A Note on Embedding Nonabelian Finite Flavor Groups in Continuous Groups

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Abstract

A nonabelian finite flavor group \(G \subset SO(3)\) can have double covering \(G' \subset SU(2)\) such that \(G \not\subset G'\). This situation is not contradictory, but quite natural, and we give explicit examples such as \(G = D_n, G' = Q_{2n}\) and \(G = T, G' = T'\). This observation can be crucial in particle theory model building.

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Introduction.

One promising direction for extending the standard model of particle theory lies in using a flavor symmetry $G_F$ which commutes with the gauge group. The flavor symmetry can lead to predictions for the quarks and lepton mixing angles. Typically $G_F$ is a finite nonabelian group.

The standard model gauge group $G_{SM}$ is comprised of the direct product of continuous groups $G_{SM} \equiv SU(3) \times SU(2) \times U(1)$. The flavor group $G_F$ is itself often a subgroup of a Lie group: $G_F \subset G_L$. Here we focus on the cases $G_L = SO(3)$ and $G_L = SU(2)$.

There can be, as we shall discuss, a pair of candidates $G^i_F$ ($i = 1, 2$) which have the following set of seemingly contradictory properties.

- $G^1_F \subset SO(3)$
- $G^2_F \subset SU(2)$
- $G^2_F$ is the double covering of $G^1_F$ just as $SU(2)$ for $SO(3)$.
- $G^1_F \not\subset G^2_F$

The apparent contradiction in the last statement is only that, since $SO(3) \not\subset SU(2)$! The situation described is not uncommon. Indeed, we will show that $G^1_F \subset G^2_F$ only when $G^1_F$ is abelian. First we will set the stage by considering a few explicit examples, including an infinite series of $SU(2)$ subgroups, after a brief and self-contained survey of the small nonabelian groups.

Nonabelian groups with order $g \leq 31$

Some of our examples will be taken from nonabelian groups with order $g \leq 31$. There are 45 such groups. Of the nonabelian finite groups, the best known are perhaps the permutation groups $S_N$ (with $N \geq 3$) of order $N$! The smallest non-abelian group is $S_3$ ($\equiv D_3$), the symmetry of an equilateral triangle with respect to all rotations in a three dimensional sense. This group is a member of two disparate infinite series, $S_N$ and $D_N$, distinct for $N > 3$. Both have elementary geometrical significance since the symmetric permutation group $S_N$ is the symmetry of the N-plex in N dimensions while the dihedral group $D_N$ is the symmetry including inversion of the planar N-agon in 3 dimensions.

As a family symmetry, the $S_N$ series (and the even-parity series $A_N$) becomes less interesting as the order and the dimensions of the representations increase, because $S_N$ contains
transformations which independently exchange elements of the fundamental $N$-plet. Only $S_3$ and $S_4$ are of any interest as symmetries associated with the particle spectrum [1]. Also, the order (number of elements) of the $S_N$ groups grow factorially with $N$, and the order of the dihedral groups increase only linearly with $N$ and their irreducible representations are all one- and two- dimensional. This is reminiscent of the representations of the electroweak $SU(2)_L$ used in Nature.

Each $D_N$ is a subgroup of $O(3)$ and has a counterpart double dihedral group $Q_{2N}$, of order $4N$, which is a subgroup of the double covering $SU(2)$ of $SO(3)$.

With only the use of $D_N$, $Q_{2N}$, $S_N$ and the tetrahedral group $T = A_4$ (of order 12, the even permutations subgroup of $S_4$), alone and in direct products with abelian groups, we find 32 of the 45 nonabelian groups up to order 31. (Note that $D_6 \simeq Z_2 \times D_3, D_{10} \simeq Z_2 \times D_5$ and $D_{14} \simeq Z_2 \times D_7$)

| $g$ | $D_3 \equiv S_3$ | $D_4, Q = Q_4$ | $D_5$ | $D_6, Q_6, T$ | $D_7$ | $D_8, Q_8, Z_2 \times D_4, Z_2 \times Q$ | $D_9, Z_3 \times D_3$ | $D_{10}, Q_{10}$ | $D_{11}$ | $D_{12}, Q_{12}, Z_2 \times D_6, Z_2 \times Q_6, Z_2 \times T, Z_3 \times D_4, D_3 \times Q, Z_4 \times D_3, S_4$ | $D_{13}$ | $D_{14}, Q_{14}$ | $D_{15}, D_5 \times Z_3, D_3 \times Z_5$ |

There remain thirteen others formed by “twisted products” of abelian factors. (Twisted products are a way of constructing certain group extensions; a group extension is a way of describing a group in terms of a normal subgroup and its factor group. We will see later that double covers are also a form of group extension.) Only certain such “twistings” are permissible, namely (completing all $g \leq 31$)
It can be shown that these thirteen exhaust the classification of all inequivalent nonabelian groups up to order thirty-one [2].

Of these 45 nonabelian groups, the dihedrals \((D_N)\) and double dihedrals \((Q_{2N})\), of order 2N and 4N respectively, form the simplest sequences. In particular, they can be realized as subgroups of \(O(3)\) and \(SU(2)\) respectively, the two simplest nonabelian continuous groups. A finite group is completely described by its presentation as a multiplication table, but it is often more convenient to use a more compact description. One equivalent and compact representation is the character table, and for physical applications this is preferred because it also summarizes the vector representations of the group. For the \(D_N\) and \(Q_{2N}\), the tables for Kronecker products, as derivable from the character tables, are simple to express in general [3]. These tables, whose role is to describe the decomposition of tensor products of irreducible vector representations (irreps) as a direct sum of irreps, may be degenerate among different finite groups which have distinct character tables. The character table, on the other hand, just like the presentation as a multiplication table contains the complete information about a group.

\[ SU(2) \text{ as the double cover of } SO(3) \]

Continuous groups which have the same structure for small transformations, described by the Lie algebra, can have various global forms distinguished by their discrete subgroups. For compact Lie groups, each simple Lie algebra corresponds to one global form which is connected and simply-connected; other global forms are obtained through taking a quotient by some subgroup of the center of the group, constructing an extension by a discrete symmetry of the group which is not continuously related to the identity, or some combination of such operations.

In particular, since \(SU(2)\) is the double cover of \(SO(3)\), their Lie algebras are isomorphic. Conversely, \(SO(3)\) is obtained from \(SU(2)\) if we identify pairs of elements which differ by a \(2\pi\) rotation. The irreps of \(SO(3)\) have dimensions equal to odd integers \(1, 3, 5,...\) while the irreps of \(SU(2)\) appear with all positive integers \(1, 2, 3,...\). In particular \(SU(2)\) has defining irrep the 2 which is a spinor and the corresponding rotations change sign after any odd number of \(2\pi\) rotations becoming the identity only after an integer multiple of \(4\pi\) rotations. This is what we mean by saying that \(SU(2)\) is the double cover of \(SO(3)\): the group manifold is topologically \(S^3/\mathbb{Z}_2\) while that of \(SU(2)\) is \(S^3\).
Finite subgroups of $SO(3)$ and $SU(2)$

Let us consider an explicit example. The tetrahedral group $T$ is the symmetry of a regular tetrahedron. As such it is a subgroup of the rotations in three space dimensions: $T \subset SO(3)$. From the visualization it is isomorphic to the even permutations of four objects so $T \subset S_4$. The irreps of $T$ are $1, 1, 2, 3$.

Next consider the binary tetrahedral group $T'$. This group is more challenging to visualize as a symmetry. From standard treatises we learn that $T'$ is the double cover of $T$. The irreps of $T'$ are $1, 1, 2, 2, 2, 3$ with the table for Kronecker products shown in Table 1.

Table 1

Table for Kronecker products of $T'$ irreps.

| $\otimes$ | $1_1$ | $1_2$ | $1_3$ | $2_1$ | $2_2$ | $2_3$ | $3$ |
|-----------|-------|-------|-------|-------|-------|-------|-----|
| $1_1$     | $1_1$ | $1_2$ | $1_3$ | $2_1$ | $2_2$ | $2_3$ | $3$ |
| $1_2$     | $1_2$ | $1_3$ | $1_1$ | $2_2$ | $2_3$ | $2_1$ | $3$ |
| $1_3$     | $1_3$ | $1_1$ | $1_2$ | $2_3$ | $2_1$ | $2_2$ | $3$ |
| $2_1$     | $2_1$ | $2_2$ | $2_3$ | $1 + 3$ | $1’ + 3$ | $1'' + 3$ | $2_1 + 2_2 + 2_3$ |
| $2_2$     | $2_2$ | $2_3$ | $2_1$ | $1’ + 3$ | $1'' + 3$ | $1 + 3$ | $2_1 + 2_2 + 2_3$ |
| $2_3$     | $2_3$ | $2_1$ | $2_2$ | $1'' + 3$ | $1 + 3$ | $1’ + 3$ | $2_1 + 2_2 + 2_3$ |
| $3$       | $3$   | $3$   | $3$   | $2_1 + 2_2 + 2_3$ | $2_1 + 2_2 + 2_3$ | $2_1 + 2_2 + 2_3$ | $1_1 + 1_2 + 1_3 + 3 + 3$ |

We note that the group $T$ has irreps $1, 1, 2, 3$ which are a subset of those of $T'$ and that their table for Kronecker products shown in Table 2 is contained within that (Table 1) for $T'$.

Table 2

Table for Kronecker products of $T$ irreps.

| $\otimes$ | $1$ | $1'$ | $1''$ | $3$ |
|-----------|-----|------|-------|-----|
| $1$       | $1$ | $1'$ | $1''$ | $3$ |
| $1'$      | $1'$ | $1''$ | $1$   | $3$ |
| $1''$     | $1''$ | $1$   | $1'$  | $3$ |
| $3$       | $3$ | $3$   | $3 + 1' + 1'' + 3 + 3$ | $3$ |
With all this circumstantial evidence the reader might well suspect from a physical perspective that $T \subseteq T'$. The present paper will show, however, that this is interestingly incorrect and will discuss in detail why $T \not\subseteq T'$. This can first be straightforwardly confirmed by using the representation of $T$ in $\text{SO}(3)$ to construct the corresponding rotations in $\text{SU}(2)$. In $\text{SU}(2)$ these elements do not close by themselves, because a $2\pi$ rotation is $-1$. Instead, $T'$ has two subsets of elements which correspond to $T$ and $-T$ (pointwise).

From a more mathematical point of view, the binary tetrahedral group $T'$, and the tetrahedral group $T$ live in a short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow T' \rightarrow T \rightarrow 1$$

which does not split, hence $T'$ is not a semidirect product of $\mathbb{Z}_2 \times T$, i.e., $T'$ has no subgroup isomorphic to $T$ [4].

Upon further scrutiny we realize that this is also a reflection of the following property. Since the group $T$ has no 2-dimensional irrep, its embedding in $\text{SU}(2)$ can only be such that the defining 2 of $\text{SU}(2)$ contains two one-dimensional representations of $T$. In that case, only an abelian subgroup of $T$ is represented (because any sum of one-dimensional representations is abelian), and all irreps of $\text{SU}(2)$ will also contain only one-dimensional irreps of $T$. There can exist no nontrivial embedding of $T$ in $\text{SU}(2)$. Instead $T$ fits perfectly in $\text{SO}(3)$ as expected. Similarly $T'$ cannot be embedded in $\text{SO}(3)$ but embeds regularly in $\text{SU}(2)$.

For the purposes of particle theory model building $T'$ acts as if it contains $T$ as a subgroup only because Table 1 contains Table 2. Strictly speaking $T \not\subseteq T'$ and any group properties which are more subtle and which must rely on characters will possibly reflect this crucial fact. From another perspective $T'$ acts as if it contains $T$ as a subgroup only because $T'$ is a central extension of $T$. The element $-1$ is added to $T$ in a nontrivial and consistent way, such that it commutes with all of the elements of $T$, and the order of each element is doubled.

Is this a peculiar property only of $\text{SO}(3) \supset T \not\subseteq T' \subset \text{SU}(2)$? One remark may help clarify the possibilities: $\text{SU}(2)$ has exactly one element of order 2 (which is the matrix $-1$). Therefore it cannot contain any subgroup with more than one element of order 2 (which rules out all of the dihedral groups), or any subgroup whose element of order 2 does not commute with other elements.

We may quickly cite an infinite number of further examples. The dihedral groups $D_n$ are subgroups of $\text{SO}(3)$ and their double covers are the dicyclic groups $Q_{2n}$ which are subgroups of $\text{SU}(2)$. The irreps for $e.g.$ $Q_8$ are $1_1, 1_2, 1_3, 1_4, 2_1, 2_2, 2_3$ with the table for Kronecker products given in Table 3.
For comparison we note that $Q_8$ is the double cover of $D_4$ which has irreps $1_1, 1_2, 1_3, 1_4, 2$. Their table for Kronecker products is shown as Table 4.

**Table 3**

Table for Kronecker products of $Q_8$ irreps.

| $\otimes$ | $1_1$ | $1_2$ | $1_3$ | $1_4$ | $2_1$ | $2_2$ | $2_3$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|
| $1_1$     | $1_1$ | $1_2$ | $1_3$ | $1_4$ | $2_1$ | $2_2$ | $2_3$ |
| $1_2$     | $1_2$ | $1_1$ | $1_4$ | $1_3$ | $2_3$ | $2_2$ | $2_1$ |
| $1_3$     | $1_3$ | $1_4$ | $1_1$ | $1_2$ | $2_1$ | $2_2$ | $2_3$ |
| $1_4$     | $1_4$ | $1_3$ | $1_2$ | $1_1$ | $2_3$ | $2_2$ | $2_1$ |
| $2_1$     | $2_1$ | $2_3$ | $2_1$ | $2_3$ | $1_1 + 1_3 + 2_2$ | $2_1 + 2_3$ | $1_2 + 1_4 + 2_2$ |
| $2_2$     | $2_2$ | $2_2$ | $2_2$ | $2_2$ | $2_1 + 2_3$ | $1_1 + 1_2 + 1_3 + 1_4$ | $2_1 + 2_3$ |
| $2_3$     | $2_3$ | $2_1$ | $2_3$ | $2_1$ | $1_2 + 1_4 + 2_2$ | $2_1 + 2_3$ | $1_1 + 1_3 + 2_2$ |

We see that Table 4 is contained in Table 3 and that, from [2], $D_4 \not\subset Q_8$.

Thus the case $T \not\subset T'$ is not unique yet it is important to recall this relationship of $T$ to $T'$ when building models of particle theory based on the binary tetrahedral group.
Direct Products, Semidirect Products and Central Extensions

A general group extension can be represented by a short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

which expresses the conditions that $H$ is a normal subgroup of $G$ and $G/H = K$. We can distinguish three classes of extensions from the rest: direct products, semidirect products, and central extensions.

Direct products are extensions of a trivial sort: $H$ and $K$ are both realized as subgroups of $G$, and the elements of each subgroup commute with those of the other; more concisely, $G = HK, [H, K] = 1$.

A semidirect product is one in which the first property holds, but the multiplication in $H$ is twisted by an action of the elements of $K$:

$$g = h_1 k_1 h_2 k_2 = h_1 (k_1 h_2 k_1^{-1})k_1 k_2 = h_1 h_2^{k_1} k_1 k_2$$

where the notation $h_2^{k_1}$ is defined by the parenthesized expression.

A central extension is characterized by the property that $H$ is a subgroup of the center of $G$, $Z(G)$ (which consists of elements commuting with all of $G$). The point is that an extension can be both central and a semidirect product only if it is in fact a direct product: since $H \subset Z(G), K \subset G$ implies $[H, K] = 1$. Thus $SU(2)$ is a central extension of $SO(3)$, and if $K \subset SO(3)$, the corresponding extension $K' \subset SU(2)$ contains $K$ only if $K$ is abelian.

Concerning our explicit examples, $T$ is not a subgroup of $T'$ because $SO(3)$ is not a subgroup of $SU(2)$; $T'$ is a double cover of $T$ because $SU(2)$ is a double cover of $SO(3)$; $T'$ is a central extension of $T$ because $SU(2)$ is a central extension of $SO(3)$; $T$ is a central quotient of $T'$ because $SO(3)$ is a central quotient of $SU(2)$. The same statements hold mutatis mutandis replacing $T, T'$ respectively by $D_n, Q_{2n}$.
Discussion

The present discussion can stand on its own as an original contribution to the mathematics of group theory.

It is also germane to model building for quarks and leptons. As an example, consider the $T'$ model of quark masses and neutrino masses and mixings discussed in [3]. There the leptons are in singlets and triplets of $T'$, while the quarks are in singlets and doublets. If for some reason one wished to break the $T'$ quark mass relations without disrupting the neutrino sector, i.e., without disturbing tribimaximal mixing (TBM), one could hope to give a VEV to a scalar in a nontrivial $T'$ irrep. The only possibility would appear to be a $T'$ doublet VEV. However, VEVs for doublets do not leave an unbroken $A_4$ symmetry to protect TBM. Furthermore, not all $T'$ irreps can be decomposed into $A_4$ irreps, as can easily be seen from studying the character tables of the two groups. (As a specific example, the 2' is such a $T'$ irrep.) This means that in the $T'$ model there is no consistent way to reduce the quark symmetry, while temporarily preserving TBM. Other models can be analysed in a similar fashion.

As evidenced by [5], the groups $D_n, Q_{2n}, T, T', etc.$ have been used extensively as flavor symmetries in the literature. Many of these papers contain too hasty statements which can in the future be avoided so thereby the present article contributes to embedding nonabelian finite flavor groups in continuous groups.

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