HARMONIOUS COLORING ON DOUBLE STAR GRAPH FAMILIES

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Abstract. In this paper we investigate the harmonious chromatic number for the central graph, middle graph, total graph and line graph of double star graph $K_{1,n,n}$ denoted by $C(K_{1,n,n})$, $M(K_{1,n,n})$, $T(K_{1,n,n})$ and $L(K_{1,n,n})$ respectively. We prove that for the line graph of double star graph, the harmonious chromatic number and the achromatic number are equal.

1. Preliminaries

The central graph $C(G)$ of a graph $G$ is formed by adding an extra vertex on each edge of $G$, and then joining each pair of vertices of the original graph which were previously non-adjacent.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of $G$, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds: (i) $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$. (ii) $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

The total graph $T(G)$ of $G$ has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in $G$.

The line graph $L(G)$ of $G$ denoted by $L(G)$ is the graph with vertices are the edges of $G$ with two vertices of $L(G)$ adjacent whenever the corresponding edges of $G$ are adjacent.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing $n$ pendant vertices. It has $2n+1$ vertices and $2n$ edges. Let $V(K_{1,n,n}) = \{v\} \cup \{v_1,v_2,\ldots,v_n\} \cup \{u_1,u_2,\ldots,u_n\}$ and $E(K_{1,n,n}) = \{e_1,e_2,\ldots,e_n\} \cup \{s_1,s_2,s_3,\ldots,s_n\}$.

A harmonious coloring of a simple graph $G$ is proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

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An achromatic coloring of a graph $G$ is a proper vertex coloring of $G$ in which every pair of colors appears on at least one pair of adjacent vertices. The achromatic number of $G$ denoted $\chi_c(G)$, is the greatest number of colors in an achromatic coloring of $G$.

A collection of articles in harmonious coloring and achromatic coloring can be found in the bibliography [2].

2. Harmonious Coloring on central graph of double star graph

Proposition 2.1. For any double star graph $K_{1,n,n}$, the harmonious chromatic number,

$$\chi_H(C(K_{1,n,n})) = 3n + 1.$$  

Figure 1: Double star graph $K_{1,n,n}$     Figure 2: Central graph of double star graph $C(K_{1,n,n})$.

Proof. By definition of central graph, let the edge $vv_i$ and $v_iu_i$ ($1 \leq i \leq n$) be subdivided by the vertices $e_i(1 \leq i \leq n)$ and $s_i(1 \leq i \leq n)$ in $C(K_{1,n,n})$. Clearly $V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. The vertices $v_i(1 \leq i \leq n)$ induce a clique of order $n$ (say $K_n$) and the vertices $v, u_i(1 \leq i \leq n)$ induce a clique of order $n + 1$ (say $K_{n+1}$) in $C(K_{1,n,n})$ respectively. The number of edges in $C(K_{1,n,n})$ is $3n + 2n^2$. Thus we have $\chi_H(C(K_{1,n,n})) \geq 3n + 1$.

All the vertices $v, e_i, v_i, u_i$ are mutually at distance at most 2, and so must all have distinct colors and the vertices $s_i$ could have the $n$ colors used on $e_i$ but in different order.

Now consider the vertex set $V(C(K_{1,n,n}))$ and the color class $C = \{c_i : 1 \leq i \leq 3n+1\}$, assign a proper harmonious coloring to $C(K_{1,n,n})$ as follows: For $(1 \leq i \leq n)$, assign the color $c_i$ for $u_i$ and assign the color $c_{n+1+i}$ to $v$. For $(1 \leq i \leq n)$, assign the color $c_{n+1+i}$ for $e_i$. For $(1 \leq i \leq n)$,
assign the color $c_{2n+1+i}$ for $v_i$. For $(1 \leq i \leq n-1)$, assign the color $c_{2n+1}$ for $s_i$ and assign the color $c_{n+2}$ to $s_n$. Therefore, $\chi_H(C(K_{1,n,n})) \leq 3n+1$. Hence $\chi_H(C(K_{1,n,n})) = 3n+1$.

3. Harmonious coloring on middle graph and total graph of double star graph

**Proposition 3.1.** For any double star graph $K_{1,n,n}$, the harmonious chromatic number,

$$\chi_H(M(K_{1,n,n})) = 2n+1, \forall n \geq 3.$$ 

![Figure 3: Middle graph of double star graph $M(K_{1,n,n})$.](image)

**Proof.** By definition of middle graph, each edge $vv_i$ and $v_iu_i(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices $u_i$ and $s_i$ in $M(K_{1,n,n})$ and the vertices $v,e_1,e_2,\ldots,e_n$ induce a clique of order $n+1$ (say $K_{n+1}$) in $M(K_{1,n,n})$. i.e., $V(M(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. The number of edges in $M(K_{1,n,n})$ is $\frac{n^2+9n}{2}$. Thus we have $\chi_H(M(K_{1,n,n})) \geq 2n+1$.

The lower bound can be established from the fact that $2n+1$ vertices $v,e_i,v_i$ are all mutually at distance at most 2. Hence the $2n+1$ vertices $v,v_i,e_i$ require distinct colors. The vertices $u_i$ can be colored with the same colors as $e_i$, but in a different order.

The following $(2n+1)$ - coloring for $M(K_{1,n,n})$ is harmonious: For $(1 \leq i \leq n)$, assign the color $c_i$ for $e_i$ and assign $c_{n+1}$ to $v$. For $(1 \leq i \leq n)$, assign the color $c_{n+1+i}$ for $v_i$. For $(1 \leq i \leq n-1)$, assign the color $c_{n+2+i}$ for $s_i$ and assign the color $c_{n+2}$ to $s_n$. For $(1 \leq i \leq n-2)$, assign the color $c_{i+2}$ for $u_i$ and assign the colors $c_1$ to $u_{n-1}$ and $c_2$ to $u_n$. Therefore, $\chi_H(M(K_{1,n,n})) \leq 2n+1$. Hence $\chi_H(M(K_{1,n,n})) = 2n+1$. Note that $\chi_H(M(K_{1,1,1})) = 4$ and $\chi_H(M(K_{1,2,2})) = 6$.

**Proposition 3.2.** For any double star graph $K_{1,n,n}$, the harmonious chromatic number,

$$\chi_H(T(K_{1,n,n})) = 2n+1.$$
Figure 4: Total graph of double star graph $T(K_1,n,n)$.

**Proof.** By the definition of total graph, we have $V(T(K_1,n,n)) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$, in which the vertices $v, e_1, e_2,\ldots,e_n$ induce a clique of order $n + 1$ (say $K_{n+1}$). The number of edges in $T(K_1,n,n)$ is $\frac{n^2 + 13n}{2}$. Thus we have $\chi_H(T(K_1,n,n)) \geq 2n + 1$.

The lower bound can be established from the fact that $2n + 1$ vertices $v, e_i, v_i$ are all mutually at distance at most 2. The following $(2n + 1)$-coloring for $M(K_1,n,n)$ is harmonious: For $(1 \leq i \leq n)$, assign the color $c_i$ for $e_i$ and assign $c_{n+1}$ to $v$. For $(1 \leq i \leq n)$, assign the color $c_{n+1+i}$ for $v_i$. For $(1 \leq i \leq n-1)$, assign the color $c_{n+2+i}$ for $s_i$ and assign the color $c_{n+2}$ to $s_n$. For $(1 \leq i \leq n-2)$, assign the color $c_{i+2}$ for $u_i$ and assign the colors $c_1$ to $u_{n-1}$ and $c_2$ to $u_n$.

Therefore, $\chi_H(T(K_1,n,n)) \leq 2n + 1$. Hence $\chi_H(T(K_1,n,n)) = 2n + 1$. □

4. Harmonious and achromatic coloring on line graph of double star graph

**Proposition 4.1.** For any double star graph $K_1,n,n$, the harmonious chromatic number,

$$\chi_H(L(K_1,n,n)) = n + 1.$$ 

**Proof.** By the definition of line graph, each edge of $K_1,n,n$ taken to be as vertex in $L(K_1,n,n)$. The vertices $e_1, e_2,\ldots,e_n$ induce a clique of order $n$ in $L(K_1,n,n)$, i.e., $V(L(K_1,n,n)) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.
\{s_i : 1 \leq i \leq n\}. The number of edges in \(L(K_{1,n,n})\) is \(\frac{n^2 + n}{2}\). Thus we have \(\chi_H(L(K_{1,n,n})) \geq n + 1\).

Now consider the vertex set \(V(L(K_{1,n,n}))\) and color class \(C = \{c_1, c_2, \ldots, c_n, c_{n+1}\}\), assign a proper harmonious coloring to \(L(K_{1,n,n})\) as follows: For \((1 \leq i \leq n)\), assign the color \(c_i\) for \(e_i\). For \((1 \leq i \leq n)\), assign the color \(c_{n+1}\) to \(s_i\). Therefore, \(\chi_H(L(K_{1,n,n})) \leq n + 1\). Hence \(\chi_H(L(K_{1,n,n})) = n + 1\).

**Theorem 4.2** ([5]). *For any double star graph \(K_{1,n,n}\), the achromatic number,*

\[\chi_c(L(K_{1,n,n})) = n + 1.\]

**Proof.** As the number of edges in \(L(K_{1,n,n})\) is \(\frac{n^2 + n}{2} < \binom{n + 2}{2}\). Thus we have \(\chi_c(L(K_{1,n,n})) \geq n + 1\). Now consider the vertex set \(V(L(K_{1,n,n}))\) and the color class \(C = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}\}\), assign the achromatic coloring to \(L(K_{1,n,n})\) as follows: For \((1 \leq i \leq n)\), assign the color \(c_i\) for \(e_i\). For \((1 \leq i \leq n)\), assign the color \(c_{n+1}\) to \(s_i\). Therefore, \(\chi_c(L(K_{1,n,n})) \leq n + 1\). Hence \(\chi_c(L(K_{1,n,n})) = n + 1\).

5. Main theorem

Now we characterize the graph for which the harmonious chromatic number and achromatic number are same. The proof of the Main Theorem follows from Proposition 4.1 and Theorem 4.2.

**Theorem 5.1.** *For any double star graph \(K_{1,n,n}\),*

\[\chi_c(L(K_{1,n,n})) = \chi_H(L(K_{1,n,n})) = n + 1.\]

6. Conclusion

In this present paper, we have proved for the line graph of double star graph, the harmonious chromatic number and the achromatic number are equal. As a motivation this work can be extended by classifying the different families of graphs for which these two numbers are equal.

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