Local distinguishability based genuinely quantum nonlocality without entanglement

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Abstract
Recently, Halder et al (2019 Phys. Rev. Lett. 122 040403) proposed the concept of strong nonlocality without entanglement: an orthogonal set of fully product states in multipartite quantum systems that is locally irreducible for every bipartition of its subsystems. Due to the complexity of the problem, most results are limited to tripartite systems. Here we consider a weaker form of nonlocality which is called local distinguishability based genuine nonlocality. A set of orthogonal multipartite quantum states is said to be genuinely nonlocal if it is locally indistinguishable for every bipartition of the subsystems. In this work, we study how to construct sets of orthogonal product states which are genuinely nonlocal. Firstly, we present a set of product states with simple structure in bipartite systems that is locally indistinguishable. After that, based on a simple observation, we present a general method to construct genuinely nonlocal sets of multipartite product states by using those sets that are genuinely nonlocal but with less parties. As a consequence, we obtain that genuinely nonlocal sets of fully product states exist in any \( L \) parties systems \( \bigotimes_{i=1}^{L} \mathbb{C}^{d_i} \) provided \( L \geq 3 \) and \( d_i \geq 3 \) for all \( i \).

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1. Introduction

Quantum states discrimination plays a fundamental role in quantum information processing. It is well known that a set of quantum states can be perfectly distinguished by positive operation value measurement if and only if these states are pairwise orthogonal [1]. In a multipartite setting, due to the physical obstacles, sometimes we cannot take a global measurement but only can use local operations with classical communication (LOCC). Bennett et al [2] presented examples of orthogonal product states that are indistinguishable under LOCC and named such a phenomenon as quantum nonlocality without entanglement. The nonlocality here is in the sense that there exists some quantum information that could be inferred from the outcome of a global measurement but cannot be read from local correlations of the subsystems. A set of orthogonal states which is indistinguishable under LOCC is also called as being locally indistinguishable or nonlocal. The local indistinguishability has been practically applied in quantum cryptography primitives such as data hiding [3, 4] and secret sharing [5–7].

Since the work of Bennett et al [2], the problem of local discrimination of quantum states has attracted much attention. The maximally entangled states and the product states, as being two extreme sets among the pure states, their local distinguishabilities are the most attractive. Here we present an incomplete list of the results about the local distinguishability of maximally entangled states [8–22] and product states [2, 23–43]. Another direction of related research is to study how much resource of entanglement are needed in order to distinguish quantum states which are locally indistinguishable [44–48]. In addition, the unextendible product bases is a class of sets which are known to be locally indistinguishable [23, 25]. Here an unextendible product basis is a set of incomplete orthonormal product states whose complementary space has no product state [25, 26, 49–51].

Recently, Halder et al [52] introduced a stronger form of local indistinguishability, i.e. local irreducibility. A set of multipartite orthogonal quantum states is said to be locally irreducible if it is not possible to locally eliminate one or more states from the set while preserving orthogonality of the postmeasurement states. Under this setting, they proposed the concept strong nonlocality without entanglement. A set of orthogonal multipartite product states is called to be strong nonlocality if it is locally irreducible for every bipartition of its subsystems. They provided the first two examples of strongly nonlocal sets of product states in $C^3 \otimes C^3 \otimes C^3$ and $C^4 \otimes C^4 \otimes C^4$ and raised the questions of how to extend their results to multipartite quantum systems and sets of unextendible product bases [52]. Quite recently, Zhang and Zhang [53] extended the concept of strong nonlocality to more general settings. However, there are only a few sets which have been proven to be strongly nonlocal [52–57].

Most of the known results are limited in considering the tripartite quantum setting. Here we study a form of nonlocality called genuine nonlocality whose nonlocality is lying between the local distinguishability based nonlocality and the local irreducibility based strong nonlocality. A set of orthogonal multipartite quantum states is said to be genuinely nonlocal if it is locally indistinguishable for every bipartition of its subsystems (this definition is proposed by Rout et al in the abstract of reference [54], however, the definition of genuine nonlocality in their main text is slightly different from that in their abstract). There, the authors emphasized on the classification of the genuinely nonlocal product sets. They used two known nonlocal sets in $C^3 \otimes C^3$ to construct a genuine nonlocal set in $C^4 \otimes C^4 \otimes C^4$ in proposition 1 of reference [54] and pointed out that such a construction of genuinely nonlocal set is straightforward for
an arbitrary number of parties. One should note that the local dimensions of their constructing sets are always higher than those of the known ones in order to make sure that the states in the constructed set are mutually orthogonal. Whether genuinely nonlocal set can be constructed from those with less parties but the same local dimensions? Moreover, are there genuinely nonlocal sets of fully product states for more general multipartite quantum systems? In this paper, we tend to address these problems.

The rest of this article is organized as follows. In section 2, we give some necessary notations and the definition of the genuinely nonlocal sets. In section 3, we present a construction of locally indistinguishable set of product states in bipartite systems. In section 4, we present a general method to construct genuinely nonlocal set of multipartite product states. Finally, we draw a conclusion and present some interesting problems in section 5.

2. Definition and notations

In this section, we introduce some necessary notations and definition. Throughout this paper, if $\mathcal{H}$ is an Hilbert space of dimensional $d$, we always assume that $\{|\psi_j\rangle\}_{j=1}^N$ is its computational basis. First, we give a review of the definition of genuinely nonlocal product sets from reference [54].

Definition 1 (Genuinely nonlocal product set). A multipartite orthogonal product set $\mathcal{S}_{\text{gnl}} \equiv \{|\psi_j\rangle|\psi_j\rangle = \otimes_{j=1}^p |\alpha_j\rangle, j = 1,2,\ldots,N \} \subseteq \bigotimes_{j=1}^p \mathbb{C}^d$ is called genuinely nonlocal if the states in $\mathcal{S}_{\text{gnl}}$ cannot be perfectly distinguished by LOCC across every bipartition of the subsystems.

We make a simple comparison between this concept and the strongly nonlocal product set. A set of orthogonal multipartite product states is said to be strongly nonlocal if it is locally irreducible for every bipartition of its subsystems. Yuan et al [56] showed that the local irreducibility of $\mathcal{S}_{\text{gnl}}$ for every $(1, n - 1)$ bipartition implies the local irreducibility of $\mathcal{S}_{\text{gnl}}$ for every $(k, n - k)$ bipartition with $1 \leq k \leq n - 1$. Therefore, in order to prove that $\mathcal{S}_{\text{gnl}}$ is strongly nonlocal it is enough to show the local irreducibility of $\mathcal{S}_{\text{gnl}}$ for every $(1, n - 1)$ bipartition. However, similar implication does not hold when we consider the genuinely nonlocal product sets. In fact, the following set in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ are locally indistinguishable for partitions $A|BCD, B|CDA, C|DAB, D|ABC$ (see observation 1).

$$\mathcal{S} \equiv \{ |\theta_i\rangle_A |\xi_i\rangle_B |2\rangle_C |2\rangle_D, |2\rangle_A |2\rangle_B |\theta_i\rangle_c |\xi_i\rangle_D | i = 1,2,\ldots,8 \},$$

where the set of states

$$|\theta_1\rangle |\xi_1\rangle = |1\rangle (|1\rangle + |2\rangle), \quad |\theta_2\rangle |\xi_2\rangle = |1\rangle (|1\rangle - |2\rangle), \quad |\theta_3\rangle |\xi_3\rangle = (|1\rangle + |2\rangle) |3\rangle,$$

$$|\theta_4\rangle |\xi_4\rangle = (|1\rangle - |2\rangle) |3\rangle, \quad |\theta_5\rangle |\xi_5\rangle = |3\rangle (|2\rangle + |3\rangle), \quad |\theta_6\rangle |\xi_6\rangle = |3\rangle (|2\rangle + |3\rangle),$$

$$|\theta_7\rangle |\xi_7\rangle = (|2\rangle + |3\rangle) |1\rangle, \quad |\theta_8\rangle |\xi_8\rangle = (|2\rangle - |3\rangle) |1\rangle,$$

are locally indistinguishable in $\mathbb{C}^3 \otimes \mathbb{C}^3$ [2]. Since the states in $\{|\theta_i\rangle |\xi_i\rangle \}_i \subseteq \{|2\rangle |2\rangle \}$ are mutually orthogonal, the set $\mathcal{S}$ is locally distinguishable for bipartition $AB|CD$. Therefore, to prove a set to be genuinely nonlocal, we need to check the local indistinguishability for every bipartition of the subsystems. As a consequence, there is a bit difference between strongly nonlocal set and genuinely nonlocal set.

Now we introduce more notations which will be used in the next section and the first part of section 4. For any integer $d \geq 2$, we denote $U(d)$ to be the set of all unitary matrices of
dimensional $d$. We will use the following subset of unitary matrices

$$U_{FL}(d) := \{(h_i)_{d}^{d} | h_{ik} \in U(d); h_{ik} \neq 0, k = 1, \ldots, d\}.$$ 

That is, the set of $d$ dimensional unitary matrices whose elements on the first and last rows are all nonzero.

Let $d \geq 3$ be an integer and $\mathcal{H}$ be a Hilbert space of dimension $d$. Assume that $B = (|1\rangle, \ldots, |d\rangle)$ is an $d$-tuple of vectors in $\mathcal{H}$ and these $d$ vectors are consisting of an orthonormal basis of $\mathcal{H}$. And we call $B$ an ordered orthonormal basis of $\mathcal{H}$. For any $H = (h_{jk})_{j=1}^{d-1} \in U_{FL}(d - 1)$, we define two operations on $\mathcal{H}$ with respect to $B$

$$H_B^{(U)} := \sum_{j=1}^{d-1} \sum_{k=1}^{d-1} h_{jk} |j\rangle \langle k|, H_B^{(D)} := \sum_{j=1}^{d-1} \sum_{k=1}^{d-1} h_{jk} |j+1\rangle \langle k+1|.$$ 

That is, under the computational basis $\{|1\rangle, \ldots, |d\rangle\}$, their matrix representations are as follows

$$H_B^{(U)} = \begin{bmatrix} H & 0_{d(d-1)\times 1} \\ 0_{1\times (d-1)} & 0 \end{bmatrix},$$

$$H_B^{(D)} = \begin{bmatrix} 0 & 0_{1\times (d-1)} \\ 0_{(d-1)\times 1} & H \end{bmatrix}.$$ 

We call them the up extension and down extension of $H$ with respect to $B$ respectively.

### 3. Locally indistinguishable set of bipartite product states

Motivated by the constructions of nonlocal sets of product states in reference [42], the following theorem is a generalized version of their results but the proof is more elegant.

**Theorem 1** Let $x, y \geq 3$ be integers and $X \in U_{FL}(x - 1)$, $Y \in U_{FL}(y - 1)$. Let $\mathcal{H}_A(\mathcal{H}_B)$ be an Hilbert space of dimension $x(y)$ with an ordered orthonormal basis $A = (|1\rangle_A, \ldots, |x\rangle_A)$ ($B = (|1\rangle_B, \ldots, |y\rangle_B)$). The following $2(x + y) - 4$ product states in $\mathcal{H}_A \otimes \mathcal{H}_B$ are locally indistinguishable (see figure 1)

$$|\psi_i\rangle := |1\rangle_A \otimes (Y_B^{(U)}|i\rangle_B),$$

$$|\psi_{y+1+i}\rangle := (X_A^{(D)}|j\rangle_A) \otimes |y+1\rangle_B,$$

$$|\psi_{x+y+3+i}\rangle := |x\rangle_A \otimes (Y_B^{(D)}|k\rangle_B),$$

$$|\psi_{x+2y+4+i}\rangle := (X_A^{(D)}|l\rangle_A) \otimes |1\rangle_B,$$

where $1 \leq i \leq y - 1, 1 \leq j \leq x - 1, 2 \leq k \leq y, 2 \leq l \leq x$.

**Proof.** Suppose Alice starts with a measurement $\{M_a^iM_a^j\}_{a=1}^S$. The postmeasurement states should be orthogonal to each other, i.e.

$$\langle \psi_i|M_a^iM_a^j \otimes I_B|\psi_j\rangle = 0, \text{ for } i \neq j.$$ 

Let $M := M_a^iM_a^j$. Suppose its matrix representation under the ordered basis $A$ is $(m_{ij})_{i,j=1}^x$. Then one finds

$$M = \sum_{i=1}^x \sum_{j=1}^x m_{ij} |i\rangle_A \langle j|.$$
Because \( M_A \otimes I_p |\psi_i\rangle \) is orthogonal to the set of states \( \{ M_A \otimes I_p |\psi_{x+y^3-4+l}\rangle \mid l = 2, 3, \ldots, x \} \), we have the following equations
\[
A \langle l | M X_A^{(l)} | l \rangle_A = 0, \quad l = 2, 3, \ldots, x,
\]
which are equivalent to the matrix equality [symmetry of the constructed states, Bob can only start with a trivial measurement. Meanwhile, the first row of \( X \) pairing these two vectors, one can derive [\( M_A \) invertible, we have \( m_{12}, \ldots, m_{1k} \) is of the form \( \alpha = 1, 2, \ldots, x \), we have the following equations
\[
\left[ M \right]_{x \times x} \left[ M = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right.
\]
where \( \alpha = 1, 2, \ldots, x \).]

Because the set of states \( \{ M_A \otimes I_p |\psi_{-1+j}\rangle \mid j = 1, 2, \ldots, x - 1 \} \) are pairwise orthogonal to each other, we have the following equations
\[
A \langle j_1 | X_A^{(l)} M X_A^{(l)} | j_2 \rangle_A = 0, \quad \text{for } 1 \leq j_1 \neq j_2 \leq x - 1.
\]
If we define \( M^{(a)} := \{m_{ij}\}_{i,j=1}^{a-1} \), the above equalities are equivalent to \( X^T M^{(a)} X = \text{diag}(\alpha_1, \ldots, \alpha_{x-1}) \). Since \( X \) is a unitary matrix, we have
\[
M^{(a)} X = X \text{ diag}(\alpha_1, \ldots, \alpha_{x-1}).
\]
Suppose that \( X = (X_{ij})_{i,j=1}^{x-1} \). Then the first row of \( M^{(a)} X \) is \( [m_{11} X_{11}, \ldots, m_{1x} X_{1(x-1)}] \). Meanwhile, the first row of \( X \text{ diag}(\alpha_1, \ldots, \alpha_{x-1}) \) is \( [\alpha_1 X_{11}, \ldots, \alpha_{x-1} X_{1(x-1)}] \). Comparing these two vectors, one can derive \( [\alpha_1, \alpha_2, \ldots, \alpha_{x-1}] = [m_{11}, m_{11}, \ldots, m_{1x}] \) as \( X_{11}, X_{12}, \ldots, X_{1(x-1)} \) are all nonzero. Hence
\[
M^{(a)} = X \text{ diag}(\alpha_1, \ldots, \alpha_{x-1}) X^T = m_{11} I_{x-1}.
\]
Similarly, if we define \( M^{(d)} := \{m_{ij}\}_{i,j=2}^{x} \), using the orthogonal relations among the states in \( \{ M_A \otimes I_p |\psi_{x+y^3-4+l}\rangle \mid l = 2, 3, \ldots, x \} \), we can get \( M^{(d)} = m_{x1} I_{x-1} \). Therefore, the Hermitian matrix \( M \) is of the form \( m_{11} I_x \). That is, Alice can only start with trivial measurement. By the symmetry of the constructed states, Bob can only start with a trivial measurement. \( \square \)
4. Genuinely nonlocal set of multipartite product states

Rout et al pointed out genuinely nonlocal sets do exist in an arbitrary number of parties [54]. However, it is not known such a construction of genuinely nonlocal set exist in an arbitrary quantum system. As any set of orthogonal product states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is locally distinguishable [26], a necessary condition for an orthogonal set of fully product states in $\bigotimes_{i=1}^{L} \mathbb{C}^{d_i}$ to be genuinely nonlocal is $d_i \geq 3$ for all $i$. In this section, we show that there always exists some genuinely nonlocal set of fully product states in $\bigotimes_{i=1}^{L} \mathbb{C}^{d_i}$ if the previous necessary condition is fulfilled. That is,

**Theorem 2** Let $L \geq 3$ and $d_i \geq 3 (i = 1, 2, \ldots, L)$ be integers. Then there always exists an orthogonal set of fully product states in $\bigotimes_{i=1}^{L} \mathbb{C}^{d_i}$ that is genuinely nonlocal.

This conclusion can be derived from theorems 1 and 3, proposition 1 of this paper and the genuinely nonlocal set of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ constructed in reference [54]. Note that the construction here is general and does not rely on the result of theorem 1 when $L \geq 4$. The starting point of our constructions of genuinely nonlocal sets is the following observation.

Note that if $S = \{|\phi_k\rangle_A |\theta_k\rangle_B\}_{k=1}^{N}$ is $A|B$ locally indistinguishable, then $S' = \{|\phi_k\rangle_A |\theta_k\rangle_B\} |\varphi\rangle_{A_2} |\theta\rangle_{B_2}\}_{k=1}^{N}$ is also $A_2|B_2$ locally indistinguishable. Otherwise, in the first setting, Alice and Bob can prepare the ancillary qudit states as $|\varphi\rangle_{A_1}$, $|\theta\rangle_{B_1}$ respectively on themselves sides.
and using the latter distinguish strategy to locally distinguish the set of states in $S$. Moreover, we have the following observation (see also in reference [53])

**Observation 1** Let $S = \{ |\Psi_k \rangle \}_{k=1}^n$ be a nonlocal product set shared between Alice and Bob. Consider the set $\mathcal{S} := \{ |\Psi_1 \rangle \otimes \Theta_0 A_k \otimes A_n \}$, where $|\Theta_0 \rangle A_k \otimes A_n$ and $|\Psi_1 \rangle B_n \otimes B_n$ are some fully product states with some of the subsystems $\{ A_i \}_{i=1}^n$ and $\{ B_i \}_{i=1}^n$. The resulting set $S$ is also nonlocal between $\mathcal{H}_A \otimes (\otimes_{i=1}^n \mathcal{H}_A)$ and $\mathcal{H}_B \otimes (\otimes_{i=1}^n \mathcal{H}_B)$.

### 4.1. Genuinely nonlocal set of product states of tripartite systems

With the observation 1 at hand, we show how to use the special structure of nonlocal sets in theorem 1 to construct genuinely nonlocal sets of product states in tripartite systems.

**Theorem 3** Let $x, y, \leq 3, y \leq 4$ be integers and $X, Y, Z$ belong to $UFL(x-1), UFL(y-1)$ and $UFL(z-1)$ respectively. Let $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ be Hilbert spaces of dimension $x, y, z$ respectively. Suppose that $A = \{ 1 \}, \ldots, | x \rangle A, B = \{ 1 \}, \ldots, | y \rangle B$, and $C = \{ 1 \}, \ldots, | z \rangle C$ are ordered orthonormal bases with respect to $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$. The following $2x + 4y + 2z - 8$ product states in $C^x \otimes C^y \otimes C^z$ are pairwise orthogonal and they form a set of product states which is genuinely nonlocal (see figure 1)

$$|\Psi_i \rangle := |1 \rangle A \otimes (Y_B^U D_k) |1 \rangle_B \otimes |1 \rangle C,$$

$$|\Psi_{y+1} \rangle := (X_A^U D_k) |1 \rangle A \otimes |y \rangle_B \otimes |1 \rangle C,$$

$$|\Psi_{y+y+1} \rangle := |1 \rangle A \otimes (Y_B^D D_k) |1 \rangle_B \otimes |1 \rangle C,$$

$$|\Psi_{x+y-4+1} \rangle := (X_A^D D_k) |1 \rangle A \otimes |1 \rangle_B \otimes |1 \rangle C,$$

where $1 \leq i \leq y - 1, 1 \leq j \leq x - 1, 2 \leq k \leq y, 2 \leq l \leq y$ and (see figure 3)

$$|\Phi_i \rangle := |2 \rangle A \otimes |1 \rangle_B \otimes (Z_C^U D_k) |1 \rangle C,$$

$$|\Phi_{z-1-k} \rangle := |2 \rangle A \otimes (Y_B^U D_k) |1 \rangle_B \otimes |1 \rangle C,$$

$$|\Phi_{y+z-3+k} \rangle := |2 \rangle A \otimes |y \rangle_B \otimes (Z_C^D D_k) |1 \rangle C,$$

$$|\Phi_{y+z-2-4+i} \rangle := |2 \rangle A \otimes (Y_B^D D_k) |1 \rangle_B \otimes |1 \rangle C,$$

where $1 \leq i \leq z - 1, 1 \leq j \leq y - 1, 2 \leq k \leq z, 2 \leq l \leq y$. Here the ordered bases $B'$ and $C'$ are defined as follows

$\mathcal{B}' := \{ |1 \rangle_B, |2 \rangle_B, \ldots, |y-1 \rangle_B, |y \rangle_B \}, \quad \mathcal{C}' := \{ |1 \rangle_C, |2 \rangle_C, \ldots, |z \rangle_C \},$

where $|1 \rangle_B = |2 \rangle_B = |3 \rangle_B = |1 \rangle_B$, $|y-1 \rangle_B = |y \rangle_B$, $|y \rangle_B = |y-1 \rangle_B$, $|j \rangle_B = |j \rangle_B$ for $3 \leq j \leq y - 1$ and $|1 \rangle_C = |2 \rangle_C = |3 \rangle_C = |C \rangle$ for $3 \leq j \leq z$.

**Proof.** Note that $\text{span}_C \{ |\Psi_1 \rangle, |\Psi_2 \rangle, \ldots, |\Psi_{2x+y-4} \rangle \}$ is equal to the linear space spanned by

$\mathcal{S}_B := \{ |j \rangle_B |1 \rangle C \mid i \in \{ 1, x \} or j \in \{ 1, y \} \}$

while $\text{span}_C \{ |\Phi_1 \rangle, |\Phi_2 \rangle, \ldots, |\Phi_{2x+y-4} \rangle \}$ is equal to the linear space spanned by

$\mathcal{S}_B := \{ |2 \rangle_A (|j \rangle_B |k \rangle_C \mid j \in \{ 1, y \} or k \in \{ 1, z \} \} = \{ |2 \rangle_A (|j \rangle_B |k \rangle_C \mid j \in \{ 2, y - 1 \} or k \in \{ 2, z \} \}. $
As $S_{\Phi} \cap S_{\Psi} = \emptyset$ and $S_{\Psi} \subseteq \{|i\rangle_A | j\rangle_B | k\rangle_C \mid 1 \leq i \leq x, 1 \leq j \leq y, 1 \leq k \leq z\}$ which is an orthonormal basis of $H_A \otimes H_B \otimes H_C$, we have $\langle \Psi_u | \Psi_v \rangle = 0$ for integers $u, v$ with $1 \leq u \leq 2x + 2y - 4$, $1 \leq v \leq 2y + 2z - 4$. Therefore, the states in $\{|\Psi_u\rangle\}_{u=1}^{2x+2y-4} \cup \{|\Phi_v\rangle\}_{v=1}^{2y+2z-4}$ are pairwise orthogonal (figure 4 is more intuitive for the orthogonality).

To prove that the set of states we construct are genuinely nonlocal. There are only three ways to separate $ABC$ into two sets. That is, $A|BC$, $B|CA$, and $C|AB$. By theorem 1 and observation 1, the set $\{|\Psi_u\rangle\}_{u=1}^{2x+2y-4}$ is locally indistinguishable as the partitions $A|BC$ and $B|CA$. And the set $\{|\Phi_v\rangle\}_{v=1}^{2y+2z-4}$ is locally indistinguishable as the partitions $B|CA$ and $C|AB$. Hence the given set is genuinely nonlocal.

\[\square\]

4.2. Constructing genuinely nonlocal set from known ones

In the following, we begin to strengthen the results in [37] where they constructed locally indistinguishable multipartite product states from known bipartite ones. The following two
propositions enhance their results to genuine nonlocality settings. The constructions are general as there is no assumption on the locally indistinguishable sets.

**Proposition 1** Let $L \geq 4$ be an integer and $d_i \geq 3$ for all $1 \leq i \leq L$. Let $S_i = \{|\psi_j^{(i)}\rangle \rangle \psi_j^{(i)}\rangle |1\rangle |1\rangle |\ldots |2\rangle \}^n_j$ be sets of product states that are locally indistinguishable for $1 \leq i \leq L - 1$. Then the union of the following sets (see figure 5)

\[
S_1 = \{|\psi_j^{(1)}\rangle \rangle \psi_j^{(1)}\rangle |1\rangle |1\rangle |\ldots |2\rangle \}^n_j,
S_2 = \{|\psi_j^{(2)}\rangle \rangle \psi_j^{(2)}\rangle |2\rangle |1\rangle |\ldots |1\rangle \}^n_j,
S_3 = \{|\psi_j^{(3)}\rangle \rangle \psi_j^{(3)}\rangle |1\rangle |2\rangle |\ldots |1\rangle \}^n_j,
\vdots
S_{L-1} = \{|\psi_j^{(L-1)}\rangle \rangle \psi_j^{(L-1)}\rangle |1\rangle |\ldots |2\rangle |\ldots |1\rangle \}^n_j,
\]

is also a genuinely nonlocal set of product states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_L}$.

**Proof.** To distinguish the states of $S_1$, by observation 1, the first two parties must come together and perform a global measurement. Similarly, to distinguish the states of $S_i$, the 1th and $(i + 1)$th parties must come together and perform a global measurement. Therefore, all the parties must come together to distinguish all the states in equation (2). Hence such a set of states is genuinely nonlocal.

Lots of works have shown that there exist different kinds of locally indistinguishable sets of product states in bipartite systems $\mathbb{C}^m \otimes \mathbb{C}^n$ ($m, n \geq 3$). Based on these results and proposition 1, one can conclude that there always exists a genuinely nonlocal set of product states in $\bigotimes_{i=1}^{L} \mathbb{C}^{d_i}$ whenever $L \geq 4$ and $d_i \geq 3$.

**Proposition 2** Let $L \geq 5$ be an integer and $d_i \geq 3$ for all $1 \leq i \leq L$. Let $S_i := \{|\psi_j^{(i)}\rangle \rangle \psi_j^{(i)}\rangle |1\rangle |\ldots |2\rangle \}^{n_j}$ be sets of product states that are locally indistinguishable for $1 \leq
Figure 6. States structure corresponding to the proposition 2. The states in $S_i$ are different only at the subsystems labeled by $i$ and $(i + 1)$. For example, as $S_2$ is defined as $S_2 = \{ |1\rangle \psi(j_1) |\phi(j_2) |2\rangle |1\rangle |1\rangle \cdots |1\rangle \}$, all states in $S_2$ have the same local part, except the second subsystem and the third subsystem. Moreover, to distinguish the states in $S_i$, the two parties $i$ and $(i + 1)$ must come together and perform a global measurement.

Figure 7. States structure corresponding to the proposition 3. The states in $S_i$ are different only at the subsystems labeled by $1$, $2i$ and $2i + 1$. For example, as $S_1$ is defined as $S_1 = \{ |\psi(j_1) \rangle |\phi(j_2) \rangle |\chi(j_3) \rangle |1\rangle |1\rangle \cdots |1\rangle \}$, all states in $S_1$ have the same local part, except the first three subsystems. Moreover, to distinguish the states in $S_i$, the three parties $1$, $2i$ and $2i + 1$ must come together and perform a global measurement.

$i \leq L - 1$. Then the union of the following sets (see figure 6)

\[
S_1 = \{ |\psi(j_1) \rangle \psi(j_2) \rangle |\phi(j_1) \rangle |\phi(j_2) \rangle |2\rangle |1\rangle |1\rangle \cdots |1\rangle \}^{n_1}_{j=1},
S_2 = \{ |1\rangle \psi(j_1) \rangle |\phi(j_2) \rangle |\phi(j_1) \rangle |2\rangle |1\rangle |1\rangle \cdots |1\rangle \}^{n_2}_{j=1},
S_3 = \{ |1\rangle |1\rangle \psi(j_1) \rangle |\phi(j_2) \rangle |\phi(j_1) \rangle |2\rangle |1\rangle |1\rangle \cdots |1\rangle \}^{n_3}_{j=1},
\]

\[\vdots\]

\[
S_{L-1} = \{ |2\rangle |1\rangle |1\rangle |1\rangle \cdots |1\rangle |\psi((L-1)\rangle |\phi((L-1)\rangle |2\rangle |1\rangle \cdots |1\rangle \}^{n_{L-1}}_{j=1}
\]

is also a genuinely nonlocal set of product states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_L}$.

To construct multipartite genuinely nonlocal sets, instead of using bipartite nonlocal product states, we can also start with some known genuinely nonlocal sets of product states in tripartite systems.

**Proposition 3** Let $L \geq 3$ be an integer. Let $\{ |\psi(j) \rangle |\phi(j) \rangle |\chi(j) \rangle \}^{n}_{j=1} \subseteq \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$ be a set of product states that is genuinely nonlocal. Then the union of the following sets (see figure 7)

\[
S_1 = \{ |\psi(j) \rangle |\phi(j) \rangle |\chi(j) \rangle |1\rangle |1\rangle |1\rangle \cdots |1\rangle |2\rangle |1\rangle \}^{n_1}_{j=1},
S_2 = \{ |\psi(j) \rangle |2\rangle |2\rangle |\phi(j) \rangle |\chi(j) \rangle |1\rangle \cdots |1\rangle |2\rangle |1\rangle \}^{n_2}_{j=1},
S_3 = \{ |\psi(j) \rangle |1\rangle |1\rangle |2\rangle |2\rangle |\phi(j) \rangle |\chi(j) \rangle |1\rangle \cdots |1\rangle \}^{n_3}_{j=1},
\]

\[\vdots\]

\[
S_L = \{ |\psi(j) \rangle |1\rangle |1\rangle |1\rangle \cdots |2\rangle |2\rangle |\phi(j) \rangle |\chi(j) \rangle \}^{n_L}_{j=1},
\]
is also a genuinely nonlocal set of product states in $\otimes_{i=1}^{2L+1} \mathcal{H}_{A_i}$, where $\mathcal{H}_{A_i} = \mathbb{C}^3$.

In fact, the above constructions can be extended to much more general settings. Let $L \geq 3$ be an integer and $\mathcal{P} := \{1, 2, 3, \ldots, L\}$. Let $\mathcal{H} := \otimes_{j \in \mathcal{P}} \mathcal{H}_{j}$ be an $L$-parties quantum systems. Suppose there are $s$ proper subsets of $\mathcal{P}$: $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s$ and we denote $\overline{\mathcal{P}}_i := \mathcal{P} \setminus \mathcal{P}_i$ for each $i$. We make the following assumptions:

(a) $S_i = \{\ket{\psi_j}_{\mathcal{P}_i}\}_{j=1}^{\eta_i}$ is a genuinely nonlocal product set in $\mathcal{H}_{\mathcal{P}_i} := \otimes_{j \in \mathcal{P}_i} \mathcal{H}_{j}$ for each $i \in \{1, 2, \ldots, s\}$.

(b) There is a fully product state $\ket{\Phi_i}_{\mathcal{P}_i} \in \otimes_{j \notin \mathcal{P}_i} \mathcal{H}_{j}$ for each $i$ such that the states in the union of the sets $S_i = \{\ket{\psi_j}_{\mathcal{P}_i}\}_{j=1}^{\eta_i}$ (1 $\leq i \leq s$) are mutually orthogonal to each other.

We use the notation $\mathcal{P} := (\mathcal{P}, \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s\})$. For each $\mathcal{P}$, we attach it a graph $G_{\mathcal{P}} = (V_{\mathcal{P}}, E_{\mathcal{P}})$ defined as follows: its vertex set is $V_{\mathcal{P}} = \mathcal{P}$ and its edge set is

$$E_{\mathcal{P}} = \bigcup_{i=1}^{s} \{(u_i, v_i) | u_i, v_i \in \mathcal{P}_i \text{ and } u_i \neq v_i\}.$$

**Theorem 4** Under the notation and assumptions of the two paragraphs previous and suppose that $G_{\mathcal{P}}$ is connected, then the set $S := \bigcup_{i=1}^{s} S_i$ is a genuinely nonlocal set of product states in $\otimes_{j \in \mathcal{P}} \mathcal{H}_{j}$.

**Proof.** Suppose not, there exist a nontrivial bipartition of $\mathcal{P}$, say $U \cup V$ (both $U$ and $V$ are nonempty subset of $\mathcal{P}$), such that the set $S$ is locally distinguishable when considering as a set of bipartite states in $(\otimes_{j \in U} \mathcal{H}_{j}) \otimes (\otimes_{j \in V} \mathcal{H}_{j})$. As the connectivity of $G_{\mathcal{P}}$, there must exist some edge $(u, v) \in E_{\mathcal{P}}$ which connects the two sets $U$ and $V$, i.e. $u \in U$ and $v \in V$. By the definition of $E_{\mathcal{P}}$, there exist some $i$ such that $u, v \in \mathcal{P}_i$. However, the set $S_i$ is genuinely nonlocal in $\mathcal{H}_{\mathcal{P}_i}$ by the assumption (a) above. So it is locally indistinguishable for the partition $(U \cap \mathcal{P}_i) \setminus (V \cap \mathcal{P}_i)$

of $\mathcal{P}_i$ as both $U \cap \mathcal{P}_i$ and $V \cap \mathcal{P}_i$ are nonempty. By observation 1, the set $S_i$ is locally indistinguishable in the bipartite systems $(\otimes_{j \in U} \mathcal{H}_{j}) \otimes (\otimes_{j \in V} \mathcal{H}_{j})$. However, $S_i \subseteq S$ implies that $S$ must be locally indistinguishable as bipartite states $(\otimes_{j \in U} \mathcal{H}_{j}) \otimes (\otimes_{j \in V} \mathcal{H}_{j})$. Hence we obtain a contradiction. So the set $S$ must be genuinely nonlocal.

**Example 1** Let $\mathcal{P} := \{1, 2, 3, 4, 5\}$, $\mathcal{P}_1 := \{1, 2, 3\}$, $\mathcal{P}_2 := \{3, 4\}$, $\mathcal{P}_3 := \{4, 5\}$. Let $S_1 = \{\ket{\psi_1}_{1} \ket{\phi_2}_{2} \ket{\chi_3}_{3}\}_{i=1}^{N}$ be any genuinely local set of product states in $\mathbb{C}^{3} \otimes \mathbb{C}^{3} \otimes \mathbb{C}^{3}$. Set $S_2 =$
\{\{\theta\}_j\}_{j=1}^8 \subseteq \mathbb{C}^3 \otimes \mathbb{C}^3 \text{ and } S_3 = \{\{\theta\}_j\}_{j=1}^8 \subseteq \mathbb{C}^3 \otimes \mathbb{C}^3 \} \text{ (here } \{\{\theta\}_j\}_{j=1}^8 \text{ is the set of product states defined by equation (1) which has been shown to be locally indistinguishable). Set } |\phi_1\rangle_{\mathbb{P}_1} := |2\rangle_4 |2\rangle_s, |\phi_2\rangle_{\mathbb{P}_2} := |2\rangle_1 |2\rangle_2 |1\rangle_s \text{ and } |\phi_3\rangle_{\mathbb{P}_3} := |1\rangle_3 |2\rangle_2 |2\rangle_4. \text{ That is, the set } S \text{ is the union of the following sets}

\begin{align*}
S_1 &= \{|\psi\rangle|\phi\rangle|\chi\rangle|2\rangle|2\rangle\}_{j=1}^N, \\
S_2 &= \{|2\rangle|2\rangle|\theta\rangle|\xi\rangle|1\rangle\}_{j=1}^8, \\
S_3 &= \{|1\rangle|2\rangle|2\rangle|\theta\rangle|\xi\rangle\}_{j=1}^8.
\end{align*}

One can check that the states in S are mutually orthogonal as |2\rangle|2\rangle \perp |\theta\rangle|\xi\rangle. And we can easily draw the graph $G_\Psi$ determined by $\mathbb{P} := (\mathbb{P}, \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3\})$ (see figure 8). From this figure, $G_\Psi$ is connected. Therefore, we can conclude that the set S is genuinely nonlocal in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

**Remark.** Although the set S is genuinely nonlocal, but it is not strongly nonlocal. In fact, for the partition \{1, 2, 3\} \{4, 5\} of the subsystems, the 4th and 5th subsystems can perform the measurement \{M_1 = |2\rangle_4 \otimes |2\rangle_5 |2\rangle_4, M_2 = I_3 \otimes I_3 - M_1.\} If the outcome is ‘1’, we can conclude that the state must belong to the set $S_1$. If the outcome is ‘2’, we can conclude that the state must belong to the set $S_2 \cup S_3$. That is, the set S is reducible for this bipartition. Hence, by definition, it is not strongly nonlocal.

5. Conclusion and discussion

We study a strong form of locally indistinguishable set of fully product states called genuinely nonlocal set. We generalize the results of locally indistinguishable product states in bipartite systems in reference [42] but provide a much more elegant proof. Based on a simple observation, we extend the results of Zhang et al in reference [37] to the cases of genuinely nonlocal sets. Moreover, we extend these results to a much more general setting by relating the construction of genuinely nonlocal sets with the connectivity of some graphs. As a consequence, we can show that there always exists some genuinely nonlocal set of fully product states in $\bigotimes_{i=1}^L \mathbb{C}^d_i$ provided $d_i \geq 3$ for all $i$. One should note that the genuinely nonlocal set we constructed here maybe locally reducible under the concept introduced in reference [52]. Therefore, it is interesting to find some method to characterize the locally irreducible settings.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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