New Identities In Universal Osborn Loops *†

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Abstract

A question associated with the 2005 open problem of Michael Kinyon (Is every Osborn loop universal?), is answered. Two nice identities that characterize universal (left and right universal) Osborn loops are established. Numerous new identities are established for universal (left and right universal) Osborn loops like CC-loops, VD-loops and universal weak inverse property loops. Particularly, Moufang loops are discovered to obey the new identity \( y(x^{-1}u) \cdot u^{-1}(xu) = y(xu) \cdot u^{-1}(x^{-1}u) \) surprisingly. For the first time, new loop properties that are weaker forms of well known loop properties like inverse property, power associativity and diassociativity are introduced and studied in universal (left and right universal) Osborn loops. Some of them are found to be necessary and sufficient conditions for a universal Osborn to be 3 power associative. For instance, four of them are found to be new necessary and sufficient conditions for a CC-loop to be power associative. A conjugacy closed loop is shown to be diassociative if and only if it is power associative and has a weak form of diassociativity.

1 Introduction

The isotopic invariance of varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops have been of interest to researchers in loop theory in the recent past. These types of identities were first named by Fenyves [18] and [17] in the 1960s and later on in this 21st century by Phillips and Vojtěchovský [32], [33] and [25]. Among such are Etta Falconer [15] and [16] which investigated isotopy invariants in quasigroups. Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. For an overview of the theory of loops, readers may check [30], [8], [10], [12], [19], [34].

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Consider \((G, \cdot)\) and \((H, \circ)\) been two distinct groupoids (quasigroups, loops). Let \(A, B\) and \(C\) be three bijective mappings, that map \(G\) onto \(H\). The triple \(\alpha = (A, B, C)\) is called an \textit{isotopism} of \((G, \cdot)\) onto \((H, \circ)\) if and only if
\[
xA \circ yB = (x \cdot y)C \quad \forall x, y \in G.
\]
So, \((H, \circ)\) is called a groupoid (quasigroup, loop) \textit{isotope} of \((G, \cdot)\).

If \(C = I\) is the identity map on \(G\) so that \(H = G\), then the triple \(\alpha = (A, B, I)\) is called a \textit{principal isotopism} of \((G, \cdot)\) onto \((G, \circ)\) and \((G, \circ)\) is called a \textit{principal isotope} of \((G, \cdot)\).

Eventually, the equation of relationship now becomes
\[
x \cdot y = xA \circ yB \quad \forall x, y \in G
\]
which is easier to work with. But if \(A = R_g\) and \(B = L_f\) where \(R_x : G \to G\), the right \textit{translation} is defined by \(yR_x = y \cdot x\) and \(L_x : G \to G\), the left \textit{translation} is defined by \(yL_x = x \cdot y\) for all \(x, y \in G\), for some \(f, g \in G\), the relationship now becomes
\[
x \cdot y = xR_g \circ yL_f \quad \forall x, y \in G
\]
or
\[
x \circ y = xR_g^{-1} \cdot yL_f^{-1} \quad \forall x, y \in G.
\]
With this new form, the triple \(\alpha = (R_g, L_f, I)\) is called an \textit{f, g-principal isotopism} of \((G, \cdot)\) onto \((G, \circ)\), \(f\) and \(g\) are called \textit{translation elements} of \(G\) or at times written in the pair form \((g, f)\), while \((G, \circ)\) is called an \textit{f, g-principal isotope} of \((G, \cdot)\).

The last form of \(\alpha\) above given rises to an important result in the study of loop isotopes of loops.

\textbf{Theorem 1.1.} (Bruck [8])

Let \((G, \cdot)\) and \((H, \circ)\) be two distinct isotopic loops. For some \(f, g \in G\), there exists an \textit{f, g-principal isotope} \((G, \ast)\) of \((G, \cdot)\) such that \((H, \circ) \cong (G, \ast)\).

With this result, to investigate the isotopic invariance of an isomorphic invariant property in loops, one simply needs only to check if the property in consideration is true in all \(f, g\)-principal isotopes of the loop. A property is \textit{isotopic invariant} if whenever it holds in the domain loop i.e. \((G, \cdot)\) then it must hold in the co-domain loop i.e. \((H, \circ)\) which is an isotope of the formal. In such a situation, the property in consideration is said to be a \textit{universal property} hence the loop is called a \textit{universal loop} relative to the property in consideration as often used by Nagy and Strambach [28] in their algebraic and geometric study of the universality of some types of loops. For instance, if every isotope of a ”certain” loop is a ”certain” loop, then the formal is called a \textit{universal ”certain” loop}. So, we can now restate Theorem [1.1] as :

\textbf{Theorem 1.2.} Let \((G, \cdot)\) be a ”certain” loop where ”certain” is an isomorphic invariant property. \((G, \cdot)\) is a universal ”certain” loop if and only if every \(f, g\)-principal isotope \((G, \ast)\) of \((G, \cdot)\) has the ”certain” loop property. □
From the earlier discussions, if \((H, \circ) = (G, \cdot)\) then the triple \(\alpha = (A, B, C)\) is called an autotopism where \(A, B, C \in \text{SYM}(G, \cdot)\), the set of all bijections on \((G, \cdot)\) called the symmetric group of \((G, \cdot)\). Such triples form a group \(\text{AUT}(G, \cdot)\) called the autotopism group of \((G, \cdot)\).

Bol-Moufang type of quasigroups (loops) are not the only quasigroups (loops) that are isomorphic invariant and whose universality have been considered. Some others are weak inverse property loops (WIPLs) and cross inverse property loops (CIPLs). The universality of WIPLs and CIPLs have been addressed by Osborn [29] and Artzy [1] respectively. In 1970, Basarab [4] later continued the work of Osborn of 1961 on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs ([2]) in 1967. Osborn [29], while investigating the universality of WIPLs discovered that a universal WIPL \((G, \cdot)\) obeys the identity

\[yx \cdot (zE_y \cdot y) = (y \cdot xz) \cdot y \quad \forall \ x, y, z \in G\]  

where \(E_y = L_yL_y^\lambda = R_y^{-1}R_y^\rho = L_yR_yL_y^{-1}R_y^{-1}\) and \(y^\lambda\) and \(y^\rho\) are respectively the left and right inverse elements of \(y\).

Eight years after Osborn’s [29] 1960 work on WIPL, in 1968, Huthnance Jr. [20] studied the theory of generalized Moufang loops. He named a loop that obeys (1) a generalized Moufang loop and later on in the same thesis, he called them M-loops. On the other hand, he called a universal WIPL an Osborn loop and this same definition was adopted by Chiboka [11]. Basarab dubbed a loop \((G, \cdot)\) satisfying the identity:

\[x(yz \cdot x) = (x \cdot yE_x) \cdot zx \quad \forall \ x, y, z \in G\]

an Osborn loop where \(E_x = R_xR_x^{-1} = (L_xL_x^\lambda)^{-1} = R_yL_xL_x^{-1}R_y^{-1}\).

It will look confusing if both Basarab’s and Huthnance’s definitions of an Osborn loop are both adopted because an Osborn loop of Basarab is not necessarily a universal WIPL (Osborn loop of Huthnance). So in this work, Huthnance’s definition of an Osborn loop will be dropped while we shall stick to that of Basarab which was actually adopted by Kinyon [21] and the open problem we intend to solve is relative to Basarab’s definition of an Osborn loop and not that of Huthnance. Huthnance [20] was able to deduce some properties of \(E_x\) relative to (1). \(E_x = E_{x^\lambda} = E_{x^\rho}\). So, since \(E_x = R_xR_x^{-1}\), then \(E_x = E_{x^\lambda} = R_{x^\lambda}R_x\) and \(E_x = (L_xL_x^{-1})^{-1}\). So, we now have two identities equivalent to identities (1) and (2) defining an Osborn loop.

\[\text{OS}_0 : \ x(yz \cdot x) = x(yE_x \cdot x) \cdot zx\]  
\[\text{OS}_1 : \ x(yz \cdot x) = x(yx^\rho \cdot x) \cdot zx\]

Although Basarab [4] and [7] considered universal Osborn loops but the universality of Osborn loops was raised as an open problem by Michael Kinyon in 2005 at a conference tagged ”Milehigh Conference on Loops, Quasigroups and Non-associative Systems” held at
the University of Denver, where he presented a talk titled "A Survey of Osborn Loops". The present authors have been able to find a counter example to prove that not every Osborn loop is universal (in a different paper, submitted for publication) thereby putting the open problem to rest. Kinyon [21] further raised the question concerning the problem by asking if there exists a 'nice' identity that characterizes a universal Osborn loop.

In this study, a question associated with the 2005 open problem of Michael Kinyon (Is every Osborn loop universal?), is answered. Two nice identities that characterize universal (left and right universal) Osborn loops are established. Numerous new identities are established for universal Osborn loops like CC-loops, VD-loops and universal weak inverse property loops. Particularly, Moufang loops are discovered to obey the new identity \( y(x^{-1}u) \cdot u^{-1}(xu) = [y(xu) \cdot u^{-1}](x^{-1}u) \) surprisingly. For the first time, new loop properties that are weaker forms of well known loop properties like inverse property, power associativity and diassociativity are introduced and studied in universal (left and right universal) Osborn loops. Some of them are found to be necessary and sufficient conditions for a universal Osborn to be 3 power associative. For instance, four of them are found to be new necessary and sufficient conditions for a CC-loop to be power associative. A conjugacy closed loop is shown to be diassociative if and only if it is power associative and has a weak form of diassociativity.

2 Preliminaries

Let \( G \) be a non-empty set. Define a binary operation \((\cdot)\) on \( G \).

If \( x \cdot y \in G \) for all \( x, y \in G \), then the pair \((G, \cdot)\) is called a groupoid or Magma.

If each of the equations:

\[
a \cdot x = b \quad \text{and} \quad y \cdot a = b
\]

has unique solutions in \( G \) for \( x \) and \( y \) respectively, then \((G, \cdot)\) is called a quasigroup.

A quasigroup is therefore an algebra having a binary multiplication \( x \cdot y \) usually written \( xy \) which satisfies the conditions that for any \( a, b \) in the quasigroup the equations

\[
a \cdot x = b \quad \text{and} \quad y \cdot a = b
\]

have unique solutions for \( x \) and \( y \) lying in the quasigroup.

If there exists a unique element \( e \in G \) called the identity element such that for all \( x \in G \), \( x \cdot e = e \cdot x = x \), \((G, \cdot)\) is called a loop. We write \( xy \) instead of \( x \cdot y \), and stipulate that \( \cdot \) has lower priority than juxtaposition among factors to be multiplied. For instance, \( x \cdot yz \) stands for \( x(yz) \).

It can now be seen that a groupoid \((G, \cdot)\) is a quasigroup if it’s left and right translation mappings are bijections or permutations. Since the left and right translation mappings of a loop are bijective, then the inverse mappings \( L_x^{-1} \) and \( R_x^{-1} \) exist. Let

\[
x \backslash y = yL_x^{-1} = y\mathbb{L}_x \quad \text{and} \quad x/y = xR_y^{-1} = x\mathbb{R}_y
\]
and note that
\[ x \setminus y = z \iff x \cdot z = y \quad \text{and} \quad x/y = z \iff z \cdot y = x. \]
Hence, \((G, \setminus)\) and \((G, /)\) are also quasigroups. Using the operations \((\setminus)\) and \((/),\) the definition of a loop can be stated as follows.

**Definition 2.1.** A loop \((G, \cdot, /, \setminus, e)\) is a set \(G\) together with three binary operations \((\cdot),\) \((/),\) \((\setminus)\) and one nullary operation \(e\) such that

(i) \(x \cdot (x \setminus y) = y, (y/x) \cdot x = y\) for all \(x, y \in G,\)

(ii) \(x \setminus (x \cdot y) = y, (y \cdot x)/x = y\) for all \(x, y \in G\) and

(iii) \(x \setminus x = y/y\) or \(e \cdot x = x\) for all \(x, y \in G.\)

We also stipulate that \((/),\) \((\setminus)\) have higher priority than \((\cdot)\) among factors to be multiplied. For instance, \(x \cdot y/z\) and \(x \cdot y \setminus z\) stand for \(x(y/z)\) and \(x \cdot (y \setminus z)\) respectively.

In a loop \((G, \cdot)\) with identity element \(e,\) the left inverse element of \(x \in G\) is the element \(xJ_\lambda = x^\lambda \in G\) such that
\[ x^\lambda \cdot x = e \]
while the right inverse element of \(x \in G\) is the element \(xJ_\rho = x^\rho \in G\) such that
\[ x \cdot x^\rho = e. \]

**Definition 2.2.** A loop \((Q, \cdot)\) is called:

(a) a 3 power associative property loop (3-PAPL) if and only if \(xx \cdot x = x \cdot xx\) for all \(x \in Q.\)

(b) a left self inverse property loop (LSIPL) if and only if \(x^\lambda \cdot xx = x\) for all \(x \in Q.\)

(c) a right self inverse property loop (RSIPL) if and only if \(xx \cdot x^\rho = x\) for all \(x \in Q.\)

(d) a self automorphic inverse property loop (SFAIPL) if and only if \((xx)^\rho = x^\rho xx\) for all \(x \in Q.\)

(e) a self weak inverse property loop (SWIPL) if and only if \(x \cdot (xx)^\rho = x^\rho\) for all \(x \in Q.\)

(f) a left \(^1\)bi-self inverse property loop (\(L^1\)BSIPL) if and only if \(x^\lambda(xx \cdot x) = xx\) for all \(x \in Q.\)

(g) a left \(^2\)bi-self inverse property loop (\(L^2\)BSIPL) if and only if \(x^\lambda(x \cdot xx) = xx\) for all \(x \in Q.\)

**Definition 2.3.** Let \((Q, \cdot)\) be a loop and let \(w_1(q_1, q_2, \ldots, q_n)\) and \(w_2(q_1, q_2, \ldots, q_n)\) be words in terms of variables \(q_1, q_2, \ldots, q_n\) of the loop \(Q\) with equal lengths \(N (N \in \mathbb{N}, \ N > 1)\) such that the variables \(q_1, q_2, \ldots, q_n\) appear in them in equal number of times. \(Q\) is called a \(N^{m_1, m_2, \ldots, m_n}_{w_1(r_1, r_2, \ldots, r_n)=w_2(r_1, r_2, \ldots, r_n)}\) loop if it obeys the identity \(w_1(q_1, q_2, \ldots, q_n) = w_2(q_1, q_2, \ldots, q_n)\) where \(m_1, m_2, \ldots, m_n \in \mathbb{N}\) represent the number of times the variables \(q_1, q_2, \ldots, q_n \in Q\) respectively appear in the word \(w_1\) or \(w_2\) such that the mappings \(q_1 \mapsto r_1, q_1 \mapsto r_1, \ldots, q_n \mapsto r_n\) are assumed, \(r_1, r_2, \ldots, r_n \in \mathbb{N}.\)
In this study, we shall concentrate on when \( N = 4 \).

The identities describing the most popularly known varieties of Osborn loops are given below.

**Definition 2.4.** A loop \((Q, \cdot)\) is called:

(a) (Basarab \cite{7}) a VD-loop if and only if \( (\cdot)_x = (\cdot)^{L_x^{-1}R_x} \) and \( x(\cdot) = (\cdot)^{R_x^{-1}L_x} \) i.e. \( R_x^{-1}L_x \in PS_\lambda(Q, \cdot) \) with companion \( c = x \) and \( L_x^{-1}R_x \in PS_\rho(Q, \cdot) \) with companion \( c = x \) for all \( x \in Q \) where \( PS_\lambda(Q, \cdot) \) and \( PS_\rho(Q, \cdot) \) are respectively the left and right pseudo-automorphism groups of \( Q \).

(b) a Moufang loop if and only if the identity \( (xy) \cdot (zx) = (x \cdot yz)x \) holds in \( Q \).

(c) a conjugacy closed loop (CC-loop) if and only if the identities \( x \cdot yz = (xy)/x \cdot xz \) and \( zy \cdot x = zx \cdot x \setminus (yx) \) hold in \( Q \).

(d) a universal WIPL if and only if the identity \( x(yx)^{\rho} = y^{\rho} \) or \( (xy)^{\lambda}x = y^{\lambda} \) holds in \( Q \) and all its isotopes.

All these four varieties of Osborn loops are universal. CC-loops, and VD-loops are G-loops. G-loops are loops that are isomorphic to all their loop isotopes. Kunen \cite{27} has studied them.

**Definition 2.5.** Let the triple \( \alpha = (A, B, C) \) be an isotopism of the groupoid \((G, \cdot)\) onto a groupoid \((H, \circ)\).

(a) If \( \alpha = (A, B, B) \), then the triple is called a left isotopism and the groupoids are called left isotopes.

(b) If \( \alpha = (A, B, A) \), then the triple is called a right isotopism and the groupoids are called right isotopes.

(c) If \( \alpha = (A, I, I) \), then the triple is called a left principal isotopism and the groupoids are called left principal isotopes.

(d) If \( \alpha = (I, B, I) \), then the triple is called a right principal isotopism and the groupoids are called right principal isotopes.

A loop is a left (right) universal ”certain” loop if and only if all its left (right) isotopes are ”certain” loops.

**Theorem 2.1.** Let \((G, \cdot)\) and \((H, \circ)\) be two distinct left (right) isotopic loops with the former having an identity element \( e \). For some \( g \mid f \in G \), there exists an \( e, g \mid (f, e)\)-principal isotope \((G, *)\) of \((G, \cdot)\) such that \((H, \circ) \cong (G, *)\).

**Proof.** The proof of this is similar to that of Theorem III.2.1 of \cite{30}.

**Theorem 2.2.** Let \((G, \cdot)\) be a ”certain” loop where ”certain” is an isomorphic invariant property. \((G, \cdot)\) is a left (right) universal ”certain” loop if and only if every left (right) principal isotope \((G, *)\) of \((G, \cdot)\) has the ”certain” loop property.

**Proof.** Use Theorem 2.1.
3 Main Results

Theorem 3.1. A loop \((Q, \cdot, \backslash, /)\) is a universal Osborn loop if and only if it obeys the identity
\[
\text{os}_0
\]
\[
x \cdot u \{ (yz) / v \cdot [u \backslash (xv)] \} = (x \cdot u \{ [y(u / (u \backslash (xv))) / v] \cdot [u \backslash (xv)] \}) / v \\

\text{or}
\]
\[
x \cdot u \{ (yz) / v \cdot [u \backslash (xv)] \} = (x \cdot u \{ [y(u / (u \backslash (xv))) / v] \cdot [x \backslash (uv)] \}) / v \cdot u [[(uz) / v \cdot (u \backslash (xv))]]
\]

Proof. Let \(Q = (Q, \cdot, \backslash, /)\) be an Osborn loop with any arbitrary principal isotope \(Q = (Q, \triangle, \backslash, /)\) such that
\[
x \triangle y = xR^{-1}_{v} \cdot yL^{-1}_{u} = (x / v) \cdot (u \backslash y) \forall u, v \in Q.
\]

If \(Q\) is a universal Osborn loop then, \(Q\) is an Osborn loop. \(Q\) obeys identity \(\text{os}_0\) implies
\[
x \triangle [(y \triangle z) \triangle x] = \{ x \triangle [(y \triangle x') \triangle x] \} \triangle (z \triangle x)
\]
where \(x' = xJ_{x'}\) is the left inverse of \(x\) in \(Q\). The identity element of the loop \(Q\) is \(uv\). So,
\[
x \triangle y = xR^{-1}_{v} \cdot yL^{-1}_{u} \text{ implies } y' \triangle y = yR^{-1}_{v} \cdot yL^{-1}_{u} = uv \text{ implies }
\]
\[
y'R^{-1}_{v}R_{yL^{-1}_{u}} = uv \text{ implies } yJ_{v} = (uv)R^{-1}_{yL^{-1}_{u}}, R_{v} = (uv)R^{-1}_{(u \backslash y)}R_{v} = [(uv) / (u \backslash y)]v.
\]

Thus, using the fact that
\[
x \triangle y = (x / v) \cdot (u \backslash y),
\]
\(Q\) is an Osborn loop if and only if
\[
(x / v) \cdot u \{ [(y / v) \cdot (u \backslash z)] / v \cdot (u \backslash x) \} = ((x / v) \cdot u \{ [(y / v) \cdot (u \backslash z)] / v \cdot (u \backslash x) \}) / v \cdot u [(z / v) / (u \backslash x)].
\]

Do the following replacements:
\[
x' = x / v \Rightarrow x = x'v, \ z' = u \backslash z \Rightarrow z = uz', \ y' = y / v \Rightarrow y = y'v
\]
we have
\[
x' \cdot u \{ (y'z') / v \cdot [u \backslash (x'v)] \} = (x' \cdot u \{ [(y' / v) \cdot (u \backslash (x'v))] / v \cdot [u \backslash (x'v)] \}) / v \cdot u [(z' / v) / (u \backslash (x'v))].
\]

This is precisely identity \(\text{os}_0\) by replacing \(x', y'\) and \(z'\) by \(x, y\) and \(z\) respectively.

The proof of the converse is as follows. Let \(Q = (Q, \cdot, \backslash, /)\) be an Osborn loop that obeys identity \(\text{os}_0\). Doing the reverse process of the proof of the necessary part, it will be observed that equation \([5]\) is true for any arbitrary \(u, v\)-principal isotope \(Q = (Q, \Delta, \backslash, /)\) of \(Q\). So, every \(f, g\)-principal isotope \(Q\) of \(Q\) is an Osborn loop. Following Theorem \([1, 2]\), \(Q\) is a universal Osborn loop if and only if \(Q\) is an Osborn loop.

The proof for the second identity is done similarly by using identity \(\text{os}_1\). \(\square\)
Lemma 3.1. Let $Q$ be a loop with multiplication group $\text{Mult}(Q)$. $Q$ is a universal Osborn loop if and only if the triple $(\alpha(x, u, v), \beta(x, u, v), \gamma(x, u, v)) \in \text{AUT}(Q)$ or the triple $(R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}) = R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}})$ is obtained from identity OS$_0$ or OS$_1$ of Theorem 3.1.

Proof. This is obtained from identity OS$_0$ or OS$_1$ of Theorem 3.1.

Theorem 3.2. Let $Q$ be a loop with multiplication group $\text{Mult}(Q)$. If $Q$ is a universal Osborn loop, then the triple $(\gamma(x, u, v)R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}) = R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}})$ is obtained from identity OS$_0$ or OS$_1$ of Theorem 3.1.

Proof. Theorem 3.1 will be employed. Let $z = e$ in identity OS$_0$, then

$x \cdot u \{ y/v \cdot [u \setminus (xv)] \} = (x \cdot u \{ y(u \setminus [(uv)/v]u \}) \cdot v \cdot u \{ [(uv)/v]u \})$.

So, identity OS$_0$ can now be written as

$x \cdot u \{ y/z \cdot [u \setminus (xv)] \} = \{ x \cdot u \{ y[u \setminus [(uv)/v]u \}) \cdot v \cdot u \{ [(uv)/v]u \}) \} \cdot u \{ [(uv)/v]u \}$.

From where we obtain $(\gamma(x, u, v)R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}) = R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}})$ is obtained from identity OS$_0$ or OS$_1$ of Theorem 3.1.

Lemma 3.2. Let $(Q, \cdot, \setminus, /)$ be a universal Osborn loop. The following identities are satisfied:

$y \{ u \setminus [(uv)/v]u \} = (y \cdot [x(u)]u \cdot v$.

Furthermore, $\{ u \setminus [(uy \cdot u)(uu \cdot u)]u \} / u \cdot u^p = y$, $uu \setminus (uu \cdot u) = (u \cdot uu)u$,

$v^\lambda \cdot u \{ (v^\lambda \cdot u)(uv) \} = (v^\lambda \cdot u)(uv) \cdot v$, $v^\lambda \cdot v^\lambda = (v^\lambda)(v^\lambda) = v^\lambda \cdot v^p$,

are also satisfied.

Proof. To prove these identities, we shall make use of the three autotopisms in Lemma 3.1 and Theorem 3.2. In a quasigroup, any two components of an autotopism uniquely determine the third. So equating the first components of the three autotopisms, it is easy to see that

$\alpha(x, u, v) = \gamma(x, u, v)R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}) = R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}}R_{\text{ult}})$.

The establishment of the identities OS$_{0,1}$, OS$_{0,1,1}$ and OS$_{0,1,2}$ follows by using the bijections appropriately to map an arbitrary element $y \in Q$ as follows:
\[ \alpha(x, u, v) = R_{[v \backslash (xv)]} R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_u \gamma(x, u, v) \] implies that
\[ R_{(u \backslash ((u/v) \cup (xv)))} R_{[v \backslash (xu)]} L_x R_v = \]
\[ R_{(u \backslash ((u/v) \cup (xv)))} \gamma(x, u, v) R_v = R_{[u \backslash (xv)]} R_v R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_u \gamma(x, u, v) R_v \] which gives
\[ R_{(u \backslash ((u/v) \cup (xv)))} = R_{[u \backslash (xv)]} R_v R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_v. \]

So, for any \( y \in Q \),
\[ y R_{(u \backslash ((u/v) \cup (xv)))} = y R_{[u \backslash (xv)]} R_v R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_v \] implies that
\[ y \{ u \backslash ((u/v) \cup (xv)) \} \} = \{(y[u \backslash (xv)])/v \cdot [x \backslash (uv)] \}/[u \backslash (xv)] \cdot v. \]

OSI\(_{01}\) Consider
\[ \alpha(x, u, v) = \gamma(x, u, v) R_{(u \backslash ((u/v) \cup (xv)))}, \] then for all \( y \in Q \),
\[ y \alpha(x, u, v) = y R_{[u \backslash ((u/v) \cup (xv))] u} R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} L_x R_v = y \gamma(x, u, v) R_{(u \backslash ((u/v) \cup (xv)))} = \]
\[ y R_{[u \backslash (xv)]} R_v R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_v. \] Consequently,
\[ \{ x \cdot u \backslash (\{(y/u \backslash ((u/v) \cup (xv)))/v \cdot u \backslash (xv)\}) \}/v = \{ x \cdot u \backslash (y/v \cdot u \backslash (xv)) \}/(u \backslash (u/v) \cup (xv))). \]

Now replace \( y/v \) by \( y \) and post-multiply both sides by \( (u \backslash (u/v) \cup (xv))) \) to get
\[ \{ x \cdot u \backslash (\{(y/v) \cdot u \backslash ((u/v) \cup (xv)))/v \cdot u \backslash (xv)\}) \}/v = \{ (u \backslash (u/v) \cup (xv))) \} = x \cdot u \backslash (y \cdot u \backslash (xv)). \]

Again, let \( z = u \backslash (xv) \) which implies that \( x = (uz)/v \) and hence we now have
\[ \{(uz)/v \cdot u \backslash (\{(y/v) \cdot u \backslash (z)/v)) \}/v \cdot (u \backslash (u/v) z) = (uz)/v \cdot u \backslash (yz). \]

OSI\(_{01.2}\) Consider
\[ R_{[u \backslash (xv)]} R_v R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_v \gamma(x, u, v) R_v = \gamma(x, u, v) R_{(u \backslash ((u/v) \cup (xv)))} \] then for all \( y \in Q \),
\[ y R_{[u \backslash (xv)]} R_v R_{[x \backslash (uv)]} R_{[u \backslash (xv)]} R_v \gamma(x, u, v) R_v = y \gamma(x, u, v) R_{(u \backslash ((u/v) \cup (xv)))} \] results in
\[ \{(u[y \backslash (xv)])/v \cdot [x \backslash (uv)] \}/[u \backslash (xv)] \cdot v \} \gamma(x, u, v) /v = (y \gamma(x, u, v) /u \backslash (u/v) \cup (xv))) \] which is equivalent to the equation below after substituting the value of \( \gamma(x, u, v) \) and post-multiply both sides by \( v \):
\[ x \cdot u \backslash (\{(y/v \cdot u \backslash (xv)) /v \cdot [u \backslash (xv)] \}) = \]
\[ [x \cdot u \backslash (y/v \cdot u \backslash (xv))]/(u \backslash (u/v) \cup (xv))) \cdot v. \]

Do the replacement \( z = u \backslash (xv) \) \( \Rightarrow \) \( x = (uz)/v \) to get
\[ (uz)/v \cdot u \backslash (y/z)/v \cdot (uz)/v \backslash (uv)) = [(uz)/v \cdot u \backslash (y/v \cdot z)]/(u \backslash (u/v) z) \cdot v. \]

Now, replace \( y \) by \( yv \) to get
\[ (uz)/v \cdot u \backslash (yv \cdot z)/v \cdot (uz)/v \backslash (uv)) = [(uz)/v \cdot u \backslash (yz)]/(u \backslash (u/v) z) \cdot v. \]
Identity OSI$_{01,1,1}$ is deduced from identity OSI$_{01,1}$ while identities OSI$_{01,2,1}$ and OSI$_{01,2,2}$ are deduced from identity OSI$_{01,2}$. The other identities are gotten from OSI$_{01,1,1}$ and OSI$_{01,2,2}$.

**OSI$_{01,1,1}$** Put $u = v$ in identity OSI$_{01,1}$ to get
\[
\{(uz)/u \cdot u\((\{yu\}(u\((uu)/z\)u))\}/u \cdot z\} = (uz)/u \cdot u\((yz)\).
\]
Now replace $z$ by $uz$ to get
\[
\{(u \cdot uz)/u \cdot u\((\{yu\}(u\((uu)/uz\)u))\}/u \cdot uz\} = (u \cdot uz)/u \cdot u\((y \cdot uz)\).
\]
Then, substitute $z = w^p$ and compute to have
\[
\{u\((\{yu\}(uu \cdot u))\}/u\} \cdot w^p = u\((y)\).
\]
Replacing $y$ by $uy$, finally have
\[
\{u\((\{uy \cdot u\}(uu \cdot u))\}/u\} \cdot w^p = y.
\]

**OSI$_{01,2,1}$** Substitute $z = w^p$ in identity OSI$_{01,2}$ to get
\[
v^\Lambda \cdot u\[(yu \cdot w^p)/v \cdot [v^\Lambda\((uv)\)] = [v^\Lambda \cdot u\((yuw^p)/(u\((u/v)w^p))\) \cdot v.
\]

**OSI$_{01,2,2}$** Put $u = e$ in identity OSI$_{01,2,1}$ to get
\[
v^\Lambda(y \cdot v^\Lambda\(v) = (v^\Lambda y)/v^\Lambda \cdot v.
\]
By putting $y = e$ in identity OSI$_{01,1,1}$, we have $uu \cdot u\((uu \cdot u) = (u \cdot uu)u$. Also, substitute $y = v$ into identity OSI$_{01,2,2}$ and use the fact that $x^\Lambda x = x^\Lambda xx$ to get $v^\Lambda \cdot (v \cdot v^\Lambda\(v) = v^\Lambda \cdot v = (v^\Lambda \cdot vv)v$ and $v(v^p \cdot v\((v^p) = v^\Lambda \cdot v^p$.

**Lemma 3.3.** A universal Osborn loop is a 3-PAPL if and only if it is a $4^1_{11,11-111}$ and a $4^1_{11,11-11\cdot1}$ loop.

**Proof.** In Lemma 3.2 it was shown that $uu \cdot u\((uu \cdot u) = (u \cdot uu)u$ in a universal Osborn loop. The necessary and sufficient parts are easy to prove using this identity. □

**Lemma 3.4.** In a universal Osborn loop $Q$, the following are equivalent.

1. $Q$ is a 3-PAPL.
2. $Q$ is a $4^1_{11,11-111}$ loop and a $4^1_{11,11-11\cdot1}$ loop.
3. $Q$ is a LSIPL.
4. $Q$ obeys the identity $v[v^\Lambda \cdot (v \cdot v^\Lambda\(v)] = v^\Lambda\(v \cdot v.$
5. $Q$ is a $4^1_{12,22-122}$ loop.
6. $Q$ is a $4_{11-11}^{1}$ loop.

Proof. This is established by using the identities $uu \cdot u \setminus (uu \cdot u) = (u \cdot uu)u$ and $v^\lambda \cdot (v \cdot v^\lambda \setminus v) = (v^\lambda \cdot vv)v$ of Lemma 3.2, Lemma 3.7, Lemma 3.9, Lemma 3.10.

Corollary 3.1. In a universal Osborn loop, the $4_{11-11}^{1}$ and $4_{11-11}^{1}$ loop properties are equivalent.

Proof. This follows from Lemma 3.4.

Corollary 3.2. A universal Osborn loop that is a LSIPL or RSIPL or 3-PAPL or $4_{12-22}^{1,3}$ or $4_{11-11}^{1}$ loop is a $L^2$ BSIPL and a $L^1$ BSIPL.

Proof. This is established by using Corollary 3.4, Lemma 3.8, and Lemma 3.9.

Theorem 3.3. A loop $(Q, \cdot, \setminus, /)$ is a left universal Osborn loop if and only if it obeys the identity

$$x \cdot [(y \cdot zv) / v \cdot (xv)] = (x \cdot \{[y([v/(xv)]v)] / v \cdot (xv)]\} / v \cdot [z \cdot xv]$$

or

$$x \cdot [(y \cdot zv) / v \cdot (xv)] = x \cdot [(y \cdot xv) / v \cdot (x \setminus v)] / v \cdot [z(xv)].$$

Proof. The procedure of the proof of this theorem is similar to the procedure used to prove Theorem 3.1 by just using the arbitrary left principal isotope $\Omega = (Q, \Delta, \setminus, /)$ such that

$$x \cdot y = xR_0^{-1} \cdot y = (x / v) \cdot y \forall v \in Q.$$

Lemma 3.5. Let $Q$ be a loop with multiplication group $\text{Mult}(Q)$. $Q$ is a left universal Osborn loop if and only if the triple $(\alpha(x, v), \beta(x, v), \gamma(x, v)) \in \text{AUT}(Q)$ or $(R_{(xv)}R_{(xv)}\beta(x, v)) / \beta(x, v) = R_{(xv)}L_x \beta(x, v) = R_{(xv)R_{(xv)}}L_x$ and $\gamma(x, v) = R_{(xv)}L_x$ are elements of $\text{Mult}(Q)$.

Proof. This is obtained from identity $O_0^\lambda$ or $O_1^\lambda$ of Theorem 3.3.

Theorem 3.4. Let $Q$ be a loop with multiplication group $\text{Mult}(Q)$. If $Q$ is a left universal Osborn loop, then the triple $(\gamma(x, v)R_{(xv)}, \beta(x, v), \gamma(x, v)) \in \text{AUT}(Q)$ for all $x, v \in Q$ where $\beta(x, v) = R_{(xv)}L_x$ and $\gamma(x, v) = R_{(xv)}L_x$ are elements of $\text{Mult}(Q)$.

Proof. This follows by using identity $O_0^\lambda$ or $O_1^\lambda$ of Theorem 3.3 the way identity $O_0^\lambda$ or $O_1^\lambda$ of Theorem 3.1 was used to prove Theorem 3.2.
Lemma 3.6. Let \((Q, \cdot, \backslash, /)\) be a left universal Osborn loop. The following identities are satisfied:

\[
y\{ [v/(xv)]v \} = \{ [y(xv)]v / (xv) \cdot v, \quad \text{and} \quad \}
\]

\[
y\{ (yv / zv) / v \cdot zv \} = \{ [y(\cdot zv)] / (v^\lambda \cdot zv) \cdot v \}
\]

Furthermore, \(v^\lambda \{ [(vv)/(vv)] / v \cdot v^\lambda \} = v^\lambda y, \quad \{ [v(v/vzv)] / v \cdot v^\lambda (zv) = z \cdot zv, \quad \}

\[
v\{ (yy \cdot vv) / v \} = [v(y \cdot vv)] / (v^\lambda \cdot vv) \cdot v, \quad \{ v[v(v/vzv)] / (v \cdot vv) / (v^\lambda \cdot vv) \cdot v, \}
\]

\[
v\{ [(vv)/(vv)] / v \cdot v^\lambda \} = [v(v/vzv)] / (v^\lambda \cdot vv) \cdot v, \quad \}
\]

\[
v^\lambda [y \cdot v^\lambda / v] = (v^\lambda y) / (v^\lambda \cdot v), \quad v \cdot v = v^\lambda \cdot v \text{ and } vv \cdot vv = v^\lambda \cdot v 
\]

are also satisfied.

Proof. To prove these identities, we shall make use of the three automorphisms in Lemma 3.5 and Theorem 3.4. In a quasigroup, any two components of an automorphism uniquely determine the third. So equating the first components of the three automorphisms, it is easy to see that

\[
\alpha(x, v) = \gamma(x, v)\mathbb{R}_{[v^\lambda, xv]} = R_{[xv]}\mathbb{R}_v R_{[x \backslash v]}\mathbb{R}_{[xv]} R_v \gamma(x, v)\mathbb{R}_v.
\]

The establishment of the identities \(\text{OSI}^\lambda_{01}, \text{OSI}^\lambda_{01.1}\) and \(\text{OSI}^\lambda_{01.2}\) follows by using the bijections appropriately to map an arbitrary element \(y \in Q\) as follows:

\(\text{OSI}^\lambda_{01}\)

\[
\alpha(x, v) = R_{[xv]}\mathbb{R}_v R_{[x \backslash v]}\mathbb{R}_{[xv]} R_v \gamma(x, v)\mathbb{R}_v \text{ implies that }
\]

\[
R_{[[v/(xv)]v]v} R_{[xv]} L_x \mathbb{R}_v = R_{[[v/(xv)]v]v} \gamma(x, v)\mathbb{R}_v = R_{[xv]} \mathbb{R}_v R_{[x \backslash v]} \mathbb{R}_{[xv]} R_v \gamma(x, v)\mathbb{R}_v
\]

which gives \(R_{[[v/(xv)]v]v} = R_{[xv]} \mathbb{R}_v R_{[x \backslash v]} \mathbb{R}_{[xv]} R_v\).

So, for any \(y \in Q\), \(y R_{[[y/(yxv)]v]v} = y R_{[xv]} \mathbb{R}_v R_{[x \backslash v]} \mathbb{R}_{[xv]} R_v\) implies that

\[
y\{ [v/(xv)]v \} = \{ [y(xv)] / v \cdot (x \backslash v) \} / (xv) \cdot v
\]

\(\text{OSI}^\lambda_{01.1}\) Consider

\[
\alpha(x, v) = \gamma(x, v)\mathbb{R}_{[v^\lambda, xv]}, \text{ then for all } y \in Q,
\]

\[
y \alpha(x, v) = y R_{[[y/(xv)]v]v} R_{[xv]} L_x \mathbb{R}_v = y \gamma(x, v)\mathbb{R}_{[v^\lambda, xv]} = y \mathbb{R}_v R_{[xv]} L_x \mathbb{R}_{[v^\lambda, xv]}.
\]

Consequently, \(\{ x \cdot \{ y [(v/(xv)]v) / v \cdot x \} \} / / v = \{ x \cdot (y / v \cdot xv) \} / [v^\lambda \cdot xv].\)

Now replace \(y / v\) by \(y\) and post-multiply both sides by \([v^\lambda \cdot xv]\) to get

\[
\{ x \cdot \{ y [(v/(xv)]v) / v \cdot x \} \} / v \cdot [v^\lambda \cdot xv] = \{ x \cdot (y \cdot xv) \}.
\]
\( \text{OSI}_{01.2} \) Consider

\[
R_{[x\backslash v]}R_{[x\backslash v]}y \gamma(x, v)R_v = \gamma(x, v)R_{[v \cdot x \backslash v]}, \quad \text{then for all } y \in Q,
\]

\[
yR_{[x\backslash v]}R_{[x\backslash v]}R_v y \gamma(x, v)R_v = y \gamma(x, v)R_{[v \cdot x \backslash v]} \quad \text{results in}
\]

\[
(\left\{ [[y(xv)] / v \cdot (xv)] / (xv) \cdot v \right\} \gamma(x, v) / v = \left( y \gamma(x, v) / v \right) / [v^\lambda \cdot xv]
\]

which is equivalent to the equation below after substituting the value of \( \gamma(x, v) \) and post-multiply both sides by \( v \):

\[
x\{ [[y(xv)] / v \cdot (xv)] / (xv) \cdot v \} = (x \cdot [y / v \cdot (xv)]) / [v^\lambda \cdot xv] \cdot v.
\]

Now, replace \( y \) by \( yv \) to get \( x\{ [[y(xv)] / v \cdot (xv)] / (xv) \cdot v \} = (x[y \cdot (xv)]) / [v^\lambda \cdot xv] \cdot v. \)

Identities \( \text{OSI}_{01.1.1}^{\lambda} \) and \( \text{OSI}_{01.1.2}^{\lambda} \) are deduced from identity \( \text{OSI}_{01.1}^{\lambda} \). Identities \( \text{OSI}_{01.2.1}^{\lambda} \) and \( \text{OSI}_{01.2.4}^{\lambda} \) are deduced from identity \( \text{OSI}_{01.2}^{\lambda} \) while identities \( \text{OSI}_{01.2.2}^{\lambda} \) and \( \text{OSI}_{01.2.3}^{\lambda} \) are deduced from identity \( \text{OSI}_{01.2.1}^{\lambda} \). The other identities are gotten from \( \text{OSI}_{01.1.1}^{\lambda} \).

\( \text{OSI}_{01.1.1}^{\lambda} \) Simply put \( z = v^\lambda \) in identity \( \text{OSI}_{01.1}^{\lambda} \) to get identity \( \text{OSI}_{01.1.1}^{\lambda} \).

\( \text{OSI}_{01.1.2}^{\lambda} \) Simply put \( y = e \) in identity \( \text{OSI}_{01.1}^{\lambda} \) to get identity \( \text{OSI}_{01.1.2}^{\lambda} \).

\( \text{OSI}_{01.2.1}^{\lambda} \) Simply put \( z = v \) in identity \( \text{OSI}_{01.2}^{\lambda} \) to get identity \( \text{OSI}_{01.2.1}^{\lambda} \).

\( \text{OSI}_{01.2.2}^{\lambda} \) Substitute \( y = e \) in identity \( \text{OSI}_{01.2.1}^{\lambda} \) to get identity \( \text{OSI}_{01.2.2}^{\lambda} \).

\( \text{OSI}_{01.2.3}^{\lambda} \) Substitute \( y = v \) in identity \( \text{OSI}_{01.2.1}^{\lambda} \) to get identity \( \text{OSI}_{01.2.3}^{\lambda} \).

\( \text{OSI}_{01.2.4}^{\lambda} \) Simply put \( z = v^\lambda \) in identity \( \text{OSI}_{01.2}^{\lambda} \) to get identity \( \text{OSI}_{01.2.4}^{\lambda} \).

By putting \( y = e \) in identity \( \text{OSI}_{01.1.1}^{\lambda} \), we have \( \{v^\lambda / [v(vv)] / v\} / v \cdot v^\lambda = v^\lambda \) which implies \( v^\lambda / [v(vv)] / v = v, \) hence, \( v(vv) = (v^\lambda \cdot v) \cdot v. \)

Again, by putting \( y = v \) in identity \( \text{OSI}_{01.1.1}^{\lambda} \), we have \( \{v^\lambda / [(vv)(vv)] / v\} / v \cdot v^\lambda = e \) which implies \( v^\lambda / [(vv)(vv)] / v = v^\lambda, \) hence, \( vv \cdot vv = v^\lambda \cdot (v^\lambda v) \cdot v. \)

**Lemma 3.7.** A left universal Osborn loop is a LSIPL if and only if it is a 3 PAPL.

**Proof.** This is proved by using the identity \( v \cdot vv = v^\lambda \cdot v \cdot v \) in Lemma 3.6.

---

**Lemma 3.8.** A left universal Osborn loop \((Q, \cdot, \backslash, /)\) is a \( 4_{11-11=(11-1)1}^1 \) loop if and only if it obeys the identity \( v^\lambda (vv \cdot v) = v^\lambda v. \)

**Proof.** This is proved by using the identity \( vv \cdot vv = v^\lambda (v^\lambda v) \cdot v \) in Lemma 3.6.

---

**Corollary 3.3.** A left universal Osborn loop \((Q, \cdot, \backslash, /)\) that is a \( 4_{11-11=(11-1)1}^1 \) loop is a \( L^1 \) BSIPL if and only if it is a LSIPL. Hence, it is a \( L^2 \) BSIPL.

**Proof.** This follows from Lemma 3.8 by using the fact that in an Osborn loop, \( x^\lambda = x \mapsto x^\lambda \cdot xx. \)
Lemma 3.9. A left universal Osborn loop is a LSIPL if and only if it is a $4^{1,3}_{12:22=(1-22)2}$ loop.

Proof. This is proved by using the identity OSI$_{01,2.1}^3$ of Lemma 3.6.

Lemma 3.10. A left universal Osborn loop is a LSIPL if and only if it is a $4^1_{11:11=(1-11)1}$ loop.

Proof. This is proved by using the identity OSI$_{01,2.3}^1$ of Lemma 3.6.

Lemma 3.11. Let $G$ be a left universal Osborn loop. The following are equivalent.

1. $G$ is a LSIPL and a $4^{1,3}_{12:22=(1-22)2}$ loop.
2. $G$ is a left alternative property loop.
3. $G$ is a Moufang loop.

Proof. This is proved by using the identity OSI$_{01,2.1}^3$ of Lemma 3.6.

Lemma 3.12. A left universal Osborn loop $(Q, \cdot, \backslash, /)$ that is a $4^{2,2}_{12:12=(12-12)2}$ or $4^{2,2}_{12:12=(1-21)2}$ loop obeys the identity $[y(yy \cdot y^\rho)]y = y \cdot yy$.

Proof. This is proved by using the identity OSI$_{01,2.2}^2$ of Lemma 3.6.

Lemma 3.13. In an Osborn loop, the following properties are equivalent. LSIP, RSIP, $|J_\lambda| = 2$, $|J_\rho| = 2$ and $J_\rho = J_\lambda$.

Proof. This can be proved by using the facts that in an Osborn loop, $J^2_\rho : x \mapsto xx \cdot x^\rho$ and $J^2_\lambda : x \mapsto x^\lambda \cdot xx$.

Corollary 3.4. In a left universal Osborn loop, the following properties are equivalent. LSIP, RSIP, 3-PAP, $J_\rho = J_\lambda$, $4^{1,3}_{12:22=(1-22)2}$ and $4^1_{11:11=(1-11)1}$ properties.

Proof. Use Lemma 3.13, Lemma 3.7, Lemma 3.9 and Lemma 3.10.

Corollary 3.5. Let $L$ be a CC-loop. The following are equivalent.

1. $L$ is a power associativity loop.
2. $L$ is a 3-PAPL.
3. $L$ obeys $x^\rho = x^\lambda$ for all $x \in L$.
4. $L$ is a LSIPL.
5. $L$ is a RSIPL.
6. $L$ is a $4^{1,3}_{12:22=(1-22)2}$ loop.
7. $L$ is a $4^1_{11:11=(1-11)1}$ loop.
Proof. The proof of the equivalence of the first three is shown in Lemma 3.20 of [26] and mentioned in Lemma 1.2 of [31]. The proof of the equivalence of the last four and the first three can be deduced from the last result of Corollary 3.4.

Corollary 3.6. A CC-loop is a diassociative loop if and only if it is a power associative loop and a $4^{1\cdot 3}_{12\cdot 22} = (12\cdot 2)^2$ loop.

Proof. The proof of this follows from Corollary 3.5 and Lemma 3.11.

Theorem 3.5. A loop $(Q, \cdot, \\backslash, /)$ is a right universal Osborn loop if and only if it obeys the identity
\[
(ux) \cdot \backslash (yz) \cdot (ux) = ((ux) \cdot \backslash (y) \cdot (ux)) \cdot \backslash (xy) \quad \text{or} \quad \rho_0^{OS}(ux) \cdot \backslash (yz) \cdot (ux) = ((ux) \cdot \backslash (y) \cdot (ux)) \cdot \backslash (xy).
\]

Proof. The procedure of the proof of this theorem is similar to the procedure used to prove Theorem 3.1 by just using the arbitrary right principal isotope $Q = (Q, \triangledown, \check{\triangledown}, \check{\triangledown})$ such that
\[
x \triangledown y = x \cdot y L_u^{-1} = x \cdot (u \backslash y) \quad \forall u \in Q.
\]

Lemma 3.14. Let $Q$ be a loop with multiplication group $\text{Mult}(Q)$. $Q$ is a right universal Osborn loop if and only if the triple $(\alpha(x, u), \beta(x, u), \gamma(x, u)) \in \text{AUT}(Q)$ or the triple $(\rho_0^{OS}(ux) \cdot \backslash (y) \cdot (ux)), \beta(x, u), \gamma(x, u)) \in \text{AUT}(Q)$ for all $x, u \in Q$ where $\alpha(x, u) = R_{[u\backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \gamma(x, u), \beta(x, u) = L_u R_{[u \backslash x]} L_x, \beta(x, u) = R_{[u \backslash x]} L_u L_x$ are elements of $\text{Mult}(Q)$.

Proof. This is obtained by using identity $\rho_0^{OS}$ or $\rho_1^{OS}$ of Theorem 3.5.

Theorem 3.6. Let $Q$ be a loop with multiplication group $\text{Mult}(Q)$. If $Q$ is a right universal Osborn loop, then the triple $(\gamma(x, u) \mathbb{R}_{(u \backslash x)}, \beta(x, u), \gamma(x, u)) \in \text{AUT}(Q)$ for all $x, u \in Q$ where $\beta(x, u) = L_u R_{[u \backslash x]} L_u L_x$ and $\gamma(x, u) = R_{[u \backslash x]} L_u L_x$ are elements of $\text{Mult}(Q)$.

Proof. This follows by using identity $\rho_0^{OS}$ or $\rho_1^{OS}$ in Theorem 3.5 the way identity $\rho_0^{OS}$ or $\rho_1^{OS}$ was used in Theorem 3.1.
Lemma 3.15. Let \((Q, \cdot, \setminus, /)\) be a right universal Osborn loop. The following identities are satisfied:

\[
y\{u\setminus(u/x)\} = \{y\setminus[(ux)\setminus u]\}/x, \quad \{u\setminus u/[(yz)[(uz)\setminus u]]\}z = (uz)\cdot u\setminus(yz) \quad \text{and} \\
\{uz\cdot u\setminus(yu/z)\}z = (uz)\cdot u\setminus(yz).
\]

Furthermore, \((uz)\cdot u\setminus\{z^\lambda(u\setminus(u/z))\}z = (uz)\cdot u^\rho\), \((uu)\cdot u\setminus(u^\lambda u^\rho\cdot u)u = uu\cdot u^\rho)\), \((uu)\cdot u\setminus(u^\lambda(u\setminus u^\rho))\}\cdot u^\rho\}z = u\setminus(u^\rho u^\rho), \text{ and} \)

\[
\{uz\cdot u\setminus\{z^\lambda(u\setminus(u/z))\}z = (uz)\cdot u\setminus(z^\rho z)\}z = (zz)\cdot z\setminus(z^\rho z).
\]

are also satisfied.

Proof. To prove these identities, we shall make use of the three autotopisms in Lemma 3.14 and Theorem 3.6. In a quasigroup, any two components of an autotopism uniquely determine the third. So equating the first components of the three autotopisms, it is easy to see that

\[
\alpha(x, u) = \gamma(x, u)R(x/\setminus x) = R_{[u\setminus x]}R_{[x\setminus u]}R_{[u\setminus x]}\gamma(x, u).
\]

The establishment of the identities \(\text{OSI}^p_{01}, \text{OSI}^p_{01.1} \) and \(\text{OSI}^p_{01.2}\) follows by using the bijections to map an arbitrary element \(y \in Q\) as follows:
\( \alpha(x, u) = R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \gamma(x, u) \) implies that
\[
R_{[u \backslash (u/(u \backslash x))]} R_{[u \backslash x]} \mathbb{L}_{u} L_{x} = R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \gamma(x, u) = R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} R_{[u \backslash x]} \mathbb{L}_{u} L_{x}
\]
which gives \( R_{[u \backslash (u/(u \backslash x))]} = R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \mathbb{R}_{[u \backslash x]} \). So, for any \( y \in Q \),
\[
y R_{[u \backslash (u/(u \backslash x))]} = y R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \gamma(x, u) = \{ [(y z)(\{(u z)\})]\} / z.
\]
Let \( z = u \backslash x \) so that \( x = u z \). Thus, \( y R_{[u \backslash (u/(u \backslash x))]} = \{(y u \backslash x)\}[x \backslash u]/[u \backslash x] \).

\( \text{OSI}_{01} \)

Consider
\[
\alpha(x, u) = \gamma(x, u) \mathbb{R}_{[u \backslash x]}, \text{ then for all } y \in Q,
\]
\[
y \alpha(x, u) = y R_{[u \backslash (u/(u \backslash x))]} R_{[u \backslash x]} \mathbb{L}_{u} L_{x} = y \gamma(x, u) \mathbb{R}_{[u \backslash x]} R_{[u \backslash x]} \mathbb{L}_{u} L_{x} \mathbb{R}_{[u \backslash x]}.
\]
Consequently, \( x \cdot u \backslash \{(y u \backslash x)\}[u \backslash x]\} (u \backslash x) = x \cdot u \backslash y R_{[u \backslash x]} \).

Post-multiply both sides by \( (u \backslash x) \) to get
\[
\{x \cdot u \backslash \{(y u \backslash x)\}[u \backslash x]\} (u \backslash x) = x \cdot u \backslash y R_{[u \backslash x]} \).
\]
Again, let \( z = u \backslash x \) which implies that \( x = u z \) and hence we now have
\[
\{(u z) \cdot u \backslash \{(y u \backslash z)\} z\} z = (u z) \cdot u \backslash (y z).
\]

\( \text{OSI}_{01.2} \)

Consider
\[
R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \gamma(x, u) = \gamma(x, u) \mathbb{R}_{[u \backslash x]}, \text{ then for all } y \in Q,
\]
\[
y R_{[u \backslash x]} R_{[x \backslash u]} \mathbb{R}_{[u \backslash x]} \gamma(x, u) = y \gamma(x, u) \mathbb{R}_{[u \backslash x]} \mathbb{R}_{[u \backslash x]} \gamma(x, u) \text{ results in}
\]
\[
\{(y u \backslash x)\} (x \backslash u) \mathbb{L}_{u} L_{x} = [y \gamma(x, u)] / (u \backslash x)
\]
which is equivalent to the equation below after substituting the value of \( \gamma(x, u) \) and post multiplying by \( (u \backslash x) \):
\[
x \cdot u \backslash \{(y u \backslash x)\}[u \backslash x]\} (u \backslash x) = x \cdot u \backslash y R_{[u \backslash x]} \).
\]
Do the replacement \( z = u \backslash x \Rightarrow x = u z \) to get \( \{(u z) \cdot u \backslash \{(y z)\} (u z)\} z = (u z) \cdot u \backslash (y z) \).

Identities \( \text{OSI}_{01.1.1} \), \( \text{OSI}_{01.1.2} \) and \( \text{OSI}_{01.1.5} \) are deduced from identity \( \text{OSI}_{01.1} \). \( \text{OSI}_{01.1.2} \) is deduced from \( \text{OSI}_{01.1.1} \), \( \text{OSI}_{01.1.4} \) is deduced from \( \text{OSI}_{01.1.3} \) while identities \( \text{OSI}_{01.1.6} \) and \( \text{OSI}_{01.1.7} \) are deduced from \( \text{OSI}_{01.1.5} \) by doing the following:

\( \text{OSI}_{01.1.1} \)

Put \( y = z^\lambda \) in identity \( \text{OSI}_{01.1} \).

\( \text{OSI}_{01.1.2} \)

Substitute \( z = u \) in identity \( \text{OSI}_{01.1.1} \).

\( \text{OSI}_{01.1.3} \)

Put \( y = z \) in identity \( \text{OSI}_{01.1.2} \).

\( \text{OSI}_{01.1.4} \)

Put \( z = u^\rho \) in identity \( \text{OSI}_{01.1.3} \).
OSI$^\rho_{01.1.5}$ Put $y = z^\rho$ in identity OSI$^\rho_{01.1}$.

OSI$^\rho_{01.1.6}$ Put $u = z^\lambda$ in identity OSI$^\rho_{01.1.5}$.

OSI$^\rho_{01.1.7}$ Put $u = z$ in identity OSI$^\rho_{01.1.5}$.

Identities OSI$^\rho_{01.2.1}$, OSI$^\rho_{01.2.4}$, OSI$^\rho_{01.2.6}$ and OSI$^\rho_{01.2.7}$ are deduced from identity OSI$^\rho_{01.2}$.

Lemma 3.16. A right universal Osborn loop $(Q, \cdot, \backslash, /)$ is a RSIPL if and only if it obeys the identity $u^\lambda w^\rho \cdot u = u(uw)^\rho$.

Proof. This is proved by using the identity OSI$^\rho_{01.1.2}$ in Lemma 3.15.

Lemma 3.17. A right universal Osborn loop $(Q, \cdot, \backslash, /)$ is a RSIPL if and only if it obeys the identity $u^\rho w^\rho = u[u \backslash (u^\rho u \cdot w^\rho) \cdot w^\rho]$.

Proof. This is proved by using the identity OSI$^\rho_{01.1.4}$ in Lemma 3.15.

Lemma 3.18. A right universal Osborn loop $(Q, \cdot, \backslash, /)$ is a RSIPL if and only if it is a $z^\lambda \backslash [z^\rho z^\lambda \cdot z] \cdot z = z^\lambda \backslash (z^\rho z)$.

Proof. This is proved by using the identity OSI$^\rho_{01.1.6}$ of Lemma 3.15.
Lemma 3.19. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) obeys the identity \(zz \cdot z^\lambda = z\) if and only if it obeys the identity \([zz \cdot z^\rho z]z = zz \cdot z^{(\rho z)}\).

Proof. This is proved by using the identity OSI\(_{01.1.8}\) of Lemma 3.15.

Corollary 3.7. If a right universal Osborn loop \((Q, \cdot, \backslash, /)\) obeys the identity \([zz \cdot z^\rho]z = zz \cdot z^{(\rho z)}\) then, it is a SFAIPL if and only if it is a SWIPL.

Proof. This is proved by using the identity OSI\(_{01.1.8}\) of Lemma 3.15.

Lemma 3.20. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) with the RSIP is a SFAIPL if and only if it obeys the identity \(u \cdot u[u \cdot u^\rho \cdot u^\rho] = u^\rho\).

Proof. This is proved by using Lemma 3.16, Lemma 3.17 and Lemma 3.13.

Lemma 3.21. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) with the RSIP obeys the identity \(u \cdot u^\rho = (uu)^\rho\).

Proof. This is proved by using the identity OSI\(_{01.2.2}\) of Lemma 3.15.

Corollary 3.8. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) with the RSIP is a SFAIPL and \(|J_f| = 6\).

Proof. This is achieved by using Lemma 3.21 and Lemma 3.13. The second claim can be deduced from the fact in [Page 18, [21]] that SFAIPL implies \(x^{ \rho \rho \rho \rho \rho \rho} = x\).

Lemma 3.22. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) obeys the identity \(uu^\lambda \cdot u^\rho = u^\lambda\) if and only if it obeys the identity \(u = (uu^\lambda) \cdot u((uu^\lambda)^\rho)\).

Proof. This is proved by using the identity OSI\(_{01.2.3}\) of Lemma 3.15.

Lemma 3.23. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) obeys the identity \(u^\rho u = uu^\lambda\) if and only if it obeys the identity \(u \cdot u^\lambda u^\rho = u^\rho\).

Proof. This is proved by using the identity OSI\(_{01.2.8}\) of Lemma 3.15.

Lemma 3.24. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) obeys the identity \(u^\rho u = uu^\lambda\) if and only if it obeys the identity \(\{(uu) \cdot u[u \cdot uu^\rho (uu) \cdot uu]\}u = uu \cdot u^\lambda\).

Proof. This is proved by using the identity OSI\(_{01.2.8}\) of Lemma 3.15.

Corollary 3.9. A right universal Osborn loop \((Q, \cdot, \backslash, /)\) that obeys the identity \(u^\rho u = uu^\lambda\) and the RSIP is a SWIPL.

Proof. This can be deduced from Lemma 3.24.
4 Concluding Remarks and Future Studies

Identities OSI₀¹, OSI₀₁,...; OSI₀¹⁺, OSI₀₁⁺,... and OSI₀¹⁻, OSI₀₁⁻,... are all newly discovered identities that are true in universal, right universal and left universal Osborn loops respectively. So they are all obeyed by any Moufang loop, extra loop, CC-loop, universal WIPL and VD-loop. This is a good news for CC-loop which has just received a tremendous growth increase by the works of Kinyon, Kunen, Drapal, Phillips e.t.c and especially for VD-loops which is yet to grow in study compared to CC-loops. We hope VD-loops will catch the attention of researchers with the newly found identities. A trilling observation in this study is the fact that identities OSI₀¹⁺ and OSI₀¹⁻ are of the forms

\[ y(x^{-1}v) \cdot v^{-1} \cdot (xv) = [y(xv) \cdot v^{-1}](x^{-1}v) \] and

\[ y\{u^{-1}((uv)(v^{-1}x^{-1} \cdot u))v\} = \{(y[u^{-1}(xv)])v^{-1} \cdot x^{-1}(uv)]v^{-1}x^{-1} \cdot u\} \cdot v. \]

respectively, in a Moufang loop or extra loop. If a Moufang or extra loop is of exponent 2 then, the first identity will be obviously true. Basarab [5] has shown that an Osborn loop of exponent 2 is an abelian group. So it is not wise to study identity OSI₀¹⁺ for a loop of exponent 2 e.g. Steiner loops, but identity OSI₀¹⁻ can be studied for such a loop.

According to Phillips [31], a chain of five prominent varieties of CC-loops are: (1) groups, (2) extra loops, (3) WIP PACC-loops, (4) PACC-loops and (5) CC-loops. He was able to axiomatize the variety of WIP PACC-loops. With our new loop properties that are weaker forms of well known loop properties like inverse property, power associativity and diassociativity, we now have subvarieties of varieties of CC-loops mentioned above. It will be interesting to axiomatize some of them e.g. SWIP PACC-loops. These new algebraic properties give more insight into the algebraic properties of universal Osborn loops. Particularly, it can be used to fine tune some recent equations on CC-loop as shown in works of Kunen, Kinyon, Phillips and Drapal; [24, 22, 23], [13, 14], [26].

The continuation of this study will switch to the notations of Bryant and Schneider [9] for principal isotopes of quasigroups (loops) and use their results to deduce more algebraic equations for universal Osborn loops.

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