Non-perturbative determination of $Z_A^{\text{stat}}$ in quenched QCD\textsuperscript{*}†

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We non-perturbatively calculate the renormalization factor of the static axial vector current in $O(a)$ improved quenched lattice QCD. Its scale dependence is mapped out in the Schrödinger functional scheme by means of a recursive finite-size scaling technique, taking the continuum limit in each step. We also obtain $Z_A^{\text{stat}}$ for Wilson fermions in order to renormalize existing unimproved data on $F_B^{\text{bare}}$ non-perturbatively.

1. INTRODUCTION

The decay constant $F_B$, governing the leptonic decay of a B-meson, is a quantity of much phenomenological interest. However, as heavy flavours on the lattice escape a direct numerical treatment, the static approximation represents a valuable alternative, since this effective theory has simplified dynamics and describes the large mass limit of the theory. Yet the problems of this framework have been twofold in the past. Owing to the infinitely heavy b-quark, (i) the renormalization properties of the static theory are different, i.e. the renormalization constant $Z_A^{\text{stat}}$ of the axial current in $(A_R^{\text{stat}})_0 = Z_A^{\text{stat}}(\mu) \psi_d(0)\gamma_5 \psi_b^{\text{stat}}$ becomes scale ($\mu$) dependent, thereby entailing an additional uncertainty, and (ii) MC computations of the matrix element itself are difficult.

Here, we solve (i) by matching through the renormalization group invariant (RGI) operator: The non-perturbatively calculated $\mu$-dependence of any renormalized matrix element of $A_R^{\text{stat}}$, $\Phi = \Phi^{\text{stat}} = \langle f | (A_R^{\text{stat}})_{0} | i \rangle$, together with the value of $Z_A^{\text{stat}}$ at some (low) $\mu$, yields a factor leading to the RGI $\Phi$. It is then related to the ‘matching’ scheme, which approximates the relativistic theory up to $1/m$-corrections. For a more detailed explanation of the overall strategy we refer to \cite{1} and for a full report on our work to \cite{2}.

2. STRATEGY

Our determination of $Z_A^{\text{stat}}$ and its running with $\mu$ uses the Schrödinger functional (SF) \cite{3} as intermediate scheme \cite{3}. The procedure resembles ALPHA’s quark mass renormalization \cite{3}:

1. Choose a proper combination $X$ of SF correlation functions such that, with $X_R = Z_A^{\text{stat}} X$ finite and $X = X^{(0)} + X^{(1)} g_0^2 + \cdots$, the condition $X_R \equiv X^{(0)}$ non-perturbatively defines $Z_A^{\text{stat}}$, running with the SF’s box size $L = 1/\mu$.

2. Map out the $L$-dependence recursively by employing the finite-size step scaling function (SSF) $\sigma_X^{\text{stat}}(u, a/L)$, $\sigma_X^{\text{stat}}(u, a/L) = \frac{Z_X^{\text{stat}}(g_0, 2L/a)}{Z_X^{\text{stat}}(g_0, L/a)}$, $u \equiv \bar{g}_S(L)$, (1)

computed on the lattice and extrapolated to the continuum as illustrated in fig. \cite{3}.

3. Evolve $Z_A^{\text{stat}}$ from initially large $L$ (low $\mu$) to small $L$ (high $\mu$) by repeatedly applying $[\sigma_A^{\text{stat}}]^{-1}$; from there, continue with the perturbative high-energy behaviour to arrive at (scale and scheme independent) RGI matrix elements

\begin{equation}
\Phi_{\text{RGI}} = \Phi(\mu) \times \left( \frac{2 b_0 \bar{g}^2}{-\gamma_0/2b_0} \right) \times \exp \left\{ - \int_0^{\bar{g}} \frac{dg}{\bar{g}} \left[ \frac{\sigma(g) - \gamma_0}{b_0 \bar{g}} \right] \right\}
\end{equation}

of $(A_R^{\text{stat}})_0$, with $\bar{g} = \bar{g}_S(L = 1/\mu)$ and universal coefficients $b_0 = 11/(4\pi^2)$ and $\gamma_0 = -1/(4\pi^2)$.
Finally, for sufficiently high $\mu$, the RG invariant $\Phi_{\text{RGI}}$ can be converted to the ‘matching’ scheme via a perturbative factor $\Phi_{\text{match}}(\mu)/\Phi_{\text{RGI}}$.

Figure 1. Lattice step scaling function and its continuum extrapolation for some selected $u$.

3. SURVEY OF THE RESULTS

The simulation and analysis as well as the SF setup are basically analogous to [5], except that the boundary coefficient $c_t$ is set to 2-loop [6]. While the inclusion of static quarks and their impact on $O(a)$ improvement are described in [7], we only recall the heavy quark lattice action [8],

$$S_h[U, \bar{\psi}_h, \psi_h] = a^4 \sum_x \bar{\psi}_h(x) \nabla^*_0 \psi_h(x)$$

($\nabla^*_0$ the backward derivative’s time component, i.e. static quarks propagate only forward in time), and the $O(a)$ improvement term of $A^{\text{stat}}_0$ in

$$(A^{\text{stat}}_0) = A^{\text{stat}}_0 + a c^{\text{stat}}_0 \bar{\psi}_1 \gamma_5 \gamma_\tau \left( \nabla_j + \nabla^*_j \right) \psi_h,$$

where the 1–loop $c^{\text{stat}}_0 = -1/(4\pi)$ [3] is used.

Although in ref. [7] a suitable renormalization condition for $A^{\text{stat}}_0$ in the SF scheme has already been formulated, we slightly modify it in the present non-perturbative context for reasons of numerical precision. We impose instead

$$X(u, a/L) \equiv \left. \frac{f^{\text{stat}}_A(L/2)}{[f_1 \times f^{\text{hh}}_1(L/2)]^{1/4}} \right|_{\sigma^2 = u},$$

at vanishing quark mass. $f^{\text{stat}}_A$ is a correlator between a static-light pseudoscalar boundary source and $A^{\text{stat}}_0$ in the bulk, $f_1$ between two light-quark boundary sources, and $f^{\text{hh}}_1$ denotes a correlator in $x_3$–direction of Wilson lines between static-light boundary sources, which is efficiently evaluated in MC by multi-hit [2]. Note that the boundary wave function renormalizations and a linearly divergent mass counterterm cancel in eq. (3).

Fig. 2 displays the continuum SSF after $a \to 0$ extrapolation of $\Sigma^{\text{stat}}_A$ together with an interpolating fit. $\sigma^{\text{stat}}_A$ being known, it connects $\Phi(\mu)$ in the low-energy regime at $\mu_m = 1/L_m = (2L_{\text{max}})^{-1}$ step-by-step with the perturbative domain, and once $\mu$ is large enough for perturbative running to set in, we find by numerical integration in eq. (2):

$$\Phi(\mu)/\Phi_{\text{RGI}} = 1.088(8) \quad \text{at} \quad \mu = \mu_m.$$ (6)

In fig. 3 we compare the numerically computed running of the static axial current in the SF scheme with perturbation theory, where also $\Lambda L_{\text{max}} = 0.211$ from ref. [5] enters the analysis.

3.1. The total renormalization factor

We still need to relate $(A^{\text{stat}}_0) / (\Phi_{\text{RGI}})$ in the SF scheme compared to perturbation theory.

$$\Phi(\mu)/\Phi_{\text{RGI}}$$

Figure 3. Running matrix element of $(A^{\text{stat}}_0)$ in the SF scheme compared to perturbation theory.
to translate any bare matrix element $\Phi_{\text{bare}}(g_0)$ of $A_0^{\text{stat}}$ into the RGI one, $\Phi_{\text{RGI}}$, then reads

$$Z_\Phi(g_0) = \frac{\Phi_{\text{RGI}}}{\Phi(\mu)} \bigg|_{\mu=\mu_m} \times Z_A^{\text{stat}}(g_0, L/a) \bigg|_{L=L_m} \quad (7)$$

with $\Phi_{\text{RGI}}/\Phi(\mu_m)$ a universal part independent of the regularization and $Z_A^{\text{stat}}(g_0, L/a)\big|_{L=L_m}$ that depends on it. $Z_\Phi$ will be given explicitly in $[2]$.

### 3.2. Wilson data on $F_B$ revisited

Since we now also have the SF renormalization factor $Z_A^{\text{stat}}(g_0, L/a)\big|_{L=L_m}$ for unimproved Wilson fermions ($c_{\text{sw}}=0$) available, it may be combined with the universal part $\Phi_{\text{RGI}}/\Phi(\mu)$ at $\mu = \mu_m$ and the perturbative conversion factor $\Phi_{\text{match}}(\mu)/\Phi_{\text{RGI}}$ at scale $\mu = m_{b,\text{MS}}$ in order to confront it with tadpole-improved estimates of the FNAL group $[10]$ on the $Z$–factor, which equals $\Phi_{\text{match}}(m_{b,\text{MS}})/\Phi_{\text{bare}}(g_0)$ in our notation. As it turns out (cf. $[1]$), they deviate significantly in the relevant range of $g_0$, which reveals the importance of non-perturbative renormalization.

Just so, we non-perturbatively renormalize existing Wilson data $[10]$ on $\Phi_{\text{bare}} = F_{B_{\text{bare}}} \sqrt{m_B}$,

$$Z_{A,\text{SF}}^{\text{stat}} \bigg|_{L_m} \times r_0^{3/2} \Phi_{\text{bare}} = r_0^{3/2} F_{\text{SF}} \bigg|_{\mu_m}, \quad (8)$$

and extrapolate them to $a = 0$ in fig. $[3]$ discarding the rightmost point ($\beta=5.7$). With $r_0 = 0.5$ fm, this results in $F_{B_{\text{bare}}} = 261(46)$ MeV at $\mu = m_{b,\text{MS}}$ in the ‘matching’ scheme, containing all errors apart from quenching.

## 4. CONCLUSIONS

We have performed the scale dependent renormalization of $A_0^{\text{stat}}$ by constructing a non-perturbative RG in the SF scheme, and agreement with perturbation theory at large scales is demonstrated. The renormalization factors needed to get the associated RG invariants are computed with good numerical accuracy $[2]$, which is a crucial prerequisite for a controlled determination of $F_B$ in the static limit.

Our continuum extrapolation that uses unimproved data from the literature and also quite large lattice spacings leaves much room to improve on the present result $F_{B_{\text{bare}}} = 261(46)$ MeV.

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