Abstract

We consider the problem of estimating the state and unknown input for a large class of nonlinear systems subject to unknown exogenous inputs. The exogenous inputs themselves are modeled as being generated by a nonlinear system subject to unknown inputs. The nonlinearities considered in this work are characterized by multiplier matrices that include many commonly encountered nonlinearities. We obtain a linear matrix inequality (LMI), that, if feasible, provides the gains for an observer which results in certified $\mathcal{L}_2$ performance of the error dynamics associated with the observer. We also present conditions which guarantee that the $\mathcal{L}_2$ norm of the error can be made arbitrarily small and investigate conditions for feasibility of the proposed LMIs.

1 Introduction

Exogenous unknown inputs acting on a dynamical system (plant) can result in compromised safety and degraded performance. One way to protect a system against such unknown attacks is by employing unknown input observers (UIOs), as reported in [1] and [2]. Common estimation frameworks for systems in which one assumes stochastic models for the unknown exogeneous input include Kalman filtering [3] and minimum variance filters [4]. For unknown exogeneous inputs where underlying statistics are not available and cannot be guessed, methods that have proven effective include: adaptive estimation [5], sliding mode observers [6, 7], and observers that minimize the system’s input-output gain such as $\mathcal{H}_\infty$ observers [8–11]. Recent work has produced many effective methods for generating unknown input observers for nonlinear systems; see for example [12–19].

A common underlying assumption in many of the cited works is that the unknown exogeneous input is bounded. One way of relaxing that assumption is by using an extended state observer, that is, by appending the exogenous input to the system state. Exogenous input estimation via an extended state observer has been successful in various practical systems, including robotic systems [20], electric drive systems [21], power electronics [22], and avionics [23]. These exogenous inputs could be completely unknown, or partially unknown. In this paper, we refer to partially unknown inputs as exogenous inputs that have been generated by a completely unknown input acting on in [24,25]. Although prior investigations into extended state observer design for estimation with
unknown exogenous inputs has yielded useful results [26–28], the authors assume that the inputs are bounded, and tackle linear systems or linearized versions of nonlinear systems; the sparsity of results on observers for nonlinear systems with unknown deterministic exogenous inputs motivates the present paper.

In this paper, we propose extended state observers to estimate the state and the unknown exogenous input for nonlinear systems whose nonlinearities satisfy so-called incremental quadratic constraints [29,30]. Such nonlinearities encompass a wide range of nonlinearities including globally Lipschitz, one-sided Lipschitz, monotonic and other commonly occurring nonlinearities. Also, the exogenous input can be unbounded. Observer design is based on a linear matrix inequality which we demonstrate is satisfied by a large class of commonly encountered nonlinear systems. The observers guarantee that the input-output system from exogenous input to observer error is $L^2$-stable with a specific gain; for linear systems this gain is an upper bound on the $\mathcal{H}_\infty$ norm of the system. We also present conditions which guarantee that an arbitrarily small $L^2$ gain can be achieved.

2 Problem statement

2.1 Systems under consideration

Consider a nonlinear time-varying system (the plant) described by

\begin{align}
\dot{x} &= Ax + B_f f_1(t, y, q_1) + g_x(t, y) + Bw, \\
q_1 &= C_{q_1} x + D_{q_1} f_1(t, y, q_1) + g_{q_1}(t, y) + D_{q_1}w, \\
y &= Cx + D_f f_1(t, y, q_1) + g_y(t, y) + Dw. 
\end{align}

(1a), (1b), (1c)

Here, $t \in \mathbb{R}$ is the time variable, $x(t) \in \mathbb{R}^{n_x}$ is the state, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and $w(t) \in \mathbb{R}^{n_w}$ models the disturbance input and the measurement noise combined into one term; we refer to it as the exogenous input; this is unknown at every $t$. The vector $f_1(t, y, q_1) \in \mathbb{R}^{n_f_1}$ models nonlinearities of known structure, but because this term depends on the state $x$ (through $q_1$), it cannot be instantaneously determined from measurements. The vector $q_1 \in \mathbb{R}^{n_{q_1}}$ is a state-dependent argument of the nonlinearity $f_1$. The vectors $g_x(t, y) \in \mathbb{R}^{n_x}$, $g_q(t, y) \in \mathbb{R}^{n_{q_1}}$ and $g_y(t, y) \in \mathbb{R}^{n_y}$ represent nonlinearities which can be calculated instantaneously from measurements. An example is $g_x(t, y) = u(t)$ where $u(t)$ is a control input. All the matrices are constant and of appropriate dimensions.

We consider the general case in which the exogenous input $w$ is generated by the following nonlinear exogenous input model:

\begin{align}
\dot{x}_m &= A_m x_m + B_{mf} f_2(t, q_2) + B_m v, \\
q_2 &= C_{q_2} x_m + D_{q_2} f_2(t, q_2) + D_{q_1} v, \\
w &= C_m x_m + D_{wf} f_2(t, q_2) + D_m v, 
\end{align}

(2a), (2b), (2c)

where $x_m(t) \in \mathbb{R}^{n_m}$ is a exogenous input model state, $v(t) \in \mathbb{R}^{n_v}$ is another unknown exogenous input signal and $f_2$ is a known nonlinearity.

**Definition 1 ($L^2$ signal).** We say that a signal $s(\cdot) : [t_0, \infty) \to \mathbb{R}^p$ is $L^2$ if $\int_{t_0}^{\infty} \|s(t)\|^2 \, dt$ is finite where $\|s(t)\|$ is the usual Euclidean norm of $s(t)$ and we define its $L^2$ norm by

\begin{align}
\|s(\cdot)\|_2 = \left( \int_{t_0}^{\infty} \|s(t)\|^2 \, dt \right)^{\frac{1}{2}}. 
\end{align}

(3)

When we say that a signal is bounded, we mean that it is an $L^2$ signal.
Remark 1. The model (2) is used to reflect partial knowledge regarding the unknown input \( w \). For example, if \( w \) is an unknown input with an unknown derivative which is \( L_2 \) it can be described with model (2) with a bounded \( v \); specifically,

\[
\dot{x}_m = v, \quad w \equiv x_m
\]

where \( v = \dot{w} \) is \( L_2 \). An example is

\[
w(t) = a + \ln(1 + bt)
\]

where \( a \) and \( b \) are unknown constants.

In this paper, we characterize nonlinearities via their incremental multiplier matrices.

**Definition 2** (Incremental Multiplier Matrices). A symmetric matrix \( M \in \mathbb{R}^{(n_q+n_f) \times (n_q+n_f)} \) is an incremental multiplier matrix (\( \delta \text{MM} \)) for \( f \) if it satisfies the following incremental quadratic constraint (\( \delta \text{QC} \)) for all \( t \in \mathbb{R} \), \( y \in \mathbb{R}^{n_y} \) and \( q_1, q_2 \in \mathbb{R}^{n_q} \):

\[
\begin{bmatrix}
\Delta q \\
\Delta f
\end{bmatrix}^\top M \begin{bmatrix}
\Delta q \\
\Delta f
\end{bmatrix} \geq 0,
\]

(5)

where \( \Delta q \triangleq q_1 - q_2 \) and \( \Delta f \triangleq f(t, y, q_1) - f(t, y, q_2) \).

The utility of characterizing nonlinearities using incremental multipliers is that our observer design strategy applies to a broad class of nonlinear systems. \( \delta \text{MM} \) for many common nonlinearities are provided in [29,30].

### 2.2 Problem statement

Ideally, we wish to obtain observers that provide an estimate of \( x \) and \( w \). To this end we define the augmented state

\[
\xi \triangleq \begin{bmatrix}
x \\
x_m
\end{bmatrix}
\]

(6)

and look for observers to obtain an estimate \( \hat{\xi} \) of \( \xi \). With

\[
\begin{bmatrix}
\hat{x} \\
\hat{x}_m
\end{bmatrix} = \hat{\xi}
\]

\( \hat{x} \) and \( \hat{x}_m \) will be the observer estimates of \( x \) and \( x_m \), respectively. An estimate of \( w \) can be achieved if \( D_{wf} = D_m = 0 \). In this case, an estimate of the unknown input \( w \) is given by

\[
\hat{w} = C_m \hat{x}
\]

This occurs in the special case when \( w \) has a bounded derivative; see Remark 1.

Let

\[
e = \hat{\xi} - \xi
\]

(7)

denote the estimation error and suppose that

\[
z = He
\]

(8)

is a user-defined performance output associated with the observer where \( z \in \mathbb{R}^{n_z} \). As we demonstrate below, a proposed observer generates an error system that can be described by

\[
\dot{e} = F(t, e, v), \quad \mathbf{z} = G(t, e, v).
\]

(9a, 9b)

We want this system to have the following performance with performance level \( \gamma \).
Definition 3. Let $\gamma$ be a non-negative real scalar. The input-output system (9) is globally uniformly $\mathcal{L}_2$-stable with performance level $\gamma$ if it has the following properties.

(P1) Global uniform exponential stability with zero input. The zero-input system ($v \equiv 0$) is globally uniformly exponentially stable about the origin.

(P2) Global uniform boundedness of the error state. For every initial condition $e(t_0) = e_0$, and every $\mathcal{L}_2$ unknown input $v(\cdot)$, there exists $\beta_1(e_0, \|v(\cdot)\|_2)$ such that

$$\|e(t)\| \leq \beta_1(e_0, \|v(\cdot)\|_2)$$

for all $t \geq t_0$.

(P3) Output response. For every initial condition $e(t_0) = e_0$, and every $\mathcal{L}_2$ unknown input $v(\cdot)$, there exists $\beta_2(e_0, \|v(\cdot)\|_2)$ such that

$$\|z(\cdot)\|_2 \leq \beta_2(e_0, \|v(\cdot)\|_2)$$

and

$$\beta_2(0, \|v(\cdot)\|_2) \leq \gamma \|v(\cdot)\|_2.$$

3 Proposed observers

With the augmented state $\xi$ given by (6), we obtain the augmented plant:

$$\dot{\xi} = A\xi + B_f f + \tilde{g}_\xi + B v$$

(10a)

$$q = C_q \xi + D_q f + \tilde{g}_q + D_q v$$

(10b)

$$y = C \xi + D f + \tilde{g}_y + D v,$$

(10c)

where

$$f(t, y, q) \equiv \begin{bmatrix} f_1(t, y, q_1) \\ f_2(t, q_2) \end{bmatrix} \quad \tilde{g}_s(t, y) = \begin{bmatrix} g_*(t, y) \\ 0 \end{bmatrix}$$

with $q = [q_1^\top \ q_2^\top]^\top$ and

$$A = \begin{bmatrix} A & BC_m \\ 0 & A_m \end{bmatrix}, \quad B_f = \begin{bmatrix} B_f & BD_{wf} \\ 0 & B_{mf} \end{bmatrix},$$

(11a)

$$B = \begin{bmatrix} BD_m \\ B_m \end{bmatrix}, \quad C_q = \begin{bmatrix} C_{q_1} & D_{q_1} C_m \\ 0 & C_{q_2} \end{bmatrix},$$

(11b)

$$D_q f = \begin{bmatrix} D_q f_1 \\ D_q f_2 \\ 0 \\ D_{q_2} \end{bmatrix}, \quad D = DD_m,$$

(11c)

$$D_q = \begin{bmatrix} D_q f_1 \\ D_q \end{bmatrix}, \quad C = \begin{bmatrix} C & DC_m \end{bmatrix},$$

(11d)

$$D_f = \begin{bmatrix} D_f & DD_{wf} \end{bmatrix}.$$

(11e)

In view of the above augmented plant, we propose the following observer:

$$\dot{\hat{\xi}} = A\hat{\xi} + B_f f(t, q) + \tilde{g}_\xi(t, y) + L_1(\hat{y} - y)$$

(12a)

$$\dot{\hat{q}} = C_q \hat{\xi} + D_q f(t, \hat{q}) + \tilde{g}_q(t, y) + L_2(\hat{y} - y)$$

(12b)

$$\dot{\hat{y}} = C \hat{\xi} + D f(t, \hat{q}) + \tilde{g}_y(t, y),$$

(12c)
where \( \hat{\xi} \) is an estimate of the augmented state \( \xi \). Basically, the proposed observer is a copy of the augmented plant along with two correction terms \( L_1(\hat{y} - y) \) and \( L_2(\hat{y} - y) \).

Observer gains \( L_1, L_2 \) that yield the desired performance can be obtained using the following result.

**Theorem 1.** Consider the augmented plant (10) along with performance output given by (8). Suppose that there exist matrices \( P = P^\top > 0, Y, L_2, \) an incremental multiplier matrix \( M \) for \( f \), and scalars \( \alpha, \mu_1 > 0, \mu_2 \geq 0 \) such that

\[
\Xi + \Gamma^\top M \Gamma \preceq 0 \tag{13}
\]

where

\[
\Xi = \begin{bmatrix} \Phi_\alpha + \mu_1 H^\top H & PB_f + YD_f & -PB - YD \\ * & 0 & 0 \\ * & * & -\mu_2 I \end{bmatrix},
\]

with

\[
\Phi_\alpha = PA + A^\top P + YC + C^\top Y^\top + 2\alpha P, \tag{14}
\]

and

\[
\Gamma = \begin{bmatrix} C_q + L_2C & D_{qf} & -D_q - L_2D \\ 0 & I & 0 \end{bmatrix}.
\]

Consider now observer (12) with gains

\[
L_1 = P^{-1}Y \tag{15}
\]

and \( L_2 \). Then, for any initial condition \( e(t_0) = e_0 \) with \( t_0 \in \mathbb{R} \) and \( e_0 \in \mathbb{R}^{n_e} \) and any \( L_2 \) exogenous input \( v(\cdot) : [t_0, \infty) \to \mathbb{R}^{n_v} \),

\[
\|z(\cdot)\|_2 \leq \sqrt{\beta_2/\mu_1}\|e_0\| + \sqrt{\mu_2/\mu_1}\|v(\cdot)\|_2 \tag{16}
\]

and

\[
\|e(t)\| \leq e^{-\alpha(t-t_0)}\sqrt{\beta_1/\beta_2}\|e_0\| + \sqrt{\beta_1/\mu_2}\|v(\cdot)\|_2 \tag{17}
\]

for all \( t \geq t_0 \). Hence the error dynamics with performance output \( z = He \) are \( L_2 \)-stable with performance level

\[
\gamma = \sqrt{\mu_2/\mu_1} \tag{18}
\]

A proof is given in Section 5.

**Remark 2.** Note that, with \( \alpha \) and \( L_2 \) fixed, the matrix inequalities in Theorem 1 are linear in \( Y, P, M, \) and \( \mu_1, \mu_2 \). Only the structure of \( M \) has to be determined \textit{a priori} for the given nonlinearity \( f \); its exact value is obtained by solving the LMI (13).

**Remark 3.** Although in the inequality (13) we require \( L_2 \) to be fixed, the problem can be reposed with variable \( L_2 \). In fact, the entirety of Section IV in [30] is devoted to computing \( L_1 \) and \( L_2 \) simultaneously using convex programming, by exploiting the structure of the incremental multiplier matrices for the given nonlinearity.
Remark 4. To get optimal estimation performance, one can let $\mu_1 = 1$ and formulate the generalized eigenvalue problem

$$\mu_2^* = \arg \min \mu_2 \text{ subject to: (13)}$$

(19)

to obtain a minimal $\gamma$ while line searching over $\alpha$ in some bounded set $[0, \alpha_{\max}]$.

Remark 5. Recall the class of inputs discussed in Remark 1. Recalling Definition 3 (P3), it follows from Theorem 1 that, for zero initial state, a proposed observer results in

$$\|z(\cdot)\|_2 \leq \gamma \|\dot{w}(\cdot)\|_2.$$ 

Thus, $w$ can be unbounded. The bound on $\|\dot{w}(\cdot)\|_2$ does not explicitly need to be known by the designer in order to construct the observer. However, if known, then a bound on the performance output can be calculated.

3.1 Existence of observers with desired $L_2$ performance

Here, we present conditions which guarantee the existence of observers whose error dynamics are $L_2$-stable.

Lemma 1. Suppose that there exist matrices $P = P^\top \succ 0$, $Y$, $L_2$, an incremental multiplier matrix $M$ for $f$, and a scalar $\bar{\alpha} > 0$ such that

$$\begin{bmatrix} \Phi_{\bar{\alpha}} & PB_f + YD_f \\ * & 0 \end{bmatrix} + \bar{\Gamma}^\top M \bar{\Gamma} \preceq 0,$$

(20)

and

$$\bar{\Gamma} = \begin{bmatrix} C_q + L_2 C \\ 0 \end{bmatrix} + L_2 D_f.$$

Then, for any performance output $z = He$ and any positive $\alpha < \bar{\alpha}$, there exist positive scalars $\mu_1, \mu_2$ such that (13) holds.

Proof. Suppose (20) holds. Choosing any positive $\alpha < \bar{\alpha}$, there exist positive scalars $\mu_1, \mu_2$ such that $N > 0$ and

$$\mu_1 H^\top H + \Xi_{13} N^{-1} \Xi_{13}^\top \preceq 2(\bar{\alpha} - \alpha)P,$$

(21)

where

$$N = \mu_2 I - \begin{bmatrix} D_q + L_2 D \\ 0 \end{bmatrix}^\top M \begin{bmatrix} D_q + L_2 D \\ 0 \end{bmatrix},$$

(22)

and $\Xi_{13} = PB + YD$. Using Schur complements, (13) is equivalent to

$$\begin{bmatrix} \Phi_{\alpha} + \mu_1 H^\top H + \Xi_{13} N^{-1} \Xi_{13}^\top & PB_f + YD_f \\ * & 0 \end{bmatrix} + \bar{\Gamma}^\top M \bar{\Gamma} \preceq 0,$$

(23)

It follows from (21) that

$$2\alpha P + \mu_1 H^\top H + \Xi_{13} N^{-1} \Xi_{13}^\top \preceq 2\bar{\alpha} P.$$

Thus,

$$\Phi_{\alpha} + \mu_1 H^\top H + \Xi_{13} N^{-1} \Xi_{13}^\top \preceq \Phi_{\bar{\alpha}}$$

and (13) holds. \qed
In characterizing a solution to a problem in terms of LMI’s one must show that the LMI’s are feasible for a significant class of systems. Here we show that this is the case for the LMIs presented here. For example, consider the case in which $D_qf = 0$ and $f$ is globally Lipschitz in the sense that

$$\|f(t, y, \tilde{q}) - f(t, y, q)\| \leq \kappa \|\tilde{q} - q\|$$

for all $t, y, q, \tilde{q}$ for some $\kappa > 0$. Here we claim that if $\kappa$ is sufficiently small then, then there is a solution to LMI (20) if $(A, C)$ and $(A_m, C_m)$ are detectable and the following condition is satisfied.

**Condition 1.** The matrix

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

has full column rank for every eigenvalue $\lambda$ of $A_m$ with non-negative real part.

Recall that a pair $(C, A)$ is detectable if the matrix

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has full column rank for every $\lambda \in \mathbb{C}$ with non-negative real part.

To prove the above claim, we first note that an incremental multiplier matrix for $f$ is given by

$$M = \begin{bmatrix} I & 0 \\ 0 & -\kappa^{-2}I \end{bmatrix}$$

and, with $L_2 = 0$, (20) reduces to

$$\begin{bmatrix} \Phi_\alpha & PB_f + YD_f + C_q^T C_q \\ * & -\kappa^{-2}I \end{bmatrix} \preceq 0$$

where $\Phi_\alpha$ is given by (14). This is equivalent to

$$\Phi_\alpha + \kappa^2 (PB_f + YD_f + C_q^T C_q)(PB_f + YD_f + C_q^T C_q)^T \preceq 0$$

If $\Phi_\alpha \prec 0$, the above inequality is satisfied when $\kappa > 0$ is sufficiently small. It follows from Lemmas 2 and 3 (given later) that, if $(A, C)$ and $(A_m, C_m)$ are detectable and Condition 1 holds then, there exist matrices $P = P^\top > 0$ and $Y$ such that $\Phi_\alpha \prec 0$.

### 3.2 Estimating with arbitrarily small error

Here, we provide conditions which guarantee that one can estimate the plant state and exogenous input to any arbitrary accuracy, that is, for any performance output $z = H e$, one can achieve any desired level of performance $\gamma > 0$. The result also provides a method of computing observer gain matrices $L_1$ and $L_2$ to achieved the desired performance.

**Theorem 2.** Suppose there exist matrices $P = P^\top > 0$, $Y$, $L_2$, $F$ an incremental multiplier matrix $M$ for $f$, and a positive scalar $\bar{\alpha}$ such that (20) holds and

$$PB + YD = C^T F^T$$

$$D_q + L_2 D = 0$$

$$C^T F^T F D_f = 0, \quad C^T F^T F D_f = 0$$

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Consider any matrix $H \in \mathbb{R}^{n \times n}$ and any performance level $\gamma > 0$. Considering any positive $\alpha < \bar{\alpha}$, choose $\mu_1 > 0$ to satisfy
\begin{equation}
\mu_1 H^T H \preceq 2(\bar{\alpha} - \alpha)P
\end{equation}
and choose $\zeta$ to satisfy
\begin{equation}
\zeta \geq 1/(2\mu_1 \gamma^2)
\end{equation}
Consider now observer (12) with gains $L_2$ and
\begin{equation}
L_1 = P^{-1} \left( Y - \zeta C^T F^T \right)
\end{equation}
Then, for any initial condition $e(t_0) = e_0$ with $t_0 \in \mathbb{R}$ and $e_0 \in \mathbb{R}^n$, and any $L_2$ input $v(\cdot) : [t_0, \infty) \to \mathbb{R}^n$, inequalities (16) and (17) hold for all $t \geq t_0$. Hence the error dynamics with performance output $z = He$ are $L_2$-stable with performance level $\gamma$.

Proof. Consider any matrix $H \in \mathbb{R}^{n \times n}$ and any scalar $\gamma > 0$. Letting $\mu_2 = \gamma^2 \mu_1$, we have $\zeta \geq 1/2\mu_2$ and we now show that (13) holds with $Y$ replaced with
\begin{equation}
\tilde{Y} = Y - \zeta C^T F^T F
\end{equation}
We saw from the proof of Lemma 1 that (13) (with $\tilde{Y}$ replacing $Y$) is equivalent to
\begin{equation}
\begin{bmatrix}
\bar{\Phi} & PB + \tilde{Y}D_f \\
0 & 0
\end{bmatrix} + \bar{\Gamma}^T M \bar{\Gamma} \preceq 0,
\end{equation}
where
\begin{align*}
\bar{\Phi} &= PA + A^T P + \tilde{Y}C + C^T \tilde{Y}^T + 2\alpha P + \mu_1 H^T H + \Xi_{13} N^{-1} \Xi_{13}^T \\
&\leq PA + A^T P + YC + C^T Y^T - 2\zeta C^T F^T FC + 2\bar{\alpha} P + \Xi_{13} N^{-1} \Xi_{13}^T \\
&= \Phi_{\tilde{\alpha}} - 2\zeta C^T F^T FC + \Xi_{13} N^{-1} \Xi_{13}^T
\end{align*}
with $\Phi_{\tilde{\alpha}}$ given by (14) and
\begin{align*}
\Xi_{13} &= PB + \tilde{Y}D \\
&= PB + YD - \zeta C^T F^T FD \\
&= C^T F
\end{align*}
The last two equalities follow from (25c) and (25a). Also, using (22) and (25b), $N = \mu_2 I$. Note that
\begin{equation}
\tilde{Y}D_f = YD_f - \zeta C^T F^T FD_f = YD_f
\end{equation}
We now obtain that
\begin{align*}
\Xi_{13} N^{-1} \Xi_{13}^T &= \mu_2^{-1} C^T F^T FC \\
&\leq 2\zeta C^T F^T FC
\end{align*}
and $\bar{\Phi} \preceq \Phi_{\tilde{\alpha}}$. It now follows from (20) and (32) that (13) holds. The proof is completed by invoking Theorem 1.

Remark 6. Theorem 2 implies that, for any $H$, $H\xi$ can be estimated to arbitrary accuracy. That is, for any given $\varepsilon > 0$ there exists a corresponding observer of the form (12) that is $L_2$-stable with performance level $\varepsilon/\|v(\cdot)\|$. From Definition 3, we deduce that, for zero initial state, $\|He(\cdot)\| \leq \varepsilon$. 

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4 Linear error dynamics

Consider plant (1) with $f_1 = 0$ and disturbance model (2) with $f_2 = 0$, that is,
\[ \dot{x} = Ax + g_x(t,y) + Bw, \quad y = Cx + g_y(t,y) + Dw, \]  
(33a)
and
\[ \dot{x}_m = A_m x_m + B_m v, \quad w = C_m x_m + D_m v. \]  
(34a)

The corresponding observer (12) simplifies to
\[ \dot{\hat{\xi}} = A\hat{\xi} + \tilde{g}_\xi(t,y) + L_1 (\hat{y} - y) \]  
(35a)
\[ \hat{y} = C\hat{\xi} + \tilde{g}_y(t,y). \]  
(35b)

which only involves the observer gain matrix $L_1$. The error dynamics resulting from this observer are described by
\[ \dot{e} = (A + L_1 C)e - (B + L_1 D)v. \]  
(36)

Herein, we obtain simple conditions guaranteeing the existence of an observer gain $L_1$ which yields the desired behavior. First we need a preliminary lemma.

**Lemma 2.** A pair $(A, C)$ is detectable if and only if there are matrices $\mathcal{P} = \mathcal{P}^\top > 0$, $\mathcal{Y}$ and a scalar $\bar{\alpha} > 0$ such that
\[ \Phi_{\bar{\alpha}} \prec 0 \]  
(37)
where $\Phi_{\bar{\alpha}}$ is given by (14).

**Proof.** Detectability of $(A, C)$ is equivalent to the existence of a matrix $L_1$ such that $A + L_1 C$ is Hurwitz, that is, all of its eigenvalues have negative real part. By Lyapunov theory this is equivalent to the existence of a matrix $\mathcal{P} = \mathcal{P}^\top > 0$ such that
\[ \mathcal{P}(A + L_1 C) + (A + L_1 C)^\top \mathcal{P} + 2I = 0. \]
Choosing $\bar{\alpha} > 0$ such that $\bar{\alpha}\mathcal{P} < I$ results in
\[ \mathcal{P}(A + L_1 C + \bar{\alpha}I) + (A + L_1 C + \bar{\alpha}I)^\top \mathcal{P} < 0, \]  
(38)
that is, (37) with $\mathcal{Y} = \mathcal{P}L_1$. Conversely, if (38) holds, then, by Lyapunov theory, $A + L_1 C$ is Hurwitz.

**Lemma 3.** Suppose that $(A, C)$ and $(A_m, C_m)$ are detectable and Condition 1 holds. Then $(A, C)$ is detectable.

**Proof.** The pair $(A, C)$ is detectable if and only if
\[ H(\lambda) = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} A - \lambda I & BC_m \\ 0 & A_m - \lambda I \end{bmatrix} \]
has full column rank for every eigenvalue $\lambda$ of $A$ with non-negative real part. Note that $\lambda$ is an eigenvalue of $A$ if and only if it is an eigenvalue of $A$ or $A_m$. 


Suppose that $H(\lambda)$ does not have full column rank. Then there is a non-zero vector $\xi = [x^\top \ x_m^\top]^\top$ such that $H(\lambda)\xi = 0$, that is,

\begin{align}
(A - \lambda I)x + BC_m x_m &= 0 \quad (39) \\
Cx + DC_m x_m &= 0 \quad (40) \\
(A_m - \lambda I)x_m &= 0. \quad (41)
\end{align}

If $x_m = 0$ then $x \neq 0$ and the above equations imply that $(A - \lambda I)x = 0$ and $Cx = 0$. Thus $\lambda$ is an eigenvalue of $A$ and the matrix $[A - \lambda I \ C]$ does not have full column rank. Since $(C, A)$ is detectable, the real part of $\lambda$ must be negative.

If $x_m \neq 0$, equation (41) implies that $\lambda$ is an eigenvalue of $A_m$. If $C_m x_m = 0$ then the matrix $[A_m - \lambda I \ C_m]$ does not have full column rank. Since $(A_m, C_m)$ is detectable, the real part of $\lambda$ must be negative. If $C_m x_m \neq 0$, equations (39) and (40) imply that

$$
\begin{bmatrix}
A - \lambda I & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
x \\
C_m x_m
\end{bmatrix} = 0.
$$

that is the matrix $[A - \lambda I \ B \ C \ D]$ does have full column rank. Since $\lambda$ is an eigenvalue of $A_m$, $\lambda$ must have negative real part.

Thus, we have shown that if $H(\lambda)$ does not have full column rank then the real part of $\lambda$ is negative. Hence $H(\lambda)$ has full column rank whenever the real part of $\lambda$ is non-negative and $(A, C)$ is detectable.

\[\square\]

**Theorem 3.** Suppose that $(A, C)$ and $(A_m, C_m)$ are detectable and Condition 1 holds Then, for any performance output $z = H e$ there exists an observer gain $L_1$ such that the observer error dynamics (36) are $\mathcal{L}_2$-stable with some performance level $\gamma$.

**Proof.** Note that (20) of Lemma 1 with $B_f = 0$ and $M = 0$ is equivalent to (37). Hence, using Theorem 1, Lemma 1 and Lemma 2 we only need to show that $(A, C)$ is detectable. This follows from Lemma 3.

\[\square\]

## 4.1 Estimating with arbitrarily small error

First we have the following result from [31].

**Lemma 4.** Suppose $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_w}$ and $C \in \mathbb{R}^{n_y \times n_x}$. Then there exist matrices $P = P^\top > 0$, $\mathcal{V}$ and $\mathcal{F}$ such that

\begin{align}
PA + A^\top P + \mathcal{V}C + C^\top \mathcal{V}^\top &< 0 \quad (42) \\
B^\top P &= \mathcal{F}C \quad (43)
\end{align}

if and only if

\begin{align}
\text{rank } CB &= \text{rank } B \quad (44)
\end{align}

and

\begin{align}
\text{rank } \begin{bmatrix}
A - \lambda I & B \\
C & 0
\end{bmatrix} &= n_x + \text{rank } B \quad (45)
\end{align}

for all $\lambda \in \mathbb{C}$ with non-negative real part.
The following result provides conditions that, when satisfied, ensure the existence of observers of the form (35) that generates error dynamics that are $L_\infty$-stable with any arbitrary performance level $\gamma > 0$.

**Lemma 5.** Suppose $DD_m = 0$ and

$$CBD_m + DC_mB_m, \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}, \begin{bmatrix} A_m - \lambda I & B_m \\ C_m & D_m \end{bmatrix}$$

have full column rank for all $\lambda \in \mathbb{C}$ with non-negative real part. Consider any matrix $H \in \mathbb{R}^{n_x \times n_x}$ and any performance level $\gamma > 0$. Then there exist matrices $P = P^\top > 0$, $Y$ and $F$ such that (42) and (43) hold. Choose $\mu_2 > 0$ and $\zeta$ to satisfy (26) and (27). Then the observer (35) with gain given by (28) generates error dynamics with performance output $z = He$ that are $L_2$-stable with performance level $\gamma$.

**Proof.** We use Theorem 2 and Lemma 4. Since $B_f = D_f = 0$ and considering $M = 0$, (20) reduces to (37). The existence of $\bar{\alpha} > 0$ such that (37) holds is equivalent to (42) of Lemma 4. Also $D_q = 0$ and $D = DD_m = 0$; this implies that (25a)-(25c) reduce to (43). Since $CBD_m + DC_mB_m$ has full column rank, condition (44) holds. Also, $B$ must have full column rank, that is, $n_v$. To verify condition (45) of Lemma 4, consider any $\lambda \in \mathbb{C}$ with non-negative real part. Then

$$\begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A - \lambda I & 0 & B' \\ 0 & I & 0 \\ C & 0 & D \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & A_m - \lambda I & B_m \\ 0 & C_m & D_m \end{bmatrix}.$$

As a consequence of the hypotheses of the lemma, the two matrices on the right-hand side of the second equality have maximum column rank; hence $\begin{bmatrix} A - \lambda I & B' \\ C & 0 \end{bmatrix}$ has maximum column rank, that is, $n_\xi + n_v$ which equals $n_\xi + \text{rank} B$. condition (45). Invoking Theorem 2 and Lemma 4 concludes the proof.

4.1.1 **Connection to classical rank conditions**

Consider the classical linear case of the linear system, $\dot{x} = Ax + Bw$, $y = Cx$, and $w = v$. This is described by (33) and (34) with $D = 0$, $D_m = I$ and $A_m, B_m, C_m$ vanish. Hence,

$$CBD_m + DC_mB_m = CB, \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A - \lambda I & B' \\ C & D \end{bmatrix}, \begin{bmatrix} A_m - \lambda I & B_m \\ C_m & D_m \end{bmatrix} = -\lambda I.$$

Consequently, the conditions in Lemma 5 reduce to the requirements that $CB$ and $\begin{bmatrix} A - \lambda I & B' \\ C & 0 \end{bmatrix}$ have full column rank for all $\lambda \in \mathbb{C}$ with non-negative real part. With $B$ full column rank, these are exactly the classical conditions for state estimation to an arbitrary degree of accuracy; see [31].
5 Proof of Theorem 1

First we need the following result.

**Lemma 6.** Consider a system described by (9) with state $e(t) \in \mathbb{R}^{n_e}$, input $v(t) \in \mathbb{R}^{n_v}$ and performance output $z(t) \in \mathbb{R}^{n_z}$. Suppose there exists a differentiable function $V : \mathbb{R}^{n_e} \to \mathbb{R}$ and scalars $\alpha, \beta_1, \beta_2, \mu_1 > 0$ and $\mu_2 \geq 0$ such that

$$\beta_1 \|e\|^2 \leq V(e) \leq \beta_2 \|e\|^2$$

(46)

and

$$DV(e) F(t, e, v) \leq -2\alpha V(e) - \mu_1 \|G(t, e, v)\|^2 + \mu_2 \|v\|^2$$

(47)

for all $t \in \mathbb{R}$, $e \in \mathbb{R}^{n_e}$ and $v \in \mathbb{R}^{n_v}$, where $DV$ denotes the derivative of $V$. Then, for any initial condition $e(t_0) = e_0$ with $t_0 \in \mathbb{R}$ and $e_0 \in \mathbb{R}^{n_e}$ and any $L_2$ exogenous input $v(\cdot) : [t_0, \infty) \to \mathbb{R}^{n_v}$, inequalities (16) and (17) hold for all $t \geq t_0$. Hence, system (9) is globally uniformly $L_2$-stable with performance level $\gamma = \sqrt{\mu_2/\mu_1}$.

**Proof.** Consider any initial condition $e(t_0) = e_0$ and any $L_2$ exogenous input $v(\cdot) : [t_0, \infty) \to \mathbb{R}^{n_v}$. Recalling (47), the time-derivative of $V(e)$ evaluated along a corresponding trajectory of (9) satisfies

$$\frac{dV(e(t))}{dt} = DV(e(t)) \dot{e}(t) = DV(e(t)) F(t, e(t), v(t))$$

$$\leq -2\alpha V(e(t)) - \mu_1 \|z(t)\|^2 + \mu_2 \|v(t)\|^2$$

(48)

$$\leq -2\alpha V(e(t)) + \mu_2 \|v(t)\|^2$$

(49)

for all $t \geq t_0$. Hence,

$$V(e(t)) \leq e^{-2\alpha(t-t_0)}V(e_0) + \mu_2 \int_{t_0}^{t} e^{-2\alpha(t-T)} \|v(\tau)\|^2 \, d\tau$$

$$\leq e^{-2\alpha(t-t_0)} \beta_2 \|e_0\|^2 + \mu_2 \|v(\cdot)\|_2^2$$

Since $\beta_1 \|e\|^2 \leq V(e)$ we see that

$$\|e(t)\|^2 \leq e^{-2\alpha(t-t_0)} (\beta_2 / \beta_1) \|e_0\|^2 + (\mu_2 / \beta_1) \|v(\cdot)\|_2^2,$$

(50)

from which it follows that

$$\|e(t)\| \leq e^{-\alpha(t-t_0)} \sqrt{\beta_2 / \beta_1} \|e_0\| + \sqrt{\mu_2 / \beta_1} \|v(\cdot)\|_2.$$

(51)

To demonstrate (16), note that (48) implies that

$$\mu_1 \|z(t)\|^2 \leq -\frac{dV(e(t))}{dt} + \mu_2 \|v(t)\|^2,$$

(52)

which, upon integrating from $t_0$ to any $t \geq t_0$ results in

$$\mu_1 \int_{t_0}^{t} \|z(\tau)\|^2 \, d\tau \leq V(e_0) + \mu_2 \|v(\cdot)\|_2^2.$$

Hence, for all $t \geq t_0$,

$$\int_{t_0}^{t} \|z(\tau)\|^2 \, d\tau \leq \mu_1^{-1} V(e_0) + \mu_2 / \mu_1 \|v(\cdot)\|_2^2$$

$$\leq \beta_2 / \mu_1 \|e_0\|^2 + \mu_2 / \mu_1 \|v(\cdot)\|_2^2.$$

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Hence, system (54) satisfies the hypotheses of Lemma 6 with scalars $\alpha, \mu > 0$, such that
\[
\begin{bmatrix}
\mathcal{P}\tilde{A} + \tilde{A}^T\mathcal{P} + 2\alpha \mathcal{P} + \mu_1 H^T H \\
\tilde{B}_f \\
0
\end{bmatrix} + \tilde{\Gamma}^T M \tilde{\Gamma} \leq 0,
\]
(56)
where
\[
\tilde{\Gamma} = \begin{bmatrix}
\tilde{C}_q & \tilde{D}_{qf} & \tilde{D}_q \\
0 & I & 0
\end{bmatrix}.
\]
(57)
Then, for any initial condition $e(t_0) = e_0$ with $t_0 \in \mathbb{R}$ and $e_0 \in \mathbb{R}^{n_e}$ and any $L_2$ input $v(\cdot) : [t_0, \infty) \to \mathbb{R}^{n_v}$, inequalities (16) and (17) hold for all $t \geq t_0$ with $\beta_1 = \lambda_{\min}(\mathcal{P})$ and $\beta_2 = \lambda_{\max}(\mathcal{P})$. Hence, system (54) is $L_2$-stable with performance level $\gamma = \sqrt{\mu_2/\mu_1}$.

**Proof.** We will show that system (54)-(55) satisfies the hypotheses of Lemma 6 with $V(e) = e^T \mathcal{P} e$ . This choice of $V$ satisfies the Rayleigh inequality
\[
\lambda_{\min}(\mathcal{P})\|e\|^2 \leq V(e) \leq \lambda_{\max}(\mathcal{P})\|e\|^2
\]
for all $e \in \mathbb{R}^{n_e}$. Hence, (46) holds with $\beta_1 = \lambda_{\min}(\mathcal{P}) > 0$ and $\beta_2 = \lambda_{\max}(\mathcal{P})$. For system (54)-(55),
\[
F(t, e, v) = \tilde{A} e + \tilde{B}_f \tilde{f} + \tilde{B} v.
\]
Therefore,
\[
\mathcal{D}V(e)F(t, e, v) = 2e^T \mathcal{P}(\tilde{A} e + \tilde{B}_f \tilde{f} + \tilde{B} v),
\]
\[
= e^T (\mathcal{P}\tilde{A} + \tilde{A}^T\mathcal{P}) e + 2e^T \tilde{B}_f \tilde{f} + 2e^T \tilde{B} v.
\]
(58)
Recalling the description of $\tilde{q}$ in (55), we see that $\tilde{\Gamma} \begin{bmatrix} e^T & \tilde{f}^T & v^T \end{bmatrix} = \begin{bmatrix} \tilde{q}^T & \tilde{f}^T \end{bmatrix}$. Hence, pre-and post-multiplying the matrix inequality (56) by $\begin{bmatrix} e^T & \tilde{f}^T & v^T \end{bmatrix}$ and its transpose results in
\[
\mathcal{D}V(e)F(t, e, v) + 2\alpha V + \mu_1 \|z\|^2 - \mu_2 \|v\|^2 + \begin{bmatrix} \tilde{q}^T \\
\tilde{f}^T \\
M \begin{bmatrix} \tilde{q} \\
\tilde{f} \end{bmatrix}
\end{bmatrix} \leq 0.
\]
It now follows from (55) that
\[
\mathcal{D}V(e)F(t, e, v) \leq -2\alpha V(e) + \mu_1 \|z\|^2 - \mu_2 \|v\|^2,
\]
that is, (47) holds. Using Lemma 6, we are done. \[\square\]
5.1 Proof of Theorem 1

With the estimation error given by (7), it follows from (12) and (10) that the observer error dynamics are given by

\[
\dot{e} = (A + L_1C)e + (B_f + L_1D_f)\hat{f} - (B + L_1D)v,
\]

(59a)

\[
\dot{\hat{f}} = f(t, y, q + \tilde{q}) - f(t, y, q),
\]

(59b)

\[
\tilde{q} = (C_q + L_2C)e + D_qf\hat{f} - (D_q + L_2D)v.
\]

(59c)

That is, it is described by (54) and satisfies (55) with

\[
\tilde{A} = A + L_1C, \quad \tilde{B}_f = B_f + L_1D_f, \quad \tilde{B} = -(B + L_1D),
\]

\[
\tilde{C}_q = C_q + L_2C, \quad \tilde{D}_q = D_qf + L_2D_f, \quad \tilde{D}_q = -(D_q + L_2D).
\]

Recalling that \(L_1 = \mathcal{P}^{-1}\mathcal{Y}\), we see that (13) is the same as (56). The desired result now follows from Lemma 7.

6 Numerical Example

We employ a modified model of the active magnetic bearing system investigated in [32]. The modification includes disturbance inputs and measurement noise to illustrate the unknown input observer capabilities and to make the problem more challenging than the one considered in our previous work [30]. The model is given by

\[
\dot{x} = \begin{bmatrix} x_2 + w_1 \\ x_3 + x_3|x_3| \\ w_2 \end{bmatrix}, \quad y = \begin{bmatrix} x_1 + 0.1w_1 \\ x_2 \end{bmatrix},
\]

(60)

which is in the form of (1) with \(f_1(t, y, q_1) = q_1|q_1| \text{ and } q_1 = x_3\). Considering

\[
w_1(t) = 1/\sqrt{1 + t} \quad \text{and} \quad w_2(t) = \log(1 + t),
\]

\(w\) is unbounded in the \(L_2\) sense, but \(\dot{w}\) is bounded. Also \(w_2\) is unbounded in the usual sense. Hence \(w\) can be modelled by (4) where \(v = \dot{w}\). Any matrix of the form

\[
M = \kappa \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

with any \(\kappa \geq 0\) is incremental multiplier matrix for \(f_1\). Note that we will solve for \(\kappa\): we only know the form of \(M\), the parameter \(\kappa\) is an optimization variable. We choose \(z = w\), which implies that we are interested in obtaining a good estimate of \(w\) and are ready to accept lower accuracy when reconstructing \(x\). Thus, \(z = w\). We fix \(L_2 = \begin{bmatrix} 0 & -110 \end{bmatrix}\) and solve (19) with a line search to find an optimal \(\alpha\). We get \(\alpha^* = 0.710, \kappa = 1.6 \times 10^6, \) and \(\mu_2 = 0.08\). We test our proposed observer on system (60) with the initial conditions

\[
x(0) = \begin{bmatrix} -2.7247 \\ 10.9842 \\ -2.7787 \end{bmatrix},
\]

and \(\hat{\xi}(0) = 0\).

The response of the proposed observer is shown in Figure 1. Note that the unknown input \(w_2\) is monotonically increasing, yet from Figure 1[C-D], we observe that the estimates of the unbounded unknown inputs are very accurate; this is to be expected since \(\gamma\) is small.
Figure 1: [A] State estimation error $e_x = \hat{x} - x$ of the nonlinear system in (60). [B] Unknown input estimation error. The convergence of the norm of $e_w = \hat{w} - w$ is illustrated. [C, D] Unknown inputs (blue) and their estimates (dashed red).

7 Conclusions

This paper provides an LMI based approach to the design of observers for estimating the state and unknown exogenous input for a wide range of nonlinear systems. The resulting input-output system from exogenous input to estimation error is $L_2$ stable with a gain that can be pre-specified and computed via standard toolboxes.

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