Generalized $pp$-wave solutions on product of Ricci-flat spaces

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Abstract

A multidimensional gravitational model with several scalar fields and form fields is considered. A wide class of generalized $pp$-wave solutions defined on a product of $n + 1$ Ricci-flat spaces is obtained. Certain examples of solutions (e.g. in supergravitational theories) are singled out. For special cone-type internal factor spaces the solutions are written in Brinkmann form. An example of $pp$-wave solution is obtained using Penrose limit of a solution defined on a product of two Einstein spaces.

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1 Introduction

Plane-fronted, parallel gravitational waves (pp-waves) described by the metrics

\[ g = dx^+ \otimes dx^- + dx^- \otimes dx^+ + H(x^-, x)dx^- \otimes dx^- + \sum_{i=1}^{m} dx^i \otimes dx^i, \quad (1.1) \]

with certain smooth functions \( H(x^-, x) \) become rather popular objects of investigations (see [1]-[25] and refs. therein). The metric (1.1) has a covariantly constant null Killing vector \( \partial/\partial x^+ \) and obey the vacuum Einstein equation when \( \sum_{i=1}^{m} \frac{\partial^2}{\partial x^i \partial x^-} H = 0 \). In four dimensions it is the most general solution to Einstein equations with covariantly constant null vector [3].

The metrics (1.1) also appeared as exact solutions of string theory [1, 2, 3] (e.g. with non-trivial dilaton and null 3-form backgrounds) due to vanishing of all higher order terms in string (\( \beta \)-function) equations of motion. Despite of the triviality of all scalar invariants constructed on powers of the Riemann tensor and its derivatives the metric (1.1) is in general singular [18, 19]. For \( H(x^-, x) = A_{ij}(x^-)x^ix^j \) the metric is regular. In the case of constant \( A_{ij} \) we are led to so called Cahen-Wallach (CW) spaces [5]. These spaces contain maximally supersymmetric plane waves in eleven-dimensional [4, 7] and ten-dimensional type IIB [8] supergravity (see also [20]). It was observed in [9, 10] that these solutions may be also obtained as Penrose limits [11, 12] of the \( AdS_p \times S^{D-p} \) type solutions.

It was shown in [13, 14, 15] that superstring theory on pp-wave background can be solved in the light-cone gauge. In [16] a sector of \( N = 4 \) super-Yang-Mills dual to string theory on certain pp-wave background was identified (see also [25]), allowing for stringy tests of the \( AdS/CFT \) correspondence.

For \( H(x^-, x) = A_{ij}(x^-)x^ix^j \) the solution (1.1) may be also rewritten in Rosen coordinates [12, 10, 24]

\[ g = du \otimes dv + dv \otimes du + \sum_{i=1}^{m} C_{ij}(u)dy^i \otimes dy^j. \quad (1.2) \]

For diagonal matrix \((C_{ij}(u)) = (C_i(u)\delta_{ij})\) the metric has a natural generalization to a chain of Ricci-flat spaces. In this paper we obtain a rather general class of solutions defined on product of \( n+1 \) Ricci-flat spaces for the multidimensional gravitational model with fields of forms and scalar fields. These solutions follow just from the equations of motion.
The paper is organized as follows. In Sect. 2 we outline the general approach with arbitrary forms and dilaton fields on a product of \((n+1)\) manifolds. In Sect. 3 we obtain \(pp\)-wave solutions defined on a product of \((n+1)\) Ricci-flat spaces. Here the integrability problem is also discussed. Several examples of solutions are presented in Section 4. In Section 5 a (generalized) Brinkmann form of solution for special (cone-type) Ricci-flat factor-spaces is suggested. Here an example of \(pp\)-wave solution is obtained using Penrose limit of a solution defined on product of two Einstein spaces \(31\).

2 The model

We consider the model governed by the action

\[
S = \int_M d^Dz \sqrt{|g|} \left\{ R[g] - 2\Lambda - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta \right. \tag{2.1}
\]

\[
- \sum_{a \in \Delta} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)] (F^a)^2 \right\},
\]

where \(g = g_{MN} dz^M \otimes dz^N\) is the metric, \(\varphi = (\varphi^\alpha) \in \mathbb{R}^l\) is a vector from dilatonic scalar fields, \((h_{\alpha\beta})\) is a non-degenerate symmetric \(l \times l\) matrix \((l \in \mathbb{N})\), \(\theta_a \neq 0\),

\[
F^a = dA^a = \frac{1}{n_a!} F^a_{M_1...M_{n_a}} dz^{M_1} \wedge ... \wedge dz^{M_{n_a}} \tag{2.2}
\]

is a \(n_a\)-form \((n_a \geq 2)\) on a \(D\)-dimensional manifold \(M\), \(\Lambda\) is a cosmological constant and \(\lambda_a\) is a 1-form on \(\mathbb{R}^l\): \(\lambda_a(\varphi) = \lambda_{aa} \varphi^a\), \(a \in \Delta\); \(\alpha = 1, ..., l\). In (2.1) we denote \(|g| = |\det(g_{MN})|\),

\[
(F^a)^2 = F^a_{M_1...M_{n_a}} F^a_{N_1...N_{n_a}} g^{M_1 N_1} ... g^{M_{n_a} N_{n_a}}, \tag{2.3}
\]

\(a \in \Delta\), where \(\Delta\) is some finite set. In the models with one time all \(\theta_a = 1\) when the signature of the metric is \((-1, +1, ..., +1)\).

The equations of motion corresponding to (2.1) have the following form

\[
R_{MN} = Z_{MN} + \frac{2\Lambda}{D-2} g_{MN}, \tag{2.4}
\]

\[
\triangle[g] \varphi^\alpha - \sum_{a \in \Delta} \frac{\lambda_a}{n_a!} e^{2\lambda_a(\varphi)} (F^a)^2 = 0, \tag{2.5}
\]

\[
\nabla_{M_1}[g](e^{2\lambda_a(\varphi)} F^a_{M_1...M_{n_a}}) = 0, \tag{2.6}
\]
$a \in \Delta; \alpha = 1, \ldots, l$. In (2.5) $\lambda^a = h^{\alpha \beta} \lambda_{\beta a}$, where $(h^{\alpha \beta})$ is a matrix inverse to $(h_{\alpha \beta})$. In (2.4) $Z_{MN} = Z_{MN}[\varphi] + \sum_{a \in \Delta} \theta_\alpha e^{2 \lambda_\alpha(\varphi)} Z_{MN}[F^a, g],$ (2.7)

where

$$Z_{MN}[\varphi] = h^{\alpha \beta} \partial_M \varphi^\alpha \partial_N \varphi^\beta,$$ (2.8)

$$Z_{MN}[F^a, g] = \sum_{n} \frac{1}{n!} \left[ \frac{n_a - 1}{2 - D} g_{MN}(F^a)^2 + n_a F^a_{MN,M_{2n-1}} F^a_{MN,M_{2n-2}} \right].$$ (2.9)

In (2.5) and (2.6) $\triangle[g]$ and $\nabla[g]$ are Laplace-Beltrami and covariant derivative operators respectively corresponding to $g$.

**Multi-index notations.** Let us consider the manifold

$$M = M_0 \times M_1 \times \ldots \times M_n.$$ (2.10)

We denote $d_i = \dim M_i \geq 1; i = 0, \ldots, n$. $D = \sum_{i=0}^n d_i$. Let $g^i = g^i_{m,n_i}(y_i) dy^m_i \otimes dy^{n_i}_i$ be a metric on the manifold $M_i, i = 1, \ldots, n$.

Here we use the notations of our previous papers [28]. Let any manifold $M_i$ be oriented and connected. Then the volume $d_i$-form

$$\tau_i = \text{dvol}(g^i) \equiv \sqrt{|g^i(y_i)|} \; dy^1_i \wedge \ldots \wedge dy^{d_i}_i, \quad (2.11)$$

and the signature parameter

$$\varepsilon_i \equiv \text{sign}(\det(g^i_{m,n_i})) = \pm 1$$ (2.12)

are correctly defined for all $i = 1, \ldots, n$.

Let $\Omega$ be a set of all non-empty subsets of $\{1, \ldots, n\}$. For any $I = \{i_1, \ldots, i_k\} \in \Omega, i_1 < \ldots < i_k$, we denote

$$\tau(I) \equiv \hat{\tau}_{i_1} \wedge \ldots \wedge \hat{\tau}_{i_k}, \quad (2.13)$$

$$\varepsilon(I) \equiv \varepsilon_{i_1} \times \ldots \times \varepsilon_{i_k}, \quad (2.14)$$

$$M_I \equiv M_{i_1} \times \ldots \times M_{i_k}, \quad (2.15)$$

$$d(I) \equiv \sum_{i \in I} d_i, \quad (2.16)$$

where $d_i$ is both, the dimension of the oriented manifold $M_i$ and the rank of the volume form $\tau_i$ and $\hat{\tau}_i$ is the pullback of $\tau_i$ to the manifold $M$: $\hat{\tau}_i = p_i^* \tau_i$, where $p_i : M \rightarrow M_i$, is the canonical projection, $i = 1, \ldots, n$. 


We also denote by
\[ \delta^i_I = \sum_{j \in I} \delta^i_j \]  \hspace{1cm} (2.17)
the indicator of \( i \) belonging to \( I \): \( \delta^i_I = 1 \) for \( i \in I \) and \( \delta^i_I = 0 \) otherwise.

3 General solutions

3.1 Solutions governed by one equation

Let \( M_0 \) be an open domain in \( \mathbb{R}^2 \) equipped with a flat metric \( g^0 = du \otimes dv + dv \otimes du \) where \( u, v \) are coordinates.

Let us consider a plane wave metric on the manifold (2.10) of the following form
\[ g = \hat{g}^0 + \sum_{i=1}^n \exp(2\phi^i(u))\hat{g}^i, \]  \hspace{1cm} (3.1)
where \( g^i \) is a metric on \( M_i, i = 1, \ldots, n \).

Here and in what follows \( \hat{g}^i = p^*_i g^i \), is the pullback of the metric \( g^i \) to the manifold \( M \) by the canonical projection: \( p_i : M \to M_i, i = 0, \ldots, n \).

The fields of forms and scalar fields are also chosen in the \( u \)-dependent form
\[ F^a = \sum_{I \in \Omega_a} d\Phi^{(a,I)}(u) \wedge \tau(I), \]  \hspace{1cm} (3.2)
\[ \varphi^a = \varphi^a(u). \]  \hspace{1cm} (3.3)

where \( \Omega_a \subset \Omega \) are non-empty subsets, satisfying the relations \( d(I) = n_a - 1 \) for all \( I \in \Omega_a, a \in \Delta \).

The substitution of fields from (3.1), (3.2) and (3.3) into equations of motion (2.4)-(2.6) lead to the following relations
\[ \text{Ric}[\hat{g}^i] = 0, \]  \hspace{1cm} (3.4)
\[ \Lambda = 0, \]  \hspace{1cm} (3.5)
\[ -\sum_{i=1}^n d_i[\ddot{\phi}^i + (\dot{\phi}^i)^2] = h_{ab} \dot{\varphi}^a \varphi^b + \sum_{s \in S} \varepsilon_s \exp[-2U^s(\phi, \varphi)](\dot{\varphi}^s)^2 \]  \hspace{1cm} (3.6)
\[ i = 1, \ldots, n. \]  In derivation of eqs. (3.4)-(3.6) the formulas for Ricci-tensor and Z-tensor (2.7) from Appendix were used.
Here and in what follows, Ric[\(g^i\)] is the Ricci-tensor corresponding to \(g^i\), \(i = 1, \ldots, n\), and \(\dot{X} \equiv dX/du\). The (brane) index set \(S\) consists of elements \(s = (a_s, I_s)\), where \(a_s \in \Delta\) and \(I_s \in \Omega_{a_s}\) are “color” and “brane” indices, respectively. The electric \(U\)-covectors and \(\epsilon\)-symbols are defined as follows

\[
U^s = U^s(\phi, \varphi) = -\lambda_{a_s}(\varphi) + \sum_{i \in I_s} d_i \phi^i, \quad (3.7)
\]

\[
\epsilon_s = \epsilon(I_s) \theta_{a_s} \quad (3.8)
\]

for \(s = (a_s, I_s) \in S\).

Thus, we get from equations of motion that cosmological term is zero and all spaces \((M_i, g^i)\) \((i = 1, \ldots, n)\) are Ricci-flat. We also are led to the second order differential equation (3.6) on logarithms of scale-factors \(\phi^i\), scalar fields \(\varphi^a\) and “brane” scalar fields \(\Phi_s\).

The solution (3.4)-(3.6) is valid without imposing of any restrictions (on brane configurations) or intersection rules. (Compare with the solutions from [28] and references therein).

### 3.1.1 “Electro-magnetic” form of solution

Now we show that more general composite ”electro-magnetic” ansatz

\[
F^a = \sum_{I \in \Omega_{a,e}} d \Phi^{(a,e,I)}(u) \wedge \tau(I) + \sum_{J \in \Omega_{a,m}} e^{-2\lambda_{a}(\varphi)} \ast (d \Phi^{(a,m,J)}(u) \wedge \tau(J)), \quad (3.9)
\]

(instead of composite electric one from (3.2)) will not give new solutions. Here \(\ast = \ast[g]\) is the Hodge operator on \((M, g)\), \(\Omega_{a,e}, \Omega_{a,m} \subset \Omega\) are non-empty subsets, satisfying the relations: \(d(I) = n_a - 1\) for all \(I \in \Omega_{a,e}\) and \(d(J) = D - n_a - 1\) for all \(J \in \Omega_{a,m}\), \(a \in \Delta\).

Indeed, due to relations (5.26) from [30] we get

\[
\ast (d \Phi \wedge \tau(I)) = P_I (\ast_0 d \Phi) \wedge \tau(\bar{I}) = P_I d \Phi \wedge \tau(\bar{I}), \quad (3.10)
\]

\[
P_I = \varepsilon(I) \mu(I) \exp\left[\sum_{j=1}^{n} d_j \phi^j - 2 \sum_{i \in I_s} d_i \phi^i\right], \quad (3.11)
\]

where \(\bar{I} = \{1, \ldots, n\} \setminus I\) is ”dual” set, \(\mu(I) = \pm 1\) is defined by the formula \(\tau(\bar{I}) \wedge du \wedge \tau(I) = \mu(I) du \wedge \tau(\{1, \ldots, n\})\) and \(\ast_0 = \ast[g^0]\) is the Hodge operator
on \((M_0, g^0)\), obeying \(* d\Phi = d\Phi\) for \(\Phi = \Phi(u)\). \(^2\)

Using (3.10) any ”magnetic” monom may be rewritten in the electric form

\[
\exp(-2\lambda_a(\varphi)) \ast (d\Phi^{(a,m,J)}) \wedge \tau(J)) = (d\bar{\Phi}^{(a,m,J)}) \wedge \tau(J),
\]

(3.12)

where \(d\bar{\Phi}^{(a,m,J)} = \exp(-2\lambda_a(\varphi(u))) P_J(u) d\Phi^{(a,m,J)}\), \(J \in \Omega_{a,m}\).

**Remark.** A rather simple way to verify that fields of forms (3.2) obey the ”Maxwell” equations (2.6) written in the form \(d \ast (e^{2\lambda_a(\varphi)} F^a) = 0\) is to use the relation (3.10).

### 3.2 The integrability problem

Here we consider the problem of integrability of equation (3.6) that may be rewritten in equivalent form in terms of scale factors \(f_i = \exp(\phi_i)\) as following

\[
- \sum_{i=1}^n d_i \frac{\ddot{f}_i}{f_i} = h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta + \sum_{s \in S} \varepsilon_s \exp[2\lambda_s(\varphi)](\dot{\Phi}^s)^2 \prod_{i \in I_s} f_i^{-2d_i}.
\]

(3.13)

Let us consider the following problem: for given scale factors \(f_2, \ldots, f_n\), scalar fields \(\varphi^\alpha\) and brane scalar fields \(\Phi^s\) to find \(f_1\) satisfying eq. (3.13). The solution of this problem is equivalent to finding of all metrics (2.2) obeying equations of motion (2.4)-(2.6) for given fields of forms (3.2) and scalar fields (3.3).

**Remark.** Here we do not consider another trivial task: for given scale factors \(f_1, \ldots, f_n\) and scalar fields \(\varphi^\alpha\) to find all brane scalar fields \(\Phi^s\) satisfying eq. (3.13).

Denoting \(\Psi = f_1(u)\) we get the following non-linear equation

\[
\ddot{\Psi} = A(u)\Psi + B(u)\Psi^{1-2d_1},
\]

(3.14)

where function \(A = A(u)\) and \(B = B(u)\) are defined as follows

\[
A = -\frac{1}{d_1} \sum_{i=2}^n d_i \frac{\ddot{f}_i}{f_i} + h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta +
\]

(3.15)

\(^2\)Here we use the following definition for Hodge dual:

\[
(\ast \omega)_{M_1 \ldots M_{D-k}} = \frac{|g|^{1/2}}{k!} \varepsilon_{M_1 \ldots M_{D-k} N_1 \ldots N_k} \omega^{N_1 \ldots N_k},
\]

where \(\text{rank}(\omega) = k\). We also put \(\varepsilon_{uv} = 1\).
\[
\sum_{s \in S \setminus S_1} \varepsilon_s \exp[2\lambda_{as}(\varphi)](\hat{\Phi}^s)^2 \prod_{i \in I_s} f_i^{-2d_i} \bigg],
\]
\[B = -\frac{1}{d_1} \sum_{s \in S_1} \varepsilon_s \exp[2\lambda_{as}(\varphi)](\hat{\Phi}^s)^2 \prod_{i \in I_s \setminus \{1\}} f_i^{-2d_i}. \quad (3.16)\]

Here we denote
\[S_1 \equiv \{s \in S : 1 \in I_s\}. \quad (3.17)\]
The subset \(S_1 \subset S\) describes all branes that "cover" the space \(M_1\).

A non-linear equation (3.14) with \(d_1 \neq 0, 1\) was considered by Reid in [27], where a special subclass of solutions was obtained for certain functions \(B(u) = B(u, d_1)\).

### 3.2.1 One factor-space is "free"

Let us suppose that one of factor spaces, say \(M_1\), is not "occupied" by branes, i.e.
\[1 \notin I_s \quad (3.18)\]
for all \(s \in S\).

In this case \(B = 0\) and we get a very familiar (from quantum mechanics) linear equation
\[\ddot{\Psi} = A(u)\Psi, \quad (3.19)\]
where function \(A = A(u)\) is defined in (3.15) with \(S_1 = \emptyset\). For certain "potentials" \(A(u)\) one can find explicit general solutions to (Schroedinger-type) eq. (3.19).

### 4 Special solutions

#### 4.1 Sin-type solutions.

Let us consider a special class of solutions with \(sin\)-type dependent scale factors
\[g = du \otimes dv + dv \otimes du + \sum_{i=1}^{n} f_i^2 \hat{g}_i, \quad (4.1)\]
\[F^a = \sum_{s \in S} \delta_a^s q_s e^{-\lambda_{as}(\varphi)} (\prod_{i \in I_s} f_i^{d_i}) du \wedge \tau(I_s), \quad (4.2)\]
\[\varphi^a = \text{const}, \quad (4.3)\]
where
\[ f_i = c_i \sin(\omega_i u + \omega_i^0), \quad (4.4) \]
c_i > 0, \( \omega_i, \omega_i^0 \) and \( q_s \) are constants \((i = 1, \ldots, n, s = (a_s, I_s) \in S)\) obeying the following relation
\[ \sum_{i=1}^{n} d_i \omega_i^2 = \sum_{i \in I_s} \varepsilon_s q_s^2. \quad (4.5) \]

These special solutions (with \( \varepsilon_s = +1, \theta_a = +1 \)) have important applications in supergravity and string theory.

In what follows we consider two examples of these \( \sin \)-type solutions.

### 4.2 Solutions in IIB supergravity.

Let us consider a solution in IIB-supergravity with two 4-dimensional Ricci-flat spaces \( (M_i, g_i) \), \( i = 1, 2, \) of Euclidean signature.

We consider a sector with 5-form and put \( \varphi = 0 \). We also put \( I_1 = \{1\}, I_2 = \{2\}; c_1 = 1, \omega_1 = 1, \omega_1^0 = 0 \) \((i = 1, 2)\) and \( q_s = \pm 2, s \in \{s_1, s_2\} \).

In this case the solution (4.1), (4.2) reads as follows
\[ g = du \otimes dv + dv \otimes du + \sum_{i=1}^{2} \sin^2(u) \hat{g}^i, \quad (4.6) \]
\[ F[5] = \pm 2(\sin^4(u))[du \wedge \hat{\tau}_1 + du \wedge \hat{\tau}_2], \quad (4.7) \]
with \( F[5] = \ast F[5] \) (see (3.10)).

For flat spaces \((M_i, g_i)\), \( i = 1, 2, \) we get a well-known supersymmetric solution from [8], written in the Rosen representation.

### 4.3 Solutions in D = 11 supergravity.

Now we consider a solution in \( D = 11 \)-supergravity [26] with two Ricci-flat spaces \((M_i, g_i)\), \( i = 1, 2, \) of Euclidean signature and dimensions \( d_1 = 3 \) and \( d_2 = 6, \) respectively. Here the first space \((M_1, g^1)\) is obviously flat, since it is 3-dimensional Ricci-flat space.

Let \( I_1 = \{1\} \) (i.e. one brane "living" on \( M_1 \) is considered); \( c_1 = 1/2, c_2 = 1, \omega_1 = 1, \omega_2 = 1/2, \omega_2^0 = 0 \) \((i = 1, 2)\) and \( q_s = \pm 3/\sqrt{2}, s = s_1 \).

Then the solution (4.1), (4.2) reads as follows
\[ g = du \otimes dv + dv \otimes du + \frac{1}{4} \sin^2(u) \hat{g}^1 + \sin^2(\frac{1}{2} u) \hat{g}^2, \quad (4.8) \]
\[ F_{[4]} = \pm \frac{3}{\sqrt{2}}(\sin^3(u))du \wedge \hat{\tau}_1, \quad (4.9) \]

For flat space \((M_2, g^2)\) we are led to supersymmetric solution from \([7]\), written in the Rosen representation.

### 4.4 Solutions with constant scale factors

A special class of solutions occurs when all scale factor are constant. In this case relation (4.10) implies

\[ 0 = h_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta + \sum_{s \in S} \varepsilon_s \exp[-2U_s(\phi, \varphi)](\dot{\Phi}^s)^2. \quad (4.10) \]

Usually, in all substantial (supergravitational) examples the matrix \((h_{\alpha\beta})\) is positive definite and all \(\varepsilon_s\) are positive (for pseudo-Euclidean signature \((-\,+,\ldots,\,+)\) of \(g\)). Hence, it follows from (4.10) that in this case all scalar fields are constant and all fields of forms are zero. But one may obtain non-trivial solutions when \((h_{\alpha\beta})\) is not positive definite one, or, when some of \(\varepsilon_s\) are negative. Such solutions occur in twelve dimensional model \([32]\), corresponding to \(F\)-theory and in so-called \(BD\)-models \([34]\) in dimension \(D \geq 12\).

### 5 Solutions with special factor spaces

#### 5.1 Brinkmann form of special solutions

In this Section we show that the solutions under consideration may be written in generalized ”Brinkmann form” when special ”internal” factor spaces are chosen.

Let \(M_i = R_+ \times N_i\) and

\[ g^i = dr_i \otimes dr_i + r_i^2 \hat{h}_i, \quad (5.1) \]

where \((h_i, N_i)\) is Einstein space of dimension \(d_i - 1\) and \(\text{Ric}[h_i] = (d_i - 2)h_i\). Clearly, all metrics (5.1) are Ricci-flat.

For cone-type metrics (5.1) the relations (3.1) and (3.2) may be rewritten as follows

\[ g = du \otimes d\bar{v} + d\bar{v} \otimes du - \left( \sum_{i=1}^n \lambda_i(u) r_i^2 \right) du \otimes du + \sum_{i=1}^n \hat{g}^i, \quad (5.2) \]
\[ F^a = \sum_{s \in S} \delta_a^s d\bar{\Phi}^s(u) \wedge \tau(I_s), \quad (5.3) \]

and eq. (3.6) reads

\[ \sum_{i=1}^{n} d_i \lambda_i = h_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta + \sum_{s \in S} \varepsilon_s \exp[2\lambda_s(\varphi)](\dot{\Phi}^s)^2. \quad (5.4) \]

The metric (5.2) may be obtained if one substitute into original metric the internal metric \( g^i = dR_i \otimes dR_i + R_i^2 \hat{h}_i \), and then make redefinition of coordinates

\[ \bar{v} = v + \frac{1}{2} \sum_{i=1}^{n} h_i(u) R_i^2, \quad (5.5) \]
\[ r_i = f_i(u) R_i, \quad h_i = -f_i \dot{f}_i, \quad (5.6) \]

\( i = 1, \ldots, n \). In (5.4) \( \lambda_i = -\ddot{f}_i / f_i \) and \( \dot{\Phi}^s = \hat{\Phi}^s \prod_{i=1}^{d_i} f_i^{-d_i}, \quad i = 1, \ldots, n; \quad s \in S. \)

It should be noted that the class of supersymmetric pp-wave solutions in ten dimensional IIB supergravity obtained by Maldacena and Maoz (see also [22]) has a non-empty intersection with the family of our solutions for \( D = 10 \).

### 5.2 Penrose limit of a solution on product of two Einstein spaces

Some of pp-wave solutions may be obtained from generalized Freund-Rubin-type solutions [31] (defined on product of Einstein spaces) using the Penrose limit. Here we consider an example of such procedure.

Let us consider the special solution from [31] defined on product of two Einstein spaces \( (\bar{M}_i, \bar{g}_i) \),

\[ \text{Ric}(\bar{g}_i) = \xi_i \bar{g}_i, \quad (5.7) \]

\( i = 1, 2 \), with equal dimensions \( d_1 = d_2 = d + 2 \).

The static Freund-Rubin-type solution with one non-zero form reads

\[ g = \hat{g}_1 + \hat{g}_2, \quad (5.8) \]
\[ F^a = Q_1 \hat{\tau}_1 + Q_2 \hat{\tau}_2, \quad (5.9) \]
\[ \varphi^\alpha = 0, \quad (5.10) \]
\[ \xi_1 = -Q^2, \quad \xi_2 = Q^2, \quad (5.11) \]
where $Q_1^2 = Q_2^2 = Q^2$.

Let $\bar{M}_i = T_i \times \mathbb{R} \times N_i$, where $T_i \subset \mathbb{R}$ are intervals. We consider Einstein metrics on $\bar{M}_1$ and $\bar{M}_2$, respectively,

\[
\bar{g}_1 = (d\rho \otimes d\rho - \cosh^2(\rho) dt \otimes dt + \sinh^2(\rho) \hat{h}_1) R^2, \quad (5.12)
\]

\[
\bar{g}_2 = (d\theta \otimes d\theta - \cos^2(\theta) d\psi \otimes d\psi + \sin^2(\theta) \hat{h}_2) R^2, \quad (5.13)
\]
generated by Einstein spaces $(h_i, N_i)$ (of dimension $d$) obeying $\text{Ric}[h_i] = (d - 1) h_i$, $i = 1, 2$. Here $R^2 = (d + 2)/Q^2$.

Introducing new variables

\[
\bar{x}^{\mp} = (t \pm \psi)/2, \quad x^- = \bar{x}^-/\mu, \quad x^+ = -\bar{x}^+ 2\mu R^2, \quad (5.14)
\]

\[
r_1 = \rho R, \quad r_2 = \theta R, \quad (5.15)
\]
where $\mu \neq 0$ and taking the Penrose limit $R \to +\infty$, we get

\[
g = dx^+ \otimes dx^- + dx^- \otimes dx^+ - \mu^2 (r_1^2 + r_2^2) dx^- \otimes dx^- + \hat{g}_1 + \hat{g}_2, \quad (5.16)
\]

\[
F^n = \mu \sqrt{d + 1} dx^- \wedge (\delta_1 \hat{\tau}_1 + \delta_2 \hat{\tau}_2), \quad (5.17)
\]
where $\delta_i = \pm 1$ ($\delta_i = -\text{sign} Q_i$), $\tau_i = d\text{vol}(g_i)$ and $g_i = dr_i \otimes dr_i + r_i^2 \hat{h}_i$.

Ricci-flat metrics on $M_i = \mathbb{R} \times N_i$, $i = 1, 2$.

\section{Discussions}

In this paper we obtained exact solutions describing plane waves on the product of $(n + 1)$ Ricci-flat spaces for the gravitational model with fields of forms and (dilatonic) scalar fields. The solutions are given by the relations (3.1)-(3.6) and may be considered as a composite generalization of well-known $pp$-wave solutions in supergravitational theories.

The general solutions are defined up to solutions of the second order differential equation (3.6). When i) one of factor spaces is not "occupied" by branes then the problem is reduced to Schroedinger-type eq. (3.19). In the case ii) when all spaces are occupied by branes we are led to non-linear equation (3.14) (Reid equation). An interesting question here is to find a possible "chaotic" behaviour among solutions of eq. (3.14), as it sometimes takes place in "cosmology" with $p$-branes [35, 36]. In the first case i) the "chaotic" behaviour seems to be absent for $pp$-wave solutions (see (3.19)) in agreement with Kasner-like (non-oscillating) behaviour near the singularity for $S$-brane cosmology [37] (when one space is "free").
Another interesting problem is related to classification of (fractional) supersymmetric configurations among the solutions under consideration. On this way the earlier results of the paper \cite{38} may be used.

Here we considered as an example a solution from \cite{31} ("Freund-Rubin" type solution) defined on product of two Einstein spaces and showed the appearance of a \( pp \)-wave solution in the Penrose limit. An interesting problem is to investigate all possible Penrose limits of the composite p-branes solutions on product of Einstein spaces from \cite{31} (we remind that certain solutions from \cite{31} appear in the "near-horizon" limit of solutions with harmonic functions and Ricci-flat internal spaces \cite{30, 28}).

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Appendix

A Ricci and Riemann tensors

The non-zero (identically) Ricci tensor components for the metric (3.1) are the following (see Appendix in \cite{28})

\[
R_{uu}[g] = - \sum_{i=1}^{n} d_i [\ddot{\phi}^i + (\dot{\phi}^i)^2],
\]

\[
R_{mni}[g] = R_{mni}[g^i],
\]

\( i = 1, \ldots, n. \)

The scalar curvature for (3.1) reads

\[
R[g] = \sum_{i=1}^{n} e^{-2\phi^i} R[g^i].
\]
The non-zero (identically) components of the Riemann tensor corresponding to the metric (3.1) have the following form

\[
R_{um_{i}u_{n}}[g] = - R_{m_{i}um_{i}}[g] = - R_{um_{i}u_{n}}[g] = \\
R_{m_{i}um_{i}[g]} = - \exp(2\phi^{i})g_{m_{i}n_{i}}[\dot{\phi}^{i} + (\dot{\phi}^{i})^{2}], \quad \text{(A.4)}
\]

\[
R_{m_{i}u_{n}u_{i}}[g] = \exp(2\phi^{i})R_{m_{i}u_{n}u_{i}}[g], \quad \text{(A.5)}
\]

\[
i = 1, \ldots, n.
\]

**B  Product of forms**

For two forms \( F_{1} \) and \( F_{2} \) of rank \( r \) on \((M, g) \) (\( M \) is a manifold and \( g \) is a metric on it) we use notations

\[
(F_{1} \cdot F_{2})_{MN} \equiv (F_{1})_{M_{1}M_{2} \ldots M_{r}}(F_{2})_{N}^{M_{2} \ldots M_{r}}, \quad (B.6)
\]

\[
F_{1}F_{2} \equiv (F_{1} \cdot F_{2})_{M}^{M_{1}}(F_{1})_{M_{1}M_{2} \ldots M_{r}}(F_{2})_{M_{1}M_{2} \ldots M_{r}}, \quad (B.7)
\]

It may be verified (see also formulas in Appendix from \[28\]) that

\[
\mathcal{F}(a,I) \mathcal{F}(a,J) = 0, \quad (B.8)
\]

\( I, J \in \Omega_{a} \), where

\[
\mathcal{F}(a,I) = d\Phi^{(a,I)}(u) \wedge \tau(I). \quad (B.9)
\]

Hence for composite fields

\[
F^{a} = \sum_{I \in \Omega_{a}} \mathcal{F}(a,I), \quad (B.10)
\]

we get

\[
(F^{a})^{2} = 0, \quad (B.11)
\]

\( a \in \Delta. \)

Let us consider the tensor \( F^{a} \cdot F^{a} = (F^{a} \cdot F^{a})_{MN}dz^{M} \otimes dz^{N} \) for composite \( F^{a} \) from (B.10). One verify can that

\[
(\mathcal{F}(a,I) \cdot \mathcal{F}(a,J))_{MN} = 0, \quad (B.12)
\]

for \( I \neq J, I, J \in \Omega_{a}, a \in \Delta. \) This may done using the relations from Appendix of \[28\] for "non-dangerous" intersection \( d(I \cap J) \neq d(I) - 1 \) and relation (4.14) from \[30\] for "dangerous" intersection \( d(I \cap J) = d(I) - 1. \)
Now we put $I = J \in \Omega_a$. The only non-zero (identically) components of the tensor $(\mathcal{F}^{(a,I)} \cdot \mathcal{F}^{(a,I)})$ are the following ones (see Appendix in [28])

$$(\mathcal{F}^{(a,I)} \cdot \mathcal{F}^{(a,I)})_{uu} = (n_a - 1)! \varepsilon(I) \exp[-2 \sum_{i \in I} d_i \phi^i](\dot{\Phi}^{(a,I)})^2. \tag{B.13}$$

It should be noted, that the key ingredient in verification of formulas (B.10) and (B.12) (and some other formulas from Section 2) is the following obvious relation $g^{0,\mu\nu} \partial_\mu \Phi \partial_\nu \Phi' = 0$ for 2-metric $g^0 = du \otimes dv + dv \otimes du$ and functions $\Phi = \Phi(u)$, $\Phi' = \Phi'(u)$.

It follows from (B.8), (B.12) and (B.13) that the only non-zero (identically) components of $Z$-tensor (2.7) read as follows

$$Z_{uu} = h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta + \sum_{a \in \Delta} \theta_a \sum_{I \in \Omega_a} \varepsilon(I) \exp[2\lambda_a(\varphi) - 2 \sum_{i \in I} d_i \phi^i](\dot{\Phi}^{(a,I)})^2. \tag{B.14}$$

This relation coincides with the right hand side of the equation (3.6).

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