On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes

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Abstract: In this paper, we study the phase structure and equilibrium state space geometry of charged topological dilaton black holes in $(n + 1)$-dimensional anti-de Sitter spacetime. By considering the pairs of parameters $(P \sim V)$ and $(Q \sim U)$ as variables, we analyze the phase structure and critical phenomena of black holes and discuss the relation between the two kinds of critical phenomena. We find that the phase structures and critical phenomena drastically depend on the cosmological constant $l$ (or the static electric charge $Q$ of the black holes), dimensionality $n$ and dilaton field $\Phi$. 
1. Introduction

Black hole physics is a subject at the intersection of general relativity, quantum mechanics and statistical physics and field theory. This makes the subject receive a lot of attention. Black holes have been used as the laboratory of many kinds of theories, specially the thermodynamics of black holes plays an important role\cite{1, 2, 3, 4, 5, 6}. The thermodynamic properties of black holes have been studied for several years, although the exact statistical explanation of black hole thermodynamics is still lacked.

It shows that black holes also have the standard thermodynamic quantities, such as temperature, entropy, even possess abundant phase structures like hawking-Page phase transition\cite{7} and the critical phenomena similar to ones in the ordinary thermodynamic system. What is more interesting is the research on charged, non-rotating RN-AdS black hole, which shows that there exists phase transition similar to the van der Waals-Maxwell vapor-liquid phase transition\cite{8, 9, 10}.

Motivated by the AdS/CFT correspondence\cite{11}, where the transitions have been related with the holographic superconductivity\cite{12, 13}, the subject that the phase transitions of black holes in asymptotically anti-de-Sitter (AdS) spacetime, has received considerable attention\cite{14, 15, 16, 17, 18}. The underlying microscopic statistical interaction of the black holes is also expected to be understood via the study of the gauge theory living on the boundary in the gauge/gravity duality.

The studies\cite{19, 20, 21, 22, 23, 24, 25, 26} on the phase transition and critical phenomena of black holes in AdS spacetime indicate the black holes are similar to the van der Waals vapor-liquid system. The $(Q \sim U)$ (where $Q$ is the static electric charge, $U$ is the electrostatic potential on the horizon) phase diagram of black holes in AdS spacetime is almost the same as the $(P \sim V)$ phase diagram in van der Waals vapor-liquid system.
Among the gravity theories with higher derivative curvature terms, the Gauss-Bonnet (GB) gravity has some special features and gives rise to some interesting effects on the thermodynamics of black holes in AdS space \[27, 28, 29, 30, 31, 32\]. The phase structure of a GB-AdS black hole was briefly studied in \[27, 33\]. And in the grand canonical ensemble, the local and global thermal phase structure of a charged asymptotically AdS black hole with both GB and quartic field strength corrections were thoroughly researched \[34\]. In \[35\] the phase transition and critical phenomena of d-dimensional charged GB-AdS black hole is analyzed. The consequence show that the phase structure and critical temperature, critical electric charge, critical electrostatic potential of the black hole all depend on the cosmological constant \(\Lambda\) and the dimension of spacetime. The \((Q \sim U)\) critical conditions of d-dimensional charged GB-AdS black hole agree with the \((P \sim V)\) ones in van der Waals vapor-liquid system.

Recently, many interests focus on the studies of critical behaviors of AdS black holes \[36, 37, 38, 39\] by considering cosmological constant as the thermal pressure

\[
P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi} \frac{1}{l^2},
\]

and corresponding conjugate thermal volume as

\[
V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,J_k}.
\]

The complete analog model of vapor-liquid system for black holes is established in \[36\]. In \[40\] the relation (1.1) in higher dimensional spherically symmetric AdS spacetime first proposed which lays the foundation for the research of black holes thermodynamics.

Theoretically, if regarding the black holes in AdS spacetime as thermodynamic systems the corresponding critical behaviors and phase transition should exist. However, until now the statistical explanation of black hole thermodynamics is still lack. Therefore it is a meaningful work to discuss the relations of thermodynamic properties for all kinds of black holes in AdS spacetime. This may help to recognize further black hole entropy, temperature, heat capacity and may help to improve the geometric theory of black holes thermodynamics.

A scalar field called the dilaton appears in the low energy limit of string theory. The presence of the dilaton field has important consequences on the causal structure and the thermodynamic properties of black holes. Thus much interest has been focused on the study of the dilaton black holes in recent years \[41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55\].

In this paper we study the phase transition and critical behaviors of \((n+1)\) dimensional charged topological dilaton AdS black hole. Firstly we consider the cosmological constant as thermodynamic pressure and the conjugate thermodynamic volume. We find that the phase structure and critical phenomena are dependent on the static electrostatic charge \(Q\), dimension \(n\) and dilaton field \(\Phi\). The critical exponents are the same as the ones in van der Waals vapor-liquid system. Secondly we consider the pair of conjugate parameters \((Q \sim U)\) as the thermodynamic variables and study the phase transition and critical behaviors of \((n+1)\) dimensional charged topological dilaton AdS black hole again. The results show that
the phase structure and critical phenomena are dependent on the cosmological constant $\Lambda$, dimension $n$ and dilaton field $\Phi$. The critical exponents are also the same as the ones in van der Waals vapor-liquid system. Thus the two approaches are equivalent because of the same phase diagrams and critical behavior.

The paper is arranged as follows: in the next section we simply introduce the $(n+1)$-dimensional charge Dilaton AdS black hole. In section 3 we will consider the parameters $(P \sim V)$ and $(Q \sim U)$ respectively and discuss the phase structure and critical phenomena of black holes. We will make some concluding remarks in section 4. (we use the units $G_{n+1} = \hbar = k_B = c = 1$)

2. Charged Dilaton Black Holes in Anti-de Sitter Space

The Einstein-Maxwell-Dilaton action in $(n + 1)$-dimensional $(n \geq 3)$ spacetime is [49,50]

$$S = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left( R - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) - e^{-4\alpha\Phi/(n-1)} F_{\mu\nu} F^{\mu\nu} \right), \quad (2.1)$$

where the dilaton potential is expressed in terms of the dilaton field and its coupling to the cosmological constant:

$$V^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{\alpha}{2} e^{-4\alpha\Phi/(n-1)} F_{\lambda\eta} F^{\lambda\eta}, \quad (2.2)$$

$$\nabla_\mu \left( e^{-4\alpha\Phi/(n-1)} F^{\mu\nu} \right) = 0, \quad (2.3)$$

where $R$ is the Ricci scalar curvature, $\Phi$ is the dilaton field and $V(\Phi)$ is a potential for $\Phi$, $\alpha$ is a constant determining the strength of coupling of the scalar and electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $A_\mu$ is the electromagnetic potential.

The topological black hole solutions take the form [38,49,50]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r)d\Omega^2_{k,n-1}, \quad (2.4)$$

where

$$f(r) = -\frac{k(n-2)(\alpha^2 + 1) b^{-2\gamma} r^{2\gamma}}{(\alpha^2 - 1)(\alpha^2 + n - 2)} - \frac{m}{r^{(n-1)(1-\gamma) - 1}} + \frac{2q^2(\alpha^2 + 1) b^{-2(n-2)\gamma}}{(n-1)(\alpha^2 + n - 2)} r^{2(n-2)(\gamma - 1)}, \quad (2.5)$$

$$R(r) = e^{2\alpha\Phi/(n-1)}, \quad \Phi(r) = \frac{(n-1)\alpha}{2(1 + \alpha^2)} \ln \left( \frac{b}{r} \right), \quad (2.6)$$

with $\gamma = \alpha^2/(\alpha^2 + 1)$. The cosmological constant is related to spacetime dimension $n$ by

$$\Lambda = -\frac{n(n-1)}{2l^2}, \quad (2.7)$$
where $b$ is an arbitrary constant and $l$ denotes the AdS length scale. In the above expression, $m$ appears as an integration constant and is related to the ADM (Arnowitt-Deser-Misner) mass of the black hole. According to the definition of mass due to Abbott and Deser [56, 57], the mass of the solution (2.5) is

$$M = \frac{b^{(n-1)\gamma}(n-1)\omega_{n-1}}{16\pi(a^2 + 1)} m.$$  \hfill (2.8)

the electric charge is

$$Q = \frac{q\omega_{n-1}}{4\pi},$$  \hfill (2.9)

where $\omega_{n-1}$ represents the volume of constant curvature hypersurface described by $d\Omega^2_{k,n-1}$. The Hawking temperature of the topological black hole on the outer horizon $r_+$ can be calculated using the relation

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi},$$  \hfill (2.10)

where $\kappa$ is the surface gravity. It can easily show that

$$T = -\frac{(a^2 + 1)}{2\pi(n-1)} \left( \frac{k(n-2)(n-1)b^{-2\gamma}}{2(a^2 - 1)} r_+^{2\gamma - 1} + Ab^{2\gamma}r_+^{1-2\gamma} + q^2b^{-2(n-2)\gamma}r_+^{(2n-3)(\gamma-1)-\gamma} \right)$$

$$= \frac{k(n-2)(a^2 + 1)b^{-2\gamma}}{2\pi(a^2 + n - 2)} r_+^{2\gamma - 1} + \frac{(n - a^2)m}{4\pi(a^2 + 1)} r_+^{(n-1)(\gamma-1)}$$

$$- \frac{q^2(a^2 + 1)b^{-2(n-2)\gamma}}{\pi(a^2 + n - 2)} r_+^{(2n-3)(\gamma-1)-\gamma}. \hfill (2.11)$$

Topological black hole entropy

$$S = \frac{b^{(n-1)\gamma}(n-1)r_+^{(n-1)(1-\gamma)}}{4}.$$  \hfill (2.12)

The electric potential

$$U = \frac{q b^{(3-n)\gamma}}{r_+^{\lambda}},$$  \hfill (2.13)

where $\lambda = (n - 3)(1 - \gamma) + 1$. From $f(r_+) = 0$ and (2.8), we obtain

$$M = \frac{q^2b^{\gamma(3-n)}(a^2 + 1)\omega_{n-1}}{8\pi(a^2 + n - 2)} r_+^{\lambda} - \frac{n(n-1)(a^2 + 1)b^{\gamma(n+1)}\omega_{n-1}}{16\pi l^2(a^2 - n)} r_+^{n-\gamma(n+1)}$$

$$- \frac{k(n-2)(n-1)(a^2 + 1)b^{\gamma(n-3)}\omega_{n-1}}{16\pi(a^2 - 1)(a^2 + n - 2)} r_+^\lambda. \hfill (2.14)$$

One may then regard the parameters $S$, $Q$ and $P$ as a complete set of extensive parameters for the mass $M(S,Q,P)$ and define the intensive parameters conjugate to $S$, $Q$ and $P$. These quantities are the temperature, the electric potential and volume.

$$T = \left( \frac{\partial M}{\partial S} \right)_{Q,P}, \quad U = \left( \frac{\partial M}{\partial Q} \right)_{S,P}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{Q,S}. \hfill (2.15)$$
where
\[
P = \frac{n(n-1)}{16\pi l^2}, \quad V = \frac{(\alpha^2 + 1)b^{\gamma(n+1)}\omega_{n-1}}{(\alpha^2 - n)} r_+^{n-\gamma(n+1)}. \tag{2.16}
\]

It is a matter of straightforward calculation to show that the quantities calculated by Eq. (2.16) for the temperature, and the electric potential coincide with Eqs. (2.11) and (2.13). Thus, the thermodynamics quantities satisfy the first law of thermodynamics
\[
dM = TdS + UdQ + V dp. \tag{2.17}
\]

The thermodynamic quantities above, energy \(M\), entropy \(S\), temperature \(T\), volume \(V\), pressure \(P\), electrostatic potential \(U\) and electric charge \(Q\) satisfy Smarr formula:
\[
M = \frac{(n-1)(1-\gamma)}{\lambda} TS + UQ - \frac{n-\lambda}{\lambda} VP. \tag{2.18}
\]

In what follows we concentrate on analyzing the phase transition of the \((n+1)\)dimensional charged topological dilaton AdS black hole system in the extended phase space while we treat the black hole charge \(Q\) as a fixed external parameter, or the cosmological constant is an invariant parameter, not a thermodynamic variable. We shall find that an even more remarkable coincidence with the Van der Waals fluid is realized in this case.

3. Critical behaviour

3.1 \(Q\) is an invariant parameter

For a fixed charge \(q\), Eq. (2.11) translates into the equation of state for a charged topological dilaton black hole, \(P = P(V, T)\)
\[
P = \frac{T(n-1)b^{-2\gamma}r_+^{2\gamma-1}}{4(\alpha^2 + 1)} + \frac{k(n-1)(n-2)b^{-4\gamma}r_+^{2(\gamma-1)}}{16\pi(\alpha^2 - 1)} + \frac{Q^2 b^{2(1-n)\gamma}2\pi}{\omega_{n-1}^2} r_+^{2(n-1)(\gamma-1)}
\]
\[
= \left(\frac{V(n-\alpha^2)}{(\alpha^2 + 1)\omega_{n-1}b^{\gamma(n+1)}}\right)^{1/(n-\gamma(n+1))}. \tag{3.1}
\]

Where \(P\) and \(V\) are given by (2.10), \(V\) is the thermodynamic volume, given in terms of the event horizon radius \(r_+\), \(T\) is the black hole temperature, and \(Q\) its charge. The Van der Waals equation
\[
(P + \frac{a}{v^3})(v - \bar{b}) = kT, \tag{3.2}
\]

Here, \(v = V/N\) is the specific volume of the fluid, \(P\) its pressure, \(T\) its temperature, and \(k\) is the Boltzmann constant.

Comparing with the Van der Waals equation, (3.2), we conclude that we should identify the specific volume \(v\) of the fluid with the horizon radius of the black hole as
\[
v = \frac{4(\alpha^2 + 1)b^{2\gamma}}{(n-1)} r_+^{1-2\gamma}. \tag{3.3}
\]
In \((n+1)\) dimensions, the equation of \((3.1)\) reads

\[
P = \frac{T}{v} + \frac{k(n-2)(\alpha^2+1)^2}{\pi(n-1)(\alpha^2-1)v^2} + \frac{Q^2b^{2(1-n)\gamma}2\pi}{\omega_{n-1}^2} \left( \frac{v(n-1)}{4(\alpha^2+1)b^{2\gamma}} \right)^{2(n-1)(\gamma-1)/(1-2\gamma)}
\]

\[
= \frac{T}{v} - \frac{A}{v^2} + \frac{B}{v^2(n-1)(1-\gamma)/(1-2\gamma)},
\]

where

\[
A = \frac{k(n-2)(\alpha^2+1)^2}{\pi(n-1)(1-\alpha^2)}, \quad B = \frac{Q^2b^{2(1-n)\gamma}2\pi}{\omega_{n-1}^2} \left( \frac{4(\alpha^2+1)b^{2\gamma}}{(n-1)} \right)^{2(n-1)(\gamma-1)/(1-2\gamma)}.
\]

Critical points occur at points of inflection in the \(P-V\) diagram, where

\[
\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0.
\]

Substituting \((3.4)\) into \((3.6)\) we can derive the critical volume, temperature and pressure:

\[
v_c^{x-2} = \frac{(x-1)xB}{2A}, \quad T_c = \frac{2A}{v_c} - \frac{xB}{v_c^{x-1}} = \frac{2A(x-2)}{(x-1)} \left( \frac{2A}{(x-1)xB} \right)^{1/(x-2)},
\]

\[
P_c = \frac{A}{v_c^2} - \frac{B}{v_c^x}(x-1) = \frac{A(x-2)}{x} \left( \frac{2A}{(x-1)xB} \right)^{2/(x-2)},
\]

where

\[
x = \frac{2(n-1)(1-\gamma)}{(1-2\gamma)}.
\]

Van der Waals equation critical temperature, volume and pressure are:

\[
v_c^{x-2} = \frac{(x-1)xB}{2A}, \quad T_c = \frac{2A}{v_c} - \frac{xB}{v_c^{x-1}} = \frac{2A(x-2)}{(x-1)} \left( \frac{2A}{(x-1)xB} \right)^{1/(x-2)},
\]

\[
T_c = \frac{8a}{27b}, \quad v_c = 3b, \quad P_c = \frac{a}{27b^2}.
\]

From \((3.7)\), \((n+1)\)-dimensional charged topological dilaton black hole correspond to

\[
\tilde{b} = \frac{x}{4(x-1)} \left( \frac{(x-1)xB}{2A} \right)^{1/(x-2)}, \quad a = \frac{27Ax(x-2)}{16(x-1)^2}, \quad \rho_c = \frac{P_cv_c}{T_c} = \frac{x-1}{2x}.
\]

Therefore, the critical temperature, volume and pressure of \((n+1)\)-dimensional charged topological dilaton black hole are

\[
v_c = \frac{4(x-1)}{x} \tilde{b}, \quad T_c = \frac{8a}{27b}, \quad P_c = \frac{a}{27b^2}.
\]
Figure 1: $p - v$ diagram of $(n+1)$-dimensional charge Dilaton AdS black hole in $(n = 3, \alpha = 0)$, $(n = 3, \alpha = 0.6)$ and $(n = 10, \alpha = 0.6)$ respectively. The temperature of isotherms decreases from top to bottom. The three upper dashed lines correspond to the “ideal gas” one-phase behaviour for $T > T_c$, the critical isotherm $T = T_c$ is denoted by the thick solid line, lower (red) solid lines correspond to two-phase state occurring for $T < T_c$. We have set $Q = 1, b = 1, k = 1$. The behaviour for $n > 3$ and $\alpha > 0$ is qualitatively.

Figure 2: The $T_c - \alpha$ diagram shows the influence of dilaton field $\alpha$ on the critical temperature with different spacetime dimensions.

In Fig.1, (a) and (b) show the influence of dilaton field $\alpha$ on the isothermal curves with the same spacetime dimension. (b) and (c) represent the influence of spacetime dimension $n$ on the isothermal curves with the same dilaton field.

To calculate the critical exponent $\alpha$ we consider the entropy $S$, (2.12), as a function of $T$ and $V$. Using (3.1) we have

$$S = S(T, V) = \frac{b^{-\gamma(n-1)/(n-\gamma(n+1))}}{4} \omega^{(1-2\gamma)/(n-\gamma(n+1))} \left( \frac{V(n - \alpha^2)}{(\alpha^2 + 1)} \right)^{(n-1)(1-\gamma)/(n-\gamma(n+1))}.$$  \hspace{1cm} (3.12)

Since this is independent of $T$, we have $C_V = T \left( \frac{\partial S}{\partial T} \right)_V = 0$ and hence $\tilde{\alpha} = 0$. Defining specific
variables
\[ p = \frac{P}{P_c}, \quad v = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}. \] (3.13)

Thus Eq.(3.4) turns into
\[ p = \frac{\tau}{v} \left( \frac{2x}{x - 1} - \frac{1}{v^2} \frac{x}{x - 2} + \frac{1}{v^x} \frac{2}{x - 1} \right), \] (3.14)

One can write Eq.(3.14) in the form of van der Waals
\[ (x - 1)v \left( p + \frac{1}{v^2} \frac{x}{x - 2} \right) - \frac{2}{v^{x - 1}} \frac{2}{x - 2} = 2x \tau. \] (3.15)

In Eq.(3.14) there exist no constants which depend on the properties of matter, but there is the quantity \( x \) which is dependent on \( n \) and \( \gamma \). Therefore, when \( n \) and \( \gamma \) are the same, the equation can be simplified to
\[ p = \frac{\tau}{\rho_c} v + f(v), \] (3.16)
where \( \rho_c \) stand for the critical ratio, which is given by (3.10).
\[ f(v) = \frac{1}{v^2} \frac{x}{x - 2} + \frac{1}{v^x} \frac{2}{x - 1} \frac{2}{x - 2} \] (3.17)

where \( \rho_c \) stands for the critical ratio. Expanding this equation near the critical point
\[ \tau = t + 1, \quad v = (\omega + 1)^{1/q}, \] (3.18)
where \( \tilde{q} > 0 \), and using the fact that from the definition of the critical point we have
\[ \frac{1}{\rho_c} + f(1) = 1, \quad \rho_c f'(1) = 1, \quad \rho_c f''(1) = -2, \] (3.19)
And so obtain
\[ p = 1 + \frac{t}{\rho_c} - \frac{t \omega}{\tilde{q} \rho_c} - C \omega^3 + O(t \omega^2, \omega^4), \] (3.20)
where \( C = \frac{1}{\tilde{q}} \left( \frac{1}{\rho_c} - \frac{f''(1)}{6} \right) \). Differentiating the series for a fixed \( t < 0 \) we get
\[ dP = -P_c \left( \frac{t}{\tilde{q} \rho_c} + 3C \omega^2 \right) d\omega. \] (3.21)

Employing Maxwell’s equal area law, see, e.g., [35, 36, 38], while denoting \( \omega_g \) and \( \omega_l \) the ‘volume’ of small and large black holes, we get the following two equations:
\[ p = 1 + \frac{t}{\rho_c} - \frac{t \omega_l}{\tilde{q} \rho_c} - C \omega_l^3 = 1 + \frac{t}{\rho_c} - \frac{t \omega_g}{\tilde{q} \rho_c} - C \omega_g^3, \]
\[ 0 = \int_{\omega_l}^{\omega_g} \omega dP = \int_{\omega_l}^{\omega_g} \omega \left( \frac{t}{\tilde{q} \rho_c} + 3C \omega^2 \right) d\omega. \] (3.22)
The unique non-trivial solution is
\[ \omega_g = -\omega_l = \sqrt{-\frac{2l}{3C\tilde{q}\rho_c}} \propto (-t)^{1/2}, \] (3.23)
which implies that the degree of the coexistence curve
\[ \beta = 1/2. \] (3.24)

To calculate the exponent \( \tilde{\gamma} \), we use again (3.22), to get
\[ \kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \propto \frac{1}{P_c} (\tilde{q}\rho_c + 3C\omega^2). \] (3.25)
The set \( \omega = 0 \), we get
\[ \kappa_T \propto \frac{1}{P_c} \frac{\tilde{q}\rho_c}{t}. \] (3.26)

Thus the isothermal compressibility exponent
\[ \tilde{\gamma} = 1. \] (3.27)

Finally, the ‘shape of the critical isotherm’ \( t = 0 \) is given by (3.23), i.e.,
\[ p - 1 = -C\omega^3, \] (3.28)

Therefore the critical exponent
\[ \delta = 3. \] (3.29)

According to Eq. (3.11), when the electric charge of black holes is invariant, the equation of state of the charged topological dilaton AdS black hole in any dimension can be expressed as the form of Van der Waals equation. The critical exponents are the same as the ones for Van der Waals fluid.

In particular, the law of corresponding states, (3.14), takes the form (3.16). Taking \( \tilde{q} = \frac{n-\gamma(n+1)}{1-2\gamma} \) (in which cases \( \omega = \frac{V}{V_c} - 1 \)). We obtain the expansion (3.21) with
\[ C = \frac{x(1 - 2\gamma)^3}{3[n - \gamma(n + 1)]^3}, \] (3.30)
and so the discussion above applies.

### 3.2 \( l \) is an invariant parameter

In the case of \( l \) invariant, substituting Eq. (2.13) into (2.11), one can derive
\[ T = \frac{(\alpha^2 + 1) k(n + 2) b^{-2\gamma}}{4\pi (1 - \alpha^2)} \left( \frac{q'b^{(3-n)\gamma}}{\lambda U} \right)^{(2\gamma-1)/\lambda} + \frac{n(\alpha^2 + 1) b^{2\gamma}}{4\pi l^2} \left( \frac{q'b^{(3-n)\gamma}}{\lambda U} \right)^{(1-2\gamma)/\lambda}, \]
The critical points should satisfy the conditions

\[ a \text{ where } k_0. \]  

This result holds for any values of \( k \) and \( \gamma \). In the neighborhood of the critical points,

\[ \frac{\partial q}{\partial U} = \frac{(2\gamma - 1)}{\lambda U} = 0. \]  

Substituting Eq. (3.31) into Eq. (3.33), one can obtain the critical electric charge, critical electrostatic potential and critical temperature:

\[ q_c^2 = -\frac{\mathcal{A} + \mathcal{B}D}{C' D^{(2\gamma - 3\gamma - n + 2)/(1 - 2\gamma)}} = \frac{k(n - 2)}{2(x - 1)} b^{2(3 - n)\gamma} D^{\lambda/(1 - 2\gamma)} = E \]

\[ \lambda U_c = E^{1/2} b^{(3 - n)\gamma} D^{-\lambda/(2 - 4\gamma)}, \]

\[ T_c = \mathcal{A}D^{-1/2} + \mathcal{B}D^{1/2} + \mathcal{C}' \mathcal{A}' + \mathcal{B}'D \frac{\mathcal{C}'}{\mathcal{C}'} D^{-1/2} = \frac{k(n - 2)}{2\pi(1 - 2\gamma)(x - 1)} b^{-2\gamma} (x - 2) D^{-1/2}, \]  

where

\[ \mathcal{A}' = \mathcal{A}(1 - 2\gamma), \quad \mathcal{B}' = \mathcal{B}(2\gamma - 1), \quad \mathcal{C}' = \mathcal{C}'[(2n - 3)(\gamma - 1) - \gamma], \quad D = \frac{k(n - 2)}{2n(n - 1)} b^{-2\gamma} (x - 2). \]

In Fig.3, (a) and (b) show the influence of dilaton field \( \alpha \) on the isothermal curves with the same spacetime dimension. (b) and (c) represent the influence of spacetime dimension \( n \) on the isothermal curves with the same dilaton field.

Expanding around a critical point \( \tau = \beta_c, u = \frac{U}{U_c}, \vartheta = \frac{q}{q_c} \). Eq. (3.31) can be expressed as

\[ \frac{1}{\tau_{\beta_c}} = \mathcal{A}D^{-1/2} \left( \frac{\vartheta}{u} \right)^{(2\gamma - 1)/\lambda} + \mathcal{B}D^{1/2} \left( \frac{\vartheta}{u} \right)^{(1 - 2\gamma)/\lambda} + \mathcal{C}' \mathcal{A}' + \mathcal{B}'D \frac{\mathcal{C}'}{\mathcal{C}'} \vartheta^2 \left( \frac{\vartheta}{u} \right)^{(2n - 3)(\gamma - 1) - \gamma}/\lambda, \]  

Near the critical points, defining \( t = \tau - 1, \omega = u - 1 \) and substituting them into the above equation, we can obtain

\[ \vartheta = \sum_{m,n=0} a_{mn} t^m \omega^n, \]  

where \( a_{mn} = \left. \frac{\partial^{(m+n)\vartheta}}{\partial t^m \partial \omega^n} \right|_{t=0} \). According to Eq. (3.31) and (3.33), \( a_{00} = 1, a_{01} = 0, a_{02} = 0 \). This result holds for any values of \( k \) and \( \gamma \).
Figure 3: $q - U$ diagram of $(n + 1)$-dimensional charge Dilaton AdS black hole in $(n = 3, \alpha = 0)$, $(n = 3, \alpha = 0.6)$ and $(n = 10, \alpha = 0.6)$ respectively. The temperature of isotherms decreases from top to bottom. The three lower dashed lines correspond to the “ideal gas” one-phase behaviour for $T > T_c$, the critical isotherm $T = T_c$ is denoted by the thick solid line, upper (red) solid lines correspond to two-phase state occurring for $T < T_c$. We have set $b = 1, k = 1$ and the cosmological constant is given by Eqs.(3.5), (3.7). The behaviour for $n > 3$ and $\alpha > 0$ is qualitatively.

we have (3.36). The values of $\omega$ on either side of the coexistence curve can be found from the conditions that along the isotherm, Employing Maxwell’s equal area law, see, e.g., [33,35], while denoting $\omega_g$ and $\omega_l$ the ‘volume’ of small and large black holes, we get the following two equations:

\[ \vartheta(\omega_g) = \vartheta(\omega_l), \]  
\[ 0 = \int_{\omega_l}^{\omega_g} (\omega + 1) d\vartheta. \]  

From the first condition (3.37), we derive

\[ a_{11} t (\tilde{\omega}_g + \tilde{\omega}_l) + a_{21} t^2 (\tilde{\omega}_g + \tilde{\omega}_l) + a_{12} t (\tilde{\omega}_g^2 - \tilde{\omega}_l^2) + a_{03} (\tilde{\omega}_g^3 + \tilde{\omega}_l^3) + o(t\omega^2, \omega^4) = 0, \]  

where we have denoted $\tilde{\omega}_l = -\omega_l$ and $\tilde{\omega}_g = \omega_g$ for different phases. The second condition (3.38) reduces to

\[ a_{11} t (\tilde{\omega}_g + \tilde{\omega}_l) + a_{21} t^2 (\tilde{\omega}_g + \tilde{\omega}_l) + \frac{1}{2} (a_{11} + 2a_{12}) t (\tilde{\omega}_g^2 - \tilde{\omega}_l^2) + a_{03} (\tilde{\omega}_g^3 + \tilde{\omega}_l^3) + o(t\omega^2, \omega^4) = 0. \]  

The unique non-trivial solution is

\[ \tilde{\omega}_g^2 = \tilde{\omega}_l^2 = -\frac{1}{a_{03}} (a_{11} t + a_{21} t^2). \]  

yielding

\[ \tilde{\omega}_g = \tilde{\omega}_l \sim \sqrt{-\frac{a_{11}}{a_{03}}} t \sim t^{1/2} \quad \Rightarrow \quad \beta = \frac{1}{2}. \]
To calculate the exponent $\hat{\gamma}$, we use again (3.36), to get

$$\kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T \propto \frac{1}{P_c a_{11} t} \quad \Rightarrow \quad \hat{\gamma} = 1.$$  

(3.43)

Let $t = 0$, (3.40) will be of the form

$$\vartheta = a_{03} \omega^3 + o(\omega^4),$$  

(3.44)

Thus we get the degree of the critical isotherm

$$\delta = 3.$$  

(3.45)

The heat capacity at fixed $U$ is

$$C_U = T \left( \frac{\partial S}{\partial T} \right)_U = T \left( \frac{\partial S}{\partial q} \right)_U \left( \frac{\partial T}{\partial q} \right)_U^{-1},$$  

(3.46)

From Eq.(2.12) and Eq.(3.31)

$$\left( \frac{\partial S}{\partial q} \right)_U = \frac{b^{(n-1)\gamma} \omega_{n-1} (n-1)(1-\gamma)}{\lambda} \left( \frac{q b^{(3-n)\gamma}}{\lambda U} \right)^{(n-1)(1-\gamma)/\lambda} \frac{1}{q},$$

$$\left( \frac{\partial T}{\partial q} \right)_U = -\hat{A} \frac{(1-2\gamma)}{\lambda q} \left( \frac{q b^{(3-n)\gamma}}{\lambda U} \right)^{(2\gamma-1)/\lambda} + \hat{B} \frac{(1-2\gamma)}{\lambda q} \left( \frac{q b^{(3-n)\gamma}}{\lambda U} \right)^{(1-2\gamma)/\lambda}$$

$$+ \hat{C} q \frac{(1-2\gamma)}{\lambda} \left( \frac{q b^{(3-n)\gamma}}{\lambda U} \right)^{(2n-3)(\gamma-1)/\lambda}$$

$$= \frac{k(n-2)(x-2)b^{-2\gamma}}{2\pi \lambda q_c x(x-1)} D^{-1/2},$$  

(3.47)

When $n \neq 2$, and $x \neq 2$, Eq.(3.47) is nonzero and $C_U$ is non-singular at the critical points. Thus the heat capacity exponent

$$\hat{\alpha} = \hat{\alpha}' = 0.$$  

(3.48)

When the cosmological constant is invariant we conclude that the thermodynamic exponents associated with the charged topological dilaton AdS black holes in any dimension $n \geq 3$ coincide with those of the Van der Waals fluid.

4. Discussion and Conclusions

In Sec.3 we discussed the phase structure and critical phenomena of charged topological dilaton AdS black holes in the case of $k = 1$ and the cases of electric charge $Q$ and the cosmological constant $l$ are invariant respectively. We obtain two pairs of critical temperature and critical pressure (critical electric charge) and critical volume (critical electric potential),
which are represented by Eq.(3.7) and (3.34). Below we will analyze the relations between the two pairs of critical quantities.

In the case of invariant electric charge \( Q \), from critical pressure Eq.(3.7) we know that

\[
\frac{1}{l_c} = 4 \left( \frac{\pi A(x-2)}{n(n-1)x} \right)^{1/2} \left( \frac{2A}{(x-1)xB} \right)^{(x-2)/2}. \tag{4.1}
\]

Substituting Eq.(4.1) into the critical temperature Eq.(3.35) derived from the case of invariant cosmological constant \( l \), one can get

\[
T_c = \tilde{A}D^{-1/2} + \tilde{B}D^{1/2} + \tilde{C} \tilde{A}' + \tilde{B}'D \tilde{C}'D^{-1/2} = \frac{k(n-2)b^{-2\gamma}(x-2)}{2\pi(1-2\gamma)(x-1)}D^{-1/2}
\]

\[
= \frac{k(n-2)(x-2)}{2\pi(1-2\gamma)l(x-1)} \left( \frac{2n(n-1)}{k(n-2)(x-2)} \right)^{1/2} = \frac{2A(x-2)}{(x-1)} \left( \frac{2A}{(x-1)xB} \right)^{(x-2)/2}. \tag{4.2}
\]

From Eq.(4.2) and (3.7) we can find that the two pair of critical temperatures are the same. Because the critical temperature, critical pressure and critical volume in Eq.(3.7) are all dependent on \( B \), and \( B \) is the function of electric charge of black hole, the critical quantities should be the function of electric charge of black hole. The critical temperature and critical electric charge in Eq.(3.34) are the function of \( l \). The relations between the both quantities are given by Eq.(4.1), thus critical pressure and critical volume can also be expressed as the function of cosmological constant. The critical electric potential from Eq.(3.34)

\[
U_c = \frac{1}{\lambda} E^{1/2} b^{(3-n)\gamma} D^{-\lambda/(2-4\gamma)} = \frac{1}{\lambda} \left( \frac{k(n-2)}{2(x-1)} \right)^{1/2}. \tag{4.3}
\]

From Eq.(4.3), the critical electric potential is dependent on the spacetime dimension \( n \) and dilaton field \( \gamma \) and is independent of the cosmological constant \( l \).

From above we find that when the relation (4.1) is satisfied for the charged topological dilaton AdS black hole the phase transition like van der Waals vapor-liquid one will turn up.

The critical temperature, critical pressure, critical volume and critical electric potential are given by Eq.(3.7) and (3.34). The critical exponents are the same as the ones in van der Waals vapor-liquid phase transition.

Because of the relation (4.1), according to the critical temperature, critical pressure and critical volume derived in Sec.3 we can obtain the critical electric charge and critical electric potential.

From Eq.(4.1)

\[
B = \frac{2A}{(x-1)x} \left( \frac{16\pi^2 A(x-2)}{n(n-1)x} \right)^{(x-2)/2}. \tag{4.4}
\]

Substituting Eq.(4.3) into the above equation, one can get

\[
q_c^2 = \frac{k(n-2)}{2(x-1)} b^{2(n-3)\gamma} D^{\lambda/(1-2\gamma)}. \tag{4.5}
\]
which agrees with Eq.(3.35). From Eq.(3.3) and (3.7), the horizon of the black hole corresponding to the critical points is

\[ r_{+c} = \left( \frac{(x - 1)x B}{2A} \right)^{\lambda/(x-2)(1-2\gamma)} \left( \frac{(n-1)(1-\gamma)b^{2\gamma}}{4} \right)^{\lambda/(1-2\gamma)} \] (4.6)

Substituting Eq.(4.5) into (2.13), one can obtain the consistent critical electric potential with Eq.(4.3). Thus for the \((n+1)\)-dimensional charged topological dilaton AdS black hole, in Sec.3 and Sec.4 we can both derive the critical temperature, critical volume, critical pressure, critical electric potential and critical electric charge. For the two pairs of parameters \((P-V)\) or \((Q-U)\), the phase structure and critical phenomena of black hole are the same as the ones in van der Waals vapor-liquid system.

Due to \(A \propto k\), the critical temperature, critical pressure, critical electric potential and critical electric charge derived above will tend to zero, however the critical volume will tend to infinity. Therefore when \(k = 0\), for the \((n+1)\)-dimensional charged topological dilaton AdS black hole there no exist similar phase transition to the one in the van der Waals system.

When \(k = -1\), the results are complicated. The critical temperature \(T_c < 0\), critical electric charge and electric potential are imaginary numbers. \(v_c \propto (-1)^{1/(x-2)}\), \(P_c \propto -(-1)^{2/(x-2)}\), which is dependent on the value of \(x\). Whether this process can happen or not? If happens, what physical mechanism it should correspond? These questions should be studied further.

In this paper we studied the phase structure and critical phenomena of the \((n+1)\)-dimensional charged topological dilaton AdS black holes. We find that the phase structure in the canonical ensemble significantly depends on the parameter \(k\), dimensionality \(n\), dilaton field \(\gamma\) and cosmological constant \(l\) or the electric charge \(q\) of black hole. We consider Hawking temperature, electric charge, the cosmological constant, electric potential and volume as the state parameters and analyzed the phase structure and the critical phenomena. We do not discuss the effects of dilaton field \(\gamma\) and \(b\) in Eq.(2.4) on the phase structure and the critical phenomena and will leave this for further work.

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