Unitary chiral dynamics of two hadrons in a finite volume: the $K D$, $\eta D_s$ system and the $D_{s^*0}(2317)$ resonance

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Abstract. We investigate the $K D$ and $\eta D_s$ system in a finite volume and study the properties of the $D_{s^*0}(2317)$ resonance, which is generated in this coupled channel system. We calculate the energy levels in a cubic box and considering them as synthetic lattice data we solve the inverse problem of determining the bound states and phase shifts in the infinite volume. We observe that it is possible to obtain accurate $K D$ phase shifts and the position of the $D_{s^*0}(2317)$ state from the synthetic lattice data considered and that a careful analysis of the finite volume data can shed some light on the nature of the $D_{s^*0}(2317)$ resonance as a $K D$ molecule or otherwise.

1. Introduction

Quantum Chromodynamics (QCD) is considered nowadays as the theory for the strong interactions, with the quarks being the degrees of freedom. However, while at high energies, because of the asymptotic freedom of the quarks, the theory has been successfully tested by the experiment, the situation is completely different at low or intermediate energies due to the confinement regime of QCD, which makes perturbative methods non applicable.

It is interesting to notice that in the low energy region the QCD spectrum exhibits a very peculiar fact: the presence of an isospin triplet (pions) with a mass much smaller than the rest of the QCD states. An extension to SU(3) can be done by considering the lowest octet of pseudoscalar states, i.e., $\pi$, $K$, and $\eta$. This fact can be understood by the presence in the QCD Lagrangian of a chiral symmetry for the light quark sector ($u$, $d$, $s$) which spontaneously breaks down and generates, through the Goldstone theorem, the aforementioned states (see, for example, Refs. [1, 2, 3, 4]).

A theory which considering these facts describes successfully the interaction between hadrons is the so called chiral perturbation theory or $\chi PT$. This theory consist in a series of Lagrangians in a power momentum expansion which is valid up to some high energy scale, $\Lambda_{\chi PT}$, with $\Lambda_{\chi PT} \sim 1$ GeV. Hence, the $\chi PT$ expansion is valid for momenta $p \ll \Lambda_{\chi PT} \sim 1$GeV. Although $\chi PT$ is a powerful tool at low energies, its convergence is limited to a narrow interval (for example, in the study of the meson-meson scattering it is possible to reach energies around 0.5 GeV, while in the meson-baryon scattering one can goes up to the threshold region). This means that one of the most interesting facts of the strong interacting phenomena, which is the
formation of resonances, cannot be studied with $\chi PT$ alone. One could think in increasing the energy region of applicability of the theory just by including higher orders terms in the Lagrangian. However, this has the disadvantage of losing the predictive power of the theory, since the number of free parameters increases tremendously with the order: the leading order $\chi PT$ Lagrangian has basically no free parameters, while in the next-to-leading order there are 12 parameters and if one goes to the next-next-to-leading order then there are more than 100. But, even then a resonance cannot be obtained by summing a series perturbatively. As a consequence, the development of non-perturbative techniques is needed in order to extend the energy region of applicability of the theory, but without losing its predictive power. A simple way of accomplish this issue is considering the lowest order chiral Lagrangians and implement the theory with unitarity, which has given rise to the so mentioned unitary chiral perturbation theory ($U \chi PT$).

In the last decade, two-body meson-baryon and meson-meson systems, as for example, $\bar{K}N$, $\pi\Sigma$, $\pi\Lambda$, $\pi\pi$, $KK$, $\pi\eta$, etc., have been studied extensively with $U \chi PT$ and dynamical generation of many baryon and meson resonances in these systems has been found, for instance, the $\Lambda(1405)$, $\Lambda(1520)$, $N^*(1535)$, etc., in the baryon sector and the $f_0(980)$, $a_0(980)$, $\sigma(600)$, etc., in the meson sector [5, 6, 7, 8, 9, 10, 11, 12, 13]. The $U \chi PT$ theory has also been used to study three body hadron systems, like two pseudoscalars and a baryon, two pseudoscalars and a vector meson, three pseudoscalars, or clusters of a hadron and a resonance formed from the interaction of two hadrons and generation of many baryon and mesons states have been found [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Recently, the $U \chi PT$ has also been extended to investigate the interaction of two hadrons in a finite volume [25, 26, 27, 28, 29]. As it is well known, one of the challenges of lattice QCD calculations is to determine the spectra of mesons and baryons. However, in the case of resonances the main difficulty is that they do not correspond to isolated energy levels in the spectrum of the QCD Hamiltonian on the lattice. Thus, an additional effort has to be done. In the case of one or two channels, the problem is formally solved under the framework of Lüscher [30, 31, 32, 33], which relates the measured discrete value of the energy in a finite volume to the scattering phase shift at the same energy, for the same system in the infinite volume.

In the present case we study the $KD$, $\eta D_s$ system, in which the $D_{s*0}(2317)$ state is formed [34, 35, 36, 37], in a finite volume following the model developed in Ref. [28]. We start with the “phenomenological” potential of Ref. [37] and use the $KD$, $\eta D_s$ channels to evaluate the energy levels in a box using periodic boundary conditions. Using this energy levels to generate synthetic lattice data, we face the inverse problem of extracting from them the $KD$ phase shifts in the infinite volume and the position of the $D_{s*0}(2317)$. We show that with a few data from the lattice (about ten) corresponding to two levels of the box with various values of $L$ (the box size), we can reproduce the results in the infinite volume, both for the energy of the bound state and the $KD$ phase shifts, with a good accuracy.

2. Formalism

The scattering matrix in the $U \chi PT$ is obtained by solving the Bethe-Salpeter equation

$$T(E) = [1 - V(E)G(E)]^{-1}V(E),$$

with $E$ the energy of the system in the center of mass frame, $V$ represents the matrix for the transition potentials between the channels and $G$, in our case, the loop function of two meson propagators, which is defined as

$$G = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\epsilon} \frac{1}{(P - q)^2 - M^2 + i\epsilon}$$

(2)
with \( m \) and \( M \) the masses of the two mesons and \( P \) the four momentum of the external meson-meson system. This loop function is divergent and needs to be regularized. In the dimensional regularization scheme, this integral is evaluated and for a meson-meson system gives the result [38, 39]

\[
G_i^D(E) = \frac{1}{(4\pi)^2} \left\{ a_i(\mu) + \log \frac{m_i^2}{\mu^2} + \frac{M_i^2 - m_i^2 + s}{2s} \log \frac{M_i^2}{m_i^2} \\
+ \frac{Q_i(\sqrt{s})}{\sqrt{s}} \log \left( s - (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) + \log \left( s + (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) \\
- \log \left( -s + (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) - \log \left( -s - (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) \right\},
\]

where \( s = E^2 \), \( Q_i \) is the on-shell momentum of the particles in the \( i \)th channel, \( \mu \) a regularization scale and \( a_i(\mu) \) a subtraction constant (note that there is only one independent parameter, because a change in \( \mu \) can be absorbed into \( a_i \)).

In a finite box of side length \( L \), instead of integrating over the energy states of the infinite volume as done in Eq. (2), we have to perform a sum over the discrete momenta given by the periodic boundary conditions, i.e., \( \vec{q}_L = (2\pi/L)\vec{n} \) with \( \vec{n} \in \mathbb{Z}^3 \). In this case, the loop function associated with the one of Eq. (3) in a finite volume is given by (see Ref. [28] for more details)

\[
\tilde{G}_i(E) = G_i^D(E) + \lim_{q_{\text{max}} \to \infty} \left[ \frac{1}{L^3} \sum_{q_L} I_i(q_L) - \int_{q<q_{\text{max}}} \frac{d^3q}{(2\pi)^3} I_i(q) \right]
\]

with

\[
I_i(x) = \frac{1}{2\omega_i(x)\omega'_i(x)} \frac{\omega_i(x) + \omega'_i(x)}{E^2 - (\omega_i(x) + \omega'_i(x))^2 + i\epsilon},
\]

\[
\omega_i(x) = \sqrt{m_i^2 + x^2}, \quad \omega'_i(x) = \sqrt{M_i^2 + x^2},
\]

\[
x = \vec{q}_L, \vec{q}.
\]

In this way, to determine the \( T \) matrix in the box we just need to replace the \( G \) function in Eq. (1) by Eq. (4). The eigenenergies of the box correspond to energies that produce poles in the \( T \) matrix. Thus we search for these energies by looking for zeros of the determinant

\[
\det(1 - \tilde{V} \tilde{G}) = 0.
\]

In Fig. 1 we show the first five energy levels obtained for a cubic box and for different values of \( L \). For large values of \( L \) the lowest energy level in the box should converge to the value of the energy of the bound state of the infinite volume case. This is a standard feature for the ground level bound state in the lattice. As it can be seen in Fig. 1, we find that with values of \( L = 3 \ m^{-1} \) there is already a good convergence to the value in the infinite volume.

### 3. The inverse problem

Let us face now the problem of determining the pole position of the \( D_{s0}(2317) \) state and the \( KD \) phase shift in the infinite volume from the energy levels shown in Fig. 1. To do this, we take the first two energy levels and consider 10 points to which we associate an error of ± 10
Figure 1. Energy levels for the $KD$, $\eta D_s$ system as functions of the side length of the box, $L$. We call the data generated in this form as synthetic lattice data. Then we perform a fit to these synthetic data by solving Eq. (6) with $KD$ and $\eta D_s$ as coupled channels and under the assumption that we have a potential with the same energy dependence than the one obtained in Ref. [37], i.e.,

$$V_{ij} = a_{ij} + b_{ij} [s - (m_K + M_D)^2],$$

with $m_K$ and $M_D$ the mass of the Kaon and $D$ meson, respectively. We look for the minimum value of $\chi^2$, $\chi^2_{\text{min}}$, and obtain a set of parameters for $a_{ij}, b_{ij}$. Then we generate random sets of the parameters close to those of the minimum, such that $\chi^2$ is only increased below $\chi^2_{\text{min}} + 1$. In Fig. 2 we show the results of the levels reconstructed from the best fits to the data, with a band corresponding to the random choices of parameters satisfying the condition $\chi^2 < \chi^2_{\text{min}} + 1$.

Figure 2. Energy levels as functions of the side length of the box $L$, reconstructed from fits to the “data” of Fig. 1 using the potential of Eq. (7). The band corresponds to different choices of parameters within errors.

With the different potentials obtained from the fits we can solve Eq. (1) and determine the $T$ matrix in infinite volume and, thus, calculate the phase shift for the $KD$ channel. The results
for the $KD$ phase shift for total isospin 0 and s-wave, $\delta^0_0$, are shown in Fig. 3 and, as it can be seen, the agreement with the exact result (solid line) is quite good.

![Figure 3](image-url)

**Figure 3.** Phase shifts for $KD$ scattering derived from the coupled channels unitary approach of Ref. [37] (solid line). The band corresponds to the results obtained from the fits to the “data” of Fig. 1 using the potential of Eq. (7) with two channels.

Since the $\eta D_s$ channel is far away from the region at which the resonance is generated, we may consider the possibility of explaining the synthetic lattice data considered by taking into account only the $KD$ channel. In Fig. 4 we can see the results obtained for the $KD$ phase shifts from the fits to the synthetic lattice data of Fig. 1 with one channel potential. We observe that for low energies the results are very similar to those obtained with two channels. However, as the energy increases, the results with two channels are closer to the exact solution.

![Figure 4](image-url)

**Figure 4.** Phase shifts for $KD$ scattering derived from the coupled channels unitary approach of Ref. [37] (solid line). The band corresponds to using the fits to the “data” of Fig. 1 using the potential of Eq. (7) with only the $KD$ channel.

Apart from the phase shift, we can also determine from the fits the energy of the bound state in infinite volume, getting good results with one or with two channels: $E = 2317 \pm 5$ MeV in both cases.
4. The bound state and its nature
A very interesting issue is to investigate the possibility of determining the nature of the $D_{s^*0}(2317)$ from the lattice data. For this purpose we can use a sum rule whose origin stems from normalization to unity of the wave function of the bound state and which states [40, 41]

$$\sum_i g_i^2 \frac{dG_{ii}}{ds} \Bigg|_{E=E_\alpha} = -1$$

where

$$g_i^2 = \lim_{s \to s_R} (s - s_R)T_{ii}$$

are the residues of the $T_{ii}$ scattering matrices (coupling squared) at the pole of the bound state, $\sqrt{s_R}$, and $G_{ii}$ are the propagator of the two particles of the corresponding channels (the loop function $G^{D^0}$ that we use here). Note, however, that there is a caveat in our formulation, since in the derivation of Eq. (8) an energy independent potential is assumed, while we are now producing potentials with a moderate energy dependence. Taking into account the energy dependence of Eq. (7) we find that approximately one should have now (see Ref. [28] for more details)

$$\sum_i g_i^2 \left\{ \frac{b_{ii}}{a_{ii} + b_{ii}(s - (m_D + m_K)^2)} + \frac{dG_{ii}}{ds} \right\} = -1.$$  

One can see from the derivation in Ref. [40] that each term in Eq. (8) accounts (with reversed sign) for the probability of the bound state to be a bound state of the pair of particles of the channel considered. In the case that we had the coupling of the bound state to another hypothetical elementary particle outside the space of pair of particles considered, there would be an extra term in the sum of Eq. (8), $-Z$, where $Z$ would account for the overlap of the bound state with this hypothetical genuine particle. The diversion of the sum of Eq. (8) with respect to -1 would indicate the amount of the bound state which cannot be considered a bound state of the two particles.

Obviously with the synthetic lattice data which we have produced the answer is trivial since the state was dynamically generated from the interaction of the $KD$, $\eta D_s$ channels and Eq. (10) is fulfilled up to the level of 3%. But if the real lattice results were different than those obtained here the answer might be different. For example, there is the possibility that the data might be such that one could find a gross deviation from having $KD$ as the large component of the $D_{s^*0}(2317)$ wave function, reflecting the fact that the $D_{s^*0}(2317)$ would be a genuine state, not generated by the $KD$ interaction. For this purpose we have made a test introducing a different potential where a CDD pole (Castillejo, Dalitz, Dyson) [42] is introduced by hand. The potential in just one channel would now be

$$V = V_M + \frac{g_{CDD}^2}{s - s_{CDD}},$$

where $V_M$ is assumed to be energy independent and $g_{CDD}^2$, $s_{CDD}$ are the parameters of the CDD pole. In this case, the sum rule for this particular energy dependence read as (see Ref. [28] for
more details)

\[-g_2^2 \frac{dG}{ds} = \frac{1}{-dV^{-1} \left( \frac{dG}{ds} \right)^{-1} + 1} = 1 - Z\]

\[= \frac{1}{-g_{CDD}^2 G^2 \left( s - s_{CDD} \right)^2 \frac{dG}{ds} + 1} \tag{12}\]

Using the potential of Eq. (11) we try to fit the synthetic lattice data of Fig. 1 (obtained with the coupled channels approach of Ref. [37]). The results are shown in Fig. 5 and, as we can see, the fit to the data is quite good. In Fig. 6 we show the results for the phase shifts associated with the CDD potential, which coincide with those obtained with the two channel potential up to energies around 2400 MeV. After this, there is a large dispersion of the results and they divert significantly from those obtained with the two channel approach. Certainly, the use of an additional “lattice” level would put big constraints in the phase shifts in that region helping us decide between the two options.

However, it is interesting to analyze the results obtained: the CDD pole is found at the energy of about 2500 MeV, far away from the energy at which the state $D_{s^0}(2317)$ is formed. If one restricts oneself to low energies, the fit with the CDD would be acceptable. Yet, the fact that the CDD pole has appeared so far away from the energy of the $D_{s^0}(2317)$ state indicates that the data do not want a CDD pole being responsible for this state. The CDD pole far away in this case simply generates a smooth energy dependent potential in the region of the low energies. The interesting thing is that if we calculate now $Z$ from Eq. (12) we find that $Z \sim 0.15$, indicating that the bound state is basically a $KD$ bound state, with the precision that the limited low energy data provide. A more precise determination would require the consideration of an additional level of the box.

5. Conclusions
We have studied the $KD$, $\eta D_s$ system in a finite volume and generate the energy levels of the system in a cubic box of side length $L$. By assuming that the results obtained for these energy
levels would correspond to results given by lattice calculations, we have faced the problem of determining the \( KD \) phase shift in infinite volume as well as the pole position and nature of the \( D_{s^0}(2317) \), a state generated in the \( KD, \eta D_s \) system. We found that the two channel method proposed by us works well, but if we restrict ourselves to low energies of the \( KD \) system we also see that the one channel analysis is quite good, but as we approach the threshold of \( \eta D_s \), the two channel analysis provides better phase shifts. We have also investigated whether we could say something from the lattice results concerning the nature of the bound state obtained. We found that indeed it was possible to induce some information on the nature of this state as basically a molecule of \( KD \), by inspecting the spectra in finite volume. Alternatively we tried to analyze the lattice data in terms of a CDD potential that could accommodate a non molecular state and we could find a good fit to the data but with an artificial pole at an energy very far away, such that the potential was essentially constant in the region of interest and the \( Z \) value for overlap with a genuine state was small, compatible with zero within the precision that the data allow.

All these exercises show that the data provided by QCD lattice calculations on energy levels in a box contain the information to decide on the nature of the bound state \( D_{s^0}(2317) \).

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