RANDOM EXPONENTIAL ATTRACTOR FOR STOCHASTIC DISCRETE LONG WAVE-SHORT WAVE RESONANCE EQUATION WITH MULTIPLICATIVE WHITE NOISE

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Abstract. We mainly consider the existence of a random exponential attractor (positive invariant compact measurable set with finite fractal dimension and attracting orbits exponentially) for stochastic discrete long wave-short wave resonance equation driven by multiplicative white noise. Firstly, we prove the existence of a random attractor of the considered equation by proving the existence of a uniformly tempered pullback absorbing set and making an estimate on the “tails” of solutions. Secondly, we show the Lipschitz property of the solution process generated by the considered equation. Finally, we prove the existence of a random exponential attractor of the considered equation, which implies the finiteness of fractal dimension of random attractor.

1. Introduction. It is well known that the existence of attractor (including the global attractor, pullback attractor, uniform attractor, exponential attractor) and the estimate of its dimension are two main topics in analyzing the asymptotic behavior of deterministic or stochastic infinite-dimensional dynamic systems. Since late 1970s, the research on the theory of attractors for dynamic systems has made substantial progress. The global attractors can describe the asymptotic dynamics of deterministic autonomous dynamical systems, see [6]. The uniform attractors and pullback attractors capture asymptotic behavior of deterministic non-autonomous equations.
dynamical systems. In reality, dynamic systems are often affected by some external random factors. These random effects are not only introduced to compensate the defects in deterministic models, but also to explain the intrinsic phenomena. Recently, random dynamic systems have drawn much attention from researchers due to their wide range of applications. The random attractor is one of most important tools which are used to determine long-time behavior of solutions for autonomous or non-autonomous random dynamic systems, see [1, 12, 4, 3, 10, 9, 2, 15, 16]. However, random attractors attract the orbit slowly compared to exponential attractors, see [13], which are not conducive to numerical estimation and application.

In order to solve this problem, Shirikyan and Zelik in [8] introduced the concept of random exponential attractor and Zhou in [14] proposed some sufficient conditions on how to construct a random exponential attractor for stochastic lattice systems on a separable Banach space of infinite sequences and proved the existence of random exponential attractors for first order lattice systems.

The long wave-short wave resonance equations are an important model in nonlinear science, and they arise naturally in the interaction of surface waves with both gravity and capillary modes present, and also in the analysis of internal waves as well as Rossby waves, see [5]. In fact, the discrete long wave-shortwave resonance equation can be regarded as an “approximation” to the corresponding continuous PDEs, that is, the long wave-short wave resonance equation. Zhao and Zhou in [11] proved the existence of a compact kernel section of non-autonomous long wave-short wave resonance equation on infinite lattice. Liang and Zhu in [7] estimated the fractal dimension of the kernel section obtained in [11].

In this paper, we are interested in the random exponential attractor for discrete long wave-short wave resonance equation driven by multiplicative white noise as:

\[
\begin{cases}
\dot{u}_m - (Au)_m - u_m v_m + i\alpha u_m = f_m(t) + au_m \circ \dot{\omega}^{(1)}(t), m \in \mathbb{Z}^N, \\
\dot{v}_m + \beta v_m + \lambda(B(|u|^2))_m = g_m(t) + bv_m \circ \dot{\omega}^{(2)}(t), m \in \mathbb{Z}^N,
\end{cases}
\tag{1.1}
\]

with initial values

\[u_m(\tau) = u_{r,m}, v_m(\tau) = v_{r,m}, m = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N, \tau \in \mathbb{R}, t > \tau, \tag{1.2}\]

where $\mathbb{Z}^N$ denotes the product of $N$ integer sets; $\alpha$, $\beta$, $\lambda$ are positive constants; $i$ is the unit of the imaginary number ($i^2 = -1$); $u_m(t) \in \mathbb{C}$, $v_m(t) \in \mathbb{R}$ ($\mathbb{C}$, $\mathbb{R}$ are the sets of complex and real numbers, respectively), $u = (u_m(t))_{m \in \mathbb{Z}^N}$, $v = (v_m(t))_{m \in \mathbb{Z}^N}$, $Au = (Au(t)m)_{m \in \mathbb{Z}^N}$, $B(|u|^2) = (|B(|u(t)|^2))_{m \in \mathbb{Z}^N}$; the definitions of $A, B$ are the same as those in (3.1)-(3.5); $f(t) = (f_m(t))_{m \in \mathbb{Z}^N}$, $g(t) = (g_m(t))_{m \in \mathbb{Z}^N} : \mathbb{R} \to l^2$ are bounded continuous functions (the definition of $l^2$ is given below); $a, b \in \mathbb{R}$; $\omega^{(1)}, \omega^{(2)}$ are mutually independent two-side real-value Wiener process on probability space $(\mathbf{\Omega}, \mathcal{F}, P)$, where $\mathbf{\Omega} = \{\omega \in C(\mathbb{R}, \mathbb{R}^N) : \omega(0) = 0\}$, $\mathcal{F}$ is a Borel $\sigma$-algebra induced by the compact open topology of $\mathbf{\Omega}$, $P$ is the Wiener measure on $(\mathbf{\Omega}, \mathcal{F})$, $\circ$ represents a Stratonvich product.

2. Preliminaries. In this section we review the basic concepts and the sufficient conditions of random exponential attractors in [14].

Let $(\mathbf{\Omega}, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ be an ergodic metric dynamical system on probability space $(\mathbf{\Omega}, \mathcal{F}, P)$, where $\{\theta_t : \mathbf{\Omega} \to \mathbf{\Omega}, t \in \mathbb{R}\}$ is a family of measure preserving transformations such that $(t, \omega) \to \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, $\theta_0$ is the identity on $\mathbf{\Omega}$, $\theta_{s+t} = \theta_s \theta_t$ for all $s, t \in \mathbb{R}$. In addition, if for any $F \in \mathcal{F}$, provided $P(\theta_t^{-1} F \Delta F) = 0$, it holds that $P(F) = 0$ or 1 for all $t \in \mathbb{R}$.

Let $X$ be a separable Banach space with Borel $\sigma$-algebra $\mathcal{B}(X)$. A mapping
A family of nonempty bounded subsets of $X$ is a collection of all tempered families of nonempty bounded subsets of $X$, that is, for any family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$, it holds that for every $\varepsilon > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$, $\lim_{\tau \to -\infty} e^{\varepsilon t} || D(\tau + t, \theta \omega)||_X = 0$, where $||D||_X = \sup_{u \in D} ||u||$. For any $u_\tau \in X$ and $\omega \in \Omega$, the subset $\{\Phi(t, \tau, \omega)u_\tau : t \in [\tau, \infty)\} \subset X$ is called a random trajectory starting form $u_\tau$ at initial time $\tau \in \mathbb{R}$ for $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$. Recall that the distance between a point $u \in X$ and a subset $F \subset X$ is given by $d(u, F) = \inf_{v \in F} ||v||_X$. The Hausdorff and symmetric distances between two subsets are defined by, respectively,

$$d_h(F_1, F_2) = \sup_{u \in F_1} d(u, F_2), \quad d_s(F_1, F_2) = \max\{d_h(F_1, F_2), d_h(F_2, F_1)\}, \quad \forall F_1, F_2 \subset X.$$

**Definition 2.1.** A family $\{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ of subsets of $X$ is called a random exponential attractor in $\mathcal{D}(X)$ for the continuous cocycle $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta \omega\}_{\omega \in \Omega})$ if there is a set of full measure $\Omega \in \mathcal{F}$ such that for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, it holds that

1. Compactness: $\mathcal{A}(\tau, \omega)$ is compact in $X$ and measurable in $\omega$,
2. Finite-dimensionality: there exists a random variable $\zeta, (\zeta_\omega < \infty)$ such that

$$\sup_{\tau \in \mathbb{R}} \dim \mathcal{A}(\tau, \omega) \leq \zeta_\omega < \infty,$$

where $\dim \mathcal{A}(\tau, \omega) = \limsup_{\varepsilon \to 0^+} \frac{\ln N_{\varepsilon}(\mathcal{A}(\tau, \omega))}{-\ln \varepsilon}$ is the fractal dimension of $\mathcal{A}(\tau, \omega)$ and $N_{\varepsilon}(\mathcal{A}(\tau, \omega))$ is the minimal number of balls with radius $\varepsilon$ covering $\mathcal{A}(\tau, \omega)$ in $X$,

1. Positive invariance: $\Phi(t, \tau - t, \theta t, \omega)\mathcal{A}(\tau - t, \theta t, \omega) \subseteq \mathcal{A}(\tau, \omega)$ for all $t \geq 0$,
2. Exponential attraction: there exist a constant $a > 0$, such that for any $B \in \mathcal{D}(X)$, there exist random variables $t_B(\tau, \omega) \geq 0, Q(\tau, \omega, ||B||_X) \geq 0$ satisfying,

$$d_h(\Phi(t, \tau - t, \theta t, \omega)B(\tau - t, \theta t, \omega), \mathcal{A}(\tau, \omega)) \leq Q(\tau, \omega, ||B||_X)e^{-at}, \quad t \geq t_B(\tau, \omega).$$

Let $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ be a continuous cocycle on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta \omega\}_{\omega \in \Omega})$. Assume that there exist a family of tempered close random subsets $\{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ of $X$ satisfying the following conditions: for any fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

1. (H1) there exists a tempered random variable $R_{\omega}$ (independent of $\tau$) such that

$$\sup_{\tau \in \mathbb{R}} \sup_{u, v \in \mathcal{A}(\tau, \omega)} ||u - v||_X \leq R_{\omega} < \infty$$

and $R_{\omega}$ is continuous in $t$ for all $t \in \mathbb{R}$;
2. (H2) positive invariance: $\Phi(t, \tau - t, \theta t, \omega)\mathcal{A}(\tau - t, \theta t, \omega) \subseteq \mathcal{A}(\tau, \omega)$ for all $t \geq 0$;
3. (H3) $\{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ is pullback absorbing in the sense that for any set $B \in \mathcal{D}(X)$, there exists $T_B = T_B(\tau, \omega) \geq 0$ such that

$$\Phi(t, \tau - t, \theta t, \omega)B(\tau - t, \theta t, \omega) \subseteq \mathcal{A}(\tau, \omega), \quad \forall t \geq T_B;$$

(H4) there exist a positive numbers $t_0, \delta$, random variables $S_0(\omega) \geq 0$ and a $N$-dimensional projector $P_N : X \to P_N X$ $(\dim(P_N X) = N)$ such that for any $\tau \in$
\( \mathbf{R} \), \( \omega \in \Omega \) and any \( u, v \in \chi(\tau, \omega) \),
\[
\| P_N \Phi(t_0, \tau, \omega) u - P_N \Phi(t_0, \tau, \omega) v \|_X \leq e^{\int_0^{t_0} S_{0}(\theta, \omega) ds} \| u - v \|_X,
\]
(2.2)
\[
\| (I - P_N) \Phi(t_0, \tau, \omega) u - (I - P_N) \Phi(t_0, \tau, \omega) v \|_X
\leq (e^{\int_0^{t_0} S_{1}(\theta, \omega) ds} + \delta e^{\int_0^{t_0} S_{0}(\theta, \omega) ds}) \| u - v \|_X,
\]
(2.3)
where \( t_0, \delta, N \) are independent of \( (\tau, \omega) \), but \( \delta, N \) may depend on \( t_0 \).
(H5) \( t_0, \delta, S_0(\omega), S_1(\omega) \) satisfy:
\[
\begin{cases}
-\infty \leq E[S_1(\omega)] < 0, \
0 \leq E[S_1^2(\omega)] < \infty, \ i = 0, 1, \\
0 < \delta \leq \min \left\{ \frac{1}{8}, e^{-\frac{\delta}{\ln 2}(3E[S_0^2(\omega)]+E[S_1^2(\omega)])} \right\}.
\end{cases}
\]
(2.4)

Then we introduce the following result.

**Proposition 2.2.** (see [14]) Suppose conditions (H1)-(H5) hold. Then \( \{ \Phi(t, \tau, \omega) \}_{t \geq 0, \tau \in \mathbf{R}, \omega \in \Omega} \) possesses a random exponential attractor \( \{ A(\tau, \omega) \}_{\tau \in \mathbf{R}, \omega \in \Omega} \) with properties: for any \( \tau \in \mathbf{R} \) and \( \omega \in \Omega \),
\( (1) \ A(\tau, \omega)(\subseteq \chi(\tau, \omega)) \) is a compact set of \( X \);
\( (2) \ \Phi(t, \tau, \omega) A(\tau, \omega) \subseteq A(t + \tau, \theta \omega), \ \forall t \geq 0; \)
\( (3) \ \text{dim}_f A(\tau, \omega) \leq \frac{2N \ln \left( \frac{3N}{\delta} \right) + 1}{\ln \frac{1}{4}} < \infty; \)
\( (4) \) for any set \( B \in D(X) \), there exists a random variable \( \overline{T}_{\omega} \geq 0 \) and a tempered random variable \( b_{\omega} > 0 \) such that
\[
d_{h}(\Phi(t, \tau, \omega) B(\tau, \omega), A(t + \tau, \theta \omega)) \leq \overline{b}_{\omega} e^{-\frac{\ln 4 t}{400}}, \ t \geq T_B + \overline{T}_{\omega}.
\]
(2.5)

3. Mathematical preparation. We make the following provisions for this paper: for given positive integer \( N \), let
\[
L^2 = \{ u = (u_m)_{m \in \mathbf{Z}^N} : m = (m_1, m_2, \cdots, m_N) \in \mathbf{Z}^N, u_m \in \mathbf{R}, \sum_{m \in \mathbf{Z}^N} u_m^2 < +\infty \}
\]
and
\[
L^2 = \{ u = (u_m)_{m \in \mathbf{Z}^N} : m = (m_1, m_2, \cdots, m_N) \in \mathbf{Z}^N, u_m \in \mathbf{C}, \sum_{m \in \mathbf{Z}^N} |u_m|^2 < +\infty \}.
\]
For convenience, let \( H \) denote the Hilbert space \( L^2 \) or \( L^2 \). The norms and the inner product is defined as: \( \forall u = (u_m)_{m \in \mathbf{Z}^N}, \ v = (v_m)_{m \in \mathbf{Z}^N} \in H \),
\[
(u, v) = \sum_{m \in \mathbf{Z}^N} u_m \overline{v}_m, \ |u|^2 = (u, u) = \sum_{m \in \mathbf{Z}^N} |u_m|^2,
\]
where \( \overline{v}_m \) is the conjugate complex number of \( v_m \).

Define linear operator \( A : H \to H \) as:
\[
A = A_1 + A_2 + \cdots + A_N, \ B = B_1 + B_2 + \cdots + B_N,
\]
(3.1)
and for every \( u = (u_m)_{m \in \mathbf{Z}^N} \in H, \ m = (m_1, m_2, \cdots, m_N) \in \mathbf{Z}^N, \ j = 1, 2, \cdots, N, \)
\[
(A_j u)_m = - u_{(m_1, m_2, \cdots, m_j+1, \cdots, m_N)} + 2u_{(m_1, m_2, \cdots, m_j, \cdots, m_N)} - u_{(m_1, m_2, \cdots, m_j-1, \cdots, m_N)},
\]
(3.2)
and there exist linear operators \( B \) and \( B^* : H \to H \) defined by:
\[
(B_j u)_m = u_{(m_1, m_2, \cdots, m_j+1, \cdots, m_N)} - u_{(m_1, m_2, \cdots, m_j, \cdots, m_N)},
\]
(3.3)
\[ (B_j^* u)_m = u_{(m_1, m_2, \ldots, m_{j-1}, \ldots, m_N)} - u_{(m_1, m_2, \ldots, m_j, \ldots, m_N)}, \quad (3.4) \]

such that
\[ A_j = B_j B_j^* = B_j^* B_j, \quad (3.5) \]

where \( B_j^* \) is the conjugate operator of \( B \). In addition, simple computation shows that for \( u = (u_m)_{m \in \mathbb{Z}^N} \in \textbf{H} \), \( m = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N \), \( j = 1, 2, \ldots, N \), there holds
\[ \|B_j u\|^2 \leq 4\|u\|^2. \quad (3.6) \]

Let \( \textbf{E} = l^2 \times L^2 \). For \( \psi^{(j)} = (u^{(j)}, y^{(j)}) = ((u^{(j)}_m), (y^{(j)}_m))_{m \in \mathbb{Z}^N} \in \textbf{E} \), \( j = 1, 2 \), the norm and the product is given by
\[ \langle \psi^{(1)}, \psi^{(2)} \rangle_{\textbf{E}} = \langle u^{(1)}, u^{(2)} \rangle + \langle v^{(1)}, v^{(2)} \rangle = \sum_{m \in \mathbb{Z}^N} (u^{(1)}_m v^{(2)}_m) + (v^{(1)}_m u^{(2)}_m), \quad (3.7) \]

\[ \|\psi\|_{\textbf{E}}^2 = \langle \psi, \psi \rangle_{\textbf{E}} = \|u\|^2 + \|v\|^2. \quad (3.8) \]

According to (3.1)-(3.7), system (1.1)-(1.2) can be rewritten as the following vector form
\[
\begin{cases}
i u_t - Au - uv + i\alpha u = f(t) + au \circ \omega^{(1)}(t), \\
i v_t + \beta v + \lambda B |u|^2 = g(t) + bv \circ \omega^{(2)}(t),
\end{cases}
\quad (3.9)
\]

with initial values
\[ u(\tau) = (u_{\tau, m})_{m \in \mathbb{Z}^N} = u_\tau, \quad v(\tau) = (v_{\tau, m})_{m \in \mathbb{Z}^N} = v_\tau, \quad \tau > 0. \quad (3.10)\]

We consider the following Ornstein-Uhlenbeck process (see [3.9]): for \( j = 1, 2 \),
\[ dz^{(j)} + z^{(j)} dt = dw^{(j)}(t, \omega), \quad z^{(j)}(-\infty) = 0, \quad (3.11) \]

where \( w^{(j)}(t, \omega) = \omega^{(j)}(t) \) for \( \omega \in \Omega, t \in \mathbf{R} \). Denote by \( z^{(j)}(\theta t, \omega) \), \( \omega \in \Omega, t \in \mathbf{R} \), an Ornstein-Uhlenbeck process on ergodic metric dynamical system \((\Omega, F, P, \{\theta_t\}_{t \in \mathbf{R}})\), and solves the equation (3.11), respectively.

Denote
\[ \rho_1(\omega) = az^{(1)}(\omega), \quad \rho_2(\omega) = bz^{(2)}(\omega) \quad (3.12) \]

The following properties about \( z^{(j)}(\theta t, \omega) \) and \( \rho_j(\theta t, \omega) \) holds: for \( \omega \in \Omega \) and \( j = 1, 2 \),
\[ \lim_{t \to \pm \infty} \frac{\rho_j(\theta t, \omega)}{t} = \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \rho_j(\theta s, \omega) ds = 0, \quad (3.13) \]

\[ \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t |z^{(j)}(\theta s, \omega)|^r ds = E[|z^{(j)}(\theta s, \omega)|^r] = \frac{\Gamma(1 + r)}{\sqrt{\pi}^r}, \quad \forall r > 0, \quad s \in \mathbf{R}, \quad (3.14) \]

\[ E[e^{\epsilon z^{(j)}(\theta s, \omega)}] \leq \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}}, \quad \forall s \in \mathbf{R}, \quad |\epsilon| \leq 1, \quad (3.15) \]

\[ E[e^{\epsilon \int_0^{t+s} |z^{(j)}(\theta t, \omega)| ds}] \leq e^{\epsilon t}, \quad \forall t \in \mathbf{R}, \quad t \geq 0, \quad (3.16) \]

where \( \Gamma(\cdot) \) is the Gamma function.

By the above properties of \( \rho_1(\theta t, \omega) \) and \( \rho_2(\theta t, \omega) \), we obtain
\[ \lim_{t \to \pm \infty} e^{-C_2 t + J_0(2p_1(\theta t, \omega) + 2p_2(\theta t, \omega) + 2C_1 e^{-2p_2(\theta t, \omega) ds})} = 0, \quad (3.17) \]
where $C_1$, $C_2$ both are constants given below.

Let $x = u e^{i \rho_1(\theta, \omega)}$, $y = v e^{-p_2(\theta, \omega)}$, and then we have $|x| = |u|$. So system (3.9)-(3.10) can be written as

$$
\begin{cases}
\dot{x} + iAx + i \rho_1(\theta, \omega)x + \alpha x = -ie^{i \rho_1(\theta, \omega)} f(t(t) - ie^{i \rho_2(\theta, \omega)} xy, \\
\dot{y} + \beta y - \rho_2(\theta, \omega)y = e^{-p_2(\theta, \omega)} g(t) - \lambda e^{-p_2(\theta, \omega)} B(|x|^2),
\end{cases}
$$

(3.18)

with initial values

$$
x_m(\tau) = x_{\tau, m}, \quad y_m(\tau) = y_{\tau, m}, \quad m = (m_1, m_2, \cdots, m_m) \in \mathbb{Z}^N, \quad \tau \in \mathbb{R}, \quad t > \tau. \quad (3.19)
$$

Set $\phi = (x, y)^T$, and the system (3.18)-(3.19) can be rewritten as

$$
\dot{\phi} + L\phi = F(\phi, \theta, \omega),
$$

(3.20)

with initial values

$$
\phi_\tau = (x_\tau, y_\tau) = (x_{\tau, m}, y_{\tau, m})_{m \in \mathbb{Z}^N}, \quad t > \tau,
$$

(3.21)

where

$$
L = \begin{pmatrix}
iA + i \rho_1(\theta, \omega)I + \alpha I & 0 \\
0 & (\beta - \rho_2(\theta, \omega))I
\end{pmatrix},
$$

$$
F(\phi, \theta, \omega) = \begin{pmatrix}
-ie^{i \rho_1(\theta, \omega)} f(t) - ie^{i \rho_2(\theta, \omega)} xy \\
e^{p_2(\theta, \omega)} g(t) - \lambda e^{-p_2(\theta, \omega)} B(|x|^2)
\end{pmatrix}.
$$

Lemma 3.1. For $\forall \omega \in \Omega$, $F(\phi, \theta, \omega)$ is continuous in $t$ and $\phi$; for $\forall \omega \in \Omega$, $t \in [0, T], \ T > 0$, $F(\phi, \theta, \omega)$ is locally Lipschitz in $\phi$.

Proof. It is easy to know that for $\forall \omega \in \Omega$, $F(\phi, \theta, \omega)$ is continuous in $t$ and $\phi$. Let $Q$ be a bounded set in $E$, there exists a positive constant $C$ depending on $Q$ such that for $\phi^{(j)} = (x^{(j)}, y^{(j)})^T \in Q, j = 1, 2$, and for any $\omega \in \Omega, \ t \in [0, T]$, we have

$$
\begin{align*}
\|F(\phi^{(1)}, \theta, \omega) - F(\phi^{(2)}, \theta, \omega)\|_E^2 & = \|\| - ie^{\rho_2(\theta, \omega)}(x^{(1)}_1 y^{(1)}_1 - x^{(2)}_1 y^{(2)}_1)\|_E^2 \\
& + \| - \lambda e^{-\rho_2(\theta, \omega)}(B(|x^{(1)}|^2) - B(|x^{(2)}|^2))\|_E^2 \\
& \leq e^{2\rho_2(\theta, \omega)}\|x^{(1)}_1 (y^{(1)}_1 - y^{(2)}_1)\|_E^2 + 2\lambda e^{-\rho_2(\theta, \omega)}\|B(|x^{(1)}_1|^2) - B(|x^{(2)}_1|^2)\|_E^2 \\
& \leq e^{2\rho_2(\theta, \omega)}\|x^{(1)}_1 (y^{(1)}_1 - y^{(2)}_1)\|_E^2 + \|y^{(2)}_1 (x^{(1)}_1 - x^{(2)}_1)\|_E^2 \\
& + 2\lambda e^{-2\rho_2(\theta, \omega)}\|B(|x^{(1)}_1|^2) - B(|x^{(2)}_1|^2)\|_E^2 \\
& \leq 2e^{2\rho_2(\theta, \omega)}\|x^{(1)}_1 (y^{(1)}_1 - y^{(2)}_1)\|_E^2 + 2\lambda e^{-\rho_2(\theta, \omega)}\|B(|x^{(1)}_1|^2) - B(|x^{(2)}_1|^2)\|_E^2 \\
& \leq 2\lambda e^{-2\rho_2(\theta, \omega)}\|B(|x^{(1)}_1|^2) - B(|x^{(2)}_1|^2)\|_E^2 \\
& \leq 2\lambda e^{-2\rho_2(\theta, \omega)}(\|x^{(1)}_1\|^2 + |x^{(2)}_1|^2)\|\|x^{(1)}_1 - x^{(2)}_1\|_E^2 \\
& \leq 2\lambda e^{-2\rho_2(\theta, \omega)}(\|x^{(1)}_1\|^2 + |x^{(2)}_1|^2)\|\|x^{(1)}_1 - x^{(2)}_1\|_E^2 \\
& \leq 2\lambda e^{-2\rho_2(\theta, \omega)}(\|x^{(1)}_1\|^2 + |x^{(2)}_1|^2)\|\|x^{(1)}_1 - x^{(2)}_1\|_E^2
\end{align*}
$$

(3.22)

that is to say, for $\forall \omega \in \Omega, \ t \in [0, T], \ T > 0$, $F(\phi, \theta, \omega)$ is locally Lipschitz in $\phi$. □

Lemma 3.2. The solution of (3.20)-(3.21) corresponding to initial data $u(\tau) = u(\tau, m)_{m \in \mathbb{Z}^N} = u_\tau$, $v(\tau) = v(\tau, m)_{m \in \mathbb{Z}^N} = v_\tau$ satisfies

$$
\|u(t, \tau, \omega; u_\tau(\omega))\|^2 \leq e^{-\alpha(t-\tau)}\|u_\tau\|^2 + \frac{1}{\alpha} \|f\|^2, \quad \forall t \geq \tau.
$$

(3.25)

Proof. Taking the imaginary part of the inner product of the first equation of (3.9) with $u$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \alpha \|u\|^2 = -\text{Im}(f(t), u) \leq \frac{\alpha}{2} \|u\|^2 + \frac{1}{2\alpha} \|f\|^2, \quad \forall t \geq \tau,
$$

(3.26)

i.e.,

$$
\frac{d}{dt} \|u\|^2 + \alpha \|u\|^2 \leq \frac{1}{\alpha} \|f\|^2, \quad \forall t \geq \tau.
$$

(3.27)
Applying Gronwall’s inequality to (3.27), we have
\[ \|u(t, \tau, \omega, u_\tau(\omega))\|^2 \leq e^{-\alpha(t-\tau)}\|u_\tau\|^2 + \frac{1}{\alpha}\|f\|^2, \quad \forall t \geq \tau. \] (3.28)
Moreover, from \( x = u e^{i\phi_1(\theta_1\omega)} \), we have
\[ \|x(t, \tau, \omega, u_\tau(\omega))\|^2 = \|u(t, \tau, \omega, u_\tau(\omega)) e^{i\phi_1(\theta_1\omega)}\|^2 \leq \|u(t, \tau, \omega, u_\tau(\omega))\|^2 + 1 \leq e^{-\alpha(t-\tau)}\|u_\tau\|^2 + \frac{1}{\alpha}\|f\|^2 + 1, \quad \forall t \leq \tau, \quad \omega \in \Omega, \] (3.29)
where \( \|f\|^2 = \sup_{t \in \mathbb{R}} |f(t)|^2 \).

**Theorem 3.3.** Let \( \psi = (u, v)^T \). For every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and any initial data \( \phi_\tau \in E \) (or \( \psi_\tau = (u_\tau, v_\tau)^T \in E \)), the system (3.20)-(3.21) admits a unique solution \( \phi(t, \tau, \omega, \phi_\tau(\omega)) \in L^2(\Omega; C[\tau, \tau + T]; E) \) for any \( T > 0 \) with \( \phi(t, \tau, \omega, \phi_\tau(\omega)) = \phi_\tau(\omega) \), being continuous in \( \phi_\tau \). Moreover, the solution \( \phi(t, \tau, \omega, \phi_\tau(\omega)) \) of the system (3.20)-(3.21) generates a continuous random dimensional system \( \Phi \) on \( \mathbb{E} \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_1)_{r \in \mathbb{R}}) \) with state space \( \mathbb{E} \) defined by
\[ \Phi(t, \tau, \omega, \phi_\tau(\omega)) = \phi(t + \tau, \tau, \theta_{-\tau}\omega, \phi_\tau(\theta_{-\tau}\omega)), \quad \forall t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega. \] (3.30)
In addition, for \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\( \psi(t, \tau, \omega, \phi_\tau(\omega)) = H^{-1}(1)\omega) \phi(t, \tau, \omega, H^{-1}(1)\theta_{-\tau}\omega, \phi_\tau(\theta_{-\tau}\omega)) \) generates a continuous random dimensional system \( \Psi \) on \( \mathbb{E} \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_1)_{r \in \mathbb{R}}) \) associated with (3.9)-(3.10), defined by
\[ \Psi(t, \tau, \omega, \phi_\tau(\omega)) = \psi(t + \tau, \tau, \theta_{-\tau}\omega, \phi_\tau(\theta_{-\tau}\omega)), \quad \forall t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \] (3.31)
where
\[ H(\theta_1\omega) = \begin{pmatrix} e^{i\phi_1(\theta_1\omega)} I & 0 \\ 0 & e^{-\phi_2(\theta_1\omega)} I \end{pmatrix}. \]
Therefore, system \( \Psi \) and system \( \Phi \) are equivalent to each other. In the following, we just consider the system \( \Phi \).

4. **Existence of random attractor.** Firstly, we have the existence of a uniformly tempered measurable \( \mathcal{D}(E) \)-pullback absorbing set for \( \Psi \).

**Lemma 4.1.** The system (3.20)-(3.21) has a tempered random variable \( M_0(\omega) \geq 0 \) (independent of \( \tau \)),
\[ M_0^2(\omega) = \frac{4}{\beta} \|g\|^2 K_0(\omega), \quad \text{where} \quad K_0(\omega) = \int_{-\infty}^0 e^{-2\phi_2(\theta_1\omega) + C_2(t + f_0 K(\theta_1\omega) ds dl}, \] (4.1)
where \( \|f\|^2 = \sup_{t \in \mathbb{R}} \|f(t)\|^2, \quad \|g\|^2 = \sup_{t \in \mathbb{R}} \|g(t)\|^2, \) such that \( M_0^2(\theta_1\omega) \) is continuous in \( t \), and the family of balls centered at \( \theta \) with radius \( M_0^2(\omega) \):
\[ B_0 = \{ B_0(\tau, \omega) = B_0(0, M_0(\omega)) = \{ \phi \in \mathbb{E} : \|\phi\| \leq M_0(\omega) \}| \tau \in \mathbb{R}, \omega \in \Omega \} \] (4.2)
is a measurable \( \mathcal{D}(E) \)-pullback absorbing set for \( \Phi \). That is to say, for any \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( B \in \mathcal{D}(E) \), there exists a \( T_B(\tau, \omega) \geq 0 \) such that the solution \( \phi(\tau, \tau - t, \theta_{-\tau}\omega, \phi_{\tau-\tau}(\theta_{-\tau}\omega)) \) of (3.20)-(3.21) with \( \phi_{\tau-\tau}(\theta_{-\tau}\omega) \in B(\tau - t, \theta_{-\tau}\omega) \) satisfies\( E \)
\[ \|\phi(\tau, \tau - t, \theta_{-\tau}\omega, \phi_{\tau-\tau}(\theta_{-\tau}\omega))\| \leq M_0(\omega), \quad \forall t \geq T_B(\tau, \omega), \] (4.3)
where $T_B(\tau, \omega)$ is uniform for $\varepsilon$ in a bounded interval of $\mathbb{R}$. Particularly, there exists a $T_{B_0}(\omega) \geq 0$ (independent of $\tau$) such that
\[
\phi(r, \tau - t, \theta_{r - \tau}, B_0(\theta_{r - \tau})) \subset B_0(\theta_{t - \tau}), \forall t \geq T_{B_0}(\omega), \quad r \geq \tau - t.
\]

**Proof.** Keeping the real part of the inner product of (3.20) with $\phi(r) = \phi(r, \tau - t, \theta_{r - \tau}, x_{r - \tau}(\theta_{r - \tau}))$ in $E$, we obtain that
\[
\frac{1}{2} \frac{d}{dt} \|\phi\|_E^2 + \text{Re}(L\phi, \phi)_E = \text{Re}(F(\phi, \theta_{r - \tau}), \phi)_E.
\]

Firstly, for $r \geq \tau - t$, we have
\[
\text{Re}(L\phi, \phi)_E = \alpha \|x\|^2 + \beta \|y\|^2 - \text{Im}(\rho_1(\theta_{r - \tau})x, x) - \text{Re}(\rho_2(\theta_{r - \tau})y, y), \quad (4.6)
\]
\[
\text{Re}(F(\phi, \theta_{r - \tau}), \phi)_E = -\text{Im}(e^{\rho_1(\theta_{r - \tau})} f(t), x) + (e^{-\rho_2(\theta_{r - \tau})} g(t), y) - (\lambda e^{-\rho_2(\theta_{r - \tau})} B(|x|^2), y).
\]

Secondly, we estimate the terms on the right-side of (4.7) as following.
\[
-\text{Im}(e^{\rho_1(\theta_{r - \tau})} f(t), x) \leq \frac{1}{4\alpha} \|f\|^4 + 1 + \frac{\alpha}{2} \|x\|^2,
\]
\[
(e^{-\rho_2(\theta_{r - \tau})} g(t), y) \leq \frac{\beta}{4} \|y\|^2 + \frac{\beta}{\beta} e^{-2\rho_2(\theta_{r - \tau})} \|g\|^2,
\]
\[
(\lambda e^{-\rho_2(\theta_{r - \tau})} B(|x|^2), y) \leq \frac{\beta}{4} \|y\|^2 + \frac{4N\lambda^2}{\beta} e^{-2\rho_2(\theta_{r - \tau})} \|x\|^2.
\]

By (3.29) and (4.7)-(4.10) we obtain
\[
\text{Re}(F(\phi, \theta_{r - \tau}), \phi)_E \leq \frac{\alpha}{2} \|x\|^2 + \frac{\beta}{4} \|y\|^2 + C_1 e^{-2\rho_2(\theta_{r - \tau})} \|x\|^2 + \frac{1}{2\alpha} \|f\|^4 + 1 + \frac{\beta}{2} e^{-2\rho_2(\theta_{r - \tau})} \|g\|^2,
\]

where $C_1$ is a constant depending on $N, \beta$, and $\lambda$.

Combining (4.5), (4.6) with (4.11), we obtain
\[
\frac{d}{dt} \|\phi\|_E^2 \leq (-C_2 + K(\theta_{r - \tau})) \|\phi\|_E^2 + \frac{1}{2\alpha} \|f\|^4 + 1 + \frac{\beta}{\beta} e^{-2\rho_2(\theta_{r - \tau})} \|g\|^2,
\]

where $C_2 = \min\{\alpha, \beta\}$, and $K(\theta_{r - \tau}) = 2\rho_1(\theta_{r - \tau}) + 2\rho_1(\theta_{r - \tau}) + C_1 e^{-2\rho_2(\theta_{r - \tau})}$.

By applying Gronwall’s inequality to (4.12) on $[\tau - t, \tau - \omega]$, we have
\[
\|\phi(\tau)\|_E^2 \leq e^{-C_2 t + \int_0^\tau K(\theta_{r - \tau})ds}\|\phi_{\tau - t}(\theta_{r - \tau})\|_E^2 + \frac{1}{2\alpha} \|f\|^4 + 1 + \frac{\beta}{\beta} e^{-2\rho_2(\theta_{r - \tau})} \|g\|^2.
\]

For any $\phi_{\tau - t}(\theta_{r - \tau}) \leq B(\tau - t, \theta_{r - \tau})$, we have
\[
\lim_{t \to +\infty} e^{-C_2 t + \int_0^\tau K(\theta_{r - \tau})ds} \|\phi_{\tau - t}(\theta_{r - \tau})\|_E^2 = 0
\]

and
\[
\lim_{t \to +\infty} \frac{1}{2\alpha} \|f\|^4 + 1 + \int_\tau^\tau e^{-C_2 (\tau - t)} + \int_\tau^\tau K(\theta_{r - \tau})ds dl = 0. \quad (4.15)
\]

Set
\[
M_1^2(\omega) = \frac{2}{\beta} \|g\|^2 \int_0^\tau e^{-2\rho_2(\theta_{r - \tau})} + C_2 t + \int_\tau^\tau K(\theta_{r - \tau})ds dl < +\infty.
\]

Then for any $\delta > 0$, we find
\[
\lim_{t \to +\infty} e^{-\delta t} M_1^2(\theta_{r - \tau}) = 0.
\]

Let $M_0^2(\omega) = 2M_1^2(\omega)$. By (3.13), $M_0(\theta_{r - \tau})$ is a tempered random variable and $M_0(\theta_{r - \tau})$ is continuous in $t$.
Lemma 4.2. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon \geq 0$, there exist $T(\tau, \omega, \varepsilon) \geq 0$ and $I_0(\tau, \omega, \varepsilon) \in \mathbb{N}$ such that the solution $\phi(r, \tau-t, \theta_{\tau-\omega}, \phi_{\tau-t}(\theta_{\tau-\omega}))$, $(r \geq \tau-t, t \geq 0)$ of the system (3.20)-(3.21) with $\phi_{\tau-t} \in B_0(\theta_{\tau-\omega})$ satisfies
\begin{equation}
\sum_{\|m\| \geq M(\tau, \omega, \varepsilon)} |\phi_m(\tau, \tau-t, \theta_{\tau-\omega}, \phi_{\tau-t}(\theta_{\tau-\omega}))|^2_{E} \leq \varepsilon, \; \forall t \geq T_0(\tau, \omega, \varepsilon). \tag{4.18}
\end{equation}

Proof. Choose a smooth increasing function $p \in C^1(\mathbb{R}^+, \mathbb{R})$, satisfying
\begin{equation}
p(s) = 0, \; 0 \leq s \leq 1,
0 \leq p(s) \leq 1, \; 1 \leq s \leq 2,
\{ p(s) = 1, \; s \geq 2, \}
\|p'(s)\| \leq c_0, \; c_0 \in \mathbb{R}^+, \; s \in \mathbb{R}^+. \tag{4.19}
\end{equation}

Let $\phi(t) = (x, y)^T = (x_m, y_m)_{m \in \mathbb{Z}^N}$ be the solution of the system (3.20)-(3.21), and $M \in \mathbb{N}$ be a positive integer. Let
\begin{equation}
\varphi = (\varphi_m)_{m \in \mathbb{Z}^N} = (\mu, \bar{\mu})^T, \; \varphi_m = (\mu_m, \bar{\mu}_m)^T = (p(\|m\|_M) x_m, p(\|m\|_M) y_m)^T. \tag{4.20}
\end{equation}
where $\|m\| = \max\{|m_d|, d = 1, 2, \cdots, N\}$.

Taking the real part of the inner product of the system (3.20)-(3.21) with $\varphi$, we have that for $r \geq \tau - s$,
\begin{equation}
\Re(\dot{\phi}, \varphi)_E + \Re(L \phi, \varphi)_E = \Re(F(\phi, \theta_{r-\tau} \omega), \varphi)_E. \tag{4.21}
\end{equation}

We now estimate the terms in (4.21). Firstly,
\begin{align}
\Re(L \phi, \varphi)_E &= \Re((i A x + i p_1(\theta_{r-\tau} \omega)x + ax, \beta y - p_2(\theta_{r-\tau} \omega)y)^T, (\mu, \bar{\mu})^T)_E \\
&= \alpha(x, \mu) + \beta(y, \bar{\mu}) - \Re(p_1(\theta_{r-\tau} \omega)x, \mu) - \Re(p_2(\theta_{r-\tau} \omega)y, \bar{\mu}). \tag{4.22}
\end{align}

Since
\begin{align}
-\Re(Ax, \mu) &= -\Re(Bx, B\mu) \\
&= -\Re \sum_{m \in \mathbb{Z}^N} \sum_{j=1}^N (x_{mj} - \mu_m) (p(\|m\|_M) x_{mj} - p(\|m\|_M) \mu_m) \\
&= \Re \sum_{m \in \mathbb{Z}^N} \sum_{j=1}^N (p(\|m\|_M) x_{mj} \overline{\mu_m} - p(\|m\|_M) \overline{x_{mj}} \mu_m) \\
&\geq - \sum_{m \in \mathbb{Z}^N} \sum_{j=1}^N |p(\|m\|_M)| |x_{mj}| |\mu_m| \\
&\geq - \frac{N c_0}{M} \|x\|^2,
\end{align}
where $m_{j1} = (m_1, m_2, \cdots, m_j + 1, \cdots, m_N)$, $\|m_{j1}\| = \max\{|m|, m_j + 1\}$, then, we have
\begin{align}
\Re(L \phi, \varphi)_E &\geq - \frac{N c_0}{M} \|x\|^2 + \alpha \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) |x_m|^2 + \beta \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) y_m^2 \\
&\quad - p_1(\theta_{r-\tau} \omega) \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) |x_m|^2 \\
&\quad - p_2(\theta_{r-\tau} \omega) \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) y_m^2 \\
&\geq - \frac{N c_0}{M} \|x\|^2 + C_2 \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) |\phi_m|_E^2 \\
&\quad - p_1(\theta_{r-\tau} \omega) \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) |x_m|^2 \\
&\quad - p_2(\theta_{r-\tau} \omega) \sum_{m \in \mathbb{Z}^N} p(\|m\|_M) y_m^2. \tag{4.24}
\end{align}
Secondly,

\[
\text{Re}(F(\phi, \theta_{r-\tau}), \varphi)_{\mathbb{E}} = -\text{Im}(e^{i\rho_1(\theta_{r-\tau})} f(r), \mu) + (e^{-\rho_2(\theta_{r-\tau})} g(r), \bar{\mu}) - (\lambda e^{-\rho_2(\theta_{r-\tau})} B(|x|^2), \bar{\mu}),
\]

(4.25)

since

\[
\begin{align*}
-\text{Im}(e^{i\rho_1(\theta_{r-\tau})} f(r), \mu) = & -\text{Im} \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) m f(r) e^{i\rho_1(\theta_{r-\tau})} \\
\leq & -\text{Im} \sum_{|m| \geq M} \frac{1}{\alpha_0} |m| + \frac{\alpha}{2} \sum_{|m| \geq M} p(\frac{|m|}{M}) |x_m|^2 \\
\leq & \frac{1}{\alpha_0} \sum_{|m| \geq M} (|f_m|^4 + 1) + \frac{\alpha}{2} \sum_{|m| \geq M} p(\frac{|m|}{M}) |x_m|^2 \\
\end{align*}
\]

(4.26)

\[
\begin{align*}
(e^{-\rho_2(\theta_{r-\tau})} g(r), \bar{\mu}) \leq & \frac{\beta}{4} \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) y_m^2 + \frac{1}{\beta} e^{-2\rho_2(\theta_{r-\tau})} \sum_{|m| \geq M} |g_m|^2, \\
- (\lambda e^{-\rho_2(\theta_{r-\tau})} B(|x|^2), \bar{\mu}) \leq & \frac{\beta}{4} \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) y_m^2 + \frac{1}{\beta} e^{-2\rho_2(\theta_{r-\tau})} \sum_{|m| \geq M} p(\frac{|m|}{M}) |x_m|^4. \\
\end{align*}
\]

(4.27) (4.28)

From (4.26)-(4.28), we obtain

\[
\text{Re}(F(\phi, \theta_{r-\tau}), \varphi)_{\mathbb{E}} \leq \frac{1}{\alpha_0} \sum_{|m| \geq M} (|f_m|^4 + 1) + \frac{\alpha}{2} \sum_{|m| \geq M} p(\frac{|m|}{M}) |x_m|^2 \\
+ \frac{\beta}{4} \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) y_m^2 + \frac{1}{\beta} e^{-2\rho_2(\theta_{r-\tau})} \sum_{|m| \geq M} |g_m|^2 + \frac{2N_c}{\beta} e^{-2\rho_2(\theta_{r-\tau})} \sum_{|m| \geq M} p(\frac{|m|}{M}) |x_m|^4. \\
\]

(4.29)

Taking (4.21), (4.24), (4.29) into account, we have

\[
\frac{d}{dt} \left( \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) |\phi_m|^2_{\mathbb{E}} \right) \leq (-C_2 + K(\theta_{r-\tau})) \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) |\phi_m|^2_{\mathbb{E}} \\
+ \frac{2N_c}{\beta} |\phi|^2_{\mathbb{E}} + \frac{1}{\alpha_0} \sum_{|m| \geq M} (|f_m|^4 + 1) \\
+ \frac{1}{\beta} e^{-2\rho_2(\theta_{r-\tau})} \sum_{|m| \geq M} |g_m|^2, \\
\]

(4.30)

Applying Gronwall’s inequality to (4.30) on \([\tau - t, \tau]\), then we obtain

\[
\begin{align*}
\sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) |\phi_m(\tau)|^2_{\mathbb{E}} \leq e^{C_2 t + \int_{\tau}^{\infty} K(\theta_{r-\tau}) ds} \sum_{m \in \mathbb{Z}^N} p(\frac{|m|}{M}) |\phi_m,\tau-t(\theta_{r-\tau})|^2_{\mathbb{E}} \\
+ \frac{2N_c}{M} \int_{\tau}^{\infty} e^{-C_2 t} (t-l) + \int_{\tau}^{\infty} K(\theta_{r-\tau}) ds \|\phi(l, \tau-t, \theta_{r-\tau}, \phi_{\tau-t(\theta_{r-\tau})})\|^2_{\mathbb{E}} dl \\
+ \frac{1}{\beta} \sum_{|m| \geq M} (|f_m|^4 + 1) \int_{\tau}^{\infty} e^{-C_2 t} + \int_{\tau}^{\infty} K(\theta_{r-\tau}) ds dl \\
+ \frac{1}{\beta} \sum_{|m| \geq M} |g_m|^2 \int_{-\infty}^{0} e^{-2\rho_2(\theta_{r-\tau})} - C_2 t + \int_{\tau}^{\infty} K(\theta_{r-\tau}) ds dl. \\
\end{align*}
\]

(4.31)
Next we estimate each terms on the right-hand side of (4.31). Firstly, by the (3.14) and (4.13), we obtain
\[
e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} \sum_{m \in \mathbb{Z}^{N}} p\left(\frac{\|m\|}{M}\right) |\phi_{m, \tau-t}(\theta_{-\tau})|_{E}^{2} \
\leq e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} \|\phi_{t-\tau}(\theta_{-\tau})\|_{E}^{2} \
\leq e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} (e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} \|\phi_{t-\tau}(\theta_{-\tau})\|_{E}^{2}) \\
+ \frac{\alpha}{25} \|f\|^{4} + 1 \int_{\tau-t}^{T} e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} dl \\
e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} \sum_{m \in \mathbb{Z}^{N}} p\left(\frac{\|m\|}{M}\right) |\phi_{m, \tau-t}(\theta_{-\tau})|_{E}^{2} \leq 1.
\]
(4.33)

Secondly, by (4.13), we have
\[
\frac{2Nc_{0}}{M} \int_{\tau-t}^{T} e^{-C_{2}(\tau-t)} \int_{\tau-t}^{T} K(\theta_{t-\tau})ds \|\phi(l, \tau-t, \theta_{-\tau}, \phi_{t-\tau}(\theta_{-\tau}))\|_{E}^{2} dl \\
\leq \frac{2Nc_{0}}{M} \|f\|^{4} + 1 \int_{\tau-t}^{T} \int_{-\infty}^{0} e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} dl \\
+ \frac{2Nc_{0}}{M} \|g\|^{2} \int_{\tau-t}^{T} \int_{0}^{\infty} e^{-2\rho_{2}(\theta_{t-\tau}) + \int_{0}^{t} K(\theta_{t-\tau})ds} dl.
\]
(4.34)

From (3.14), there exist $M_{1}(\tau, \varepsilon, \omega) \in \mathbb{N}$ and $T_{2}(\tau, \varepsilon, \omega)$, such that for $t \geq T_{2}(\tau, \varepsilon, \omega)$ and $M > M_{1}(\tau, \varepsilon, \omega)$, we have
\[
\frac{2Nc_{0}}{M} \int_{\tau-t}^{T} e^{-C_{2}(\tau-t)} \int_{\tau-t}^{T} K(\theta_{t-\tau})ds \|\phi(l, \tau-t, \theta_{-\tau}, \phi_{t-\tau}(\theta_{-\tau}))\|_{E}^{2} dl \leq \frac{\varepsilon}{4}.
\]
(4.35)

Thirdly, since $f \in C_{b}(\mathbb{R}, l^{2})$ and by (3.14), there exists $M_{2}(\tau, \varepsilon, \omega) \in \mathbb{N}$, such that for $M > M_{2}(\tau, \varepsilon, \omega)$, we have
\[
\frac{1}{\alpha} \sum_{\|m\| \geq M} (|f_{m}|^{4} + 1) \int_{-\infty}^{0} e^{-C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} dl \leq \frac{\varepsilon}{4}.
\]
(4.36)

Since $g \in C_{b}(\mathbb{R}, l^{2})$, by (3.14), there exists $M_{3}(\tau, \varepsilon, \omega) \in \mathbb{N}$, such that for $M > M_{3}(\tau, \varepsilon, \omega)$, we have
\[
\frac{2}{\beta} \sum_{\|m\| \geq M} |m|^{2} \int_{-\infty}^{0} e^{-2\rho_{2}(\theta_{t-\tau}) - C_{2}t + \int_{0}^{t} K(\theta_{t-\tau})ds} dl \leq \frac{\varepsilon}{4}.
\]
(4.37)

In summary, letting
\[
T_{0}(\tau, \varepsilon, \omega) = \max\{T_{1}(\tau, \varepsilon, \omega), T_{2}(\tau, \varepsilon, \omega)\},
\]
\[
M(\tau, \varepsilon, \omega) = \max\{M_{1}(\tau, \varepsilon, \omega), M_{2}(\tau, \varepsilon, \omega), M_{3}(\tau, \varepsilon, \omega)\},
\]
we obtain
\[
\sum_{\|m\| \geq M(\tau, \varepsilon, \omega)} |\phi_{m}(\tau, \tau-t, \theta_{-\tau}, \phi_{t-\tau}(\theta_{-\tau}))|_{E}^{2} \\
\leq \sum_{\|m\| \in \mathbb{Z}^{N}} |\phi_{m}(\tau, \tau-t, \theta_{-\tau}, \phi_{t-\tau}(\theta_{-\tau}))|_{E}^{2} \\
\leq \varepsilon.
\]
(4.38)

The results restated in Theorem 4.3 and Corollary 4.4 (see[14, 2]).
The continuous cocycle $\Phi$ associated with (3.20)-(3.21) has a unique random attractor $R = \{R(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega} \in \mathcal{D}(E)$ and $R(\tau, \omega) \in B_0(\omega)$ for any $\tau \in \mathbb{R}, \omega \in \Omega$.

Corollary 4.4 For every $\tau \in \mathbb{R}, \omega \in \Omega, \nu > 0$ and $I \in \mathbb{N}$, there exists $\bar{T}_I(\omega) > 0$ (independent of $\tau$) such that the solution $\phi(r, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))$ ($r \geq \tau - t$) of (3.9)-(3.10) with $\phi_{\tau-t}(\theta_{-\tau} \omega) \in B_0(\theta_{-t} \omega)$ satisfies

$$
\sum_{m \in \mathbb{Z}^n} p_t(\|m\|) \norm[\phi_m(\tau, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))_E^2 \
\leq \nu + (\frac{C_3}{T} + \gamma_I)K_0(\omega), \forall t \geq \bar{T}_I(\omega), I \in \mathbb{N},
$$

where $C_3 = \frac{2N\nu}{\beta} \norm{g}_E^2, \gamma_I = \sup_{\tau \in \mathbb{R}} \sum_{\|m\| \geq I} |g_m(\tau)|^2$.

5. Random exponential attractor. From Theorem 4.4, we know that the cocycle $\Phi$ associated with (3.20)-(3.21) has a random attractor $R = \{R(\tau, \omega)_{\tau \in \mathbb{R}, \omega \in \Omega} \in \mathcal{D}(E)$.

Choose a fixed positive number $\nu = \nu_0 > 0$ in (4.39) small enough such that

$$
\beta + 8N\lambda^2 \nu_0 4\sqrt{\pi} + 3e \frac{3C_2}{8} - \frac{|a| + |b|}{\sqrt{\pi}}.
$$

(5.1)

For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, set

$$
\chi_1(\tau, \omega) = \bigcup_{s \geq \max\{t, t_0(\omega), \bar{T}_n(\omega)\}} \phi(\tau, \tau - s, \theta_{-\tau} \omega, B_0(\theta_{-s} \omega)) \subseteq B_0(\omega).
$$

(5.2)

By Remark 2.2 in [14], Lemma 4.1 and equation (4.4), the family of closed sets $\{\chi_1(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies (H1)-(H3).

Moreover, by Corollary 4.4 and definition (5.2) of $\chi_1(\tau, \omega)$, it holds that for any $\tau \in \mathbb{R}, \omega \in \Omega$ and $\bar{\phi} = (\bar{\phi}_m)_{m \in \mathbb{Z}^n} \in \chi_1(\tau, \omega)$, we have

$$
\sum_{\|m\| \geq 2I} \bar{\phi}_m^2 \leq \nu_0 + (\frac{C_3}{T} + \gamma_I)K_0(\omega) \forall I \in \mathbb{N}.
$$

(5.3)

We now show that $\{\chi_1(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies (H4)-(H5). Once this is done, we obtain the existence of a random exponential attractor for $\Phi$ by Theorem 2.2. To this end, by the cocycle property and continuity of $\Phi$, it is sufficient to prove that $\{\chi_1(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies (H4)-(H5).

Let $\phi(r, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))$ ($r \geq \tau - t$) be a solution of (3.20)-(3.21) with $\phi_{\tau-t}(\theta_{-\tau} \omega) \in \chi_1(\tau - t, \theta_{-t} \omega)$. By the cocycle property of $\Phi$, (4.4) and (5.3), we have

$$
\|\phi(r, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))_E^2 \leq M_0(\theta_{-\tau} \omega), \forall r \geq \tau - t
$$

(5.4)

and

$$
\sum_{I \geq 2I} \|\phi(r, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))_E^2 \leq \nu_0 + (\frac{C_3}{T} + \gamma_I)K_0(\theta_{-\tau} \omega) \forall r \geq \tau - t, I \in \mathbb{N}.
$$

(5.5)

We prove that $\Phi$ has the Lipschitz property on $\chi_1(\tau, \omega)$.

Lemma 5.1. For every $\tau \in \mathbb{R}, \omega \in \Omega, \tau \geq 0$ and $\phi_{\tau-t}(\theta_{-\tau} \omega) \in \chi_1(\tau - t, \theta_{-t} \omega), j = 1, 2$, there exists a random variable $S_1(\omega) > 0$ such that

$$
\|\phi(\tau, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega)) - \phi(\tau, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))_E \leq e^{\int_{0}^{t} S_1(\theta_{-\tau} \omega) ds} \|\phi(\tau, \tau - t, \theta_{-\tau} \omega, \phi_{\tau-t}(\theta_{-\tau} \omega))_E.
$$

(5.6)
Proof. For every \( \tau \in \mathbb{R}, \omega \in \Omega, \ t \geq 0 \) and \( \phi_{\tau-t}^{(j)}(\theta_{-\tau}\omega) \in \chi_1(\tau-t, \theta_{-t}\omega) \), \( j = 1, 2 \), letting
\[
\phi^{(j)}(r) = \phi(r, \tau-t, \theta_{-\tau}\omega, \phi_{\tau-t}^{(j)}(\theta_{-\tau}\omega)), \tag{5.7}
\]
\[
\xi(r) = \phi^{(1)}(r) - \phi^{(2)}(r) = (u^{(1)}(r) - u^{(2)}(r), v^{(1)}(r) - v^{(2)}(r))^T, \tag{5.8}
\]
then for \( r \geq \tau-t \),
\[
\begin{cases}
\dot{\xi} + L\xi = F(\xi, \theta_{-\tau}\omega), \\
\xi(\tau, \omega) = \phi^{(1)}_\tau(\omega) - \phi^{(2)}_\tau(\omega).
\end{cases} \tag{5.9}
\]
By (5.4), it holds that
\[
\|\phi^{(1)}(r)\|_E \leq M_0(\theta_{-\tau}\omega), \ \|\phi^{(2)}(r)\|_E \leq M_0(\theta_{-\tau}\omega), \ \forall r \geq \tau-t. \tag{5.10}
\]
Taking the real part of the inner product of (5.9) with \( \xi(r) \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\xi\|_E^2 + \text{Re}(L\xi, \xi)_E = \text{Re}(F(\xi, \theta_{-\tau}\omega), \xi)_E. \tag{5.11}
\]
Now, we estimate (5.11). Similar to (4.6),
\[
\text{Re}(L(\xi(r), \xi(r))_E = \alpha \|x^{(1)}(r) - x^{(2)}(r)\|^2 + \beta \|y^{(1)}(r) - y^{(2)}(r)\|^2
- \text{Im}(\rho_1(\theta_{-\tau}\omega))(x^{(1)}(r) - x^{(2)}(r)), (x^{(1)}(r) - x^{(2)}(r)))
- \text{Re}(\rho_2(\theta_{-\tau}\omega))(y^{(1)}(r) - y^{(2)}(r)), (y^{(1)}(r) - y^{(2)}(r))), \tag{5.12}
\]
by \( \phi_{j, -t}^{(j)}(\theta_{-\tau}\omega) \in \chi_1(\tau-t, \theta_{-\tau}\omega), \ j = 1, 2 \), and the boundedness of \( \chi_1(\tau-t, \theta_{-\tau}\omega) \), we hold that there exists a constant \( L_1 > 0 \), such that
\[
\text{Re}(F(\xi, \theta_{-\tau}\omega), \xi)_E \\
= -((\lambda e^{-\rho_2(\theta_{-\tau}\omega)}B(|x^{(1)}|^2) - \lambda e^{-\rho_2(\theta_{-\tau}\omega)}B(|x^{(2)}|^2), y^{(1)} - y^{(2)})) \tag{5.13}
\]
\[
\leq \frac{\beta}{4} \|y^{(1)} - y^{(2)}\|^2 + \frac{1}{\beta} \lambda^2 e^{-2\rho_2(\theta_{-\tau}\omega)} \|B(|x^{(1)}|^2) - B(|x^{(2)}|^2)\|^2 \
\leq \frac{\beta}{4} \|y^{(1)} - y^{(2)}\|^2 + \frac{4\lambda^2 L_1}{\beta} e^{-2\rho_2(\theta_{-\tau}\omega)} \|x^{(1)} - |x^{(2)}\|^2} \
= \left(\frac{\beta}{4} + \frac{4\lambda^2 L_1}{\beta} e^{-2\rho_2(\theta_{-\tau}\omega)}\right) \|\xi\|^2. \tag{5.13}
\]
Combining (5.11), (5.12) with (5.13), we obtain
\[
\frac{d}{dt} \|\xi\|_E^2 \leq 2S_1(\theta_{-\tau}\omega) \|\xi\|_E^2, \tag{5.14}
\]
where \( S_1(\omega) = |\rho_1| + |\rho_2| + \frac{\beta}{4} + \frac{4\lambda^2 L_1}{\beta} e^{-2\rho_2(\omega)}. \)

Applying Gronwall’s inequality to (5.14) on \([\tau - t, \tau], \ r \geq \tau-t \), we have
\[
\|\xi(\tau-t, \theta_{-\tau}\omega, \xi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \leq e^{2S_1(\theta_{-\tau}\omega)\tau-t} \|\xi(\tau-t, \tau-t, \theta_{-\tau}\omega, \xi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2. \tag{5.15}
\]
Setting \( r = \tau, \ (5.6) \) holds.

\[\square\]

**Lemma 5.2.** For any \( \tau \in \mathbb{R}, \omega \in \Omega, \ t \geq 0 \), there exist random variables \( S_2(\omega) > 0, \ S_3(\omega) > 0 \) such that for any \( I \in \mathbf{N} \) and \( \phi_{\tau-t}^{(j)}(\theta_{-\tau}\omega) \in \chi_1(\tau-t, \theta_{-t}\omega) \), \( j = 1, 2 \),
\[
\sum_{\substack{m \leq 4I}} |\phi_m(\tau-t, \theta_{-\tau}\omega, \phi_{\tau-t}^{(j)}(\theta_{-\tau}\omega)) - \phi_m(\tau-t, \theta_{-t}\omega, \phi_{\tau-t}^{(j)}(\theta_{-\tau}\omega))|_E^2 \leq e^{\int_{\tau-t}^{\tau} S_2(\theta_{-\tau}\omega)ds} \|\phi_{\tau-t}^{(1)}(\theta_{-\tau}\omega) - \phi_{\tau-t}^{(2)}(\theta_{-\tau}\omega)\|_E^2 \tag{5.16}
\]
and
\[
\sum_{\|m\|>4t} |\phi_m(\tau, \tau - t, \theta_{\tau - \omega}, \phi_{\tau - t}^{(1)}(\theta_{\tau - \omega})) - \phi_m(\tau, \tau - t, \theta_{\tau - \omega}, \phi_{\tau - t}^{(2)}(\theta_{\tau - \omega}))|^2_E \leq (e^{-C_2t + f^0_s} S_3(\theta_{\omega}) ds + \delta_1 e^{-f^0_s} S_2(\theta_{\omega}) ds) ||\phi_{\tau - t}^{(1)}(\theta_{\tau - \omega}) - \phi_{\tau - t}^{(2)}(\theta_{\tau - \omega})||_E.
\]
(5.17)

Proof. Let \(\phi_j, j = 1, 2,\) and \(\xi\) be the same as defined in the proof of Lemma 5.1. For \(m \in \mathbb{Z}^N\), let \(q_m = p_{\|m\|_M} \xi_m,\) \(q = (q_m)_{m \in \mathbb{Z}^N}\). Taking the real part of the inner product of (5.9) with \(q\) in \(E\), we have that for \(r \geq \tau - t,\)
\[
\text{Re}(\xi, q)_E + \text{Re}(L\xi, q)_E = \text{Re}(F(\phi^{(1)}, \theta_{\tau - \omega}) - F(\phi^{(2)}, \theta_{\tau - \omega}), q)_E.
\]
(5.18)
We now estimate the terms in (5.18). Firstly, similar to (4.24)
\[
\text{Re}(L\xi, q)_E \geq -\frac{N_c\omega}{M^2} \|x^{(1)} - x^{(2)}\|^2 + (C_2 - \rho_1(\theta_{\tau - \omega}) - \rho_2(\theta_{\tau - \omega})) \sum_{m \in \mathbb{Z}^N} p_{\|m\|_M} |\xi_m|^2_E,
\]
(5.19)
Secondly, similar to (4.29)
\[
\text{Re}(F(\phi^{(1)}, \theta_{\tau - \omega}) - F(\phi^{(2)}, \theta_{\tau - \omega}), q)_E \leq \frac{\beta}{4} \sum_{m \in \mathbb{Z}^N} p_{\|m\|_M} |y^{(1)} - y^{(2)}|^2 + 4\frac{N_c \omega^2}{\beta} e^{-2\rho_2(\theta_{\tau - \omega})} \sum_{m \in \mathbb{Z}^N} p_{\|m\|_M} ((x^{(1)}_m)^2 + (x^{(2)}_m)^2) |x^{(1)}_m - x^{(2)}_m|^2.
\]
(5.20)
By (5.5), for \(I \in N\)
\[
\sum_{\|m\| \geq 2I} ((x^{(1)}_m)^2 + (x^{(2)}_m)^2) \leq \frac{2}{\beta} \rho_0 + \frac{2}{\beta} (\frac{C_3}{I} + \gamma_I) K_0(\theta_{\tau - \omega}),
\]
(5.21)
then, for \(M \geq 2I,\) we have
\[
\text{Re}(F(\phi^{(1)}, \theta_{\tau - \omega}) - F(\phi^{(2)}, \theta_{\tau - \omega}), q)_E \leq \left(\frac{\beta}{4} + \frac{8N_c \omega^2}{\beta^2} e^{-2\rho_2(\theta_{\tau - \omega})} + \frac{8N_c \omega^2}{\beta^2} e^{-2\rho_2(\theta_{\tau - \omega})}(\frac{C_3}{I} + \gamma_I) K_0(\theta_{\tau - \omega})\right) \times \sum_{m \in \mathbb{Z}^N} p_{\|m\|_M} |\xi_m|^2_E.
\]
(5.22)
Combining (5.18) with (5.19), (5.22), for \(M \geq 2I,\) we obtain
\[
\frac{d}{d_m} \sum_{m \in \mathbb{Z}^N} p_{\|m\|_M} |\xi_m|^2 + 2(C_2 - \rho_1(\theta_{\tau - \omega}) - \rho_2(\theta_{\tau - \omega}) - K_1(\theta_{\tau - \omega}))
\times \sum_{m \in \mathbb{Z}^N} p_{\|m\|_M} |\xi_m|^2_E \leq \left(\frac{N_c \omega^2}{\beta^2} + \frac{16N_c \omega^2}{\beta^2} e^{-2\rho_2(\theta_{\tau - \omega})}(\frac{C_3}{I} + \gamma_I) K_0(\theta_{\tau - \omega})\right) ||\xi(r)||^2_E
\leq C_4 \frac{1}{I} + \gamma_I (1 + e^{-2\rho_2(\theta_{\tau - \omega})} K_0(\theta_{\tau - \omega})) e^{2 \int_{\tau - t}^\infty \left(\frac{C_3}{I} + \gamma_I\right) K_0(\theta_{\tau - \omega}) ds} ||\xi(\tau - t)||^2_E,
\]
where
\[
K_1(\omega) = \frac{\beta}{4} + \frac{8N_c \omega^2 \rho_0}{\beta^2} e^{-2\rho_2(\omega)},
\]
(5.24)
\[
C_4 = \max\{(Nc_0 + \frac{16N_c \omega^2}{\beta^2})C_3, \frac{16N_c \omega^2}{\beta^2}\},
\]
(5.25)
By applying Gronwall’s inequality to (5.23) on $[\tau - t, \tau)(t \geq 0)$, we have that for $M \geq 2I$,
\[
\begin{align*}
\sum_{m \in \mathbb{Z}^N} p(\|m\|_M)\|\xi(\tau, t, \theta, \omega) - \xi_{\tau-t}(\theta, \omega)\|_E^2 \\
\leq \sum_{m \in \mathbb{Z}^N} p(\|m\|_M)\|\xi(\tau, t, \theta, \omega) - \xi_{\tau-t}(\theta, \omega)\|_E^2 \\
\times e^{\int_{\tau-t}^\tau (-C_2 + p_1(\theta, \omega) + p_2(\theta, \omega) + K_1(\theta, \omega))ds} \\
\times e^{\int_{\tau-t}^\tau C_4(1 + e^{-2p_2(\theta, \omega)}K_0(\theta, \omega))ds} \\
\leq e^{\int_{\tau-t}^\tau (-C_2 + p_1(\theta, \omega) + p_2(\theta, \omega) + K_1(\theta, \omega))ds} \\
\times e^{\int_{\tau-t}^\tau C_4(1 + e^{-2p_2(\theta, \omega)}K_0(\theta, \omega))e^{2C_2}dr}.
\end{align*}
\]

Since $\sqrt{t} \leq e^x$ for all $x \geq 0$, it follows that
\[
\begin{align*}
\int_{\tau-t}^\tau C_4^2(1 + e^{-2p_2(\theta, \omega)}K_0(\theta, \omega))e^{2C_2r}dr \\
\leq (\int_{\tau-t}^\tau C_4^2(1 + e^{-2p_2(\theta, \omega)}K_0(\theta, \omega))^2dr)^{1/2}(\int_{\tau-t}^\tau e^{4C_2r}dr)^{1/2} \\
\leq \frac{1}{2\sqrt{C_2}} e^{\int_{\tau-t}^\tau C_4^2(1 + e^{-2p_2(\theta, \omega)}K_0(\theta, \omega))^2dr}.
\end{align*}
\]

By (5.26),
\[
\begin{align*}
&\sum_{\|m\| \geq 4I} |\xi_m(\tau, t, \theta, \omega) - \xi_{\tau-t}(\theta, \omega)|_E^2 \\
\leq &\sum_{m \in \mathbb{Z}^N} p(\|m\|_M)\|\xi_m(\tau, t, \theta, \omega) - \xi_{\tau-t}(\theta, \omega)\|_E^2 \\
\leq &\sum_{m \in \mathbb{Z}^N} p(\|m\|_M)\|\xi_m(\tau, t, \theta, \omega) - \xi_{\tau-t}(\theta, \omega)\|_E^2 \\
\leq &\left(\int_{\tau-t}^\tau (-C_2 + 2S_2)ds + \delta_I e^{\int_{\tau-t}^\tau C_4^2(1 + e^{-2p_2(\theta, \omega)}K_0(\theta, \omega))^2dr}\right)^{1/2}.
\end{align*}
\]

where
\[
\delta_I = \frac{1}{\sqrt{4C_2}} \sqrt{1 + \gamma_I}.
\]

\[
S_2(\omega) = C_2 + |p_1(\omega)| + |p_2(\omega)| + K_1(\omega) + \frac{1}{2}C_4^2(1 + e^{-2p_2(\omega)}K_0(\omega))^2
\]

and
\[
S_3(\omega) = |p_1(\omega)| + |p_2(\omega)| + K_1(\omega).
\]

That is to say, (5.17) holds. From Lemma 5.1, it follows that (5.16) holds.

**Lemma 5.3.** For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, it holds that
\[
0 \leq E[S_3(\omega)] \leq \frac{3C_2}{8}, \quad E[S_j^2(\omega)] < \infty, \quad j = 2, 3.
\]

**Proof.** By the property of Gamma function $\Gamma$,
\[
\Gamma(0) = \Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \Gamma(n) = (n - 1)!, \quad \Gamma(r + 1) = r \Gamma(r), \quad \forall r \in \mathbb{R}.
\]

It follows from (3.15) that
\[
E[K_1(\omega)] \leq \frac{\beta}{4} + \frac{8\Lambda \lambda_0}{\beta \bar{\rho}} \frac{1 + (4 \pi + 3e)^{1/2}}{\sqrt{3}} \leq \frac{3C_2}{8} - \frac{|a_1| + |b|}{\sqrt{\pi}}.
\]

By (3.14), (5.31) and (5.34),
\[
E[S_3(\omega)] = E[|p_1(\omega)|] + E[|p_2(\omega)|] + E[K_1(\omega)] \leq \frac{3C_2}{8}.
\]

By (5.31) and (5.34), we have
\[
S_3^2(\omega) \leq 3p_1^2(\omega) + 3p_2^2(\omega) + 3K_1^2(\omega),
\]

\[
E[S_3^2(\omega)] \leq \frac{3C_2}{8}.
\]
\[ E[K_2^2(\omega)] \leq \frac{a^2}{8} + \left( \frac{32N^2\chi_1^2}{\beta^2\gamma^2} \right) E[e^{-4\rho_2(\omega)}] \]
\[ \leq \frac{a^2}{8} + \left( \frac{32N^2\chi_1^2}{\beta^2\gamma^2} \right) \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}} C_0, \]  
(5.37)

thus
\[ E[S_2^2(\omega)] \leq \frac{3(a^2 + b^2)}{2} + 3C_5 < \infty. \]  
(5.38)

By (5.30), there exists a constant \( C_6 \) such that
\[ S_2^2(\omega) \leq C_6(1 + |\rho_1(\omega)|^2 + |\rho_2(\omega)|^2 + K_1^2(\omega) + e^{-8\rho_2(\omega)} + K_0^2(\omega)), \]  
(5.39)

and by (4.16), there exists a constant \( C_7 \) such that
\[ E[K_0^2(\omega)] = E[\int_{-\infty}^{0} e^{-2\rho_2(\theta_2) + C_2 t + f_0^\omega K(\theta_2, \omega)ds} \]  
\[ \leq C_7. \]  
(5.40)

By (5.40),
\[ E[S_2^2(\omega)] \leq C_6(1 + \frac{a^2 + b^2}{2} + C_5 + 4\sqrt{\pi} + 3e + C_7) < +\infty. \]  
(5.41)

\[ \square \]

Lemma 5.4. For any \( \tau \in \mathbb{R}, \ \omega \in \Omega \), it hold that
\[ \lim_{t \to 0} \sup_{\delta \in \chi_1(\tau, \omega)} \| \Phi(t, \tau, \omega) - \psi \|_E = 0, \]
\[ \lim_{t \to 0} \sup_{u \in \chi_1(\tau - t, \theta - t \omega)} \| \Phi(0, \tau - t, \theta - t \omega) - \psi \|_E = 0. \]  
(5.42)

Proof. From the estimate for \( \psi \in \chi_1(\tau, \omega) \) and \( t \geq 0 \)
\[ \| \Phi(t, \tau, \omega) - \phi \|_E^2 \]
\[ \leq 9t \int_0^{\tau+t} (||Ax(r)||^2 + ||\rho_1(\theta_{t-r} \omega)x(r)||^2 + ||\alpha x(r)||^2)dr \]
\[ + 9t \int_0^{\tau+t} (||\rho_2(\theta_{t-r} \omega)f(r)||^2 + ||\rho_2(\theta_{t-r} \omega)xy||^2)dr \]
\[ + 9t \int_0^{\tau+t} (||\rho_2(\theta_{t-r} \omega)y(r)||^2)dr \]
\[ + 9t \int_0^{\tau+t} (||\rho_2(\theta_{t-r} \omega)g(r)||^2 + ||\lambda e^{-\rho_2(\theta_{t-r} \omega)}B(x)^2||^2)dr \]
\[ \leq 9t \int_0^t (16N + |\rho_1(\theta_{t-r} \omega)|^2 + \lambda^2 M_1^2(\theta_{t-r} \omega))dr \]
\[ + 9t \int_0^t (||f||^2 + ||\rho_2(\theta_{t-r} \omega)M_2(\theta_{t-r} \omega)||^2)dr \]
\[ + 9t \int_0^t (\beta^2 + |\rho_2(\theta_{t-r} \omega)|^2)M_0^2(\theta_{t-r} \omega)dr \]
\[ + 9t \int_0^t (e^{-2\rho_2(\theta_{t-r} \omega)})^2||g||^2 + 4N\lambda^2 e^{-2\rho_2(\theta_{t-r} \omega)}M_0^2(\theta_{t-r} \omega)dr \]
\[ \to 0 \quad (t \to 0), \]

it follows that, \( \lim_{t \to 0} \sup_{\phi \in \chi_1(\tau, \omega)} \| \Phi(t, \tau, \omega) - \phi \|_E^2 = 0 \). Similarly, \( \lim_{t \to 0} \sup_{\phi \in \chi_1(\tau, \omega)} \| \Phi(0, \tau - t, \theta - t \omega) - \phi \|_E^2 = 0 \). \[ \square \]

Theorem 5.5. \( \{ \Phi(t, \tau, \omega) \}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega} \) possesses a random exponential attractor \( \{ \mathcal{K}(\tau, \omega) \}_{\tau \in \mathbb{R}, \omega \in \Omega} \) with properties: for any \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
(i) \( \mathcal{R}(\tau, \omega) \subseteq \mathcal{K}(\tau, \omega) \subseteq \chi_1(\tau, \omega) \) and \( \mathcal{K}(\tau, \omega) \) is a compact set of \( E \);
(ii) \( \Phi(t, \tau, \omega)\mathcal{K}(\tau, \omega) \subseteq \mathcal{K}(t + \tau, \theta t \omega) \) for all \( t \geq 0 \);
(iii) there exists a finite integer \( I_0 \in \mathbb{N} \) such that
\[ \dim_f \mathcal{R}(\tau, \omega) \leq \dim_f \mathcal{K}(\tau, \omega) \leq \frac{2(8I_0 + 1) \ln(\sqrt{8I_0 + 1} + 1)}{\ln 4} < \infty; \]  
(5.44)
(iv) for any set $B \in \mathcal{D}(E)$, there exist a random variable $\overline{T}_\omega \geq 0$ and a tempered random variable $b_\omega$ such that

$$d_h(\Phi(t, \tau, \omega)B(\tau, \omega), K(t + \tau, \theta_\omega)) \leq b_\omega e^{-\frac{\alpha \ln \frac{4}{3} t}{64 \ln \frac{16}{3}}} \quad t \geq T_B + \overline{T}_\omega; \quad (5.45)$$

(v) for any $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\lim_{t \searrow 0} d_s(K(\tau + t, \theta_\omega), K(\tau, \omega)) = 0, \quad \lim_{t \searrow 0} d_h(K(\tau - t, \theta_\omega), K(\tau, \omega)) = 0. \quad (5.46)$$

Proof. From the proof of Lemma 5.3,

$$-C_2 < E[-C_2 + S_3(\omega)] \leq -C_2 + \frac{3C_2}{8} = -C_2 = 0, \quad (5.47)$$

take $t = t_0$ in (5.16) and (5.17). By

$$0 < \frac{2 \ln \frac{16}{3}}{C_2} \leq t_0 = \frac{2 \ln \frac{4}{16}}{E[-C_2 + S_3(\omega)]} < \frac{16 \ln \frac{16}{3}}{C_2} < +\infty, \quad (5.48)$$

then we have

$$-\frac{C_2}{8 \ln \frac{16}{3}} \leq -\frac{1}{4t_0} < -\frac{C_2}{64 \ln \frac{16}{3}} < 0. \quad (5.49)$$

From (5.41)

$$0 < 3E[S_2^2(\omega)] + E[S_3^2(\omega)] < +\infty. \quad (5.50)$$

By (5.50) and (5.51),

$$0 < e^{-\frac{1}{4t_0} t_0^2 (3E[S_2^2(\omega)] + E[S_3^2(\omega))]} < +\infty. \quad (5.51)$$

Comparing (2.52) in [9] and (5.28), we see that

$$0 < \delta = 2\delta_I = \frac{2}{\sqrt{AC_2}} \sqrt{\frac{1}{7} + \gamma_I}. \quad (5.52)$$

Let

$$\tilde{\gamma} = \min \left\{ \frac{1}{8}, e^{-\frac{1}{4t_0} t_0^2 (3E[S_2^2(\omega)] + E[S_3^2(\omega))] + E[S_2^2(\omega))]} \right\} \in (0, +\infty) \quad (5.53)$$

be a bounded fixed positive number. By $\lim_{t \rightarrow +\infty} \frac{1}{t} = 0$, it then follows from (5.53) that there exists a finite integer $I_0 \in \mathbb{N}$ such that $0 < 2\delta I_0 < \tilde{\gamma}$. Then Theorem 2.4 and Theorem 2.2 in [14] assure the statements in Theorem 5.5.

\[\square\]

Remark 1. Consider following non-autonomous long wave-short wave resonance equations on $\mathbb{R}^3$:

$$\begin{cases}
  iu_t + u_{xx} - uv + i\alpha u = f(x, t) + a u \circ \dot{\omega}^{(1)}(t), \quad x \in \mathbb{R}^3, \quad t > \tau, \\
  v_t + \beta v + \lambda(|u|^2)_x = g(x, t) + b v \circ \dot{\omega}^{(2)}(t), \quad x \in \mathbb{R}^3, \quad t > \tau,
\end{cases} \quad (5.54)$$

with the initial conditions

$$u(x, \tau) = u_\tau(x), \quad v(x, \tau) = v_\tau(x), \quad x \in \mathbb{R}^3, \quad \tau \in \mathbb{R}, \quad \tau > 0. \quad (5.55)$$

Similarly, the system (5.54)-(5.55) has a random exponential attractor as in Theorem 5.5.
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