{K^2 + M^2}} \frac{K}{\sqrt{K^2 + M^2}} \lim_{M/P \to \infty} \frac{P}{\sqrt{P^2 + M^2}} .$$

(23)

Thus, $\varphi_{X,P}(x)$ propagates like a classical (point-like) particle of the same mass if $M \gg D$. 


3. Wave-packet momentum in Minkowski spacetime

Making use of $[\hat{\Phi}(x), \hat{\Phi}(x')] = i \Delta(x - x')\hat{1}$, where $\Delta(x - x')$ is the commutator function, we find the stress-tensor expectation value in the single-particle state $|\varphi_{X,P}\rangle$:

$$\langle \hat{\Theta}_{\mu\nu} \rangle = 2 \partial_{(\mu} \varphi_{X,P} \partial_{\nu)} \varphi_{X,P} - \frac{1}{3} \eta_{\mu\nu} (|\partial \varphi_{X,P}|^2 - M^2 |\varphi_{X,P}|^2),$$

(24)

where we have omitted the vacuum stress tensor, as this does not depend on the wave packet. The energy and momentum, which are ascribed to the wave packet, are given by

$$\langle p_t(t) \rangle \equiv \int_t d^3x \langle \hat{T}^t_t(x) \rangle = \left( K_2 \left( \frac{M^2}{D^2} \right) / K_1 \left( \frac{M^2}{D^2} \right) \right) P \xrightarrow{M/D \to \infty} P_t,$$

(25a)

$$\langle p_i(t) \rangle \equiv \int_t d^3x \langle \hat{T}^t_i(x) \rangle = \left( K_2 \left( \frac{M^2}{D^2} \right) / K_1 \left( \frac{M^2}{D^2} \right) \right) P \xrightarrow{M/D \to \infty} P_i.$$  

(25b)

The packet $\varphi_{X,P}(x)$ is thus characterised by the four-momentum like a classical (point-like) particle of the same mass and three-momentum if $M \gg D$.

C. Covariant wave packet in de-Sitter spacetime

The covariant phase factor suggests that a single-particle wave packet in curved space must be coordinate-independent. The wave packet should then have the following structure:

$$\varphi_{X,P}(x) = \int \frac{d^4K}{(2\pi)^3} \theta(K^T) \delta(K_A K^A - M^2) F_K(\varphi_{X,K}(x),$$

(26)

where the index $A$ refers to the tangent frame at $X$ with the vierbein $e_A^M(X)$, and

$$\phi_{X,K}(x) \xrightarrow{x \text{ close to } X} \exp(iK \cdot \sigma),$$

(27)

where $\sigma$ is a shorthand notation of the geodetic distance $\sigma(x,X)$, so that

$$K \cdot \sigma \equiv K_A \sigma^A = e_A^M(X) e_N^A(X) K_M \sigma^N = K_M \sigma^M,$$

(28)

where

$$\sigma^M \equiv \nabla^M \sigma(x,X) = g^{MN}(X) \partial_N \sigma(x,X)$$

(29)

is a vector of length equal to the distance along the geodesic between $x$ and $X$, tangent to it at $X$, and oriented in the direction from $x$ to $X$ [22]. Note, in general, $\phi_{X,K}(x)$ is a function of dimensionless combinations of the curvature tensor at $X$ with $K_M$ and $\sigma_M$.

An exact expression of the covariant wave packet can be found in de-Sitter spacetime due to its symmetries. We look for a solution of the scalar-field equation in the form

$$\phi_{X,K}(x) = \frac{1}{4 \pi i} \frac{\sqrt{2H^2} \sigma}{\sinh \sqrt{2H^2} \sigma} \frac{\phi(K \cdot \sigma, R \sigma)}{\sqrt{K \cdot \sigma^2 - 2K^2 \sigma}},$$

(30)
where the Ricci scalar $R = -12H^2$. Substituting $\phi_{X,K}(x)$ in the field equation and using

$$\sigma_{,\mu}^\mu = 1 + 3\sqrt{2H^2\sigma} \coth\sqrt{2H^2\sigma}, \quad (31a)$$

$$\sigma^\mu (K\cdot\sigma)_{,\mu} = K\cdot\sigma, \quad (31b)$$

$$\frac{(K\cdot\sigma)^\mu}{\sqrt{2H^2\sigma}} \left[ \coth\sqrt{2H^2\sigma} - \frac{\sqrt{2H^2\sigma}}{\sinh^2\sqrt{2H^2\sigma}} \right], \quad (31c)$$

$$\frac{(K\cdot\sigma)^\mu (K\cdot\sigma)_{,\mu}}{2H^2\sigma} K^2 + \frac{K\cdot\sigma^2}{2\sigma} \left[ 1 - \frac{2H^2\sigma}{\sinh^2\sqrt{2H^2\sigma}} \right], \quad (31d)$$

we obtain

$$\left( \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \zeta^2} + \frac{\gamma(1 - \gamma)}{\sinh^2 \eta} \right) \phi(\eta, \zeta) = 0 \quad \text{with} \quad \gamma \equiv \frac{1}{2} (1 - i\sqrt{4\nu^2 - 1}), \quad (32)$$

where we have introduced new variables

$$\eta \equiv \ln \tanh \left[ \frac{\sqrt{2H^2\sigma}}{2} \right], \quad (33a)$$

$$\zeta \equiv \ln \left[ \frac{\sqrt{K\cdot\sigma^2 - 2K^2\sigma - K\cdot\sigma}}{\sqrt{2K^2\sigma}} \right], \quad (33b)$$

such that $\eta \in (-\infty, 0)$ and $\zeta \in [0, +\infty)$ if $\sigma > 0$ and $-K\cdot\sigma \geq \sqrt{2K^2\sigma}$ are fulfilled.

1. **Locally plane-wave solutions in de-Sitter spacetime**

The equation (32) has infinitely many solutions. One of them reads

$$\phi_\nu(\eta, \zeta) \equiv \int_{-\infty}^{+\infty} dp e^{(ip-1)\zeta} (\phi_{\nu,ip-1}(\eta) - \phi_{\nu,1-ip}(\eta)), \quad (34)$$

where by definition

$$\phi_{\nu,ip-1}(\eta) \equiv e^{(ip-1)(\eta+\ln \nu)} \frac{\Gamma[2 - \gamma - ip] \Gamma[1 + \gamma - ip]}{\Gamma[1 - ip]} \frac{\Gamma[1 - \gamma; 2 - ip]}{1 - e^{2\eta}} \binom{2F1}{\gamma, 1 - \gamma; 2 - ip, 1}{1 - e^{2\eta}}. \quad (35)$$

The coefficient to depend only on $p$ and $\nu$ has been chosen from the following argument. In the observable Universe, $M \gg H_0$ holds for the massive fields of the Standard Model of elementary particle physics. It is an empirical fact that collider physics is well described by the Minkowski plane-wave solutions. We must, therefore, obtain a plane-wave solution $e^{iK\cdot\sigma}$ for $\phi_{X,K}(x)$ if $H \ll M$ and $H^2|\sigma| \ll 1$ are satisfied. Specifically, if $\nu \equiv M/H \to \infty$, then we find from the definition of the hypergeometric function and 8.328.2 in [23] that

$$\binom{2F1}{\gamma, 1 - \gamma; c; -|x|} \quad \xrightarrow{\nu \to \infty} \quad \Gamma[c] (\nu \sqrt{|x|})^{1-c} J_{c-1} (2\nu \sqrt{|x|}), \quad (36)$$
where \( J_{e-1}(z) \) is the Bessel function of the first kind. This result agrees with 10.16.10 in [24]. Employing 9.131 in [23], we find that

\[
2F_1\left[\gamma, 1-\gamma; 2-ip; \frac{1}{1-e^{2\eta}}\right] \xrightarrow{\nu \to \infty} \frac{\Gamma[1-ip]}{\Gamma[2-\gamma-ip] \Gamma[1+\gamma-ip]} \frac{\pi i(ip-1)}{\sinh(\pi p)} e^{(1-ip)\eta} \nu^{1-ip} \frac{1}{1-e^{2\eta}} \nu \frac{1}{1-e^{2\eta}} J_{ip-1}\left(\frac{2\nu e^{\eta}}{\sqrt{1-e^{2\eta}}}\right) + e^{\pi p} \left(1-e^{2\eta}\right)^{\frac{1-ip}{2}} J_{1-ip}\left(\frac{2\nu e^{\eta}}{\sqrt{1-e^{2\eta}}}\right).
\]

(37)

If we consider \( H^2|\sigma| \ll 1 \), then \( \eta \) approaches \(-\infty\), i.e. we are allowed to set \( 1-e^{2\eta} \) to unity in this limit, whereas \( 2\nu e^{\eta} \) turns into \( \sqrt{2M^2\sigma} \). Having used 10.4.8 in [24], we obtain

\[
\phi_\nu(\eta, \zeta) \xrightarrow{H \ll M} \frac{H^2|\sigma| \ll 1}{H \ll M} \rightarrow -2\pi \int_{-\infty}^{+\infty} dp \ (ip-1) e^{(ip-1)\zeta} e^{-\frac{\pi}{i} p} H_{ip-1}^{(2)}\left(\sqrt{2M^2\sigma}\right) ,
\]

(38)

where \( H_{ip-1}^{(2)}(z) \) is the Hankel function of the second kind. With the help of 8.421.2 in [23], we have that

\[
\phi_\nu(\eta, \zeta) \xrightarrow{H^2|\sigma| \ll 1}{H \ll M} \rightarrow 4\pi i \sqrt{2M^2\sigma} \sinh \zeta e^{-i\sqrt{2M^2\sigma} \cosh \zeta} .
\]

(39)

Substituting this result into (30), we find

\[
\phi_{X,K}(x) \xrightarrow{H^2|\sigma| \ll 1}{H \ll M} \rightarrow e^{iK\cdot\sigma} ,
\]

(40)

as has been required.

The integral over \( p \) in (31) can actually be exactly evaluated. Specifically, we obtain from 15.6.7 in [24] and 6.422.12 in [23] that

\[
\phi_\nu(\eta, \zeta) = \lim_{w \to 0} \partial_w \left( \Phi_\nu(\eta, w+\zeta) - \Phi_\nu(\eta, w-\zeta) \right) ,
\]

(41)

where by definition

\[
\Phi_\nu(\eta, \zeta) = -2\sqrt{2\pi\nu} \sqrt{-e^{\xi} \sinh \eta} e^{-\frac{\pi}{4} \nu} e^{-\nu e^{\xi} \cosh \eta} K_{\frac{1}{2}}\left(\nu e^{\xi} \sinh \eta\right) .
\]

(42)

Considering first the limit \( \nu \to \infty \) and then \( H^2|\sigma| \ll 1 \), we have \( \phi_{X,K}(x) \to e^{iK\cdot\sigma} \), where we have used the uniform expansion of \( K_{\nu}(\nu z) \) for \( \nu \to \infty \) obtained in [26].

The \( \phi_{\nu,1-ip}(\eta) \)-dependent part of the integrand in (34) vanishes in the limit \( \nu \to 0 \). We therefore consider in the case \( \nu \equiv M/H = 0 \) that

\[
\phi_0(\eta, \zeta) \equiv 2 \lim_{\nu \to 0} \left[ \lim_{\nu \to +\infty} \int_{-\infty}^{+\infty} dp \ e^{(ip-1)\zeta} \phi_{\nu,ip-1}(\eta) \right] = 2 \lim_{\nu \to 0} \left[ \lim_{\nu \to +\infty} \int_{-\infty}^{+\infty} dp \ e^{(ip-1)(\zeta+\eta+\ln i\nu)} \ \Gamma[2-ip] \right] .
\]

(43)

Making use of 3.328 in [23], we obtain in the massless \((M = 0)\) case that

\[
\phi_{X,K}(x) = \frac{2}{\cosh \sqrt{2H^2\sigma} + 1} \exp \left( iK\cdot\sigma \tanh \frac{1}{2} \sqrt{2H^2\sigma} \right) .
\]

(44)

Note that \( \phi_{X,K}(x) \) turns into the standard plane-wave solution \( e^{iK\cdot\sigma} \) as in Minkowski spacetime if \( H^2|\sigma| \ll 1 \) holds.
2. In-in and in-out propagators in de-Sitter spacetime

To explain a non-trivial structure of the $\eta$-dependent integrand in (34), we need to compute the Wightman function that might be related to this solution. In Minkowski spacetime, $H = 0$, the two-point function can be found as follows:

$$W(x, X)\big|_{H=0} = \int \frac{d^4K}{(2\pi)^3} \theta(K^T) \delta(K^2 - M^2) e^{iK \cdot \sigma}.$$  \hspace{1cm} (45)

In the de-Sitter universe, the correlation function may be defined via the same formula with $e^{iK \cdot \sigma}$ replaced by $\phi_{X,K}(x)$, where $K$ belongs to the cotangent space at $X$:

$$W(x, X) = \frac{1}{4\sqrt{2H^2\sigma}} \int_{-\infty}^{+\infty} dp \left( \phi_{\nu, ip-1}(\eta) - \phi_{\nu, 1-ip}(\eta) \right) \int \frac{d^3K}{\sqrt{K^2 + M^2}} \frac{e^{ip\zeta}}{\sinh \zeta}.$$  \hspace{1cm} (46)

Note, in the second line, one can replace $+p$ by $-p - 2i$ in the $e^{ip\zeta}$-dependent part of the integrand to get the delta function $\delta(p)$ from the integral over $q$, by taking into account that residues at $\pm \mu - 3i/2$ and $\mp \mu - i/2$ cancel each other and $\phi_{\nu, \pm ip\pm 1}(\eta)$ vanishes exponentially in the limit $\text{Re} \, p \to \pm \infty$.

Therefore, $\phi_{\nu}(\eta, \zeta)$ might be related to the Wightman function of the Chernikov-Tagirov aka Bunch-Davies state [11, 12]. If we assume that $\text{Im}(\cosh\sqrt{2H^2\sigma}) < 0$, then the correlation function (46) turns into the in-in propagator. It was, however, argued in [25] that the in-out propagator should be considered in non-linear quantum field models in de-Sitter spacetime. This type of the Feynman propagator is associated with $\phi_{\nu}(+\eta, \zeta) + e^{-\pi i\gamma} \phi_{\nu}(-\eta, \zeta)$.

In the massless case, $M = 0$, the integration over the Fourier parameter $K$ in the formula of the Wightman function gives the result (46) if taken in the limit $\nu \to 0$. The discontinuity $\phi_{0}(\eta, \zeta) \neq \phi_{\nu \to 0}(\eta, \zeta)$ is also present in the integration over the Fourier parameter. Namely, the integral over $K$ in the first line of (46) is proportional to the delta function $\delta(p)$ if $\nu = 0$, while not if $\nu > 0$.

3. Gaussian wave packet in de-Sitter spacetime

According to our suggestion, the Gaussian wave packet is given by

$$\varphi_{X,P}(x) = N \int \frac{d^4K}{(2\pi)^3} \theta(K^T) \delta(K^2 - M^2) e^{-\frac{P \cdot K}{2D^2} \phi_{X,K}(x)},$$  \hspace{1cm} (47)
where \( D \) is the momentum variance and \( \mathcal{N} \) needs to be determined from the normalisation condition

\[
-i \int_{\Sigma} d\Sigma^\mu (x) \left( \varphi_{X,P}(x) \nabla_\mu \varphi_{X,P}(x) - \varphi_{X,P}(x) \nabla_\mu \varphi_{X,P}(x) \right) = 1, \tag{48}
\]

where \( \Sigma \) is a Cauchy surface. Since the wave packet \( \varphi_{X,P}(x) \) vanishes as \(|\Delta x|^{-3}\) for large \(|\Delta x|\) and is a solution of the scalar-field equation, the normalisation factor does not depend on the Cauchy surface. Therefore, it generically holds \( \mathcal{N} = \mathcal{N}(M, D, H) \) (see fig. left).

Plugging \( \phi_{X,K}(x) \) found above into (47) and assuming that \( M/D \neq 0 \), we obtain

\[
\varphi_{X,P}(x) = \frac{iM^2 \mathcal{N}}{4\nu(2\pi)^3} \int_{-\infty}^{+\infty} dw \sinh w e^{-\frac{M^2}{2D^2} \cosh w} \frac{\Phi_\nu(\eta, w + v) - \Phi_\nu(\eta, w - v)}{\csch \eta \sinh v}, \tag{49}
\]

where by definition

\[
v \equiv \ln \left[ \frac{\sqrt{P\cdot\sigma^2 - 2M^2\sigma - P\cdot\sigma}}{\sqrt{2M^2\sigma}} \right]. \tag{50}
\]

The integral over \( w \) in (49) seems not to be generically tractable. Still, it can be “simplified” with the help of 8.432.1 in [23] and the first formula on p. 86 in [27].

In Secs. III B 2 and III B 3, we have learned that the Gaussian wave packet in Minkowski spacetime behaves kinematically as a classical point-like particle if its mass \( M \) is much larger than its momentum variance \( D \). Considering \( M \gg D \) in (49), we observe that the integrand is extremely suppressed for \(|w| \gtrsim 1\). Therefore, if we multiply that integrand by \( \exp(-\frac{1}{2}w) \), then \( \varphi_{X,P}(x) \) remains essentially unchanged if \( M \gg D \). However, this modified integral can be exactly evaluated by using 6.653.2 in [23]. Specifically, we have

\[
\tilde{\varphi}_{X,P}(x) \equiv \frac{iM^2 \mathcal{N}}{4\nu(2\pi)^3} \int_{-\infty}^{+\infty} dw \sinh w e^{-\frac{M^2}{2D^2} \cosh w} \frac{\Phi_\nu(\eta, w + v) - \Phi_\nu(\eta, w - v)}{\csch \eta \sinh v}
\]

\[
= \frac{2H^2 \mathcal{N} c_+ c_-}{(2\pi)^3 (c_+ - c_-)} \left( \sqrt{c_+} \partial_{b_+} (K_{ip}(\chi_+) K_{ip}(\chi_+)) - \sqrt{c_-} \partial_{b_-} (K_{ip}(\chi_-) K_{ip}(\chi_-)) \right), \tag{51}
\]

where by definition

\[
\chi_+ \equiv \frac{1}{2} \left( a_+ - b_+ \right)^{\frac{1}{2}} \left( (a_+ + b_+ + 2c_+)^{\frac{1}{2}} + (a_+ + b_+ - 2c_+)^{\frac{1}{2}} \right), \tag{52a}
\]

\[
\chi_- \equiv \frac{1}{2} \left( a_+ - b_+ \right)^{\frac{1}{2}} \left( (a_+ + b_+ + 2c_+)^{\frac{1}{2}} - (a_+ + b_+ - 2c_+)^{\frac{1}{2}} \right), \tag{52b}
\]

and

\[
a_+ = \frac{M^2}{2D^2} + b_+, \tag{53a}
\]

\[
b_+ = i\nu e^{\pm v} \cosh \eta, \tag{53b}
\]

\[
c_+ = i\nu e^{\pm v} \sinh \eta. \tag{53c}
\]
FIG. 1. Left: Numerical evaluation of the normalisation factor $N(M, D, H)$. This plot shows the ratio $N(M, D, H)/N(M, D, 0)$, where $N(M, D, 0)$ is the Minkowski-spacetime normalisation factor (see equation (20)). Right: Numerical evaluation of $\langle z(t) \rangle$, where the black solid curve corresponds to the classical trajectory $z(t)$ (see equation (54)).

Numerical computations with $\varphi_{X,P}(x)$ and $\tilde{\varphi}_{X,P}(x)$ give us the same results within numerical error bars. However, it is worth mentioning at this point that the integrand in (49) is highly oscillatory. Presumably, this circumstance makes it non-trivial to do numerics with $\varphi_{X,P}(x)$ if used its integral form.

4. Wave-packet position in de-Sitter spacetime

Without loss of generality, we intend to consider a free motion with the initial conditions $X = 0$ and $P = (\sqrt{M^2 + P_2^2}, 0, 0, P)$ (in the tangent frame at $X$). The position of a classical particle of the mass $M$ in this case reads

$$z(t) = \frac{1}{PH} \left( \sqrt{M^2 + P^2} - \sqrt{M^2 + e^{-2Ht}P^2} \right),$$

where $x(t) = y(t) = 0$ due to spatial-translation symmetry of the flat de-Sitter universe.

In analogy to the Minkowski case, the wave-packet position should follow from

$$\langle z(t) \rangle \equiv -i \int_S d\Sigma^\mu(x) z \left( \varphi_{X,P}(x) \nabla_\mu \overline{\varphi}_{X,P}(x) - \overline{\varphi}_{X,P}(x) \nabla_\mu \varphi_{X,P}(x) \right)$$

$$= -ie^{3Ht} \int_d^3 x \left( \varphi_{X,P}(x) \partial_t \overline{\varphi}_{X,P}(x) - \overline{\varphi}_{X,P}(x) \partial_t \varphi_{X,P}(x) \right),$$

whereas $\langle x(t) \rangle = \langle y(t) \rangle = 0$ due to the invariance of $\varphi_{X,P}(x)$ under rotations around $z$-axis. It should be mentioned that $\varphi_{X,P}(x)$ is spherically symmetric if $P = 0$. In this special case, we obtain $z(t) = \langle z(t) \rangle = 0$. In general, we numerically find that $\langle z(t) \rangle$ matches the classical trajectory if $M \gg D \gg H$ (see fig. right).
1.0 1.2 1.4 1.6 1.8

\[ \langle p_z(t) \rangle = \frac{1}{M_0 + \frac{2}{3}M_0 M \langle p_z(t) \rangle^2} - \frac{1}{3} \left( \frac{1}{2} \langle \nabla \phi_{X,P} \rangle^2 \right) \]

FIG. 2. Left: Numerical evaluation of \( \langle p_z(t) \rangle \). The solid straight line corresponds to the classical result (see equation (56b)). Right: Numerical evaluation of \( \langle p_t(t) \rangle \). This plot shows \( \langle p_t(t) \rangle / p_t(t) \), where \( p_t(t) \) corresponds to the classical energy (equation (56a)) with \( P \) replaced by the wave-packet momentum \( \langle p_z(t) \rangle \). In the case of \( M/H = 1 \), the ratio \( \langle p_t(t) \rangle / p_t(t) \) appears to be oscillating around 1.165 at \( Ht \in \{6, 7, \ldots, 19, 20\} \) with the amplitude 0.018.

5. Wave-packet momentum in de-Sitter spacetime

The four-momentum of the classical particle is given by

\[ p^t(t) = \sqrt{M^2 + e^{-2Ht}P^2}, \]
\[ p^z(t) = e^{-2Ht}P, \]  

(56a)

(56b)

where \( p^z(t) = p^y(t) = 0 \) due to the initial conditions considered and the spatial-translation symmetry of flat de-Sitter spacetime.

Making use of the commutator function in de-Sitter spacetime, one finds that the stress-tensor expectation value in the single-particle state \( |\phi_{X,P} \rangle \) reads

\[ \langle \hat{\Theta}_{\mu\nu} \rangle = 2\nabla_{(\mu} \phi_{X,P} \nabla_{\nu)} \phi_{X,P} - \frac{1}{3} \nabla_{\mu} \nabla_{\nu} |\phi_{X,P}|^2 - \frac{1}{3} g_{\mu\nu} \left( |\nabla \phi_{X,P}|^2 - (M^2 - H^2)|\phi_{X,P}|^2 \right), \]

(57)

where we have omitted the vacuum contribution, as this does not depend on the wave packet. It is straightforward to show that this stress tensor is covariantly conserved:

\[ \nabla^\mu \langle \hat{\Theta}_{\mu\nu} \rangle = 0. \]

(58)

Since \{\partial_t\} are three Killing vectors of flat de-Sitter spacetime, the momentum-conservation law holds, namely

\[ \langle p_t(t) \rangle \equiv \int_{\Sigma} d^3x \langle \dot{\phi}_t(x) \rangle = a^3(t) \int_{t} d^3x \langle \dot{\phi}_t(x) \rangle \]

(59)

does not depend on the Cauchy surface \( \Sigma \) (see fig. 2 left). Yet, the wave-packet energy,

\[ \langle p_t(t) \rangle \equiv \int_{\Sigma} d^3x \langle \dot{\phi}_t(x) \rangle = a^3(t) \int_{t} d^3x \langle \dot{\phi}_t(x) \rangle \]

(60)
depends generically on cosmic time:

\[
\frac{d}{dt} \langle p_t(t) \rangle = -H \langle p_t(t) \rangle + 2M^2Ha^3(t) \int d^3x |\varphi_{x,P}(t,x)|^2,
\]  

(61)

where we have used (58) and \( \langle \hat{\Theta}_{\mu \nu} \rangle \to 0 \) at spatial infinity. We find numerically that \( \langle p_t(t) \rangle \) approaches the classical result (56a) if \( M \gg D \gg H \) (see fig. 2, right).

IV. DISCUSSION

Elementary particles are described by wave packets in quantum theory. A wave packet in Minkowski spacetime is usually constructed through the superposition of positive-frequency plane-wave solutions of a given field equation [18]. This wave packet can in turn be associated with an asymptotic state used in the definition of \( S \)-matrix. But, the plane-wave solutions may exist only locally in non-flat spacetimes. The basic question is then how to construct a wave packet to describe a free elementary particle in the Universe.

In flat de-Sitter spacetime, one believes that the exact solution (3a) is appropriate for the definition of elementary particles at past infinity, while the solution (3b) is usually suggested for the description of particles at future infinity. The superposition of each of these can certainly be used to construct a Gaussian wave packet. The three-momentum of these packets are given by the Minkowski result (25b). Hence, we should assume \( M \gg D \) for each packet. In addition, we should assume \( D \gg H \), otherwise these wave packets have spatial support to be larger than the cosmological extent of de-Sitter spacetime. In general, any wave packet should be well-localised within the Hubble scale in order to describe an elementary particle. Repeating numerical calculations made in the previous section, we find that these Gaussian wave packets propagate along a curve which approaches the geodesic (54) if \( M \gg D \gg H \). In this case, the energy of the packets also approaches the classical result (56a).

Still, the adiabatic wave packets cannot be locally represented through the superposition of plane waves and depend on coordinates used to parametrise the de-Sitter hyperboloid. All these mean that the adiabatic wave packets are locally described by phase factors which may differ from \( e^{-iM\tau} \), where \( \tau \) is the proper time. In particular, their on-mass-shell phase factors depend explicitly on the three-momentum. Specifically, taking the same initial conditions as in the previous section, we find that that difference becomes more pronounced if we increase the ratio \( P/M \) for fixed \( M/D = 10 \) and \( M/H = 100 \). This turns out to be counter-intuitive, because high-energy physics should not depend on the space-time curvature. This property of the adiabatic wave packets may, thereby, lead to different results for the flavour oscillation and for the quantum interference induced by gravity.

The main goal of this article was to derive a wave packet which is coordinate-independent and locally reduces to the plane-wave superposition at any space-time point, no matter if that is at past or future cosmic infinity or in between. The main result of this article is that we have found such a solution in the de-Sitter universe. Moreover, we have shown that this
FIG. 3. The absolute value of the wave packet \( \varphi_{X,P}(x) \) as a function of time in Minkowski spacetime (top panel) and in de-Sitter spacetime (bottom panel), assuming that \( M/D = 10, M/H = 100 \gg 1 \) and the initial conditions \( X = 0 \) with \( P = (\sqrt{2}M, 0, 0, M) \). In de-Sitter spacetime, the packet evolves as in flat spacetime for small values of \( Ht \), in accordance with the equivalence principle. At later times, \( Ht \gtrsim 0.1 \), the Minkowski and de-Sitter wave packets behave differently. Note that the probability to find a scalar-field particle on the classical geodesic is maximal in both cases.

covariant wave packet propagates like a classical particle of the same mass \( M \), assuming that \( M \gg D \gg H \) (see also fig. 3). From another side, if we take \( M \gg D \sim H \) or \( M \sim H \gg D \), then the packet behaves highly non-classically.

This wave-packet solution, \( \varphi_{X,P}(x) \), gives rise to the particle-annihilation operator

\[
\hat{a}(\varphi_{X,P}) = +i \int \Sigma d\Sigma^\mu(x) \left( \overline{\varphi_{X,P}}(x) \nabla_\mu \Phi(x) - \Phi(x) \nabla_\mu \overline{\varphi_{X,P}}(x) \right).
\]  

(62)

This operator has two basic properties, namely it does not depend on the Cauchy surface \( \Sigma \) and on the coordinates \( x \) used to parametrise the de-Sitter hyperboloid. The former property comes from, first, the absence of non-linear terms in the scalar-field equation and, second, the localisation of \( \varphi_{X,P}(x) \) on the Cauchy surface \( \Sigma \). The latter property is due to the covariant character of the Klein-Gordon product and the wave-packet solution \( \varphi_{X,P}(x) \). Thus, \( \hat{a}(\varphi_{X,P}) \) defines a coordinate-independent quantum vacuum \( \langle \hat{a}(\varphi_{X,P})|\Omega \rangle = 0 \) in de-Sitter spacetime, while its Hermitian conjugate defines a covariant particle state \( \langle |\varphi_{X,P} \rangle = \hat{a}^\dagger(\varphi_{X,P})|\Omega \rangle \).
The quantum state $|\Omega\rangle$ is a no-covariant-particle state which still may be non-empty with respect to the de-Sitter particles which have been introduced in Sec. II. To clarify this issue, the Bogolyubov coefficients need to be computed:

$$\alpha_{X,P}(k) \equiv -i e^{3Ht} \int_t d^3x \left( \varphi_{X,P}(x) \partial_t \bar{\varphi}_{k,-\infty}(x) - \bar{\varphi}_{k,-\infty}(x) \partial_t \varphi_{X,P}(x) \right),$$

$$\beta_{X,P}(k) \equiv -i e^{3Ht} \int_t d^3x \left( \bar{\varphi}_{X,P}(x) \partial_t \varphi_{k,-\infty}(x) - \varphi_{k,-\infty}(x) \partial_t \bar{\varphi}_{X,P}(x) \right),$$

which are time-independent due to the spatial localisation of $\varphi_{X,P}(x)$. Having used the same initial conditions for $X$ and $P$ as in the previous section, we numerically find for $k = P$ that $\alpha_{X,P}(k)$ is time-independent, whereas $\beta_{X,P}(k)$ changes with time and $|\beta_{X,P}(k)| \ll |\alpha_{X,P}(k)|$. The same calculations with $\varphi_{k,-\infty}(x)$ replaced by $\varphi_{k,+\infty}(x)$ yield the Bogolyubov coefficients which are time-independent. These observations might mean that $|\Omega\rangle$ is unitarily equivalent to the state $|dS\rangle$. In any case, $\varphi_{X,P}(x)$ is related to the 2-point function in the de-Sitter state, as shown in Sec. III C 2. Specifically, $\varphi_{X,P}(x)$ is proportional to that function if $D \to \infty$. This turns out to be analogous to the Minkowski case, namely the packet (21) is also proportional to the Minkowski 2-point function if the momentum variance of the wave packet is infinite. In both cases, the proportionality coefficient is given by the normalisation factor.

The de-Sitter universe is not only one curved spacetime which is of physical interest from the viewpoint of elementary particle physics. For example, black-hole spacetimes serve as a non-trivial background for probing predictions of quantum theory. Field-equation solutions which are commonly employed to define particles depend explicitly on global symmetries of a given black-hole geometry [28]. Still, the observable Universe can locally be approximated by such a geometry only nearby a black hole this geometry is supposed to describe. Thereby, black-hole global symmetries are local for the Universe. This circumstance poses a question why those global symmetries should be “preferred” with respect to local Poincaré symmetry, taking into account that both are non-exact in the Universe. Since local Poincaré symmetry is well-known to play a crucial role in elementary particle physics [4], one might actually need to re-consider those solutions which are employed to define elementary particles in black-hole spacetimes. The reason is that those solutions like $\varphi_{k,-\infty}(x)$ and $\varphi_{k,+\infty}(x)$ give rise to wave packets which cannot be everywhere represented locally through the superposition of plane waves, as required, for example, by the equivalence principle.

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[1] E. Schrödinger, Physica 6 (1939) 899.
[2] L. Parker, Phys. Rev. **183** (1969) 1057.

[3] L. Parker, D. Toms, *Quantum Field Theory in Curved Spacetime. Quantized Fields and Gravity* (Cambridge University Press, 2009).

[4] S. Weinberg, *Quantum Theory of Fields* (Cambridge University Press, 1995).

[5] W. Pauli, *Zur älteren und neueren Geschichte des Neutrinos*, in *Aufsätze und Vorträge über Physik und Erkenntnistheorie* (Friedr. Vieweg & Sohn, Braunschweig, 1961).

[6] C.M. Will, Living Rev. Relativ. **9** (2006) 3.

[7] R. Colella, A.W. Overhauser, S.A. Werner, Phys. Rev. Lett. **34** (1975) 1472.

[8] L. Stodolsky, Gen. Rel. Grav. **11** (1979) 391.

[9] P.R. Anderson, E. Mottola, Phys. Rev. D **89** (2014) 104038.

[10] P.R. Anderson, E. Mottola, D.H. Sanders, Phys. Rev. D **97** (2018) 065016.

[11] N.A. Chernikov, E.A. Tagirov, Ann. Inst. H. Poincaré A **9** (1968) 109.

[12] T.S. Bunch, P.C.W. Davies, Proc. R. Soc. A **360** (1978) 117.

[13] O. Nachtmann, Commun. Math. Phys. **6** (1967) 1.

[14] O. Nachtmann, Österr. Akad. Wiss., Math.-Naturw. Kl., Abt. II **176** (1968) 363.

[15] J. Bros, H. Epstein, U. Moschella, JCAP **02** (2008) 003; Ann. Henri Poincaré A **11** (2010) 611.

[16] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (W.H. Freeman and Co., 1973).

[17] V.F. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, 2005).

[18] C. Itzykson, J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill Inc., 1980).

[19] D.V. Naumov, V.A. Naumov, J. Phys. G: Nucl. Part. Phys. **37** (2010) 105015.

[20] D.V. Naumov, Phys. Part. Nucl. Lett. **10** (2013) 642.

[21] H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cimento **1** (1955) 205.

[22] B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, 1965).

[23] I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series, and Products* (7th Edition, Elsevier Inc., 2007).

[24] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, *NIST Handbook of Mathematical Functions* (Cambridge University Press, 2010).

[25] A.M. Polyakov, Nucl. Phys. B **797** (2008) 199; Nucl. Phys. B **834** (2010) 316.

[26] C.B. Balogh, Bull. Amer. Math. Soc. **72** (1966) 40; SIAM J. Appl. Math. **15** (1967) 1315.

[27] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (3rd Edition, Springer-Verlag, 1966).

[28] B.S. DeWitt, Phys. Rep. **19** (1975) 295.