A single oscillating bubble in liquids with high Mach number

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1. Introduction

Bubble dynamics is widely existing in contemporary science and technology (e.g. cavitation erosion in hydraulic machinery [1,2], underwater explosion [3–5], biomedical applications [6–11], ultrasonic cleaning [12–15] and sonochemistry [16–19]). The violent expansion and collapse of bubbles play an important role in the applications of bubble dynamics. For example, the sharp collapse of the bubble will seriously affect the safe and efficient operation of hydraulic machinery [1,2]. The destructive collapse of the bubbles generated by the underwater explosion can cause considerable damage to the surrounding targets [3–5]. And, the jet generated by the rapid collapse of the bubble can effectively destroy kidney and bladder stones [8,9]. For the above situations, the radial velocity of the bubble wall is great, which can be described by the bubble wall Mach number. Moreover, the influence of the liquid compressibility is strong and should be fully considered for high accuracy. Therefore, in the present paper, the accuracy of different equations with high Mach number is numerically investigated in detail.

Previous researches on the bubble wall motion equations with high Mach number can be classified as the zero-order, the first-order and the second-order equations (in terms of the bubble wall Mach number). First, Rayleigh and Plesset [20,21] derived the zero-order equation (called “Rayleigh-Plesset equation”) by ignoring the compressibility of the liquids. Then, more and more researchers paid attention to the compressibility of the liquids and have made many important theoretical breakthroughs in the derivation of the first-order equations. Among them, Herring et al. [3,5] derived the equation (called “Herring equation”) by employing the acoustic approximation, which assumes the sound velocity in the liquid as a constant. Moreover, Keller et al. [22–24] derived the equation (called “Keller-Miksis equation”) by replacing the Laplace equation with the wave equation of the liquid velocity potential. Furthermore, based on a simplified version of the matched asymptotic expansions technique, Prosperetti and Leuzzi [25] derived the one-parameter family of the bubble wall motion equation with the first-order Mach number, in which Herring equation and Keller-Miksis equation are both special cases. Meanwhile, it is found that the revised Keller-Miksis equation expressed by enthalpy in the first-order equation family has the highest accuracy in all the tested cases [25].

The researches of the second-order equations have also made the following progress. Gilmore [4] derived the equation (called “Gilmore equation”) only containing a part of the second-order Mach number terms based on the Kirkwood Bethe hypothesis, and replaced the pressure with the more accurate enthalpy. Moreover, Tomita and Shima [26]...
derived the equation by the Poincaré-Lighthill-Kuo method. Base on the work of Tomita and Shima, Fujikawa and Akamatsu [27] developed the equation by considering the effects of the non-equilibrium condensation of steam, heat conduction and discontinuity of phase interface temperature. Furthermore, based on a rigorous version of the matched asymptotic expansions method, Lezzi and Prosperetti [28] derived the two-parameter family of the bubble wall motion equation with the second-order Mach number expressed by enthalpy and is widely applied due to its high accuracy [29,30]. In addition, the first-order equation is also widely employed because of its simple form [31,32]. However, the accuracy is the crucial basis for selecting equations with high Mach number. Unfortunately, the predictions (e.g. the bubble radius, the bubble wall velocity, the bubble wall acceleration, the dissipated power and the radiation pressure) by the first-order and the second-order equations with high Mach number have not been compared in detail. As a result, the prediction accuracy and the applicable conditions of the two equations have not been revealed.

The dissipated power during bubble oscillation is an important topic in modelling bubble dynamic behavior [33-36]. Moreover, the difference between the dissipated power predicted by the first-order and the second-order equations is essential for the applications of bubble dynamics. For example, in the phenomenon of acoustic cavitation, the pressure and temperature conditions attained during the strong collapse of bubbles can cause chemical reaction, which can convert methanol in wastewater into hydrogen. And, the dissipated power can be calculated according to the bubble wall motion equation to obtain the efficiency of hydrogen production [37]. In addition, previous researches about the dissipated power have experienced the process from linear to nonlinear and from ignoring the compressibility of the liquid to considering compressibility [38-42]. The dissipated power during bubble oscillation is nonlinear and pressure dependent. However, the widely applied linear model developed by Prosperetti and Commander [38] is only valid for small amplitude oscillation. Lousinard improved this model to be pressure dependent, but it was only obtained based on Rayleigh-Plesset equation without considering compressibility [39]. After that, Sojahrood [40-42] continues to expand the dissipated power model based on the first-order equation with considering the compressibility of liquid, but their model is only effective for the case with low Mach number. However, the prediction of the dissipated power by the second-order equation with high Mach number has not been fully revealed. Therefore, in the present paper, the dissipated power predicted by the first-order and the second-order equations is investigated and compared quantitatively.

In the present paper, in terms of the bubble oscillation characteristics, the radiation pressure and the dissipated power, the predictions of the first-order and the second-order equations for a free oscillating bubble were numerically investigated and compared in detail. The chapter arrangement is introduced as follows. In section 2, the theoretical model of a free oscillating bubble in liquids is introduced. In section 3, the bubble wall motion equation is verified by the classical numerical simulation results and the experimental data. In section 4, the predictions of the first-order and the second-order equations are investigated and compared in terms of the bubble oscillation characteristics. In section 5, the influences of the initial bubble radius are discussed. In section 6, the comparisons between the two equations on the prediction of the dissipated power are demonstrated. In section 7, the influences of the ambient pressure on the predictions of the two equations are discussed quantitatively in terms of the bubble oscillation characteristics and the radiation pressure. In section 8, the main conclusions and the prospect of follow-up work are given.

2. Theoretical model

In the present paper, the bubble oscillation characteristics, the radiation pressure and the dissipated power with high Mach number are investigated. Several essential assumptions are employed as follows: (1) The bubble always remains spherical during the quick collapse and expansion, and there is no displacement of the bubble center. (2) The liquid is compressible. (3) The mass transfer and the thermal effects at the interface between the bubble and the liquid are ignored. (4) The gas pressure in the bubble is uniform.

Through combining the continuity equation, the momentum equation, the sound velocity equation, the enthalpy equation [25] and the state equation of the water in Tait form [43], a pair of dimensionless
differential equations can be obtained as follows [28]:

\[
\left(1 - \varepsilon^2(n - 1)\left(\frac{\partial \varphi_r}{\partial r} + \frac{1}{2} \left(\frac{\partial \varphi_r}{\partial r}\right)^2\right)\right) \nabla^2 \varphi_r = \varepsilon^2 \left(\frac{\partial^2 \varphi_r}{\partial r^2} + 2 \frac{\partial \varphi_r}{\partial r} \frac{\partial^2 \varphi_r}{\partial r \partial \theta} + \frac{(\partial \varphi_r)}{\partial \theta^2}\right)
\]

(1)

\[
\frac{\partial \varphi_r}{\partial r} + \frac{1}{2} \left(\frac{\partial \varphi_r}{\partial r}\right)^2 + h_r = 0
\]

(2)

where,

\[
\varepsilon = \frac{U}{\dot{c}_w}
\]

(3)

\[
\varphi_r = \frac{p}{R_0 B}
\]

(4)

\[
\tau_r = \frac{t}{R_0/U}
\]

(5)

\[
r_r = \frac{r}{R_0}
\]

(6)

\[
h_r = \frac{h}{U^2}
\]

(7)

\[
U = \left(\frac{\dot{p}_0}{\rho_w}\right)^{1/2}
\]

(8)

\[
\rho_w = \rho_\infty + \rho_\text{gw}gH
\]

(9)

here, Eqs. (1) and (2) represent the dimensionless governing equations

\[
\left[1 - \frac{1}{6} \frac{1}{\varepsilon R_t R_r} \right] \nabla^2 \varphi_r + \frac{3}{2} \left[1 - \left(\frac{1}{3} + \lambda\right) \varepsilon R_\dot{R}_r \right] \nabla \varphi_r = \left[1 + \frac{3}{2} \left(\frac{1}{3} + \lambda\right) \varepsilon R_\dot{R}_r \right] h_{sr} + \varepsilon R \dot{h}_r.
\]

(13)

for the motion of the liquid around the bubble. \(\varepsilon\) represents the characteristic Mach number of the bubble wall motion. \(n\) represents the empirical constant with \(n = 7.15\) for the water [43]. \(\varphi_r\) represents the local velocity potential of the liquid. \(r_r\) represents the dimensionless local velocity potential of the liquid. \(t_r\) represents the dimensionless time. \(t_r\) represents the dimensionless radial coordinate. \(h_r\) represents the dimensionless radial coordinate. \(h_\text{sr}\) represents the dimensionless local specific enthalpy of the liquid. \(\nabla^2 \varphi_r\) represents the Laplace operator of \(\varphi_r\) in the spherical coordinate system with \(r, \theta\) as the variable. \(U\) represents the characteristic velocity of the bubble wall. \(\dot{c}_w\) represents the speed of sound in the liquid in infinity. \(p_\text{gw}\) represents the pressure of the liquid at the bubble wall. \(\rho_\infty\) represents the pressure of the liquid at the bubble wall. \(B\) represents the empirical constant with \(B = 3049 \times 10^5\) Pa for the water [28]. \(p_\text{gw}\) represents the gas pressure in the bubble. \(\sigma\) represents the surface tension coefficient. \(R\) represents the instantaneous bubble radius. \(\mu\) represents the viscosity of the liquid. \(\dot{R}\) represents the first derivative of \(R\) with respect to the time. \(\kappa\) represents the polytropic exponent (\(\kappa = 1.4\) for the adiabatic process).

Combining Eqs. (1)-(12), based on a simplified version of the matched asymptotic expansions technique [25], the dimensionless bubble wall motion equation with the first-order Mach number (referred to as the first-order equation for short) for a free oscillating bubble can be obtained as follows [25]:

\[
\dot{h}_{sr} = \frac{1}{\varepsilon^2 n - 1} \left[ \left(\frac{p_\text{gw} + B}{\rho_\infty + B}\right)^{\frac{\kappa}{\kappa - 1}} - 1 \right]
\]

(10)

where,

\[
p_\text{gw} = p_\text{gw} - \frac{2\sigma}{R} \frac{4\mu\dot{R}}{R}
\]

(11)

\[
p_\text{gw} = \left(p_\text{gw} + \frac{2\sigma}{R}\right) \left(R_0/R\right)^{2\kappa}
\]

(12)

here, \(h_{sr}\) represents the dimensionless specific enthalpy of the liquid at the bubble wall. \(p_\text{gw}\) represents the pressure of the liquid at the bubble wall. \(B\) represents the empirical constant with \(B = 3049 \times 10^5\) Pa for the water [28]. \(p_\text{gw}\) represents the gas pressure in the bubble. \(\sigma\) represents the surface tension coefficient. \(R\) represents the instantaneous bubble radius. \(\mu\) represents the viscosity of the liquid. \(\dot{R}\) represents the first derivative of \(R\) with respect to the time. \(\kappa\) represents the polytropic exponent (\(\kappa = 1.4\) for the adiabatic process).

Furthermore, based on a rigorous version of the matched asymptotic expansions method [28], the dimensionless bubble wall motion equation with the second-order Mach number for a free oscillating bubble can be obtained as follows [28]:
\[ \frac{1}{2} \left( 1 - \lambda \right) \dot{R}_b + \left( \frac{14}{9} + 2\lambda + \theta \right) R_\infty \dot{R}_b + \left( \frac{7}{3} \right) \dot{R}_b + \left( \frac{16}{15} + \frac{4}{3} \lambda + \theta \right) \dot{R}_b^2 + \frac{4}{3} \lambda + \theta \right] \ddot{R}_b = \frac{1}{2} \left( 1 - \lambda \right) \dot{R}_b + \frac{1}{2} \left( 1 - \lambda \right) \dot{R}_b + \frac{1}{2} \left( 1 - \lambda \right) \dot{R}_b + \frac{1}{2} \left( 1 - \lambda \right) \dot{R}_b \dot{R}_b^2 
\]

Here, \( \lambda \) and \( \theta \) are two independent arbitrary parameters with the order smaller than 1/\( \epsilon \). Eq. (15) is called the second-order dimensionless bubble wall motion equation family [28] (referred to as the second-order equation family for short). When \( \lambda = 1 \) and \( \theta = 0 \) are taken, Eq. (15) becomes the bubble wall motion equations in Tomita [26] and Fujikawa form [27] expressed by the enthalpy.

For solving Eqs. (13) and (15), ode45 with adaptive step size was adopted. And, the specific details of solution method of Eq. (15) are given in appendix. In addition, the allowable relative and absolute tolerances are 10^{-16} and 10^{-8} respectively, and the minimum time steps are 10^{-10}. The following constants are employed for simulations and subsequent analysis: \( R_0 = 10^{-4} \text{ m} \); \( \lambda = 1 \); \( \theta = 0 \); \( \rho_0 = 101325 \text{ Pa} \); \( c_\infty = 1478.2 \text{ m/s} \); \( \rho_\infty = 1000 \text{ kg/m}^3 \); \( \mu = 0.001 \text{ Pa s} \); \( \sigma = 0.0725 \text{ N/m} \).

According to the law of the liquid continuity, the bubble wall velocity cannot exceed the sound velocity of the liquid at the bubble wall. In other words, the absolute value of the bubble wall Mach number can never be greater than 1 (\( |Ma| < 1 \) [44]). Otherwise, for the accuracy of the prediction of the radius-times curves, the velocity of the bubble wall will be replaced by the sound velocity of the liquid at the bubble wall. Based on the work of Yasui, the bubble wall Mach number can be defined as follows [44]:

\[ Ma = \frac{\dot{R}_b}{c_{b,\infty}} \quad (16) \]

where

\[ c_{b,\infty} = \sqrt{\frac{7.15(p_b+B)}{\rho_{b,\infty}}} \quad (17) \]

Here, \( c_{b,\infty} \) represents the speed of sound in the liquid at the bubble wall. \( \rho_{b,\infty} \) represents the density of the liquid at the bubble wall, which can be obtained by the following equation [43]:

\[ \rho_{b,\infty} = \frac{\rho_b + B}{\rho_{\infty}} \quad (18) \]

In addition, \( Ma_{max} \) is defined as the maximum of \( |Ma| \) for the subsequent discussion.

In addition, the radiation pressure induced by the bubble oscillation can be given as follows [45]:

\[ P_{rad} = \frac{\rho U^2 R}{r_e} \left( R_\infty + 2 \dot{R}_b \right) \quad (19) \]

Fig. 1. Comparisons between the present numerical results and the classical numerical simulation results. (a) Comparisons of the maximum dimensionless radial collapse velocity and the minimum dimensionless radius at the first collapse. (b) Comparisons of the maximum dimensionless radial rebound velocity and the minimum dimensionless radius at the first rebound. \( R_{max, col} \) represents the maximum dimensionless collapse velocity of the bubble wall. \( R_{max, reb} \) represents the maximum dimensionless rebound velocity of the bubble wall. \( R_{min} \) represents the minimum dimensionless bubble radius at the end of the first bubble collapse. The crosses represent the classical numerical simulation results obtained by Lezzi and Prosperetti [28]. The stars represent the present numerical results predicted by the second-order equation. \( R_\infty(0) = 3.0 \); \( \dot{R}_b(0) = 0 \).

Fig. 2. Comparisons between the present numerical results and the experimental data on the variations of the dimensionless bubble radius with the dimensionless time. Scattered points represent the experimental results for a 0.55-lb charge of tetryl exploded at the depth of 91.44 m [22]. The solid line represents the present numerical results based on the second-order equation. \( R_\infty(0) = 3.0 \); \( \dot{R}_b(0) = 0 \).
Fig. 3. The variations of the dimensionless bubble radius versus the dimensionless time. The solid and the dashed lines refer to the first-order and the second-order equations respectively. $R_{\text{max, local}}$ represents the dimensionless local maximum bubble radius. $T_*$ represents the dimensionless length of the bubble oscillation period.

For convenience, we set $r_* = 5$ for investigation here.

3. Numerical and experimental verifications

In this section, the present numerical results predicted by the second-order equation will be verified numerically and experimentally with high Mach number.

Fig. 1 shows the comparisons between the present numerical results and the classical numerical simulation results with $R_*(0) = 4$ and $R_*(0) = 0$. In Fig. 1, $R_{\text{max, coll.}}$ represents the maximum collapse velocity of the bubble wall, $\dot{R}_{\text{max, reb.}}$ represents the maximum rebound velocity of the bubble wall, and $R_{\text{min, reb.}}$ represents the minimum dimensionless bubble radius at the end of the bubble collapse. The crosses represent the classical numerical simulation results obtained by directly solving the dimensionless governing equations Eqs. (1) and (2) [28]. The stars refer to the present numerical results predicted by the second-order equation Eq. (15). As shown in Fig. 1, by comparing with the classical numerical simulation results, it can be concluded that the prediction of the second-order equation is reliable.

Fig. 2 shows the comparisons between the present numerical results and the experimental data on the variations of the dimensionless bubble radius with the dimensionless time. The black scattered points represent the experimental results for a 0.55-lb charge of tetryl exploded at the depth of 91.44 m [22]. The solid line represents the present numerical results predicted by the second-order equation with $R_*(0) = 3$, $R_*(0) = 0$ and $H = 91.44 \text{ m}$. As shown in Fig. 2, the present numerical results are consistent with the experimental results in the bubble radius and the length of the oscillation period, especially for the first two periods. After the second collapse, the difference between the present numerical results and the experimental results gradually increases due to the significant difference of the dissipated power.

Through the above verifications of the classical numerical simulation results and the experimental data, it can be concluded that the second-order equation is reliable to predict the bubble oscillation characteristics with high Mach number.

4. Free oscillating bubble in liquids

In this section, in terms of the bubble oscillation characteristics and the radiation pressure, the differences between the first-order equation Eq. (13) and the second-order equation Eq. (15) with high Mach number for a free oscillating bubble are discussed in detail. In order to show the differences with high bubble wall Mach number significantly, the initial conditions and the ambient pressure employed in this section are $R_*(0) = 8$, $R_*(0) = 0$ and $p_\infty = 1.01 \times 10^5 \text{ Pa}$.

4.1. Time domain

Fig. 3 shows the variations of the dimensionless bubble radius ($R_*$) versus the dimensionless time. The solid and the dashed lines refer to the first-order and the second-order equations respectively. $R_{\text{max, local}}$ represents the dimensionless local maximum bubble radius. $T_*$ represents the dimensionless length of the bubble oscillation period. Due to the small difference in the first period [28], Fig. 3 starts from the end of the first period. As shown in Fig. 3, $R_*$ predicted by the two equations are significantly different, mainly reflected in $R_{\text{max, local}}$ and $T_*$. The difference indicates that the dissipated power and the resonance frequency predicted by the two equations has a significant difference.

Fig. 4. The variations of the dimensionless bubble wall radial velocity versus the non-dimensional time. The solid and the dashed lines refer to the first-order and the second-order equations respectively.

Fig. 5. The variations of the dimensionless radial acceleration of the bubble wall versus the dimensionless time. The solid and the dashed lines refer to the first-order and the second-order equations respectively. $R_{\text{max, reb.}}$ represents the maximum dimensionless radial acceleration of the bubble wall.
In addition, Fig. 4 shows the variations of the dimensionless bubble wall radial velocity (\(\dot{R}_\text{rad}\)) versus the non-dimensional time. The solid and the dashed lines refer to the first-order and the second-order equations respectively. \(R_{\text{max}\_\text{col}}\) represents the maximum collapse velocity of the bubble wall. \(R_{\text{max}\_\text{reb}}\) represents the maximum rebound velocity of the bubble wall. As shown in Fig. 4, the predictions of \(\dot{R}_\text{rad}\) by the two equations are significantly different during the first collapse and the rebound of the bubble referring to \(R_{\text{max}\_\text{col}}\), \(R_{\text{max}\_\text{reb}}\) and the length of subsequent oscillation periods. Through simulations, the dimensionless difference between \(R_{\text{max}\_\text{col}}\) predicted by the two equations is equal to 20.62%, and the dimensionless difference between \(R_{\text{max}\_\text{reb}}\) predicted by the two equations is equal to 38.55%. In the subsequent bubble oscillations, due to the reduction of bubble energy, the difference between the first-order and the second-order equations gradually decreases.

In addition, \(\dot{R}_\text{rad}\) is essential for calculating many important physical quantities (e.g. the radiation pressure \([35]\), the scattering cross section \([45]\), the Bjerknes force \([46]\) and the dissipated power \([40-42]\)). Therefore, it is necessary to carefully investigate the prediction accuracy of \(\dot{R}_\text{rad}\). Fig. 5 shows the variations of the dimensionless radial acceleration of the bubble wall versus the dimensionless time. The solid and the dashed lines refer to the first-order and the second-order equations respectively. \(R_{\text{rad}}\) represents the maximum of \(R_{\text{rad}}\) induced by bubbles oscillation. As shown in Fig. 5, compared with the first-order equation, \(R_{\text{rad}}\) predicted by the second-order equation is much smaller with the dimensionless difference equal to 129.46%. Therefore, the difference between \(R_{\text{rad}}\) predicted by the two equations is more significant than that of \(R_\text{col}\) and \(R_\text{reb}\).

Furthermore, the radiation pressure \((P_{\text{rad}})\) induced by bubbles oscillation is one of the essential physical mechanisms for the bubble-driven particle motion in the abrasive phenomenon, which causes serious cavitation damage to the surface of hydraulic machinery \([47]\). In addition, radiation pressure is also an important mechanism for bubbles to remove pollutant particles in ultrasonic cleaning \([12-15]\). Therefore, it is necessary to carefully investigate the prediction accuracy of \(P_{\text{rad}}\). Fig. 6 shows the variations of the radiation pressure versus the dimensionless time. The solid and the dashed lines refer to the first-order and the second-order equations respectively. \(P_{\text{rad}\_\text{max}}\) represents the maximum radiation pressure. As shown in Fig. 6, the radiation pressures \((P_{\text{rad}})\) predicted by the two equations are also significantly different with the dimensionless difference of \(P_{\text{rad}\_\text{max}}\) equal to 45.80%. Based on the radiation pressure expression Eq. \(\text{(19)}\), \(P_{\text{rad}\_\text{max}}\) is determined by \(R_\text{col}\), \(R_\text{reb}\) and \(\dot{R}_\text{rad}\). Through simulations, \(R_\text{col}\), \(\dot{R}_\text{rad}\) and \(\dot{R}_\text{reb}\) predicted by the first-order and the second-order equations are 0.07 and 0.09, –44.30 and –152.09, 8.92 × 10^5 and 1.11 × 10^5 respectively at the times corresponding to \(P_{\text{rad}\_\text{max}}\). Therefore, compared with \(R_\text{col}\) and \(\dot{R}_\text{reb}\), \(\dot{R}_\text{rad}\) can be considered as the dominant term for \(P_{\text{rad}\_\text{max}}\).

### 4.2. Local maximum bubble radius

Fig. 7 shows the variations of the dimensionless local maximum bubble radius \((R_{\text{max}\_\text{local}})\) versus the sequence number of the period. The black square and the red circle refer to the first-order and the second-order equations respectively. As shown in Fig. 7, due to the continuous reduction of bubble energy, \(R_{\text{max}\_\text{local}}\) decreases gradually with the increase of the period. Moreover, compared with the first-order equation, because the predicted dissipation power is smaller, \(R_{\text{max}\_\text{local}}\) predicted by the second-order equation is always much greater. Furthermore, with the increase of the period, the difference between \(R_{\text{max}\_\text{local}}\) predicted by the two equations decreases gradually.
The black solid and the red dashed lines refer to the first-order and the second-order equations respectively. The blue solid line refers to $Ma_{\text{max}} = 1$. The values marked on the ordinate represent $Ma_{\text{max}} = 1$ predicted by the first-order equation and $Ma_{\text{max}}$ predicted by the second-order equation with the same $R_*(0)$. 

**4.3. Oscillation period**

Furthermore, Fig. 8 shows the variations of the dimensionless length of the oscillation period ($T_*$) versus the sequence number of the period. The black square and the red circle refer to the first-order and the second-order equations. Moreover, the expression of the bubble collapse time derived by Rayleigh is as follows [20]:

$$t_c = \eta \sqrt{\frac{\rho_w}{\rho_v - \rho_w}} R_{\text{max}}$$  \hspace{1cm} (20)

here, $t_c$ represents the bubble collapse time, $\eta$ represents the dimensionless constant with $\eta = 0.915$ for the water [20], $p_v$ represents the saturated vapor pressure in the bubble. $R_{\text{max}}$ represents the maximum bubble radius. Based on Eq. (20), the collapse time is positively related to the maximum bubble radius. According to Fig. 7, the local maximum bubble radius predicted by the second-order equation is greater than that predicted by the first-order equation, so the period time predicted by the second-order equation is greater than that predicted by the first-order equation. Moreover, because the local maximum bubble radius decreases gradually with the increase of the sequence number of the period, the oscillation time required decreases accordingly. Moreover, the above analysis is consistent with the results shown in Fig. 8.

In summary, when the bubble wall Mach number is high, the bubble oscillation characteristics ($R_*, \dot{R}_*, \ddot{R}_*, R_{\text{max, local}}$, and $T_*$) and its induces the maximum radiation pressure ($P_{\text{rad, max}}$) predicted by the second-order equation are significantly different from those predicted by the first-order equation.

**5. Influences of the initial bubble radius**

In this section, the influences of the dimensionless initial bubble radius ($R_*(0)$) on the characteristic parameters of bubble oscillations ($Ma_{\text{max}}$, $R_{\text{max}}$, and $P_{\text{rad, max}}$) are discussed quantitatively respectively. And, the purpose of adjusting $R_*(0)$ is to change the bubble wall Mach number. In addition, based on Eq. (16), in terms of the difference between $Ma_{\text{max}}$ predicted by the first-order and the second-order equations, the valid prediction ranges of the two equations are discussed and compared in detail.

Based on Eqs. (16), Fig. 9 shows the variations of the maximum bubble wall Mach number ($Ma_{\text{max}}$) with the initial dimensionless bubble radius. The black solid and the red dashed lines refer to $Ma_{\text{max}} = 1$. In addition, the values marked on the abscissa represent $R_*(0)$ corresponding to $Ma_{\text{max}} = 1$. The values marked on the ordinate represent the $Ma_{\text{max}}$ corresponding to $R_*(0) = 6.9$. As shown in Fig. 9, with the increase of $R_*(0)$, $Ma_{\text{max}}$ predicted by the two equations gradually increase. Moreover, $Ma_{\text{max}}$ predicted by the first-order equation is always greater than that predicted by the second-order equation with the same $R_*(0)$. And, the difference gradually increases with the increase of $R_*(0)$.
Furthermore, as shown in Fig. 9, when $Ma_{\text{max}}$ is equal to 1, $R_c(0)$ corresponding to the first-order equation is 6.9, and $R_c(0)$ corresponding to the second-order equation is 7.7. Therefore, according to the principle of $Ma_{\text{max}} < 1$ [44], one can easily find that the valid prediction range of the second-order equation is much larger than that of the first-order equation.

In addition, the dimensionless differences of the maximum bubble wall radial acceleration ($\Delta R_{\text{rad}}$) and the maximum radiation pressure ($\Delta P_{\text{rad\ max}}$) predicted by the first-order and the second-order equations will be discussed respectively. $\Delta R_{\text{rad}}$ and $\Delta P_{\text{rad\ max}}$ are defined as follows:

$$\Delta R_{\text{rad}} = \frac{R_{\text{rad\ max\ 1st}} - R_{\text{rad\ max\ 2nd}}}{R_{\text{rad\ max\ 2nd}}}$$  \quad (21)$$  

$$\Delta P_{\text{rad\ max}} = \frac{P_{\text{rad\ max\ 1st}} - P_{\text{rad\ max\ 2nd}}}{P_{\text{rad\ max\ 2nd}}}$$  \quad (22)$$

here, $R_{\text{rad\ max\ 1st}}$ and $R_{\text{rad\ max\ 2nd}}$ represent the maximum dimensionless radial accelerations of the bubble wall predicted by the first-order and the second-order equations respectively. $P_{\text{rad\ max\ 1st}}$ and $P_{\text{rad\ max\ 2nd}}$ represent the maximum dimensionless radiation pressures predicted by the first-order and the second-order equations respectively.

Fig. 10 shows the variation of $\Delta R_{\text{rad\ max}}$ with the initial dimensionless bubble radius. Points $A_1$, $A_2$ and $A_3$ marked in the figure correspond to $\Delta R_{\text{rad\ max}} = 40\%, 80\%$ and $120\%$ respectively. As shown in Fig. 10, with the increase of $R_c(0)$, $\Delta R_{\text{rad\ max}}$ gradually increase and can even exceed $130\%$.

In addition, Fig. 11 shows the variation of $\Delta P_{\text{rad\ max}}$ with the initial dimensionless bubble radius. Points $B_1$ and $B_2$ marked in the figure represent $\Delta P_{\text{rad\ max}}$ up to $20\%$ and $40\%$ respectively. As shown in Fig. 11, with the increase of $R_c(0)$, $\Delta P_{\text{rad\ max}}$ significantly increases and can even exceed $45\%$.

According to the above discussion, both $\Delta R_{\text{rad\ max}}$ and $\Delta P_{\text{rad\ max}}$ gradually increase with the increase of the dimensionless initial bubble radius. This is because when the bubble deviates far from the equilibrium position, the energy stored in the initial state of the bubble is very high, which will lead to a high bubble wall Mach number at the end of the bubble collapse. Therefore, the compressibility effect is significant, which will lead to a large difference between the first-order and the second-order equations. Furthermore, $\bar{R}_c$ and $P_{\text{rad\ max}}$ are essential for the investigations of the physical mechanism of practical applications, such as the cavitation damage [47] and the ultrasonic cleaning [12-15]. Therefore, when the bubble wall Mach number is high, the second-order equation should be safely selected for simulations due to its high accuracy.

6. Quantitative analysis of the dissipated power

The accurate prediction of the dissipated power is often closely related to the application of bubble dynamics. According to previous researches, the total dissipated power mainly includes the thermal, the viscous and the radiation dissipated powers [40]. In this section, based on the expressions of the dissipated power derived by Sojahrood et al. [40], the differences between the above dissipated powers predicted by the first-order and the second-order equations are investigated respectively. The thermal dissipated power ($E_{th}$), the viscous dissipated power ($E_v$), the radiation dissipated power ($E_r$) and the total dissipated power ($E_{\text{tot}}$) are defined as follows [40]:

$$E_{th} = -\frac{1}{\tau} \int_0^\tau \left(\frac{\partial V}{\partial t}\right) dt$$  \quad (23)$$  

$$E_v = \frac{16\pi \mu}{\tau} \int_0^\tau \left(R \dot{R}^2\right) dt$$  \quad (24)$$

Fig. 12. The variations of the dissipated powers predicted by the first-order and the second-order equations with the initial dimensionless bubble radius. The solid and the dashed lines refer to the first-order and the second-order equations respectively. Subgraphs (a), (b), (c) and (d) refer to the thermal dissipated power, the viscous dissipated power, the radiation dissipated power and the total dissipated power respectively.
The variation of the dimensionless difference between the total dissipated power predicted by the first-order and the second-order equations with the initial dimensionless bubble radius. Point C marked in the figure represents the difference up to 10%.

\[ E_i = \frac{1}{\tau} \int_0^\tau \left( \frac{4\pi}{3} R^3 \left( \dot{R}^2 + R \dot{R}^2 \right) - \frac{1}{2} \rho \dot{R}^2 \right) \, dt \]

\[ \Delta E_{tot} = \frac{E_{tot,1st} - E_{tot,2nd}}{E_{tot,2nd}} \]

here, \( \tau \) represents the time of the completion of the first bubble collapse. \( V \) represents the instantaneous bubble volume, which can be obtained from the bubble radius. \( E_{tot, 1st} \) and \( E_{tot, 2nd} \) represent the total dissipated powers predicted by the first-order and the second-order equations respectively.

Fig. 12 shows the variations of the dissipated powers (\( E_{th}, E_r, E_t \) and \( E_{tot} \)) predicted by the first-order and the second-order equations with the initial dimensionless bubble radius. The solid and the dashed lines refer to the first-order and the second-order equations respectively. Subgraphs (a), (b), (c) and (d) refer to the thermal dissipated power, the viscous dissipated power, the radiation dissipated power and the total dissipated power respectively. As shown in Fig. 12, with the increase of \( R_i(0) \), both the difference on \( E_r \) and the difference on \( E_{tot} \) predicted by the two equations gradually increase. Moreover, the difference of \( E_{th} \) first increases with the increase of \( R_i(0) \), and then the difference decreases due to the decrease of \( E_{th} \) predicted by the first-order equation. And, there is almost no difference on \( E_r \). Furthermore, by comparing the values of \( E_{th}, E_r \) and \( E_t \) with same \( R_i(0) \), it can be found that the difference of \( E_r \) is the main reason for the difference of \( E_{tot} \) predicted by the two equations.

Furthermore, Fig. 13 shows the variation of the dimensionless difference between the total dissipated power predicted by the first-order and the second-order equations (\( \Delta E_{tot} \)) with the initial dimensionless bubble radius. The point C marked in the figure refers to \( \Delta E_{tot} = 10\% \) and \( R_i(0) = 5.2 \). As shown in Fig. 13, with the increase of \( R_i(0) \), the difference of \( \Delta E_{tot} \) gradually increases and can even exceed 13%.

In addition, it must be noted that the expressions of the dissipated power employed in this section was derived through the Keller-Miksis equation with first-order Mach number. Moreover, the main purpose is to show the differences between the dissipated power predicted by the two equations. For the second-order equation, the dissipated power is just an approximation.

In conclusion, when the bubble wall Mach number is high, \( \Delta E_{tot} \) predicted by the first-order and the second-order equations with \( R_i(0) \) is
Similarly, according to the significant. Moreover, the dissipated power is an important physical representation of the initial bubble radius. The black solid, red dashed and blue dotted lines represent $R_0(0) = 4$, $R_0(0) = 6$ and $R_0(0) = 8$ respectively.

In Fig. 16, the variations of the dimensionless differences between the dimensionless maximum radiation pressures predicted by the first-order and the second-order equations versus the ambient pressure with the different dimensionless initial bubble radius. The black solid, red dashed and blue dotted lines represent $R_0(0) = 4$, $R_0(0) = 6$ and $R_0(0) = 8$ respectively.

In the present paper, the second-order equation is employed for investigating a free oscillating bubble in the liquid with numerical and experimental verifications. Because the second-order equation Eq. (15) has more complete second-order terms than the Gilmore equation. Therefore, the second-order equation is investigated due to the high accuracy. For the purpose of comparisons, the revised Keller-Miksis equation up to the first-order Mach number is solved with the same conditions (e.g. the initial conditions and the ambient pressure). Furthermore, in order to investigate the influences of the bubble wall Mach number on the predictions of the two equations, the bubble wall Mach number is adjusted by changing the initial bubble radius and the ambient pressure. In terms of the bubble oscillation characteristic parameters ($R_0$, $R_0$ and $K_0$), the dissipated powers ($E_{\text{tot}}, E_v, E_t$, and $E_{\text{rad}}$) and the radiation pressure ($P_{\text{rad}}$), comparing with the predictions by the first-order equation, we can conclude that:

1. The bubble radius, the bubble wall radial velocity and the bubble wall Mach number are smaller, and there is no obvious difference in the viscous dissipated power. Moreover, the dimensionless differences of the maximum bubble wall Mach number for the first-order equation obtained is $M_{\text{max}} < 0.59$.

2. The radiation dissipated power and the total dissipated power predicted by the second-order equation with high Mach number are smaller, and there is no obvious difference in the viscous dissipated power. Moreover, the dimensionless differences of the total dissipated power increases with the increase of the bubble wall Mach number.

3. According to the principle that the bubble wall Mach number never exceed $1$ [44], the valid prediction range of the second-order equation is much larger.

This paper investigates the differences of the different bubble wall motion equations with high Mach number for a free oscillating bubble. Moreover, the forced oscillating bubbles are also widely employed in biomedical [6] and chemical application [48] because of the unique physical complexity [49,50]. Therefore, in the future, the equations employed to predict the oscillation characteristics of the forced bubbles will be investigated in detail.

8. Conclusion

In the present paper, the second-order equation is employed for investigating a free oscillating bubble in the liquid with numerical and experimental verifications. Because the second-order equation Eq. (15) has more complete second-order terms than the Gilmore equation. Therefore, the second-order equation is investigated due to the high accuracy. For the purpose of comparisons, the revised Keller-Miksis equation up to the first-order Mach number is solved with the same conditions (e.g. the initial conditions and the ambient pressure). Furthermore, in order to investigate the influences of the bubble wall Mach number on the predictions of the two equations, the bubble wall Mach number is adjusted by changing the initial bubble radius and the ambient pressure. In terms of the bubble oscillation characteristic parameters ($R_0$, $R_0$ and $K_0$), the dissipated powers ($E_{\text{tot}}, E_v, E_t$, and $E_{\text{rad}}$) and the radiation pressure ($P_{\text{rad}}$), comparing with the predictions by the first-order equation, we can conclude that:

1. The bubble radius, the bubble wall radial velocity and the bubble wall Mach number are smaller, and there is no obvious difference in the viscous dissipated power. Moreover, the dimensionless differences of the maximum bubble wall Mach number for the first-order equation obtained is $M_{\text{max}} < 0.59$.

2. The radiation dissipated power and the total dissipated power predicted by the second-order equation with high Mach number are smaller, and there is no obvious difference in the viscous dissipated power. Moreover, the dimensionless differences of the total dissipated power increases with the increase of the bubble wall Mach number.

3. According to the principle that the bubble wall Mach number never exceed $1$ [44], the valid prediction range of the second-order equation is much larger.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. The numerical solution method of the second-order equation

In this appendix, the specific details of the numerical solution method of the second-order equation Eq. (15) will be shown in detail.

In order to clearly express $R$, as a function of $R_*$ and $\dot{R}_*$, according to Eq. (15), the quadratic equation of $\dot{R}_*$ can be clearly expressed as follows:

$$a_1 \dot{R}_* + b_1 + a_2 \dot{R}_{**} = b_2 + b_1 - a_1 \dot{R}_*.$$  \hspace{1cm} (29)

where

$$a_1 = \left[ 1 - (1 + \lambda) e R_* + \left( \frac{14}{5} + 2 \lambda + \theta \right) e R_*^2 \right] R_*$$  \hspace{1cm} (30)

$$a_2 = e R_*^2$$  \hspace{1cm} (31)

$$b_1 = e R_* \left[ 1 - (1 + \lambda) e R_* \right] \frac{1}{e^2} \frac{1}{\rho_0 + B} \frac{1}{\rho_0 + B} \left( \frac{4 \mu U}{R_* R_*} \right)$$  \hspace{1cm} (32)

$$b_2 = \left[ 1 + (1 - \lambda) e R_* + \theta e R_*^2 \right] \frac{1}{e^2} \frac{1}{\rho_0 + B} \left( \frac{(\rho_0 + B)}{\rho_0 + B} \right)^{-1}$$  \hspace{1cm} (33)

$$b_3 = e R_* \left[ 1 - (1 + \lambda) e R_* \right] \frac{1}{e^2} \frac{1}{\rho_0 + B} \left( \frac{(\rho_0 + B)}{\rho_0 + B} \right)^{-1}$$  \hspace{1cm} (34)

$$b_3 = e R_* \left[ 1 - (1 + \lambda) e R_* \right] \frac{1}{e^2} \frac{1}{\rho_0 + B} \left( \frac{(\rho_0 + B)}{\rho_0 + B} \right)^{-1}$$  \hspace{1cm} (35)

Here, $a_1$, $a_2$, $a_3$, $b_1$, $b_2$ and $b_3$ are functions of $R_*$ and $\dot{R}_*$.

Two solutions of Eq. (29) can be obtained as follows:

$$\dot{R}_* = \frac{-(a_1 + a_2) \pm \sqrt{(a_1 + a_2)^2 - 4a_3 (b_1 - b_2 - b_3)}}{2a_2}$$  \hspace{1cm} (36)

$$\dot{R}_* = \frac{-(a_1 + a_2) - \sqrt{(a_1 + a_2)^2 - 4a_3 (b_1 - b_2 - b_3)}}{2a_2}$$  \hspace{1cm} (37)

Here, we need to choose one of Eqs. (36) and (37) to numerically solve Eq. (15).

Then, Eq. (15) can be readily divided into a system of equations consisting of two ordinary differential equations by variable substitution ($y_1 = R_*$ and $y_2 = \dot{R}_*$).

$$\ddot{y}_1 = y_2$$  \hspace{1cm} (38)

$$2a_3 \ddot{y}_2 + (a_1 - a_2) y_2 + (b_1 - b_2 - b_3) = f(y_1, y_2)$$  \hspace{1cm} (39)

By setting the initial conditions of $y_1$ and $y_2$, Eqs. (38) and (39) can be numerically solved by ode45 with adaptive step size. So, $R_*$, $\dot{R}_*$, $\ddot{R}_*$, and the corresponding $t$ can be obtained.

Within the range of parameters involved in the present paper, Eqs. (36) and (37) are tried to solve Eqs. (38) and (39) respectively. However, when Eq. (37) is selected to solve Eqs. (38) and (39), compared with the calculation results of the first-order equation, the second-order results refer to Eq. (37) do not accord with the physical characteristics of the bubble oscillations. Therefore, Eq. (36) is selected to solve the Eqs. (38) and (39) by ode 45.

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