Existence Result for a Superlinear Fractional Navier Boundary Value Problems

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Abstract. In this paper, we study the following fractional Navier boundary value problem

\[
\begin{cases}
D^\beta(D^\alpha u)(x) = u(x)g(u(x)), \quad x \in (0, 1), \\
\lim_{x \to 0} x^{1-\beta} D^\alpha u(x) = -a, \quad u(1) = b,
\end{cases}
\]

where \( \alpha, \beta \in (0, 1] \) such that \( \alpha + \beta > 1 \), \( D^\beta \) and \( D^\alpha \) stand for the standard Riemann–Liouville fractional derivatives and \( a, b \) are nonnegative constants such that \( a + b > 0 \). The function \( g \) is a nonnegative continuous function in \((0, \infty)\) that is required to satisfy some suitable integrability condition. Using estimates on the Green’s function and a perturbation argument, we prove the existence of a unique positive continuous solution, which behaves like the unique solution of the homogeneous problem.

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1. Introduction

Recently, many papers on fractional differential equations have been studied extensively by many researchers. The motivation for those works stems from the fact that fractional equations serve as an excellent tool to describe many phenomena in various fields of science and engineering such as viscoelasticity, electrochemistry, control theory, porous media, electromagnetism and other fields. Also, it provides an excellent tool to describe the hereditary properties of various materials and processes. Concerning the development of theory methods and applications of fractional calculus, we refer to [5, 8–12, 14, 16, 23, 24, 26, 28] and the references therein for discussions of various applications.

The theory of fractional differential equations with various boundary conditions has been developed very quickly and the investigation for the
existence, uniqueness and asymptotic behavior of positive continuous solutions attracted a considerable attention of researchers (see [1–4,6,7,13,15,17–19,21,22,25,29,30] and the references therein).

In [18], Mâagli et al. studied the following initial value problem:

$$\begin{cases}
D^\alpha u(x) = -p(x)u^\sigma, & x \in (0, 1), \\
\lim_{x \to 0^+} x^{1-\alpha}u(x) = 0,
\end{cases}$$

(1.1)

where $\alpha \in (0, 1)$, $\sigma < 1$ and $p$ is a nonnegative continuous function in $(0, 1)$ satisfying some appropriate conditions related to the Karamata class $K$ (see Definition 4 below). By a potential theory approach associated with $D^\alpha$ and some technical tools relying to Karamata regular variation theory, the authors proved the existence, uniqueness and asymptotic behavior of a positive solution to problem (1.1) in the weighted space of continuous functions $C_{1-\alpha}([0, 1])$.

Later, in [19], Mâagli and Dhifli studied the following sublinear fractional Navier boundary value problem:

$$\begin{cases}
D^\beta(D^\alpha u)(x) = -p(x)u^\sigma, & x \in (0, 1), \\
\lim_{x \to 0^+} x^{1-\beta}D^\alpha u(x) = 0, & u(1) = 0,
\end{cases}$$

(1.2)

where $\sigma \in (-1, 1)$, $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and $p$ is a nonnegative continuous function in $(0, 1)$. Under some appropriate conditions on the function $p$ and using the Schäfer fixed point theorem, the authors proved the existence of a unique positive solution to problem (1.2). Further, based on the asymptotic behavior for the Green function and some technical tools relying on Karamata regular variation theory, the authors gave a global asymptotic behavior of such solutions to problem (1.2).

Inspired by the above-mentioned papers, we aim at studying in this paper the following superlinear fractional Navier boundary value problem:

$$\begin{cases}
D^\beta(D^\alpha u)(x) = u(x)g(u(x)), & x \in (0, 1), \\
\lim_{x \to 0^+} x^{1-\beta}D^\alpha u(x) = -a, & u(1) = b,
\end{cases}$$

(1.3)

where $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and $a$ and $b$ are nonnegative constants such that $a + b > 0$. The nonlinear term $g(t)$ is required to be a nonnegative continuous function on $[0, \infty)$ satisfying some appropriate conditions related to the class of functions $K_{\alpha,\beta}$ defined as follows.

**Definition 1.** Let $\alpha, \beta \in (0, 1]$. A Borel measurable function $q$ in $(0, 1)$ belongs to the class $K_{\alpha,\beta}$ if $q$ satisfies the following:

$$\int_0^1 r^{\alpha-1}(1 - r)^{\alpha+\beta-1}|q(r)|dr < \infty.$$

We use the properties of this class to investigate an existence result for the fractional Navier boundary value problem (1.3). To state our main result in this paper, we need to introduce some convenient notations. Throughout this paper, we denote $\mathcal{B}((0, 1))$ the set of Borel measurable functions in $(0, 1)$ and $\mathcal{B}^+(0, 1))$ the set of nonnegatives ones. We use $C_r([0, 1])$ to denote the
set of continuous functions $f$ on $(0,1]$ such that $x \to x^r f(x)$ is continuous on $[0,1]$.

For $q \in \mathcal{B}((0,1))$, we denote

$$k_q := \sup_{x,t \in (0,1)} \int_0^1 \frac{H(x,r)H(r,t)}{H(x,t)} |q(r)| dr,$$

(1.4)

where $H(x,t)$ is the Green function of the operator $u \to -D^\beta(D^\alpha u)$ in $(0,1)$, with boundary conditions $\lim_{x \to 0^+} x^{1-\beta} D^\alpha u(x) = 0$ and $u(1) = 0$. We will prove that if $q \in \mathcal{K}_{\alpha,\beta}$, then $k_q < \infty$.

We denote by $\omega$ the unique solution of the following homogenous problem:

$$\begin{cases}
D^\beta(D^\alpha u)(x) = 0, & x \in (0,1), \\
\lim_{x \to 0^+} x^{1-\beta} D^\alpha u(x) = -a, & u(1) = b.
\end{cases}$$

(1.5)

We can easily verify that for $x \in (0,1)$

$$\omega(x) := ah_1(x) + bh_2(x),$$

where

$$h_1(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} x^{\alpha-1}(1 - x^\beta) \quad \text{and} \quad h_2(x) = x^{\alpha-1}.$$ 

(1.6)

Finally, a combination of the following hypotheses on the term $g$ is required:

(\textbf{H}_1) $g$ is a nonnegative continuous function in $[0,\infty)$.

(\textbf{H}_2) There exists a nonnegative function $q \in \mathcal{K}_{\alpha,\beta} \cap C((0,1))$ satisfying:

(i) $q(t) \leq t^{-\mu+1-\alpha} L(t)$ for $t$ near $0$ with $\mu \leq 1$ and $L \in \mathcal{K}$ satisfying

$\int_0^\alpha t^{-\mu} L(t) dt < \infty$.

(ii) $\kappa_q \leq \frac{1}{2^\alpha}$.

(iii) For each $x \in (0,1)$, the map $t \to t(q(x) - g(t\omega(x)))$ is nondecreasing on $[0,1]$.

(\textbf{H}_3) The function $t \to tg(t)$ is nondecreasing on $[0,\infty)$.

As a typical example of the function satisfying (\textbf{H}_1)–(\textbf{H}_3), we quote $g(t) = t^\sigma$, where $\sigma \geq 0$.

Our main result is the following.

\textbf{Theorem 1.} Assume (\textbf{H}_1)–(\textbf{H}_2), then problem (1.3) has a positive solution $u$ in $C_{1-\alpha}([0,1])$ satisfying

$$c_0 \omega(x) \leq u(x) \leq \omega(x), \quad x \in (0,1),$$

(1.7)

where $c_0$ is a constant in $(0,1)$.

Moreover, this solution is unique if hypothesis (\textbf{H}_3) is also satisfied.

Observe that in Theorem 1, we obtain a positive solution $u \in C_{1-\alpha}([0,1])$ to problem (1.3) whose behavior is not affected by the perturbed term. That is, it behaves like the solution $\omega$ of the homogeneous problem (1.5).

Our paper is organized as follows. In Sect. 2, we give some estimates on $H(x,t)$. In Sect. 3, for a given function $q \in \mathcal{K}_{\alpha,\beta}$ with $\kappa_q \leq \frac{1}{2^\alpha}$, we construct the Green function $H(x,t)$ of the perturbed operator $u \to -D^\beta(D^\alpha u) + q(x)u$ with boundary conditions $\lim_{x \to 0^+} x^{1-\beta} D^\alpha u(x) = 0$ and $u(1) = 0$ and we
derive some of its properties. Exploiting these results, we prove our main result in Sect. 4.

2. Definitions and Preliminary Results

2.1. Fractional Calculus

For the convenience of the reader, we recall in the following some basic definitions and elementary properties of fractional calculus (For more details, see [9,23,26]).

**Definition 2.** The Riemann–Liouville fractional integral of order \( \alpha > 0 \) of a function \( f : (0,1) \rightarrow \mathbb{R} \) is given by

\[
I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,
\]

provided that the right-hand side is pointwise defined on \((0,1)\).

**Definition 3.** The Riemann–Liouville fractional derivative of order \( \alpha > 0 \) of a function \( f \in \mathcal{B}((0,1)) \) is given by

\[
D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_0^x (x-t)^{n-\alpha-1} f(t)dt = \left( \frac{d}{dx} \right)^n I^{n-\alpha} f(x),
\]

where \( n = [\alpha] + 1 \) and \([\alpha]\) mean the integer part of the number \( \alpha \), provided that the right-hand side is pointwise defined on \((0,1)\).

**Lemma 1** ([9,26]).

(i) Let \( \alpha, \beta > 0 \) and \( 0 < a < 1 \). Let \( f \in L^1((0,a)) \cap C((0,a)) \), then we have

\[
D^\alpha I^\beta f(x) = f(x) \quad \text{for} \ x \in [0,a],
\]

\[
I^\beta D^\alpha f(x) = I^{\alpha+\beta} f(x) \quad \text{for} \ x \in [0,a], \ \beta + \alpha \geq 1.
\]

(ii) Let \( \alpha > 0 \) and \( 0 < a < 1 \). Let \( f \in L^1((0,a)) \), then

\[
D^\alpha f(x) = 0 \quad \text{if and only if} \quad f(x) = \sum_{k=1}^n c_k x^{\alpha-k},
\]

where \( n \) is the smallest integer greater than or equal to \( \alpha \) and \((c_1, \ldots, c_n) \in \mathbb{R}^n\).

(iii) Let \( \alpha > 0 \) and \( 0 < a < 1 \). Let \( f \) such that \( D^\alpha f \in L^1((0,a)) \cap C((0,a)) \), then

\[
I^\alpha D^\alpha f(x) = f(x) + \sum_{k=1}^n c_k x^{\alpha-k},
\]

where \( n \) is the smallest integer greater than or equal to \( \alpha \) and \((c_1, \ldots, c_n) \in \mathbb{R}^n\).
2.2. Karamata Class \( \mathcal{K} \)

In this subsection, we introduce the Karamata class \( \mathcal{K} \) and we recall some fundamental properties of functions belonging to this class.

**Definition 4.** The class \( \mathcal{K} \) is the set of Karamata functions \( L \) defined on \( (0, \eta] \) by

\[
L(t) := c \exp \left( \int_{t}^{\eta} \frac{z(s)}{s} \, ds \right)
\]

for some \( \eta > 1 \), where \( c > 1 \) and \( z \in C([0, \eta]) \) such that \( z(0) = 0 \).

**Remark 1.** It is clear that a function \( L \) is in \( \mathcal{K} \) if and only if \( L \) is a positive function in \( C^1((0, \eta]) \) for some \( \eta > 1 \), such that \( \lim_{t \to 0^+} \frac{iL'(t)}{L(t)} = 0 \).

As a typical example of function belonging to the class \( \mathcal{K} \), we quote

\[
L(t) = \prod_{j=1}^{m} \left( \log \left( \frac{w}{t} \right) \right)^{\xi_j},
\]

where \( \xi_j \) are real numbers, \( \log_j x = \log \circ \log \ldots \log x \ (j \text{ times}) \) and \( w \) is a sufficiently large positive real number such that \( L \) is defined and positive on \( (0, \eta] \) for some \( \eta > 1 \).

**Lemma 2 ([20, 27]).** Let \( \gamma \in \mathbb{R} \) and \( L \) be a function in \( \mathcal{K} \) defined on \( (0, \eta] \). Then, we have that

(i) if \( \gamma > -1 \), then \( \int_0^\eta s^\gamma L(s) \, ds \) converges and \( \int_t^\eta s^\gamma L(s) \, ds \sim t^{1+\gamma L(t)} \) as \( t \to 0^+ \);

(ii) if \( \gamma < -1 \), then \( \int_0^\eta s^\gamma L(s) \, ds \) diverges and \( \int_t^\eta s^\gamma L(s) \, ds \sim t^{1+\gamma L(t)} \) as \( t \to 0^+ \).

**Lemma 3 [19].** Let \( \alpha, \beta \in (0, 1] \). Let \( f \in C((0, 1)) \) such that the map \( t \to (1-t)^{\alpha+\beta-1}f(t) \) is integrable and \( |f(t)| \leq t^{-\delta}L(t) \) for \( t \) near 0, with \( \delta \leq 1 \) and \( L \in \mathcal{K} \) satisfying \( \int_0^\eta t^{-\delta}L(t) \, dt < \infty \). Then the function \( x \to \Gamma^\delta f(x) \in C((0,1)) \cap L^1((0,1)) \) and \( \lim_{x \to 0} x^{1-\beta} \Gamma^\delta f(x) = 0 \).

### 2.3. Estimates on the Green’s Function \( H(x, t) \)

This subsection is devoted to give estimates on the Green function \( H(x, t) \). From [19], the Green function of the operator \( u \to -D^2(D^\alpha u) \) in \( (0, 1) \), with boundary conditions \( \lim_{x \to 0^+} x^{1-\beta}D^\alpha u(x) = 0 \) and \( u(1) = 0 \), is given explicitly by

\[
H(x, t) = \frac{x^{\alpha-1}(1-t)^{\alpha+\beta-1} - ((x-t)^+)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)},
\]

where \( \Gamma \) is the Euler gamma function and for \( r \in \mathbb{R}, r^+ = \max(r, 0) \).

First, we recall in the following lemma due to [19] some estimates on the Green function \( H(x, t) \) and properties of the associated potential kernel defined by

\[
Vf(x) = \int_0^1 H(x, t)f(t) \, dt, \quad \text{for } f \in B^+((0, 1)) \quad \text{and} \quad x \in (0, 1).
\]
Lemma 4 [19]. Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and $\varphi \in \mathcal{B}^+((0, 1))$. Then we have

(i) For $(x, t) \in (0, 1) \times (0, 1)$, we have:

$$\frac{\alpha + \beta - 1}{\beta \Gamma(\alpha + \beta)} K(x, t) \leq H(x, t) \leq \frac{1}{\Gamma(\alpha + \beta)} K(x, t),$$

(2.2)

where $K(x, t) := x^{\alpha - 1}(1 - t)^{\alpha + \beta - 2}(1 - \max(x, t))$.

In particular,

$$\frac{\alpha + \beta - 1}{\beta \Gamma(\alpha + \beta)} x^{\alpha - 1}(1 - x)(1 - t)^{\alpha + \beta - 1} \leq H(x, t) \leq \frac{1}{\Gamma(\alpha + \beta)} x^{\alpha - 1}(1 - t)^{\alpha + \beta - 2}\min(1 - t, 1 - x).$$

(2.3)

(ii) For any $\varphi \in \mathcal{B}^+((0, 1))$, the function $x \to V \varphi(x) = \int_0^1 H(x, t) \varphi(t)dt$ belongs to $C_{1-\alpha}([0, 1])$ if and only if $\int_0^1 (1 - t)^{\alpha + \beta - 1} \varphi(t)dt < \infty$.

Next, we establish the following property of $H(x, t)$.

**Proposition 1.** We have, for $x, r, t \in (0, 1)$,

$$\frac{H(x, r)H(r, t)}{H(x, t)} \leq \frac{\beta}{(\alpha + \beta - 1)\Gamma(\alpha + \beta)} r^{\alpha - 1}(1 - r)^{\alpha + \beta - 1}. \quad \text{(2.4)}$$

**Proof.** Using Lemma 4 (i), we have for each $x, r, t \in (0, 1)$,

$$\frac{H(x, r)H(r, t)}{H(x, t)} \leq \frac{\beta r^{\alpha - 1}(1 - r)^{\alpha + \beta - 2}}{(\alpha + \beta - 1)\Gamma(\alpha + \beta)} \frac{(1 - \max(x, r))(1 - \max(t, r))}{1 - \max(x, t)}.$$

We claim that

$$\frac{(1 - \max(x, r))(1 - \max(t, r))}{1 - \max(x, t)} \leq 1 - r. \quad \text{(2.5)}$$

Indeed, by symmetry, we may assume that $x \leq t$. Then we deduce that

$$\frac{(1 - \max(x, r))(1 - \max(t, r))}{1 - t} \leq 1 - \max(x, r)$$

$$\leq 1 - r.$$

Now, by using (2.5), we obtain the required result.

Next, we recall that $\omega(x) := ah_1(x) + bh_2(x)$, where

$$h_1(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} x^{\alpha - 1}(1 - x^\beta) \quad \text{and} \quad h_2(x) = x^{\alpha - 1}, \quad \text{for } x \in (0, 1).$$

**Proposition 2.** Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and let $q \in \mathcal{B}((0, 1))$. Then we have

(i)

$$\kappa_q \leq \frac{\beta}{(\alpha + \beta - 1)\Gamma(\alpha + \beta)} \int_0^1 r^{\alpha - 1}(1 - r)^{\alpha + \beta - 1}|q(r)|dr, \quad \text{(2.6)}$$

where $\kappa_q$ is given by (1.4).
In particular, if \( q \in \mathcal{K}_{\alpha, \beta} \), then \( \kappa_q < \infty \).

(ii) For \( x \in (0,1] \), we have

\[
\int_0^1 H(x,t)h_1(t)|q(t)|dt \leq \kappa_q h_1(x) \tag{2.7}
\]

and

\[
\int_0^1 H(x,t)h_2(t)|q(t)|dt \leq \kappa_q h_2(x). \tag{2.8}
\]

In particular for \( x \in (0,1] \),

\[
\int_0^1 H(x,t)\omega(t)|q(t)|dt \leq \kappa_q \omega(x). \tag{2.9}
\]

**Proof.** Let \( q \) be a function in \( \mathcal{B}((0,1)) \).

(i) The inequality (2.6) follows immediately from (1.4) and (2.4).

(ii) Since for each \( x, t \in (0,1) \), we have \( \lim_{r \to 0} \frac{H(t,r)}{H(x,r)} = \frac{h_1(t)}{h_1(x)} \), then we deduce by Fatou’s lemma and (1.4) that

\[
\int_0^1 H(x,t)\frac{h_1(t)}{h_1(x)}|q(t)|dt \leq \liminf_{r \to 0} \int_0^1 H(x,t)\frac{H(t,r)}{H(x,r)}|q(t)|dt \leq \kappa_q,
\]

which implies that for \( x \in (0,1) \),

\[
\int_0^1 H(x,t)h_1(t)|q(t)|dt \leq \kappa_q h_1(x).
\]

Similarly, we prove inequality (2.8) by observing that

\[
\lim_{r \to 1} \frac{H(t,r)}{H(x,r)} = \frac{h_2(t)}{h_2(x)}.
\]

Inequality (2.9) follows from (2.7) to (2.8) and the fact that \( \omega(x) = ah_1(x) + bh_2(x) \). \( \square \)

**Proposition 3** [19]. Let \( \alpha, \beta \in (0,1] \) such that \( \alpha + \beta > 1 \). Let \( f \in C((0,1)) \) such that the map \( t \to (1-t)^{\alpha+\beta-1}f(t) \) is integrable and \( |f(t)| \leq t^{-\delta}L(t) \) near 0, with \( \delta \leq 1 \) and \( L \in \mathcal{K} \) satisfying

\[
\int_0^1 t^{-\delta}L(t)dt < \infty.
\]

Then \( Vf \) is the unique solution in \( C_{1-\alpha}([0,1]) \) of the boundary value problem:

\[
\begin{aligned}
D^{\beta}(D^{\alpha}u(x)) &= -f(x), & x &\in (0,1), \\
\lim_{x \to 0^+} x^{1-\beta}D^{\alpha}u(x) &= 0, & u(1) &= 0.
\end{aligned}
\]

\[
\tag{2.10}
\]

3. **Green’s Function of the Operator** \(-D^{\beta}(D^{\alpha}u) + q(x)u\)**

In this section, we will prove that the operator \(-D^{\beta}(D^{\alpha}u) + q(x)u\) has a Green function for small nonnegative function \( q \) in \( \mathcal{K}_{\alpha, \beta} \). To this end, we need the following preliminary result. Let \( \alpha, \beta \in (0,1] \) such that \( \alpha + \beta > 1 \) and let \( q \in \mathcal{B}^+((0,1)) \). For \( (x,t) \in (0,1] \times (0,1] \), put \( H_0(x,t) = H(x,t) \) and

\[
H_n(x,t) = \int_0^1 H(x,r)H_{n-1}(r,t)q(r)dr, \quad n \geq 1.
\]

\[
\tag{3.1}
\]
Now, let $\mathcal{H} : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ be defined by
\[
\mathcal{H}(x, t) = \sum_{n=0}^{\infty} (-1)^n H_n(x, t).
\]
(3.2)

For $f \in B^+((0, 1))$ and $x \in (0, 1)$, we put
\[
V_q f(x) := \int_0^1 \mathcal{H}(x, t) f(t) dt.
\]

Then, we have the following.

Lemma 5. Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and let $q$ be a nonnegative function in $K_{\alpha, \beta}$ with $\kappa_q < 1$. Then for $(x, t) \in (0, 1] \times (0, 1]$, we have

(i) For each $n \in \mathbb{N}$,
\[
H_n(x, t) \leq \kappa_n^q H(x, t).
\]
(3.3)

In particular, $\mathcal{H}(x, t)$ is well defined on $(0, 1] \times [0, 1]$.

(ii) For each $n \in \mathbb{N}$, we have
\[
L_n x^{\alpha-1} (1-x)(1-t)^{\alpha+\beta-1} \leq H_n(x, t) \leq R_n x^{\alpha-1} (1-t)^{\alpha+\beta-2} \min(1-t, 1-x),
\]
(3.4)

where
\[
L_n = \left( \frac{\alpha + \beta - 1}{\beta \Gamma(\alpha + \beta)} \right)^{n+1} \left( \int_0^1 r^{\alpha-1} (1-r)^{\alpha+\beta} q(r) dr \right)^n
\]
and
\[
R_n = \frac{1}{\Gamma(\alpha + \beta)^{n+1}} \left( \int_0^1 r^{\alpha-1} (1-r)^{\alpha+\beta-1} q(r) dr \right)^n.
\]

(iii) For $(x, t) \in (0, 1] \times [0, 1]$, we have
\[
\int_0^1 \mathcal{H}(x, r) H(r, t) q(r) dr = \int_0^1 H(x, r) \mathcal{H}(r, t) q(r) dr.
\]
(3.5)

Proof. The assertions (i) and (ii) are obtained by simple induction. Let us prove (iii). Let $n \geq 0$ and $x, t, r \in (0, 1]$. By (3.3), we have
\[
0 \leq H_n(x, t) H(r, t) q(r) \leq \kappa_n^q H(x, r) H(r, t) q(r).
\]
Hence, the $\sum_{n \geq 0} \int_0^1 H_n(x, r) H(r, t) q(r) dr$ converges. So by the dominated convergence theorem, we deduce that
\[
\int_0^1 \mathcal{H}(x, r) H(r, t) q(r) dr = \sum_{n=0}^{\infty} (-1)^n \int_0^1 H_n(x, r) H(r, t) q(r) dr
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \int_0^1 H(x, r) H_n(r, t) q(r) dr
\]
\[
= \int_0^1 H(x, r) \mathcal{H}(r, t) q(r) dr.
\]
\[\square\]
**Proposition 4.** Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and let $q$ be a nonnegative function in $\mathcal{K}_{\alpha, \beta}$ with $\kappa_q < 1$. Then the function $(x, t) \mapsto x^{1-\alpha} \mathcal{H}(x, t)$ is continuous on $[0, 1] \times [0, 1]$.

*Proof.* Firstly, we claim that for $n \in \mathbb{N}$, the function $(x, t) \mapsto x^{1-\alpha} H_n(x, t)$ is continuous on $[0, 1] \times [0, 1]$. Indeed, from (2.1) the function $(x, t) \mapsto x^{1-\alpha} H_0(x, t)$ is continuous on $[0, 1] \times [0, 1]$.

Assume that the function $(x, t) \mapsto x^{1-\alpha} H_{n-1}(x, t)$ is continuous on $[0, 1] \times [0, 1]$.

Using Lemmas 5(i) and 4(i), we have for all $(x, t) \in [0, 1] \times [0, 1]$ and $q \in \mathcal{K}_{\alpha, \beta}$, we deduce by the dominated convergence theorem that the function $(x, t) \mapsto x^{1-\alpha} H_n(x, t)$ is continuous on $[0, 1] \times [0, 1]$. This proves our claim.

Now, by using again Lemmas 5(i) and 4(i), we have for each $x, t \in [0, 1]$,

$$x^{1-\alpha} H_n(x, t) \leq \frac{1}{\Gamma(\alpha + \beta)^2} \frac{1}{(1-t)\alpha + \beta - 2} \times \min(1 - t, 1 - x) r^{\alpha - 1} (1 - r)^{\alpha + \beta - 1} q(r).$$

This implies that the series $\sum_{n \geq 0} (-1)^n H_n(x, t)$ uniformly converges on $[0, 1] \times [0, 1]$ and therefore the function $(x, t) \mapsto x^{1-\alpha} H(x, t)$ is continuous on $[0, 1] \times [0, 1]$. The proof is completed. \square

**Lemma 6.** Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$ and let $q$ be a nonnegative function in $\mathcal{K}_{\alpha, \beta}$ with $\kappa_q \leq \frac{1}{2}$. Then for $(x, t) \in (0, 1] \times [0, 1]$, we have

$$(1 - \kappa_q) \mathcal{H}(x, t) \leq \mathcal{H}(x, t) \leq H(x, t). \tag{3.6}$$

*Proof.* Since $\kappa_q \leq \frac{1}{2}$, we deduce from Lemma 5(i) that

$$|\mathcal{H}(x, t)| \leq \sum_{n=0}^{\infty} \kappa_q^n \mathcal{H}(x, t) = \frac{1}{1 - \kappa_q} H(x, t). \tag{3.7}$$

On the other hand, from the expression of $\mathcal{H}$ we have

$$\mathcal{H}(x, t) = H(x, t) - \sum_{n=0}^{\infty} (-1)^n H_{n+1}(x, t). \tag{3.8}$$

Since the series $\sum_{n \geq 0} \int_0^1 H(x, r) H_n(r, t) q(r) dr$ is convergent, we deduce by (3.8) and (3.2) that
\[ \mathcal{H}(x, t) = H(x, t) - \sum_{n=0}^{\infty} (-1)^n \int_0^1 H(x, r)H_n(r, t)q(r)dr \]

That is,
\[ \mathcal{H}(x, t) = H(x, t) - \int_0^1 H(x, r)\left(\sum_{n=0}^{\infty} (-1)^n H_n(r, t)\right)q(r)dr. \]  

On the other hand, since
\[ V(qH(., t))(x) \leq 1 - \kappa q \]
we deduce that
\[ \mathcal{H}(x, t) \geq H(x, t) - \kappa q \]
\[ \mathcal{H}(x, t) \geq H(x, t) - V(qH(., t))(x) \geq (1 - \kappa q)H(x, t). \]  

So it follows that 0 \leq \mathcal{H}(x, t) \leq H(x, t) and by (3.9)
\[ \mathcal{H}(x, t) \geq H(x, t) - V(qH(., t))(x) \geq (1 - \kappa q)H(x, t). \]

\[ \square \]

**Corollary 1.** Let \( \alpha, \beta \in (0,1) \) such that \( \alpha + \beta > 1 \) and let \( q \) be a nonnegative function in \( K_{\alpha,\beta} \) with \( \kappa_q \leq \frac{1}{2} \). Let \( f \in B^+((0,1)) \), then
\[ V_q f(x) \in C_{1-\alpha}([0,1]) \text{ if and only if } \int_0^1 (1-t)^{\alpha+\beta-1}f(t)dt < \infty. \]

Next, we will prove that the kernel \( V_q \) satisfies the following resolvent equation.

**Lemma 7.** Let \( \alpha, \beta \in (0,1) \) such that \( \alpha + \beta > 1 \) and let \( q \) be a nonnegative function in \( K_{\alpha,\beta} \) with \( \kappa_q \leq \frac{1}{2} \). Let \( f \in B^+((0,1)) \), then \( V_q f \) satisfies the following resolvent equation:
\[ V f = V_q f + V_q(qV f) = V_q f + V(qV_q f). \]  

In particular, if \( V(qf) < \infty \), we have
\[ (I - V_q(q.))(I + V(q.))f = (I + V(q.))(I - V_q(q.))f = f. \]

**Proof.** Let \((x, t) \in (0,1) \times (0,1)\), then by (3.9), we have
\[ H(x, t) = \mathcal{H}(x, t) + V(q\mathcal{H}(., t))(x), \]
which implies by the Fubini–Tonelli theorem that for \( f \in B^+((0,1)) \),
\[ V f(x) = \int_0^1 (\mathcal{H}(x, t) + V(q\mathcal{H}(., t))(x))f(t)dt \]
\[ = V_q f(x) + V(qV_q f)(x). \]
On the other hand, by Lemma 5(iii) and again the Fubini–Tonelli theorem, we have
\[
\int_0^1 \int_0^1 \mathcal{H}(x,r)H(r,t)q(r)f(t)drdt = \int_0^1 \int_0^1 H(x,r)\mathcal{H}(r,t)q(r)f(t)drdt.
\]
That is,
\[
V_q(qVf)(x) = V(qV_qf)(x).
\]
So, we obtain
\[
Vf = V_qf + V_q(qVf) = V_qf + V(qV_qf).
\]

\[\square\]

**Proposition 5.** Let \( \alpha, \beta \in (0,1] \) such that \( \alpha + \beta > 1 \) and let \( q \) be a nonnegative function in \( K_{\alpha,\beta} \cap C((0,1)) \) satisfying (i) and (ii) in \((H_2)\). Let \( f \in C^+(0,1) \) such that the map \( t \to (1-t)^{\alpha+\beta-1}f(t) \) is integrable and \( f(t) \leq t^{-\delta}L(t) \) near 0, with \( \delta \leq 1 \) and \( L \in K \) satisfying \( \int_0^1 t^{-\delta}L(t)dt < \infty \). Then, \( V_qf \in C_{1-\alpha}(0,1] \) and it is the unique nonnegative solution of the perturbed problem
\[
\begin{align*}
D^\beta(D^\alpha)u(x) - q(x)u(x) &= -f(x), \quad x \in (0,1), \\
\lim_{x \to 0^+} x^{1-\beta}D^\alpha u(x) &= u(1) = 0,
\end{align*}
\]
satisfying
\[
(1-\kappa_q)Vf \leq V_qf \leq Vf.
\]

**Proof.** Since by Corollary 1 the function \( t \to V_qf(t) \) is in \( C_{1-\alpha}(0,1] \), it follows that the function \( t \to q(t)V_qf(t) \) is continuous on \((0,1)\).

Using (3.6) and Lemma 4(i), there exists a nonnegative constant \( c \) such that
\[
V_qf(x) \leq Vf(x) \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^1 (1-t)^{\alpha+\beta-2} \min(1-t,1-x)f(t)dt \\
\leq \frac{1}{\Gamma(\alpha + \beta)} x^{\alpha-1} \int_0^1 (1-t)^{\alpha+\beta-1}f(t)dt \\
\leq cx^{\alpha-1}.
\]
Therefore,
\[
\int_0^1 (1-t)^{\alpha+\beta-1}q(t)V_qf(t)dt \leq c \int_0^1 t^{\alpha-1}(1-t)^{\alpha+\beta-1}q(t)dt < \infty
\]
and \( q(t)V_qf(t) \leq ct^{-\alpha}q(t) \leq ct^{-\mu}L(t) \) for \( t \) near 0, where \( \mu \leq 1 \) and \( L \in K \) satisfies \( \int_0^1 t^{-\mu}L(t)dt < \infty \). Hence by using Proposition 3, we conclude that the function \( u = V_qf = Vf - V(qV_qf) \) satisfies
\[
\begin{align*}
D^\beta(D^\alpha)u(x) &= -f(x) + q(x)u(x), \quad x \in (0,1), \\
\lim_{x \to 0^+} x^{1-\beta}D^\alpha u(x) &= u(1) = 0.
\end{align*}
\]
It remains to prove the uniqueness. Assume that there exists another nonnegative solution \( v \) in \( C_{1-\alpha}(0,1] \) of problem (3.12) satisfying (3.13).
We remark that the function $t \rightarrow q(t)v(t)$ is continuous on $(0, 1)$ and by (3.13) and (2.3) we have
\[
\int_0^1 (1-t)^{\alpha+\beta-1}q(t)v(t)dt \leq \int_0^1 (1-t)^{\alpha+\beta-1}q(t)Vf(t)dt \\
\leq c \int_0^1 t^{\alpha-1}(1-t)^{\alpha+\beta-1}q(t)dt < \infty.
\]
Moreover, we have
\[
q(t)v(t) \leq q(t)Vf(t) \leq ct^{\alpha-1}q(t) \leq ct^{-\mu}L(t) \quad \text{for } t \text{ near } 0,
\]
where $\mu \leq 1$ and $L \in K$ satisfies $\int_0^\eta t^{-\mu}L(t)dt < \infty$.

It follows by Proposition 3, the function $\tilde{v} := v + V(qv)$ satisfies
\[
\begin{cases}
D\beta(D\alpha)\tilde{v}(x) = -f(x), & x \in (0, 1), \\
\lim_{x \to 0^+} x^{1-\beta}D\alpha\tilde{v}(x) = \tilde{v}(1) = 0.
\end{cases}
\]
From the uniqueness in Proposition 3, we deduce that $\tilde{v} := v + V(qv) = Vf$.

Hence,
\[
(I + V(q.))(v - u) = 0.
\]
Now, since for $x \in (0, 1]$, there exists a nonnegative constant $c$ such that $Vqf(x) \leq Vf(x) \leq ch_2(x)$, where $h_2$ given by (1.6), it follows by (2.8) that
\[
V(q|v-u|) \leq 2cV(qh_2) \leq 2c\kappa_qh_2 < \infty.
\]
So by (3.11), we deduce that $u = v$. \hfill \Box

4. Proof of Theorem 1

Consider $a \geq 0$ and $b \geq 0$ such that $a + b > 0$. Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$, and $q \in K_{\alpha,\beta} \cap C((0, 1))$ be such that $(H_2)$ is satisfied.

Let
\[
\Lambda := \{u \in B^+((0, 1)) : (1 - \kappa_q)\omega \leq u \leq \omega\},
\]
where $\omega(x) = ah_1(x) + bh_2(x)$, $h_1$ and $h_2$ are defined by (1.6).

Define the operator $T$ on $\Lambda$ by
\[
Tu = \omega - V_q(q\omega) + V_q((q - g(u))u).
\]
By (3.6) and (2.9), we have
\[
V_q(q\omega) \leq V(q\omega) \leq \kappa_q\omega \leq \omega.
\] (4.1)

Using $(H_2)$, we get
\[
0 \leq g(u) \leq q \quad \text{for all } u \in \Lambda.
\] (4.2)

Next, we prove that $\Lambda$ is invariant under $T$. Indeed, using (4.1) and (4.2), we have for $u \in \Lambda$,
\[
Tu \leq \omega - V_q(q\omega) + V_q(qu) \leq \omega
\]
and
\[
Tu \geq \omega - V_q(q\omega) \geq (1 - \kappa_q)\omega.
\]
Next, we will prove that the operator $T$ is nondecreasing on $\Lambda$. Indeed, let $u, v \in \Lambda$ be such that $u \leq v$. By $(H_2)$, the function $t \to t(q(x) - g(t))$ is nondecreasing on $[0, 1]$, for $x \in (0, 1)$.

Then, we obtain

$$Tv - Tu = V_q([v(q - g(v)) - u(q - g(u))]) \geq 0.$$ 

Now, define the sequence $(u_n)$ by $u_0 = (1 - \kappa_q)\omega$ and $u_{n+1} = Tu_n$ for $n \in \mathbb{N}$.

Since $T\Lambda \subseteq \Lambda$, we have $u_1 = Tu_0 \geq u_0$ and, by the monotonicity of $T$, we deduce that

$$(1 - \kappa_q)\omega = u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq \omega.$$ 

Hence by $(H_1)$–$(H_2)$ and the dominated convergence theorem, we deduce that the sequence $(u_n)$ converges to a function $u \in \Lambda$ satisfying

$$u = (I - V_q(q_\cdot))\omega + V_q((q - g(u))u),$$

that is,

$$(I - V_q(q_\cdot))u = (I - V_q(q_\cdot))\omega - V_q(ug(u)).$$

By (2.9), we have $V(qu) \leq V(q\omega) \leq \omega < \infty$; then applying the operator $(I + V(q_\cdot))$ on both sides of the above equality and using (3.10) and (3.11), we deduce that $u$ satisfies

$$u = \omega - V(ug(u)).$$

(4.3)

It remains to prove that $u$ is a solution of problem (1.3). Using (4.2) and (1.6), we have

$$0 \leq u(t)g(u(t)) \leq q(t)\omega(t) \leq \max \left( \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)}a, b \right) t^{\alpha - 1}q(t).$$

(4.4)

Therefore,

$$\int_0^1 (1 - t)^{\alpha + \beta - 1}u(t)g(u(t))dt \leq \max \left( \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)}a, b \right)$$

$$\int_0^1 t^{\alpha - 1}(1 - t)^{\alpha + \beta - 1}q(t)dt < \infty.$$ 

So, by Lemma 4(ii) the function $x \to V(ug(u))(x)$ is in $C_{1-\alpha}([0, 1])$. This implies by (4.3), $u \in C_{1-\alpha}([0, 1])$. Now, we remark that the function $t \to (1 - t)^{\alpha + \beta - 1}u(t)g(u(t))$ is continuous and integrable on $(0, 1)$. Moreover, by (4.2) and $(H_2)$, we have for $t$ near 0

$$0 \leq u(t)g(u(t)) \leq \max \left( \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)}a, b \right) t^{-\mu}L(t),$$

where $\mu \leq 1$ and $L \in K$ satisfies $\int_0^1 t^{-\mu}L(t)dt < \infty$.

Then, we deduce by Proposition 3 that $u$ is a solution of (1.3).

Finally, suppose that hypotheses $(H_3)$ is satisfied and let us show that problem (1.3) has a unique solution satisfying (1.7). Assume that $v$ is another nonnegative solution in $C_{1-\alpha}([0, 1])$ to problem (1.3) satisfying the inequality (1.7).
Since \( v \leq \omega \), we deduce by (4.2) that
\[
0 \leq v(t)g(v(t)) \leq q(t)\omega(t) \leq \max \left( \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} a, b \right) t^{\alpha-1}q(t).
\]
This implies that, the function \( t \to (1-t)^{\alpha+\beta-1}v(t)g(v(t)) \in L^1((0,1)) \cap C((0,1)) \) and for \( t \) near 0, we have
\[
0 \leq v(t)g(v(t)) \leq \max \left( \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} a, b \right) t^{-\mu}L(t),
\]
where \( \mu \leq 1 \) and \( L \in \mathcal{K} \) satisfies \( \int_0^1 t^{-\mu}L(t)dt < \infty \). Let \( \bar{v} := v + V(vg(v)) \).

By Proposition 3, we have
\[
\begin{cases}
D^\beta(D^\alpha)\bar{v}(x) = 0, x \in (0,1), \\
\lim_{x \to 0^+} x^{1-\beta}D^\alpha\bar{v}(x) = -a, \quad \bar{v}(1) = b.
\end{cases}
\]
Hence,
\[
v = \omega - V(vg(v)). \tag{4.5}
\]

Now, let be \( h : (0,1) \to \mathbb{R} \) defined by
\[
h(t) := \begin{cases}
\frac{v(t)g(v(t))-u(t)g(u(t))}{v(t)-u(t)}, & \text{if } v(t) \neq u(t), \\
0, & \text{if } v(t) = u(t).
\end{cases}
\]

By \((H_3)\), \( h \in \mathcal{B}^+((0,1)) \) and from (4.3) and (4.5), we deduce that
\[
(I + V(h.))(v - u) = 0.
\]

From \((H_2)\), we have \( h \leq q \). So by using (2.9), we deduce that
\[
V(h|v - u|) \leq V(vg(v)) + V(u\sigma(u)) \leq 2V(q\omega) \leq 2\kappa q \omega < \infty.
\]
Hence by (3.11), we conclude that \( u = v \). This completes the proof.

To illustrate our result proved in Theorem 1, we give the following example.

**Example 1.** Let \( \alpha, \beta \in (0,1] \) such that \( \alpha + \beta > 1 \), and let \( \sigma \geq 1 \) and \( a \geq 0 \), \( b \geq 0 \) such that \( a+b > 0 \). Then for nonnegative small \( \lambda \), the following problem
\[
\begin{cases}
D^\beta(D^\alpha u)(x) = \lambda u^\sigma(x), 0 < x < 1, \\
\lim_{x \to 0^+} x^{1-\beta}D^\alpha u(x) = -a, \quad u(1) = b,
\end{cases}
\]
has a unique positive solution \( u \) in \( C_{1-\alpha}([0,1]) \) satisfying
\[
c\omega(x) \leq u(x) \leq \omega(x) \quad \text{for } x \in (0,1),
\]
where \( 0 < c < 1 \).

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