Schützenberger Products in a Category

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Abstract The Schützenberger product of monoids is a key tool for the algebraic treatment of language concatenation. In this paper we generalize the Schützenberger product to the level of monoids in an algebraic category $\mathcal{D}$, leading to a uniform view of the corresponding constructions for monoids (Schützenberger), ordered monoids (Pin), idempotent semirings (Klíma and Polák) and algebras over a field (Reutenauer). In addition, assuming that $\mathcal{D}$ is part of a Stone-type duality, we derive a characterization of the languages recognized by Schützenberger products.

1 Introduction

Since the early days of automata theory, it has been known that regular languages are precisely the languages recognized by finite monoids. This observation is the origin of algebraic language theory. One of the classical and ongoing challenges of this theory is the algebraic treatment of the concatenation of languages. The most important tool for this purpose is the Schützenberger product $M \diamond N$ of two monoids $M$ and $N$, introduced in \cite{two.prop}. Its key property is that it recognizes all marked products of languages recognized by $M$ and $N$. Later, Reutenauer \cite{two.prop/one.prop} showed that $M \diamond N$ is the “smallest” monoid with this property: any language recognized by $M \diamond N$ is a boolean combination of such marked products.

In the past decades, the original notion of language recognition by finite monoids has been refined to other algebraic structures, namely to ordered monoids by Pin \cite{one.prop}, to idempotent semirings by Polák \cite{eight.prop}, and to associative algebras over a field by Reutenauer \cite{zero.prop}. For all these structures, a Schützenberger product was introduced separately \cite{four.prop,seven.prop,zero.prop}. Moreover, Reutenauer’s characterization of the languages recognized by Schützenberger products has been adapted to ordered monoids and idempotent semirings, replacing boolean combinations by positive boolean combinations \cite{four.prop} and finite unions \cite{seven.prop}, respectively.

This paper presents a unifying approach to Schützenberger products, covering the aforementioned constructions and results as special cases. Our starting point is the observation that all the algebraic structures appearing above (monoids, ordered monoids, idempotent semirings, and algebras over a field $K$) are monoids

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interpreted in some variety \( \mathcal{D} \) of algebras or ordered algebras, viz. \( \mathcal{D} = \) sets, posets, semilattices, and \( \mathbb{K} \)-vector spaces, respectively. Next, we note that these categories \( \mathcal{D} \) are related to the category \( \mathcal{S}\text{-Mod} \) of modules over some semiring \( \mathcal{S} \). Indeed, semilattices and vector spaces are precisely modules over the two-element idempotent semiring \( \mathcal{S} = \{0, 1\} \) and the field \( \mathcal{S} = \mathbb{K} \), respectively. And every set or poset freely generates a semilattice (i.e. a module over \( \{0, 1\} \)), viz. the semilattice of finite subsets or finitely generated down-sets. Precisely speaking, each of the above categories \( \mathcal{D} \) admits a monoidal adjunction

\[
\begin{array}{c}
\mathcal{S}\text{-Mod} \\
\xrightarrow{\top} \\
\vdash \\
\xleftarrow{F} \\
\mathcal{D}
\end{array}
\]

for some semiring \( \mathcal{S} \), where \( U \) is a forgetful functor and \( F \) is a free construction.

In this paper we introduce the Schützenberger product at the level of an abstract monoidal adjunction \((1.1)\): for any two \( \mathcal{D} \)-monoids \( M \) and \( N \), we construct a \( \mathcal{D} \)-monoid \( M \circ N \) that recognizes all marked products of languages recognized by \( M \) and \( N \) (Theorem 4.8), and prove that \( M \circ N \) is the “smallest” \( \mathcal{D} \)-monoid with this property (Theorem 4.13). Further, we derive a characterization of the languages recognized by \( M \circ N \) in the spirit of Reutenauer’s theorem \([21]\). To this end, we consider another variety \( \mathcal{C} \) that is dual to \( \mathcal{D} \) on the level of finite algebras. For example, for \( \mathcal{D} = \) sets we choose \( \mathcal{C} = \) boolean algebras, since Stone’s representation theorem gives a dual equivalence between finite boolean algebras and finite sets. We then prove that every language recognized by \( M \circ N \) is a “\( \mathcal{C} \)-algebraic combination” of languages recognized by \( M \) and \( N \) and their marked products (Theorem 4.16). The explicit use of duality makes our proof conceptually different from the original ones.

By instantiating \((1.1)\) to the proper adjunctions, we recover the Schützenberger product for monoids, ordered monoids, idempotent semirings and algebras over a field, and obtain a new Schützenberger product for algebras over a commutative semiring. Moreover, our Theorems 4.8 and 4.16 specialize to the corresponding results \([14, 17, 21]\) for (ordered) monoids and idempotent semirings. In the case of \( \mathbb{K} \)-algebras, Theorem 4.16 appears to be a new result. Apart from that, we believe that the main contribution of our paper is the identification of a categorical setting for language concatenation. We hope that the generality and the conceptual nature of our approach can contribute to an improved understanding of the various ad hoc constructions and separate results appearing in the literature.

**Related work.** In recent years, categorical approaches to algebraic language theory have been a growing research topic. The present paper is a natural continuation of \([2]\), where we showed that the construction of syntactic monoids works at the level of \( \mathcal{D} \)-monoids in any commutative variety \( \mathcal{D} \), allowing for a uniform treatment of syntactic (ordered) monoids, idempotent semirings and algebras over a field. The systematic use of duality in algebraic language theory originates in the work of Gehrke, Grigorieff, and Pin \([11]\), who interpreted Eilenberg’s variety theorem in terms of Stone duality. In our papers \([3, 10]\) we extended their approach to an abstract Stone-type duality, leading to a uniform view of several Eilenberg-type
Theorems for regular languages. See also [24] for related duality-based work. Recently, Bojańczyk [7] proposed to use *monads* instead of monoids to get a categorical grasp on languages beyond finite words. By combining this idea with our duality framework, we established in [9,25] a variety theorem that covers most Eilenberg-type correspondences known in the literature, e.g. for languages of finite words, infinite words, words on linear orderings, trees, and cost functions.

2 Preliminaries

In this paper we study monoids and language recognition in algebraic categories. The reader is assumed to be familiar with basic universal algebra and category theory; see the Appendix for a toolkit. We call a variety $\mathcal{V}$ of algebras or ordered algebras *commutative* if, for any two algebras $A, B \in \mathcal{V}$, the set $[A, B]$ of morphisms from $A$ to $B$ forms an algebra of $\mathcal{V}$ with operations taken pointwise in $B$. Our applications involve the commutative varieties $\text{Set}$ (sets), $\text{Pos}$ (posets, as join-semilattices with $0$), $K\text{-Vec}$ (vector spaces over a field $K$) and $\mathbb{S}\text{-Mod}$ (modules over a commutative semiring $\mathbb{S}$ with $0, 1$). Note that $\text{JSL}$ and $K\text{-Vec}$ are special cases of $\mathbb{S}\text{-Mod}$ for $\mathbb{S} = \{0, 1\}$, the two-element semiring with $1 + 1 = 1$, and $\mathbb{S} = K$, respectively.

**Notation 2.1.** Let $\mathcal{A} / two.prop, \mathcal{R} / two.prop, \mathcal{C} / two.prop, \mathcal{D} / two.prop$ always denote commutative varieties of algebras or ordered algebras. We write $\Psi = \Psi_{\mathcal{D}} : \text{Set} \to \mathcal{D}$ for the left adjoint to the forgetful functor $[-] : \mathcal{D} \to \text{Set}$; thus $\Psi X$ is the free algebra of $\mathcal{D}$ over $X$. For simplicity, we assume that $X$ is a subset of $|\Psi X|$ and the universal map $X \to |\Psi X|$ is the inclusion. Denote by $1_{\mathcal{D}} = \Psi 1$ the free one-generated algebra.

**Example 2.2.** (1) For $\mathcal{D} = \text{Set}$ or $\text{Pos}$ we have $\Psi X = X$ (discretely ordered).

(2) For $\mathcal{D} = \text{JSL}$ we get $\Psi X = (\mathcal{P}_X, \cup)$, the semilattice of finite subsets of $X$.

(3) For $\mathcal{D} = \mathbb{S}\text{-Mod}$ we have $\Psi X = \mathbb{S}(X)$, the $\mathbb{S}$-module of all finite-support functions $X \to \mathbb{S}$ with sum and scalar product defined pointwise.

**Definition 2.3.** Let $A, B, C \in \mathcal{D}$. By a *bimorphism* from $A, B$ to $C$ is meant a function $f : |A| \times |B| \to |C|$ such that the maps $f(a, -) : |B| \to |C|$ and $f(-, b) : |A| \to |C|$ carry morphisms of $\mathcal{D}$ for every $a \in |A|$ and $b \in |B|$. An *tensor product* of $A$ and $B$ is a universal bimorphism $t_{A,B} : |A| \times |B| \to |A \otimes B|$, in the sense that for any bimorphism $f : |A| \times |B| \to |C|$ there is a unique $f' : A \otimes B \to C$ in $\mathcal{D}$ with $f' \circ t_{A,B} = f$. We denote by $a \otimes b$ the element $t_{A,B}(a, b) \in |A \otimes B|$. In $\text{Set}$ and $\text{Pos}$ we have $A \otimes B = A \times B$. In $\mathbb{S}\text{-Mod}$, $A \otimes B$ is the usual tensor product of $\mathbb{S}$-modules, and $t_{A,B}$ is the universal $\mathbb{S}$-bilinear map.

**Remark 2.5.** (1) Tensor products exist in any commutative variety $\mathcal{D}$, see [5].

(2) $\otimes$ is associative and has unit $1_{\mathcal{D}}$; that is, there are natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C), \quad \rho_A : A \otimes 1_{\mathcal{D}} \cong A, \quad \lambda_A : 1_{\mathcal{D}} \otimes A \cong A.$$

(3) Given $f : A \to C$ and $g : B \to D$ in $\mathcal{D}$, denote by $f \otimes g : A \otimes B \to C \otimes D$ the morphism induced by the bimorphism $|A| \times |B| \xrightarrow{f \times g} |C| \times |D| \xrightarrow{|C \otimes D|}$. 

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Definition 2.6. A \( \mathcal{D} \)-monoid is a triple \((M, 1, \bullet)\) where \( M \) is an object of \( \mathcal{D} \) and \(|M|, 1, \bullet\) is a monoid whose multiplication \(|M| \times |M| \to |M|\) is a bimorphism of \( \mathcal{D} \). A morphism \( h: (M, 1_M, \bullet_M) \to (N, 1_N, \bullet_N)\) of \( \mathcal{D} \)-monoids is a morphism \( h: M \to N \) in \( \mathcal{D} \) with \( h(1_M) = 1_N \) and \( h(m \bullet_M m') = h(m) \bullet_N h(m') \) for \( m, m' \in |M| \). We denote the category of \( \mathcal{D} \)-monoids by \( \text{Mon}(\mathcal{D}) \).

Example 2.7. Monoids in \( \mathcal{D} = \text{Set} \), \( \text{Pos} \), \( \text{JSL} \) and \( \mathbb{S} \)-\text{Mod} are precisely monoids, ordered monoids, idempotent semirings, and associative algebras over \( \mathbb{S} \).

Proposition 2.8 (see [1]). The free \( \mathcal{D} \)-monoid on a set \( \Sigma \) is carried by \( \Psi \Sigma^* \in \mathcal{D} \), the free algebra in \( \mathcal{D} \) on the set \( \Sigma^* \) of finite words over \( \Sigma \). Its multiplication extends the concatenation of words in \( \Sigma^* \), and its unit is the empty word \( \varepsilon \).

Example 2.9. (1) In \( \mathcal{D} = \text{Set} \) or \( \text{Pos} \) we have \( \Psi \Sigma^* = \Sigma^* \) (discretely ordered).

(2) In \( \mathcal{D} = \text{JSL} \) we have \( \Psi \Sigma^* = \mathcal{P} \Sigma^* \), the idempotent semiring of all finite languages over \( \Sigma \) w.r.t. union and concatenation of languages.

(3) In \( \mathcal{D} = \text{Mod}(\mathbb{S}) \) we get \( \Psi \Sigma^* = \mathbb{S}[\Sigma] \), the \( \mathbb{S} \)-algebra of all polynomials \( \Sigma_n^* w_i \), (equivalently, finite-support functions \( c: \Sigma^* \to \mathbb{S} \)) w.r.t. the usual sum, scalar product and multiplication of polynomials.

Remark 2.10. Since the multiplication \( \bullet: |M| \times |M| \to |M| \) of a \( \mathcal{D} \)-monoid \((M, 1, \bullet)\) forms a bimorphism, it corresponds to a morphism \( \mu_M: M \otimes M \to M \) in \( \mathcal{D} \), mapping \( m \otimes m' \in |M| \otimes |M| \) to \( m \bullet m' \in |M| \). Likewise, the unit \( 1 \in |M| \) corresponds to the morphism \( \iota_M: 1_M \to M \) sending the generator of \( 1_M \) to 1. We can thus represent a \( \mathcal{D} \)-monoid \((M, 1, \bullet)\) as the triple \((M, \iota_M, \mu_M)\).

Remark 2.11. For any two \( \mathcal{D} \)-monoids \( M \) and \( N \), the tensor product \( M \otimes N \) in \( \mathcal{D} \) carries a \( \mathcal{D} \)-monoid structure with unit \( 1 \otimes 1 \) and multiplication \((M \otimes N) \otimes (M \otimes N) \to (M \otimes M) \otimes (N \otimes N) \) and \( \mu_M \otimes \mu_N \), see e.g. [19]. Equivalently, the unit of \( M \otimes N \) is the element \( 1_M \otimes 1_N \), and the multiplication is determined by \( (m \otimes n) \bullet (m' \otimes n') = (m \bullet m') \otimes (n \bullet n') \).

Definition 2.12. A monoidal functor \((G, \theta): \mathcal{C} \to \mathcal{D}\) is a functor \(G: \mathcal{C} \to \mathcal{D}\) with a morphism \( \theta_1: 1_\mathcal{D} \to G 1_\mathcal{C} \) and morphisms \( \theta_{A,B}: G(A) \otimes G(B) \to G(A \otimes B) \) natural in \( A, B \in \mathcal{C} \) such that the following squares commute (omitting indices):

\[
\begin{array}{ccc}
(GA \otimes GB) \otimes GC & \longrightarrow & GA \otimes (GB \otimes GC) \\
\downarrow_{G(A \otimes B) \otimes GC} & & \downarrow_{GA \otimes (GB \otimes GC)} \\
G((A \otimes B) \otimes C) & \longrightarrow & GA \otimes G((B \otimes C)) \\
\end{array}
\]

\[
\begin{array}{ccc}
GA \otimes (GB \otimes GC) & \longrightarrow & GA \otimes (G1_\mathcal{C}) \\
\downarrow_{G \otimes G(\theta)} & & \downarrow_{G \otimes G(\theta)} \\
GA \otimes G(1_\mathcal{C}) & \longrightarrow & GA \otimes GA \\
\end{array}
\]

Given another monoidal functor \((G', \theta'): \mathcal{C} \to \mathcal{D}\), a natural transformation \( \varphi: G \to G' \) is called monoidal if the following diagrams commute:

\[
\begin{array}{ccc}
GA \otimes GB & \longrightarrow & G' A \otimes G'B \\
\downarrow_{\varphi_{A \otimes B}} & & \downarrow_{\varphi'_{A \otimes B}} \\
G((A \otimes B) \otimes C) & \longrightarrow & G'((A \otimes B) \otimes C) \\
\end{array}
\]

\[
\begin{array}{ccc}
GA \otimes (GB \otimes GC) & \longrightarrow & G' (GA \otimes (GB \otimes GC)) \\
\downarrow_{G(\varphi \otimes \theta)} & & \downarrow_{G' \varphi \otimes \theta'} \\
GA \otimes G(1_\mathcal{C}) & \longrightarrow & G' (G1_\mathcal{C}) \\
\end{array}
\]

\[
\begin{array}{ccc}
G(A \otimes B) & \longrightarrow & G'(A \otimes B) \\
\downarrow_{\varphi_{A \otimes B}} & & \downarrow_{\varphi'_{A \otimes B}} \\
G((A \otimes B) \otimes C) & \longrightarrow & G'((A \otimes B) \otimes C) \\
\end{array}
\]
The importance of monoidal functors is that they preserve monoid structures:

\[ \text{Definition} \]

Example \( [\text{two.prop}] \) (two.prop)

Example \( [\text{two.prop}] \)

Remark \( [\text{two.prop}] \)

Lemma \( [\text{two.prop}] \)

Example \( [\text{two.prop}] \)

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The importance of monoidal functors is that they preserve monoid structures:

**Lemma 2.14.** Let \((G, \theta): \mathcal{C} \to \mathcal{D}\) be a monoidal functor. Then \(G\) lifts to the functor \(\overline{G}: \text{Mon}(\mathcal{C}) \to \text{Mon}(\mathcal{D})\) mapping a \(\mathcal{C}\)-monoid \((M, \iota, \mu)\) to the \(\mathcal{D}\)-monoid \((GM, 1_{\mathcal{D}} \overset{\theta}{\to} G1_{\mathcal{D}} \overset{G\iota}{\to} GM, GM \otimes GM \overset{\theta}{\to} G(M \otimes M) \overset{\overline{G}\iota}{\to} GM)\), and a \(\mathcal{C}\)-monoid morphism \(h\) to \(Gh\).

**Example 2.15.** (1) \(\mathcal{P}_f: \text{Set} \to \text{JSL}\) lifts to the functor \(\mathcal{P}_f: \text{Mon}(\text{Set}) \to \text{Mon}(\text{JSL})\) that maps a monoid \(M\) to the semiring \(\overline{\mathcal{P}}_fM\) of finite subsets of \(M\), with union as addition, and multiplication \(XY = \{x y: x \in X, y \in Y\}\).

(2) \(\mathcal{D}_f: \text{Pos} \to \text{JSL}\) lifts to \(\mathcal{D}_f: \text{Mon}(\text{Pos}) \to \text{Mon}(\text{JSL})\); mapping an ordered monoid \(M\) to the semiring \(\overline{\mathcal{D}}_f(M)\) of finitely generated down-sets of \(M\), with union as addition, and multiplication \(XY = \downarrow\{x y: x, y \in Y\}\).

**Lemma 2.16.** Let \((G, \theta): \mathcal{A} \to \mathcal{D}\) and \((H, \sigma): \mathcal{B} \to \mathcal{C}\) be monoidal functors. Then the composite \(HG: \mathcal{A} \to \mathcal{C}\) is a monoidal functor w.r.t. \(H(\theta) \circ \sigma_1: 1_{\mathcal{A}} \to HG(1_{\mathcal{A}})\) and \(H(\theta_{A,B}) \circ \sigma_{G,A,GB}: HGA \otimes HGB \to HG(A \otimes B)\).

**Definition 2.17.** A monoidal adjunction between \(\mathcal{C}\) and \(\mathcal{D}\) is an adjunction \(F \dashv U: \mathcal{C} \to \mathcal{D}\) such that \(U\) and \(F\) are monoidal functors and the unit \(\eta: 1_{\mathcal{D}} \to UF\) and counit \(\varepsilon: UF \to 1_{\mathcal{D}}\) are monoidal natural transformations.

**Example 2.18.** \(\text{id} \dashv \text{id}: \mathcal{D} \to \mathcal{D}\) is a monoidal adjunction. We call the latter the monoidal adjunction of \(\mathcal{D}\).

**Remark 2.19.** If \((H \dashv V: \mathcal{C} \to \mathcal{D}, \eta', \varepsilon')\) and \((G \dashv U: \mathcal{B} \to \mathcal{A}, \eta, \varepsilon)\) are monoidal adjunctions, so is the composite adjunction \((HG \dashv UV: \mathcal{C} \to \mathcal{A}, U\eta'G \circ \eta, \varepsilon' \circ He\varepsilon)\). Here \(HG\) and \(UV\) are the composites of Lemma 2.16.

**Definition 2.20.** A monoidal adjunction \(F \dashv U: \mathcal{C} \to \mathcal{D}\) is called a concrete monoidal adjunction if its composite with the monoidal adjunction of \(\mathcal{D}\) is the monoidal adjunction of \(\mathcal{C}\).
3 Languages and Algebraic Recognition

In this section we set the scene for our categorical approach to Schützenberger products. For the rest of this paper let us fix a commutative variety $\mathcal{D}$ of algebras or ordered algebras, a commutative semiring $\mathbb{S} = (\mathbb{S}, +, \cdot, 0, 1)$, and a concrete monoidal adjunction $F \dashv U: \mathbb{S}\text{-Mod} \to \mathcal{D}$ with unit $\eta: \text{Id} \to UF$. Thus we have the diagram of functors below, where $U$ and $F$ are the lifted functors, see Lemma 2.14: the vertical functors are the forgetful functors, and $\Psi$ and $\mathbb{S}(-)$ are the left adjoints to the forgetful functors of $\mathcal{D}$ and $\mathbb{S}\text{-Mod}$, see Example 2.2.

\[
\begin{array}{ccc}
\mathbb{S}\text{-Alg} & \xrightarrow{\mathcal{V}} & \text{Mon}(\mathcal{D}) \\
\downarrow & & \downarrow \\
\mathbb{S}\text{-Mod} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\
\end{array}
\]

Example 3.1. In our applications we will choose the concrete monoidal adjunctions listed below. (The third and last column will be explained later.)

| $\mathbb{S}$ | $\mathcal{C}$ | $\mathcal{D}$ | $\mathbb{S}\text{-Mod} \xrightarrow{\mathcal{F}} \mathcal{D}$ | $\mathcal{D}$-monoids | $M \circ N$ carried by |
|-------------|------------|-------------|----------------|---------------------|---------------------|
| $\{0, 1\}$ | BA         | Set         | $\text{JSL} \xrightarrow{\text{Id}} \text{Set}$ | monoids             | $M \times \mathcal{P}(M \times N) \times N$ |
| $\{0, 1\}$ | DL         | Pos         | $\text{JSL} \xrightarrow{\text{Id}} \text{Pos}$ | ord. monoids        | $M \times \mathcal{D}(M \times N) \times N$ |
| $\{0, 1\}$ | JSL        | JSL         | $\text{JSL} \xrightarrow{\text{Id}} \text{JSL}$ | id. semirings       | $M \times (M \times N) \times N$ |
| $\{0, 1\}$ | K-Vec      | K-Vec       | $\text{K-Vec} \xrightarrow{\text{Id}} \text{K-Vec}$ | K-algebras          | $M \times (M \times N) \times N$ |
| $\{0, 1\}$ | ?          | $\mathbb{S}\text{-Mod}$ | $\mathbb{S}\text{-Mod} \xrightarrow{\text{Id}} \mathbb{S}\text{-Mod}$ | $\mathbb{S}$-algebras | $M \times (M \times N) \times N$ |

Notation 3.2. We can view the semiring $\mathbb{S}$ as (i) an $\mathbb{S}$-algebra $\mathbb{S}_{\text{Alg}} \in \mathbb{S}\text{-Alg}$ with scalar product given by this multiplication of $\mathbb{S}$, (ii) a $\mathcal{D}$-monoid $\mathbb{S}_{\text{Mon}} \in \text{Mon}(\mathcal{D})$ (by applying $U$ to $\mathbb{S}_{\text{Alg}}$), (iii) an $\mathbb{S}$-module $\mathbb{S}_{\text{Mod}} \in \mathbb{S}\text{-Mod}$ (by applying the forgetful functor to $\mathbb{S}_{\text{Alg}}$) and (iv) an object $\mathbb{S}_{\mathcal{D}}$ of $\mathcal{D}$ (by applying $U$ to $\mathbb{S}_{\text{Mod}}$). The $\mathcal{D}$-monoid $\mathbb{S}_{\text{Mon}}$ is carried by the object $\mathbb{S}_{\mathcal{D}}$, and its multiplication is a morphism of $\mathcal{D}$ that we denote by $\sigma: \mathbb{S}_{\mathcal{D}} \otimes \mathbb{S}_{\mathcal{D}} \to \mathbb{S}_{\mathcal{D}}$. For ease of notation we will usually drop the indices and simply write $\mathbb{S}$ for $\mathbb{S}_{\mathcal{D}}$, $\mathbb{S}_{\text{Mod}}$, etc.

Definition 3.3. (1) A language (a.k.a. a formal power series) over a finite alphabet $\Sigma$ is a map $L: \Sigma^* \to \mathbb{S}$. Denote by $L_\mathcal{D}: \Psi \Sigma^* \to \mathbb{S}$ the adjoint transpose of $L$ w.r.t. the adjunction $\Psi \dashv [-]: \mathcal{D} \to \text{Set}$. A $\mathcal{D}$-monoid morphism $f: \Psi \Sigma^* \to M$ recognizes $L$ if there is a morphism $p: M \to \mathbb{S}$ in $\mathcal{D}$ with $L_\mathcal{D} = p \circ f$. In this case, we also say that $M$ recognizes $L$ (via $f$ and $p$).

(2) The marked Cauchy product of two languages $K, L: \Sigma^* \to \mathbb{S}$ w.r.t. a letter $a \in \Sigma$ is the language $KaL: \Sigma^* \to \mathbb{S}$ with $(KaL)(u) = \sum_{u=\text{var}} K(v) \cdot L(w)$. 


For $S = \{0, 1\}$, a language $L: \Sigma^* \to \{0, 1\}$ corresponds to a classical language $L \subseteq \Sigma^*$ by taking the preimage of 1. Under this identification, we have $KaL = \{vw : v \in K, w \in L\}$. Our concept of language recognition by $D$-monoids originates in [2] and specializes to several related notions from the literature:

**Example 3.4.** (1) $D = \text{Set}$ with $S = \{0, 1\}$: a map $p: M \to \{0, 1\}$ corresponds to a subset $p^{-1}[1] \subseteq M$. Thus a monoid morphism $f: \Sigma^* \to M$ recognizes the language $L \subseteq \Sigma^*$ iff $L$ is the preimage under $f$ of some subset of $M$. This is the classical notion of language recognition by a monoid, see e.g. [16].

(2) $D = \text{Pos}$ with $S_{\text{Pos}} = \{0 < 1\}$: given an ordered monoid $M$, a monotone map $p: M \to \{0, 1\}$ defines an upper set $p^{-1}[1] \subseteq M$. Hence a monoid morphism $f: \Sigma^* \to M$ recognizes $L \subseteq \Sigma^*$ iff $L$ is the preimage under $f$ of some upper set of $M$. This notion of recognition is due to Pin [15].

(3) $D = \text{JSL}$ with $S_{\text{JSL}} = \{0 < 1\}$: for any idempotent semiring $M$, a semilattice morphism $p: M \to \{0, 1\}$ defines an ideal $I = p^{-1}[0]$, i.e. a nonempty downset closed under joins. Hence a language $L \subseteq \Sigma$ is recognized by a semiring morphism $f: \bigwedge \Sigma^* \to M$ via $p$ iff $\bigwedge L = \Sigma^* \cap f^{-1}[I]$. Here we identify $\Sigma^*$ with the set of all singleton languages $\{w\}, w \in \Sigma^*$. This is the concept of language recognition by idempotent semirings introduced by Polak [18].

(4) $D = \text{S-Mod}$: given an $S$-algebra $M$, a formal power series $L: \Sigma^* \to S$ is recognized by $f: S[\Sigma] \to M$ via $p: M \to S$ iff $L_{\text{S-Mod}} = p \circ f$. This notion of recognition is due to Reutenauer [20]. If $S$ is a commutative ring, the power series recognizable by $S$-algebras of finite type (i.e. $S$-algebras whose underlying $S$-module is finitely generated) are precisely rational power series.

## 4 The Schützenberger Product

We are ready to introduce the Schützenberger product for $D$-monoids. Fix two $D$-monoids $(M, 1, \cdot)$ and $(N, 1, \cdot)$, and write $xy$ for $x \cdot y$. Our goal is to construct a $D$-monoid $M \ast N$ that recognizes all marked products of languages recognized by $M$ and $N$, and is the “smallest” such $D$-monoid (Theorems [1.8], [4.13], [1.16]).

**Construction 4.1.** As a preliminary step, we define a $D$-monoid $M \ast N$ as follows. Call a family $\{f_i: A \to B_i\}_{i \in I}$ in $D$ separating if the morphism $f: A \to \prod_i B_i$ with $f(a) = (f_i(a))_{i \in I}$ is injective (resp. order-reflecting when $D$ is a variety of ordered algebras).

Any family $\{f_i\}$ yields a separating family $\{f'_i: A' \to B_i\}_{i \in I}$ by factorizing $f = m \circ \pi$ with $\pi$ surjective and $m$ injective (resp. order-reflecting), and setting $f'_i := p_i \circ m$, where $p_i$ is the projection. Now consider the family of all morphisms $\sigma \circ (p \otimes q): M \otimes N \to S$, where $p: M \to S$ and $q: N \to S$ are arbitrary morphisms in $D$. Applying the above construction to this family $\{\sigma \circ (p \otimes q)\}_{p,q}$ gives an algebra $M \ast N$ in $D$, a surjective morphism $\pi: M \otimes N \to M \ast N$, and a separating family $\{p \ast q: M \ast N \to S\}_{p,q}$, making the following diagram commute for all $p$ and $q$:

$$
\begin{array}{ccc}
M \otimes N & \xrightarrow{\pi} & M \ast N & \xrightarrow{p \ast q} & S \\
\xrightarrow{\sigma} & S \otimes S & & S \\
\end{array}
$$

(4.1)
Notation 4.2. For any $m \in |M|$ and $n \in |N|$, we write $m * n$ for the element $\pi(m \odot n) \in |M * N|$.

Lemma 4.3. There exists a (unique) $\mathcal{D}$-monoid structure on $M * N$ such that $\pi: M \otimes N \to M * N$ is a $\mathcal{D}$-monoid morphism. The multiplication is determined by $(m * n) \cdot (m' * n') = (mm') * (nn')$, and the unit is $1 * 1$.

Example 4.4. For $\mathcal{D} = \text{Set}$, $\text{Pos}$ or $\mathbf{K}\text{-Vec}$, the family $\{\sigma \circ (p \otimes q)\}_{p,q}$ is already separating, and therefore $M * N = M \otimes N$ and $p * q = \sigma \circ (p \otimes q)$. For $\mathcal{D} = \text{JSL}$ and in case $M$ and $N$ are finite idempotent semirings, we can describe the idempotent semiring $M * N$ as follows. For any subset $X \subseteq M \times N$, let $[X] \subseteq M \times N$ consist of those elements $(m,n) \in M \times N$ such that, for all ideals $I \subseteq M$ and $J \subseteq N$ with $m \notin I$ and $n \notin J$, there exists some $(x,y) \in X$ with $x \notin I$ and $y \notin J$. This gives us the closure operator $X \mapsto [X]$ on the power set of $M \times N$ in $\text{JSL}$. One can show that $M * N$ is isomorphic to the idempotent semiring of all closed subsets of $M \times N$, with sum and product defined by $[X] \lor [Y] = [X \cup Y]$ and $[X][Y] = [XY]$, where $XY = \{ xy : x \in X, y \in Y \}$.

Definition 4.5. The Schützenberger product of $M$ and $N$ is the $\mathcal{D}$-monoid $M \diamond N$ carried by the product $M \times UF(M \times N) \times N$ in $\mathcal{D}$ and equipped with the following monoid structure: representing elements $(m,a,n) \in |M| \times |F(M \times N)| \times |N|$ as upper triangular matrices $\begin{pmatrix} m & a \\ 0 & n \end{pmatrix}$, the multiplication and unit are given by

$$
\begin{pmatrix} m & a \\ 0 & n \end{pmatrix} \cdot \begin{pmatrix} m' & a' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} mm' & a' + a \cdot \eta(1 * n') \\ 0 & nn' \end{pmatrix}
$$

and

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Here $\eta: M \times N \to UF(M \times N)$ is the universal map, and the sum, product and 0 in the upper right components are taken in the $\mathcal{S}$-algebra $\overline{F}(M \times N)$.

Lemma 4.6. $M \diamond N$ is a well-defined $\mathcal{D}$-monoid, and the product projections $\pi_M: M \diamond N \to M$ and $\pi_N: M \diamond N \to N$ are $\mathcal{D}$-monoid morphisms.

Example 4.7. For the categories and adjunctions of Example 3.1, we recover four notions of Schützenberger products known in the literature, and obtain a new Schützenberger product for $\mathcal{S}$-algebras:

1. $\mathcal{D} = \text{Set}$: given monoids $M$ and $N$, the monoid $M \diamond N$ is carried by the set $M \times P(M \times N) \times N$, with multiplication and unit

$$
\begin{pmatrix} m & X'' \\ 0 & n \end{pmatrix} \cdot \begin{pmatrix} m' & X'' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} mm' & mX'' \cup Xn' \\ 0 & nn' \end{pmatrix}
$$

and

$$
\begin{pmatrix} 1 & \emptyset \\ 0 & 1 \end{pmatrix},
$$

where $mX'' = \{ (m,y,z) : (y,z) \in X'' \}$ and $Xn' = \{ (y,zn') : (y,z) \in X \}$. This is the original construction of Schützenberger [22].

2. $\mathcal{D} = \text{Pos}$: for ordered monoids $M$ and $N$, the ordered monoid $M \diamond N$ is carried by the poset $M \times \mathcal{D}_f(M \times N) \times N$ with multiplication and unit

$$
\begin{pmatrix} m & X'' \\ 0 & n \end{pmatrix} \cdot \begin{pmatrix} m' & X'' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} mm' & \downarrow(mX'' \cup Xn') \\ 0 & \uparrow(nn') \end{pmatrix}
$$

and

$$
\begin{pmatrix} 1 & \emptyset \\ 0 & 1 \end{pmatrix}.
$$

This construction is due to Pin [17].
(3) \( \mathcal{D} = \text{JSL} \): given idempotent semirings \( M \) and \( N \), the idempotent semiring \( M \circ N \) is carried by the semilattice \( M \times (M \ast N) \times N \). If \( M \) and \( N \) are finite, \( M \ast N \) is the idempotent semiring of closed subsets of \( M \times N \) by Example 4.4, and the multiplication and unit of \( M \circ N \) are given by

\[
(mX) (m'X') = \begin{pmatrix} m & m' & [mX' \cup Xn'] \\ 0 & n & n' \end{pmatrix} \quad \text{and} \quad (1 0 0).
\]

For the finite case, this construction is due to Klíma and Polák [14].

(4) \( \mathcal{D} = \mathbf{K} \text{-Vec} \): given \( \mathbf{K} \)-algebras \( M \) and \( N \), the \( \mathbf{K} \)-algebra \( M \circ N \) is carried by the vector space \( M \times (M \otimes N) \times N \) with multiplication and unit

\[
(mz) (m'z') = \begin{pmatrix} mm' & mz' + zn' \\ 0 & nn' \end{pmatrix} \quad \text{and} \quad (1 0 0),
\]

where \( mz' = (mm_0) \otimes n_0 \) for \( z' = m_0 \otimes n_0 \), and extending via bilinearity for arbitrary \( z \); similarly for \( zn' \). This construction is due to Reutenauer [20].

(5) \( \mathcal{D} = \mathbf{S} \text{-Mod} \): given \( \mathbf{S} \)-algebras \( M \) and \( N \), the \( \mathbf{S} \)-algebra \( M \circ N \) is carried by the \( \mathbf{S} \)-module \( M \times (M \ast N) \times N \) with multiplication and unit

\[
(mz) (m'z') = \begin{pmatrix} mm' & mz' + zn' \\ 0 & nn' \end{pmatrix} \quad \text{and} \quad (1 0 0),
\]

where \( mz' = (mm_0) \ast n_0 \) for \( z' = m_0 \ast n_0 \), and similarly for \( zn' \). This example specializes to (3) and (4) by taking \( \mathbf{S} = \{0, 1\} \) and \( \mathbf{S} = \mathbf{K} \), respectively, but appears to be a new construction for other semirings \( \mathbf{S} \).

The following theorem gives the key property of \( M \circ N \).

**Theorem 4.8.** Let \( K, L : \Sigma^* \to \mathbf{S} \) be languages recognized by \( M \) and \( N \), respectively. Then \( M \circ N \) recognizes the languages \( K \), \( L \) and \( KaL \) for all \( a \in \Sigma \).

Next, we aim to show that \( M \circ N \) is the “smallest” \( \mathcal{D} \)-monoid satisfying the statement of the above theorem. This requires further assumptions on our setting.

**Notation 4.9.** Recall from (4.4) the morphism \( p \ast q : M \ast N \to \mathbf{S} \). We denote its adjoint transpose w.r.t. the adjunction \( F \dashv U \) by \( \overline{p \ast q} : F(M \ast N) \to \mathbf{S} \).

**Assumptions 4.10.** From now on, suppose that:

(i) \( \mathcal{D} \) is locally finite, i.e. every finitely generated algebra of \( \mathcal{D} \) is finite.

(ii) Epimorphisms in \( \mathcal{D} \) and \( \mathbf{S} \text{-Mod} \) are surjective.

(iii) \( \mathcal{D}(M, \mathbf{S}) \), \( \mathcal{D}(N, \mathbf{S}) \), and \( \{ U(\overline{p \ast q}) : UF(M \ast N) \to \mathbf{S} \}_{p : M \to \mathbf{S}, q : N \to \mathbf{S}} \) are separating families of morphisms in \( \mathcal{D} \).

(iv) There is a locally finite variety \( \mathcal{E} \) of algebras such that the full subcategories \( \mathcal{E}_f \) and \( \mathcal{D}_f \) on finite algebras are dually equivalent. We denote the equivalence functor by \( E : \mathcal{D}_f^0 \cong \mathcal{E}_f \).

(v) The semiring \( \mathbf{S} \) is finite, and \( E(\mathbf{S}) \cong 1_\mathcal{E} \).
Let us indicate the intuition behind our assumptions. First, (i) and (ii) imply that $M \circ N$ is finite if $M$ and $N$ are. This is important, as one is usually interested in language recognition by finite $\mathcal{D}$-monoids. (iii) expresses that the semiring $\mathbb{S}$ has enough structure to separate elements of $M$, $N$ and $UF(M \ast N)$, the three components of the Schützenberger product $M \circ N$, by suitable morphisms into $\mathbb{S}$. This technical condition on $\mathbb{S}$ is the crucial ingredient for proving the “smallness” of $M \circ N$ (Theorem [1,13]). Finally, the variety $\mathcal{V}$ in (iv) and (v) will be used to determine, via duality, the algebraic operations to express languages recognized by $M \circ N$ in terms of languages recognized by $M$ and $N$ (Theorem [1,16]).

Example 4.11. The categories and adjunctions of Example [3,11]-(4) satisfy our assumptions. Here we briefly sketch the dualities; see [13] for details.

(1) For $\mathcal{D} = \text{Set}$, choose $\mathcal{C} = \text{BA}$ (Boolean algebras). Stone duality [13] gives a dual equivalence $E: \text{Set} \cong \text{BA}$. Mapping a finite set to the Boolean algebra of all subsets.

(2) For $\mathcal{D} = \text{Pos}$, choose $\mathcal{C} = \text{DL}$ (Distributive lattices with 0, 1). Birkhoff duality [6] gives a dual equivalence $E: \text{Pos} \cong \text{DL}$. Mapping a finite poset to the lattice of all down-sets.

(3) For $\mathcal{D} = \text{JSL}$, choose $\mathcal{C} = \text{JSL}$. The dual equivalence $E: \text{JSL} \cong \text{JSL}$. Mapping a finite semilattice $(X, \vee)$ to its opposite semilattice $(X, \wedge)$, see [13].

(4) For $\mathcal{D} = \text{K-Vec}$, $K$ a finite field, choose $\mathcal{C} = \text{K-Vec}$. The dual equivalence $E: \text{K-Vec} \cong \text{K-Vec}$. Mapping a space $X$ to its dual space $X^\ast = \text{hom}(X, K)$.

Notation 4.12. For any $\mathcal{D}$-monoid morphism $f: \Psi \Sigma^\ast \to M \circ N$, put

$L_{M,N}(f) := \{ K, L, KaL \mid a \in \Sigma, \pi_M \circ f \text{ recognizes } K, \pi_N \circ f \text{ recognizes } L \}$

Theorem 4.13. Let $f: \Psi \Sigma^\ast \to M \circ N$ and $e: \Psi \Sigma^\ast \to P$ be two $\mathcal{D}$-monoid morphisms. If $e$ is surjective and recognizes all languages in $L_{M,N}(f)$, then there exists a unique $\mathcal{D}$-monoid morphism $h: P \to M \circ N$ with $h \circ e = f$.

Using our duality framework, this theorem can be rephrased in terms of language operations. Recall that $E(\mathbb{S}) \cong 1_{\mathcal{C}}$ by Assumption [1,10]. Putting $O_{\mathcal{C}} := E(1_{\mathcal{C}})$, we obtain a bijection $i: \mathbb{S} \cong \mathcal{D}(1_{\mathcal{C}}, \mathbb{S}) \cong \mathbb{C}(E(\mathbb{S}), E(1_{\mathcal{C}})) \cong \mathbb{C}(1_{\mathcal{C}}, O_{\mathcal{C}}) \cong [O_{\mathcal{C}}]$.

Definition 4.14. For any $n$-ary operation symbol $\gamma$ in the signature of $\mathcal{C}$ and languages $L_1, \ldots, L_n: \Sigma^\ast \to S$, the language $\gamma(L_1, \ldots, L_n): \Sigma^\ast \to S$ is given by $\gamma(L_1, \ldots, L_n)(u) := i^{-1}(\gamma^{O_{\mathcal{C}}}(i(L_1u), \ldots, i(L_nu))$. The operations $\gamma$ are called the $\mathcal{C}$-algebraic operations on the set of languages over $\Sigma$.

Example 4.15. $O_{\text{BA}} \cong \{ 0, 1 \}$ is the two-element Boolean algebra, and the $\text{BA}$-algebraic operations are precisely the Boolean operations (union, intersection, complement, $\emptyset, \Sigma^\ast$) on languages. For example, the operation symbol $\lor$ induces the language operation $(K \lor L)(u) = K(u) \lor L(u)$ corresponding to the union of languages. Similarly, for $\mathcal{C} = \text{DL}$ we get union, intersection, $\emptyset, \Sigma^\ast$, for $\mathcal{C} = \text{JSL}$ we get union and $\emptyset$, and for $\mathcal{C} = \text{K-Vec}$ we get sum, scalar product and $\emptyset$.

All our constructions and results so far apply to arbitrary $\mathcal{D}$-monoids. However, in the following theorem we need to restrict to finite $\mathcal{D}$-monoids. Recall that the
derivatives of a language $L$: $\Sigma^* \to S$ are the languages $a^{-1}L$, $La^{-1}$: $\Sigma^* \to S$ (where $a \in \Sigma$) defined by $(a^{-1}L)(u) = L(au)$ and $(La^{-1})(u) = L(ua)$.

**Theorem 4.16.** Let $M$ and $N$ be finite $\mathcal{G}$-monoids and $f: \Psi \Sigma^* \to M \circ N$ be a $\mathcal{G}$-monoid morphism. Then every language recognized by $f$ lies in the closure of $L_{M,N}(f)$ under the $\mathcal{G}$-algebraic operations and derivatives.

Our proof uses the *Local Variety Theorem* of 1: for any finite set $V$ of recognizable languages closed under $\mathcal{G}$-algebraic operations and derivatives, there is a finite $\mathcal{G}$-monoid recognizing precisely the languages of $V$. Coincidentally, for each of our categories of Example 3.1.1)-(4) it suffices to take the closure of $L_{M,N}(f)$ under $\mathcal{G}$-algebraic operations, as this set is already derivative-closed. For example, for $\mathcal{G} = K$-Vec we have $a^{-1}(KaL) = (a^{-1}K)aL + K(\varepsilon)L$, i.e. $a^{-1}(KaL)$ is a linear combination of languages in $L_{M,N}(f)$ and thus lies in the closure of $L_{M,N}(f)$ under $K$-Vec-operations. For $\mathcal{G} = Set, Pos$ and JSL, Theorem 4.16 then gives

**Corollary 4.17 (Reutenauer [21], Pin [17], Klíma and Polák [14]).** Let $M$ and $N$ be finite monoids [ordered monoids, idempotent semirings]. Then any language recognized by the Schützenberger product $M \circ N$ is a boolean combination [positive boolean combination, finite union] of languages of the form $K$, $L$ and $KaL$, where $K$ is recognized by $M$, $L$ is recognized by $N$, and $a \in \Sigma$.

For $\mathcal{G} = K$-Vec, we obtain a new result for formal power series:

**Corollary 4.18.** Let $M$ and $N$ be finite algebras over a finite field $K$. Then any language recognized by $M \circ N$ is a linear combination of power series of the form $K$, $L$ and $KaL$, where $K$ is recognized by $M$, $L$ is recognized by $N$, and $a \in \Sigma$.

5 Conclusions and Future Work

We presented a uniform approach to Schützenberger products for various algebraic structures. Our categorical framework encompasses all known instances of Schützenberger products in the setting of regular languages. Two related constructions are the Schützenberger products for $\omega$-semigroups [8] (dealing with $\omega$-languages), and for boolean spaces with internal monoids [12] (dealing with non-regular languages). Neither of these structures are monoids in the categorical sense, and thus are not covered by our present setting. The use of monads as in [7],[9],[25] might pave the way to extending the scope of our work.

Since our main focus in the present paper was to establish the categorical setting, we restricted to binary Schützenberger products $M \circ N$. For (ordered) monoids and semirings, a non-trivial $n$-ary generalization of the Schützenberger product is known [7],[18],[23], and we aim to adapt our results to arbitrary $n$.

**References**

1. Adámek, J., Milius, S., Myers, R., Urbat, H.: Generalized Eilenberg Theorem I: Local Varieties of Languages. In: Muscholl, A. (ed.) Proc. FoSSaCS’14. LNCS, vol. 8412, pp. 366–380. Springer (2014), full version: [http://arxiv.org/pdf/1501.02834v1.pdf](http://arxiv.org/pdf/1501.02834v1.pdf)
2. Adámek, J., Milius, S., Urbat, H.: Syntactic monoids in a category. In: Proc. CALCO'15, LIPIcs, Schloss Dagstuhl–Leibniz-Zentrum für Informatik (2015)
3. Adámek, J., Myers, R., Milius, S., Urbat, H.: Varieties of languages in a category. In: LICS'15, IEEE (2015)
4. Ballester-Bolínches, A., Cosme-López, E., Rutten, J.: The dual equivalence of equations and coequations for automata. Inform. and Comp. 244, 49–75 (2015)
5. Banaschewski, B., Nelson, E.: Tensor products and bimorphisms. Canad. Math. Bull. 19, 385–402 (1976)
6. Birkhoff, G.: Rings of sets. Duke Mathematical Journal 3(3), 443–454 (1937)
7. Bojańczyk, M.: Recognisable languages over monads. In: Proc. DLT’15, LNCS, vol. 9168, pp. 1–13. Springer (2015), http://arxiv.org/abs/1502.04898
8. Carton, O.: Mots infinis, ω-semigroupes et topologie. Tech. rep., Université Paris 7 (1993), report LITP-TH 93-08
9. Chen, L.T., Adámek, J., Milius, S., Urbat, H.: Profinite monads, profinite equations and Reiterman’s theorem. In: Jacobs, B., Löding, C. (eds.) Proc. FoSSaCS’16. LNCS, vol. 9634. Springer (2016), http://arxiv.org/abs/1511.02147
10. Chen, L.T., Urbat, H.: A fibrational approach to automata theory. In: Proc. CALCO'15, LIPIcs, Schloss Dagstuhl–Leibniz-Zentrum für Informatik (2015)
11. Gehrke, M., Grigorieff, S., Pin, J.E.: Duality and equational theory of regular languages. In: ICALP’08, Part II. LNCS, vol. 5126, pp. 246–257. Springer (2008)
12. Gehrke, M., Petrisan, D., Reggio, L.: The schützenberger product for syntactic spaces (2016), to appear in Proc. ICALP’16. Preprint: http://arxiv.org/abs/1603.08264
13. Johnstone, P.T.: Stone spaces. Cambridge University Press (1982)
14. Klíma, O., Polák, L.: On Schützenberger products of semirings. In: Gao, Y., Lu, H., Seki, S., Yu, S. (eds.) DLT’10, pp. 279–290. LNCS, Springer (2010)
15. Pin, J.E.: A variety theorem without complementation. Russ. Math. 39, 80–90 (1995)
16. Pin, J.E.: Mathematical foundations of automata theory (October 2015), available at http://www.liafa.jussieu.fr/~jep/PDF/MPRI/MPRI.pdf
17. Pin, J.E.: Algebraic tools for the concatenation product. Theor. Comput. Sci. 292, 317–342 (2003)
18. Polák, L.: Syntactic semiring of a language. In: Sgall, J., Pultr, A., Kolman, P. (eds.) Proc. MFCS’01. LNCS, vol. 2136, pp. 611–620. Springer (2001)
19. Porst, H.E.: On categories of monoids, comonoids, and bimonoids. Quasistones Math. 31, 127–139 (2008)
20. Reutenauer, C.: Sérries formelles et algèbres syntactiques. J. Algebra 66, 448–483 (1980)
21. Reutenauer, C.: Sur les variétés de langages et de monoïdes. In: Theor. Comput. Sci., 4th GI-Conference. LNCS, vol. 67, pp. 260–265. Springer (1979)
22. Schützenberger, M.P.: On finite monoids having only trivial subgroups. Inform. and Control 8, 190–194 (1965)
23. Straubing, H.: A generalization of the Schützenberger product of finite monoids. Theor. Comp. Sci. 13(2), 137 – 150 (1981)
24. Uramoto, T.: Semi-galois categories: The classical Eilenberg variety theory, to appear in Proc. LICS’16. Preprint: http://arxiv.org/abs/1512.04389
25. Urbat, H., Adámek, J., Chen, L.T., Milius, S.: One Eilenberg theorem to rule them all (2016), preprint: http://arxiv.org/abs/1602.05831
This appendix provides all omitted proofs, as well as additional details for our examples. We start with a review of concepts from universal algebra and category theory.

A Algebraic Toolkit

A.1. Varieties of algebras. Fix a finitary signature $\Gamma$, i.e. a set of operation symbols with finite arities. A $\Gamma$-algebra is a set $A$ equipped with an operation $\gamma^A : A^n \to A$ for each $n$-ary $\gamma \in \Gamma$, and a morphism of $\Gamma$-algebras is a map preserving these operations. Quotients and subalgebras of $\Gamma$-algebras are represented by surjective resp. injective morphisms. A variety of algebras is a class of $\Gamma$-algebras closed under quotients, subalgebras, and products. Equivalently, a variety is class of $\Gamma$-algebras specified by equations $s = t$ between $\Gamma$-terms.

A.2. Varieties of ordered algebras. An ordered $\Gamma$-algebra is a poset $A$ equipped with a monotone operation $\gamma^A : A^n \to A$ for each $n$-ary $\gamma \in \Gamma$, and a morphism of ordered $\Gamma$-algebras is a monotone map preserving these operations. Quotients of ordered algebras are represented by surjective morphisms, and subalgebras by order-reflecting morphisms $m$ (i.e. $mx \leq my$ iff $x \leq y$). A variety of ordered algebras is a class of ordered $\Gamma$-algebras closed under quotients, subalgebras, and products. Equivalently, a variety is class of ordered $\Gamma$-algebras specified by inequations $s \leq t$ between $\Gamma$-terms.

In the following, let $\mathcal{D}$ always denote a variety of algebras or ordered algebras.

A.3. Commutative varieties. A variety $\mathcal{D}$ of algebras or ordered algebras is commutative if, for any $A \in \mathcal{D}$ and any $n$-operation symbol $\gamma \in \Gamma$, the corresponding operation $\gamma^A : |A|^n \to |A|$ carries a morphism $\gamma^A : A^n \to A$ of $\mathcal{D}$. Equivalently, for any two algebras $A, B \in \mathcal{D}$, the set $[A, B]$ of morphisms from $A$ to $B$ forms an algebra of $\mathcal{D}$ under the pointwise $\Gamma$-operations, i.e. $[A, B]$ carries a subalgebra of $B^{|[A]}$, the $|A|$-fold power of $B$.

A.4. Congruences and stable preorders.

1. A congruence on a $\Gamma$-algebra $A$ is an equivalence relation $\equiv$ on $A$ such that for all $n$-ary operations $\gamma \in \Gamma$ and elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

\[
a_i \equiv b_i \ (i = 1, \ldots, n) \ \text{implies} \ \gamma^A(a_1, \ldots, a_n) \equiv \gamma^A(b_1, \ldots, b_n).
\]

The set $A/\equiv$ of equivalence classes carries a $\Gamma$-algebra structure defined by

\[
\gamma^{A/\equiv}([a_1], [a_n]) := [\gamma^A(a_1, \ldots, a_n)],
\]

and the projection map $\pi : A \to A/\equiv$, $a \mapsto [a]$, is a surjective morphism of $\Gamma$-algebras.

2. Let $(\Gamma, \leq)$ be an ordered $\Gamma$-algebra. A stable preorder on $A$ is a preorder $\preceq$ on $A$ such that (i) $a \leq b$ implies $a \preceq b$, and (ii) for all $n$-ary operations $\gamma \in \Gamma$ and elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

\[
a_i \preceq b_i \ (i = 1, \ldots, n) \ \text{implies} \ \gamma^A(a_1, \ldots, a_n) \preceq \gamma^A(b_1, \ldots, b_n).
\]
For any stable preorder, the equivalence relation \( \equiv := \leq \cap \geq \) forms a congruence on \( A \) in the sense of (1), and \( A / \equiv \) becomes an ordered \( I \)-algebra by setting \( [a] \leq [a'] \) iff \( a \leq a' \). We write \( A / \prec \) for this ordered algebra. The projection map \( \pi : A \to A / \leq, a \mapsto [a] \), is a surjective morphism of ordered \( I \)-algebras.

A.5. Separating families. A family \( \{ f_i : A \to B_i \}_{i \in I} \) of morphisms in \( \mathcal{D} \) is separating if the morphism \( f : A \to \prod_i B_i \) with \( f(a) = (f_i(a))_{i \in I} \) is injective (resp. order-reflecting if \( \mathcal{D} \) is a variety of ordered algebras). Equivalently, for any two elements \( a, a' \in A \) with \( a \neq a' \) (resp. \( a \nleq a' \)), there exists an \( i \in I \) with \( f_i(a) \neq f_i(a') \) (resp. \( f_i(a) \nleq f_i(a') \)). Suppose that, for each \( i \in I \), another separating family \( \{ g_{i,j} : B_i \to C_{i,j} \}_{j \in J_i} \) is given. Then the combined family \( \{ g_{i,j} \circ f_i : A \to C_{i,j} \}_{i \in I,j \in J_i} \) is also separating.

A.6. Factorization systems. Any variety \( \mathcal{D} \) of algebras or ordered algebras has the factorization system of surjective and injective (resp. order-reflecting) morphisms. This means that (i) any morphism \( h : A \to B \) has a factorization \( h = m \circ e \) with \( e \) surjective and \( m \) injective (resp. order-reflecting), and (ii) the diagonal fill-in property holds: given a commutative square as displayed below with \( e \) surjective and \( m \) injective (order-reflecting), there is a unique morphism \( d \) making both triangles commutative:

\[
\begin{array}{ccc}
D & \xrightarrow{e} & C \\
g \downarrow & & \downarrow h \\
A & \xrightarrow{m} & B_i
\end{array}
\]

The diagonal fill-in property generalizes to families of morphisms: suppose that \( e, g, h_i \) and \( m_i \) \( (i \in I) \) are morphisms with \( h_i \circ e = m_i \circ g \) for all \( i \). If \( e \) is surjective and the family \( \{ m_i \}_{i \in I} \) is separating, then there exists a unique morphism \( d \) with \( d \circ e = g \) and \( m_i \circ d = h_i \) for all \( i \):

\[
\begin{array}{ccc}
D & \xrightarrow{e} & C \\
g \downarrow & & \downarrow h_i \\
A & \xrightarrow{m_i} & B_i
\end{array}
\]

A.7. Tensor products. Let \( \mathcal{D} \) be commutative variety of algebras or ordered algebras

(1) Let \( A, B, C \in \mathcal{D} \). By a bimorphism from \( A, B \) to \( C \) is meant a function \( f : |A| \times |B| \to |C| \) such that the maps \( f(a, -) : |B| \to |C| \) and \( f(-, b) : |A| \to |C| \) carry morphisms of \( \mathcal{D} \) for every \( a \in |A| \) and \( b \in |B| \). A tensor product of \( A \) and \( B \) is a universal bimorphism \( t_{A,B} : |A| \times |B| \to |A \otimes B| \), i.e. for any bimorphism \( f : |A| \times |B| \to |C| \) there is a unique \( f' : A \otimes B \to C \) in \( \mathcal{D} \) with \( f' \circ t_{A,B} = f \). We write \( a \otimes b \) for the element \( t_{A,B}(a, b) \in |A \otimes B| \).
(2) Tensor products exist in any variety \( \mathcal{D} \). Let us indicate how to construct them in the case where \( \mathcal{D} \) is a variety of ordered algebras. Given \( A, B \in \mathcal{D} \), form the free algebra \( \Psi(|A| \times |B|) \) in \( \mathcal{D} \) generated by the set \( |A| \times |B| \). (For simplicity, we assume that \( |A| \times |B| \) is a subset of \( \Psi(|A| \times |B|) \).) Form the smallest stable preorder \( \preceq \) on \( \Psi(|A| \times |B|) \) containing all inequations of the form

\[
(a, \gamma(b_1, \ldots, b_n)) \leq \gamma((a, b_1), \ldots, (a, b_n)) \\
\gamma((a, b_1), \ldots, (a, b_n)) \leq (a, \gamma(b_1, \ldots, b_n)) \\
(\gamma(a_1, \ldots, a_n), b) \leq (\gamma((a_1, b), \ldots, (a_n, b)) \\
(\gamma((a_1, b), \ldots, (a_n, b)) \leq (\gamma(a_1, \ldots, a_n), b)
\]

where \( \gamma \in \Gamma \) is an \( n \)-ary operation symbol, \( a, a_1, \ldots, a_n \in A \) and \( b, b_1, \ldots, b_n \in B \). Then the tensor product of \( A \) and \( B \) is given by

\[
A \otimes B := \Psi(|A| \times |B|)/\preceq
\]

and the universal bimorphism

\[
t_{A,B} := (|A| \times |B| \mapsto \Psi(|A| \times |B|) \xrightarrow{\pi} A \otimes B = \Psi(|A| \times |B|)/\preceq),
\]

the composite of the inclusion map and the projection. In particular, \( A \otimes B \) is generated by the elements \( a \otimes b \) with \( a \in |A| \) and \( b \in |B| \). The construction of \( A \otimes B \) for unordered algebras is analogous: just replace inequations and stable preorders by equations and congruences.

(3) For any \( A, B, C \in \mathcal{D} \) there is a natural bijective correspondence between

(i) morphisms from \( A \otimes B \) to \( C \),
(ii) bimorphisms from \( A, B \) to \( C \), and
(iii) morphisms from \( A \) to \( [B, C] \).

Indeed, the correspondence of (i) and (ii) follows from the universal property of the tensor product. Further, any bimorphism \( f : |A| \times |B| \rightarrow |C| \) defines a morphism

\[
\lambda f : A \rightarrow [B, C], \quad (\lambda f)(a)(b) = f(a, b),
\]

and the map \( f \mapsto \lambda f \) gives the bijective correspondence between (ii) and (iii).

(4) Up to isomorphism, \( \otimes \) is associative, commutative and has unit \( 1_\mathcal{D} \). More precisely, for any three objects \( A, B, C \in \mathcal{D} \) there are natural isomorphisms \( \alpha_{A,B,C}, \sigma_{A,B}, \rho_A, \lambda_A \) making the following squares commute:

\[
\begin{array}{c}
|A \otimes B) \otimes C| \xrightarrow{\alpha_{A,B,C}} |A \otimes (B \otimes C)| \\
\downarrow{t_{A \otimes B, C}(t_{A,B} \times C)} \downarrow{t_{A,B \otimes C}(A \times B, C)} \\
(|A| \times |B|) \times |C| \xrightarrow{\sigma_{A,B,C}} |A \times (B \times C)|
\end{array}
\]

\[
\begin{array}{c}
|A \otimes B| \xrightarrow{\sigma_{A,B}} |B \otimes A| \\
\downarrow{t_{A,B}} \downarrow{t_{A,B}} \\
|A| \times |B| \xrightarrow{\sigma_{A,B}} |B \times A|
\end{array}
\]

\[
\begin{array}{c}
|A| \otimes |1_\mathcal{D}| \xrightarrow{\rho_A} |A| \\
\downarrow{t_{1_\mathcal{D}, A}} \downarrow{t_{1_\mathcal{D}, A}} \\
|1_\mathcal{D} \otimes A| \xrightarrow{\lambda_A} |A|
\end{array}
\]

\[
\begin{array}{c}
|1_\mathcal{D}| \otimes |1_\mathcal{D}| \xrightarrow{1_\mathcal{D} \otimes 1_\mathcal{D}} |1_\mathcal{D}| \\
\downarrow{\pi_A} \downarrow{\pi_A} \\
|1_\mathcal{D}| \times |1_\mathcal{D}| \xrightarrow{1_\mathcal{D} \times 1_\mathcal{D}} |1_\mathcal{D}|
\end{array}
\]
where \( \alpha' \) and \( \sigma' \) are the canonical bijections, and \( \pi_A \) and \( \pi'_A \) are the projection maps. We shall often omit indices and write \( t \) for \( t_{A,B} \), \( \alpha \) for \( \alpha_{A,B,C} \), etc.

(5) A \( \mathcal{D} \)-monoid \( (M, \iota, \mu) \) is an object \( M \in \mathcal{D} \) equipped with two morphisms \( \iota : 1_\mathcal{D} \to M \) and \( \mu : M \otimes M \to M \) such that the following diagrams commute:

\[
(M \otimes M) \otimes M \xrightarrow{\alpha} M \otimes (M \otimes M) \\
M \otimes M \xrightarrow{\mu} M \quad M \otimes M \xrightarrow{\iota} M \\
M \otimes M \xrightarrow{\iota \otimes M} M \otimes M \xrightarrow{\rho} 1_\mathcal{D} \otimes M
\]

A morphism between \( \mathcal{D} \)-monoids \( (M, \iota_M, \mu_M) \) and \( (N, \iota_N, \mu_N) \) is a morphism \( h : M \to N \) in \( \mathcal{D} \) such that the following square commutes:

\[
\begin{array}{ccc}
M \otimes M & \xrightarrow{\mu_M} & M \\
\downarrow h \otimes h & & \downarrow h \iota_M \\
N \otimes N & \xrightarrow{\mu_N} & N
\end{array}
\]

Due to \( \otimes \) representing bimorphisms, the notion of \( \mathcal{D} \)-monoids and their morphisms given here is equivalent to the set-theoretic one of Definition 2.6.

### B Categorical Toolkit

We assume familiarity with basic concepts from category theory, like categories, functors, natural transformations, and (co-)limits (see e.g. [4]). However, we recall here some definitions and facts concerning adjunctions.

**B.1. Adjunctions.** Let \( \mathcal{A} \to \mathcal{X} \) be a functor between categories \( \mathcal{A} \) and \( \mathcal{X} \). Suppose that there exists, for each \( X \in \mathcal{X} \), an object \( FX \in \mathcal{A} \) and a morphism \( \eta_X : X \to UFX \) in \( \mathcal{X} \) with the following universal property: for any morphism \( f : A \to UB \) in \( \mathcal{X} \) with \( B \in \mathcal{B} \), there is a unique \( \xi : FX \to B \) in \( \mathcal{B} \) (called the adjoint transpose of \( f \)) with \( U(\xi) \circ \eta_X = f \). In this case the object map \( X \mapsto FX \) extends uniquely to a functor \( F : \mathcal{X} \to \mathcal{A} \) such that \( \eta : \text{Id}_\mathcal{X} \to UF \) becomes a natural transformation, and we say that the functors \( U \) and \( F \) form an adjunction (commonly denoted by \( F \dashv U : \mathcal{A} \to \mathcal{X} \)). The functor \( U \) is the right adjoint, \( F \) the left adjoint, and the natural transformation \( \eta \) the unit of the adjunction. \( \eta \) induces another natural transformation \( \varepsilon : FU \to \text{Id}_\mathcal{A} \) with components \( \varepsilon_A := \overline{\eta}_{UAX} \), called the counit of the adjunction. The universal property gives rise to an isomorphism \( \mathcal{A}(FX, A) \cong \mathcal{B}(X, U A) \) natural in \( X \in \mathcal{X} \) and \( A \in \mathcal{A} \).

An important fact about adjunctions is that right adjoints preserve limits, and dually left adjoints preserve colimits (and thus, in particular, epimorphisms).

A typical source of adjunctions are free constructions in algebra. For any variety \( \mathcal{D} \) of algebras or ordered algebras, the forgetful functor \( | - | : \mathcal{D} \to \text{Set} \)
Schützenberger Products in a Category

(mapping an algebra to its underlying set) has the left adjoint $\Psi : \text{Set} \to \mathcal{D}$ that maps a set $X$ to the free algebra $\Psi X$ in $\mathcal{D}$ generated by $X$. The unit $\eta_X : X \to |\Psi X|$ is the inclusion of generators. The freeness of $\Psi X$ amounts exactly to the universal property of $\eta_X$.

### B.2. Composition of adjunctions

The composite of two adjunctions

$$\left(F \dashv U : \mathcal{A} \to \mathcal{B}, \eta, \varepsilon \right) \quad \text{and} \quad \left(G \dashv V : \mathcal{B} \to \mathcal{C}, \eta', \varepsilon' \right)$$

is the adjunction $FG \dashv VU : \mathcal{A} \to \mathcal{C}$ with unit $\eta' \eta X : X \to VUFGX$ and counit $FGVUA \xrightarrow{\varepsilon \varepsilon' A} FA \xrightarrow{\varepsilon A} A (A \in \mathcal{A})$.

### B.3. Yoneda lemma (weak form)

Let $\mathcal{A}$ be a category and $A \in \mathcal{A}$. The hom-functor $\mathcal{A}(A,-) : \mathcal{A} \to \text{Set}$ maps an object $B \in \mathcal{A}$ to the set $\mathcal{A}(A,B)$ of morphisms from $A$ to $B$, and a morphism $f : B \to B'$ to the function $\mathcal{A}(A,f) : \mathcal{A}(A,B) \to \mathcal{A}(A,B')$, $g \mapsto f \circ g$.

The hom-functor determines objects of $\mathcal{A}$ up to isomorphism: any natural isomorphism $\theta : \mathcal{A}(A,-) \cong \mathcal{A}(A',-)$ with $A, A' \in \mathcal{A}$ yields an isomorphism $\theta_A(id_A) : A' \cong A$.

### C Monoidal Adjunctions

Here we present some well-known facts about monoidal functors and adjunctions. Since these facts appear scattered throughout the literature or are folklore in category theory, we sketch the proofs for some statements for the convenience of the reader. In the following, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be commutative varieties of (ordered) algebras; we remark that all concepts treated in this section can be introduced in a more general form for monoidal categories, see e.g. [4].

Recall from Definition 2.12 the notion of a monoidal functor and a monoidal natural transformation. Recall also from A.7 that $\mathcal{D}$-monoids $(M,1,\cdot)$ (see Definition 2.6) can be represented as triples $(M, \iota, \mu)$.

**Lemma C.1.** Any monoidal functor $(G, \theta) : \mathcal{A} \to \mathcal{B}$ lifts to a functor $\overline{G} : \text{Mon}(\mathcal{A}) \to \text{Mon}(\mathcal{B})$ such that the following diagram commutes, where $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are the forgetful functors:

$$\text{Mon}(\mathcal{A}) \xrightarrow{\overline{G}} \text{Mon}(\mathcal{B})$$

$$\begin{array}{ccc}
\text{Mon}(\mathcal{A}) & \xrightarrow{U_{\mathcal{A}}} & \mathcal{A} \\
\xrightarrow{\overline{G}} & & \xrightarrow{U_{\mathcal{B}}} \\
\mathcal{B} & \xrightarrow{G} & \mathcal{B}
\end{array}$$

Explicitly, $\overline{G}$ maps an $\mathcal{A}$-monoid $(M, \iota, \mu)$ to the $\mathcal{B}$-monoid

$$(G(M), 1_{\mathcal{B}} \xrightarrow{\theta} G1_{\mathcal{A}} \xrightarrow{G\iota} GM, GM \circ GM \xrightarrow{G\mu} G(M \circ M))$$
and an \( \mathscr{A} \)-monoid morphism \( f \) to \( G_f \). Moreover, any monoidal natural transformation \( \varphi : (G, \theta) \to (H, \sigma) \) yields a natural transformation \( \overline{\varphi} : \overline{G} \to \overline{H} \) with components
\[
\overline{\varphi}_{(M, \iota, \mu)} = \varphi_M : \overline{G}(M, \iota, \mu) \to \overline{H}(M, \iota, \mu).
\]

**Proof.** 1. It is straightforward to show that \((GM, G \circ \theta, G \circ \theta_{M, M})\) is a \( \mathscr{B} \)-monoid. For example, associativity is established by the commutative diagram below. Part (1) commutes since \( G \) is monoidal; (2) commutes since \((M, \iota, \mu)\) is a monoid; (3) and (4) commute because \( \theta_{A,B} : GA \otimes GB \to G(A \otimes B) \) is natural in \( A \) and \( B \).
\[
\begin{array}{ccc}
(GM \otimes GM) \otimes GM & \xrightarrow{\alpha_{GM,GM,GM}} & GM \otimes (GM \otimes GM) \\
\theta_{M,M,GM} \downarrow & & \downarrow \theta_{M,M,GM} \\
(GM \otimes M) \otimes GM & \xrightarrow{\alpha_{GM,GM}} & GM \otimes (GM \otimes M) \\
\theta_{M,G,M} \downarrow & & \downarrow \theta_{M,G,M} \\
1GM \otimes GM & \xrightarrow{\theta_{M,M}} & GM \otimes M \\
\theta_{M,M} \downarrow & & \downarrow \theta_{M,M} \\
G(M \otimes M) & \xrightarrow{G\alpha_{GM,GM}} & G(M \otimes (M \otimes M)) \\
\theta_{M,M} \downarrow & & \downarrow \theta_{M,M} \\
G(M \otimes M) & \xrightarrow{G\mu} & G(M \otimes M) \\
\theta_{M,M} \downarrow & & \downarrow \theta_{M,M} \\
GM & \xrightarrow{G\mu} & GM
\end{array}
\]

The unit laws follow in a similar way.

Assume that \( f : (M, \mu, \mu_M) \to (N, \iota_N, \mu_N) \) is an \( \mathscr{A} \)-monoid morphism. Then the following diagram commutes:
\[
\begin{array}{ccc}
GM \otimes GM & \xrightarrow{\theta_{M,M}} & G(M \otimes M) \\
Gf \otimes Gf & \xrightarrow{Gf \otimes f} & Gf \\
GN \otimes GN & \xrightarrow{\theta_{N,N}} & G(N \otimes N) \\
Gf & \xrightarrow{Gf} & GN
\end{array}
\]

where the right square commute because \( f \) is a \( \mathscr{D} \)-monoid morphism and the left square commutes as \( \theta_{M,M} \) is natural. Together with the corresponding diagram for the preservation of the unit, this shows that \( Gf : \overline{G}(M, \iota_M, \mu_M) \to \overline{G}(N, \iota_N, \mu_N) \) is a \( \mathscr{B} \)-monoid morphism.

2. To show that every monoidal natural transformation \( \varphi \) lifts to a natural transformation from \( \overline{G} \) to \( \overline{H} \), it suffices to show that \( \varphi_M \) is a \( \mathscr{B} \)-monoid morphism for every \( \mathscr{A} \)-monoid \((M, \iota, \mu)\). The preservation of the multiplication follows...
from the following diagram:

\[
\begin{array}{ccc}
GM \otimes GM & \xrightarrow{\psi_M \otimes \psi_M} & HM \otimes HM \\
\downarrow \theta_{M,M} & & \downarrow \sigma_{M,M} \\
G(M \otimes M) & \xrightarrow{\psi_{M \otimes M}} & H(M \otimes M) \\
\downarrow G\mu & & \downarrow H\mu \\
GM & \xrightarrow{\psi_M} & HM
\end{array}
\]

where the upper square uses that \( \psi \) is a monoidal natural transformation and the lower square is the naturality of \( \psi \). Similarly for the preservation of the unit.

\[\square\]

**Lemma C.2.** For any two monoidal functors \((G, \theta): \mathcal{A} \to \mathcal{B}\) and \((H, \sigma): \mathcal{B} \to \mathcal{C}\) the composite \(HG\) becomes a monoidal functor via

\[
(H\theta \circ \sigma)_{A,B} = H(GA) \otimes H(GB) \xrightarrow{\sigma_{A,B}} H(GA \otimes GB) \xrightarrow{H\theta_{GA,GB}} H(G(A \otimes B))
\]

\[
G\theta \circ \sigma = 1_{\mathcal{C}} \xrightarrow{\sigma} H1_{\mathcal{B}} \xrightarrow{G\theta} HG1_{\mathcal{A}}.
\]

**Proof.** The naturality of \((H\theta \circ \sigma)_{A,B}\) follows from the naturality of \(\sigma_{A,B}\) and \(H\theta_{GA,GB}\). It remains to verify the diagrams in Definition \[\text{(2.12)}\]. For example, for the left
In (1) we use that $H$ is monoidal, in (2) that $G$ is monoidal, in (3) the naturality of $\theta$, and in (4) the naturality of $\sigma$. Similarly for the other two diagrams in Definition 2.12.

**Example C.3.** (1) For any commutative variety $\mathcal{D}$ of algebras or ordered algebras, the forgetful functor $|-|: \mathcal{D} \to \textbf{Set}$ is a monoidal functor w.r.t. the maps

$$i: 1 \mapsto |1_{\mathcal{D}}| = |\Psi 1| \quad \text{and} \quad t_{A,B}: |A| \times |B| \to |A \otimes B|$$

where $i$ is the inclusion of the generator. Indeed, by Definition 2.12 we need to verify that the following diagrams commute for all $A, B \in \mathcal{D}$:
The left adjoint \( \Psi : \mathcal{D} \to \mathcal{C} \) to \( - \) : \( \mathcal{D} \to \mathbf{Set} \) is also monoidal. Indeed, observe that for any two sets \( X \) and \( Y \) we have the following bijections (natural in \( A \in \mathcal{D} \), cf. \( \Delta_7 \)):

\[
\mathcal{D}(\Psi(X \times Y), A) \cong \mathbf{Set}(X \times Y, |A|)
\]

\[
\cong \mathbf{Set}(X, |A|^Y)
\]

\[
\cong \mathbf{Set}(X, [\Psi Y, A])
\]

\[
\cong \mathcal{D}(\Psi X, [\Psi Y, A])
\]

\[
\cong \mathcal{D}(\Psi X \otimes \Psi Y, A).
\]

The Yoneda lemma, see \( \Delta_3 \), gives a natural isomorphism \( \Psi(X \times Y) \cong \Psi X \otimes \Psi Y \), mapping a pair \( (x, y) \in X \times Y \) to \( x \otimes y \in [\Psi X \otimes \Psi Y] \). Its inverse \( \theta_{X,Y} : \Psi X \otimes \Psi Y \cong \Psi(X \times Y) \) together with the morphism \( \theta = \text{id} : 1_{\mathcal{D}} \to 1_{\mathcal{C}} \), makes \( \Psi \) a monoidal functor, i.e. the following diagrams commute for all sets \( X, Y, Z \):

\[
\begin{array}{ccc}
(\Psi X \otimes \Psi Y) \otimes \Psi Z & \xrightarrow{\alpha} & \Psi X \otimes (\Psi Y \otimes \Psi Z) \\
\theta \otimes \Psi Z & \downarrow & \Psi(\theta \otimes \Psi Z) \\
\Psi(X \times Y) \otimes \Psi Z & \cong & \Psi X \otimes (\Psi Y \times Z) \\
\cong & \Psi((X \times Y) \times Z) & \cong \Psi(X \times (Y \times Z)) \\
\Psi((X \times Y) \otimes Z) & \xrightarrow{\Psi \alpha} & \Psi(X \times (Y \times Z)) \\
\end{array}
\]

This follows directly from the definitions of \( \theta, \alpha, \lambda, \rho \). For example, both legs of the left-hand diagram map an element \((x \otimes y) \otimes z \in [\Psi X \otimes \Psi Y] \otimes \Psi Z \) (with \( x \in X, y \in Y, z \in Z \)) to \((x, (y, z)) \in X \times (Y \times Z) \in \Psi(X \times (Y \times Z)) \). Since the elements \((x \otimes y) \otimes z \) generate the algebra \( [\Psi X \otimes \Psi Y] \otimes \Psi Z \), see \( \Delta_7 \) this shows that the diagram commutes. Similarly for the other two diagrams.

(3) The adjunction \( \Psi \dashv | - | : \mathcal{D} \to \mathbf{Set} \) is monoidal (see Definition \( \Delta_2, \Delta_17 \)). To see this, we need to show that the unit \( \eta : \text{Id}_{\mathbf{Set}} \to |\Psi| \) and the counit \( \varepsilon : \Psi \circ | - | : \mathcal{D} \to \mathbf{Set} \) are monoidal natural transformations. We only prove that \( \varepsilon \) is monoidal, since the proof for \( \eta \) is similar. Note first that by \( \Psi \circ | - | : \mathcal{D} \to \mathcal{D} \) is meant the composite monoidal functor in the sense of Lemma \( \Delta_2, \Delta_15 \). Thus
the associated morphisms are

\[ \Psi|A| \otimes \Psi|B| \xrightarrow{\theta_{|A||B|}} \Psi(|A| \times |B|) \xrightarrow{\psi_{A,B}} \Psi|A \otimes B| \]

and

\[ 1_{\mathcal{D}} \xrightarrow{id = \eta_1} \Psi 1 \xrightarrow{\psi_1} \Psi|1_{\mathcal{D}}|. \]

To show that \( \varepsilon \) is a monoidal natural transformation, we need to verify that the following two diagrams commute, cf. Definition \ref{Def:MonoidalNatTrans}.

\[ \Psi|A| \otimes \Psi|B| \xrightarrow{\varepsilon_A \otimes \varepsilon_B} A \otimes B \]

\[ \Psi(|A| \times |B|) \]

\[ \theta_{A,B} \]

\[ \Psi|A \otimes B| \xrightarrow{\varepsilon_{A \otimes B}} A \otimes B \]

But this follows immediately from the definitions of \( \varepsilon \), \( \theta \) and \( i \). Both legs of the left diagram map an element \( a \otimes b \in \Psi|A| \otimes \Psi|B| \) (with \( a \in |A| \) and \( b \in |B| \)) to \( a \otimes b \). Since \( \Psi|A| \otimes \Psi|B| \) is generated by these elements, this shows that the left diagram commutes. In the right diagram, both legs map the generator of \( 1_{\mathcal{D}} \) to itself, and thus both legs are the identity morphism.

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(1) An $S$-module $A$ and the corresponding algebra $UA$ in $\mathcal{D}$ have the same underlying set, i.e. $|A|_S = |UA|_\mathcal{D}$. Likewise, an $S$-module morphism $f : A \to B$ and the corresponding morphism $Uf : UA \to UB$ in $\mathcal{D}$ have the same underlying function.

(2) Every bimorphism from $A, B$ to $C$ in $\mathbb{S}$-Mod is also a bimorphism from $UA, UB$ to $UC$ in $\mathcal{D}$. To see this, suppose that $f : |A|_S \otimes |B|_S \to |C|_S$ is a bimorphism of $\mathbb{S}$-Mod, and let $f' : A \otimes B \to C$ be the corresponding $S$-linear map. Consider the diagram below:

$$
\begin{array}{ccc}
|UA \otimes UB|_\mathcal{D} & \xrightarrow{f'} & |UC|_\mathcal{D} \\
\left|\begin{array}{c}
|A|_S \otimes |B|_S \\
\end{array}\right| & \xrightarrow{\theta'} & \left|\begin{array}{c}
|C|_S \\
\end{array}\right|
\end{array}
$$

Here $\theta'$ and $\theta$ denote the universal bimorphisms. The left part commutes because the monoidal functor $|-|_S$ is the composite of the monoidal functors $U$ and $|-|_\mathcal{D}$ (as $U$ is part of a concrete monoidal adjunction); cf. Lemma C.2 and Example 2.13 (1). The upper right square commutes because $f'$ and $Uf'$ have the same underlying function, see (1). The triangle is the definition of $f'$. It follows that $f : |UA|_\mathcal{D} \times |UB|_\mathcal{D} \to |UC|_\mathcal{D}$ is a bimorphism from $UA, UB$ to $UC$ in $\mathcal{D}$, being the composite of the $\mathcal{D}$-bimorphism $\theta$ with the $\mathcal{D}$-morphism $Uf' \circ \theta'$. The upper right square commutes because $f'$ and $Uf'$ have the same underlying function, see (1). The triangle is the definition of $f'$. It follows that $f : |UA|_\mathcal{D} \times |UB|_\mathcal{D} \to |UC|_\mathcal{D}$ is a bimorphism from $UA, UB$ to $UC$ in $\mathcal{D}$, being the composite of the $\mathcal{D}$-bimorphism $\theta$ with the $\mathcal{D}$-morphism $Uf' \circ \theta'$. The upper right square commutes because $f'$ and $Uf'$ have the same underlying function, see (1). The triangle is the definition of $f'$. It follows that $f : |UA|_\mathcal{D} \times |UB|_\mathcal{D} \to |UC|_\mathcal{D}$ is a bimorphism from $UA, UB$ to $UC$ in $\mathcal{D}$, being the composite of the $\mathcal{D}$-bimorphism $\theta$ with the $\mathcal{D}$-morphism $Uf' \circ \theta'$.

(3) As a consequence of (2), for any $S$-algebra $A$ the multiplication $|A| \times |A| \to |A|$ is a bimorphism from $UA, UA$ to $UA$. Moreover, the sum $|A| \times |A| \xrightarrow{+} |A|$ carries a morphism $+ : A \times A \to A$ in $\mathbb{S}$-Mod because the latter is a commutative variety. Applying $U$, we obtain a morphism $+ : UA \times UA \to UA$ in $\mathcal{D}$. Analogously for the scalar product $\lambda \cdot - : A \to A$ ($\lambda \in S$).

**Remark D.3.** For any two morphisms $p : M \to S$ and $q : N \to S$ in $\mathcal{D}$ and elements $m \in |M|$ and $n \in |N|$, we have

$$
(p \circ (- \cdot m)) \otimes (q \circ (- \cdot n)) = (p \otimes q) \circ (- \cdot (m \otimes n)) \quad (D.1)
$$

where $- \cdot m : M \to M$, $- \cdot n : N \to N$ and $- \cdot (m \otimes n) : M \otimes N \to M \otimes N$ are morphisms of $\mathcal{D}$ because $M, N$ and $M \otimes N$ are $\mathcal{D}$-monoids (i.e. their multiplication is a $\mathcal{D}$-bimorphism). Indeed, for any $m' \in |M|$ and $n' \in |N|$ we have

$$
[p \circ (- \cdot m)) \otimes (q \circ (- \cdot n))](m' \otimes n') = p(m' \cdot m) \otimes q(n' \cdot n) = p \otimes q((m' \otimes n') \cdot (m \otimes n)) = [(p \otimes q) \circ (- \cdot (m \otimes n))](m' \otimes n').
$$

In the first step we use the definition of the tensor product of two morphisms, see Remark 2.5. In the second step, we use the definition of the multiplication in...
The proof of Theorem A.7. The last step is obvious. Since the elements \( m' \otimes n' \) generate \( M \otimes N \), see Remark 2.1. this proves (D.4).

Proof (Lemma 4.3). We prove the case where \( \mathcal{D} \) is a variety of ordered algebras, the unordered case being analogous. Consider the preorder \( \preceq \) on \( M \otimes N \) defined \( x \preceq y \iff \pi(x) \preceq \pi(y) \). By A.4 we only need to show that \( \preceq \) is a stable preorder of the \( \mathcal{D} \)-monoid \( M \otimes N \) (cf. Remark 2.1); then, since \( \pi \) is surjective,

\[
\pi(x) \bullet \pi(y) := \pi(x \bullet y) \quad (x, y \in [M \otimes N])
\]
gevens a well-defined \( \mathcal{D} \)-monoid structure on \( M \ast N \) making \( \pi \) a \( \mathcal{D} \)-monoid morphism.

In particular for \( x = m \otimes n \) and \( y = m' \otimes n' \), the multiplication is given by

\[
(m \ast n) \bullet (m' \ast n') = \pi(m \otimes n) \bullet \pi(m' \otimes n') = \pi((m \otimes n) \bullet (m' \otimes n')) = \pi((mm') \otimes (nn')) = (mm') \ast (nn')
\]

Thus let us show that \( \preceq \) is indeed stable. Clearly \( x \preceq y \) in \( M \otimes N \) implies \( x \preceq y \) because \( \pi \) is monotone. Also, \( \preceq \) is stable w.r.t. all operations of \( \mathcal{D} \), since \( \pi \) is morphism of \( \mathcal{D} \). It remains to show that \( x \preceq y \) implies \( x \bullet z \preceq y \bullet z \) and \( z \bullet x \preceq z \bullet y \) (equivalently, \( \pi(x) \preceq \pi(y) \) implies \( \pi(x \bullet z) \preceq \pi(y \bullet z) \) and \( \pi(z \bullet x) \preceq \pi(z \bullet y) \)) for all \( x, y, z \in [M \otimes N] \). We may assume that \( z = m \otimes n \) for some \( m \in |M| \) and \( n \in |N| \); since \( M \otimes N \) is generated by these elements, see A.7 and \( \bullet \) is a \( \mathcal{D} \)-bimorphism, this implies the statement for all \( z \). So suppose that \( \pi(x) \preceq \pi(y) \). By (D.4) this implies that

\[
\sigma \circ (p \otimes q)(x) \preceq \sigma \circ (p \otimes q)(y)
\]

evens for all \( p : M \to S \) and \( q : N \to S \) in \( \mathcal{D} \).

In particular we get, for all \( p \) and \( q \),

\[
\sigma \circ ((p \circ (- \cdot m)) \otimes (q \circ (- \cdot n)))(x) \preceq \sigma \circ ((p \circ (- \cdot m)) \otimes (q \circ (- \cdot n)))(y)
\]

using that \( - \cdot m : M \to M \) and \( - \cdot n : N \to N \) are morphisms of \( \mathcal{D} \) since the multiplication of \( M \) resp. \( N \) is a bimorphism. Equivalently, by (D.4),

\[
\sigma \circ (p \otimes q) \circ (- \cdot (m \otimes n))(x) \preceq \sigma \circ (p \otimes q) \circ (- \cdot (m \otimes n))(y)
\]

Thus, since \( z = m \otimes n \),

\[
\sigma \circ (p \otimes q)(x \bullet z) \preceq \sigma \circ (p \otimes q)(y \bullet z).
\]

for all \( p \) and \( q \). By the definition of \( \pi \), this means precisely that \( \pi(x \bullet z) \leq \pi(y \bullet z) \).

The proof of \( \pi(z \bullet x) \leq \pi(z \bullet y) \) is symmetric. \( \Box \)

**Lemma D.4.** For any \( S \)-algebra \( A \) and any two \( \mathcal{D} \)-monoid morphisms \( f : M \to UA \) and \( g : N \to UA \), the product \( M \times UA \times N \) in \( \mathcal{D} \) carries a \( \mathcal{D} \)-monoid structure with unit \((1, 0, 1)\) and multiplication

\[
(m, a, n)(m', a', n') := (mn', f(m) \cdot a' + a \cdot g(n'), nn').
\]
That \( \pi_M \times \pi_N : A^{f,g} \rightarrow M \) and \( \pi_N : A^{f,g} \rightarrow N \) are \( \mathcal{D} \)-monoid morphisms.

Proof. That \((1,0,1)\) is the unit is clear since \( f(1) = 1 \) and \( g(1) = 1 \). For associativity, we compute

\[
\begin{align*}
(m,a,n)(m',a',n')(m'',a'',n'') &= (mm', f(m) \cdot a' + a \cdot g(n'), mn')(m'', a'', n'') \\
&= ((mm')n'', f(mm') \cdot a'' + [f(m) \cdot a' + a \cdot g(n') \cdot g(n'') \cdot (mm')n'') \\
&= (m(m'm''), f(m) \cdot [f(m') \cdot a'' + a' \cdot g(n'') + a \cdot g(n'')]) + a \cdot g(n''), n'(n''n'')) \\
&= (m,a,n)(m''', f(m') \cdot a'' + a' \cdot g(n''), n'n'') \\
&= (m,a,n)(m',a',n')(m'',a'',n'') 
\end{align*}
\]

It remains to verify that the multiplication of \( A^{f,g} \) is a \( \mathcal{D} \)-bimorphism. For simplicity, let us just prove that for any binary operation \( \gamma \) in the signature of \( \mathcal{D} \), the multiplication preserves \( \gamma \) in the right component. Indeed, we have

\[
\begin{align*}
(m,a,n)[\gamma((m',a',n'),(m'',a'',n''))] &= (m,a,n)(\gamma(m',m''), \gamma(a',a''), \gamma(n',n'')) \\
&= (m\gamma(m',m''), f(m) \cdot \gamma(a',a'') + a \cdot g(\gamma(n',n'')), n\gamma(n',n'')) \\
&= (\gamma(mm',mm''), \gamma(f(m) \cdot a', f(m) \cdot a'') + a \cdot \gamma(g(n'), g(n''))), \gamma(mm'',a,a'')) \\
&= (\gamma(mm',mm''), \gamma(f(m) \cdot a' \cdot a \cdot g(n'), f(m) \cdot a'' + a \cdot g(n''))), \gamma(mm'',a,a'')) \\
&= \gamma((mm', f(m) \cdot a' + a \cdot g(n'), nn'), (mm'', f(m) \cdot a'' + a \cdot g(n''), nn'')) \\
&= \gamma((m,a,n)(m',a',n'), (m,a,n)(m'',a'',n'')) 
\end{align*}
\]

Explanation of the individual steps: (1) Definition of the operation \( \gamma \) in the product \( M \times UA \times N \) in \( \mathcal{D} \). (2) Definition of the multiplication in \( A^{f,g} \). (3) In the first and third component we use that the multiplication of \( M \) and \( N \) is a \( \mathcal{D} \)-bimorphism; in the second component we use that the multiplication of \( A \) is a \( \mathcal{D} \)-bimorphism on \( UA \), see Remark D.2, and moreover that \( g \) is a \( \mathcal{D} \)-morphism. (4) Again we use that the multiplication of \( A \) is a bimorphism on \( UA \). (5) + is a morphism of \( \mathcal{D} \), see Remark D.2, (6) Definition of the operation \( \gamma \) in the product \( M \times UA \times N \) in \( \mathcal{D} \). (7) Definition of the multiplication in \( A^{f,g} \).

That \( \pi_M \) is a \( \mathcal{D} \)-monoid morphism follows from the computation

\[
\pi_M((m,a,n)(m',a',n')) = \pi_M(mm', f(m) \cdot a' + a \cdot g(n'), mn') \\
\overset{m'n'm}{=} \pi_M(m,a,n)\pi_M(m',a',n').
\]

Analogously for \( \pi_N \).

\[\square\]

Proof (Lemma D.4). Consider the following morphisms in \( \mathcal{D} \):

\[
f \equiv (M \rightarrow M \otimes 1_{\mathcal{D}} \overset{M \otimes \mathcal{D}}{\rightarrow} M \otimes N \overset{\pi}{\rightarrow} M \ast N \overset{\partial}{\rightarrow} UF(M \ast N))
\]
\[ g \equiv (N \xrightarrow{\lambda} 1 \otimes N \xrightarrow{\iota_M \otimes \iota_N} M \otimes N \xrightarrow{\pi} M * N \xrightarrow{\eta} UF(M * N)) \]

Note that \( \pi \) is a \( D \)-monoid morphism by Lemma 4.3. \( \bar{\eta} = \eta : M * N \rightarrow UF(M * N) \) is a \( D \)-monoid morphism by Lemma 2.14 and that \( \rho, \lambda, M \otimes \iota_N \) and \( \iota_M \otimes N \) are \( D \)-monoid morphisms follows easily from the definition of the monoid structure on tensor products, see Remark 2.13. Thus \( f \) and \( g \) and \( D \)-monoid morphisms. Applying Lemma 1.14 to the \( S \)-algebra \( A = F(M * N) \) and the morphisms \( f \) and \( g \) yields the Schützenberger product \( M \circ N = [F(M * N)]f,g \).

**Notation D.5.** We denote the product projections in \( D \) by

\[
\begin{array}{ccc}
M & \xrightarrow{\pi_M} & M \circ N \\
\downarrow & & \downarrow \pi_{MN} \\
UF(M \ast N) & \xrightarrow{\pi_N} & N
\end{array}
\]

For any \( D \)-monoid morphism \( f : \Psi \Sigma^* \rightarrow M \circ N \), we put

\[ f_M := \pi_M \circ f : \Psi \Sigma^* \rightarrow M, \quad f_N := \pi_N \circ f : \Psi \Sigma^* \rightarrow N, \]

and

\[ f_{MN} := \pi_{MN} \circ f : \Psi \Sigma^* \rightarrow UF(M \ast N). \]

Recall that we put \( m \ast n := \pi(m \otimes n) \) for \( m \in |M| \) and \( n \in |N| \), and that \( \eta : M \ast N \rightarrow UF(M \ast N) \) denotes the universal map.

The following lemma appears in [21] for the case \( D = \text{Set} \):

**Lemma D.6.** For any \( D \)-monoid morphism \( f : \Psi \Sigma^* \rightarrow M \circ N \) and \( u \in \Sigma^* \) we have

\[ f_{MN}(u) = \sum_{u = vaw \in \Sigma^*} \eta(f_M(v) \ast 1) \cdot f_{MN}(a) \cdot \eta(1 \ast f_N(w)), \]

where the sum ranges over all factorizations \( u = vaw \) with \( a \in \Sigma \) and \( v, w \in \Sigma^* \).

**Proof.** The proof is by induction on the length of \( u \). For \( u = \varepsilon \), we have \( f(\varepsilon) = (1, 0, 1) \) since \( f \) is a \( D \)-monoid morphism, and thus \( f_{MN}(\varepsilon) = 0 \) (the empty sum). Now suppose that the formula holds for some \( u \in \Sigma^* \), and consider a word \( ub \) with \( b \in \Sigma \). Then

\[
f_{MN}(ub) = \pi_{MN}(f(ub)) \tag{1}
= \pi_{MN}(f(u)f(b)) \tag{2}
= \eta(f_M(u) \ast 1) \cdot f_{MN}(b) + f_M(u) \cdot \eta(1 \ast f_N(b)) \tag{3}
= \eta(f_M(u) \ast 1) \cdot f_{MN}(b) \]
\[
+ \sum_{a = vaw} \eta(f_M(v) \ast 1) \cdot f_{MN}(a) \cdot \eta(1 \ast f_N(w)) \cdot \eta(1 \ast f_N(b)) \tag{4}
= \eta(f_M(u) \ast 1) \cdot f_{MN}(b)
\]
\[ + \sum_{v=w} \eta(f_M(v) \cdot 1) \cdot f_MN(a) \cdot \eta(1 \cdot f_N(w)) \cdot \eta(1 \cdot f_N(b)) \quad (5) \]

\[ = \eta(f_M(u) \cdot 1) \cdot f_MN(b) \cdot \eta(1 \cdot f_N(\varepsilon)) + \sum_{u=v\neq w} \eta(f_M(v) \cdot 1) \cdot f_MN(a) \cdot \eta(1 \cdot f_N(wb)) \quad (6) \]

\[ = \sum_{ub=v\neq w'} \eta(f_M(v) \cdot 1) \cdot f_MN(a) \cdot \eta(1 \cdot f_N(w')) \quad (7) \]

Explanation of the individual steps: (1) Definition of \( f_MN \). (2) \( f \) is a \( \mathcal{D} \)-monoid morphism. (3) Definition of the multiplication in \( M \circ N \). (4) Induction hypothesis. (5) Distributive law in the K-algebra \( \mathcal{P}(M \ast N) \). (6) Definition of the multiplication in \( M \ast N \), \( \eta = \eta : M \ast N \rightarrow UF(M \ast N) \) is a \( \mathcal{D} \)-monoid morphism (see 2.14) and \( f_N \) is a \( \mathcal{D} \)-monoid morphism. (7) Any factorization \( ub = vaw' \) has either \( w' = wb \) for some \( w \), or \( a = b \) and \( w' = \varepsilon \). \( \square \)

**Notation D.7.** (1) For any two morphisms \( p : M \rightarrow S \) and \( q : N \rightarrow S \) in \( \mathcal{D} \), we denote by \( \overrightarrow{pq} : F(M \ast N) \rightarrow S \) the adjoint transpose of \( p \ast q : M \ast N \rightarrow S \).

(2) Recall from Definition 3.3 the morphism \( L_{\mathcal{D}} : \Psi \Sigma^* \rightarrow S \) in \( \mathcal{D} \) corresponding to a language \( L : \Sigma^* \rightarrow S \). Since \( L \) and \( L_{\mathcal{D}} \) agree on \( \Sigma^* \), we usually drop the index and write \( L \) for \( L_{\mathcal{D}} \).

**Proof (Theorem 4.8).** Let \( g : \Psi \Sigma^* \rightarrow M \) and \( h : \Psi \Sigma^* \rightarrow N \) be \( \mathcal{D} \)-monoid morphisms recognizing \( K \) resp. \( L \). Thus there exist morphisms \( p : M \rightarrow S \) and \( q : N \rightarrow S \) in \( \mathcal{D} \) with \( K = p \circ g \) and \( L = q \circ h \). Fix a letter \( a \in \Sigma \). We define a \( \mathcal{D} \)-monoid morphism \( f : \Psi \Sigma^* \rightarrow M \circ N \) that recognizes the languages \( K \), \( L \) and \( KaL \).

(1) Let \( f : \Psi \Sigma^* \rightarrow M \circ N \) be the unique \( \mathcal{D} \)-monoid morphism defined on letters \( b \in \Sigma \) by

\[ f(b) = \begin{cases} (g(b), 1, h(b)), & \text{if } b = a \\ (g(b), 0, h(b)), & \text{if } b \neq a. \end{cases} \]

Then \( f_M(b) = g(b) \) for all \( b \in \Sigma \) and therefore \( f_M = g \), since any \( \mathcal{D} \)-monoid morphism with domain \( \Psi \Sigma^* \) is determined by its values on the generators \( \Sigma \). Similarly, we have \( f_N = h \). Since \( f_MN(a) = 1 \) and \( f_MN(b) = 0 \) for \( b \neq a \), Lemma 4.6 gives, for all \( u \in \Sigma^* \),

\[ f_MN(u) = \sum_{v=u} \eta(g(v) \cdot h(w)). \quad (D.2) \]

(2) \( f \) recognizes the language \( K \) via the morphism \( p \circ \pi_M \), since

\[ K = p \circ g = p \circ f_M = (p \circ \pi_M) \circ f. \]

Analogously, \( f \) recognizes \( L \) via \( q \circ \pi_N \).
(3) We show that $f$ recognizes the language $KaL$ via the morphism

$$s \equiv (M \circ N \xrightarrow{\pi_N} UF(M \ast N) \xrightarrow{U(s') \pi} \Sigma).$$

Indeed, for all $u \in \Sigma^*$ we have

$$s \circ f(u) = \overline{p \ast q}(f_{MN}(u)) \quad \text{(def. } s \text{ and } f_{MN})$$

$$= \overline{p \ast q} \left( \sum_{u = v \cdot w} \eta(g(v) \ast h(w)) \right) \quad \text{(D.2)}$$

$$= \sum_{u = v \cdot w} \overline{p \ast q}(\eta(g(v) \ast h(w))) \quad \text{ (def. } \overline{p \ast q})$$

$$= \sum_{u = v \cdot w} [p \ast q](g(v) \ast h(w)) \quad \text{ (def. } p \ast q)$$

$$= \sum_{u = v \cdot w} \sigma(p(g(v)) \otimes q(h(w))) \quad \text{ (def. } p \otimes q)$$

$$= \sum_{u = v \cdot w} \sigma(K(v) \otimes L(w)) \quad \text{ (def. } \sigma)$$

$$= \sum_{u = v \cdot w} (KaL)(u) \quad \text{ (def. } KaL)$$

□

From now on, we suppose that the Assumptions 4.10 hold.

**Remark D.8.** The Assumptions 4.10(i) and (ii) have the following consequences:

1. $F$ preserves finite objects. Indeed, let $D$ be a finite object of $\mathcal{D}$, and express $D$ as a quotient $e : \Psi X \to D$ for some finite set $X$. Since the left adjoint $F$ preserves epimorphisms, see B.3, the morphism $F e : \mathcal{P}_f X = F \Psi X \to FD$ is an epimorphism in $\mathcal{S} \text{-Mod}$ and therefore surjective by Assumption 4.10(ii). Since $\mathcal{P}_f X$ is finite, so is $FD$.

2. $U$ also preserves finite objects by Remark D.2.

3. Consequently the Schützenberger product of two finite $\mathcal{D}$-monoids $M$ and $N$ is finite. To see this, recall from A.7 that the tensor product $M \otimes N$ is generated by the finite set $|M| \times |N|$ and is thus finite by Assumption 4.10(i). Therefore the quotient $M \ast N$ of $M \otimes N$ is also finite. Since $F$ and $U$ preserve finite objects by (1) and (2), $UF(M \ast N)$ is finite. Thus $M \circ N$ is finite, being carried by the finite object $M \times UF(M \ast N) \times N$.

**Definition D.9.** (1) The set of languages over $\Sigma$ forms an $\mathcal{S}$-algebra w.r.t. to the sum, scalar product and Cauchy product of languages, defined by

$$(K + L)(u) = K(u) + L(u), \quad (\lambda L)(u) = \lambda L(u), \quad (KL)(u) = \sum_{u = v \cdot w} K(v) \cdot L(w)$$
for languages $K, L : \Sigma^* \to S$, $\lambda \in S$, and $u \in \Sigma^*$. Identifying a letter $a \in \Sigma$ with the language $a : \Sigma^* \to S$ that sends $a$ to 1 and all other words to 0, the marked Cauchy product $KaL$ is thus the Cauchy product of the languages $K, a$ and $L$.

(2) The derivatives of a language $L : \Sigma^* \to S$ are the languages $a^{-1}L, La^{-1} : \Sigma^* \to S$ (for $a \in \Sigma$) with

\[
a^{-1}L(u) = L(au) \quad \text{and} \quad La^{-1}(u) = L(ua).
\]

**Lemma D.10.** Let $f : \Psi \Sigma^* \to M$ be a $\mathcal{D}$-monoid morphism. Then the set of languages recognized by $f$ is closed under sum, scalar products, and derivatives.

**Proof.** (1) **Closure under derivatives.** Suppose that $L : \Sigma^* \to S$ is a language recognized by $M$, i.e. there is a morphism $p : M \to S$ in $\mathcal{D}$ with $L = p \circ f$. We claim that $f$ recognizes the left derivative $a^{-1}L$ via the morphism $p' := p \circ (f(a) \bullet -)$. (Note that $f(a) \bullet - : M \to M$ is a morphism of $\mathcal{D}$ because $\bullet$ is a $\mathcal{D}$-bimorphism.) Indeed, we have for all $u \in \Sigma^*$:

\[
p'(f(u)) = p(f(a) \bullet f(u)) = p(f(au)) = L(au) = (a^{-1}L)(u).
\]

and thus $L = p' \circ f$, as claimed. Analogously for right derivatives.

(2) **Closure under sums.** Let $K$ and $L$ be two languages recognized by $f$, i.e. $K = p \circ f$ and $L = q \circ f$ for morphisms $p, q : M \to S$ in $\mathcal{D}$. Denote by $\overline{p}, \overline{q} : FM \to S$ the adjoint transposes of $p$ and $q$ in $\mathbb{S}\text{-Mod}$ (w.r.t. the adjunction $F \dashv U$). Since $\mathbb{S}\text{-Mod}$ forms a commutative variety, we have the morphism $\overline{p + q} : FM \to S$ in $\mathbb{S}\text{-Mod}$, defined by $[\overline{p + q}](x) = \overline{p}(x) + \overline{q}(x)$. Then $K + L$ is recognized by the morphism $U(\overline{p + q}) \circ \eta : M \to S$, since for all $u \in \Sigma^*$,

\[
[\overline{p + q}](\eta(f(u))) = \overline{p}(\eta(f(u)))+\overline{q}(\eta(f(u))) = p(f(u))+q(f(u)) = K(u)+L(u).
\]

Thus $K + L = U(\overline{p + q}) \circ \eta \circ f$, i.e. $f$ recognizes $K + L$.

(3) **Closure under scalar product.** Analogous to (2).

\[\square\]

**Proof (Theorem 4.13).** We first establish two preliminary technical results (steps (1) and (2)).

(1) Consider the commutative diagram below, where $t$ is the adjoint transpose of the universal bimorphism $t$; note that $S^*(-) = F\Psi$ since the adjunction $F \dashv U : \mathbb{S}\text{-Mod} \to \mathcal{D}$ is concrete.

\[
\begin{array}{cccccc}
|\mathbb{S}(\{M\} \times \{N\})| & \xrightarrow{\tau_\Psi} & |F(M \otimes N)| & \xrightarrow{F\pi} & |F(M * N)| \\
\eta \downarrow & & \eta \downarrow & & \eta \downarrow \\
|\Psi(|M| \times |N|)| & \xrightarrow{\tau} & |M \otimes N| & \xrightarrow{\pi} & |M * N| \\
|M| \times |N| & \xrightarrow{t} & & & &
\end{array}
\]
Since $F \dashv U$ is a concrete monoidal adjunction, precomposing the unit \( \eta : \mathcal{D} \to \mathcal{S}([M] \times [N]) \) with the unit \([M] \times [N] \to \mathcal{D} \to \mathcal{S}([M] \times [N]) \) of the adjunction \( \mathcal{S}(-) \dashv -| \) gives the unit \([M] \times [N] \to \mathcal{S}([M] \times [N]) \) of the adjunction \( \mathcal{S}(-) \dashv -| \) : \( \mathcal{S}\text{-Mod} \to \mathcal{Set} \). Thus the \( \mathcal{S}\text{-linear map} F\pi \circ FT \) maps a generator \((m,n)\) of \( \mathcal{S}([M] \times [N]) \) to \( \eta(m \ast n) \). Since the left adjoint \( F \) preserves epimorphisms, \( F\pi \) and \( FT \) are surjective by Assumption \[4.10\](ii). It follows that every element of the \( \mathcal{S}\text{-module} F(M \ast N) \) can be expressed as a linear combination \( \sum_{j=1}^{n} \lambda_{j} \eta(m_{j} \ast n_{j}) \) with \( \lambda_{j} \in S, m_{j} \in |M| \) and \( n_{j} \in |N| \).

(2) Let \( p : M \to S \) and \( q : N \to S \) be two morphisms in \( \mathcal{D} \), and let \( L \) be the language recognized by \( f \) via \( U(p \circ q) \circ \pi_{MN} \), i.e.

\[
L := U(p \circ q) \circ \pi_{MN} \circ f = U(p \circ q) \circ f_{MN}. \tag{D.3}
\]

By (1), each element \( f_{MN}(a) \in |F(M \ast N)| \) with \( a \in \Sigma \) can be expressed as a linear combination

\[
f_{MN}(a) = \sum_{j=1}^{n_{a}} \lambda_{j}^{a} \eta(m_{j}^{a} \ast n_{j}^{a}). \tag{D.4}
\]

with \( \lambda_{j}^{a} \in S, m_{j}^{a} \in |M| \) and \( n_{j}^{a} \in |N| \). For \( a \in \Sigma \) and \( j = 1, \ldots, n_{a} \), put

\[
L_{M}^{a,j} := p \circ (\bullet \circ m_{j}^{a}) \circ f_{M} \quad \text{and} \quad L_{N}^{a,j} := q \circ (n_{j}^{a} \bullet \circ \cdot) \circ f_{N}, \tag{D.5}
\]

where \( \bullet \circ m_{j}^{a} : M \to M \) and \( n_{j}^{a} \bullet \circ \cdot : N \to N \) are morphisms in \( \mathcal{D} \) because the monoid multiplication of \( M \) resp. \( N \) is a \( \mathcal{D}\text{-bimorphism} \). Then \( L \) can be expressed as the following linear combination of languages (cf. Definition \[D.9\]):

\[
L = \sum_{a \in \Sigma} \sum_{j=1}^{n_{a}} \lambda_{j}^{a} (L_{M}^{a,j} \circ L_{N}^{a,j}). \tag{D.6}
\]

To prove this, we compute for all \( u \in \Sigma^{*} \):

\[
L(u) = p \circ q(f_{MN}(u)) \tag{1}
\]

\[
= p \circ q \left( \sum_{a} \sum_{u \in \Sigma^{*}} \eta(f_{M}(v) \ast 1) \cdot f_{MN}(a) \cdot \eta(1 \ast f_{N}(w)) \right) \tag{2}
\]

\[
= p \circ q \left( \sum_{a} \sum_{u \in \Sigma^{*}} \eta(f_{M}(v) \ast 1) \cdot \left( \sum_{j=1}^{n_{a}} \lambda_{j}^{a} \eta(m_{j}^{a} \ast n_{j}^{a}) \right) \cdot \eta(1 \ast f_{N}(w)) \right) \tag{3}
\]

\[
= p \circ q \left( \sum_{a} \sum_{u \in \Sigma^{*}} \sum_{j=1}^{n_{a}} \lambda_{j}^{a} \eta(f_{M}(v) \ast 1) \cdot \eta(m_{j}^{a} \ast n_{j}^{a}) \cdot \eta(1 \ast f_{N}(w)) \right) \tag{4}
\]

\[
= p \circ q \left( \sum_{a} \sum_{u \in \Sigma^{*}} \sum_{j=1}^{n_{a}} \lambda_{j}^{a} \eta((f_{M}(v)m_{j}^{a}) \ast (n_{j}^{a}f_{N}(w))) \right) \tag{5}
\]
Thus, by the assumptions on \( L \)
\[ \text{Moreover, for any pair of morphisms } \]
\[ \text{each } \exists \text{ morphism } \]
\[ \text{recognized by } \]
\[ \text{For any } \]
\[ \text{distribution of languages.} \]
\[ \text{product distributes over sums.} \]
\[ \text{(multiplication in } \]
\[ \text{distributes over sum and scalar product.} \]
\[ \text{Apply equation } (D. \]
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Explanation of the individual steps: (1) Definition of \( L \). (2) Lemma \( D.6 \).
(3) Apply equation \( D.4 \). (4) The multiplication in the \( S \)-algebra \( F(M \ast N) \) distributes over sum and scalar product. (5) Use the definition of the multiplication in \( M \ast N \), and the fact that \( \eta = \pi : UF (M \ast N) \) is a \( \mathcal{D} \)-monoid morphism by Lemma \( 2.14 \). (6) Interchange sums, and use that the scalar product distributes over sums. (7) \( \overline{p \ast q} \) is an \( S \)-linear map. (8) Definition of \( \overline{p \ast q} \). (9) Definition of \( p \ast q \). (10) Definition of \( \sigma \) and \( p \otimes q \). (11) Definition of \( L_{M}^{a,j} \) and \( L_{N}^{a,j} \). (12) Definition of marked Cauchy product and scalar product of languages.

(3) We are prepared to prove the theorem. Suppose that \( e : \Psi \Sigma^{*} \twoheadrightarrow P \) is a surjective \( \mathcal{D} \)-monoid morphism satisfying the assumptions of the theorem. For any \( p : M \rightarrow \mathcal{S} \) in \( \mathcal{D} \), the language \( \langle \Sigma^{*} \rightarrow \Psi \Sigma^{*} \xrightarrow{\pi_{M}} M \ast N \rightarrow \mathcal{S} \rangle \) is recognized by \( \pi_{M} \circ f \) and hence, by assumption, also by \( e \). Consequently there exists a morphism \( h^{p} : P \rightarrow \mathcal{S} \) in \( \mathcal{D} \) with \( h^{p} \circ e = p \circ \pi_{M} \circ f \). Analogously, for each \( q : N \rightarrow \mathcal{S} \) there exists a morphism \( h^{q} : P \rightarrow \mathcal{S} \) in \( \mathcal{D} \) with \( h^{q} \circ e = q \circ \pi_{N} \circ f \). Moreover, for any pair of morphisms \( p : M \rightarrow \mathcal{S} \) and \( q : N \rightarrow \mathcal{S} \) in \( \mathcal{D} \), the language \( L_{M}^{a,j} \) and \( L_{N}^{a,j} \) of \( D.3 \) are recognized by \( f_{M} = \pi_{M} \circ f \) resp. \( f_{N} = \pi_{N} \circ f \). Thus, the assumptions on \( e \), the marked product \( L_{M}^{a,j} \ast L_{N}^{a,j} \) is recognized by \( e \) for every \( a \in \Sigma \) and \( j = 1, \ldots, n_{a} \). Therefore, by Lemma \( D.10 \) \( e \) recognizes \( L \). That is, there exists a morphism \( h^{p,q} : P \rightarrow \mathcal{S} \) in \( \mathcal{D} \) with \( h^{p,q} \circ e = L \) (\( = U(\overline{p \ast q}) \circ \pi_{MN} \circ f \)).
To summarize, we have the following commutative diagrams for all $p : M \to S$ and $q : N \to S$:

\[
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{e} & P \\
\downarrow{f} & & \downarrow{h^p} \\
M \circ N & \xrightarrow{\pi_M} & M \\
\end{array}
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{e} & P \\
\downarrow{f} & & \downarrow{h^q} \\
M \circ N & \xrightarrow{\pi_N} & N \\
\end{array}
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{e} & P \\
\downarrow{f} & & \downarrow{h^{p,q}} \\
M \circ N & \xrightarrow{\pi_{MN}} & UF(M \ast N) \\
\end{array}
\]

Since the family $\{\pi_M, \pi_N, \pi_{MN}\}$ of product projections is separating, and using Assumption 4.10(iii), the combined family

\[
\{M \circ N \xrightarrow{\pi_M} M \xrightarrow{p} S\}_{p:M \to S} \\
\cup \{M \circ N \xrightarrow{\pi_N} N \xrightarrow{q} S\}_{q:N \to S} \\
\cup \{M \circ N \xrightarrow{\pi_{MN}} UF(M \ast N) \xrightarrow{U(p \ast q)} S\}_{p:M \to S, q:N \to S}
\]

is separating, see \[\text{A.5}\]. Thus diagonal fill-in, see \[\text{A.6}\], gives a unique $h : P \to M \circ N$ in $\mathcal{D}$ with $h \circ e = f$. Moreover, since both $e$ and $f$ are $\mathcal{D}$-monoid morphisms and $e$ is surjective, $h$ is a $\mathcal{D}$-monoid morphism.

\[\square\]

**Definition D.11.** A language is called recognizable if it is recognized by some finite $\mathcal{D}$-monoid. Denote by $\text{Rec}(\Sigma)$ the set of $\mathcal{D}$-recognizable languages over the alphabet $\Sigma$.

**Remark D.12.** Recognizable languages are precisely the regular languages $L : \Sigma^* \to S$, i.e. languages accepted by some finite Moore automaton with output set $S$; see \[\text{P.25}\][Lemma G.1].

In \[\cite{D.1}\] we established (working with a subset of our present Assumptions 4.10) the following results:

**Theorem D.13 (Adámek, Milius, Myers, Urbat \[\cite{D.1}\]).** $\text{Rec}(\Sigma)$ forms an algebra in the variety $\mathcal{C}$ w.r.t to the $\mathcal{C}$-algebraic operations on languages. Moreover, $\text{Rec}(\Sigma)$ is closed under derivatives, and the maps $a^{-1}(-)$ and $(-)a^{-1}$ on $\text{Rec}(\Sigma)$ are morphisms of $\mathcal{C}$, i.e. the $\mathcal{C}$-algebraic operations preserve derivatives.

**Theorem D.14 (Adámek, Milius, Myers, Urbat \[\cite{D.1}\]).** For any finite set $V_\Sigma$ of recognizable languages over $\Sigma$, the following statements are equivalent:

(i) $V_\Sigma$ is closed under the $\mathcal{C}$-algebraic operations and derivatives.

(ii) There exists a finite $\mathcal{D}$-monoid $P$ and a surjective $\mathcal{D}$-monoid morphism $e : \Psi \Sigma^* \to P$ such that $e$ recognizes precisely the languages in $V_\Sigma$. 
Thus every derivative of
Theorem D.13 gives a \( \mathcal{D} \)-monoid morphism \( h : P \to M \circ N \) with \( h \circ e = f \). Then every language recognized by \( f \) (say via the morphism \( s : M \circ N \to \mathcal{S} \) in \( \mathcal{D} \)) is also recognized by \( e \) (via the morphism \( s \circ h \)), and therefore lies in \( \mathcal{V}_\Sigma \). \( \square \)

**Remark D.15.** For the categories \( \mathcal{C}/\mathcal{D} \) of Example D.11 one can drop the closure under derivatives in Theorem D.16 in each case, the closure \( \mathcal{V}_\Sigma \) of \( L_{M,N}(f) \) under \( \mathcal{C} \)-algebraic operations is already closed under derivatives. To see this, note that by Theorem D.13 it suffices to show that every derivative of a language in \( L_{M,N}(f) \) lies in \( \mathcal{V}_\Sigma \). The latter is clear for the languages \( K \) and \( L \) recognized by \( \pi_M \circ f \) resp. \( \pi_N \circ f \); by Lemma D.10 their derivatives are even elements of \( L_{M,N}(f) \). Now consider the languages of the form \( KaL \) in \( L_{M,N}(f) \). One easily verifies that

\[
b^{-1}(KaL) = \begin{cases} (b^{-1}K)aL, & b \neq a; \\ (a^{-1}K)aL + K(\varepsilon)L, & b = a, \end{cases}
\]

and analogously for right derivatives. In the case \( \mathcal{D} = \text{K-Vec} \), this shows that any derivative of \( KaL \) is a linear combination (i.e. a \( \text{K-Vec} \)-algebraic combination) of languages in \( L_{M,N}(f) \), and thus lies in \( \mathcal{V}_\Sigma \). For the other examples (\( \mathcal{D} = \text{Set, Pos, JSL} \) with \( \mathcal{S} = \{0,1\} \)), the above case \( b = a \) states that

\[
a^{-1}(KaL) = \begin{cases} (a^{-1}K)aL & \varepsilon \notin L; \\ (a^{-1}K)aL \cup L & \varepsilon \in L. \end{cases}
\]

Thus every derivative of \( KaL \) is a finite union of languages in \( L_{M,N}(f) \) and therefore lies in \( \mathcal{V}_\Sigma \), since the union is part of the \( \mathcal{C} \)-algebraic operations for \( \mathcal{D} = \text{Set, Pos, JSL} \).

**E  Details for the Examples**

**Details for Example D.13.**
(1) See Example C.3
(2) This is a special case of Example C.3 with \( \mathcal{D} = \text{JSL} \) and \( \Psi = \mathcal{P}_f \) (cf. Example 2.2). Explicitly, \( \mathcal{P}_f \) is monoidal w.r.t. the isomorphism \( \theta_{X,Y} : \mathcal{P}_f X \otimes \mathcal{P}_f Y \cong \mathcal{P}_f(X \times Y) \) whose inverse \( \theta_{X,Y}^{-1} \) maps \( \{(x_1,y_1),\ldots,(x_n,y_n)\} \subseteq X \times Y \) to \( (\forall_{j=1}^n x_j \otimes y_j) \in X \otimes Y \), and the identity morphism \( \theta_1 = id : \text{JSL} \to \mathcal{P}_f \).
(3) Let us first verify that \( D_f \) is a left adjoint to \( U \). The unit is given by the monotone map \( \eta_X : X \to UD_f(X) \) with \( \eta(x) = \downarrow \{x\} \). Given a monotone map \( h : X \to UA \) into a semilattice \( A \), let \( \overline{h} : D_f(X) \to A \) be the function that maps a finitely generated down-set \( S = \downarrow S_0 \subseteq X \) to the finite join \( \bigvee S = \bigvee S_0 \subseteq |A| \). Clearly \( \overline{h} \) is a semilattice morphism and satisfies \( U(\overline{h}) \circ \eta_X = h \).

Moreover, since every finitely generated down-set is a finite union of one-generated downsets \( \downarrow \{x\} \), \( \overline{h} \) is uniquely determined by this property.

To show that \( U \) is monoidal, observe that for any two semilattices \( A \) and \( B \), the universal bimorphism \( t_{A,B} : |A| \times |B| \to |A \otimes B| \) is monotone, since it preserves joins in each component (and is thus monotone in each component). Thus we can view \( t_{A,B} \) as a morphism \( t_{A,B}^* : UA \times UB \to U(A \otimes B) \) in \( \text{Pos} \), and in complete analogy to Example 2.18 one can show that \( U \) is monoidal w.r.t. the maps \( t_{A,B}^* \) and the unit \( \eta : 1 \mapsto UF1 = 1_{JSL} \), where \( 1 = 1_{\text{Pos}} \) is the one-element poset. Similarly, the proof that \( D_f \) is monoidal is analogous to Example 2.18, with \( \text{Set} \) replaced by \( \text{Pos} \) and \( |-|, \psi \) by \( U, D_f \). Explicitly, \( D_f \) is monoidal w.r.t. the isomorphism \( \theta^X_{X,Y} : D_fX \otimes D_fY \cong D_f(X \times Y) \) whose inverse \( (\theta^X_{X,Y})^{-1} \) maps a down-set \( \{(x_1, y_1), \ldots, (x_n, y_n)\} \subseteq X \times Y \) to \((\text{max}_{j = 1}^{n}(x_j \uparrow y_j)) \in X \otimes Y \), and the identity morphism \( \theta_1 = id : 1_{JSL} \to D_f1 \).

**Details for Example 2.18.** The adjunction \( \Psi |-| : \mathcal{D} \to \text{Set} \) is monoidal by Example C.3. The proof given in that example also works for \( U \dashv D_f : JSL \to \text{Pos} \), replacing \( \mathcal{D} \) by \( \text{JSL} \) and \( \text{Set} \) by \( \text{Pos} \). That \( \text{Id} \dashv \text{Id} : \mathcal{D} \to \mathcal{D} \) is a monoidal adjunction is trivial.

**Details for Example 2.18.** (3), (4), (5) are trivially concrete monoidal adjunctions. So is \( \mathcal{P}_f |-| : JSL \to \text{Set} \), since the monoidal adjunction of \( \text{Set} \) is the identity adjunction \( \text{Id} \dashv \text{Id} : \text{Set} \to \text{Set} \). Concerning \( \mathcal{D}_f \dashv U : JSL \to \text{Pos} \), we clearly have \( \Psi_{\text{Pos}} \circ U = \mathcal{P}_f \) and \( \mathcal{D}_f = \mathcal{D}_f \circ \Psi_{\text{Pos}} \). Since also the units and counits of these three adjunctions compose accordingly, the composite of the adjunction \( \mathcal{D}_f \dashv U \) with \( \Psi_{\text{Pos}} \dashv |-| : \mathcal{D} \) is the adjunction \( \mathcal{P}_f |-| : \mathcal{D} \).

**Details for Example 4.4.**

(1) For \( \mathcal{D} = \text{Set} \) or \( \text{Pos} \) and \( S = \{0,1\} \), the family \( \{ M \times N \xrightarrow{\sigma_{p,q}} \{0,1\} \} \) is separating, where \( \sigma(m,n) = m \cdot n \) is the multiplication of \( S \). This proves only for \( \mathcal{D} = \text{Pos} \), the argument for \( \mathcal{D} = \text{Set} \) being analogous. Let \( (m,n) \) and \( (m',n') \) be two elements of \( M \times N \) with \( (m,n) \not\leq (m',n') \), say \( m \not\leq m' \). Choose \( p : M \to \{0,1\} \) to be the monotone map with \( p(x) = 0 \) iff \( x \leq m' \), and \( q : N \to \{0,1\} \) to be the constant map on \( 1 \). Then

\[
\sigma \circ (p \times q)(m,n) = p(m) \cdot q(n) = 1 \cdot 1 = 1
\]

and

\[
\sigma \circ (p \times q)(m',n') = p(m') \cdot q(n') = 0 \cdot 1 = 0,
\]

so \( \sigma \circ (p \times q)(m,n) \not\leq \sigma \circ (p \times q)(m',n') \). This shows that the family \( \{ \sigma \circ (p \times q) \}_{p,q} \) is separating.
(2) Similarly, for $\mathcal{D} = \textbf{K-Vec}$ the family \( \{ M \otimes N \xrightarrow{p \otimes q} K \otimes K \xrightarrow{\sigma} K \}_{p,q} \) is separating. To see this, recall that if $M$ and $N$ are a vector spaces with bases \( \{ b_i \}_{i \in I} \) resp. \( \{ c_j \}_{j \in J} \), then the tensor product $M \otimes N$ has the basis \( \{ b_i \otimes c_j \}_{i \in I, j \in J} \). It suffices to show that, for any element $x \in |M \otimes N|$ with $x \neq 0$, there are linear maps $p : M \to K$ and $q : N \to K$ with $\sigma(p \otimes q)(x) \neq 0$. Suppose that $x = \sum_{i \in I, j \in J} \lambda_{i,j} (b_i \otimes c_j)$ with $\lambda_{i,j} \in K$. Since $x \neq 0$, there exist $i_0 \in I$ and $j_0 \in J$ with $\lambda_{i_0,j_0} \neq 0$. Let $p : M \to K$ and $q : N \to K$ be the linear maps defined by

\[
p(b_i) = \begin{cases} 1, & i = i_0; \\ 0, & i \neq i_0,
\end{cases}
\quad \text{and} \quad
q(c_j) = \begin{cases} 1, & j = j_0; \\ 0, & j \neq j_0.
\end{cases}
\]

Then

\[
\sigma \circ (p \otimes q)(x) = \sum_{i,j} \lambda_{i,j} [\sigma \circ (p \otimes q)(b_i \otimes c_j)] = \sum_{i,j} \lambda_{i,j} (p(b_i) \cdot q(c_j)) = \lambda_{i_0,j_0} \neq 0.
\]

Here we use the linearity of $\sigma$ and $p \otimes q$ in the first step, the definition of $\sigma$ and $p \otimes q$ in the second step, and the definition of $p$ and $q$ in the last step.

(3) If $\mathcal{D} = \text{JSL}$ and $M$ and $N$ are finite idempotent semilattices, we can describe $M \ast N$ as follows. Consider the surjective semilattice morphism

\[
eq (\mathcal{P}_f(|M| \times |N|) \xrightarrow{\overline{t}} M \otimes N \xrightarrow{\pi} M \ast N),
\]

where $\overline{t}$ is the adjoint transpose of the universal bimorphism $t : |M| \times |N| \to |M \otimes N|$, cf. (3.4). Let \( \equiv \subseteq \mathcal{P}_f(|M| \times |N|) \times \mathcal{P}_f(|M| \times |N|) \) be the kernel of $\epsilon$, i.e. $X \equiv Y$ if $\pi \circ \overline{t}(X) = \pi \circ \overline{t}(Y)$. By the definition of $\pi$, the latter means precisely that $\sigma \circ (p \otimes q) \circ \overline{t}(X) = \sigma \circ (p \otimes q) \circ \overline{t}(Y)$ for all semilattice morphisms $p : M \to \{0,1\}$ and $q : N \to \{0,1\}$. Since such morphisms correspond to ideals, see Example (3.4), we have $X \equiv Y$ iff, for all ideals $I \subseteq M$ and $J \subseteq N$,

\[
\exists (m,n) \in X : m \notin I \land n \notin J \quad \leftrightarrow \quad \exists (m',n') \in Y : m' \notin I \land n' \notin J.
\]

Since $\equiv$ is a semilattice congruence on $\mathcal{P}_f(|M| \times |N|)$, being the kernel of a semilattice morphism, $X \equiv Y$ and $X \equiv Z$ implies $X = X \cup Z \equiv Y$ and $X \equiv X \cup \overline{Z} \equiv X$. Thus, for every $X \subseteq |M| \times |N|$ there exists a largest set $[X] \subseteq |M| \times |N|$ with $X \equiv [X]$, viz. the union of all $Y$ with $X \equiv Y$. It follows that $X \mapsto [X]$ defines a closure operator on $\mathcal{P}_f(|M| \times |N|)$, and that every equivalence class of $\equiv$ contains a unique closed subset of $|M| \times |N|$ (viz. the union of all sets in the equivalence class). One easily verifies that $[X]$ consists of those elements $(x,y) \in |M| \times |N|$ such that, for all ideals $I \subseteq M$ and $J \subseteq N$,

\[
x \notin I \land y \notin J \quad \Rightarrow \quad \exists (m,n) \in X : m \notin I \land n \notin J.
\]

Since $\equiv$ is the kernel of $\pi \circ \overline{t}$, we have $M \ast N \cong \mathcal{P}_f(|M| \times |N|)/\equiv$. Identifying the equivalence classes of $\equiv$ with the closed subsets of $M \times N$, the join in the idempotent semiring $M \ast N$ is given by $[X] \vee [Y] = [X \cup Y]$, and the multiplication (see Lemma (4.3) by $[X][Y] = [XY]$.}
Details for Example 4.11. Clearly Assumption 4.10(i) holds for our examples \( \mathcal{D} = \text{Set}, \text{Pos}, \text{JSL} \) and \( \text{K-Vec} \) (\( K \) a finite field). Also (ii) is well-known in all these cases. For \( \text{Set}, \text{Pos} \) and \( \text{K-Vec} \), see e.g. [1] Example 7.40]. For \( \text{JSL} \), see [2]. Concerning (iii), that \( \mathcal{D}(M, S) \) and \( \mathcal{D}(N, S) \) are separating is easy to verify in all cases. Also, that \( \{ U(p \circ q) \}_{p,q} \) forms a separating family is trivial for \( \mathcal{D} = \text{JSL} \) and \( \text{K-Vec} \), since here \( U = \text{Id} \) and \( U(p \circ q) = p \circ q \), and the morphisms \( p \circ q \) are separating by definition.

It remains to consider the cases \( \mathcal{D} = \text{Set} \) and \( \mathcal{D} = \text{Pos} \). We only treat \( \text{Pos} \), the argument for \( \text{Set} \) being the discrete special case. We need to show that the family of monotone maps

\[
\{ p \times q : \mathcal{D}_f(M \times N) \to \{0, 1\} \}_{p: M \to \{0, 1\}, q: N \to \{0, 1\}}
\]

is separating, where \( S = \{0, 1\} \) (considered as a poset) is ordered by \( 0 < 1 \). Note that \( p \times q \) maps a finitely generated down-set \( X \subseteq M \times N \) to 1 iff there exists a pair \( (m, n) \in X \) with \( p(m) = 1 \) and \( q(n) = 1 \). Let \( X \not\subseteq Y \) be two finitely generated down-sets of \( M \times N \). Choose an element \( (m, n) \in X \setminus Y \), and define monotone maps \( p: M \to \{0, 1\} \) and \( q: N \to \{0, 1\} \) by

\[
p(x) = 1 \text{ iff } x \geq m \quad \text{resp.} \quad q(y) = 1 \text{ iff } y \geq n.
\]

Then we get

\[
\overline{p \times q}(X) = 1 \not\leq 0 = \overline{p \times q}(Y),
\]

i.e. \( p \times q \) separates \( X \) and \( Y \), as desired.

References

1. J. Adámek, H. Herrlich and G. Strecker. Abstract and Concrete Categories. Wiley and Sons (1990).
2. F. Borceux. Handbook of Categorical Algebra: Volume 2, Categories and Structures. Cambridge University Press, 1994.
3. A. Horn and N. Kimura. The category of semilattices. Algebra Universalis 1, 26–38 (1971)
4. S. Mac Lane. Categories for the Working Mathematician. 2nd ed., Springer (1998).