A DUALITY METHOD IN PREDICTION THEORY OF MULTIVARIATE STATIONARY SEQUENCES

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Abstract. Let $W$ be an integrable positive Hermitian $q \times q$-matrix valued function on the dual group of a discrete abelian group $G$ such that $W^{-1}$ is integrable. Generalizing results of T. Nakazi and of A. G. Miamee [N] and M. Pourahmadi [MP] for $q = 1$ we establish a correspondence between trigonometric approximation problems in $L^2(W)$ and certain approximation problems in $L^2(W^{-1})$. The result is applied to prediction problems for $q$-variate stationary processes over $G$, in particular, to the case $G = \mathbb{Z}$. 
1. Introduction

In 1984 T. Nakazi \[N\] introduced a new idea into prediction theory of univariate weakly stationary sequences. Under the additional assumption that such a sequence has an absolutely continuous spectral measure and its spectral density \(w\) is such that \(w^{-1}\) exists and is integrable he related approximation problems in \(L^2(w)\) to certain approximation problems in \(L^2(w^{-1})\). His method opened a way for him to give an elegant proof of Szegő’s infimum formula and to obtain a partial solution of a certain prediction problem which will be called Nakazi’s prediction problem in the present paper. A. G. Miyamee and M. Pourahmadi \[MiP\] pointed out that the essence of Nakazi’s method consists in a certain duality between the Hilbert spaces \(L^2(w)\) and \(L^2(w^{-1})\). This way they found a unified approach to several ‘classical’ prediction problems, and they generalized some of Nakazi’s results, partially even to more general harmonizable stable sequences. The papers \[Mi, CMiP\] contain further completions of these results.

The aim of the present paper is the application of Nakazi’s duality method to \(q\)-variate (\(q \in \mathbb{N}\), the set of positive integers) weakly stationary processes over discrete abelian groups \(G\). Under the assumption that the inverse \(W^{-1}\) of the \(q \times q\)-matrix valued spectral density \(W\) of such a process is integrable we establish duality relations between the left Hilbert modules \(L^2(W)\) and \(L^2(W^{-1})\). Section three of our paper contains the general results. The further sections are devoted to applications.

In Section four we obtain some well-known prediction results by quite different proof methods. We compute the one-point interpolation error matrix and derive Yaglom’s interpolation recipe under the mentioned assumption on \(W\). Moreover, Section four contains our generalization of Nakazi’s proof of Szegő’s infimum formula to the multivariate situation. This way we obtain a special case of a result due to V. N. Zasukhin \[Z\] as well as to H. Helson and D. Lowdenslager \[HelL, Thm. 8\].

If \(G = \mathbb{Z}\) is the abelian group of integers and the index set \(S\) of the known values consists of all negative integers and the set of integers \(\{1, 2, \ldots, n\}\) for some \(n \in \mathbb{N}\) then the arising prediction problem is called Nakazi’s prediction problem. In Section five we solve this problem under the additional assumption that \(\log \det W\) is integrable. We will see that in case \(W^{-1}\) is integrable the result is an easy consequence of our duality results. The more general case in which merely \(\log \det W\) is integrable can be solved by approximation procedures. Section five also contains some straightforward multivariate generalizations of univariate results of \[MiP\].

2. Preliminaries

Let \(G\) be a discrete abelian group with neutral element 0, \(G^*\) be its dual group, and \(\lambda\) be the normalized Haar measure on \(G^*\), i.e. \(\lambda(G^*) = 1\). Relations between measurable functions on \(G^*\) are to be understood as relations which hold true almost everywhere (abbreviated to “a.e.”) with respect to (abbreviated to “w.r.t.”) \(\lambda\). Integration is always ment to be done over \(G^*\). For a subset \(S\) of \(G\) set \(S^0 = S \cup \{0\}\) and \(S^c = G \setminus S^0\). For \(q \in \mathbb{N}\) denote by \(M_q\) the algebra of all complex-valued \(q \times q\)-matrices. The zero matrix and the identity matrix of \(M_q\) are denoted by 0 and \(I\), respectively. For a non-empty subset \(S\) of \(G\) let \(T(S)\) be the left \(M_q\)-module of all \(M_q\)-valued trigonometric polynomials
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with frequencies of \( S \). Functions of \( T(\{0\}) \), i.e. constants, and elements of \( M_q \) will be identified and denoted by the same symbols.

If \( A \in M_q \) then the symbol \( A^* \) stands for the adjoint of \( A \), det \( A \) for its determinant, \( \text{tr} A \) for its normalized trace and \( |A| \) for its normalized euclidean norm, i.e. \( |A|^2 = \text{tr}(AA^*) \).

If \( A \) is regular then \( A^{-1} \) denotes its inverse. The set of all Hermitian matrices will be equipped with Loewner’s semi-ordering. In particular, a maximum or minimum of a subset of the set of Hermitian \( q \times q \)-matrices is to be understood w.r.t. that semi-ordering.

The derived Hilbert space norm on \( L^1(M) \) is the result of the orthogonal projection of \( H \) onto the orthogonal complement of \( M \). A detailed study of the geometry of Hilbert \( M_q \)-modules can be found in [GH], for applications to prediction theory cf. [WMa].

Let \( W_q(G^*) \) be the set of \( M_q^2 \)-valued functions of \( L^1 \) and \( \tilde{W}_q(G^*) = \{ W \in W_q(G^*) : W^{-1} \in L^1 \} \). For \( W \in W_q(G^*) \) the symbol \( L^2(W) \) denotes the left Hilbert \( M_q \)-module of (equivalence classes of) \( M_q \)-valued functions \( F \) on \( G^* \) such that \( \text{tr}(FWF^*) \) is integrable w.r.t. \( \lambda \). The \( M_q \)-valued inner product on \( L^2(W) \) is defined by

\[
(F,G) = \int FWG^* \ d\lambda ,
\]

the corresponding scalar product by

\[
(F,G) = \text{tr}(F,G) , \ F,G \in L^2(W).
\]

The derived Hilbert space norm on \( L^2(W) \) is denoted by \( \| \cdot \| \). For a more general construction see [Ros].

We recall that if \( M \) is a (closed) submodule of \( L^2(W) \) and \( F \in L^2(W) \) there exists a unique function \( F_M \in M \) such that \( (F - F_M, F - F_M) \) is the minimum of the set \( \{(F - G, F - G) : G \in M\} \). The element \( F_M \) is the result of the orthogonal projection of \( F \) onto \( M \) w.r.t. both \( (\cdot, \cdot) \) and \( (\cdot, \cdot') \), i.e. \( (F - F_M, G) = 0 \) for any \( G \in M \). The orthogonal complement of \( M \) is equal to the submodule \( \{ H \in L^2(W) : (H, G) = 0 \text{ for all } G \in M \} \).

The function \( (F - F_M) \) is the result of the orthogonal projection of \( F \) onto the orthogonal complement of \( M \). A detailed study of the geometry of Hilbert \( M_q \)-modules can be found in [GH], for applications to prediction theory cf. [WMa].

Considering \( L^2(W) \) as the spectral domain of a \( q \)-variate weakly stationary process over \( G \) we can formulate linear prediction problems of such a process as trigonometric approximation problems in \( L^2(W) \). If \( S \subseteq G \) let \( M(S) \) be the closure of \( T(S) \) in \( L^2(W) \) and \( R_S \) be the orthogonal projection onto \( M(S) \). If \( S \subseteq G \setminus \{0\} \) the determination of \( P_S I \) as well as of the prediction error matrix

\[
\Delta_S = (I - P_S I, I - P_S I) = (I - P_S I, I)
\]

is of fundamental importance in the prediction theory of \( q \)-variate weakly stationary processes over \( G \). This kind of determination problem is sometimes called the general prediction problem. There exist more or less complete solutions of it for some special choices for the set \( S \). For an introduction to prediction theory of \( q \)-variate weakly stationary processes we refer to [Ros].
3. Duality relations

Let \( W \in \mathcal{W}_q(G^*) \). Then along with \( L^2(W) \) one can consider the left Hilbert \( M_q \)-module \( L^2(W^{-1}) \). The symbols \((\cdot, \cdot), \langle \cdot, \cdot \rangle, \| \cdot \|, \mathcal{M}(S), P_S \) and \( \Delta_S \) referring to \( L^2(W) \) will be replaced by \((\cdot, \cdot)\sim, \langle \cdot, \cdot \rangle\sim, \| \cdot \|\sim, \mathcal{M}(S), P_S \sim \) and \( \Delta_S \sim \), respectively, for the denotation of the corresponding objects related to \( L^2(W^{-1}) \). It turns out that a certain duality between \( L^2(W) \) and \( L^2(W^{-1}) \) is helpful if we study the general prediction problem in \( L^2(W) \).

**Lemma 3.1.** Let \( W \in \mathcal{W}_q(G^*) \). Then the mapping

\[
F \mapsto F W, \, F \in L^2(W),
\]

is an isometric isomorphism of \( L^2(W) \) onto \( L^2(W^{-1}) \).

**Proof:** If \( F, G \in L^2(W) \), we obviously have \((F, G) = (FW, GW)_\sim \). If \( H \in L^2(W^{-1}) \) then \( HW^{-1} \in L^2(W) \) and \( HW^{-1}W = H \).

Another immediate consequence of the integrability of \( W^{-1} \) is the point of view on \( L^2(W) \) and on \( L^2(W^{-1}) \) as on subspaces of \( L^1 \). This can be seen from the inequality

\[
(3.1) \quad \int |F| \, d\lambda = \int |FW^{1/2}W^{-1/2}| \, d\lambda \leq \int |FW^{1/2}| \cdot |W^{-1/2}| \, d\lambda \\
\leq \left( \int |FW^{1/2}|^2 \, d\lambda \right)^{1/2} \left( \int |W^{-1}| \, d\lambda \right)^{1/2} < \infty, \, F \in L^2(W).
\]

If \( \mathcal{M} \) is a submodule of \( L^2(W) \) then the submodule

\[
\mathcal{M}^d = \left\{ G \in L^2(W^{-1}) : \int FG^* \, d\lambda = 0, \, F \in \mathcal{M} \right\} \subseteq L^2(W^{-1})
\]

is said to be its dual. Note, that \( \mathcal{M}^d \) is the orthogonal complement in \( L^2(W^{-1}) \) of the submodule \( \mathcal{M}W = \{ FW : F \in \mathcal{M} \} \). For a subset \( S \subseteq G \setminus \{0\} \) the symbol \( \tilde{I}_S \) denotes the orthogonal projection of \( I \) onto \( \mathcal{M}(S^0)^d \subseteq L^2(W^{-1}) \).

**Theorem 3.2.** Let \( W \in \mathcal{W}_q(G^*) \). Then for subsets \( S \subseteq G \setminus \{0\} \)

\[
(3.2) \quad P_S I = I - (I - \tilde{I}_S, I)_{\sim}^{-1}(I - \tilde{I}_S)W^{-1}
\]

and

\[
(3.3) \quad \Delta_S = (I - \tilde{I}_S, I)_{\sim}^{-1} = (I - \tilde{I}_S, I - \tilde{I}_S)_{\sim}^{-1}.
\]

In order to prove Theorem 3.2 we need the following lemma.

**Lemma 3.3.** Let \( F \in \mathcal{M}(S^0) \). Then \( F \in \mathcal{M}(S) \) if and only if \( \int F \, d\lambda = 0 \).

**Proof:** If \( F \in \mathcal{T}(S) \) then \( \int F \, d\lambda = 0 \). If \( F \in \mathcal{M}(S) \) we approximate it by elements of \( \mathcal{T}(S) \), and by (3.1) we obtain \( \int F \, d\lambda = 0 \). If \( F \in \mathcal{M}(S^0) \) it can be written in the form \( F = A + G \), where \( A \in \mathcal{M} \) and \( G \in \mathcal{M}(S) \). By the result just proved we have \( \int F \, d\lambda = A \). Consequently, \( \int F \, d\lambda = 0 \) implies \( F = G \in \mathcal{M}(S) \). \( \square \)

**Proof of Theorem 3.2:** Since \( \tilde{I}_S \in \mathcal{M}(S^0)^d \) we have \( \int \tilde{I}_S \, d\lambda = \int \tilde{I}_S I^* \, d\lambda = 0 \) and hence, \( (I - \tilde{I}_S, W)_{\sim} = \int (I - \tilde{I}_S) \, d\lambda = I \). By a generalization of the Cauchy-Schwarz inequality the inequality \( (W, W)_{\sim}^{-1} = (I - \tilde{I}_S, W)_{\sim}(W, W)_{\sim}^{-1}(I - \tilde{I}_S, W)_{\sim} \leq (I - \tilde{I}_S, I - \tilde{I}_S)_{\sim} \).
Therefore the right hand side of \((3.2)\) is well-defined. Let us denote it by \(F_S\). Since \((I - \tilde{I}_S) \in \mathcal{M}(S^0)W\) we conclude \(F_S \in \mathcal{M}(S^0)\) by Lemma 3.3. Because \(\int F_S \, d\lambda = I - (I - \tilde{I}_S, I)^{-1}(I - \tilde{I}_S, I) = 0\) Lemma 3.3 implies
\[ F_S \in \mathcal{M}(S). \]

Let \(F \in \mathcal{M}(S)\). Then \((I, FW) = \int F^* \, d\lambda = 0\) by Lemma 3.3. Therefore, the equality \((I - F_S, F) = (I - \tilde{I}_S, I)^{-1}(I - \tilde{I}_S, FW) = (I - \tilde{I}_S, I)^{-1}(I, FW) = 0\) yields. By a combination with (3.4) we get \(F_S = P_SI\). To obtain formula (3.3) simply replace one term according to \((3.2)\) in \((2.1)\).

From Theorem 3.2 we can derive a general result which seems not have immediate applications to prediction theory of multivariate stationary processes, however which can be considered as a certain generalization of the univariate case.

Let \(M_q^f\) be the set of all \(q \times q\)-matrices whose determinant equals to one. For \(S \subseteq G \setminus \{0\}\) define
\[ \delta_S = \inf\{\|A - F\|^2 : A \in M_q^f, F \in \mathcal{M}(S)\}. \]

**Lemma 3.4.** Let \(W \in \mathcal{W}_q(G^*)\) and \(S\) be a subset of \(G \setminus \{0\}\). Then
\[ \delta_S = [\det(\Delta_S)]^{1/q}. \]

**Proof:** We have the equality \([\det((I - F, I - F)]^{1/q} = \inf\{\text{tr}((I - F, I - F)B) : B \in M_q^f \cap M_q^f, F \in \mathcal{M}(S)\}, cf. \[HoJ\], Section 7.8, Problem 19\]. Since \(B = A^*A\) for some \(A \in M^f\) we obtain \([\det((I - F, I - F)]^{1/q} = \inf\{\|A - AF\|^2 : A \in M^f, F \in \mathcal{M}(S)\}, \) and therefore
\[ [\det(\Delta_S)]^{1/q} = \inf\{[\det(\Delta_S)]^{1/q} : \Delta_S \in \mathcal{M}_q \cap M_q^f, F \in \mathcal{M}(S)\}. \]

\[ = \inf\{\|A - AF\|^2 : A \in M_q^f, F \in \mathcal{M}(S)\} \]

\[ = \delta_S. \]

**Remark 3.5.** Since a matrix \(A \in M_q^f\) can be written in the form \(A = UB\) for a unitary matrix \(U\) and for a matrix \(B \in (M_q^f \cap M_q^f)\) the value of \(\delta_S\) does not change if \(A\) runs through \((M_q^f \cap M_q^f)\) on the right-hand side of \((3.5)\). Moreover, if \(A\) exhausts the larger set of all \(q \times q\)-matrices with \(|\det A| = 1\) we get the same result.

Combining Theorem 3.2 and Lemma 3.4 we get the following assertion.

**Theorem 3.6.** Let \(W \in \tilde{W}_q(G^*)\). Then for subsets \(S \subseteq G \setminus \{0\}\) we have the equality
\[ \delta_S = (\inf\{\|A - G\|^2 : A \in M_q^f, G \in \mathcal{M}(S^0)\})^{-1}. \]

**Proof:** In a similar way to that one taken in the proof of Lemma 3.4 we derive the equality
\[ \inf\{\|A - G\|^2 : A \in M_q^f, G \in \mathcal{M}(S^0)\} = [\det(\min\{(I - G, I - G) : G \in \mathcal{M}(S^0)\})]^{1/q} \]

\[ = [\det(I - \tilde{I}_S, I - \tilde{I}_S)]^{1/q}. \]
By (3.3) and (3.6) the right-hand end of this equality equals to $\delta_S^{-1}$. □

Another duality relation can be derived by a simple adaptation of a chain of equalities mentioned in the proof of [C, Thm. 4.1] to the multivariate situation.

**Theorem 3.7.** Let $W \in \tilde{W}_q(G^*)$. Then for subsets $S \subseteq G \setminus \{0\}$ one has

$$\inf\{\|A - P_S I\| : A \in M_q, \; \text{tr} A = 1\} = \|I - \tilde{I}_S\|^{-1}. \tag{3.8}$$

**Proof:** We have the following chain of equalities

$$\inf\{\|A - P_S I\|^2 : A \in M_q, \; \text{tr} A = 1\} =$$

$$\inf\{\|A - T\|^2 : A \in M_q, \; \text{tr} A = 1, \; T \in T(S)\} =$$

$$\inf\left\{\frac{||T||^2}{|\text{tr}(T d\lambda)|^2} : T \in T(S^0), \; \text{tr} \left(\int T d\lambda\right) \neq 0\right\} =$$

$$\left(\sup\left\{\frac{(TW, I)}{||TW||^2} : T \in T(S^0), \; \text{tr} \left(\int T d\lambda\right) \neq 0\right\}\right)^{-1} =$$

$$\|\tilde{P}_{S^0} I\|^{-2} =$$

$$\|I - \tilde{I}_S\|^{-2}. \tag{3.9}$$

For applications of the preceding theorems a description of the space $\mathcal{M}(S^0)^d$ is needed. The identification of $\mathcal{M}(S^0)^d$ and of $\tilde{\mathcal{M}}(S^c)$ would be desirable from the point of view of prediction theory. Clearly,

$$\tilde{\mathcal{M}}(S^c) \subseteq \mathcal{M}(S^0)^d.$$  

However, whether equality really holds or not, seems to be a difficult problem, in general. It is related to basis properties of characters. MIAMEE and POURAHMADI [MiP] discussed this question and gave particular answers to it in case $q = 1$, cf. also [Mi]. In contrast to these careful investigations, in the proofs of [C, Thm. 4.1] and of [CMHP, Thm. 1] the equality $\tilde{\mathcal{M}}(S^c) = \mathcal{M}(S^0)^d$ seems to be used in a rather general situation, but without any explanation how to prove it.

We call a subset $S \subseteq G$ to be $G$-exact if for any $q \in \mathbb{N}$ and any $W \in \tilde{W}_q(G^*)$ the set identity $\tilde{\mathcal{M}}(G \setminus S) = \mathcal{M}(S)^d$ holds. Because of the symmetry of this definition we immediately obtain the following fact.

**Lemma 3.8.** The set $S$ is $G$-exact if and only if the set $G \setminus S$ is $G$-exact.

The next result demonstrates that the alteration of a $G$-exact set $S$ by finitely many elements does not have any influence on its $G$-exactness.

**Theorem 3.9.** Let $S$ be a subset of $G$ and $g \in G \setminus S$. Then $S$ is $G$-exact if and only if $S \cup \{g\}$ is $G$-exact.
Proof: Without loss of generality we may assume that $0 \not\in S$ and $g = 0$. Suppose $S$ to be $G$-exact. By (3.3) we have the chain of set inclusions $\mathcal{M}(S^c) \subseteq \mathcal{M}(S^0)^d \subseteq \mathcal{M}(S)^d \subseteq \tilde{\mathcal{M}}(G \setminus S)$. Thus, if $G \in \mathcal{M}(S^0)^d$ it can be decomposed as $G = A + H$ for some $A \in \mathcal{M}_q$ and $H \in \mathcal{M}(S^c)$. So $G - H = A \in \mathcal{M}(S^0)^d$ and hence, $A = 0$ and $G = H \in \tilde{\mathcal{M}}(S^c)$. The latter means that $S^0$ is $G$-exact. To derive the $G$-exactness of $S$ from the $G$-exactness of $S^0$ one has to combine the result just proved and Lemma 3.8. 

Corollary 3.10. If either $S$ or $G \setminus S$ are finite subsets of $G$ then $S$ is $G$-exact.

Proof: Define $\mathcal{M}(\emptyset) = \{0\}$, so the empty set is $G$-exact by definition. With this setting the assertion follows from Theorem 3.9 and Lemma 3.8. 

If $G = \mathbb{Z}$ then its dual group $G^*$ can be identified with the numerical interval $[-\pi, \pi)$, where the addition of elements is understood to be done mod $2\pi$. The characters of $G^* = [-\pi, \pi)$ can be described as the set of functions $e_k(t) = e^{ikt}$, $t \in [-\pi, \pi)$, $k \in \mathbb{Z}$. A univariate version of the assertion stated thereafter was given in [MiP, Lemma 3.6], compare with [Mi, Thm. 3.1] for a univariate extension of this assertion to $L^p$-spaces.

Theorem 3.11. The set $\mathbb{N}$ is $\mathbb{Z}$-exact.

Proof: By (3.2) we have $\tilde{\mathcal{M}}(\mathbb{Z} \setminus \mathbb{N}) \subseteq \mathcal{M}(\mathbb{N})^d$. Let $G \in \mathcal{M}(\mathbb{N})^d$ be orthogonal to $\tilde{\mathcal{M}}(\mathbb{Z} \setminus \mathbb{N})$. Then

$$(3.10) \quad \int e_k G^* \, d\lambda = 0 \, , \ k \in \mathbb{N} ,$$

and

$$(3.11) \quad \int e_k W^{-1} G^* \, d\lambda = 0 \, , \ k \in \mathbb{Z} \setminus \mathbb{N} .$$

By (3.10) the element $G^*$ belongs to the Hardy space $H^1$ (of $\mathcal{M}_q$-valued functions), and (3.11) implies that $GW^{-1}$ belongs to $H^1$, too. Therefore, $GW^{-1}G^*$ is an element of $H^{1/2}$ taking values in the set of non-negative Hermitian matrices. According to [ST] the function $GW^{-1}G^*$ is a constant function. However, by (3.10) and (3.11) the index zero Fourier coefficient of $GW^{-1}G^*$ equals to zero, so $GW^{-1}G^* = 0$ and $G = 0$. 

4. Some applications

From the general assertions of section three we can easily derive some prediction results for multivariate stationary sequences. Many of them are well-known, however here they are drawn from a different context.

Theorem 4.1. (cf. [Heb], Thm. 4.3] or [Mi, Cor. 2.9])

Let $W \in \mathcal{W}_q(G^*)$ and $S = G \setminus \{0\}$. Then

$$P_S I = I - \left( \int W^{-1} \, d\lambda \right)^{-1} W^{-1} ,$$

$$\Delta_S = \left( \int W^{-1} \, d\lambda \right)^{-1} \quad \text{and}$$

$$\delta_S = \left[ \det \left( \int W^{-1} \, d\lambda \right) \right]^{-1/q} .$$
Theorem 3.2 and Lemma 3.4.

Remark 4.3. From (4.2) we obtain

\[ q \]

Let \( \Gamma \) (4.3) system of equations (4.1) and (4.3) can be written in the form

\[
\sum_{k=1}^{n} C_{j-k} D_k^* = C_j, \quad j = 1, \ldots, n.
\]

Then

\[
P_{S_1} I = I - \left( I - \sum_{k=1}^{n} D_k e_k, I \right)^{-1} \left( I - \sum_{k=1}^{n} D_k e_k \right) W^{-1},
\]

(4.2)

\[
\Delta_{S_1} = \left( I - \sum_{k=1}^{n} D_k e_k, I \right)^{-1}
\]

and

\[
\delta_{S_1} = \left[ \det \left( I - \sum_{k=1}^{n} D_k e_k, I \right) \right]^{-1/q}.
\]

Proof: According to Corollary 3.10 the set \( S_1 \) is \( \mathbb{Z} \)-exact. Elementary calculations show that \( P_{S_1} I = \sum_{k=1}^{n} D_k e_k \). Knowing this the result is an immediate consequence of Theorem 3.2 and Lemma 3.4.

Remark 4.3. From (4.2) we obtain

\[
\sum_{k=1}^{n} C_{-k} D_k^* = C_0 - \Delta_{S_1}^{-1}.
\]

Let \( \Gamma_n \) be the Toeplitz matrix \( \Gamma_n = (C_{j-k})_{j,k=1,\ldots,n}, \quad n \in \mathbb{N} \). It is not hard to see that the system of equations (1.1) and (1.3) can be written in the form

\[
\begin{pmatrix}
-I \\
D_1^* \\
\vdots \\
D_n^*
\end{pmatrix} = \begin{pmatrix}
-I \\
-D_{S_1}^{-1} \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \text{i.e.} \quad \begin{pmatrix}
-I \\
D_1^* \\
\vdots \\
D_n^*
\end{pmatrix} = \Gamma_{n+1}^{-1} \begin{pmatrix}
-D_{S_1}^{-1} \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Comparing the first \( q \times q \)-blocks from both the sides we get \( I = \Omega_n \Delta_{S_1}^{-1} \), and therefore \( 1 = \det(\Omega_n) \det(\Delta_{S_1}^{-1}) \). Here \( \Omega_n \) denotes the \( q \times q \)-matrix in the left upper corner of the matrix \( \Gamma_{n+1}^{-1} \). Using a well-known rule for computing minors of inverse matrices together with the fact that the \( nq \times nq \)-matrix in the right lower corner of \( \Gamma_{n+1} \) equals \( \Gamma_n \), we get

\[
\det(\Delta_{S_1}) = \frac{\det(\Omega_n)}{\det(\Gamma_{n+1})}.
\]
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For the special case where \( q = 1 \) \textsc{Nakazi} gave an elegant proof of Szegö’s infimum formula, cf. \[N\,\text{Cor. 4}\]. \textsc{Zasukhin} \[Z\], \textsc{Helson} and \textsc{Lowdenslager} \[HelL\,\text{Thm. 8}\] stated a version of Szegö’s theorem for non-negative Hermitian matrix-valued measures. Using the results of section three we can adapt Nakazi’s proof to the multivariate situation to obtain the result by Zasukhin and Helson-Lowdenslager for absolutely continuous non-negative Hermitian matrix-valued measures.

\textbf{Theorem 4.4.} Let \( G = \mathbb{Z} \) and \( W \) be a function of \( L^1 \), whose values are non-negative Hermitian matrices. Then

\[
\delta_N = \exp \left\{ \frac{1}{q} \cdot \int \log \det W \, d\lambda \right\},
\]

where the right-hand side of (4.4) has to be interpreted as zero in case the expression \( \log \det W \) is not integrable.

\textbf{Proof:} First assume that \( W \in \tilde{W}_q(\mathbb{R}) \). If \( A \in M_q^1 \) and \( T \in T(\mathbb{N}) \) from the inequality between the arithmetic and the geometric means and from the fact that \( \int \log |\det(A - T)| \, d\lambda \geq \log |\det A| = 0 \) we obtain

\[
\|A - T\|^2 = \int \text{tr}[(A - T)W(A - T)^*] \, d\lambda \\
\geq \exp \left\{ \int \log \text{tr}[(A - T)W(A - T)^*] \, d\lambda \right\} \\
\geq \exp \left\{ \int \log \{\det[(A - T)W(A - T)^*]\}^{1/q} \, d\lambda \right\} \\
\geq \exp \left\{ \frac{1}{q} \cdot \int \log \det W \, d\lambda \right\}.
\]

Hence

\[
(4.5) \quad \delta_N \geq \exp \left\{ \frac{1}{q} \cdot \int \log \det W \, d\lambda \right\}. \]

Similarly,

\[
(4.6) \quad \tilde{\delta}_{N^c} \geq \exp \left\{ \frac{1}{q} \cdot \int \log \det(W^{-1}) \, d\lambda \right\}.
\]

By Theorem 3.11 and Theorem 3.9 the set \( \mathbb{N}^0 \) is \( \mathbb{Z} \)-exact, and by (3.7) the equality \( \delta_N = \tilde{\delta}_{N^c}^{-1} \) turns out. Comparing this with (4.3) and (4.4) we get the desired result under the additional assumption that \( W^{-1} \in L^1 \). The general assertion can be derived from this partial one by a standard approximation argument demonstrated in \[N\,\text{pp. 260-261}\] for the case \( q = 1 \).

\( \square \)

From Theorem 4.4 we obtain a somewhat surprising result.

\textbf{Corollary 4.5.} Let \( G = \mathbb{Z} \) and \( W \in \mathcal{W}_q(\mathbb{R}) \). Then

\[
[\det(\Delta_N)]^{1/q} = \inf \left\{ \int [\det[(I - T)W(I - T)^*]]^{1/q} \, d\lambda : T \in T(\mathbb{N}) \right\}.
\]
Proof: By (3.6) and (4.4) we conclude \[\det(\Delta_N)^{1/q} = \exp\left\{1/q \cdot \int \log \det W \, d\lambda\right\}.\] On the other hand, \[
\exp\left\{1/q \cdot \int \log \det W \, d\lambda\right\} = \inf \left\{\int \left[\det([I - T]W(I - T)^*)\right]^{1/q} \, d\lambda : T \in \mathcal{T}(N)\right\},
\] cf. [K, Cor.].

5. Nakazi’s prediction problem

Another application of the results of Section three gives some information about the multivariate version of Nakazi’s prediction problem. Let \(G = \mathbb{Z}\) and \(S\) be of the form
\[
S_2 = \mathbb{N}^c \cup \{1, 2, \ldots, n\}
\] for some \(n \in \mathbb{N}\). Assume the function \(W\) belongs to \(\mathcal{W}_q([\pi, \pi])\) and is such that \(\log \det W\) is integrable. The set of such functions is denoted by \(\mathcal{W}_q([\pi, \pi])\) in the sequel. Let \(\Phi\) be the unique outer function of the Hardy space \(H^2\) (of functions with values in \(M_q\)) such that
\[
W = \Phi^* \Phi \quad \text{and} \quad \int \Phi \, d\lambda \in M_q.
\]
The function \(\Phi^{-1}\) is also outer, but it does not belong to \(H^2\), in general. Let \(B_j\) be the \(j\)-th Taylor coefficient of \(\Phi^{-1}\), \(j \in \mathbb{N}^0\).

Lemma 5.1. (cf. [MiP, Cor. 3.7] for \(q = 1\))

Let \(W \in \mathcal{W}_q([\pi, \pi]).\) Then
\[
P_{S_2} = I - \sum_{j=0}^n (B_j B_j^*)^{-1} \cdot \sum_{j=0}^n B_j e_j (\Phi^*)^{-1}
\]
and
\[
\Delta_{S_2} = \left(\sum_{j=0}^n B_j B_j^*\right)^{-1}.
\]

Proof: By Lemma 3.8 and Theorems 3.3 and 3.11 the set \(S_2\) is \(\mathbb{Z}\)-exact. Taking into account Theorem 3.2 and the fact that the outer function \(\Phi^{-1}\) belongs to \(H^2\) as soon as \(W \in \mathcal{W}_q([\pi, \pi])\), the proof consists of straightforward calculations which will be omitted.

Our goal is to establish (5.3) and (5.4) under the weaker assumption that \(\log \det W\) is integrable. This can be done by an approximation procedure. Our approach is similar to that one presented in the proof of [CMiP, Th. 3] for the case \(q = 1\). However, since we wish to compute not only the prediction error as done there but also the orthogonal projection, we give a complete proof of our generalized result.

Let \(W_m = W + 1/m \cdot I, \ m \in \mathbb{N}\). Denote by \((.,.)_m\) the \(M_q\)-valued inner product of \(L^2(W_m)\), by \(\mathcal{M}_m(S_2)\) the closure of \(\mathcal{T}(S_2)\) in \(L^2(W_m)\), and by \(P_{S_2}^{(m)}\) the orthogonal projection of \(L^2(W_m)\) onto \(\mathcal{M}_m(S_2)\). Note that \(L^2(W_m)\) can be considered as a subset of \(L^2(W)\), and that the inclusion
\[
\mathcal{M}_m(S_2) \subseteq \mathcal{M}(S_2), \quad m \in \mathbb{N},
\]
Let $\Phi$ holds. Furthermore,
\begin{equation}
\lim_{m \to \infty} (I - P^{(m)}_{S_2} I, I - P^{(m)}_{S_2} I)_m = (I - P_{S_2} I, I - P_{S_2} I).
\end{equation}

Let $\Phi_m$ be the corresponding outer factor of $W_m$, i.e.
\begin{equation}
W_m = \Phi^*_m \Phi_m \quad \text{and} \quad \int \Phi_m d\lambda \in M^*_q, \ m \in \mathbb{N}.
\end{equation}

The following result is an easy consequence of [DLE], so the proof is omitted.

**Lemma 5.2.** Let $W \in \mathcal{W}_q([-\pi, \pi])$. Then $\lim_{m \to \infty} \Phi_m = \Phi$ w.r.t. the topology of $L^2$.

Let $B_{jm}$ be the $j$-th Taylor coefficient of $\Phi_m^{-1}$. Set
\begin{equation}
T_m = \left( \sum_{j=0}^n B_{jm} B^*_j \right)^{-1} \cdot \sum_{j=0}^n B_{jm} e_j
\end{equation}
and
\begin{equation}
T = \left( \sum_{j=0}^n B_j B^*_j \right)^{-1} \cdot \sum_{j=0}^n B_j e_j.
\end{equation}

**Lemma 5.3.** Let $W \in \mathcal{W}_q([-\pi, \pi])$. Then there exists a subsequence \(\{m_l\}_{l \in \mathbb{N}}\) of $\mathbb{N}$ such that
\begin{equation}
\lim_{l \to \infty} \| I - T(\Phi^*)^{-1} - P^{(m_l)}_{S_2} I \| = 0.
\end{equation}

**Proof:** Lemma 5.2 implies that there exists a subsequence $\{m_l\}_{l \in \mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{l \to \infty} \Phi_{m_l} = \Phi$ a.e.. Since
\begin{equation}
\| (\Phi^*_{m_l})^{-1} - (\Phi_*)^{-1} \|^2 \leq \int |\Phi - \Phi_{m_l}|^2 |\Phi^{-1}|^2 |W^{1/2}|^2 d\lambda = \int |\Phi - \Phi_{m_l}|^2 |W^{1-1}|^1 d\lambda
\end{equation}
and $|\Phi - \Phi_{m_l}|^2 |W^{1-1}| \leq 2 \cdot (|\Phi|^2 + |\Phi_{m_l}|^2) |W^{1-1}| \leq 4$ Lebesgue’s dominated convergence theorem yields
\begin{equation}
\lim_{l \to \infty} \| (\Phi^*_{m_l})^{-1} - (\Phi^*)^{-1} \| = 0.
\end{equation}

Applying Lemma 5.1 we can write
\begin{equation}
P^{(m_l)}_{S_2} I = \| T_{m_l}(\Phi^*_{m_l})^{-1} - T(\Phi^*)^{-1} \|
\leq \| T_{m_l}(\Phi^*_{m_l})^{-1} - T_{m_l}(\Phi^*)^{-1} \| + \| T_{m_l}(\Phi^*)^{-1} - T(\Phi^*)^{-1} \|.
\end{equation}

Lemma 5.2 implies that for every $j \in \mathbb{N}^0$ the $j$-th Taylor coefficient of $\Phi_m$ tends to the $j$-th Taylor coefficient of $\Phi$ as $m$ tends to infinity. Therefore, $\lim_{l \to \infty} T_{m_l} = T$ uniformly on $[-\pi, \pi)$, and the second summand on the right hand side of (5.7) tends to zero as $l$ tends to infinity. Also the first summand there tends to zero because of (5.7) and the uniform boundedness of the set $\{T_{m_l} : l \in \mathbb{N}\}$.

**Theorem 5.4.** Let $G = \mathbb{Z}$, $S_1$ be the set described at (5.1) and $W \in \mathcal{W}_q([-\pi, \pi])$. Then
\begin{equation}
P_{S_2} I = I - \left( \sum_{j=0}^n B_j B^*_j \right)^{-1} \cdot \sum_{j=0}^n B_j e_j (\Phi^*)^{-1},
\end{equation}
where
\begin{equation}
P^{(m)}_{S_2} I = \frac{1}{m} \int \Phi_m \ d\lambda.
\end{equation}
\( \Delta_{S_2} = \left( \sum_{j=0}^{n} B_j B_j^* \right)^{-1} \)

and

\( \delta_{S_2} = \left[ \det \left( \sum_{j=0}^{n} B_j B_j^* \right) \right]^{-1/q} \).

**Proof:** From (5.6) and Lemma 5.3 we get

\[ \|I - P_{S_2} I\| \geq \lim_{l \to \infty} \|I - P_{S_2}^{(m_l)} I\| = \|I - (I - T(\Phi^*))^{-1}\|. \]

If we take (5.3) into account, we can conclude that \((I - T(\Phi^*))^{-1} \in \mathcal{M}(S_2)\). This yields (5.9). To obtain (5.10) combine formula (5.9) with (2.1). Finally, (5.11) is a consequence of (3.6).

We conclude our paper by stating the prediction error matrix \(\Delta_S\) for sets \(S\) of the form

\( S_3 = \mathbb{N}^c \cup \{n\} \)

and

\( S_4 = \mathbb{N}^c \setminus \{-n\} \)

for some \(n \in \mathbb{N}\). The univariate versions of the results below can be found at [CMIP, Th. 5 and 6]. Let \(A_j\) be the \(j\)-th Taylor coefficient of the outer function \(\Phi\) which has the properties (5.2), \(j \in \mathbb{N}_0\).

**Theorem 5.5.** Let \(G = \mathbb{Z}, S_3\) be the set described at (5.13) and \(W \in \mathcal{W}_q^\prime([-\pi, \pi])\). Then

\( \Delta_{S_3} = A_0^* \left( I - A_n \left( \sum_{j=0}^{n} A_j^* A_j \right)^{-1} A_n^* \right) A_0. \)

If \(A_n\) is regular then

\( \Delta_{S_3} = A_0^* (A_n^*)^{-1} \left( \sum_{j=0}^{n-1} A_j^* A_j \right) \left( \sum_{j=0}^{n} A_j^* A_j \right)^{-1} A_n^* A_0. \)

We omit the proof since (5.14) can be obtained by a straightforward generalization of the proof given for [CMIP, Th. 5] to the multivariate case, and (5.15) follows by some simple matrix computations from (5.14).

**Theorem 5.6.** Let \(G = \mathbb{Z}, S_4\) be the set described at (5.13) and \(W \in \mathcal{W}_q^\prime([-\pi, \pi])\). Then

\( \Delta_{S_4} = A_0 \left( I - B_n \left( \sum_{j=0}^{n} B_j^* B_j \right)^{-1} B_n^* \right) A_0^*. \)

If \(B_n\) is regular then

\( \Delta_{S_4} = A_0 (B_n^*)^{-1} \left( \sum_{j=0}^{n} B_j^* B_j \right) \left( \sum_{j=0}^{n-1} B_j^* B_j \right)^{-1} B_n^* A_0^*. \)
Proof: Since the set $S_4$ is $Z$-exact by Theorem 3.11, Theorem 3.9 and Corollary 3.10 and since the set $S_4'$ is the reflection of $S_3$ about the origin, the result immediately follows from the Theorems 3.2 and 5.5 whenever $W \in \tilde{W}_q([-\pi, \pi])$. If merely $W \in W_q([-\pi, \pi])$ approximate $W$ by the sequence \(\{W + 1/m \cdot I\}_{m \in \mathbb{N}}\). \(\square\)

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