INFINITE-TIME RUIN PROBABILITY OF A RENEWAL RISK MODEL WITH EXPONENTIAL LÉVY PROCESS INVESTMENT AND DEPENDENT CLAIMS AND INTER-ARRIVAL TIMES

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ABSTRACT. We investigate the infinite-time ruin probability of a renewal risk model with exponential Lévy process investment and dependent claims and inter-arrival times. Assume that claims and corresponding inter-arrival times form a sequence of independent and identically distributed copies of a random pair \((X, T)\) with dependent components. When the product of the claims and the discount factors of the corresponding inter-arrival times are heavy tailed, we establish an asymptotic formula for the infinite-time ruin probability without any restriction on the dependence structure of \((X, T)\).

1. Introduction. Consider a renewal risk model in which claims, \(\{X_n\}_{n \geq 1}\), constitute a sequence of independent, identically distributed (i.i.d.), and positive random variables (r.v.s) with generic random variable \((r.v.)\) \(X\), common distribution \(F\) such that \(\bar{F}(x) = 1 - F(x) > 0\) for all \(x > 0\), and their arrival times \(\{\tau_n\}_{n \geq 1}\) constitute a renewal counting process

\[
N(t) = \sum_{n=1}^{\infty} 1[\tau_n \in [0,t)], t \geq 0,
\]

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where $1_A$ denotes the indicator function of the set $A$. For later use, we write $\tau_0 = 0$. To avoid triviality, we assume that $\tau_1$ is a nonnegative and non-degenerate at 0. Denote the renewal function of $\{N(t)\}_{t \geq 0}$ as

$$
\lambda_t = EN(t) = \sum_{n=1}^{\infty} \mathbb{P}\{\tau_n \leq t\}.
$$

The inter-arrival times $\{T_n\}_{n \geq 1}$, $T_n = \tau_n - \tau_{n-1}$, form a sequence of i.i.d. and positive r.v.s with generic r.v. $T$. Hence, the total amount of claims up to time $t \geq 0$ can be written as

$$
S(t) = \sum_{n=1}^{N(t)} X_n
$$

with $S(t) = 0$ when $N(t) = 0$. The total amount of premiums accumulated up to $t \geq 0$ is denoted by $C(t) = ct$, where $c$ is a positive constant.

The complete independence of $\{X_n\}_{n \geq 1}$ and $\{T_n\}_{n \geq 1}$ is far unrealistic with the increasing complexity of economic environment. For example, if the deductible retained to the insured is raised, then the inter-arrival time will increase and the claim sizes would decrease because small losses will be ruled out and retained by the insured. Hence, many researchers introduced dependence structures between $\{X_n\}_{n \geq 1}$ and $\{T_n\}_{n \geq 1}$ into renewal risk models. We refer readers to Asimit and Badescu [1], Li et al. [14], Li [15], Fu and Ng [9], among many others. Motivated by this, we introduce a type of dependence structure as follows:

**Assumption 1.** $\{(X_n, T_n)\}_{n \geq 1}$ are i.i.d. copies of $(X, T)$ with dependent components.

Assume that an insurer can make investment in risk-free and risky assets. The price processes of the risk-free and risky assets, respectively, satisfy

$$
R_0(t) = e^{rt} \text{ and } R_1(t) = e^{L(t)}, \quad t > 0,
$$

where $r > 0$ is the risk-free interest rate. $\{L(t)\}_{t \geq 0}$ is a Lévy process. That is to say, $L(0) = 0$, $\{L(t)\}_{t \geq 0}$ has independent and stationary increments, and is right continuous with left limit. Let $(\gamma, \sigma^2, \nu)$ be the characteristic triple of $\{L(t)\}_{t \geq 0}$, where $\gamma \in \mathbb{R}, \sigma \geq 0$ and Lévy measure $\nu$ satisfies $\nu(0) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty$. For the general theory of Lévy processes, see Sato [17] and Cont and Tankov [6]. Suppose that the insurer continuously invests a constant fraction $\theta \in (0, 1)$ of its surplus in the risky asset and invests the remaining surplus in the risk-free asset (see e.g. Emmer et al. [7] and Emmer and Klüppelberg [8]). This strategy is classical in financial portfolio optimization (see Korn [13], Section 2.1). The fraction $\theta$ is called the investment strategy.

With this investment strategy $\theta \in (0, 1)$, we can define the portfolio price process:

$$
R_\theta(t) = e^{L_\theta(t)}, \quad t > 0; \quad R_\theta(0) = 1.
$$

By Lemma 2.5 in Emmer and Klüppelberg [8], $\{L_\theta(t)\}_{t \geq 0}$ is also a Lévy process with characteristic triple $(\gamma_\theta, \sigma_\theta^2, \nu_\theta)$, where

$$
\gamma_\theta = \gamma \theta + (1 + \theta) \left( r + \frac{\sigma^2}{2} \theta \right) + \int_{\mathbb{R}} (\log(1 + \theta (e^x - 1))) 1_{\{\log(1 + \theta (e^x - 1)) \leq 1\}} - \theta x 1_{\{|x| \leq 1\}} \nu(dx),
$$
\[ \sigma_\theta^2 = \theta^2 \sigma^2, \]
\[ \nu_\theta(A) = \nu(\{x \in \mathbb{R} : \log(1 + \theta(e^x - 1)) \in A\}) \]
for any Borel set \( A \subseteq \mathbb{R} \).

Define the Laplace exponents of the process \{\( L(t) \)\} \( t \geq 0 \) and \{\( L_\theta(t) \)\} \( t \geq 0 \) by
\[ \psi(s) = \log \mathbb{E}[e^{-sL(1)}] \quad \text{and} \quad \psi_\theta(s) = \log \mathbb{E}[e^{-sL_\theta(1)}]. \]

If \( \psi(s) < \infty \) then \( \mathbb{E}[e^{-sL(t)}] = e^{t\psi(s)} < \infty \) for all \( t \geq 0 \) (see Sato [17], Theorem 25.17). For \( \theta \in (0, 1) \), the proof of Lemma 4.1 in Klüppelberg and Kostadinova [12] show that \( \psi_\theta(s) \) is finite for any \( s \geq 0 \), and if \( 0 < \mathbb{E}[L(1)] < \infty \) and either \( \sigma > 0 \) or \( \nu((\infty, 0)) > 0 \), then there exists a unique \( \kappa_\theta \) such that \( \psi_\theta(\kappa_\theta) = 0 \). Thus, we can obtain
\[ \psi_\theta(\kappa) < 0 \quad \text{for any} \quad 0 < \kappa < \kappa_\theta, \quad (1) \]
which can be found more detail in Guo and Wang [10]. The discount factor from \( t = \tau_n \) to \( t = 0 \), by which a future asset price at time \( \tau_n \) must be multiplied in order to obtain the present value at time 0, can be denoted by \( R_\theta^{-1}(\tau_n) = e^{-L_\theta(\tau_n)} \).

Hence, the discount factor of \( T_n \) (i.e., from \( \tau_n \) to \( \tau_{n-1} \)) is \( e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)} \), which has identical distribution with \( e^{-L_\theta(T_n)} \).

Although insurance processes and investment processes may be weakly dependent in economic environment, we assume that they are independent, which allows for a very explicit analysis of the integrated risk process. We refer reader to Klüppelberg and Kostadinova [12]. Thus, we have:

**Assumption 2.** \( L_\theta(t) \) is independent of \( S(t) \).

Then, by Lemma 2.2 of Klüppelberg and Kostadinova [12], we can obtain the integrated risk process (IRP)
\[ U_\theta(0) = x, U_\theta(t) = e^{L_\theta(t)} \left( x + \int_0^t e^{-L_\theta(v)} (cdv - dS(v)) \right), t > 0, \quad (2) \]
where \( x > 0 \) is the initial surplus of the insurer. Denote the discounted net loss process by
\[ V_\theta(t) = x - U_\theta(t)e^{-L_\theta(t)} = \int_0^t e^{-L_\theta(v)} (dS(v) - cdv), t \geq 0. \quad (3) \]

Now we can define the infinite-time ruin probability of IRP (2) by
\[ \Psi(x) = \mathbb{P}\left\{ \inf_{0 \leq t < \infty} U_\theta(t) < 0|U_\theta(0) = x \right\} = \mathbb{P}\left\{ \sup_{0 \leq t < \infty} V_\theta(t) > x \right\}. \quad (4) \]

By Assumption 1, \( X_n \) and \( e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)} \) are dependent. Without the consideration of specific dependence structures, we make the following assumption:

**Assumption 3.** The common distribution of \( \{X_n e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)}\}_{n \geq 1} \), denoted by \( H \), belongs to the consistent variation class \( \mathcal{C} \).

Trivially, if \( X_n \) is independent of \( T_n \) and has a distribution belonging to \( \mathcal{C} \), we can derive that for \( \kappa_\theta > J_\theta^+ \),
\[ \mathbb{E}\left[ (e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)})^{\kappa_\theta} \right] = \mathbb{E}e^{-\kappa_\theta L_\theta(T_n)} = \mathbb{E}e^{T_n \psi_\theta(\kappa_\theta)} = 1. \]

Then, Lemma 3.3 shows that \( H \) belongs to \( \mathcal{C} \). If \( (X_n, T_n) \) follows a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution, i.e.,
\[ \mathbb{P}\{X_n \leq x, T_n \leq y\} = \mathbb{P}\{X_n \leq x\} \mathbb{P}\{T_n \leq y\} (1 + \vartheta \mathbb{P}\{X_n > x\} \mathbb{P}\{T_n > y\}) \], \( \vartheta < 1 \).
Suppose that $X_n$ has a distribution belonging to $\mathcal{C}$. By relation (4.9) of Chen [4] (p. 1041), we have
\[ \mathbb{P}\{X_ne^{L_0(T_n)} > x\} = \mathbb{P}\{X_n e^{-L_0(T_n)} > x\} \sim \mathbb{P}\{X_n e^{-L_0(T_n^*)} > x\}, \]
where $T_n^*$ is a positive r.v. independent of $X_n$ and distributed by
\[ \mathbb{P}\{T_n^* \leq y\} = (1 - \vartheta)\mathbb{P}\{T_n \leq y\} + \vartheta \mathbb{P}\{T_n \leq y\}, y > 0. \]
Also, we can derive that for $\kappa > J_{\vartheta}^b$,
\[ \mathbb{E}e^{-\kappa L_0(T_n^*)} = \mathbb{E}e^{\tau_n \psi_b(\kappa)} = 1. \]
Then, by Lemma 3.3, the distribution of $X_ne^{-L_0(T_n^*)}$ belongs to $\mathcal{C}$. Hence, $H$ belongs to $\mathcal{C}$. The similar discussion can be done for the regular variation class ($\mathcal{R}_{-\alpha}$) if we use Lemma 3.4.

In this paper, we obtain an asymptotic estimate for $\Psi(x)$ under the Assumption 1-3 and apply the estimate to the special case that $H$ belongs to $\mathcal{R}_{-\alpha}$. Notice that we do not make any special assumption on the dependence structure of $(X,T)$. The remaining part of this paper is organized as follows. In Section 2, we introduce some notations and state our main results. In Section 3, we provide some lemmas and prove the main results of the paper.

2. Notations and main results. In this paper, $C$ represents a positive constant without relation to $x$ and may vary from place to place. Hereafter, all limit relations are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup a(x)/b(x) \leq 1$ and write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$. Also, we write $a(x) \asymp b(x)$ if $0 < \liminf a(x)/b(x) \leq \limsup a(x)/b(x) < \infty$.

In order to facilitate subsequent expression, we denote
\[ Z_{[s,t]} = \int_s^t e^{-L_0(v)} dv, \quad 0 \leq s \leq t \quad \text{and} \quad Z_{[0,\infty]} = \int_0^\infty e^{-L_0(v)} dv. \]

Now we recall several classes of heavy-tailed distributions. A distribution $F$ belongs to the dominated variation class (denoted by $\mathcal{D}$) if $\overline{F}(x) = 1 - F(x) > 0$ for all $x \in \mathbb{R}$ and
\[ \limsup_{x \to \infty} \frac{F(xy)}{F(x)} < \infty \text{ for any } 0 < y < 1. \]

A distribution $F$ belongs to the long-tailed class (denoted by $\mathcal{L}$) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and
\[ \lim_{x \to \infty} \frac{F(x + y)}{F(x)} = 1 \text{ for any } y > 0. \]

A distribution $F$ belongs to the consistent variation class (denoted by $\mathcal{C}$) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and
\[ \limsup_{y \to 1} \lim_{x \to \infty} \frac{F(xy)}{F(x)} = 1, \text{ or equivalently, } \lim_{y \to 1} \liminf_{x \to \infty} \frac{F(xy)}{F(x)} = 1. \]

A distribution $F$ belongs to the regular variation class (denoted by $\mathcal{R}_{-\alpha}$) if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and
\[ \lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha} \text{ for some } \alpha > 0 \text{ and all } y > 0. \]
It is well known that $\mathcal{R}_{-\alpha} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L}$.

Besides that, the upper Matuszewska index $\mathcal{J}_F^+$ and lower Matuszewska index $\mathcal{J}_F^-$ (see Bingham et al. [2], Ch.2.1) are used. It is well known that $\mathcal{J}_F^+ < \infty$ if $F \in \mathcal{D}$ and $\mathcal{J}_F^+ = \mathcal{J}_F^- = \alpha$ if $F \in \mathcal{R}_{-\alpha}$.

Now, we state the main results.

**Theorem 2.1.** Consider the IRP ([2]) under Assumption 1-3. If $0 < \mathbb{E}[L_0(1)] < \infty$ and there exists $\kappa_0 > 0$ such that $\psi_0(\kappa_0) = 0$ and $0 < \mathcal{J}_H^- \leq \mathcal{J}_H^+ < \kappa_0$, then it holds that

$$
\Psi(x) \sim \sum_{n=1}^{\infty} \mathbb{P}\{X_n e^{-L_0(\tau_n)} > x\} = \int_{0}^{\infty} \mathbb{P}\{X_n e^{-L_0(s)} > x\} d\lambda_s. \quad (5)
$$

**Theorem 2.2.** Replacing “$H \in \mathbb{C}$” and “$0 < \mathcal{J}_H^- \leq \mathcal{J}_H^+ < \kappa_0$” with “$H \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \kappa_0$” among the conditions of Theorem 2.1, we have

$$
\Psi(x) \sim \overline{\mathcal{F}}(x) \sum_{n=1}^{\infty} \{\mathbb{E}e^{\tau_1 \psi_0(\alpha)}\}^{n-1} = \frac{\overline{\mathcal{H}}(x)}{1 - \mathbb{E}e^{\tau_1 \psi_0(\alpha)}}. \quad (6)
$$

3. **Proofs of the main results.** By convention, an empty sum is 0 and an empty product is 1.

3.1. **Some lemmas.** By Proposition 2.2.1 in Bingham et al. [2], for a distribution $F \in \mathcal{D}$ and arbitrarily fixed $p > \mathcal{J}_F^+$, there exist positive constants $C_p$ and $D_p$ such that

$$
\frac{\mathcal{F}(y)}{\mathcal{F}(x)} \leq C_p \left(\frac{x}{y}\right)^p \quad (7)
$$

holds for all $x \geq y \geq D_p$. Fixing the variable $y$ leads to

$$
x^{-p} = o(\mathcal{F}(x)) \text{ for any } p > \mathcal{J}_F^+. \quad (8)
$$

The following fundamental lemmas will be used.

**Lemma 3.1.** Let $X$ and $Y$ be two independent and nonnegative random variables, where $X$ is distributed by $F$. If $F \in \mathcal{D}$, then for arbitrarily fixed $\delta > 0$ and $p > \mathcal{J}_F^+$, there exists a positive constant $C$ without relation to $\delta$ and $Y$ such that for all large $x$,

$$
\mathbb{P}(XY > \delta x \mid Y) \leq C \mathcal{F}(x) \left(\delta^{-p} Y^p 1_{[Y \geq \delta]} + 1_{[Y < \delta]} \right). \quad (9)
$$

*Proof.* See Lemma 3.2 in Heyde and Wang [11].

**Lemma 3.2.** Let $X$ and $Y$ be two independent and nonnegative random variables, where $X$ is distributed by $F$. If $F \in \mathcal{D}$ with $\mathcal{J}_F^+ > 0$, then, for any fixed $\delta > 0$ and $0 < p_1 < \mathcal{J}_F^- \leq \mathcal{J}_F^+ < p_2 < \infty$, there exists a positive constant $C$ without relation to $\delta$ and $Y$, such that for all large $x$,

$$
\mathbb{P}(XY > \delta x \mid Y) \leq C \mathcal{F}(x) \left(\delta^{-p_1} Y^{p_1} + \delta^{-p_2} Y^{p_2}\right). \quad (10)
$$

*Proof.* See Lemma 3 in Guo and Wang [10].

**Lemma 3.3.** Let $X$ and $Y$ be two independent random variables, $X$ be real-valued and distributed by $F$ and $Y$ be nonnegative and nondegenerate at 0. If $F \in \mathcal{C}$ and $\mathbb{E}Y^{p} < \infty$ for some $p > \mathcal{J}_F^+$, then the distribution of $XY$ belongs to $\mathcal{C}$ and $\mathbb{P}(XY > x) \asymp \mathcal{F}(x)$.
Proof. See Lemma 2.4 and Lemma 2.5 in Wang et al. [18].

Lemma 3.4. Let $X$ and $Y$ be two independent random variables, $X$ be real-valued and distributed by $F$ and $Y$ be nonnegative and nondegenerate at 0. If $F \in R_\alpha$ for some $0 < \alpha < \infty$ and $EY^p < \infty$ for some $p > \alpha$. Then, the distribution of $XY$ belongs to $R_\alpha$ and $P(XY > x) \sim F(x)EY^\alpha$.

Proof. The complete proof can be found in Breiman [3] or Cline and Samorodnitsky [2].

The following lemmas will be used in the proof of Theorem 2.1

Lemma 3.5. Under the conditions of Theorem 2.1

$$P \left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x, \bigcap_{n=1}^{k} \left\{ X_n e^{-L_\theta(\tau_n)} \leq (1 - \delta)x - x^\mu \right\} \right\} \leq o(P(x)) \quad (9)$$

holds for any fixed $k \in \mathbb{N}^+, 0 < \delta < 1, J_H^+/\kappa_\theta < \mu < 1$.

Proof. We can derive

$$= P \left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x, \bigcap_{n=1}^{k} \left\{ X_n e^{-L_\theta(\tau_n)} \leq (1 - \delta)x - x^\mu \right\}, \bigcup_{l=1}^{k} \left\{ X_l e^{-L_\theta(\tau_1)} > \frac{(1 - \delta)x}{k} \right\} \right\}$$

$$\leq \sum_{l=1}^{k} \sum_{n=1,n\neq l}^{k} P \left\{ X_l e^{-L_\theta(\tau_1)} > \frac{(1 - \delta)x}{k}, X_n e^{-L_\theta(\tau_n)} > x^\mu \right\} \quad \quad \text{(10)}$$

Take any fixed $\pi$ satisfying $J_H^+/\kappa_\theta < \pi < \mu < 1$. In the last inequality of (10), if $l < n$, we denote $Y_\pi = e^{-L_\theta(\pi)} \mathbb{1}_{\{X_n e^{-L_\theta(\tau_n)} - L_\theta(\tau_n) e^{-L_\theta(\pi)} > \frac{\mu - \pi}{\mu - 1}\}}$ and get that for all large $x$,

$$P \left\{ X_l e^{-L_\theta(\tau_1)} > \frac{(1 - \delta)x}{k}, X_n e^{-L_\theta(\tau_n)} > \frac{x^\mu}{k - 1} \right\}$$

$$= P \left\{ X_l e^{-L_\theta(\tau_1)} > \frac{(1 - \delta)x}{k}, X_n e^{-L_\theta(\tau_n)} > \frac{x^\mu}{k - 1}, e^{L_\theta(\tau_n) - L_\theta(\tau_1)} > x^\pi \right\}$$

$$+ P \left\{ X_l e^{-L_\theta(\tau_1)} > \frac{(1 - \delta)x}{k}, X_n e^{-L_\theta(\tau_n)} > \frac{x^\mu}{k - 1}, e^{L_\theta(\tau_n) - L_\theta(\tau_1)} \leq x^\pi \right\}$$

$$\leq P \left\{ e^{L_\theta(\tau_n) - L_\theta(\tau_1)} > x^\pi \right\} + P \left\{ e^{L_\theta(\tau_n) - L_\theta(\tau_1)} e^{-L_\theta(\tau_1)} > \frac{1 - \delta}{k}, X_n e^{-L_\theta(\tau_n) - L_\theta(\tau_1)} > \frac{x^\mu - \pi}{k - 1} \right\}$$
Combining (12), (8) and (11), we get that for $l < n$

Lemma 3.6. Under the conditions of Theorem 2.1, there exists $k^* \in \mathbb{N}^+$ such that

$$\mathbb{P}\left\{ \sum_{n=k+1}^{\infty} X_n e^{-L_\delta(\tau_n)} > \delta x \right\} \lesssim o(H(x))$$

holds for any fixed $k \geq k^*$ and $0 < \delta < 1$.

Proof. Take some $k'$ such that $\sum_{n=k+1}^{\infty} \frac{1}{n^{\eta+\eta'}} < 1$ holds for any $k \geq k'$ and fixed $\eta > 0$. By Lemma 3.2, we can derive that for any $p_1, p_2$ satisfying $0 < p_1 < \mathbb{J}_H \leq \mathbb{J}_H^+ < p_2 < \kappa_\theta$ and all large $x$,

$$\mathbb{P}\left\{ \sum_{n=k+1}^{\infty} X_n e^{-L_\delta(\tau_n)} > \delta x \right\}$$
Lemma 3.8. If

Lemma 3.7.

Proof.

holds for any fixed \( k \) different definition of Laplace exponent. See Lemma 2.1 in Maulik and Zwart [16]. The modifications come from the

Proof.

holds for any \( k \)

\[ n \leq \frac{1}{i} \sum_{n=1}^{\infty} E \left[ \eta \left( 1+\eta \right) p_i e^{-p_i L_\theta (\tau_n)} \right] \]  

\[ \leq C \mathcal{H}(x) \sum_{n=1}^{\infty} E \left[ n \left( 1+\eta \right) p_i e^{-p_i L_\theta (\tau_n)} \right] \]

(16)

holds for any \( k \geq k' \) and fixed \( \eta > 0 \). By [1], we get \( \mathbb{E} e^{-p_i L_\theta (\tau_i)} = e^{\eta \psi_i(p_i)} < 1, i = 1, 2 \). Thus,

\[ \sum_{n=1}^{\infty} E \left[ n \left( 1+\eta \right) p_i e^{-p_i L_\theta (\tau_n)} \right] = \sum_{n=1}^{\infty} n \left( 1+\eta \right) p_i \left[ \mathbb{E} e^{-p_i L_\theta (\tau_i)} \right]^{n-1} < \infty, i = 1, 2. \]  

(17)

Hence, there exists \( k^* > k' \) such that for any \( k \geq k^* \),

\[ \mathbb{P} \left\{ \sum_{n=k+1}^{\infty} X_n e^{-L_\theta (\tau_n)} > \delta x \right\} \leq o(\mathcal{H}(x)). \]

Lemma 3.7. If \( p > 0, \psi_\theta(p) < 0 \) and \( 0 < \mathbb{E}[L_\theta(1)] < \infty \), then \( \mathbb{E} Z_0^p < \infty \).

Proof. See Lemma 2.1 in Maulik and Zwart [16]. The modifications come from the different definition of Laplace exponent.

Lemma 3.8. Under the conditions of Theorem 2.1, there exists \( k^* \in \mathbb{N}^+ \) such that

\[ \sum_{n=k+1}^{\infty} \mathbb{P} \left\{ X_n e^{-L_\theta (\tau_n)} > x \right\} \leq o(\mathcal{H}(x)) \]  

(18)

holds for any fixed \( k \geq k^* \).

Proof. By Lemma 3.2 and (17), there exists \( k^* \) such that for any fixed \( k \geq k^* \),

\[ \sum_{n=k+1}^{\infty} \mathbb{P} \left\{ X_n e^{-L_\theta (\tau_n)} > x \right\} \]

\[ = \sum_{n=k+1}^{\infty} \mathbb{P} \left\{ X_n e^{L_\theta (\tau_n)-L_\theta (\tau_n)} e^{-L_\theta (\tau_n)} > x \right\} \]

\[ \leq C \mathcal{H}(x) \sum_{n=k+1}^{\infty} E \left[ e^{-p_1 L_\theta (\tau_n-1)} + e^{-p_2 L_\theta (\tau_n-1)} \right] \]

\[ \leq C \mathcal{H}(x) \sum_{n=k+1}^{\infty} E \left[ n \left( 1+\eta \right) p_1 e^{-p_1 L_\theta (\tau_n-1)} + n \left( 1+\eta \right) p_2 e^{-p_2 L_\theta (\tau_n-1)} \right] \]

\[ \leq o(\mathcal{H}(x)). \]
3.2. **Proof of Theorem 2.1.** After rewriting the expression \([4]\), we have

\[
\Psi(x) = \mathbb{P}\left\{ \sup_{0 \leq t < \infty} \sum_{n=1}^{\infty} X_n e^{-L_\theta(\tau_n)} 1_{[\tau_n \leq t]} - cZ_{[0,t]} > x \right\}.
\]

It is clear that for any fixed \(k \in \mathbb{N}^+\),

\[
\mathbb{P}\left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} - cZ_{[0,\infty]} > x \right\} \leq \Psi(x) \leq \mathbb{P}\left\{ \sum_{n=1}^{\infty} X_n e^{-L_\theta(\tau_n)} > x \right\}.
\]

(19)

Firstly, we deal with the upper bound. For any fixed \(0 < \delta < 1\) and \(k \in \mathbb{N}^+\), we get

\[
\Psi(x) \leq \mathbb{P}\left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x \right\} + \mathbb{P}\left\{ \sum_{n=k+1}^{\infty} X_n e^{-L_\theta(\tau_n)} > \delta x \right\}
\]

\[
:= P_1 + P_2.
\]

(20)

For \(P_1\), we can derive that for any fixed \(\mu\) satisfying \(\Upsilon^+ / \kappa_\theta < \mu < 1\),

\[
P_1 = \mathbb{P}\left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x, \bigcup_{n=1}^{k} \{ X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x - x^\mu \} \right\}
\]

\[
+ \mathbb{P}\left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x, \bigcap_{n=1}^{k} \{ X_n e^{-L_\theta(\tau_n)} \leq (1 - \delta)x - x^\mu \} \right\}
\]

\[
\leq \sum_{n=1}^{k} \mathbb{P}\left\{ X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x - x^\mu \right\}
\]

\[
+ \mathbb{P}\left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > (1 - \delta)x, \bigcap_{n=1}^{k} \{ X_n e^{-L_\theta(\tau_n)} \leq (1 - \delta)x - x^\mu \} \right\}
\]

\[
:= P_{11} + P_{12}.
\]

(21)

By \([1]\), we can get that for any \(p\) satisfying \(\Upsilon^+ / \kappa_\theta < p < \kappa_\theta\),

\[
\mathbb{E} e^{-pL_\theta(\tau_{n-1})} = \mathbb{E} e^{\tau_{n-1} - \psi(p)} < 1.
\]

Then, by Lemma \([3.3]\) the distribution of

\[
X_n e^{-L_\theta(\tau_n)} = X_n e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_{n})} e^{-L_\theta(\tau_{n-1})}
\]

belongs to \(\mathcal{C}\) and

\[
\mathbb{P}\{ X_n e^{-L_\theta(\tau_n)} > x \} \sim \overline{\mathcal{H}}(x).
\]

(22)

Thus, we obtain

\[
\lim_{\delta \downarrow 0} \lim_{x \to \infty} P_{11} \sim \sum_{n=1}^{k} \mathbb{P}\left\{ X_n e^{-L_\theta(\tau_n)} > x \right\},
\]

(23)

where we notice \((1 - \delta)x - x^\mu = x(1 - \delta - x^{\mu-1})\) and \((1 - \delta - x^{\mu-1}) \uparrow 1\). By Lemma \([3.5]\) we have

\[
P_{12} \lesssim o(\overline{\mathcal{H}}(x)).
\]

(24)

By Lemma \([3.6]\) there exists \(k^*\) such that for any fixed \(k \geq k^*\),

\[
P_{2} \lesssim o(\overline{\mathcal{H}}(x)).
\]

(25)
Combining \((20), (21), (23) - (25)\), there exists \(\delta^* > 0\) such that for \(\delta = \delta^*\) and \(k = k^*\),

\[
\Psi(x) \lesssim \sum_{n=1}^{\infty} \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\} + o(H(x)) \\
\lesssim \sum_{n=1}^{\infty} \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\},
\]

(26)

where we used \((22)\) in the last step.

Secondly, we deal with the lower bound. For any fixed \(k \in \mathbb{N}^+\), we have

\[
\Psi(x) \geq \mathbb{P} \left\{ \sum_{n=1}^{k} X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} \right\} \\
\geq \mathbb{P} \left\{ \bigcup_{n=1}^{k} \left\{ X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} \right\} \right\} \\
\geq \sum_{n=1}^{k} \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} \right\} \\
- \sum_{n=1}^{k} \sum_{l=1,l \neq n}^{k} \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x, X_l e^{-L_\theta(\tau_l)} > x \right\} \\
:= L_{11} - L_{12}.
\]

(27)

For \(L_{11}\), we can derive that for any fixed \(p\) and \(\mu\) satisfying \(J_H^+ < p < \kappa_\theta\) and \(J_H^+ / p < \mu < 1\),

\[
\mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} \right\} \\
\geq \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} + Z_{[0,\infty)} \right\} \\
\geq \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} + Z_{[0,\infty)} > x^\mu \right\} \\
\geq \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x + cZ_{[0,\infty)} > x^\mu \right\} \\
:= L_{11} - L_{12}.
\]

(28)

Because the distribution of \(X_n e^{-L_\theta(\tau_n)}\) belongs to \(\mathcal{C}\), we can get

\[
L_{11} \sim \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\},
\]

(29)

where we notice \(x + cZ^\mu = x(1 + cx^{\mu-1})\) and \((1 + cx^{\mu-1}) \downarrow 1\). Taking any fixed \(\pi\) satisfying \(J_H^+ / p < \pi < \mu < 1\) and using the method used in \((11)\), we can obtain that for all large \(x\),

\[
L_{12} = \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x, Z_{[0,\infty)} + x^\mu, Z_{[\tau_{n-1}, \tau_n]} > x^\pi \right\} \\
+ \mathbb{P} \left\{ X_n e^{-L_\theta(\tau_n)} > x, Z_{[0,\infty)} + x^\mu, Z_{[\tau_{n-1}, \tau_n]} \leq x^\pi \right\} \\
\leq \mathbb{P} \left\{ Z_{[0,\infty)} > x^\mu \right\} \\
+ \mathbb{P} \left\{ X_n e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)} e^{-L_\theta(\tau_{n-1})} > x, Z_{[0,\tau_{n-1}]} + Z_{[\tau_n, \infty)} > x^\mu - x^\pi \right\}
\]
\[
\begin{align*}
\mathbb{E} Z_p^\infty & \leq \frac{\mathbb{E} Z_p^x}{x^{p+\mu}} + C\mathcal{H}(x) \mathbb{E} \left[ e^{-\kappa_\theta L_\theta(\tau_{n-1})} 1_{\{Z_{[0,\tau_{n-1}]} + Z_{(\tau_{n-1},\infty)} > x^{\mu} - x^\pi\}} \right. \\
& \left. \quad + \ 1_{\{Z_{[0,\tau_{n-1}]} + Z_{(\tau_{n-1},\infty)} > x^{\mu} - x^\pi\}} \right],
\end{align*}
\]

where \(X_n e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)}\) is independent of \(e^{-L_\theta(\tau_{n-1})} 1_{\{Z_{(0,\tau_{n-1})} + Z_{\tau_n,\infty) > x^{\mu} - x^\pi\}}\).

From (11), we have \(\psi_\theta(p) < 0\) for \(0 < \int_1^\infty < p < \kappa_\theta\). Recalling the condition \(0 < \mathbb{E}[L_\theta(1)] < \infty\) and using Lemma 3.7, we can obtain

\[
\mathbb{E} Z_p^{[0,\infty)} < \infty.
\]

Then, by (12) and (8), we can get

\[
L_{12} \lesssim o(\mathcal{H}(x)).
\] (30)

For \(L_2\), by (13), (14) and

\[
P \left\{ X_n e^{-L_\theta(\tau_n)} > x, X_l e^{-L_\theta(\tau_l)} > x \right\}
\leq P \left\{ X_n e^{-L_\theta(\tau_n)} > \left(\frac{1 - \delta}{k}\right) x, X_l e^{-L_\theta(\tau_l)} > \frac{x^{\mu}}{k - 1} \right\},
\]

we have

\[
L_2 \lesssim o(\mathcal{H}(x)).
\] (31)

Combining (27) - (31) and using Lemma 3.8, there exists \(k^*\) such that for \(k = k^*\),

\[
\Psi(x) \geq \sum_{n=1}^k P \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\} + o(\mathcal{H}(x))
= \left(\sum_{n=1}^\infty - \sum_{n=k+1}^\infty \right) P \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\} + o(\mathcal{H}(x))
\geq \sum_{n=1}^\infty P \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\} + o(\mathcal{H}(x))
\geq \sum_{n=1}^\infty P \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\},
\] (32)

where we used (22) in the last step.

By (20) and (32), we have

\[
\Psi(x) \sim \sum_{n=1}^\infty P \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\}
= \sum_{n=1}^\infty \int_0^\infty P \left\{ X_n e^{-L_\theta(s)} > x \right\} P\{\tau_n \in ds\}
= \int_0^\infty P \left\{ X_n e^{-L_\theta(s)} > x \right\} d\lambda_s.
\]

\[\square\]
3.3. **Proof of Theorem 2.2.** If $H \in \mathcal{R}_{-\alpha}$, we can get $\mathbb{J}_H = \mathbb{J}^+_H = \alpha$. Then, by $\mathcal{R}_{-\alpha} \subset \mathcal{C}$ and Theorem 2.1, we can get

$$
\Psi(x) \sim \sum_{n=1}^{\infty} P \left\{ X_n e^{-L_\theta(\tau_n)} > x \right\}. 
$$

By (1), we can get that for any $p$ satisfying $\alpha < p < \kappa_\theta$,

$$
\mathbb{E} e^{-p L_\theta(\tau_{n-1})} = \mathbb{E} e^{\tau_{n-1} \psi_\theta(p)} < 1.
$$

Thus, by Lemma 3.4, the distribution of $X_n e^{-L_\theta(\tau_n)} = X_n e^{L_\theta(\tau_{n-1}) - L_\theta(\tau_n)} e^{-L_\theta(\tau_{n-1})}$

belongs to $\mathcal{R}_{-\alpha}$ and

$$
P \{ X_n e^{-L_\theta(\tau_n)} > x \} \sim \mathbb{H}(x) \mathbb{E} e^{-\alpha L_\theta(\tau_{n-1})}
= \mathbb{H}(x) (\mathbb{E} e^{-\alpha L_\theta(\tau_1)})^{n-1}
= \mathbb{H}(x) (\mathbb{E} e^{\tau_1 \psi_\theta(\alpha)})^{n-1}.
$$

Substituting (34) into (33), we can get

$$
\Psi(x) \sim \mathbb{H}(x) \sum_{n=1}^{\infty} (\mathbb{E} e^{\tau_1 \psi_\theta(\alpha)})^{n-1} = \frac{\mathbb{H}(x)}{1 - \mathbb{E} e^{\tau_1 \psi_\theta(\alpha)}},
$$

where $\mathbb{E} e^{\tau_1 \psi_\theta(\alpha)} < 1$. □

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