Irreducible decompositions of the elasticity tensor under the linear and orthogonal groups and their physical consequences

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Abstract. We study properties of the fourth rank elasticity tensor \(C\) within linear elasticity theory. First \(C\) is irreducibly decomposed under the linear group into a “Cauchy piece” \(S\) (with 15 independent components) and a “non-Cauchy piece” \(A\) (with 6 independent components). Subsequently, we turn to the physically relevant orthogonal group, thereby using the metric. We find the finer decomposition of \(S\) into pieces with \(9+5+1\) and of \(A\) into those with \(5+1\) independent components. Some reducible decompositions, discussed earlier by numerous authors, are shown to be inconsistent. — Several physical consequences are discussed. The Cauchy relations are shown to correspond to \(A = 0\). Longitudinal and transverse sound waves are basically related by \(S\) and \(A\), respectively.

1. Introduction

The constitutive law in linear elasticity theory for an anisotropic homogeneous body, the generalized Hooke law, postulates a linear relation between the two second-rank tensor fields, the stress \(\sigma^{ij}\) and the strain \(\varepsilon_{kl}\) (see [17, 23, 18, 13]):

\[
\sigma^{ij} = C^{ijkl} \varepsilon_{kl} \,.
\]

In standard linear elasticity, the stress and the strain tensors are assumed to be symmetric. Consequently the elasticity tensor has the so called minor symmetries:

\[
C^{ij|kl} = C^{ij|lk} = 0 \,.
\]

Another restriction for the elasticity tensor is based on the energy consideration. The energy density of a deformed material is expressed as \(W = \frac{1}{2} \sigma^{ij} \varepsilon_{ij}\). When the Hooke law (2) is substituted here, this expression takes the form

\[
W = \frac{1}{2} C^{ijkl} \varepsilon_{ij} \varepsilon_{kl} \,.
\]

1 The Bach parenthesis () and [] denote symmetrization and antisymmetrization, respectively: \((ij) := \{ij + ji\}/2\) and \([ij] := \{ij − ji\}/2\), see [21].
The right-hand side of (3) involves only those combinations of the elasticity tensor components which are symmetric under permutation of the first and the last pairs of indices. Consequently, the so-called major symmetry,

$$C_{ijkl} - C_{klij} = 0$$

is assumed. In 3-dimensional space, a fourth rank tensor with the symmetries (2) and (4) has 21 independent components.

In the literature on elasticity, a special decomposition of $C_{ijkl}$ into two tensorial parts is frequently used, see, for example, Cowin [5], Campanella & Tonton [4], Podio-Guidugli [20], Weiner [25], and Haussühl [10]. It is obtained by symmetrization and antisymmetrization of the elasticity tensor with respect to its two middle indices:

$$M_{ijkl} := C_{i[jk]l}, \quad N_{ijkl} := C_{i[k]j]l}, \quad \text{with} \quad C_{ijkl} = M_{ijkl} + N_{ijkl}.$$ (5)

It is straightforward to show [15] that $M$ and $N$ fulfill the major symmetry (2) but not the minor symmetries (4). Moreover, the tensor $M$ can be further decomposed. Accordingly, the reducible decomposition in (5) does not correspond to a direct sum decomposition of the vector space defined by $C$. The vector spaces of $M$ and $C$ both turn out to be 21-dimensional, that of $N$ is 6-dimensional. Thus, the tensors $M$ and $N$ are auxiliary quantities but, due to the lack of the minor symmetries, they do not represent elasticity tensors, that is, they cannot be used to characterize a certain material elastically. These quantities are placeholders without direct physical interpretation. However, in the elasticity literature some physical interpretations of the tensors $M$ and $N$ are offered. Still, these results are inconsistent as we showed earlier in [15].

In this paper, we present various decompositions of the elasticity tensor based on group-theoretic arguments and discuss some physical applications of these decompositions. The organization of the paper is as follows: In Sec.2 we discuss the relation between the decomposition of a tensor and the group of transformations acted on the basic vector space. Sec.3 is devoted to the $GL(3)$ decomposition of the elasticity tensor. It is based on the Young diagram technique. Similarly, in Sec.4 we treat the case of the $SL(3)$ and in Sec.5 that of the $SO(3)$. In Sec.6 we delineate some physical applications of the irreducible decomposition described.

2. Groups of transformations and corresponding decompositions of tensors

In a precise algebraic treatment, a tensor must be viewed as a multi-linear map from a vector space $V$, or the dual of it $V^*$, to the field of real numbers, rather than as a collection of real valued components.

A contravariant tensor of rank $p$ is defined (see [7], p.59) as a multi-linear map from the Cartesian product of $p$ copies of $V^*$ into the field $\mathbb{R}$,

$$T: V^* \times V^* \times \cdots \times V^* \to \mathbb{R}.$$ (6)

The set of all tensors $T$ of the rank $p$ compose a vector space by itself, say $\mathcal{T}$. The dimension of this tensor space is equal to $n^p$, that is, a 4th rank tensor in 3 dimensions (3d) provides for $\mathcal{T}$ $3^4 = 81$ dimensions. As a basis in $\mathcal{T}$, we can take the tensor products of the basis elements of $V$,

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}.$$ (7)

Accordingly, an arbitrary contravariant tensor of rank $p$ can be expressed as

$$T = T^{i_{i_1} i_{i_2} \cdots i_p} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}.$$ (8)
Thus, the properties of the tensor $T$ in (8), in particular its proper decomposition, is related to the group of general linear transformations $GL(3)$ acting on the basis of $V$. This group encompasses all non-singular $3 \times 3$ matrices.

The vector space $V$ has a basis $e_i$ and a dual cobasis $\vartheta^i$. Under the $GL(3)$, the cobasis transforms as $\vartheta'^i = L^i_j \vartheta^j$. We can attribute a volume element to this cobasis,

$$\epsilon := \frac{1}{3!} \epsilon_{ijk} \vartheta^i \wedge \vartheta^j \wedge \vartheta^k;$$

(9)

here $\epsilon_{ijk}$ is the totally antisymmetric Levi-Civita permutation symbol, which is $\epsilon_{ijk} = \pm 1$ for even or odd permutations of 123, respectively, and is $\epsilon_{ijk} = 0$ otherwise. The primed volume element $\epsilon'$ for the transformed cobasis $\vartheta'^i$ is defined analogously. If we substitute $\vartheta'^i = L^i_j \vartheta^j$, we find

$$\epsilon' = \frac{1}{3!} \epsilon_{ijk'} L^i_j L^j_k \vartheta^i \wedge \vartheta^j \wedge \vartheta^k.$$

(10)

Since $\epsilon_{ijk'} = \epsilon_{ijk} = 0, \pm 1$ is numerically invariant under linear basis transformations, the volume element transforms as a scalar density,

$$\epsilon' = (\det L)^{-1} \epsilon.$$

(11)

If we require that $\det L = +1$, then the volume element is an invariant (or scalar) and the $GL(3)$ reduces to the special linear group $SL(3)$.

Furthermore, we know from elasticity theory that the description of a deformation is based on the concept of the distance within a continuum. A deformation is meant to be the change of the distances between nearby points. However, the (Euclidean) distance $ds^2 = g := g_{ij} dx^i \otimes dx^j$ as such, with $g_{ij} = \text{diag}(1, 1, 1)$, is, by definition, invariant under the Euclidean group $T(3) \times SO(3)$, the semidirect product of the translations $T(3)$ with the rotations $SO(3)$. Accordingly, at one point in a continuum, the special orthogonal group $SO(3)$ is the one relevant for the representations of the tensors in elasticity theory.

The $SO(3)$ corresponds to all orthogonal matrices (reciprocal equals to the transpose) with determinant $+1$. If we allow also parity transformations, we arrive at the $O(3)$, the orthogonal group, and, if we drop the orthogonality requirement, eventually at the $GL(3)$. Alternatively, we can keep parity invariance in the first step and then arrive at the $GL(3)$ via the special linear group $SL(3)$, which has determinant $+1$:

$$GL(3) \leftarrow \begin{cases} SL(3), \det = +1 \\ O(3), \det = \pm 1 \end{cases} \leftarrow SO(3).$$

(12)

Incidentally, between the set of subgroups of $GL(3)$ we will use only those that do not preserve any spatial direction. For physical problems that involve some special direction, elastic material in exterior magnetic field, for instance, smaller subgroups such as $GL(2)$ are also relevant.

The process of going down from the $GL(3)$ to the $SO(3)$ is described in detail in Schouten [22], Chap.III. On the level of the $O(3)$, we can define a scalar volume element$^2$ according to $\eta := \sqrt{\det(g_{ik})} \epsilon$. The transformations of the $O(3)$ are volume preserving. The $\epsilon$ is a premetric concept, whereas the $\eta$ requires the existence of a metric $g$.

We can collect there results in a table:

| space | volume element | line element | transformation | group |
|-------|----------------|--------------|----------------|-------|
| $V$   | $\epsilon$    | $\vartheta^i$| $\det L \neq 0$| $GL(3)$|
| $V$   | $\epsilon$    | $g$          | $\det L = +1$ | $SL(3)$|
| $V$   | $\eta$        | $g$          | $\det L = \pm 1$| $O(3)$|
| $V$   | $\eta$        | $g$          | $\det L = +1$ | $SO(3)$|

$^2$ This is also discussed in some detail, for example, in [12], Secs.A.1.9 and C.2.3, respectively.
Table 1. The different groups involved in the irreducible decomposition of the elasticity tensor.

Although the group $SO(3)$ is highly relevant in elasticity theory, it is convenient to provide the decomposition of the elasticity tensor relative to the bigger groups and to arrive at the $SO(3)$ only at the last stage. In this case, we are able to identify the different origins of the irreducible parts and to derive the stratified hierarchy of the invariants. Such a method is also useful from the technical point of view because we need to use different algebraic methods for different groups of transformations. We begin with the decomposition relative to the biggest group $GL(3)$.

3. $GL(3)$-decomposition

3.1. Young decomposition

Let us recall some basic facts about 4th rank tensors and their Young decomposition:

1) Covariant tensor of 4th rank over the vector space $V$ (with $\dim V = 3$) is a multi-linear map from the Cartesian product of 4 copies of the vector space $V$ into the real numbers (see [7], p.58),

$$ T : V \times V \times V \times V \rightarrow \mathbb{R} . \quad (13) $$

2) The set of the 4th rank tensors constitutes a new vector space $\mathcal{T}$ called a 4th rank tensor space. Its dimension is equal to $3^4 = 81$.

3) The tensor space $\mathcal{T}$ can be decomposed into the direct sum of its subspaces that are invariant under the action of the group $GL(3, \mathbb{R})$.

4) Due to the well-known Schur-Weyl duality, the irreducible decomposition of the space of the 4th rank tensors under $GL(3, \mathbb{R})$ corresponds to the irreducible decomposition of the permutation group $S_4$.

5) A known practical way to derive the irreducible decomposition of $S_4$ is by use of Young diagrams.

6) A generic 4th rank tensor over the 3d tensor space can be decomposed into a direct sum of four subspaces. This decomposition is depicted by the Young diagrams, see [3] or [8],

$$ \square \otimes \square \otimes \square \otimes \square = \square \oplus \square \oplus \square \oplus \square . \quad (14) $$

The left-hand side describes a generic 4th rank tensor. On the right-hand side, the diagrams represent the 4 subspaces.

7) The diagrams in (14) come with a weight factor $f^\lambda$, called the dimension of the irreducible representation $\lambda$ of the permutation group $S_4$. For a diagram of the shape $\lambda$, the $f^\lambda$-factor is calculated by the use of the hook-length formula,

$$ f^\lambda = p! \prod_{(\alpha, \beta) \in \lambda} \frac{1}{\text{hook}(\alpha, \beta)} . \quad (15) $$

The hook length $\text{hook}(\alpha, \beta)$ of the element $(\alpha, \beta)$ of the diagram is defined as the number of squares directly below or to the right of $(\alpha, \beta)$ (counting $(\alpha, \beta)$ once). Thus, for the diagrams depicted in (14),

$$ f^{\lambda_1} = 1 , \quad f^{\lambda_2} = 3 , \quad f^{\lambda_3} = 2 , \quad f^{\lambda_4} = 3 , \quad f^{\lambda_5} = 1 . \quad (16) $$
8) The dimensions of the irreducible decomposition of $GL(n, \mathbb{R})$ are calculated by the Stanley hook-content formula

$$\dim V^\lambda = \prod_{(\alpha, \beta) \in \lambda} (n + \alpha - \beta)_{\text{hook}(\alpha, \beta)}.$$  

Consequently, for the diagrams depicted in (14) and for the dimension $n = 3$ of the vector space, we have

$$\dim V^{\lambda_1} = 15, \quad \dim V^{\lambda_2} = 15, \quad \dim V^{\lambda_3} = 6, \quad \dim V^{\lambda_4} = 3.$$  

9) Every diagram represents a subspace of the tensor space $\mathcal{T}$,

$$\mathcal{T} = (1)\mathcal{T} \oplus (2)\mathcal{T} \oplus (3)\mathcal{T} \oplus (4)\mathcal{T}.$$  

These subspaces intersect only at zero. Moreover, they all are mutually non-isomorphic. Hence we have a direct sum decomposition of the tensor space. The dimension of the initial tensor space is separated into the dimension of the subspaces as follows:

$$\dim \mathcal{T} = \sum_i f^{\lambda_i} \times \dim V^{\lambda_i}.$$  

The dimension of the tensor space is distributed between the subspaces according to

$$81 = 1 \times 15 + 3 \times 15 + 2 \times 6 + 3 \times 3.$$  

10) The decomposition (21) is unique but reducible. In accordance with (16), the subspaces $(2)\mathcal{T}$, $(3)\mathcal{T}$, and $(4)\mathcal{T}$ can be decomposed still further into sub-subspaces:

$$(2)\mathcal{T} = \left( (2,1)\mathcal{T} \oplus (2,2)\mathcal{T} \oplus (2,3)\mathcal{T} \right),$$

$$(3)\mathcal{T} = \left( (3,1)\mathcal{T} \oplus (3,2)\mathcal{T} \right),$$

$$(4)\mathcal{T} = \left( (4,1)\mathcal{T} \oplus (4,2)\mathcal{T} \oplus (4,3)\mathcal{T} \right).$$

These decompositions are irreducible but not unique.

3.2. Irreducible decomposition of $C^{ijkl}$

The elasticity tensor is not a general 4th rank tensor; rather, it carries its minor and major symmetries. Accordingly, we are looking for a decomposition of an invariant subspace of the tensor space $\mathcal{T}$ into a direct sum of its sub-subspaces,

$$\mathcal{C} = \alpha (1)\mathcal{C} \oplus \beta (2)\mathcal{C} \oplus \gamma (3)\mathcal{C} \oplus \delta (4)\mathcal{C},$$

where

$$\alpha = 0, 1; \quad \beta = 0, 1, 2, 3; \quad \gamma = 0, 1, 2; \quad \delta = 0, 1, 2, 3.$$  

Using the minor and major symmetries, we find $\alpha = 1$ and $\delta = 0$. Since $\dim \mathcal{C} = 21$, we find as a unique solution of (25),

$$\alpha = \gamma = 1; \quad \beta = \delta = 0.$$  

Thus, we proved
**Proposition 1:** The decomposition
\[ C_{ijkl} = S_{ijkl} + A_{ijkl}, \]  
with
\[ S_{ijkl} := \frac{1}{3}(C_{ijkl} + C_{iklj} + C_{iljk}), \quad \text{and} \quad A_{ijkl} := \frac{1}{3}(2C_{ijkl} - C_{ilkj} - C_{iklj}), \]  
is irreducible and unique.

**Proposition 2:** The partial tensors satisfy the minor symmetries,
\[ S_{[ij]kl} = S_{ij[kl]} = 0, \quad \text{and} \quad A_{[ij]kl} = A_{ij[kl]} = 0, \]  
and the major symmetry,
\[ S_{ijkl} = S_{klij} \quad \text{and} \quad A_{ijkl} = A_{klij}. \]  

**Proposition 3:** The irreducible decomposition of \( C \) signifies the reduction of the tensor space \( C \) into a direct sum of two subspaces \( S \subset C \) (for the tensor \( S \)) and \( A \subset C \) (for the tensor \( A \)),
\[ C = S \oplus A. \]  
In particular, we have
\[ \dim C = 21, \quad \dim S = 15, \quad \dim A = 6. \]

**Proposition 4:** The irreducible piece \( A_{ijkl} \) of the elasticity tensor is a fourth rank tensor. Alternatively, it can be represented as a symmetric second rank tensor density
\[ \Delta_{mn} := \frac{1}{4} \epsilon_{mil} \epsilon_{njk} A_{ijkl}, \]  
where \( \epsilon_{ijk} = 0, \pm 1 \) denotes the Levi-Civita permutation symbol. The proof of this proposition is given in the next section.

4. **SL(3)-decomposition**

Since the elasticity tensor \( C_{ijkl} \) satisfies the symmetries (2) and (4), most of its contractions with \( \epsilon_{ijk} \) vanish. Indeed, for the completely symmetric part the contraction in two indices is identically zero,
\[ \epsilon_{mij} S_{ijkl} = 0. \]

The second part of the elasticity tensor yields a non-vanishing contraction,
\[ K_{m}^{jk} := \frac{1}{2} \epsilon_{mil} A_{ijkl}. \]

This tensor \( K_{i}^{jk} \) has vanishing traces and is antisymmetric in the upper indices,
\[ K_{i}^{ik} = 0, \quad K_{k}^{jk} = 0, \quad K_{j}^{(jk)} = 0. \]

Thus, \( K_{i}^{jk} \) has 6 independent components, exactly like the initial tensor \( A_{ijkl} \).
Because of the antisymmetry of $K_{i\, jk}$, we do not lose anything if we contract the upper indices with $\epsilon_{njk}$:

$$\Delta_{mn} := \frac{1}{2} \epsilon_{njk} K_{m\, jk} = \frac{1}{4} \epsilon_{mnl} \epsilon_{njk} A^{ijkl}.$$  

(37)

We can check that this tensor is symmetric

$$\Delta_{[mn]} = 0.$$  

(38)

Thus, it has the same 6 independent components. This proves the Proposition 4.

Summing up: there are no additional $SL(3)$ invariants of the elasticity tensor, and this tensor cannot be decomposed further under the action of the $SL(3)$. Moreover, we derived a representation of $A^{ijkl}$ in terms of $\Delta_{mn}$. Under the action of the special linear group $SL(3)$, the latter quantity is an ordinary tensor.

5. $SO(3)$-decomposition

Since for the elasticity tensor the invariance of the volume element does not yield additional tensor decompositions, we skip the group $O(3)$ and pass directly to its subgroup $SO(3)$. Consider a vector space $W$ endowed with a metric tensor $g^{ij}$. The norm and the scalar product of vectors are defined now in the conventional way by $(u, v) = g_{ij} u^i v^j$ and $|v| = (v, v)^{1/2}$. In order to preserve the scalar product, we must restrict ourselves to the orthogonal group $O(3)$. Invariance of the volume element $\eta$ is guaranteed if we restrict ourselves to the group $SO(3)$. Now we can use the metric tensor $g_{ij}$ and its inverse $g^{ij}$.

From the contraction of the metric with the totally symmetric Cauchy part $S^{ijkl}$, a unique symmetric second rank tensor and its scalar contraction can be constructed,

$$S^{ij} := g_{kl} S^{ijkl} \quad \text{and} \quad S := g_{ij} g_{kl} S^{ijkl}.$$  

(39)

Define the traceless part of the tensor $S^{ij}$ as

$$S^{\#}^{ij} := S^{ij} - \frac{1}{3} S g^{ij}, \quad \text{with} \quad g_{ij} S^{\#}^{ij} = 0.$$  

(40)

Now we turn to the decomposition of the tensor $S^{ijkl}$. We define the subtensors

$$(2)S^{ijkl} := \alpha S^{(ij \, g^{kl})} \quad \text{and} \quad (3)S^{ijkl} := \beta S g^{(ij \, g^{kl})}.$$  

(41)

We denote the remaining part as

$$(1)S^{ijkl} := S^{ijkl} - (2)S^{ijkl} - (3)S^{ijkl}.$$  

(42)

Now we require the tensor $(1)S^{ijkl}$ to be traceless. This yields,

$$\alpha = \frac{6}{7}, \quad \beta = \frac{1}{5}.$$  

(43)

Hence we obtain the unique irreducible decomposition of the tensor $S^{ijkl}$ into three pieces,

$$S^{ijkl} = (1)S^{ijkl} + (2)S^{ijkl} + (3)S^{ijkl}.$$  

(44)

These pieces are invariant under the action of the group $SO(3)$.

In order to decompose the non-Cauchy part $A^{ijkl}$, it is convenient to use its representation by the tensor density $\Delta_{ij}$. The latter is irreducibly decomposed to a scalar and traceless parts.

$$\Delta_{ij} = \Delta_{ij} + \frac{1}{3} \Delta g_{ij}, \quad \text{where} \quad \Delta := g^{ij} \Delta_{ij}.$$  

(45)
Consequently, we obtain the decomposition of $A^{ijkl}$ into two independent parts

$$A^{ijkl} = (1)A^{ijkl} + (2)A^{ijkl},$$

(46)

where the scalar and the traceless parts are given by

$$(2)A^{ijkl} := 2 \frac{1}{3} \Delta \left( g^{ij} g^{kl} - g^{il} g^{jk} \right)$$

and

$$(1)A^{ijkl} := A^{ijkl} - (2)A^{ijkl},$$

(47)

respectively. This way we derived a composition of the elasticity tensor into five irreducible parts

$$C^{ijkl} = \left( (1)S^{ijkl} + (2)S^{ijkl} + (3)S^{ijkl} \right) + \left( (1)\Delta^{ijkl} + (2)\Delta^{ijkl} \right).$$

(48)

Between the first parentheses we collected the terms corresponding to the Cauchy part, the second parentheses enclose the non-Cauchy terms. The dimension of the tensor space of the elasticity tensor is decomposed into the sum of the corresponding dimensions of the subspaces,

$$21 = (9 + 5 + 1) + (5 + 1).$$

(49)

Thus, $C^{ijkl}$ can be represented by one totally traceless and totally symmetric 4th rank tensor $^{(1)}C^{ijkl}$ plus two symmetric traceless 2nd rank tensors $S^{ij}, \Delta_{ij}$ plus two scalars $S, \Delta$. This decomposition is unique and irreducible under the action of the $SO(3, \mathbb{R})$.

It is quite remarkable that the same decomposition was derived Backus [1] (see also Baerheim [2]) in a rather different way. In our group-theoretical treatment, we obtained all irreducible pieces in a covariant form. Moreover, we derived a tree of the independent invariances and their relations to different transformation groups. We describe this stratified structure in the following diagram:

![Diagram](image)

*Figure 1.* The decomposition tree of the elasticity tensor $C_{ijkl}$. Note that $\Delta_{ij}$ is a tensor (and *not* a density) under the $SL(3)$.

6. Applications

Some non-trivial applications were recently discussed by us in [11, 15] (for the analogous case in electrodynamics, see [14]):
6.1. Cauchy relations
The Cauchy relations are given by
\[ N_{ijkl} = 0 \quad \text{or} \quad C_{ijkl} = C_{ikjl}, \tag{50} \]
for their history, see Todhunter [24]. The representation in (50) is widely used in the elasticity literature, see, for example, [9, 5, 6, 4, 20, 25]. In [11, 15], we presented an alternative form of these conditions, namely
\[ A_{ijkl} = 0 \quad \text{or} \quad 2C_{ijkl} - C_{ilkj} - C_{iklj} = 0. \tag{51} \]
The last equation can be presented in a more economical form,
\[ \Delta_{mn} = 0. \tag{52} \]
The basic difference between (50) on the one hand and (51) or (52) at the other hand is that our conditions (51) or (52) are represented in an irreducible form. This can have decisive consequences.

For most materials the Cauchy relations do not hold, even approximately. In fact, the elasticity of a generic anisotropic material is described by the whole set of the 21 independent components, it is not restricted to the set of 15 independent components obeying the Cauchy relations. This fact seems to nullify the importance of the Cauchy relations for modern solid state theory and leave them only as historical artifact.

However, a lattice-theoretical approach to the elastic constants shows, see Leibfried [16], that the Cauchy relations are valid provided (i) the interaction forces between the atoms or molecules of a crystal are central forces, as, for instance, in rock salt, (ii) each atom or molecule is a center of symmetry, and (iii) the interaction forces between the building blocks of a crystal can be well approximated by a harmonic potential, see Perrin [19]. Accordingly, a study of the violations of the Cauchy relations yields important information about the intermolecular forces of elastic bodies. Accordingly, one should look for the deviation \( A_{ijkl} \) of the elasticity tensor \( C_{ijkl} \) from its Cauchy part \( S_{ijkl} \).

6.2. Acoustic waves
If \( u_i \) is the displacement field, the propagation of acoustic waves in anisotropic media is determined by the following equation (\( \rho = \text{mass density} \)):
\[ \rho g^{il} \ddot{u}_l - C_{ijkl} \partial_j \partial_k u_l = 0. \tag{53} \]
With a plane wave ansatz \( u_i = U_i e^{i(\zeta j x^j - \omega t)} \), we obtain a system of three homogeneous algebraic equations
\[ (v^2 g^{ij} - \Gamma^{ij}) U_j = 0, \tag{54} \]
where the Christoffel tensor \( \Gamma^{ij} := (1/\rho) C^{ijkl} n_j n_k \) and of the phase velocity \( v := \omega/\zeta \) are used. Substituting the irreducible SA-decomposition into the Christoffel tensor, we obtain
\[ \Gamma^{ij} = S^{ij} + A^{ij}, \tag{55} \]
where the Cauchy and non-Cauchy Christoffel tensors are defined by
\[ S^{ij} := S^{ijkl} n_j n_k = S^{li}, \quad \text{and} \quad A^{ij} := A^{ijkl} n_j n_k = A^{li}. \tag{56} \]
Acoustic wave propagation in an elastic medium is an eigenvector problem, see (54), with the phase velocity $v^2$ as the eigenvalues. In general, three distinct real positive solutions correspond to three independent waves $(1) U_l$, $(2) U_l$, and $(3) U_l$, called acoustic polarizations. For isotropic materials, there are three pure polarizations: one longitudinal (or compression) wave with $\vec{U} \times \vec{n} = 0$ and two transverse (or shear) waves with $\vec{U} \cdot \vec{n} = 0$.

In general, for anisotropic materials, three pure modes do not exist. The identification of the pure modes and the condition for their existence is an interesting problem. Let us show how the irreducible decomposition of the elasticity tensor, which we applied to the Christoffel tensor, can be used here.

**Proposition 5:** Let $n^i$ denote an allowed direction for the propagation of a compression wave. Then the velocity $v_L$ of this wave in the direction of $n^i$ is determined only by the Cauchy part of the elasticity tensor:

$$v_L = \sqrt{S^{ij} n_i n_j}.$$  \hspace{1cm} (57)

Moreover, for a medium with a given elasticity tensor, all three purely polarized waves (one longitudinal and two transverse) can propagate in the direction $\vec{n}$ if and only if

$$S^{ijkl} n_k n_l n_i = S^{kl} n_k n_l n_i.$$  \hspace{1cm} (58)

Accordingly, for a given medium, the directions of the purely polarized waves depend on the Cauchy part of the elasticity tensor alone. In other words, two materials, with the same Cauchy parts $S^{ijkl}$ of the elasticity tensor but different non-Cauchy parts $A^{ijkl}$, have the same pure wave propagation directions and the same longitudinal velocity.

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