A new fractional derivative involving the normalized sinc function without singular kernel

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Abstract

In this paper, a new fractional derivative involving the normalized sinc function without singular kernel is proposed. The Laplace transform is used to find the analytical solution of the anomalous heat-diffusion problems. The comparative results between classical and fractional-order operators are presented. The results are significant in the analysis of one-dimensional anomalous heat-transfer problems.

Key words: Fractional derivative, anomalous heat diffusion, integral transform, analytical solution

1 Introduction

In recent years, fractional derivatives (FDs) in the sense of Caputo type have used to describe anomalous behaviors of diffusive phenomena in mathematical physics involving different kernels, such as the power-law [1], exponential
[2], Mittag-Leffler [3], stretched exponential [4], and stretched Mittag-Leffler [5] functions. For example, the fractional diffusion-wave, in the power-law function kernel was considered in [6]. The numerical solution for the space-fractional diffusion equation was presented in [7]. The Cauchy problem for the time-fractional diffusion equations was investigated in [8]. With the help of the FD involving the exponential-function kernel, the heat-diffusion problem with respect to a non-singular fading memory was proposed in [9]. The heat transfer problem within the non-singular second grade fluid was discussed in [10]. The non-singular unsteady flow of the ordinary couple stress fluid was studied in [11]. With the use of the FD involving the stretched Mittag-Leffler-function kernel, the Irving–Mullineux oscillator [12] and the Allen-Cahn equation [13] were also analyzed. For more details see [14,15].

FDs in the sense of Riemann–Liouville type were developed in [16,17,18]. We can mention the studies about not only the Fokker-Planck [19] and the diffusion [20,21,22,23] equations, but also the wave propagation [24]. Furthermore, the Chen’s system of the Riemann–Liouville type [25], the monotone iterative method for neutral fractional differential equations [26] and the time-fractional-order Harry-Dym equation [27] were also discussed. Readers can find the more details about the distinct versions of FDs in [28].

The normalized sinc function, structured by Whittaker in [29], and its properties were considered in [30]. Furthermore, the Fourier [31], Laplace [31] and Sumudu [32] transforms of the NSF were formulated. However, the FD involving the normalized sinc function without singular kernel has not proposed. Motivated by the idea, the present article derives a new FD with respect to the normalized sinc function without singular kernel. Furthermore, based on the new concept it is considered the applications in one-dimensional anomalous heat-transfer problems.

The structure of the present paper is as follows. In Section 2, a new FD with respect to the normalized sinc function without singular kernel is presented. In Section 3, the anomalous heat-diffusion models and their solutions are analyzed by means of the Laplace transform. Finally, the conclusion is outlined in Section 4.

2 Preliminaries, definitions and integral transforms

In this section, we derive the FD involving the normalized sinc function without singular kernel.
2.1 A new FD involving the normalized sinc function without singular kernel

**Definition 1** The normalized sinc function is defined by [29,30]:

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x},
\]

where \(x \in \mathbb{R}\).

If \(\varphi(x)\) is any smooth function with compact support, where \(x \in \mathbb{R}\), then [30]

\[
\lim_{\varpi \to 0} \frac{1}{\varpi} \sin \left( \frac{x}{\varpi} \right) = \lim_{\varpi \to 0} \frac{\sin \left( \frac{\varpi x}{\varpi} \right)}{\varpi} = \delta(x),
\]

where \(\sin c(0) = 1\),

\[
\lim_{\varpi \to 0} \int_{-\infty}^{\infty} \varphi(x) \frac{\sin \left( \frac{x}{\varpi} \right)}{\varpi} dx = \varphi(0).
\]

**Definition 2** Let \(\Pi(x) \in H^1(a,b)\) and \(b > a\). A new FD involving the normalized sinc kernel of the function \(\Pi(\mu)\) of order \(\varpi\) \((\varpi \in (0,1))\) is defined as:

\[
a D^{(\varpi)}_\mu \Pi(\mu) = \frac{\varpi \varphi(\varpi)}{1-\varpi} \int_{a}^{\mu} \sin \left( -\frac{\varpi (\mu - x)}{1-\varpi} \right) \Pi^{(1)}(x) \, dx,
\]

where \(a \in (-\infty, \mu)\), and \(\varphi(\varpi)\) is a normalization constant depending on \(\varpi\) such that \(\varphi(0) = \varphi(1) = 1\).

Following Eq.(1), we obtain

\[
\lim_{\varpi \to 0} \frac{1}{\varpi} \sin \left( \frac{\mu - x}{\varpi} \right) = \lim_{\varpi \to 0} \frac{\sin \left( \frac{\pi (\mu - x)}{\varpi} \right)}{\pi (\mu - x)} = \delta(\mu - x),
\]

where \(\varphi(x)\) is any smooth function with compact support where \(x \in \mathbb{R}\) such that

\[
\lim_{\varpi \to 0} \int_{-\infty}^{\infty} \varphi(x) \frac{1}{\varpi} \sin \left( \frac{x - \mu}{\varpi} \right) dx = \varphi(\mu).
\]

Thus, we have

\[
\lim_{\varpi \to 0} a D^{(\varpi)}_\mu \Pi(\mu) = \lim_{\varpi \to 0} \frac{\varpi \varphi(\varpi)}{1-\varpi} \int_{a}^{\mu} \sin \left( -\frac{\varpi (\mu - x)}{1-\varpi} \right) \Pi^{(1)}(x) \, dx
\]

\[=
\left( \lim_{\varpi \to 0} \varphi(\varpi) \right) \int_{a}^{\mu} \delta(\mu - x) \Pi^{(1)}(x) \, dx
\]

\[=
\Pi^{(1)}(x).
\]
When
\[
\lim_{\omega \to 1} \varphi (\omega) \left[ \operatorname{sinc} \left( -\frac{\omega (\mu - x)}{1 - \omega} \right) \right] = \lim_{\omega \to 1} \varphi (\omega) \frac{\omega}{1 - \omega} \frac{\sin \left( \frac{\pi (\mu - x)}{1 - \omega} \right)}{\pi (\mu - x)} = \lim_{\omega \to 1} 1,
\]
we have
\[
\lim_{\omega \to 1} a D^{(\omega)} \mu (\mu) = \lim_{\omega \to 1} \frac{\mu}{1 - \omega} \int_a^\mu \operatorname{sinc} \left( -\frac{\omega (\mu - x)}{1 - \omega} \right) \Pi^{(1)} (x) \, dx
\]
\[
= \lim_{\omega \to 1} \int_a^\mu \Pi^{(1)} (x) \, dx
\]
\[
= \Pi (\mu) - \Pi (a).
\]  

For \( n \geq 1 \) and \( \omega \in (0, 1) \), the FD \( D^{(n+\omega)} \mu (\mu) \) of order \( n + \omega \) is defined as:
\[
a D^{(n+\omega)} \mu (\mu) := a D^{(n)} \mu \left( a D^{(\omega)} \mu (\mu) \right).
\]  

**Property 1 (T1)** \( a D^{(\omega)} \mu \theta = 0 \), where \( \theta \) is a constant;

\( (T2) \ a D^{(\omega)} \mu \mu = \frac{\omega \varphi (\omega)}{1 - \omega} \int_0^\mu \operatorname{sinc} \left( -\frac{\omega (\mu - x)}{1 - \omega} \right) \, dx. \)

**Proof.** We have from Eq.(5) that
\[
a D^{(\omega)} \mu \theta = \frac{\omega \varphi (\omega)}{1 - \omega} \int_0^\mu \operatorname{sinc} \left( -\frac{\omega (\mu - x)}{1 - \omega} \right) \theta^{(1)} \, dx = 0.
\]  

We have, by using the definition Eq.(5),
\[
a D^{(\omega)} \mu = \frac{\omega \varphi (\omega)}{1 - \omega} \int_0^\mu \operatorname{sinc} \left( -\frac{\omega (\mu - x)}{1 - \omega} \right) \, dx = \frac{\omega \varphi (\omega)}{1 - \omega} \int_0^\mu \operatorname{sinc} \left( \frac{\omega x}{1 - \omega} \right) \, dx.
\]  

\[
2.2 \text{ Integral transforms of the new FD involving the normalized sinc function without singular kernel}
\]

Here, we have [31]
\[
\mathcal{R} \{ \operatorname{sinc} (x) \} = \mathcal{R} \left\{ \frac{\sin (\pi x)}{\pi x} \right\} = \sqrt{\frac{1}{2\pi}} H (\pi - |\xi|)
\]  

\[\square\]
such that

\[ \mathcal{N} \left\{ \text{sinc} \left( -\frac{\omega x}{1-\omega} \right) \right\} = \mathcal{N} \left\{ \frac{\sin \left( -\frac{\omega x}{1-\omega} \right)}{-\frac{\omega x}{1-\omega}} \right\} = -\sqrt{\frac{1-\omega}{2\pi}} H \left( -\frac{\omega x}{1-\omega} - |\xi| \right) = \sqrt{\frac{1-\omega}{2\pi}} H \left( \frac{\omega x}{1-\omega} + |\xi| \right), \]

(15)

where \( \mathcal{N} \) is the Fourier transform operator [31], and \( H(x) \) is the Heaviside function [31].

The Fourier transform of Eq.(5) can be written as

\[ \mathcal{N} \left\{ 0 D^{(\omega)}_{\mu} \Pi (\mu) \right\} = \mathcal{N} \left\{ \frac{\omega x (\omega)}{1-\omega} \int_{0}^{\mu} \text{sinc} \left( -\frac{\omega (\mu-x)}{1-\omega} \right) \Pi^{(1)} (x) \, dx \right\} = \frac{\omega x (\omega)}{1-\omega} \mathcal{N} \left\{ \text{sinc} \left( -\frac{\omega x}{1-\omega} \right) \right\} \mathcal{N} \left\{ \Pi^{(1)} (x) \right\} = \frac{\omega x (\omega)}{1-\omega} \left[ \sqrt{\frac{1-\omega}{2\pi}} H \left( \frac{\omega \pi}{1-\omega} + |\xi| \right) \right] [i \xi \Pi (\xi)] = i \xi \sqrt{\frac{1}{2\pi}} \varphi (\omega) H \left( \frac{\omega \pi}{1-\omega} + |\xi| \right) \Pi (\xi), \]

(16)

where \( \mathcal{N} \left\{ \Pi (\mu) \right\} = \Pi (\xi). \)

Similarly, we have [31]

\[ \mathfrak{I} \left\{ \text{sinc} (x) \right\} = \mathfrak{I} \left\{ \frac{\sin (\pi x)}{\pi x} \right\} = \frac{1}{\pi} \tan^{-1} \left( \frac{\pi}{s} \right) \]

(17)

such that

\[ \mathfrak{I} \left\{ \text{sinc} \left( -\frac{\omega x}{1-\omega} \right) \right\} = \mathfrak{I} \left\{ \frac{\sin \left( -\frac{\omega x}{1-\omega} \right)}{-\frac{\omega x}{1-\omega}} \right\} = \frac{1}{\omega x (1-\omega)} \tan^{-1} \left( \frac{\omega x}{s} \right) \]

(18)

where \( \mathfrak{I} \) is the Laplace transform operator [31].

From Eq.(18) the Laplace transform of Eq.(5) can be given by:

\[ \mathfrak{I} \left\{ 0 D^{(\omega)}_{\mu} \Pi (\mu) \right\} = \mathfrak{I} \left\{ \frac{\omega x (\omega)}{1-\omega} \int_{0}^{\mu} \text{sinc} \left( -\frac{\omega (\mu-x)}{1-\omega} \right) \Pi^{(1)} (x) \, dx \right\} = \frac{\omega x (\omega)}{1-\omega} \mathfrak{I} \left\{ \text{sinc} \left( -\frac{\omega x}{1-\omega} \right) \right\} \mathfrak{I} \left\{ \Pi^{(1)} (x) \right\} = \frac{\omega x (\omega)}{\pi} \tan^{-1} \left( \frac{\omega x}{s} \right) (s \Pi (s) - \Pi (0)), \]

(19)

where \( \mathfrak{I} \left\{ \Pi (\mu) \right\} = \Pi (s). \)
As a direct result, we have \([32]\)

\[
\mathcal{R} \{ \text{sinc} (x) \} = \mathcal{R} \left\{ \frac{\sin (\pi x)}{\pi x} \right\} = \frac{\tan^{-1} (\pi \zeta)}{\pi \zeta}
\]  

such that

\[
\mathcal{R} \left\{ \text{sinc} \left( -\frac{\omega}{1 - \omega} x \right) \right\} = \Im \left\{ \frac{\sin \left( -\frac{\omega}{1 - \omega} \pi x \right)}{-\frac{\omega \pi x}{1 - \omega}} \right\} = \frac{\tan^{-1} \left( \frac{\omega \pi \zeta}{1 - \omega} \right)}{-\frac{\omega \pi \zeta}{1 - \omega}},
\]

where \(\mathcal{R}\) is the Sumudu transform operator \([32]\).

Thus, we have from Eq.(13) that

\[
\mathcal{R} \left\{ 0D_{\mu}^{(\omega)} \Pi (\mu) \right\} = \mathcal{R} \left\{ \frac{\omega}{1 - \omega} \int_0^{\mu} \text{sinc} \left( -\frac{\omega}{1 - \omega} \pi x \right) \Pi^{(1)} (x) \, dx \right\} = \frac{\omega}{1 - \omega} \mathcal{R} \left\{ \text{sinc} \left( -\frac{\omega}{1 - \omega} x \right) \right\} \mathcal{R} \left\{ \Pi^{(1)} (x) \right\} = \frac{\omega}{\pi \zeta} \tan^{-1} \left( \frac{\omega \pi \zeta}{1 - \omega} \right) \left( \Pi(\zeta) - \Pi(0) \right) \zeta,
\]

where \(\mathcal{R} \{ \Pi (\mu) \} = \Pi (\zeta)\).

3 Modelling the anomalous heat-diffusion problems

In this section, we model the anomalous heat-diffusion problems involving fractional-time and -space derivatives of the normalized sinc function without singular kernel.

Example 1

The anomalous heat-diffusion within the fractional-time derivative of the normalized sinc function without singular kernel is written as:

\[
0D_{\tau}^{(\omega)} \Pi (\mu, \tau) = \kappa \frac{\partial^2 \Pi (\mu, \tau)}{\partial \mu^2}, \quad \mu > 0, \ \tau > 0,
\]

subjected to the initial and boundary conditions:

\[
\Pi (\mu, 0) = 0, \ \mu > 0,
\]

\[
\Pi (0, \tau) = \lambda (\tau), \ \tau > 0,
\]

\[
\Pi (\mu, \tau) \to 0, \ \text{as} \ \mu \to \infty, \ \tau > 0,
\]

where \(\kappa\) is the thermal diffusivity.
With the aid of Eq.(19), Eq.(23) can be transferred into
\[ \frac{\varphi (\omega)}{\pi} \tan^{-1} \left( \frac{\omega \pi}{1-\omega} \right) (s\Pi (\mu, s) - \Pi (\mu, 0)) = \kappa \frac{d^2\Pi (\mu, s)}{d\mu^2}. \]  
(27)

From Eq.(24) we have the following:
\[ \frac{d^2\Pi (\mu, s)}{d\mu^2} = \frac{\varphi (\omega)s}{\pi\kappa} \tan^{-1} \left( \frac{\omega \pi}{1-\omega} \right) \Pi (\mu, s), \]  
(28)

which leads to
\[ \Pi (\mu, s) = \Omega_1 \exp (-\mu \sqrt{H}) + \Omega_2 \exp (\mu \sqrt{H}), \]  
(29)

where \( \Omega_1 \) and \( \Omega_2 \) are two unknown constants and
\[ H = \frac{\varphi (\omega)s}{\pi\kappa} \tan^{-1} \left( \frac{\omega \pi}{1-\omega} \right). \]  
(30)

In view of Eq.(25) and Eq.(26), we have
\[ \Omega_2 = 0 \]  
(31)

such that
\[ \Pi (\mu, s) = \lambda (s) \exp (-\mu \sqrt{H}), \]  
(32)

where \( \Im \{\lambda (\mu)\} = \lambda (s) \).

Thus, the Laplace transform solution of Eq.(23) is:
\[ \Pi (\mu, s) = \lambda (s) \exp \left( -\frac{\varphi (\omega)s}{\pi\kappa} \tan^{-1} \left( \frac{\omega \pi}{1-\omega} \right) \mu \right). \]  
(33)

**Example 2**

The anomalous heat-diffusion within the fractional-space derivative of the normalized sinc function without singular kernel is
\[ \frac{\partial \Pi (\mu, \tau)}{\partial \tau} = \kappa_0 D^{(1)}_\mu \left( 0 D^{(\omega)}_\mu \Pi (\mu, \tau) \right), \mu > 0, \tau > 0, \]  
(34)

with the initial and boundary conditions:
\[ \Pi (\mu, 0) = 0, \mu > 0, \]  
(35)
\[ \Pi (0, \tau) = \lambda (\tau), \tau > 0, \]  
(36)
\[ \Pi (\mu, 0) \to 0, \text{ as } \mu \to \infty, \tau > 0, \]  
(37)
where $\kappa$ is the thermal diffusivity, and

$$
0D^{(1)}_\mu \left(0D^{(w)}_\mu \Pi(\mu, \tau)\right) = \frac{\varphi(\varpi)}{1-\varpi} \frac{\partial}{\partial \mu} \int_0^\mu \text{sinc} \left(-\varpi \frac{\mu-x}{1-\varpi}\right) \Pi^{(1)}(x, \tau) \, dx.
$$

With the help of Eq.(19) and Eq.(35), Eq.(34) can be written as:

$$
0D^{(1)}_\mu \left(0D^{(w)}_\mu \Pi(\mu, s)\right) = \frac{s}{\kappa} \Pi(\mu, s),
$$

where

$$
0D^{(1)}_\mu \left(0D^{(w)}_\mu \Pi(\mu, s)\right) = \frac{\varphi(\varpi)}{1-\varpi} \frac{\partial}{\partial \mu} \int_0^\mu \text{sinc} \left(-\varpi \frac{\mu-x}{1-\varpi}\right) \Pi^{(1)}(x, s) \, dx.
$$

By the integration of Eq.(39) we have

$$
0D^{(w)}_\mu \Pi(\mu, s) = \frac{\varphi(\varpi)}{1-\varpi} \int_0^\mu \text{sinc} \left(-\varpi \frac{\mu-x}{1-\varpi}\right) \Pi^{(1)}(x, s) \, dx = \int_0^\mu \frac{s}{\kappa} \Pi(x, s) \, dx + \Theta,
$$

where $\Theta$ is a constant.

By taking the Sumudu transform operator with $\mu$ and $\Theta = 0$, we have

$$
\frac{\varphi(\varpi)}{1-\varpi} \int_0^\mu \text{sinc} \left(-\varpi \frac{\mu-x}{1-\varpi}\right) \Pi^{(1)}(x, s) \, dx = \int_0^\mu \frac{s}{\kappa} \Pi(x, s) \, dx,
$$

which implies that

$$
\frac{\varphi(\varpi)}{\pi^2 \zeta^2} \tan^{-1} \left(\frac{\varphi \pi \zeta}{1-\varpi}\right) \left(\Pi(\zeta, s) - \Pi(0, s)\right) = \frac{s \zeta}{\kappa} \Pi(\zeta, s).
$$

From Eq.(35) and Eq.(43), we have the following:

$$
\frac{\varphi(\varpi)}{\pi^2 \zeta^2} \tan^{-1} \left(\frac{\varphi \pi \zeta}{1-\varpi}\right) \left(\Pi(\zeta, s) - \lambda(s)\right) = \frac{s \zeta}{\kappa} \Pi(\zeta, s).
$$

Thus, we have

$$
\Pi(\zeta, s) = \frac{\varphi(\varpi) \lambda(s)}{\pi \zeta^2} \tan^{-1} \left(\frac{\varphi \pi \zeta}{1-\varpi}\right) - \frac{s \zeta}{\kappa}
$$

where $\mathcal{R}\{\Pi(\mu, s)\} = \Pi(\zeta, s)$ represents the Sumudu transform operator [32].

From Eq.(45), the Laplace transform solution of Eq.(23) is:

$$
\Pi(\mu, s) = \mathcal{R}^{-1}\left\{\frac{\varphi(\varpi) \lambda(s)}{\pi \zeta^2} \tan^{-1} \left(\frac{\varphi \pi \zeta}{1-\varpi}\right) - \frac{s \zeta}{\kappa}\right\},
$$

where $s = \mu \pi \zeta$.
where $R^{-1}\{\Pi(\zeta,s)\} = \Pi(\mu,s)$ represents the inverse Sumudu transform operator [30].

When $\tau = 0$, Eq.(33) and Eq.(46) become the Laplace transform solution of the classical heat-diffusion equation [33]:

$$\Pi(\mu,s) = \lambda(s) \exp \left(-\sqrt{\frac{s}{\kappa\mu}}\right).$$

which is in agreement with the result in [31].

4 Conclusions

In the present study, we addressed a new FD in respect to the normalized sinc function without singular kernel. Moreover, the Fourier, Laplace and Sumudu transforms of the FD operator and the Laplace–transform solutions of the anomalous heat-diffusion equations were considered. The analytical solutions of the classical and anomalous heat-diffusion equations in the form of the Laplace transform were also compared. The new formulation may be used to support a new perspective for describing the anomalous behaviors in mathematical physics.

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