Solving The Beam Deflection Problem Using Al-Tememe Transforms

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Abstract

In this paper, an enhancement to the beam deflection problem is performed through the substitution of \( q(x) \) by \( \frac{1}{x^4} \). This substitution is performed to reduce the beam load intensity, also the enhanced beam deflection problem is solved using two new transforms, which are complex AL-Tememe and AL-Tememe transforms. The results (solutions) from complex AL-Tememe and AL-Tememe transforms are compared to each other; both transforms have the ability to solve the enhanced problem of the beam deflection.

Keywords: Complex AL-Tememe transform; AL-Tememe transform; deflection of the beam, differential equations; famous function; Inverse of AL-Tememe transform; Inverse of complex AL-Tememe transform; uniform distributed load.

I. Introduction

The beam deflection problem is widely discussed in many books [IV,VI,VII,VIII,XI], where many methods are used to solve that problem, however the use of AL-Tememe and complex AL-Tememe transforms never discussed before. AL-Tememe and complex AL-Tememe are two transforms that emerged at 2016 and 2018 respectively, these transforms can solve some types of differential equations, which can be used in many scientific fields, such as physics, engineering and bio-medical signal processing [X,II,III,V]. In this paper, the problem of deflection of beam is solved using complex AL-
Tememe and AL-Tememe transforms, and the solutions from these transforms are compared.

II. Basic Concepts

It is necessary to mention some relevant definitions, functions, properties and theorems to make the calculations clearer.

Definition of complex AL-Tememe transform \([X]\):

A complex AL-Tememe transform for the function \(f(x), x > 1\) is defined by the integral:

\[
T^c[f(x)] = \int_1^\infty x^{-ip} f(x) dx = F(ip).
\]

Such that this integral is convergent in \([1, \infty]\), \(p\) is a positive constant, and \(x^{-ip}\) is the kernel of this transform and \(i = \sqrt{-1}\).

Definition of inverse complex AL-Tememe transform \([X]\):

If \(T^c[f(x)] = F(ip)\) represents a complex AL-Tememe transform of \(f(x)\), then \(f(x)\) is said to be the inverse AL-Tememe transform and it can be written by: \(f(x) = T^{-1}c(F(ip))\).

Property of complex AL-Tememe transform \([2]\):

A complex AL-Tememe transform linear:

\[
T^c(Af(x) \pm BT^c(g(x)) = AT^c(f(x)) \pm BT^c(g(x)).
\]

Where \(A\) and \(B\) are constants, the function \(f(x)\) and \(g(x)\) are defined when \(x > 1\).

Complex AL-Tememe transform of some famous function \([X]\):

I. \(T^c(1) = \frac{1}{-1+ip}\)

II. \(T^c(x^n) = \frac{1}{ip-(n+1)}\), \(n \in R\)

Inverse of complex AL-Tememe transform of famous function \([X]\):

I. \(T^{-1}c\left(\frac{1}{-1+ip}\right) = 1\).

II. \(T^{-1}c\left(\frac{1}{ip-(n+1)}\right) = x^n\), \(n \in R\).

III. \(T^{-1}c\left(\frac{1}{ip-1}\right) = \ln(x)\).

Theorem \([X]\):

Let \(y(x)\) be defined function for \(x > 1\), and its derivatives \(y'(x), y''(x), \cdots, y^n(x)\) exist, then:

\[
T^c[x^n y^{(n)}(x)] = -y^{(n-1)}(1) - (ip - n)y^{(n-2)}(1) - \cdots - (ip - n)(ip - (n - 1))\cdots(ip - 2)y(1) + (ip - n)(ip - (n - 1))\cdots(ip - 1)F(ip) \quad n \in Z^+.
\]

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Definition of AL-Tememe transform [I]:

Al-Tememe Transform for the function $f(x) ; x > 1$ is defined by the following integral

$$T[f(x)] = \int_{1}^{\infty} x^{-p} f(x) \, dx = F(p).$$

Such that this integral is convergent in some region, $p$ is a positive constant, and $x^{-p}$ the kernel of Al-Tememe Transform.

Definition of inverse AL-Tememe transform [I]:

Let $f(x)$ be a function where $x > 1$ and $T[f(x)] = F(p)$, $f(x)$ is said to be an inverse for Al-Tememe Transform and written as $T^{-1}[F(p)] = f(x)$, where $T^{-1}$ returns the transform to the original function.

Propriety of AL-Tememe transform [I]:

The transformation is characterized by the linear propriety, that is:

$$T[Af(x) \pm Bg(x)] = AT[f(x)] \pm BT[g(x)]$$

where $A$ and $B$ are constants, the functions $f(x)$ and $g(x)$ are defined when $x > 1$.

Table of selected AL-Tememe transforms [I]

| Region of convergence | $F(p)$ | $F(p)_{\infty} = \int_{1}^{\infty} x^{-p} f(x) \, dx$ |
|-----------------------|--------|------------------------------------------------------|
| I. $k, k = \text{Constant}$ | $\frac{k}{p - 1}$ | $p > 1$ |
| II. $x^a, a \in R$ | $\frac{1}{p - (a + 1)}$ | $p > a + 1$ |
| III. $\ln x$ | $\frac{1}{(p - 1)^2}$ | $p > 1$ |
| IV. $x^a \ln x, a \in R$ | $\frac{1}{[p - (a + 1)]^2}$ | $p > a + 1$ |

**Deflection of the Beam Problem [IX]**

If a beam of length $L$ with rectangular cross section and homogenous elastic material (e.g. steel) is considered as shown in figure (1).

- And if a load is applied to the beam in vertical plane through the axis of symmetry (the x-axis), the beam is going to bent.
- If a cross-section of the beam cutting the elastic curve in $p$ and the neutral surface in the line $AA'$. 

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Then the bending moment $M$ about $AA'$ is given by Bernoulli-Euler law.

$$M = \frac{EI}{R} (3.1)$$

Where:

- $E =$ modulus of elasticity of the beam.
- $I =$ moment of inertia of the cross-section $AA'$.
- $R =$ radius of curvature of the elastic curve at $p(x, y)$.

If the deformation of the beam is small, the slope of the elastic curve is also small so that it is possible to neglect $\left(\frac{1}{R}\right)^2$ in the formula $R = \frac{1}{\frac{d^2y}{dx^2}}$.

- For small deflection, $\frac{1}{R} = \frac{1}{\frac{d^2y}{dx^2}}$.

Hence, (3.1) bending moment $M = EI \frac{d^2y}{dx^2}$.

- Shear force $= \frac{dM}{dx} = EI \frac{d^3y}{dx^3}$.
- Intensity of loading $= \frac{d^2M}{dx^2} = EI \frac{d^4y}{dx^4} = q(x)$.
- The sum of moments about any section due to external forces on the left of the section, if anti-clockwise is taken as positive and if clockwise is taken as negative.

The most important supports corresponding boundary conditions are:

1. Simply supported as shown in figure (2):

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No deflection and bending moment exist. Then:
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\[ y(0) = 0, \ y''(0) = 0 \, . \]
\[ y(l) = 0, \ y''(l) = 0 \, . \]

II. Completed at \( x = 0 \), free at \( x = l \). as shown in figure (3).

- At \( x = 0 \), the deflection and slope of the beam are both zero. At \( x = 1 \), there is no bending moment and shear force. We have, \( y(0) = y'(0) = 0, \ y''(l) = y'''(l) = 0. \)

III. Clamped at both ends: The deflection and slope of the beam being both zero, then: \( y(0) = 0, \ y'(0) = 0 \, . \)
\[ y(l) = 0, \ y'(l) = 0 \, . \]

IV. The Deflection of a Beam Carrying Uniform Distributed Load

Assume that a uniform loaded beam of length \( L \) is supported at both ends, as shown in figure (4). The deflection \( y(x) \) is a function of horizontal position \( x \), it is given by the differential equation:
\[ \frac{d^4 y}{dx^4} = \frac{1}{El} q(x) \quad (4.1) \]

Where \( q(x) \) is the load per unit length at point \( x \). It is assumed in this problem that \( q(x) = q \) (\( q \) is a constant).

The boundary conditions are:
I. No deflection at \( x = 0 \) and \( x = l \).
II. No bending moment of the beam at \( x = 0 \) and \( x = l \).

\[ y(0) = 0, \ y(l) = 0 \, \text{no deflection at} \ x = 0 \ \text{and} \ x = l \]
\[ y''(0) = 0, \ y''(l) = 0 \, \text{no bending moment at} \ x = 0 \ \text{and} \ x = l \]
Solving the deflection of a beam carrying uniform distributed load using complex AL-Tememe transform

Complex AL-Tememe transform is used to solve the problem of deflection for a beam that carries a uniform distributed load. After substituting each $q(x)$ by $\frac{1}{x^4}$ equation (4.1) becomes:

$$x^4 y^{(4)} - \frac{1}{E I} = 0 \quad y(1) = 0, y''(1) = 0$$

By taking a complex AL-Tememe transform to both sides:

$$T^c (x^4 y^{(4)}) - T^c \left( \frac{1}{E I} \right) = 0,$$

$$-y'''(1) - (ip - 4)y''(1) - (ip - 4)(ip - 3)y'(1) - (ip - 4)(ip - 3)(ip - 2)y(1) + (ip - 4)(ip - 3)(ip - 2)(ip - 1)T^c(y) - \frac{1}{E I} T^c(1) = 0.$$

Then:

$$T^c(y) = \frac{y'''(1)}{(ip - 4)(ip - 3)(ip - 2)(ip - 1)} + \frac{y'(1)}{(ip - 2)(ip - 1)} + \frac{1}{E I} \frac{1}{(ip - 2)(ip - 1)}.$$

By taking the inverse of a complex AL-Tememe transform to both sides:

$$T^{-1}(y) = \frac{y'''(1)}{(ip - 4)(ip - 3)(ip - 2)(ip - 1)} + T^{-1} \left( \frac{y'(1)}{(ip - 2)(ip - 1)} \right) + T^{-1} \left( \frac{1}{E I} \right).$$

Now, we take

$$\frac{1}{E I} = A \frac{1}{ip - 4} + B \frac{1}{ip - 3} + C \frac{1}{ip - 2} + D \frac{1}{ip - 1}.$$

After simple computations, we get:

$$A = \frac{1}{6}, B = -\frac{1}{2}, C = \frac{1}{2}, D = -\frac{1}{6}.$$

Then:

$$T^{-1} \left( \frac{1}{(ip - 2)(ip - 1)} \right) = T^{-1} \left( \frac{1}{ip - 2} \right) + T^{-1} \left( -\frac{1}{ip - 3} \right) + T^{-1} \left( \frac{\frac{1}{6}}{ip - 1} \right).$$

Also, we take

$$\frac{1}{ip - 2} = \frac{A}{ip - 2} + \frac{B}{ip - 1}.$$ 

After simple computations:

$$A = 1, \text{ and } B = -1.$$

Then:

$$T^{-1} \left( \frac{1}{(ip - 2)(ip - 1)} \right) = T^{-1} \left( \frac{1}{ip - 2} \right) + T^{-1} \left( \frac{1}{ip - 1} \right) = (x - 1)y'.$$

As well as, we take

$$\frac{1}{(ip - 4)(ip - 3)(ip - 2)(ip - 1)^2} = \frac{A}{ip - 4} + \frac{B}{ip - 3} + \frac{C}{ip - 2} + \frac{D}{ip - 1}.$$ 

After, simple computations:

$$A = \frac{1}{10}, B = -\frac{1}{4}, C = \frac{1}{2}, D = -\frac{11}{36}, E = -\frac{1}{6}.$$

Then:
Also, we take
\[ y = \left(\frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{1}{6}\right) y'''(1) + (x - 1) y'(1) + \left(\frac{1}{18} x^3 - \frac{1}{4} x^2 + \frac{1}{2} x - \frac{11}{36} - \frac{1}{6} \ln(x)\right) \cdot \frac{1}{EI}. \]
To use the boundary condition \( y''(l) = 0 \), and by taking the second derivative of (4.2) then:
\[ y(x) = \frac{1}{24EI} x^4 - \frac{1}{EI} x^3 + \frac{l^3}{24EI} x \quad (4.3). \]
The above equation gives the deflection of the beam at a distance \( x \).
To find the maximum deflection, put \( x = \frac{l}{2} \) in equation (4.3).

Solving the deflection of a beam carrying uniform distributed load using AL-Tememe transform

AL-Tememe transform is used to solve the problem of deflection for a beam that carrying uniform distributed load. After substituting each \( q(x) \) by \( \frac{1}{x^4} \) equation (4.1) becomes:
\[ x^4 y^{(4)} - \frac{1}{EI} = 0 \quad y(1) = 0, y''(1) = 0 \]
By taking AL-Tememe transform to both sides:
\[ T\left(x^4 y^{(4)} - \frac{1}{EI}\right) = 0, \]
\[ -y'''(1) - (p - 4) y''(1) - (p - 4)(p - 3)y'(1) - (p - 4)(p - 3)(p - 2)y(1) + \]
\[ (p - 4)(p - 3)(p - 2)(p - 1) T(y) - \frac{1}{EI} T(1) = 0. \]
\[ T(y) = y'''(1) + y''(1) + \frac{1}{EI} \cdot \frac{1}{(p - 4)(p - 3)(p - 2)(p - 1)^2}. \]
By taking the inverse of AL-Tememe transform to both sides:
\[ y = T^{-1} \left(\frac{1}{(p - 4)(p - 3)(p - 2)(p - 1)}\right) + \frac{1}{EI} T^{-1} \left(\frac{1}{(p - 4)(p - 3)(p - 2)(p - 1)^2}\right). \]
Now, we take
\[ \frac{1}{(p - 4)(p - 3)(p - 2)(p - 1)} = \frac{A}{p - 4} + \frac{B}{p - 3} + \frac{C}{p - 2} + \frac{D}{p - 1}. \]
After simple computations, we get:
\[ A = \frac{1}{6}, B = -\frac{1}{2}, C = \frac{1}{2}, D = -\frac{1}{6}. \]
Then
\[ T^{-1} \left(\frac{1}{(p - 4)(p - 3)(p - 2)(p - 1)}\right) = T^{-1} \left(\frac{1}{4} \frac{1}{p - 4}\right) + T^{-1} \left(-\frac{1}{2} \frac{1}{p - 3}\right) + T^{-1} \left(\frac{1}{2} \frac{1}{p - 2}\right) + T^{-1} \left(-\frac{1}{6} \frac{1}{p - 1}\right). \]
Also, we take

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\[
\frac{1}{(p-2)(p-1)} = \frac{A}{p-2} + \frac{B}{p-1}.
\]

After simple computations, we have:
\[A = 1, \text{ and } B = -1.\]

As well as, we take:
\[
\frac{1}{(p-4)(p-3)(p-2)(p-1)^2} = \frac{A}{p-4} + \frac{B}{p-3} + \frac{C}{p-2} + \frac{D}{p-1} + \frac{E}{(p-1)^2}.
\]

After, simple computations, we have:
\[A = \frac{1}{18}, B = -\frac{1}{4}, C = \frac{1}{2}, D = -\frac{11}{36}, E = -\frac{1}{6}.\]

Now:
\[
T^{-1} \left[ \frac{y'''}(1)}{(p-4)(p-3)(p-2)(p-1)^2} \right] = (\frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{1}{6}) y'''.
\]

Also, \[T^{-1} \left[ \frac{y'(1)}{(p-2)(p-1)} \right] = (x - 1) y'.\]

And, \[\frac{1}{El} T^{-1} \left[ \frac{1}{(p-4)(p-3)(p-2)(p-1)^2} \right] = \left[ \frac{1}{8} x^3 - \frac{1}{4} x^2 + \frac{1}{2} x - \frac{11}{36} - \frac{1}{6} \ln x \right] \frac{1}{El}.\]

Finally, \[y = \left[ \frac{1}{16} x^3 - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{1}{6} \right] y''' + (x - 1) y' + \left[ \frac{1}{8} x^3 - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{11}{36} - \frac{1}{6} \ln x \right] \frac{1}{El}.\]

V. Conclusions

There are many solutions to the beam deflection problem, however Al-Tememe transforms (Al-Tememe and Complex Al-Tememe) are never used before to solve this problem. The previous computations solved the beam deflection problem through the reduction of load that provided over the beam, by dividing the beam deflection equation by \( x^4 \) to become \( y^{(4)} = \frac{1}{x^4 El} \). Both transforms gave the same results therefore it is possible to use either of them to solve the beam deflection problem.

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