Random Matrix Theory by the Supersymmetry Method Beyond the Sigma-Model

Vladan Lučić
Physics Department, Northeastern University
Boston, MA 02115, USA
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Abstract

The leading correction to the smoothed connected energy density-density correlation function is obtained for the large energy difference, within the context of the Gaussian Random Matrix Theory. In order to achieve this result, the supersymmetry method is extended beyond the sigma-model, to include small quadratic fluctuations around the saddle point. Special care is taken to avoid the potential divergence arising from the unbounded nature of the saddle point. Also, in the small energy difference regime, the leading correction to a two point correlation function is obtained.

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1 Introduction

The Random Matrix Theory (RMT) has its roots in an idea of Wigner that statistical mechanics should be used to analyze highly excited energy levels of large nuclei [1]. This concept was further developed by Dyson [2] to become what is now known as RMT. Since then, it has been applied to numerous problems related to complex quantum systems in different areas of physics [3].

In RMT, the Hamiltonian $H$ is represented by a matrix of dimension $N$, for $N$ large. The matrix is completely random, except that it satisfies a certain symmetry. The ensemble average of any function $f(H)$ of the Hamiltonian is calculated by multiplying the function by a probability distribution and averaging over all possible matrices that possess the required symmetry:

$$\langle f(H) \rangle = \frac{1}{Z} \int \mathcal{D}H f(H) \exp\{-NTr[V(H)]\},$$  \hspace{1cm} (1)

where $Z$ is the partition function:

$$Z = \int \mathcal{D}H \exp\{-NTr[V(H)]\}. \hspace{1cm} (2)$$

RMT has been utilized for various purposes. For example, in condensed matter physics of disordered systems, the Hamiltonian, or a part of it, can often be taken to be random, except for some physical symmetries. Assuming that a particular physical quantity does not depend on the details of the system, it can be calculated using RMT. The problem of dealing with a complicated (or unknown) Hamiltonian of the physical system is replaced in RMT by averaging over an ensemble of random matrices which possess a certain symmetry, as required by the symmetry of the original Hamiltonian. Commonly used symmetries of the ensemble are unitary, orthogonal and symplectic. The problems where RMT has been applied to include conductance fluctuations in mesoscopic systems [4], persistent current of disordered metal rings [5] and the motion of a quantum particle in a thick wire [6]. A similar approach is taken in analyzing the low energy behavior of QCD, only here the ensemble considered possesses one of the chiral symmetries: chiral orthogonal, chiral unitary, or chiral symplectic [7].
Another incentive for exploring RMT comes from its relation to both classical and quantum chaotic systems. On the basis of numerical evidence, it was strongly conjectured that certain statistical properties of quantum chaotic spectra are universal and identical to those predicted by Gaussian ensembles of RMT \[8\]. The connection was further elaborated in [4] and proved in semiclassical limit in [10]. Another important problem in chaos, the transition from classical integrable to chaotic behavior, has been investigated by relating classical chaos to the size of a band of an ensemble of banded random matrices [11]. Banded ensembles have been constructed to describe the level fluctuations of a particular type of anharmonic oscillator with a parameter that is responsible for the transition from integrable to non-integrable behavior [12].

In all the applications of RMT mentioned above, one deals with Gaussian ensembles. The potential \( V \) in (1) has a simple quadratic form (5), which is equivalent to the requirement that different elements of matrix \( H \) be uncorrelated. It is interesting to investigate how additional terms in the potential of the non-Gaussian ensembles affect different correlators, or, in other words, what are the quantities calculated within RMT that do not depend on the form of the potential, but on the symmetries alone. Universal properties of correlations for non-Gaussian ensembles and their relation to the Gaussian ensembles were studied and the correlations calculated, in the context of matrix models, in [13] using the loop equations method. The same properties were investigated in the context of disordered systems in [14] using the orthogonal polynomials. The conjecture made there was clarified in [15], while the rigorous treatment was given in [16]. Other methods, such as the diagrammatic method [17], the functional derivative method [18] and the supersymmetry method [19] were also introduced and used for similar problems.

On the other hand, RMT can be extended to matrix models which have been studied for their own sake. Matrix models, for example, has been used to describe 2D gravity. The partition function of the matrix models is obtained from the RMT partition function by adding a kinetic term to the potential and summing over all
Feynman diagrams, while in 2D gravity, the partition function is obtained by summing over all surfaces. The potential term generates Feynman diagrams, which can be regarded as the inverse lattice of a discretized surface of arbitrary genus. More precisely, a Feynman diagram is the inverse lattice of the discretized surface. The ensemble average of RMT can now be regarded as a sum over all diagrams (surfaces). Furthermore, when \( N \) is large, the partition function has a perturbative expansion in \( N^{-1} \), each term corresponding to the contribution of all surfaces of a fixed genus. It has been shown that the dominant contribution comes from a sphere (genus 0) \([20]\), so the approximation in which only the leading term in the expansion is used is called spherical or planar approximation. The genus 1 contribution to the partition function and the one point correlator have been calculated in \([21]\), using the loop equations. Good reviews of different aspects of matrix models are given in \([22]\).

While the original approach used in RMT was the method of orthogonal polynomials \([3]\), the supersymmetry method, pioneered by Efetov \([23]\), has been also proven to be very useful, especially when the coupling to external sources is needed \([4]\). It is essentially an extension of Anderson’s replica trick \([24]\). A very thorough and detailed review of the method can be found in \([25]\).

The results cited above were concerned with the leading order contributions to different RMT correlators. The leading order correction (genus one contribution, in the language of matrix models) to the one-point correlation function for the Hermitian matrix model has been calculated in \([26]\), while the iterative procedures to calculate other corrections (higher genus contributions) were presented for Hermitian \([27]\), complex \([28]\), and orthogonal and symplectic \([29]\) matrix models. Recently, there has been significant interest in finding the corrections within the context of disordered systems. Orthogonal polynomials have been used to calculate leading corrections to one and two-point correlation functions for the complex RMT \([30]\). The supersymmetry method has also been used to calculate corrections to various quantities within RMT: the energy-energy correlation in the regime where the energy difference
is small \cite{31}, the level curvature distribution \cite{32} and the eigenfunction amplitudes
distribution \cite{33}.

In this paper, the leading order correction to two-point correlators, in the $N^{-1}$ ex-
pansion, is calculated using the extension of the supersymmetry method. Within the
usual supersymmetry method, the partition function is expanded around the saddle
point (the saddle point manifold, to be precise), the energy difference is considered
small ($\sim N^{-1}$), and only the contribution coming from the saddle point is considered.
This simplification leads to the so-called zero-dimensional non-linear $\sigma$-model. Here,
the method is extended beyond the $\sigma$-model: both to accommodate for the large en-
ergy difference ($\sim N^0$) and to include small fluctuations around the saddle point. By
extending the method for the large energy difference the leading contribution to the
two point correlators can be obtained (in the large energy regime), while the inclusion
of the fluctuations allows the calculation of the corrections in both regimes. As it
is customary in the large energy regime, the smoothed energy density-density corre-
lation function is considered \cite{14}; that is, the correlator is averaged over the energy
scale of the order of few $N^{-1}$'s, so that the terms oscillating with the frequency of
the order $N$ go to zero. The leading order contribution of the (smoothed) connected
energy density-density correlation function obtained in this text agrees with the result
obtained using the method of orthogonal polynomials, \cite{14}. In the small energy differ-
ence regime, the correction to the connected two point correlator, $\langle G^+(E_1)G^-(E_2)\rangle^c$, is
obtained. The leading term of the correlator is also calculated and it is found to
agree with the well known result which was obtained using the supersymmetry, as
well as other methods.

Furthermore, the convergence of the method is proved. Particular care has to be
taken because of the potential divergence arising from the unbounded nature of the
saddle point manifold. The singularity structure of Green’s functions is used in order
to resolve the ambiguities arising from the double-valued nature of the saddle point.
As it is usual in the supersymmetry method, the Gaussian ensemble is considered in
the limit when \( N \to \infty \) and the energies are taken not too close to the ends of the spectrum. For the sake of the definiteness, in this text, the ensemble with the unitary symmetry is considered.

The approach used in this text was to follow the presentation of the supersymmetry method as given in an excellent introductory paper by Zuk [34] and to modify it when needed. Some parts of the calculations were done using the symbolic software package Dill [35] as well as additional programs written in the Mathematica programming language [36].

The paper is organized as follows: the important steps of the usual supersymmetry formalism are given in section 2. The definitions and conventions related to the Grassman variables and the integration over the Grassman variables are given in appendices [A] and [B], respectively. The formalism is modified to include the corrections in section [3], while the correlation functions are calculated and the convergence of the method is proved in section [4], using the results presented in appendix [C].

2 Short Review of the Supersymmetry Method

In this section, the main points of the supersymmetry method are summarized. This is needed in order to show how the extended supersymmetry method is constructed and to prove its convergence. We begin by defining the relevant quantities. The quantity of main interest here, the connected two point correlation function, is given by:

\[
\langle G^+(E_1)G^\pm(E_2) \rangle^c = \langle G^+(E_1)G^\pm(E_2) \rangle - \langle G^+(E_1) \rangle \langle G^\pm(E_2) \rangle,
\]

where:

\[
G^\pm(E) = \text{Tr} \frac{1}{E - H \pm i \epsilon},
\]

and the averaging \( \langle \ldots \rangle \) is performed over Gaussian unitary ensemble; therefore, it is given by [1] with the potential \( V \) given by:

\[
V(H) = \frac{1}{2\lambda^2} H^2,
\]
where $\lambda$ is an arbitrary constant.

The density of states is given by:

$$\rho = \frac{i}{2\pi N} \left[ G^+(E) - G^-(E) \right],$$

(6)

so that the connected part of the ensemble averaged energy density-density correlation function is given by:

$$C(E_1, E_2) = \langle \rho(E_1) \rho(E_2) \rangle - \langle \rho(E_1) \rangle \langle \rho(E_2) \rangle = \frac{1}{2\pi^2 N^2} \text{Re} \left[ \langle G^+(E_1) G^-(E_2) \rangle^c - \langle G^+(E_1) G^+(E_2) \rangle^c \right].$$

(7)

The density of states satisfies Wigner’s semi-circle law [37]:

$$\rho(E) = \frac{1}{2\pi \lambda^2} \sqrt{4\lambda^2 - E^2},$$

(8)

so that the $N$ energy eigenstates of $H$ are confined to a region between $2\lambda$ and $-2\lambda$.

In order to begin with the supersymmetry formalism, let us consider now a generating function:

$$Z(\varepsilon) = \text{Det}^{-1} [D + J(\varepsilon)]$$

$$= \exp\{-\text{Trg} \ln [D + J(\varepsilon)]\}$$

(9)

where $\text{Det}$ and $\text{Trg}$ denote the graded (super) determinant and the graded (super) trace respectively [38] and are taken over all indices. The symbol $\text{trg}$ is used when the summation is to be taken over $p$ and $\alpha$ indices only. (For a complete introduction to anti-commutative (Grassman) variables one can consult [39] and [25].) The definitions and conventions used in this article related to the Grassman variables are given in appendix A. $D$ and $J$ are $4N \times 4N$ graded matrices, diagonal in both upper and lower indices, the only non-vanishing elements being:

$$D^0_0 = D^1_1 = (E_1 + i\varepsilon_p) 1_N - H,$$

(10)

$$J^\alpha_\alpha(\varepsilon) = (-)^\alpha \varepsilon_p 1_N.$$

(11)
It is then easy to show that:

$$G(E_p) = -\frac{1}{2} \frac{\partial}{\partial \varepsilon_p} Z(\varepsilon) \bigg|_{\varepsilon=0}$$

(12)

$$G(E_1 + i\varepsilon_1)G(E_2 + i\varepsilon_2) = \frac{1}{4} \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2} Z(\varepsilon) \bigg|_{\varepsilon=0}$$

(13)

The two point Green’s functions, $G^+(E_1)G^-(E_2)$ and $G^+(E_1)G^+(E_2)$, are obtained from the above equation by taking $\varepsilon_1 = \varepsilon = -\varepsilon_2$ and $\varepsilon_1 = \varepsilon = \varepsilon_2$ respectively, for $\varepsilon$ positive and infinitesimal. Therefore $Z(\varepsilon)$ is indeed the generating function for the two point Green’s functions. In this case, the matrix $D + J$ consists of two diagonal blocks, corresponding to the two Green’s functions. The two blocks contain the information about the respective energies as well as the sides of the branch cut of the Green’s function from which the energy approaches the real axis. If the ensemble averaged product of more Green’s functions were to be calculated, additional blocks would have to be added to $D + J$.

Some important properties of the integration over Grassman variables are given in appendix B. Using the Gaussian super-integral (73), the generating function (9), can be ensemble averaged and shown to be:

$$\langle Z(\varepsilon) \rangle = \int D\phi D\bar{\phi} \left\langle \exp \left[-i \sum_{p,\alpha,\mu} \bar{\phi}_p^\alpha(\mu) H \phi_p^\alpha(\mu) \right] \right\rangle \times \exp \left[i \sum_{p,\alpha,\mu} \bar{\phi}_p^\alpha(\mu)(E + \frac{\omega}{2} L + i\varepsilon_1 I_1 + i\varepsilon_2 I_2 + J) \phi_p^\alpha(\mu) \right],$$

(14)

where the average energy $E = (E_1 + E_2)/2$, and the difference between the energies $\omega = E_1 - E_2$ have been introduced and $L$, $I_1$ and $I_2$ are diagonal in $p$ and $\alpha$ indices:

$$L_{pp'}^{\alpha\alpha'} = (-)^{p-1} \delta^{\alpha\alpha'} \delta_{pp'}, \quad (I_i)_{pp'}^{\alpha\alpha'} = \delta^{\alpha\alpha'} \delta_{pp'} \delta_i^p.$$  

(15)

At this point, it is clear that due to the Grassman variables, the generating function is normalized by the definition (8) $(Z(0) = 1)$ which makes it possible to interchange the integration over $\phi$’s and the ensemble averaging (integration over the random matrix $H$) and, eventually, to perform the ensemble averaging completely.
In the case where a generating function is not normalized, it has to be normalized first by dividing by the corresponding partition function and then ensemble averaged. Typically, both the partition function and the generating function are integrals, so it is a very nontrivial task to perform the ensemble averaging.

The procedure is continued using:

\[
\langle \exp \left[ -i \sum_{p,\alpha,\mu} \bar{\phi}_p^\alpha(\mu) H \phi_p^\alpha(\mu) \right] \rangle = \exp \left( -\frac{\lambda^2}{2N} \text{trg} \ S^2 \right), \quad S^{\alpha\alpha'}_{pp'} = \sum_{\mu} \phi_p^\alpha(\mu) \bar{\phi}_{p'}^{\alpha'}(\mu).
\]  

(16)

Then the ensemble average over the random Hamiltonian can be completely performed to give:

\[
\langle Z(\varepsilon) \rangle = \int \mathcal{D}\phi \bar{\phi} \exp \left[ -\frac{\lambda^2}{2N} \text{trg} \ S^2 + i\text{trg} \ (E + \frac{\omega}{2} L + i\epsilon_1 I_1 + i\epsilon_2 I_2 + J) S \right].
\]  

(17)

Obviously, the price that has to be paid for carrying out the ensemble averaging completely is the introduction of the quartic terms \( (S^2) \) in the integrand. The way to proceed is to eliminate these terms, using the Hubbard-Stratonovich transformation, as follows.

The function \( W \) is defined by:

\[
iW(\sigma, S) = \frac{\lambda^2}{2N} \text{trg} \ (S - i\frac{N}{\lambda} \sigma)^2 + \text{trg} \ (\epsilon_1 I_1 + \epsilon_2 I_2)(S - i\frac{N}{\lambda} \sigma),
\]  

(18)

where \( \sigma \) is an arbitrary \( 4 \times 4 \) super-matrix having the same symmetry properties as \( S \). Using the parametrization for \( \sigma \) introduced below, and as long as \( N \) is large, it follows that:

\[
\int \mathcal{D}\sigma e^{iW(\sigma, S)} = \int \mathcal{D}\sigma e^{iW(\sigma, 0)} = 1.
\]  

(19)

The second identity is easy to check once the parametrization for \( \sigma \) is introduced \([24]–[37]\). After the previous identity is inserted in \([17]\) and the shift of variables: \( \sigma \to \sigma - \frac{1}{\lambda} (\frac{\omega}{2} L + J) \) is performed, the averaged generating function takes the form:

\[
\langle Z(\varepsilon) \rangle = \int \mathcal{D}\sigma \exp \left\{ -\frac{N}{2} \text{trg} \ \sigma^2 - \frac{N\omega}{2\lambda} \text{trg} \ \sigma L - \frac{iN}{\lambda} \text{trg} \ (\epsilon_1 I_1 + \epsilon_2 I_2) \sigma - N\text{trg} \ \ln(E - \lambda \sigma) - \frac{N}{\lambda} \text{trg} \left[ \sigma + \frac{\omega}{2\lambda} L + \frac{i}{\lambda} (\epsilon_1 I_1 + \epsilon_2 I_2) \right] J \right\}
\]  

(20)
completing the Hubbard-Stratonovich transformation.

At this point, it is important to notice that not only have the quartic terms disappeared from the generating functional, but in addition the number of integration variables have decreased from $4N$ complex variables in (14) to those spanned by the $4 \times 4$ complex super-matrix $\sigma$ in (20). Not all entries in $\sigma$ are independent, because it has to satisfy certain symmetry properties. Therefore, in order to see what exactly the integration variables are, the symmetries of the super-matrix $\sigma$ have to be investigated first and then the parametrization of $\sigma$ (the parameters being the independent variables) consistent with the symmetries can be found. The detailed account of the symmetries and the parametrization leading to the independent variables of the super-matrix $\sigma$ for the Gaussian orthogonal ensemble is given in [25]. The analysis for the case needed here, the Gaussian unitary ensemble, is similar and is clearly stated in [34]; what follows is only a short description of the results.

In order to satisfy (19), and having in mind the definition of $W$ given in (18), the super-matrices $\sigma$ and $S$ have to have the same symmetry. From the definition of $S$, (16) it follows that: $S^\dagger = s S s$, so: $\sigma^\dagger = s \sigma s$ also, leading to the decomposition:

$$\sigma = T^{-1} P_d T,$$

(21)

where $4 \times 4$ super-matrix $T$ has to satisfy:

$$T^{-1} = s T^\dagger s,$$

(22)

and $P_d$ has to be a diagonal super-matrix with real boson-boson and imaginary fermion-fermion elements. The last condition has to be changed slightly in order to achieve convergence of integrals in (19) and make $P_d$ belong to the saddle point of the exponent in (20). Instead of (21), $\sigma$ is decomposed as:

$$\sigma = T^{-1} (\sigma_0 + N^{-1/2} P_d) T.$$

(23)

The change amounts to a simple shift of the diagonal matrix $P_d$ by a constant diagonal $4 \times 4$ super-matrix $\sigma_0$ determined below. $P_d$ now has real boson-boson and imaginary fermion-fermion elements. Also, $P_d$ has been rescaled for future convenience.
Due to its defining property (22), the boson-boson part of the super-matrix $T$ spans the (non-compact) group $SU(1, 1)$ while the fermion-fermion part spans the (compact) group $SU(2)$. Further, $T$ can be written as $T = RT_0$, where $[R, L] = 0$, so that $R$ is block-diagonal in the lower indices, and the blocks are denoted by $R_1$ and $R_2$. The symmetry of the matrix $T$ implies: $T_0^{-1} = sT_0^s$, $R_1^{-1} = R_1^1$ and $R_2^{-1} = kR_2^1k$. This gives the final decomposition of $\sigma$:

$$\sigma = T_0^{-1}(\sigma_0 + N^{-1/2}\delta P)T_0 = T_0^{-1}P T_0,$$

(24)

where $\delta P = R^{-1}P dR$. Thus $\delta P$ is a block-diagonal super-matrix with real boson-boson and imaginary fermion-fermion eigenvalues. The decomposition of $\sigma$ (24), can be regarded as splitting of the manifold spanned by the independent variables of $\sigma$ to the space spanned by $\delta P$ and the coset-manifold defined by $T_0$. Due to the change of variables, the original measure $D\sigma$ becomes:

$$D\sigma = I(P)DP D\mu(T_0),$$

(25)

where $D\mu(T_0)$ is the invariant measure on the coset manifold and $I(P)$ is the Jacobian of the transformation $I(P)$:

$$I(P) = \left[ \frac{(\Lambda_0^1 - \Lambda_2^0)(\Lambda_1^1 - \Lambda_2^1)}{(\Lambda_0^0 - \Lambda_2^1)(\Lambda_1^0 - \Lambda_2^0)} \right]^2,$$

(26)

and $\Lambda_\alpha^p$ are the eigenvalues of $P_p$. The derivation of the last expression is straightforward; useful hints are given in similar derivations of [40] and [23].

Following its symmetry, the matrix $T_0$ can be decomposed as:

$$T_0 = U^{-1}T_d U,$$

(27)

where:

$$U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad u = \exp \begin{pmatrix} 0 & -\eta^* \\ \eta & 0 \end{pmatrix}, \quad v = \exp \begin{pmatrix} 0 & -i\rho^* \\ i\rho & 0 \end{pmatrix},$$

(28)
\( \eta \) and \( \rho \) are Grassman variables, and

\[
T_d = \begin{pmatrix}
\sqrt{1+\mu_0\bar{\mu}_0} & 0 & \mu_0 & 0 \\
0 & \sqrt{1+\mu_1\bar{\mu}_1} & 0 & \mu_1 \\
\bar{\mu}_0 & 0 & \sqrt{1+\mu_0\bar{\mu}_0} & 0 \\
0 & \bar{\mu}_1 & 0 & \sqrt{1+\mu_1\bar{\mu}_1}
\end{pmatrix}
\] (29)

\[
\mu_0 = -e^{i\phi_0} \sinh \frac{\theta_0}{2}, \quad \bar{\mu}_0 = -e^{-i\phi_0} \sinh \frac{\theta_0}{2}
\]

\[
\mu_1 = ie^{i\phi_1} \sin \frac{\theta_1}{2}, \quad \bar{\mu}_1 = ie^{-i\phi_1} \sin \frac{\theta_1}{2}.
\] (30)

\( \phi_0, \phi_1 \) and \( \theta_1 \) are bounded (from 0 to \(2\pi\)), while \( \theta_0 \) is unbounded.

It is also useful to introduce:

\[
\lambda_0 \equiv \cosh \theta_0 = 2\mu_0\bar{\mu}_0 + 1, \quad 1 \leq \lambda_0 < \infty
\]

\[
\lambda_1 \equiv \cos \theta_1 = 2\mu_1\bar{\mu}_1 + 1, \quad -1 \leq \lambda_1 \leq 1.
\] (31)

With the change of variables stated above, the invariant measure on the coset manifold is found to be:

\[
D\mu(T_0) = \frac{1}{(\lambda_0 - \lambda_1)^2} d\lambda_0 d\lambda_1 d\phi_0 d\phi_1 d\eta d\eta^* d\rho d\rho^*. \] (32)

It is obvious from the previous equations that the symmetries of the super-matrix \( T_d \) are \( SU(1,1)/U(1) \) for the boson-boson part and \( SU(2)/U(1) \) for the fermion-fermion part. Some parameters of the original \( SU(1,1) \times SU(2) \) of \( T \) have been absorbed in the degrees of freedom other than those of \( T_d \), nevertheless the non-compact degree of freedom can be identified as \( \lambda_0 \).

Finally, due to the symmetries of its factors, \( \delta P \) is decomposed to:

\[
\delta P = \begin{pmatrix}
\delta P_1 & 0 \\
0 & \delta P_2
\end{pmatrix},
\] (33)

where:

\[
\delta P_1 = \exp \begin{pmatrix}
0 & \zeta_1^* \\
-\zeta_1 & 0
\end{pmatrix} \begin{pmatrix}
a_1 & 0 \\
0 & ib_1
\end{pmatrix} \exp \begin{pmatrix}
0 & -\zeta_1^* \\
\zeta_1 & 0
\end{pmatrix}
\]

\[
\delta P_2 = \exp \begin{pmatrix}
0 & i\zeta_2^* \\
-i\zeta_2 & 0
\end{pmatrix} \begin{pmatrix}
a_2 & 0 \\
0 & ib_2
\end{pmatrix} \exp \begin{pmatrix}
0 & -i\zeta_2^* \\
i\zeta_2 & 0
\end{pmatrix}
\] (34)
\( \zeta_1 \) and \( \zeta_2 \) being Grassman variables and \( a_1, a_2, b_1 \) and \( b_2 \) ranging from \(-\infty\) to \( \infty \). The measure on \( \delta P \) is easily computed to be:

\[
D\ P = -\frac{1}{(a_1 - ib_1)^2(a_2 - ib_2)^2}da_1da_2db_1db_2d\zeta_1d\zeta^*_1d\zeta_2d\zeta^*_2.
\]  

(35)

Using the above stated parametrization for \( \delta P \) and the definition of \( I(P) \), (26) it follows that:

\[
I(P) = \left[ \frac{(\sigma_{01} - \sigma_{02} + N^{-1/2}(a_1 - a_2))(\sigma_{01} - \sigma_{02} + N^{-1/2}i(b_1 - b_2))}{(\sigma_{01} - \sigma_{02} + N^{-1/2}(a_1 - ib_2))(\sigma_{01} - \sigma_{02} + N^{-1/2}(ib_1 - a_2))} \right]^2,
\]

(36)

where the \( \sigma_{0p} \) are the saddle points determined in the next section. Combining (25), (32) and (36), the final expression for the measure on the \( \sigma \)-manifold is given by:

\[
D\sigma = -I(P)\frac{d\lambda_0d\lambda_1d\phi_0d\phi_1d\eta^*d\eta d\rho d\rho^* da_1da_2db_1db_2d\zeta_1d\zeta^*_1d\zeta_2d\zeta^*_2}{(\lambda_0 - \lambda_1)^2(a_1 - ib_1)^2(a_2 - ib_2)^2},
\]

(37)

with \( I(P) \) given above (36).

Therefore, an exact integral expression for the ensemble averaged partition function (20) has been found in this section. Also determined is the parametrization of the super-matrix \( \sigma \) in terms of the independent integration variables (24), (28), (29), (31), (33) and (34), and the expression for the measure in terms of the integration variables (37). The connected part of the ensemble averaged density-density correlation function, the quantity of the main interest in this article, is determined from the derivatives of the averaged partition function, as follows from (7) and (13).

### 3 Supersymmetry Formalism Beyond the \( \sigma \)-model

In this section, the calculation of the averaged partition function is continued using the saddle point method. Contrary to the usual supersymmetry method, where \( \omega \) is taken to be small (of the order of \( N^{-1} \)) and only the highest order terms in \( N \) are retained, both the leading and some sub-leading order terms are kept in this article. In other words, in addition to the contribution coming from the saddle point as in the
standard supersymmetry method, the small quadratic fluctuations, massive modes, are also considered here.

The saddle point of the exponent in (20) can be obtained as a solution of the following equation:

\[
\frac{\delta}{\delta \sigma} \left[ \frac{1}{2} \text{trg} \sigma^2 + \frac{\omega}{2\lambda} \text{trg} \sigma \lambda + \frac{i}{\lambda} \text{trg} (\epsilon_1 I_1 + \epsilon_2 I_2) \sigma + \text{trg} \ln(E - \lambda \sigma) \right] = 0.
\]

(38)
The solution of the previous equation is not unique. If the saddle point is required to be diagonal, the diagonal elements \(\sigma_{0p}\) have to satisfy the following equation:

\[
\sigma_{0p} - \frac{\lambda}{E - \lambda \sigma_{0p}} + (-)^{p-1} \frac{\omega}{2\lambda} + (-)^{p-1} \frac{i\epsilon_p}{\lambda} = 0.
\]

(39)
The solution, for both values of \(p\), is given by:

\[
\sigma_{0p} = E' + (-)^p \omega' - 2i\epsilon'_p \pm i \sqrt{1 - \left[ E' - (-)^p \omega' + 2i\epsilon'_p \right]^2},
\]

(40)
where \(E, \omega\) and \(\epsilon_p\) has been rescaled:

\[
E' = \frac{E}{2\lambda}, \quad \omega' = \frac{\omega}{4\lambda} \quad \text{and} \quad \epsilon'_p = \frac{\epsilon_p}{2\lambda}.
\]

(41)

\(\sigma_{0p}, (p = 1, 2)\) can be regarded as the two factors of the correlator \(\langle G(E_1)G(E_2)\rangle\).

Furthermore, for \(\epsilon_p\) positive or negative in (40), \(G^+(E_p)\) or \(G^- (E_p)\) respectively, are obtained, as shown below. The Green’s function \(G(z)\) as defined by (4) (\(z\) complex), has a branch cut for \(z\) real and \(-2\lambda \leq z \leq 2\lambda\). That means that the points \(G(E + i\epsilon) \equiv G^+(E)\) and \(G(E - i\epsilon) \equiv G^- (E)\), for \(-2\lambda \leq E \leq 2\lambda\), belong to different Riemann surfaces, so that the sign of the small imaginary part \(i\epsilon\) determines the correct surface. On the other hand, the square in (40) also has a branch cut. Furthermore, it is again the sign of the \(\epsilon'_p\) under the square root that determines the sign of the imaginary part of the square root and consequently the Riemann surface. Therefore, the branch cut structure of the Green’s function arises from the branch cut of the (square root in the) expression for the saddle point \(\sigma_{0p}\). As in the standard \(\sigma\)-model, the convergence can only be achieved if the \(\pm\) sign in front of the square
root corresponds to $G^+(E_p)$ [23]. The convergence of the extended model, presented here, is proved in the next section. Therefore, the solutions of (39) are:

$$
\sigma_{01}^{\pm} = E' - \omega' - \epsilon' \frac{E' + \omega'}{\sqrt{1 - (E' + \omega')^2}} - i \sqrt{1 - (E' + \omega')^2} - i \epsilon'
$$

$$
\sigma_{02}^{\pm} = E' + \omega' - \epsilon' \frac{E' - \omega'}{\sqrt{1 - (E' - \omega')^2}} \pm i \sqrt{1 - (E' - \omega')^2} \pm i \epsilon'
$$

(42)

Eventually, $\epsilon$ (in the equations above) will be set to zero, but the convergence properties have to be determined before. Further, the diagonal saddle point is defined as:

$$
\sigma_0 = \text{diag}(\sigma_{01}, \sigma_{01}, \sigma_{02}, \sigma_{02}).
$$

(43)

In the standard supersymmetry method, $\omega$ is taken to be small, so that the second term in the equation for the saddle point, (38), is neglected. In that case, a matrix of the form $T_0^{-1}\sigma_0 T_0$ also satisfies the saddle point condition. This is because the first and the last terms are obviously unchanged by the transformation $\sigma_0 \mapsto T_0^{-1}\sigma_0 T_0$ and the third is small because it is proportional to $\epsilon$. Consequently, the saddle point is actually a manifold, parametrized by the degrees of freedom of matrix $T_0$.

The approach taken in this paper is that $\omega$ is not neglected. Using the parametrization introduced in the previous section: (28), (29) and (31), the transformation of the second term of the saddle point equation (38) can be calculated:

$$
\text{trg } T_0^{-1}\sigma_0 T_0 L = (\sigma_{01} - \sigma_{02})(\lambda_0 - \lambda_1) \neq \text{trg } \sigma_0 L.
$$

(44)

This lifts the degeneracy of the saddle point. If $\omega$ is small, $\omega \sim N^{-1}$, the manifold spanned by the matrix $\sigma$ is almost flat in the direction of the matrix $T_0$ compared to the directions of the other degrees of freedom (matrix $\delta P$). Consequently, the manifold $T_0^{-1}\sigma_0 T_0$ can still be considered to be a saddle point manifold. Therefore it makes sense to integrate over it (over the almost flat direction of the matrix $T_0$), provided that the following constraint is observed. Since $\lambda_0$, one of the degrees of freedom of the matrix $T_0$, is unbounded (28)-(31), its upper bound should be small.
compared to \( N \) in order that the second, degeneracy lifting, term of the saddle point equation (B8) stays small compared to the other terms. The upper bound of \( \lambda_0 \) is still taken to be infinity, provided that in the limiting procedure \( N \to \infty \) first and only then the limit of the upper bound of \( \lambda_0 \) is taken.

In the large \( \omega \) regime, the degeneracy of the saddle point manifold is lifted fully, so that the degrees of freedom of both \( T_0 \) and \( \delta P \) can be considered as fluctuations around the saddle point.

The decomposition of the matrix \( \sigma \) given in (24) should now be much more transparent: it is simply the expansion around a saddle point. \( N^{-1/2} T_0^{-1} \delta P T_0 \) are the fluctuations around the saddle point and \( T_0 \) represents the saddle point manifold or the fluctuations, depending on the regime. The factor \( N^{-1/2} \) is introduced because it is customary in the saddle point method to suppress the fluctuations by the inverse square root of the coupling constant [41]. In effect, it gives another limiting procedure – the infinite bounds of the bosonic degrees of freedom of the matrix \( \delta P \) are of the order \( N^{1/2} \).

Both \( \langle G^+(E_1)G^+(E_2) \rangle \) and \( \langle G^+(E_1)G^-(E_2) \rangle \) are calculated from \( \langle Z(\varepsilon) \rangle \); the difference lies in the sign of \( \varepsilon_2 \) and in the saddle points (42), used in the expression for \( \langle Z(\varepsilon) \rangle \). Neither \( \varepsilon_1 \) nor \( \varepsilon_2 \) play any role in the further calculations related to \( \langle G^+(E_1)G^+(E_2) \rangle \), so they can be set to zero. However, they are still needed for \( \langle G^+(E_1)G^-(E_2) \rangle \). Both facts can be expressed using \( \omega^{+'} \) with the understanding that:

\[
\omega^{+'} = \begin{cases} 
\omega' + i\epsilon' & \text{for the calculation of } \langle G^+(E_1)G^-(E_2) \rangle \\
\omega' & \text{for the calculation of } \langle G^+(E_1)G^+(E_2) \rangle
\end{cases}
\] (45)

Expanding the exponent of the averaged partition function, (20), around the saddle point (24), keeping the terms to the order of \( N^0 \) in the expansion of the logarithm, and bearing in mind the definitions for \( \sigma_0 \) (43) and \( J \) (11), it follows that:

\[
\langle Z(\varepsilon) \rangle = \int \mathcal{D}\sigma \exp\{-2N\omega^{+'} \text{trg} T_0^{-1}(\sigma_0 + N^{-1/2}\delta P)T_0 L - \frac{1}{2} \text{trg} \left[ 1 - (\sigma_0 + 2\omega^{+'} L)^2 \right] (\delta P)^2 + 2N^{1/2}\omega^{+'} \text{trg} L\delta P - \}
\]

16
The above equation makes it clear that the degrees of freedom of \( \sigma \), contained in the parametrization of \( \delta P \), correspond to the massive fluctuations around the saddle manifold. Only the quadratic fluctuations are kept in the previous equation. The others are suppressed by the factor \( N \) to negative powers. Furthermore, using the explicit value for \( \sigma_0 \) (42), the fluctuations become massless at the end of the spectrum, indicating that the procedure fails for the energies close to the end of the spectrum.

In the standard supersymmetry method, it is assumed that \( \omega' \sim N^{-1} \) and only the highest order terms in the exponent of the partition function (46) are kept. Further, all terms proportional to \( \omega \) in the expressions for the saddle point (42) are neglected. As a consequence, the massive modes decouple and are integrated out. The expression for the averaged partition function obtained by the above approximations defines the non-linear zero-dimensional \( \sigma \) model. The leading order term of the connected part of the two point correlation functions can be calculated from such partition function (25) [34]. If the corrections to that result are needed, or the same correlator have to be calculated for \( \omega' \sim 1 \), other terms have to be considered also, as in (46) and (42). It is the interaction between the modes parameterizing the matrix \( T_0 \) and the massive modes, \( \delta P \) that give rise to the non-leading order terms.

### 4 The Correlation Functions

The correlation functions that have to be calculated are obtained from the second derivative of the partition function with respect to the sources, as can be seen from (7) and (13). Differentiating (46) and setting the sources to zero, the expression whose evaluation is the main aim in this section is obtained:

\[
I = 4 \langle G(E_1) G(E_2) \rangle = \left. \frac{\partial^2}{\partial \bar{\epsilon}_1 \partial \bar{\epsilon}_2} \langle Z(\epsilon) \rangle \right|_{\bar{\epsilon} = 0} = \int \mathcal{D} \sigma K_{11} K_{22} e^{B+F}, \tag{47}
\]
where:
\[ K_{pp} = \frac{N}{\lambda} \text{trg} \left[ T_{0}^{-1}(\sigma_0 + N^{-1/2}\delta P)T_0 + 2\omega^+L \right] k, \quad (p = 1, 2), \quad k^{\alpha\alpha'} = (-)^{\alpha} \delta^{\alpha\alpha'} \]

\[ B + F = -2N\omega^+\text{trg} T_{0}^{-1}(\sigma_0 + N^{-1/2}\delta P)T_0L + 2N^{1/2}\omega^+\text{trg} L\delta P - \]
\[ \frac{1}{2} \text{trg} \left[ 1 - (\sigma_0 + 2\omega^+L)^2 \right] (\delta P)^2. \]  

The measure is given by (37), \( B \) is the part of the exponent that does not contain any Grassman variables and \( F \) is the Grassman dependent part of the exponent.

The calculation of the above expression begins with the evaluation of the exponent:

\[ B = -2N\omega^+(\sigma_{01} - \sigma_{02})(\lambda_0 - \lambda_1) - \]
\[ 2N^{1/2}\omega^+ \left[ (\lambda_0 - 1)(a_1 - a_2) - i(\lambda_1 - 1)(b_1 - b_2) \right] - \]
\[ \frac{1}{2} \left[ 1 - (\sigma_{01} + 2\omega^+)^2 \right] (a_1^2 + b_1^2) - \frac{1}{2} \left[ 1 - (\sigma_{02} - 2\omega^+)^2 \right] (a_2^2 + b_2^2) \]

\[ F = 2N^{1/2}\omega^+(\lambda_0 - \lambda_1) \left[ (a_1 - ib_1)(\eta^* - \zeta_1^*)(\eta - \zeta_1) + (a_2 - ib_2)(\rho^* - \zeta_2^*)(\rho - \zeta_2) \right]. \]

It is understood that \( \sigma_{0p} \) take value: \( \sigma_{0p}^{\pm} \) depending on which one of the two two-point correlators, \( \langle G^+(E_1)G^+(E_2) \rangle \), is calculated.

Next, the Grassmanian part of the exponent is expanded, it is multiplied by the rest of the integrand of (17) and the product expanded in terms of the Grassman variables. The integration over the Grassman variables is done using Wegner’s theorem [40]. The special case of the theorem needed here is presented in appendix B.

The important steps of the integration over the Grassman variables are summarized in appendix C, leading to the conclusion that among all terms in the expansion of the integrand of (17) in terms of the powers of Grassman variables, only two contribute to the connected correlator, \( \langle G(E_1)G(E_2) \rangle^c \): the term with factor \( \eta^*\eta^*\rho \), denoted by: \( \langle G(E_1)G(E_2) \rangle^c_{(\xi=0)} \); and the term with factor \( \eta^*\eta^*\rho^*_1\zeta_1^*\zeta_2^* \), denoted by: \( \langle G(E_1)G(E_2) \rangle^c_{(\max)} \):

\[ \langle G(E_1)G(E_2) \rangle^c = \langle G(E_1)G(E_2) \rangle^c_{(\xi=0)} + \langle G(E_1)G(E_2) \rangle^c_{(\max)}. \]
If the appropriate terms of (47) are extracted, the integration over the Grassman variables can be completed. Using the explicit expression for the measure, (37), and integrating over \( \phi_0 \) and \( \phi_1 \), the following expression for the connected correlator is obtained:

\[
N^{-2} \langle G(E_1)G(E_2) \rangle^c_{\zeta=0} = \frac{1}{4\lambda^2} \int_1^\infty d\lambda_0 \int_{-1}^1 d\lambda_1 \left( \sigma_{01} - \sigma_{02} \right)^2 e^{-2N\omega'(\lambda_0 - \lambda_1)(\sigma_{01} - \sigma_{02})} \tag{53}
\]

\[
N^{-2} \langle G(E_1)G(E_2) \rangle^c_{\text{max}} = \frac{-1}{4\lambda^2(2\pi)^2} \int_1^\infty d\lambda_0 \int_{-1}^1 d\lambda_1 \int_{-\infty}^\infty \frac{da_1da_2db_1db_2I(P)I_{\text{max}}e^B}{(\lambda_0 - \lambda_1)^2(a_1 - ib_1)^2(a_2 - ib_2)^2} \tag{54}
\]

where \( I(P) \) and \( B \) are given in (26) and (30), respectively, and:

\[
I_{\text{max}} = (a_1 - ib_1)(a_2 - ib_2)(\lambda_0 - \lambda_1)^2 \left\{ 2N^{-1} - 4\omega^{+'}(a_1 + a_2 - ib_1 - ib_2)^2 + 8\omega^{+'} \lambda_1 \left[ \sigma_{01} - \sigma_{02} + iN^{-1/2}(b_1 - b_2) \right] + 4N\omega^{+'} \lambda_1^2 \left[ \sigma_{01} - \sigma_{02} + iN^{-1/2}(b_1 - b_2) \right]^2 - 8N\omega^{+'} \lambda_0 \lambda_1 \left[ \sigma_{01} - \sigma_{02} + N^{-1/2}(a_1 - a_2) \right] \right\} - (55)
\]

The final step is the integration of (53) and (54) over the ordinary variables. The crucial problem at this point is the convergence of the integrals. Clearly, from (50), the integrals over \( a_1, a_2, b_1 \) and \( b_2 \) are Gaussian, therefore convergent. Furthermore, since \( \lambda_0 - \lambda_1 \geq 0 \) over the entire integration range, the integrals in (53) and (54) are convergent as long as the real part of \(-2N\omega'(\sigma_{01} - \sigma_{02})(\lambda_0 - \lambda_1)\) is negative. The other terms in the exponent (50) are small, since they contain the massive fluctuations: \( a_1, a_2, b_1, b_2 < N^{1/2} \). Only in that case does the perturbative expansion around the saddle point make sense. The convergence properties, as well as the final calculations, are given separately for the two regimes in the next two sub-sections.

### 4.1 Large \( \omega \) regime

In the large \( \omega \) regime (\( \omega' \) of the order of, but smaller than 1) one looks for the smoothed correlators – all terms oscillating with frequencies of the order \( N \) are neglected. (More
precisely, the correlator is averaged over the scale few times bigger than $N^{-1}$.) The idea is that the oscillations are due to the discrete energy spectrum when $N$ is finite (isolated singularities of the Green’s function), which should disappear when $N \to \infty$ (the singularities merge to form a branch cut). The integrations over $\lambda_0$ and $\lambda_1$ in each of the expressions for the two parts of the connected correlator (53) and (54) give four terms. The only non-oscillating term comes from the boundary values $\lambda_0 = 1 = \lambda_1$. Nevertheless, even the neglected oscillating terms have to be finite in order for the procedure to be valid. In order to motivate the forthcoming discussion, one can easily see that the connected two point correlator (52), remains the same even if some of the massive modes, $a_1, a_2, b_1$ or $b_2$, are scaled by an arbitrary complex constant $c$, provided that the real part of the quadratic term in the exponent remains negative. It may be surprising, although it is easy to check, that, considering the same equations and bearing in mind the relation between $\mu_p\bar{\mu}_p$ and $\lambda_p$ (31), if both $\mu_0\bar{\mu}_0$ and $\mu_1\bar{\mu}_1$ are scaled by $c$, or equivalently $\lambda_p \mapsto c\lambda_p + 1 - c$, the term that survives smoothing, after the integrations over $\lambda_0$ and $\lambda_1$ are done, does not explicitly depend on $c$. Therefore, $c$ can be chosen such that $\text{Re } c \omega^+(\sigma^+_0 \mp \sigma^+_1 \mp) > 0$ is positive infinitesimal so that all terms are convergent, while the non-vanishing term is unaffected by the scaling. In effect, what has been done is to rotate the contour of the integration of $\lambda_0$ and $\lambda_1$ in the complex plane, while keeping the branch-cut structure of the Green’s function intact. The rotation is justified because the integrands of (53) and (54) are polynomials in $\lambda_0$ and $\lambda_1$, so the only poles are at infinity. Furthermore, from the discussion given in section 3, the upper limit of $\lambda_0$ has to be understood as large but finite. It is set to infinity only after the limit $N \to \infty$ is taken.

The above procedure may resemble the one used in the orthogonal polynomials method, in which the same connected correlator was calculated at infinity in the complex plane and then analytically continued to the real axis, making sure that the appropriate branch-cut is obtained \cite{15}. Since the correlator in that case is calculated at infinity, the oscillatory parts do not exist: the function is smoothed by construction.
Similarly, the scaling by an arbitrary constant $c$ given in this article, introduces small negative parts in the exponentials of the oscillating terms, setting them to zero.

The following are the final expressions for the smoothed connected correlator $\langle G^+(E_1)G^\pm(E_2) \rangle$ in the large $\omega$ regime, obtained by the straightforward integration of (53) and (54).

\[
N^{-2}\langle G^+(E_1)G^\pm(E_2) \rangle^c = \frac{1}{4\lambda^2} \frac{1}{(2N\omega')^2}
\] (56)

\[
N^{-2}\langle G^+(E_1)G^\pm(E_2) \rangle^c_{\text{max}} = \frac{1}{\lambda^2} \text{Re} \left\{ \frac{1}{N^2(\sigma_{01}^+ - \sigma_{02}^+)^2} \left[ \frac{1}{1 - (\sigma_{01}^+ + 2\omega')^2} - \frac{1}{1 - (\sigma_{02}^+ - 2\omega')^2} \right] \right. \\
\left. + \frac{5}{N^4} \frac{[1 - (\sigma_{01} + 2\omega')^2]^2 - 6[1 - (\sigma_{01} + 2\omega')^2][1 - (\sigma_{02} - 2\omega')^2] + [1 - (\sigma_{02} - 2\omega')^2]^2}{(\sigma_{01} - \sigma_{02})^6 [1 - (\sigma_{01} + 2\omega')^2]^3 [1 - (\sigma_{02} - 2\omega')^2]^3} \right\}
\] (57)

In order to derive the above relations, the denominator of $I(P)$, (26), has been expanded in terms of the power series. The validity of the expansion is based on the fact that only the small quadratic fluctuations are considered, that is, the massive modes are smaller than $N^{1/2}$ over the entire integration range.

Using (7) and (42), the smoothed ensemble-averaged energy density-density correlation function, in the large $N$ limit and for: $E_1 - E_2 \sim N^0$, provided the energies are not close to the endpoints of the spectrum, is found to be:

\[
C(E_1, E_2) = \frac{-1}{2\pi^2 N^2} \frac{4\lambda^2 - E_1 E_2}{(E_1 - E_2)^2} \sqrt{4\lambda^2 - E_1^2} \sqrt{4\lambda^2 - E_2^2} + \frac{40\lambda^6}{N^4 \omega^6} \frac{U^2(SU - T) - (\frac{\omega}{4\lambda})^2(3SU - T)}{\left(1 - (E_1/2\lambda)^2\right)^2 \left(1 - (E_2/2\lambda)^2\right)^3},
\] (58)

where:

\[
S = -4(4\lambda^2 - E_1^2)(4\lambda^2 - 2E_1^2) - 4(4\lambda^2 - E_2^2)(4\lambda^2 - 2E_2^2) + 244(4\lambda^2 - E_1^2)(4\lambda^2 - E_2^2) \] (59)

\[
T = 24(2\lambda)^2(4\lambda - E_1^2)(4\lambda - E_2^2) E_1 E_2 \] (60)

\[
U = 4\lambda^2 - E_1 E_2. \] (61)
It should be noted that the usual supersymmetry method (zero-dimensional $\sigma$ model) does not give a result in this regime. The leading order of the above result is identical to the result obtained by Brézin and Zee [14] giving another confirmation of the correctness of the above stated convergence prescription. As for the sub-leading term, it is questionable if any of the neglected parts, such as higher than quadratic fluctuations, may give contribution of the same order. Nevertheless, it is correct within the current approximation, namely small quadratic fluctuations around the saddle point.

4.2 Small $\omega$ regime

Let us consider the convergence of $\langle G^+(E_1)G^-(E_2) \rangle$ first. Again, the convergence is achieved if the real part of $2N\omega' \sigma_{01} - \sigma_{02} (\lambda_0 - \lambda_1)$ is positive. From (42), for small positive $\epsilon$ and for small but finite $\omega'$ (of the order $N^{-1}$), it follows that:

$$\text{Re}\omega' \sigma_{01}^\pm - \sigma_{02}^\pm = -2\omega'^2 \left(1 + \frac{\epsilon}{(1 - \epsilon^2)^{3/2}}\right) + 2\epsilon(1 - E^2)^{1/2}. \quad (62)$$

Therefore the convergence can be achieved by taking the $N \to \infty$ limit first, so that $\omega \to 0$ while $\epsilon$ is still positive, making the above expression positive. Only then can $\epsilon$ be set to zero giving the required correlator. Using the procedure just stated, the terms coming from the upper bound of the integration over $\lambda_0$ in (53) and (54) vanish after the limit $N \to \infty$ is taken. The integration over $\lambda_1$ gives the non-exponential terms (from $\lambda_1 = 1$) and the purely oscillatory terms ($\lambda_1 = 1$), rendering the integration convergent. Note that in the usual supersymmetry method ($\sigma$-model) the terms proportional to $\omega$ in the expressions for $\sigma_0$, (42), are neglected and consequently the integrals are immediately convergent. The above analysis was necessary for the calculations beyond the $\sigma$-model, but could also be thought of as a clarification of the usual procedure.

The integrations needed in (53) and (54) are now straightforward, with the fol-
lowing result:

\[ N^{-2}\langle G^+(E_1)G^-(E_2) \rangle_{\xi=0}^c = \frac{1}{4\lambda^2} \frac{1}{(2N\omega')^2} \left[ 1 - e^{8iN\omega'\sqrt{1-E'}} \right] \]  

(63)

\[ N^{-2}\langle G^+(E_1)G^-(E_2) \rangle_{\xi=0}^c \]

\[ \frac{-1}{16\lambda^2} \frac{1}{(2N\omega')^2} \left[ \frac{1}{1 - (\sigma_{01}^+ - 2\omega')^2} \right] \left[ \frac{1}{1 - (\sigma_{02}^+ - 2\omega')^2} \right] e^{8iN\omega'\sqrt{1-E'}} \]

(64)

\[ \frac{1}{1 - (\sigma_{01}^+ - 2\omega')^2} \left[ \frac{1}{1 - (\sigma_{02}^+ - 2\omega')^2} \right] e^{8iN\omega'\sqrt{1-E'}} \]

(65)

In order to derive the above relations, the denominator of \( I(P) \), (26), as well as the factor \( \exp\{-4iN^{1/2}\omega'(b_1 - b_2)\} \) appearing after the \( \lambda_1 \) integration in (54), have been expanded in terms of the power series. The validity of the expansions is based on the fact that only the small quadratic fluctuations are considered, that is, the massive modes are smaller than \( N^{1/2} \) over the entire integration range.

Using the expressions for the saddle points (42) the final expression is obtained:

\[ N^{-2}\langle G^+(E_1)G^-(E_2) \rangle_{\xi=0}^c = \frac{-1}{16\lambda^2} \frac{1}{N^2} \left[ \frac{1}{1 - E'} - \left[ \frac{1}{1 - E'} + 16\omega'^2 N^2 \right] e^{8iN\omega'\sqrt{1-E'}} \right] \]

(65)

The equations (63) and (65) together constitute the final result for the connected two point correlation function \( \langle G^+(E_1)G^-(E_2) \rangle \) in the small \( \omega \) regime. While the former is a well known result obtained using the standard supersymmetry method, among other methods, the latter is the leading order correction to the result.

5 Conclusion

One of the methods used in RMT is the supersymmetry method. The usual procedure includes some approximations and leads to the zero dimensional \( \sigma \)-model. In this
article, the Gaussian unitary ensemble is considered and the supersymmetry method is extended to keep some of the previously neglected terms. The partition function for the extended method is given in (46). It contains the $\sigma$-model terms, as well as the lower order terms (quadratic fluctuations around it).

In the large energy difference regime, the extended supersymmetry method is used to calculate the connected energy density-density correlation function (7). The usual degeneracy of the saddle point is lifted fully. Using a symmetry of the model (invariance under rescaling of the variables $\mu_p\bar{\mu}_p$) the potentially divergent integrals can be made finite. As a consequence the smoothed connected energy density-density correlation function is calculated, (58). Both the leading order term, which agrees with the known result [14] obtained by the method of the orthogonal polynomials, and the highest order corrections in the $N^{-1}$ expansion are contained in the result. Furthermore, both contributions can be seen to come from terms that are neglected in the usual supersymmetry approach.

As for the small energy difference regime, the degeneracy of the saddle point is only slight. The contributions from the (non-degenerate) saddle point, as well as from the quadratic fluctuations are taken into account. Convergence can be achieved provided that the right limiting procedure is used. As the result, the leading correction to the well known result for $\langle G^+(E_1)G^-(E_2)\rangle^c$ is calculated (63).

A natural extension of the work presented here would be to consider other ensembles (orthogonal, symplectic, . . .). Also, the application of the modified supersymmetry method to non-Gaussian ensembles seems worth investigating.

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A Grassman algebra: definitions and conventions

Grassman variables, $\eta_1$ and $\eta_2$ are anti-commuting:

$$\eta_1 \eta_2 = - \eta_2 \eta_1. \quad (66)$$

The complex-conjugate of a Grassman variable is defined as:

$$(\eta_1 \eta_2)^* = \eta_1^* \eta_2^*, \quad \eta^{**} = - \eta. \quad (67)$$

A general super-vector $\varphi$ is defined as:

$$\varphi = (z_1, \eta_1, z_2, \eta_2, \ldots) \quad (68)$$

where $z_p$ and $\eta_p$ are vectors with ordinary and Grassman elements respectively. The supervector $\varphi$ used in this article (14) and below) has the form:

$$\varphi = (\varphi_0^0, \varphi_1^1, \varphi_0^0, \varphi_2^1), \quad (69)$$

where $\varphi_\alpha^\mu_p$ is a $N$-dimensional vector consisting of ordinary ($\alpha = 0$), or Grassman ($\alpha = 1$) elements $\varphi_\alpha^\mu_p(\mu)$ with $\mu = (1, \ldots, N)$. Also, $\bar{\varphi}$ is defined as $\bar{\varphi} = \varphi^\dagger s$ where $s = (1_N, 1_N, -1_N, 1_N)$, or explicitly:

$$\bar{\varphi} = (\varphi_1^0, \varphi_1^1, -\varphi_0^0, \varphi_2^1). \quad (70)$$

If $a$ and $b$ are ordinary (Bosonic) and $\alpha$ and $\beta$ are Grassman (Fermionic) variables or matrices, the super-transpose and Hermitian adjoint of a super-matrix $A$ are:

$$A = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \quad A^T = \begin{pmatrix} a^T & \beta^T \\ -\alpha^T & b^T \end{pmatrix} \quad \text{and} \quad A^\dagger = A^{*T}. \quad (71)$$

$a, b, \alpha$ and $\beta$ are called boson-boson, fermion-fermion, boson-fermion and fermion-boson blocks. The super-matrix $A$ was given in what it is called a boson-fermion notation. A super-matrix in a general form can be obtained from a super-matrix in the boson-fermion block notation by interchanging some of its rows and columns, so that
it appears to consist of blocks, each block having the boson-fermion block form. The definitions for the super-transpose and Hermitian adjoint are to be applied by putting the matrix in boson-fermion notation first and then using the above definitions.

Elements, or blocks, of both super-vectors and super-matrices used in this article have their superscripts denoting grading (0 for ordinary and 1 for Grassman variables), while the subscripts denote blocks, each having the boson-fermion block form.

B Grassman integration and Wegner’s theorem

In this appendix, the integration over the Grassman variables is defined, the Gaussian super-integral is given and a special case of Wegner’s integral theorem, needed in this article, is presented.

Grassmanian integration is defined as:

\[ \int d\eta = 0, \quad \int d\eta \eta = (2\pi)^{-1/2} \]

(72)

It is easy to extend Gaussian integration to include both ordinary and Grassman variables:

\[ \int_{-\infty}^{\infty} \prod_{k=1}^{N} idz_k dz_k^* \int \prod_{k=1}^{N} d\eta_k d\eta_k^* e^{-\phi^\dagger A \phi} = \detg^{-1} A, \]

(73)

where \( \phi = (z, \eta) \) is a 2\(N\)-dimensional supervector, \( A \) is a super-matrix in the boson-fermion notation and \( \detg \) is the graded (super) determinant [38]. The above integral is commonly referred to as the Gaussian superintegral.

A property of the Grassman integration, called Wegner’s integral theorem, deserves clarification. Although it is not often explained, it is important for the calculations in this article. The motivation for the theorem comes from considering the following simple Gaussian superintegral:

\[ I \equiv \int dp dq d\eta d\eta^* e^{-\frac{1}{2}(p^2 + q^2) + m^*} = 1 \]

(74)

On the other hand, performing the following change of variables:

\[ p = u + \zeta^* \zeta(u - iv), \quad \eta = \zeta^* (u - iv), \]
\[ iq = iv - \zeta^* \zeta (u - iv), \quad \eta^* = \zeta (u - iv) \] (75)

and including the Jacobian factor: \((u - iv)^{-2}\), the integral becomes:

\[ I = \int dudvd\zeta d\zeta^* (u - iv)^{-2} e^{-\frac{1}{2}(u^2 + v^2)} . \] (76)

Although there is no Grassman dependence in the integrand, the integral is not zero because the integration over the ordinary variables is singular. The correct result was already given in (74). In many cases, when an integral of the form of (76) is encountered, it is neither easy, nor convenient to transform it to the form of (74). The theorem gives the prescription on how to evaluate such an integral. Instead of the full theorem, only a special case of the theorem, relevant for the calculations presented in this article, is now presented. A more precise formulation is given in [40].

Consider an integral of the form:

\[ I = \int D\mu(Q) f(Q), \] (77)

where \( Q \) is a super-matrix that can be parametrized as a coset space: \( Q = U^{-1} \Lambda U \). \( U \) contains Grassman and \( \Lambda \) contains ordinary variables, while \( D\mu(Q) \) has a singularity at the origin of the coset space. \( f \) can be expanded as \( f(Q) = f_0 + f_{\text{max}} + f_{\text{other}} \), where \( f_0 \) and \( f_{\text{max}} \) are the terms in the expansion of \( f(Q) \) in the powers of Grassman variables that contain no Grassman variables and the product of all other terms in the expansion. If \( f_{\text{other}} \) goes to zero at the origin, the integral \( I \) can be evaluated to give:

\[ I = \int D\mu(Q) f_{\text{max}} + f_0 \bigg|_{\Lambda = \Lambda_0} , \] (78)

where \( \Lambda_0 \) is the origin of the coset space.

C Integration Over Grassman Variables

The integration over Grassman variables in the integral (47) is outlined in this appendix. As a result, the terms in the Grassman expansion of the integrand relevant
for the connected correlators, (7), are identified.

The integration over the Grassman variables of the matrix $T_0 (\eta, \eta^*, \rho$ and $\rho^*)$ can be done by expanding the integrand in terms of those variables using the obvious notation:

$$I = \int \mathcal{D}\sigma (f_0 + f_{\text{max}} \eta^* \eta \rho + f_{\text{other}}),$$

(79)

$f_{\text{other}}$ vanishes at the origin of the $T_0$ coset $\lambda_0 = 1 = \lambda_1$ (the $\phi_{0,1}$ are not important because they decouple). The Wegner’s theorem applies, giving:

$$I = \int \mathcal{D}P I(P) \bigg|_{T_0=1} + \int \mathcal{D}\sigma f_{\text{max}}.$$

(80)

The first term of the above equation can be easily computed using (47), (48) and (49):

$$\int \mathcal{D}P I(P) \bigg|_{T_0=1} = 4 N^2 \lambda^2 (\sigma_{01} + 2 \omega^{+'})(\sigma_{02} - 2 \omega^{+'}).$$

(81)

On the other hand, $\langle G(E_p) \rangle$ can be calculated starting from (12) and using (46), (48) and (49). Wegner’s theorem is used again to show that the only contribution comes from the term that has no Grassman variables and gives:

$$\langle G(E_p) \rangle = - \frac{N}{\lambda} (\sigma_{0p} - (-)^p 2 \omega^{+'}).$$

(82)

Therefore, the first term in (80) (divided by 4) is exactly the disconnected part of the correlator $\langle G(E_1) \rangle \langle G(E_2) \rangle$. Therefore the connected part (3), is simply:

$$\langle G(E_1)G(E_2) \rangle^c = \frac{1}{4} \int \mathcal{D}\sigma f_{\text{max}} \equiv \frac{1}{4} I^c.$$

(83)

Next, $I^c$ is integrated over the Grassman variables contained in $\delta P$: $\zeta_1$, $\zeta_1^*$, $\zeta_2$, and $\zeta_2^*$. The procedure is the same as above, only this time it has to be done separately for $\delta P_1$ and $\delta P_2$ because they are independent and both of them have a coset structure. The origins of the respective coset spaces are given by: $a_1 = 0 = b_1$ and $a_2 = 0 = b_2$, so that $I^c$ can be expanded as:

$$I^c = \int \mathcal{D}\sigma \left[ g_{p0} + g_{p_{\text{max}}} \zeta_p^* \zeta_p + g_{p_{\text{other}}} \right],$$

(84)
in obvious notation, for both $p = 1$ and $p = 2$. Both $g_{p other}$ vanish at the respective origins, so Wegner’s theorem applies. Consequently, the connected correlation function receives only two contributions: one from the term with factor $\eta^* \eta \rho^* \rho$ evaluated at the origin of $\delta P$ coset (denoted by subscript ($\zeta = 0$) ) and the other from the term with all Grassman variables (denoted by subscript (max)). Therefore the above statement can be written as:

$$\langle G(E_1)G(E_2) \rangle^c = \langle G(E_1)G(E_2) \rangle^c_{(\zeta=0)} + \langle G(E_1)G(E_2) \rangle^c_{(max)}.$$  \hspace{1cm} (85)

The above equation specifies how to perform the integration over the Grassman variables needed in section 4.
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