Symplectic reflection algebras in positive characteristic as Ore extensions

Emily Norton

Department of Mathematics, TU Kaiserslautern, Kaiserslautern, Germany

ABSTRACT
We investigate PBW deformations $H_{\lambda}$ of $k[x,y] \rtimes G$ where $G$ is the cyclic group of order $p$ and the ground field $k$ has characteristic $p$. The algebras $H_{\lambda}$ are a version of symplectic reflection algebras that only exist in positive characteristic. They also admit a presentation as Ore extensions over $k[x] \otimes kG$, and the combinatorics of the derivation used in this presentation is related to André and Eulerian polynomials. We find the center of $H_{\lambda}$ and classify the simple modules. We also study a version of $H_{\lambda}$ in which $G$ is replaced by an elementary abelian $p$-group $G'$.

ARTICLE HISTORY
Received 29 January 2018
Revised 10 October 2019
Communicated by Ellen Kirkman

KEYWORDS
Skew group algebras; quadratic algebras; Ore extensions; deformations; symplectic reflection algebras; elementary abelian $p$-groups; André polynomials; modular representations in defining characteristic

2010 MATHEMATICS SUBJECT CLASSIFICATION
16S32; 16S35; 16S37; 16S80; 16W70; 16G99

1. Introduction

The goal of this paper is to study symplectic reflection algebras associated to a cyclic group of order $p$, and more generally an elementary abelian $p$-group, when the characteristic of the ground field is also $p$. The family of algebras $H_{\lambda}$ as $\lambda$ ranges through $kG$ can be expressed as Ore extensions over their commutative subring $R := k[x] \otimes kG$; this construction makes it possible to find the center of the algebras as well as to draw conclusions about the structure of the algebras and their modules.

For any $\lambda$, the algebra $H_{\lambda}$ is Noetherian because $R$ is Noetherian, and an Ore extension of a Noetherian ring is Noetherian. The powers of $x^p$ (central elements) generate an infinite descending chain of two-sided ideals and thus $H_{\lambda}$ is neither left nor right Artinian. It is not a domain since $kG$, which embeds into $H_{\lambda}$, is not a domain. It follows from the PBW theorem that $H_{\lambda}$ has GK-dimension equal to that of $k[x,y] \rtimes G$, and hence has GK-dimension 2. $H_{\lambda}$ has infinite global dimension. The Jacobson radical of $H_{\lambda}$ is 0, so $H_{\lambda}$ is semiprimitive (but not semisimple, since it is not Artinian). The only idempotents $H_{\lambda}$ possesses are 0 and 1. The last two facts together imply that no simple $H_{\lambda}$-module has a projective cover.

The algebra $H_{\lambda}$ for a fixed $\lambda$ is always a finite module over its center. This is not obvious and takes some work to establish. Contrast this with the story when $kG$ is semisimple (say $k = \mathbb{C}$):
the associated symplectic reflection algebra \( H_{t,c} \) has an important subalgebra called the spherical subalgebra, and the Satake isomorphism states that the center of \( H_{t,c} \) coincides with the center of the spherical subalgebra [2]. The spherical subalgebra is defined as \( e \mathcal{H}_{t,c} \) where \( e \) is the idempotent \( \frac{1}{|G|} \sum_{g \in G} g \). When \( \text{char} \, k = p \) divides the order of \( G \), such an idempotent does not exist and there is no spherical subalgebra of \( H_{t,c} \). Nonetheless the center \( Z_\lambda \) of \( H_{t,c} \) is large. The degrees of the generators of \( Z_\lambda \) depend on whether \( \lambda \) has a constant term or not. These cases correspond, respectively, to the \( t = 1 \) and \( t = 0 \) cases for symplectic reflection algebras in characteristic 0. In characteristic 0, the center of \( H_{t,c} \) is \( \mathbb{C} \) if \( t = 1 \) and is large if \( t = 0 \). What we see in the algebras \( H_{t,c} \) is that in the \( t = 1 \) setting the center \( Z_\lambda \) is isomorphic to a polynomial ring in two variables, generated by elements of degrees \( p \) and \( p^2 \); in the \( t = 0 \) case, \( Z_\lambda \) is generated by elements of degrees \( 2, p, \) and \( p \) with a singular relation between the first two generators.

Simple representations of \( H_\lambda \) are all finite-dimensional and are parametrized by \( k \times k \). The simples are obtained as quotients of “Verma modules” \( \Delta_x \), \( x \in k \), which are defined by inducing a character of the maximal commutative subring \( \mathbb{R} \subset H_\lambda \). However, each \( \Delta_x \) has a one-parameter family of maximal submodules, and so a one-parameter family of simple quotients. Each simple module corresponds to a different choice of central character \( (x, \beta) \in k^2 \).

Here is an outline of the contents of the sections of this paper. Section 2 defines the algebras \( H_\lambda \) in imitation of the definition of a symplectic reflection algebra, and observes that \( H_\lambda \) satisfies a PBW theorem. Then we show that \( H_\lambda \) is an Ore extension \( \mathbb{R}[y; \delta] \) of its commutative subring \( \mathbb{R} \) with respect to the derivation \( \delta = \lambda \partial_x + xg \partial_y \). In the Appendix A, Section 6, we investigate the combinatorics of more general derivations \( \delta = f(x,y) \partial_x + xg \partial_y \) for an arbitrary polynomial \( f(x,y) \in \mathbb{R} \) and establish a result used in proving that the center of \( H_\lambda \) is large. In Section 3 we construct explicit generators for the center of \( H_\lambda \) in both the \( t = 0 \) and \( t = 1 \) cases, showing that \( H_\lambda \) is always a finite module over its center. Section 4 describes simple modules and Verma modules over \( H_\lambda \). In contrast to what happens in characteristic 0, the geometry of the center does not completely determine the dimensions of the simple modules, and the deformation parameter \( \lambda \in kG \) has an effect on the outcome: when \( t = 0 \) the Azumaya and smooth loci coincide only when \( \lambda \in \text{Rad} kG \). When \( t = 1 \) the center is smooth but the dimension of simple modules drops over the 0-fiber of the projection \( \text{Mod} H_{t,c} \to \text{Spec} \mathbb{R} \). Section 5 generalizes the construction of \( H_{t,c} \) from the cyclic group \( G \) to an elementary \( p \)-group \( E = (\mathbb{Z}/p)^\ell \). For \( r > 1 \), we first consider the case when \( [x,y] \in E \), which reduces to the case \( \lambda = g = (1 \quad 0 \quad 1 \quad 1) \). We prove that the center of \( H_g \) in this case is generated by a quadratic element \( x^2 - 2D\partial^2(g), x^p \), and \( y^\partial - y^\partial(g) = \prod_{i \in \Xi} \prod_{a = 1}^p (y - (\zeta^a + a)x) \) where \( D = \frac{1}{\zeta_i} \sum g_i \partial_i \), \( \zeta_i \) an \( \mathbb{F}_p \) basis of \( \mathbb{F}_q \), and where \( \Xi \) consists of all elements of \( \mathbb{F}_q \) orthogonal to \( \mathbb{F}_p \subset \mathbb{F}_q \) thought of as the \( \mathbb{F}_p \)-linear subspace spanned by 1. That is, the big central element is the product over all \( e \in E \) of \( e \cdot y \), taken in a certain order. We then look at simple \( H_g \)-representations and show that for all \( r > 1 \), the Azumaya locus and the smooth locus of the center coincide. As for the center of \( H_\lambda \) in the case of an arbitrary parameter \( \lambda \in kE \), we show that in the \( t = 0 \) case that \( x^2 - 2D\partial^2(\lambda), x^p \), and \( y^\partial - y^\partial(g) \) generate \( Z_\lambda \), while in the \( t = 1 \) case, \( x^p \) and \( \prod_{a \in \mathbb{F}_p} (y^\partial - y^\partial((g)) - ax^p) \) generate \( Z_\lambda \). We then look at the structure of Verma modules \( \Delta_x \) and show that simple modules obtained as quotients of \( \Delta_x \) either have dimension \( p \) or 1, depending on whether \( \lambda \) is invertible or a zero-divisor; simple quotients of \( \Delta_x \) when \( \lambda \neq 0 \) have dimension equal to the degree of the big central element.

2. Presentations of the algebra \( H_\lambda \) and the PBW theorem

The name “symplectic reflection algebra” derives from a phenomenon that is not necessarily true in characteristic \( p \): a finite subgroup \( G \) of \( \text{SL}(2, \mathbb{C}) \) acting on \( \mathbb{C}^2 \) which preserves the symplectic...
Remark 2.1. The equation
\[ \text{ural action of} \]
studied rational Cherednik algebras in characteristic
possible to define the symplectic reflection algebra associated to
finite over its center.

appears in [10]; a criterion has since been proved for a wider class of algebras called Drinfeld
any element of the algebra
by the action of

Theorem 2.2. (PBW theorem).

Note that the algebra
asymmetric roles in

Remark 2.1. The equation \( x = gy^{-1} - y \) holds in \( H_\lambda \), so the algebra \( H_\lambda \) is generated by \( y \) and \( g \).

For any \( \lambda \in kG, H_\lambda \) is a filtered algebra with the filtration given by assigning \( x \) and \( y \) degree 1
and \( G \) degree 0. Let \( \mathfrak{gr} H_\lambda \) be the associated graded algebra of \( H_\lambda \) with respect to this filtration.

\[ H_\lambda := \frac{k\langle x, y \rangle \rtimes G}{(xy - yx - \lambda)}. \]

Theorem 2.2. (PBW theorem). For any \( \lambda \in kG, \mathfrak{gr} H_\lambda = H_0. \)

A proof by Shepler and Witherspoon of the PBW theorem for certain deformations of skew
group rings using Bergman’s Diamond Lemma [4] includes the case of \( H_\lambda \) [13, Theorem 3.1].
A criterion to check the PBW theorem holds for Drinfeld Hecke algebras pertains to \( H_\lambda \) and
appears in [10]; a criterion has since been proved for a wider class of algebras called Drinfeld
orbifold algebras (of which Drinfeld Hecke algebras are a special case) by Shepler and
Witherspoon [14, Theorem 3.1]. The PBW theorem also follows from the fact that \( H_\lambda \) admits a
description as an Ore extension, as we explain below. The PBW theorem allows us to write
any element of the algebra \( H_\lambda \) in normal form: any element of \( H_\lambda \) can be written uniquely as a
k-linear combination of monomials \( y^i x^j g^k \).

For any \( \alpha \in k^\times \) we have \( H_\alpha \cong H_{\alpha^2}. \) The case \( c_0 \neq 0 \) is called the \( t = 1 \) case, while the case \( c_0 = 0 \)
is called the \( t = 0 \) case. As happens with symplectic reflection algebras, the center and the represen-
tation theory of \( H_\lambda \) differ according to the cases \( t = 0 \) and \( t = 1 \). However, in both cases \( H_\lambda \) is
finite over its center.
2.1. $H_2$ as an Ore extension

Another way to view $H_2$ is as an Ore extension of its commutative subring $R := k[x] \otimes k[G] \cong kG[x]$. Given an injective ring homomorphism $\sigma : R \to R$ and a $\sigma$-twisted derivation $\delta : R \to R$, that is, a $k$-linear map $\delta$ satisfying $\delta(r_1 r_2) = r_1 \delta(r_2) + \delta(r_1) \sigma(r_2)$, the (left) Ore extension $R[y; \sigma, \delta]$ of $R$ with respect to $\sigma$ and $\delta$ is defined to be $R[y]$ as a free right $R$-module subject to the multiplication rule $ry = y\sigma(r) + \delta(r)$ for all $r \in R$. If $\sigma$ is the identity then $\delta$ may be any derivation of $R$ and the resulting Ore extension is simply denoted $R[y; \delta]$. Ore extensions of the form $R[y; \delta]$ are called differential operator extensions, since the variable $y$ acts on $R$ via the derivation $\delta$ and thus $R[y; \delta]$ is an algebra of differential operators on $R$ [8].

Proposition 2.3. $H_2 \cong R[y; \delta]$ with $R = k[x] \otimes k[G]$ and $\delta$ the derivation given by $\delta(x) = \lambda, \delta(g) = xg$, where $g$ is the generator of $G$ as in Section 2.

Proof. First, $\delta$ respects the relations defining $R$, namely $\delta(xg) = x^2g + \lambda g = \delta(gx)$ and $\delta(g^p) = pxg^p = 0 = \delta(1)$. Next, $y$ satisfies the defining relations $xy = yx + \delta(x) = yx + \lambda$ and $gy = yg + \delta(g) = yg + xg$ in $R[y; \delta]$, and these are the defining relations of $y$ with $x$ and $g$ in $H_2$. Thus we have homomorphisms $\phi : H_2 \to R[y; \delta]$ and $\psi : R[y; \delta] \to H_2$ given by sending $g$ to $g$, $x$ to $x$, and $y$ to $y$, such that $\psi \circ \phi = \text{Id}_{H_2}$ and $\phi \circ \psi = \text{Id}_{R[y; \delta]}$. \hfill \Box

Corollary 2.4. The PBW theorem holds: $\text{gr} H_2 = H_0 = k[x,y] \rtimes kG$.

Proof. Any Ore extension $R[y; \sigma, \delta]$ over a $k$-algebra $R$ is isomorphic to $R[y]$ as a $k$-vector space. \hfill \Box

3. The center of $H_2$

3.1. Powers of $\delta$ on $g$ and on $x$: partition recurrence diagrams

Consider the effect of $\delta := \lambda \partial_x + xg\partial_g$ applied successively to the element $g \in kG[x]$. It is easy to check that $\delta^n(g)$ is divisible by $g$ for all $n \geq 0$ and that, considered as a polynomial in $kG[x]$, it has degree $n$; moreover, only the coefficients of $x^{n-2k}$, $n-2k \geq 0$, are nonzero. Set $D := g\partial_g$. For a given monomial $C_{n,m}x^ng$ in $\delta^n(g)$ the coefficient $C_{n,m} \in kG$ obeys the recurrence relation $C_{0,0} = 1, C_{n,m} = 0$ if $n < m$ or $m < 0$, $C_{n,m} = (D + 1)(C_{n-1,m-1}) + (m + 1)\lambda C_{n-1,m+1}$ for $n > 0$. The $C_{n,m}$ are polynomials in $\lambda, D(\lambda), D^2(\lambda), \ldots, D^{p-2}(\lambda)$ and may be written linear combinations of partitions: assign $\lambda$ the value 1, $D(\lambda)$ the value 2, and so forth, so that $\lambda^a D(\lambda)^{a_2} \cdots D^{p-2}(\lambda)^{a_{p-2}}$ is encoded by the partition $\{(p-2)^{a_{p-2}}, \ldots, 2^{a_2}, 1^{a_1}\}$. The monomials produced by the recurrence equation for $C_{n,m}$, and thus the powers of $\delta$ on $g$, may be represented by a directed graph whose vertices are the monomials $C_{n,m}$ and whose $kG$-labeled edges denote the recurrence relation. A given vertex $C_{n,m}$ is the sum over the vertices $C_{i,j}$ at the tails of its incoming arrows with the operations labeling the incoming arrows applied to the $C_{i,j}$ [12].

The diagram below represents $\delta^n(g)$, $n \geq 0$. The node in the $n$th row from the top and $m$th column to the right picks out the $kG$-coefficient of $x^mg$ in $\delta^n(g)$. Both $n$ and $m$ start from 0. Multiplication by $\lambda$ corresponds to appending the partition by 1, that is, $\lambda \cdot (p-2)^{a_{p-2}}, \ldots, 2^{a_2}, 1^{a_1}) = ((p-2)^{a_{p-2}}, \ldots, 2^{a_2}, 1^{a_1} + 1)$. Southeast arrows indicate the application of $D + 1$ and we omit their labels. Application of $D$ to a partition corresponds to differentiating that partition part by part using the Leibniz rule as if it were the product of its parts and where $D(i) = i + 1 \mod p - 1$. That is, $D((p-2)^{a_{p-2}}, \ldots, 2^{a_2}, 1^{a_1}) = a_{p-2}((p-2)^{a_{p-2}-1}, \ldots, 2^{a_2}, 1^{a_1+1}) + \ldots + a_2((p-2)^{a_{p-2}-1}, \ldots, 2^{a_2+1}, 1^{a_1}) + a_1((p-2)^{a_{p-2}-1}, \ldots, 2^{a_2+1}, 1^{a_1+1})$. For simplicity, we have assumed $D^{p-1}(\lambda) = \lambda$ (the $t = 0$ case) to illustrate the rule; otherwise an additional scalar term gets produced if $a_{p-2}$ is nonzero.
Figure 1. Recurrence diagram for $\delta^n(g)$.

Compare this with the diagram for $\delta^n(\lambda) = \delta^{n+1}(x)$, starting from $\delta^0(\lambda) = \delta(x) = \lambda$. The $kG$-coefficient $\tilde{C}_{n,m}$ of $x^m$ in $\delta^n(\lambda)$ obeys the recurrence relation $\tilde{C}_{0,0} = \lambda$, $\tilde{C}_{n,m} = 0$ if $n < m$ or $m < 0$, $\tilde{C}_{n,m} = D(\tilde{C}_{n-1,m-1}) + (m+1)\lambda \tilde{C}_{n-1,m+1}$ for $n > 0$. Again, southeast arrows denote the application of $D$. The diagram replicates the subdiagram of $\delta^n(g)$ where all partitions whose sum isn’t the maximum possible for their node are erased, and the top diagonal is omitted. Thus the statement that $\delta^n(g) = \delta^{n-1}(x)g + x^p g = \delta^{n-2}(\lambda)g + x^p g$ is equivalent to the statement that the only partitions that survive in $\delta^n(g)$ mod $p$ are those whose sum is the maximum possible at those nodes, namely $p - 1$. For partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$, define $\mu \sqcup \nu$ to be $(\mu_1, \mu_2, \ldots, \nu_1, \nu_2, \ldots)$ with the parts rearranged if necessary so that the result is a partition. Now define the homogenization of a node $\sum_x c_x x$ in row $n$ and column $m$ of the $\delta^n(g)$ recurrence diagram as the linear combination of partitions $\sum_x c_x (x \sqcup (n + 1 - |x|))$ (Figures 1 and 2).

Lemma 3.1. The triangle of $kG$-coefficients for $\delta^n(\lambda)$, $n \geq 0$, is the homogenization of the triangle of $kG$-coefficients for $\delta^n(g)$, $n \geq 0$.

Proof. We will find a recurrence relation for the homogenization of $\delta^n(g)$ and show that it is the same as the recurrence relation for $\delta^n(\lambda)$ with the same initial condition.

Write $C_{n,k} = \sum_{j \leq n} \sum_{\# \text{parts}(P) = j} c_P P$ where $C_{n,k}$ is the coefficient of $x^m g$ in $\delta^n(g)$ and $c_P \in k$. For a partition $P = (P_1, \ldots, P_{j\lambda})$ appearing in $C_{n,k}$ denote $\overline{P} = P \sqcup (n + 1 - j)$ where $j = |P|$. The homogenization of $C_{n,k}$ is $\overline{C_{n,k}} := \sum_{j \leq n} \sum_{\# \text{parts}(P) = j} c_P \overline{P}$. Then $D(\overline{C_{n,k-1}}) = D(C_{n,k-1}) + \overline{C_{n,k-1}}$.

Moreover, it is clear from the definition that $(k + 1)\lambda \overline{C_{n,k+1}} = (k + 1)\lambda C_{n,k+1}$. Since $C_{n,k}$ obeys the recurrence relation $C_{n+1,k} = D(C_{n,k-1}) + C_{n,k-1} + (k + 1)\lambda C_{n,k+1}$ it follows that $\overline{C_{n,k}}$ obeys the recurrence relation $\overline{C_{n+1,k}} = D(\overline{C_{n,k-1}}) + \overline{C_{n,k-1}} + (k + 1)\lambda \overline{C_{n,k+1}}$ which is the same recurrence relation satisfied by $\overline{C_{n,k}}$ for the coefficients of $\delta^n(\lambda)$. Moreover, $\overline{C_{0,0}}$ is the empty partition and thus $\overline{C_{0,0}} = 1 = \tilde{C}_{0,0}$ so that the two sequences share the same initial condition. It follows that $\overline{C_{n,m}} = \tilde{C}_{n,m}$ for all $n, m$, that is, $\delta^n(\lambda)$ is the homogenization of $\delta^n(g)$. □
3.2. Generators and relations for the center of $H_k$

Let $Z_k$ denote the center of $H_k$. The main result of this section shows that $Z_k$ is a finite module over its center.

Lemma 3.2. Let $r \in R$.

(1) We have $ry^n = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} \delta^i(r)$ [8].

(2) We have $ry^p = y^p r + \delta^p(r)$.

(3) The $p$th power of $\delta$, $\delta^p$, is a derivation.

(4) We have $x^p \in Z_k$.

(5) We have $r \in Z_k$ if and only if $\delta(r) = 0$.

Proof. (1) Easily proved by induction.

(2) Follows from (1).

(3) For any $r, s \in R$, we have $\delta^p(rs) = [rs, y^p] = r[s, y^p] + [r, y^p]s = r\delta^p(s) + \delta^p(r)s$.

(4) We have $\delta(x^p) = p^\lambda x^{p-1} = 0$.

(5) The ring $R$ is commutative, and by definition, $ry = yr + \delta(r)$. Thus $ry = yr$ if and only if $\delta(r) = 0$.

Theorem 3.3.

(1) ($t = 0$ case). Let $\lambda \in kG$ have zero constant term. Then the center of $H_\lambda$ is generated by $x^2 - 2D^p - 2(\lambda)$, $x^p$, and $y^p - y^{\delta^p(\lambda)}$ and

$Z_\lambda \cong \frac{k[A, B, C]}{(A^p - B^2)}$.

(2) ($t = 1$ case). Let $\lambda \in kG$ have nonzero constant term. Then the center of $H_\lambda$ is generated by $x^p$ and $y^p - y^{\delta^p(\lambda)} = \prod_{a=1}^{p} (Y - ax^p)$ where $Y := y^p - y^{\delta^p(\lambda)}$, and

$Z_\lambda \cong k[B, D]$.
Proof. Step 1. First we show that if \( \lambda \) has constant term \( c_0 = 0 \) then \( x^2 - 2(g \partial_y)^{p-2}(\lambda) \in \mathbb{Z}_\lambda \), while if \( c_0 \neq 0 \) then \( \mathbb{Z}_\lambda \cap \mathbb{R} = k[x^p] \). If \( c_0 = 0 \), \((g \partial_y)^{p-1}(\lambda) = \lambda \). Then \( \delta \left((g \partial_y)^{p-2}(\lambda)\right) = x\lambda \), and consequently \( \delta \left(x^2 - 2(g \partial_y)^{p-2}(\lambda)\right) = 2x\lambda - 2x\lambda = 0 \). If \( c_0 \neq 0 \) then \( k[x^p] \subseteq \mathbb{Z}_\lambda \) by Lemma 3.2. We want to show that \( \mathbb{Z}_\lambda \subseteq k[x^p] \). First, we will give an unrigorous proof based on the idea that \( H_t \) for \( t = 1 \) deforms the semidirect product of the Weyl algebra with \( kG \). Then we will provide a computation that certifies the result.

The first argument that \( \mathbb{Z}_\lambda \subseteq k[x^p] \). We may think of \( H_t \) at \( t = 1 \) as a deformation of \( A_1 \times kG \), the semidirect product of the Weyl algebra with \( kG \). We fix \( c_0 = 1 \), and the deformation parameter is \((c_1, ..., c_{p=1})\): when we set all \( c_i = 0 \) for \( i = 1, ..., p - 1 \) we recover \( A_1 \times kG \). Suppose \( f \in \mathbb{R} \) is of minimal degree such that \( f \in \mathbb{Z}_\lambda \). Setting all \( c_i = 0 \) for \( i = 1, ..., p - 1 \) in the defining relation of \( H_t \), we should obtain a central element \( f_0 \in \mathbb{Z}_1 = k[x, y^p - y^p x^{p-1}] \subset A_1 \times kG \). The top degree term of \( f \) must then be a multiple of \( p \). Thus \( f = x^{pn} + f' \) for some \( n \in \mathbb{N} \) and some \( f' \in \mathbb{R} \) with \( f' \) of strictly smaller degree than \( f \). But \( x^{pn} \) is central in \( H_t \), so \( f' = f - x^{pn} \) is central, contradicting the minimality of \( f \). This shows that \( \mathbb{Z}_\lambda \subseteq k[x^p] \), concluding the proof.

The second argument that \( \mathbb{Z}_\lambda \subseteq k[x^p] \). Let \( f = f(x, g) = x^n + w_{n-1}x^{n-1} + w_{n-2}x^{n-2} + ... + w_0, w_j \in kG \), be an element of \( \mathbb{Z}_\lambda \). Indeed the top degree term of any central element has coefficient in \( k \), as the image of \( f(x, g) \) under the associated graded map belongs to \( \mathbb{Z}_0 \cong k[x, y^p - y^p x^{p-1}] \), and then without loss of generality \( f(x, g) \) is monic. By Lemma 3.2, \( \delta(f) = 0 \). We have (up to an isomorphism of algebras) \( \lambda = 1 + \sum_{i=1}^{p-1} c_i g^i \). We may write \( \delta = \partial_x + \left(\sum_{i=1}^{p-1} c_i g^i\right) \partial_x + \lambda \partial_y \). Consider the top degree term of \( \delta(\lambda) \). If \( w_{n-1} \notin k \) then it is given by \( g \partial_y(w_{n-1})x^n \neq 0 \). Since \( \delta(\lambda) = 0 \) we must therefore have \( w_{n-1} \in k \). Then the top degree term of \( \delta(\lambda) \) is given by \( nx^{n-1} + n\left(\sum_{i=1}^{p-1} c_i g^i\right)x^{n-1} + g \partial_y(w_{n-2})x^{n-1} \). Of these three summands, the first summand \( nx^{n-1} \) is the only one not divisible by \( g \). Since \( \delta(\lambda) = 0 \), this implies that \( p \) divides \( n \). Now we argue as in the first argument that \( f = x^{pm} + f' \) where \( \deg f' < pm \) and \( f' \in \mathbb{Z}_\lambda \). By downwards induction on degree, \( f' \in k[x^p] \). We conclude that \( \mathbb{Z}_\lambda \subseteq k[x^p] \).

Step 2. The PBW theorem imposes restrictions on the generators of \( \mathbb{Z}_\lambda \). If \( Z \in \mathbb{Z}_\lambda \), then the image of \( Z \) under the associated graded map is an element of the center of \( \mathbb{H}_0 \). The center of \( \mathbb{H}_0 \) is \( \mathbb{Z}_0 := k[x, y^p - y^p x^{p-1}] \). If \( Z \in \mathbb{Z}_\lambda \), \( Z \notin \mathbb{R} \), and \( \deg(Z) = p \), this means that \( Z \) must be of the form \( y^p - y^p x^{p-1} + \text{l.o.t.}'s \). Moreover, for \( C = y^p - y^p f(x, g) \) to belong to the center would have to mean that \( f(x, g) = \frac{\delta(g)}{\delta(g)} \), because \( y^p = y^p g + \delta^p(g) \) and \( y^p g = y^p + \delta(g) \). On the other hand, if \( \lambda \) has a nonzero constant term then \( \mathbb{Z}_\lambda \) is like a deformation of \( H_1 \), and the center of \( H_1 = A_1 \times kG \) consists of the \( G \)-invariants of the center of the Weyl algebra:

\[
\mathbb{Z}_1 = Z(A_1)^G = k[x^p, y^p] = k[x^p, y^p - y^p x^{p-1}] = k[x^p, y^p - y^p x^{p-1}]^G.
\]

We will have a central element \( D = y^p - y^p h(x, g) \) of degree \( p^2 \) in \( \mathbb{Z}_\lambda \) if \( h(x, g) = \frac{\delta^2(g)}{\delta(g)} \), as this is the condition needed for \( g \) and \( D \) to commute.

In order to show that, we check that \( \delta(g) \) divides \( \delta^r(g) \), and thus \( \frac{\delta(g)}{\delta(g)} \in R \). We have \( \delta(g) = xg \) and \( \delta^2(g) = x^2g + \lambda g \). Suppose by induction that \( g \) divides \( \delta^n(g) \), and write \( \delta^n(g) = rg \) for some \( r \in R \). Then \( \delta^{n+1}(g) = \delta(r)g + r\delta(g) = \delta(r)g + rxg \) is also divisible by \( g \). As for divisibility by \( x \), suppose by induction that \( \delta^n(g) = x^n + x^{n-2}f_{n-2}(g) + \ldots + x^f(x) \) for some polynomials \( f_i(g) \in kG \), and where \( e = 0 \) or \( 1 \) depending on whether \( n \) is even or odd; i.e. suppose the only powers of \( x \) appearing in \( \delta^n(g) \) are congruent to \( n \) mod 2. Applying \( \delta \) to a monomial \( x^f(g) \)
gives \( \delta\left(x^kf_k(g)\right) = kx^{k-1}\delta(x)f_k(g) + x^{k+1}g\delta_k(f_k(g)) \) and thus

\[
\delta^{n+1}(g) = x^{n+1} + x^{n-1}\left(f_{n-2}(g) + n\lambda f_n(g)\right) + x^{n-3}\left(f_{n-4} + (n-2)\lambda f_{n-2}(g)\right) + \ldots
\]

It follows that \( x \) divides \( \delta^n(g) \) if and only if \( n \) is odd. Since \( p \) is odd, it then follows that \( \frac{\delta'(g)}{\delta(g)} \in R \).

**Step 3.** Let \( C_{n,k} \) denote the coefficient of \( x^k \) in \( \delta^n(g) \), and let \( \tilde{C}_{n,k} \) denote the coefficient of \( x^k \) in \( \delta^n(\lambda) \). Then \( \tilde{C}_{p-2,k} \equiv C_{p,k} \mod p \) for \( k = 1, 3, \ldots, p-2 \) since \( \delta(g) = \delta^{p-2}(\lambda)g + x^p g \) by Proposition A3. Moreover, \( \tilde{C}_{p,k} = D(\lambda)C_{p,k} \) for \( k = 1, 3, \ldots, p-2, p \) by Lemma 3.1 and the observation that only the partitions in \( C_{p,k} \) the sum of whose parts is maximal have nonzero coefficients \( \mod p \).

Suppose \( t = 0 \). We claim that \( \frac{\delta'(g)}{\delta(g)} = 0 \). We have \( \delta(g) = xg \) and \( \frac{\delta'(g)}{xg} = C_{p,1} + C_{p,3}x^2 + C_{p,5}x^4 + \ldots + C_{p,p-2}x^{p-3} + x^{p-1} \). Then

\[
\delta\left(\frac{\delta'(g)}{\delta(g)}\right) = (D(C_{p,1}) + 2\lambda C_{p,3})x + (D(C_{p,3}) + 4\lambda C_{p,5})x^3 + \ldots + (D(C_{p,p-2}) + (p-1)\lambda)x^{p-2}
\]

\[
= (D(\tilde{C}_{p-2,1}) + 2\lambda \tilde{C}_{p-2,3})x + (D(\tilde{C}_{p-2,3}) + 4\lambda \tilde{C}_{p-2,5})x^3 + \ldots + (D(\tilde{C}_{p-2,p-2}) + (p-1)\lambda)x^{p-2}.
\]

What we must show, then, is that \( D(\tilde{C}_{p-2,k}) = -(k+1)\lambda \tilde{C}_{p-2,k+2} \) for \( k = 1, 3, 5, \ldots, p-2 \). This is true for \( k = 1 \): the linear recursion formula for \( \tilde{C}_{n,k} \) implies that \( \tilde{C}_{p-1,0} = \lambda \tilde{C}_{p-2,1} \). Applying the result of Lemma 3.1 on the one hand, and the linear recursion relation on the other,

\[
D(\lambda)\tilde{C}_{p-2,1} = \tilde{C}_{p,1} = D\left(\tilde{C}_{p-1,0}\right) + 2\lambda \tilde{C}_{p-1,2}
\]

\[
= D(\lambda)\tilde{C}_{p-2,1} + \lambda D\left(\tilde{C}_{p-2,1}\right) + 2\lambda D\left(\tilde{C}_{p-2,1}\right) + 3\lambda \tilde{C}_{p-2,3}
\]

and so \( \lambda D(\tilde{C}_{p-2,1}) = -2\lambda^2 \tilde{C}_{p-2,3} \). Now pretend that \( \lambda \) is invertible so that we can cancel a \( \lambda \) on both sides to get \( D(\tilde{C}_{p-2,1}) = -2\lambda \tilde{C}_{p-2,3} \). But now observe that the case that \( \lambda \) is invertible proves it, in fact, for \( \lambda \) a zero-divisor as well, since the coefficients of the partitions are the same no matter what \( \lambda \) is.

From \( D(\tilde{C}_{p-2,1}) = -2\lambda \tilde{C}_{p-2,3} \) it follows that \( \tilde{C}_{p-1,2} = \lambda \tilde{C}_{p-2,3} \): indeed, if \( D(\tilde{C}_{p-2,k}) = -(k+1)\lambda \tilde{C}_{p-2,k+2} \) then we have the formula \( \tilde{C}_{p-1,k+1} = D(\tilde{C}_{p-2,k}) + (k+2)\lambda \tilde{C}_{p-2,k+2} \). This implies that \( \tilde{C}_{p-1,k+1} = \lambda \tilde{C}_{p-2,k+2} \). We illustrate this with the following diagram:

\[
\begin{array}{ccccccc}
\tilde{C}_{p-2,1} & D & \lambda & \tilde{C}_{p-2,3} & D & \lambda & \tilde{C}_{p-2,5} & \ldots \\
D & 2\lambda & \tilde{C}_{p-2,1} & 3\lambda & \tilde{C}_{p-2,3} & 4\lambda & \tilde{C}_{p-2,5} & \ldots \\
D(\lambda) & \tilde{C}_{p-1,0} & D(\lambda) & \tilde{C}_{p-1,2} & D(\lambda) & \tilde{C}_{p-1,4} & D(\lambda) & \tilde{C}_{p-1,6} & \ldots
\end{array}
\]
Induction takes care of \( k = 3, 5, \ldots, p - 4 \). If \( D\left(\tilde{C}_{p-2,k}\right) = -(k+1)\tilde{C}_{p-2,k+2} \) then \( \tilde{C}_{p-1,k+1} = \tilde{C}_{p-2,k+2} \), from which it follows by turn, by the same calculation just done, that \( D\left(\tilde{C}_{p-2,k+2}\right) = -(k+3)\tilde{C}_{p-2,k+4} \).

The only subtlety depending on the assumption \( t = 0 \) or \( t = 1 \) arises with the term \( D\left(\tilde{C}_{p-2,p-2}\right) + (p-1)\tilde{\lambda} \). We have \( \tilde{C}_{p-2,p-2} = D^{p-2}(\tilde{\lambda}) \) and thus \( D\left(\tilde{C}_{p-2,p-2}\right) = D^{p-1}(\tilde{\lambda}) \). The assumption that \( \tilde{\lambda} \) does not contain a constant term, so \( t = 0 \), implies that \( D^{p-1}(\tilde{\lambda}) = \tilde{\lambda} \). In this case, then, \( D\left(\tilde{C}_{p-2,p-2}\right) - \tilde{\lambda} = 0 \) and it follows that \( \delta\left(\frac{\delta^p(g)}{\delta(g)}\right) = 0 \). This implies that \( y^p - y^\frac{\delta^p(g)}{\delta(g)} \) is a polynomial.

In the case \( t = 1 \) where \( \lambda \) contains a constant term, assumed after rescaling to be 1, we then have \( D\left(\tilde{C}_{p-2,p-2}\right) = D^{p-1}(\tilde{\lambda}) = \tilde{\lambda} - 1 \) from which it follows that \( \delta\left(\frac{\delta^p(g)}{\delta(g)}\right) = -x^p \). Then \( y^p - y^\frac{\delta^p(g)}{\delta(g)} \) is a polynomial. We claim that \( \prod_{a=1}^{p}(Y-a\tilde{x}) \in \mathbb{Z}[\tilde{\lambda}] \). The element clearly commutes with \( g \) and we must show it commutes with \( y \). Let \( \bar{y} := Y, \bar{x} := x^p, \bar{g} := y \). We know that \( [\bar{g}, \bar{y}] = -\bar{x} \bar{g} \) and \( [\bar{g}, \bar{x}] = [\bar{x}, \bar{y}] = 0 \), and we want to show that \( \bar{g} \) commutes with \( \prod_{a=1}^{p}(\bar{y} - a\bar{x}) \). But the relations between \( \bar{x}, \bar{y}, \) and \( \bar{g} \) are exactly those defining \( k[\bar{x}, \bar{y}] \rtimes (\bar{g}) \) where \( \bar{g} = g^{-1} \). The center of \( k[\bar{x}, \bar{y}] \rtimes (\bar{g}) \), equal to the \( \bar{g} \)-invariants of \( k[\bar{x}, \bar{y}] \), contains \( \prod_{a=1}^{p}(\bar{y} - a\bar{x}) = \prod_{a=1}^{p}(\bar{y} - (a-1)\bar{x}) = \bar{g} \cdot \prod_{a=1}^{p}(\bar{y} - a\bar{x}) \). This concludes the proof of the theorem about the center of \( H_2 \).

3.3. The special case \( \lambda = g \) and the André polynomials

When \( \lambda = g \) there are other nice explicit formulas for the central generator involving \( y^p \):

\[
C = y^p - y^\frac{\delta^p(g)}{\delta(g)} = y^p - y(x^2 - 2g) = y^p - \sum_{k=0}^{p-1} \frac{(2k-1)!!}{k!} xy^{p-2k}g^k = \prod_{a=1}^{p}(y - ax)
\]

Moreover, in the case \( \lambda = g, \delta^n(g) \) has a significance in combinatorics: \( \delta^n(g) = A_n(g, x) \) is the \( n \)th André polynomial. The André polynomials are a sequence of bivariate polynomials which, when evaluated at \( (1, 1) \), produce the Euler numbers \( 1, 1, 2, 5, 16, 61, \ldots \) (OEIS A000111). The Euler numbers enumerate alternating permutations in the symmetric group \( S_n \), that is, permutations \( \sigma \) such that \( \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots \). There is a monograph about the André polynomials and some variations on them by Foata and Schützenberger [7]. The André in question is Désiré André, a French mathematician of the late nineteenth century who discovered that the generating function for the number of alternating permutations in \( S_n \) is \( \sec x + \tan x \). No connection has previously been made between the André polynomials and the algebra \( H_2 \), or more broadly \( H_z \) (Table 1). We are interested in the integer sequence formed by the coefficients of the André polynomials A094503. After reindexing, we visualize the triangles of numbers as forming square arrays. The reason for visualizing the sequence in the square array is that the \( n \)th row keeps track
of the coefficients of monomials of total degree \( N - 2n \) in \( x \) and \( y \) in the expansion of \((x + y)^N\) in \( H_g \). More precisely, the coefficient of \( y^{N-2n-k}x^k g^n \) is then given by \( \binom{N}{k} A_{n,k} \). Everything can be done over \( \mathbb{Z} \), thinking of \( g \) as infinite-order, generating a cyclic semigroup (isomorphic to \( \mathbb{N} \)) acting on \( k(x,y) \). The \( n \)th knight’s-move antidiagonal, starting from the top row and walking two spaces left and one down, two spaces left and one down, … until it reaches the leftmost column, reads off the coefficients of the \( n \)th André polynomial as above.

Each array is a directed graph with vertices at integer lattice points in the lower right quadrant of the plane and with edges given by the vectors \((1, 0)\) and \((1, 1)\) between the vertices, with multiplicities. Note that the square array arising from \( d_n g \) coincides, for \( k = g \), with that for \( d_m g \), but starts with the coefficient of \( d_1 x \) in the upper left corner (at the origin). The labeled arrows in the array show the linear recursion relation for the sequence as follows: if vertex \( v \) is at the head of \( k \) arrows with tails at \( w_1, \ldots, w_k \), and those arrows are labeled with \( a_1, \ldots, a_k \) respectively, then \( v = a_1 w_1 + \cdots + a_k w_k \). The numbers \( A_{n,k} \) in the sequence are placed at integer lattice points \((k, -n)\) in the lower right quadrant of the plane, starting with the initial value of \( A_{0,0} = 1 \) at \((0,0)\). The numbers above the arrows can be interpreted as multiplicities of arrows, so that the value of an integer in the sequence placed at \((k, -n)\) equals the number of paths from \((0,0)\) to \((k, -n)\) in the graph [12] (Figure 3).

**Proposition 3.4.** The integers \( A_{n,k} \) satisfy the linear and quadratic recursion relations

\[
A_{n,k} = (n+1)A_{n,k-1} + (k+1)A_{n-1,k+1}
\]

\[
A_{n,k} = A_{n,k-1} + \sum_{l=0}^{n-1} \sum_{m=0}^{k} \left( \frac{2n+k-1}{2l+m} \right) A_{l,m} A_{n-1-l,k-m}
\]

with initial condition \( A_{0,0} = 1, A_{n,k} = 0 \) if \( n < 0 \) or if \( k < 0 \) and \( n > 0 \). The proposition can be proved directly by finding the coefficients of \((x+y)^{N+1} = (x+y)(x+y)^N = (x+y)^N(x+y)\) by identifying the coefficient of \( y^{N-2k-n}x^n g^k \) in \((x+y)^N\) as \( \binom{N}{2k+n} A_{n,k} \) by induction and then computing which terms combine to give each coefficient in \((x+y)^{N+1}\).

### 3.3.1. Integer sequences from \( \delta^n \) when \( \delta = g \partial_x + x g \partial_y \)

The polynomials \( \delta^n(g) \) in \( H_\lambda \) for arbitrary \( \lambda \) can be thought of as generalizations of the André polynomials. When \( \delta^n(g) \) is viewed as a polynomial in \( x \) and \( g \partial_x \)-powers of \( \lambda \), its coefficients
sum to the \( n \)th Euler number; when \( \lambda = g \), this polynomial gives back the \( n \)th André polynomial, since \((g\partial_k)^r(g) = g\). These polynomials are already interesting when \( \lambda = g^a \) for \( a > 1 \), \( a \in \mathbb{Z} \). This section discusses positive integer sequences related to the André numbers which result from replacing the differential operator \( \delta = g\partial_k + xg\partial_g \) with \( \delta = g^a\partial_k + xg\partial_g \).

For the first example, consider \( \delta = g^2\partial_k + xg\partial_g \). This should be something special, since if we thought of \( g \) as being the shadow of a generator of a polynomial ring where \( g \), like \( x \) and \( y \), had degree 1, then \( \delta \) would be homogeneous and the resulting algebra would be graded. We notice that in both polynomial sequences, the row sums of the integer coefficients are factorials (Table 2).

**Proposition 3.7.** When \( \delta = g^2\partial_k + xg\partial_g \), \( \delta^n(x)(1,1) = \delta^n(g)(1,1) = n! \).

**Proof.** We have 
\[
\delta^n(g) = \sum_{i=0}^{n-1} \binom{n-1}{i} \delta^i(x) \delta^{n-1-i}(g).
\]
Applying induction and evaluating both sides at \((1,1)\), we get
\[
\delta^n(g)(1,1) = \sum_{i=0}^{n-1} \binom{n-1}{i} i!(n-1-i)! = \sum_{i=0}^{n-1} (n-1)! = n!
\]
The quadratic recurrence relation \( \delta^n(x) = 2 \sum_{i=1}^{n-1} \binom{n-2}{i-1} \delta^i(x) \delta^{n-1-i}(x) \) likewise gives \( \delta^n(x)(1,1) = n! \). 

**Remark 3.8.** It is interesting to compare Proposition 3.7 to a similar result in [3], where a different partitioning of factorials is also obtained from certain coefficients appearing in powers of \( \delta \) in the setting of an Ore extension of \( k[x] \).

We keep track of the coefficients arising from \( \delta^n(g) \) as a Pascal-style array given by a directed graph whose arrows multiply the node at the tail by the value labeling the arrow, and where a node is the sum of the quantities carried into it by the heads of the incoming arrows:

A similar Pascal-style array produces a visualization of \( \delta^n(x) \):

Now replace 2 with any \( a \in \mathbb{N} \) and consider \( \delta = g^a\partial_k + xg\partial_g \). The linear recursion relation for the Pascal-style array arising from \( \delta^n(g) \) then becomes
by induction that $f$ numbers

The sequence $T_n$ arises from

Recall that $d \equiv \frac{32x^2g^4 + 416x^2g^6 + 272xg^6}{64x^6g^2 + 1824x^6g^4 + 2880x^4g^6 + 272g^8}$

... 

while the linear recursion relation for the array arising from $\delta^n(x)$ is

Recall that $\delta^m(g)$ obeys a quadratic recurrence because $\delta(g) = xg$:

The sequence $T_{n,k}$ also obeys a quadratic recurrence relation. After a change of variables, and looking at each monomial in $\delta^m(g)$, it follows that

On the other hand, the sequence $U_{n,k}$ obtained from $\delta^m(x)$ also satisfies a quadratic recurrence relation but one that does not involve $T_{n,k}$:

Setting $a = 1$ in this equation recovers the quadratic recurrence equation defining the André numbers $\{A_{n,k}\}$. As the initial values $U_{0,0}$ and $T_{0,0}$ for $\{U_{n,k}\}$ and $\{A_{n,k}\}$ are both 1, it follows by induction that $U_{n,k} = a^{n+k}A_{n,k}$. So the sequence $U_{n,k}$ arising from $\delta^m(x)$ simply looks like:

| $\delta^0(g)$ | $\delta^1(g)$ | $\delta^2(g)$ | $\delta^3(g)$ | $\delta^4(g)$ | $\delta^5(g)$ | $\delta^6(g)$ | $\delta^7(g)$ | $\delta^8(g)$ |
|---|---|---|---|---|---|---|---|---|
| $g$ | $xg$ | $x^2g + g^2$ | $x^3g + 5xg^2 + 2g^3$ | $x^4g + 18x^2g^2 + 5g^4$ | $x^5g + 58x^3g^2 + 61xg^3$ | $x^6g + 179x^4g^2 + 479x^2g^4 + 61g^5$ | $x^7g + 543x^6g^3 + 3111x^4g^5 + 1385x^2g^7$ | $...$ |
| $\delta^m(g)$ | | | | | | | | |

| $T_{n,k} = (an + 1)T_{n,k-1} + (k + 1)T_{n-1,k+1}$ |
|---|

| $U_{n,k} = a(n + 1)U_{n-1,k+1} + (k + 1)U_{n-1,k+1}$ |

... 

**Corollary 3.9.** Returning to the example of $xy = yx + g^2$ and the triangular array for $\delta^n(x)$, it follows that weighted row sums of André numbers partition the factorials:
\[ n! = 2^n A_{0,n} + 2^{n-2} A_{1,n-2} + 2^{n-4} A_{2,n-4} + \ldots + 2^{n-2\lfloor \frac{n}{2} \rfloor} A_{\lfloor \frac{n}{2} \rfloor,n-2\lfloor \frac{n}{2} \rfloor}. \]

**Remark 3.10.** Since \( n! \) is the order of the symmetric group \( S_n \) and \( A_{k,n-2k} \) counts some type of permutation ("André permutation") in \( S_n \) [7], there may be a combinatorial meaning to the formula above in the \( a = 2 \) case. In the case \( a = 3 \), the leftmost column of the square array corresponding to the sequence \( \delta^n(g) \) is OEIS sequence A126151: 1, 6, 96, 2976, 151416, .... When \( a = 4 \), the row sums \( \delta^n(x)(1, 1) \) (knight’s-move diagonal sums in the square version) count the number of minimax trees on \( n \) vertices (A080795). Minimax trees, a family of recursively generated binary trees, were defined and studied by Foata and Han and are closely related to André numbers [6].

### 4. Representations of \( H_{\lambda} \)

#### 4.1. Simple \( H_{\lambda} \)-modules in the \( t = 0 \) case

All simple modules over \( H_{\lambda} \) are finite-dimensional because \( H_{\lambda} \) is finite over its center.\(^1\) We follow the strategy of [9] and identify simple \( H_{\lambda} \)-modules on which the center \( Z_{\lambda} \) acts by a fixed character \( (x, \beta) \in \text{Spec} Z_{\lambda} \) with simple modules over a finite-dimensional algebra \( \overline{H}_{\lambda;x,\beta} \) obtained as the quotient of \( H_{\lambda} \) by the two-sided ideal where the central generators are set equal to corresponding scalars. This process is called central reduction, and the geometry of the center gives important information about simple modules and their dimensions.

Points in the affine space \( \text{Spec} Z_{\lambda} \) correspond to central characters. Following [9] we consider the following subsets of \( \text{Spec} Z_{\lambda} \):

- The smooth locus or nonsingular locus: where the Jacobian of the relations among the generators defining \( \text{Spec} Z_{\lambda} \) is non-degenerate;
- The Azumaya locus: a point is in the Azumaya locus if the simple module on which \( H_{\lambda} \) acts with the corresponding central character is of the maximal possible dimension;
- The locus of points for which \( H_{\lambda} \) has a unique simple module on which it acts with that central character.

In characteristic 0, the smooth locus, the Azumaya locus, and the third locus (which we will call the "one-to-one locus") coincide in the case of a wreath product symplectic reflection algebra with \( t = 0 \) [9, Section 2.6]. In characteristic \( p \), these three loci may fail to coincide. We will see that there is a unique simple module for every point in \( \text{Spec} Z_{\lambda} \); thus, the one-to-one locus always strictly contains the smooth locus. The smooth locus is \( \text{Spec} Z_{\lambda} \) minus a line. The Azumaya locus is \( \text{Spec} Z_{\lambda} \) when \( \lambda \) is invertible. Thus when \( \lambda \) is invertible the Azumaya and one-to-one loci coincide. The Azumaya locus coincides with the smooth locus when \( \lambda \) is a 0-divisor.

#### 4.1.1. Simple representations of \( H_g \)

We consider the case \( \lambda = g \) first. The center \( Z_g \) of \( H_g \) is generated by \( X = x^2 - 2g, Y = x^p \), and \( Z = y^p - yX^{p-1} \) with the relation \( X^p + 2 = Y^2 \). Take \( (x, \beta) \in \mathbb{F}_p^2 \). Then \( m_x := (X - \sqrt{2} + 2, Y - \sqrt{2}, Z - \beta) \) is a maximal two-sided ideal of \( Z_g \). Let \( \overline{H}_{g;x,\beta} := H_g/m_x H_g \) be the algebra \( H_g \) subject to the additional relations \( x^p = \sqrt{2}, x^2 - 2g = \sqrt{2} - 2, y^p - y(\sqrt{2} - 2)^{p-1} = \beta \). Simple \( \overline{H}_{g;x,\beta} \)-modules correspond to simple \( H_g \)-modules with central character \( (x, \beta) \).

---

\(^1\)See [9, Section 2.5], where the statement is attributed to Dixmier.
For any \((x, \beta) \in \mathbb{F}_p^2\), \(\{x'y'|0 \leq i, j \leq p - 1\}\) is a \(k\)-basis for \(H_{g;x,\beta}\). Form the left module

\[ S_{x,\beta} := \mathbb{H}_{g;x,\beta}/\left(\mathbb{H}_{g;x,\beta}(x - \sqrt[p]{x}) + \mathbb{H}_{g;x,\beta}(g - 1)\right). \]

**Theorem 4.1.** The module \(S_{x,\beta}\) is simple of dimension \(p\). The Azumaya locus and the nonsingular locus of \(H_g\) do not coincide.

**Proof.** It is clear that \(1, y, y^2, \ldots, y^{p-1}\) is a basis for \(S_{x,\beta}\). Let \(f(y) = y^d + a_1 y^{d-1} + \ldots + a_d \in k[y]\), \(d < p\). We will show \(f(y)\) generates \(S_{x,\beta}\) by induction on \(d\). If \(d = 0\) this is obviously true.

Suppose first that \(x \neq 0\). Consider \(gf(y) = g \cdot f(y)g = g \cdot f(y) = (x + y)^d + a_1(x + y)^{d-1} + a_2(x + y)^{d-2} + \ldots + a_d\). In expanding \((x + y)^d\), the term \(dy^d = dx^{d-1} \sqrt[p]{x} \neq 0\) appears since \(x, d \neq 0 \mod p\). Therefore \(gf(y) = f(y) + h(y)\) where \(h(y) \in k[y]\) has leading term \(d y^{d-1}\), and so \(h(y)\) is an element of degree \(d - 1\) in the cyclic submodule generated by \(f(y)\). By induction, \(h(y)\) generates \(S_{x,\beta}\) and thus so does \(f(y)\).

Next, suppose \(x = 0\). If \(d > 1\), \(gf(y) - f(y) = \left(\frac{d}{2}\right)y^{d-2}\) plus lower order terms. By induction, \(h(y) = gf(y) - f(y)\) generates \(S_{0,\beta}\) and thus so does \(f\). On the other hand, if \(d = 1\) and \(f(y)\) generates a proper submodule then \(yf(y)\) is of degree 2, thus is a cyclic vector, so \(f(y)\) is too.

Since \(S_{x,\beta}\) is simple of dimension \(p\) for any \((x, \beta)\), the Azumaya locus is \(\text{Spec}(\mathbb{Z}_g)\). However, the nonsingular locus is \(\text{Spec}(\mathbb{Z}_g) \setminus \{(x, \beta)|x = 0\}\).

For simple modules in the singular locus, we may write explicit matrices representing the action of \(g, x,\) and \(y\) on the basis \(\{1, y, \ldots, y^{p-1}\}\). Let \(\delta = g\partial_x + xg\partial_y\) and let \(A_0(x, g) = g, A_1(x, g) = xg, A_2(x, g) = x^2 + g^2, \ldots, A_n(x, g) = \delta(A_{n-1}(x, g))\) be the André polynomials. Set \(A_n = A_n(0, 1)\). Note that for \(n\) odd, \(A_n = 0\). For \(\beta \in k\), the representing matrices are:

\[
g = \begin{pmatrix} 1 & 0 & A_2 & 0 & A_4 & 0 & A_6 & 0 & \ldots \\ 0 & 1 & 0 & \left(\frac{3}{2}\right)A_2 & 0 & \left(\frac{5}{4}\right)A_4 & 0 & \left(\frac{7}{6}\right)A_6 & \ldots \\ 0 & 0 & 1 & 0 & \left(\frac{4}{2}\right)A_2 & 0 & \left(\frac{6}{4}\right)A_4 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \left(\frac{5}{2}\right)A_2 & 0 & \left(\frac{7}{4}\right)A_4 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & \left(\frac{6}{2}\right)A_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & \end{pmatrix},
\]

\[
x = \begin{pmatrix} 0 & 1 & 0 & A_2 & 0 & A_4 & 0 & \ldots \\ 0 & 0 & 2 & 0 & \left(\frac{4}{3}\right)A_2 & 0 & \left(\frac{6}{5}\right)A_4 & \ldots \\ 0 & 0 & 0 & 3 & 0 & \left(\frac{5}{3}\right)A_2 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 4 & 0 & \left(\frac{6}{3}\right)A_2 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ \end{pmatrix},
\]

\[
y = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & \beta \\ 1 & 0 & 0 & 0 & \ldots & 0 & T_{\frac{\beta}{2n}} \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ \end{pmatrix}.
\]

In the matrix \(y\), \(T_j\) denotes the \(j\)th reduced tangent number modulo \(p\): see sequence A002105 in the OEIS which begins \(T_1 = 1, T_2 = 1, T_3 = 4, T_4 = 34, T_5 = 496, \ldots, T_n = 2^n(2^{2n} - 1)^{|B_{2n}|}/n\) where \(B_j\) is the \(j\)th Bernoulli number.
4.1.2. Finite-dimensional quotients of $H_x$

The simple modules over $H_x$ may be found by the same procedure of central reduction that was used for $H_y$, though we’ll shift gears and use Verma modules for $H_x$ to define the simple modules in the end. The description of simple modules over $H_x$ depends on whether or not $\lambda \in kG$ is invertible. Write $\lambda = \sum_{i=1}^{p-1} c_i g^i$. Then $\lambda^p = \sum_{i=1}^{p-1} c_i^p$, so $\lambda$ is invertible if $p$ does not divide $c_i := \sum_{i=1}^{p-1} c_i$, while $\lambda$ is nilpotent if $c_i$ is divisible by $p$. If $\lambda \in kG$ is invertible then the simple modules for $H_x$ behave the same as those for $H_y$: all simple representations of $H_x$ are $p$-dimensional and the smooth locus of the center of the algebra is strictly contained in its Azumaya locus which equals $\text{Spec } Z$. When $\lambda$ is a zero-divisor, the simple modules behave differently at the singular locus of the center.

Consider the quotient of $H_x$ by its two-sided ideal $(g - 1)$:

**Proposition 4.2.** If $\lambda$ is a zero-divisor then $\frac{H_x}{(g - 1)} \cong k[y]$. If $\lambda$ is invertible then $\frac{H_x}{(g - 1)} \cong 0$.

**Proof.** Let $\phi : H_x \to \frac{H_x}{(g - 1)}$. Then $\phi(y) = \phi(yg) = \phi(yg + xg) = \phi(y) + \phi(x)$, so $\phi(x) = 0$. Therefore $c_x \phi(1) = \phi(\lambda) = \phi([x, y]) = [\phi(x), \phi(y)] = 0$. If $\lambda$ is invertible then $c_x \neq 0$ and thus $\frac{H_x}{(g - 1)} = 0$. \hfill $\Box$

**Corollary 4.3.** $H_x$ has a 1-dimensional representation if and only if $\lambda$ is nilpotent, in which case $H_x$ has a distinct 1-dimensional representation for each $\beta \in k$.

**Proof.** Let $\beta \in k$. If $\lambda$ is nilpotent, we apply the map $\phi : H_x \to k[y]$ above, then postcompose with evaluation of $y$ at $\beta$. In this way we obtain a one-dimensional representation of $H_x$ on which $y$ acts by $\beta$, $x$ acts by $0$, and $g$ acts by $1$. For the converse, suppose that $\lambda$ is invertible. The group $G$ must act by a character on any 1-dimensional representation, but the cyclic group of order $p$ has only the trivial character over a field of the same characteristic. Therefore $\lambda$ must act by the scalar $c_x \neq 0$ if $\lambda$ is invertible, so $xy - yx = \lambda$ must act by $c_x \neq 0$. However, $xy - yx$ acts by $0$ on any 1-dimensional representation, contradiction. \hfill $\Box$

Note that $x$ must act by $0$ on any 1-dimensional representation, because of the relation $gy = xg + yg$. Fix $\lambda \in kG$. Given a finite-dimensional representation $V$ of $H_x$, the center of $H_x$ must act by scalars on it, and we call the scalars by which the generators of the center act the “central character.” If $Y$ acts by $\alpha$ then $X$ must act by $\sqrt{\alpha^2}$ and so we simply write $(\alpha, \beta)$ instead of $(\sqrt{\alpha^2}, \alpha, \beta)$, and then modules over $H_x$ with central character $(\alpha, \beta)$ correspond to modules over the finite-dimensional algebra

$$\overline{H}_{\lambda, \alpha, \beta} := \frac{H_x}{(X - \sqrt{\alpha^2}, Y - \alpha, Z - \beta)}.$$ 

**Proposition 4.4.** $H_x$ is a finite-dimensional module over its center of generic rank $p^2$.

**Proof.** The point $(\alpha, \beta)$ is nonsingular point in $\text{Spec } Z$ when $\alpha \neq 0$, in which case $x = \frac{1}{2} x^p = \frac{1}{2} (x^2)^{p-1}$ and as $x^2$ is linearly dependent on $kG$, then $x$ is too. Thus, for a smooth point $(\alpha, \beta)$ the algebra $\overline{H}_{\lambda, \alpha, \beta}$ is of dimension $p^2$. \hfill $\Box$

While $\overline{H}_{\lambda, \alpha, \beta}$ has dimension $p^2$ for all $\alpha \neq 0$, the minimal possible dimension of $\overline{H}_{\lambda, 0, \beta}$ is $p^2$ and the maximal possible dimension is $2p^2$, and both of these bounds are realized as $\lambda$ ranges through $kG$ and are the only dimensions occurring. In fact, we will show that the set of $\lambda$ such
that the quotient algebras \( \mathbb{H}_{\lambda,0,\beta} \) of \( H_\lambda \) have the maximal possible dimension consists of exactly those \( \lambda \) which are zero-divisors in \( kG \). On the other hand, if \( \lambda \) is invertible then \( \dim(\mathbb{H}_{\lambda,0,\beta}) = p^2 \).

Consider the three relations defining \( \mathbb{H}_{\lambda,0,\beta} \) as a quotient of \( H_\lambda \): \( x^2 - 2 \sum_{a=1}^{p-1} \frac{c_a}{a} g^a + 2 \sum_{a=1}^{p-1} \frac{c_a}{a} g^a = 0, \) \( x^p = 0, \) and \( y^p - y \left( x^2 - 2 \sum_{a=1}^{p-1} \frac{c_a}{a} g^a \right)^{\frac{p^2}{2}} = \beta. \) Because of the first relation, the third relation becomes a polynomial in \( k[y] \). Moreover, \( x^2 \) is dependent on \( kG \) but \( x \) is not. Thus the dimension of \( \mathbb{H}_{\lambda,0,\beta} \) is at most 2\( p^2 \). Writing \( x^2 = 2 \sum_{a=1}^{p-1} \frac{c_a}{a} (g^a - 1), \) and observing that \( 0 = x^p + 1 = (x^2)^{\frac{p^2}{2}} \), it follows that the dimension of \( \mathbb{H}_{\lambda,0,\beta} \) is 2\( p^2 \) if and only it holds in \( kG \) that \( \left( \sum_{a=1}^{p-1} \frac{c_a}{a} (g^a - 1) \right)^{\frac{p^2}{2}} = 0. \)

Recall that the set of zero-divisors in \( kG \) are the elements belonging to the augmentation ideal which has a basis \( (g - 1), (g^2 - 1), \ldots, (g^{p-1} - 1) \). If \( \mu \) is a zero-divisor then \( \mu^p = 0. \) Recall that \( g\partial_g \left( \sum_{a=0}^{p-1} m_a g^a \right) = \sum_{a=0}^{p-1} am_a g^a. \)

**Proposition 4.5.** Let \( \mu := \sum_{a=0}^{p-1} m_a (g^a - 1) \) be an arbitrary zero-divisor of \( kG. \) Then \( \mu^{p-1} \neq 0 \) if and only if \( g\partial_g (\mu) \) is invertible. If \( g\partial_g (\mu) \) is a zero-divisor then \( \mu^{p-1} = 0. \)

**Proof.** Suppose that \( \lambda := g\partial_g (\mu) \) is invertible. Then \( \lambda^{-1} g\partial_g (\mu^p) = n \mu^{p-1} \) and thus \( \left( \lambda^{-1} g\partial_g \right)^{p-2} (\mu^p) = (p-1)! \mu \neq 0 \iff \mu \neq 0. \) So \( \mu^{p-1} \neq 0. \)

Now suppose \( \lambda = g\partial_g (\mu) \) is a zero-divisor. Assume \( p > 3. \) It is easy to verify the statement for \( p = 3 \) by hand. First, we claim that \( \mu^{p-1} = 0. \) The element \( g\partial_g (\mu^{p-1}) = -\lambda \mu^{p-2} \) must square to 0 as \( 2p - 4 > p \Rightarrow \mu^{2p-4} = 0. \) Observe that any element \( \zeta \in kG \) such that \( \zeta^2 = 0 \) generates an ideal that is a one-dimensional representation of \( G. \) But \( G \) has a unique one-dimensional representation spanned by \( 1 + g + \ldots + g^{p-1} \). Thus \( \zeta^2 = 0 \) implies \( \zeta = C \cdot (1 + g + \ldots + g^{p-1}) \) for some \( C \in k. \) Therefore \( g\partial_g (\mu^{p-1}) \) must be a scalar multiple of \( 1 + g + g^2 + \ldots + g^{p-1}, \) which is impossible unless \( C = 0, \) since the result of \( g\partial_g \) applied to anything cannot contain a constant term. So \( g\partial_g (\mu^{p-1}) = 0. \) It follows that \( \mu^{p-1} \) is a constant. Since \( \mu \) is a zero-divisor, this implies \( \mu^{p-1} = 0. \)

This implies \( \left( \mu^{p-1} \right)^2 = 0 \) and thus \( \mu^{p-1} = C \cdot (1 + g + g^2 + \ldots + g^{p-1}) \) for some constant \( C \) (apparently given by a homogenous polynomial \( f \) of degree \( \frac{p^2}{2} \) in \( k[x_1, \ldots, x_{p-2}] \) evaluated at \( c_1, \ldots, c_{p-2} \) with \( c_i = \frac{4\mu_i}{n} \)). Since \( g \) acts trivially on the line spanned by \( (1 + g + \ldots + g^{p-1}) \) and the sum of the coefficients of \( \mu \) is equal to 0, we have \( \mu^{p+1} = \mu \cdot \mu^{p-1} = C \mu \cdot (1 + g + g^2 + \ldots + g^{p-1}) = 0. \)

**Corollary 4.6.** If \( \lambda \) is a zero-divisor, then the central quotient algebras \( \mathbb{H}_{\lambda,0,\beta} \) of \( H_\lambda \), that is those defined over the singular line of \( \text{Spec} Z(H_\lambda) \), are of the maximal possible dimension, namely \( 2p^2. \) If \( \lambda \) is invertible then the algebras \( \mathbb{H}_{\lambda,0,\beta} \) are of dimension \( p^2 \).

**Proof.** Let \( \mu = \sum_a \frac{c_a}{a} (g^a - 1). \) Then \( g\partial_g (\mu) = \sum_{a=1}^{p-1} c_a g^a = \lambda, \) and \( 2\mu = x^2 \) in \( \mathbb{H}_{\lambda,0,\beta}. \) By Proposition 4.5 and the defining relations of \( \mathbb{H}_{\lambda,0,\beta} \), it holds that (i) \( \lambda \) is a zero-divisor if and only if \( \mu^{p+1} = 0 \) if and only if the equations \( (x^2)^{\frac{p+1}{2}} = 0, \) \( (x^2)^{p+1} = 0, \) etc., do not impose relations in \( kG; \) and (ii) \( \lambda \) is invertible if and only if \( \mu^{p-1} \neq 0 \) if and only if the equations \( (x^2)^{p-1} = 0, \) \( (x^2)^{p+1} = 0, \) etc., impose \( p-1 \) relations in \( kG. \)
4.1.3. Simple modules as quotients of Verma modules

Define Verma modules for $H_\lambda$ as

$$\Delta_\alpha := H_\lambda/(H_\lambda(x-\alpha) + H_\lambda(g-1))$$

as $\alpha$ ranges through $k$. As a left $H_\lambda$-module, $\Delta_\alpha$ is isomorphic to $k[y]$. When $t = 0$, $\Delta_\alpha$ contains an infinite descending chain of submodules generated by $(C - \beta)^n$ for each $n \in \mathbb{N}$, $C = y^p - \frac{\delta(g)}{\delta(g)}$. The description of the Verma modules implies:

Lemma 4.7. Let $f(y) \in \Delta_\alpha$ generate a submodule of $\Delta_\alpha$. Then $f(y)$ generates a proper submodule if and only if $g \cdot f(y) = f(y)$ and $x \cdot f(y) = cf(y)$ for some $c \in k$.

Proof. We may assume the degree of $f(y)$ in $y$ is minimal over elements of $H_\lambda f(y) \subset \Delta_\alpha$. Multiplication by $y$ always raises the degree. For any $r \in R$, $rf = fr +$ lower order terms, so the “only if” is clear. Next, suppose $r \in R$ acts by a scalar on $f(y)$. Writing $h \in H_\lambda$ in its normal form $h = \sum y^i r_i$, $r_i \in R$, the degree of $hf(y)$ is at least that of $f(y)$ unless $hf(y) = 0$ since $r_i$ multiplies $f(y)$ by a scalar and $y^i$ raises the degree.

Proposition 4.8. Suppose $\lambda \neq 0$ does not contain a constant term. There is a two-parameter family of distinct simple $H_\lambda$-modules $S_{\alpha, \beta}, \alpha, \beta \in k$. If $\lambda$ is invertible, all simple $H_\lambda$-modules are $p$-dimensional, while if $\lambda$ is a zero-divisor then the simple $H_\lambda$-modules over the smooth locus are $p$-dimensional while the simple modules over the singular locus are $1$-dimensional.

Proof. Suppose $\alpha \neq 0$. Then $\delta(g) = \alpha \neq 0$ in $\Delta_\alpha$. Suppose $f(y) = y^n + a_{n-1}y^{n-1} + \ldots \in \Delta_\alpha, 0 < n < p$. Then $g \cdot f(y) = \left(\sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} y^{n-i} \delta'(g) + a_{n-1} \left(\sum_{i=1}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} y^{n-1-i} \delta'(g) + \ldots = y^n + a_{n-1}y^{n-1} + \left(\begin{pmatrix} n \\ 1 \end{pmatrix} \alpha y^{n-1} + a_{n-2}y^{n-2} + \ldots \right.$ Since $\alpha \neq 0$ and $\left(\begin{pmatrix} n \\ 1 \end{pmatrix} = n \neq 0$ as $n < p$, $g$ cannot fix $f(y)$ and so, by the lemma, $f(y)$ cannot generate a proper ideal.

When $\alpha = 0$, $g \cdot (y - \beta) = y - \beta$ and $x \cdot (y - \beta) = c_i$ where $c_i$ is the sum of the coefficients of $\lambda$. Thus $y - \beta$ generates a proper submodule if and only if $c_i = 0 \mod p$.

If $\alpha = 0$ and $\lambda$ is not a zero-divisor then $\delta(x) = c_i \neq 0$, and by the same argument given in the $\alpha \neq 0$ case but with $x$ in place of $g$, $x$ cannot act by a scalar on any polynomial of degree less than $p$, and so there can be no proper submodule generated in degree less than $p$.

Since the shift by any $\beta \in k$ of the central element $Y = y^p - yx^{p-1} - \ldots$ commutes with $x$ and $g$, the image $Y - \beta$ in $\Delta_\alpha$ generates a maximal submodule of $\Delta_\alpha$ when $\alpha \neq 0$ or when $\alpha = 0$ and $\lambda$ is invertible.

Remark 4.9. Let $J_\beta$ be the submodule of $\Delta_\alpha$ generate by $(Y - \beta)$, where $Y$ is the central element $y^p - yx^{p-1} - \ldots$. Over the smooth locus, and over the singular locus when $\lambda$ is an invertible element of $kG$, the simple modules are of the form $\Delta_\alpha/J_\beta$. Now suppose that $\lambda \in \text{Rad} kG$. Let $I_\beta \subset \Delta_0$ be the submodule of $\Delta_0$ generated by $(y - \beta)$. Over the singular locus when $\lambda$ is a zero-divisor the simple modules then have the form $\Delta_0/I_\beta$.

The representation theory of $H_\lambda$ has consequences for the structure of $H_\lambda$. Recall that an element $v$ in an algebra $H$ is called normal if $vH = Hv$, that is, the left and right ideals generated by $v$ coincide. The description of the Verma modules implies:

Corollary 4.10. The only normal elements of $H_\lambda$ are the elements of the center $Z_\lambda$.

Proof. Suppose $v \in H_\lambda$ is normal, and look at its image $\overline{v} \in \Delta_\alpha$. Since $vH_\lambda = H_\lambda v, \overline{v}$ generates a proper submodule of $\Delta_\alpha$. But all submodules of $\Delta_\alpha, \alpha \neq 0$, are generated by powers of $Y$ and their
scalar translates. So \( \nu \) would have to be of the form \( \nu = f(Y) + F \) with for some \( f \in k[Y] \) and \( F \in H_{\bar{k}}(x - z) + H_{\bar{k}}(g - 1) \). Since this must hold for all \( z, F \) must be in \( H_{\bar{k}}(g - 1) \), \( F = h(g - 1) \) for some \( h \in H_{\bar{k}} \). Since the difference of normal elements is normal, \( F \) must be normal. However, \( h(g - 1)y = hy(g - 1) + hgx = yh(g - 1) + [h, y](g - 1) + hgx \). Then \( H_{\bar{k}}h(g - 1) = h(g - 1)H_{\bar{k}} \) implies \([h, y](g - 1) + hgx \in H_{\bar{k}}h(g - 1)\) and in particular \([h, y](g - 1) + hgx \in H_{\bar{k}}(g - 1)\), so that we must have \( h \in H_{\bar{k}}(g - 1) \). Writing \( h = h'(g - 1) \) and repeating the argument, we obtain \( h' \in H_{\bar{k}}(g - 1) \), and so on; after the \( p \)th iteration, we have \( F \in H_{\bar{k}}(g - 1)^p = 0 \). \( \square \)

**Proposition 4.11.** The Jacobson radical \( \text{Rad} H_{\bar{k}} = 0 \).

**Proof.** Take \( z \in \text{Rad} H_{\bar{k}} \) and write \( z = y^0f_0(x, g) + y^{n - 1}f_{n - 1}(x, g) + \ldots + f_0(x, g) \). Since \( \text{Rad} H_{\bar{k}} \) is a two-sided ideal of \( H_{\bar{k}}, gzg^{-1} - z \in \text{Rad} H_{\bar{k}} \). Recall that \( R = kG[x] \). Suppose first that \( n = 0 \), so that \( z \in \text{Rad} H_{\bar{k}} \cap R \). Then \( z \in \text{Rad} R \), implying that \( g - 1 \) divides \( z \). Take a simple module \( S \) of dimension \( p \) and consider the action of \( z \) on the basis element \( y \) of \( S \). Since \( g - 1 \) act by \( 0 \) on the basis element \( 1 \) of \( S \), \( (g - 1) \cdot y = x \cdot 1 \). It follows that \( z \cdot y = q(x) \cdot 1 = 0 \) for some polynomial \( q(x) \in k[x] \) which does not depend on \( S \). Therefore \( q(x) = 0 \) for every \( x \in k^x \), and since \( k \) is algebraically closed, this implies that \( q(x) = 0 \). Therefore \( z = 0 \).

Next, suppose that \( n > 0 \). In this case, \( z \in Z_{\bar{k}} \) if and only if \( gzg^{-1} - z = 0 \). Suppose first that \( z \in Z_{\bar{k}} \). Then \( z \) is a polynomial in \( x^2 - 2Dp^{-1}(\lambda), x^d, \) and \( Y = y^p - y^{S_{\lambda}(g)}(g) \). On the one hand, \( z \in \text{Rad} H_{\bar{k}} \) means that \( z \) annihilates any irreducible representation \( S_{\alpha, \beta} \). On the other hand, \( z \) acts on \( S_{\alpha, \beta} \) by evaluating at \( x = \alpha, g = 1 \), and \( Y = \beta \). Note that \( z \) evaluated at \( g = 1 \) is not zero if \( z \neq 0 \). So, \( z \) acts by a two-variable polynomial in variables \( x \) and \( Y \), evaluated at \( x = \alpha \) and \( Y = \beta \). This cannot be \( 0 \) for all choices of \( (\alpha, \beta) \in k^2 \) unless \( z = 0 \). It follows that \( Z_{\bar{k}} \cap \text{Rad} H_{\bar{k}} = 0 \).

Next, suppose that \( z \notin Z_{\bar{k}} \). Then \( gzg^{-1} - z \neq 0 \) has \( y \)-degree smaller than \( n = \deg z \). We may then use induction on \( n \) with the argument above as the base case. \( \square \)

An element \( e \) in a ring \( A \) is called idempotent if \( e^2 = e \). The existence of idempotents or lack there-of is important for understanding the structure of a ring and its modules.

**Lemma 4.12.** Let \( R = k[x] \otimes kG \). Then \( R \) contains no idempotents besides \( 0 \) and \( 1 \).

**Proof.** Suppose that \( f(x, g) = x^d\lambda_n + x^{n - 1}\lambda_{n - 1} + \ldots + x\lambda_1 + \lambda_0 \) is idempotent and that \( f(x, g) \neq 0, 1 \). If \( n = 0 \) so that \( f(x, g) = \lambda_0 \in kG \), then \( f \) cannot be idempotent as the \( p \)th power of any element of \( kG \) belongs to \( k \) and the only idempotents in \( k \) are \( 0 \) and \( 1 \). If \( n > 0 \), we note that \( \lambda_0^2 = \lambda_0 \) and therefore \( \lambda_0 = 0 \) or \( 1 \). If \( \lambda_0 = 0 \) then there exists a \( d > 0 \) such that \( x^d \) divides \( f(x, g) \) and \( x^{d + 1} \) does not divide \( f(x, g) \). But then \( x^{ad} \) divides \( f(x, g)^2 = f(x, g) \), a contradiction. If \( \lambda_0 = 1 \), it follows easily by induction that the coefficient of \( x^{ad} \) in \( f(x, g) \) is \( 0 \) for each \( m = 1, 2, \ldots, n \). \( \square \)

**Proposition 4.13.** For any \( \lambda \in kG \), the only central idempotents in \( H_{\bar{k}} \) are \( 0 \) and \( 1 \).

**Proof.** By [11], if \( A \) is a ring then the central idempotents in a differential operator extension \( A[y; \delta] \) coincide with the central idempotents in \( A \). Now apply Proposition 2.3, Lemma 4.12. \( \square \)

### 4.1.4. Self-extensions of one-dimensional simple modules

**Proposition 4.14.** Suppose \( \lambda \in kG \) is a zero-divisor. Observe that \( g - 1 \) divide \( \lambda \). Consider a simple \( H_{\bar{k}} \)-module \( S \) of dimension \( 1 \). Suppose \( (g - 1)^2 \) does not divide \( \lambda \). Then \( \text{Ext}^i_{H_{\bar{k}}}(S, S) = k \) for \( i = 0, 1, \text{Ext}^2_{H_{\bar{k}}}(S, S) = k^2 \), and \( \text{Ext}^j_{H_{\bar{k}}}(S, S) = k^3 \) for all \( j \geq 3 \). If \( (g - 1)^2 \) divides \( \lambda \) then \( \text{Ext}^i_{H_{\bar{k}}}(S, S) = k, \text{Ext}^2_{H_{\bar{k}}}(S, S) = k^2, \text{Ext}^3_{H_{\bar{k}}}(S, S) = k^3, \) and \( \text{Ext}^i_{H_{\bar{k}}}(S, S) = k^4 \) for all \( i \geq 3 \).
Proof. The assumption that $S$ is one-dimensional means that $S = S_{0, \beta}$ for some $\beta \in k$, i.e. $x$ acts on $S$ by 0, $y$ by $\beta$, and $g$ by 1. We claim that the following complex is a free resolution of $S$:

\[\begin{array}{ccc}
\cdots & B H_{\xi}^{\oplus 4} & A H_{\xi}^{\oplus 4} \xrightarrow{B} H_{\xi}^{\oplus 4} A H_{\xi}^{\oplus 4} \\
A := \left( \begin{array}{ccc}
g - 1 & x & 0 \\
0 & \sum_a g^a & y \bar{\beta} \\
0 & 0 & \sum_a g^a \\
0 & 0 & 0 & 0 & g - 1
\end{array} \right) & R := \left( \begin{array}{ccc}
g - 1 & x & 0 \\
0 & \sum_a g^a & y \bar{\beta} \\
0 & 0 & \sum_a g^a \\
0 & 0 & 0 & 0 & g - 1
\end{array} \right) & \left( \begin{array}{ccc}
x & 0 & 0 \\
0 & g - 1 & x \\
0 & 0 & \sum_a g^a \\
0 & 0 & 0 & 0 & g - 1
\end{array} \right)
\end{array}\]

It is easily checked to be a complex. To see that the complex is exact, first observe that the annihilator of $\lambda$ is the left ideal $H_{\xi}(\sum_a g^a)$. Secondly, we use the identities in $(g - 1)^{p - 1} = \sum_a g^a = g \sum_a g^a$, $(\sum_a g^a)^2 = 0$, and $g(1 - g)^{p - 2} = \sum_a g^a = g \partial_x (\sum_a g^a)$. Applying $\text{Hom}_{H_{\xi}}(-, S)$ to the resolution gives the complex

\[0 \to k \xrightarrow{\lambda} k^{\oplus 3} \xrightarrow{\lambda^2} k^{\oplus 4} \xrightarrow{\lambda^3} k^{\oplus 4} \xrightarrow{\lambda^4} k^{\oplus 4} \xrightarrow{\lambda^5} \cdots\]

Here $\tilde{B}$ denotes the matrix with all 0’s except for $\frac{\lambda^2}{g - 1}$ in the lower left corner. The maps 0 and $\tilde{B}$ continue in alternation indefinitely to the right. Now there are two cases:

Case I. If $\lambda$ is divisible by $g - 1$ but not by $(g - 1)^2$ then $\frac{\lambda^2}{g - 1}$ is invertible. Then at $k^{\oplus 3}$ the kernel is 1-dimensional, so that $\text{Ext}_{H_{\xi}}^1(S, S) = k$. The image of the $3 \times 4$ matrix is $k^{\oplus 2}$ in the first two components of $k^{\oplus 4}$, so that $\text{Ext}_{H_{\xi}}^2(S, S) = k \oplus k$. At every $k^{\oplus 4}$ from there onwards, either the kernel is 4-dim and the image 1-dim, or the image is 0-dim and the kernel 3-dim.

Case II. If $\lambda$ is divisible by $(g - 1)^2$ then $\frac{\lambda^2}{g - 1}$ is divisible by $g - 1$, hence must act by 0 on $S$, and so every map above is the zero map except the $3 \times 4$ matrix, which has 2-dimensional kernel and 1-dimensional image. \hfill \square

Corollary 4.15. In the $t = 0$ case, the algebras $H_{\xi}$ have infinite global dimension.

4.2. Simple $H_{\xi}$-modules in the $t = 1$ case

The description of simple left $H_{\xi}$-modules follows more or less the same pattern when $\lambda$ contains a constant term ($t = 1$) as when $\lambda$ does not ($t = 0$). The simple modules are the quotients of left Verma modules $\Delta_x = H_{\xi}/(H_{\xi}(x - \alpha) + H_{\xi}(g - 1)) \cong k[x]$ by maximal submodules (which are necessarily cyclic). As in the $t=0$ case, there is a two-parameter family of simple modules over $H_{\xi}$.

The description of simple modules for $H_{\xi}$ when $t = 1$ follows from Lemma 5.13.

Proposition 4.16. Let $\lambda \in kG$ be a polynomial with constant term equal to 1 and consider the $H_{\xi}$-modules $\Delta_x = H_{\xi}/(H_{\xi}(x - \alpha) + H_{\xi}(g - 1))$. For $\alpha \neq 0$, the simple modules are $p^2$-dimensional. When $\alpha = 0$ there are two cases:

\[\cdots \to B H_{\xi}^{\oplus 4} \xrightarrow{A} H_{\xi}^{\oplus 4} \xrightarrow{B} H_{\xi}^{\oplus 4} \xrightarrow{A} H_{\xi}^{\oplus 4} \xrightarrow{B} \cdots\]
• If \( \lambda \) is a zero-divisor then \( y - \beta, \beta \in k \), generates a maximal submodule of \( \Delta_0 \) and so the simple modules obtained from \( \Delta_0 \) are 1-dimensional.

• If \( \lambda \) is invertible then \( Y - \beta := y^\theta - y^{\theta(g)/\delta(g)} - \beta \) generates a maximal submodule and so the simple modules obtained from \( \Delta_0 \) are \( p \)-dimensional.

**Proof.** If \( \alpha \neq 0 \) the subset of \( g \)-fixed elements of \( \Delta_4 \) is \( k[Y] \). We have \( x \cdot Y = Yx + [\alpha, Y] = Yx + [gxy^\theta - y, Y] = g^{-1} \). If \( \alpha \neq 0 \) the submodule generated by \( Y \) contains \( 1 \); likewise, for \( f(Y) \), deg \( f < p \), by downwards induction. The submodules generated by the images of the central elements \( y^\theta - y^{\theta(g)/\delta(g)} \) are therefore the maximal ones.

When \( \alpha = 0 \) then \( g \cdot (Y - \beta) = Y - \beta, x \cdot (y - \beta) = c_i \) where \( c_i \) is the sum of coefficients of \( \lambda \); thus \( y \) generates a proper submodule if and only if \( \beta \) is a zero-divisor. Otherwise, by the same argument as in the \( t = 0 \) case, \( f(Y) \) does not generate a proper submodule for any \( f \) of degree less than \( p \). However, since \( x \cdot (Y - \beta) = Yx - \beta x = 0 \) and \( g \cdot (Y - \beta) = (Y - \beta)g = Y - \beta \), the element \( Y - \beta \) generates a maximal submodule. \( \square \)

### 4.3. Other notable representations of \( \mathbf{H}_\lambda \)

There are several natural infinite-dimensional representations of \( \mathbf{H}_\lambda \), beyond the most obvious one of \( \mathbf{H}_\lambda \) as a module over itself.

#### 4.3.1. The differential operator representation

Consider \( \mathbf{R} \) as a representation of \( \mathbf{H}_\lambda = \mathbf{R}[y; \delta] \) on which \( r \in \mathbf{R} \) acts by left multiplication while \( y \) acts as \( y \cdot r = -\delta(r) \).

**Lemma 4.17.** The left module \( \mathbf{R} \) is not a faithful representation of \( \mathbf{H}_\lambda \).

**Proof.** By Theorem 3.3, \( \mathbf{H}_\lambda \) has a central element of the form \( Z = y^n - y^mF(x, g) \) for some \( n, m \in \mathbb{N} \). Then \( F(x, g) \) belongs to the kernel of \( \delta \) since it commutes with \( y \). For any \( r \in \mathbf{R} \), then it holds that \( Z \cdot r = Zr \cdot 1 = rZ \cdot 1 = -r(\delta^m(1) - \delta^m(F(x, g))) = 0 \). \( \square \)

The representation \( \mathbf{R} \) contains an infinite chain of nested sub-representations \( \mathbf{R}(x^m) \), \( n \in \mathbb{N} \). In the case \( t = 0 \), the ideals \( \mathbf{R}(x^2 - 2g\partial^2(g)) \), \( n \in \mathbb{N} \), also generate sub-representations: since \( \delta(x^2) = 0 \), and \( \delta(x^2 - 2g\partial^2(g)) = 0 \) when \( t = 0 \), it holds for any \( r \in \mathbf{R} \) that \( y \cdot r = -\delta(r)x^m + y \cdot r(x^2 - 2g\partial^2(g)) = -\delta(r)(x^2 - 2g\partial^2(g)) \).

#### 4.3.2. The Weyl representation

For any \( \lambda \not\in \text{Rad} kG \), the Weyl algebra is a module for \( \mathbf{H}_\lambda \). Let \( c_i \in k^\times \) be the sum of the coefficients of \( \lambda \). Set

\[
A_{1, c_i} = \frac{k\langle x, y \rangle}{(xy - yx - c_i)} \cong \frac{k\langle x, y \rangle}{(xy - yx - 1)}.
\]

Then \( A_{1, c_i} \) is an \( \mathbf{H}_\lambda \)-module. The action is given by \( y \) acting by left multiplication and \( x \) and \( g \) acting as they do on \( \mathbf{H}_\lambda \), then evaluating the resulting polynomials at \( g = 1 \). Recall that \( \delta = \lambda \partial_x + xg\partial_x \). The action is given by \( y \cdot y^n = y^{n+1}x^m, \) and \( r \cdot y^n = \sum_{i=0}^n \binom{n}{i} y^{n-i}x^m \delta^i(r) \) for all \( r \in \mathbf{R} \). The Weyl representation \( A_{1, c_i} \) can be realized as the quotient \( A_{1, c_i} \cong \mathbf{H}_\lambda / \mathbf{H}_\lambda (g - 1) \) of
H, by the left ideal H,(g − 1). All of the modules \( \Delta_z \) are quotients of the Weyl representation: 
\[
\Delta_z = H_z/(H_z(g − 1) + H_z(x - z)) \cong A_{1, e_i}/A_{1, e_i}(x - z).
\]
Consequently, every simple representation also arises as a quotient of \( A_{1, e_i} \).

\[\text{4.3.3. The polynomial representation}\]
If \( \lambda \in \text{Rad} kG \) then the sum of the coefficients of \( \lambda \) is zero mod \( p \). The polynomial algebra \( k[x, y] \)
admits an \( H_z \)-module structure via the same action given for the Weyl representation above. Then \( k[x, y] \cong H_z/H_z(g - 1) \), and every \( \Delta_z \) and hence every simple module \( S_{x, \beta} \) is a quotient of \( k[x, y] \) as an \( H_z \)-module.

\[\text{5. The algebras } H_z \text{ for } E = G'\]
Let \( E = (\mathbb{Z}/p)^r \) be an elementary abelian \( p \)-group and realize \( E \) as the unipotent subgroup of \( SL(2, \mathbb{F}_q) \), \( q = p^r \):
\[
E = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\}.
\]
Let \( \xi_1, \xi_2, \ldots, \xi_r \) be a basis for \( \mathbb{F}_q \) as an \( \mathbb{F}_p \)-vector space, so that 
\[
g_i := \begin{pmatrix} 1 & \xi_i \\ 0 & 1 \end{pmatrix}
\]
is the generator of the \( i \)th copy of \( \mathbb{Z}/p \) and together \( g_1, \ldots, g_r \) generate \( E \). We will always take \( \xi_1 = 1 \) so that \( g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \xi_2 = \xi \) where \( \xi \) generates \( \mathbb{F}_q^\times \) and \( \xi_i = \xi^{r-1} \). Let \( k = \mathbb{F}_q \). The group algebra \( kE \) is a truncated polynomial ring in the \( r \) variables given by the generators \( g_i \):
\[
kE \cong k[g_1, g_2, \ldots, g_r]/(g_r^p - 1)^{r}.
\]
Let \( g_\partial g = g_1 \partial_{g_1} + \xi g_2 \partial_{g_2} + \xi^2 g_3 \partial_{g_3} + \cdots + \xi^{r-1} g_r \partial_{g_r} \). By \( \partial_{g_i} \) we mean partial differentiation with respect to the variable \( g_i \), thinking of the elements of \( kE \) as polynomials in \( r \) variables of degree at most \( p - 1 \) in each variable. For any \( \lambda \in kE \), the algebra 
\[
H_\lambda := \frac{k(x, y) \rtimes E}{(xy - yx - \lambda)}
\]
admits a presentation as a differential operator Ore extension over its commutative subring \( R := k[x] \otimes kE \) as \( H_\lambda = R[y; \delta] \) where \( \delta := \lambda \partial_x + xy \partial_y \). This means that for any \( r \in R, yr = yr + \delta(r) \).
As with the algebras \( H_z \) for \( G \) the cyclic group of order \( p \) studied so far in this paper, the family of algebras \( H_\lambda \) for \( E \) consists of PBW deformations of the skew group ring \( H_0 = k[x, y] \rtimes E \) analogous to certain symplectic reflection algebras, namely those deformations of \( \mathbb{C}[x, y] \rtimes G \) where \( G \) is a finite subgroup of \( SL(2, \mathbb{C}) \) and where \( [x, y] \) takes a value \( C \) in the center of the group algebra of \( G \).

In our work so far, we have observed a dichotomy between the cases \( t = 0 \) and \( t = 1 \) for the algebras \( H_\lambda \). The action of the operator \( g_\partial g \) is key: the two cases \( t = 0 \) and \( t = 1 \) correspond to whether or not \( \lambda \) lies in the image of \( g_\partial g \), that is, whether or not \( (g_\partial g)^{q-1}(\lambda) = \lambda \). If we write \( \bar{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}/p)^r \) and \( \lambda = \sum \bar{a} \xi_1^{a_1} \xi_2^{a_2} \cdots \xi_r^{a_r} \) then the condition \( (g_\partial g)^{q-1}(\lambda) = \lambda \) is equivalent to requiring that we have \( c_s = 0 \) whenever \( a_1 + \xi_2 a_2 + \cdots + \xi_r a_r \equiv 0 \) mod \( p \); since the \( \xi_i \) are linearly independent over \( \mathbb{F}_p \) this is no more than the requirement that the coefficient of 1 is 0. Thus we expect that the algebras \( H_\lambda \) constructed from \( (\mathbb{Z}/p)^r \) will likewise fall into two families, determined by whether \( c_0 = 0 \) or \( c_0 \neq 0 \). We will work with the \( t = 0 \) case from now on.
Thus for the rest of this section, \( \lambda \in kE \) is a nonzero polynomial in \( g_i \), \( i = 1, \ldots, r \), of degree at most \( p - 1 \) in each variable and whose constant term is 0.

### 5.1. Some remarks about the center of \( H_\lambda \)

Let \( Z_\lambda \) denote the center of \( H_\lambda \). We expect that \( Z_\lambda \) is isomorphic to the quotient of a polynomial ring on three variables by one relation (between the first and second variables), and that \( H_\lambda \) is of generic rank \( qp \) as a module over \( Z_\lambda \). We first identify those generators of \( Z_\lambda \) belonging to the commutative subalgebra \( R = k[x] \otimes kE \). To check an element \( r \in R \) commutes with \( g \) it is sufficient to check that \( \delta(r) = 0 \), since \( ry = yr + \delta(r) \) for all \( r \in R \) by the definition of a differential operator extension.

As in the case of a cyclic group of order \( p \), \( x^p \in Z_\lambda \). Next, since we have assumed that \( \lambda \) does not have a constant term, the operator \( g \partial_k \) has an inverse on \( \lambda \): \( (g \partial_k)^{-1}(\lambda) = \lambda \) and we write \( (g \partial_k)^{-1}(\lambda) \) for \( (g \partial_k)^{-2}(\lambda) \). This allows us to identify a second, quadratic central element of \( H_\lambda \) contained in the subalgebra \( R \):

**Lemma 5.1.** \( x^2 - 2(g \partial_k)^{-1}(\lambda) \in Z_\lambda \).

**Proof.** \( \delta\left(x^2 - 2(g \partial_k)^{-1}(\lambda)\right) = 2x\lambda - 2xg \partial_k (g \partial_k)^{-1}(\lambda) = 2x\lambda - 2x\lambda = 0. \)

Let \( A = x^2 - 2(g \partial_k)^{-1}(\lambda) + 2 \sum_{\pi} (a_1 + \zeta_2 a_2 + \ldots + \zeta_n a_n)^{-1} e_\pi \), and \( B = x^p \). Note that \( A^p = B^2 \). Recall that \( H_\lambda \) is filtered by degree in \( x \) and \( y \) with both variables assigned degree 1 and \( E \) put in degree 0. Then the associated graded algebra with respect to this filtration is \( \text{gr} H_\lambda = H_0 = k[x,y] \rtimes E \). The center \( Z_0 \) of \( H_0 \) is the subalgebra of \( E \)-invariants in \( k[x,y] \):

\[ Z_0 = k[x,y]^E = k[x,y^l - yx^{d-1}] \]

There is a natural homomorphism from \( Z_\lambda \) to \( Z_0 \) but it is not necessarily onto. We wish to find a central element which deforms the second generator \( y^l - yx^{d-1} \) of \( Z_0 \), that is, a central element whose top degree term is \( y^l - yx^{d-1} \). A computation shows:

**Lemma 5.2.** If \( Z \in H_\lambda \) commutes with \( y \) and \( Z \) is of the form \( y^l - yx^{d-1} + \text{lower order terms} \) then \( Z \) is of the form

\[ Z = y^l - y(x^{d-1} + x^{d-3}y_3 + x^{d-5}y_5 + \ldots) \]

where \( y_i \in kE \) satisfy \( g \partial_k(y_i) = (i - 2)y_{i-2} \).

**Lemma 5.3.** If \( C \in H_\lambda \) commutes with \( y \) and \( g_1 \) and \( C = y^l - yx^{d-1} + \text{lower order terms} \) then \( C = y^l - y^l \frac{\delta g_1}{\delta g_1} \).

**Proof.** By the previous lemma, \( C = y^l - yF, F \in k[x] \otimes kG \). Then \( g_1 C = g_1(y^l - yF) = y^l g_1 + \delta g_1 F = (y^l - yF) g_1 = C g_1 \) so \( \delta g_1 - \delta g_1 F = 0. \) Note that \( \delta g_1 = x g_1 \) is not a zero divisor and it divides \( \delta g_1 \); thus we can solve for \( F \).

### 5.2. The center \( Z_0 \) of \( H_0 \)

The degree \( q \) generator of \( Z_0 \) may be expressed as \( y^l - yx^{d-1} = \prod_{a \in F_q} (y - ax) \). A natural question is whether the expression \( \prod_{a \in F_q} (y - ax) \) gives a central element of \( H_\lambda \) if the factors are taken in the right order. In general, this fails, but it turns out to work when \( \lambda \in G \). Consider the
case that \([x, y] = g\) where \(g \in E\) is an element of the group different from the identity. All of the \(q - 1\) algebras obtained this way are isomorphic:

**Lemma 5.4.** For any \(x \in \mathbb{F}_q^x\), set \(g_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\) and \(H_{g_x} := \begin{pmatrix} k(x, y) y \Xi \mathbb{F} \end{pmatrix}. Then \(H_g \cong H_{g_x}\).

**Proof.** Define an algebra homomorphism \(\phi_x : H_g \to H_{g_x}\) such that \(\phi_x \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\) for any \(\beta \in \mathbb{F}_q, \phi_x(x) = \sqrt{x}z, and \phi_x(y) = \frac{1}{\sqrt{y}}y\). Then \(\phi_x^{-1} = \phi_x\).

**Proposition 5.5.** Take \(q = p\) and \(\lambda = g := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\). Then the third generator of \(Z_g\) is \(C := \prod_{n=1}^p (y - ax)\).

Note that the order in which the product is taken matters; let indices going from bottom to top correspond to multiplicands going from left to right.

**Proof.** Using the observation that \([y - ax, y] = -a\) together with the Leibniz rule, we compute the commutator of \(C\) and \(y\). Since \(\sum_{a \in \mathbb{F}_p, a = 0, |C, y| = \sum_{a \in \mathbb{F}_p} (-a) \prod_{b=1}^{p-1} (y - bx) \prod_{c=j+1}^{p-1} (y - (c-1)x)g = \sum_{a \in \mathbb{F}_p} (-a) \prod_{b=1}^{p-1} (y - bx)g = 0\). As for the commutator of \(C\) with \(g\), we compute \([g, C] = \sum_{a \in \mathbb{F}_p} (-a) \prod_{b=1}^{p-1} (y - bx)g\prod_{c=a}^{p-1} (y - cx)g = p \prod_{a=1}^{p-1} (y - ax)g\prod_{a=1}^{p-1} (y - cx)g = 0\).

**Corollary 5.6.** Take \(q = p\) and \(\lambda = g_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\) for any \(1 < n < p\). Then the third generator of \(Z_{g_n}\) is \(C := \prod_{n=1}^p (y - nax)\).

An analogous statement holds for \(r > 1\). From now on, we take \(r > 1, q = p^r\), and \(E = (\mathbb{Z}/p)^r; 1, \zeta, \zeta^2, ..., \zeta^{r-1}\) is an \(\mathbb{F}_p\) basis for \(\mathbb{F}_q\) and we set \(\Xi := \langle \zeta, \zeta^2, ..., \zeta^{r-1} \rangle_{\mathbb{F}_p}\) to be the \(\mathbb{F}_p\)-linear span of the \(\zeta_i, i > 1\). Form \(H_g := H_{\lambda}\) with \(\lambda := g_1 = [x, y]\) where \(g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\).

**Proposition 5.7.** The element \(C = \prod_{\zeta \in \Xi} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) \((y - (\zeta + a)x) \in H_g\) commutes with \(y\) and \(g_1\).

Note that the outer product over \(\zeta\) does not require any particular order but the innermost product over the variable \(a\) should be read with a increasing from left to right.

**Proof.** For a given \(\zeta\), the terms involving \(\zeta\) appear in succession with \(a\)'s increasing by 1 from left to right. When we apply the Leibniz rule to \(C\) to compute \([C, y]\), it has the effect of picking out one term \((y - (\zeta + a)x)\) at a time and replacing it with \(g_1\) which then moves past the remaining terms to its right decreasing the number \(a\) in each of them by 1 (since the terms involve subtracting \(ax\)). This does not affect any terms with \(\zeta' \in \Xi\) which appeared to the left of \(\zeta\); moreover, the effect on all terms \(\zeta'' \in \Xi\) which appear to the right of \(\zeta\) is the same for each for value of \(a = 1, ..., p\). There are \(p\) terms with \(\zeta\). Deleting one term and replacing it with \(g_1\) always produces \((y - (\zeta + 1)x) (y - (\zeta + 2)x) \cdots (y - (\zeta + (p - 1)x))\) once \(g_1\) is moved all the way to the right of the terms involving \(\zeta\). This yields \(p\) identical summands for each \(\zeta \in \Xi\) whose sum is 0.

Similarly, applying the Leibniz rule to compute \([g_1, C]\) has the effect of picking out one term \((y - (\zeta + a)x)\) at a time and replacing it with \(xg_1\). For a fixed \(\zeta \in \Xi\), this produces \(\Pi_{\zeta \in \Xi} \Pi_{a} (y - (\zeta' + a)x) \left( \sum_{b \in \mathbb{F}_p} b \left( \Pi_{a=1}^{p-1} (y - (\zeta + a)x) \right) xg_1 \right) \Pi_{a' > \zeta} \Pi_{b} (y - (\zeta'' + a)x) = 0.\)

**Corollary 5.8.** When \(\lambda = g_1\), we have \(C = y^t - y^{\partial^i_{(g_1)}}\). Moreover, either \(C\) is the third generator of the center \(Z_{g_1}\), or there is no such third generator of degree \(q\).
Proof. We have seen that $C$ commutes with $y$ and $g_1$; also, the top degree term of $C$ is $y^d - yx^{d-1}$. By Lemma 5.3, $C = y^d - y \frac{\delta^i(g)}{\delta^i(g)}$. On the other hand, suppose a generator $\tilde{C}$ of the center $\mathbb{Z}_g$ exists with $\tilde{C}$ of degree $q$. Then $\tilde{C}$ maps to $y^d - yx^{d-1}$ in the associated graded map $\text{gr} : H_g \to H_0$ (the map chops off all lower order terms). Since $\tilde{C}$ is central, it commutes with $y$ and $g_1$. It follows from Lemma 5.3 that $C = \tilde{C}$. 

Our aim now is to show that $C$ is indeed central, and so we must show that $C$ commutes with the other generators $g_i$ of $E$.

Example 5.9. Let $p = 3$ and $r = 2$, and let $C = \prod_{a=1}^3 \prod_{b=1}^3 (y - (a\xi + b)x)$ where $1, \xi$ are an $\mathbb{F}_3$-basis for $\mathbb{F}_9$ over $\mathbb{F}_3$. Then $C \in \mathbb{Z}_g$.

Proof. Fix $a$. A calculation shows that $\prod_{b=1}^3 (y - (a\xi + b)x) = y^3 - yx^2 - yg - \prod_{b=1}^3 (a\xi + b)x^3$. Set $Z = y^3 - yx^2 - yg$ and note that $Z = y^3 - yA$ where $A = x^2 - 2y\delta(g)z$ is the quadratic central element of $H_g$. Since $x^3 \in \mathbb{Z}_g$ we see that each of these 3 subproducts of $C$ commute with each other: $C = \left(Z - \prod_{b=1}^3 (\xi + b)x^3\right)\left(Z - \prod_{b=1}^3 (2\xi + b)x^3\right)Z$. The effect of multiplication by $g_2$ is to cyclically permute these three factors. Since they commute, this preserves $C$: $g_2C = \prod_{a\in\mathbb{F}_p} \prod_{b=1}^3 \left(y - ((a-1)\xi + b)x\right)g_2 = Z\left(Z - \prod_{b=1}^3 (\xi + b)x^3\right)\left(Z - \prod_{b=1}^3 (2\xi + b)x^3\right)g_2 = Cg_2$. □

The example indicates that the $p^r-1$ products $Y_{\xi} := \prod_{a=1}^p (y - (\xi + a)x)$ should commute with one another for any $p$. Set $Y := \prod_{a=1}^p (y - ax)$. Since $[x, y] = g_1$, $Y$ is simply the element $y^p - \frac{\delta^i(g)}{\delta^i(g)} = y^p - y(x^2 - 2g)z$. On the one hand, the effect of $g_i$ on $C$ is to permute the $Y_{\xi}$; for any $g_\xi = \left(\begin{array}{c} 1 \\ \xi \\ 1 \end{array}\right) \in E, g_\xi \cdot Y = Y_{\xi}$. On the other hand, $g_\xi \cdot Y = g_\xi \cdot \left(y^p - y \frac{\delta^i(g)}{\delta^i(g)}\right) = (y + \xi x)^{p^2} - (y + \xi x)(x^2 - 2g)^{p^2-1}$ and thus $Y_{\xi} = \prod_{a\in\mathbb{F}_p} (y - (a - \xi)x) = (y + \xi x)^p - (y + \xi x)^2$. 

Proposition 5.10. We have $Y_{\xi} = Y - \prod_{a\in\mathbb{F}_p} (a - \xi)x^p = Y + (\xi^p - \xi)x^p$.

Proof. Let $A_{k,j}$ be the coefficient of $x^jg^k$ in $\delta^{n-2k}(g)$ in the algebra $H_g$ when $E = G = < g >$. We claim that the $\mathbb{F}_q$-coefficient of $y^{n-2k}x^{\ell}g^k$ in $(y + \xi x)^n$ is given by $\left(\begin{array}{c} n \\ 2k + j \end{array}\right)T_{k,j}$ where $T_{k,j}$ is defined by the linear recursive formula $T_{0,0} = 1$, $T_{k,j} = (\xi + k)T_{k-1,j-1} + (j + 1)T_{k-1,j+1}$, where $T_{k,j} = 0$ if $k < 0$ or $j < 0$. The coefficients $T_{k,j}$ form a square array that’s a variation on the André numbers $A_{k,j}$, the coefficients of André polynomials. The formula is easily checked by writing $(\xi x + y)^n = (\xi x + y)(\xi x + y)^{n-1}$ and expanding out $(\xi x + y)^{n-1}$ according to the formula by induction. Thus $Y_{\xi}$ will have the desired form if and only if (1) $T_{0,p}$, the coefficient of $x^p$, is $\xi^p$, and (2) for each $k \geq 1$, $T_{k,p-2k} = \xi A_{k,p-2j}$ where $A_{k,j}$ is the $k, j$th André number (mod $p$).

For any $j \geq 0$, it follows immediately from the recursive formula that $T_{0,j} = \xi^j$, proving (1) holds. We prove that (2) holds by induction. First, assume $k = 1$: by induction on $j$, $T_{1,j} = \left(\begin{array}{c} j+2 \\ 2 \end{array}\right)\xi^{j+1} + \left(\begin{array}{c} j+2 \\ 3 \end{array}\right)\xi^j + \ldots + \left(\begin{array}{c} j+2 \\ j+1 \end{array}\right)\xi^2 + \xi$. Therefore, $T_{1,p-2} = \left(\begin{array}{c} p \\ 2 \end{array}\right)\xi^{p-1} + \ldots + \left(\begin{array}{c} p \\ p-1 \end{array}\right)\xi^2 + \xi = \xi^p$ (mod $p$). Since $A_{1,p-2} = 1$ mod $p$ this establishes (2) for $k = 1$. For the induction step, assume that $T_{k,p-2k} = \xi A_{k,p-2k}$. Then consider the element in the sequence whose incoming arrows come from $T_{k,p-2k}$ and $T_{k+1,p-2k-2}$, namely $T_{k+1,p-2k-1}$:

$$T_{k+1,p-2k-1} = (\xi + k + 1)T_{k+1,p-2k-2} - 2kT_{k,p-2k} = (\xi + k + 1)T_{k+1,p-2k-2} - 2k\xi T_{k,p-2k}.$$
The next ingredient is a quadratic recursive formula for $T_{k,j}$ provable by induction:

$$T_{k,j} = \zeta T_{k,j-1} + \zeta \sum_{l=0}^{j-1} \sum_{m=0}^{k-1} \binom{2k+j-1}{2l+m} T_{l,m} A_{k-1-l,j-m}$$

When $2k+j-1 = p$, all but one of the binomial coefficients in the sum vanishes, yielding $T_{k,p-2k+1} = \zeta T_{k,p-2k-2} + \zeta A_{k-1,p-2k+1}$. Replacing $k$ with $k+1$, we have $T_{k+1,p-2k-1} = \zeta T_{k+1,p-2k-2} + \zeta A_{k,p-2k}$. Setting the two expressions for $T_{k+1,p-2k-1}$ equal to each other, we have $(\zeta + k + 1)T_{k+1,p-2k-2} - 2\zeta k T_{k,p-2k} = \zeta T_{k+1,p-2k-2} + \zeta A_{k,p-2k}$. Solving for $T_{k+1,p-2k-2}$, we then get $T_{k+1,p-2k-2} = \frac{\zeta}{k+1} A_{k,p-2k} = \zeta A_{k+1,k-2k-2}$. The last equality uses the fact that $A_{k,p-2k} = \frac{(k-1)!}{k!}$ mod $p$ for each $A_{k,p-2k}$ along the $k$th knight's-move antidiagonal of the sequence. That numerical formula can be proven on its own by comparing the two recursive definitions of $A_{k,j}$ and using induction on $k$. It also follows, after reindexing, from the formula $g \partial_g(C_{p,k}) = -(k+1)\lambda C_{p,k+2}$.

\begin{proof}
Applying the isomorphism of Lemma 5.4 gives an analogous formula for $H_g$, where $[x,y] = g_x$.
\end{proof}

\begin{corollary}
We have $Z_{g_r} \cong \frac{k[X,Y,Z]}{(x^2-2g \partial_g^{-1}(g_x), Y = x^p, Z = \prod_{i=1}^g (y - z(\zeta + a)x))}$.  
\end{corollary}

### 5.3. Simple representations over $H_g$ when $E=\mathbb{Z}/p$, $r > 1$

Define the left $H_g$-module $\Delta_0 := H_g/(H_g x + \sum_{i=1}^g H_g(g_i - 1))$. It is infinite-dimensional and has $k$-basis $1, y, y^2, ...$.

\begin{proposition}
For each $\beta \in k$, $Y - \beta = y^p - y(x^2 - 2g \partial_g^{-1}(g_x), Y = x^p)$ generates a maximal left ideal of $\Delta_0$. Therefore, the simple modules over the singular locus are $p$-dimensional.
\end{proposition}

\begin{proof}
First, by a minimality argument, it is clear that the module of any $f \in \Delta_0$. Second, any $g \in E$ acts trivially on $Y - \beta$: $g(Y - \beta) = (Y - \beta)g + cXg$ for some $c \in k$, and $g = 1$ and $x^p = 0$ in $\Delta_0$, so $gY = Y$ in $\Delta_0$. Moreover, $x$ acts by 0 on $Y - \beta$ since $x = g_1 y g_1^{-1} - y$ commutes with $Y - \beta$. By Lemma 4.7, $Y - \beta$ then generates a proper submodule. It holds for $G$ cyclic that $x, y, g_1$ do not preserve any larger submodule of $\Delta_0$, thus also for $E$.

Over the smooth locus of the center, simple modules are $q$-dimensional. They are quotients of Verma modules $\Delta_x := H_x/(H_x (x - \alpha) + \sum_{i=1}^g H_x(g_i - 1))$, whose simple quotient $\Delta_x^\alpha$ realize those simple module on which $x^p$ acts by $x^p$. Then $x^p \in k^\alpha$.

\begin{proposition}
The element $C - \beta = \prod_{i=1}^g Y_{g_i} - \beta$ generates a maximal submodule of $\Delta_x$. Thus, the dimension of the simple modules over the smooth locus is $q$.
\end{proposition}

\begin{proof}
By repeating the arguments in the case $E = G$, it is clear that a maximal submodule $M \subset \Delta_x$ must be cyclically generated by an element of degree a power of $p$; moreover, for $g_1$ to fix the generator it must be of the form $f = (Y - \beta_1)(Y - \beta_2) \cdots (Y - \beta_d) - \beta$. Thus the degree of $f$ is $dp$ and it may be supposed this is the minimal degree of any element of $M$. Then if $\eta \not\in \mathbb{F}_p$, $\eta^p - \eta \neq 0$ and $g_0 \cdot f = (Y - \beta_1 - (\eta^p - \eta) \alpha\eta)(Y - \beta_2 - (\eta^p - \eta) \alpha\eta) \cdots (Y - \beta_d - (\eta^p - \eta) \alpha\eta)$ (note that $\alpha\eta$ may be replaced with its image $\alpha\eta$ in $\Delta_x$ because it is central). Then either $g_0 \cdot f = f$ in $\Delta_x$
or \( f - g_0 \cdot f \in M \) is a polynomial of smaller degree, contradicting minimality. But the former is only possible if for every \( i, \beta_i = \beta_j - (\eta^p - \eta)x^p \) for some \( j \). As this is true for every \( \eta, d = p^{r-1} \) and the \( \beta_i \) must range over all \( \eta \) in the \( \mathbb{F}_p \)-span of \( \xi_i \), \( i = 1, \ldots, r - 1 \). It follows that

\[
 f = \prod_{\eta} Y_{\eta} - \beta = C - \beta. \tag*{\square}
\]

**Remark 5.15.** To summarize, simple modules over \( H_\lambda \) are parametrized by pairs \( \{(\alpha, \beta)\} \in k^2 \). The simple modules over the smooth locus of the center correspond to pairs with \( \alpha \neq 0 \); in this case, the simple modules are \( q \)-dimensional. When \( \alpha = 0 \) the corresponding simple modules lie above the singular locus of the center and have dimension \( p \). Thus when \( q = p^r \) with \( r > 1 \) the Azumaya locus and the smooth locus coincide for \( H_\lambda \), unlike what happens when \( q = p \).

### 5.4. The center of \( H_\lambda \) when \( \lambda \in kE \)

**Theorem 5.16.** Let \( E = (\mathbb{Z}/p)^r \subset SL(2, \mathbb{F}_q) \), \( q = p^r \), \( g_i \) the generator of the \( i \)th copy of \( G \), \( \lambda = \sum_{i=1}^r \sum_{j=0}^{p-1} c_i g_i^j \in kE \), and \( H_\lambda = \frac{k[A, B, C]}{\langle A^p - B^r \rangle} \). Scale \( \lambda \) so that \( c_{0,0} \), the coefficient of 1 in \( \lambda \), is either 0 or 1, and let \( Z_\lambda \) denote the center of \( H_\lambda \). Then:

- \( (t = 0 \text{ case}) \). When \( c_{0,0} = 0 \) then \( Z_\lambda \) is generated by \( A := x^2 - 2D^{r-1}(\lambda), B := x^p, C := y^q - y^{\delta^a(g_i)} \), which have degrees 2, \( p \), and \( q \), respectively, and \( Z_\lambda \cong \frac{k[A, B, C]}{\langle A^p - B^r \rangle} \).
- \( (t = 1 \text{ case}) \). When \( c_{0,0} = 0 \), then \( Z_\lambda \) is generated by \( B = x^p \) and \( C := \prod_{a \in \mathbb{F}_p} (C - ax^q) \) which have degrees \( p \) and \( pq \), respectively, where \( C \) is as in the \( t = 0 \) case. Then \( Z_\lambda \cong k[B, C] \).

**Proof.** This follows quite easily from the combinatorics of \( \delta \). Take \( \xi \) any generator of \( \mathbb{F}_q^\times \), \( \xi_i = \xi^{i-1} \) a basis for \( \mathbb{F}_q \) over \( \mathbb{F}_p \), and \( g_i = \left( \begin{array}{cc} 1 & \xi_i \\ 0 & 1 \end{array} \right) \). Set \( D := \sum_{i=1}^r \xi_i g_i D_i \). Then the \( r \) trees for \( \delta^a(g_i) \) all look almost identical to that for \( \delta^a(g_i) \) in the cyclic case when polynomials in powers \( D^m(\lambda) \) are translated into partitions and \( g_i \) is not written in the tree, except that each partition is “homogenized” by some power of \( \xi_i \), so that \( D^j(\lambda) \) has degree \( j + 1 \) and \( \xi_i \) has degree 1 then every partition in \( \delta^a(g_i) \) has degree \( n \) as a polynomial in \( \xi_i, \lambda, D(\lambda), ..., D^{r-2}(\lambda) \) for \( n \) up to \( q \). As for \( \delta^a(\lambda) \), its tree will be identical to what it was in the cyclic case, so long as the sizes of parts in the partitions are not reduced mod \( p - 1 \) but are allowed to grow.

The coefficients \( C_{p^r,k} \) at all but the rightmost node \( C_{p^r,p^r} \) in \( \delta^a(g_i) \), the \( p \)th row of the tree for \( \delta^a(g_i) \), which survive mod \( p \) belong to those partitions whose parts sum to the maximal possible number: \( p^r - 1 \). Thus all \( C_{p^r,k} \), \( k < p \) are divisible by the same power of \( \xi_i \), namely \( \xi_i^p \) implying \( D(C_{p^r,k}) = \xi_i D^p \xi_i^{k+2} \). It is easy to see that \( C_{p^r, p^r} = \xi_i D^p \xi_i^{p^r} \) and \( C_{p^r, p^r} = \xi_i^p \) and thus \( \delta \left( \frac{\delta^a(g_i)}{\delta^a(g_i)} \right) = (D^{p^r-1}(\lambda) - \xi_i^{p^r-1} \lambda) x^{p^r} \). It follows that in the \( t = 0 \) case,

\[
 \delta \left( \frac{\delta^a(g_i)}{\delta^a(g_i)} \right) = (D^{p^r-1}(\lambda) - \lambda) x^{p^r} = 0 \quad \text{while in the } \ t = 1 \text{ case,} \\
 \delta \left( \frac{\delta^a(g_i)}{\delta^a(g_i)} \right) = (D^{p^r-1}(\lambda) - \lambda) x^{p^r} = -x^q. \]

And so, in the \( t = 0 \) case, \( C = y^q - y^{\delta^a(g_i)} \) is central. In the \( t = 1 \) case, \( C, x^q, \) and \( y \) generate a subalgebra isomorphic to \( k[\tilde{x}, \tilde{y}] \) with \( \tilde{y} = C, \tilde{x} = x^q, \) and \( \tilde{g} = y, \) so \( \prod_{a \in \mathbb{F}_p} (C - ax^q) \in Z_\lambda. \) \( \square \)

### 5.5. Verma modules and simple modules for \( H_\lambda, \lambda \in kE \)

Form the left Verma modules \( \Delta_x = H_\lambda /(H_\lambda (x - z) + \sum_{i=1}^r H_\lambda (g_i - 1)) \) for each \( x \in k \). The spectrum of the center of \( H_\lambda \) is singular over \( x = 0 \) when \( \lambda \) has no constant term, while in the \( t = 1 \)
case $Z_\epsilon$ is smooth everywhere since then $Z_\epsilon$ is a polynomial ring. Nonetheless even in the $t=1$ case there is a difference between the sizes of simple quotients of $\Delta_0$ and those of $\Delta_x$ for $x \neq 0$.

Recall that $\Delta_x$ is the fiber over $x=\alpha$ of the projection $\text{Mod} H_\epsilon \to \text{Spec} R, R = k[x] \otimes kE$. The 0-fiber $\Delta_0$ coincides with the singular locus when $t=0$. $\Delta_x$ is cyclic and isomorphic to $k[y]$ as a $k$-vector space. We look for maximal submodules of $\Delta_x$. Clearly the big central element generates a submodule, and this bounds the degrees of simples by the degree of the big central element: $q$ when $t=0$ and $pq$ when $t=1$. These upper bounds are realized over the generic fiber $\Delta_x$.

**Theorem 5.17.** Consider $H_\epsilon$ for $E$. In both the $t=0$ and $t=1$ cases, simple $H_\epsilon$-modules over the 0-fiber are $p$-dimensional if $\lambda \notin \text{Rad} kE$, and 1-dimensional if $\lambda \in \text{Rad} kE$. Simple modules over the generic fiber (that is, simple quotients of $\Delta_x$ for $x \neq 0$) have dimension $q$ when $t=0$ and dimension $pq$ when $t=1$.

**Proof.** Consider $\Delta_0$ when $\lambda \in \text{Rad} kE$. Then for any $\beta \in k$, $g_\beta$ fixes $y-\beta$ for each $i$ and $x$ annihilates $y-\beta$ so $y-\beta$ generates a maximal submodule of $\Delta_0$; this is true in both the $t=0$ and $t=1$ cases.

By the arguments we’ve made previously in this type of theorem, a maximal submodule needs to be generated by an element of degree a power of $p$. Set $Y = y^{p} - y^{\frac{\partial^p (g_i)}{\partial (g_i)}}$. By the results of Section 5.4, $[y, Y] = y(D^{p-1}(\lambda) - \lambda)x^p$ and thus $x \cdot Y = Yx + [x, Y] = Yx + [g_\beta]y^{p-1} - y, Y] = Yx - x^{p+1}(D^{p-1}(\lambda) - \lambda)|_{g_\beta=1}$. Thus $x \cdot Y$ is a scalar multiple of $Y$ if $x = 0$. Furthermore, recursion relations imply that $\partial^p (g_i) - \xi^p g_i x^p = \xi (\partial^p (g_i) - x^p g_i)|_{g_\beta=1}$. Therefore $[g_\beta, Y] = \partial^p (g_i) - \partial (g_i) = \xi (\partial^p (g_i) - \partial (g_i)) = \xi (\partial (g_i) - \partial (g_i))$. It follows that each $g_i$ fixes $Y$ in $\Delta_0$. Then by Lemma 4.7, $Y$, and more generally $Y - \beta$ for any $\beta \in k$, generates a proper submodule if $x = 0$. This proves the statement in both the $t=1$ and $t=0$ cases for the dimensions of simple modules over the 0-fiber.

Suppose $x \neq 0$. From the computation of the action of $g_i$, above, it is evident that $g_i$ does not fix $Y$ when $x \neq 0$, since $\xi^p - \xi \neq 0$. By the same argument given in Proposition 5.14, $g_i$ does not fix any polynomial in $Y$ of degree less than $p^*$. Same thing if $Y$ is replaced by $y^{p} - y^{\frac{\partial^p (g_i)}{\partial (g_i)}}$. On the other hand, we know that in the $t=0$ case, the image of $C = y^{p} - y^{\frac{\partial^p (g_i)}{\partial (g_i)}}$ in $\Delta_x$ generates a submodule. So it must be maximal, and the simples over the smooth locus are $q$-dimensional in the $t=0$ case. In the $t=1$ case, $g_{1} \cdot C = C - xy^{p} = C - x^{2}y^{p}$, so $C$ does not generate a proper submodule. Likewise, argue by downwards induction on degree in $y$ that $g_{1}$ will not fix $f(C)$ if $f \in k[C]$ is of degree less than $p$.

**Appendix A: Combinatorics of certain derivations in two variables**

**Powers of $\delta = f(x, g)^{\partial / \partial x} + xg^{\partial / \partial x}$ on $g$ in differential operator extensions of $kG[x]$.**

The symplectic-reflection-type algebras $H_{1}$, viewed instead as Ore extensions over their commutative subring $R = k[x] \otimes kG$, fit into a larger family of differential operator Ore extensions of $R$. Namely, we may consider what happens more generally when $k = \mathbb{F}_p$, $G$ is the cyclic group of order $p$, and the derivation $\delta$ takes the form $\delta = f(x, g)^{\partial / \partial x} + xg^{\partial / \partial x}$ with $f(x, g) \in R$. Specifying $xg^{\partial / \partial x}$ in the formula for $\delta$ means that these algebras are all quotients of $k(x, y) \rtimes G$, so that in all the noncommutative algebras produced by such $\delta$, the group action is preserved.

**Lemma A1.** Let $\delta = f^{\partial / \partial x} + xg^{\partial / \partial x}$ where $f \in k[x, g]/(g^{p} - 1)$. Then for any $n \geq 1$, $g$ divides $\delta^n (g)$ and $\delta^n (g) = x^n g + x^{n-2} \delta \left( \frac{\delta (g)}{x} \right) g + x^{n-3} \delta \left( \frac{\delta^2 (g)}{x} \right) g + \ldots + x \delta \left( \frac{\delta^{n-1} (g)}{x} \right) g + \delta \left( \frac{\delta^n (g)}{x} \right) g$. Here is the beginning of the sequence $\delta^n (g)$ written in terms of $\delta$-powers of $f$ and ordinary powers of $x$. 


We visualize the powers of delta in a recurrence diagram. The \( n \)th row corresponds to \( \delta^n \) and the \( m \)th column corresponds to the coefficient of \( x^{mg} \) as a polynomial in \( f, \delta(f), \delta^2(f), \ldots \). The arrows indicate an operation by the label above the arrow applied to the node at the tail of the arrow. The outputs of arrows with the same target are added resulting in the node at their heads.

**Notation A2.** Write \( x := 2^{a_2}3^{a_3}4^{a_4} \cdots (j+2)^{a_{j+2}} \) as shorthand for the monomial \( f^{a_2}\delta(f)^{a_3}\delta^2(f)^{a_4}\cdots\delta^j(f)^{a_{j+2}} \). We write \( |x| = 2a_2 + \ldots + (j+2) a_{j+2} \), and write \( x \vdash |x| \) to say \( x \) is a partition of \( |x| \). The exponent \( a_j \) in a partition denotes that part \( j \) being repeated \( a_j \) times; for example, \( 2^33^4 \) denotes the partition \( 2, 2, 2, 3, 4, 4, 4, 4 \). Note that in this section we are using the convention that partitions are non-decreasing instead of non-increasing (Figure A1).
**Proposition A3.** It holds that \( \delta^n(g) = \delta^{n-2}(f)g + x^fg = \delta^{n-1}(x)g + x^fg \).

**Proof.** Let \( F_j \) be the \( kG[x] \)-sequence of polynomials in \( f, \delta(f), \delta^2(f), \ldots \) that appears along the left vertical edge of the diagram above, so \( F_0 = f, F_1 = \delta(f), F_2 = \delta^2(f) + 3f^2, \ldots \). We have \( \delta^n(g) = x^fg + \binom{n}{2} F_2x^{n-2}g + \binom{n}{3} F_3x^{n-3}g + \binom{n}{4} F_4x^{n-4}g + \ldots \). Since all binomial coefficients save \( \binom{p}{p} \) and \( \binom{0}{0} \) vanish mod \( p \), it follows that \( \delta^n(g) = \delta^{n-2}(f)g + x^fg \). The sequence \( F_n \) satisfies the recursion \( F_n = \delta(F_{n-1}) + (n-1)fF_{n-2} \).

The partitions that appear in \( F_n \) are all the partitions of \( n \) with \( k-1 \) parts, for all \( k > 0 \) such that \( n - 2k \geq 0 \). This follows by induction on \( n \) together with the recursion formula for \( F_n \); \( \delta \) raises \( |x| \) by 1 while multiplication by \( f \) corresponds increases the number of parts of \( x \) equal to 2 by one. Moreover, the sums \( S(n) := (\text{sum of the coefficients of } F_n) \) form the sequence A000296: \( S_n \) counts the number of ways \( n \) people can arrange themselves into cliques, where a clique contains at least two people. Write \( F_n \) as the linear combination of the partitions associated to the monomials appearing in \( F_n \). The coefficient of \( x = (2^{a_1}, ..., n^{a_n}) \) in \( n \) counts the number of ways \( n \) people can be divided up into \( a_2 \) cliques of size 2, \( a_3 \) cliques of size 3, \ldots. We obtain:

\[
F_n = \sum_{\lambda | n} \binom{n}{a_1 \geq 0, a_2 > 0, \ldots} \cdot \left( \frac{2^{a_1}!3^{a_2}! \cdots n^{a_n}!}{(a_2!a_3! \cdots a_n)!} \right) (2^{a_1}, 3^{a_2}, \ldots, n^{a_n})
\]

If \( n = p \) is prime then all but one of the coefficients is divisible by \( p \), yielding \( F_n \equiv \delta^{n-2}(\lambda) \equiv \delta^{n-1}(x) \) mod \( p \). \( \square \)

We owe the idea to use partitions as a shorthand for polynomials in \( \delta \)-powers of \( f \) to a paper on Ore extensions of \( k[x] \) [3], and our formulas are modeled on theirs. It is interesting to compare the combinatorics of \( \delta = h(x) \partial_x \) for Ore extensions \( k[x; \partial] \) with the combinatorics of \( \delta = f(x, g) \partial_x + xg \partial_x \) for Ore extensions of \( R[y; \partial] \) as above. For instance, in the table of coefficients following Corollary 9.5 in [3], some of the numbers appearing are familiar from the case of \( \lambda = g \): the sequence composed of the second entry from the right in each row of their table is the sequence of Eulerian numbers A000295, which is the second row of the diagram of \( \lambda \) numbers. The sequence composed of the third entry from the right in each row of their table is the sequence of reduced tangent numbers A0002105, which is the leftmost column of the diagram of \( \lambda \) numbers. Moreover, the sum of the entries in the \( n \)th row of the table in [3] is \( (n-1)! \) [3], Proposition 9.7; by comparison, when \( \lambda = g^2 \) then the sum of the coefficients of \( \delta^n(g) \) for \( \delta = \lambda \partial_\lambda + xg \partial_x \) is \( n! \) (Figure A2).

**Acknowledgements**

I would like to thank Victor Ginzburg for suggesting the problem and for useful conversations, and the anonymous referee for suggesting many improvements to the text.

**References**

[1] Balagovic, M., Chen, H. (2013). Representations of rational Cherednik algebras in positive characteristic. *J. Pure Appl. Algebra* 217(4):716–740. DOI: 10.1016/j.jpaa.2012.09.015.

[2] Bellamy, G. (2016). Symplectic reflection algebras. In *Noncommutative Algebraic Geometry*. Math. Sci. Res. Inst. Publ., 64, New York: Cambridge University Press, pp. 167–238.

[3] Benkart, G., Lopes, S. A., Ondrus, M. (2013). A parametric family of subalgebras of the Weyl algebra II. Irreducible modules. In *Recent Developments in Algebraic and Combinatorial Aspects of Representation Theory*. Contemp. Math., 602, Amer. Math. Soc., Providence, RI, pp. 73–98.

[4] Bergman, G. M. (1978). The diamond lemma for ring theory. *Adv. Math.* 29(2):178–218. DOI: 10.1016/0001-8708(78)90010-5.

[5] Etingof, P., Ginzburg, V. (2002). Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.* 147(2):243–348. DOI: 10.1007/s002220010171.

[6] Foata, D., Han, G. (2001). Arbres minimaux et polynômes d’André. *Adv. Appl. Math.* 27(2–3):367–389. DOI: 10.1006/aama.2001.0740.

[7] Foata, D., Schützenberger, M.-P. (1971). Nombres d’Euler et permutations alternantes. Technical Report. University of Florida.

[8] Goodearl, K. R., Warfield, R. B. Jr. (2004). *An Introduction to Noncommutative Noetherian Rings, Volume 61* of London Mathematical Society Student Texts. 2nd ed. Cambridge: Cambridge University Press.

[9] Gordon, I., Smith, S. F. (2004). Representations of symplectic reflection algebras and resolutions of deformations of symplectic quotient singularities. *Math. Ann.* 330(1):185–200. DOI: 10.1007/s00208-004-0545-y.
[10] Griffeth, S. (2010). Towards a combinatorial representation theory for the rational Cherednik algebra of type $G(r,p,n)$. *Proc. Edinb. Math. Soc.* 53(2):419–445. DOI: 10.1017/S0013091508000904.

[11] Kamal, A. A. M. (1992). Idempotents in polynomial rings. *Acta Math. Hung.* 59(3–4):355–363. DOI: 10.1007/BF00050898.

[12] Lando, S. K. (2003). *Lectures on Generating Functions, Volume 23 of Student Mathematical Library.* Providence, RI: American Mathematical Society. Translated from the 2002 Russian original by the author.

[13] Shepler, A., Witherspoon, S. (2012). Drinfeld Orbifold algebras. *Pacific J. Math.* 259(1):161–193. DOI: 10.2140/pjm.2012.259.161.

[14] Shepler, A., Witherspoon, S. (2015). PBW deformations of skew group algebras in positive characteristic. *Algebr. Represent. Theor.* 18(1):257–280. DOI: 10.1007/s10468-014-9492-9.