Oscillating Fubini instantons in curved space

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September 16, 2014

Abstract

Fubini instanton is a bounce solution which describes the decay of a vacuum state located at the top of the tachyonic potential via the tunneling without barrier. We investigate various types of Fubini instantons of a self-gravitating scalar field in a tachyonic quartic potential. With gravity taken into account, we show there exist various types of unexpected solutions including oscillating bounce solutions. In this work, we present numerically oscillating Fubini bounce solutions in de Sitter (dS) and anti-de Sitter (AdS) spaces. We also construct the parametric phase diagrams of the solutions. Of particular significance is that there always exist solutions in all parameter spaces in AdS space. The regions are divided depending on the number of oscillations. On the other hand, dS space allows solutions with codimension-one in parameter spaces. We numerically evaluate semiclassical exponents which give the finite tunneling probabilities.

PACS numbers: 04.62.+v, 98.80.Cq

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1 Introduction

A bounce solution represents an unstable nontopological configuration that corresponds to the saddle point configuration of the Euclidean action. The second derivative of the Euclidean action around the bounce has one negative eigenvalue which is related to the imaginary part of the energy. The bounce solution describes the decay of a metastable vacuum state and determines semiclassical exponent of the tunneling probability \[1, 2\]. One can use this probability to calculate the lifetime of the metastable vacuum state \[3, 4\].

There are three different kinds of bounce solutions, which have become a remarkable Euclidean solution as applied to cosmology \[5, 6, 7, 8, 9, 10\]. One corresponds to a vacuum bubble. The mechanism was introduced to describe a phase transition \[1, 11\] without gravity. The formalism was developed with gravity \[12\] and extended to the case with an arbitrary vacuum energy \[13\]. The possible types of the true vacuum bubbles in dS background space and general background were studied in Ref. \[14, 15\], in which six types of the true vacuum bubble were analyzed in more detail. For the nucleation of a false vacuum bubble in the true vacuum background, the nucleation of a large false vacuum bubble in dS space was originally obtained in Ref. \[16\]. The possible types of the false vacuum bubbles were investigated. The false vacuum solutions only with compact geometry are possible in Einstein gravity \[15\]. The bounce solutions mediating tunneling between the degenerate vacua was also studied \[17, 18\].

Another corresponds to oscillating bounce solutions oscillating around the minimum of the inverted potential. The crossing number of the potential barrier by the oscillating solutions is denoted as \(n\). With this convention an ordinary bounce solution corresponds to \(n = 1\). The existence of oscillating solutions, \(n > 1\), is highly probable if an ordinary bounce solution. Little investigation has been carried out on the physical meaning of the oscillating solutions in Lorentzian spacetime \[10\]. The study on the existence of the Euclidean solution deserves to receive attention in this stage. We review tunneling problem in quantum mechanics. The action of the particle oscillating \(n\) times around the minimum of the inverted potential should be \(n\) times of the ordinary bounce solution. Hence \(n = 1\) contribution dominates the path integral \[19\]. In four dimension, there is a damping term in the scalar field equation in the absence of gravity. Therefore we can not expect the existence of oscillating solutions with \(O(4)\) symmetry. There can only exist an ordinary bounce solution without oscillation. If the gravity is taken into account, the situation changes drastically. For instance, the role of a damping term in the scalar field equation can be changed to an anti-damping term if dS region is included during the transition. The oscillating solution in dS background was first studied in Ref. \[17\], in which the authors found the solution of oscillating scalar field in a fixed background. This is a good approximation when the variation of the potential during the transition is much smaller than the value of the top of the potential. The oscillating solutions in the symmetric double potential was obtained in general background space, in which the condition for the existence of the oscillating solution was analyzed. Moreover the solution without oscillation representing the tunneling from the local maximum of the symmetric double well potential to one of minimums of the potential was obtained in AdS space \[10\]. Originally the tunneling without barrier was
studied in the flat potentials [20, 21, 22]. Recently, the vacuum decay from the flat Minkowski to AdS space was studied as a tunneling without barrier [23]. The oscillating instantons as homogeneous tunneling channels was also studied [24].

The other corresponds to the so-called Hawking-Moss (HM) instanton [9]. The solution describes the scalar field jumping simultaneously onto the top of the potential barrier. In general, probabilities of oscillating instantons are smaller than those of CdL instantons and hence it was the reason why oscillating instantons are overlooked in Einstein gravity. However, if we add correction terms to the gravity sector, then this may change the status of oscillating instantons. One interesting example is non-linear massive gravity [25, 26, 27]. In this case, the correction term from the gravity sector may enhance HM instantons (and hence probably oscillating instantons, too) though we need further investigation.

The Fubini instanton [28, 29, 30] describes the decay of a metastable vacuum state by tunneling instead of a rolling down on the tachyonic potential consisting of a quartic term only. In a scale invariant Lagrangian field theory, the instanton introduces a fundamental scale of hadron phenomena by means of a dilatation noninvariant vacuum state. The conformal invariance of the scalar field theory allows the existence of the Euclidean solution with arbitrary size and the same probability. The explicit form of the solution is known as \( \Phi(\eta) = \sqrt{\frac{8}{3}} \frac{b}{\eta^2 + b^2} \) and the Euclidean action for the decay probability is \( S_E = \frac{8\pi^2}{3\lambda} \) [28]. If the scale invariance breaks down due to the existence of the mass term a bounce solution does not exist. In other words, the particle can not have enough energy to climb the hill up to \( \Phi = 0 \) [31]. The Fubini instanton is a one-parameter family of bounce solutions representing tunneling without a barrier interpolating the state at the top of the potential to an arbitrary state. The solutions could be considered as a ball composed of only a thick wall except for one point at the center of the solution with an arbitrary state lower than the outer vacuum state, unlike a vacuum bubble that composed of an inside part with a lower vacuum state and a wall. When the gravity is taken into account, the conformal invariance is broken. However, the instanton solution was studied in a conformally invariant models in a fixed background without the backreaction [32, 33, 34].

We have shown numerical solutions of Fubini instanton in the initial flat and AdS spaces for the potential with only the quartic term [35]. We have also shown numerically there exist solutions for the potential with both a quartic and a quadratic term irrespective of the value of the cosmological constant. We obtained the solutions with \( Z_2 \) symmetry in dS background. Recently, the oscillating bounce solutions under flat potential barriers was extensively studied in dS space [36], in which the authors analyzed the variety of solutions using instanton diagram [37]. In the present paper, we investigate oscillating Fubini instantons in AdS and dS spaces constructing the parametric phase diagram.

In the model of cosmology, the first picture of the inflationary multiverse scenario was proposed to make the universe scenario without cosmological singularity problem. The picture has the interesting property of self-reproducing or regenerating exponentially expanding universe. In this scenario, the universe as a whole consists of different parts of inflationary domains or an infinite number of mini-universes (bubbles) [38]. The universe could be eternal into the future.
A very large class of inflationary scenarios for both small and large field inflationary model have been analyzed leading to a regime called eternal inflation, in which once the inflation can start, it never stop globally. It seems to stop locally. The scenario is composed of regions separated by more than observable universe or a Hubble volume without correlation \[39, 40, 41\]. Two scenarios could be combined to the eternally inflating multiverse scenario.

The cosmic landscape of string theory is the design that involves a huge number of different metastable and stable vacua, in which the low-energy property of the string theory could be approximated by a set of fields and a potential. The space of all string theory vacua or these fields is called the landscape \[42, 43, 44\]. One of vacua has its laws of physics and constants of nature. In the landscape, the vacuum energy has a huge number of chances to obtain the appropriate value like in our universe if the scenario could be realized. On the other hand, the simplest representative of \(d = 4, N = 8\) supergravity from M-theory could have a dS maximum as the potential \(U = \Lambda(2 - \cosh \sqrt{2}\Phi)\), which is unbounded from below \[3, 4, 45, 46\]. We are not sure whether or not the above scenarios could be combined to one scenario. Anyway, if there exist different states corresponding to metastable and stable vacua, the tunneling could be interesting phenomena. To simply things, we could assume the potential has a lot of vacuum states. One of them could have a very high hill. Then the vicinity of the top of the hill could be approximated as a tachyonic potential.

The instanton solutions has renewed interest in the AdS/CFT correspondence \[47, 48, 49\]. The bulk scalar field \(\Phi\) is dual to a dimension one operator \(\mathcal{O}\) in the boundary. The Fubini instanton under a tachyonic potential in AdS bulk could be related to an instanton solution under a tachyonic potential in the boundary conformal field theory. The ambiguity is what kind of the instanton in the boundary corresponds to the bulk Fubini instanton. With the above point of view, one could introduce the potential with identifying \(\mathcal{O}\) with \(\Phi^2\) in the boundary. The double trace term \(\mathcal{O}^2\) is relevant to the perturbation. The triple trace term could preserve conformal invariance and be relevant to the tunneling. The tachyonic potential consists of the sixth order term in the dual field theory, \(u = -\frac{2}{3}\varphi^6\). The corresponding solution, \(\varphi = \left(\frac{3}{2\sigma}\right)^{1/4} \left(\frac{b}{\eta^2 + b^2}\right)^{1/2}\), is Fubini instanton in three dimension \[28\].

The tunneling process is quantum phenomenon where a particle can penetrate through a finite potential barrier. The simplest case in quantum tunneling is one-dimensional problem, which is extensively studied. However, the extension of the problem to higher dimension is not straightforward. If the gravity takes into account, the situation is much more complicated. Therefore, the tunneling phenomenon including the effect of gravity is worthwhile to be studied in more detail. The purpose of this paper is to investigate further this tunneling process by finding the diversity of tunneling solutions and construct the parametric phase diagrams of oscillating bounce solutions.

The outline of this paper is as follows: In the next section we set up the basic framework for this work. We explain the boundary conditions for the numerical solutions. We employ the potential with only the quartic self-interaction term. In Sec. 3 we present numerical solutions including oscillating bounce solutions. For the decay probability, we evaluate the
action difference which is the leading semiclassical exponent between that of bounce solution and the background by numerical calculation. In Sec. 4 we construct the parametric phase diagram in AdS and dS spaces. In the parametric phase diagrams, the solutions occupy the area composed of two parameters in AdS space, while the solutions occupy the line in dS space. In the final section, we summarize and discuss our results.

2 Set up

The law of exponential decay is a good approximation to describe quantum tunneling phenomena. In the semiclassical approximation, the decay probability of a metastable vacuum state is represented as $Ae^{-B}$. This is the imaginary part of the ground state energy. In this expression the prefactor $A$ is a functional determinant evaluated from the Gaussian integral over fluctuations around the classical solution \[2, 50, 51, 52\]. The exponent $B = S_{bs} - S_{bg}$ is the difference between the Euclidean action of the bounce solution and the background action. We are interested in finding the exponent $B$.

We consider the tunneling phenomena in Einstein gravity minimally coupled to a scalar field which possess a tachyonic potential. To explore the phenomena, we consider the action

$$S = \int_{\mathcal{M}} \sqrt{-g} d^4 x \left[ \frac{R}{2\kappa} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - U(\Phi) \right] + \oint_{\partial \mathcal{M}} \sqrt{-h} d^3 x \frac{K - K_0}{\kappa},$$

where $g = \text{det} g_{\mu\nu}$, $\kappa \equiv 8\pi G$, $R$ denotes the scalar curvature of the spacetime $\mathcal{M}$, and $h$ is the determinant of the first fundamental form of the boundary $\partial \mathcal{M}$ for the metric $g_{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. The second term on the right-hand side is the so-called York-Gibbons-Hawking (YGH) boundary term \[53, 54\]. Here we adopt the sign conventions in Ref. \[55\].

We consider the tachyonic potential with only a quartic self-interaction term as in Ref. \[35\]

$$U(\Phi) = -\frac{\lambda}{4} \Phi^4 + U_o,$$

where the coupling constant $\lambda > 0$. $U_o$ is related to the cosmological constant as $\Lambda = \kappa U_o$, such that background space will be dS, flat and AdS depending on the values of $U_o$. This potential is unbounded from below on either side of the center.

We assume an initial field configuration on the top of the potential, in which the field expectation value is spatially homogeneous and equal to zero. This configuration on top of the tachyonic quartic potential can be a metastable vacuum state \[28\]. The field has a finite probability to leave the top to an arbitrary state inhomogeneously by tunneling instead of a rolling down on the tachyonic potential. In what follows, we explore the transition process through the nucleation of Fubini instanton. We employ the Euclidean path integral approach for the transition probability. The semiclassical approximation leads to the classical equation of motion of a single particle.
For both the geometry and the scalar field, we assume Euclidean $O(4)$-symmetry for the dominant contribution to the decay probability

$$ds^2 = d\eta^2 + \rho(\eta)^2 \left[ d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right].$$

(3)

Then, the resulting $\Phi$ and $\rho$ depend only on $\eta$. The field equations turn out to be

$$\Phi'' + \frac{3\rho'}{\rho} \Phi' = -\frac{d}{d\Phi}(-U), \quad \rho'' = -\frac{\kappa}{3} \rho \left( \Phi'^2 + U \right),$$

(4)

and the Hamiltonian constraint is given by

$$\rho'^2 - \frac{1}{3} \rho^2 \left( \frac{1}{2} \Phi'^2 - U \right) = 0,$$

(5)

where the prime denotes differentiation with respect to $\eta$. The first equation in Eq. (4) is formally equal to a one-particle equation of motion in the inverted potential in Newtonian mechanics. The second term on the left-hand side can be interpreted as a damping term. It can play the role of an anti-damping term if $\rho'$ is negative as in dS space.

To solve the equations of motion, we should impose appropriate boundary conditions in accord with our purpose. In the absence of gravity, the boundary conditions of the Fubini instanton are $d\Phi/d\eta|_{\eta=0} = 0$ and $\Phi|_{\eta=\infty} = 0$ as in Ref. [28]. The solutions exist an arbitrary $\Phi = \Phi_0$ due to the scale invariance. The first condition is for the solution being regular at the origin. The second condition is for the requirement to describe the outside state of the solution, i.e. the initial background configuration. This makes the decay probability finite. We can interpret the equation of motion as follows: The particle starts with the zero velocity at $\Phi = \Phi_0$ and rolls down to the bottom in the inverted potential. Finally, the particle stops at $\Phi = 0$ at $\eta = \infty$ without any oscillation. In the presence of gravity, we should impose two additional boundary conditions for $\rho(\eta)$. The geometry is noncompact for flat or AdS space and compact for dS space which causes us to choose the different type of boundary conditions depending on the convenience. If we consider flat or AdS space, we can impose boundary conditions as follows [10]:

$$\rho|_{\eta=0} = 0, \quad \frac{d\rho}{d\eta}|_{\eta=0} = 1, \quad \frac{d\Phi}{d\eta}|_{\eta=0} = 0, \quad \text{and} \quad \Phi|_{\eta=\eta_{\text{max}}} = 0,$$

(6)

where $\eta_{\text{max}}$ is the maximum value of $\eta$. It is infinite, $\eta_{\text{max}} = \infty$, for flat and AdS spaces while finite for dS space. The first condition is for a geodesically complete space. The second condition stems from Eq. (5).

If we consider dS space, we can impose boundary conditions specified at $\eta = 0$ and $\eta = \eta_{\text{max}}$ as follows:

$$\rho|_{\eta=0} = 0, \quad \rho|_{\eta=\eta_{\text{max}}} = 0, \quad \frac{d\Phi}{d\eta}|_{\eta=0} = 0, \quad \text{and} \quad \frac{d\Phi}{d\eta}|_{\eta=\eta_{\text{max}}} = 0.$$

(7)

The conditions at $\eta = \eta_{\text{max}}$ are needed to make closed dS space. Analytic solutions are not known in the presence of gravity we employ the numerical computation. To solve the Euclidean
field Eqs. (4) and (5) numerically, we make dimensionless variables as in Ref. [35]. In this procedure, the parameter $\kappa$ corresponds to the ratio between Planck mass and the mass scale in the theory. In what follows, we employ dimensionless variables without a tilde. For this procedure we choose the initial values of $\Phi(\eta_{\text{initial}})$, $\Phi'(\eta_{\text{initial}})$, $\rho(\eta_{\text{initial}})$, and $\rho'(\eta_{\text{initial}})$ at $\eta_{\text{initial}} = 0 + \epsilon$ for $\epsilon \ll 1$ as follows:

\[
\begin{align*}
\Phi(\epsilon) & \simeq \Phi_o - \frac{\epsilon^2}{8} \Phi_o^3 + \cdots, \\
\Phi'(\epsilon) & \simeq -\frac{\epsilon}{4} \Phi_o^3 + \cdots, \\
\rho(\epsilon) & \simeq \epsilon + \cdots, \\
\rho'(\epsilon) & \simeq 1 + \cdots.
\end{align*}
\]  

(8)

The initial value of $\Phi'$ is taken to be positive in the present work. Once we specify the initial position $\Phi_o$, then all other quantities can be exactly determined from Eq. (8).

To get the tunneling probability, we only need to consider the bulk part of the Euclidean action. The contribution from the YGH boundary term between the bounce solution and background cancels out each other. The bulk action is evaluated as follows:

\[
S_E = \int_M \sqrt{g_E} d^4x_E \left[ -\frac{R_E}{2\kappa} + \frac{1}{2} \Phi'^2 + U \right] = 2\pi^2 \int \rho^3 d\eta [-U],
\]

(9)

where we used $R_E = 6[1/\rho^2 - \rho''/\rho^2 - \rho''''/\rho]$, Eq. (4), and Eq. (5). We define the “Euclidean energy” as $E_\xi = 2\pi^2 \rho^3 \xi$, where the volume energy density is $\xi = -U$ in Eq. (9). As a result, the semiclassical exponent is as follows:

\[
B = 2\pi^2 \left[ \int_0^{\eta_{bs}(\bar{\rho})} \rho^3 d\eta [-U(bs)] - \int_0^{\eta_{bg}(\bar{\rho})} \rho^3 d\eta [-U(bg)] \right],
\]

(10)

where $\eta_{bs}(\bar{\rho})$ is the arrival time at $\bar{\rho}$, which is the radius of a bounce solution, and $\eta_{bg}(\bar{\rho})$ is the time at $\bar{\rho}$ for the background. For the case of dS space, the length of an evolution parameter is finite due to the topology of 4-sphere. Thus, the probability should be finite. For the cases of AdS space or flat space, the length of an evolution parameter is infinite. If $\eta_{bs}(\bar{\rho})$ is finite, the outside part cancels out each other. Therefore, the tunneling probability should be finite. The vacuum bubble solutions also correspond to this case. If $\eta_{bs}(\bar{\rho})$ is infinite as in the case of Fubini instanton in the absence of gravity, we should check in more detail whether the probability is exponentially suppressed or finite.

### 3 Various types of numerical solutions

In this section, we numerically solve the coupled equations of the scalar field and gravity. We concentrate on various types of numerical solutions both in AdS and dS spaces. Three quantities, $\kappa$, $U_o$, and $\Phi_o$, are used as numerical parameters in the present paper. They correspond to
the reduced Newtonian gravitational constant, the maximum value of the potential indicating
the initial background state located at $\Phi = 0$, and the initial value of a scalar field in numerical
computation, respectively. In the absence of gravity, there is only one parameter $\Phi_o$, regardless
of $U_o$, and $\kappa = 0$. The initial value $\Phi_o$ is related to the size of a solution. There are bounce
solutions with an arbitrary value of $\Phi_o$ without oscillation. The parameter $U_o$ is most important
among parameters because that is related to the cosmological constant as $\Lambda = \kappa U_o$, such that
background space is dS, flat and AdS depending on the values of $U_o$. The maximum value of the
evolution parameter $\eta$, the Euclidean time, is also determined by the values of $U_o$. It has finite
value for dS space, $\eta_{max} =$ finite, while it has infinite value for AdS and flat space, $\eta_{max} = \infty$.
Thus, the different boundary conditions are employed depending on the convenience. The sec-
ond term in the scalar field equation is a damping term in flat and AdS spaces, while it can be
anti-damping term in dS space. Consequentially, the types of solutions depend on the values
of $U_o$.

3.1 Computational Methods

In order to solve the coupled equations of motion numerically, we employ the 4th-order Runge-
Kutta method with the Euclidean evolution parameter step size of $10^{-7}$. The purpose of
this numerical computation is to find several types of solutions with respect to depending
parameters. We expect that the general behavior of the solutions does not change irrespective
of the particular numerical method.

We have to cut the Euclidean time for the numerical calculation because the evolution
parameter is infinite in AdS space. To specify the solutions, we should check how many times
the scalar field passing $\Phi = 0$ which implies the number of oscillation during the limited
Euclidean time. We show several solutions with the different number of oscillations and discuss
about the sufficient time which specifies the solutions.

For the case in dS space, the evolution parameter is finite. Therefore, we impose boundary
conditions (7). In the numerical computations, the cases with only specific parameter values can
satisfy given boundary conditions. Other cases, which do not satisfy given boundary conditions,
are diverging to the positive or negative infinity at $\eta_{max}$ as shown in Fig. 1. We take $\kappa = 0.10$
and $U_o = 1.00$. Suppose we have two initial values $\Phi_1$ and $\Phi_2$ which are diverging to positive
and negative infinity, respectively. Then, we expect that there is the initial value $\Phi_0$ between
$\Phi_1$ and $\Phi_2$ which does not diverge to infinity (see the green line in Fig. 1(a)). This is the so-
called undershoot-overshoot procedure for the case in dS space of the present work. There
is no instanton solution stopped at the point $\Phi = 0$ in dS space, while there exists the solution
stopped near the point $\Phi = 0$ (see Fig. 1). As a result, the tunneling occurs from near the top
to a certain state in dS space.
Figure 1: (color online). Figure represents the behaviors of scalar fields with different initial values which is to describe the undershoot-overshoot procedure for the case in dS space. We take $\kappa = 0.10$ and $U_o = 1.00$.

### 3.2 Numerical results

We first divide into two types of solutions giving rise to the totally different behaviors which are the cases in AdS and dS spaces. We take $\kappa = 0.30$ and $U_o = -0.30$. We consider five cases with different initial positions of $\Phi$. The numerical solutions for $\Phi$ and $\rho$ in AdS space are shown in Fig. 2. Figure 2(a) presents the solutions for $\Phi$ with respect to $\eta$. The solutions are asymptotically approaching the value $\Phi = 0$. Figure 2(b) presents the solutions for $\rho$. The curves move upwards with increasing value of $\Phi_o$. The general behavior of the numerical solution can be easily understood if one thinks of the shape of the solution in a fixed AdS space as $\rho = \sqrt{\frac{3}{\Lambda}} \sinh \sqrt{\frac{\Lambda}{3}} \eta$. Figure 2(c) presents $\Phi'$ versus $\Phi$. Each trajectory begins with zero velocity as $\Phi' = 0$. The velocity rapidly increases to the maximum and then decreases linearly up to the turning point. Figure 2(d) presents the Euclidean energy $E_\xi$ evaluated at constant $\eta$. The first peak is due to non-trivial contributions stemming from the potential and kinetic energy. The increase of the Euclidean energy with respect to $\eta$ is due to the increase of $\rho$ in AdS space. In each figure, we denote the oscillating solutions as $s_i$ with the subscript index $i$ which implies the number of oscillation. In this figure, the black, green and sky blue lines correspond to the solutions with the oscillation 1, 2, and 3 times, respectively. We also denote the oscillating boundary solution as $b_j$ which is a marginal solution of $s_j$ solutions because the $b_j$ solution is the boundary solution between $s_j$ and $s_{j+1}$ solutions. In this figure, the red and blue lines correspond to the marginal solutions with the oscillation 1 and 2 times, respectively.
Figure 2: (color online). (a) Numerical solutions for $\Phi$, (b) solutions for $\rho$, (c) phase diagram of $\Phi'$ versus $\Phi$, and (d) Euclidean energy $E_\xi$ evaluated at constant $\eta$ in AdS space. We take $\kappa = 0.30$ and $U_0 = -0.30$.

| Type | $\kappa$ | $U_0$ | $\Phi_0$ | $S^{bs}$ | $S^{bg}$ | $B$ |
|------|----------|-------|---------|---------|---------|-----|
| $s_1$ | | $-2.00000$ | $1.34 \times 10^{228}$ | | | $0.07 \times 10^{228}$ |
| $b_1$ | | $-3.69897$ | $1.45 \times 10^{228}$ | | | $0.18 \times 10^{228}$ (45) |
| $s_2$ | $0.30$ | $-0.30$ | $-6.00000$ | $1.84 \times 10^{228}$ | $1.27 \times 10^{228}$ | $0.57 \times 10^{228}$ |
| $b_2$ | | $-7.95711$ | $3.59 \times 10^{228}$ | | | $1.32 \times 10^{228}$ (1345) |
| $s_3$ | | $-10.00000$ | $27.44 \times 10^{28}$ | | | $26.17 \times 10^{228}$ |

Table 1: Parameter choices and probabilities of the solutions plotted in Fig. 2.

We summarize the parameter choices of five solutions in Table 1. We carry out the action integral in the range of $0 \leq \eta \leq 1000$ numerically. If the field stops at $\Phi = 0$ within the finite time, the outside part of the solution and corresponding part of the background are canceled each other at the same radius (see Eq. (10)). However, in the present numerical work, it is difficult to decide the exact size of some solutions. Thus, we straightforwardly compute the action and then the difference $B$ has got an approximate behavior $\delta (\sinh^3 \eta) = 3 \sinh^2 \eta \cosh \eta$.
which cause the divergence at infinity as in [35]. We expect that the action difference has a finite value after this minor error being corrected. For the marginal solutions the field stops at $\Phi = 0$ essentially within the finite time. Thus, we carried out the canceling procedure at the same radius. The quantities in the parenthesis represent the values after canceling procedure.

For the case of the tachyonic top in AdS space, one important comment is that the subtracted Euclidean action between the solution and the background diverges in general unless we finely tune the shape of the potential. Let us consider the general case:

$$U = -U_o - \frac{1}{2} m^2 \Phi^2 + \cdots.$$  

(11)

Around the boundary, the approximate behavior of the scalar field becomes

$$\Phi \simeq A_1 \exp \left[ \left( -\frac{3}{2} + \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \right) H \eta \right] + A_2 \exp \left[ \left( -\frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \right) H \eta \right],$$  

(12)

where $H^2 \simeq \kappa U_o/3$ and $A_{1,2}$ are constants [56]. According to Ref. [57], the condition of the converging subtracted action is $A_1 = 0$. On the other hand, if the tachyonic top only has the quartic term, then approximately

$$\Phi \simeq A_1 + A_2 \exp[-3H\eta].$$  

(13)

This is not so strange because we discuss the behavior of the solution around the tachyonic top (hence around the boundary of AdS space). Because of the global shape of the potential, $A_1$ should be controlled to be zero for every numerical solutions of quartic potentials. Therefore, for the special cases when the tachyonic top is sufficiently gentle, we may expect that the tunneling probability still non-zero.

Moreover, (even though this is not the case) there are hopes to regularize Euclidean actions using the no-boundary regulator [58]. This is also worthwhile to investigate further for quartic potentials, while we only have restricted for the quadratic potentials in our paper. We postpone this issue for future works.

One of the things which need to be further investigated in Fig. 2(a) is that the solutions seem to show being slowly approaching to $\Phi = 0$. Therefore, it is doubtful that all the solutions are really going to zero. In order to observe the asymptotic behaviors of $\Phi$, we have to look at the log-log plot of $\Phi$ and $\eta$ and check the small region representing the behavior of the solution near at $\Phi' = 0$ and $\Phi = 0$.

We take $\kappa = 0.30$ and $U_0 = -1.00$. The nine $s_1$ solutions and one $b_1$ solution corresponding to the purple line are plotted in Fig. 3. Figure 3(a) presents the log-log plot of $\Phi$ and $\eta$. We can see that the $s_1$ solutions which are starting from $\Phi = 0.5$ to $\Phi = 4.5$ converge on a certain linear line. Using this property, we approximate a scalar field for the late-time behavior as

$$\log |\Phi| = -A \log \eta + B,$$  

(14)

10
where the $A$ and $B$ are positive constants. The above equation is reduced to

$$|\Phi| = B' \eta^{-A},$$

where $B' = e^B$ which is a constant. Therefore, we can see that the $s_1$ solutions with above late-time behaviors obey Eq. (15) and the scalar field goes to zero when $\eta$ goes to infinity. Especially, the late-time behavior of the $b_1$ solution corresponding to the purple line is totally different from those of other solutions. The slope of the line tends to increase to approximately $\log \eta = \text{constant}$ as $\eta$ increases, which means that the $b_1$ solution seems to reach at $\Phi = 0$ and $\Phi' = 0$ within a finite time. Similarly, $s_1$ solutions make an attractor solution, which means that all solutions except $b_1$ solution are converged one curve exponentially approaching to $\Phi = 0$ and $\Phi' = 0$ in Fig. 3(b). Thus, $s_1$ solutions converge to the origin point when $\eta$ goes to infinity. Moreover, this analysis implies that the number of oscillation is determined before the scalar field reach to zero. Thus, if the scalar field shows the late-time behavior, we can stop to check the number of oscillation before $\eta$ goes to infinity. This is the reason why we are safe to cut the Euclidean time when we count the number of oscillation.

For the tunneling probability, we straightforwardly compute the action difference and summarize the numerical values in Table I. The marginal solutions are clear to have a finite probability because the field seems to reach at $\Phi = 0$ within a finite time. $s_j$ solutions are not clear to have a finite probability at this point because the field seems to reach at $\Phi = 0$ when $\eta$ goes to infinity.

Figures 4 and 5 present the peculiar $Z_2$-asymmetric solutions with specific parameter values in dS space. We denote the $Z_2$ asymmetric solutions as $a_{iL}$ and $a_{iR}$. The subscript index $i$ means the number of oscillation, $L$ and $R$ means that the absolute value of $\Phi_0$ is less or greater than the $\Phi_0$ which corresponds to $Z_2$-symmetric solution, which is denoted as $z_i$ solution. In these figures, the red and green lines correspond to the $a_{iL}$ and $a_{iR}$ solutions, respectively.
Figure 4: (color online). (a) Numerical solutions for $\Phi$, (b) solutions for $\rho$, (c) phase diagram of $\Phi'$ versus $\Phi$, and (d) Euclidean energy $E_\xi$ evaluated at constant $\eta$ of $Z_2$ asymmetric cases in dS space.

The reason why we call this pair of solution as $Z_2$-asymmetric is as follows: if one solution starts from $\Phi_1$ at $\eta = 0$ and it ends $\Phi_2$ at $\eta = \eta_{\text{max}}$, then the other solution will starts $\Phi_2$ at $\eta = 0$ and it will ends $\Phi_1$ at $\eta = \eta_{\text{max}}$. Thus, when we take a transformation as $\eta \rightarrow \eta_{\text{max}} - \eta$, the solution will transform as $a_{iL,iR} \rightarrow a_{iR,iL}$.

Figure 4(a) presents the numerical solutions for $\Phi$. The field with the green line starts at $4.31876$ with zero velocity, is rolling down the hill, monotonically pass the point $\Phi = 0$, and arrive at $-0.22912$. The field with the red line starts at $-0.22912$ with zero velocity, monotonically passes the point $\Phi = 0$, is rolling up the hill due to the anti-damping term, and arrive at $4.31876$. There seems to be no Fubini instanton solution stopped at the point $\Phi = 0$ in dS space. They are the most similar solutions corresponding to a Fubini instanton in the absence of gravity. Using Eqs. (1) and (2) we can estimate the equations with asymptotic values. At the final stage, $\frac{d(U)}{d\Phi} \sim -\Phi^3_{fs}$ and $\frac{3\rho'}{\rho} \Phi' \sim -\frac{3}{4} \Phi^3_{fs}$, which give $\Phi'' \sim -\frac{1}{4} \Phi^3_{fs}$. For the cases of flat and AdS spaces, all of terms in the equation go to zero at the final stage, respectively. Figure 4(b) presents the numerical solutions for $\rho$. Figure 4(c) presents the phase diagram of $\Phi'$ versus $\Phi$. Using the phase diagram method in Ref. [35], the first stage of the red curve has got the form $\Phi' \sim -\sqrt{\frac{\lambda}{2}(\Phi^4_o - \Phi^4)}$, the second stage corresponds to $d\Phi'/d\Phi = 0$, and the third stage corresponds to $d\Phi'/d\Phi = c$, i.e. a positive constant. Figure 4(d) presents the Euclidean energy $E_\xi$ evaluated at constant $\eta$. 
Figure 5: (color online). (a) Numerical solutions for $\Phi$, (b) solutions for $\rho$, (c) phase diagram of $\Phi'$ versus $\Phi$, and (d) Euclidean energy $E_\xi$ evaluated at constant $\eta$ of $Z_2$ asymmetric cases in dS space.

Figure 5(a) presents the numerical solutions for $\Phi$. The field with the green line starts at 7.60997 with zero velocity, is rolling down the hill, passes the point $\Phi = 0$ three times, is rolling down the hill, and arrive at $-6.33115$. The field with the red line starts at $-6.33115$ with zero velocity, is rolling down the hill, passes the point $\Phi = 0$ three times, is rolling down the hill, and arrive at 7.60997. Figure 5(b) presents the numerical solutions for $\rho$. Figure 5(c) presents the phase diagram of $\Phi'$ versus $\Phi$. The first stage of the red curve has got the form $\Phi' \simeq \sqrt{\frac{\lambda}{2}}(\Phi_o^4 - \Phi^4)$, the second stage corresponds to $d\Phi'/d\Phi = 0$, the third stage corresponds to $d\Phi'/d\Phi = -c$, i.e. a negative constant, the fourth stage corresponds to the behavior near the origin point, the fifth stage corresponds to $d\Phi'/d\Phi = c$, the sixth stage corresponds to $d\Phi'/d\Phi = 0$, and the seventh stage corresponds to $\Phi' \simeq -\sqrt{\frac{\lambda}{2}}(\Phi_o^4 - \Phi^4)$. The upper box in the same figure shows the magnification of the small region representing behavior of the curves. Figure 5(d) presents the Euclidean energy $E_\xi$ evaluated at constant $\eta$. The red lines are overlapped in Figs. 5(b) and 5(d). We summarize the parameter choices and probabilities of several $Z_2$-asymmetric solutions in Table 2.
Table 2: Parameter choices and probabilities of $Z_2$-asymmetric solutions

| Type | $\kappa$ | $U_0$ | $\Phi_0$ | $S^{cs}$ | $S^{bg}$ | $B$ |
|------|----------|-------|----------|----------|----------|-----|
| $a_{1L}$ | 0.10 | 1.00 | $-0.22912$ | $-23654$ | $-23687$ | 33 |
| $a_{1R}$ | | | $4.31876$ | $-23654$ | | 33 |
| $a_{3L}$ | 0.05 | 0.05 | $-6.33115$ | $-189486$ | $-189496$ | 10 |
| $a_{3R}$ | | | $7.60997$ | $-189486$ | | 10 |

4 Parametric phase diagram

In this section, we construct the parametric phase diagram of various solutions with respect to the parameters $\kappa$, $U_0$ and $\Phi_0$. There are several types of solutions including oscillating solutions as we discussed in Sec. 3. We investigate the properties of those solutions and analyze the solutions using the properties.

4.1 Computational Methods

The purpose of this numerical computation is to construct the parametric phase diagram with respect to depending parameters. We employ the matrix plot to generate a plot that gives a visual representation of the values of elements in a matrix. Figure 6 presents the matrix plot for visual representation of the solutions including oscillating solutions in AdS and dS spaces. We divide into two parametric phase diagrams depending on the values of $U_0$. The $X$-axis in each plot corresponds to $\Phi_0$, which is from 0 to $-14.00$ and $Y$-axis corresponds to $U_0$ however the range is different with respect to the values of $U_0$. 

Figure 6: Matrix plots in (a) AdS space with white, gray and black colors which are correspond to the different number of oscillation such as 1, 2, and 3, respectively, and in (b) dS space with white and black colors which are correspond to the direction of divergence such as negative and positive, respectively.
Figure 6(a) presents an example how to get the parametric phase diagram in AdS space. The Y-axis is from 0 to $-2.00$ where the direction is downward. We first choose the parameter $\kappa = 0.30$, and divide X-axis and Y-axis as 100 pieces. In this case, $\eta$ grows up to infinity however we are safe to check the number of oscillation during the finite Euclidean time as we discussed at Sec. 3.1. We count the number of oscillation for every point in the phase diagram. In general, this process can only recognize the $s_i$ solutions. Thus, we need to take a more elaborated approach to find the $b_j$ solutions and get the data more dense rather than Fig. 6(a). For the case in AdS space, there exist infinite number of solutions occupying the area composed of two parameters $U_o$ and $\Phi_o$ with given value of $\kappa$ as shown in Fig. 6(a) which are extended to those with all parameter spaces if the oscillating solutions are allowed.

Similarly with the previous figure, Figure 6(b) presents an example how to get the parametric phase diagram in dS space. The Y-axis is from 2.50 to 0 where the direction is downward. We choose the parameter $\kappa$ same as in AdS space and divide X-axis as 100 pieces and Y-axis as 125 pieces. In this case, $\eta$ is terminated at $\eta_{\text{max}}$. Therefore, we can calculate whole procedure numerically. We check the divergence at all points using the shooting method, i.e. undershoot-overshoot procedure. The result is illustrated in Fig. 6(b). Because this process can show only approximated points of solutions, we should gather more dense data in every area of color changing region. For the case in a dS space, it is very difficult to find the solution corresponding to that very Fubini instanton. If the oscillating solutions with both $Z_2$-symmetric and asymmetric solutions are allowed, there exist infinite number of solutions occupying the line composed of two parameters with given value of $\kappa$ as shown in Fig. 6(b).

### 4.2 Numerical results

We consider four cases with different values of $\kappa$. We plot the parametric phase diagrams in dS and AdS spaces. Figure 7 presents a phase diagram with respect to the parameters $U_o$ and $\Phi_o$ with given values of $\kappa$ in AdS space. In this figure, we consider only up to $n = 3$ oscillating solutions. There are solutions with oscillation more than $n = 3$ near the $U_o = 0$. However, we do not display these solutions because it is very difficult to gather all solutions with many oscillations at the extremely small region of $U_o$. We take $\kappa = 0.05, 0.10, 0.30, $ and $0.50$ for (a), (b), (c), and (d), respectively. The $U_o$ is from $-2.0 \times 10^{-4}$ to $-2.00$ and the $\Phi_o$ is from 0 to $-14.00$. The black lines with different $\kappa$’s used in Fig. 7 correspond to the $b_1$ solutions and the red lines correspond to the $b_2$ solutions for the given range of $U_0$’s. The points out of lines are $s_i$ solutions with the oscillation $i$ times. Consequentially, the solutions occupy the area composed of two parameters, $U_o$ and $\Phi_o$ in AdS space. It can occupy the volume composed of all three parameter set.

In this figure, we can observe the behavior of the marginal solutions with respect to the parameter values. When $\kappa$ is increasing, $b_j$ solutions move to the left and upward which means that the solution tends to have more number of oscillation for the same $\Phi_o$ when $U_o$ decreases. However, it lead to restrict the numbers of oscillation quickly as the horizontal lines appeared. From the horizontal behavior of $b_2$ solution, we can see that there is the minimum value of $U_o$.
which restricts the number of oscillation as \( n = 3 \); i.e. if we take \( U_o = -1.00 \) then the solution can oscillate up to \( n = 3 \). From this observation, we can expect that all \( b_j \) solutions have the same property. Moreover, the initial values \( \Phi_o \)s which give the \( b_j \) solutions are approaching to zero as \( U_0 \) goes to zero. It is not clearly seen from Fig. 7. Therefore, we have to check the behavior shown in the figure using the log-log plot of the absolute values of \( U_0 \) and \( \Phi_o \).

Figure 8 shows the log-log scale with the same marginal solutions used in Fig. 7. Left four curves correspond to \( b_1 \), while right four curves correspond to \( b_2 \). Because the \( b_1 \) solutions and \( b_2 \) solutions with respect to \( U_o \) and \( \Phi_o \) are linear in the vicinity of \( U_o = 0 \) and \( \Phi_o = 0 \), each of the solution has a relation between these two parameters as follows:

\[
\log |U_0| = A \log |\Phi_0| + B \quad \Rightarrow \quad U_0 = \pm e^B |\Phi_0|^A,
\]

where \( A \) and \( B \) are positive constant. We take the \(-\) sign for our case and the other sign for \( \Phi_o > 0 \). We can easily see that \( b_1 \) solution and \( b_2 \) solution approach to the point \( U_0 = 0 \) and \( \Phi_o = 0 \) from Eq. (16). From this numerical result, we can expect that all \( b_j \) solutions behave like a \( b_1 \) solution.

Figure 9 presents parametric phase diagrams of the selected solutions with respect to parameters in dS space [36]. We also do not display the solutions with oscillation more than
Figure 8: (color online). Parametric phase diagram of the log-log scale of $|U_o|$ and $\Phi_o$ with four different values of $\kappa'$ in AdS space.

$n = 4$ for $Z_2$-symmetric solution because of the same reason as in AdS space. We illustrate $z_1$ solutions with the red line, $z_2$ solutions with the green line, $z_3$ solutions with the blue line, $z_4$ solutions with the green line, $a_1$ solutions with the black line and $a_3$ solutions with the gray line. There are bifurcation points for $Z_2$-asymmetric solutions \cite{17, 36}. We can see that there exists the bifurcation point for $a_3$ solution, i.e. for the $n > 1$ branch. If we take smaller values of $\kappa$, the bifurcation points could be observed for the higher $n$ branches. The $U_o$ is from $10^{-2}$ to 2.50 and $\Phi_o$ is from 0 to $-14.00$ almost same as in Fig. 7. The solutions located only on the presented lines that is different from the case in AdS space. The points out of the presented lines do not satisfy the boundary conditions.

In this figure, we can observe the behavior of $z_i$ solutions with respect to the parameter values. The oscillation number is suppressed as $U_0$ or $\kappa$ is increased, i.e. if we take $U_o = 1.50$ and $\kappa = 0.30$ then $z_1$ solution is only possible. The strange thing is that the $a_{iL}$ solution and $a_{iR}$ solution are collapsing to $z_i$ solution for some specific values of $U_o$. These $Z_2$-asymmetric solutions are shown only up to some $U_o$ and this value is decreased as $\kappa$ is increased. At this point, $a_{iL}$ solution and $a_{iR}$ solution meet each other with $z_i$ solution.

We may expect that there will be $a_i$ solutions with odd number of oscillation greater than $n = 3$. However, it is very difficult to find solutions because the solutions are highly suppressed to the vicinity of $U_o = 0$. Furthermore, if $\Phi_o$ is low enough with high $\kappa$ value, the solutions are localized to specific values of $U_o$. 

17
Figure 9: (color online). Parametric phase diagrams in dS space with (a) $\kappa = 0.05$, (b) $\kappa = 0.10$, (c) $\kappa = 0.30$, and (d) $\kappa = 0.50$, respectively.

5 Summary and Discussion

If the potential has a large number of stable and metastable vacuum state including a very high hill, the vicinity of the top of the hill could be approximated as a tachyonic potential. Then, it would be worthwhile to be investigated the issue on the tunneling without barrier.

In this paper, we have investigated various Fubini instantons of a self-gravitating scalar field under a tachyonic quartic potential. The Fubini instanton is a bounce solution which describes the decay of a vacuum state located at the top of the potential to an arbitrary state as the tunneling without barrier. The Fubini instantons could be considered as a ball consisted of only a thick wall except for one point at the center of the solution with an arbitrary lower energy state.

There are three different kinds of bounce solutions. They are vacuum bubbles, oscillating bounce solutions, and the HM instanton, respectively. Oscillating bounce solutions and HM instanton are possible only if gravity is taken into account. In this paper, we have investigated oscillating Fubini bounce solutions and constructed the parametric phase diagrams.

The Euclidean bounce trajectory $\Phi_b(\eta)$ is constrained by the specific boundary conditions
at $\eta = 0$ and $\eta = \eta_{\text{max}}$. The evolution parameter $\eta_{\text{max}}$ is finite for dS space, while it is infinite for flat and AdS spaces. For the tunneling probability, we reduced the action integral only to have the potential term for the semiclassical exponent. In AdS space, the late-time behavior of marginal solutions, $b_j$ solutions, is totally different from those of other solutions, $s_j$ solutions. The behavior of a marginal solution indicates the field can reach at $\Phi = 0$ essentially within a finite time, while other solutions reach at $\Phi = 0$ when $\eta$ goes to infinity. The late-time behaviors could affect the tunneling probabilities. There is no doubt that the case of the marginal solution can give the finite exponent $B$. For the cases of other solutions, it is not clear whether to have a finite exponent $B$ at this point. This issue in AdS space remains to be explored. In dS space, there exist $Z_2$-symmetric solutions [35, 36]. This is because the geometry of Euclidean dS space has a topology of 4-sphere and the potential also has $Z_2$ symmetry. We have shown that there exist $Z_2$-asymmetric solutions including $a_3$ solution. This one belongs to very peculiar type of solutions. One of the remarkable things is that there does not exist the solution stopped at the point $\Phi = 0$. There exist solutions stopped near the point $\Phi = 0$. In other words, the tunneling seems to occur from the point near the top of the potential to a certain state in dS space. It might be described by using thermal interpretation [59]. Another interpretation is that dS spacetime has a repulsive property which may prevent a full stop at $\Phi = 0$.

We have presented the parametric phase diagrams of the oscillating solutions with respect to three parameters ($\kappa$, $U_o$, and $\Phi_o$) in AdS and dS spaces. Of particular significance is that there always exist solutions in all parameter space in AdS space. The regions are divided depending on the number of oscillations. In other words, the solutions occupy the area composed of two parameters ($U_o$ and $\Phi_o$) in AdS space. The solutions occupy the volume composed of three parameters if include all parameter set. On the other hand, dS space allows solutions with codimension-one in parameter space. In other words, the solutions occupy the line composed of two parameters and the area composed of three parameters. Therefore, the solutions are more rich in AdS space than those in dS space. We think that dS has compact geometry and the corresponding boundary conditions. Thus, the solution space is restricted from those specific characteristics.

From these parametric phase diagrams, we can estimate a parametric phase diagram or type of solutions in the absence of gravity by limiting $\kappa$ to be zero. For the solutions in dS space, we have taken a different kind of boundary conditions due to the compact geometry of dS space. Thus, we take $\kappa$ to be zero from the solutions in AdS space. In this case, $b_1$ solution goes to right and downward direction when we take $\kappa$ to be zero. Then, $s_1$ solution space is expanded by this effect and finally we can expect that whole phase space will be covered by $s_1$ solution at $\kappa = 0$. This expectation gives the exactly same result which is obtained by Fubini in the absence of gravity [28].
Acknowledgements

We would like to thank Remo Ruffini, Sung-Won Kim, Hyung Won Lee, Myeong-Gu Park, and S.-S. Xue for their hospitality at the 13th Italian-Korean Symposium on Relativistic Astrophysics in Seoul, Korea, 15-19 Jul 2013. We would like to thank Misao Sasaki for his hospitality during our visit to Yukawa Institute for Theoretical Physics, Kyoto University. We would like to thank Kyoung Yee Kim and Kyung Kiu Kim for helpful discussions and comments. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. 2014R1A2A1A01002306). WL was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A2043908). DY was supported by the JSPS Grant-in-Aid for Scientific Research (A) No. 21244033. DY also would like to thank Leung Center for Cosmology and Particle Astrophysics (LeCosPA) of National Taiwan University (103R4000).
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