On multiplicative Chung–Diaconis–Graham process *

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Annotation.

We study the lazy Markov chain on $\mathbb{F}_p$ defined as $X_{n+1} = X_n$ with probability $1/2$ and $X_{n+1} = f(X_n) \cdot \varepsilon_{n+1}$, where $\varepsilon_n$ are random variables distributed uniformly on $\{\gamma, \gamma^{-1}\}$, $\gamma$ is a primitive root and $f(x) = \frac{x}{\gamma - 1}$ or $f(x) = \text{ind}(x)$. Then we show that the mixing time of $X_n$ is $\exp(O(\log p/\log \log p))$. Also, we obtain an application to an additive–combinatorial question concerning a certain Sidon–type family of sets.

1 Introduction

The Chung–Diaconis–Graham process [5] is the random walk on $\mathbb{F}_p$ (or more generally on $\mathbb{Z}/n\mathbb{Z}$ for composite $n$) defined as

$$X_{j+1} = aX_j + \varepsilon_{j+1},$$

(1)

where $a \in \mathbb{F}_p^*$ is a fixed residue and the random variables $\varepsilon_j$ are independent and identically distributed (in the original paper [5] the variables $\varepsilon_j$ were distributed uniformly on $\{-1, 0, 1\}$ and $a = 2$). This process was studied extensively, see papers [3], [5]–[9] and so on. In our article we are interested in the following characteristic of $X_n$, which is called the mixing time. The definition is

$$t_{\text{mix}}(\varepsilon) := \inf \left\{ n : \max_{A \subseteq \mathbb{F}_p} \left| P(X_n \in A) - \frac{|A|}{p} \right| \leq \varepsilon \right\}.$$  

Usually one takes a concrete value of the parameter $\varepsilon$, e.g., $\varepsilon = 1/4$ and below we will say about $t_{\text{mix}} := t_{\text{mix}}(1/4)$. Simple random walk on $\mathbb{F}_p$ has the mixing time $t_{\text{mix}}$ of order $p^2$, see [15] and it was shown in [5] (also, see recent paper [16]) that the mixing time of process (1) is at most $O(\log p \cdot \log \log p)$. Hence the Chung–Diaconis–Graham process gives an example of a speedup phenomenon, i.e., a phenomenon of increasing the time of the convergence. In [7] it was studied a more general non–linear version of the Chung–Diaconis–Graham process, defined as

$$X_{j+1} = f(X_j) + \varepsilon_{j+1},$$

(2)

where $f$ is a bijection on $\mathbb{F}_p$. In particular, it was proved that for rational functions of bounded degree (defined correctly at poles, see [7]) the mixing time is

$$t_{\text{mix}}(1/4) = O(p^{1+\varepsilon}), \quad \forall \varepsilon > 0.$$  

(3)

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Perhaps, the right answer for process (2) is \( t_{mix} = O(\log p) \) but it was obtained in the only case \( f(x) = 1/x \) for \( x \neq 0 \) and \( f(0) = 0 \), see [8]. The proof is based on \( \text{SL}_2(\mathbb{F}_p) \)--actions methods from paper [2]. In [3] it was asked whether other explicit examples of Markov chains with low mixing time could be provided.

Our paper is devoted to a multiplicative form of Chung–Diaconis–Graham process. Multiplicative variants of the process were studied in [1], [10], [11], [14] and in other papers. Consider the family of functions

\[
f_{\alpha, \beta}^*(x) = \frac{x}{\alpha x + \beta},
\]

(4)

where \( \alpha, \beta \neq 0 \). Most of our results below do not depend on \( \alpha, \beta \), so we will not write these parameters in such cases. In Theorems 1, 12 we need \( f_{\alpha, \beta}^*(x) \) be a bijection, so we put \( f_{\alpha, \beta}^*(-\beta/\alpha) := 1/\alpha \). In turn Theorems 1, 12 not depend on a particular choice of \( (\alpha, \beta) \) and one can consider \( \alpha = 1, \beta = -1 \), say, and write \( f_{\alpha}^*(x) := f_{1,-1}^*(x) \). Let us formulate a particular case of our main result.

**Theorem 1** Let \( p \) be a prime number and \( \gamma \in \mathbb{F}_p^\ast \) be a primitive root. Also, let \( \varepsilon_j \) be the random variables distributed uniformly on \( \{\gamma, \gamma^{-1}\} \). Consider the lazy Markov chain \( X_0 \neq x_0, X_1, \ldots, X_n, \ldots \) defined by

\[
X_{j+1} = \begin{cases} 
  f_{\alpha}^*(X_j) \cdot \varepsilon_{j+1}, & \text{with probability } 1/2, \\
  X_j, & \text{with probability } 1/2.
\end{cases}
\]

Then for any \( c > 0 \) and any \( n = c \exp(\log p / \log \log p) \) one has

\[
\|P_n - U\| := \frac{1}{2} \max_{A \subseteq \mathbb{F}_p} \left| P(X_n \in A) - \frac{|A|}{p-1} \right| \leq e^{-O(c)}.
\]

The same is true for the chain \( X_{j+1} = f_{\alpha}^*(X_j) \cdot \varepsilon_{j+1} \), where \( \varepsilon_j \) denote the random variables distributed uniformly on \( \{1, \gamma^{-1}, \gamma\} \).

In other words, the mixing time of our Markov chain is \( \exp(O(\log p / \log \log p)) \). By a similar method we obtain the same bound for another chain with \( f_{\alpha}^*(x) = \text{ind}(x) \) and for the chain of form (2) with \( f(x) = \exp(x) \), see Theorem 15 and formulae (27), (28) below. As a byproduct we show that in the case \( f(x) = x^2 \) and \( p \equiv 3 \pmod{4} \) the mixing time of (2) is, actually, \( O(p \log p) \), see Remark 14.

Our approach is not analytical as in [7] but it uses some methods from Additive Combinatorics and Incidence Geometry. In particular, we apply some results on growth in the affine group \( \text{Aff}(\mathbb{F}_p) \). The core of our article has much more in common with papers [2], [23] than with [7] but we extensively use the general line of the proof from this paper. From additive–combinatorial point of view the main innovation is a series of asymptotic formulae for the incidences of points and lines, which were obtained via the action of \( \text{Aff}(\mathbb{F}_p) \), see the beginning of section 3. The author hopes that such formulae are interesting in its own right. It is well–known see, e.g., [2], [17], [18], [24], [25], [26], [28] that Incidence Geometry and the sum–product phenomenon sometimes work better than classical analytical methods and that is why it is possible to break the square–root barrier, which corresponds to natural bound [3] (for details see Theorem 13 and the proofs of Theorems 12, 15).
It turns out that the same method is applicable to a purely additive–combinatorial question on Sidon sets. Sidon sets is a classical subject of Combinatorial Number Theory, see, e.g., survey [19]. Recall that a subset \( S \) of an abelian group \( G \) with the group operation \( \ast \) is called \( g \)-Sidon set if for any \( z \neq 1 \) the equation \( z = x \ast y^{-1} \), where \( x, y \in S \) has at most \( g \) solutions. If \( g = 1 \), then we arrive to the classical definition of Sidon sets [27]. Having an arbitrary set \( A \subseteq G \), we write \( \text{Sid}^+(A) \) for size of the maximal (by cardinality) Sidon subset of the set \( A \). It is known [13] (also, see [22]) that for any subset \( A \) of our abelian group \( G \) the following estimate takes place

\[ \text{Sid}^+(A) \gg \sqrt{|A|} \]

and Klurman and Pohoata [12] asked about possibility to improve the last bound, having two different operations on a ring \( G \). In [26] the author obtains

**Theorem 2** Let \( A \subseteq \mathbb{F} \) be a set, where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{F}_p \) (in the prime field case suppose, in addition, that \( |A| < \sqrt{p} \), say). Then there are some absolute constants \( c > 0, K \geq 1 \) such that

\[ \max\{\text{Sid}^+_K(A), \text{Sid}^\times_K(A)\} \gg |A|^{1/2+c}. \tag{5} \]

On upper bounds for (5), see [20] and [26]. Notice that \( \text{Sid}^\times_K(A) = \text{Sid}^\times_K(\log(A)) \) and \( \text{Sid}^+_K(A) = \text{Sid}^+_K(\exp(A)) \) for \( A \subseteq \mathbb{R}^+ \), say. Hence it is possible to rewrite bound (5) in terms of the only operation. We now consider a general question, which was mentioned by A. Warren during CANT–2021 conference [32].

**Problem.** Let \( f, g \) be some ‘nice’ (say, convex or concave) functions. Is it true that for any set \( A \subseteq \mathbb{R}^+ \), say, one has

\[ \max\{\text{Sid}^+_K(A), \text{Sid}^+_K(f(A))\}, \quad \max\{\text{Sid}^\times_K(A), \text{Sid}^\times_K(g(A))\} \gg |A|^{1/2+c} ? \]

Here \( c > 0, K \geq 1 \) are some absolute constants. What can be said for \( K \) exactly equals one and for a certain \( c > 0 \) ?

In this paper we obtain an affirmative answer in the case of \( g(x) = x + 1 \) and \( f(x) = \exp(x) \), where in the case of \( \mathbb{F}_p \) the latter function is defined as \( \exp(x) := g^x \) and \( g \) is a fixed primitive root.

**Theorem 3** Let \( A \subseteq \mathbb{F} \) be a set, where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{F}_p \) (in the prime field case suppose, in addition, that \( |A| < \sqrt{p} \)). Then there are some absolute constants \( c > 0, K \geq 1 \) such that

\[ \max\{\text{Sid}^\times_K(A), \text{Sid}^\times_K(A + 1)\} \gg |A|^{1/2+c}, \tag{6} \]

and

\[ \max\{\text{Sid}^+_K(A), \text{Sid}^+_K(\exp(A))\} \gg |A|^{1/2+c}, \tag{7} \]

On the other hand, for any integer \( k \geq 1 \) there is \( A \subseteq \mathbb{F} \) with

\[ \max\{\text{Sid}^\times_K(A), \text{Sid}^\times_K(A + 1)\} \ll k^{1/2}|A|^{3/4}. \tag{8} \]

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2 Definitions and preliminaries

By $G$ we denote an abelian group. Sometimes we underline the group operation writing $+$ or $\times$ in the considered quantities (as the energy, the representation function and so on, see below). Let $F$ be the field $\mathbb{R}$ or $F = F_p = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$. Let $F^* = F \setminus \{0\}$.

We use the same capital letter to denote set $A \subseteq F$ and its characteristic function $A : F \to \{0, 1\}$. Given two sets $A, B \subseteq G$, define the sumset of $A$ and $B$ as

$$A + B := \{a + b : a \in A, b \in B\}.$$  

In a similar way we define the difference sets and higher sumsets, e.g., $2A - A$ is $A + A - A$. We write $\sqcup$ for a direct sum, i.e., $|A + B| = |A||B|$. For an abelian group $G$ the Plünnecke–Ruzsa inequality (see, e.g., [30]) holds stating

$$|nA - mA| \leq \left(\frac{|A + A|}{|A|}\right)^{n+m} \cdot |A|,$$  

where $n, m$ are any positive integers. It follows from a more general inequality contained in [16] that for arbitrary sets $A, B, C \subseteq G$ one has

$$|B + C + X| \leq \frac{|B + X|}{|X|} \cdot |C + X|,$$  

where $X \subseteq A$ minimize the quantity $|B + X|/|X|$. We use representation function notations like $r_{A+B}(x)$ or $r_{A-B}(x)$ and so on, which counts the number of ways $x \in G$ can be expressed as a sum $a + b$ or $a - b$ with $a \in A$, $b \in B$, respectively. For example, $|A| = r_{A-A}(0)$.

For any two sets $A, B \subseteq G$ the additive energy of $A$ and $B$ is defined by

$$E(A, B) = E^+(A, B) = \{|(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - b_1 = a_2 - b_2|\}.$$  

If $A = B$, then we simply write $E(A)$ for $E(A, A)$. More generally, for sets (functions) $A_1, \ldots, A_{2k}$ belonging an arbitrary (noncommutative) group $G$ and $k \geq 2$ define the energy $T_k(A_1, \ldots, A_{2k})$ as

$$T_k(A_1, \ldots, A_{2k}) = \{|(a_1, \ldots, a_{2k}) \in A_1 \times \cdots \times A_{2k} : a_1 a_2^{-1} \cdots a_{k-1} a_k^{-1} = a_{k+1} a_{k+2} \cdots a_{2k-1} a_{2k}^{-1}\}|.$$  

In the abelian case put for $k \geq 2$

$$E^+_k(A) = \sum_{x} r_{A-A}^k(x) = \sum_{\alpha_1, \ldots, \alpha_{k-1}} |A \cap (A + \alpha_1) \cap \cdots \cap (A + \alpha_{k-1})|^2.$$  

Clearly, $|A|^k \leq E^+_k(A) \leq |A|^{k+1}$. Also, we write $\hat{E}_k^+(A) = \sum_x r_{A+A}^k(x)$.

By ord($x$) denote the multiplicative order of an element of $x \in F_p^*$ and let ind($x$) is defined as $x = g^{\text{ind}(x)}$, where $g$ is a fixed primitive root of $F_p^*$. It is convenient for us to think that the function ind($x$) takes values from 1 to $p - 1$ and hence ind($x$) is defined on $F_p^*$. In a similar way, we denote by exp($x$) : $F_p^* \to F_p^*$, the function exp($x$) = $g^x$, where $x \in F_p^*$. Let $\text{Aff}(F)$ be the group
of transformations $x \to ax + b$, where $a \in \mathbb{F}^*$, $b \in \mathbb{F}$. Sometimes we write $(a, b) \in \text{Aff}(\mathbb{F})$ for the map $x \to ax + b$.

The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. When the constants in the signs depend on a parameter $M$, we write $\ll_M$ and $\gg_M$. All logarithms are to base 2. If we have a set $A$, then we will write $a \ll b$ or $b \gg a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. Let us denote by $\lfloor n \rfloor$ the set $\{1, 2, \ldots, n\}$.

We now mention several useful results, which we will appeal in the text. We start with a result from [25].

**Lemma 4** Let $f_1, \ldots, f_{2k} : G \to \mathbb{C}$ be any functions. Then

$$T_{2k}^2(f_1, \ldots, f_{2k}) \leq \prod_{j=1}^{2k} T_k(f_j), \quad (13)$$

and $\|f\| := T_k(f)^{1/2k} \geq \|f\|_{2k}, k \geq 2$ is a norm of a function $f : G \to \mathbb{C}$.

The next result on collinear quadruples $Q(A)$ was proved in [18]. We rewrite the asymptotic formula for $Q(A)$ in the following convenient form.

**Lemma 5** Let $A \subseteq \mathbb{F}_p$ be a set and $f_A(x) = A(x) - |A|/p$. Then

$$\sum_{l \in \text{Aff}(\mathbb{F}_p)} \left| \sum_x f_A(x)f_A(lx) \right|^4 \ll |A|^5 \log |A|,$$

where the summation over $l$ in the last formula is taken over all affine transformations.

Finally, we need a simplified version of [23, Theorem 5].

**Theorem 6** Let $A, B \subseteq \mathbb{F}_p$ be sets, $|AB| \leq M|A|$, $k \geq 2$, and $|B| \gg k M^{2k+1}$. Then

$$T_{2k}^+(A) \lesssim_k M^{2k+1} \left( \frac{|A|^{2k+1}}{p} + |A|^{2k+1-1} \cdot |B|^{-\frac{k-1}{2}} \right). \quad (14)$$

3 The proof of the main result

We start with our counting Proposition 7. Let $\mathcal{P}, \mathcal{L} \subseteq \mathbb{F}_p \times \mathbb{F}_p$ be a set of points and a set of lines, correspondingly. The number of incidences between $\mathcal{P}$ and $\mathcal{L}$ is

$$I(\mathcal{P}, \mathcal{L}) := |\{(q, l) \in \mathcal{P} \times \mathcal{L} : q \in l\}|. \quad (15)$$
**Proposition 7** Let $A, B \subseteq \mathbb{F}_p$ be sets and $\mathcal{L}$ be a set of affine transformations. Then for any positive integer $k$ one has

$$\mathcal{I}(A \times B, \mathcal{L}) - \frac{|A||B||\mathcal{L}|}{p} \ll \sqrt{|A||B||\mathcal{L}|} \cdot (T_{2^k}(\mathcal{L})|A|\log |A|)^{1/2^{k+2}}. \quad (16)$$

**Proof.** We have

$$\mathcal{I}(A \times B, \mathcal{L}) = \frac{|A||B||\mathcal{L}|}{p} + \sum_{x \in B} \sum_{l \in \mathcal{L}} f_A(lx) = \frac{|A||B||\mathcal{L}|}{p} + \sigma.$$  

To estimate the error term $\sigma$ we use the Hölder inequality several times as in [17], [21] and obtain

$$\sigma^2 \leq |B| \sum_h r_{\mathcal{L}^{-1}\mathcal{L}}(h) \sum_x f_A(x)f_A(hx),$$

and further

$$\sigma^{2k} \leq |B|^{2k-1}|A|^{2k-1} \sum_h r_{(\mathcal{L}^{-1}\mathcal{L})^{2k-1}}(h) \sum_x f_A(x)f_A(hx).$$

Finally, applying Lemma 5 and the Hölder inequality one more time, we derive

$$\sigma^{2k+2} \ll |B|^{2k+1}|A|^{2k+1} \left( \sum_h r_{(\mathcal{L}^{-1}\mathcal{L})^{2k-1}}(h) \right)^3 \cdot |A|^5 \log |A| \ll$$

$$\ll |B|^{2k+1}|A|^{2k+1} T_{2^k}(\mathcal{L})|\mathcal{L}|^{2^{k+1}} \cdot |A|^5 \log |A|$$

as required. \qed

The main advantage of bound (16) of Proposition 7 is that we have an asymptotic formula for the number of incidences $\mathcal{I}(A \times B, \mathcal{L})$ (and the set $\mathcal{L}$ can be rather small) but not just upper bounds for $\mathcal{I}(P, \mathcal{L})$ as in [28]. An asymptotic formula for the quantity $\mathcal{I}(P, \mathcal{L})$ was known before in the specific case of large sets (see [31] or estimate (26) below) and in the case of Cartesian products but with large sets of lines, see [24] and [28].

In the next lemma we estimate the energy $T_k(\mathcal{L})$ for a concrete family of lines which will appear in the proofs of the results of our paper.

**Lemma 8** Let $A, B \subseteq \mathbb{F}_p$ be sets, and $\mathcal{L} = \{(a, b) : a \in A, b \in B\} \subseteq \text{Aff}(\mathbb{F}_p)$. Then for any $k \geq 2$ one has

$$T_k(\mathcal{L}) \leq |A|^{2k-1} T^+_{k}(B). \quad (17)$$
Proof. Let us consider the case of even $k$ and for odd $k$ the arguments are similar. One has $\mathcal{L}^{-1}\mathcal{L} = \{(a/c, (b-d)/c) : a, c \in A, b, d \in B\}$. Considering $T_{2k}(\mathcal{L})$, we arrive to two equations. The first one is

$$\frac{a_1 \ldots a_k}{c_1 \ldots c_k} = \frac{a'_1 \ldots a'_k}{c'_1 \ldots c'_k}. \quad (18)$$

If we fix all variables $a_1 \ldots a_k, a'_1 \ldots a'_k, c_1 \ldots c_k, c'_1 \ldots c'_k \in A$, then the number of the solutions to the second equation is $T_{2k}(\alpha_1 B, \ldots, \alpha_{2k} B)$, where $\alpha_1, \ldots, \alpha_{2k} \in \mathbb{F}_p^*$ are some elements of $A$ depending on the fixed variables. The last quantity is at most $T_{2k}^+(B)$ by Lemma 3. Returning to (18), we obtain the required inequality.

Now we can obtain our first driving result.

**Theorem 9** Let $A, B, X_1, Y_1, Z_1 \subseteq \mathbb{F}_p^*$ be sets, $A = XY_1$, $B = XY_2$, $|A| = |X||Y_1|/K_*$, $|B| = |X||Y_2|/K_*$, and $|XZ| \leq K|X|$, $|ZZ| \leq K|Z|$. Suppose that $|Z| \geq p^\delta$ for a certain $\delta \gg \log^{-1}\left(\frac{\log \tilde{p}}{\log K}\right)$. Then for a certain $k \ll \delta^{-1}$ the following holds

$$|\{(a, b) \in A \times B : a := f_*(b)\}| - \frac{K^2 K_*^2 |A||B|}{p} \ll K^2 K_*^2 \tilde{K} \sqrt{|A||B|} \cdot p^{-\frac{1}{10k}}. \quad (19)$$

**Proof.** Let $\sigma$ be the quantity from the left–hand side of (19). Also, let $Q_1 = AZ$, $Q_2 = BZ$. Then $|Q_1| \leq |XZ||Y_1| \leq K|X||Y_1| = KK_*|A|$ and, similarly, for $Q_2$. We have

$$|Z|^2 \sigma \leq |\{(q_1, q_2, z_1, z_2) \in Q_1 \times Q_2 \times Z^2 : q_1/z_1 := f_*(q_2/z_2)\}|.$$

Using the definition of the function $f_*$, we arrive to the equation

$$\frac{q_1}{z_1} = \frac{q_2}{\alpha q_2 + \beta z_2} \quad \Rightarrow \quad \frac{z_1}{q_1} - \frac{\beta z_2}{q_2} = \alpha. \quad (20)$$

The last equation can be interpreted as points/lines incidences with the set of lines $\mathcal{L} = Z \times Z$, any $l \in \mathcal{L}$ has the form $l : z_1 X - \beta z_2 Y = \alpha$ and the set of points $\mathcal{P} = Q_1^{-1} \times Q_2^{-1}$. Applying Proposition 7, we obtain for any $k$

$$\sigma - \frac{|Q_1||Q_2|}{p} \ll |Z|^{-1} \sqrt{|Q_1||Q_2|} \cdot (T_{2k}(\mathcal{L})|Q_1| \log |Q_1|)^{1/2k^2 + 2}.$$

Using our bounds for sizes of the sets $Q_1, Q_2$, combining with Lemma 5 and Theorem 3 we get

$$\sigma - \frac{K^2 K_*^2 |A||B|}{p} \ll KK_* \tilde{K} \sqrt{|A||B|} \cdot \left(KK_* |A||Z|^{-k+1/2}\right)^{1/2k^2 + 2}$$

provided $|Z| \geq_k \tilde{K}^{2k+1}$ and $|Z|^{k+1} \ll p^2$. Taking $|Z|^k \sim p$, we satisfy the second condition and obtain

$$\sigma - \frac{K^2 K_*^2 |A||B|}{p} \ll K^2 K_*^2 \tilde{K} \sqrt{|A||B|} \cdot p^{-\frac{1}{10k}}.$$

Choosing $k \sim 1/\delta$, we have the condition $|Z|^k \sim p$ and the assumption $\delta \gg \log^{-1}\left(\frac{\log p}{\log K}\right)$ implies that the inequality $|Z| \geq_k \tilde{K}^{2k+1}$ takes place. \qed
Remark 10 One can increase the generality of Theorem 9 considering different sets \(X_1, X_2, Z_1, Z_2\) such that \(|X_1Z_1| \leq K_1|X_1|, |X_2Z_2| \leq K_2|X_2|\) and so on. We leave the proof of this generalization to the interested reader.

Corollary 11 Let \(g\) be a primitive root and \(I, J \subseteq \mathbb{F}_p^*\) be two geometric progressions with the same base \(g\) such that
\[
\exp(C \log p / \log \log p) \ll |I| = |J| \leq p/2,
\]
where \(C > 0\) is an absolute constant. Then
\[
|\{(a, b) \in I \times J : a := f_*(b)\}| \leq (1 - \kappa)|I|,
\]
where \(\kappa > 0\) is an absolute constant.

Proof. Let \(I = a \cdot \{1, g, \ldots, g^n\}, J = b \cdot \{1, g, \ldots, g^n\}\), where \(n = |I| = |J|\). We apply Theorem 9 with \(A = I, B = J, Y_1 = \{a\}, Y_2 = \{b\}, X = \{1, g, \ldots, g^n\}, K_1 = 1\) and \(Z = \{1, g, \ldots, g^m\}\), where \(m = \lfloor cn\rfloor, c = 1/4\). Then \(K_1 \leq 1 + \epsilon\) and \(\tilde{K} < 2\). By formula (19), we obtain
\[
|\{(a, b) \in I \times J : a := f_*(b)\}| - \frac{(1 + \epsilon)^2|I||J|}{p} \ll |I| \cdot p^{-1/\epsilon^2}.
\]
We have \((1 + \epsilon)^2|I||J|/p \leq 2|I|/p\) because \(n \leq p/2\). Recalling that \(k \sim 1/\delta\) and \(\delta \gg (\log \log p)^{-1}\), we derive estimate (22) thanks to our condition (21). This completes the proof. \(\square\)

Now we are ready to prove Theorem 1 from the introduction, which we formulate in a slightly general form. In our arguments we use some parts of the proof from [7].

Theorem 12 Let \(p\) be a prime number and \(\gamma \in \mathbb{F}_p^*\) be an element of order at least
\[
\exp(\Omega(\log p / \log \log p)).
\]
Also, let \(\varepsilon_j\) be the random variables distributed uniformly on \(\{\gamma^{-1}, \gamma\}\). Consider the lazy Markov chain \(0 \neq X_0, X_1, \ldots, X_n, \ldots\) defined by
\[
X_{j+1} = \begin{cases} f_*(X_j) \cdot \varepsilon_{j+1} & \text{with probability } 1/2, \\ X_j & \text{with probability } 1/2. \end{cases}
\]
Then for an arbitrary \(c > 0\) and for any \(n = c \exp(\log p / \log \log p)\) one has
\[
\|P_n - U\| := \frac{1}{2} \max_{A \subseteq \mathbb{F}_p} \left| \mathbb{P}(X_n \in A) - \frac{|A|}{p-1} \right| \leq e^{-O(c)}.
\]
The same is true for the chain \(X_{j+1} = f_*(X_j) \cdot \varepsilon_{j+1}\), where \(\varepsilon_j\) denote the random variables distributed uniformly on \(\{1, \gamma^{-1}, \gamma\}\).

Let \(P\) be an ergodic Markov chain on a \(k\)-regular directed graph \(G = G(V, E)\). Let \(h(G)\) be the Cheeger constant
\[
h(G) = \min_{S \subseteq V} \frac{e(S, S^c)}{|S| - k|S|}, \tag{23}
\]
where \(e(S, S^c)\) is the number of edges between \(S\) and the complement of \(S\). We need a result from [4] (a more compact version is [7, Theorem 4.1]).
Theorem 13 Let $P$ be an ergodic Markov chain on a graph $G = G(V, E)$. Consider the lazy chain $X_0, X_1, \ldots, X_n, \ldots$ with transition matrix $(I + P)/2$, and starting from a certain deterministic $X_0$. Then for any $c > 0$ and any $n = ch(G)^{-2} \log |V|$ one has

$$\max_{A \subseteq V} \left| \frac{P(X_n \in A) - |A|}{|V|} \right| \leq e^{-O(c)}.$$

In our case $G = G(V, E)$ with $V = \mathbb{F}_p^*$ and $x \rightarrow y$ iff $y = f_s(x)\gamma^{\pm 1}$. Thus our task is to estimate the Cheeger constant of $G$. Take any $S, |S| \leq p/2$ and write $S$ as the disjoint union $S = \bigsqcup_{j \in J} G_j$, where $G_j$ are geometric progressions with step $\gamma^2$. Here and below we use the fact that $\mathbb{F}_p^*$ is cyclic, isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$ and generated by a fixed primitive root $g$. Consider $z, z\gamma, z\gamma^2$, where $z \in S$ is a right endpoint (if it exists) of some $G_j$. Then $z\gamma^2 \in S^c$ and $z, z\gamma^2$ are connected with $f^{-1}_s(z\gamma)$. The point $f^{-1}_s(z\gamma)$ belongs either $S$ or $S^c$ but in any case we have an edge between $S$ and $S^c$. Let $J = J_0 \bigsqcup J_1$, where for $j \in J_0$ the set $G_j$ has no the right endpoint and $J_1 = J \setminus J_0$. Clearly, $|J_0| \leq 2|S|/\text{ord}(\gamma)$. By the argument above

$$2h(G) \geq \frac{|J_1|}{|S|} \geq \frac{|J|}{|S|} - \frac{2}{\text{ord}(\gamma)}. \tag{24}$$

We want to obtain another lower bound for $h(G)$, which works better in the case when $J$ is small. Put $L = |S|/|J|$ and let $\omega \in (0, 1)$ be a small parameter, which we will choose later. One has $\sum_{j \in J} |G_j| = |S|$ and hence $\sum_{j : |G_j| \geq \omega L} |G_j| \geq (1 - \omega)|S|$. Splitting $G_j$ up into intervals of length exactly $L_\omega := \omega L/2$, we see that the rest is at most $(1 - 2\omega)|S|$. Hence we have obtained some geometric progressions $G'_i, i \in I$, having lengths $L_\omega$ and step $\gamma^2$ and such that $\sum_{i \in I} |G'_i| \geq (1 - 2\omega)|S|$. Put $S' = \bigsqcup_{i \in I} G'_i$ and let $\Omega = S \setminus S', |\Omega| \leq 2\omega|S|$. In other words, we have $S' = XY, |S'| = |X||Y| \geq (1 - 2\omega)|S|$, where $X = [1, \gamma^2, \ldots, \gamma^{2(L_\omega-1)}]$ and $Y$ is a certain set of multiplicative shifts. Clearly,

$$2h(G) \geq \frac{e(S, S^c)}{|S|} \geq 1 - \frac{e(S, S)}{|S|} \geq 1 - 8\omega - \frac{e(S', S')}{|S|}. \tag{25}$$

Put $Z = [1, \gamma^2, \ldots, \gamma^{2(L_\omega-1)}]$, where $L_\omega' = [cL_\omega], c = 1/4$. We have $|ZZ| < 2|Z|$. Also, by the assumption the element $\gamma$ has order at least $\exp(\Omega(\log p/\log \log p))$. Using Theorem 9 with $K = 1 + c, \delta = 2, k \sim 1/\delta$ and taking $\delta \geq C(\log \log p)^{-1}$ for sufficiently large constant $C > 0$, we get

$$\frac{e(S', S')}{|S|} < \frac{25|S'|}{16p} < p^{-\frac{1}{16}} \leq \frac{1}{32}.$$

Recalling that $|S'| \leq |S| \leq p/2$, we derive

$$\frac{e(S', S')}{|S|} < \frac{25}{32} + \frac{1}{32} = \frac{13}{16}.$$

Substituting the last formula into (25), taking sufficiently large $p$ and choosing $\omega = 2^{-8}$, say, we have $h(G) \geq 1/32$. We need to check the only condition of Theorem 9, namely, $|Z| \geq p^\delta$. If not, then $|S'|/|J| = L \ll L_\omega \ll |Z| < p^\delta \sim \exp(\Omega(\log p/\log \log p))$,
and hence $|J| \gg |S| \exp(-O(\log p/\log \log p))$. But then by (24) and our assumption ord($\gamma$) = $\exp(\Omega(\log p/\log \log p))$, we see that in any case $h(G) \gg \exp(-O(\log p/\log \log p))$. Combining the last bound for the Cheeger constant and Theorem 13 we derive $n \leq \exp(O(\log p/\log \log p))$.

The last part of Theorem 12 follows by the same method, combining with the arguments from [3] and [7, Section 4.3]. We need to ensure that the bijection $f_*(f_*^{-1}(\cdot, \gamma)) : \mathbb{F}_p^* \to \mathbb{F}_p^*$ has the same form as in (24) (with our usual convention that $f_*(-\beta/\alpha) = 1/\alpha$ of course). It can be check via a direct calculation or thanks to the fact that $f_*$ corresponds to the standard action of a lower–triangular matrix in $GL_2(\mathbb{F}_p)$. This completes the proof of Theorem 12.

Remark 14 Consider lazy Markov chain (2) with $f(x) = x^2$ and $p \equiv 3 \pmod{4}$. Using the same argument as in the proof of Theorem 12 we need to have deal with the equation $y + a = f(x + b) = x^2 + 2bx + b^2$, where $a, b$ belong to some arithmetic progression $P$ and $x, y$ are from a disjoint union of $J$ arithmetic progressions, see details in [7] (strictly speaking, now the stationary distribution is not uniform and, moreover, our graph is not regular which requires to have a modification of definition (23)). Then last equation can be interpreted as points/lines incidences with the set of lines $\mathcal{L}$ of the form $Y = 2bX + (b^2 - a)$ and the set of points $\mathcal{P} = (y - x^2, x)$. Using the main result from [3] (also, see [24]), we obtain

$$|\mathcal{I}(\mathcal{P}, \mathcal{L}) - |\mathcal{P}| |\mathcal{L}|/p| \leq \sqrt{|\mathcal{P}| |\mathcal{L}|/p}. \quad (26)$$

By formula (26) and the calculations as above (see details in [7, Section 4.2]) we have an expander if $|S|/J \sim |P| \gg \sqrt{p}$. If the last inequality does not holds, then $J \gg |S|/\sqrt{p}$ and by an analogue of formula (24), we obtain $h(G) \gg 1/\sqrt{p}$. Hence in view of Theorem 13 we see that the mixing time is $O(p \log p)$.

The method of the proof of Theorem 12 (and see Remark 14) allows us to produce easily some lazy Markov chains on $\mathbb{F}_p^*$ with the mixing time $O(p \log p)$, e.g.,

$$X_{j+1} = \begin{cases} \text{ind} \ (X_j) \cdot \epsilon_{j+1} & \text{with probability } 1/2, \\ X_j & \text{with probability } 1/2 \end{cases} \quad (27)$$

($X_0 \neq 0$) or as in (2) with $f(x) = \exp(x)$, namely,

$$X_{j+1} = \begin{cases} \exp (X_j) + \epsilon_{j+1} & \text{with probability } 1/2, \\ X_j & \text{with probability } 1/2 \end{cases} \quad (28)$$

Indeed, in the first chain we arrive to the equation $ya = \text{ind}(x) + \text{ind}(b)$ and in the second one to $y + b = \exp(x) \cdot \exp(a)$. Both equations correspond to points/lines incidences. Let us underline one more time that our functions ind($x$), $\exp(x)$ are defined on $\mathbb{F}_p^*$ but not on $\mathbb{F}_p$. In reality, one has much better bound for the mixing time of two Markov chains above.

Theorem 15 Let $p$ be a prime number and $\gamma \in \mathbb{F}_p^*$. Then the mixing time of Markov chain (28) is $\exp(O(\log p/\log \log p))$. If, in addition, the order of $\gamma$ is $\exp(\Omega(\log p/\log \log p))$, then the mixing time of Markov chain (27) is $\exp(O(\log p/\log \log p))$. 


Proof. Our arguments follow the same scheme as the proofs of Theorem 9 and Theorem 12. In both cases we need to estimate the energy \( T_{2k} \) of the set of affine transformations \( L \) of the form \( x \to gx + r \), where coefficients \( g \in \Gamma \) and \( r \in P \) belongs to a geometric and an arithmetic progression of size \( \sqrt{|L|} \), respectively. An application of Lemma 3 is useless because \( T_{2k}(P) \) is maximal. Nevertheless, we consider the set \( L^{-1}L \) and notice that any element of \( L^{-1}L \) has the form \( x \to g_2/g_1x + (r_2 - r_1)/g_1 \), where \( g_1, g_2 \in \Gamma \) and \( r_1, r_2 \in P \). Now in view of the arguments of Lemma 3 our task is to estimate \( |\Gamma|^{2^{k+1}-1}|P|^{2^{k+1}T_{2k}(Q/\Gamma)} \), where \( Q = P - P \). Write \( W = Q/\Gamma \) and notice that \( |Q| < 2|P| \). Taking \( X \subseteq \Gamma^{-1} \) as in inequality (10) and applying this inequality with \( A = \Gamma^{-1}, B = \Gamma^{-1} \) and \( C = Q \), we see that

\[
|WX| = |Q/\Gamma \cdot X| \leq 2|Q/\Gamma| = 2|W|.
\]

Increasing the constant 2 to \( O(1) \) in the formula above, one can easily assume (or see [30]) that for a certain \( Y \) the following holds \( |Y| \geq |\Gamma|/2 \). Applying Theorem 6 with \( A = W \) and \( B = Y \), we obtain

\[
T_{2k}(W) \lesssim_k \frac{|W|^{2^{k+1}}}{p} + |W|^{2^{k+1}-1} \cdot |\Gamma|^{-rac{k-1}{2}}.
\]

Here we need to assume that \( |\Gamma| \gtrsim_k 1 \). Hence arguing as in Lemma 3 and using the trivial bound \( |W| \leq |L| \), we get

\[
T_{2^{k+1}}(L) \lesssim_k |\Gamma|^{2^{k+1}-1}|P|^{2^{k+1}} \left( \frac{|L|^{2^{k+1}}}{p} + |L|^{2^{k+1}-1} \cdot |\Gamma|^{-rac{k-1}{2}} \right) \ll |L|^{2^{k+2}} \cdot |\Gamma|^{-rac{k+5}{2}},
\]

provided \( |L| \gtrsim_k 1 \) and \( |\Gamma|^{k+3} \ll p^2 \). After that we apply the same argument as in the proof of Theorem 12. \qed

4 Combinatorial applications

We now obtain an application of the developed technique to Sidon sets and we follow the arguments from [26]. We need Lemma 3, Lemma 7 and Theorem 4 from this paper.

Lemma 16 Let \( A \subseteq G \) be a set. Then for any \( k \geq 2 \) one has

\[
\text{Sid}_{3k-3}(A) \gg \left( \frac{|A|^{2k}}{E_k(A)} \right)^{1/(2k-1)}, \quad \text{and} \quad \text{Sid}_{2k-2}(A) \gg \left( \frac{|A|^{2k}}{E_k(A)} \right)^{1/(2k-1)}. \tag{29}
\]

Lemma 17 Let \( A \subseteq G \) be a set, \( A = B + C \), and \( k \geq 1 \) be an integer. Then

\[
\text{Sid}_k(A) \leq \min\{|C|\sqrt{k|B|} + |B|, |B|\sqrt{k|C|} + |C|\}.
\]
Theorem 18 Let $A \subseteq \mathbb{G}$ be a set, $\delta, \varepsilon \in (0, 1]$ be parameters, $\varepsilon \leq \delta$.

1) Then there is $k = k(\delta, \varepsilon) = \exp(O(\varepsilon^{-1} \log(1/\delta)))$ such that either $E_k(A) \leq |A|^{k+\delta}$ or there is $H \subseteq \mathbb{G}$, $|H| \gtrsim |A|^{(1-\varepsilon)}$, $|H + H| \ll |A|^\varepsilon |H|$ and there exists $Z \subseteq \mathbb{G}$, $|Z||H| \ll |A|^{1+\varepsilon}$ with

$|(H + Z) \cap A| \gg |A|^{1-\varepsilon}$.

2) Similarly, either there is a set $A' \subseteq A$, $|A'| \gg |A|^{1-\varepsilon}$ and $P \subseteq \mathbb{G}$, $|P| \gtrsim |A|^{\delta}$ such that for all $x \in A'$ one has $r_{A-P}(x) \gg |P||A|^{-\varepsilon}$ or $E_k(A) \leq |A|^{k+\delta}$ with $k \ll 1/\varepsilon$.

To have deal with the real setting we need the famous Szemerédi–Trotter Theorem \[29\].

Theorem 19 Let $\mathcal{P}$, $\mathcal{L}$ be finite sets of points and lines in $\mathbb{R}^2$. Then

$$ I(\mathcal{P}, \mathcal{L}) \ll (|\mathcal{P}| |\mathcal{L}|)^{2/3} + |\mathcal{P}| + |\mathcal{L}|. $$

Now we are ready to prove Theorem 1. Take any $\delta < 1/2$, e.g., $\delta = 1/4$ and let $\varepsilon \leq \delta/4$ be a parameter, which we will choose later. In view of Lemma 18 we see that $E_k^x(A) \leq |A|^{k+\delta}$ implies

$$ \text{Sid}_{3k-3}^x(A) \gg |A|^{\frac{1}{1+\frac{2\varepsilon}{2k-4}}} = |A|^{\frac{1}{1+\frac{1}{2k-4}}} \quad (30) $$

and we are done. Here $k = k(\varepsilon)$. Otherwise there is $H \subseteq \mathbb{F}$, $|H| \gtrsim |A|^{(1-\varepsilon)} \gtrsim |A|^{\delta/2}$, $|HH| \ll |A|^\varepsilon |H|$ and there exists $Z \subseteq \mathbb{F}$, $|Z||H| \ll |A|^{1+\varepsilon}$ with $|(H \cdot Z) \cap A| \gg |A|^{1-\varepsilon}$. Here the product of $H$ and $Z$ is direct. Put $A_s = (H \cdot Z) \cap A$, $|A_s| \gg |A|^{1-\varepsilon}$ and we want to estimate $E^{x}_{k+1}(A_s + 1)$ or $E^{x}_{k+1}(A_s + 1)$ for large $l$. After that having a good upper bound for $E^{x}_{k+1}(A_s + 1)$ or $E^{x}_{k+1}(A_s + 1)$, we apply Lemma 15 again to find large multiplicative Sidon subset of $A_s$.

First of all, notice that in view of (30), one has

$$ |HA^{−1}_{s}| \ll |HH^{−1}| |Z| \ll |A|^{2\varepsilon} |H||Z| \ll |A|^{1+3\varepsilon}. $$

In other words, the set $A^{−1}_{s}$ almost does not grow after the multiplication with $H$. Let $Q = HA^{−1}_{s}$, $|Q| \ll |A|^{1+3\varepsilon}$ and also let $M = |A|^{\varepsilon}$. Secondly, fix any $\lambda \neq 0, 1$. The number of the solutions to the equation $a_1/a_2 = \lambda$, where $a_1, a_2 \in A_s + 1$ does not exceed

$$ \sigma_\lambda := |H|^{−2}|\{ h_1, h_2 \in H, q_1, q_2 \in Q : (h_1/q_1 + 1)/(h_2/q_2 + 1) = \lambda \}|. $$

The last equation has form (20), namely,

$$ \frac{h_1}{q_1} - \lambda \frac{h_2}{q_2} = \lambda - 1 $$

and can be interpreted as a question about the number of incidences between points and lines. For each $\lambda \neq 0, 1$ the quantity $\sigma_\lambda$ can be estimated as

$$ \sigma_\lambda \ll |H|^{−2} \cdot |Q||H|^{2−\kappa} \ll |A|^{1+3\varepsilon} |H|^{−\kappa} \quad (31) $$
Similarly to the proof of Theorem 8 above (in the case $F = \mathbb{R}$ the same is true thanks to Theorem 9). Here $\kappa = \kappa(\delta) > 0$. Indeed, by our assumption $|A| < \sqrt{p}$, Theorem 6 Proposition 7 and Lemma 8 we have

$$\sigma_\lambda - \frac{|Q|^2}{p} \lesssim |Q||H|^{-1/2}(|Q|T_2^+(H))^{1/2r+2} \lesssim |Q|\sqrt{M}M^3|A||H|^{-\frac{r+1}{2}} 1^{2r+2} \quad (32)$$

provided $|H| \gtrsim_r M^{2r+1}$ and $|H|^{r+1} \ll p$. Here $r$ is a parameter and we take $r \sim 1/\delta$ to satisfy the second condition. To have the first condition just take $\epsilon^{2r+1} \ll \delta$ (in other words, $\epsilon \ll \exp(-\Omega(1/\delta))$) and we are done because $|H| \gg |A|^{\delta/2}$.

Further using $|H| \gg |A|^{\delta/2}, |A| \gg |A|^{1-\epsilon}$ and choosing any $\epsilon \ll \delta \kappa/100$, we obtain after some calculations and formula (31) that $\sigma_\lambda \ll |A|^{1-\delta \kappa/4}$. Hence taking sufficiently large $l \gg (\delta \kappa)^{-1}$, we derive

$$\hat{E}_l^{\times}(A) = \sum_{\lambda} |l^{l+1}_{A_\lambda} A_\lambda(\lambda) \ll |A_\lambda|^{l+1} + (|A_\lambda|^{1-\delta \kappa/2})^2 |A_\lambda|^{l+1} + |A_\lambda|^{l+2-\delta \kappa/2} \ll |A_\lambda|^{l+1}.$$

Applying Lemma 13 and choosing $\epsilon \ll l^{-1}$, we see that

$$\text{Sid}^\times_{2l}(A) \gg \text{Sid}^\times_{2l}(A_\lambda) \gg |A_\lambda|^{l+1} \gg |A|^{(1+\epsilon)(l+1)\frac{1}{2(2l+1)}} = |A|^{\frac{1}{2} + \frac{1-2(2l+1)}{2(2l+1)}} \gg |A|^{\frac{1}{2} + c},$$

where $c = c(\delta) > 0$ is an absolute constant. We have obtained bound (5) of Theorem 2.

As for estimate (7), we use the same argument as above but now our analogue of the quantity $\sigma_\lambda$ is $\exp(q_1) - \exp(q_2) \exp(h_2) = \lambda$, where $q_1, q_2 \in Q = A_\lambda + H, h_1, h_2 \in H$ (we use the notation above). The last equation can be treated as points/lines incidences with the set of lines $x_1 \exp(h_1) - y \exp(h_2) = \lambda, |L| = |H|^2$ and the correspondent set of points $P$ of size $|Q|^2$. Then analogues of bounds (31), (32) take place and we are done.

It remains to obtain estimate (8) of the theorem. For any sets $X_1, X_2, X_3$ consider the set $R[X_1, X_2, X_3]$

$$R[X_1, X_2, X_3] = \left\{ \frac{x_1 - x_3}{x_2 - x_3} : x_1, x_2, x_3 \in X, x_2 \neq x_3 \right\}.$$

If $X_1 = X_2 = X_3 = X$, then we put $R[X_1, X_2, X_3] = R[X]$. One can check that $1 - R[X_1, X_2, X_3] = R[X_1, X_3, X_2]$. For $F = \mathbb{R}$ or $F = \mathbb{F}_p$ we put $X = P, A = R[X]$, where $P = \{1, \ldots, n\}, P - \{n, \ldots, n\}$ and let $n < \sqrt{p}$ in the case of $\mathbb{F}_p$. Then $A$ is contained in $P/P := B \cdot C$ and in view of Lemma 17 any multiplicative $k$-Sidon subset of $A$ has size at most $O(\sqrt{k}|A|^{3/4})$ because as one can check $|A| \gg |P|^2$. Further $1 - A = A$ and hence the same argument is applicable for the set $1 - A$. It remains to notice that $\text{Sid}^\times(X) = \text{Sid}^\times(-X)$ for any set $X$. Finally, let us make a remark that there is an alternative (but may be a little bit harder) way to obtain estimate (8). Indeed, consider $R[\Gamma]$, where $\Gamma \subseteq \mathbb{F}_p^n, |\Gamma| < \sqrt{p}$ is a multiplicative subgroup (we consider the case $F = \mathbb{F}_p$, say). One can notice that $R[\Gamma] = (\Gamma - 1)/(\Gamma - 1)$ and repeat the argument above.

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