SEIBERG-WITTEN-CASSON INVARIANT OF HOMOLOGY
$S^1 \times S^3$ WITH CIRCLE ACTION

DAOYUAN HAN

Abstract. In this paper we shall compute the Mrowka-Ruberman-Saveliev invariant introduced in [17] for the case when the manifold admits a free circle action.

1. Introduction

The Mrowka-Ruberman-Saveliev invariant [17] defined for 4-manifolds with $b_2^+ = 0$ is the count of irreducible solutions plus an index of Dirac operator over end-periodical manifold. Several special cases have been computed by others, for example, when $X$ is of the form $S^1 \times Y$ and $Y$ is of the homology type of the three sphere $S^3$, Mrowka, Ruberman, Saveliev proved using Lim’s work in [9] that it coincides with Casson’s invariant. The authors of [17] computed this invariant in other cases, like mapping tori. Moreover, we want to verify a special case of the conjecture made in the paper [17], which states that the Mrowka-Ruberman-Saveliev invariant is the same as the invariant define by Furuta and Ohta in [5], which is considered as another generalization of Casson’s invariant to 4-manifold.

We shall begin this section by reviewing the definition of the Mrowka-Ruberman-Saveliev invariant introduced in [17]. Let $X$ be an integral homology $S^1 \times S^3$, then define

$$\lambda_{SW}(X) = \#\mathcal{M}(X, g, \beta) - \omega(X, g, \beta),$$

where $\#\mathcal{M}(X, g, \beta)$ is the count of irreducible solutions in the Seiberg-Witten moduli space over $X$ equipped with metric $g$ and perturbation $\beta$ and $\omega(X, g, \beta)$ is a correction term so that $\lambda_{SW}(X)$ is independent of $g$ and $\beta$. The correction term $\omega(X, g, \beta)$ is defined as

$$\omega(X, g, \beta) = \text{index}D^+(Z_+, g, \beta) + \text{sign}(Z)/8,$$

where $Z_+ = Z \cup W_1 \cup W_2 \cup \cdots$, with each copy of $W_i$ a cobordism formed by cutting along a 3-submanifold $M$ representing generator of $H_3(X)$, and $Z$ is a spin 4-manifold with boundary $M$. $D^+(Z_+, g, \beta)$ is the Dirac operator over $Z_+$ equipped with $\text{Spin}$-structure extending that over $W$ to $Z$. And it is proved in [17] that this correction term is independent of $Z$ and the way to extend the $\text{Spin}$ structure.

This invariant can be treated as a lift of Rohlin’s invariant by Theorem A in [17], which can also be considered as a generalization of Theorem 1.2 by Chen in [16], where an integer invariant $\alpha(Y)$ for homology sphere $Y$ is defined and equal to Rohlin’s invariant mod 2. The Chen’s invariant in [16] is also defined as a combination of Seiberg-Witten invariant and index correction term.

The proof that $\lambda_{SW}$ is well-defined in [17] uses the blown-up of SW-equation and shows at first that for generic metric and perturbation $(g, \beta)$, the pair is regular, meaning that the corresponding blown-up moduli space has no reducible solution.
The moduli space under regular pair of \((g, \beta)\) is a zero dimensional manifold by computing the virtual dimension. Considering a path of such regular pairs, it’s proved that the corresponding parametrized moduli space is a 1-dimensional manifold with boundary \(M(X, g_0, \beta_0) \cup M(X, g_1, \beta_1) \cup M^0_I\) where \(M^0_I\) denotes the path components approaching reducibles. Thus the change of Seiberg-Witten invariants can be expressed as the count of points in \(M^0_I\). Then the remainder of the proof shows that there is a 1-1 correspondence between the change of correction terms along the same path \((g_t, \beta_t)\) and \(M^0_I\). It is achieved by expressing the change of correction terms as a spectral flow of certain path of Dirac operators over \(X\). This new path of Dirac operators over \(X\) is derived from Laplace-Fourier transform of the end-periodic Dirac operators over \(Z^+\). Note that the spectral flow only changes when a Dirac operator on the path has nontrivial kernel and it remains to be seen that the parameters where the kernel is nontrivial are in 1-1 correspondence with the points in \(M^0_I\).

When the manifold admits a free circle action, the Seiberg-Witten invariant and the correction term can be computed more explicitly. We assume that the circle action induces a \(S^1\)-bundle \(\pi: X \to Y\) whose Euler number \(e = 1\). The submanifold \(M\) of \(X\) representing the homology generator of \(H_3(X; \mathbb{Z})\), fibers over a 2-surface \(\Sigma\). It is shown by Baldridge [4] that in this case, the Seiberg-Witten invariant of \(X\) can be related to the 3 dimensional Seiberg-Witten invariant of \(Y\), and this 3-dimensional Seiberg-Witten invariant can be further related to Alexander polynomial of a knot, surgery on which of \(S^3\) gives \(Y\). The correction term on the other hand, can be computed by using a special neck-stretching metric in [10] over \(Z^+\) with the effect of stretching \(M \times [0, R]\) by letting \(R\) be sufficiently large, and we can then use the index formula for cylindrical end manifold to compute the correction term. Thus we have the following theorem,

**Theorem 1.1.** Let \(X\) be a smooth 4-manifold of integral homology \(S^1 \times S^3\) with a free circle action such that \(X\) is circle bundle over \(Y\) and \(H_3(Y) = H_3(S^1 \times S^3)\). Then there exists a pair \((g_X, \beta)\) such that \(#M(X, g_X, \beta) = \Delta_Y^*(1)\), where \(\Delta_Y(t)\) denotes the normalized Alexander polynomial and the correction term \(\omega(X, g_X, \beta) = 0\).

Note that when the infinite cyclic cover \(\tilde{X}\) has the same homology as \(S^3\), we have \(#M(X, g, \beta) = 0\) as \(\Delta_Y(t)\) is trivial. Thus we can verify the following conjecture made in [17] in the case when \(X\) admits a free circle action.

**Conjecture 1.2 ([17]).** For any smooth oriented homology oriented 4-manifold \(X\) with the \(\mathbb{Z}[\mathbb{Z}]\)-homology of \(S^1 \times S^3\), one has \(\lambda_{SW}(X) = -\lambda_{FO}(X)\).

Here the \(\lambda_{FO}(X)\) denotes the Furuta-Ohta invariant introduced in [8] defined by counting the points in the moduli space of irreducible ASD connections on a trivial \(SU(2)\) bundle \(P \to X\) and is zero when the assumptions in Theorem [11] are satisfied [15].

2. Seiberg-Witten invariant

2.1. Moduli space over circle bundle. Let \(X\) be a smooth 4-manifold admitting a free circle action and the circle bundle \(\pi: X \to Y\) has Euler number 1. We can equip \(X\) with a metric of the form \(g_X = \eta \otimes \eta \oplus \pi^* g_Y\) where \(g_Y\) is a any metric on \(Y\)
and \(i\eta\) is a connection 1-form of the circle bundle \(\pi : X \to Y\). Under these settings, Scott Baldridge proved in [4] that the \(Spin^c\)-structures \(\xi\) for which \(SW_X(\xi) \neq 0\) are pulled back from the ones on \(Y\) and the moduli space of \(Y\) equipped with the metric \(g_Y\) is homeomorphic (or orientation preserving diffeomorphic for well chosen metric and perturbation) to a component of the moduli space of \(X\) equipped with the metric \(g_X = \eta \otimes \eta \otimes \pi^* g_Y\).

**Theorem 2.1.** (Baldridge [4]) The pullback map induces a homeomorphism

\[
\pi'^* : \mathcal{M}^*(Y, g_Y, \delta) \to \mathcal{N}^*(X, g_X, \pi^* (\delta)^+) .
\]

There exists pairs \((g_Y, \delta)\) such that the two moduli spaces are smooth and \(\pi'^*\) is an orientation-preserving diffeomorphism.

To get an idea of the proof of this theorem, we consider the projection map \(\pi : X \to Y\) which induces a map between moduli space \(\pi^* : \mathcal{M}^*(Y, g_Y) \to \mathcal{M}^*(X, g_X)\). Given a proper perturbation 2-form \(\delta\) on \(Y\), and pull-back perturbation 2-form \(\pi^* (\delta)\) then \(\pi'^* : \mathcal{M}^*(Y, g_Y, \delta) \to \mathcal{M}^*(X, g_X, \pi^* (\delta)^+)\) is a map between smooth moduli spaces. The map \(\pi'^*\) is injective. This can be seen by considering two pairs of solutions of Seiberg-Witten equation over \(Y\) which are pulled back to solutions over \(X\), \((A, \Phi), (A', \Phi')\) which differ by a gauge transformation \(g \in \text{Map}(X, S^1)\). It remains to check that \(g\) is a pull-back from a gauge transformation \(g' \in \text{Map}(Y, S^1)\). It’s not hard to see \(g\) can be viewed as a section of \(\pi'^*(\text{End}(\text{det}(S)))\) where \(S\) is the spinor bundle over \(Y\). The connection \(\nabla_{\text{End}}\) on the bundle \(\text{End}(\pi'^*(W))\) satisfies

\[
(\nabla_{T}^\text{End}) g(\Phi) = \nabla_T^\Lambda(g\Phi) - g\nabla_T^\Lambda(\Phi) = 0 ,
\]

where \(T\) is a vertical vector field of unit length along the fiber, because \(g\Phi = \Phi'\) is a pull-back from spinor over \(Y\). By ellipticity of the first Seiberg-Witten equation \(D_A \Phi = 0\) as a function of \(\Phi\), we know \(\Phi \neq 0\) on a dense open subset. Therefore \(\nabla_{T}^\text{End} g = 0\) meaning \(g\) is constant along the fiber. This shows \(g\) is a pull-back from a gauge transformation \(g' \in \text{Map}(Y, S^1)\).

As in [4], the image of \(\pi^* : \mathcal{M}^*(Y, g_Y, \delta) \to \mathcal{M}^*(X, g_X, \pi^* (\delta)^+)\) is denoted by \(\mathcal{N}^*(X, g_X, \pi^* (\delta)^+)/\), which is the component in \(\mathcal{M}^*(X, g_X, \pi^* (\delta)^+)\) with \(Spin^c\) structures pulled back from \(Y\). To prove that \(\pi'^*\) is a diffeomorphism, we need a description of the tangent space to the moduli space at a solution \(S_0\). This is done by considering the deformation complex at \(S\) and identifying the tangent space to \(S\) with \(H^1_S\), the first cohomology group of the complex. It’s proved in [4] that \(\pi'^*(H^1_{S_0}) = H^1_S\), where \(S\) is an irreducible solution over \(X\) and \(S_0\) is a solution over \(Y\). In addition, we can see that \(\pi : X \to Y\) preserves the homology orientation. Given an ordered base of \(H^1(Y; \mathbb{R})\), we can use Gysin sequence to see that \(H^1(X; \mathbb{R})\) is isomorphic to \(H^1(Y; \mathbb{R})\), so an orientation in \(H^1(Y; \mathbb{R})\) gives one in \(H^1(X; \mathbb{R})\). Note that the homology orientation for \(X\) is an orientation for the vector space \(H^1(X; \mathbb{R}) \oplus H^0(X; \mathbb{R}) = H^1(X; \mathbb{R})\) when \(X\) is a homology \(S^3 \times S^1\).

2.2. **Seiberg-Witten invariant for \(b_1(Y) = 1\).** The Baldridge theorem above helps us understand SW-invariants over total space of circle bundle in terms of those over the base space. In this subsection, we will focus on the 3-dimensional SW-invariant over the base space of the circle bundle \(\pi : X \to Y\). Note that the Baldridge theorem has no restriction on \(b_1\). When \(b_1(Y) > 1\), the SW-invariant is a diffeomorphism invariant while in the case when \(b_1(Y) = 1\), there is a chamber
structure and we have two invariants $SW_Y^+$ and they are related by the following fundamental wall-crossing formula by Meng and Taubes

**Theorem 2.2.** (Meng-Taubes [11]) Let $Y$ be the homology $S^2 \times S^1$ obtained from 0-framed surgery on a knot $K \subset S^3$. Then

$$SW_Y^+ \cdot (t - t^{-1})^2 = \Delta_K(t^2),$$

where $t = t_T$ for the generator $T$ of $H^2(Y; \mathbb{Z}) = \mathbb{Z}$ satisfying $T \cdot \lambda = 1$.

When $Y$ is homology $S^2 \times S^1$, there is no torsion element in $H_1(Y)$, the spin$^c$-structures $s$ over $Y$ are classified by $c_1(s) := c_1(det(S)) \in H^2(Y, \mathbb{Z})$. We know $c_1(s)$ is an even class for it is an integral lift of Stiefel-Whitney class $w_2$. So there is a 1-1 correspondence between $k \in \mathbb{Z}$ and spin$^c$-structures $s_k$ with $c_1(s_k) = 2k$. The pullback spin$^c$-structure $\pi^*s_k$ over $X$ are equivalent if $X$ is homology $S^3 \times S^1$, we will denote this unique spin$^c$ structure by $\xi_0$. In view of Theorem 2.1 the Seiberg-Witten invariant of the spin$^c$-structure $\xi_0$ over $X$ is equal to the sum of the invariants $SW_Y(s_k)$ over all the spin$^c$-structures on $Y$.

In general, there is a small-perturbation Seiberg-Witten invariant defined for 3-manifold $Y$ with $b_1(Y) = 1$. It is defined using Seiberg-Witten equation with an exact perturbation. In the case when $b_1(Y) = 1$, the existence of reducible solution gives $FA = \delta$ where $\delta$ is the perturbation 2-form. This condition gives a codimension 1 "wall" in $H^2(Y; \mathbb{R})$ since it’s equivalent to $(2\pi c_1(s) + \delta) \cdot \lambda = 0$ for a generator of $H^1(Y; \mathbb{R})$ dual to the orientation of $H_1(Y; \mathbb{R})$. When the perturbation form $\delta$ is an exact 2-form, the small-perturbation Seiberg-Witten invariant $SW^0_Y(s_k)$ [7] is well defined for $Y$ with $b_1(Y) = 1$. To see that the Seiberg-Witten invariant of the spin$^c$-structure $\xi_0$ over $X$ with parameter $(g_Y, \pi^*(\delta)^+)$ is equal to the sum of the invariants $SW^0_Y(s_k)$ over all the spin$^c$-structures on $Y$ with parameter $(g_Y, \delta)$, we need to verify first that both sides are well defined under suitable choice of $(g_Y, \delta)$. Consider the exact perturbation $\delta = da \in \Omega^2(Y; \mathbb{R})$, the pull-back $\pi^*\delta = \pi^*(da) = d\pi^*a \in \Omega^2(X; \mathbb{R})$ to $X$ is a $S^1$-invariant exact perturbation 2-form after projecting to self-dual component. Since $Y$ is three dimensional, we know the expected dimension of the moduli space is 0 and in addition, we can find metric and exact perturbation $(g_Y, \delta)$ so that the moduli space is smooth without any reducible solution. In terms of the deformation complex associated with the gauge action and Seiberg-Witten equation

$$0 \to \Omega^0(Y; i\mathbb{R}) \xrightarrow{\delta^0} \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S) \xrightarrow{\delta^1} \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S) \to 0,$$

where the first map $\delta^0$ at a solution $(A_0, \Phi_0)$ is given by the derivative of gauge group action,

$$\delta^0(\gamma) = (2d\gamma, -\gamma \Phi_0),$$

and the second map $\delta^1$ at a solution $(A_0, \Phi_0)$ is given by

$$\delta^1(a, \phi) = (* (da - \frac{1}{2} \sigma(\Phi_0, \phi)), D_{A_0} \phi + \frac{1}{2} a \cdot \Phi_0),$$

we know that the $H^1_{(A_0, \Phi_0)} = H^1_{(A_0, \Phi_0)} = H^2_{(A_0, \Phi_0)} = 0$ for the complex above by our assumption on $(g_Y, \delta)$. We have a corresponding complex on $X$

$$0 \to \Omega^0(X; i\mathbb{R}) \to \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+) \to \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^-) \to 0$$
at a solution \((A, \Phi) = \pi^*(A_0, \Phi_0)\) defined in a similar way. By Baldridge’s theorem in [1], we know that under the parameter \((g_X, \pi^*(\delta)^+)\), \(\pi\) induces an isomorphism between \(H^1_{(A, \Phi)}\) and \(H^1_{(A_0, \Phi_0)}\), thus \(H^1_{(A, \Phi)} = 0\). When \(X\) is a 4-manifold with free circle action, the expect dimension of the moduli space is 0 by direct computation. So at each irreducible solution \((A, \Phi) \in \mathcal{M}^*(X, g_X, \pi^*(\delta)^+)\), \(H^0_{(A, \Phi)} = H^1_{(A, \Phi)} = 0\). Each equivalence class of solution in \(\mathcal{M}^*(X, g_X, \pi^*(\delta)^+)\) is then an isolated point with smooth neighborhood modeled on the zero of the Kuranishi map \(H^1_{(A, \Phi)} \to H^2_{(A, \Phi)}\). So there is a well-defined number (not an invariant) \(SW_X(\xi_0, g_X, \pi^*(\delta)^+)\) defined by taking the algebraic count of points in \(\mathcal{M}^*(X, g_X, \pi^*(\delta)^+)\).

Now using the small-perturbation Seiberg-Witten invariant, the sum of Seiberg-Witten invariant over all \(spin^c\)-structures \(\xi_k\) on \(Y\) is equal to

\[
\sum_{k \in \mathbb{Z}} SW^0_Y(\xi_k) = \sum_{k \in \mathbb{Z}} a_{1+|k|} + 2a_{2+|k|} + 3a_{3+|k|} + \cdots ,
\]

where \(a_i\)’s on the right hand side are coefficients of the normalized Alexander polynomial of \(Y\). It’s not hard to check that the right hand side is the \(\Delta_Y^c(1)\). Therefore, by the discussion above, we know that the Seiberg-Witten invariant of the \(spin^c\)-structure \(\xi_0\) over \(X\) with parameter \((g_X, \pi^*(\delta)^+)\) is equal to \(\Delta_Y^c(1)\).

3. Correction Term

3.1. Neck Stretching Operation. The correction term can be simplified by using the neck stretching operation discussed in detail in [10]. Let \(M\) be the 3-submanifold representing the Poincare dual to the generator of \(H^1(X; \mathbb{Z})\). The metric on \(X\) induces a metric on \(M\) by restriction. Assuming that the metric \(g_X\) is a product in a neighborhood \([-\epsilon, \epsilon] \times M\), \(\epsilon > 0\). Consider the manifold “with long neck”

\[X_R = W \cup ([0, R] \times M),\]

where \(W\) is the cobordism obtained by cutting \(X\) along \(Y\). It’s prove in [10] that under certain assumptions, this long neck manifold \(X_R\) with metric \(g_R\) obtained by gluing metric \(g_X|_W\) and product metric on the cylinder \([0, R] \times M\) can be used to compute the correction term.

**Theorem 3.1** ([10]).

\[\omega(X_R, g_R) = \text{index} \mathcal{D}^+(Z_+(M), g, \beta) + \sigma(Z)/8,\]

where \(Z_+(M) = Z \cup ([0, \infty) \times M)\) and \(Z\) is a spin 4-manifold with \(\partial Z = Y\). It remains to check that the metric \(g_X = \pi^*(g_Y) + \eta \otimes \eta\) used in computing the Seiberg-Witten invariant satisfies the following assumption from [10].

**Assumption 3.2.** The Dirac operator

\[\mathcal{D}^+(W_\infty, g_\infty) : L^2(W_\infty; S^+) \to L^2(W_\infty; S^-)\]

is invertible, where \(W_\infty = ((-\infty, 0] \times M) \cup W \cup ([0, +\infty) \times M)\) and \(g_\infty\) is the metric on \(W_\infty\) induced by \(g_X\).

This metric in the above assumption exists in the case when \(X\) is an integral homology \(S^1 \times S^3\) by the Theorem 10.3 in [10]. So in the following sections, we will
focus on computing $\omega(X_R, g_R) = \text{index} D^+ (Z_+(M), g, \beta) + \sigma(Z)/8$. Using Atiyah-Patodi-Singer index theorem [3], the correction term can be computed as

$$\omega(X_R, g_R) = \text{index} D^+ (Z_+(M), g, \beta) + \sigma(Z)/8$$

$$= \left( \int_Z \hat{A}(p) - \frac{1}{2} h_D - \frac{1}{2} \eta_D(M) \right) + \frac{1}{8} \left( \int_Z L(p) - \eta_{\text{Sign}}(M) \right)$$

$$= -\frac{1}{2} h_D - \frac{1}{2} \eta_D(M) - \frac{1}{8} \eta_{\text{Sign}}(M).$$

Here $h_D := \dim \ker(D^+_M)$.

3.2. Eta Invariants of Dirac Operator. Let $M$ be the restriction of the circle bundle $X \to Y$ to a closed surface $\Sigma$ which generates $H_2(Y; \mathbb{Z})$. Equip $\Sigma$ with a constant sectional curvature metric $g_\Sigma$ such that Vol$(\Sigma) = \pi$. The induced metric on $M$ by restriction can be written as $g_M = \pi^* g_\Sigma \oplus \eta \otimes \eta$ and using this metric we can split $T^* M = \langle \eta \rangle \oplus \pi^* T^* \Sigma$ orthogonally. By rescaling the length of the fiber, we can form a family of metrics, parametrized by fiber length, by $g_M = \pi^* g_\Sigma \oplus \eta_r \otimes \eta_r$,

where $\eta_r = r \eta$. For each $g_M$, there exists a Levi-Civita connection $\nabla^r$ which can be written in simple matrix form in well-chosen local frames. In [12], the local orthonormal frame for $T^* M = \langle \eta_i \rangle \oplus \pi^* T^* \Sigma$ is chosen to be $(\eta_i, \eta^1, \eta^2)$ so that $\eta^i = \pi^* \theta^i$, $i = 1, 2$ where $\theta^i$ is a local orthonormal frame of $T^* \Sigma$ satisfying

$$d \theta^1 = \kappa \theta^1 \wedge \theta^2$$

and

$$d \theta^2 = 0.$$

The existence of this local frame comes from the classification of space forms. The connection 1-form under this local frame can be written in matrix form as

$$\omega_r = \begin{pmatrix}
0 & -r \eta^2 & -r \eta^1 \\
r \eta^2 & 0 & r \eta_r - \kappa \eta^1 \\
r \eta^1 & -r \eta_r + \kappa \eta^1 & 0
\end{pmatrix}.$$

In [12], Nicolaescu studied the Dirac operators of type $D_r$ associated to the connection with local connection 1-form of the above form when $r$ is small using the adiabatic limit technique. Note that Theorem 3.1 holds when we use the partial rescaling metrics $g_X = \pi^*(g_Y) + r^2 \eta \otimes \eta$ for arbitrarily small positive $r$. To see this, we use a result in [6] on the asymptotic behavior of spectrum of $D_r$. Let $\{\lambda_r\}$ denote the spectrum of $D_r$, by Dai’s result of Theorem 1.5 in [6], $\lambda_r$ is analytic on $r$ and either $|\lambda_r| \geq \frac{1}{r} \lambda_0 \gg 0$ for $r$ sufficiently small or has the asymptotic formula below as

$$|\lambda_r| \sim \lambda_1 + \lambda_2 r + ...$$

and when $\lambda_1 \neq 0$, the spectrum of $D_r$ satisfies

$$|\lambda_r| \geq \frac{1}{2} |\lambda_1| \text{ when } r \text{ is sufficiently close to 0.}$$

It remains to deal with the case when $\lambda_1 = 0$, in which case $\lambda_r$ decays at least linearly in $r$. It’s sufficient to show that the first eigenvalue estimate Proposition 7.1 in [11] holds uniformly for $r$ when $r$ is small. By the same idea in the proof of Proposition 7.1 and the result in [6], the linear operator $T_{+,r}(\lambda, R) : V_i(Y_2) \oplus
Proposition 3.4. \( r \) independent of \( D \) is replaced with \( \ker D \) is an isomorphism for each \( D \). Now we can choose \( \epsilon > 0 \) uniformly bounded below by (3.2). Under this assumption, the polynomials \( \lambda \epsilon \geq R \geq 0 \) has no eigenvalues in the interval \([0, \frac{\lambda}{\lambda/\epsilon}] \). By (3.4), we have that since \( \lambda / \lambda_0, r \) decays to \( \lambda \) at the rate of polynomial \( P(r) \) by (3.2), then by (3.3), we have \( \lambda / \lambda_0, r \leq \epsilon_2 \), which implies that \( \lambda \leq \epsilon_2 \cdot P(r) \). By (3.3), we have \( \lambda_0, R \geq \epsilon_0 \), so \( R \geq \epsilon_0 / \lambda_0, r = \epsilon_0 / P(r) \). 

Proof. Using (3.3) and (3.4), and the asymptotic formula (3.2), we have that since \( \lambda_0, r \) decays to \( \lambda \) at the rate of polynomial \( P(r) \) by (3.2), then by (3.4), we have \( \lambda / \lambda_0, r \leq \epsilon_2 \), which implies that \( \lambda \leq \epsilon_2 \cdot P(r) \). By (3.3), we have \( \lambda_0, R \geq \epsilon_0 \), so \( R \geq \epsilon_0 / \lambda_0, r = \epsilon_0 / P(r) \). 

Under the assumption that the Dirac operator on the base satisfies \( \ker D_Y = 0 \), then by Theorem 1.5 in [23], we know \( \lambda_1 \in \text{spec}(D_Y \otimes \ker D_{S^1}) \), thus \( \lambda_1 \neq 0 \) in (3.2). Under this assumption, the polynomials \( \epsilon_2(r) \) and \( R_0(r) \) can be chosen to be independent of \( r \), and furthermore the first eigenvalue estimate is uniform in the fiber length \( r \). See [10].

Proposition 3.4. Assuming that the spin Dirac operator 
\[
D^+ : L^2(Z) \oplus \bigoplus (L^2(W_i)) \rightarrow L^2(W^\infty; S^+)
\]
is an isomorphism for each \( r \) and the Levi-Civita Dirac operator on the base satisfies \( \ker D_Y = 0 \). Then there exists constants \( R_0 > 0 \) and \( \epsilon_1 > 0 \) such that for any \( R \geq R_0 \), the operator 
\[
\Delta_R = D^+ D^+ : L^2(X; S^+) \rightarrow L^2(X; S^+)
\]
has no eigenvalues in the interval \([0, \epsilon_1^2] \).

Using Proposition 3.4, we can see Theorem 3.1 holds for \( g_X^r \) with arbitrarily small \( r > 0 \) by checking the proof in [10]. In step 6 of the proof in [10],
\[
K : L^2(Z) \oplus \bigoplus (L^2(W_i)) \rightarrow L^2(Z) \oplus \bigoplus (L^2(W_i)) \oplus \bigoplus V_+(M^+) \oplus \bigoplus V_-(M^-)
\]
sending \( \phi_0 \oplus (\phi_1, \phi_2, ... \phi_n) \) to 
\[
0 \oplus e^{-R \pi \cdot \phi_1} |M^+_{\phi_1}| \oplus e^{-R \pi \cdot \phi_2} |M^+_{\phi_2}| \oplus (e^{-R \pi \cdot \phi_0} + e^{-R \pi \cdot \phi_1}) \oplus (e^{-R \pi \cdot \phi_1} + e^{-R \pi \cdot \phi_2}).
\]
Now we can choose \( r = r_1 \) chosen above, then when \( g_X^r \) is replaced with \( g_X^{r_1} \), \( D \) is replaced with \( D_{r_1,M} \). If the minimum absolute value of eigenvalues of \( D \) is uniformly bounded below by \( \epsilon_1 > 0 \) for all sufficiently small \( r > 0 \), then we can
find a sufficiently large $R$ so that $e^{-Rt}$ is sufficiently small, so the same argument works. The following theorem gives a formula of the $\eta(D_r)$.

Theorem 3.5 (12). For all $0 < r \ll r_0$, we have

$$\frac{1}{2}\eta(D_r) = \frac{l}{12} - \text{Sign}(l)h_{1/2} + \frac{l}{12}(\bar{r}^4 - r^2).$$

Here $h_{1/2}$ is the dimension of global holomorphic sections of $K^1_\Sigma$, the square root of canonical bundle over $\Sigma$ and $l$ is the Euler number of the circle bundle $M \to \Sigma$. The proof in [12] by Nicolaescu is done by studying the variation of $\eta(D_r)$ as follows: Let $\xi_r = \frac{1}{2}(\eta(D_r) + h(D_r))$ where $h(D_r) = \dim \ker(D_r|M)$, then by Atiyah-Patodi-Singer index theorem, we can get a variation formula for $\xi_r$ in terms of spectral flow by studying the Dirac operator $D_u$ on cylinder $[0,1] \times M$ equipped with metric $g = du^2 \oplus g_{r(u)}$, $u$ is a coordinate on $[0,1]$, and $\nabla$ is the Levi-Civita connection of $g$. We have

$$\xi_{r_1} - \xi_{r_0} = SF(D_{r(u)}) + \int_{[0,1] \times M} \hat{A}(\nabla).$$

According to [1], $D_{r(u)}$ can be chosen to be invertible for each $u$, so the term $SF(D_{r(u)}) = 0$. The remaining term

$$\int_{[0,1] \times M} \hat{A}(\nabla)$$

can be explicitly computed by using Chern-Simons transgression form

$$T \hat{A}(\nabla^{r(0)}, \nabla^{r(1)}) = \frac{d + 1}{2} \int_0^1 \hat{A}(\omega, \Omega_t) dt,$$

where $\omega = \nabla^{r(0)} - \nabla^{r(1)}$ and $\Omega_t$ is curvature form of $\nabla^{r(0)} + t\omega$. We have

$$\int_{[0,1] \times M} \hat{A}(\nabla) = \int_M T \hat{A}(\nabla^{r(0)}, \nabla^{r(1)}) = \frac{d + 1}{2} \int_0^1 \hat{A}(\omega, \Omega_t) dt,$$

which follows from a general lemma below

Lemma 3.6. Let $F : g \times g \times \cdots \times g \to \mathbb{R}$ be a $k$-linear function on Lie algebra of $G$ and $F$ is invariant under adjoint action of $G$ on $g$. Given a linear path of connection 1-form $\omega_t = \omega_0 + t\alpha$ on a principal $G$-bundle $P$, $\Omega_t = d\omega_t + \omega_t \wedge \omega_t$ is the curvature 2-form of $\omega_t$, then

$$\frac{d}{dt}F(\Omega_t, ..., \Omega_t) = k dF(\alpha, \Omega_t, ..., \Omega_t).$$

Proof. By definition

$$\Omega_t = d\omega_t + \omega_t \wedge \omega_t$$

$$= d\omega_0 + t d\alpha + (\omega_0 + t\alpha) \wedge (\omega_0 + t\alpha)$$

$$= \Omega_0 + t d\alpha + t\omega_0 \wedge \alpha + t\alpha \wedge \omega_0 + t^2 \alpha \wedge \alpha$$

$$d\Omega_t = d\omega_t \wedge \omega_t - \omega_t \wedge d\omega_t$$

$$= (\Omega_t - \omega_t \wedge \omega_t) \wedge \omega_t - \omega_t \wedge (\Omega_t - \omega_t \wedge \omega_t)$$

$$= [\Omega_t, \omega_t].$$
Using linearity of $F$, we have \( \frac{d}{dt} F(\Omega_t, \ldots, \Omega_t) = k F(\alpha + [\omega_t, \alpha], \Omega_t, \ldots, \Omega_t) \), and
\[
dF(\alpha, \Omega_t, \ldots, \Omega_t) = F(\alpha + [\omega_t, \alpha], \Omega_t, \ldots, \Omega_t) + (k - 1) F(\alpha, [\omega_t, \Omega_t], \ldots, \Omega_t).
\]

Since $F$ is invariant under adjoint action,
\[
F([\omega_t, \alpha], \Omega_t, \ldots, \Omega_t) - (k - 1) F(\alpha, [\omega_t, \Omega_t], \ldots, \Omega_t) = 0,
\]
it’s immediate to get
\[
\frac{d}{dt} F(\Omega_t, \ldots, \Omega_t) = k dF(\alpha, \Omega_t, \ldots, \Omega_t).
\]

\[\square\]

### 3.3. Eta Invariants of Signature Operator

In [13], Ouyang computed the $\eta$-invariant of signature operator for circle bundles over surface $\Sigma$. In fact, he proved a more general theorem when $\Sigma$ is orbifold.

**Theorem 3.7** ([13]). Let $p : E \to \Sigma$ be a complex line bundle over surface $\Sigma$. Equip the fiber with metric $\tilde{g}$ and let $\tilde{\nabla}$ be a $\tilde{g}$ preserving connection in $E$. Assume the curvature $\tilde{R}$ is constant on $F$. Then the $\eta$-invariant of the circle bundle of radius $r$ is given by
\[
\eta(S_rE) = \frac{2}{3} l \left\{ \frac{\pi r^2}{\text{Vol}(\Sigma)} \chi - \left( \frac{\pi r^2}{\text{Vol}(\Sigma)} \right)^2 l^2 \right\} + \frac{1}{3} l - \text{Sign}(l),
\]
where $l$ is the Euler number of the line bundle $E \to \Sigma$, $\chi$ is the Euler characteristic of $\Sigma$.

We can check that the corresponding disk bundle of the circle bundle $M \to \Sigma$ equipped with connection $\eta$ and the metric $g_M = g_F \oplus \pi^* g_\Sigma = \eta \otimes \eta \oplus \pi^* g_\Sigma$ satisfies the conditions of the theorem above. First extend the metric from $M \to \Sigma$ to its disk bundle $E \to \Sigma$ by setting
\[
g_E = dr^2 + r^2 g_F + \pi^* g_\Sigma = \pi \otimes \eta + \pi^* g_\Sigma.
\]

The connection $\tilde{\nabla}$ can be defined to be of the form
\[
\tilde{\nabla} = d \oplus \pi^*(\nabla^\Sigma),
\]
where $\nabla^\Sigma$ is the Levi-Civita connection of $g_\Sigma$. In fact, for any local vector fields $X, Y, Z$ on the fiber of $E \to \Sigma$, we have
\[
\tilde{\nabla}_Z(\eta \otimes \eta(X, Y)) = (d + i\eta)(Z)(\eta(X)\eta(Y)) = (Z\eta(Y))\eta(X) + \eta(X)(Z\eta(Y)) + 2i\eta(Z)\eta(X)\eta(Y) = \eta(\tilde{\nabla}_Z X)\eta(Y) + \eta(X)\eta(\tilde{\nabla}_Z Y) = \eta \otimes \eta(\tilde{\nabla}_Z X, Y) + \eta \otimes \eta(X, \tilde{\nabla}_Z Y).
\]

Therefore, we can see that $\tilde{\nabla}$ is compatible with the fiber metric. The curvature tensor of $\tilde{\nabla}$, $\tilde{R}$ is pulled back from the curvature tensor $R$ of $\nabla^\Sigma$, so it is invariant along the fiber.
4. Result

It’s not hard to see from [3.5] and [3.6] that

\[
\omega(X, g_X, \beta) = -\frac{1}{2} h_D - \frac{1}{2} \eta(D, g) - \frac{1}{8} \eta_{\text{sign}}(M) = -\frac{1}{2} h_D + h_{1/2}
\]

Note here in \( \omega(X, g_X, \beta) \) we use the Levi-Civita connection of \( g_X \) to define the \( \eta \)-invariant of Dirac operator, however, in the definition of Seiberg-Witten invariant, the connection we used is circle bundle compatible connection of the form \( \nabla = d \otimes \pi^*(\nabla^Y) \). The idea to solve this problem is to consider a path of connections \( \nabla^t, t \in [0, 1] \) connecting the Levi-Civita connection and the bundle compatible connection \( \tilde{\nabla} \) such that \( \nabla^t \) is compatible with \( g_X \) for each \( t \in [0, 1] \), the associated Dirac operators \( D_{A^t}^r \) at time \( t \) can be viewed as a compact perturbation of \( D_{A^0}^r \), so have the same index.

Consider a path of connections \( \nabla^t \) by generalizing the method in [12] to 4-manifolds: first define a sequence of bundle metrics parameterized by the length of fiber \( g_X^{(r)} = r^2 \eta \otimes \eta \otimes \pi^* g_Y \) where \( \eta \) be the globally defined connection 1-form of length 1 with respect to the metric \( g_X = g_X^{(1)} \). Then we complete \( r \eta \) to form a local orthonormal coframe of the form \( \{ e^0 = r \eta, e^1, e^2, e^3 \} \) and let \( \{ e_0, e_1, e_2, e_3 \} \) be the corresponding local dual orthonormal frame with respect to the metric \( g_X^{(r)} \). We define as in [12] a family of bundle maps \( L_t : TX \to TX \) locally by

\[
e_0 \to t e_0, \quad e_i \to e_i, \quad i = 1, 2, 3
\]

where \( e_i \) is the vector field of the free circle action defined earlier. \( L_t \) defines an isometry from \( (TX, g_X^{(rt)}) \) to \( (TX, g_X^{(r)}) \) for \( r > 0 \) and \( t \in (0, 1] \). Now the connection defined by

\[
\nabla^{r,t} = L_t \nabla^t L_t^{-1}
\]

is compatible with \( g_X^{(r)} \). To see this, let \( X, Y, Z \) be local vector field on \( X \) and compute the derivative of \( g^{(r)}(Y, Z) \) in the direction \( X \).

\[
X g^{(r)}(Y, Z) = X g^{(rt)}(L_t^{-1}Y, L_t^{-1}Z) = g^{(rt)}(\nabla_X^{(r)} L_t^{-1}Y, L_t^{-1}Z) + g^{(rt)}(L_t^{-1}Y, \nabla_X^{(r)} L_t^{-1}Z)
\]

We will choose \( \nabla^{r,t}, t \in [0, 1] \) as our path of connections. Using the local frame defined earlier, we can write down the matrix of connection 1-form \( \omega \) as follows

\[
\omega_{r,t} = \begin{bmatrix}
0 & r a_{12}^{(t)} e^2 + r a_{13}^{(t)} e^3 & -r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 & -r a_{21}^{(t)} e^1 - r a_{23}^{(t)} e^3 \\
-r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 & 0 & -r a_{12}^{(t)} e^2 + r a_{13}^{(t)} e^3 & -r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 \\
-r a_{12}^{(t)} e^1 + r a_{23}^{(t)} e^2 & -r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 & 0 & -r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 \\
-r a_{12}^{(t)} e^1 + r a_{23}^{(t)} e^2 & -r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 & -r a_{12}^{(t)} e^2 - r a_{13}^{(t)} e^3 & 0
\end{bmatrix},
\]

where \( a_{ij}^{(t)} = t a_{ij} \) and \( a_{ij} \) is defined by

\[
d\eta = e^1 \wedge (a_{12} e^2 + a_{13} e^3) + e^2 \wedge (-a_{12} e^1 + a_{13} e^3) + e^3 \wedge (-a_{12} e^1 - a_{23} e^2).
\]

The connection 1-form matrix of \( \tilde{\nabla} \) is

\[
\omega = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \omega^1 & \omega^2 & \omega^3 \\
-\omega^1 & 0 & \omega^2 & 0 \\
-\omega^1 & -\omega^2 & 0 & 0
\end{bmatrix}.
\]
We can see from the connection matrix above that $\nabla^{r,t} \to \tilde{\nabla}$ as $t \to 0$. When $t = 1$, $L_t = id$, $\nabla^{r,t}$ is just the Levi-Civita connection of $g_X^{(r)}$. The path of corresponding Dirac operators can be written down as

**Lemma 4.1.**

$$D^r_t A = D_A - \frac{1}{2} r^2 t^2 \sigma(\eta \wedge d\eta),$$

where $D^r_t A$ is the Dirac operator associated to the Levi-Civita connection $\nabla^{r,t}$ and $D_A$ is the Dirac operator associated to the connection $\tilde{\nabla}$.

**Proof.** The proof is essentially the same as the proof given in [4]. It follows by writing down the local connection 1-form matrix $\omega^{r,t}$ and $\tilde{\omega}$ for $\nabla^{r,t}$ and $\tilde{\nabla}$ respectively using a local frame as we did above and take the difference 1-form $\omega = \omega^{r,t} - \tilde{\omega} \in \Omega^1(\mathfrak{so}(T^*X))$. Then the local difference of the two corresponding Dirac operators $D^r_A$ and $D_A$ can be written as the Clifford multiplication by $\omega$

$$D^r_A - D_A = \sigma(\omega)$$

here $\omega \in \Omega^1(\mathfrak{so}(T^*X)) \cong \Omega^1(\Lambda^2 T^*X)$, where the latter is the space of 1-forms with value in the exterior square of $T^*X$. Using the isomorphism

$$(a^k_j) \mapsto \frac{1}{2} \sum_{j < k} a^j_k e^j \wedge e^k$$

from $\mathfrak{so}(4)$ to $\Lambda^2 T^*X$, we can write $\omega$ as an element in $\Omega^1(\Lambda^2 T^*X)$

$$\omega = \frac{1}{2} \sum_{i=1}^3 e^i \otimes (r \eta) \wedge \iota_{e_i}(d(r \eta)) + \frac{1}{2} r \eta \wedge d(r \eta),$$

then

$$\sigma(\omega) = -\frac{1}{2} r^2 t^2 \sigma(\eta \wedge d\eta). \quad \Box$$

It remains to show that the index of $D^r_A$ is unchanged along the path $t = 0$ to $t = 1$. As we can see from the lemma above, $D^r_A$ can be thought of as zero order perturbation of $D_A$ and by the theory in compact operator, it’s sufficient to prove the following lemma

**Proposition 4.2.** If $\omega \in \Omega^1(\Lambda^2 T^*X)$, and $i$ is the Sobolev embedding $L^2_1(W^\infty; S^+) \subset L^2(W^\infty; S^-)$, $\sigma(\omega)$ is the Clifford multiplication, then the composition

$$i \circ \sigma(\omega) : L^2_1(W^\infty; S^+) \to L^2(W^\infty; S^-)$$

is compact.

**Proof.** The Sobolev inequality may fail for non-compact manifold. So $i$ may not be a compact operator in general. We use instead the Laplace-Fourier transform introduced in [17]. Consider the following diagram,

$$\begin{align*}
L^2_1(W^\infty; S^+) \xrightarrow{\Psi} & L^2(W^\infty; S^-) \\
L^2_1(X; S^+) \xrightarrow{\hat{\Psi}} & L^2(X; S^-) \\
\end{align*}$$

\begin{align*}
\Psi & \\
\hat{\Psi} & \\
\end{align*}$$

\begin{align*}
\xrightarrow{F} & \\
\xrightarrow{F} & \\
\end{align*}$$

\begin{align*}
\psi & \\
\hat{\psi} & \\
\end{align*}$$

\begin{align*}
\xrightarrow{F} & \\
\xrightarrow{F} & \\
\end{align*}$$
Here $Ψ$ is the composition $i \circ σ(ω)$ defined above, and $F$ is the Laplace-Fourier transform. To prove the compactness of $Ψ$, consider a bounded sequence of sections $\{u_k\} \in L^2_1(W^\infty; S^+)$ and we need to prove $\{Ψ(u_k)\}$ has a convergent subsequence. To show this, we apply Laplace-Fourier transform to $\{u_k\}$ and get a sequence $\{F(u_k)\}$ of sections in $L^2_1(X; S^+)$, which is bounded by proposition 4.1 in [17]. By direct computation, $Ψ$ has the same form as $Ψ = i \circ σ(ω) : L^2_1(X; S^+) \to L^2(X; S^-)$, which is a compact operator when $X$ is compact by the Rellich theorem. So $\{Ψ(F(u_k))\}$ has a convergent subsequence. We obtain a corresponding subsequence by taking the inverse transform as is defined in [17]

$$v_k(x + n) = \frac{1}{2\pi i} \int \frac{e^{-\mu(f(x)+n)}Ψ(F(u_k))(x)}{I(ν)} \, dy.$$

We can prove that $v_k(x)$ is convergent by showing that the inverse Laplace transform

$$L^2(X; S^-) \to L^2(W^\infty; S^-)$$

is bounded. This can be seen by

$$\int_{W^\infty} |g| \cdot |v_k| \, dx = \frac{1}{2\pi i} \int \int_{W^\infty} |g| \cdot \left| \int \frac{e^{-\mu(f(x)+n)}Ψ(F(u_k))(x)}{I(ν)} \, dx \right| \, dx$$

$$\leq \frac{1}{2\pi i} \int \int_{W^\infty} |g| \cdot \|Ψ(F(u_k))(x)\|_{L^2(X)} \cdot \left( \int \left| e^{-\mu(f(x)+n)} \right|^2 \right)^{1/2}$$

$$\leq \frac{1}{2\pi i} \|g\|_{L^2(W^\infty)} \cdot \left( \int \left| e^{-\mu(f(x)+n)} \right|^2 \right)^{1/2} \cdot \|Ψ(F(u_k))\|_{L^2(X)}$$

and the fact that the integral

$$\int_{W^\infty} \int |e^{-\mu(f(x)+n)}|^2 < \infty$$

In particular, we can use the above result to prove the correction term (4.1) is 0.

$$ω(X, gX, β) = -\frac{1}{2}h_D + h_{1/2} = -\frac{1}{2}h_D + h_{1/2}$$

and in [12], Nicolescu claimed the last term is 0.

REFERENCES

1. Ammann, Bernd.; Dahl, Mattias.; Humbert, Emmanuel. Surgery and harmonic spinors. Adv. Math. 220 (2009), 523539.
2. Atiyah, Michael; Hirzebruch, Friedrich. Spin-manifolds and group actions. 1970 Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) pp. 18-28 Springer, New York
3. Atiyah, M. F.; Patodi, V. K.; Singer, I. M. Spectral asymmetry and Riemannian geometry. I. Math. Proc. Cambridge Philos. Soc. 77 (1975), 43-69.
4. Baldridge, Scott Jeremy, Seiberg-Witten Invariants, Orbitfolds, and Circle Actions. Trans. Amer. Math. Soc. 355 (2003), no. 4, 1669-1697
5. Chern, S. S.; Hirzebruch, F.; Serre, J.-P. On the index of a fibered manifold. Proc. Amer. Math. Soc. 8 (1957), 587-596.
6. Dai, Xianzhe. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. J. Amer. Math. Soc. 4 (1991), no. 2, 265321.
7. Fintushel, Ronald; Stern, Ronald J. Knots, links, and 4-manifolds. Invent. Math. 134 (1998), no. 2, 363-400.
8. Furuta, Mikio, Ohta, Hiroshi, *Differentiable structures on punctured 4-manifolds*, Topology Appl. 51 (1993), no. 3, 291-301.
9. Lim, Yuhan, *The equivalence of Seiberg-Witten and Casson invariants for homology 3-spheres*, Math. Res. Lett. 6 (1999), no. 5-6, 631-643
10. Lin, Jianfeng; Ruberman, Daniel; Saveliev, Nikolai, *A Splitting Theorem for the Seiberg-Witten Invariant of a Homology $S^1 \times S^3$*. arXiv:1702.04117 [math.GT]
11. Meng, Guowu; Taubes, Clifford Henry, *SW=Milnor torsion*, Math. Res. Lett. 3 (1996), no. 5, 661-674.
12. Nicolaescu, Liviu I. *Eta invariants of Dirac operators on circle bundles over Riemann surfaces and virtual dimensions of finite energy Seiberg-Witten moduli spaces*. Israel J. Math. 114 (1999), 61-123.
13. Ouyang, Mingqing *Geometric invariants for Seifert Fibered 3-manifold*. Trans. Amer. Math. Soc. 346 (1994), no. 2, 641659.
14. Rohlin, V. A. *New results in the theory of four-dimensional manifolds*. (Russian) Doklady Akad. Nauk SSSR (N.S.) 84, (1952). 221-224.
15. Ruberman, Daniel; Saveliev, Nikolai, *Casson-Type Invariants in Dimension Four*. arXiv:math/0501090
16. Taubes, Clifford Henry. *Casson’s invariant and gauge theory*. J. Differential Geom. 31 (1990), no. 2, 547-599.
17. Tomasz Mrowka, Daniel Ruberman, Nikolai Saveliev, *Seiberg-Witten Equations, End-Periodic Dirac Operators, and a Lift of Rohlin’s Invariant*, J. Differential Geom. Volume 88, Number 2 (2011), 333-377
18. Weimin Chen. *Casson’s invariant and Seiberg-Witten gauge theory*. Turkish J. Math. 21 (1997), no. 1, 6181.

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02453

Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015

E-mail address: dah517@lehigh.edu