Two Minicourses on Analytic Microlocal Analysis

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Abstract: These notes correspond roughly to the two minicourses prepared by the authors for the workshop on Analytic Microlocal Analysis, held at Northwestern University in May 2013. The first part of the text gives an elementary introduction to some global aspects of the theory of metaplectic FBI transforms, while the second part develops the general techniques of the analytic microlocal analysis in exponentially weighted spaces of holomorphic functions.

In memory of Lars Gårding and Lars Hörmander
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Chapter 1

Introduction to metaplectic FBI transforms

1.1 Introduction

The metaplectic Fourier-Bros-Iagolnitzer (FBI) transform allows one to pass from the standard Hilbert space $L^2(\mathbb{R}^n)$ to an exponentially weighted space of holomorphic functions on $\mathbb{C}^n$. Such transforms occur under various other names in the literature, such as the Bargmann, Segal, Gabor, and wave packet transforms, and from the general point of view of microlocal analysis, these can all be viewed as Fourier integral operators with complex phase. In this part of the text, the connection to analytic microlocal analysis will be emphasized, and we shall therefore refer to these transforms as FBI transforms, as they were used by J. Bros and D. Iagolnitzer to give a definition of the analytic wave front set. Pseudodifferential operators can be transported to the FBI transform side, and in this way, one obtains some flexible and powerful techniques for their analysis, particularly in the analytic case. In this chapter, we give an elementary introduction to the theory of metaplectic FBI transforms. In Section 1.2 we discuss aspects of the geometry of positive complex Lagrangian planes and some closely related complex canonical transformations, following Appendix A of [1] and Chapter 11 of [15]. In Section 1.3, following [17], [18], we introduce metaplectic FBI transforms, derive a representation for the Bergman projection and establish the unitarity of the FBI transform between $L^2(\mathbb{R}^n)$ and a suitable weighted space of holomorphic functions on $\mathbb{C}^n$. See also [11], [19]. Section 1.4 is concerned with pseudodifferential operators on the FBI transform side. We discuss their mapping properties and prove the metaplectic Egorov theorem, finishing with a brief discussion of the case of pseudodifferential
operators with holomorphic symbols. Our presentation here follows [17] and [18] closely.

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1.2 Complex symplectic linear algebra. Positivity

We shall work in the complex space \( C^{2n} = C^n_x \times C^n_\xi \), which is equipped with the complex symplectic (2,0)-form

\[
\sigma = \sum_{j=1}^{n} dx_j \wedge d\xi_j, \quad (x, \xi) \in C^{2n}.
\]

(1.2.1)

The form \( \sigma \) is non-degenerate and closed, and we can write

\[
\sigma(X,Y) = JX \cdot Y, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X,Y \in C^{2n}.
\]

(1.2.2)

Here and in what follows we shall use the complex bilinear scalar product on \( C^k \), given by \( X \cdot Y = \sum_{j=1}^{k} X_j Y_j \).

The corresponding real 2-forms

\[
\text{Re} \ \sigma = \frac{\sigma + \bar{\sigma}}{2}, \quad \text{Im} \ \sigma = \frac{\sigma - \bar{\sigma}}{2i}.
\]

(1.2.3)

are closed and non-degenerate, and hence give rise to real symplectic structures on \( C^{2n} \).

**Definition 1.2.1** A complex linear map \( \kappa : C^{2n} \to C^{2n} \) is called a complex canonical transformation if

\[
\sigma(\kappa(X), \kappa(Y)) = \sigma(X,Y), \quad X,Y \in C^{2n}.
\]

(1.2.4)

If \( \kappa : C^{2n} \to C^{2n} \) is a complex canonical transformation, then \( \kappa \) preserves the complex volume form \( \sigma^n/n! \) on \( C^{2n} \), and therefore \( \det \kappa = 1 \). If \( n = 1 \), the converse is also true.

Let us consider the following configuration: Let \( \Sigma \subseteq C^{2n} \) be a real subspace which is \( I \)-Lagrangian in the sense that \( \dim_R \Sigma = 2n \) and \( \text{Im} \ \sigma|_\Sigma = 0 \). Assume also that
$\Sigma$ is $R$-symplectic: Re $\sigma|_{\Sigma}$ is non-degenerate. Such a subspace is automatically maximally totally real, $\Sigma \cap i\Sigma = \{0\}$, and we can write

$$C^{2n} = \Sigma \oplus i\Sigma.$$  

Let $\Gamma = \Gamma_{\Sigma} : C^{2n} \rightarrow C^{2n}$ be the unique antilinear map such that $\Gamma|_{\Sigma} = 1$. Clearly, we have

$$\sigma(\Gamma X, \Gamma Y) = \overline{\sigma(X, Y)}, \quad X, Y \in C^{2n}. \quad (1.2.5)$$

**Examples.**

1. $\Sigma = R^{2n}$, $\Gamma X = \bar{X}$, the complex conjugation.

2. Let $\Phi$ be a real valued quadratic form on $C^n$, such that the Levi matrix, $\partial_{\bar{x}} \partial_x \Phi = (\partial_{\bar{x}} \partial_x \Phi)^n_{j,k=1}$, is non-degenerate.

Let us set

$$\Sigma = \Lambda_{\Phi} := \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi}{\partial x_{k}}(x) \right) ; \ x \in C^n \right\}. \quad (1.2.6)$$

We claim that the linear subspace $\Sigma$ is I-Lagrangian and R-symplectic. Indeed, using $x \in C^n$ to parametrize $\Lambda_{\Phi}$, we get

$$\sigma|_{\Lambda_{\Phi}} = \sum_{k=1}^{n} d \left( \frac{2}{i} \frac{\partial \Phi}{\partial x_{k}} \right) \wedge dx_k = \sum_{j,k=1}^{n} \frac{2}{i} \frac{\partial^2 \Phi}{\partial \bar{x}_j \partial x_k} d\bar{x}_j \wedge dx_k. \quad (1.2.7)$$

Using only the fact that $\Phi$ is real, we see that $\sigma|_{\Lambda_{\Phi}}$ is real, so that $\Lambda_{\Phi}$ is I-Lagrangian. Since the Levi form of $\Phi$ is non-degenerate, (1.2.7) also shows that $\sigma|_{\Lambda_{\Phi}}$ is non-degenerate.

Let us now describe the involution $\Gamma|_{\Lambda_{\Phi}}$ explicitly. We have

$$\Phi(x) = \frac{1}{2} \Phi'''_{xx} x \cdot x + \Phi''_{xx} x \cdot \bar{x} + \frac{1}{2} \Phi''_{\bar{x}\bar{x}} \bar{x} \cdot \bar{x}, \quad (1.2.8)$$

and therefore,

$$\Lambda_{\Phi} = \left\{ \left( x, \frac{2}{i} (\Phi''_{xx} x + \Phi''_{\bar{x}\bar{x}} \bar{x}) \right) ; \ x \in C^n \right\}. \quad (1.2.9)$$

Using that $\Gamma_{\Lambda_{\Phi}}(X + iY) = X - iY$, $X, Y \in \Lambda_{\Phi}$, we see that $\Gamma = \Gamma_{\Lambda_{\Phi}}$ is given by

$$\left( y, \frac{2}{i} (\Phi''_{xx} y + \Phi''_{\bar{x}\bar{x}} \bar{y}) \right) \mapsto \left( x, \frac{2}{i} (\Phi''_{xx} x + \Phi''_{\bar{x}\bar{x}} \bar{y}) \right) \quad (1.2.10)$$
Notice that the map $\Phi''_{xx}$ is well-defined since $\det (\Phi''_{xx}) \neq 0$.

Now let $\Lambda \subseteq \mathbb{C}^{2n}$ be a $\mathbb{C}$-Lagrangian subspace, i.e. a complex linear subspace such that $\dim_{\mathbb{C}} \Lambda = n$ and $\sigma|\Lambda = 0$. If $\Sigma \subseteq \mathbb{C}^{2n}$ is $I$-Lagrangian, $R$-symplectic as above, with the associated involution $\Gamma$, we can introduce the Hermitian form

$$b(X, Y) = \frac{1}{i} \sigma(X, \Gamma Y), \quad (X, Y) \in \Lambda \times \Lambda. \quad (1.2.11)$$

Here the Hermitian property, $\overline{b(X, Y)} = b(Y, X)$, follows from (1.2.5).

**Remark.** When $\Sigma = \mathbb{R}^{2n}$, the Hermitian form (1.2.11) was introduced in [9]. The general case was considered in [15].

**Proposition 1.2.2** The form $b$ is non-degenerate if and only if the subspaces $\Lambda$ and $\Sigma$ are transversal, i.e. $\Lambda \cap \Sigma = \{0\}$.

**Proof:** Consider the radical of $b$,

$$\text{Rad} (b) = \{X \in \Lambda; b(X, Y) = 0 \text{ for all } Y \in \Lambda\}.$$

If $0 \neq X \in \text{Rad} (b)$, then $\sigma(\Gamma X, Y) = 0$ for all $Y \in \Lambda$, and therefore, $\Gamma X \in \Lambda$, since $\Lambda$ is Lagrangian. We see, using the fact that $\Gamma$ is an antilinear involution, that the vectors $(1/2) (X + \Gamma X)$ and $(1/2i) (X - \Gamma X)$ both belong to $\Lambda \cap \Sigma$, and at least one of them is $\neq 0$, so that $\Lambda \cap \Sigma \neq \{0\}$. Conversely, $\Lambda \cap \Sigma \subseteq \text{Rad} (b)$, and the result follows. \[\square\]

**Example 1.2.3**

Let $\Sigma = \mathbb{R}^{2n}$ and assume that $\Lambda$ is transversal to the fiber $F = \{(0, \xi); \xi \in \mathbb{C}^{n}\}$, $\Lambda \cap F = \{0\}$. Then necessarily, $\Lambda = \Lambda_{\varphi}$ is of the form $\xi = \varphi'(x) = \varphi'' x$, where $\varphi$ is a holomorphic quadratic form on $\mathbb{C}^{n}_{x}$. We can compute the form $b$ explicitly using this representation of $\Lambda$. When $X = (x, \varphi'' x) \in \Lambda$, we get, using (1.2.11),

$$\frac{1}{2} b(X, X) = (\text{Im} \varphi'') \cdot x \cdot \bar{x}. \quad (1.2.12)$$

Here

$$\text{Im} \varphi'' = \frac{1}{2i} \left( \varphi'' - (\varphi'')^* \right).$$
Definition 1.2.4 Let $\Lambda \subseteq \mathbb{C}^{2n}$ be $C$-Lagrangian and let $\Sigma \subseteq \mathbb{C}^{2n}$ be $I$-Lagrangian, $R$-symplectic, with the involution $\Gamma$. We say that $\Lambda$ is $\Sigma$-positive (negative) if the Hermitian form $b$ is positive definite (negative definite) on $\Lambda$.

Proposition 1.2.5 Let $\Sigma = \mathbb{R}^{2n}$. Then $\Lambda$ is $\Sigma$-positive if and only if $\Lambda = \Lambda_\varphi$, where $\text{Im} \varphi'' > 0$.

Proof: If $\Lambda = \Lambda_\varphi$ with $\text{Im} \varphi'' > 0$, then in view of (1.2.12), we see that $\Lambda$ is $\Sigma$-positive. Conversely, if $\Lambda$ is $\Sigma$-positive, then $\Lambda$ is transversal to the fiber $F$, so that $\Lambda = \Lambda_\varphi$, and Example 1.2.3 applies again. \Box

Proposition 1.2.6 The set $\{\Lambda \subseteq \mathbb{C}^{2n}; \Lambda$ is $C$-Lagrangian and $\Lambda$ is $\Sigma$-positive\} is a connected component in the set of all $C$-Lagrangian spaces that are transversal to $\Sigma$.

Proof: After applying a suitable linear complex canonical transformation, we may assume that $\Sigma = \mathbb{R}^{2n}$. Proposition 1.2.5 shows then that the set of all $\Sigma$-positive $C$-Lagrangian spaces is a connected (even convex) and open subset of the set of all $C$-Lagrangian spaces that are transversal to $\Sigma$. It is also closed, for if $\Lambda$ is a $C$-Lagrangian space transversal to $\Sigma$, such that the form $b$ is positive semi-definite on $\Lambda$, then $b$ is necessarily positive definite on $\Lambda$, in view of Proposition 1.2.2. We conclude that the set of all $\Sigma$-positive $C$-Lagrangian spaces is a component in the set of all $C$-Lagrangian spaces that are transversal to $\Sigma$. \Box

Let us return to the situation where $\Sigma = \Lambda_\Phi$, with $\Phi$ being a real quadratic form on $\mathbb{C}^n_x$. Assume that the Levi form of $\Phi$ is positive definite,

$$
\sum_{j,k=1}^n \frac{\partial^2 \Phi}{\partial \bar{x}_j \partial x_k} \bar{\xi}_j \xi_k > 0, \quad \forall 0 \neq \xi \in \mathbb{C}^n, \quad (1.2.13)
$$

i.e. the quadratic form $\Phi$ is strictly pluri-subharmonic.

Proposition 1.2.7 The fiber $F = \{(0, \eta); \ \eta \in \mathbb{C}^n\}$ is $\Lambda_\Phi$-negative.

Proof: Using (1.2.10) we see that $\Gamma(0, \eta) = (x, \xi)$, where $\xi = \frac{2}{i} \Phi'' x, \ \eta = \frac{2}{i} \Phi'' \bar{x}$, which implies that

$$
\frac{1}{i} \sigma((0, \eta), (x, \xi)) = \frac{1}{i} \eta \cdot x = -2 \Phi''_{xx} \bar{x} \cdot x \leq -\frac{1}{C} |x|^2 \leq -\frac{1}{C} |\eta|^2.
$$
Now the space $\Gamma(F) : \xi = 2\Phi''_{xx}x = \frac{1}{i} \partial_x (\Phi''_{xx}x \cdot x)$ is C-Lagrangian and $\Lambda_{\Phi}$-positive. Let us write

$$\Phi(x) = \Phi_{plh}(x) + \Phi_{herm}(x),$$

where

$$\Phi_{plh}(x) = \text{Re} (\Phi''_{xx}x \cdot x)$$

is the pluri-harmonic part, and

$$\Phi_{herm}(x) = \Phi''_{xx}x \cdot \bar{x}$$

is the positive definite Hermitian part. Using that

$$\partial_x (\Phi''_{xx}x \cdot x) = 2 \partial_x \Phi_{plh}(x),$$

we conclude that $\Gamma(F)$ is of the form $\Lambda_{\Phi_{plh}}$, where $\Phi(x) - \Phi_{plh}(x) \sim |x|^2$.

**Proposition 1.2.8** Assume that $\partial_x \partial_{\bar{x}} \Phi > 0$. A C-Lagrangian space $\Lambda$ is $\Lambda_{\Phi}$-positive if and only if $\Lambda = \Lambda_{\tilde{\Phi}}$, where $\tilde{\Phi}$ is pluri-harmonic quadratic and $\Phi - \tilde{\Phi} \sim |x|^2$.

**Proof:** If $\tilde{\Phi}$ is pluri-harmonic quadratic and $\Phi - \tilde{\Phi} > 0$ then clearly, $\Lambda_{\tilde{\Phi}}$ is C-Lagrangian and transversal to $\Lambda_{\Phi}$. It follows that the set

$$\{\Lambda_{\tilde{\Phi}}; \tilde{\Phi} \text{ pluri-harmonic}, \Phi - \tilde{\Phi} > 0\}$$

is an open connected subset of the set of all C-Lagrangian spaces that are transversal to $\Lambda_{\Phi}$. It is also closed, for if $\tilde{\Phi}$ is pluri-harmonic, $\Phi - \tilde{\Phi} \geq 0$, and $\Lambda_{\tilde{\Phi}}$ is transversal to $\Lambda_{\Phi}$, then the quadratic form $\Phi - \tilde{\Phi}$ is necessarily positive definite. (The transversality forces a non-strict inequality to become strict.) It follows that the set $\{\Lambda_{\tilde{\Phi}}; \tilde{\Phi} \text{ pluri-harmonic}, \Phi - \tilde{\Phi} > 0\}$ is a connected component of the set of all C-Lagrangian spaces that are transversal to $\Lambda_{\Phi}$. It contains $\Lambda_{\Phi_{plh}}$, as we saw above, which is $\Lambda_{\Phi}$-positive. An application of Proposition 1.2.6 allows us to conclude the proof. \[\Box\]

**Example.** Let $\Sigma = \mathbb{R}^{2n}$, and let $\Lambda_{\pm} \subseteq \mathbb{C}^{2n}$ be C-Lagrangian spaces such that $\Lambda_+$ is positive and $\Lambda_-$ is negative, with respect to $\Sigma$. Let us verify that there exists a holomorphic quadratic form $\varphi(x, y)$ on $\mathbb{C}^n_x \times \mathbb{C}^n_y$ such that

$$\det \varphi''_{xy} \neq 0, \quad \text{Im} \varphi''_{yy} > 0,$$

(1.2.14)
and such that the complex linear canonical transformation
\[ \kappa_\varphi : \mathbb{C}^{2n} \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in \mathbb{C}^{2n} \]
satisfies
\[ \kappa_\varphi(\Lambda_+) = \{(x, 0); x \in \mathbb{C}^n\}, \quad (1.2.15) \]
and
\[ \kappa_\varphi(\Lambda_-) = \{(0, \xi); \xi \in \mathbb{C}^n\}. \quad (1.2.16) \]

When showing the existence of the quadratic form \( \varphi(x, y) \), let us recall from Proposition 1.2.5 that \( \Lambda_\pm \) has the form
\[ \eta = F_\pm y, \]
where \( F_\pm \) is a complex symmetric matrix such that \( \pm \operatorname{Im} F_\pm > 0 \). Looking for \( \varphi \) in the form
\[ \varphi(x, y) = \frac{1}{2}Ax \cdot x + Bx \cdot y + \frac{1}{2}Cy \cdot y, \]
where the matrices \( A \) and \( C \) are symmetric and \( B \) is bijective, we observe first that (1.2.16) is equivalent to the fact that
\[ \kappa^{-1}_\varphi(\{(0, \xi); \xi \in \mathbb{C}^n\}) = \{(y, -Cy); y \in \mathbb{C}^n\} = \Lambda_-, \]
so we must have
\[ C = -F_. \quad (1.2.17) \]
The second condition in (1.2.14) is then satisfied, and we also see that
\[ \kappa^{-1}_\varphi(\{(x, 0); x \in \mathbb{C}^n\}) = \{(y, -Bx - Cy); Ax + B^t y = 0\} = \{(-B^-1 A) x, -B x + C(B^{-1} A) x\}. \quad (1.2.18) \]

In order to have (1.2.15), the matrix \( A \) should necessarily be bijective, and we assume that this is the case. Writing \( y = -(B^{-1} A)x, x = -A^{-1} B^t y \), we then get from (1.2.18),
\[ \kappa^{-1}_\varphi(\{(x, 0); x \in \mathbb{C}^n\}) = \{(y, B A^{-1} B^t y - C(B^t)^{-1} A A^{-1} B^t y)\}
\[ = \{(y, (B A^{-1} B^t - C) y)\}. \]
The condition (1.2.15) therefore holds precisely when
\[ BA^{-1} B^t - C = F_. \quad (1.2.19) \]
Using (1.2.17), we may rewrite (1.2.19) in the form
\[ BA^{-1} B^t = F_+ - F_. \]
and observe that the matrix $F_+ - F_-$ is invertible, since $\text{Im} (F_+ - F_-) > 0$. It follows that $A^{-1} = B^{-1}(F_+ - F_-)(B^t)^{-1}$, and choosing the invertible symmetric matrix $A$ in the form

$$A = B^t (F_+ - F_-)^{-1} B,$$

we achieve (1.2.15). The general solution to (1.2.15), (1.2.16), satisfying (1.2.14), is therefore of the form

$$\varphi(x, y) = \frac{1}{2} B^t (F_+ - F_-)^{-1} Bx \cdot x + Bx \cdot y - \frac{1}{2} F_- y \cdot y.$$ 

Here $B$ is an arbitrary invertible matrix.

### 1.3 Metaplectic FBI transforms and Bergman kernels

Last time we discussed the geometry of complex Lagrangian planes in the complexified phase space and that motivated us to look at complex canonical transformations of the form

$$\kappa_\varphi : \mathbb{C}^{2n} \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in \mathbb{C}^{2n}.$$ 

Here $\varphi$ is a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y$ such that

$$\det \varphi''_{x y} \neq 0, \quad \text{Im} \varphi''_{y y} > 0.$$  

(1.3.1)

**Definition 1.3.1** The metaplectic Fourier-Bros-Iagolnitzer (FBI) transform associated to the quadratic form $\varphi$ satisfying (1.3.1) is the operator

$$T : \mathcal{S}'(\mathbb{R}^n) \rightarrow \text{Hol}(\mathbb{C}^n),$$

(1.3.2)

given by

$$Tu(x; h) = Ch^{-\frac{an}{2}} \int e^{i\varphi(x, y)/h} u(y) \, dy, \quad 0 < h \leq 1.$$  

(1.3.3)

To understand the growth properties of the entire function $Tu$ in the complex domain, let us set

$$\Phi(x) = \sup_{y \in \mathbb{R}^n} (\text{Im} \varphi(x, y)).$$

(1.3.4)

Since $\text{Im} \varphi''_{y y} > 0$, we see that the supremum in (1.3.4) is achieved at a unique point $y(x) \in \mathbb{R}^n$, which is the unique critical point of the function

$$\mathbb{R}^n \ni y \mapsto -\text{Im} \varphi(x, y).$$
It follows that
\[ \Phi(x) = \text{vc}_{y \in \mathbb{R}^n} (-\text{Im} \varphi(x, y)) = -\text{Im} \varphi(x, y(x)), \]  
(1.3.5)
and by Taylor’s formula, we can write, for \( y \in \mathbb{R}^n \),
\[ -\text{Im} \varphi(x, y) = \Phi(x) - \frac{1}{2} \text{Im} \varphi''_{yy}(y - y(x)) \cdot (y - y(x)) \leq \Phi(x) - \frac{1}{C} |y - y(x)|^2. \]
It is therefore clear that for some \( M > 0 \) depending on the order of the distribution \( u \), we have
\[ |Tu(x; h)| \leq C h^{-M} e^{\Phi(x)/h}, \quad x \in \mathbb{C}^n. \]  
(1.3.6)
We also observe that the quadratic form \( \Phi(x) = \sup_{y \in \mathbb{R}^n} (-\text{Im} \varphi(x, y)) \) is pluri-subharmonic, being the supremum of a family of pluri-harmonic quadratic forms.

**Example.** Let \( \varphi(x, y) = \frac{i}{2} (x - y)^2 \). Then \( \Phi(x) = \frac{1}{2} (\text{Im} x)^2 \), and the canonical transformation \( \kappa_\varphi \) is given by
\[ \kappa_\varphi(y, \eta) = (y - i\eta, \eta). \]

**Remark.** In microlocal analysis, microlocal properties of \( u \in \mathcal{S}'(\mathbb{R}^n) \) near \((y, \eta) \in T^* \mathbb{R}^n \backslash \{0\} \) can be characterized using local properties of the holomorphic function \( Tu \) near \( \pi_x \left( \kappa_\varphi(y, \eta) \right) \in \mathbb{C}^n \). Here \( \pi_x : \mathbb{C}_{x, \xi}^{2n} \ni (x, \xi) \to x \in \mathbb{C}^n \) is the natural projection map. We refer to [15] and to Section 2.6 of Chapter 2 of this text for further details. In this elementary discussion, we shall only be concerned with global aspects of the metaplectic FBI transforms.

The following proposition indicates that there is a dictionary between the real side and the FBI transform side, where \( \mathbb{R}^{2n} \) corresponds to the linear manifold
\[ \Lambda_\Phi = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi}{\partial x}(x) \right) ; \ x \in \mathbb{C}^n \right\} \subseteq \mathbb{C}^{2n}. \]  
(1.3.7)

**Proposition 1.3.2** The complex canonical transformation
\[ \kappa_\varphi : \mathbb{C}^{2n} \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in \mathbb{C}^{2n} \]  
(1.3.8)
maps \( \mathbb{R}^{2n} \) bijectively onto \( \Lambda_\Phi \). The quadratic form \( \Phi \) introduced in (1.3.4) is strictly pluri-subharmonic.
Proof: We claim that for any \( x \in \mathbb{C}^n \) there is a unique \((y(x), \eta(x)) \in \mathbb{R}^{2n}\) such that \( \pi_x \circ \kappa_{\varphi}(y(x), \eta(x)) = x \). Indeed, if \( y \in \mathbb{R}^n \), then \( \varphi'_y(x, y) \) is real if and only if \( \nabla_y(-\text{Im} \, \varphi(x, y)) = 0 \), in other words, if and only if \( y = y(x) \), the critical point in \((1.3.5)\). The claim follows with \( \eta(x) = -\varphi'_y(x, y(x)) \). We let next \( \xi(x) \in \mathbb{C}^n \) be such that \( \kappa_{\varphi}(y(x), \eta(x)) = (x, \xi(x)) \), i.e. \( \xi(x) = \varphi'_x(x, y(x)) \). Writing \( \Phi(x) = -\text{Im} \, \varphi(x, y(x)) = \frac{i}{2} \left( \varphi(x, y(x)) - \overline{\varphi(x, y(x))} \right) \), we check, using the fact that \( \varphi'_y(x, y(x)) \) and \( y(x) \) are real that
\[
\xi(x) = \frac{2}{i} \frac{\partial \Phi}{\partial x}(x). \tag{1.3.9}
\]
It follows that \( \kappa_{\varphi}(\mathbb{R}^{2n}) = \Lambda_\Phi \), and since \( \sigma|_{\mathbb{R}^{2n}} \) is non-degenerate, we obtain that \( \sigma|_{\Lambda_\Phi} \) is non-degenerate, or equivalently, the Levi form \( \partial_\bar{x} \partial_x \Phi \) is non-degenerate. Since we already know that \( \Phi \) is pluri-subharmonic, we conclude that \( \Phi \) is strictly pluri-subharmonic. \( \square \)

We shall now establish the following basic result, concerning the mapping properties of the FBI transform on \( L^2(\mathbb{R}^n) \).

**Theorem 1.3.3** If \( C > 0 \) is suitably chosen in \((1.3.3)\), then \( T \) is unitary,
\[
T : L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\Phi/h} L(dx)) \cap \text{Hol}(\mathbb{C}^n).
\]
Here \( L(dx) \) is the Lebesgue measure on \( \mathbb{C}^n \).

As a preparation for the proof, let us first derive an expression for the orthogonal (Bergman) projection:
\[
\Pi : L^2_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n),
\]
where \( L^2_\Phi(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi/h} L(dx)) \) and \( H_\Phi(\mathbb{C}^n) \subseteq L^2_\Phi(\mathbb{C}^n) \) is the closed subspace of holomorphic functions. Let \( \psi(x, y) \) be the unique holomorphic quadratic form on \( \mathbb{C}^n_x \times \mathbb{C}^n_y \) such that \( \psi(x, \bar{x}) = \Phi(x) \). Here we may notice that the anti-diagonal \( \{(x, \bar{x}); x \in \mathbb{C}^n\} \) is maximally totally real \( \subseteq \mathbb{C}^n_x \times \mathbb{C}^n_y \). Explicitly, we have
\[
\psi(x, y) = \frac{1}{2} \Phi''_{xx} x \cdot x + \Phi''_{xy} x \cdot y + \frac{1}{2} \Phi''_{yy} y \cdot y,
\]
so that in particular, \( \psi''_{xy} = \Phi''_{xx} \) is non-degenerate. It also follows that when \( y = \bar{x} \), we have
\[
\partial_\bar{y} \psi = \partial_\bar{x} \Phi, \quad \partial_x \psi = \partial_x \Phi. \tag{1.3.10}
\]
These observations have the following useful consequence:

\[ 2\text{Re} \psi(x, y) - \Phi(x) - \Phi(y) = -\Phi''_{xx}(y-x) \cdot (y-x) \sim -|y-x|^2, \quad (1.3.11) \]
on \( \mathbb{C}^n \times \mathbb{C}^n \). Here the last conclusion follows since \( \Phi \) is strictly pluri-subharmonic, and to verify the first equality in (1.3.11) it suffices to Taylor expand the quadratic functions \( y \mapsto \Phi(y) \) and \( y \mapsto \psi(x, \bar{y}) \) at the point \( y = x \), and exploit (1.3.10) to obtain some cancelations.

**Proposition 1.3.4** The orthogonal projection \( \Pi : L^2_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n) \) is given by

\[ \Pi u(x) = \frac{2^n \det \psi''_{xy}}{(\pi h)^n} \int_{\mathbb{C}^n} e^{2\psi(x, \bar{y})/h} u(y) e^{-2\Phi(y)/h} L(dy). \quad (1.3.12) \]

**Proof:** Let \( \Pi \) be the operator given in (1.3.12). To see that

\[ \Pi = \mathcal{O}(1) : L^2_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n), \quad (1.3.13) \]

we consider the reduced kernel

\[ \tilde{\Pi}(x, y) = e^{-\Phi(x)/h} \Pi(x, y) e^{\Phi(y)/h}, \quad (1.3.14) \]

and observe that thanks to (1.3.11), we have

\[ \left| \tilde{\Pi}(x, y) \right| \leq \frac{C}{h^n} e^{-|x-y|^2/Ch}. \]

The uniform boundedness of \( \Pi \) on \( L^2_\Phi \) is therefore a consequence of Schur’s lemma, and since the range of \( \Pi \) consists of holomorphic functions, the property (1.3.13) follows. The selfadjointness of \( \Pi \) on \( L^2_\Phi \) follows since \( \psi(x, \bar{y}) = \psi(y, \bar{x}) \). We finally need to show the reproducing property of \( \Pi \),

\[ \Pi u = u, \quad u \in H_\Phi(\mathbb{C}^n). \quad (1.3.15) \]

To see (1.3.15), we start by establishing the Fourier inversion formula in the complex domain,

\[ u(x) = \frac{1}{(2\pi h)^n} \int_{\Gamma(x)} e^{\hat{\psi}(x-y) \cdot \theta} u(y) dy \wedge d\theta, \quad u \in H_\Phi(\mathbb{C}^n). \quad (1.3.16) \]

Here \( dy \wedge d\theta \) is a \( (2n, 0) \)-form in \( \mathbb{C}^n \times \mathbb{C}^n \), and the integration in (1.3.16) is carried out over the \( 2n \)-dimensional contour (chain) \( \Gamma(x) \), parametrized by \( y \in \mathbb{C}^n \) and given by

\[ \Gamma(x) : \mathbb{C}^n \ni y \mapsto (y, \theta) \in \mathbb{C}^n \times \mathbb{C}^n, \quad \theta = \frac{2}{i} \frac{\partial \Phi}{\partial x} (x) + iC(x-y). \quad (1.3.17) \]
Here $C \gg 1$ is large enough. We have

$$dy \wedge d\theta|_{\Gamma(x)} = \left(\frac{C}{i}\right)^n dy \wedge d\bar{y}$$ (1.3.18) is real and non-vanishing, and it what follows we shall tacitly assume that the orientation on $\Gamma(x)$ has been chosen so that the form in (1.3.18) is a positive multiple of the Lebesgue measure on $\mathbb{C}^n$. Let us also notice that the unique critical point of the function $\mathbb{C}^n \times \mathbb{C}^n \ni (y,\theta) \mapsto -\operatorname{Im}(x-y) \cdot \theta + \Phi(y)$ is given by $y = x$, $\theta = \frac{2}{i} \frac{\partial \Phi}{\partial x}(x)$, and the contour $\Gamma(x)$ passes through the critical point for all $C$. To see (1.3.16), we first observe that the contour $\Gamma(x)$ is good [32], since along $\Gamma(x)$, we have in view of Taylor’s formula,

$$\Re \left(i(x-y) \cdot \theta + \Phi(y) - \Phi(x)\right) \leq -|x-y|^2,$$

provided that $C > 1$ is large enough. The integral in (1.3.16) therefore converges absolutely for all $u \in \text{Hol}(\mathbb{C}^n)$ such that $|u(x)| \leq O_h(1) e^{\Phi(x)/h}$, for some $N_0 > 0$, and in particular, for all $u \in H_\Phi$. We also notice that it is independent of $C \gg 1$, in view of Stokes’ formula.

Using (1.3.17), we see that the right hand side in (1.3.16) is given by

$$\frac{2^n C^n}{(2\pi h)^n} \int e^{-C|x-y|^2/h} e^{\frac{2}{i} \frac{\partial \Phi}{\partial x}(x-y) u(y)} L(dy).$$ (1.3.19)

Here the Gaussian

$$\mathbb{C}^n \ni y \mapsto \frac{C^n}{(\pi h)^n} e^{-C|y|^2/h}$$

is spherically symmetric of integral one, and therefore, by the mean value theorem for holomorphic functions, here applied to the function

$$y \mapsto e^{\frac{2}{i} \frac{\partial \Phi}{\partial x}(x-y) u(y)},$$

we conclude that the expression (1.3.19) is equal to $u(x)$ — see also Lemma 7.3.11 in [10]. This establishes the validity of (1.3.16), and we may observe that the argument given above is in some sense simpler than the usual proof of Fourier’s inversion formula in the real domain, since all the integrals involved converge absolutely, thanks to the choice of a family of good contours, such as $\Gamma(x)$ above.

We shall now finish the proof of Proposition 1.3.4 by passing from (1.3.16) to (1.3.12). To this end, we make a linear complex change of variables $\theta \mapsto w$, given by

$$\theta = \frac{2}{i} \frac{\partial \psi}{\partial x} \left(\frac{x+y}{2}, w\right) = \frac{2}{i} \left(\Phi'' \left(\frac{x+y}{2}\right) + \Phi'' \left(\frac{x+y}{2}\right) w\right).$$
It follows, since $\psi$ is quadratic, that

$$2(\psi(x, w) - \psi(y, w)) = i(x - y) \cdot \theta,$$

and we get therefore from (1.3.16),

$$u(x) = \frac{1}{(2\pi\hbar)^n} \int_{\tilde{\Gamma}(x)} e^{\frac{2}{i}(\psi(x, w) - \psi(y, w))} \left(\frac{2}{i}\right)^n (\det \Phi_x) u(y) \, dy \wedge dw. \tag{1.3.20}$$

Here $\tilde{\Gamma}(x)$ is the natural image of $\Gamma(x)$, so that $(y, w) \in \tilde{\Gamma}(x)$ precisely when $(y, \theta) \in \Gamma(x)$. The contour $\tilde{\Gamma}(x)$ is good in the sense that along $\tilde{\Gamma}(x)$, we have

$$2 \Re (\psi(x, w) - \psi(y, w)) + \Phi(y) - \Phi(x) \leq -|x - y|^2,$$

and another good contour $\hat{\Gamma}(x)$ is given by $w = \bar{y}$. Indeed, we have in view of (1.3.11),

$$2 \Re (\psi(x, \bar{y}) - \psi(y, \bar{y})) + \Phi(y) - \Phi(x) \leq -\frac{1}{C} |x - y|^2.$$

The good contour $\hat{\Gamma}(x)$ is homotopic to $\tilde{\Gamma}(x)$, with the homotopy being within the set of good contours, and we conclude, in view of Stokes’ formula, that

$$u(x) = \frac{\det \Phi_x}{i^n(\pi\hbar)^n} \int_{\hat{\Gamma}(x)} e^{\frac{2}{i}(\psi(x, w) - \psi(y, w))} u(y) \, dy \wedge dw = \Pi u. \tag{1.3.21}$$

This completes the proof of Proposition 1.3.4. \Box

We shall return to the proof of Theorem 1.3.3, where, without loss of generality, we may assume that

$$\varphi''_{xx} = \Re \varphi''_{yy} = 0,$$

so that we can write

$$\varphi(x, y) = Ax \cdot y + \frac{i}{2} By \cdot y, \quad B > 0, \quad \det A \neq 0. \tag{1.3.22}$$

We shall first show that $T : L^2(\mathbb{R}^n) \rightarrow H_\Phi(\mathbb{C}^n)$ is an isometry. To this end, we observe that $Tu(A^{-1}x; \hbar)$ is equal to $C \hbar^{-3n/4}$ times the semiclassical Fourier-Laplace transform of $u(y)e^{-By \cdot y/2\hbar}$, and therefore, by Parseval’s formula,

$$\int |Tu(A^{-1}x; \hbar)|^2 \, dRe x = (2\pi\hbar)^n C^2 \hbar^{-3n/2} \int e^{-By \cdot y/\hbar} e^{-2i\pi x \cdot y/\hbar} |u(y)|^2 \, dy.$$
Next, a computation using (1.3.22) shows that
\[ \Phi(x) = \frac{1}{2} B^{-1} \text{Im} (Ax) \cdot \text{Im} (Ax), \]  
(1.3.23)
and therefore
\[
\int \int |Tu(A^{-1}x; h)|^2 e^{-2\Phi(A^{-1}x)/h} L(dx)
= (2\pi)^n C^2 h^{-n/2} \int \int e^{-(By \cdot y + 2\xi \cdot y + B^{-1}(\xi + By))} |u(y)|^2 \, dy \, d\xi.
\]
We have \( By \cdot y + 2\xi \cdot y + B^{-1}(\xi + By) \cdot (\xi + By) \), and therefore the integral with respect to \( \xi \) in the right hand side is equal to \((\pi h)^{n/2} (\det B)^{1/2}\). On the other hand, the left hand side is given by \( |\det A|^2 \| Tu \|_{H_\Phi}^2 \), so that we get
\[ |\det A|^2 \| Tu \|_{H_\Phi}^2 = 2^n \pi^{n/2} C^2 (\det B)^{1/2} \| u \|_{L^2}^2. \]
Choosing \( C = 2^{-n/2} \pi^{-3n/4} (\det B)^{-1/4} |\det A| > 0 \),
(1.3.24)
we conclude that \( T : L^2(\mathbb{R}^n) \rightarrow H_\Phi(\mathbb{C}^n) \) is an isometry.

We shall finally show that \( TT^* = 1 \) on \( H_\Phi(\mathbb{C}^n) \). Here the Hilbert space adjoint \( T^* \) of \( T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{C}^n) \) is given by
\[
T^* v(y) = C h^{-3n/4} \int e^{-i\phi^*(x,y)/h} v(x) e^{-2\Phi(x)/h} L(dx),
\]  
(1.3.25)
where \( \phi^*(x,y) = \overline{\phi(x,y)} \) is the holomorphic extension of \( \mathbb{R}^n_x \times \mathbb{R}^n_y \ni (x,y) \mapsto \overline{\phi(x,y)} \).

We get, for \( v \in \text{Hol}(\mathbb{C}^n) \), such that \( |v(x)| \leq O_{N_h}(1) |\langle x \rangle|^N \), for all \( N \),
\[ (TT^*) v(x) = C^2 h^{-3n/2} \int \int e^{i(\phi(x,y) - \phi^*(\bar{w},y))/h} v(w) e^{-2\Phi(w)/h} L(dw) \, dy. \]  
(1.3.26)
The integral with respect to \( y \) can be computed by exact stationary phase and we get, writing \( q(x, \bar{w}, y) = \phi(x,y) - \phi^*(\bar{w}, y) \),
\[
\int e^{iq(x,\bar{w},y)/h} \, dy = h^{n/2} \left( \det \frac{q_{yy}''}{2\pi i} \right)^{-1/2} e^{i\text{vc}_y q(x,\bar{w},y)/h}. \]  
(1.3.27)
Here
\[
\text{vc}_y q(x, z, y) = \text{vc}_y (\phi(x, y) - \phi^*(z, y)) \]  
(1.3.28)
is a holomorphic quadratic form on $\mathbb{C}^n \times \mathbb{C}^n$, and when $z = \bar{x}$, we see using (1.3.22) that the unique critical point $y$ in (1.3.28) is real and that (1.3.28) is equal to $\Phi(x)$. It follows that

$$\frac{i}{2} \text{vc}_y (\varphi(x, y) - \varphi^*(z, y)) = \psi(x, z),$$

and using also that $q''_{yy} = 2iB$, we obtain from (1.3.27) that

$$\int e^{iq(x, \bar{w}, y)/h} dy = h^{n/2} \pi^{n/2} (\det B)^{-1/2} e^{2\psi(x, \bar{w})/h}.$$

Returning to (1.3.26) and recalling the explicit expression for the constant $C$ in (1.3.24), we see that

$$(TT^*)v(x) = C^2 h^{-3n/2} h^{n/2} (\det B)^{-1/2} \int e^{2\psi(x, \bar{w})/h} v(w) e^{-2\Phi(w)/h} L(dw)$$

$$= 2^{-n} (\det B)^{-1} |\det A|^2 \int e^{2\psi(x, \bar{w})/h} v(w) e^{-2\Phi(w)/h} L(dw) = (\Pi v)(x) = v(x),$$

where the penultimate equality follows from Proposition 1.3.4. Here we have also used that $\det \Phi''_{xx} = 4^{-n} |\det A|^2 (\det B)^{-1}$, in view of (1.3.23). The proof of Theorem 1.3.3 is complete.

### 1.4 Pseudodifferential operators on FBI transform side

Let $\Phi$ be a strictly pluri-subharmonic quadratic form on $\mathbb{C}^n$, and let us recall the linear IR-manifold $\Lambda_{\Phi} \subset \mathbb{C}^n \times \mathbb{C}^n$, defined in (1.3.7). Introduce

$$S(\Lambda_{\Phi}) = \{ a \in C^\infty(\Lambda_{\Phi}); \partial^\alpha a = O(1), \ \forall \alpha \} \quad (1.4.1)$$

Here we identify $\Lambda_{\Phi}$ linearly with $\mathbb{C}^n$ via the projection map $\Lambda_{\Phi} \ni (x, \xi) \mapsto x \in \mathbb{C}^n$. If $a \in S(\Lambda_{\Phi})$ and $u \in \text{Hol}(\mathbb{C}^n)$ is such that $u = \mathcal{O}_{h, N}(1) \langle x \rangle^{-N} e^{\Phi(x)/h}$, for all $N \geq 0$, we put

$$\text{Op}_h^w(a) u(x) = \frac{1}{(2\pi h)^n} \int_{\Gamma(x)} e^{\frac{i}{h} (x-y) \cdot \theta} a \left( \frac{x+y}{2}, \theta \right) u(y) dy \wedge d\theta. \quad (1.4.2)$$
Here $\Gamma(x)$ is the only possible integration contour given by

$$\theta = \frac{2}{i} \frac{\partial \Phi}{\partial x} \left( \frac{x+y}{2} \right).$$

Along $\Gamma(x)$, we get, by Taylor’s formula,

$$\text{Re} \left( i(x - y) \cdot \theta \right) - \Phi(x) + \Phi(y) = \left\langle x - y, \nabla \Phi \left( \frac{x+y}{2} \right) \right\rangle_{\mathbb{R}^{2n}} - \Phi(x) + \Phi(y) = 0,$$

and let us notice also that

$$dy \wedge d\theta \mid_{\Gamma(x)} = \frac{1}{i^n} \det (\Phi''_{xx}) dy \wedge d\bar{y}.$$

It follows that the integral in (1.4.2) converges absolutely, and for a suitable constant $C \neq 0$, we may write,

$$\text{Op}_{w}^{w}(a)u(x) = \frac{C}{h^n} \int K(x,y)u(y) L(dy), \quad (1.4.3)$$

where

$$K(x,y) = e^{\frac{2}{h^2} (x-y) \cdot \frac{\partial \Phi}{\partial x} \left( \frac{x+y}{2} \right) + \frac{2}{i} \frac{\partial \Phi}{\partial x} \left( \frac{x+y}{2} \right)}.$$

It follows that $\partial_{x} K(x,y) = \partial_{y} K(x,y)$, and using an integration by parts we conclude that the function $\text{Op}_{w}^{w}(a)u(x)$ is holomorphic, since $u$ is.

**Theorem 1.4.1** Let $a \in S(\Lambda \Phi)$. The operator $\text{Op}_{w}^{w}(a)$ extends to a bounded operator: $H_{\Phi}(\mathbb{C}^{n}) \rightarrow H_{\Phi}(\mathbb{C}^{n})$, whose norm is $O(1)$, as $h \rightarrow 0^+.$

**Proof:** Following [18], we shall prove this result by means of a contour deformation argument. When $0 \leq t \leq 1$, let $\Gamma_{t}(x)$ be the $2n$-dimensional contour, given by

$$\theta = \frac{2}{i} \frac{\partial \Phi}{\partial x} \left( \frac{x+y}{2} \right) + it \frac{x-y}{\langle x-y \rangle}.$$

(1.4.4)

We also introduce the $(2n+1)$-dimensional contour $G(x) \subset \mathbb{C}^{n}_{y} \times \mathbb{C}^{n}_{\theta}$, given by

$$G(x) = \bigcup_{0 \leq t \leq 1} \Gamma_{t}(x).$$
We would like to replace the contour $\Gamma(x) = \Gamma_0(x)$ by $\Gamma_1(x)$ in (1.4.2), and to that end, we let $\tilde{a} \in C^\infty(\mathbb{C}^{2n})$ be an almost holomorphic extension of $a \in S(\Lambda_\Phi)$, so that $\text{supp}(\tilde{a}) \subseteq \Lambda_\Phi + \text{neigh}(0, \mathbb{C}^{2n})$, all derivatives of $\tilde{a}$ are bounded, $\tilde{a}|_{\Lambda_\Phi} = a$, and
\[
|\partial_x \tilde{a}(x, \xi)| \leq O_N(1) \left| \xi - \frac{2 \partial \Phi}{i \partial x}(x) \right|^N, \tag{1.4.5}
\]
for all $N \geq 0$. Let us recall that to construct $\tilde{a}$, we may first make a complex linear change of coordinates to replace $\Lambda_\Phi$ by $\mathbb{R}^{2n}$ and consider the problem of constructing an almost holomorphic extension of $a \in C^\infty(\mathbb{R}^{2n})$, with $\partial^\alpha a \in L^\infty(\mathbb{R}^{2n})$ for all $\alpha$.

To this end, following the classical construction by Hörmander, explained in [4], we set
\[
\tilde{a}(X + iY) = \sum_{|\alpha| \geq 0} \frac{\partial^\alpha a(X)}{\alpha!} (iY)^\alpha \chi(t|\alpha| Y), \tag{1.4.6}
\]
where $\chi \in C^\infty(\mathbb{R}^{2n})$, $\chi = 1$ near 0, and $t_j \to \infty$ sufficiently rapidly. Returning to (1.4.2), we get by Stokes’ formula, assuming that $u \in \text{Hol}(\mathbb{C}^n)$, with $u(x) = \mathcal{O}_{h,N}(1) x^{-N} e^{\Phi(x)/h}$, for all $N \geq 0$,
\[
\mathcal{O}_h(\Phi)(a)u = I_1 u + I_2 u, \tag{1.4.7}
\]
where
\[
I_1 u(x) = \frac{1}{(2\pi h)^n} \int_{\Gamma_1(x)} e^{\frac{i}{h}(x-y) \cdot \partial \tilde{a}} \left( \frac{x+y}{2}, \theta \right) u(y) dy \wedge d\theta, \tag{1.4.8}
\]
and
\[
I_2 u(x) = \frac{1}{(2\pi h)^n} \int_{\mathcal{G}(x)} dy, \theta \left( e^{\frac{i}{h}(x-y) \cdot \partial \tilde{a}} \left( \frac{x+y}{2}, \theta \right) u(y) \right) \wedge dy \wedge d\theta. \tag{1.4.9}
\]
We have $dy \wedge d\theta|_{\Gamma_1(x)} = \mathcal{O}(1)L(dy)$, and it follows from (1.4.4) that the reduced kernel of $I_1$ satisfies
\[
|e^{-\Phi(x)/h} I_1(x, y)e^{\Phi(y)/h}| \leq \frac{C}{h^n} e^{-\frac{|x-y|^2}{h(x-y)}}. \tag{1.4.10}
\]

In order to conclude that $I_1 = \mathcal{O}(1) : L^2_\Phi(\mathbb{C}^n) \to L^2_\Phi(\mathbb{C}^n)$, in view of Schur’s lemma, it suffices to check that
\[
\frac{1}{h^n} \int e^{-\frac{|x|^2}{h(x)}} L(dx) = \mathcal{O}(1), \tag{1.4.11}
\]
which is easily seen by considering the integrals over the regions where $|x| \leq 1$ and $|x| \geq 1$. When estimating the contribution of $I_2$, we write

$$d_y, \theta \left( e^{\frac{i}{\hbar}(x-y) \cdot \hat{a}} \left( \frac{x+y}{2}, \theta \right) u(y) \right) \wedge dy \wedge d\theta$$

$$= e^{\frac{i}{\hbar}(x-y) \cdot \hat{a}} u(y) \partial_{y, \theta} \left( \frac{x+y}{2}, \theta \right) \wedge dy \wedge d\theta,$$

and notice that in view of (1.4.5), we have along $G(x)$,

$$\partial_{y, \theta} \left( \frac{x+y}{2}, \theta \right) \wedge dy \wedge d\theta = O_N(1)$$

It follows that the reduced kernel of $I_2$ satisfies

$$|e^{-\Phi(x)/\hbar} I_2(t, x, y) e^{\Phi(y)/\hbar}| \leq C \hbar^{-n} e^{-\frac{t|x-y|^2}{h(x-w)\hbar t N}} \frac{|x-y|^N}{(x-y)^N},$$

and by an application of Schur’s lemma, we see that in order to control the norm of the operator

$$I_2 : L^2_\Phi(C^n) \to L^2_\Phi(C^n),$$

it suffices to estimate

$$\frac{1}{\hbar^n} \int e^{-\frac{t|x|^2}{\hbar N}} \frac{|x|^N}{(x)^N} L(dx),$$

uniformly in $t \in [0, 1]$. In doing so, we consider first the contribution of the region where $|x| \leq 1$. We get

$$\frac{1}{\hbar^n} \int_{|x| \leq 1} e^{-\frac{t|x|^2}{\hbar N}} \frac{|x|^N}{(x)^N} L(dx) = O(1) \hbar^{-n} \int_0^1 e^{-\frac{t}{\hbar N}} t^{N+2n-1} dt$$

$$\leq O(1) \hbar^{-n} \int_0^1 e^{-ts^2} t^{N/2+n} s^{N+2n-1} ds = O(1) \hbar^{N/2} t^{N/2-n} = O(h^{N/2}),$$

uniformly in $t \in [0, 1]$, for $N$ large enough. Next, the contribution of the integral over the region $|x| \geq 1$ does not exceed a constant times

$$\hbar^{-n} \int_{|x| \geq 1} e^{-\frac{t|x|^2}{2\hbar} / t N} L(dx) = O(1) \hbar^{-n} \int_1^\infty e^{-\frac{t}{2\hbar} t^{N+2n-1} dr}$$

$$= O(1) \hbar^n t^{N-2n} \int_{t/h}^{\infty} e^{-t/2} t^{2n-1} d\rho = O(1) \hbar^n t^{N-2n} O \left( \left( 1 + \frac{t}{\hbar} \right)^{-M} \right).$$

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for all $M \geq 0$. If $t \leq h^{1/2}$, we use the factor $t^{N-2n}$ to get the bound $O(h^{N/2})$, while for $t \geq h^{1/2}$, we use the factor

$$O\left(\left(1 + \frac{t}{h}\right)^{-M}\right) = O(h^{M/2}),$$

to get the bound $O(h^{n+M/2})$. We conclude, in view of (1.4.7) that

$$\text{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int \int_{\Gamma_1(x)} e^{i(x-y) \cdot \theta \cdot a_h} \left(\frac{x+y}{2}, \theta\right) u(y) \, dy \wedge d\theta + Ru, \quad (1.4.10)$$

where

$$R = O(h^{\infty}) : L^2_\Phi(C^n) \to L^2_\Phi(C^n).$$

This completes the proof. \qed

We shall next discuss the link between the $h$-pseudodifferential operators on the FBI transform side and the semiclassical Weyl quantization on $\mathbb{R}^n$. We have the following metaplectic Egorov theorem.

**Theorem 1.4.2** Let $T : L^2(\mathbb{R}^n) \to H_\Phi(C^n)$ be a metaplectic FBI transform with the associated canonical transformation

$$\kappa_T : \mathbb{R}^{2n} \to \Lambda_\Phi.$$

If $a \in S(\Lambda_\Phi)$ then we have

$$T^* \text{Op}_h^w(a)_T = \text{Op}_h^w(a \circ \kappa_T).$$

Here the operator in the right hand side is the $h$-Weyl quantization of the symbol $a \circ \kappa_T \in S(1)$ on $\mathbb{R}^n$.

**Proof:** The starting point is the following fact that can be verified by means of an explicit computation: let $l$ be a real linear form on $\mathbb{R}^{2n}$ and let $k$ be the linear form on $\Lambda_\Phi$ such that $k \circ \kappa_T = l$. Then we have on $S(\mathbb{R}^n)$,

$$\text{Op}_h^w(k) \circ T = T \circ \text{Op}_h^w(k). \quad (1.4.11)$$

In the computation, it is convenient to use that if $k(x, \xi) = x^* \cdot x + \xi^* \cdot \xi$, $x, \xi \in \mathbb{C}^n$, then

$$\text{Op}_h^w(k) = k(x, hD_x) = x^* \cdot x + \xi^* \cdot hD_x,$$
and there is a similar formula for \( \text{Op}^w_\hbar(\ell) \). Now let us recall from [4] that the first order operator \( \ell(x, hD_x) = \text{Op}^w_\hbar(\ell) \) is essentially selfadjoint on \( L^2(\mathbb{R}^n) \) from \( \mathcal{S}(\mathbb{R}^n) \), and

\[
e^{i\ell(x, hD_x)/\hbar} = \text{Op}^w_\hbar(e^{i\ell(x, \xi)/\hbar}).
\] (1.4.12)

It follows from (1.4.11) and the unitarity of \( T \) that \( k(x, hD_x) \) is essentially selfadjoint on \( H_\Phi(\mathbb{C}^n) \) from \( T\mathcal{S}(\mathbb{R}^n) \), and therefore, the corresponding unitary groups are intertwined by \( T \),

\[
e^{ik(x, hD_x)/\hbar} \circ T = T \circ e^{il(x, hD_x)/\hbar}.
\]

Here we claim that in analogy with (1.4.12), we have

\[
e^{ik(x, hD_x)/\hbar} = \text{Op}^w_\hbar(e^{ik(x, \xi)/\hbar}),
\] (1.4.13)

where the right hand side is still given by the contour integral in (1.4.2). Indeed, let us write, for \( u \in T\mathcal{S}(\mathbb{R}^n) \),

\[
\text{Op}^w_\hbar(e^{ik(x, \xi)/\hbar}) u(x) = \frac{1}{(2\pi\hbar)^n} \int \int_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y+\xi^*) \cdot \theta + x^* \cdot \left( \frac{y+\xi^*}{2} \right)} u(y) \, dy \wedge d\theta.
\] (1.4.14)

Here by Stokes’ theorem, the integration contour can be deformed to the following,

\[
\theta = \frac{2}{i} \frac{\partial \Phi}{\partial x}(x) + iC(x-y+\xi),
\]

for \( C \gg 1 \) large enough, and the expression (1.4.14) becomes

\[
\frac{2^n C^n}{(2\pi\hbar)^n} \int e^{-C|x-y+\xi|^2/\hbar} e^{\frac{2}{\hbar}(x-y+\xi^*) \cdot \frac{\partial \Phi}{\partial x}(x) + \frac{i}{\hbar} x^* \cdot \left( \frac{y+\xi^*}{2} \right)} u(y) \, L(dy),
\]

which, by the mean value theorem for holomorphic functions, is equal to

\[
x \mapsto e^{\frac{i}{\hbar} x^* \cdot \xi} e^{\frac{2}{\hbar}(x-y+\xi^*) \cdot \frac{\partial \Phi}{\partial x}(x) + \frac{i}{\hbar} x^* \cdot \left( \frac{y+\xi^*}{2} \right)} u(y) = e^{ik(x, hD_x)/\hbar} u(x).
\]

This establishes (1.4.13) and therefore, we get

\[
\text{Op}^w_\hbar(e^{ik(x, \xi)}) \circ T = T \circ \text{Op}^w_\hbar(e^{i\ell(x, \xi)}).
\] (1.4.15)

If \( a \in \mathcal{S}(\Lambda_\Phi) \) and \( b \in \mathcal{S}(\mathbb{R}^{2n}) \) are related by \( b = a \circ \kappa_T \), then by Fourier’s inversion formula, we can represent \( a \) and \( b \) as superpositions of bounded exponentials of the form \( e^{ik(x, \xi)/\hbar} \) and \( e^{i\ell(x, \xi)/\hbar} \), respectively. Here the linear forms \( k \) and \( \ell \) are related by \( \ell = k \circ \kappa_T \), and passing to the \( h \)–Weyl quantizations, we get, in view of (1.4.15),

\[
\text{Op}^w_\hbar(a) \circ T = T \circ \text{Op}^w_\hbar(b).
\] (1.4.16)
A density argument allows us to complete the proof. □

We shall finally make some remarks concerning pseudodifferential operators with holomorphic symbols, referring to [15], as well as to the second part of this text, for a much more extensive discussion. Let us assume that \(a(x, \xi)\) is a holomorphic bounded function in a region of the form \(\Lambda_\Phi + W \subset C^n_x \times C_\xi^n\). Here \(W\) is a bounded open neighborhood of \(0 \in C^{2n}\). It follows from the proof of Theorem 1.4.1 that in this case we have, for \(u \in H_\Phi(\mathbb{C}^n)\),

\[
\text{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int \int_{\Gamma_C(x)} e^{\frac{i}{h}(x-y)\cdot \theta} a \left( \frac{x+y}{2}, \theta \right) u(y) \, dy \wedge d\theta, \tag{1.4.17}
\]

where the contour \(\Gamma_C(x)\) is given by

\[
\theta = \frac{2}{i} \frac{\partial \Phi}{\partial x} \left( \frac{x+y}{2} \right) + \frac{i}{C} \frac{(x-y)}{\langle x-y \rangle}, \tag{1.4.19}
\]

and \(C > 0\) is large enough fixed, so that \(\Gamma_C(x) \subset \Lambda_\Phi + W\). The holomorphy of the symbol allows us to consider weight functions different from \(\Phi\) as well, and study boundedness properties of \(\text{Op}_h^w(a)\) in the corresponding exponentially weighted spaces.

Following [18], we have the following result.

**Theorem 1.4.3** Let \(\tilde{\Phi} \in C^{1,1}(\mathbb{C}^n)\) be such that \(\tilde{\Phi}(x) = \Phi(x) + f(x)\), where \(f \in C^{1,1}_0(\mathbb{C}^n)\) is such that \(\| \nabla f \|_{L_\infty}, \| \nabla^2 f \|_{L_\infty}\) are sufficiently small. We then have a uniformly bounded operator

\[
\text{Op}_h^w(a) = O(1) : H_{\tilde{\Phi}}(\mathbb{C}^n) \to H_{\tilde{\Phi}}(\mathbb{C}^n). \tag{1.4.18}
\]

Here we set \(H_{\tilde{\Phi}}(\mathbb{C}^n) = \text{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\tilde{\Phi}/h} L(dx))\).

**Proof:** We make a deformation to the new contour and set

\[
\text{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int \int_{\tilde{\Gamma}_C(x)} e^{\frac{i}{h}(x-y)\cdot \theta} a \left( \frac{x+y}{2}, \theta \right) u(y) \, dy \wedge d\theta, \tag{1.4.19}
\]

where

\[
\tilde{\Gamma}_C(x) = \frac{2}{i} \frac{\partial \tilde{\Phi}}{\partial x} \left( \frac{x+y}{2} \right) + \frac{i}{C} \frac{x-y}{\langle x-y \rangle}. \tag{1.4.20}
\]
Along the contour \( \tilde{\Gamma}_C(x) \), we have
\[
- \tilde{\Phi}(x) + \text{Re} \left( i(x-y) \cdot \theta \right) + \tilde{\Phi}(y) = - \tilde{\Phi}(x) + \left\langle x-y, \nabla \tilde{\Phi} \left( \frac{x+y}{2} \right) \right\rangle_{\mathbb{R}^{2n}} + \tilde{\Phi}(y) - \frac{1}{C} \frac{|x-y|^2}{\langle x-y \rangle}
\]
and applying Taylor’s formula we see that this expression does not exceed
\[
O(1) \| f'' \|_{L^\infty} \frac{|x-y|^2}{\langle x-y \rangle} - \frac{1}{C} \frac{|x-y|^2}{\langle x-y \rangle} \leq - \frac{1}{2C} \frac{|x-y|^2}{\langle x-y \rangle},
\]
provided that \( \| f'' \|_{L^\infty} \) is small enough. The proof can therefore be concluded as before, by an application of Schur’s lemma. \( \square \)

Remark. Let us notice that \( H_{\tilde{\Phi}}(\mathbb{C}^n) = H_{\Phi}(\mathbb{C}^n) \) as linear spaces, with the norms being equivalent, but not uniformly as \( h \to 0^+ \). We observe also that the Lipschitz IR-manifold \( \Lambda_{\tilde{\Phi}} \) is close to \( \Lambda_{\Phi} \), in the sense of Lipschitz graphs.

It turns out that the natural symbol associated to the operator in (1.4.18) is \( a \big|_{\Lambda_{\tilde{\Phi}}} \). Indeed, we have the following fundamental quantization-multiplication formula, due to [16], [2].

**Proposition 1.4.4** We have
\[
(\text{Op}_h^w(a)u, v)_{H_{\tilde{\Phi}}} = \int a \left( x, \frac{2}{i} \frac{\partial \tilde{\Phi}}{\partial x}(x) \right) u(x)\overline{v(x)e^{-\frac{i}{2}\tilde{\Phi}(x)}} L(dx) + \mathcal{O}(h)\| u \|_{H_{\tilde{\Phi}}} \| v \|_{H_{\tilde{\Phi}}},
\]
for \( u, v \in H_{\tilde{\Phi}}(\mathbb{C}^n) \).

**Proof:** We represent the operator \( \text{Op}_h^w(a) \) as in (1.4.19) with the contour (1.4.20), and Taylor expand \( a \), writing \( \xi(x) = \frac{2}{i} \frac{\partial \tilde{\Phi}}{\partial x}(x) \),
\[
a \left( \frac{x+y}{2}, \theta \right) = a(x, \xi(x)) + (\partial_\xi a)(x, \xi(x))(\theta - \xi(x))
\]
\[
+ (\partial_x a)(x, \xi(x)) \left( \frac{y-x}{2} \right) + \mathcal{O}(|y-x|^2) + \mathcal{O}(|\theta - \xi(x)|^2).
\]
Here the remainder terms are both $O(|x - y|^2)$ along the contour $\tilde{C}_C(x)$, and therefore, in view of Schur’s lemma, their contribution gives rise to an operator of the norm $O(h) : H^{2}_{\Phi}(C^n) \to L^2_{\Phi}(C^n)$. Next, observing that the term $(\partial_x a)(x, \xi(x)) \left( \frac{u - x}{2} \right)$ drops out, when passing to the quantizations, we conclude that

$$Op_h^w(a) = a(x, \xi(x)) + (\partial_\xi a)(x, \xi(x)) \cdot (hD_x - \xi(x)) + R,$$

where

$$R = O(h) : H^{2}_{\Phi}(C^n) \to L^2_{\Phi}(C^n).$$

It remains to estimate the integral

$$\int (\partial_\xi a)(x, \xi(x)) \left( \left( hD_{x_j} - \xi_j(x) \right) u(x) \right) \overline{v(x)} e^{-2\tilde{\Phi}(x)/h} L(dx), \quad 1 \leq j \leq n,$$

and since the function $(\partial_\xi a)(x, \xi(x))$ is Lipschitz, we can integrate by parts in (1.4.21), getting $O(h)||u||_{H^{2}_{\Phi}} ||v||_{H^{2}_{\Phi}}$ plus the term

$$\int (\partial_\xi a)(x, \xi(x)) u(x) \overline{v(x)} \left( -hD_{x_j} - \xi_j(x) \right) e^{-2\tilde{\Phi}(x)/h} L(dx) = 0.$$

This completes the proof. □

We shall finish with the following general idea suggested by the discussion above: given an $h$–pseudodifferential operator of the form $Op_h^w(a)$, with $a$ holomorphic in a tubular neighborhood of $\Lambda_\Phi$, try to find an IR-manifold $\Lambda_{\tilde{\Phi}}$ close to $\Lambda_\Phi$ so that the operator

$$Op_h^w(a) : H^{2}_{\tilde{\Phi}}(C^n) \to H^{2}_{\tilde{\Phi}}(C^n)$$

acquires some improved properties, such as the invertibility, ellipticity, normality, etc. We refer to the works [3], [5], [6], [7], [8], [13], [14], where implementations of this idea have led to some precise results in the spectral theory of semiclassical non-selfadjoint operators. It may also be interesting to compare this idea with the recent developments around Carleman estimates with limiting Carleman weights for second order elliptic differential operators, see [12].

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Chapter 2

Analytic microlocal analysis using holomorphic functions with exponential weights

2.1 Introduction

There are several approaches to analytic microlocal analysis:

• One very natural approach consists in adapting the classical theory of pseudodifferential operators on the real domain to the analytic category. The basic calculus was developed by L. Boutet de Monvel and P. Krée [3]. K.G. Andersson [1] and L. Hörmander [13] studied propagation of analytic singularities. The work [13] also introduced the analytic wave front set of distributions, a corresponding notion in the framework of hyperfunctions had previously been introduced by M. Sato (see [26]). The two works [1], [13] use special sequences of cutoff functions, remedying for the lack of analytic functions with compact support. Such special sequences have an earlier history, see L. Ehrenpreis [6], S. Mandelbrojt [19, 20]. The book [36] of F. Treves gives the theory of analytic pseudodifferential operators, with the help of such cutoffs.

• A second approach is based on the representation of distributions and more generally hyperfunctions as sums of boundary values of holomorphic functions. The main work in this direction is the one of M. Sato, T. Kawai and M. Kashiwara [26].

• A third approach is to work with Fourier transforms that have been modified by the introduction of a Gaussian (avoiding the use of the special cut-
offs mentioned above. Such transforms come under different names: FBI, Bargmann-Segal, Gabor, wavepacket ... transforms. Microlocal properties are now described in terms of exponential growth/decay of the transformed functions. In the context of analytic microlocal analysis they were introduced and used by D. Iagolnitzer, H. Stapp [16], J. Bros, Iagolnitzer [4]. This is the method we follow here. See [32, 21].

The aim of this part of the text is to explain the basic ingredients in the approach of [32], that was preceded by some work on propagation of analytic singularities for boundary value problems, see [30]. The main observation is that an FBI-transform produces holomorphic functions whose exponential growth rate reflect the regularity and that such transforms are Fourier integral operators with complex phase functions. This leads to a calculus of Fourier integral operators and pseudodifferential operators in the complex domain via an Egorov theorem. In this calculus oscillatory integrals are systematically replaced by contour integrals, leading to “Cauchy integral operators”.

This part of the text will split roughly into 4 unequal parts:

- In Sections 2.2–2.5 we discuss pseudodifferential operators and Fourier integral operators acting on exponentially weighted spaces of holomorphic functions.
- In Sections 2.6, 2.7 we introduce FBI (generalized Bargmann-) transforms and the analytic wave front set of a distribution.
- The sections 2.8, 2.9 are devoted to some applications: propagation of singularities, construction of exponentially accurate quasi-modes for non-self-adjoint differential operators.
- In Section 2.10 we give a very quick review of related developments.

2.2 Classical analytic symbols and pseudodifferential operators.

Let $\Omega \subset \mathbb{C}^n$ be open, $\phi \in C(\Omega; \mathbb{R})$. By definition, the function $u = u(z; h)$ on $\Omega \times ]0, h_0[ \subset H^{\text{loc}}_\phi(\Omega)$ if

- $u(\cdot; h) \in \text{Hol}(\Omega)$, for all $h$, where $\text{Hol}(\Omega)$ denotes the space of holomorphic functions on $\Omega$. 


\( \forall K \in \Omega, \varepsilon > 0, \exists C > 0 \) such that \( |u(z;h)| \leq C e^{(\phi(z) + \varepsilon)/h}, z \in K \).

When \( u \in H_0(\Omega) \), we say that \( u \) is an analytic symbol. When \( u = \mathcal{O}(h^{-m}) \) locally uniformly on \( \Omega \), we say that \( u \) is of finite order \( m \in \mathbb{R} \).

We frequently identify equivalent elements of \( H^\text{loc}_\phi(\Omega) \), where the equivalence \( u \sim v \) of \( u, v \in H^\text{loc}_\phi(\Omega) \) means that there exists \( \epsilon > 0 \), such that \( u - v \in H^\text{loc}_\phi(\Omega) \).

When \( \Omega \) is pseudoconvex and the weights are pluri-subharmonic, we can represent equivalence classes by functions \( u \in L^2_{\text{loc}}(\Omega) \) for which

\[
|e^{-\phi/h}u|_{L^2(K)} \leq C \varepsilon, \quad \varepsilon > 0, \quad K \in \Omega.
\]

Indeed by applying Hörmander’s method of solving the \( \bar{\partial} \) equation it is easy to make such a function \( u \) holomorphic by adding a correction which is locally exponentially small compared to \( e^{\phi/h} \).

By \( H_{\phi,x_0} \) we denote the intersection of all spaces \( H_{\phi}(\Omega) \) where \( \Omega \) is a small neighborhood of \( x_0 \in \mathbb{C}^n \) and \( \phi \) is defined in some fixed neighborhood of \( x_0 \). We have a corresponding equivalence relation.

**Classical analytic symbols** (Boutet de Monvel, Krée [3]). We restrict the attention to symbols of order 0. Let \( a_k \in \text{Hol}(\Omega), \ k = 0, 1, \ldots \) and assume that for every \( \tilde{\Omega} \in \Omega, \exists C = C_{\tilde{\Omega}} > 0 \) such that

\[
|a_k(z)| \leq C^{k+1}k^k, \quad z \in \tilde{\Omega}.
\]  (2.2.1)

\( a = \sum_{0}^{\infty} a_k(z)h^k \) is called a classical analytic symbol.

We have a realization of \( a \) on \( \tilde{\Omega} \) by

\[
a_{\tilde{\Omega}}(z; h) = \sum_{0 \leq k \leq (eC_{\tilde{\Omega}}h)^{-1}} a_k(z)h^k.
\]

For \( 0 \leq k \leq (eC_{\tilde{\Omega}}h)^{-1} \) we have

\[
|a_k(z)|h^k \leq C_{\tilde{\Omega}}(C_{\tilde{\Omega}}h)^k \leq C_{\tilde{\Omega}}e^{-k},
\]

so the defining sum above converges geometrically and \( |a_{\tilde{\Omega}}(z; h)| \leq C_{\tilde{\Omega}}e/(e - 1) \).

If \( \tilde{\Omega} \supseteq \tilde{\Omega} \) is another relatively compact subset of \( \Omega \), then \( a_{\tilde{\Omega}} \) and \( a_{\tilde{\Omega}} \) are equivalent on \( \tilde{\Omega} \). It is sometimes convenient to consider classical symbols of the form

\[
a = \sum_{0}^{\infty} a_k(z)h^k, \quad a_k \in \text{Hol}(\Omega)
\]
without the growth condition \((2.2.1)\).

Let

\[
p(x, \xi; h) = \sum_{0}^{\infty} h^k p_k(x, \xi), \quad q(x, \xi; h) = \sum_{0}^{\infty} h^k q_k(x, \xi)
\]

be classical symbols defined near \((x_0, \xi_0) \in C^{2n}\). Denote by \(p(x, hD; h)\), \(q(x, hD; h)\) the corresponding formal pseudodifferential operators. The formal composition of \(p\) and \(q\) is defined by

\[
p#q = \sum_{\alpha \in \mathbb{N}^n} \frac{h^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi; h) D_x^\alpha q(x, \xi; h),
\]

which is a finite sum for each power of \(h\). Here, we use standard PDE-notation,

\[
D_x = i^{-1} \partial_x,
\]

\[
\partial_\xi^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = |\alpha|_{\ell_1} = \alpha_1 + \ldots + \alpha_n, \quad \text{for} \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.
\]

When \(p, q\) are polynomials in \(\xi\), the differential operators \(p(x, hD; h)\), \(q(x, hD; h)\) are well defined and

\[
p(x, D_x; h) \circ q(x, hD_x; h) = (p#q)(x, hD; h).
\]

If \(r\) is a third symbol, also polynomial in \(\xi\), it follows that

\[
(p#q)#r = p#(q#r). \tag{2.2.2}
\]

In general, we can approximate \(p, q, r\) with finite Taylor polynomials at any given point and see that we still have \((2.2.2)\).

To \(p\), we associate

\[
A(x, \xi, hD_x; h) = p(x, \xi + hD_x; h) = \sum_{\alpha} \frac{h^\alpha}{\alpha!} \partial_\xi^\alpha p(x, \xi) D_x^\alpha = \sum_{k=0}^{\infty} h^k A_k(x, \xi, D_x),
\]

where

\[
A_k = \sum_{\nu+|\alpha|=k} \frac{1}{\alpha!} (\partial_\xi^\nu p_\nu)(x, \xi) D_x^\alpha \tag{2.2.3}
\]

is a differential operator of order \(\leq k\).
Formally, \( A = e^{-ix\cdot \xi/h} \circ p(x, hD_x; h) \circ e^{ix\cdot \xi/h} \) which is exact and well defined, when \( p \) is a polynomial in \( \xi \). Let \( B = q(x, \xi + hD_x; h) = \sum_0^\infty h^\ell B_\ell \). Then \( C = A \circ B \) is well defined by \( C = \sum_k C_k B_k \), where \( C_k = \sum_{\ell=0}^k A_k B_\ell \). By Taylor approximation with polynomials in \( \xi \), we see that
\[
C = r(x, \xi + hD_x; h), \quad \text{if} \quad r = p \# q.
\]

**Quasi-norms** Let \( \Omega_t \subseteq C^{2n}, 0 \leq t \leq t_0, t_0 > 0 \) be a family of open neighborhoods of a point \((x_0, \xi_0)\) such that
\[
(y, \xi) \in \Omega_s \quad \text{and} \quad |x - y|_{\ell_\infty} < t - s \implies (x, \xi) \in \Omega_t,
\]
whenever \( 0 \leq s \leq t \leq t_0 \). Here,
\[
|x|_{\ell_\infty} = \sup |x_j|, \quad x = (x_1, \ldots, x_n) \in C^n.
\]

Then \( D_x^\alpha \) is a bounded operator: \( B(\Omega_t) \to B(\Omega_s) \) where \( B(\Omega) \) denotes the space of bounded holomorphic functions on \( \Omega \). Moreover, by the Cauchy inequalities,
\[
\|D_x^\alpha\|_{t,s} := \|D_x^\alpha\|_{L(B(\Omega_t), B(\Omega_s))} \leq \frac{\alpha!}{(t-s)^{\|\alpha\|}} \leq \frac{C_0^{\alpha!} |\alpha|^{\|\alpha\|}}{(t-s)^{\|\alpha\|}},
\]
for some constant \( C_0 > 0 \).

If \( \Omega_{t_0} \) is a relatively compact subset of the domain of definition of \( p \), then on \( \Omega_{t_0} \),
\[
|\partial_\xi^\alpha p \nu| \leq C^{1+|\alpha|} \nu^{\|\alpha\|} \alpha!.
\]

Hence, with a new constant
\[
\|\frac{1}{\alpha!} \partial_\xi^\alpha p D_x^\alpha\|_{t,s} \leq C^{1+|\alpha|} \nu^{\|\alpha\|} \frac{|\alpha|^{\|\alpha\|}}{(t-s)^{\|\alpha\|}}.
\]

The number of terms in \( (2.2.3) \) is \( \leq (1 + k)^{n+1} \), so with a new constant \( C > 0 \), we have
\[
\|A_k\|_{t,s} \leq C^{k+1} \frac{k^k}{(t-s)^k}, \quad 0 \leq s < t \leq t_0.
\]  

(2.2.4)

Conversely, if \( p \) is a classical symbol such that \( (2.2.4) \) holds for some \( C > 0 \), then \( p \) is a classical analytic symbol near \((x_0, \xi_0)\). In fact, since \( p_k = A_k(1) \), we get for some new \( C > 0 \) that
\[
\sup_{\Omega_{t_0/2}} |p_k| \leq C^{k+1} k^k.
\]  

(2.2.5)
Put $f(A) = (f_k(A))_{k=0}^\infty$, where $f_k(A)$ is the smallest constant $\geq 0$ such that

$$\|A_k\|_{t,s} \leq f_k(A) k^k (t-s)^{-k}, \quad 0 \leq s < t \leq t_0.$$  

When (2.2.4) holds, $f_k(A)$ is of at most exponential growth.

Let $B = \sum_0^\infty h^k B_k$ be an operator of the same type, so that $B_k$ is a differential operator of order $\leq k$.

**Lemma 2.2.1** If $C = A \circ B$, then $f_k(C) \leq \sum_{\nu+\mu=k} f_{\nu}(A) f_{\mu}(B)$ or in other terms, $f(C) \leq f(A) \ast f(B)$.

**Proof:** We have $C_k = \sum_{\nu+\mu=k} A_\nu \circ B_\mu$ and for $0 \leq s < r < t \leq t_0$:

$$\|A_\nu \circ B_\mu\|_{t,s} \leq f_{\nu}(A) f_{\mu}(B) \frac{\nu^\nu \mu^\mu}{(r-s)^\nu (t-r)^\mu}.$$  

Choose $r$ such that

$$r-s = \frac{\nu}{\nu+\mu}(t-s), \quad t-r = \frac{\mu}{\nu+\mu}(t-s).$$

Then

$$\|A_\nu \circ B_\mu\|_{t,s} \leq f_{\nu}(A) f_{\mu}(B) \frac{(\nu+\mu)^{\nu+\mu}}{(t-s)^{\nu+\mu}},$$

$$\|C_k\|_{t,s} \leq \left( \sum_{\nu+\mu=k} f_{\nu}(A) f_{\mu}(B) \right) \frac{k^k}{(t-s)^k}.$$  

For $\rho > 0$, put

$$\|A\|_\rho = \sum_0^\infty \rho^k f_k(A).$$

Then (2.2.4) holds iff $\|A\|_\rho < \infty$ for $\rho > 0$ small enough.

**Lemma 2.2.2** Let $C = A \circ B$. If $\|A\|_\rho$, $\|B\|_\rho < \infty$, then $\|C\|_\rho < \infty$ and we have $\|C\|_\rho \leq \|A\|_\rho \|B\|_\rho$.

**Proof:** By Lemma 2.2.1, we have pointwise with respect to $k$:

$$(\rho^k f_k(C))_0^\infty \leq (\rho^k f_k(A))_0^\infty \ast (\rho^k f_k(B))_0^\infty$$

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and we have the corresponding inequality for the \( \ell^1 \)-norms. \(\square\)

If \( p(x, \xi; h) \) is a classical symbol on a neighborhood of \( \overline{\Omega}_t \), we put \( \| p \|_\rho = \| A \|_\rho \). If \( p \) is a classical analytic symbol then there exists \( \rho > 0 \) such that \( \| p \|_\rho < \infty \) and similarly for \( q \) corresponding to \( B \). Since \( p \# q \) corresponds to \( A \circ B \), we obtain \( \| p \# q \|_\rho \leq \| p \|_\rho \| q \|_\rho \) and we conclude that \( p \# q \) is a classical analytic symbol in \( \Omega_t \).

Next we give a semi-classical formulation of a fundamental result of L. Boutet de Monvel, P. Krée [3]:

**Theorem 2.2.3** Let \( p \) be an elliptic classical analytic symbol \((p_0 \neq 0)\) on a neighborhood of \( \overline{\Omega}_t \) and let \( q \) be the classical symbol given by \( p \# q = 1 \). Then \( q \) is a classical analytic symbol in \( \Omega_t \).

**Proof:** Let \( q_0 = 1/p_0 \), so that \( q_0 \) is a classical analytic symbol. Then \( p \# q_0 = 1 - r \) where \( r \) is a classical analytic symbol of order \(-1\) in the sense that its asymptotic expansion starts with a term in \( h \). Consequently \( \| r \|_\rho < 1/2 \) if \( \rho > 0 \) is small enough.

We have
\[
q = q_0 \# (1 + r + r \# r + \ldots),
\]
so that
\[
\| q \|_\rho \leq \| q_0 \|_\rho (1 + \| r \|_\rho + \| r \|_\rho^2 + \ldots) \leq 2\| q_0 \|_\rho < \infty.
\]
\(\square\)

### 2.3 Stationary phase – steepest descent

Let \( B = B_{R^n}(0, 1) \) be the open unit ball in \( \mathbb{R}^n \) and put
\[
\tilde{B} = \{ \lambda x; x \in B, \lambda \in \mathbb{C}, |\lambda| \leq 1 \}.
\]

**Theorem 2.3.1** There exist a constant \( C > 0 \) depending only on the dimension, such that for all \( N \in \mathbb{N}, 0 < h \leq 1, u \in \text{Hol (neigh} (\tilde{B})),
\[
\int_B e^{-x^2/(2h)} u(x) dx = \sum_{\nu = 0}^{N-1} (2\pi)^{\frac{n}{2}} h^{\frac{n}{2} + \nu} \frac{1}{\nu!} \left( \frac{1}{2} \Delta \right)^\nu u(0) + R_N(h),
\]

where
\[
|R_N(h)| \leq Ch^{\frac{n}{2} + N} (N + 1)^2 N! 2^N \sup_{\tilde{B}} |u(z)|.
\]
We omit the proof and refer to [32], Chapter 2.

**Example 3.2.** Consider

\[ J(h) = \left( \frac{h}{2\pi} \right)^n \int \int_{|\xi| \leq C_1, \xi = -C_2} e^{-ix \cdot \xi/h} u(x, \xi) dx d\xi. \]

Then,

\[
J(h) = \sum_{k=0}^{N-1} \frac{1}{k!} \left( \frac{h}{i} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)^k u(0,0) + R_N(h)
\]

\[
= \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} \left( \frac{h}{i} \right)^{|\alpha|} (h^{\alpha} \partial^\alpha u)(0,0) + R_N(h),
\]

\[
|R_N(h)| \leq C(n)(N + 1)^n N! \left( \frac{h}{C_1^2 C_2} \right)^N \sup_{|x| \leq C_1, |\xi| \leq C_1 C_2} |u(x, \xi)|.
\]

This follows from Theorem 2.3.1, some calculations and the following three observations:

- \( \Gamma : \xi = (C_2/i)x \) is a maximally totally real subspace of \( C^{2n} \), hence \( \simeq \mathbb{R}^{2n} \) after a complex linear change of coordinates.

- The restriction of \( e^{-ix \cdot \xi/h} \) to \( \Gamma \) is equal to \( e^{-C_2|x|^2/h} \).

- The corresponding restriction of \( i^{-1} \partial_x \cdot \partial_\xi \) is equal to

\[
\frac{1}{i} \partial_x \cdot \frac{i}{C_2} \partial_\xi = \frac{1}{4C_2} \Delta_{Re x, Im x}.
\]

**Non-quadratic case.** The holomorphic version of the Morse lemma is the following:

**Lemma 2.3.2** Let \( \phi \in \text{Hol} (\text{neigh} (z_0, \mathbb{C}^n)) \), \( \phi'(z_0) = 0 \), \( \det \phi''(z_0) \neq 0 \). Then there exist local holomorphic coordinates \( z_1, ..., z_n \), centered at \( z_0 \) such that

\[ \phi(z) = \phi(z_0) + \frac{1}{2}(z_1^2 + ... + z_n^2). \]

The main ingredient in the standard proof of the Morse lemma in the real smooth category is the implicit function theorem in the same category. To get the proof of the holomorphic Morse lemma it suffices to use the holomorphic implicit function theorem.
Theorem 2.3.3  Let $0 \in V \Subset U \subset \mathbb{C}^n$, $V$, $U$ open, $\phi \in \text{Hol}(U)$, $\phi(0) = 0$, $\phi'(0) = 0$, $\phi''(0)$ non degenerate. Assume that $\text{Re}\phi \geq 0$ on $V_\mathbb{R} := V \cap \mathbb{R}^n$, $\text{Re}\phi > 0$ on $\partial V_\mathbb{R}$, $\phi'(x) \neq 0$ on $V_\mathbb{R} \setminus \{0\}$. Then, for every $C > 0$ large enough, there exists a constant $\varepsilon > 0$ such that for every $u \in \text{Hol}(U)$,

$$
\int_{V_\mathbb{R}} e^{-\phi(x)/h} u(x) dx = \sum_{0 \leq k \leq 1/(C h)} (2\pi h)^{\frac{k}{2}} \frac{h^k}{k!} \left( \frac{1}{2} \tilde{\Delta} \right)^{-\frac{k}{2}} \left( \frac{u}{J} \right)(0) + R(\lambda),
$$

where

$$
|R(h)| \leq \frac{1}{\varepsilon} e^{-\frac{\pi}{h}} \sup_U |u(z)|, \quad 0 < h \leq 1.
$$

Here, $\tilde{\Delta}$ denotes the Laplacian in the Morse coordinates, $J = \det \frac{d\phi}{dz}$, $J(0) = (\det \phi''(0))^{\frac{1}{2}}$, with the choice of the branch of the square root that tends to 1, when we deform $\phi''(0)$ to 1 in the space of invertible symmetric matrices with real part $\geq 0$.

**Proof:** Up to an exponentially small modification, we may replace the integral by

$$
I_\chi = \int_{\mathbb{R}^n} e^{-\phi(x)/h} u(x) \chi(x) dx, \quad \chi \in C^\infty_0(V_\mathbb{R}),
$$

$$
\text{supp} (1 - \chi) \subset \text{small neighborhood of } \partial V_\mathbb{R}.
$$

Make a first contour deformation $\Gamma_\delta : V_\mathbb{R} \ni x \mapsto x + \delta \overrightarrow{\phi'(x)}$, $0 \leq \delta \leq \delta_0 \ll 1$. Along $\Gamma_\delta$ we have

$$
\phi(z) = \phi(x) + \delta |\phi'(x)|^2 + \mathcal{O}(\delta^2 |\phi'(x)|^2) \geq \frac{\delta}{C} |z|^2,
$$

when $\delta_0$ is small enough.

Let $G$ be the $(n + 1)$-dimensional contour formed by the union of the $\Gamma_\delta$ for $0 \leq \delta \leq \delta_0$. Then Stokes’ formula gives (with $\chi$ denoting also a suitable smooth extension to the complex domain),

$$
I_\chi = \int_{\Gamma_{\delta_0}} e^{-\phi(z)/h} u(z) \chi(z) dz - \int_G d(e^{-\phi/h} u(z) \chi(z)) dz.
$$

The last integral is equal to

$$
\int_{G \cap \text{neigh} (\partial V_\mathbb{R})} e^{-\phi(z)/h} u(z) \overline{\partial} \chi(z) \wedge dz.
$$
When estimating the integral over $\Gamma_{\delta_0}$, we can restrict the attention to a small neighborhood of 0 and then use Morse coordinates for which $\phi = \frac{1}{2} |\tilde{z}|^2$. Since $\Re \phi \simeq |\tilde{z}|^2$ along $\Gamma_{\delta_0}$, we see that $\Gamma_{\delta_0}$ must be of the form $\tilde{y} = k(\tilde{x}) \ (\tilde{z} = \tilde{x} + i\tilde{y})$, where $|k'| \leq \theta < 1$, $k(0) = 0$. (Use the implicit function theorem, to see that the projection $\Gamma_{\delta_0} \ni \tilde{z} \mapsto \tilde{x}$ is a diffeomorphism near 0.) The last step is then to deform the contour $\tilde{y} = k(\tilde{x})$ to $\tilde{y} = 0$ in the simplest possible way and to apply Theorem 2.3.1. □

2.4 Contour integrals and Fourier transforms

a. Remarks about real quadratic forms on $\mathbb{C}^n$. Let $q$ be a real quadratic form on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Let $\text{sign} (q) = (m_+, m_-)$ where $m_\pm = m_\pm (q)$ are given by

$$q = \sum_{j=1}^{m_+} \xi_j^2 - \sum_{j=m_++1}^{m_+ + m_-} \xi_j^2,$$

for suitable real-linear coordinates on $\mathbb{C}^n$. We know that $m_+$ ($m_-$) is the largest possible dimension of a real-linear subspace on which $q$ is positive (negative) definite.

Using the complex structure, put $J q(x) = q(ix)$, so that $J^2 q = q$ (since $q$ is even).

Notice that $q$ is pluriharmonic iff $J q = -q$.

We say that $q$ is Levi if $J q = q$.

In general we have the decomposition

$$q = h + \ell = 2 \Re \left( \sum a_{j,k} \bar{z}_j z_k \right) + \sum b_{j,k} \bar{z}_j z_k,$$

where $h = (1 - J)q/2$ is pluri-harmonic and $\ell = (1 + J)q/2$ is Levi.

**Proposition 2.4.1** Let $q$ be a pluri-subharmonic quadratic form on $\mathbb{C}^n$. Then

(a) $m_+(q) \geq m_-(q)$

(b) If $q$ is non-degenerate of signature $(n, n)$, then the same fact holds for every pluri-subharmonic quadratic form $\tilde{q} \leq q$.

**Proof:** The pluri-subharmonicity of $q$ means that $\ell \geq 0$. (a) Let $L \subset \mathbb{C}^n$ be a real-linear subspace of dimension $m_- = m_-(q)$ such that $q|_L < 0$. Use the decomposition $q = h + \ell$. Then $h(x) = q(x) - \ell(x) < 0$ for $0 \neq x \in L$. Consequently, $h(ix) > 0$, so $q(ix) = h(ix) + \ell(ix) > 0$. Thus $q$ is positive definite on the $m_-$-dimensional
space $iL$, so $m_+ \geq m_-$. (b) Now assume that $m_+ = m_- = n$. Let $\tilde{q} \leq q$ be pluri-subharmonic and choose the subspace $L$ as in (a). Then $\tilde{q}$ is negative definite on $L$ so $m_-(\tilde{q}) \geq m_-(q) = n$ and from the part (a) of the proposition we conclude that $\tilde{q}$ has signature $(n,n)$. \hfill \square

b. Fundamental lemma.

**Lemma 2.4.2** Let $\phi \in C^\infty(\text{neigh}((0,0), \mathbb{C}^{n+k}); \mathbb{R})$ be pluri-subharmonic. Assume that $\nabla_y \phi(0,0) = 0$ and that $\nabla^2_y \phi(0,0)$ is nondegenerate of signature $(k,k)$. For $x \in \text{neigh}(0, \mathbb{C}^n)$, let $y(x) \in \text{neigh}(0, \mathbb{C}^k)$ be the unique critical point of $\phi(x,\cdot)$, so that $y(x)$ is a smooth function of $x$ by the implicit function theorem. Then the critical value of $y \mapsto \phi(x,y)$,

$$\Phi(x) = \phi(x,y(x)) = \text{vc}_y \phi(x,y)$$

is pluri-subharmonic. If $\tilde{\phi} \leq \phi$ is pluri-subharmonic with $\tilde{\phi}(0,0) = \phi(0,0)$, then $\nabla^2_y \tilde{\phi}(0,0)$ is also non-degenerate of signature $(k,k)$ and

$$\text{vc}_y \tilde{\phi}(x,y) \leq \text{vc}_y \phi(x,y), \text{ for } x \in \text{neigh}(0, \mathbb{C}^n).$$

**Proof:** Let $L \subset \mathbb{C}^k$ be a subspace of real dimension $k$ such that $\nabla^2_y \phi(0,0)|_L < 0$. Then $\nabla^2_y \phi(0,0)|_{iL} > 0$. For $t \in \text{neigh}(0, iL)$, put $L_t = t + L$, so that the $\Gamma_t$ form a foliation of a neighborhood of $0 \in \mathbb{C}^k$. Then, it is well known that

$$\phi(x,y(x)) = \inf_{t} \sup_{y \in \Gamma_t} \phi(x,y), \text{ x \in \text{neigh}(0, \mathbb{C}^n).}$$

If $\tilde{\phi} \leq \phi$ as in the statement of the lemma, we have $\nabla^2_y \tilde{\phi}(0,0)|_L < 0$, so $\nabla^2_y \tilde{\phi}(0,0)$ is non-degenerate of signature 0. Then $y \mapsto \tilde{\phi}(x,y)$ has a non-degenerate critical point $\tilde{y}(x)$ and we have the same mini-max formula as for $\phi$:

$$\tilde{\phi}(x,y(x)) = \inf_{t} \sup_{y \in \Gamma_t} \tilde{\phi}(x,y), \text{ x \in \text{neigh}(0, \mathbb{C}^n).}$$

It is then clear that $\tilde{\phi}(x,\tilde{y}(x)) \leq \phi(x,y(x))$. Replacing $\phi$, $\tilde{\phi}$ by their quadratic Taylor polynomial $\phi^{(2)}(x,y)$, $\tilde{\phi}^{(2)}(x,y)$ at (0,0), and the critical points by their linear Taylor polynomials $y^{(1)}(x)$ and $\tilde{y}^{(1)}(x)$, we see that $\phi^{(2)}(x,y^{(1)}(x))$, $\tilde{\phi}^{(2)}(x,\tilde{y}^{(1)}(x))$ are the quadratic Taylor polynomials of $\phi(x,y(x))$, $\tilde{\phi}(x,\tilde{y}(x))$. Taking $\tilde{\phi}^{(2)}$ pluri-harmonic it is clear that $\tilde{\phi}^{(2)}(x,\tilde{y}^{(1)}(x))$ is pluri-harmonic and $\leq \phi^{(2)}(x,y^{(1)}(x))$, so the latter is pluri-subharmonic. This shows that $\text{vc}_y \phi(x,y)$ has a positive semi-definite Levi form at 0. The same argument now works with 0 replaced by any other point in $\text{neigh}(0, \mathbb{C}^n)$ and we get the desired plurisubharmonicity. \hfill \square
c. Contour integration. Let $\phi(y) \in C^\infty(\text{neigh}(0, C^k); \mathbb{R})$. Assume that 0 is a “col” for $\phi$ in the sense that $\nabla_y \phi(0) = 0$ and $\nabla_y^2 \phi(0)$ is non-degenerate of signature $(k, k)$. Consider a smooth contour $\Gamma : \text{neigh}(0, \mathbb{R}^k) \to \text{neigh}(0, C^k)$ with $\Gamma(0) = 0$, $d\Gamma$ injective. We say that $\Gamma$ is a good contour if

$$\phi(y) - \phi(0) \leq -\frac{1}{C} |y|^2, \ y \in \Gamma.$$ 

If $u \in H_{\phi,0}$ i.e. an element of $H_{\phi}(\text{neigh}(0, C^k))$, then

$$I_{\Gamma}(h) = e^{-\phi(0)/h} \int_{\Gamma} u(y; h) dy$$

is well-defined up to an exponentially small ambiguity (and also up to a factor $\pm$ depending on a choice of orientation, that we shall simply forget). As we have seen, a second good contour passing through 0 can be deformed to $\Gamma$ within the set of such good contours.

Now take $\phi(x, y) \in C^\infty(\text{neigh}((0,0), C^{n+k}); \mathbb{R})$ with $\phi(0, y)$ as above. If $\Gamma$ is a good contour for the latter function and $u \in H_{\phi,(0,0)}$, then by deforming $\Gamma$ into an $x$-dependent good contour for $\phi(x, \cdot)$, we see that

$$U(x; h) = \int_{\Gamma} u(x, y; h) dy$$

is a well defined element of $H_{\Phi,0}$, where $\Phi(x) = v_c y \phi(x, y)$.

When working with differential forms of other degrees, we may be interested in other signatures than $(k, k)$. Also, for instance when composing Fourier integral operators, one is frequently in the situation of integrating along a good contour with respect to one group of variables and then for the resulting integral we want a good contour for the last group of variables. The following discussion (that we state only for quadratic forms) shows that this will always work as well as one can possibly hope for.

This has nothing to do with the complex structure, so we consider a decomposition $x = (x', x'') \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-d}$, $x'' \in \mathbb{R}^d$. Let $q$ be a quadratic form on $\mathbb{R}^n$ such that $q''(x'') := q(0, x'')$ is a non-degenerate quadratic form on $\mathbb{R}^d$. Then $x'' \mapsto q(x', x'')$ has a unique critical point $x'' = x''(x')$ depending linearly on $x'$. Consequently, the corresponding critical value $q'(x') = q(x', x''(x'))$ is a quadratic form on $\mathbb{R}^{n-d}$. Let $(m_+(q), m_-(q))$ be the signature of $q$ and denote the signatures of $q'$ and $q''$ similarly. Then by assumption, $m_+(q'') + m_-(q'') = d$. 

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Proposition 2.4.3 Under the above assumptions we have
\[ m_+ (q) = m_+ (q') + m_+ (q''), \quad m_- (q) = m_- (q') + m_- (q''). \] (2.4.1)
If \( L'_-, \ L''_- \) are subspaces of \( \mathbb{R}^n \) of dimension \( m_- (q') \) and \( m_- (q'') \) respectively such that \( q'|_{L'_-}, \ q''|_{L''_-} \) are negative definite, and we put \( L_- = \{(x', x''(x') + x''); x' \in L'_-, x'' \in L''_- \} \), then \( q|_{L_-} \) is negative definite.

Proof: After the change of variables \( x' = \tilde{x}', \ x'' = x''(\tilde{x}') + \tilde{x}'' \), we are reduced to the case when \( x''(\tilde{x}') \equiv 0 \). This means (after dropping the tildes on the new variables) that \( q(x) = q'(x') + q''(x'') \) and the conclusion follows. \( \square \)

d. Application to Fourier transforms. Let \( \phi \in C^\infty (\text{neigh} (x_0, \mathbb{C}^n); \mathbb{R}) \) be pluri-subharmonic with \( \phi''(x_0) \) non-degenerate of signature \( (n, n) \). Let \( \xi_0 = \frac{2}{i} \partial x \phi (x_0) \).

For \( \xi \in \text{neigh} (\xi_0, \mathbb{C}^n) \), we put
\[ \phi^*(\xi) = \text{vc}_x (\phi (x) + \text{Im} (x \cdot \xi)), \]
where the critical point \( x = x(\xi) \) is given by
\[ \xi = \frac{2}{i} \partial x \phi (x), \quad x(\xi_0) = x_0. \]

Guided by the Fourier inversion formula (that we shall study below), we look at
\[ (y, \xi) \mapsto -\text{Im} (x \cdot \xi) + \text{Im} (y \cdot \xi) + \phi (y) \]
which is pluri-subharmonic with the critical point \( y = x, \ \xi = \frac{2}{i} \partial x \phi (x) \) and the corresponding critical value \( \phi (x) \). The critical point is non-degenerate of signature \( (2n, 2n) \) since we have the good contour
\[ \Gamma_R (x) : \xi = \frac{2}{i} \partial x \phi (x) + iR(x - y), \quad |x - y| < r, \]
parametrized by \( y \in B_{\mathbb{C}^n} (x, r) \). Indeed by Taylor expanding, we get:
\[ -\text{Im} ((x - y) \cdot \xi) + \phi (y) = \phi (x) - (R - O(1))|x - y|^2, \quad (y, \xi) \in \Gamma_R (x). \]
with the “\( O(1) \)” uniform in \( R \). Hence \( \Gamma_R \) is a good contour for \( R \) large enough and \( r > 0 \) small enough.
Applying Proposition 2.4.3, we now see that
\[ \xi \mapsto - \text{Im} (x \cdot \xi) + \phi^*(\xi) \]
has a non-degenerate critical point \( \xi = \xi(x) \) of signature \((n, n)\) at \( \xi(x) = \frac{2}{i} \partial_x \phi(x) \) and
\[ \phi(x) = \psi_\xi \left(- \text{Im} (x \cdot \xi) + \phi^*(\xi) \right). \]
This is a standard inversion formula for Legendre transforms when viewing \( \phi^* \) as the Legendre transform of \( \phi \).

Using a good contour, we can define the Fourier transform
\[ \mathcal{F}u(\xi; h) = \int_{\Gamma_\xi} e^{-ix \cdot \xi/h} u(x; h) \, dx \in H_{\phi^*, \xi^*}. \]

For \( v \in H_{\phi^*, \xi^*} \), we put
\[ \mathcal{G}v(x; h) = \frac{1}{(2\pi h)^n} \int_{\Gamma_x} e^{ix \cdot \xi/h} v(\xi) \, d\xi, \]
where \( \Gamma_x^* \) is a good contour such that
\[ \phi^*(\xi) - \text{Im} (x \cdot \xi) - \phi(x) \leq - \frac{1}{C} |\xi - \xi(x)|^2, \quad \xi(x) = \frac{2}{i} \partial_x \phi(x). \]

**Proposition 2.4.4** For \( u \in H_{\phi, x_0} \), we have \( u = \mathcal{G}\mathcal{F}u \) in \( H_{\phi_0, x_0} \) (up to equivalence).

**Proof:** We have
\[ \mathcal{G}\mathcal{F}u(x) = \frac{1}{(2\pi h)^n} \int_{\Gamma_x} \int_{\Gamma_\xi} e^{i(x-y) \cdot \xi/h} u(y) \, dy \, d\xi \, (\text{iterated integral}). \]

Along the composed contour we have (cf Proposition 2.4.3)
\[ -\text{Im} (x \cdot \xi) + \phi^*(\xi) \leq \phi(x) - \frac{1}{C} |\xi - \xi(x)|^2, \quad \xi \in \Gamma_x^*, \]
\[ \text{Im} (y \cdot \xi) + \phi(y) \leq \phi^*(\xi) - \frac{1}{C} |y - x(\xi)|^2, \quad y \in \Gamma_\xi, \]
so
\[ -\text{Im} ((x-y) \cdot \xi) + \phi(y) \leq \phi(x) - \frac{1}{C} (|\xi - \xi(x)|^2 + |y - x(\xi)|^2). \]
The composed contour is a good contour like $\Gamma_R$.

Thus, up an exponentially small error, we can replace the composed contour by $\Gamma_R$ for $R$ large enough and get

$$\frac{1}{(2\pi h)^n} \iiint_{\Gamma_R(x)} e^{i(x-y)\cdot \xi/h} u(y) dy d\xi = \left( \frac{R}{i 2\pi h} \right)^n \iint_{|x-y|<r} e^{\frac{i}{h}(x-y)\cdot \partial_x \phi(x)-\frac{R}{h}|x-y|^2} u(y) dy \wedge d\overline{y} = (1 + O(e^{-Rr^2/h})) u(x)$$

by the spherical mean-value property for holomorphic functions. □

## 2.5 Pseudodifferential operators and Fourier integral operators

Let $a(x, y, \theta; h)$ be an analytic symbol defined near $(x_0, x_0, \xi_0) \in \mathbb{C}^{3n}$, so that $a \in H_{0,(x_0,x_0,\xi_0)}$. Let $\phi \in C^\infty(\text{neigh}(x_0, \mathbb{C}^n); \mathbb{R})$ with $(2/i)\partial_x \phi(x_0) = \xi_0$. For $u \in H_{\phi,x_0}$, we define $Au \in H_{\phi,x_0}$ by

$$Au(x; h) = \frac{1}{(2\pi h)^n} \iiint_{\Gamma(x)} e^{i(x-y)\cdot \theta/h} a(x, y, \theta; h) u(y; h) dy d\theta,$$

where $\Gamma(x) = \Gamma_R(x)$ is the good contour introduced at the end of the preceding section so that (for $R$ large enough)

$$e^{-\phi(x)/h} \left| e^{i(x-y)\cdot \theta/h} \right| e^{\phi(y)/h} \leq e^{-\frac{1}{h}(R-O(1))|x-y|^2}$$

along $\Gamma(x)$. It follows that

$$Au(x; h) = A_{\Gamma} u(x; h) = \int k_{\Gamma}(x, y; h) u(y) L(dy),$$

where

$$|k_{\Gamma}(x, y; h)|e^{(-\phi(x)+\phi(y))/h} \leq C_{\Gamma} h^{-n} e^{-\frac{1}{h}(R-O(1))|x-y|^2}.$$

$A_{\Gamma}$ is uniformly bounded $L^2_{\phi,x_0} \rightarrow L^2_{\phi,x_0}$. Here, we assume for simplicity that $|a(x, y, \theta; h)| \leq O(1)$. Without that assumption we would need to insert a factor $C_{\epsilon} e^{\epsilon/h}$ to the right in the last estimate and the boundedness statement about $A_{\Gamma}$ has to be modified accordingly.
We define the symbol of $A$ by

$$\sigma_A(x, \xi; h) = e^{-ix \cdot \xi/h}A(e^{i(\cdot) \cdot \xi/h}), \quad (x, \xi) \in \text{neigh} ((x_0, \xi_0), \mathbb{C}^{2n}).$$

The method of stationary phase gives

$$\sigma_A(x, \xi; h) \equiv \sum_{|\alpha| \leq 1/(Ch)} \frac{1}{\alpha!} (\partial_\xi^\alpha D_x a)(x, x, \xi; h)$$

and this is (a realization of) a classical analytic symbol when $a$ is a classical analytic symbol. Clearly $\sigma_A \equiv a$ when $a$ does not depend on $y$.

**Lemma 2.5.1** Assume that $\sigma_A = 0$ in $H_{0,(x_0,\xi_0)}$. Then $\exists b \in H_{0,(x_0,x_0,\xi_0)}$ with values in the $(n-1)$-forms in $\theta$ such that

$$e^{i(x-y) \cdot \theta/h}a(x, y, \theta)d\theta \equiv ih\theta (e^{i(x-y) \cdot \theta/h}d\theta), \quad \text{in } H_{-\text{Im} ((x-y) \cdot \theta),(x_0,x_0,\xi_0)}.$$

Applying the Stokes formula along the good contour, it then follows that $A = 0$ as an operator in $H_{\phi,x_0}$.

**Proof:** By a simple change of variables,

$$(2\pi h)^n \sigma_A(x, \eta) = \int \int e^{-iy \cdot \eta/h} a(x, x - y, \theta; h) e^{i(y \cdot \theta/h} dy d\theta$$

$$= \mathcal{F}_{(y, \theta) \rightarrow (\eta, \theta^*)}(u)(\eta, 0; h) = v(x, \eta, 0; h),$$

where $x$ is treated as a parameter and $v := \mathcal{F}_{(y, \theta) \rightarrow (\eta, \theta^*)}(u)$.

We have $u \in H_\phi$, $v \in H_{\phi^*}$, $\phi = -\text{Im} (y \cdot \theta)$, $\phi^* = \text{Im} (\eta \cdot \theta^*)$ and we observe that $\phi$ and $\phi^*$ are pluri-harmonic. Now $v(x, \eta, 0; h) = 0$ and Taylor’s formula gives

$$v(x, \eta, \theta^*; h) = \sum_{1}^{n} \hat{v}_j(x, \eta, \theta^*; h) \theta^*_j, \quad \hat{v}_j \in H_{\phi^*},$$

and $\hat{v}_j$ depend holomorphically on $x$. By Fourier inversion

$$u(x, y, \theta; h) = \sum_{1}^{n} hD_\theta v_j \text{ in } H_\phi, \quad v_j \in H_\phi,$$

so $v_j = b_j(x, y, \theta; h)e^{iy \cdot \theta/h}$, $b_j \in H_0$. Going back to the original variables, we get the identity in the lemma. \qed
General remarks about Fourier integral operators. Let

$$\phi(z, y, \theta) \in C^2(\text{neigh } ((z_0, y_0, \theta_0), C^{n_x+n_y+n_\theta}; \mathbb{R}), \ f \in C^2(\text{neigh } (y_0, C^n); \mathbb{R})$$

be pluri-subharmonic and assume that \((y, \theta) \mapsto \phi(z, y, \theta) + f(y)\) has a col at \((y_0, \theta_0)\).

If \(a \in H_{\phi,(z_0,y_0,\theta_0)},\) we can define \(A : H_{f,y_0} \to H_{g,z_0}\) by

$$Au(z; h) = \int_{\Gamma_1(z)} a(z, y, \theta; h)u(y)dyd\theta,$$

where \(g(z) = v_{c_y,\theta}(\phi(z, y, \theta) + f(y))\) and \(\Gamma_1(z)\) is a good contour.

Let \(b(x, z, w; h) \in H_{\psi,(x_0,z_0,w_0)},\ x \in C^n_x\) and assume that \(\psi, g\) fulfill the same assumptions as \(\phi, f\). Then for \(v \in H_{g,z_0}\), we define \(Bv \in H_{k,x_0}\) by

$$Bv(x; h) = \int_{\Gamma_2(x)} b(x, z, w; h)v(z)dzdw,$$

where \(\Gamma_2(x)\) and \(k(x)\) denote a good contour and the critical value respectively, for \((z, w) \mapsto \psi(x, z, w) + g(z)\).

We can then define \(B \circ A : H_{f,y_0} \to H_{k,x_0}\) by

$$B \circ Au(x; h) = \iiint_{\Gamma(x)} b(x, z, w)a(z, y, \theta)u(y)dyd\theta dzdw,$$

where \(\Gamma(x)\) is the composed contour given by \((z, w) \in \Gamma_2(x), (y, \theta) \in \Gamma_1(z)\). It is a good contour for

\((z, w, y, \theta) \mapsto \psi(x, z, w) + \phi(z, y, \theta) + f(y)\).

Now assume that

\((z, w) \mapsto \psi(x_0, z, w) + \phi(z, y_0, \theta_0)\) \hspace{1cm} (2.5.1)

has a col at \((z_0, w_0)\). Let \(F(x, y, \theta)\) be the critical value when \((z, y, \theta)\) varies near \((x_0, y_0, \theta_0)\). Then \(F\) is pluri-subharmonic, and knowing that \((z, w, y, \theta) \mapsto \psi + \phi + f\) has col, we see that

\((y, \theta) \mapsto F(x, y, \theta) + f(y)\) \hspace{1cm} (2.5.2)

has a col. Hence, if \(\Gamma_3(x, y, \theta)\) is a good contour for (2.5.1) and \(\Gamma_4(x)\) a good contour for (2.5.2), the composed contour

\[\tilde{\Gamma}(x) : (y, \theta) \in \Gamma_4(x), \ (z, w) \in \Gamma_3(x, y, \theta)\]
is good for

$$(z, w, y, \theta) \mapsto \psi(x, z, w) + \phi(z, y, \theta) + f(y).$$

By Stokes, we can replace $\Gamma(x)$ in the formula for $B \circ Au(x)$ by $\tilde{\Gamma}(x)$ and write

$$B \circ Au(x; h) = \int\int\int_{\Gamma_4(x)} b(x, z, w) a(z, y, \theta) dz dw$$

$$= \int\int_{\Gamma_3(y, \theta)} b(x, z, w) a(z, y, \theta) dz dw$$

$$= c(x, y, \theta) \in H_F,$$

This remark can be applied to the case when $A, B$ are pseudodifferential operators

**Theorem 2.5.2** Let $A, B : H_{\phi,x_0} \to H_{\phi,x_0}$ be two pseudodifferential operators. Then $B \circ A$ is a pseudodifferential operator with symbol

$$\sigma_{B \circ A}(x, \xi; h) = \sum_{|\alpha| \leq \frac{1}{2n}} \frac{1}{\alpha!} h^{\frac{|\alpha|}{2n}} \partial_\xi^\alpha \sigma_B(x, \xi; h) D_x^\alpha \sigma_A(x, \xi; h).$$

### 2.6 FBI-transforms and analytic wavefront sets

Let $\phi \in \text{Hol}(\text{neigh}((x_0, y_0), C^{2n})), y_0 \in \mathbb{R}^n$ and assume that

$$\phi'_y(x_0, y_0) = -\eta_0 \in \mathbb{R}^n, \quad \text{Im} \phi''_{yy}(x_0, y_0) > 0,$$

$$\det \phi''_{xy}(x_0, y_0) \neq 0. \quad (2.6.1)$$

Let $a(x, y; h)$ be an elliptic classical analytic symbol defined near $(x_0, y_0)$ and let $\chi \in C^\infty_0(\text{neigh}(y_0, \mathbb{R}^n))$ be equal to one near $y_0$. If $u \in D'(\mathbb{R}^n)$ (or just defined in a neighborhood of the support of $\chi$), we put

$$Tu(x; h) = \int e^{i\phi(x,y)/h} a(x, y; h) \chi(y) u(y) dy, \quad x \in \text{neigh}(x_0, C^n). \quad (2.6.2)$$

**Proposition 2.6.1** $Tu \in H_{\Phi}(\text{neigh}(x_0))$, where

$$\Phi = \sup_{y \in \text{neigh}(y_0, \mathbb{R}^n)} -\text{Im} \phi(x, y) \in C^\infty(\text{neigh}(x_0, C^n); \mathbb{R}).$$
This is evident since $R^n \ni y \mapsto -\text{Im } \phi(x, y)$ has a non-degenerate maximum at $y = y(x) \in \text{neigh } (y_0, R^n)$.

Introduce

$$\Lambda_\Phi = \{(x, \frac{2}{i} \partial_x \Phi(x)); x \in \text{neigh } (x_0, C^n)\}$$

Then (and here we only use that $\Phi$ is real and smooth), the restriction to $\Lambda_\Phi$ of the complex symplectic 2-form $\sigma = \sum d\xi_j \wedge dx_j$ is real, so $\Lambda_\Phi$ is an I-Lagrangian manifold, i.e. a Lagrangian manifold for the real symplectic form $\text{Im } \sigma$.

**Proposition 2.6.2** $\Lambda_\Phi = \kappa_T(R^{2n})$, where

$$\kappa_T: \text{neigh } ((y_0, \eta_0)) \ni (y, -\phi'_y(x, y)) \mapsto (x, \phi'_x(x, y)) \in \text{neigh } ((x_0, \xi_0))$$

is the complex canonical transformation associated to $T$, when viewed as a Fourier integral operator. Here $(x_0, \xi_0) = \kappa_T(y_0, \eta_0) = (x_0, (2/i)\partial_x \Phi(x_0))$. In particular $\sigma|_{\Lambda_\Phi}$ is real and non-degenerate. ($\Lambda_\Phi$ is I-Lagrangian and R-symplectic.) Further, $\Phi$ is strictly pluri-subharmonic.

**Proof:** The real critical point of $-\text{Im } \phi(x, \cdot)$ is characterized by the property that $\eta(x) := -\phi'_y(x, y(x))$ is real. Further,

$$\frac{2}{i} \partial_x \Phi(x) = \frac{2}{i}(\partial_x (-\text{Im } \phi))(x, y(x)) = \phi'_x(x, y(x)).$$

Hence $\Lambda_\Phi$ is contained in $\kappa_T(R^{2n})$ and the two manifolds have the same dimension so they have to coincide (near $(x_0, \xi_0)$).

We then know that

$$\sigma|_{\Lambda_\Phi} = \sum_1^n d\left(\frac{2}{i} \partial_x \Phi(x)\right) \wedge dx_j = \frac{2}{i} \sum_1^n \sum_1^n \partial_{x_k} \partial_{x_j} \Phi dx_k \wedge dx_j$$

is non-degenerate, so the Levi-form of $\Phi$ is non-degenerate. Since $\Phi$ by definition is the supremum of the family of pluri-harmonic functions $x \mapsto -\text{Im } \phi(x, y)$ we know that $\Phi$ is pluri-subharmonic and hence strictly pluri-subharmonic.

For $y \in R^n$ (close to $y_0$) let

$$\Gamma_y = \{x \in C^n; y(x) = y\} = \pi_x \kappa_T(T^*_y R^n),$$

where $\pi_x: C^{2n}_{x,\xi} \to C^n_x$ is the natural projection, so that $\Gamma_y$ is of real dimension $n$ and the $\Gamma_y$ form a foliation of $\text{neigh } (x_0, C^n)$. $\Gamma_y$ is totally real: $T_x \Gamma_y \cap iT_x \Gamma_y = 0$, $\forall x \in \Gamma_y$. In fact, $T_x \Gamma_y = \{t_x \in C^n; \partial_{y_{2x}}t_x \in R^n\}$.
For every fixed real $y$:

$$\Phi(x) + \text{Im} \phi(x, y) = -\text{Im} \phi(x, y(x)) + \text{Im} \phi(x, y) \asymp \text{dist}(x, \Gamma_y)^2. \quad (2.6.3)$$

Since $x \mapsto -\text{Im} \phi(x, y)$ is pluri-harmonic, this gives another proof of the fact that $\Phi(x)$ is strictly pluri-subharmonic.

**Exercise** Explore the standard case of Bargmann transforms with $\phi(x, y) = i(x - y)^2/2$.

**Exercise** Let $f(y)$ be analytic near $y_0$, real valued on the real domain and with $f'(y_0) = \eta_0$. Show that

$$T(e^{if/h}) = h^{n/2}c(x; h)e^{ig(x)/h},$$

where $c(x; h)$ is a classical analytic symbol of order 0 and

$$g(x) = \text{vc}_{y \in \text{neigh}(y_0, \mathbb{C}^n)}(\phi(x, y) + f(y))$$

is holomorphic, $\Lambda_g := \{(x, g'((x)))\} = \kappa_T(\Lambda_f)$ where $\Lambda_f$ is defined as $\Lambda_g$.

Let $(\Lambda_f)_R = \Lambda_f \cap \mathbb{R}^{2n}$. Show that $-\text{Im} g \leq \Phi$ and that more precisely,

$$\Phi(x) + \text{Im} g(x) \asymp \text{dist}(x, \pi_x(\kappa_T((\Lambda_f)_R)))^2. \quad (2.6.4)$$

Observe also that $\pi_x(\kappa_T((\Lambda_f)_R))$ is transversal to $\Gamma_y$.

Assume that $\eta_0 \neq 0$. For $x \in \text{neigh}(x_0)$, write

$$(y(x), \eta(x)) = (y(x), -\partial_y \phi(x, y(x))) \in T^*\mathbb{R}^n \setminus 0,$$

where $y(x)$ is the local real maximum of $-\text{Im} \phi(x, \cdot)$. Also, we have

$$(y(x), \eta(x)) = \kappa_T^{-1}(x, 2i\partial_x \Phi(x)).$$

**Definition 2.6.3** Let $u$ be a distribution defined near $y_0$, independent of $h$. We say that $(y(x), \eta(x)) \notin \text{WF}_a(u)$ if $Tu = 0$ in $H_{\Phi, x}$.

We shall see that this defines a closed conic subset $\text{WF}_a(u)$ of $T^*(\text{neigh}(y_0, \mathbb{R}^n)) \setminus 0$, independent of the choice of $T$.

In order to prove that the definition does not depend on the choice of $T$ we would like to construct “the inverse $T^{-1}$”. However, this can never succeed completely.
since $Tu$ only carries microlocal information about $u$ near $(y_0, \eta_0)$. We can however give meaning to this inverse on certain smaller spaces and that will suffice to be able to describe a second FBI-transform $\tilde{T}u$ in terms of $Tu$.

Put

$$Sv(x; h) = h^{-n} \int e^{-i\phi(z,x)/h} b(z, x; h) v(z) dz,$$  \hspace{1cm} (2.6.5)

where $b$ is an elliptic classical analytic symbol of order 0, defined near $(x_0, y_0)$. Formally,

$$STu(x; h) = h^{-n} \int e^{(\phi(z,x)+\phi(z,y))/h} b(z, x; h, y; h) a(z, y; h) u(y) dy dz$$  \hspace{1cm} (2.6.6)

and we can apply the Kuranishi trick (change of variables in $z$) to see that formally

$$STu(x; h) = \frac{1}{(2\pi h)^n} \int e^{(x-y)\theta} c(x, y, \theta; h) u(y) dy d\theta,$$  \hspace{1cm} (2.6.7)

where $c$ is an elliptic classical analytic symbol of order 0, defined near $(y_0, y_0, \eta_0)$. According to Lemma 2.5.1 and the previously given definition of the symbol of a pseudodifferential operator, we can replace $c$ by $\tilde{c}(x, \theta; h)$, independent of $y$ and still elliptic to get a new pseudodifferential operator which has the same action on expressions as in the last exercise above.

Let $\tilde{d}$ satisfy $\tilde{d} \# \tilde{c} = 1$. Then

$$\tilde{d}(x, hD_x; h) \circ ST = 1$$

when acting on functions as in the exercise. On the other hand we can apply stationary phase to get formally

$$\tilde{d}(x, hD; h) Sv = h^{-n} \int e^{-i\phi/h} \tilde{b}v(z) dz =: \tilde{S}v(x; h)$$

Our compositions are well defined and hence associative when restricted to expressions as in the exercise and we therefore get

$$\tilde{S}T = 1.$$ 

Dropping the tildes, we have shown that we can find $S$ of the form (2.6.5) such that

$$ST = 1.$$
when acting on expressions as in the exercise.

When trying to define \( S_v(x; h) \) for \( v \in H_\Psi \), we would like to have a contour \( \Gamma \) in \( z \) space such that
\[
\Im \phi(z, x) + \Phi(z) \leq 0, \quad z \in \Gamma,
\]
with strict inequality near the boundary. In view of (2.6.3) the best possible choice in general is \( \Gamma = \Gamma_x \) and we then just achieve equality.

If however \( v \in H_\Psi \), where \( \Psi - \Phi \asymp -\dist (z, \tilde{\Gamma})^2 \) and \( \tilde{\Gamma} \) is a real manifold of dimension \( n \) transversal to \( \Gamma_x \), then \( S_v \) is well-defined. In particular if \( u \) is as in the exercise, \( v = Tu \), this is the case with \( \Psi = -\Im g \), so \( S_v \) is well-defined up to an exponentially small ambiguity, and we get \( S_v \equiv u \) in \( H_{-\Im f} \).

Let
\[
\tilde{T}u(x; h) = \int e^{i\phi(x, y)/h} \tilde{a}(x, y; h)u(y)dy
\]
be a second FBI-transform with \( \tilde{\phi}, \tilde{a} \) defined near \((\tilde{x}_0, y_0)\) and with \( -\tilde{\phi}'(\tilde{\xi}_0, y_0) = \eta_0 \).

Then formally
\[
\tilde{T}Sv(x; h) = h^{-n} \int e^{i\left(\tilde{\phi}(x, y) - \phi(z, y)\right)}/h} \tilde{a}(x, y; h) a(z, y; h)u(y)dydz. \tag{2.6.8}
\]

This is a Fourier integral operator with associated canonical transformation \( \kappa_{\tilde{T}} \circ \kappa_T^{-1} \), mapping \( \Lambda_\Phi \) to \( \Lambda_{\tilde{\Phi}} \), and it follows from this observation, or by direct verification, that

\[
(y, z) \mapsto -\Im \tilde{\phi}(x, y) + \Im \phi(z, y) + \Phi(z) =: F
\]
has a non-degenerate critical point, given by the conditions
\[
(z, \frac{2}{i} \partial_z \Phi(z)) = \kappa_T(y, \eta), \quad (x, \frac{2}{i} \partial_x \tilde{\Phi}(x)) = \kappa_{\tilde{T}}(y, \eta),
\]
where \( (y, \eta) \) is real \( y = y(z) = \tilde{y}(x), \eta = \eta(z) = \tilde{\eta}(z) \).

Next, we show that there is a good contour for (2.6.8): As a first attempt, we take \( y \in \mathbb{R}^n, z \in \Gamma_y \). Along that contour we have
\[
F(y, z) - \tilde{\Phi}(x) = -(\tilde{\Phi}(x) + \Im \tilde{\phi}(x, y)) \asymp -|y - \tilde{y}(x)|^2.
\]

1A general local theory for Fourier integral operators can be developed in the spirit of Section 2.5. See [32], Chapter 11.
Thus the contour is “almost good”. Since our critical point is non-degenerate, it is then clear that we can make a small deformation and find a good contour. In conclusion

\[ \tilde{T}S \text{ is a well-defined Fourier integral operator } H_{\Phi,x_0} \to H_{\tilde{\Phi},\tilde{x}_0}. \]

**Proposition 2.6.4** For \( x \in \text{neigh}(x_0), \tilde{x} \in \text{neigh}(\tilde{x}_0) \) related by

\[ \tilde{\kappa}^{-1}_T(\tilde{x}, (2/i)\partial \tilde{\Phi}(\tilde{x})) = \kappa^{-1}_T(x, (2/i)\partial \Phi(x)), \]

the following two statements are equivalent:

1) \( \tilde{\mathcal{T}}u = 0 \) in \( H_{\tilde{\Phi},\tilde{x}} \).

2) \( Tu = 0 \) in \( H_{\Phi,x} \).

**Proof:** Take \( x = x_0, \tilde{x} = \tilde{x}_0 \) for simplicity. Let \( \chi \in C^\infty_0(\text{neigh}(\eta_0, \mathbb{R}^n)) \) be equal to one near \( \eta_0 \). Without loss of generality, we may assume that the distribution \( u \) has compact support in a neighborhood of \( y_0 \). Then from the (classical!) Fourier inversion formula,

\[ u(x) = \frac{1}{(2\pi h)^n} \int e^{ix\cdot\eta/h} \mathcal{F}u(\eta)d\eta, \]

and contour deformations, we see that

\[ Tu = T\chi(hD_y)u \text{ in } H_{\Phi,x_0}, \quad \tilde{T}u = \tilde{T}\chi(hD_y)u \text{ in } H_{\tilde{\Phi},\tilde{x}_0}. \]

On the other hand \( v = \chi(hD_y)u \) is a superposition of plane waves (special cases of states as in the last exercise), so

\[ \chi(hD_y)u = ST\chi(hD_y)u + \mathcal{O}(e^{-1/C\hbar}), \]

where now

\[ Sv(y) = \int_{\Gamma_y} e^{-i\phi(x,y)/\hbar} b(x, y; h)v(x)dx. \]

Consequently,

\[ \tilde{T}\chi(hD_y)u = \tilde{T} \circ ST\chi(hD_y)u \text{ in } H_{\tilde{\Phi},\tilde{x}_0}. \]
Here, for each plane wave in $\chi(hD_y)u$, we can make a contour deformation to the good contour discussed above for the Fourier integral operator $\tilde{T}S$ and putting everything together, we get

$$\tilde{T}u = (\tilde{T}S)(Tu) \text{ in } H_{\Phi,\tilde{x}_0}.$$  

Since the Fourier integral operator $\tilde{T}S$ maps $H_{\Phi,x_0} \to H_{\tilde{\Phi},\tilde{x}_0}$, we see that $\tilde{T}u = 0$ in $H_{\tilde{\Phi},\tilde{x}_0}$ if $Tu = 0$ in $H_{\Phi,x_0}$. The converse implication also holds. □

This shows that the definition of $WF_a(u)$ does not depend on the choice of $T$. By a simple dilation in $h$ we then see that it is a conic subset of $T^*X \setminus 0$ (if $X \subset \mathbb{R}^n$ is the open set where $u$ is defined). Another basic property of the analytic wavefront set is given by

**Proposition 2.6.5** We have

$$\pi_y(WF_a(u)) = \text{Sing Supp}_a(u),$$

where the right hand side denotes the analytic singular support, i.e. the complement in $X$ of the largest open subset where $u$ is real analytic.

**Idea of the proof.** We start by using a resolution of the identity of the form $1 = \int_{T^*\mathbb{R}^n} \pi_\alpha \, d\alpha$ where $\pi_\alpha$ is a Gaussian Fourier integral operator “concentrated at $\alpha$”. If $y_0 \notin \pi_y(WF_a(u))$, then a simple adaptation of the proof above shows that $\pi_\alpha u$ decays exponentially when $\alpha_y$ tends to infinity while $\alpha_y$ is confined to a small neighborhood of $y_0$. (Here we write $\alpha = (\alpha_y, \alpha_\eta).$)

### 2.7 Egorov’s theorem and elliptic regularity.

Let $\tilde{P}(y, D_y) = \sum_{|\alpha| \leq m} a_\alpha(y)D_y^\alpha$ be a differential operator with analytic coefficients, defined on an open set $X \subset \mathbb{R}^n$. Let $T$ be an FBI-transform as above. Then we have the Egorov theorem which states that there exists a pseudodifferential operator with classical analytic symbol, $P(x, hD_h; h) : H_{\Phi,x_0} \to H_{\Phi,x_0}$ such that

$$PTu = Th^m \tilde{P}u \text{ in } H_{\Phi,x_0}$$

when $u \in \mathcal{D}'(X)$ is independent of $h$. Indeed, we can take $P = Th^m \tilde{P}S$. For the leading symbols, we have the relation

$$p \circ \kappa_T = \tilde{p}. \quad (2.7.1)$$
Theorem 2.7.1 In the above situation, let \( u \in D'(X) \) be independent of \( h \) and assume that \( \tilde{P}u \) is analytic on \( X \). Then \( WF_a(u) \subset \tilde{p}^{-1}(0) \).

Proof: Let \((y_0, \eta_0) \in T^*X \setminus 0\) be a point where \( \tilde{p}(y_0, \eta_0) \neq 0 \) and assume that \((y_0, \eta_0) \notin WF_a(\tilde{P}u)\) (which is a weaker assumption than in the theorem). We choose \( T \) adapted to the point \((y_0, \eta_0)\). Then

\[
PTu = 0 \text{ in } H_{\Phi,x_0} \text{ and } p(x_0, \frac{2}{t} \partial_x \Phi(x_0)) \neq 0.
\]

Let \( Q(x, \xi; h) \) be a classical analytic symbol \( Q \sim \sum_0^\infty h^k q_k(x, \xi) \) such that

\[
Q\# P = 1 \text{ near } (x_0, \xi_0).
\]

Correspondingly, we have \( Q(x, hD; h) : H_{\Phi,x_0} \to H_{\Phi,x_0} \) so that

\[
Q(x, hD; h) \circ P(x, hD; h) = 1 : H_{\Phi,x_0} \to H_{\Phi,x_0}.
\]

Apply this to \( Tu \):

\[
Tu = QPTu = 0 \text{ in } H_{\Phi,x_0}.
\]

Hence \((y_0, \eta_0) \notin WF_a(u)\). We have thus shown that \( WF_a(u) \subset WF_a(\tilde{P}u) \cup \tilde{p}^{-1}(0) \) which is a stronger statement than in the theorem. \( \square \)

For the notes of a course of more than 3 hours, it would here be the natural place to discuss the method of non-characteristic deformations and the Kawai-Kashiwara theorem about propagation of analytic regularity for micro-hyperbolic operators. See [32], Chapter 10.

### 2.8 Analytic WKB and quasi-modes

Let \( P(x, hD; h) \) be a classical analytic pseudodifferential operator of order 0, defined near \((0, \xi_0) \in \mathbb{C}^{2n}\), such that the leading symbol satisfies

\[
p(0, \xi_0) = 0, \partial_{\xi_n} p(0, \xi_0) \neq 0.
\]

Let \( \phi \in \text{Hol (neigh } (0, \mathbb{C}^n)) \) solve the eikonal problem

\[
p(x, \phi'(x)) = 0, \phi'(0) = \xi_0.
\]

(2.8.1)

Let \( H \) be the hypersurface \( x_n = 0 \). We use the standard notation \( x = (x', x_n) \in \mathbb{C}^n \).
Theorem 2.8.1 Let \( v(x; h), w(x'; h) \) be classical analytic symbols of order 0 defined near 0 in \( \mathbb{C}^n \) and \( \mathbb{C}^{n-1} \) respectively. Then there exists a classical analytic symbol \( u(x; h) \) defined near 0 in \( \mathbb{C}^n \) such that
\[
e^{-i\phi(x)/h} \circ P \circ e^{i\phi/h} u = hv, \quad u|_H = w. \tag{2.8.2}
\]

Proof: We may assume that \( w = 0 \). Also \( e^{-i\phi(x)/h} \circ P \circ e^{i\phi/h} \) is a classical analytic pseudodifferential operator of order 0 with leading symbol \( p(x, \phi_x'(x) + \xi) \), so we may assume that \( \phi = 0, p(x, 0) = 0 \). After a change of variables, which does not modify \( H \), we may also assume that \( \partial_\xi p(x, 0) = 0, \partial_\xi n p = i \), or in other words, \( p(x, \xi) = i\xi_n + O(\xi^2) \).

Writing \( P = \sum_0^\infty h^k p_k(x, \xi), p_0 = p \), the first equation in (2.8.2) becomes
\[
\partial_{x_n} u + p_1(x, 0) u(x; h) + \frac{1}{h} A u = v
\]
\[
A = \sum_{k+|\alpha|\geq 2} \frac{h^k}{\alpha!} (\partial_{\xi}^zp_k)(x, 0)(hD_x)^\alpha = \sum_{k=2}^\infty h^k A_k,
\tag{2.8.3}
\]
where \( A \) has the same general properties as in Section 2.2. Assume for simplicity that \( p_1(x, 0) = 0 \) (which otherwise can be achieved by conjugation).

Let \( \Omega = \{ x \in \mathbb{C}^n; \frac{|x'|}{R} + \frac{|x_n|}{r} < 1 \} \), where \( R, r > 0 \) are small enough so that we stay in the domains of definition of the various symbols and operators. For \( 0 \leq t \leq r \), we define \( \Omega_t \subset \mathbb{C}^n \) by
\[
\frac{|x'|}{R - \frac{Rt}{r}} + \frac{|x_n|}{r} < 1.
\]

Let \( a \in \text{Hol}(\Omega_0) \) have the property that for some \( k > 1 \):
\[
\sup_{\Omega_t} |a| \leq C(a, k)t^{-k}, \quad 0 < t \leq r.
\]

Put
\[
\partial_{x_n}^{-1} a(x) = \int_0^{x_n} a(x', y_n)dy_n.
\]
Then
\[
\sup_{\Omega_t} |a| \leq C(a, k) \int_0^{+\infty} s^{-k}ds = \frac{C(a, k)}{(k-1)t^{k-1}}.
\]

Let \( a = \sum_2^\infty a_k h^k \) be a classical analytic symbol of order \(-2\) such that
\[
\sup_{\Omega_t} |a_k| \leq \frac{f(a, k)k^k}{t^k}, \quad 0 < t \leq r,
\tag{2.8.4}
\]

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where $k \mapsto f(a, k)$ grows at most exponentially. Then,

$$b := (h\partial_{x_n})^{-1}a = \sum_1^\infty b_k h^k, \ b_k = \partial_{x_n}^{-1}a_{k+1},$$

$$\sup_{\Omega_t} b_k \leq \frac{f(a, k+1)(k+1)^{k+1}}{kt^k} \leq 2ef(a, k+1)\frac{k^k}{tk}. $$

Hence, $f(b, k) \leq 2ef(a, k + 1)$, when defining $f(b, k)$ as in (2.8.4).

Put

$$||a||_\rho = \sum_2^\infty f(a, k)\rho^k, \ ||b||_\rho = \sum_1^\infty f(b, k)\rho^k.$$ 

Then

$$||b||_\rho \leq \frac{2e}{\rho}||a||_\rho.$$ (2.8.5)

The problem (2.8.2), (2.8.3), with $w = 0$ and $p_1(x, 0) = 0$, can be written

$$u + (h\partial_{x_n})^{-1}Au = h(h\partial_{x_n})^{-1}v =: \tilde{v},$$ (2.8.6)

where $\tilde{v}$ is a classical analytic symbol of order 0. Defining $||A||_\rho$ as in Section 2.2 with respect to the family $\Omega_t$, we have

$$||Au||_\rho \leq ||A||_\rho ||u||_\rho \leq O(\rho^2)||u||_\rho,$$

when $\rho$ is small enough. Hence by (2.8.5),

$$||(h\partial_{x_n})^{-1}Au||_\rho \leq O(1)\rho||u||_\rho.$$ 

We then see from (2.8.6) that $||u||_\rho < \infty$ when $\rho > 0$ is small enough and we conclude that $u$ is an analytic symbol in $\Omega_0$. □

We next discuss quasimodes for non-self-adjoint differential operators in the semi-classical limit. Let

$$P = P(x, hD_x; h) = \sum_{|\alpha| \leq m} a_\alpha(x; h)(hD_x)^\alpha$$

be a semi-classical differential operator defined on an open set $\Omega \subset \mathbb{R}^n$. Assume that

$$a_\alpha(x; h) \sim \sum_{0}^\infty a^k_\alpha(x)h^k$$ (2.8.7)
are (realizations of) classical analytic symbols. The semi-classical principal symbol of $P$ is then
\begin{equation}
 p(x, \xi) = \sum_{\lvert \alpha \rvert \leq m} a_{\alpha}(x)\xi^\alpha.
\end{equation}

Let $(x_0, \xi_0) \in T^*\Omega$ be a point where
\begin{equation}
 p(x_0, \xi_0) = 0, \quad \frac{1}{2i}\{p, \overline{p}\}(x_0, \xi_0) > 0.
\end{equation}

Here, $\{a, b\} = a'_\xi \cdot b'_x - a'_x \cdot b'_\xi$ denotes the Poisson bracket of two sufficiently smooth functions $a(x, \xi)$, $b(x, \xi)$. The following result, in a different non-semi-classical formulation is due to Hörmander [11, 12] in the smooth setting and goes back to Sato-Kawai-Kashiwara [26] in the analytic case. See [5] for references and direct proofs in the semi-classical formalism.

**Theorem 2.8.2** There exist an analytic function $\phi(x)$ and a classical analytic symbol $b(x; h)$ of order 0, defined in a neighborhood of $x_0$ such that
\begin{align}
 \phi(x_0) &= 0, \quad \phi'(x_0) = \xi_0, \quad (2.8.10) \\
P(x, \phi'(x)) &= 0, \quad x \in \text{neigh } (x_0, \Omega), \quad (2.8.11) \\
\text{Im } \phi''(x_0) &> 0, \quad (2.8.12) \\
P(\chi(x)b(x; h)e^{i\phi(x)/h}) &= O(1)e^{-\frac{1}{Ch}}, \quad C = C_\chi > 0, \quad (2.8.13)
\end{align}

if $\chi \in C_0^\infty(\text{neigh } (x_0, \Omega))$ is equal to 1 near $x_0$ and has its support sufficiently close to $x_0$,\begin{equation}
 \left\| \chi be^{i\phi/h} \right\|_{L^2} = h^{n/4}(1 + O(e^{-1/(Ch)})).
\end{equation}

As usual, it follows from the proof that the conclusion remains uniformly valid if we replace $P$ by $P - z$ for $z \in \text{neigh } (0, C)$. More generally the conclusion is valid for $P - z$ for $z \in \text{neigh } (z_0, C)$, if we replace the condition $p(x_0, \xi_0) = 0$ by $p(x_0, \xi_0) = z_0$ in (2.8.9).

When $P$ can be realized as a closed operator on $L^2(\Omega)$ or on $L^2(M)$ for some manifold containing $\Omega$, then we conclude that $\|(P - z)^{-1}\| \geq e^{1/(Ch)}/C$ for some $C > 0$ and for $z \in \text{neigh } (z_0, C) \setminus \sigma(P)$, where $\sigma(P)$ denotes the spectrum of $P$. Notice that $i^{-1}\{p, \overline{p}\}$ is the semi-classical principal symbol of the commutator $h^{-1}[P, P^*]$, so $P$ is non-normal.

When $P$ is a fixed elliptic operator in the classical sense, with analytic $h$-independent coefficients, the result with some obvious modifications applies to $P - z$ when $z$ tends to infinity in a narrow sector.
We refer to [5] for a fuller discussion of the spectral aspects.

**Proof of Theorem 2.8.2.** The assumption (2.8.9) implies that \( p'(x_0, \xi_0) \neq 0 \). The existence of analytic solutions to (2.8.10), (2.8.11) then follows from complex Hamilton-Jacobi theory or simply from the Cauchy-Kowalevksa theorem. More precisely, if \( H \) is a complex hypersurface in \( x \)-space that passes through \( x_0 \) transversally to \( p'(x_0, \xi_0) \cdot \partial_x \) and \( \psi \) is holomorphic on \( \text{neigh} (x_0, H) \) with \( d\psi = \xi_0 \cdot dx|_H \) at \( x_0 \), then (2.8.10), (2.8.11) has a solution \( \phi \) such that \( \phi|_H = \psi \), unique near \( x_0 \).

For (2.8.12) we recall a geometric characterization by Hörmander [14]. Let \( \Lambda_\phi \) be the complex Lagrangian manifold defined near \( (x_0, \xi_0) \) by \( \xi = \phi'(x) \) where \( \phi(x) \) is holomorphic near \( x_0 \) and \( \phi'(x_0) = \xi_0 \). Then,

- \( (2.8.12) \implies \)
  \[
  \frac{1}{i} \sigma(t, \bar{t}) > 0, \quad \forall t \in T_{x_0, \xi_0}(\Lambda_\phi) \setminus \{0\},
  \]
  where we view the symplectic form \( \sigma \) as an alternate bilinear form.

- If \( \Lambda \) is a complex Lagrangian manifold containing \( (x_0, \xi_0) \) such that (2.8.15) holds, then after restricting \( \Lambda \) to a small neighborhood of \( (x_0, \xi_0) \), we get \( \Lambda = \Lambda_\phi \), where \( \phi \) is holomorphic near \( x_0 \) and satisfies (2.8.10), (2.8.12).

The geometric formulation of the problem (2.8.10)–(2.8.12) is then to find a complex Lagrangian manifold \( \Lambda \subset \Gamma := p^{-1}(0) \) which contains \( (x_0, \xi_0) \) and is strictly positive in the sense of (2.8.15). Notice that the strict positivity of \( \Lambda \) at \( (x_0, \xi_0) \) implies that \( \Lambda \) intersects \( T^*\Omega \) transversally at \( (x_0, \xi_0) \).

Here \( \Gamma = p^{-1}(0) \) denotes the complex hypersurface and we recall that \( H_p \) is tangent to \( \Gamma \). We also know by elementary symplectic geometry that \( H_p \) is everywhere tangent to \( \Lambda \).

Let \( \Sigma = p^{-1}(0) \cap \text{neigh} ((x_0, \xi_0), T^*\Omega) \) be the real characteristic manifold. It is symplectic and of codimension 2. Let \( \Sigma^C \subset \text{neigh} ((x_0, \xi_0), \mathbb{C}^{2n}) \) denote its complexification. It is a complex symplectic manifold of codimension 2 in \( \mathbb{C}^{2n} \), given by the equations \( p(\rho) = 0, p^*(\rho) = 0 \), where \( p^*(\rho) = p(\overline{\rho}) \). The assumption (2.8.9) implies that \( \Sigma^C \) is a complex hypersurface in \( \Gamma \), given there by the equation \( p^*(\rho) = 0 \), transversal to \( H_p \), since \( H_p p^* = \{p, \overline{p}\} \neq 0 \).

It is now clear that the complex Lagrangian manifolds \( \Lambda \) with \( (x_0, \xi_0) \in \Lambda \subset \text{neigh} ((x_0, \xi_0), \Gamma) \) coincide near that point with the ones of the form

\[
\{\exp (z H_p)(\rho'); \rho' \in \Lambda', \; z \in D(0, \varepsilon)\},
\]

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where $\varepsilon > 0$ is small and $\Lambda'$ is a complex Lagrangian submanifold of $\Sigma^\mathbb{C}$ containing $(x_0, \xi_0)$. By the Darboux theorem, $\Sigma, \Sigma^\mathbb{C}$ can locally be identified with $\mathbb{R}^{2(n-1)}$, $\mathbb{C}^{2(n-1)}$, and we see that $\Lambda$ is strictly positive at $(x_0, \xi_0)$ iff $\Lambda'$ is. Indeed, a general $t \in T_{(x_0, \xi_0)}\Lambda$ is of the form $t = t' + \overline{z}H_p(x_0, \xi_0)$, for $t' \in T_{(x_0, \xi_0)}\Lambda'$, $z \in \mathbb{C}$ and since $\sigma(t', H_p) = \sigma(t', \overline{H_p}) = 0$, we get

$$\frac{1}{2i} \sigma(t, \overline{t}) = \frac{1}{2i} \sigma(t', \overline{t'}) + \frac{|z|^2}{2i} \sigma(H_p, \overline{H_p})$$

$$= \frac{1}{2i} \sigma(t', \overline{t'}) + \frac{|z|^2}{2i} \{p, \overline{p}\} \approx |t'|^2 + |z|^2 \approx |t|^2.$$

Now there are plenty of strictly positive Lagrange manifolds $\Lambda' \subset \Sigma^\mathbb{C}$ passing through $(x_0, \xi_0)$ and hence there are plenty of strictly positive Lagrange manifolds $\Lambda \subset \Gamma$ containing that point. This means that we have plenty of solutions to the problem (2.8.10)–(2.8.12).

We choose one such solution $\phi(x)$ and apply Theorem 2.8.1 to conclude that there exists an elliptic classical analytic symbol $b(x; h) \sim \sum_0^\infty b_k(x)h^k$ such that formally,

$$P(x, hD; h)(b(x; h)e^{i\phi(x)/h}) = 0, \quad x \in \text{neigh} (x_0, \Omega).$$

This means that (if $b$ also denotes a realization as in Theorem 2.8.2)

$$P(x, hD_x; h)(be^{i\phi/h}) = O(e^{-1/(Ch)})e^{i\phi/h}.$$ 

From 2.8.12 we see that $e^{i\phi(x)/h}$ is exponentially decaying on the real domain away from any fixed neighborhood of $x_0$. Thus, if $\chi$ is a cutoff as in the statement of the theorem,

$$P(\chi be^{i\phi/h}) = O(e^{-1/(Ch)}).$$

By analytic stationary phase,

$$\|\chi be^{i\phi/h}\|^2_{L^2} = h^\tau c(h),$$

where $c(h) \sim c_0 + c_1h + \ldots$ is a positive elliptic analytic symbol. Applying the quasinorms of Section 2.2 (that simplify a lot since the family $\Omega_t$ is absent), we see that $c^{-1/2}$ is a classical analytic symbol. Replacing $b$ with $c^{-1/2}b$, we get 2.8.13, 2.8.14.
2.9 Propagation of regularity along a real bicharacteristic strip

Let $P$ be a differential operator with analytic coefficients on an open set $X \subset \mathbb{R}^n$. Let $p$ be the principal symbol. The following theorem is due to N. Hanges [9]. It improves the classical propagation theorem of L. Hörmander [13] and Sato, Kawai and Kashiwara [26] for operators of real principal type in that it only requires one real bicharacteristic strip. See also [10].

**Theorem 2.9.1** Assume that $H_p = p' \xi \cdot \partial_x - p'' \eta \cdot \partial_\eta$ has a real integral curve $\gamma : [a, b] \to p^{-1}(0) \cap T^*X \setminus 0$, $a < b$. If $u \in \mathcal{D}'(X)$, $WF_a(Pu) \cap \gamma([a, b]) = \emptyset$, then $\gamma([a, b])$ is either contained in, or disjoint from $WF_a(u)$.

The proof uses a WKB-construction and the variant we give here is slightly different from the one in Chapter 9 in [32].

If $dp$ vanishes at some point of $\gamma$, then $\gamma$ is reduced to a point and the statement in the theorem becomes trivial. Hence, we may assume that $dp \neq 0$ along $\gamma$.

**Theorem 2.9.2** Assume that $p(y_0, \eta_0) = 0$, $dp(y_0, \eta_0) \neq 0$. Then we can find an FBI-transform $T$ defined near $(y_0, \eta_0)$ such that $hD_{x_n}Tu = Th^mPu$ in $H_{\Phi,x_0}$ for $u \in \mathcal{D}'(X)$ independent of $h$.

**Proof:** We start with the phase.

**Lemma 2.9.3** There exists an FBI-phase $\phi(x, y)$, defined near $(x_0, y_0)$ such that

$$\partial_{x_n}\phi = p(y, -\partial_y\phi(y)).$$

**Proof:** We put

$$\phi(x', 0, y) = \frac{i}{2}(x' - y')^2 - \eta_0y_n + iC(y_n - y_0)^2,$$

and choose $x_0 = (y'_0 - i\eta'_0, 0)$. Here $C$ will be chosen with $\text{Re}\, C > 0$. Then $\phi'_y((x'_0, 0), y_0) = -\eta_0$ and we let $\phi(x, y)$ be the corresponding local solution of (2.9.1). Then $\phi$ fulfills the first two conditions in (2.6.1). In order to have $\det \phi''_{xy}(x_0, y_0) \neq 0$, we may assume, after a change of coordinates in $y$, that

$$\partial_{y_n}p(y_0, \eta_0) \neq 0,$$

or $[\partial_{y}p(y_0, \eta_0) = 0$ and $\partial_{y_n}p(y_0, \eta_0) \neq 0]$.

Then we can find $C$ with $\text{Re}\, C > 0$ such that

$$\partial_{y_n}(p(y, -\partial_y\phi)) \neq 0 \text{ at } (x_0, y_0).$$

(2.9.2)

Now the following statements are equivalent:
\begin{itemize}
\item \( \det \phi''_{xy}(x_0, y_0) \neq 0, \)
\item \( y \mapsto \partial_x \phi \) has bijective differential at \( x = x_0, y = y_0, \)
\item \( y \mapsto (\partial_x \phi, p(y, -\partial_y \phi)) \) has bijective differential at \( x = x_0, y = y_0, \)
\item \((2.9.2)\)
\end{itemize}

The last equivalence follows from
\[
\det \phi''_{x',y'} \neq 0, \quad \phi''_{y',x'} = 0 \text{ at } (x_0, y_0).
\]
Thus \( \phi \) is an FBI-phase.

We can now finish the proof of the last theorem. Take \( \phi \) as in the lemma. It suffices to choose \( a \) in \((2.6.2)\) such that
\[
(hD_n - h^n P^t(y, D_y)) (e^{i\phi(x,y)/h} a(x, y; h)) = 0,
\]
which we can solve locally as in the preceding section with a prescribed \( a(x', 0, y; h) \).

\[\square\]

**Proof of Hanges’ theorem**: We may decompose \([a, b]\) into finitely many short intervals, each being covered by one FBI transform. Thus we may assume that \( \gamma([a, b]) \) is contained in a small neighborhood of \((y_0, \eta_0)\). Let \( T \) be a corresponding FBI transform as in the last theorem. Then \( \kappa_T \circ \gamma \) is an integral curve in \( \Lambda_{\Phi} \) of \( H_{\xi_n} = \partial_{x_n} \) on which \( \xi_n \) vanishes. Assume for simplicity that \( x_0 = 0 \). Then we know that
\[
\frac{2}{i} \partial_x \Phi(0, t) = \xi_0 = (\xi'_0, 0)
\]
and consequently \( \Phi(x) = -\operatorname{Im} (x' \cdot \xi'_0) + O(x'^2) \).

By the intertwining property and the fact that \( \gamma([a, b]) \) is disjoint from \( \mathrm{WF}_{a}(P u) \), we know that
\[
hD_n T u = 0 \text{ in } H_{\Phi}(\mathrm{neigh}(\{0\} \times [a, b], C^n)),
\]
so by integration,
\[
Tu = v(x') + O(e^{-\operatorname{Im}(x' \cdot \xi'_0)/h - c/h}) \text{ near } \{0\} \times [a, b].
\]
Consequently, if \( Tu = 0 \) in \( H_{\Phi, \gamma(t)} \) for some \( t \in [a, b] \) we have the same fact for all \( t \in [a, b] \). In other words, if \( \gamma(t) \not\in \mathrm{WF}_{a}(u) \) for some \( t \in [a, b] \), the same must hold for all \( t \in [a, b] \).
2.10 Related results and developments

The work [32] was the natural continuation of a series of works on the propagation of singularities for solutions of boundary value problems of order 2 and higher in the analytic category, [27, 28, 29, 31, 30, 25] In the case of second order operators, the main result here is that the analytic wavefront set for solutions to homogenous problems is a union of maximally extended analytic rays (and a more general microhyperbolic propagation theorem for operators of higher order). This is analogous to the corresponding result in the $C^\infty$ by M. Taylor, R. Melrose, G. Eskin, V. Ivrii, culminating in [22, 23], stating that the ordinary $C^\infty$ wavefront set of solutions to the homogeneous problem is a union of maximally extended $C^\infty$-rays. Such rays have (with the exception of some slightly pathological cases) unique extensions while analytic rays have non-unique extensions from points where they are tangential to the boundary and the domain is concave in the ray direction so that the complement, that we may call “the obstacle”, is convex in the same direction. Roughly, analytic rays may glide along the boundary into the $C^\infty$ shadow region.

The methods used another kind of FBI-transforms, closely related to Gaussian resolutions of the identity. In [32] such resolutions still play a role, while in the present text, we have eliminated them completely. It would have been nice if there had been time and energy to revisit the boundary propagation in [32] with the improved methods there.

G. Lebeau [18] explored the propagation of singularities for the wave equation outside a strictly convex obstacle in the whole scale of Gevrey spaces $G^s$ that interpolate between the smooth and the analytic functions and found that the essential difference between the two types of propagations appears at the value $s = 3$. See also [17].

A related area is that of analytic hypoellipticity for non-elliptic operators. Here F. Treves [35] and later D. Tartakoff [34] established analytic hypoellipticity for operators of the type $\Box_b$ that degenerate to order 2 on a symplectic submanifold of the real cotangent space. The approach of Treves is based on a full fledged machinery of analytic pseudodifferential operators and reductions to model-like cases while the one of Tartakoff is restricted to a more special class of operators and uses very sophisticated iterated a priori-estimates to gain control of high order derivatives directly. G. Métivier [24] in a still very long paper generalized the results to operators with multiple characteristics following the general approach of Treves.

In [33] the second author gave a short proof of Métivier’s result as well as some generalizations. We refer to [8, 7] for some related results. The method of [33] is
that of subelliptic deformations: After an FBI-transform we work in a space $H^\text{loc}_{\Phi_0}$ for some strictly plurisubharmonic weight $\Phi_0$ and the given subelliptic operator satisfies an a priori-estimate in that space. We then look for a small deformation $\Phi \approx \Phi_0$ such that $P$ satisfies a nice a priori estimate also in $H^\text{loc}_{\Phi}$ and such that $\Phi < \Phi_0$ where we want to obtain analytic regularity and $\Phi \geq \Phi_0$ near the boundary of a neighborhood of those points. A variant of the method used when we have micro-hyperbolicity, is to make deformations such that the operator on the FBI-side is elliptic on $\Lambda_{\Phi}$, $\Phi > \Phi_0$ in a region where we want to gain analytic regularity and such that on the boundary of a slightly larger region we have that $\Phi > \Phi_0$ only at points where already have analytic regularity by assumption. The deformation of weights on the FBI-side corresponds to a local deformation $\kappa^{-1}_T(\Lambda_{\Phi})$ of the real phase space $T^*\Omega$ (locally equal to $\kappa^{-1}_T(\Lambda_{\Phi_0})$). See [32, 29].

In the theory of scattering poles (resonances) and other branches of spectral theory for non-self-adjoint (pseudo-)differential operators, many works rely on phase space deformations which are now global. Since this activity started later we simply refer to some of the works which also include some of those devoted to other global questions: [37]–[66].
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