Particle propagation and effective space-time in Gravity’s Rainbow

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Basing on the results obtained in our previous study on Gravity’s Rainbow, we determine the quantum corrections to the space-time metric for the Schwarzschild and the de Sitter background, respectively. We analyze how quantum fluctuations alter these metrics inducing modifications on the propagation of test particles. Significantly enough we find that quantum corrections can become relevant not only for particles approaching the Planck energy but, due to the one loop contribution, even for low-energy particles as far as Planckian length scales are considered. We briefly compare our results with others obtained in similar studies and with the recent experimental OPERA announcement of superluminal neutrino propagation.

I. INTRODUCTION

It is almost more than a decade that the idea of modifying the Lorentz-Poincaré symmetry at the Planck scale has been considered in a systematic way. The pioneering idea is known under the name of Doubly Special Relativity, according to which the modifications of the Lorentz-Poincaré symmetry should occur also preserving the relativity principle, i.e. preserving the equivalence of all inertial observers [1-4]. Actually, in the most studied doubly-special-relativity proposals, the Lorentz sector of the Poincaré symmetry is modified only for its action on the energy-momentum space of the test particles. This action in fact becomes nonlinear and Planck-scale dependent. The resulting deformed symmetry produces deformed Casimirs and, as a consequence, deformed energy-momentum dispersion relations of the type

\[ E^2 g_1^2 \left( \frac{E}{E_P} \right) - p^2 g_2^2 \left( \frac{E}{E_P} \right) = m^2, \]

where \( g_1 \left( \frac{E}{E_P} \right) \) and \( g_2 \left( \frac{E}{E_P} \right) \) are two arbitrary functions whose contributions become relevant only at Planckian energies. In order to reproduce the usual behavior at low energies we must have

\[ \lim_{\frac{E}{E_P} \to 0} g_1 \left( \frac{E}{E_P} \right) = 1 \quad \text{and} \quad \lim_{\frac{E}{E_P} \to 0} g_2 \left( \frac{E}{E_P} \right) = 1. \]

In [5] these type of relativistic symmetries have been applied to general relativity. The resulting gravity model, known as Gravity’s Rainbow, produces a correction to the space-time metric that becomes significant as soon as the particle energy/momentum approaches the Planck energy/momentum, being otherwise these quantum corrections Planck-scale suppressed. In a recent paper [7] we have discussed the modifications induced by the Gravity’s Rainbow of [5] on the zero-point gravitational energy. In particular it has been shown that certain classes of deformed dispersion relations lead to a finite Zero Point Energy (ZPE) for the gravitational field and induce a finite cosmological constant (see also [8-11]), avoiding therefore the traditional procedures of regularization and renormalization. Procedures that can be avoided even in Noncommutative geometry [6]. An interesting aspect is that from the Einstein’s Field Equations, \( G_{\mu\nu} = 8\pi GT_{\mu\nu} \) (with \( c = 1 \)), written in an orthonormal reference frame for a spherically symmetric space-time

\[ ds^2 = -\exp\left(-2\Phi(r)\right) \frac{dt^2}{g_1^2 \left( \frac{E}{E_P} \right)} + \frac{dr^2}{\left( 1 - \frac{b(r)}{r} \right) g_2^2 \left( \frac{E}{E_P} \right)} + \frac{r^2}{g_2^2 \left( \frac{E}{E_P} \right)} (d\theta^2 + \sin^2 \theta d\phi^2), \]
we obtain the following set of equations

\[ \rho(r) = \frac{1}{8\pi G} \frac{b'}{r^2}, \] (4)

\[ p_r(r) = \frac{1}{8\pi G} \left[ 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r} - \frac{b}{r^3} \right], \] (5)

\[ p_t(r) = \frac{1}{8\pi G} \left( 1 - \frac{b}{r} \right) \left[ \Phi'' + \left( \Phi' \right)^2 \right. \]
\[ \left. - \frac{b'r - b}{2r(r-b)} \Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r} \right]. \] (6)

\( \Phi(r) \) is termed the redshift function, while \( b(r) \) is the shape function subject to the only condition \( b(r_t) = r_t \). \( \rho(r) \) is the energy density, \( p_r(r) \) is the radial pressure, and \( p_t(r) \) is the lateral pressure measured in the orthogonal direction to the radial direction. Among these equations if we focus our attention on Eq.(4) and we impose that \( \rho(r) = \Lambda(r) \) \( \frac{8\pi G}{8\pi G} \), we find that it must be

\[ b_{qc}(r) = b_{qc}(+\infty) + \int_{+\infty}^{r} \Lambda(r')r'^2 dr'. \] (8)

Therefore, we expect that the quantum-induced cosmological constant obtained in Ref.[7], can be considered as an energy density source leading to an effective metric

\[ b_{eff}(r) = b_{cl}(r) + b_{qc}(r), \] (9)

where \( b_{cl}(r) \) is the classical shape function, and where \( b_{qc}(r) \) accounts for the quantum corrections. In this paper we extend the analysis of Ref.[7] to find the distortion of the classical metric caused by quantum fluctuations. Such a distortion will produce effects on the motion of some test particle in the considered background which, in principle can be measured. The paper is organized as follows. In section II we outline the procedure we used to calculate the effective metric from the quantum induced cosmological fluctuations. In section III we study the (quantum) induced space-time metric for a Schwarzschild background. In section IV we discuss the implications of the Schwarzschild effective metric on a orbiting test particle. In section V we analyze the effects of the quantum corrections on a de Sitter space-time, briefly discussing possible implications for particles propagating on a de Sitter metric. In Section VI we discuss our conclusions.

II. INDUCED COSMOLOGICAL CONSTANT AND EFFECTIVE METRIC

The procedure followed in this work relies heavily on the formalism outlined in Ref.[7], where the graviton one loop contribution to the cosmological constant in a background of the form (3) was computed. Rather than reproduce the formalism, we shall refer the reader to Ref.[7] for details, when necessary. However, for self-completeness and self-consistency, we present here a brief outline of the formalism used. In this paper, rather than working with energy density as in Ref.[7], we shall integrate over the whole hypersurface \( \Sigma \) to obtain enough information for the effective metric. The idea is to consider the Wheeler-De Witt equation as a formal eigenvalue equation where the cosmological constant divided by the Newton’s constant is the desired eigenvalue. The expectation value we are interested is \( (\kappa = 8\pi G) \)

\[ \frac{\langle \Psi | \int_{\Sigma} d^3x \Delta \Sigma | \Psi \rangle}{V \langle \Psi | \Psi \rangle} = -\frac{\Lambda_c}{\kappa}, \] (10)

where

\[ \Lambda_\Sigma = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{gR}}{(2\kappa) g^2(E)} \] (11)

and \( V \) is the volume. However if we use the line element \( (3) \), the expectation value transforms into

\[ \frac{g^3_\Sigma(E) \langle \Psi | \int_{\Sigma} d^3x \Delta \Sigma | \Psi \rangle}{V \langle \Psi | \Psi \rangle} = -\frac{\Lambda_c}{\kappa}. \] (12)
with
\[
\lambda_\Sigma = (2\kappa) \frac{g_1^2(E)}{g_2^2(E)} \tilde{G}_{ijkl} \tilde{x}^{ij} \tilde{x}^{kl} \frac{\sqrt{g}}{2(\kappa) g_2(E)}
\]
(13)

where
\[
G_{ijkl} = \frac{1}{2\sqrt{g}} (g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) = \frac{\tilde{G}_{ijkl}}{g_2(E)}
\]
(14)

With the help of Gaussian trial wave functionals, a solution of Eq.(12) has been obtained in Ref.[7] to one loop, whose form is
\[
\frac{\Lambda}{8\pi G} = \frac{1}{3\pi^2} \sum_{i=1}^{2} \int_{E_*}^{+\infty} E_i g_1(E) g_2(E) \frac{d}{dE_i} \sqrt{\frac{E_i^2}{g_2(E) - m_i^2(r)}}^3 dE_i,
\]
(15)

where
\[
\begin{align*}
m_1^2(r) &= \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{3}{2r^2} \frac{b(r)}{r} - \frac{3}{2r^3} b(r) \\
m_2^2(r) &= \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{1}{2r^2} \frac{b(r)}{r} + \frac{3}{2r^3} b(r)
\end{align*}
\]
(16)

and \(E_*\) is the value that leads to a vanishing argument of the square root. To be more explicit and to evaluate Eqs.(15), we need to specify the form of \(g_1(E/E_P)\) and \(g_2(E/E_P)\). Following the ansatz of [7] we choose:
\[
g_1 \left( \frac{E}{E_P} \right) = \left(1 + \beta \frac{E}{E_P} + \delta \frac{E^2}{E^2_P} + \gamma \frac{E^3}{E^3_P} \right) \exp(-\alpha \frac{E^2}{E^2_P}) \quad g_2 \left( \frac{E}{E_P} \right) = 1.
\]
(17)

The aim of this paper is to study the effective metric induced by quantum fluctuations of the gravitational field obtained in Eq.(8). We argue that this effective rainbow metric presents some relevant novelties with respect to the original (tree-level) rainbow metric mainly due to the fact that the Planck scale corrections become significant not only for high energy particles but can become significant also for low-energy particles. We can recognize two relevant cases:

a) \(m_1^2 (r) = -m_2^2 (r) = m_0^2 (r)\),

b) \(m_1^2 (r) = m_2^2 (r) = m_0^2 (r)\).

When condition a) is satisfied (for example in the Schwarzschild, Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter cases in proximity of the throat), Eq.(15) can be rearranged in the following way:
\[
\frac{\Lambda}{8\pi GE_P} = \frac{1}{16\pi^2 r^{7/2} e^{-x^2 \alpha}} \left\{ -\sqrt{\pi} \big[15\gamma + 4x^2 \alpha^2 (\beta + 3x^2 \gamma) + 6\alpha (\beta + 2x^2 \gamma) \big] - 25/2 e^{x^2 \alpha}\left(1 + e^{x^2 \alpha}\right)x^4 K_0\left(\frac{x^2 \alpha}{2}\right) \\
+e^{x^2 \alpha} \sqrt{\pi} \left[2\alpha (3 - x^2 \alpha) \beta + 15 - 4x^2 \alpha (3 - x^2 \alpha) \right] \gamma \right \} \right. \\
+2e^{x^2 \alpha} x\sqrt{\alpha} \left( -6\alpha\beta - 15\gamma + 2x^2 \alpha + 2\alpha xK_1\left(\frac{x^2 \alpha}{2}\right) \right) \left[ -2(\alpha + 2\beta) \cosh\left(\frac{x^2 \alpha}{2}\right) + x^2 \alpha \delta \sinh\left(\frac{x^2 \alpha}{2}\right) \right]
\]
(18)

where \(x = \sqrt{m_0^2 (r)/E_P^2}\), \(\beta_1 \equiv \beta\) and where \(K_0(x)\) and \(K_1(x)\) are the Bessel functions and \(erf(x)\) is the error function. On the other hand, when condition b) is satisfied (for example in Minkowski, de Sitter and anti-de Sitter cases), Eq.(15) becomes
\[
\frac{\Lambda}{8\pi GE_P} = \frac{1}{8\pi^2 r^{7/2} e^{-x^2 \alpha}} \left\{ -\sqrt{\pi} \left[15\gamma + 4x^2 \alpha^2 (\beta + 3x^2 \gamma) + 6\alpha (\beta + 2x^2 \gamma) \right] + \\
+2e^{x^2 \alpha} x^3 \alpha^{3/2} \left[-\alpha \delta x^2 K_0\left(\frac{x^2 \alpha}{2}\right) - (4\delta + \alpha (2 + x^2 \delta) \right) K_1\left(\frac{x^2 \alpha}{2}\right) \right]\right\}
\]
(19)

Even though we have all orders equations (18) and (19), more insight can be gained considering the limiting cases. For the purposes of our analysis the case of interests are those of \(x \ll 1\) and \(x \gg 1\). For the case \(x \gg 1\) the expression (18) reduces to:
\[
\frac{\Lambda}{8\pi GE_P} = A_1 x + A_2 x^{-1} + O \left(x^{-3}\right)
\]
(20)
where

\[ A_1 = - \frac{(2\sqrt{\pi}\alpha^{3/2} + 4\alpha \beta + 8\gamma + 3\sqrt{\pi\alpha} \delta)}{8\pi^2 \alpha^3} \]  \hspace{1cm} (21)

and

\[ A_2 = - \frac{16\alpha \beta + 48\gamma + 3\sqrt{\pi\alpha}(2\alpha + 5\delta)}{32\pi^2 \alpha^3}. \]  \hspace{1cm} (22)

It is straightforward to see that the coefficient \( A_1 \) must be set to zero to have a finite result. Instead for \( x \ll 1 \) we get:

\[
\frac{\Lambda}{8\pi GE_P} = B_1 + \left[ B_2 - \frac{1}{8\pi^2} \ln \left( \frac{\alpha x^2}{4} \right) \right] x^4 + O(x^5)
\]  \hspace{1cm} (23)

where

\[ B_1 = \frac{8\alpha^{3/2} + 6\sqrt{\pi} \alpha \beta + 15\sqrt{\pi} \gamma + 16\sqrt{\alpha} \delta}{8\pi^2 \alpha^{7/2}} \]  \hspace{1cm} (24)

and

\[ B_2 = \frac{(1 + 2\gamma_E)\alpha^{3/2} - 2\sqrt{\pi} \alpha \beta - \sqrt{\pi} \gamma - 2\sqrt{\alpha} \delta}{16\pi^2 \alpha^{3/2}}. \]  \hspace{1cm} (25)

Concerning Eq. (19) for \( x \gg 1 \) we get:

\[
\frac{\Lambda}{8\pi GE_P} = -e^{-x^2} \alpha \left( \frac{\gamma}{2\pi^{3/2} \alpha^{3/2}} x^4 + \frac{\delta}{2\pi^{3/2} \alpha^{3/2}} x^3 \right) + O(e^{-x^2} x^2),
\]  \hspace{1cm} (26)

whereas for \( x \ll 1 \) we find:

\[
\frac{\Lambda}{8\pi GE_P} = C_1 + C_2 x^2 + \left[ C_3 - \frac{1}{8\pi^2} \ln \left( x^2 \alpha/4 \right) \right] x^4 + O(x^5)
\]  \hspace{1cm} (27)

where:

\[ C_1 = \frac{-8\alpha^{3/2} - 6\sqrt{\pi} \alpha \beta - 15\sqrt{\pi} \gamma - 16\sqrt{\alpha} \delta}{8\pi^2 \alpha^{7/2}}, \]  \hspace{1cm} (28)

\[ C_2 = \frac{4\alpha^{3/2} + 2\sqrt{\pi} \alpha \beta + 3\sqrt{\pi} \gamma + 4\sqrt{\alpha} \delta}{8\pi^2 \alpha^{5/2}} \]  \hspace{1cm} (29)

and

\[ C_3 = \frac{-\alpha^{3/2} - 2\gamma_E \alpha^{3/2} + 2\sqrt{\pi} \alpha \beta + \sqrt{\pi} \gamma + 2\sqrt{\alpha} \delta}{16\pi^2 \alpha^{3/2}}. \]  \hspace{1cm} (30)

We can fix our attention on particular forms of spherically-symmetric metrics: the Schwarzschild geometry and the de Sitter geometry. We begin with the Schwarzschild geometry.

### III. THE SCHWARZSCHILD CASE

The Schwarzschild’s metric is described by \( b(r) = R_S = 2GM \) and from Eq. (19) we obtain

\[
\begin{align*}
m_1^2(r) &= \frac{6}{r^2} \left( 1 - \frac{R_S}{r} \right) - \frac{3R_S}{2r^3} = \frac{6}{r^2} \left( 1 - \frac{3}{4} \frac{R_S}{r} \right) \\
m_2^2(r) &= \frac{6}{r^2} \left( 1 - \frac{R_S}{r} \right) + \frac{3R_S}{2r^3} = \frac{6}{r^2} \left( 1 - \frac{1}{4} \frac{R_S}{r} \right)
\end{align*}
\]  \hspace{1cm} (31)
If we restrict our attention to the range $R_S < r < 5R_S/4$, which will be denoted as a “short range” approximation (SR), we fall into the case a) and we find

$$m_1^2(r) = -m_2^2(r) = -m_0^2(r) = \frac{3R_S}{2r^3}$$

with

$$x = \sqrt{\frac{m_0^2(r)}{E_p^2}} = \left( \frac{3R_SL_p^2}{2r^3} \right)^{1/2}$$

where we have introduced the Planck length $L_p$. For small $x$, Eq.(23) becomes

$$\frac{\Lambda}{8\pi G E_p^4} = B_1 + \left[ B_2 - \frac{1}{8\pi^2} \ln \left( \frac{3\alpha R_SL_p^2}{8r^3} \right) \right] \left( \frac{3RSL_p^2}{2r^3} \right)^2 + O \left( \frac{RSL_p^2}{r^3} \right)^{5/2}.$$  (34)

In particular, using the values $\alpha = 1/4, \beta = -2/(3\sqrt{\pi}), \gamma = \delta = 0$ found in [7], the parameters $B_1$ and $B_2$ in Eq.(23) become $B_1 = 0$ and $B_2 = -(9 + 2\gamma_E)/ (16\pi^2)$ so that the effective metric reduces to

$$b_{eff}(r) = R_S + \frac{3R_p^2L_p^2}{4\pi r^3} \left[ \ln \left( \frac{3R_SL_p^2}{32r^3} \right) + \gamma_E + \frac{7}{2} \right] - \frac{3L_p^2}{4\pi R_S} \left[ \ln \left( \frac{3R_S}{32R_p^2} \right) + \gamma_E + \frac{7}{2} \right] + O \left( \frac{L_p^3}{R_S^2} \right).$$  (35)

Note that the use of expression (23) is appropriate whenever $R_S \gg L_p$. However in case $B_1$ was not nought, then

$$b_{eff}(r) \simeq R_S + 8\pi B_1 \frac{r^3}{L_p^3} + \frac{3R_p^2L_p^2}{4\pi r^3} \left[ \ln \left( \frac{3R_SL_p^2}{32r^3} \right) + \gamma_E + \frac{7}{2} \right] - \frac{3L_p^2}{4\pi R_S} \left[ \ln \left( \frac{3R_S}{32R_p^2} \right) + \gamma_E + \frac{7}{2} \right] - 8\pi B_1 \frac{R_p^3}{L_p^2} + O \left( \frac{L_p^3}{R_S^2} \right).$$  (36)

Another interesting case is $x \gg 1$ or $L_p \gg R_S$ (sub-Planckian wormhole). The correct expression to use is given by Eq.(20) and the effective metric becomes

$$b_{eff}(r) = R_S + \frac{32}{9\sqrt{\pi}L_p^2} \left( \frac{2}{3R_S L_p^2} \right)^{1/2} r^{9/2} + O \left( r^{9/2} R_S^{-1/2} L_p^{-3} \right).$$  (37)

We now fix our attention on the other range of approximation, namely when $r \gg R_S$. In this case we are in the “long range” approximation (LR) and the effective masses are distinct. In this approximation both $m_1(r)$ and $m_2(r)$ are positive that it means that we are dealing with a $\Lambda(x)$ given by Eq.(19)

$$\Lambda(x_1, x_2) = \frac{\Lambda(x_1)}{2} + \frac{\Lambda(x_2)}{2},$$

where

$$x_1 = \sqrt{\frac{m_1^2(r)}{E_p^2}} \quad \text{and} \quad x_2 = \sqrt{\frac{m_2^2(r)}{E_p^2}}.$$  (39)

Since we are in the LR approximation, we can claim that $x_1, x_2 \ll 1$ and we can use the expansion of Eq.(27). Thus explicitly:

$$\frac{\Lambda}{8\pi G E_p^4} = C_1 + \frac{C_2}{2} (x_1^2 + x_2^2) + \frac{C_3}{2} (x_1^4 + x_2^4) - \frac{1}{16\pi^2} \left[ \ln (x_1^2\alpha/4) x_1^4 + \ln (x_2^2\alpha/4) x_2^4 \right] + O(x_1^5, x_2^5).$$  (40)

Setting properly the parameters i.e. using the values $\alpha = 1/4, \beta = -2/(3\sqrt{\pi})$ and adopting the simple choice $\gamma = \delta = 0$ we would obtain at the leading order $\Lambda(r) \simeq 1/r^2$ and thus $b_{eff}(r) \simeq R_S + 64/\pi r$ i.e. we would get an increasing large $b_{eff}(r)$ that would be incompatible with observations. To maintain compatibility with observations we have to request that also $C_2 = 0$ that fixes the parameters $\gamma = 2/(9\sqrt{\pi})$ and $\delta = -5/12$. This choice leads to

$$\frac{\Lambda(r)}{8\pi G E_p^4} = \frac{36}{\pi^2} \ln \left( \frac{r}{L_p} \right) \left( \frac{L_p^2}{r^2} \right)^2 + O \left( \frac{L_p^2}{r^2} \right)^2$$  (41)
which, in terms of $b_{\text{eff}}(r)$ means

$$b_{\text{eff}}(r) = R_S + \frac{288}{\pi} \frac{L_P^2}{r} \ln \left( \frac{r}{L_P} \right) + O \left( \frac{L_P^2}{r} \right). \quad (42)$$

The different expressions of the effective metric in the different régimes can be used to see the effects on a test particle moving on a background of the form [3]. Even in this case, we will fix our attention on the Schwarzschild and the de Sitter space-time, respectively. We begin with the Schwarzschild background.

IV. IMPLICATIONS FOR TEST PARTICLE ON A SCHWARZSCHILD BACKGROUND

Using the appropriate form of the effective metric, we consider the modifications induced on the orbital motion and onto the effective potential.

A. The orbital motion

In Gravity’s Rainbow the motion is still geodetic but the geodetic equation (see [12, 13]) acquires, by means of the connection, the dependency on the energy $E$ of the test particle:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(E) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \quad (43)$$

It is easy to check that the quantity

$$g_{\mu\nu}(E) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\mu \quad (44)$$

defines a constant of the motion, being $\mu = 1$ for massive particles (time-like geodesics) and $\mu = 0$ for massless particles (null-like geodesics). Using [3] with [4] provide us the following equation

$$-\exp[-2\Phi(r)] \frac{i^2}{g_1^2(E)} + \frac{i^2}{(1 - b_{\text{eff}}(r)) g_2^2(E)} + \frac{r^2}{g_2^2(E)} \left[ \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right] = -\mu. \quad (45)$$

Assuming an equatorial plane $\theta = \pi/2$ and using the following constants of the motion

$$\epsilon = \frac{\exp[-2\Phi(r)]}{g_1^2(E)} t, \quad l = r^2 \dot{\phi}, \quad \mu = 0 \quad (46)$$

Eq.(45) can be written in the form:

$$\epsilon^2 g_1^2(E) g_2^2(E) \frac{i^2}{(1 - b_{\text{eff}}(r))} - \frac{i^2}{1 - b_{\text{eff}}(r)} - \frac{l^2}{r^2} = g_2^2(E) \mu, \quad (47)$$

where we have used the relation $\exp[-\Phi(r)] = \sqrt{1 - b(r)/r}$ for the Schwarzschild case. Then in term of the variable $u = 1/r$, being $\dot{r} = -udu/d\phi$ and differentiating with respect to $\phi$, we obtain the equation of the orbit

$$\frac{d^2 u}{d\phi^2} + u - b_{\text{eff}} \left( \frac{u^{-1}}{2} \right) \left[ 3u^2 + \frac{\mu g_2^2(E)}{l^2} \right] + b_{\text{eff}} \left( \frac{u^{-1}}{2} \right) \left[ u + \frac{\mu g_2^2(E)}{u l^2} \right] = 0. \quad (48)$$

We can evaluate Eq.(48) using the form of $b_{\text{eff}}$ given by (42). The leading corrections to the usual general relativistic result are of order of $\delta = L_P^2 u/R_S \ln (uL_P)$. For the planets of the solar system one gets the greatest correction for Mercury orbit: $\delta \simeq 10^{-82}$ that it means that the quantum effects are largely negligible. Notice that in the case of photons, being $\mu = 0$, the Binet equation takes the form

$$\frac{d^2 u}{d\phi^2} + u \left[ 1 - \frac{3}{2} b_{\text{eff}} \left( \frac{u^{-1}}{2} \right) u \right] + \frac{u}{2} b_{\text{eff}}^\prime = 0, \quad (49)$$

so that quantum corrections modify the orbit only through the loop effects (i.e. by the modification induced in $b_{\text{eff}}$). At the tree-level, being $b_{\text{eff}} = R_S$ and $b_{\text{eff}}^\prime = 0$, there are no corrections to the orbit with respect to the standard general relativistic case, in agreement with what is found in Ref.[13].
B. Photon time delay in radial motion.

For massless particles ($\mu = 0$) in radial motion using equation (45) one obtains

$$\frac{dr}{dt} = \exp[-\Phi(r)] \frac{g_2(E)}{g_1(E)} \sqrt{1 - \frac{b_{eff}(r)}{r}}.$$  \hfill (50)

Now let us consider two photons, both emitted at $r = r_e$ and detected at $r = r_d$. The first photon be an high-energy photon, the second one be a low-energy photon ($E \approx 0$). Then the time delay of detection between the two is given by

$$\Delta t = \left[ \frac{g_1(E)}{g_2(E)} - 1 \right] \int_{r_e}^{r_d} \frac{\exp[\Phi(r)]}{\sqrt{1 - \frac{b_{eff}(r)}{r}}} dr. \hfill (51)$$

This last formula holds for a general spherical symmetric metric of the type (3). In particular for the Schwarzschild case, being $b_{eff}(r) = R_S + b_{qc}(r)$, Eq. (51) takes the form

$$\Delta t \simeq \Delta t_1 + \Delta t_2,$$  \hfill (52)

where

$$\Delta t_1 = \left[ \frac{g_1(E)}{g_2(E)} - 1 \right] \int_{r_e}^{r_d} \frac{dr}{1 - \frac{R_S}{r}}, \hfill (53)$$

$$\Delta t_2 = \left[ \frac{g_1(E)}{g_2(E)} - 1 \right] \int_{r_e}^{r_d} \frac{b_{qc}(r) dr}{r^2 \left(1 - \frac{R_S}{r} \right)^2}, \hfill (54)$$

and where we have neglected terms of order $b_{qc}^2(r)$ or higher. Formula (53) can be easily evaluated. Being in our assumptions, at the lowest order in the Planck scale, $g_2(E) = 1$, $g_1(E) \simeq 1 + \beta E/E_P$ and $\beta = -2/(3\sqrt{\pi})$, we get

$$\Delta t_1 = - \frac{2}{3\sqrt{\pi}} \frac{E}{E_P} r_d \left[ 1 - \frac{r_e}{r_d} + \frac{R_S}{r_d} \ln \left( \frac{r_d/R_S - 1}{r_e/R_S - 1} \right) \right]. \hfill (55)$$

To analytically evaluate the integral (54) we have to specify the exact form of $b_{QC}(r)$. For instance, under the hypothesis of validity of Eq. (42), we get

$$\Delta t_2 \simeq - \frac{2}{3\sqrt{\pi}} \frac{288}{\pi^2} \frac{E}{E_P} r_d^2 \left[ 1 - \frac{1}{r_d} \ln \left( \frac{r_e}{r_d} \right) - \frac{1}{r_d} \ln \left( \frac{1}{r_d} \right) \right]. \hfill (56)$$

Notice that, whereas the tree level contribution grows with the distance between the detector and the source, the quantum fluctuations given by $\Delta t_2$ are more and more suppressed as the distance from the source increases. To provide a numerical estimation, if we considered two photons emitted at the same time in $r_e \approx 10km$ from a star in the center of our galaxy ($r_d \approx 3 \cdot 10^{14} ly$), we would have $\Delta t (\approx \Delta t_1) \approx 1s$ already with $E_2 \approx 10^{-12} E_P$, and the effects of quantum fluctuations, being $\Delta t_2 \approx 10^{-86}s$, would be largely negligible.

C. The effective potential

We can also study the modifications induced on the effective potential by the quantum fluctuation of the metric with respect to the classical general relativistic case. Let us notice that by the standard procedure Eq. (47) can be rearranged in the form

$$i^2 + V^2(r) = c^2 g_1^2(E) g_2^2(E), \hfill (57)$$

where now

$$V(r) = \sqrt{\left( g_2^2(E) + \frac{l_2^2}{r^2} \right) \left( 1 - \frac{b_{eff}(r)}{r} \right)} = V_{GR}(r) \sqrt{1 - \frac{b_{qc}(r)}{r - R_S}}, \hfill (58)$$
and where
\[ V_{GR}(r) = \sqrt{\left(1 + \frac{l^2}{r^2}\right) \left(1 - \frac{R_S}{r}\right)} \]  
(59)
is the classical general-relativistic effective potential. In Eq. (58) we have assumed \( g_2(E) = 1 \) and we have defined \( b_{eff}(r) = R_S + h_{qc}(r) \). To explicitly evaluate \( b_{eff}(r) \) we have to introduce \( h_{eff}(r) \) in the formula. For instance, in the range in which Eq. (52) holds \( (r \gg R_S) \) we find
\[ V(r) \simeq V_{GR}(r) \left[1 - \frac{144 L_P^4 \ln (r/L_P)}{\pi r^2} \right]. \]  
(60)

We notice that Planck-scale corrections begin to manifest at the one-loop level, as far as space-time is probed at Planckian scales (i.e. at scales at which the terms in \( L_P^2 \) become relevant). It is also interesting to note here that if one considered the scattering between two masses in a gravitational potential, at the lowest order in Planck scale, one would find, according to Ref. [14], the Donoghue’s potential
\[ V_{Donoghue}(r) = -\frac{GM}{r} \left[1 - \frac{G(M + m)}{rc^2} + \frac{127 L_P^4}{60 \pi^2 r^2} \right]. \]  
(61)

In the Donoghue’s potential the leading quantum corrections are of the form \( L_P^2/r^2 \), while in our case they are of order \( L_P^2/r^2 \ln (r/L_P) \). Once the analysis has been made for the Schwarzschild background it is not difficult to extend it to other spherically symmetric cases. In the next section, we will analyze the de Sitter background.

V. THE DE SITTER CASE

The de Sitter case written in static coordinates is simply described by \( b(r) = \Lambda r^3/3 \). In this situation the effective masses of Eq. (16) take the form
\[ m_1^2(r) = m_2^2(r) = \frac{6}{r^2} - \Lambda, \quad r \in (0, r_C] \]  
(62)
with \( r_C = \sqrt{3/\Lambda} \). Defining the dimensionless variable
\[ x = \frac{L_P}{r} \sqrt{6 - \Lambda r^2}, \]  
(63)
we can use expansion (27), assuming \( r \gg L_P \) and \( \Lambda r^2 = O(1) \), to obtain
\[ \frac{\Lambda}{8\pi G E_P} = C_1 + C_2 \left(\frac{L_P}{r}\right)^2 (6 - \Lambda r^2) + \left\{ C_3 - \frac{1}{8\pi^2} \log \left[\left(\frac{L_P}{r}\right)^2 (6 - \Lambda r^2)^2 \alpha/4\right]\right\} \left(\frac{L_P}{r}\right)^4 (6 - \Lambda r^2)^2 + O \left(\frac{L_P}{r}\right)^5. \]  
(64)

Again substituting in Eq. (5) we get the quantum-corrected de Sitter parameter
\[ b_{eff}(r) = \frac{\Lambda + C_1 r^3 + r \left(6 - \frac{1}{3} \Lambda^2\right) C_2 L_P^2}{3 \times 3} + \frac{L_P}{r} \left\{ \frac{9}{\pi^2} \log \left[\frac{3L_P \sqrt{\alpha}}{r}\right] - 9 \left(4C_3 + \frac{1}{\pi^2}\right) + r^2 \Lambda \left[ -12C_3 + \frac{9}{2\pi^2} + \frac{3}{\pi^2} \log \left(3L_P \sqrt{\alpha}\right) \right]\right\} + O[\Lambda]^2. \]
(65)

If we assume compatibility with the Schwarzschild-background case then we have to set \( C_1 \simeq C_2 \simeq 0 \) so that at the leading order we would have
\[ \Lambda_{eff} \simeq \Lambda + 27 \frac{L_P}{r^4} \left\{ \frac{1}{\pi^2} \log \left[\frac{3L_P \sqrt{\alpha}}{r}\right] - \left(4C_3 + \frac{1}{\pi^2}\right) + r^2 \Lambda \left[ -12C_3 + \frac{9}{2\pi^2} + \frac{3}{\pi^2} \log \left(3L_P \sqrt{\alpha}\right) \right]\right\} + O[\Lambda^2, L_P^5/r^4]. \]  
(66)

If instead we admit the possibility \( C_1 \) and \( C_2 \) to assume values that are different from those assumed on the Schwarzschild background, at the leading order we would find
\[ \Lambda_{eff} \simeq \Lambda + C_1 + C_2 \frac{L_P^2}{r^2} (18 - r^2 \Lambda). \]  
(67)
Notice that from Eqs. \[66-67\] follows that even starting from exactly \(\Lambda = 0\), one can obtain a nonvanishing cosmological constant induced by the quantum fluctuations, depending on the parameters of the rainbow functions. Thus we could have an effective de Sitter space-time starting from a bare Minkowski space-time. We can also have the situation where quantum fluctuations cancel the nonvanishing \(\text{"bare cosmological constant" (}\Lambda \neq 0\) providing \(\Lambda_{eff} \simeq 0\). In this last case we would have an effective Minkowski space-time starting from a bare de Sitter space-time. Finally it could be also possible that quantum fluctuations be able to transform a de Sitter space-time into an anti-de Sitter space-time or vice versa. All these cases seem to suggest how ZPE be a source of a topology change\[15\].

A. Implications for photons propagating on a de Sitter background

To study the motion of photons on a de Sitter background one can follow the strategy outlined in the previous section for the Schwarzschild metric. In particular photon time delay for radial motion can be inferred directly from Eq.\[68\] with the simple prescription of using the form of \(b_{eff}(r)\) given by Eq.\[66\]. However we observe here that the metric

\[
ds^2 = -(1 - \frac{\Lambda_{eff} r^2}{3}) \frac{dt^2}{g_1^2(E)} + \frac{dr^2}{g_1(E) g_2(E)} + \frac{r^2}{g_2^2(E)} (dg^2 + \sin^2 \theta d\phi^2) ,
\]

is expressed in terms of "static coordinates" while for our purposes it is better to write it in terms of "flat coordinates". Indeed flat coordinates appear more natural from a phenomenological perspective since they allow to associate the motion of a detector to a given-comoving position. The change between "static" and "flat" coordinates can be obtained in Gravity’s Rainbow by means of the map

\[
t' = t + \frac{1}{2 \sqrt{\Lambda_{eff}/3}} g_1(E) \log \left(1 - \frac{\Lambda_{eff} r^2}{3}\right),
\]

\[
\rho = \frac{r}{g_2(E)} \exp \left(-t' \sqrt{\frac{\Lambda_{eff}/3}{g_1(E)}}\right).
\]

In terms of the variables \(\rho = \sqrt{x^2 + y^2 + z^2}, t'\) given by Eqs.\[69-70\], the metric \[68\] becomes

\[
ds^2 = -\frac{dt'^2}{g_1^2(E)} + \exp \left[2 \frac{g_2(E)}{g_1(E)} \sqrt{\frac{\Lambda_{eff}/3}{t'}}\right] (dx^2 + dy^2 + dz^2),
\]

from which one can easily deduce the photon equation of motion

\[
x_{dS}(t) = e^{\sqrt{\Lambda_{eff}/3} t_0} - e^{-\sqrt{\Lambda_{eff}/3} t_0} ,
\]

where according to \[17\] we have assumed \(g_2(E) = 1\).

Following Ref.\[10\] let us now consider two photons emitted at the same time \(t = -t_0\) at \(x_{dS} = 0\). The first photon be a low energy photon \((E \ll E_P)\) and the second one be a Planckian photon \((E \sim E_P)\). Both photons are assumed to be detected at a later time in \(x_{dS}\). We expect to detect the two photons with a time delay \(\Delta t\) given by the solution of the equation

\[
x_{dS}^{E \ll E_P}(0) = x_{dS}^{E \sim E_P}(\Delta t),
\]

that implies

\[
\Delta t \simeq \frac{g_1(E)}{E_P} e^{\sqrt{\Lambda_{eff}/3} t_0} - e^{-\sqrt{\Lambda_{eff}/3} t_0} \simeq \frac{E}{E_P} t_0 \left(1 + \sqrt{\Lambda_{eff}/3}\right),
\]

where we have used \[17\] for the rainbow functions.

At the lowest order the formula \[74\] agrees with the corresponding formula of Ref.\[10\] in which the Planck scale comes into play by means of a parameter \(w = f(H_{LP})\) appearing in the quantum-de Sitter group. The result for the time delay found in Ref.\[10\] is

\[
\Delta t \simeq p(1 - e^{2H(t_0)}/(2H^2))w,
\]
where $H = \sqrt{\Lambda/3}$. A first difference between (74) and (75) regards the origin of the Planck parameter for the time delay that in (75) is connected to the quantum deformation of the classical de Sitter group “$w$” whereas in (74) comes from the rainbow deformation “$\beta$”. A second key difference between (74) and (75) is in $\Lambda_{eff}$. Indeed according to Eqs. (66)–(67) the effective cosmological constant includes quantum corrections: this further dependence on $L_P$ is not present in (75).

### B. Quantum corrections to Minkowski space-time

The Minkowski limit can be derived directly from Eq. (65) by assuming a vanishing bare cosmological constant ($\Lambda = 0$). In this case one obtains for the space-time metric parameter

$$b_{eff}(r) = \frac{C_1}{3} r^3 + 6 r C_2 L_P^2 + \frac{9 L_P^2}{r} \left\{ \frac{1}{\pi^2} \log \left( \frac{3 L_P^2 \sqrt{\Lambda}}{r} \right) - \left( 4 C_3 + \frac{1}{\pi^2} \right) \right\},$$

and consequently, for the cosmological constant

$$\Lambda_{eff}(r) \simeq C_1 + 18 C_2 L_P^2 + \frac{27 L_P^2}{r^2} \left\{ \frac{1}{\pi^2} \log \left( \frac{3 L_P^2 \sqrt{\Lambda}}{r} \right) - \left( 4 C_3 + \frac{1}{\pi^2} \right) \right\},$$

from which it is easily seen that, at the lowest order, the quantum corrections to a bare Minkowski space-time can in principle transform it into either a de Sitter or an anti-de Sitter space-time, depending on the signs and on the values of the constants $C_1$ and $C_2$ (which in turn depend on the parameters of the deformed dispersion relations). If we assume $C_1 \simeq C_2 \simeq 0$, according to what we have found for the Schwarzschild background, we get

$$\Lambda_{eff}(r) \simeq \frac{27 L_P^4}{r^4} \left\{ \frac{1}{\pi^2} \log \left( \frac{3 L_P^2 \sqrt{\Lambda}}{r} \right) - \left( 4 C_3 + \frac{1}{\pi^2} \right) \right\}.$$

From Eq. (78) follows that quantum corrections are actually more likely to turn Minkowski space-time into an anti-de Sitter space-time rather than into a de Sitter one. To evaluate how rainbow effects and cosmological-constant-induced quantum fluctuations affect time delay for a particle in motion in a Minkowski space-time, we can use directly formulas (52)–(54) with the prescription of fixing $R_S = 0$. It can be interesting here to compare the time delay derived from (52)–(54) with the experimental time delay whose measure has very recently been announced by [19]. In [19] a negative time delay $\Delta t_{exp} \simeq -60\,\text{ns}$ has been claimed for $E \simeq 17\,\text{GeV}$ neutrinos propagating over a distance $(r_d - r_e) \simeq 730\,\text{km}$ with respect to the time of propagation of light (i.e. of low-energy photons). Substituting in (52)–(54) the values of the parameters corresponding to the experimental settings of [19], and treating the neutrinos as massless particles, we find

$$\Delta t \simeq \Delta t_1 \simeq \frac{\beta E}{E_P} \left( r_d - r_e \right) \approx -10^{-12}\,\text{ns} \approx -10^{-14}\,\Delta t_{exp},$$

being in our case $\beta = -2/(3\sqrt{\pi})$. Thus our formulas with the value of $\beta$ fixed according to the procedure followed in [2] predict a time delay $10^{-14}$ times smaller than the announced experimental one, and with the opposite sign (i.e. our formulas predict high-energy neutrinos to be slower than low-energy photons), neither the quantum fluctuations of the metric encoded in $\Delta t_2 (\ll \Delta t_1)$ are able to qualitatively change this conclusion. It is worth noticing that in order to account for the proper sign allowing the superluminal neutrino propagation of [19], and in order to get the right time delay $\Delta t \simeq \Delta t_{exp}$, we should assume $\beta \approx +10^{14}$. This huge-positive and unnatural value of $\beta$ however would also be in contrast with other experimental data such us those relative to the time delay observed in TeV flares coming from active galaxies [21], that should be various orders of magnitude greater than measured.

### VI. CONCLUSIONS

We have found that in Gravity’s Rainbow quantum fluctuations can significantly modify the structure of the tree level space-time metric and, consequently, the propagation of test particles. The modifications induced on the metric depend on the parameters that deform the energy-momentum dispersion relation. The request of recovering the classical limit at large distances from the source constrains the classes of admissible deformation of the dispersion relation. A result that is common to all the spherical symmetric backgrounds that we have analyzed is that the metric is Planck scale deformed also in the case one should consider low-energy particles as long as Planck scale distances...
are involved. In fact the quantum effects modify the metric in two different ways. The first way is by means of the dependency on the Planck scale of the rainbow functions. This type of modification is significant only as soon as the energy of the test particle approaches the Planck energy. The second way is through the quantum fluctuations of the metric that in our approach are finite. This other type of modification becomes significant independently on the energy of the test particle and manifests on Planck length scale even for low-energy particles. For the Schwarzschild background we have found that the quantum effects alter the orbit of the test particles (even those of massless test particles) and not only the time of propagation, as instead happens at the tree level (see Ref. [13]). The leading quantum correction at large distances \( r \) from a source, whose Schwarzschild radius is \( R_S \), is of order \( \frac{L_p^2}{(R_S r)} \ln (r/L_P) \) with respect to the usual general relativistic correction. This means that a direct observation of a modification of the orbit is largely out of reach. Instead the leading effect on the time delay between two photons being amplified by the distance from the source could be more effective with respect to the possibility of being experimentally revealed. However the effect of fluctuations of the metric appear to be largely negligible with respect to the leading tree level effect given by the rainbow functions. Finally our calculation of the effective potential has shown that the gravitational potential is modified (again also for low energy test particles) but only at Planckian length scales. The type of modifications induced on the potential is, at the leading order, similar to others already appeared in literature and can be inef fluent in astrophysical situations in which ultra-high densities are reached \([17,18]\). Concerning the de Sitter case our analysis has shown that quantum fluctuations of the de Sitter metric can greatly influence the effective value of the cosmological constant. In particular quantum fluctuations could be able to change a de Sitter space time into an anti-de Sitter one or even into a Minkowski one. It is also likely to get an effective anti-de Sitter, rather than a de Sitter, space-time starting from a tree level Minkowski space-time. The analysis of the time delay of photons in the de Sitter case has lead to a result that is qualitatively similar to another already appeared in literature in the different framework of the quantum groups. The analysis of the time delay of photons in Minkowski space-time, also accounting for the quantum fluctuations of the metric, has however lead us to a result that compared with the experimental delay announced in \([19]\) is roughly 14 orders of magnitude smaller, and with the opposite sign. Thus we cannot account for the experimental data reported in \([19]\) unless we do not assume for the parameter deforming the energy/momentum dispersion relation the value \( \beta \approx +10^{14} \), that however (see also \([20]\)) would remain in contrast with other experimental data as e.g. those of Ref. \([21]\). However, if the OPERA measure reveals correct, we do not have to forget that our analysis is completely based on wave functionals that obey Bose-Einstein statistics, while neutrinos obey the Fermi-Dirac statistics that it means that the \( \beta \) parameter can assume the correct sign to obtain superluminal neutrinos.

[1] G. Amelino-Camelia, Int. J. Mod. Phys. D11, 35 (2002), gr-qc/0012051
[2] G. Amelino-Camelia, Phys. Lett. B 510, 255 (2001), hep-th/0012238
[3] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88, 190403 (2002), hep-th/0112090.
[4] J. Kowalski-Glikman, Lect. Notes Phys. 669, 131 (2005), hep-th/0405273.
[5] J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004), gr-qc/0305055.
[6] R. Garattini and P. Nicolini, Phys. Rev. D 83, 064021 (2011), arXiv:1006.5418 [gr-qc].
[7] R. Garattini and G. Mandanici, Phys. Rev. D 83, 084021 (2011), arXiv:1102.3803 [gr-qc].
[8] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[9] R. Garattini, TSPU Vestnik 44 N 7, 72 (2004), gr-qc/0409016.
[10] R. Garattini, J. Phys. A 39, 6393 (2006), gr-qc/0510061.
[11] R. Garattini, J. Phys. Conf. Ser. 33, 215 (2006), gr-qc/0510062.
[12] Y. Ling, S. He, and H.-b. Zhang, Mod. Phys. Lett. A 22, 2931 (2007), gr-qc/0609130.
[13] C. Leiva, J. Saavedra, and J. Villanueva, Mod. Phys. Lett. A 24, 1443 (2009), 0808.2601.
[14] J. F. Donoghue, Phys. Rev. D 50, 3874 (1994), gr-qc/9405057.
[15] A. DeBenedictis, R. Garattini and F. S.N. Lobo Phys. Rev. D 78, 104003 (2008); arXiv:0808.0839 [gr-qc].
[16] A. Marciano et al., JCAP 1006, 030 (2010), 1004.1110.
[17] G. Amelino-Camelia, N. Loret, G. Mandanici, and F. Mercati, (2009), 0906.2016.
[18] G. Amelino-Camelia, N. Loret, G. Mandanici, and F. Mercati, Int. J. Mod. Phys. D 19, 2385 (2010), 1007.0851.
[19] OPERA, (2011), 1109.4897.
[20] G. Amelino-Camelia et al., (2011), 1109.5172.
[21] S. Biller et al., Phys. Rev. Lett. 83, 2108 (1999), gr-qc/9810044.