THE SCHEME OF LIFTINGS AND APPLICATIONS

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Abstract. We study the locus of the liftings of a homogeneous ideal $H$ in a polynomial ring over any field. We prove that it can be endowed with a structure of scheme $L_H$ by applying the constructive methods of Gröbner bases, for any fixed term order. Indeed, this structure does not depend on the term order, since it can be defined as the scheme representing the functor of liftings of $H$. We also provide an explicit isomorphism between the schemes corresponding to two different term orders.

Our approach allows to embed $L_H$ in a Hilbert scheme as a locally closed subscheme, with the consequence that its radical locus is an open subset. Moreover, we show that over an infinite field every ideal defining a Cohen-Macaulay scheme of codimension two has a radical lifting, giving in particular an answer to an open question posed by L. G. Roberts.

Introduction

In this paper we consider the lifting problem as proposed in terms of ideals first in [9] and then in [19] and, equivalently, in terms of $K$-algebras by Grothendieck (e.g. [19] and the references therein or [5]). Many authors have investigated this interesting problem, sometimes also describing particular lifting procedures to construct algebraic varieties with specific properties (see [15, 11, 9, 17, 19, 16, 14] and the references therein).

We propose and use a new approach that is based on the theory of representable functors. Indeed, we define the functor of liftings of a homogeneous polynomial ideal $H$ and show that it is representable, in the perspective given by [13] for Gröbner strata and according to the point of view of [2]. Our approach is constructive and we compute the scheme of liftings of $H$, i.e. the scheme that parameterises the liftings of $H$ and represents the functor, by a reformulation of a result of [6] in terms of Gröbner bases.

An almost immediate result of the application of our approach, together with the features of Gröbner strata, is that a scheme of liftings can be embedded in a Hilbert scheme, with the consequence that its radical locus is an open subset.

Moreover, we are able to prove that every ideal defining a Cohen-Macaulay scheme of codimension two has a radical lifting. This result is particularly significant in the context of the study of radical liftings, because of the lack of information endured until now and highlighted in [17] in the case of polynomial homogeneous ideals in three variables.

The paper is organized in the following way. Referring to [8, 18, 12, 13], in Section 2 we recall definitions and main features of Gröbner strata. Moreover, we give an improvement of [12, Theorem 4.7] (Theorem 2.2) which will be useful to embed the scheme of liftings of a homogeneous ideal in a Hilbert scheme.

In Section 3, we define the functor of liftings of a homogeneous polynomial ideal and introduce the constructive tool we use to represent it, i.e. a reformulation of [6, Theorem 2000 Mathematics Subject Classification. 13P10, 14B10, 14M05.

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by means of Gröbner bases (Theorem 3.2). In Section 4, we prove that a functor of liftings is representable, thus obtaining that the construction of the scheme of liftings that arises from Theorem 3.2 does not depend on the fixed term order, up to isomorphisms (Theorems 4.2 and 5.1). Moreover, we describe how we embed the scheme of liftings of a homogeneous polynomial ideal in a Hilbert scheme and, then, deduce that its radical locus is an open subset (Corollary 4.6). This fact gives a contribution to a question posed in [14, Remark at pag. 332]. In Section 5, we give an explicit construction of the above isomorphisms (Theorem 5.3).

In Section 6, exploiting the Hilbert-Burch Theorem and the potentiality of Gröbner deformations, we conceive a constructive method to show that every Cohen-Macaulay scheme of codimension two has a radical lifting (Theorem 6.6). In particular, we provide an answer to a question posed by L. G. Roberts in [17] about the existence of radical liftings for homogeneous polynomial ideals in three variables: our result gives an affirmative answer to this question in case the ideals are saturated. Moreover, although in general a scheme of liftings is not reduced and irreducible, we find that, if \( H \) is a saturated homogeneous polynomial ideal defining a Cohen-Macaulay scheme of codimension two, then the scheme of liftings of \( H \) is isomorphic to an affine space, (Theorem 6.4).

All the results we present are based on constructive arguments. Hence, the last section is devoted to give explicative examples of the constructive methods we introduce and apply in this paper.

1. Generalities

A term is a power product \( x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \). Let \( \mathbb{T}_x \) and \( \mathbb{T}_{x,x_n} \) be the set of terms in the variables \( x = \{x_0, \ldots, x_{n-1}\} \) and \( x, x_n \), respectively.

We consider the variables ordered as \( x_0 > x_1 > \cdots > x_n \). We denote by \( \max(x^\alpha) \) the biggest variable that appears in \( x^\alpha \) and, analogously, \( \min(x^\alpha) \) is the smallest variable that appears in \( x^\alpha \). The degree of a term is \( \deg(x^\alpha) = \sum \alpha_i = |\alpha| \).

**Definition 1.1.** For a given term order \( \prec \) on \( \mathbb{T}_x \), will denote \( \prec_n \) the corresponding *degreverse term order* in \( \mathbb{T}_{x,x_n} \), namely the graded term order such that for two terms \( x^\alpha \) and \( x^\beta \) in \( \mathbb{T}_{x,x_n} \) of the same degree, \( x^\alpha \prec_n x^\beta \) if \( \alpha_n > \beta_n \) or \( \alpha_n = \beta_n \) and \( \frac{\alpha}{x_n} \leq \frac{\beta}{x_n} \).

We will always consider commutative rings with unit such that \( 1 \neq 0 \) and every morphism will preserve the unit.

Let \( K \) be a field. From now on, we will denote the polynomial ring \( K[x_0, \ldots, x_{n-1}] \) by \( K[x] \) and the polynomial ring \( K[x_0, \ldots, x_n] \) by \( K[x, x_n] \). For any \( K \)-algebra \( A \), \( A[x] \) will denote the polynomial ring \( A \otimes_K K[x] \) and \( A[x, x_n] \) will denote \( A \otimes_K K[x, x_n] \). Obviously, \( A[x] \) is a subring of \( A[x, x_n] \), hence the following notations and assumption will be stated for \( A[x, x_n] \) but will hold for \( A[x] \) too.

For every \( a \in A \), we set \( \deg(a) = 0 \) and, hence, for every \( f \in A[x, x_n] \), \( \deg(f) = \max\{\deg(x^\alpha) | x^\alpha \in \text{Supp}(f)\} \). In this way \( K[x] \) and \( A[x, x_n] \) are *standard graded algebras*.

For any non-zero homogeneous polynomial \( f \in A[x, x_n] \), the *support* of \( f \) is the set \( \text{Supp}(f) \) of terms that appear in \( f \) with a non-zero coefficient and \( \text{coeff}(f) \) is the set of such coefficients. The *head term* of a polynomial \( f \) is the maximum of \( \text{Supp}(f) \) w.r.t. a fixed term order.

**Remark 1.2.** As for the graded reverse lexicographic term order (degrevlex, for short), which is a particular degréverse term order, also for every degréverse term order, we
have that if the head term of a homogeneous polynomial \( f \) is divisible by \( x_n^r \) then the polynomial \( f \) is divisible by \( x_n^r \).

A monomial ideal is generated by terms. We denote by \( j \) a monomial ideal in \( K[x] \) and by \( J \) the monomial ideal \( j \cdot K[x, x_n] \). Note that \( j \) and \( J \) have the same monomial basis, that we denote by \( B_j \). We denote by \( \mathcal{N}(j) \) the sous-escalier of \( j \), that is the set of terms in \( T_x \) not belonging to \( j \). Analogously, we have \( \mathcal{N}(J) \subseteq T_{x, x_n} \).

2. Background: Gröbner strata

In what follows, we will always consider homogeneous polynomials and ideals. Moreover, the polynomial ideals that we consider have and are always given by monic Gröbner bases. This is a key point for the use of functors we will introduce, because of the following two facts:

(i) there is no ambiguity when using the terminology “initial ideal”, because the head terms of the polynomials in the Gröbner bases we consider have coefficient 1;
(ii) if \( \varphi : A \to B \) is a morphism of \( K \)-algebras and \( I \) is an ideal in \( A[x] \) (or \( A[x, x_n] \)) generated by a monic Gröbner basis \( G_I \), then \( I \otimes_A B \) is generated by \( \varphi(G_I) \) which is again a monic Gröbner basis, with the same head terms [2]. In other words, monic Gröbner bases have a good behavior with respect to the extension of scalars. Recall that the polynomials of a reduced Gröbner basis are monic by definition.

Now, let \( J \) be any monomial ideal in \( A[x, x_n] \) and \( \sigma \) be a term order on \( T_{x, x_n} \). Given an ideal \( I \) in \( A[x, x_n] \), we will denote by \( \text{in}_\sigma(I) \) the initial ideal of \( I \) w.r.t. \( \sigma \).

**Definition 2.1.** [12] The family of the homogeneous ideals \( I \subseteq A[x, x_n] \) having the monomial ideal \( J \) as initial ideal with respect to \( \sigma \), is called a Gröbner stratum and denoted by \( \text{St}^\sigma_{\sigma,J}(A) \).

By construction, the ideals belonging to a Gröbner stratum share the same Hilbert function. Further, \( \text{St}^\sigma_{\sigma,J} \) is a representable functor between the category of \( K \)-algebras and that of sets. We call its representing scheme Gröbner stratum scheme and denote it by \( \text{St}^\sigma_{\sigma,J} \). Now, we briefly recall the construction of the representing scheme \( \text{St}^\sigma_{\sigma,J} \) and some of its main features, for which anyway we refer to [12].

In order to compute \( \text{St}^\sigma_{\sigma,J} \), we consider a set of polynomial \( \mathcal{G} \) of the following shape:

\[
\mathcal{G} = \{ F_\alpha = x^\alpha + \sum_{x^\gamma x^\delta \sigma} C_{\alpha \gamma} x^\gamma : \text{Ht}(F_\alpha) = x^\alpha \in B_J \} \subset K[C][x, x_n]
\]

where \( C \) is a compact notation for the set of new variables \( C_{\alpha \gamma} \).

Denote by \( a \) the ideal in \( K[C] \) generated by the coefficients of the complete reductions of the \( S \)-polynomials \( S(F_\alpha, F_\beta) \) with respect to \( \mathcal{G} \). By [12, Proposition 3.5], the ideal \( a \) depends only on \( J \) and \( \sigma \) because it can be defined in an equivalent intrinsic way and defines the affine scheme \( \text{St}^\sigma_{\sigma,J} \) which represents the functor \( \text{St}^\sigma_{\sigma,J} \).

If in particular \( J \) is a strongly stable saturated ideal and \( \sigma \) is the degrevlex term order, then we have

\[
\text{St}^\sigma_{\sigma,J} \simeq \text{St}^\sigma_{\sigma,J_{\geq m}},
\]

for every integer \( m \) [12, Proposition 4.11]. Now, we observe that this last result holds under weaker hypotheses on \( J \) and on \( \sigma \), by the following more general version of [12, Theorem 4.7].
Theorem 2.2. Let $J \subset A[x, x_n]$ be a monomial ideal with Hilbert polynomial $p(t)$ and assume that $B_J \subset \mathbb{T}_{x}$ and that $\prec_n$ is a degreverse term order. Then
\[ \text{St}_{J^n} \cong \text{St}_{J_{\geq m}}^n, \text{ for every integer } m. \]
Hence $\text{St}_{J^n}$ can be embedded in the Hilbert scheme $\mathcal{Hilb}_p^\mathbb{N}$ as a locally closed subscheme.

Proof. We show that the functors $\text{St}_{J^n_{\geq m}}$ and $\text{St}_{J^n}$ are isomorphic. As a consequence, their representing scheme are isomorphic too. Recall that we always consider monic Gröbner bases. We denote by $G_I = \{ f_\alpha \}_\alpha$ the reduced Gröbner basis of $I \in \text{St}_{J^n}(A)$. Then the ideal $I_{\geq m}$ has Gröbner basis (in general non reduced) $W_{I_{\geq m}}$, made up of the polynomials $f_\alpha x^n$ with $|\gamma| = d_\alpha := \max\{0, m - |\alpha|\}$. Observe that the polynomials in $W_{I_{\geq m}}$ are still monic. Furthermore, $W_{I_{\geq m}}$ contains the set of polynomials $\overline{G_I} = \{ f_\alpha x^m \; | \; f_\alpha \in G_I \}$, $\overline{G_I}$ is contained also in the reduced Gröbner basis $G_{I_{\geq m}}$ of $I_{\geq m}$. Obviously, from $\overline{G_I}$ we can recover the polynomials of $G_I$ and of $I$ too.

Hence, there is a well-defined natural transformation of functors
\[ \begin{array}{c}
\text{St}_{J^n_{\geq m}} \rightarrow \text{St}_{J^n} \\
I \mapsto I_{\geq m}.
\end{array} \]
Indeed, for every $K$-algebra morphism $\varphi : A \rightarrow B$ and ideal $I \in \text{St}_{J^n}(A)$ we obtain that $I_{\geq m} \otimes_A B = (I \otimes A)_{\geq m}$ since in $B[x_0, \ldots, x_n]$ these two ideals are both generated by $\varphi(W_{I_{\geq m}})$.

The above natural transformation of functors is actually an isomorphism, with inverse
\[ \begin{array}{c}
\text{St}_{J^n_{\geq m}} \rightarrow \text{St}_{J^n} \\
L \mapsto (L : x_n^\infty).
\end{array} \]
Indeed, consider the reduced Gröbner basis of an ideal $L \in \text{St}_{J^n_{\geq m}}$: for every term $x_\alpha \in B_J$, this basis contains a polynomial $h_\alpha$ whose head term is $x_\alpha x_\alpha^{d_\alpha}$. Since $\prec_n$ is a degreverse term order, we have that $h_\alpha / x_\alpha^{d_\alpha} \in (L : x_\alpha^n)$, hence $(L : x_\alpha^n)$ contains the set of generators of an ideal whose initial ideal is $J$. Since no element in the basis of $J$ is divisible by $x_n$, we have in $\prec_n(L : x_\alpha^n) = J$ and, hence, $(L : x_\alpha^n) \in \text{St}_{J^n_{\geq m}}(A)$.

Arguing again on the polynomials in $L$, which are monic and with head terms of kind $x_\alpha x_\alpha^{d_\alpha}$ for every $x_\alpha \in B_J$, it is immediate that for every $K$-algebra morphism $\varphi : A \rightarrow B$ we obtain that $(L \otimes_A B) : x_\alpha^n = (L : x_\alpha^n) \otimes_A B \in \text{St}_{J^n_{\geq m}}(B)$, where the inclusion “$\subseteq$” is a standard fact (e.g. [1, Exercise 1.18]). The opposite one is due to the facts that the extension of scalars does not modify the head terms and that $x_n$ is not a zero-divisor for $J$. Indeed, let $f \otimes_A B$ be an element of $L \otimes_A B$ and $x_n^k$ be the maximal power of $x_n$ by which the head term of $f$ is divisible. Then, $x_n^k$ is also the maximal power of $x_n$ by which $f$ is divisible (see Remark 1.2) and the same happens for $f \otimes_A B$, because the extension of scalars does not modify the head terms, being the corresponding polynomials monic. In conclusion, letting $\tilde{f} := f / x_n^k$, we have that $\tilde{f} \otimes_A B$ belongs to $(L \otimes_A B) : x_\alpha^n$, thus it belongs to $(L : x_\alpha^n) \otimes_A B$.

The last assertion is a direct consequence of the previous one and [12, Theorem 6.3].

Remark 2.3. If $J \subset A[x, x_n]$ is a monomial ideal with $B_J \subset K[x]$ then $J$ is saturated, but the vice versa is false. We could prove the analogous of Theorem 2.2 for every ideal $J'$ such that $(J')^{sat} = J$ with some further details in the proof. \hfill \Box
3. The functor of liftings of a homogeneous polynomial ideal

In this section, first we recall what a lifting of a fixed homogeneous ideal \( H \subseteq K[x] \) with respect to \( x_n \) is, or simply \( x_n \)-lifting of \( H \), referring to [9, 19, 14]. Then, following the perspective of [13], we introduce a functorial description of these liftings.

**Definition 3.1.** Let \( H \) be a homogeneous ideal of \( K[x] \) and \( A \) be a \( K \)-algebra. A homogeneous ideal \( I \) of \( A[x, x_n] \) is called a lifting of \( H \) with respect to \( x_n \) or a \( x_n \)-lifting of \( H \) if the following conditions are satisfied:

(a) the indeterminate \( x_n \) is a non-zero divisor in \( A[x, x_n]/I \);
(b) \((I, x_n)/(x_n) \cong H \) or, equivalently,
(b') \( H = (g(x_0, x_1, \ldots, x_{n-1}, 0) : \forall g \in I) \).

For every homogeneous ideal \( H \subseteq K[x] \) and for every \( K \)-algebra \( A \) we consider the set
\[
\mathbb{L}_H(A) = \{ I \subseteq A[x, x_n] : I \text{ is a } x_n \text{-lifting of } H \}.
\]

Next result is a reformulation of [6, Theorem 2.5] in terms of Gröbner bases.

**Theorem 3.2.** Let \( A \) be a \( K \)-algebra, \( H \) a homogeneous ideal of \( K[x] \) and \( I \) a homogeneous ideal of \( A[x, x_n] \). Then, the following conditions are equivalent:

(i) the ideal \( I \) belongs to \( \mathbb{L}_H(A) \);
(ii) the reduced Gröbner basis of \( I \) w.r.t. a degreverse term order \( \prec_n \) on \( A[x, x_n] \) is \( \{f_\alpha + g_\alpha \}_\alpha \), where \( \{f_\alpha \}_\alpha \) is the reduced Gröbner basis of \( H \) w.r.t. the restriction of \( \prec \) to \( K[x] \) and \( g_\alpha \) is a homogeneous polynomial in \( A[x, x_n] \) of degree \( \deg(f_\alpha) \) divisible by \( x_n \).

**Proof.** To prove that (ii) implies (i) first we observe that \( x_n \) is not a zero-divisor in \( A[x, x_n]/I \), because the head terms of the polynomials \( f_\alpha + g_\alpha \) are not divisible by \( x_n \) and these polynomials form a Gröbner basis w.r.t. \( \prec \). Moreover, for every \( \alpha \), \((f_\alpha + g_\alpha)(x_0, \ldots, x_{n-1}, 0) = f_\alpha \), hence \( H = (g(x_0, x_1, \ldots, x_{n-1}, 0) : \forall g \in I) \).

Vice versa, if \( I \) is a \( x_n \)-lifting of \( H \), then \( x_n \) is not a zero-divisor in \( A[x, x_n]/I \). Thus, the reduced Gröbner basis of \( I \) w.r.t. \( \prec_n \) is exactly of the described type. \( \square \)

**Proposition 3.3.** If \( \phi : A \to B \) is a \( K \)-algebra morphism, then for every \( I \in \mathbb{L}_H(A) \), the ideal \( I \otimes_A B \) belongs to \( \mathbb{L}_H(B) \).

**Proof.** Let \( G \) be the reduced Gröbner basis of \( H \) with respect to a term order \( \prec \) and \( \mathcal{G} \) be the corresponding (monic) reduced Gröbner basis of \( I \in \mathbb{L}_H(A) \) w.r.t. \( \prec_n \), in the sense of Theorem 3.2. We have
\[
G = \{f_\alpha\}_\alpha, \quad \mathcal{G} = \{f_\alpha + g_\alpha\}_\alpha = \{f_\alpha + \sum_{x_n x^\gamma \in \mathcal{N}(J)_{|\alpha|}} c_{\alpha \gamma} x_n x^\gamma \}, \quad c_{\alpha \gamma} \in A.
\]

The ideal \( I \otimes_A B \) is then generated by
\[
\phi(\mathcal{G}) = \{f_\alpha + \sum_{x_n x^\gamma \in \mathcal{N}(J)_{|\alpha|}} \phi(c_{\alpha \gamma}) x_n x^\gamma \}_\alpha.
\]

Since the direct sum of free modules is preserved by extension of scalars, \( \phi(\mathcal{G}) \) is still a Gröbner basis with respect to \( \prec_n \) and it fulfills condition (ii) of Theorem 3.2. Hence \( I \otimes_A B \) is a lifting of \( H \) in \( B[x] \). \( \square \)
The previous result allows us to define a functor between the category of \( K \)-algebras and the category of sets as follows.

**Definition 3.4.** The functor of liftings of \( H \)

\[
L_H : K \text{-Alg} \to \text{Set}
\]

associates to every \( K \)-algebra \( A \) the set \( L_H(A) \) and to every morphism of \( K \)-algebras \( \phi : A \to B \) the map

\[
L_H(\phi) : L_H(A) \to L_H(B) \quad I \mapsto I \otimes_A B.
\]

**Remark 3.5.** In [19], the definition of \( x_n \)-lifting is actually given by a more general version of condition (b'). Indeed, the ideal \( H \) can be replaced by its image via an automorphism \( \theta \) of \( K[x] \) as a graded \( K \)-algebra. Anyway, our study of \( x_n \)-liftings by the functor \( L_H \) includes this more general situation because, if we extend the automorphism \( \theta \) to \( A[x, x_n] \) by fixing \( \theta(x_n) = x_n \), we can apply \( \theta \) also to the ideal \( I \) and have that \( I \) is an \( x_n \)-lifting of \( H \) if and only if \( \theta(I) \) is a \( x_n \)-lifting of \( \theta(H) \).

More precisely, we will define the scheme of liftings, which will represent the functor of liftings, as a closed subscheme of a suitable affine space. The action of an automorphism \( \theta \) induces an isomorphism among the schemes of liftings of \( H \) and \( \theta(H) \), respectively.

### 4. Representability of the functor \( L_H \)

Given the homogeneous ideal \( H \) in \( K[x] \) and its reduced Gröbner basis \( G = \{f_\alpha\}_\alpha \) w.r.t. \( \prec \), for every \( x^\alpha \in J = \text{in}_\prec(H) \) we define

\[
(4.1) \quad g_\alpha := \sum_{x^\gamma \in N(J)_\alpha} C_{\alpha\gamma} x^\gamma, \quad G = \{f_\alpha + g_\alpha\}_\alpha.
\]

where the \( C_{\alpha\gamma} \)'s are new variables. We set \( C = \{\text{coeff}(g_\alpha)\}_\alpha = \{C_{\alpha\gamma}\}_{\alpha,\gamma} \) and fix a term order on the monomials of \( K[C] \) by which we extend the term order \( \prec_n \) to an elimination term order of the variables \( x, x_n \) in \( K[C][x, x_n] \). For simplicity, we keep on using the notation \( \prec_n \) for this elimination term order on \( K[C][x, x_n] \).

**Proposition 4.1.** Let \( H, G, \mathcal{G}, \prec \) and \( \prec_n \) be as above and consider an ideal \( \mathfrak{h} \) in \( K[C] \) with Gröbner basis \( \mathcal{H} \). The followings are equivalent:

(i) \( G \cup \mathcal{H} \) is a Gröbner basis in \( K[C][x, x_n] \);

(ii) \( \mathfrak{h} \) contains the coefficients of all the polynomials in the ideal \((G)K[C][x, x_n] \) that are reduced modulo \( \text{in}_\prec(G) \);

(iii) \( \mathfrak{h} \) contains all the coefficients of every complete reduction of \( S(f_\alpha + g_\alpha, f_{\alpha'} + g_{\alpha'}) \) with respect to \( G \).

**Proof.** It is sufficient to repeat the arguments of [12, Proposition 3.5]. \( \square \)

**Theorem 4.2.** Given a homogeneous ideal \( H \subseteq K[x] \) and a term order \( \prec \), let \( \mathfrak{h} \) be the smallest ideal in \( K[C] \) satisfying the conditions of Proposition 4.1. The affine scheme \( \text{Spec}(K[C]/\mathfrak{h}) \) represents the functor \( L_H \).

**Proof.** This is a consequence of Proposition 4.1. It is sufficient to observe that by Theorem 3.2 an ideal \( I \) in \( A[x, x_n] \) belongs to \( L_H(A) \) if and only if it has a Gröbner basis with respect to \( \prec_n \) of the shape of (4.1), where the parameters \( C_{\alpha\gamma} \) are replaced by constants \( c_{\alpha\gamma} \in A \), which satisfy the conditions in \( \mathfrak{h} \). The choice of these coefficients \( c_{\alpha\gamma} \) is unique, since the
reduced Gröbner basis of \( I \) is, and it corresponds to a \( K \)-algebra morphism \( K[C]/\mathfrak{a} \rightarrow A \), i.e. to a scheme morphism \( \text{Spec}(A) \rightarrow L_H \).

**Definition 4.3.** For every homogeneous ideal \( H \subset K[x] \), the representing scheme of the functor \( L_H \) is called the *scheme of liftings of \( H \)* and is denoted by \( L_H \).

Given a homogeneous ideal \( H \subset K[x] \) and a term order \( \prec \), let \( j \) be the initial ideal of \( H \) with respect to \( \prec \), \( f(t) \) be the Hilbert function of \( K[x]/H \) and \( p(t) \) be the Hilbert polynomial of the Hilbert function \( F(t) = \sum_{i=0}^{t} f(i) \).

**Corollary 4.4.** In the above settings, \( L_H \) is a closed subfunctor of \( \text{St}_{\prec}^n \) and, hence, \( L_H \) is a locally closed subscheme of the Hilbert scheme \( \text{Hilb}^n_{p(t)} \).

**Proof.** It is sufficient to compute the ideal defining \( \text{St}_{\prec}^n \) and the reduced Gröbner basis of \( H \). Then, we consider the linear section of \( \text{St}_{\prec}^n \) obtained by the polynomials \( C_{\alpha\gamma} - c_{\alpha\gamma} \), where \( c_{\alpha\gamma} \) is the coefficient of \( x^\gamma \) in the polynomial \( f_\alpha \) of the Gröbner basis of \( H \). Then \( L_H \) is a closed subscheme of \( \text{St}_{\prec}^n \), hence by Theorem 2.2 a locally closed subscheme of \( \text{Hilb}^n_{p(t)} \). \( \square \)

**Remark 4.5.** Observe that, similarly to the Hilbert scheme, the scheme \( L_H \) parameterises not only the \( x_n \)-liftings of \( H \) according to Definition 3.1, which corresponds to closed points of \( L_H \), but also families of \( x_n \)-liftings.

We end this section, with the following result about the radical locus of \( L_H \).

**Corollary 4.6.** For every homogeneous ideal \( H \subset K[x] \), the radical locus of \( L_H \) is open in \( L_H \).

**Proof.** By Corollary 4.4, \( L_H \) can be embedded in a Hilbert scheme and we can conclude that the locus of radical ideals of \( L_H \) is an open subscheme of \( L_H \), because the radical locus of a Hilbert scheme is open (see [10, Théorème (12.2.1)]). \( \square \)

5. **Explicit construction of the isomorphisms**

The functorial approach used in Sections 3 and 4 immediately leads to the following fact.

**Theorem 5.1.** For every homogeneous ideal \( H \subset K[x] \), the scheme of liftings \( L_H \) is independent on the term order on \( K[x] \) used to construct it.

**Proof.** For every term order on \( K[x] \), the scheme we obtain following Theorem 4.2 represents \( L_H \). By Yoneda Lemma such a scheme is unique, up to isomorphisms. \( \square \)

However, if we consider two different term orders \( \prec \) and \( \prec' \) on \( K[x] \), the functorial approach does not provide the isomorphism between the two schemes that we obtain by Proposition 4.1 and Theorem 4.2 starting from the two term orders.

Now, we explicitly construct this isomorphism. Let \( B \) and \( B' \) be the monomial bases of the initial ideals \( j := \text{in}_\prec(H) \) and \( j' := \text{in}_{\prec'}(H) \), respectively, and let

\[
G =: \{ f_\alpha \mid x^\alpha \in B \} \quad \text{and} \quad G' =: \{ f'_\beta \mid x^\beta \in B' \}
\]

be the reduced Gröbner bases of \( H \) w.r.t. \( \prec \) and \( \prec' \), respectively. Then, we have

\[
f'_\beta = \sum h_{\alpha\beta} f_\alpha, \quad \forall f'_\beta \in G'.
\]
Using a fixed term order on the monomials of $K[C]$, we extend the term orders $\prec_n$ and $\prec_h'$ on $K[C][x, x_n]$ to elimination term orders of the variables $x, x_n$. For simplicity, we keep on using the notations $\prec_n$ and $\prec_h'$ for these elimination term order on $K[C][x, x_n]$, which are the same when restricted to $K[C]$.

We define $G := \{f \alpha + g \alpha\}$, where $g \alpha := \sum x \alpha \gamma \in N(J) C\alpha\gamma x_n x^\gamma$, and $G' := \{f' \beta + g' \beta\}$, where $g' \beta := \sum x \alpha \gamma \in N(J) C\beta\gamma x_n x^\gamma$.

Observe that the set of initial terms of $G$ with respect to $\prec_n$ is exactly $B$ and the set of initial terms of $G'$ with respect to $\prec_h'$ is exactly $B'$: this is due to the fact that the monomials in the supports of $g \alpha$ and $g' \beta$ are divisible by $x_n$, by construction.

Let $\mathcal{H} \subset K[C]$ be the reduced Gröbner basis w.r.t. $\prec_n$ of the smallest ideal $\mathfrak{h}$ such that $G \cup \mathcal{H}$ is a Gröbner basis w.r.t. $\prec_n$ in $K[C][x, x_n]$. The ideal $\mathfrak{h}$ is that described in Theorem 4.2 and can be computed by any of the procedures suggested by Proposition 4.1. Hence $\mathfrak{h} = (\mathcal{H}) \subset K[C]$ is exactly the ideal that defines $L_H$.

Using the relations (5.1), for every $x \beta \in B'$ we define

$$p' \beta := \sum h \alpha \beta(f \alpha + g \alpha) = f' \beta + \sum h \alpha \beta g \alpha \in K[C][x, x_n].$$

**Lemma 5.2.** The set of polynomials $\{p' \beta\}_\beta \cup \mathcal{H} \subseteq K[C][x, x_n]$ is a monic Gröbner basis w.r.t. $\prec_h'$ of $(G \cup \mathcal{H})$.

**Proof.** By construction, we have $\{p' \beta\}_\beta \cup \mathcal{H} \subseteq (G \cup \mathcal{H})$, hence $(\{p' \beta\}_\beta \cup \mathcal{H}) \subseteq (G \cup \mathcal{H})$. We now prove the other inclusion and furthermore that $\{p' \beta\}_\beta \cup \mathcal{H}$ is a Gröbner Basis of $(G \cup \mathcal{H})$ w.r.t. $\prec_h'$, showing that every homogeneous polynomial $f$ in $(G \cup \mathcal{H})$ can be reduced to 0 by $\{p' \beta\}_\beta \cup \mathcal{H}$ using $\prec_n$. We proceed by induction on the degree of $f$ w.r.t. the variables $x, x_n$.

The 0-degree modules $(\{p' \beta\}_\beta \cup \mathcal{H})_0$ and $(G \cup \mathcal{H})_0$ are both equal to $\mathcal{H}$, which is a Gröbner basis w.r.t. $\prec_n$ and $\prec_n'$, since they are the same term order when restricted to $K[C]$. We now assume $(\{p' \beta\}_\beta \cup \mathcal{H})_{m-1} = (G \cup \mathcal{H})_{m-1}$ and prove the same for degree $m$.

We consider $f \in (G \cup \mathcal{H})_m$, and start the reduction by using the polynomials in $\{p' \beta\}$:

$$f \xrightarrow{(p' \beta)} p, \quad \text{with } \text{Supp}(p) \subseteq N(J') \text{ and } p = x_n \cdot p'. $$

We observe that, by the construction of the polynomials $p' \beta$, the polynomial $p$ belongs to $(G \cup \mathcal{H})_m$ and $x_n$ is not a 0-divisor in $K[C][x, x_n]/(G \cup \mathcal{H})$: hence $p'$ belongs to $(G \cup \mathcal{H})_{m-1}$.

By the inductive hypothesis we have $p' \xrightarrow{(p' \beta \cup \mathcal{H})} 0$, more precisely $p'$ is reduced to 0 by $\mathcal{H}$, because $\text{Supp}(p') \subseteq N(J')$. Hence, we get $p \xrightarrow{\mathcal{H}} 0$, as desired.

We consider the unique reduced Gröbner basis we obtain from $\{p' \beta\} \cup \mathcal{H}$ by interreducing the polynomials $p' \beta$ and denote by $q' \beta$ the reduced forms of the polynomials $p' \beta$:

$$q' \beta = f' \beta + x_n m \beta, \quad \text{with } \text{Supp}(m \beta) \subset N(J').$$

By comparing the polynomials $q' \beta$ and $f' \beta + g' \beta$ we obtain a $K$-algebra morphism

$$\phi: K[C'][x, x_n] \to K[C][x, x_n]$$

such that $\phi(x^\gamma) = x^\gamma$ and $\phi(C'_{\beta \gamma})$ is the coefficient of $x^\gamma$ in $q' \beta$.

Fix a term order on $K[C']$ and extend $\prec_n$ and $\prec_h'$ to $K[C'][x, x_n]$ as done previously for $K[C][x, x_n]$. We consider the set of polynomials $\mathcal{H}' \subset K[C']$ which is the reduced
Gröbner basis with respect to \(\prec_h'\) of the smallest ideal such that \(G' \cup H'\) is a Gröbner basis w.r.t. \(\prec_h'\) in \(K[\mathbb{C}'][[x, x_n]]\). This ideal is \(\mathfrak{h}'\), as in Theorem 4.2.

**Theorem 5.3.** The \(K\)-algebra morphism \(\phi\) induces an isomorphism between \(K[\mathbb{C}']/\mathfrak{h}'\) and \(K[\mathbb{C}]/\mathfrak{h}\).

**Proof.** We first observe that \(\phi(\mathcal{H}') \subseteq (\mathcal{H})\). Indeed, by Proposition 4.1(ii), \((\mathcal{H}')\) contains the coefficients of every polynomial of kind \(\sum \ell_\beta(f'_\beta + g'_\beta)\) with \(\text{Supp}(\sum \ell_\beta(f'_\beta + g'_\beta)) \subseteq \mathcal{N}(J')\) and it is sufficient to observe that

\[
\phi(\sum \ell_\beta(f'_\beta + g'_\beta)) = \sum \phi(\ell_\beta)p'_\beta, \quad \text{Supp}(\sum \phi(\ell_\beta)p'_\beta) \subseteq \mathcal{N}(J').
\]

With the analogous construction done for \(\phi\) and starting from the relations \(f_\alpha = \sum h'_{\alpha \beta}f'_{\beta}\), we obtain a \(K\)-algebra homomorphism

\[
\psi: K[\mathbb{C}][x, x_n] \rightarrow K[\mathbb{C}'][x, x_n]
\]

such that \(\psi(\mathcal{H}) \subseteq (\mathcal{H}')\). Hence, it makes sense to restrict \(\phi\) and \(\psi\) to \(K[\mathbb{C}']/(\mathcal{H}')\) and \(K[\mathbb{C}]/(\mathcal{H})\).

We define \(\chi := \phi \circ \psi: K[\mathbb{C}][x, x_n] \rightarrow K[\mathbb{C}][x, x_n]\). We finally show that its restriction \(\overline{\chi}: K[\mathbb{C}]/(\mathcal{H}) \rightarrow K[\mathbb{C}]/(\mathcal{H})\) is the identity.

It is sufficient to observe that the polynomial \(C_{\alpha \gamma} - \chi(C_{\alpha \gamma})\) belongs to \((\mathcal{H})\) for every \(\alpha\) and every \(\gamma\). Indeed, \(\chi(f_\alpha + g_\alpha) = f_\alpha + \chi(g_\alpha)\). By construction of \(\phi\) and \(\psi\), \(f_\alpha + \chi(g_\alpha)\) belongs to \((\mathcal{G} \cup \mathcal{H})\). In particular, \(f_\alpha + g_\alpha - (f_\alpha + \chi(g_\alpha)) = g_\alpha - \chi(g_\alpha)\) is a polynomial in \((\mathcal{G} \cup \mathcal{H})\) with \(\text{Supp}(g_\alpha - \chi(g_\alpha)) \subseteq \mathcal{N}(J)\). Hence, by Theorem 4.1, the coefficients of \(g_\alpha - \chi(g_\alpha)\) belong to \((\mathcal{H})\). Then, in \(K[\mathbb{C}]/(\mathcal{H})\) we have \(C_{\alpha \gamma} = \overline{\chi}(C_{\alpha \gamma})\), hence \(\overline{\chi}\) is the identity on \(K[\mathbb{C}]/(\mathcal{H})\).

In Example 7.2 we will exhibit an explicit computation of the isomorphism described in Theorem 5.3. Observe that this isomorphism \(K[\mathbb{C}]/\mathfrak{h} \simeq K[\mathbb{C}']/\mathfrak{h}'\) is constructed from \(\phi: K[\mathbb{C}] \rightarrow K[\mathbb{C}']\) which in general is not an isomorphism.

6. **Liftings of a Cohen-Macaulay scheme of codimension two**

Given a homogeneous ideal \(H \subset K[x]\) and a \(x_n\)-lifting \(I \subset A[x, x_n]\) of \(H\), let \(G_H = \{f_\alpha\}_\alpha\) be the reduced Gröbner basis of \(H\) w.r.t. a term order \(\prec\) and \(G_I = \{f_\alpha + g_\alpha\}_\alpha\) be the reduced Gröbner basis of \(I\) w.r.t. \(\prec_h\), as in Theorem 3.2. We denote by \(j\) the initial ideal \(\text{in}_\prec(H)\) and by \(J\) the initial ideal \(\text{in}_{\prec_h}(I)\). Recall that these ideals have the same monomial basis \(B_j\).

**Lemma 6.1.** If the variable \(x_n\) is replaced by 0 in a first syzygy \(S = (s_\alpha)_\alpha\) of \(G_I\), then we get a first syzygy of \(G_H\).

**Proof.** Recall that the head terms of the polynomials of \(G_H\) are the same as the head terms of the polynomials of \(G_I\) and the polynomials \(g_\alpha\) are all divisible by \(x_n\). If for every \(\alpha\) we distinguish the part of \(s_\alpha\) divisible by \(x_n\) setting \(s_\alpha = s'_\alpha + x_n s''_\alpha\), where \(x_n\) does not occur in \(s'_\alpha\), we have

\[
\sum_\alpha (s'_\alpha + x_n s''_\alpha)(f_\alpha + g_\alpha) = 0,
\]

hence

\[
\sum_\alpha (s'_\alpha f_\alpha) + \sum_\alpha (x_n s''_\alpha f_\alpha) + \sum_\alpha (s'_\alpha + x_n s''_\alpha)g_\alpha = 0
\]
and $x_n$ does not occur in the first sum, but the second and the third sums are divisible by $x_n$. Thus, we have $\sum \alpha (s'_\alpha f_\alpha) = 0$.

Now, assume that $H$ is the (saturated) defining ideal of a Cohen-Macaulay scheme $X \subset \mathbb{P}^{n-1}_K$ of codimension two. Then, there is a (graded) free resolution of the following type:

$$0 \to K[x]^a \overset{\psi_2}{\to} K[x]^{a+1} \overset{\psi_1}{\to} K[x] \to K[x]/H \to 0, \tag{6.1}$$

and the Hilbert-Burch Theorem guarantees that the $a \times a$ minors of the $a \times (a+1)$ matrix of the homomorphism $\psi_2$ in (6.1) form a set of generators for $H$. Further, we can observe that the rows of this matrix generate the first syzygies module of $H$. Vice versa, a 2-codimensional scheme defined by the $a \times a$ minors of such a matrix of type $a \times (a+1)$ is a Cohen-Macaulay scheme (e.g., see [7, Theorem 20.15 and the paragraph that follows the proof]).

If the field $K$ is infinite and $\prec$ is the degrevlex term order, then, up to a suitable linear change of coordinates, the variables $x_2, \ldots, x_{n-1}$ form a regular sequence for $K[x]/H$ and also for $K[x]/j$. Indeed, being $K[x]/H$ Cohen-Macaulay of codimension two, by the graded prime avoidance lemma, we can find a regular sequence for $K[x]/H$ consisting of $n-2$ linear forms $l_2, \ldots, l_{n-1}$. After a suitable linear change of coordinates $\phi$ which sets $l_2 \mapsto x_2, \ldots, l_{n-1} \mapsto x_{n-1}$, by [4, Lemma (2.2)] we obtain that the initial ideal of $\phi(H)$ is generated by terms which are not divisible by $x_2, \ldots, x_{n-1}$ (see also Remark 1.2). In particular, $K[x]/\in_{\prec}(H)$ is Cohen-Macaulay. From now, we assume that $K$ is an infinite field, hence we can also suppose that $x_2, \ldots, x_{n-1}$ form a regular sequence for $K[x]/H$ and, thus, for $K[x]/j$.

In the following, we will call a lifting of a minimal free resolution of $j$ every complex obtained lifting the syzygies of $B_1$ in the usual sense of the theory of Gröbner bases.

**Lemma 6.2.** Let $K$ be an infinite field and $\prec$ be the degrevlex term order. Then there is a free resolution of $H$ (resp. $I$) of type (6.1) obtained by lifting a minimal free resolution of $j$ (resp. $J$).

**Proof.** We show that a free resolution of type (6.1) can be constructed starting from the reduced Gröbner basis of $H$. We can do the same for $I$, because also the ideal $I$ is the saturated defining ideal of a 2-codimensional Cohen-Macaulay closed subscheme (in $\mathbb{P}^n$, in this case), by the definition of $x_n$-lifting.

Being $K[x]/j$ Cohen-Macaulay of codimension two, by the Hilbert-Burch Theorem we can consider a minimal free resolution of the following type, constructed starting form $B_i$:

$$0 \to K[x]^a \overset{\psi_2'}{\to} K[x]^{a+1} \overset{\psi_1'}{\to} K[x] \to K[x]/j \to 0 \tag{6.2}$$

where the rows of the matrix $M_i$ of $\psi_2'$ form a minimal set of generators of the first syzygies of $B_i$.

By the properties of Gröbner bases, we obtain a set of generators of the first syzygies of $G_H$ by lifting the rows of $M_i$ and obtaining a new matrix $M_H$. Hence, lifting the resolution (6.2), we get a complex

$$K[x]^a \overset{\psi_2}{\to} K[x]^{a+1} \overset{\psi_1}{\to} K[x] \to K[x]/H \to 0, \tag{6.3}$$

that is exact in $K[x]^{a+1}$ by construction, thus the rank of the homomorphism $\psi_2$ is equal to $a$. This means that the rows of $M_H$ are independent and $\psi_2$ is injective. So, we obtain a free resolution of type (6.1) for $H$ by lifting a minimal free resolution of $j$. \qed
Remark 6.3. By Lemma 6.2, the rows of the matrix of the homomorphism $\psi_2$ in (6.1) can be constructed by lifting the syzygies of the analogous matrix for $j$. Moreover, they form a set of minimal generators of the module of the first syzygies of $G_H$ (respectively, of $G_I$). As a consequence of Lemma 6.1, the matrix of the homomorphism $\psi_2$ in (6.1) of $H$ is obtained from the analogous matrix of $I$, letting $x_n = 0$.

Theorem 6.4. The $x_n$-liftings of a saturated homogeneous ideal $H \subset K[x_0, \ldots, x_{n-1}]$ defining a Cohen-Macaulay scheme of codimension two form an affine space over an infinite field $K$.

Proof. Consider the matrix $M$ obtained by the matrix $M_H$ of $\psi_2$ in (6.1) adding to each entry of $M_H$ a linear combination of the terms divisible by $x_n$ of the appropriate degree. By Lemmas 6.1 and 6.2 and Remark 6.3, it is enough to observe that the $a \times a$ minors of the matrix $M$ generate an $x_n$-lifting of $H$ and that all the $x_n$-liftings of $H$ are of this type. Hence, a $x_n$-lifting of $H$ depends only on the coefficients of the linear combinations of terms divisible by $x_n$ added to the entries of $M_H$, i.e. the $x_n$-liftings of $H$ are parameterised on an affine space whose dimension is the number of the terms that occur in these linear combinations. □

Remark 6.5. When $H$ defines a Cohen-Macaulay scheme of codimension two, Theorem 6.4 gives a new parameterisation of the $x_n$-liftings of $H$ by an affine space $\mathbb{A}^q$ that is isomorphic to $L_H$, because also this affine space $\mathbb{A}^q$ represents the functor $L_H$.

Theorem 6.6. Every saturated homogeneous ideal $H$ defining a 2-codimensional Cohen-Macaulay scheme over an infinite field $K$ has a radical $x_n$-lifting. In particular, this is true for every saturated homogeneous ideal defining a 0-dimensional scheme in $\mathbb{P}^2$.

Proof. For every $x^a \in B_j$, let $f_a := x^a + \sum_\gamma c_{\alpha, \gamma} x^\gamma$ the polynomial of the reduced Gröbner basis $B_H$ with head term $x^a$. Then, let $\omega = [\omega_1, \ldots, \omega_{n-1}]$ be a weight vector such that the polynomials $f_\alpha(t) := x^a + \sum_\gamma c_{\alpha, \gamma} t^{\omega(\alpha - \gamma)} x^\gamma$ generate a flat family $\{H(t)\}_t$ of Gröbner deformations from $H$ to $j$ (e.g. [3]). Recall that the ideal of this family corresponding to a non-null value $t$ of $t$ is isomorphic to $H$ by the automorphism $\phi_t$ of $\mathbb{P}^{n-1}$ given by $x_0 \mapsto t^{-\omega_0} x_0, \ldots, x_n \mapsto t^{-\omega_n} x_n$.

For $H(t)$ there is a resolution of type (6.1). Let $M_{H(t)}$ be the matrix of the corresponding homomorphism $\psi_2$. Observe that each addend of the entries of $M_{H(t)}$ is divisible by $t$, except for the addends that form the syzygies of $j$, according to Lemma 6.2. Let $N$ be a radical lifting of $j$, which exists by [11] or [9, Th. 2.2] or [19, Th. 8]. Moreover, let $M_N$ be the matrix corresponding to the homomorphism $\psi_2$ in a free resolution of $N$ of type (6.1). Observe that each addend of the entries of $M_N$ is divisible by $x_n$, except for the addends that form the syzygies of $j$, according to Lemma 6.2. Moreover, let $M_j$ the analogous matrix for $j$.

Now, consider the matrix $M(t) := M_{H(t)} + M_N - M_j$ and let $\{I(t)\}_t$ be the family of the ideals generated by the maximal minors of $M(t)$. Hence, $M(t)$ is the matrix of the homomorphism $\psi_2(t)$ of a resolution

$$0 \longrightarrow (K[x][t])^{a+1} \xrightarrow{\psi_2(t)} (K[x][t])^{a+1} \xrightarrow{\psi_1(t)} K[x][t] \longrightarrow K[x][t]/I(t) \longrightarrow 0.$$ 

By construction, $\{I(t)\}_t$ is a flat family of $x_n$-liftings of $\{H(t)\}_t$ (by Theorem 6.4) parameterised by an affine line $\mathbb{A}^1$. Hence, it embeds in a Hilbert scheme.

By the already cited [10, Théoréme (12.2.1)], we know that the radical locus of a Hilbert scheme is open. Hence, there is an open subset $U$ of $\mathbb{A}^1$ corresponding to reduced schemes.
Moreover, $U$ is not empty because it contains $t = 0$. Due to the fact that the field $K$ is infinite, there is at least another value $\bar{t} \neq 0$ belonging to $U$, so $I(\bar{t})$ is reduced. Then, $H$ has the radical $x_n$-lifting obtained from $I(\bar{t})$ by the automorphism $\varphi_{\bar{t}^{-1}}$ applied to $I(\bar{t})$. \hfill $\square$

7. Examples

Example 7.1. We study the $x_3$-liftings of the lex-segment ideal

$$J := \langle x_0^2, x_0x_1, x_0x_2, x_1^2 \rangle \subseteq K[x_0, x_1, x_2],$$

finding a scheme having two distinct irreducible components, one of which is made of non-radical liftings. This is the same example as [14, Example 3.3], after a change of coordinates allowing us to consider a monomial ideal. In [14], the authors explicitely compute a set of equations defining $L_J$ with a different technique: they impose conditions on the syzygies of a set of polynomials generating a $x_3$-lifting of $J$. In general, the algorithm they use leads to a different set of conditions from the ones we compute by Theorem 3.2 and Proposition 4.1. However, on this example, the equations defining the scheme $L_J$ obtained in [14] are the same as those that we obtain with our strategy. Here we just briefly expose the equations and we focus on the structure of $L_J$.

Starting from the set $G = \{ f_1 + g_1, f_2 + g_2, f_3 + g_3, f_4 + g_4 \}$ with

$$f_1 = x_0^2, \quad g_1 = C_1x_0x_3 + C_2x_1x_3 + C_3x_2x_3 + C_4x_1^2,$$
$$f_2 = x_0x_1, \quad g_2 = C_5x_0x_3 + C_6x_1x_3 + C_7x_2x_3 + C_8x_1^2,$$
$$f_3 = x_0x_2, \quad g_3 = C_9x_0x_3 + C_{10}x_1x_3 + C_{11}x_2x_3 + C_{12}x_1^2,$$
$$f_4 = x_1^2, \quad g_4 = C_13x_0x_3 + C_{14}x_1x_3 + C_{15}x_2x_3 + C_{16}x_1^2,$$

we impose that $G$ is a Gröbner basis in $K[x_0, x_1, x_2, x_3]$ with initial ideal $J$ w.r.t. the degreverse term order that coincides with the deglex term order on $K[x_0, x_1, x_2]$. In this way, we obtain the ideal $H = (C_6 - C_{11}, C_2, C_7, C_3, -C_{10}C_5 - C_{11}C_9 - C_{12}, -C_{14}C_5 + C_5^2 + C_{13}C_{11} - C_{13}C_1 - C_{15}C_9 - C_{16}, -C_5C_{11} - C_8, -C_{11}^2 + C_1C_{11} + C_4, -C_{10}C_{13}, -2C_{11}C_{10} + C_1C_{10}, 2C_{10}C_5 - C_{10}C_{14}, -C_{10}C_{15}).$

We can use the first 8 generators of $H$ to eliminate the 8 variables $C_2, C_3, C_4, C_6, C_7, C_8, C_{12}, C_{16}$. After eliminating these variables from the polynomials in $G$, we obtain an embedding of the scheme of liftings in $A^8$ given by the ideal

$$(C_{10}) \cap (C_{13}, C_{15}, C_1 - 2C_{11}, C_{14} - 2C_5) \tag{7.1}$$

Hence, the scheme of liftings $L_J$ has two irreducible components: $L_1$, the hyperplane in $A^8$ given by the ideal $(C_{10})$, and $L_2$, the linear subspace of dimension 4 in $A^8$ given by the ideal $(C_{13}, C_{15}, C_1 - 2C_{11}, C_{14} - 2C_5)$. We will now explicitely compute the radical locus of $L_J$, which by Corollary 4.6 is an open subset of $L_J$, hence we compute the radical locus of both the components $L_1$ and $L_2$.}

The first component $L_1$ parameterises the liftings that are generated by polynomials of the following type:

$$f_1 + g_1 = (x_0 + C_{11}x_3)(x_0 + (C_1 - C_{11})x_3),$$
$$f_2 + g_2 = (x_1 + C_5x_3)(x_0 + C_{11}x_3),$$
$$f_3 + g_3 = x_1^2 + C_{13}x_0x_3 + C_{14}x_1x_3 + C_{15}x_2x_3 + (C_{14}C_5 - C_{13}C_{11} - C_5^2 + C_{13}C_1 + C_{15}C_9)x_3^2,$$
$$f_4 + g_4 = (x_2 + C_9x_3)(x_0 + C_{11}x_3).$$

By easy computations we can see that we obtain non reduced ideals only if either $C_1 - 2C_{11} = 0$ or $(C_{14} - 2C_5)^2 + 4C_{13}(C_1 - 2C_{11}) = C_{15} = 0$. 


The second irreducible component $L_2$ of $L_J$ parameterises the liftings generated by polynomials of the following type:

\[ f_1 + g_1 = (x_0 + C_{11}x_3)^2, \]
\[ f_2 + g_2 = (x_1 + C_{14}x_3)(x_0 + C_{11}x_3), \]
\[ f_3 + g_3 = (x_1 + C_{14}x_3)^2, \]
\[ f_4 + g_4 = 2x_0x_2 + 2C_0x_0x_3 + 2C_{10}x_1x_3 + 2C_{11}x_2x_3 + (2C_{11}C_9 + C_{10}C_{14})x_3^2. \]

Each ideal of this type corresponds to a double structure over the line $x_0 + C_{11}x_3 = 2x_1 + C_{14}x_3 = 0$, hence the radical locus of $L_2$ is empty.

**Example 7.2.** In this example, we apply the construction arising from Theorem 3.2 and Proposition 4.1 with the degrevlex and the degrlex term orders, respectively, to compute two different $K$-algebras that both define the affine scheme representing the functor of liftings of an ideal $H$, thanks to Theorem 4.2. Then, we find an explicit isomorphism between them.

Take the homogeneous ideal $H = (x_0^2, x_0x_1, x_1^4 + x_1x_3^2) \subset K[x_0, x_1, x_2]$, with $x_0 > x_1 > x_2$. The reduced Gröbner basis of $H$ w.r.t. the degrevlex term order is $G = \{ f_1 = x_0^2, f_2 = x_0x_1, f_3 = x_1^4 + x_0x_3^2 \}$. The reduced Gröbner basis of $H$ w.r.t. the degrlex term order is $G' = \{ f'_1 = x_0^2, f'_2 = x_0x_1, f'_3 = x_0x_3^2 + x_1^4, f'_4 = x_1^4 \}$. We get $G := \{ f_1 + g_1, f_2 + g_2, f_3 + g_3 \}$ with

\[
\begin{align*}
g_1 &= C_1x_0x_3 + C_2x_1x_3 + C_3x_2x_3 + C_4x_3^2, \\
g_2 &= C_5x_0x_3 + C_6x_1x_3 + C_7x_2x_3 + C_8x_3^2, \\
g_3 &= C_9x_0^2x_3 + C_{10}x_1^2x_2x_3 + C_{11}x_0x_2^2x_3 + C_{12}x_1x_2x_3 + C_{13}x_2x_3 + C_{14}x_1^2x_3^2 + \\
&\quad C_{15}x_0x_2x_3^2 + C_{16}x_1x_2x_3^2 + C_{17}x_2x_3^2 + C_{18}x_0x_3^2 + C_{19}x_1x_3^2 + C_{20}x_2x_3 + C_{21}x_4^2.
\end{align*}
\]

Observe that, starting from $G$, we construct the scheme $L_H$ in an affine space of dimension 21. We will soon see that the scheme $L_H$ is smaller than this ambient space. The set $G$ is a Gröbner basis w.r.t. the degrlex term order, modulo the following ideal in $K[C]$:

\[ h = (C_2, C_7, C_{13} - C_1 + C_6, -C_6C_5 + C_8, -C_6^2 + C_1C_6 - C_4, C_{17} - C_{11}C_1 - C_{12}C_5 + C_6C_{11}, C_6C_{15} + C_{10}C_5^2 - C_{16}C_5 + C_{20} - C_{15}C_1, -C_9C_5^3 - C_{19}C_5 + C_{21} + C_5^4 + C_6C_{18} - C_{18}C_1 + C_{14}C_5^2). \]

Using this set of generators of $h$, we can eliminate several of the variables $C$, more precisely there are only 12 free variables left. Hence, in this case $L_H$ is an affine space of dimension 12. Using elimination, the polynomials of $G$ become

\[
\begin{align*}
f_1 + g_1 &= (x_0 + C_6x_3)(C_5x_3 + x_1), \\
f_2 + g_2 &= (x_0 + C_6x_3)(-C_6x_3 + C_1x_3 + x_0), \\
f_3 + g_3 &= x_1^4 + x_0x_2^2 + C_{10}x_1^2x_2x_3 + C_{11}x_0x_2^2x_3 + C_{12}x_1x_2x_3 + C_{15}x_0x_2x_3^2 + \\
&\quad C_{16}x_1x_2x_3^2 + C_5x_1^2x_3 + C_{14}x_1^2x_3^2 + C_{18}x_0x_3^2 + C_{19}x_1x_3^2 + (C_1 - C_6)x_3^2x_3 + \\
&\quad (C_{15}C_1 + C_{10}C_5 - C_{10}C_5^2 - C_6C_{15})x_2x_3^3 + \\
&\quad (C_{12}C_5 + C_{11}C_1 - C_6C_{11})x_2^2x_3^3 + \\
&\quad (C_9C_5^3 + C_{19}C_5 - C_6C_{18} + C_{18}C_1 - C_{14}C_5^2 - C_5^4)x_4^4.
\end{align*}
\]
The polynomials of the reduced Gröbner basis of \((G)\) modulo \(\mathfrak{h}\) w.r.t. to the deglex term order (in terms of the variables \(C\)) are:

\[
\begin{align*}
    p'_1 &= x_0^2 + C_1 x_0 x_3 + (C_1 C_6 - C_0^2) x_3^2, \\
    p'_2 &= x_0 x_1 + C_5 x_0 x_3 + C_5 x_1 x_3 + C_6 C_5^2 x_3^2, \\
    p'_3 &= x_0 x_2^3 + x_1^4 + C_{10} x_1^2 x_2 x_3 + C_{11} x_0 x_2 x_3 + C_{12} x_1 x_2 x_3 + C_{15} x_0 x_2 x_3^2 + C_{16} x_1 x_2 x_3^2 + C_9 x_1^3 x_3 + C_{14} x_1^2 x_3^2 + C_{18} x_0 x_3^3 + C_{19} x_1 x_3^3 + (C_{15} C_1 + C_{16} C_5 - C_{10} C_2^2 - C_6 C_{15}) x_2 x_3^2 + (-C_6 + C_1) x_3^3 x_3 + (C_{12} C_5 + C_{11} C_1 - C_6 C_{11}) x_2^2 x_3^2 + (C_9 C_5^3 + C_{19} C_5 - C_6 C_{18} + C_{18} C_1 - C_{14} C_2^2 - C_3^2) x_3^3 + (C_5 C_1 C_1 - 2 C_5 C_5 C_5) x_3^2 x_3^2 + (C_5 C_1 C_1 - C_9 C_3^2 - C_4^2 + 2 C_9 C_5 - 2 C_6 C_{18} - C_4 C_{2}^2) x_1 x_3^3 + (C_5 + C_6) x_4 x_3^3 + (C_5 C_9 + C_4) x_5 x_3^2 + (C_5 C_{14} + C_9) x_1^2 x_3^3 + (C_5 C_3 - 2 C_6 C_5) x_3^2 x_3^2 + (2 C_{12} C_5 + C_{11} C_1 - 2 C_6 C_{11}) x_2 x_1 x_3 + (C_5 C_15 + C_16) x_2^2 x_3^3 + (2 C_16 C_5 - 2 C_6 C_{15} + C_{15} C_1 - C_6 C_5^2) x_2 x_1 x_3^3 + C_{12} x_1^3 x_2 x_3 + C_{10} x_1^3 x_2 x_3 + (-2 C_6 + C_1) x_3^2 x_1 x_3 + (C_5 C_4^5 - C_{14} C_2^2 - 2 C_5 C_5 C_6 + C_{15} C_5^2 + C_5 C_{18} C_1 - C_6^2) x_3^5.
\end{align*}
\]

Starting from the Gröbner basis \(G'\), consider \(G' = \{f'_1 + g'_1, f'_2 + g'_2, f'_3 + g'_3, f'_4 + g'_4\}\), where \(g'_i \in K[D][x, x_n]\) as in formula (4.1) with the variables \(D\) in place of \(C\). We highlight that \(|D| = 39 \neq |C|\).

Now, we are going to construct \(L_H\) into an affine space of dimension 39. The set \(G'\) is a Gröbner basis w.r.t. the deglex term order, modulo the ideal \(\mathfrak{h}' = (D_9 + D_{25} - D_{22}, D_{10} - D_{23}, -D_{24} + D_{12}, -D_6 + D_{13} - D_{25}, D_{25} - 2 D_{13} + D_{1}, D_7 - D_{26}, -D_2 - D_3, -D_{29}, D_5 D_{22} - D_5^2 - D_{27} + D_{14}, D_5 D_{23} - D_{28} + D_{16}, -D_5 D_{24} - D_{13} D_{11} + D_{17}, 2 D_5 D_{24} + D_{11} D_{25} - D_{30}, D_5 D_{25} - D_{6} D_{13} + D_8, D_5 D_{25} - D_{31}, -D_5 D_{25} + D_2 D_{13} - D_{34}, -D_3, D_4 D_{2}^5 + D_5 D_{5} D_{11} - D_{35}, -D_5 D_{22} + D_3^2 + D_5 D_{27} - D_{32} + D_9, 2 D_5^2 D_{23} - D_{13} D_{15} - D_5 D_{28} + D_{20}, -3 D_5^2 D_{23} + D_{15} D_{25} + 2 D_5 D_{28} - D_{34}, -D_{36}, -2 D_{23} D_5^2 + D_5 D_{25} D_{15} + D_2 D_5^2 - D_{38}, -3 D_5^2 D_{22} + 4 D_4^4 + 2 D_5^2 D_{27} - D_{13} D_{18} - D_5 D_{32} + D_{21}, 4 D_2 D_{22} - 5 D_5^4 - 3 D_5^2 D_{27} + D_{18} D_{25} + 2 D_5 D_{32} - D_{37}, 3 D_2 D_5^2 - 4 D_5^2 - 2 D_7 D_5^3 + D_2 D_5 D_{18} + D_{32} D_5^2 - D_{39}) \subseteq K[D]\).

Again, we can eliminate several of the variables \(D\) using this set of generators of \(\mathfrak{h}'\). After elimination, there are again 12 free variables \(D\) left.

By comparing the coefficients of the polynomials \(f'_1 + g'_1, f'_2 + g'_2, f'_3 + g'_3, f'_4 + g'_4\) with those of \(p'_1, p'_2, p'_3, p'_4\), we can construct a \(K\)-algebra morphism \(\phi\) between \(K[D]/\mathfrak{h}'\) and \(K[C]/\mathfrak{h}\). Since we are considering \(K[D]/\mathfrak{h}'\), it is enough to give the images under \(\phi\) of the 12 free variables \(D\):

\[
\begin{align*}
    \phi(D_5) &= C_5, & \phi(D_{11}) &= C_{11}, & \phi(D_{13}) &= C_6 + C_1, \\
    \phi(D_{15}) &= C_{15}, & \phi(D_{18}) &= C_{18}, & \phi(D_{22}) &= C_5 + C_9, \\
    \phi(D_{23}) &= C_{10}, & \phi(D_{24}) &= C_{12}, & \phi(D_{25}) &= -2 C_6 + C_1, \\
    \phi(D_{27}) &= C_5 C_9 + C_{14}, & \phi(D_{28}) &= C_5 C_{10} + C_{16}, & \phi(D_{32}) &= C_5 C_{14} + C_{19}.
\end{align*}
\]

The morphism \(\phi\) is exactly the one of Theorem 5.3, hence it is an isomorphism between \(K[D]/\mathfrak{h}'\) and \(K[C]/\mathfrak{h}\).

**Example 7.3.** In this example, we describe the scheme of liftings of a given homogeneous saturated ideal defining a Cohen-Macaulay scheme of codimension two using the results of Section 6 and also compute the open subset of the radical liftings.
We consider the saturated ideal \( H = (x_0^2 - x_1^3, x_0x_1 + 2x_1^2, x_1^3) \) in \( K[x_0, x_1, x_2] \) with \( x_0 > x_1 > x_2 \), defining a zero dimensional scheme in \( \mathbb{P}^2 \), hence an Cohen-Macaulay scheme of codimension two. The scheme defined by \( H \) is non reduced and its support is a point, since the saturated ideal is not radical. The initial ideal of \( H \) w.r.t. degrevlex is \( j = (x_0^2, x_0x_1, x_1^3) \). In order to obtain a radical \( x_3 \)-lifting of \( H \) in \( K[x_0, x_1, x_2, x_3] \) following the proof of Theorem 6.6, we first construct a radical \( x_3 \)-lifting \( N \) of \( j \) obtained by a so-called distraction [11]: \( N = (x_0(x_0 + x_3), x_0x_1, x_1(x_1 + x_3)(x_1 - x_3)) \). Observe that this ideal is not radical when \( \text{char}(K) = 2 \); in this case we will consider a different ideal \( N \), at the end of this example.

We now write down the matrices associated to \( j \), \( H \) and \( N \) and we consider the ideal \( I \) which corresponds to the matrix \( M_I = M_H + M_N - M_j \):

\[
M_I = \begin{pmatrix}
1 & -x_0 & 0 \\
0 & x_1^2 & 0 \\
0 & x_1^2 & -x_0
\end{pmatrix}
\quad M_H = \begin{pmatrix}
x_1 & -x_0 + 2x_1 & -3 \\
0 & x_1^2 & -x_0 - 2x_1
\end{pmatrix}
\quad M_N = \begin{pmatrix}
x_1 & -x_0 - x_3 & 0 \\
0 & x_1^2 - x_3^2 & 0 \\
0 & x_1^2 & -x_0
\end{pmatrix}
\quad M_I = \begin{pmatrix}
x_1 & -x_0 + 2x_1 - x_3 & -3 \\
0 & x_1^2 - x_3^2 & -x_0 - 2x_1
\end{pmatrix}
\]

Note that the matrix \( M_H \) determines the isomorphism \( \psi_2 \) of a free resolution of type (6.1) which is not minimal, in this case.

The generators of the ideal \( I \) are the minors of maximal order of the matrix \( M_I \):

\[
I = (x_0^2 - x_1^3 + x_0x_3 + 2x_1x_3 - 3x_3^2, x_0x_1 + 2x_1^2, x_1^3 - x_1x_3^2).
\]

We can easily verify that \( I \) is a radical lifting of \( H \) assuming that \( \text{char}(K) \neq 13 \); indeed,

\[
I = (x_0 + 2x_3, x_1 - x_3) \cap (x_0 + x_3, 2x_1 - x_3) \cap (x_0^2 + x_0x_3 - 3x_3^2, x_1)
\]

and the discriminant of \( x_0^2 + x_0x_3 - 3x_3^2 \) is 13.

On the other hand, if \( \text{char}(K) = 13 \), we consider the weight vector \( \omega = [3, 2, 0, 0] \) which makes every term in \( J \) of degree 2 or 3 bigger than every term of the same degree in \( \mathcal{N}(J) \). As in the proof of Theorem 6.6, we construct the ideal

\[
I(t) = (x_0^2 - x_1^2 + t^{-3}x_0x_3 + 2t^{-3}x_1x_3 - 3t^{-4}x_3^2, x_0x_1 + 2x_1^2, x_1^3 - t^{-4}x_1x_3^2).
\]

We replace \( t \) by a random integer, for instance \( t = 7 \), and we obtain for \( \text{char}(K) = 13 \) the decomposition

\[
I(7) = (x_0 + 2x_1, x_1 - 2x_3) \cap (x_0 + 2x_1, x_1 + 2x_3) \cap (x_0 + x_3, x_1) \cap (x_0 + 4x_3, x_1).
\]

When the field \( K \) has characteristic 2, we compute a \( x_3 \)-lifting of \( J \) assuming \( |K| \geq 3 \) (for example, \( K \) could be the algebraically closure of \( \mathbb{Z}_2 \)) and letting \( \chi \) be any element of \( K \) different from 0, 1. Thus, we obtain the following radical \( x_3 \)-lifting of \( J \)

\[
N = (x_0(x_0 + x_3), x_0x_1, x_1(x_1 + x_3)(x_1 + \chi x_3)) = (x_0^2 + x_0x_3, x_0x_1, x_1^3 + (\chi + 1)x_1^2x_3 + \chi x_3^2).
\]

In this case, the ideal \( H \) becomes \( (x_0^2 + x_1^2, x_0x_3, x_1^3) \) and the matrices of syzygies are

\[
M_H = \begin{pmatrix}
x_1 & x_0 & 1 \\
0 & x_1 & 0 \\
0 & x_0 & x_1
\end{pmatrix}
\quad M_N = \begin{pmatrix}
x_1 & x_0 + x_3 \\
0 & x_1^2 + (\chi + 1)x_1x_3 + \chi x_3^2 \\
0 & x_0
\end{pmatrix}.
\]
Anyway, in this case we take the matrix $M = \begin{pmatrix} x_1 & x_0 + x_3 \\ 0 & x_1^2 + (\chi + 1)x_1x_3 + \chi x_2^3 & x_0 \end{pmatrix}$ whose maximal minors define the following radical $x_3$-lifting of $H$:

$$I = (x_1(x_1^2 + (\chi + 1)x_1x_3 + \chi x_2^3), x_1x_0, x_0x_0 + x_0x_3 + x_1^2 + (\chi + 1)x_1x_3 + \chi x_2^3) = (x_1, \chi x_3^2 + x_0x_0 + x_0x_3) \cap (x_0, \chi x_3x + x_1) \cap (x_0, x_1 + x_3).$$

**Example 7.4.** Consider the saturated monomial ideal $J = (x_0^2, x_0x_1, x_2^2)$ in $K[x_0, x_1, x_2]$, with $K$ of characteristic 0. In this example, we study the radical locus of the $x_3$-liftings of the ideal $J$. In order to construct the scheme of liftings of $J$ by Theorem 3.2 and Proposition 4.1, we start from the polynomials

$$f_1 + g_1 = x_0^2 + C_1x_0x_3 + C_2x_1x_3 + (C_5C_9 + C_1C_6 + C_2C_{10} - C_6^2)x_3^2,$$

$$f_2 + g_2 = x_0x_1 - C_5x_0x_3 + C_6x_1x_3 + (-C_6C_5 - C_2C_9)x_3^2,$$

$$f_3 + g_3 = x_1^2 + C_9x_0x_3 + C_{10}x_1x_3 + (-C_{10}C_5 + C_9C_1 - C_6C_9 - C_5^2)x_3^2.$$

Hence, we construct the scheme $L_J$ as a subscheme of $\mathbb{A}^{12}$. We compute a set of generators for the smallest ideal $\mathfrak{h} \subset K[C]$ such that $\{f_1 + g_1, f_2 + g_2, f_3 + g_3\}$ is a Gröbner basis modulo $\mathfrak{h}$. The set of generators we obtain allow the elimination of 6 parameters, while the other 6 are free.

If $C_2 \neq 0$, we obtain $x_1 = -\frac{1}{C_2}x_0^2 - C_4x_0 + (\frac{C_2}{C_2} - C_5 - C_1x_0C_6 - C_1x_0x_3^2)$ from the polynomial $(f_1 + g_1)(x_0, x_1, x_2, 1)$ and replacing $x_1$ in $(f_2 + g_2)(x_0, x_1, x_2, 1)$ we get a degree 3 polynomial whose discriminant with respect to the variable $x_0$ is:

$$\Delta = 16C_5C_1C_{10}C_6 - 36C_2C_9C_1C_5 - 18C_2C_9C_1C_{10} - 4C_2^2C_5C_{10} - 16C_2^2C_6^2 + 4C_2C_5^3 + 27C_2^2C_9^2 + 32C_5^3C_2 + 16C_5C_{10}C_6 + 72C_5C_2C_6C_6 + 16C_5^2C_1C_6 + 48C_1C_5C_6^2 + 36C_2C_{10}C_9C_6 + 4C_2^2C_1C_6 - 24C_9C_2C_6^2 + -4C_1^2C_2^2 - C_1^2C_2C_6 + 4C_1^2C_9 + 48C_2C_2C_1C_6 + 24C_5C_2C_{10}^2 - 32C_5C_6^2.$$

If $C_9 \neq 0$, the analogous argument applied on $(f_3 + g_3)(x_0, x_1, x_2, 1)$ used to eliminate $x_0$ leads to the same discriminant $\Delta$. Finally, assuming that $C_2 = C_9 = 0$ gives a polynomial $(f_2 + g_2)(x_0, x_1, x_2, 1)$ that decomposes as $(x_0 + C_6)(x_1 - C_5)$. We study separately the two components corresponding to these two factors. Replacing $x_0$ by $-C_6$ we find that $f_2 + g_2$, and $f_3 + g_3$ vanish, while $f_1 + g_1$ becomes a degree 2 polynomial in $x_0$ with discriminant $(C_1 - 2C_6)^2$. Analogously, replacing $x_1$ by $C_5$ we get $(C_{10} + 2C_5)^2$ as the discriminant of $f_3 + g_3$. Then, the locus of non-radical $x_3$-liftings is defined by $\Delta$ also when $C_2 = C_9 = 0$, since $\Delta = (C_1 - 2C_6)^2(C_{10} + 2C_5)^2$ in $K[C]/(C_2, C_9)$.

Summing up, the radical locus of the scheme of liftings $L_J$ is the open subset complementary to the closed one defined by $(\Delta)$.

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