Unveiling the spectrum of inspiralling binary black holes

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The measurement of sub-leading modes of gravitational wave signals from compact binary coalescences is of central importance for astrophysics and fundamental physics with wide-ranging implications. The detection of subdominant modes is very challenging as they carry much less power compared to the dominant quadrupole mode, and are visible only for binaries consisting of unequal masses with an orbital geometry that is not face-on. In this Letter, we present a novel method to combine the energy from multiple events observed in interferometric gravitational wave detectors. The events are stacked using time-frequency representations of the data, making use of the set of intrinsic parameters (mass, spin, and orbital parameters) inferred from the measurement of the dominant quadrupole mode. Stacking the events enhances the signal-to-noise ratio of the subdominant modes of the inspiral part of the signals, thereby increasing the chances of their detection. Our studies suggest that there is a ≥ 95% chance of detecting these subdominant modes at a false alarm probability of 1% from O(250) events observed simultaneously in two advanced-LIGO detectors, a target that may be easily achieved soon.

Introduction—The first detection of gravitational wave (GW) from a merging binary black hole (BH) [1] has ushered in a new era in observational astronomy and fundamental physics. From current estimates of the rate of binary black hole mergers, one expects future gravitational wave detectors to observe a large number of events which can reveal the diversity in population of compact binaries. Among compact binaries with precession and orbital eccentricity, an important class of sources that has eluded us thus far is the one which shows the signatures of sub-dominant modes of gravitational waveforms.

According to general relativity (GR), the dominant contribution to the gravitational waveform emitted by a compact binary comes from the quadrupole. While higher multipoles do contribute to the gravitational waveform, their amplitudes are suppressed [2, 3]. The strength of subdominant corrections also depend on the orientation of the binary with respect to the observer’s line of sight (zero for ‘face-on’ binaries) and the mass ratio of the binary constituents (zero for equal mass systems). Hence it is extremely challenging to detect the presence of these sub-dominant modes in the data. Nevertheless, the detection of higher modes is of immense scientific importance that could pave the way for new tests of GR [4], resolve the two states of gravitational wave polarization [5], and (for individual neutron star-blackhole binaries [6]) measure the inclination angle of the compact binary and thereby constrain possible jets [6, 7].

The present generation of interferometric GW observatories are biased towards detecting comparable-mass inspiraling binaries in the face-on or face-off oriented to the line of sight. As such, they are unlikely to detect higher-order modes from a single observation. However, a suitable combination of several sources could unravel these weak signals as shown here. Several algorithms have been developed in earlier studies to combine the post-inspiral, merger-ringdown signals from BBH systems by exploiting the constant ringdown frequency of all modes. These include coherent mode stacking [8] in the ringdown regime where individual signals are rescaled with reference to a base event such that the target mode in each of them has the same frequency. The constructive addition of signals in this manner enhances the overall amplitude of a chosen sub-leading ringdown mode. Similar transformations have also been attempted in the time-frequency domain [9] where merger/post-merger signals from different sources are synchronized through a series of changes so as to maximize the overlap between the dominant mode of the transformed signals. It also results in a semi-coherent stacking of the sub-dominant modes. Tests of GR with higher-order modes of ringdown signals from multiple BBH observations have also been posited [10, 11] using Bayesian model selection methods.

The crucial impediment in combining inspiral signals is the time-varying instantaneous frequency, which changes quite rapidly in the late-inspiral stages. As such, no method has been proposed till date to coherently combine the ‘inspiral-only’ part of signals from compact binaries. We seek to address this issue in this Letter.

Signal model—The GW wave signal \( h(t) \) propagating along an arbitrary direction \((t, \phi_0)\) in the source frame, can be decomposed over the spin-weighted spherical harmonic basis (with spin-weight \(-2\)) as:

\[
\begin{align*}
  h(t; t, \phi_0, \vec{\lambda}) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(-1)^m}{(2\ell+1)^{3/2}} Y_{\ell m}(t, \phi_0) h_{\ell m}(t; \vec{\lambda}),
\end{align*}
\]

where, \( h_{\ell m}(t; \vec{\lambda}) = A_{\ell m}(t; \vec{\lambda}) e^{i \Phi_{\ell m}(t; \vec{\lambda})} \) represents the \((\ell, m)\) mode of the signal described by the corresponding amplitude \( A_{\ell m}(t; \vec{\lambda}) \) and phase \( \Phi_{\ell m}(t; \vec{\lambda}) \); and where \( \vec{\lambda} \equiv \{m_1, m_2, \vec{s}_1, \vec{s}_2\} \) represents the set of intrinsic parameters consisting of the binary black hole system’s component masses \( m_{1,2} \) and spins \( s_{1,2} \). In particular, for non-precessing spinning BHs, the inspiral phase of an arbitrary \((\ell, m)\) mode can be expressed in terms of the phase of the \((2, 2)\) mode alone: \( \Phi_{\ell m}(t; \vec{\lambda}) = m/2 \Phi_{22}(t; \vec{\lambda}) \). The instantaneous frequency of each
mode is related to the time-derivative of the phase of the $(2,2)$ mode such that $f_{mn}(t; \tilde{\lambda}) = \Phi_{mn} = m/2 f_{22}(t; \tilde{\lambda})$, allowing us to define an arbitrary time-frequency ‘track’ scaled with respect to the trajectory of the $(2,2)$ track,

$$f_\alpha(t; \tilde{\lambda}) = \alpha f_{22}(t; \tilde{\lambda}), \quad (2)$$

where $\alpha$ is a nonzero positive scaling factor. One can obtain the specific track corresponding to the $(\ell, \pm m)$ harmonic mode of the signal from Eq. (2) by setting $\alpha = m/2$. This relationship between the phase of the harmonic modes $h_{mn}(t; \tilde{\lambda})$ of a GW signal, valid over the inspiral and merger regime, is of vital importance to the method presented in this Letter. Using a time-frequency spectrogram of the signals, this relation is leveraged for collecting the signal energy along such tracks parameterised by the scaling parameter $\alpha$, thereby decoupling the different modes of the GW signal. It is to be noted that all the $\ell \geq m$ modes of the signal follow the same track for $\alpha = m/2$. However, the energy along such a track is dominated by the $(\ell, m = \pm \ell)$ mode.

The spectrogram $\tilde{x}(\tau,f)$ of a given time series $x(t)$ is defined as the absolute square of its Stockwell transformation (S-transform) [12]:

$$\tilde{x}(\tau,f) = \left| \int_{-\infty}^{\infty} x(t) w(t-\tau,f) e^{-2\pi i ft} dt \right|^2, \quad (3)$$

where a frequency dependent Gaussian analyzing window $w(t-\tau,f) = |f|/\sqrt{2\pi} e^{-\left((\alpha-\tau)^2/2\right)}$ is used to improve the resolution of the signals in the time-frequency domain.

We adopt the following notation: the whitened detector data time-series around the $j$th detected event is denoted by $y_j(t) = n_j(t) + s(t; \tilde{\lambda}_j)$: consisting of ‘ideal’ detector noise $n_j(t)$ around the event having a normal distribution $\mathcal{N}(0,1)$; and a purported residual gravitational wave signal $s(t; \tilde{\lambda}_j)$ consisting of only the sub-dominant harmonics of the signal with intrinsic parameters $\tilde{\lambda}_j$ as determined from the measurement of the dominant $(2,2)$ quadrupole mode. In other words, $y_j(t)$ is constructed by subtracting off the best-fit $(2,2)$ mode of the signal from the data, and now contains the sub-dominant signal modes embedded in noise. Their corresponding spectrograms, calculated using Equation 3 are denoted by $\tilde{y}_j, \tilde{n}_j$ and $\tilde{s}_j$ respectively. The aLIGO power spectral density [13] is used to whiten the data and signals unless stated otherwise.

From observational data we can only access $y_j(t)$: data samples from a few tens of seconds away from the epoch of detection are assumed to contain no astrophysical GW signal, and provide representative samples of the noise-only time series $n_j(t)$. Similarly the residual signal $s(t; \tilde{\lambda}_j)$, allegedly embedded in this noise, is constructed by subtracting the $(2,2)$ mode from the full signal using an appropriate waveform model that incorporates higher-order modes.

The residual signal model $S_j(\alpha) \in \mathbb{R}^4$ is calculated from $\tilde{s}_j(\tau,f)$ by varying the scaling parameter $\alpha_{\text{min}} \leq \alpha \leq \alpha_{\text{max}}$ in $d$-discrete steps and summing over the pixels along a time-frequency arc given by Eq. 2:

$$S_j(\alpha) = \sum_{t_\tau = t_c - \Delta \tau}^{t_\tau = t_c + \Delta \tau} \tilde{s}_j(\tau,f = \alpha f_{22}(\tau; \tilde{\lambda}_j)), \quad (4)$$

leading up to the epoch $t_c$ at which the orbiting masses reach the innermost stable circular orbit (ISCO). The time durations of the tracks are taken to be $\Delta \tau = 0.5 \, s$. The noise model $N_j(\alpha)$ and the residual data model $Y_j(\alpha)$ for the $j$th event over the scaling parameter $\alpha$ are calculated in a similar manner from their respective spectrograms by substituting $\tilde{s}_j$ on the RHS of Equation 4 with $\tilde{n}_j$ and $\tilde{y}_j$ respectively.

We illustrate the residual signal model $S_j(\alpha)$ in Fig. 1 for three non-spinning BBH systems. A clear peak at $\alpha = \sqrt{2}/2$ is observed for all three systems corresponding to the energy present in the most significant subdominant $(3,3)$ mode. In contrast, the peaks at $\alpha = 3/2, 2\sqrt{3}/2$ and $\sqrt{2}/2$ are much smaller in proportion to the relative energy in these modes. The height of these peaks depend on the signal parameters and sensitivity of the detector whereas the peak-widths result from the finite time-frequency resolution of $\tilde{s}_j(\tau,f)$. For a hypothetical spectrogram having an arbitrarily small pixel resolution, one would expect $S_j(\alpha)$ to be composed of a sum of four Dirac-6 functions located at $\alpha = [0.5, 1.5, 2.0, 2.5]$. The peaks of $S_j(\alpha)$ from all the three events having different intrinsic parameters can be seen to occur at the same value of $\alpha$; indicating the possibility of stacking the events over the $\alpha$ parameter, thereby enhancing the detectability of the subdominant modes.

**Noise model**— As the ideal additive detector noise $n_j(t)$ follows a normal distribution $\mathcal{N}(0,1)$, the real and imaginary values of its S-transform at a time-frequency pixel $(\tau,f)$ can be shown to follow a Gaussian distribution $\mathcal{N}(0,1/\sqrt{\pi})$. As a result the spectrogram $\tilde{n}_j(\tau,f)$ follows a Gamma distribution $\Gamma(1,1/\sqrt{\pi})$.

By construction, each $N_j(\alpha)$ is a random variable whose distribution can be calculated from the convolution of many Gamma distributions. Noting that the limits of the summation in Equation 4 invariably result in summing over a very large number of pixels to arrive at the noise model, we appeal to the central limit theorem to deduce that $N_j(\alpha)$ follows a Gaussian distribution. This assertion is attested by the $p$-values obtained from the Kolmogorov-Smirnov test applied on each ele-
ment of \( N_j(\alpha) \). However, the elements of \( N_j(\alpha) \) are not independent random variables as evident from nonzero off-diagonal components in the noise covariance matrix 

\[
\Sigma_j(\alpha, \alpha') = E \left[ (N_j(\alpha) - \mu_j(\alpha))(N_j(\alpha') - \mu_j(\alpha'))^T \right],
\]

which can be numerically computed as outlined in Supplementary Material A. The Cholesky factor of its inverse: \( \Sigma_j^{-1} = L_j^T L_j \), can be used to construct the decorrelated noise model \( L_j N_j(\alpha) \). This construction will be used in subsequent analysis for a binary hypothesis test framework to detect signals in additive, correlated Gaussian noise.

**Combining multiple BBH observations**— The ‘combined data model’ \( Y(\alpha) \) is constructed by stacking the decorrelated data models \( Y_j(\alpha) \) for each of the observations over the \( \alpha \) parameter:

\[
Y(\alpha) = \sum_{j=1}^{n_0} L_j \left( Y_j(\alpha) - \mu_j(\alpha) \right) / \sqrt{\mu_0}. \tag{5}
\]

This construction ensures that in the absence of any residual signal from sub-leading harmonics in the data, \( Y(\alpha) \) has an identity covariance matrix. Similarly, decorrelated residual signal models \( S_j(\alpha) \) are stacked to get the ‘combined signal model’ \( S(\alpha) = \sum_{j=1}^{n_0} L_j S_j(\alpha) / \sqrt{\mu_0} \).

A binary hypotheses test is formulated on the combined residual data model \( Y(\alpha) \) to distinguish the null hypothesis \( \mathcal{H}_0 \) that the combined data model is consistent with detector noise, from the alternative \( \mathcal{H}_A \) which asserts that \( Y(\alpha) \) favours the presence of signals from sub-leading harmonics of GW signals.

The logarithmic likelihood ratio \( \Lambda \) measuring the ratio of probabilities \( p(Y(\alpha) | \mathcal{H}_1) / p(Y(\alpha) | \mathcal{H}_0) \) defined as:

\[
\Lambda = \langle Y(\alpha) | S(\alpha) \rangle / \frac{1}{2} \left| S(\alpha) \right|^2, \tag{6}
\]

can be used to distinguish between the two. Here \( \langle \cdot | \cdot \rangle \) denotes the inner-product between two vectors and \( \left| \cdot \right| \) denotes the norm.

For the null hypothesis \( \mathcal{H}_0 \), \( \Lambda \) can be shown to follow a Gaussian distribution: \( \mathcal{N}(\gamma^2 / 2, \gamma^2) \), where \( \gamma = \left| S(\alpha) \right| \) is the norm of the combined signal model vector \( S(\alpha) \). This motivates us to define a detection statistic \( \beta \) as:

\[
\beta = \left\{ \Lambda + \gamma^2 / 2 \right\} / \gamma, \tag{7}
\]

which ensures that the probability distribution \( p(\beta | \mathcal{H}_0) \equiv \mathcal{N}(0,1) \). Conversely, for the alternate hypothesis \( \mathcal{H}_1 \), the average of the detection statistics (over a large number of noise realizations) can be shown to be \( \langle \beta \rangle = \gamma \).

A threshold \( \beta^* \) corresponding to a nominal fixed false-alarm probability of 1% can be calculated from the distribution \( p(\beta | \mathcal{H}_0) \) by solving \( \int_{-\infty}^{\beta^*} p(\beta | \mathcal{H}_0) d\beta = 1% \) for \( \beta^* \). For ideal noise, we find \( \beta^* = 2.325 \) above which, one rejects the hypothesis \( \mathcal{H}_0 \) in favour of \( \mathcal{H}_1 \).

A subtlety involving combining the events can be illustrated by considering only two events with indices \( j = 1, 2 \) such that \( \gamma_1 > \gamma_2 \), where \( \gamma_j = \left| S_j(\alpha) \right| \) is the norm of the \( j \)th signal model. From the combined data model: \( Y(\alpha) = \sum_{j=1}^{2} L_j \left( Y_j(\alpha) - \mu_j(\alpha) \right) / \sqrt{2} \) for these two events, one can verify that their average combined detection statistic \( \langle \beta \rangle \) cannot exceed that obtained from the single (louder) event unless \( \gamma_2 / \gamma_1 \geq (\sqrt{2} - 1) \); thereby invalidating the advantage expected from stacking the events. This can be generalized for \( n_0 \) events assumed to be first arranged in a descending order of their norms \( \left| S_j(\alpha) \right| \) such that \( \gamma_1 > \gamma_2 > \cdots > \gamma_{n_0} \). One chooses to ‘optimally’ combine \( p \leq n_0 \) events where:

\[
p = \arg \max_{j \leq n_0} \left\{ \frac{\sum_{i=1}^{j} \gamma_i}{j} \right\}^2. \tag{8}
\]

This leads to the maximum possible \( \langle \beta \rangle \) for the combined data model.

In Fig S4 of Section D of the Supplemental Material, we show that the average detection statistic \( \langle \beta \rangle \propto \sqrt{n_0} \) when all \( n_0 \) identical events are combined using the method presented here. Thereby, the fully coherent nature of stacking the higher-order modes from multiple BBH observations is established. In contrast, combining the events in a Bayesian model selection study through the product of the Bayes factor of each event leads to a \( \sim n_0^{1/4} \) scaling of the SNR as shown in Ref [8].

**Prospects in Advanced LIGO**— We investigate the possibility of observing the sub-leading harmonics of GW signals from multiple BBH observations in aLIGO-like detectors by performing a Monte Carlo simulation with a set of non-spinning binary black hole systems drawn from an astrophysical population.

We assume a uniform merger rate density of 53 Gpc\(^{-3}\)yr\(^{-1}\) in the co-moving volume for stellar-mass black holes as inferred from aLIGO’s O1 and O2 observing runs [15]. The component masses are chosen between 5 \( \leq m_{1,2} / M_\odot \leq 50 \) with the primary mass \( m_1 \) chosen from a power-law distribution \( pm_1 \propto m_1^{\kappa} \) where \( \kappa = 2.3 \), and \( m_2 \) chosen from a uniform distribution \( pm_2 \sim U(5,m_1) \). To account for cosmological expansion, the component masses are converted to the detector frame by a multiplicative factor \( (1+z) \), where \( z \) denotes the redshift of the sources. The \( \Lambda \)CDM cosmological model (with parameters as provided in the Planck2015 [16]...
data release) are used for redshift - luminosity distance conversions. Further, the sources are assumed to be uniformly distributed over the celestial sphere up to a redshift of $z = 1.4$, and their inclination with respect to the line of sight are taken to be isotropically distributed. BBH systems with optimal quadrupole-mode SNR $\rho_{22} \geq 8$ (calculated at aLIGO design sensitivity using inspiral-merger-ringdown template) are considered as they are expected to be detectable in aLIGO search pipelines.

These distributions and selection criteria were applied to generate a playground-set of 2500 events, expected to be observed in aLIGO from $\sim 1.5$ years of continuous data acquisition. GW strains containing sub-dominant modes were generated using the SEOBNRv2HM [14] waveform model for each of the playground events and injected in synthetic Gaussian noise to mimic aLIGO data. Thereafter, single ($\beta_j$) and combined ($\beta$) detection statistic were calculated from the data.

In Fig. 2, we show the resulting distribution of the single-event detection statistic $p(\beta_j|H_1)$ obtained from all the events in the playground set, noting that $\langle \beta_j \rangle = 0.12$. By integrating the distribution: $\int_{-\infty}^{\infty} p(\beta_j|H_1) \, d\beta_j$, we conclude that there is a 7% probability of detecting the sub-leading harmonics from single events in aLIGO by this method. We also obtain the distribution $p(\beta|H_1)$ of the combined detection statistic after stacking 500 such events expected to be observed within 3.5 months of data acquisition of aLIGO at design sensitivity. For each subset of $n_0 = 500$ events chosen at random from the playground set through a bootstrapping procedure, we optimally stack $p \leq 500$ events using the algorithm presented earlier in Eq. (8). By integrating this distribution: $\int_{-\infty}^{\infty} p(\beta|H_1) \, d\beta$ we find that this leads to a detection probability of 95%.

In Fig. 3, we quantify the chances of detecting the subdominant modes of GW signals detection by varying the number of combined events $n_0$ in a single detector. As expected, the detection probability $P_D$ (calculated at 1% false-alarm) grows monotonically with increasing number of stacked events, reaching 95% when 500 events are combined and 99% for 750 events respectively.

The results presented here are for observations in a single aLIGO-like detector. In practice, BBH mergers are detected in coincidence across 2 or more detectors. It can be argued that by treating them independently, the required number of events to reach a certain threshold of detection probability may be reduced by factors of $\sim 2$ (double coincident detection) or more! This implies that $\mathcal{O}(250)$ events with $\rho_{22} \geq 8$ observed in double coincidence may be adequate to detect subdominant modes. This is a target that may be achieved very soon.

**Implementation in LIGO data**— Finally, we highlight some practical issues of stacking events detected in the network of advanced LIGO-Virgo detectors during first and second observation run [17].

a. Our analysis relies on data residuals obtained by subtracting the best-fit (2,2) mode waveform from the data. For this, it is imperative to have accurate estimates of the source parameters including coalescence time $t_c$ and phase $\phi_c$.

b. Accurate signal parameters are needed to construct the residual signal models (Equation 4) for which, one may choose the median of their posterior distributions obtained from parameter estimation using the dominant quadrupole mode [18]. In such studies, intrinsic parameters (such as chirp mass) are relatively well estimated in comparison to extrinsic parameters such as luminosity distance, inclination angle, sky location etc. which have a large error bars. The large error bars over extrinsic parameters translate to an unresolved overall amplitude of the residual signal models $S_j(\alpha)$ and in turn, an unreliable signal norm $\gamma_j$. As shown before, it is advantageous to stack only those signals whose norms are 'comparable'. The uncertainty in the signal norm thus lead to the delicate problem of choosing the right signals to combine! A brute-force strategy involving sampling the posterior parameter distributions for the highest norm of the residual signal can be computationally challenging.

c. Under the circumstances, it may be advantageous to construct the residual signal model from only the leading subdominant (3,3) mode of a non-precessing waveform model (e.g. SEOBNRv4HM [19]) by setting their intrinsic and extrinsic parameters to the median values of their respective posterior distributions. It follows from Equation 7 that the uncertainty of the overall amplitude has no effect on the single or combined event detection statistic for such a model. We have verified this through rigorous Monte-Carlo simulations.

d. Ranking the events by $\rho_{22}$ and mass-ratio $q$ could help: it is prudent to start from systems with highest $q$ to combine or stack the events.

For the event GW151012: using the residual data provided in the public data release [20] accompanying a recent paper [21] on testing GR using BBH signals in the GWTC-1 catalogue [17], and after combining
this event as seen in data from Hanford and Livingston interferometers independently, we find a combined $\beta = 3.95$, at a false alarm probability less than 0.3%! The threshold $\beta^* = 2.85$ corresponding to $P_{FA} = 1\%$ is calculated using LIGO strain data [22] proximal to the event epoch. Note that this is higher than the threshold obtained earlier for ideal noise and alludes the event epoch. Note that this is higher than the non-Gaussian traits present in real LIGO data.

Further details on determination of $p(\beta|H_0)$ from real LIGO strain data can be found in Section C of the Supplemental Material.

In future, we would like to incorporate the above points for efficiently stacking BBH events recorded in advanced LIGO and Virgo detectors. We would also like to extend this framework for inspiral-merger-ringdown waveforms and the possibility of testing general relativity from higher-order harmonics.

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In this document, we provide detailed calculation of certain crucial results used in the main text. For clarity, we define the notations used in this paper.

| Symbol | Description |
|--------|-------------|
| $\tilde{x}$ | S-transform of the time-series $x(t)$ |
| $\hat{x}$ | S-spectrum of $x(t)$ |
| $\hat{\sim}$ | Fourier transform of $x(t)$ |
| $\mathcal{N}(\mu, \sigma^2)$ | Gaussian distribution with mean $\mu$ and variance $\sigma^2$ |
| $\Gamma(a, b)$ | Gamma distribution with shape parameter $a$ and rate parameter $b$ |
| $\bar{\lambda}_j$ | Parameters of the $j$-th event |
| $\beta_j$ (β) | Single (Combined) event detection statistic |

### A. Estimation of the noise characteristics

The noise in the LIGO Like detectors is assumed to be approximately stationary and Gaussian with zero mean. With this assumption, the noise is fully characterized by the one-sided power spectral density, $S_n(f)$, such that $E[\tilde{n}_d(f) \tilde{n}_d(f')] = \frac{1}{2} \delta(f - f') S_n(f)$, where $E[.]$ denotes the ensemble average, and $\tilde{n}_d(f)$ represents the Fourier transform of the detector output $n_d(t)$. This allows us to produce the Whitened Gaussian Noise (WGN) time-series $n(t)$, from the detector strain $n_d(t)$ such that (in the frequency domain), $\tilde{n}(f) = \tilde{n}_d(f)/\sqrt{S_n(f)}$. Thus $n(t)$ follows a Gaussian distribution with zero mean and unit variance i.e. $n(t) \sim \mathcal{N}(0, 1)$.

The S-transform of $n(t)$ is defined as:

$$
\tilde{n}(\tau, f) = \int_{-\infty}^{\infty} n(t) \left| \frac{f}{\sqrt{2\pi}} e^{-\frac{(t-\tau)^2}{2f^2}} e^{-2\pi if t} dt, \quad (S1)
$$

where $g(t - \tau, f) = \left| \frac{f}{\sqrt{2\pi}} e^{-\frac{(t-\tau)^2}{2f^2}} \right|$ is the analyzing Gaussian window.

The S-transform of a Gaussian time-series will follow a complex Gaussian distribution since it is a linear transformation, where both the real and imaginary parts of $\tilde{n}(\tau, f)$ follow Gaussian distributions with same variance and zero mean. Further, the S-spectrum ($\bar{\tilde{n}}(\tau, f) = |n(\tau, f)|^2$) is the quadrature summation of two Gaussian random variables, which will follow a Gamma distribution.

The noise model $N_j(\alpha)$ is constructed by summing many (typically, several thousands) time-frequency pixels of $n$ along trajectories that are scaled with respect to $f_{22}(\tau, \bar{\lambda})$. This implies that the probability distribution over $N_j(\alpha)$ is a convolution of several thousand Gamma random variables. In this regime the well-known central limit theorem ensures that $N_j(\alpha)$ can be approximated by a Gaussian distribution.

FIG. S1. Covariance matrix $\Sigma_j(\alpha, \alpha')$ of the noise-model $N_j(\alpha)$, computed over the range $0.2 \leq \alpha, \alpha' \leq 3.2$. The subscript $j$ corresponds to the parameters $\bar{\lambda}_j$ of a BBH system non-spinning BBH system with component masses $[35, 13]M_\odot$.

Not only are the pixels along a track correlated with each other, but the sumation of pixels along two nearby tracks are also highly correlated. In order to characterize this correlation, we estimate the covariance matrix numerically from the ensemble average of several realizations of $N_j(\alpha)$, such that

$$
\Sigma_j(\alpha, \alpha') = E \left[ (N_j(\alpha) - \mu_j(\alpha))(N_j(\alpha') - \mu_j(\alpha'))^T \right],
$$

where $\mu_j(\alpha)$ is the ensemble average of $N_j(\alpha)$.

An example of a numerically estimated covariance matrix for a nonspinning BBH system is shown in Fig. S1. It is clearly seen that the covariance matrix is nondiagonal, especially the off-diagonal elements near the principal diagonal are comparable to the diagonal elements. In order to diagonalize the covariance matrix (and thus detrend the data), we employ a computationally efficient method of Cholesky decomposition (factorization) of $\Sigma_j^{-1}$. Such a decomposition is guaranteed by the fact that both the covariance matrix ($\Sigma_j$) and its inverse ($\Sigma_j^{-1}$) are symmetric and positive definite. The Cholesky decomposition of $\Sigma_j^{-1}$ factors it into a product $\Sigma_j^{-1} = L_j L_j^T$, where $L_j$ is a unique lower triangular matrix with positive diagonal entries. Now, the whitened (detrended) transformation of $N_j(\alpha)$ is given by $L_j N_j(\alpha)$, whose covariance matrix is identity. To check the agreement, we computed corresponding $p$-value using Kolmogorov-Smirnov test, which yields $p = 0$ for the alternative hypothesis.
B. Details of binary hypothesis testing for composite signal model

Here we discuss the binary hypothesis testing in additive correlated Gaussian noise for detecting the subdominant modes of a single BBH merger event.

As defined in Equation 4, the residual-signal model $S_j(\alpha)$ is calculated from $\tilde{s}_j(\tau, f)$; by varying the scaling parameter $\alpha_{\text{min}} \leq \alpha \leq \alpha_{\text{max}}$ in $d$-discrete steps and summing over the time-frequency pixels along well-defined arc. As such, $S_j(\alpha) \equiv \tilde{S}$ can be considered to be a vector in a $d$-dimensional Euclidean vector space $\mathbb{R}^d$. Similarly, the noise-model and the observational data-model can also be treated as vectors in $\mathbb{R}^d$.

Let $\tilde{N} \sim \mathcal{N}(0, \Sigma_{d \times d})$ be correlated Gaussian random vector in a $d$-dimensional vector space. The observed data $\tilde{Y}$ is either $\tilde{N}$ or $\tilde{N} + \tilde{S}$ depending on whether the null hypothesis ($H_0$) or alternative hypothesis ($H_1$) is true. Thus, the likelihood of $\tilde{Y}$ under the two hypotheses are given by:

$$p(\tilde{Y} \mid H_0) = \frac{\exp \left[ -\frac{1}{2} \tilde{Y}^T \Sigma^{-1} \tilde{Y} \right]}{(2\pi)^{d/2} |\Sigma|^{1/2}},$$

$$p(\tilde{Y} \mid H_1) = \frac{\exp \left[ -\frac{1}{2} (\tilde{Y} - \tilde{S})^T \Sigma^{-1} (\tilde{Y} - \tilde{S})^T \right]}{(2\pi)^{d/2} |\Sigma|^{1/2}},$$

(S3)

where $|\Sigma^{-1}|$ denotes the determinant of $\Sigma^{-1}$. Thus, logarithmic likelihood ratio is given by

$$\Lambda = \tilde{Y}^T \Sigma^{-1} \tilde{S} - \frac{1}{2} \tilde{S}^T \Sigma^{-1} \tilde{S}. \tag{S4}$$

Incorporating the Cholesky decomposition of $\Sigma^{-1} = L^T L$ in Eq. (S4), one can write

$$\Lambda = (L \tilde{Y})^T (L \tilde{S}) - \frac{\gamma^2}{2}, \tag{S5}$$

where $\gamma^2 = \tilde{S} \Sigma^{-1} \tilde{S} = \left\| \tilde{S} \right\|^2$ is the norm of the signal embedded in noise.

Now, we focus on constructing a detection statistic from $\Lambda$. If the null hypothesis $H_0$ is true, $(\tilde{Y} = \tilde{N})$, then $\Lambda$ from Eq. (S5) becomes $\Lambda = (L \tilde{N})^T (L \tilde{N}) - \gamma^2/2$. Thus, $(L \tilde{N})^T (L \tilde{N})$ becomes a summation of $d$ Gaussian random variables. For a sequence of mutually independent Gaussian random variables $x_1, x_2, \ldots, x_d$ with means $\mu_1, \mu_2, \ldots, \mu_d$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_d^2$, their linear combination $\sum_{i=1}^d c_i x_i$ can be shown to follow the Gaussian distribution with mean $\sum_{i=1}^d \mu_i$ and variance $\sum_{i=1}^d c_i^2 \sigma_i^2$. Thus, $L \tilde{N}^T$ follows $\mathcal{N}(0, \left\|L\tilde{S}\right\|^2)$ distribution. Therefore $\Lambda$ follows the Gaussian distribution:

$$\Lambda \sim \mathcal{N}(-\gamma^2/2, \gamma^2), \tag{S6}$$

when $H_0$ is True. Similarly, one can show that $\Lambda \sim \mathcal{N}(\gamma^2/2, \gamma^2)$ when $H_1$ is true. Motivated by these results, we define a new detection statistic,

$$\beta = (\Lambda + \gamma^2/2) / \gamma, \tag{S7}$$

which follows $\mathcal{N}(0, 1)$ when $H_0$ is true, is independent of the signal model. On the other hand, when $H_1$ is true, then contrary to expectation, $\beta$ follows $\mathcal{N}(\gamma, \sigma^2 > 1)$ with a variance greater than unity. The distribution of $\beta$ is spread out with variance more than unity due to cross-terms between signal and noise in the S-spectrum.

In Fig. S2, we show the distribution of $\beta$ for two cases from a set of noise realizations: (a) no signal present in noise ($H_0$ is true), depicted by the red histogram, and (b) an identical signal embedded in many different noise realizations ($H_1$ is true), depicted by the blue histogram. For the first case, it is clearly seen that red histogram has an excellent agreement with the probability density function $\mathcal{N}(0, 1)$ (green-dashed line). On the other hand, for second case, the mean ($\beta$) of the blue histogram agrees with the norm $\gamma$ of the injected signal but the variance is larger than 1.

C. Estimating the background in LIGO data

The computation of the noise covariance matrix is essential to obtain the robust detection statistic. In Sec. A, we discussed the derivation of the covariance matrix for synthetic whitened Gaussian noise. Here, we outline a similar approach for estimating the covariance matrix for real LIGO data.

First, we fetch 4000 s of LIGO strain data surrounding (but excluding) the event merger time. GW strain data for all the events detected during first and second observation runs are available from the Gravitational Wave Open Science Center [22]. Here we assume that LIGO data collected several tens of seconds away from the event epoch contains no GW signal and are representative of the noise in the detector. This data is then divided into 1000 equal-length chunks of 4 s each. Next, we whiten each chunk using their respective PSDs estimated using the standard Welch method. Finally, we
follows the event. On the other hand, the blue-solid line represents the normalized Gaussian function with zero mean unit variance for the case of ideal Gaussian noise. LIGO O1 data proximal to the GW150914 event, a few tens of seconds away from the merger epoch is used for this analysis as it serves as representative detector noise samples.

calculate the noise models $N_j(\alpha)$ from the S-spectrum of each chunk, and compute the covariance matrix using Eq. (S2). The tracks used to calculate the noise models are constructed using the source properties of detected BBH systems. We access them from the catalog of compact binary mergers observed by LIGO and Virgo during O1 and O2 (GWTC-1) [17].

In Fig. S3, we show that the histogram of $p(\beta|H_0)$ follows a Gaussian distribution, $\mathcal{N}(0,1.23^2)$, for a stretch of LIGO O1 data surrounding the GW150914 event. On the other hand, the blue-solid line refers to the ideal case for whitened Gaussian noise, which follows $\mathcal{N}(0,1)$. Due to the increase in noise variance, we require louder events in LIGO to detect it at the same false alarm probability. In particular, the threshold $\beta^*$ becomes 2.85 instead of 2.325 at a fixed false alarm probability of 1%. We also did the same analysis using the other LIGO data surrounding the GWTC-1 events and found that the variance of $p(\beta|H_0)$ has a maximum value of 1.32, which in turn implies a detection threshold of $\beta^* = 3.0$ to detect sub-dominant modes at a fixed false alarm probability of 1%.

D. Demonstrating the coherent nature of stacking

Here we demonstrate that our method can stack the energies of the subleading modes coherently over the $\alpha$ parameter. We show that the mean of the combined detection statistic increases as $n_0^{1/2}$ after stacking $n_0$ identical events, where the average is obtained over various noise realizations. The combined statistic using Eq. (7) and Eq. (6) can be written as:

$$\beta = \frac{\langle Y(\alpha) | S(\alpha) \rangle}{\gamma}, \quad (S8)$$

where $Y(\alpha) = \sum_{j=1}^{n_0} L_j (Y_j(\alpha) - \mu_j(\alpha)) / \sqrt{n_0}$ is the combined data model, and $S(\alpha) = \sum_{j=1}^{n_0} L_j S_j(\alpha) / \sqrt{n_0}$ is the combined signal model having a norm $\gamma = \|S(\alpha)\| = \langle S(\alpha) | S(\alpha) \rangle^{1/2}$. If all the events are identical then $\gamma = n_0^{1/2} \gamma_j$, where $\gamma_j$ is the signal norm of the single event. When the signal is present in the data then $Y(\alpha) = \sum_{j=1}^{n_0} L_j (Y_j(\alpha) + S_j(\alpha) - \mu_j(\alpha)) / \sqrt{n_0}$, and the numerator of Eq. (S8) becomes

$$\langle Y(\alpha) | S(\alpha) \rangle = \frac{1}{n_0} \left( \sum_{j=1}^{n_0} L_j (Y_j(\alpha) - \mu_j(\alpha)) \right) \left( \sum_{j=1}^{n_0} L_j S_j(\alpha) \right) + \frac{1}{n_0} \left( \sum_{j=1}^{n_0} L_j S_j \right) \left( \sum_{j=1}^{n_0} L_j S_j(\alpha) \right).$$

For identical events, the second term of the RHS above is $n_0 \gamma_j^2$, and the combined statistic in Eq. (S8) can be expressed as:

$$\beta = \frac{1}{n_0^{1/2} \gamma_j} \left( \sum_{j=1}^{n_0} L_j (Y_j(\alpha) - \mu_j(\alpha)) \right) \left( \sum_{j=1}^{n_0} L_j S_j(\alpha) \right) + n_0^{1/2} \gamma_j \quad (S9)$$

On the other hand, the first term:

$$\frac{1}{n_0^{1/2}} \left( \sum_{j=1}^{n_0} L_j (Y_j(\alpha) - \mu_j(\alpha)) \right) \left( \sum_{j=1}^{n_0} L_j S_j(\alpha) \right)$$

is the summation of Gaussian random variables. Using the properties of independent Gaussian random variable defined in Sec. B), it can shown that the first term has mean zero. Therefore the mean value of the combined detection statistic $\langle \beta \rangle = n_0^{1/2} \gamma_j$ for $n_0$ identical events. Recalling that the mean of the single-event detection statistic is equal to it’s signal norm $\gamma_j$, we have $\langle \beta \rangle / \langle \beta_j \rangle = \sqrt{n_0}$.

In Fig. S4, we stack a number of identical events (embedded in ideal Gaussian noise) and compare the ratio $\langle (\beta)/\langle \beta_j \rangle \rangle^2$ with the analytical result obtained above. The agreement between the two shows that the stacking method presented in this paper combines the events coherently with an increase of the statistic by a factor of $n_0^{1/2}$. 

![FIG. S3. Distribution $p(\beta|H_0)$ determined from the histogram of the detection statistic in absence of signal using LIGO data during O1 run. The red-dashed line is fitted with the histogram using a normalized Gaussian function, and the blue-solid line represents the normalized Gaussian function with zero mean unit variance for the case of ideal Gaussian noise. LIGO O1 data proximal to the GW150914 event, a few tens of seconds away from the merger epoch is used for this analysis as it serves as representative detector noise samples.](Image)

![FIG. S4. The plot of $\langle (\beta)/\langle \beta_j \rangle \rangle^2$ versus number of events follows a straight line with unit slope, where all the events are identical. This implies that the stacking algorithm is coherent where increment of average detection static after stacking $n_0$ identical events grows as $\sqrt{n_0}$.](Image)