Optimal Petrov–Galerkin Spectral Approximation Method for the Fractional Diffusion, Advection, Reaction Equation on a Bounded Interval

Xiangcheng Zheng\(^1\) · V. J. Ervin\(^2\) · Hong Wang\(^1\)

Received: 31 March 2020 / Revised: 29 October 2020 / Accepted: 11 November 2020 / Published online: 16 January 2021

© Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract

In this paper we investigate the numerical approximation of the fractional diffusion, advection, reaction equation on a bounded interval. Recently the explicit form of the solution to this equation was obtained. Using the explicit form of the boundary behavior of the solution and Jacobi polynomials, a Petrov–Galerkin approximation scheme is proposed and analyzed. Numerical experiments are presented which support the theoretical results, and demonstrate the accuracy and optimal convergence of the approximation method.

Keywords

Fractional diffusion equation · Petrov–Galerkin · Jacobi polynomials · Spectral method · Weighted Sobolev spaces

Mathematics Subject Classification

65N30 · 35B65 · 41A10 · 33C45

1 Introduction

Of interest in this paper is the approximation of the solution to the fractional diffusion, advection, reaction equation

\[
\mathcal{L}_\alpha^r u(x) + b(x)Du(x) + c(x)u(x) = f(x), \quad x \in I, \quad (1.1)
\]

subject to

\[
u(0) = u(1) = 0, \quad (1.2)
\]

where

\[
\mathcal{L}_\alpha^r u(x) := -D(rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)x})Du(x), \quad (1.3)
\]
and I := (0, 1), 1 < α < 2, 0 ≤ r ≤ 1, c(x) − 1/2Db(x) ≥ 0, D denotes the usual derivative operator, \( D^\alpha \) the \( \alpha \)-order left fractional derivative operator, and \( D^{\alpha r} \) the \( \alpha \)-order right fractional derivative operator, defined by:

\[
D^\alpha u(x) := D_0D_x^{-(2-\alpha)} Du(x) = D \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{1}{(x-s)^{\alpha-1}} Du(s) ds, \tag{1.4}
\]

\[
D^{\alpha r} u(x) := D_1D_x^{-(2-\alpha)} Du(x) = D \frac{1}{\Gamma(2-\alpha)} \int_x^1 \frac{1}{(s-x)^{\alpha-1}} Du(s) ds. \tag{1.5}
\]

In recent years fractional differential equations have received increased attention as they have been used in modeling a number of physical phenomena such as contaminant transport in ground water flow [4], viscoelasticity [29], image processing [7,15], turbulent flow [29,35], and chaotic dynamics [41].

There are two important properties that distinguish a fractional order differential equation from its integer order counterpart. Firstly, as can be noted from (1.3), fractional differential equations are nonlocal in nature. Secondly, the solution of fractional differential equations (typically) have a lack of regularity at the boundary of the domain. Finite difference methods [10,27,34,37,38], finite element methods [14,23,28,39], discontinuous Galerkin methods [40], and mixed methods [8,26], have all been developed for fractional differential equations. These methods typically exhibit slow convergence due to the lack of regularity of the solution at the boundary. In [22,24] an enriched subspace was given for one sided fractional differential equations, where the boundary behavior of the solution was included in the finite element trial space. Mao and Shen in [33] extended the work of Gui and Babuška in [17] to establish that, for an assumed boundary behavior of the solution, a geometrically spaced mesh with increasing polynomial degree trial function on the subintervals resulted in an exponential rate of convergence for the approximation. For a special class of self-adjoint fractional differential equations a spectral approximation scheme was presented in [42] using a special class of functions, polyfractonomials. Spectral methods, exploiting a special property satisfied by fractional diffusion operator applied to Jacobi polynomials (see (2.16)) has been particularly effective for the approximation of the solution to fractional diffusion equations [9,13,25,30–32,43,44].

Three recent papers have established the explicit form of the solution to fractional diffusion, advection, reaction equation on a bounded domain in \( \mathbb{R}^1 \). In [20], Hao and Zhang studied the case for \( r = 1/2 \), for which \( \mathcal{L}_r^\alpha \) is a symmetric operator. The general fractional diffusion reaction, equation was investigated by Hao et al. in [19]. (As commented by the authors in their summary, the regularity results obtained in [19] are not optimal.) The work in these papers was extended in [12] to the general fractional diffusion, advection, reaction equation. The solution was shown to have the form \( u(x) = (1-x)^{r-\beta} x^\beta \phi(x) \), where \( \phi \) is contained in the weighted Sobolev space \( H^{\alpha+r\beta} (\alpha-\beta, \beta) (1) \) (defined in Sect. 2), where \( \beta \) and \( \bar{s} \) are explicit functions of \( \alpha, r \), and the regularity of the right hand side function, \( f \) (see Theorems 2.2 and 2.3 below). Of particular note is that for the fractional diffusion, reaction problem, and the fractional diffusion, advection, reaction problem, the regularity of the solution \( u \) is bounded, regardless of the regularity of \( f \). This boundedness in the regularity of \( u \) is not the case for the fractional diffusion, advection, reaction equation on \( \mathbb{R} \), as was recently established by Ginting and Li in [16].

The numerical approximation scheme presented below is accurate as, using [12], the precise boundary behavior of the solution is incorporated into the approximate solution. Additionally, using the special property of the fractional diffusion operator applied to Jacobi
Polynomials (see (2.16))

\[ L_t^\alpha \omega_k(x) \hat{G}_k^{(\alpha, \beta)}(x) = \lambda_k \hat{G}_k^{(\beta, \alpha)}(x), \]

and that \( \{\hat{G}_k^{(\alpha, \beta)}\}_{k=0}^\infty \) is a basis for \( H^\alpha_{(\alpha, \beta)}(I) \), the approximation scheme using Jacobi polynomial is efficient in that if the solution is \( C^\infty(I) \) (very rarely the case) the approximation converges exponentially. If the solution has bounded regularity (typically the case) the approximation converges optimally at an algebraic rate of convergence.

This paper is organized as follows. In the following section definitions, notation, and several known results are summarized. Section 3 contains the Petrov–Galerkin weak formulation for (1.1), (1.2), and establishes the existence and uniqueness of its solution. The analysis follows the work of Jin et al. in [24], wherein the lower order terms are handled using the Petree-Tartar Lemma. The approximation scheme is given in Sect. 4, and associated error estimates derived. Numerical experiments are presented in Sect. 5.

## 2 Notation and Properties

Jacobi polynomials have an important connection with fractional order diffusion equations [2,13,30,31]. We briefly review their definition and some of their important properties [1,36].

**Usual Jacobi Polynomials**, \( P_n^{(a,b)}(t) \), on \((-1, 1)\).

**Definition:** \( P_n^{(a,b)}(t) := \sum_{m=0}^n p_{n,m} (t-1)^{(n-m)}(t+1)^m \), where

\[ p_{n,m} := \frac{1}{2^n} \binom{n+a}{m} \binom{n+b}{n-m}. \]  \hspace{1cm} (2.1)

**Orthogonality:**

\[ \int_{-1}^{1} (1-t)^a (1+t)^b P_j^{(a,b)}(t) P_k^{(a,b)}(t) dt = \begin{cases} 0, & k \neq j, \\ \|P_j^{(a,b)}\|^2, & k = j \end{cases}, \]

where \( \|P_j^{(a,b)}\| = \left( \frac{2}{(2j+a+b+1)} \frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma((j+1)\Gamma(j+a+b+1))} \right)^{1/2} \). \hspace{1cm} (2.2)

In order to transform the domain of the family of Jacobi polynomials to \([0, 1]\), let \( t \rightarrow 2x - 1 \) and introduce \( G_n^{(a,b)}(x) = P_n^{(a,b)}(t(x)) \). From (2.2),

\[ \int_{-1}^{1} (1-t)^a (1+t)^b P_j^{(a,b)}(t) P_k^{(a,b)}(t) dt \]

\[ = \int_0^1 (1-x)^a 2^b x^b P_j^{(a,b)}(2x-1) P_k^{(a,b)}(2x-1) 2 dx \]

\[ = 2^{a+b+1} \int_0^1 (1-x)^a 2^b x^b G_j^{(a,b)}(x) G_k^{(a,b)}(x) dx \]

\[ = \begin{cases} 0, & k \neq j, \\ 2^{a+b+1} \|G_j^{(a,b)}\|^2, & k = j \end{cases}. \]

where \( \|G_j^{(a,b)}\| = \left( \frac{1}{(2j+a+b+1)} \frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma((j+1)\Gamma(j+a+b+1))} \right)^{1/2} \). \hspace{1cm} (2.3)
From [30, equation (2.19)] we have that
\[
\frac{d^k}{dt^k} p_n^{(a,b)}(t) = \frac{\Gamma(n + k + a + b + 1)}{2^k \Gamma(n + a + b + 1)} p_{n-k}^{(a+k,b+k)}(t). \tag{2.4}
\]
Hence,
\[
\frac{d^k}{dx^k} G_n^{(a,b)}(x) = \frac{\Gamma(n + k + a + b + 1)}{\Gamma(n + a + b + 1)} G_{n-k}^{(a+k,b+k)}(x). \tag{2.5}
\]
Note that, from Stirling’s formula, we have that
\[
\lim_{n \to \infty} \frac{\Gamma(n + \sigma)}{\Gamma(n)n^{\sigma}} = 1, \quad \text{for } \sigma \in \mathbb{R}. \tag{2.6}
\]
For compactness of notation, let
\[
\omega^{(a,b)} = \omega^{(a,b)}(x) := (1 - x)^a x^b. \tag{2.7}
\]
We let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and use \( y_n \sim n^p \) to denote that there exists constants \( c \) and \( C > 0 \) such that, as \( n \to \infty \), \( c n^p \leq |y_n| \leq C n^p \). Additionally, we use \( a \ll b \) to denote that there exists a constant \( C \) such that \( a \leq C b \).

For \( t \in \mathbb{R} \), \([t]\) is used to denote the largest integer that is less than or equal to \( t \), and \([t]\) is used to denote the smallest integer that is greater than or equal to \( t \).

**Function space \( L^2_\sigma(I) \).**

For \( \sigma(x) > 0, \ x \in (0,1) \), let
\[
L^2_\sigma(I) := \{ f(x) : \int_0^1 \sigma(x)f(x)^2 \, dx < \infty \}. \tag{2.8}
\]
Associated with \( L^2_\sigma(0,1) \) is the inner product, \((\cdot, \cdot)_\sigma\), and norm, \(\| \cdot \|_\sigma\), defined by
\[
(f, g)_\sigma := \int_0^1 \sigma(x)f(x)g(x) \, dx, \quad \text{and} \quad \| f \|_\sigma := (\langle f, f \rangle_\sigma)^{1/2}.
\]

The set of orthogonal polynomials \( \{ G_j^{(a,b)} \}_{j=0}^{\infty} \) form an orthogonal basis for \( L^2_{\omega^{(a,b)}}(I) \), and \( \hat{G}_j^{(a,b)} := G_j^{(a,b)}/\| G_j^{(a,b)} \| \), \( \{ \hat{G}_j^{(a,b)} \}_{j=0}^{\infty} \) form an orthonormal basis for \( L^2_{\omega^{(a,b)}}(I) \).

Without a subscript, \((\cdot, \cdot)\) denotes the usual \( L^2(I) \) inner product.

**Function space \( H^s_{\omega^{(a,b)}}(I) \).**

The weighted Sobolev spaces \( H^s_{\omega^{(a,b)}}(I) \) differ from the usual \( H^s(I) \) spaces in that the associated norms apply a polynomial weight at each endpoint of \( I \), namely, \( x^b \) and \( (1-x)^a \). These weights increase with the order of the derivative. We give two equivalent definitions for the \( H^s_{\omega^{(a,b)}}(I) \) spaces. In the first definition the spaces \( H^s_{\omega^{(a,b)}}(I) \), for \( 0 < s \neq \mathbb{N} \), are defined by the \( K \)-method of interpolation. The second definition is based on the decay rate of the coefficients of a function expanded in terms of the Jacobi polynomials \( G_j^{(a,b)}(x) \). Both definitions are useful, and used in the analysis below. The equivalence of the spaces is discussed in [12].

**Definition** [Based on the \( K \)-method of interpolation.] Following Babuška and Guo [3], and Guo and Wang [18], we introduce the weighted Sobolev spaces \( H^s_{\omega^{(a,b)}}(I) \).

**Definition 2.1** Let \( s, a, b \in \mathbb{R}, \ s \geq 0, \ a, b > -1 \). Then
\[
H^s_{\omega^{(a,b)}}(I) := \left\{ v : \| v \|^2_{\cdot,\omega^{(a,b)}} := \sum_{j=0}^{s} \| D^j v \|_{\omega^{(a+j,b+j)}}^2 < \infty \right\}. \tag{2.9}
\]
Definition (2.9) is extended to \( s \in \mathbb{R}^+ \) using the \( K \)-method of interpolation. For \( s < 0 \) the spaces are defined by (weighted) \( L^2 \) duality.

**Definition:** [Based on the decay rate of Jacobi polynomial coefficients.] Next we define function spaces in terms of the decay rate of the Jacobi coefficients of their member functions. Given \( v \), let

\[
v_j = \int_0^1 \omega^{(a,b)}(x) v(x) \hat{G}^{(a,b)}_j(x) \, dx. \tag{2.10}
\]

Note that for \( v \in L^2_{\omega(a,b)}(I) \),

\[
v(x) = \sum_{j=0}^{\infty} v_j \hat{G}^{(a,b)}_j(x). \tag{2.11}
\]

**Definition 2.2** Let \( s, a, b \in \mathbb{R} \), \( a, b > -1 \), \( L^2_{(a,b)}(I) := L^2_{\omega(a,b)}(I) \), and \( v_j \) be given by (2.10). Then, define

\[
H^s_{(a,b)}(I) := \{ v : \sum_{j=0}^{\infty} (1 + j^2)^s v_j^2 < \infty \} \tag{2.12}
\]

as the \((a, b)\)-weighted Sobolev space of order \( s \).

**Theorem 2.1** [12, Theorem 4.1] The spaces \( H^s_{(a,b)}(I) \) and \( H^s_{\omega(a,b)}(I) \) coincide, and their corresponding norms are equivalent.

With the structure of the \( H^s_{(a,b)}(I) \) spaces, and properties (2.5) and (2.3), it is straightforward to show that \( D \) is a bounded mapping from \( H^s_{(a,b)}(I) \) onto \( H^{s-1}_{(a+1, b+1)}(I) \).

**Lemma 2.1** [12, Lemma 4.5] For \( s, a, b \in \mathbb{R} \), \( a, b > -1 \), the differential operator \( D \) is a bounded mapping from \( H^s_{(a,b)}(I) \) onto \( H^{s-1}_{(a+1, b+1)}(I) \).

For convenience, from hereon we use \( H^s_{(a,b)}(I) \) to represent the spaces \( H^s_{\omega(a,b)}(I) \) and \( H^s_{(a,b)}(I) \).

**Definition: Condition A** The parameters \( \alpha, \beta, \) and \( r \), and constant \( c^*_s \) satisfy: \( 1 < \alpha < 2 \), \( \alpha - 1 \leq \beta \), \( \alpha - \beta \leq 1 \), \( 0 \leq r \leq 1 \)

\[
c^*_s = \frac{\sin(\pi \alpha)\sin(\pi \beta)}{\sin(\pi (\alpha - \beta)) + \sin(\pi \beta)}, \tag{2.13}
\]

where \( \beta \) is determined by

\[
r = \frac{\sin(\pi \beta)}{\sin(\pi (\alpha - \beta)) + \sin(\pi \beta)}. \tag{2.14}
\]

For compactness of notation, for \( \alpha \) and \( r \) defined in (1.1) and \( \beta \) defined in (2.14) we introduce

\[
\omega(x) := \omega^{(a-\beta, \beta)}(x) = (1-x)^{\alpha-\beta} x^\beta, \quad \text{and} \quad \omega^*(x) := \omega^{(\beta, \alpha-\beta)}(x) = (1-x)^\beta x^{\alpha-\beta}. \tag{2.15}
\]

Additionally, we use \( \langle \cdot, \cdot \rangle_\omega \) to denote the weighted \( L^2 \) duality pairing between functions if \( H^{1-s}_{(a-\beta, \beta)}(I) \) and \( H^s_{(a-\beta, \beta)}(I) \).

From [13,21],

\[
L^a_r \omega(x) \hat{G}^{(a-\beta, \beta)}_k(x) = \lambda_k \hat{G}^{(\beta, a-\beta)}_k(x), \quad \text{where} \quad \lambda_k = -c^*_s \frac{\Gamma(k+1+a)}{\Gamma(k+1)}, \quad k = 0, 1, 2, \ldots, \tag{2.16}
\]
and \( c^+ \) given by (2.13). Also, using (2.6), \( \lambda_k \sim k^\alpha \).

Let \( S_N \) denote the space of polynomials of degree less than or equal to \( N \). We define the weighted \( L^2 \) orthogonal projection \( P_N : L^2_w(I) \rightarrow S_N \) by the condition
\[
(v - P_N v, \phi_N) = 0, \quad \forall \phi_N \in S_N.
\]
(2.17)

Note that \( P_N v = \sum_{j=0}^N v_j \mathcal{G}_j^{(\alpha, \beta)}(x) \), where \( v_j = \int_0^1 \omega(x) v(x) \mathcal{G}_j^{(\alpha, \beta)}(x) \, dx \).

**Lemma 2.2** [18, Theorem 2.1] For \( \mu \in \mathbb{N}_0 \) and \( v \in H^\mu(I) \), with \( 0 \leq \mu \leq t \), there exists a constant \( C \), independent of \( N \), \( \alpha \) and \( \beta \) such that
\[
\lVert v - P_N v \rVert_{H^\mu(I)} \leq C N^{\mu - t} \lVert v \rVert_{H^\mu(I)}.
\]
(2.18)

**Remark** In [18] (2.18) is stated for \( t \in \mathbb{N}_0 \). The result extends to \( t \in \mathbb{R}^+ \) using interpolation.

The regularity of the solution to (1.1) can be influenced by the regularity of \( b(x) \) and \( c(x) \).

The following lemma enables us to insulate the influence of these terms.

Introduce the space \( W^{k, \infty}_w(I) \) and its associated norm, defined for \( k \in \mathbb{N}_0 \), as
\[
W^{k, \infty}_w(I) := \left\{ f : (1 - x)^{j/2} x^{j/2} D^j f(x) \in L^\infty(I), \quad j = 0, 1, \ldots, k \right\}.
\]
(2.19)

\[
\lVert f \rVert_{W^{k, \infty}_w} := \max_{0 \leq j \leq k} \lVert (1 - x)^{j/2} x^{j/2} D^j f(x) \rVert_{L^\infty(I)}.
\]
(2.20)

The subscript \( w \) denotes the fact that \( W^{k, \infty}_w(I) \) is a weaker space than \( W^{k, \infty}(I) \) in that the derivative of functions in \( W^{k, \infty}_w(I) \) may be unbounded at the endpoints of the interval.

**Lemma 2.3** [12, Lemma 7.1] Let \( s \geq 0 \), \( \alpha \), \( \beta > -1 \), \( k \geq s \), and \( f \in W^{k, \infty}_w(I) \). For
\[
(i) \ g \in H^s(I) \text{ then } f g \in H^s(I), \quad \text{and for}
\]
(2.21)

\[
(ii) \ g \in H^{-s}(I) \text{ then } f g \in H^{-s}(I).
\]
(2.22)

**Theorem 2.2** [12, Theorem 7.1] Let \( s \geq -\alpha \), \( \beta \) be determined by **Condition A**, \( c \in W^{[\min(s, \alpha + (\alpha - \beta) + 1, \alpha + \beta + 1)], \infty}_w(I) \) satisfying \( c(x) \geq 0 \) and \( f \in H^{-\alpha/2}(I) \cap H^s(I) \).

Then there exists a unique solution \( u(x) = (1 - x)^{\alpha - \beta} x^\beta \phi(x) \), with \( \phi(x) \in H^{\alpha + \min(s, \alpha + (\alpha - \beta) + 1, \alpha + \beta + 1)}(I) \), to
\[
L^a u(x) + c(x) u(x) = f(x), \quad x \in I, \quad \text{subject to } u(0) = u(1) = 0.
\]
(2.24)

The inclusion of an advection term can significantly reduced the regularity of the solution.

**Theorem 2.3** [12, Theorem 7.2] Let \( s \geq -\alpha \), \( \beta \) be determined by **Condition A**, \( b, c \in W^{[\min(s, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1)], \infty}_w(I) \) satisfying \( c(x) - 1/2 Db(x) \geq 0 \), and \( f \in H^{-\alpha/2}(I) \cap H^s(I) \).

Then there exists a unique solution \( u(x) = (1 - x)^{\alpha - \beta} x^\beta \phi(x) \), with \( \phi(x) \in H^{\alpha + \min(s, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1)}(I) \), to
\[
L^a u(x) + b(x) Du(x) + c(x) u(x) = f(x), \quad x \in I, \quad \text{subject to } u(0) = u(1) = 0.
\]
(2.26)

Introduce \( \tilde{s} \) defined by
\[
\tilde{s} := \begin{cases} 
\min\{s, \alpha + (\alpha - \beta) + 1, \alpha + \beta + 1\}, \text{ if } b = 0 \text{ (see Theorem 2.2)} \\
\min\{s, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1\}, \text{ if } b \neq 0 \text{ (see Theorem 2.3)}.
\end{cases}
\]
(2.27)
3 Weak Formulation

In place of (1.1), (1.2), we consider the following problem.

Given $f \in H^{a/2}_\omega (I)$, and $b$ and $c$ satisfying

$$b \in W^{|\min(s, a+(\alpha - \beta) - 1, \alpha + \beta - 1)|+1, \infty}_w (I), \quad c \in W^{|\min(s, a+(\alpha - \beta) - 1, \alpha + \beta - 1)|, \infty}_w (I),$$

$$c(x) - 1/2Db(x) \geq 0, \quad x \in I,$$

(3.1)
determine $\phi \in H^{a/2}_\omega (I)$ such that $u(x) = \omega(x) \phi(x)$ satisfies

$$\langle L^a_x u + b Du + cu, \psi \rangle_{\omega^s} = \langle f, \psi \rangle_{\omega^s}, \quad \forall \psi \in H^{a/2}_\omega (I).$$

Remark The assumption on $b(\cdot)$ is stronger than that required for Theorem 2.3, and that of $f(\cdot)$ is weaker. This extra regularity for $b(\cdot)$ is needed in the proof of Lemma 3.5, where Theorem 2.3 is applied to the adjoint of equation (2.26) (see (3.22)).

Note that the formulation (3.2) has different test and trial spaces. With this in mind we recall the Banach-Nečas-Babuška theorem.

Theorem 3.1 [11, Pg. 85, Theorem 2.6] Let $H_1$ and $H_2$ denote two real Hilbert spaces, $B(\cdot, \cdot) : H_1 \times H_2 \to \mathbb{R}$ a bilinear form, and $F : H_2 \to \mathbb{R}$ a bounded linear functional on $H_2$. Suppose there are constants $C_1 < \infty$ and $C_2 > 0$ such that

$$(i) \ |B(w, v)| \leq C_1 \|w\|_{H_1} \|v\|_{H_2}, \quad \text{for all } w \in H_1, \ v \in H_2,$$

$$(ii) \ \sup_{0 \neq v \in H_2} \frac{|B(w, v)|}{\|v\|_{H_2}} \geq C_2 \|w\|_{H_1}, \quad \text{for all } w \in H_1,$$

$$(iii) \ \sup_{w \in H_1} |B(w, v)| > 0, \quad \text{for all } v \in H_2, \ v \neq 0.$$

Then there exists a unique solution $w_0 \in H_1$ satisfying $B(w_0, v) = F(v)$ for all $v \in H_2$. Further, $\|w_0\|_{H_1} \leq C_2 \|F\|_{H_2}$.

For $f \in H^{a/2}_\omega (I)$, and $b$ and $c$ satisfying (3.1), let $B : H^{a/2}_\omega \times H^{a/2}_\omega \to \mathbb{R}$, and $F : H^{a/2}_\omega \to \mathbb{R}$ be defined by

$$B(\phi, \psi) := \langle L^a_x \omega \phi + b \omega \phi + c \omega \phi, \psi \rangle_{\omega^s},$$

$$F(\psi) := \langle f, \psi \rangle_{\omega^s}.$$

3.1 Continuity of $B(\cdot, \cdot)$

In order to establish that $B(\cdot, \cdot)$ is well defined and continuous we need to determine which $H^{s}_{(a,b)} (I)$ space $\omega \phi$ lies in.

The $H^{s}_{(a,b)} (I)$ space a function $f$ lies in is determined by its behavior at: (i) the left endpoint ($x = 0$), (ii) the right endpoint ($x = 1$), and (iii) away from the endpoints. In order to separate the consideration of the endpoint behaviors, following [6], we introduce the following function space $H^{s}_{(a,b)} (J)$. Let $J := (0, 3/4)$, and

$$\Lambda^* := \left\{ (x, y) : \frac{2}{3} x < y < \frac{3}{2} x, \ 0 < x < \frac{1}{2} \right\} \cup \left\{ (x, y) : \frac{3}{2} x - \frac{1}{2} < y < \frac{2}{3} x + \frac{1}{3}, \ 1/2 \leq x < 3/4 \right\}$$

$$:= \Lambda \cup \Lambda_1$$

(see Fig. 1).
Introduce the semi-norm and norm
\[ |f|_{H^s(\gamma)}^2(J) := \int_{\Lambda} \int_{\Lambda} \frac{|D^{|s|} f(x) - D^{|s|} f(y)|^2}{|x - y|^{1+2(s-|s|)}} \, dy \, dx + \int_{\Lambda_1} \int_{\Lambda_1} \frac{|D^{|s|} f(x) - D^{|s|} f(y)|^2}{|x - y|^{1+2(s-|s|)}} \, dy \, dx \]
\[ := |f|_{H^s(\gamma)(\Lambda)}^2 + |f|_{H^s(\gamma)(\Lambda_1)}^2, \]
and
\[ \|f\|_{H^s(\gamma)(\Lambda)}^2 := \begin{cases} \sum_{j=0}^s \|D^j f\|_{L^2(\Lambda)}^2, & \text{for } s \in \mathbb{N}_0 \\ \sum_{j=0}^{[s]} \|D^j f\|_{L^2(\Lambda)}^2 + |f|_{H^s(\gamma)(\Lambda_1)}^2, & \text{for } s \in \mathbb{R}^+ \setminus \mathbb{N}_0 \end{cases}, \]
where \( \|g\|_{L^2(\gamma)(J)}^2 := \int_J x^{\gamma} g^2(x) \, dx \).

Then, \( H^s(\gamma)(J) := \{ f \mid f \text{ is measurable and } \|f\|_{H^s(\gamma)(J)} < \infty \} \).

**Note:** A function \( f(x) \) is in \( H^s_{(a,b)}(I) \) if and only if \( f(\frac{3}{4} x) \in H^s_{(b)}(J) \) and \( f(\frac{3}{4} (1 - x)) \in H^s_{(a)}(J) \).

From [12] we have the following theorem.

**Theorem 3.2** [12, Theorem 6.4] Let \( n \leq s < n + 1, n \in \mathbb{N}_0, p \geq n, \mu > -1, \) and \( \psi \in H^s_{(\mu)}(I) \). Then \( x^p \psi \in H^t_{(\sigma)}(J) \) provided
\[ 0 \leq t \leq s, \sigma + 2p \geq \mu, \sigma + 2p - t > -1, \text{ and } \sigma + 2p + t \geq \mu + s. \quad (3.8) \]

Additionally, when (3.8) is satisfied, there exists \( C > 0 \) (independent of \( \psi \)) such that
\[ \|x^p \psi\|_{H^t_{(\sigma)}(J)} \leq C \|\psi\|_{H^s_{(\mu)}(I)}. \]

**Lemma 3.1** The terms \( \langle L^a_\phi \omega \phi, \psi \rangle_{\omega^*}, \langle b D \omega \phi, \psi \rangle_{\omega^*} \) and \( \langle c \omega \phi, \psi \rangle_{\omega^*} \) are well defined. Additionally, there exists \( C > 0 \) such that for \( \phi(x) \in H^s_{\omega^2}(I) \) and \( \psi(x) \in H^s_{\omega^2}(I) \)
\[ |B(\phi, \psi)| = |\langle L^a_\phi \omega \phi + b D \omega \phi + c \omega \phi, \psi \rangle_{\omega^*}| \leq C \|\phi\|_{H^s_{\omega^2}(I)} \|\psi\|_{H^s_{\omega^2}(I)}. \quad (3.9) \]

**Proof** We begin by considering the \( \langle b D \omega \phi, \psi \rangle_{\omega^*} \) term.
From Theorem 3.2, with $s = \alpha/2$, $\mu = \beta$, $p = \beta$, and choosing $\sigma = \alpha - \beta - 1$ we have that $t \leq \alpha/2$. Hence for $\phi_0 \in H^{\alpha/2}_\beta(I)$, $x^\beta \phi_0(x) \in H^{\alpha/2}_{\alpha - \beta - 1}(I)$, with $\|x^\beta \phi_0(x)\|_{H^{\alpha/2}_{\alpha - \beta - 1}} \lesssim \|\phi_0(x)\|_{H^{\alpha/2}_\beta}$.

Again, using Theorem 3.2, with $s = \alpha/2$, $\mu = \alpha - \beta$, $p = \alpha - \beta$, and choosing $\sigma = \beta$ we have that $t \leq \alpha/2$. Hence for $\phi_1 \in H^{\alpha/2}_{\alpha - \beta}(I)$, $x^{\alpha - \beta} \phi_1(x) \in H^{\alpha/2}_{\beta - 1}(I)$ with $\|x^{\alpha - \beta} \phi_1(x)\|_{H^{\alpha/2}_{\beta - 1}} \lesssim \|\phi_1(x)\|_{H^{\alpha/2}_{\alpha - \beta}}$.

Combining the above two applications of Theorem 3.2 we have that for $\phi \in H^{\alpha/2}_\omega(I)$, $\omega \phi \in H^{\alpha/2}_{\omega - 1}(I)$ with

$$\|\omega \phi\|_{H^{\alpha/2}_{\omega - 1}(I)} \lesssim \|\phi\|_{H^{\alpha/2}_\omega(I)} \cdot \tag{3.10}$$

A similar application of Theorem 3.2 establishes that for $\phi \in H^{\alpha/2}_{\omega^*}(I)$, $\omega \phi \in H^{\alpha/2}_{\omega^*}(I)$ with

$$\|\omega \phi\|_{H^{\alpha/2}_{\omega^*}(I)} \lesssim \|\phi\|_{H^{\alpha/2}_{\omega^*}(I)} \cdot \tag{3.11}$$

From (3.10) and Lemma 2.1 we have that $D \omega \phi \in H^{\alpha/2-1}_{(\beta, \alpha - \beta)}(I)$ with $\|D \omega \phi\|_{H^{\alpha/2-1}_{(\beta, \alpha - \beta)}} \lesssim \|\phi\|_{H^{\alpha/2}_\omega}$. Thus, with the assumption on $b$ and using Lemma 2.3,

$$\langle b D \omega \phi, \psi \rangle_{\omega^*} \leq \|D \omega \phi\|_{H^{\alpha/2-1}_{(\beta, \alpha - \beta)}} \|b \psi\|_{H^{1-\alpha/2}_{(\beta, \alpha - \beta)}} \lesssim \|\phi\|_{H^{\alpha/2}_{\omega^*}} \|\psi\|_{H^{1-\alpha/2}_{\omega^*}}, \tag{3.12}$$

$$\lesssim \|\phi\|_{H^{\alpha/2}_{\omega^*}} \|\psi\|_{H^{\alpha/2}_{\omega^*}}, \tag{3.13}$$

where in the last step we have used that $1 - \alpha/2 \leq \alpha/2$.

For $\|\phi\| \in H^{\alpha/2}_{\omega^*}(I)$ and $\|\psi\| \in H^{\alpha/2}_{\omega^*}(I)$, using (3.11) and the assumption on $c$,

$$\langle c \omega \phi, \psi \rangle_{\omega^*} = \int_I \omega^s(x) c(x) \omega(x) \phi(x) \psi(x) \, dx \leq \|\omega^{1/2} \omega^{1/2}\|_{L^\infty} \int_I \omega^{1/2}(x) \phi(x) \omega^{1/2}(x) c(x) \psi(x) \, dx \leq \|\phi\|_{L^2_{\omega}} \|c \psi\|_{L^2_{\omega^*}} \lesssim \|\phi\|_{H^{\alpha/2}_{\omega^*}} \|\psi\|_{H^{1-\alpha/2}_{\omega^*}} \cdot \tag{3.14}$$

$$\lesssim \|\phi\|_{H^{\alpha/2}_{\omega^*}} \|\psi\|_{H^{\alpha/2}_{\omega^*}} \cdot \tag{3.15}$$
For \( \phi(x) = \sum_{i=0}^{\infty} \phi_i \hat{G}^{(\alpha-\beta, \beta)}_i(x) \in H^\alpha/2_\omega(I) \) and \( \psi(x) = \sum_{j=0}^{\infty} \psi_j \hat{G}^{(\beta, \alpha-\beta)}_j(x) \in H^{\alpha/2}_\omega(I) \), using (2.16)

\[
\langle L^\alpha_\omega \phi, \psi \rangle_{\omega^*} = \left( \sum_{i=0}^{\infty} -c^*_i \lambda_i \phi_i \hat{G}^{(\beta, \alpha-\beta)}_i(x), \sum_{j=0}^{\infty} \psi_j \hat{G}^{(\beta, \alpha-\beta)}_j(x) \right)_{\omega^*}
\]

\[
\leq -c^*_i \sum_{k=0}^{\infty} \lambda_k \phi_k \psi_k \sim \sum_{k=0}^{\infty} k^\alpha \phi_k \psi_k
\]

\[
\leq \left( \sum_{k=0}^{\infty} k^\alpha \phi_k \right)^{1/2} \left( \sum_{k=0}^{\infty} k^\alpha \psi_k \right)^{1/2}
\]

\[
\leq \left( \sum_{k=0}^{\infty} (1 + k^2)^{\alpha/2} \phi_k \right)^{1/2} \left( \sum_{k=0}^{\infty} (1 + k^2)^{\alpha/2} \psi_k \right)^{1/2}
\]

\[
\leq \|\phi\|_{H^{\alpha/2}_\omega} \|\psi\|_{H^{\alpha/2}_\omega}, \text{ using (2.12).}
\]

Combining (3.13), (3.15) and (3.17) we obtain (3.9).

\[\square\]

### 3.2 Conditions (3.4) and (3.5)

For the case \( r = 1/2 \) we have \( \alpha - \beta = \beta = \alpha/2 \) and, consequently, \( \omega = \omega^* \). In this case for \( \psi = \phi \)

\[
\langle b D(\omega \phi) + c \omega \phi, \psi \rangle_{\omega} = \int_{-1}^{1} \omega (b D(\omega \phi) + c \omega \phi) \phi \, dx
\]

\[
= \int_{-1}^{1} b \frac{1}{2} (D(\omega \phi))^2 + c (\omega \phi)^2 \, dx
\]

\[
= \int_{-1}^{1} \left( c - \frac{1}{2} Db \right) (\omega \phi)^2 \, dx.
\]

Proceeding as in (3.16), for \( \psi = \phi \) and \( \omega^* = \omega \),

\[
\langle L^\alpha_{1/2} (\omega \phi), \phi \rangle_{\omega} \sim \sum_{k=0}^{\infty} k^\alpha \phi_k^2 \sim \sum_{k=0}^{\infty} (1 + k^2)^{\alpha/2} \phi_k^2
\]

\[
\sim \|\phi\|^2_{H^{\alpha/2}_{(\alpha/2, \alpha/2)}}.
\]

Hence for \( (c - \frac{1}{2} Db) \geq 0 \), combining (3.18) and (3.19) we have that \( B(\cdot, \cdot) \) is coercive on \( H^{\alpha/2}_{(\alpha/2, \alpha/2)} \times H^{\alpha/2}_{(\alpha/2, \alpha/2)} \). Then, from the Lax-Milgram, we have the following lemma.

**Lemma 3.2** For \( 1 < \alpha < 2 \) and \( r = 1/2 \), given \( f \in H^{-\alpha/2}_{(\alpha/2, \alpha/2)}(I) \) and \( b(x) \) and \( c(x) \) satisfying (3.1), there exists a unique solution \( u(x) = (1 - x)^{\alpha/2} \phi(x) \) to (3.2), with \( \phi \in H^{\alpha/2}_{(\alpha/2, \alpha/2)}(I) \) satisfying \( \|\phi\|_{H^{\alpha/2}_{(\alpha/2, \alpha/2)}(I)} \lesssim \|f\|_{H^{-\alpha/2}_{(\alpha/2, \alpha/2)}(I)} \).

This special case of (3.2) corresponding to \( r = 1/2 \) has been thoroughly investigated by Hao and Zhang in [20].

\[\copyright\ Springer\]
For the general case, \((r \neq \frac{1}{2})\), to show (3.4) and (3.5), and hence establish the well posedness of the formulation, following an approach by Jin et al. in [24], we use the Petree–Tartar Lemma.

**Lemma 3.3** [11, Pg. 469] (Petree-Tartar). Let \(X, Y, Z\) be three Banach spaces. Let \(A \in \mathcal{L}(X; Y)\) be an injective operator and let \(T \in \mathcal{L}(X; Z)\) be a compact operator. If there exists \(c_1 > 0\) such that \(c_1 \|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z\), then \(\text{Im}(A)\) is closed; equivalently, there is \(c_2 > 0\) such that
\[
\forall x \in X, \quad c_2 \|x\|_X \leq \|Ax\|_Y . \tag{3.20}
\]

To relate the Petree-Tartar Lemma to the formulation (3.2), with \(b, c\) satisfying (3.1), let \(X = H^{\alpha/2}_\omega(I), \ Y = Z = H^{-\alpha/2}_\omega(I), \)
\[
A : X \to Y \text{ be defined by } A\phi := L_\alpha^\omega \omega \phi + b \ D \omega \phi + c \ \omega \phi, \text{ and}
T : X \to Z \text{ be defined by } T\phi := -(b \ D \omega \phi + c \ \omega \phi) .
\]
That \(A \in \mathcal{L}(X; Y)\) follows from its definition and the continuity of \(B(\cdot, \cdot)\). To establish the injectivity of \(A\), consider \(u_1 \in X\) and let \(u_2 = \omega \phi_2\) would satisfy
\[
L_\alpha^\omega (u_1 - u_2)(x) + b(x) \ D (u_1 - u_2)(x) + c(x) (u_1 - u_2)(x) = 0 \in H^{-\alpha/2}(I) \cap H^{1-\alpha/2}(I),
\]
with \((u_1 - u_2)(0) = (u_1 - u_2)(1) = 0\). Theorem 2.3 would then implies \((u_1 - u_2)(x) = 0\), i.e., \(u_1 = u_2 \iff \phi_1 = \phi_2\). Hence \(A\) is injective on \(Y\).

The fact that \(T \in \mathcal{L}(X; Z)\) follows from its definition and (3.13) and (3.15). Also, from (3.12) and (3.14) we have that \(T : H^{\alpha/2}_\omega(I) \to H^{1-\alpha/2}_\omega(I)\) is bounded. As \(H^{\alpha/2}_\omega(I)\) is compactly embedded in \(H^{s}_\omega(I)\) for \(s > t\), [12, pg. 10, Remark 2], since \(1 - \alpha/2 > -\alpha/2\), it follows that \(T \in \mathcal{L}(X; Z)\) is a compact operator.

Let \(\phi(x) = \sum_{i=1}^{\infty} \phi_i \hat{G}_i^{(\alpha-\beta, \beta)}(x) \in H^{\alpha/2}_\omega(I)\) and \(\psi(x) = \sum_{i=1}^{\infty} \phi_i \hat{G}_i^{(\beta, \alpha-\beta)}(x) \in H^{\alpha/2}_\omega(I)\). Note that \(\|\phi\|_{H^{\alpha/2}_\omega(I)} = \|\psi\|_{H^{\alpha/2}_\omega(I)}\). Then,
\[
\|\phi\|^2_{H^{\alpha/2}_\omega(I)} = \sum_{i=0}^{\infty} (1 + i^2)^{\alpha/2} \phi_i^2 \leq \sum_{i=0}^{\infty} \lambda_i \phi_i^2 = (L_\omega^\alpha \omega \phi, \psi)_{\omega^*} = (L_\omega^\alpha \omega \phi + b \ D \omega \phi + c \ \omega \phi, \psi)_{\omega^*} + (- (b \ D \omega \phi + c \ \omega \phi), \psi)_{\omega^*}.
\]
Using \(\|\phi\|_{H^{\alpha/2}_\omega(I)} = \|\psi\|_{H^{\alpha/2}_\omega(I)}\), we obtain that there exists \(c_1 > 0\) such that
\[
c_1 \|\phi\|_X \leq \|A\phi\|_Y + \|T\phi\|_Z .
\]
Then, applying the Petree-Tartar Lemma, it follows that there exists \(c_2 > 0\) such that
\[
C_2 \|\phi\|_X \leq \|A\phi\|_Y, \text{ i.e., } C_2 \|\phi\|_{H^{\alpha/2}_\omega(I)} \leq \|A\phi\|_{H^{-\alpha/2}_\omega(I)} . \tag{3.21}
\]

**Lemma 3.4** For \(B(\cdot, \cdot)\) defined by (3.6), the condition (ii) given by (3.4) is satisfied.
Lemma 3.5 For $B(\cdot, \cdot)$ defined by (3.6), the condition (iii) given by (3.5) is satisfied.

Proof The adjoint problem to (3.2) is: Given $g \in H_{\omega}^{-\alpha/2}(I)$, determine $\psi \in H_{\omega}^{\alpha/2}(I)$ such that $\psi(x) = \omega^*(x)\psi(x)$ satisfies

$$\langle L_{(1-r)}^\alpha v - b Dv + (c - Db)v, \phi \rangle_\omega = \langle g, \phi \rangle_\omega, \quad \forall \phi \in H_{\omega}^{\alpha/2}(I).$$

(3.22)

Observe that the advection coefficient $(-b)$, and the reaction coefficient $(c - Db)$, satisfy the assumptions of (3.2).

In relation to Theorem 2.3, the weak form corresponds to the fractional diffusion, advection, reaction equation: Given $\tilde{g} \in H^{-\alpha/2}(I) \cap H_{\omega}^{-\alpha/2}(I)$ determine $v(x)$ satisfying

$$L_{(1-r)}^\alpha v(x) - b(x) Dv(x) + (c(x) - Db(x))v(x) = \tilde{g}(x), \quad x \in I, \quad \text{subject to } v(0) = v(1) = 0.$$  

(3.23)

Note that for the weak formulation (3.22), $g$ may be chosen in $H_{\omega}^{-\alpha/2}(I)$, whereas Theorem 2.3 requires the RHS, $\tilde{g}$, to be in $H^{-\alpha/2}(I) \cap H_{\omega}^{-\alpha/2}(I)$. Also, note that properties (ii) and (iii) of Theorem 3.1 are similar (property (ii) a stronger condition), where the supremum is taken over one function space with the element in the other function space fixed.

For $B^*(\psi, \phi) := \langle L_{1-r}^\alpha \omega^* \psi + b D\omega^* \psi + c \omega^* \psi, \phi \rangle_\omega$

$$= \langle L_{1-r}^\alpha \omega \phi + b D\omega \phi + c \omega \phi, \psi \rangle_\omega = B(\phi, \psi).$$

An analogous argument as used to establish condition (ii) can be applied to $B^*(\cdot, \cdot)$ to obtain

$$\sup_{0 \neq w \in H_{\omega}^{\alpha/2}(I)} \frac{|B^*(v, w)|}{\|w\|_{H_{\omega}^{\alpha/2}(I)}} \geq \tilde{C}_2 \|v\|_{H_{\omega}^{\alpha/2}(I)}$$

$$\iff \sup_{0 \neq w \in H_{\omega}^{\alpha/2}(I)} \frac{|B(w, v)|}{\|w\|_{H_{\omega}^{\alpha/2}(I)}} \geq \tilde{C}_2 \|v\|_{H_{\omega}^{\alpha/2}(I)} \quad \text{for all } v \in H_{\omega}^{\alpha/2}(I),$$

$$\Rightarrow \sup_{w \in H_{\omega}^{\alpha/2}(I)} |B(w, v)| > 0 \quad \text{for all } v \in H_{\omega}^{\alpha/2}(I), \quad v \neq 0.$$  

(Recall that in establishing condition (ii) Theorem 2.3 is only used with RHS function equal to 0 in establishing the injectivity of the operator $A$.)

Combining Lemmas 3.1, 3.4 and 3.5 with Theorem 3.1 we obtain the following.

Theorem 3.3 There exists a unique solution $\phi$ to (3.2), satisfying $\|\phi\|_{H_{\omega}^{\alpha/2}(I)} \leq \frac{1}{\tilde{C}_2} \|f\|_{H_{\omega}^{-\alpha/2}(I)}$.

Proof First, note that $F$ defined by (3.7) satisfies

$$\|F\| = \sup_{0 \neq \psi \in H_{\omega}^{\alpha/2}(I)} \frac{|F(\psi)|}{\|\psi\|_{H_{\omega}^{\alpha/2}(I)}} = \sup_{0 \neq \psi \in H_{\omega}^{\alpha/2}(I)} \frac{|\langle f, \psi \rangle_{\omega^*}|}{\|\psi\|_{H_{\omega}^{\alpha/2}(I)}}$$

$$\leq \sup_{0 \neq \psi \in H_{\omega}^{\alpha/2}(I)} \frac{\|f\|_{H_{\omega}^{\alpha/2}(I)} \|\psi\|_{H_{\omega}^{\alpha/2}(I)}}{\|\psi\|_{H_{\omega}^{\alpha/2}(I)}} = \|f\|_{H_{\omega}^{\alpha/2}(I)}.$$  

\(\Box\)
Hence, $F$ defines a bounded linear functional. The existence and uniqueness of $\phi$ then follows from combining Lemmas 3.1, 3.4 and 3.5 with Theorem 3.1. To obtain the bound for $\|\phi\|_{H_0^{\alpha/2}(I)}$, from Lemma 3.4

$$\|\phi\|_{H_0^{\alpha/2}(I)} \leq \frac{1}{C_2} \sup_{0 \neq \psi \in H_0^{\alpha/2}} \frac{|B(\phi, \psi)|}{\|\psi\|_{H_0^{\alpha/2}(I)}} = \frac{1}{C_2} \sup_{0 \neq \psi \in H_0^{\alpha/2}} \frac{|\langle f \cdot \psi \rangle \omega^s|}{\|\psi\|_{H_0^{\alpha/2}(I)}} \leq \frac{1}{C_2} \sup_{0 \neq \psi \in H_0^{\alpha/2}} \frac{\|f\|_{H_0^{-\alpha/2}(I)} \|\psi\|_{H_0^{\alpha/2}(I)}}{\|\psi\|_{H_0^{\alpha/2}(I)}} = \frac{1}{C_2} \|f\|_{H_0^{-\alpha/2}(I)} \cdot (3.24)$$

**Corollary 3.1** For $f \in H_{\omega^s}^s(I)$, $s \geq -\alpha/2$, and $b$ and $c$ satisfying (3.1), there exists $C > 0$ such that with $\phi$ given by (3.2) satisfies

$$\|\phi\|_{H_{\omega^s}^{\alpha/2}(I)} \leq C \|f\|_{H_{\omega^s}^s(I)} \cdot (3.25)$$

**Proof** The proofs of Theorems 2.2 and 2.3 use a boot strapping argument. The first part of the proof establishes that, for $f \in H^{-\alpha/2}(I)$, the existence and uniqueness of a solution $u(x) = \omega(x) \phi(x)$, where $\phi \in L^2(I)$, which then implies that $u \in H_0^{\alpha/2}(I)$. The subsequent (finite) steps in the proofs iteratively improve the regularity of $\phi$ (boot strapping argument), until the optimum regularity of $\phi$ is obtained.

In view of Theorem 3.3, for $f \in H_0^{\alpha/2}(I)$, $s \geq -\alpha/2$, there exists $\phi \in H_0^{\alpha/2}(I)$ satisfying (3.2). Consequently, for $u(x) = \omega(x) \phi(x)$, using Theorem 3.2, $u \in H_0^{\alpha/2}(I)$. Repeating the boot strapping argument used in the proofs of Theorems 2.2 and 2.3 results in $u(x) = \omega(x) \phi(x)$ satisfying (2.26), with $\phi \in H_0^{\alpha/2}(I)$, where $\tau$ is defined in (2.27). The norm estimate (3.25) follows from that at each of the (finite number of) steps in the boot strapping argument the terms on the right hand side are bounded by a constant times $\|f\|_{H_0^{\alpha/2}(I)}$. □

**Remark 3.1** Comparing Corollary 3.1 with Theorems 2.2 and 2.3, for Corollary 3.1: (i) the regularity condition for $b$ is stronger, (ii) the condition on $f$ is weaker, and (iii) a bound for $\phi$ is not given in Theorems 2.2 and 2.3.

**Remark 3.2** A corresponding weak formulation to (3.2) can be given for $u$, and subsequent analysis performed. As the unknown in our computational algorithm is $\phi_N$ we have chosen to present the analysis in terms of $\phi$.

### 4 Approximation Scheme

As $\{\tilde{G}_j^{(b)}\}_{j=0}^\infty$ is a basis for $H_0^{\alpha/2}(I)$, let $X_N := \text{span}\{\tilde{G}_j^{(b)}(x) : j = 0 \ldots N\} \subset H_0^{\alpha/2}(I)$, and $Y_N := \text{span}\{\tilde{G}_j^{(b)}(x) : j = 0 \ldots N\} \subset H_0^{\alpha/2}(I)$. Corresponding to (3.2) we have the following approximation scheme.

**Given** $f \in H_0^{-\alpha/2}(I)$, and $b$ and $c$ satisfying (3.1), determine $\phi_N \in X_N$ such that $u_N(x) = \omega(x) \phi_N(x)$ satisfies

$$\langle \mathcal{L}_x^N \omega(x) \phi_N(x) + b(x) D\omega(x) \phi_N(x) + c(x) \omega(x) \phi_N(x), \psi_N \rangle_{\omega^s} = \langle f, \psi_N \rangle_{\omega^s}, \ \forall \psi_N \in Y_N \cdot (4.1)$$

The following lemma is used to establish the well posedness of (4.1).
Lemma 4.1 There exists $C_3 > 0$, such that for $N$ sufficiently large,

$$
\sup_{0 \neq \phi_N \in Y_N} \frac{|B(\phi_N, \psi_N)|}{\|\psi_N\|_{H^{\alpha/2}_0(I)}} \geq C_3 \|\phi_N\|_{H^{\alpha/2}_0(I)} \quad \forall \phi_N \in X_N.
$$  \hspace{1cm} (4.2)

**Proof** Let $\phi_N \in X_N$. For $\psi \in H^{\alpha/2}_0(I)$, let $\psi_N = \sum_{i=0}^{N} \psi_i G_i^{(\beta, \alpha-\beta)}(x)$. Using Lemma 3.4,

$$
C_2 \|\phi_N\|_{H^{\alpha/2}_0(I)} \leq \sup_{0 \neq \psi \in H^{\alpha/2}_0(I)} \frac{B(\phi_N, \psi)}{\|\psi\|_{H^{\alpha/2}_0(I)}} \leq \sup_{0 \neq \psi \in H^{\alpha/2}_0(I)} \frac{B(\phi_N, \psi) + B(\phi_N, \psi - \psi_N)}{\|\psi\|_{H^{\alpha/2}_0(I)}}
$$

where in the last step we have used $\langle \mathcal{L}_2^\alpha \phi_N, \psi - \psi_N \rangle_{\omega^*} = 0$.

From (3.12) and (3.14), and using (2.18),

$$
|\langle b(x) D_\omega(x) \phi_N(x) + c(x) {\omega}(x) \phi_N(x) \rangle, \psi - \psi_N |_{\omega^*} \leq C \|\phi_N\|_{H^{\alpha/2}_0} \|\psi - \psi_N\|_{H^{\alpha/2}_{1-\alpha/2}} \leq C \|\phi_N\|_{H^{\alpha/2}_0} N^{1-\alpha} \|\psi\|_{H^{\alpha/2}_0}.
$$

Combining (4.3) and (4.4), for $N$ sufficiently large we obtain (4.2). \hfill\Box

**Theorem 4.1** There exists a unique $\phi_N \in H^{\alpha/2}_0(I)$ satisfying (4.1). In addition, for $C_3$ given in (4.2), $\|\phi_N\|_{H^{\alpha/2}_0(I)} \leq \frac{1}{C_3} \|f\|_{H^{-\alpha/2}_0(I)}$.

**Proof** For $\phi_N = \sum_{j=0}^{N} c_j \widehat{G}_j^{(\alpha-\beta, \beta)}(x)$, from (4.1), the constants $c_j$ are determined from

$$
\mathcal{A} c = b, \quad \text{where} \quad \mathcal{A}_i = B(\widehat{G}_i^{(\alpha-\beta, \beta)}, \widehat{G}_j^{(\beta, \alpha-\beta)}), \quad \text{and} \quad b_i = \langle f(x), \widehat{G}_i^{(\beta, \alpha-\beta)}(x) \rangle_{\omega^*},
$$

for $0 \leq i, j \leq N$. Condition (4.2) implies the invertibility of the square matrix $\mathcal{A}$, and hence the uniqueness of $\phi_N$ satisfying (4.1). The bound for $\phi_N$ is obtained in an analogous manner to the bound for $\phi$ in (3.24). \hfill\Box

For $\phi_N$ given by (4.1) we have the following error bound.

**Lemma 4.2** There exists $C > 0$ such that for $\phi$ satisfying (3.2) and $\phi_N$ satisfying (4.1)

$$
\|\phi - \phi_N\|_{H^{\alpha/2}_0(I)} \leq C \inf_{\zeta_N \in X_N} \|\phi - \zeta_N\|_{H^{\alpha/2}_0(I)}.
$$

**Proof** Note that for $\zeta_N \in X_N$, using (4.2),

$$
C_3 \|\phi_N - \zeta_N\|_{H^{\alpha/2}_0(I)} \leq \sup_{\psi_N \in X_N, \psi_N \neq 0} \frac{|B(\phi_N - \zeta_N, \psi_N)|}{\|\psi_N\|_{H^{\alpha/2}_0(I)}} = \sup_{\psi \neq 0} \frac{|\langle f, \psi \rangle_{\omega^*} - B(\zeta_N, \psi_N)|}{\|\psi_N\|_{H^{\alpha/2}_0(I)}}
$$

$$
= \sup_{\psi_N \in X_N, \psi_N \neq 0} \frac{|B(\phi - \zeta_N, \psi_N)|}{\|\psi_N\|_{H^{\alpha/2}_0(I)}} \quad \text{(using (3.2))}
$$

$$
\leq \sup_{\psi_N \in X_N, \psi_N \neq 0} \frac{C_1 \|\phi - \zeta_N\|_{H^{\alpha/2}_0(I)} \|\psi_N\|_{H^{\alpha/2}_0(I)}}{\|\psi_N\|_{H^{\alpha/2}_0(I)}} = C_1 \|\phi - \zeta_N\|_{H^{\alpha/2}_0(I)}.
$$

\hfill\Box
With the triangle inequality and (4.6), we obtain

\[ \| \phi - \phi_N \|_{H^{\alpha/2}_\omega(I)} \leq \| \phi - \xi \|_{H^{\alpha/2}_\omega(I)} + \| \xi_N - \phi_N \|_{H^{\alpha/2}_\omega(I)} \leq (1 + C_1) \| \phi - \xi \|_{H^{\alpha/2}_\omega(I)}. \]

As \( \xi_N \in X_N \) is arbitrary, then (4.5) follows. \( \square \)

Combining Lemma 4.2 with Lemma 2.2 and Theorems 2.2 and 2.3 we obtain the following error estimate.

**Corollary 4.1** For \( f \in H^{s}_\omega(I) \), \( s \geq -\alpha/2 \), and \( b \) and \( c \) satisfying (3.1), there exists \( C > 0 \) such that for \( \phi \) satisfying (3.2) and \( \phi_N \) satisfying (4.1)

\[ \| \phi - \phi_N \|_{H^{\alpha/2}_\omega(I)} \leq C N^{-\tilde{\alpha}/2} \| \phi \|_{H^{\tilde{\alpha}/2}_\omega(I)} \leq C N^{-\tilde{\alpha}/2} \| f \|_{H^{\tilde{\alpha}/2}_\omega(I)}. \]  

**Proof** From Corollary 3.1 we have that \( \phi \) satisfies \( \phi \in H^{\tilde{\alpha}/\alpha}_\omega(I) \). Then, applying Lemma 2.2, with \( \mu = \alpha/2 \) and \( t = \tilde{s} + \alpha \), and using Corollary 3.1, we obtain (4.7). \( \square \)

An estimate for \( \| \phi - \phi_N \|_{L^2_\omega(I)} \) can be obtained using a Aubin-Nitsche type argument.

**Corollary 4.2** For \( f \in H^{s}_\omega(I) \), \( s \geq -\alpha/2 \), and \( b \) and \( c \) satisfying (3.1), there exists \( C > 0 \) such that for \( \phi \) satisfying (3.2) and \( \phi_N \) satisfying (4.1)

\[ \| \phi - \phi_N \|_{L^2_\omega(I)} \leq C N^{-\tilde{\alpha}/\alpha} \| \phi \|_{H^{\tilde{\alpha}/\alpha}_\omega(I)} \leq C N^{-\tilde{\alpha}/\alpha} \| f \|_{H^{\tilde{\alpha}/\alpha}_\omega(I)}. \]  

**Proof** Introduce \( \psi \in H^{\alpha/2}_\omega(I) \) satisfying

\[ \mathcal{L}^{\alpha/2}_{(1-r)\omega^*} \psi - b D \omega^* \psi + (c - Db) \omega^* \psi = \phi - \phi_N. \]

As \( (\phi - \phi_N) \in L^2_\omega(I) \), analogous to (3.25), we have that

\[ \| \psi \|_{H^{\alpha}_\omega(I)} \leq C \| \phi - \phi_N \|_{L^2_\omega(I)}. \]  

Then,

\[ \| \phi - \phi_N \|_{L^2_\omega} = \| (\phi - \phi_N) \|_{L^{\alpha/2}_\omega} = \| (\phi - \phi_N) \mathcal{L}^{\alpha/2}_{\omega} \psi - b D \omega^* \psi + (c - Db) \omega^* \psi \|_{\omega} \]

\[ = (L^{\alpha/2}_\omega \omega \phi - \phi_N) + b D \omega \phi - \phi_N \]  

\[ + c \omega \phi - \phi_N \| \psi \geq B((\phi - \phi_N), \psi) \]

\[ \leq C_1 \| \phi - \phi_N \|_{H^{\alpha/2}_\omega} \| \psi - \eta_N \|_{H^{\alpha/2}_\omega}, \]

\[ \leq C N^{-\alpha/2} \| \phi \|_{H^{\tilde{\alpha}/\alpha}_\omega} N^{-\alpha/2} \| \psi \|_{H^{\tilde{\alpha}/\alpha}_\omega} \]  

\[ \leq C N^{-\alpha/2} \| \phi \|_{H^{\tilde{\alpha}/\alpha}_\omega} \| \phi - \phi_N \|_{L^2_\omega}. \]

Finally, dividing through by \( \| \phi - \phi_N \|_{L^2_\omega} \) and using (3.25) we obtain (4.8). \( \square \)

**Error estimate for \( u - u_N \).**

The weighted \( L^2_{\omega^{-1}} \) error estimate for \( u - u_N \), where \( u_N := \omega \phi_N \), follows easily from the definitions of \( u_N \) and the \( L^2_{\omega^{-1}} \) norm, and the estimate (4.8). The proof of the estimate for \( u - u_N \) in the \( H^{\alpha/2}_\omega \) norm is not so straightforward. The following lemma is helpful in establishing the \( H^{\alpha/2}_\omega \) error estimate.

**Lemma 4.3** Let \( 0 \leq \mu \leq 1 \). For \( \xi \in H^\mu_\omega(I) \), then \( z := \omega \xi \in H^{\mu\tilde{\alpha}/\alpha}_\omega(I) \), with, for some \( C > 0 \),

\[ \| z \|_{H^{\mu\tilde{\alpha}/\alpha}_\omega(I)} \leq C \| \xi \|_{H^{\mu}_\omega(I)}. \]  

(4.10)
Proof For this proof it is convenient to use the definition of the $H_{(a,b)}^{s}(I)$ spaces given by (2.9).
Let $\mu = 0$, and $\zeta \in H_{\omega}^{0}(I) = L_{\omega}^{2}(I)$. Then, for $z = \omega \zeta$
\[\|z\|_{H_{\omega}^{0}(I)}^{2} = \|z\|_{L_{\omega}^{2}(I)}^{2} = \int_{1}^{1} (1-x)^{-(\alpha-\beta)}x^{-\beta} (\omega \zeta)^{2} \, dx = \|\zeta\|_{L_{\omega}^{2}(I)}^{2} = \|\zeta\|_{H_{\omega}^{0}(I)}^{2}. \] (4.11)

Next, for $\mu = 1$, let $\zeta \in C^{\infty}(I) \subset H_{\omega}^{1}(I)$, and let $z = \omega \zeta$. Note that $Dz \sim (1-x)^{\alpha-\beta}x^{\beta-1}(\zeta(x) + (1-x)^{\alpha-\beta-1}x^{\beta}\zeta(x) + (1-x)^{\alpha-\beta}x^{\beta}D\zeta(x)$, and
\[\int_{1}^{1} (1-x)^{-(\alpha-\beta)+1}x^{-\beta+1} (Dz)^{2} \, dx \sim \int_{0}^{1/2} (1-x)^{\alpha-\beta+1}x^{\beta-1}\zeta(x)^{2} \, dx + \int_{1/2}^{1} (1-x)^{\alpha-\beta+1}x^{\beta+1}\zeta(x)^{2} \, dx \]
\[+ \int_{0}^{1/2} (1-x)^{\alpha-\beta+1}x^{\beta+1}D\zeta(x)^{2} \, dx \]
\[=: \mathcal{I}_{1} + \mathcal{I}_{2} + \|D\zeta\|_{L_{(1-x)(\alpha-\beta)+1,\beta+1}(I)}^{2}. \] (4.12)

To bound $\mathcal{I}_{1}$ and $\mathcal{I}_{2}$ in terms of $\|\zeta\|_{L_{\omega}^{2}(I)}$ and $\|D\zeta\|_{L_{(1-x)(\alpha-\beta)+1,\beta+1}(I)}$ we use Hardy’s inequality [5, Lemma 3.2].
\[
\mathcal{I}_{1} = \int_{0}^{1/2} x^{\beta-1}\zeta(x)^{2} \, dx + \int_{1/2}^{1} x^{\beta+1}\zeta(x)^{2} \, dx \lesssim \int_{0}^{1/2} x^{\beta+1}(D\zeta(x))^{2} \, dx \]
\[+ \int_{0}^{1/2} x^{\beta+1}\zeta(x)^{2} \, dx + \int_{0}^{1/2} x^{\beta+1}\zeta(x)^{2} \, dx \]
\[\lesssim \|\zeta\|_{L_{\omega}^{2}(I)}^{2} + \|D\zeta\|_{L_{(1-x)(\alpha-\beta)+1,\beta+1}(I)}^{2}. \] (4.13)

An analogous argument yields
\[\mathcal{I}_{2} \lesssim \|\zeta\|_{L_{\omega}^{2}(I)}^{2} + \|D\zeta\|_{L_{(1-x)(\alpha-\beta)+1,\beta+1}(I)}^{2}. \] (4.15)

Combining (4.11), (4.12), (4.14), and (4.15), we obtain
\[\|z\|_{H_{\omega}^{1}(I)} \leq C \|\zeta\|_{H_{\omega}^{0}(I)}. \] (4.16)

Estimate (4.16) extends to $\zeta \in H_{\omega}^{1}(I)$, using the density of $C^{\infty}(I)$ in $H_{\omega}^{1}(I)$.
Finally, estimate (4.10) then follows from (4.11) and (4.16) using interpolation. □
Table 1  Experiment 1: $\alpha = 1.60$, $r = 0.20$, $b(x) = 0$, $c(x) = 5$, and $f(x) = 1$

| $N$ | $\|u - u_N\|_{L^2_{\omega}}$ | $\kappa$ | $\|u - u_N\|_{H^{\alpha/2}_{\omega}}$ | $\kappa$ |
|-----|-----------------|--------|-----------------|--------|
| 6   | 1.05E-04        | 5.36E-04 |
| 8   | 2.52E-05        | 4.97   | 1.56E-04        | 4.30   |
| 10  | 8.62E-06        | 4.81   | 6.22E-05        | 4.11   |
| 12  | 3.61E-06        | 4.77   | 2.97E-05        | 4.06   |
| 14  | 1.74E-06        | 4.76   | 1.59E-05        | 4.05   |
| Pred.| 4.87           | 4.07   |

Fig. 2  The plot of the reference solution $u_{40}(x)$ (left), and the plot of the errors for Experiment 1

Corollary 4.3  For $H^s_{\omega}(I)$, $s \geq -\alpha/2$, and $b$ and $c$ satisfying (3.1), there exists $C > 0$ such that for $u$ determined from (3.2) and $u_N$ determined from (4.1)

$$\|u - u_N\|_{L^2_{\omega}(I)} \leq C N^{-(\tilde{s} + \alpha)} \|f\|_{H^{\tilde{s}}_{\omega}(I)}, \quad (4.17)$$

$$\|u - u_N\|_{H^{\alpha/2}_{\omega}(I)} \leq C N^{-(\tilde{s} + \alpha/2)} \|f\|_{H^{\tilde{s}}_{\omega}(I)}. \quad (4.18)$$

Proof  As commented above, (4.17) follows from the definition of $u_N$ and (4.2). The estimate (4.18) follows from (4.10) (with $z = u - u_N$, $\zeta = \phi - \phi_N$) and (4.7).

5 Numerical Experiments

In this section we present three numerical experiments to investigate the approximation of (1.1), (1.2) using (4.1). We compare the approximation errors with those predicted by Corollary 4.2.

For the numerical experiments we use $f(x) = 1$ and $f(x) = \begin{cases} 0, & 0 < x \leq 1/2, \\ 1, & 1/2 < x < 1 \end{cases}$. For these choices of $f$ the true solution is unknown. In order to be able to compute a convergence
Table 2  Experiment 2: $\alpha = 1.40$, $r = 0.40$, $b(x) = e^x$, $c(x) = 5 + \sin(x)$, and $f(x) = 1$

| $N$ | $\|u - u_N\|_{L^2_\omega}$ | $\kappa$ | $\|u - u_N\|_{H^{\alpha/2}_\omega}$ | $\kappa$ |
|-----|----------------|--------|----------------|--------|
| 12  | 5.67E−03       | 3.57E−02 |          |        |
| 14  | 4.11E−03       | 2.08    | 2.82E−02   | 1.53   |
| 16  | 3.09E−03       | 2.15    | 2.27E−02   | 1.62   |
| 18  | 2.38E−03       | 2.21    | 1.86E−02   | 1.71   |
| 20  | 1.87E−03       | 2.26    | 1.54E−02   | 1.80   |
| Pred. |              | 2.41    |              | 1.71   |

Fig. 3  The plot of the reference solution $u_{40}(x)$ (left), and the plot of the errors for Experiment 2

rate for the approximation a very accurate approximation (using $N = 40$) is used as the reference solution. For the computational experiments the entries of the coefficient matrices, which require the evaluation of integrals of weighted products of Jacobi polynomials on 1, are evaluated using the Legendre-Gauss quadrature rule with 200 nodes. This ensures sufficient accuracy in order to accurately measure the error associated with the approximation scheme (4.1). We evaluate the norms of the error using the norms associated with Definition 2.2.

The numerical convergence rate, $\kappa$, corresponding to $\|u_{40} - u_N\|_{\text{norm}} \lesssim N^{-\kappa}$, is presented in the tables together with the errors. Also included are plots of the reference solution $u_{40}$, and the error $u_{40} - u_N$.

In Experiment 1 the data is symmetric about $x = 1/2$. However the operator is not symmetric ($r = 0.2$), corresponding to a preferred diffusion toward $x = 1$ over diffusion toward $x = 0$. This is reflected in the solution being slightly skewed toward $x = 1$ (see Fig. 2). In Experiment 2 the larger value of $r$ ($r = 0.3$), together with a left-to-right drift (advection) term results in a solution highly skewed to the right (see Fig. 3). For Experiment 3, with the diffusion and drift parameters as used in Experiment 2, the source term is taken to be zero for $x \in (0, 1/2)$ and one for $x \in (1/2, 1)$. This data results again in a solution highly skewed to the right (see Fig. 4).
Typically when approximating a function which is itself, or its derivative, singular at a point $x_s$, the error in the approximation will be significantly larger in a neighborhood of $x_s$. In the approximation scheme studied herein the correct endpoint behavior of the solution is built into the approximation. Figures 2, 3 and 4 contain plots of the error for the approximations. In Experiments 1 and 2 the largest errors occur at the right hand endpoint, $x = 1$. Notable is that the errors in a neighborhood of $x = 1$ are the same order of magnitude as the errors across the interval. For Experiment 3 the largest errors occur in a neighborhood of the discontinuity in the source term, around $x = 1/2$.

Experiment 1. Fractional diffusion, reaction equation with $C^\infty(I)$ data For this experiment we use $\alpha = 1.60$, $r = 0.20$, $b(x) = 0$, $c(x) = 5$, and $f(x) = 1$. Theorem 2.2 states that even with $C^\infty(I)$ data the regularity of the solution is bounded. For this data $\beta = 0.93$, and $\tilde{s} = \min\{\infty, \alpha + (\alpha - \beta) + 1, \alpha + \beta + 1\} = 3.27$. Corollary 4.3 predicts that $\|u - u_N\|_{L^2_{\omega^{-1}}} \sim N^{-4.87}$ and $\|u - u_N\|_{H^{s/2}_{\omega^{-1}}(I)} \sim N^{-4.07}$. The numerical convergence rates for the errors are presented in Table 1, and are in good agreement with the predicted rates. A plot of the reference solution and plots of the errors are given in Fig. 2.

Experiment 2. Fractional diffusion, advection, reaction equation with $C^\infty(I)$ data For this experiment we use $\alpha = 1.40$, $r = 0.40$, $b(x) = e^x$, $c(x) = 5 + \sin(x)$, and $f(x) = 1$.  

Table 3 Experiment 3: $\alpha = 1.70$, $r = 0.30$, $b(x) = 2$, $c(x) = 5$

| $N$  | $\|u - u_N\|_{L^2_{\omega^{-1}}}$ $\kappa$ | $\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}}$ $\kappa$ |
|------|--------------------------------|----------------------------------|
| 12   | 3.71E−04  2.10                  | 4.27E−03  1.38                  |
| 14   | 2.69E−04  1.91                  | 3.45E−03  1.26                  |
| 16   | 2.08E−04  2.18                  | 2.92E−03  1.50                  |
| 18   | 1.61E−04  2.02                  | 2.44E−03  1.40                  |
| 20   | 1.30E−04  1.91                  | 2.11E−03  1.26                  |
| Pred.| 2.20       | 1.35                             |

Fig. 4 The plot of the reference solution $u_{40}(x)$ (left), and the plot of the errors for Experiment 3
As previously commented, even with $C^\infty(I)$ data the regularity of the solution is bounded. In addition, comparing Theorems 2.2 and 2.3, the presence of an advection term results in reduced regularity of the solution of the fractional diffusion, advection, reaction equation to that of the fractional diffusion, reaction equation. For this data $\beta = 0.93$, and $\bar{\gamma} = \min\{\infty, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1\} = 1.01$. Corollary 4.3 predicts that $\|u - u_N\|_{L^2_{\omega - 1}(I)} \sim N^{-2.41}$ and $\|u - u_N\|_{H^{\alpha/2}_{\omega - 1}(I)} \sim N^{-1.71}$. The numerical convergence rates for the errors are presented in Table 2, and are in good agreement with the predicted rates. A plot of the reference solution and plots of the errors are given in Fig. 3.

**Experiment 3. Fractional diffusion, advection, reaction equation with $f \in H^{1/2-\epsilon}_{w^*}(I)$**

For this experiment we use $\alpha = 1.70$, $r = 0.30$, $b(x) = 2$, $c(x) = 5$, and $f(x) = \{0, 0 < x \leq 1/2, 1, 1/2 < x < 1\}$. In this case the regularity of the solution is limited by the the regularity of $f$. For this data $\beta = 0.91$, and $\bar{\gamma} = \min\{1/2 - \epsilon, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1\} = 1/2 - \epsilon$. Corollary 4.3 predicts that $\|u - u_N\|_{L^2_{\omega - 1}(I)} \sim N^{-2.2}$ and $\|u - u_N\|_{H^{\alpha/2}_{\omega - 1}(I)} \sim N^{-1.35}$. The numerical convergence rates for the errors are presented in Table 3, and are in good agreement with the predicted rates. A plot of the reference solution and plots of the errors are given in Fig. 4.

**Acknowledgements** This work was partially funded by the ARO MURI Grant W911NF-15-1-0562, by the National Science Foundation under Grants DMS-1620194 and DMS-2012291, and by a SPARC Graduate Research Grant from the Office of the Vice President for Research at the University of South Carolina. All data generated or analyzed during this study are included in this published article.

**References**

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Volume 55 of National Bureau of Standards Applied Mathematics Series. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington (1964)

2. Acosta, G., Borthagaray, J.P., Bruno, O., Maas, M.: Regularity theory and high order numerical methods for the (1-d)-fractional Laplacian. Math. Comput. 87, 1821–1857 (2018)

3. Babuška, I., Guo, B.: Direct and inverse approximation theorems for the fractional Sobolev spaces. I. Approximability of functions in the weighted Sobolev spaces. SIAM J. Numer. Anal. 39(5):1512–1538 (2001/02)

4. Benson, D.A., Wheatcraft, S.W., Meerschaert, M.M.: The fractional-order governing equation of Lévy motion. Water Resour. Res. 36(6), 1413–1424 (2000)

5. Bernardi, C., Dauge, M., Maday, Y.: Polynomials in the Sobolev World. Preprint IRMAR 07-14, Université de Rennes 1 (2007)

6. Bernardi, C., Dauge, M., Maday, Y.: Polynomials in weighted Sobolev spaces: basics and trace liftings. Internal Report 92039, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris (1992)

7. Buades, A., Coll, B., Morel, J.M.: Image denoising methods. A new nonlocal principle. SIAM Rev. 52(1):113–147 (2010). Reprint of “A review of image denoising algorithms, with a new one” [MR2162865]

8. Chen, H., Wang, H.: Numerical simulation for conservative fractional diffusion equations by an expanded mixed formulation. J. Comput. Appl. Math. 296, 480–498 (2016)

9. Chen, S., Shen, J., Wang, L.-L.: Generalized Jacobi functions and their applications to fractional differential equations. Math. Comput. 85(300), 1603–1638 (2016)

10. Cui, M.: Compact finite difference method for the fractional diffusion equation. J. Comput. Phys. 228(20), 7792–7804 (2009)

11. Ern, A., Guermond, J.-L.: Theory and Practice of Finite Elements. Applied Mathematical Sciences, vol. 159. Springer, New York (2004)

12. Ervin, V.J.: Regularity of the solution to fractional diffusion, advection, reaction equations. arXiv:1911.03261 (2019)
13. Ervin, V.J., Heuer, N., Roop, J.P.: Regularity of the solution to 1-D fractional order diffusion equations. Math. Comput. 87, 2273–2294 (2018)
14. Ervin, V.J., Roop, J.P.: Variational formulation for the stationary fractional advection dispersion equation. Numer. Methods Partial Differ. Equ. 22(3), 558–576 (2006)
15. Gatto, P., Hesthaven, J.S.: Numerical approximation of the fractional Laplacian via hp-finite elements, with an application to image denoising. J. Sci. Comput. 65(1), 249–270 (2015)
16. Ginting, V., Li, Y.: On the fractional diffusion–advection–reaction equation in R. Fract. Calc. Appl. Anal. 22(4), 1039–1062 (2019)
17. Gui, W., Babinčka, I.: The h, p and h-p versions of the finite element method in 1 dimension. II. The error analysis of the h- and h-p versions. Numer. Math. 49(6), 613–657 (1986)
18. Guo, B.-Y., Wang, L.-L.: Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces. J. Approx. Theory 128(1), 1–41 (2004)
19. Hao, Z., Lin, G., Zhang, Z.: Error estimates of a spectral Petrov–Galerkin method for two-sided fractional reaction–diffusion–reaction equations. Appl. Math. Comput. 374, 125045 (2020)
20. Hao, Z., Zhang, Z.: Optimal regularity and error estimates of a spectral galerkin method for fractional advection–diffusion–reaction equations. SIAM J. Numer. Anal. 58(1), 211–233 (2020)
21. Jia, L., Chen, H., Ervin, V.J.: Existence and regularity of solutions to 1-D fractional order diffusion equations. Electron. J. Differ. Equ. 93, 1–21 (2019)
22. Jin, B., Lazarov, R., Lu, X., Zhou, Z.: A simple finite element method for boundary value problems with a Riemann–Liouville derivative. J. Comput. Appl. Math. 293, 94–111 (2016)
23. Jin, B., Lazarov, R., Pasciak, J., Rundell, W.: Variational formulation of problems involving fractional order differential operators. Math. Comput. 84(296), 2665–2700 (2015)
24. Jin, B., Lazarov, R., Zhou, Z.: A Petrov–Galerkin finite element method for fractional convection–diffusion equations. SIAM J. Numer. Anal. 54(1), 481–503 (2016)
25. Li, C., Zeng, F., Liu, F.: Spectral approximations to the fractional integral and derivative. Fract. Calc. Appl. Anal. 15(3), 383–406 (2012)
26. Li, Y., Chen, H., Wang, H.: A mixed-type Galerkin variational formulation and fast algorithms for variable-coefficient fractional diffusion equations. Math. Methods Appl. Sci. 40(14), 5018–5034 (2017)
27. Liu, F., Anh, V., Turner, I.: Numerical solution of the space fractional Fokker–Planck equation. In: Proceedings of the International Conference on Boundary and Interior Layers—Computational and Asymptotic Methods (BAIL 2011), vol. 166, pp. 209–219 (2004)
28. Liu, Q., Liu, F., Turner, I., Anh, V.: Finite element approximation for a modified anomalous subdiffusion equation. Appl. Math. Model. 35(8), 4103–4116 (2011)
29. Mainardi, F.: Fractional calculus: Some basic problems in continuum and statistical mechanics. In: Fractals and Fractional Calculus in Continuum Mechanics (Udine. 1996), Volume 378 of CISM Courses and Lectures, pp. 291–348. Springer, Vienna (1997)
30. Mao, Z., Chen, S., Shen, J.: Efficient and accurate spectral method using generalized Jacobi functions for solving Riesz fractional differential equations. Appl. Numer. Math. 106, 165–181 (2016)
31. Mao, Z., Em-Karniadakis, G.: A spectral method (of exponential convergence) for singular solutions of the diffusion equation with general two-sided fractional derivative. SIAM J. Numer. Anal. 56(1), 24–49 (2018)
32. Mao, Z., Shen, J.: Efficient spectral-Galerkin methods for fractional partial differential equations with variable coefficients. J. Comput. Phys. 307, 243–261 (2016)
33. Mao, Z., Shen, J.: Spectral element method with geometric mesh for two-sided fractional differential equations. Adv. Comput. Math. 44(3), 745–771 (2018)
34. Meerschaert, M.M., Tadjeran, C.: Finite difference approximations for fractional advection–dispersion flow equations. J. Comput. Appl. Math. 172(1), 65–77 (2004)
35. Shlesinger, M.F., West, B.J., Klafter, J.: Lévy dynamics of enhanced diffusion: application to turbulence. Phys. Rev. Lett. 58(11), 1100–1103 (1987)
36. Szegő, G.: Orthogonal polynomials. American Mathematical Society, Providence, R.I., 4th edition, 1975. American Mathematical Society, Colloquium Publications, vol. XXIII
37. Tadjeran, C., Meerschaert, M.M.: A second-order accurate numerical method for the two-dimensional fractional diffusion equation. J. Comput. Phys. 220(2), 813–823 (2007)
38. Wang, H., Basu, T.S.: A fast finite difference method for two-dimensional space-fractional diffusion equations. SIAM J. Sci. Comput. 34(5), A2444–A2458 (2012)
39. Wang, H., Yang, D.: Wellposedness of variable-coefficient conservative fractional elliptic differential equations. SIAM J. Numer. Anal. 51(2), 1088–1107 (2013)
40. Xu, Q., Hesthaven, J.S.: Discontinuous Galerkin method for fractional convection–diffusion equations. SIAM J. Numer. Anal. 52(1), 405–423 (2014)
41. Zaslavsky, G.M., Stevens, D., Weitzner, H.: Self-similar transport in incomplete chaos. Phys. Rev. E (3) 48(3), 1683–1694 (1993)
42. Zayernouri, M., Karniadakis, G.E.: Fractional Sturm–Liouville eigen-problems: theory and numerical approximation. J. Comput. Phys. 252, 495–517 (2013)
43. Zheng, X., Ervin, V.J., Wang, H.: Spectral approximation of a variable coefficient fractional diffusion equation in one space dimension. Appl. Math. Comput. 361, 98–111 (2019)
44. Zheng, X., Ervin, V.J., Wang, H.: An indirect finite element method for variable-coefficient space-fractional diffusion equations and its optimal-order error estimates. Commun. Appl. Math. Comput. 2(1), 147–162 (2020)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.