A Cauchy-Kowalevski theorem for inframonogenic functions

Helmuth R. Malonek∗,1, Dixan Peña Peña∗,2
and Frank Sommen†,3

∗Department of Mathematics, Aveiro University,
3810-193 Aveiro, Portugal
†Department of Mathematical Analysis, Ghent University,
9000 Gent, Belgium
1e-mail: hrmalon@ua.pt
2e-mail: dixanpena@ua.pt; dixanpena@gmail.com
3e-mail: fs@cage.ugent.be

Abstract

In this paper we prove a Cauchy-Kowalevski theorem for the functions
satisfying the system \( \partial_x f \partial_x = 0 \) (called inframonogenic functions).

Keywords: Inframonogenic functions; Cauchy-Kowalevski theorem.
Mathematics Subject Classification: 30G35.

1 Introduction

Let \( \mathbb{R}_{0,m} \) be the \( 2^m \)-dimensional real Clifford algebra constructed over the
orthonormal basis \( (e_1, \ldots, e_m) \) of the Euclidean space \( \mathbb{R}^m \) (see [3]). The
multiplication in \( \mathbb{R}_{0,m} \) is determined by the relations \( e_j e_k + e_k e_j = -2\delta_{jk} \),
\( j, k = 1, \ldots, m \), where \( \delta_{jk} \) is the Kronecker delta. A general element of \( \mathbb{R}_{0,m} \)
is of the form
\[
a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},
\]

1 accepted for publication in Mathematical Journal of Okayama University
where for $A = \{ j_1, \ldots, j_k \} \subset \{ 1, \ldots, m \}$, $j_1 < \cdots < j_k$, $e_A = e_{j_1} \cdots e_{j_k}$. For the empty set $\emptyset$, we put $e_\emptyset = 1$, the latter being the identity element.

Notice that any $a \in \mathbb{R}_{0,m}$ may also be written as $a = \sum_{k=0}^{m} a_k$ where $[a]_k$ is the projection of $a$ on $\mathbb{R}^{(k)}_{0,m}$. Here $\mathbb{R}^{(k)}_{0,m}$ denotes the subspace of $k$-vectors defined by

$$\mathbb{R}^{(k)}_{0,m} = \left\{ a \in \mathbb{R}_{0,m} : a = \sum_{|A|=k} a_A e_A, \quad a_A \in \mathbb{R} \right\}.$$  

Observe that $\mathbb{R}^{m+1}$ may be naturally identified with $\mathbb{R}^{(0)}_{0,m} \oplus \mathbb{R}^{(1)}_{0,m}$ by associating to any element $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ the “paravector” $x = x_0 + \overline{\alpha} = x_0 + \sum_{j=1}^{m} x_j e_j$.

Conjugation in $\mathbb{R}_{0,m}$ is given by

$$\overline{a} = \sum_{A} a_A \overline{e_A},$$

where $\overline{e_A} = \overline{e_{j_k} \cdots e_{j_1}}$, $\overline{e_j} = -e_j$, $j = 1, \ldots, m$. One easily checks that $\overline{ab} = \overline{a} \overline{b}$ for any $a, b \in \mathbb{R}_{0,m}$. Moreover, by means of the conjugation a norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,m}$ by putting

$$|a|^2 = \overline{a} a = \sum_{A} a_A^2.$$  

Let us denote by $\partial_x = \partial_{x_0} + \overline{\partial_\overline{x}} = \partial_{x_0} + \sum_{j=1}^{m} e_j \partial_{x_j}$ the generalized Cauchy-Riemann operator and let $\Omega$ be an open set of $\mathbb{R}^{m+1}$. According to \[11\], an $\mathbb{R}_{0,m}$-valued function $f \in C^{2}(\Omega)$ is called an inframonogenic function in $\Omega$ if and only if it fulfills in $\Omega$ the “sandwich” equation $\partial_x f \partial_x = 0$.

It is obvious that monogenic functions (i.e. null-solutions of $\partial_x$) are inframonogenic. At this point it is worth remarking that the monogenic functions are the central object of study in Clifford analysis (see \[2, 4, 5, 7, 8, 9, 10, 14\]). Furthermore, the concept of monogenicity of a function may be seen as the higher dimensional counterpart of holomorphy in the complex plane.

Moreover, as

$$\Delta_x = \sum_{j=0}^{m} \partial^2_{x_j} = \partial_x \overline{\partial_x} = \overline{\partial_x} \partial_x,$$

every inframonogenic function $f \in C^{4}(\Omega)$ satisfies in $\Omega$ the biharmonic equation $\Delta^2 f = 0$ (see e.g. \[11, 13, 12, 15\]).

This paper is intended to study the following Cauchy-type problem for the inframonogenic functions. Given the functions $A_0(\overline{x})$ and $A_1(\overline{x})$ analytic
in an open and connected set $\Omega \subset \mathbb{R}^m$, find a function $F(x)$ inframonogenic in some open neighbourhood $\tilde{\Omega}$ of $\Omega$ in $\mathbb{R}^{m+1}$ which satisfies

$$F(x)|_{x_0=0} = A_0(\underline{x}),$$

$$\partial_{x_0} F(x)|_{x_0=0} = A_1(\underline{x}).$$

(1) \hspace{1cm} (2)

2 Cauchy-type problem for inframonogenic functions

Consider the formal series

$$F(x) = \sum_{n=0}^{\infty} x_0^n A_n(\underline{x}).$$

(3)

It is clear that $F$ satisfies conditions (1) and (2). We also see at once that

$$\partial_x (x_0^n A_n) \partial_{\underline{x}} = n(n-1)x_0^{n-2} A_n + nx_0^{n-1}(\partial_{\underline{x}} A_n + A_n \partial_{\underline{x}}) + x_0^n \partial_{\underline{x}} A_n \partial_{\underline{x}}.$$  

We thus get

$$\partial_x F \partial_x = \sum_{n=0}^{\infty} x_0^n \left( (n+2)(n+1)A_{n+2} + (n+1)(\partial_{\underline{x}} A_{n+1} + A_{n+1} \partial_{\underline{x}}) + \partial_{\underline{x}} A_n \partial_{\underline{x}} \right).$$

From the above it follows that $F$ is inframonogenic if and only if the functions $A_n$ satisfy the recurrence relation

$$A_{n+2} = -\frac{1}{(n+2)(n+1)} \left( (n+1)(\partial_{\underline{x}} A_{n+1} + A_{n+1} \partial_{\underline{x}}) + \partial_{\underline{x}} A_n \partial_{\underline{x}} \right), \quad n \geq 0.$$  

(4)

It may be easily proved by induction that

$$A_n = \frac{(-1)^{n+1}}{n!} \left( \sum_{j=0}^{n-2} \partial_{\underline{x}}^{n-j-1} A_0 \partial_{\underline{x}}^{j+1} + \sum_{j=0}^{n-1} \partial_{\underline{x}}^{n-j-1} A_1 \partial_{\underline{x}}^j \right), \quad n \geq 2.$$  

(4)

We now proceed to examine the convergence of the series (3) with the functions $A_n$ ($n \geq 2$) given by (4). Let $\underline{y}$ be an arbitrary point in $\Omega$. Then there exist a ball $B(\underline{y}, R(\underline{y}))$ of radius $R(\underline{y})$ centered at $\underline{y}$ and a positive constant $M(\underline{y})$, such that

$$|\partial_{\underline{x}}^{n-j} A_s(\underline{x}) \partial_{\underline{x}}^j| \leq M(\underline{y}) \frac{n!}{R^n(\underline{y})}, \quad \underline{x} \in B(\underline{y}, R(\underline{y})), \quad j = 0, \ldots, n, \quad s = 0, 1.$$  

(4)
It follows that
\[ |A_n(x)| \leq M(y)^n R^n(y) \frac{n + R(y) - 1}{R^n(y)}, \quad x \in B(y, R(y)), \]
and therefore the series converges normally in
\[ \tilde{\Omega} = \bigcup_{y \in \Omega} (-R(y), R(y)) \times B(y, R(y)). \]

Note that \( \tilde{\Omega} \) is a \( x_0 \)-normal open neighbourhood of \( \Omega \) in \( \mathbb{R}^{m+1} \), i.e. for each \( x \in \tilde{\Omega} \) the line segment \( \{x + t : t \in \mathbb{R}\} \cap \tilde{\Omega} \) is connected and contains one point in \( \tilde{\Omega} \).

We thus have proved the following.

**Theorem 1** The function \( CK[A_0, A_1] \) given by
\[
CK[A_0, A_1](x) = A_0(x) + x_0 A_1(x)
- \sum_{n=2}^{\infty} \left( \frac{(-x_0)^n}{n!} \left( \sum_{j=0}^{n-2} \partial_x^{n-j-1} A_0(x) \partial_x^{j+1} + \sum_{j=0}^{n-1} \partial_x^{n-j-1} A_1(x) \partial_x^j \right) \right)
\]
is inframonogenic in a \( x_0 \)-normal open neighbourhood of \( \Omega \) in \( \mathbb{R}^{m+1} \) and satisfies conditions (7)-(8).

It is worth noting that if in particular \( A_1(x) = -\partial_x A_0(x) \), then
\[
CK[A_0, -\partial_x A_0](x) = \sum_{n=0}^{\infty} \frac{(-x_0)^n}{n!} \partial_x^n A_0(x),
\]
which is nothing else but the left monogenic extension (or \( CK \)-extension) of \( A_0(x) \). Similarly, it is easy to see that \( CK[A_0, -A_0 \partial_x](x) \) yields the right monogenic extension of \( A_0(x) \) (see [2, 5, 13, 16, 17]).

Let \( P(k) \ (k \in \mathbb{N}_0 \text{ fixed}) \) denote the set of all \( \mathbb{R}_{0,m} \)-valued homogeneous polynomials of degree \( k \) in \( \mathbb{R}^m \). Let us now take \( A_0(x) = P_k(x) \in P(k) \) and \( A_1(x) = P_{k-1}(x) \in P(k-1) \). Clearly,
\[
CK[P_k, P_{k-1}](x) = P_k(x) + x_0 P_{k-1}(x)
- \sum_{n=2}^{k} \left( \frac{(-x_0)^n}{n!} \left( \sum_{j=0}^{n-2} \partial_x^{n-j-1} P_k(x) \partial_x^{j+1} + \sum_{j=0}^{n-1} \partial_x^{n-j-1} P_{k-1}(x) \partial_x^j \right) \right),
\]

since the other terms in the series vanish. Moreover, we can also claim that $\mathcal{C}K[P_k, P_{k-1}](x)$ is a homogeneous inframonogenic polynomial of degree $k$ in $\mathbb{R}^{m+1}$.

Conversely, if $P_k(x)$ is a homogeneous inframonogenic polynomial of degree $k$ in $\mathbb{R}^{m+1}$, then $P_k(x)|_{x_0=0} \in \mathcal{P}(k)$, $\partial_{x_0} P_k(x)|_{x_0=0} \in \mathcal{P}(k-1)$ and obviously $\mathcal{C}K[P_k|_{x_0=0}, \partial_{x_0} P_k|_{x_0=0}](x) = P_k(x)$.

Call $\mathcal{I}(k)$ the set of all homogeneous inframonogenic polynomials of degree $k$ in $\mathbb{R}^{m+1}$. Then $\mathcal{C}K[., .]$ establishes a bijection between $\mathcal{P}(k) \times \mathcal{P}(k-1)$ and $\mathcal{I}(k)$.

It is easy to check that

$$P_k(x) = P_k(\partial_u) \frac{(x, u)^k}{k!}, \quad P_k(x) \in \mathcal{P}(k),$$

where $P_k(\partial_u)$ is the differential operator obtained by replacing in $P_k(u)$ each variable $u_j$ by $\partial_{u_j}$. Therefore, in order to characterize $\mathcal{I}(k)$, it suffices to calculate $\mathcal{C}K[\langle x, u \rangle^k e_A, 0]$ and $\mathcal{C}K[0, \langle x, u \rangle^{k-1} e_A]$ with $u \in \mathbb{R}^m$.

A simple computation shows that

$$\mathcal{C}K[\langle x, u \rangle^k e_A, 0](x) = \langle x, u \rangle^k e_A - \sum_{n=2}^{k} \binom{k}{n} (-x_0)^n \langle x, u \rangle^{k-n} \left( \sum_{j=0}^{n-2} u^{n-j-1} e_A u^{j+1} \right),$$

$$\mathcal{C}K[0, \langle x, u \rangle^{k-1} e_A](x) = x_0 \langle x, u \rangle^{k-1} e_A - \frac{1}{k} \sum_{n=2}^{k} \binom{k}{n} (-x_0)^n \langle x, u \rangle^{k-n} \left( \sum_{j=0}^{n-1} u^{n-j-1} e_A u^j \right).$$

Acknowledgments

The second author was supported by a Post-Doctoral Grant of Fundação para a Ciência e a Tecnologia, Portugal (grant number: SFRH/BPD/45260/2008).

References

[1] S. Bock and K. Gürlebeck, *On a spatial generalization of the Kolosov-Muskhelishvili formulae*, Math. Methods Appl. Sci. 32 (2009), no. 2, 223–240.
[2] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*, Research Notes in Mathematics, 76, Pitman (Advanced Publishing Program), Boston, MA, 1982.

[3] W. K. Clifford, *Applications of Grassmann’s Extensive Algebra*, Amer. J. Math. 1 (1878), no. 4, 350–358.

[4] J. Cnops and H. Malonek, *An introduction to Clifford analysis*, Textos de Matemática, Série B, 7, Universidade de Coimbra, Departamento de Matemática, Coimbra, 1995.

[5] R. Delanghe, F. Sommen and V. Souček, *Clifford algebra and spinor-valued functions*, Mathematics and its Applications, 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[6] K. Gürlebeck and U. Kähler, *On a boundary value problem of the biharmonic equation*, Math. Methods Appl. Sci. 20 (1997), no. 10, 867–883.

[7] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers*, Wiley and Sons Publ., 1997.

[8] V. V. Kravchenko, *Applied quaternionic analysis*, Research and Exposition in Mathematics, 28, Heldermann Verlag, Lemgo, 2003.

[9] V. V. Kravchenko and M. V. Shapiro, *Integral representations for spatial models of mathematical physics*, Pitman Research Notes in Mathematics Series, 351, Longman, Harlow, 1996.

[10] H. R. Malonek, *Selected topics in hypercomplex function theory*, Clifford algebras and potential theory, 111–150, Univ. Joensuu Dept. Math. Rep. Ser., 7, Univ. Joensuu, Joensuu, 2004.

[11] H. R. Malonek, D. Peña Peña and F. Sommen, *Fischer decomposition by inframonogenic functions*, accepted for publication in CUBO, A Mathematical Journal.

[12] V. V. Meleshko, *Selected topics in the history of the two-dimensional biharmonic problem*, Appl. Mech. Rev. 56 (2003), no. 1, 33-85.

[13] D. Peña Peña, *Cauchy-Kowalevski extensions, Fueter’s theorems and boundary values of special systems in Clifford analysis*, Ph.D. Thesis, Ghent University, Ghent, 2008 (available at http://hdl.handle.net/1854/11636).

[14] J. Ryan, *Basic Clifford analysis*, Cubo Mat. Educ. 2 (2000), 226–256.
[15] L. Sobrero, *Theorie der ebenen Elastizität unter Benutzung eines Systems hyperkomplexer Zahlen*, Hamburg. Math. Einzelschriften, Leipzig, 1934.

[16] F. Sommen, *Monogenic functions on surfaces*, J. Reine Angew. Math. 361 (1985), 145–161.

[17] F. Sommen and B. Jancewicz, *Explicit solutions of the inhomogeneous Dirac equation*, J. Anal. Math. 71 (1997), 59–74.