The Weibull Birnbaum-Saunders Distribution: Properties and Applications

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Abstract

Based on the Weibull-G class (Bourguignon et al., 2014), we introduce and study a new four-parameter continuous model called the Weibull Birnbaum-Saunders distribution. We derive, various mathematical properties of the new distribution including expansions for the density function, hazard function, explicit expressions for the moments, moment generating function, quantile function, order statistics and their moments, mean deviations and reliability. We estimate the model parameters using the maximum likelihood method of estimation and determine the observed information matrix. By means of two real data sets, we illustrate the usefulness of the new distribution. For these data, the new model provides a better fits than the beta Birnbaum-Saunders, McDonald Birnbaum-Saunders, Kumaraswamy Birnbaum-Saunders, exponentiated Birnbaum-Saunders and Birnbaum-Saunders models.

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1 Introduction

The continuous Binrbaum-Saunders (BS) distribution was first introduced by Birnbaum and Saunders (1969). This distribution is also commonly known as the fatigue life distribution and it was originally derived from a model that shows the total time that passes until that some type of cumulative damage, produced by the development and growth of a dominant crack, surpasses a threshold value and causes the material specimen to fail. Desmond (1985) provided a more general derivation based on a biological model and strengthened the physical justification for the use of this distribution.

A random variable $T$ following the BS distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, denoted by $T \sim BS (\alpha, \beta)$, is defined by

$$T = \beta \left[ \frac{\alpha Z}{2} + \left\{ \left( \frac{\alpha Z}{2} \right)^2 + 1 \right\}^{\frac{1}{2}} \right]^2,$$

where $Z$ is a standard normal random variable. The cumulative distribution function (cdf) of $T$ is

$$G(t) = \Phi (v), \ t > 0,$$

where $\Phi (\cdot)$ is the standard normal distribution function, $v = \alpha^{-1} \rho (t/\beta)$ and $\rho (z) = z^{1/2} - z^{-1/2}$. The probability density function (pdf) corresponding to (1) is given by

$$g(t) = \kappa (\alpha, \beta) t^{-3/2} (t + \beta) \exp \left\{ -\tau(t/\beta) \right\}, \ t > 0,$$

where $\kappa (\alpha, \beta) = \exp (\alpha^2) / (2\alpha \sqrt{2\pi \beta})$ and $\tau (z) = z + z^{-1}$. The $k$th moments of $T$ is given by (see Rieck, 1999)

$$E(T^k) = \beta^k I(\alpha, \beta),$$

where

$$I(\alpha, \beta) = \frac{K_{k+1/2}(\alpha^2) + K_{k-1/2}(\alpha^2)}{2K_{1/2}(\alpha^2)},$$

and the function $K_v(z)$ denotes the modified Bessel function of the third kind with $v$ representing its order and $z$ the argument. The parameter $\beta$ is the median of the BS distribution, because $G(\beta) = \Phi (0) = 1/2$.

Since the BS distribution was proposed, it has received much attention in the literature. This attention for the BS distribution is due to its many
attractive properties and its relationship with the normal distribution. For further details on the BS distribution, see Johnson et al. (1995, pp. 651–662). The BS distribution has been used in several research areas such as, economics, insurance, environmental and medical data, see Leiva et al. (2007, 2008b, 2009, 2010a,b, 2011), Podlaski (2008), Bhatti (2010), Ahmed et al. (2010) and Vilca et al. (2010).

Various generalizations and extensions of the BS distribution have been proposed in the statistical literature. For example, Díaz-García and Leiva (2005), Vilca and Leiva (2006), Gómes et al. (2009), Guiraud et al. (2009), Leiva et al. (2009), Leiva et al. (2010), Cordeiro and Lemonte (2011), Saulo et al. (2012), Lemonte (2013), Cordeiro et al. (2013) and Martínez-Flórez et al. (2014). Cordeiro and Lemonte (2011), using the beta-G class (Eugene et al., 2002), proposed a extension of BS distribution named as the beta BS (BBS) distribution. Saulo et al. (2012), based on the work of Cordeiro and Castro (2011), defined the Kumaraswamy-BS (KwBS ). Lemonte (2013), based on the scheme introduced by Marshall and Olkin (1997), defined the Marshall–Olkin extended BS distribution with three-parameters. Cordeiro et al. (2013 ) adopted the McDonald-G class (Alexander et al., 2012) to define the McDonald-BS (McBS) distribution with five positive parameters. Recently, Martínez-Flórez (2014) proposed an extension of the BS distribution based on the asymmetric alpha-power family of distributions (see Pewsey et al., 2012). In this study, a new four-parameter extension for the BS distribution is proposed.

Recently, Bourguignon et al. (2014) proposed an interesting method of adding a new parameter to an existing G distribution. The resulting distribution, known as the Weibull-G distribution, includes the original distribution as a special case and gives more flexibility to model various types of data. Let $G(t, \xi)$ be a continuous baseline distribution with density $g$ depends on a parameter vector $\xi$ and the Weibull cdf $F_W(w) = 1 - e^{-aw}$ (for $w > 0$) with positive parameters $a$ and $b$. Bourguignon et al. (2014) replaced the argument $w$ by $G(w, \xi) / \overline{G}(w, \xi)$ where $\overline{G}(w, \xi) = 1 - G(w, \xi)$, and defined the cdf of their class by

$$F(t; a, b, \xi) = ab \int_0^{[G(t, \xi) / \overline{G}(t, \xi)]} w^{b-1} e^{-aw} dw$$

$$= 1 - e^{-a\left([G(t, \xi) / \overline{G}(t, \xi)]^b\right)}.$$  (5)
Then, the Weibull-G density function is given by

\[
f (t; a, b, \xi) = abg (t, \xi) \left[ \frac{G (t, \xi)^{b-1}}{G (t, \xi)^{b+1}} \right] e^{-a \left[ \frac{G (t, \xi)}{G (t, \xi)^{b+1}} \right]^b}.
\]  

(6)

Some Weibull-G distributions were discussed in recent literature. Tahir et al. (2014) defined the Weibull-Dagum, Weibull-Pareto, Weibull-Lomax distributions by taking \( G \) to be the cdf of the Dagum, Pareto and Lomax distributions, respectively, and studied some of their properties. In a similar way, we propose and study the Weibull BS (WBS) distribution based on equations (4) and (5).

The rest of the paper is organized as follows. In Section 2, we introduce the WBS distribution and plot its density and hazard rate functions. In Section 3, we derive useful expansions for its density and cumulative distributions. Structural properties such as the ordinary moments, generating function, quantile function and quantile measures are derived in Section 4. The density of the order statistics and their moments are determined in Section 5. In Sections 6 and 7, we obtain mean deviations and the reliability, respectively. In Section 8, we discuss maximum likelihood estimation of the WBS parameters from one observed sample and derive the observed information matrix. Two applications are presented in Section 9 to show the usefulness of the new distribution for fatigue life modeling. Concluding remarks are given in Section 10.

2 The WBS distribution

By replacing (1) in (5) we obtain a new four-parameter distribution, called WBS, with cdf given by

\[
F (t) = 1 - e^{-a \left[ \frac{\Phi (v)}{1 - \Phi (v)} \right]^b}.
\]  

(7)

By replacing (1) and (2) in (6), we obtain the WBS density function with four positive parameters \( \alpha, \beta, a \) and \( b \), say WBS(\( \alpha, \beta, a, b \)), given by

\[
f (t) = abk (\alpha, \beta) t^{-3/2} (t + \beta) \exp \left\{ -\frac{\tau (t/\beta)}{2\alpha^2} \right\} \frac{\Phi (v)^{b-1}}{[1 - \Phi (v)]^{b+1}} e^{-a \left[ \frac{\Phi (v)}{1 - \Phi (v)} \right]^b},
\]  

(8)

where \( a > 0 \) and \( b > 0 \) are two additional shape parameters. It is clear that the BS distribution is not a special case of WBS.
The reliability function or survival function of the WBS distribution, denoted by $R$, is a continuous monotone and decreasing function, given by

$$R(x) = e^{-a \left[ \Phi(v) \right]^b}.$$  

The hazard rate function, also known as the conditional failure rate in reliability, of the WBS distribution is

$$h(x) = b \kappa(\alpha, \beta) t^{-3/2} (t + \beta) \frac{\Phi(v)^{b-1}}{[1 - \Phi(v)]^{b+1}} \exp \left\{ -\frac{\tau (t/\beta)}{2\alpha^2} \right\}.$$  

The reversed hazard rate function of the new distribution are given by

$$h^r(x) = ab \kappa(\alpha, \beta) t^{-3/2} (t + \beta) \exp \left\{ -\frac{\tau (t/\beta)}{2\alpha^2} \right\} \frac{\Phi(v)^{b-1}}{[1 - \Phi(v)]^{b+1}} \left( e^{a \left[ \Phi(v) \right]^b} - 1 \right).$$  

Plots of pdf and hazard rate function of the WBS distribution for selected values of the parameters are given in Figure 1 and Figure 2, respectively. From these figures, we observe that the density function and hazard rate function can take various forms, depending on the parameter values. It is clear that the shapes of the WBS distribution are much more flexible than the BS distribution.

Figure 1. Plots of density functions of WBS for different values of parameters.
3 Expansion for the density function

In this section, we derive expansions for the pdf and cdf of the WBS distribution that are useful to study its statistical properties. We have

\[ f(t) = abg(t) \frac{\Phi(v)^{b-1}}{[1 - \Phi(v)]^{b+1}} e^{-a\left[\frac{v}{1 - \Phi(v)}\right]^b}, \]

By using the power series for the exponential function, we obtain

\[ e^{-a\left[\frac{v}{1 - \Phi(v)}\right]^b} = \sum_{k=0}^{\infty} \frac{(-1)^k a^k}{k!} \frac{\Phi(v)^{bk}}{[1 - \Phi(v)]^{bk}}. \]
Then, we have

\[ f(t) = abg(t) \sum_{k=0}^{\infty} \frac{(-1)^k a^k}{k!} \Phi(v)^{b(k+1)-1} [1 - \Phi(v)]^{-[b(k+1)+1]} . \]

Since \( 0 < \Phi(v) < 1 \), for \( t > 0 \), using the binomial theorem, we have

\[ [1 - \Phi(v)]^{-[b(k+1)+1]} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b(k+1) + 1 + j)}{\Gamma(b(k+1) + 1 + j)} \Phi(v)^j , \]

Therefore

\[ f(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} h_{b(k+1)+j}(t) , \tag{9} \]

where

\[ h_{b(k+1)+j}(t) = (b(k+1) + j) g(t) \Phi(v)^{b(k+1)+j-1} \]

is the exponentiated BS density function with power parameter \( b(k+1) + j \) and the weights \( w_{k,j} \) are given by

\[ w_{k,j} = \frac{(-1)^k b a^{k+1} \Gamma(b(k+1) + 1 + j)}{k! j! (b(k+1) + j - 1) \Gamma(b(k+1) + 1)} , \]

with

\[ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} = 1 . \]

By integrating Equation (9), we obtain

\[ F(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} \Phi(v)^{b(k+1)+j-1} . \tag{10} \]

Then, Equation (9) shows that the WBS density can be written in terms of a mixture of exponentiated BS density function. Hence, some mathematical properties of the WBS distribution can follow directly from those properties of the exponentiated BS distribution. For example, the ordinary, inverse and factorial moments, generating function and characteristic function of the WBS distribution can be obtained directly from the exponentiated BS distribution.
4 Properties of the WBS distribution

In this section, we give some statistical properties of the WBS distribution including the moments, generating function and the quantile function.

4.1 Moments

The \( p \)th moment of the WBS random variable \( T \) can be derived from the probability weighted moments (see Cordeiro and Lemonte, 2011) of the BS distribution formally defined for \( p \) and \( r \) non-negative integers by

\[
\tau_{p,r} = \int_0^\infty t^p g(t) \Phi(v)^r \, dt. \tag{11}
\]

The integral (11) can be easily computed numerically in software such as MAPLE, MATLAB, MATHEMATICA, Ox and R. From Cordeiro and Lemonte (2011), we have an alternative representation to compute \( \tau_{p,r} \) that is

\[
\tau_{p,r} = \frac{\beta^p}{2^r} \sum_{j=1}^r \sum_{k_1,\ldots,k_j=0} A(k_1,\ldots,k_j) \times \sum_{m=0}^{2s_j+j} (-1)^m \binom{2s_j+j}{m} I\left(p + \frac{2s_j+j-2m}{2},\alpha\right),
\]

where \( s_j = k_1 + \ldots + k_j \), \( A(k_1,\ldots,k_j) = \alpha^{-2s_j-j} a_{k_1} \ldots a_{k_j} \),

\[
a_k = (-1)^k 2^{1-2k} \left[\sqrt{\pi} (2k+1) k!\right]^{-1}
\]

and \( I(p + (2s_j+j-2m)/2,\alpha) \) is calculated from Equation (4) in terms of the modified Bessel function of the third kind.

Let \( T \) be a random variable having the WBS pdf (7). The \( p \)th moment of \( T \) can be written from Equation (9) as

\[
E(T^p) = \sum_{k=0}^\infty \sum_{j=0}^\infty w_{k,j} (b(k+1) + j) \tau_{p, (b(k+1)+j-1)},
\]

where \( \tau_{p, (b(k+1)+j-1)} \) is obtained from (11). The \( r \)th moment can be computed numerically in any symbolic software, e.g. MAPLE, MATLAB and MATHEMATICA by taking in the sum a large number of summands. These algebraic software have currently the ability to deal with analytic expressions of formidable size and complexity.
4.2 Moment generating function

Let $T$ be a random variable having the WSB density function (7). The moment generating function of $T$, say $M(s) = E\left(e^{sx}\right)$, is an alternative specification of its probability distribution. We have

$$M(s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} \int_0^\infty e^{st} h_{b(k+1)+j}(t) \, dt,$$

Expanding the exponential in Taylor series, we have

$$M(s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} w_{k,j} \frac{s^p}{p!} \int_0^\infty t^p h_{b(k+1)+j-1}(t) \, dt.$$

Then

$$M(s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} w_{k,j} \frac{T_{p,b(k+1)+j-1}}{p!} s^p.$$

4.3 Quantile function, simulation and quantile measures

Let $T$ be a random variable having the WBS cdf (7). The quantile function of $T$, say $Q(u) = F^{-1}(u)$, can be easily obtained as

$$Q(u) = \beta \rho^{-1} \left( \alpha \Phi \left( 1 + \left[ -\frac{1}{a} \log \left( 1 - u \right) \right]^{\frac{1}{b}} \right) \right),$$

where $\rho^{-1}$ is the inverse of $\rho$.

Therefore, it is easy to simulate the WBS distribution. Let $U$ be a continuous uniform variable on the unit interval $(0, 1]$. Thus, using the inverse transformation method, the random variable $T$ given by

$$T = Q(U) = \beta \rho^{-1} \left( \alpha \Phi \left( 1 + \left[ -\frac{1}{a} \log \left( 1 - U \right) \right]^{\frac{1}{b}} \right) \right), \quad (12)$$

has the WBS distribution. Equation (12) may be used to generate random numbers from the WBS distribution when the parameters are known.

The shortcomings of the classical kurtosis measure are well-known. There are many heavy-tailed distributions for which this measure is infinite, so it
becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical kurtosis for several distributions. The Bowley skewness (see Kenney and Keeping, 1963) is based on quartiles

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(2/4)}{Q(3/4) - Q(1/4)},$$

and the Moors kurtosis (see Moors, 1998) is based on octiles:

$$M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

These measures are less sensitive to outliers and they exist even for distributions without moments.

5 Order statistics

In this section, the distribution of the $k$th order statistic for the WBS distribution are presented. The order statistics play an important role in reliability and life testing. Let $T_1, \ldots, T_n$ be a simple random sample from WBS distribution with cdf and pdf as in (7) and (8), respectively. Let $T_{1:n} \leq \cdots \leq T_{n:n}$ denote the order statistics obtained from this sample. In reliability literature, the $k$th order statistic, say $T_{k:n}$, denote the lifetime of an $(n-k+1)$–out–of–$n$ system which consists of $n$ independent and identically components. Then the pdf of $T_{k:n}$ is given by

$$f_{k,n}(t) = \frac{f(t)}{B(k, n-k+1)} F(t)^{k-1} [1 - F(t)]^{n-k}, \text{ for } k = 1, \ldots, n.$$ 

Since $0 < F(t) < 1$, for $t > 0$, then by using the binomial series expansion of $[1 - F(t)]^{n-k}$ given by

$$[1 - F(t)]^{n-k} = \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} F(t)^r,$$

we obtain

$$f_{k,n}(t) = \frac{1}{B(k, n-k+1)} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} f(t) F(t)^{k+r-1}. \quad (13)$$
From (6), we have

\[ F(t)^{k+r-1} = \left[ 1 - e^{-a\Phi(v)b} \right]^{k+r-1}. \]

By using the binomial series expansion, we obtain

\[ F(t)^{k+r-1} = \sum_{i=0}^{\infty} (-1)^i \binom{k+r-1}{i} e^{-a\Phi(v)b} [1 - \Phi(v)]^i. \]  

(14)

By inserting (8) and (14) in (13), we obtain

\[ f_{k,n}(t) = \frac{ab\kappa(\alpha, \beta) t^{-3/2} (t + \beta)}{B(k, n - k + 1)} \frac{\Phi(v)^{b-1}}{[1 - \Phi(v)]^b+1} \exp \left\{ -\frac{\tau (t/\beta)}{2\alpha^2} \right\} \]

\[ \times \sum_{r=0}^{n-k} \sum_{i=0}^{\infty} (-1)^{i+r} \binom{k+r-1}{i} \binom{n-k}{r} e^{-a(i+1)\Phi(v)b} [1 - \Phi(v)]^{b+1}. \]

By using the power series for the exponential function again, we have

\[ e^{-a(i+1)\Phi(v)b} = \sum_{l=0}^{\infty} \frac{(-1)^l (ai + a)^l}{l!} \Phi(v)^{bl}. \]

Then

\[ f_{k,n}(t) = \frac{g(t)}{B(k, n - k + 1)} \sum_{r=0}^{n-k} \sum_{i=0}^{\infty} (-1)^{i+r} \binom{k+r-1}{i} \binom{n-k}{r} \sum_{l=0}^{\infty} \frac{(-1)^l (ai + a)^l}{l!} \Phi(v)^{bl+b+1} [1 - \Phi(v)]^{-(b(l+1)+1)} \]

Since \( 0 < \Phi(v) < 1 \), for \( t > 0 \) and \( (b(l+1)+1) > 0 \), then

\[ [1 - \Phi(v)]^{-(b(l+1)+1)} = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(b(l+1)+1+j)}{j! \Gamma(b(l+1)+1)} \Phi(v)^j. \]

Therefore, the pdf of the \( k \)th order statistic for WBS distribution is

\[ f_{k,n}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \nu_{k,j} h_{b(k+1)+j}(t), \]  

(15)
where \( h_{b(k+1)+j}(t) = (b(l+1) + j) g(t) \Phi (v)^{b(l+1)+j-1} \) is the exponentiated BS density function with power parameter \( b(k+1) + j \) and \( w_{k,j} \) are given by

\[
\begin{align*}
v_{k,j} &= \frac{1}{B(k, n-k+1)} \sum_{r=0}^{n-k} \sum_{i=0}^{\infty} (-1)^{i+r+j} (-1)^l (ai + a)^l \times \binom{n-k}{r} \binom{k+r-1}{i} \frac{\Gamma (b(l+1)+1+j)}{\Gamma (b(l+1)+1) b(l+1)+j} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{k,j} (b(l+1)+j) \tau_{p,b(l+1)+j-1}.
\end{align*}
\]

Equation (15) reveals that the density function of the WBS order statistics is a linear combination of the exponentiated BS densities. Then, we can easily obtain the mathematical properties for \( T_{k,n} \). For example, the \( p \)th moment of \( T_{k,n} \) is

\[
E \left( T_{k,n}^p \right) = \int_0^{\infty} t^p f_{k,n}(t) dt = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{k,j} \int_0^{\infty} t^p h_{b(k+1)+j}(t) dt
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} v_{k,j} (b(l+1)+j) \tau_{p,b(l+1)+j-1}.
\]

### 6 Mean deviations

The mean deviations of \( T \) about the mean and about the median can be used as measures of spread in a population. They are given by

\[
\delta_1 = E \left( |T - \mu_1'| \right) = \int_0^{\infty} |T - \mu_1'| f(t) dt
\]

and

\[
\delta_2 = E \left( |T - m| \right) = \int_0^{\infty} |t - m| f(t) dt,
\]

respectively, where the mean \( \mu_1' \) is calculated from Equation (11) and the median \( m \) is given by \( m = Q(1/2) \). The measures \( \delta_1 \) and \( \delta_2 \) can be expressed as

\[
\delta_1 = 2\mu_1' F(\mu_1') - J(\mu_1') \quad \text{and} \quad \delta_2 = E(\left|T - m\right|) = \mu_1' - 2J(m).
\]
where \( F(t) = \Phi(c) \), \( J(q) = \int_0^q t \phi(t) \, dt \), and \( c = \alpha^{-1} \rho (\mu_1/\beta) \). From Equation (9), \( J(q) \) can be written as

\[
J(q) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} \int_0^q t \phi (b(k+1)+j) \, dt
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} \phi(q, b(k+1)+j),
\]

where

\[
\phi(q, b(k+1)+j) = \int_0^q t \phi(b(k+1)+j)(t) \, dt.
\]

From Cordeiro and Lemonte (2011), we have

\[
\phi(q, r) = \kappa(\alpha, \beta) \sum_{j=1}^{r} \binom{r}{j} \sum_{k_1, \ldots, k_j=0}^{\infty} \beta^{-(2s_j+j)/2} A(k_1, \ldots, k_j) \sum_{m=0}^{2s_j+j} (-\beta)^m
\]

\[
\times \left( \binom{2s_j+j}{m} \int_0^q t^{(2s_j+j-2m-1)/2} (t+\beta) \exp \left\{ -\frac{\tau (t/\beta)}{2\alpha^2} \right\} \, dt \right).
\]

Let

\[
D(p, q) = \int_0^q t^p \exp \left\{ -\frac{(t/\beta + \beta/t)}{2\alpha^2} \right\} \, dt
\]

\[
= \beta^{p+1} \int_0^{q/\beta} u^p \exp \left\{ -\frac{u + u^{-1}}{2\alpha^2} \right\} \, du.
\]

From Terras (1981), we can write

\[
D(p, q) = 2\beta^{p+1} K_{p+1}(\alpha^{-2}) - q^{p+1} K_{p+1} \left( \frac{\beta}{2\alpha^2 \beta}, \frac{\beta}{2\alpha^2 q} \right),
\]

where, \( K_v(z_1, z_2) \) is the incomplete Bessel function with order \( v \) and arguments \( z_1 \) and \( z_2 \). Then, we obtain

\[
\phi(q, r) = \frac{\kappa(\alpha, \beta)}{2^r} \sum_{j=1}^{r} \binom{r}{j} \sum_{k_1, \ldots, k_j=0}^{\infty} \beta^{-(2s_j+j)/2} A(k_1, \ldots, k_j) \sum_{m=0}^{2s_j+j} (-\beta)^m
\]

\[
\times \left( \binom{2s_j+j}{m} \left\{ D \left( \frac{2s_j+j-2m+1}{2}, q \right) + \beta D \left( \frac{2s_j+j-2m-1}{2}, q \right) \right\} \right),
\]

which can be calculated from the function \( D(p, q) \). Hence, we can use this expression for \( \phi(q, r) \) to compute \( J(q) \). From Equation (16), we obtain the Bonferroni and Lorenz curves defined by \( L_F(t) = J(q)/E(T) \) and \( B_F(t) = \frac{L_F(t)}{t} \), respectively. These curves have applications in economics, reliability, demography, insurance and medicine.
7 Reliability

In the stress-strength modelling, \( R = \mathbb{P}(T_2 < T_1) \) is a measure of component reliability when it is subjected to random stress \( T_2 \) and has strength \( T_1 \). The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever \( T_1 > T_2 \). The parameter \( R \) is referred to as the reliability parameter. This type of functional can be of practical importance in many applications. In this Section, we derive the reliability \( R \) when \( T_1 \) and \( T_2 \) have independent WBS(\( \alpha, \beta, a_1, b_1 \)) and WBS(\( \alpha, \beta, a_2, b_2 \)) distributions. The pdf of \( T_1 \) can be written from Equations (9) as

\[
f_1(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{1k,j} (b_1 (k + 1) + j) g(t) \Phi (v)^{b_1 (k+1)+j-1},
\]

where

\[
w_{1k,j} = \frac{(-1)^k b_1 a_1^{k+1} \Gamma (b_1 (k + 1) + 1 + j)}{k! j! (b_1 (k + 1) + j - 1) \Gamma (b_1 (k + 1) + 1)}.
\]

The cdf of \( T_2 \) can be written from (10) as

\[
F_2(t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} w_{2l,m} \Phi (v)^{b_2 (l+1)+m-1},
\]

where

\[
w_{2l,m} = \frac{(-1)^l b_2 a_2^{l+1} \Gamma (b_2 (l + 1) + 1 + m)}{l! m! (b_2 (l + 1) + m - 1) \Gamma (b_1 (l + 1) + 1)}.
\]

We have

\[
R = \mathbb{P}(T_2 < T_1) = \int_0^{\infty} f_1(t) F_2(t) dt.
\]

Then

\[
R = \sum_{k,j,l,m=0}^{\infty} w_{1k,j} w_{2l,m} (b_1 (k + 1) + j) \tau_0 b_1 (k+1) + b_2 (l+1) + j + m - 2.
\]

8 Estimation of model parameters

In this section, we present maximum likelihood estimate (MLE) of the parameters of the WBS distribution. Let \( X_i \) be a random variable following
with unknown parameter vector $\xi = (\alpha, \beta, a, b)^T$. Suppose that the data consist of $n$ independent observations $x_i$, for $i = 1, \ldots, n$. The likelihood function is given by

$$L(\xi) = (ab \kappa(\alpha, \beta))^n \prod_{i=1}^n \left( \frac{t_i^{3/2} (t_i + \beta) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}} e^{-\frac{t_i(\beta/\alpha)}{2\alpha^2}} - a \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b \right)$$

(17)

The values of the parameters that maximize the likelihood function also maximize the log-likelihood function, denoted by $\ell(\xi)$. Taking the logarithm of Equation(17), we obtain

$$\ell(\xi) = n \left[ \log a + \log b + \log \kappa(\alpha, \beta) \right] - \frac{3}{2} \sum_{i=1}^n \log t_i + \sum_{i=1}^n \log (t_i + \beta)$$

$$- \frac{1}{2\alpha^2} \sum_{i=1}^n \tau \left( \frac{t_i}{\beta} \right) + (b - 1) \sum_{i=1}^n \log [\Phi(v_i)]$$

$$- (b + 1) \sum_{i=1}^n \log [1 - \Phi(v_i)] - a \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b.$$

Now, taking the partial derivatives of the above log-likelihood function with respect to $\alpha$, $\beta$, $a$ and $b$, we obtain the components of the score vector

$$U_a(\xi) = \frac{n}{a} - \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b,$$

$$U_b(\xi) = \frac{n}{b} + \sum_{i=1}^n \log [\Phi(v_i)] - \sum_{i=1}^n \log [1 - \Phi(v_i)]$$

$$- a \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right],$$

$$U_\alpha(\xi) = -\frac{n}{\alpha} \left( 1 + \frac{2}{\alpha^2} \right) + \frac{1}{\alpha^2} \sum_{i=1}^n \left( \frac{t_i}{\beta} + \frac{\beta}{t_i} \right) - \frac{b - 1}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i)}{\Phi(v_i)}$$

$$- \frac{b + 1}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{ab}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}},$$
and
\[
U_\beta (\xi) = -\frac{n}{2\beta} + \sum_{i=1}^{n} \frac{1}{t_i + \beta} + \frac{1}{2\beta\alpha^2} \sum_{i=1}^{n} \left( \frac{t_i}{\beta} - \frac{\beta}{t_i} \right) - \frac{(b - 1)}{2\beta\alpha} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{t_i/\beta} \right) \phi (v_i)}{\Phi (v_i)} - \frac{(b + 1)}{2\beta\alpha} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{t_i/\beta} \right) \phi (v_i)}{1 - \Phi (v_i)} + \frac{ab}{2\beta\alpha} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{t_i/\beta} \right) \phi (v_i) \Phi (v_i)^{b-1}}{[1 - \Phi (v_i)]^{b+1}},
\]

where \( \phi (\cdot) \) is the standard normal density function. The MLE \( \hat{\xi} = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})^T \) of \( \xi = (\alpha, \beta, a, b)^T \) is obtained by solving the non-linear likelihood equations \( U_a (\xi) = 0 \), \( U_b (\xi) = 0 \), \( U_\alpha (\xi) = 0 \) and \( U_\beta (\xi) = 0 \). These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton–Raphson type algorithm to obtain the estimate \( \hat{\xi} \).

Under the usual regularity conditions, the known asymptotic properties of the maximum likelihood method ensure that
\[
\sqrt{n} \left( \hat{\xi} - \xi \right) \overset{d}{\to} \mathcal{N}_4 \left( \mathbf{0}, \mathbf{I}^{-1} (\xi) \right), \quad \text{as } n \to \infty,
\]

where \( \overset{d}{\to} \) means the convergence in distribution and \( \mathcal{N}_4 \left( \mathbf{0}, \mathbf{I}^{-1} (\xi) \right) \) is the multivariate normal distribution with mean vector \( \mathbf{0} = (0, 0, 0, 0)^T \) and \( 4 \times 4 \) covariance matrix \( \mathbf{I}^{-1} (\xi) \), with \( \mathbf{I} (\xi) \) is the unit expected information matrix. With this result, confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions are easily constructed. The observed Fisher information matrix \( \mathbf{I} (\xi) \) given by
\[
\mathbf{I} (\xi) = \begin{pmatrix} I_{aa} & I_{ab} & I_{aa} & I_{a\beta} \\ I_{ab} & I_{bb} & I_{ba} & I_{b\beta} \\ I_{aa} & I_{ba} & I_{aa} & I_{a\beta} \\ I_{a\beta} & I_{b\beta} & I_{a\beta} & I_{b\beta} \end{pmatrix},
\]

where the elements are given in the Appendix.

The approximate \( 100(1 - \eta)\% \) two-sided confidence intervals for \( \alpha, \beta, a \) and \( b \) are given by
\[
\hat{a} \pm Z_{1-\frac{\eta}{2}} \sqrt{\text{var} (\hat{a})}, \quad \hat{\beta} \pm Z_{1-\frac{\eta}{2}} \sqrt{\text{var} (\hat{\beta})},
\]
\[
\hat{b} \pm Z_{1-\frac{\eta}{2}} \sqrt{\text{var} (\hat{b})} \quad \text{and} \quad \hat{\alpha} \pm Z_{1-\frac{\eta}{2}} \sqrt{\text{var} (\hat{\alpha})},
\]
respectively, where \( Z_a \) is the upper \( a \)th quantile of the standard normal distribution and the \( \text{var}(\cdot) \)'s are the diagonal elements of \( I^{-1}(\hat{\xi}) \).

9 Application

In this section, we present two applications of the WBS distribution in two real data sets to illustrate its usefulness. The first real data set corresponds to an uncensored data set from Nichols and Padgett (2006) on breaking stress of carbon fibres (in Gba). The data are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. The second real data set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes (see Proschan, 1963). The data are given as follows: 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

We compare the results with the BBS, MwBS, McBS, EBS and BS distributions (see Section 1). We estimate the model parameters of the distributions by the method of maximum likelihood. We use the statistical software R for all the computations (Ihaka and Gentleman, 1996). Many maximization methods exist in R Packages such as NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), SANN (Simulated-Annealing), NM (Nelder-Mead) and L-BFGS-B.

The model selection is carried out using the AIC (Akaike information criterion), the CAIC (consistent Akaike information criteria), the BIC (Bayesian
information criterion), the HQIC (Hannan-Quinn information criterion) and K-S (Kolmogorov-Smirnov statistic). These statistics are given by

\[
AIC = -2\hat{\ell} + 2p, \quad BIC = -2\hat{\ell} + p \log (n),
\]

\[
CAIC = -2\hat{\ell} + \frac{2pn}{n-p-1}, \quad HQIC = -2\hat{\ell} + 2p \log (\log (n)),
\]

and

\[
K-S = \max_{1 \leq i \leq n} \left\{ F(t_i) - \frac{i-1}{n}, \frac{i}{n} - F(t_i) \right\},
\]

where \( \hat{\ell} \) denotes the log-likelihood function evaluated at the MLEs, \( p \) is the number of parameters and \( n \) is the sample size. Also, we use the Cramér–von Mises (\( W^* \)) and Anderson–Darling (\( A^* \)) statistics (for more detail on \( W^* \) and \( A^* \) see Chen and Balakrishnan, 1995). The better distribution to fit the data corresponds to smaller values of \(-\hat{\ell}, AIC, BIC, CAIC, HQIC, K-S, W^* \) and \( A^* \).

Tables 1 and 3 list the MLEs and the corresponding standard errors (in parentheses) of the model parameters for the first data set and for the second data set respectively. The statistics \(-\hat{\ell}, AIC, BIC, CAIC, HQIC, K-S, W^* \) and \( A^* \) are listed in tables 2 and 4 for all the distribution. These results show that the WBS distribution has the lowest values of \(-\hat{\ell}, AIC, BIC, CAIC, HQIC, K-S, W^* \) and \( A^* \). Therefore, the WBS distribution gives an excellent fit than the others models for both data sets. In order to assess if the model is appropriate, the plots of the densities together with the data histogram and cdfs with empirical distribution function are given in Figures 3 (for the first data set) and 4 (for the second data set). From these plots, we can conclude that the WBS model yield the best fits and hence can be adequate for modeling these data.

| Distribution | \( \alpha \) | \( \beta \) | \( a \) | \( b \) | \( c \) |
|--------------|--------------|--------------|--------|--------|---------|
| WBS          | 4.1267       | 0.2332       | 21.4238| 28.8716| –       |
| BBS          | 1.0781       | 44.2830      | 0.2182 | 271.1031| –       |
| KwBS         | 4.6235       | 0.1657       | 17.1037| 21.1157| –       |
| McBS         | 5.7341       | 0.0467       | 17.8754| 5.9197 | 43.4241 |
| EBS          | 0.0991       | 5.3261       | 0.0201 | –      | –       |
| BS           | 0.4622       | 2.3660       | –      | –      | –       |
Table 2: The statistics: -log-likelihood, AIC, CAIC, BIC, HQIC, K-S, $W^*$ and $A^*$ for the first data set.

| Distribution | $-\ell$     | AIC    | CAIC   | BIC    | HQIC   | K-S    | $W^*$  | $A^*$  |
|--------------|-------------|--------|--------|--------|--------|--------|--------|--------|
| WBS          | 132.25      | 280.71 | 272.71 | 306.38 | 277.74 | 0.0089 | 0.0222 | 0.1208 |
| McBS         | 136.17      | 282.34 | 282.67 | 298.52 | 288.89 | 0.0311 | 0.0662 | 0.3907 |
| KwBS         | 140.02      | 288.04 | 288.26 | 300.99 | 286.67 | 0.0332 | 0.0683 | 0.4002 |
| BBS          | 143.86      | 295.61 | 295.83 | 308.56 | 294.23 | 0.0923 | 0.1199 | 0.6845 |
| EBS          | 145.75      | 297.51 | 297.64 | 307.22 | 301.44 | 0.1264 | 0.1943 | 1.0488 |
| BS           | 150.06      | 304.12 | 304.18 | 310.59 | 306.74 | 0.1816 | 0.2976 | 1.6182 |

Figure 3. (a) Observed histogram and the fitted pdfs (b) Empirical cdf and the fitted cdfs; for the first data set.

Table 3: MLEs for the second data set.

| Distribution | $\alpha$  | $\beta$  | $a$     | $b$     | $c$     |
|--------------|-----------|----------|---------|---------|---------|
| WBS          | 1.1412    | 2.7370   | 0.4838  | 1.8206  | $-$     |
| BBS          | 1.5487    | 9.5752   | 1.7053  | 0.3941  | $-$     |
| KwBS         | 2.0459    | 1.1598   | 18.2213 | 22.3679 | $-$     |
| McBS         | 3.1486    | 0.1447   | 8.5211  | 2.9547  | 30.5547 |
| EBS          | 1.0243    | 21.8791  | 0.0847  | $-$     | $-$     |
| BS           | 1.5147    | 41.3240  | $-$     | $-$     | $-$     |
Table 4: The statistics: -log-likelihood, AIC, CAIC, BIC, HQIC, K-S, $W^*$ and $A^*$ for the second data set.

| Distribution | $-\ell$  | AIC    | CAIC   | BIC    | HQIC   | K-S    | $W^*$ | $A^*$ |
|--------------|----------|--------|--------|--------|--------|--------|-------|-------|
| WBS          | 1016.12  | 2040.24| 2040.66| 2050.66| 2044.46| 0.0120 | 0.0087| 0.0912|
| McBS         | 1021.26  | 2052.53| 2053.17| 2065.55| 2067.80| 0.0287 | 0.0527| 0.2294|
| KwBS         | 1027.98  | 2063.97| 2064.39| 2074.39| 2068.19| 0.0332 | 0.0556| 0.2340|
| BBS          | 1035.38  | 2076.75| 2077.00| 2084.57| 2079.91| 0.0991 | 0.0873| 0.5789|
| EBS          | 1038.48  | 2082.96| 2083.21| 2090.78| 2086.13| 0.2301 | 0.1149| 0.8551|
| BS           | 1041.84  | 2087.68| 2087.80| 2092.89| 2089.79| 0.3216 | 0.1258| 0.9478|

Figure 4. (a) Observed histogram and the fitted pdfs (b) Empirical cdf and the fitted cdfs; for the second data set.

10 Conclusions

We have introduced, based on the Weibull–G class (Bourguignon et al., 2014), a new four-parameter distribution called the Weibull-Birnbaum-Saunders (WBS) distribution that extends the Birnbaum-Saunders (BS) distribution. A mathematical treatment of the new distribution including expansions for the density function, moments, generating function, order statistics, quantile
function, mean deviations and reliability have been provided. The estimation of the parameters has been approached by maximum likelihood and the observed Fisher information matrix has been derived. Two data sets are used to illustrate the application of the proposed models and the result are compared to other existing models. The results of these applications suggest that the new model provides consistently better fit than other models. We hope that the proposed distribution may attract wider applications in survival analysis for modeling positive real data sets.

11 Appendix

The elements of the observed Fisher information matrix $I(\xi)$ consist of the expected values of the second partial derivatives of the negative log-likelihood. The elements of the $I(\xi)$ are

$$I_{aa} = \frac{n}{a^2},$$

$$I_{ab} = \sum_{i=1}^{n} \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right]^b \log \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right],$$

$$I_{aa} = -\frac{b}{\alpha} \sum_{i=1}^{n} \frac{v_i \phi(v_i) \Phi(v_i)^{b-1}}{[1-\Phi(v_i)]^{b+1}},$$

$$I_{a\beta} = -\frac{b}{2\beta} \sum_{i=1}^{n} \tau \left( \frac{\tau_i}{\beta} \right) \phi(v_i) \Phi(v_i)^{b-1},$$

$$I_{bb} = \frac{n}{b^2} + a \sum_{i=1}^{n} \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right]^b \log \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right]^2,$$

$$I_{ba} = \sum_{i=1}^{n} \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right]^b \log \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right],$$

$$I_{ba} = \frac{1}{\alpha} \sum_{i=1}^{n} \frac{v_i \phi(v_i)}{\Phi(v_i)} + \frac{1}{\alpha} \sum_{i=1}^{n} \frac{v_i \phi(v_i)}{1-\Phi(v_i)} - \frac{a}{\alpha} \sum_{i=1}^{n} \frac{v_i \phi(v_i) \Phi(v_i)^{b-1}}{[1-\Phi(v_i)]^{b+1}} \left\{ 1 + b \log \left[ \frac{\Phi(v_i)}{1-\Phi(v_i)} \right] \right\},$$

[21]
\[
I_{b\beta} = \frac{1}{2\beta\alpha} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{\frac{t_i/\beta}{\Phi(v_i)}} \right) \phi(v_i)}{\Phi(v_i)} + \frac{1}{2\beta\alpha} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{\frac{t_i/\beta}{1 - \Phi(v_i)}} \right) \phi(v_i)}{1 - \Phi(v_i)}
- \frac{1}{2\beta\alpha} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{\frac{t_i/\beta}{1 - \Phi(v_i)}} \right) \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}} \left\{ 1 + b \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right] \right\},
\]

\[
I_{\alpha\alpha} = -\frac{n}{\alpha^2} - \frac{6n}{\alpha^4} + \frac{3}{\alpha^4} \sum_{i=1}^{n} \left( \frac{t_i}{\beta} + \beta \right) - \frac{2(b-1)}{\alpha^2} \sum_{i=1}^{n} \frac{v_i \phi(v_i)}{\Phi(v_i)} + \frac{2(b+1)}{\alpha^2} \sum_{i=1}^{n} \frac{v_i \phi(v_i)}{1 - \Phi(v_i)} - \frac{(b+1)}{\alpha^3} \sum_{i=1}^{n} \frac{v_i^2 \phi(v_i)}{1 - \Phi(v_i)} - \frac{(b-1)}{\alpha^2} \sum_{i=1}^{n} \frac{v_i^2 \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}} - \frac{ab}{\alpha^2} \sum_{i=1}^{n} \frac{v_i \phi(v_i)^2 \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+2}},
\]

\[
I_{\alpha\beta} = \frac{1}{\alpha^3 \beta} \sum_{i=1}^{n} \left( \frac{t_i}{\beta} - \beta \right) - \frac{2(b-1)}{\alpha^2} \sum_{i=1}^{n} \left\{ \frac{\alpha v_i \phi(v_i)}{\Phi(v_i)} + \frac{v_i^2 \phi(v_i)}{\Phi(v_i)} - \frac{\alpha v_i^2 \phi(v_i)^2}{\Phi(v_i)^2} \right\}
+ \frac{(b-1)}{\beta \alpha^2} \sum_{i=1}^{n} \left\{ \frac{\alpha v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{v_i^2 \phi(v_i)}{1 - \Phi(v_i)} - \frac{\alpha v_i^2 \phi(v_i)^2}{[1 - \Phi(v_i)]^2} \right\}
- \frac{ab}{\beta \alpha^2} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{\frac{t_i/\beta}{1 - \Phi(v_i)}} \right) \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}}
+ \frac{ab}{\beta \alpha^2} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{\frac{t_i/\beta}{1 - \Phi(v_i)}} \right) \phi(v_i) \Phi(v_i)^{b-2}}{[1 - \Phi(v_i)]^{b+1}}
+ \frac{ab}{\beta \alpha^2} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{\frac{t_i/\beta}{1 - \Phi(v_i)}} \right) \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+2}},
\]

and
\[ I_{\beta\beta} = -\frac{n}{2\beta^2} + \sum_{i=1}^{n} \frac{1}{(t_i + \beta)^2} + \frac{1}{2\alpha^2\beta} \sum_{i=1}^{n} t_i - \frac{(b-1)}{2\alpha\beta^2} \sum_{i=1}^{n} \tau \left( \frac{\sqrt{t_i/\beta}}{\Phi(v_i)} \right) \]

\[ + \frac{(b+1)}{2\alpha\beta^2} \sum_{i=1}^{n} \tau \left( \frac{\sqrt{t_i/\beta}}{1 - \Phi(v_i)} \right) \]

\[ + \frac{(b-1)}{4\alpha\beta^2} \sum_{i=1}^{n} \left\{ \frac{-\alpha v_i \phi(v_i)}{\Phi(v_i)} + \frac{v_i \tau \left( \sqrt{t_i/\beta} \right)^2 \phi(v_i)}{\alpha \Phi(v_i)^2} + \frac{v_i \tau \left( \sqrt{t_i/\beta} \right)^2 \phi(v_i)^2}{\alpha \Phi(v_i)^2} \right\} \]

\[ - \frac{(b+1)}{4\alpha\beta^2} \sum_{i=1}^{n} \left\{ \frac{-\alpha v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{v_i \tau \left( \sqrt{t_i/\beta} \right)^2 \phi(v_i)}{\alpha \left[ 1 - \Phi(v_i) \right]} - \frac{v_i \tau \left( \sqrt{t_i/\beta} \right)^2 \phi(v_i)^2}{\alpha \left[ 1 - \Phi(v_i) \right]^2} \right\} \]

\[ + \frac{ab}{2\alpha\beta^2} \sum_{i=1}^{n} \tau \left( \frac{\sqrt{t_i/\beta}}{\Phi(v_i)} \right) \Phi(v_i)^{b-1} \]

\[ - \frac{ab}{2\alpha^2\beta^2} \sum_{i=1}^{n} \left[ \tau \left( \frac{\sqrt{t_i/\beta}}{\Phi(v_i)} \right)^2 \phi(v_i)^2 \Phi(v_i)^{b-1} \right] \]

\[ + \frac{ab(b-1)}{2\alpha\beta^2} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{t_i/\beta} \right)^2 \phi(v_i)^2 \Phi(v_i)^{b-1}}{\left[ 1 - \Phi(v_i) \right]^{b+1}} \]

\[ - \frac{ab(b+1)}{2\alpha\beta^2} \sum_{i=1}^{n} \frac{\tau \left( \sqrt{t_i/\beta} \right)^2 \phi(v_i)^2 \Phi(v_i)^{b-1}}{\left[ 1 - \Phi(v_i) \right]^{b+2}}. \]

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