SELF-SIMILAR SOLUTIONS FOR THE EMERGENCE OF ENERGY-VARYING SHOCK WAVES FROM PLANE-PARALLEL ATMOSPHERES

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ABSTRACT

We present a self-similar solution to describe the propagation of a shock wave whose energy is deposited or lost at the front. The propagation of the shock wave in a medium having a power-law density profile and the expansion of the medium to a vacuum after the shock breakout are both described with a Lagrangian coordinate. The Chapman-Jouquet detonation is found to accelerate the medium most effectively. The results are compared with some numerical simulations in the literature. We derive the fractions of the deposited/lost energy at the shock front in some specific cases, which will be useful when applying this solution to actual phenomena.

Subject headings: hydrodynamics — shock waves — supernovae: general

1. INTRODUCTION

The self-similar solution is a powerful method of describing flows of a fluid involving strong shock waves. In a supernova (SN) explosion, a shock wave propagates in the stellar interior toward the surface. This process can be regarded as a self-similar problem as long as the density distribution has a power-law profile with respect to the distance from the center or the surface.

In fact, self-similar solutions for this problem have been discovered and widely known (see, e.g., Zel’ dovich & Raizer 2002). Sakurai (1960) discovered a self-similar solution for the emergence of a strong shock wave from a medium with a power-law density distribution, which was first posed by Gundelman & Frank-Kamenetski (1956). This solution describes distributions of the velocity, pressure, and density of the matter with an Eulerian coordinate until the shock wave reaches the surface. After that, the solution uses a Lagrangian coordinate. Matzner & McKee (1999) invented another description for a whole evolution of this solution with a Lagrangian coordinate.

However, these solutions do not completely reflect on the circumstances. In reality, thermal energy can be deposited and/or lost across a shock front (referred to as the energy variation in this paper). For example, some nuclear reactions can take place in the region behind the shock front when the temperature thereof becomes sufficiently high. This happens in a detonation wave and might be realized in a Type Ia SN (SN Ia), which is thought to be a thermonuclear explosion of a white dwarf. The thermonuclear explosion model for SNe Ia was originally proposed by Hoyle & Fowler (1960) and several models for the propagation of the flame front have been discussed, for example, the “deflagration model” (see Ivanova et al. 1974; Nomoto et al. 1976), the “delayed detonation model” (see Khokhlov 1991a; Yamaoka et al. 1992; Woosley & Weaver 1994), and the “pulsating delayed detonation model” (Khokhlov 1991b). In particular, the (pulsating) delayed detonation models suggest that the detonation wave is generated somewhere in the outer layer of a white dwarf and propagates toward the surface. In addition, a carbon detonation model of a white dwarf with the mass of $\sim 1 M_\odot$ was proposed (Shigeyama et al. 1992) for a peculiar SN Ia, SN 1990N. The detonation wave propagates toward the surface in this model as well. Furthermore, thermal energy generated behind the shock front can be transported through the front near the stellar surface where the optical depth is close to unity. In this region, photons diffuse out at velocities close to the speed of light, which is larger than the shock velocity. In other words, the energy of the shock wave can decrease by the radiative cooling. Chevalier (1976) calculated the effect of the cooling using a radiation-hydrodynamics code.

This kind of phenomena in some SNe Ia accelerates a considerable amount of matter to sufficiently high energies to synthesize light elements Li, Be, and B through spallation reactions (see Fields et al. 2002; Nakamura & Shigeyama 2004; Nakamura et al. 2006). Thus, the solution derived in this paper might be useful in investigating the chemical evolution of light elements in galaxies.

Barenblatt & Sivashinskii (1970) studied the treatment of the energy variation at the shock front in the framework of self-similar solutions and showed that it can be treated by modifying the jump conditions at the front to impose the condition that the energy deposition or loss per unit mass be proportional to the internal energy. In a real shock acceleration problem, this condition is rarely satisfied. Whether the energy is deposited by nuclear reactions or lost by radiation, it tends to have a characteristic energy per nucleon or a characteristic timescale. Thus, the self-similarity of the associated flow might appear in very limited space and/or time if any.

In § 2 we review the procedure of Barenblatt (1996) to treat the energy variation. In § 3 we formulate the problem following Matzner & McKee (1999). The integrations of the governing equations are found to be reduced to an eigenvalue problem. The Appendix gives a method for determining the eigenvalue. The results are presented in § 4. Furthermore, the asymptotic behaviors of the velocity, pressure, and density of the ejecta are derived. The energy distributions are deduced from the exponents of these physical quantities in § 5. We conclude this paper in § 6.

2. TREATMENT OF ENERGY VARIATION

In this section, we review the procedure for treating the energy deposition or loss at the shock front introduced in Barenblatt (1996). The shock propagates in the ideal gas in which density is $\rho_0$. We assume that the gas ahead of the shock front is stationary.
and neglect its pressure. The gas behind the shock front has the velocity $u_f$, pressure $p_f$, and density $\rho_f$. Then, the energy conservation law across the shock front propagating at the speed $U$ is described as

$$\rho_f (u_f - U) \left[ \frac{\gamma p_f}{\gamma - 1} \frac{u_f - U}{p_f} \right] + \frac{(u_f - U)^2}{2} = -\frac{\rho_0 U^3}{2}. \quad (1)$$

Using the mass conservation law

$$\rho_f (u_f - U) = -\rho_0 U \quad (2)$$

and the momentum conservation law

$$\rho_f (u_f - U) u_f + p_f = 0, \quad (3)$$

we obtain

$$\rho_f (u_f - U) \left[ \frac{\gamma p_f}{\gamma - 1} \frac{u_f^2}{p_f} + \frac{u_f^2}{2} - q \right] + p_f u_f = 0, \quad (4)$$

where we have added an energy flux term $-\rho_f (u_f - U) q$ to the left-hand side of equation (1). Here the energy source term is denoted by $q$. This source term needs to be proportional to the internal energy to guarantee the self-similarity of the problem. Thus, the source term is expressed as

$$q = \frac{\gamma e - \gamma}{(\gamma e - 1)(\gamma - 1)} \frac{p_f}{\rho_f} \quad (5)$$

by introducing the effective adiabatic exponent $\gamma_e$ (Barenblatt & Sivashinskii 1970). The physical meaning of $\gamma_e$ is as follows. (1) If $\gamma_e > \gamma$, energy is deposited at the front. (2) If $\gamma_e = \gamma$, the situation is the same as the classical very intense explosion problem. (3) If $\gamma_e < \gamma$, energy is lost at the shock front. Using the effective adiabatic exponent $\gamma_e$, we transform equation (4) into

$$\rho_f (u_f - U) \left[ \frac{\gamma p_f}{\gamma - 1} \frac{u_f^2}{\rho_f} + \frac{u_f^2}{2} \right] + p_f u_f = 0. \quad (6)$$

This expression has the same form as the classical very intense explosion problem, except $\gamma$ is replaced with $\gamma_e$. Therefore, the Rankine-Hugoniot relations for a strong shock become

$$u_f = \frac{2}{\gamma_e + 1} U, \quad (7)$$

$$p_f = \frac{2}{\gamma_e + 1} \rho_0 U^2, \quad (8)$$

$$p_f = \frac{\gamma_e + 1}{\gamma_e - 1} \rho_0. \quad (9)$$

In other words, we can treat the energy variation by varying $\gamma_e$.

3. FORMULATION

In this section, we derive equations describing the self-similar motion of the stellar matter. Following Matzner & McKee (1999), we treat the shock emergence with a Lagrangian coordinate.

3.1. Basic Equations

We first define the space coordinate $x$ as the distance of the fluid element from the stellar surface and the time coordinate $t$ as the time measured from the moment when the shock reaches the surface. The initial position of each fluid element is denoted by $x_0$. The basic equations to describe the evolution of the density $\rho$, velocity $u$, and pressure $p$ of the fluid are formulated with the Lagrangian coordinate $x_0$ as

$$\frac{\partial x}{\partial x_0} = \frac{\rho_0}{\rho}, \quad (10)$$

$$\frac{\partial u}{\partial t} = \frac{1}{\rho_0} \frac{\partial p}{\partial x_0}, \quad (11)$$

$$p\gamma^{-1} = p_f \gamma_f^{-1}. \quad (13)$$

We assume that the stellar envelope has a power-law density profile,

$$\rho_0(x) = \begin{cases} k_1 x_0^\alpha, & \text{for } x_0 \geq 0, \\ 0, & \text{for } x_0 < 0, \end{cases} \quad (14)$$

Fig. 1.—Profile of the flow variable $S(\eta)$ for $\alpha = 3.0, \gamma = 4/3, \gamma_e = 11/3, 2.0, 4/3, 1.2, 1.01$. 

| $\alpha$ | 11/3 | 3.0 | 2.0 | 1.5 | 4/3 | 1.2 | 1.1 | 1.01 | 1.001 | 1.0001 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 3.0 | 1.0000 | 0.9525 | 0.7915 | 0.6407 | 0.5572 | 0.4608 | 0.3484 | 0.1212 | 0.03733 | 0.01092 |
| 1.5 | 0.5000 | 0.4725 | 0.4000 | 0.3271 | 0.2863 | 0.2386 | 0.1822 | 0.06507 | 0.02050 | 0.006113 |
where \( k_1 \) and \( \alpha \) are constants. If the envelope of the star is radiative, the power-law exponent is \( \alpha = 3 \). On the other hand, \( \alpha = 1.5 \) corresponds to the convective envelope. The time when each fluid element experiences the shock is denoted by \( t_0(<0) \). By defining the variable \( m \) as

\[
m = \int_0^{x_0} \rho_0(y) dy,
\]

the coordinates \( x, x_0, \) and \( t_0 \) are labeled by \( m \). Then, we define the similarity variable \( \eta \) as

\[
\eta = \frac{t}{t_0(m)},
\]

and the flow variable \( S(\eta) \) as the position of a fluid element normalized with its initial position, i.e.,

\[
S(\eta) = \frac{x(m)}{x_0(m)}.
\]

Both of these variables \( \eta(m) \) and \( S(\eta) \) become unity when the shock hits the fluid element with \( m \), whereas \( \eta \) becomes zero when the shock reaches the surface. The phase with \( 0 < \eta < 1 \) corresponds to the evolution of shocked fluid elements until the shock breakout, and the phase with \( \eta < 0 \) to the expansion of the stellar matter to a vacuum.
where the prime denotes the derivative with respect to $\eta$. Then the pressure is also expressed in terms of $S(\eta)$ as

$$p = p_f \left( \frac{p}{p_f} \right)^\gamma = \frac{2}{\gamma_e + 1} \left( \frac{\gamma_e - 1}{\gamma_e + 1} \right)^\gamma \rho_0 U^2 \left[ S(\eta) - (\lambda + 1) \eta S'(\eta) \right]^{-\gamma},$$

and the velocity

$$u = (1 + \lambda) S' U.$$

As a result, solving equations (10)–(13) reduces to the integration of a second-order ordinary differential equation for the flow variable $S(\eta)$. The method of determining the function $S(\eta)$ is described in the Appendix.

4. RESULTS

4.1. Distributions of the Physical Variables

The distributions of the flow variable $S(\eta)$, velocity, and density of the stellar matter for $\alpha = 3$ and various $\gamma_e$ values are shown in Figures 1, 2, and 3, respectively. The same distributions, but for $\alpha = 1.5$ are shown in Figures 4–6. Each line in these figures traces the time evolution of each fluid element after the arrival of the shock front (at $\eta = 1$). At the same time, each line in the range of $0 < \eta < 1$ represents the spatial distribution of the shocked fluid element before the shock breakout, while lines with $\eta < 0$ represent the spatial distribution after the shock breakout. The results show that the energy variation immediately behind the shock front enhances the acceleration after the shock breakout rather than before the shock breakout (Figs. 2 and 5). All the numerical results suggest that the velocities converge to constant values and that the density distributions converge to power-law profiles as $t \to \infty$. Such asymptotic behaviors are discussed in § 4.2.

4.2. Asymptotic Behavior

From the numerical results, the derivative of the flow variable $S'$ is found to converge to a constant value; thus, the asymptotic behavior of $S$ should be

$$S \propto \eta.$$

Using this relation, we derive the power-law exponents of velocity, pressure, and density of the ejecta with respect to the variable $x_0$ as

$$u \propto x_0^{-\lambda}, \quad p \propto x_0^{\gamma_e - 2\lambda - (1 + \lambda)\gamma}, \quad \rho \propto x_0^{\gamma_e + \lambda + 1}.$$  

### TABLE 2

| $\gamma_e$ | 11/3 | 3.0 | 2.0 | 1.5 | 4/3 | 1.2 | 1.1 | 1.01 | 1.001 | 1.0001 |
|------------|------|-----|-----|-----|-----|-----|-----|------|-------|--------|
| 3........... | 0    | 1.000 | 1.095 | 1.417 | 1.719 | 1.886 | 2.078 | 2.303 | 2.758 | 2.925 | 2.978 |
| +\infty     | 3.667 | 3.698 | 3.806 | 3.906 | 3.962 | 4.026 | 4.101 | 4.253 | 4.308 | 4.326 |
| 1.5.......... | 0    | 0.5000 | 0.5550 | 0.7000 | 0.8459 | 0.9275 | 1.023 | 1.136 | 1.370 | 1.459 | 1.488 |
| +\infty     | 2.500 | 2.518 | 2.567 | 2.615 | 2.642 | 2.674 | 2.712 | 2.790 | 2.820 | 2.829 |
The values of the exponents of $p$ and $\rho$ for various $\gamma_c$ are shown in Tables 2 and 3. From these relations, we obtain

$$\rho \propto u^{-(\alpha+\lambda+1)/\lambda}. \quad (26)$$

Thus, the density distribution in the free expansion phase can be obtained by substituting $u = \sqrt{\gamma_c} t$ into this expression. The exponents in this equation are also shown in Table 4.

The late-time behavior for the case including the energy variation is essentially the same as that of the adiabatic case ($\gamma_c = \gamma$), except for the value of the exponents.

4.3. Chapmann-Jouguet Detonation

We concentrate on the flow called the “Chapmann-Jouguet detonation,” which satisfies the Chapmann-Jouguet condition; the velocity of the gas relative to the shock front is equal to the local velocity of sound. Barenblatt & Sivashinskii (1970) showed that this condition can be expressed as the relation between $\gamma$ and $\gamma_c$. We review the procedure and apply it to this study.

From the Rankine-Hugoniot relations (eqs. [7]–[9]), the velocity of the gas relative to the shock front is

$$|u_f - U| = \frac{\gamma - 1}{\gamma_c + 1} U, \quad (27)$$

and the velocity of sound is

$$\sqrt{\frac{\gamma p_f}{\rho_f}} = \frac{\sqrt{2\gamma (\gamma - 1)}}{\gamma_c + 1} U. \quad (28)$$

Equating the right-hand sides of equations (27) and (28), the condition is expressed as

$$\gamma_c = 2\gamma + 1. \quad (29)$$

Using this expression, $T_f$ and $T_s$ in equations (A17) and (A20) are found to take the same value. This means that the singular point is located at the shock front. Then by equating $R$ defined in equation (A8) at the shock front to that at the singular point, we obtain the analytic expression for $\lambda$ as

$$\lambda = \frac{\alpha}{3}. \quad (30)$$

On the other hand, the following relation is derived from equations (14) and (18),

$$U \propto \rho_0^{-\lambda/\alpha}. \quad (31)$$

In Chapmann-Jouguet detonation, substitution of the relation in equation (30) yields

$$U \propto \rho_0^{-1/3}, \quad (32)$$

which is independent of $\alpha$. Therefore, this relation can be applied to an arbitrary power-law density profile. For a shock wave without energy variation ($\gamma_c = \gamma$), substitution of the values in Table 1 yields

$$U \propto \rho_0^{-0.19}, \quad (33)$$

which is also independent of $\alpha$.

4.4. Energy Loss Limit

Here we consider the situation in which $\gamma_c \to 1$. However, the density diverges for the case in which $\gamma_c$ is exactly unity. Therefore, introducing the parameter $\delta$ as

$$\delta = \gamma_c - 1, \quad (34)$$

we treat the behavior of the eigenvalue $\lambda$ when $\delta$ is close to zero. Figure 7 shows the behavior in the range of $10^{-4} \leq \delta \leq 10^{-2}$ for $\gamma = 4/3$, $\alpha = 3.0, 1.5$. In this range, the following empirical relations are derived by interpolation;

$$\lambda = \begin{cases} 1.3\delta^{0.51}, & \text{for } \alpha = 3.0, \\ 0.65\delta^{0.50}, & \text{for } \alpha = 1.5. \end{cases} \quad (35)$$

| $\alpha$ | $t$ | $\gamma_c$ |
|---------|-----|-----------|
| 3............ | 0 | $-3.000$ | $-3.150$ | $-3.790$ | $-4.683$ | $-5.384$ | $-6.511$ | $-8.611$ | $-24.76$ | $-24.76$ | $-274.8$ |
|         | $+\infty$ | $-5.000$ | $-5.199$ | $-6.054$ | $-7.244$ | $-8.178$ | $-9.681$ | $-12.48$ | $-34.01$ | $-108.2$ | $-367.4$ |
| 1.5............ | 0 | $-3.000$ | $-3.174$ | $-3.750$ | $-4.586$ | $-5.240$ | $-6.288$ | $-8.234$ | $-23.05$ | $-73.16$ | $-245.4$ |
|         | $+\infty$ | $-6.000$ | $-6.291$ | $-7.250$ | $-8.644$ | $-9.733$ | $-11.48$ | $-14.72$ | $-39.42$ | $-122.9$ | $-409.9$ |

| $\alpha$ | $t$ | $\gamma_c$ |
|---------|-----|-----------|
| 3............ | 0 | $-3.000$ | $-3.150$ | $-3.790$ | $-4.683$ | $-5.384$ | $-6.511$ | $-8.611$ | $-24.76$ | $-24.76$ | $-274.8$ |
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These formulae suggest that when the shock becomes radiative, the shock velocity tends to be independent of the initial density. Thus, the shock is no longer accelerated due to radiative energy loss and propagates at a constant velocity.

5. ENERGY SPECTRUM

In this section, we derive the energy spectrum from the asymptotic behavior of physical quantities described above. The energy spectrum is defined as the mass whose energy is larger than \( \epsilon \):

\[
M(> \epsilon) = \int_{\epsilon}^{\infty} \rho \, dx.
\]  

(36)

From the relation in equation (25) and \( \epsilon = u^2/2 \), the dependence on \( \epsilon \) is found to be

\[
M(> \epsilon) \propto \epsilon^{-(\alpha+1)/2}.
\]  

(37)

The values of the power-law exponent for various \( \gamma_e \) are shown in Table 5. Here \( \gamma \) is assumed to be 4/3. The results for \( \gamma_e = 2\gamma + 1 \) give the upper limit for the exponent, because \( \gamma_e \) takes the maximum value for the Chapman-Jouguet detonation, and then \( \lambda \) yields the maximum. The exponents increase as the energy is deposited at the shock front and converge to the upper limit in the Chapman-Jouguet detonation. On the other hand, the energy loss decreases the exponents, and the exponents diverge at \( \gamma_e = 1 \), related to the situation when all the internal energy is lost at the shock front. This result implies that the energy distribution of cosmic rays for the case in which \( \gamma_e > \gamma \) (energy deposition) becomes harder than that for the case in which \( \gamma_e = \gamma \).

6. CONCLUSIONS AND DISCUSSION

In this paper we have derived a self-similar solution to describe the propagation of a shock wave whose energy varies at the shock front. Using the solution, we describe the propagation of the shock wave in a plane-parallel medium having a power-law density profile. The motion of the ejecta after the shock breakout is derived. The asymptotic behavior of the ejecta is also derived, and then the energy spectrum in the free expansion phase is deduced. Considering the case for \( \gamma_e = 2\gamma + 1 \), we derive the power-law exponent of the energy spectrum analytically, which gives the upper limit for it.

The calculation of the radioactive energy input model for SNe Ia (see Colgate & McKee 1969) showed that the explosion of an \( n = 3 \) polytrope leads to the outer density profile

\[
\rho \propto u^{-7},
\]  

(38)

which corresponds to the case \( \gamma_e \approx 1.6 \) in this work. Then, the total energy per unit mass at the shock front becomes

\[
E_{\text{tot}} = \left( \frac{1}{\gamma_e - 1} + \frac{1}{\gamma - 1} \right) \frac{p_f}{\rho_f} \approx \frac{14}{3} \frac{p_f}{\rho_f},
\]  

(39)

and the energy source per unit mass becomes

\[
q \approx \frac{4}{3} \frac{p_f}{\rho_f},
\]  

(40)

which is \( \approx 30\% \) of the total energy at the front. For the Chapman-Jouguet detonation, the outer density profile is

\[
\rho \propto u^{-5},
\]  

(41)

which corresponds to

\[
E_{\text{tot}} \approx \frac{27}{8} \frac{p_f}{\rho_f},
\]  

(42)

and

\[
q \approx \frac{21}{8} \frac{p_f}{\rho_f}.
\]  

(43)

This implies that \( \approx 78\% \) of the total energy at the front is deposited by nuclear reactions.

On the other hand, the shock wave in a SN II suffers from radiative cooling. Chevalier (1982) showed that the outer density profile of the ejecta of SNe II is steeper than that of SNe I and adopted the relation

\[
\rho \propto u^{-12}.
\]  

(44)

We assume that the progenitor of the SN II is a red giant, whose stellar envelope is convective (\( \alpha = 1.5 \)). The relation

**TABLE 5**

| \( \alpha \) | 11/3 | 3.0 | 2.0 | 1.5 | 4/3 | 1.2 | 1.1 | 1.01 | 1.001 | 1.0001 |
|--------------|------|-----|-----|-----|-----|-----|-----|------|--------|--------|
| 3............ | −2.000 | −2.100 | −2.527 | −3.122 | −3.589 | −4.340 | −5.740 | −16.50 | −53.58 | −183.2 |
| 1.5........... | −2.500 | −2.645 | −3.125 | −3.822 | −4.367 | −5.141 | −6.862 | −19.21 | −60.96 | −204.5 |
in equation (44) corresponds to the case in which \( \gamma_c \approx 1.18 \) in
this work, which reduces to

\[
E_{\text{tot}} \approx \frac{77}{9} \frac{p}{f}, \quad (45)
\]

\[
q \approx -\frac{23}{9} \frac{p}{f}, \quad (46)
\]

denominator becomes zero. In other words, the eigenvalue
then \( \sim 30\% \) of the total energy is lost at the shock front by the
radiative cooling.

Here we consider some applicable limits of this solution. First,
from the comparisons with some numerical simulations, the treat-
ment of the energy variation in this self-similar solution is found to
be valid. However, this solution cannot determine the coefficients
of physical variables, because we cannot express \( k_2 \) in equation
(18) in terms of physical parameters specifying the phenome-
on. As a consequence, we cannot obtain absolute values for
the physical variables. Second, we consider the behavior of the
energy source \( q \) near the surface, where the total energy per unit
mass diverges, because its density becomes zero. Simultaneously,
the energy source diverges to \( \pm \infty \) for the energy deposition/loss.
We assume that the energy deposition is caused by some nuclear
reactions. Then the energy generation rate must decrease near the
surface, because the density decreases while the shock front accel-
terates toward the surface. Therefore, this model becomes unreal-
istic near the surface. On the other hand, for the energy loss, the
situation is realistic. As the total energy diverges, the energy source
also diverges to \( +\infty \). This situation is understood as the radiative
diffusion deprives the stellar matter of the diverging energy.

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APPENDIX

DERIVATION OF DISTRIBUTIONS

A1. FORMULATION

From the basic equations of gas dynamics (eqs. [11] and [12]), the equation of motion is expressed as

\[
d^2x \over dt^2 = -\frac{1}{\rho} \frac{dp}{dx} = -\frac{1}{\rho_0} \frac{dp}{dx_0}. \quad (A1)
\]

Using the flow variable \( S(\eta) \), the second derivative of \( x \) yields

\[
d^2x \over dt^2 = \frac{x_0}{l_0} S''(\eta). \quad (A2)
\]

Substitution of equation (22) into the right-hand side of equation (A1) yields

\[
\frac{1}{\rho_0} \left( \frac{dp}{dx_0} \right)_x = \frac{2}{\gamma_c + 1} \left( \frac{\gamma_c - 1}{\gamma_c + 1} \right)^{\gamma} \frac{k_2 x_0^{-\alpha}}{\gamma} \frac{d}{dx_0} \left[ \left( \frac{S(\eta)}{(1 + \lambda) S'(\eta)} \right)^{\gamma-1} \right]
\]

\[
= \frac{2}{\gamma_c + 1} \left( \frac{\gamma_c - 1}{\gamma_c + 1} \right)^{\gamma} \frac{k_2 x_0^{-2\lambda}}{\gamma} \left\{ \frac{\alpha - 2\lambda}{S - (1 + \lambda) S'} - \gamma(1 + \lambda) \frac{\lambda S' + (1 + \lambda) \eta S''}{[S - (1 + \lambda) S']^{\gamma+1}} \right\} \quad (A3)
\]

Therefore, the equation of motion is rewritten in terms of \( S(\eta) \) as

\[
(1 + \lambda)^2 S'' = -\frac{2}{\gamma_c + 1} \left( \frac{\gamma_c - 1}{\gamma_c + 1} \right)^{\gamma} \left\{ \frac{\alpha - 2\lambda}{S - (1 + \lambda) S'} - \gamma(1 + \lambda) \frac{\lambda S' + (1 + \lambda) \eta S''}{[S - (1 + \lambda) S']^{\gamma+1}} \right\}, \quad (A4)
\]

and we obtain the following ordinary differential equation,

\[
S'' = \frac{2(\gamma_c - 1)}{(1 + \lambda)^2} \frac{\alpha - 2\lambda}{2\gamma(\gamma_c - 1)^{\gamma - 1} - (\gamma_c + 1)^{\gamma+1} + [S - (1 + \lambda) S']^{\gamma+1}}. \quad (A5)
\]

The distribution of physical variables are obtained by solving this equation under the initial conditions

\[
S(1) = 1, \quad S'(1) = \frac{2}{(1 + \lambda)(\gamma_c + 1)}. \quad (A6)
\]

A2. DETERMINATION OF EIGENVALUES

Equation (A5) can be integrated to \( \eta \to -\infty \) only for the eigenvalue \( \lambda \), because the derivative \( S'' \) diverges at a singular point where its
denominator becomes zero. In other words, the eigenvalue \( \lambda \) is determined so that its numerator and denominator from the right-hand side
of equation (A5) vanish at the singular point. Setting the denominator of the left-hand side of equation (A5) to zero, it is found that the value of \( \eta \) at the singular point is determined by the relation

\[
\eta = \left[ \frac{(\gamma_e + 1)^{\gamma + 1}}{2\gamma(\gamma_e - 1)^2} \right]^{1/2} \left[ S - (1 + \lambda)S' \right]^{(\gamma + 1)/2}.
\]  

(A7)

Therefore, \( \eta \) takes a positive value at the singular point. We determine the eigenvalue \( \lambda \) as follows. Introducing new variables \( R \) and \( T \) as

\[
\eta^{2/(1+\gamma)}R = S,
\]

(A8)

\[
\eta^{2/(1+\gamma)}T = S - (1 + \lambda)S',
\]

(A9)
equation (A5) is converted to two first-order differential equations by the following steps. First, \( S' \) is rewritten with \( R \) and \( T \) as

\[
S' = \frac{1}{1 + \lambda} \eta^{(1-\gamma)/(1+\gamma)}(R - T).
\]

(A10)

Then, its derivative is

\[
S'' = \frac{1}{1 + \lambda} \eta^{-2\gamma/(1+\gamma)} \left[ \frac{1 - \gamma}{1 + \gamma} (R - T) + \frac{dR}{d\ln \eta} - \frac{dT}{d\ln \eta} \right].
\]

(A11)

Substitution of equations (A9)–(A11) into equation (A5) yields

\[
(1 + \lambda) \left[ \frac{1 - \gamma}{1 + \gamma} (R - T) + \frac{dR}{d\ln \eta} - \frac{dT}{d\ln \eta} \right] = 2(\gamma_e - 1)^\gamma \frac{[\alpha + (\gamma - 2)\lambda]T - \gamma \lambda R}{2\gamma(\gamma_e - 1)^\gamma - [(\gamma_e + 1)T]^{\gamma + 1}}.
\]

(A12)

On the other hand, the derivative of equation (A8) is

\[
S' = \eta^{(1-\gamma)/(1+\gamma)} \left( \frac{2}{1 + \gamma} R + \frac{dR}{d\ln \eta} \right).
\]

(A13)

Combining this expression with equation (A10), we obtain

\[
(1 + \lambda) \frac{dR}{d\ln \eta} = \left[ 1 - \frac{2(1 + \lambda)}{1 + \gamma} \right] R - T,
\]

(A14)

and substitution of this equation into equation (A12) yields

\[
(1 + \lambda) \frac{dT}{d\ln \eta} = -\lambda R - 2 \frac{[\alpha + (\gamma - 2)\lambda]T - \gamma \lambda R}{1 + \gamma} \left( \frac{2(\gamma - 1)}{2\gamma(\gamma_e - 1)^\gamma - [(\gamma_e + 1)T]^{\gamma + 1}} \right) \left( 1 - \frac{2(1 + \lambda)}{1 + \gamma} \right) R - T \right)^{-1}.
\]

(A15)

Dividing equation (A15) by equation (A14), we obtain the following ordinary differential equation;

\[
\frac{dT}{dR} = - \left\{ \lambda R + 2 \frac{[\alpha + (\gamma - 2)\lambda]T - \gamma \lambda R}{1 + \gamma} \left( \frac{2(\gamma - 1)}{2\gamma(\gamma_e - 1)^\gamma - [(\gamma_e + 1)T]^{\gamma + 1}} \right) \left( 1 - \frac{2(1 + \lambda)}{1 + \gamma} \right) R - T \right\}^{-1}.
\]

(A16)

The initial conditions for the variables \( R \) and \( T \) are derived from its definition in equations (A8) and (A9) and the condition in equation (A6) as

\[
R_s = \frac{1}{\gamma_e + 1}, \quad T_s = \frac{\gamma_e - 1}{\gamma_e + 1}.
\]

(A17)

For an arbitrary \( \lambda \), the integration of equation (A16) from the values of the shock front \( (R_f, T_f) \) cannot necessarily reach the other boundary, because the derivative \( dT/dR \) diverges. To avoid this situation, we require that the denominator and the numerator of equation (A16) vanish at the same point, which leads to the following conditions:

\[
[\alpha + (\gamma - 2)\lambda]T - \gamma \lambda R = 0,
\]

(A18)

\[
2\gamma \left( \frac{\gamma_e - 1}{\gamma_e + 1} \right)^\gamma - (1 + \gamma_e)T^{\gamma + 1} = 0.
\]

(A19)

Therefore, we determine the eigenvalue \( \lambda \) so that equation (A16) can be integrated from \( (R_f, T_f) \) to \( (R_s, T_s) \), given by

\[
R_s = \frac{\alpha + (\gamma - 2)\lambda}{\gamma \lambda} \left[ \frac{2\gamma}{\gamma_e + 1} \left( \frac{\gamma_e - 1}{\gamma_e + 1} \right)^\gamma \right]^{1/(\gamma + 1)}, \quad T_s = \left[ 2\gamma \left( \frac{\gamma_e - 1}{\gamma_e + 1} \right)^\gamma \right]^{1/(\gamma + 1)}.
\]

(A20)
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