Research Article

On the Fractional Derivative of Dirac Delta Function and Its Application

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The Dirac delta function and its integer-order derivative are widely used to solve integer-order differential/integral equation and integer-order system in related fields. On the other hand, the fractional-order system gets more and more attention. This paper investigates the fractional derivative of the Dirac delta function and its Laplace transform to explore the solution for fractional-order system. The paper presents the Riemann-Liouville and the Caputo fractional derivative of the Dirac delta function, and their analytic expression. The Laplace transform of the fractional derivative of the Dirac delta function is given later. The proposed fractional derivative of the Dirac delta function and its Laplace transform are effectively used to solve fractional-order integral equation and fractional-order system, the correctness of each solution is also verified.

1. Introduction

The Dirac delta function $\delta(t)$ was introduced by the theoretical physicist Paul Dirac to develop tools for the quantum field theory [1]. Till now, it is widely used to describe the impulse phenomenon in physics, mathematics, and engineering fields [2–5]. In control theory and signal process, the Dirac delta function is a basic typical input or testing signal to study the system’s response [6, 7]. The integer-order derivative of the Dirac delta function $\delta(t)$, i.e., $i$th order derivative $\delta^{(i)}(t)$ $(i \in \mathbb{N})$ is also well-studied and finds great significance in applications of related fields [8, 9]. The Laplace transform of the Dirac delta function $\delta(t)$ and its integer-order derivative $\delta^{(i)}(t)$ are $\mathcal{L}[\delta(t)] = 1$ and $\mathcal{L}[\delta^{(i)}(t)] = s^i, (i \in \mathbb{N})$, respectively [10]. They are effectively used to solve various kinds of integer-order differential/integral equation and integer-order system.

On the other hand, the fractional-order system which is characterized with “process memory” and “historical heredity” gets more and more attention [11–15]. Many different definitions of integrals and fractional derivatives, such as the $\psi$-Hilfer integral and the $\psi$-Hilfer fractional derivative, are proposed and find great significance in applications [16–18]. Just as solving integer-order integral/differential equation and system, it will be significant to investigate the fractional derivative of The Dirac delta function $\delta(t)$, i.e., $\psiD^\alpha_D \delta(t)$ and its Laplace transform $\mathcal{L}[\psiD^\alpha_D \delta(t)]$ for exploring the solution for fractional-order integral/differential equation or fractional-order system.

Research on the fractional derivative of the Dirac delta function $\psiD^\alpha_D \delta(t)$ is just beginning so far.

From the point of view of viscoelasticity, the literature [19] showed that the memory function of complex materials is the fractional derivative of the Dirac delta function, and the Laplace transform of the fractional derivative of the Dirac delta function is derived. However, the literature [19] only discussed the Riemann-Liouville fractional derivative, and the fractional derivative order discussed in [19] is limited in $\mathbb{R}^+$. 
In this paper, we explore the Riemann-Liouville fractional derivative and the Caputo fractional derivative of the Dirac delta function under the widely used sense, and the order of the fractional derivative is generalized to R. Comparing with [19], our research method is simpler while results are more general.

This paper is organized as follows: Section 2 Basic Preparation and Section 3 investigates the fractional derivative of Dirac delta function, including the Riemann-Liouville fractional derivative of \( \delta(t) \), i.e., \( R_{0}^{D} \delta(t) \) and the Caputo fractional derivative of \( \delta(t) \), i.e., \( C_{0}^{D} \delta(t) \). Section 4 studies the Laplace transform of the fractional derivative of Dirac delta function; Section 5 presents application with examples to verify the correctness of the theorems proposed in the paper. Section 6 briefly summarizes the main conclusions of the paper.

\[
R_{0}^{D} f(t) = \begin{cases} 
\frac{1}{\Gamma(-\alpha)} \int_{a}^{t} (t-\tau)^{-\alpha-1} f(\tau) d\tau, & \alpha < 0, \\
\frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau, & 0 < m - 1 < \alpha < m, m \in \mathbb{N}^+.
\end{cases}
\]

Supposing \( \mathcal{L}[f(t)] = F(s) \), the Laplace transform of \( R_{0}^{D} f(t) \) is as follows:

**Property 1.** \( \mathcal{L}[R_{0}^{D} f(t)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} F^{(k)}(0) \), \( 0 \leq m - 1 < \alpha < m \).

The Laplace transform of \( C_{0}^{D} f(t) \) only involves the initial value of the integer-order derivative of the function \( f(t) \), which brings great convenience to its application in engineering.

**Property 2.** \( \mathcal{L}[C_{0}^{D} f(t)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \), \( 0 \leq m - 1 < \alpha < m \).

The Dirac delta function \( \delta(t) \) and its \( i \)-th order derivative \( \delta^{(i)}(t) (i \in \mathbb{N}) \) belong to a generalized function. \( \delta(t) \) is generally defined as [1]:

**Definition 4.** \( \delta(t) = \begin{cases} 0, & t \neq 0 \\
1, & t = 0 \end{cases}, \int_{-\infty}^{\infty} \delta(t) dt = 1. \)

### 2. Basic Preparation

The fractional derivative of function \( f \), i.e., \( _{a}D_{t}^{\alpha} f \) is the basic tool used in fractional calculus and related applications. The basis for its definition is an arbitrary-order integral \( _{a}D_{t}^{\alpha} \), namely definition 1 given below [11].

**Definition 1.** \( _{a}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(p)} \int_{a}^{t} (t-\tau)^{\alpha-p-1} f(\tau) d\tau (p > 0) \).

With Definition 1 as the basis, the full definition of the function \( f \)'s Riemann-Liouville fractional derivative \( _{a}R_{t}^{D} \) and its Caputo fractional derivative \( _{a}C_{t}^{D} \) are as follows:

**Definition 2.** The Riemann-Liouville fractional derivative of the continuous function \( _{a}R_{t}^{D} f \) is as follows:

\[
_{a}R_{t}^{D} f(t) = \begin{cases} 
\frac{1}{\Gamma(-\alpha)} \int_{a}^{t} (t-\tau)^{-\alpha-1} f(\tau) d\tau, & \alpha < 0, \\
\frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau, & 0 \leq m - 1 < \alpha < m, m \in \mathbb{N}^+.
\end{cases}
\]

**Definition 3.** If the function \( f \) has continuous derivatives up to the \( m \)-order, then its Caputo fractional derivative \( _{a}C_{t}^{D} f \) is as follows:

\[
_{a}C_{t}^{D} f(t) = \begin{cases} 
\frac{1}{\Gamma(-\alpha)} \int_{a}^{t} (t-\tau)^{-\alpha-1} f(\tau) d\tau, & \alpha < 0, \\
\frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & 0 \leq m - 1 < \alpha < m, m \in \mathbb{N}^+.
\end{cases}
\]

Main useful properties of \( \delta(t) \) include [8, 10]:

**Property 3.** Suppose \( f \) is continuous within the interval \([a, b]\) containing \( 0 \), then we have \( \int_{a}^{b} \delta(t) f(t) dt = f(0) \), particularly, \( \int_{a}^{b} \delta(t) dt = 1 \).

**Property 4.** The \( i \)-th derivative of \( \delta(t) \), \( \delta^{(i)}(t) = 0, t \neq 0, (i \in \mathbb{N}) \). If the function \( f \) is continuous and \( i \)-th differentiable at \( t = 0 \), within the interval \([a, b]\), then \( \int_{a}^{b} \delta(t) f^{(i)}(t) dt = (-1)^{i} f^{(i)}(0) \).

**Property 5.** Laplace transform of \( \delta^{(i)}(t) \), \( \mathcal{L}[\delta^{(i)}(t)] = s^i, (i \in \mathbb{N}) \).
3. The Fractional Derivative of Dirac Delta Function

Corresponding to the above two definitions of fractional derivative, the Riemann-Liouville fractional derivative of \( \delta(t) \), i.e., \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \) and the Caputo fractional derivative of \( \delta(t) \), i.e., \( \frac{d^\alpha}{dt^\alpha} \delta(t) \) could be defined as in Definition 2 and Definition 3. In fact, we can give the analytic expression for them further. Theorem 1 gives the analytic formula of \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \).

**Theorem 1.** \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = 1/\Gamma(-\alpha)t^{\alpha+1}, (\alpha \notin \mathbb{N}). \)

**Proof.** According to the value of \( \alpha \notin \mathbb{N}: \)

(1) When \( \alpha < 0 \), Definition 2, Definition 4, and Property 3 result in:

\[
\frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (t-\tau)^{-\alpha-1} \delta(\tau) d\tau = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (t-\tau)^{-\alpha-1} d\tau = \frac{1}{\Gamma(-\alpha)} t^{\alpha+1}.
\]

(2) When \( 0 \leq m-1 < \alpha < m, m \in \mathbb{N}^* \), Definition 1, Definition 2, and Property 3 result in:

\[
\frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = \frac{d^m}{dt^m} \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^\infty (t-\tau)^{m-\alpha-1} \delta(\tau) d\tau = \frac{1}{\Gamma(m-\alpha)} \int_0^\infty (t-\tau)^{m-\alpha-1} d\tau = \frac{1}{\Gamma(m-\alpha)} t^{\alpha+1}.
\]

Remark 1. Theorem 2 shows that the analytic formulas of \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \) is the same as that of \( \frac{d^m}{dt^m} \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \). Denoting \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = 1/\Gamma(-\alpha)t^{\alpha+1}, (\alpha \notin \mathbb{N}) \) as \( \delta_{\alpha} \delta(t) \); thus, \( \frac{d^m}{dt^m} \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) = 1/\Gamma(-\alpha)t^{\alpha+1}, (\alpha \notin \mathbb{N}) \). We will use \( \delta_{\alpha} \delta(t) \) to replace \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \) or \( \frac{d^m}{dt^m} \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \) below. The different forms of Definition 2 and Definition 3 allow us to adopt suitable one as the definition of \( \delta_{\alpha} \delta(t) \) when dealing with specific problem related to \( \delta_{\alpha} \delta(t) \).

4. The Laplace Transform of the Fractional Derivative of Dirac Delta Function

The Laplace transform is of great importance in solving differential/integral equation and system analysis. Theorem 3 gives the Laplace transform of \( \frac{\partial^\alpha}{\partial t^\alpha} \delta(t) \).

**Theorem 3.** \( \mathcal{L}[\frac{\partial^\alpha}{\partial t^\alpha} \delta(t)] = s^\alpha, \alpha \in \mathbb{R}. \)

**Proof.** We will prove it by classifying the different values of \( \alpha \).

(1) When \( \alpha < 0 \), \( \mathcal{L}[\frac{\partial^\alpha}{\partial t^\alpha} \delta(t)] = \mathcal{L}[\frac{1}{\Gamma(-\alpha)} \int_0^\infty (t-\tau)^{-\alpha-1} \delta(\tau) d\tau] = (1/\Gamma(-\alpha)) \mathcal{L}[\tau^{-\alpha-1} \delta(t)] = (1/\Gamma(-\alpha)) \mathcal{L}[\tau^{-\alpha-1} \cdot \delta(t)]. \)
Because \(-\alpha - 1 > -1\), thus, \(\mathcal{L}[t^{\alpha-1}] = (\Gamma(-\alpha)/s^\alpha)\). Combining Property 5, we further obtain:
\[
\mathcal{L}[\alpha D^\alpha_t \delta(t)] = \frac{1}{\Gamma(-\alpha)} \cdot \frac{\Gamma(-\alpha)}{s^\alpha} = s^\alpha. \tag{7}
\]

(2) When \(\alpha = m \in \mathbb{N}\), it is known that \(\mathcal{L}[\delta^{(m)}(t)] = s^m\), \((m \in \mathbb{N})\) from Property 5.

(3) When \(0 \leq m - 1 < \alpha < m, m \in \mathbb{N}^+\), we adopt Definition 3 as the definition of \(\alpha D^\alpha_t \delta(t)\); thus,
\[
\mathcal{L}[\alpha D^\alpha_t \delta(t)] = \mathcal{L} \left[ \int_0^t (t - \tau)^{m-\alpha-1} \delta^{(m)}(\tau) d\tau \right]
= \frac{1}{\Gamma(m - \alpha)} \cdot \frac{\Gamma(m - \alpha)}{s^{m-\alpha}} \cdot s^m = s^\alpha. \tag{8}
\]

Because \(0 \leq m - 1 < \alpha < m, m \in \mathbb{N}^+, -1 < m - \alpha - 1 < 0\); thus, we can obtain:
\[
\mathcal{L}[\alpha D^\alpha_t \delta(t)] = \frac{1}{\Gamma(m - \alpha)} \cdot \frac{\Gamma(m - \alpha)}{s^{m-\alpha}} \cdot s^m = s^\alpha. \tag{9}
\]

Remark 2. Theorem 3 proves \(\mathcal{L}[\alpha D^\alpha_t \delta(t)] = s^\alpha, \alpha \in \mathbb{R}\), while Property 5 gives \(\mathcal{L}[\delta^{(i)}(t)] = s^i, i \in \mathbb{N}\). Theorem 3 extends the Laplace transform of the function \(\delta^{(i)}(t)\) \((i \in \mathbb{N})\), to the Laplace transform of \(\alpha D^\alpha_t \delta(t)\) \((\alpha \in \mathbb{R})\). Obviously, the Laplace transforms of \(\delta^{(i)}(t)\) \((i \in \mathbb{N})\) and that of \(\alpha D^\alpha_t \delta(t)\) \((\alpha \in \mathbb{R})\) are consistent in the form, the Laplace transform of \(\alpha D^\alpha_t \delta(t)\) \((\alpha \in \mathbb{R})\) is the generalization of the Laplace transform of the integer-order derivative of \(\delta(t)\).

Theorem 4 could be derived from implementing the inverse Laplace transform to Theorem 3.

**Theorem 4.** \(\mathcal{L}^{-1}[s^\alpha] = \alpha D^\alpha_t \delta(t), \alpha \in \mathbb{R}\).

**Remark 3.** The conclusion \(\mathcal{L}^{-1}[s^\alpha] = \mathcal{L}^{-1}[1/s^{\alpha}] = \mathcal{L}^{-1}[t^{\alpha-1}/\Gamma(-\alpha)]\) is well known and often used [20]. By introducing the nonclassical function \(\delta(t)\) and its fractional derivative \(\alpha D^\alpha_t \delta(t)\), the inverse Laplace transform of \(s^\alpha, \alpha \in \mathbb{R}\) is obtained.

**Remark 4.** It is well-known that the role of the Laplace transform in differential/integral equation and system analysis is significant. In classical cases, objects involved in the Laplace transform are often the rational proper fraction of \(s\) [21]. For example, to the transfer function essentially discussed in control theory, the case that the power of the polynomial of \(s\) in the numerator is higher than the power of the polynomial of \(s\) in the denominator is nearly ignored. Therefore, Theorem 4 allows us to implement Laplace transform to general rational fraction. This provides a new way for the solution of differential equations and the analysis of control systems, especially of the singular systems.

**5. Application with Examples**

**Example 1 (Solving fractional Volterra integral equation of the first kind [22]).** Investigate the following integral equation:
\[
\frac{1}{\Gamma(p)} \int_0^t (t - \tau)^{p-1} y(\tau) d\tau = kt^m, \quad (p > 0, m > 1).
\tag{10}
\]

Obviously, the left hand of (10) is the \(p\)-order integral of the function \(y(\tau)\), implementing the Laplace transform on the both side of the equation results in: \(Y(s)/s^p = k\Gamma(m + 1)/s^{m+1}\). Consequently,
\[
Y(s) = k\Gamma(m + 1)s^{p-m-1}.
\tag{11}
\]

The solution of (10) could be derived by implementing the inverse Laplace transform on (11) with Theorem 4:
\[
y(t) = k\Gamma(m + 1)\alpha D^\alpha_t \delta(t). \tag{12}
\]

We verify the correctness of (12) according to \(p - m - 1 > 0\) and \(p - m - 1 < 0\) respectively.

(a) In case of \(p - m - 1 > 0\), \(\alpha D^\alpha_t \delta(t)\) represents \((p - m - 1)\) order derivative. Taking \(p = m + 2\) as an example, in this case:
\[
y(t) = k\Gamma(m + 1)\alpha D^\alpha_t \delta(t) = k\Gamma(m + 1)\delta'(t).
\tag{13}
\]

Substituting it into the left hand of (10) yields:
\[
\frac{1}{\Gamma(p)} \int_0^t (t - \tau)^{p-1} y(\tau) d\tau = \frac{k\Gamma(m + 1)}{\Gamma(m + 2)} \int_0^t (t - \tau)^{m+1} \delta'(\tau) d\tau
= \frac{k(m + 1)}{(m + 1) \times (-1) \times (-1) \times t^m} = kt^m.
\tag{14}
\]

(b) In case of \(p - m - 1 < 0\), \(\alpha D^\alpha_t \delta(t)\) represents \((p - m - 1)\) order integration. Taking \(p = 1, m = 5/2\) as an example, in this case:
\[
y(t) = k\Gamma\left(\frac{5}{2} + 1\right)\alpha D^{2.5}_t \delta(t).
\tag{15}
\]

Figure 1 plots the curve of \(y(t)\) with \(k = 2\).
Verifying: substituting (15) into the left hand of (10) yields:

\[
\frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau) d\tau = k \Gamma \left( \frac{5}{2} + 1 \right) \int_0^t D_\tau^{-1.5} \delta(\tau) d\tau
\]

\[
= k \Gamma \left( \frac{5}{2} + 1 \right) \int_0^t \frac{1}{\Gamma(2.5)} t^{1.5} d\tau
\]

\[
= k \Gamma \left( 2.5 + 1 \right) \frac{1}{\Gamma(2.5)} t^{1.5} d\tau
\]

\[
= k \Gamma \left( 2.5 + 1 \right) \frac{t^{2.5}}{2.5 \times \Gamma(2.5)} = kt^{1.5}. \tag{16}
\]

Example 2 (Computation of convolution of power functions). Compute the convolution of power functions:

\[
f(t) = \int_0^t (t-\tau)^{\alpha} \tau^\beta d\tau \quad (\alpha, \beta > -1). \tag{17}
\]

Implementing the Laplace transform on the both side of (17), we get

\[
F(s) = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}} \cdot \frac{\Gamma(\beta + 1)}{s^{\beta+1}} = \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{s^{\alpha+\beta+2}} = \Gamma(\alpha + 1) \Gamma(\beta + 1) s^{-\alpha-\beta-2}. \tag{18}
\]

Implementing the inverse Laplace transform on the both side of (18) with Theorem 4, we have

\[
f(t) = \Gamma(\alpha + 1) \Gamma(\beta + 1) D_t^{-1-\alpha-\beta-2} \delta(t). \tag{19}
\]

Verifying: for instance, taking \( \alpha = 1, \beta = 0.5 \) results in

\[
f(t) = \int_0^t (t-\tau) \sqrt{\tau} d\tau. \]

Computation of \( f(t) \)

\[
f(t) = \int_0^t (t-\tau) \sqrt{\tau} d\tau = \int_0^t (\sqrt{\tau^3} - \sqrt{\tau^2}) d\tau
\]

\[
= \left[ \frac{2}{3} \sqrt{\tau^3} - \frac{2}{5} \sqrt{\tau^2} \right]_0^t = \frac{4}{15} t^{3/2}. \tag{20}
\]

Investigating (19), we have the analytic form of \( f(t) \):

\[
f(t) = \Gamma(2) \Gamma(1.5) D_t^{1-0.5-2} \delta(t) = \frac{\Gamma(1.5)}{\Gamma(3.5)} t^{-2.5} = \frac{4}{15} t^{5/2}. \tag{21}
\]

Obviously, the correctness of (19) is confirmed.

Note that this kind of convolution is not easy to compute for the vast major value of \( \alpha \) and \( \beta \) by traditional way, such as integration by substitution.

Example 3 (Solution for fractional singular system [23]). Find the solution for the next fractional singular system:

\[
\begin{bmatrix}
0 & 1 & C^0.5 P \tau(t) = x(t) +
0 & -1 & u(t), x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.
\end{bmatrix} \tag{22}
\]

Implementing the Laplace transform on the both side of (22) yields:

\[
\begin{bmatrix}
0 & 1 & C^0.5
0 & -1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\tau \delta \chi
\end{bmatrix} = x(s) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} U(s). \tag{23}
\]

After transposition and sorting out:

\[
\begin{bmatrix}
0 & 1 & C^0.5 - I
0 & -1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\tau \delta \chi
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} x_0
-1
1
\end{bmatrix} U(s). \tag{24}
\]

Thus,

\[
X(s) = \left( \begin{bmatrix}
0 & 1 \\ 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} - I
\end{bmatrix} \right)^{-1} \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} x_0
-1
1
\end{bmatrix} U(s) \right). \tag{25}
\]

\[
\begin{bmatrix}
0 & 1 \\ 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} - I
\end{bmatrix} \] could be expanded by the Neumann series [24], namely

\[
\left( \begin{bmatrix}
0 & 1 \\ 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} - I
\end{bmatrix} \right)^{-1} = - \sum_{k=0}^{\infty} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} x_0
-1
1
\end{bmatrix} U(s). \tag{26}
\]

\[
\begin{bmatrix}
0 & 1 \\ 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} - I
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix}
\delta^{0.5} x_0
-1
1
\end{bmatrix} U(s).
\]
Taking side of (27) with Theorem 4, we obtain
tðtÞ, \( t \in \mathbb{R} \).

The correctness of Veri in (28) into the right side of (29-1) brings us
\( 0 = x_2(t) + u(t) \) (29-2).

The correctness of \( x_2(t) = -u(t) \) is obvious from (29-2). Taking \( x_2(t) = -u(t) \), \( x_2(0) = -u(0) \) into account, substituting \( x_1(t) \) in (28) into the right side of (29-1) brings us
\[
-C_0 D_t^{0.5} u(t) = -C_0 D_t^{0.5} \delta(t)x_{20} + u(t) - C_0 D_t^{0.5} u(t) \\
-C_0 D_t^{0.5} \delta(t)u(0) - u(t) \\
= -C_0 D_t^{0.5} u(t) - C_0 D_t^{0.5} \delta(t)(x_{20} + u(0)).
\]

Since \( x_{20} = -u(0) \), thus, \( x_{20} + u(0) = 0 \). The right side of (16-1) is changed into \(-C_0 D_t^{0.5} u(t) \), which is exactly equal to the left side of (16-1). Figure 2 plots the curves of \( x(t) \) with \( x_{20} = 1 \).

6. Conclusion

This paper investigated the fractional derivative of the Dirac delta function \( \delta(t) \), its Laplace transform and their application. It is proved that \( C_0 D_t^{\alpha} \delta(t) \) is equal to \( C_0 D_t^{\alpha} \delta(t) \), and we further have \( C_0 D_t^{\alpha} \delta(t) = C_0 D_t^{\alpha} \delta(t) = 1/\Gamma(1-\alpha)t^{\alpha-1}, \) \( \alpha \in \mathbb{R} \). Consequently, we denote them as \( C_0 D_t^{\alpha} \delta(t) \). The paper generalized the Laplace transform of \( \mathcal{L}[\delta^{(i)}(t)] = s^i \), \( i \in \mathbb{N} \) of \( \mathcal{L}[C_0 D_t^{\alpha} \delta(t)] = s^\alpha, \) \( \alpha \in \mathbb{R} \) to the fractional volterra integral equation, computation of convolution of power functions, and the solution for fractional singular system.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interests in this work.

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