Wave Equations for Invariant Infeld-van der Waerden Wave Functions for Photons and Their Physical Significance

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Abstract

The inner structure of the γε-formalisms of Infeld and van der Waerden admits the occurrence of spin-tensor electromagnetic fields which bear invariance under the action of the generalized Weyl gauge group. A concise derivation of the wave equations for such fields is carried out explicitly along with the construction of a set of torsionless covariant-derivative expressions. It is emphatically pointed out that the integration of the wave equations arising herein may under certain circumstances produce significant insights into the situation concerning the description of some physical properties of the cosmic microwave background.

1 Introduction

One of the striking features of the γε-formalisms of Infeld and van der Waerden [1-3] is related to the fact that spacetime curvature structures generally carry wave functions for photons of opposite handednesses in an inextricable manner [3, 4]. Loosely speaking, such wave functions amount in the case of either formalism to contracted curvature contributions that enter locally into spinor decompositions of Maxwell bivectors. Traditionally, the presence of intrinsically geometric electromagnetic fields appears to be bound up with a single condition upon the metric spinors for the γ-formalism, which remains formally unaltered when the action of the generalized Weyl gauge group [5] is effectively implemented. In spacetimes that admit nowhere-vanishing Weyl spinor fields, background photons turn out to interact with underlying gravitons. However,
the relevant coupling patterns are strictly borne by the wave equations that control the electromagnetic propagation [4].

Any electromagnetic curvature spinors for the $\gamma$-formalism behave as spin tensors under the action of the Weyl group, whilst typical $\varepsilon$-formalism contributions are correspondingly looked upon as pairs of gauge-invariant spin-tensor densities having appropriate weights and antiweights. Remarkably enough, rearranging index configurations adequately, produces the occurrence in both formalisms of wave functions for photons that bear an invariant spin-tensor character. This notable property of the formalisms was brought forward for the first time in Ref. [3]. It was utilized thereabout mainly to facilitate setting out the entire system of electromagnetic wave equations that should be tied in with the formalisms.

In the present paper, we derive in a concise way the wave equations for the invariant spin-tensor fields we have just allowed for above, along with a set of interesting covariant-derivative expressions. We believe that, under the standard cosmological circumstances [6, 7], these wave equations may enable one to gain fresh insights into the situation concerning the description of some physical properties of the microwave background of the universe. Actually, it is this aspect of our work which most strongly ensures the relevance of the whole presentation.

We will have necessarily to recall in Section 2 the definitions and prescriptions that are of immediate interest to us. The differential expressions and wave equations are deduced together in Section 3. A few remarks on the work are made in Section 4. Many of the conventions adopted in Ref. [3] will be used throughout the work, but we shall occasionally explain some of them. In particular, the effect on any index block of the actions of the symmetry and antisymmetry operators is indicated by surrounding the involved indices with round and square brackets, respectively.

## 2 Basic formulae

Wave functions for geometric photons are defined in either formalism as

$$\phi_{AB} \doteq \frac{i}{2} \omega_{ABC}^C, \quad \phi_{A'B'} \doteq \frac{i}{2} \omega_{A'B'C}^C,$$

with the $\omega$-spinors entering the contracted configurations

$$\omega_{ABC}^C = \omega_{(AB)}^C \doteq \frac{1}{2} W_{ABA'}^{A'C} C,$$

and\(^1\)

$$\omega_{A'B'C}^C = \omega_{(A'B')}^C \doteq \frac{1}{2} W_{A'B'A}^{A'C} C.$$

\(^1\)In the $\varepsilon$-formalism, the $\phi$-fields taken up by Eqs. (1) are gauge-invariant spin-tensor densities of weight and antiweight $-1$. 2
The $W$-objects of Eqs. (2) and (3) may be thought of as arising from the commutator structure
\[ [\nabla_{AA'}, \nabla_{BB'}] \zeta^C \doteq W_{AA'BB'M}^C \zeta^M, \tag{4} \]
with $\zeta^C$ being an arbitrary spin vector, and $\nabla_a$ denoting some torsion-free covariant-derivative operator for the formalism eventually taken into consideration. In each formalism, the overall curvature spinors of $\nabla_a$ are thus carried by the expansion
\[ W_{AA'BB'CD} = M_{A'B'} \omega_{ABCD} + M_{AB} \omega_{A'B'CD}, \tag{5} \]
where the kernel letter $M$ stands for either $\gamma$ or $\varepsilon$.

The electromagnetic curvature spinors for both formalisms are taken to fulfill the relationships
\[ \omega_{ABC}^C = 2i \nabla_C (\Phi^C_B) = 2i \nabla_C (\Phi^C_B), \tag{6} \]
together with the prescriptions
\[ F_{AA'BB'} = -(M_{AB} \nabla_C (\Phi_B^C) + M_{A'B'} \nabla_C (\Phi_B^C)), \tag{7} \]
and
\[ \frac{i}{2} W_{abc}^C = F_{ab} = 2 \nabla_{[a} \Phi_{b]} \tag{8}. \]

It is shown in Ref. [3] that the formalisms usually bear the same quantity $\Phi_a$, which comes into play as an affine electromagnetic potential that satisfies the Weyl principle of gauge covariance.

3 Derivative expressions and wave equations

The Weyl group has to be taken as one and the same in both formalisms. Indeed, the only unprimed-index configuration that accounts for invariant wave functions for photons in the $\gamma$-formalism is provided by $\phi_A^B$. On the other hand, the spin-density character inherently carried by the metric spinors for the $\varepsilon$-formalism [1], implies that the respective field $\phi_A^B$ should be regarded as an invariant spin-tensor wave function as well. Consequently, we can readily write down in either formalism the simple expression
\[ \nabla_a \phi_A^B = \partial_a \phi_A^B - \vartheta_{aA}^C \phi_C^B + \vartheta_{aC}^B \phi_A^C, \tag{9} \]
where $\vartheta_{aA}^C$ accordingly denotes an admissible spin affinity. It follows that, manipulating the indices of the $\vartheta$-terms of Eq. (9), after some easy calculations, we obtain
\[ \nabla_a \phi_B^A = \partial_a \phi_B^A - \vartheta_{a(AC)} M_{BD}^C \phi_D^B + \vartheta_{a(B)} M_{DC}^A \phi_A^D, \tag{10} \]

The parts $\omega_{AB(CD)}$ and $\omega_{AB(CD)}$ constitute the gravitational spin curvature for either $\nabla_a$ (for further details, see Ref. [3]).
with the kernel letter $M$ bearing the same meaning as before. In fact, the $\varepsilon$-formalism versions of $\vartheta_{n(BC)}$ and $\vartheta_{s(BC)}$ show up, respectively, as invariant spin-tensor densities of weights $-1$ and $+1$, whence both of the $\vartheta$-pieces of Eq. (10) really bear gauge invariance in either formalism.

In both formalisms, the field equation for $\phi_{A}^{B}$ must be spelt out as the massless-free-field statement

$$\nabla^{AB'}\phi_{A}^{B} = 0. \quad (11)$$

Therefore, making use of the splitting [3]

$$\nabla^{C}A'\nabla^{'AA'}\phi_{A}^{B} = \Delta^{AC}\phi_{A}^{B} - \frac{1}{2}M^{AC}\Box\phi_{A}^{B}, \quad (12)$$

and taking account of the (symmetric) derivative

$$\Delta^{AB}\phi_{A}^{C} = \frac{R}{6}M^{BD}\phi_{D}^{C} - \omega^{(ABCD)}\phi_{A}^{H}M_{HD}, \quad (13)$$

with $R$ being the pertinent Ricci scalar, we arrive at the equation

$$\left(\Box + \frac{R}{3}\right)\phi_{A}^{B} = (-2)\Psi_{ABCD}^{C}D^{-D}, \quad (14)$$

with the definitions

$$\Psi_{ABCD} = \omega^{(ABCD)}, \quad (15)$$

and

$$\Box = \nabla^{CC'}\nabla^{-CC'}. \quad (16)$$

The $\Psi$-spinor of Eq. (15) represents one of the wave functions for gravitons [6-8]. It should be stressed that the legitimacy in either formalism of Eq. (14), stems essentially from the common invariant spin-tensor character of $\phi_{A}^{B}$.

Any procedures associated to the implementation of specific techniques for solving Eq. (14) would become considerably simplified if the situations being entertained were set upon conformally flat spacetimes. Under such a circumstance, one would just deal in either formalism with

$$\left(\Box + \frac{R}{3}\right)\phi_{A}^{B} = 0. \quad (17)$$

At this stage, we could call upon the invariant distributional methods of Ref. [9] to treat systematically Eq. (17) in conjunction with the standard Friedmann-Robertson-Walker cosmological model [7]. We will probably elaborate upon this issue elsewhere.

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3In both formalisms, the operator $\Delta^{AB}$ amounts to $\nabla^{AC}\nabla^{B}C'$. It behaves in the $\varepsilon$-formalism as an invariant spin-tensor density of weight $+1$. 

4
4 Conclusions and outlook

Within manifestly cosmological frameworks, it might be expected on the basis of the invariant geometric character of either $\phi_A^B$ that the key properties of the cosmic microwave background should be naturally described by solutions of Eq. (17) subject to suitably prescribed boundary conditions. In this connection, the following expression for the energy-momentum tensor for either formalism

$$ T_{AA'B'B'} = \frac{1}{2\pi} \phi_A^B \phi_{A'B'}. $$

would presumably be helpful for achieving present-time values of the energy and linear momentum of the radiation.

We should observe that the right-hand side of Eq. (10) supplies elementary index-displacement rules for covariant derivatives of gauge-invariant fields for both formalisms. In practice, one could additionally invoke the useful relation

$$ \partial_\alpha(BC) = \frac{1}{2} \left( S^{bD'}_{(B} \partial_{C)} D' g_{ab} + S^{D'}_{[b} \partial_{|a|} S^{b}_{C]} \right), $$

which absorbs some $\gamma\varepsilon$-connecting objects along with a spacetime metric tensor carrying the signature $(+ - - -)$.

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