Null cone preserving maps, causal tensors and algebraic Rainich theory

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Abstract

A rank-$n$ tensor on a Lorentzian manifold whose contraction with $n$ arbitrary causal future directed vectors is non-negative is said to have the dominant property. These tensors, up to sign, are called causal tensors, and we determine their general mathematical properties in arbitrary dimension $N$. Then, we prove that rank-2 tensors which map the null cone on itself are causal tensors. Previously it has been shown that, to any tensor field $A$ on a Lorentzian manifold there is a corresponding “superenergy” tensor field $T\{A\}$ (defined as a quadratic sum over all Hodge duals of $A$) which always has the dominant property. Here we prove that, conversely, any symmetric rank-2 tensor with the dominant property can be written in a canonical way as a sum of $N$ superenergy tensors of simple forms. We show that the square of any rank-2 superenergy tensor is proportional to the metric in dimension $N \leq 4$, and that the square of the superenergy tensor of any simple form is proportional to the metric in arbitrary dimension. Conversely, we prove in arbitrary dimension that any symmetric rank-2 tensor $T$ whose square is proportional to the metric must be a causal tensor and, up to sign, the superenergy of a simple $p$-form, and that the trace of $T$ determines the rank $p$ of the form. This generalises, both with respect to the dimension $N$ and the rank...
\( p \), the classical algebraic Rainich conditions, which are necessary and sufficient conditions for a metric to originate algebraically in some physical field. Furthermore, it has the important geometric interpretation that the set of superenergy tensors of simple forms is precisely the set of tensors which leave the null cone invariant and preserve its time orientation. It also means that all involutory Lorentz transformations can be represented as superenergy tensors of simple forms, and that any rank-2 superenergy tensor is the sum of at most \( N \) conformally involutory Lorentz transformations. Non-symmetric null cone preserving maps are shown to have a symmetric part with the dominant property and are classified according to the null eigenvectors of the skew-symmetric part. We therefore obtain a complete classification of all conformal Lorentz transformations and singular null cone preserving maps on any Lorentzian manifold of any dimension.
1 Introduction

The Bel-Robinson tensor \([2, 4]\), a rank-4 tensor constructed from the Weyl curvature
tensor and its dual (it has only one dual in *four* dimensions), was until some ten years
ago not a widely known tensor outside part of the general relativity community. That
it has the dominant property —the contraction with any four causal future directed
vectors is non-negative— was certainly known \([26]\), and many relations to gravita-
tional energy were found (see e.g. \([35]\) and references therein). Its precise physical
meaning was, and still is, however not clear, and it is possible that no fundamental
physical interpretation can be given. Thus, interest in the Bel-Robinson tensor was
limited. This all changed with the work of Christodoulou and Klainerman on the global
non-linear stability of Minkowski spacetime \([13]\). (Bel-Robinson estimates were in fact
previously considered in the works by Friedrich on hyperbolic formulations of the field
equations, see \([14]\).) It became clear that the Bel-Robinson tensor is
mathematically a very useful quantity, its positivity (the dominant property) and divergence properties
being the main reasons. Today, the tensor is established as a key ingredient in many
mathematical studies of Einstein’s vacuum equations, see e.g. \([18, 32]\) and references
therein.

Considering this rise of interest in the Bel-Robinson tensor, it is remarkable that
the Bel tensor seems virtually unknown. This is the full Riemann curvature tensor
analogue of the Bel-Robinson tensor, so it is constructed from the Riemann tensor and
its duals \([3, 35]\). A fundamental fact is that also the Bel tensor has the dominant
property \([6, 8, 35]\), and it is essentially the only tensor with this property one can
construct from the Riemann tensor. Its divergence can often be controlled if some
suitable field equations for the matter are given, and it should therefore be the natural
candidate to replace the Bel-Robinson tensor if the full Einstein’s equations are studied.

More recently, it was discovered that this way of constructing a tensor with the
dominant property from a given tensor and its duals is universal \([34, 35]\). Given any
tensor field \(A\) on a Lorentzian manifold of arbitrary dimension, one can always in
an essentially unique way construct from \(A\) a corresponding tensor \(T\{A\}\) with the
dominant property \([1, 10, 35]\). It is perhaps unfortunate that, by historical reasons,
\(T\{A\}\) has become to be known as the *superenergy* tensor of \(A\), as this terminology
may have prevented attention from those studying differential equations on curved
manifolds. Superenergy tensors provide a very natural and geometric way to define
norms (including Sobolev norms) and inner products (corresponding to the positive
norms) on Lorentzian manifolds. Like the Bel-Robinson tensor, there is no need of a
physical interpretation of \(T\{A\}\) for it to be mathematically useful.

A first example of how the general superenergy tensors can be used in this sense was
given in \([6]\), where causal propagation of fields on Lorentzian manifolds was studied.
generalising techniques from [16, 9]. Note that for energy-momentum tensors (symmetric rank-2 tensors) the dominant property, first introduced in [28], is usually called the dominant energy condition [16]. Such tensors map the future cone on itself, something we refer to as a causal map or causal tensor. Superenergy tensors have also been used to construct new conserved quantities [35, 36], and to study the propagation of shock-waves [35].

In this paper we develop the mathematical structure of tensors having the dominant property and prove some new basic results about superenergy tensors. We prove that the product $T_{ac}T_{bc}$ of the superenergy tensor $T_{ab}\{A\}$ of a simple form $A$ is always proportional to the metric. This is further shown to be true for any arbitrary rank-2 superenergy tensor in dimensions $N \leq 4$.

While any superenergy tensor has the dominant property, we prove that any symmetric rank-2 tensor with the dominant property can be written as a sum of $N$ superenergy tensors of simple forms in a canonical way. We also present some geometrical interpretation of these forms and emphasize that, in this sense, superenergy tensors of simple forms are the basic building blocks of positive or causal quantities.

The classical Rainich conditions [31, 23], sometimes referred to as RMW (Rainich-Misner-Wheeler) theory or already unified theory, are necessary and sufficient conditions in 4 dimensions for an energy-momentum tensor to originate in a Maxwell field. One may also express this as saying that they are conditions on a metric, which then via the Ricci tensor and Einstein’s equations give the energy-momentum tensor. The algebraic Rainich conditions as stated in [26] are that the energy-momentum tensor is trace-free, satisfies the dominant energy condition, and has a square that is proportional to the metric. The Rainich conditions have also been generalised to cover some other physical situations (e.g. [10, 20, 24, 25, 27]).

Here, we prove a much more general result, namely that in $N$ dimensions any symmetric rank-2 tensor with a square proportional to the metric must be the superenergy tensor of a simple $p$-form. We prove that the trace can only have certain discrete values related to the rank $p$ of this form. This result, being an equivalence, has an important geometrical interpretation. It says that on any Lorentzian manifold of any dimension, the set of superenergy tensors of simple forms is precisely the set of tensors which leave the null cone invariant and preserve its time orientation. This also leads to an extended algebraic Rainich theory which includes the previously known results as special cases. It also has the interesting implication that all symmetric (i.e. involutory) Lorentz transformations can be represented as superenergy tensors of simple forms. Furthermore, the combination with the previous results proves that any superenergy tensor is the sum of at most $N$ conformally involutory Lorentz transformations. We also study non-symmetric null cone preserving maps, which are proven to
have a symmetric part with the dominant property, and classify them according to the null eigenvectors of its skew-symmetric part. All this provides a complete classification of all conformal Lorentz transformations as well as the singular null cone preserving maps in any Lorentzian manifold of arbitrary dimension.

In our notation we sometimes use indices on the tensors. These indices may be considered as abstract indices in the sense of Penrose and Rindler [26], and it is clear that all results are geometric and independent of any basis or coordinate system. We will also use the standard arrows for vectors and boldface characters for 1-forms. The tensor and exterior products are denoted by \( \otimes \) and \( \wedge \) respectively. As usual, (square) round brackets enclosing any set of indices indicate (anti) symmetrization. Equalities by definition are denoted by \( \equiv \). The symbol \( \blacksquare \) is used to mark the end of proofs. We shall use the signature \(+, -, \ldots, -\) of the metric. Note that this is the opposite of [35].

The outline of the paper is as follows. In section 2 we develop some general mathematical properties of tensors with the dominant property and in section 3 we recall the definition of superenergy tensors and prove certain new results for superenergy tensors of simple forms. We also extend the previous definition to \( N \)-forms in \( N \) dimensions and motivate why their superenergy is essentially the metric. Various properties of null cone preserving maps, their relation to superenergy tensors, and how these are used to construct any tensor satisfying the dominant energy condition are described in section 4. The classification of the (conformally) non-involutory Lorentz transformation is then given in section 5 while that of the (conformally) involutory ones and the generalised algebraic Rainich conditions with their important geometrical consequences are presented in section 6.

## 2 The dominant property: causal tensors

We assume that we work on an \( N \)-dimensional manifold \( V_N \) endowed with a Lorentzian metric \( g_{ab} \) and that a time-orientation has been chosen. Most of our considerations are algebraic and are implicitly assumed to hold in a point \( x \in V_N \); of course, they can be straightforwardly translated to tensor fields. We begin by giving the basic definition.

**Definition 2.1** A tensor \( T_{a_1 \ldots a_r} \) is said to have the **dominant property** if

\[
T_{a_1 \ldots a_r} u^{a_1}_1 \ldots u^{a_r}_r \geq 0
\]

for any set \( u^{a_1}_1, \ldots, u^{a_r}_r \) of causal future-pointing vectors. The set of tensors with the dominant property will be denoted by \( \mathcal{D} \mathcal{P} \). By \( -\mathcal{D} \mathcal{P} \) we mean the set of tensors \( T_{a_1 \ldots a_r} \) such that \( -T_{a_1 \ldots a_r} \in \mathcal{D} \mathcal{P} \).
We will see below, in Properties 2.3 and 2.4 respectively, that the definition of $\mathcal{DP}$ implies in fact that the strict inequality holds if the future-pointing $u_1^a \ldots u_r^a$ are all timelike, and that the use of only null vectors $u_1^a \ldots u_r^a$ is also enough.

By a natural extension, the non-negative real numbers are also considered to have the dominant property: $\mathbb{R}^+ \subset \mathcal{DP}$. Rank-1 tensors with the dominant property are simply the future-pointing causal vectors (while those in $-\mathcal{DP}$ are the past-directed ones). For rank-2 tensors, the dominant property was introduced by Plebański \[28\] in General Relativity and is usually called the dominant energy condition \[16\] because it is a requirement for physically acceptable energy-momentum tensors. The elements of $\mathcal{DP}$ could thus be termed as “future tensors”, and those of $\mathcal{DP} \cup -\mathcal{DP}$ will be called “causal tensors”. As in the case of past- and future-pointing vectors, any statement concerning $\mathcal{DP}$ has its counterpart concerning $-\mathcal{DP}$, and they will be taken as obvious unless otherwise stated.

The basic properties of tensors in the class $\mathcal{DP}$ are given in what follows. First of all, the class is closed under linear combinations with non-negative coefficients as well as under tensor products \[35\].

**Property 2.1** If $T^{(i)}_{a_1 \ldots a_r} \in \mathcal{DP}$ and $\alpha_i \in \mathbb{R}^+ \ (i = 1, \ldots, n)$ then $\sum_{i=1}^n \alpha_i T^{(i)}_{a_1 \ldots a_r} \in \mathcal{DP}$.

Moreover, if $T^{(1)}_{a_1 \ldots a_r}$, $T^{(2)}_{a_1 \ldots a_r} \in \mathcal{DP}$ then $\left( T^{(1)} \otimes T^{(2)} \right)_{a_1 \ldots a_r \sigma} \in \mathcal{DP}$.

**Proof:** This is an immediate consequence of the definition of $\mathcal{DP}$.

Given any tensor $T_{a_1 \ldots a_r} \in \mathcal{DP}$, one can immediately construct many other tensors in $\mathcal{DP}$ by simply permuting the indices, as is obvious from Definition 2.1. Then, we also have (see Section 5 in \[35\])

**Lemma 2.1** If $T_{a_1 \ldots a_r} \in \mathcal{DP}$, then for any set of non-negative constants $c_\sigma$ the family of tensors $\sum_{\sigma} c_\sigma T_{a_\sigma(1) \ldots a_\sigma(r)}$ belongs to $\mathcal{DP}$ where the sum is over all permutations $\sigma(1), \ldots, \sigma(r)$ of $(1, \ldots, r)$. In particular, any symmetric part of $T_{a_1 \ldots a_r}$ is in $\mathcal{DP}$.

**Proof:** Given that $T_{a_1 \ldots a_r} \in \mathcal{DP}$ for any permutation $\sigma(1), \ldots, \sigma(r)$ the first part follows from Property 2.1. Since any symmetric part is in fact a linear combination of such terms with particular positive coefficients $c_\sigma$ the Lemma is proven.

It must be remarked that, sometimes, linear combinations $\sum_{\sigma} c_\sigma T_{a_\sigma(1) \ldots a_\sigma(r)}$ with some negative coefficients $c_\sigma$ may also be in $\mathcal{DP}$. On the other hand, we also have

**Lemma 2.2** If $T_{a_1 \ldots a_r} \neq 0$ is antisymmetric in any pair of indices, then $T_{a_1 \ldots a_r}$ cannot be in $\mathcal{DP} \cup -\mathcal{DP}$.

**Proof:** Assume, for instance, that $T_{a_1 a_2 \ldots a_r} = -T_{a_2 a_1 \ldots a_r}$. Then, for any future-pointing $u_1^a, u_2^a, \ldots, u_r^a$, the scalars $T_{a_1 a_2 \ldots a_r} u_1^{a_1} u_2^{a_2} \ldots u_r^{a_r}$ and $T_{a_1 a_2 \ldots a_r} u_2^{a_1} u_1^{a_2} \ldots u_r^{a_r}$ have opposite signs. This implies that a constant sign cannot be maintained.
Property 2.2 \( T_{a_1...a_r} \in \mathcal{DP} \iff u^{a_i}T_{a_1a_2...a_r} \in \mathcal{DP} \) for all future-pointing vectors \( \vec{u} \).

Proof: Again this is trivial from Definition 2.1.

Of course, this can be equally proven for the contraction of \( \vec{u} \) with any index of \( T_{a_1...a_r} \). The previous property can be generalized to show that the class \( \mathcal{DP} \) is also closed under tensor products with one contraction applied. To that end, we introduce the following products for any two tensors \( T_{a_1...a_r} \) and \( T_{a_1...a_s} \):

\[
(T^{(1)}_{i\times j} T^{(2)}_{a_1...a_{r+s-2}}) a_1...a_{r+s-2} \equiv T^{(1)}_{a_1...a_{r-1}ba_{r+1}...a_{r+s-2}} T^{(2)}_{a_r...a_{r+j-2}a_{r+j+1}...a_{r+s-2}}
\]

where the contraction is taken with the \( i^{th} \) index of the first tensor and the \( j^{th} \) of the second. There are of course many different products \( \times_j \) depending on where the contraction is made.

Lemma 2.3 For all \( i = 1,...,r \) and all \( j = 1,...,s \), if \( T_{a_1...a_r}, t_{a_1...a_s} \in \mathcal{DP} \) or if \( T_{a_1...a_r}, t_{a_1...a_s} \in -\mathcal{DP} \), then \( (T \times_j t)_{a_1...a_{r+s-2}} \in \mathcal{DP} \); and if \( T_{a_1...a_r} \in \mathcal{DP} \) and \( t_{a_1...a_s} \in -\mathcal{DP} \) then \( (T \times_j t)_{a_1...a_{r+s-2}} \in -\mathcal{DP} \) and \( (t \times_j T)_{a_1...a_{r+s-2}} \in -\mathcal{DP} \).

Proof: If \( T_{a_1...a_r}, t_{a_1...a_s} \in \mathcal{DP} \), or if they are in \( -\mathcal{DP} \), then, by Property 2.2,

\[
v_b \equiv T_{a_1...a_{r-1}ba_{r+1}...a_{r+s-2}} a_1...a_{r-1} u_{i-1}...u_{r-1},
\]

\[
w_b \equiv t_{a_{r+j-2}ba_{r+j+1}...a_{r+s-2}} u_{i}...u_{r+j-2} u_{r+j+1}...u_{r+s-2}
\]

are causal vectors with the same time orientation for any set \( a_1...a_{r+s-2} \) of future-pointing vectors. Hence

\[
(T \times_j t)_{a_1...a_{r+s-2}} a_1...a_{r+s-2} = v_b \equiv w_b \geq 0
\]

and the first result follows. The other is similar.

Corollary 2.1 If for some \( i,j \), \( 0 \neq (T \times_j T)_{a_1...a_{2r-2}} \in -\mathcal{DP} \), then \( T_{a_1...a_r} \) cannot be in \( \mathcal{DP} \cup -\mathcal{DP} \).

Proof: For if \( T_{a_1...a_r} \) were in either \( \mathcal{DP} \) or \( -\mathcal{DP} \), by Lemma 2.3 \((T \times_j T)_{a_1...a_{2r-2}} \) should be in \( \mathcal{DP} \). Notice that \( \mathcal{DP} \cap -\mathcal{DP} = \{0\} \).

Corollary 2.2 \( T_{a_1...a_r} \in \mathcal{DP} \iff (T \times_j t)_{a_1...a_{r+s-2}} \in \mathcal{DP} \) for all \( t_{a_1...a_s} \in \mathcal{DP} \).

Proof: The implication from left to right is in Lemma 2.3. The converse can be proved by taking in particular \( s = 1 \) and using Property 2.2.

Corollary 2.2 is the evident generalization of the well-known fact that a causal vector \( \vec{u} \) is future-pointing if and only if \( u_i v^i \in \mathbb{R}^+ \) for all future-pointing vectors \( \vec{v} \).

The concept of “positivity” does not capture all that is behind the definition of \( \mathcal{DP} \), and the terminology dominant property (or dominant energy condition) is preferable because the pure time component dominates any other component in orthogonal bases.
Lemma 2.4: \( T_{a_1...a_r} \in \mathcal{DP} \iff T_{0...0} \geq |T_{a_1...a_r}| \) for all \( \alpha_1,...,\alpha_r \in \{0,1,...,N-1\} \), where \( T_{a_1...a_r} \) are the components of \( T_{a_1...a_r} \) with respect to any orthonormal basis \( \{\vec{e}_0, \vec{e}_1,...,\vec{e}_{N-1}\} \) with a future-pointing timelike \( \vec{e}_0 \).

Proof: See Lemma 4.1 in [35].

Corollary 2.3: If \( T_{a_1...a_r} \in \mathcal{DP} \) and \( T_{a_1...a_r}u^{a_1}...u^{a_r} = 0 \) for a timelike vector \( \vec{u} \), then \( T_{a_1...a_r} = 0 \).

Proof: By choosing the sign \( \epsilon \) we have that \( \vec{e}_0 = \epsilon \vec{u}/(u_a u^a) \) is unit and future-pointing. Thus, by Lemma 2.4, all components of \( T_{a_1...a_r} \) vanish in any orthonormal basis including \( \vec{e}_0 \), which means that \( T_{a_1...a_r} \) is the zero tensor.

Corollary 2.4: If \( T_{a_1...a_r} \in \mathcal{DP} \) and \( T_{a_1...a_r}u^{a_r} = 0 \) for a timelike vector \( \vec{u} \), then \( T_{a_1...a_r} = 0 \).

Definition 2.1 involves all causal future-pointing vectors, but in fact the class \( \mathcal{DP} \) can be equally characterized by using timelike vectors exclusively, or also only null vectors. Concerning the timelike case we have:

Property 2.3: A tensor \( T_{a_1...a_r} \neq 0 \) is in \( \mathcal{DP} \) \( \iff \) \( T_{a_1...a_r}u^{a_1}...u^{a_r} > 0 \) for any set \( u^{a_1},...,u^{a_r} \) of timelike future-pointing vectors.

Proof: The implication from right to left follows by continuity. Conversely, first for rank 1, \( T_{a_1}u^{a_1}_1 = 0 \) for \( T_{a_1} \in \mathcal{DP} \) would imply that \( T_{a_1} \) and \( u^{a_1}_1 \) are parallel and null as this is the only way two causal vectors can be orthogonal. Thus, if \( u^{a_1}_1 \) is timelike and \( T_{a_1} \neq 0 \) then \( T_{a_1}u^{a_1}_1 > 0 \). Suppose now that the property has been proved for rank-\((r-1)\) tensors and define \( \tau_{a_1...a_{r-1}} = T_{a_1...a_r}u^{a_r} \). By Property 2.2 \( \tau_{a_1...a_{r-1}} \in \mathcal{DP} \) and hence, if \( T_{a_1...a_r}u^{a_1}_1...u^{a_{r-1}}_1 = 0 \) then \( \tau_{a_1...a_{r-1}}u^{a_1}_1...u^{a_{r-1}}_1 = 0 \) which in turn, by Corollary 2.3 would imply that \( \tau_{a_1...a_{r-1}} = T_{a_1...a_r}u^{a_r} = 0 \). But then Corollary 2.4 would lead to \( T_{a_1...a_r} = 0 \). Thus the result follows by induction on \( r \).

In order to give the characterization with null vectors we first need a basic result stating that future-pointing null vectors are the basic “building blocks” of all future-pointing vectors, i.e. rank-1 tensors in \( \mathcal{DP} \). In section 4 we shall generalize this by identifying the analogous building blocks of rank-2 tensors in \( \mathcal{DP} \).

Lemma 2.5: Given a future-pointing timelike vector \( \vec{u} \) and a future-pointing null vector \( \vec{n} \), there is another future-pointing null vector \( \vec{\bar{n}} \) such that \( \vec{u} = c\vec{k} + \vec{\bar{n}} \) where \( c = u^a u_a/(2u^a u_a) > 0 \).

Proof: See, e.g., [3].

We can now show that in order to check that a tensor is in \( \mathcal{DP} \) it is sufficient to check it for null vectors. This is very helpful because obviously it is easier to work with null vectors exclusively rather than with both null and timelike vectors.
Property 2.4 \( T_{a_1...a_r} \in \mathcal{DP} \iff T_{a_1...a_r}k^{a_1}_{1}...k^{a_r}_{r} \geq 0 \) for any set \( k^{a_1}_{1}, ..., k^{a_r}_{r} \) of future-pointing null vectors.

Proof: By Lemma 2.3, \( T_{a_1...a_r}u^{a_1}_{1}...u^{a_r}_{r} \) with \( s \leq r \) timelike vectors can be written as a sum with positive coefficients of \( 2^s \) terms of the type \( T_{a_1...a_r}k^{a_1}_{1}...k^{a_r}_{r} \) involving null vectors only and the result follows immediately.

Now we can prove a partial but important converse of the Lemma 2.3.

Proposition 2.1 \( 0 \neq (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) for some \( i = 1, \ldots, r \implies T_{a_1...a_r} \in \mathcal{DP} \cup -\mathcal{DP} \).

Proof: For any set of timelike future pointing vectors \( u^{a_1}_{1}, ..., u^{a_{r-1}}_{r-1} \) define \( v_{b} \) as in (2), of Lemma 2.3. Now, if \( (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) then \( v_{b}b^{b} \geq 0 \) which implies that \( v_{b} \) is causal, either future- or past-pointing. To see that the time orientation of these \( v_{b} \) is consistent take any other arbitrary set of timelike future-pointing vectors \( a^{a_1}_{1}, ..., a^{a_{r-1}}_{r-1} \) and define \( \tilde{v}_{b} \) analogously to (3). As \( (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) and is not zero by assumption, using Property 2.3 we have that \( v_{b}\tilde{v}_{b}^{b} > 0 \), so that \( v_{b} \) and \( \tilde{v}_{b} \) have the same time orientation. As all the vectors \( a^{a_1}_{1}, ..., a^{a_{r-1}}_{r-1} \) and \( a^{a_1}_{1}, ..., a^{a_{r-1}}_{r-1} \) are arbitrary and future pointing, this means that \( T_{a_1...a_r} \) or \( -T_{a_1...a_r} \) is in \( \mathcal{DP} \).

Corollary 2.5 \( 0 \neq (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) for some \( i = 1, \ldots, r \implies (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) for all \( i, j = 1, \ldots, r \).

Proof: If \( 0 \neq (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) for some \( i = 1, \ldots, r \) then by Proposition 2.1 \( eT_{a_1...a_r} \in \mathcal{DP} \) with \( e^{2} = 1 \), but then by Lemma 2.3 \( (T_{i} \times T)_{a_1...a_{2r-2}} \in \mathcal{DP} \) for all \( i, j = 1, \ldots, r \).

In the last two results, Proposition 2.1 and Corollary 2.5, the special case with \( (T_{i} \times T)_{a_1...a_{2r-2}} = 0 \), which is in \( \mathcal{DP} \), has been excluded. Similar results apply for this extreme case, but they need some refinement.

Proposition 2.2 \( (T_{i} \times T)_{a_1...a_{2r-2}} = 0 \) for some \( i = 1, \ldots, r \iff \) there is a null vector \( \tilde{k} \) such that \( T_{a_1...a_r} = k_{a_1}t_{a_1...a_{r-1}a_{r-1}...a_r} \) for some tensor \( t_{a_1...a_{r-1}} \).

Proof: Using the same notation as in Proposition 2.1 we have that all the \( v_{b} \) are causal and furthermore, as \( v_{b}\tilde{v}_{b}^{b} = 0 \), all of them are orthogonal to each other. This means that all of them must be proportional to a null vector \( v_{b} \propto k_{b} \), and the result follows. The converse is immediate.

Therefore, it can happen that \( (T_{i} \times T)_{a_1...a_{2r-2}} = 0 \) and therefore is in \( \mathcal{DP} \) and yet neither \( T_{a_1...a_r} \) nor \( -T_{a_1...a_r} \) is in \( \mathcal{DP} \). It is enough that \( t_{a_1...a_{r-1}} \notin \mathcal{DP} \cup -\mathcal{DP} \). Nevertheless, we have the following result.
Corollary 2.6 \((T_i \times_i T)_{a_1...a_{2r-2}} = 0\) for all \(i = 1, \ldots, r \iff T_{a_1\ldots a_r} = k_{a_1} \ldots n_{a_r}\) for a set of \(r\) null vectors \(\vec{k}, \ldots, \vec{n}\).

Therefore, \((T_i \times_i T)_{a_1...a_{2r-2}} = 0\) for all \(i = 1, \ldots, r \Rightarrow T_{a_1...a_r} \in \mathcal{DP} \cup -\mathcal{DP}\).

Proof: The first part follows from repeated application of Proposition 2.3. Then, depending on how many of the null vectors \(\vec{k}, \ldots, \vec{n}\) are future-pointing, either \(T_{a_1...a_r}\) or \(-T_{a_1...a_r}\) is in \(\mathcal{DP}\).

Corollary 2.7 Assume that \(T_{a_1...a_r}\) is completely symmetric. Then, \((T_i \times_j T)_{a_1...a_{2r-2}} = 0 \iff T_{a_1...a_r} = f k_{a_1} \ldots n_{a_r}\) for some future-pointing null vector \(\vec{k}\).

Proof: If \(T_{a_1...a_r}\) is completely symmetric then all the products \((T_i \times_j T)_{a_1...a_{2r-2}}\) are the same. Thus, from Corollary 2.6 and the symmetry of \(T_{a_1...a_r}\) the result follows.

Of course, under the assumptions of Corollary 2.7, \(T_{a_1...a_r} \in \mathcal{DP} \cup -\mathcal{DP}\). On the other hand, we have

Corollary 2.8 If \(T_{a_1...a_r}\) is completely antisymmetric and \((T_i \times_i T)_{a_1...a_{2r-2}} = 0\) then \(T_{a_1...a_r} = 0\).

Proof: If \(T_{a_1...a_r}\) is completely antisymmetric again all the products \((T_i \times_j T)_{a_1...a_{2r-2}}\) are the same. Thus, from Corollary 2.6 and the antisymmetry of \(T_{a_1...a_r}\) the only possibility is \(T_{a_1...a_r} = 0\).

3 Superenergy tensors

In the previous section we have defined the set \(\mathcal{DP}\) and analyzed its general properties. However, we must still face the question of how general is the class \(\mathcal{DP}\) and how we can build such causal tensors. Actually this has been already solved and the result is that, given an arbitrary tensor \(A_{c_1\ldots c_m}\), there is a general procedure to construct its “positive square”: a tensor quadratic in \(A_{c_1\ldots c_m}\) and with the dominant property. This general procedure was introduced in \((34)\) and extensively considered in \((35)\), and the positive tensors thus constructed receive the generic name of “super-energy tensors” (due to historical reasons \((35)\)). In what follows, we recall here the definition of a general superenergy tensor (see section 3 of \((35)\)).

Let \(A_{c_1\ldots c_m}\) be an arbitrary rank-\(m\) tensor. Let \([n_1]\) denote the set of indices containing \(c_1\) and all other indices \(c_j\) such that \(A_{c_1\ldots c_m}\) is anti-symmetric in \(c_1c_j\). The number \(n_1\) is the number of indices in \([n_1]\). Then \([n_2]\) is the next set formed from anti-symmetries with \(c_2\) (or \(c_3\) if \(c_2\) is already in \([n_1]\) and so on). Note that \(1 \leq n_i \leq N\) for each \(i\). In this way \(c_1, \ldots, c_m\) are divided into \(r\) blocks \([n_1], \ldots, [n_r]\)
with $n_1 + \ldots + n_r = m$. We can therefore consider $A_{c_1 \ldots c_m}$ as an $r$-fold $(n_1, \ldots, n_r)$-form and we write $A_{c_1 \ldots c_m} = A_{[n_1] \ldots [n_r]}$.

There are $2^r$ different (multiple) Hodge duals of $A_{[n_1] \ldots [n_r]}$. The dual with respect to the block $[n_1]$ is denoted $A_{[N-n_1][n_2] \ldots [n_r]}$, with respect to $[n_2]$ by $A_{[n_1][N-n_2] \ldots [n_r]}$, the dual with respect to $[n_1]$ and $[n_2]$ by $A_{[N-n_1][N-n_2][n_3] \ldots [n_r]}$ and so on. Note that different duals may be tensors of different rank but all duals are $r$-fold forms. Denote by $(A_P)[\ldots]\ , \ P = 1, 2, \ldots , 2^r$, all the possible duals, where $P \equiv 1 + s_1 + 2s_2 + \ldots + 2^{r-1}s_r$ with $s_i = 0$ if there is no dual with respect to the block $[n_i]$ and $s_i = 1$ if there is a dual with respect to this block (so $A_1 = A$ and $A_{2^r}$ is the $r$-fold form where the dual has been taken with respect to all blocks).

In order to define the superenergy tensor of $A_{c_1 \ldots c_m}$ we need a product $\odot$ of an $r$-fold form by itself resulting in a $2r$-tensor. Let $A_{c_1 \ldots c_m}$ be the tensor obtained by permuting the indices in $A_{c_1 \ldots c_m}$ so that the $n_1$ first indices in $\tilde{A}_{c_1 \ldots c_m}$ are precisely the indices in the block $[n_1]$, the following $n_2$ indices are the ones in $[n_2]$ and so on. Now define the product $\odot$ by

$$\quad (A \odot A)_{a_1 b_1 \ldots a_r b_r} = \left( \prod_{\gamma=1}^{r} \frac{(-1)^{n_\gamma-1}}{(n_\gamma - 1)!} \right) \tilde{A}_{a_1 c_2 \ldots c_{n_1} \ldots a_r d_2 \ldots d_{n_r}} \tilde{A}_{b_1 c_2 \ldots c_{n_1} \ldots b_r d_2 \ldots d_{n_r}}. \quad (2)$$

From each block in $A_{[n_1] \ldots [n_r]}$ two indices are obtained in $(A \odot A)_{a_1 b_1 \ldots a_r b_r}$. We can form $(A_P \odot A_P)_{a_1 b_1 \ldots a_r b_r}$ for any $P$ but observe that $A_{[n_1] \ldots [n_r]}$ could contain $N$-blocks (with dual 0-blocks) for which the expression (2) has no meaning. Therefore, assuming $1 \leq n_i \leq N - 1$ for all $i$, we make the following definition.

**Definition 3.1**  The superenergy tensor of $A_{c_1 \ldots c_m}$ is defined to be

$$T_{a_1 b_1 \ldots a_r b_r} \{A\} = \frac{1}{2} \sum_{P=1}^{2^r} (A_P \odot A_P)_{a_1 b_1 \ldots a_r b_r}.$$

Observe that any dual $A_P$ of the original tensor $A = A_1$ generates the same superenergy tensor. We note that

$$T_{a_1 b_1 \ldots a_r b_r} \{A\} = T_{(a_1 b_1) \ldots (a_r b_r)} \{A\}$$

and if $A_{[n_1] \ldots [n_r]}$ is symmetric with respect to two blocks $[n_\gamma]$ and $[n_\xi]$, then $T_{a_1 b_1 \ldots a_r b_r}$ is symmetric with respect to the pairs $a_\gamma b_\gamma$ and $a_\xi b_\xi$. Important is also the property that $T_{a_1 b_1 \ldots a_r b_r} \{A\} = 0$ if and only if $A_{c_1 \ldots c_m} = 0$.

Another property of superenergy tensors is:

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The signs in this formula arise here because of the different choice of signature with respect to $[35]$. 

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Property 3.1 If $A_{[n_1]...[n_r]} = B_{[n_1]...[n_s]} C_{[n_{s+1}]...[n_r]}$ for some $s$- and $(r-s)$-folded forms $B_{[n_1]...[n_s]}$ and $C_{[n_{s+1}]...[n_r]}$, then

$$T_{a_1b_1...a rb_r} \left\{ B_{[n_1]...[n_s]} C_{[n_{s+1}]...[n_r]} \right\} = T_{a_1b_1...a sb_s} \left\{ B_{[n_1]...[n_s]} \right\} T_{a_{s+1}b_{s+1}...a rb_r} \left\{ C_{[n_{s+1}]...[n_r]} \right\} .$$

Proof: This follows at once from Definition 3.1 because the $2^s$ terms and the $2^{r-s}$ terms of $T_{a_1b_1...a sb_s} \left\{ B_{[n_1]...[n_s]} \right\}$ and $T_{a_{s+1}b_{s+1}...a rb_r} \left\{ C_{[n_{s+1}]...[n_r]} \right\}$, respectively, produce precisely the $2^r$ terms needed. ■

Definition 3.2 An $r$-fold form $A_{[n_1]...[n_r]}$ is said to be decomposable if there are $r$ forms $
abla^{(r)}_{a_1...a_{r-1}} = \nabla^{(r)}_{[a_1...a_{r-1}]} (r = 1...r)$ such that $A_{[n_1]...[n_r]} = \left( \nabla^{(1)} \otimes \ldots \otimes \nabla^{(r)} \right)_{[n_1]...[n_r]}$.

Corollary 3.1 If $A_{[n_1]...[n_r]}$ is decomposable, then

$$T_{a_1b_1...a rb_r} \left\{ A_{[n_1]...[n_r]} \right\} = T_{a_1b_1} \left\{ \nabla^{(1)}_{[n_1]} \right\} \ldots T_{a rb_r} \left\{ \nabla^{(r)}_{[n_r]} \right\} .$$

Proof: This is evident from Property 3.1 and Definition 3.2. ■

The last result shows that rank-2 superenergy tensors may be used as basic set to build up more general superenergy tensors in many occasions. Actually, we will later be interested (for other important reasons) in rank-2 tensors, specially those in $\mathcal{D}P$.

It is remarkable that, after expanding all duals in the Definition 3.1, one obtains an explicit expression for the general superenergy tensor which is independent of the dimension $N$, see [35]. In the case of a general $p$-form $\Omega_{a_1...a_p}$, the rank-2 superenergy tensor becomes [35]

$$T_{ab} \{ \Omega_{[p]} \} = \frac{(-1)^{p-1}}{(p - 1)!} \left[ \Omega_{a_2...a_p} \Omega_b^{a_2...a_p} - \frac{1}{2p} (\Omega \cdot \Omega) g_{ab} \right] .$$

(3)

Here we have used a notation which will be useful on many occasions: for any two tensors of the same rank $a_{a_1...a_j}$ and $b_{a_1...a_j}$, we write $a_{a_1...a_j} b_{a_1...a_j} \equiv a \cdot b$, i.e. we have contracted over all indices in order.

In the Definition 3.1 we assumed that there were no $N$-blocks. The expression (3) however is perfectly well defined for an $N$-form. If $\Omega_{a_1...a_N} = f \eta_{a_1...a_N}$ where $\eta$ is the canonical volume form and $f$ a scalar, then (3) gives

$$T_{ab} \{ \Omega_{[N]} \} = \frac{1}{2} f^2 g_{ab} .$$

(4)

If we combine (4) with Property 3.1 the Definition 3.1 is naturally extended to include $N$-blocks.
Definition 3.3  The superenergy tensor of \( A_{c_1...c_m} = \Omega_{a_1...a_N} = f\eta_{a_1...a_N} \) is defined to be
\[
T_{a_1b_1...a_{r-1}b_{r-1}a_r b_r} \{ A \} = \frac{1}{2} f^{2} T_{a_1b_1...a_{r-1}b_{r-1}} \{ B \} g_{a_r b_r}
\]

This definition is to be understood recursively, if there are \( q \) \([N]\)-blocks one continues until a tensor \( T_{a_1b_1...a_{r-q}b_{r-q}} \{ B \} \) given by Definition 3.1 is obtained.

We note that the tensor obtained by taking the dual of \( A_{a_1...a_r} \) with respect to the \( N \)-block to get a 0-block does not have the same superenergy tensor as \( A_{a_1...a_r} \) has, the difference being the \( g_{a_r b_r} \). This is a special situation only for \( N \)-blocks and it is the price one has to pay to extend the definition. The advantages however will be seen in a more consistent presentation of several definitions and results, the first being the following definition.

Definition 3.4  The set \( \mathcal{SE} \) is the set of all superenergy tensors according to Definitions 3.1 and 3.3. By \( -\mathcal{SE} \) we denote the set of tensors such that \( -T_{a_1...a_r} \in \mathcal{SE} \). The sets \( \mathcal{SE}_n \) and \( -\mathcal{SE}_n \) will denote the classes of rank-\( n \) tensors in \( \mathcal{SE} \) and \( -\mathcal{SE} \), respectively.

The metric is an essential element in this set. In [6] it was shown that the metric \( g_{ab} \) is not a superenergy tensor of any \( p \)-form \( \Omega_{[p]} \) with \( 1 \leq p \leq N - 1 \) so without the extended definition elements of the form \( f^{2} g_{ab} \) would have to be added artificially to \( \mathcal{SE} \).

A fundamental result is that superenergy tensors always have the dominant property.

Theorem 3.1  \( \mathcal{SE} \subset \mathcal{DP} \).

Proof:  The first proof for 4 dimensions was given in [1] and used spinors. In arbitrary dimension the first proof is in [8] while a proof that uses Clifford algebras and which is also valid in arbitrary dimension was presented in [30]. Of course, \( -\mathcal{SE} \subset -\mathcal{DP} \).

It is important to remark that the superenergy tensor \( T_{a_1b_1...a_{r}b_{r}} \{ A \} \) and its derived tensors by permutation of indices are the only (up to linear combinations) tensors quadratic in \( A_{c_1...c_m} \) and with the dominant property [35]. Therefore, there is a unique (up to a proportionality factor) completely symmetric tensor in \( \mathcal{DP} \) which is quadratic in \( A_{c_1...c_m} \), and this is simply \( T_{(a_1b_1...a_{r}b_{r})} \{ A \} \) [35].

In \( N = 4 \), the superenergy tensor of a 2-form \( F_{ab} = F_{[ab]} \) is its Maxwell energy-momentum tensor, and the superenergy tensor of an exact 1-form \( d\phi \) has the form

\[ T_{a_1b_1...a_{r}b_{r}} \{ A \} \]
of the energy-momentum tensor for a massless scalar field $\phi$. If we compute the superenergy tensor of the Riemann tensor, which is a double symmetrical (2,2)-form, we get the so-called Bel tensor $[3]$. The superenergy tensor of the Weyl curvature tensor is the well-known Bel-Robinson tensor $[2, 4]$. For these and other interesting physical examples see $[6, 35, 36]$. The dominant property of the Bel-Robinson tensor was used by Christodoulou and Klainerman $[13]$ in their study of the global stability of Minkowski spacetime, and in $[9]$ to study the causal propagation of gravity in vacuum.

In order to study relations between $SE_2$ and $DP$, and to see how $SE_2$ builds up $DP$, we prove now some results for rank-2 tensors. First, we need a very simple Lemma to fix the notation.

**Lemma 3.1** For any $T_{ab}$ we have that $(T_1 \times_1 T)_{ab}$ and $(T_2 \times_2 T)_{ab}$ are symmetric. Furthermore, if $s_{ab} = s_{(ab)}$ and $t_{ab} = t_{(ab)}$ are symmetric, then

$$(s_1 \times_1 t)_{ab} = (s_1 \times_2 t)_{ab} = (s_2 \times_1 t)_{ab} = (s_2 \times_2 t)_{ab} \equiv (s \times t)_{ab};$$

if $F_{ab} = F_{[ab]}$ and $G_{ab} = G_{[ab]}$ are antisymmetric, then

$$(F_1 \times_1 G)_{ab} = -(F_1 \times_2 G)_{ab} = -(F_2 \times_1 G)_{ab} = (F_2 \times_2 G)_{ab} \equiv (F \times G)_{ab};$$

finally in the mixed case

$$(s_1 \times_1 F)_{ab} = -(s_1 \times_2 F)_{ab} = (s_2 \times_1 F)_{ab} = -(s_2 \times_2 F)_{ab} \equiv (s \times F)_{ab}$$

so that in the three cases the simple $\times$-notation will be used.

**Proof:** First, $(T_1 \times_1 T)_{ab} = T_{ca}T^c_b$ which is obviously symmetric in $ab$, and analogously for $2\times_2$. If $s_{ab}, r_{ab}$ are symmetric then $s_{ca}r^c_b = s_{ac}r^c_b = s_{ac}r^c_b$ and similarly for the other cases.

**Lemma 3.2** If $F_{ab} = F_{[ab]} \neq 0$ is a 2-form, then $(F \times F)_{ab} \notin DP$.

**Proof:** If $(F \times F)_{ab}$ were in $DP$ and non-zero, then from Proposition 2.1, $F_{ab}$ should be in $DP \cup -DP$, which is impossible due to Lemma 2.2. If $(F \times F)_{ab} = 0$, then from Corollary 2.8 it follows that $F_{ab} = 0$.

Notice that, still, $(F \times F)_{ab}$ can certainly be in $-DP$.

**Proposition 3.1** In dimension $N \leq 4$, $T_{ab} \in SE_2 \Longrightarrow (T \times T)_{ab} = h^2 g_{ab}$. On the other hand, if $N > 4$ there exist tensors $T_{ab} \in SE_2$ such that $(T \times T)_{ab}$ is not proportional to the metric.
Proof: If $T_{ab} = h g_{ab}$ then $(T \times T)_{ab} = h^2 g_{ab}$ so the property is trivial. As the superenergy of a $p$-form is the same as the superenergy of its dual $(N-p)$-form, we just have to confirm the proposition for 1-forms for $N \leq 3$, and for 1-forms and 2-forms for $N = 4$. By (3), the superenergy tensor of a 1-form $J_a$ in any dimension $N$ is

$$T_{ab}\{J[1]\} = J_a J_b - \frac{(J \cdot J)}{2} g_{ab}$$

and this gives

$$\left( T\{J[1]\} \times T\{J[1]\} \right)_{ab} = \frac{(J \cdot J)^2}{4} g_{ab}$$

so $(T\{J[1]\} \times T\{J[1]\})_{ab}$ is proportional to the metric in any dimension. For a 2-form $F_{ab}$, the superenergy tensor reads

$$T_{ab}\{F[2]\} = -F_{ac} F_b^c + (1/4)(F \cdot F) g_{ab}$$

which again, by (3), holds in any $N$. Now, if $N = 4$, and only in this case, a very well-known result is (see, e.g., [21, 22, 23, 26, 31])

$$\left( T\{F[2]\} \times T\{F[2]\} \right)_{ab} = \frac{1}{16} \left[ (F \cdot F)^2 + (F \cdot *F)^2 \right] g_{ab} \quad \text{if } N = 4$$

where $*F_{ab}$ is the 2-form dual to $F_{ab}$ in 4 dimensions. This is the basis of the Rainich theory [31, 23, 26] and, as was pointed out by Lovelock [21, 22], formula (7) is an explicit example of a dimensionally-dependent identity, being valid only in $N = 4$.

Not even by changing the proportionality factor on the righthand side the above expression (7) holds in $N > 4$. To check it, we can construct explicit counterexamples. Let $\{e_0, e_1, ..., e_{N-1}\}$ be an orthonormal basis and let $F_{ab} = (e_0 \wedge e_1)_{ab} + (e_2 \wedge e_3)_{ab}$. Then the computation of (3) gives $T_{ab}\{F[2]\} = (e_0 \otimes e_0)_{ab} - (e_1 \otimes e_1)_{ab} + (e_2 \otimes e_2)_{ab} + (e_3 \otimes e_3)_{ab}$ from where one immediately obtains $\left( T\{F[2]\} \times T\{F[2]\} \right)_{ab} = (e_0 \otimes e_0)_{ab} - (e_1 \otimes e_1)_{ab} - (e_2 \otimes e_2)_{ab} - (e_3 \otimes e_3)_{ab}$ which is (proportional to) the metric in 4 but not higher dimension.

Thus, for $N \leq 4$, $\mathcal{SE}_2$ is the set of tensors with the property that $(T \times T)_{ab}$ is proportional to the metric, but this is not true for $N > 4$. The natural question arises of which super-energy tensors satisfy this property in arbitrary $N$. This is going to be answered now, and in a more complete way in the next section. The generalization of the algebraic Rainich condition (3) will be dealt with in the last section.

Recall that a $p$-form $\Omega_{a_1...a_p} = \Omega_{[a_1...a_p]}$ is called simple [33, 26] if it is a product of $p$ linearly independent 1-forms $\omega^1, ..., \omega^p$, i.e. $\Omega_{a_1...a_p} = (\omega^1 \wedge ... \wedge \omega^p)_{a_1...a_p}$. By
standard techniques, the set $\omega^1, \ldots, \omega^p$ can be chosen to be orthogonal by simply taking the appropriate linear combinations $\omega^i = a^i_j \omega^j$ with $\det(a^i_j) = 1$, because $(\omega^1 \wedge \ldots \wedge \omega^p)_{a_1 \ldots a_p} = (\omega^1 \wedge \ldots \wedge \omega^p)_{a_1 \ldots a_p}$. A $p$-form $\Omega_{[p]}$ is simple if and only if $\Omega_{[N-p]}$ is simple, and if and only if $(\Omega_{[p]} 1 \times 1 \Omega_{[N-p]})_{a_1 \ldots a_{N-2}} = 0$, see e.g. [33, 20].

**Definition 3.5** We denote by $SS$ the set of superenergy tensors of simple $p$-forms. Observe that $SS \subset SE_2$. We define $-SS$ as usual.

**Proposition 3.2** $T_{ab} \in SS \implies (T \times T)_{ab} = h^2 g_{ab}$.

**Proof:** From the proof of Proposition [3.1] the result is already proved for $fg_{ab}$ and for the superenergy tensor $T_{ab}\{\Omega_{[1]}\}$ of any 1-form. Using (9) for the superenergy tensor of a general $p$-form $\Omega_{a_1 \ldots a_p}$, a straightforward calculation gives

$$[(p-1)!]^2 \left(T\{\Omega_{[p]}\} \times T\{\Omega_{[p]}\}\right)_{ab} =$$

$$= \Omega^a_{a_2 \ldots a_p} \Omega^b_{b_2 \ldots b_p} \Omega^{c_2 \ldots b_p} - \frac{\Omega \cdot \Omega}{p} \Omega^a_{a_2 \ldots a_p} \Omega^b_{b_2 \ldots b_p} + \frac{(\Omega \cdot \Omega)^2}{4p^2} g_{ab}$$

Now, in the first term $\Omega^a_{a_2 \ldots a_p} \Omega^b_{b_2 \ldots b_p} \Omega^{c_2 \ldots b_p} = \Omega^a_{b_2 \ldots b_p} \Omega^b_{c_2 \ldots b_p} \Omega^{c_2 \ldots b_p}$. By [33] (p.23) (or [26] (p.165)) we have $\Omega^a_{b_2 \ldots b_p} \Omega^b_{c_2 \ldots b_p} = (1/p) \Omega^a_{b_2 \ldots b_p} \Omega^b_{c_2 \ldots b_p}$ if and only if $\Omega_{a_1 \ldots a_p}$ is simple. Thus, in this case we are left with

$$\left(T\{\Omega_{[p]}\} \times T\{\Omega_{[p]}\}\right)_{ab} = \frac{(\Omega \cdot \Omega)^2}{(2p)^2} g_{ab} \quad (8)$$

which proves the proposition. ■

**Proposition 3.3** If $\Omega_{a_1 \ldots a_p} = (\omega^1 \wedge \ldots \wedge \omega^p)_{a_1 \ldots a_p}$ is a simple $p$-form, then $\bar{\omega}^1, \ldots, \bar{\omega}^p$ are eigenvectors, all with the same eigenvalue $[-1]^{p-1} (\Omega \cdot \Omega) / (2p)!$, of its superenergy tensor $T_{ab}\{\Omega_{[p]}\} \in SS$.

**Proof:** The case $p = N$ is trivial so assume $p < N$. The dual $\Omega_{[N-p]}$ of $\Omega_{[p]}$ is obviously orthogonal to any of the $\bar{\omega}^1, \ldots, \bar{\omega}^p$. But the superenergy tensors $T_{ab}\{\Omega_{[p]}\}$ and $T_{ab}\{\Omega_{[N-p]}\}$ are identical, so that using the explicit expression (9) for $T_{ab}\{\Omega_{[N-p]}\}$ and contracting with $\bar{\omega}^i$ for $i = 1, \ldots, p$ we get

$$\omega^i_a T^a_b\{\Omega_{[N-p]}\} = \omega^i_a T^a_b\{\Omega_{[p]}\} = \frac{(-1)^{p-1}}{2p!} (\Omega \cdot \Omega) \omega^i_b$$
for any $i$.

Recall finally that a $p$-form $\Omega_{[p]}$ is called null if it is simple and $(\Omega \cdot \Omega) = 0$. Then, $\Omega_{[p]}$ defines canonically a null direction $\vec{k}$ such that $(\vec{k} \wedge \Omega_{[p]})_{a_1 \ldots a_{p+1}} = 0$ and $(\vec{k} \wedge \Omega_{[N-p]})_{a_1 \ldots a_{N-p+1}} = 0$. Equivalently, $\Omega_{[p]}$ can be written in the form $\Omega_{[p]} = (\vec{k} \wedge \omega_2^\wedge \ldots \wedge \omega_p)_{a_1 \ldots a_p}$ where the $(p-1)$ 1-forms $\omega_2^\wedge \ldots \wedge \omega_p$ are mutually orthogonal and orthogonal to $\vec{k}$.

**Definition 3.6** We denote by $\mathcal{NS}$ the set of superenergy tensors of null $p$-forms. Obviously, $\mathcal{NS} \subset \mathcal{SS} \subset \mathcal{SE}_2 \subset \mathcal{DP}$. $-\mathcal{NS}$ is defined as usual.

**Corollary 3.2** $T_{ab} \in \mathcal{NS} \implies (T \times T)_{ab} = 0$ and $T_{ab} = f k_a k_b$ where $\vec{k}$ is null and $f > 0$.

**Proof:** As $T_{ab} = T_{ab} \{\Omega_{[p]}\}$ for a null $p$-form $\Omega_{[p]}$, so that $(\Omega \cdot \Omega) = 0$, from (8) we get $(T \times T)_{ab} = 0$. Furthermore, as $\Omega_{a_1 \ldots a_p} = (\vec{k} \wedge \omega_2^\wedge \ldots \wedge \omega_p)_{a_1 \ldots a_p}$ where $k_a$ is its canonical null direction, a simple calculation produces $T_{ab} = f k_a k_b$ where $f \equiv (-1)^{p-1}(\omega_2 \cdot \omega_2) \ldots (\omega_p \cdot \omega_p)$ which is positive.

**Corollary 3.3** $T_{ab} \in \mathcal{SS} \setminus \mathcal{NS} \implies T_{ab}$ has $N$ independent eigenvectors, $p$ of them with eigenvalue $[(\Omega \cdot \Omega)]/(2p!)$ and $(N-p)$ of them with the opposite eigenvalue. $T_{ab} \in \mathcal{NS} \implies T_{ab}$ has $N-1$ independent eigenvectors, all with zero eigenvalues. It has a unique null eigenvector.

**Proof:** Again $p = N$ is trivial so assume $p < N$. As $\Omega_{[N-p]}$ is simple if $\Omega_{[p]}$ is simple, and as $T_{ab} \{\Omega_{[p]}\} = T_{ab} \{\Omega_{[N-p]}\}$, the $(N-p)$ 1-forms that generate $\Omega_{[N-p]}$ are also eigenvectors of $T_{ab} \{\Omega_{[p]}\}$. From (8) one immediately finds the eigenvalues $(-1)^p(\Omega \cdot \Omega)/(2p!)$. The case $T_{ab} \in \mathcal{NS}$ is trivial from Corollary 3.2.

### 4 Maps preserving the null cone and $\mathcal{DP}$

In this section we are going to show two important properties of the set $\mathcal{SS}$: on one hand, its elements are the basic building blocks of all rank-2 tensors in $\mathcal{DP}$, and on the other they define maps which leave the null cone invariant. The converse of this result also holds but is left for the last section.

**Definition 4.1** We say that $T^{ab}$ defines a null-cone preserving map if $k^a T^{ab}$ is null for any null vector $\vec{k}$. A map that preserves the null cone is said to be orthochronous.
(respectively time reversal) if it keeps (resp. reverses) the cone’s time orientation, and is called proper, improper or singular if \( \det(T_a^b) \) is positive, negative, or zero, respectively. If the map is proper and orthochronous then it is called restricted. A null-cone preserving map is involutory if \( T_a^b = (T^{-1})_a^b \), and bi-preserving if \( T_a^b k_b \) is also null for any null 1-form \( k \).

Most of the above terminology is taken from that of Lorentz transformations, see e.g. [26]. Notice that involutory null-cone preserving maps are necessarily non-singular.

In order to characterize all these maps and relate them to \( SS \) we first recall a simple result.

**Lemma 4.1** \( T_{(ab)} = fg_{ab} \iff T_{ab}k^ak^b = 0 \) for any \( k^a \) that is null.

**Proof:** The implication from left to right is trivial. Conversely, if \( T_{ab}k^ak^b = 0 \), take an orthonormal basis \( \{ \vec{e}_0, \vec{e}_1, \ldots, \vec{e}_{N-1} \} \) with a timelike \( \vec{e}_0 \). Using first as null \( \vec{k} \) the vectors \( \vec{e}_0 \pm \vec{e}_i \) for \( i = 1, \ldots, N-1 \) one immediately deduces \( T_{(0i)} = 0 \) and \( T_{00} + T_{ii} = 0 \) for each \( i \). Using then as null \( \vec{k} \) the vectors \( \vec{e}_0 \pm \cos \alpha \vec{e}_i \pm \sin \alpha \vec{e}_j \) for \( i, j = 1, \ldots, N-1 \) one gets \( T_{(ij)} = 0 \) for all \( i \neq j \).

The following Lemma gives important geometrical interpretations to some results.

**Lemma 4.2** \( (T_2 \times T_2)_{ab} = fg_{ab} \iff T_a^b \) is a null cone preserving map.

**Proof:** The basic formula is \( (T_2 \times T_2)_{ab} = (T_{ac}k^a)(T_{bc}k^b) \). If \( (T_2 \times T_2)_{ab} = fg_{ab} \) then for any null \( k^b \) we have that \( T_{ab}k^a \) must be null. Conversely, if \( T_{ab}k^a \) is null for any \( k^a \) that is null, and given that \( (T_2 \times T_2)_{ab} \) is symmetric according to Lemma 3.1, then by Lemma 4.1 \( (T_2 \times T_2)_{ab} \) must be proportional to the metric.

**Corollary 4.1** If \( (T_2 \times T_2)_{ab} = fg_{ab} \) then \( f \geq 0 \) (for \( N \neq 2 \)).

**Proof:** From Lemma 4.1 we know that \( T_{ab}k^a \) is null for any \( k^a \) that is null. If \( f \) were negative, then for any null and future-pointing vectors \( k^a \) and \( n^b \) we would have \( (T_{ac}k^a)(T_{bc}n^b) = f k_c n^c < 0 \), so that any two null vectors of type \( T_{ab}k^a \) and \( T_{ab}n^a \) would have opposite time orientations. But this is evidently impossible for all the null vectors of type \( T_{ab}k^a \) unless there are only two, that is, \( N = 2 \).

Similar results can be shown for the product \( (T_1 \times T_1)_{ab} \). However, they are mainly redundant because of the following

**Lemma 4.3** In \( N > 2 \), \( (T_2 \times T_2)_{ab} = fg_{ab} \neq 0 \iff f > 0 \) and \( (T_1 \times T_1)_{ab} = fg_{ab} \iff T_{ab} \) defines a non-singular null-cone preserving map. A fortiori, all non-singular maps preserving the null cone are automatically bi-preserving, proportional to an \( N \)-dimensional Lorentz transformation, and in \( DP \cup -DP \).
Proof: If \((T_2 \times_2 T)_{ab} = fg_{ab} \neq 0\), from Corollary 4.1 we get \(f > 0\) so that we can define \(L_{ab} \equiv \frac{1}{\sqrt{f}} T_{ab}\) and the condition becomes \(g_{cd}L_a^cL_b^d = g_{ab}\). This means that \(L_{ab}\) is a Lorentz transformation (ergo non-singular), which as is well-known also satisfies \(g_{cd}L_a^cL_b^d = g_{ab}\), see e.g. [33, 26]. This is exactly \((T_1 \times_1 T)_{ab} = fg_{ab}\). Now, a reasoning identical to that in the proof of Lemma 4.2 implies that \(T_{ab}k^b\) is null for any \(k^a\) that is null, that is, \(T_{ab}\) is bi-preserving. Finally, as \(f > 0\), \((T_2 \times_2 T)_{ab} \in \mathcal{DP}\) so that from Proposition 2.2 \(T_{ab} \in \mathcal{DP} \cup -\mathcal{DP}\). The singular case must be treated separately because of some minor subtleties.

Lemma 4.4 (a) \((T_2 \times_2 T)_{ab} = 0 \iff T_{ab}^b\) is a singular null cone preserving map \(\iff T_{ab} = s^a k_b\) where \(k_b\) is null. 
(b) \((T_2 \times_2 T)_{ab} = (T_1 \times_1 T)_{ab} = 0 \iff T_{ab}^b\) is a singular null-cone bi-preserving map \(\iff T_{ab} = n^a k_b\) where \(n^a\) and \(k_b\) are null and \(T_{ab} \in \mathcal{DP} \cup -\mathcal{DP}\).

Proof: From Lemma 4.2 we know that \((T_2 \times_2 T)_{ab} = 0\) if and only if the map defined by \(T_{ab}^b\) preserves the null cone, and by Lemma 4.3 this map must be singular. Thus, from Proposition 2.2 there exists a null \(k_b\) such that \(T_{ab} = s^a k_b\). This proves (a). Then, (b) follows from Corollary 2.2 in a similar way.

Corollary 4.2 The tensors in \(SS \setminus NS\) (respectively in \(-SS \setminus -NS\)) are proportional to involutory orthochronous (resp. time-reversal) Lorentz transformations. The tensors in \(NS\) (resp. \(-NS\)) define singular orthochronous (resp. time-reversal) null-cone bi-preserving maps.

Proof: This follows at once from Proposition 3.2, Corollary 3.2, Lemmas 4.2, 4.4, and the fact that if \(T_{ab}^b\) is involutory then by Lemma 4.3 it must coincide with an involutory Lorentz transformation \(T_{ab}^b = L_{ab}^b\), which are symmetric \(L_{ab} = L_{ba}\) [26]. If a null-cone preserving map is non-symmetric (ergo not proportional to an involutory Lorentz transformation if non-singular), then it can be divided into its symmetric and anti-symmetric parts:

\[ T_{ab} \equiv S_{ab} + F_{ab}, \quad S_{ab} \equiv T_{(ab)}, \quad F_{ab} \equiv T_{[ab]} \, . \]

Notice that, by definition, if \(T_{ab}^b\) is proportional to an involutory Lorentz transformation then \(T_{[ab]} = 0\) and (up to sign) \(T_{(ab)} \in \mathcal{DP}\) (later we shall prove that, in fact, \(T_{(ab)} \in SS\), see Theorem 5.1). The general characterization is (see [11, 12, 26] for \(N = 4\)):

Lemma 4.5 The symmetric and antisymmetric parts of \(T_{ab}^b\) satisfy

\[ (S \times S)_{ab} + (F \times F)_{ab} = fg_{ab}, \quad (S \times F)_{(ab)} = 0 \] (9)

if and only if \(T_{ab}^b\) defines a null cone bi-preserving map. Furthermore, \(S_{ab} \in \mathcal{DP} \cup -\mathcal{DP}\).
Proof: If \((T_2 \times 2) T_{ab} = (T_1 \times 1) T_{ab} = fg_{ab}\) then

\[
S_{ac} S_b^c + S_{ac} F_b^c + F_{ac} S_b^c + F_{ac} F_b^c = fg_{ab}
\]
\[
S_{ac} S_b^c - S_{ac} F_b^c - F_{ac} S_b^c + F_{ac} F_b^c = fg_{ab}
\]

and by adding and subtracting these two equations the expressions (9) are obtained. Moreover, due to Lemmas 4.3 and 4.4 (b) we know that \(T_{ab} \in \mathcal{DP} \cup -\mathcal{DP}\). Then, from Lemma 2.1 it follows that \(S_{ab} \in \mathcal{DP} \cup -\mathcal{DP}\). 

Recall that, from elementary considerations, any eigenvector of a 2-form \(F_{ab}\) with non-zero eigenvalue must be null. If there is one such eigenvector, then there are exactly two of them with non-zero eigenvalues of opposite signs, and any other eigenvector must be spacelike. Thus, if there is a timelike eigenvector then all eigenvectors have zero eigenvalue. The possible number of null eigenvectors for a 2-form is: (i) if \(N = 2\), there are exactly two of them with nonzero eigenvalues of opposite sign; (ii) if \(N = 3\), there can be \(0, 1, 2\) null eigenvectors; if there are \(2\) then both of them have non-zero eigenvalues of opposite sign. (iii) for \(N > 3\) and even, say \(N = 2n (n \geq 2)\), there can be either \(1, 2, 4, \ldots, 2(n-1) = N-2\) null eigenvectors (the only odd number in the list is \(1\)). (iv) for \(N > 3\) and odd, say \(N = 2n + 1\), there can be either \(0, 1, 2, 3, \ldots, 2n-1 = N-2\) null eigenvectors (the only even numbers in the list are \(0, 2\)). In all cases, if there is only one null eigenvector its eigenvalue is zero. This case includes the null 2-forms.

**Lemma 4.6** If \(T_{ab}^b\) defines a null cone bi-preserving map then:

(a) every null eigenvector of its symmetric part \(T_{(ab)}\) is also a null eigenvector of its antisymmetric part \(T_{[ab]}\);

(b) every eigenvector with non-zero eigenvalue of \(T_{[ab]}\) is also a null eigenvector of \(T_{(ab)}\).

(c) In the singular case, \(T_{ab} = k_a n_b\), and \(k^b\) and \(n^b\) (which may coincide if \(T_{[ab]} = 0\)) are the null eigenvectors of both \(T_{ab}\) and \(T_{[ab]}\).

(d) every null eigenvector \(k^a\) with zero eigenvalue of \(T_{[ab]}\) is either a null eigenvector of \(T_{(ab)}\) or there is another independent null eigenvector \(n^b\) with vanishing eigenvalue of \(T_{[ab]}\) such that the timelike 2-plane generated by \(\{k, n\}\) contains two eigenvectors of both \(T_{(ab)}\) and \(T_{[ab]}\), one of them spacelike the other timelike, with opposite eigenvalues.

(e) every timelike eigenvector of its symmetric part \(T_{(ab)}\) is either an eigenvector also of \(T_{[ab]}\) or there are two null vectors which are simultaneously eigenvectors of both \(T_{(ab)}\) and \(T_{[ab]}\) with non-zero eigenvalues.

(f) if \(T_{[ab]}\) has a timelike eigenvector then there is a common timelike eigenvector for \(T_{[ab]}\) and \(T_{(ab)}\).

(g) Furthermore, if \(T_{[ab]} \neq 0\) then \(T_{(ab)} \notin SS \cup -SS\) (for \(N \neq 2\)).
Proof: Let us start with the null eigenvectors. From equations (9) we get for any null vector $k^a$

$$
(k^a S_{ac})(k^b S_{bc}) + (k^a F_{ac})(k^b F_{bc}) = 0,
$$

(10)

$$
(k^a S_{ac})(k^b F_{bc}) = 0.
$$

(11)

Thus, if $k^a$ is an eigenvector of $S_{ab}$ then by (10) $k^a F_{ac}$ is null and obviously orthogonal to
$k^c$ so that they must be proportional $k^a F_{ac} \propto k^c$. This proves (a). If $k^a$ is an eigenvector
of $F_{ab}$ then by (10) $k^a S_{ac}$ is null, and by (11) it is orthogonal to $k^a F_{ac} = \lambda k^c$. Hence,
if $\lambda \neq 0$ then $k^a S_{ac} \propto k^c$, which proves (b). The statement (c) for the singular case
follows immediately from Lemma 4.4 (b). It remains the case with $k^a F_{ac} = 0$. In this
case from (9) we get

$$
(k^a S_{ac}) F_{bc} = 0,
$$

(12)

$$
(k^a S_{ac}) S_{bc} = f k_b
$$

so that $k^a S_{ac}$ is also a null eigenvector of $F_{ab}$ with zero eigenvalue. If $k^a S_{ac} \equiv n^c$ and
$k^c$ are not colinear, that is $k \wedge n \neq 0$, then $n^b \pm \sqrt{f} k^b$ are eigenvectors of $S_{ab}$ with
eigenvalues $\pm \sqrt{f}$, respectively. From (c) we know that $f \neq 0$, so obviously one of these
vectors is timelike and the other spacelike, and both of them are eigenvectors with zero
eigenvalue of $F_{ab}$. This proves (d).

Concerning timelike eigenvectors, let $\bar{u}$ be unit and such that $S_{ab} u^b = \lambda u_a$. Con-
stracting relations (9) with $\bar{u}$ we get

$$
(u^a F_{ac}) F_{bc} = (f - \lambda^2) u_b,
$$

(12)

$$
S_b^c (u^a F_{ac}) = \lambda (u^a F_{ab}).
$$

Thus, either $p_c \equiv u^a F_{ab}$ vanishes or it is spacelike (for it is orthogonal to $u^c$). In the
latter case from (12) we have $(p \cdot p) = f - \lambda^2 < 0$ and the two null vectors $\bar{u} \pm \sqrt{\lambda} u^b$ are eigenvectors of $S_{ab}$ with
eigenvalues $\pm \sqrt{\lambda}$, respectively. From (c) we know that $f \neq 0$, so obviously one of these
vectors is timelike and the other spacelike, and both of them are eigenvectors with zero
eigenvalue of $F_{ab}$. This proves (e). To prove (f), let $\bar{u}$ be unit and such that $F_{ab} u^b = 0$. Then, from (9) it follows

$$
(u^a S_{ac}) S_{bc} = f u_b,
$$

(13)

$$
F_b^c (u^a S_{ac}) = 0.
$$

From Lemma 4.7 we know that $S_{ab} \in \mathcal{DP} \cup -\mathcal{DP}$, and then $v_c \equiv u^a S_{ac}$ is causal. In
fact, contracting the first relation in (13) with $u^b$ we deduce $(v \cdot v) = f$, so that $\bar{v}$
must be timelike, as otherwise $f$ would vanish which is impossible due to (c) above.
Then, using (13) is easy to check that the two vectors $\bar{v} \pm \sqrt{\lambda} u^b$ are eigenvectors of both
$S_{ab}$ and $F_{ab}$, with eigenvalues $\pm \sqrt{\lambda}$ respectively, one of them timelike and the other
spacelike.

Finally, to prove (g), if $\epsilon S_{ab}$ is in $\mathcal{SS}$ for $\epsilon = 1$ or $-1$, then by Proposition 3.2
$$(S \times S)_{ab} = \hbar^2 g_{ab}$$
so that from (9) we have $(F \times F)_{ab} = (f - \hbar^2) g_{ab}$. But then Corollary 4.4 and Lemma 3.2 imply that $F_{ab} = 0$ unless $N = 2$. 

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Thus, the maps preserving the null cone have a symmetric part which is in $\mathcal{DP}$ and either in $\mathcal{SS}$ (if $F_{ab} = 0$, see Theorem 6.1) or not (if $F_{ab} \neq 0$), in the second case algebraically determined by the antisymmetric part of the map and its null eigenvalues. Hence, in order to classify all these maps we only need to know the structure of tensors in $\mathcal{DP}_2$ (defined as the rank-2 tensors in $\mathcal{DP}$) in relation with $\mathcal{SS} \subset \mathcal{DP}_2$ and with the null eigenvectors. Curiously enough, this result is the analogue to Lemma 2.5 but for rank-2 symmetric tensors ($\mathcal{DP}_2$ and $\mathcal{SS}$ playing the role analogous to causal and null future-pointing vectors, respectively): we now show that all symmetric tensors in $\mathcal{DP}_2$ can be written as sums of terms in $\mathcal{SS}$. This means that the elements in $\mathcal{SS}$ can be used to build up $\mathcal{DP}_2$, and a fortiori $\mathcal{SE}_n$. Furthermore, each term of $\mathcal{SS}$ in the sum is related in a precise way to the null eigenvectors of the tensor in $\mathcal{DP}_2$. More precisely, we have:

**Theorem 4.1** In $N$ dimensions, any symmetric rank-2 tensor $S_{ab} \in \mathcal{DP}_2$ can be written

$$S_{ab} = \sum_{p=1}^{N} T_{ab}\{\Omega_{[p]}\}$$

where $T_{ab}\{\Omega_{[p]}\} \in \mathcal{SS}$ are the superenergy tensors of simple $p$-forms $\Omega_{[p]}$, $p = 1, \ldots, N$ such that for $p > 1$ they have the structure $\Omega_{[p]} = k^1 \wedge \ldots \wedge k^p$ where $k^1, \ldots, k^p$ are appropriate null 1-forms. The number of tensors in the sum (14) and the structure of the $\Omega_{[p]}$ depend on the particular $S_{ab}$ as follows: if $S_{ab}$ has $N - m \geq 1$ null eigenvectors $k^1, \ldots, k^{N-m}$ then at least $T_{ab}\{\Omega_{[N-m]}\}$, with $\Omega_{[N-m]} = k^1 \wedge \ldots \wedge k^{N-m}$, must appear in the sum, and possibly terms with $p > N - m$. If it has no null eigenvectors, then at least $T_{ab}\{\Omega_{[1]}\}$ appears in the sum, and possibly terms with $p > 1$, and $\Omega_{[1]}$ is the timelike eigenvector of $S_{ab}$.

**Remark:** As already stated, the superenergy tensor of the dual of a $p$-form ($p < N$) is identical with that of the $p$-form itself. Thus, in the sum (14) there are two superenergy tensors of 1-forms, namely $T_{ab}\{\Omega_{[1]}\}$ and $T_{ab}\{\Omega^*_{[1]}\} = T_{ab}\{\Omega_{[1]}\}$, but the first one is the superenergy tensor of a causal 1-form and the second of a spacelike 1-form. This is an essential difference. Similar remarks apply to the 2-forms $\Omega_{[2]}$ and $\Omega_{[N-2]}$, and so on. The choice of simple $p$-forms taken in Theorem 4.1 is such that $(-1)^{p-1}(\Omega_{[p]} \cdot \Omega_{[p]}) > 0$ for $p = 2, \ldots, N - 1$, and $\Omega_{[1]}$ is causal.

**Proof:** Recall that for a symmetric tensor $S_{ab}$ any two eigenvectors with different eigenvalues must be orthogonal. Then, any two linearly independent null eigenvectors of a symmetric tensor must have the same eigenvalue.

We divide up in cases depending on the number of null eigenvectors of $S_{ab}$. Suppose that $S_{ab}$ has $N$ linearly independent null eigenvectors. All their eigenvalues must be equal to some constant, say $\alpha$, and $\alpha \geq 0$ as $S_{ab} \in \mathcal{DP}$. The $N$ null eigenvectors
span all tangent vectors so we get $S_{ab} = \alpha g_{ab} = T_{ab}\{\sqrt{2\alpha}\eta|N\}$ where $\eta$ is the volume $N$-form.

Suppose now that the Theorem is proven for the case with $(N - m) + 1$ linearly independent null eigenvectors and assume that $S_a^b$ has $N - m \geq 2$ linearly independent null eigenvectors, $\vec{k}^{(1)}, \ldots, \vec{k}^{(N - m)}$ say, all with eigenvalue $\beta$. With $(\vec{k}^1 \wedge \ldots \wedge \vec{k}^{N - m}) = r^1 \wedge \ldots \wedge r^m$, all $\vec{r}^{(i)}$ must be spacelike as they are orthogonal to all $\vec{k}^{(i)}$ and $N - m \geq 2$ (a vector which is orthogonal to two null vectors must be spacelike). We have $S_a^b k_a^{(i)} r_b^{(i)} = \beta k^{(i)} \cdot r^{(i)} = 0$ so $S_a^b r_b^{(i)}$ is orthogonal to all $\vec{k}^{(i)}$ and hence $S_a^b r_b^{(i)} \in \text{Span}\{\vec{r}^{(1)}, \ldots, \vec{r}^{(m)}\} \equiv V_m$. Therefore $S_a^b$ is a symmetric map on $V_m$ which is a Euclidean space. There are then $m$ orthonormal $(y^{(1)} \cdot y^{(2)} = -\delta^{ij})$ eigenvectors $\vec{y}^{(1)}, \ldots, \vec{y}^{(m)}$ to $S_a^b$ in $V_m$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. We can assume $\lambda_i \geq \lambda_j$ for $i = 2, \ldots, m$. Now, by Proposition 3.3 and Corollary 3.3, $T_{ab}\{\Omega_{[N - m]}\}$, where $\Omega_{[N - m]} = \vec{k}^1 \wedge \ldots \wedge \vec{k}^{N - m}$, has also the null eigenvectors $\vec{k}^{(1)}, \ldots, \vec{k}^{(N - m)}$ with some eigenvalue $\gamma > 0$ and it has the spacelike eigenvectors $\vec{y}^{(1)}, \ldots, \vec{y}^{(m)}$ with eigenvalues $-\gamma$. Define $\tau_{ab} = S_{ab} + (\lambda_1 - \beta)T_{ab}\{\Omega_{[N - m]}\}/2\gamma$. Then $\tau_a b \vec{k}^{(j)} = (\beta + \lambda_1)k_a^{(j)}/2$ and $\tau_a b \gamma^{(1)} = (\beta + \lambda_1)\gamma^{(1)}/2$. Take $u_a \in V_m$ with $u_a u^a = 1$ and let $\vec{k} = a_u + \gamma^{(1)}$. Then $\vec{k} \notin V_m$, $\vec{k} \vec{k}^a = 0$ and $\tau_a b \vec{k} = (\beta + \lambda_1)k_a/2$. Hence $\tau_{ab}$ has $(N - m) + 1$ linearly independent null eigenvectors $\vec{k}^{(1)}, \ldots, \vec{k}^{(N - m)}, \vec{k}$. Note that $S_a^b (u_a \pm y_b^{(i)}) = \beta u_a \pm \lambda_i y_b^{(i)}$ are future-pointing since $S_a^b \in \mathbb{D} \mathbb{P}$, and therefore $\beta \geq |\lambda_i|$ for all $i$. To show that $\tau_{ab} \in \mathbb{D} \mathbb{P}$, then use that an arbitrary future-pointing null vector $N_a$ can be written $N_a = K_a + Y_a$ where $K_a = a_1 k_a^{(1)} + \ldots + a_{N - m} k_a^{(N - m)}$ and $Y_a = b_1 y_a^{(1)} + \ldots + b_m y_a^{(m)}$, and where $K_a K^a = \gamma^2 \gamma^2 = 0$. Then $\tau_a b N_a = (\beta + \lambda_1)K_a/2 + \lambda_1 b_1 y_a^{(1)} + \ldots + \lambda_m b_m y_a^{(m)} + (\beta - \lambda_1)Y_a/2$ which has the squared length $(\tau_a b k_a)(\tau_a b k_a) = (\lambda_1 - \lambda_2)(\beta + \lambda_2)b_2^2 + \ldots + (\lambda_1 - \lambda_m)(\beta + \lambda_m)b_m^2 \geq 0$. Thus, $S_{ab} = (\beta - \lambda_1)T_{ab}\{\Omega_{[N - m]}\}/2\gamma + \tau_{ab} = T_{ab}\{\sqrt{(\beta - \lambda_1)}/2\gamma\Omega_{[N - m]}\} + \tau_{ab}$ has the form required by the induction hypothesis so the statement of the theorem holds for the cases with at least 2 linearly independent null eigenvectors.

Next consider the case with precisely one null eigenvector $\vec{k}$, with eigenvalue $\beta$. Take a set $\{\vec{r}^{(1)}, \ldots, \vec{r}^{(N - 2)}\}$ of linearly independent spacelike vectors, all orthogonal to $\vec{k}$. Again we have $S_a^b k_a \vec{r}^{(i)} = 0$ so $S_a^b \vec{r}^{(i)}$ is orthogonal to $\vec{k}$. For those $i$ that $S_a^b \vec{r}^{(i)} = \mu_i k_a$ for some real number $\mu_i$, define $\vec{r}^{(i)} = \vec{r}^{(i)} - \mu_i k_a/\beta$ so $S_a^b \vec{r}^{(i)} = 0$, $\vec{r}^{(i)}$ is spacelike, and $\vec{r}^{(i)} k_a = 0$. For those $i$ that $S_a^b \vec{r}^{(i)}$ is already spacelike, let $\vec{r}^{(i)} = \vec{r}^{(i)}$. Let $\vec{n}$ be the other future-pointing null vector orthogonal to $V_{(N - 2)} = \text{Span}\{\vec{r}^{(1)}, \ldots, \vec{r}^{(N - 2)}\}$ and normalised by $k_a n_a = 1$. As $S_a^b r_b^{(i)} \in V_{(N - 2)}$ we have $n_a (S_a^b r_b^{(i)}) = 0$ which means that $S_a^b n_b$ is orthogonal to all $\vec{r}^{(i)}$. Thus, $S_a^b n_b = \beta n_a + \gamma k_a$ with $\gamma > 0$. Define $\tau_{ab} = S_{ab} - \gamma k_a k_b$. Then $\tau_{ab} k_b = \beta k_a$ and $\tau_{ab} n_b = \beta n_a$ so
\(\vec{k}\) and \(\vec{n}\) are two linearly independent null eigenvectors of \(\tau_{ab}\). To show \(\tau_{ab} \in \mathcal{DP}\) we use as above that in \(V_{(N-2)}\) there is an orthonormal basis of eigenvectors \(\{\vec{y}^{(1)}, \ldots, \vec{y}^{(N-2)}\}\) where \(\tau_{ab} y_{b}^{(i)} = \lambda_{i} y_{a}^{(i)}\). For any \(c > 0\), \(\ell_{a} = c k_{a} + n_{a}/2c + y_{a}^{(i)}\) is future-pointing and null, and therefore \(0 \leq (S_{ab}^{\ell_{b}})(S_{a}^{\ell_{b}c}) = \beta^{2} + \beta \gamma / 2c - \lambda_{i}^{2}\). As \(c\) can be taken arbitrary large we get \(\beta^{2} \geq \lambda_{i}^{2}\). An arbitrary future-pointing null vector can be written \(N_{a} = a_{1} k_{a} + a_{2} n_{a} + b_{1} y_{a}^{(1)} + \ldots + b_{N-2} y_{a}^{(N-2)}\) where \(2a_{1} a_{2} - b_{1}^{2} - \ldots - b_{N-2}^{2} = 0\). We find \((\tau_{ab} N_{b})(\tau_{ac} N_{c}) = b_{1}^{2}(\beta^{2} - \lambda_{i}^{2}) + \ldots + b_{N-2}^{2}(\beta^{2} - \lambda_{N-2}^{2}) \geq 0\) and conclude that \(\tau_{ab} \in \mathcal{DP}\). Thus, \(S_{ab} = \gamma k_{a} k_{b} + \tau_{ab} = T_{ab}(\sqrt{\gamma k_{[1]}}) + \tau_{ab}\) has the required form which proves the case with one null eigenvector.

Finally we consider the case with no null eigenvector. If there exist null vectors \(\vec{k}\) and \(\vec{n}\) such that \(S_{ab} k_{a} = n_{a}\) then, as \(S_{ab} \in \mathcal{DP}\), \(S_{ab} n_{a} = \beta k_{b}\) for some \(\beta > 0\). Then \(\vec{n} + \sqrt{\beta} \vec{k}\) is a timelike eigenvector which normalised we denote by \(\vec{u}\). Otherwise, if all null vectors are mapped on timelike vectors then again \(S_{ab}\) has a (unit) timelike eigenvector \(\vec{u}\). Thus we have a unit timelike eigenvector \(\vec{u}\) with eigenvalue \(\lambda_{0}\), and on \(\{\vec{u}\}^{\perp}\) there is an ON-basis \(\{\vec{y}^{(1)}, \ldots, \vec{y}^{(N-1)}\}\) of eigenvectors with eigenvalues \(\lambda_{1}, \ldots, \lambda_{N-1}\). As \(\vec{u} + \vec{y}^{(i)}\) is future-pointing and null \(S_{ab} y_{b}^{(i)}\) must be future-pointing which implies \(\lambda_{0} \geq |\lambda_{i}|\) for all \(i = 1, \ldots, N-1\). Assume \(\lambda_{1} \geq \lambda_{i}\) for \(i = 2, \ldots, N-1\) and define \(\tau_{ab} = S_{ab} - (\lambda_{0} - \lambda_{1})(u_{a} u_{b} - \beta g_{a b})\). Then \(\tau_{a} y_{b}^{(i)} \pm y_{b}^{(1)} = \frac{1}{2}(\lambda_{0} + \lambda_{1})(u_{a} \pm y_{a}^{(1)})\) so \(\tau_{ab}\) has two null eigenvectors. To show that \(\tau_{ab} \in \mathcal{DP}\), let \(c_{1}^{2} + \ldots + c_{N-1}^{2} = 1\); then \(N_{a}(c_{1}, \ldots, c_{N-1}) = (u_{a} + c_{1} y_{a}^{(1)} + \ldots + c_{N-1} y_{a}^{(N-1)})\) is proportional to an arbitrary future-pointing null vector. One finds \((\tau_{ab} N_{b})(\tau_{ac} N_{c}) = \sum c_{i}^{2}(\lambda_{1} - \lambda_{i})(\lambda_{0} + \lambda_{i}) \geq 0\), so \(S_{ab} = T_{ab}(\sqrt{\lambda_{0} - \lambda_{1} u_{[1]}}) + \tau_{ab}\) has the right properties and this finishes the proof. ■

Remarks: Recall that by Lemma 2.5 a future-pointing causal vector can be written as a sum of two future-pointing null vectors in infinitely many ways. In the same manner, a symmetric \(S_{ab} \in \mathcal{DP}_{2}\) can be expressed as a sum of \(N\) elements of \(SS\) in many ways. As an example, let \(\{e_{a}\}\) be an orthonormal basis. Then, by \[\text{(1)}\], one easily find relations such as

\[
T_{ab}\{e_{0}\} + T_{ab}\{e_{1}\} = T_{ab}\{e_{0} \wedge e_{2}\} + T_{ab}\{e_{1} \wedge e_{2}\};
\]

\[
\alpha T_{ab}\{e_{1}\} + \beta T_{ab}\{e_{2}\} = \frac{1}{2} g_{ab} + (\alpha - \beta) T_{ab}\{e_{1}\} + \beta T_{ab}\{e_{1} \wedge e_{2}\}. \tag{15}
\]

In Theorem 4.4 however, we construct the representation of \(S_{ab} \in \mathcal{DP}_{2}\) in a canonical way in which the simple \(p\)-forms \(\Omega_{[p]}\) are constructed from the null eigenvectors of \(S_{ab}\).

5 Non-symmetric null-cone preserving maps

We are now prepared to present the classification of the general conformally non-involuntary null-cone preserving maps, which follows directly from the Theorem 4.4 and
the Lemma 4.6. Given that the results are elementary but the number of different cases is increasing with the dimension $N$, we will restrict ourselves to the low-dimension cases in full, but this will show the way one has to follow as well as the general ideas which serve for a general $N$. As the singular case has been already solved, in this section we only deal with the non-singular conformally non-involutory maps, so that $T_{[ab]} \neq 0$. The conformally involutory ones are left for the next section.

**Case** $N = 2$. The simplest case is a 2-dimensional Lorentzian manifold. In this case there are only two independent null directions, say $\ell$ and $k$, and we can always write $T_{[ab]} = (\ell \wedge k)_{ab} = \mu \eta_{ab}$. Both $\vec{\ell}$ and $\vec{k}$ are null eigenvectors of $T_{[ab]}$ with non-zero eigenvalue, and then due to Lemma 4.6 (b), they are also null eigenvectors of $T_{(ab)}$. Using then Theorem 4.1 the only possibility is that $T_{(ab)} = \alpha g_{ab}$. Thus, we have

**Corollary 5.1** In $N = 2$, the maps proportional to non-involutory Lorentz transformations are given by $T_{ab} = \alpha g_{ab} + \mu \eta_{ab}$ with arbitrary $\alpha$ and $\mu$ such that $\alpha^2 - \mu^2 \neq 0$. They are proper (resp. improper) if $\alpha^2 - \mu^2 > 0$ (resp. $< 0$), and orthochronus (resp. time-reversal) if $\alpha > |\mu|$ (resp. $\alpha < -|\mu|$).

Notice that in this particular case, an arbitrary 2-form $\mu \eta_{ab}$ defines an improper null cone bi-preserving map. This is the only possibility in which a 2-form can preserve the null cone, and it appears as an exceptional case as follows from Corollary 4.1 and Lemmas 4.2 and 4.3.

Before we proceed with the non-trivial cases $N > 2$, we need some simple lemmas. From Corollary 3.3 we know that if $T_{ab} \in SS \setminus NS$ then the tangent space can be decomposed as $T_x(V_N) = E^+ \oplus E^-$ where $E^+$ is $p$-dimensional, $E^-$ is $(N-p)$-dimensional, and both $E^\pm$ are eigensubspaces of $T_{ab}$ with opposite eigenvalues.

**Lemma 5.1** If $F_{[2]}$ is a simple 2-form and $T_{ab} \in SS \setminus NS$, then $(T \times F)_{(ab)} = 0$ if and only if $F_{ab}$ lies entirely in either $\Lambda_2(E^+)$ or $\Lambda_2(E^-)$.

**Proof:** $F = \theta_1 \wedge \theta_2$ for some 1-forms $\theta_1$ and $\theta_2$. Obviously $\vec{\theta}_1 = \vec{\theta}_1^+ + \vec{\theta}_1^-$, $\vec{\theta}_2 = \vec{\theta}_2^+ + \vec{\theta}_2^-$ with $\vec{\theta}_1^+, \vec{\theta}_2^+ \in E^+$ and $\vec{\theta}_1^-, \vec{\theta}_2^- \in E^-$. A straightforward computation gives then

$$
(T \times F)_{(ab)} = 2\lambda \left[ (\theta_1^- \otimes \theta_2^+ + \theta_1^+ \otimes \theta_2^-)_{(ab)} - (\theta_1^+ \otimes \theta_2^- + \theta_1^- \otimes \theta_2^+)(ab) \right]
$$

where $\lambda$ is the eigenvalue for $E^+$. Then, the condition $(T \times F)_{(ab)} = 0$ holds if and only if either $\vec{\theta}_1^+ = \vec{\theta}_2^+ = 0$ or $\vec{\theta}_1^- = \vec{\theta}_2^- = 0$.

Similarly, from Corollary 3.3 if $T_{ab} \{\Omega_{[p]}\} \in NS$, there is an $(N-1)$-dimensional subspace $E^0$ of eigenvectors with zero eigenvalue generated by $\{\vec{k}, \vec{\omega}_2, \ldots, \vec{\omega}_{N-1}\}$, where $\vec{k}$ is the canonical null direction of the null $\Omega_{[p]}$.
Lemma 5.2 If $F_{[2]}$ is a simple 2-form and $T_{ab} \in NS$, then $(T \times F)_{(ab)} = 0 \iff (T \times F)_{ab} = 0 \iff F_{ab}$ lies entirely in $\Lambda^2(E^0)$.

Proof: Set $F = \theta_1 \wedge \theta_2$ as before and choose $\vec{n}$ null, independent of $k$, and orthogonal to all $\{\vec{\omega}_2, \ldots, \vec{\omega}_{N-1}\}$. Obviously $\vec{\theta}_1 = \vec{\theta}_1^0 + C_1 \vec{n}$, $\vec{\theta}_2 = \vec{\theta}_2^0 + C_2 \vec{n}$ with $\vec{\theta}_1^0, \vec{\theta}_2^0 \in E^0$. As $T_{ab} = f_{ka}k_b$, we have

$$(T \times F)_{ab} = f(k \cdot n)k_a(C_1 \theta_2^+ - C_2 \theta_1^+)$$

and given that $\theta_1$ and $\theta_2$ are linearly independent, the vanishing of this (or of its symmetric part) gives $C_1 = C_2 = 0$, and conversely.

The notation of Lemma 4.6 for $T_{(ab)} = S_{ab}$ and $T_{[ab]} = F_{ab}$ is used in the remaining of this section.

Case $N = 3$. There are three possibilities, as $F_{ab}$ can have 0, 1, or 2 null eigenvectors.

(a) If $F_{ab}$ has no null eigenvector, then it is proportional to the dual of a unit timelike vector $\vec{u}$, i.e. $F_{[2]} = \mu u^*$. Due to Lemma 4.6 (a), $S_{ab}$ has no null eigenvectors, and due to Lemma 4.6 (f), $\vec{u}$ is timelike eigenvector also of $S_{ab}$. Thus, Theorem 4.1 allows us to write

$$T_{ab} = \beta T_{ab} \{u_{[1]}\} + \gamma T_{ab} \{\Omega_{[2]}\} + \alpha g_{ab} + F_{ab}.$$ 

Using Lemma 5.1 one has $(S \times F)_{(ab)} = \gamma \left( T_{ab} \{\Omega_{[2]}\} \times F \right)_{(ab)}$ and here the term in brackets is non-vanishing due again to Lemma 5.1. Thus, the second condition (9) implies $\gamma = 0$. With this, it is easily checked that the first condition in (9) leads to $\mu^2 = 2\alpha \beta$. Thus, we obtain

$$T_{ab} = \beta T_{ab} \{u_{[1]}\} + \alpha g_{ab} \pm \sqrt{2\alpha \beta} \left( u^* \right)_{[2]}.$$ 

These maps are proper and orthochronous if $\alpha > 0$, and improper and time-reversal if $\alpha < 0$.

(b) If $F_{ab}$ has one null eigenvector $\vec{k}$, then $F_{[2]}$ is null and can be written $F = \mu k \wedge p$ with $(k \cdot p) = 0$. Lemma 4.6 (d) implies that $\vec{k}$ is also a null eigenvector of $S_{ab}$, and this is unique for $S_{ab}$ due to Lemma 4.6 (a). So, again Theorem 4.1 tells us that

$$T_{ab} = \beta T_{ab} \{k_{[1]}\} + \gamma T_{ab} \{\Omega_{[2]}\} + \alpha g_{ab} + F_{ab}.$$ 

Analogously to case (a) above, Lemmas 5.1 and 5.2 lead to $\gamma = 0$, and the first relation in (9) gives again $\mu^2 = 2\alpha \beta$. Hence

$$T_{ab} = \beta T_{ab} \{k_{[1]}\} + \alpha g_{ab} \pm \sqrt{2\alpha \beta} \left( k^* \right)_{[2]}.$$ 

(17)
Notice that this can be considered a limit case of (16) when \( \vec{u} \) becomes null.

(c) If \( F_{ab} \) has two independent null eigenvectors \( \vec{k} \) and \( \vec{n} \), then they necessarily have non-zero eigenvalues, and by Lemma 4.6(b) they are also eigenvectors of \( S_{ab} \), which cannot have more null eigenvectors due to Lemma 4.6(a). Thus, by Theorem 4.1

\[
T_{ab} = \beta T_{ab} \left\{ (k \wedge n)_{[2]} \right\} + \alpha g_{ab} + \mu (k \wedge n)_{ab}.
\]

The computation of (3) leads now simply to \( \mu^2 = 2 \alpha \beta \). In summary,

\[
T_{ab} = \beta T_{ab} \left\{ (k \wedge n)_{[2]} \right\} + \alpha g_{ab} \pm \sqrt{2 \alpha \beta} (k \wedge n)_{ab}.
\]

Observe that this case can be rewritten as

\[
T_{ab} = \beta T_{ab} \left\{ p_{[1]} \right\} + \alpha g_{ab} \pm \sqrt{2 \alpha \beta} \left( p_{*} \right)_{ab}.
\] (18)

where \( p_{*} \) is spacelike and defined by \( p \equiv \star (k \wedge n) \). Hence, the combination of (16-18) proves the following

**Corollary 5.2** In \( N = 3 \), the maps proportional to non-involutory Lorentz transformations are given by

\[
T_{ab} = \beta T_{ab} \left\{ \Sigma_{[1]} \right\} + \alpha g_{ab} \pm \sqrt{2 \alpha \beta} \left( \Sigma_{*} \right)_{ab}
\]

where \( \alpha \) and \( \beta \) are arbitrary with \( \alpha \beta > 0 \) and \( \Sigma_{[1]} \) is any 1-form. These maps leave none, one or two null directions invariant if \( \Sigma_{[1]} \) is time-, light-, or space-like, respectively.

---

**Case** \( N = 4 \). Now there are just two possibilities: either \( F_{ab} \) has one or two null eigenvectors.

(a) If \( F_{ab} \) has one null eigenvector \( \vec{k} \), then \( F_{[2]} \) is null, \( F = \mu k \wedge p \) with \( (k \cdot p) = 0 \). Due to Lemma 4.6(d) and (a) this is also the unique null eigenvector of \( S_{ab} \) so that from Theorem 4.1

\[
T_{ab} = \beta T_{ab} \left\{ k_{[1]} \right\} + \delta T_{ab} \left\{ \Omega_{[2]} \right\} + \gamma T_{ab} \left\{ \Omega_{[3]} \right\} + \alpha g_{ab} + F_{ab}.
\]

with \( \Omega_{[2]} \) and \( \Omega_{[3]} \) having the form \( k \wedge n \) and \( k \wedge n \wedge \ell \), respectively, for null \( n \) and \( \ell \). Lemmas 5.1 and 5.2 imply that the second equation in (3) reads

\[
\delta \left( T \left\{ \Omega_{[2]} \right\} \times F \right)_{(ab)} + \gamma \left( T \left\{ \Omega_{[3]} \right\} \times F \right)_{(ab)} = 0
\]

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which, as $\vec{p}$ cannot be linear combination of $\vec{k}$ and $\vec{n}$, becomes

$$-(\delta \lambda + \gamma \lambda)k_{(a}p_{b)} + \gamma k_{(a}T_{b)c} \left\{ \Omega_{[3]} \right\} p^c = 0$$

where $\lambda$ and $\tilde{\lambda}$ are positive as they are proportional to $-(\Omega_{[2]} \cdot \Omega_{[2]})$ and $(\Omega_{[3]} \cdot \Omega_{[3]})$, respectively. The above expression can only be satisfied with non-negative $\delta \gamma \geq 0$ if $\delta = 0$ and $T_{bc} \left\{ \Omega_{[3]} \right\} p^c = \lambda p_b$. This also implies that $p \wedge (k \wedge n \wedge \ell) = 0$ and we can write

$$T_{ab} = \beta T_{ab} \left\{ k_{[1]} \right\} + \gamma T_{ab} \left\{ (k \wedge n \wedge \ell)_{[3]} \right\} + \alpha g_{ab} + F_{ab}.$$  

The remaining condition in (9) implies in particular that $\alpha \gamma = 0$, so that two possibilities arise (assuming that $\vec{p}$ is unit): $\alpha = 0$ and then $\mu^2 = \beta \gamma$; or $\gamma = 0$ and $\mu^2 = 2 \alpha \beta$.

In summary, by setting $q \equiv \ast (k \wedge n \wedge p)$, we have

$$T_{ab} = \beta T_{ab} \left\{ k_{[1]} \right\} + \gamma T_{ab} \left\{ (k \wedge n \wedge \ell)_{[3]} \right\} + \alpha g_{ab} + F_{ab}.$$  

Observe that in both cases one can replace $T_{ab} \left\{ k_{[1]} \right\}$ by $T_{ab} \left\{ (k \wedge p)_{[2]} \right\}$, because $F_{[2]}$ is null. Furthermore, the above expressions (19-20) are valid for arbitrary $N$ so they are proportional to Lorentz transformations in any $V_N$ (where $\vec{q}$ is just any spacelike vector orthogonal to both $\vec{k}$ and $\vec{p}$).

(b) If $F_{ab}$ has two null eigenvectors $\vec{k}$ and $\vec{n}$, then

$$F_{[2]} = \mu_1 (k \wedge n)_{[2]} + \mu_2 (k \wedge n)_{[2]}$$

with $\mu_1^2 + \mu_2^2 \neq 0$. If $\mu_1 \neq 0$, from Lemma 4.4 (a), (b) and (d), $\vec{k}$ and $\vec{n}$ are the two null eigenvectors of $S_{ab}$ and we can write in principle, from Theorem 4.1,

$$T_{ab} = \gamma T_{ab} \left\{ \Omega_{[2]} \right\} + \beta T_{ab} \left\{ \Omega_{[3]} \right\} + \alpha g_{ab} + F_{ab}.$$

with $\Omega_{[2]}$ and $\Omega_{[3]}$ having the form $k \wedge n$ and $k \wedge n \wedge \ell$, respectively, for null $\ell$. Lemmas 5.1 and 5.2 imply that the second equation in (4) leads to $\beta \mu_2 = 0$. Solving these two possibilities we arrive at

$$T_{ab} = \gamma T_{ab} \left\{ (k \wedge n)_{[2]} \right\} + \alpha g_{ab} \pm \sqrt{2 \alpha \gamma} \left[ \cos \theta (k \wedge n) + \sin \theta \ast (k \wedge n) \right]_{ab},$$  

$$T_{ab} = 2 \alpha T_{ab} \left\{ (k \wedge n)_{[2]} \right\} + \beta T_{ab} \left\{ (k \wedge n \wedge \ell)_{[3]} \right\} + \alpha g_{ab} \pm \sqrt{2 \alpha (\beta + 2 \alpha) (k \wedge n)_{ab}}$$

where $\theta$ is arbitrary.
If $\mu_1 = 0$, there also arises the possibility given by Lemma $4.6$ (d), (f) that $S_{ab}$ has a timelike eigenvector $\vec{u}$ and a spacelike one $\vec{p}$ with $k \wedge n = u \wedge p$, such that from Theorem $4.1$ one has in principle

$$T_{ab} = \beta T_{ab} \{u[1]\} + \delta T_{ab} \{\Omega[2]\} + \gamma T_{ab} \{\Omega[3]\} + \alpha g_{ab} + F_{ab}.$$ 

with $\Omega[2] \wedge p \neq 0$ and $\Omega[3] \equiv p^\ast$. However, $\vec{u}$ and $\vec{p}$ have opposite eigenvalues due to Lemma $4.6$ (d), from where we get $\alpha = 0$. Then, from Lemma $5.1$ and the second equation in (9) it follows that $\delta = 0$ too. Finally, taking $\vec{u}$ and $\vec{p}$ unit, the first relation in (9) leads to $\mu_2^2 = 2\beta\gamma$ so that

$$T_{ab} = \beta T_{ab} \{u[1]\} + \gamma T_{ab} \{p[1]\} \pm \sqrt{2\beta\gamma}(u \wedge p)_{ab}. \quad (23)$$

**Corollary 5.3** In $N = 4$, the maps proportional to non-involutory Lorentz transformations are given by $(9), (23)$. 

These results were obtained for the restricted case in [11, 12], and in general in [26] using spinors. The case given by (22) may seem not included in the solution presented in [26], but this is apparent. In fact, one can rewrite (22) by using the identity (15) as

$$2\alpha T_{ab} \{p[1]\} + (2\alpha + \beta) T_{ab} \{q[1]\} \pm \sqrt{2\alpha(\beta + 2\alpha)(k \wedge n)_{ab}}$$

where $q \equiv *(k \wedge n \wedge \ell)$ and $p \wedge q \equiv *(k \wedge n)$, and this last form is certainly included in the cases given in [26].

The number of possibilities and the complexity of the equations increase with $N$, but the reasonings and techniques are always simple and the same: application of Lemmas $4.6$, 5.1 and 5.2 and Theorem $4.1$ to the equations (3). The details will be omitted here but, as an illustrative example, we present the general solution for arbitrary odd dimension $N = 2n + 1$.

**Case $N = 2n + 1$, $(n \geq 2)$.** Let $\{\vec{e}_0, \ldots, \vec{e}_{2n}\}$ be an orthonormal basis. Then, the maps proportional to non-involutory Lorentz transformations are in one of the following cases:

1. $$T_{ab} = \beta_1 T_{ab} \{v[1]\} + \sum_{j=2}^{n} \beta_j T_{ab} \{(e_{2j-1} \wedge \ldots \wedge e_{2n})_{[2j-1]}\} + \alpha g_{ab} + \pm \mu_1 (v \wedge e_2)_{ab} \pm \sum_{j=2}^{n} \mu_j (e_{2j-1} \wedge e_{2j})_{ab}$$

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where $\alpha, \beta_1, \ldots, \beta_n$ are arbitrary, the $\mu_i$ are given, for all $i = 1, \ldots, n$ by

$$
\mu_i^2 = (\beta_1 + \ldots + \beta_i)(\beta_{i+1} + \ldots + \beta_n + 2\alpha),
$$

and $v$ is a causal 1-form equal to $e_0$ if $T_{ab}$ leaves no null direction invariant, and to $e_0 + e_1$ if it leaves exactly one null direction ($\vec{v}$) invariant.

$$
(2) \quad T_{ab} = \beta_1 T_{ab} \left\{ (e_0 \wedge e_1)_{[2]} \right\} + \sum_{j=2}^{n} \beta_j T_{ab} \left\{ (e_{2j-1} \wedge \ldots \wedge e_{2n})_{[2j-1]} \right\} + \alpha g_{ab} +
\pm \mu_1 (e_0 \wedge e_1)_{ab} \pm \sum_{j=2}^{n} \mu_j (e_{2j-1} \wedge e_{2j})_{ab}
$$

where now

$$
\mu_2 = \beta_1(\beta_2 + \ldots + \beta_n + 2\alpha),
$$

and for all $j = 2, \ldots, n$,

$$
\mu_j = (\beta_2 + \ldots + \beta_j)(\beta_{j+1} + \ldots + \beta_n + 2\alpha - \beta_1).
$$

Generically, this leaves 2 null directions invariant, 3 if $\beta_1 = 0$, and in general $2j + 1$ null directions invariant if $\beta_1 = \ldots = \beta_j = 0$.

$$
(3) \quad \text{Those cases which effectively reduce to low-dimensional cases, such as for instance}
\quad T_{ab} = 2\alpha T_{ab} \left\{ (e_0 \wedge e_1)_{[2]} \right\} + \beta T_{ab} \left\{ (e_2)_{[2_2]} \right\} + \alpha g_{ab} \pm \sqrt{2\alpha(\beta + 2\alpha)}(e_0 \wedge e_1)_{ab}
$$

which is the analogue of (22) and has two invariant null directions. And similarly for the appropriate generalizations of (19-20) and (23).

6 Symmetric null-cone preserving maps and algebraic Rainich conditions

We are now going to prove an important result: the converses of Proposition 3.2, Lemma 4.4 and Corollary 4.2 hold. One can also intrinsically determine the rank $p$ of the $p$-form generating the tensor in $SS$. More precisely

**Theorem 6.1** In $N$ dimensions, if $T_{ab}$ is symmetric and $(T \times T)_{ab} = fg_{ab}$ then:

(a) $f = 0 \implies T_{ab} \in NS \cup -NS$ and $T_{ab} = \beta k_a k_b$ for a null $k_b$.

(b) $f \neq 0 \implies T_{ab} \in SS \cup -SS$ and, for some $p \in \{1, \ldots, N\}$, $\epsilon T_{a}^{\alpha} = (2p - N)\sqrt{f}$ with $\epsilon = \pm 1$. Moreover, $\epsilon T_{a}^{\alpha} = (2p - N)\sqrt{f} \iff \epsilon T_{ab} = T_{ab} \{\Omega[p]\}$ where $T_{ab} \{\Omega[p]\}$ is the superenergy tensor of a simple $p$-form $\Omega[p]$ of the type used in Theorem 4.4.
Proof: By Corollary 4.1, \( T_{ab} \) symmetric and \( (T \times T)_{ab} = f_{gab} \) implies that \( f \geq 0 \). Thus \( \sqrt{f} \) is well defined as the positive square root of \( f \). Lemmas 4.3 and 4.4 give then \( \epsilon T_{ab} \in \mathcal{D} \mathcal{P} \). Then by Theorem 4.3 and using \( T_{ab}\{\Omega[N]\} = \alpha g_{ab} \), \( \epsilon T_{ab} = T_{ab}\{\Omega[1]\} + \ldots + T_{ab}\{\Omega[N-1]\} + \alpha g_{ab} \). We have to verify that only one term or proportional terms can remain if \( (T \times T)_{ab} = f_{gab} \). We have \( (T \times T)_{ab} = (\alpha g_{ca} + T_{ca}\{\Omega[1]\} + \ldots + T_{c(a}\{\Omega[N-1]\})(\alpha g_{bc} + T_{b}\{\Omega[1]\} + \ldots + T_{b}\{\Omega[N-1]\}) \). This expression contains four types of terms: \( \alpha^2 g_{ca} g_{bc} \), \( \alpha g_{ca} T_{bc}\{\Omega[1]\} \), \( T_{ca}\{\Omega[1]\} T_{bc}\{\Omega[1]\} \) and \( T_{ca}\{\Omega[1]\} T_{bc}\{\Omega[1]\} \) with \( p \neq q \). By Theorem 5.1 and Lemma 2.3 every term is in \( \mathcal{D} \mathcal{P} \). If \( (T \times T)_{ab} = f_{gab} \) then \( (T \times T)_{ab} k^a k^b = 0 \) for every null vector \( k^a \). Since each term is non-negative this means that every term must be zero when contracted with \( k^a k^b \). By Proposition 3.2 this is satisfied by \( \alpha^2 g_{ca} g_{bc} \) and \( T_{ca}\{\Omega[1]\} T_{bc}\{\Omega[1]\} \) since they are proportional to \( g_{ab} \). Take then \( \alpha g_{ca} T_{bc}\{\Omega[1]\} k^a k^b = 0 \). This means that \( \alpha T_{ab}\{\Omega[1]\} k^a k^b = 0 \) which is impossible for every null vector \( k^a \) unless \( \alpha = 0 \) or \( \Omega[p] = 0 \). Thus \( \alpha \neq 0 \) then \( \Omega[p] = 0 \) for all \( p = 1, \ldots, N-1 \) and \( \epsilon T_{ab} = \alpha g_{ab} \) with \( \alpha = \sqrt{f} \) and \( \epsilon T_{a}^a = N \sqrt{f} \).

If \( \alpha = 0 \) we need to study the implications of \( T_{ca}\{\Omega[1]\} T_{bc}\{\Omega[1]\} k^a k^b = 0 \) for every null vector \( k^a \). By Proposition 3.2 and Lemma 4.2 \( T_{ca}\{\Omega[1]\} k^a \) and \( T_{bc}\{\Omega[1]\} k^b \) are null vectors and hence they must be parallel, \( T_{ca}\{\Omega[1]\} k^a = \beta(T_{ca}\{\Omega[1]\} k^a) \). Contraction with another null vector \( n^a \) gives \( \beta(n) = \beta(k) \) because of symmetry. Therefore, there is some constant \( \beta \) such that \( T_{ca}\{\Omega[1]\} k^a = \beta T_{ca}\{\Omega[1]\} k^a \) for all null vectors, and extending by linear combinations to all vectors we must have \( T_{ca}\{\Omega[1]\} = \beta T_{ca}\{\Omega[n]\} \), or that one of them is zero. In total, all non-zero \( T_{ab}\{\Omega[n]\} \), \( p = 1, \ldots, N-1 \), should be proportional, but this is not possible because as seen in the proof of Theorem 4.1 each of them has a strict different number of null eigenvectors. Then \( \epsilon T_{ab} = T_{ab}\{\Omega[p]\} \) for some \( p \) and by (8) \( (T \times T)_{ab} = [(\Omega \cdot \Omega)^2/(2p!)] g_{ab} \) so \( f = [(\Omega \cdot \Omega)/(2p!)]^2 \).

\[
\epsilon T_{a}^a = T_{a}^a\{\Omega[p]\} = \frac{(-1)^{p-1}}{(p-1)!}(1 - N/2p)(\Omega \cdot \Omega) = \frac{(2p - N)}{2p!}\Omega \cdot \Omega = (2p - N) \sqrt{f}
\]

which, as \( p = 1, \ldots, N-1 \), gives the values \((2 - N)\sqrt{f}, \ldots, (N - 2)\sqrt{f}\). Finally, the particular case (a), with \( f = 0 \), follows immediately from the above or directly from Lemma 4.4 (b).

Combining Proposition 3.2, Lemma 4.4, Corollary 4.2 and Theorem 5.1, we immediately obtain the following important result.

**Theorem 6.2** In \( N \) dimensions, if \( T_{ab} \) is symmetric then \( (T \times T)_{ab} = f_{gab} \iff T_{ab} \in \mathbb{S} \mathbb{S} \cup -\mathbb{S} \mathbb{S} \). This means that the tensors in \( \mathbb{S} \mathbb{S} \setminus \mathbb{N} \mathbb{S} \) (respectively in \(-\mathbb{S} \mathbb{S} \setminus -\mathbb{N} \mathbb{S}\)) are precisely those proportional to involutory orthochronous (resp. time-reversal) Lorentz transformations. The symmetric tensors in \( \mathbb{N} \mathbb{S} \) (resp. \(-\mathbb{N} \mathbb{S}\)) are exactly the symmetric singular orthochronous (resp. time-reversal) null-cone bi-preserving maps.
Theorems 6.1 and 6.2 provide a complete characterization of the conformally involutory null-cone preserving maps. Its classification also follows from the proof of Proposition 3.3 and Corollary 3.3, for we know that eigenvalues of $T_{ab}\{\Omega[p]\} \in SS$ are equal to $(-1)^{p-1}(\Omega \cdot \Omega)/(2p!)$ while $(N-p)$ are equal to $(-1)^{p}(\Omega \cdot \Omega)/(2p!)$. If an odd number of these are negative, $T_{ab}\{\Omega[p]\}$ is an improper null cone preserving map, otherwise a proper one. If $\Omega_a$ is a spacelike 1-form, then one eigenvalue is negative so $T_{ab}\{\Omega[1]\}$ is improper and can be interpreted as a reflection in the hyperplane orthogonal to $\Omega_a$. If $\omega_a$ is a timelike 1-form, then $(N-1)$ eigenvalues are negative and $T_{ab}\{\omega[1]\}$ is proper in odd dimensions and improper in even dimensions. It can be interpreted as a reflection in the line parallel to $\omega_a$. For other $p$-forms one can develop the corresponding geometrical interpretations.

Proposition 3.1 and Theorem 6.2 also imply:

**Corollary 6.1** If $N \leq 4$ then $SE_2 = SS$, i.e. in 2, 3, and 4 dimensions, $SE_2$ is precisely the set of symmetric tensors leaving invariant the null cone with its time orientation. Thus, if $N \leq 4$, $SE_2$ is constituted by all tensors proportional to involutory orthochronous Lorentz transformations plus the symmetric singular orthochronous null-cone bi-preserving maps. $-SE_2$ gives the time-reversal case.

For $N \leq 3$ this is trivial. For $N = 4$ this also means that the energy-momentum of any Maxwell field is proportional to an involutory orthochronous (and proper) Lorentz transformation, and coincides with the energy-momentum of some (possibly another) Maxwell field corresponding to a simple 2-form. This is well known and related to the duality rotations [26, 29]. With $N = 4$ and $p = 2$ in Theorem 6.1 we can state this as

**Corollary 6.2** In 4 dimensions, a tensor $T_{ab}$ is (up to sign) algebraically the energy-momentum tensor of a Maxwell field (a 2-form) if and only if $(T \times T)_{ab} = fg_{ab}$ and $T_{aa} = 0$.

These are the classical algebraic Rainich conditions [23, 31] (see also [13, 17, 19, 26]). They are necessary and sufficient conditions for a spacetime metric to originate algebraically (via Einstein’s equations) in a Maxwell field, i.e. a way of determining the physics from the geometry.

By Theorem 6.1 we can find generalisations to arbitrary dimension and to many different physical fields. In order to show the possibilities of our results, we can derive the following algebraic Rainich conditions. For a scalar field (compare with the partial results in [23, 12, 13, 27] for $N = 4$) we have

**Corollary 6.3** In $N$ dimensions, a tensor $T_{ab}$ is algebraically the energy-momentum tensor of a minimally coupled massless scalar field $\phi$ if and only if $(T \times T)_{ab} = fg_{ab}$ and $T_a^a = \beta \sqrt{T_{ab}T^{ab}}/N$ where $\beta = \pm (N-2)$. Moreover, $d\phi$ is spacelike if $\beta = N-2$ and $T_a^a \neq 0$, timelike if $\beta = 2 - N$ and $T_a^a \neq 0$, and null if $T_a^a = 0$. For $N \leq 4$ we have
Proof: Recall that \( T_{ab} = \nabla_a \phi \nabla_b \phi - (1/2)(\nabla \phi \cdot \nabla \phi) g_{ab} \) which is exactly \( T_{ab} \{ \nabla [1] \phi \} \in SS \) so \( (T \times T)_{ab} = f_{gab} \). We get \( T_{a}^a = (2-N)(\nabla \phi \cdot \nabla \phi)/2 \) and \( T_{ab} = N(\nabla \phi \cdot \nabla \phi^2)/4 \) so \( T_{a}^a = \pm (N-2)\sqrt{T_{ab}T_{ab}/N} \) with a plus sign if \( \nabla \phi \cdot \nabla \phi < 0 \) and a minus sign if \( \nabla \phi \cdot \nabla \phi > 0 \). Conversely, if \( (T \times T)_{ab} = f_{gab} \) then \( T_{ab} = N.f \). If \( T_{a}^a = \pm (N-2)\sqrt{T_{ab}T_{ab}/N} \) then by Theorem 5.1 \( T_{ab} \) is the superenergy tensor of a null 1-form in case \( f = 0 \). If \( f \neq 0 \) and \( T_{a}^a = (2-N)\sqrt{f} \) then \( T_{ab} \) is the superenergy tensor of a timelike 1-form. If \( f \neq 0 \) and \( T_{a}^a = (N-2)\sqrt{f} \) then \( T_{ab} \) is the superenergy tensor of an \( (N-1) \)-form of the type used in Theorem 1 which is the same as the superenergy tensor of its dual spacelike 1-form.

We can also generalize the algebraic Rainich conditions for a perfect fluid as given by Coll and Ferrando \([1]\) to the case of general \( N \).

**Corollary 6.4** In \( N \) dimensions, a tensor \( T_{ab} \) is algebraically the energy-momentum tensor of a perfect fluid satisfying the dominant energy condition if and only if

\[
T_{ab} = \frac{\lambda}{2} g_{ab} + \mu T_{ab} \{ v[1] \}
\]

where \( \lambda, \mu \geq 0 \) and \( v_b \) is timelike (so that \( T_{ab} \{ v[1] \} \) is intrinsically characterized as a tensor in \( SS \) according to its trace, see previous Corollary). The velocity vector of the fluid, its energy density and pressure are given by \( u^b = v^b/(v \cdot v), \rho = (\mu(v \cdot v) + \lambda)/2 \) and \( P = (\mu(v \cdot v) - \lambda)/2 \), respectively.

**Proof:** Recall that a perfect fluid has the Segre type \( \{ 1, (1 \ldots 1) \} \), so that

\[
T_{ab} = (\rho + P)u_a u_b - Pg_{ab}
\]

where \( (u \cdot u) = 1 \). Thus, if \([24]\) holds it is obvious that \( T_{ab} \) takes the form \([25]\). Conversely, if \([25]\) holds, then \( T_{ab} - T_{ab} \{ u[1] \} \) has every null \( k^b \) as eigenvector, as can be trivially checked. Therefore, \( T_{ab} - T_{ab} \{ u[1] \} \) is proportional to the metric, and the proportionality factor is obtained from the \( T_{a}^a \).

In fact, we can get the conditions as stated in \([1]\) generalized for \( N \) dimensions as follows. From \([24]\) we get

\[
(T \times T)_{ab} = \lambda \mu T_{ab} \{ v[1] \} + \frac{\mu^2 (v \cdot v)^2 + \lambda^2}{4} g_{ab} = \lambda T_{ab} + \frac{\mu^2 (v \cdot v)^2 - \lambda^2}{4} g_{ab} = \lambda T_{ab} + \rho P g_{ab}
\]

and also \( N(T \times T)_{a}^a - (T_{a}^a)^2 \geq 0 \), \( T_{a}^a \leq \frac{\lambda}{2} \) and \( T_{ab} w^a w^b \geq \lambda/2 \) for all timelike \( w^a \).

As another example, let us consider the case of dust \( (P = 0 \) perfect fluids). Of course, this case can be deduced from the previous one by setting \( P = 0 \). However, in dimension \( N = 5 \) some stronger results can be derived. To see it, recall that any
2-form $F_{[2]}$ with no null eigenvector can only exist in odd dimension $N = 2n + 1$, and must take the form

$$F_{[2]} = \mu_1(e_1 \wedge e_2) + \ldots + \mu_n(e_{2n-1} \wedge e_{2n})$$

where \(\{e_0, e_1, \ldots, e_{2n}\}\) is an orthonormal basis and \(\mu_i (i = 1, \ldots, n)\) are non-zero constants. Thus, in the particular case that all the \(\mu_i\)'s are equal we get for the superenergy tensor \((\ref{eq:superenergy})\) of such an \(F_{[2]}\)

$$T_{ab}\{F_{[2]}\} = \frac{\mu_1^2}{2} [n e_0 \otimes e_0 + (2 - n)(e_1 \otimes e_1 + \ldots + e_{2n} \otimes e_{2n})]_{ab}. \quad (26)$$

**Corollary 6.5** In 5 dimensions, \(T_{ab}\) is algebraically the energy-momentum tensor of a dust, that is \(T_{ab} = \rho u_a u_b\), where \((u \cdot u) = 1\) and \(\rho \geq 0\), if and only if \(T_{ab}\) is the s-e tensor of a 2-form \(F_{[2]}\) with no null eigenvector having \(\mu_2 = \mu_1\).

**Proof:** This is the case \(n = 2 (\implies N = 5)\) of the previous formula \((\ref{eq:superenergy})\), identifying \(u = e_0\) and \(\rho = \mu_1^2\). Notice that the timelike direction \(u\) is intrinsically defined by \(\ast(F_{[2]} \wedge F_{[2]})\).

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