Noncommutative Residue and sub-Dirac Operators for Foliations

Jian Wang, Yong Wang*

School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.China

Abstract

In this paper, we define lower dimensional volumes associated to sub-Dirac operators for foliations. In some cases, we compute these lower dimensional volumes. We also prove the Kastler-Kalau-Walze type theorems for foliations with or without boundary. As a corollary, we give an explanation of the gravitational action for the Robertson-Walker space \([a,b] \times_f M^3\).

Keywords: Lower-dimensional volumes; Noncommutative residue; sub-Dirac Operators; Foliations.

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1. Introduction

The noncommutative residue, found in [1], [2] and [3], plays a prominent role in noncommutative geometry. In [4], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Moreover, in [5], Connes proved that the noncommutative residue on a compact manifold \(M\) coincided with the Dixmier’s trace on pseudodifferential operators of order \(-\dim M\). Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which was called Kastler-Kalau-Walze Theorem now. Kastler gave a brute-force proof of this theorem [6]. In [7], Kalau and Walze also gave a proof of this theorem by using normal coordinates. In [8], Ponge explained how to define “lower dimensional” volumes of any compact Riemannian manifold as the integrals of local Riemannian invariants and dealt with the lower dimensional volumes in even dimension. For spin manifolds with boundary and the associated Dirac operators, Wang defined and computed lower dimensional volumes and got a Kastler-Kalau-Walze type theorem in [9], [10] and [11]. In [12], Liu and Wang derived a Kastler-Kalau-Walze theorem for foliations. In [13], we got a Kastler-Kalau-Walze type theorem associated to nonminimal operators by heat equation asymptotics on compact manifolds without boundary.

The warped product \([a,b] \times_f M^3\) with the metric \(dt^2 + f(t)^2 g_{TM}\) is an important space in physics. Here \(M\) maybe is not spin. One of the motivations is to give a Kastler-Kalau-Walze type theorem for this manifold with boundary. We note that \([a,b] \times_f M^3\) is a special foliation with spin leave \([a,b]\). Since \([a,b] \times_f M^3\) is not spin, we consider sub-Dirac operators for foliations with spin leave instead of Dirac operators. In this paper, we define lower dimensional volumes associated to sub-Dirac operators for foliations. In some cases, we compute these lower dimensional volumes. We also prove the Kastler-Kalau-Walze type theorems for foliations with or without boundary. As a corollary, we give an explanation of the gravitational action for the Robertson-Walker space \([a,b] \times_f M^3\).

This paper is organized as follows: In Section 2, we recall the sub-Dirac operators and define the lower dimensional volumes associated to sub-Dirac operators for foliation with spin leave. In Section 3, for 4-dimensional compact foliations with boundary and the associated sub-Dirac operators, we compute the lower dimensional volumes \(Vol^{(1,1)}_{4}\), \(Vol^{(1,1)}_{5}\) and get the Kastler-Kalau-Walze type theorems in these

*Corresponding author. Email address: wangy581@nenu.edu.cn (Yong Wang)
Email address: wangj068@gmail.com (Jian Wang)

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case. In Section 4, we compute the lower dimensional volume \( V_{\text{dir}}^{(2,2)} \) associated to sub-Dirac operators for foliations. In section 5, we compute the lower dimensional volume \( V_{\text{dir}}^{(2,5)} \) associated to sub-Dirac operators for foliations. In section 6, we compute the lower dimensional volumes and the spectral action for the Robertson-Walker space \([a, b] \times_f M^3\).

2. Lower-Dimensional Volumes associated to sub-Dirac operators for Foliations

In this section, we shall restrict our attention to the sub-Dirac operators for foliations. Let \((M, F)\) be a closed foliation and \(M\) has spin leave, \(g^F\) be a metric on \(F\). Let \(g^{TM}\) be a metric on \(TM\) which restricted to \(g^F\) on \(F\). Let \(F^\perp\) be the orthogonal complement of \(F\) in \(TM\) with respect to \(g^{TM}\). Then we have the following orthogonal splitting

\[
TM = F \oplus F^\perp,
\]
\[
g^{TM} = g^F \oplus g^{F^\perp},
\]
(2.1)

where \(g^{F^\perp}\) is the restriction of \(g^{TM}\) to \(F^\perp\).

Let \(P, P^\perp\) be the orthogonal projection from \(TM\) to \(F, F^\perp\) respectively. Let \(\nabla^{TM}\) be the Levi-Civita connection of \(g^{TM}\) and \(\nabla^F\) (resp. \(\nabla^{F^\perp}\)) be the restriction of \(\nabla^{TM}\) to \(F\) (resp. \(F^\perp\)). Without loss of generality, we assume \(F\) is oriented, spin and carries a fixed spin structure. Furthermore, we assume \(F^\perp\) is oriented and we do not assume that \(\text{dim}F\) and \(\text{dim}F^\perp\) are even. By assumption, we may write

\[
\nabla^F = P\nabla^{TM} P,
\]
\[
\nabla^{F^\perp} = P^\perp\nabla^{TM} P^\perp.
\]
(2.2)

Let \(S(F)\) be the bundle of spinors associated to \((F, g^F)\). For any \(X \in \Gamma(F)\), denote by \(c(X)\) the Clifford action of \(X\) on \(S(F)\). The exterior algebra bundle of \(F^\perp\) is defined by \(\wedge(F^\perp, \ast)\). Then \(\wedge(F^\perp, \ast)\) carries a canonically induced metric \(g^{\wedge(F^\perp, \ast)}\) from \(g^{F^\perp}\). For any \(U \in \Gamma(F^\perp)\), let \(U^* \in \Gamma(F^\perp, \ast)\) be the corresponding dual of \(U\) with respect to \(g^{F^\perp}\). The Clifford action of \(U\) is defined by

\[
(c(U)U^*) = U^* \wedge -i_U,
\]
\[
\tilde{c}(U) = U^* \wedge +i_U,
\]
(2.3)

where \(U^*\wedge\) and \(i_U\) are the exterior and inner multiplications.

Let \(S(F) \otimes \wedge(F^\perp, \ast)\) be the tensor product of \(S(F)\) and \(\wedge(F^\perp, \ast)\). For \(X \in \Gamma(F), U \in \Gamma(F^\perp)\), the operators \(c(X), c(U)\) and \(\tilde{c}(U)\) are anticomute which extend naturally to \(S(F) \otimes \wedge(F^\perp, \ast)\). For \(s_1 \in S(F)\) and \(s_2 \in \wedge(F^\perp, \ast)\), we assume that

\[
(c(X)c(U))(s_1 \otimes s_2) = c(X)s_1 \otimes c(U)s_2;
\]
\[
(c(U)c(X))(s_1 \otimes s_2) = -c(X)s_1 \otimes c(U)s_2.
\]
(2.4)

Moreover, the connections \(\nabla^F (\nabla^{F^\perp})\) lift to \(S(F) (\wedge(F^\perp, \ast))\) naturally denoted by \(\nabla^{S(F)} (\nabla^{\wedge(F^\perp, \ast)})\) respectively. Then \(S(F) \otimes \wedge(F^\perp, \ast)\) carries the induced tensor product connection

\[
\nabla^{S(F) \otimes \wedge(F^\perp, \ast)} = \nabla^{S(F)} \otimes \text{Id}_{\wedge(F^\perp, \ast)} + \text{Id}_{S(F)} \otimes \nabla^{\wedge(F^\perp, \ast)}.
\]
(2.5)

Then we can define \(S \in \Omega(T^*M) \otimes \Gamma(\text{End}(TM))\)

\[
\nabla^{TM} = \nabla^F + \nabla^{F^\perp} + S.
\]
(2.6)

For any \(X \in \Gamma(TM)\), \(S(X)\) exchanges \(\Gamma(F)\) and \(\Gamma(F^\perp)\) and is skew-adjoint with respect to \(g^{TM}\). Let \(\{f_i\}_{i=1}^p\) be an oriented orthonormal basis of \(F\), we define

\[
\tilde{\nabla}^F = \nabla^{S(F) \otimes \wedge(F^\perp, \ast)} + \frac{1}{2} \sum_{j=1}^p \sum_{s=1}^q < S(.)f_j, h_s > c(f_j)c(h_s).
\]
(2.7)
where the vector bundle $F^\perp$ might well be non-spin. An application of definition 2.2 in [14] shows that the following sub-Dirac operator.

**Definition 2.1.** Let $D_F$ be the operator mapping from $\Gamma(S(F) \otimes (F^{\perp})^*)$ to itself defined by

$$D_F = \sum_{i=1}^{p} c(f_i) \tilde{\nabla}^F_{f_i} + \sum_{s=1}^{q} c(h_s) \tilde{\nabla}^F_{h_s}. \quad (2.8)$$

From (2.19) in [14], we shall make use of the Bochner Laplacian $\triangle^F$ stating that

$$\triangle^F := -\sum_{i=1}^{p} (\tilde{\nabla}^F_{f_i})^2 - \sum_{s=1}^{q} (\tilde{\nabla}^F_{h_s})^2 + \tilde{\nabla}^F_{\sum_{i=1}^{p} \nabla^T_{f_i} f_i} + \tilde{\nabla}^F_{\sum_{s=1}^{q} \nabla^T_{h_s} h_s}. \quad (2.9)$$

Let $r_M$ be the scalar curvature of the metric $g^{TM}$. Let $R^F$ be the curvature tensor of $F^\perp$. From Theorem 2.3 in [14], we have the following Lichnerowicz formula for $D_F$.

**Theorem 2.2.** [14] The following identity holds

$$D_F^2 = \triangle^F + \frac{r_M}{4} + \frac{1}{4} \sum_{i=1}^{p} \sum_{s=1}^{q} \left< R^F (f_i, h_s) h_s, h_s \right> c(f_i) c(h_s) \tilde{c}(h_s) \tilde{c}(h_s)$$

$$+ \frac{1}{8} \sum_{i,j=1}^{p} \sum_{s,t=1}^{q} \left< R^F (f_i, f_j) h_t, h_s \right> c(f_i) c(f_j) \tilde{c}(h_s) \tilde{c}(h_t)$$

$$+ \frac{1}{8} \sum_{s,t,r,u=1}^{q} \left< R^F (h_r, h_t) h_s, h_s \right> c(h_r) c(h_u) \tilde{c}(h_u) \tilde{c}(h_t). \quad (2.10)$$

In order to get a Kastler-Kalau-Walze type theorem for foliations, Liu and Wang [12] considered the noncommutative residue of the $-n + 2$ power of the sub-Dirac operator, and got the following Kastler-Kalau-Walze type theorem for foliations.

**Theorem 2.3.** [12] Let $(M^n, F)$ be a compact even-dimensional oriented foliation with spin leave and codimension $q$, and $D_F$ be the sub-Dirac operator; then

$$\lim_{\varepsilon \to 0} \varepsilon \text{Res}(D_F^{n+2}) \text{ is proportional to } \int_M [k^F + \Phi(\omega)] d\text{vol}_g.$$  

Similarly, we have

**Theorem 2.4.** Let $(M^n, F)$ be a compact even-dimensional oriented foliation with spin leave and codimension $q$, and $D_F$ be the sub-Dirac operator, then

$$W_{\text{Res}}(D_F^{n+2}) = \tilde{c}_0 \int_M r_M d\text{vol}_g, \quad (2.11)$$

where $\tilde{c}_0 = -\frac{1}{6(\frac{n-2}{2})! \times (4\pi)^{\frac{n}{2}}} \dim[S(F) \otimes (F^{\perp})^*], \dim[S(F) \otimes (F^{\perp})^*] = 2\frac{n}{2} + q$ (resp., $2\frac{n}{2} - q$) when $p$ is even (resp., odd).

**Remark 2.5.** Let $(M^n, F)$ be a compact even-dimensional oriented foliation with spin leave and codimension $q$, and $D_F$ be the sub-Dirac operator. When $p = n$ and $q = 0$, then $D_F$ is the Dirac operator and we get the classical Kastler-Kalau-Walze theorem for the Dirac operators. When $p = 0$ and $q = n$, then $D_F$ is the de-Rham Hodge operator and we get the classical Kastler-Kalau-Walze theorem for the de-Rham Hodge operator.

Let us now consider the lower dimensional volumes of foliations. The lower dimensional volume of a compact Riemannian manifold $(M^n, g)$ without boundary was defined in [8]. Let $(M^n, F)$ be a compact oriented foliation with spin leave and $D_F$ be the associated sub-Dirac operator. Similarly to Proposition 2.3 and Proposition 3.2 in [8], the definition of the lower dimensional volumes for foliations is given as follows.
Definition 2.6. The lower dimensional volume of \((M^n, F)\) is defined by

\[
\text{Vol}^{(k)}_{(n,p)}(M, F) := W\text{res}(D_F^{-k}).
\] (2.12)

Proposition 2.7. Let \((M^n, F)\) be a compact foliation without boundary, then

1. \(\text{Vol}^{(k)}_{(n,p)}(M, F)\) vanishes when \(k\) is odd and \(n\) is even, or \(k\) is even and \(n\) is odd;
2. when \(k\) is even and \(n\) is even, we have

\[
\text{Vol}^{(k)}_{(n,p)}(M, F) = v_{n,k} \int_M a_{n-k} \text{d}v_g(x), \quad v_{n,k} = \frac{k}{n} (2\pi)^{\frac{\nu-n}{2}} \frac{\Gamma\left(\frac{n}{2} + 1\right)^k}{\Gamma\left(\frac{k}{2} + 1\right)};
\] (2.13)

3. when \(k\) is odd and \(n\) is odd, we have

\[
\text{Vol}^{(k)}_{(n,p)}(M, F) = v_{n,k} \int_M a_{n-k} \text{d}v_g(x), \quad v_{n,k} = \frac{n}{k} \frac{(2\pi)^{\frac{\nu-n}{2}} \Gamma\left(\frac{n}{2} + 1\right)^k}{\Gamma\left(\frac{k}{2} + 1\right)},
\] (2.14)

where \(a_{n-k}\) is a linear combination of complete contractions of weight \(n - k\) of covariant derivatives of the curvature tensor. The coefficients of this linear combination depend only on \(n - k\).

As a consequence we see that the lower dimensional volumes for foliation are integrals of local Riemannian invariants. The definition of the lower dimensional volumes for any foliation is defined as follows.

Definition 2.8. Let \((M^n, F)\) be a compact foliation. Then for \(k = 1, \ldots, n\), the \(k\)th dimensional volume of \((M^n, g)\) is:

1. If \(k\) is even and \(n\) is even, or \(k\) is odd and \(n\) is odd,

\[
\text{Vol}^{(k)}_{(n,p)}(M, F) = v_{n,k} \int M a_{n-k} \text{d}v_g(x);
\] (2.15)

2. If \(k\) is odd and \(n\) is even, or \(k\) is even and \(n\) is odd,

\[
\text{Vol}^{(k)}_{(n,p)}(M, F) = 0.
\] (2.16)

where \(a_{n-k}\) is the coefficient of \(t^{\frac{n-k}{2}}\) in the heat kernel asymptotics for sub-Dirac operator \(D_F^2\).

Now, we compute the lower dimension volumes for 4-dimension foliations. Hence from definition 2.8, we are going to compute the coefficients \(a_{n-k}\). From Theorem 4.1.6 in [15], we obtain the first three coefficients of the heat trace asymptotics

\[
a_0(D_F^2) = (4\pi)^{-\frac{n}{2}} \int_M \text{tr}(\text{Id}) \text{dvol},
\] (2.17)

\[
a_2(D_F^2) = (4\pi)^{-\frac{n}{2}} \int_M \text{tr}(r_M + 6E)/6 \text{dvol},
\] (2.18)

\[
a_4(D_F^2) = \frac{(4\pi)^{-\frac{n}{2}}}{360} \int_M \text{tr}[-12R_{ijkl} + 5R_{ijkl} R_{ijkl} - 2R_{ijkl} R_{ijkl} + 2R_{ijkl} R_{ijkl} - 60R_{ijkl} E + 180E^2 + 60E_{kk} + 30\Omega_{ij} \Omega_{ij}] \text{dvol},
\] (2.19)
Combining (2.27), (2.28) and (2.29), we obtain
\[ -E = \frac{r_M}{4} + W = \frac{r_M}{4} + \frac{1}{4} \sum_{i=1}^{2p} \sum_{s,t,l=1}^{q} \left\langle R^{F^+}(f_i, h_r)h_t, h_s \right\rangle c(f_i)c(h_r)\tilde{c}(h_t) \]
\[ + \frac{1}{8} \sum_{i,j=1}^{2p} \sum_{s,t,l=1}^{q} \left\langle R^{F^+}(f_i, f_j)h_t, h_s \right\rangle c(f_i)c(f_j)\tilde{c}(h_t) \]
\[ + \frac{1}{8} \sum_{s,t,r,l=1}^{q} \left\langle R^{F^+}(h_r, h_l)h_t, h_s \right\rangle c(h_r)c(h_l)\tilde{c}(h_t), \] (2.20)

and
\[ \Omega_{ij} = \tilde{\nabla}_{e_i} \nabla_{e_j} - \tilde{\nabla}_{e_j} \nabla_{e_i} - \tilde{\nabla}_{[e_i, e_j]}, \] (2.21)

with \( e_i \) is \( f_i \) or \( h_s \).

Since \( \dim[S(F) \otimes \Lambda(F^+, \ast)] = 2^p + q \) and \( n = 2p + q \), we have
\[ a_0(D^2_F) = \frac{1}{2^p \pi^{p+\frac{3}{2}}} \int_M \text{dvol.} \] (2.22)

We note that the trace of the odd degree operator is zero, thus cyclicity of the trace and Clifford relations yield
\[ \text{tr}(c(f_i)) = 0; \ \text{tr}(c(f_i)c(f_j)) = 0 \ \text{for} \ i \neq j; \]
\[ \text{tr}(c(h_r)c(h_l)\tilde{c}(h_t)) = 0, \ \text{for} \ r \neq l. \] (2.23)

Moreover
\[ \text{tr}E = -2^{p+q} \cdot \frac{r_M}{4}, \] (2.24)

and
\[ a_2(D^2_F) = -\frac{1}{12 \cdot 2^p \pi^{p+\frac{3}{2}}} \int_M r_M \text{dvol.} \] (2.25)

Let \( I_1, I_2, I_3 \) denote respectively the last three terms in (2.20). Hence
\[ \text{tr}(E^2) = \text{tr}\left(\frac{r_M^2}{16} + W^2\right) = \text{tr}\left(\frac{r_M^2}{16} + I_1^2 + I_2^2 + I_3^2\right), \] (2.26)
\[ \text{tr}(I_1^2) = \frac{1}{16} \sum_{i,i'=1}^{2p} \sum_{r,s',t,t'=1}^{q} \left\langle R^{F^+}(f_i, h_r)h_{t'}, h_s \right\rangle \left\langle R^{F^+}(f_{i'}, h_{r'})h_t, h_s \right\rangle \]
\[ \times \text{tr}[c(f_i)c(h_r)\tilde{c}(h_s)c(f_{i'})c(h_{r'})\tilde{c}(h_t)], \] (2.27)

Similar to (2.23), we have
\[ \text{tr}[c(f_i)c(h_r)\tilde{c}(h_s)c(h_{r'})\tilde{c}(h_t)] = -\delta_{r,t'}^t \delta_{s,t'}^s \text{tr}_{\Lambda(F^+, \ast)}[\tilde{c}(h_s)c(h_t)\tilde{c}(h_{r'})c(h_{r'})]. \] (2.28)

Considering \( t \neq s, \ t' \neq s' \), then
\[ \text{tr}_{\Lambda(F^+, \ast)}[\tilde{c}(h_s)c(h_t)\tilde{c}(h_{r'})c(h_{r'})] = (\delta_{r,t'}^t \delta_{s,t'}^s - \delta_{r,t}^s \delta_{s,t'}^s) \text{tr}(2^{p+q}). \] (2.29)

Combining (2.27), (2.28) and (2.29), we obtain
\[ \text{tr}(I_1^2) = \frac{2^{p+q} + 2p}{8} \sum_{i=1}^{2p} \sum_{s,t,r,l=1}^{q} \left\langle R^{F^+}(f_i, h_r)h_t, h_s \right\rangle^2. \] (2.30)
Similarly we have
\[
\text{tr}(I^2_2) = \frac{2^{p+q}}{16} \sum_{i,j=1}^{2p} \sum_{s,t=1}^{q} \left\langle R^{F^+}(f_i, f_j) h_t, h_s \right\rangle^2;
\] (2.31)
\[
\text{tr}(I^2_3) = \frac{2^{p+q}}{16} \sum_{s,t,r,l=1}^{q} \left\langle R^{F^+}(h_r, h_l) h_t, h_s \right\rangle^2.
\] (2.32)

Hence in this case
\[
\text{tr}E^2 = \frac{2^{p+q}}{16} \frac{2}{M} + \frac{2^{p+q}}{16} ||R^{F^+}||^2,
\] (2.33)
where
\[
||R^{F^+}||^2 = \frac{2}{2} \sum_{i=1, r,s,t=1}^{2p} \sum_{j=1}^{q} \left\langle R^{F^+}(f_i, h_r) h_t, h_s \right\rangle^2 + \frac{2}{2} \sum_{i,j=1}^{2p} \sum_{s,t=1}^{q} \left\langle R^{F^+}(f_i, f_j) h_t, h_s \right\rangle^2
\]
\[
+ \sum_{s,t,r,l=1}^{q} \left\langle R^{F^+}(h_r, h_l) h_t, h_s \right\rangle^2.
\] (2.34)

Let us now compute \(\text{tr}[\Omega_{ij} \Omega_{ij}]\) in a local coordinate \(\{e_1, e_2, e_3, e_4\}\). Without loss of generality, we assume \(M\) is spin and \(\nabla\) is the standard twisted connection on the twisted spinors bundle \(S(TM) \otimes S(F^+)\). A simple computation shows
\[
\Omega_{ij} = R^{S(TM)}(e_i, e_j) \otimes \text{Id}_{S(F^+)} + \text{Id}_{S(TM)} \otimes R^{S(F^+)}(e_i, e_j)
\]
\[
= -\frac{1}{4} R^M_{ijkl} c(e_k) c(e_l) \otimes \text{Id}_{S(F^+)} - \frac{1}{4} \text{Id}_{S(TM)} \otimes \left\langle R^{F^+}(e_i, e_j) h_s, h_t \right\rangle c(h_s) c(h_t).
\] (2.35)

Similar to the computations of (2.24), we have
\[
\text{tr}[\Omega_{ij} \Omega_{ij}] = -\frac{2^{p+q}}{8}(R^M_{ijkl} + ||R^{F^+}||^2).
\] (2.36)

Substituting into (2.21) and by the divergence theorem, we have
\[
a_4(D^2_F) = \frac{1}{360 \cdot 2^{p+q} \pi^3} \int_M \left( \frac{5}{4} R^2 - 2 R_{ijkl} R_{ijkl} - \frac{7}{4} R^2_{ijkl} + \frac{15}{2} ||R^{F^+}||^2 \right) \text{dvol}.
\] (2.37)

Then we obtain the lower dimensional volumes of foliations.

**Theorem 2.9.** Let \((M^n, F)\) be a compact \(n\)-dimensional oriented foliation with spin leave and codimension \(q\), and \(D_F\) be the sub-Dirac operator, then we have
\[
\text{Vol}^{(n-4)}_{(n, 2p)}(M, F) = \frac{\nu_{n,n-4}}{360 \cdot 2^{p+q} \pi^4} \int_M \left( \frac{5}{4} R^2 - 2 R_{ijkl} R_{ijkl} - \frac{7}{4} R^2_{ijkl} + \frac{15}{2} ||R^{F^+}||^2 \right) \text{dvol};
\] (2.38)
\[
\text{Vol}^{(n-2)}_{(n, 2p)}(M, F) = \frac{-\nu_{n,n-2}}{12 \cdot 2^{p+q} \pi^4} \int_M r_M \text{dvol};
\] (2.39)
\[
\text{Vol}^{(n)}_{(n, 2p)}(M, F) = \frac{-\nu_{n,n}}{2^{p+q} \pi^4} \int_M \text{dvol}.
\] (2.40)

**Remark 2.10.** In Theorem 2.9, we assume that \(\dim F = 2p\). When \(\dim F\) is odd, we can get the similar results.

3. A Kastler-Kalau-Walze Type Theorem for foliations with boundary

In this section, we compute the lower dimension volume for 4-dimensional foliations with boundary and get a Kastler-Kalau-Walze type formula in this case.
3.1. A Kastler-Kalau-Walze Type Theorem for 4-dimensional foliations with boundary

Let $M$ be a $n$-dimensional foliation with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2,$$

(3.1)

where $g^{\partial M}$ is the metric on $\partial M$; $h(x_n) \in C^\infty([0,1]) = \{g|[0,1]|g \in C^\infty((-\varepsilon,1))\}$ for some sufficiently small $\varepsilon > 0$ and satisfied $h(x_n) > 0$, $h(0) = 1$, where $x_n$ denotes the normal directional coordinate. Let $p_1, p_2$ be nonnegative integers and $p_1 + p_2 \leq n$. Then Wang pointed out the definition of lower volumes for compact connected manifolds with boundary in [14]. Similarly, we define

**Definition 3.1.** Lower-dimensional volumes of compact connected foliations with boundary are defined by

$$V_{\Omega}^{(p_1,p_2)}(M,F) := \widetilde{Wres}[\pi^+D_F^{-p_1} \circ \pi^+D_F^{-p_2}].$$

(3.2)

where $\widetilde{Wres}$ denotes the noncommutative residue for manifolds with boundary in [14].

Denote by $\sigma(D_F)$ the $l$-order symbol of an operator $D_F$. Similarly to (2.1.4)-(2.1.8) in [10], we obtain

$$\widetilde{Wres}[\pi^+D_F^{-p_1} \circ \pi^+D_F^{-p_2}] = \int_M \int_{[\xi]=1} \text{trace}_{S(F) \otimes (F \perp \cdot \cdot \cdot)}[\sigma(-n)(D_F^{-p_1} - p_2)](\xi)dx + \int_{\partial M} \Phi,$$

(3.3)

where

$$\Phi = \int_{[\xi']=1} \int_{-\infty}^{+\infty} \sum_{j=0}^{\infty} \frac{(-1)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\text{trace}_{S(F) \otimes (F \perp \cdot \cdot \cdot)}[\partial_{\xi'}^{\alpha} \partial_{\xi}^{p_1} \sigma(D_F^{-p_1})(x',0,\xi',\xi_n) \times \partial_{\xi}^{k} \partial_{\xi_n}^{l} \sigma(D_F^{-p_2})(x',0,\xi',\xi_n)]dx' \cdot dx,$$

(3.4)

and the sum is taken over $r - k + |\alpha| + l - j - 1 = -n$, $r \leq -p_1$, $l \leq -p_2$.

Since $[\sigma(-n)(D_F^{-p_1} - p_2)]|_M$ has the same expression with the case of without boundary in [10], so locally we can use Theorem 2.9 to compute the first term. Now let us give explicit formulas for the volume in dimension 4. Where $v_{4,2} = \frac{(2\pi)^{3}}{4} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{1}{2\pi\sqrt{2}}$. An application of Theorem 2.9 shows that

$$\int_M \int_{[\xi]=1} \text{trace}_{S(TM)}[\sigma(-4)(D_F^{-1})](\xi)dx = -\frac{1}{24\sqrt{2} \cdot 2 \pi \Gamma^{\frac{3}{2}} \Gamma^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}}} \int_M \text{vol}_M dx,$$

(3.5)

Therefore, we only need to compute $\int_{\partial M} \Phi$.

Let us now turn to compute the symbol expansion of $D_F^{-1}$. Recall the definition of the sub-Dirac operator $D_F$ in (2.12). Let $\nabla^TM$ denote the Levi-civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\varepsilon_1, \cdots, \varepsilon_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\text{\nabla}^TM(e_1, \cdots, e_n) = (\varepsilon_{1}, \cdots, \varepsilon_{n})(\omega_{s,t}).$$

(3.6)

Let $c(\varepsilon_i)$ denote the Clifford action. Let $g^{ij} = g(dx_i, dx_j)$ and

$$\nabla^T_{\partial M} \partial_j = \sum_k \Gamma^k_{ij} \partial_k; \quad \Gamma^k = g^{ij} \Gamma^k_{ij}.$$  

(3.7)

Let the cotangent vector $\xi = \sum \xi_j dx_j$ and $\xi^j = g^{ij} \xi_i$. We shall make use of the following convention on the ranges of indices, $i, j, k, l \in F$ and $s, t, r, u \in F^\perp$, and we shall agree that repeated indices are summed over

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the respective ranges. From (2.8) and (2.12), we obtain the sub-Dirac operator

\[
D_F = \sum_{i=1}^{p} c(f_i) \overline{\nabla} F_{f_i} + \sum_{s=1}^{q} c(h_s) \overline{\nabla} F_{h_s}
\]

\[
= \sum_{i=1}^{p} c(f_i) \left( \overline{\nabla}_{f_i} S(F) \otimes \mathrm{Id}_{F(F^\perp \perp)} + \mathrm{Id}_{S(F)} \otimes \nabla_{f_i} (F^\perp \perp) + \frac{1}{2} \sum_{i,j=1}^{p} \sum_{s=1}^{q} (S(f_i)f_j, h_s)c(f_j)c(h_s) \right)
\]

\[
+ \sum_{s=1}^{q} c(h_s) \left( \overline{\nabla}_{h_s} S(F) \otimes \mathrm{Id}_{F(F^\perp \perp)} + \mathrm{Id}_{S(F)} \otimes \nabla_{h_s} (F^\perp \perp) + \frac{1}{2} \sum_{s,t=1}^{q} \sum_{i=1}^{p} (S(h_s)h_t, f_j)c(h_t)c(f_j) \right)
\]

\[
= \left( \sum_{i}^{p} c(f_i)f_i - \frac{1}{4} \sum_{k,l} \omega_{k,l}(f_i)c(f_k)c(f_l) \right) \otimes \mathrm{Id}_{F(F^\perp \perp)}
\]

\[
+ \sum_{i}^{p} c(f_i) \left( f_i + \frac{1}{4} \sum_{r,t} \omega_{r,t}(f_i)[\tilde{e}(h_r)\tilde{e}(h_t) - c(h_r)c(h_t)] \right)
\]

\[
+ \left( h_s - \frac{1}{4} \sum_{k,l} \omega_{k,l}(h_s)c(f_k)c(f_l) \right) \otimes \sum_{s=1}^{q} c(h_s)
\]

\[
+ \mathrm{Id}_{S(F)} \otimes \sum_{s=1}^{q} c(h_s) \left( h_s + \frac{1}{4} \sum_{r,t} \omega_{r,t}(h_s)[\tilde{e}(h_r)\tilde{e}(h_t) - c(h_r)c(h_t)] \right)
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{p} \sum_{s=1}^{q} (\nabla_{f_i}^{TM} f_j, h_s)c(f_i)c(f_j)c(h_s) + \frac{1}{2} \sum_{s,t=1}^{q} \sum_{i=1}^{p} (\nabla_{h_s}^{TM} h_t, f_i)c(h_s)c(h_t)c(f_i)
\]

\[
= \sum_{i}^{p} c(f_i)f_i + \sum_{s=1}^{q} c(h_s)h_s - \frac{1}{4} \sum_{i,k,l} \omega_{k,l}(f_i)c(f_k)c(f_l) \otimes \mathrm{Id}_{F(F^\perp \perp)}
\]

\[
- \frac{1}{4} \sum_{s,k,l} \omega_{k,l}(f_i)c(f_k)c(f_l) \otimes \mathrm{Id}_{F(F^\perp \perp)}
\]

\[
+ \mathrm{Id}_{S(F)} \otimes \sum_{s=1}^{q} c(h_s) \left[ \tilde{e}(h_s)[\tilde{e}(h_s) - c(h_s)] \right)
\]

\[
+ \mathrm{Id}_{S(F)} \otimes \sum_{s=1}^{q} c(h_s) \left[ \tilde{e}(h_s)[\tilde{e}(h_s) - c(h_s)] \right)
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{p} \sum_{s=1}^{q} (\nabla_{f_i}^{TM} f_j, h_s)c(f_i)c(f_j)c(h_s) + \frac{1}{2} \sum_{s,t=1}^{q} \sum_{i=1}^{p} (\nabla_{h_s}^{TM} h_t, f_i)c(h_s)c(h_t)c(f_i).
\]
Then from (3.8), we have
\[\begin{align*}
\sigma_1(D_F) &= \sqrt{-1}c(\xi), \\
\sigma_0(D_F) &= -\frac{1}{4} \sum_{i,k,l} \omega_{k,l}(f_i)c(f_i)c(f_k)c(f_l) \otimes \text{Id}_{\Lambda(F^\perp \cdot)} \\
&\quad - \frac{1}{4} \sum_{i,k,l} \omega_{k,l}(f_i)c(f_k)c(f_l)c(h_s) \otimes \text{Id}_{\Lambda(F^\perp \cdot)} \\
&\quad + \text{Id}_{S(F)} \otimes \frac{1}{4} \sum_{i,k,l} \omega_{k,l}(f_i)c(f_i)[c(h_r)c(h_t) - c(h_r)c(h_t)] \\
&\quad + \text{Id}_{S(F)} \otimes \frac{1}{4} \sum_{i,k,l} \omega_{k,l}(h_s)c(h_s)[c(h_r)c(h_t) - c(h_r)c(h_t)] \\
&\quad + \frac{1}{2} \sum_{i,j=1}^p \sum_{s,t=1}^q (\nabla^TM f_i h_s c(f_i) c(f_j) c(h_s) + \frac{1}{2} \sum_{i,j=1}^p \sum_{s,t=1}^q (\nabla^TM h_t f_i c(h_s) c(h_t) c(f_i)).
\end{align*}\] (3.10)

By Lemma 1 in [10] and Lemma 2.1 in [10], for any fixed point \(x_0 \in \partial M\), we can choose the normal coordinates \(U\) of \(x_0 \in \partial M\) (not in \(M\)). By the composition formula and (2.2.11) in [10], we obtain

**Lemma 3.2.** Let \(D_F\) be the sub-Dirac operator associated to \(g\) on \(\Gamma(S(F) \otimes \Lambda(F^\perp \cdot))\). Then
\[\begin{align*}
\sigma_1(D_F^{-1}) &= \sqrt{-1}c(\xi), \\
\sigma_0(D_F^{-1}) &= \frac{c(\xi)p_0(\xi)}{\|\xi\|^4} + \frac{c(\xi)}{\|\xi\|^6} \sum_j c(dx_j)[\partial_{x_j}[c(\xi)]|\xi|^2 - c(\xi)\partial_{x_j}[|\xi|^2]],
\end{align*}\] (3.11) (3.12)

where \(p_0 = \sigma_0(D_F)\).

As in [10], we take normal coordinates in boundary and we get orthonormal frames \(\{\bar{e}_1, \ldots, \bar{e}_{n-1}\}\). Note that \(\bar{e}_i\) is not \(f_i\) or \(h_s\) in general. We assume \(dx_n = h^*_n \in \Gamma(F^\perp \cdot)\). When \(dx_n\) is not in \(\Gamma(F^\perp \cdot)\), we can prove it in a similar way.

Let us now turn to compute \(\Phi\) (see formula (3.4) for definition of \(\Phi\)). Since the sum is taken over \(-r - \ell + k + j + |\alpha| = 3\), \(r, \ell \leq -1\), then we have the following five cases:

**case a) I) \(r = -1, \ell = -1, k = j = 0, |\alpha| = 1\)**

From (3.4) we have
\[\begin{align*}
\text{case a) I) } &= - \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha| = 1} \text{trace}[\partial_{x_0}^2 \bar{x}_0^{-1} \sigma_1(D_F^{-1}) \times \partial_{x_0}^2 \bar{x}_0^{-1} \sigma_1(D_F^{-1})](x_0) d\bar{x}_0 \sigma(\xi') dx', \\
\end{align*}\] (3.13)

By Lemma 2.2 in [10], for \(i < n\), then
\[\begin{align*}
\partial_{x_i} \sigma_1(D_F^{-1})(x_0) &= \partial_{x_i} \left(\frac{-\text{Id}_{S(F)} c(\xi)}{|\xi|^2}\right)(x_0) = \frac{\sqrt{-1} \text{Id}_{S(F)} c(\xi)(x_0)}{|\xi|^2} - \frac{\sqrt{-1} \text{Id}_{S(F)} c(\xi)(|\xi|^2)(x_0)}{|\xi|^4} = 0,
\end{align*}\] (3.14)

so **case a) I)** vanishes.

**case a) II) \(r = -1, \ell = -1, k = |\alpha| = 0, j = 1\)**

From (3.4) we have
\[\begin{align*}
\text{case a) II) } &= - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha| = 1} \text{trace}[\partial_{x_0} \bar{x}_0^{-1} \sigma_1(D_F^{-1}) \times \partial_{x_0} \bar{x}_0^{-1} \sigma_1(D_F^{-1})](x_0) d\bar{x}_0 \sigma(\xi') dx', \label{eq:3.15}
\end{align*}\] (3.15)
From equation (3.16), (3.20) and (3.21), one sees that

\[ \partial_{\xi_n} \sigma_1(D_F^{-1}) = \sqrt{-1} \left( -\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right); \quad (3.16) \]

and

\[ \partial_{x_n} \sigma_1(D_F^{-1})(x_0) = \frac{\sqrt{-1} \partial_{x_n} c(\xi')(x_0)}{|\xi|^2} - \frac{\sqrt{-1} c(\xi')^2 h'(0)}{|\xi|^4}. \quad (3.17) \]

By (2.1) in [10] and the Cauchy integral formula, we obtain

\[ \pi^+_{\xi_n} \left[ \frac{c(\xi)}{|\xi|^2} \right](x_0) |_{|\xi'|=1} = \pi^+_{\xi_n} \left[ \frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right] = \frac{1}{2\pi i} \lim_{\eta_n \to 0} \int_{\Gamma^+} \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^2 (\xi_n - \eta_n)^2} d\eta_n = \left[ \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^4 (\xi_n - \eta_n)^3} \right] |_{\eta_n = i} = \frac{ic(\xi')}{4(\xi_n - i)} \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}. \quad (3.18) \]

Similarly,

\[ \pi^+_{\xi_n} \left[ \sqrt{-1} \partial_{x_n} c(\xi') \right] (x_0) |_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)}; \quad (3.19) \]

Combining (3.17), (3.18) and (3.19), we have

\[ \pi^+_{\xi_n} \partial_{x_n} \sigma_1(D_F^{-1})(x_0) |_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1} h'(0) \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]. \quad (3.20) \]

By the relation of the Clifford action and trAB = trBA, we have the equalities:

\[ \text{tr}[c(\xi')c(dx_n)] = 0; \quad \text{tr}[c(dx_n)^2] = -8; \quad \text{tr}[c(\xi')^2](x_0) |_{|\xi'|=1} = -8; \]

\[ \text{tr}[\partial_{x_n} c(\xi')c(dx_n)] = 0; \quad \text{tr}[\partial_{x_n} c(\xi')c(\xi')(x_0)] |_{|\xi'|=1} = -4h'(0). \quad (3.21) \]

From equation (3.16), (3.20) and (3.21), one sees that

\[ h'(0) \text{tr} \left\{ \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right] \times \left[ \frac{6\xi_n c(dx_n) + 2c(\xi')}{(1 + \xi_n^2)^2} - \frac{8\xi_n^2 c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^3} \right] \right\} (x_0) |_{|\xi'|=1} = -8h'(0) \frac{-2i\xi_n^2 - \xi_n + i}{(\xi_n - i)^4 (\xi_n + i)^3}. \quad (3.22) \]

Similarly, we have

\[ -\sqrt{-1} \text{tr} \left\{ \left[ \frac{\partial_{x_n} c(\xi')(x_0)}{2(\xi_n - i)} \right] \times \left[ \frac{6\xi_n c(dx_n) + 2c(\xi')}{(1 + \xi_n^2)^2} - \frac{8\xi_n^2 c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^3} \right] \right\} (x_0) |_{|\xi'|=1} = -4\sqrt{-1} h'(0) \frac{3\xi_n^2 - 1}{(\xi_n - i)^4 (\xi_n + i)^3}. \quad (3.23) \]
Combining (3.22) and (3.23), we obtain

\[ \begin{align*}
\text{case a) II} &= -\int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{2ih'(0)(\xi_n - i)^2}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\
&= -2ih'(0) \Omega_3 \int_{D_+} \frac{1}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n dx' \\
&= -2ih'(0) \Omega_3 2\pi i \left( \frac{1}{(\xi_n + i)^3} \right)_{|\xi_n = i} dx' \\
&= -\frac{3}{4\pi} h'(0) \Omega_3 dx'.
\end{align*} \]

(3.24)

\[ \text{case a) III} \quad r = -1, \quad l = -1, \quad j = |\alpha| = 0, \quad k = 1 \]

From (3.4) we have

\[ \begin{align*}
\text{case a) III} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \frac{\partial z_n}{\partial \xi_n} \pi_{\xi_n}^{+} \sigma_{-1}(D_{F}^{-1}) \times \partial \xi_n, \partial_{x_n} \sigma_{-1}(D_{F}^{-1}) \right] (x_0) dx' d\xi_n \sigma(\xi') dx',
\end{align*} \]

(3.25)

Then an application of Lemma 2.2 in \[10\] shows

\[ \begin{align*}
\partial_{\xi_n} \partial_{x_n} q_{-1}(x_0) |_{|\xi'|=1} &= -\sqrt{-1} h'(0) \left[ \frac{c(dx_n)}{|\xi|^4} - 4 \xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] - \frac{2\pi \sqrt{-1}}{\Omega_3} \partial_{\xi_n} c(\xi')(x_0),
\end{align*} \]

(3.26)

and

\[ \begin{align*}
\partial_{\xi_n} \pi_{\xi_n}^{+} q_{-1}(x_0) |_{|\xi'|=1} = \frac{c(\xi') + i c(dx_n)}{2(\xi_n - i)^2}.
\end{align*} \]

(3.27)

Similarly to (3.22) and (3.23), we have

\[ \begin{align*}
\text{tr} \left\{ \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \right\} & \times \sqrt{-1} h'(0) \left[ \frac{c(dx_n)}{|\xi|^4} - 4 \xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] \\
= 4h'(0) \left( \frac{i - 3\xi_n}{(\xi_n - i)^4(\xi_n + i)^3} \right).
\end{align*} \]

(3.28)

Therefore, case a) III = \( \frac{3}{4\pi} h'(0) \Omega_3 dx' \).

\[ \text{case b) } r = -2, \quad l = -1, \quad k = j = |\alpha| = 0 \]

From (3.4) we have

\[ \begin{align*}
\text{case b) } &= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^{+} \sigma_{-2}(D_{F}^{-1}) \times \partial \xi_n, \sigma_{-1}(D_{F}^{-1}) \right] (x_0) dx' d\xi_n \sigma(\xi') dx',
\end{align*} \]

(3.30)

By Lemma 2.1 and Lemma 2.2 in \[10\], we obtain

\[ \begin{align*}
\partial_{\xi_n} \sigma_{-1}(D_{F}^{-1})(x_0) |_{|\xi'|=1} &= \sqrt{-1} \left[ \frac{c(dx_n)}{1 + \xi_n^2} + 2 \xi_n c(\xi') + \frac{2c^2 c(dx_n)}{(1 + \xi_n^2)^2} \right] = \left( \frac{i - i\xi_n^2}{1 + \xi_n^2} \right) c(dx_n) - 2i \xi_n c(\xi'),
\end{align*} \]

(3.31)

and

\[ \sigma_{-2}(D_{F}^{-1})(x_0) = \frac{c(\xi_n) \rho_n(x_0) c(\xi)}{|\xi_n^4|} + \frac{c(\xi)}{|\xi|^6} c(dx_n) [\partial_{x_n} c(\xi')(x_0)] |\xi|^2 - c(\xi) h'(0) |\xi_n^3| \partial_{\alpha M}. \]

(3.32)
From (3.32), one sees that
\[
\pi^+_{\xi_n} \sigma_2(D_F^{-1})(x_0)|_{\xi'|=1} = \pi^+_{\xi_n} \left[ c(\xi)p_0(x_0) + c(\xi) c(dx_n) \partial_{x_n} [c(\xi')](x_0) \right] - h'(0) \pi^+_{\xi_n} \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right]
\]
\(\quad := B_1 - B_2,\) \hspace{1cm} (3.33)
where
\[
B_2 = h'(0) \pi^+_{\xi_n} \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right]
\]
\[
= h'(0) \pi^+_{\xi_n} \left[ \frac{-\xi_n^2 c(dx_n)^2 - 2\xi_n c(\xi') + c(dx_n)}{(1 + \xi_n^2)^3} \right] \big|_{\|h\|=i}
\]
\[
= \frac{h'(0)}{2} \left[ \frac{-\eta_n^2 c(dx_n) - 2\eta_n c(\xi') + c(dx_n)}{(\eta_n + i)^3(\xi_n - \eta_n)} \right] \big|_{\|h\|=i}
\]
\[
= \frac{h'(0)}{2} \left[ \frac{c(dx_n) + c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^2} [ic(\xi') - c(dx_n)] \right]. \hspace{1cm} (3.34)
\]
By (3.31) and (3.34), we have
\[
\text{tr} [B_2 \times \partial_{\xi_n} \sigma_1(D_F^{-1})(x_0)]|_{\xi'|=1} = \frac{\sqrt{-1}}{2} \frac{-\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2} \text{tr} [\sigma_1(F) \otimes (F^\perp \cdot)] [id], \hspace{1cm} (3.35)
\]
where \(\text{tr} [\sigma_1(F) \otimes (F^\perp \cdot)] [id] = 8.\) Hence
\[
\text{tr} [B_2 \times \partial_{\xi_n} \sigma_1(D_F^{-1})(x_0)]|_{\xi'|=1} = h'(0) \frac{\xi_n - \xi_n - 4}{(\xi_n - i)^2(\xi_n + i)^2}; \hspace{1cm} (3.36)
\]
Similarly to (3.18), we have
\[
B_1 = \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') p_0 c(\xi') + i\xi_n c(dx_n) p_0 c(dx_n) \right.
\]
\[
\left. + (2 + i\xi_n) c(\xi') c(dx_n) \partial_{x_n} c(\xi') + ic(dx_n) p_0 c(\xi') + ic(\xi') p_0 c(dx_n) - i\partial_{x_n} c(\xi') \right]
\]
\(\quad := C_1 + C_2,\) \hspace{1cm} (3.37)
where
\[
C_1 := \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') p_0 c(\xi') + i\xi_n c(dx_n) p_0 c(dx_n) + ic(dx_n) p_0 c(\xi') + ic(\xi') p_0 c(dx_n) \right], \hspace{1cm} (3.38)
\]
\[
C_2 := \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') c(dx_n) \partial_{x_n} c(\xi') - i\partial_{x_n} c(\xi') \right]. \hspace{1cm} (3.39)
\]
Combining (3.31) and (3.39), we have
\[
\text{tr} [C_2 \times \partial_{\xi_n} \sigma_1(D_F^{-1})(x_0)]|_{\xi'|=1} = h'(0) \frac{\xi_n^2 - \xi_n - 2}{(\xi_n - i)^2(\xi_n + i)^2}; \hspace{1cm} (3.40)
\]
On the other hand, let \(c(\xi') = \sum_{j=1}^{p} a_j c(f_j) + \sum_{j=1}^{q} b_j c(h_j) \quad (a_j^2 + b_j^2 = 1), \) \(c(dx_n) = c(h_q).\) By the trace identity \(\text{tr}(AB) = \text{tr}(BA), \) \(\text{tr}(A \otimes B) = \text{tr}(A) \cdot \text{tr}(B),\) and the relation of the Clifford action, we obtain
\[
\text{tr} [c(f_1) c(f_2) c(h_a) c(h_q)] = \text{tr}(c(f_1) c(f_2)) \cdot \text{tr}(c(h_a) c(h_q)) = \delta_{12}^q 2^{n-q}; \hspace{1cm} (3.41)
\]
and
\[
\text{tr}[c(h_s)c(h_t)c(h_i) - c(h_r)c(h_s)c(h_t)] = -\text{tr}[c(h_s)c(h_r)c(h_i)c(h_t)] = -(\delta_i^s\delta_j^r - \delta_j^s\delta_i^r)2^q.
\] 

Similarly,
\[
\text{tr}[c(f_s)c(f_t)c(f_i)c(f_j)] = (\delta_i^k\delta_j^l - \delta_j^k\delta_i^l)2^p;
\text{tr}[c(h_s)c(h_t)c(f_i)c(f_j)] = \delta_i^s\delta_j^t2^p+q;
\text{tr}[c(h_s)[c(h_r)c(h_t) - c(h_r)c(h_i)]c(h_u)] = -(\delta_i^s\delta_u^r - \delta_u^s\delta_i^r)2^q;
\text{tr}[c(f_i)c(f_t)c(h_s)c(h_u)] = \delta_i^s\delta_j^t2^p+q,
\]

the other is zero.

Combining (3.31), (3.38), and (3.41)-(3.43), we have
\[
\text{tr}[C_1 \times \partial_\xi \sigma_{-1}(D_\xi^{-1})(x_0)]|_{\xi' = 1}
= -\frac{1}{4(\xi_0 - i)^2(1 + \xi_0^2)} \left\{ \left( (2 + i\xi_0)c(\xi')p_0c(\xi') + i\xi_0c(dx_n)p_0c(dx_n) \right)
+ i(dx_n)p_0c(\xi') + ic(\xi')p_0c(dx_n) \right\} \{x_0\}|_{\xi' = 1}
= -\frac{1}{4(\xi_0 - i)^2(1 + \xi_0^2)} \left[ \left\{ -2(\xi_0^2 - 4\xi_0 + 2i)\text{tr}[p_0c(dx_n)] + (-2\xi_0^2 + 4i\xi_0 + 2)\text{tr}[p_0c(\xi')] \right\} \{x_0\}|_{\xi' = 1}
+ \frac{i}{2(\xi_0 - i)^2(\xi_0 + i)^2} \left[ \sum_{j=1}^q a_j \left( -\frac{1}{4} \sum_{i,j} \omega_{i,j}(h_s(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l)) + \frac{1}{2} \sum_{j=1}^q (\nabla^TM_{f_i}f_i, h_q) \right) \times \text{tr}_{(S(F) \otimes \Lambda(\xi'))}[\text{id}](x_0) \right] |_{\xi' = 1}
+ \frac{2i}{(\xi_0 - i)^2(\xi_0 + i)^2} \left[ \sum_{s,q} \omega_{s,q}(h_s) + \sum_{s=1}^p (\nabla^TM_{f_i}f_i, h_q) \right] \{x_0\}|_{\xi' = 1}
+ \frac{2}{(\xi_0 - i)^2(\xi_0 + i)^2} \left[ \sum_{j=1}^p a_j \sum_{j=1}^q \left( -\frac{1}{4} \sum_{i,j} \omega_{i,j}(h_s(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l)) + \frac{1}{2} \sum_{j=1}^q (\nabla^TM_{f_i}f_i, h_s) \right) \right] \{x_0\}|_{\xi' = 1}
\]

where we have used the fact that in normal coordinates, $\sum_{i=1}^{p+q} \langle \nabla^TM_{\xi_i}\xi_i, d\xi_n \rangle(x_0) = \left( \sum_{i=1}^{p+q} \langle \nabla^TM_{f_i}f_i, \xi_i \rangle + \sum_{i=1}^{p+q} \langle \nabla^TM_{s_i}s_i, \xi_i \rangle \right)(x_0) = 0$. 

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Combining (3.35), (3.40) and (3.44), we obtain

\[\text{case b) } = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \pi_{\xi_n}^{\perp} \sigma_{-2}(D_{F}^{-1}) \times \partial_{\xi_n} \sigma_{-1}(D_{F}^{-1}) (x_{0}) d\xi_{n} \sigma(\xi') dx'\]
\[= -i \int_{|\xi'|=1}^{+\infty} \text{trace} [(C_{1} + C_{2} - B_{2}) \times \partial_{\xi_n} \sigma_{-1}(D_{F}^{-1})] (x_{0}) d\xi_{n} \sigma(\xi') dx'\]
\[= -i \int_{|\xi'|=1}^{+\infty} \frac{-2ih'(0)}{(\xi_{n} - i)(\xi_{n} + i)^{2}} (x_{0}) d\xi_{n} \sigma(\xi') dx'\]
\[= \frac{3}{4} h'(0) \pi \Omega_{3} dx'. \quad (3.45)\]

\[\text{case c) } r = -1, \ l = -2, \ k = j = |\alpha| = 0\]

From (3.4) we have

\[\text{case c) } = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \pi_{\xi_n}^{\perp} \sigma_{-1}(D_{F}^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D_{F}^{-1}) (x_{0}) d\xi_{n} \sigma(\xi') dx'. \quad (3.46)\]

From (3.11) and (3.12), we have

\[\pi_{\xi_n}^{\perp} \sigma_{-1}(D_{F}^{-1}) (x_{0})|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_{n})}{2(\xi_{n} - i)}, \quad (3.47)\]

and

\[\partial_{\xi_n} \sigma_{-2}(D_{F}^{-1}) (x_{0})|_{|\xi'|=1}\]
\[= \frac{1}{(1 + \xi_{n}^{2}) \xi_{n}} \left[ (2\xi_{n} - 2\xi_{n}^{3}) c(dx_{n}) p_{0} c(dx_{n}) + (1 - 3\xi_{n}^{2}) c(dx_{n}) p_{0} c(\xi') \right.\]
\[\left. + (1 - 3\xi_{n}^{2}) c(\xi') p_{0} c(dx_{n}) - 4\xi_{n} c(\xi') p_{0} c(\xi') + (3\xi_{n}^{2} - 1) \partial_{x_n} c(\xi') - 4\xi_{n} c(\xi') c(dx_{n}) \partial_{x_n} c(\xi') \right.\]
\[\left. + 2h'(0) c(\xi') + 2h'(0) \xi_{n} c(dx_{n}) \right] + 6\xi_{n} h'(0) c(dx_{n}) c(\xi) \sigma(\xi'). \quad (3.48)\]

Similarly to (3.44), we obtain

\[\text{trace} \pi_{\xi_n}^{\perp} \sigma_{-1}(D_{F}^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D_{F}^{-1}) (x_{0})|_{|\xi'|=1} = h'(0) \frac{-6}{(\xi_{n} - i)^{3}(\xi_{n} + i)^{2}} + h'(0) \frac{24i \xi_{n}}{(\xi_{n} - i)^{3}(\xi_{n} + i)^{2}}. \quad (3.49)\]

Hence

\[\text{case c) } = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \pi_{\xi_n}^{\perp} \sigma_{-1}(D_{F}^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D_{F}^{-1}) (x_{0}) d\xi_{n} \sigma(\xi') dx'\]
\[= -i \int_{|\xi'|=1}^{+\infty} \left[ h'(0) \frac{-6}{(\xi_{n} - i)^{3}(\xi_{n} + i)^{2}} + h'(0) \frac{24i \xi_{n}}{(\xi_{n} - i)^{3}(\xi_{n} + i)^{2}} \right] (x_{0}) d\xi_{n} \sigma(\xi') dx'\]
\[= \int_{|\xi'|=1}^{+\infty} \left[ h'(0) \frac{6i}{(\xi_{n} - i)^{3}(\xi_{n} + i)^{2}} + h'(0) \frac{24i \xi_{n}}{(\xi_{n} - i)^{3}(\xi_{n} + i)^{2}} \right] (x_{0}) d\xi_{n} \sigma(\xi') dx'\]
\[= \frac{3}{4} h'(0) \pi \Omega_{3} dx'. \quad (3.50)\]

Since \(\Phi\) is the sum of the case a, b and c, so is zero. Then we have
Theorem 3.3. Let $M$ be a 4-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^M$ as above, and $D_F$ be the sub-Dirac operator on $\Gamma(S(F) \otimes (F^\perp \cdot))$, then

$$Vol_4^{(1,1)}(M) = -\frac{1}{24\sqrt{2} \cdot 2^p \pi^{p + \frac{1}{2}} + 1} \int_M r_M \text{d}vol. \tag{3.51}$$

where $r_M$ be the scalar curvature of the foliation.

Remark 3.4. Let $M$ be a 4-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^M$ as above, and $D_F$ be the sub-Dirac operator. When $n = p$, we get Theorem 2.5 in [10]. When $n = q$, we get Theorem 3.1 in [10].

3.2. The gravitational action for 4-dimensional foliation with boundary

Firstly, we recall the Einstein-Hilbert action for manifolds with boundary in [10],

$$I_{Gr} = \frac{1}{16\pi} \int_M sdvol_M + 2\int_{\partial M} Kdvol_{\partial M} := I_{Gr,i} + I_{Gr,b}, \tag{3.52}$$

where

$$K = \sum_{1 \leq i,j \leq n - 1} K_{i,j}g^M_{i,j}; \quad K_{i,j} = -\Gamma^n_{i,j}, \tag{3.53}$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2 in [10], $K_{i,j}(x_0) = -\Gamma^n_{i,j}(x_0) = -\frac{1}{2}h'i(0)$, when $i = j < n$, otherwise is zero. For $n = 4$, we obtain

$$K(x_0) = \sum_{i,j} K_{i,j}(x_0)g^M_{i,j}(x_0) = \sum_{i=1}^3 K_{i,i}(x_0) = -\frac{3}{2}h'(0). \tag{3.54}$$

So

$$I_{Gr,b} = -3h'(0)Vol_{\partial M}. \tag{3.55}$$

Let $M$ be 4-dimensional foliation with boundary and $P, P'$ be two pseudodifferential operators with transmission property (see [10] on $M$. From (4.4) in [10], we have

$$\pi^+ P \circ \pi^+ P' = \pi^+ (PP') + L(P, P') \tag{3.56}$$

and $L(P, P')$ is leftover term which represents the difference between the composition $\pi^+ P \circ \pi^+ P'$ in Boutet de Monvel algebra and the composition $PP'$ in the classical pseudodifferential operators algebra. By (3.4), we define locally

$$\text{res}_{1,1}(P, P') := -\frac{1}{2} \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi^+ \sigma_{-1}(P) \times \partial_{\xi_n} \sigma_{-1}(P')]d\xi_n \sigma(\xi')dx'; \tag{3.57}$$

$$\text{res}_{2,1}(P, P') := -i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\pi^+ \sigma_{-1}(P) \times \partial_{\xi_n} \sigma_{-1}(P')]d\xi_n \sigma(\xi')dx'. \tag{3.58}$$

Hence, they represent the difference between the composition $\pi^+ P \circ \pi^+ P'$ in Boutet de Monvel algebra and the composition $PP'$ in the classical pseudodifferential operators algebra partially. Then

$$\text{case a) II} = \text{res}_{1,1}(D_F^{-1}, D_F^{-1}); \quad \text{case b) } = \text{res}_{2,1}(D_F^{-1}, D_F^{-1}). \tag{3.59}$$

Now, we assume $\partial M$ is flat, then $\{dx_i = e_i\}$, $g^M_{i,j} = \delta_{i,j}$, $\partial_{x_i} g^M_{i,j} = 0$. So $\text{res}_{1,1}(D_F^{-1}, D_F^{-1})$ and $\text{res}_{2,1}(D_F^{-1}, D_F^{-1})$ are two global forms locally defined by the above oriented orthonormal basis $\{dx_i\}$. From case a) II and case b), we have:
Theorem 3.5. Let M be a 4-dimensional flat compact connected foliation with the boundary ∂M and the metric $g^M$ as above, and $D_F$ the sub-Dirac operator on $\Gamma(S(F) \otimes (F^{*,*}))$, then

$$\int_{\partial M} \text{res}_{1,1}(D_F^{-1}, D_F^{-1}) = \frac{\pi}{4} \Omega_3 \text{Gr}_{b};$$

(3.60)

$$\int_{\partial M} \text{res}_{2,1}(D_F^{-1}, D_F^{-1}) = -\frac{1}{4} \pi \Omega_3 \text{Gr}_{b}.$$  

(3.61)

3.3. Computations of $\overline{\text{Wres}}[(\pi + D_F^{-1})^2]$ for 3-dimensional Foliations

For an odd dimensional manifolds with boundary, as in Theorem 2.9 and (3.3), we have the formula

$$\overline{\text{Wres}}[(\pi + D_F^{-1})^2] = \int_{\partial M} \Phi.$$  

(3.62)

From (3.4), when $n = 3$, $r - k - |\alpha| + l - j - 1 = -3$, $r, l \leq -1$, so we get $r = l = -1$, $k = |\alpha| = j = 0$, then

$$\Phi = \int_{\xi' = 1}^{+\infty} \int_{\xi' = -\infty}^{\xi'} \text{tr}[s_{(TM)}]s_{-1}^{+}(D_F^{-1})(x', 0, \xi', \xi_n) \times \partial_{\xi_n} s_{-1}(D_F^{-1})(x', 0, \xi', \xi_n)]d\xi_3 s(\xi')dx'.$$

(3.63)

Similar to (3.20), by Lemma 3.2, we have

$$\sigma_{-1}^{+}(D^{-1})|_{\xi' = 1} = \frac{\sqrt{-1}\xi'(\xi') + ic(dx_n)}{2i(\xi_n - i)};$$

(3.64)

and

$$\partial_{\xi_n} s_{-1}(D^{-1})|_{\xi' = 1} = \frac{\sqrt{-1}c(dx_n)}{1 + \xi_n^2} - \frac{2\sqrt{-1}c(\xi)}{(1 + \xi_n^2)^2}.$$  

(3.65)

We take the coordinates as in Section 3. Locally $S(TM)|_B \cong \tilde{U} \times \Lambda_{G^{even}}^2(2)$. Let $\{f_1, f_2\}$ be an orthonormal basis of $\Lambda_{G^{even}}^2(2)$ and we will compute the trace under this basis. Similarly to (3.21), we have

$$\text{tr}[c(\xi')c(dx_n)] = 0; \quad \text{tr}[c(dx_n)^2] = -4; \quad \text{tr}[c(\xi')^2](x_0)|_{\xi' = 1} = -4.$$  

(3.66)

Combining (3.3) (3.4) and (3.5), we have

$$\text{tr}[s_{-1}^{+}(D^{-1}) \times \partial_{\xi_n} s_{-2}(D^{-1})](x_0)|_{\xi' = 1} = \frac{1}{(\xi_n + i)(\xi_n - i)}.$$  

(3.67)

By (4.3), (4.6) and the Cauchy integral formula, we obtain

$$\Phi = i\pi \Omega_2 \text{vol}_{\partial M} = 2i\pi^2 \text{vol}_{\partial M},$$  

(3.68)

where vol$_{\partial M}$ denotes the canonical volume form of $\partial M$.

Theorem 3.6. Let M be a 3-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^M$ as above, and $D_F$ the sub-Dirac operator on $\Gamma(S(F) \otimes (F^{*,*}))$, then

$$\overline{\text{Wres}}[(\pi + D^{-1})^2] = 2i\pi^2 \text{Vol}_{\partial M},$$  

(3.69)

where Vol$_{\partial M}$ denotes the canonical volume of $\partial M$.

4. A Kastler-Kalau-Walze Type Theorem for 6-dimensional foliations with boundary

In this section, We compute the lower dimensional volume Vol$_{\partial}^{(2,2)}$ for 6-dimensional foliations with boundary and get a Kastler-Kalau-Walze type theorem in this case.
4.1. A Kastler-Kalau-Walze Type Theorem for 6-dimensional foliations with boundary

Since $|σ_n(D_F^{−p_1−p_2})|_M$ has the same expression with the case of without boundary in \[10\], so locally we can use Theorem 2.4 to compute the first term. Now let us give explicit formulas for the volume in dimension 6. Let $\text{dim}(S(F)) = l$, then

$$\int_M \int_{|ξ| = 1} \text{trace}_{S(TM)}[σ_6(D_F^{2−2})]σ(ξ)dx = -\frac{l \times 2^q}{6 \times (4\pi)^l} \int_M r_M d\text{vol}_g. \tag{4.1}$$

Hence, we only need to compute $\int_{\partial M} \Phi$.

Firstly, we give the symbol expansion of $D_F^{−2}$. Recall the definition of the Dirac operator $D_F$. Let $\nabla^TM$ denote the Levi-civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, the connection matrix $(ω_{s,t})$ is defined by

$$\nabla^TM(\tilde{e}_1, \cdots, \tilde{e}_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)(ω_{s,t}). \tag{4.2}$$

Let $c(\tilde{e}_i)$ denotes the Clifford action. Let $g^{ij} = g(dx_i, dx_j)$ and

$$\nabla^TM_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k; \quad Γ^k = g^{ij}Γ^k_{ij}. \tag{4.3}$$

Let the cotangent vector $ξ = \sum ξ_i dx_i$ and $ξ^i = g^{ij}ξ_i$. By the composition formula of pseudodifferential operators in \[13\] and direct computations, we obtain

**Lemma 4.1.** Let $D_F$ be the sub-Dirac operator associated to $g$ on $Γ(S(F) \otimes (F^{1,*}))$. Then

$$σ_{−2}(D_F^{−2}) = |ξ|^{−2}; \tag{4.4}$$

$$σ_{−3}(D_F^{−2}) = -\sqrt{−1}|ξ|^{−4}ξ_k(Γ^k − 2α^k \otimes \text{Id}_{S(F)} \otimes 2\bar{s}^k) - \frac{1}{2}q\sum_{j=1}^q (\nabla^TM_{\partial_k} f_j, h_s)c(f_j)c(h_s)$$

$$- \frac{1}{2}q\sum_{j=1}^q (∑_{s,t=1}^2 (\nabla^TM_{\partial_k} f_j, h_s)c(f_j)c(h_s))$$

$$- √(−1)|ξ|^{−6}2ξ^i ξ^j ξ^k \partial_i \partial_j g^{ij}. \tag{4.5}$$

where $σ_k = -\frac{1}{2}q\sum_k ω_k, i(∂_k)c(f_k)c(f_i)$, $σ_k = \frac{1}{2}q\sum_k ω_{r,t}(∂_k)[c(h_r)c(h_t) − c(h_t)c(h_r)].$

**Proof.** In the fixed orthonormal frame $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, the Bochner Laplacian $Δ^F$ stating that in \[18\],

$$Δ^S(F)\otimes(F^{1,*}) = -\sum_{i,j} g^{ij}(x)\left(\nabla^S(F)\otimes(F^{1,*})\nabla^S(F)\otimes(F^{1,*}) - \sum_k Γ^k_{ij} \nabla^S(F)\otimes(F^{1,*})\right). \tag{4.6}$$

Let $σ_i = -\frac{1}{2}q\sum_k ω_k, i(∂_k)c(f_k)$, $σ_i = \frac{1}{2}q\sum_k ω_{r,t}(∂_k)[c(h_r)c(h_t) − c(h_t)c(h_r)]$ and $σ^i = g^{ij}σ_j$. 

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Combining (2.5), (2.10) and (4.6), we have
\[
D_F^2 = -\sum_{i,j} g^{ij}(x) \left\{ \left[ \partial_i + \sigma_i \otimes \text{Id}_{\text{H}(F)} + \text{Id}_{\text{H}(F)} \otimes \tilde{\sigma}_i + \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{q} <S(\partial_j)f_j, h_s > c(f_j)c(h_s) \right] \right. \\
\times \left[ \partial_j + \sigma_j \otimes \text{Id}_{\text{H}(F)} + \text{Id}_{\text{H}(F)} \otimes \tilde{\sigma}_j + \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{q} <S(\partial_j)f_j, h_s > c(f_j)c(h_s) \right] \\
- \sum_{k} h_{i,j} \left[ \partial_k + \sigma_k \otimes \text{Id}_{\text{H}(F)} + \text{Id}_{\text{H}(F)} \otimes \tilde{\sigma}_k + \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{q} <S(\partial_k)f_j, h_s > c(f_j)c(h_s) \right] \right. \\
\left. \right) + \frac{r_M}{4} \sum_{i,j} \sum_{s,t=1}^{p} \left< R^{F^+}(f_i, h_r)h_t, h_s \right> c(f_i)c(h_r)c(h_s)c(h_t) \\
+ \frac{1}{8} \sum_{j=1}^{p} \sum_{s,t,r,u=1}^{q} \left< R^{F^+}(f_i, f_j)h_t, h_s \right> c(f_i)c(f_j)c(h_s)c(h_t) \\
+ \frac{1}{8} \sum_{s,t,r,u=1}^{q} \left< R^{F^+}(h_r, h_t)h_s, h_u \right> c(h_r)c(h_t)c(h_u)c(h_t). \quad (4.7)
\]

From (4.5) in [13], we have
\[
\sigma_2(D_F^2)\sigma_2(D_F^2) = 1; \quad \sigma_1(D_F^2)\sigma_2(D_F^2) + \sigma_2(D_F^2)\sigma_3(D_F^2) + \sum_j \partial_j \sigma_2(D_F^2)D_{x_j}\sigma_2(D_F^2) = 0. \quad (4.8)
\]

Then the Lemma follows. 

Since \( \Phi \) is a global form on \( \partial M \), so for any fixed point \( x_0 \in \partial M \), we can choose the normal coordinates \( U \) of \( \partial M \) (not in \( M \)) and compute \( \Phi(x_0) \) in the coordinates \( \tilde{U} = U \times [0,1) \subset M \) and the metric \( \frac{1}{\pi(x_0)} g^{\partial M} + dx_n^2 \). For details, see Section 2.2.2 in [10].

Let us now turn to compute \( \Phi \) (see formula (3.4) for the definition of \( \Phi \)), since the sum is taken over \( -r-l+k+j+|\alpha| = -5, \ r, l \leq -2 \), then we have the following five cases:

**case a) I)** \( r = -2, \ l = -2, \ k = j = 0, \ |\alpha| = 1 \)

From (3.4) we have
\[
\text{case a) I) = } \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} [\partial^2_{\xi_n} \sigma_2(D_F^2) \times \partial^2_{\xi_n} \sigma_2(D_F^2)](x_0)d\xi_n \sigma(\xi')d\xi', \quad (4.9)
\]

By Lemma 2.2 in [10], for \( i < n \), then
\[
\partial_{x_i} \sigma_2(D_F^2)(x_0) = \partial_{x_i}(|\xi'|^2)(x_0) = -\frac{\partial_{x_i}(|\xi'|^2)(x_0)}{|\xi'|^4} = 0, \quad (4.10)
\]

so case a) I) vanishes.

**case a) II)** \( r = -2, \ l = -2, \ k = |\alpha| = 0, \ j = 1 \)

From (3.4) we have
\[
\text{case a) II) = } \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} [\partial^2_{\xi_n} \sigma_2(D_F^2) \times \partial^2_{\xi_n} \sigma_2(D_F^2)](x_0)d\xi_n \sigma(\xi')d\xi', \quad (4.11)
\]

An application of Lemma 2.2 in [10], shows that 
\[
\partial_{x_n} \sigma_2(D_F^2)(x_0)|_{\xi'|=1} = -\frac{h'(0)}{(1 + \xi_n^2)^2}. \quad (4.12)
\]
By (2.1.1) in [10] and the Cauchy integral formula, we obtain

\[ \pi_n^+ \partial_n \sigma_2(D_{F}^{-2})(x_0) |_{\xi' = 1} = -h'(0) \frac{1}{2\pi i} \lim_{u \to 0} \int_{\Gamma^+} \frac{1}{(\eta_n - i)^2} d\eta_n \]

\[ = h'(0) \frac{i\xi_n + 2}{4(\xi_n - i)^2}, \quad (4.13) \]

and

\[ \partial_n^2 (|\xi'|^{-2})(x_0) = -2 + 6\xi_n^2 \quad (1 + \xi_n^2)^3. \quad (4.14) \]

We note that

\[ \int_{-\infty}^{\infty} \frac{i\xi_n + 2}{(\xi_n - i)^2} x \frac{-1 + 3\xi_n^2}{(1 + \xi_n^2)^3} d\xi_n \]

\[ = \int_{\Gamma^+} \frac{3i\xi_n^3 + 6\xi_n^2 - i\xi_n - 2}{(\xi_n - i)^3(\xi_n + i)^3} d\xi_n \]

\[ = \frac{2\pi i}{4!} \left[ \frac{3i\xi_n^3 + 6\xi_n^2 - i\xi_n - 2}{(\xi_n + i)^4} \right] \bigg|_{\xi_n = i} \]

\[ = \frac{5\pi}{16} \quad (4.15) \]

Since \( n = 2p + q = 6, \ \text{tr}_{(S^4)\otimes \Lambda (F^+ \cdot \cdot)}[\mathcal{id}] = 1 \times 2^q \). Combining (4.11) and (4.15), we have case a) II) = \(-\frac{5}{6\pi} \cdot 1 \times 2^q h'(0) \Omega_4 dx' \), where \( \Omega_4 \) is the canonical volume of \( S^4 \).

case a) III) \( r = -2, \ l = -2 \ j = |\alpha| = 0, \ k = 1 \)

By (3.4) and an integration by parts, we obtain

\[ \text{case a) III) } = -\frac{1}{2} \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial_n \pi^+_{\xi_n} \sigma_2(D_{F}^{-2}) \times \partial_n \sigma_2(D_{F}^{-2})(x_0)]d\xi_n \sigma(\xi')dx' \]

\[ = \frac{1}{2} \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial^2_n \pi^+_{\xi_n} \sigma_2(D_{F}^{-2}) \times \partial_n \sigma_2(D_{F}^{-2})(x_0)]d\xi_n \sigma(\xi')dx'. \quad (4.16) \]

From Lemma 2.2 in [10], we have

\[ \partial^2_n \pi^+_{\xi_n} \sigma_2(D_{F}^{-2})(x_0) |_{|\xi'| = 1} = \frac{-i}{(\xi_n - i)^3}. \quad (4.17) \]

Combining (4.12) and (4.17), we have

\[ \text{case a) III) } = 4ih'(0) \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \int_{\Gamma^+} \frac{1}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_n \sigma(\xi')dx' = \frac{5}{64} \cdot 2^q h'(0) \Omega_4 dx'. \quad (4.18) \]

Thus the sum of case a) II) and case a) III) is zero.

case b) \( r = -2, \ l = -3, \ k = j = |\alpha| = 0 \)

By (3.4) and an integration by parts, we get

\[ \text{case b) } = -i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial_n \pi^+_{\xi_n} \sigma_2(D_{F}^{-2}) \times \sigma_2(D_{F}^{-2})(x_0)]d\xi_n \sigma(\xi')dx' \]

\[ = i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial_n \pi^+_{\xi_n} \sigma_2(D_{F}^{-2}) \times \sigma_3(D_{F}^{-2})(x_0)]d\xi_n \sigma(\xi')dx'. \quad (4.19) \]
By Lemma 2.2 in [10], we have

\[ \partial_{\xi_n} \pi_n^+ \sigma_{-2}(D_F^{-2})(x_0) |_{\xi' = 1} = \frac{i}{2(\xi_n - i)^2} \quad (4.20) \]

In the normal coordinate, \( g^{ij}(x_0) = \delta_i^j \) and \( \partial_{x_k}(g^{\alpha\beta})(x_0) = 0 \), if \( j < n; = h'(0)\delta_\beta^\alpha \), if \( j = n \). So by Lemma A.2 in [10], we have \( \Gamma^n(x_0) = \frac{3}{2}h'(0) \) and \( \Gamma^k(x_0) = 0 \) for \( k < n \). Let

\[ \sigma_{-3}(D_F^{-2}) := A_1 + A_2, \quad (4.21) \]

where

\[ A_1 = \sqrt{-1}[\xi^{-6}_i]^{2\xi - 2} \xi \partial_j g^{\alpha\beta}; \quad A_2 = -\sqrt{-1}\xi^{-4} \xi_k \left( \Gamma^k + \frac{1}{2} \sum_{k,l} \omega_{k,l}(\partial^k)c(f_k)c(f_l) \otimes \text{Id}_{\Lambda(F^+)} \right) \]

\[ -\text{Id}_{S(F)} \otimes \frac{1}{2} \sum_{r,t} \omega_{r,t}(\partial^k)[\partial(h_r)c(h_t) - c(h_r)c(h_t)] - \sum_{i,j=1}^q \left( \sum_{i=1}^q \sum_{s=1}^q \omega_{r,t}(\partial^i)2^{p+q} - \sum_{r} \omega_{r,t}(\partial^i)2^{p+q} \right). \quad (4.23) \]

Then

\[ \text{tr}[\partial_{\xi_n} \pi_n^+ \sigma_{-2}(D_F^{-2}) 	imes A_1] = \frac{i}{2(\xi_n - i)^2} \times -\frac{2h'(0)\xi_n}{(1 + x_n^2)^3}. \quad (4.24) \]

By the trace identity \( \text{tr}(AB) = \text{tr}(BA), \text{tr}(A \otimes B) = \text{tr}(A) \cdot \text{tr}(B) \) and the relation of the Clifford action, we have

\[ \text{tr}[c(f_k)c(f_l)] = -\delta_{lk}2^p, \quad \text{tr}[\partial(h_r)c(h_t) - c(h_r)c(h_t)] = 2\delta_{rt}2^q. \quad (4.25) \]

Then

\[ \text{tr}[\partial_{\xi_n} \pi_n^+ \sigma_{-2}(D_F^{-2}) \times A_2] = \frac{\xi_n}{2(\xi_n - i)^2} \left( \frac{5}{2} h'(0) + \frac{1}{2} \sum_k \omega_{k,k}(\partial^i)2^{p+q} - \sum_r \omega_{r,r}(\partial^i)2^{p+q} \right). \]

\[ = \frac{5h'(0)\xi_n}{4(\xi_n - i)^2}, \quad (4.26) \]

where we have used the fact that when \( k = l \) and \( r = t \), \( \sum_k \omega_{k,l}(\partial^i) = \sum_r \omega_{r,t}(\partial^i) = 0 \).

Hence in this case,

\[ \text{case b)} = \left[ \frac{1}{2(\xi_n - i)^2} \times (A_1 + A_2) \right] d\xi_n \sigma(\xi') dx' \]

\[ = \frac{i}{4} \right[ \frac{5h'(0)}{4} \times 2^q \Omega_4 \int_{-i}^{+\infty} \frac{5x_n^3 + 9x_n}{(\xi_n - i)^2} d\xi_n dx' \]

\[ = \frac{ih'(0)}{4} \times 2^q \Omega_4 \int_{-i}^{+\infty} \frac{5x_n^3 + 9x_n}{(\xi_n - i)^2} (4)_{\xi_n = i} dx' \]

\[ = \frac{15i}{64} \times 2^q h'(0) \Omega_4 dx' \quad (4.27) \]

\[ \text{case c)} \quad r = -3, \quad l = -2, \quad k = j = |\alpha| = 0 \]

From (3.4) we have

\[ \text{case c)} = -i \int_{\xi'|1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\pi_n^+ \sigma_{-3}(D_F^{-2}) \times \partial_{\xi_n} \sigma_{-2}(D_F^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.28) \]
By the Leibniz rule, trace property and "++" and "−−" vanishing after the integration over $\xi_n$ in (3), then
\[
\int_{-\infty}^{+\infty} \text{trace}[\pi_n^+ \sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n
= \int_{-\infty}^{+\infty} \text{tr}[\sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[\pi_n^+ \sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n
= \int_{-\infty}^{+\infty} \text{tr}[\sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[\sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n
= \int_{-\infty}^{+\infty} \text{tr}[\sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n + \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_2(D_F^{-2}) \times \sigma_3(D_F^{-2})]d\xi_n + \int_{-\infty}^{+\infty} \text{tr}[\pi_n^+ \sigma_3(D_F^{-2}) \times \sigma_2(D_F^{-2})]d\xi_n
= \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_2(D_F^{-2}) \times \sigma_3(D_F^{-2})]d\xi_n + \int_{-\infty}^{+\infty} \text{tr}[\pi_n^+ \sigma_3(D_F^{-2}) \times \partial_{\xi_n} \sigma_2(D_F^{-2})]d\xi_n. 
\] (4.29)

Then we have
\[\text{case c) = case b}) - i \int_{|\xi'|=1}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_2(D_F^{-2}) \times \sigma_3(D_F^{-2})]d\xi_n \delta(\xi')dx'. \] (4.30)

In order to compute case c), we only need compute the last term in (4.30).

From (4.4), one sees that
\[\partial_{\xi_n} \sigma_2(D_F^{-2})(x_0)|_{|\xi'|=1} = - \frac{2\xi_n}{(\xi_n^2 + 1)^2}. \] (4.31)

Similarly to case b, we obtain
\[-i \int_{|\xi'|=1}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_2(D_F^{-2}) \times \sigma_3(D_F^{-2})]d\xi_n \delta(\xi')dx'
= -i \int_{|\xi'|=1}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_2(D_F^{-2}) \times (A_1 + A_2)]d\xi_n \delta(\xi')dx'
= h'(0)\tilde{l} \times 2^q \Omega_4 \int_{\Gamma^+} \frac{5\xi_n^4 + 9\xi_n^2}{(\xi_n - i)^{4}(\xi_n + i)^{4}} d\xi_n dx'
= h'(0)\tilde{l} \times 2^q \Omega_4 \frac{2\pi i}{4!} \frac{5\xi_n^4 + 9\xi_n^2}{(\xi_n + i)^{4}} \bigg|_{\xi_n = i} dx'
= \frac{15}{32} \times 2^{4}\pi h'(0)\Omega_4 dx'. \] (4.32)

Combining (4.27), (4.30) and (4.32), we have the sum of case b and case c) is zero. Now $\Phi$ is the sum of the cases a), b) and c), so is zero. Hence we conclude that

**Theorem 4.2.** Let $M$ be a 6-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^M$ as above, and $D_F$ be the sub-Dirac operator on $\Gamma(S(F) \otimes (F^{-1})^*)$, then

\[\text{Vol}_\square^{(2,2)}(M, F) = - \frac{i \times 2^q}{6 \times (4\pi)^{4}} \int_M r_M d\nu l_y. \] (4.33)

where $r_M$ be the scaler curvature of the foliation and $\text{dim}(S(F)) = \tilde{l}$.

**Remark 4.3.** Let $M$ be a 6-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^M$ as above, and $D_F$ be the sub-Dirac operator. When $n = p$, we obtain Theorem 1 in [3].
4.2. The gravitational action for 6-dimensional manifolds with boundary

Let $M$ be 6-dimensional manifolds with boundary and $P, P'$ be two pseudodifferential operators with transmission property on $\tilde{M}$. Motivated by (4) in [2], we define locally

$$\text{res}_{2,2}(P, P') := -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} [\partial_{\xi_n} \pi_{\xi_n} \sigma_{-2}(P) \times \partial_{\xi_n} \sigma_{-2}(P')] d\xi_n \sigma(\xi') dx';$$  \hspace{1cm} (4.34)

$$\text{res}_{2,3}(P, P') := -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n} \sigma_{-2}(P) \times \partial_{\xi_n} \sigma_{-3}(P')] d\xi_n \sigma(\xi') dx'. \hspace{1cm} (4.35)$$

Combining (4.34) and (4.35), we have

\begin{align*}
\text{case a) II) } &= \text{res}_{2,2}(D_{F}^{-2}, D_{F}^{-2}); \quad \text{case b) } = \text{res}_{2,3}(D_{F}^{-2}, D_{F}^{-2}). \hspace{1cm} (4.36)
\end{align*}

Now, we assume $\partial M$ is flat, then $\{dx_i = e_i\}$, $g_{ij}^{BM} = \delta_{i,j}$, $\partial x_i g_{ij}^{BM} = 0$. So $\text{res}_{2,2}(D_{F}^{-2}, D_{F}^{-2})$ and $\text{res}_{2,3}(D_{F}^{-2}, D_{F}^{-2})$ are two global forms locally defined by the aboved oriented orthonormal basis $\{dx_i\}$. From case a) II) and case b), we have

**Theorem 4.4.** Let $M$ be a 6-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^{M}$ as above, and $D_{F}$ be the sub-Dirac operator on $\Gamma(S(F) \otimes (F_{\perp}, \cdot))$. Assume $\partial M$ is flat, then

\begin{align*}
\int_{\partial M} \text{res}_{2,2}(D_{F}^{-2}, D_{F}^{-2}) &= \frac{1}{64} \hat{l} \times 2q \Omega_{4} I_{\text{Gr}, b}; \\ 
\int_{\partial M} \text{res}_{2,3}(D_{F}^{-2}, D_{F}^{-2}) &= \frac{3}{64} \hat{l} \times 2q \Omega_{4} I_{\text{Gr}, b}. \hspace{1cm} (4.37) (4.38)
\end{align*}

5. A Kastler-Kalau-Walze Type Theorem for 5-dimensional foliations with boundary

5.1. A Kastler-Kalau-Walze Type Theorem for 5-dimensional foliations with boundary

First of all, for 5-dimensional foliations with boundary, we compute $\text{Vol}_{5}^{(2,2)}$. From Theorem 2.9, we have

$$\text{Wres}([\pi^{+} D_{F}^{-2})^{2}] = \int_{\partial M} \Phi. \hspace{1cm} (5.1)$$

From (3.4), when $n = 5$, $r - k - |\alpha| + l - j - 1 = -5$, $r, l \leq -2$, so we get $r = l = -2$, $k = |\alpha| = j = 0$, then

$$\Phi = \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} [\pi_{\xi_n}^{+} (D_{F}^{-2})(x', 0, \xi', \xi_n) \times \partial_{\xi_n} \sigma_{-2}(D_{F}^{-2})(x', 0, \xi', \xi_n)] d\xi_n d\xi(\xi') dx'. \hspace{1cm} (5.2)$$

An application of Lemma 2.2 in [10], shows that

$$\pi_{\xi_n}^{+} \sigma_{-2}(x_0)|_{|\xi'| = 1} = \frac{1}{2i(\xi_n - i)}. \hspace{1cm} (5.3)$$

By (3.21) and $\text{tr}(S(F) \otimes (F_{\perp}, \cdot)) [\text{id}] = \hat{l} \times 2q$, we obtain

$$\text{Vol}_{5}^{(2,2)} = \frac{\pi}{8} \hat{l} \times 2q \Omega_{3} \text{Vol}_{\partial M}. \hspace{1cm} (5.4)$$

By $I_{\text{Gr}, b} = -4h'(0) \text{Vol}_{\partial M}$, we have

**Theorem 5.1.** Let $M$ be a 5-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^{M}$ as above, and $D_{F}$ be the sub-Dirac operator on $\Gamma(S(F) \otimes (F_{\perp}, \cdot))$, then

$$\text{Vol}_{5}^{(2,2)} = \text{Wres}([\pi^{+} D_{F}^{-2})^{2}] = \frac{\pi}{8} \hat{l} \times 2q \Omega_{3} \text{Vol}_{\partial M}; \hspace{1cm} (5.5)$$

$$I_{\text{Gr}, b} = \frac{32i h'(0)}{\hat{l} \times 2q \Omega_{3}} \text{Wres}([\pi^{+} D_{F}^{-2})^{2}]. \hspace{1cm} (5.6)$$

where $\text{Vol}_{\partial M}$ denotes the canonical volume of $\partial M$. 22
5.2. A Kastler-Kalau-Walze Type Theorem for 5-dimensional Manifolds with boundary

In this section, we compute the lower dimension volume for 5-dimension spin manifolds with boundary and get a Kastler-Kalau-Walze type formula in this case.

Let $M$ be an $n$-dimensional compact oriented connected manifold with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)}g^\partial M + dx_n^2,$$  \hspace{1cm} (5.7)

where $g^\partial M$ is the metric on $\partial M$: $h(x_n) \in C^\infty([0,1)) = \{g|_{[0,1]}|g \in C^\infty((-\varepsilon, 1))\}$ for some sufficiently small $\varepsilon > 0$ and satisfied $h(x_n) > 0$, $h(0) = 1$, where $x_n$ denotes the normal directional coordinate. Let $D$ be the Dirac operator associated to $g^M$ on $C^\infty(S(TM))$. Let $p_1, p_2$ be nonnegative integers and $p_1 + p_2 \leq n$.

Then Wang pointed out the related definition between the volumes of compact connected manifolds and the Wodzicki residue in [16].

**Definition 5.2.** Lower-dimensional volumes of compact connected manifolds with boundary are defined by

$$Vol_n^{(p_1, p_2)} := \frac{Wres[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]}{23}.$$  \hspace{1cm} (5.8)

Denote by $\sigma_l(D)$ the $l$-order symbol of an operator $D$. Combining (2.1.4)-(2.1.8) of [10], we obtain

$$Wres[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi| = 1} \text{trace}_{S(TM)}[\sigma_{-n}(D^{-p_1} \circ D^{-p_2})]dx + \int_{\partial M} \Phi,$$  \hspace{1cm} (5.9)

where

$$\Phi = \int_{|\xi| = 1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{\alpha(j+k+1)}}{\alpha!} \text{trace}_{S(TM)}[\partial^j_{x_n} \partial^k_{\xi_n} \sigma^+(D^{-p_1})(x', 0, \xi', \xi_n)]$$

$$\times \partial^j_{x_n} \partial^k_{\xi_n} \sigma_l(D^{-p_2})(x', 0, \xi', \xi_n) dx_n \sigma(\xi') dx',$$  \hspace{1cm} (5.10)

and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -n$, $r \leq -p_1$, $\ell \leq -p_2$.

Since $[\sigma_{-n}(\Delta_{-p_1} \circ \Delta_{-p_2})]|_M$ has the same expression with the case of without boundary in [10], so locally we can use Theorem 2.9 to compute the first term. Let us now give explicit formulas for the $Vol_n^{(1)}$ in dimension 5.

Since $\Gamma(\frac{1}{2} + 1) = \frac{1}{2} \Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{5}{2} + 1) = \frac{3}{2} \Gamma(\frac{3}{2}) + 1 = \frac{5}{4} - \frac{1}{4}$, $\Gamma(\frac{1}{2} + 1) = \frac{5\sqrt{\pi}}{4}$, we get

$$v_{5,1} = \frac{1}{2} \frac{(-3\Gamma(5\frac{1}{2}) + 1)}{16 \pi^2} \frac{\Gamma\left(\frac{5}{2} + 1\right)}{\Gamma\left(\frac{7}{2} + 1\right)} = \frac{2}{5} \frac{\pi^4}{\sqrt{\pi}} \frac{\sqrt{30}}{20 \sqrt{\pi}}.$$  \hspace{1cm} (5.11)

Therefore, by using Proposition 2.3 in [8], we see that in dimension 5

$$\int_M \int_{|\xi| = 1} \text{trace}_{S(TM)}[\sigma_{-5}(D^{-2-1})]dx = -\frac{\pi \sqrt{30}}{240 \sqrt{2\pi}} \int_M r_M dvol,$$  \hspace{1cm} (5.12)

where we have used the fact that $\int_M \Delta_j k d\nu_j(x) = \int_M g(\nabla k, \nabla 1)d\nu_j(x) = 0$. Hence we only need to compute $\int_{\partial M} \Phi$.

Since $\Phi$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $u$ of $x_0$ in $\partial M$ and compute $\Phi(x_0)$ in the coordinates $U = U \times [0, 1] \subset M$ and the metric $g^M = h(x_n)g^\partial M + dx_n^2$. Firstly, we recall the symbol expansion of $D^{-2}$ and $D^{-1}$. By Lemma 1 in [8] and Lemma 2.1 in [10], we have
Lemma 5.3. \([9, 11]\) Let \(D\) be the Dirac operator associated to \(g\) on the spinors bundle \(S(TM)\). Then

\[
\sigma_{-1}(D^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}; \\
\sigma_{-2}(D^{-1}) = \frac{c(\xi)p(x)(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^2}\sum_j c(dx_j)[\partial x_j][\xi(\xi)]|\xi|^2 - c(\xi)\partial x_j(|\xi|^2);
\]

(5.13)

where \(p_0 = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(\partial_i)c(\xi)c(\xi_i)\).

Let us now turn to compute \(\Phi\) (see formula (5.10) for definition of \(\Phi\)). Since the sum is taken over \(r - \ell + k + j + |\alpha| = 4\), \(r \leq -2, \ell \leq -1\), then we have the following five cases:

**Case a (I):** \(r = -2, \ell = -1, k = j = 0, |\alpha| = 1\)

From (5.10) we have

\[
\text{Case a (I)} = -\int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial^0_{\xi_1} \pi_{\xi_1}^+ \sigma_{-2}(D^{-2}) \times \partial^0_{\xi_1} \sigma_{-1}(D^{-1})](x_0) \text{d}\xi_0 \sigma(\xi') \text{d}x'.
\]

Then an application of Lemma 2.2 in \([10]\) shows, for \(i < n\)

\[
\partial_{\xi_1} \sigma_{-1}(D^{-1})(x_0) = \partial_{\xi_1} \left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2}\right)(x_0) = \frac{\sqrt{-1}c(\xi)\partial x_1(|\xi|^2)}{|\xi|^4} = 0,
\]

so **Case a (I) vanishes.**

**Case a (II):** \(r = -2, \ell = -1, k = |\alpha| = 0, j = 1\)

From (5.10) we have

\[
\text{Case a (II)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_1} \pi_{\xi_1}^+ \sigma_{-2}(D^{-2}) \times \partial^0_{\xi_1} \sigma_{-1}(D^{-1})](x_0) \text{d}\xi_0 \sigma(\xi') \text{d}x'.
\]

By (2.2.16) in \([10]\), we have

\[
\partial_{\xi_1} \sigma_{-2}(D^{-2})(x_0)|_{|\xi'|=1} = -\frac{h'(0)}{(1 + \xi_2^2)^2}.
\]

(5.20)

By the Cauchy integral formula we obtain

\[
\pi_{\xi_1}^+ \xi_1 \left(\frac{1}{1 + \xi_1^2}\right)^2(x_0)|_{|\xi'|=1} = \frac{1}{2\pi i} \lim_{u \to 0} \int_{\Gamma^+} \frac{1}{(\eta_n - i)^2} \text{d}\eta_n = -\frac{\xi_0 + 2}{4(\xi_n - i)^2}.
\]

(5.21)

Then

\[
\pi_{\xi_1}^+ \partial_{\xi_1} \sigma_{-2}(D^{-2})(x_0)|_{|\xi'|=1} = h'(0)\frac{\xi_0 + 2}{4(\xi_n - i)^2},
\]

(5.22)

and

\[
\partial^0_{\xi_1} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \sqrt{-1} \left(\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_0^2 c(\xi)}{|\xi|^6}\right).
\]

(5.23)

Since \(n = 5\), \(\text{tr}(\text{id}) = \dim(S(TM)) = 4\). Locally \(S(TM)|_{\tilde{G}} \cong \tilde{U} \times \wedge\vec{C}\frac{n+1}{2}\)), and \(\text{clc}(n) \hookrightarrow \text{clc}(n+1) \cong \text{Hom}(\wedge\vec{C}\frac{n+1}{2})\). Take a spin frame field \(\sigma : \tilde{U} \to \text{Spin}(M)\) such that \(\pi\sigma = (\xi_1 e_0, \ldots, \xi_6 e_6)\) where \(\pi : \text{Spin}(M) \to O(M)\) is a double covering, then \(\{(\sigma f_1), 1 \leq i \leq 6\}\) is an orthonormal frame of \(S(TM)|_{\tilde{G}}\). Let \(\{E_1, \ldots, E_n\}\) be the canonical basis of \(\mathbb{R}^n\) and \(c(E_i) \in \text{clc}(n) \cong \text{Hom}(\wedge\vec{C}\frac{n+1}{2}, \wedge\vec{C}\frac{n+1}{2})\) be the Clifford action. In the following, since the
global form $\Phi$ is independent of the choice of the local frame, so we can compute $\text{tr}_{S(TM)}$ in the frame $\{[\sigma, f_i]_1 \leq i \leq 6\}$.
By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we obtain the equalities:
\[
\text{tr}[c(\epsilon')] = 0; \quad \text{tr}[c(dx_n)] = 0; \quad \text{tr}[c(\xi)]|_{\epsilon' = 1} = 0. \tag{5.24}
\]
Combining (5.21), (5.22) and (5.23), we obtain
\[
\text{trace}[\partial_{x_n} \pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \partial_{x_n} \sigma_{-1}(D^{-1})](x_0)|_{\epsilon' = 1} = 0.
\]

Therefore Case a (II) = 0.

Case a (III): $r = -2, \ell = -1, j = |\alpha| = 0, k = 1$
From (5.10) we have
\[
\text{Case a (III)} = -\frac{1}{2} \int_{|\epsilon'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \times \partial_{x_n} \sigma_{-1}(D^{-1})](x_0) d\xi_n \sigma(\epsilon') dx'. \tag{5.26}
\]

An application of Lemma 2.2 in [10] shows
\[
\partial_{x_n} \tau_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{\epsilon' = 1} = -\sqrt{-\lambda'}(0) \int_{|\epsilon'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \times \partial_{x_n} \sigma_{-1}(D^{-1})](x_0) d\xi_n \sigma(\epsilon') dx'. \tag{5.27}
\]
and
\[
\partial_{x_n} \pi^+_{\xi_n} \sigma_{-2}(D^{-2})(x_0)|_{\epsilon' = 1} = \frac{i}{2(\xi_n - i)^2}. \tag{5.28}
\]
From (5.24), (5.27) and (5.28), we have
\[
\text{trace}[\partial_{x_n} \pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \times \partial_{x_n} \sigma_{-2}(D^{-1})](x_0)|_{\epsilon' = 1} = 0.
\]

Hence Case a (III) = 0.

Case b: $r = -2, \ell = -2, k = j = |\alpha| = 0$
From (5.10) we have
\[
\text{Case b} = -i \int_{|\epsilon'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \times \partial_{x_n} \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\epsilon') dx'. \tag{5.30}
\]
By (2.2.16) in [11], we obtain
\[
\pi^+_{\xi_n} \sigma_{-2}(D^{-2})(x_0)|_{\epsilon' = 1} = \frac{1}{2(\xi_n - i)^2}. \tag{5.31}
\]
and
\[
\partial_{x_n} \sigma_{-2}(D^{-1})(x_0)|_{\epsilon' = 1} = \frac{1}{(1 + \xi_n^2)^3} [2(\xi_n - \xi_n^3) c(dx_n) p_0 c(dx_n) + (1 - 3\xi_n^2)c(dx_n) p_0 c(\xi')
+ (1 - 3\xi_n^2) c(\xi') p_0 c(dx_n) - 6\xi_n c(\xi') p_0 c(\xi')
+ (3\xi_n^2 - 1) \partial_{x_n} c(\xi') - 4\xi_n c(\xi') c(dx_n) \partial_{x_n} c(\xi')
+ 2h'(0) c(\xi') + 2h'(0) \xi_n c(dx_n)] + 6\xi_n h'(0) c(\xi) c(dx_n)c(\xi) \tag{5.32}
\]
Notice $p_0 = -\frac{1}{4} \sum_{x,t} \omega_{x,t} (\partial c(c\bar{c}))$. By the relation of the Clifford action and $\text{tr} AB = \text{tr} BA$, we obtain

$$\text{tr}[c(\xi') c(dx_n) \partial x_n, c(\xi')] = \text{tr}[c(\xi') e_0) (c(dx_n) e_0) \partial x_n, c(\xi') e_0]$$
$$= \text{tr}[c(\xi') c(dx_n) \partial x_n, c(\xi')] = \text{tr}[\partial x_n, c(\xi') (c(\xi') e_0) c(dx_n)]$$
$$= - \text{tr}[\partial x_n, (c(\xi') c(dx_n) c(\xi')] = - \text{tr}[c(\xi') c(dx_n) \partial x_n, c(\xi')], \quad (5.33)$$

then $\text{tr}[c(\xi') c(dx_n) \partial x_n, c(\xi')] = 0$.

Then Case b = 0.

Case c: $r = -3, \ell = -1, k = j = |\alpha| = 0$.

From (5.10) we have

$$\text{trace}[\pi^+_\xi \sigma_{-3}(D^{-2}) \times \partial x_n, \sigma_{-1}(D^{-1})(x_0)] |_{\xi' = 1} = 0. \quad (5.34)$$

Now, in the normal coordinate, $g^{ij}(x_0) = \delta^{ij}$ and $\partial_j (g^\alpha\beta)(x_0) = 0$, if $j < n; = h^0(\xi') \delta_0^\alpha$, if $j = n$. So by Lemma A.2 in [16], we have $\Gamma^n(x_0) = \frac{1}{2} h^0(0)$ and $\Gamma^k(x_0) = 0$, for $k < n$. By (20) in [1], we have

$$\sigma_{-3}(D^{-2})(x_0)|_{\xi' = 1} = - \sqrt{-1} [\xi'^{-1} \xi_k (e^k - 2 \delta^k)(x_0)|_{\xi' = 1} \sqrt{-1} \xi'^{-1} 2 \xi_k \xi_k \partial_{\xi^\alpha}(g^\alpha\beta)(x_0)|_{\xi' = 1}$$
$$= - \frac{1}{(1 + \xi_0^2)^2} \sum_{k<n} \xi_k c(e_k) c(e_n) + \frac{5}{2} h^0(0) \xi_n - \frac{2 \xi^0(0) \xi_n}{(1 + \xi_0^2)^2}. \quad (5.35)$$

We note that $\int_{|\xi'| = 1} \xi_1 \cdots \xi_{2q+1} \xi(\xi') = 0$, so the first term in (5.36) has no contribution for computing Case c).

By the Cauchy integral formula, we get

$$\pi^+_\xi \frac{\xi_n}{(1 + \xi_n^2)^2}(x_0)|_{\xi' = 1} = - \frac{1}{2\pi i} \lim_{u \to 0} \int_{|\eta_n + i \xi_n| = \frac{1}{4} \xi_n} \frac{\eta_n}{(\eta_n - i)^2} \frac{(\eta_n + i \xi_n - \eta_n)^2}{(\eta_n + i \xi_n - \eta_n)} d\eta_n$$
$$= \left[ \frac{\eta_n}{(\eta_n + i \xi_n - \eta_n)} \right]_{|\eta_n = i} = - i \xi_n \frac{4(\xi_n - i)^2}{4(\xi_n - i)^2}, \quad (5.37)$$

and

$$\pi^+_\xi \frac{\xi_n}{(1 + \xi_n^2)^2}(x_0)|_{\xi' = 1} = - \frac{1}{2\pi i} \lim_{u \to 0} \int_{|\eta_n + i \xi_n| = \frac{1}{4} \xi_n} \frac{\eta_n}{(\eta_n - i)^2} \frac{(\eta_n + i \xi_n - \eta_n)^2}{(\eta_n + i \xi_n - \eta_n)} d\eta_n$$
$$= \left[ \frac{\eta_n}{(\eta_n + i \xi_n - \eta_n)} \right]_{|\eta_n = i} = \frac{3(35 + 47 i \xi_n - 25 \xi_n^2 - 5 i \xi_n^3)}{32 \xi_n^2 - i \xi_n^2}, \quad (5.38)$$
Theorem 5.4. Let $M$ be a 5-dimensional compact connected manifold with the boundary $\partial M$. Then

$$\pi^+_{\xi_0} \sigma_{-3}(D^{-2})(x_0)|_{\xi'|=1} = -\frac{5h'(0)\xi_n}{8(\xi_n - i)^2} \frac{3h'(0)(35 + 47i\xi_n - 25\xi_n^2 - 5i\xi_n^3)}{16(\xi_n - i)^5}. \quad (5.39)$$

By (5.13), we have

$$\delta_{\xi_n} \sigma_{-1}(D^{-1})(x_0)|_{\xi'|=1} = \sqrt{-1} \left( \frac{c(dx_n)}{\xi'^2} - \frac{2c_0c(\xi)}{\xi'^4} \right). \quad (5.40)$$

Combining (5.39) and (5.40), we obtain

$$\text{trace}[\pi^+_{\xi_0} \sigma_{-3}(D^{-2}) \times \delta_{\xi_n} \sigma_{-1}(D^{-1})(x_0)|_{\xi'|=1} = \left[ -\frac{5h'(0)\xi_n}{8(\xi_n - i)^2} \frac{3h'(0)(35 + 47i\xi_n - 25\xi_n^2 - 5i\xi_n^3)}{16(\xi_n - i)^5} \right] \times \left( \frac{\text{tr}[c(dx_n)]}{\xi'^2} \frac{2\xi_n \text{tr}[c(\xi)]}{\xi'^4} \right) \quad (5.41)$$

Then

$$\text{Case } c = -i \int_{\xi'|=1}^{+\infty} \int_{-\infty}^0 \text{trace}[\delta_{\xi_n} \sigma_{-1}(D^{-1}) \times \sigma_{-1}(D^{-1})(x_0)] d\xi_n \sigma(\xi') dx' = 0. \quad (5.42)$$

Since $\Phi$ is the sum of the case $a$, $b$ and $c$, so is zero. Then we have

**Theorem 5.4.** Let $M$ be a 5-dimensional compact connected manifold with the boundary $\partial M$ and the metric $g^M$ as above, and $D$ the Dirac operator on $S(TM)$, then

$$\text{Vol}_{3}^{(2,1)}(M) = -\frac{\pi \sqrt{30}}{240 \sqrt{2\pi^2 \pi^p \pi^d}} \int_M r_M statevol, \quad (5.43)$$

where $r_M$ be the scalar curvature.

Let us now consider the Einstein-Hilbert action for 5-dimensional manifolds with boundary. Let $P, P'$ be two pseudodifferential operators with transmission property (see [18]) on $S(TM)$. Motivated by (5.10), we define locally

$$\text{res}_{2,1}(P, P') := -\frac{1}{2} \int_{\xi'|=1}^{+\infty} \int_{-\infty}^0 \text{trace}[\delta_{\xi_n} \pi^+_{\xi_0} \sigma_{-2}(P^{-2}) \delta_{\xi_n} \sigma_{-1}(P^{-1})(x_0)] d\xi_n \sigma(\xi') dx'; \quad (5.44)$$

$$\text{res}_{2,2}(P, P') := -i \int_{\xi'|=1}^{+\infty} \int_{-\infty}^0 \text{trace}[\pi^+_{\xi_0} \sigma_{-2}(P^{-2}) \delta_{\xi_n} \sigma_{-2}(P^{-1})(x_0)] d\xi_n \sigma(\xi') dx'. \quad (5.45)$$

From (5.26) and (5.30), we have

**Case a (II) = res}_{2,1}(D^{-2}, D^{-1}); \text{ Case b = res}_{2,2}(D^{-2}, D^{-1}). \quad (5.46)**

Without loss of generality, we may assume that $\partial_M$ is flat, then $\{dx_i = e_i\}$, $g^{ij} = \delta_{ij}$, $\partial_x g_{ij} = 0$. So $\text{res}_{2,1}(D^{-2}, D^{-1})$ and $\text{res}_{2,2}(D^{-2}, D^{-1})$ are two global forms locally defined by the aboved oriented orthonormal basis $dx_i$. Hence by **Case a (II)** and **Case b**, we have

**Theorem 5.5.** Let $M$ be a 5-dimensional compact manifold with boundary $\partial_M$ and the metric $g^M$ as above, and $D$ the Dirac operator on $S(TM)$. Assume $\partial_M$ is flat, then

$$\int_{\partial_M} \text{res}_{2,1}(D^{-2}, D^{-1}) = \int_{\partial_M} \text{res}_{2,2}(D^{-2}, D^{-1}) = 0. \quad (5.47)$$

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Nextly, for 4-dimensional spin manifolds with boundary, we compute $\text{Vol}^{(2,1)}_4$. By (4) in [9], we have

$$\widetilde{\text{Wres}}[\pi^+ D^{-2} \circ \pi^+ D^{-1}] = \int_{\partial M} \Phi. \quad (5.48)$$

When $n = 4$, in (5.10), $r - k - |\alpha| + l - j - 1 = -3$, $r \leq -2$, $l \leq -1$, so we get $r = -2$, $l = -1$, $k = |\alpha| = j = 0$, then

$$\Phi = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \times \partial_{\xi_n} \sigma_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi') d\xi'. \quad (5.49)$$

From (5.13), we obtain

$$\partial_{\xi_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \sqrt{-1} \left( \frac{e(dx_n)}{|\xi'|^2} - \frac{2\xi_n e(\xi)}{|\xi'|^4} \right). \quad (5.50)$$

Combining (5.31) and (5.50), we obtain

$$\text{trace}[\pi^+_{\xi_n} \sigma_{-2}(D^{-2}) \times \partial_{\xi_n} \sigma_{-1}(D^{-1})](x_0)|_{|\xi'|=1} = 0. \quad (5.51)$$

Therefore $\text{Vol}^{(2,1)}_4 = 0$.

**Remark 5.6.** In fact, we may generalize Theorem 5.4 and Theorem 5.5 to the foliation case.

Then we have

**Theorem 5.7.** Let $M$ be a 5-dimensional compact connected foliation with the boundary $\partial M$ and the metric $g^M$ as above, and $D_F$ be the sub-Dirac operator, then

$$\text{Vol}^{(2,1)}_5(M) = -\frac{\pi \sqrt{30}}{240 \sqrt{\pi^2 \pi^2 + \pi}} \int_M r_M d\text{vol}, \quad (5.52)$$

where $r_M$ be the scalar curvature.

6. The Lower Volume for the Robertson-Walker Space

One very important family of cosmological models in general relativity is the family of Robertson-Walker space-times:

$$L^4_I(f, c) := (I \times M, g^c_I), \quad g^c_I = -dt^2 + f^2(t)g^c, \quad (6.1)$$

with a warped product Lorentzian metric $g^c_I$ defined on the product of an open interval $I$ and a Riemannian 3-manifold $(S, g_c)$ of constant sectional curvature $c$.

Let $M = I \times_f M$ be a Riemannian manifold with the metric $g_f = dt^2 + f^2(t)g^M$. Now we compute the lower dimension volumes for 4-dimensional foliations $\tilde{M}$ with spin leave $I$. Let $\mathcal{L}(I)$ and $\mathcal{L}(M)$ be the set of lifts of vector fields on $I$ and $M$ to $I \times_f M$ respectively. For $q \in M$, the horizontal leaf $\eta^{-1}(M)$ is a totally geodesic submanifold isometric to $I$ with scalar factor $1/h(p)$. For $p \in I$, $\pi^{-1}(p)$ is a totally umbilical submanifold that is homothetically isomorphic to $M$ with scalar factor $1/f(p)$. The submanifolds $\pi^{-1}(p) = \{p\} \times F$, $p \in I$ and $\eta^{-1}(q) = I \times \{q\}$, $q \in M$ are called fibers and leaves respectively. A vector field on $M$ is called vertical if it is always tangent to fibers; and horizontal if it is always orthogonal to fibers. We use the corresponding terminology for individual tangent vectors as well.

Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of tangent spaces of $M$ onto the subspaces of horizontal and vertical vectors, respectively. We use the same letters to denote the horizontal and vertical distributions. On the warped product $I \times_f M$, denote by $\partial_t$ the lift of the standard vector field $d/dt$ on $I$ of $I \times_f M$, so we have $\partial_t \in \mathcal{L}(I)$.

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For a vector field $V$ on $I \times fM$, we decompose $V$ into a sum
\[ V = \varphi V \partial_t + \hat{V}, \]  
(6.2)
where $\varphi V = \langle V, \partial_t \rangle$ and $\hat{V}$ is the vertical component of $V$ that is orthogonal to $\partial_t$. By Proposition 2.2 and Proposition 2.4 in [13], we have

**Lemma 6.1.** Let $\tilde{M} = I \times fM$ be a Riemannian manifold with the metric $g_f = dt^2 + f^2(t)g$. For vector fields $X, Y$ in $\mathcal{L}(M)$, then
\begin{align*}
(1) & \tilde{\nabla}_{\partial_t} \partial_t = 0, \\
(2) & \tilde{\nabla}_{\partial_t} X = \tilde{\nabla}_X \partial_t = (\ln f)'X, \\
(3) & \nabla_X Y = \nabla^M_X Y - \frac{g(X, Y)}{f} \text{grad}(f).
\end{align*}
(6.3)
(6.4)
(6.5)

**Lemma 6.2.** For vector fields $X, Y, Z$ in $\mathcal{L}(M)$, the curvature tensor $\tilde{R}$ of $\tilde{M}$ satisfies
\begin{align*}
(1) & \tilde{R}(\partial_t, X)\partial_t = \frac{f''}{f}X, \\
(2) & \tilde{R}(X, \partial_t)Y = \langle X, Y \rangle \frac{f''}{f} \partial_t, \\
(3) & \tilde{R}(X, Y)\partial_t = 0, \\
(4) & \tilde{R}(X, Y)Z = R^M(X, Y)Z - \frac{\langle \text{grad}(f), \text{grad}(f) \rangle}{f^2} \{ (X, Z)Y - (Y, Z)X \}.
\end{align*}
(6.6)
(6.7)
(6.8)
(6.9)

Let $M$ be a foliation with boundary $\partial M$. Let $\psi \in \Gamma(S(F) \otimes \wedge(F^\perp, \ast))$, we impose the Dirichlet boundary conditions $\psi |_{\partial M} = 0$. With the Dirichlet boundary conditions in [20], we have the heat trace asymptotics for $t \to 0$
\[ \text{tr}(e^{-tD^2_F}) \sim \sum_{n \geq 0} \frac{t^{-m}}{\Gamma\left(\frac{m}{2}\right)} a_n(D^2_F). \]

When $\dim \tilde{M} = 4$, by (18) in [21], one uses the Seely-deWitt coefficients $a_n(D^2_F)$ and $t = \wedge^{-2}$ to obtain an asymptotics for the spectral action
\[ I = \text{tr} \tilde{F} \left( \frac{D^2_F}{\wedge^2} \right) \sim \wedge^4 F_4 a_0(D^2_F) + \wedge^3 F_3 a_1(D^2_F) + \wedge^2 F_2 a_2(D^2_F) + \wedge F_1 a_3(D^2_F) + \wedge^0 F_0 a_4(D^2_F) \quad \text{as } \wedge \to \infty, \]  
(6.10)
where
\[ F_k := \frac{1}{\Gamma\left(\frac{k}{2}\right)} \int_0^\infty \tilde{F}(s)s^{\frac{k}{2}-1}ds. \]  
(6.11)

Let $N = e_m$ be the inward pointing unit normal vector on $\partial \tilde{M}$ and $e_i, 1 \leq i \leq m - 1$ be the orthonormal frame on $T(\partial M)$. Let $L_{ab} = (\nabla_{e_a} e_b, N)$ be the second fundamental form and indices $\{a, b, \cdots\}$ range from
1 through $m - 1$. By Theorem 1.1 in [20], we obtain the first five coefficients of the heat trace asymptotics

\begin{align}
    a_0(D_F^2) &= (4\pi)^{-\frac{d}{2}} \int_M \text{tr}(\text{Id}) \text{dvol}_M, \\
    a_1(D_F^2) &= -4^{-1}(4\pi)^{-\frac{d(m-1)}{2}} \int_{\partial M} \text{tr}(\text{Id}) \text{dvol}_{\partial M}, \\
    a_2(D_F^2) &= (4\pi)^{-\frac{d}{2}} \frac{6^{-1}}{360} \left\{ \int_M \text{tr}(r_M + 6E) \text{dvol}_M + 2 \int_{\partial M} \text{tr}(L_{aa}) \text{dvol}_{\partial M} \right\}, \\
    a_3(D_F^2) &= -4^{-1}(4\pi)^{-\frac{d}{2}} \frac{96^{-1}}{360} \left\{ \int_{\partial M} \left[ \text{tr}(96E + 16r_M + 8R_{aNaN} + 7L_{aa}L_{bb} \\
    &\quad - 10L_{ab}L_{ab}) \text{dvol}_{\partial M} \right] \right\}, \\
    a_4(D_F^2) &= \frac{(4\pi)^{-\frac{d}{2}}}{360} \frac{2^{-p+q}}{12 \cdot 2^{p+q}} \left\{ \int_M \text{tr} \left[ -12R_{ijij},bb + 5R_{ijij}R_{ikklt} - 2R_{ijik}R_{ikj} + 2R_{ijikl}R_{ijkl} \\
    &\quad - 60R_{ijij}E + 180E^2 + 60E_{jk} + 30\Omega_{ij} \text{dvol}_M \right] \\
    &\quad + \int_{\partial M} \left[ \text{tr} \left[ -120E;N - 18r_M;N + 120EL_{aa} + 20r_ML_{aa} + 4R_{aNaN}L_{bb} \\
    &\quad - 12R_{ab}L_{ab},bb + 4R_{ab}L_{ac} + 24L_{aa},bb + 40/21L_{aa}L_{bc}L_{cc} \\
    &\quad - 88/7L_{ab}L_{ab}L_{cc} + 320/21L_{ab}L_{bc}L_{ac} \text{dvol}_{\partial M} \right] \right\}. 
\end{align}

By (2.5), (2.19) and the divergence theorem for manifolds with boundary, we obtain

\begin{align}
    a_0(D_F) &= \frac{1}{2^p \pi^{p+\frac{1}{2}}} \int_M \text{dvol}_M, \\
    a_1(D_F) &= -4^{-1}(4\pi)^{-\frac{d}{2}} \frac{2^{p+q}}{12 \cdot 2^{p+q}} \int_{\partial M} \text{dvol}_{\partial M}, \\
    a_2(D_F) &= \frac{1}{12 \cdot 2^{p+q}} \left\{ \int_M r_M \text{dvol}_M + 4 \int_{\partial M} L_{aa} \text{dvol}_{\partial M} \right\}, \\
    a_3(D_F) &= -4^{-1}(4\pi)^{-\frac{d}{2}} \frac{96^{-1}}{360} \left\{ \int_{\partial M} \left[ (-8r_M + 8R_{aNaN} + 7L_{aa}L_{bb} \\
    &\quad - 10L_{ab}L_{ab}) \text{dvol}_{\partial M} \right] \right\}, \\
    a_4(D_F) &= \frac{(4\pi)^{-\frac{d}{2}}}{360} \frac{2^{-p+q}}{12 \cdot 2^{p+q}} \left\{ \int_M \left[ 5 \frac{1}{4} - 2R_{ijij}R_{ikj} - \frac{7}{4} R_{ikj}^2 + \frac{15}{2} \left| R^F \right|^2 \right] \text{dvol}_M \\
    &\quad + \int_{\partial M} \left[ \text{tr} \left[ (-5r_M,bb - 10r_ML_{aa} + 4R_{aNaN}L_{bb} - 12R_{ab}L_{ab},bb + 4R_{ab}L_{ac} + 24L_{aa},bb + 40/21L_{aa}L_{bc}L_{cc} \\
    &\quad - 88/7L_{ab}L_{ab}L_{cc} + 320/21L_{ab}L_{bc}L_{ac} \text{dvol}_{\partial M} \right] \right\}. 
\end{align}

Consider $\tilde{M} = I \times f M$ be a Riemannian manifold with the metric $g_f = dt^2 + f^2(t)g^M$. As in [10], we take normal coordinates in boundary and we get orthonormal frame $\{\partial_i, e_1, e_2, e_3\}$. In the sequel we let $L_{aa} = \langle \nabla e_a, e_a, \partial_i \rangle$ be the second fundamental form and $\tilde{R}_{ijkl} = \langle \tilde{R}(e_i, e_j) e_k, e_l \rangle$ be the components of the curvature tensor in local coordinates in $\mathbb{R}^3$. Then we obtain

\begin{align}
    L_{aa} &= \langle \nabla e_a, e_a, \partial_i \rangle = -\delta_a^i (ln f)', \\
    \tilde{R}_{aNaN} &= \langle \tilde{R}(e_a, \partial_i) e_a, \partial_i \rangle = 3 \frac{f''}{f}. 
\end{align}
Similarly we obtain
\[ L_{bb} = -3(ln f)' \quad L_{aa} L_{bb} = 9 \left( \frac{f'}{f} \right)^2 \quad L_{ab} L_{ab} = 3 \left( \frac{f'}{f} \right)^2 \]
\[ L_{aa} L_{bb} L_{cc} = -27 \left( \frac{f'}{f} \right)^3 \quad L_{ab} L_{ab} L_{cc} = -9 \left( \frac{f'}{f} \right)^3 \quad L_{ab} L_{bc} L_{ac} = -3 \left( \frac{f'}{f} \right)^3 \]
\[ \tilde{R}_{ijkl} \tilde{R}_{ijkl} = \tilde{R}^M_{ijkl} \tilde{R}^M_{ijkl} + 12 \left( \frac{f''}{f} \right)^2 \]
\[ \tilde{R}_{ijkl} \tilde{R}_{ijkl} = (\tilde{R}^M_{ijkl})^2 + 12 \left( \frac{f''}{f} \right)^2 \]
\[ \tilde{R}_{ijkl} \tilde{R}_{ijkl} = -3 \left( \frac{f'}{f} \right)^2 \quad \tilde{R}_{ijkl} \tilde{R}_{ijkl} = 3 \left( \frac{f'}{f} \right)^2 \]
\[ r_M = \frac{R_M}{f} + 6 \left( \frac{f''}{f} \right)^2 \]
\[ ||R^{F+}||^2 = \sum_{s,t,r,l=1}^n (R^M_{rlls})^2 \quad (6.24) \]

Then we obtain

**Theorem 6.3.** Let \( \tilde{M} = I \times_f M \) be a compact 4-dimensional oriented foliation with spin leaf, then the spectral action for sub-Dirac operators

\[ a_0(D_F) = \frac{1}{2 \pi^p + 2} \int_{\tilde{M}} \text{dvol}_{\tilde{M}}, \]
\[ a_1(D_F) = -4^{-1} (4\pi)^{-1} \left( \frac{m-3}{2} + 2p + q \right) \int_{\tilde{M}} \text{dvol}_{\tilde{M}}, \]
\[ a_2(D_F) = \frac{1}{12 \cdot 2^{p+q + 1}} \left[ - \int_{\tilde{M}} \left( \frac{R_M}{f} + 6 \left( \frac{f''}{f} \right)^2 \right) \text{dvol}_{\tilde{M}} - 12 \int_{\partial \tilde{M}} (ln f)' \text{dvol}_{\partial \tilde{M}} \right], \]
\[ a_3(D_F) = -\frac{1}{384} (4\pi)^{-1} \left( \frac{m-3}{2} + 2p + q \right) \int_{\tilde{M}} \left( -8 \frac{R_M}{f^2} - 24 \frac{f''}{f} - 15 \left( \frac{f'}{f} \right)^2 \right) \text{dvol}_{\partial \tilde{M}}, \]
\[ a_4(D_F) = \left( \frac{4\pi}{30} \right)^{-1} \left\{ \int_{\tilde{M}} \left( \frac{5}{2} \frac{R_M}{f} + 6 \left( \frac{f''}{f} \right)^2 \right)^2 - 2 R^M_{ijkl} R^M_{ijkl} + \frac{23}{4} (R^M_{ijkl})^2 - 45 \left( \frac{f''}{f} \right)^2 \right\} \text{dvol}_{\tilde{M}} \]
\[ + \int_{\partial \tilde{M}} \text{tr} \left( \left( \frac{102}{f^3} + 30 \frac{f'}{f} + 12 \frac{f''}{f} \right) r_M - 306 \left( \frac{f''}{f} \right)^2 - 378 \left( \frac{f''}{f} \right) \left( \frac{f'}{f} \right)^2 + 180 (\frac{f'}{f})^4 \right)^2 \]
\[ + 628 \left( \frac{f''}{f} \right)^2 \text{dvol}_{\partial \tilde{M}} \right\}, \quad (6.29) \]

Nextly, Consider \( \tilde{M} = S^1 \times_f M^n \) be a Riemannian manifold with the metric \( g_f = dt^2 + f^2(t) g^M \). As in \( 10 \), we take normal coordinates in boundary and we get orthonormal frame \( \{ \partial_t, e_1, e_2, e_3 \} \). By (2.34), we get \( ||R^{F+}||^2 = \sum_{s,t,r,l=1}^n (R^M_{rlls})^2 \). Then by (6.24) and Theorem 2.9, we obtain

**Theorem 6.4.** Let \( \tilde{M} = S^1 \times_f M^n \) be a Robertson-Walker space, then

\[ \text{Vol}^{(n-3)}_{\tilde{M}} (M, F) = \frac{v_{n+1,n-3}}{360 \cdot 2^p + 4} \int_{\tilde{M}} \left( \frac{5}{4} \left( \frac{R_M}{f^2} + 6 \left( \frac{f''}{f} \right)^2 \right) + 2 R^M_{ijkl} R^M_{ijkl} \right) \text{dvol}_{\tilde{M}} \]
\[ + \frac{23}{4} (R^M_{ijkl})^2 - 45 \left( \frac{f''}{f} \right)^2 \text{dvol}_{\tilde{M}}; \]
\[ \text{Vol}^{(n-1)}_{\tilde{M}} (M, F) = -\frac{v_{n+1,n-1}}{12 \cdot 2^p + 4} \int_{\tilde{M}} \left( \frac{R_M}{f^2} + 6 \left( \frac{f''}{f} \right)^2 \right) \text{dvol}_{\tilde{M}} \]
\[ \text{Vol}^{(n+1)}_{\tilde{M}} (M, F) = \frac{v_{n+1,n+1}}{2^p + 4} \int_{\tilde{M}} f^4 \text{dvol}_{\tilde{M}}. \]

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when $k$ is even and $n$ is even, $v_{n,k} = \frac{k}{n(2\pi)} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}$; when $k$ is odd and $n$ is odd, $v_{n,k} = \frac{k}{n} \frac{\Gamma\left(\frac{k-2n+1}{2}\right)}{\pi} \times \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}$.

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