What is the right price of a European option in an incomplete market?

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Dedicated to the memory of Mark H. A. Davis, whose pioneering and innovative work inspired us all

Abstract

We study option prices in financial markets where the risky asset prices are modelled by jump diffusions. For simplicity we put the risk free asset price equal to 1. Such markets are typically incomplete, and therefore there are in general infinitely many arbitrage-free option prices in these markets.

We consider in particular European options with a terminal payoff $F$ at the terminal time $T$, and propose that the right price of such an option is the initial wealth needed to make it possible to generate by a self-financing portfolio a terminal wealth which is as close as possible to the payoff $F$ in the sense of variance.

We show that such an optimal initial wealth $\tilde{z}$ with corresponding optimal portfolio $\tilde{\pi}$ exist and are unique. We call $\tilde{z}$ the \textit{minimal variance price of $F$} and denote it by $p_{mv}(F)$. In the classical Black-Scholes market this price coincides with the classical Black-Scholes option price.

If the coefficients of the risky asset prices are deterministic, we show that

$$p_{mv}(F) = E_{Q}\left[F\right],$$

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for a specific equivalent martingale measure (EMM) $Q^*$. This shows in particular that the minimal variance price is free from arbitrage.

The, for the general case we apply a suitable maximum principle of optimal stochastic control to relate the minimal variance price $\hat{z} = p_{mv}(F)$ to the Hamiltonian and its adjoint processes, and we show that, under some conditions, $\hat{z} = p_{mv}(F) = E_{Q_0}[F]$ for any $Q_0$ in a family $M_0$ of EMMs, described by the set of solutions of a system of linear equations.

Finally, We illustrate our results by looking at specific examples.

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1 Introduction

Recent the economic crises, including impacts from the Covid 19, has given researchers in mathematical finance a motivation to study new models for financial markets. In particular, it has become more relevant than ever to include default time or jumps that come from Lévy processes. However, such markets are typically incomplete and there are infinitely many EMMs and hence also infinitely many arbitrage-free prices for European options. Therefore it is natural to ask what is the best arbitrage-free option pricing rule to use in such incomplete markets? Or, equivalently, if $F$ is the terminal payoff of the option (and the interest rate is 0), what is the best EMM $Q$ to choose for the pricing rule that

"price of option" = $E_Q[F]$?

This is a question that has been studied by many researchers, and several approaches and answers have been provided. They are all different from what we are proposing.

Föllmer and Schweizer [FS] discuss optimal hedging in incomplete markets and introduce the concept of a risk-minimizing martingale measure, which they relate to optimality of strategies, but not directly to option pricing.

Delbaen and Schachermayer [DS] consider continuous markets and propose to use as the option pricing measure the EMM obtained by minimizing the $L^2$-norm of its Radon-Nikodym derivative with respect to the original probability measure $P$.

Frittelli [F] deals with a general market model and suggests to use the EMM with minimal entropy. He justifies this by relating it to the marginal utility of terminal wealth in an exponential utility maximization problem. Sufficient conditions for a martingale measure to be entropy-minimizing are given by Grandits and Reinlaender [GR], using the theory of BMO-martingales.

Hodges and Neuberger [HN] introduce the utility indifference pricing principle, saying that the price $p$ at time 0 of an option with terminal payoff $F$ should be such that a seller of the option is indifferent, with respect to a given utility function, to the following two options:

(i) either receiving that price $p$ for the option and at time 0, then trade optimally with that added initial wealth and paying out the amount $F$ at the terminal time $T$, or

(ii) not selling the option at all and just trade optimally without any payout at the terminal time $T$. There are many papers dealing with the use of this pricing principle. For a survey see Henderson and Hobson [HH] and the references therein. In the case of exponential utility, this pricing rule turns out to be related to pricing by means of the minimal entropy measure. See Davis and Yoshikawa [DY].
In this paper, we propose a different answer:
As a motivation, recall the well-known result stating that if the terminal payoff \( F \) of the European option is replicable, then the unique arbitrage-free option price of the option is the initial wealth \( z = X(0) \) needed for the replication, i.e. to make \( X(T) = F \) a.s., where \( X(t) \) denotes the wealth at time \( t \), \( 0 \leq t \leq T \). Therefore we argue that in general the right price of a European option with terminal payoff \( F \) (replicable or not) should be the initial wealth \( z = X(0) \) needed to make it possible to generate by means of an admissible portfolio \( \pi \) a terminal wealth \( X_{z,\pi}(T) \) which is as close as possible to \( F \) in the sense of variance. i.e. \( \hat{z}, \hat{\pi} \) minimizes

\[
(z, \pi) \mapsto E[(X_{z,\pi}(T) - F)^2],
\]

where \( E[\cdot] = E_P[\cdot] \) denotes expectation with respect to the underlying probability measure \( P \).

The paper is organized as follows:

In Section 2 we show that such an optimal initial wealth \( z = \hat{z} \) with corresponding optimal portfolio \( \pi = \hat{\pi} \) exist and are unique. We call \( \hat{z} \) the minimal variance price of \( F \) and denote it by \( p_{mv}(F) \). In the classical Black-Scholes market this price coincides with the classical Black-Scholes option price. To find this optimal initial wealth \( \hat{z} \) we use two methods:

First, in Section 3 we formulate the problem as a Stackelberg game and find its unique solution using stochastic calculus. In particular, we show that under some conditions there is a unique EMM \( Q^* \) which gives the solution \( \hat{z} \) of the minimal variance problem, in the sense that \( \hat{z} = p_{mv}(F) = E_{Q^*}[F] \). This shows in particular that the minimal variance price is free from arbitrage.

Second, in Section 4 we use a suitable version of the maximum principle for optimal stochastic control to relate the EMMs \( Q \) such that \( \hat{z} = E_Q[F] \) to the Hamiltonian and the corresponding adjoint processes \( p(t), q(t), r(t, \zeta) \). Then we show that, under some conditions, \( \hat{z} = p_{mv}(F) = E_{Q_0}[F] \) for any \( Q_0 \) in a family \( \mathbb{M}_0 \) of EMMs, described by the set of solutions of a system of linear equations.

Finally, in section 5 we give examples to illustrate our results.

## 2 The general incomplete market case

In this section we present our financial market model, and we give a brief survey of some of the fundamental concepts and results from the theory of pricing of European options.

Consider a financial market with two investment possibilities:
(i) A risk free asset, with unit price $S_0(t) = 1$ for all $t$.

(ii) A risky asset, with unit price $S(t)$ at time $t$ given by

$$dS(t) = S(t^-) \left[ \alpha(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right], \quad S(0) > 0. \quad (2.1)$$

For simplicity of notation, we will write $S(t)$ instead of $S(t^-)$ in the following.

Here $\alpha(t) \in \mathbb{R}, \sigma(t) = (a_1(t), ..., a_m(t)) \in \mathbb{R}^m, \gamma(t, \zeta) = (b_1(t, \zeta), ..., b_k(t, \zeta)) \in \mathbb{R}^k, \text{ and } B(t) = (B_1(t), ..., B_m(t))' \in \mathbb{R}^m \text{ and } \tilde{N}(dt, d\zeta) = (\tilde{N}_1(dt, d\zeta), ..., \tilde{N}(dt, d\zeta))' \in \mathbb{R}^k \text{ are independent Brownian motions and compensated Poisson random measures, respectively, on a complete filtered probability space } (\Omega, \mathcal{F}, \mathbb{F}) = \{\mathcal{F}_t\}_{t \geq 0}, P). \text{ We are using the matrix notation, i.e.}

$$\sigma(t) dB(t) : = \sum_{i=1}^m \sigma_i(t) dB_i(t), \quad \gamma(t, \zeta) \tilde{N}(dt, d\zeta) = \sum_{j=1}^k \gamma_j(t, \zeta) \tilde{N}_j(dt, d\zeta),$$

and we assume that $\gamma(t, \zeta) > -1$ and

$$\sigma^2(t) + \int_{\mathbb{R}^*} \gamma^2(t, \zeta) \nu(d\zeta) > 0 \text{ for a.a. } t, \text{ a.s.} \quad (2.2)$$

For simplicity we assume throughout this paper that all the coefficients $\alpha, \sigma$ and $\gamma$ are bounded $\mathbb{F}$-predictable processes. The results in this paper can easily be extended to an arbitrary number of risky assets, but since the features of incomplete markets we are dealing with, can be fully illustrated by one jump diffusion risky asset only, we will for simplicity concentrate on this case in the following. We emphasize however, that we deal with an arbitrary number $m$ of independent Brownian motions and an arbitrary number $k$ of independent Poisson random measures in the representation (2.1).

Let $z \in \mathbb{R}$ be a given initial wealth and let $\pi(t) \in \mathbb{R}$ be a self-financing portfolio, representing the fraction of the total wealth $X(t) = X_{z,\pi}(t)$ invested in the risky asset at time $t$. Then the corresponding wealth dynamics is given by the following linear stochastic differential equation (SDE) with jumps

$$dX(t) = X(t)\pi(t) \left[ \alpha(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right], \quad X(0) = z. \quad (2.3)$$

Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $\{B(s)\}_{s \leq t}$ and $\{N(s, \zeta)\}_{s \leq t}$. We recall the general pricing problem (of a European option):
Let $F \in L^2(P)$ be a given $\mathcal{F}_T$-measurable random variable, representing the payoff at the terminal time $T$ written on the contract. If you own the contract, you are entitled to get this payoff $F$ at time $T$. For example, $F$ might be linked to the terminal value of the risky asset, i.e. $F = h(S(T))$ for some function $h$. If somebody comes to you and offer you this contract, how much would you be willing to pay for it now, at time $t = 0$?

To answer this question the buyer, who does not want to risk losing money, will argue as follows:

If the price of the option is $z$, then to buy it I need to borrow $z$ in the bank, thereby starting with an initial wealth $-z$. Then it should be possible for me to find a portfolio $\varphi$ such that the corresponding final wealth $X_{-z,\varphi}(T)$ with the contract payoff $F$ added, gives me a non-negative net wealth, a.s. In other words, the buyer’s price of $F$, denoted by $p_b(F)$, is defined by

$$p_b(F) = \sup\{z; \text{ there exists } \varphi \in \mathcal{A} \text{ such that } X_{-z,\varphi}(T) + F \geq 0 \text{ a.s.}\}.$$ 

Similarly, the seller of the contract does not want the risk of losing money either, so he/she argues as follows: If I receive a payment $y$ for the contract, it should be possible for me to find a portfolio $\psi$, such that the corresponding wealth at time $T$ amount is big enough to pay the contract obligation $F$ at the terminal time $T$, a.s. In other words, the seller’s price of $F$, denoted by $p_s(F)$, is defined by

$$p_s(F) = \inf\{y: \text{ there exists } \psi \in \mathcal{A} \text{ such that } X_{y,\psi}(T) \geq F \text{ a.s.}\}.$$ 

In general the two prices $p_b(F)$ and $p_s(F)$ do not coincide. In fact, we have the following well-known result:

Let $\mathbb{M}$ denote the set of all EMMs for $S(\cdot)$.

Then

$$p_b(F) \leq E_Q[F] \leq p_s(F),$$

for all $Q \in \mathbb{M}$. See e.g. Øksendal and Sulem [ØS].

A portfolio $\pi$ is called an *arbitrage* if it can generate from an initial wealth $X(0) = z = 0$ a terminal wealth $X_{0,\pi}(T) \geq 0$ a.s. with $P[X_{0,\pi}(T) > 0] > 0$. In that sense, an arbitrage is a kind of money machine; it can generate a profit with no risk. Markets with arbitrage cannot exist for a reasonable length of time. Therefore option prices which lead to arbitrage are not accepted in any financial market model.

An option price $p(F) \in \mathbb{R}$ is called *arbitrage-free* if it does not give an arbitrage to neither the buyer nor the seller. It is well-known and easy to prove that for any $Q \in \mathbb{M}$ the price $E_Q[F]$ is arbitrage-free.
Even though the buyer’s price and the seller’s price are different in general, they can be the same in some cases. For example, it is well-known that if $F$ is replicable, in the sense that there exists $z \in \mathbb{R}$ and $\pi \in \mathcal{A}$ (the set of admissible portfolios), such that

$$X_{z,\pi}(T) = F \text{ a.s.},$$

then we have equality in (2.4), i.e,

$$p_b(F) = E_Q[F] = p_s(F), \text{ for all } Q \in \mathcal{M}. \quad (2.5)$$

This is a special case, because in a general incomplete market there are many payoffs $F$ which are not replicable. Only if the market is complete, are all $F$ are replicable, and in that case there is only one EMM $Q$. In the next section we will prove a generalization of (2.5). See Theorem 4.3 and Corollary 4.4.

Thus the interesting case is when the market is incomplete, and then there is usually a big gap between $p_s$ and $p_b$. Then one needs additional requirements to find the "best" price. The purpose of this paper is to introduce a stochastic control approach to the pricing of an option with payoff $F$. Specifically, to each initial wealth $z \in \mathbb{R}$ and each portfolio $\pi \in \mathcal{A}$, we associate the quadratic cost functional

$$J(z, \pi) = E\left[\frac{1}{2} (X_{z,\pi}(T) - F)^2\right].$$

Then we consider the following problem:

**Problem 2.1** Find the optimal initial wealth $\hat{z} \in \mathbb{R}$ and the optimal portfolio $\hat{\pi} \in \mathcal{A}$, such that

$$\inf_{z,\pi} J(z, \pi) = J(\hat{z}, \hat{\pi}). \quad (2.6)$$

Heuristically, this means that we define the price of the option with payoff $F$ to be the initial wealth $\hat{z}$ needed to get the terminal wealth $X(T)$ as close as possible to $F$ in quadratic mean by an admissible portfolio.

In other words, we make the following definition:

**Definition 2.2** Suppose that $(\hat{z}, \hat{\pi}) \in \mathbb{R} \times \mathcal{A}$ is the unique solution of the stochastic control problem Problem 2.1. Then we define the minimal variance price of $F$, denoted by $p_{mv}(F)$, by

$$p_{mv}(F) = \hat{z}.$$ 

Our main results are the following:
We prove that the solution of Problem 2.1 exists and is unique, and we find explicit expressions for the optimal portfolio \( \hat{\pi} \), the optimal initial wealth \( \hat{z} = p_{mv}(F) \) and the corresponding optimal wealth \( \hat{X} \). See Theorem 3.1, Theorem 3.3 and Theorem 3.5 (i).

If the coefficients of the risky asset price \( S(t) \) are deterministic, we prove that \( \hat{z} = E_{Q^*}[F] \) for a specific EMM \( Q^* \) (see Theorem 3.5 (ii)). In particular, this shows that \( \hat{z} \) is an arbitrage-free price in this case.

We study the general case of Problem 2.1 by means of the maximum principle for optimal control, and we describe a family \( M_0 \) of an EMM such that the value \( \hat{z} \) found in Theorem 3.5 (i) is of the form \( \hat{z} = E_{\hat{Q}}[F] \) for all EMM \( \hat{Q} \in M_0 \). See Theorem 4.7. In particular, this shows that our price \( \hat{z} = p_{mv}(F) \) coincides with the Black-Scholes option price if the market is complete.

3 Existence and uniqueness of the optimal initial wealth

In this section we prove the existence and the uniqueness of solutions of Problem 2.1.

3.1 Equivalent martingale measures (EMMs)

Since an EMMs play a crucial role in our discussion, we start this section by recalling that an important group of measures \( Q \in \mathcal{M} \) can be described as follows (we refer to Chapter 1 in [OS], for more details):

Let \( \theta_0(t) \) and \( \theta_1(t, \zeta) > -1 \) be \( \mathcal{F} \)-predictable processes such that

\[
\alpha(t) + \theta_0(t)\sigma(t) + \int_{\mathbb{R}^*} \theta_1(t, \zeta)\gamma(t, \zeta)\nu(d\zeta) = 0, \quad t \geq 0. \tag{3.1}
\]

Define the local martingale \( Z(t) = Z_{\theta_0,\theta_1}(t) \), by

\[
dZ(t) = Z(t) \left[ \theta_0(t)dB(t) + \int_{\mathbb{R}^*} \theta_1(t, \zeta)\tilde{N}(dt, d\zeta) \right], \quad Z(0) = 1, \tag{3.2}
\]

i.e.,

\[
Z(t) = \exp \left( \int_0^t \theta_0(s)dB(s) - \frac{1}{2} \int_0^t \theta_0^2(s)ds + \int_0^t \int_{\mathbb{R}^*} \{\ln(1 + \theta_1(s, \zeta)) - \theta_1(s, \zeta)\} \nu(d\zeta)ds \right.
\]

\[
+ \int_0^t \int_{\mathbb{R}^*} \ln(1 + \theta_1(s, \zeta))\tilde{N}(ds, d\zeta). \tag{3.3}
\]
Suppose that $Z$ is a true martingale. A sufficient condition for this to hold is
\[
E \left[ \exp \left( \frac{1}{2} \int_0^T \theta_0^2(s) ds + \int_0^T \int_{\mathbb{R}^r} \theta_1^2(s, \zeta) N(ds, d\zeta) \right) \right] < \infty. \tag{3.4}
\]
See Kallsen and Shiryaev [KaSh]. Then the measure $Q^{\theta_0, \theta_1}$ defined by
\[
dQ^{\theta_0, \theta_1}(\omega) = Z(T)dP(\omega) \text{ on } \mathcal{F}_T \tag{3.5}
\]
is in $\mathcal{M}$.

### 3.2 The optimal portfolio

We may regard the minimal variance problem (Problem 2.1) as a Stackelberg game, in which the first player chooses the initial wealth $z$, followed by the second player choosing the optimal portfolio $\pi$ based on this initial wealth. Knowing this response $\pi = \pi_z$ from the follower, the first player chooses the initial wealth $\tilde{z}$ which leads to a response $\pi = \pi_{\tilde{z}}$ which is optimal, in the sense that $J(\tilde{z}, \pi_{\tilde{z}}) \leq J(z, \pi)$ over all admissible pairs $(z, \pi)$. To this end, in this subsection we first proceed to find the optimal portfolio based on a given initial wealth $z$.

Accordingly, assume as before that the wealth process $X(t) = X_z, \pi(t)$, corresponding to an initial wealth $z$ and a self-financing portfolio $\pi$, is given by
\[
dX(t) = X(t)\pi(t) \left[ \alpha(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}^r} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right], \quad X(0) = z. \tag{3.6}
\]

Suppose the terminal payoff $F \in L^2(P, \mathcal{F}_T)$ has the form $F = F(T)$, where the martingale $F(t) := E[F | \mathcal{F}_t], t \in [0, T]$ has the Itô-Lévy representation
\[
dF(t) = \beta(t) dB(t) + \int_{\mathbb{R}^r} \kappa(t, \zeta) \tilde{N}(dt, d\zeta), \quad E[F] = F_0.
\]
for some (unique) $\mathbb{F}$-predictable processes $\beta(t) \in L^2(\lambda \times P), \kappa(t, \zeta) \in L^2(\lambda \times \nu \times P)$. Then by the Itô formula for jump diffusions (see e.g. Theorem 1.14 in Øksendal and Sulem [OS]), we get
\[
d(X(t)F(t)) = X(t) dF(t) + F(t) dX(t) + d[X, F]_t
= X(t)[\beta(t) dB(t) + \int_{\mathbb{R}^r} \kappa(t, \zeta) \tilde{N}(dt, d\zeta)]
+ X(t) F(t)[\pi(t) \alpha(t) dt + \pi(t) \sigma(t) dB(t) + \int_{\mathbb{R}^r} \pi(t) \gamma(t, \zeta) \tilde{N}(dt, d\zeta)]
+ X(t)[\pi(t) \alpha(t) \beta(t) dt + \int_{\mathbb{R}^r} \pi(t) \gamma(t, \zeta) \kappa(t, \zeta) \tilde{N}(dt, d\zeta)]
+ \int_{\mathbb{R}^r} \pi(t) \gamma(t, \zeta) \kappa(t, \zeta) \nu(d\zeta) dt. \tag{3.7}
\]
Hence
\[
E[X(T)F(T)] = zF_0 + \int_0^T E\left[X(t)\left\{F(t)\pi(t)\alpha(t) + \pi(t)\sigma(t)\beta(t)
\right.\right.
\]
\[
+ \int_{\mathbb{R}^*} \pi(t)\gamma(t, \zeta)\kappa(t, \zeta)\nu(d\zeta)\left.\right\}\left.\right\} dt.
\]
(3.8)

Similarly,
\[
E[X^2(T)] = z^2 + \int_0^T E\left[X^2(t)\left\{2\pi(t)\alpha(t) + \pi^2(t)\sigma^2(t) + \int_{\mathbb{R}^*} \pi^2(t)\gamma^2(t, \zeta)\nu(d\zeta)\right\} dt\right],
\]
and
\[
E[F^2(T)] = F_0^2 + \int_0^T E\left[\beta^2(t) + \int_{\mathbb{R}^*} \kappa^2(t, \zeta)\nu(d\zeta)\right] dt.
\]

This gives
\[
J(\pi) = E\left[\frac{1}{2}(X(T) - F)^2\right] = \frac{1}{2}\left(E[X^2(T)] - 2E[X(T)F(T)] + E[F^2(T)]\right)
\]
\[
= \frac{1}{2}(z - F_0)^2 + \frac{1}{2}E\left[\int_0^T \left\{2X^2(t)\pi(t)\alpha(t) + X^2(t)\pi^2(t)\left(\sigma^2(t) + \int_{\mathbb{R}^*} \gamma^2(t, \zeta)\nu(d\zeta)\right)
\right.
\]
\[
- 2X(t)\pi(t)\left(F(t)\alpha(t) + \sigma(t)\beta(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\kappa(t, \zeta)\nu(d\zeta)\right)
\]
\[
+ \beta^2(t) + \int_{\mathbb{R}^*} \kappa^2(t, \zeta)\nu(d\zeta)\right\} dt\right].
\]

We can minimize \(J(\pi)\) by minimizing the \(dt\)-integrand pointwise for each \(t\). This gives the following result:

**Theorem 3.1** a) For given initial value \(X(0) = z > 0\) the portfolio \(\hat{\pi}_z\) which minimizes
\[
\pi \mapsto E\left[\frac{1}{2}(X_{\pi}(T) - F)^2\right]
\]
is given in feedback form with respect to \(X(t)\) by
\[
\hat{\pi}(t) = \pi(t, X(t)) = \frac{F(t)\alpha(t) + \sigma(t)\beta(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\kappa(t, \zeta)\nu(d\zeta) - X(t)\alpha(t)}{X(t)\left(\sigma^2(t) + \int_{\mathbb{R}^*} \gamma^2(t, \zeta)\nu(d\zeta)\right)}.
\]
(3.9)

or, equivalently,
\[
\hat{\pi}(t)X(t) = G(t)[X(t) - F(t)] + \frac{\sigma(t)\beta(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\kappa(t, \zeta)\nu(d\zeta)}{\sigma^2(t) + \int_{\mathbb{R}^*} \gamma^2(t, \zeta)\nu(d\zeta)},
\]
(3.10)
where
\[ G(t) = -\alpha(t) \left( \sigma^2(t) + \int_{\mathbb{R}^+} \gamma^2(t, \zeta) \nu(d\zeta) \right)^{-1}. \tag{3.11} \]

b) Given an initial value \( z > 0 \), the corresponding optimal wealth \( X_{\bar{F}} = \tilde{X}(t) \) solves the SDE

\[
d\tilde{X}(t) = \tilde{X}(t) \tilde{\pi}(t, \tilde{X}(t)) \left[ \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^+} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right]
= \frac{F(t)\alpha(t) + \sigma(t)\beta(t) + \int_{\mathbb{R}^+} \gamma(t, \zeta) \kappa(t, \zeta) \nu(d\zeta) - X(t)\alpha(t)}{\sigma^2(t) + \int_{\mathbb{R}^+} \gamma^2(t, \zeta) \nu(d\zeta)} \times \\
\times \left[ \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^+} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right]. \tag{3.12} \]

**Remark 3.2** Consider the special case when \( F \) is a deterministic constant. Then \( F(t) = F = E[F] \) for all \( t \), and \( \beta = \kappa = 0 \). Hence the optimal portfolio is given in feedback form by

\[ \tilde{\pi}(t)X(t) = G(t)[X(t) - F]. \]

Assume, for example, that \( \alpha(t) > 0 \). Then \( G(t) < 0 \) and we see that if \( X(t) < F \) then \( \tilde{\pi}X(t) > 0 \) and hence the optimal portfolio pushes \( X(t) \) upwards towards \( F \). Similarly, if \( X(t) > F \) then \( \tilde{\pi}(t)X(t) < 0 \) and the optimal push of \( X(t) \) is downwards towards \( F \). This is to be expected, since the portfolio tries to minimize the terminal variance \( E[(X(T) - F)^2] \).

### 3.3 Explicit solution of the corresponding optimal wealth

Writing \( X = \tilde{X} \) for notational simplicity, the equation (3.12) is of the form,

\[
dx(t) = C(t) d\Lambda_t + X(t) d\Gamma_t, \quad X(0) = z, \tag{3.13} \]

where

\[
C(t) = \frac{F(t)\alpha(t) + \sigma(t)\beta(t) + \int_{\mathbb{R}^+} \gamma(t, \zeta) \kappa(t, \zeta) \nu(d\zeta)}{\sigma^2(t) + \int_{\mathbb{R}^+} \gamma^2(t, \zeta) \nu(d\zeta)}, \tag{3.14} \]

\[
d\Lambda_t = \alpha(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}^+} \gamma(t, \zeta) \tilde{N}(dt, d\zeta), \tag{3.15} \]

\[
d\Gamma_t = \sigma_1(t) dt + \sigma_1(t) dB(t) + \int_{\mathbb{R}^+} \gamma_1(t, \zeta) \tilde{N}(dt, d\zeta), \tag{3.16} \]

with
\[
\alpha_1(t) = G(t)\alpha(t), \quad \sigma_1(t) = G(t)\sigma(t), \quad \gamma_1(t, \zeta) = G(t)\gamma(t, \zeta). \tag{3.17} \]

We rewrite (3.13) as

\[
dx(t) - X(t) d\Gamma_t = C(t) d\Lambda_t, \tag{3.18} \]
and multiply this equation by a process of the form

\[ Y_t = Y_t^{(\rho, \lambda, \theta)} = \exp \left( A_t^{(\rho, \lambda, \theta)} \right), \quad (3.19) \]

with

\[ A_t^{(\rho, \lambda, \theta)} = \int_0^t \rho(s) \, ds + \int_0^t \lambda(s) \, dB(s) + \int_0^t \int_{\mathbb{R}^*} \theta(s, \zeta) \, \tilde{N}(ds, d\zeta), \quad (3.20) \]

where \( \rho, \lambda \) and \( \theta \) are processes to be determined.

Then (3.18) gets the form

\[ Y_t dX(t) - Y_t X(t) \, d\Gamma_t = Y_t C(t) \, d\Lambda_t. \quad (3.21) \]

We want to choose \( \rho, \lambda \) and \( \theta \) such that \( Y_t \) becomes an integrating factor, in the sense that

\[ d(Y_t X(t)) = Y_t dX(t) - Y_t X(t) \, d\Gamma_t + \text{terms not depending on } X. \quad (3.22) \]

To this end, note that by the Itô formula for Lévy processes, we have

\[
\begin{align*}
    dY_t &= Y_t \left[ \rho(t) \, dt + \lambda(t) \, dB(t) \right] + \frac{1}{2} Y_t \lambda^2(t) \, dt \\
    &+ \int_{\mathbb{R}^*} \{ \exp(A_t + \theta(t, \zeta)) - \exp(A_t) \} \nu(d\zeta) \, dt \\
    &+ \int_{\mathbb{R}^*} \{ \exp(A_t + \theta(t, \zeta)) - \exp(A_t) \} \tilde{N}(dt, d\zeta) \\
    &= Y_t \left[ \rho(t) + \frac{1}{2} \lambda^2(t) + \int_{\mathbb{R}^*} \left( e^{\theta(t, \zeta)} - 1 - \theta(t, \zeta) \right) \nu(d\zeta) \right] \, dt \\
    &+ \lambda(t) \, dB(t) + \int_{\mathbb{R}^*} \left( e^{\theta(t, \zeta)} - 1 \right) \tilde{N}(dt, d\zeta). 
\end{align*}
\]

Therefore, again by the Itô formula, using (3.13),

\[
\begin{align*}
    d(Y_t X(t)) &= Y_t dX(t) + X(t) \, dY_t + d[X,Y]_t \\
    &= Y_t dX(t) + Y_t X(t) \left[ \rho + \frac{1}{2} \lambda^2 + \int_{\mathbb{R}^*} \left( e^{\theta(t, \zeta)} - 1 - \theta(t, \zeta) \right) \nu(d\zeta) \right] \, dt \\
    &+ \lambda(t) \, dB(t) + \int_{\mathbb{R}^*} \left( e^{\theta(t, \zeta)} - 1 \right) \tilde{N}(dt, d\zeta) \\
    &+ \gamma_1(t, \zeta) Y_t X(t) \left[ \lambda(t) \sigma_1(t) + \int_{\mathbb{R}^*} \left( e^{\theta(t, \zeta)} - 1 \right) \gamma_1(t, \zeta) \nu(d\zeta) \right] \, dt \\
    &+ \int_{\mathbb{R}^*} \left( e^{\theta(t, \zeta)} - 1 \right) \gamma_1(t, \zeta) \tilde{N}(dt, d\zeta) + Y_t C(t) \, dK_t, 
\end{align*}
\]
where
\[
dK_t = \left\{ \lambda (t) \sigma (t) + \int_{\mathbb{R}^*} \left( e^{\theta(t,\zeta)} - 1 \right) \gamma (t, \zeta) \nu (d\zeta) \right\} dt + \int_{\mathbb{R}^*} \left( e^{\theta(t,\zeta)} - 1 \right) \gamma (t, \zeta) \tilde{N} (dt, d\zeta) .
\]

This gives
\[
d (Y_t X (t)) - Y_t dX (t) + Y_t X (t) d\Gamma_t
\]
\[
= Y_t X (t) \left[ \left\{ \rho + \alpha_1 + \frac{1}{2} \lambda^2 + \lambda \sigma_1 + \int_{\mathbb{R}^*} \left( e^{\theta(t,\zeta)} - 1 - \theta (t, \zeta) \right) \nu (d\zeta) \right\} dt \right.
\]
\[
+ \left( \lambda (t) + \sigma_1 (t) \right) dB (t) + \int_{\mathbb{R}^*} \left( e^{\theta(t,\zeta)} - 1 \right) \gamma_1 (t, \zeta) \nu (d\zeta) dt
\]
\[
+ \int_{\mathbb{R}^*} \left\{ \left( e^{\theta(t,\zeta)} - 1 \right) (1 + \gamma_1 (t, \zeta)) + \gamma_1 (t, \zeta) \right\} \tilde{N} (dt, d\zeta)
\]
\[
+ Y_t C (t) dK_t .
\]

Choose \( \theta (t, \zeta) = \hat{\theta} (t, \zeta) \), such that
\[
\left( e^{\theta(t,\zeta)} - 1 \right) (1 + \gamma_1 (t, \zeta)) + \gamma_1 (t, \zeta) = 0,
\]
i.e.
\[
\hat{\theta} (t, \zeta) = - \ln (1 + \gamma_1 (t, \zeta)) .
\]

Next, choose \( \lambda (t) = \hat{\lambda} (t) \) such that
\[
\hat{\lambda} (t) = - \sigma_1 (t) .
\]

Finally, choose \( \rho (t) = \hat{\rho} (t) \), such that
\[
\hat{\rho} (t) = - \left[ \alpha_1 (t) + \frac{1}{2} \sigma_1^2 (t) - \sigma_1^2 + \int_{\mathbb{R}^*} \left( e^{\theta(t,\zeta)} - 1 - \theta(t, \zeta) + (e^{\theta(t,\zeta)} - 1) \gamma_1 (t, \zeta) \right) \nu (d\zeta) \right]
\]
\[
= - \left[ \alpha_1 (t) - \frac{1}{2} \sigma_1^2 (t) + \int_{\mathbb{R}^*} \left( \ln (1 + \gamma_1 (t, \zeta)) - \gamma_1 (t, \zeta) \right) \nu (d\zeta) \right] .
\]

Then
\[
\hat{A}_t := A_t (\hat{\rho}, \hat{\lambda}, \hat{\theta})
\]
\[
= - \left[ \int_0^t \left\{ \alpha_1 (s) - \frac{1}{2} \sigma_1^2 (s) + \int_{\mathbb{R}^*} \left( \ln (1 + \gamma_1 (s, \zeta)) - \gamma_1 (s, \zeta) \right) \nu (d\zeta) \right\} ds
\]
\[
+ \int_0^t \sigma_1 (s) dB (s) + \int_0^t \int_{\mathbb{R}^*} \ln (1 + \gamma_1 (s, \zeta)) \hat{N} (ds, d\zeta) \right] ,
\]

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with \( \tilde{Y}_t = Y_t^{(\hat{\rho}, \hat{\lambda}, \hat{\theta})} = \exp(\hat{A}_t) \) we have, by (3.25)

\[
d\left( \tilde{Y}_t X (t) \right) - \tilde{Y}_t dX (t) + \tilde{Y}_t X (t) d\Gamma_t = \tilde{Y}_t C (t) dK_t.
\]

Substituting this into (3.21), we get

\[
d\left( \tilde{Y}_t X (t) \right) - \tilde{Y}_t C (t) dK_t = \tilde{Y}_t C (t) d\Lambda_t,
\]

which we integrate to, since \( \tilde{Y}_0 = 1 \),

\[
\tilde{Y}_t X (t) = z + \int_0^t \tilde{Y}_s C (s) d(K_s + \Lambda_s).
\]

Solving for \( X (t) \), we obtain the following:

**Theorem 3.3** With initial value \( z \) the corresponding optimal wealth process \( \tilde{X}_z (t) \) is given by

\[
\tilde{X}_z (t) = z \tilde{Y}_t^{-1} + \tilde{Y}_t^{-1} \int_0^t \tilde{Y}_s C (s) d(K_s + \Lambda_s)
\]

\[
= z \exp(-\hat{A}_t) + \exp(-\hat{A}_t) \int_0^t \exp(\hat{A}_s) C (s) d(K_s + \Lambda_s).
\]

In particular, note that

\[
dz \tilde{X}_z (t) = \exp(-\hat{A}_t).
\]

### 3.4 The optimal initial wealth and the option price \( \hat{\gamma} \)

Completing the Stackelberg game, we now proceed to find the initial wealth \( \hat{\gamma} \) which leads to a response \( \hat{\pi} = \pi_{\hat{\gamma}} \) which is optimal for Problem 2.1 in the sense that \( J(\hat{\gamma}, \hat{\pi}) \leq J(z, \pi) \) over all pairs \((z, \pi)\).

To this end, choose \( z \in \mathbb{R} \) and let \( \hat{\pi}_{\hat{\gamma}} \) be the corresponding optimal portfolio given by (3.9) and let \( \tilde{X}_{\hat{\gamma}} \) be the corresponding optimal wealth process given by (3.12) and (3.32). Then

\[
\inf_{z, \pi} J(z, \pi) = \inf_{z, \pi} E \left[ \frac{1}{2} (X_{z, \pi} - F)^2 \right] = \inf_z E \left[ \frac{1}{2} (X_{z, \hat{\pi}_{\hat{\gamma}}} (T) - F)^2 \right] = \inf_z E \left[ \frac{1}{2} (\tilde{X}_{\hat{\gamma}} (T) - F)^2 \right].
\]

Note that, if we define

\[
R_t := \exp(-\hat{A}_t) = \exp \left[ \int_0^t \left\{ \alpha_1 (s) - \frac{1}{2} \sigma_1^2 (s) + \int_{\mathbb{R}^*} \left( \ln(1 + \gamma_1 (s, \zeta)) - \gamma_1 (s, \zeta) \right) \nu (d\zeta) \right\} ds
\]

\[
+ \int_0^t \sigma_1 (s) dB (s) + \int_0^t \int_{\mathbb{R}^*} \ln(1 + \gamma_1 (s, \zeta)) \tilde{N} (ds, d\zeta),
\]

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and

\[ Z_t^* := \exp \left( - \int_0^t \alpha_1(s) ds \right) R_t = \exp \left[ \int_0^t \left\{ - \frac{1}{2} \sigma_1^2(s) + \int_{\mathbb{R}^*} \log(1 + \gamma_1(s, \zeta)) - \gamma_1(s, \zeta) \nu(d\zeta) \right\} ds \right. \]
\[ + \int_0^t \sigma_1(s) dB(s) + \int_0^t \int_{\mathbb{R}^*} \log(1 + \gamma_1(s, \zeta)) \tilde{N}(ds, d\zeta), \]

then we can verify by the Itô formula that

\[ dR_t = R_t \left( \alpha_1(t) dt + \sigma_1(t) dB(t) + \int_{\mathbb{R}^*} \gamma_1(t, \zeta) \tilde{N}(dt, d\zeta) \right) \]
\[ = R_t G(t) \left( \alpha(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right) \]
\[ = R_t G(t) S^{-1}(t) dS(t), \]

and

\[ dZ_t^* = Z_t^* G(t) \left( \sigma(t) dB(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right). \] (3.34)

**Proposition 3.4** Assume that \( Z_t^* \) is a martingale. (See (3.3).) Define

\[ dQ^*(\omega) = Z_T^*(\omega) dP(\omega) \text{ on } \mathcal{F}_T. \] (3.35)

Then \( Q^* \) is an EMM for \( S(\cdot) \).

**Proof.** To see this, we verify that the coefficients \( \theta_0(t) := G(t) \sigma(t) \) and \( \theta_1(t, \zeta) := G(t) \gamma(t, \zeta) \) satisfy condition (3.1):

\[
\alpha(t) + \theta_0(t) \sigma(t) + \int_{\mathbb{R}^*} \theta_1(t, \zeta) \gamma(t, \zeta) \nu(d\zeta) = \alpha(t) \left( \frac{\alpha(t)}{\sigma^2(t) + \int_{\mathbb{R}^*} \gamma^2(t, \zeta) \nu(d\zeta)} \sigma^2(t) \right) - \frac{\alpha(t)}{\sigma^2(t) + \int_{\mathbb{R}^*} \gamma^2(t, \zeta) \nu(d\zeta)} \int_{\mathbb{R}^*} \gamma^2(t, \zeta) \nu(d\zeta) \]
\[ = \alpha(t) - \alpha(t) = 0, \quad t \geq 0. \] (3.36)

\[ \square \]

Using this we obtain the following, which is the main result in this section:
**Theorem 3.5**  
(i) The unique minimal variance price $\hat{z} = p_{mv}(F)$ of a European option with terminal payoff $F$ at time $T$ is given by

$$
\hat{z} = \frac{E\left[ \exp(-2\hat{A}_T) \left( F - \int_0^T \exp(\hat{A}_s) C(s) \, d(K_s + \Lambda_s) \right) \right]}{E[\exp(-2\hat{A}_T)]},
$$

(3.37)

where $\hat{A}_T$ is given by (3.30), $C(s), \Lambda_s$ are given by (3.14), (3.15) respectively, and $K$ is given by (3.25).

(ii) Assume that the coefficients $\alpha(t), \sigma(t)$ and $\gamma(t, \zeta)$ are bounded and deterministic. Then

$$
\hat{z} = p_{mv}(F) = E_{Q^*}[F],
$$

(3.38)

where $Q^*$ is the EMM measure given by (3.35).

**Proof.**

(i) To minimize $J_0(z) := E\left[ \frac{1}{2}(\hat{X}_z(T) - F)^2 \right]$ with respect to $z$ we note by (3.32) and (3.33) that

$$
\frac{d}{dz} J_0(z) = E\left[ (\hat{X}_z(T) - F) \frac{d}{dz} \hat{X}_z(T) \right]
= E[(\hat{X}_z(T) - F) \exp(-\hat{A}_T)]
= E\left[ (z \exp(-\hat{A}_T) + \exp(-\hat{A}_T) \int_0^T \exp(\hat{A}_s) C(s) \, d(K_s + \Lambda_s) - F) \exp(-\hat{A}_T) \right].
$$

(3.39)

This is 0 if and only if (3.37) holds.

(ii) By (3.39), we get

$$
E[\hat{X}_z(T) \exp(-\hat{A}_T)] = E[F \exp(-\hat{A}_T)],
$$

i.e.

$$
E\left[ \hat{X}_z(T) \exp\left( - \int_0^T \alpha_1(s) \, ds \right) Z_T^z \right] = E\left[ F \exp\left( - \int_0^T \alpha_1(s) \, ds \right) Z_T^z \right].
$$

If the $\alpha_1$ is deterministic, we can cancel out the factor $\exp\left( - \int_0^T \alpha_1(s) \, ds \right)$ and (3.38) follows.

**Remark 3.6**  
Note that in the special case when $F$ is a deterministic constant, we have $C(t) = 0$ and we get from (3.37) that $\hat{z} = E[F] = F$, with corresponding optimal portfolio $\hat{\pi} = 0$. 

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Remark 3.7  An important question is: Is \( \hat{z} \) an arbitrage-free price of \( F \)?

If the coefficients are deterministic, we know that the answer is yes, by Theorem 3.5 (ii). But in the general case in Theorem 3.5 (i) this is not clear. In the next section we will give, under some conditions, an affirmative answer to this question, by proving that

\[
\hat{z} = E_{Q_0}[F],
\]

for any \( Q_0 \in \mathcal{M}_0 \), where \( \mathcal{M}_0 \) is a nonempty subset of \( \mathcal{M} \). As pointed out in the Introduction this implies in particular that \( \hat{z} \) is arbitrage-free.

Example 3.8  (European call option) We give some details about how to proceed if we want to compute the minimal variance price \( \hat{z} = p_{\text{mv}}(F) \) explicitly in the case of a European call option:

(i) Note that the term \( C(s) \) in Theorem 3.5 depends on the coefficients \( \beta \) and \( \kappa \) in the Itô representation of \( F \). These coefficients can for example be found by using the generalized Clark-Ocone formula for Lévy processes, extended to \( L^2(P) \). See Theorem 12.26 in [DØP].

Let us find these coefficients in the case of a European call option, where

\[
F = (S(T) - K)^+,
\]

where \( K \) is a given exercise price. In this case \( F(\omega) \) represents the payoff at time \( T \) (fixed) of a (European call) option which gives the owner the right to buy the stock with value \( S(T, \omega) \) at a fixed exercise price \( K \). Thus if \( S(T, \omega) > K \) the owner of the option gets the profit \( S(T, \omega) - K \) and if \( S(T, \omega) \leq K \) the owner does not exercise the option and the profit is 0. Hence in this case

\[
F(\omega) = (S(T, \omega) - K)^+.
\]

Thus, we may write

\[
F(\omega) = f(S(T, \omega)),
\]

where

\[
f(x) = (x - K)^+.
\]

The function \( f \) is not differentiable at \( x = K \), so we cannot use the chain rule directly to evaluate \( D_t F \). However, we can approximate \( f \) by \( C^1 \) functions \( f_n \) with the property that

\[
f_n(x) = f(x) \quad \text{for} \quad |x - K| \geq \frac{1}{n},
\]

and

\[
0 \leq f'_n(x) \leq 1 \quad \text{for all} \ x.
\]
Putting
\[ F_n(\omega) = f_n(S(T, \omega)), \]
we see
\[ D_t F(\omega) = \lim_{n \to +\infty} D_t F_n(\omega), \]
Then if the coefficients \(\alpha, \sigma, \gamma\) of the risky asset price \(S\) are deterministic, we get
\[
\beta(t) = E[D_t F|F_t], \quad \kappa(t, \zeta) = E[D_{t,\zeta} F|F_t],
\]
where \(D_t F\) and \(D_{t,\zeta} F\) denote the generalized Malliavin derivatives (also called the Hida-Malliavin derivative) of \(F\) at \(t\) and \((t, \zeta)\) respectively, with respect to \(B(\cdot)\) and \(N(\cdot, \cdot)\), respectively. Combining this with the chain rule for the Hida-Malliavin derivative and the Markov property of the process \(S(\cdot)\), we obtain the following for \(\beta\):
\[
\beta(t) = E_{S_0} \left[ \chi_{[K, \infty)}(S(T)) \sigma S(T) | F_t \right]
= E_{S(t)} \left[ \chi_{[K, \infty)}(S(T-t)) \sigma S(T-t) | F_t \right]. \tag{3.40}
\]
To find the corresponding result for \(\kappa\) we first use the chain rule for \(D_{t,\zeta}\) (Theorem 12.8 in Di Nunno et al [DØP]) and get
\[
D_{t,\zeta} S(T) = D_{t,\zeta} \left[ S_0 \exp \left( \alpha T - \frac{1}{2} \sigma^2 T + \sigma B(T) + \int_{\mathbb{R}^1} \left( \log(1 + \gamma(\zeta)) - \gamma(\zeta) \nu(d\zeta) \right) T \right.ight.
\quad \left. + \int_{t}^{T} \int_{\mathbb{R}^1} \ln(1 + \gamma(\zeta)) \tilde{N}(ds,d\zeta) \right] = S(T) \gamma(\zeta).
\]
Then we obtain
\[
\kappa(t, \zeta) = E_{S_0} \left[ \chi_{[K, \infty)}(S(T) + D_{t,\zeta} S(T)) - \chi_{[0,T]}(S(T)) | F_t \right]
= E_{S_0} \left[ \chi_{[K, \infty)}(S(T) + \gamma(\zeta) S(T)) - \chi_{[0,T]}(S(T)) | F_t \right]
= E_{S(t)} \left[ \chi_{[K, \infty)}(S(T-t) + \gamma(\zeta) S(T-t)) - \chi_{[K, \infty)}(S(T-t)) \right], \tag{3.41}
\]
where in general \(E^y[h(S(u))]\) means \(E[h(S^y(u))]\), i.e. expectation when \(S\) starts at \(y\).

**(ii)** Assume that the coefficients \(\alpha, \sigma\) and \(\gamma\) of the process \(S\) are deterministic and bounded.

To compute numerically the minimal variance price
\[
\hat{z} = p_{mv}(S(T) - K^+^+) = E \left[ (S(T) - K)^+ Z_T^+ \right]
\]
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of a European call option with payoff
\[ F = (S(T) - K)^+ = E[F] + \int_0^T \beta(t) dB(t) + \int_{\mathbb{R}^+} \kappa(t, \zeta) \tilde{N}(dt, d\zeta), \]
where \( \beta, \kappa \) are given by (3.40), (3.41), respectively, we use the Itô formula combined with (3.34) to obtain
\[ \hat{z} = E[FZ^*_T] = E[F] + \int_0^T G(t) \left\{ \sigma(t) E[Z^*_t \beta(t)] + \int_{\mathbb{R}^+} \gamma(t, \zeta) E[\kappa(t, \zeta) Z^*_t] \nu(d\zeta) \right\} dt, \]
where \( G(t) \) is given by (3.11), i.e.
\[ G(t) = -\alpha(t) \left( \sigma^2(t) + \int_{\mathbb{R}^+} \gamma^2(t, \zeta) \nu(d\zeta) \right)^{-1}. \] (3.42)

(iii) Alternatively, in some cases it may be convenient to use the Fourier transform in the computation, as follows:
Recall that if \( \eta_t \in L^2(P) \) is a Lévy process with the representation
\[ \eta_t = \alpha_0 t + \sigma_0 B(t) + \int_{\mathbb{R}^+} \zeta \tilde{N}(t, d\zeta), \]
where \( \alpha_0 \) and \( \sigma_0 \) are constants, then
\[ E[e^{iu\eta_t}] = e^{i\Psi(u)}, \]
where
\[ \Psi(u) = i\alpha_0 u - \frac{1}{2} \sigma_0^2 u^2 + \int_{\mathbb{R}^+} (e^{iu\zeta} - 1 - iu\zeta) \nu(d\zeta), \]
\( \nu \) being the Lévy measure of \( \eta \). Combining this with the Fourier transform inversion
\[ f(x) = \frac{1}{\sqrt{2\pi}} \hat{f}(y) e^{iyx} dy, \]
where in general
\[ \hat{f}(y) = \frac{1}{\sqrt{2\pi}} f(y) e^{-iyx} dx \]
is the Fourier transform of \( f \), we see that we can obtain explicit expressions for expected values of the type
\[ E[f(\eta_t)] = E \left[ \frac{1}{\sqrt{2\pi}} \hat{f}(y) e^{iy\eta} dy \right] = \frac{1}{\sqrt{2\pi}} \hat{f}(y) E[e^{iy\eta} dy] = \frac{1}{\sqrt{2\pi}} \hat{f}(y) e^{i\Psi(y)} dy, \]
and similarly (by extending to 2 dimensions) for \( \hat{z} = E[(S(T) - K)^+ Z^*_T] \). We omit the details.
Remark 3.9 (The price at time \( t \)) We have discussed above the option price \( \hat{z} \) at time \( t = 0 \). If we want the option price at time \( t \), we take the conditional expectation with respect to the right EMM \( Q^* \). Hence, if we consider an European call option and we assume that the coefficients in the stock price \( \alpha = \alpha(t), \sigma = \sigma(t) \) and \( \gamma(\zeta) = \gamma(t, \zeta) \) are deterministic and bounded, we may write
\[
\hat{z}(t) := E_{Q^*} \left[ (S(T) - K)^+ | \mathcal{F}_t \right].
\]
By the Markov property, we have
\[
\hat{z}(t) = E_{Q^*}^{S(t)} [S(T - t)] = u(T - t, S(t)),
\]
where \( u(t, s) \) satisfies the Feynman-Kac formula
\[
\begin{align*}
\frac{\partial}{\partial t} u(t, s) &= Lu(t, s), \quad 0 \leq t \leq T, \\
u(t, s) &= (s - K)^+.
\end{align*}
\]
Here \( L \) is the integro-differential operator acting on \( s \), given by
\[
L u(t, s) = \alpha(t) \frac{\partial u}{\partial s}(t, s) + \frac{1}{2} \sigma(t)^2 \frac{\partial^2 u}{\partial s^2}(t, s)
\]
\[+ \int_{\mathbb{R}^*} \left\{ u(t, s + s\pi \gamma(\zeta)) - u(t, s) - s\gamma(\zeta) \frac{\partial u}{\partial s}(t, s) \right\} v(d\zeta).
\]

4 Finding the minimal variance option price by means of the maximum principle and equivalent martingale measures

In this section we relate the minimal variance option price \( \hat{z} \) found in Section 2 to pricing by means of EMMs. We will do this by approaching Problem 2.6 by means of stochastic control theory. Since the given \( \mathcal{F}_T \)-measurable random variable \( F \) is not Markovian, we cannot use classical dynamic programming to solve it. However, we can adapt the maximum principle method in Agram et al [ABO] to our situation. Thus we define the Hamiltonian \( H \) by
\[
H(t, x, z, \pi, p, q, r) = x \pi \alpha p + x \pi \sigma q + \int_{\mathbb{R}^*} x \pi \gamma(\zeta) r(\zeta) \nu(d\zeta),
\]
and we define the associated adjoint BSDE for the adjoint processes \( (p, q, r) \) by
\[
\begin{align*}
dp(t) &= -\pi(t) \left[ \alpha(t)p(t) + \sigma(t)q(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta) r(t, \zeta) \nu(d\zeta) \right] dt \\
+ q(t) dB(t) + \int_{\mathbb{R}^*} r(t, \zeta) N(dt, d\zeta), \\
p(T) &= X(T) - F.
\end{align*}
\]
Then, by extending Theorem 2.4 in [ABO] to jumps, we have:
Theorem 4.1  Suppose \( \hat{z}, \hat{\pi} \) is a solution of the problem (2.6), with associated solutions \((\hat{p}, \hat{q}, \hat{r})\) of the corresponding BSDE (4.2). Then

\[
\hat{p}(0) + \nabla_z \hat{H}(t) = 0 \text{ at } z = \hat{z},
\]

and

\[
\nabla_\pi \hat{H}(t) = 0 \text{ at } \pi = \hat{\pi},
\]

i.e.,

\[
\hat{p}(0) = 0, \tag{4.3}
\]

and

\[
\hat{X}(t) \left[ \alpha(t)\hat{p}(t) + \sigma(t)\hat{q}(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\hat{r}(t, \zeta)\nu(\,d\zeta) \right] = 0. \tag{4.4}
\]

Note that if \( z = 0 \) then \( X(t) = 0 \) for all \( \pi \) and all \( t \), which is a trivial special case. Thus we assume that \( z \neq 0 \) and this gives \( X(t) \neq 0 \) for all \( \pi \) and \( t \) and then (4.3) and (4.4) give

\[
\alpha(t)\hat{p}(t) + \sigma(t)\hat{q}(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\hat{r}(t, \zeta)\nu(\,d\zeta) = 0. \tag{4.5}
\]

and

\[
d\hat{p}(t) = \hat{q}(t)dB(t) + \int_{\mathbb{R}^*} \hat{r}(t, \zeta)\tilde{N}(dt, \,d\zeta), \quad \hat{p}(0) = 0 \text{ and } \hat{p}(T) = \hat{X}(T) - F. \tag{4.6}
\]

We have proved:

Theorem 4.2  Suppose there exists an optimal control \((\hat{z}, \hat{\pi})\) of the problem (2.6). Then the corresponding forward-backward system, consisting of

\[
d\hat{X}(t) = \hat{X}(t)\hat{\pi}(t) \left[ \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\hat{N}(dt, \,d\zeta) \right], \quad \hat{X}(0) = \hat{z}, \tag{4.7}
\]

\[
d\hat{p}(t) = \hat{q}(t)dB(t) + \int_{\mathbb{R}^*} \hat{r}(t, \zeta)\tilde{N}(dt, \,d\zeta), \quad \hat{p}(T) = \hat{X}(T) - F, \tag{4.8}
\]

satisfies the equations

\[
\hat{p}(0) = 0,
\]

\[
\alpha(t)\hat{p}(t) + \sigma(t)\hat{q}(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\hat{r}(t, \zeta)\nu(\,d\zeta) = 0.
\]

This implies the following option pricing result:
Theorem 4.3 (Option pricing theorem 1) Suppose $(\hat{z}, \hat{u})$ is an optimal control for the problem (2.6), with corresponding solutions $\hat{X} = X_{\hat{z}, \hat{u}}$ and $(\hat{p}, \hat{q}, \hat{r})$ of (4.7) and (4.8), respectively. Then for all $Q \in \mathbb{M}$ the minimal variance price of $F$ is given by

$$\hat{z} = p_{mv}(F) = E_Q[F] + E_Q \left[ \int_0^T \hat{q}(t)dB(t) + \int_0^T \int_{\mathbb{R}^*} \hat{r}(t, \zeta)\tilde{N}(dt, d\zeta) \right], \text{ for all } Q \in \mathbb{M}.$$  \hfill (4.9)

Proof. By (4.8) in Theorem 4.3 we have, for all $Q \in \mathbb{M},$

$$E_Q[\hat{p}(T)] = E_Q[\hat{X}(T) - F] = E_Q[\hat{X}(T)] - E_Q[F] = \hat{z} - E_Q[F].$$

Hence, by (4.8),

$$\hat{z} = E_Q[F] + E_Q[\hat{p}(T)] = E_Q[F] + E_Q \left[ \int_0^T \hat{q}(t)dB(t) + \int_0^T \int_{\mathbb{R}^*} \hat{r}(t, \zeta)\tilde{N}(dt, d\zeta) \right].$$

□

From this we deduce the following result, which shows in particular that if the market is complete, then the minimal variance price agrees with the price given by the classical Black-Scholes formula:

Corollary 4.4 (Generalized Black-Scholes formula) Let $\hat{X}$ be as in Theorem 4.3. Suppose that $\hat{X}(T) = F$ a.s.

Then

$$\hat{z} = p_{mv}(F) = E_Q[F], \text{ for all } Q \in \mathbb{M}.$$  

Proof. If $\hat{X}(T) = F$ a.s. then clearly $(\hat{z}, \hat{\pi})$ is an optimal pair for the problem (2.6). Therefore, by (4.8) we have $\hat{p}(T) = 0$ and hence, since $\hat{p}$ is an $(P, \mathbb{F})$-martingale,

$$p(t) = E[p(T)|\mathcal{F}_t] = 0,$$

for all $t$. But then $\hat{q}(t) = \hat{r}(t, \zeta) = 0$ for all $t, \zeta$, and the result follows from (4.9). □. We now make the following definition:

Definition 4.5 $\mathbb{M}_0$ is the set of measures $Q \in \mathbb{M}$ which are also equivalent martingale measures for the process $\hat{p}(\cdot)$.

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Then we obtain the following pricing result:

**Theorem 4.6 (Option pricing theorem 2)** Let \( \tilde{z}, \tilde{\pi}, \tilde{q}, \tilde{r} \) be as in Theorem 4.3. Then we have

\[
\tilde{z} = p_{mv}(F) = E_{Q_0}[F],
\]

for all \( Q_0 \in M_0 \).

**Proof.** If we apply (4.9) to \( Q_0 \in M_0 \) we get \( \tilde{z} = p_{mv}(F) = E_{Q_0}[F] \), because

\[
E_{Q_0}\left[ \int_0^T \tilde{q}(t)dB(t) + \int_0^T \tilde{r}(t, \zeta)\tilde{N}(dt, d\zeta) \right] = 0.
\]

\( \square \)

The following result illustrates what the measures in \( M_0 \) may look like:

**Theorem 4.7 (Option pricing theorem 3)** Suppose the risky asset price \( S(t) \) is given as in (2.1), but with \( m = k = 1 \), i.e.

\[
dS(t) = S(t)\left[ \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^*} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right], \quad S(0) > 0.
\]

Specifically, suppose that \( (\theta_0(t), \theta_1(t, \zeta)) \) solves the following system of two equations

\[
\begin{align*}
\alpha(t) + \theta_0(t)\sigma(t) + \int_{\mathbb{R}^*} \theta_1(t, \zeta)\gamma(t, \zeta)\nu(d\zeta) &= 0, \quad t \geq 0, \\
\theta_0(t)\tilde{q}(t) + \int_{\mathbb{R}^*} \theta_1(t, \zeta)\tilde{r}(t, \zeta)\nu(d\zeta) &= 0, \quad t \geq 0.
\end{align*}
\]

Define the process \( \tilde{Z}(t) = Z_{\theta_0, \theta_1}(t) \) by

\[
d\tilde{Z}(t) = \tilde{Z}(t)\left[ \theta_0(t)dB(t) + \int_{\mathbb{R}^*} \theta_1(t, \zeta)\tilde{N}(dt, d\zeta) \right], \quad \tilde{Z}(0) = 1,
\]

i.e.,

\[
\tilde{Z}(t) = \exp\left( \int_0^T \theta_0(s)dB(s) - \frac{1}{2} \int_0^T \theta_0^2(s)ds \\
+ \int_0^T \int_{\mathbb{R}^*} \left\{ \ln(1 + \theta_1(s, \zeta)) - \theta_1(s, \zeta) \right\} \nu(d\zeta)ds + \int_0^T \int_{\mathbb{R}^*} \ln(1 + \theta_1(s, \zeta))\tilde{N}(ds, d\zeta) \right).
\]

Assume that \( \tilde{Z} \) is a martingale. (See \[3.4\].) Then the measure \( \tilde{Q}_0 := \tilde{Q}_{\theta_0, \theta_1} \) defined by

\[
d\tilde{Q}_0(\omega) = \tilde{Z}(T)dP(\omega) \quad \text{on} \quad \mathcal{F}_T
\]

is in \( M_0 \), and

\[
p_{mv}(F) = \tilde{z} = E_{Q_0}[F].
\]
5 Examples

In this section we illustrate the results above by considering some examples.

5.1 The classical Black-Scholes market

We consider the classical Black-Scholes market, with \( N = 0, m = 1 \) in the market model (2.1), (2.3). Then we have a complete financial market with the following two investment possibilities:

(i) A risk free asset, with unit price \( S_0(t) = 1 \) for all \( t \).

(ii) A risky asset, with unit price \( S(t) \) at time \( t \) given by

\[
dS(t) = S(t) [\alpha(t)dt + \sigma(t)dB(t)], \quad S(0) > 0.
\]

Note that with \( N(t, \zeta) = 0 \) and \( \sigma(t) \) bounded away from 0 for all \( t \), (3.1) gets the form

\[
\alpha(t) + \theta_0(t)\sigma(t) = 0,
\]

which has the unique solution

\[
\theta_0(t) = -\frac{\alpha(t)}{\sigma(t)}.
\]

This gives, by (3.2),

\[
Z(t) = Z^{\theta_0}(t) = \exp\left( \int_0^t \theta_0(s)dB(s) - \frac{1}{2} \int_0^t \theta_0^2(s)ds \right),
\]

and, assuming that \( Z \) is a martingale,

\[
dQ^{\theta_0}(\omega) = Z^{\theta_0}(T)dP(\omega)
\]

is the unique element in \( \mathcal{M} \). Therefore the unique option price in this case is

\[
p_0(F) = p_s(F) = E_{Q^{\theta_0}}[F],
\]

which is the celebrated Black-Scholes formula.

We now compare this with what we get by using the stochastic control approach of Section 4: Since, \( N(t,.) = 0 \) for all \( t \), (4.11) gets the form

\[
\alpha(t) + \theta_0(t)\sigma(t) = 0.
\]

\[
-\frac{\alpha(t)}{\sigma(t)} = \theta_0(t).
\]
From (4.12), we have
\[ \hat{p}(t) = \hat{q}(t) = 0. \]
Therefore, by (4.3) and (4.6), we get
\[ \hat{X}(T) - F = \hat{p}(T) = 0. \]
By (4.3), this gives
\[ \hat{X}(T) = X_{\hat{z}}(T) = \hat{p}(T) = 0. \]

Since \( Q^{\theta_0} \) is a martingale measure for \( X_{\hat{z}}(T) \), we conclude that
\[ \hat{z} = X_{\hat{z}}(0) = E_{Q^{\theta_0}}[X_{\hat{z}}(T)] = E_{Q^{\theta_0}}[F]. \]

We conclude that in this case the optimal control \( \hat{\pi}(t) \) is the replicating portfolio for \( F \), and then optimal initial wealth \( \hat{z} \in \mathbb{R} \) is the \( Q^{\theta_0} \)-expectation of \( F \). Thus the minimum variance price \( p_{mv}(F) \) coincides with the classical option price in this case.

### 5.2 A continuous incomplete market

Consider the case with no jumps \( (N = 0) \) and with two Brownian motions, \( B_1(t), B_2(t) \). Then the price process is given by
\[ dS(t) = S(t)[\alpha(t)dt + a_1(t)dB_1(t) + a_2(t)dB_2(t)], \quad S(0) > 0, \quad (5.1) \]
and equations (4.11), (4.12) become
\[ \alpha(t) + x_1 a_1(t) + x_2 a_2(t) = 0, \quad t \geq 0, \quad (5.2) \]
\[ x_1 \hat{q}_1(t) + x_2 \hat{q}_2(t) = 0, \quad t \geq 0. \quad (5.3) \]

where we for simplicity have put \( \hat{\theta}_0(t) = (x_1, x_2) \). This is a linear system of two equations with the two unknowns \( x_1, x_2 \). This system has a unique solution \( (x_1, x_2) = \theta_0(t) \) if and only if \( \sigma_1(t) \hat{q}_2(t) - \sigma_2(t) \hat{q}_1(t) \neq 0 \). We conclude that

**Corollary 5.1** (a) In the market \( (5.1) \) there is a unique \( \hat{Q}_0 \in \mathcal{M}_0 \) if and only if \( \sigma_1(t) \hat{q}_2(t) - \sigma_2(t) \hat{q}_1(t) \neq 0 \), and if the process \( Z = \hat{Z}_0 \) defined by (3.2) is a martingale, then the minimal variance price of \( F \) is given by
\[ p_{mv}(F) = E_{\hat{Q}_0}[F], \quad \text{with } d_{\hat{Q}_0} = \hat{Z}_0(T)dP, \]
where \( \hat{\theta}_0 = (x_1, x_2) \) is the unique solution of the system \( (5.2), (5.3) \).

(b) If the coefficients \( \alpha, \sigma_1, \sigma_2 \) are deterministic, we can apply Theorem 3.2 (ii) to conclude that
\[ p_{mv}(F) = E_{Q^*}[F], \]
where \( dQ^* = Z^*(T)dP \) is given by \( (3.35) \).
5.3 A pure jump incomplete market

Suppose the risky asset price $S(t)$ is given by

$$dS(t) = S(t)[\alpha(t)dt + \int_{\mathbb{R}^+} \gamma(t, \zeta) \tilde{N}(dt, d\zeta)], \quad S(0) > 0. \quad (5.4)$$

Then the wealth process satisfies

$$d\hat{X}(t) = \hat{X}(t)\hat{\pi}(t)\left[\alpha(t)dt + \int_{\mathbb{R}^+} \gamma(t, \zeta) \tilde{N}(dt, d\zeta)\right], \quad \hat{X}(0) = \hat{z}. \quad (5.5)$$

Here there is no Brownian motion component and only one compensated Poisson random measure (i.e. $k = 1$), but we assume that $N$ has at least two possible jump sizes, i.e. that the Lévy measure $\nu$ is not a point mass. Then the market is not complete, because there are several solutions $\theta_1(t, \zeta)$ of the equation (3.1) (or (4.11)), which now has the form

$$\alpha(t) + \int_{\mathbb{R}^+} \theta_1(t, \zeta) \gamma(t, \zeta) \nu(d\zeta) = 0. \quad (5.6)$$

To find possible elements $\hat{Q}$ of $\mathcal{M}_0$ we combine (5.6) with (4.11), which now reduces to

$$\int_{\mathbb{R}^+} \theta_1(t, \zeta) \hat{r}(t, d\zeta) \nu(d\zeta) = 0. \quad (5.7)$$

This gives the following result:

**Corollary 5.2** (a) Suppose there exists a solution $\theta_1(t, \zeta) = \hat{\theta}_1(t, \zeta)$ of the two equations (5.6), (5.7) and that the corresponding $Z^{\hat{\theta}_1}$ defined by (3.2) is a martingale. Then the minimal variance price of $F$ in the market (5.4) is given by

$$p_{mv}(F) = E_{\hat{Q}}[F], \quad \text{where } d\hat{Q} = Z^{\hat{\theta}_1}(T)dP. \quad (5.8)$$

(b) If the coefficients $\alpha, \gamma$ are deterministic, we can apply Theorem 3.5 (ii) to conclude that

$$p_{mv}(F) = E_{Q^*}[F] \quad (5.8)$$

where $dQ^* = Z^*dP$ is given by (3.33).

5.4 Merton type markets

Finally, consider the Merton type markets, driven by a Brownian motion $B(t)$ and a jump process being the standard Poisson process $N(t)$ with intensity $\lambda > 0$, which implies that
the Lévy measure $\nu$ is just the point mass at 1, $\delta_1$. Then the corresponding compensated Poisson random measure will be

$$\tilde{N}(dt, d\zeta) = \delta_1(\zeta)dN(t) - \lambda\delta_1(\zeta)dt,$$

and the price process is then given by

$$dS(t) = S(t)[\alpha(t)dt + \sigma(t)dB(t) + \gamma(t, 1)(dN(t) - \lambda dt)], \quad S(0) > 0. \quad (5.9)$$

with corresponding wealth process

$$d\hat{X}(t) = \hat{X}(t)\hat{\pi}(t)[\alpha(t)dt + \sigma(t)dB(t) + \gamma(t, 1)(dN(t) - \lambda dt)], \quad \hat{X}(0) = \hat{z}. \quad (5.10)$$

Then the equations (4.11), (4.12) get the form

$$\alpha(t) + x(t)\sigma(t) + y(t)\gamma(t, 1) = 0, \quad t \geq 0, \quad (5.11)$$

$$x(t)\hat{q}(t) + y(t)\hat{r}(t, 1) = 0, \quad t \geq 0. \quad (5.12)$$

where we for simplicity have put $x(t) = \hat{\theta}_0(t), y(t) = \hat{\theta}_1(t, \zeta)$. This system has a unique solution $(x(t), y(t))$ if and only if

$$\sigma(t)\hat{r}(t, 1) - \hat{q}(t)\gamma(t, 1) \neq 0.$$

**Corollary 5.3** (a) Suppose there exists a solution $x(t) = \hat{\theta}_0(t), y(t) = \hat{\theta}_1(t, \zeta)$ of the two equations (5.11), (5.12) and that the corresponding $Z^{\hat{\theta}_1}$ defined by (3.2) is a martingale. Then the minimal variance price of $F$ in the market (5.4) is given by

$$p_{mv}(F) = E_{\hat{Q}}[F], \quad \text{where } d\hat{Q} = Z^{\hat{\theta}_1}(T)dP.$$

(b) If the coefficients $\alpha, \sigma, \gamma$ are deterministic, we can apply Theorem 3.5 (ii) to conclude that

$$p_{mv}(F) = E^{\hat{Q}^*}[F],$$

where $dQ^* = Z^*(T)dP$ is given by (3.35).

5.4.1 Pure jump market

$$dS(t) = S(t)[\alpha(t)dt + \gamma(t, 1)(dN(t) - \lambda dt)], \quad S(0) > 0. \quad (5.13)$$

with corresponding wealth process

$$d\hat{X}(t) = \hat{X}(t)\hat{\pi}(t)[\alpha(t)dt + \gamma(t, 1)(dN(t) - \lambda dt)], \quad \hat{X}(0) = \hat{z}.$$
Then the equations (4.11), (4.12) get the form
\[
\alpha(t) + y(t)\gamma(t, 1) = 0, \quad t \geq 0,
\]
\[
y(t)\tilde{r}(t, 1) = 0, \quad t \geq 0.
\]
This system has a unique solution \(\tilde{r}(t, 1) = 0, y(t) = \hat{\theta}(t, 1) = -\frac{\alpha(t)}{\gamma(t, 1)}\). Assume that
\[
\frac{\alpha(t)}{\gamma(t, 1)} < 1, \quad \text{for all } t.
\]
Define
\[
dQ(\omega) = Z(T, \omega)dP(\omega) \quad \text{on } \mathcal{F}_T,
\]
where
\[
Z(t) = \exp \left( \int_0^t \lambda \ln(1 + \hat{\theta}(s, 1)) - \hat{\theta}(s, 1)ds + \int_0^t \ln(1 + \hat{\theta}(s, 1))(dN(s) - \lambda ds) \right) = \exp \left( \int_0^t -\lambda \hat{\theta}(s, 1)ds + \int_0^t \ln(1 + \hat{\theta}(s, 1))dN(s) \right).
\]
Suppose that \(Z\) is a martingale. Then \(Q\) is the unique EMM for the process (5.13). Hence the unique no-arbitrage price of an option with payoff \(F\) in this market is
\[
p(F) = E_Q[F].
\]

### 5.4.2 Merton mixed type market

Assume that the risky asset price \(S(t)\) is given by
\[
dS(t) = S(t) \left[ \alpha_0 dt + \sigma_0 dB(t) + \int_{\mathbb{R}^*} \gamma_0 \tilde{N}(dt, d\zeta) \right], \quad S(0) = S_0 > 0,
\]
where \(\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt\), with \(N(dt, d\zeta) = dN(t)\delta_1(d\zeta)\). Here \(N\) is a Poisson process with intensity \(\lambda > 0\), \(\delta_1\) is the Dirac measure (unit point mass) at 1 and \(\alpha_0, \sigma_0 > 0\) and \(\gamma_0 > -1\) are given constants. This equation has the following explicit solution (see e.g. Example 1.15 in [OS])
\[
S(t) = S_0 \exp \left( \left\{ \alpha_0 - \frac{1}{2} \sigma_0^2 - \lambda \gamma_0 \right\} t + \sigma_0 B(t) + \ln(1 + \gamma_0)N(t) \right).
\]
In this case the process \(Z^*_t\) given by (3.34) gets the form
\[
dZ^*_t = Z_t \left[ G\sigma_0 dB(t) + G \int_{\mathbb{R}^*} \gamma_0 \tilde{N}(dt, d\zeta) \right], \quad Z^*_0 = 1,
\]
28.
where, by (3.11),

\[ G = -\frac{\alpha_0}{\sigma_0^2 + \lambda\gamma_0}. \]

This equation has the solution

\[ Z^*_t = \exp \left( \left\{ -\frac{1}{2}G^2\sigma_0^2 - \lambda G\gamma_0 \right\} t + G\sigma_0 B(t) + G\gamma_0 N(t) \right). \]

By Theorem 3.5 (ii) the minimal variance price \( \hat{z} \) of a contract with payoff \( F \) at time \( T \) is

\[ \hat{z} = E_{Q^*}(F) = E[FZ^*_T]. \]

Let us assume that \( F \) has the Itô representation

\[ F = F_0 + \phi_0 B(T) + \psi_0 N(T), \quad F_0 > 0. \]

Then by the Itô isometry we get that

\[ \hat{z} = E[FZ^*_T] = F_0 + G(\sigma_0\phi_0 + \lambda\gamma_0\psi_0)T. \]

6 Summary

- In this paper we introduce a new principle for pricing of European options, based on optimal control and the minimal variance principle. Our pricing principle states the following:

  The minimal variance price \( p_{mv}(F) \) of a given contingent claim \( F \) is the initial wealth \( \hat{z} \) needed to make it possible to generate a terminal wealth \( X(T) \) which minimizes the variance of the difference between \( X(T) \) and the option payoff \( F \) at the terminal time \( T \). In other words, \((\hat{z}, \hat{\pi})\) is a minimum point of the map

  \[ (z, \pi) \mapsto \frac{1}{2}E[(X_{z,\pi}(T) - F)^2]. \]

- We regard this minimizing problem as a Stackelberg game and prove that the minimum exists and is unique, and we find explicit expressions for the optimal portfolio \( \hat{\pi} \), the optimal initial wealth \( \hat{z} = p_{mv}(F) \) and the corresponding optimal wealth process \( \hat{X}(t); 0 \leq t \leq T \).

- If the coefficients of the risky asset price \( S(t) \) are deterministic, we prove that

  \[ \hat{z} = E_{Q^*}[F] \]

for a specific EMM \( Q^* \), which is the minimal variance option pricing measure. In particular, this shows that \( \hat{z} \) is an arbitrage-free price in this case.
Next, we study the general case of this stochastic control problem by means of the maximum principle for optimal control, and we describe a family $M_0$ of EMM’s such that the value $\tilde{z}$ found in above is of the form $\tilde{z} = E_{\tilde{Q}}[F]$ for all EMM $\tilde{Q} \in M_0$. In particular, this shows that our price $\tilde{z} = p_{ms}(F)$ coincides with the Black-Scholes option price if the market is complete.

Finally, we give several examples to illustrate our results.

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