A TANNAKIAN APPROACH TO
DIMENSIONAL REDUCTION OF PRINCIPAL BUNDLES

LUIS ÁLVEZ-CÓNSUL, INDRANIL BISWAS, AND OSCAR GARCÍA-PRADA

To Ugo Bruzzo on his 60th birthday

Abstract. Let $P$ be a parabolic subgroup of a connected simply connected complex semisimple Lie group $G$. Given a compact Kähler manifold $X$, the dimensional reduction of $G$-equivariant holomorphic vector bundles over $X \times G/P$ was carried out by the first and third authors [2]. This raises the question of dimensional reduction of holomorphic principal bundles over $X \times G/P$. The method used for equivariant vector bundles does not generalize to principal bundles. In this paper, we adapt to equivariant principal bundles the Tannakian approach of Nori, to describe the dimensional reduction of $G$-equivariant principal bundles over $X \times G/P$, and to establish a Hitchin–Kobayashi type correspondence. In order to be able to apply the Tannakian theory, we need to assume that $X$ is a complex projective manifold.

1. Introduction

Dimensional reduction is a very powerful construction in the context of gauge theory, both in physics and geometry. Many important gauge-theoretic equations appear as symmetric solutions of fundamental equations for connections, like the instanton equations on Riemannian 4-manifolds. Examples of these are the Bogomol’nyi equations for magnetic monopoles in 3 dimensions and the vortex equation in 2 dimensions, as well as many other important integrable systems and soliton equations. In the context of model building, some physicists have applied for a long time the method of ‘coset-space dimensional reduction’ in the construction of gauge unified theories (see, e.g., [15, 7, 20, 26]).

In [16, 17, 11, 1] the dimensional reduction techniques were brought into the context of holomorphic vector bundles over Kähler manifolds, to study the dimensional reduction of stable $\text{SL}(2, \mathbb{C})$-equivariant bundles over $X \times \mathbb{P}^1$ and the corresponding Hermitian–Yang–Mills equations, where $X$ is a compact Kähler manifold and $\mathbb{P}^1$ is the Riemann sphere. In [2] this construction was generalised to $G$-equivariant vector bundles on $X \times G/P$, where $G$ is a connected simply connected complex semisimple Lie group and $P \subset G$ is a parabolic subgroup. These gave rise to quiver bundles with relations, that is representations of a quiver with relations in the category of vector bundles, where the quiver with relations $(Q, K)$ is determined by $P$. In particular when $X$ is a point, one has a description of
homogeneous vector bundles over $G/P$ in terms of representations of the quiver with relations in the category of vector spaces.

In this paper we undertake the task of generalizing [2] to holomorphic principal $H$-bundles over $X \times G/P$, where $H$ is a complex reductive group. Since a $G$-equivariant holomorphic principal $H$-bundle over $X \times G/P$ is equivalent to a $P$-equivariant holomorphic principal $H$-bundle over $X$ (for the trivial $P$-action on $X$), the problem reduces then to studying the dimensional reduction of such bundles over $X$. To do this we take the Tannakian point of view of Nori [21], regarding a principal $H$-bundle as a functor from the category of representations of $H$ to the category of vector bundles. To apply this to our situation, we first show in Section 2 that the category of representations of $(Q, K)$ in complex vector spaces has a structure of neutral Tannakian category. The construction of an identity object, dual objects and tensor products is extended to the category of $(Q, K)$-bundles in Section 3, where we also show that $P$-equivariant holomorphic principal $H$-bundles on $X$ are in bijection with $H$-torsors in the category of $(Q, K)$-bundles over $X$, that is strict exact faithful tensor functors from the category of $H$-modules to the category $(Q, K)$-bundles. For our Tannakian approach we need to assume that $X$ is projective. In Section 4 we define stability of an $H$-torsor in the category of $(Q, K)$-bundles, and show that the semistability (resp. polystability) of such a torsor is equivalent to the semistability (resp. polystability) of the corresponding $G$-equivariant holomorphic $H$-bundle on $X \times G/P$. We complete our work by showing that the polystability of an $H$-torsor is equivalent to the existence of solutions of the quiver vortex equations on the $(Q, K)$-bundle corresponding to the adjoint representation of $H$.

To the knowledge of the authors, this is the first paper where a Tannakian approach to dimensional reduction is adopted. Another more direct approach to describe $P$-equivariant holomorphic bundles on $X$ in terms of pairs consisting of a holomorphic principal bundle over $X$ with a reduced structure group and certain ‘Higgs fields’ is indeed possible. We plan to come back to this and its relation to our Tannakian approach in a future paper.

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### 2. Homogeneous vector bundles and quiver modules

This section is devoted to a Tannakian description of holomorphic equivariant vector bundles over a flag manifold $G/P$, in terms of representations of a quiver with relations.

Throughout this paper, $G$ is a connected simply connected semisimple complex affine algebraic group, and $P \subset G$ is a parabolic subgroup with unipotent radical $R_u(P) \subset P$. The Lie algebras of $R_u(P) \subset P \subset G$ are denoted $\mathfrak{u} \subset \mathfrak{p} \subset \mathfrak{g}$, respectively. We also fix a Levi subgroup $L$ of $P$, that is, a maximal connected reductive subgroup of $P$.

#### 2.1. Induction and reduction for homogeneous vector bundles

Let $G/P$ be the quotient of $G$ under the right $P$-action $G \times P \to G, (g, p) \mapsto gp$ (its points are the orbits $gP$, for $g \in G$). It is a flag variety, i.e., a complex projective manifold, with a transitive holomorphic left $G$-action $G \times G/P \to G/P, (g, g'P) \mapsto gg'P$. A homogeneous holomorphic vector bundle on $G/P$ is a holomorphic vector bundle $E$ on $G/P$, with
an equivariant holomorphic $G$-action, that is, a holomorphic $G$-action via holomorphic vector-bundle automorphisms such that $g \cdot v \in E_{g'P}$ for $g, g' \in G, v \in E_{gP}$.

In Proposition 2.9 we will provide an algebraic description of homogeneous holomorphic vector bundle in terms of quivers with relations. Our first step for this description is provided by a standard process of induction and reduction.

**Lemma 2.1.** There is an equivalence of categories between the homogeneous holomorphic vector bundles on $G/P$, with $G$-equivariant holomorphic vector-bundle maps, and the (holomorphic finite-dimensional) representations of $P$, with $P$-equivariant linear maps.

**Proof.** One first observes that the projection $G \to G/P$ is a holomorphic principal $P$-bundle. Then the equivalence maps a homogeneous holomorphic vector bundle $E$ on $G/P$ into its fibre $E_o$ over the base point $o = P \in G/P$ with isotropy group $P$, and a representations $V$ of $P$ into its associated holomorphic vector bundle $E = G \times^P V$, with $G$-action induced by left multiplication of $G$ on itself. See, e.g., [2, §1.1.1] for details. □

### 2.2. Quivers, relations, and their modules.

To fix notation, here we recall a few relevant definitions from the theory of quiver modules (see, e.g., [6] for an introduction to this topic). A quiver $Q$ is a pair of sets $Q_0$ and $Q_1$, together with two maps $t, h: Q_1 \to Q_0$. In this paper, we do not impose the usual restriction that the sets $Q_0$ and $Q_1$ are finite. The elements of $Q_0$ and $Q_1$ are called the vertices and the arrows of the quiver, respectively. An arrow $a \in Q_1$ is represented pictorially as $a: v \to w$, where $v = ta$ and $w = ha$ are called the tail and the head of the quiver, respectively. Using the convention that arrows compose as maps, from right to left, a non-trivial path in $Q$ of length $\ell \geq 1$ is a sequence $p = a_1 \cdots a_\ell$ of arrows $a_i \in Q_1$ that compose, that is, $ha_{i-1} = ta_i$ for all $1 \leq i \leq \ell$. It is represented as

\begin{equation}
(2.1) \quad p: \bullet \overleftarrow{a_\ell} \cdots \overleftarrow{a_2} \bullet \overrightarrow{a_1} \overrightarrow{tp},
\end{equation}

where $tp = ta_1$ and $hp = ha_\ell$ are called the tail and the head of the path $p$. The trivial path $e_v$ at a vertex $v \in Q_0$ consists of the zero-length path at vertex $v$ with no arrows, so it has tail and head $te_v = he_v = v$. A (complex) relation of $Q$ with tail $v$ and head $w$ is a finite formal sum

\begin{equation}
(2.2) \quad r = c_1 p_1 + \cdots + c_k p_k,
\end{equation}

of paths $p_1, \ldots, p_k$ such that $tp_i = v$ and $hp_i = w$ for all $1 \leq i \leq k$, with coefficients $c_i \in \mathbb{C}$. A quiver with relations is a pair $(Q, K)$ consisting of a quiver $Q$ and a set $K$ of relations of $Q$. A (complex) linear representation of $Q$, or a $Q$-module for short, is a pair $\langle V, \varphi \rangle$ consisting of a set $V$ of finite-dimensional vector spaces $V_v$, indexed by the vertices $v \in Q_0$, and a set $\varphi$ of linear maps $\varphi_a: V_{ta} \to V_{ha}$, indexed by the arrows $a \in Q_1$. Since $Q_0$ may be infinite, we impose the condition that $V_v = 0$ for all but finitely many $v \in Q_0$. A path $p$ of $Q$ determines a linear map, for each $Q$-module $\langle V, \varphi \rangle$,

\[ \varphi(p): V_{tp} \to V_{hp}, \]

developed as $\varphi(p) = \varphi_{a_\ell} \cdots \varphi_{a_1}$ for a non-trivial path $[2.1]$, and as the identity $\varphi(p) = \text{Id}_{V_v}: V_v \to V_v$ for a trivial path $p = e_v$ at a vertex $v \in Q_0$. Such a $Q$-module $\langle V, \varphi \rangle$ satisfies a relation $[2.2]$ if the linear map $\varphi(r) := c_1 \varphi(p_1) + \cdots + c_k \varphi(p_k)$ is zero.

Quiver modules form a category $\text{mod}(Q)$, where a morphism $f: \langle V, \varphi \rangle \to \langle V', \varphi' \rangle$ between two $Q$-modules is a set of linear maps $f_v: V_v \to V'_v$, indexed by the vertices
v \in Q_0$, such that $\varphi'_a \circ f_{ta} = f_{ha} \circ \varphi_a$ for all $a \in Q_1$. Morphism composition in this category is defined by vertexwise composition of linear maps. In this paper, we will be interested in certain full subcategory

$$\text{mod}(Q, K) \subset \text{mod}(Q)$$

of $(Q, K)$-modules, that is, $Q$-modules that satisfy a fixed set of relations $K$ of $Q$.

2.3. The quiver with relations associated to a parabolic subgroup. Here we review a construction of a quiver with relations $(Q, K)$ associated to $P$, such that homogeneous holomorphic vector bundles over $G/P$ are equivalent to $(Q, K)$-modules (see Remark 2.4). We follow the construction by the first and third authors [2], as it is well suited for dimensional reduction, but we should emphasize that other constructions exist (see, e.g., [9, 18, 22]). Recall that $G$ is a complex semisimple simply connected affine algebraic group, $P \subseteq G$ is a parabolic subgroup, with unipotent radical $R_u(P)$, $L \subset P$ is a Levi subgroup, and $u \subset p \subset g$ are the Lie algebras of $R_u(P) \subset P \subset G$, respectively (see Section 2.1). Given two (complex finite-dimensional) vector spaces $U$ and $V$, $\text{Hom}(U, V)$ is the vector space of linear maps $U \to V$, and $V^* = \text{Hom}(V, \mathbb{C})$ is the dual vector space of $V$. If $U$ and $V$ are (holomorphic) $L$-representations, then $\text{Hom}(U, V)$ and $V^* = \text{Hom}(V, \mathbb{C})$ become $L$-representations (for the trivial $L$-action on $\mathbb{C}$); in this case, $V^L \subset V$ is the subspace of $L$-invariant vectors, and $\text{Hom}_L(U, V) = \text{Hom}(U, V)^L \subset \text{Hom}(U, V)$ is the space of $L$-equivariant linear maps.

Let $\Lambda^*_P$ be the set of all isomorphism classes of irreducible complex algebraic representations of $L$. Note that if $T \subset G$ is a Cartan subgroup such that $T \subset L$, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$, then the lattice $\Lambda \subset \mathfrak{t}^*$ of integral weights of $T$ parametrizes isomorphism classes of irreducible complex algebraic representations of $T$, and $\Lambda^*_P$ can be identified with a fundamental chamber in $\Lambda$ for the reductive Lie group $L$. From this viewpoint, $\Lambda^*_P$ is identified with the set of dominant weights of $L$.

We fix an irreducible representation $M_\lambda$ in the isomorphism class $\lambda$, for each $\lambda \in Q_0$. For each $\lambda, \mu, \nu \in \Lambda^*_P$, we define vector spaces

\begin{align}
(2.3a) & \quad A_{\mu \lambda} := \text{Hom}_L(u \otimes M_\lambda, M_\mu), \\
(2.3b) & \quad B_{\mu \lambda} := \text{Hom}_L(\wedge^2 u \otimes M_\lambda, M_\mu),
\end{align}

where $u$ is regarded as an $L$-module, and linear maps

\begin{align}
(2.4a) & \quad \psi_{\mu \lambda} : A_{\mu \lambda} \longrightarrow B_{\mu \lambda}, \\
(2.4b) & \quad \psi_{\mu \nu \lambda} : A_{\mu \nu} \otimes A_{\nu \lambda} \longrightarrow A_{\mu \lambda},
\end{align}

obtained by restriction (see [2 Lemma 1.2]) of the linear maps

\begin{align*}
\psi_{\mu \lambda} : \text{Hom}_L(u \otimes M_\lambda, M_\mu) & \longrightarrow \text{Hom}_L(\wedge^2 u \otimes M_\lambda, M_\mu) \\
\psi_{\mu \nu \lambda} : \text{Hom}(u \otimes M_\mu, M_\lambda) \otimes \text{Hom}(u \otimes M_\mu, M_\nu) & \longrightarrow \text{Hom}(\wedge^2 u \otimes M_\lambda, M_\mu),
\end{align*}

(denoted with the same symbols), given by

$$\psi_{\mu \lambda}(a)(e, e') := -a([e, e']), \quad \psi_{\mu \nu \lambda}(a'' \otimes a') = a'' \wedge a',$$
for all \( a \in \text{Hom}(u \otimes M_\lambda, M_\mu) \), \( a' \in \text{Hom}(u \otimes M_\lambda, M_\nu) \), \( a'' \in \text{Hom}(u \otimes M_\nu, M_\mu) \), \( e, e' \in u \).

Note that in the definition of \( \psi_{\mu\lambda} \) and \( \psi_{\mu\nu\lambda} \), we are using the identifications

\[
\text{Hom}(u \otimes U, V) = \text{Hom}(u, \text{Hom}(U, V)) = u^* \otimes \text{Hom}(U, V),
\]

\[
\text{Hom}(\wedge^2 u \otimes U, V) = \text{Hom}(\wedge^2 u, \text{Hom}(U, V)) = \wedge^2 u^* \otimes \text{Hom}(U, V)
\]

for vector spaces \( U \) and \( V \), and in the definition of \( \psi_{\mu\nu\lambda} \), the exterior product is

\[
a'' \wedge a' := (s'' \wedge s') \otimes (f'' \circ f')
\]

for \( a' = s' \otimes f', a'' = s'' \otimes f'' \), with \( s', s'' \in u^* \) and \( f', f'' \in \text{Hom}(M_\lambda, M_\nu), f'' \in \text{Hom}(M_\nu, M_\mu) \).

Let \( \{a^{(i)}_{\mu\lambda} | i = 1, \ldots, n_{\mu\lambda}\} \) and \( \{b^{(p)}_{\mu\lambda} | p = 1, \ldots, m_{\mu\lambda}\} \) be bases of \( A_{\mu\lambda} \) and \( B_{\mu\lambda} \), respectively, where \( n_{\mu\lambda} = \dim A_{\mu\lambda} \) and \( m_{\mu\lambda} = \dim B_{\mu\lambda} \), for all \( \lambda, \mu \in \Lambda^+_P \). For all \( \lambda, \mu, \nu \in \Lambda^+_P \), \( 1 \leq i \leq n_{\nu\lambda}, 1 \leq j \leq n_{\mu\nu}, 1 \leq k \leq n_{\mu\lambda} \), we expand the vectors

\[
\psi_{\mu\lambda}(a^{(i)}_{\mu\lambda}), \psi_{\mu\nu\lambda}(a^{(j)}_{\mu\nu} \otimes a^{(i)}_{\nu\lambda}) \in B_{\mu\lambda}
\]

in the given basis of \( B_{\mu\lambda} \) as follows, with coefficients \( c^{(k,p)}_{\mu\lambda}, c^{(j,i,p)}_{\mu\nu\lambda} \in \mathbb{C} \):

\[
(2.5) \quad \psi_{\mu\lambda}(a^{(k)}_{\mu\lambda}) = \sum_{p=1}^{m_{\mu\lambda}} c^{(k,p)}_{\mu\lambda} b^{(p)}_{\mu\lambda}, \quad \psi_{\mu\nu\lambda}(a^{(j)}_{\mu\nu} \otimes a^{(i)}_{\nu\lambda}) = \sum_{p=1}^{m_{\mu\lambda}} c^{(j,i,p)}_{\mu\nu\lambda} b^{(p)}_{\mu\lambda}.
\]

**Definition 2.2.** The *quiver with relations* \((Q, \mathcal{K})\) associated to \( P \) is defined as follows.

(a) The quiver \( Q \) has vertex set \( Q_0 = \Lambda^+_P \), i.e., the set of isomorphism classes of irreducible representations of \( L \), and arrow set

\[
Q_1 = \{a^{(i)}_{\mu\lambda} | \lambda, \mu \in Q_0, 1 \leq i \leq n_{\mu\lambda}\}.
\]

The tail and head maps \( t, h : Q_1 \rightarrow Q_0 \) are defined by

\[
t(a^{(i)}_{\mu\lambda}) = \lambda, \quad h(a^{(i)}_{\mu\lambda}) = \mu.
\]

(b) The set of relations is \( \mathcal{K} = \{r^{(p)}_{\mu\lambda} | \lambda, \mu \in Q_0, 1 \leq p \leq m_{\mu\lambda}\} \), with

\[
r^{(p)}_{\mu\lambda} := \sum_{\nu \in Q_0} \sum_{i=1}^{n_{\nu\lambda}} \sum_{j=1}^{n_{\mu\nu}} c^{(j,i,p)}_{\mu\nu\lambda} a^{(j)}_{\mu\nu} a^{(i)}_{\nu\lambda} + \sum_{k=1}^{n_{\mu\lambda}} c^{(k,p)}_{\mu\lambda} a^{(k)}_{\mu\lambda},
\]

where \( a^{(j)}_{\mu\nu} a^{(i)}_{\nu\lambda} \) denotes the path \( \mu \leftarrow \nu \leftarrow \lambda \) defined by the arrows \( a^{(j)}_{\mu\nu} \) and \( a^{(i)}_{\nu\lambda} \).

To simplify some notation, given a \( Q \)-module \((V, \varphi)\), the linear map \( \varphi_a : V_\lambda \rightarrow V_\mu \) corresponding to the arrow \( a = a^{(i)}_{\mu\lambda} \in A_{\mu\lambda} \) will be denoted

\[
(2.6) \quad \varphi^{(i)}_{\mu\lambda} := \varphi_a : V_\lambda \rightarrow V_\mu.
\]

With Definition 2.2, we have the following result, where \( \text{Rep}(P) \) is the category of (complex finite-dimensional) representations of \( P \), or \( P \)-modules for short, in which the morphisms are the \( P \)-equivariant linear maps.

**Theorem 2.3** ([2] Theorem 1.4). Let \((Q, \mathcal{K})\) be the quiver with relations associated to the group \( P \). Then there is an equivalence of categories

\[
(2.7) \quad \text{Rep}(P) \cong \text{mod}(Q, \mathcal{K}).
\]
Remark 2.4. Since $P$-modules are equivalent to homogeneous holomorphic vector bundles over $G/P$ (see Lemma 2.1), Theorem 2.3 implies that the latter are also equivalent to $(Q,K)$-modules ([2, Corollary 1.13]).

For the purposes of this paper, it will be useful to sketch some ingredients of the proof of Theorem 2.3. We begin observing that there is an equivalence

$$
\{ \text{all but finitely many } Q, \}$$

and the choice of basis \( \{ a_{\mu \lambda}^{(i)} \}_{i=1}^{n_{\mu \lambda}} \) of \( A_{\mu \lambda} \) induces isomorphisms

$$
(2.9) \quad \mathbb{C}^{n_{\mu \lambda}} \otimes \text{Hom}(V_\lambda, V_\mu) \xrightarrow{\cong} A_{\mu \lambda} \otimes \text{Hom}(V_\lambda, V_\mu) \xrightarrow{\cong} \text{Hom}_L(u \otimes V_\lambda \otimes M_\lambda, V_\mu \otimes M_\mu),
$$

for all \( \lambda, \mu \in Q_0 \), where the left-hand space parametrizes sets \( \varphi_{\mu \lambda} = \{ \varphi_{\mu \lambda}^{(i)} \}_{i=1}^{n_{\mu \lambda}} \) of linear maps \( \varphi_{\mu \lambda}^{(i)} \in \text{Hom}(V_\lambda, V_\mu) \), for \( 1 \leq i \leq n_{\mu \lambda} \), the left-hand isomorphism maps \( \varphi_{\mu \lambda} \) into

$$
\varphi_{\mu \lambda} := \sum_{i=1}^{n_{\mu \lambda}} a_{\mu \lambda}^{(i)} \otimes \varphi_{\mu \lambda}^{(i)} \in A_{\mu \lambda} \otimes \text{Hom}(V_\lambda, V_\mu),
$$

and the right-hand isomorphism follows directly from the definition of \( A_{\mu \lambda} \). Let \( \mathbb{V} \) be the \( L \)-module associated to \( \mathbb{V} \) via (2.3). Then there is an isomorphism

$$
(2.10) \quad \mathcal{R}_Q(\mathbb{V}) \xrightarrow{\cong} \mathcal{R}_{L,u}(\mathbb{V}),
$$

obtained taking the direct sum of the isomorphisms (2.9) for all \( \lambda, \mu \in Q_0 \), where the representation space of \( Q \) on a \( Q_0 \)-graded module \( \mathbb{V} \) is

$$
(2.11) \quad \mathcal{R}_Q(\mathbb{V}) := \bigoplus_{\lambda,\mu \in Q_0} \mathbb{C}^{n_{\mu \lambda}} \otimes \text{Hom}(V_\lambda, V_\mu) \cong \bigoplus_{\lambda,\mu \in Q_0} A_{\mu \lambda} \otimes \text{Hom}(V_\lambda, V_\mu),
$$

and the space of \( L \)-equivariant maps from \( u \) into \( \text{End} \mathbb{V} \), for any \( L \)-module \( \mathbb{V} \), is

$$
(2.12) \quad \mathcal{R}_{L,u}(\mathbb{V}) := \text{Hom}_L(u \otimes \mathbb{V}, \mathbb{V}) \cong \text{Hom}_L(u, \text{End} \mathbb{V}).
$$

Note that the data of a pair \( (\mathbb{V}, \varphi) \), where \( \mathbb{V} \) is a \( Q_0 \)-graded vector space and \( \varphi \in \mathcal{R}_Q(\mathbb{V}) \), is equivalent to a \( Q \)-module. In fact, the isomorphisms (2.10) determine an equivalence

$$
(2.13) \quad \text{Rep}(L,u) \cong \text{mod}(Q)
$$

where the left-hand side is the category in which an object is a pair \( (\mathbb{V}, \tau) \) consisting of an \( L \)-module \( \mathbb{V} \) and an \( L \)-equivariant linear map \( \tau : u \to \text{End} \mathbb{V} \), and a morphism \( (\mathbb{V}, \tau) \to (\mathbb{V}', \tau') \) is an \( L \)-equivariant linear map \( f : \mathbb{V} \to \mathbb{V}' \) such that \( f \circ \tau(e) = \tau'(e) \circ f \) for all \( e \in u \). The right-hand side of (2.13) is the category of \( Q \)-modules (with no relations). To prove (2.7), we observe now that an object of \( \text{Rep}(P) \), i.e. a group morphism
\( \rho: P \to \text{GL}(V) \), is equivalent to an object \((V, \tau)\) of \(\text{Rep}(L, u)\) satisfying commutation relations, where \(V\) has \(L\)-action \(\rho|_L: L \to \text{GL}(V)\), \(\tau: u \to \text{End} V\) is given by
\[
\tau = d\rho|_u
\]
(d \(\rho\): \(p \to \text{End} V\) being the differential), and the commutation relations are
\[
\tau([e, e']) = [\tau(e), \tau(e')],
\]
for all \(e, e' \in u\). Then (2.7) follows because one can show that imposing (2.15) on an object \((V, \tau)\) of \(\text{Rep}(L, u)\) corresponds precisely to imposing the set of relations \(K\) on the \(Q\)-module \((V, \varphi)\) corresponding to \((V, \tau)\) via (2.13) (see the proof of [2, Theorem 1.4]).

2.4. A Tannakian structure on modules over the quiver with relations associated to a parabolic subgroup. The category \(\text{mod}(Q, K)\) of modules over an arbitrary quiver with relations \((Q, K)\) is always abelian, but it may have no further sensible algebraic structure in general, such as a tensor product, unless we impose some conditions on \((Q, K)\). In this section, we construct a structure of neutral Tannakian category on \(\text{mod}(Q, K)\), when \((Q, K)\) is the quiver with relations associated to \(P\) (see, e.g., [12] for research on related algebraic structures, namely Hopf quivers).

We start with a short review of the notion of a neutral Tannakian category (see, e.g., [13, 14, 21, 25, 28] for details). A tensor category is a category \(\mathcal{C}\) with a bifunctor \(\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}\), called the tensor product, two natural isomorphisms \(A \otimes (B \otimes C) \to (A \otimes B) \otimes C\) and \(A \otimes B \to B \otimes A\), for all objects \(A, B, C\) of \(\mathcal{C}\), called an associativity constraint and a commutativity constraint, satisfying certain axioms (including the pentagon and the hexagon axioms), and with an object \(1\), called the identity object, and an isomorphism \(1 \to 1 \otimes 1\), such that \(1 \otimes -: \mathcal{C} \to \mathcal{C}\) is an equivalence of categories. A tensor functor \((\mathcal{F}, c): \mathcal{C} \to \mathcal{C}'\) between tensor categories consists of a functor \(\mathcal{F}: \mathcal{C} \to \mathcal{C}'\) and a natural isomorphism \(c_{A,B}: \mathcal{F}(A) \otimes \mathcal{F}(B) \to \mathcal{F}(A \otimes B)\) satisfying canonical axioms involving the tensor products. A morphism \(\lambda: (\mathcal{F}, c) \to (\mathcal{F}', c')\) between two tensor functors \((\mathcal{F}, c), (\mathcal{F}', c'): \mathcal{C} \to \mathcal{C}'\) is a natural transformation \(\lambda: \mathcal{F} \to \mathcal{F}'\) satisfying canonical axioms involving tensor products. The space of these morphisms is denoted \(\text{Hom}^\otimes(\mathcal{F}, \mathcal{F}')\).

A tensor category is rigid if all its objects admit duals, i.e. there is a contravariant functor \((-)^\vee: \mathcal{C}^{op} \to \mathcal{C}\), and a natural isomorphism \(\text{ev}_A: A^\vee \otimes A \to 1\) for all objects \(A\) of \(\mathcal{C}\), inducing natural isomorphisms \(\text{Hom}(B, A^\vee) \to \text{Hom}(B \otimes A, 1)\) for all \(A, B\) (then the category has inner Hom objects \(\text{Hom}(A, B) = B \otimes A^\vee\)). It is known that for all tensor functors \((\mathcal{F}, c), (\mathcal{F}', c'): \mathcal{C} \to \mathcal{C}'\), if \(\mathcal{C}\) and \(\mathcal{C}'\) are rigid, then every morphism \(\lambda: (\mathcal{F}, c) \to (\mathcal{F}', c')\) is an isomorphism (see, e.g., [14, Proposition 1.13]). In this case, the group of tensor automorphisms of any tensor functor \((\mathcal{F}, c): \mathcal{C} \to \mathcal{C}'\) is \(\text{Aut}^\otimes(\mathcal{F}) = \text{Hom}^\otimes(\mathcal{F}, \mathcal{F})\), with group multiplication given by vertical composition of natural transformations. In Definition 2.5, \(\text{Vect}_\mathbb{C}\) is the tensor category of (finite-dimensional) complex vector spaces.

**Definition 2.5** (see, e.g., [14, Definition 2.19]). A neutral Tannakian category (over \(\mathbb{C}\)) is pair \((\mathcal{C}, \mathcal{F})\) consisting of a rigid abelian tensor category such that \(\mathbb{C} = \text{End}(1)\), and an exact faithful linear tensor functor \(\mathcal{F}: \mathcal{C} \to \text{Vect}_\mathbb{C}\), called the fibre functor.

**Example 2.6.** For any complex affine group scheme \(\mathcal{G}\), the category \(\text{Rep}(\mathcal{G})\) of (complex finite-dimensional) representations of \(\mathcal{G}\) is a neutral Tannakian category, with the fibre functor \(\mathcal{F}_\mathcal{G}: \text{Rep}(\mathcal{G}) \to \text{Vect}_\mathbb{C}\) mapping a \(\mathcal{G}\)-module into its underlying vector space.
By the following fundamental duality theorem of Tannaka–Grothendieck–Saavedra, roughly speaking, a neutral Tannakian category is a category with enough added structure that it is equivalent to the category of representations of an affine group scheme, with the neutral Tannakian structure of Example 2.6.

**Theorem 2.7** (see, e.g., [14, Theorem 2.11]). Let \((\mathcal{C}, \mathcal{F})\) be a neutral Tannakian category. Then the group \(\mathcal{G} = \text{Aut}^{\oplus}(\mathcal{F})\) of tensor automorphisms of the fibre functor is an affine algebraic complex group scheme, and the fibre functor \(\mathcal{F}\) determines an equivalence \((\mathcal{C}, \mathcal{F}) \cong (\text{Rep}(\mathcal{G}), \mathcal{F}_\mathcal{G})\).

Suppose now \(P \subseteq G\) is a parabolic subgroup, as in Section 2.1 and \((Q, \mathcal{K})\) is the quiver with relations associated to \(P\), as in Section 2.3. By Theorem 2.7 we can transfer the structure of neutral Tannakian category from \(\text{Rep}(P)\) to \(\text{mod}(Q, \mathcal{K})\) via the equivalence of Theorem 2.3 so \(\text{mod}(Q, \mathcal{K})\) becomes a neutral Tannakian category such that the group of tensor automorphisms of the fibre functor is isomorphic to \(P\). Our next task is to construct this structure of neutral Tannakian category on \(\text{mod}(Q, \mathcal{K})\). Since \(\text{mod}(Q, \mathcal{K})\) is an abelian category, it suffices to define an identity object, suitable dualization and tensor product operations, and a fibre functor on \(\text{mod}(Q, \mathcal{K})\).

Since \(\text{Rep}(L)\) has identity object \(\mathbb{C}\) with the trivial \(L\)-action, we define the identity object \(1\) of \(\text{mod}(Q, \mathcal{K})\) as the \(Q\)-module \((V, \varphi)\) given by \(V_1 = 0\) if \(\lambda \neq \lambda_0\), \(V_{\lambda_0} = \mathbb{C}\), \(\varphi_a = 0\), for all \(\lambda \in Q_0\), \(a \in Q_1\), where \(\lambda_0 \in Q_0\) is the isomorphism class of the trivial \(L\)-module \(M_{\lambda_0} := \mathbb{C}\). The space \(\text{Hom}(U, V)\) of linear maps \(f: U \to V\) between two \(L\)-representations \(U\) and \(V\) is canonically an \(L\)-representation (this is the inner \(\text{Hom}\) in \(\text{Rep}(L)\)). In particular, the dual of \(V\) is the \(L\)-representation \(V^* = \text{Hom}(V, \mathbb{C})\). To construct the dualization functor on \(\text{mod}(Q, \mathcal{K})\), we will use the involution

\[
Q_0 \longrightarrow Q_0: \lambda \mapsto \lambda^*,
\]

where \(\lambda^* \in Q_0\) is the isomorphism class of the dual irreducible \(L\)-module \(M^*_\lambda\), for all \(\lambda \in Q_0\), and the 1-dimensional vector spaces

\[
(2.16) \quad C_\lambda := \text{Hom}_L(M_\lambda, M^*_\lambda),
\]

for all \(\lambda \in Q_0\). Note that \(\dim C_\lambda = 1\), by Schur’s Lemma, and we could have chosen the irreducible representations \(M_\lambda\), for all \(\lambda \in Q_0\), so that either \(M_\lambda = M^*_\lambda\) or \(M_\lambda = M^*_\lambda\), for all \(\lambda \in Q_0\), in which case \(C_\lambda\) would be canonically isomorphic to \(\mathbb{C}\). However, to make several natural isomorphisms below become slightly clearer, we will refrain from making this choice — this has the effect that there is no canonical isomorphism \(C_\lambda \cong \mathbb{C}\), so the isotypical decomposition of \(M^*_\lambda\) becomes a canonical \(L\)-equivariant isomorphism

\[
(2.17) \quad M^*_\lambda = C_\lambda \otimes M_\lambda.
\]

It follows now from (2.3a) and (2.17) that there are canonical isomorphisms

\[
(2.18) \quad A_{\lambda^*\mu^*} \cong C_\mu \otimes A_{\mu \lambda} \otimes C^*_\lambda,
\]

obtained by composing the transposition isomorphisms \(\text{Hom}(M^*_\mu, M^*_\lambda) \cong \text{Hom}(M^*_\mu, M^*_\lambda)\) with the isomorphisms (given by evaluation) \(M^*_\lambda = C_\lambda \otimes M_\lambda\) and \(M^*_\mu = C_\lambda \otimes M^*_\mu\). This isomorphism induces the following linear isomorphisms for all \(\lambda, \mu \in Q_0\) and any (finite-dimensional) vector spaces \(V_\lambda\) and \(V_\mu\), with \(A_{\mu \lambda}\) given by (2.3a):

\[
(2.19) \quad A_{\mu \lambda} \otimes \text{Hom}(V^*_\lambda \otimes C_\lambda, V^*_\mu \otimes C_\mu) \cong (C_\mu \otimes A_{\mu \lambda} \otimes C^*_\lambda) \otimes \text{Hom}(V^*_\lambda, V^*_\mu)
\]

\[
\cong A_{\lambda^*\mu^*} \otimes \text{Hom}(V^*_\mu, V^*_\lambda),
\]
where again we have used the transposition isomorphisms $\text{Hom}(V_{\lambda^*}, V_{\mu^*}) \cong \text{Hom}(V_{\mu^*}, V_{\lambda^*})$.

Let $(V, \varphi)$ be a $(Q, K)$-module. Then its dual $(Q, K)$-module

\[(V, \varphi)^\vee = (V^\vee, \varphi^\vee)\]

is constructed as follows. For each $\lambda \in Q_0$,

\[(V^\vee, \varphi^\vee)_\lambda := V_{\lambda^*}^* \otimes C_\lambda.\]

For each $\lambda, \mu \in Q_0$, let $\varphi_{\lambda^* \mu^*} \in A_{\lambda^* \mu^*} \otimes \text{Hom}(V_{\mu^*}, V_{\lambda^*})$ be the vector corresponding to the set $\varphi_{\lambda^* \mu^*} = \{\varphi_{\lambda^* \mu^*}^{(i)}\}_{i=1}^{n_{\lambda^* \mu^*}}$ via the left-hand isomorphism (2.9), with $\varphi_{\lambda^* \mu^*}^{(i)} : V_{\mu^*} \to V_{\lambda^*}$ as in (2.6). Let $\varphi_{\mu \lambda}^\vee \in A_{\lambda^* \mu^*} \otimes \text{Hom}(V_{\mu^*}^\vee, V_{\lambda^*}^\vee)$ be the vector corresponding to

\[\varphi_{\mu \lambda}^\vee (i) : V_{\lambda^*}^\vee \to V_{\mu^*}^\vee\]

via (2.19) (note the negative sign). Then the set $\varphi_{\mu \lambda}^\vee = \{\varphi_{\mu \lambda}^\vee (i)\}_{i=1}^{n_{\mu \lambda}}$ of linear maps

\[\varphi_{\mu \lambda}^\vee : V_{\lambda^*}^\vee \to V_{\mu^*}^\vee\]

corresponds to $\varphi_{\mu \lambda}^\vee$ via the left-hand isomorphism (2.9), with $V$ replaced by $V^\vee$.

To construct a tensor product on $\text{mod}(Q, K)$, we will use the multiplicity vector spaces

\[C_{\mu \nu}^\lambda := \text{Hom}_L(M_\lambda, M_\mu \otimes M_\nu),\]

for all $\lambda, \nu, \nu' \in Q_0$. In other words, the $L$-module $M_\mu \otimes M_\nu$ has isotopical decomposition

\[M_\mu \otimes M_\nu = \bigoplus_{\lambda \in Q_0} C_{\mu \nu}^\lambda \otimes M_\lambda.\]

Let $(V, \varphi)$ and $(V', \varphi')$ be two $(Q, K)$-modules. Then their tensor product

\[(U, \psi) = (V \otimes V', \varphi \otimes \varphi') := (V, \varphi) \otimes (V', \varphi')\]

is constructed as follows. For each $\lambda \in Q_0$,

\[U_{\lambda} = (V \otimes V')_{\lambda} := \bigoplus_{\mu, \nu \in Q_0} V_{\mu} \otimes V_{\nu} \otimes C_{\mu \nu}^\lambda.\]

For each $\alpha, \beta, \alpha', \beta' \in Q_0$, the sets $\varphi_{\beta \alpha} = \{\varphi_{\beta \alpha}^{(i)}\}_{i=1}^{n_{\beta \alpha}}$ and $\varphi_{\beta' \alpha'} = \{\varphi_{\beta' \alpha'}^{(i)}\}_{i=1}^{n_{\beta' \alpha'}}$ correspond via (2.23) to $L$-equivariant linear maps

\[\varphi_{\beta \alpha} : u \otimes V_{\alpha} \otimes M_{\alpha} \longrightarrow V_{\beta} \otimes M_{\beta}, \quad \varphi_{\beta' \alpha'} : u \otimes V_{\alpha}' \otimes M_{\alpha'} \longrightarrow V_{\beta'} \otimes M_{\beta'},\]

respectively, that induce other ones given by

\[\varphi_{\beta \alpha, \alpha'} := \varphi_{\beta \alpha} \otimes \text{Id}_{V_{\alpha}' \otimes M_{\alpha}'} : u \otimes V_{\alpha} \otimes M_{\alpha} \otimes V_{\alpha}' \otimes M_{\alpha}' \longrightarrow V_{\beta} \otimes M_{\beta} \otimes V_{\alpha}' \otimes M_{\alpha}',\]

\[\varphi_{\beta' \alpha', \alpha} := \text{Id}_{u \otimes M_{\alpha}} \otimes \varphi_{\beta' \alpha'} : u \otimes V_{\alpha} \otimes M_{\alpha} \otimes V_{\alpha}' \otimes M_{\alpha}' \longrightarrow u \otimes V_{\alpha} \otimes M_{\alpha} \otimes V_{\beta'} \otimes M_{\beta'},\]

respectively obtained tensoring by $V_{\alpha}' \otimes M_{\alpha}'$ and $V_{\alpha} \otimes M_{\alpha}$, where (2.25) implies

\[u \otimes V_{\alpha} \otimes M_{\alpha} \otimes V_{\alpha}' \otimes M_{\alpha}' = \bigoplus_{\lambda \in Q_0} u \otimes V_{\alpha} \otimes V_{\alpha}' \otimes C_{\alpha \alpha'}^\lambda \otimes M_{\lambda};\]

\[V_{\beta} \otimes M_{\beta} \otimes V_{\alpha}' \otimes M_{\alpha}' = \bigoplus_{\mu \in Q_0} V_{\beta} \otimes V_{\alpha}' \otimes C_{\beta \alpha'}^\mu \otimes M_{\mu};\]

\[V_{\alpha} \otimes M_{\alpha} \otimes V_{\beta'} \otimes M_{\beta'} = \bigoplus_{\mu \in Q_0} V_{\alpha} \otimes V_{\beta'} \otimes C_{\alpha \beta'}^\mu \otimes M_{\mu}.\]
Hence \( \varphi_{\beta,\alpha'} \) and \( \varphi'_{\beta',\alpha''} \) admit decompositions

\[
\varphi_{\beta,\alpha'} = \sum_{\lambda, \mu \in Q_0} \varphi_{\beta,\alpha'}^{\mu \lambda}, \quad \varphi'_{\beta',\alpha''} = \sum_{\lambda, \mu \in Q_0} \varphi'_{\beta',\alpha''}^{\mu \lambda},
\]

where

\[
\varphi_{\beta,\alpha'}^{\mu \lambda} : u \otimes V_{\alpha} \otimes V'_{\alpha'} \otimes C_{\alpha \alpha'}^{\lambda} \otimes M_{\lambda} \to V_{\beta} \otimes V'_{\alpha'} \otimes C_{\beta \alpha'}^{\mu}, \\
\varphi'_{\beta',\alpha''}^{\mu \lambda} : u \otimes V_{\alpha} \otimes V'_{\alpha'} \otimes C_{\alpha \alpha'}^{\lambda} \otimes M_{\lambda} \to V_{\alpha} \otimes V'_{\beta'} \otimes C_{\alpha \beta'}^{\mu}. 
\]

Since \( \varphi_{\beta,\alpha'}^{\mu \lambda} \) and \( \varphi'_{\beta',\alpha''}^{\mu \lambda} \) are \( L \)-equivariant maps, they may be viewed as linear maps

\[
\varphi_{\beta,\alpha'}^{\mu \lambda} : V_{\alpha} \otimes V'_{\alpha'} \otimes C_{\alpha \alpha'}^{\lambda} \to A_{\mu \lambda} \otimes V_{\beta} \otimes V'_{\alpha'} \otimes C_{\beta \alpha'}^{\mu}, \\
\varphi'_{\beta',\alpha''}^{\mu \lambda} : V_{\alpha} \otimes V'_{\alpha'} \otimes C_{\alpha \alpha'}^{\lambda} \to A_{\mu \lambda} \otimes V_{\alpha} \otimes V'_{\beta'} \otimes C_{\alpha \beta'}^{\mu},
\]

via (2.29). Adding them together for all \( \alpha, \alpha', \beta, \beta' \in Q_0 \), we obtain another linear map

\[
\psi_{\mu \lambda} = \sum_{\alpha, \alpha' \in Q_0} \varphi_{\beta,\alpha'}^{\mu \lambda} + \sum_{\alpha, \alpha' \in Q_0} \varphi'_{\beta',\alpha''}^{\mu \lambda} : V_{\alpha} \otimes V'_{\alpha'} \otimes C_{\alpha \alpha'}^{\lambda} \to A_{\mu \lambda} \otimes \bigoplus_{\beta, \beta' \in Q_0} V_{\beta} \otimes V'_{\beta'} \otimes C_{\beta \beta'}^{\mu},
\]

where

\[
U_{\lambda} = \bigoplus_{\alpha, \alpha' \in Q_0} V_{\alpha} \otimes V'_{\alpha'} \otimes C_{\alpha \alpha'}^{\lambda}, \quad U_{\mu} = \bigoplus_{\beta, \beta' \in Q_0} V_{\beta} \otimes V'_{\beta'} \otimes C_{\beta \beta'}^{\mu}
\]

and hence \( \psi_{\mu \lambda} : U_{\lambda} \to A_{\mu \lambda} \otimes U_{\mu} \). Then the set \( \psi_{\mu \lambda} = \{ \psi_{\mu \lambda}^{(i)} \}_{i=1}^{n_{\mu \lambda}} \) of linear maps

\[
\psi_{\mu \lambda}^{(i)} = (\varphi \otimes \varphi')_{\mu \lambda}^{(i)} : U_{\lambda} \to U_{\mu}
\]

corresponds to \( \psi_{\mu \lambda} \) via the left-hand isomorphism (2.29), with \( V \) replaced by \( U \).

The category \( \text{mod}(Q, \mathcal{K}) \) is also equipped with a fibre functor

\[
\mathcal{F}_{Q, \mathcal{K}} : \text{mod}(Q, \mathcal{K}) \to \text{Vect}_\mathbb{C}, \quad (V, \varphi) \mapsto \bigoplus_{\lambda \in Q_0} V_{\lambda} \otimes M_{\lambda}.
\]

**Lemma 2.8.** The pair \( (\text{mod}(Q, \mathcal{K}), \mathcal{F}_{Q, \mathcal{K}}) \) is a neutral Tannakian category, and there is an equivalence of neutral Tannakian categories

\[
(\text{mod}(Q, \mathcal{K}), \mathcal{F}_{Q, \mathcal{K}}) \cong (\text{Rep}(P), \mathcal{F}_P).
\]

Therefore the affine algebraic complex group scheme \( \text{Aut}(\mathcal{F}_{Q, \mathcal{K}}) \) is identified with \( P \).

**Proof.** In view of Theorem 2.3, we need to check that the equivalence (2.7) identifies the dualization operation, the tensor product operation, and the fibre functor on the categories \( \text{mod}(Q, \mathcal{K}) \) and \( \text{Rep}(P) \). To check that the dualization operations coincide, we fix a \( (Q, \mathcal{K}) \)-module \( (V, \varphi) \), with dual \( (V, \varphi)^\vee = (V^\vee, \varphi') \). Let \( V \) and \( V'^\vee \) be the \( L \)-modules associated to \( V \) and \( V^\vee \) (as in (2.28)), and \( \rho : P \to \text{GL}(V) \) the \( P \)-module corresponding to \( (V, \varphi) \). Then (2.17) and (2.21) give canonical \( L \)-equivariant isomorphisms

\[
V_{\lambda}^\vee \otimes M_{\lambda} \cong V_{\lambda}^* \otimes M_{\lambda}^*.
\]
for all $\lambda \in Q_0$, so the $L$-modules $V$ and $V'$ are related by another canonical isomorphism
\[
V' = \bigoplus_{\lambda \in Q_0} V^\lambda \otimes M_\lambda \cong \bigoplus_{\lambda \in Q_0} V^\lambda_* \otimes M^*_\lambda = V^*.
\]
Now, the $P$-module $\rho: P \to \GL(V)$ determines an object $(V, \tau)$ of $\Rep(L, u)$ (see (2.13)), where the Lie-algebra representation $\tau: u \to \End V$ is given by $\tau = d\rho|_u$ (see (2.14)), whereas its dual $P$-module $\rho^*: P \to \GL(V^*)$ determines the dual object $(V^*, \tau^*)$ of $\Rep(L, u)$, where the dual Lie-algebra representation $\tau^*: u \to \End V^*$ is given by
\[
\tau^*(e) = -\tau(e)^* \quad \text{for all } e \in u \quad (\tau(e)^*: V^* \to V^* \text{ being the dual map of } \tau(e): V \to V).
\]
Comparing (2.32) with (2.22), it now follows from the construction of the maps (2.23) that the object $(\tau^*)$ of $\Rep(L, u)$ corresponds to the $Q$-module $(V, \varphi)^\vee$. Note that the $Q$-module $(V, \varphi)^\vee$ satisfies the set of relations $K$, because it corresponds to the $P$-module $\rho^*: P \to \GL(V^*)$.

To check that the tensor product operations coincide, we fix $(Q, K)$-modules $(V, \varphi)$ and $(V', \varphi')$. Let $(U, \psi) = (V, \varphi) \otimes (V', \varphi')$ be their tensor product, so $U = V \otimes V'$ is given by (2.21). Then (2.25) implies that the $L$-modules $V, V'$ and $U$ associated (as in (2.8)) to the $Q_0$-graded vector spaces $V, V'$ and $U = V \otimes V'$, respectively, satisfy
\[
U = V \otimes V'.
\]
Let $\rho: P \to \GL(V)$ and $\rho': P \to \GL(V')$ be the $P$-modules corresponding to the $(Q, K)$-modules $(V, \varphi)$ and $(V', \varphi')$, respectively. They determine objects $(V, \tau)$ and $(V', \tau')$ of $\Rep(L, u)$ (see (2.13)), with $\tau = d\rho|_u, \tau' = d\rho'|_u$ (see (2.14)), and their tensor product $\rho'' := \rho \otimes \rho': P \to \GL(U)$ determines another one $(U, \tau'')$, with $\tau'' = d\rho''|_u$ given by
\[
\tau''(e) = \tau(e) \otimes \Id_{V'} + \Id_V \otimes \tau(e),
\]
for all $e \in u$. Comparing (2.33) with (2.28), it now follows from the construction of the maps (2.29) that the object $(U, \tau'')$ of $\Rep(L, u)$ corresponds to the $Q$-module $(U, \psi)$. The fact that the $Q$-module $(U, \psi)$ satisfies the set of relations $K$ follows because it corresponds to the tensor product $P$-module $\rho': P \to \GL(U)$.

Finally, the identification of the fibre functors follows by comparing (2.8) and (2.30). □

Note that the inner Hom spaces of the rigid tensor categories $\Rep(P)$ and $\mod(Q, K)$, corresponding to each other via the equivalence (2.7), are respectively given by
\[
\Hom(V, V') \cong V' \otimes V^*, \quad \Hom((V, \varphi), (V', \varphi')) = (V', \varphi') \otimes (V, \varphi)^\vee.
\]

We are now ready to prove the main result of this section.

**Proposition 2.9.** There is an equivalence of neutral Tannakian categories between the category of holomorphic homogeneous vector bundles on $G/P$ and the category of $(Q, K)$-modules with the fibre functor $\mathcal{F}_{Q, K}$.

**Proof.** This follows from Lemmas 2.1 and 2.8 since the equivalence in Lemma 2.1 is an equivalence of neutral Tannakian categories. In the category of holomorphic homogeneous vector bundles $E$ over $G/P$, the identity object is the trivial homogeneous vector bundle $C_{G/P} := G \times P C = G/P \times C$, and the fibre functor $\mathcal{F}_{G/P}$ is given by
\[
\mathcal{F}_{G/P}(E) = E_o,
\]
where $o = P \in G/P$ is the base point of $G/P$, with isotropy group $P$. □
3. EQUIVARIANT PRINCIPAL BUNDLES AND TORSORS IN QUIVER BUNDLES

Building on the equivalences of Lemma 2.8 and Proposition 2.9 in this section we adapt Nori's functor approach to principal bundles, to obtain a Tannakian description of equivariant principal bundles over $X \times G/P$. This viewpoint will be useful, in Section 4, in the understanding via dimensional reduction to $X$ of the algebro-geometric (semi/poly)stability condition for equivariant holomorphic principal bundles on $X \times G/P$.

3.1. NOTATION. Throughout this paper, $X$ is a smooth irreducible complex projective variety, $H$ is a complex connected reductive affine algebraic group, and $\text{Rep}(H)$ is the category of $H$-modules, that is, (holomorphic finite-dimensional complex) representations of $H$ and their $H$-equivariant linear maps. As in Section 2, $G$ is a connected simply connected semisimple complex affine algebraic group, and $P \subset G$ is a parabolic subgroup, with unipotent radical $R_u(P) \subset P$. The Lie algebras of $R_u(P) \subset P \subset G$ are denoted $u \subset p \subset g$, respectively. We also fix a Levi subgroup $L=L(P) \subset P$ of $P$, that is, a maximal connected reductive subgroup of $P$. Recall that any two Levi subgroups are conjugate by some element of $R_u(P) \subset P$ [19, p. 185, Theorem].

Given a complex manifold $M$ with a holomorphic left $G$-action, a $G$-equivariant holomorphic principal $H$-bundle over $M$ is a holomorphic principal $H$-bundle $\pi : E'_H \to M$ on $M$, together with a holomorphic left $G$-action $\rho : G \times E'_H \to E'_H$ on the total space of $E'_H$ that commutes with $\pi$, and such that for all $(g,p) \in G \times M$, the map $\rho_{g,p} : E'_p \to E'_{g \cdot p}$ induced by this action is an isomorphism of $H$-manifolds, where $E'_p = \pi^{-1}(p)$. In this paper, we are concerned with $G$-equivariant holomorphic principal $H$-bundles on the $G$-manifold $M = X \times G/P$. The group $G$ acts on $X \times G/P$ by the trivial action on $X$ and the left-translation action induced by left multiplication on the flag variety $G/P$. If $X$ is a point, we recover the notion of homogeneous holomorphic principal $H$-bundle over $G/P$ [8], that is, a holomorphic principal $H$-bundle $E'_H \to G/P$, together with a holomorphic left $G$-action on $E'_H$ that lifts the left-translation action of $G$ on $G/P$. If, furthermore, $H = \text{GL}(r,\mathbb{C})$ for some integer $r > 0$, then $E'_H$ is equivalent to a rank-$r$ homogeneous holomorphic vector bundle over $G/P$ [2], that is, a $G$-equivariant holomorphic vector bundle $E'$ over $G/P$ (see Section 2.1).

3.2. INDUCTION AND REDUCTION OF EQUIVARIANT PRINCIPAL BUNDLES. We start constructing an equivalence between the groupoid of $G$-equivariant holomorphic principal $H$-bundles on $X \times G/P$, and the groupoid of $P$-equivariant holomorphic principal $H$-bundles on the $P$-variety $X$, with $P$ acting trivially on $X$ (as usual, a groupoid is a category in which every arrow is invertible). The morphisms of the former groupoid are $G$-equivariant isomorphisms of holomorphic principal $H$-bundles. The morphisms of the latter groupoid are $P$-equivariant isomorphisms of holomorphic principal $H$-bundles.

**Lemma 3.1.** There is an equivalence between the groupoid of $G$-equivariant holomorphic principal $H$-bundles on $X \times G/P$, and the groupoid of $P$-equivariant holomorphic principal $H$-bundles on $X$. 

Proof. For any $G$-equivariant holomorphic principal $H$-bundle $E_H'$ on $X \times G/P$, the restriction of $E_H'$ to the slice $X \cong X \times P/P \subset X \times G/P$ is a $P$-equivariant holomorphic principal $H$-bundle on $X$. Conversely, given a $P$-equivariant holomorphic principal $H$-bundle $E_H$ on $X$, consider the pullback $p_1^*E_H \to X \times G$, where $p_1: X \times G \to X$ is the canonical projection. The left-translation action of $G$ on itself and the trivial action of $G$ on $X$ together define an action of $G$ on $X \times G$. Since $p_1^*E_H$ is a pullback from $X$, this action of $G$ on $X \times G$ has a canonical lift to a left action of $G$ on $p_1^*E_H$.

The right-translation action of $P$ on $G$ and the trivial action of $P$ on $X$ together produce an action of $P$ on $X \times G$. Using the action of $P$ on $E_H$, we get a lift of this action of $P$ on $X \times G$ to an action of $P$ on $p_1^*E_H$. The quotient $(p_1^*E_H)/P$ for this action of $P$ on $p_1^*E_H$ is a principal $H$-bundle on $X \times G/P$. The left action of $G$ on $p_1^*E_H$ constructed earlier descends to an action of $G$ on the principal $H$-bundle $(p_1^*E_H)/P \to X \times G/P$ (this is because the left-translation action of $G$ on itself commutes with the right-translation action of $P$ on $G$). \qed

In the situation of the above lemma, we say a $G$-equivariant holomorphic principal $H$-bundle $E_H'$ on $X \times G/P$ is induced by the corresponding $P$-equivariant holomorphic principal $H$-bundle $E_H$ on $X$, denoted

\begin{equation}
E_H' = G \times^P E_H \longrightarrow X \times G/P.
\end{equation}

According to Lemma 3.1, our next task is to give an appropriate description of $P$-equivariant holomorphic principal $H$-bundles over $X$. Since we will apply Tannakian methods in this paper, this will be achieved using to the category of $P$-equivariant holomorphic vector bundles over $X$. The latter category has been described in a combinatorial way using representations of a quiver with relations [2], as reviewed in Section 2.2.

3.3. Equivariant vector bundles and quiver bundles. Let $Q$ be a quiver. Replacing vector spaces by holomorphic vector bundles in the definition of $Q$-module given in Section 2.2, we obtain the category $\text{Vect}_X(Q)$ of holomorphic quiver bundles. More precisely,

- a holomorphic $Q$-bundle on $X$ is a pair $(E, \phi)$ consisting of a set $E$ of holomorphic vector bundles $E_v$ on $X$, indexed by the vertices $v \in Q_0$, and a set $\phi$ of holomorphic vector-bundle homomorphisms $\phi_a: E_{ia} \to E_{ha}$, indexed by the arrows $a \in Q_1$.

- a homomorphism $f: (E, \phi) \to (E', \phi')$ of holomorphic $Q$-bundles is a set of homomorphisms $f_v: E_v \to E'_v$ of holomorphic vector bundles, indexed by the vertices $v \in Q_0$, such that $\phi'_a \circ f_{ia} = f_{ha} \circ \phi_a$ for all $a \in Q_1$.

Let $\mathcal{K}$ be a set of relations of $Q$. In this section, we will be interested in the full subcategory

$$
\text{Vect}_X(Q, \mathcal{K}) \subset \text{Vect}_X(Q)
$$

of holomorphic $(Q, \mathcal{K})$-bundles on $X$, that is, holomorphic $Q$-bundles that satisfy the set of relations $\mathcal{K}$, where $(E, \phi)$ satisfies a relation (2.2) if $\phi(r) := c_1\phi(p_1) + \cdots + c_k\phi(p_k)$ is zero, with a path $p$ of $Q$ determining a holomorphic vector-bundle homomorphism $\phi(p): E_{tp} \to E_{hp}$, given by $\phi(p) = \phi_{a_k} \circ \cdots \circ \phi_{a_1}$ for a non-trivial path (2.1), and by the identity $\phi(p) = \text{Id}_{E_v}: E_v \to E_v$ for the trivial path $p = e_v$ at any $v \in Q_0$.

Now, let $P \subsetneq G$ be a parabolic subgroup, and $(Q, \mathcal{K})$ the quiver with relations associated to $P$ (see Definition 2.2). Then the category $\text{Vect}_X(Q, \mathcal{K})$ of holomorphic $(Q, \mathcal{K})$-bundles
has an identity object, a dualization functor and a tensor product bifunctor, denoted
\((-)^\vee\colon \text{Vect}_X(Q, \mathcal{K})^{\text{op}} \to \text{Vect}_X(Q, \mathcal{K}),\)
\(\otimes\colon \text{Vect}_X(Q, \mathcal{K}) \times \text{Vect}_X(Q, \mathcal{K}) \to \text{Vect}_X(Q, \mathcal{K}),\)
respectively, constructed exactly as in Section 2.4, replacing vector spaces by holomorphic vector bundles (see (2.20) and (2.26)).

**Theorem 3.2.** There are equivalences of categories, preserving the identity objects, dualization and tensor product, between

- the category of \(G\)-equivariant holomorphic vector bundles on \(X \times G/P\),
- the category of \(P\)-equivariant holomorphic vector bundles on \(X\),
- the category of holomorphic \((Q, \mathcal{K})\)-bundles on \(X\).

**Proof.** This follows as in the part of the proof of Lemma 2.6 and Proposition 3.2 concerning the identity objects, the dualization and the tensor product, with [2, Lemma 2.8 and Theorem 2.5] playing the roles of Lemma 2.1 and Theorem 2.3, respectively. \(\Box\)

**3.4. Equivariant principal bundles as torsors in quiver bundles.** Let \((Q, \mathcal{K})\) be the quiver with relations associated to the parabolic subgroup \(P \subseteq G\). In the following definition (cf., e.g., [27, p. 82]), the adjectives “strict exact faithful tensor” applied to a functor mean the four abstract conditions \(F_1–F_4\) listed by Nori [21, p. 77] (in particular, the functor must be compatible with the operations of taking dual and tensor product).

**Definition 3.3.** An \(H\)-torsor in holomorphic \((Q, \mathcal{K})\)-bundles over \(X\) is a strict exact faithful tensor functor
\[(3.2) \quad \mathcal{E}\colon \text{Rep}(H) \to \text{Vect}_X(Q, \mathcal{K}).\]

**Theorem 3.4.** There is a bijective correspondence between the \(P\)-equivariant holomorphic principal \(H\)-bundles on \(X\) and the \(H\)-torsors in holomorphic \((Q, \mathcal{K})\)-bundles over \(X\).

**Proof.** Let \(E_H\) be a \(P\)-equivariant holomorphic principal \(H\)-bundle over \(X\). For each \(H\)-module \(W \in \text{Rep}(H)\), we have the associated holomorphic vector bundle
\[E_W = E_H(W) := E_H \times^H W \to X.\]

The action of \(P\) on \(E_W\) and the trivial action of \(P\) on \(W\) together produce an action of \(P\) on \(E_H \times W\). This action descends to an action of \(P\) on \(E_W\). Now by Theorem 3.2, the \(P\)-equivariant holomorphic vector bundle \(E_W\) produces an object \(\mathcal{E}(W)\) of \text{Vect}_X(Q, \mathcal{K}).

For the converse direction, let \(\mathcal{E}\) be an \(H\)-torsor in holomorphic \((Q, \mathcal{K})\)-bundles over \(X\), i.e. a strict exact faithful tensor functor, as in (3.2). Let \text{Vect}_X be the category of holomorphic vector bundles on \(X\). Define the functor
\[(3.3) \quad S\colon \text{Vect}_X(Q, \mathcal{K}) \to \text{Vect}_X, \quad (E, \phi) \mapsto \bigoplus_{\lambda \in Q_0} E_\lambda \otimes M_\lambda.\]

Then the functor \(S \circ \mathcal{E}\colon \text{Rep}(H) \to \text{Vect}_X\) produces an algebraic principal \(H\)-bundle over \(X\) [21, Lemma 2.3 and Proposition 2.4]. This principal \(H\)-bundle over \(X\) will be denoted by \(E_H(\mathcal{E})\). Our aim is to construct an action of \(P\) on the principal \(H\)-bundle \(E_H(\mathcal{E})\). Let
\[p_X\colon X \times P \to X\]
be the canonical projection onto the first factor. Let $\text{Vect}_{X \times P}$ be the category of holomorphic vector bundles on $X \times P$. Consider the functor

$$p_X^* \circ S \circ \mathcal{E} : \text{Rep}(H) \rightarrow \text{Vect}_{X \times P}$$

that maps any $W \in \text{Rep}(H)$ into the pulled back vector bundle $p_X^*(S \circ \mathcal{E}(W))$. As above, this functor defines a principal $H$-bundle on $X \times P$ [21, Lemma 2.3 and Proposition 2.4]; this principal $H$-bundle on $X \times P$ will be denoted by $\widetilde{E}_H(\mathcal{E})$. The principal $H$-bundle $\widetilde{E}_H(\mathcal{E})$ is evidently identified with the pullback $p_X^*E_H(\mathcal{E})$.

For any $W \in \text{Rep}(H)$, the vector bundle $p_X^*(S \circ \mathcal{E}(W))$ has a natural automorphism

$$A_W : p_X^*(S \circ \mathcal{E}(W)) \rightarrow p_X^*(S \circ \mathcal{E}(W)),$$

which is constructed as follows: by Theorem 3.2 the vector bundle $S \circ \mathcal{E}(W)$ on $X$ is equipped with an action of $P$; this action of $P$ on $S \circ \mathcal{E}(W)$ produces an automorphism of the pullback $p_X^*(S \circ \mathcal{E}(W))$. More precisely, the automorphism sends any $w \in (p_X^*(S \circ \mathcal{E}(W)))(x, g)$ to the image of $w$ under the action of $g \in P$.

Now we have a functor

$$A : \text{Rep}(H) \rightarrow \text{Vect}_{X \times P}$$

that sends any $W \in \text{Rep}(H)$ to $p_X^*(S \circ \mathcal{E}(W))$, and sends any homomorphism of $H$-modules $\rho : W_1 \rightarrow W_2$ to $A_{W_2} \circ \rho' \circ A_{W_1}^{-1}$, where $\rho' : p_X^*(S \circ \mathcal{E}(W_1)) \rightarrow p_X^*(S \circ \mathcal{E}(W_2))$ is the image of $\rho$ by the functor $p_X^* \circ S \circ \mathcal{E}$.

The functor $A$ defined above produces an automorphism of the principal $H$-bundle $p_X^*E_H(\mathcal{E})$. This automorphism defines a morphism

$$\psi : P \times E_H(\mathcal{E}) \rightarrow E_H(\mathcal{E}),$$

which commutes with the right action of $H$. To show that $\psi$ is an action of $P$, take a faithful $H$-module $W$. Consider the principal $\text{GL}(W)$-bundle $E_{\text{GL}(W)}(\mathcal{E}) = E_H(\mathcal{E}) \times^H \text{GL}(W)$ over $X$ obtained by extending the structure group of $E_H(\mathcal{E})$ using the homomorphism $\rho_W : H \rightarrow \text{GL}(W)$ given by the action of $H$ on $W$. Since $E_H(\mathcal{E})$ is embedded in the total space $E_{\text{GL}(W)}(\mathcal{E})$ (as $\rho_W$ is injective), and $\psi$ is the restriction of a $P$-action on this principal $\text{GL}(W)$-bundle $E_{\text{GL}(W)}(\mathcal{E})$, we conclude that $\psi$ defines a $P$-action on $E_H(\mathcal{E})$. $\Box$

Combining Theorem 3.4 with Lemma 3.1 we obtain the following.

**Corollary 3.5.** There is a bijective correspondence between the $G$-equivariant principal $H$-bundles on $X \times G/P$ and the $H$-torsors in holomorphic $(Q, K)$-bundles over $X$.

4. **Dimensional reduction for equivariant principal bundles**

In this section, we study algebro-geometric (semi/poly)stability conditions for equivariant holomorphic principal $H$-bundles on $X \times G/P$, their dimensional reduction to $X$, via the correspondences of Lemma 3.1, Theorem 3.4 and Corollary 3.5 and the corresponding gauge-theoretic equations. Throughout this section, we use the notation of Section 3.1 so $X$ is a smooth irreducible complex projective variety, $H$ is a complex connected reductive affine algebraic group, $G$ is a connected simply connected semisimple complex affine algebraic group, $P \subseteq G$ is a parabolic subgroup, and $L \subseteq P$ is a Levi subgroup. The Lie
algebras of $L \subset P \subset G$ are denoted $I \subset \mathfrak{p} \subset \mathfrak{g}$, respectively, and $(Q, \mathcal{K})$ is the quiver with relations associated to $P$.

4.1. Semistable and polystable $G$-equivariant principal bundles on $X \times G/P$. Choose a Cartan subalgebra $t \subset I \subset \mathfrak{g}$, and a system $S$ of simple roots of $\mathfrak{g}$ with respect to $t$, such that all the negative roots of $\mathfrak{g}$ with respect to $t$ are roots of $\mathfrak{p}$. Let $\Delta_+(t)$ be the set of positive roots of $\mathfrak{g}$ with respect to $t$ and $S$ that are not roots of $I$ (this notation is motivated in [2 §4.2.1]), and $\Sigma \subset S$ the set of simple roots of $\mathfrak{g}$ that are not roots of $\mathfrak{p}$. Let $K \subset G$ be a maximal compact Lie subgroup. Then the $K$-invariant Kähler 2-forms on the complex $G$-manifold $K/(K \cap P) \cong G/P$ are parametrized by elements $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in \Sigma}$ of $\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ and $\mathbb{R}^\Sigma_{>0}$ denote the set of positive real numbers and the set of maps (4.1) $\varepsilon: \Sigma \rightarrow \mathbb{R}_{>0}, \alpha \mapsto \varepsilon_\alpha,$ respectively (see [2 p. 38, Lemma 4.8]). The Kähler form on $G/P$ associated to $\varepsilon$ will be denoted $\omega_\varepsilon$. We equip the complex manifolds $X$, $G/P$ and $X \times G/P$ with fixed Kähler 2-forms $\omega$, $\omega_\varepsilon$ and $p_1^*\omega + p_2^*\omega_\varepsilon$, respectively, where $p_i$ is the projection of $X \times G/P$ to the $i$-th factor. We define the degree of a holomorphic vector bundle $E$ on $X$ by the formula (4.2) $\deg E = \frac{2\pi}{(n-1)! \text{Vol}(X)} \int_X c_1(E) \wedge \omega^{n-1},$ where $n = \dim_{\mathbb{C}} X$ and $\text{Vol}(X) = \int_X \omega^n/n!$ is the volume of $X$. We use this normalization convention also for the degree of torsion-free coherent sheaves on $X$, and for the degree of torsion-free coherent sheaves on $X \times G/P$ with respect to the Kähler form $p_1^*\omega + p_2^*\omega_\varepsilon$.

We now recall the definition of a semistable principal $H$-bundle (see [24 23 5]). Let $Z_0(H)$ be the connected component of the center of $H$ containing the identity element.

**Definition 4.1.** A holomorphic principal $H$-bundle $E_H$ on $X \times G/P$ is called **semistable** (respectively, **stable**) if for every quadruple $(B, U, E_B, \chi)$, where

- $B$ is a proper parabolic subgroup of $H$,
- $\iota: U \hookrightarrow X \times G/P$ is a non-empty Zariski open subset such that the complement $U^c$ is of complex codimension at least two,
- $E_B \subset E_H|_U$ is a holomorphic reduction of structure group to $B$ over the open subset $U$, and
- $\chi$ is a strictly anti-dominant character of $B$ trivial over $Z_0(H)$,

the inequality $\deg(\iota_* E_B(\chi)) \geq 0$ (respectively, $\deg(\iota_* E_B(\chi)) > 0$) holds, where $E_B(\chi)$ is the holomorphic line bundle over $U$ associated to the principal $B$-bundle $E_B$ for the character $\chi$.

It should be clarified that since the complex codimension of $U^c$ is at least 2, the degree $\deg(\iota_* E_B(\chi))$ of the direct image $\iota_* E_B(\chi)$ is well-defined.

**Definition 4.2.** A $G$-equivariant holomorphic principal $H$-bundle $E_H$ on $X \times G/P$ is called **equivariantly semistable** (respectively, **equivariantly stable**) if the condition in Definition 4.1 for semistability (respectively, stability) holds for all quadruples $(B, U, E_B, \chi)$ with the extra condition that both $U \subset X \times G/P$ and $E_B \subset E_H|_U$ are $G$-invariant.

**Lemma 4.3.** A $G$-equivariant holomorphic principal $H$-bundle $E_H$ on $X \times G/P$ is semistable if and only if it is equivariantly semistable.
Proof. If $E_H$ is semistable, then clearly it is equivariantly semistable. To prove the converse, assume that $E_H$ equivariantly semistable. Let $E_B \subset E_H|_U$ be the Harder–Narasimhan reduction of $E_H$ (see [4, Theorem 1]). From the uniqueness of the Harder–Narasimhan reduction it follows that both $U$ and $E_B$ are preserved by the actions of $G$ on $X \times G/P$ and $E_H$ respectively. Since $E_H$ is equivariantly semistable, from the conditions on the Harder–Narasimhan reduction it follows immediately that $B = H$. This implies that $E_H$ is semistable. (See also [8, Lemma 4.1].) □

Let $E_H$ be a holomorphic principal $H$-bundle over $X \times G/P$. A holomorphic reduction $E_B \subset E_H$ of structure group to some parabolic subgroup $B \subset H$ is called admissible if

$$\deg(E_B(\chi)) = 0$$

for each character $\chi$ of $B$ which is trivial on $Z_0(H)$, where $E_B(\chi)$ is the line bundle over $X \times G/P$ associated to the principal $B$-bundle $E_B$ for the character $\chi$.

**Definition 4.4.** A holomorphic principal $H$-bundle $E_H$ on $X \times G/P$ is called polystable if either $E_H$ is stable, or there is a proper parabolic subgroup $B \subset H$ and a holomorphic reduction of structure group

$$E_{L(B)} \subset E_H$$

of $E_H$ to a Levi subgroup $L(B) \subset B$ over $X \times G/P$, such that the following two conditions are satisfied:

- the principal $L(B)$-bundle $E_{L(B)}$ is stable, and
- the reduction of structure group of $E_H$ to $B$, obtained by extending the structure group of $E_{L(B)}$ using the inclusion of $L(B)$ in $B$, is admissible.

**Definition 4.5.** A $G$-equivariant holomorphic principal $H$-bundle $E_H$ on $X \times G/P$ is called equivariantly polystable if either $E_H$ is equivariantly stable, or there is a proper parabolic subgroup $B \subset H$ and a $G$-equivariant holomorphic reduction of structure group

$$E_{L(B)} \subset E_H$$

of $E_H$ to a Levi subgroup $L(B) \subset B$ over $X \times G/P$ such that the following two conditions are satisfied:

- the principal $L(B)$-bundle $E_{L(B)}$ is equivariantly stable, and
- the reduction of structure group of $E_H$ to $B$, obtained by extending the structure group of $E_{L(B)}$ using the inclusion of $L(B)$ in $B$, is admissible.

**Lemma 4.6.** A $G$-equivariant holomorphic principal $H$-bundle $E_H$ on $X \times G/P$ is polystable if and only if it is equivariantly polystable.

Proof. The proof is identical to the proof of Lemma 4.2 of [8]; we omit the details. □

4.2. Semistable and polystable torsors in holomorphic quiver bundles. Our definition of semistability and polystability of $H$-torsors in $(Q, \mathcal{K})$-bundles over $X$ will rely on the corresponding notions for quiver bundles. To define the latter, we need to enlarge the category of holomorphic $(Q, \mathcal{K})$-bundles, considering the so-called $(Q, \mathcal{K})$-sheaves on $X$. To do this, we simply replace the notion of holomorphic vector bundle by coherent sheaf of $\mathcal{O}_X$-modules in the corresponding definitions of Section 3.3 (see [2, §2.1] for details). In particular, a $Q$-sheaf on $X$ is a pair $(E, \phi)$ consisting of a set $E$ of coherent sheaves $E_\lambda$
on $X$, indexed by $\lambda \in Q_0$, and a set $\phi$ of sheaf homomorphisms $\phi_a : E_{ta} \to E_{ha}$, indexed by $a \in Q_1$. Then $Q$-sheaves form an abelian category. Note that if a $Q$-sheaf satisfies the set of relations $\mathcal{K}$, then so do all its $Q$-subsheaves. A $Q$-sheaf $(E, \phi)$ is called torsion free if so is $E_\lambda$, for all $\lambda \in Q_0$.

Fix a Kähler form $\omega$ on $X$, and $\varepsilon$ as in (4.1). For each $\lambda \in Q_0$, we define the numbers
\[ n_\lambda := \dim_\mathbb{C} M_\lambda \in \mathbb{N}, \]
\[ \tau'_\lambda := -n_\lambda \sum_{\alpha \in \Delta_+(t)} \varepsilon_\alpha^{-1} \kappa(\lambda, \alpha^\vee) \in \mathbb{R}, \]
(cf. [2, p. 36, (4.10)]), where $M_\lambda$ is any irreducible $L$-module in the isomorphism class $\lambda$ (see Section 2.3), $\kappa$ is the Killing form on $\mathfrak{g}$, $\alpha^\vee := 2\alpha/\kappa(\alpha, \alpha)$, and $\varepsilon_\alpha$ is defined by
\[ \varepsilon_\alpha := \sum_{\beta \in \Sigma} \varepsilon_{\beta} \kappa(\lambda_\beta, \alpha^\vee), \]
for $\alpha \in \Delta_+(t) \setminus \Sigma$. The $\tau'$-degree and the $\tau'$-slope of a torsion-free $Q$-sheaf $(E, \phi)$ on $X$ are
\[ \deg_{\tau'}(E, \phi) := \sum_{\lambda \in Q_0} (n_\lambda \deg E_\lambda - \tau'_\lambda \rk E_\lambda), \]
\[ \mu_{\tau'}(E, \phi) := \frac{\deg_{\tau'}(E, \phi)}{\sum_{\lambda \in Q_0} n_\lambda \rk E_\lambda}, \]
respectively, where the degree is defined as in (4.2), and $\rk E_\lambda$ is the rank of $E_\lambda$.

A torsion-free $Q$-sheaf $(E, \phi)$ on $X$ is called semistable if all its non-zero $Q$-subsheaves $(E', \phi')$ satisfy
\[ \mu_{\tau'}(E', \phi') \leq \mu_{\tau'}(E, \phi). \]
It is called stable if this inequality is strict for all proper $Q$-subsheaves, and polystable if it is a direct sum of stable $Q$-sheaves of the same $\tau'$-slope (see [2, Definition 4.19]).

**Definition 4.7.** An $H$-torsor $\mathcal{E} : \text{Rep}(H) \to \text{Vect}_X(Q, \mathcal{K})$ is called polystable (respectively, semistable) if the holomorphic $(Q, \mathcal{K})$-bundle $\mathcal{E}(W)$ is polystable (respectively, semistable) for every indecomposable $H$-module $W$.

The above definitions of $\tau'$-degree and $\tau'$-slope are motivated by the following.

**Lemma 4.8.** Let $E$ be a $P$-equivariant holomorphic vector bundle $E$ over $X$, and
\[ E' := G \times^P E \]
the induced $G$-equivariant holomorphic vector bundle over $X \times G/P$ (see Theorem 3.2). Then the degree and the slope of $E'$ with respect to the Kähler form $p_1^*\omega + p_2^*\omega_\mathcal{E}$ are respectively computed by the $\tau'$-degree and the $\tau'$-slope of the holomorphic $(Q, \mathcal{K})$-bundle $(E, \phi)$ over $X$ associated to $E$.
\[ \deg E' = \deg_{\tau'}(E, \phi), \quad \mu(E') = \mu_{\tau'}(E, \phi). \]

**Proof.** Fix an irreducible $L$-module $M_\lambda$ in the isomorphism class $\lambda$, for each $\lambda \in Q_0$, as in Section 2.3. Then $M_\lambda$ can also be considered as an irreducible $P$-module, with the unipotent radical $R_u(P) \subset P$ acting trivially (see [2, §1.1.3]). Let $\mathcal{O}_\lambda = G \times^P M_\lambda$ be its
associated homogeneous holomorphic vector bundle over $G/P$ (see Lemma 2.1). By \cite{2, p. 36, Lemma 4.15} and the definition of $\tau'_\lambda$, the slope of $\mathcal{O}_\lambda$ with respect to $\omega_\varepsilon$ is
\[ \mu_\varepsilon(\mathcal{O}_\lambda) := \frac{\deg_\varepsilon \mathcal{O}_\lambda}{\text{rk} \mathcal{O}_\lambda} = -\frac{\tau'_\lambda}{n_\lambda} \]
where we use the normalization convention in (1.2) for the degree $\deg_\varepsilon \mathcal{O}_\lambda$ of $\mathcal{O}_\lambda$ with respect to $\omega_\varepsilon$. Furthermore, applying the induction process of Theorem 3.2 (cf. \cite{2, Lemma 2.8 and Theorem 2.5}) to the image of (3.3), we see that $E'$ has underlying vector bundle
\[ E' = \bigoplus_{\lambda \in \mathbb{Q}_0} p_1^* E_\lambda \otimes p_2^* \mathcal{O}_\lambda, \]
so it has rank and degree (with respect to $p_1^* \omega + p_2^* \omega_\varepsilon$) given by
\[ \text{rk}(E') = \sum_{\lambda \in \mathbb{Q}_0} \text{rk}(p_1^* E_\lambda \otimes p_2^* \mathcal{O}_\lambda) = \sum_{\lambda \in \mathbb{Q}_0} n_\lambda \text{rk} E_\lambda, \]
\[ \text{deg} E' = \sum_{\lambda \in \mathbb{Q}_0} \text{deg}(p_1^* E_\lambda \otimes p_2^* \mathcal{O}_\lambda) = \sum_{\lambda \in \mathbb{Q}_0} n_\lambda (\text{deg} E_\lambda + \mu_\varepsilon(\mathcal{O}_\lambda) \text{rk} E_\lambda) = \text{deg}_{\tau'}(E', \phi), \]
respectively. The result about the slope now follows by diving these quantities. \qed

4.3. Dimensional reduction, semistability and polystability.

**Theorem 4.9.** Let $E'_H$ be a $G$-equivariant holomorphic principal $H$-bundle over $X \times G/P$. Let $E : \text{Rep}(H) \to \text{Vect}_X(Q, K)$ be the corresponding $H$-torsor in quiver bundles over $X$. Then $E'_H$ is polystable (respectively, semistable) with respect to the Kähler form $p_1^* \omega + p_2^* \omega_\varepsilon$ if and only if $E$ is polystable (respectively, semistable).

**Proof.** Let $E_H$ be the $P$-equivariant principal $H$-bundle over $X$ corresponding to $E$ and $E'_H$ in Theorem 3.4 and Corollary 3.5. Let $W$ be an $H$-module. Then the functorial correspondence
\[ E'_H \cong G \times^P E_H \]
of equivariant principal bundles of Lemma 3.1 (see (3.1)) induces another one
\[ E'_W \cong G \times^P E_W, \]
where $E_W$ and $E'_W$ are the $P$-equivariant holomorphic vector bundle on $X$ associated to the principal bundle $E_H$, and the $G$-equivariant holomorphic vector bundle on $X \times G/P$ associated to the principal bundle $E'_H$, respectively, for the $H$-module $W$, that is
\[ E_W := E_H \times^H W, \quad E'_W := E'_H \times^H W. \]

We will apply the identification (4.4) to the indecomposable $H$-submodules of the Lie algebra $\text{Lie}(H)$ of $H$. More precisely, the adjoint representation of $H$ has a decomposition
\[ \text{Lie}(H) = \bigoplus_{i=1}^\ell W_i \]
into a direct sum of indecomposable $H$-modules. Since $H$ is reductive, we know that each $W_i$ is self-dual, meaning the $H$-module $W_i^*$ is isomorphic to $W_i$ for all $1 \leq i \leq \ell$. Then
\[ \text{ad}(E'_H) = \bigoplus_{i=1}^\ell E'_i, \]
where $\text{ad}(E'_H) := E'_H(\text{Lie}(H))$ and $E'_i := E'_H(W_i)$ are the vector bundles over $X \times G/P$ associated to $E'_H$ for the $H$-modules $\text{Lie}(H)$ and $W_i$, respectively. Note that the vector bundle $(E'_i)^*$ is isomorphic to $E'_i$, because the $H$-module $W_i$ is self-dual, so in particular,

\begin{equation}
\deg(E'_i) = 0.
\end{equation}

Assume now that $\mathcal{E}$ is polystable. To prove that the principal $H$-bundle $E'_H$ is polystable, note that if $W$ is an indecomposable $H$-module, then $E'_W$ is polystable. This follows because the holomorphic $(Q,K)$-bundle $\mathcal{E}(W)$ is polystable in this case (by Definition 4.7), so $E'_W \cong G \times^P E_W$ is equivariantly polystable by [2, p. 40, Theorem 4.21], and hence $E'_W$ is polystable by Lemma 4.6. In particular, each $E'_i$ is polystable, because each $W_i$ is indecomposable. Hence the adjoint vector bundle $\text{ad}(E'_H)$ is polystable (by (4.6) and (4.7)), and therefore so is the principal $H$-bundle $E'_H$, by [5, p. 224, Corollary 3.8].

To prove the converse, assume $E'_H$ is polystable. Let $W$ be an indecomposable $H$-module. As $W$ is indecomposable, $Z_0(H)$ acts on $W$ as multiplication through a character of $Z_0(H)$, so the associated vector bundle $E'_W$ is polystable (by [5, p. 224, Theorem 3.9]), and hence the holomorphic $(Q,K)$-bundle $\mathcal{E}(W)$ is polystable too, by [2, p. 40, Theorem 4.21] and Lemma 4.6. Thus the $P$-equivariant principal $H$-bundle $E_H$ is polystable.

The proof for semistability is very similar.

Assume $\mathcal{E}$ is semistable. To prove that the principal $H$-bundle $E'_H$ is semistable, we first observe that if $W$ is an indecomposable $H$-module, then $E'_W$ is semistable. This follows because the associated $P$-equivariant vector bundle $E'_W$ corresponds to a semistable $(Q,K)$-module $\mathcal{E}(W)$ in this case, so it follows as above, from [2, p. 40, Theorem 4.21] and Lemma 4.3 that the associated vector bundle $E'_H(W)$ is semistable. In particular, each vector bundle $E'_i$ in (4.7) is semistable, because $W_i$ is indecomposable. Hence we conclude from (4.6) and (4.7) that the adjoint vector bundle $\text{ad}(E'_H)$ is semistable, and therefore so is the principal $H$-bundle $E'_H$ is semistable, by [5, p. 214, Proposition 2.10].

To prove the converse, assume $E'_H$ is semistable. To prove that $\mathcal{E}$ is semistable, in view of [2, p. 40, Theorem 4.21] and Lemma 4.3 it suffices to show that for every indecomposable $H$-module $W$, the associated vector bundle $E'_H(W)$ is semistable. If the Kähler class on $X \times G/P$ is rational, then $E'_H(W)$ is semistable by [23, p. 285, Theorem 3.18]. For the general Kähler class, the semistability of $E'_H(W)$ follows from [4, p. 700, Lemma 5]. This completes the proof of the proposition. □

4.4. Dimensional reduction and the vortex equations. A hermitian metric on a holomorphic $(Q,K)$-bundle $(E, \phi)$ over $X$ is a set $h = \{h_\lambda\}_{\lambda \in Q_0}$ of $(C^\infty)$ hermitian metrics $h_\lambda$ on $E_\lambda$, indexed by the vertices $\lambda \in Q_0$ (where we set $h_\lambda = 0$ for all $\lambda$ such that $E_\lambda = 0$). For each arrow $a \in Q_1$, the homomorphism $\phi_a : E_{ta} \rightarrow E_{ha}$ has a $(C^\infty)$ adjoint

$\phi_a^* : E_{ha} \rightarrow E_{ta}$

with respect to the hermitian metrics $h_{ta}$ and $h_{ha}$ on $E_{ta}$ and $E_{ha}$, respectively. Let $\Lambda$ be the adjoint of multiplication of differential forms on $X$ by the Kähler form $\omega$. Let $F_{h_\lambda}$ be the curvature of the Chern connection of the hermitian metric $h_\lambda$ on $E_\lambda$, for all $\lambda \in Q_0$ such that $E_\lambda \neq 0$. A hermitian metric $h$ on $(E, \phi)$ satisfies the quiver vortex equations if

$$
n_\lambda \sqrt{-1} \Lambda F_{h_\lambda} + \sum_{a \in h^{-1}(\lambda)} \phi_a \phi_a^* - \sum_{a \in t^{-1}(\lambda)} \phi_a^* \phi_a = \tau_\lambda \text{Id}_{E_\lambda},$$

where $\text{ad}(E'_H) := E'_H(\text{Lie}(H))$ and $E'_i := E'_H(W_i)$ are the vector bundles over $X \times G/P$ associated to $E'_H$ for the $H$-modules $\text{Lie}(H)$ and $W_i$, respectively. Note that the vector bundle $(E'_i)^*$ is isomorphic to $E'_i$, because the $H$-module $W_i$ is self-dual, so in particular,
for all \( \lambda \in Q_0 \) such that \( E_\lambda \neq 0 \), where \( \text{Id}_{E_\lambda} : E_\lambda \rightarrow E_\lambda \) is the identity map (cf. [3, §2.1]).

Let \( \mathcal{E} : \text{Rep}(H) \rightarrow \text{Vect}_X(Q, \mathcal{K}) \) be an \( H \)-torsor in holomorphic \((Q, \mathcal{K})\)-bundles over \( X \), and \( E_H \) the corresponding \( P \)-equivariant holomorphic principal \( H \)-bundle on \( X \). We define the **adjoint holomorphic \((Q, \mathcal{K})\)-bundle** of \( \mathcal{E} \) as the holomorphic \((Q, \mathcal{K})\)-bundle \( \text{ad}(\mathcal{E}) \) on \( X \) corresponding to the \((P\)-equivariant\) adjoint holomorphic vector bundle

\[
\text{ad}(E_H) = E_H(\text{Lie}(H)) = E_H \times^H \text{Lie}(H)
\]

of \( E_H \) via Theorem 3.2 (\( \text{Lie}(H) \) being the adjoint representation of \( H \)). In other words,

\[
\text{ad}(\mathcal{E}) := \mathcal{E}(\text{Lie}(H)) \in \text{Vect}_X(Q, \mathcal{K}).
\]

**Theorem 4.10.** An \( H \)-torsor \( \mathcal{E} : \text{Rep}(H) \rightarrow \text{Vect}_X(Q, \mathcal{K}) \) in holomorphic \((Q, \mathcal{K})\)-bundles over \( X \) is polystable if and only if its adjoint holomorphic \((Q, \mathcal{K})\)-bundle \( \text{ad}(\mathcal{E}) \) admits a hermitian metric \( h \) that satisfies the quiver vortex equations.

**Proof.** Let \( E_H \) and \( E'_H = G \times^P E_H \) be the \( P \)-equivariant principal \( H \)-bundle on \( X \) and the \( G \)-equivariant principal \( H \)-bundle on \( X \times G/P \) corresponding to \( \mathcal{E} \), respectively (see (3.1)). As in the proof of Theorem 4.9, using the adjoint \( H \)-representation \( \text{Lie}(H) \) and the indecomposable \( H \)-modules \( W_i \) appearing in its decomposition (4.3), the principal bundles \( E_H \) and \( E'_H \) induce \( P \)-equivariant vector bundles over \( X \), and \( G \)-equivariant vector bundles over \( X \times G/P \), respectively by

\[
\text{ad}(E_H) = E_H \times^H \text{Lie}(H) = \bigoplus_{i=1}^\ell E_i, \quad \text{ad}(E'_H) = E'_H \times^H \text{Lie}(H) = \bigoplus_{i=1}^\ell E'_i,
\]

\[
E_i = E_H \times^H W_i, \quad E'_i = E'_H \times^H W_i,
\]

that are related as in (4.3), that is,

\[
\text{ad}(E'_H) \cong G \times^P \text{ad}(E_H), \quad E'_i \cong G \times^P E_i.
\]

Furthermore, the decomposition (4.3) induces another one of the adjoint \((Q, \mathcal{K})\)-bundle

\[
(4.8) \quad \text{ad}(\mathcal{E}) = \mathcal{E}(\text{Lie}(H)) = \bigoplus_{i=1}^\ell (E_i, \phi_i),
\]

with \((E_i, \phi_i) := \mathcal{E}(W_i)\) for all \( 1 \leq i \leq \ell \), where Lemma 4.8 and (4.7) imply

\[
(4.9) \quad \mu_{\tau'}(E_i, \phi_i) = \mu(E'_i) = 0.
\]

Assume now that \( \mathcal{E} \) is polystable. Then \((E_i, \phi_i) = \mathcal{E}(W_i)\) is polystable, because \( W_i \) is indecomposable, so \( \text{ad}(\mathcal{E}) \) is polystable, by (4.8) and (4.9). Hence \( \text{ad}(\mathcal{E}) \) admits a hermitian metric \( h \) that satisfies the quiver vortex equations, by [2, p. 41, Theorem 4.24].

To prove the converse, assume that \( \text{ad}(\mathcal{E}) = \mathcal{E}(\text{Lie}(H)) \) admits a hermitian metric \( h \) that satisfies the quiver vortex equations. Then the \( Q \)-bundle \( \text{ad}(\mathcal{E}) \) is polystable, by [2, p. 41, Theorem 4.24], and hence the vector bundle \( \text{ad}(E'_H) \) is polystable, by [2, p. 40, Theorem 4.21], because \( \text{ad}(\mathcal{E}) \) corresponds to the adjoint bundle \( \text{ad}(E'_H) \) of \( E'_H \) via Theorem 3.2. Hence \( E'_H \) is polystable, by [5, p. 224, Corollary 3.8], and therefore we conclude from Theorem 4.9 that \( E_H \) is polystable. \( \square \)
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Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Calle Nicolás Cabrera, 13-15, Campus Cantoblanco UAM, 28049 Madrid, Spain

E-mail address: l.alvarez-consul@icmat.es

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Calle Nicolás Cabrera, 13-15, Campus Cantoblanco UAM, 28049 Madrid, Spain

E-mail address: oscar.garcia-prada@icmat.es