FROBENIUS PROJECTIVE AND AFFINE GEOMETRY
OF VARIETIES IN POSITIVE CHARACTERISTIC

YASUHIRO WAKABAYASHI

Abstract. The goal of the present paper is to lay the foundations for a theory of projective and affine structures on higher-dimensional varieties in positive characteristic. This theory deals with Frobenius-projective and Frobenius-affine structures, which have been previously investigated only in the case where the underlying varieties are curves. As the first step in expanding the theory, we prove various basic properties on Frobenius-projective and Frobenius-affine structures and study (the positive characteristic version of) the classification problem, starting with S. Kobayashi and T. Ochiai, of varieties admitting projective or affine structures.

In the first half of the present paper, we construct bijective correspondences with several types of generalized indigenous bundles; one of them is defined in terms of Berthelot’s higher-level differential operators. We also prove the positive characteristic version of Gunning’s formulas, which give necessary conditions on Chern classes for the existence of Frobenius-projective or Frobenius-affine structures respectively. The second half of the present paper is devoted to studying, from the viewpoint of Frobenius-projective and Frobenius-affine structures, some specific classes of varieties, i.e., projective and affine spaces, abelian varieties, curves, and surfaces. For example, it is shown that the existence of Frobenius-projective structures of infinite level gives a characterization of ordinariness for abelian varieties. Also, we prove that any smooth projective curve of genus $g > 1$ admits a Frobenius-projective structure of infinite level, which induces a representation of the stratified fundamental group.

Contents

0. Introduction 3
0.1. Review of complex projective and affine structures 3
0.2. Previous works related to Frobenius-projective and Frobenius-affine structures 4
0.3. Results in the present paper 5
0.4. Organization of the paper 11
0.5. Notation and Conventions 13
1. $F^N$-projective and $F^N$-affine structures 13
1.1. Frobenius twists and relative Frobenius morphisms 13
1.2. $F^N$-projective and $F^N$-affine structures of level $N < \infty$ 14
1.3. $F^N$-projective and $F^N$-affine structures of level $N = \infty$ 15
1.4. Purity theorem 16
1.5. Kodaira-Spencer map associated to a flat principal bundle 17

Y. Wakabayashi: Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, JAPAN;
e-mail: wkbysh@math.titech.ac.jp;
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Key words: projective structure, affine structure, indigenous bundle, oper, uniformization, positive characteristic, $p$-curvature, vector bundle, $F$-divided sheaf, stratified bundle, Tango structure, $D$-module, curve, surface, abelian variety
1.6. Universal Kodaira-Spencer map
1.7. \( F^N \)-(affine-)indigenous structures
2. Indigenous and affine-indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
2.1. Differential operators of level \( m \)
2.2. Kodaira-Spencer map associated to a \( \mathcal{D}_X^{(m)} \)-module
2.3. Indigenous and affine-indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
2.4. From (affine-)indigenous \( \mathcal{D}_X^{(0)} \)-modules to (affine-)indigenous bundles
2.5. Twisting by invertible \( \mathcal{D}_X^{(N-1)} \)-modules
2.6. Comparison with differential operators
3. Dormant indigenous and affine-indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
3.1. \( p \)-curvature and \( F \)-divided sheaves
3.2. Dormant indigenous and affine-indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
3.3. Comparison via projectivization I
3.4. Comparison via projectivization II
3.5. Rigidification by theta characteristics
3.6. Comparison via rigidification
3.7. Chern class formula
4. Tango structures
4.1. Extensions of the sheaf of locally exact 1-forms
4.2. Tango structures on a variety
4.3. From Tango structures to dormant affine-indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
4.4. Comparison with dormant affine-indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
4.5. Dual affine connections
5. Case 1: Projective and affine spaces
5.1. Uniqueness of \( F^N \)-projective structures on projective spaces
5.2. Nonuniqueness of \( F^N \)-affine structures on affine spaces
5.3. Characterization of projective spaces I (Stratified fundamental group)
5.4. Characterization of projective spaces II (Embedded rational curves)
6. Case 2: Abelian varieties
6.1. Pull-back via étale coverings
6.2. Dual abelian variety and Cartier operator
6.3. Diagonal dormant (affine-)indigenous \( \mathcal{D}_X^{(N-1)} \)-modules
6.4. Explicit description in terms of \( p \)-adic Tate module
6.5. Descent via étale coverings
7. Case 3: Curves
7.1. A rank \( p^N \) vector bundle associated with a line bundle
7.2. Comparison with Quot schemes
7.3. Existence of \( F^N \)-projective structures
7.4. Finiteness for genus 2 curves
7.5. \( F^N \)-affine structures on curves
8. Case 4: Surfaces
8.1. Products of varieties
8.2. Surfaces admitting an \( F^N \)-projective \( (F^N\text{-affine}) \) structure
8.3. Hyperelliptic surfaces
8.4. Properly elliptic surfaces
0. Introduction

The goal of the present paper is to lay the foundations for a theory of projective and affine structures on higher-dimensional varieties in positive characteristic. This theory deals with Frobenius-projective and Frobenius-affine structures (and several structures equivalent to them respectively), which has been investigated only in the case where the underlying varieties are curves. Such structures on a curve provide us with interesting connections between exotic phenomena appearing in positive characteristic geometry and various things discussed in other fields, e.g., representation theory, combinatorics, etc. The point that enables those connections is the correspondence with certain flat bundles, called dormant indigenous bundles (or dormant PGL₂-opers). With that in mind, our study could be placed in a higher-dimensional generalization of the theory of dormant indigenous bundles. We expect that the geometry of Frobenius-projective and Frobenius-affine structures will expand in a meaningful way by comparing with what has been shown in the one-dimensional case and with the theory of projective and affine structures on complex manifolds. As the first step in expanding the theory, we prove in the present paper various basic properties on Frobenius-projective and Frobenius-affine structures and study the classification problem of varieties admitting such structures.

0.1. Review of complex projective and affine structures.

Before continuing to explain our study in the present paper, let us briefly review what is a projective (resp., an affine) structure defined on a complex manifold. A projective (resp., an affine) structure on a complex manifold of dimension $n > 0$ is a maximal system of local coordinates in the analytic topology modeled on the complex projective space $\mathbb{P}^n_C$ (resp., the complex affine space $\mathbb{A}^n_C$) such that on any two overlapping coordinate patches, the change of coordinates may be described as a projective transformation on $\mathbb{P}^n_C$ (resp., an affine transformation on $\mathbb{A}^n_C$). On a complex manifold equipped with a projective (resp., an affine) structure, there is a projective (resp., an affine) geometry, in the spirit of F. Klein’s Erlangen program, that locally agrees with the geometry of $\mathbb{P}^n_C$ (resp., $\mathbb{A}^n_C$).

Projective and affine structures on complex manifolds arise in many areas of mathematics and are still actively being investigated. For example, such additional structures on Riemann surfaces (i.e., complex manifolds of dimension 1) play a major role in understanding the framework of uniformization theorem. An important consequence of the uniformization theorem for Riemann surfaces is that any closed Riemann surface is isomorphic either to the complex projective line $\mathbb{P}^1_C$, or to a quotient of the complex affine line $\mathbb{A}^1_C$ (⊆ $\mathbb{P}^1_C$) by a discrete group of translations (i.e., a 1-dimensional complex torus), or to a quotient of the 1-dimensional complex hyperbolic space $\mathbb{H}^1_C$ (⊆ $\mathbb{P}^1_C$) by a torsion-free discrete subgroup of $\text{SU}(1, 1) \cong \text{SL}(2, \mathbb{R})$. This implies that by collecting various local inverses of a universal covering map (from $\mathbb{P}^1_C$, $\mathbb{A}^1_C$, or $\mathbb{H}^1_C$), we have a canonical projective structure on each closed Riemann surface. Also, it is shown that 1-dimensional complex tori are the only closed Riemann surfaces admitting an affine structure.
But, what about the case of higher-dimensional manifolds? In the case of higher dimensions, there are nontrivial obstructions for the existence of a projective structure. In fact, not all complex manifolds have such a structure. This fact can be verified, e.g., by a very useful formula providing a necessary condition on Chen classes for the existence of a projective structure, which was proved by R. C. Gunning (cf. [26], Part II, §8, Theorem 5). S. Kobayashi and T. Ochiai (cf. [47], [48]) studied and classified compact complex Kähler(-Einstein) manifolds admitting a projective structure (or more generally, a projective connection, which is an infinitesimal version of a projective structure). The standard examples of them are as follows:

- the \( n \)-dimensional projective space \( \mathbb{P}^n_\mathbb{C} \),
- all étale quotients of \( n \)-dimensional complex tori,
- all compact quotients of the \( n \)-dimensional complex hyperbolic space \( \mathbb{H}^n_\mathbb{C} \) by torsion-free discrete subgroup of \( \text{SU}(n, 1) \subseteq \text{PGL}_{n+1}(\mathbb{C}) \).

Each manifold in the second admits an affine structure. In the case of dimension 1, the respective examples in this list correspond to the closed Riemann surfaces of genus 0, 1, and \( > 1 \) in order. A result by S. Kobayashi and T. Ochiai (cf. [17], Corollary 5.3 and Remark preceding that corollary) asserts that the above standard examples are the only compact Kähler-Einstein manifolds admitting a projective connection. In dimension 2, a compact complex surface admits a projective structure if and only if it is one of the standard examples in the case of \( n = 2 \) (cf. [34], §1, Main Theorem). On the other hand, it was proved (cf. [34], Theorem 7.1) that there is one more example in dimension 3, namely étale quotients of smooth modular families of false elliptic curves parametrized by a Shimura curve (cf. [34]). See, e.g., [84], [44], and [17], for other related studies in higher dimensions.

0.2. Previous works related to Frobenius-projective and Frobenius-affine structures.

Then, is it possible to develop geometry in positive characteristic based on the same viewpoint? One thing to keep in mind is that due to its analytic formulation, the definitions of projective and affine structure cannot be adopted, in positive characteristic, as they are. Indeed, if we had defined them in a naive fashion, there would be no other than trivial examples of varieties having such structures. By taking this unfortunate fact into account, we need to deal with appropriate replacements. The positive characteristic analogues discussed in the present paper are what we call \( F^N \)-projective and \( F^N \)-affine structures, where \( N \) denotes either a positive integer or \( N = \infty \). Both notions were introduced by Y. Hoshi (cf. [30], [31]) in the case where \( N < \infty \) and the underlying variety is a curve. That is to say, (the higher-dimensional extensions of) \( F^N \)-projective and \( F^N \)-affine structures are our central focus, and the study in the present paper can be said to be an attempt to understand what facts, such as differences from the case of complex manifolds, can be shown about them.

Notice that we have a notion that should be mentioned here, i.e., a \( \text{PGL}_2 \)-oper, also known as an indigenous bundle. It is originally defined on a Riemann surface as a \( \mathbb{P}^1 \)-bundle, or equivalently a principal \( \text{PGL}_2 \)-bundle, equipped with a connection and a global section satisfying a strict form of Griffiths transversality. This notion is also defined for algebraic curves in any characteristic because of the algebraic nature of its formulation. We know that on a smooth projective curve over \( \mathbb{C} \), or equivalently a closed Riemann surface, there exists a bijective correspondence between \( \text{PGL}_2 \)-opers and projective structures. This correspondence may be given by taking local coordinate charts determined by local solutions to the differential equation corresponding to each \( \text{PGL}_2 \)-oper. But, of course, it doesn’t work in characteristic \( p > 0 \).
because general \( \text{PGL}_2 \)-opers do not always have local solutions. If we want to have reasonable \( \text{PGL}_2 \)-opers in this sense, an invariant called \( p \)-\textit{curvature} plays an essential role. Recall that the \( p \)-curvature of a connection is defined as the obstruction to the compatibility of \( p \)-power structures that appear in certain associated spaces of infinitesimal symmetries. A \( \text{PGL}_2 \)-oper on a curve is called \textit{dormant} if its \( p \)-curvature vanishes identically. Since a dormant \( \text{PGL}_2 \)-oper has a full set of solutions locally in the Zariski topology, it induces, in the same manner as the complex case, an atlas of \textit{étale} coordinate charts, specifying an \( F^1 \)-projective structure. By this construction, we have a bijective correspondence between dormant \( \text{PGL}_2 \)-opers and \( F^1 \)-projective structures. Also, \( F^1 \)-affine structures on a curve correspond bijectively to what we call \textit{dormant (generic) Miura} \( \text{PGL}_2 \)-opers.

There are previous studies on dormant \( \text{PGL}_2 \)-opers and dormant Miura \( \text{PSL}_2 \)-opers, as follows. Various properties of dormant \( \text{PGL}_2 \)-opers and their moduli space were discussed in the context of \( p \)-adic Teichmüller theory developed by S. Mochizuki (cf. \cite{64}). As discussed in that study, an understanding of the moduli, e.g., deformations and degenerations of dormant \( \text{PGL}_2 \)-opers, provides some important perspectives. In fact, a certain factorization property according to clutching morphisms on relevant moduli spaces determines a topological quantum field theory (cf. \cite{77}). It creates a remarkable connection with Gromov-Witten theory of certain Quot schemes (cf. \cite{40, 36, 76}) and combinatorics of edge-colored graphs, as well as of convex rational polytopes (cf. \cite{53, 77, 78}). This connection enable us to give an effective way to solve the enumerative problem of dormant \( \text{PGL}_2 \)-opers (cf. \cite{75, 77}). Some of these facts are generalized to the theory of dormant \( G \)-opers for a semisimple algebraic group \( G \) (cf. e.g., \cite{39, 40, 76, 77, 81}).

On the other hand, by a result of \cite{81}, dormant Miura \( \text{PGL}_2 \)-opers correspond bijectively to \textit{Tango structures}. A Tango structure is a certain line bundle on an algebraic curve and brings various sorts of pathological phenomena in positive characteristic. As an application of the theory of dormant Miura \( \text{PGL}_2 \)-opers developed in \cite{81}, we achieve a detailed understanding of the moduli space of algebraic surfaces in positive characteristic violating the Kodaira vanishing theorem. Also, the main result of \cite{80} describes dormant Miura \( \text{PGL}_2 \)-opers, or equivalently Tango structures, by means of solutions to the Bethe ansatz equations for Gaudin model.

\section{Results in the present paper.}

In what follows, we shall describe some (relatively important) results of the present paper. Let \( p \) be a prime, \( k \) an algebraically closed field of characteristic \( p \), and \( X \) a smooth projective variety over \( k \) of dimension \( n > 0 \). For a positive integer \( N \), denote by \( X^{(N)} \) the \( N \)-th relative Frobenius twist of \( X \) (cf. \S 1.1), i.e., the base-change of \( X \) via the \( N \)-th iterate of the Frobenius automorphism of \( k \). Denote also by \( (\text{PGL}_{n+1})^{(N)}_X \) the sheaf on \( X^{(N)} \) represented by \( \text{PGL}_{n+1} \), which we think of as a sheaf on \( X \) via the underlying homeomorphism of the \( N \)-th relative Frobenius morphism \( F^{(N)}_{X/k} : X \to X^{(N)} \) (cf. \S 1.1). Then, we define a \textit{Frobenius-projective structure of level} \( N \), or an \( F^N \)-\textit{projective structure} for short, on \( X \) (cf. Definition 1.2.1 for the precise definition) as a maximal collection of étale coordinate charts on \( X \) forming a torsor modeled on \( (\text{PGL}_{n+1})^{(N)}_X \). This is a central notion of the present paper that we regard as a positive characteristic analogue of a projective structure defined on a complex manifold. The local sections of \( (\text{PGL}_{n+1})^{(N)}_X \) play the same role as projective transformations, which appears in the definition of a projective structure. Moreover, by an \( F^\infty \)-\textit{projective structure}, we shall mean a compatible collection of \( F^N \)-projective structures for various \( N \). In similar manners,
we can define the respective “affine” versions, i.e., the notions of an $F^N$-affine structure for every $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$.

The main purpose of the first half of the present paper is to examine the relationship with several other structures generalizing dormant PGL$_2$-opers on a curve. The first one is called a dormant indigenous (resp., affine-indigenous) $D_{X}^{(N-1)}$-module, where $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, defined by using Berthelot’s sheaf of differential operators $D_{X}^{(N-1)}$ of level $N-1$ (but the case of infinite level, i.e., the sheaf $D_{X}^{(\infty)}$, is due to Grothendieck). On the other hand, the second one is called an $F^N$-indigenous (resp., $F^N$-affine-indigenous) structure, i.e., a certain principal bundle on $X^{(N)}$ in the étale topology equipped with an additional reduction structure. Both kinds of structures satisfy strict forms of Griffiths transversality, which capture their essential features. Regarding them, we provide certain bijective correspondences, as described in Theorem A later.

Before describing the statement of the theorem, we shall comment on the advantage of introducing the various structures of infinite level, not only those of finite levels. A key ingredient in the study of $F^\infty$-projective structures is the Tannakian fundamental group associated with the category of $D_{X}^{(\infty)}$-modules, or equivalently the category of $F$-divided sheaves, which we call the stratified fundamental group and denote by $\pi_{1}^{str}(X)$. That is to say, the category of representations of $\pi_{1}^{str}(X)$ classifies $D_{X}^{(\infty)}$-modules. In particular, the dormant indigenous $D_{X}^{(\infty)}$-module corresponding, via a bijection obtained in Theorem A, to each $F^\infty$-projective structure specifies a PGL$_{n+1}$-representation of $\pi_{1}^{str}(X)$ up to conjugation; it may be thought of as the monodromy representation of this $F^\infty$-projective structure, and enable us to obtain a much better understanding of related things from the viewpoint of algebraic groups. This is a remarkable consequence of Theorem A below, being a part of Proposition 3.3.2, Theorem 3.4.3, Corollary 3.4.5, Propositions 3.6.1 and 3.6.5. (We refer to [30], [31] for the case where $N < \infty$ and $\dim(X) = 1$.)

**Theorem A.**

Let $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$ and let $X$ be a smooth projective variety over $k$ of dimension $n$ ($> 0$).

(i) Suppose that $p \nmid (n + 1)$. Then, we obtain the following commutative diagram of sets in which all the maps are bijective:

\[
\begin{array}{ccc}
\mathcal{D}_{X}^{(N-1)} & \xrightarrow{\zeta_{N}} & F^{N}\text{-Proj}_{X} \\
\downarrow & & \downarrow \\
F^{N}\text{-Ind}_{X} & \xrightarrow{\zeta_{N}} & F^{N}\text{-Proj}_{X} \\
\end{array}
\]

where
- $F^{N}\text{-Proj}_{X}$ (cf. (14) and (19)) denotes the set of $F^{N}$-projective structures on $X$;
- $F^{N}\text{-Ind}_{X}$ (cf. (46) and (56)) denotes the set of isomorphism classes of $F^{N}$-indigenous structures on $X$;
- $\mathcal{D}_{X}^{(N-1)}$ (cf. (129)) denotes the set of $\mathbb{G}_{m}$-equivalence classes (cf. Definition 3.3.1) of dormant indigenous $D_{X}^{(N-1)}$-modules.

(Note that the bijectivity of $\zeta_{N}$ holds without imposing the assumption $p \nmid (n + 1)$.)
(ii) We obtain the following commutative diagram of sets in which all the maps are bijective:

\[
\begin{array}{ccc}
\mathcal{O}_{X,N}^{\text{zar}} & \xrightarrow{\sim} & \mathcal{T}\tan_{X,N} \\
\mathcal{A} & \xrightarrow{\sim} & \mathcal{A} \\
F^N\text{-Aff}_{X} & \xrightarrow{\sim} & F^N\mathcal{A}\text{id}_{X}
\end{array}
\]

where
- \(F^N\text{-Aff}_{X}\) (cf. (14) and (19)) denotes the set of \(F^N\text{-affine}\) structures on \(X\);
- \(F^N\mathcal{A}\text{id}_{X}\) (cf. (40) and (56)) denotes the set of isomorphism classes of \(F^N\text{-affine-indigenous}\) structures on \(X\);
- \(\mathcal{O}_{X,N}^{\text{zar}}\) (cf. (129)) denotes the set of \(\mathbb{G}_m\)-equivalence classes of dormant affine-indigenous \(D_{X-N-1}\) modules;
- \(\mathcal{T}\tan_{X,N}\) (cf. (192) and (197)) denotes the set of Tango structures of level \(N\) on \(X\).

As an application of this theorem, we prove Theorem B described below. This is a positive characteristic analogue of Gunning’s formula, which provides necessary conditions on Chern classes for the existence of a projective (resp., an affine) structure on complex manifolds (cf. [26], Part II, §8, Theorem 5). Gunning’s formula implies that if a Kähler manifold carries a projective (resp., an affine) structure, then the Chern numbers are proportional to those of the projective (resp., affine) space of the same dimension. We can prove the corresponding assertion in positive characteristic by examining the underlying vector bundle of a dormant indigenous (resp., affine-indigenous) \(D_{X-N-1}\)-module; this result is very helpful in characterizing higher-dimensional varieties admitting an \(F^N\text{-projective}\) (resp., \(F^N\text{-affine}\)) structure.

**Theorem B** (cf. Theorem 3.7.1 for the full statement).

Let \(N \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}\) and let \(X\) be a smooth projective variety over \(k\) admitting an \(F^N\text{-projective}\) (resp., \(F^N\text{-affine}\)) structure. We shall denote by \(c^{\text{crys}}_l(X)\) (for \(l > 0\)) the \(l\)-th crystalline Chern class of \(X\). Then, for each \(l > 0\), the equality

\[
c^{\text{crys}}_l(X) = \frac{1}{(n+1)^l} \cdot \binom{n+1}{l} \cdot c^{\text{crys}}_1(X)^l \quad \text{(resp., } c^{\text{crys}}_l(X) = 0)\]

holds in the \(2l\)-th crystalline cohomology \(H^{2l}_{\text{crys}}(X/W)\) of \(X\) over \(W := \text{the ring of Witt vectors over } k\).

In the second half of the present paper, we will study \(F^N\text{-projective}\) and \(F^N\text{-affine}\) structures (and related structures) defined on varieties in the following four classes individually:

- Projective and affine spaces (cf. §5);
- Abelian varieties (cf. §6);
- Curves (cf. §7);
- Surfaces (cf. §8).
For each class in the above list, we will try to understand which varieties have an $F^N$-projective or $F^N$-affine structure and how these structures can be described in a somewhat explicit manner. These questions are based on the classification problem, starting with S. Kobayashi and T. Ochiai, of complex manifolds admitting projective and affine structures. The most typical examples are projective and affine spaces. They have globally defined coordinate charts taking values in themselves, which define the trivial $F^N$-projective and $F^N$-affine structures respectively for each $N \in \mathbb{Z}_{>0} \cup \{\infty\}$. The projective space $\mathbb{P}^n$ (with $p \nmid (n+1)$) has only the trivial $F^\infty$-projective structure, but the situation is different for the affine space $\mathbb{A}^n$, i.e., there are various nontrivial $F^N$-affine structures on $\mathbb{A}^n$. This fact is an exotic phenomenon in positive characteristic because the complex affine space has only the trivial affine structure.

In the following theorem, we describe what facts will be shown about infinite level. In particular, we prove some characterizations of projective spaces in terms of $F^\infty$-projective structure. These characterizations can be regarded as the positive characteristic versions of [47], Theorem 4.4, and [34], Theorem 4.1 (or the main theorem in [84]).

**Theorem C** (cf. Proposition 5.1.1, Theorems 5.3.1, 5.4.1, and Remark 5.3.2).

Let $n$ be a positive integer. Then, the following assertions hold:

(i) The projective space $\mathbb{P}^n$ admits exactly one $F^\infty$-projective structure, but admits no $F^\infty$-affine structures. Moreover, the affine space $\mathbb{A}^n$ admits an $F^\infty$-affine structure, in particular, admits an $F^\infty$-projective structure.

(ii) Let $X$ be a smooth projective variety over $k$ of dimension $n$. Suppose that $X$ admits an $F^\infty$-projective structure and moreover one of the following conditions is fulfilled:
   - $X$ contains a rational curve;
   - the étale fundamental group $\pi_1^\text{ét}(X)$ of $X$ is trivial;
   - the stratified fundamental group $\pi_1^\text{str}(X)$ of $X$ is trivial.

Then, $X$ is isomorphic to the projective space $\mathbb{P}^n$. In particular, $\mathbb{P}^n$ is the only Fano variety admitting an $F^\infty$-projective structure.

The next class of varieties that we discuss is the class of abelian varieties. One important result of our study is, as described in the following theorem, that the existence of an $F^\infty$-projective (or $F^\infty$-affine) structure gives a characterization of ordinarity for abelian varieties.

**Theorem D** (cf. Corollary 6.3.3, Theorem 6.4.1).

Let $X$ be an abelian variety over $k$ of dimension $n$. Then, the following assertions hold:

(i) $X$ admits an $F^\infty$-projective (resp., $F^\infty$-affine) structure if and only if $X$ is ordinary in the usual sense.

(ii) Suppose further that $X$ is ordinary. Write

$$\mathbb{B}_{X,\infty} := \left\{ (b_i)_{i=1}^n \in (T_p X^\vee)^n \mid \land_i b_i \in \det(T_p X^\vee)^\times \right\}$$

(4)
(cf. (319)), where $X^\vee$ denotes the dual abelian variety of $X$ and $T_p X^\vee$ denotes the $p$-adic Tate module of $X^\vee$. Notice that this set is endowed with a natural action of the symmetric group of $n$ letters $\mathfrak{S}_n$ by permutations of factors, hence we have the orbit set $\mathbb{B}_{X,\infty}/\mathfrak{S}_n$. Then, we obtain the following commutative square diagram in which both the upper and lower horizontal arrows are bijective:

$$
\begin{array}{c}
\mathbb{B}_{X,\infty}/\mathfrak{S}_n \\
\downarrow \text{quotient} \\
\delta \mathbb{B}_{X,\infty}/\mathfrak{S}_n \\
\downarrow \\
F_\infty^{\text{-Aff}} X
\end{array} \xrightarrow{\iota_\infty}\begin{array}{c}
F_\infty^{\text{-Proj}} X,
\end{array}
$$

where $F_\infty^{\text{-Proj}} X$ and $F_\infty^{\text{-Aff}} X$ are as in Theorem A and $\delta \mathbb{B}_{X,\infty}/\mathfrak{S}_n$ denotes the quotient set of $\mathbb{B}_{X,\infty}/\mathfrak{S}_n$ divided by the equivalent relation (321). In particular, the forgetting map

$$
\iota_\infty : F_\infty^{\text{-Aff}} X \to F_\infty^{\text{-Proj}} X
$$

(cf. (21)) is surjective and moreover gives an $n$-to-1 correspondence between $F_\infty$-affine structures and $F_\infty$-projective structures on $X$.

We then consider $F^N$-projective and $F^N$-affine structures on smooth projective curves. As mentioned earlier, the study of the case $N = 1$ has been investigated in the previous works under the correspondence with dormant (Miura) PGL$_2$-opers. Besides that case, not much is known about those of higher levels. In the present paper, we mainly focus on one property, that is, the existence of an $F^N$-projective structure for every $N \in \mathbb{Z}_{>0} \cup \\{\infty\}$. To this end, we construct, for a positive integer $N$, a bijection between $F^N$-projective structures on a smooth projective curve $X$ and $k$-rational points of a certain Quot-scheme, denoted by $\mathcal{Q}^{2\Omega}_{N, \Theta}$, classifying subsheaves of a rank $p^N$ vector bundle on $X$. This bijection is nothing but the higher-level generalization of the bijection constructed in [40], Theorem 5.4.1, and [75], Proposition 4.3. Then, the desired existence property can be deduced from a nonemptiness assertion on Quot-schemes due to A. Hirschowitz (cf. [40], Theorem 2.3.1, or [29]). We describe the resulting fact in the following Theorem E. Also, an attempt will be made to understand the finiteness of $F^N$-projective structures for $N < \infty$. We expect that there are only finitely many $F^N$-projective structures on a prescribed smooth projective variety $X$ (cf. Conjecture 7.4.3). This assertion is already known only in the case where $N = 1$ and $X$ is a curve. In the present paper, we show that this is true for every $N < \infty$ when $X$ is a curve of genus 2 (cf. Theorem 7.4.2).

**Theorem E** (cf. Theorem 7.2.2, Corollary 7.3.4 Propositions 7.5.1, 7.5.2 for the full statements).

*Suppose that $p (= \text{char}(k)) > 2$, and let $X$ be a smooth projective curve over $k$ and denote by $g$ the genus of $X$. Then, the following assertions hold:*  

(i) *Suppose that $g = 0$. Then, $X$ admits an $F_\infty$-projective structure, but admit no $F_\infty$-affine structures.*  

(ii) *Suppose that $g = 1$. Then, $X$ admits an $F_\infty$-projective (resp., $F_\infty$-affine) structure if and only if $X$ is ordinary.*
(iii) Suppose that \( g > 1 \). Then, \( X \) admits an \( F^\infty \)-projective structure, but admit no \( F^\infty \)-affine structures. Moreover, let \( N \) be a positive integer and fix a theta characteristic \( \Theta \) of \( X \). Then, there exists a canonical bijection of sets

\[
F^N\text{-Proj}_X \cong \mathcal{Q}^{2,0}_{N,\Theta}(k),
\]

where \( F^N\text{-Proj}_X \) is as in Theorem A and \( \mathcal{Q}^{2,0}_{N,\Theta} \) (cf. (350)) denotes the Quot-scheme over \( k \) classifying \( \mathcal{O}_{X^{(N)}} \)-submodules \( F \) of \( F^{(N)}_{X/k^*}(\Theta^\vee) \) with \( \text{rank}(F) = 2 \) and \( \text{det}(F) \cong \mathcal{O}_X \).

The final examples that we deal with are smooth projective surfaces. In their three fundamental articles [67], [11], [12], E. Bombieri and D. Mumford established the Kodaira-Enriques classification of projective smooth minimal surfaces in positive characteristic. According to this classification, we try, by using, e.g., the Chern class formula (cf. Theorem B), to understand which surfaces admit an \( F^N \)-projective or \( F^N \)-affine structure. (If \( N = \infty \), then it suffices to consider minimal surfaces because nonminimal surfaces cannot have any \( F^\infty \)-projective structure according to Theorem C, (ii).) A key observation regarding this question is, as in the case of curves, that the classification of surfaces admitting an \( F^N \)-projective or \( F^N \)-affine structure with \( N = \infty \) seems to be almost the same as the corresponding classification of complex surfaces, but we can find various exotic examples for \( N \neq \infty \). For example, if \( N < \infty \), then the product of two curves of genus \( > 1 \) having Tango functions of level \( N \) (cf. the comment preceding Proposition 7.5.2 and [32], Theorem 3) forms a general type surface admitting an \( F^N \)-projective structure; this surface satisfies (cf. Remark 8.1.3) an unexpected equality on the Chern classes when compared to the case of complex surfaces. In the following Theorem F we describe results only for \( N = \infty \) obtained in the present paper. At the time of writing the present paper, the classification is unfortunately not complete because some details are not filled. Especially, the author does not know much about surfaces of general type. As asserted in Theorem F, a minimal surface of general type admitting an \( F^\infty \)-projective structure must satisfy the equality \( c_1^{\text{cris}}(X)^2 - 3 \cdot c_2^{\text{cris}}(X) = 0 \) of crystalline Chern classes. By taking account of a certain fact on complex surfaces of general type (cf. Remark 8.5.2), we expect that this equality also becomes, under some reasonable conditions, a sufficient condition for the existence of an \( F^\infty \)-projective structure.

**Theorem F** (cf. Corollary 8.5.1).

Suppose that \( p \) (\( = \text{char}(k) \)) \( > 3 \), and let \( X \) be a smooth projective surface over \( k \). Then, \( X \) admits no \( F^\infty \)-projective structures unless \( X \) is minimal. Moreover, if \( X \) is minimal, then we have the following assertions:

(i) Suppose that \( X \) is a rational surface. Then, \( X \) admits an \( F^\infty \)-projective structure if and only if \( X \) is isomorphic to the projective plane \( \mathbb{P}^2 \). Moreover, there are no \( F^\infty \)-affine structures on any rational surface.

(ii) Suppose that \( X \) is a nonrational ruled surface. Then, \( X \) admits no \( F^\infty \)-projective structures.

(iii) Suppose that \( X \) is an Enriques surface. Then, \( X \) admits no \( F^\infty \)-projective structures.

(iv) Suppose that \( X \) is a hyperelliptic surface. Then, \( X \) admits an \( F^\infty \)-projective (resp., \( F^\infty \)-affine) structure if and only if \( X \) is ordinary in the sense of Definition 8.3.1.

(v) Suppose that \( X \) is a K3 surface. Then, \( X \) admits no \( F^\infty \)-projective structures.
(vi) Suppose that $X$ is an abelian surface. Then, $X$ admits an $F^\infty$-projective (resp., $F^\infty$-affine) structure if and only if $X$ is ordinary.

(vii) Suppose that $X$ is a properly elliptic surface. Then, $X$ admits no $F^\infty$-projective structures if every multiple fiber of the elliptic fibration of $X$ is tame and has multiplicity prime to $p$.

(viii) Suppose that $X$ is a surface of general type. Then, $X$ admits no $F^N$-projective (resp., $F^\infty$-affine) structures unless the equality $c_1^{\text{crys}}(X)^2 - 3c_2^{\text{crys}}(X) = 0$ holds (resp., the equalities $c_1^{\text{crys}}(X) = c_2^{\text{crys}}(X) = 0$ hold) in the crystalline cohomology $H^*_\text{crys}(X/W)$ of $X$.

0.4. Organization of the paper.

The present paper is organized as follows. In §1, we first introduce the notions of $F^N$-projective and $F^N$-affine structure defined on a smooth variety of any arbitrary dimension (cf. Definitions 1.2.1 and 1.3.1). A related purity theorem is proved in the middle of that section (cf. Theorem 1.4.1); this result concerns restriction of $F^N$-projective and $F^N$-affine structures on a variety to its open subscheme of codimension $\geq 2$. After that, we make the definitions of $F^N$-indigenous and $F^N$-affine-indigenous structure (cf. Definitions 1.7.1, 1.7.3). By means of some properties of Kodaira-Spencer maps proved before these definitions, we construct, at the end of §1 (cf. (62)), a map $\zeta_N^\triangledown \Rightarrow \bullet$ (resp., $\zeta_N^\triangledown \Rightarrow \bullet$) from the set of $F^N$-projective (resp., $F^N$-affine) structures to the set of $F^N$-indigenous (resp., $F^N$-affine-indigenous) structures; this map is one of the maps appearing in the diagram of Theorem A (i) (resp., (ii)).

In §2, we first recall Berthelot’s ring of differential operators $D_X^{(m)}$ of finite level $m \geq 0$ defined on a smooth variety $X$. As in the case of flat principal bundles, the notion of Kodaira-Spencer map is defined for $D_X^{(m)}$-modules (equipped with a subbundle). (We refer to Remark 2.2.2 for the relationship between these two notions of Kodaira-Spencer maps.) By using this, we make the definitions of indigenous and affine-indigenous $D_X^{(N-1)}$-module (cf. Definitions 2.3.1 and 2.3.2). At the end of §2, it is observed that indigenous $D_X^{(0)}$-modules equipped with a prescribed line subbundle correspond bijectively to certain differential operators between line bundles (cf. Proposition 2.6.1).

At the beginning of §3, we recall some properties on $p$-$m$-curvature of $D_X^{(m)}$-modules ($m \geq 0$). The definitions of dormant (affine-)indigenous $D_X^{(N-1)}$-module and $\mathbb{G}_m$-equivalence relation are given after that (cf. Definitions 3.2.1 and 3.3.1). The main part of that section is to give two maps $\zeta_N^\triangledown \Rightarrow \triangledown$, $\zeta_N^\triangledown \Rightarrow \bullet$ (resp., $\zeta_N^\triangledown \Rightarrow \triangledown$, $\zeta_N^\triangledown \Rightarrow \bullet$) appearing in Theorem A (cf. (131)) and prove the bijectivities of these maps. Except for $N = \infty$ in the non-resp’d portion, the bijectivities are proved in §3.4 (cf. Theorem 3.4.3). A key idea to finish the proof of the remaining case (i.e., $N = \infty$) is rigidification of dormant (affine-)indigenous $D_X^{(N-1)}$-module by means of a (generalized) theta characteristic (cf. Definition 3.5.1). Once we fix such an additional data, each dormant indigenous $D_X^{(\infty)}$-module may be expressed as the projective limit of a compatible collection of dormant indigenous $D_X^{(N-1)}$-modules for various $N$. This implies the desired bijectivity (cf. Proposition 3.6.5). Thus, we complete the proof of assertion (i) in Theorem A. The final subsection is devoted to proving Theorem B i.e., the Chern class formulas for varieties admitting $F^N$-projective or $F^N$-affine structures (cf. Theorem 3.7.1).
In § 4, we generalize the notion of a Tango structure to higher levels and higher-dimensional varieties. The definition is given in Definitions 4.2.1 (for $N < \infty$) and 4.2.3 (for $N = \infty$). Then, we construct the map $h : \mathfrak{M} \to \mathfrak{M}$ (cf. (217)) in the diagram of Theorem A (ii). The bijectivity of this map is proved in Theorem 4.4.1. By this result, we complete the proof of Theorem A (ii).

After that, we define the notion of a dual affine connection on a smooth variety $X$, as a certain $D^{(N-1)}$-action on the sheaf $\Omega_X$ of 1-forms (cf. Definition 4.5.1). This kind of object will be easier to understand and handle than Tango structures, as well as other equivalent structures. But, as proved in Proposition 4.5.3, dual affine connections in fact correspond bijectively to Tango structures, or equivalently Frobenius-affine structures, under certain conditions.

The second half of the present paper begins from § 5. In that section, we explicitly describe the trivial $F^N$-projective structure (and the various equivalent structures) on a projective space and prove the uniqueness assertion (cf. Proposition 5.1.1), which is the first portion of Theorem C (i). Also, the case of affine spaces, i.e., the second portion of Theorem C (i), is studied in detail (cf. Proposition 5.2.4). After recalling the definition of stratified fundamental group, we prove Theorem 5.3.1 i.e., a part of Theorem C (ii). The remaining portion of Theorem C (ii), is given at the end of that section. Thus, the proof of Theorem C is completed there.

The first subsection of § 6 concerns the construction of pull-backs by an étale morphism of the various structures obtained so far. In particular, it is shown (cf. Proposition 6.1.3) that the existence of $F^N$-projective structures, as well as $F^N$-affine structures, is closed under replacing the underlying variety with its étale covering. Next, we deal with abelian varieties in detail. Assertion (i) of Theorem D (and assertion (ii) of Theorem E) is proved in Corollary 6.3.3 by applying an explicit description of the stratified fundamental group of an abelian variety. Also, we show that the set of Tango structures of level $\infty$ is in bijection with the set of dual affine connections of level $\infty$ (cf. Corollary 6.3.4). After that, we describe explicitly the respective sets of $F^N$-projective and $F^N$-affine structures in terms of (the mod $p^N$ reduction of) the $p$-adic Tate module of the dual abelian variety (cf. Theorem 6.4.1). These descriptions yield lower bounds of the numbers of them for $N < \infty$ (cf. Corollary 6.4.2). Finally, we construct quotients, via a Galois étale covering, of $F^N$-projective and $F^N$-affine structures that are equivariant with respect to the Galois group.

The main objects of § 7 are algebraic curves. The first half of that section is devoted to proving the existence assertion of $F^N$-projective structures on a smooth projective curve $X$ of genus $> 1$ for a positive integer $N$. We construct a bijection between the set $\mathfrak{M}_{X,N,\Theta}$ of dormant $D^{(N-1)}_X$-modules on $X$ (rigidified by a fixed theta characteristic $\Theta$ of $X$) and the set of $k$-rational points of a certain Quot-scheme (cf. Theorem 7.2.2), which induces the last assertion of Theorem E (iii). After proving some properties on this Quot-scheme, we conclude (cf. Corollary 7.3.4) that the set $\mathfrak{M}_{X,N,\Theta}$, including the case of $N = \infty$, is nonempty, i.e., prove the first assertion of Theorem E (iii). Then, we try to consider the finiteness of $F^N$-projective structures. The relevant conjecture is formulated in Conjecture 7.4.3 and we solve this conjecture for every genus 2 curve (cf. Theorem 7.4.2). Some properties on $F^N$-affine structure on a (not necessarily projective) smooth curve are discussed in the final subsection. In particular, we there prove the remaining portion of (iii) (cf. Proposition 7.5.2) in Theorem E. Thus, the proof of Theorem E is completed. (Assertion (i) was already obtained in Theorem C.)

In the first subsection of § 8, we construct $F^N$-affine structures on the product of varieties by using $F^N$-affine structures on the respective factors of the product. This implies (cf. Proposition 8.1.2) that the existence of $F^N$-affine structure is closed under forming products of the
underlying varieties. Then, we discuss the classification problem of surfaces admitting $F^N$-projective and $F^N$-affine structures according to the Kodaira-Enriques classification. To do this, the Chern class formula, i.e., Theorem 3 plays an essential role in our discussion. Indeed, this formula and Theorem 4 help us to know whether or not there exist an $F^\infty$-projective ($F^N$-affine) structures on each surface in many classes. But, the classes of hyperelliptic surfaces and properly elliptic surfaces require detailed discussions, which are made in §8.3 and §8.4 respectively. As a consequence, we conclude Theorem F (cf. Corollary 8.5.1).

0.5. Notation and Conventions.

Throughout the present paper, we fix a prime number $p$ and an algebraically closed field $k$ of characteristic $p$.

By a variety (over $k$), we mean a connected integral scheme of finite type over $k$. Moreover, by a curve (resp., a surface), we mean a variety of dimension 1 (resp., 2). Unless stated otherwise, we will always be working over $k$; for example, products of varieties will be taken over $k$, i.e., $X_1 \times X_2 := X_1 \times_k X_2$. Also, for each variety $X$, $\Omega_X$ (resp., $T_X$) denotes the sheaf of 1-forms (resp., the sheaf of vector fields) on $X$ relative to $k$; i.e., $\Omega_X := \Omega_{X/k}$ (resp., $T_X := T_{X/k} = \Omega_{X/k}^\vee$). Given a smooth variety $X$, we can construct the canonical bundle $\omega_X$ of $X$ (over $k$) by setting $\omega_X := \text{det}(\Omega_X) = \bigwedge^{\dim(X)} \Omega_X$.

For each positive integer $n$, denote by $\mathbb{P}^n$ (resp., $\mathbb{A}^n$) the $n$-dimensional projective (resp., affine) space over $k$. The $k$-rational points of $\mathbb{P}^n$ correspond bijectively to the ratios $[a_0 : a_1 : \cdots : a_n]$ (with $a_0, \cdots, a_n \in k$ and $(a_0, \cdots, a_n) \neq (0, \cdots, 0)$). In the present paper, we identify $\mathbb{A}^n$ with the open subscheme of $\mathbb{P}^n$ consisting of the elements $\{[1 : a_1 : \cdots : a_n] \mid a_1, \cdots, a_n \in k\}$.

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1. $F^N$-PROJECTIVE AND $F^N$-AFFINE STRUCTURES

In this section, some notions introduced in [30] and [31] are generalized to higher dimensions. We first consider Frobenius-projective and Frobenius-affine structures on smooth projective varieties (cf. Definitions 1.2.1), including those of infinite level (cf. Definition 1.3.1). Also, in terms of connections on principal bundles, we introduce the higher-dimensional versions of Frobenius-indigenous and Frobenius-affine-indigenous structures of (in)finite level (cf. Definitions 1.7.1 (ii) and 1.7.3). Bijective correspondences between these structures will be constructed in §3 (cf. Corollary 3.4.5).

Let us fix a positive integer $n$ and a smooth variety $X$ over $k$ of dimension $n$.

1.1. Frobenius twists and relative Frobenius morphisms.
W shall write \( f : X \to \text{Spec}(k) \) for the structure morphism of \( X/k \) and \( F_X \) for the absolute Frobenius endomorphism of \( X \). For each positive integer \( N \), the \( N \)-th Frobenius twist of \( X \) over \( k \) is, by definition, the base-change \( X^{(N)} (:= X \times_k F_k^N) \) of \( X \) via the \( N \)-th iterate \( F_k^N \) of \( F_k \) \((= F_{\text{Spec}(k)}) : \text{Spec}(k) \to \text{Spec}(k) \)\). Denote by \( f^{(N)} : X^{(N)} \to \text{Spec}(k) \) the structure morphism of \( X^{(N)} \). The \( N \)-th relative Frobenius morphism of \( X \) over \( k \) is the unique morphism \( F_{X/k}^{(N)} : X \to X^{(N)} \) over \( k \) that makes the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Spec}(k)} & \text{Spec}(k) \\
\downarrow{f} & & \downarrow{f^{(N)}} \\
X^{(N)} & \xrightarrow{\text{id} \times F_k^N} & X^{(N)}
\end{array}
\]

For convenience, we often use the notation \( X^{(0)} \) to denote \( X \) itself.

Given a connected smooth algebraic group \( G \) over \( k \), we denote by \( G_X \) the Zariski sheaf of groups on \( X \) represented by \( G \). Moreover, given a positive integer \( N \), we shall write

\[
G_X^{(N)} := (F_{X/k}^{(N)})^{-1}(G_X^{(N)}) \subseteq G_X.
\]

The various Frobenius morphisms \( \{ F_{X(k)}^{(N)} \}_{N \in \mathbb{Z}_{>0}} \) give rise to a sequence of inclusions

\[
G_X \supseteq G_X^{(1)} \supseteq G_X^{(2)} \supseteq \cdots \supseteq G_X^{(N)} \supseteq \cdots.
\]

1.2. \( F^N \)-projective and \( F^N \)-affine structures of level \( N < \infty \).

Denote by \( \text{GL}_{n+1} \) (resp., \( \text{PGL}_{n+1} \)) the general (resp., projective) linear group over \( k \) of rank \( n+1 \), and we shall use the notation \( (\cdot)^{-1} \) to denote the natural quotient \( \text{GL}_{n+1} \twoheadrightarrow \text{PGL}_{n+1} \).

The group \( \text{PGL}_{n+1} \) can be identified with the automorphism group of \( \mathbb{P}^n \) in such a way that

\[
\overline{A}([a_0 : \cdots : a_n]) := [a_0 : \cdots : a_n] \overline{A}
\]

for each \( A \in \text{GL}_{n+1} \) and \([a_0 : \cdots : a_n] \in \mathbb{P}^n \). Also, denote by

\[
\text{PGL}_{n+1}^{\mathbb{A}}
\]

the subgroup of \( \text{PGL}_{n+1} \) consisting of automorphisms \( h \) of \( \mathbb{P}^n \) with \( h(\mathbb{A}^n) = \mathbb{A}^n \). That is to say, each element of \( \text{PGL}_{n+1}^{\mathbb{A}} \) may be expressed as

\[
\begin{pmatrix}
a & 0 \\
ta & A
\end{pmatrix}
\]

for some \( a \in \mathbb{G}_m \), \( a \in \mathbb{A}^n \), and \( A \in \text{GL}_n \).

Let us write

\[
\mathcal{P}^\text{ét}_X \quad (\text{resp., } \mathcal{A}^\text{ét}_X)
\]

for the Zariski sheaf of sets on \( X \) that assigns, to each open subscheme \( U \) of \( X \), the set of étale \( k \)-morphisms from \( U \) to \( \mathbb{P}^n \) (resp., \( \mathbb{A}^n \)). We shall write \( \mathcal{P}^\text{ét}_X := \mathcal{P}^\text{ét}_X \quad (\text{resp., } \mathcal{A}^\text{ét}_X := \mathcal{A}^\text{ét}_X) \) if there is no fear of confusion. The graph of a local section \( \phi : U \to \mathbb{P}^n \) (resp., \( \phi : U \to \mathbb{A}^n \)) of this sheaf, which will be denoted by \( \Gamma_\phi : U \to U \times \mathbb{P}^n \) (resp., \( \Gamma_\phi : U \to U \times \mathbb{A}^n \)), defines a local
section of the trivial $\mathbb{P}^n$-bundle $X \times \mathbb{P}^n \rightarrow X$ (resp., the trivial $\mathbb{A}^n$-bundle $X \times \mathbb{A}^n \rightarrow X$) on $X$.

For each positive integer $N$, the sheaf $\mathcal{P}^\mathrm{et}$ has a $(\operatorname{PGL}_{n+1})_X^{(N)}$-action defined as follows. Let $U$ be an open subscheme of $X$, $\phi: U \rightarrow \mathbb{P}^n$ an element of $\mathcal{P}^\mathrm{et}(U)$, and $\frob\in H^0(U, \mathcal{O}^\otimes_{\mathbb{P}^n}(-2))$ an element of $(\operatorname{PGL}_{n+1})_X^{(N)}(U)$ ($= \operatorname{PGL}_{n+1}(U^{(N)}) \subseteq \operatorname{PGL}_{n+1}(U)$). Then, one verifies that the composite
\begin{equation}
\frob(\phi): U \rightarrow \mathbb{P}^n \rightarrow \mathbb{P}^n \rightarrow \mathbb{P}^n
\end{equation}
belongs to $\mathcal{P}^\mathrm{et}(U)$. The assignment $(\frob, \phi) \mapsto \frob(\phi)$ defines a $(\operatorname{PGL}_{n+1})_X^{(N)}$-action on $\mathcal{P}^\mathrm{et}$. These actions for various $N$ are compatible with the sequence of inclusions \([10]\) in the case where $G = \operatorname{PGL}_{n+1}$. In a similar manner, one can define a $(\operatorname{PGL}_{n+1})_X^{(N)}$-action on the sheaf $\mathcal{A}^\mathrm{et}$ that is compatible with the $(\operatorname{PGL}_{n+1})_X^{(N)}$-action on $\mathcal{P}^\mathrm{et}$ via the natural inclusions $\mathcal{A}^\mathrm{et} \rightarrow \mathcal{P}^\mathrm{et}$ and $(\operatorname{PGL}_{n+1})_X^{(N)} \rightarrow (\operatorname{PGL}_{n+1})_X^{(N)}$.

In what follows, we introduce the definitions of a Frobenius-projective structure and a Frobenius-affine structure (of finite level) on a smooth projective variety of any arbitrary dimension. If the underlying variety has one dimension, then they have been already defined by Y. Hoshi (cf. \[30\], §2, Definition 2.1 and \[31\], §2 Definition 2.1).

**Definition 1.2.1.** Let $\mathcal{S}^\triangledown$ be a subsheaf of $\mathcal{P}^\mathrm{et}$ (resp., $\mathcal{A}^\mathrm{et}$) and $N$ a positive integer. Suppose that $\mathcal{S}^\triangledown$ is closed under the $(\operatorname{PGL}_{n+1})_X^{(N)}$-action on $\mathcal{P}^\mathrm{et}$ (resp., the $(\operatorname{PGL}_{n+1})_X^{(N)}$-action on $\mathcal{A}^\mathrm{et}$) and forms a $(\operatorname{PGL}_{n+1})_X^{(N)}$-torsor (resp., a $(\operatorname{PGL}_{n+1})_X^{(N)}$-torsor) on $X$ with respect to the resulting $(\operatorname{PGL}_{n+1})_X^{(N)}$-action (resp., $(\operatorname{PGL}_{n+1})_X^{(N)}$-action) on $\mathcal{S}^\triangledown$. Then, we say that $\mathcal{S}^\triangledown$ is a Frobenius-projective (resp., Frobenius-affine) structure of level $N$ on $X$, or an $F^N$-projective (resp., $F^N$-affine) structure on $X$, for short.

Let $N$ be a positive integer. Denote by
\begin{equation}
F^N-\operatorname{proj}_X \quad \text{(resp., } F^N-\operatorname{Aff}_X)\end{equation}
the set of $F^N$-projective (resp., $F^N$-affine) structures on $X$. If $\mathcal{S}^\triangledown$ is an $F^N$-affine structure on $X$, then the smallest subsheaf of $\mathcal{P}^\mathrm{et}$ that contains $\mathcal{S}^\triangledown$ ($\subseteq \mathcal{A}^\mathrm{et} \subseteq \mathcal{P}^\mathrm{et}$) and is closed under the $(\operatorname{PGL}_{n+1})_X^{(N)}$-action forms an $F^N$-projective structure $\iota^\triangledown_N(\mathcal{S}^\triangledown)$ on $X$. The resulting assignment $\mathcal{S}^\triangledown \mapsto \iota^\triangledown_N(\mathcal{S}^\triangledown)$ gives a map of sets
\begin{equation}
\iota^\triangledown_N: F^N-\operatorname{Aff}_X \rightarrow F^N-\operatorname{proj}_X.
\end{equation}

1.3. $F^N$-projective and $F^N$-affine structures of level $N = \infty$.

Let $N'$ be another positive integer with $N \leq N'$ and $\mathcal{S}^\triangledown$ an $F^{N'}$-projective (resp., $F^{N'}$-affine) structure on $X$. Denote by
\begin{equation}
\mathcal{S}^\triangledown|^{(N)}
\end{equation}
the smallest subsheaf of $\mathcal{P}^\mathrm{et}$ (resp., $\mathcal{A}^\mathrm{et}$) which contains $\mathcal{S}^\triangledown$ and is closed under the $(\operatorname{PGL}_{n+1})_X^{(N)}$-action (resp., the $(\operatorname{PGL}_{n+1})_X^{(N)}$-action). One verifies that $\mathcal{S}^\triangledown|^{(N)}$ forms an $F^N$-projective (resp., $F^N$-affine) structure on $X$. We refer to $\mathcal{S}^\triangledown|^{(N)}$ as the $N$-th truncation of $\mathcal{S}^\triangledown$. The resulting
assignments $S^\diamond \mapsto S^\diamond|^{(N)}$ for various positive integers $N$, $N'$ with $N' > N$ give a projective system of sets

\[(17) \quad \cdots \rightarrow F^N\text{-Proj}_X \rightarrow \cdots \rightarrow F^3\text{-Proj}_X \rightarrow F^2\text{-Proj}_X \rightarrow F^1\text{-Proj}_X \]

(resp., $\cdots \rightarrow F^N\text{-Aff}_X \rightarrow \cdots \rightarrow F^3\text{-Aff}_X \rightarrow F^2\text{-Aff}_X \rightarrow F^1\text{-Aff}_X$).

**Definition 1.3.1.**

A Frobenius-projective (resp., Frobenius-affine) structure of level $\infty$ on $X$, or simply an $F^\infty$-projective (resp., $F^\infty$-affine) structure on $X$, is a collection

\[(18) \quad S^\diamond_{\infty} := \{S^\diamond_N\}_{N \in \mathbb{Z}_{>0}},\]

where each $S^\diamond_N$ ($N \in \mathbb{Z}_{>0}$) is an $F^N$-projective (resp., $F^N$-affine) structure with $S^\diamond_{N+1}|^{(N)} = S^\diamond_N$.

Denote by

\[(19) \quad F^\infty\text{-Proj}_X \quad (\text{resp., } F^\infty\text{-Aff}_X)\]

the set of $F^\infty$-projective (resp., $F^\infty$-affine) structures on $X$. Then, $F^\infty\text{-Proj}_X$ (resp., $F^\infty\text{-Aff}_X$) may be naturally identified with the limit of the projective system (17), i.e.,

\[(20) \quad \lim_{N \in \mathbb{Z}_{>0}} F^N\text{-Proj}_X = F^\infty\text{-Proj}_X \quad \text{(resp., } \lim_{N \in \mathbb{Z}_{>0}} F^N\text{-Aff}_X = F^\infty\text{-Aff}_X)\].

The collection of maps $\{i^\diamond_N\}_{N \in \mathbb{Z}_{>0}}$ (cf. (15)) is compatible with the projective systems of sets $\{F^N\text{-Proj}_X\}_{N \in \mathbb{Z}_{>0}}$ and $\{F^N\text{-Aff}_X\}_{N \in \mathbb{Z}_{>0}}$. Hence, this collection induces a map of sets

\[(21) \quad i^\diamond_{\infty} : F^\infty\text{-Aff}_X \rightarrow F^\infty\text{-Proj}_X.\]

**1.4. Purity theorem.**

In this subsection, we observe one property of Frobenius-projective and Frobenius-affine structures, as described in Theorem 1.4.1 below. (This result will not be used in the rest of the present paper.) Let $U$ be an open subscheme of $X$ and let $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$. If $S^\diamond$ is an $F^N$-projective (resp., $F^N$-affine) structure on $X$, then its restriction $S^\diamond|_U$ to $U$ forms an $F^N$-projective (resp., $F^N$-affine) structure on $U$. Hence, the assignment $S^\diamond \mapsto S^\diamond|_U$ gives a map of sets

\[(22) \quad F^N\text{-Proj}_X \rightarrow F^N\text{-Proj}_U \quad (\text{resp., } F^N\text{-Aff}_X \rightarrow F^N\text{-Aff}_U).\]

(More generally, if $U$ is a variety together with an etale morphism $U \rightarrow X$, then we can construct the pull-backs of $F^N$-projective and $F^N$-affine structures on $X$. See §6.1 for the detailed discussion.) Regarding this map, the following purity theorem holds.

**Theorem 1.4.1.**

Let $X$ be a smooth variety over $k$ and $U$ an open subscheme of $X$. Assume that the codimension of $X \setminus U$ in $X$ is $\geq 2$. Then, for $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, each $F^N$-projective (resp., $F^N$-affine) structure on $U$ extends uniquely to an $F^N$-projective (resp., $F^N$-affine) structure on $X$. That is to say, under the above assumption, the map (22) becomes bijective.
Proof. Since the proof of the resp’d assertion is entirely similar, we only prove the non-resp’d assertion. Moreover, by (20), it suffices to consider the case where $N < \infty$.

Let $\mathcal{S}^\oplus$ be an $F^N$-projective structure on $U$. Also, let us take an open subscheme $V$ of $X$ and an étale $k$-morphism $\phi : U \cap V \to \mathbb{P}^n$, which is a section of $\mathcal{S}^\oplus$ over $U \cap V$. Notice that any $k$-morphism $W \to \mathbb{P}^n$ (for a variety $W$) is locally determined by a unique $n$-tuple of sections in $\mathcal{O}_W$. Hence, since the codimension of $V \setminus U \cap V$ in $V$ is $\geq 2$, $\phi$ extends uniquely to a morphism $\phi_V : V \to \mathbb{P}^n$ (cf. [25], Exposé X, §3, Lemme 3.5). It follows from the Zariski-Nagata purity theorem (cf. [25], Exposé X, §3, Théorème 3.4) that $\phi_V$ is étale. The various sections $\phi_V$ of $\mathcal{P}_X^\oplus$ constructed from $V$ and $\phi$ in this manner defines a unique subsheaf $\mathcal{S}_X^\oplus$ ($\subseteq \mathcal{P}_X^\oplus$) with $\mathcal{S}_X^\oplus|_U = \mathcal{S}^\oplus$.

To complete the proof, we shall prove the claim that this subsheaf forms a $(\text{PGL}_{n+1})_X^{(N)}$-torsor. Let $V$ be as above and $\phi_V : V \to \mathbb{P}^n$ an étale $k$-morphism, i.e., a section of $\mathcal{S}^\oplus$ over $V$. Denote by $\phi := \phi_V|_{U \cap V}$ the restriction of $\phi_V$ to $U \cap V$.

First, let us take an element $\mathcal{A}_V$ of $(\text{PGL}_{n+1})_X^{(N)}(V)$. Write $\mathcal{A} := \mathcal{A}_V|_{U \cap V}$. Since $\mathcal{A}_V(\phi_V)|_{U \cap V} = \mathcal{A}(\phi)$ (cf. (13)), which lies in $\mathcal{S}^\oplus$, the uniqueness of the extension $\phi \leadsto \phi_V$ implies that $\mathcal{A}_V(\phi_V) \in \mathcal{S}_X^\oplus(V)$. Hence, $\mathcal{S}_X^\oplus$ is closed under the $(\text{PGL}_{n+1})_X^{(N)}$-action.

Next, let $\phi_V' : V \to \mathbb{P}^n$ be another étale $k$-morphism and write $\phi' := \phi_V|_{U \cap V}$. Since $\mathcal{S}^\oplus$ ($= \mathcal{S}_X^\oplus|_V$) is a $(\text{PGL}_{n+1})_U^{(N)}$-torsor, there exists uniquely an element $\mathcal{A} \in (\text{PGL}_{n+1})_U^{(N)}(U \cap V)$ ($= \text{PGL}_{n+1}((U \cap V)^{(N)})$) with $\mathcal{A}(\phi') = \phi$. The natural quotient $\text{GL}_{n+1} \to \text{PGL}_{n+1}$ forms a principal $\mathbb{G}_m$-bundle, so the fiber product $E := \text{GL}_{n+1} \times_{\text{PGL}_{n+1}, \mathcal{A}} (U \cap V)^{(N)}$ specifies a principal $\mathbb{G}_m$-bundle (or equivalently, a line bundle) on $(U \cap V)^{(N)}$. Denote by $A$ the $\mathcal{E}$-rational point of $\text{GL}_{n+1}$ determined by the second projection $\mathcal{E} \to \text{GL}_{n+1}$. By [25], Exposé X, §3, Lemme 3.5, there exists uniquely (up to isomorphism) a principal $\mathbb{G}_m$-bundle $\mathcal{E}_V$ on $V^{(N)}$ extending $\mathcal{E}$. The codimension of $\mathcal{E}_V \setminus \mathcal{E}$ in $\mathcal{E}_V$ is $\geq 2$, so $A$ extends uniquely to an element $A_V \in \text{GL}_{n+1}(\mathcal{E}_V)$. Since $A_V : \mathcal{E}_V \to \text{GL}_{n+1}$ is compatible with the respective $\mathbb{G}_m$-actions of $\mathcal{E}_V$ and $\text{GL}_{n+1}$, it induces a well-defined element $\mathcal{A}_V \in \text{PGL}_{n+1}(V^{(N)})$ ($= (\text{PGL}_{n+1})_X^{(N)}(V)$). Because of the uniqueness of the extension $\phi \leadsto \phi_V$, $\mathcal{A}_V$ turns out to be a unique element satisfying the equality $\mathcal{A}_V(\phi_V) = \phi_V$. Thus, we complete the proof of the claim, and consequently, completes the proof of the theorem. \hfill \square

1.5. Kodaira-Spencer map associated to a flat principal bundle.

Let $G$ be a connected smooth affine algebraic group over $k$ with Lie algebra $\mathfrak{g}$, and $\pi : \mathcal{E} \to X$ a principal $G$-bundle on $X$ in the étale topology. Denote by $\text{Ad}_\mathcal{E}$ the adjoint vector bundle associated with $\mathcal{E}$, i.e., the vector bundle on $X$ corresponding to the principal $\text{GL}(\mathfrak{g})$-bundle obtained from $\mathcal{E}$ via change of structure group by the adjoint representation $G \to \text{GL}(\mathfrak{g})$. The differential of $\pi$ gives rise to the following short exact sequence:

$$0 \to \text{Ad}_\mathcal{E} \to \tilde{T}_\mathcal{E} \xrightarrow{d_\mathcal{E}} T_X \to 0,$$

where $\tilde{T}_\mathcal{E}$ denotes the subsheaf $\pi_*((T\mathcal{E})^G)$ of $G$-invariant sections of $\pi_*T\mathcal{E}$ (cf. [76], §1.2, (31)). Both $T_X$ and $\tilde{T}_\mathcal{E}$ have structures of Lie algebra $[-,-]$ and the surjection $d_\mathcal{E}$ is compatible with these structures. By a connection on $\mathcal{E}$, we mean an $\mathcal{O}_X$-linear morphism $\nabla_\mathcal{E} : T_X \to \tilde{T}_\mathcal{E}$ such that $d_\mathcal{E} \circ \nabla_\mathcal{E} = \text{id}_{T_X}$. 

\[ \text{(23)} \]
Let $\nabla_\mathcal{E}$ be a connection on $\mathcal{E}$. Recall that the curvature of $(\mathcal{E}, \nabla_\mathcal{E})$ is, by definition, the $O_X$-linear morphism

$$\psi(\mathcal{E}, \nabla_\mathcal{E}) : \bigwedge^2 \mathcal{T}_X \rightarrow \tilde{\mathcal{T}}_\mathcal{E}$$

given by $\partial_1 \wedge \partial_2 \mapsto [\nabla_\mathcal{E}(\partial_1), \nabla_\mathcal{E}(\partial_2)] - \nabla_\mathcal{E}([\partial_1, \partial_2])$ for any local sections $\partial_1, \partial_2 \in \mathcal{T}_X$. We say that $\nabla_\mathcal{E}$ is flat (or integrable) if $\psi(\mathcal{E}, \nabla_\mathcal{E}) = 0$. By a flat $G$-bundle on $X$, we mean a pair $(\mathcal{E}, \nabla_\mathcal{E})$ consisting of a principal $G$-bundle $\mathcal{E}$ on $X$ (in the étale topology) and a flat connection $\nabla_\mathcal{E}$ on $\mathcal{E}$.

The trivial $G$-bundle on $X$ is, by definition, the product $X \times G$ equipped with the natural $G$-action, where the first projection $\text{pr}_1 : X \times G \rightarrow X$ is regarded as the structure morphism of this principal $G$-bundle. Since $\tilde{T}_{X \times G} \cong \mathcal{T}_X \oplus O_X \otimes g$, the first inclusion $\mathcal{T}_X \hookrightarrow \mathcal{T}_X \oplus O_X \otimes g$ determines a connection

$$\nabla^{\text{triv}}_{X \times G} : \mathcal{T}_X \rightarrow \tilde{T}_{X \times G}$$
on $X \times G$. We shall refer to $\nabla^{\text{triv}}_{X \times G}$ as the trivial connection on $X \times G$.

Recall from [76], §3.3, (217), that, for each principal $G$-bundle $\mathcal{G}$ on $X^{(1)}$, there exists a canonical flat connection $\nabla^{\text{can}}_{F^{(1)}_{X/k}(\mathcal{G})}$ on $F^{(1)}_{X/k}(\mathcal{G})$, i.e., the pull-back of $\mathcal{G}$ via $F^{(1)}_{X/k} : X \rightarrow X^{(1)}$. More generally, if $\mathcal{G}$ is a principal $G$-bundle on $X^{(N)}$ for a positive integer $N$, then we shall define a canonical connection

$$\nabla^{\text{can}}_{F^{(N)}_{X/k}(\mathcal{G})}$$
on the pull-back $F^{(N)}_{X/k}(\mathcal{G})$ to be the connection $\nabla^{\text{can}}_{F^{(1)}_{X/k}(F^{(N-1)}_{X/k}(\mathcal{G}))}$ under the natural identification $F^{(1)}_{X/k}(F^{(N-1)}_{X/k}(\mathcal{G})) \cong F^{(N)}_{X/k}(\mathcal{G})$. In the case where $\mathcal{G} = X^{(N)} \times G$, i.e., the trivial $G$-bundle, the equality $\nabla^{\text{can}}_{F^{(N)}_{X/k}(X^{(N)} \times G)} = \nabla^{\text{triv}}_{X \times G}$ holds.

Let $H$ be an algebraic subgroup of $G$. Given a principal $H$-bundle $\mathcal{E}_H$ on $X$, we write $\mathcal{E}_H \times^H G$ for the principal $G$-bundle on $X$ obtained from $\mathcal{E}_H$ by change of structure group via the natural inclusion $H \hookrightarrow G$. The natural inclusion $\mathcal{E}_H \hookrightarrow \mathcal{E}_H \times^H G$ induces an $O_X$-linear injection $\tilde{T}_{\mathcal{E}_H} \hookrightarrow \tilde{T}_{\mathcal{E}_H \times^H G}$. We shall regard $\tilde{T}_{\mathcal{E}_H}$ as an $O_X$-submodule of $\tilde{T}_{\mathcal{E}_H \times^H G}$ via this injection. If $\mathcal{E}_H$ has a connection $\nabla_{\mathcal{E}_H} : \mathcal{T}_X \rightarrow \tilde{T}_{\mathcal{E}_H}$, then the composite $\mathcal{T}_X \xrightarrow{\nabla_{\mathcal{E}_H}} \tilde{T}_{\mathcal{E}_H} \hookrightarrow \tilde{T}_{\mathcal{E}_H \times^H G}$ specifies a connection on $\mathcal{E}_H \times^H G$. In this way, changing structure group by $H \hookrightarrow G$ gives an assignment from flat $H$-bundles to flat $G$-bundles.

Recall that an $H$-reduction on a principal $G$-bundle $\mathcal{E}$ is a principal $H$-bundle $\mathcal{E}_H$ together with an isomorphism $\mathcal{E}_H \times^H G \cong \mathcal{E}$ of principal $G$-bundles.

**Definition 1.5.1.**

Let $(\mathcal{E}, \nabla_\mathcal{E}, \mathcal{E}_H)$ be a triple consisting of a principal $G$-bundle on $X$, a connection $\nabla_\mathcal{E}$ on $\mathcal{E}$, and an $H$-reduction on $\mathcal{E}$. The Kodaira-Spencer map associated to $(\mathcal{E}, \nabla_\mathcal{E}, \mathcal{E}_H)$ is the $O_X$-linear composite

$$\text{KS}(\mathcal{E}, \nabla_\mathcal{E}, \mathcal{E}_H) : \mathcal{T}_X \xrightarrow{\nabla_\mathcal{E}} \tilde{T}_\mathcal{E} \twoheadrightarrow \tilde{T}_\mathcal{E} / \tilde{T}_{\mathcal{E}_H},$$

where the second arrow denotes the natural quotient.
If $\mathcal{E}_H$ is an $H$-reduction on the trivial $G$-bundle $X \times G$, then we shall write

$$\text{KS}_{\mathcal{E}_H} : \mathcal{T}_X \to \tilde{\mathcal{T}}_{X \times G}/\tilde{\mathcal{T}}_H$$

for the Kodaira-Spencer map associated to the triple $(X \times G, \nabla^{\text{triv}}_{X \times G}, \mathcal{E}_H)$ and refer to it as the **Kodaira-Spencer map associated to** $\mathcal{E}_H$.

### Remark 1.5.2.

Let $(\mathcal{E}, \nabla_{\mathcal{E}}, \mathcal{E}_H)$ be as in the above definition. By differentiating the inclusion $\mathcal{E}_H \to \mathcal{E}$, we obtain the following morphism of short exact sequences:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ad}_{\mathcal{E}_H} & \longrightarrow & \tilde{\mathcal{T}}_{\mathcal{E}_H} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0 \\
& & \downarrow \text{incl.} & & \downarrow \text{incl.} & & \downarrow \text{id}_{\mathcal{T}_X} & & \\
0 & \longrightarrow & \text{Ad}_{\mathcal{E}} & \longrightarrow & \tilde{\mathcal{T}}_{\mathcal{E}} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0.
\end{array}
$$

This morphism induces an isomorphism

$$\text{Ad}_{\mathcal{E}}/\text{Ad}_{\mathcal{E}_H} \cong \tilde{\mathcal{T}}_{\mathcal{E}}/\tilde{\mathcal{T}}_{\mathcal{E}_H}.$$  

Thus, by passing to this isomorphism, we can think of the Kodaira-Spencer map $\text{KS}_{(\mathcal{E}, \nabla_{\mathcal{E}}, \mathcal{E}_H)}$ as an $\mathcal{O}_X$-linear morphism $\mathcal{T}_X \to \text{Ad}_{\mathcal{E}}/\text{Ad}_{\mathcal{E}_H}$.

### 1.6. Universal Kodaira-Spencer map.

Let $[\infty]$ be the point $[1 : 0 : 0 : \cdots : 0]$ in $\mathbb{P}^n$, and write

$$\text{PGL}_{n+1}^\infty$$

for the subgroup of $\text{PGL}_{n+1}$ consisting of elements fixing $[\infty]$. For a principal $\text{PGL}_{n+1}$-bundle $\mathcal{E}$ on $X$, one verifies that giving a $\text{PGL}_{n+1}^\infty$-reduction of $\mathcal{E}$ is equivalent to giving a global section $X \to \mathbb{P}_X^n$ of the $\mathbb{P}^n$-bundle $\mathbb{P}_X^n$ corresponding to $\mathcal{E}$.

We shall write

$$\text{PGL}_{n+1}^\infty,\infty := \text{PGL}_{n+1}^\infty \cap \text{PGL}_{n+1}^\text{A} \subseteq \text{PGL}_{n+1}.$$  

The assignment given by $\overline{A} \mapsto -\overline{A}([\infty])$ for any $\overline{A} \in \text{PGL}_{n+1}$ defines an isomorphism

$$\text{PGL}_{n+1}/\text{PGL}_{n+1}^\infty \cong \mathbb{P}^n$$

(resp., $\text{PGL}_{n+1}^\text{A}/\text{PGL}_{n+1}^\text{A,\infty} \cong \mathbb{A}^n$).

By the composite surjection

$$\pi : \text{PGL}_{n+1} \mapsto \text{PGL}_{n+1}/\text{PGL}_{n+1}^\infty \mapsto \mathbb{P}^n$$

(resp., $\pi^\text{A} : \text{PGL}_{n+1}^\text{A} \mapsto \text{PGL}_{n+1}^\text{A}/\text{PGL}_{n+1}^\text{A,\infty} \mapsto \mathbb{A}^n$),

$\text{PGL}_{n+1}$ (resp., $\text{PGL}_{n+1}^\text{A}$) may be thought of as a principal $\text{PGL}_{n+1}^\infty$-bundle on $\mathbb{P}^n$ (resp., a principal $\text{PGL}_{n+1}^\text{A,\infty}$-bundle on $\mathbb{A}^n$); for convenience, we denote this principal bundle by

$$\mathcal{E}_{\text{univ}}^\text{red}$$

(resp., $\mathcal{E}_{\text{univ}}^\text{red}^\text{A}$).
Moreover, it forms a $PGL_{n+1}^\infty$-reduction (resp., a $PGL_{n+1}^{\hat{A},\infty}$-reduction) on the trivial $PGL_{n+1}$-bundle $\mathbb{P}^n \times PGL_{n+1}$ (resp., $\mathbb{A}^n \times PGL_{n+1}^\hat{A}$) via the morphism

\[(\pi, \text{id}_{PGL_{n+1}}) : \mathcal{E}_{\text{red}}^{\text{univ}} \to \mathbb{P}^n \times PGL_{n+1} \quad \text{(resp.,} \quad (\pi^\hat{A}, \text{id}_{PGL_{n+1}^\hat{A}}) : \mathcal{E}_{\text{red}}^{\text{univ}} \to \mathbb{A}^n \times PGL_{n+1}^\hat{A}) \]  

This reduction has a universal property described as follows: if we are given a $PGL_{n+1}^\infty$-reduction (resp., a $PGL_{n+1}^{\hat{A},\infty}$-reduction) $\mathcal{E}_{\text{red}}$ on the trivial bundle $U \times PGL_{n+1}$ (resp., $U \times PGL_{n+1}^\hat{A}$) over a $k$-scheme $U$, there exists a unique $k$-morphism $f : U \to \mathbb{P}^n$ (resp., $f : U \to \mathbb{A}^n$) such that the pull-back $\mathcal{E}_{\text{red}} \times_{\mathbb{P}^n} fU$ (resp., $\mathcal{E}_{\text{red}}^{\text{univ}} \times_{\mathbb{A}^n} fU$) is isomorphic to $\mathcal{E}_{\text{red}}$. The morphism $f$ may be constructed as the quotient by the $PGL_{n+1}^\infty$-actions of the equivariant composite $\mathcal{E}_{\text{red}} \to U \times PGL_{n+1} \to PGL_{n+1}$. By construction, $\mathcal{E}_{\text{red}}^{\text{univ}}$ corresponds to a global section of the trivial $\mathbb{P}^n$-bundle $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ defined as the diagonal embedding $\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$.

**Proposition 1.6.1.**

The Kodaira-Spencer map

\[\text{KS}_{\mathcal{E}_{\text{red}}}^{\text{univ}} : T_{\mathbb{P}^n} \to \tilde{T}_{\mathbb{P}^n \times PGL_{n+1}} / \tilde{T}_{\mathcal{E}_{\text{red}}}^{\text{univ}} \quad \text{(resp.,} \quad \text{KS}_{\mathcal{E}_{\text{red}}}^{\text{univ}} : T_{\mathbb{A}^n} \to \tilde{T}_{\mathbb{A}^n \times PGL_{n+1}^\hat{A}} / \tilde{T}_{\mathcal{E}_{\text{red}}}^{\text{univ}})\]

is associated to $\mathcal{E}_{\text{red}}^{\text{univ}}$ (resp., $\mathcal{E}_{\text{red}}^{\text{univ}}$) is an isomorphism.

**Proof.** We only consider the non-resp’d assertion because the proof of the resp’d assertion is entirely similar.

To prove the non-resp’d assertion, we shall construct the inverse to $\text{KS}_{\mathcal{E}_{\text{red}}}^{\text{univ}}$. Let us take an affine open subscheme $U$ of $\mathbb{P}^n$ and a $PGL_{n+1}^\infty$-reduction $\mathcal{E}_{\text{red}}$ on the trivial bundle $U \times PGL_{n+1}$. Write $U_\varepsilon := U \times \text{Spec}(k[[\varepsilon]])$. It follows from well-known generalities on deformation theory that the set $\Gamma(U, A_{dU \times PGL_{n+1}})$ is in bijection with the set of automorphisms of the $PGL_{n+1}$-bundle $U_\varepsilon \times PGL_{n+1}$ inducing the identity morphism of $U \times PGL_{n+1}$ via reduction modulo $\varepsilon$. For each such automorphism $\gamma$, the pull-back via $\gamma$ of the trivial deformation $(\mathcal{E}_{\text{red}})_\varepsilon$ of $\mathcal{E}_{\text{red}}$ determines a $PGL_{n+1}^\infty$-reduction $\mathcal{E}_{\text{red}}^\gamma$ of $U_\varepsilon \times PGL_{n+1}$. One verifies that any $PGL_{n+1}^\infty$-reduction of $U_\varepsilon \times PGL_{n+1}$ inducing $\mathcal{E}_{\text{red}}$ via reduction is obtained in this way. Moreover, $\mathcal{E}_{\text{red}}^\gamma$ is the trivial deformation if and only if $\gamma$ lies in $\Gamma(U, A_{d\mathcal{E}_{\text{red}}}^{\text{univ}})$. Hence, if $\text{Def}_{\mathcal{E}_{\text{red}}}^{\text{univ}}$ denotes the set of deformations of the $PGL_{n+1}^\infty$-reduction $\mathcal{E}_{\text{red}}$ over $U$, then the assignment $\gamma \mapsto \mathcal{E}_{\text{red}}^\gamma$ determines a bijection

\[\Gamma(U, A_{dU \times PGL_{n+1}}) / \Gamma(U, A_{d\mathcal{E}_{\text{red}}}^{\text{univ}}) \cong \text{Def}_{\mathcal{E}_{\text{red}}}^{\text{univ}}.\]

Since $U$ is affine, the domain of this bijection may be identified with $\Gamma(U, A_{dU \times PGL_{n+1}} / A_{d\mathcal{E}_{\text{red}}}^{\text{univ}})$. On the other hand, the universal property of $\mathcal{E}_{\text{red}}^{\text{univ}}$ described before this proposition yields a canonical bijection $\text{Def}_{\mathcal{E}_{\text{red}}}^{\text{univ}} \to \Gamma(U, T_{\mathbb{P}^n})$. Hence, (33) may be thought of as a bijection

\[\Gamma(U, A_{dU \times PGL_{n+1}} / A_{d\mathcal{E}_{\text{red}}}^{\text{univ}}) \cong \Gamma(U, T_{\mathbb{P}^n}).\]

By considering various open subschemes $U$ of $\mathbb{P}^n$, we obtain, from (39), an $O_{\mathbb{P}^n}$-linear isomorphism

\[A_{dU \times PGL_{n+1}} / A_{d\mathcal{E}_{\text{red}}}^{\text{univ}} \cong T_{\mathbb{P}^n}.\]
It follows from the various constructions involved that this isomorphism gives the inverse to the composite

\[
\mathcal{E} \xrightarrow{KS_{E}^{\text{univ}}_{\text{red}}} \tilde{T}^{n} \times \text{PGL}_{n+1}/\tilde{T}^{n}_{\text{red}} \xrightarrow{\sim} \text{Ad}_{\mathcal{E}^{\text{univ}}_{\text{red}}}
\]

(cf. Remark 1.5.2 for the second arrow). Hence, $KS_{E}^{\text{univ}}_{\text{red}}$ turns out to be an isomorphism. This completes the proof of the proposition.

The above proposition implies the following corollary.

**Corollary 1.6.2.**

Let $X$ be a smooth variety over $k$ and $\phi : X \rightarrow \mathbb{P}^{n}$ (resp., $\phi : X \rightarrow \mathbb{A}^{n}$) a $k$-morphism. Denote by $E_{\phi}^{\text{red}}$ the $\text{PGL}_{n+1}^{\infty}$ reduction (resp., the $\text{PGL}_{\mathbb{A}^{n}+1}^{\infty}$ reduction) of the trivial bundle $X \times \text{PGL}_{n+1}$ (resp., $X \times \text{PGL}_{\mathbb{A}^{n}+1}$) associated to the graph $\Gamma_{\phi} : X \rightarrow X \times \mathbb{P}^{n}$ (resp., $\Gamma_{\phi} : X \rightarrow X \times \mathbb{A}^{n}$) of $\phi$. Then, the following square diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{T}^{\text{X}} & \xrightarrow{\phi^{\ast}} & \phi^{\ast}(\mathcal{T}^{\text{P}^{n}}) \\
\downarrow & \downarrow & \downarrow \\
\tilde{T}^{n} \times \text{PGL}_{n+1}/\tilde{T}^{n}_{\text{red}} & \xrightarrow{\sim} & \phi^{\ast}(\tilde{T}^{n} \times \text{PGL}_{n+1}/\tilde{T}^{n}_{\text{red}})
\end{array}
\]

where the lower horizontal arrow denotes the morphism obtained by differentiating the equivariant morphisms $X \times \text{PGL}_{n+1} \rightarrow \mathbb{P}^{n} \times \text{PGL}_{n+1}$ and $E_{\phi}^{\text{red}} \rightarrow E_{\phi}^{\text{univ}}$ (resp., $X \times \text{PGL}_{n+1}^{\mathbb{A}^{n}} \rightarrow \mathbb{A}^{n} \times \text{PGL}_{n+1}^{\mathbb{A}^{n}}$ and $E_{\phi}^{\text{red}} \rightarrow E_{\phi}^{\text{univ}}$) induced from $\phi$. In particular, $\phi$ is étale if and only if the Kodaira-Spencer map $KS_{E_{\phi}^{\text{red}}}$ associated to $E_{\phi}^{\text{red}}$ is an isomorphism.

### 1.7. $F^{N}$-(affine-)indigenous structures.

In the following definition, we shall introduce higher-dimensional generalizations of indigenous bundles (cf., e.g., [75], Definition 2.1) and Frobenius-(affine-)indigenous structures (cf. [30], Definition 3.4, [31], Definition 3.3). Let $X$ be a smooth variety over $k$, as before.

**Definition 1.7.1.**

(i) An **indigenous** (resp., **affine-indigenous**) bundle on $X$ is a triple

\[
(\mathcal{E}, \nabla_{\mathcal{E}}, \mathcal{E}_{\text{red}}),
\]

where the lower horizontal arrow denotes the morphism obtained by differentiating the equivariant morphisms $X \times \text{PGL}_{n+1} \rightarrow \mathbb{P}^{n} \times \text{PGL}_{n+1}$ and $E_{\phi}^{\text{red}} \rightarrow E_{\phi}^{\text{univ}}$ (resp., $X \times \text{PGL}_{n+1}^{\mathbb{A}^{n}} \rightarrow \mathbb{A}^{n} \times \text{PGL}_{n+1}^{\mathbb{A}^{n}}$ and $E_{\phi}^{\text{red}} \rightarrow E_{\phi}^{\text{univ}}$) induced from $\phi$. In particular, $\phi$ is étale if and only if the Kodaira-Spencer map $KS_{E_{\phi}^{\text{red}}}$ associated to $E_{\phi}^{\text{red}}$ is an isomorphism.
where \((\mathcal{E}, \nabla_\mathcal{E})\) is a flat PGL\(_{n+1}\)-bundle (resp., a flat PGL\(^\mathcal{A}_{n+1}\)-bundle) on \(X\) in the étale topology and \(\mathcal{E}_{\text{red}}\) is a PGL\(_{n+1}^\infty\)-reduction (resp., a PGL\(^\mathcal{A}_{n+1}\)-reduction) of \(\mathcal{E}\), such that the Kodaira-Spencer map \(\text{KS}(\mathcal{E}, \nabla_\mathcal{E}, \mathcal{E}_{\text{red}})\) associated to \((\mathcal{E}, \nabla_\mathcal{E}, \mathcal{E}_{\text{red}})\) is an isomorphism.

(ii) Let \(N\) be a positive integer. An \(F^N\)-indigenous (resp., \(F^N\)-affine-indigenous) structure on \(X\) is a pair

\[
\mathcal{E}^{\bullet} := (\mathcal{E}^\nabla, \mathcal{E}_{\text{red}})
\]

consisting of a principal PGL\(_{n+1}\)-bundle (resp., a principal PGL\(^\mathcal{A}_{n+1}\)-bundle) \(\mathcal{E}^\nabla\) on \(X^{(N)}\) and a PGL\(_{n+1}^\infty\)-reduction (resp., a PGL\(^\mathcal{A}_{n+1}\)-reduction) \(\mathcal{E}_{\text{red}}\) of the pull-back \(F_{X/k}^{(N)}(\mathcal{E}^\nabla)\) such that the triple

\[
(F_{X/k}^{(N)}(\mathcal{E}^\nabla), \nabla^\text{can}_{F_{X/k}^{(N)}(\mathcal{E}^\nabla)}, \mathcal{E}_{\text{red}})
\]

(cf. \([20]\) for the definition of \(\nabla^\text{can}_{F_{X/k}^{(N)}(\mathcal{E}^\nabla)}\)) forms an indigenous (resp., affine-indigenous) bundle on \(X\).

**Remark 1.7.2.**

In the case where \(X\) is a curve, the notion of an \(F^1\)-indigenous (resp., \(F^1\)-affine-indigenous) structure on \(X\) is equivalent to the notion of a dormant PGL\(_2\)-oper (resp., a dormant generic Miura PGL\(_2\)-oper) on \(X\). See, e.g., \([76], [81]\) for the precise definitions and fundamental studies of dormant (generic Miura) PGL\(_2\)-opers.

One can define, in the evidence manner, the notion of an isomorphism between indigenous (resp., affine-indigenous) bundles, as well as the notion of an isomorphism between \(F^N\)-indigenous (resp., \(F^N\)-affine-indigenous) structures. For example, an isomorphism of indigenous bundles from \((\mathcal{E}, \nabla_\mathcal{E}, \mathcal{E}_{\text{red}})\) to \((\mathcal{E}', \nabla'_{\mathcal{E}'}, \mathcal{E}'_{\text{red}})\) is defined as an isomorphism \(\mathcal{E} \overset{\sim}{\rightarrow} \mathcal{E}'\) of principal PGL\(_{n+1}\)-bundles that is compatible with the respective connections \(\nabla_\mathcal{E}, \nabla'_{\mathcal{E}'}\) and with the respective PGL\(_{n+1}^\infty\)-reduction structures \(\mathcal{E}_{\text{red}}, \mathcal{E}'_{\text{red}}\).

For each positive integer \(N\), we shall denote by

\[
F^N\mathfrak{Ind}_X \ (\text{resp., } F^N\mathfrak{AInd}_X)
\]

the set of isomorphism classes of \(F^N\)-indigenous (resp., \(F^N\)-affine-indigenous) structures on \(X\).

If \(\mathcal{E}^{\bullet} := (\mathcal{E}^\nabla, \mathcal{E}_{\text{red}})\) is an \(F^N\)-affine-indigenous structure on \(X\), then the pair

\[
\iota_N(\mathcal{E}^{\bullet}) := (\mathcal{E}^\nabla \times^{\text{PGL}_{n+1}^\mathcal{A}} \text{PGL}_{n+1}, \mathcal{E}_{\text{red}} \times^{\text{PGL}_{n+1}^\mathcal{A}} \text{PGL}_{n+1}^\infty)
\]

forms an \(F^N\)-indigenous structure on \(X\). The resulting assignment \(\mathcal{E}^{\bullet} \mapsto \iota_N(\mathcal{E}^{\bullet})\) determines a map of sets

\[
\iota_N : F^N\mathfrak{AInd}_X \rightarrow F^N\mathfrak{Ind}_X.
\]

Next, let \(N'\) be a positive integer with \(N \leq N'\) and \(\mathcal{E}^{\bullet} := (\mathcal{E}^\nabla, \mathcal{E}_{\text{red}})\) an \(F^{N'}\)-indigenous (resp., \(F^{N'}\)-affine-indigenous) structure on \(X\). Then, the pair

\[
\mathcal{E}^{\bullet}^{(N)} := (F_{X^{(N')}/k}^{(N')-N}(\mathcal{E}^\nabla), \mathcal{E}_{\text{red}})
\]

forms an \(F^N\)-indigenous (resp., \(F^N\)-affine-indigenous) structure on \(X\). We shall refer to \(\mathcal{E}^{\bullet}^{(N)}\) as the \(N\text{-th truncation}\) of \(\mathcal{E}^{\bullet}\). Each isomorphism of \(F^{N'}\)-indigenous (resp., \(F^{N'}\)-affine-indigenous) structures induces an isomorphism of \(F^N\)-indigenous (resp., \(F^N\)-affine-indigenous)
structures via truncation. Hence, the assignment $\mathcal{E}^{\bullet} \rightarrow \mathcal{E}^{\bullet} |^{(N)}$ defines a well-defined map of sets

$$(-)^{(N)} : F^{N'} \text{Ind}_X \rightarrow F^N \text{Ind}_X \quad \text{(resp., } F^{N'} \text{Ind}_X \rightarrow F^N \text{Ind}_X \text{)},$$

which makes the following square diagram commute:

$$\begin{array}{ccc}
F^{N'} \text{Ind}_X & \xrightarrow{(-)^{(N)}} & F^N \text{Ind}_X \\
\downarrow & & \downarrow \\
F^{N' - \text{Ind}_X} & \xrightarrow{(-)^{(N)}} & F^N \text{Ind}_X.
\end{array}$$

Moreover, taking truncations gives a projective system of sets

$$\cdots \rightarrow F^N \text{Ind}_X \rightarrow \cdots \rightarrow F^3 \text{Ind}_X \rightarrow F^2 \text{Ind}_X \rightarrow F^1 \text{Ind}_X$$

(resp., $\cdots \rightarrow F^{N'} \text{Ind}_X \rightarrow \cdots \rightarrow F^{N'} \text{Ind}_X \rightarrow F^{N} \text{Ind}_X \rightarrow F^{N-1} \text{Ind}_X$).

Now, we introduce Frobenius-(affine-)indigenous structures of infinite level, as follows.

**Definition 1.7.3.**

(i) An $F^\infty$-indigenous (resp., $F^\infty$-affine-indigenous) structure on $X$ is a collection

$$\mathcal{E}^{\bullet} := \{(\mathcal{E}^{\bullet}_{N}, \alpha_{N})\}_{N \in \mathbb{Z}_{\geq 0}},$$

where each $\mathcal{E}^{\bullet}_{N}$ ($N \in \mathbb{Z}_{\geq 0}$) denotes an $F^N$-indigenous (resp., $F^N$-affine-indigenous) structure on $X$ and $\alpha_{N}$ denotes an isomorphism $\mathcal{E}^{\bullet}_{N+1} |^{(N)} \sim \mathcal{E}^{\bullet}_{N}$ of $F^N$-indigenous (resp., $F^N$-affine-indigenous) structures.

(ii) Let $\mathcal{E}^{\bullet}_{i} := \{(\mathcal{E}^{\bullet}_{i,N}, \alpha_{i,N})\}_{N \in \mathbb{Z}_{\geq 0}}$ ($i = 1, 2$) be $F^\infty$-indigenous (resp., $F^\infty$-affine-indigenous) structures. Then, an isomorphism from $\mathcal{E}^{\bullet}_{1}$ to $\mathcal{E}^{\bullet}_{2}$ is defined to be a collection

$$\{\eta_{N}\}_{N \in \mathbb{Z}_{\geq 0}},$$

where each $\eta_{N}$ denotes an isomorphism $\mathcal{E}^{\bullet}_{1,N} \sim \mathcal{E}^{\bullet}_{2,N}$ of $F^N$-indigenous (resp., $F^N$-affine-indigenous) structures such that for each $N$ the following square diagram is commutative:

$$\begin{array}{ccc}
\mathcal{E}^{\bullet}_{1,N+1} & \xrightarrow{\alpha_{1,N}} & \mathcal{E}^{\bullet}_{1,N} \\
\downarrow & & \downarrow \\
\mathcal{E}^{\bullet}_{2,N+1} & \xrightarrow{\alpha_{2,N}} & \mathcal{E}^{\bullet}_{2,N}.
\end{array}$$

**Remark 1.7.4.**

In the terminology of [16], Definition 9, we can identify each $F^\infty$-indigenous (resp., $F^\infty$-affine-indigenous) structure with an $F$-divided PGL$_{n+1}$-torsor (resp., an $F$-divided PGL$^h_{n+1}$-torsor) $\{\mathcal{E}_{N}\}_{N \in \mathbb{Z}_{\geq 0}}$, where each $\mathcal{E}_{N}$ denotes a principal PGL$_{n+1}$-bundle (resp., a principal PGL$^h_{n+1}$-bundle) on $X^{(N)}$, together with a PGL$^\infty_{n+1}$-reduction (resp., a PGL$^h_{n+1}$-reduction) $\mathcal{E}_{\text{red}}$ on $\mathcal{E}_{0}$ such that $\mathcal{K}S_{(\mathcal{E}_{0}^{\text{can}}_{X,k}^{(1)} \mathcal{E}_{\text{red}})}$ is an isomorphism.
Moreover, because of the commutativity of (51), the collection of maps 

\[
\lim_{N \in \mathbb{Z}_{>0}} F^{N, \mathcal{I}_{X}} = F^{\infty, \mathcal{I}_{X}} \quad \text{(resp., } \lim_{N \in \mathbb{Z}_{>0}} F^{N, \mathcal{A}_{X}} = F^{\infty, \mathcal{A}_{X}} \text{)}.
\]

Moreover, because of the commutativity of (51), the collection of maps \(\{i_N\}_{N \in \mathbb{Z}_{>0}}\) (cf. (48)) determines a map of sets

\[
i_{\infty} : F^{\infty, \mathcal{A}_{X}} \rightarrow F^{\infty, \mathcal{I}_{X}}.
\]

In what follows, we shall construct a map from \(F^{N, \text{Proj}_{X}}\) (resp., \(F^{N, \text{Aff}_{X}}\)) to \(F^{N, \mathcal{I}_{X}}\) (resp., \(F^{N, \mathcal{A}_{X}}\)) for each \(N \in \mathbb{Z}_{>0} \cup \{\infty\}\). To this end, it suffices to consider the case of \(N \neq \infty\) because the case of \(N = \infty\) can be constructed by taking the projective limit of the resulting maps for finite \(N\)'s via the identifications (20) and (57).

Let \(S^{\mathcal{I}}\) be an \(F^{\mathcal{I}}\)-projective (resp., \(F^{\mathcal{A}}\)-affine) structure on \(X\), where \(N\) is a positive integer. Its underlying torsor specifies a principal \(\text{PGL}_{n+1}\)-bundle (resp., a principal \(\text{PGL}_{\infty}^{A}\)-bundle) \(E^{\mathcal{I}}\) on \(X^{(N)}\) via the underlying homomorphism of \(F^{(N)}_{X/k}\). The graphs \(\Gamma_{\phi} : U \rightarrow U \times \mathbb{P}^{n}\) (resp., \(\Gamma_{\phi} : U \rightarrow U \times \mathbb{A}^{n}\)) associated to various local sections \(\phi : U \rightarrow \mathbb{P}^{n}\) (resp., \(\phi : U \rightarrow \mathbb{A}^{n}\)) in \(S^{\mathcal{I}}\) may be glued together to construct a \(\text{PGL}_{n+1}\)-reduction (resp., a \(\text{PGL}_{\infty}^{A}\)-reduction) \(E_{\text{red}}\) of \(F^{(N)}_{X/k}\). Hence, we obtain a triple

\[
(F^{(N)}_{X/k}(E^{\mathcal{I}}), \nabla^{\text{can}}_{F^{(N)}_{X/k}(E^{\mathcal{I}})}), E_{\text{red}}).
\]

Lemma 1.7.5.

The triple (59) forms an indigenous (resp., affine-indigenous) bundle on \(X\).

Proof. Let us choose a local section \(\phi : U \rightarrow \mathbb{P}^{n}\) (resp., \(\phi : U \rightarrow \mathbb{A}^{n}\)) in \(S^{\mathcal{I}}\), where \(U\) denotes an open subscheme of \(X\), such that \(E^{\mathcal{I}}\) may be trivialized over \(U\). Then, since

\[
\nabla^{\text{can}}_{F^{(N)}_{X/k}(U^{(N)} \times \text{PGL}_{n+1})} = \nabla^{\text{triv}}_{U \times \text{PGL}_{n+1}} \quad \text{(resp., } \nabla^{\text{can}}_{F^{(N)}_{X/k}(U^{(N)} \times \text{PGL}_{\infty}^{A})} = \nabla^{\text{triv}}_{U \times \text{PGL}_{\infty}^{A}}),
\]

there exists an identification between the restriction of (59) to \(U\) and the triple

\[
(U \times \text{PGL}_{n+1}, \nabla^{\text{triv}}_{U \times \text{PGL}_{n+1}}, E_{\text{red}}|U) \quad \text{(resp., } (U \times \text{PGL}_{\infty}^{A}, \nabla^{\text{triv}}_{U \times \text{PGL}_{\infty}^{A}}, E_{\text{red}}|U)).
\]

According to Corollary 1.6.2, the étaleness of \(\phi\) implies that the Kodaira-Spencer map \(K_{E_{\text{red}}|U}\) associated to \(E_{\text{red}}|U\) is an isomorphism. By applying this discussion to various \(\phi\)'s, we see that the Kodaira-Spencer map associated with the triple (59) is an isomorphism, that is to say, the triple (59) forms an indigenous bundle. This completes the proof of the lemma. \(\square\)

By the above lemma, the pair

\[
\zeta_{N}^{\mathcal{I}} : (S^{\mathcal{I}}) := (E^{\mathcal{I}}, E_{\text{red}}) \quad \text{(resp., } \zeta_{N}^{\mathcal{A}} : (S^{\mathcal{A}}) := (E^{\mathcal{A}}, E_{\text{red}}))
\]

turns out to form an \(F^{N}\)-indigenous (resp., \(F^{N}\)-affine-indigenous) structure on \(X\). The resulting assignment \(S^{\mathcal{I}} \mapsto \zeta_{N}^{\mathcal{I}}(S^{\mathcal{I}})\) (resp., \(S^{\mathcal{A}} \mapsto \zeta_{N}^{\mathcal{A}}(S^{\mathcal{A}})\)) is compatible, in the evident
sense, with truncation to lower levels. It follows that this assignment may be extended to the
case of infinite level. Thus, for every \( N \in \mathbb{Z}_{>0} \cup \{ \infty \} \), we have obtained a map of sets
\begin{equation}
\zeta^\circ_N : F^N,\mathbf{Proj}_X \to F^N,\mathbf{Ind}_X \quad \text{(resp., } k\zeta^\circ_N : F^N,\mathbf{Aff}_X \to F^N,k\mathbf{Ind}_X \text{)}.
\end{equation}

2. INDIGENOUS AND AFFINE-ININDIGENOUS \( D^{(N-1)}_X \)-MODULES

In this section, we describe \( F^N \)-projective and \( F^N \)-affine structures, as well as \( F^N \)-indigenous
and \( F^N \)-affine-indigenous structures, in terms of vector bundles and Berthelot’s ring of differential
operators of finite level.

Let \( n \) be a positive integer and \( X \) a smooth variety over \( k \) of dimension \( n \).

2.1. Differential operators of level \( m \).

First, we quickly recall (cf. [3, 4]) the ring of differential operators of (in)finite level. For
each \( m = 0, 1, 2, \cdots, \infty \), denote by \( D^{(m)}_{X,l} \) the sheaf of differential operators of level \( m \) and order
at most \( l \) (hence \( D^{(0)}_X = \mathcal{O}_X \)), and write
\begin{equation}
D^{(m)}_X := \bigcup_{l \geq 0} D^{(m)}_{X,l}.
\end{equation}

This sheaf acts on \( \mathcal{O}_X \) and forms a sheaf of (noncommutative) \( k \)-algebras. We shall write
\( tD^{(m)}_X \) (resp., \( rD^{(m)}_X \)) for the sheaf \( D^{(m)}_X \) endowed with a structure of \( \mathcal{O}_X \)-module arising from
left (resp., right) multiplication by sections in \( D^{(m)}_{X,0} \) (= \( \mathcal{O}_X \)).

In what follows, let us explain the local description of these structures. We shall fix a local
coordinate system \( t := (t_1, \cdots, t_n) \) of \( X \) defined on an open subscheme \( U \subseteq X \). Given an
\( n \)-tuple of nonnegative integers \( h := (h_1, \cdots, h_n) \in \mathbb{Z}_{\geq 0}^n \), we write \( t_h := \prod_{i=1}^n t_i^{h_i} \in \Gamma(U, \mathcal{O}_U) \).
Also, given an integer \( r \), we write \( q_r \) for the unique integer satisfying \( r = q_r \cdot p^m + l \) with
\( 0 \leq l < p^m \). We set \( q_r^m := 0 \) if \( m = \infty \). Then, the \( \mathcal{O}_U \)-module \( tD^{(m)}_X |_U \) is freely generated by the symbols \( \partial^{(\underline{r})} := \partial^{(r_1)} \cdots \partial^{(r_n)} \) for various \( \underline{r} := (r_1, \cdots, r_n) \in \mathbb{N}^n \), and the action of \( D^{(m)}_X \) on
\( \mathcal{O}_X \) is determined by
\begin{equation}
\partial^{(\underline{r})} (t^h) = q_r! \cdot \left( l^h \right) \cdot t^{h-R} \left( := \prod_{i=1}^n q_r! \cdot \left( h_i \right)^{r_i} \cdot r_i^{h_i - r_i} \right).
\end{equation}

Moreover, the multiplication in \( D^{(m)}_X \) is given by
\begin{equation}
\partial^{(\underline{r})} \cdot \partial^{(\underline{r}')} = \left( \frac{\underline{r} + \underline{r}'}{\underline{r}} \right) \cdot \partial^{(\underline{r} + \underline{r}')} \quad \text{and} \quad \partial^{(\underline{r})} \cdot f = \sum_{\underline{s} \leq \underline{r}} \binom{\underline{r}}{\underline{s}} \cdot \partial^{(\underline{s})} (f) \cdot \partial^{(\underline{r} - \underline{s})}
\end{equation}
for any \( f \in \Gamma(U, \mathcal{O}_X) \), where
\begin{equation}
\left\{ a + b \atop a \right\} := \frac{q_{a+b}!}{q_a! \cdot q_b!} \quad \text{and} \quad \left( \frac{\underline{l}}{\underline{l}} \right) := \left( \frac{\underline{l}_1}{\underline{l}_2} \right) \cdot \left( \underline{l}_2 \right)^{-1}.
\end{equation}
Given a nonnegative integer \( r < p^{m+1} \) and \( i = 1,\ldots,n \), we shall write \( \partial_i^{[r]} := \frac{1}{q_i^r} \cdot \partial_i^{(r)} \). Then, \( D^{(m)}_X \) is locally generated as an \( \mathcal{O}_X \)-algebra by the sections \( \partial_i^{[p^l]} \left( = \partial_i^{(p^l)} \right) \) for various \( l \leq m \). \( D^{(0)}_X \) is, in particular, locally generated by \( \partial_1 \left( := \partial_1^{[p^0]} \right), \ldots, \partial_n \left( := \partial_n^{[p^0]} \right) \) and \( D^{(\infty)}_X \) is Grothendieck’s ring of differential operators (cf. [23], § 16.8). For each pair of nonnegative integers \( (m_1, m_2) \) with \( m_1 \leq m_2 \), there exists a natural \( k \)-linear morphism \( D^{(m_1)}_X \to D^{(m_2)}_X \) given by \( \frac{\partial|^{[l]}}{\partial|^{[l]}} \to \frac{\partial|^{[l]}}{\partial|^{[l]}} \), which restricts to an isomorphism \( D^{(m_1)}_{X,l} \to D^{(m_2)}_{X,l} \) for each \( l < p^{m_1} \). The sheaves \( D^{(m)}_X \) \( (m \geq 0) \) and the morphisms \( D^{(m_1)}_X \to D^{(m_2)}_X \) form an inductive system with

\[
\lim_{m \to 0} D^{(m)}_X \cong D^{(\infty)}_X.
\]

By a (left) \( D^{(m)}_X \)-module, we shall mean a pair \((\mathcal{V}, \nabla\mathcal{V})\) consisting of a vector bundle (i.e., a locally free coherent sheaf of finite rank) \( \mathcal{V} \) on \( X \) and an \( \mathcal{O}_X \)-linear morphism \( \nabla\mathcal{V} : \mathcal{L}D^{(m)}_X \to \mathcal{E}nd_k(\mathcal{V}) \) of sheaves of \( k \)-algebras, i.e., a left \( D^{(m)}_X \)-action \( \nabla\mathcal{V} \) on \( \mathcal{V} \), where \( \mathcal{E}nd_k(\mathcal{V}) \) denotes the sheaf of locally defined \( k \)-linear endomorphisms of \( \mathcal{V} \) endowed with a structure of \( \mathcal{O}_X \)-module given by left multiplication. An invertible \( D^{(m)}_X \)-module is defined to be a \( D^{(m)}_X \)-module \((\mathcal{L}, \nabla\mathcal{L})\) such that \( \mathcal{L} \) is a line bundle. Given two \( D^{(m)}_X \)-modules \( (\mathcal{V}_i, \nabla\mathcal{V}_i) \) \( (i = 1, 2) \), we define a morphism from \((\mathcal{V}_1, \nabla\mathcal{V}_1)\) to \((\mathcal{V}_2, \nabla\mathcal{V}_2)\) as an \( \mathcal{O}_X \)-linear morphism \( \mathcal{V}_1 \to \mathcal{V}_2 \) that is compatible with the respective \( D^{(m)}_X \)-actions \( \nabla\mathcal{V}_1, \nabla\mathcal{V}_2 \). Denote by \( \nabla\mathcal{V}^{\text{triv}} \) the \( D^{(m)}_X \)-action on \( \mathcal{O}_X \) recalled above. Thus, we obtain the trivial (invertible) \( D^{(m)}_X \)-module

\[
(\mathcal{O}_X, \nabla\mathcal{V}^{\text{triv}}).
\]

Given a vector bundle \( \mathcal{V} \) on \( X \), we always suppose that the tensor product \( \mathcal{D}^{(m)}_{X,l} \otimes \mathcal{V} := r\mathcal{D}^{(m)}_{X,l} \otimes \mathcal{V} \) (resp., \( \mathcal{V} \otimes \mathcal{D}^{(m)}_{X,l} : = \mathcal{V} \otimes l\mathcal{D}^{(m)}_{X,l} \)) is endowed with a structure of \( \mathcal{O}_X \)-module arising from that on \( l\mathcal{D}^{(m)}_{X,l} \) (resp., \( r\mathcal{D}^{(m)}_{X,l} \)).

Let \( m' \) be another element of \( \mathbb{Z} \cup \{ \infty \} \) with \( m \leq m' \) and let \((\mathcal{V}, \nabla\mathcal{V})\) be a \( D^{(m')}_X \)-module. Then, the composite \( \nabla\mathcal{V}^{(m)} : l\mathcal{D}^{(m)}_X \to l\mathcal{D}^{(m')}_X \to \mathcal{E}nd_k(\mathcal{V}) \) specifies a \( D^{(m')}_X \)-action on \( \mathcal{V} \). That is to say, we obtain a \( D^{(m')}_X \)-module

\[
(\mathcal{V}, \nabla\mathcal{V}^{(m)}).
\]

We shall refer to \((\mathcal{V}, \nabla\mathcal{V}^{(m)})\) as the \( m \)-th truncation of \((\mathcal{V}, \nabla\mathcal{V})\).

If \((\mathcal{V}_i, \nabla\mathcal{V}_i) \) \( (i = 1, 2) \) are \( D^{(m)}_X \)-modules, then the tensor product \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) has a canonical structure of \( D^{(m)}_X \)-action

\[
\nabla\mathcal{V}_{1,2} : l\mathcal{D}^{(m)}_X \to \mathcal{E}nd_k(\mathcal{V}_1 \otimes \mathcal{V}_2),
\]

determined, under the local description discussed above, by

\[
\partial_i^{[r]}(v_1 \otimes v_2) = \sum_{j=0}^r \partial_i^{[j]}(v_1) \otimes \partial_i^{[r-j]}(v_2)
\]

for any \( r < p^{m+1}, i = 1, 2, \ldots, n \) and any local sections \( v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2 \). In a similar way, we can define the dual \((\mathcal{V}^\vee, \nabla\mathcal{V}^\vee)\) of a \( D^{(m)}_X \)-module \((\mathcal{V}, \nabla\mathcal{V})\).
**Remark 2.1.1.**
Recall from [42], (1.0), the definition of a \((k-)\)connection on a vector bundle. For a vector bundle \(\mathcal{V} \) on \(X\), a \textbf{connection} on \(\mathcal{V}\) in the sense of loc. cit. is defined to be a \(k\)-linear map \(\nabla'_{\mathcal{V}} : \mathcal{V} \to \Omega_X \otimes \mathcal{V}\) satisfying that \(\nabla'_{\mathcal{V}}(av) = da \otimes v + a \cdot \nabla'_{\mathcal{V}}(v)\) for any local sections \(a \in \mathcal{O}_X\), \(v \in \mathcal{V}\). If \(\nabla'_{\mathcal{V}}\) is a connection on \(\mathcal{V}\), then the \textbf{curvature} of \((\mathcal{V}, \nabla'_{\mathcal{V}})\) is defined as the \(\mathcal{O}_X\)-linear morphism
\[
\psi_{(\mathcal{V}, \nabla'_{\mathcal{V}})} : \bigwedge^2 \mathcal{T}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V}) \quad (\subseteq \mathcal{E}nd_k(\mathcal{V}))
\]
given by \(\partial_1 \wedge \partial_2 \mapsto [\nabla''_{\mathcal{V}}(\partial_1), \nabla''_{\mathcal{V}}(\partial_2)] - \nabla''_{\mathcal{V}}([\partial_1, \partial_2])\) for any local sections \(\partial_1, \partial_2 \in \mathcal{T}_X\), where \(\nabla''_{\mathcal{V}}\) denotes the morphism \(\mathcal{T}_X \to \mathcal{E}nd_k(\mathcal{V})\) induced naturally from \(\nabla'_{\mathcal{V}}\). We shall say that a pair \((\mathcal{V}, \nabla'_{\mathcal{V}})\) as above is \textbf{flat} (or \textbf{integrable}) if \(\psi_{(\mathcal{V}, \nabla'_{\mathcal{V}})} = 0\). By a \textbf{flat bundle}, we shall mean such a pair \((\mathcal{V}, \nabla'_{\mathcal{V}})\) with \(\nabla'_{\mathcal{V}}\) flat. It is well-known that giving a \(\mathcal{D}_X^{(0)}\)-module is equivalent to giving a flat bundle on \(X\), and moreover, equivalent to giving a flat \(\text{GL}_{n+1}\)-bundle in the sense of §1.5 (cf. [78], §4.2 for the detailed discussion).

### 2.2. Kodaira-Spencer map associated to a \(\mathcal{D}_X^{(m)}\)-module.

In this subsection, we shall consider the Kodaira-Spencer map associated with a \(\mathcal{D}_X^{(m)}\)-module together with a subbundle. Let \(m\) be a nonnegative integer.

**Definition 2.2.1.**
Let \((\mathcal{V}, \nabla_{\mathcal{V}})\) be a \(\mathcal{D}_X^{(m)}\)-module and \(\mathcal{U}\) a subbundle of \(\mathcal{V}\). Then, the \textbf{Kodaira-Spencer map} associated with the triple \((\mathcal{V}, \nabla_{\mathcal{V}, \mathcal{U}})\) is defined as the \(\mathcal{O}_X\)-linear morphism
\[
\text{KS}_{(\mathcal{V}, \nabla_{\mathcal{V}, \mathcal{U}})} : \mathcal{T}_X \to \mathcal{U}^\prime \otimes (\mathcal{V}/\mathcal{U}) \quad (= \mathcal{H}om_{\mathcal{O}_X}(\mathcal{U}, \mathcal{V}/\mathcal{U}))
\]
induced, in the natural manner, from the composite
\[
\text{KS}_{(\mathcal{V}, \nabla_{\mathcal{V}, \mathcal{U}})} : \mathcal{U} \xrightarrow{\text{incl.}} \mathcal{V} \xrightarrow{\nabla_{\mathcal{V}}^{(0)}} \Omega_X \otimes \mathcal{V} \xrightarrow{\text{surj.}} \Omega_X \otimes (\mathcal{V}/\mathcal{U})
\]
(which is verified to be \(\mathcal{O}_X\)-linear), where \(\nabla_{\mathcal{V}}^{(0)}\) denotes the connection on \(\mathcal{V}\) corresponding to the \(\mathcal{D}_X^{(0)}\)-action \(\nabla_{\mathcal{V}}^{(0)}\) (cf. (69) and Remark 2.1.1). Notice that if \(\text{KS}_{(\mathcal{V}, \nabla_{\mathcal{V}, \mathcal{U}})}\) is an isomorphism and \(\mathcal{U}\) is a line bundle, then (since \(\mathcal{T}_X\) forms a rank \(n\) vector bundle) \(\mathcal{V}\) turns out to be a vector bundle of rank \(n+1\).

**Remark 2.2.2.**
In this remark, we compare two notions of Kodaira-Spencer map defined in Definitions 1.5.1 and 2.2.1. Let \((\mathcal{V}, \nabla_{\mathcal{V}})\) be a \(\mathcal{D}_X^{(0)}\)-module such that \(\mathcal{V}\) is of rank \(n+1\), and \(\mathcal{U}\) a line subbundle of \(\mathcal{V}\). Denote by \((\mathcal{E}, \nabla_{\mathcal{E}})\) the flat \(\text{GL}_{n+1}\)-bundle on \(X\) corresponding to \((\mathcal{V}, \nabla_{\mathcal{V}})\) (cf. Remark 2.1.1). If we write \(\text{GL}_{n+1}^\infty := \text{GL}_{n+1} \times_{\text{PGL}_{n+1}} \text{PGL}^\infty_{n+1} \subseteq \text{GL}_{n+1}\), then \(\mathcal{U}\) specifies a \(\text{GL}^\infty_{n+1}\)-reduction \(\mathcal{E}_{\text{red}}\) on \(\mathcal{E}\). Also, denote by \((\mathcal{E}, \nabla_{\mathcal{E}})\) the flat \(\text{PGL}_{n+1}\)-bundle induced by \((\mathcal{E}, \nabla_{\mathcal{E}})\), and by \(\mathcal{E}_{\text{red}}\) the \(\text{PGL}^\infty_{n+1}\)-reduction on \(\mathcal{E}\) determined by \(\mathcal{E}_{\text{red}}\). Observe that there exists a canonical isomorphism \(\text{Ad}_{\mathcal{E}} \simeq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V})\), which restricts to an isomorphism from \(\text{Ad}_{\mathcal{E}_{\text{red}}}\) to
the $\mathcal{O}_X$-submodule of $\text{End}_{\mathcal{O}_X}(\mathcal{V})$ consisting of locally defined $\mathcal{O}_X$-linear endomorphisms $f$ with $f(U) \subseteq U$. Hence, we have a diagram consisting of isomorphisms

$$
\begin{array}{ccc}
\text{Ad}_{\mathcal{E}/\mathcal{E}_{\text{red}}} & \sim & \text{Hom}_{\mathcal{O}_X}(U, \mathcal{V}/U) \\
\mathcal{T}_{\mathcal{E}/\mathcal{T}_{\text{red}}} & \sim & \mathcal{T}_{\mathcal{E}/\mathcal{T}_{\text{red}}},
\end{array}
$$

by which we can identify both the Kodaira-Spencer maps $\mathcal{K}_S(\mathcal{E},\mathcal{V},\mathcal{T}_{\text{red}})$ and $\mathcal{K}_S(\mathcal{E},\mathcal{V},\mathcal{T}_{\text{red}})$ (in the sense of Definition 1.5.1) as $\mathcal{O}_X$-linear morphisms $\mathcal{T}_X \to \text{Hom}_{\mathcal{O}_X}(U, \mathcal{V}/U)$. Taking this into consideration, one verifies that the morphisms $\mathcal{K}_S(\mathcal{E},\mathcal{V},\mathcal{T}_{\text{red}})$ coincide with $\mathcal{K}_S(\mathcal{V},\mathcal{V},\mathcal{N})$ at least up to composition with an automorphism of the vector bundle $\text{Hom}_{\mathcal{O}_X}(U, \mathcal{V}/U)$. In particular, $\mathcal{K}_S(\mathcal{V},\mathcal{V},\mathcal{N})$ is an isomorphism if and only if $\mathcal{K}_S(\mathcal{E},\mathcal{V},\mathcal{T}_{\text{red}})$, or equivalently $\mathcal{K}_S(\mathcal{E},\mathcal{V},\mathcal{T}_{\text{red}})$, is an isomorphism.

The following lemma will be used in the proof of Proposition 1.3.1.

**Lemma 2.2.3.**

Let $(\mathcal{V}, \nabla_{\mathcal{V}}, U)$ be as in Definition 2.2.1. Here, we can consider $(\mathcal{V}/U)^\vee$ and $U^\vee$ as, respectively, a subbundle and a quotient bundle of the dual vector bundle $\mathcal{V}^\vee$ of $\mathcal{V}$. Denote by $\tilde{\nabla}_{\mathcal{V}}$ the connection on $\mathcal{V}^\vee$ induced naturally from $\nabla_{\mathcal{V}}$. More precisely, $\tilde{\nabla}_{\mathcal{V}}$ is the $k$-linear morphism $\mathcal{V}^\vee \to \mathcal{O}_X \otimes \mathcal{V}^\vee$ given by $s \mapsto d \circ s - (\text{id}_{\mathcal{O}_X} \otimes s) \circ \nabla_{\mathcal{V}}$ for each local section $s \in \mathcal{V}^\vee$.

Now, let us consider the following composite isomorphism:

$$
(\mathcal{V}/U)^{\vee \vee} \otimes U^\vee \xrightarrow{c \otimes \text{id}_U} (\mathcal{V}/U) \otimes U^\vee \xrightarrow{\nabla_{\mathcal{V}} \otimes U^\vee} U^\vee \otimes (\mathcal{V}/U),
$$

where $c$ denotes the natural identification $(\mathcal{V}/U)^{\vee \vee} \cong (\mathcal{V}/U)$. Then, under the identification of $(\mathcal{V}/U)^{\vee \vee} \otimes U^\vee$ with $U^\vee \otimes (\mathcal{V}/U)$ via this isomorphism, we have the equality

$$
\mathcal{K}_S(\mathcal{V},\mathcal{V},(\mathcal{V}/U)^\vee) = -\mathcal{K}_S(\mathcal{V},\mathcal{V},U^\vee)
$$

of $\mathcal{O}_X$-linear morphisms $\mathcal{T}_X \to U^\vee \otimes (\mathcal{V}/U)$ $(= (\mathcal{V}/U)^{\vee \vee} \otimes U^\vee)$.

**Proof.** Let us consider the morphisms $\mathcal{K}_S'(\mathcal{V},\mathcal{V},\mathcal{N}) : U \to \Omega_X \otimes (\mathcal{V}/U)$ and $\mathcal{K}_S'(\mathcal{V},\mathcal{V},(\mathcal{V}/U)^\vee) : (\mathcal{V}/U)^\vee \to \Omega_X \otimes U^\vee$ (cf. (1.3.1)) inducing the Kodaira-Spencer maps $\mathcal{K}_S(\mathcal{V},\mathcal{V},\mathcal{N})$ and $\mathcal{K}_S(\mathcal{V},\mathcal{V},(\mathcal{V}/U)^\vee)$ respectively. If we are given any local sections $s \in (\mathcal{V}/U)^\vee$ and $t \in U$, then the following equalities hold:

$$
\mathcal{K}_S'(\mathcal{V},\mathcal{V},(\mathcal{V}/U)^\vee)(s)(t) = ds(t) - s(\overline{\nabla_{\mathcal{V}}(t)})
\begin{align*}
&= ds(t) - s \circ \mathcal{K}_S'(\mathcal{V},\mathcal{V},U^\vee)(t) \\
&= -s \circ \mathcal{K}_S'(\mathcal{V},\mathcal{V},U^\vee)(t),
\end{align*}
$$

where $\overline{(-)}$ denotes the natural quotient $\mathcal{V} \to \mathcal{V}/U$. This implies the required equality (77).
2.3. Indigenous and affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules.

We shall introduce indigenous and affine-indigenous $\mathcal{D}_X^{(N)}$-modules defined on a smooth variety $X$ of any dimension. Let $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ and let $X$ be as before.

**Definition 2.3.1.**

(i) An *indigenous* $\mathcal{D}_X^{(N-1)}$-module is a triple

\[(79) \quad \mathcal{V}^\diamond := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})\]

consisting of a $\mathcal{D}_X^{(N-1)}$-module $(\mathcal{V}, \nabla_\mathcal{V})$ and a line subbundle $\mathcal{N}$ of $\mathcal{V}$ such that the Kodaira-Spencer map $\text{KS}_{(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})} : T_X \to \text{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{V}/\mathcal{N})$ is an isomorphism.

(ii) Let $\mathcal{V}_i^\diamond := (\mathcal{V}_i, \nabla_{\mathcal{V},i}, \mathcal{N}_i)$ ($i = 1, 2$) be two indigenous $\mathcal{D}_X^{(N-1)}$-modules. Then, an *isomorphism* from $\mathcal{V}_1^\diamond$ to $\mathcal{V}_2^\diamond$ is defined as an isomorphism $\eta : \mathcal{V}_1 \sim \mathcal{V}_2$ of vector bundles that is compatible with the respective $\mathcal{D}_X^{(N-1)}$-actions $\nabla_{\mathcal{V},1}, \nabla_{\mathcal{V},2}$ and satisfies the equality $\eta(\mathcal{N}_1) = \mathcal{N}_2$.

**Definition 2.3.2.**

(i) An *affine-indigenous* $\mathcal{D}_X^{(N-1)}$-modules is a quadruple

\[(80) \quad A\mathcal{V}^\diamond := (\mathcal{V}, \mathcal{N}, \delta)\]

consisting of an indigenous $\mathcal{D}_X^{(N-1)}$-module $(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ and a left inverse morphism $\delta : \mathcal{V} \to \mathcal{N}$ to the natural inclusion $\mathcal{N} \hookrightarrow \mathcal{V}$ (i.e., the composite $\mathcal{N} \hookrightarrow \mathcal{V} \xrightarrow{\delta} \mathcal{N}$ coincides with the identity morphism of $\mathcal{N}$) such that $\text{Ker}(\delta) (\subseteq \mathcal{V})$ is closed under the $\mathcal{D}_X^{(N-1)}$-action $\nabla_\mathcal{V}$.

(ii) Let $A\mathcal{V}_i^\diamond := (\mathcal{V}_i, \nabla_{\mathcal{V},i}, \mathcal{N}_i, \delta_i)$ ($i = 1, 2$) be two affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules. Then, an *isomorphism* from $A\mathcal{V}_1^\diamond$ to $A\mathcal{V}_2^\diamond$ is defined as an isomorphism $\eta : (\mathcal{V}_1, \nabla_{\mathcal{V},1}, \mathcal{N}_1) \sim (\mathcal{V}_2, \nabla_{\mathcal{V},2}, \mathcal{N}_2)$ of indigenous $\mathcal{D}_X^{(N-1)}$-modules that is compatible with $\delta_1$ and $\delta_2$, i.e., satisfies the equality $\delta_2 \circ \eta = \eta|_{\mathcal{N}_1} \circ \delta_1$.

In what follows, we shall prove two propositions concerning indigenous $\mathcal{D}_X^{(N-1)}$-modules.

**Proposition 2.3.3.**

Let $(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ be a triple consisting of a $\mathcal{D}_X^{(N-1)}$-module $(\mathcal{V}, \nabla_\mathcal{V})$ and a line subbundle $\mathcal{N}$ of $\mathcal{V}$. Denote by $\text{KS}^\circ_{(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})}$ the $\mathcal{O}_X$-linear composite

\[(81) \quad \text{KS}^\circ_{(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})} : \mathcal{D}_X^{(N-1)} \otimes \mathcal{N} \xrightarrow{\text{incl.}} \mathcal{D}_X^{(N-1)} \otimes \mathcal{V} \xrightarrow{\mathcal{V}} \mathcal{V},\]

where the second arrow denotes the morphism induced from $\nabla_\mathcal{V}$. Then, the triple $(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ forms an indigenous $\mathcal{D}_X^{(N-1)}$-module if and only if $\text{KS}^\circ_{(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})}$ is an isomorphism.

**Proof.** The assertion follows from the various definitions involved. \qed

**Proposition 2.3.4.**

Let $\mathcal{V}_i^\diamond := (\mathcal{V}_i, \nabla_{\mathcal{V},i}, \mathcal{N}_i)$ ($i = 1, 2$) be indigenous $\mathcal{D}_X^{(N-1)}$-modules. Then, the following assertions hold:
(i) Denote by $\text{Isom}(V_1^\diamond, V_2^\diamond)$ (resp., $\text{Isom}(N_1, N_2)$) the set of isomorphisms of indigenous $D_1^{(N-1)}$-modules (resp., $O_X$-modules) from $V_1^\diamond$ (resp., $N_1$) to $V_2^\diamond$ (resp., $N_2$). Then, the assignment $\eta \mapsto \eta|_{N_1}$ gives an injection of sets

$$\text{Isom}(V_1^\diamond, V_2^\diamond) \hookrightarrow \text{Isom}(N_1, N_2).$$

(ii) Suppose that $X$ is proper over $k$, or more generally, the equality $\Gamma(X, O_X) = k$ holds. Moreover, suppose that $V_1 = V_2 = V$ and $N_1 = N_2 = N$, where $V$ and $N$ denote, respectively, a vector bundle and a line bundle on $X$. Then, $\nabla_{V,1} = \nabla_{V,2}$ if and only if $V_1^\diamond \simeq V_2^\diamond$. Moreover, there exist canonical identifications

$$\text{Isom}(V_1^\diamond, V_2^\diamond) = k^\times = \text{Isom}(N_1, N_2)$$

and the map (82) coincides with the identity map of $k^\times$ under these identifications.

**Proof.** First, we shall consider assertion (i). Let $\eta$ be an isomorphism $V_1^\diamond \simeq V_2^\diamond$. Since $\eta$ preserves the respective $D_1^{(N-1)}$-actions $\nabla_{V,1}$ and $\nabla_{V,2}$, the following square diagram is commutative:

$$\begin{array}{ccc}
D_1^{(N-1)} \otimes N_1 & \xrightarrow{id \otimes \eta|_{N_1}} & D_1^{(N-1)} \otimes N_2 \\
\downarrow \sim & & \downarrow \sim \\
V_1 & \xrightarrow{\sim} & V_2 \\
\end{array}$$

By Proposition 2.3.3, both the right-hand and left-hand vertical arrows are isomorphisms. Hence, the commutativity of this diagram implies that $\eta$ is uniquely determined by its restriction $\eta|_{N_1}$ to $N_1$. This completes the proof of assertion (i).

Next, we shall consider the former assertion of (ii). The “only if” part is clear. To prove the “if” part, we suppose that there exists an isomorphism $\eta : V_1^\diamond \simeq V_2^\diamond$. Because of the assumption on $X$, i.e, $\Gamma(X, O_X) = k$, the automorphism $\eta|_{N}$ of the line bundle $N$ may be given by multiplication by some $a \in k^\times$. By the commutativity of the diagram (84), $\eta$ turns out to be given by multiplication by $a$. But, such an automorphism cannot preserve the respective $D_1^{(N-1)}$-actions $\nabla_{V,1}$, $\nabla_{V,2}$ unless $\nabla_{V,1} = \nabla_{V,2}$. This proves the former assertion of (ii).

Finally, the latter assertion of (ii) follows immediately from the argument discussed so far. Thus, we finish the proof of the proposition. \qed

**Remark 2.3.5.**

In this remark, we shall examine the determinant of vector bundles associated with indigenous and affine-indigenous $D_1^{(N-1)}$-modules.

Note that the sheaf $D_1^{(N-1)}$ forms a rank $n + 1$ vector bundle obtained as an extension of $D_1^{(N-1)}/D_1^{(N-1)} (= T_X)$ by $D_1^{(N-1)} (= O_X)$. Hence, we obtain natural isomorphisms

$$\det(D_1^{(N-1)}) \simeq \det(D_1^{(N-1)}) \otimes \det(D_1^{(N-1)}/D_1^{(N-1)}) \simeq \det(T_X) \simeq \omega_X.$$
If $\mathcal{V}^\vartriangleleft := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ is an indigenous $\mathcal{D}_X^{(N-1)}$-module, then the isomorphism $\text{KS}^\vartriangleleft_{(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})} : \mathcal{D}_X^{(N-1)} \otimes \mathcal{N} \xrightarrow{\sim} \mathcal{V}$ (cf. (41)) induces, via taking determinants, the following composite isomorphism

$$\text{det}(\mathcal{V}) \xrightarrow{\sim} \text{det}(\mathcal{D}_X^{(N-1)} \otimes \mathcal{N}) \xrightarrow{\sim} \text{det}(\mathcal{D}_X^{(N-1)} \otimes \mathcal{N}^{\otimes(n+1)}) \xrightarrow{\sim} \omega_X^\mathcal{V} \otimes \mathcal{N}^{\otimes(n+1)}.$$

In particular, the determinant $\text{det}(\mathcal{V})$ is isomorphic to the line bundle $\omega_X^\mathcal{V} \otimes \mathcal{N}^{\otimes(n+1)}$.

Also, let $\mathcal{V}^\vartriangleright := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N}, \delta)$ be an affine-indigenous $\mathcal{D}_X^{(N-1)}$-module. By passing to $\text{KS}^\vartriangleright_{(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})}$, we can identify the inclusion $\mathcal{N} \hookrightarrow \mathcal{V}$ with the natural inclusion $\mathcal{N} \left( = \mathcal{D}_X^{(N-1)} \otimes \mathcal{N} \right) \hookrightarrow \mathcal{D}_X^{(N-1)} \otimes \mathcal{N}$. This implies that the surjection $\delta : \mathcal{V} \twoheadrightarrow \mathcal{N}$ allows us to identify $\text{Ker}(\delta)$ with $(\mathcal{D}_X^{(N-1)} \otimes \mathcal{N})/(\mathcal{D}_X^{(N-1)} \otimes \mathcal{N})$. Hence, we have the following composite isomorphism

$$\text{Ker}(\delta) \xrightarrow{\sim} (\mathcal{D}_X^{(N-1)} \otimes \mathcal{N})/(\mathcal{D}_X^{(N-1)} \otimes \mathcal{N}) \xrightarrow{\sim} (\mathcal{D}_X^{(N-1)}/\mathcal{D}_X^{(N-1)}) \otimes \mathcal{N} \xrightarrow{\sim} \mathcal{T}_X \otimes \mathcal{N},$$

which induces an isomorphism

$$\text{det}(\text{Ker}(\delta)) \xrightarrow{\sim} (\text{det}(\mathcal{T}_X \otimes \mathcal{N}) =) \omega_X^\mathcal{V} \otimes \mathcal{N}^{\otimes(n)}.$$  

2.4. From (affine-)indigenous $\mathcal{D}_X^{(0)}$-modules to (affine-)indigenous bundles.

Let $\mathcal{N}, \mathcal{N}' \in \mathbb{Z}_{>0} \cup \{\infty\}$ with $\mathcal{N} \leq \mathcal{N}'$. Also, let $\mathcal{V}^\vartriangleleft := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ (resp., $\mathcal{V}^\vartriangleright := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N}, \delta)$) be an indigenous (resp., affine-indigenous) $\mathcal{D}_X^{(N-1)}$-module. Then, the collection of data

$$\mathcal{V}^\vartriangleleft|_{\mathcal{N}} := (\mathcal{V}, \nabla_\mathcal{V}|_{(N-1)}, \mathcal{N}) \quad \text{(resp., } \mathcal{V}^\vartriangleright|_{\mathcal{N}} := (\mathcal{V}, \nabla_\mathcal{V}|_{(N-1)}, \mathcal{N}, \delta))$$

forms an indigenous (resp., affine-indigenous) $\mathcal{D}_X^{(N-1)}$-module. We shall refer to $\mathcal{V}^\vartriangleleft|_{\mathcal{N}}$ (resp., $\mathcal{V}^\vartriangleright|_{\mathcal{N}}$) as the $\mathcal{N}$-th truncation of $\mathcal{V}^\vartriangleleft$ (resp., $\mathcal{V}^\vartriangleright$). Giving an (affine-)indigenous $\mathcal{D}_X^{(\infty)}$-module is equivalent to giving a system of (affine-)indigenous $\mathcal{D}_X^{(N-1)}$-modules $(\mathcal{V}^\vartriangleleft|_{\mathcal{N}})$ together with isomorphisms $\mathcal{V}^\vartriangleleft|_{N+1} \sim \mathcal{V}^\vartriangleleft|_{N}$ defined for all $\mathcal{N} \in \mathbb{Z}_{>0}$.

In what follows, we shall construct indigenous and affine-indigenous bundles by means of, respectively, indigenous and affine-indigenous $\mathcal{D}_X^{(0)}$-modules. Let $\mathcal{V}^\vartriangleleft := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ be an indigenous $\mathcal{D}_X^{(0)}$-module. Then, $(\mathcal{V}, \nabla_\mathcal{V})$ induces a flat $\text{PGL}_{n+1}$-bundle $(\mathcal{V}^\text{PGL}, \nabla_\mathcal{V}^\text{PGL})$ on $X$ by change of structure group via the natural quotient $\text{GL}_{n+1} \twoheadrightarrow \text{PGL}_{n+1}$. Moreover, the line subbundle $\mathcal{N}$ determines a $\text{PGL}_{n+1}$-reduction $\mathcal{V}^\text{red}$ of $\mathcal{V}^\text{PGL}$. Let us write

$$\mathcal{V}^\vartriangleleft \mapsto \mathcal{\mathcal{\bullet}} := (\mathcal{V}^\text{PGL}, \nabla_\mathcal{V}^\text{PGL}, \mathcal{V}^\text{PGL}_{\text{red}}).$$

It follows from the discussion in Remark [2.2.2] that the Kodaira-Spencer map $\text{KS}_{\mathcal{V}^\vartriangleleft \mapsto \mathcal{\mathcal{\bullet}}}$ associated with this triple is an isomorphism, that is to say, $\mathcal{V}^\vartriangleleft \mapsto \mathcal{\mathcal{\bullet}}$ forms an indigenous bundle on $X$. We shall refer to $\mathcal{V}^\vartriangleleft \mapsto \mathcal{\mathcal{\bullet}}$ as the indigenous bundle associated with $\mathcal{V}^\vartriangleleft$.

Next, let $\mathcal{V}^\vartriangleright := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N}, \delta)$ be an affine-indigenous $\mathcal{D}_X^{(0)}$-module. By the above discussion, we obtain the triple $(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ in $\mathcal{V}^\vartriangleright$. Also, the surjection $\delta$ defines a $\text{PGL}_{n+1}$-reduction $\mathcal{V}^\mathcal{\mathcal{\bullet}}$ of $\mathcal{V}^\text{PGL}$. Since $\text{Ker}(\delta)$ is closed under the $\mathcal{D}_X^{(0)}$-action $\nabla_\mathcal{V}$, there exists a connection $\nabla_\mathcal{V}^\mathcal{\mathcal{\bullet}}$ on $\mathcal{V}^\mathcal{\mathcal{\bullet}}$ inducing $\nabla_\mathcal{V}^\text{PGL}$ by change of structure group via the inclusion
PGL_{n+1} \to PGL_n. The fiber product ν^A_{\text{red}} := ν^PGL_{\text{red}} \times_{PGL} ν^A forms a PGL_{n+1}^\infty-reduction on ν^A. One verifies that the triple
\begin{equation}
\nu^\wedge := (\nu^A, \nabla^A, \nu^A_{\text{red}})
\end{equation}
specifies an affine-indigenous bundle on X. We shall refer to \nu^\wedge as the affine-indigenous bundle associated with \nu^\wedge.

2.5. Twisting by invertible \mathcal{D}_X^{(N-1)}-modules.

Let \mathcal{N} \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}. Also, let \nu^\wedge := (\nu, \nabla, \mathcal{N}) (resp., \nu^\wedge := (\nu, \nabla, \mathcal{N}, \delta)) be an indigenous (resp., affine-indigenous) \mathcal{D}_X^{(N-1)}-module and (\mathcal{L}, \nabla_\mathcal{L}) an invertible \mathcal{D}_X^{(N-1)}-module. Since KS_{(\nu, \nabla, \mathcal{N})} is an isomorphism, the Kodaira-Spencer map KS_{(\nu, \nabla, \mathcal{N})} is associated to the triple (\nu \otimes \mathcal{L}, \nabla \otimes \nabla_\mathcal{L}, \mathcal{N} \otimes \mathcal{L}) (cf. (70) for the definition of \nabla \otimes \nabla_\mathcal{L}) is an isomorphism. In particular, the collection
\begin{equation}
\nu^\wedge_{(\mathcal{L}, \nabla_\mathcal{L})} := (\nu \otimes \mathcal{L}, \nabla \otimes \nabla_\mathcal{L}, \mathcal{N} \otimes \mathcal{L})
\end{equation}
forms an indigenous (resp., affine-indigenous) \mathcal{D}_X^{(N-1)}-module. In the case of \mathcal{N} = 1, the indigenous bundle \nu^\wedge_{(\mathcal{L}, \nabla_\mathcal{L})} (resp., the affine-indigenous bundle \nu^\wedge_{(\mathcal{L}, \nabla_\mathcal{L})}) associated with \nu^\wedge_{(\mathcal{L}, \nabla_\mathcal{L})} (resp., \nu^\wedge_{(\mathcal{L}, \nabla_\mathcal{L})}) is canonically isomorphic to \nu^\wedge (resp., \nu^\wedge).

Regarding the underlying vector bundles of indigenous and affine-indigenous \mathcal{D}_X^{(N-1)}-modules, we prove the following assertion.

Lemma 2.5.1.

For each \mathcal{L}_1, 2, suppose that we are given \mathcal{N}_i \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\} and an indigenous \mathcal{D}_X^{(N_i-1)}-module \nu^\wedge_i := (\nu_i, \nabla_{\nu,i}, \mathcal{N}_i) (resp., an affine-indigenous \mathcal{D}_X^{(N_i-1)}-module \nu^\wedge_i := (\nu_i, \nabla_{\nu,i}, \mathcal{N}_i, \delta_i)). In the non-resp’d assertion, we suppose further that \mathcal{N}_1 = \mathcal{N}_2 = 1.

Proof. By replacing \nu^\wedge_i (resp., \nu^\wedge_i) with \nu^\wedge_i|^{(1)} (resp., \nu^\wedge_i|^{(1)}) we may assume that \mathcal{N}_1 = \mathcal{N}_2 = 1.

First, we shall consider the non-resp’d assertion. It follows from Proposition 2.3.3 that, for each \mathcal{L}_1, 2, the vector bundle \nu_i may be identified with \mathcal{D}_X^{(0)}(\nu_i, \nabla_{\nu_i, \mathcal{N}_i}) via an isomorphism \nu_i \sim \nabla^\nu \otimes \mathcal{N}_i^{\otimes(n+1)}. Denote by \nu^\wedge_{\nu_{1, \mathcal{L}}} the connection on \omega_{\nu_i} \otimes \mathcal{N}_i^{\otimes(n+1)} induced, after taking determinants, from \nabla_{\nu_{1, \mathcal{L}}} via the isomorphism det(\nu_{1, \mathcal{L}}) \sim \omega_{\nu_i} \otimes \mathcal{N}_i^{\otimes(n+1)} (cf. (55)). Then, tensoring \nu^\wedge_{\nu_{1, \mathcal{L}}} with the dual of \nu^\wedge_{\nu_{2, \mathcal{L}}} yields a connection \nabla on the line bundle (\mathcal{N}_1 \otimes \mathcal{N}_2) \otimes \mathcal{N}_i^{\otimes(n+1)} \otimes (\omega_{\nu_i} \otimes \mathcal{N}_i^{\otimes(n+1)})^\nu.

Because of the assumption \mathcal{N} \not\mid n+1, one can find uniquely a connection \nu^\wedge_{\nu_{1, \mathcal{L}}} on \mathcal{L} := \mathcal{N}_1 \otimes \mathcal{N}_2 such that the (n+1)-st tensor product of \nu^\wedge_{\nu_{1, \mathcal{L}}} coincides with \nabla. In fact, if \nabla may be described locally as \nabla = d + A for some \mathcal{A} \in \Gamma(X, \Omega_X) under a local trivialization \mathcal{L} \cong \Omega_X, then we can write \nabla_{\nu_{1, \mathcal{L}}} = d + A. The tensor product of \nu^\wedge_{\nu_{1, \mathcal{L}}} and \nabla_{\nu_{1, \mathcal{L}}}(more precisely, the connection

\nabla := (\nu^A, \nabla_A, \nu^A_{\text{red}})
corresponding to $\nabla_{\mathcal{V},2}$ specifies a structure of $\mathcal{D}_{X}^{(0)}$-action $\nabla_{\mathcal{V} \otimes L,2}$ on $\mathcal{V}_{2} \otimes L$. Since $\mathcal{V}_{2}^{\otimes}$ forms an indigenous bundle, the Kodaira-Spencer map
\begin{equation}
\eta := K_{S_{\mathcal{V}_{2} \otimes L,\nabla_{\mathcal{V} \otimes L,2}}}: \mathcal{D}_{X,1}^{(0)} \otimes (\mathcal{N}_{2} \otimes \mathcal{L}) \to \mathcal{V}_{2} \otimes \mathcal{L}
\end{equation}
(cf. (81)) is an isomorphism satisfying that $\eta(\mathcal{D}_{X,0}^{(0)} \otimes (\mathcal{N}_{2} \otimes \mathcal{L})) = \mathcal{N}_{2} \otimes \mathcal{L}$. Here, consider the natural identifications
\begin{equation}
\mathcal{V}_{1} \left( = \mathcal{D}_{X,1}^{(0)} \otimes \mathcal{N}_{1} \right) \cong \mathcal{D}_{X,1}^{(0)} \otimes (\mathcal{N}_{2} \otimes \mathcal{L}), \quad \mathcal{N}_{1} \left( = \mathcal{D}_{X,0}^{(0)} \otimes \mathcal{N}_{1} \right) \cong \mathcal{D}_{X,0}^{(0)} \otimes (\mathcal{N}_{2} \otimes \mathcal{L}).
\end{equation}
Under these identifications, $\eta$ specifies a required isomorphism. This completes the proof of the non-resp’ed assertion.

Next, we shall consider the resp’ed assertion. For each $i = 1, 2$, denote by $\nabla_{\mathcal{N},i}$ the $\mathcal{D}_{X}^{(0)}$-action on $\mathcal{N}_{i}$ induced from $\nabla_{\mathcal{V},i}$ via the surjection $\delta_{i}$. Also, denote by $\nabla_{\mathcal{L}}$ the structure of $\mathcal{D}_{X}^{(0)}$-action on $\mathcal{L} := \mathcal{N}_{1} \otimes \mathcal{N}_{2}$ defined as the tensor product of $\nabla_{\mathcal{N},1}$ and the dual of $\nabla_{\mathcal{N},2}$. Just as (93), the Kodaira-Spencer map associated to the underlying indigenous $\mathcal{D}_{X}^{(0)}$-module of $\mathcal{A}\mathcal{V}_{\otimes (\mathcal{L},\nabla_{\mathcal{L}})}$ specifies an isomorphism $\eta: \mathcal{V}_{1} \cong \mathcal{V}_{2} \otimes \mathcal{L}$ with $\eta(\mathcal{N}_{1}) = \mathcal{N}_{2} \otimes \mathcal{L}$. This completes the proof of assertion (ii). \hfill \Box

2.6. Comparison with differential operators.

In this last subsection, we make a bijective correspondence between certain differential operators and indigenous $\mathcal{D}_{X}^{(0)}$-modules (cf. [82], §3.4, Proposition 3.6, for the case where $X$ is a curve). But, we note that the discussion in this subsection will not be used in the rest of the present paper.

Let $\mathcal{V}_{1}$, $\mathcal{V}_{2}$ be vector bundles on $X$ and $m$ a nonnegative integer. By a differential operator of order $\leq m$ from $\mathcal{V}_{1}$ to $\mathcal{V}_{2}$, we shall mean an $\mathcal{O}_{X}$-linear morphism $D : \mathcal{V}_{1} \to \mathcal{V}_{2} \otimes \mathcal{D}_{X,m}^{(0)}$. We shall write
\begin{equation}
\mathcal{D}_{X,m}(\mathcal{V}_{1}, \mathcal{V}_{2})
\end{equation}
for the set of such morphisms. If $D$ is an element of $\mathcal{D}_{X,m}(\mathcal{V}_{1}, \mathcal{V}_{2})$, then the composite
\begin{equation}
\Sigma(D) : \mathcal{V}_{1} \xrightarrow{D} \mathcal{V}_{2} \otimes \mathcal{D}_{X,m}^{(0)} \to \mathcal{V}_{2} \otimes S^{m}(\mathcal{T}_{X})
\end{equation}
is called the principal symbol of $D$, where $S^{m}(\mathcal{T}_{X})$ denotes the $m$-th symmetric product of $\mathcal{T}_{X}$ over $\mathcal{O}_{X}$ and the second arrow denotes the morphism obtained from the natural quotient $\mathcal{D}_{X,m}^{(0)} \to \left( \mathcal{D}_{X,m}^{(0)} / \mathcal{D}_{X,m-1}^{(0)} \cong \right) S^{m}(\mathcal{T}_{X})$.

Now, let $\mathcal{N}$ be a line bundle on $X$. For each nonnegative integer $m$, we shall denote by
\begin{equation}
\mathcal{D}_{X,m,N}
\end{equation}
the sheaf on $X$ given by assigning $U \mapsto \mathcal{D}_{X,m,N}(N^{\vee}|_{U}, N^{\vee}|_{U} \otimes S^{m}(\mathcal{O}_{U}))$ for each open subscheme $U$ of $X$. This sheaf has naturally a structure of $\mathcal{O}_{X}$-module. Taking principal symbols gives rise to the following short exact sequence of $\mathcal{O}_{X}$-modules:
\begin{equation}
0 \to \mathcal{D}_{X,1,N} \to \mathcal{D}_{X,2,N} \xrightarrow{\Sigma} \mathcal{H}om_{\mathcal{O}_{X}}(N^{\vee}, N^{\vee} \otimes S^{2}(\mathcal{O}_{X}) \otimes S^{2}(\mathcal{T}_{X})) \to 0.
\end{equation}
Next, let us consider the composite injection
\begin{equation}
\mathcal{O}_{X} \hookrightarrow \mathcal{E}nd_{\mathcal{O}_{X}}(N^{\vee} \otimes S^{2}(\mathcal{T}_{X})) \cong \mathcal{H}om_{\mathcal{O}_{X}}(N^{\vee}, N^{\vee} \otimes S^{2}(\mathcal{O}_{X}) \otimes S^{2}(\mathcal{T}_{X}))
\end{equation}
where the first arrow denotes the diagonal embedding. By pulling-back the sequence \( [98] \) via this composite injection, we obtain an \( \mathcal{O}_X \)-submodule \( \text{Diff}^{\Sigma^{-1}(\mathcal{O}_X)} \) of \( \text{Diff}^{X,2,N} \) fitting into the following short exact sequence:

\[
0 \rightarrow \text{Diff}^{X,1,N} \rightarrow \text{Diff}^{\Sigma^{-1}(\mathcal{O}_X)} \rightarrow \mathcal{O}_X \rightarrow 0.
\]

We shall write \( \Sigma \) again for the third arrow \( \text{Diff}^{\Sigma^{-1}(\mathcal{O}_X)} \rightarrow \mathcal{O}_X \) in this sequence, and write

\[
\text{Diff}^{\Sigma=1}_{X,2,N} := \Gamma(X, \Sigma^{-1}(1)) \left( \subseteq \Gamma(X, \text{Diff}^{\Sigma^{-1}(\mathcal{O}_X)}) \right),
\]

e.g., the set of second order differential operators from \( \mathcal{N}^\vee \) to \( \mathcal{N}^\vee \otimes S^2(\Omega_X) \) with unit principal symbol. If

\[
\mathcal{U}_{X,N} \in \text{Ext}^1(\mathcal{O}_X, \text{Diff}^{X,1,N}) \left( \cong H^1(X, \text{Diff}^{X,1,N}) \right)
\]
denotes the extension class of \([100]\), then \( \mathcal{U}_{X,N} = 0 \) if and only if \( \text{Diff}^{\Sigma=1}_{X,2,N} \neq \emptyset \).

Here, denote by

\[
\mathcal{ID}_{X,N}
\]

the set of isomorphism classes of objects in the groupoids defined as follows:

- The objects are indigenous \( \mathcal{D}^{(0)}_X \)-modules \( (\mathcal{V}', \nabla_{\mathcal{V}'}, \mathcal{N}') \) with \( \mathcal{N}' = \mathcal{N} \),
- The morphisms from \( (\mathcal{V}', \nabla_{\mathcal{V}'}, \mathcal{N}') \) to \( (\mathcal{V}'', \nabla_{\mathcal{V}''}, \mathcal{N}) \) are isomorphisms of \( \mathcal{D}^{(0)}_X \)-modules inducing the identity morphism of \( \mathcal{N} \).

Then, the following assertion holds.

**Proposition 2.6.1.**

There exists a canonical bijection of sets

\[
\mathcal{ID}_{X,N} \sim \text{Diff}^{\Sigma=1}_{X,2,N}.
\]

Moreover, if \( \mathcal{U}_{X,N} = 0 \), then \( \mathcal{ID}_{X,N} \) is nonempty and forms an affine space modeled on the \( k \)-vector space \( \Gamma(X, \text{Diff}^{X,1,N}) \).

**Proof.** Since the latter assertion follows from the above discussion (cf. the short exact sequence \([100]\)), it suffices to consider the former assertion.

First, let us construct a map from \( \mathcal{ID}_{X,N} \) to \( \text{Diff}^{\Sigma=1}_{X,2,N} \). Let \( \mathcal{V}^\diamond := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N}) \) be an indigenous \( \mathcal{D}^{(0)}_X \)-module representing an element of \( \mathcal{ID}_{X,N} \). The morphism \( \text{KS}_{\mathcal{V}^\diamond} : \mathcal{D}^{(0)}_{X,1} \otimes \mathcal{N} \rightarrow \mathcal{V} \) (cf. \([81]\)) is an isomorphism, and the resulting composite

\[
\mathcal{D}^{(0)}_{X,2} \otimes \mathcal{N} \xrightarrow{\text{incl}} \mathcal{D}^{(0)}_{X} \otimes \mathcal{V} \xrightarrow{\nabla_{\mathcal{V}}} \mathcal{V} \xrightarrow{(\text{KS}_{\mathcal{V}^\diamond})^{-1}} \mathcal{D}^{(0)}_{X,1} \otimes \mathcal{N}
\]

specifies a split surjection of the short exact sequence

\[
0 \rightarrow \mathcal{D}^{(0)}_{X,1} \otimes \mathcal{N} \rightarrow \mathcal{D}^{(0)}_{X,2} \otimes \mathcal{N} \rightarrow S^2(\mathcal{T}_X) \otimes \mathcal{N} \rightarrow 0.
\]

This split surjection corresponds to a split injection \( S^2(\mathcal{T}_X) \otimes \mathcal{N} \hookrightarrow \mathcal{D}^{(0)}_{X,2} \otimes \mathcal{N} \). Moreover, it corresponds naturally to an \( \mathcal{O}_X \)-linear morphism

\[
D_{\mathcal{V}^\diamond} : \mathcal{N}^\vee \rightarrow (\mathcal{N}^\vee \otimes S^2(\Omega_X)) \otimes \mathcal{D}^{(0)}_{X,2},
\]
which is verified to be an element of $\Diff_{X,2,N}^{\Sigma=1}$. The morphism $D_{V^\partial}$ depends only on the isomorphism class of $V^\partial$, and hence, the assignment $V^\partial \mapsto D_{V^\partial}$ yields a well-defined map of sets

$$\mathcal{D}_{X,N} \to \Diff_{X,2,N}^{\Sigma=1}.$$ 

Next, we shall construct its inverse. Let $D : N^\vee \to (N^\vee \otimes S^2(\Omega_X)) \otimes D_{X,2}^{(0)}$ be an element of $\Diff_{X,2,N}^{\Sigma=1}$. It determines naturally an $\mathcal{O}_X$-linear morphism $D' : S^2(T_X) \otimes N \to D_{X,2}^{(0)} \otimes N' \left( \subseteq D_{X,2}^{(0)} \otimes \mathcal{N} \right)$. Here, note that $D_{X,2}^{(0)} \otimes \mathcal{N}$ has a structure of $D_{X,2}^{(0)}$-action arising from left multiplication. Let $V_D$ denote the quotient of $D_{X,2}^{(0)} \otimes \mathcal{N}$ by the smallest $\mathcal{O}_X$-submodule that is closed under the $D_{X,2}^{(0)}$-action and contains $\text{Im}(D')$. $V_D$ admits a $D_{X,2}^{(0)}$-action $\nabla_{V,D} : \mathcal{D}_{X,2}^{(0)} \to \mathcal{E}nd_k(V_D)$ arising, via the natural quotient $D_{X,2}^{(0)} \otimes \mathcal{N} \to V_D$, from the $D_{X,2}^{(0)}$-action on $D_{X,2}^{(0)} \otimes \mathcal{N}$. Since $\Sigma(D) = 1$, the composite $D_{X,1}^{(0)} \otimes \mathcal{N} \hookrightarrow D_{X,2}^{(0)} \otimes \mathcal{N} \to V_D$ turns out to be an isomorphism. By passing to this composite isomorphism, we can think of $\mathcal{N} \left( = D_{X,0}^{(0)} \otimes \mathcal{N} \subseteq D_{X,1}^{(0)} \otimes \mathcal{N} \right)$ as a line subbundle of $V_D$. By construction, the resulting triple

$$(109)\quad V_D^\partial := (V_D, \nabla_{V,D}, \mathcal{N})$$

forms an indigenous $D_{X,2}^{(0)}$-module belonging to $\mathcal{D}_{X,N}$. Moreover, one verifies that the assignment $D \mapsto V_D^\partial$ defines the inverse to the map (108). This completes the proof of the proposition. \hfill $\square$

### 3. Dormant indigenous and affine-indigenous $D_{X}^{(N-1)}$-modules

In this section, we consider dormant indigenous and affine-indigenous $D_{X}^{(N-1)}$-modules, which are characterized by means of an invariant associated with each flat bundle called $p$-(N – 1)-curvature. The central argument is to construct correspondences with the additional structures on a variety introduced in §1 (cf. Proposition 3.3.2 Theorem 3.4.3 Corollary 3.4.5 and Theorem 3.6.4). As an application, we provide necessary conditions on Chern classes for the existence of Frobenius-projective or Frobenius-affine structures respectively (cf. Theorem 3.7.1).

Let $n$ be a positive integer and $X$ a smooth variety over $k$ of dimension $n$.

#### 3.1. $p$-curvature and $F$-divided sheaves.

Let $m$ be a nonnegative integer. First, recall from [51] the definition of $p$-$m$-curvature and Cartier’s theorem for $D_{X}^{(m)}$-modules. Denote by $\mathcal{K}_{X}^{(m)}$ the kernel of the morphism $\nabla_{\mathcal{O}_X}^{\text{triv}(m)} : \mathcal{I}D_{X}^{(m)} \to \mathcal{E}nd_k(\mathcal{O}_X)$ defining the trivial $D_{X}^{(m)}$-action on $\mathcal{O}_X$ (cf. (68)). If $(\mathcal{V}, \nabla_{\mathcal{V}})$ is a $D_{X}^{(m)}$-module, then the composite

$$(110)\quad p_{\psi^{(m)}_{(\mathcal{V}, \nabla_{\mathcal{V}})}} : \mathcal{K}_{X}^{(m)} \hookrightarrow \mathcal{I}D_{X}^{(m)} \to \mathcal{E}nd_k(\mathcal{O}_X)$$

is called the $p$-$m$-curvature of $(\mathcal{V}, \nabla_{\mathcal{V}})$ (cf. [51], Definition 3.1.1).
Given a $\mathcal{D}^{(m)}_X$-module $(\mathcal{V}, \nabla_{\mathcal{V}})$, we shall write
\begin{equation}
\text{Sol}(\nabla_{\mathcal{V}})
\end{equation}
for the subsheaf of $\mathcal{V}$ on which $\mathcal{D}^{(m)}_X$ acts as zero, where $\mathcal{D}^{(m)}_X$ denotes the kernel of the canonical morphism $\mathcal{D}^{(m)}_X \to \mathcal{O}_X$. We shall refer to $\text{Sol}(\nabla_{\mathcal{V}})$ as the sheaf of horizontal sections of $(\mathcal{V}, \nabla_{\mathcal{V}})$. This sheaf may be thought of as an $\mathcal{O}_{X^{(m+1)}}$-module via the underlying homeomorphism of $F^{(m+1)}_{X/k}$.

If $\mathcal{U}$ is a vector bundle on $X^{(m+1)}$, then its pull-back $F^{(m+1)*}_{X/k}(\mathcal{U})$ via $F^{(m+1)}_{X/k}$ admits naturally a structure of $\mathcal{D}^{(m)}_X$-action
\begin{equation}
\nabla^{\text{can}(m)}_{F^{(m+1)*}_{X/k}(\mathcal{U})} : \mathcal{D}^{(m)}_X \to \mathcal{E}nd_k(F^{(m)*}_{X/k}(\mathcal{U})),
\end{equation}
for which the pair $(F^{(m+1)*}_{X/k}(\mathcal{U}), \nabla^{\text{can}(m)}_{F^{(m+1)*}_{X/k}(\mathcal{U})})$ forms a $\mathcal{D}^{(m)}_X$-module with vanishing $p$-$m$-curvature.

The $\mathcal{D}^{(m)}_X$-action $\nabla^{\text{can}(m)}_{F^{(m+1)*}_{X/k}(\mathcal{U})}$ may be given, under the local description of $\mathcal{D}^{(m)}_X$ introduced in §2.1, by $\partial^\mathcal{U}(x \otimes v) := \partial^\mathcal{U}(x) \otimes v$ for each local section $x \otimes v \in \mathcal{O}_X \otimes (F^{(m+1)}_{X/k})^{-1}(\mathcal{O}_{X^{(m+1)}})$.

According to [51], Corollary 3.2.4, the assignments $\mathcal{U} \mapsto (F^{(m+1)*}_{X/k}(\mathcal{U}), \nabla^{\text{can}(m)}_{F^{(m+1)*}_{X/k}(\mathcal{U})})$ and $(\mathcal{V}, \nabla_{\mathcal{V}}) \mapsto \text{Sol}(\nabla_{\mathcal{V}})$ determine an equivalence of categories
\begin{equation}
\left(\text{the category of vector bundles on } X^{(m+1)}\right) \cong \left(\text{the category of } \mathcal{D}^{(m)}_X\text{-modules with vanishing } p\text{-}m\text{-curvature}\right),
\end{equation}
which is compatible with the formations of tensor products and extensions.

Recall (cf. [16], §2.2) that an $F$-divided sheaf on $X$ is a collection
\begin{equation}
\{(\mathcal{E}_m, \alpha_m)\}_{m \in \mathbb{Z}_{\geq 0}},
\end{equation}
where each pair $(\mathcal{E}_m, \alpha_m)$ consists of a vector bundle $\mathcal{E}_m$ on $X^{(m)}$ and an $\mathcal{O}_{X^{(m)}}$-linear isomorphism $\alpha_m : F^{(1)*}_{X^{(m)}}(\mathcal{E}_{m+1}) \cong \mathcal{E}_m$. $F$-divided sheaves form a category, in which the morphisms from $\{(\mathcal{E}_m, \alpha_m)\}_m$ to $\{(\mathcal{E}_m', \alpha_m')\}_m$ are the collections of $\mathcal{O}_{X^{(m)}}$-linear morphisms $\mathcal{E}_m \to \mathcal{E}_m'$ which commute with the $\alpha_m$’s and $\alpha_m'$’s in the obvious sense. Given two $F$-divided sheaves $\{(\mathcal{E}_m, \alpha_m)\}_m$, $\{(\mathcal{E}_m', \alpha_m')\}_m$, we can construct the tensor product $\{(\mathcal{E}_m \otimes \mathcal{E}_m', \alpha_m \otimes \alpha_m')\}_m$ of them.

For an $F$-divided sheaf $\{(\mathcal{E}_m, \alpha_m)\}_m$ on $X$, $\mathcal{E}_0$ admits a canonical $\mathcal{D}^{(\infty)}_X$-action. In fact, the $\mathcal{D}^{(m)}_X$-action $\nabla^{\text{can}(m)}_{F^{(m+1)*}_{X/k}(\mathcal{E}_{m+1})}$ (cf. (12)) on $F^{(m+1)*}_{X/k}(\mathcal{E}_{m+1})$ induces a $\mathcal{D}^{(m)}_X$-action on $\mathcal{E}_0$ (with vanishing $p$-$m$-curvature) via the composite isomorphism
\begin{equation}
F^{(m)*}_{X/k}(\alpha_m) \circ \cdots \circ F^{(1)*}_{X/k}(\alpha_1) \circ \alpha_0 : F^{(m+1)*}_{X/k}(\mathcal{E}_{m+1}) \cong \mathcal{E}_0.
\end{equation}

The collection of these actions for various $m$ yields, via (67), a $\mathcal{D}^{(\infty)}_X$-action
\begin{equation}
\nabla^{\text{can}(\infty)}_{\mathcal{E}_0} : \mathcal{D}^{(\infty)}_X \to \mathcal{E}nd_k(\mathcal{E}_0)
\end{equation}
on $\mathcal{E}_0$. By Katz’s theorem appearing in [20], Theorem 1.3, the assignment \(((\mathcal{E}_m, \alpha_m))_m \mapsto (\mathcal{E}_0, \nabla^{\text{can}}_{\mathcal{E}_0})\) determines an equivalence of categories:

\[(117) \begin{pmatrix}
\text{the category of } \\
F\text{-divided sheaves on } X
\end{pmatrix} \sim \begin{pmatrix}
\text{the category of } \\
\mathcal{D}_X^{(\infty)}\text{-modules}
\end{pmatrix},
\]

which is compatible with the formations of tensor products and extensions. Moreover, for each $m \geq 0$, the following diagram of categories is 2-commutative:

\[(118) \begin{pmatrix}
\text{the category of } \\
F\text{-divided sheaves on } X
\end{pmatrix} \sim \begin{pmatrix}
\text{the category of } \\
\mathcal{D}_X^{(\infty)}\text{-modules}
\end{pmatrix}
\]

For convenience, we always refer to any $\mathcal{D}_X^{(\infty)}$-module as having vanishing $p\text{-}\infty\text{-curvature.}

### 3.2. Dormant indigenous and affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules.

We shall introduce dormant indigenous and affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules, as follows.

**Definition 3.2.1.**

Let $N$ be a positive integer and $\mathcal{V}^{\circ} := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N})$ (resp., $A\mathcal{V}^{\circ} := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N}, \delta)$) an indigenous (resp., affine-indigenous) $\mathcal{D}_X^{(N-1)}$-module. Then, we shall say that $\mathcal{V}^{\circ}$ (resp., $A\mathcal{V}^{\circ}$) is **dormant** if the $p\text{-}(N-1)$-curvature of $(\mathcal{V}, \nabla_{\mathcal{V}})$ vanishes identically. In the case of infinite level, we shall refer to any indigenous (resp., affine-indigenous) $\mathcal{D}_X^{(\infty)}$-module as being **dormant**, for convenience.

The following proposition will be useful when we try to understand which varieties have an indigenous or affine-indigenous $\mathcal{D}_X^{(\infty)}$-module.

**Proposition 3.2.2.**

Let $\mathcal{N}$ be a line bundle on $X$. Then, the following assertions hold:

(i) Suppose that there exists a dormant indigenous $\mathcal{D}_X^{(\infty)}$-module of the form $\mathcal{V}^{\circ} := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N})$. Then, the order of the line bundle $\omega_X^\vee \otimes \mathcal{N}^{\otimes(n+1)}$ in the Neron-Severi group of $X$ is finite of order prime to $p$. In particular, $\omega_X^\vee \otimes \mathcal{N}^{\otimes(n+1)}$ is numerically trivial.

(ii) Suppose that there exists a dormant affine-indigenous $\mathcal{D}_X^{(\infty)}$-module of the form $A\mathcal{V}^{\circ} := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N}, \delta)$. Then, both $\omega_X$ and $\mathcal{N}$ are numerically trivial.

**Proof.** Assertion (i) follows from [20], §1, Theorem 1.8. In fact, by passing to the isomorphism $\det(\mathcal{V}) \sim \omega_\mathcal{V}^\vee \otimes \mathcal{N}^{\otimes(n+1)}$ (cf. [86]), we can obtain a structure of $\mathcal{D}_X^{(\infty)}$-action $\nabla_{\omega_\mathcal{V}^\vee \otimes \mathcal{N}^{\otimes(n+1)}}$ on the line bundle $\omega_\mathcal{V}^\vee \otimes \mathcal{N}^{\otimes(n+1)}$ corresponding to the determinant of $\nabla_{\mathcal{V}}$.

Next, we shall consider assertion (ii). Since $\text{Ker}(\delta)$ is closed under the $\mathcal{D}_X^{(\infty)}$-action $\nabla_{\mathcal{V}}$, $\mathcal{N}$ has a structure of $\mathcal{D}_X^{(\infty)}$-action $\nabla_{\mathcal{N}}$ induced by $\nabla_{\mathcal{V}}$ via the quotient $\delta : \mathcal{V} \twoheadrightarrow \mathcal{N}$. Let $\nabla_{\mathcal{N}^{\otimes(n+1)}}$ denote the $\mathcal{D}_X^{(\infty)}$-action on $\mathcal{N}^{\otimes(n+1)}$ defined as the $(n+1)$-st tensor product of $\nabla_{\mathcal{N}}$. Then,
tensoring $\nabla_{N^\otimes(n+1)}$ and the dual of $\nabla_{\omega^\otimes N^\otimes(n+1)}$ obtained in the proof of (i) yield a $\mathcal{D}_X^{(\infty)}$-action $\nabla_{\omega_X}$ on $\omega_X$. Thus, the proof of the assertion is completed by applying again \[20\], § 1, Theorem 1.8, to both $(N, \nabla_N)$ and $(\omega_X, \nabla_{\omega_X})$. \qed

For each $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, denote by
\begin{equation}
\mathcal{D}_{X,N}^{\text{Zas}...} \quad \text{(resp., $\mathcal{A}\mathcal{D}_{X,N}^{\text{Zas}...}$)}
\end{equation}
the set of isomorphism classes of dormant indigenous (resp., dormant affine-indigenous) $\mathcal{D}_X^{(N-1)}$-modules. In the rest of this section, we study the relationships between dormant (affine-)indigenous $\mathcal{D}_X^{(N-1)}$-modules and the additional structures on a variety introduced in § 1. To do this, we construct maps relating them, as follows.

**From (affine-)indigenous $\mathcal{D}_X^{(N-1)}$-modules to $F^N$-(affine-)indigenous structures:**

Let $N$ be a positive integer and $\mathcal{V}^\triangledown := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ a dormant indigenous $\mathcal{D}_X^{(N-1)}$-module. Since $(\mathcal{V}, \nabla_\mathcal{V})$ has vanishing $p-(N-1)$-curvature, the sheaf of horizontal sections $\mathcal{S}\text{ol}(\nabla_\mathcal{V})$ forms a rank $(n+1)$ vector bundle on $X^{(N)}$ and the morphism $F_{X/k}^{(N)*}(\mathcal{S}\text{ol}(\nabla_\mathcal{V})) \sim \mathcal{V}$ induced from the inclusion $\mathcal{S}\text{ol}(\nabla_\mathcal{V}) \hookrightarrow \mathcal{V}$, considered as a morphism of $\mathcal{O}_{X^{(N)}}$-modules, is an isomorphism (cf. \[113\]). Let $\mathcal{V}^{\text{PGL}}$ denote the principal $\text{PGL}_{n+1}$-bundle on $X^{(N)}$ induced by $\mathcal{S}\text{ol}(\nabla_\mathcal{V})$ via projectivization. Then, the principal $\text{PGL}_{n+1}$-bundle on $X$ associated to $\mathcal{V}$, denoted by $\mathcal{V}^{\text{PGL}}$, is isomorphic to the pull-back $F_{X/k}^{(N)*}(\mathcal{V}^{\text{PGL}})$. It follows that $\mathcal{V}^{\text{PGL}}$ admits a canonical connection $\nabla_{\mathcal{V}^{\text{PGL}}}^{\text{can}} := \nabla_{F_{X/k}^{(N)*}(\mathcal{V}^{\text{PGL}})}^{\text{can}}$ (cf. \[26\]). On the other hand, the line subbundle $\mathcal{N} (\subseteq \mathcal{V})$ determines a $\text{PGL}_{n+1}$-reduction $\mathcal{V}_{\text{red}}^{\text{PGL}}$ of $\mathcal{V}^{\text{PGL}}$. Since $\text{KS}_{\mathcal{V}^\triangledown}$ is an isomorphism, the Kodaira-Spencer map $\text{KS}_{\mathcal{V}^{\triangledown}, \mathcal{V}^{\text{PGL}}, \mathcal{V}^{\text{PGL}}_{\text{red}}}$ is an isomorphism because of the discussion in Remark \[2.2.2\]. As a consequence, the pair
\begin{equation}
\mathcal{V}^{\triangledown\Rightarrow \triangledown} := (\mathcal{V}^{\text{PGL}}, \mathcal{V}_{\text{red}}^{\text{PGL}})
\end{equation}
forms an $F^N$-indigenous structure on $X$.

Next, suppose further that we have been given a left inverse morphism $\delta : \mathcal{V} \rightarrow \mathcal{N}$ to the inclusion $\mathcal{N} \hookrightarrow \mathcal{V}$ for which the quadruple $\mathcal{A}\mathcal{V}^{\triangledown} := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N}, \delta)$ forms a dormant affine-indigenous $\mathcal{D}_X^{(N-1)}$-module. The $\mathcal{D}_X^{(N-1)}$-action $\nabla_\mathcal{V}$ restricts to a $\mathcal{D}_X^{(N-1)}$-action $\nabla_{\text{Ker}(\delta)}$ on $\text{Ker}(\delta)$ with vanishing $p-(N-1)$-curvature. Then, the sheaf of horizontal sections $\mathcal{S}\text{ol}(\nabla_{\text{Ker}(\delta)})$ determines, via projectivization, a $\text{PGL}_{n+1}$-reduction $\mathcal{V}_{\text{red}}^{\text{A}}$ of $\mathcal{V}^{\text{PGL}}$. Since $\text{PGL}_{n+1}^{\text{A,\infty}} = \text{PGL}_{n+1}^\infty \times_{\text{PGL}_{n+1}} \text{PGL}_{n+1}^{\text{A}}$, the intersection of the two reductions $F_{X/k}^{(N)*}(\mathcal{V}_{\text{red}}^{\text{A}})$, $\mathcal{V}_{\text{red}}^{\text{PGL}}$ of $\mathcal{V}^{\text{PGL}} \cong F_{X/k}^{(N)*}(\mathcal{V}^{\text{PGL}})$ specifies a $\text{PGL}_{n+1}^{\text{A,\infty}}$-reduction $\mathcal{V}_{\text{red}}^{\text{A}}$. One verifies that the resulting pair
\begin{equation}
\mathcal{A}\mathcal{V}^{\triangledown\Rightarrow \triangledown} := (\mathcal{V}^{\text{A}}, \mathcal{V}_{\text{red}}^{\text{A}})
\end{equation}
forms an $F^N$-affine-indigenous structure on $X$.

The assignment $\mathcal{V}^{\triangledown} \mapsto \mathcal{V}^{\triangledown\Rightarrow \triangledown}$ (resp., $\mathcal{A}\mathcal{V}^{\triangledown} \mapsto \mathcal{A}\mathcal{V}^{\triangledown\Rightarrow \triangledown}$) constructed above is compatible with the formation of truncation to lower levels. Hence, by considering this assignment for every $N \in \mathbb{Z}_{>0}$, we also obtain an $F^\infty$-indigenous (resp., $F^\infty$-affine-indigenous) structure $\mathcal{V}^{\triangledown\Rightarrow \triangledown}$ (resp., $\mathcal{A}\mathcal{V}^{\triangledown\Rightarrow \triangledown}$) on $X$ by means of a dormant indigenous (resp., dormant affine-indigenous) $\mathcal{D}_X^{(\infty)}$-module $\mathcal{V}^{\triangledown}$ (resp., $\mathcal{A}\mathcal{V}^{\triangledown}$). In any case of $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, we shall refer to $\mathcal{V}^{\triangledown\Rightarrow \triangledown}$ (resp., $\mathcal{A}\mathcal{V}^{\triangledown\Rightarrow \triangledown}$)
\( \hat{\mathcal{V}}^{\bowtie} \) as the \( F^N \)-indigenous (resp., \( F^N \)-affine-indigenous) structure associated with \( \mathcal{V} \) (resp., \( \hat{\mathcal{V}}^{\bowtie} \)). The assignment \( \mathcal{V} \mapsto \mathcal{V}^{\bowtie} \) (resp., \( \hat{\mathcal{V}}^{\bowtie} \mapsto \hat{\mathcal{V}}^{\bowtie} \)) gives a map of sets
\[
(122) \quad \zeta_N^{\bowtie} = : X \otimes_{\mathbb{Z}_{\leq 0}} \to F^N \cdot \operatorname{End}_X \quad \left( \text{resp., } \hat{\zeta}_N^{\bowtie} = : \hat{X} \otimes_{\mathbb{Z}_{\leq 0}} \to F^N \cdot \operatorname{End}_X \right).
\]
In the case of \( N = 1 \), this assignment may be thought of as a restriction of the assignment \( \mathcal{V} \mapsto \mathcal{V}^{\bowtie} \) (resp., \( \hat{\mathcal{V}}^{\bowtie} \mapsto \hat{\mathcal{V}}^{\bowtie} \)) constructed in §2.4.

From (affine-)indigenous \( \mathcal{D}^{(N-1)} \)-modules to \( F^N \)-projective (\( F^N \)-affine) structures:

Let \( \mathcal{N} \) and \( \mathcal{V} \) be as above. Just as in the above discussion, we obtain the principal \( \operatorname{PGL}_{n-1} \)-bundles \( \mathcal{V}^{\nabla \operatorname{PGL}} \) and \( \mathcal{V}^{\operatorname{PGL}} \) associated with \( \operatorname{Sol}(\nabla) \) and \( \mathcal{V} \) respectively. Denote by \( \mathbb{P}_V^{\nabla} \) and \( \mathbb{P}^n_V \) the \( \mathbb{P}^n \)-bundles corresponding to \( \mathcal{V}^{\nabla \operatorname{PGL}} \) and \( \mathcal{V}^{\operatorname{PGL}} \) respectively. The \( \operatorname{PGL}_{n+1} \)-reduction \( \mathcal{V}_{\text{red}}^{\operatorname{PGL}} \) determined by \( \mathcal{N} \) corresponds to a global section \( \sigma : X \to \mathbb{P}^n_V \). Let us take an open subscheme \( U \) of \( X \) such that \( \operatorname{Sol}(\nabla) \) as well as \( \mathcal{V}^{\operatorname{PGL}} \), may be trivialized over the open subscheme \( U(N) \) of \( X(N) \). Fix a trivialization \( \tau : \nabla_{U(N)} \xrightarrow{\sim} U(N) \times \operatorname{PGL}_{n+1} \) of \( \nabla^{\operatorname{PGL}} \) over \( U(N) \), inducing a trivialization \( \tau_P : \mathbb{P}^n_V \xrightarrow{\sim} U(N) \times \mathbb{P}^n \). Thus, we obtain the composite
\[
(123) \quad \phi_r : U \xrightarrow{\sigma|_U} \mathbb{P}^n_V |_U \xrightarrow{\tau_F^P} \mathbb{P}^n \xrightarrow{\text{pr}_2} \mathbb{P}^n,
\]
where \( \tau_F^P \) denotes the isomorphism obtained, under the natural identification \( F^{(N)}_{X/k}(\mathbb{P}^n_V) \xrightarrow{\sim} \mathbb{P}^n_V \), as the pull-back of \( \tau_P \) via \( F^{(N)}_{X/k} \). Since \( \mathcal{K}_V^{\bowtie} \) is an isomorphism, it follows from Corollary [1.6.2] (and the discussion in Remark [2.2.2]) that \( \phi \) is étale. Let \( \mathcal{V}^{\bowtie} \) denote the set of étale morphisms \( \phi_r \) constructed in this way for various trivializations \( \tau \). The set \( \mathcal{V}^{\bowtie} \) has a structure of \( \operatorname{PGL}_{n+1}(U(N)) \)-torsor with respect to the \( \operatorname{PGL}_{n+1}(U(N)) \)-action defined by \( \phi_r \mapsto \phi_{g \sigma r} \) for each \( g \in \operatorname{PGL}_{n+1}(U(N)) \). Hence, the sheaf associated to the assignment \( U \mapsto \mathcal{V}^{\bowtie \psi}(U) \) specifies a subsheaf
\[
(124) \quad \mathcal{V}^{\nabla \psi} \subset \mathcal{V}^{\bowtie \psi}
\]
of \( \mathcal{P}_{X}^{\psi} \), which forms an \( F^N \)-projective structure on \( X \).

Next, suppose further that we are given a left inverse morphism \( \delta : \mathcal{V} \to \mathcal{N} \) to the inclusion \( \mathcal{N} \hookrightarrow \mathcal{V} \) for which \( \hat{\mathcal{V}}^{\bowtie} = (\mathcal{V}, \nabla, \delta) \) forms a dormant affine-indigenous \( \mathcal{D}^{(N-1)} \)-module. Let \( \nabla_{\text{Ker}(\delta)} \), \( \nabla^{\nabla A} \), and \( \nabla_{\text{red}}^{\nabla A} \) be as before, and write \( \mathbb{A}^{\nabla}_{\nabla A} \) and \( \mathbb{A}_V^{\nabla A} \) for the \( \mathbb{A}^n \)-bundles corresponding to \( \nabla^{\nabla A} \) and \( F^{(N)}_{X/k}(\nabla^{\nabla A}) \) respectively. By an argument entirely similar to the above argument, we can obtain an étale morphism \( U \to \mathbb{A}^n \) from a sufficiently small open subscheme \( U \) of \( X \) and a local trivialization \( \tau : \nabla^{\nabla A}|_{U(N)} \xrightarrow{\sim} U(N) \times \operatorname{PGL}^{\bowtie}_{n+1}(\nabla^{\nabla A}) \) of \( \nabla^{\nabla A} \). Moreover, the sheaf associated with the assignment from each such \( U \) to the set of étale morphisms \( U \to \mathbb{A}^n \) constructed in this way specifies a subsheaf
\[
(125) \quad \mathbb{A}^{\nabla \bowtie \psi}
\]
of \( \mathbb{A}_X^{\nabla} \), which forms an \( F^N \)-affine structure on \( X \).

The assignment \( \mathcal{V} \mapsto \mathcal{V}^{\bowtie \psi} \) (resp., \( \hat{\mathcal{V}}^{\bowtie} \mapsto \hat{\mathcal{V}}^{\bowtie \psi} \)) is compatible with the formation of truncation to lower levels. Hence, by considering this assignment for every \( N \in \mathbb{Z}_{>0} \), we also obtain an \( F^\infty \)-projective (resp., \( F^\infty \)-affine) structure \( \mathcal{V}^{\bowtie \psi} \) (resp., \( \hat{\mathcal{V}}^{\bowtie \psi} \)) by means of a dormant indigenous (resp., dormant affine-indigenous) \( \mathcal{D}^{(\infty)}_X \)-module \( \mathcal{V}^{\bowtie} \) (resp., \( \hat{\mathcal{V}}^{\bowtie} \)). In any
case of $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, we shall refer to $\mathcal{V}^\diamondsuit \Rightarrow \bigodot$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit \Rightarrow \bigodot$) as the $F^N$-projective (resp., $F^N$-affine) structure associated with $\mathcal{V}^\diamondsuit$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit$). Thus, we obtain a map of sets

\begin{equation}
\zeta_N^\diamondsuit \Rightarrow \bigodot : \mathcal{D}^\text{zaz}_{X,N} \to F^N\text{-Proj}_X \quad \text{(resp., $\mathcal{A}\zeta_N^\diamondsuit \Rightarrow \bigodot : \mathcal{A}\mathcal{D}^\text{zaz}_{X,N} \to F^N\text{-Aff}_X$)}.
\end{equation}

given by $\mathcal{V}^\diamondsuit \mapsto \mathcal{A}\mathcal{V}^\diamondsuit \Rightarrow \bigodot$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit \mapsto \mathcal{A}\mathcal{V}^\diamondsuit \Rightarrow \bigodot$).

It follows from the various definitions involved that the following diagrams consisting of the maps obtained so far are commutative:

\begin{equation}
\begin{array}{ccc}
\mathcal{D}^\text{zaz}_{X,N} & \xrightarrow{\zeta_N^\diamondsuit \Rightarrow \bigodot} & F^N\text{-Proj}_X \\
\alpha \mathcal{D}^\text{zaz}_{X,N} & \xrightarrow{\mathcal{A}\zeta_N^\diamondsuit \Rightarrow \bigodot} & F^N\text{-Ind}_X \\
\end{array}
\end{equation}

3.3. Comparison via projectivization I.

We shall introduce certain equivalence relations in $\mathcal{D}^\text{zaz}_{X,N}$ and $\mathcal{A}\mathcal{D}^\text{zaz}_{X,N}$. Moreover, we prove (cf. Proposition 3.3.2) that the resulting sets of equivalence classes are in bijection with $F^N\text{-Proj}_X$ and $F^N\text{-Aff}_X$ respectively. Let us fix $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$.

**Definition 3.3.1.**

For each $i = 1, 2$, let $\mathcal{V}^\diamondsuit_1 := (\mathcal{V}_i, \nabla_{V,i}, \mathcal{N}_i)$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit_1 := (\mathcal{V}_i, \nabla_{V,i}, \mathcal{N}_i, \delta_i)$) be a dormant indigenous (resp., dormant affine-indigenous) $\mathcal{D}^{(N-1)}_X$-module. Then, we shall say that $\mathcal{V}^\diamondsuit_1$ and $\mathcal{V}^\diamondsuit_2$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit_1$ and $\mathcal{A}\mathcal{V}^\diamondsuit_2$) are $\mathcal{G}_m$-equivalent if there exists an invertible $\mathcal{D}^{(N-1)}_X$-module $(\mathcal{L}, \nabla_\mathcal{L})$ with vanishing $p-(N-1)$-curvature such that $(\mathcal{V}^\diamondsuit_1)_{\mathcal{L}, \nabla_\mathcal{L}}$ (resp., $(\mathcal{A}\mathcal{V}^\diamondsuit_1)_{\mathcal{L}, \nabla_\mathcal{L}}$) (cf. (92)) is isomorphic to $\mathcal{V}^\diamondsuit_2$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit_2$). If $\mathcal{V}^\diamondsuit_1$ and $\mathcal{V}^\diamondsuit_2$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit_1$ and $\mathcal{A}\mathcal{V}^\diamondsuit_2$) are $\mathcal{G}_m$-equivalent, then we write

\begin{equation}
\mathcal{V}^\diamondsuit_1 \sim \mathcal{V}^\diamondsuit_2 \quad \text{(resp., $\mathcal{A}\mathcal{V}^\diamondsuit_1 \sim \mathcal{A}\mathcal{V}^\diamondsuit_2$)}.
\end{equation}

The relation $\sim$ in fact defines an equivalence relation in $\mathcal{D}^\text{zaz}_{X,N}$ (resp., $\mathcal{A}\mathcal{D}^\text{zaz}_{X,N}$). Write

\begin{equation}
\mathcal{D}^\text{zaz}_{X,N} / \sim \quad \text{(resp., $\mathcal{A}\mathcal{D}^\text{zaz}_{X,N} / \sim$)}
\end{equation}

for the quotient set of $\mathcal{D}^\text{zaz}_{X,N}$ (resp., $\mathcal{A}\mathcal{D}^\text{zaz}_{X,N}$) by this equivalence relation. If $\mathcal{V}^\diamondsuit$ (resp., $\mathcal{A}\mathcal{V}^\diamondsuit$) is a dormant indigenous (resp., dormant affine-indigenous) $\mathcal{D}^{(N-1)}_X$-module, then we denote by $[\mathcal{V}^\diamondsuit]$ (resp., $[\mathcal{A}\mathcal{V}^\diamondsuit]$) the image of $\mathcal{V}^\diamondsuit$ in $\mathcal{D}^\text{zaz}_{X,N}$ (resp., the image of $\mathcal{A}\mathcal{V}^\diamondsuit$ in $\mathcal{A}\mathcal{D}^\text{zaz}_{X,N}$). If $(\mathcal{L}, \nabla_\mathcal{L})$ is an invertible $\mathcal{D}^{(N-1)}_X$-module with vanishing $p-(N-1)$-curvature, then we have $\zeta_N^\diamondsuit \Rightarrow \bigodot ([\mathcal{V}^\diamondsuit]) \cong \zeta_N^\diamondsuit \Rightarrow \bigodot (\mathcal{V}^\diamondsuit_{\mathcal{L}, \nabla_\mathcal{L}})$ and $\mathcal{A}\zeta_N^\diamondsuit \Rightarrow \bigodot ([\mathcal{A}\mathcal{V}^\diamondsuit]) \cong \mathcal{A}\zeta_N^\diamondsuit \Rightarrow \bigodot ((\mathcal{A}\mathcal{V}^\diamondsuit)_{\mathcal{L}, \nabla_\mathcal{L}})$ (resp., $\mathcal{A}\zeta_N^\diamondsuit \Rightarrow \bigodot (\mathcal{A}\mathcal{V}^\diamondsuit_{\mathcal{L}, \nabla_\mathcal{L}})$ and $\mathcal{A}\zeta_N^\diamondsuit \Rightarrow \bigodot (\mathcal{V}^\diamondsuit_{\mathcal{L}, \nabla_\mathcal{L}})$ (cf. (92)). It follows that the assignments $[\mathcal{V}^\diamondsuit] \mapsto$
ζ^\diamondsuit : (\mathcal{Y}^\diamondsuit) \text{ and } [\mathcal{Y}^\diamondsuit] \mapsto \zeta_N^\diamondsuit (\mathcal{Y}^\diamondsuit) \text{ (resp., } [\mathcal{X}^\diamondsuit] \mapsto \zeta_N^\diamondsuit (\mathcal{X}^\diamondsuit) \text{ and } [\mathcal{X}^\diamondsuit] \mapsto \zeta_N^\diamondsuit (\mathcal{X}^\diamondsuit)) \text{ determine well-defined maps}

\begin{equation}
\zeta_N^\diamondsuit : \mathcal{D}_{X,N}^{\text{Zar}} \to F^N-\text{Proj}_X \quad \text{and} \quad \zeta_N^\diamondsuit : \mathcal{D}_{X,N}^{\text{Zar}} \to F^N-\text{Proj}_X
\end{equation}

(resp., \zeta_N^\diamondsuit : \mathcal{D}_{X,N}^{\text{Zar}} \to F^N-\text{Aff}_X \quad \text{and} \quad \zeta_N^\diamondsuit : \mathcal{D}_{X,N}^{\text{Zar}} \to F^N-\text{Aff}_X)

respectively. Moreover, the commutative diagrams in (127) induce the following commutative diagrams:

\begin{equation}
\begin{array}{ccc}
\mathcal{D}_{X,N}^{\text{Zar}} & \xrightarrow{\zeta_N^\diamondsuit} & F^N-\text{Proj}_X \\
\uparrow \mathcal{D}_{X,N}^{\text{Zar}} & & \downarrow \mathcal{D}_{X,N}^{\text{Zar}} \\
\mathcal{D}_{X,N}^{\text{Zar}} & \xrightarrow{\zeta_N^\diamondsuit} & F^N-\text{Proj}_X,
\end{array}
\end{equation}

Proposition 3.3.2.

For each \( N \in \mathbb{Z}_{>0} \), the maps \( \zeta_N^\diamondsuit : \mathcal{D}_{X,N}^{\text{Zar}} \to F^N-\text{Proj}_X \) and \( \zeta_N^\diamondsuit : \mathcal{D}_{X,N}^{\text{Zar}} \to F^N-\text{Aff}_X \) are bijective.

Proof. We only consider the bijectivity of \( \zeta_N^\diamondsuit \) because the case of \( \zeta_N^\diamondsuit \) can be proved by an entirely similar argument. To complete the proof, we shall construct the inverse to the map \( \zeta_N^\diamondsuit \). Let us take an arbitrary \( F^N \)-projective structure \( \mathcal{S}^\diamondsuit \) on \( X \). Write (\( \mathcal{E}^\nabla, \mathcal{E}^\text{red} \)) := \( \zeta_N^\diamondsuit (\mathcal{S}^\diamondsuit) \). Note that the principal \( \text{PGL}_{n+1} \)-bundle \( \mathcal{E}^\nabla \) may be, by construction, trivialized locally in the Zariski topology. Hence, there exists a rank \( n+1 \) vector bundle \( \mathcal{U} \) on \( X^{(N)} \), inducing \( \mathcal{E}^\nabla \) via projectivization. We thus obtain a \( \mathcal{D}_X^{(N-1)} \)-module \( (F^{(N)}(\mathcal{U}), N^{\text{can}}(N-1)(F^{(N)}(\mathcal{U}))) \) (cf. (112)) with vanishing \( p-(N-1) \)-curvature. Moreover, the \( \text{PGL}_{n+1}^{\infty} \)-reduction \( \mathcal{E}_\text{red} \) of \( (F^{(N)}(\mathcal{U})) \) determines a line subbundle \( \mathcal{N} \) of \( F^{(N)}_{X/k}(\mathcal{U}) \). Since \( (\mathcal{E}^\nabla, \mathcal{E}^\text{red}) \) forms an \( F^N \)-indigenous structure, the Kodaira-Spencer map \( \mathcal{K}^\nabla_{F^{(N)}(\mathcal{U}), N^{\text{can}}(N-1)} \) turns out to be an isomorphism (cf. Remark 2.2.2). That is to say, the triple

\begin{equation}
\mathcal{S}^\diamondsuit := ((F^{(N)}(\mathcal{U}), N^{\text{can}}(N-1)), \mathcal{N})
\end{equation}

specifies a dormant indigenous \( \mathcal{D}_X^{(N-1)} \)-module. One verifies immediately that the resulting assignment \( \mathcal{S}^\diamondsuit \mapsto \mathcal{S}^\diamondsuit \) defines the inverse to \( \zeta_N^\diamondsuit \). This completes the proof of the proposition. \( \square \)

3.4. Comparison via projectivization II.

In this subsection, we prove (cf. Theorem 3.4.3) the bijectivities of \( \zeta_N^\diamondsuit \) (for \( N \in \mathbb{Z}_{>0} \)) and \( \zeta_N^\diamondsuit \) (for \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \)). To this end, we will apply Lemmas 3.4.1 and 3.4.2 described below. A point of the proof for \( \zeta_N^\diamondsuit \) is to show that the underlying \( \text{PGL}_{n+1} \)-bundle of each dormant \( F^N \)-indigenous bundle may be trivialized locally in the Zariski topology not only in
the étale topology. Until the end of Lemma 3.4.2, we assume that \( X \) is quasi-projective over \( k \).

For each smooth affine algebraic group \( G \) over \( k \), we denote by \( \check{H}^1_{\text{ét}}(X,G) \) the first Čech cohomology set of \( X \) with coefficients in \( G \) with respect to the étale topology. This pointed set may be identified with the set of isomorphism classes of principal \( G \)-bundles on \( X \), where the distinguished point is represented by the trivial \( G \)-bundle. Then, since we assumed that \( X \) is quasi-projective, there exists a coboundary map

\[
\gamma : \check{H}^1_{\text{ét}}(X, \text{PGL}_{n+1}) \to H^2_{\text{ét}}(X, \mathbb{G}_m)
\]

induced by the short exact sequence

\[
1 \to \mathbb{G}_m \to \text{GL}_{n+1} \to \text{PGL}_{n+1} \to 1
\]

(cf. [59], Chap. III, §2, Theorem 2.17, and Chap. IV, §2, Theorem 2.5). Given a principal \( \text{PGL}_{n+1} \)-bundle \( \mathcal{E} \) on \( X \), we shall denote by \([\mathcal{E}]\) the element of \( \check{H}^1_{\text{ét}}(X, \text{PGL}_{n+1}) \) representing \( \mathcal{E} \).

**Lemma 3.4.1.**

The coboundary map

\[
\check{H}^1_{\text{ét}}(X, \text{PGL}_n(\infty)) \to H^2_{\text{ét}}(X, \mathbb{G}_m)
\]

induced (in the same manner as the map “d” in [59], Chap. IV, §2, Theorem 2.5, the proof of Step 3) by the short exact sequence

\[
1 \to \mathbb{G}_m \to \text{GL}^\infty_{n+1} \to \text{PGL}^\infty_{n+1} \to 1
\]

is the zero morphism.

**Proof.** Let \( \pi_1 : \text{GL}^\infty_{n+1} \to \mathbb{G}_m \), \( \pi_2 : \text{GL}^\infty_{n+1} \to \text{GL}_n \), and \( \pi_3 : \text{PGL}^\infty_{n+1} \to \text{GL}_n \) be the homomorphisms given, respectively, by

\[
\pi_1\left(\begin{pmatrix} a & \mathbf{a} \\ 0 & A \end{pmatrix}\right) = a, \quad \pi_2\left(\begin{pmatrix} a & \mathbf{a} \\ 0 & A \end{pmatrix}\right) = A, \quad \pi_3\left(\begin{pmatrix} a & \mathbf{a} \\ 0 & A \end{pmatrix}\right) = a^{-1} \cdot A
\]

for any \( a \in \mathbb{G}_m, \mathbf{a} \in \mathbb{A}^n \), and \( A \in \text{GL}_n \). These homomorphisms fit into the following morphism of short exact sequences:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{G}_m & \stackrel{\text{incl.}}{\longrightarrow} & \text{GL}^\infty_{n+1} & \stackrel{\text{proj.}}{\longrightarrow} & \text{PGL}^\infty_{n+1} & \longrightarrow & 1 \\
1 & \longrightarrow & \mathbb{G}_m & \stackrel{a \mapsto (a, \Delta(a))}{\longrightarrow} & \mathbb{G}_m \times \text{GL}_n & \stackrel{(a, A) \mapsto a^{-1}.A}{\longrightarrow} & \text{GL}_n & \longrightarrow & 1,
\end{array}
\]

where the upper horizontal sequence is (133) and \( \Delta \) denotes the diagonal embedding \( \mathbb{G}_m \hookrightarrow \text{GL}_n \). It induces a commutative square diagram

\[
\begin{array}{ccc}
\check{H}^1_{\text{ét}}(X, \text{PGL}^\infty_{n+1}) & \stackrel{\gamma_1}{\longrightarrow} & H^2_{\text{ét}}(X, \mathbb{G}_m) \\
\check{H}^1_{\text{ét}}(\pi_3) & \downarrow{id} & \downarrow{id} \\
\check{H}^1_{\text{ét}}(X, \text{GL}_n) & \stackrel{\gamma_2}{\longrightarrow} & H^2_{\text{ét}}(X, \mathbb{G}_m).
\end{array}
\]

Observe here that the lower horizontal sequence in (133) splits by taking the first projection \( \mathbb{G}_m \times \text{GL}_n \twoheadrightarrow \mathbb{G}_m \) as a split surjection. Hence, \( \gamma_2 = 0 \), which implies \( \gamma_1 = \gamma_2 \circ \check{H}^1_{\text{ét}}(\pi_3) = 0 \). This completes the proof of the lemma. \( \square \)
Lemma 3.4.2.
Let \( N \) be a positive integer and \( \mathcal{E}^\nabla \) a principal \( \text{PGL}_{n+1} \)-bundle on \( X^{(N)} \) with \( \gamma([F_{X/k}^n(\mathcal{E}^\nabla)]) = 0 \). Suppose further that \( p \nmid (n+1) \). Then, there exists a principal \( \text{GL}_{n+1} \)-bundle on \( X^{(N)} \) inducing \( \mathcal{E}^\nabla \) via projectivization.

Proof. First, we shall prove the following claim:

For each positive integer \( N \), the endomorphism \( f_{\mathbb{G}_m} \) of \( \tilde{H}^2_{\acute{e}t}(X, \mathbb{G}_m) \) induced by the \( N \)-th relative Frobenius morphism \( F_{\mathbb{G}_m/k}^N \) of \( \mathbb{G}_m \) restricts to a bijective endomorphism of the subset \( \text{Im}(\gamma) \subseteq \tilde{H}^2_{\acute{e}t}(X, \mathbb{G}_m) \).

Let us consider the following morphism of short exact sequences:

\[
\begin{align*}
1 & \longrightarrow \mu_{n+1} \longrightarrow \text{SL}_{n+1} \longrightarrow \text{PSL}_{n+1} \longrightarrow 1 \\
\downarrow \text{incl.} & \quad \downarrow \text{incl.} & \quad \downarrow \text{incl.} \\
1 & \longrightarrow \mathbb{G}_m \longrightarrow \text{GL}_{n+1} \longrightarrow \text{PGL}_{n+1} \longrightarrow 1,
\end{align*}
\]

where \( \mu_{n+1} \) denotes the group of \((n+1)\)-st roots of unity. The right-hand vertical arrow \( \text{PSL}_{n+1} \to \text{PGL}_{n+1} \) is an isomorphism because of the assumption \( p \nmid n+1 \). This implies that \( \gamma \) can be decomposed as

\[
\tilde{H}^1_{\acute{e}t}(X, \text{PGL}_{n+1}) (\cong \tilde{H}^1_{\acute{e}t}(X, \text{PSL}_{n+1})) \xrightarrow{\gamma_1} \tilde{H}^2_{\acute{e}t}(X, \mu_{n+1}) \xrightarrow{\gamma_2} \tilde{H}^2_{\acute{e}t}(X, \mathbb{G}_m),
\]

where \( \gamma_1 \) denotes the coboundary map induced by the upper horizontal arrow in (140) and \( \gamma_2 \) denotes the map induced by the natural inclusion \( \mu_{n+1} \hookrightarrow \mathbb{G}_m \). In particular, the inclusion relation \( \text{Im}(\gamma) \subseteq \text{Im}(\gamma_2) \) holds. Since \( \tilde{H}^2_{\acute{e}t}(X, \mu_{n+1}) \) is finite (cf. [59], Chap. III, §2, Corollary 2.10, and Chap. VI, §5, Corollary 5.5), the set \( \text{Im}(\gamma_2) \), as we as \( \text{Im}(\gamma) \), turns out to be finite. Next, note that the diagram (140) is compatible with the \( N \)-th relative Frobenius morphisms on the various algebraic groups appearing there. It follows that \( f_{\mathbb{G}_m}(\text{Im}(\gamma)) \subseteq \text{Im}(\gamma) \). Moreover, if \( f_{\mu_{n+1}} \) denotes the endomorphism of \( \tilde{H}^2_{\acute{e}t}(X, \mu_{n+1}) \) induced by the \( N \)-th relative Frobenius morphism \( F_{\mu_{n+1}/k}^N \) of \( \mu_{n+1} \), then the surjection \( \tilde{H}^2_{\acute{e}t}(X, \mu_{n+1}) \to \text{Im}(\gamma_2) \) is compatible with \( f_{\mu_{n+1}} \) and \( f_{\mathbb{G}_m}|_{\text{Im}(\gamma_2)} \). Since \( F_{\mu_{n+1}/k}^N \) is an an isomorphism, \( f_{\mu_{n+1}} \) is bijective. This implies from the finiteness of \( \text{Im}(\gamma_2) \) that \( f_{\mathbb{G}_m}|_{\text{Im}(\gamma_2)} \) is bijective. Thus, the endomap \( f_{\mathbb{G}_m}|_{\text{Im}(\gamma)} \) of \( \text{Im}(\gamma) \) is bijective, so this completes the proof of the claim.

Now, let us back to the proof of the lemma. Let \( \mathcal{E}^\nabla \) be a principal \( \text{PGL}_{n+1} \)-bundle on \( X^{(N)} \) with \( \gamma([F_{X/k}^n(\mathcal{E}^\nabla)]) = 0 \). If \( \mathcal{E}^\nabla F \) denotes the principal \( \text{PGL}_{n+1} \)-bundle on \( X \) corresponding to \( \mathcal{E}^\nabla \) via \( \text{id}_X \times F_k^N : X^{(N)} \cong X \), then \( f_{\mathbb{G}_m}(\gamma([\mathcal{E}^\nabla])) = \gamma([F_{X/k}^n(\mathcal{E}^\nabla)]) = 0 \). It follows from the above claim that the equality \( \gamma(\mathcal{E}^\nabla F) = 0 \) holds. Since the sequence

\[
\tilde{H}^1_{\acute{e}t}(X, \text{GL}_{n+1}) \longrightarrow \tilde{H}^1_{\acute{e}t}(X, \text{PGL}_{n+1}) \xrightarrow{\gamma} \tilde{H}^2_{\acute{e}t}(X, \mathbb{G}_m)
\]

induced by (134) is exact (cf. [59], Chap. IV, §2, Theorem 2.5, Step 3), there exists a \( \text{GL}_{n+1} \)-bundle \( F \) on \( X \) inducing \( \mathcal{E}^\nabla F \) via projectivization. Hence, the \( \text{GL}_{n+1} \)-bundle on \( X^{(N)} \) corresponding to this \( \text{GL}_{n+1} \)-bundle via \( \text{id}_X \times F_k^N \) satisfies the required conditions. This completes the proof of the assertion. \( \square \)

Theorem 3.4.3.
Let \( X \) be a smooth variety over \( k \) of dimension \( n \). Then, the following assertions hold:
(i) Suppose that $p \nmid (n+1)$ and $X$ is quasi-projective over $k$. Then, for each $N \in \mathbb{Z}_{>0}$, the map $\zeta_N^{\circ \Rightarrow } : \mathcal{D}_{X,N}^{\text{aza}} \to F^N \cdot \text{Ind}_X$ is bijective.

(ii) For each $N \in \mathbb{Z}_{>0} \cup \{\infty\}$, the map $\zeta_N^{\circ \Rightarrow } : \mathcal{D}_{X,N}^{\text{aza}} \to F^N \cdot \text{Ind}_X$ is bijective.

Proof. First, we shall consider the surjectivity of $\zeta_N^{\circ \Rightarrow }$ asserted in (i). Let $\mathcal{E}^{\bullet} := (\mathcal{E}^\nabla, \mathcal{E}_{\text{red}})$ be an $F^N$-indigenous structure on $X$. Since $F_X^{(N)}(\mathcal{E}^\nabla)$ has a $\text{PGL}_{n+1}$-reduction $\mathcal{E}_{\text{red}}$, the element $[F_X^{(N)}(\mathcal{E}^\nabla)]$ of $\hat{H}^1_{\text{et}}(X, \text{PGL}_{n+1})$ represented by $F_X^{(N)}(\mathcal{E}^\nabla)$ comes from $\hat{H}^1_{\text{et}}(X, \text{PGL}_{n+1})$. Hence, the result of Lemma 3.4.2 implies that $\gamma([F_X^{(N)}(\mathcal{E}^\nabla)]) = 0$. It follows from Lemma 3.4.2 that there exists a vector bundle $\mathcal{V}^\nabla$ on $X^{(N)}$ inducing $\mathcal{E}^\nabla$ via projectivization. Write $\mathcal{V} := F_X^{(N)}(\mathcal{V}^\nabla)$ and $\nabla^\text{can}(N-1) := \nabla^\text{can}(N-1)$. According to the discussion in Remark 2.2.2, the triple $\mathcal{V}^{\circ \Rightarrow } := (\mathcal{V}, \nabla^\text{can}(N-1), \mathcal{N})$ forms a dormant indigenous $\mathcal{D}_X^{(N-1)}$-module, where $\mathcal{N}$ denotes the line bundle determined by the $\text{PGL}_{n+1}$-reduction $\mathcal{E}_{\text{red}}$ of $F_X^{(N)}(\mathcal{E}^\nabla)$. One verifies that $\mathcal{V}^{\circ \Rightarrow } \cong \mathcal{E}^{\bullet}$, and this completes the proof of the surjectivity of $\zeta_N^{\circ \Rightarrow }$.

Next, we shall prove the injectivity of $\zeta_N^{\circ \Rightarrow }$. Let $\mathcal{V}^{\circ \Rightarrow } := (\mathcal{V}_1, \nabla_{\mathcal{V}_{1}}, \mathcal{N}_1)$ be dormant indigenous $\mathcal{D}_X^{(N-1)}$-modules. Also, suppose that there exists an isomorphism $\alpha : \mathcal{V}_1^{\circ \Rightarrow } \cong \mathcal{V}_2^{\circ \Rightarrow }$. Then, $\alpha$ induces an isomorphism between the $\mathbb{P}^m$-bundles associated with $\text{Sol}(\nabla_{\mathcal{V}_1})$ and $\text{Sol}(\nabla_{\mathcal{V}_2})$. It follows (cf. [27], Chap. II, § 7, Exercise 7.9) that we find a line bundle $\mathcal{L}$ on $X^{(N)}$ and an isomorphism $\text{Sol}(\nabla_{\mathcal{V}_1}) \sim \mathcal{L} \otimes \text{Sol}(\nabla_{\mathcal{V}_2})$ inducing this isomorphism of $\mathbb{P}^m$-bundles. Hence, in order to complete the proof, we can assume (after twisting $\mathcal{V}_1^{\circ \Rightarrow }$ by $(F_X^{(N)}(\mathcal{L}), \nabla^\text{can}(N-1))$) that there exists an $\mathcal{O}_{X^{(N)}}$-linear isomorphism $\alpha^\nabla : \text{Sol}(\nabla_{\mathcal{V}_1}) \sim \text{Sol}(\nabla_{\mathcal{V}_2})$ inducing $\alpha$ via projectivization. Let $\alpha^\nabla : (\mathcal{V}_1, \nabla_{\mathcal{V}_1}) \cong (\mathcal{V}_2, \nabla_{\mathcal{V}_2})$ denote the isomorphism of $\mathcal{D}_X^{(N-1)}$-modules corresponding to $\alpha^\nabla$ via the equivalence of categories (113). Since $\alpha$ preserves the respective $\text{PGL}_{n+1}$-reductions in $\mathcal{V}_1^{\circ \Rightarrow }$ and $\mathcal{V}_2^{\circ \Rightarrow }$, the equality $\alpha^\nabla(\mathcal{N}_1) = \mathcal{N}_2$ holds. Thus, $\alpha^\nabla$ specifies an isomorphism $\mathcal{V}_1^{\circ \Rightarrow } \cong \mathcal{V}_2^{\circ \Rightarrow }$ of indigenous $\mathcal{D}_X^{(N-1)}$-modules. This completes the proof of the injectivity of $\zeta_N^{\circ \Rightarrow }$, and consequently, completes the proof of assertion (i).

Next, we shall prove assertion (ii). In the following, we only consider the case of $N \neq \infty$ for convenience of discussion. The case of $N = \infty$ can be proved by applying inductively the proofs for $N \neq \infty$. Now, let us consider the surjectivity of $\zeta_N^{\circ \Rightarrow } = \bullet$ (with $N \neq \infty$). Write $\text{GL}_{n+1}^\mathcal{A} := \text{GL}_{n+1} \times \text{PGL}_{n+1}$ $\text{PGL}_{n+1}^\mathcal{A} (\subseteq \text{GL}_{n+1})$. This algebraic group fits into the following natural short exact sequence:

$$1 \longrightarrow \mathfrak{g}_m \longrightarrow \text{GL}_{n+1}^\mathcal{A} \longrightarrow \text{PGL}_{n+1}^\mathcal{A} \longrightarrow 1.$$  

(143)

The homomorphism $\text{PGL}_{n+1}^\mathcal{A} \to \text{GL}_{n+1}^\mathcal{A}$ given by $\begin{pmatrix} a & 0 \\ \frac{1}{a} & \frac{1}{a} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ a^{-1} & a^{-1} \end{pmatrix}$ specifies a split injection $\sigma$ of this sequence with $\sigma(\text{PGL}_{n+1}^\mathcal{A}) \subseteq \text{GL}_{n+1}^\mathcal{A}$, where $\text{GL}_{n+1}^\mathcal{A} := \text{GL}_{n+1} \times \text{PGL}_{n+1}$ $\text{PGL}_{n+1}^\mathcal{A}$. Now, let $\mathcal{E}^{\bullet} := (\mathcal{E}^\nabla, \mathcal{E}_{\text{red}})$ be an $F^N$-affine-indigenous structure on $X$. The principal $\text{PGL}_{n+1}^\mathcal{A}$-bundle $\mathcal{E}^\nabla$ determines, via change of structure group by $\sigma$, a vector bundle $\mathcal{V}^\nabla$ on $X^{(N)}$ together with a surjection $\delta^\nabla : \mathcal{V}^\nabla \to N^\nabla$ onto a line bundle $N^\nabla$. Here, we shall
write \( \mathcal{V} := F_{X/k}^{(N)*} (\mathcal{V}^\nabla) \), \( \delta := F_{X/k}^{(N)*} (\delta^\nabla) \), and \( \nabla_{\mathcal{V}}^\can(N-1) := \nabla_{F_{X/k}^{(N)*} (\mathcal{V})}^\can(N-1) \) (cf. [112]). The \( GL_{n+1}^A \)-
reduction \( E_{\text{red}} \times_{PGL_{n+1}^A,\sigma} GL_{n+1}^A \) of \( F_{X/k}^{(N)*} (\mathcal{E}^\nabla) \times_{PGL_{n+1}^A,\sigma} GL_{n+1}^A \) corresponds to a line subbundle \( \mathcal{N} \) of \( \mathcal{V} \) such that the composite \( \mathcal{N} \hookrightarrow \mathcal{V} \xrightarrow{\delta} F_{X/k}^{(N)*} (\mathcal{N}^\nabla) \) is an isomorphism. By passing to this composite, we identify \( \mathcal{N} \) with \( F_{X/k}^{(N)*} (\mathcal{N}^\nabla) \) and regard \( \delta \) as a left inverse of the natural inclusion \( \mathcal{N} \to \mathcal{V} \). It is verified that the quadruple \( \mathcal{A}^\nabla \mathcal{V} := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N}, \delta) \) forms a dormant affine-indigenous \( D_{X}^{(N-1)} \)-modules with \( \mathcal{A}^\nabla \mathcal{V} // \mathcal{A} \cong \mathcal{E} // \mathcal{A} \). This implies the desired surjectivity of \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} \).

The injectivity of \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} \) follows from an argument similar to the proof of the injectivity of \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} \) discussed above. This completes the proof of the theorem. \( \square \)

By the above theorem, we have the following corollaries.

**Corollary 3.4.4.**

Let \( X \) be a smooth variety over \( k \) of dimension \( n \) and \((\mathcal{E}^\nabla, \mathcal{E}_{\text{red}})\) an \( F^N \)-indigenous (resp., \( F^N \)-affine-indigenous) structure on \( X \), where \( N \in \mathbb{Z}_{>0} \sqcup \{ \infty \} \). In the non-resp’d assertion, we suppose that \( N \neq \infty, p \nmid (n+1) \), and \( X \) is quasi-projective over \( k \). Then, for any other \( F^N \)-indigenous (resp., \( F^N \)-affine-indigenous) structure \((\mathcal{E}^\nabla, \mathcal{E}'_{\text{red}})\) on \( X \), there exists an isomorphism \( F_{X/k}^{(N)*} (\mathcal{E}^\nabla) \cong F_{X/k}^{(N)*} (\mathcal{E}'^\nabla) \) of principal \( PGL_{n+1}^A \)-bundles (resp., principal \( PGL_{n+1}^A \)-bundles) that is compatible with the respective \( PGL_{n+1}^\infty \)-reductions (resp., \( PGL_{n+1}^{A,\infty} \)-reductions) \( \mathcal{E}_{\text{red}} \) and \( \mathcal{E}'_{\text{red}} \). Moreover, both \( \mathcal{E}^\nabla \) and \( \mathcal{E}_{\text{red}} \) may be trivialized locally in the Zariski topology.

**Proof.** The assertion follows from the surjectivity of \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} \) (resp., \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} \)) and Lemma 2.5.1 \( \square \)

**Corollary 3.4.5.**

Let \( X \) be a smooth variety over \( k \) of dimension \( n \) and let \( N \in \mathbb{Z}_{>0} \sqcup \{ \infty \} \). Then, the following assertions hold:

(i) Suppose that \( p \nmid (n+1) \) and \( X \) is quasi-projective over \( k \). Then, the map \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} : F^N \text{-Proj}_X \to F^N \text{-Ind}_X \) is bijective.

(ii) The map \( \kappa^\nabla \mathcal{O}_{N} // \mathcal{A} : F^N \text{-Aff}_X \to F^N \text{-Ind}_X \) is bijective.

**Proof.** Assertions (i) and (ii) follow from Theorem 3.4.3 the commutativities of the diagrams in [131]. Notice that the case of \( N = \infty \) is reduced to the case of \( N < \infty \) by using the identifications [20] and [57]. \( \square \)

### 3.5. Rigidification by theta characteristics.

We shall consider rigidification of dormant indigenous and affine-indigenous \( D_{X}^{(N-1)} \)-modules carried out by fixing their determinants by means of a (generalized) theta characteristic. By a **theta characteristic** of \( X \), we shall mean a line bundle \( \Theta \) on \( X \) together with an isomorphism \( \Theta \otimes (n+1) \cong \omega_X \). This notion is well-known and has been studied in the case where the underlying variety \( X \) is a curve. In fact, there always exists a theta characteristic of any smooth curve. Also, as we will discuss later, projective spaces, affine spaces, and abelian varieties are other typical examples admitting a theta characteristic. In the following, the notion of a theta
characteristic will be generalized so that we can apply the same argument to a wider class of varieties.

Let us fix an element \( N \) of \( \mathbb{Z}_{>0} \cup \{\infty\} \).

**Definition 3.5.1.**

An \( F^N \)-theta characteristic of \( X \) is a pair

\[
\Theta := (\Theta, \nabla),
\]

where \( \Theta \) denotes a line bundle on \( X \) and \( \nabla \) denotes a \( D_{X}^{(N-1)} \)-action on \( \omega_X^\vee \otimes \Theta^{\otimes(n+1)} \) with vanishing \( p-(N-1) \)-curvature.

**Remark 3.5.2.**

Let \( N' \) be an element of \( \mathbb{Z}_{>0} \cup \{\infty\} \) with \( N' \geq N \), and suppose that we are given an \( F^{N'} \)-theta characteristic \( \Theta := (\Theta, \nabla) \) of \( X \). Then, the \( N \)-th truncation \( \Theta, \nabla|^{(N)} \) (cf. (69)) forms an \( F^{N} \)-theta characteristic of \( X \). In particular, if \( X \) admits an \( F^{\infty} \)-theta characteristic, then there exists an \( F^{N} \)-theta characteristic of \( X \) for any positive integer \( N \). By abuse of notation, we also use the notation \( \Theta \) to denote \( (\Theta, \nabla|^{(N)}) \).

**Remark 3.5.3.**

Suppose that \( X \) admits a theta characteristic \( \Theta \). Then, for any \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \), the pair \( (\Theta, \nabla^{\text{triv}(N-1)}) \) (cf. (68)) forms an \( F^{N} \)-theta characteristic on \( X \), where \( \omega_X^\vee \otimes \Theta^{\otimes(n+1)} \) is identified with \( \Omega_X \) by means of the fixed isomorphism \( \Theta^{\otimes(n+1)} \rightarrow \Omega_X \). In this way, each theta characteristic specifies an \( F^N \)-theta characteristic. In particular, any smooth variety admitting a theta characteristic admits an \( F^{\infty} \)-theta characteristic. More generally, if \( \omega_X \) is \( (n+1) \)-divisible in the Neron-Severi group of \( X \), then one can verifies that \( X \) admits an \( F^{\infty} \)-theta characteristic because of the equivalence of categories (117) and the \( p \)-divisibility of \( \text{Pic}^0(X) \), i.e., the identity component of the Picard scheme.

**Remark 3.5.4.**

Suppose that \( p \nmid (n+1) \). Then, any smooth variety \( X \) of dimension \( n \) admits an \( F^N \)-theta characteristic for any positive integer \( N \) even when \( X \) has no theta characteristics. In what follows, we shall show this fact by constructing an explicit example. Because of the assumption \( p \nmid (n+1) \iff p^N \nmid (n+1) \), we can find a pair of integers \( (a, b) \) with \( a(n+1) - 1 = bpN \). Then, we obtain a sequence of natural isomorphisms

\[
(145) \quad \omega_X^\vee \otimes (\omega_X^{\otimes a})^{\otimes(n+1)} \sim \omega_X^{a(n+1)-1} \left(= \omega_X^{\otimes bpN} \right) \sim F_X^{(N)*}(\omega_X^{\otimes b}).
\]

Let \( \nabla_{a,b} \) denote the \( D_{F_X^{(N)*}(\omega_X^{\otimes a})}^{(N-1)} \)-action on \( \omega_X^\vee \otimes (\omega_X^{\otimes a})^{\otimes(n+1)} \) corresponding, via this composite isomorphism, to \( \nabla_{F_X^{(N)*}(\omega_X^{\otimes a})}^{\text{can}(N-1)} \) (cf. (112)). Then, the resulting pair

\[
(146) \quad (\omega_X^{\otimes a}, \nabla_{a,b})
\]

forms an \( F^N \)-theta characteristic of \( X \), as desired.
Now, suppose that we are given an $F^N$-theta characteristic $\Theta := (\Theta, \nabla)$ of $X$. For each $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, we shall write

$$\uparrow \mathcal{D}^{\text{zaz...}}_{X,N,\Theta}$$

for the subset of $\mathcal{D}^{\text{zaz...}}_{X,N}$ consisting of isomorphism classes of dormant indigenous $\mathcal{D}_X^{(N-1)}$-modules $\mathcal{V}^\diamond := (\mathcal{V}, \nabla, \mathcal{N})$ satisfying the two conditions (a) and (b) described as follows:

(a) $\mathcal{V} = \mathcal{D}_X^{(N-1)} \otimes \Theta$ and $\mathcal{N} = \Theta \left( = \mathcal{D}_X^{(N-1)} \otimes \Theta \subseteq \mathcal{D}_X^{(N-1)} \otimes \Theta \right)$;

(b) The determinant of $\nabla$ corresponds to $\nabla$ via the isomorphism $\det(\mathcal{V}) \cong \omega_X^\diamond \otimes \mathcal{N}^{\otimes (n+1)}$

If we take an $F^N$-theta characteristic of the form $(\Theta, \nabla^{\text{triv}(N-1)})$ for some theta characteristic $\Theta$, then, for simplicity, we write

$$\uparrow \mathcal{D}^{\text{zaz...}}_{X,N,\Theta} := \uparrow \mathcal{D}^{\text{zaz...}}_{X,N,(\Theta, \nabla^{\text{triv}(N-1)})}$$

Also, for each $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, we shall write

$$\uparrow \mathcal{A}^{\text{zaz...}}_{X,N}$$

for the subset of $\mathcal{A}^{\text{zaz...}}_{X,N}$ consisting of isomorphism classes of dormant affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules $\mathcal{A}^{\text{zaz...}}_{X,N}$ satisfying the two conditions (c) and (d) described as follows:

(c) $\mathcal{V} = \mathcal{D}_X^{(N-1)} \otimes \mathcal{O}_X$ and $\mathcal{N} = \mathcal{O}_X \left( = \mathcal{D}_X^{(N-1)} \otimes \mathcal{O}_X \subseteq \mathcal{D}_X^{(N-1)} \otimes \mathcal{O}_X \right)$;

(d) The $\mathcal{D}_X^{(N-1)}$-action on $\mathcal{O}_X$ induced from $\nabla$ via the surjection $\delta : \mathcal{V} \rightarrow \mathcal{N}$ ($= \mathcal{O}_X$) coincides with $\nabla^{\text{triv}(N-1)}$.

By restricting $\zeta^\diamond = \bowtie$ and $\zeta^\diamond = \bowtie$ (resp., $\mathcal{A}^{\text{zaz...}}_{X,N,\Theta}$ and $\mathcal{A}^{\text{zaz...}}_{X,N,\Theta}$), we obtain maps

$$\uparrow \zeta^\diamond = \bowtie : \uparrow \mathcal{D}^{\text{zaz...}}_{X,N,\Theta} \rightarrow F^N \text{-Ind}_X \quad \text{and} \quad \uparrow \zeta^\diamond = \bowtie : \uparrow \mathcal{A}^{\text{zaz...}}_{X,N,\Theta} \rightarrow F^N \text{-Proj}_X$$

(respectively $\uparrow \mathcal{A}^{\text{zaz...}}_{X,N,\Theta} : \uparrow \mathcal{A}^{\text{zaz...}}_{X,N,\Theta} \rightarrow F^N \text{-Aff}_X$) respectively. Moreover, the commutative diagrams in (127) induce the following commutative diagrams:

Proposition 3.5.5.
Assume that $\Gamma(X, \mathcal{O}_X) = k$. (For example, this assumption is satisfied if $X$ is proper over $k$.) In the following non-resp’d assertion, suppose further that we are given an $F^N$-theta characteristic $\Theta$ of $X$. Then, the following assertions hold:
(i) The set $\dD_{X, N}^{2x\ldots}$ (resp., $\dA_{X, N}^{2x\ldots}$) may be identified with the set (not the set of isomorphism classes) of dormant indigenous (resp., dormant affine-indigenous) $D_X^{(N-1)}$-modules satisfying the conditions (a) and (b) (resp., (c) and (d)) described before. That is to say, there exists exactly one dormant indigenous (resp., dormant affine-indigenous) $D_X^{(N-1)}$-module representing each element of $\dD_{X, N}^{2x\ldots}$ (resp., $\dA_{X, N}^{2x\ldots}$).

(ii) The assignment $V^{\diamond} \mapsto \{V^{\diamond}(i)\}_{i \in \mathbb{Z}_{>0}}$ (resp., $\Lambda V^{\diamond} \mapsto \{\Lambda V^{\diamond}(i)\}_{i \in \mathbb{Z}_{>0}}$) defines a bijection of sets

$$\tag{152} \dD_{X, N}^{2x\ldots} \cong \lim_{\rightarrow \in \mathbb{Z}_{>0}} \dD_{X, N}^{2x\ldots} \left(\text{resp., } \dA_{X, N}^{2x\ldots} \cong \lim_{\rightarrow \in \mathbb{Z}_{>0}} \dA_{X, N}^{2x\ldots}\right).$$

Proof. Assertion (i) follows directly from Proposition 2.3.4. Assertion (ii) follows from assertion (i) together with the identification $\lim_{\rightarrow \in \mathbb{Z}_{>0}} D_X^{(N-1)} \cong D_X^{(\infty)}$ (cf. [17]).

3.6. Comparison via rigidification.

In what follows, we prove (cf. Theorem 3.6.4) that the set of rigidified dormant indigenous (resp., affine-indigenous) $D_X^{(N-1)}$-modules correspond bijectively to the set of isomorphism classes of $F^N$-indigenous (resp., $F^N$-affine-indigenous) structures.

Proposition 3.6.1.

(i) Suppose that $p \nmid (n+1)$, $N \in \mathbb{Z}_{>0}$, and $X$ is quasi-projective over $k$. Then, for each $F^N$-theta characteristic $\Theta := (\Theta, \nabla)$ of $X$, the composite

$$\zeta_{N, \Theta}^{\nabla \mapsto (-)} : \dD_{X, N}^{2x\ldots} \xrightarrow{\text{incl}} \dD_{X, N}^{2x\ldots} \xrightarrow{\text{quot}} \dD_{X, N}^{2x\ldots}$$

is injective. If, moreover, $X$ is projective over $k$, or more generally, satisfies the equality $\Gamma(X, \mathcal{O}_X) = k$, then this composite is bijective.

(ii) For any $N \in \mathbb{Z}_{>0} \cup \{\infty\}$, the composite

$$\zeta_{N, \Theta}^{\nabla \mapsto (-)} : \dA_{X, N}^{2x\ldots} \xrightarrow{\text{incl}} \dA_{X, N}^{2x\ldots} \xrightarrow{\text{quot}} \dA_{X, N}^{2x\ldots}$$

is injective.

Proof. First, we shall prove the former assertion of (i), i.e., the injectivity of $\zeta_{N, \Theta}^{\nabla \mapsto (-)}$. Let $\mathcal{V}_i := (\mathcal{V}_i, \nabla_{\mathcal{V}_i}, \mathcal{N}_i)$ ($i = 1, 2$) be dormant indigenous $D_X^{(N-1)}$-modules classified by $\dD_{X, N}^{2x\ldots}$ (hence $\mathcal{V}_i = D_X^{(N-1)} \otimes \Theta$ and $\mathcal{N}_i = \Theta$) such that $[\mathcal{V}_1^{\nabla}] = [\mathcal{V}_2^{\nabla}]$. By the definition of $\otimes\mathcal{G}_{\alpha}$, there exists a collection $\{\mathcal{L}, \Theta, \nabla, \eta\}$ consisting of an invertible $D_X^{(N-1)}$-module $(\mathcal{L}, \nabla, \eta)$ with vanishing $p$-$(N-1)$-curvature and an isomorphism $\eta : \mathcal{V}_1^{\nabla} \cong (\mathcal{V}_2^{\nabla})_{(\mathcal{L}, \nabla, \eta)}$. This isomorphism $\eta$ restricts to an isomorphism $\mathcal{N}_1 := \Theta \cong \mathcal{N}_2 := \Theta \otimes \mathcal{L}$, which implies that $\mathcal{L} \cong \mathcal{O}_X$. According to the equivalence of categories $\{113\}$, $\text{Sol}((\nabla))$ forms a line bundle on $X^{(N)}$ with $\text{Sol}((\nabla)) \cong \mathcal{O}_X$. It follows from Lemma 3.6.2 below that

$$\tag{155} \text{Sol}((\nabla)) \otimes p^{\alpha N} \cong \mathcal{O}_X^{(N)}.$$
On the other hand, the determinant of $\eta$ gives an isomorphism
\begin{equation}
\omega^\vee_X \otimes \Theta^{(n+1)} \cong \det(\Lambda_1) \sim \omega^\vee_X \otimes \Theta^{(n+1)} \otimes \mathcal{L}^{\otimes (n+1)} \cong \det(\Lambda_2 \otimes \mathcal{L})
\end{equation}
compatible with the respective $\mathcal{D}_X^{(N-1)}$-actions $\nabla$ and $\nabla \otimes \nabla^{\otimes (n+1)}_L$, where $\nabla^{\otimes (n+1)}_L$ denotes the $(n+1)$-st tensor product of $\nabla_L$. This implies that $(\mathcal{L}^{\otimes (n+1)} \otimes \nabla^{\otimes (n+1)}_L) \cong (\mathcal{O}_X \otimes \nabla^{\text{triv}(N-1)}_L)$. Since the equivalence of categories [113] is compatible with taking tensor products, we see
\begin{equation}
\text{Sol}(\nabla_L^{\otimes (n+1)}) \cong \text{Sol}(\nabla^{\text{triv}(N-1)}_L) \cong \mathcal{O}_X^{(N)}.
\end{equation}
It follows from [155], [157], and the assumption $(p^{nN}, n+1) = 1$ that $\text{Sol}(\nabla_L) \cong \mathcal{O}_X^{(N)}$, so $(\mathcal{L}, \nabla_L)$ is isomorphic to the trivial $\mathcal{D}_X^{(N-1)}$-module $(\mathcal{O}_X, \nabla^{\text{triv}(N-1)}_L)$. Consequently, $\mathcal{V}_1$ turns out to be isomorphic to $\mathcal{V}_1^{\otimes}$. This completes the proof of assertion (i).

Next, we shall consider the latter assertion of (i), i.e., the bijectivity of $\zeta^\dagger_{N \otimes \mathcal{D}}$ under the assumption $\Gamma(X, \mathcal{O}_X) = k$. Let $\mathcal{E}^\bullet := (\mathcal{E}^\vee, \mathcal{E}_{\text{red}})$ be an $F^N$-indigenous structure on $X$. By Theorem 3.4.3, there exists a dormant indigenous $\mathcal{D}_X^{(N-1)}$-module $\mathcal{V}^\otimes := (\mathcal{V}, \nabla_V, \mathcal{N})$ with $\mathcal{V}^{\otimes} \cong \mathcal{E}^\bullet$. Consider the following sequence of natural isomorphisms:
\begin{equation}
\omega^\vee_X \otimes \Theta^{(n+1)} \otimes \det(\Lambda)^\vee \cong \omega^\vee_X \otimes \Theta^{(n+1)} \otimes (\omega^\vee_X \otimes \mathcal{N}^{\otimes (n+1)})^\vee \sim \mathcal{L}^{\otimes (n+1)},
\end{equation}
where $\mathcal{L} := \Theta \otimes \mathcal{N}^\vee$ and the first isomorphism follows from [80]. By passing to the composite of them, we obtain a $\mathcal{D}_X^{(N-1)}$-action $\nabla_{\mathcal{L}^{\otimes (n+1)}}$ on $\mathcal{L}^{\otimes (n+1)}$ corresponding to the tensor product of $\nabla$ and the dual of the determinant of $\nabla_V$. It follows from Lemma 3.6.3 below that there exists a $\mathcal{D}_X^{(N-1)}$-action $\nabla_L$ with vanishing $p^N(N-1)$-curvature whose $(n+1)$-st tensor product coincides with $\nabla_{\mathcal{L}^{\otimes (n+1)}}$. Thus, we obtain a dormant indigenous $\mathcal{D}_X^{(N-1)}$-module $\mathcal{V}^{\otimes}_{(\mathcal{L}, \nabla_L)}$ (cf. [92]). If we write $(\mathcal{V}, \nabla_V, \mathcal{N}') := \mathcal{V}^{\otimes}_{(\mathcal{L}, \nabla_L)}$, then the line bundle $\mathcal{N}' := \mathcal{N} \otimes (\Theta \otimes \mathcal{N}^\vee)$ can be identified with $\Theta$ via the natural isomorphism. We also identify $\mathcal{V}'$ with $\mathcal{D}_X^{(N-1)} \otimes \Theta$ via the isomorphism $\mathcal{E}^{\otimes}_{(\mathcal{L}, \nabla_L)}$ (cf. [81]). Under these identifications, $\mathcal{V}^{\otimes}_{(\mathcal{L}, \nabla_L)}$ satisfies the condition (a) described before. Moreover, it follows from the construction of $\nabla_L$ that $\mathcal{V}^{\otimes}_{(\mathcal{L}, \nabla_L)}$ also satisfies the condition (b). Since $\zeta_N^{\otimes \mathcal{L}}(\mathcal{V}^{\otimes}_{(\mathcal{L}, \nabla_L)}) \cong \zeta_N^{\otimes \mathcal{E}^\bullet}(\mathcal{V}^{\otimes}) \cong \mathcal{E}^\bullet$, the desired surjectivity has been proved. Thus, we finish the proof of assertion (i).

Assertion (ii) follows from an argument similar to the argument in the proof of Theorem 3.4.3 (ii).

The following two lemmas were used in the above proposition.

Lemma 3.6.2 (cf. [22], §6.5).
Let $f : X \to X'$ be a finite and faithfully flat morphism between varieties over $k$ and $\mathcal{M}$ a line bundle on $X'$. (In the context of this subsection, we consider the case where $X' = X^{(N)}$ and $f = F_{X/k}^{(N)}$ for a positive integer $N$.) If $f^*(\mathcal{M}) \cong \mathcal{O}_X$, then we have $\mathcal{M}^{\otimes d} \cong \mathcal{O}_{X'}$, where $d$ denotes the degree of $f$.

Proof. Since $f$ is finite and faithfully flat of degree $d$, the direct image $f_*(\mathcal{O}_X)$ of $\mathcal{O}_X$ forms a rank $d$ vector bundle on $X'$. Hence, we have the following sequence of isomorphisms:
\begin{equation}
\det(f_*(\mathcal{O}_X)) \sim \det(f_*(f^*(\mathcal{M}))) \sim \det(\mathcal{M} \otimes f_*(\mathcal{O}_X)) \sim \mathcal{M}^{\otimes d} \otimes \det(f_*(\mathcal{O}_X)),
\end{equation}
where the first arrow follows from the assumption, i.e., \( f^*(\mathcal{M}) \cong \mathcal{O}_X \), and the second arrow follows from the projection formula. This implies that \( \mathcal{M}_\otimes \cong \mathcal{O}_X \), as desired. \( \square \)

**Lemma 3.6.3.**

Suppose that \( X \) is projective over \( k \), or more generally, satisfies the equality \( \Gamma(X, \mathcal{O}_X) = k \). Let \( N \) be a positive integer and \( l \) a positive integer prime to \( p \). Also, let \( \mathcal{L} \) be a line bundle on \( X \) and \( \nabla_{\mathcal{L}_\otimes} \) a \( D_X^{(N-1)} \)-action on \( \mathcal{L}_\otimes \) with vanishing \( p \)-(\( N-1 \))-curvature. Then, there exists uniquely a \( D_X^{(N-1)} \)-action \( \nabla_{\mathcal{L}} \) on \( \mathcal{L} \) with vanishing \( p \)-(\( N-1 \))-curvature whose \( l \)-th tensor product \( \nabla_{\mathcal{L}}^\otimes \) coincides with \( \nabla_{\mathcal{L}_\otimes} \).

**Proof.** Let us consider the Kummer exact sequence

\[
\begin{align*}
1 & \longrightarrow \mu_l \xrightarrow{\text{incl.}} \mathbb{G}_m \xrightarrow{(-)^l} \mathbb{G}_m \xrightarrow{} 1,
\end{align*}
\]

where \( \mu_l \) denotes the group of \( l \)-th roots of unity. Since the equality \( \Gamma(X, \mathbb{G}_m) = k^\times \) holds by assumption, the above sequence remains exact after applying the functor \( \Gamma(X, -) \). Thus, the Kummer sequence induces the following exact sequence:

\[
\begin{align*}
0 & \longrightarrow \check{H}_\text{et}^1(X, \mu_l) \longrightarrow \check{H}_\text{et}^1(X, \mathbb{G}_m) \longrightarrow \check{H}_\text{et}^1(X, \mathbb{G}_m) \longrightarrow \check{H}_\text{et}^2(X, \mu_l).
\end{align*}
\]

The \( N \)-th relative Frobenius morphisms of \( \mu_l, \mathbb{G}_m \) are compatible with the morphisms in (160) and hence induce the following endomorphism of the sequence (161):

\[
\begin{align*}
0 & \longrightarrow \check{H}_\text{et}^1(X, \mu_l) \longrightarrow \check{H}_\text{et}^1(X, \mathbb{G}_m) \xrightarrow{\xi^*} \check{H}_\text{et}^1(X, \mathbb{G}_m) \longrightarrow \check{H}_\text{et}^2(X, \mu_l).
\end{align*}
\]

where both the leftmost and rightmost vertical arrows are isomorphisms because of the assumption \( (p, l) = 1 \). Denote by \([\mathcal{L}], [\mathcal{L}_\otimes], \) and \([\text{Sol}(\nabla_{\mathcal{L}_\otimes})] \) the elements of \( \check{H}_\text{et}^1(X, \mathbb{G}_m) \) represented by \( \mathcal{L}, \mathcal{L}_\otimes, \) and \( \text{Sol}(\nabla_{\mathcal{L}_\otimes}) \) respectively. By definition, the equalities \( \xi^{-1}([\mathcal{L}]) = [\mathcal{L}_\otimes] \) and \( \xi([\text{Sol}(\nabla_{\mathcal{L}_\otimes})]) = [\mathcal{L}_\otimes] \) hold. One verifies from a routine diagram-chasing argument that there exists a line bundle \( \mathcal{L}_0 \) uniquely determined up to isomorphism, such that the element \([\mathcal{L}_0] \) of \( \check{H}_\text{et}^1(X, \mathbb{G}_m) \) represented by \( \mathcal{L}_0 \) satisfies the equalities \( \xi^{-1}([\mathcal{L}_0]) = [\text{Sol}(\nabla_{\mathcal{L}_\otimes})] \) and \( \xi([\mathcal{L}_0]) = [\mathcal{L}] \).

Because of the equality \( \Gamma(X, \mathbb{G}_m) = k^\times \), one can find isomorphisms \( \eta_1 : \mathcal{L}_0^\otimes \xrightarrow{\sim} \text{Sol}(\nabla_{\mathcal{L}_\otimes}) \) and \( \eta_2 : F_{X/k}^*(\mathcal{L}_0) \xrightarrow{\sim} \mathcal{L} \) making the following square diagram commute:

\[
\begin{align*}
F_{X/k}^*(\mathcal{L}_0^\otimes) & \xrightarrow{\sim} F_{X/k}^*(\text{Sol}(\nabla_{\mathcal{L}_\otimes})) \\
\downarrow \quad & \quad \downarrow \\
F_{X/k}^*(\mathcal{L}_0) & \xrightarrow{\sim} \mathcal{L}.
\end{align*}
\]

where the vertical arrows denote the natural isomorphisms. We shall denote by \( \nabla_{\mathcal{L}} \) the \( D_X^{(N-1)} \)-action on \( \mathcal{L} \) corresponding to the \( D_X^{(N-1)} \)-action \( \nabla_{\text{can}}^{(N-1)} \) on \( F_{X/k}^*(\mathcal{L}_0) \) (cf. (12)) via \( \eta_2 \). It follows from the commutativity of (163) that \( \nabla_{\mathcal{L}} \) satisfies the required conditions. Moreover,
such a $\mathcal{D}_X^{(N-1)}$-action $\nabla_{\mathcal{L}}$ does not depend on the choice of $(\eta_1, \eta_2)$, i.e., is uniquely determined; this is because any gauge transformation of $\nabla_{\mathcal{L}}$ may be given by multiplication by some element of $k^\times$ ($= \Gamma(\mathcal{O}^\times_\mathcal{X})$). This completes the proof of the lemma. 

\begin{flushright}
$\square$
\end{flushright}

**Theorem 3.6.4.**

Let $N \in \mathbb{Z}_{>0} \cup \{\infty\}$. Then, the following assertions hold:

(i) Suppose that $p \nmid (n + 1)$ and $X$ is projective over $k$. Then, for each $F^N$-theta characteristic $\Theta := (\Theta, \nabla)$ of $X$, the maps $\iota^\Theta_{\Omega} : \mathcal{D}_{X,N,\Theta} \to F^N\text{-}\mathfrak{Ind}_X$ and $\iota^\Theta_{\Omega} : \mathcal{D}_{X,N,\Theta} \to F^N\text{-}\mathfrak{Aff}_X$ are bijective.

(ii) The maps $\iota^\Theta_{\Omega} : \mathcal{D}_{X,N} \to F^N\text{-}\mathfrak{Ind}_X$ and $\iota^\Theta_{\Omega} : \mathcal{D}_{X,N} \to F^N\text{-}\mathfrak{Aff}_X$ are bijective.

**Proof.** Assertions (i) and (ii) follow from Theorem 3.4.3 and Proposition 3.6.1. Notice that the case of $N = \infty$ in assertion (i) follows from the cases of $N < \infty$ together with the identifications $\lim_{N \in \mathbb{Z}_{>0}} F^N\text{-}\mathfrak{Ind}_X = F^\infty\text{-}\mathfrak{Ind}_X$ and $\lim_{N \in \mathbb{Z}_{>0}} \mathcal{D}_{X,N,\Theta} = \mathcal{D}_{X,\infty,\Theta}$ given by (57) and (152) respectively. In fact, the maps $\iota^\Theta_{\Omega}$ and $\iota^\Theta_{\Omega}$ are, by construction, compatible with truncation to lower levels. 

\begin{flushright}
$\square$
\end{flushright}

By means of the above theorem, we have the following assertion concerning the comparisons between $(\iota^\Theta_{\Omega})\mathcal{D}_{X,\infty}$ and $\lim_{N \in \mathbb{Z}_{>0}} (\iota^\Theta_{\Omega})\mathcal{D}_{X,N}$.

**Proposition 3.6.5.**

The following assertions hold:

(i) Suppose that $p \nmid (n + 1)$, $X$ is projective over $k$, and $X$ admits an $F^\infty$-theta characteristic. Then, the natural map

$$\mathcal{D}_{X,\infty} \to \lim_{N \in \mathbb{Z}_{>0}} \mathcal{D}_{X,N}$$

is bijective. In particular, the maps $\zeta^\Theta_{\infty} : \mathcal{D}_{X,\infty} \to F^\infty\text{-}\mathfrak{Ind}_X$ and $\zeta^\Theta_{\infty} : \mathcal{D}_{X,\infty} \to F^\infty\text{-}\mathfrak{Aff}_X$ are bijective.

(ii) The natural map

$$F^\infty\text{-}\mathfrak{Ind}_X \to \lim_{N \in \mathbb{Z}_{>0}} F^N\text{-}\mathfrak{Ind}_X$$

is bijective. In particular, the maps $\zeta^\Theta_{\infty} : \mathcal{D}_{X,\infty} \to F^\infty\text{-}\mathfrak{Ind}_X$ and $\zeta^\Theta_{\infty} : \mathcal{D}_{X,\infty} \to F^\infty\text{-}\mathfrak{Aff}_X$ are bijective. (Note that the bijectivity of $\zeta^\Theta_{\infty}$ was already proved in Theorem 3.4.3 (ii).)

**Proof.** We shall prove the former assertion of (i). Let us fix an $F^\infty$-theta characteristic $\Theta := (\Theta, \nabla)$ of $X$. Denote by $\zeta^\Theta_{\infty} \lim$ the map $\mathcal{D}_{X,\infty} \to \lim_{N \in \mathbb{Z}_{>0}} \mathcal{D}_{X,N}$ in question. The surjectivity of $\zeta^\Theta_{\infty} \lim$ follows immediately from the bijectivities of $\iota^\Theta_{\Omega}$. For various $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ (cf. Theorem 3.6.4), and the fact that the equality $\lim_{N \in \mathbb{Z}_{>0}} \iota^\Theta_{\Omega} = (\lim_{N \in \mathbb{Z}_{>0}} \iota^\Theta_{\Omega}) \circ \zeta^\Theta_{\infty} \lim$ holds under the identification $\lim_{N \in \mathbb{Z}_{>0}} F^N\text{-}\mathfrak{Ind}_X = F^\infty\text{-}\mathfrak{Ind}_X$ (cf. (57)).
Next, we shall consider the injectivity of $\xi^\otimes_{\infty \Rightarrow \lim}$. Let $\mathcal{V}_i^\otimes := (\mathcal{V}_i, \nabla_{\mathcal{V},i}, N_i)$ ($i = 1, 2$) be dormant indigenous $\mathcal{D}_X^{(\infty)}$-modules such that $[\mathcal{V}_1^\otimes(N)] = [\mathcal{V}_2^\otimes(N)]$ in $\mathfrak{M}_{X,N}$ for every $N \in \mathbb{Z}_{>0}$. Then, for each $N \in \mathbb{Z}_{>0}$, there exist an invertible $\mathcal{D}_X^{(N-1)}$-module $(\mathcal{L}_N, \nabla_{\mathcal{L},N})$ with vanishing $p$-$\left(N - 1\right)$-curvature and an isomorphism

$$\eta_N : \mathcal{V}_1^\otimes(N) \xrightarrow{\sim} (\mathcal{V}_2^\otimes(N)) \otimes (\mathcal{L}_N, \nabla_{\mathcal{L},N}).$$

The isomorphism $\eta_N$ restricts to an isomorphism $(\mathcal{V}_1 \supseteq) N_1 \xrightarrow{\sim} N_2 \otimes \mathcal{L}_N$ ($\subseteq \mathcal{V}_2 \otimes \mathcal{L}_N$), which gives an isomorphism

$$\mathcal{L}_N \xrightarrow{\sim} N_1 \otimes N_2^\vee.$$

Also, $\eta_N$ induces, via taking determinants, an isomorphism

$$\eta_N^{\det} : \det(\mathcal{V}_1) \xrightarrow{\sim} \det(\mathcal{V}_2 \otimes \mathcal{L}_N) \left( \cong \det(\mathcal{V}_2) \otimes \mathcal{L}_N^{\otimes (n+1)} \right).$$

For each $i = 1, 2$, let $\nabla_{\mathcal{V},i}^{\det}$ denote the $\mathcal{D}_X^{(N-1)}$-action on $\det(\mathcal{V}_i)$ induced naturally from $\nabla_{\mathcal{V},i}$. Then, $\eta_N^{\det}$ gives an isomorphism of $\mathcal{D}_X^{(N-1)}$-modules

$$\left(\mathcal{L}_N^{\otimes (n+1)}, \nabla_{\mathcal{L},N}^{\otimes (n+1)}\right) \xrightarrow{\sim} \left(\det(\mathcal{V}_1) \otimes \det(\mathcal{V}_2)^\vee, \nabla_{\mathcal{V},1}^{\det} \otimes \left(\nabla_{\mathcal{V},2}^{\det}\right)^\vee\right),$$

where $\nabla_{\mathcal{L},N}^{\otimes (n+1)}$ denotes the $(n+1)$-st tensor product of $\nabla_{\mathcal{L},N}$. This isomorphism is compatible with the $(n+1)$-st tensor product of (167) under the composite of natural isomorphisms

$$\left(\mathcal{N}_1 \otimes \mathcal{N}_2^\vee\right)^{\otimes (n+1)} \xrightarrow{\sim} \mathcal{N}_1^{\otimes (n+1)} \otimes \left(\mathcal{N}_2^{\otimes (n+1)}\right)^\vee \xrightarrow{\sim} \left(\omega_X \otimes \det(\mathcal{V}_1)\right) \otimes \left(\omega_X \otimes \det(\mathcal{V}_1)^\vee\right) \xrightarrow{\sim} \det(\mathcal{V}_1) \otimes \det(\mathcal{V}_2)^\vee,$$

where the second arrow arises from (86). Hence, the uniqueness portion of Lemma 3.6.3 implies that, for another positive integer $N'$ with $N' > N$, the $N'$-th truncation of $(\mathcal{L}_{N'}$, $\nabla_{\mathcal{L},N'}$) obtained in the same manner as above is isomorphic to $(\mathcal{L}_N, \nabla_{\mathcal{L},N})$. It follows that the collection $\{(\mathcal{L}_N, \nabla_{\mathcal{L},N})\}_{N \in \mathbb{Z}_{>0}}$ yields a $\mathcal{D}_X^{(\infty)}$-module $(\mathcal{L}_\infty, \nabla_{\mathcal{L},\infty})$ such that, for each $N \in \mathbb{Z}_{>0}$, the $N$-th truncations of $\mathcal{V}_1^\otimes$ and $\mathcal{V}_2^\otimes$ are isomorphic via $\eta_N$. By taking Proposition 2.3.4 (ii) into account, we can obtain an isomorphism $\mathcal{V}_1^\otimes \xrightarrow{\sim} (\mathcal{V}_2^\otimes)_{\otimes(\mathcal{L}_\infty, \nabla_{\mathcal{L},\infty})}$ determined by the collection $\{\eta_N\}_{N \in \mathbb{Z}_{>0}}$ after possibly composing each $\eta_N$ with multiplication by some element of $k^\times$. Thus, $\mathcal{V}_1^\otimes$ and $\mathcal{V}_2^\otimes$ are $\mathbb{G}_m$-equivalent. This completes the proof of the desired injectivity.

The latter assertion follows directly from the former assertion, Proposition 3.3.2 and Theorem 3.4.3 (i).

Finally, assertion (ii) follows from Proposition 3.3.2, Theorem 3.4.3 (ii), and the identification $\lim_{-N \in \mathbb{Z}_{>0}} F^N \cdot \mathfrak{J} \mathfrak{n} \mathfrak{d}_X = F^\infty \cdot \mathfrak{J} \mathfrak{n} \mathfrak{d}_X$ (cf. (3.7)). This completes the proof of the proposition. 

\[\square\]

3.7. Chern class formula.

In this subsection, we prove (cf. Theorem 3.7.1 below) necessary conditions on Chern classes for the existence of $F^N$-projective or $F^N$-affine structures. For each vector bundle $\mathcal{V}$ on $X$ and each $l = 0, 1, \ldots$, $\text{rank}(\mathcal{V})$, we write $c_l(\mathcal{V})$ for the $l$-th Chern class of $\mathcal{V}$, which is defined as an element of $CH^l(X)$, i.e., the Chow group of codimension-$l$ cycles modulo rational equivalence.
Also, write $c^\text{crys}_i(V)$ for the $l$-th crystalline Chern class of $V$ (cf. [3] or [21], Chap. III, Theorem 1.1.1), which is defined as an element of $H^2_{\text{crys}}(X/W)$. Here, $H^2_{\text{crys}}(X/W)$ denotes the 2l-th crystalline cohomology group of $X$ over $W$, where $W$ denotes the ring of Witt vectors over $k$. As usual, we set $c_1(X) := c_1(\mathcal{T}_X)$ and $c^\text{crys}_1(X) := c^\text{crys}_1(\mathcal{T}_X)$.

**Theorem 3.7.1.**

Let $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$ and let $X$ be a smooth quasi-projective variety over $k$ admitting an $F^N$-projective (resp., $F^N$-affine) structure. In the non-resp’d statement, we suppose further that $p \nmid (n+1)$. Then, the following assertions hold:

(i) In the case where $N \neq \infty$, the equality

\[(171) \quad c_l(X) = \frac{1}{(n+1)^l} \cdot \binom{n+1}{l} \cdot c_1(X)^l \quad (\text{resp., } c_l(X) = 0)\]

holds after reduction modulo $p^N$ (resp., modulo $p^l$) for each positive integer $l$. In particular, if $X$ admits an $F^\infty$-projective (resp., $F^\infty$-affine) structure, then the above equality holds after reduction modulo $p^N$ (resp., module $p^l$) for any positive integers $N$ and $l$. Moreover, the same assertions hold for $c^\text{crys}_i(X)$’s.

(ii) In the case where $N = \infty$ and $X$ is projective over $k$, the equality $c_l(X)$ in $H^2_{\text{crys}}(X/W)$, where “$c_1(X)$” and “$c_l(X)$” are replaced by $c^\text{crys}_1(X)$ and $c^\text{crys}_l(X)$ respectively, holds without any reduction for every positive integer $l$.

**Proof.** We shall consider assertion (i). The second assertion of (ii) follows directly from the first one and their crystalline versions can be proved in similar ways. Hence, it suffices to consider the first assertion.

Let us prove first the non-resp’d portion by induction on $l$. The case of $l = 1$ is clear. Next, suppose that the assertion with $l$ replaced by any $l' < l$ has been proved. By assumption, $X$ admits an $F^N$-projective structure $\mathcal{S}^\bowtie$. It follows from Proposition 3.3.2 that there exists a dormant indigenous $\mathcal{D}^{(N-1)}_{X/k}$-module $\mathcal{V}^\bowtie := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N})$ with $\mathcal{V}^\bowtie \cong \mathcal{S}^\bowtie$. Since $(\mathcal{V}, \nabla_{\mathcal{V}})$ vanishing $p$-($N - 1$)-curvature, we have $F^{(N)}_{X/k}(\text{Sol}(\nabla_{\mathcal{V}})) \cong \mathcal{V}$ (cf. (113)). If $\text{Sol}(\nabla_{\mathcal{V}})^F$ denotes the vector bundle on $X$ corresponding to $\text{Sol}(\nabla_{\mathcal{V}})$ via the isomorphism $\text{id}_X \times F^k_{X/k} : X^{(N)} \cong X$. Then, $F^{(N)}_{X/k}(\text{Sol}(\nabla_{\mathcal{V}})^F) \cong \mathcal{V}$, and hence, the equality $c_l(\mathcal{V}) = p^l \cdot c_l(\text{Sol}(\nabla_{\mathcal{V}})^F)$ holds (cf. [20], §2, the proof of Lemma 2.1). In particular, the equality

\[(172) \quad c_l(\mathcal{V}) = 0\]

holds after reduction modulo $p^N$. By Proposition 2.3.3, $\mathcal{V}$ may be identified with $\mathcal{D}^{(N-1)}_{X,k} \otimes \mathcal{N}$ via $\text{KS}^\bowtie(\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N})$ and we have

\[(173) \quad c_l(\mathcal{V}) = c_l(\mathcal{D}^{(N-1)}_{X,k} \otimes \mathcal{N}).\]

The vector bundle $\mathcal{D}^{(N-1)}_{X,k} \otimes \mathcal{N}$ fits into the following short exact sequence:

\[(174) \quad 0 \rightarrow \mathcal{N} \rightarrow \mathcal{D}^{(N-1)}_{X,0} \otimes \mathcal{N} \rightarrow \mathcal{D}^{(N-1)}_{X,1} \otimes \mathcal{N} \rightarrow \mathcal{T}_X \otimes \mathcal{N} \rightarrow \mathcal{D}^{(N-1)}_{X,k} \otimes \mathcal{N} \rightarrow 0\]

It follows that

\[(175) \quad c_l(\mathcal{D}^{(N-1)}_{X,1} \otimes \mathcal{N}) = \sum_{i+j=l} c_i(\mathcal{N})c_j(\mathcal{T}_X \otimes \mathcal{N}) = c_l(\mathcal{T}_X \otimes \mathcal{N}) + c_1(\mathcal{N}) \cdot c_{l-1}(\mathcal{T}_X \otimes \mathcal{N}).\]
Also, consider the equality

\[
(176) \quad c_l(T_X \otimes \mathcal{N}) = \sum_{i=0}^{l} \binom{n-i}{l-i} \cdot c_i(X) c_1(\mathcal{N})^{l-i}
\]

resulting from [19], Chap. 3, §3.2, Example 3.2.2. By combining (172), (173), (175), and (176), we obtain the following sequence of equalities in the Chow group $CH^l(X)$ modulo $p^N$:

\[
(177) \quad 0 \overset{(172)}{=} c_l(\mathcal{V}) \overset{(173)}{=} c_l(D_{X,1}^{(N-1)} \otimes \mathcal{N}) \overset{(175)}{=} c_l(T_X \otimes \mathcal{N}) + c_1(\mathcal{N}) c_{l-1}(T_X \otimes \mathcal{N}) \overset{(176)}{=} \sum_{i=0}^{l} \binom{n-i}{l-i} \cdot c_i(X) c_1(\mathcal{N})^{l-i} + \sum_{i=0}^{l-1} \binom{n-i}{l-1-i} \cdot c_i(X) c_1(\mathcal{N})^{l-i} = c_l(X) + \sum_{i=0}^{l-1} \binom{n+1-i}{l-i} \cdot c_i(X) c_1(\mathcal{N})^{l-i}.
\]

This composite equality in the case of $l = 1$ reads

\[
(178) \quad c_1(X) = -(n+1) c_1(\mathcal{N}).
\]

Moreover, by (177), (178), and the induction assumption, the following sequence of equalities holds:

\[
(179) \quad c_l(X) \overset{(177)}{=} - \sum_{i=0}^{l-1} \binom{n+1-i}{l-i} \cdot c_i(X) c_1(\mathcal{N})^{l-i} \overset{(178)}{=} - \sum_{i=0}^{l-1} \binom{n+1-i}{l-i} \cdot \left(\frac{-1}{n+1}\right)^{l-i} \cdot c_i(X) c_1(X)^{l-i} \overset{(\text{induction})}{=} - \sum_{i=0}^{l-1} \binom{n+1-i}{l-i} \cdot \frac{1}{(n+1)^i} \cdot \binom{n+1}{i} \cdot \left(\frac{-1}{n+1}\right)^{l-i} \cdot c_1(X)^i c_1(X)^{l-i} = c_1(X)^l \cdot \frac{1}{(n+1)^l} \cdot \sum_{i=0}^{l-1} (-1)^{l+1-i} \binom{n+1-i}{l-i} \cdot \binom{n+1}{i} = c_1(X)^l \cdot \frac{1}{(n+1)^l} \cdot \binom{n+1}{l} \cdot \left(1 - \sum_{i=0}^{l} (-1)^{l-i} \left(\frac{l}{l-i}\right)\right) = c_1(X)^l \cdot \frac{1}{(n+1)^l} \cdot \binom{n+1}{l}.
\]

This proves the non-resp’d equality of (171).

Next, we shall prove the resp’d portion. By assumption, $X$ admits an $F^N$-affine structure $\mathcal{S}$. It follows from Proposition 3.3.2 that there exists a dormant affine-indigenous $\mathcal{D}_{X}^{(N-1)}$-module $^\wedge \mathcal{V} := (\mathcal{V}, \nabla_{\mathcal{V}}, \mathcal{N}, \delta)$ with $^\wedge \mathcal{V} \cong \mathcal{S}$.

The $\mathcal{D}_{X}^{(N-1)}$-action $\nabla_{\mathcal{V}}$ induces $\mathcal{D}_{X}^{(N-1)}$-actions on $\text{Ker}(\delta)$ and $\mathcal{N}$ via the natural inclusion $\text{Ker}(\delta) \hookrightarrow \mathcal{V}$ and the surjection $\delta : \mathcal{V} \twoheadrightarrow \mathcal{N}$ respectively; we shall denote these $\mathcal{D}_{X}^{(N-1)}$-actions by $\nabla_{\text{Ker}(\delta)}$ and $\nabla_{\mathcal{N}}$ respectively. Let us identity $\text{Ker}(\delta)$
with $\mathcal{T}_X \otimes \mathcal{N}$ via the composite isomorphism \([87]\). Then, the tensor product of $\nabla_{\text{Ker}(\delta)}$ and the dual of $\nabla_N$ yields a $\mathcal{D}_X^{(N-1)}$-action $\nabla_{\mathcal{T}_X}$ on $\mathcal{T}_X \cong (\mathcal{T}_X \otimes \mathcal{N}) \otimes \mathcal{N}'$. Since $\nabla_V$ has vanishing $p$-$(N - 1)$-curvature, both $\nabla_{\text{Ker}(\delta)}$ and $\nabla_N$ has vanishing $p$-$(N - 1)$-curvature, and hence, $\nabla_{\mathcal{T}_X}$ has vanishing $p$-$(N - 1)$-curvature. It follows that there exists a vector bundle $\mathcal{W}$ on $X$ with $F^N_{\mathcal{X}}(\mathcal{W}) \cong \mathcal{T}_X$. This implies $c_i(X) := c_i(\mathcal{T}_X) = p^N \cdot c_i(\mathcal{W})$ (cf. \([20]\), §2, the proof of Lemma 2.1). In particular, the equality $c_i(X) = 0$ holds modulo $p^N$, as desired. This completes the proof of assertion (i).

Finally, assertion (ii) follows from assertion (i) and the fact that $H^2_{\text{crys}}(X/W)$ is finitely generated over $W$ (cf. \([2]\), ChapVII, §1, Corollaire 1.1.2). This completes the proof of the theorem. \(\Box\)

4. Tango structures

In this section, we generalize so-called Tango structures to higher dimensions and study the relationship with dormant affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules.

Let $n$ be a positive integer and $X$ a smooth variety over $k$ of dimension $n$.

4.1. Extensions of the sheaf of locally exact 1-forms.

To begin with, we introduce a certain vector bundle, denoted by $\mathcal{B}_X^{(N)}$, which may be thought of as a generalization of the sheaf of locally exact 1-forms. The generalization of Tango structure that we discuss in the next subsection will be defined as a certain subbundle of this vector bundle.

For each positive integer $N$, we shall denote by $\mathcal{B}_X^{(N)}$ the cokernel of the morphism $F_{X/k}^{(N)} : \mathcal{O}_{X^{(N)}} \to F_{X/k}^{(N)}(\mathcal{O}_X)$ induced by $F_{X/k}^{(N)}$. Then, $\mathcal{B}_X^{(N)}$ is a vector bundle on $X^{(N)}$ of rank $p^nN - 1$ fitting into the following short exact sequence of $\mathcal{O}_{X^{(N)}}$-modules:

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{O}_{X^{(N)}} & \xrightarrow{F_{X/k}^{(N)}} & F_{X/k}^{(N)}(\mathcal{O}_X) & \xrightarrow{F_{X/k}^{(N)}} & \mathcal{B}_X^{(N)} & \longrightarrow & 0,
\end{array}
\]

where $F_{X/k}^{(N)}$ denotes the natural surjection. Let $d$ denote the morphism $F_{X/k}^{(N)}(\mathcal{O}_X) \to F_{X/k}^{(N)}(\Omega_X)$ obtained as the direct image of the universal derivation $\mathcal{O}_X \to \Omega_X$. Since $d(\mathcal{O}_{X^{(N)}}) = 0$, the morphism $d$ factors through $F_{X/k}^{(N)}$ and hence induces a morphism

\[
\gamma_X^{(N)} : \mathcal{B}_X^{(N)} \to F_{X/k}^{(N)}(\Omega_X).
\]

In particular, $\mathcal{B}_X^{(1)}$ may be identified with the sheaf of locally exact 1-forms on $X$ and $\gamma_X^{(1)}$ coincides, under this identification, with the natural inclusion.

Next, let $N'$ be another positive integer with $N' > N$. Then, there exists a canonical surjection

\[
\delta_{X,N,N'} : \mathcal{B}_X^{(N')} \to F_{X^{(N')/k}}^{(N'-N)}(\mathcal{B}_X^{(N)}).
\]
Lemma 4.1.1.

Suppose that $X$ is Frobenius split (or $F$-split) if the short exact sequence \((180)\) of the case where $N = 1$ splits.

Proof. Since $X$ is Frobenius split, the Frobenius twists $X^{(m)}$ (for $m = 0, 1, 2, \ldots, N - 1$) are Frobenius split. That is to say, for each $m$, there exists a split surjection $\delta_m : F_{X^{(m)}}(O_X) \rightarrow O_{X^{(m+1)}}$ of the natural inclusion $O_{X^{(m+1)}} \hookrightarrow F_{X^{(m)}}(O_X)$. Then, the composite \[(185)\]

\[\delta_N^2 := \delta_{N-1} \circ F_{X^{(N-1)}}(O_X) \rightarrow \cdots \circ F_{X^{(2)}}(O_X) \rightarrow O_{X^{(N)}}\]

determines a split surjection of \((180)\). In the same manner, we obtain a split surjection $\tilde{\delta}_N$ of \((180)\) with $N$ replaced by $N'$; let $\tilde{\delta}_N : B^{(N')} \rightarrow F_{X^{(N')}}(O_X)$ be the corresponding split injection. Let us consider the following morphism of short exact sequences:

\[(186)\]

\[0 \rightarrow F_{X^{(N')}}(O_X) \rightarrow F_{X^{(N')}}(O_X) \rightarrow 0\]

where the upper horizontal sequence denotes the direct image via $F_{X^{(N')}}(O_X)$ of the sequence \((180)\) and the lower horizontal sequence denotes \((183)\). By the commutativity of the left-hand square in this diagram, the composite

\[(187)\]

\[B^{(N')}_{X} \rightarrow F_{X^{(N')}}(O_X) \rightarrow F_{X^{(N')}}(O_X) \rightarrow B^{(N')}_{X} \rightarrow 0\]

turns out to specify a split surjection of \((184)\). This completes the proof of the lemma. \qed
4.2. Tango structures on a variety.

Now, we make the definition of a Tango structure of level \( N \in \mathbb{Z}_{>0} \sqcup \{\infty\} \) on a smooth variety of any dimension. In the case where \( N = 1 \) and \( X \) is a smooth projective curve of genus \( > 1 \), the following definition is nothing but the usual definition of a Tango structure (cf. e.g., [81], Definition 5.1.1). First, let us fix \( N \in \mathbb{Z}_{>0} \).

**Definition 4.2.1.**

A **Tango structure of level** \( N \) on \( X \) is defined as a subbundle \( \mathcal{U}^\bullet \) of \( \mathcal{B}_X^{(N)} \) such that the morphism

\[
\gamma_{\mathcal{U}^\bullet} : F_{X/k}^{(N)*}(\mathcal{U}^\bullet) \to \Omega_X
\]

(188)

responding, via the adjunction relation “\( F_{X/k}^{(N)*}(-) \dashv F_{X/k*}^{(N)}(-) \)”, to the composite

\[
\gamma_{\mathcal{U}^\bullet} : \mathcal{U}^\bullet \xrightarrow{\text{incl.}} \mathcal{B}_X^{(N)} \xrightarrow{\gamma_X^{(N)}} F_{X/k*}^{(N)}(\Omega_X)
\]

(189)

is an isomorphism.

**Lemma 4.2.2.**

*Let \( N' > N \) and \( \mathcal{U}^\bullet \left( \subseteq \mathcal{B}_X^{(N')} \right) \) a Tango structure of level \( N' \) on \( X \). Also, let us consider the morphism

\[
F_{X(k)}^{(N'-N)*}(\mathcal{U}^\bullet) \to \mathcal{B}_X^{(N)}
\]

(190)

corresponding to the composite \( \mathcal{U}^\bullet \xrightarrow{\text{incl.}} \mathcal{B}_X^{(N')} \xrightarrow{\delta_{X,N,N'}} F_{X(k)}^{(N'-N)*}(\mathcal{B}_X^{(N)}) \) via the adjunction relation “\( F_{X(k)}^{(N'-N)*}(-) \dashv F_{X(k)*}^{(N'-N)}(-) \)”. Then, it is injective and its image specifies a Tango structure of level \( N \) on \( X \)***

*Proof. Let us consider the composite

\[
\left( F_{X/k}^{(N)*}(\mathcal{U}^\bullet) \right) \cong F_{X(k)}^{(N)*}(F_{X(k)}^{(N'-N)*}(\mathcal{U}^\bullet)) \to F_{X/k}^{(N)*}(\mathcal{B}_X^{(N)}) \to \Omega_X,
\]

(191)

where the first arrow denotes the pull-back of (190) via \( F_{X/k}^{(N)} \) and the second arrow denotes the morphism corresponding to \( \gamma_X^{(N)} \) (cf. (181)) via the adjunction relation “\( F_{X/k}^{(N)*}(-) \dashv F_{X/k*}^{(N)}(-) \)”. Since \( \mathcal{U}^\bullet \) is a Tango structure of level \( N' \), this composite turns out to be an isomorphism. It follows that the first arrow in the composite (191) is injective, and hence, (190) is verified to be injective because of the faithful flatness of \( F_{X/k}^{(N)} \). Moreover, the latter assertion follows from the bijectivity of (191).\( \square \)

Denote by

\[
\text{Tan}_{X,N}
\]

(192)

the set of Tango structures on \( X \) of level \( N \).

Given another positive integer \( N' \) with \( N' > N \) and a Tango structure \( \mathcal{U}^\bullet \) of level \( N' \) on \( X \), we shall write

\[
\mathcal{U}^\bullet|_{X^{(N)}} := \text{Im}(F_{X(k)}^{(N'-N)*}(\mathcal{U}^\bullet) \to \mathcal{B}_X^{(N)})
\]

(193)
for the Tango structure of level $N$ obtained from $U^\bullet$ in the way described in Lemma 4.2.2 and refer to it as the $N$-th truncation of $U^\bullet$. By construction, there exists a canonical isomorphism
\begin{equation}
F_{X/k}^{(N-N)^*} (U^\bullet) \xrightarrow{\sim} U^\bullet|^{(N)},
\end{equation}
i.e., the morphism $\langle 190 \rangle$ with the codomain restricted to $U^\bullet|^{(N)}$. The resulting assignments $U^\bullet \mapsto U^\bullet|^{(N)}$ for various pairs $(N,N')$ with $N < N'$ give rise to a projective system of sets
\begin{equation}
\cdots \to \text{Tan}_{X,N} \to \cdots \to \text{Tan}_{X,3} \to \text{Tan}_{X,2} \to \text{Tan}_{X,1}.
\end{equation}

**Definition 4.2.3.**
A Tango structure of level $\infty$ on $X$ is a collection
\begin{equation}
U^\bullet_{\infty} := \{ U^\bullet_N \}_{N \in \mathbb{Z}_{>0}},
\end{equation}
where each $U^\bullet_N$ denotes a Tango structure of level $N$, such that $U^\bullet_{N+1}|^{(N)} = U^\bullet_N$ for any $N \in \mathbb{Z}_{>0}$.

Denote by
\begin{equation}
\text{Tan}_{X,\infty}
\end{equation}
the set of Tango structures of level $\infty$ on $X$. Then, this set may be naturally identified with the limit of the projective system $\langle 195 \rangle$, i.e., we have
\begin{equation}
\text{Tan}_{X,\infty} = \lim_{\longleftarrow \substack{N \in \mathbb{Z}_{>0}}} \text{Tan}_{X,N}.
\end{equation}

### 4.3. From Tango structures to dormant affine-indigenous $D_X^{(N-1)}$-modules.

Let $N$ be a positive integer. In what follows, we shall construct dormant affine-indigenous $D_X^{(N-1)}$-modules by means of Tango structures of level $N$.

First, let us consider the pull-back
\begin{equation}
0 \to O_X \xrightarrow{F_{X/k}^{(N)*}} F_{X/k}^{(N)*} (F_{X/k}^{(N)*} (O_X)) \xrightarrow{F_{X/k}^{(N)*} (F_{X/k}^{(N)*})} F_{X/k}^{(N)*} (B_X^{(N)}) \to 0
\end{equation}
of the sequence $\langle 180 \rangle$ via $F_{X/k}^{(N)}$. The morphism
\begin{equation}
\alpha^* : F_{X/k}^{(N)*} (F_{X/k}^{(N)*} (O_X)) \to O_X
\end{equation}
corresponding, via the adjunction relation "$F_{X/k}^{(N)*} (-) \dashv F_{X/k}^{(N)*} (-)$", to the identity morphism of $F_{X/k}^{(N)*} (O_X)$ specifies a split surjection of $\langle 199 \rangle$. This split surjection determines a decomposition
\begin{equation}
F_{X/k}^{(N)*} (F_{X/k}^{(N)*} (O_X)) \xrightarrow{\sim} F_{X/k}^{(N)*} (B_X^{(N)}) \oplus O_X.
\end{equation}
By using this decomposition, we shall consider $F_{X/k}^{(N)*} (B_X^{(N)})$ (resp., $O_X$) as a subbundle (resp., a quotient bundle) of $F_{X/k}^{(N)*} (F_{X/k}^{(N)*} (O_X))$. Denote by
\begin{equation}
\text{KS}_\star : F_{X/k}^{(N)*} (B_X^{(N)}) \to \Omega_X \otimes O_X \ (= \Omega_X)
\end{equation}
inclusion of short exact sequences

\[ (F^{(N)}_{X/k}(F^*_{X/k}(O_X)), \nabla^{\text{can}(N-1)}_{F^*_{X/k}(O_X)}, F^{(N)}_{X/k}(B^N_X)) \]

(cf. (112) for the definition of \( \nabla^{\text{can}(\cdot)} \)).

Now, let \( U \subseteq B^N_X \) be a Tango structure of level \( N \) on \( X \). Denote by \( G_U \) the inverse image of \( U \) via the quotient \( F^{(N)}_{X/k}(O_X) \rightarrow B^N_X \) (cf. (180)). Hence, we have an inclusion of short exact sequences

\[ 0 \rightarrow O_{X(N)} \rightarrow G_U \rightarrow U \rightarrow 0 \]

(204)

Consider the pull-back via \( F^{(N)}_{X/k} \) of this diagram:

\[ 0 \rightarrow O_X \rightarrow F^{(N)}_{X/k}(G_U) \rightarrow F^{(N)}_{X/k}(U) \rightarrow 0 \]

(205)

By taking account of this morphism, we see that the decomposition (201) restricts to a decomposition

\[ F^{(N)}_{X/k}(G_U) \cong F^{(N)}_{X/k}(U) \oplus O_X. \]

(206)

of \( F^{(N)}_{X/k}(G_U) \). Moreover, it induces a decomposition of its dual

\[ F^{(N)}_{X/k}(G_U)^\vee \cong O_X \oplus F^{(N)}_{X/k}(U)^\vee. \]

(207)

This decomposition allows us to regard \( O_X \) (i.e., the first factor of the right-hand side) as a line subbundle of \( F^{(N)*}_{X/k}(G_U)^\vee \). Moreover, note that the dual

\[ \delta : F^{(N)*}_{X/k}(G_U)^\vee \rightarrow O_X \]

(208)

of the natural inclusion \( O_X \hookrightarrow F^{(N)*}_{X/k}(G_U) \) is compatible with the respective \( D_X^{(N-1)} \)-actions

\( \nabla^{\text{can}(N-1)}_{F^{(N)*}_{X/k}(G_U)^\vee} \) and \( \nabla^{\text{triv}(N-1)}_{O_X} \).

**Proposition 4.3.1.**

The quadruple

\[ (F^{(N)*}_{X/k}(G_U)^\vee, \nabla^{\text{can}(N-1)}_{F^{(N)*}_{X/k}(G_U)^\vee}, O_X, \delta) \]

(209)
forms a dormant affine-indigenous $D_{X}^{(N-1)}$-module classified by $\mathcal{I}_{[X]}^{\mathbb{Z}_{\infty}}$ (cf. (149)). Moreover, the resulting assignment $\mathcal{U}^{\bullet} \to h\mathcal{U}^{\Delta} \equiv \mathcal{O}_{X}$ is compatible with truncation to lower levels.

**Proof.** Let us first prove the former assertion. To this end, we shall show that the Kodaira-Spencer map $KS_{\mathcal{U}^{\bullet} \to \mathcal{O}}$ associated to the triple

$$\mathcal{U}^{\Delta} := (F_{X/k}^{(N)}(\mathcal{G}_{\mathcal{U}})^{\vee}; \nabla_{F_{X/k}^{(N-1)}}^{\text{can}(N)}(\mathcal{G}_{\mathcal{U}})^{\vee}; \mathcal{O}_{X})$$

is an isomorphism. According to the decomposition (200), we consider $F_{X/k}^{(N)}(\mathcal{U}^{\bullet})$ and $\mathcal{O}_{X}$ as, respectively, a subbundles and a quotient bundle of $F_{X/k}^{(N)}(\mathcal{G}_{\mathcal{U}})$. Let $KS_{\mathcal{U}^{\bullet} \to \mathcal{O}} : F_{X/k}^{(N)}(\mathcal{U}^{\bullet}) \to \Omega_{X}$ (cf. (74)) be the morphism inducing the Kodaira-Spencer map associated to the triple $(F_{X/k}^{(N)}(\mathcal{G}_{\mathcal{U}}), \nabla_{F_{X/k}^{(N)}}^{\text{can}(N)}(\mathcal{G}_{\mathcal{U}}); F_{X/k}^{(N)}(\mathcal{U}^{\bullet}))$. Since the decomposition (200) was defined as a restriction of (201), the equality

$$KS_{\mathcal{U}^{\bullet} \to \mathcal{O}}^{\ast} = KS_{\mathcal{U}^{\bullet}}^{\ast} \circ \iota$$

holds, where $\iota$ denotes the natural inclusion $F_{X/k}^{(N)}(\mathcal{U}^{\bullet}) \hookrightarrow F_{X/k}^{(N)}(\mathcal{B}_{X}^{(N)})$. Hence, it follows from Lemma 4.3.2 described below that

$$KS_{\mathcal{U}^{\bullet} \to \mathcal{O}}^{\ast} = \nu \circ F_{X/k}^{(N)}(\gamma_{X}^{(N)}) \circ \iota$$

(cf. (181) for the definition of $\gamma_{X}^{(N)}$), where the morphism $\nu$ will be introduced in that lemma. But, the right-hand side of this equality coincides (up to sign) with the morphism $\gamma_{\mathcal{U}^{\bullet}}^{\ast}$ (cf. (188)). Since $\mathcal{U}^{\bullet}$ defines a Tango structure of level $N$ (i.e., $\gamma_{\mathcal{U}^{\bullet}}^{\ast}$ is an isomorphism), this fact implies that the left-hand side $KS_{\mathcal{U}^{\bullet} \to \mathcal{O}}^{\ast}$ of the above equality is an isomorphism. By applying Lemma 2.2.3 to $\mathcal{U}^{\Delta} \equiv \mathcal{O}_{X}$, $KS_{\mathcal{U}^{\bullet} \to \mathcal{O}}^{\ast}$ turns out to be an isomorphism. This completes the proof of assertion (i).

Finally, assertion (ii) follows from the various definitions involved. This completes the proof of the assertion.  

The following lemma was used in the proof of the above proposition.

**Lemma 4.3.2.**
The morphism $KS_{\mathcal{U}^{\bullet}}^{\ast}$ coincides with the composite

$$F_{X/k}^{(N)}(\mathcal{B}_{X}^{(N)}) \xrightarrow{F_{X/k}^{(N)}(\gamma_{X}^{(N)})} F_{X/k}^{(N)}( \Omega_{X}) \xrightarrow{\nu} \Omega_{X},$$

where $\nu$ denotes the morphism corresponding, via the adjunction relation "$F_{X/k}^{(N)}(-) \to F_{X/k}^{(N)}(-)^{\ast}$", to the automorphism of $F_{X/k}^{(N)}(\Omega_{X})$ given by multiplication by $(-1)$.

**Proof.** By the local nature of the assertion, one may replace $X$ by $\text{Spec}(k[[x_{1}, \ldots, x_{n}]]$ and $\Omega_{X}$ by the $\mathcal{O}_{X}$-module given by the free $k[[x_{1}, \ldots, x_{n}]]$-module $\bigoplus_{i=1}^{n} k[[x_{1}, \ldots, x_{n}]]dx_{i}$. (But, we still use the notation $X$ to denote $\text{Spec}(k[[x_{1}, \ldots, x_{n}]]$ for simplicity.) Observe that for each $\mathcal{O}_{X}$-module $\mathcal{M}$ obtained from some $k[[x_{1}, \ldots, x_{n}]]$-module $M$, we have

$$\Gamma(X, F_{X/k}^{(N)}(\mathcal{M})) = k[[x_{1}, \ldots, x_{n}]] \otimes_{k[[x_{1}^{N}, \ldots, x_{n}^{N}]}} M.$$
For each \( f \in k[[x_1, \cdots, x_n]] \) (\( = \Gamma(X, \mathcal{O}_X) \)), write \( \overline{f} \) for its image via the quotient \( F_X/(N)_k: F_X/(N)/k(\mathcal{O}_X) \to B_X/(N) \) (cf. (180)). If \( \alpha^\flat: F_X/(N)^\flat(\mathcal{O}_X) \to F_X/(N)^\flat (\mathcal{O}_X) \) denotes the split injection of (199) corresponding to \( \alpha^\sharp \) (cf. (200)), then it is given by assigning \( 1 \otimes f \mapsto 1 \otimes f - f \otimes 1 \) for any \( f \in k[[x_1, \cdots, x_n]] \) under the expression (214) in the case of \( M = \mathcal{O}_X \). Also, the canonical connection \( \nabla_{\mathcal{O}_X}^{\text{can}}(\mathcal{O}_X) \) (cf. (26) and Remark (2.1.1)) on \( F_X^\flat(\mathcal{O}_X) \) is given by \( a \otimes b \mapsto da \otimes b \) for any \( a, b \in k[[x_1, \cdots, x_n]] \). Hence, for each \( f \in k[[x_1, \cdots, x_n]] \), the following equalities hold:

\[
(\text{id}_X \otimes \alpha^\sharp)(\nabla_{\mathcal{O}_X}^{\text{can}}(\mathcal{O}_X)(\alpha^\flat(1 \otimes \overline{f}))) = (\text{id}_X \otimes \alpha^\sharp)(\nabla_{\mathcal{O}_X}^{\text{can}}(\mathcal{O}_X)(1 \otimes f - f \otimes 1)) = (\text{id}_X \otimes \alpha^\sharp)(-df \otimes 1) = -df.
\]

This shows that the image of \( (1 \otimes \overline{f}) \) via the morphism \( \text{KS}'_\star \) is \(-df\). On the other hand, we have

\[
\nu(F_X^\flat(\gamma_X^\flat)(1 \otimes \overline{f})) = \nu(1 \otimes df) = -df.
\]

Thus, \( \text{KS}'_\star \) coincides with the composite \( \nu \circ F_X/\gamma_X^\flat \), as desired. \( \square \)

### 4.4. Comparison with dormant affine-indigenous \( D_X^{(N-1)} \)-modules.

The assignment \( \mathcal{U}\mapsto \mathcal{A}\mathcal{U}\mapsto \mathcal{A}\mathcal{U}^{\otimes} \) resulting from Proposition (4.3.1) determines a map of sets

\[
\mathcal{A}\mathcal{U}^{\otimes}: \mathcal{T} \to \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}} \subseteq \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}}
\]

which is compatible with truncation to lower levels. If \( X \) is proper over \( k \), or more generally, satisfies the equality \( \Gamma(X, \mathcal{O}_X) = k \), then we have \( \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}} \subseteq \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}} \) (cf. Proposition (3.5.5) (ii)); hence, the collection of maps \( \{ \mathcal{A}\mathcal{U}^{\otimes} \}_{N \in \mathbb{Z}_{>0}} \) determines a map of sets

\[
\mathcal{A}\mathcal{U}^{\otimes}: \mathcal{T} \to \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}} \subseteq \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}}
\]

#### Theorem 4.4.1.

For each positive integer \( N \), the map \( \mathcal{A}\mathcal{U}^{\otimes} \) constructed above is bijective. If, moreover, \( X \) is proper over \( k \), or more generally, satisfies the equality \( \Gamma(X, \mathcal{O}_X) = k \), then the map \( \mathcal{A}\mathcal{U}^{\otimes} \) is bijective.

**Proof.** Since \( \mathcal{A}\mathcal{U}^{\otimes} \) is compatible with truncation to lower levels, it suffices to prove the former assertion, i.e., the case of \( N \neq \infty \). To this end, we shall construct the inverse to the map \( \mathcal{A}\mathcal{U}^{\otimes} \).

Let \( \mathcal{U}^{\otimes} := (\mathcal{D}_{X,1}^{(N-1)}, \nabla_{\mathcal{U}}, \mathcal{O}_X, \delta) \) be an element of \( \mathcal{A}\mathcal{J}^{\mathcal{A} \mathcal{U}^{\otimes}} \). Denote by \( \nabla_{\mathcal{V}} \) the \( \mathcal{D}_{X,1}^{(N-1)} \)-action on the dual \( (\mathcal{D}_{X,1}^{(N-1)})^\vee \) induced by \( \nabla_{\mathcal{U}} \). \( \nabla_{\mathcal{V}} \) has vanishing \( p-(N-1) \)-curvature, so the natural morphism \( F_X/(N) \circ (\text{Sol}(\nabla_{\mathcal{V}})) \to (\mathcal{D}_{X,1}^{(N-1)})^\vee \) is an isomorphism. The dual \( \delta^\vee : \mathcal{O}_X \to (\mathcal{D}_{X,1}^{(N-1)})^\vee \)}
of $\delta$ preserves the respective $\mathcal{D}^{(N)}_{X}$-actions $\nabla^{\text{triv}(N-1)}_{\mathcal{O}_{X}}, \nabla^{\vee}_{\mathcal{U}}$ and induces, via taking sheaves of horizontal sections, an injection

\[(219) \quad \mathcal{O}_{X}^{(N)} \hookrightarrow \text{Sol}(\nabla^{\vee}_{\mathcal{U}}).\]

If $\mathcal{U}^\bullet$ denotes its cokernel, then we have a canonical identification

\[(220) \quad \text{Ker}(\delta)^{\vee} = F^{(N)}_{X/k}(\mathcal{U}^\bullet).\]

Since $(\mathcal{D}^{(N-1)}_{X,1}, \nabla_{\mathcal{U}}, \mathcal{O}_{X})$ defines an indigenous $\mathcal{D}^{(N-1)}_{X}$-module, it follows from Lemma 2.2.3 that the Kodaira-Spencer map $\text{KS}_{\mathcal{U}^{\vee}} : T_{X} \to \text{Ker}(\delta)^{\vee}$ associated to the triple $\mathcal{U}^{\vee} := ((\mathcal{D}^{(N-1)}_{X,1})^{\vee}, \nabla^{\vee}_{\mathcal{U}}, \text{Ker}(\delta)^{\vee})$ is an isomorphism; denote by

\[(221) \quad \text{KS}_{\mathcal{U}^{\vee}} : F^{(N)*}_{X/k}(\mathcal{U}^\bullet) \sim \Omega_{X}
\]

the isomorphism induced, in the natural manner, from $\text{KS}_{\mathcal{U}^{\vee}}$ under the identification (220).

Next, let us consider the composite $F^{(N)*}_{X/k}(\text{Sol}(\nabla^{\vee}_{\mathcal{U}})) \sim (\mathcal{D}^{(N-1)}_{X,1})^{\vee} \to \mathcal{O}_{X}$, where the second arrow denotes the dual of the natural inclusion $\mathcal{O}_{X} \leftarrow (\mathcal{D}^{(N-1)}_{X,1}) \to \mathcal{D}^{(N-1)}_{X,1}$. By the adjunction relation “$F^{(N)*}_{X/k}(-) \dashv F^{(N)}_{X/k*}(-)$”, this composite determines a morphism

\[(222) \quad \text{Sol}(\nabla^{\vee}_{\mathcal{U}}) \to F^{(N)}_{X/k*}(\mathcal{O}_{X}).\]

Then, there exists uniquely a morphism $\eta : \mathcal{U}^\bullet \to \mathcal{B}^{(N)}_{X}$ fitting into the following morphism of short exact sequences:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{X}^{(N)} & \xrightarrow{219} & \text{Sol}(\nabla^{\vee}_{\mathcal{U}}) & \longrightarrow & \mathcal{U}^\bullet & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{O}_{X}^{(N)} & \xrightarrow{222} & F^{(N)}_{X/k*}(\mathcal{O}_{X}) & \longrightarrow & F^{(N)}_{X/k*}(\mathcal{U}^\bullet) & \longrightarrow & 0.
\end{array}
\]

Let us pull back it by $F^{(N)}_{X/k}$ to obtain the following morphism of short exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_{X} & \xrightarrow{\delta^{\vee}} & (\mathcal{D}^{(N-1)}_{X,1})^{\vee} & \longrightarrow & F^{(N)*}_{X/k}(\mathcal{U}^\bullet) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{O}_{X} & \xrightarrow{\eta} & F^{(N)*}_{X/k*}(\mathcal{O}_{X}) & \longrightarrow & F^{(N)*}_{X/k*}(\mathcal{B}^{(N)}_{X}) & \longrightarrow & 0.
\end{array}
\]

It follows from the definition of the morphism (222) that this morphism of sequences is compatible with the respective split surjections $(\mathcal{D}^{(N-1)}_{X,1})^{\vee} \to \mathcal{O}_{X}$ (defined as the dual of the natural inclusion $\mathcal{O}_{X} \leftarrow (\mathcal{D}^{(N-1)}_{X,1})$ and $\alpha^{\vee}$ (cf. 200)). Hence, the composite $\text{KS}_{\mathcal{U}^\bullet} \circ F^{(N)*}_{X/k}(\eta)$ (cf. (202) for the definition of $\text{KS}_{\mathcal{U}^\bullet}$) coincides with $\text{KS}_{\mathcal{U}^{\vee}}$, and hence, is an isomorphism. On the other hand, by Lemma 4.3.2, this composite corresponds (up to sign) to the composite $\mathcal{U}^\bullet \xrightarrow{\eta} \mathcal{B}^{(N)}_{X} \xrightarrow{\gamma_{X}} F^{(N)}_{X/k*}(\mathcal{O}_{X})$ via the adjunction relation “$F^{(N)*}_{X/k}(-) \dashv F^{(N)}_{X/k*}(-)$”. This implies that $\mathcal{U}^\bullet$ specifies a Tango structure of level $N$ on $X$. By construction, the resulting assignment $\overset{\circ}{\lambda}_{N}^\mathcal{U} \to \mathcal{U}^\bullet$ is verified to determine the inverse to $\overset{\circ}{\lambda}_{N}^\mathcal{U} \to \mathcal{U}^\bullet$. This completes the proof of the theorem. \qed
4.5. Dual affine connections.

In this subsection, we introduce dual $F^N$-affine connections and study the relationship with Tango structures of level $N$. Let us fix an element $N$ of $\mathbb{Z}_{>0} \sqcup \{\infty\}$.

**Definition 4.5.1.**

A dual affine connection of level $N$ on $X$ is a $D(N-1)$-action $\nabla$ on $\Omega_X$ with vanishing $p-(N-1)$-curvature such that the sheaf of horizontal sections $\text{Sol}(\nabla|^{(1)})$ with respect to its 1-st truncation $\nabla^{(1)}$ is contained in $B_X^{(1)} (\subseteq \Omega_X)$.

Denote by $\mathcal{AC}_{X,N}^{\vee}$ the set of dual affine connections of level $N$ on $X$. If $N'$ is another positive integer with $N' > N$, then the assignment $\nabla \mapsto \nabla^{(1)}$ defines a map of sets $\mathcal{AC}_{X,N}^{\vee} \to \mathcal{AC}_{X,N'}^{\vee}$. The collection $\{\mathcal{AC}_{X,N}^{\vee}\}_{N \in \mathbb{Z}_{>0}}$ forms a projective system and we have

$$\mathcal{AC}_{X,\infty}^{\vee} = \lim_{\leftarrow} \mathcal{AC}_{X,N}^{\vee},$$

Let $U^\bullet \left( \subseteq B_X^{(N)} \right)$ be a Tango structure of level $N$ on $X$. Denote by

$$\nabla_{U^\bullet} \mapsto \star : D^{(N-1)}_X \to \mathcal{G}_{\Omega_X},$$

the structure of $D^{(N-1)}_X$-action on $\Omega_X$ corresponding to $F^{(N)}_{X/k}(U^\bullet)$ (cf. (12)) via $\gamma^{-1} : F^{(N)}_{X/k}(U^\bullet) \to \Omega_X$ (cf. (88)). By construction, the sheaf of horizontal sections $\text{Sol}(\nabla_{U^\bullet}^{\star})$ with respect to $\nabla_{U^\bullet}^{\star}|^{(1)}$ coincides with $U^\bullet|^{(1)}$, which is contained in $B_X^{(1)}$. Thus, $\nabla_{U^\bullet}^{\star}$ forms a dual affine connection of level $N$ on $X$. The resulting assignment $U^\bullet \mapsto \nabla_{U^\bullet}^{\star}$ defines a map of sets

$$\nabla_{U^\bullet}^{\star} : \mathcal{Tan}_{X,N} \to \mathcal{AC}_{X,N}^{\vee}.$$

Next, let

$$\mathcal{Tan}_{X,N}^{-\vee} \text{ (resp., } \mathcal{AC}_{X,N}^{-\vee} \text{)}$$

the subset of $\mathcal{Tan}_{X,N}$ (resp., $\mathcal{AC}_{X,N}^{\vee}$) consisting of Tango structures $U^\bullet$ with $\Gamma(X^{(1)}, (U^\bullet|^{(1)})^\vee) = 0$ (resp., dual affine connections $\nabla_{U^\bullet}$ of level $N$ with $\Gamma(X^{(1)}, \text{Sol}(\nabla_{U^\bullet}|^{(1)})^\vee) = 0$). Then, the map $\nabla_{U^\bullet}^{\star}$ restricts to a map

$$\nabla_{U^\bullet}^{\star} : \mathcal{Tan}_{X,N}^{-\vee} \to \mathcal{AC}_{X,N}^{-\vee},$$

which fits into the following cartesian square diagram:

\[
\begin{array}{ccc}
\mathcal{Tan}_{X,N}^{-\vee} & \xrightarrow{\nabla_{U^\bullet}^{\star}} & \mathcal{AC}_{X,N}^{-\vee} \\
\text{incl} & & \text{incl} \\
\mathcal{Tan}_{X,N} & \xrightarrow{\nabla_{U^\bullet}^{\star}} & \mathcal{AC}_{X,N}^{\vee} \\
\end{array}
\]
Lemma 4.5.2.
Let $N, N'$ be positive integers with $N < N'$ and $\mathcal{U}_N^\bullet$ a Tango structure of level $N$ on $X$. Suppose that $X$ is Frobenius split and there exists a vector bundle $\mathcal{U}^N$ on $X^{(N')}$ together with an isomorphism of $\mathcal{O}_X^{(N')}$-modules $\alpha : F_{X^{(N')}/k}^{(N'-N)}(\mathcal{U}^N) \approx \mathcal{U}_N^\bullet$. Then, there exists a pair
\begin{equation}
(\mathcal{U}_N^\bullet, \alpha')
\end{equation}
consisting of a Tango structure $\mathcal{U}_N^\bullet$ of level $N'$ on $X$ with $\mathcal{U}_N^\bullet|^{(N')} = \mathcal{U}_N^\bullet$ and an isomorphism of $\mathcal{O}_X^{(N')}$-modules $\alpha' : \mathcal{U}^N \approx \mathcal{U}_N^\bullet$ which makes the following square diagram commute:
\begin{equation}
\begin{array}{ccc}
F_{X^{(N')}/k}^{(N'-N)}(\mathcal{U}^N) & \xrightarrow{\alpha} & \mathcal{U}_N^\bullet \\
\downarrow F_{X^{(N')}/k}^{(N'-N)}(\alpha') & & \downarrow \text{id} \\
F_{X^{(N')}/k}^{(N'-N)}(\mathcal{U}_N^\bullet) & \xleftarrow{\approx} & \mathcal{U}_N^\bullet|^{(N')}
\end{array}
\end{equation}

If, moreover, $\Gamma(X^{(N)}, (\mathcal{U}_N^\bullet)^\vee) = 0$, then such a pair $(\mathcal{U}_N^\bullet, \alpha')$ is uniquely determined.

Proof. First, we shall prove the former assertion. Let us observe that $\alpha$ can be decomposed as the composite
\begin{equation}
\alpha : F_{X^{(N'-N)}/k}^{(N'-N)}(\mathcal{U}^N) \xrightarrow{\alpha^+} F_{X^{(N'-N)}/k}^{(N'-N)}(\mathcal{U}_N^\bullet) \xrightarrow{\text{id}^+} \mathcal{U}_N^\bullet,
\end{equation}
where $\alpha^+ : \mathcal{U}^N \to F_{X^{(N'-N)}/k}^{(N'-N)}(\mathcal{U}_N^\bullet)$ and $\text{id}^+$ are the morphisms corresponding, via the adjunction relation “$F_{X^{(N'-N)}/k}^{(N'-N)}(-) \dashv F_{X^{(N'-N)}/k}^{(N'-N)}(-)$”, to $\alpha$ and the identity morphism of $F_{X^{(N'-N)}/k}^{(N'-N)}(\mathcal{U}_N^\bullet)$ respectively. Since $\alpha$ is an isomorphism and $F_{X^{(N'-N)}/k}^{(N'-N)}$ is faithfully flat, this decomposition implies that $\alpha^+$ is injective. Now, let us choose a split injection
\begin{equation}
\sigma : F_{X^{(N')}/k^*}^{(N'-N)}(\mathcal{B}_X^{(N)}) \hookrightarrow \mathcal{B}_X^{(N')}
\end{equation}
of the short exact sequence (184); such a morphism exists because of Lemma 4.4.1 and the assumption that $X$ is Frobenius split. Thus, we obtain the composite of injections
\begin{equation}
\gamma_{\mathcal{U}^N} : \mathcal{U}^N \xrightarrow{\alpha^+} F_{X^{(N'-N)}/k}^{(N'-N)}(\mathcal{U}_N^\bullet) \xrightarrow{\text{id}^+} \mathcal{B}_X^{(N')},
\end{equation}
where the second arrow is obtained from applying the functor $F_{X^{(N'-N)}/k^*}^{(N'-N)}(-)$ to the natural inclusion $\mathcal{U}_N^\bullet \hookrightarrow \mathcal{B}_X^{(N')}$. The morphism $\gamma_{\mathcal{U}^N}^+ : F_{X/k}^{(N')}(\mathcal{U}^N) \to \Omega_X$ associated with the composite
\begin{equation}
\gamma_{\mathcal{U}^N}^+ : \mathcal{U}^N \xrightarrow{\gamma_{\mathcal{U}^N}} \mathcal{B}_X^{(N')} \xrightarrow{\gamma_X^{(N')}} F_{X/k^*}^{(N')}(\Omega_X)
\end{equation}
coincides with the composite $\gamma_{\mathcal{U}^N}^+ \circ F_{X/k}^{(N')}(\alpha)$ (cf. (188)), and hence, is an isomorphism. Thus, the pair
\begin{equation}
(\text{Im}(\gamma_{\mathcal{U}^N}^+), \gamma_{\mathcal{U}^N}^+)
\end{equation}
consisting of the image $\text{Im}(\gamma_{\mathcal{U}^N}^+)$ of $\gamma_{\mathcal{U}^N}^+$ and the morphism $\gamma_{\mathcal{U}^N}$ regarded as an isomorphism $\mathcal{U}^N \approx \text{Im}(\gamma_{\mathcal{U}^N}^+)$ specifies a required pair.
Next, we shall prove the latter assertion. Since the sequence (184) splits (cf. Lemma 4.1.1), it induces the following short exact sequence of $k$-vector spaces:

$$
(239) \quad 0 \longrightarrow \text{Hom}(U^{\vee}, B^{(N'-N)}_{X(N)}) \longrightarrow \text{Hom}(U^{\vee}, B^{(N)}_{X}) \xrightarrow{\delta_{\text{Hom}}^{B}} \text{Hom}(U^{\vee}, F^{(N'-N)}_{X(N)/k}(B^{(N)}_{X})) \longrightarrow 0,
$$

where $\text{Hom}(-, -) := \text{Hom}_{X(N')}(-, -)$. Denote by $\alpha_{B}^{\perp}$ the composite $U^{\vee} \rightarrow F^{(N'-N)}_{X(N)/k}(B^{(N)}_{X})$ of the first and second arrows in (239). If we are given a pair $(U^{\bullet}, \alpha')$ satisfying the requirements, then the composite $U^{\vee} \xrightarrow{\alpha'} U^{\bullet}_{N'} \hookrightarrow B^{(N)}_{X}$ specifies an element in the inverse image $(\delta_{\text{Hom}}^{B})^{-1}(\alpha_{B}^{\perp})$ of the element $\alpha_{B}^{\perp} \in \text{Hom}(U^{\vee}, F^{(N'-N)}_{X(N)/k}(B^{(N)}_{X}))$ via $\delta_{\text{Hom}}^{B}$. In this way, the set of such pairs may be embedded into the affine space $(\delta_{\text{Hom}}^{B})^{-1}(\alpha_{B}^{\perp})$ modeled on the $k$-vector space $\text{Ker}(\delta_{\text{Hom}}^{B})$.

To verify the equality $\text{Hom}(U^{\vee}, B^{(N'-N)}_{X(N)}) = 0$. To this end, let us consider the short exact sequence

$$
(240) \quad 0 \longrightarrow \mathcal{O}_{X(N)} \longrightarrow F^{(N'-N)}_{X(N)/k}(\mathcal{O}_{X(N)}) \longrightarrow B^{(N'-N)}_{X(N)} \longrightarrow 0
$$

defined as (180) with $X$ and $N$ replaced by $X(N)$ and $N' - N$ respectively. Since $X$, as well as $X(N)$, is Frobenius split by assumption, this sequence splits (cf. the proof in Lemma 4.1.1). Thus, (240) induces, after tensoring with $U^{\vee}$, the following short exact sequence:

$$
(241) \quad 0 \longrightarrow \Gamma(X(N), U^{\vee}) \longrightarrow \Gamma(X(N'), U^{\vee} \otimes F^{(N'-N)}_{X(N)/k}(\mathcal{O}_{X(N)})) \longrightarrow \Gamma(X(N'), U^{\vee} \otimes B^{(N'-N)}_{X(N)}) \longrightarrow 0.
$$

Let us observe that

$$
(242) \quad \Gamma(X(N'), U^{\vee} \otimes F^{(N'-N)}_{X(N)/k}(\mathcal{O}_{X(N)})) \simeq \Gamma(X(N'), F^{(N'-N)}_{X(N)/k}(F^{(N'-N)}_{X(N)/k}(U^{\vee})))
$$

$$
\simeq \Gamma(X(N'), F^{(N'-N)}_{X(N)/k}(U^{\bullet}_{N}^{\vee}))
$$

$$
\simeq \Gamma(X(N'), (U^{\bullet}_{N}^{\vee}))
$$

$$
\simeq 0,
$$

where the first isomorphism follows from the projection formula, the second isomorphism follows from $\alpha'$, and the last “$\simeq$” follows from the assumption. This observation and the exactness of (241) implies the equality

$$
(243) \quad \text{Hom}(U^{\vee}, B^{(N'-N)}_{X(N)}) \left( = \Gamma(X(N'), U^{\vee} \otimes B^{(N'-N)}_{X(N)}) \right) = 0.
$$

This completes the proof of the lemma. \qed

By using the above lemma, we can prove the following assertion.

**Proposition 4.5.3.**

The following assertions hold:

(i) The map $\h_{\zeta_{1}}^{\bullet \rightarrow \bullet} \rightarrow \bullet$ is bijective.

(ii) Let $N \in \mathbb{Z}_{>0} \cup \{\infty\}$, and suppose that $X$ is Frobenius split. Then, the map $\h_{\zeta_{N}}^{\bullet \rightarrow \bullet} \rightarrow \bullet$ is surjective and the map $\otimes \h_{\zeta_{N}}^{\bullet \rightarrow \bullet} \rightarrow \bullet$ is bijective.
Proof. Assertion (i) follows directly from the various definitions involved and the equivalence of categories \([113]\). The surjectivity of \(A \star \rightarrow \star\) asserted in (ii) follows from assertion (i) and the former assertion of Lemma \([4.5.2]\) where the pair \((N, N')\) is taken to be \((1, N)\). The bijectivity of \(\star \rightarrow \star\) follows from both the former and latter assertions of Lemma \([4.5.2]\). Notice that assertion (ii) for infinite level can be proved by applying inductively that lemma to the case where \((N, N')\) is taken to be \((N, N + 1)\). □

5. Case 1: Projective and affine spaces

In this section, we study \(F^N\)-projective and \(F^N\)-affine structures (as well as the related structures discussed so far) on projective and affine spaces respectively. Since projective and affine spaces have global coordinate charts, there exist the trivial constructions among these structures. In the case of projective spaces, we will show (cf. Proposition 5.1.1) the uniqueness of \(F^N\)-projective structures of prescribed level \(N\). On the other hand, it will be verified (cf. Proposition 5.2.3) that there exist many \(F^N\)-affine structures on affine spaces; this is an exotic phenomenon that occurs in positive characteristic. The main results of this section provide characterizations of projective spaces from the viewpoint of \(F^\infty\)-structures (cf. Theorems 5.3.1 and 5.4.1). These characterizations can be thought of as positive characteristic analogues of the corresponding results for complex manifolds, proved in \([17]\), \([84]\), and \([34]\).

In what follows, let \(n\) be a positive integer.

5.1. Uniqueness of \(F^N\)-projective structures on projective spaces.

Let \(N\) be a positive integer. To begin with, we construct the trivial \(F^N\)-projective (resp., \(F^N\)-affine) structure on the projective (resp., affine) space \(\mathbb{P}^n\) (resp., \(\mathbb{A}^n\)), as well as construct the corresponding \(F^N\)-indigenous (resp., \(F^N\)-affine-indigenous) structure and dormant indigenous \(D^{(N-1)}\mathbb{P}^n\) (resp., dormant affine-indigenous \(D^{(N-1)}\mathbb{A}^n\)).

\(F^N\)-projective and \(F^N\)-affine structures: We shall set

\[
S_{N,\text{triv}}^\triangledown \quad \text{(resp., } \, A S_{N,\text{triv}}^\triangledown) \]

(244)

to be the subsheaf of \(\mathcal{P}^\text{\acute{e}t}_{\mathbb{P}^n}\) (resp., \(\mathcal{A}^\text{\acute{e}t}_{\mathbb{A}^n}\)) which, to any open subscheme \(U\) of \(\mathbb{P}^n\) (resp., \(\mathbb{A}^n\)), assigns the set

\[
S_{N,\text{triv}}^\triangledown(U) := \left\{ \overline{\alpha}(\phi_U) \in \mathcal{P}^\text{\acute{e}t}_{\mathbb{P}^n}(U) \mid \overline{\alpha} \in (\text{PGL}_{n+1})_{\mathbb{P}^n}(U) \right\} 
\]

(245)

(resp., \( \, A S_{N,\text{triv}}^\triangledown(U) := \left\{ \overline{\alpha}(\phi_U) \in \mathcal{A}^\text{\acute{e}t}_{\mathbb{A}^n}(U) \mid \overline{\alpha} \in (\text{PGL}_{n+1})_{\mathbb{A}^n}(U) \right\} \),

where \(\phi_U\) denotes the natural open immersion \(U \hookrightarrow \mathbb{P}^n\) (resp., \(U \hookrightarrow \mathbb{A}^n\)). That is to say, \(S_{N,\text{triv}}^\triangledown\) (resp., \( \, A S_{N,\text{triv}}^\triangledown\)) is obtained as the smallest subsheaf of \(\mathcal{P}^\text{\acute{e}t}_{\mathbb{P}^n}\) (resp., \(\mathcal{A}^\text{\acute{e}t}_{\mathbb{A}^n}\)) that is closed under the \((\text{PGL}_{n+1})_{\mathbb{P}^n}\)-action (resp., the \((\text{PGL}_{n+1})_{\mathbb{A}^n}\)-action) and contains the global section determined by the identity morphism of \(\mathbb{P}^n\) (resp., \(\mathbb{A}^n\)). Then, \(S_{N,\text{triv}}^\triangledown\) (resp., \( \, A S_{N,\text{triv}}^\triangledown\)) specifies
an $F^N$-projective (resp., $F^N$-affine) structure on $\mathbb{P}^n$ (resp., $\mathbb{A}^n$). The formation of $S_N^{\vee}$ (resp., $^\wedge S_N^{\vee}$) is compatible with truncation to lower levels, so the collection

$$S_{\infty, \text{triv}}^{\vee} := \{S_N^{\vee}\}_{N \in \mathbb{Z}_{>0}} \quad (\text{resp., } ^\wedge S_{\infty, \text{triv}}^{\vee} := \{^\wedge S_N^{\vee}\}_{N \in \mathbb{Z}_{>0}})$$

forms an $F^\infty$-projective (resp., $F^\infty$-affine) structure on $\mathbb{P}^n$ (resp., $\mathbb{A}^n$).

$F^N$-indigenous and $F^N$-affine-indigenous structures:

Next, it follows from Proposition 1.6.1 that the pair

$$\mathcal{E}_{N, \text{triv}}^{\bullet} := (\mathbb{P}^n(N) \times \text{PGL}_{n+1}, \mathcal{E}_{\text{red}}^{\text{univ}}) \quad (\text{resp., } ^\wedge \mathcal{E}_{\text{N, triv}}^{\bullet} := (\mathbb{A}^n(N) \times \text{PGL}_{n+1}, ^\wedge \mathcal{E}_{\text{red}}^{\text{univ}}))$$

(cf. (35) for the definitions of $\mathcal{E}_{\text{red}}^{\text{univ}}$ and $^\wedge \mathcal{E}_{\text{red}}^{\text{univ}}$) specifies an $F^N$-indigenous (resp., $F^N$-affine-indigenous) structure on $\mathbb{P}^n$ (resp., $\mathbb{A}^n$). Moreover, if

$$\alpha_{N, \text{triv}} : F_{\mathbb{P}^n(N)/k}^{(1)*}(\mathbb{P}^n(N+1) \times \text{PGL}_{n+1}) \sim \mathbb{P}^n(N) \times \text{PGL}_{n+1}$$

$$\quad \left(\text{resp., } ^\wedge \alpha_{N, \text{triv}} : F_{\mathbb{A}^n(N)/k}^{(1)*}(\mathbb{A}^n(N+1) \times \text{PGL}_{n+1}) \sim \mathbb{A}^n(N) \times \text{PGL}_{n+1}\right)$$

denotes the natural isomorphism of principal PGL$_{n+1}$-bundles (resp., principal PGL$_{n+1}^\wedge$-bundles), then the collection

$$\mathcal{E}_{\infty, \text{triv}}^{\bullet} := \{(\mathcal{E}_{N, \text{triv}}, \alpha_{N, \text{triv}})\}_{N \in \mathbb{Z}_{>0}} \quad (\text{resp., } ^\wedge \mathcal{E}_{\infty, \text{triv}}^{\bullet} := \{^\wedge (\mathcal{E}_{N, \text{triv}}, ^\wedge \alpha_{N, \text{triv}})\}_{N \in \mathbb{Z}_{>0}})$$

forms an $F^\infty$-indigenous (resp., $F^\infty$-affine-indigenous) structure on $\mathbb{P}^n$ (resp., $\mathbb{A}^n$).

Dormant indigenous and affine-indigenous $D^{(N-1)}$-modules:

We shall observe that there exists a canonical theta characteristic on the projective space $\mathbb{P}^n := \text{Proj}(k[x_0, x_1, \ldots, x_n])$ defined as follows. Let

$$\eta_0 : \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}$$

be the $\mathcal{O}_{\mathbb{P}^n}$-linear injection given by $w \mapsto \sum_{i=0}^n w x_i \cdot e_i$ for each local section $w \in \mathcal{O}_{\mathbb{P}^n}(-1)$, where $(e_0, \ldots, e_n)$ is the standard basis of $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}$. The natural short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\eta_0} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0) \rightarrow 0$$

induces a composite isomorphism

$$\mathcal{O}_{\mathbb{P}^n}(-1) \sim \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))^\vee \otimes \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}) \sim \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))^\vee,$$

where the second arrow arises from the isomorphism $\mathcal{O}_{\mathbb{P}^n} \sim \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)})$ given by $1 \mapsto e_0 \wedge \cdots \wedge e_n$. Also, observe that the composite

$$\mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\eta_0} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{d_{\mathbb{P}^n}^{(n+1)}} \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \rightarrow \Omega_{\mathbb{P}^n} \otimes (\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))$$

is $\mathcal{O}_{\mathbb{P}^n}$-linear and induces an isomorphism of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$\mathcal{O}_{\mathbb{P}^n}(-1) \otimes (\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))^\vee \sim \Omega_{\mathbb{P}^n}.$$
Thus, the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)\otimes (n+1)$ is an isomorphism; we shall identify $\mathcal{O}_{\mathbb{P}^n}(-1)\otimes (n+1)$ as an isomorphism, and hence, (253) shows that, for each $N$, $\mathcal{D}^{(N-1)}_{\mathbb{P}^n,1}$ with $\mathcal{O}_{\mathbb{P}^n(-1)}$ and $\mathcal{D}^{(N-1)}_{\mathbb{P}^n,1}$, and $\mathcal{O}_{\mathbb{P}^n(-1)}$ and $\mathcal{D}^{(N-1)}_{\mathbb{P}^n,1}$, respectively. Moreover, if $p \not\mid (n+1)$, then we have

$$\mathcal{D}^{(N-1)}_{\mathbb{P}^n,1, N, \mathcal{O}_{\mathbb{P}^n(-1)}(-1)} = \{ \mathcal{D}_{\mathbb{P}^n, N, \mathcal{O}_{\mathbb{P}^n(-1)}(-1)} \}.$$
Proof. We shall prove assertion (i). The former assertion follows directly from the various definitions involved. To prove the latter assertion, let us recall that $\zeta_N^\otimes \Rightarrow \bullet; \bar{\zeta}_N^\otimes \Rightarrow \otimes; \zeta_{N,\mathcal{O}_\mathbb{P}^n(-1)}$, and $\gamma_{N,\mathcal{O}_\mathbb{P}^n(-1)}$ are bijective (cf. Theorem \textbf{3.1.3}, Corollary \textbf{3.4.5}, and Theorem \textbf{3.6.4}). Thus, the problem is reduced to proving the claim that the set $\mathcal{D}_{\mathcal{O}_{X,N},\mathcal{O}_\mathbb{P}^n(-1)}$ consists exactly of $\mathcal{V}_{N,\mathbb{P}^n}$. Moreover, by the identification $\mathcal{D}_{\mathcal{O}_{X,N},\mathcal{O}_\mathbb{P}^n(-1)} \cong \lim_{\leftarrow N \in \mathbb{Z}_{\geq 0}} \mathcal{D}_{\mathcal{O}_{X,N},\mathcal{O}_\mathbb{P}^n(-1)}$ resulting from Proposition \textbf{3.9.5} (ii), it suffices to consider the case where $N \neq \infty$.

Let $\mathcal{V}^\otimes := (\mathcal{D}_{\mathcal{O}_{X,1}}(N-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1), \nabla_{\mathcal{V}}, \mathcal{O}_\mathbb{P}^n(-1))$ be a dormant indgenous $\mathcal{D}_{\mathcal{O}_{X,1}^N}(N-1)$-module representing an element of $\mathcal{D}_{\mathcal{O}_{X,N},\mathcal{O}_\mathbb{P}^n(-1)}^\otimes$. We identify the underlying vector bundle $\mathcal{D}_{\mathcal{O}_{X,1}}^\otimes \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ of $\mathcal{V}^\otimes$ with $\mathcal{O}_{\mathbb{P}^1}^{(\mathbb{P}^n)}(n+1)$ via (257). Since $\nabla_{\mathcal{V}}$ has vanishing $p(N-1)$-curvature, the natural morphism $\alpha : F_{\mathbb{P}^n/k}(\mathcal{D}_{\mathcal{O}_{X,1}}(N-1)) \to \mathcal{O}_{\mathbb{P}^1}^{(\mathbb{P}^n)}(n+1)$ is an isomorphism. According to Lemma \textbf{5.1.2} described below, there exists an isomorphism $\beta_0 : \mathcal{D}_{\mathcal{O}_{X,1}}(N-1) \cong \mathcal{O}_{\mathbb{P}^1}^{(\mathbb{P}^n)}(n+1)$. Denote by $\beta : F_{\mathbb{P}^n/k}(\mathcal{D}_{\mathcal{O}_{X,1}}(N-1))$ the pull-back of $\beta_0$ by $F_{\mathbb{P}^n/k}(N)$, where we identify $F_{\mathbb{P}^n/k}(N)$ with $\mathcal{O}_{\mathbb{P}^1}$ by passing to the natural isomorphism induced from $F_{\mathbb{P}^n/k}(N)$. Because of the properness of $X$, the automorphism $\alpha \circ \beta^{-1}$ of $\mathcal{O}_{\mathbb{P}^1}^{(\mathbb{P}^n)}(n+1)$ may be described as an element of $\text{GL}_{n+1}(k)$. Hence, there exists an automorphism $\gamma_0$ of $\mathcal{O}_{\mathbb{P}^n}(n+1)$ inducing $\alpha \circ \beta^{-1}$ via pull-back by $F_{\mathbb{P}^n/k}$. The pull-back of $\gamma_0 \circ \beta_0 : \mathcal{D}_{\mathcal{O}_{X,1}}(N-1)$ by $F_{\mathbb{P}^n/k}((\mathbb{P}^n)(n+1))$, i.e., $\mathcal{V}_{N,\mathbb{P}^n}^\otimes = \mathcal{V}_{N,\mathbb{P}^n}^\otimes \circ \beta_0$. This completes the proof of assertion (i).

Next, assertion (ii) follows from the resp’d portion of Theorem \textbf{3.7.1} (i), together with the following equalities in $CH^1(\mathbb{P}^1)$ (i.e., $\mathbb{P}^n$ isomorphic to a direct sum of finite copies of the trivial line bundle) if and only if the pull-back $F_{\mathbb{P}^n/k}^*(\mathcal{F})$ is trivial:

\begin{equation}
\tag{260}
c_1(\mathbb{P}^n) = c_1(\omega_{\mathbb{P}^n}) = c_1(\mathcal{O}_{\mathbb{P}^n}(n+1)) = (n+1) \cdot c_1(\mathcal{O}_{\mathbb{P}^n}(1)).
\end{equation}

This completes the proof of the proposition. \hfill \Box

The following lemma was used in the proof of the above proposition.

\textbf{Lemma 5.1.2.}

Let $N$ be a positive integer and $\mathcal{F}$ a vector bundle on $\mathbb{P}^n(N)$. Then, $\mathcal{F}$ is trivial (i.e., isomorphic to a direct sum of finite copies of the trivial line bundle) if and only if the pull-back $F_{\mathbb{P}^n/k}^*(\mathcal{F})$ is trivial.

\textbf{Proof.} It suffices to consider the “if” part because the inverse direction is clear. In the following discussion, we shall prove the triviality of $\mathcal{F}$ under the assumption that $F_{\mathbb{P}^n/k}^*(\mathcal{F})$ is trivial. Let us take an arbitrary rational curve in $\mathbb{P}^n(N)$, which may be obtained as the $N$-th Frobenius twist $C(N)$ for some rational curve $C$ in $\mathbb{P}^n$. Write $f : \mathbb{P}^1 \to \mathbb{P}^n$ for the normalization of $C$. Hence, its base-change $f(N) : \mathbb{P}^1/(\mathbb{P}^1) \to \mathbb{P}^n/(\mathbb{P}^n) := \mathbb{P}^n$ gives the normalization of $C(N)$. According to \textbf{8}, Theorem 1.1 (or \textbf{74}, Theorem 1.1), the problem is reduced to proving the claim that the pull-back $f(N)^*(\mathcal{F})$ of $\mathcal{F}$ via $f(N)$ is trivial. (We can apply the result of \textbf{8} because $\mathbb{P}^n$ is separably rationally connected, as mentioned in \textbf{45}, Chap.IV, §3, Example 3.2.6.) Recall the Birkhoff-Grothendieck’s theorem, asserting that any vector bundle on $\mathbb{P}^1$ is
The following assertions hold:

Proposition 5.2.2.  

isomorphic to a direct sum of line bundles. Hence, we have a decomposition

\[ f^{(N)}(\mathcal{F}) \cong \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^{(N)}(a_j)}, \]

where \( m := \text{rank}(\mathcal{F}) \) and \( a_1 \leq a_2 \leq \cdots \leq a_m \). This decomposition implies that

\[ f^*(f^{(N)}_{X/k}(\mathcal{F})) \cong f^*(\mathcal{O}^{\otimes m}_{\mathbb{P}^{(N)}}) \cong \mathcal{O}_{\mathbb{P}^{1}}(p^N a_j). \]

On the other hand, the triviality assumption on \( F_{\mathbb{P}^{n}/k}^{(N)}(\mathcal{F}) \) implies that

\[ f^*(\mathcal{O}^{\otimes m}_{\mathbb{P}^{1}}) \cong \mathcal{O}_{\mathbb{P}^{1}}(p^N a_j). \]

By (262) and (263), we can immediately verify the equalities \( a_i = 0 \) (for all \( i = 1, 2, \cdots, m \)). Consequently, \( f^{(N)}(\mathcal{F}) \) turns out to be trivial. This completes the proof of the lemma. \( \square \)

5.2. Nonuniqueness of \( F^{N} \)-affine structures on affine spaces.

Next, we deal with \( F^{N} \)-affine structures on affine spaces. Unlike the case of \( \mathbb{P}^{n} \), we will see that there are many \( F^{N} \)-affine structures other than the trivial one (if \( N > 1 \)). This fact can be thought of as an exotic phenomenon in positive characteristic because complex affine spaces have exactly one affine structure. Before describing our result, let us introduce and study global \( F^{N} \)-projective and \( F^{N} \)-affine structures, as follows.

Let \( X \) be a smooth variety over \( k \) of dimension \( n \) and let \( N \in \mathbb{Z}_{>0} \sqcup \{ \infty \} \).

Definition 5.2.1.  

Let \( S^{\oslash} \) be an \( F^{N} \)-projective (resp., \( F^{N} \)-affine) structure on \( X \). If \( N < \infty \), then we shall say that \( S^{\oslash} \) is global if \( S^{\oslash} \) has a global section, or equivalently, the principal \( \text{PGL}_{n+1} \)-bundle (resp., the principal \( \text{PGL}_{n+1} \)-bundle) on \( X^{(N)} \) determined by \( S^{\oslash} \) is trivial. Also, if \( N = \infty \), then we shall say that \( S^{\oslash} \) is global if, for any \( N' \in \mathbb{Z}_{>0}, \) the \( N' \)-th truncation \( S^{\oslash}_{N'|^{(N')}} \) is global.

Regarding global \( F^{N} \)-projective and \( F^{N} \)-affine structures, we have the following proposition.

Proposition 5.2.2.  

The following assertions hold:

(i) \( X \) admits a global \( F^{N} \)-projective (resp., \( F^{N} \)-affine) structure if and only if there exists an étale \( k \)-morphism \( X \to \mathbb{P}^{n} \) (resp., \( X \to \mathbb{A}^{n} \)).

(ii) Suppose that \( X \) is proper over \( k \). Then, \( X \) is isomorphic to \( \mathbb{P}^{n} \) if \( X \) admits a global \( F^{N} \)-projective structure. Moreover, \( X \) admits no global \( F^{N} \)-affine structures.

(iii) If \( X \) admits a global \( F^{N} \)-projective (resp., \( F^{N} \)-affine) structure \( S^{\oslash}_{N} \), then, for any \( N' \geq N \), \( X \) also admits a global \( F^{N'} \)-projective (resp., \( F^{N'} \)-affine) structure \( S^{\oslash}_{N'} \) with \( S^{\oslash}_{N'|^{(N')}} = S^{\oslash}_{N'} \).

Proof. Assertion (i) follows from the definition of a global \( F^{N} \)-projective (\( F^{N} \)-affine) structure. In fact, given an étale \( k \)-morphism \( \phi : X \to \mathbb{P}^{n} \) (resp., \( \phi : X \to \mathbb{A}^{n} \)), we obtain a global \( F^{N} \)-projective (resp., \( F^{N} \)-affine) structure \( S^{\oslash}_{\phi} \) on \( X \) defined as the subsheaf of \( \mathcal{P}^{\text{ét}}_{X} \) (resp., \( \mathcal{A}^{\text{ét}}_{X} \))
which, to each open subscheme $U$ of $X$, assigns the set of étale morphisms of the form $\overline{A}(\phi|_U)$ (cf. (13)) for some $\overline{A} \in (\text{PGL}_{n+1})^X(U)$ (resp., $\overline{A} \in (\text{PGL}^A_{n+1})^X(U)$).

The former assertion of (ii) follows from assertion (i) and the well-known fact that $\mathbb{P}^n$ has no nontrivial étale coverings. In fact, the properness on $X$ implies that an étale morphism $X \to \mathbb{P}^n$ defining a global $F^N$-projective structure becomes an étale covering, and hence, an isomorphism. Moreover, the latter assertion can be verified as follows. Suppose that $X$ admits a global $F^N$-affine structure, which contains an étale morphism of $\mathbb{A}^n$. Since $\mathbb{A}^n$ is affine, the image of $\phi$ must be a point. But, it contradicts the fact that any étale morphism of varieties is open.

Finally, assertion (iii) follows from assertion (i) together with the construction of $S^0_n$ described above. \hfill $\square$

Now, let us discuss the affine space $\mathbb{A}^n$. Consider $\mathbb{A}^n (\subseteq \mathbb{P}^n)$ as the affine scheme associated to the polynomial ring $R_0 := k[t_1, \cdots, t_n]$, where $t_i := x_i/x_0$ ($i = 1, \cdots, n$). Also, for a positive integer $N$, we identify $\mathbb{A}^{n(N)}$ with the affine scheme associated with the subring $R_N := k[t_1^N, \cdots, t_n^N]$ of $R_0$ and $F_n^{(N)}$ with the morphism $\mathbb{A}^n \to \mathbb{A}^{n(N)}$ induced by the inclusion $R_N \hookrightarrow R_0$.

We shall set

$$(264) \quad \text{End}^{\text{et}}_A := \left\{ (f_i)_{i=1}^n \in R_0^n \left| \det (\partial f_i/\partial t_j)_{ij} \in k^\times \right. \right\}.$$ 

A $\text{PGL}^A_{n+1}(R_N)$-action on $\text{End}^{\text{et}}_A$ is defined as follows. Take $\overline{A} \in \text{PGL}^A_{n+1}(R_N)$ and $\overline{f} := (f_i)_i \in \text{End}^{\text{et}}_A$. The element $\overline{A}$ may be described uniquely as $\begin{pmatrix} 1 & 0 \\ t \mathbf{a} & A' \end{pmatrix}$ for some $\mathbf{a} := (g_i)_{i=1}^n \in R_N^n$ and $A' \in \text{GL}_n(R_N)$. Denote by $\overline{A}(\overline{f})$ the element of $R_0^n$ defined as

$$(265) \quad \overline{A}(\overline{f}) := (f_1^{\overline{A}}, \cdots, f_n^{\overline{A}}), \quad \text{where} \quad f_i^{\overline{A}} := g_i + \sum_{j=1}^n a_{ij} \cdot f_j \quad (i = 1, \cdots, n).$$

Then, the following equalities hold:

$$(266) \quad \det \left( \partial f_i^{\overline{A}}/\partial t_j \right)_{ij} = \det \left( \sum_{i=1}^n a_{ii} \cdot (\partial f_i/\partial t_j) \right)_{ij} = \det(A') \cdot \det(\partial f_i/\partial t_j)_{ij},$$

where the rightmost side belongs to $k^\times = (R_0^\times)$ because of the assumptions on $A'$ and $\overline{f}$. This implies that $\overline{A}(\overline{f})$ belongs to $\text{End}^{\text{et}}_A$. The resulting assignment $(\overline{A}, \overline{f}) \mapsto \overline{A}(\overline{f})$ is verified to define a $\text{PGL}^A_{n+1}(R_N)$-action on $\text{End}^{\text{et}}_A$. In particular, we obtain the orbit set

$$(267) \quad \text{End}^{\text{et}}_A / \text{PGL}^A_{n+1}(R_N)$$

of $\text{End}^{\text{et}}_A$ with respect to this action.

**Proposition 5.2.3.**
Let $N$ be a positive integer. Then, the following assertions hold:

(i) Any $F^N$-projective (resp., $F^N$-affine) structure on $\mathbb{A}^n$ is global. In particular, the map

$$(268) \quad F^N \cdot \text{Proj}_{\mathbb{A}^n} \to F^N \cdot \text{Proj}_{\mathbb{A}^n} \quad \text{(resp.,} \quad F^N \cdot \text{Aff}_{\mathbb{A}^n} \to F^N \cdot \text{Aff}_{\mathbb{A}^n} \text{)}$$

(for any $N' > N$) given by truncation is surjective.
(ii) There exists a canonical bijection of sets

\[ \text{End}_{\mathbb{A}^n}/\text{PGL}_{n+1}(R_N) \overset{\sim}{\rightarrow} F^N-\text{Aff}_{\mathbb{A}^n}. \]  

Proof. First, we shall prove the former assertion of (i). We only consider the resp’d assertion because the proof of the non-resp’d assertion is simpler. Let \( S^\diamond \) be an \( F^N \)-affine structure on \( \mathbb{A}^n \). According to Proposition 5.2.2, we can find a dormant affine-indigenous \( D_{(n-1)} \)-module \( A^\diamond := (V, \nabla_V, \mathcal{N}, \delta) \) of level \( N \) on \( \mathbb{A}^n \) with \( \lambda^\diamond \mathcal{N} \rangle \rangle \mathcal{N} \rangle = S^\diamond. \) The vector bundle \( \text{Sol}(\nabla_V) \) on \( \mathbb{A}^{n(N)} \) together with the subbundle \( \text{Sol}(\nabla_V|_{\text{Ker}(\delta)}) \) corresponds, via projectivization, to the principal \( \text{PGL}_{n+1} \)-bundle determined by \( S^\diamond \). Recall here that every vector bundle on \( \mathbb{A}^n \) is trivial by the Quillen-Suslin theorem, and that every extension of vector bundles split since \( \mathbb{A}^n \) is affine. Hence, the inclusion \( \text{Sol}(\nabla_V|_{\text{Ker}(\delta)}) \hookrightarrow \text{Sol}(\nabla_V) \) may be identified with the inclusion \( \mathcal{O}_{\mathbb{A}^n} \hookrightarrow \mathcal{O}_{\mathbb{A}^n}^{(n+1)} \) into the latter \( n \) factors. This implies that the principal \( \text{PGL}_{n+1} \)-bundle determined by \( S^\diamond \) is trivial, i.e., \( S^\diamond \) is global. This completes the proof of the former assertion of (i). The latter assertion of (i) follows from the former assertion and Proposition 5.2.2 (iii).

Next, we shall consider assertion (ii). Let us take \( \overline{f} := (f_i)_i \in \text{End}_{\mathbb{A}^n}^{st} \). Since \( \det(\partial f_i/\partial t_j)_{ij} \in k^\times \), the endomorphism \( \phi_{\overline{f}} \) of \( \mathbb{A}^n \) given by \( t_i \mapsto f_i \) \( (i = 1, \ldots, n) \) is étale. In other words, \( \phi_{\overline{f}} \) specifies a global section of \( \mathcal{A}_{\mathbb{A}^n}^{st} \). Denote by \( S^\diamond_{\overline{f}} \) the global \( F^N \)-affine structure on \( \mathbb{A}^n \) generated by \( \phi_{\overline{f}} \). To be precise, \( S^\diamond_{\overline{f}} \) denotes the subsheaf of \( \mathcal{A}_{\mathbb{A}^n}^{st} \) determined by assigning

\[ U \mapsto \{ \overline{A}(\phi_{\overline{f}}|_U) \mid \overline{A} \in (\text{PGL}_{n+1}(\mathbb{A}^n))^\times(U) \} \]

(270) \[ \text{End}_{\mathbb{A}^n}^{st} \rightarrow F^N-\text{Aff}_{\mathbb{A}^n}. \]

By the definition of the \( \text{PGL}_{n+1}(R_N) \)-action on \( \text{End}_{\mathbb{A}^n}^{st} \), one verifies that two elements \( \overline{f}_1, \overline{f}_2 \) of \( \text{End}_{\mathbb{A}^n}^{st} \) specify the same \( \text{PGL}_{n+1}(R_N) \)-orbit if and only if \( S_{\overline{f}_1}^\diamond = S_{\overline{f}_2}^\diamond \). Hence, the map (270) factors through the quotient \( \text{End}_{\mathbb{A}^n}^{st} \rightarrow \text{End}_{\mathbb{A}^n}^{st}/\text{PGL}_{n+1}(R_N) \) and the resulting map \( \text{End}_{\mathbb{A}^n}^{st}/\text{PGL}_{n+1}(R_N) \rightarrow F^N-\text{Aff}_{\mathbb{A}^n} \) is injective. This map is also surjective by the former assertion of (i). Consequently, we have obtained the desired bijection. This completes the proof of the proposition. \[ \square \]

By means of the above proposition, we can conclude the nonuniqueness of \( F^N \)-affine structures on \( \mathbb{A}^n \) for \( N > 1 \), as follows.

Proposition 5.2.4.

Let \( N \in \mathbb{Z}_{>0} \cup \{ \infty \} \). Then, the following assertions hold:

(i) The following equalities hold:

\[ \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \overset{\mathcal{O} \mathcal{V}}{\mathcal{A} \mathcal{N}} = \overset{\text{triv}}{\mathcal{A} \mathcal{S}}, \quad \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \overset{\mathcal{O} \mathcal{V}}{\mathcal{A} \mathcal{N}} \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \overset{\mathcal{O} \mathcal{V}}{\mathcal{A} \mathcal{N}} = \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \overset{\mathcal{O} \mathcal{V}}{\mathcal{A} \mathcal{N}}. \]  

(ii) Suppose that \( (n, N) = (1, 1) \). Then, we have

\[ F^1-\text{Aff}_{\mathbb{A}^1} = \{ \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \}, \quad F^1-\text{Ind}_{\mathbb{A}^1} = \{ \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \}, \]

\[ \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} = \{ \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \}, \quad \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} = \{ \overset{\text{triv}}{\mathcal{A}_{\mathbb{A}^1}} \}. \]
(iii) Suppose that \( N \neq 1 \). Then, the sets \( F^N \text{-Aff}_{A^n}, F^N \text{-Ind}_{A^n}, \text{Aff}_A^{\Delta_{n,N}}, \text{Ind}_A^{\Delta_{n,N}}, \) and \( \text{Aff}_A^{\Delta_{n,N},0_{\text{triv}}(-1)} \) are (nonempty and) not singletons.

**Proof.** Assertion (i) follows from the various definitions involved.

Next, we shall prove the equalities (272) asserted in (ii). By Theorem 3.4.3 Corollary 3.4.5 and Theorem 3.6.1, it suffices to show that the set \( F^1 \text{-Aff}_{A^1} \) consists only of \( \text{Aff}_A^{\Delta^0_{1,\text{triv}}} \). It follows from Proposition 5.2.3 (ii), that \( F^1 \text{-Aff}_{A^1} \) corresponds bijectively to the set \( \text{End}_{A^1}^{\text{triv}}(\text{Rep}_k^A(R_1)) \). This set is nonempty because it has an element corresponding to \( A \). Let \( t \in A \) and the category \( \text{Rep}_k^A \). As an example, we know (cf. [20], Theorem 2.2) that the stratified fundamental group of \( X \) is trivial, then \( F^1 \text{-Aff}_{A^1} \) is a singleton. The proof of assertion (ii) was completed. Then, by the definition of the \( \text{PGL}_{A^1}^A(R_1) \)-action on \( \text{End}_{A^1}^{\text{triv}}(\text{Rep}_k^A) \), we can verify that \( f \) and \( t \) specify the same \( \text{PGL}_{A^1}^A(R_1) \)-orbit. Consequently, \( \text{End}_{A^1}^{\text{triv}}(\text{Rep}_k^A) \), as well as \( F^1 \text{-Aff}_{A^1} \), is a singleton. The proof of the proposition is completed.

5.3. Characterization of projective spaces I (Stratified fundamental group).

Let \( X \) be a smooth variety over \( k \) of dimension \( n \) and let \( N \in \mathbb{Z}_{>0} \cup \{ \infty \} \). Here, we shall recall the notion of stratified fundamental group. Denote by \( \text{Str}(X) \) the category of \( \mathcal{D}_X^{\infty} \)-modules, which is equivalent to the category of \( F \)-divided sheaves on \( X \) (cf. [17]) and forms an abelian \( k \)-linear rigid tensor category. Once we choose a \( k \)-rational point \( x : \text{Spec}(k) \to X \), the assignment \( (\mathcal{V}, \nabla|_\mathcal{V}) \mapsto x^*(\mathcal{V}) \) defines a functor \( \omega_x : \text{Str}(X) \to \text{Vec}_k \), where \( \text{Vec}_k \) denotes the category of finite dimensional \( k \)-vector spaces. The category \( \text{Str}(X) \) together with the functor \( \omega_x \) forms a neutral Tannakian category (cf. [16], §2.2). In particular, there exists a pro-algebraic group \( k \)-scheme \( \pi_1^{\text{str}}(X, x) \) such that \( \omega_x \) induces an equivalence between \( \text{Str}(X) \) and the category \( \text{Rep}_k(\pi_1^{\text{str}}(X, x)) \) of finite dimensional \( k \)-representations of \( \pi_1^{\text{str}}(X, x) \). The pro-algebraic group \( \pi_1^{\text{str}}(X, x) \) will be referred to as the **stratified fundamental group** of \( X \) (at \( x \)). We sometimes use the notation \( \pi_1^{\text{str}}(X) \) to denote \( \pi_1^{\text{str}}(X, x) \) if there is no fear of confusion. As an example, we know (cf. [20], Theorem 2.2) that the stratified fundamental group of \( \mathbb{P}^n \) is trivial, i.e., every \( \mathcal{D}^{\infty}_{\mathbb{P}^n} \)-module is isomorphic to a direct sum of finite copies of \( (\mathcal{O}_{\mathbb{P}^n}, \nabla^{\text{triv}}(\infty)) \).

Then, we prove the following assertion, which is the positive characteristic version of [17], Theorem 4.4.

**Theorem 5.3.1.**

Let \( X \) be a smooth projective variety over \( k \) of dimension \( n \) admitting an \( F^\infty \)-projective structure on \( X \). If \( \pi_1^{\text{str}}(X) \) is trivial, then \( X \) is isomorphic to \( \mathbb{P}^n \).

**Proof.** Suppose that there exists an \( F^\infty \)-projective structure \( \mathcal{S}_N^{\infty} := \{ \mathcal{S}_N^{\infty} \}_{N \in \mathbb{Z}_{>0}} \) on \( X \). For each \( N \in \mathbb{Z}_{>0} \), we shall denote by \( \mathcal{E}_N \) the principal \( \text{PGL}_{n+1} \)-bundle on \( X^{(N)} \) determined by \( \mathcal{S}_N^{\infty} \). If we are given an arbitrary faithful representation \( \text{PGL}_{n+1} \to \text{GL}_l \) (for \( l > 0 \)), then the \( F \)-divided sheaf corresponding to the collection \( \{ \mathcal{E}_N \times_{\text{PGL}_{n+1}} \text{GL}_l \}_{N \in \mathbb{Z}_{>0}} \) is trivial because of
the assumption $\pi_1^{\text{str}}(X) = 1$. It follows from [8], §3.4, Lemma 3.5, (ii), that the principal $\text{PGL}_{n+1}$-bundle $E_N$ is trivial for every $N \in \mathbb{Z}_{>0}$. (Note that the result in loc. cit. is asserted under the assumption that $X$ is separably rationally connected. But, this assumption was not used in its proof.) Hence, $S^\circ$ turns out to be global, so the assertion follows from Proposition 5.2.2 (ii).

\[\square\]

**Remark 5.3.2.**
We shall recall the relationship between $\pi_1^{\text{str}}(X, x)$ and the étale fundamental group $\pi_1^{\text{et}}(X) := \pi_1^{\text{et}}(X, x)$ of $X$ at $x$. According to [16], §2.4, Proposition 13, there exists a natural quotient homomorphism of group schemes $\pi_1^{\text{str}}(X) \to \pi_1^{\text{et}}(X)$, where we regard $\pi_1^{\text{et}}(X)$ as a constant group scheme. In particular, $\pi_1^{\text{et}}(X)$ is trivial if $\pi_1^{\text{str}}(X)$ is trivial. Moreover, it follows from [18], Theorem 1.1, that the converse is true if $X$ is a projective smooth variety. This result may be thought of as a positive characteristic analogue of the Malcev-Grothendieck theorem (cf. [55], [24]). By taking account of this fact and the above theorem, we conclude (cf. Theorem C) that any $n$-dimensional smooth projective variety admitting an $F^\infty$-projective structure with trivial $\pi_1^{\text{et}}(X)$ is isomorphic to the projective space $\mathbb{P}^n$.

Also, according to [15], §3, Corollaire 3.6, any separably rationally connected variety has trivial $\pi_1^{\text{et}}(X)$. Hence, the above theorem remains true when the assumption “$\pi_1^{\text{str}}(X)$ is trivial” is replaced by “$X$ is separably rationally connected”. See §2 in loc. cit. for the definition of a separably rationally connected variety.

Let us consider representations of the stratified fundamental group associated with $F^\infty$-projective and $F^\infty$-affine structures. For algebraic groups $G, H$ over $k$, then we write

\[\text{Out}(G, H) := \text{Hom}(G, H)/\text{Inn}(H),\]

where $\text{Hom}(G, H)$ denotes the set of homomorphisms from $G$ to $H$ and $\text{Inn}(H)$ denotes the group of inner automorphisms of $H$, acting on $\text{Hom}(G, H)$ in the natural manner; we shall call each element of $\text{Out}(G, H)$ as an **outer homomorphism** from $G$ to $H$.

Let $X$ be a smooth projective variety over $k$ of dimension $n$ and $x$ a $k$-rational point of $X$. First, let $S^\circ$ be an $F^\infty$-projective structure on $X$. Suppose that $p \nmid (n + 1)$ and $X$ admits an $F^\infty$-theta characteristic. According to Proposition 3.6.3 (i), we see that the map $\zeta_\infty : \mathfrak{D}_X \to \mathcal{F}^{\text{str}}_X$ is bijective. In particular, there exists a dormant indigenous $\mathcal{D}_X^{(\infty)}$-module $\mathcal{V}^\bigcirc := (\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N})$ with $\zeta_\infty^{\bigcirc} (\mathcal{V}^\bigcirc) = S^\circ$. The $\mathcal{D}_X^{(\infty)}$-module $(\mathcal{V}, \nabla_\mathcal{V})$ determines the $k$-vector space $x^*(\mathcal{V})$ endowed with a $\pi_1^{\text{str}}(X, x)$-action. By choosing a basis of $x^*(\mathcal{V})$, we obtain a representation $\pi_1^{\text{str}}(X, x) \to \text{GL}_{n+1}$. The class in $\text{Out}(\pi_1^{\text{str}}(X, x), \text{PGL}_{n+1})$ represented by the composite $\pi_1^{\text{str}}(X, x) \to \text{GL}_{n+1} \to \text{PGL}_{n+1}$ does not depend on the choices of both $\mathcal{V}^\bigcirc$ and the basis of $x^*(\mathcal{V})$ (i.e., depend only on $S^\circ$). Thus, we obtain a well-defined outer homomorphism

\[\rho_{S^\circ} \in \text{Out}(\pi_1^{\text{str}}(X, x), \text{PGL}_{n+1}).\]

Next, let $S^\circ$ be an $F^\infty$-affine structure on $X$. According to Proposition 3.6.3 (ii), there exists a dormant affine-indigenous $\mathcal{D}_X^{(\infty)}$-module $\mathcal{A}\mathcal{V}^\bigcirc := (\mathcal{V}, \nabla_\mathcal{V}, \delta)$ with $\zeta_\infty^{\bigcirc} (\mathcal{A}\mathcal{V}^\bigcirc) = S^\circ$. Just as in the above discussion, the $\mathcal{D}_X^{(\infty)}$-module $(\mathcal{V}, \nabla_\mathcal{V})$ induces, up to conjugation, a homomorphism $\pi_1^{\text{str}}(X, x) \to \text{GL}_{n+1}$. Moreover, the $\mathcal{D}_X^{(\infty)}$-submodule $(\text{Ker}(\delta), \nabla_\mathcal{V}|_{\text{Ker}(\delta)})$ determines a $\text{PGL}_{n+1}$-reduction of this homomorphism, i.e., a homomorphism $\pi_1^{\text{str}}(X, x) \to \text{PGL}_{n+1}$. According to Proposition 3.6.5, (i), we see that the map $\text{Hom}(\text{Ker}(\delta), \nabla_\mathcal{V})$ is bijective. In particular, there exists a dormant affine-indigenous $\mathcal{D}_X^{(\infty)}$-module $\mathcal{A}\mathcal{V}^\bigcirc := (\mathcal{V}, \nabla_\mathcal{V}, \delta)$ with $\zeta_\infty^{\bigcirc} (\mathcal{A}\mathcal{V}^\bigcirc) = S^\circ$. Just as in the above discussion, the $\mathcal{D}_X^{(\infty)}$-module $(\mathcal{V}, \nabla_\mathcal{V})$ induces, up to conjugation, a homomorphism $\pi_1^{\text{str}}(X, x) \to \text{GL}_{n+1}$. Moreover, the $\mathcal{D}_X^{(\infty)}$-submodule $(\text{Ker}(\delta), \nabla_\mathcal{V}|_{\text{Ker}(\delta)})$ determines a $\text{PGL}_{n+1}$-reduction of this homomorphism, i.e., a homomorphism $\pi_1^{\text{str}}(X, x) \to \text{PGL}_{n+1}$. According to Proposition 3.6.5, (i), we see that the map $\text{Hom}(\text{Ker}(\delta), \nabla_\mathcal{V})$ is bijective. In particular, there exists a dormant affine-indigenous $\mathcal{D}_X^{(\infty)}$-module $\mathcal{A}\mathcal{V}^\bigcirc := (\mathcal{V}, \nabla_\mathcal{V}, \delta)$ with $\zeta_\infty^{\bigcirc} (\mathcal{A}\mathcal{V}^\bigcirc) = S^\circ$. Just as in the above discussion, the $\mathcal{D}_X^{(\infty)}$-module $(\mathcal{V}, \nabla_\mathcal{V})$ induces, up to conjugation, a homomorphism $\pi_1^{\text{str}}(X, x) \to \text{GL}_{n+1}$. Moreover, the $\mathcal{D}_X^{(\infty)}$-submodule $(\text{Ker}(\delta), \nabla_\mathcal{V}|_{\text{Ker}(\delta)})$ determines a $\text{PGL}_{n+1}$-reduction of this homomorphism, i.e., a homomorphism $\pi_1^{\text{str}}(X, x) \to \text{PGL}_{n+1}$.

\[\rho_{S^\circ} \in \text{Out}(\pi_1^{\text{str}}(X, x), \text{PGL}_{n+1}).\]
Associated with the \( n \)-dimensional \( \mathbb{P}^n \)-projective structure, if \( f \) is a compact Kähler manifold admitting a projective structure, then it is isomorphic to the projective space of the same dimension. According to the main theorem in [84], a complex Fano variety admitting a projective structure is isomorphic to the projective space of the same dimension. Also, we can obtain an \( \mathbb{P}^n \)-projective structure. According to the main theorem in [84], a complex Fano variety admitting a projective structure is isomorphic to the projective space of the same dimension. Also, we can obtain an \( \mathbb{P}^n \)-projective structure. According to the main theorem in [84], a complex Fano variety admitting a projective structure is isomorphic to the projective space of the same dimension. Also, we can obtain an \( \mathbb{P}^n \)-projective structure.

\[ \rho_{\mathcal{S}^\lor} \in \text{Out}(\pi_{1}^\str(X, x), \text{PGL}^\h_n). \]

We shall refer to \( \rho_{\mathcal{S}^\lor} \) (resp., \( \rho_{\mathcal{S}^\lor}^\h \)) constructed above as the monodromy representation associated with the \( F^\infty \)-projective (resp., \( F^\infty \)-affine) structure \( \mathcal{S}^\lor \).

### 5.4. Characterization of projective spaces II (Embedded rational curves).

Next, let us prove another characterization of projective spaces using the existence of an \( F^\infty \)-projective structure. According to the main theorem in [84], a complex Fano variety admitting a projective structure is isomorphic to the projective space of the same dimension. Also, we can see its generalization in [34], Theorem 4.1, saying that if a compact Kähler manifold admitting a projective structure (or more generally, a projective connection) contains a rational curve, then it is isomorphic to the projective space. The crucial step for this result is to show that \( f^*(\mathcal{T}_X) \) is ample for an arbitrary nontrivial morphism \( f: \mathbb{P}^1 \to X \). Once this is proved, we can use a result of S. Mori (cf. [65]) to conclude the assertion. By applying a similar argument, we obtain the positive characteristic analogue of this assertion, as follows.

**Theorem 5.4.1.**

*Let \( X \) be a smooth projective variety over \( k \) of dimension \( n \) admitting an \( F^\infty \)-projective structure. If \( X \) contains a rational curve, then \( X \) is isomorphic to \( \mathbb{P}^n \). In particular, \( \mathbb{P}^n \) is the only Fano variety of dimension \( n \) admitting an \( F^\infty \)-projective structure.*

**Proof.** Since any Fano variety contains a rational curve (cf. [17], Chap. V, Theorem 1.6, (1.6.1)), it suffices to prove the former assertion.

Let \( C \) be any rational curve in \( X \) and \( f: \mathbb{P}^1 \to X \) its normalization. According to Mori’s theorem (cf. [65]; or [64], Theorem 4.2), the problem is reduced to proving the claim that \( f^*(\mathcal{T}_X) \) is ample. Since any vector bundle on \( \mathbb{P}^1 \) decomposes into a finite direct sum of line bundles, we can write

\[ f^*(\mathcal{T}_X) \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i), \]

where \( a_1 \leq a_2 \leq \cdots \leq a_n \). Since the differential \( \mathcal{T}_C \) (\( \cong \mathcal{O}_{\mathbb{P}^1}(2) \)) \( \to f^*(\mathcal{T}_X) \) of \( f \) is injective, the inequality

\[ a_n \geq 2 \]

holds. Now, let us take an \( F^\infty \)-projective structure \( \mathcal{S}^\lor \) on \( X \). The \( F^\infty \)-indigenous structure induced from \( \mathcal{S}^\lor \) determines an \( F \)-divided \( \text{PGL}^\infty_n \)-torsor \( \mathcal{E}_\infty := \{ \mathcal{E}_N \}_{N \in \mathbb{Z}_{\geq 0}} \) on \( X \) together with a \( \text{PGL}^\infty_n \)-reduction \( \mathcal{E}_{\text{red}} \) on \( \mathcal{E}_0 \) (cf. Remark [17,4]). By pulling-back this data via \( f \), we obtain an \( F \)-divided \( \text{PGL}^\infty_n \)-torsor \( f^*(\mathcal{E}_\infty) \) on \( \mathbb{P}^1 \) and a \( \text{PGL}^\infty_n \)-reduction \( f^*(\mathcal{E}_{\text{red}}) \) on \( f^*(\mathcal{E}_0) \).

It is immediately verified from the fact that \( \pi_{1}^\str(\mathbb{P}^1) = 1 \) that \( f^*(\mathcal{E}_\infty) \) is trivial, i.e., obtained from the rank \((n + 1)\) trivial \( F \)-divided sheaf on \( \mathbb{P}^1 \), or equivalently, from the rank \((n + 1)\) trivial \( \mathcal{D}_{\mathbb{P}^1}^\infty \)-module \((\mathcal{O}_{\mathbb{P}^1}(n+1), (\nabla_{\text{triv}(\infty)})^{(n+1)}) \) (cf. [117]). Denote by \( \mathcal{N} \) the line subbundle of \( \mathcal{O}_{\mathbb{P}^1}(n+1) \) determined by \( \mathcal{E}_{\text{red}} \). Then, we have the following sequence of isomorphisms:

\[ f^*(\mathcal{T}_X) \cong f^*(\text{Ad}_{\mathcal{E}_0})/f^*(\text{Ad}_{\mathcal{E}_{\text{red}}}) \cong \text{Ad}_{f^*(\mathcal{E}_0)}/\text{Ad}_{f^*(\mathcal{E}_{\text{red}})} \cong \mathcal{N}^\lor \otimes (\mathcal{O}_{\mathbb{P}^1}(n+1)/\mathcal{N}), \]
where the first arrow denotes the pull-back of the Kodaira-Spencer map $\text{KS}_{\xi_0, (\text{can}(0), \xi_{\text{red}})}^{(E_0, \nabla_{\text{can}(0)})_{F_X/k}(E_1)}$ (composed with the inverse to (30)) and the last arrow follows from the discussion in Remark 2.2.2. The composite of these isomorphisms induces the following exact sequence:

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \rightarrow f^*(\mathcal{T}_X) \otimes \mathcal{N} \left( \cong \bigoplus_{i=1}^n \mathcal{N} \otimes \mathcal{O}_{\mathbb{P}^1}(a_i) \right) \rightarrow 0.$$  

The injectivity of the second arrow in (279) implies the inequality $l := \deg(\mathcal{N}) \leq 0$. On the other hand, the surjectivity of the third arrow implies the inequality

$$a_i + l \geq 0$$

for any $i = 1, \cdots, n$. By the decomposition (276) and the exactness of (279), we have

$$0 = \deg(\mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)}) (279) = \deg(\mathcal{N}) + \deg(f^*(\mathcal{T}_X) \otimes \mathcal{N}) (276) = l(n+1) + \sum_{i=1}^n a_i.$$  

Because of the inequality (277) and (281), the integer $l$ must be negative. Hence, the inequality (280) implies the inequality $a_i > 0$, so $f^*(\mathcal{T}_X)$ is ample. This completes the proof of the assertion. □

6. Case 2: Abelian varieties

In this section, we study $F^N$-projective and $F^N$-affine structures, as well as dormant indigenous and affine-indigenous $\mathcal{D}^{(N-1)}$-modules, on abelian varieties. One of the main results asserts (cf. Corollary 6.3.3) that the existence of an $F^\infty$-projective (or $F^\infty$-affine) structure gives a characterization of ordinarity for abelian varieties. Moreover, it is shown (cf. Propositions 6.3.1 (iii) and 6.3.2 (iii)) that, for each ordinary abelian variety $X$, dormant indigenous and affine-indigenous $\mathcal{D}^{(\infty)}_X$-modules can be decomposed into direct sums of invertible $\mathcal{D}^{(\infty)}_X$-modules. As a consequence, we can describe explicitly the respective sets of $F^\infty$-projective and $F^\infty$-affine structures on $X$ in terms of the $p$-adic Tate module of its dual abelian variety (cf. Theorem 6.4.1).

In what follows, let $n$ be a positive integer.

6.1. Pull-back via étale coverings.

Before discussing the study on abelian varieties, let us consider constructions of the pull-backs by an étale covering of various structures introduced before. Let $X$ and $Y$ be smooth varieties over $k$ of dimension $n$ and $\pi : Y \rightarrow X$ an étale $k$-morphism.

$F^N$-projective and $F^N$-affine structures:

Given each étale morphism $\phi : U \rightarrow \mathbb{P}^n$ (resp., $\phi : U \rightarrow \mathbb{A}^n$) from an open subscheme $U$ of $X$ to $\mathbb{P}^n$ (resp., $\mathbb{A}^n$), we obtain the étale composite $\phi \circ \pi : \pi^{-1}(U) \rightarrow \mathbb{P}^n$ (resp., $\phi \circ \pi : \pi^{-1}(U) \rightarrow \mathbb{A}^n$); the assignment $\phi \mapsto \phi \circ \pi$ yields a morphism

$$\pi^{-1}(\mathcal{P}^\text{ét}_X) \rightarrow \mathcal{P}^\text{ét}_Y \ (\text{resp., } \pi^{-1}(\mathcal{A}^\text{ét}_X) \rightarrow \mathcal{A}^\text{ét}_Y)$$
of sheaves on $Y$. If $\mathcal{S}^\triangledown$ is an $F^N$-projective (resp., $F^N$-affine) structure on $X$ for $N \in \mathbb{Z}_{>0}$, then we shall denote by

$$\pi^!(\mathcal{S}^\triangledown)$$

the subsheaf of $\mathcal{P}_Y^{\text{et}}$ (resp., $\mathcal{A}^{\text{et}}_Y$) defined to be the smallest subsheaf that is closed under the $(\text{PGL}_{n+1})^\text{(N)}_X$-action (resp., $(\text{PGL}_{n+1})^\text{(N)}_X$-action) and contains the image of $\pi^{-1}(\mathcal{S}^\triangledown)$ via [282]. One verifies that $\pi^!(\mathcal{S}^\triangledown)$ forms an $F^N$-projective (resp., $F^N$-affine) structure on $Y$. Moreover, since the formation of $\pi^!(\mathcal{S}^\triangledown)$ is compatible with truncation to lower levels, we obtain, from each $F^\infty$-projective (resp., $F^\infty$-affine) structure $\mathcal{S}^\triangledown := \{\mathcal{S}_N^\triangledown\}_{N \in \mathbb{Z}_{>0}}$, an $F^\infty$-projective (resp., $F^\infty$-affine) structure $\pi^!(\mathcal{S}^\triangledown) := \{\pi^!(\mathcal{S}_N^\triangledown)\}_{N \in \mathbb{Z}_{>0}}$. In any case of $N \in \mathbb{Z}_{>0} \cup \{\infty\}$, we shall refer to $\pi^!(\mathcal{S}^\triangledown)$ as the \textbf{pull-back} of $\mathcal{S}^\triangledown$ via $\pi$. If $Y$ is an open subscheme of $X$ and $\pi : Y \to X$ is the open embedding, then the pull-back $\pi^!(\mathcal{S}^\triangledown)$ via $\pi$ is nothing but the restriction $\mathcal{S}^\triangledown|_Y$ discussed in §1.4.

\textbf{$F^N$-indigenous and $F^N$-affine indigenous structures:}

Next, let $N$ be a positive integer and $\mathcal{E}^\bullet := (\mathcal{E}^{\triangledown}, \mathcal{E}_{\text{red}})$ an $F^N$-indigenous (resp., $F^N$-affine-indigenous) structure on $X$. Denote by $\pi^!(\mathcal{E})$ the pull-back of $\mathcal{E} := F^{\text{(N)}^*}_{X/k}(\mathcal{E}^{\triangledown})$, i.e., the principal $(\text{PGL}_{n+1})^\text{(N)}_X$-bundle (resp., the principal $(\text{PGL}_{n+1})^\text{(N)}_X$-bundle) on $X$ obtained by pulling-back $\mathcal{E}$ via $\pi$, and by $\pi^!(\mathcal{E}^{\triangledown})$ the pull-back of $\mathcal{E}^{\triangledown}$ via the base-change $\pi^{\text{(N)}} : Y^{\text{(N)}} \to X^{\text{(N)}}$ of $\pi$. Also, denote by $\pi^!(\mathcal{E}_{\text{red}})$ the $\text{PGL}_{n+1}^\infty$-reduction (resp., the $\text{PGL}_{n+1}^\infty$-reduction) on $\pi^!(\mathcal{E})$ defined as the pull-back of $\mathcal{E}_{\text{red}}$. The differential of $\pi$ induces natural identifications

$$\pi^!(\mathcal{E}^{\triangledown}) \to \pi^!(\mathcal{E}) \to \pi^!(\mathcal{E}_{\text{red}})$$

making the following diagram commute:

$$\begin{array}{ccc}
\tilde{\mathcal{T}}_{\pi^!(\mathcal{E}_{\text{red}})} & \xrightarrow{\text{incl.}} & \tilde{\mathcal{T}}_{\pi^!(\mathcal{E})} \\
\downarrow d_3 & & \downarrow d_2 \\
\pi^!(\tilde{\mathcal{T}}_{\mathcal{E}_{\text{red}}}) & \xrightarrow{\text{incl.}} & \pi^!(\tilde{\mathcal{T}}_{\mathcal{E}}) \\
\downarrow & & \downarrow \\
\pi^!(\mathcal{E}_{\text{red}}) & \xrightarrow{d_{\pi^!(\mathcal{E})}} & \pi^!(\mathcal{E}) \\
\downarrow & & \downarrow \\
\pi^!(\mathcal{T}_X) & \xrightarrow{d_{\pi^!(\mathcal{E})}} & \pi^!(\mathcal{T}) \\
\end{array}$$

The pull-back $\pi^!(\nabla^{\text{can}}_{F^{\text{(N)}^*}_{X/k}(\mathcal{E}^{\triangledown})}) : \pi^!(\mathcal{T}_X) \to \pi^!(\tilde{\mathcal{T}}_{\mathcal{E}})$ of $\nabla^{\text{can}}_{F^{\text{(N)}^*}_{X/k}(\mathcal{E}^{\triangledown})}$ specifies a flat connection on $\pi^!(\mathcal{E})$ under the identifications $d_1$ and $d_2$. By construction, it coincides with $\nabla^{\text{can}}_{F^{\text{(N)}^*}_{Y/k}(\pi^!(\mathcal{E}^{\triangledown}))}$. Hence, we obtain the following commutative square diagram:

$$\begin{array}{ccc}
\mathcal{T}_Y & \xrightarrow{\tilde{\mathcal{T}}_{\pi^!(\mathcal{E})}} & \tilde{\mathcal{T}}_{\pi^!(\mathcal{E})} \\
\downarrow d_1 & & \downarrow \\
\pi^!(\mathcal{T}_X) & \xrightarrow{d_{\pi^!(\mathcal{E})}} & \pi^!(\mathcal{T}) \\
\end{array}$$

where the right-hand vertical arrow is induced by both $d_2$ and $d_3$, the upper horizontal arrow denotes the Kodaira-Spencer map associated to $(\pi^!(\mathcal{E}), \nabla^{\text{can}}_{F^{\text{(N)}^*}_{Y/k}(\pi^!(\mathcal{E}^{\triangledown}))}, \pi^!(\mathcal{E}_{\text{red}}))$, and the
lower horizontal arrow denotes the pull-back via \( \pi \) of the Kodaira-Spencer map associated to \((\mathcal{E}, \nabla^\text{can}_{F^N_{X/k}(\mathcal{E}^\nabla)}, \mathcal{E}_\text{red})\), which is an isomorphism. It follows that the pair

\[(287) \quad \pi^*(\mathcal{E}^\bullet) := (\pi^*(\mathcal{E}^\nabla), \pi^*(\mathcal{E}_\text{red}))\]

specifies an \( F^N \)-indigenous (resp., \( F^N \)-affine-indigenous) structure on \( Y \). The formation of \( \pi^*(\mathcal{E}^\bullet) \) is compatible with truncation to lower levels, so we can construct an \( F^{\infty} \)-indigenous (resp., \( F^{\infty} \)-affine-indigenous) structure \( \pi^*(\mathcal{E}^\bullet) \) on \( Y \) associated, via pull-back by \( \pi \), with each \( F^{\infty} \)-indigenous (resp., \( F^{\infty} \)-affine-indigenous) structure \( \mathcal{E}^\bullet \) on \( X \). In any case of \( \mathcal{N} \in \mathbb{Z}_{>0} \cup \{ \infty \} \), we shall refer to \( \pi^*(\mathcal{E}^\bullet) \) as the pull-back of \( \mathcal{E}^\bullet \) via \( \pi \).

**Dormant indigenous and affine-indigenous \( \mathcal{D}^{(N-1)}_X \)-modules:**

Next, let \( \mathcal{N} \in \mathbb{Z}_{>0} \cup \{ \infty \} \) and let \( \mathcal{V}^\nabla := (\mathcal{V}, \nabla, \mathcal{N}) \) (resp., \( \mathcal{A} \mathcal{V}^\nabla := (\mathcal{V}, \nabla, \mathcal{N}, \delta) \)) be a dormant indigenous (resp., affine-indigenous) \( \mathcal{D}^{(N-1)}_X \)-module. Recall that \( \pi^*(\mathcal{V}) \) admits a \( \mathcal{D}^{(N-1)}_Y \)-action induced by \( \nabla_\mathcal{V} \). In fact, the \( \pi^*(\mathcal{D}^{(N-1)}_X) \)-action \( \pi^*(\nabla_\mathcal{V}) \) on \( \pi^*(\mathcal{V}) \) yields that action via the natural morphism \( \mathcal{D}^{(N-1)}_Y \to \pi^*(\mathcal{D}^{(N-1)}_X) \), which is an isomorphism because of the étaleness of \( \pi \). The resulting \( \mathcal{D}^{(N-1)}_Y \)-module \( (\pi^*(\mathcal{V}), \pi^*(\nabla_\mathcal{V})) \) has vanishing \( p-(N-1) \)-curvature. Also, the pull-back \( \pi^*(\mathcal{N}) \) of \( \mathcal{N} \) specifies a line subbundle of \( \pi^*(\mathcal{V}) \). Since the pull-back via \( \pi \) of the Kodaira-Spencer map \( \text{KS}(\mathcal{V}, \nabla_\mathcal{V}, \mathcal{N}) \) may be identified with \( \text{KS}(\pi^*(\mathcal{V}), \pi^*(\nabla_\mathcal{V}), \pi^*(\mathcal{N})) \), the collection

\[(288) \quad \pi^*(\mathcal{V}^\nabla) := (\pi^*(\mathcal{V}), \pi^*(\nabla_\mathcal{V}), \pi^*(\mathcal{N})) \quad \text{(resp., } \pi^*(\mathcal{A} \mathcal{V}^\nabla) := (\pi^*(\mathcal{V}), \pi^*(\nabla_\mathcal{V}), \pi^*(\mathcal{N}), \pi^*(\delta)))\]

forms a dormant indigenous (resp., affine-indigenous) \( \mathcal{D}^{(N-1)}_X \)-module. We refer to \( \pi^*(\mathcal{V}^\nabla) \) (resp., \( \pi^*(\mathcal{A} \mathcal{V}^\nabla) \)) as the pull-back of \( \mathcal{V}^\nabla \) (resp., \( \mathcal{A} \mathcal{V}^\nabla \)) via \( \pi \). One verifies that if two dormant indigenous (resp., affine-indigenous) \( \mathcal{D}^{(N-1)}_X \)-modules are \( \mathbb{G}_m \)-equivalent (cf. Definition 3.3.1), then their pull-backs via \( \pi \) are \( \mathbb{G}_m \)-equivalent as dormant indigenous (resp., affine-indigenous) \( \mathcal{D}^{(N-1)}_Y \)-modules. In particular, we obtain a well-defined class

\[(289) \quad \pi^*([\mathcal{V}^\nabla]) := [\pi^*(\mathcal{V}^\nabla)] \in \mathcal{J}^{\text{z}}_{X,N} \quad \text{(resp., } \pi^*([\mathcal{A} \mathcal{V}^\nabla]) := [\pi^*(\mathcal{A} \mathcal{V}^\nabla)] \in \mathcal{A} \mathcal{J}^{\text{z}}_{X,N})\]

for each \([\mathcal{V}^\nabla] \in \mathcal{J}^{\text{z}}_{X,N} \) (resp., \([\mathcal{A} \mathcal{V}^\nabla] \in \mathcal{A} \mathcal{J}^{\text{z}}_{X,N}) \). Finally, note that if \( \mathcal{A} \mathcal{V}^\nabla \) is an element of \( \mathcal{J}^{\text{z}}_{X,N} \), then the pull-back \( \pi^*(\mathcal{A} \mathcal{V}^\nabla) \) belongs to \( \mathcal{A} \mathcal{J}^{\text{z}}_{Y,N} \).

**Tango structures of level \( N \):**

Let \( U^\bullet \subseteq \mathcal{E}^{(N)}_X \) be a Tango structure of level \( N \) on \( X \), where \( N \) is a positive integer. Since \( \pi \) is étale, the commutative square

\[(290) \quad \xymatrix{ X \ar[r]^{F^{(N)}_{X/k}} & X^{(N)} \ar[d]^{\pi^{(N)}} \cr Y \ar[r]_{F^{(N)}_{Y/k}} & Y^{(N)} \ar[u]_{\pi} } \]

is cartesian. Hence, we obtain the commutativity of the following square diagram:

\[
\begin{array}{ccc}
\pi^*(\mathcal{O}_{X(N)}) & \xrightarrow{\pi^*(F_X^{(N)\sharp})} & \pi^*(F_X^{(N)*}(\mathcal{O}_X)) \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y(N)} & \xrightarrow{F_Y^{(N)\sharp}} & F_Y^{(N)*}(\mathcal{O}_Y),
\end{array}
\]

(291)

where, for each morphism \( f : Z_1 \to Z_2 \) of varieties, we shall denote by \( f^2 \) the morphism \( \mathcal{O}_{Z_2} \to f_*(\mathcal{O}_{Z_1}) \) of sheaves induced \( f \). Taking the cokernels of both the upper and lower horizontal arrows in this square gives an isomorphism \( \pi^*(\mathcal{B}_X^{(N)}) \sim \mathcal{B}_Y^{(N)} \) (cf. (180)).

**Lemma 6.1.1.**

The image of the composite

\[
\pi^*(\mathcal{U}) \hookrightarrow \pi^*(\mathcal{B}_X^{(N)}) \sim \mathcal{B}_Y^{(N)},
\]

where the first arrow denotes the pull-back of the inclusion \( \mathcal{U} \hookrightarrow \mathcal{B}_X^{(N)} \), forms a Tango structure of level \( N \) on \( Y \).

**Proof.** Since (290) is cartesian, there exists a canonical isomorphism

\[
\eta : \pi^*(\mathcal{B}_X^{(N)}) \xrightarrow{\sim} F_Y^{(N)}(\pi^*(\mathcal{O}_X)).
\]

The following diagram is verified to be commutative:

\[
\begin{array}{ccc}
\pi^*(F_X^{(N)*}(\mathcal{U})) & \xrightarrow{\pi^*(F_X^{(N)\sharp}(\gamma_{\mathcal{U}}))} & \pi^*(F_X^{(N)*}(\mathcal{O}_X)) \\
\downarrow & & \downarrow \\
F_Y^{(N)*}(\pi^*(\mathcal{O}_X)) & \xrightarrow{\pi^*(\alpha_X)} & \pi^*(\mathcal{O}_X)
\end{array}
\]

(294)

where \( \alpha_{\mathcal{U}} \) (for \( \square = X, Y \)) denotes the morphism \( F_{\square/k}^{(N)*}(\Omega_{\square}) \to \Omega_{\square} \) corresponding to the identity of \( F_{\square/k}^{(N)*}(\mathcal{O}_{\square}) \) via the adjunction relation “\( F_{\square/k}^{(N)*}(-) \dashv F_{\square/k}^{(N)*}(-) \)”. The composite of the upper horizontal arrows is an isomorphism, the commutativity of this diagram implies that the composite of the lower horizontal arrows is an isomorphism. That is to say, \( \pi^*(\mathcal{U}) \) turns out to specify a Tango structure of level \( N \) on \( Y \), as desired. \( \square \)
For convenience, we write \( \pi^*(U^\bullet) := \pi^{(N)*}(U^\bullet) \). The formation of \( \pi^*(U^\bullet) \) is compatible with truncation to lower levels. Hence, each Tango structure \( \{U^\bullet\}_{N \in \mathbb{Z}_{>0}} \) of level \( \infty \) on \( X \) induces a Tango structure \( \pi^*(U^\bullet) := \{\pi^*(U_N^\bullet)\}_{N \in \mathbb{Z}_{>0}} \) of level \( \infty \) on \( Y \). In any case of \( N \in \mathbb{Z}_{>0} \sqcup \{\infty\} \), we shall refer to \( \pi^*(U^\bullet) \) as the pull-back of \( U^\bullet \) via \( \pi \).

Regarding the various pull-backs obtained above, we have the following propositions.

**Proposition 6.1.2.**
Let \( N \in \mathbb{Z}_{>0} \sqcup \{\infty\} \) and \( X, Y \) be smooth varieties over \( k \). Suppose that there exists an étale \( k \)-morphism \( Y \to X \) and that \( X \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure. Then, \( Y \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure.

**Proof.** The assertion is a direct consequence of forming pull-backs of \( F^N \)-projective (resp., \( F^N \)-affine) structures. \( \square \)

**Proposition 6.1.3.**
Let \( N \in \mathbb{Z}_{>0} \sqcup \{\infty\} \) and let \( X, Y \) be smooth varieties over \( k \). Suppose that there exists an étale \( k \)-morphism \( Y \to X \) and that \( X \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure. Then, \( Y \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure.

**Proof.** The assertion follows from the various definitions involved. \( \square \)

### 6.2. Dual abelian variety and Cartier operator

Now, let \( X \) be an abelian variety over \( k \) of dimension \( n \). We know that the vector bundle \( \Omega_X \), as well as \( T_X \), is trivial. Also, the equality \( \dim_k(\Gamma(X, \Omega_X)) = n \) holds, and each global section of \( \Omega_X \) is invariant under the translation by any point of \( X \). For each \( u \in k \), we shall write

\[
\mu_u : \mathcal{O}_X \to \mathcal{O}_X
\]

for the \( \mathcal{O}_X \)-linear endomorphism of \( \mathcal{O}_X \) given by multiplication by \( u \). Since \( X \) is proper over \( k \), any \( \mathcal{O}_X \)-linear endomorphism of \( \mathcal{O}_X \) may be obtained as such an endomorphism.

Denote by \( X^\vee := \text{Pic}^0(X) \) the dual abelian variety of \( X \), i.e., the identity component of the Picard scheme of \( X \). For each positive integer \( N \), we write \( T_p^{(N)} := \text{Pic}^0(X)[p^N] \) for the group of \( k \)-rational points in the \( p^N \)-torsion subgroup scheme \( X^\vee[p^N] \) of \( X^\vee \). Also, we set

\[
T_p^{(\infty)} := \lim_{\substack{\longrightarrow \\ N \in \mathbb{Z}_{>0}}} T_p^{(N)} \quad (= T_p X^\vee),
\]

i.e., the \( p \)-adic Tate module of \( X^\vee \) (cf. [68], Chap.IV, §18, p.159, Definition). If \( g_X \) denotes the \( p \)-rank of \( X \), then the group \( T_p^{(N)} \) for \( N \in \mathbb{Z}_{>0} \) (resp., \( T_p^{(\infty)} \)) forms a rank \( g_X \) free module over \( \mathbb{Z}/p^N\mathbb{Z} \) (resp., \( \mathbb{Z}_p \)).

Given a pair of nonnegative integers \( (m, m') \) with \( m' > m \), we shall denote by

\[
\text{Ver}_{X}^{(m') \to (m)} : X^{(m')\vee} \to X^{(m)\vee}
\]

the morphism of group schemes given by pulling-back line bundles on \( X^{(m')} \) via \( F^{(m'-m)}_{X^{(m)}/k} \). If \( N \) is a positive integer, then the multiplication \( [p^N]_{X^\vee} : X^\vee \to X^\vee \) by \( p^N \) coincides with the
composite \( \text{Ver}^{(N)}_{X,k} \circ F^{(N)}_{X,k} \). Hence, the \( N \)-th relative Frobenius morphism \( F^{(N)}_{X,k} \) restricts to a bijection of sets
\[
T^{(N)}_p \cong (\text{Ver}^{(N)}_{X,k})^{-1}(\mathcal{O}_X).
\]

In what follows, we recall the relationship between \( \mathcal{D}^{(N)}_X \)-actions on \( \mathcal{O}_X \) and elements of \( T^{(N)}_p \). Let \( N \) be a positive integer and \( a \) an element of \( T^{(N)}_p \). The image of \( a \) via \( (298) \) specifies a line bundle \( \mathcal{L} \) on \( X^{(N)} \) with \( F^{(N)}_{X/k}(\mathcal{L}) \cong \mathcal{O}_X \). The \( \mathcal{D}^{(N)}_X \)-action \( \nabla_{\mathcal{L}}^{\text{con}(N-1)} \) (cf. (112)) corresponds, via an isomorphism \( F^{(N)}_{X/k}(\mathcal{L}) \cong \mathcal{O}_X \), to a \( \mathcal{D}^{(N-1)}_X \)-action
\[
\nabla_a : \mathcal{D}^{(N-1)}_X \rightarrow \text{End}_k(\mathcal{O}_X)
\]
on \( \mathcal{O}_X \) with vanishing \( p \,(N-1) \)-curvature. The formation of \( \nabla_a \) is independent of the choice of \( F^{(N)}_{X/k}(\mathcal{L}) \). This is because such an isomorphism is uniquely determined up to composition with \( \mu_a \) for some \( u \in k^\times = \Gamma(Z,\mathcal{O}_X^\times) \) and \( \nabla_a \) is invariant under the gauge transformation by \( \mu_a \). In the case of \( N = \infty \), each element \( a := (a_N)_{N \in \mathbb{Z}_{>0}} \in T^{(\infty)}_p \left( = \lim_{\leftarrow N \in \mathbb{Z}_{>0}} T^{(N)}_p \right) \) associates a \( \mathcal{D}^{(\infty)}_X \)-action \( \nabla_a \) on \( \mathcal{O}_X \) defined as the limit of \( \nabla_{a,N} \)'s. The equivalences of categories (113) and (117) imply that, for \( N \in \mathbb{Z}_{>0} \cup \{ \infty \} \), any \( \mathcal{D}^{(N)}_X \)-action on \( \mathcal{O}_X \) with vanishing \( p \,(N-1) \)-curvature may be expressed as \( \nabla_a \) for a unique \( a \in T^{(N)}_p \). In particular, given \( a_1, a_2 \in T^{(N)}_p \), we have the following equivalences:
\[
(300) \quad a_1 = a_2 \iff (\mathcal{O}_X, \nabla_{a_1}) \cong (\mathcal{O}_X, \nabla_{a_2})
\]
\[ \iff \text{there exists a nonzero morphism } (\mathcal{O}_X, \nabla_{a_1}) \rightarrow (\mathcal{O}_X, \nabla_{a_2}). \]

Next, let us focus on the case of \( N = 1 \). Denote by \( \mathcal{Z}^{(1)}_X \) the sheaf of closed 1-forms on \( X \), often regarded as an \( \mathcal{O}_{X(1)} \)-module via the underlying homeomorphism of \( F^{(1)}_{X,k} \); it fits into the following short exact sequence:
\[
(301) \quad 0 \rightarrow \mathcal{B}^{(1)}_X \xrightarrow{\text{incl.}} \mathcal{Z}^{(1)}_X \xrightarrow{C_X} \Omega^{(1)}_X \rightarrow 0,
\]
where \( C_X \) denotes the Cartier operator of \( X \). By identifying \( \Omega_X \) with \( \Omega^{(1)}_X \) via the isomorphism \( F_k \times \text{id}_X : X^{(1)} (= k \times_k X) \xrightarrow{\sim} X \), we often think of \( C_X \) as a \( p^{-1} \)-linear morphism from \( \mathcal{Z}^{(1)}_X \) (\( \subseteq \Omega_X \)) to \( \Omega_X \). The notation \( C_X \) will be used to denote the \( p^{-1} \)-linear map \( (\Gamma(X, \Omega_X) \supseteq \Gamma(X, \mathcal{Z}^{(1)}_X) \rightarrow \Gamma(X, \Omega_X) \) induced by \( C_X \). We shall refer to each element \( a \) of \( \Gamma(X, \mathcal{Z}^{(1)}_X) \) with \( C_X(a) = a \) as a \( C_X \)-invariant element. Write
\[
\Gamma(X, \Omega_X)^{C_X}
\]
for the set of \( C_X \)-invariant elements, forming an \( \mathbb{F}_p \)-vector space.

Given an element \( a \) of \( \Gamma(X, \mathcal{Z}^{(1)}_X) \), we shall set
\[
(303) \quad \nabla_a : \mathcal{O}_X \rightarrow \Omega_X
\]
to be the connection on \( \mathcal{O}_X \) expressed as \( \nabla_a := d + a \), where \( d \) denotes the universal derivation \( \mathcal{O}_X \rightarrow \Omega_X \). This connection is flat because of the closedness of \( a \). Also, it is invariant under the translation by any point of \( X \). Moreover, any connection on \( \mathcal{O}_X \) may be expressed as \( \nabla_a \) for a unique \( a \in \Gamma(X, \Omega_X) \). By [33], Proposition 7.1.2, we have the equivalence
\[
(304) \quad \nabla_a \text{ has vanishing } p \text{-curvature } \iff a = C_X(a) \quad (\text{i.e., } a \in \Gamma(X, \Omega_X)^{C_X}).
\]
Hence, the assignment $a \mapsto \text{Sol}(\nabla_a)$ composed with the inverse of the bijection \([298]\) for $N = 1$ gives an $\mathbb{F}_p$-linear bijection
\[
\Gamma(X, \Omega_X)^{C_\times} \simeq T_p^{(1)}.
\]

Finally, we shall suppose that the abelian variety $X$ is ordinary. Then, the equality $H^i(X, B_X^{(1)}) = 0$ holds for $i = 0, 1$ (cf. \[10, 33\]). The exactness of \(301\) implies
\[
\dim_k(\Gamma(X, \mathcal{Z}_X^{(1)})) = \dim_k(\Gamma(X, \Omega_X^{(1)})) (= \dim_k(\Gamma(X, \Omega_X)) = n).
\]
That is to say, the natural inclusion $\mathcal{Z}_X^{(1)} \hookrightarrow \Omega_X$ induces the equality $\Gamma(X, \mathcal{Z}_X^{(1)}) = \Gamma(X, \Omega_X)$. The resulting $p^{-1}$-linear endomorphism $C_X$ of the $k$-vector space $\Gamma(X, \Omega_X)$ is bijective. Moreover, $\Gamma(X, \Omega_X)$ has a basis consisting of $C_X$-invariant elements, which are invariant under the translation by any point of $X$. If $(b_i)$ is such a basis of $\Gamma(X, \Omega_X)$, then an $n$-tuple $(b'_i) \in \Gamma(X, \Omega_X)^n$ specifies a $C_X$-invariant basis if and only if there exists $A \in \text{GL}_n(\mathbb{F}_p)$ with $(b'_1, \ldots, b'_n) = (b_1, \ldots, b_n)A$.

### 6.3. Diagonal dormant (affine-)indigenous $\mathcal{D}_X^{(N-1)}$-modules.

Let us consider dormant indigenous and affine-indigenous $\mathcal{D}_X^{(N-1)}$-modules expressed as a direct sum of invertible $\mathcal{D}_X^{(N-1)}$-modules. In the case of $N = \infty$, we prove (cf. Propositions 6.3.1 (iii), and 6.3.2 (iii)) that any dormant indigenous or affine-indigenous $\mathcal{D}_X^{(\infty)}$-module can be obtained as one of them. As a consequence, the existence of an $F^\infty$-projective (or an $F^\infty$-affine) structure gives a characterization of ordinarity for abelian varieties (cf. Corollary 6.3.3).

We shall set
\[
X^\vee(p) := \lim \left( \cdots \xrightarrow{\text{Ver}_X^{(3)} \Rightarrow (2)} X^{(2)} \xrightarrow{\text{Ver}_X^{(2)} \Rightarrow (1)} X^{(1)} \xrightarrow{\text{Ver}_X^{(1)} \Rightarrow (0)} X^{(0)} \right).
\]

Also, denote by Diag$(X^\vee(p)(k))$ (cf. \[33, \S \ 2.2\]) the diagonal group associated with $X^\vee(p)(k)$, considered as an abstract group. Each element $([\mathcal{L}_N])_{N \in \mathbb{Z}_{>0}}$ of $X^\vee(p)(k)$ determines a character of Diag$(X^\vee(p)(k))$, i.e., a homomorphism
\[
\text{Char}((([\mathcal{L}_N])_N) : \text{Diag}(X^\vee(p)(k)) \to \mathbb{G}_m.
\]

Here, recall from \[16\] the study on the stratified fundamental groups of abelian varieties. For $X$ as above, the stratified fundamental group $\pi_1^{\text{str}}(X)$ of $X$ at a fixed base point is abelian; if $\pi^{\text{diag}}$ and $\pi^{\text{unip}}$ denote its diagonal and unipotent parts respectively, then $\pi_1^{\text{str}}(X)$ decomposes as the direct sum $\pi^{\text{unip}} \oplus \pi^{\text{diag}}$. On the one hand, the unipotent part $\pi^{\text{unip}}$ is isomorphic to $T_pX$, the $p$-adic Tate module of $X$. On the other hand, the diagonal part $\pi^{\text{diag}}$ is isomorphic to Diag$(X^\vee(p)(k))$. Thus, the pro-algebraic group $\pi_1^{\text{str}}(X)$ decomposes as the direct sum
\[
\pi_1^{\text{str}}(X) \simeq T_pX \oplus \text{Diag}(X^\vee(p)(k)).
\]

By construction, this decomposition is functorial with respect to morphisms between abelian varieties.

In the following proposition, we describe explicitly dormant indigenous $\mathcal{D}_X^{(\infty)}$-modules classified by the set $\prod \mathbb{Z}^{\text{str}}_{X,\infty,\mathcal{O}_X}$, where we consider the trivial line bundle $\mathcal{O}_X$ as a theta characteristic of $X$ by fixing an isomorphism $\theta : \mathcal{O}_X^{\otimes (n+1)} (= \mathcal{O}_X) \xrightarrow{\sim} \omega_X$. Let us fix an element $N$ of $\mathbb{Z}_{>0} \sqcup \{ \infty \}$. 
Proposition 6.3.1.
Given an element \( \mathbf{a} := (a_i)_{i=1}^{n+1} \) of \( (T_p^N)^{n+1} \), we shall write

\[
\mathcal{O}_\mathbf{a}^\triangledown := (\mathcal{O}_X^\oplus(n+1), \bigoplus_{i=1}^{n+1} \nabla_{a_i}, \text{Im}(\Delta))
\]

(cf. \((\ref{eq:define_nabla})\)) for the definition of \( \nabla(-) \), where \( \Delta \) denotes the diagonal embedding \( \mathcal{O}_X \hookrightarrow \mathcal{O}_X^\oplus(n+1) \).

Then, the following assertions hold:

(i) The collection \( \mathcal{O}_\mathbf{a}^\triangledown \) forms a dormant indigenous \( \mathcal{D}_X^{(N-1)} \)-module if and only if the \( n \)-tuple \( (a_1 - a_{n+1}, a_2 - a_{n+1}, \ldots, a_n - a_{n+1}) \) forms a basis of \( T_p^N \). In particular, \( X \) admits an \( FN \)-projective structure of the form \( \mathcal{O}_a^\triangledown \) if and only if \( X \) is ordinary.

(ii) Suppose that \( \mathcal{O}_\mathbf{a}^\triangledown \) forms a dormant indigenous \( \mathcal{D}_X^{(N-1)} \)-module. By passing to the isomorphism \( \text{KS}_{\mathcal{O}_a^\triangledown} \) (cf. \((\ref{eq:KodairaSpencer})\)), we shall identify \( \mathcal{D}_X^{(N-1)} \) with \( \mathcal{O}_X^\oplus(n+1) \), i.e., the underlying vector bundle of \( \mathcal{O}_a^\triangledown \). Then, \( \mathcal{O}_a^\triangledown \) specifies an element of \( \mathcal{D}_X^{\infty} \) if and only if the equality \( \sum_{i=1}^{n+1} a_i = 0 \) holds.

(iii) Let us consider the case of \( N = \infty \). Then, for any element \( \mathcal{V}^\triangledown \) of \( \mathcal{D}_X^{\infty} \), there exists \( \mathbf{a} \in (T_p^\infty)^{n+1} \) with \( \mathcal{O}_\mathbf{a}^\triangledown \cong \mathcal{V}^\triangledown \). Moreover, such an \( (n+1) \)-tuple \( \mathbf{a} \) is uniquely determined up to permutation.

Proof. First, we shall prove the former assertion of (i). In order to complete the proof, it suffices to consider the case where \( N = 1 \). Let us fix an identification \( \Gamma(X, \mathcal{O}_X) = \bigoplus_{j=1}^{n} k \cdot e_j \), where \( (e_1, \ldots, e_n) \) is a basis of \( \Gamma(X, \mathcal{O}_X) \). Under the isomorphism \((\ref{eq:identification})\), we regard each \( a_i \) \((i = 1, \ldots, n)\) as an element of \( \Gamma(X, \mathcal{O}_X)^{\oplus} \), which may be expressed as \( a_i = \sum_{j=1}^{n} a_{ij} e_j \) for some \( a_{ij} \in k \) \((j = 1, \ldots, n)\). Let \( (\partial_1, \ldots, \partial_n) \in \Gamma(X, \mathcal{T}_X)^n \) be the dual basis of \( (e_1, \ldots, e_n) \). Also, we write \( \nabla := \bigoplus_{i=1}^{n} \nabla_{a_i} \) for simplicity. Then, the Kodaira-Spencer map \( \text{KS}_{\mathcal{O}_a^\triangledown} \) (cf. \((\ref{eq:KodairaSpencer})\)) is an isomorphism (i.e., \( \mathcal{O}_a^\triangledown \) forms a dormant indigenous \( \mathcal{D}_X^{(N-1)} \)-module) if and only if the elements

\[
(311) \quad (\Delta(1), \nabla(\partial_1)(\Delta(1)), \ldots, \nabla(\partial_n)(\Delta(1)))
\]

forms a basis of \( k^{n+1} \left( = \Gamma(X, \mathcal{O}_X^\oplus(n+1)) \right) \). Observe that \( \Delta(1) = (1, \ldots, 1), \nabla(\partial_j)(\Delta(1)) = (a_{1j}, \ldots, a_{(n+1)j}) \) \( (\text{for each } j = 1, \ldots, n) \), and

\[
(312) \quad \begin{vmatrix}
1 & \cdots & 1 \\
a_{11} & \cdots & a_{(n+1)1} \\
\vdots & & \vdots \\
a_{1n} & \cdots & a_{(n+1)n}
\end{vmatrix} = (-1)^n \begin{vmatrix}
a_{11} - a_{(n+1)1} & \cdots & a_{n1} - a_{(n+1)1} \\
\vdots & & \vdots \\
a_{1n} - a_{(n+1)n} & \cdots & a_{nn} - a_{(n+1)n}
\end{vmatrix}.
\]

Hence, the elements \((311)\) forms a basis if and only if the right-hand side of \((312)\) is nonzero, that is to say, the \( n \)-tuple \( (a_1 - a_{n+1}, a_2 - a_{n+1}, \ldots, a_n - a_{n+1}) \) forms a basis of \( \Gamma(X, \mathcal{O}_X)^{\oplus} \). This completes the proof of the former assertion. The latter assertion follows from the former assertion together with the fact that \( g_X \) coincides with the rank of \( T_p^N \).

Assertion (ii) follows from the equality \( \det(\bigoplus_{i=1}^{n+1} \nabla_{a_i}) = \nabla_{\sum_{i=1}^{n+1} a_i} \) and the equivalence \((\ref{eq:equivalence})\).

Finally, we shall consider assertion (iii). Let \( \mathcal{V}^\triangledown := (\mathcal{D}_X^{\infty}, \nabla_\mathcal{V}, \mathcal{O}_X) \) be an element of \( \mathcal{D}_X^{\infty} \). Denote by \( \rho \) the \( \text{GL}_{n+1} \)-representation associated with the \( \mathcal{D}_X^{(\infty)} \)-module \( (\mathcal{D}_X^{(\infty)}, \nabla_\mathcal{V}) \) (cf. \S 5.3), which is well-defined up to conjugation. Under the identification \((\ref{eq:identification})\), we regard
ρ as a homomorphism \( T_pX \oplus \text{Diag}(X^\vee(p))(k) \) \( \to \text{GL}_{n+1} \). Since \( \text{GL}_{n+1} \) is of finite type over \( k \) and \( T_pX \) is pro-finite, the image of \( \rho|_{T_pX} \) is finite. Hence, there exists an abelian variety \( Y \) and a Galois étale covering \( \pi : Y \to X \) such that the restriction of \( \rho \) to \( T_pY \) (\( \subseteq T_pX \)) is trivial. This means that the pull-back \( (\pi^*(\mathcal{D}^{(\infty)}_{X,1}), \pi^*(\nabla_Y)) \) decomposes as a direct sum \( \bigoplus_{i=1}^{n+1} (\mathcal{L}_i, \nabla_i) \) of invertible \( \mathcal{D}^{(\infty)}_X \)-modules. Since \( \pi^*(\mathcal{D}^{(\infty)}_{X,1}) \) may be obtained as an extension of the trivial vector bundle \( \mathcal{O}_Y^\oplus = (\pi^*(\mathcal{D}^{(\infty)}_{X,1})/\pi^*(\mathcal{D}^{(\infty)}_{X,0}) = \pi^*(\mathcal{T}_X) \) by the trivial line bundle \( \mathcal{O}_Y = (\pi^*(\mathcal{D}^{(\infty)}_{X,0})) \), each factor \( \mathcal{L}_i \) of \( \bigoplus_{i=1}^{n+1} \mathcal{L}_i = \pi^*(\mathcal{D}^{(\infty)}_{X,1}) \) is trivial. One verifies from this fact that the pull-back \( \pi^*(\mathcal{V}^\diamond) \) (cf. (288)), being a dormant indigenous \( \mathcal{D}^{(\infty)}_X \)-module, is isomorphic to that of the form (310). By assertion (i), \( Y \) turns out to be ordinary. It follows that \( X \) is ordinary, and hence, admits an \( F^\infty \)-projective structure of the form \( \mathcal{O}_a^\diamond \) because of assertion (i) again. In particular, \( \mathcal{D}^{(\infty)}_{X,1} \) can be trivialized via KS\(^\bullet\)\(\mathcal{O}_X^\diamond \), so the natural morphism \( \Gamma(X, \mathcal{D}^{(\infty)}_{X,1}) \to \Gamma(Y, \pi^*(\mathcal{D}^{(\infty)}_{X,1})) \) is bijective. This implies that if we fix an identification \( \eta : \bigoplus_{i=1}^{n+1} (\mathcal{O}_Y, \nabla_i) \xrightarrow{\sim} (\pi^*(\mathcal{D}^{(\infty)}_{X,1}), \pi^*(\nabla_Y)) \) of \( (\pi^*(\mathcal{D}^{(\infty)}_{X,1}), \pi^*(\nabla_Y)) \), then the element of \( \Gamma(Y, \pi^*(\mathcal{D}^{(\infty)}_{X,1}))^{n+1} \) defining \( \eta \) is invariant under the action of \( \text{Aut}(Y/X) \). Hence, each \( (\mathcal{O}_Y, \nabla_i) \) descends to an invertible \( \mathcal{D}^{(\infty)}_X \)-module of the form \( (\mathcal{O}_X, \nabla'_i) \) and the isomorphism \( \eta \) descends to an isomorphism \( \eta' : \bigoplus_{i=1}^{n+1} (\mathcal{O}_X, \nabla'_i) \xrightarrow{\sim} (\mathcal{D}^{(\infty)}_{X,1}, \nabla_Y) \). The \( \mathcal{D}^{(\infty)}_X \)-action \( \nabla'_i \) \( (i = 1, \cdots, n + 1) \) may be expressed as \( \nabla'_i = \nabla_{a'_i} \) for a unique \( a'_i \in T_p^{(\infty)} \). The inclusion \( \mathcal{O}_X \xhookrightarrow{} \mathcal{O}_X^{\oplus(n+1)} \) corresponding, via \( \eta' \), to the natural inclusion \( \mathcal{O}_X \to \mathcal{D}^{(\infty)}_{X,1} \) may be expressed as \( \mu_b := (\mu_{b_1}, \cdots, \mu_{b_{n+1}}) \) (cf. (295)) for some \( b = (b_i)_{i=1}^{n+1} \in k^{n+1} \). Since the triple \( (\mathcal{O}_X^{\oplus(n+1)}, \bigoplus_{i=1}^{n+1} \nabla_{a'_i}, \text{Im}(\mu_b)) \) \( (\equiv \mathcal{V}^\diamond) \) forms an indigenous \( \mathcal{D}^{(\infty)}_X \)-module, all \( b_i \)'s must be nonzero. After composing \( \eta' \) with \( \bigoplus_{i=1}^{n+1} \mu_{b_i}^{-1} \), we can assume that \( b = (1, \cdots, 1) \), i.e., \( \mu_b = \Delta \). Consequently, \( \mathcal{V}^\diamond \) is isomorphic to \( \mathcal{O}_a^\diamond \), where \( a' = (a'_1, \cdots, a'_{n+1}) \). Because of the equivalence (300), such an \( (n+1) \)-tuple \( a' \) is verified to be unique up to permutation. This completes the proof of assertion (iii).

The following proposition is the “affine” version of the above assertion.

**Proposition 6.3.2.**

Given an element \( a := (a_i)_{i=1}^{n+1} \) of \( (T_p^{(N)})^{n+1} \) and a surjection \( \delta : \mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_X \), we shall write

\[
\mathcal{O}^\diamond_{a,\delta} := (\mathcal{O}_X^{\oplus(n+1)}, \bigoplus_{i=1}^{n+1} \nabla_{a_i}, \text{Im}(\Delta), \delta),
\]

where \( \Delta \) is as in Proposition 6.3.1. Then, the following assertions hold:

(i) The collection \( \mathcal{O}^\diamond_{a,\delta} \) forms a dormant affine-indigenous \( \mathcal{D}^{(N-1)}_X \)-module if and only if the \( n \)-tuple \( (a_1 - a_{n+1}, a_2 - a_{n+1}, \cdots, a_n - a_{n+1}) \) forms a basis of \( T_p^{(N)} \) and \( \delta \) coincides with the \( j \)-th projection \( \text{pr}_j : \mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_X \) for some \( j \in \{1, \cdots, n+1\} \). In particular, \( X \) admits an \( F^N \)-affine structure of the form \( \mathcal{O}^\diamond_{a,\delta} \) if and only if \( X \) is ordinary.

(ii) Let us fix \( j \in \{1, \cdots, n+1\} \). Suppose that \( \delta = \text{pr}_j \) and the collection

\[
\mathcal{O}^\diamond_{a,j} := \mathcal{O}^\diamond_{a,\text{pr}_j}
\]
forms a dormant affine-indigenous $\mathcal{D}_X^{(N-1)}$-module. By passing to $K_S^{\otimes}$, where $\mathcal{O}_a^{\otimes} := (\mathcal{O}_X^{\otimes(n+1)}, \bigoplus_{i=1}^{n+1} \nabla_{a_i}, \text{Im}(\Delta))$, we shall identify $\mathcal{O}_X^{\otimes(n+1)}$ with $\mathcal{D}_X^{(N-1)}$. Then, $\mathcal{K}_a^{\otimes}$ specifies an element of $t^\infty \mathcal{O}_{X,N}^{2zz}$ if and only if the equality $a_j = 0$ holds.

(iii) Let us consider the case of $N = \infty$. Then, for any element $\mathcal{K}_a^{\otimes} \mathcal{O}_{X,\infty}$, there exists a basis $\mathfrak{a} := (a_1, \ldots, a_n)$ of $T_p(\infty)$ with $\mathcal{O}_X^{\otimes(\mathfrak{a},0),n+1} \cong \mathcal{K}_a^{\otimes} \mathcal{O}_{X,\infty}$. Moreover, such an $n$-tuple $\mathfrak{a}$ is uniquely determined up to permutation.

Proof. The assertion follows immediately from Proposition 6.3.1 and the equivalences (300).

Corollary 6.3.3.
Let $X$ be an abelian variety over $k$. Then, $X$ admits an $F^\infty$-projective (resp., $F^\infty$-affine) structure if and only if $X$ is ordinary.

Proof. The assertion follows from Proposition 6.3.1 (resp., Proposition 6.3.2), (i) and (iii).

Corollary 6.3.4.
Let $X$ be an ordinary abelian variety over $k$. Then, the following assertions hold:

(i) The map

$$\mathcal{K}_a^{\otimes} \otimes \mathcal{O}_X^{\otimes} : \mathfrak{T} \to \mathfrak{K}_{X,\infty}$$

(cf. (228)) is bijective.

(ii) For $i = 1, 2$, let $\mathcal{S}_i^{\ominus}$ be an $F^\infty$-affine structure on $X$ and denote by $\nabla^{\nabla}_i$ the dual affine connection of level $\infty$ induced by $\mathcal{S}_i^{\ominus}$ via the diagram in Theorem A, (ii). Then, $\mathcal{S}_1^{\ominus} = \mathcal{S}_2^{\ominus}$ if and only if $(\Omega_X, \nabla^{\nabla}_1) \cong (\Omega_X, \nabla^{\nabla}_2)$ as $\mathcal{D}_X^{(\infty)}$-modules.

Proof. First, we shall consider assertion (i). Let us take an arbitrary Tango structure $\mathcal{U}^{\infty}$ in $\mathfrak{T}_{X,\infty}$. Denote by $\mathcal{V}^{\ominus}$ the dormant affine-indigenous $\mathcal{D}_X^{(\infty)}$-module classified by $t^\infty \mathcal{O}_{X,\infty}$ corresponding to $\mathcal{U}^{\infty}$. According to Proposition 6.3.2 (iii), $\mathcal{V}^{\ominus}$ may be expressed as

$$\mathcal{O}_X^{\otimes(n+1)} \bigoplus \nabla_{a_i} \nabla^{\text{triv}(\infty)} \text{Im}(\Delta), \text{pr}_{n+1}$$

for some basis $(a_i)_{i=1}^{n}$ of $T_p(\infty)$. In particular, each $a_i$ is nonzero, which implies the equality $\Gamma(X^{(1)}, \text{Sol}(\nabla_{a_i}^{(0)})) = 0$ (cf. (300)). The dual affine connection $\nabla^{\nabla}_i$ of level $\infty$ associated with $\mathcal{U}^{\infty}$ is, by construction, given by $\bigoplus_{i=1}^{n} \nabla_{a_i}$ under a suitable identification $\Omega_X \cong \mathcal{O}_X^{\otimes n}$. Hence, we have

$$\Gamma(X^{(1)}, \text{Sol}(\nabla^{\nabla}_{\Omega_X}^{(1)}))^{\nabla} = \bigoplus_{i=1}^{n} \Gamma(X^{(1)}, \text{Sol}(\nabla_{a_i}^{(0)}))^{\nabla} = 0,$$

i.e., $\nabla^{\nabla}_{\Omega_X}$ belongs to $\mathfrak{K}_{X,\infty}$ (cf. (229)). It follows that the image of $\mathcal{K}_a^{\otimes} \mathcal{O}_{X,\infty}$ lies in $\mathfrak{K}_{X,\infty}^{\otimes}$. Since $X$ is Frobenius split because of the ordinariness assumption (cf. [58], Lemma 1.1), the assertion follows from Proposition 4.3.3 (ii).

Assertion (ii) follows from the equivalences (300) and the description of a dual affine connection of level $\infty$ discussed in the proof of (i).
6.4. Explicit description in terms of $p$-adic Tate module.

We shall describe dormant indigenous and affine-indigenous $D_X^{(\infty)}$-modules by using the $p$-adic Tate module of $X^\vee$.

In this subsection, we assume that $p \mid (n+1)$. Let $N$ be an element of $\mathbb{Z}_{>0} \cup \{ \infty \}$. To begin with, we shall define a map of sets

$$\uparrow \ell_N : \uparrow A^Y_{X,N} \to \uparrow A^Z_{X,N}$$

as follows. Let $\downarrow Y_{X,N} := (D^{(N-1)}_{X,1}, \nabla_Y, \mathcal{O}_X, \delta)$ be an element of $\uparrow A^Y_{X,N}$. Denote by $\nabla_{\text{det}(\text{Ker}(\delta))}$ the $D^{(N-1)}_X$-action on $\mathcal{O}_X$ ($= \det(\nabla_Y) = \det(\text{Ker}(\delta))$) defined as the determinant of the restriction $\nabla_Y|_{\text{Ker}(\delta)}$ of $\nabla_Y$ to $\text{Ker}(\delta)$. It follows from Lemma 3.6.3 that there exists a unique $D^{(N-1)}_X$-action $\nabla_{\text{det}(\text{Ker}(\delta))}^{1/(n+1)}$ on $\mathcal{O}_X$ whose $(n+1)$-st tensor product coincides with $\nabla_{\text{det}(\text{Ker}(\delta))}$. One verifies that $\downarrow Y_{X,N} \otimes (\mathcal{O}_X, \nabla_{\text{det}(\text{Ker}(\delta))}^{1/(n+1)})$ (cf. (321)) belongs to $\uparrow A^Z_{X,N,\mathcal{O}_X}$. Thus, we define $\uparrow \ell_N$ to be the map given by assigning $\downarrow Y_{X,N} \mapsto \downarrow Y_{X,N} \otimes (\mathcal{O}_X, \nabla_{\text{det}(\text{Ker}(\delta))}^{1/(n+1)})$.

Now, we shall set

$$\mathbb{B}_{X,N} := \left\{ (b_i)_{i=1}^n \in (T_p^{(N)})^n \; \mid \; \forall_i b_i \in \det(T_p^{(N)})^\times \right\}.$$

In the case where $X$ is ordinary (or equivalently, $\mathbb{B}_{X,N}$ is nonempty) and $N < \infty$, the number of elements in $\mathbb{B}_{X,N}$ may be calculated by the following formula:

$$\sharp(\mathbb{B}_{X,N}) = p^{n(n-1)/2} \cdot \prod_{i=1}^n (p^i - 1).$$

(This calculation will be used in the proof of Corollary 6.4.2 described later.) Notice that the set $\mathbb{B}_{X,N}$, where $N \in \mathbb{Z}_{>0} \cup \{ \infty \}$, has an action of the symmetric group $\mathfrak{S}_n$ of $n$ letters by permutations of factors, i.e., in such a way that $(b_i)_i \mapsto (b_{\sigma(i)})_i$ for any $\sigma \in \mathfrak{S}_n$. In particular, we obtain the orbit set $\mathbb{B}_{X,N}/\mathfrak{S}_n$.

Next, let us consider the equivalent relation “$\sim$” in $\mathbb{B}_{X,N}/\mathfrak{S}_n$ generated by the relations

$$\delta \sim (b_1, b_2, \ldots, b_n) \sim (-b_1, b_2 - b_1, \ldots, b_n - b_1)$$

for all representatives $(b_1, \ldots, b_n) \in \mathbb{B}_{X,N}$. Denote by

$$\delta \mathbb{B}_{X,N}/\mathfrak{S}_n$$

the quotient set of $\mathbb{B}_{X,\infty}/\mathfrak{S}_n$ by this equivalence relation. The natural quotient $\mathbb{B}_{X,N}/\mathfrak{S}_n \to \delta \mathbb{B}_{X,N}/\mathfrak{S}_n$ turns out to be an $(n+1)$-to-1 correspondence.

Theorem 6.4.1.

For each $b := (b_1, \ldots, b_n) \in (T_p^{(N)})^n$, we shall write $b' := (b, 0) - \sum_{i=1}^n b_i \cdot (1, 1, \ldots, 1) \in (T_p^{(N)})^{n+1}$. Then, the assignments $b \mapsto \mathcal{O}_{b'}^\delta$ and $b \mapsto \mathcal{O}_{(b, 0), n+1}^\delta$ (cf. (314)) determine well-defined injections

$$\uparrow \ell_N : \delta \mathbb{B}_{X,N}/\mathfrak{S}_n \to \uparrow A^Z_{X,N,\mathcal{O}_X}, \quad \uparrow A^Y_{N,\mathcal{O}_X} : \mathbb{B}_{X,N}/\mathfrak{S}_n \to \uparrow A^Z_{X,N}.$$
respectively, and the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{B}_{X,N}/\mathcal{S}_n & \xrightarrow{\tau_{\mathbb{B}_{X,N}/\mathcal{S}_n}} & \mathcal{D}_{X,N} \\
\downarrow & & \downarrow \sim \\
\delta\mathbb{B}_{X,N}/\mathcal{S}_n & \xrightarrow{\tau_{\delta\mathbb{B}_{X,N}/\mathcal{S}_n}} & \mathcal{D}_{X,N,O_X} \\
\end{array}
\]

(324)

where the leftmost vertical arrow denotes the natural quotient. If, moreover, \( N = \infty \), then the two maps in (323) are bijective. In particular, the map \( \iota_\infty : F^\infty\text{-}\text{Aff}_{X,N} \to F^\infty\text{-}\text{Proj}_X \) defines an \((n+1)\)-to-1 correspondence between the sets \( F^\infty\text{-}\text{Aff}_{X,N} \) and \( F^\infty\text{-}\text{Proj}_X \).

**Proof.** The assertion follows from the various definitions involved and Propositions 6.3.1 and 6.3.2.

**Corollary 6.4.2.**

Let \( N \) be a positive integer and \( X \) an ordinary abelian variety of dimension \( n \) over \( k \). Then, \( X \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure that is invariant under the translation by any point of \( X \). Moreover, the following inequality holds:

\[
\sharp(F^N\text{-}\text{Proj}_X) \geq \frac{1}{(n+1)!} \cdot p^{n(N-1)+\frac{n(n-1)}{2}} \cdot \prod_{i=1}^{n}(p^i - 1)
\]

resp., \( \sharp(F^N\text{-}\text{Aff}_X) \geq \frac{1}{n!} \cdot p^{n(N-1)+\frac{n(n-1)}{2}} \cdot \prod_{i=1}^{n}(p^i - 1) \).

**Proof.** The assertion follows from Theorem 6.4.1 and (320).

**Remark 6.4.3.**

In this remark, we shall describe dormant affine-indigenous \( \mathcal{D}_X^{(\infty)} \)-module in terms of representations of the stratified fundamental group. (The case of dormant indigenous \( \mathcal{D}_X^{(\infty)} \)-modules is entirely similar.)

Let us take an element \( b \in T_p^{(\infty)} \). The collection \( \{\text{Sol}(\nabla_b^{(m)})\}_{m \in \mathbb{Z}_{\geq 0}} \) (cf. (299) for the definition of \( \nabla_{(-)} \)) specifies an element of \( X^\vee(p)(k) \). It defines a character

\[
\text{Char}(b) := \text{Char}(\{\text{Sol}(\nabla_b^{(m)})\}_{m} : \text{Diag}(X^\vee(p)(k)) \to \mathbb{G}_m)
\]

(cf. (308)). Now, let \( b := (b_i)_{i=1}^n \) be an element of \( \mathbb{B}_{X,\infty} \), and denote by \( \mathcal{S}^\Diamond \) the \( F^\infty \)-affine structure corresponding to the dormant affine-indigenous \( \mathcal{D}_X^{(\infty)} \)-module \( \mathcal{O}_{(b,0),n+1}^{\Diamond} \). (According to Proposition 6.3.2 (iii), any \( F^\infty \)-affine structure can be obtained in this way.) Then, the
monodromy representation $\rho_{S^\circ}$ (cf. (275)) may be represented by the composite homomorphism

\[(327)\]
\[
\pi_1^{\text{str}}(X) \xrightarrow{\text{309}} T_p(X) \oplus \text{Diag}(X^\vee(p)(k)) \\
\xrightarrow{\text{pr}_2} \text{Diag}(X^\vee(p)(k)) \\
\xrightarrow{(\text{Char}(b_i))_{i=1}^n} \mathbb{G}_m^n \\
\rightarrow \text{PGL}_{n+1}^k,
\]

where the last arrow denotes the homomorphism given by $(a_1, \cdots, a_n) \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$.

6.5. Descent via étale coverings.

In this final subsection, we shall discuss constructions of the quotients via a Galois étale covering of various structures.

Let $Z$ be a smooth variety over $k$ of dimension $n$ and $\pi : X \rightarrow Z$ a Galois étale covering with Galois group $G$. For each $g \in G$, denote by $\iota_g$ the automorphism of $X$ corresponding to $g$. Given a sheaf $F$ on $Z$ endowed with a $G$-action, we write $\mathcal{F}^G$ for the subsheaf of $\mathcal{F}$ consisting of $G$-invariant sections.

$F^N$-projective and $F^N$-affine structures:

For each $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, we shall denote by

\[(328)\]
\[
F^N\text{-proj}^G_X \quad (\text{resp., } F^N\text{-aff}^G_X)
\]

the set of $G$-invariant elements of $F^N\text{-proj}_X$ (resp., $F^N\text{-aff}_X$), i.e., the subset consisting of $F^N$-projective (resp., $F^N$-affine) structures $S^\circ$ such that $\iota_g^*(S^\circ) = S^\circ$ for any $g \in G$. In what follows, let us construct a map from $F^N\text{-proj}^G_X$ (resp., $F^N\text{-aff}^G_X$) to $F^N\text{-proj}^G_Z$ (resp., $F^N\text{-aff}^G_Z$).

Observe that for each $g \in G$, the automorphism $\iota_g$ induces an isomorphism $\iota_g^*: \iota_g^{-1}(\mathcal{P}^\text{ét}_X) \xrightarrow{\sim} \mathcal{P}^\text{ét}_X$ (resp., $\iota_g^*: \iota_g^{-1}(\mathcal{A}^\text{ét}_X) \xrightarrow{\sim} \mathcal{A}^\text{ét}_X$). The isomorphisms $\iota_g^*$'s (resp., $\iota_g^*$'s) determine a $G$-action on the direct image $\pi_*(\mathcal{P}^\text{ét}_X)$ (resp., $\pi_*(\mathcal{A}^\text{ét}_X)$). Then, the morphism $\mathcal{P}^\text{ét}_Z \rightarrow \pi_*(\mathcal{P}^\text{ét}_X)$ (resp., $\mathcal{A}^\text{ét}_Z \rightarrow \pi_*(\mathcal{A}^\text{ét}_X)$) corresponding to (282) via the adjunction relation "$\pi^{-1}(-) \dashv \pi_*(-)$" induces an isomorphism

\[(329)\]
\[
\mathcal{P}^\text{ét}_Z \xrightarrow{\sim} \pi_*(\mathcal{P}^\text{ét}_X)^G \quad (\text{resp., } \mathcal{A}^\text{ét}_Z \xrightarrow{\sim} \pi_*(\mathcal{A}^\text{ét}_X)^G).
\]

Now, let $N$ be a positive integer and let us take an element $S^\circ$ of $F^N\text{-proj}^G_X$ (resp., $F^N\text{-aff}^G_X$). Write

\[(330)\]
\[
S^\circ_G := \pi_*(S^\circ)^G,
\]

where the $G$-action on $\pi_*(S^\circ)$ is given by restricting $\iota_g^*$'s (resp., $\iota_g^*$'s) to $S^\circ$. Then, it may be thought of as a subsheaf of $\mathcal{P}^\text{ét}_Z$ (resp., $\mathcal{A}^\text{ét}_Z$) via (329), and forms an $F^N$-projective (resp.,
forms a dormant indigenous structure on $Z$. The resulting assignment $\mathcal{S}^{\Theta} \mapsto \mathcal{S}_{/G}^{\Theta}$ determines a bijection of sets

$$F^N\text{-Proj}_{X}^G \cong F^N\text{-Proj}_{Z}, \quad \text{resp., } F^N\text{-Aff}_{X}^G \cong F^N\text{-Aff}_{Z},$$

whose inverse is given by pull-back $\pi^*(-)$. Since the formation of $\mathcal{S}_{/G}^{\Theta}$ is compatible with truncation to lower levels, the construction of this map may be extended to the case of $N = \infty$. In any case of $N \in \mathbb{Z}_{>0} \cup \{\infty\}$, we shall refer to $\mathcal{S}_{/G}^{\Theta}$ as the quotient of $\mathcal{S}^{\Theta}$ by the $G$-action.

**Dormant indigenous and affine-indigenous $D_X^{(N)}$-modules:**

Next, let us study quotients of dormant indigenous and affine-indigenous $D_X^{(N)}$-modules. Suppose that $X$ satisfies the equality $\Gamma(X, \mathcal{O}_X) = k$. In the following non-resp’d discussion, we fix a theta characteristic $\Theta$ of $X$ that is invariant, in the evident sense, under the $G$-action on $X$. (The following discussion can be generalized to the situation where we deal with an $F^N$-theta characteristic, not just a theta characteristic.) Then, $\Theta$ descends to a theta characteristic $\Theta^G$ of $Z$, i.e., $\Theta^G := \pi_*(\Theta)^G$. For each $N \in \mathbb{Z}_{>0} \cup \{\infty\}$, we shall write

$$\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta}^{\mathbb{Z}_{\geq -r}\cdots G} \quad \text{resp., } \dagger \mathfrak{A}\mathcal{O}_{X,N}^{\mathbb{Z}_{\geq -r}\cdots G}$$

for the set of $G$-invariant elements in $\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta}^{\mathbb{Z}_{\geq -r}\cdots G}$ (resp., $\dagger \mathfrak{A}\mathcal{O}_{X,N\cdots G}$), i.e., the subset consisting of dormant indigenous (resp., affine-indigenous) $D_X^{(N-1)}$-modules $(\mathbb{A})\mathcal{V}^{\Theta}$ with $\pi_*(\mathbb{A})\mathcal{V}^{\Theta} = (\mathbb{A})\mathcal{V}^{\Theta}$ for every $g \in G$ (cf. Proposition 3.5.5).

In what follows, we shall construct a map from $\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta}^{\mathbb{Z}_{\geq -r}\cdots G}$ (resp., $\dagger \mathfrak{A}\mathcal{O}_{X,N\cdots G}$) to $\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta^G}$ (resp., $\dagger \mathfrak{A}\mathcal{O}_{X,N\cdots G}$). Let $\mathcal{V}^{\Theta} := (D_{X,1}^{(N-1)} \otimes \Theta, \nabla_{\mathcal{V}, \Theta})$ be an element in $\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta}^{\mathbb{Z}_{\geq -r}\cdots G}$. One verifies from, e.g., the local description of $D_X^{(N-1)}$ discussed in §2.7 that the morphism $D_{Z}^{(N-1)} \otimes \Theta^G \rightarrow \pi_*(D_{X}^{(N-1)} \otimes \Theta)^G$ induced by $\pi$ is an isomorphism. By means of this isomorphism, the $D_X^{(N-1)}$-action $\nabla_{\mathcal{V}}$ induces a $D_{Z}^{(N-1)} := \pi_*(D_{X}^{(N-1)})^G$-action $\nabla_{\mathcal{V}/G}$ on $D_{Z}^{(N-1)} \otimes \Theta^G$. Under the natural identifications $\pi^*(D_{Z}^{(N-1)}) \rightarrow D_X^{(N-1)}$ and $\pi^*(\Theta^G) \rightarrow \Theta$, the pull-back via $\pi$ of the Kodaira-Spencer map $\text{KS}_{(D_{Z}^{(N-1)} \otimes \Theta^G, \nabla_{\mathcal{V}/G}, \Theta^G)}$ may be identified with $\text{KS}_{(D_{X}^{(N-1)} \otimes \Theta, \nabla_{\mathcal{V}, \Theta})}$. Since $\text{KS}_{(D_{X}^{(N-1)} \otimes \Theta, \nabla_{\mathcal{V}, \Theta})}$ is an isomorphism, the faithful flatness of $\pi$ implies that $\text{KS}_{(D_{Z}^{(N-1)} \otimes \Theta^G, \nabla_{\mathcal{V}/G}, \Theta^G)}$ is an isomorphism, that is to say, the collection

$$\mathcal{V}_{/G}^{\Theta} := (D_{Z}^{(N-1)} \otimes \Theta^G, \nabla_{\mathcal{V}/G}, \Theta^G)$$

forms a dormant indigenous $D_{Z}^{(N-1)}$-module. We shall refer to $\mathcal{V}_{/G}^{\Theta}$ as the quotient of $\mathcal{V}^{\Theta}$ by the $G$-action. Moreover, given a dormant affine-indigenous $D_X^{(N-1)}$-module $\mathbb{A}\mathcal{V}^{\Theta}$ classified by $\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta}^{\mathbb{Z}_{\geq -r}\cdots G}$, we can define, in a similar manner, the quotient $\mathbb{A}\mathcal{V}_{/G}^{\Theta}$ of $\mathbb{A}\mathcal{V}^{\Theta}$ by the $G$-action. The resulting assignment $\mathcal{V}^{\Theta} \mapsto \mathcal{V}_{/G}^{\Theta}$ (resp., $\mathbb{A}\mathcal{V}^{\Theta} \mapsto \mathbb{A}\mathcal{V}_{/G}^{\Theta}$) defines a bijection

$$\dagger \mathfrak{I}\mathcal{O}_{X,N,\Theta}^{\mathbb{Z}_{\geq -r}\cdots G} \cong \dagger \mathfrak{I}\mathcal{O}_{Z,N,\Theta^G}^{\mathbb{Z}_{\geq -r}\cdots G}, \quad \text{resp., } \dagger \mathfrak{A}\mathcal{O}_{X,N\cdots G}^{\mathbb{Z}_{\geq -r}\cdots G} \rightarrow \dagger \mathfrak{A}\mathcal{O}_{Z,N,\Theta^G}^{\mathbb{Z}_{\geq -r}\cdots G},$$

whose inverse is given by pull-back via $\pi$.

By the above discussions, we have obtained quotients of $F^N$-projective and $F^N$-affine structures, as well as dormant indigenous and affine-indigenous $D_X^{(N-1)}$-modules, via a Galois étale
covering. Also, we can define, in an evident manner, the quotients of \( F^N \)-indigenous and \( F^N \)-affine-indigenous structures, as well as Tango structures. Just as the assertion of Proposition 6.1.2, the formations of such quotients are respectively compatible with various maps appearing in Theorem A (i) and (ii). We will omit the details of this fact. At any rate, the construction of pull-back for \( F^N \)-projective and \( F^N \)-affine structures gives the following assertion.

**Proposition 6.5.1.**
Let \( N \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \), and let \( X \) and \( Z \) be smooth varieties over \( k \). Suppose that there exists a Galois étale covering \( X \to Z \) with Galois group \( G \) and that \( X \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure invariant under the \( G \)-action, i.e., that belongs to \( F^N \text{-Proj}^G_X \) (resp., \( F^N \text{-Aff}^G_X \)). Then, \( Z \) admits an \( F^N \)-projective (resp., \( F^N \)-affine) structure.

Also, by means of Theorem 6.4.1, we can describe explicitly \( F^\infty \)-projective and \( F^\infty \)-affine structure on a quotient of an ordinary abelian variety, as follows.

**Proposition 6.5.2.**
Let \( n \) be a positive integer with \( p \nmid (n + 1) \). Also, let \( Z \) be a smooth variety over \( k \) such that there exists an ordinary abelian variety \( X \) over \( k \) of dimension \( n \) and a Galois étale covering \( \pi : X \to Z \) with Galois group \( G \). Let us consider the \( G \)-action on the set \( \delta \backslash \mathbb{B}_{X,\infty}/\mathfrak{S}_n \) (resp., \( \mathbb{B}_{X,\infty}/\mathfrak{S}_n \)) induced by the natural \( G \)-action on \( T^\infty_p \) (resp., \( \mathbb{B}_{X,\infty}/\mathfrak{S}_n \)) of \( \mathfrak{S}_n \)-equivariant elements. Then, the composite bijection \( \tilde{\iota}_\infty^\mathbb{B} \circ \tilde{\iota}_\infty^\mathbb{B} \propto \mathfrak{S}_n^\mathbb{B} / \mathfrak{S}_n^\mathbb{B} \mapsto F^\infty - \text{Proj}^G_X \) (resp., \( \tilde{\iota}_\infty^\mathbb{B} \circ \tilde{\iota}_\infty^\mathbb{B} \propto \mathfrak{S}_n^\mathbb{B} / \mathfrak{S}_n^\mathbb{B} \mapsto F^\infty - \text{Aff}^G_X \)) obtained in Theorem 6.4.1 restricts, via the bijection \( F^\infty - \text{Proj}^G_X \propto F^\infty - \text{Proj}^G_Z \) (resp., \( F^\infty - \text{Aff}^G_X \propto F^\infty - \text{Aff}^G_Z \)) displayed in (331), to a bijection
\[
(335) \quad \tilde{\iota}_\infty^\mathbb{B} \propto \mathfrak{S}_n^\mathbb{B} / \mathfrak{S}_n^\mathbb{B} : \delta \backslash \mathbb{B}_{X,\infty}/\mathfrak{S}_n \mapsto F^\infty - \text{Proj}^G_Z \]
In particular, \( Z \) admits an \( F^\infty \)-projective (resp., \( F^\infty \)-affine) structure if and only if there exists a \( G \)-invariant element of \( \delta \backslash \mathbb{B}_{X,\infty}/\mathfrak{S}_n \) (resp., \( \mathbb{B}_{X,\infty}/\mathfrak{S}_n \)).

**Proof.** The assertion follows from Theorem 6.4.1 and the fact that the arrows in the diagram (324) are respectively \( G \)-equivariant. (Notice that, in the non-resp’d portion, the formations of \( \mathfrak{S}_X^{\mathbb{B}, \infty} \), \( \mathfrak{S}_X^{\mathbb{B}, \infty} \), and \( \mathfrak{S}_X^{\mathbb{B}, \infty} \) do not depend on the choice of \( \theta : \mathcal{O}_X^{(n+1)} \mapsto \omega_X \) defining the trivial theta characteristic \( \mathcal{O}_X \). Hence, pulling-back dormant indigenous \( \mathfrak{D}_X^{\infty} \)-modules via \( \iota_g \) for various \( g \in G \) gives a \( G \)-action on \( \mathfrak{S}_X^{\mathbb{B}, \infty} \), and the maps \( \mathfrak{S}_X^{\mathbb{B}, \infty} \), \( \mathfrak{S}_X^{\infty} \) turn out to be \( G \)-equivariant.)

7. Case 3: Curves

The purpose of this section is to develop the study of \( F^N \)-projective and \( F^N \)-affine structures on algebraic curves of higher levels, especially of infinite level. The level 1 case has been studied in many literatures (cf. [30], [31], [64], [75], [76], [80], and [82]) from various points of view.
For example, the moduli space of such objects has been investigated in the context of p-adic Teichmüller theory. As a remarkable result in that theory, it was shown that any smooth projective curve admits a dormant $F^1$-projective structure, but has only finitely many such structures. In this section, we focus mainly on the higher-level version of this result.

Let $X$ be a smooth projective curve over $k$ and suppose that $p > 2$. Also, let $N$ be a positive integer.

7.1. A rank $p^N$ vector bundle associated with a line bundle.

First, we deal with a certain rank $p^N$ vector bundle, which will be denoted by $A_{N,(-)}$ below, associated with each line bundle. As we will prove later, any dormant indigenous $D_X^{(N-1)}$-module can be embedded into this vector bundle; this fact yields a bijective correspondence between dormant indigenous $D_X^{(N-1)}$-modules and $k$-rational points of a certain Quot scheme (cf. Theorem 7.2.2).

Write $C_X^{(N)}$ for the $O_X^{(N)}$-linear composite
\[(336) \quad C_X^{(N)} := C_X^{(N-1)} \circ \cdots \circ F^{(N-2)}_{X(k^s)}(C_{X(k^s)}) \circ F^{(N-1)}_{X(k^s)}(C_X) : F^{(N)}_{X(k^s)}(\Omega_X) \to \Omega_{X^{(N)}},
\]
where $C_X^{(m)}$ (for each $m = 0, 1, \cdots, N-1$) denotes the Cartier operator of $X^{(m)}$. Also, given a line bundle $L$ on $X$, we shall write
\[(337) \quad A^\mathbb{P}_{N,L} := L \otimes \Omega_X^{(p^N-1)}, \quad A_{N,L} := F^{(N)}_{X(k^s)}(F^{(N)}_{X(k^s)}(L)).
\]

The sheaf $A_{N,L}$, being a rank $p^N$ vector bundle on $X$, has a structure of $D_X^{(N-1)}$-action
\[(338) \quad \nabla A_{N,L} := \nabla^{can(N-1)}_{F^{(N)}_{X(k^s)}(L)}
\]
(cf. (112)) with vanishing $p-(N-1)$-curvature.

In what follows, we construct an $O_X$-linear morphism from $A^\mathbb{P}_{L}$ to $A_{L}$. Let us consider the duality pairing
\[(339) \quad F^{(N)}_{X(k^s)}(L) \otimes F^{(N)}_{X(k^s)}(\mathcal{L}^\vee \otimes \Omega_X) \to \Omega_{X^{(N)}}
\]
which is the $O_X^{(N)}$-linear morphism given by $v \otimes (u \otimes \delta) \mapsto C_X^{(N)}(vu \delta)$ for any local sections $v \in \mathcal{L}$, $u \in \mathcal{L}^\vee$, and $\delta \in \Omega_X$. We shall denote by $\gamma$ the following composite:
\[(340) \quad \gamma : F^{(N)}_{X(k^s)}(\mathcal{L}^\vee \otimes \Omega_X^{(p^N-1)}) \xrightarrow{\gamma} F^{(N)}_{X(k^s)}(\mathcal{L}^\vee \otimes \Omega_X \otimes F^{(N)}_{X(k^s)}(\Omega_{X^{(N)}})) \xrightarrow{\gamma} F^{(N)}_{X(k^s)}(\mathcal{L}^\vee \otimes \Omega_X) \otimes \Omega_{X^{(N)}} \xrightarrow{\gamma} F^{(N)}_{X(k^s)}(\mathcal{L}^\vee),
\]
where the second arrow follows from the projection formula and the third arrow denotes the morphism arising from (339).

**Lemma 7.1.1.**

Let us fix a local coordinate $x \in O_X$, which gives a local identification $\Omega_X = O_X dx$, and a local generator $v$ of $\mathcal{L}$. Write $v^\vee$ for the local generator of $\mathcal{L}^\vee$ defined as the dual of $v$. Then, the
morphism $\gamma$ is an isomorphism whose local description may be given by
\[
\gamma(x^\alpha \cdot v^\vee \otimes (dx)^{(1-p^N)})(x^\beta \cdot v) = \begin{cases} 
 x^{(\alpha+\beta+1)/p^N-1} & \text{if } (\alpha + \beta + 1)/p^N \text{ is an integer,} \\
 0 & \text{if otherwise}
\end{cases}
\]
for each $\alpha, \beta = 0, 1, \ldots, p^N - 1$.

Proof. The assertion follows from an argument entirely similar to the argument in the proof of [57], §2, Proposition 5. \qed

The inverse $\gamma^{-1}$ to the isomorphism $\gamma$ corresponds, via the adjunction relation “$F_X^{(N)*}(-) \dashv F_X^{(N)}(-)$”, to a morphism
\[
\mathcal{A}_{N,L}^\vee \left( = F_X^{(N)*}(F_X^{(N)}(L))\right) \to (\mathcal{A}_{N,L}^\alpha)^\vee \left( = L^\vee \otimes \Omega_X^{\otimes (1-p^N)} \right).
\]
Hence, its dual determines a morphism
\[
\xi_L : \mathcal{A}_{N,L}^\xi \to \mathcal{A}_{N,L},
\]
as desired. Since $\gamma$ is nonzero and $X$ is a smooth curve, the resulting morphism $\xi_L$ turns out to be injective. We often consider $\mathcal{A}_{N,L}^\xi$ as an $\mathcal{O}_X$-submodule of $\mathcal{A}_{N,L}$ via this injection $\xi_L$.

**Lemma 7.1.2.**

The composite
\[
(\mathcal{D}_X^{(N-1)} \otimes \mathcal{A}_{N,L}^\xi) \hookrightarrow \mathcal{D}_X^{(N-1)} \otimes \mathcal{A}_{N,L} \xrightarrow{\nabla_{A_{N,L}}} \mathcal{A}_{N,L}
\]
is an isomorphism, where the first arrow denotes the tensor product of the natural inclusion $\mathcal{D}_X^{(N-1)} \hookrightarrow \mathcal{D}_X^{(N-1)}$ and $\xi_L$.

Proof. In order to complete the proof, we observe the local description of $\kappa_L$. Let us take a local coordinate $x$ of $X$, which gives a local identification $\Omega_X = \mathcal{O}_X dx$, and a local generator $v$ of $L$. One verifies from Lemma [7.1.1] that $\xi_L$ may be described locally as the assignment
\[
v \otimes (dx)^{(p^N-1)} : D := \sum_{\alpha=0}^{p^N-1} x^{p^N-\alpha-1} \otimes x^\alpha v.
\]
Let $\partial \in \mathcal{D}_X^{(N-1)}$ be the dual of $dx$ and use the local description of $\mathcal{D}_X^{(N-1)}$ recalled in §2.1.

Then, the $\mathcal{D}_X^{(N-1)}$-action $\nabla_{A_{N,L}}$ satisfies, by definition, the equality
\[
\partial^{[r]}(D) = \sum_{\alpha=0}^{p^N-1} \left( \frac{p^N - \alpha - 1}{r} \right) \cdot x^{p^N-\alpha-r-1} \otimes x^\alpha v
\]
for each $r = 1, \ldots, p^N - 1$. It follows that the the collection $D, \partial^{[1]}(D), \partial^{[2]}(D), \ldots, \partial^{[p^N-1]}(D)$ turns out to form a local basis of $\mathcal{A}_{N,L}$. On the other hand, each $\partial^{[r]}(D)$ coincides with the image of $\partial^{[r]} \otimes v$ via $\kappa_L$, and the collection $1 \otimes v, \partial^{[1]} \otimes v, \ldots, \partial^{[p^N-1]} \otimes v$ forms a basis of $\mathcal{D}_X^{(N-1)} \otimes L$. Hence, we conclude that $\kappa_L$ is an isomorphism. This completes the proof of the assertion. \qed
For each \( i = 0, \ldots, p^N \), we set \( \mathcal{A}_{N,\xi}^i \) to be the subbundle of \( \mathcal{A}_{N,\xi} \) defined as

\[
\mathcal{A}_{N,\xi}^i := \kappa_{\xi}(\mathcal{D}_{X/p^N-1-i}^{(N-1)} \otimes \mathcal{A}_{N,\xi}^\xi) \quad (\subseteq \mathcal{A}_{N,\xi}).
\]

Then, \( \{\mathcal{A}_{N,\xi}^i\}^p_{i=0} \) specifies a decreasing filtration on \( \mathcal{A}_{N,\xi} \) such that \( \mathcal{A}_{N,\xi}^0 = \mathcal{A}_{N,\xi} \), \( \mathcal{A}_{N,\xi}^{p^N} = 0 \), and the subquotient \( \mathcal{A}_{N,\xi}^i/\mathcal{A}_{N,\xi}^{i+1} \) (for each \( i = 0, \ldots, p^N - 1 \)) is a line bundle isomorphic to \( \Omega_X^\otimes \otimes \mathcal{L} \). The surjection

\[
\mathcal{A}_{N,\xi} \rightarrow (\mathcal{A}_{N,\xi}/\mathcal{A}_{N,\xi}^i) \cong \mathcal{L}
\]

corresponds to the identity morphism of \( F_{X/k}^{(N)}(\mathcal{L}) \) via the adjunction relation “\( F_{X/k}^{(N)}(-) \) ⊆ \( F_{X/k}^{(N)}(-) \)”. By the various definitions involved (including the definition of \( \kappa_{\xi} \)), the \( k \)-linear morphism \( \nabla_{A_{N,\xi}} : \mathcal{D}_{X}^{(N-1)} \otimes \mathcal{A}_{N,\xi} \rightarrow \mathcal{A}_{N,\xi} \) satisfies the inclusion relation

\[
\nabla_{A_{N,\xi}}(\mathcal{D}_{X_{ij}}^{(N-1)} \otimes \mathcal{A}_{N,\xi}) \subseteq \mathcal{A}_{N,\xi}^{i-j}
\]

for each pair of integers \((i, j)\) with \( 0 \leq j \leq i \leq p^N \). This implies that the subbundles \( \mathcal{A}_{N,\xi}^i \) \((i = 1, \ldots, p)\) may be characterized by the surjection \((348)\) and \( \nabla_{A_{N,\xi}} \), as described in the following lemma.

**Lemma 7.1.3.**

Let \( \nabla_{A_{N,\xi}} : \mathcal{A}_{N,\xi} \rightarrow \Omega_X \otimes \mathcal{A}_{N,\xi} \) be the connection on \( \mathcal{A}_{N,\xi} \) corresponding to \( \nabla_{A_{N,\xi}}^{(0)} \). Then, we have \( \nabla_{A_{N,\xi}}(\mathcal{A}_{N,\xi}^i) \subseteq \Omega_X \otimes \mathcal{A}_{N,\xi}^{i-1} \) for any \( i = 1, \ldots, p \), and the morphisms

\[
\mathcal{A}_{N,\xi}^i/\mathcal{A}_{N,\xi}^{i+1} \rightarrow \Omega_X \otimes (\mathcal{A}_{N,\xi}^{i-1}/\mathcal{A}_{N,\xi}^i)
\]

\((i = 1, \ldots, p-1)\) induced by \( \nabla_{A_{N,\xi}} \) are \( \mathcal{O}_X \)-linear isomorphisms. In particular, the following equalities hold:

- \( \mathcal{A}_{N,\xi}^1 = \text{Ker}(\mathcal{A}_{N,\xi} \xrightarrow{(348)} \mathcal{L}) \);
- \( \mathcal{A}_{N,\xi}^{i+1} = \text{Ker}(\mathcal{A}_{N,\xi}^i \xrightarrow{\nabla_{A_{N,\xi}} \mathcal{A}_{N,\xi}^i} \Omega_X \otimes \mathcal{A}_{N,\xi} \rightarrow \Omega_X \otimes (\mathcal{A}_{N,\xi}/\mathcal{A}_{N,\xi}^i)) \) for any \( i = 1, \ldots, p-1 \).

**Proof.** The assertion follows from \((349)\) and the fact that \( \mathcal{A}_{N,\xi}^i/\mathcal{A}_{N,\xi}^{i+1} \cong \Omega_X^\otimes \otimes \mathcal{L} \).

Next, let us fix another positive integer \( N' \) with \( N' > N \). The identity morphism of \( F_{X/k}^{(N')}(\mathcal{L}) \) induces a morphism \( F_{X/k}^{(N'-N)\ast}(F_{X/k}^{(N')}(-)) \rightarrow F_{X/k}^{(N')}(-) \) by the adjunction relation “\( F_{X/k}^{(N'-N)\ast}(-) \) ⊆ \( F_{X/k}^{(N'-N)\ast}(-) \)”. By pulling-back this morphism by \( F_{X/k}^{(N')} \), we obtain a morphism

\[
\mathcal{A}_{N',\xi} := F_{X/k}^{(N')\ast}(F_{X/k}^{(N'-N)\ast}(F_{X/k}^{(N')}(-))) \rightarrow \mathcal{A}_{N,\xi} \quad \text{(for } \mathcal{A}_{N,\xi} = F_{X/k}^{(N')\ast}(F_{X/k}^{(N')}(-)) \text{)}.
\]

One verifies that this morphism is compatible with the respective \( \mathcal{D}_{X}^{(N-1)} \)-actions \( \nabla_{A_{N',\xi}}^{(N-1)} \), \( \nabla_{A_{N,\xi}} \) and compatible with the respective surjections \( \mathcal{A}_{N',\xi} \rightarrow \mathcal{L}, \mathcal{A}_{N,\xi} \rightarrow \mathcal{L} \) onto \( \mathcal{L} \) (cf. \((348)\)). Hence, by the latter assertion in Lemma \(7.1.3\) the morphism \((351)\) turns out to be compatible with the respective filtrations \( \{\mathcal{A}_{N,\xi}^i\}^p_{i=0} \) and \( \{\mathcal{A}_{N,\xi}^i\}^p_{i=0} \). To be precise, for each \( i = 0, \ldots, p \), the morphism \((351)\) restricts to a morphism \( \mathcal{A}_{N,\xi}^i \rightarrow \mathcal{A}_{N,\xi}^i \) and the resulting morphisms

\[
\mathcal{A}_{N,\xi}^i/\mathcal{A}_{N,\xi}^{i+1} \rightarrow \mathcal{A}_{N,\xi}^i/\mathcal{A}_{N,\xi}^{i+1}
\]
(i = 0, · · · , p − 1) are isomorphisms.

7.2. Comparison with Quot schemes.

Next, we shall introduce certain Quot schemes. To do this, let us fix a theta characteristic of \( X \), i.e., a line bundle \( \Theta \) together with an isomorphism \( \Theta^\otimes 2 \xrightarrow{\sim} \Omega_X \). In particular, we obtain a rank \( p^N \) vector bundle \( A_{N,\Theta^v} \) equipped with a decreasing filtration \( \{ A_i^i_{N,\Theta^v} \}_{i=0}^{p^N} \). The subquotients \( A_i^i_{N,\Theta^v}/A_{N,\Theta^v}^{i+1} \) have natural identifications

\[
A_i^i_{N,\Theta^v}/A_{N,\Theta^v}^{i+1} \xrightarrow{\sim} \Omega^\otimes i \otimes \Theta^v \xrightarrow{\sim} \Theta^\otimes (2i-1).
\]

Denote by

\[
Q_{N,\Theta}^{2,0}
\]

the Quot-scheme classifying \( O_{X^{(N)}} \)-submodules of \( F_{X/k^*}^{(N)}(\Theta^v) \) of rank 2 and degree 0. Since \( X^{(N)} \) is a smooth curve, each sheaf classified by \( Q_{N,\Theta}^{2,0} \) is automatically locally free. The assignment \( \mathcal{G} \mapsto \det(\mathcal{G}) \) for each \( \mathcal{G} \in Q_{N,\Theta}^{2,0} \) defines a morphism

\[
\text{Det} : Q_{N,\Theta}^{2,0} \rightarrow \text{Pic}^0(X^{(N)}).
\]

We shall denote by

\[
\mathcal{Q}_{N,\Theta}^{2,0}
\]

the scheme-theoretic inverse image, via \( \text{Det} \), of the identity point \([O_{X^{(N)}}]\) of \( \text{Pic}^0(X^{(N)}) \).

Let \( \mathcal{G} \left( \subseteq F_{X/k^*}^{(N)}(\Theta^v) \right) \) be an \( O_X \)-submodule classified by \( Q_{N,\Theta}^{2,0} \). Then, it induces, via pull-back by \( F_{X/k}^{(N)} \), an \( O_X \)-submodule \( F_{X/k}^{(N)}(\mathcal{G}) \) of \( A_{N,\Theta^v} \). We shall define a decreasing filtration

\[
\{ F_{X/k}^{(N)}(\mathcal{G})^{i} \}_{i=0}^{p^N} \text{ on } F_{X/k}^{(N)}(\mathcal{G})
\]

by setting

\[
F_{X/k}^{(N)}(\mathcal{G})^{i} := F_{X/k}^{(N)}(\mathcal{G}) \cap A_i^i_{N,\Theta^v}.
\]

**Lemma 7.2.1.**

The following assertions hold:

(i) The \( O_X \)-linear composite

\[
F_{X/k}^{(N)*}(\mathcal{G}) \hookrightarrow A_{N,\Theta^v} \twoheadrightarrow A_{N,\Theta^v}/A_{N,\Theta^v}^2
\]

of the natural inclusion \( F_{X/k}^{(N)*}(\mathcal{G}) \hookrightarrow A_{N,\Theta^v} \) and the quotient \( A_{N,\Theta^v} \twoheadrightarrow A_{N,\Theta^v}/A_{N,\Theta^v}^2 \) is an isomorphism. Moreover, this composite restricts to an isomorphism \( F_{X/k}^{(N)*}(\mathcal{G})^1 \twoheadrightarrow A_{N,\Theta^v}/A_{N,\Theta^v}^2 \).

(ii) Denote by \( \nabla^\mathcal{G} \) the \( D_X^{(N-1)} \)-action on \( A_{N,\Theta^v}/A_{N,\Theta^v}^2 \) corresponding to \( \nabla^\text{can}^{(N-1)}_X \) (cf. (112)) via the composite isomorphism \( (358) \). We shall identify the line subbundle \( A_{N,\Theta^v}/A_{N,\Theta^v}^2 \) of \( A_{N,\Theta^v}/A_{N,\Theta^v}^2 \) with \( \Theta \) via \( (358) \). Then, the triple

\[
\mathcal{G}^\otimes \Theta^\otimes := (A_{N,\Theta^v}/A_{N,\Theta^v}^2, \nabla^\mathcal{G}, \Theta)
\]
forms a dormant indigenous $D_{X}^{(N-1)}$-module. If, moreover, $G$ belongs to $Q_{N,Θ}^{2,Ω}$, then it specifies an element of $\hat{\mathcal{D}}_{X,N,Θ}^{\text{zar}}$ under identifying $A_{N,Θ^\vee}/A_{N,Θ^\vee}^{2}$ with $D_{X,1}^{(N-1)} \otimes Θ$ via $KS_{\hat{\mathcal{D}}_{X,N,Θ}^{\text{zar}}} : D_{X,1}^{(N-1)} \otimes Θ \sim A_{N,Θ^\vee}/A_{N,Θ^\vee}^{2}$ (cf. (351)).

Proof. First, we consider assertion (i). If we write $gr^i := (F_{X/k}^{(N)}(G))^i/(F_{X/k}^{(N)}(G))^{i+1}$ ($i = 0, \cdots , p^N - 1$), then the inclusion $F_{X/k}^{(N)}(G) \hookrightarrow A_{N,Θ^\vee}$ induces an $O_X$-linear injection

$$gr^i \hookrightarrow A_{N,Θ^\vee}/A_{N,Θ^\vee}^{i+1}$$

into the subquotient $A_{N,Θ^\vee}/A_{N,Θ^\vee}^{i+1}$. Since $A_{N,Θ^\vee}/A_{N,Θ^\vee}^{i+1}$ is a line bundle, $gr^i$ is either trivial or a line bundle. In particular, since $F_{X/k}^{(N)}(G)$ is of rank 2, the cardinality of the set $I := \{i \mid gr^i \neq 0\}$ is exactly 2. Next, let us observe that the inclusion $F_{X/k}^{(N)}(G) \hookrightarrow A_{N,Θ^\vee}$ is compatible with the respective $D_{X}^{(N-1)}$-actions $\nabla_{F_{X/k}^{(N)}(G)}^{\text{can}}$ and $\nabla_{A_{N,Θ^\vee}}$. By this fact together with the result of Lemma 7.1.3, we see that $gr^{i+1} \neq 0$ implies $gr^i \neq 0$. Therefore, the equality $I = \{0, 1\}$ holds, and the composite (358) is an isomorphism at the generic point of $X$. On the other hand, observe the sequences of equalities

$$\text{deg}(F_{X/k}^{(N)}(G)) = p^N \cdot \text{deg}(G) = p^N \cdot 0 = 0$$

and

$$\text{deg}(A_{N,Θ^\vee}/A_{N,Θ^\vee}^{2}) = \text{deg}(A_{N,Θ^\vee}/A_{N,Θ^\vee}^{1}) + \text{deg}(A_{N,Θ^\vee}/A_{N,Θ^\vee}^{0}) = \text{deg}(Θ) + \text{deg}(Θ^\vee) = 0.$$ 

By comparing the respective degrees of $F_{X/k}^{(N)}(G)$ and $A_{N,Θ^\vee}/A_{N,Θ^\vee}^{2}$, we conclude that the composite (358) is an isomorphism. In particular, the injection $(F_{X/k}^{(N)}(G))^{1} = gr^1 \hookrightarrow A_{N,Θ^\vee}/A_{N,Θ^\vee}^{1}$, i.e., the restriction of the composite (358) to $F_{X/k}^{(N)}(G)$, is an isomorphism. This completes the proof of assertion (i).

Assertion (ii) follows immediately from assertion (i), Lemma 7.1.3 and the fact that the inclusion $F_{X/k}^{(N)}(G) \hookrightarrow A_{N,Θ^\vee}$ is compatible with the respective $D_{X}^{(N-1)}$-actions $\nabla_{F_{X/k}^{(N)}(G)}$ and $\nabla_{A_{N,Θ^\vee}}$.

Next, we discuss the relationship between $\hat{\mathcal{D}}_{X,N,Θ}^{\text{zar}}$ and $Q_{N,Θ}^{2,Ω}$.

**Theorem 7.2.2.**

The assignment $G \mapsto G^{Ω}$ determines a bijection of sets

$$\hat{\mathcal{D}}_{X,N,Θ}^{\text{zar}} \ni G^{Ω} \mapsto Q_{N,Θ}^{2,Ω}(k) \sim \hat{\mathcal{D}}_{X,N,Θ}^{\text{zar}}.$$ 

**Proof.** First, we shall consider the injectivity of $Q_{N,Θ}^{2,Ω}(Ω)$. Let us take an $O_X$-submodule $G$ of $F_{X/k}(G)$ classified by $Q_{N,Θ}^{2,Ω}(k)$. Write $\nabla^G$ for the $D_{X}^{(N-1)}$-action on $D_{X,1}^{(N-1)} \otimes Θ$ defining $G^{Ω}$. By the definition of $G^{Ω}$ (cf. Lemma 7.2.1), there exists a natural isomorphism
where the first arrow arises from the natural inclusion \( (365) \). 

Then, under the identification given by this isomorphism, the inclusion \( \mathcal{G} \hookrightarrow F_{X/k}^{(N)}(\mathcal{G}) \) is, by adjunction, determined by the composite 

\[
(365) 
F_{X/k}^{(N)}(\mathcal{G}) \cong D_{X,1}^{(N-1)} \otimes \Theta, 
\]

which is uniquely determined by \( \nabla^\Theta \). This completes the proof of the injectivity of \( \dagger \zeta_{N,\Theta}^{\Diamond} \).

Next, we shall consider the surjectivity. Let \( \mathcal{V}^\Diamond := (D_{X,1}^{(N-1)} \otimes \Theta, \nabla_\mathcal{V}, \Theta) \) be a dormant indigenous \( D_X^{(N-1)} \)-module classified by \( \dagger \mathfrak{D}_X^{\text{haz}} : \). Consider the composite \( F_{X/k}^{(N)}(\mathcal{G}) \cong D_{X,1}^{(N-1)} \otimes \Theta \rightarrow \Theta^\vee \) defined as in (365). This composite determines a morphism 

\[
(366) 
h : D_{X,1}^{(N-1)} \otimes \Theta \cong F_{X/k}^{(N)}(\mathcal{G}) \rightarrow A_{N,\Theta^\vee} \cong (F_{X/k}^{(N)}(\mathcal{G})) \]

via the adjunction relation \( \text{"}F_{X/k}^{(N)} \text{"} \) and pull-back by \( F_{X/k}^{(N)} \). In what follows, we shall prove the claim that \( h \) is injective. The morphism \( h \) is, by construction, compatible with the respective surjections \( D_{X,1}^{(N-1)} \otimes \Theta \rightarrow \Theta^\vee, A_{N,\Theta^\vee} \rightarrow (A_{N,\Theta^\vee}/A_{N,\Theta^\vee}) \cong \Theta^\vee \) onto \( \Theta^\vee \). This implies that \( h(D_{X,0}^{(N-1)} \otimes \Theta) \subseteq A_{N,\Theta^\vee} \), and \( \ker(h) \subseteq D_{X,0}^{(N-1)} \otimes \Theta \). Since \( h \) is also compatible with the respective \( D_X^{(N-1)} \)-actions \( \nabla_\mathcal{V}, \nabla_{A_{N,\Theta^\vee}}, \ker(h) \) is stabilized by \( \nabla_\mathcal{V} \). Hence, \( \ker(h) \) is contained in the kernel of the morphism \( KS_{\mathcal{V}^\Diamond} \) (cf. (74)). But, the morphism \( KS_{\mathcal{V}^\Diamond} \) is an isomorphism because of the assumption that \( \mathcal{V}^\Diamond \) forms an indigenous \( D_X^{(N-1)} \)-module. It follows that \( h \) is injective, and this completes the proof of the claim. Now, denote by \( h^{\mathcal{G}} : \mathcal{G} \rightarrow F_{X/k}^{(N)}(\mathcal{G}) \) the morphism obtained from \( h \), where the domain and codomain are restricted to the respective subsheaves of horizontal sections. The pull-back of \( h^{\mathcal{G}} \) via \( F_{X/k}^{(N)} \) may be identified with \( h \). By the faithfulness of \( F_{X/k}^{(N)} \), \( h^{\mathcal{G}} \) turns out to be injective. Moreover, since the determinant of \( \nabla_\mathcal{V} \) is trivial, \( \det(\mathcal{G}) \) is isomorphic to the trivial line bundle. Consequently, the \( \mathcal{O}_{X^{(N)}} \)-module \( \mathcal{G} \) together with the injection \( h^{\mathcal{G}} \) determines a \( k \)-rational point of \( Q_{2,\mathcal{O}}^{N,\Theta} \); it is mapped by \( \dagger \zeta_{N,\Theta}^{\Diamond} \) to the element of \( \dagger \mathfrak{D}_X^{\text{haz}} : \) represented by \( \mathcal{V}^\Diamond \). This implies that \( \dagger \zeta_{N,\Theta}^{\Diamond} \) is surjective and hence completes the proof of the assertion.

\[\square\]

7.3. Existence of \( F^N \)-projective structures.

In this subsection, we shall prove the existence assertion of \( F^N \)-projective structure on any smooth projective curve (cf. Corollary 7.3.4). To begin with, we shall examine the relationship between \( Q_{2,\mathcal{O}}^{*}(k) \) and \( Q_{2,\Theta}^{0}(k) \). Given a pair of positive integers \( (m, m') \) with \( m < m' \), we denote by 

\[
(367) 
\text{Ver}_X^{(m')}_{(m')} : \text{Pic}^0(X^{(m')}) \rightarrow \text{Pic}^0(X^{(m)}) 
\]

the morphism of group \( k \)-schemes given by \( \mathcal{L} \mapsto F_{X^{(m')}/k}^{(m'-m)}(\mathcal{L}) \) for each degree 0 line bundle \( \mathcal{L} \) on \( X^{(m')} \). It is well-known that \( \text{Ver}_X^{(m')} \) is finite and faithfully flat. In particular, each fiber
of this morphism is nonempty. Given a positive integer $N$, we write

$$\text{Ker}^{(N)\Rightarrow(0)}_X := (\text{Ver}^{(N)\Rightarrow(0)}_X)^{-1}([\mathcal{O}_X]) \neq \emptyset,$$

i.e., the set-theoretic inverse image, via the morphism $\text{Ver}^{(N)\Rightarrow(0)}_X$, of the identity element $[\mathcal{O}_X] \in \text{Pic}^0(X)$.

**Lemma 7.3.1.**

There exists a canonical bijection of sets

$$\mathcal{Q}^{2,0}_{N,\Theta}(k) \times \text{Ker}^{(N)\Rightarrow(0)}_X \sim \mathcal{Q}^{2,0}_{N,\Theta}.$$ (369)

**Proof.** Let $(\mathcal{G}, \mathcal{L})$ be an element of $\mathcal{Q}^{2,0}_{N,\Theta}(k) \times \text{Ker}^{(N)\Rightarrow(0)}_X(k)$. Denote by $h : \mathcal{G} \hookrightarrow F^{(N)}_{X/k*}(\Theta^\vee)$ the natural inclusion. The composite

$$h_{\mathcal{L}} : \mathcal{G} \otimes \mathcal{L} \xrightarrow{h \otimes \text{id}_{\mathcal{L}}} F^{(N)}_{X/k*}(\Theta^\vee) \otimes \mathcal{L} \xrightarrow{\sim} F^{(N)}_{X/k*}(\Theta^\vee \otimes F^{(N)*}_{X/k}(\mathcal{L})) \xrightarrow{\sim} F^{(N)}_{X/k*}(\Theta^\vee \otimes \mathcal{O}_X) \left(= F^{(N)}_{X/k*}(\Theta^\vee)\right)$$ (370)

determines an element of $\mathcal{Q}^{2,0}_{N,\Theta}(k)$, where the second arrow follows from the projection formula. Thus, the assignment $(\mathcal{G}, h) \mapsto h_{\mathcal{L}}$ determines a map of sets

$$\mathcal{Q}^{2,0}_{N,\Theta}(k) \times \text{Ker}^{(N)\Rightarrow(0)}_X \to \mathcal{Q}^{2,0}_{N,\Theta}(k).$$ (371)

Conversely, let $\mathcal{G}$ be an $\mathcal{O}_X$-submodule of $F^{(N)}_{X/k*}(\Theta^\vee)$ classified by $\mathcal{Q}^{2,0}_{N,\Theta}(k)$, where the natural inclusion $\mathcal{G} \hookrightarrow F^{(N)}_{X/k*}(\Theta^\vee)$ will be denoted by $h$. It follows from Lemma 7.2.1 (i), that

$$F^{(N)*}_{X/k}(\det(\mathcal{G})) \cong (A_{\mathcal{O}_X^\vee}/A_{\mathcal{O}_X^\vee}) \otimes (A_{\mathcal{O}_X^\vee}/A_{\mathcal{O}_X^\vee}) \cong \Theta^\vee \otimes \Theta \cong \mathcal{O}_X.$$ (372)

Now, consider the injective morphism $h_{\det(\mathcal{G})}(\otimes (p^{N-1})/2)$, i.e., the morphism $h_{\mathcal{L}}$ constructed above in the case where “$\mathcal{L}$” is taken to be $\mathcal{L} = \det(\mathcal{G})(\otimes (p^{N-1})/2)$. Then, we have

$$\det(\mathcal{G} \otimes \det(\mathcal{G})(\otimes (p^{N-1})/2) \cong \det(\mathcal{G}) \otimes \det(\mathcal{G})(\otimes 2(\otimes (p^{N-1})/2) \cong \det(\mathcal{G})(\otimes (p^{N}) \cong \mathcal{O}_X),$$ (373)

where the last “$\cong$” follows from (372) and Lemma 3.6.2. This implies that the pair

$$(g_{\det(\mathcal{G})}(\otimes (p^{N-1})/2), \det(\mathcal{G}))$$ (374)

determines an element of $\mathcal{Q}^{2,0}_{N,\Theta}(k) \times \text{Ker}^{(N)\Rightarrow(0)}_X$. One verifies easily that the assignment $\mathcal{G} \mapsto (g_{\det(\mathcal{G})}(\otimes (p^{N-1})/2), \det(\mathcal{G}))$ determines an inverse to the map (371). This completes the proof of the assertion. $\square$

Let $N'$ be another positive integer with $N' > N$. In what follows, we shall construct a map from $\mathcal{Q}^{2,0}_{N',\Theta}(k)$ to $\mathcal{Q}^{2,0}_{N,\Theta}(k)$. Let $\mathcal{G}$ be an $\mathcal{O}_X$-submodule of $F^{(N')}_{X/k*}(\Theta^\vee)$ classified by $\mathcal{Q}^{2,0}_{N',\Theta}$. If $h : \mathcal{G} \hookrightarrow F^{(N')}_{X/k*}(\Theta^\vee)$ denotes the natural inclusion, then it induces a morphism

$$h^{-1} : F^{(N'-N)*}_{X/(N')/k}(\mathcal{G}) \to F^{(N)}_{X/k*}(\Theta^\vee)$$ (375)

by the adjunction relation “$F^{(N'-N)*}_{X/(N')/k}(\mathcal{G}) \to F^{(N'-N)}_{X/(N')/k}(\mathcal{G})$.”
Lemma 7.3.2.

The morphism $h^i$ is injective, i.e., the image $\text{Im}(h^i) \subseteq F_{X/k*}^{(N)}(\Theta^\vee)$ of this morphism specifies an element of $Q_{N,\Theta}^{2,0}$. If, moreover, $G$ is classified by $Q_{N',\Theta}^{2,0}$, then $\text{Im}(h^i)$ specifies an element of $Q_{N,\Theta}^{2,0}$.

Proof. Observe that the following diagram is commutative:

$$
\begin{array}{c}
F_{X/k}^{(N')}^*(G) \xrightarrow{\text{id}} A_{N',\Theta^\vee} \xrightarrow{\text{surj}} A_{N',\Theta^\vee}/A_{N',\Theta^\vee}^2 \\
\downarrow \quad \downarrow 351 \quad \downarrow \\
F_{X/k}^{(N)}(F_{X/(N)/k}^{(N'-N)}(G)) \xrightarrow{h^i} A_{N,\Theta^\vee} \xrightarrow{\text{surj}} A_{N,\Theta^\vee}/A_{N,\Theta^\vee}^2,
\end{array}
$$

where the upper left-hand horizontal arrow denotes the pull-back of the inclusion $G \hookrightarrow F_{X/k*}^{(N')}((\Theta^\vee)$ by $F_{X/k}^{(N)}$ and the rightmost vertical arrow denotes the isomorphism arising from (351). By Lemma 7.2.1 (i), the composite of the upper horizontal arrows turns out to be an isomorphism. This implies that the composite of the lower horizontal arrows is an isomorphism, so $F_{X/k}^{(N)}(h^i)$ is injective. Since $F_{X/k}^{(N)}$ is faithfully flat, we see that $h^i$ is injective. This completes the proof of the lemma. \qed

By the above lemma, the assignment $G \mapsto \text{Im}(h^i)$ defines maps of sets

$$
\mu_{N'\Rightarrow N}^{0} : Q_{N',\Theta}^{2,0}(k) \to Q_{N,\Theta}^{2,0}(k), \quad \mu_{N'\Rightarrow N}^{\mathcal{O}} : Q_{N',\Theta}^{2,\mathcal{O}}(k) \to Q_{N,\Theta}^{2,\mathcal{O}}(k).
$$

By construction, the following square diagrams are commutative:

$$
\begin{array}{c}
Q_{N',\Theta}^{2,\mathcal{O}}(k) \times \text{Ker}_X^{(N'\Rightarrow(0))} \xrightarrow{369} Q_{N',\Theta}^{2,0}(k) \quad Q_{N',\Theta}^{2,\mathcal{O}}(k) \xrightarrow{\mu_{N'\Rightarrow N}^{\mathcal{O}}} Q_{N,\Theta}^{2,\mathcal{O}}(k) \\
\downarrow \sim \quad \downarrow \sim \\
\mu_{N'\Rightarrow N}^{0} \quad \mu_{N'\Rightarrow N}^{\mathcal{O}} \xrightarrow{\sim} Q_{N,\Theta}^{2,0}(k) \\
\downarrow \quad \downarrow \\
\mu_{N'\Rightarrow N}^{0} \quad \mu_{N'\Rightarrow N}^{\mathcal{O}} \xrightarrow{\sim} Q_{N,\Theta}^{2,\mathcal{O}}(k)
\end{array}
$$

Regarding the morphisms $\mu_{N'\Rightarrow N}^{0}$ and $\mu_{N'\Rightarrow N}^{\mathcal{O}}$, the following assertion holds.

Proposition 7.3.3.

Suppose that the genus $g$ of $X$ satisfies $g > 1$. Then, for each pair of positive integers $(N, N')$ with $N < N'$, both the maps $\mu_{N'\Rightarrow N}^{0}$ and $\mu_{N'\Rightarrow N}^{\mathcal{O}}$ are surjective. Moreover, for each positive integer $N$, both the sets $Q_{N,\Theta}^{2,0}(k)$ and $Q_{N,\Theta}^{2,\mathcal{O}}(k)$ are nonempty.

Proof. First, we shall prove the surjectivity of $\mu_{N'\Rightarrow N}^{0}$. Let us take an injection $h_G : G \hookrightarrow F_{X/k*}^{(N)}((\Theta^\vee)$ classified by $Q_{N,\Theta}^{2,0}$. We can compute the degree $\deg(F_{X/(N)/k*}^{(N'-N)}(G))$ of the direct image $F_{X/(N)/k*}^{(N'-N)}(G)$ of $G$, as follows. By the Riemann-Roch theorem, the following equalities
hold:

\[(379)\]
\[\chi(G) = \deg(G) + \text{rank}(G) \cdot (1 - g)\]
\[\quad (= 2(1 - g)) ,\]
\[\chi(F_{X(N)/k}^{N-N}(G)) = \deg(F_{X(N)/k}^{N-N}(G)) + \text{rank}(F_{X(N)/k}^{N-N}(G)) \cdot (1 - g)\]
\[\quad \left(= \deg(F_{X(N)/k}^{N-N}(G)) + 2 \cdot p^{N-N} \cdot (1 - g)\right).\]

As the morphism \(F_{X(N)/k}^{N-N}(G)\) is affine, the equality \(\chi(G) = \chi(F_{X(N)/k}^{N-N}(G))\) holds, and hence, the equalities \(379\) imply

\[(380)\]
\[\deg(F_{X(N)/k}^{N-N}(G)) = 2 \cdot (1 - g) \cdot (1 - p^{N-N}).\]

Now, let us apply [19], Theorem 2.3.1 (or [29], [49]), to the case where the quadruple \("(V, n, d, m)"\) is taken to be \((F_{X(N)/k}^{N-N}(G), p^{N-N}, 2 \cdot (1 - g) \cdot (1 - p^{N-N}), 2)\). Then, there exists an \(O_{X(N)}\)-submodule \(W\) of \(F_{X(N)/k}^{N-N}(G)\) of rank 2 such that

\[(381)\]
\[\mu(W) \geq \mu(F_{X(N)/k}^{N-N}(G)) - \frac{2 \cdot p^{N-N} - 2}{2 \cdot p^{N-N}} \cdot (g - 1) - \frac{\varepsilon}{2 \cdot (2 \cdot p^{N-N})},\]

where \(\varepsilon\) denotes the unique integer satisfying \(0 \leq \varepsilon \leq 2 \cdot p^{N-N} - 1\) and

\[(382)\]
\[\varepsilon + 2 \cdot (2 \cdot p^{N-N} - 2) \cdot (g - 1) \equiv 2 \cdot \deg(F_{X(N)/k}^{N-N}(G)) \mod 2 \cdot p^{N-N}.
\]

By \(380\), the condition \(382\) is equivalent to the condition that \(\varepsilon \equiv 0 \mod 2 \cdot p^{N-N}\). This implies \(\varepsilon = 0\). Also, we have

\[(383)\]
\[\mu(F_{X(N)/k}^{N-N}(G)) = \frac{\deg(F_{X(N)/k}^{N-N}(G))}{2 \cdot p^{N-N}} \geq \frac{(1 - g) \cdot (1 - p^{N-N})}{p^{N-N}}.\]

Hence, the inequality \(381\) reads

\[(384)\]
\[\deg(W) \geq 2 \cdot \left(\frac{(1 - g) \cdot (1 - p^{N-N})}{p^{N-N}} - \frac{2 \cdot p^{N-N} - 2}{2 \cdot p^{N-N}} \cdot (g - 1) - \frac{0}{2 \cdot (2 \cdot p^{N-N})}\right) = 0.
\]

After possibly replacing \(W\) with its lower modification, we can assume that \(\deg(W) = 0\). In particular, the composite injection

\[(385)\]
\[h_W : W \hookrightarrow F_{X(N)/k}^{N-N}(G) \xrightarrow{\mu_{X(N)/k}^{N-N}(h)} F_{X(N)/k}^{N-N}(G) \xrightarrow{\Theta^\vee} F_{X(k)/k}^{N-N}(\Theta^\vee) \quad (= F_{X(k)/k}^{N}(\Theta^\vee))\]

defines an element of \(O_{N', \Theta}^{2,0}(k)\). The morphism \(h : F_{X(N)/k}^{N-N}(W) \to G\) corresponding, via adjunction, to the inclusion \(W \hookrightarrow F_{X(N)/k}^{N-N}(G)\) is compatible with the respective inclusions \(h_G\), \(h_W\) (cf. \(372\)). It follows that \(h\) is injective. Moreover, by comparing the respective degrees of \(F_{X(N)/k}^{N-N}(W)\) and \(G\), we conclude that the morphism \(h\) is an isomorphism. Thus, the element of \(O_{N', \Theta}^{2,0}(k)\) represented by \(h_W\) is mapped, via \(\mu_{N', \Theta}^{2,0}\), to the element of \(O_{N, \Theta}^{2,0}(k)\) represented by \(h_G\). This completes the surjectivity of \(\mu_{N, \Theta}^{2,0}\).

The surjectivity of \(\mu_{N, \Theta}^{2,0}\) follows from that of \(\mu_{N', \Theta}^{2,0}\) and the commutativity of the left-hand square diagram in \(378\). This completes the former assertion.
Finally, by means of the former assertion, we can prove the latter assertion by induction on $N$; the base case, i.e., the case of $N = 1$ was already proved in [40], Theorem 5.4.1. This completes the proof of the assertion. □

By the above proposition, the following assertion holds.

**Corollary 7.3.4.**

Suppose that the genus $g$ of $X$ satisfies $g > 1$. Then, all the morphisms in the sequence

$$(386) \quad \mathcal{D}^{\text{zar}}_{X,N,\Theta} \to \cdots \to \mathcal{D}^{\text{zar}}_{X,2,\Theta} \to \mathcal{D}^{\text{zar}}_{X,1,\Theta}$$

obtained by truncation are surjective. Moreover, for any $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, the set $\mathcal{D}^{\text{zar}}_{X,N,\Theta}$ is nonempty. In particular, any (not necessarily projective) smooth curve admits an $F^N$-projective structure for any $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$.

**Proof.** The first and second assertions follow from Theorem [7.2.2] Proposition [7.3.3] and the commutativity of the right-hand square diagram in (378). The last assertion follows from the former two assertions and the fact that any smooth curve can be embedded into a smooth projective curve. □

**Remark 7.3.5.**

Let $X$ be a projective smooth curve of genus $g > 1$, and suppose that $X$ may be defined over the finite field $\mathbb{F}_q$ of order $q$, where $q$ denotes a power of $p$. Then, there are no dormant indigenous $\mathcal{D}_X^{(\infty)}$-modules whose underlying $\mathcal{D}_X^{(\infty)}$-module can be defined over $\mathbb{F}_q$. Indeed, let us suppose, on the contrary, that there exists a dormant indigenous $\mathcal{D}_X^{(\infty)}$-module $\mathcal{V}^\vee := (\mathcal{V}, \nabla, \mathcal{N})$ with $(\mathcal{V}, \nabla)$ defined over $\mathbb{F}_q$. Then, according to [13], Proposition 2.3, we can find an étale covering $\pi : Y \to X$ such that the pull-back $\pi^*(\mathcal{V}, \nabla)$ is trivial. If $\mathcal{S}^\vee$ denotes the $F^\infty$-projective structure associated with $\mathcal{V}^\vee$ via $\zeta_{\infty}^\vee \Rightarrow \mathcal{V}^\vee$, then its pull-back $\pi^*(\mathcal{S}^\vee)$ by $\pi$ is global (cf. Definition [5.2.1] and Proposition [6.1.2]). Hence, by Proposition [5.2.2] (ii), $Y$ turns out to be isomorphic to $\mathbb{P}^1$. But, it contradicts the assumption that $X$ has genus $g > 1$. This completes the proof of the italicized assertion above.

We conclude this subsection with one property of $F^\infty$-projective structures on a curve, i.e., the property that an $F^\infty$-projective structure is determined uniquely by the associated monodromy representation.

**Proposition 7.3.6.**

Let $X$ be a smooth projective curve over $k$ of genus $> 1$ and $x$ a $k$-rational point of $X$. Also, let $\mathcal{S}^\vee_i$ $(i = 1, 2)$ be $F^\infty$-projective structures on $X$. Then, $\mathcal{S}^\vee_1 = \mathcal{S}^\vee_2$ if and only if $\rho_{\mathcal{S}^\vee_1} = \rho_{\mathcal{S}^\vee_2}$ in $\text{Out}(\pi^{\text{str}}(X,x), \text{PGL}_2)$ (cf. (274)).

**Proof.** It suffices to consider the “if” part of the desired equivalence. We shall assume that $\rho_{\mathcal{S}^\vee_1} = \rho_{\mathcal{S}^\vee_2}$. Fix a theta characteristic $\Theta$ on $X$. Then, for each $i = 1, 2$, $\mathcal{S}^\vee_i$ corresponds to a unique dormant indigenous $\mathcal{D}_X^{(\infty)}$-modules $\mathcal{V}^\vee_i := (\mathcal{V}, \nabla, \Theta)$ classified by $\mathcal{D}^{\text{zar}}_{X,\infty,i,\Theta}$, where $\mathcal{V} := \mathcal{D}_X^{(\infty)} \otimes \Theta$. The assumed equality $\rho_{\mathcal{S}^\vee_1} = \rho_{\mathcal{S}^\vee_2}$ implies that there exist an invertible $\mathcal{D}_X^{(\infty)}$-module $(\mathcal{L}, \nabla)$ and an isomorphism of $\mathcal{D}_X^{(\infty)}$-modules $\eta : (\mathcal{V} \otimes \mathcal{L}, \nabla_{\mathcal{V}_1} \otimes \nabla_{\mathcal{L}}) \cong (\mathcal{V}, \nabla_{\mathcal{V}_2})$. 

Since $X$ has genus $>1$, $\Theta$ turns out to be the unique line subbundle in $\mathcal{V}$ of maximal degree. Hence, $\eta$ restricts to an isomorphism $\Theta \otimes \mathcal{L} \sim \Theta$. It follows that $\eta$ becomes an isomorphism $(\mathcal{V}_1^\vee)_{(\mathcal{L}, \nabla_{\mathcal{L}})} \sim \mathcal{V}_2^\vee$ (cf. (32)). Consequently, we have

\begin{equation}
S_1^\vee = \zeta_\infty \sim \mathcal{V}_1^\vee = \zeta_\infty \sim ((\mathcal{V}_1^\vee)_{(\mathcal{L}, \nabla_{\mathcal{L}})}) = \zeta_\infty \sim \mathcal{V}_2^\vee = S_2^\vee.
\end{equation}

This completes the proof of the assertion. \hfill \square

### 7.4. Finiteness for genus 2 curves.

Next, we consider finiteness of $F^N$-projective structures defined on a curve. According to a result in the previous works, there are only finitely many $F^1$-projective structures on a prescribed smooth projective curve $X$ (cf. [64], Chap. II, §2.3, Theorem 2.8). Moreover, under the correspondence between $F^1$-projective structures and dormant indigenous bundles, the main result of [75] computes explicitly the number of such structures by means of the genus “$g$” of $X$ and the characteristic “$p$” of the base field $k$. In fact, if $X$ is sufficiently general (in the moduli space of smooth projective curves of genus $g$), then the number of $F^1$-projective structure $\sharp(F^1\text{-}\text{proj}_X)$ can be calculated by the following formula:

\begin{equation}
\sharp(F^1\text{-}\text{proj}_X) = \frac{p^{g-1}}{2^{2g+1}} \cdot \prod_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\theta}{p})}.
\end{equation}

Notice that the inequality $p > 2(g - 1)$ was imposed in the statement of loc. cit.. But, by combining this result with [53], Theorem 2.1, one may remove this condition, i.e., the equality (388) holds even if $g$ is arbitrary (cf. [78], Corollary 8.11).

In the case of higher levels, the author does not know much about the finiteness of Frobenius-projective structures at the time of writing the present paper. To promote understanding of this matter even a little, we here consider the problem restricted to genus 2 curves.

Let us fix a smooth projective curve $X$ over $k$ of genus 2. The main result of this subsection is Theorem 7.4.2 described later. In order to prove it, we will use the following lemma.

**Lemma 7.4.1.**

Let $E$ be a rank 2 stable bundle on $X$ with trivial determinant. Write $\text{Ker}^{(N)}_{E}$ for the set of isomorphism classes of vector bundles $E'$ on $X^{(N)}$ with $F^{(N)}_{X/k}(E') \cong E$. (Notice that each vector bundle in $\text{Ker}^{(N)}_{E}(0)$ is immediately verified to be stable.) Then, the set $\text{Ker}^{(N)}_{E}(0)$ is finite.

**Proof.** First, we shall prove the finiteness of the subset $\text{Ker}^{(N)}_{E,0}(0)$ consisting of stable bundles $E'$ with trivial determinant. To this end, it suffices, by induction on $N$, to consider the case of $N = 1$. Let $SU_2(X)$ (resp., $SU_2(X^{(1)})$) denote the moduli space of rank 2 semistable vector bundles on $X$ (resp., $X^{(1)})$ with trivial determinant. Then, the Frobenius pull-back of vector bundles $\mathcal{V} \mapsto F^{(1)}_{X/k}(\mathcal{V})$ yields a rational morphism $\nu : SU_2(X^{(1)}) \to SU_2(X)$, which is known as the generalized Vershiebung map. Now, suppose that the fiber $\nu^{-1}([E])$ via $\nu$ of the point $[E] \in SU_2(X)$ classifying $E$ is not finite, i.e., of positive dimension. Let $J$ denote the locus in $SU_2(X^{(1)})$ classifying $S$-equivalence classes of strictly semistable bundles, i.e., vector bundles of the form $\mathcal{L} \oplus \mathcal{L}^\nu$ for some degree 0 line bundle $\mathcal{L}$. The morphism $\text{Pic}^0(X^{(1)}) \to J$ given by $\mathcal{L} \mapsto [\mathcal{L} \oplus \mathcal{L}^\nu]$ is finite and surjective, which implies that $J$ is of
Thus, the set $\text{Ker} \{ (1) \}$ (cf. Theorem 2), we have $L \oplus L'$ surjectivity of (389). Hence, since both $\text{Ker} \{ (1) \}$ is nonempty. Let us take a $k$-rational point $q \in J \cap \nu^{-1}(L)$. Since the undefined locus of $\nu$ lies outside $J$, we have $q \in J \cap \nu^{-1}(L)$. It follows that $q$ classifies a rank 2 vector bundle of the form $L \oplus L'$. But, the pull-back $F_{X/k}^*(L \oplus L') = F_{X/k}^*(2L) = F_{X/k}^*(L \oplus L')$ cannot be isomorphic to $E$ (because of its stability), which contradicts the assumption that $q \in \nu^{-1}(L)$. Thus, the set $\text{Ker} \{ (1) \} \Rightarrow (0) = \nu^{-1}(L)(k)$, as well as the set $\text{Ker} \{ (N) \Rightarrow (0) \}$, turns out to be finite.

Next, we shall prove the finiteness of $\text{Ker} \{ (N) \Rightarrow (0) \}$. Let us consider the map of sets

$$\text{Ker} \{ (N) \Rightarrow (0) \} \times \text{Ker} \{ (N) \Rightarrow (0) \} \rightarrow \text{Ker} \{ (N) \Rightarrow (0) \}$$

(cf. (388) for the definition of $\text{Ker} \{ (N) \Rightarrow (0) \}$ determined by assigning $(E, L) \mapsto E \otimes L$. One may verifies that this map is surjective. Indeed, let $E'$ be a stable bundle classified by $\text{Ker} \{ (N) \Rightarrow (0) \}$. Since $\deg(E') = \frac{1}{p_N} \cdot \deg(F_{X/k}^*(E)) = \frac{1}{p_N} \cdot \deg(E) = \frac{1}{p_N} \cdot 0 = 0$, there exists a (degree 0) line bundle $L$ on $X/N$ with $L^\otimes 2 = \text{det}(E')$. The vector bundle $E' \otimes L^\vee$, and the pair $(E' \otimes L^\vee, L)$ is mapped to $E'$ via the map (389). This implies the surjectivity of (389). Hence, since both $\text{Ker} \{ (N) \Rightarrow (0) \}$ and $\text{Ker} \{ (E) \Rightarrow (0) \}$ are finite as proved above, the set $\text{Ker} \{ (N) \Rightarrow (0) \}$ turns out to be finite. This completes the proof of the assertion.

**Theorem 7.4.2.**

(Recall the assumption that $X$ has genus 2.) For each positive integer $N$ and each theta characteristic $\Theta$ of $X$, the set $\mathcal{D}_{X,N,\Theta}^{\text{zzz}}$, as well as the set $F_{X,N,\Theta}^N \cdot \text{Proj}_X$, is finite.

**Proof.** Consider the map of sets

$$\mathcal{D}_{X,N,\Theta}^{\text{zzz}} \rightarrow \mathcal{D}_{X,1,\Theta}^{\text{zzz}}$$

given by truncation to level 1. Since we already know the finiteness of $\mathcal{D}_{X,1,\Theta}^{\text{zzz}}$ as mentioned at the beginning of this subsection, it suffices to prove the finiteness of each fiber of this map. Let us fix a dormant indigenous $D_{X}^{(0)}$-module $V := (D_{X,1,\Theta} \otimes \Theta, V_\Theta)$ classified by $\mathcal{D}_{X,1,\Theta}^{\text{zzz}}$. In particular, we obtain the set $\text{Ker} \{ (N-1) \Rightarrow (0) \}$. Next, let $E'$ be a stable bundle in $\text{Ker} \{ (N-1) \Rightarrow (0) \}$, which has, by definition, an isomorphism

$$F_{X/k}^{(N-1)*}(E') \sim \text{Sol}(V_\Theta).$$

It induces the following composite isomorphism:

$$F_{X/k}^{(N)*}(E') \sim F_{X/k}^{(N)*}(F_{X/k}^{(N-1)*}(E')) \sim F_{X/k}^{(N)*}(\text{Sol}(V_\Theta)) \sim \mathcal{D}_{X,1,\Theta}^{(0)} \otimes \Theta \sim \mathcal{D}_{X,1,\Theta}^{(N,1)},$$

where the last arrow denotes the morphism arising from the natural inclusion $\text{Sol}(V_\Theta) \hookrightarrow \mathcal{D}_{X,1,\Theta}^{(0)} \otimes \Theta$. We shall denote by $\nabla_{\Theta}$ the $\mathcal{D}_{X}^{(N)}$-action on $\mathcal{D}_{X,1,\Theta}^{(N-1)} \otimes \Theta$ corresponding to $F_{X/k}^{(N)*}(V_\Theta)$ (cf. (112)) via this composite. Note that $\nabla_{\Theta}$ does not depend on the choice of an isomorphism (391). Indeed, $\mathcal{D}_{X,1,\Theta}^{(0)} \otimes \Theta$ may be obtained as an extension of $\Theta^{\vee}$ by the degree 1 line bundle $\Theta$. By taking account of this fact and the isomorphism $F_{X/k}^{(N)*}(\text{Sol}(V_\Theta)) \sim \mathcal{D}_{X,1,\Theta}^{(0)} \otimes \Theta$, we can verify that $\text{Sol}(V_\Theta)$ is stable. Hence, the isomorphism (391) is uniquely determined up to composition
with an automorphism given by multiplication by some element of $k^\times$. This implies that $\nabla_{\mathcal{V}_E}$ is well-defined, so we obtain a collection $\mathcal{V}_E^{\omega} := (\mathcal{D}_{X,1}^{(N-1)} \otimes \Theta, \nabla_{\mathcal{V}_E}, \Theta)$. Because of the equality $\text{KS}_{\mathcal{V}_E} = \text{KS}_{\mathcal{V}_E^{\omega}}$, the collection $\mathcal{V}_E^{\omega}$ forms a dormant indigenous $\mathcal{D}_{X,1}^{(N-1)}$-module. The assignment $E' \mapsto \mathcal{V}_E^{\omega}$ gives a bijective correspondence between $\text{Ker}(\mathcal{D}(N-1)_X \otimes \Theta) \subseteq \mathcal{S}_{\text{Sol}(\nabla_{\mathcal{V}_E})}$ and the fiber of the map (390) over the point $[\mathcal{V}_E^{\omega}]$ in $\mathcal{I}_{\mathcal{D}_{X,1}^{(N-1)}}$. In fact, the inverse assignment may be given by taking sheaves of horizontal sections of dormant indigenous $\mathcal{D}_{X,1}^{(N-1)}$-modules. But, by Lemma 7.4.1, the fiber of (390) over the point $[\mathcal{V}_E^{\omega}]$ turns out to be finite. This completes the proof of the theorem.

We shall formulate, as follows, a conjecture asserting the finiteness of $F^N\text{-Proj}_X$ for an arbitrary smooth projective variety $X$. By Proposition 5.1.1 and Theorem 7.4.2, this conjecture holds for projective spaces and genus 2 curves. In addition, if this conjecture could be proved for a variety $X$, then it would be natural and interesting to ask how many $F^N$-projective structures actually exist on $X$.

**Conjecture 7.4.3.**

Let $X$ be a smooth projective variety over $k$. Then, for each positive integer $N$, the set $F^N\text{-Proj}_X$ is finite.

### 7.5. $F^N$-affine structures on curves.

Next, we mention some results on $F^N$-affine structures defined on a curve. First, let us consider the case of projective curves. We already proved (cf. Proposition 5.1.1 (i)) that a smooth projective curve of genus 0, i.e., the projective line $\mathbb{P}^1$, has no $F^N$-affine structures for any $N$. Also, by Proposition 6.3.2 (i), a smooth projective curve of genus 1 has an $F^N$-affine structure for any $N \in \mathbb{Z} > 0 \cup \{\infty\}$ if it is ordinary. Regarding supersingular genus 1 curves, we have the following assertion.

**Proposition 7.5.1.**

Let $X$ be a smooth projective curve of genus 1 over $k$. Then, the following assertions hold:

(i) Suppose that $X$ is ordinary. Then, $X$ admits an $F^N$-affine structure for any $N \in \mathbb{Z}_{>0} \cup \{\infty\}$. Moreover, for each positive integer $N$, the following equality holds:

\[ #(F^N\text{-Aff}_X) = p^{N-1}(p-1). \]

(ii) Suppose that $X$ is supersingular (i.e., not ordinary). Then, $X$ admits a unique $F^1$-affine structure but admits no $F^N$-affine structures for any $N \geq 2$.

**Proof.** First, we shall prove assertion (i). The former assertion was already proved in Proposition 6.3.2 (i). To prove the latter assertion, let us recall from [81], §5.3, Example 5.3.3, that the set $\mathcal{A}_{\mathcal{D}_{X,1}}$, or equivalently the set $\mathfrak{S}_{\text{tan}_{X,1}}$ (cf. Proposition 4.5.3 (i)), coincides with the set of nontrivial connections on $\mathcal{O}_X$ ($\cong \Omega_X$) with vanishing $p$-curvature. In particular, the sheaf of horizontal sections of such a connection is nontrivial, i.e., has no nontrivial global sections. This implies the equality $F^N\text{-Aff}_X^{(\omega)} = F^N\text{-Aff}_X$. Hence, since $X$ is Frobenius split because of the
ordinariness, it follows from Proposition 4.5.3 (ii), that the map \( ^{\Lambda}z_{N} \rightarrow \mathcal{T}an_{X,N} \rightarrow F^{N} \cdot \mathcal{A}ff_{X}^{N} \) (cf. (228)) is bijective. Moreover, the composite

\[
(394) \quad \mathbb{B}_{X,N} \overline{\mathcal{O}}_{n} \xrightarrow{\quad ^{\Lambda}z_{N} \rightarrow \circ \quad} \mathcal{D}_{X,N}^{\mathbb{Z} \cdots} \xrightarrow{\quad (^{\Lambda}z_{N} \rightarrow \circ)_{-1} \quad} \mathcal{T}an_{X,N} \xrightarrow{\quad ^{\Lambda}z_{N} \rightarrow \circ \quad} \mathcal{A}c_{X,N}^{N}
\]

(cf. (323) and (217) for the definitions of the injection \( ^{\Lambda}z_{N} \rightarrow \circ \) and the bijection \( ^{\Lambda}z_{N} \rightarrow \circ \) respectively) is verified to be bijective by examining the resulting assignment explicitly. It follows that \( ^{\Lambda}z_{N} \rightarrow \circ \) is bijective. Therefore, the inequality in the resp’ed assertion of Corollary 6.4.2 becomes an equality. This completes the proof of the latter assertion.

Next, we shall prove assertion (ii). Suppose that \( X \) is supersingular, which implies the equality

\[
(395) \quad (\mathcal{V}er^{(N)}_{X})^{-1}([\mathcal{O}_{X}]) = [\mathcal{O}_{X}^{(N)}]
\]

(cf. (297)). We shall fix a generator \( a \) of \( \Gamma(X, \Omega_{X}) \), inducing the identification \( \iota_{a} : \mathcal{O}_{X} \xrightarrow{\sim} \Omega_{X} \) given by \( v \mapsto v \cdot a \). Any dual affine connection of level \( N \) may be regarded as a \( \mathcal{D}_{X}^{(N-1)} \)-action on \( \mathcal{O}_{X} \) via this identification. But, by (395) and the equivalence of categories (113), we see that any \( \mathcal{D}_{X}^{(N-1)} \)-action on \( \Omega_{X} \) cannot define a dual affine connection of level \( N \) if it does not correspond to \( \mathcal{D}_{X}^{(N-1)} \) (cf. (68)).

Now, let us consider the former assertion of (ii). Denote by \( \nabla_{X}^{\mathbb{A}} \) the \( \mathcal{D}_{X}^{(0)} \)-action on \( \mathcal{O}_{X} \) corresponding to \( \nabla_{X}^{\mathbb{A}}(0) \). Then, we have

\[
(396) \quad \text{Sol}(\nabla_{X}^{\mathbb{A}}) = \mathcal{O}_{X}^{(1)} \cdot a = (F_{X})^{-1}(\mathcal{O}_{X}) \cdot a \subseteq \Omega_{X}.
\]

On the other hand, since \( X \) is supersingular, the \( p^{-1} \)-linear endomorphism \( C_{X} \) of \( \Gamma(X, \Omega_{X}) \) induced by the Cartier operator becomes the zero map. Hence, for any local section \( v \) in \( \mathcal{O}_{X} \), we have

\[
(397) \quad C_{X}(v^{p} \cdot a) = v \cdot C_{X}(a) = v \cdot 0 = 0.
\]

It follows from (396) and (397) that \( \text{Sol}(\nabla_{X}^{\mathbb{A}}) \subseteq \text{Ker}(C_{X}) = B_{X}^{(1)} \) (cf. (301)), that is to say, \( \nabla_{X}^{\mathbb{A}} \) forms a dual affine connection of level 1 on \( X \). We conclude that \( \nabla_{X}^{\mathbb{A}} \) is the unique dual affine connection of level 1 on \( X \). Therefore, the former assertion of (ii) follows from the bijectivity of \( ^{\Lambda}z_{1} \rightarrow \circ \) (cf. Proposition 4.5.3 (i)).

Finally, let us consider the latter assertion, i.e., the case of \( N \geq 2 \). Suppose, on the contrary, that there exists an \( F^{N} \)-affine structure on \( X \). We shall denote by \( \mathcal{U}^{\mathbb{A}}( \subseteq B_{X}^{(N)} ) \) the corresponding Tango structure of level \( N \). Recall from the above discussion that the dual affine connection of level \( N \) associated to \( \mathcal{U}^{\mathbb{A}} \) is trivial, i.e., there exists an isomorphism \( \mathcal{U}^{\mathbb{A}} \xrightarrow{\sim} \mathcal{O}_{X}^{(N)} \).

By passing to this isomorphism, we obtain a nonzero element \( t \in \Gamma(X^{(N)}, B_{X}^{(N)}) \) determined by the inclusion \( \mathcal{U}^{\mathbb{A}}(1) \subseteq B_{X}^{(N)} \). Since the dual affine connection of level 1 associated to the 1-st truncation \( \mathcal{U}^{\mathbb{A}}(1) \) coincides with \( \nabla_{X}^{\mathbb{A}} \) defined above, the image of \( t \) via the map

\[
(398) \quad \Gamma(\delta_{X,1,N}) : \Gamma(X^{(N)}, B_{X}^{(N)}) \rightarrow \Gamma(X^{(1)}, B_{X}^{(1)})
\]

induced by \( \delta_{X,1,N} \) (cf. (182)) is nonzero. On the other hand, since \( X \) is supersingular, the morphism \( H^{1}(X^{(N)}, \mathcal{O}_{X}^{(N)}) \rightarrow H^{1}(X^{(N)}, \mathcal{O}_{X}^{(N)}) \) with \( N < N' \) induced by \( F_{X}^{(N'-N)} \) is the zero map. By taking account of this fact and considering the long exact sequences of cohomology associated to the diagram (183), we see that \( \Gamma(\delta_{X,1,N}) \) must be the zero map. Thus, we obtain
a contradiction. This completes the proof of the latter assertion, and hence, the proof of the proposition.

Next, let us review the previous study of $F^N$-affine structures on curves of genus $g > 1$. Recall (cf. [66], §1) that a Tango-Raynaud curve (or a Tango curve) is a smooth projective curve of genus $g > 1$ admitting a Tango structure of level 1, or equivalently, an $F^1$-affine structure. M. Raynaud provided (cf. [73] or [66], §1.1, Example 1.3) an explicit example of a Tango-Raynaud curve. Also, we can find in [80] the construction of Tango-Raynaud curves by means of solutions to the Bethe ansatz equations over finite fields. According to [81], Theorem B, the moduli space parametrizing Tango-Raynaud curves of genus $g$ forms, if it is nonempty, an equidimensional smooth Deligne-Mumford stack of dimension $2 \cdot g - 2 + \frac{2g-2}{p}$. Regarding higher-level cases, Y. Hoshi proved (cf. [31], Theorem B) that giving an integer $N > g$ he also proved (cf. [32], Theorem 3) that if an integer $g > 1$ satisfies $p^N \mid (g-1)$ for a positive integer $N$, then there exists a smooth projective curve of genus $g$ admitting a Tango function of level $N$, or equivalently an $F^N$-affine structure. The following assertion, which is the inverse direction of this assertion, is immediately verified.

**Proposition 7.5.2** (cf. [31], Corollary 1.10).

*Let $X$ be a projective smooth curve over $k$ of genus $g > 1$. If $X$ admits an $F^N$-affine structure for a positive integer $N$, then $g - 1$ is divisible by $p^N$. In particular, $X$ has no $F^\infty$-affine structures.*

*Proof.* Since the latter assertion follows directly from the former assertion, it suffices to consider only the former assertion.

Suppose that there exists an $F^N$-affine structure on $X$ for a positive integer $N$. By passing to the bijections appearing in the diagram of Theorem A (ii), it corresponds to a Tango structure $U^\bullet$ of level $N$. Write $\nabla_{\Omega_X}^\bullet := \kappa_\Omega \otimes \Omega^\bullet(\mathcal{U}^\bullet)$ (cf. (228)), i.e., the dual affine connection of level $N$ associated with $\mathcal{U}^\bullet$. Since $\nabla_{\Omega_X}^\bullet$ has vanishing $p$-$(N-1)$-curvature, the natural morphism $F^{(N)\ast}_{X/k}(\text{Sol}(\nabla_{\Omega_X}^\bullet)) \to \Omega_X$ is an isomorphism. In particular, $\text{Sol}(\nabla_{\Omega_X}^\bullet)$ is a line bundle on $X^{(N)}$ and we have

$$\text{(399)} \quad (Z \ni) \ \deg(\text{Sol}(\nabla_{\Omega_X}^\bullet)) = \frac{1}{p^N} \cdot \deg(F^{(N)\ast}_{X/k}(\text{Sol}(\nabla_{\Omega_X}^\bullet))) = \frac{1}{p^N} \cdot \deg(\Omega_X) = \frac{2 \cdot g - 2}{p^N}.$$  

Since $p > 2$, this completes the proof of the assertion.

Finally, we shall study Frobenius-affine structures on affine curves. As proved in Proposition 5.2.1 (i), the affine line $\mathbb{A}^1$ admits an $F^N$-affine structure for any $N \in \mathbb{Z}_{>0} \cup \{\infty\}$. More generally, a curve $X$ admits an $F^N$-affine structure if there exists an étale morphism $X \to \mathbb{A}^1$ (cf. Proposition 5.2.2 (i)). The following three assertions are direct consequences of this fact and Zapponi’s results proved in [86]. Here, we shall fix $N \in \mathbb{Z}_{>0} \cup \{\infty\}$.

**Proposition 7.5.3.**

*Let $X$ be a smooth projective curve over $k$ and $S$ a nonempty finite subset of $X(k)$. Suppose that there exists an exact meromorphic 1-form on $X$ whose support is contained in $S$. Then, the affine curve $X \setminus S$ admits a global $F^N$-affine structure.*
Proof. The assertion follows from [86], Proposition 2.2. □

Proposition 7.5.4.
Let \( X \) be a smooth projective curve of genus \( g \) over \( k \) and \( q \) a \( k \)-rational point of \( X \). Suppose that one of the following two conditions is fulfilled:

(i) There exists an exact 1-form on \( X \) which is regular and nowhere vanishing outside \( q \).
(ii) The \( k \)-vector space \( \Gamma(X, \Omega_X((2g-2)q)) \) is nontrivial and the equality \( C_{X\setminus\{q\}}(\Omega_X((2g-2)q)) = 0 \) holds.

Then, the affine curve \( X \setminus \{q\} \) admits a global \( F^N \)-affine structure.

Proof. The assertion follows from [86], Theorem 2.4. □

Proposition 7.5.5.
Let \( X \) be a smooth projective curve of genus 1 over \( k \), and suppose that \( X \) is supersingular. Also, let \( q \) be a \( k \)-rational point of \( X \). Then, \( X \setminus \{q\} \) admits a global \( F^N \)-affine structure.

Proof. The assertion follows from [86], §5, Proposition 5.1. □

Example 7.5.6.
Denote by \( k(x) \) the field of fractions of \( k[x] \), i.e., the ring of polynomials in the variable \( x \). Let us consider (cf. [74], §6.4, Proposition 6.4.1) the function field \( K \) over \( k(x) \) defined by

\[
y^q + \mu \cdot y = f(x),
\]

where \( q \) \( (= p^\ell) \) denotes a power of \( p \), \( f(x) \) denotes an element of \( k[x] \), and \( \mu \) denotes a nonzero element of \( k \). Artin-Schreier curves are examples of such curves. Now, assume that \( \deg(f(x)) =: m \) \((> 0) \) is prime to \( p \). If \( X \) denotes the smooth projective curve defined by the function field \( K \), then it has a finite covering \( \pi : X \to \mathbb{P}^1 \) \((\supset A^1 = \text{Spec}(k[x])) \) determined by the natural inclusion \( k(x) \hookrightarrow K \). The genus of this curve is calculated by \( g = (q-1)(m-1)/2 \).

The inverse image of \( \infty \in \mathbb{P}^1 \setminus A^1 \) consists of one point \( \infty_X \) at which \( \pi \) is totally ramified and \( \pi \) is unramified elsewhere. Hence, \( \pi \) restricts to a finite étale covering \( X \setminus \{\infty_X\} \to A^1 \), so the affine curve \( X \setminus \{\infty_X\} \) admits a global \( F^N \)-affine structure for any \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \).

The case where \( f(x) = x^{q+1} \) and \( \mu = 1 \) is well-known as the Hermitian curve. We shall denote this curve by \( H \). The set of \( \mathbb{F}_{q^2} \)-rational points \( H(\mathbb{F}_{q^2}) \), which can be defined because \( H \) is defined over \( \mathbb{F}_q \), has exactly \( q^3 + 1 \) elements. It is known that the group of \( k \)-automorphisms \( \text{Aut}(H) \) of \( H \) is large so that it acts on \( H(\mathbb{F}_{q^2}) \) transitively. Hence, by this fact and the above discussion, we see that the curve \( H \) minus one point in \( H(\mathbb{F}_{q^2}) \) admits an \( F^N \)-affine structure for any \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \).

8. Case 4: Surfaces

The final section is devoted to studying \( F^N \)-projective and \( F^N \)-affine structures on algebraic surfaces. For this purpose, we shall make use of the classification of algebraic surfaces by
their Kodaira dimension, i.e., the Kodaira-Enriques classification. In their three fundamental articles [67], [11], [12], E. Bombieri and D. Mumford established the positive characteristic version. For many algebraic surfaces, we are also able to decide, according to this classification, whether they admit an $F^N$-projective ($F^N$-affine) structure or not (cf. Corollary 8.5.1). But, unlike the case of complex surfaces discussed in [47] and [48], we do not achieve unfortunately a complete classification of such algebraic surfaces. In particular, at the time of writing the present paper, the author does not know any example of a smooth projective surface of general type admitting an $F^\infty$-projective structure corresponding to the third class in the standard examples displayed in §0.1 (cf. Remark 8.5.2).

8.1. Products of varieties.

Before discussing the study on surfaces, we consider constructions of the products of Frobenius-affine structures, affine-indigenous structures, and Tango structures. Suppose that, for each $i = 1, 2$, we are given a smooth variety $X_i$ of dimension $n_i$ ($> 0$). Denote by $\operatorname{pr}_i: X_1 \times X_2 \to X_i$ the projection onto the $i$-th factor. Let $\rho$ be the homomorphism $\rho: \operatorname{PGL}_{n_1+1}^A \times \operatorname{PGL}_{n_2+1}^A \to \operatorname{PGL}_{n_1+n_2+1}^A$ given by

$$\rho(\begin{pmatrix} a & 0 \\ t a & A \end{pmatrix}, \begin{pmatrix} b & 0 \\ t b & B \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 \\ a^{-1} \cdot t a & a^{-1} & A \\ b^{-1} \cdot t b & O & b^{-1} \cdot B \end{pmatrix}$$

for any $a, b \in \mathbb{G}_m$, $\mathbb{A}^n$, and $A, B \in \operatorname{GL}_n$. This morphism restricts to a homomorphism $\operatorname{PGL}_{n_1+1}^{A, \infty} \times \operatorname{PGL}_{n_2+1}^{A, \infty} \to \operatorname{PGL}_{n_1+n_2+1}^{A, \infty}$.

$F^N$-affine structures:

Let $N$ be a positive integer. Suppose that, for each $i = 1, 2$, we are given an $F^N$-affine structure $\mathcal{S}_i$ on $X_i$. If $\phi_i: (X_i \supseteq) U_i \to \mathbb{A}^{n_i}$ is an étale morphism classified by $\mathcal{S}_i$, then the composite

$$\phi_1 \land \phi_2: \operatorname{pr}_1^{-1}(U_1) \cap \operatorname{pr}_2^{-1}(U_2) \xrightarrow{(\phi_1 \circ \operatorname{pr}_1, \phi_2 \circ \operatorname{pr}_2)} \mathbb{A}^{n_1} \times \mathbb{A}^{n_2} \xrightarrow{\sim} \mathbb{A}^{n_1+n_2}$$

defines an étale morphism. We shall write

$$(\mathcal{S}_1 \bigotimes \mathcal{S}_2)$$

for the smallest subsheaf of $\mathcal{P}_X^{\mathcal{S}_1 \times \mathcal{S}_2}$ that is closed under the $(\operatorname{PGL}_{n_1+n_2}^{A, \infty})$-action and contains $\phi_1 \land \phi_2$ for the various $\phi_i \in \mathcal{S}_i$ and $\phi_2 \in \mathcal{S}_2$. One verifies that $\mathcal{S}_1 \bigotimes \mathcal{S}_2$ forms an $F^N$-affine structure on $X_1 \times X_2$. The underlying $\operatorname{PGL}_{n_1+n_2+1}$-torsor of $\mathcal{S}_1 \bigotimes \mathcal{S}_2$ is isomorphic to the principal $\operatorname{PGL}_{n_1+n_2+1}$-bundle defined as the product $\operatorname{pr}_1^*(\mathcal{S}_2) \times \operatorname{pr}_2^*(\mathcal{S}_2)$ after change of structure group via $\rho$.

The formation of $\mathcal{S}_1 \bigotimes \mathcal{S}_2$ commutes with truncation to lower levels, i.e., $(\mathcal{S}_1 \bigotimes \mathcal{S}_2)^{(N')} = (\mathcal{S}_1^{(N')} \bigotimes \mathcal{S}_2^{(N')})$ for any positive integer $N'$ with $N' < N$. Hence, if $\mathcal{S}_i := \{ \mathcal{S}_i^{(N)} \}_{N \in \mathbb{Z}_{>0}}$ ($i = 1, 2$) is an $F^\infty$-affine structure on $X_i$, then we obtain an $F^\infty$-affine structure $\mathcal{S}_1 \bigotimes \mathcal{S}_2 := \{ \mathcal{S}_1^{(N)} \bigotimes \mathcal{S}_2^{(N)} \}_{N \in \mathbb{Z}_{>0}}$ on $X_1 \times X_2$. In any case of $N \in \mathbb{Z}_{>0} \cup \{ \infty \}$, we shall refer to $\mathcal{S}_1 \bigotimes \mathcal{S}_2$ as the product of $\mathcal{S}_1$ and $\mathcal{S}_2$.

$F^N$-affine indigenous structures:
Next, suppose that we are given an $F^N$-affine indigenous structure $\mathcal{E}^\bigtriangledown := (\mathcal{E}_i^\bigtriangledown, \mathcal{E}_i^\text{red})$ on $X_i$ for a positive integer $N$. Write $\mathcal{E}_i := F^N_{X_i/k}(\mathcal{E}_i^\bigtriangledown)$. Then, the product $pr_1^*(\mathcal{E}_1^\bigtriangledown) \times pr_2^*(\mathcal{E}_2^\bigtriangledown)$ (resp., $pr_1^*(\mathcal{E}_1) \times pr_2^*(\mathcal{E}_2)$) forms a principal $\text{PGL}_{n_1+1}^k \times \text{PGL}_{n_2+1}^k$-bundle on $X_1^N \times X_2^N$ (resp., on $X_1 \times X_2$). Also, the product $pr_1^*(\mathcal{E}_1^\text{red}) \times pr_2^*(\mathcal{E}_2^\text{red})$ specifies a $\text{PGL}_{n_1+1}^\infty \times \text{PGL}_{n_2+1}^\infty$-reduction of $pr_1^*(\mathcal{E}_1) \times pr_2^*(\mathcal{E}_2)$. We shall write

$$\mathcal{E}_1^\bigtriangledown \boxtimes \mathcal{E}_2^\bigtriangledown \quad \text{(resp., } \mathcal{E}_1 \boxtimes \mathcal{E}_2; \text{ resp., } \mathcal{E}_1^\text{red} \boxtimes \mathcal{E}_2^\text{red})$$

for the principal $\text{PGL}_{n_1+1}^k \times \text{PGL}_{n_2+1}^k$-bundle (resp., the principal $\text{PGL}_{n_1+2}^k$-bundle; resp., the principal $\text{PGL}_{n_2+1}^k$-bundle) obtained from $pr_1^*(\mathcal{E}_1^\bigtriangledown) \times pr_2^*(\mathcal{E}_2^\bigtriangledown)$ (resp., $pr_1^*(\mathcal{E}_1) \times pr_2^*(\mathcal{E}_2)$; resp., $pr_1^*(\mathcal{E}_1^\text{red}) \times pr_2^*(\mathcal{E}_2^\text{red})$) via change of structure group by $\rho$. Then, $\mathcal{E}_1^\text{red} \boxtimes \mathcal{E}_2^\text{red}$ specifies a $\text{PGL}_{n_1+1}^\infty$-reduction of $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ and the resulting pair

$$\mathcal{E}_1^\bullet \boxtimes \mathcal{E}_2^\bullet := (\mathcal{E}_1^\bigtriangledown \boxtimes \mathcal{E}_2^\bigtriangledown, \mathcal{E}_1^\text{red} \boxtimes \mathcal{E}_2^\text{red})$$

forms an $F^N$-affine indigenous structure on $X_1 \times X_2$. The isomorphism class of $\mathcal{E}_1^\bullet \boxtimes \mathcal{E}_2^\bullet$ depends only on the isomorphism classes of $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$.

The formation of $\mathcal{E}_1^\bullet \boxtimes \mathcal{E}_2^\bullet$ commutes with truncation to lower levels. Hence, if $\mathcal{E}_i^\bullet := \{\mathcal{E}_i^{\bullet,N}\}_{N \in \mathbb{Z}_{>0}}$ (for each $i = 1, 2$) is an $\mathcal{F}^\infty$-affine indigenous structure on $X_i$, then we obtain an $\mathcal{F}^\infty$-affine indigenous structure $\mathcal{E}_1^\bullet \boxtimes \mathcal{E}_2^\bullet := \{\mathcal{E}_1^{\bullet,N} \boxtimes \mathcal{E}_2^{\bullet,N}\}_{N \in \mathbb{Z}_{>0}}$ on $X_1 \times X_2$. In any case of $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$, we shall refer to $\mathcal{E}_1^\bullet \boxtimes \mathcal{E}_2^\bullet$ as the product of $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$.

**Dormant affine-indigenous $\mathcal{D}^{(N-1)}_{X}$-modules:**

Let $N \in \mathbb{Z}_{>0} \sqcup \{\infty\}$ and, for each $i = 1, 2$, let $\mathcal{A}^{\bigtriangledown}_i := (\mathcal{D}^{(N-1)}_{X_i,1}, \nabla_{Y,i}, \mathcal{O}_{X_i}, \delta_i)$ be a dormant affine-indigenous $\mathcal{D}^{(N-1)}_{X_i}$-module classified by $\mathcal{A}^{\text{aff}}_{X_i,N}$. By pulling-back the data in $\mathcal{A}^{\bigtriangledown}_i$ by the projection $pr_i$, one can obtain various objects and structures on $X_1 \times X_2$. Indeed, the pull-back $pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1})$ has a structure of $\mathcal{D}^{(N)}_{X_1 \times X_2}$-action $pr_i^*(\nabla_{Y,i})$ with vanishing $p-(N-1)$-curvature (cf. [I], §2.1, 2.1.6). The pull-back of the submodule $\mathcal{O}_{X_i}$ of $\mathcal{D}^{(N-1)}_{X_i,1}$ specifies a submodule $\mathcal{O}_{X_1 \times X_2}$ (or $\mathcal{O}_{X_i}$) of $pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1})$. Also, the morphism $\delta_i$ induces a left inverse $pr_i^*(\delta_i) : pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1}) \to \mathcal{O}_{X_1 \times X_2}$ to the inclusion $\mathcal{O}_{X_1 \times X_2} \hookrightarrow pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1})$. If $D_i : \mathcal{D}^{(N-1)}_{X_i \times X_2,1} \to pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1})$ denotes the morphism arising from $pr_i$, then we obtain the following short exact sequence:

$$0 \to \mathcal{D}^{(N-1)}_{X_1 \times X_2,1} \xrightarrow{(D_1,D_2)} \bigoplus_{i=1}^2 pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1}) \xrightarrow{(pr_i^*(\delta_1),-pr_i^*(\delta_2))} \mathcal{O}_{X_1 \times X_2} \to 0.$$

The surjection $pr_i^*(\delta_i) : pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1}) \to \mathcal{O}_{X_1 \times X_2}$ is compatible with the respective $\mathcal{D}^{(N-1)}_{X_i \times X_2}$-actions $pr_i^*(\nabla_{Y,i})$, $\nabla_{\mathcal{O}_{X_1 \times X_2}}$, so the direct sum $\bigoplus_{i=1}^2 pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1})$ yields a $\mathcal{D}^{(N-1)}_{X_1 \times X_2}$-action $\nabla_{Y,1} \boxtimes \nabla_{Y,2}$ on $\mathcal{D}^{(N-1)}_{X_1 \times X_2,1}$ by passing to the injection $(D_1, D_2)$. Moreover, the morphism

$$\bigoplus_{i=1}^2 pr_i^*(\mathcal{D}^{(N-1)}_{X_i,1}) \to \mathcal{O}_{X_1 \times X_2}$$
restricts to a left inverse \( \delta_1 \boxtimes \delta_2 : D^{(N-1)}_{X_1 \times X_2,1} \rightarrow O_{X_1 \times X_2} \) to the natural inclusion \( O_{X_1 \times X_2} \hookrightarrow D^{(N-1)}_{X_1 \times X_2,1} \). The resulting collection

\[
\mathcal{A}\mathcal{V}_1 \boxtimes \mathcal{A}\mathcal{V}_2 := (D^{(N-1)}_{X_1 \times X_2,1}, \nabla_1 \boxtimes \nabla_2, O_{X_1 \times X_2}, \delta_1 \boxtimes \delta_2)
\]

forms a dormant affine-indigenous \( D^{(N-1)}_{X_1 \times X_2} \) module classified by the set \( \mathcal{A}\mathcal{I}^{2_{\mathbb{Z}^+}} \).

**Tango structures of level \( N \):**

Next, we shall discuss the products of Tango structures. Let \( N \) be a positive integer. In what follows, given an \( O_{X_1}^{(N)} \)-module \( F_1 \) and an \( O_{X_2}^{(N)} \)-module \( F_2 \), we write \( F_1 \boxtimes F_2 := pr_1^*(F_1) \otimes pr_2^*(F_2) \). To begin with, let us consider the following exact sequence:

\[
0 \rightarrow O_{X_1}^{(N)} \boxtimes O_{X_2}^{(N)} \xrightarrow{f_1} F^{(N)}_{X_1/k^s}(O_{X_1}) \boxtimes F^{(N)}_{X_2/k^s}(O_{X_2}) \xrightarrow{f_2} B^{(N)}_{X_1} \boxtimes F^{(N)}_{X_2/k^s}(O_{X_2}) \boxplus F^{(N)}_{X_1/k^s}(O_{X_1}) \boxtimes B^{(N)}_{X_2} \xrightarrow{f_3} B^{(N)}_{X_1} \boxtimes B^{(N)}_{X_2} \rightarrow 0
\]

where

\[
f_1 := F^{(N)}_{X_1/k^s} \boxtimes F^{(N)}_{X_2/k^s}, \quad f_2 := (F^{(N)}_{X_1/k^s} \boxtimes \text{id}, \text{id} \boxtimes F^{(N)}_{X_2/k^s}), \quad f_3 := (\text{id} \boxtimes F^{(N)}_{X_2/k^s}, -F^{(N)}_{X_1/k^s} \boxtimes \text{id}).
\]

The \( O_{(X_1 \times X_2)}^{(N)} \)-modules \( O_{X_1}^{(N)} \boxtimes O_{X_2}^{(N)} \) and \( F^{(N)}_{X_1/k^s}(O_{X_1}) \boxtimes F^{(N)}_{X_2/k^s}(O_{X_2}) \) may be naturally identified with \( O_{(X_1 \times X_2)}^{(N)} \) and \( F^{(N)}_{X_1 \times X_2/k^s}(O_{X_1 \times X_2}) \) respectively. Under these identifications, the morphism \( f_1 \) coincides with \( F^{(N)}_{X_1 \times X_2/k} \). Hence, the sequence \((409)\) induces the following short exact sequence:

\[
0 \rightarrow B^{(N)}_{X_1 \times X_2} \rightarrow B^{(N)}_{X_1} \boxtimes F^{(N)}_{X_2/k^s}(O_{X_2}) \boxplus F^{(N)}_{X_1/k^s}(O_{X_1}) \boxtimes B^{(N)}_{X_2} \xrightarrow{f_3} B^{(N)}_{X_1} \boxtimes B^{(N)}_{X_2} \rightarrow 0.
\]

Now, for each \( i = 1, 2 \), let \( U_i^{\bullet} \subset (B^{(N)}_{X_i})^\oplus \) be a Tango structure of level \( N \) on \( X_i \). By the exactness of \((411)\), the \( O_{(X_1 \times X_2)}^{(N)} \)-submodule

\[
U_1^{\bullet} \boxtimes U_2^{\bullet} := U_1^{\bullet} \boxtimes \text{Im}(F^{(N)}_{X_2/k^s}) \boxplus \text{Im}(F^{(N)}_{X_1/k^s}) \boxtimes U_2^{\bullet}
\]

of \( B^{(N)}_{X_1} \boxtimes F^{(N)}_{X_2/k^s}(O_{X_1}) \boxplus F^{(N)}_{X_1/k^s}(O_{X_1}) \boxtimes B^{(N)}_{X_2} \) may be considered as an \( O_{(X_1 \times X_2)}^{(N)} \)-submodule of \( B^{(N)}_{X_1 \times X_2} \). By taking it into consideration, one verifies that \( U_1^{\bullet} \boxtimes U_2^{\bullet} \) specifies a Tango structure of level \( N \) on \( X_1 \times X_2 \).

The formation of \( U_1^{\bullet} \boxtimes U_2^{\bullet} \) commutes with truncation to lower levels. Hence, if \( U_i^{\bullet} := \{U_{i,N}\}_{N \in \mathbb{Z}_{>0}} \) is a Tango structure of level \( \infty \) on \( X_i \), then we obtain a Tango structure \( U_1^{\bullet} \boxtimes U_2^{\bullet} := \{U_{1,N} \boxtimes U_{2,N}\}_{N \in \mathbb{Z}_{>0}} \) of level \( \infty \) on \( X_1 \times X_2 \). In any case of \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \), we shall refer to \( U_1^{\bullet} \boxtimes U_2^{\bullet} \) as the **product** of \( U_1^{\bullet} \) and \( U_2^{\bullet} \).

Regarding the products of various structures constructed above, we have the following assertions:

**Proposition 8.1.1.**

*Let \( N \in \mathbb{Z}_{>0} \cup \{\infty\} \). Then, all the maps \( \mathcal{A}\mathcal{A}\xi^{\circ}\rightarrow \mathcal{A}\mathcal{A}\xi^{\circ}, \mathcal{A}\mathcal{A}\xi^{\circ}\rightarrow \mathcal{A}\mathcal{A}\xi^{\circ}, \mathcal{A}\mathcal{A}\xi^{\circ}\rightarrow \mathcal{A}\mathcal{A}\xi^{\circ}, \text{ and } \mathcal{A}\mathcal{A}\xi^{\circ}\rightarrow \mathcal{A}\mathcal{A}\xi^{\circ} \) are compatible with the respective formations of products constructed above.*

**Proof.** The assertion follows from the various definitions involved. \( \square \)
Proposition 8.1.2.
Let \( N \in \mathbb{Z}_{>0} \cup \{ \infty \} \). Also, let \( X_1 \) and \( X_2 \) be smooth varieties either of which admits an \( F^N \)-affine structure. Then, the product \( X_1 \times \cdots \times X_i \) admits an \( F^N \)-affine structure.

Proof. The assertion is a direct consequence of forming products of \( F^N \)-affine structures. \( \square \)

Remark 8.1.3.
Notice that even if two smooth varieties admit \( F^N \)-projective structures, their product does not necessarily admit an \( F^N \)-projective structure. For example, let \( X_i \) (\( i = 1, 2 \)) be a smooth projective curve over \( k \) of genus \( g_i \). In particular, we have \( c_1(X_i) = 2 - 2 \cdot g_i \). Since \( \mathcal{T}_{X_1 \times X_2} \cong \mathcal{T}_{X_1} \boxtimes \mathcal{T}_{X_2} \), the following sequence of equalities holds:

\[
(413) \quad c_1(X_1 \times X_2)^2 = (c_1(\text{pr}_1^*(\mathcal{T}_{X_1})) + c_1(\text{pr}_2^*(\mathcal{T}_{X_2})))^2 \\
= \text{pr}_1^*(c_1(X_1)^2) + \text{pr}_2^*(c_2(X_1)^2) + 2 \cdot \text{pr}_1^*(c_1(X_1)) \text{pr}_2^*(c_1(X_2)) \\
= 0 + 0 + 2 \cdot (2 - 2 \cdot g_1) \cdot (2 - 2 \cdot g_2) \\
= 8 \cdot (1 - g_1) \cdot (1 - g_2).
\]

Moreover, we have

\[
(414) \quad c_2(X_1 \times X_2) = c_2(\text{pr}_1^*(\mathcal{T}_{X_1})) + c_1(\text{pr}_1^*(\mathcal{T}_{X_1})) c_1(\text{pr}_2^*(\mathcal{T}_{X_2})) + c_2(\text{pr}_2^*(\mathcal{T}_{X_2})) \\
= \text{pr}_1^*(c_2(X_1)) + \text{pr}_1^*(c_1(X_1)) \text{pr}_2^*(c_1(X_2)) + \text{pr}_2^*(c_2(X_2)) \\
= 0 + (2 - 2 \cdot g_1)(2 - 2 \cdot g_2) + 0 \\
= 4 \cdot (1 - g_1) \cdot (1 - g_2).
\]

By (413) and (414), the non-resp’d equality (171) (or (416) described later) in the case where \( X \) is taken to be \( X_1 \times X_2 \) reads the equality

\[
(415) \quad (1 - g_1) \cdot (1 - g_2) = 0 \mod p^N.
\]

Hence, by Theorem 3.7.1 (i), the product \( X_1 \times X_2 \) admits an \( F^N \)-projective structure only if the equality (415) is fulfilled. On the other hand, Corollary 7.3.4 asserts that both \( X_1 \) and \( X_2 \) necessarily have \( F^N \)-projective structures.

8.2. Surfaces admitting an \( F^N \)-projective (\( F^N \)-affine) structure.

In the rest of this section, we shall study the classification problem of algebraic surfaces admitting \( F^N \)-projective (\( F^N \)-affine) structures according to the Kodaira-Enriques classification. From now on, we shall suppose that \( p > 3 \). As in the complex case, surfaces in positive characteristic that are of Kodaira dimension 0 fall into four classes. However, there exist new subclasses of Enriques surfaces in characteristic 2, and new subclasses of hyperelliptic surfaces, so-called quasi-hyperelliptic surfaces, in characteristic 2 and 3. But, by the above assumption on \( p \), we will not discuss such classes in the following discussion.

Now, let us fix a smooth surface \( X \) over \( k \). To begin with, we shall describe some consequences from the results proved so far. The following assertion is a direct consequence of the Chern class formula proved in Theorem 3.7.1.
Theorem 8.2.1. Let $N$ be an element of $\mathbb{Z}_{>0} \sqcup \{\infty\}$. Suppose that $X$ is quasi-projective and admits an $F^N$-projective (resp., $F^N$-affine) structure. Then, the following assertions hold:

(i) In the case where $N \neq \infty$, the Chern classes of $X$ satisfy that

$$c_1(X)^2 - 3 \cdot c_2(X) = 0 \quad (\text{resp., } c_1(X) = c_2(X) = 0) \quad \text{mod } p^N.$$  

(ii) In the case where $N = \infty$ and $X$ is projective, the crystalline Chern classes of $X$, being elements of $H^*_{\text{crys}}(X/W)$, satisfy (without any reduction) that

$$c_1(X)^2 - 3 \cdot c_2(X) = 0 \quad (\text{resp., } c_1(X) = c_2(X) = 0).$$

Also, by Theorems 5.3.1, 5.4.1, and Remark 5.3.2, the following assertion holds:

Theorem 8.2.2. Suppose that $X$ is projective and admits an $F^\infty$-projective structure. Moreover, one of the following conditions is fulfilled:

- $X$ contains a rational curve;
- the étale fundamental group $\pi^\text{et}_1(X)$ of $X$ is trivial.

Then, $X$ is isomorphic to the projective plane $\mathbb{P}^2$. In particular, any smooth projective surface admitting an $F^\infty$-projective structure is minimal.

As a corollary of the above theorem, we can show the following assertion, which is a positive characteristic version of [47], Corollary 4.3 (see also [26], Part III, §12, the discussion following Theorem 8).

Corollary 8.2.3. Suppose that $X$ is projective and contains a smooth rational curve $C$ with $C \cdot C \leq 0$. Then, $X$ admits no $F^\infty$-projective structures.

Proof. By the above theorem, it suffices to verify that any rational curve $C$ in $\mathbb{P}^2$ satisfies $C \cdot C > 0$. But, this is clear because any plane curve of degree $d$ is linearly equivalent to $d$ times a line $L$, and $L \cdot L = 1$ (cf. [27], Chap. V, §1, Example 1.4.2).

8.3. Hyperelliptic surfaces. We shall deal with $F^\infty$-projective and $F^\infty$-affine structures on hyperelliptic surfaces. Recall that a smooth projective surface $X$ is called hyperelliptic (or bielliptic) if its Albanese morphism is an elliptic fibration. (In the next subsection, we will recall the definition of an elliptic fibration.) Any such surface admits another elliptic fibration onto $\mathbb{P}^1$ and can be written as the étale quotient of a product of two genus 1 curves by a finite abelian group. To be precise, there exist two smooth projective genus 1 curves $C_1, C_0$ (each of which is equipped with a distinguished point 0, i.e., specifies an elliptic curve) and a finite group $G$ together with embeddings $G \hookrightarrow \text{Aut}(C_1), \alpha : G \hookrightarrow \text{Aut}(C_0)$, where $G$ acts by translations on $C_1$, such that $X$ is isomorphic to the quotient $(C_1 \times C_0)/G$. The Albanese morphism arises as the projection onto $C_1/G$ with fiber $C_0$, and the other fibration arises as the projection onto $C_0/G \cong \mathbb{P}^1$ with fiber $C_1$. According to the discussion in [1], §10.24, the (isomorphism class of the) triple $(C_1, C_0, G)$ is uniquely determined. In [1], List 10.27, we can find the complete list, i.e., the
Remark 8.3.2.

We shall say that $X$ is ordinary if both the genus 1 curves $C_1$, $C_0$ are ordinary in the usual sense and $G(X)$ is isomorphic to either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$.

Definition 8.3.1.

We shall say that $X$ is ordinary if both the genus 1 curves $C_1$, $C_0$ are ordinary in the usual sense and $G(X)$ is isomorphic to one of the following finite groups:

$$(418) \quad \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^2, \quad \mathbb{Z}/3\mathbb{Z}, \quad (\mathbb{Z}/3\mathbb{Z})^2, \quad \mathbb{Z}/4\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}), \quad \mathbb{Z}/6\mathbb{Z}.$$
(ii) $X$ admits an $F^\infty$-affine structure;
(iii) $X$ is ordinary in the sense of Definition \ref{def:ordinary}.

Proof. The implication (ii) $\implies$ (i) is clear. We shall prove (iii) $\implies$ (ii). Suppose that $X$ is ordinary and $G(X) \cong \mathbb{Z}/2\mathbb{Z}$. Then, the $G(X)$-action on $C_1 \times C_0$ is given in such a way that the generator $a$ of $\mathbb{Z}/2\mathbb{Z}$ acts on $C_1 \times C_0$ by $a \cdot (c_1, c_0) = (c_1 + a, c_0)$ (cf. \cite{[1]}, List 10.27, (a_1)). The induced $G(X)$-action on $T_p(C_1^\vee \times C_0^\vee) (= T_p(C_1^\vee) \oplus T_p(C_0^\vee))$ is given by $a \cdot (b_1, b_0) = (b_1, -b_0)$. Then, the set of $G(X)$-invariant elements $(\mathbb{B}_{X,\infty}/\mathcal{S}_2)^G(X)$ in $\mathbb{B}_{X,\infty}/\mathcal{S}_2$ coincides with the subset

$$(420) \quad \left\{ ((b_1, b_0), (b_1, -b_0)) \in (T_p(C_1^\vee) \oplus T_p(C_0^\vee))^{\otimes 2} \middle| b_i \in T_p(C_i^\vee)^\times, i = 1, 2 \right\}/\mathcal{S}_2.$$  

Since this set is nonempty, it follows from Proposition \ref{prop:affine} that $X$ admits an $F^\infty$-affine structure on $X$. The existence of an $F^\infty$-affine structure in the case of $G(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is entirely similar (so we will omit the details). We complete the proof of the implication (iii) $\implies$ (ii).

Finally, we shall prove (i) $\implies$ (iii). Suppose that $X$ admits an $F^\infty$-projective structure $\mathcal{S}^\vee$. Then, both $C_1$ and $C_0$ must be ordinary. In fact, the pull-back of $\mathcal{S}^\vee$ via the natural quotient $C_1 \times C_0 \to X$ specifies an $F^\infty$-projective structure on the abelian variety $C_1 \times C_0$. It follows from Corollary \ref{cor:ordinary} that $C_1 \times C_0$ is ordinary, or equivalently, both $C_1$ and $C_0$ are ordinary. Now, we shall suppose that $G(X)$ is neither isomorphic to $\mathbb{Z}/2\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z})^2$. For example, let us consider the case where $G(X) \cong \mathbb{Z}/3\mathbb{Z}$. In this case, the $G(X)$-action on $C_1 \times C_0$ is given by $a \cdot (c_1, c_0) = (c_1 + a, \omega(c_0))$, where $\omega$ denotes a fixed generator of $\mathbb{Z}/3\mathbb{Z}$ and $\omega := \alpha(a)$ denotes an element of $\text{Aut}^0(C_0)$ of order 3. Since $\text{Aut}^0(T_p(C_0^\vee)) = \mathbb{Z}_p^\times$, there exists an element $t$ in $\mathbb{Z}_p^\times$ of order 3 such that the automorphism of $T_p(C_0^\vee)$ induced by $\omega$ is given by $b_0 \mapsto t \cdot b_0$. That is to say, the $G(X)$-action on $T_p(C_1^\vee) \oplus T_p(C_0^\vee)$ is given by $a \cdot (b_1, b_0) = (b_1, t \cdot b_0)$. Let us take an element $b := ((b_1, b_0), (b'_1, b'_0))$ of $\mathbb{B}_{X,\infty} (\subseteq (T_p(C_1^\vee) \oplus T_p(C_0^\vee))^{\otimes 2})$, where $b_1, b'_1 \in T_p(C_1^\vee)$, $b_0, b'_0 \in T_p(C_0^\vee)$. By the definition of the equivalence relation `"$\mathcal{S}^\vee$"' (cf. (321)), the element in $\delta\mathbb{B}_{X,\infty}/\mathcal{S}_2$ represented by $b$ is $G(X)$-invariant with respect to the induced $G(X)$-action if and only if the quadruple $(b_1, b_0, b'_1, b'_0)$ coincides with one of the following quadruples:

$$(421) \quad (b_1, t \cdot b_0, b'_1, t \cdot b'_0), \quad (b'_1, t \cdot b'_0, b_1, t \cdot b_0), \quad (b_1 - b'_1, t \cdot (b_0 - b'_0), -b'_1, -t \cdot b'_0), \quad (-b_1, -t \cdot b_0, b'_1 - b_1, t \cdot (b'_0 - b_0)), \quad (-b'_1, -t \cdot b'_0, b_1 - b'_1, t \cdot (b_0 - b'_0)), \quad (b'_1 - b_1, t \cdot (b'_0 - b_0), -b_1, -t \cdot b_0).$$

But, one verifies that the quadruple $(b_1, b_0, b'_1, b'_0)$ does not coincide with any of them because $b$ forms a basis of $T_p(C_1^\vee) \oplus T_p(C_0^\vee)$. That is to say, there are no $G(X)$-invariant elements of $\delta\mathbb{B}_{X,\infty}/\mathcal{S}_2$. By Proposition \ref{prop:affine} it contradicts the existence assumption of $\mathcal{S}^\vee$. Moreover, if $G(X)$ is one of the other types (i.e., $\mathbb{Z}/3$, $(\mathbb{Z}/3\mathbb{Z})^2$, $\mathbb{Z}/4\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$, $\mathbb{Z}/6\mathbb{Z}$), we also obtain a contradiction by an entirely similar discussion. Thus, we complete the proof of (i) $\implies$ (iii), and consequently, complete the proof of the proposition.

\hfill $\Box$

8.4. Properly elliptic surfaces.

Recall that a properly elliptic surface is, by definition, an elliptic surface of Kodaira dimension 1. In this subsection, we shall prove that there are no $F^\infty$-projective structures on a properly elliptic surface under a certain condition on fibers of the elliptic fibration.

Here, we shall review some well-known facts about elliptic fibrations. Basic references for elliptic fibrations in characteristic $p$ are, e.g., \cite{[1], [11], [12]}. By an elliptic fibration, we mean
a morphism of $k$-schemes $f : X \to C$, where $C$ denotes a smooth projective curve over $k$ and $f$ is proper, flat with 1-dimensional fibers, inducing $\mathcal{O}_C \cong f_*(\mathcal{O}_X)$ and with the property that the generic fiber is a smooth curve of genus 1. We say that an elliptic fibration $f : X \to C$ is (relatively) minimal if there are no exceptional $(-1)$-curves in the fibers of this fibration.

Let $X$ be a smooth projective surface equipped with a relatively minimal elliptic fibration $f : X \to C$. Then, we obtain a decomposition

$$ (422) \quad R^1f_*(\mathcal{O}_X) = L \oplus T, $$

where $L$ is a line bundle and $T$ is a torsion sheaf on $C$. Also, there exists a finite number of closed points $b_1, \cdots, b_l \in C$ ($l \geq 0$) such that, for each $i = 1, \cdots, l$, the fiber $f^{-1}(b_i)$ over $b_i$ is multiple, i.e., $f^{-1}(b_i) = m_i P_i$, with $m_i \geq 2$ and $P_i$ an indecomposable curve of canonical type (cf. [1], Definition 7.7, for the definition of an indecomposable curve of canonical type). The fibers $f^{-1}(b_i)$ ($i = 1, \cdots, l$) are called the multiple fibers of the elliptic fibration $f : X \to C$. The support $\text{Supp}(T)$ of $T$ is contained in the set $S := \{b_1, \cdots, b_l\}$. We shall refer to each fiber $f^{-1}(b)$ over $b \in \text{Supp}(T)$ as a wild fiber or an exceptional fiber of $f$. On the other hand, we refer to each fiber $f^{-1}(b)$ over $b \in S \setminus \text{Supp}(T)$ as a tame fiber of $f$.

If $X$ is a minimal elliptic surface over $k$, i.e., a smooth projective minimal surface over $k$ admitting an elliptic fibration, of Kodaira dimension 1, then the Stein factorization of the Iitaka fibration gives the unique elliptic fibration $f : X \to C$ from $X$ onto a smooth projective curve $C$ (cf. [52], Theorem 6.3).

**Proposition 8.4.1.**

Let $X$ be a smooth projective surface equipped with a relatively minimal elliptic fibration $f : X \to C$ an elliptic fibration, where $C$ denotes a smooth projective curve over $k$. Suppose that the only multiple fibers of $f$ are tame and of type $mI_0$ (according to Kodaira’s classification of singular fibers) with $p \nmid m$, i.e., multiples of a smooth fiber with multiplicity prime to $p$. Then, there exists a tamely ramified covering $\varphi : C' \to C$ of degree prime to $p$, a family of smooth genus 1 curves $f' : X' \to C'$ over $C'$, and a morphism $\pi : X' \to X$ satisfying the following two conditions:

(i) The equality $\varphi \circ f' = f \circ \pi$ holds and the $C'$-morphism $X' \to X \times_C C'$ resulting from this equality is an isomorphism over $C' \setminus \varphi^{-1}(S)$, where $S$ denotes the set of points in $C$ over which the fiber of $f$ is multiple.

(ii) $\varphi^*(R^1f_*(\mathcal{O}_X))$ is the trivial line bundle and the natural morphism $\varphi^*(R^1f_*(\mathcal{O}_X)) \to R^1f'_*(\mathcal{O}_{X'})$ is an isomorphism.

In particular, $R^1f_*(\mathcal{O}_X)$ is a torsion line bundle of order prime to $p$.

**Proof.** Let us consider the former assertion. We shall write $S := \{b_1, \cdots, b_l\}$ ($l \geq 0$). Denote by $\eta$ the generic point of the fiber $f$, by $K$ its residue field, and by $X_\eta := f^{-1}(\eta)$. Also, for each $i = 1, \cdots, l$, denote by $m_i$ the multiplicity of the fiber $f^{-1}(b_i)$ (hence $p \nmid m_i$) and by $\hat{K}_i$ ($\supseteq K$) the fraction field of the formal completion $\hat{R}_i := \hat{O}_{C,b_i}$ of the local ring $\mathcal{O}_{C,b_i}$. Recall from [60], §3 (or, [51], §6), that there exists a cyclic extension $\hat{K}'_i/\hat{K}_i$ of degree $m_i$ by which the multiple fiber $f^{-1}(b_i)$ can be resolved; that is to say, if $\hat{R}'_i$ denotes the integral closure of $\hat{R}_i$ in $K'_i$ and $X'_i$ denotes the minimal regular model of $X_\eta \times_K \hat{K}_i$, then the closed fiber of $X'_i/\hat{R}'_i$ is non-multiple, i.e., $X'_i/\hat{R}'_i$ is a family of elliptic curves, and the natural projection $X'_i \to X \times_C \hat{R}_i$ is étale. Let us recall here the well-known structure of the maximal pro-prime-to-$p$ quotient of the abelianization of the étale fundamental group.
of $C$ with some points removed. Then, we can find a tamely ramified Galois covering $C_i'/C$ of degree prime to $p$ which is ramified outside $S \setminus \{b_i\}$ and such that for every point $b'$ in $C_i'$ over $b_i$, the canonical homomorphism $\hat{\mathcal{O}}_{C,b_i} \to \hat{\mathcal{O}}_{C',b'}$ is isomorphic to the inclusion $\hat{R}_i \hookrightarrow \hat{R}_i'$. In particular, $X_i'/\hat{R}_i'$ specifies a smooth genus 1 curve $X_i'/b'$ over $\text{Spec}(\hat{\mathcal{O}}_{C,b_i})$. Let us take one component $C'$ of $C_1' \times_C C_2' \times_C \cdots \times_C C_l'$ and denote by $\varphi : C' \to C$ (resp., $\varphi' : C_i' \to C$) the natural projection, which is a tamely ramified covering of degree prime to $p$. Then, the $k$-schemes $X_i'/b' \times_C C'$ ($i \in \{1,\ldots,l\}$, $b' \in \varphi_i^{-1}(b_i)$) and $X \times_C (C' \setminus \varphi^{-1}(S))$ may be glued together to obtain a family of elliptic curves $f' : X' \to C'$ over $C'$ satisfying the condition (i) by letting $\pi$ to be the projection $X' \to X$. In what follows, we shall prove the claim that the natural morphism

$$(423) \quad \varphi^*(\mathbb{R}^1 f_*(\mathcal{O}_X)) \to \mathbb{R}^1 f'_*(\mathcal{O}_{X'})$$

is an isomorphism. By the definition of $X'$, this morphism is immediately verified to be an isomorphism outside $\varphi^{-1}(S)$. For each point $s$ in $\varphi^{-1}(S)$, the fiber of $(423)$ over $s$ may be identified with $k$-linear morphism $H^1(f^{-1}(\varphi(s)), \mathcal{O}_{f^{-1}(\varphi(s))}) \to H^1(f'^{-1}(s), \mathcal{O}_{f'^{-1}(s)})$ induced by $\pi$ (cf. [27], Chap. III, §12, Theorem 12.11). Since $\pi$ is an étale covering of degree prime to $p$, one verifies that this $k$-linear morphism is surjective. Hence, $(423)$ is surjective at $s$ because of Nakayama’s lemma, and hence is an isomorphism at $s$. This completes the proof of the claim.

By taking account of the above discussion, the proof of the proposition is reduced, after replacing $C$ and $X$ by $C'$ and $X'$ respectively, to the case where $f : X \to C$ is a family of smooth genus 1 curves. Let $J := \text{Pic}^0(X/C)$ be the Jacobian of $X/C$, and denote by $e : C \to J$ the zero section. We know that $\mathbb{R}^1 f_*(\mathcal{O}_X) \cong e^*(\mathcal{T}_{J/C})$ (cf. [54], Proposition 1.3), where $\mathcal{T}_{J/C}$ denotes the relative tangent bundle of $J$ over $C$. Hence, in order to complete the proof, it suffices to prove that the line bundle $\mathcal{T}_{J/C}$ becomes trivial after base-change by some finite étale covering over $C$ of degree prime to $p$. Let us consider the 3-torsion subgroup scheme $J[3]$ of $J$, which is finite and étale over $C$. By the assumption $p > 3$, there exists an étale Galois covering $C' \to C$ of degree prime to $p$ such that $J[3] \times_C C' \cong (\mathbb{Z}/3\mathbb{Z})^2$. By [72], Proposition 3.2, there exists a smooth projective genus 1 curve $E$ over $k$ with $J \times_C C' \cong E \times C' =: J'$. Since the natural projection $\pi_J : J' \to J$ is étale, we have $\pi_J^*(\mathcal{T}_{J/C}) \cong \mathcal{T}_{J'/C'} \cong \mathcal{O}_{J'}$. This completes the proof of the former assertion.

The latter assertion follows directly from the former assertion and Lemma 3.6.2. \[\square\]

By using the above proposition, we prove the main result of this subsection, described as follows.

**Proposition 8.4.2.**

Let $X$ be a properly elliptic surface such that every multiple fiber of the elliptic fibration is tame and has multiplicity prime to $p$. Then, $X$ admits no $F^\infty$-projective structures.

**Proof.** Denote by $f : X \to C$ the elliptic fibration of $X$, where $C$ is a smooth projective curve over $k$ of genus $\geq 2$. Suppose that $X$ admits an $F^\infty$-projective structure $S^\infty$. It follows from Theorem 8.2.2 that $X$ does not contain any rational curve, so the only multiple fibers of $f$ are of type $m_i I_i$, i.e., multiples of a smooth fiber. By Proposition 8.4.1, $\mathcal{L} := \mathbb{R}^1 f_*(\mathcal{O}_X)$ has degree 0. Set $D := \sum_{i=1}^l (m_i - 1) F_i$, where $l$ denotes a nonnegative integer and the $F_i$’s are the multiple fibers with multiplicities $m_i$ ($i = 1, 2, \cdots, l$). According to [1], §7, Theorem 7.15,
there exists an isomorphism
\[ \omega_X \cong f^*(\mathcal{L}^\vee \otimes \Omega_C) \otimes \mathcal{O}_X(D). \]

The morphism \( f^*(\Omega_C) \to \Omega_X \) induced by \( f \) factorizes over \( f^*(\Omega_C)(D) \) and induces, by using the isomorphism \((424)\), the short exact sequence
\[ 0 \to f^*(\Omega_C) \otimes \mathcal{O}_X(D) \to \Omega_X \to f^*(\mathcal{L}^\vee) \to 0. \]

In particular, \( f^*(\mathcal{L}^\vee) \) is isomorphic to the sheaf \( \Omega_{X/C} \) divided by its torsion subgroup.

Now, let us take a smooth projective curve \( C' \) equipped with a finite and generically étale morphism \( \varphi : C' \to C \) and a \( C \)-morphism \( \iota : C' \to X \). Consider the short exact sequence
\[ 0 \to \varphi^*(\mathcal{L}) \to \iota^*(\mathcal{T}_X) \to \varphi^*(\mathcal{T}_C) \otimes \iota^*(\mathcal{O}_X(-D)) \to 0 \]

obtained as the dual of the pull-back of \((425)\) by \( \iota \). The composite of the differential \( d\iota : \mathcal{T}_{C'} \to \iota^*(\mathcal{T}_X) \) and the third arrow \( \iota^*(\mathcal{T}_X) \to \varphi^*(\mathcal{T}_C) \otimes \iota^*(\mathcal{O}_X(-D)) \) in \((426)\) is injective because it arises from \( \varphi \) via differentiation. Hence, if \( \mathcal{M} \) denotes the saturation of \( \text{Im}(d\iota) \) in \( \iota^*(\mathcal{T}_X) \), then the composite \( \varphi^*(\mathcal{L}) \hookrightarrow \iota^*(\mathcal{T}_X) \to \iota^*(\mathcal{T}_X)/\mathcal{M} \) is injective, which implies the inequality
\[ \text{deg}(\iota^*(\mathcal{T}_X)/\mathcal{M}) \geq (\text{deg}(\varphi^*(\mathcal{L})) = 0. \]

Also, since \( C \) has genus \( \geq 2 \), the sequence \((426)\) implies
\[ \text{deg}(\iota^*(\mathcal{T}_X)) = \text{deg}(\varphi^*(\mathcal{L})) + \text{deg}(\varphi^*(\mathcal{T}_C)) + \text{deg}(\iota^*(\mathcal{O}_X(-D))) < 0. \]

By \((427)\) and \((428)\), we have
\[ \text{deg}(\mathcal{M}) < 0. \]

Next, let us take a dormant indigenous \( \mathcal{D}_{X}^{(\infty)} \)-module \( \mathcal{V} := (\mathcal{V}, \nabla, \mathcal{N}) \) with \( \mathcal{V}^{\otimes_{\infty}} = \mathcal{S}^{\otimes} \). By passing to \( \text{KS}_{\mathcal{V}}^{\otimes} \) (cf. \((81)\)), we shall identify \( \mathcal{V} \) with \( \mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N} \). The pull-back \( \iota^*(\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N}) \) (= \( \iota^*(\mathcal{V}) \)) has a structure of \( \mathcal{D}_{C}^{(\infty)} \)-action \( \iota^*(\nabla) \) induced by \( \nabla \). Denote by \( \widetilde{\mathcal{M}}_{\mathcal{N}} \) the inverse image of \( \mathcal{M} \otimes \mathcal{N} \) \( \subseteq \mathcal{T}_X \otimes \mathcal{N} \) via the quotient \( \mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N} \to (\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N}) = \mathcal{T}_X \otimes \mathcal{N} \). If, moreover, \( \mathcal{U} \) denotes the \( \mathcal{D}_{C}^{(\infty)} \)-submodule of \( \iota^*(\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N}) \) (with respect to \( \iota^*(\nabla) \)) generated by the line subbundle \( \iota^*(\mathcal{N}) \) \( (= \iota^*(\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N})) \), then \( \mathcal{U} \) is, by construction, contained in \( \widetilde{\mathcal{M}}_{\mathcal{N}} \) and coincides with \( \widetilde{\mathcal{M}}_{\mathcal{N}} \) at the generic point. The quotient \( \iota^*(\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N})/\mathcal{U} \) has a structure of \( \mathcal{D}_{C}^{(\infty)} \)-action induced by \( \iota^*(\nabla) \). But, since any \( \mathcal{D}_{C}^{(\infty)} \)-module is locally free (cf. \[6\], Note 2.17), \( \iota^*(\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N})/\mathcal{U} \) forms a line bundle, and hence, \( \mathcal{U} \) must be equal to \( \widetilde{\mathcal{M}}_{\mathcal{N}} \). By taking account of the resulting \( \mathcal{D}_{C}^{(\infty)} \)-action on \( (\iota^*(\mathcal{T}_X)/\mathcal{M}) \otimes \mathcal{N} \) \( (= \iota^*(\mathcal{D}_{X,1}^{(\infty)} \otimes \mathcal{N})/\mathcal{U}) \), we see (cf. \[20\], Theorem 1.8) that the line bundle \( (\iota^*(\mathcal{T}_X)/\mathcal{M}) \otimes \mathcal{N} \) has degree 0. By this fact together with \((427)\), the inequality
\[ \text{deg}(\mathcal{N}) \leq 0 \]

holds. Recall that \( \mathcal{U} \) \( (= \widetilde{\mathcal{M}}_{\mathcal{N}}) \) may be obtained as an extension of \( \mathcal{M} \otimes \mathcal{N} \) by \( \mathcal{N} \). Then, the inequalities \((429)\) and \((430)\) implies the inequality \( \text{deg}(\mathcal{U}) < 0 \), which contradicts the fact that \( \mathcal{U} \) admits a \( \mathcal{D}_{C}^{(\infty)} \)-action, i.e., the restriction of \( \iota^*(\nabla) \) (cf. \[20\], Theorem 1.8). Thus, we conclude that \( X \) cannot have any \( F^{\infty} \)-projective structure, and this completes the proof of the proposition. \( \square \)
Remark 8.4.3.
T. Katsura and K. Ueno (cf. [41]) studied multiple fibers of elliptic surfaces in positive characteristic. For example, they studied how to reduce a wild fiber to a tame fiber and moreover to a non-multiple fiber by pulling-back the elliptic fibration to a certain covering of a given elliptic surface. In particular, if the support of a multiple fiber is an ordinary elliptic curve or of type \( mI_n \) \((n \geq 0)\), then a covering used in this procedure is taken to be an étale Galois covering with Galois group \( \mathbb{Z}/p\mathbb{Z} \). By means of this result, the above proposition may be slightly generalized to the case where the elliptic fibration has certain wild fibers. To be precise, let \( X \) be a properly elliptic surface satisfying the following two conditions, where \( f : X \to C \) denotes the elliptic fibration:

(i) If a fiber \( f^{-1}(b) \) \((b \in C)\) is tame, then its multiplicity is prime to \( p \);
(ii) If a fiber \( f^{-1}(b) \) \((b \in C)\) is wild, then it is a multiple of an ordinary elliptic curve and becomes a tame fiber with multiplicity prime to \( p \) after successive applications of Katsura and Ueno’s resolution of multiple fibers.

Then, \( X \) admits no \( F^\infty \)-projective structures. Indeed, by [41], Theorem, there exists an étale covering \( \pi : X' \to X \) over \( X \) such that \( X' \) forms a properly elliptic surface satisfying the assumption in Proposition 8.4.2. If \( X \) is assumed to have an \( F^\infty \)-projective structure, then its pull-back by \( \pi \) specifies an \( F^\infty \)-projective structure on \( X' \). But, by applying Proposition 8.4.2 to \( X' \), we obtain a contradiction. Consequently, we obtain the italicized conclusion above.

Remark 8.4.4.
In the case of \( N < \infty \), we can give an example of an \( F^N \)-affine structure on a properly elliptic surface. In fact, a simple example of a properly elliptic surface is provided by the product of two curves, one of genus 1 and the other of genus \( g \geq 2 \). If we take an ordinary genus 1 curve \( X_1 \) and a smooth projective curve \( X_2 \) of genus \( g > 1 \) admitting a Tango function of level \( N \) (cf. the comment preceding Proposition 7.5.2), then both \( X_1 \) and \( X_2 \) admit an \( F^N \)-affine structure. Therefore, by Proposition 8.1.2, the properly elliptic surface defined as the product \( X_1 \times X_2 \) admits an \( F^N \)-affine structure.

8.5. Classification of minimal surfaces.
Now, we recall the Kodaira-Enriques classification table to examine which surfaces admit an \( F^N \)-projective or \( F^N \)-affine structure. Starting from the rough classification by Kodaira dimension, minimal surfaces are divided into 11 classes (cf. [67], [11], [12]). Indeed, every smooth projective minimal surface is one of the following classes:

(i) Kodaira dimension \( -\infty \):
   - rational surfaces;
   - nonrational ruled surfaces;
(ii) Kodaira dimension \( 0 \):
   - Enriques surfaces;
   - nonclassical Enriques surfaces (only if \( \text{char}(k) = 2 \));
   - hyperelliptic surfaces;
   - quasi-hyperelliptic surfaces (only if \( \text{char}(k) = 2, 3 \));
   - K3 surfaces;
   - Abelian surfaces;
(iii) Kodaira dimension 1:
- properly elliptic surfaces;
- properly quasi-elliptic surfaces (only if char($k$) = 2, 3);
(iv) Kodaira dimension 2:
- surfaces of general type;

Since $p (= \text{char}(k)) > 3$, we will not consider nonclassical Enriques surfaces, quasi-hyperelliptic surfaces, and properly quasi-elliptic surfaces.

**Rational surfaces:**

Any minimal rational surface is isomorphic to the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $H_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ with $d \neq 1$ (cf. [52], Theorem 4.5, (ii)). For $d \neq 1$, we have $c_1(H_d)^2 = 8$, $c_2(H_d) = 4$, so $c_1(H_d)^2 - 3 \cdot c_2(H_d) = -4$. Hence, it follows from Theorem 8.2.1 (i), that, for any $N \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$, the Hirzebruch surface $H_d$ admits no $F^N$-projective structures, in particular, admits no $F^N$-affine structures. We are also able to conclude this fact for $N = \infty$ by means of Theorem 8.2.2. On the other hand, according to Proposition 5.1.1, the projective plane $\mathbb{P}^2$ admits a unique $F^N$-projective structure, but admits no $F^N$-affine structures for any $N \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$.

**Nonrational ruled surfaces:**

Any minimal nonrational surface is isomorphic to the ruled surface $\mathbb{P} (\mathcal{E})$ associated to a rank 2 vector bundle $\mathcal{E}$ on a smooth projective curve $C$ over $k$ of genus $g \neq 0$ (cf. [52], Theorem 4.5, (i)). For $C$ and $\mathcal{E}$ as above, we have $c_1(\mathbb{P}(\mathcal{E}))^2 = 8 \cdot (1 - g)$, $c_2(\mathbb{P}(\mathcal{E})) = 4 \cdot (1 - g)$, so $c_1(\mathbb{P}(\mathcal{E}))^2 - 3 \cdot c_2(\mathbb{P}(\mathcal{E})) = -4 \cdot (1 - g)$. Hence, it follows from Theorem 8.2.1 (i), that, for any positive integer $N$, the ruled surface $\mathbb{P}(\mathcal{E})$ admits an $F^N$-projective structure unless $p^N \mid (g - 1)$. Moreover, since $\mathbb{P}(\mathcal{E})$ contains a rational curve and is not isomorphic to $\mathbb{P}^2$, it admits no $F^\infty$-projective structures, in particular, admits no $F^\infty$-affine structures, because of Theorem 8.2.2.

**Enriques surfaces:**

Since $p \neq 2$, each Enriques surface $X$ satisfy the equalities $c_1(X)^2 = 0$ and $c_2(X) = 12$ (cf. [52], §7.3). Hence, we have $c_1(X)^2 - 3 \cdot c_2(X) = -36 (= -2^2 \cdot 3^2)$. This implies (cf. Theorem 8.2.1 (i)) that $X$ admits no $F^N$-projective structures, in particular, admits no $F^N$-affine structures, for any $N \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$.

**Hyperelliptic surfaces:**

As we already examined, a hyperelliptic surface $X$ admits an $F^\infty$-projective structure (or an $F^\infty$-affine structure) if and only if $X$ is ordinary in the sense of Definition 8.3.1 (cf. Proposition 8.3.3). In particular, any ordinary hyperelliptic surface admits an $F^N$-projective structure, as well as an $F^N$-affine structure, for any positive integer $N$.

**K3 surfaces:**

Any K3 surface $X$ satisfies the equalities $c_1(X)^2 = 0$ and $c_2(X) = 24$ (cf. [52], §7.2). Hence, we have $c_1(X)^2 - 3 \cdot c_2(X) = -72 (= -2^3 \cdot 3^2)$. This implies (cf. Theorem 8.2.1 (i)) that $X$ admits no $F^N$-projective structures, in particular, admits no $F^N$-affine structures, for any $N \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$.

**Abelian surfaces:**
As studied in §6, an abelian surface $X$ admits an $F^\infty$-projective (resp., $F^\infty$-affine) structure if and only if $X$ is ordinary (cf. Corollary 6.3.3). Thus, any ordinary abelian surface admits an $F^N$-projective structure, as well as an $F^N$-affine structure, for any positive integer $N$. Regarding nonordinary abelian varieties, we have proved Proposition 7.5.1, (ii), asserting that any supersingular genus 1 curve has a unique $F^1$-affine structure, but does not have any $F^N$-affine structure for any $N \geq 2$. By taking account of this fact, let us, for example, take $X := X_1 \times X_2$, where $X_1$ denotes a supersingular genus 1 curve and $X_2$ denotes an arbitrary smooth projective genus 1 curve. Then, $X$ is nonordinary, but it follows from Proposition 8.1.2 that $X$ admits an $F^1$-affine structure, in particular, admits an $F^1$-projective structure.

Properly elliptic surfaces:
As discussed in the previous subsection, we have obtained a sufficient condition for the nonexistence of $F^\infty$-projective structures on properly elliptic surfaces. In fact, if $X$ is a properly elliptic surface such that every multiple fiber of the elliptic fibration is tame and has multiplicity prime to $p$, then $X$ admits no $F^\infty$-projective structures. According to Remark 8.4.3 there is a slight generalization of this result to the case where the elliptic fibration has certain wild fibers. Regarding finite levels, we may find examples of properly elliptic surface admitting an $F^N$-affine structure for every positive integer $N$ (cf. Remark 8.4.4 and [32], Theorem 3).

General type surfaces:
According to Theorem 8.2.1, (ii), a minimal surface $X$ of general type admits an $F^N$-projective (resp., $F^N$-affine) structure for $N \in \mathbb{Z}_{>0}$ only if $c_1(X)^2 - 3 \cdot c_2(X) = 0$ (resp., $c_1(X) = c_2(X) = 0$) mod $p^N$. We can construct some examples of general type surfaces admitting an $F^N$-affine structure. In fact, given a positive integer $N$, let us take two smooth projective curves $X_1$, $X_2$ of genus $> 1$ each of which admits a Tango function of level $N$, or equivalently an $F^N$-affine structure (cf. the comment preceding Proposition 7.5.2 and [32], Theorem 3). Then, their product $X_1 \times X_2$ is of general type and admits an $F^N$-affine structure (cf. Proposition 8.1.2).

We shall summarize, as stated below, a part of the results obtained so far, but restrict ourselves to the case of infinite level, for the sake of simplicity.

**Corollary 8.5.1 (cf. Theorem F).**
Suppose that $p = \text{char}(k) > 3$, and let $X$ be a smooth projective surface over $k$. Then, $X$ admits no $F^\infty$-projective structures unless $X$ is minimal. Moreover, if $X$ is minimal, then we have the following assertions:

(i) Suppose that $X$ is a rational surface. Then, $X$ admits an $F^\infty$-projective structure if and only if $X$ is isomorphic to the projective plane $\mathbb{P}^2$. Moreover, there are no $F^\infty$-affine structures on any rational surface.

(ii) Suppose that $X$ is a nonrational ruled surface. Then, $X$ admits no $F^\infty$-projective structures.

(iii) Suppose that $X$ is an Enriques surface. Then, $X$ admits no $F^\infty$-projective structures.

(iv) Suppose that $X$ is a hyperelliptic surface. Then, $X$ admits an $F^\infty$-projective (resp., $F^\infty$-affine) structure if and only if $X$ is ordinary in the sense of Definition 8.3.1.

(v) Suppose that $X$ is a K3 surface. Then, $X$ admits no $F^\infty$-projective structures.
Suppose that $X$ is an abelian surface. Then, $X$ admits an $F^\infty$-projective (resp., an $F^\infty$-affine) structure if and only if $X$ is ordinary.

Suppose that $X$ is a properly elliptic surface. Then, $X$ admits no $F^\infty$-projective structures if every multiple fiber of the elliptic fibration of $X$ is tame and has multiplicity prime to $p$.

Suppose that $X$ is a surface of general type. Then, $X$ admits no $F^N$-projective (resp., $F^\infty$-affine) structures unless the equality $c_1^{\text{crys}}(X)^2 - 3 \cdot c_2^{\text{crys}}(X) = 0$ holds (resp., the equalities $c_1^{\text{crys}}(X) = c_2^{\text{crys}}(X) = 0$ hold) in the crystalline cohomology $H^*_{\text{crys}}(X/W)$ of $X$.

Remark 8.5.2.

We shall mention the result (viii) in the above theorem. Let $X$ be a complex compact minimal surface of general type. Then, it is well-known (see [61], [85]) that $X$ satisfies the Bogomolov-Miyaoka-Yau inequality $c_1(X) - 3 \cdot c_2(X) \leq 0$. Moreover, the equality sign holds if and only if $X$ can be obtained as a quotient of the 2-dimensional hyperbolic space $\mathbb{H}_C^2$, i.e., a manifold belonging to the third class of the standard examples displayed in §0.1 (cf. [28], [62], [85]). This implies that the equality $c_1(X) - 3 \cdot c_2(X) = 0$ holds if and only if $X$ admits a projective structure. By taking account of this fact, we expect that the necessarily condition of the existence of $F^\infty$-projective structures described in (viii) is in fact (almost) sufficient. That is to say, we conjecture that a minimal smooth projective surface of general type (satisfying some reasonable conditions) admits an $F^\infty$-projective structure if the equality $c_1^{\text{crys}}(X)^2 - 3 \cdot c_2^{\text{crys}}(X) = 0$ holds in $H^1_{\text{crys}}(X/W)$ (or the equality $c_1(X) - 3 \cdot c_2(X) = 0$ holds in $CH^2(X)$). But, at the time of writing the present paper, the author does not have any idea how to approach this conjectural assertion.

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