A Modification of the Crout’s Method to Simplify the Stress Calculation for Certain Building Structures

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Abstract. This paper proposes a precise method of obtaining the stresses in certain models widely found in building to facilitate them in the early design phases. Stresses are obtained more quickly than with other usual procedures because they are deduced from a deflection that is calculated before than the real one. The method is drawn from earlier papers that proposed possible mechanical behaviours of models to justify the operations of three classical methods (Gauss, Crout, Cholesky) that solve the equation systems of conventional equilibrium analyses. Inspired by these mechanical interpretations, the method suggests another structural response that can be justified qualitatively with Cross’ philosophy. The deflection derived is obtained as a sum of partial deflections which are determined by progressively increasing the stiffness of the bars during the balancing process without varying the original stresses. The stiffness is increased in such a way that the nodal movements of each partial deflection can be determined with few operations. Its calculation is described numerically and graphically when the model is a continuous beam and also in the case of other usual building structures. To date, the graphical version has been applied to study the behaviour of certain models as it requires only a short calculation time and the results have been found to be quite precise. As an example, the new deflection of a portico is calculated by freehand and compared with the exact result determined numerically. With this deflection, the stresses of the model are obtained by using modified Maney's equations. Based on this work, possible lines of research are suggested for developing further graphical methods that can be used to analyse other types of structure accurately.

1. Introduction
The procedure suggested in this paper is inspired by from others [1, 2, 3] that propose ways to display the classical resolutions of the systems of structural analysis equations, with different objectives: in [1] the operations of the Gaussian method are geometrised in accordance with approaches from another era, reported in recent literature [4], to facilitate its manual application; in [2, 3] qualitative ways of understanding these operations and also those of other methods (Crout and Cholesky) are proposed from the Theory of Structures to better integrate them into equilibrium analyses. Using Cross’s philosophy, structural responses are imagined that justify the numerical operations of these procedures by breaking down the nodal actions into partial load states, such as those of Figs.1a, b, c of a continuous beam. Each state is formed by an action called "active", such as $M_A$, because it causes an immediate partial deflection of the calculation, and by one or various "restrictive" actions, such as $M_B$, which prevent some nodal movements. From this perspective solving a system of equilibrium equations is seen as equivalent to obtaining the deflection one by always adding the same partial
deflections calculated differently according to the mathematical method used. These interpretations are briefly described below.

1.1. Obtaining the deflection according to Gaussian operations
It was observed in [2] that when the method transformed the system of equations such as (1) of the beam in Fig.1 into an equivalent (2), the coefficients of \( \{M\} \) became the active actions of the partial load states. It was also found that clearing \( \{\theta\} \) of (2) was equivalent to adding the nodal rotations of the partial deflections (Figs.1a,b,c) of these load states, calculated with \( \{M^a\} \).

\[
[A][\theta] = \{M\} \tag{1}
\]

\[
[A'][\theta] = \{M^a\} \tag{2}
\]

1.2. Obtaining the partial deflections according to Crout when \( A=LU_1 \) (version 1)
It was observed in [3] that the operations of systems (3) and (4) derived from factoring \( [A] \) were equivalent to calculating the deflection with (4) by adding the partial deflections of 1.1 previously determined with (3) as a function of "active" nodal movements, such as \( \alpha^C \) (Figs.1d,e,f). The rotation \( \alpha^N \) in each node \( N \) was produced by the moment \( M^a_N \) in (2).

\[
[L][\alpha^a] = \{M\} \tag{3}
\]

\[
[U_1][\theta] = \{\alpha^a\} \tag{4}
\]

1.3. Obtaining the partial deflections according to Crout when \( A=L_1U \) (version 2) and Cholesky
It was observed in [3] that the systems of equations similar to (3) and (4) derived from factoring \( [A] \) according to these two procedures obtained the partial deflections of 1.2 from others, immediately calculated, whose stresses were real and coincided with those of the deflections of (3). It was thought that both methods could be modifications of Crout (version 1) since each determined its partial deflections in a way similar to that in 1.2. With Crout (version 2) these deflections were smaller and \( \{\alpha\} \) fit with \( \{M^a\} \) of (2); and with Cholesky they were found to be larger, but in both cases they could be explained by progressively varying the stiffness of the bars during the balancing of the model. For example, Cholesky’s operations were justified on the assumption that the structure was concrete and that its bars lost stiffness through cracking. Finally, the real deflection was obtained by adding these partial deflections, previously rectified to coincide with those of (3).

\[
\begin{align*}
\text{a)} & \quad M^a_1 \quad M^a_3 \quad \alpha^a_1 \quad d) & \quad \alpha^a_2 \quad g) & \quad \alpha^a_3 = \alpha^a_4 \\
\text{b)} & \quad M^a_2 \quad M^a_3 \quad \alpha^a_3 \quad e) & \quad \alpha^a_2 \quad h) & \quad \alpha^a_2 < \alpha^a_3 \quad \alpha^a_4 < \alpha^a_5 \\
\text{c)} & \quad M^a_2 \quad \alpha^a_3 \quad f) & \quad \alpha^a_2 \quad i) & \quad \alpha^a_2 < \alpha^a_3 \quad \alpha^a_4 < \alpha^a_5
\end{align*}
\]

**Figure 1.** Partial deflections of a continuous beam according to the operations of Gauss a), b), c), of Crout (version 1) d), e), f), of the new modification of Crout g), h), i).
This paper suggests another modification of the 1.2 method that could be more suitable than the previous ones for obtaining stresses since it requires fewer operations. Stresses are calculated considering a deflection different from the real one obtained without (4), adding other partial deflections previously defined with (3) which are faster to calculate and whose stresses coincide with those of the partial deflections of the previous sections. The procedure is inspired by the mechanical interpretations of the methods of 1.3 and has so far been used to graphically analyse continuous beams and some models considered in [1] under nodal actions. In the first case, the new deflection is obtained on the basis of the knowledge that the nodes of each partial deflection rotate it (Figs.1g,h,i); in the second case (4), it is replaced by other operations that require less calculation. The presentation begins by describing the new method when applied to the previous structures and ends with an example.

2. Method
2.1. New deflection obtained when considering continuous beams
The procedure is applied to the constant inertia beam $I$ in Fig. 2a by breaking down the actions $M$ into the load states of Figs.2b,c because the justification of the operations requires that the directions of rotation of two consecutive nodes be opposite. The following describes how to obtain the new deflection for the case of Fig.2b, whose system of equations is (5).

$$\begin{bmatrix}1 & 0 & \theta_A' & 0 \\ 0 & 1 & \theta_B' & 0 \\ 0 & 0 & 1 & \theta_C' \end{bmatrix} \begin{bmatrix}A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix}M_A \\ M_B \\ 0 \end{bmatrix}$$

(5)
2.1.1. Qualitative obtaining
The new deflection in Fig. 2b is stiffer than the real one and is obtained by adding three partial deflections similar to those in Figs. 2c, j, o, the last two determined by adding auxiliary load states on the model. That of Fig. 2j is achieved by first applying the state of Fig. 2d and then that of Fig. 2e, which cancels the restrictive moment in A and produces another moment \( r \) in B. By repeating this process indefinitely, \( r \) is eliminated and a deflection is obtained by a unitary action in B whose nodal rotations are made by always rotating each node in the same direction. Adding the last state, which is similar to Fig. 2d, results in a deflection (Fig. 2j) (dashed line) depending on the rotation \( \bar{\theta}_B \) obtained in (6).

\[
\theta'_A \left(1 + r + r^2 + r^3 + \ldots \right) = \frac{\theta'_A}{1-r} = \bar{\theta}_B
\]

(6)

Assume that at this moment the stiffness of the bars connected to B increases, causing \( \bar{\theta}_B \) to become \( \bar{\theta}_A \) and the deflection (dashed line of Fig. 2j) to become smaller (continuous line). The increase in stiffness affects bars AB and BC equally, which implies that the bending moments transmitted by the rotation of B to the contiguous nodes (Fig. 2h) are real even though AB and BC deflect less. It also implies that the inertia I of these bars is transformed into \( I_{BA} \) calculated in (8). The increase in stiffness is simulated by reinforcing AB and BC with steel plates arranged as shown in Fig. 2g, which must not change either the deflection caused by the original rotation of A, which is similar to that in Fig. 2e, nor the moments produced by this rotation (Fig. 2i). This is achieved by assuming that the plates are currently warped because they are very slender. The result is the deflection in Fig. 2j (continuous line) and some AB and BC whose inertias vary depending on the direction of transmission of the flectors.

\[
I_{BA} = \frac{I}{r_{AB} \theta'_A} = IK_{BA}
\]

(7)

\[
\bar{\theta}_B \rightarrow \tilde{\theta}_A = \frac{\theta_A r_{AB} \theta'_A}{EI} = \frac{\bar{\theta}_B}{EI K_{BA}} = \frac{\bar{\theta}_B}{EI_{BA}}
\]

(8)

The partial deflection of Fig. 2o is obtained in the same way: the state of Fig. 2k is applied and then that of Fig. 2l, similar to that of Fig. 2j, which produces \( r_1 \). By repeating the procedure, the deflection of Fig. 2m is obtained (dashed line), which is reduced (continuous line) by adding new plates to the model (Fig. 2n) so that the rotation in C coincides with that of B and A without modifying the flectors. As a result, the new \( I_{CB} \) inertia when rotating C is valid (9). On the other hand, the deflection in Fig. 2c is obtained by adding other similarly justified partial deflections by turning the nodes in the opposite direction.

\[
I_{CB} = \frac{IK_{BA}}{r_{BC} \theta_B} = IK_{CB}
\]

(9)

2.1.2. Numerical and graphical output
The procedure arises from the breakdown \([A]\) of (5) into four matrices (10). The first and fourth come from having factored \([A]\) by Crout (version 1) according to (11) and the remaining ones (12) are used to adapt Crout to the procedures of 1.3 and that suggested here. This is achieved by modifying the original stiffnesses of bars AB and BC using different values of \( K_{BA} \) and \( K_{CB} \) in each case. The values of the continuous beam, which coincide with those of (7) and (9), are shown in (13).
\[
[L] \cdot [B_1] \cdot [B_2] \cdot [U_1] \cdot \{\theta\} = \{M\}
\] (10)

\[
[L] \cdot [U_1] = \begin{pmatrix}
A & 0 & 0 \\
B & x & 0 \\
0 & y & z
\end{pmatrix} \begin{pmatrix}
1 & B / A & 0 \\
0 & 1 & y / x \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
A & B & 0 \\
B & C & D \\
0 & D & E
\end{pmatrix}
\] (11)

\[
[B_1][B_2] = \begin{pmatrix}
1 & 0 & 0 \\
0 & K_{BA} & 0 \\
0 & 0 & K_{CB}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 / K_{BA} & 0 \\
0 & 0 & 1 / K_{CB}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (12)

\[
K_{BA} = A / B \quad K_{CB} = K_{BA} \cdot (x / y)
\] (13)

(10) becomes (14) when their matrices are joined in groups of two. The stresses could be obtained with the real deflection calculated with (15) and (16); however it seems preferable to use the new deflection, which is faster to determine, since each of its partial deflections (Figs.1g,h,i) is defined with an unknown \( \alpha' \) of (15). These unknowns can be obtained by performing fewer operations with (17) than when comparing (3) with (18) from (15). Finally the new deflection is calculated with these and (19), which is systematically obtained from (1) by carrying out the following steps:

\[
[C_1] \cdot [C_2] \cdot \{\theta\} = \{M\}
\] (14)

\[
[C_1] \cdot \{\alpha'\} = \{M\}
\] (15)

\[
[C_2] \cdot \{\alpha'\} = \{\alpha''\}
\] (16)

\[
[B_1] \cdot \{\alpha''\} = \{\alpha''\}
\] (17)

\[
[L] \cdot [B_1] \cdot \{\alpha''\} = \{M\}
\] (18)

\[
\begin{cases}
\varphi_c = \alpha'_c \\
\varphi_b = \alpha''_b - \varphi_c \\
\varphi_a = \alpha''_a - \varphi_b
\end{cases}
\] (19)

i) Factor \([A]\) according to the Crout method. The x, y, z coefficients are calculated numerically with (11) and obtained graphically using the procedure suggested in \([3]\) from \([A]\) represented with the area of Fig.3a. Fig.3b shows the resulting graph.

ii) Obtain the rotations \(\{\alpha''\}\) (Figs.1d,e,f) according to the Crout method. They are obtained numerically with (3) and graphically with the method in \([3]\). Fig.3c shows the resulting graph.

iii) Obtain the active rotations \(\alpha'\) of the new partial deflections (Figs.1g,h,i). They are determined numerically with (17), the result of which is (20). They are obtained graphically in Fig.3e considering the directions \(d_1\) (perpendicular to \(d\)) and \(d_2\) obtained in Fig.3b. Start by drawing the segments 1-2 and 4-5 from \(\alpha''_b\) and \(\alpha''_c\) parallel to \(d_1\), which determine 2-3 (\(=\alpha'\)) and 5-6 (Fig. 3e). Then 7-6 (\(=\alpha'\))
are obtained similarly with 5-6 and with \( d_2 \). The schematics in Fig.3f justify these operations. It can be seen that the drawing in Fig.3e is substantially smaller than the one in Fig.3d which represents the operations of (4). It can also be seen that the values of \( \{ \alpha' \} \) may be too small for them to be obtained by drawing. This disadvantage could be solved by reducing the rigidity of the node associated with the first unknown eliminated from the system.

\[
\begin{align*}
\alpha'_A &= \alpha'_a \\
\alpha'_B &= \frac{\alpha'_b}{K_{BA}} \\
\alpha'_C &= \frac{\alpha'_c}{K_{CB}}
\end{align*}
\]  

(20)

iv) Obtain the new deflection. Its nodal movements \( \{ \varphi \} \) are obtained numerically with (19) and graphically by drawing some segments over the area of Fig.3c, as carried out in Fig.5c.

**Figure 3.** Obtaining a partial deflection graphically; a), b) representation of \([ A ]\) and its factoring; c) obtaining of \( \{ \alpha'' \} \); d) obtaining of \( \{ \theta \} \) by Crout; e) obtaining of \( \{ \alpha' \} \) on the area of Fig.3c; f) justification of the drawing in Fig.3e.

2.2. New deflection: obtaining when considering other models

Systems of equations from other models with fuller matrices \([ A ]\) could be solved by interpreting them as branch-shaped graphs influenced by arcs (Fig.4a). In (21) the system (4) in Fig.4a is transformed into another system \( S \) which determines the new deflection. This is done by eliminating the rectification of the partial deflections from (4), dividing the terms of each column by the coefficient of the column located in the main diagonal. However \( \{ \varphi \} \) can be obtained earlier with (22) and (23) arising from \( S \). (22) is solved graphically in the same way but using Fig.4c, obtained with Fig.4b; (23) completes the result of (22) and is equivalent to Fig.4e derived from Fig.4d.
\[
\begin{pmatrix}
1 & F & G & H \\
0 & F & I & J \\
0 & 0 & I & K \\
0 & 0 & 0 & K
\end{pmatrix}, \quad \begin{pmatrix}
\theta_A \\
\theta_B \\
\theta_C \\
\theta_D
\end{pmatrix} = \begin{pmatrix}
\alpha_A' \\
\alpha_B' \\
\alpha_C' \\
\alpha_D'
\end{pmatrix} \rightarrow S = \begin{pmatrix}
1 & 1 & G / I & H / K \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
\varphi_A \\
\varphi_B \\
\varphi_C \\
\varphi_D
\end{pmatrix} = \begin{pmatrix}
\alpha_A' \\
\alpha_B' \\
\alpha_C' \\
\alpha_D'
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
\Delta \varphi_A \\
\Delta \varphi_B \\
\varphi_C \\
\varphi_D
\end{pmatrix} = \begin{pmatrix}
\alpha_A' \\
\alpha_B' \\
\alpha_C' \\
\alpha_D'
\end{pmatrix}
\]

\[
-G / I \\
0
\end{pmatrix}, \quad \begin{pmatrix}
\varphi_C \\
\varphi_D
\end{pmatrix} = \begin{pmatrix}
\Delta \varphi_{2A} \\
\Delta \varphi_{2B}
\end{pmatrix}
\]

**Figure 4.** Graphical analysis of a model with the modified Crout’s method; a) scheme and network; b) area representing \([A]\); c) area to determine \(\{\alpha''\}\) and \(\{\alpha'\}\); d) area to determine \(\Delta \alpha\); e) detail.

2.3. Stresses
The hyperstatic moments of a non-shifting bar \(IJ\) can be obtained by using Maney equations modified by the changes in the inertia of the bars. (24) shows those of the bar \(BC\) in Fig.2b.

\[
\begin{cases}
m_{bc} = \frac{2EI}{L_{bc}} (2\varphi_B K_{BA} + \varphi_C K_{CB}) \\
m_{cb} = \frac{2EI}{L_{bc}} (2\varphi_C K_{CB} + \varphi_B K_{BA})
\end{cases}
\]

(24)

3. Example
The new deflection of the portico in Fig.5a is calculated graphically. The drawings are made freehand on certain areas (Figs.5b,c) which prevent the draftsman from making important errors. They are formed with squares with 2 cm sides on a grid sheet to make the results as precise as possible. The Fig.5b represents \([A]\) and the operations that factor it according to Crout. Its coefficients, calculated numerically, are shown with segments drawn to the same scale as the boxes. The rest of the drawing is done in the area of Fig.5c, as a function of the moment \(M\) represented with a 1.8 cm segment.

The nodal movements \(\{\varphi\}\) of the new deflection are determined by combining some segments drawn over the area of Fig.5c and considering (19). A comparison with the values obtained numerically (Table 1) shows that they are quite precise: the error made when calculating \(\varphi_C\) cannot be considered significant as the movement is very small. On the other hand, the stresses obtained with these results and with (24) seem to achieve a similar precision, though for the moment there is not enough experience to confirm this.
Figure 5. Obtaining a deflection graphically: a) scheme; b) factoring of $[A]$ and obtaining of $d_1$ and $d_2$; obtaining of $\{\alpha\}$, $\{\alpha'\}$ and $\{\phi\}$.

Table 1. New deflection: nodal movements (as a function of $1/EI$) and stresses.

| $\phi_A$ | $\phi_B$ | $\phi_C$ | $m_{AB}$ | $m_{BA}$ | $m_{BC}$ | $m_{CB}$ |
|----------|----------|----------|----------|----------|----------|----------|
| Exact value | -0.18M  | 0.18M  | 0.07M  | 0.18M  | 0.54M  | 0.73M  | 0.56M  |
| Graphical value | -0.2M  | 0.2M  | 0.1M  | 0.2M  | 0.6M  | 0.8M  | 0.72M  |
| Error (%) | 11  | 11  | 42  | 11  | 11  | 17.8  | 26.7  |

4. Conclusions

i) Geometrising the scalar and vectorial magnitudes for numerical analysis procedures reduces application times if they are calculated freehand.

ii) Showing the operations of the analysis of certain building structures with the philosophy of Cross facilitates the development of procedures based on the mechanical behaviour of models which require less calculation to obtain the stresses.

iii) Possible lines of research are also pointed to for developing further graphic methods that can analyse other types of structure directly and accurately.

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