COUNTING THE NUMBER OF SUBGROUPS AND NORMAL SUBGROUPS OF THE GROUP $U_{2np}$, $p$ IS AN ODD PRIME

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Abstract. The aim of this paper is to compute the number of subgroups and normal subgroups of the group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e, aba^{-1} = b^{-1} \rangle$, where $p$ is an odd prime. Suppose $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ in which $p_i$’s are distinct odd primes, $\alpha_i$’s are positive integers and $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$. It is proved that the number of subgroups is $2\tau(2n) + (p - 1) \left( \tau(\frac{n}{p}) + \tau(\frac{t}{p}) \right)$, when $p \mid n$ and $2\tau(2n) + (p - 1) \lfloor \tau(t) \rfloor$, otherwise. It will be also proved that this group has $\tau(2n) + \tau(n)$ normal subgroups.

Keywords. group; subgroup; dihedral group; finite group.

1. Introduction

Cavior [1] proved that the number of subgroups of a dihedral group of order $2n$ can be computed by $\tau(n) + \sigma(n)$. After publishing this work Calhoun [2] computed the number of subgroups in certain finite groups. For more information on this problem, we encourage the readers to consult the interesting book of Tărnăuceanu [6].

Following Darafsheh and Yaghoobian [3], we define:

$$U_{2nm} = \langle a, b \mid a^{2n} = b^m = e, aba^{-1} = b^{-1} \rangle.$$ 

This group has order $2nm$ and can be written as the semi-direct product of two cyclic groups that one of them is of order $m$ and another one has order $2n$. Set $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$, where $p_i$’s are distinct odd prime numbers and $\alpha_i$’s are positive integers. Shelash [4], introduced an algorithm for computing all subgroups and normal subgroups of a finite group. Shelash and Ashrafi [5] applied this algorithm to compute the number of minimal and maximal subgroups of certain finite groups.

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Lemma 2.1, one can see that

For example if $|\Phi = 1 2 4 \leq 2^4 1 2 4 | 1 2 4 c_i |

Proof. By presentation of the group $U_{2n_p}$ we assumed that $c_0 < c_1 < \cdots < c_{\alpha -1}$, where $\alpha = \tau(\prod_{1 \leq i \leq s} p_i^{\alpha_i})$. For example if $|G| = 60$, then the order table of $G$ is as follows:

| $c_i$ | 1   | 2   | 2$^2$ |
|-------|-----|-----|------|
| $c_0$ | 1   | 2   | 4    |
| $c_1$ | 3   | 6   | 12   |
| $c_2$ | 5   | 10  | 20   |
| $c_3$ | 15  | 30  | 60   |

Throughout this paper our notations are standard and can be taken from the standard books on group theory. The function $\sigma(n)$ is defined as the summation of all divisors of $n$. Furthermore, the number of subgroups and normal subgroups of a group $G$ are denoted by $Sub(G)$ and $NSub(G)$, respectively. Our calculations are done with the aid of GAP [7].

2. Main Results

The group $U_{2n_p} = \langle a, b \mid a^{2n} = b^p = e \mid aba^{-1} = b^{-1} \rangle$ is a finite group of order $2n_p$, where $p$ is an odd prime. Suppose $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ in which $p_i$'s are distinct odd primes and $\alpha_i$'s are positive integers. For simplicity of our argument, we assume that $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$. If $p = p_k | n$ then the order of $U_{2n_p}$ is equal to $2^{r+1} p_{k}^{\alpha_k+1} \cdots p_s^{\alpha_s}$, otherwise it is $2^{r+1} p \prod_{1 \leq i \leq s} p_i^{\alpha_i}$.

Lemma 2.1. The following hold:

1. If $q$ is even then $a^q b^w = b^w a^q$;
2. If $q$ is odd then $a^q b^w = b^{-w} a^q$.

Proof. By presentation of the group $U_{2n_p}$, we have $aba^{-1} = b^{-1}$ and so if $q$ is even then $a^q b = ba^q$. Furthermore, if $q$ is odd then $a^q b = b^{-1} a^q$. Choose positive integer $w$. Then $a^q b^w = ba^q b^{w-1}$. If $q$ is even number, thus $a^q b^w = b^w a^q$. If $q$ is odd number then $a^q b^w = b^{-1} a^q b^{w-1}$, then $a^q b^w = b^{-w} a^q$. \hfill \Box

Proposition 2.1. Let $n = 2^r t$, $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ and $m = p$ be an odd prime number. Then the structure description of the group $U_{2n_p}$ is $C_t \times (C_p : C_{2^r+1})$.

Proof. Suppose $\Phi = \langle a^{2^{r+1}} \rangle$, $\Psi = \langle b \rangle$ and $\Omega = \langle a^t \rangle$ are subgroups of $U_{2n_p}$. By Lemma 2.1, one can see that $g \Phi g^{-1} = g \langle a^{2^{r+1}} \rangle g^{-1} = \langle a^{2^{r+1}} \rangle = \Phi$, for all $g \in U_{2n_p}$.  

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Thus $\Phi \leq U_{2np}$. Define $(\Psi : \Omega) = \langle b, a^i \rangle$. If $i$ is odd then,
\[
    a^i b^j (\Psi : \Omega) b^{-j} a^{-i} = a^i b^j (\langle b, a^i \rangle b^{-j} a^{-i} = \langle a^i b^j b^{-j} a^{-i}, a^i b^j a^i b^{-j} a^{-i} \rangle = \langle b, a^i b^j a^i b^{-j} a^{-i} \rangle = \langle b, a^i b^j \rangle = (\Psi : \Omega),
\]
and if $i$ is an even number,
\[
    a^i b^j (\Psi : \Omega) b^{-j} a^{-i} = a^i b^j (\langle b, a^i \rangle b^{-j} a^{-i} = \langle a^i b^j b^{-j} a^{-i}, a^i b^j a^i b^{-j} a^{-i} \rangle = \langle b, a^i b^j a^i b^{-j} a^{-i} \rangle = \langle b, a^i b^j \rangle = (\Psi : \Omega).
\]

Hence $(\Psi : \Omega)$ is a normal subgroup of $U_{2np}$. On the other hand,
\[
    \langle a^{2^{r+1}} \rangle \cap \langle b, a^i \rangle = e
\]
and
\[
    \frac{|\langle a^{2^{r+1}} \rangle \times |\langle b, a^i \rangle|}{|\langle a^{2^{r+1}} \rangle \cap \langle b, a^i \rangle|} = 2np,
\]
which completes our argument.

Lemma 2.2. The group $U_{2np}$ has the following types of subgroup:

1. The cyclic subgroups $\langle a^i \rangle$ of order $\frac{2np}{i}$, where $i \mid 2n$;
2. The subgroups $\langle a^i, b \rangle$ of order $\frac{2np}{i}$, where $i \mid 2n$;
3. The cyclic subgroups $\langle a^i b^j \rangle$, where $i \mid 2n$, $2p^k \nmid i$ and $j = 1, \ldots, p - 1$.

Proof. Set $H = \langle a^i \rangle$ and $K = \langle b \rangle$, $i \mid 2n$. By presentation of $U_{2np}$, $K$ is normal and so $HK = \langle a^i, b \rangle$ has order $\frac{2np}{i}$. The result now follows from Lemma 2.1.

Proposition 2.2. Let $n = 2^r \prod_{1 \leq i \leq s, p_i}$ be a positive integer and $p$ be an odd prime number. The following hold:

1. There is at most one subgroup of order $k$ such that $2 \mid k, 2^{r+1} \nmid k$ and $p \nmid k$;
2. If $p \mid n$, then there exists one subgroup of order $k$ such that $p^{\alpha_i+1} \mid k$;
3. There exists $p$ subgroups of order $k$ when $p \nmid k$ and $2^{r+1} \mid k$;
4. There exists $\sigma(p)$ subgroups of order $k$ when $p \mid k$ and $p^{\alpha_i+1} \nmid k$.

Proof. Our main proof will consider the following parts:

1. Suppose $p \nmid 2^h v$, $1 \leq h \leq r$, and $v \mid n$. Then $\langle a^{\frac{2^{r+1} - h}{v}} \rangle$ is a cyclic group of order $2^h v$ and the order of subgroups $\langle a^{\frac{2^{r+1} - h}{v}} b \rangle$ and $\langle a^{\frac{2^{r+1} - h}{v}} \rangle$ are not $2^h v$. We now apply Lemma 2.2 to get the result.
2. Suppose $2^{r+1} \mid k$. Since $\frac{k}{2}$ is an odd number, by Lemma 2.1 $\langle a^\frac{k}{2} b^i \rangle$ are cyclic subgroups of order $2^{r+1}v$, $1 \leq j \leq p$.

3. Consider the subgroups $\langle a^{\frac{k}{2}p} \rangle$ and $\langle a^{\frac{k}{2}p}, b \rangle$, where $1 \leq h \leq r + 1$. Since there are $p - 1$ subgroups of type $\langle a^{\frac{k}{2}p} b^j \rangle$, $1 \leq j \leq p - 1$, the number of all subgroups of order $k$ is equal to $\sigma(p)$.

Hence the result. □

**Theorem 2.1.** Let $p$ be an odd prime and $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$, where $p_i$'s are distinct odd primes, $\alpha_i$'s are positive integers and $t = \prod_{1 \leq i \leq s} p_i^{\sigma_i}$. Then the number of all subgroups of the group $U_{2np}$ is given by the following:

1. If $p \mid n$ then $\text{Sub}(U_{2np}) = 2\tau(2n) + (p - 1) \left[ \tau\left(\frac{2n}{p}\right) + \tau\left(\frac{2n}{p^2}\right) \right]$.

2. If $p \nmid n$ then $\text{Sub}(U_{2np}) = 2\tau(2n) + (p - 1) \lfloor \tau(t) \rfloor$.

**Proof.** By presentation of the group $U_{2np}$, it has $\tau(2n)$ subgroups contained in $\langle a \rangle$. Since $\langle b \rangle$ is a normal subgroup, the group $U_{2np}$ has $\tau(2n)$ subgroups of the form $H \langle b \rangle$ such that $H$ is a subgroup of $\langle a \rangle$. We now assume that $p \mid n$. By Lemma 2.2, it is enough to count the number of subgroups in the form $\langle a^ib^j \rangle$, where $i \mid 2n$, $2p^\alpha \nmid i$ and $1 \leq j \leq p - 1$. Note that $2n$ has exactly $\tau\left(\frac{2n}{p}\right) = \tau\left(\frac{2n}{p^2}\right)$ odd divisors and the number of all divisors of $2n$ such that $2p \mid i$ and $2p^\alpha \nmid i$ is equal to $\tau\left(\frac{2n}{p}\right) = \tau\left(\frac{2n}{p^2}\right)$. So the group $U_{2np}$ has exactly $(p - 1)\left[ \tau\left(\frac{2n}{p}\right) + \tau\left(\frac{2n}{p^2}\right) \right]$ subgroups, when $p \mid n$. If $p \nmid n$, then the number of subgroups of type $\langle a^ib^j \rangle$ is equal to $(p - 1)\tau\left(\frac{2n}{p}\right) = (p - 1)\tau(t)$. □

We are now ready to count the number of normal subgroups of the group $U_{2np}$.

**Lemma 2.3.** The normal subgroup of the group $U_{2np}$ has one of the following forms:

1. All cyclic subgroups $\langle a^i \rangle$ such that $2 \mid i \mid 2n$;

2. All subgroups $\langle a^i, b \rangle$, when $i \mid 2n$.

**Proof.** The first part follows from Lemma 2.1. We apply the presentation of $U_{2np}$ to prove that $\langle a^k, b \rangle$ is normal, when $k \mid 2n$. Choose the element $a^ib^j$ in $U_{2np}$. Then we have four cases for the subgroup $a^ib^j \langle a^k, b \rangle$ as follows:

1. $k$ and $i$ are even numbers. In this case $\langle a^ib^ja^{-i}, a^ib^jb^{-j}a^{-i} \rangle = \langle a^k, b \rangle$, as desired.

2. $k$ is even and $i$ is odd. Then, $\langle a^ib^ja^{-i}, a^ib^jb^{-j}a^{-i} \rangle = \langle a^k, b \rangle$ which proves our claim.
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3. $k$ and $i$ are odd numbers. This shows that $\langle a^ib^j a^kb^{-j} a^{-i}, a^ib^j b^{-j} a^{-i} \rangle = \langle a^k b^2, b \rangle = \langle a^k, b \rangle$.

4. $k$ is even and $i$ is odd. In this case, $\langle a^ib^j a^kb^{-j} a^{-i}, a^ib^j b^{-j} a^{-i} \rangle = \langle a^k b^{-2j}, b \rangle = \langle a^k, b \rangle$.

Note that $a^k$ and $a^kb^j$ has the same order, when $k$ is odd number. □

Choose $a^i \in U_{2np}$, where $i$ is an odd number. Then $a^i \langle a^ib^j \rangle a^{-i} = \langle a^ib^j a^{-i} \rangle = \langle a^ib^{-j} \rangle$. Since $\langle a^ib^{-j} \rangle \neq \langle a^ib^j \rangle$, all subgroups $\langle a^ib^j \rangle$, $1 \leq j \leq p$ and $i | 2n$, are not normal in $U_{2np}$.

**Theorem 2.2.** The number of normal subgroups in the group $U_{2np}$ is given by $NSub(U_{2np}) = \tau(2n) + \tau(n)$.

**Proof.** Let $p$ be an odd prime and $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$, where $p_i$'s are distinct odd primes, $\alpha_i$'s are positive integers and $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$. To prove the theorem, we apply Lemma 2.3. We now that each subgroup of type $\langle a^i \rangle$, $i$ is even, is normal. Since

$$\begin{align*}
\tau(2^{r+1}t) - \tau(t) &= \\
\tau(2^{r+1} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) &= (r+2)\tau(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) \\
&= (r+1)\tau(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) \\
&= \tau(2^{r+1} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) \\
&= \tau(n),
\end{align*}$$

$\tau(2^{r+1}t)$ is the number all divisors of $2n$ and $\tau(t)$ is the number of odd divisors of $2n$, $\tau(2^{r+1}t) - \tau(t) = \tau(2^n) = \tau(n)$ is the number of even divisors of $2n$. On the other hand, the number of all normal subgroups of type $\langle a^i, b \rangle$, $i | 2n$, is equal to $\tau(2^n)$. Therefore, $NSub(U_{2np}) = \tau(2n) + \tau(n)$. □

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