Compact objects in Horndeski gravity

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Horndeski gravity holds a special position as the most general extension of Einstein’s theory of general relativity with a single scalar degree of freedom and second-order field equations. Because of these features, Horndeski gravity is an attractive phenomenological playground to investigate the consequences of modifications of general relativity in cosmology and astrophysics. We present a review of the progress made so far in the study of compact objects (black holes and neutron stars) within Horndeski gravity. In particular, we review our recent work on slowly rotating black holes and present some new results on slowly rotating neutron stars.

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I. INTRODUCTION

Einstein’s theory of general relativity (GR) has passed all experimental tests in its centennial history with flying colors [1]. Most precision tests of GR in our Solar System are confined to the weak-field/slow-motion regime. An exception are binary pulsars, where the orbital motion is nonrelativistic but the individual objects have strong gravitational fields [2]. As we witness the birth of the era of gravitational-wave astronomy [3], in the coming years we can hope to test GR in its strong field regime – as in the recent detection of binary black hole (BH) mergers [4], or possibly in the future via neutron star (NS) mergers – and in its radiative regime, e.g. by searching for the additional polarizations modes of gravitational radiation predicted by competing theories. Observational and theoretical issues with Einstein’s theory – such as the unknown nature of dark matter and dark energy, the presence of curvature singularities and the search for an ultraviolet completion of GR – have motivated strong efforts to develop modified theories of gravity which differ from GR in the infrared and ultraviolet regimes, while being consistent with the stringent observational bounds at intermediate energies [5]. Testing GR and searching for signatures of any deviation from its predictions is a major goal of several areas of research, including cosmology [6], “standard” electromagnetic astronomy [7, 8], and Earth- and space-based gravitational-wave astronomy [9, 10].

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In this work we will consider a very general modification of GR known as Horndeski gravity. The theory has its origins in the 1970s with Horndeski’s attempt [11] to obtain the most general action for a scalar-tensor theory with a single scalar degree of freedom and second-order field equations. Horndeski gravity has attracted much interest recently. One motivation has been the study of scalar-tensor theories with self-tuning cosmologies [12, 13]. Horndeski’s theory was rediscovered in the context of Galileon theories, i.e., scalar-tensor models which in flat space-time have Galilean symmetry. The generalization of Galileon theories to an arbitrary number of dimensions [14] was shown to be equivalent to Horndeski gravity in four dimensions [15]. The theory can also be obtained by a Kaluza-Klein compactification of higher-dimensional Lovelock gravity [16, 17]. Tensor-multiscalar theories [18–21] and multiscalar versions of Horndeski gravity [22–25] have also been formulated, but they will not be our main focus here.

In this paper we review our current understanding of compact objects (BHs and NSs) in Horndeski gravity with a single scalar field. This topic has received increasing attention because the study of compact objects can allow us to better understand the theory and (potentially) to confront it against observations in astrophysical settings. Progress has been rapid, and a summary of the recent developments in this field seems quite timely.

The paper is organized as follows. In Sec. II we review the basic aspects of Horndeski gravity and discuss some special cases. In Sec. III we review BH solutions in the theory and their stability properties. We also review no-hair theorems, their validity and loopholes. In Sec. IV we discuss NSs, presenting some new results for slowly rotating stars. In Sec. V we point out some directions for future research.

II. OVERVIEW OF HORNDESKI’S THEORY OF GRAVITY

We start by reviewing Horndeski gravity in its modern formulation. The action of the theory reads

\[ S = \sum_{i=2}^{5} \int d^4x \sqrt{-g} L_i, \]  

(2.1)

where

\[ L_2 = G_2, \]  

(2.2a)

\[ L_3 = -G_3 \Box \phi, \]  

(2.2b)

\[ L_4 = G_4 R + G_4 X \left( [\Box \phi]^2 - \phi_{\mu\nu}^2 \right), \]  

(2.2c)

\[ L_5 = G_5 G_{\mu\nu} \phi_{\mu\nu} - \frac{G_5 X}{6} \left( [\Box \phi]^3 + 2\phi_{\mu\nu}^3 - 3\phi_{\mu\nu}^2 \Box \phi \right). \]  

(2.2d)

Here \( g_{\mu\nu} \) is the metric tensor and \( g = \det(g_{\mu\nu}) \) its determinant. The Ricci scalar and Einstein tensor associated with \( g_{\mu\nu} \) are denoted by \( R \) and \( G_{\mu\nu} \), respectively. The functions \( G_i = G_i(\phi, X) \) depend only on the scalar field \( \phi \) and its kinetic energy \( X = -\partial_\mu \phi \partial^\mu \phi/2 \). We use units such that \( m_P^2 \equiv (8\pi G)^{-1} = 1 \), where \( m_P \) is the reduced Planck mass. For brevity we have also defined the shorthand notation \( \phi_{\mu\nu} \equiv \nabla_\mu \phi \nabla_\nu \phi, \phi_{\mu\nu}^2 \equiv \phi_{\mu\nu} \phi^{\mu\nu}, \phi_{\mu\nu}^3 \equiv \phi_{\mu\nu} \phi^{\rho\alpha} \phi^{\mu\nu}_{\rho\alpha}, \) and \( \Box \phi \equiv g^{\mu\nu} \phi_{\mu\nu}. \)

An attractive feature of Horndeski gravity is its generality. The theory includes a broad spectrum of phenomenological dark energy models, as well as modified gravity theories with a single scalar degree of freedom (in this review we will not discuss “beyond Horndeski” theories [26, 27] or extensions of Horndeski including two or more scalar degrees of freedom [23–25]). Some important special limits of the theory are listed below:

1. GR is obtained by choosing \( G_4 = 1/2 \) and \( G_2 = G_3 = G_5 = 0. \)
2. When the only nonzero \( G_i \) function is \( G_4 = F(\phi) \), we recover a scalar-tensor theory with nonminimal coupling of the form \( F(\phi)R \). Consequently, Brans-Dicke theory and \( f(R) \) gravity are special cases of Horndeski gravity.
3. Einstein-dilaton-Gauss-Bonnet (EdGB) gravity, whose action is

\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{2} R + X + \xi(\phi) R_{\text{GB}}^2 \right), \]  

(2.3)

where \( R_{\text{GB}}^2 = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \) is the Gauss-Bonnet invariant, corresponds to the choices

\[ G_2 = X + 8\xi(4) X^2 (3 - \ln X), \quad G_3 = 4\xi(3) X (7 - 3 \ln X), \]  

(2.4a)

\[ G_4 = \frac{1}{2} + 4\xi(2) X (2 - \ln X), \quad G_5 = -4\xi(1) \ln X, \]  

(2.4b)

where \( R_{\alpha\beta\gamma\delta} \) and \( R_{\mu\nu} \) are the Riemann and Ricci tensors, and we have defined \( \xi^{(n)} = \partial^n \xi/\partial \phi^n \) [15].
4. A theory including nonminimal derivative coupling between the scalar field $\phi$ and the Einstein tensor $G_{\mu\nu}$ (the “John” Lagrangian in the language of the so-called “Fab Four” model [12, 13]), with action

$$S = \int d^4x \sqrt{-g} \left[ \zeta R + 2\beta X + \eta G^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2\Lambda_0 \right],$$

(2.5)

can be constructed by setting

$$G_2 = -2\Lambda_0 + 2\beta X, \quad G_4 = \zeta + \eta X, \quad G_3 = G_5 = 0,$$

(2.6)

where $\Lambda_0$, $\eta$, $\zeta$ and $\beta$ are constants. Note that a coupling of the form $G^{\mu\nu} \phi_{,\mu} \phi_{,\nu}$ can also be obtained by setting $G_5 = -\phi$ and integrating by parts [28]. This action also arises in the decoupling limit of massive gravity [29, 30].

5. The Lagrangian $L_2$ corresponds to the k-essence field [31–33]. For this reason, in some of the literature the function $G_2$ is denoted by $K$.

6. The covariant Galileon model [34] is recovered by setting $G_2 = -c_2 X$, $G_3 = -c_3 X/M^3$, $G_4 = M^2 \mu^2/2 - c_4 X^2/M^6$ and $G_5 = 3c_5 X^2/M^9$, where the $c_i$ ($i = 2, \ldots, 5$) are constants and $M$ is a constant with dimensions of mass.

Because of the generality of Horndeski gravity, a comprehensive review of compact objects would inevitably have to discuss important subclasses that have been studied for a long time, such as EdGB and $f(R)$ gravity[5]. For brevity we will focus on the subclasses that have not been reviewed in the past (i.e., the special cases 4–6 above). We will also focus on four-dimensional solutions.

III. BLACK HOLE SOLUTIONS

As mentioned in the introduction, Horndeski gravity received renewed interest because of its applications to cosmology. Only more recently BH solutions have been obtained and studied in several subclasses of the theory. We begin this section by reviewing an important no-hair theorem established by Hui and Nicolis [35], which sets tight constraints on the search for hairy BH solutions.

A. A no-hair theorem in Horndeski gravity

Hui and Nicolis [35] presented a no-hair theorem which is valid for shift-symmetric Horndeski gravity, i.e., the subclass of the Horndeski action which remains invariant under a transformation $\phi \rightarrow \phi + c$ of the scalar field, where $c$ is a constant. The theorem is applicable to vacuum, static, spherically symmetric and asymptotically flat BHs. The shift symmetry implies the existence of a conserved current $J^\mu$ for the scalar field which satisfies $\nabla_\mu J^\mu = 0$.

We assume a line element of the form

$$ds^2 = -A(r)dt^2 + B(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

(3.1)

The proof of the theorem relies on the following steps/assumptions: (i) the scalar field is assumed to have the same symmetries as the metric, implying that the only nonzero component of $J^\mu$ (if any) is the radial component $J^r$. (ii) We require $J^2 \equiv J^\mu J_\mu = (J^r)^2/B$ to remain regular at the horizon $r_h$. Since $B(r_h) = 0$, we must set $J^r = 0$ at the horizon. (iii) From $\nabla_\mu J^\mu = 0$, we obtain $\partial_r J^r + 2J^r/r = 0$, with solution $J^r r^2 = k$, where $k$ is an integration constant. As $r_h \neq 0$, condition (ii) at the horizon directly implies $k = 0$. Therefore $J^r = 0 \forall r$. (iv) It is argued that $J^r$ has the schematic form

$$J^r = B\phi' F(g, g', g'', \phi'),$$

(3.2)

where $F$ is a generic function of the metric, its derivatives (indicated by the primes) and the derivatives of the scalar field $\phi'$. Asymptotic flatness requires that $B \to 1$ and $\phi' \to 0$ at spatial infinity, while $F$ tends to a nonzero constant. The latter condition follows from the requirement that in the weak-field limit the kinetic energy be quadratic in $\phi$ and that $J^\mu \approx \partial^\mu \phi$, up to a normalization constant. Moving “inwards” towards the horizon, $\phi'$ can become nonzero, and $B$ and $F$, by continuity, remain nonzero. We then conclude that $J^r \neq 0$, contradicting the conclusion from (iii). This can be resolved by forcing $\phi' = 0 \forall r$. Consequently the scalar field must be constant, and by exploiting the shift symmetry we can set its value to zero.

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1 A no-hair theorem for the subclass 4 was obtained by Germani et al. [36] following a different strategy.
As pointed out by Sotiriou and Zhou [37], the no-hair theorem summarized above has a loophole allowing for hairy BH solutions. Furthermore, other hairy solutions can be obtained by relaxing some of the assumptions that enter the proof of the no hair theorem. BH hair is classified as either primary (described by an independent charge, e.g. a scalar charge) or secondary (depending on other charges, such as the mass $M$ of the BH). The BH solutions known so far in Horndeski gravity have secondary hair [38, 39]. The various possibilities are schematically summarized in Fig. 1, and discussed below.

1. Asymptotically flat black holes

The loophole pointed out by Sotiriou and Zhou concerns the last step of the proof. An explicit calculation of the current $J^\mu$ using the action of Eq. (2.1) reveals that the form of $J^r$ assumed by Hui and Nicolis does not necessarily hold for all shift-symmetric Horndeski theories. Explicitly, $J^r$ reads

$$J^r = -BG_{2X}\phi' + \frac{B^2\phi'^2}{2} \left( \frac{A'}{A} + \frac{4}{r} \right) G_{3X} + \frac{2B^2\phi'}{r} \left( \frac{A'}{A} - \frac{1}{Br} + \frac{1}{r} \right) G_{4X}$$

$$- \frac{2B^3\phi'^3}{r} \left( \frac{A'}{A} + \frac{1}{r} \right) G_{4XX} - \frac{B^3\phi'^2}{2r^2} A' \left( \frac{3B - 1}{B} \right) G_{5X} + \frac{A'B^4\phi'^4}{2r^2} G_{5XX},$$

(3.3)

where we note that to impose shift symmetry in the theory we must set $G_i(X, \phi) = G_i(X)$. Observe that all the terms involve powers of $\phi'$, as required by Hui and Nicolis. In particular the first term has the form (3.2) and $F \rightarrow -G_{2X}$ at spatial infinity, as required by the theorem. The other terms depend on derivatives of $A$, $B$ and/or inverse powers of $r$, and seem to vanish for large $r$, as required. In principle, however, hairy BHs could exist for theories where the functions $G_i(X)$ are chosen such that $J^r$ contains terms independent of $\phi'$, but no negative powers of $\phi'$ (cf. Fig. 1). Another alternative would be to have negative power of $\phi'$, however this generally corresponds to theories that would not admit flat space with a trivial scalar configuration as a solution, leading to violations of local Lorentz symmetry [37].

An explicit example [40] of a theory in which $J^r$ contains terms independent of $\phi'$ is EdGB theory, with a linear coupling $\xi(\phi) = \alpha\phi$ – where $\alpha$ is a constant – between the scalar field and the Gauss-Bonnet invariant in Eq. (2.3). Exploiting shift symmetry and Lovelock’s theorem, Sotiriou and Zhou showed that, in fact, this is the unique shift-symmetric subclass of Horndeski

FIG. 1. A schematic representation of the Hui-Nicolis no-hair theorem for shift-symmetric Horndeski gravity, and two possible ways of violating it.
for which this happens. For this theory, $J^r$ has the form

$$J^r_{\text{EdGB}} = -B \phi' - 4\alpha \frac{A' B (B - 1)}{A r^2}. \quad (3.4)$$

The current vanishes at infinity, however the second term allows for scalar hair growth when $J^r = 0$, i.e. $\phi'$ is nontrivial. In this theory, the hair is of the “second kind,” i.e. it depends on the mass $M$ of the BH [37, 38].

For comparison, it is instructive to write down the form of $J^r$ for a theory with nonminimal derivative coupling (cf. item 4). The nonvanishing component of the current in this case is

$$J^r_{\text{GB}} = B \phi' \left[-2\beta + \frac{2B}{\tau} \left(\frac{A'}{A} - \frac{1}{B r} + \frac{1}{r}\right) \eta\right], \quad (3.5)$$

which is of the form assumed by Hui and Nicolis, and therefore the theory does not admit asymptotically flat hairy BH solutions [36].

An alternative approach was considered by Babichev and Charmousis [41] (and further explored in Ref. [42]). For the nonminimal derivative coupling theory, the conserved current $J^\mu$ associated with the shift symmetry can be written as

$$J^\mu = (\beta g^{\mu\nu} - \eta G^{\mu\nu}) \partial_\nu \phi, \quad (3.6)$$

which opens two possibilities to satisfy the condition $J^r = 0$: (i) set the scalar field to be constant or (ii) set $\beta g^{\mu\nu} - \eta G^{\mu\nu} = 0$, and then allow $\partial_\nu \phi$ to be nonzero.

In general, this latter condition cannot be satisfied together with the regularity of $J^2$ at the horizon. Babichev and Charmousis show, however, that both conditions can be satisfied if the scalar field is allowed to be time-dependent, i.e. $\phi = \phi(t, r)$, while the background metric is still fixed by Eq. (3.1): see Fig 1. In particular, they find hairy solutions with

$$\phi(t, r) = qt + \psi(r). \quad (3.7)$$

Among the solutions constructed in this way, the non-minimally coupled theory with $\beta = \Lambda = 0$ [cf. Eq. (2.5)] admits a “stealth” solution, where a Schwarzschild BH metric supports a nontrivial, regular scalar field configuration which does not backreact on the spacetime. We stress that although $\phi(t, r)$ diverges at future infinity, it is the derivatives of $\phi(t, r)$ which appear in the action, and these remain well-behaved due to the linear dependence on $t$.

Assuming shift and reflection symmetry ($\phi \to -\phi$), which implies that $G_3 = G_5 = 0$ in the action (2.1), Kobayashi and Tanahashi [28] extended the Babichev-Charmousis approach to obtain a very general class of BH solutions which do not require specific assumptions on the form of $G_2$ and $G_4$. A key ingredient in the derivation is that $X$ is a constant. Their solutions are regular, in the sense that $X$ and $J^2$ are well-behaved.

In principle the time dependence of the scalar field could affect the $t r$ component of the field equations $\mathcal{E}_{t r}$. However, assuming diffeomorphism invariance, shift symmetry and that $\phi(t, r) = qt + \psi(r)$, Babichev et al. [43] showed that $\mathcal{E}_{t r}$ is proportional to $J^r$. As we have seen, regularity of the current $J^\mu$ at the horizon demands that $J^r = 0$ everywhere, and consequently $\mathcal{E}_{t r} = 0$.

2. Non-asymptotically flat spacetimes

To our knowledge, Rinaldi [44] was the first to explore BH solutions in the special class of Horndeski’s theory given by Eq. (2.5) with $\Lambda_0 = 0$. The scalar field was found to be imaginary because $\phi'(r)^2 < 0$ outside the horizon, which may imply an instability of the solution. Note however that because of shift symmetry the field equations of the theory never contain $\phi$, but only its derivative; in this sense, one could think of $\phi'(r)^2$ as a separate field. As shown by Anabalon et al. [45] and Minamitsuji [46], the presence of a cosmological constant $\Lambda_0$ cures this problem. Requiring that the scalar field remains real imposes certain constraints on the parameters $\Lambda_0$, $\eta$ and $\zeta$. Self-tuning BHs with de Sitter asymptotics were also obtained [41, 42]. A BH solution which asymptotically approaches a Lifshitz spacetime was also found using the Babichev-Charmousis construction, and therefore a time-dependent scalar field [47].

Some works have also considered Horndeski gravity in the presence of a Maxwell field. For instance, considering the theory with nonminimal coupling between the scalar field and the Einstein tensor with an additional Maxwell Lagrangian $\propto F_{\mu\nu} F^{\mu\nu}$, Cisterna and Erices [48] obtained electrically charged BH solutions which are asymptotically anti-de Sitter (AdS). An interesting solution is obtained when the scalar field dynamics is determined solely by the “John” Lagrangian in Eq. (2.5), i.e. when $\beta = 0$. In this case one finds a charged BH solution which is locally flat as $r \to \infty$, with an asymptotically constant electric field $E \propto \Lambda_0$ supported by the presence of the cosmological constant. Extending their previous work [41], Babichev and Charmousis also studied a more general class of Horndeski-Maxwell theories, allowing for all the possible couplings between the Maxwell and scalar field under the $U(1)$ and shift symmetries [43] and obtaining charged BH solutions. Kolyvaris et al. [49] also obtained solutions involving scalar and Maxwell fields.
C. Stationary hairy black hole solutions

All of these works considered static, spherically symmetric solutions. The construction of slowly rotating solutions was recently studied in the Hartle-Thorne formalism [50, 51], where rotation is considered as a perturbation on an otherwise static spherically symmetric background [52]. Let us consider the line element

\[ ds^2 = -A(r)dt^2 + B(r)^{-2}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - 2\omega(r)r^2 \sin^2 \theta dt d\phi, \]

where the function \( \omega \) is related with the dragging of inertial frames, and it is of the same order as the angular velocity \( \Omega \). Under the slow-rotation assumption we can write down the full set of field equations of Horndeski gravity, without assuming shift or reflection symmetries, and for a scalar field of the form (3.7). We found that for all the solutions reported by Kobayashi and Tanahashi, \( \omega \) behaves exactly as in GR, i.e. \( \omega = k_1 + k_2/r^3 \), where \( k_1 \) and \( k_2 \) are integration constants [52].

We also formulated an extension of the Hui-Nicolis no-hair theorem including both a time-dependent scalar field of the form (3.7) and slow rotation. Following the arguments outlined in Sec. III A, in our case \( J^2 \) becomes

\[ J^2 = \frac{(J_r)^2}{B} - (J_t)^2 A, \]

(3.9)

Imposing regularity at the horizon yields again \( J_r = 0 \), as long as \( J_t \) does not diverge. In general, this imposes certain restrictions on \( A \) and \( B \). In the particular case of shifts- and reflection-symmetric theories, these restrictions reduce to the condition that \( (B/A)' \) remain finite at the horizon. This time, because of the time dependence, \( \nabla_\mu J^\mu = 0 \) gives \( \partial_\mu J^\mu + 2J^\mu/r + \partial_\mu J^\mu = 0 \). An explicit calculation of \( J^t \) reveals that \( \partial_\mu J^\mu = 0 \) (a particular consequence of the linear time dependence of \( \phi \)), and therefore, as in the Hui-Nicolis case, we conclude that \( J_r^\mu = 0 \forall r \). As we mentioned before, although the scalar field depends on time, \( J^t \) does not diverge and remains finite at the horizon. This imposes certain restrictions on the choices of the functions \( G_i \).

In their study of odd-parity gravitational perturbations in the nonminimal derivative coupling theory (see Sec. III D), Cisterna et al. [53] reached the same conclusion: the frame-dragging equation (in vacuum) is the same as in GR.

In general, however, we expect that rotation will cause “bald” slowly rotating BH solutions to grow hair. At second perturbative order in the slow-rotation expansion – i.e. when we add terms of order \( \Omega^2 \) to Eq. (3.8) – the scalar field also gets corrected [37]:

\[ \phi(r, \theta) = \phi^{(0)}(r) + \phi^{(2)}(r, \theta), \]

(3.10)

where superscripts indicate the perturbative order. Therefore the scalar field will not, in general, have spherical symmetry. The \( \phi^{(2)} \) correction is likely to affect the form of \( J^r \), and one could expect that the current will no longer have the form of Eq. (3.2).

Moreover, the presence of a nontrivial component \( J^\theta \) of the current should play a role when demanding that \( J^2 \) is well-behaved at the horizon.

Two possible approaches to tackle this problem would involve considering higher-order corrections in the Hartle-Thorne scheme [54] or constructing fully numerical solutions. Both approaches have been successfully applied to rotating BHs in EdGB theory [55–59], which have nontrivial scalar hair.

In conclusion, the study of general rotating BHs in Horndeski theory remains a fairly unexplored and interesting topic.

D. Stability, quasinormal modes and collapse

In a tour de force calculation, the odd [60] and even [61] gravitational perturbations of static, spherically symmetric backgrounds were studied by Kobayashi and collaborators. When applied to particular subclasses of Horndeski gravity, their perturbation equations yield conditions preventing the appearance of ghost and gradient instabilities. The analysis of these papers assumes the scalar field to be static, and therefore the results cannot be applied to the Babichev-Charmousis [41] and Kobayashi-Tanahashi [28] solutions, where the scalar field depends linearly on time. Focusing on the shift- and reflection-symmetric sectors of the theory, Ogawa et al. [62] analyzed odd gravitational perturbations allowing the scalar field to be time-dependent. A surprising result is that solutions with \( X = \text{constant} \) [28] suffer either from ghost or gradient instabilities in the vicinity of the horizon.

Minamitsuji [63] investigated the stability of BH solutions under massless scalar perturbations in the nonminimal derivative coupling subclass [46]. The solutions are asymptotically AdS, so the calculation can be done using the same techniques used for Schwarzschild-AdS BHs [64]. The quasinormal modes can be computed, and no unstable modes were found. Considering the same BH solutions, Cisterna et al. [53] found that BHs are stable under odd-parity gravitational perturbations (see Anabalon et al. [65] for an earlier study).
Let us also remark that the gravitational collapse of the scalar field was studied by Koutsoubas et al. in the nonminimal derivative coupling theory [66].

IV. NEUTRON STARS

NSs in Horndeski gravity have received attention only very recently. Cisterna et al. [67] considered the subclass of Horndeski’s theory involving a nonminimal coupling between the scalar field and the Einstein tensor. They wrote down the generalized Tolman-Oppenheimer-Volkoff (TOV) equations and adopted the same assumptions that Babichev and Charmousis [41] used to obtain stealth BH solution, constructing asymptotically flat NS models. Numerical integration of the TOV equations revealed that, depending on the sign of the coupling constant $\eta$ in Eq. (2.5), the mass-radius relation is shifted either upwards ($\eta < 0$) or downwards ($\eta > 0$) with respect to GR. A second constant $q$ [cf. Eq. (3.7)] controls deviations from GR of the Horndeski NS model for a given value of $\eta$. Interestingly, an expansion of the pressure equation near the star’s center allows one to constrain the allowed values of $(q, \eta)$ for a given central energy density $\epsilon_c$ by demanding that the pressure monotonically decay within the star (similar considerations permit to constrain the parameter space of NSs in EdGB theory [68]).

Applying the results of Maselli et al. [52] and Cisterna et al. [53] to the vacuum exterior region, we find that the frame-dragging equation outside slowly rotating stars in this subclass of Horndeski gravity is identical to the frame-dragging equation in GR. As in the case of BHs, this is not expected to hold at higher orders in the slow-rotation expansion. This might have interesting consequences for the structure of NSs. For example, the quadrupole moment could differ from GR in the presence of a scalar field, and this may affect astrophysical observables, such as quasi-periodic oscillations (QPOs).

One of the outstanding unresolved problem in testing modified theories of gravity using NSs is how to unambiguously disentangle the uncertainties in the equation of state from the effects predicted by modified gravity [5, 69].

The “universal” (nearly equation of state independent) relations between the moment of inertia $I$ and the tidal Love number $\lambda$ ($I$-Love-$Q$ relations) found in GR [70, 71] can help alleviate this problem. These $I$-Love-$Q$ relations were studied in a broad set of modified theories of gravity [58, 71–75], and it will be interesting to see whether they hold in Horndeski gravity.

In Fig. 2 we show some preliminary results in this direction. We compute the mass, radius and moment of inertia for NSs in Horndeski gravity by numerically integrating the same stellar structure equations as in Cisterna et al. [53], generalized to include the effect of rotation at first order in the Hartle-Thorne perturbative scheme (which allows us to compute the moment of inertia $I$). For illustrative purposes, here we consider a polytropic equation of state. The effect of the unconstrained parameters $\eta$ and $Q_\infty$ on the bulk properties of the star can be large. We will present a more detailed and extensive study (including NS models in other subclasses of Horndeski gravity) in a forthcoming paper [54].

![Fig. 2. NSs in Horndeski gravity.](image)

Barausse and Yagi [76] have shown that in shift-symmetric Horndeski theories, under certain assumptions, the stellar sensitivity (which quantifies the dependence of the star’s gravitational mass on the background scalar field) vanishes. Sensitivities are important in a post-Newtonian (PN) expansion of scalar-tensor theories, since they source the leading-order emission of dipolar scalar radiation. Since the sensitivities vanish, the dipolar energy flux also vanishes in this subclass of Horndeski theory, and at leading order in the PN expansion the dynamics of the binary in shift-symmetric Horndeski gravity will be the same as in GR. The emission of gravitational waves only differs from GR (if at all) at higher PN orders. For the EdGB model considered by Sotiriou and Zhou, it was recently shown [77] that NSs in this theory do not have scalar charge, therefore the theory evades the
constraints on the presence of dipolar radiation in binary pulsars \cite{68,78}. These results are in contrast with, for instance, the popular “spontaneous scalarization” model by Damour and Esposito-Farèse \cite{79,80}, which does not possess shift-symmetry, resulting in emission of dipolar scalar radiation and allowing pulsar observations to put stringent constraints on the theory \cite{81}. It will be interesting to compute sensitivities and gravitational wave emission in Horndeski theories without shift symmetry.

V. CONCLUDING REMARKS

In this paper we reviewed our current understanding of compact objects (BHs and NSs) in Horndeski gravity. Despite of the complexity of the Horndeski action, there has been rapid progress in specific subclasses of the theory, but there are still many open problems.

Existing BH solutions were found under the assumption of shift and/or reflection symmetry. It would be interesting to find BH solutions in theories that do not satisfy these simplifying assumptions, and to study their stability properties. More in general, studies of stability and dynamics (including quasinormal mode calculations) are in their infancy. It is important to understand if BH solutions in generic Horndeski theories differ significantly from GR in terms of their structure and dynamics, and if so, whether they could leave observable imprints in astrophysical settings \cite{5,8,82,83}.

Stellar models have also been constructed in just a few special cases. We are currently working to extend these studies to more general classes of theories and more realistic equations of state \cite{54}. Our main goal is to understand whether stars in Horndeski gravity can exhibit observable deviations from GR in the strong-field regime. Some classes of Horndeski gravity may produce phenomena similar to spontaneous scalarization \cite{79,80}, producing observable signatures in (say) Advanced LIGO while being compatible with weak-field bounds: see e.g. recent work on the “asymmetron” scenario \cite{84} and massive scalar-tensor theories \cite{85–87} for similar proposals.

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\[1\] C. M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Rel. 17, 4 (2014), arXiv:1403.7377 [gr-qc].
\[2\] N. Wex, Testing Relativistic Gravity with Radio Pulsars, (2014), arXiv:1402.5594 [gr-qc].
\[3\] B. . Abbott et al. (Virgo, LIGO Scientific), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc].
\[4\] B. P. Abbott et al. (Virgo, LIGO Scientific), Tests of general relativity with GW150914, (2016), arXiv:1602.03841 [gr-qc].
\[5\] E. Berti et al., Testing General Relativity with Present and Future Astrophysical Observations, Class. Quant. Grav. 32, 243001 (2015), arXiv:1501.07274 [gr-qc].
\[6\] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Modified Gravity and Cosmology, Phys.Rept. 513, 1–189 (2012), arXiv:1106.2476 [astro-ph.CO].
\[7\] D. Psaltis, Probes and Tests of Strong-Field Gravity with Observations in the Electromagnetic Spectrum, Living Rev. Rel. 11, 9 (2008), arXiv:0806.1531 [astro-ph].
\[8\] C. Bambi, Testing black hole candidates with electromagnetic radiation, (2015), arXiv:1509.03884 [gr-qc].
\[9\] N. Yunes and X. Siemens, Gravitational-Wave Tests of General Relativity with Ground-Based Detectors and Pulsar Timing-Arrays, Living Rev.Rel. 16, 9 (2013), arXiv:1304.3473 [gr-qc].
\[10\] J. R. Gair, M. Vallisneri, S. L. Larson, and J. G. Baker, Testing General Relativity with Low-Frequency, Space-Based Gravitational-Wave Detectors, Living Rev.Rel. 16, 7 (2013), arXiv:1212.5575 [gr-qc].
\[11\] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int.J.Theor.Phys. 10, 363–384 (1974).
\[12\] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, General second order scalar-tensor theory, self tuning, and the Fab Four, Phys. Rev. Lett. 108, 051101 (2012), arXiv:1106.2000 [hep-th].
\[13\] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, Self-tuning and the derivation of a class of scalar-tensor theories, Phys. Rev. D85, 104040 (2012), arXiv:1112.4866 [hep-th].
\[14\] C. Deffayet, S. Deser, and G. Esposito-Farèse, Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors, Phys. Rev. D80, 064015 (2009), arXiv:0906.1967 [gr-qc].
