WEIGHTED MULTIPOLAR HARDY INEQUALITIES AND EVOLUTION PROBLEMS WITH KOLMOGOROV OPERATORS PERTURBED BY SINGULAR POTENTIALS

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Abstract. The main results in the paper are the weighted multipolar Hardy inequalities
\[ c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x-a_i|^2} \mu(x) dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) dx + K \int_{\mathbb{R}^N} \varphi^2 \mu(x) dx, \]
in \( \mathbb{R}^N \) for any \( \varphi \) in a suitable weighted Sobolev space, with \( 0 < c \leq c_{o, \mu} \), \( a_1, \ldots, a_n \in \mathbb{R}^N \), \( K \) constant. The weight functions \( \mu \) are of a quite general type.

The paper fits in the framework of Kolmogorov operators defined on smooth functions
\[ Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \]
perturbed by multipolar inverse square potentials, and related evolution problems. Necessary and sufficient conditions for the existence of exponentially bounded in time positive solutions to the associated initial value problem are based on weighted Hardy inequalities. For constants \( c \) beyond the optimal Hardy constant \( c_{o, \mu} \) we are able to show nonexistence of positive solutions.

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1. Introduction. The paper concerns with weighted multipolar Hardy inequalities in $\mathbb{R}^N$ for a class of weight functions $\mu$. The main motivation for our interest in Hardy inequalities is the key role that these play in the study of Kolmogorov operators

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u,$$

defined on smooth functions, perturbed by singular potentials and of the related evolution problems

$$(P) \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot,0) = u_0 \geq 0 \in L^2_\mu \end{cases}$$

where $L^2_\mu := L^2(\mathbb{R}^N, \mu(x)dx)$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. The potentials we consider are inverse square potentials of multipolar type

$$V(x) = \sum_{i=1}^n \frac{c}{|x - a_i|^2}, \quad c > 0, \quad a_1 \ldots, a_n \in \mathbb{R}^N.$$

In literature there exist reference papers in the case of Schrödinger operators with singular potentials $V(x) = \frac{c}{|x|^2}$, $c > 0$. These potentials are interesting for the criticality: they lie at a borderline case where standard theories such as the strong maximum principle and Gaussian bounds in [2] fail.

The operator $\Delta + \frac{c}{|x|^2}$ has the same homogeneity as the Laplacian. In 1984 P. Baras and J. A. Goldstein in [3] showed that the evolution problem $(P)$ with $L = \Delta$ and $V(x) = \frac{c}{|x|^2}$ admits a unique positive solution if $c \leq c_o = \left(\frac{N-2}{2}\right)^2$ and no positive solutions in the sense of distributions exist if $c > c_o$. When it exists, the solution is exponentially bounded, on the contrary, if $c > c_o$, there is the so-called instantaneous blow-up phenomenon.

The drift term in (1.1) forces to use a different technique in order to extend these results to Kolmogorov operators.

A result analogous to that stated in [3] has been obtained in 1999 by X. Cabré and Y. Martel in [5] for more general potentials $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ with a different approach. To state existence and nonexistence results we follow, as in [16, 6, 9] for the unipolar case, Cabré-Martel’s approach using the relation between the weak solution of $(P)$ and the bottom of the spectrum of the operator $-(L + V)$

$$\lambda_1(L + V) := \inf_{\varphi \in H^1_\mu \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x)dx - \int_{\mathbb{R}^N} V \varphi^2 \mu(x)dx}{\int_{\mathbb{R}^N} \varphi^2 \mu(x)dx} \right),$$

with $H^1_\mu$ suitable weighted Sobolev space.

When $\mu = 1$ Cabré and Martel showed that the boundedness of $\lambda_1(\Delta + V)$ is a necessary and sufficient condition for the existence of exponentially bounded in time positive solutions to the associated initial value problem. Later in [16, 6, 9] similar results have been extended to Kolmogorov operators perturbed by inverse square potentials with a single pole. The proof uses some properties of the operator $L$ and of its corresponding semigroup in $L^2_\mu(\mathbb{R}^N)$.

The case of the Schrödinger operator with multipolar inverse square potentials has been investigated in literature.
In particular, for the operator

$$\mathcal{L} = -\Delta - \sum_{i=1}^{n} \frac{c_i}{|x-a_i|^2},$$

$$n \geq 2, c_i \in \mathbb{R},$$

for any \( i \in \{1, \ldots, n\} \), V. Felli, E. M. Marchini and S. Terracini in [15] proved that the associated quadratic form

$$Q(\varphi) := \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx - \sum_{i=1}^{n} c_i \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2} \, dx$$

is positive if \( \sum_{i=1}^{n} c_i^+ < \frac{(N-2)^2}{4} \), \( c_i^+ = \max\{c_i, 0\} \), conversely if \( \sum_{i=1}^{n} c_i^+ > \frac{(N-2)^2}{4} \) there exists a configuration of poles such that \( Q \) is not positive. Later R. Bosi, J. Dolbeaut and M. J. Esteban in [4] proved that for any \( c \in \left(0, \frac{(N-2)^2}{4}\right)\) there exists a positive constant \( K \) such that the multipolar Hardy inequality

$$c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x-a_i|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + K \int_{\mathbb{R}^N} \varphi^2 \, dx$$

holds for any \( \varphi \in H^1(\mathbb{R}^N) \). C. Cazacu and E. Zuazua in [14], improving a result stated in [4], obtained the inequality

$$\frac{(N-2)^2}{n^2} \sum_{i,j=1}^{n} \int_{\mathbb{R}^N} \frac{|a_i-a_j|^2}{|x-a_i|^2 |x-a_j|^2} \varphi^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx,$$

for any \( \varphi \in H^1(\mathbb{R}^N) \) (see also [13] for estimates for the Hardy constant in bounded domains).

For Ornstein-Uhlenbeck type operators

$$Lu = \Delta u - \sum_{i=1}^{n} A(x-a_i) \cdot \nabla u,$$

perturbed by multipolar inverse square potentials (1.2), weighted multipolar Hardy inequalities and related existence and nonexistence results were stated in [8], with \( A \) a positive definite real Hermitian \( N \times N \) matrix, \( a_i \in \mathbb{R}^N, i \in \{1, \ldots, n\} \). In such a case, the invariant measure for these operators is the Gaussian measure

$$\mu_A(x) \, dx = \kappa e^{-\frac{1}{2} \sum_{i=1}^{n} (A(x-a_i), x-a_i)} \, dx,$$

with a normalization constant \( \kappa \).

As far as we know there are no other results in the literature concerning weighted multipolar Hardy inequalities.

In this paper, in Sections 2 and 3, we discuss multipolar weighted inequalities

$$\int_{\mathbb{R}^N} V \varphi^2 \mu(x) \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) \, dx + K \int_{\mathbb{R}^N} \varphi^2 \mu(x) \, dx, \quad \varphi \in H^1_\mu, \quad K > 0,$$

where \( V \) as in (1.2), with \( 0 < c \leq c_{o,\mu} \), and state the optimality of the constant \( c_{o,\mu} \) on the left-hand side in Section 5.

We use two different approaches to get the estimates. The first is based on the vector field method, introduced in [19] in the unipolar case when \( \mu = 1 \), and the second extends the IMS method used in [4] to the weighted case.

There is a close relation between the estimate of the bottom of the spectrum \( \lambda_1(L+V) \) and the weighted Hardy inequalities. In particular, the existence of positive solutions to (P) is related to the Hardy inequality (1.3) and the nonexistence is due to the optimality of the constant \( c_{o,\mu} \).
The main difficulties to get the inequality in the multipolar case are due to the mutual interaction among the poles. In [8] we used a technique which allowed us to overcome such difficulties in the case of the Gaussian measure, but it does not work in the setting of more general measures.

It is not immediate to generalize the vector field method to the multipolar case. In order to do this, we need to isolate the poles. We state Theorem 2.2 with assumptions on the weights which generalize in a natural way those in the unipolar case (cf. [9]). The limitation of the method is that we do not achieve the best constant $c_{o,\mu}$ on the left-hand side in the estimate.

However, the IMS method allows us to get the best constant. To our knowledge, up to now this is the unique technique which allows to achieve the optimal constant in the case of Lebesgue measure (cf. [4]). We adapt the method to the weighted case.

The technique makes use of a weighted Hardy inequality with a single pole. The need for unipolar estimate is a disadvantage compared to the vector field method and it forces us to use assumptions on $\mu$ which are a bit less general. Good weight functions $\mu$ are the ones that behave in a unipolar way near to the single pole. We use as a suitable inequality the unipolar inequality stated in [9].

A class of functions satisfying our hypotheses is shown in Section 4.

In Section 5 we get the optimality of the constant in the estimate. A crucial point is to find a suitable function $\varphi$ for which the inequality (1.3) does not hold if $c > c_{o,\mu}$. We present a function which involves only one pole reasoning as in [9]. Furthermore we adapt the way to estimate the bottom of the spectrum in [6] to the multipolar case.

We state existence and nonexistence results in Section 6 following Cabrè-Martel’s approach and, then, using multipolar weighted inequalities. So we need that the unperturbed operator $L$ generates a $C_0$-semigroup. In the case of measures of a more general type than the Gaussian one, measures which could have degeneracy in one or more points, we need to require suitable assumptions on $\mu$ to guarantee the generation of a semigroup.

The proof of Theorem 6.2 relies on certain properties of the operator $L$ and of its corresponding semigroup. We ensure that these properties hold reasoning as in [6]. To this aim we state some density results in the multipolar case.

2. Weighted multipolar Hardy inequalities via the vector field method.

Let $\mu \geq 0$ be a weight function on $\mathbb{R}^N$. We define the weighted Sobolev space $H_\mu^1 = H^1(\mathbb{R}^N, \mu(x)dx)$ as the space of functions in $L^2(\mathbb{R}^N, \mu(x)dx)$ whose weak derivatives belong to $L^2(\mathbb{R}^N, \mu(x)dx)$.

The vector field method suggests us to consider the vectorial function

$$F(x) = \sum_{i=1}^n \beta \frac{x - a_i}{|x - a_i|^2} \mu(x), \quad \beta > 0.$$ 

Let us assume the following hypotheses

$H_1)$ \quad i) $\sqrt{\mu} \in H^1_{\text{loc}}(\mathbb{R}^N)$; 
\quad ii) $\mu^{-1} \in L^1_{\text{loc}}(\mathbb{R}^N)$;

$H_2)$ there exists constants $k_1, k_2 \in \mathbb{R}$, $k_2 > 2 - N$, such that

$$\beta \sum_{i=1}^n \frac{(x - a_i)}{|x - a_i|^2} \cdot \nabla \mu \geq \left(-k_1 + \sum_{i=1}^n \frac{k_2 \beta}{|x - a_i|^2}\right) \mu.$$ 

Let us observe that under the assumption $H_1$ the space $C_c^\infty(\mathbb{R}^N)$ is dense in $H^1_\mu$ (see e.g. [22]). So we can regard $H^1_\mu$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Sobolev norm
\[ \| \cdot \|_{H^1_\mu}^2 := \| \cdot \|^2_{L^2_\mu} + \| \nabla \cdot \|^2_{L^2_\mu}. \]
The next result states a preliminary weighted Hardy inequality (cf. [4] for an analogous result obtained in a different way when $\mu = 1$).

**Theorem 2.1.** Let $r_0 = \min_{1 \leq i,j \leq n} |a_i - a_j|/2$, $N \geq 3$, $n \geq 2$. Under hypotheses $H_1$ and $H_2$ we get
\[ c_o(N+k_2) \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\varphi^2}{|x-a_i|^2} \mu(x) dx + c_o(N+k_2) \int_{\mathbb{R}^N} \sum_{i,j=1 \atop i \neq j} \frac{|a_i-a_j|^2}{|x-a_i|^2 |x-a_j|^2} \varphi^2 \mu(x) dx \]
\[ \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) dx + k_1 \int_{\mathbb{R}^N} \varphi^2 \mu(x) dx, \] (2.1)
for any $\varphi \in H^1_\mu$, where $c_o(N+k_2) := (\frac{N+k_2-2}{2})^2$.

**Proof.** By density, it is enough to prove (2.1) for every $\varphi \in C_c^\infty(\mathbb{R}^N)$.

It is immediate to verify that
\[ \int_{\mathbb{R}^N} \varphi^2 \text{div} F dx = \beta \int_{\mathbb{R}^N} \sum_{i=1}^n \left[ \frac{N-2}{|x-a_i|^2} \mu(x) + \frac{(x-a_i)}{|x-a_i|^2} \cdot \nabla \mu \right] \varphi^2 dx. \] (2.2)
As a consequence of our assumptions on $\mu$, we get $F_j, \frac{\partial F_j}{\partial x_j} \in L^1_{\text{loc}}(\mathbb{R}^N)$, where $F_j(x) = \beta \sum_{i=1}^n \frac{(x-a_i)}{|x-a_i|^2} \mu(x)$. This allows us to integrate by parts on the left-hand side in (2.2). To see this it is sufficient to apply the Hölder inequality and the classical Hardy inequality taking in mind the hypothesis $i$) in $H_1$). More precisely, for any $K$ compact set in $\mathbb{R}^N$, we get
\[ \int_K |F_j| dx \leq \beta \int_K \frac{\mu(x)}{|x-a_j|^2} dx \leq \beta \left( \int_K \frac{\mu(x)}{|x-a_j|^2} dx \right)^{\frac{1}{2}} \left( \int_K \mu(x) dx \right)^{\frac{1}{2}} \]
\[ \leq \frac{c_o(N)}{\sqrt{\text{vol}(K)}} \left( \int_K |\nabla \sqrt{\mu}|^2 dx \right)^{\frac{1}{2}} \left( \int_K \mu(x) dx \right)^{\frac{1}{2}}, \]
where $c_o(N) = \frac{(N-2)^2}{4}$ is the constant in the classical Hardy inequality.

For the locally integrability of $\frac{\partial F_j}{\partial x_j}(x) = \frac{\beta}{|x-a_j|^2} \mu(x)(1 - 2 \frac{x^2}{|x-a_j|^2}) + x_j \frac{\partial \mu}{\partial x_j}$ we observe that
\[ \int_K \left| \frac{\partial F_j}{\partial x_j} \right| dx \leq 3\beta \int_K \frac{\mu(x)}{|x-a_j|^2} dx + \beta \int_K \frac{|\nabla \mu|}{|x-a_j|^2} dx \]
\[ \leq \frac{3\beta}{c_o(N)} \int_K |\nabla \sqrt{\mu}|^2 dx + 2\beta \int_K \sqrt{\mu} |\nabla \sqrt{\mu}| dx \]
\[ \leq \frac{3\beta}{c_o(N)} \int_K |\nabla \sqrt{\mu}|^2 dx + 2\beta \left( \int_K \frac{\mu(x)}{|x-a_j|^2} dx \right)^{\frac{1}{2}} \left( \int_K |\nabla \sqrt{\mu}|^2 dx \right)^{\frac{1}{2}}. \]
and apply again the Hardy inequality on the right-hand side above.

So, through Hölder’s and Young’s inequalities, we obtain
We observe that
\[
\int_{\mathbb{R}^N} \varphi^2 \text{div} F \, dx = -2 \int_{\mathbb{R}^N} \varphi F \cdot \nabla \varphi \, dx
\] (2.3)

\[
\leq 2 \left( \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) \, dx \right)^{1/2} \left[ \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\beta (x - a_i)}{|x - a_i|^2} \varphi^2 \mu(x) \, dx \right]^{1/2}
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) \, dx + \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\beta (x - a_i)}{|x - a_i|^2} \varphi^2 \mu(x) \, dx
\]

observing that
\[
\int_{\mathbb{R}^N} \frac{\mu(x)}{|x - a_i|^2} \varphi^2 \, dx < +\infty
\]

by the condition \( i \) in \( H_1 \) and the classical Hardy inequality. From (2.2) and (2.3) we deduce
\[
\int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\beta(N - 2)}{|x - a_i|^2} \varphi^2 \mu(x) \, dx
\] (2.4)

\[
\leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) \, dx + \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\beta^2}{|x - a_i|^2} \varphi^2 \mu(x) \, dx
\]

\[
+ \int_{\mathbb{R}^N} \sum_{i,j=1}^n \frac{\beta (x - a_i) \cdot (x - a_j)}{|x - a_i|^2 |x - a_j|^2} \varphi^2 \mu(x) \, dx - \beta \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{(x - a_i) \cdot \nabla \varphi^2 \mu(x)}{|x - a_i|^2} \, dx.
\]

Now we observe that
\[
\sum_{i,j=1}^n \frac{(x - a_i) \cdot (x - a_j)}{|x - a_i|^2 |x - a_j|^2} = \sum_{i,j=1}^n \frac{|x|^2 - x \cdot a_i - x \cdot a_j + a_i \cdot a_j}{|x - a_i|^2 |x - a_j|^2}
\] (2.5)

\[
= \sum_{i,j=1}^n \frac{|x - a_i|^2 + |x - a_j|^2 - |a_i - a_j|^2}{2 |x - a_i|^2 |x - a_j|^2}
\]

\[
= \sum_{i,j=1}^n \frac{1}{2} \left( \frac{1}{|x - a_i|^2} + \frac{1}{|x - a_j|^2} - \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2} \right)
\]

\[
= (n - 1) \sum_{i=1}^n \frac{1}{|x - a_i|^2} - \frac{1}{2} \sum_{i,j=1}^n \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2}.
\]

Then, taking into account the hypothesis \( H_2 \) and using (2.5), from the estimate (2.4) it follows that
\[
\int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\varphi^2}{|x - a_i|^2} \mu(x) \, dx + \frac{\beta^2}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^n \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2} \varphi^2 \mu(x) \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) \, dx + k_1 \int_{\mathbb{R}^N} \varphi^2 \mu(x) \, dx.
\] (2.6)
The theorem is proved observing that
\[ \max_{\beta} [(N + k_2 - 2)\beta - n\beta^2] = \frac{(N + k_2 - 2)^2}{4n} \]
with \( \beta = \frac{(N + k_2 - 2)}{2n} \).

Now our aim is to estimate the second term on the left-hand side in (2.6) to get a more general Hardy’s inequality. From a mathematical point of view the principal problem is due to the square of the sum on the right-hand side in (2.3). To overcome the difficulties we are able to isolate singularities but we can not achieve the constant \( c_o(N + k_2) \).

We state the following result.

**Theorem 2.2.** Let \( r_0 = \min_{1 \leq i, j \leq n} |a_i - a_j| / 2, \ N \geq 3, \ n \geq 2. \) Then if conditions \( H_1 \) and \( H_2 \) hold, for any \( \varphi \in \mathcal{H}_1^n \) and \( c \in [0, c_o(N + k_2)], \ c_o(N + k_2) = \left( \frac{N + k_2 - 2}{2} \right)^2, \) we get

\[ c \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\varphi^2}{|x - a_i|^2} \mu(x)dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x)dx + K \int_{\mathbb{R}^N} \varphi^2 \mu(x)dx \]  
(2.7)

where \( K = K(n, c, r_0) \).

**Proof.** Arguing as in the proof of Theorem 2.1 (cf. (2.4)) we get

\[ \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\beta(N - 2)}{|x - a_i|^2} \varphi^2 \mu(x)dx \]  
(2.8)

\[ \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x)dx + \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\beta^2}{|x - a_i|^2} \varphi^2 \mu(x)dx \]

\[ + \int_{\mathbb{R}^N \setminus \bigcup_{i=1}^n B(a_k, r_0)} \sum_{i \neq j} \frac{\beta^2 (x - a_i) \cdot (x - a_j)}{|x - a_i|^2|x - a_j|^2} \varphi^2 \mu(x)dx \]

\[ + \int_{\bigcup_{k=1}^n B(a_k, r_0)} \sum_{i \neq j} \frac{\beta^2 (x - a_i) \cdot (x - a_j)}{|x - a_i|^2|x - a_j|^2} \varphi^2 \mu(x)dx \]

\[ - \beta \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{(x - a_i)}{|x - a_i|^2} \nabla \mu \varphi^2 dx =: I_1 + I_2 + I_3 + I_4 + I_5, \]

where \( B(a_k, r_0), \ k = 1, \ldots, n, \) denotes the open ball in \( \mathbb{R}^N \) of radius \( r_0 \) centered at \( a_k \).

Let us estimate \( I_3 \) and \( I_4. \) The first integral can be estimated as follows

\[ I_3 \leq \frac{\beta^2}{r_0^2} n(n - 1) \int_{\mathbb{R}^N \setminus \bigcup_{k=1}^n B(a_k, r_0)} \varphi^2 \mu(x)dx. \]  
(2.9)

For the second integral we isolate the singularities and then, using again Young’s inequality, we get

\[ I_4 \leq \sum_{k=1}^n \left( \int_{B(a_k, r_0)} \sum_{j=1}^n \frac{\beta^2}{|x - a_k||x - a_j|} \varphi^2 \mu(x)dx \right. \]
The maximum of the function $\beta$ attained in $\text{Remark 1.}$.

The integral $I_5$ can be estimated applying $H_2$.

Taking into account (2.8) and using (2.9), (2.10) we deduce that

$$\int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\beta(N + k_2 - 2) - \beta^2 [1 + \frac{\epsilon}{2}(n - 1)]}{|x - a_i|^2} \varphi^2 \mu(x) dx$$

$$\leq \int_{\mathbb{R}^N} |
abla \varphi|^2 \mu(x) dx + K \int_{\mathbb{R}^N} \varphi^2 \mu(x) dx,$$

where

$$K = \frac{\beta^2}{r_0^2} \max \left\{ n \left( n - 2 + \frac{1}{2\epsilon} \right), k_1 \right\}.$$ 

The maximum of the function $\beta \mapsto (N + k_2 - 2)\beta - \beta^2 \left[ 1 + \frac{\epsilon}{2}(n - 1) \right]$ is $c_{\epsilon}(N + k_2)$

attained in $\beta_{\text{max}} = \frac{\sqrt{c_{\epsilon}(N + k_2)}}{1 + \frac{\epsilon}{2}(n - 1)}$. So, if we set

$$c = (N + k_2 - 2)\beta - \beta^2 \left[ 1 + \frac{\epsilon}{2}(n - 1) \right]$$

we deduce from (2.11) that for $c \in \left( 0, \frac{c_{\epsilon}(N + k_2)}{1 + \frac{\epsilon}{2}(n - 1)} \right)$, for any $\epsilon > 0$ it holds

$$c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2 \mu(x) dx \leq \int_{\mathbb{R}^N} |
abla \varphi|^2 \mu(x) dx + K \int_{\mathbb{R}^N} \varphi^2 \mu(x) dx.$$ 

The relation (2.12) between $\beta$ and $c$ allow us to write $\beta$ in the following form

$$\beta_{\epsilon}^\pm = \frac{\sqrt{c_{\epsilon}(N + k_2)} \pm \sqrt{c_{\epsilon}(N + k_2) - c \left[ 1 + \frac{\epsilon}{2}(n - 1) \right]}}{1 + \frac{\epsilon}{2}(n - 1)}.$$  

**Remark 1.** Of course, the Hardy inequalities stated in Section 2 hold also in the case of the Lebesgue measure. The constants $k_1$ and $k_2$ are equal to zero.
The Theorem 2.2 works also in the case \( n=1 \), replacing the constant \((n-2)\) in (2.10) with \((n-1)\), but the situation is different when one has a single pole. The method makes it possible to achieve the constant \( c_0(N+k_2) \) on the left-hand side in (2.7).

When the measure is of Gaussian type the authors in [16] used this technique to state a weighted Hardy's inequality. In Theorem 2.2 we use it in case of a more general measure.

The method, applied to obtain the classical Hardy inequality, is well known in the case of a single pole when \( \mu = 1 \) (see [19]).

We are investigating extensions of the method to get more general estimates, as the Hardy type inequalities, motivated by the application to evolution problems with more general operators.

We remark that in the applications to evolution problems we need \( C_0 \)-semigroup generation results. Operators of a more general type, perturbed by potentials with different singularities, for which the generation of semigroups was stated, have been studied, for example, in [10, 11, 12] when \( \mu = 1 \) and in [7] in weighted spaces.

3. Weighted multipolar Hardy inequalities via the IMS method. In this section we state the weighted multipolar Hardy inequality using the so-called IMS truncation method (for Ismagilov, Morgan, Morgan-Simon, Sigal, see [20, 21]), which consists in localizing the integrals around the singularities by using a partition of unity. This method, unlike the vector field one, allows us to achieve the constant \( c_0(N+k_2) \) on the left-hand side in the inequality (2.7).

We argue as in [4] adapting the proof to the weighted case. We assume that the weight functions \( \mu \) satisfy \( H'_2 \)) in Section 2 and \( H'_2 \)) as follows:

\[ f_{\varepsilon,i} = (\varepsilon + |x-a_i|^2)^{\frac{\alpha}{2}}, \quad (N + k_2 - 2) < \alpha < 0, \quad \varepsilon > 0, \]

it holds that for any \( \varepsilon > 0 \)

\[ \frac{\nabla f_{\varepsilon,i}}{f_{\varepsilon,i}} \cdot \nabla \mu = \frac{\alpha(x-a_i)}{\varepsilon + |x-a_i|^2} \cdot \nabla \mu \leq \left( k_1 + \frac{k_2\alpha}{\varepsilon + |x-a_i|^2} \right) \mu \]

in \( B(a_i, r_0) \) for any \( i = 1, \ldots, n \).

The condition \( H'_2 \)) allow us to apply locally the weighted Hardy inequality stated in [9] in the proof of Theorem 3.1 below.

The statement of our inequality is the following.

**Theorem 3.1.** Assume hypotheses \( H_1 \) and \( H'_2 \)). Let \( N \geq 3 \), \( n \geq 2 \) and \( r_0 = \min_{1 \leq i < j \leq n} |a_i - a_j|/2 \). Then there exists a constant \( k_0 \in [0, \pi^2) \) such that

\[ c \int_{\mathbb{R}^N} \left( \sum_{i=1}^{n} \frac{\varphi^2}{|x-a_i|^2} \mu(x)dx \right) \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x)dx + \left[ \frac{k_0+(n+1)c}{r_0^2} + k_1 \right] \int_{\mathbb{R}^N} \varphi^2 \mu(x)dx, \quad (3.1) \]

for all \( \varphi \in H^1_\mu \), where \( c \in (0, c_0(N + k_2)] \) with \( c_0(N + k_2) = \left( \frac{N+k_2-2}{2} \right)^2 \).

In order to prove the Theorem via the IMS method, we need to recall the notion of partition of unity and some related lemmas.
We say that a finite family \( \{J_i\}_{i=1}^{n+1} \) of real valued functions \( J_i \in W^{1,\infty}(\mathbb{R}^N) \) is a partition of unity in \( \mathbb{R}^N \) if \( \sum_{i=1}^{n+1} J_i^2 = 1 \). Furthermore we require that
\[
\Omega_i \cap \Omega_j = \emptyset \quad \text{for any } i, j = 1, \ldots, n, \ i \neq j,
\] (3.2)
where \( \bar{\Omega}_i = \text{supp}(J_i), \ i = 1, \ldots, n. \)
Any family of this type has the following properties:
\begin{enumerate}
\item \( \sum_{i=1}^{n+1} J_i \partial_{\alpha} J_i = 0 \) for any \( \alpha = 1, \ldots, N; \)
\item \( J_{n+1} = \sqrt{1 - \sum_{i=1}^{n} J_i^2}; \)
\item \( \sum_{i=1}^{n+1} |\nabla J_i|^2 \in L^\infty(\mathbb{R}^N); \)
\item \( \sum_{i=1}^{n+1} |\nabla J_i|^2 = \sum_{i=1}^{n} |\nabla J_i|^2. \)
\end{enumerate}

By proceeding as in [4, Lemma 2], we state the following result.

**Lemma 3.2.** Let \( \{J_i\}_{i=1}^{n+1} \) be a partition of unity satisfying (3.2). For any \( \varphi \in H_{\mu}^1 \) and any \( V \in L_{\text{loc}}^1(\mathbb{R}^N) \) we get
\[
\int_{\mathbb{R}^N} (|\nabla \varphi|^2 - V \varphi^2) \mu(x)dx = \sum_{i=1}^{n+1} \int_{\mathbb{R}^N} (|\nabla (J_i \varphi)|^2 - V (J_i \varphi)^2) \mu(x)dx - \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla J_i|^2 \varphi^2 \mu(x)dx.
\]

**Proof.** We can immediately observe that
\[
\int_{\mathbb{R}^N} V \left( \sum_{i=1}^{n+1} (J_i \varphi) \right) \mu(x)dx = \int_{\mathbb{R}^N} V \left( \sum_{i=1}^{n+1} J_i^2 \right) \varphi^2 \mu(x)dx = \int_{\mathbb{R}^N} V \varphi^2 \mu(x)dx. \quad (3.3)
\]

On the other hand,
\[
\sum_{i=1}^{n+1} |\nabla (J_i \varphi)|^2 = \sum_{i=1}^{n+1} |(\nabla J_i) \varphi + (\nabla \varphi) J_i|^2 \\
= \sum_{i=1}^{n+1} |\nabla J_i|^2 \varphi^2 + \sum_{i=1}^{n+1} |\nabla \varphi|^2 J_i^2 + 2 \sum_{i=1}^{n+1} (J_i \nabla J_i) \cdot (\varphi \nabla \varphi) \\
= \sum_{i=1}^{n+1} |\nabla J_i|^2 \varphi^2 + |\nabla \varphi|^2 + \left( \sum_{i=1}^{n+1} J_i \nabla J_i \right) \cdot \nabla \varphi^2. \quad (3.4)
\]

By property a) it follows that \( \left( \sum_{i=1}^{n+1} J_i \nabla J_i \right) \cdot \nabla \varphi^2 = 0, \) then by integrating (3.4) on \( \mathbb{R}^N \) we obtain
\[
\int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x)dx = \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla (J_i \varphi)|^2 \mu(x)dx - \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla J_i|^2 \varphi^2 \mu(x)dx. \quad (3.5)
\]

From (3.3) and (3.5) we get the result. \( \square \)

In the following we set
\[
V_n(x) = \sum_{i=1}^{n} \frac{1}{|x - a_i|^2}.
\]
We recall a preliminary Lemma, stated in [4], about the case \( n = 2, \) with \( a_1 = a, \)
\( a_2 = -a \) and \( 0 < r_0 \leq |a|. \)
Lemma 3.3. There is a partition of the unity \( \{ J_i \}_{i=1}^{3} \) satisfying (3.2) with \( J_1 \equiv 1 \) on \( B(a, r_0/2) \), \( J_1 \equiv 0 \) on \( B^*(a, r_0) \), \( J_2(x) = J_1(-x) \) for any \( x \in \mathbb{R}^N, 0 < r_0 \leq |a| \), such that, for any \( c > 0 \), there exists a constant \( k_0 \in [0, \pi^2) \) for which, almost everywhere for all \( x \in \text{supp}(J_1) \cup \text{supp}(J_2) \), we have

\[
\sum_{i=1}^{3} |\nabla J_i|^2 + c J_3^2 V_2(x) = \sum_{i=1,2} |\nabla J_i|^2 + c J_3^2 V_2(x) \leq \frac{k_0 + 2c}{r_0^2}. \tag{3.6}
\]

As observed in [4], a partition of unity satisfying the hypotheses of Lemma 3.3 is given by setting

\[
J(t) := \begin{cases} 
1 & \text{if } t \leq 1/2 \\
\sin(\pi t) & \text{if } 1/2 \leq t \leq 1 \\
0 & \text{if } t \geq 1
\end{cases} \tag{3.7}
\]

and defining \( J_1(x) := J(|x - a|/r_0) \), \( J_2(x) := J(|x + a|/r_0) \), and \( J_3(x) := \sqrt{1 - J_1^2 - J_2^2} \).

Now we are able to proceed with the proof of inequality (3.1).

Proof of Theorem 3.1. Let us define the following quadratic form

\[
Q[\varphi] := \int_{\mathbb{R}^N} (|\nabla \varphi|^2 - c V_n(x) \varphi^2) \mu(x) dx, \quad \varphi \in H_\mu^1, \tag{3.8}
\]

where \( V_n(x) = \sum_{i=1}^{n} \frac{1}{|x - a_i|^2} \).

Let us consider a partition of unity \( \{ J_i \}_{i=1}^{n+1} \) satisfying (3.2) such that \( J_i(x) = J(|x - a_i|/r_0) \) for all \( x \in \mathbb{R}^N, i = 1, \ldots, n \), with \( J \) as in (3.7), \( \text{supp}(J_i) = B(a_i, r_0) \). Then \( |x - a_i| \geq r_0 \) in \( B(a_j, r_0) \) for \( i \neq j \), and \( V_n(x) \leq \frac{n}{r_0^2} \) on \( \mathbb{R}^N \setminus \bigcup_{i=1}^{n} B(a_i, r_0) \).

By virtue of Lemma 3.2 we are able to write (3.8) as follows

\[
Q[\varphi] = \sum_{i=1}^{n} Q[J_i \varphi] + R_n, \quad \varphi \in H_\mu^1, \tag{3.9}
\]

where

\[
R_n = \int_{\mathbb{R}^N} |\nabla(J_{n+1} \varphi)|^2 \mu(x) dx - c \int_{\mathbb{R}^N} V_n |J_{n+1} \varphi|^2 \mu(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^N} |\nabla J_i|^2 \varphi^2 \mu(x) dx.
\]

Thanks to the property d) we have

\[
R_n = \int_{\mathbb{R}^N} |\nabla(J_{n+1} \varphi)|^2 \mu(x) dx - c \int_{\mathbb{R}^N} V_n \left( 1 - \sum_{i=1}^{n} J_i^2 \right) \varphi^2 \mu(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{|\nabla J_i|^2}{1 - J_i^2} \varphi^2 \mu(x) dx.
\]

Moreover, using the condition (3.2) we get

\[
R_n \geq -c \int_{\mathbb{R}^N} V_n \left( 1 - \sum_{i=1}^{n} J_i^2 \right) \varphi^2 \mu(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{|\nabla J_i|^2}{1 - J_i^2} \varphi^2 \mu(x) dx.
\]

For every \( i \in \{1, \ldots, n\} \) we can apply Lemma 3.3 on \( B(a_i, r_0) \) with \( (a_i, a_j) = (-a, a) \) up to a change of coordinates for some \( j \neq i \). Considering the partition
\{J_i, J_j, \sqrt{1 - J_i^2 - J_j^2}\} and taking into account that \(J_j \equiv 0\) on \(B(a_i, r_0)\), we get

\[
R_n \geq - \frac{n}{r_0^2} \int_{B(a_i, r_0)} \left[ \frac{k_0 + 2c}{r_0^2} + c(1 - J_i^2) \left( \sum_{j \neq i} \frac{1}{|x - a_j|^2} \right) \right] \varphi^2 \mu(x) dx \tag{3.10}
\]

\[
- \frac{c}{r_0^2} \int_{R^N \setminus \bigcup_{i=1}^n B(a_i, r_0)} \varphi^2 \mu(x) dx
\]

\[
\geq - \frac{n}{r_0^2} \int_{B(a_i, r_0)} \left[ \frac{k_0 + 2c}{r_0^2} + \frac{(n-2)c}{r_0} (1 - J_i^2) \right] \varphi^2 \mu(x) dx - \frac{c}{r_0^2} \int_{R^N \setminus \bigcup_{i=1}^n B(a_i, r_0)} \varphi^2 \mu(x) dx,
\]

where \(k_0 \in [0, \pi^2]\), since we can bound \(\frac{1}{|x - a_i|^2}\) by \(\frac{1}{r_0^2}\) for all \(k \neq i, j\). Taking into account (3.9) and using locally the unipolar Hardy inequality stated in [9]

\[
c \int_{R^N} \frac{\varphi^2 \mu(x) dx}{|x - a_i|^2} \leq \int_{R^N} |\nabla \varphi|^2 \mu(x) dx + k_1 \int_{R^N} \varphi^2 \mu(x) dx, \quad \varphi \in H_\mu^1,
\]

for any \(i \in \{1, \ldots, n\}\), where \(c \in (0, c_o(N + k_2)]\) with \(c_o(N + k_2) = \left(\frac{N + k_2 - 2}{2}\right)^2\), we obtain

\[
Q[J_i \varphi] = \int_{R^N} |\nabla (J_i \varphi)|^2 \mu(x) dx - c \int_{R^N} \left( \frac{1}{|x - a_i|^2} + \sum_{j \neq i} \frac{1}{|x - a_j|^2} \right) |J_i \varphi|^2 \mu(x) dx
\]

\[
\geq - \left[ k_1 + \frac{(n-1)c}{r_0^2} \right] \int_{B(a_i, r_0)} |J_i \varphi|^2 \mu(x) dx,
\]

from which

\[
\sum_{i=1}^n Q[J_i \varphi] \geq - k_1 \sum_{i=1}^n \int_{B(a_i, r_0)} \varphi^2 \mu(x) dx - \frac{(n-1)c}{r_0^2} \sum_{i=1}^n \int_{B(a_i, r_0)} J_i^2 \varphi^2 \mu(x) dx \tag{3.11}
\]

From (3.9), (3.10) and (3.11) we deduce

\[
Q[\varphi] \geq - \sum_{i=1}^n \int_{B(a_i, r_0)} \left[ \frac{k_0 + 2c}{r_0^2} + \frac{(n-2)c}{r_0^2} (1 - J_i^2) + k_1 + \frac{(n-1)c}{r_0^2} J_i^2 \right] \varphi^2 \mu(x) dx
\]

\[
- \frac{c}{r_0^2} \int_{R^N \setminus \bigcup_{i=1}^n B(a_i, r_0)} \varphi^2 \mu(x) dx.
\]

Since

\[
k_0 + 2c + c(n-2)(1 - J_i^2) + c(n-1)J_i^2 = k_0 + cn + cJ_i^2 \leq k_0 + c(n + 1),
\]

we finally obtain

\[
Q[\varphi] \geq - \left[ \frac{k_0 + (n+1)c}{r_0^2} + k_1 \right] \int_{\bigcup_{i=1}^n B(a_i, r_0)} \varphi^2 \mu(x) dx - \frac{c}{r_0^2} \int_{R^N \setminus \bigcup_{i=1}^n B(a_i, r_0)} \varphi^2 \mu(x) dx
\]

\[
\geq - \left[ \frac{k_0 + (n+1)c}{r_0^2} + k_1 \right] \int_{R^N} \varphi^2 \mu(x) dx,
\]

from which we get inequality (3.1). \(\square\)
4. A class of weight functions. A class of weight functions satisfying hypotheses $H_1$) and $H_2$) is the following
\[
\mu(x) = e^{-\delta \sum_{j=1}^{n} \frac{|x-a_j|^m}{|x-a_1|^\gamma \cdots |x-a_n|^\gamma}}, \quad \delta \geq 0, \quad \gamma < N - 2, \quad m \leq 2. \tag{4.1}
\]
For $\gamma = 0$, $\delta \neq 0$ and $m = 2$ we get the Gaussian function.

Taking into account that out of the ball $B(a_i, r_0)$ the term $\frac{1}{|x-a_j|^m}$ is bounded and the balls are disjoined, we can see that the function $\mu$ satisfies $H_1$) if $\gamma > -N$. In order to verify $H_2$), with $\beta = -\alpha$, $\alpha < 0$, we proceed in the following way.

We observe that, if $\mu_j(x) = e^{-\delta \sum_{j=1}^{n} \frac{|x-a_j|^m}{|x-a_j|^\gamma}}$, then
\[
\nabla \mu = \frac{n}{\mu} \sum_{j=1}^{n} \nabla \mu_j = \sum_{j=1}^{n} \left(-\mu_j \frac{\delta m |x-a_j|^m}{|x-a_j|^2} \right) \frac{(x-a_j)}{|x-a_j|^2}.
\]
Starting from $H_2$) and using (2.5) we get
\[
-\alpha \sum_{i=1}^{n} \frac{(x-a_i)}{|x-a_i|^2} \cdot \nabla \mu = \frac{n}{\mu} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(-\alpha \gamma - \alpha \delta m |x-a_j|^m \right) \frac{(x-a_i) \cdot (x-a_j)}{|x-a_i|^2 |x-a_j|^2}
\]
\[
= \frac{n}{\mu} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(-\alpha \gamma - \alpha \delta m |x-a_j|^m \right) \left(1 + \frac{|x-a_j|^2 - |a_i - a_j|^2}{|x-a_j|^2} \right)
\]
\[
\leq k_1 + \sum_{i=1}^{n} \frac{k_2 \alpha}{|x-a_i|^2 |x-a_i|^2}. \tag{4.2}
\]

In $B(a_k, r_0)$, for any $k$, we isolate the term with $i = k$, so the condition $H_2$) takes the form
\[
-\alpha \gamma - \alpha \delta m |x-a_k|^m + \sum_{i \neq k} \frac{-\alpha \gamma - \alpha \delta m |x-a_i|^m}{|x-a_k|^2} + \frac{1}{|x-a_k|^2} \sum_{j \neq k} \frac{(-\alpha \gamma - \alpha \delta m |x-a_j|^m)}{|x-a_j|^2} \left(|x-a_j|^2 + |x-a_k|^2 - |a_k - a_j|^2 \right)
\]
\[
= \sum_{i \neq k} \frac{1}{|x-a_i|^2} \sum_{j \neq i} \frac{(-\alpha \gamma - \alpha \delta m |x-a_j|^m)}{|x-a_j|^2} \left(|x-a_j|^2 + |x-a_i|^2 - |a_i - a_j|^2 \right)
\]
\[
= J_1 + J_2 + J_3 + J_4 \leq k_1 + \frac{k_2 \alpha}{|x-a_k|^2} + \sum_{i \neq k} \frac{k_2 \alpha}{|x-a_i|^2}. \tag{4.3}
\]

We observe that, in $B(a_k, r_0)$,
\[
|x-a_k| \leq r_0, \quad r_0 \leq |x-a_j| \leq r_0 + |a_k - a_j| \quad \forall j \neq k
\]
then
\[
J_2 + J_4 \leq \sum_{i \neq k} \frac{k_1}{a-1} + \sum_{i \neq k} \frac{k_2 \alpha}{|x-a_i|^2}
\]
for $k_1$ large enough. On the other hand
\[
(J_1 + J_3)|x-a_k|^2 \leq -\alpha \gamma \left(1 + \frac{n-1}{2} - \frac{1}{2} \sum_{j \neq k} \frac{|a_k - a_j|^2}{|x-a_j|^2} \right) - \alpha k_2 - \frac{\delta m}{2} \sum_{j \neq k} |x-a_j|^m \left(1 - \frac{|a_k - a_j|^2}{|x-a_j|^2} \right) \tag{4.4}
\]
When \( x \) is near to the pole \( a_k \) the contribution of the other poles tends to zero.

To estimate the term with of \( |x - a_j| \) we use the relation

\[
|x - a_j| \leq |x - a_i| + |a_i - a_j| \quad \forall i, j \in \{1, \ldots, n\}.
\]

Then we get

\[
-\alpha \frac{\delta m}{2} \sum_{j \neq k} |x - a_j|^m \left( 1 - \frac{|a_k - a_j|^2}{|x - a_j|^2} \right)
\leq -\alpha \frac{\delta m}{2} \sum_{j \neq k} (|a_k - a_j|)^m \left[ 1 - \frac{|a_k - a_j|^2}{(|a_k| + |x - a_j|)^2} \right].
\]

(4.5)

If \( |x - a_k| \leq \rho, \rho \leq r_0 \), the last term in (4.5) can be estimated by

\[
-\alpha \frac{\delta m}{2} \sum_{j \neq k} (\rho + |a_k - a_j|)^m \left[ 1 - \frac{|a_k - a_j|^2}{(\rho + |a_k - a_j|)^2} \right] = -\alpha \frac{\delta m}{2} c_\rho
\]

observing that \( c_\rho \) tends to zero when \( \rho \) goes to zero. Then inequality (4.4) is satisfied for \( k_1 \) large enough, with \( \rho \) small enough, and

\[
0 \leq \gamma \leq -\frac{k_2 + \frac{\delta m}{2} c_\rho}{1 + \frac{n}{2} + c_1} \quad \text{and} \quad -\frac{k_2 + \frac{\delta m}{2} c_\rho}{1 + \frac{n}{2} + c_1} \leq \gamma < 0.
\]

where

\[
c_1 = \begin{cases} 
-\frac{1}{2} \sum_{j \neq k} \frac{|a_k - a_j|^2}{r_0 + |a_k - a_j|^2} & \text{if } \gamma > 0 \\
\frac{1}{2} \sum_{j \neq k} \frac{|a_k - a_j|^2}{r_0^2} & \text{if } \gamma < 0,
\end{cases}
\]

Far enough away from the other poles \( a_j \), with \( j \neq k \), and for \( |x - a_k| \geq \rho \), the condition \( H_2 \) is connected to the inequality

\[
-\alpha \gamma \left( 1 + \frac{n-1}{2} + c_2 \right) - \alpha k_2 - \left( \frac{k_1}{n} + c_3 \right) |x - a_k|^2 - \alpha \delta m c_4 |x - a_k|^m \leq 0
\]

(4.6)

where the constant \( c_2, c_3 \) and \( c_4 \) are so defined:

\[
c_2 = \begin{cases} 
0 \quad \text{if } \gamma > 0 \\
c_1 \quad \text{if } \gamma < 0,
\end{cases}
\]

\[
c_3 = \begin{cases} 
\frac{\gamma}{2} \frac{n-1}{r_0} + \frac{\delta m}{2} \frac{n-1}{r_0^{-m}} & \text{if } \gamma > 0 \\
\frac{\lambda}{2} \frac{m n - 1}{r_0^{-m}} & \text{if } \gamma < 0,
\end{cases}
\]

\[
c_4 = 1 + \frac{c_\rho}{2 \rho^m}.
\]

The inequalities (4.4) and (4.6) are both verified if \( k_1 \) is large enough, \( \rho \) small enough, and

\[
0 \leq \gamma \leq -\frac{k_2}{1 + \frac{n-1}{2}} \quad \text{and} \quad -\frac{k_2 + \frac{\delta m}{2} c_\rho}{1 + \frac{n}{2} + c_1} \leq \gamma < 0.
\]

In order to verify \( H_2' \) we start with the analogous of (4.2)

\[
\nabla \epsilon f_{\epsilon,i}, \frac{\nabla \mu}{\mu} = \nabla f_{\epsilon,i}, \frac{\nabla \mu_j}{\mu_j} \leq k_1 + \frac{k_2 \alpha}{\epsilon + |x - a_i|^2},
\]

(4.7)

for any \( i \in \{1, \ldots, n\} \), and reason as in the previous case.
The function Hardy inequality (3.1).

The above condition allows us to estimate the exponent \( \eta \) in the proof of Theorem 5.1 in a suitable way.

Now we can state the optimality result.

**Theorem 5.1.** In the hypotheses \( H_1 \), \( H_2 \) and \( H_3 \), the inequality (3.1) doesn’t hold for any \( \varphi \in H_1^1 \) if \( c > c_0(N + k_2) = (\frac{N+k_2-2}{2})^2 \).

**Proof.** Under conditions \( H_1 \) and \( H_2 \) we are able to state the weighted multipolar Hardy inequality (3.1).

Let us fix a pole \( a_i \) such that \( H_3 \) holds. Let \( \theta \in C_c^\infty(\mathbb{R}^N) \) a cut-off function, \( 0 \leq \theta \leq 1, \theta = 1 \) in \( B(a_i, 1) \) and \( \theta = 0 \) in \( B^c(a_i, 2) \). We introduce the function

\[
\varphi_{\varepsilon,i}(x) = (\varepsilon + |x - a_i|)^\eta \theta(x),
\]

where \( \varepsilon > 0 \) and the exponent \( \eta \) is such that

\[
\max \left\{-\sqrt{c}, -\frac{N+k_2}{2}\right\} < \eta < \min \left\{-\frac{N+k_2-2}{2}, 0\right\}.
\]

The function \( \varphi_{\varepsilon,i} \) belongs to \( H_1^1 \) for any \( \varepsilon > 0 \).

For this choice of \( \eta \) we obtain \( \eta^2 < c \), \( |x - a_i|^{2\eta} \in L^1_{\text{loc}}(\mathbb{R}^N, \mu(x)dx) \) and \( |x - a_i|^{2\eta - 2} \notin L^1_{\text{loc}}(\mathbb{R}^N, \mu(x)dx) \) by hypothesis \( H_3 \).

Let us assume that \( c > c_0(N + k_2) \). In order to state the result, we prove that the bottom of the spectrum of the operator \( -(L + V) \)

\[
\lambda_1(L + V) = \inf_{\varphi \in H_1^1(\mathbb{R}^N) \setminus \{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x)dx - c \sum_{j=1}^n \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_j|^2} \mu(x)dx}{\int_{\mathbb{R}^N} \varphi^2 \mu(x)dx} \right\}
\]

is \(-\infty\). For this purpose we estimate at first the numerator in (5.2).

\[
\int_{\mathbb{R}^N} \left( |\nabla \varphi_{\varepsilon,i}|^2 - \sum_{j=1}^n \frac{c}{|x-a_j|^2} \varphi^2_{\varepsilon,i} \right) \mu(x)dx
\]

\[
= \int_{B(a_i,1)} \left[ |\nabla(\varepsilon + |x-a_i|)^\eta|^2 - \sum_{j=1}^n \frac{c}{|x-a_j|^2}(\varepsilon + |x-a_i|)^{2\eta} \right] \mu(x)dx
\]

\[
+ \int_{B^c(a_i,1)} \left[ |\nabla(\varepsilon + |x-a_i|)^\eta|^2 - \sum_{j=1}^n \frac{c}{|x-a_j|^2}(\varepsilon + |x-a_i|)^{2\eta} \theta^2 \right] \mu(x)dx
\]

\[
\leq \int_{B(a_i,1)} \left[ \eta^2(\varepsilon + |x-a_i|)^{2\eta-2} - \frac{c}{|x-a_i|^2}(\varepsilon + |x-a_i|)^{2\eta} \right] \mu(x)dx
\]

\[
+ \eta^2 \int_{B^c(a_i,1)} (\varepsilon + |x-a_i|)^{2\eta-2} \theta^2 \mu(x)dx + \int_{B^c(a_i,1)} (\varepsilon + |x-a_i|)^{2\eta} |\nabla \theta|^2 \mu(x)dx
\]
for all \( \phi \) and nonexistence conditions using Cabré-Martel’s approach.

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In order to investigate on existence and nonexistence of positive weak solutions to the evolution problem \((P)\) using multipolar weighted Hardy’s inequalities, we need to state some preliminary results regarding the operator \( L \), its associated semigroup, and a density result in the space \( H^1_0 \). These results will allow us to state existence and nonexistence conditions using Cabré-Martel’s approach.
Let us assume that the function $\mu$ is a weight function on $\mathbb{R}^N$, $\mu > 0$. In the hypothesis $\mu \in C^{1,\lambda}_{loc}(\mathbb{R}^N)$, $\lambda \in (0, 1)$ it is known that the operator $L$ with domain
\[
D_{\text{max}}(L) = \left\{ u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \text{ for all } 1 < p < \infty, Lu \in C_b(\mathbb{R}^N) \right\}
\]
is the weak generator of a not necessarily $C_0$-semigroup in $C_b(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N} Lu(x) dx = 0$ for any $u \in C_c^\infty(\mathbb{R}^N)$, then $\mu(x)dx$ is the invariant measure for this semigroup in $C_b(\mathbb{R}^N)$. So we can extend it to a positivity preserving and analytic $C_0$-semigroup \{\(T(t)\)\}_{t \geq 0}$ on $L^2_\mu$, whose generator is still denoted by $L$ (see [18]).

In the more general setting, when the assumptions on $\mu$ allow degeneracy at some points, we require the further conditions to get that $L$ generates a semigroup. In particular we assume
\[
H_1) \mu \in C^{1,\lambda}_{loc}(\mathbb{R}^N \setminus \{a_1, \ldots, a_n\}), \quad \lambda \in (0, 1), \quad \mu \in H^1_{loc}(\mathbb{R}^N), \quad \frac{\nabla \mu}{\mu} \in L^r_{loc}(\mathbb{R}^N) \text{ for some } r > N, \text{ and } \inf_{x \in K} \mu(x) > 0 \text{ for any compact set } K \subset \mathbb{R}^N.
\]
So by [1, Corollary 3.7], we have that the closure of $(L, C^\infty_c(\mathbb{R}^N))$ on $L^2_\mu$ generates a strongly continuous and analytic Markov semigroup \{\(T(t)\)\}_{t \geq 0} on $L^2_\mu$.

For such a semigroup \{\(T(t)\)\}_{t \geq 0} and its generator $L$ there are some useful properties stated in Proposition 1 and Lemma 6.1 below. These results extend to the multipolar case the analogous results in [6]. We omit the proof of the first preliminary properties since it is the same as in [6, Proposition 2.1].

**Proposition 1.** Assume that $\mu$ satisfies $H_1)$. Then the following assertions hold:

1. $D(L) \subset H^1_\mu$.
2. For every $f \in D(L)$, $g \in H^1_\mu$ we have
\[
\int Lf g \mu(x) dx = - \int \nabla f \cdot \nabla g \mu(x) dx.
\]
3. $T(t)L^2_\mu \subset D(L)$ for all $t > 0$.

Now we prove the density of $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \ldots, a_n\})$ in $H^1_1$.

Let us set $L^p_\mu := LP(\mathbb{R}^N, \mu(x)dx)$ and $\|u\|_{p,\mu} := (\int_{\mathbb{R}^N} |u|^p \mu(x)dx)^{\frac{1}{p}}$, $1 \leq p < \infty$.

**Proposition 2.** Let $W^{1,p}_\mu = C_c^\infty(\mathbb{R}^N) \|\cdot\|_{w^{1,p}}$ where $\|u\|_{w^{1,p}_\mu} = \|u\|_{p,\mu} + \|\nabla u\|_{p,\mu}$.

If the function $\mu$ satisfies the condition
\[
\lim_{\delta \to 0} \frac{1}{\delta^p} \int_{B(a_i, \delta)} \mu(x) dx = 0 \quad \forall i \in \{1, \ldots, n\},
\]
then $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \ldots, a_n\})$ is dense in $W^{1,p}_\mu$.

**Proof.** Our aim is to approximate $u \in C_c(\mathbb{R}^N)$ with functions in $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \ldots, a_n\})$ with respect to the norm $\|\cdot\|_{w^{1,p}_\mu}$. Let

\[
\vartheta(x) = \begin{cases} 0 & \text{ in } \bigcup_{i=1}^n B(a_i, \frac{\epsilon}{2}), \\ \phi_1(|x - a_1|) & \text{ in } B(a_1, r_0) \setminus B(a_1, \frac{\epsilon}{2}), \\ \vdots & \vdots \\ \phi_n(|x - a_n|) & \text{ in } B(a_n, r_0) \setminus B(a_n, \frac{\epsilon}{2}), \\ 1 & \text{ in } \mathbb{R}^N \setminus \bigcup_{i=1}^n B(a_i, r_0), \end{cases}
\]
where \( \phi_i \in C^\infty_c(\mathbb{R}^N) \) for any \( i \in \{1, \ldots, n\} \), such that \( \phi_i = 0 \) on \( B(0, \frac{r_0}{2}) \) and \( \phi_i = 1 \) on \( B'(a_i, r_0) \).

We observe that \( \partial_k(x) = \vartheta(kx) \) belongs to \( C^\infty_c(\mathbb{R}^N \setminus \{a_1, \ldots, a_n\}) \), \( \partial_k \to 1 \) pointwisely, as \( k \to \infty \), in \( \mathbb{R}^N \setminus \{a_1, \ldots, a_n\} \) and \( ||\nabla \vartheta_k||_{\infty} \leq Ck \). So we get
\[
||u - (u\partial_k)||^{p}_{W^{1,p}} \leq C \left( ||u(1 - \partial_k)||_{p,\mu} + ||\nabla (u(1 - \partial_k))||^{p}_{p,\mu} \right).
\]
The first term on the right-hand side converges to 0 by dominated convergence as \( k \to \infty \). As regards the second one we have
\[
||\nabla (u(1 - \partial_k))||^{p}_{p,\mu} \leq C \left( \int_{\mathbb{R}^N} (1 - \partial_k)^p |\nabla u|^p \mu(x) dx + \int_{\mathbb{R}^N} |\nabla \partial_k|^p |u|^p \mu(x) dx \right)
\]
\[
\leq C \left( \int_{\mathbb{R}^N} (1 - \partial_k)^p |\nabla u|^p \mu(x) dx + k \int_{\bigcup_{i=1}^{n} B(a_i, \frac{r_0}{2})} |u|^p \mu(x) dx \right)
\]
\[
\leq C \left( \int_{\mathbb{R}^N} (1 - \partial_k)^p |\nabla u|^p \mu(x) dx + k \int_{\bigcup_{i=1}^{n} B(a_i, \frac{r_0}{2})} |u|^p \mu(x) dx \right).
\]

To get the result we observe that the first integral converges to 0 by dominated convergence, the last one by condition (6.2).

We observe that, under assumption \( H_1 \), the space \( W^{1,2}_\mu \) coincides with \( H^1 \) (see [22, Corollary 1.2]). So by Proposition 2 we deduce the density of \( C^\infty_c(\mathbb{R}^N \setminus \{a_1, \ldots, a_n\}) \) in \( H^1_{\mu} \) if the function \( \mu \) satisfies (6.2).

The next step is to prove that \( \mu \in H^1_{\mu} (\mathbb{R}^N) \) fulfills this condition.

**Proposition 3.** Let \( 1 \leq p < N \). If \( \mu \in H^1_{\mu} (\mathbb{R}^N) \) then the condition (6.2) is satisfied.

**Proof.** The function \( \mu \) belongs to \( L^p_{loc}(\mathbb{R}^N) \), where \( p^* = \frac{Np}{N-p} \) is the Sobolev exponent of \( p \). Then, for any \( i \in \{1, \ldots, n\} \),
\[
\frac{1}{\delta^p} \int_{B(a_i, \delta)} \mu(x) dx \leq \frac{1}{\delta^p} \left( \int_{B(a_i, \delta)} \mu^{p^*}(x) dx \right)^{\frac{1}{p^*}} \left( \int_{B(a_i, \delta)} dx \right)^{\frac{1}{p^*}} \leq C \delta^{\frac{N}{p^*} - p},
\]
where \( \frac{1}{p^*} + \frac{1}{(p^*)^*} = 1 \). One can easily verify that \( \frac{N}{(p^*)^*} - p > 0 \) if \( p < N \). \( \square \)

Now we prove the following lemma for compact sets contained in \( \mathbb{R}^N \setminus \{a_1, \ldots, a_n\} \) stating an estimate for weak solutions to the problem (P) (cf. [16, Theorem 2.1]). The proof is analogous to that in [6, Lemma 2.2] in the case of one singularity. For the sake of completeness we report the proof.

**Lemma 6.1.** Let \( V \in L^1_{loc}(\mathbb{R}^N) \) be a nonnegative function. Let \( u \) be a nonnegative weak solution of (P). Then, for any compact set \( K \subset \mathbb{R}^N \setminus \{a_1, \ldots, a_n\} \) and \( t > 0 \) there exists \( c(t) > 0 \) (not depending on \( V \)) such that
\[
u(t, x) \geq c(t) \int_{K} u_0 \mu(x) dx \quad \text{on} \quad K \times [0, T].
\]

**Proof.** Let \( u_0 \in C^\infty_c(\mathbb{R}^N) \), \( u_0 \geq 0 \), and let \( u \) be a nonnegative weak solution of (P). We set \( C_R = B(0, R) \setminus \bigcup_{i=1}^{n} B(a_i, 1/R) \), with \( R \) large enough, such that \( K \subset C_R \) and let \( \varphi \in C^\infty_c(C_R) \) such that \( 0 \leq \varphi \leq 1 \).

We consider the problem
\[
(P_b) \begin{cases} v_t(x, t) = L v(x, t), & \text{on} \ C_R \times [0, T], \\ v(x, t) = 0, & \text{on} \ \partial C_R, \\ v(x, 0) = \varphi u_0. \end{cases}
\]
By a classical result, since \( v(x,0) \in C^{2+\alpha}(C_R) \), then the problem \((Pb)\) admits a solution \( v \in C^{2+\alpha,1+\frac{\alpha}{2}}(C_R \times [0,T]) \). Moreover,

\[
v(x,t) = \int_{C_R} G(t,x,y)v(y,0)dy
\]

where \( G \) is a strictly positive function on \((0, +\infty) \times C_R \times C_R\).

Let \( c(t) = \min_{(x,y) \in K \times K} G(t,x,y) \). We have for any \( x \in K \)

\[
v(x,t) \geq \int_K G(t,x,y)v(y,0)dy \geq c(t) \int_K v(y,0)dy.
\]

Furthermore, \( v \) is a weak solution to \( v_t = Lv \) in \( C_R \). Then, for all \( \phi \in W^{2,1}_0(C_R \times [0,T]) \) with \( \phi(\cdot,0) \geq 0 \) having compact support with \( \phi(\cdot, T) = 0 \), we have

\[
\int_0^T \int_{C_R} v(-\partial_t \phi - L\phi) \mu(x)dx dt - \int_{C_R} (\phi u_0) \mu(x)dx = 0.
\]

Comparing with (6.1), one obtains

\[
\int_0^T \int_{C_R} (v-u)(-\partial_t \phi - L\phi) \mu(x)dx dt - \int_{C_R} (\phi u_0 - u_0 - Vu) \phi(\cdot,0) \mu(x)dx \leq 0.
\]

Let us fix \( T, R > 0, 0 \leq \psi \in C^\infty_c(C_R \times [0,T]) \) such that \( \text{supp } \psi \subset C_R \times [0,T] \). We obtain a solution \( 0 \leq \phi \in W^{2,1}_0(C_R \times (0,T)) \) to the parabolic problem

\[
\begin{aligned}
&\partial_t \phi + L\phi = -\psi, \quad \text{on } C_R \times (0,T), \\
&\phi|_{\partial C_R \times (0,T)} = 0, \\
&\phi(x,T) = 0, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

by [17, Theorem IV.9.1]. We consider the solution \( \phi \) in (6.3). Therefore,

\[
\int_0^T \int_{C_R} (v-u)\psi \mu(x)dx dt \leq 0
\]

for all \( 0 \leq \psi \in C^\infty_c(C_R \times [0,T]) \). Then,

\[
u \geq v \geq c(t) \int_{C_R} \varphi u_0 \mu(x)dx.
\]

Since the last inequality holds for any \( \varphi \in C^\infty_c(C_R) \), we get

\[
u \geq c(t) \int_{C_R} u_0 \mu(x)dx.
\]

The above results allow us to state the following theorem by proceeding as in [16, Theorem 2.1].

**Theorem 6.2.** Assume that \( \mu \) satisfies the hypothesis \( H_4 \) and \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N) \). Then the following hold:

1. If \( \lambda_1(L+V) > -\infty \), then there exists a positive weak solution \( u \in C([0, \infty),L^2_{\mu}) \) of \((P)\) satisfying

\[
\|u(t)\|_{L^2_{\mu}} \leq Me^{\omega t} \|u_0\|_{L^2_{\mu}}, \quad t \geq 0,
\]

for some constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \).

2. If \( \lambda_1(L+V) = -\infty \), then for any \( 0 \leq u_0 \in L^2_{\mu}, u_0 \neq 0 \), there exists no positive weak solution of \((P)\) satisfying (6.4).
From Theorem 3.1, Theorem 5.1 and Theorem 6.2 we get the following existence and nonexistence result.

**Theorem 6.3.** Assume that the weight function $\mu$ satisfies hypotheses $H_1 \sim H_4$ and $0 \leq V(x) \leq \sum_{i=1}^{n} \frac{c_i}{|x-a_i|^2}$, $c_i > 0$, $a_i \in \mathbb{R}^N$, $i \in \{1, \ldots, n\}$. The following assertions hold:

1) If $0 < c \leq c_0(N + k_2) = \left(\frac{N+k_2}{2}\right)^2$, then there exists a positive weak solution $u \in C([0, \infty), L^2_\mu)$ of (P) satisfying

$$
\|u(t)\|_{L^2_\mu} \leq Me^{\alpha t}\|u_0\|_{L^2_\mu}, \quad t \geq 0,
$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$, and any $u_0 \in L^2_\mu$.

2) If $c > c_0(N + k_2)$, then for any $0 \leq u_0 \in L^2_\mu$, $u_0 \neq 0$, there is no positive weak solution of (P) with $V(x) = \sum_{i=1}^{n} \frac{c_i}{|x-a_i|^2}$ satisfying (6.5).

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