Let \((W, \Pi)\) be a Riemann domain over a complex manifold \(M\) and \(w_0\) be a point in \(W\). Let \(\mathbb{D}\) be the unit disk in \(\mathbb{C}\) and \(T = \partial \mathbb{D}\). Consider the space \(S_{1,w_0}(\mathbb{D}, W, M)\) of continuous mappings \(f\) of \(T\) into \(W\) such that \(f(1) = w_0\) and \(\Pi \circ f\) extends to a holomorphic on \(\mathbb{D}\) mapping \(\hat{f}\). Mappings \(f_0, f_1 \in S_{1,w_0}(\mathbb{D}, W, M)\) are called \(h\)-homotopic if there is a continuous mapping \(f_t\) of \([0, 1]\) into \(S_{1,w_0}(\mathbb{D}, W, M)\). Clearly, the \(h\)-homotopy is an equivalence relation and the equivalence class of \(f \in S_{1,w_0}(\mathbb{D}, W, M)\) will be denoted by \([f]\) and the set of all equivalence classes by \(\eta_{1}(W, M, w_0)\).

There is a natural mapping \(\iota_{1}: \eta_{1}(W, M, w_0) \to \pi_{1}(W, w_0)\) generated by assigning to \(f \in S_{1,w_0}(\mathbb{D}, W, M)\) its restriction to \(T\). We introduce on \(\eta_{1}(W, M, w_0)\) a binary operation \(\ast\) which induces on \(\eta_{1}(W, M, w_0)\) a structure of a semigroup with unity. Moreover, \(\iota_{1}([f_1] \ast [f_2]) = \iota_{1}([f_1]) \cdot \iota_{1}([f_2])\), where \(\cdot\) is the standard operation on \(\pi_{1}(W, w_0)\). Then we establish standard properties of \(\eta_{1}(W, M, w_0)\) and provide some examples. In particular, we completely describe \(\eta_{1}(W, M, w_0)\) when \(W\) is a finitely connected domain in \(M = \mathbb{C}\) and \(\Pi\) is an identity. In particular, we show for a general domain \(W \subset \mathbb{C}\) that \([f_1] = [f_2]\) if and only if \(\iota_{1}([f_1]) = \iota_{1}([f_2])\).

1. Introduction

A closed analytic disk in a complex manifold \(M\) is a continuous mapping \(f\) of the closed unit disk \(\overline{\mathbb{D}}\) into \(M\) holomorphic on \(\mathbb{D}\). Suppose that \(W\) is an open domain in \(M\) and consider the space \(S(\overline{\mathbb{D}}, W, M)\) of all analytic disks \(f\) in \(M\) such that \(f(T) \subset W\), where \(T = \partial \mathbb{D}\).

By the Kontinuitätsatz or continuity principle of H. Kneser (1932) if \(M = \mathbb{C}^n\) and \(W\) is a domain of holomorphy, then \(f \in S(\overline{\mathbb{D}}, W, M)\) can be continuously deformed in \(S(\overline{\mathbb{D}}, W, M)\) into a constant mapping if and only if \(f \in A(\overline{\mathbb{D}}, W) = S(\overline{\mathbb{D}}, W, W)\). This fact was frequently used in complex analysis.

Expanding the continuity principle in [9] B. Jörricke introduced an equivalence relation on the path connected component of \(S(\overline{\mathbb{D}}, W, M)\) containing constant mappings and proved that the quotient of this component by this equivalence is the envelope of holomorphy of \(W\), which is generally non-schlicht or a Riemann domain over \(M\).

Our remote goal is to find out what kind of complex manifolds we obtain if we apply Jörricke’s equivalence to other connected components of \(S(\overline{\mathbb{D}}, W, M)\). And, seemingly, it is worth to do. For example, all components were used in [10] to find plurisubharmonic subextensions and the mappings in the equivalences classes were used to fill the holes in \(W\).
But before we start to work on this problem it is prudent to produce an inventory of connected components. Since Riemann domains naturally enter the picture we consider a Riemann domain \((W, \Pi)\) over a complex manifold \(M\) and redefine the space \(S(\overline{D}, W, M)\) as the space of continuous mappings \(f\) of \(\mathbb{T}\) into \(W\) such that \(\Pi \circ f\) extends to a mapping \(\hat{f} \in A(\overline{D}, M)\).

A mapping \(f\) of \(\mathbb{T}\) into \(W\) is a loop and if \(f_0, f_1 \in S(\overline{D}, W, M)\) belong to the same connected component, then they are homotopy equivalent in the space of real loops. So another interesting question is whether homotopy equivalence of loops generated by \(f_0, f_1 \in S(\overline{D}, W, M)\) implies that they belong to the same connected component in \(S(\overline{D}, W, M)\).

To approach this problem we use the analogy with classical homotopy theory. We fix a base point \(w_0 \in W\) and introduce the space \(S_{1,w_0}(\overline{D}, W, M)\) of \(f \in S(\overline{D}, W, M)\) equal to \(w_0\) at 1. We endow \(S(\overline{D}, W, M)\) with the natural topology (see Section 2) and call \(f_0, f_1 \in S_{1,w_0}(\overline{D}, W, M)\) \(h\)-homotopic if they can be connected by a continuous path in \(S_{1,w_0}(\overline{D}, W, M)\). This is an equivalence relation and the equivalence class of \(f \in S_{1,w_0}(\overline{D}, W, M)\) will be denoted by \([f]\) and the set of all equivalence classes by \(\eta_1(W, M, w_0)\). There is a natural mapping \(\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)\) generated by assigning to \(f \in S_{1,w_0}(\overline{D}, W, M)\) its restriction to \(\mathbb{T}\).

In 1989 M. Gromov published the paper [5], where he introduced elliptic manifolds and proved that the homotopic Oka principle holds for holomorphic mappings into elliptic manifolds. This principle says that \(h\)-homotopy is equivalent to topological homotopy. F. Forstneriˇc and his colleagues greatly expanded studies in this direction and their results can be found in [4]. However, elliptic or Oka manifolds are non-hyperbolic and, in general, the homotopic Oka principle fails. In our settings the Oka principle holds if and only if the mapping \(\iota_1\) is an injection.

One of the main goals of this paper is to introduce on \(\eta_1(W, M, w_0)\) a binary operation \(\ast\) and show that with this operation \(\eta_1(W, M, w_0)\) becomes a semigroup with unity. Moreover, \(\iota_1([f_1] \ast [f_2]) = \iota_1([f_1]) \ast \iota_1([f_2])\), where \(\ast\) is the standard operation on \(\pi_1(W, w_0)\). This goal is motivated, first of all, by the traditional importance of algebraic structure on analytic objects and, secondly, we needed it for applications. However, the achievement of this goal is not simple.

Since the standard concatenation of the restrictions of \(f_1\) and \(f_2\) to \(\mathbb{T}\) cannot be realized as the boundary of an analytic disk we have to develop some machinery. In Section 2 we prove general facts about the topology on \(S(\overline{D}, W, M)\). They are trivial when \(M = \mathbb{C}^n\), require some labor when \(M\) is Stein and, in the general situation, the result from [12], claiming the existence of Stein neighborhood of graphs of \(f \in A(\overline{D}, M)\), is used.

To define the \(\ast\) operation on equivalence classes of \(f_1\) and \(f_2\) we take disks \(K_1 = \{\zeta - 1| \leq 1\}\) and \(K_2 = \{\zeta + 1| \leq 1\}\) and define \(h_1(\zeta) = f_1(1 - \zeta)\) and \(h_2(\zeta) = f_2(1 + \zeta)\) on these disk. The mapping \(\Pi \circ h\), where \(h = h_1\) on \(\partial K_1\) and \(h = h_2\) on \(\partial K_2\) is holomorphic in the interior of \(K = K_1 \cup K_2\) and continuous on \(K\). So we can use Mergelyan’s theorem to approximate \(\Pi \circ h\) by holomorphic mappings \(g\) on simply connected neighborhoods \(D\) of \(K\). The major problem is how to select a point \(\zeta_0 \in \partial D\), where \(g(\zeta_0) = w_0\). Moreover, since domains \(D\) and approximations \(g\) change in the proofs the choice of \(\zeta_0\) should be made continuously. This is the main obstacle and leads to the introduction in Section 3 of access curves and the general \(h\)-homotopy theory of holomorphic mappings on planar compact sets.
This theory allows us to introduce in Section 4 the ∗ operation and prove that \( \eta_1(W, M, w_0) \) becomes a semigroup with unity. Section 5 contains examples of \( \eta_1(W, M, w_0) \) when \( W \) is an annulus in the complex plane or Riemann sphere and Section 6 is devoted to major algebraic properties of \( \eta_1(W, M, w_0) \).

In Section 7 we completely describe \( \eta_1(W, M, w_0) \) when \( W \) is a finitely connected domain in \( M = \mathbb{C} \) and \( \Pi \) is an identity. In particular, we show that \([f_1] = [f_2]\) if and only if \( \tau_1([f_1]) = \tau_1([f_2])\) and this manifests the homotopic Oka principle in a hyperbolic case. The ∗ operation and its algebraic properties play the major role in the proof of this purely analytic statement.

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2. Basic facts

A Riemann domain over a complex manifold \( M \) is a pair \((W, \Pi)\), where \( W \) is a path connected Hausdorff complex manifold and \( \Pi \) is locally biholomorphic mapping of \( W \) into \( M \). Let \( \hat{\rho} \) be a Riemann metric on \( M \). The mapping \( \Pi \) lifts this metric to \( W \) as \( \rho \).

Let \( N \) be another complex manifold and let \( K \) be a compact set in \( N \). Suppose that a set \( K' \subset K \) and \( \phi : K' \to M \) is a continuous mapping. We denote by \( A_\phi(K, M) \) the set of all continuous mappings of \( K \) into \( M \) which are holomorphic on the interior \( K^0 \) of \( K \) and are equal to \( \phi \) on \( K' \). If the set \( K' \) is empty then we denote \( A_\phi(K, M) \) by \( A(K, M) \).

If \( B \subset K \) is a compact set containing \( \partial K \), \( K' \subset B \), then by \( S_\phi(B, K, W, M) \) we denote the set of all continuous mappings \( f \) of \( B \) into \( W \) such that \( f = \phi \) on \( K' \) and there is a mapping \( \hat{f} \in A(K, M) \) coinciding with \( \Pi \circ f \) on \( \partial K \). If the set \( K' \) is empty then we denote \( S_\phi(B, K, W, M) \) by \( S(B, K, W, M) \) and if \( B = \partial K \) then we will write \( S_\phi(K, W, M) \) for \( S_\phi(\partial K, K, W, M) \). Note that if \( W = M \) and \( \Pi \) is an identity, then \( S_\phi(B, K, W, M) = A_\phi(K, M) \).

If the set \( K \) is compact and has a non-empty boundary, then the mapping \( \hat{f} \) is unique due to the following proposition which allows us to define the mapping \( \hat{\Pi} : S_\phi(B, K, W, M) \to A_\phi(K, M) \), where \( \hat{\phi} = \Pi \circ \phi \), as \( \hat{\Pi}(f) = \hat{f} \).

**Proposition 2.1.** Let \( K \subset N \) be a compact set with a non-empty boundary. If \( f, g \in A(K, M) \) are equal on \( \partial K \), then they are equal on \( K \).

**Proof.** If \( K_1 \) is a connected component of \( K^0 \) and \( z_0 \in \partial K_1 \), then \( z_0 \in \partial K \) and \( f(z_0) = g(z_0) \). We can find neighborhoods \( U \) of \( z_0 \) and \( V \) of \( f(z_0) \) biholomorphic to unit balls such that \( f(U) \) and \( g(U) \) lie in \( V \). Let \( \phi \) be a biholomorphic mapping of \( V \) onto the unit ball \( B \).

Let \( f_1 = \phi \circ f \) and \( g_1 = \phi \circ g \) be mappings on some connected component \( E \) of \( K_1 \cap U \). The function \( u(z) = \log ||f_1 - g_1|| \), considered as a function with values at \( \mathbb{R} \cup \{-\infty\} \), is subharmonic on \( E \), continuous on \( \overline{E} \), and is equal to \(-\infty\) on \( \partial E \cap \partial K \) which has non-empty relative interior in \( \partial E \). Since the set of irregular boundary points on \( \partial E \) is polar (see [5]), the set of regular points is dense in the boundary and, consequently, for the harmonic measure \( \mu_z \) relative to \( E \) and \( z \in E \) we have \( \mu_z(\partial E \cap \partial K) > 0 \). Hence \( u(z) = -\infty \) and we see that \( f = g \) on \( E \). Now standard arguments show that \( f = g \) on \( K_1 \) and, after that, on \( K \). 

We introduce the space \( T(N, W, M) \) of all triples \((B, K, f)\), where \( K \subset N \) is a compact set with a non-empty boundary, \( B \) is a compact set in \( K \) containing
$\partial K$ and $f \in S(B, K, W, M)$. We define the topology on these space by choosing a system of neighborhoods. For this we introduce some Riemann metric $d$ on $N$. If $(B, K, f) \in T(N, W, M)$ and $\Phi$ is a continuous extension of $\hat{f}$ to $N$ and $\epsilon > 0$ we define a $\Phi, \epsilon$-neighborhood of $(B, K, f)$ as a set of all triples $(A, L, g) \in T(N, W, M)$ such that the Hausdorff distance between $L$ and $K$ and between $A$ and $B$ is less than $\epsilon$ and $\rho(\hat{g}(z), \Phi(z)) < \epsilon$ for all $z \in L$.

It is easy to verify that if $U$ is a $\Phi, \epsilon$-neighborhood of $(B, K, f)$, $V$ is a $\Psi, \delta$-neighborhood of $(A, L, g)$ and $(C, D, h) \in U \cap V$, then there is a $\Lambda, \eta$-neighborhood of $(C, D, h)$ lying in $U \cap V$. Hence our choice of neighborhoods defines a topology on $T(N, W, M)$.

The set $S_\epsilon(B, K, W, M) \subset T(N, W, M)$ and we define the topology on this set as the topology relative to the topology imposed on $T(N, W, M)$. We will frequently work with triples $(\partial K, K, f)$ and to simplify notation in this case we will write a pair $(K, f)$ for $(\partial K, K, f)$. The space of all pairs $(K, f) \in T(N, W, M)$ will be denoted by $S^*(N, W, M)$.

The following example explains why we measure the distance between $\Phi$ and $\hat{g}$ and but not between $f$ and $g$. Let $N = \mathbb{C}$, $K$ is the close unit disk $\mathbb{P}$ in $\mathbb{C}$, $W = \{z \in \mathbb{C} : 1 < |z| < 2\}$, $M = \mathbb{CP}^1$, and $\Pi$ is the identity. The triples $(\partial K, K, f)$ and $(\partial K, K, g)$, where

$$f(\zeta) = \zeta \text{ and } g(\zeta) = \zeta + \frac{\epsilon}{\zeta}$$

are close on $\partial K$ when $\epsilon$ is small but $\hat{f}$ and $\hat{g}$ are not close in $A(K, M)$. We will use the notation $T(N, M, M)$ when $W = M$ and $\Pi$ is an identity. We define the mapping $\Pi_1$ of $T(N, W, M)$ into the set $T(N, M, M)$ as $\Pi_1(B, K, f) = (B, K, \Pi(f))$. It follows immediately from the definitions of the topologies involved that the mapping $\Pi_1$ is continuous.

**Lemma 2.2.** Let $(B, K, f) \in T(N, W, M)$ and let $\Phi$ be a continuous extension of $\hat{f}$ to $N$.

1. There is $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ the mapping $\Pi_1$ maps the $\Phi, \epsilon$-neighborhood of $(B, K, f)$ homeomorphically onto the $\Phi, \epsilon$-neighborhood of $(B, K, f)$.

2. There is $\delta_0 > 0$ such that if $(A, L, g)$ lies in the $\Phi, \delta_0$-neighborhood of $(B, K, f)$, then $g$ can be extended to a holomorphic mapping $\hat{g}$ into $W$ of the $\delta_0$-neighborhood $V$ of $A$ in $L$ and $\Pi \circ \hat{g} = \hat{g}$ on $V$.

3. There is $\delta_1 > 0$ such that if $(\partial L, L, g)$ lies in the $\Phi, \delta_1$-neighborhood of $(B, K, f)$ and $K \subset L$, then $g$ can be extended to a holomorphic mapping $\hat{g}$ into $W$ of the $\delta_1$-neighborhood $\partial L \cup B$ in $L$, the triple $(\partial L \cup B, L, \hat{g})$ lies in in the $\Phi, \delta_1$-neighborhood of $(B, K, f)$ and $\Pi \circ \hat{g} = \hat{g}$ on $V$.

**Proof.** (1) We define the mapping $\Pi_0 : N \times W \to N \times M$ as $\Pi_0(z, w) = (z, \Pi(w))$. It is easy to see that there is a $\eta$-neighborhood $U$ of the graph of $f$ on $B$ such that the restriction of $\Pi_0$ to $U$ is a homeomorphism of $U$ onto a neighborhood $V$ of the graph of $\hat{f}$ on $B$. There is $\delta > 0$ such that if $(x, y) \in N \times M$ and there is a point $z \in B$ such that $d(x, z) < \delta$ and $\rho(\hat{f}(z), \Phi(z)) < \delta$, then $(x, y) \in V$.

Let us take $\epsilon_0 > 0$ such that $\epsilon_0 < \delta/2$ and $\rho(\hat{f}(z), \Phi(x)) < \delta/2$ when $d(x, z) < \epsilon_0$. If $\epsilon < \epsilon_0$ and $(A, L, g)$ is in the $\Phi, \epsilon$-neighborhood of $(B, K, f)$ in $T(N, M, M)$, then for any $x \in A$ there is $z \in B$ such that $d(x, z) < \epsilon$. Hence $\rho(g(x), \hat{f}(z)) \leq \frac{\epsilon}{\epsilon_0}$.
\[ \hat{\rho}(g(x), \Phi(x)) + \hat{\rho}(\Phi(x), \hat{f}(z)) < \delta. \] Thus the points \((z, g(z)), z \in A\), are in \(V\). Then we can define the mapping \(h : B \to W \) as \(h(z) = P_W(\Pi_0^{-1}(z, g(z)))\), where \(P_W\) is the projection of \(N \times W\) onto \(W\). Clearly, \(\hat{h} = g\), the triple \((A, L, h) \in T(N, W, M)\) and \(\hat{\Pi}_1(A, L, h) = (A, L, g)\). Moreover, \((A, L, h)\) is in the \(\Phi, \varepsilon\)-neighborhood of \((B, K, f)\) in \(T(N, W, M)\).

If the triple \((A, L, g)\) is in the \(\Phi, \varepsilon\)-neighborhood of \((B, K, f)\) in \(T(N, W, M)\), then \(\Pi_1(A, L, g)\) is in the \(\Phi, \varepsilon\)-neighborhood of \((B, K, \hat{f})\) in \(T(N, M, M)\). Hence, \(\hat{\Pi}_1\) is a bijection of the \(\Phi, \varepsilon\)-neighborhood of \((B, K, f)\) onto the \(\Phi, \varepsilon\)-neighborhood of \((B, K, \hat{f})\). Since the continuity of \(\hat{\Pi}_1^{-1}\) is easy to verify we proved (1).

(2) We take \(\delta_0 = \varepsilon_0/4\), where \(\varepsilon_0\) was defined in (1). For \((A, L, g)\) in the \(\Phi, \delta_0\)-neighborhood of \((B, K, f)\) we take as \(C\) the closed \(\delta_0\)-neighborhood of \(A\) in \(L\). Then the triple \((C, L, \tilde{g})\) is in the \(\Phi, 2\delta_0\)-neighborhood of \((B, K, \hat{f})\) and by (1) there is \((C, L, \tilde{g})\) in the \(\Phi, 2\delta_0\)-neighborhood of \((B, K, f)\) such that \(\Pi_1 \circ \tilde{g} = \hat{g}\) on \(C\) and \(\tilde{g} = g\) on \(A\).

(3) The proof follows the same line of argument as in (1) using the homeomorphism \(\Pi_0\) and will be omitted. \(\square\)

The following result that allows us to lift mappings from \(M\) to \(W\) is an immediate consequence of the lemma above.

**Corollary 2.3.** For every \((B, K, f) \in T(N, W, M)\) there is \(\varepsilon > 0\) such that for any continuous path \((A_t, L_t, \hat{g}_t)\) in the \(\Phi, \varepsilon\)-neighborhood of \((B, K, \hat{f})\) there is a unique continuous path \((A_t, L_t, g_t)\) in the \(\Phi, \varepsilon\)-neighborhood of \((B, K, f)\) such that \(\hat{\Pi}_1(A_t, L_t, g_t) = (A_t, L_t, \hat{g}_t)\).

The following lemma establishes some sort of “convexity” of \(\Phi, \varepsilon\)-neighborhoods.

**Lemma 2.4.** Suppose that \(f \in S(B, K, W, M)\) and the graph \(\Gamma_f^K\) of \(\hat{f}\) on \(K\) has a Stein neighborhood in \(N \times M\). For every \(\varepsilon > 0\) there is \(\delta > 0\) such that if triples \((A, L, g_0)\) and \((A, L, g_1)\) lie in the \(\Phi, \delta\)-neighborhood of \((B, K, f)\) in \(T(N, W, M)\), then there is a neighborhood \(X\) of the interval \([0, 1] \subset \mathbb{C}\) and a continuous mapping \(G_\varepsilon : X \to S(A, L, W, M)\) such that \(G_0 = g_0, G_1 = g_1, G_\varepsilon\) lies in the \(\Phi, \varepsilon\)-neighborhood of \((K, f)\) and the mapping \(G_\varepsilon(z)\) is holomorphic in \(\varepsilon\) for all \(z \in L\). Moreover, if, additionally, a compact set \(L' \subset A\) and \(g_0|_{L'} = g_1|_{L'} = \phi\), then \(G_\varepsilon|_{L'} = \phi\) for all \(\varepsilon \in X\).

**Proof.** Firstly, let us assume that \(M\) is Stein. We choose \(\varepsilon > 0\) satisfying requirements of Lemma 2.2(1) and Corollary 2.3 and is so small that there is a compact set \(Z \subset M\) such that \(\hat{g}(L) \subset Z\) for any \((A, L, g)\) in the \(\Phi, \varepsilon\)-neighborhood \(V\) of \((B, K, \hat{f})\). Let \(F\) be an imbedding of \(M\) into \(\mathbb{C}^N\) as a complex submanifold. By [7, Theorem 8.C.8] there are an open neighborhood \(U\) of \(F(Z)\) in \(\mathbb{C}^N\) and a holomorphic retraction \(P\) of \(U\) onto \(U \cap F(M)\). Let \(\tilde{f} = F \circ \hat{f}\). Let us take \(\tau > 0\) so small that the \(\sigma\)-neighborhood \(\tilde{f}(K)\) in \(\mathbb{C}^N\) lies in \(U\) and for every \(z_1\) in this neighborhood and any point \(z_2 \in F(A)\) if \(\|z_1 - z_2\| < \sigma\) then \(\hat{\rho}(F^{-1}(P(z_1)), F^{-1}(P(z_2))) < \varepsilon\) and the interval \([z_1, z_2] \subset U\). There is \(\delta > 0\) such that \(\|F(w_1) - F(w_2)\| < \sigma\) when \(w_1, w_2 \in A\) and \(\hat{\rho}(w_1, w_2) < \delta\).

If \((A, L, g_0)\) and \((A, L, g_1)\) lie in the \(\Phi, \delta\)-neighborhood of \((B, K, f)\) in \(T(N, W, M)\) and \(\tilde{g}_t = F \circ \tilde{g}_t\), then \(\hat{\rho}(\tilde{g}_t(\zeta), \Phi(\zeta)) < \delta\) for all \(\zeta \in L\). Hence, \(\|\tilde{f} - \tilde{g}_t\| < \sigma\). If \(\tilde{h}_t = t\tilde{g}_t + (1 - t)\tilde{g}_0, 0 \leq t \leq 1,\) then
\[
\|\tilde{h}_t(\zeta) - \tilde{f}(\zeta)\| \leq t\|\tilde{f}(\zeta) - \tilde{g}_t(\zeta)\| + (1 - t)\|\tilde{f}(\zeta) - \tilde{g}_0(\zeta)\| < \sigma
\]
for all \( \zeta \in L \) and \( 0 \leq t \leq 1 \). Clearly, there is a neighborhood \( X \) of \([0,1]\) in \( \mathbb{C} \) such that this inequality holds for all \( t \in X \). Thus \( \tilde{h}_t(L) \subset U \) and we can define the mappings \( h_t = F^{-1} \circ P \circ \tilde{h}_t \). Clearly, \( \tilde{g}_0 = \tilde{h}_0 \) and \( \tilde{g}_1 = \tilde{h}_1 \) on \( K \) and by conditions on \( \sigma \) we have \( \rho(\tilde{h}_t(\zeta), \Phi(\zeta)) < \varepsilon \) on \( L \). Thus the holomorphic path \( h_t \) can be lifted to \( \mathcal{S}(A, L, W, M) \) as \( G_t \).

If, additionally, a compact set \( L' \subset A \), \( \phi : L' \to W \) is continuous and \( g_0|_{L'} = g_1|_{L'} = \phi \), then the mappings \( h_t \) also are equal \( \tilde{g}_t \) on \( L' \). Hence \( G_t|_{L'} = \phi \) for all \( t \in X \).

If \( M \) is not Stein but the graph \( \Gamma_K \) has a Stein neighborhood \( Y \) in \( N \times M \), then we replace \( M \) with \( Y \), \( W \) with \( N \times W \), \( f(\zeta) \) with \( (\zeta, f(\zeta)) \) and \( g_j(\zeta) \) with \( (\zeta, g_j(\zeta)) \) for \( j = 0, 1 \). Then the same argument shows that the lemma holds.

The following lemma allows us to shift slightly continuous paths in \( \mathcal{S}^*(N, W, M) \).

**Lemma 2.5.** Let \((B_t, K_t, f_t), 0 \leq t \leq 1\) be a continuous path in \( \mathcal{F}(N, W, M) \) such that the set \( \hat{\Gamma} = \{(t, \zeta, f_t(\zeta)), 0 \leq t \leq 1, \zeta \in K_t\} \) lies in some Stein domain \( U \subset \mathbb{C} \times N \times M \). Let \( \hat{\Phi}_t \) be some continuous extension of the mapping \( \hat{f}_t(\zeta) \) to \( \mathbb{C} \times N \). For any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \((A_t, L_t, g_t)\) is a continuous path in \( \mathcal{F}(N, W, M) \), \( w_t \) is a continuous path in \( W \) and \( \zeta_t \in A_t \) is a continuous path in \( N \), \( 0 \leq t \leq 1 \), and for all \( 0 \leq t \leq 1 \) and \( \zeta \in L_t \) the triples \((A_t, L_t, g_t)\) lie in the \( \hat{\Phi}_t \)-neighborhood of \((B_t, K_t, f_t)\) and \( \rho(\hat{g}_g(\zeta_t), w_t) < \delta \), then there is another continuous path \((A_t, L_t, h_t)\) in \( \mathcal{F}(N, W, M) \) such that \( h_t(\zeta_t) = w_t \) and \( \rho(\tilde{h}_t(\zeta), \Phi(\zeta)) < \varepsilon \). Moreover, if \( g_t(\zeta_t) = w_t \) for some \( 0 \leq t \leq 1 \) then \( h_t \equiv g_t \).

**Proof.** Let \( F \) be an imbedding of \( U \) into \( \mathbb{C}^p \) as a complex submanifold. By [7] Theorem 8.3.8 there are an open neighborhood \( Y \subset \subset \mathbb{C}^p \) of \( \Gamma \) in \( \mathbb{C}^p \) and a holomorphic retraction \( P \) of \( Y \) onto \( F(U) \cap Y \). Let \( P_M \) be the projection of \( \mathbb{C} \times N \times M \) onto \( M \). We may assume that there is a constant \( C \geq 1 \) such that \( \|F(t, \zeta, z) - F(t, \zeta, w)\| \leq C\rho(z, w) \) on \( U \) and \( \rho(P_M(F^{-1}(P(z_1)) \leq C\|z_1 - z_2\| \) when \( z_1, z_2 \in W \). We take an open set \( Y' \subset \subset Y \) containing \( F(\hat{\Gamma}) \).

There is \( \sigma > 0 \) such that \( Y' + v \subset Y \) for any \( v \in \mathbb{C}^p \) with \( \|v\| < \sigma \). Let \( U' = F^{-1}(Y' \cap F(U)) \).

Let \( \Gamma = \{(t, \zeta, f_t(\zeta)) \in \mathbb{C} \times N \times W : 0 \leq t \leq 1, \zeta \in B_t\} \). The set \( \Gamma \) is compact and there is a neighborhood \( Z \) of \( \Gamma \) such that the mapping \( \Pi_2(t, \zeta, w) = (t, \zeta, \Pi(w)) \) is a homeomorphism of \( Z \) onto an open set \( \hat{Z} \subset \mathbb{C} \times N \times M \). For every \( 0 \leq t \leq 1 \) and \( \zeta \in B_t \) the point \((t, \zeta, f_t(\zeta)) \in \hat{Z} \). Therefore there is \( \eta > 0 \) such that \((t, \zeta, z) \in \hat{Z} \) if \( \rho_{\mathbb{C}}(\zeta, \zeta_t) < \eta \) for some \( \zeta_t \in B_t \) and \( \rho(z, \hat{\Phi}_t(\zeta)) < \eta \).

Given a continuous path \((A_t, L_t, g_t)\) in \( \mathcal{F}(N, W, M) \), a continuous path \( w_t \) in \( W \) and a continuous path \( \xi_t \) in \( N \) satisfying conditions of the lemma we presume, firstly, that \( \delta \) is so small that for \( 0 \leq t \leq 1 \) the paths \((t, \xi_t, \tilde{g}_t(\zeta)) \), \( \zeta \in L_t \), and \((t, \xi_t, \tilde{w}_t) \), \( \tilde{w}_t = \Pi(w_t) \), lie in \( Y' \). Hence we can define the mappings \( \tilde{g}_t(\zeta) = F(t, \zeta, \tilde{g}_t(\zeta)) \) of \( L_t \) into \( \mathbb{C}^p \) and the path \( \tilde{w}_t = F(t, \xi_t, \tilde{w}_t) \) in \( \mathbb{C}^p \). Clearly, \( \|\tilde{w}_t - \tilde{g}_t(\zeta)\| \leq C\delta \). So if we require that \( C\delta < \sigma \) then the path \( \tilde{h}_t(\zeta) = \tilde{g}_t(\zeta) - \tilde{g}_t(\xi_t) + \tilde{w}_t, \zeta \in L_t \), lies in \( Y \) and we can define \( \tilde{h}_t(\zeta) = P_M(F^{-1}(P(\tilde{g}_t(\zeta)))) \). Note that \( \tilde{h}_t \equiv \tilde{g}_t \) if \( g_t(\zeta_t) = w_t \).

Since \( \|\tilde{h}_t(\zeta) - \tilde{g}_t(\zeta)\| \leq C\delta \) for \( 0 \leq t \leq 1 \) and \( \zeta \in L_t \) we see that \( \rho(\tilde{h}_t(\zeta), \tilde{g}_t(\zeta)) \leq C^2\delta \). Hence \( \rho(\tilde{h}_t(\zeta), f_t(\zeta)) \leq (1 + C^2)\delta \). So if we require that \( (1 + C^2)\delta < \min\{\eta, \varepsilon\} \) then the points \((t, \xi_t, \tilde{h}_t(\zeta)) \in \hat{Z} \) when \( 0 \leq t \leq 1 \) and \( \zeta \in \partial L_t \). Let \( P_W \) be the projection of \( \mathbb{C} \times N \times W \) onto \( W \) and \( h_t(\zeta) = P_W \circ \Pi_2^{-1}(t, \zeta, \hat{h}_t(\zeta)) \) for \( \zeta \in \partial L_t \).

Then \( \Pi \circ h_t = \hat{h}_t \) and \( h_t(\xi_t) = w_t \). □
We say that \( f, g \in S_\partial(B, K, W, M) \) are \( h, \phi \)-homotopic or \( f \sim^h_\phi g \) if there is a continuous path connecting \( f \) and \( g \) in \( S_\partial(B, K, W, M) \). The relation \( \sim^h_\phi \) is evidently an equivalence and we will call the equivalence class of \( f \) by the \( h \)-homotopic type relative to the base \( \phi \) of \( f \) and denote by \([f]_\phi\). The set of equivalence classes will be denoted by \( H_\partial[B, K, W, M] \) or \( H_\partial[K] \) and if \([f]_\phi = [g]_\phi\) then we say that \( f \) and \( g \) are \( h, \phi \)-homotopic or \( h \)-homotopic.

As the following corollary shows the homotopic type is a continuous functions on \( S_\partial(B, K, W, M) \) provided the existence of Stein neighborhoods for the graphs.

**Corollary 2.6.** Let \( f, g \in S_\partial(B, K, W, M) \) and the graphs of \( f \) and \( g \) have Stein neighborhoods in \( N \times M \). If there are sequences \( \{f_j\} \) and \( \{g_j\} \) converging to \( f \) and \( g \) respectively in \( S_\partial(B, K, W, M) \) and such that \( f_j \sim^h_\phi g_j \), then \( f \sim^h_\phi g \).

**Proof.** By Lemma 2.4 there is \( j_0 \) such that \( f_j \sim^h_\phi f \) and \( g_j \sim^h_\phi g \) when \( j \geq j_0 \). Since the relation \( \sim^h_\phi \) is transitive we see that \( f \sim^h_\phi g \). \( \square \)

3. Homotopic Types of Holomorphic Mappings of Planar Compact Sets

Throughout this section \( K \) will denote a connected compact set in \( \mathbb{C} \) with the connected complement. Let \( \zeta_0 \in \partial K \), \( \zeta' = \{\zeta_0\} \), a base point \( w_0 \in W \) and \( \phi(\zeta_0) = w_0 \). We will denote \( S_\partial(B, K, W, M) \) by \( S_{\zeta_0,w_0}(B, K, W, M) \). It is rather difficult to describe the set \( H_\partial[B, K, W, M] = H_{\zeta_0,w_0}[B, K, W, M] \) even in this case.

To get some information we construct in this section a mapping of this set into the set \( H_{\zeta_0,w_0}([T, T], W, M) \). Two facts will help us to do this: firstly, by Corollary 4.4 in [12] any mapping \( f \in A(K, M) \) can be approximated by holomorphic mappings on neighborhoods of \( K \) and, secondly, by Theorem 3.1 in [12] the graph \( \Gamma'_{\zeta'} \) of \( f \) on \( K \) has a basis of Stein neighborhoods in \( \mathbb{C} \times M \).

Let \( D \) be a Jordan domain, i.e., a domain bounded by a Jordan curve (a homeomorphic image of a circle). Let \( \zeta_0 \in \partial D \) and let \( \zeta_1 \) be a point in \( D \). We will associate with \( D, \zeta_0 \) and \( \zeta_1 \) a unique conformal mapping \( e_{D,\zeta_0,\zeta_1} \) of the unit disk \( \mathbb{D} \) onto \( D \) which maps 1 to \( \zeta_0 \) and 0 to \( \zeta_1 \). If \( g \in S_{\zeta_0,w_0}(\mathbb{D}, W, M) \) then we let \( h_{D,\zeta_0,\zeta_1} = g \circ e_{D,\zeta_0,\zeta_1} \) and denote by \( \{g, \zeta_0\} \) the equivalence class of \( h_{D,\zeta_0,\zeta_1} \) in \( h_{1,w_0}([\mathbb{D}, W, M]) \). The choice of the point \( \zeta_1 \) does not influence \( \{g, \zeta_0\} \) because the group of conformal automorphisms of \( \mathbb{D} \) with a fixed point on the boundary is contractible and in the future we will remove \( \zeta_1 \) from notation.

We will need to construct continuous paths \( (\partial D_t, \partial D_t, f_t) \). In general, it is much more difficult to shift compact sets than the mappings. But when \( D_t \) is a Jordan domain, then the notion of Radó continuity described below is very helpful.

Suppose that we have a family of Jordan domains \( D_t \subset \mathbb{C}, 0 \leq t \leq 1 \), such that a neighborhood of a point \( \zeta \) belongs to the intersection of all \( D_t \). Such a family is **Radó continuous** if the family of conformal mappings \( \phi_t \) of \( \mathbb{D} \) onto \( D_t \) such that \( \phi_t(0) = \zeta \) and \( \phi_t'(0) > 0 \) is continuous on \( \overline{\mathbb{D}} \times [0, 1] \). (By a theorem of Carathéodory the mappings \( \phi_t \) extend to \( \overline{\mathbb{D}} \) as its homeomorphisms onto \( \overline{D_t} \).) A result of Radó (see [13] or [6] Theorem II.5.2) claims, in particular, that a family of Jordan domains \( D_t \subset \mathbb{C} \) is Radó continuous if and only if for every \( t_0 \in [0, 1] \) there are homeomorphisms \( \Psi(t, \zeta) \) of \( \partial D_{t_0} \) onto \( \partial D_t \) converging uniformly to identity on \( \partial D_{t_0} \) as \( t \to t_0 \).

Suppose that \( D_t, 0 \leq t \leq 1 \), is a Radó continuous family of Jordan domains and \( \zeta_t \) is a continuous path in \( \mathbb{C} \) such that \( \zeta_t \in \partial D_t \). Let \( \psi_t \) be conformal mappings of \( \mathbb{D} \) onto \( D_t \) such that \( \psi_t(0) = \zeta_t \). Then this family is also continuous
on $\overline{D} \times [0,1]$. Indeed, if $0 \leq t_0 \leq 1$ then, rotating $C$ if necessary, we may assume that $\psi_t''(0) > 0$ and $\psi_t = \phi_w$. If $\xi_t \in \partial D$ and $\phi_t(\xi_t) = \zeta_t$, then $\xi_t \to 1$ as $t \to t_0$. Hence $\psi_t$ differs from $\phi_t$ by a rotation by a small angle and this angle goes to 0 as $t \to t_0$.

As the proof of the following lemma demonstrates the notion of Rado continuity allows us to shift at least Jordan domains.

**Lemma 3.1.** Let $(B, K, f) \in T(C, W, M)$ and $w_0 \in W$. There is $\delta > 0$ such that if:

1. $D_0 \subset \subset D_1$ are Jordan domains and $K \subset D_1$;
2. $(\partial D_1 \cup B, \overline{D_1}, g_1)$ and $(\partial D_0, \overline{D_0}, g_0)$ lie in the $\Phi, \delta$-neighborhood of $(B, K, f)$ in $T(C, W, M)$;
3. $\zeta_0 \in \partial D_0$ and $\zeta_1 \in \partial D_1$ and $g_0(\zeta_0) = g_1(\zeta_1) = w_0$;
4. there is a continuous curve $\gamma : [0,1] \to \overline{D_1 \setminus D_0}$ of diameter less than $\delta$ and such that $\gamma(0) = \zeta_0$, $\gamma(1) = \zeta_1$ and $\gamma(t) \in D_1 \setminus D_0$, $0 < t < 1$,

then $\{g_1, \zeta_1\} = \{g_0, \zeta_0\}$.

**Proof.** For some $\varepsilon > 0$ and the triple $(B, K, f)$ we choose $\eta > 0$ as $\delta$ in Lemma 2.4. Then for the chosen $\eta$ let us choose $0 < \sigma < \eta$ so that we can use Lemma 2.5 with $\varepsilon$ replaced by $\eta$ and $\delta$ by $\sigma$.

Suppose that $\delta > 0$ is already chosen. If $\zeta \in \overline{D_1 \setminus D_0}$ then $d(\zeta, K) < \delta$ because $\overline{D_1}$ lies in the $\delta$-neighborhood of $K$. But $K$ lies in the $\delta$-neighborhood of $D_0$ and, therefore, $d(\zeta, D_0) < 2\delta$. Since $\zeta \notin D_0$, $d(\zeta, \partial D_0) < 2\delta$ and, since $\partial D_0$ lies in the $\delta$-neighborhood of $B$ we see that $d(\zeta, B) < 3\delta$.

Our first requirement on $\delta$ is that $3\delta$ should be less than $\delta_0$ in Lemma 2.2(2). Then $g_1$ extends to $\overline{D_1 \setminus D_0}$. We will denote this extension also by $g_1$.

Let $\Theta$ be a conformal mapping of $D_1 \setminus D_0$ onto the annulus $A(r_0, 1) = \{\zeta \in \mathbb{C} : r_0 < |\zeta| < 1\}$ mapping $\partial D_0$ onto the unit circle. We define the intermediate domains $D_t$ as bounded domains with boundaries equal to $\Theta^{-1}(\{\zeta = (1 - r_0)t + r_0\})$. The domains $D_t$ are simply connected and the family $D_t$ is Rado continuous. To prove the latter statement we note that as homeomorphisms $\Psi_t$ of $\partial D_t$ onto $\partial D_0$ we can take preimages under the mapping $\Theta$ of the radial correspondences between circles in $A(r_0, 1)$. We will reparameterize this family letting $D_t := D_s$, $\gamma(t) \in \partial D_s$, $t \in [0,1]$. Then the new family is still Rado continuous.

Let us set $A_t = \partial D_t \cup (B \cap \overline{D_t})$ and consider the path $(A_t, D_t, h_t)$, where $h_t$ are restriction of $g_1$ to $A_t$. The set $\overline{D_t} \subset \overline{D_1}$ and, therefore, lies in the $\delta$-neighborhood of $K$. In its turn $K$ lies in the $\delta$-neighborhood of $D_0$ which lies in $D_1$. So the Hausdorff distance between $\overline{D_1}$ and $K$ is less than $\delta$. We know that $A_t$ lies in the $3\delta$-neighborhood of $B$. If $\zeta \in B \setminus D_t$ then $\zeta \in \overline{D_1 \setminus D_0}$ and, by above, $d(\zeta, \partial D_0) < 2\delta$. Hence $d(\zeta, A_t) < 2\delta$. So the Hausdorff distance between $A_t$ and $B$ is less than $3\delta$. Consequently, the path $(A_t, D_t, h_t)$ lies in the $\Phi, 3\delta$-neighborhood of $(B, K, f)$.

Our second requirement for $\delta$ is that $2\delta < \sigma$ and $\rho(g_1(\zeta), w_0) < \sigma$ when $\zeta \in \overline{D_1 \setminus D_0}$ and $d(\zeta, \zeta_1) < \delta$. Then $\rho(h_t(\gamma(t)), w_0) < \sigma$ and by Lemma 2.5 we can replace the path $(A_t, D_t, h_t)$ with the path $(A_t, D_t, p_t)$ in the $\Phi, \eta$-neighborhood of $(B, K, f)$ such that $p_t(\gamma(t)) = w_0$.

The triple $(A_0, \overline{D_0}, p_0) = (\partial D_0 \cup (B \cap D_0), \overline{D_0}, p_0)$ is the $\Phi, \eta$-neighborhood of $(B, K, f)$. So the triple $(\partial D_0, \overline{D_0}, p_0)$ is in the same neighborhood. By Lemma 2.4
there is a continuous path \((\partial D_\delta, \overline{D}_\delta, q_\delta)\) in the \(\Phi, \varepsilon\)-neighborhood of \((B, K, f)\) connecting \((\partial D_\delta, \overline{D}_\delta, p_\delta)\) and \((\partial D_\delta, \overline{D}_\delta, g_\delta)\) and such that \(q_\delta(\zeta_0) = w_0\). Concatenation of these two paths provides a continuous path \((\partial G_t, \overline{G}_t, g_t)\) connecting \((\partial D_\delta, \overline{D}_\delta, g_\delta)\) and \((\partial D_1, \overline{D}_1, g_1)\) such that the family of Jordan domains \(G_t\) is Radó continuous. We let \(\zeta_t = \gamma(t)\) on the first part of his path and \(\zeta_t = \zeta_0\) on the second part.

By the theorem of Radó the path \((T, \overline{B}, g_t \circ e_{G_t, \zeta_t, \zeta_0})\) is also continuous. Hence, \(\{g_1, \zeta_1\} = \{g_0, \zeta_0\}\).

Let \(\gamma : [0, 1] \to \mathbb{C}\) be a continuous curve such that \(\gamma(t) \in \mathbb{C} \setminus K\) when \(t > 0\) and \(\gamma(0) = \zeta_0\). Such curves will be called *access* curves to \(K\) at \(\zeta_0\). In the terminology of the prime ends theory it means that the point \(\zeta_0\) is accessible in \(\mathbb{C} \setminus K\). If \(D\) is a domain which meets \(\gamma\) we let \(\zeta_{D, \gamma} = \gamma(s_{D, \gamma})\), where \(s_{D, \gamma} = \inf\{t : \gamma(t) \in \partial D\}\).

If \(D\) is a smooth Jordan domain containing \(K, D\) meets \(\gamma\) at \(\xi_1\), a pair \((\overline{D}, g) \in S^*(\mathbb{C}, W, M)\) and \(g(\xi_1) = w_0\), then we say that the triple \((\overline{D}, g, \xi_1)\) is a \(\Phi, \varepsilon\)-approximation of \(f \in S_{g_0, \varepsilon_0}(B, K, W, M)\) if \((\overline{D}, g)\) lies in the \(\Phi, \varepsilon\)-neighborhood of \((K, f)\). We will write \((\overline{D}, g)\) for \((\overline{D}, g, \zeta_{D, \gamma})\) and say that \((\overline{D}, g)\) is a \(\Phi, \varepsilon\)-approximation of \(f\) with respect to \(\gamma\).

The following proposition asserts the existence of \(\Phi, \varepsilon\)-approximations for every \(\Phi\) and \(\varepsilon\).

**Proposition 3.2.** Let \(f \in S_{g_0, \varepsilon_0}(B, K, W, M)\), let \(\Phi\) be a continuous extension of \(\tilde{f}\) to \(\mathbb{C}\) as a mapping to \(M\) and let \(\gamma\) be an access curve to \(K\) at \(\zeta_0\). Then for every \(\varepsilon > 0\) there is a \(\Phi, \varepsilon\)-approximation \((\overline{D}, g, \zeta_1)\) of \(f\), where \(\zeta_1\) is any point in \(\partial D \cap \gamma\).

**Proof.** By Lemma 2.5 for every \(\varepsilon > 0\) there is \(\delta > 0\) such that if \((\overline{D}, h)\) lies in the \(\Phi, \delta\)-neighborhood of \((K, f)\) and \(\rho(h(\zeta_1), w_0) < \delta\) for some \(\zeta_1 \in \partial D\), then there is a mapping \(g \in S_{\xi_1, \varepsilon_0}(\overline{D}, W, M)\) such that \((\overline{D}, g)\) lies in the \(\Phi, \varepsilon\)-neighborhood of \((K, f)\). The set \(\overline{\Gamma} = \{(t, \zeta, \tilde{f}(\zeta)), 0 \leq t \leq 1, \zeta \in K\}\) has a Stein neighborhood in \(\mathbb{C} \times N \times M\). By Corollary 4.4 from [12] for every \(\delta > 0\) there is a smooth Jordan neighborhood \(D\) of \(K\) and a mapping \(h \in S(\overline{D}, M, M)\) such that \((\overline{D}, h)\) lies in the \(\Phi, \delta\)-neighborhood of \((K, \tilde{f})\). Taking \(\delta < \varepsilon_0\) in Lemma 2.5 we can lift \(h\) to \(S(\overline{D}, W, M)\) as \(h\).

We may assume that \(\delta\) is so small that if \(\partial D\) meets \(\gamma\) at \(\zeta_1\) then \(\rho(h(\zeta_1), w_0) < \delta\). Shifting \(h\) with Lemma 2.5 we get the triple \((\overline{D}, g, \zeta_1)\) providing the needed approximation.

The following proposition asserts that if the pair \((\overline{D}, g)\) is a sufficiently good approximation of some \((B, K, f) \in S_{g_0, \varepsilon_0}(B, K, W, M)\) then \(\{g, \zeta_{D, \gamma}\}\) does not depend on \(D\) and \(g\).

**Proposition 3.3.** Let \(f \in S_{g_0, \varepsilon_0}(B, K, W, M)\) and let \(\gamma\) be an access curve to \(K\) at \(\zeta_0\). There is \(\delta > 0\) such that if and \((\overline{D}_0, g_0)\) and \((\overline{D}_1, g_1)\) are \(\Phi, \delta\)-approximations of \((K, f)\) such that \(\partial D_0\) and \(\partial D_1\) meet \(\gamma\), then \(\{g_0, \zeta_{D_0, \gamma}\} = \{g_1, \zeta_{D_1, \gamma}\}\).

**Proof.** Let us take \(\delta\) less than \(\delta/2\) from Lemma 3.1 and \(\varepsilon\) from Proposition 3.2. Suppose that \(s_{D_0, \gamma} < s_{D_1, \gamma}\). We take a Jordan domain \(D \subset \subset D_0 \cap D_1\) containing \(K\) such that the restriction of the curve \(\gamma\) to \([s_{D_0, \gamma}, s_{D_1, \gamma}]\) lies outside of \(\overline{D}\). Let \(t_1 = \sup\{t : \gamma(t) \in D\}\) and \(\xi_1 = \gamma(t_1)\). Then the restriction \(\gamma_1\) of the curve \(\gamma\) to \([t_1, s_{D_1, \gamma}]\) lies in \(\overline{D}_1 \setminus D\). By Proposition 3.2 we can find a \(\Phi, \delta\)-approximation \((D, g, \xi_1)\) of \((K, f)\). By Lemma 3.1 \(\{g, \xi_1\} = \{g_1, \zeta_{D_1, \gamma}\}\). If we replace \(\gamma_1\) with the
restriction $\gamma_2$ of the curve $\gamma$ to $[t_1, s_{D_0}, \gamma]$ the same reasoning shows that $\{g, \xi_1\} = \{g_0, \zeta_{D_0, \gamma}\}$.

Consequently, for $f \in S_{\zeta_0, w_0}(K, W, M)$ there is a $\Phi, \varepsilon$-neighborhood of $(K, f)$ such that the class $\{g, \zeta_{D, \gamma}\}$ is the same for all pairs $(D, g)$ in this neighborhood and it will be denoted by $[f, \gamma]$.

The following result shows that $[f, \gamma]$ continuously depends on $(B, K, f)$.

**Theorem 3.4.** Let $f \in S_{\zeta_0, w_0}(B, K, W, M)$ and let $\gamma$ be an access curve to $K$ at $\zeta_0$. For any continuous extension $\Phi$ of $f$ there is $\eta > 0$ such that if a triple $(\partial L, L, g)$ lies in the $\Phi, \eta$-neighborhood of $(B, K, f)$ in $T(\mathbb{C}, W, M)$ the point $\zeta_0 \in \partial L$, $g(\zeta_0) = w_0$ and $\gamma$ is an access curve to $L$, then $[f, \gamma] = [g, \gamma]$.

**Proof.** Let us take $\delta$ satisfying Lemma 3.1. We take $\eta < \min\{\delta_0, \delta_1, \delta/2\}$, where $\delta_0$ and $\delta_1$ are taken from Lemma 3.2 and find a pair $\Phi(D_0, f_0)$, where $D_0$ is a Jordan domain containing $K$ in the $\Phi, \eta$-neighborhood $Y$ of $(K, f)$ such that $\{f_0, \zeta_{D_0, \gamma}\} = [f, \gamma]$. By Lemma 3.2, the mapping $f_0$ extends to the $\delta_0$-neighborhood of $B$ so that the triple $(\partial D_0 \cup B, D_0, f_0)$ lies in the $\Phi, \eta$-neighborhood of $(B, K, f)$.

We assume that $\eta$ is so small that the diameter of $\gamma$ in $D_0$ is less than $\delta$. Then we take a continuous extension $\Psi$ of $g$ and $\sigma > 0$ such that $\Psi, \sigma$-neighborhood $V$ of $(L, g)$ lies in $Y$. There is a pair $\Phi(D_1, g_0)$, where $D_1 \subset D_0$ is a Jordan domain containing $L$, such that $\{g_0, \zeta_{D_1, \gamma}\} = [g, \gamma]$. Since the Hausdorff distances between $\partial L$ and $B$ and between $\partial L$ and $\partial D_1$ is less than $\eta$, the Hausdorff distance between $\partial D_1$ and $B$ is less than $2\eta$. So the triple $(\partial D_1, D_1, g_1)$ is in the $\Phi, 2\eta$-neighborhood of $(B, K, f)$.

Now we take a Jordan domain $D \subset D_0 \cap D_1$ containing $L$ such that the restriction of the curve $\gamma$ to $[s_{D_1}, \gamma, s_{D_0}]$ lies outside of $D$. Let $t_1 = \sup\{t : \gamma(t) \in D\}$ and $\xi_1 = \gamma(t_1)$. Then the restriction $\gamma_1$ of the curve $\gamma$ to $[t_1, s_{D_0}, \gamma]$ lies in $D_1 \setminus D$. By Proposition 3.2 we can find a $\Phi, \delta$-approximation $(D, h, \xi_1)$ of $(L, g)$ which is also in $V$. By Lemma 3.1, $\{h, \xi_1\} = \{f_0, \zeta_{D_0, \gamma}\}$. If we replace $\gamma_1$ with the restriction $\gamma_2$ of the curve $\gamma$ to $[t_1, s_{D_1}, \gamma]$ the same reasoning shows that $\{h, \xi_1\} = \{g_0, \zeta_{D_1, \gamma}\}$. Thus $[f, \gamma] = [g, \gamma]$.

In the future we will mostly use the space $S^*(\mathbb{C}, W, M)$ and the following corollary, which is an immediate consequence of the preceding theorem, is rather useful.

**Corollary 3.5.** Let $(K_t, f_t)$ be a continuous curve in $S^*(\mathbb{C}, W, M)$, $0 \leq t \leq 1$. Suppose that for all $t \in [0, 1]$ the point $\zeta_0 \in \partial K_t$, $f_t(\zeta_0) = w_0$ and $\gamma$ is an access curve to $K_t$ at $\gamma(0) = \zeta_0$. Then $[f_t, \gamma] = [f_0, \gamma]$ for all $t \in [0, 1]$.

Let $f \in S_{\zeta_0, w_0}(K, W, M)$ and let $\gamma$ be an access curve to $K$ at $\zeta_0$. By Corollary 3.5 if $f \sim^\phi \gamma$ then $[f, \gamma] = [g, \gamma]$. Hence the mapping $I_{\gamma : [f]_\phi \to [f, \gamma]}$ of $h_{\zeta_0, w_0}(K, W, M)$ into $\mathcal{H}_{1, (w_0)\mathbb{C}, W, M} = \eta_1(W, M, w_0)$ is well-defined. If $f \in S_{1, w_0}(\mathbb{C}, W, M)$ let $\iota(f)$ be the loop $f|_T$ in $W$. Clearly, if $f \sim^\phi_{1, w_0} g$ in $\eta_1(W, M, w_0)$, then $\iota(f)$ and $\iota(g)$ are homotopic in $\pi_1(W, w_0)$. Hence the mapping $\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)$ is also well-defined.

The role of an access curve $\gamma$ is to choose where 1 is mapped by $e_{D, \zeta_0, \zeta_1}$. As the following example shows that this choice is important. Let $M = \mathbb{C}$, $W = \mathbb{C} \setminus \{(|\zeta + 4| \leq 1) \cup \{|\zeta - 4| \leq 1\}\}$, $w_0 = 0$ and $\Pi(\zeta) = \zeta$. Let $K = \{|\zeta + 4| \leq 1\}$...
\[ \{ -2, 0 \} \cup \{ \zeta - 4 \leq 2 \} \] and \( f : K \to M \) is defined as \( f(\zeta) = \zeta \). Let \( \zeta_0 = 0 \), \( \gamma_1(t) = it \) and \( \gamma_2(t) = -it \), \( t \geq 0 \). Let \( D \) be a smooth Jordan domain containing \( K \) such that the Hausdorff distance between \( D \) and \( K \) is less than \( \delta > 0 \) and \( \partial D \) meets the imaginary axis in two points \( \sigma \) and \( -i\sigma \), \( \sigma > 0 \). For small \( \delta > 0 \) we let \( g_1(\zeta) = \zeta - i\sigma \) and \( g_2(\zeta) = \zeta + i\sigma \). Then the pairs \((\overline{D}, g_1)\) and \((\overline{D}, g_2)\) can be as good approximations of \((K, f)\) as we want. But the loops \( g_1(e^{D-\sigma}(e^{i\theta})) \) and \( g_2(e^{D-\sigma}(e^{i\theta})) \), \( 0 \leq \theta \leq 2\pi \), are not equivalent in \( \pi_1(W, w_0) \) and, consequently, \( [g_1, \gamma_1] \neq [g_2, \gamma_2] \).

Two access curves \( \gamma_1 \) and \( \gamma_2 \) are *equivalent* if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( 0 < t_1, t_2 < \delta \) then the points \( \gamma_1(t_1) \) and \( \gamma_2(t_2) \) can be connected by a continuous curve \( \alpha \) in \( \mathbb{D}(\zeta_0, \varepsilon) \setminus K \). In the terminology of the prime ends theory (see [2]) it means that curves \( \gamma_1 \) and \( \gamma_2 \) determine the same prime end. In particular, if \( K \) is bounded by a Jordan curve (a homeomorphic image of a circle) then by a theorem of Carathéodory all access curves at any point of \( \partial K \) are equivalent.

The following result provides some information on the dependence of \( I_\gamma \) of \( \gamma \).

**Proposition 3.6.** Let \( f \in \mathcal{S}_{\zeta_0, w_0}(K, W, M) \), \( \zeta_0 \in \partial K \) and let \( \gamma \) be an access curve to \( K \) at \( \zeta_0 \). Then:

1. if \( \gamma_0 \) and \( \gamma_1 \) are equivalent access curves then \( I_{\gamma_0} = I_{\gamma_1} \);
2. if \( K \) is the closure of a Jordan domain then \( [f, \gamma] = \{ f, \zeta_0 \} \).

**Proof.** (1) We take \( \delta \) from Lemma 3.1 and find \( \Phi, \delta \)-approximations \((D_0, f_0)\) and \((D_1, f_1)\) of \((K, f)\) such that the diameter of \( \gamma_0 \) and \( \gamma_1 \) in \( D_0 \) and \( D_1 \) respectively is less than \( \delta/3 \) and \( I_{\gamma_0}(f) = \{ f_0, \zeta_{D_0, \gamma_0} \} \) and \( I_{\gamma_1}(f) = \{ f_1, \zeta_{D_1, \gamma_1} \} \). Then we connect \( \gamma_0(t_0) \) and \( \gamma_1(t_1) \) by a curve \( \gamma_2 \) in \( \mathbb{D}(\zeta_0, \sigma) \subset D_0 \cap D_1 \) and with \( \sigma < \delta/3 \).

We take a Jordan domain \( D \subset D_0 \cap D_1 \) such that \( K \subset D \) and \( \gamma_2 \) does not meet \( D \) and let \( t_2 = \sup \{ t : \gamma_1(t) \in D \} \). Let \((D, g)\) be a \( \Phi, \delta \)-approximation of \((K, f)\) such that \( g(\gamma_1(t_2)) = w_0 \). By Lemma 3.1 \( \{ f_1, \zeta_{D_1, \gamma_1} \} = \{ g, \gamma_1(t_2) \} \).

Let \( \gamma_3 \) be the curve which follows \( \gamma_0 \) from \( s_{D_0, \gamma_1} \) down to \( t_0 \), then \( \gamma_2 \) until it reaches \( \gamma_1 \) and then \( \gamma_1 \) down to \( \gamma_1(t_2) \). The diameter of this curve is less than \( \delta \) and it lies in \( D_0 \setminus D \). Again by Lemma 3.1 \( \{ f_0, \zeta_{D_0, \gamma_0} \} = \{ g, \gamma_1(t_2) \} \) and (1) is proved.

(2) is an immediate consequence of Lemma 3.1 \( \square \).

The mapping \( I_\gamma \) need not to be surjective as the following result shows.

**Theorem 3.7.** If \( f \in \mathcal{S}_{\zeta_0, w_0}(K, W, M) \) and \( f(K) \subset W \), then \( f \) is \( h \)-homotopic to the constant mapping \( c \equiv w_0 \). In particular, if \( K \) has no interior then \( \mathcal{H}(K, W, M) \) consists of one element.

**Proof.** If \( f \in \mathcal{S}_{\zeta_0, w_0}(K, W, M) \) is not equal to \( c \) then we assume, firstly, that \( f \) extends holomorphically to a neighborhood \( U \) of \( K \). Since \( f \) maps \( K \) into \( W \) we can fix a Stein neighborhood \( V \) of the graph of \( f \) in \( U \times W \). Then we choose a smooth Jordan domain \( D \) containing \( K \) such that \( f \) extends holomorphically to \( \overline{D} \), \( f \) maps \( D \) into \( W \), the graph \( \Gamma \) of \( f \) on \( \overline{D} \) lies in \( V \) and \( \rho(w_0, f(\gamma(t))) < \varepsilon \) when \( 0 \leq t \leq s_{D, \gamma} \) where the precise value of \( \varepsilon > 0 \) will be determined later.

Then we consider the continuous family \( \phi_t, 0 \leq t < s_D \), of conformal automorphisms of \( D \) which move \( \zeta_0 \) onto \( \gamma(t) \) and leave \( \zeta_{D, \gamma} \) in place. As the result the compact sets \( K_t = \phi_t(K) \) will converge uniformly to \( \zeta_{D, \gamma} \), the mappings \( g_t = f \circ \phi_t \) will converge uniformly to the constant mapping \( g = f(\zeta_D) \) and \( \rho(g_t(\zeta_0), w_0) < \varepsilon \).
Now if $F$ is an imbedding of $V$ into $\mathbb{C}^N$ and $U$ is a neighborhood of $F(\Gamma)$ with the retraction $P$ on $F(V)$, then we require $\varepsilon > 0$ to be so small that the mappings
\[ \tilde{f}_t(\zeta) = F(\zeta, g_t(\zeta)) - F(\zeta_0, g_t(\zeta_0)) + F(\zeta_0, w_0) \] map $K$ into $U$. Then the continuous path $f_t = F_W \circ P \circ \tilde{h}_t$ connects $f$ and $c$.

For the general mapping $f$ we note that by Theorem 4.3 in [12] it can be approximated by holomorphic mappings on neighborhoods of $K$ whose restriction to $K$ belongs to $\mathcal{S}_{\zeta_0,w_0}(K,W,M)$ and then the result follows from Corollary 2.6.

If $K$ has no interior then any $f \in \mathcal{S}_{\zeta_0,w_0}(K,W,M)$ maps $K$ into $W$ and the result follows from the statement above. 

However, $I_{\gamma}$ is surjective if $K$ has a non-empty interior.

**Theorem 3.8.** If $K$ has a non-empty interior then the mapping $I_{\gamma}$ is surjective.

**Proof.** Suppose that $f \in \mathcal{S}_{1,w_0}(\mathbb{D},W,M)$. We want to show that there is $h \in \mathcal{S}_{\zeta_0,w_0}(K,W,M)$ such that $I_{\gamma}(h) = [f]_{1,w_0}$. The set $\hat{\Gamma} = \{(t,\zeta,\tilde{f}(\zeta)), 0 \leq t \leq 1, \zeta \in \mathbb{D}\}$ has a Stein neighborhood in $\mathbb{C} \times N \times M$. We take some extension of $\tilde{f}$ to $\mathbb{C}$ and for some $\varepsilon > 0$ find $\delta > 0$ so we can use Lemma 2.5.

We take $\delta$ so small that $f$ extends to $A = \{1 - \delta \leq |\zeta| \leq 1\}$ as a mapping in $A(A,W)$. We choose a smooth Jordan domain $D$ containing $K$ and meeting $\gamma$. Mapping $D \setminus K$ onto an annulus $A_r = \{r_1 < |\zeta| < 1\}$ we introduce smooth Jordan domains $D_r$ whose boundaries a preimages of the circles of radius $r_1 < s \leq 1$ under this mapping.

Let $\zeta_1$ be an interior point of $K$ and let $\phi_s$ be a conformal mapping of $D_s$ onto $\mathbb{D}$ moving $\zeta_1$ into 0 and $\zeta_{D_s,\gamma}$ to 1. Let us show that for every $\delta > 0$ there is $r_1 < s_0 < 1$ such that for all $r_1 < s \leq s_0$ the mappings $\phi_s$ move $\partial K$ into $A$. If not and there are a decreasing sequence $\{s_j\}$ converging to $r_1$ and a sequence $\{\xi_j\} \subset \partial K$ such that $|\phi_{s_j}(\xi_j)| < 1 - \delta$ then we may assume that $\{\phi_{s_j}(\xi_j)\}$ converges to $\xi \in \mathbb{D}(0,1 - \delta)$. By a theorem of Carathéodory (see [6] Theorem II.5.1) the sequence $\{\phi_{s_j}^{-1}\}$ converges uniformly on compacta to a conformal mapping $\phi$ of $\mathbb{D}$ onto the connected component of the interior of $K$ containing $\zeta_1$. But then
\[ \phi(\xi) = \lim_{j \to \infty} \phi_{s_j}^{-1}(\phi_{s_j}(\xi_j)) = \lim_{j \to \infty} \xi_j \in \partial K \]
and this contradiction refutes this possibility.

Hence we may presume that $f(\phi_s(\gamma(t)))$ is defined when $r_1 < s \leq s_0$. Let us show that for any $\delta > 0$ there is $r_1 < s_1 < s_0$ such that $\rho(f(\phi_s(\gamma(t))),w_0) < \delta$ when $0 \leq t \leq s_{D_s,\gamma}$ and $r_1 < s \leq s_1$. If there are a decreasing sequence $\{s_j\}$ converging to $r_1$ and a sequence of points $0 \leq t_j \leq s_{D_{s_j},\gamma}$ such that $\rho(f(\phi_{s_j}(\gamma(t_j))),w_0) \geq \delta$, then we denote by $\gamma_j$ the restriction of $\gamma$ to $[0,s_{D_{s_j},\gamma}]$. Then the harmonic measure of $\phi_{s_j}(\gamma_j)$ in $\mathbb{D}$ with respect to 0 is greater or equal to some $\varepsilon > 0$ while the harmonic measures of $\gamma_j$ in $D_{s_j}$ with respect to $\zeta_1$ tend to 0 as $j \to \infty$. Since the harmonic measures are preserved by biholomorphisms, we see that the diameter of $\gamma_j$ tends to 0 and $\rho(f(\phi(\gamma_j(t_j))),w_0) < \delta$ when $j$ is large.

For $r_1 < s \leq s_1$ we set $L_s = \phi_{s_1}(\overline{D_s})$ and let $g_s$ to be the restrictions of $f$ to $L_s$. We set $L_{r_1} = \phi_{s_1}(K)$ and $g_{r_1}$ to be the restriction of $f$ to $L_{r_1}$. Clearly $(L_s,g_s)$ is a continuous path in $\mathcal{S}^* \mathcal{S}(\mathbb{C},W,M)$. Let $\xi_s = \zeta_{D_{s_1},\gamma}$. By Lemma 2.5 there is another continuous path $(L_s,h_s)$ such that $h_s(\phi_{s_1}(\xi_s)) = w_0$. Moreover, $L_{s_1} = \overline{D}$ and $h_{s_1} = f$. 

12
By continuity the class \( \{ h_s, \xi_s \} \) stays constant when \( r_1 < s \leq s_1 \) and, therefore, is equal to \([f]\), \( p_s = h_s \circ \varphi_s \), then we see that \( \{ p_s, \xi_0, \xi_s \} = [f] \) when \( r_1 < s \leq s_1 \). If \( h = p_{r_1} \), then pairs \( (D, p_s) \) converge to \((K, h) \) in \( S^*(\mathbb{C}, W, M) \) and we see that \( I_s(h) = [f] \).

The following result allows us to modify compact sets and will be used as a major tool.

**Lemma 3.9.** Suppose that \( K \) is the union of disjoint compact sets \( K_1 \) and \( K_2 \), which are connected and have connected complements, and a simple curve \( \alpha : [0, 1] \to \mathbb{C} \), which connects \( K_1 \) and \( K_2 \) and \( \alpha \cap K_1 = \{ b = \alpha(1) \} \), \( \alpha \cap K_2 = \{ a = \alpha(0) \} \). Let \( \gamma \) be an access curve to \( K \) at \( a \) and let \( f \in \mathcal{S}_{a, w_0}(K, W, M) \), \( f(b) = w_1 \). There is a connected compact set \( L \) consisting of \( K_2 \), a curve \( \beta(t) = \alpha(t) \), \( 0 \leq t \leq t_0 \leq 1 \), and a closed disk \( D \) attached to \( \beta(\partial D) \) and \( g \in \mathcal{S}_{a, w_0}(L, W, M) \) such that \( g(\beta(t_0)) = w_1 \). \([f] = [g] \). \([g] = [g]_{\beta, D, \gamma} \) and \( \{ g, \beta, D, \gamma \} = [f]_{\alpha, K_1, \gamma} \).

**Proof.** By the definition of \([f]_{\alpha, K_1, \gamma} \) the pair \((K_1, f|_{K_1})\) has a \( \Phi, \varepsilon \)-approximation \((\Omega, h)\), where \( \Omega \) is a smooth Jordan domain containing \( K_1 \), such that \( \Omega \) does not meet \( K_2 \), \( h(\xi_{\Omega, \alpha}) = w_1 \) and \( \{ h, \xi_{\Omega, \alpha} \} = [f|_{K_1}, \alpha] \), where \( \xi_{\Omega, \alpha} = \alpha(t_0) \) and \( t_0 = \min \{ t : \alpha(t) \in \partial \Omega \} \). Let \( \eta \) be taken from Theorem 3.4. We may assume that \( \varepsilon \) is so small that we can extend \( h \) continuously to the curve \( \beta \) so that \( \rho(h(t), f(t)) < \eta \) and \( h(a = w_0) = (\beta \cup \Omega, h) \) lies in the \( \Phi, \eta \)-neighborhood of \((\alpha \cup K_1, f|_{\alpha \cup K_1}) \). If we extend \( h \) to \( K_2 \) as \( f \), then the pair \((K_2 \cup \beta \cup \Omega, h)\) is in the \( \Phi, \eta \)-neighborhood of \((K, f) \). By Theorem 3.4 \([f, \nu] = [h, \nu], [h]|_{\beta, D, \gamma} = [f]|_{\beta, D, \gamma} \) and \([h]|_{\beta, D, \gamma} = [f]|_{\beta, D, \gamma} \).

Now we take a disk \( D \subset \Omega \) such that \( \xi_{\Omega, \alpha} \in \partial D \) but \( D \subset \Omega \cup \{ \xi_{\Omega, \alpha} \} \). The set \( \Omega \setminus D \) is conformally equivalent to the strip \( \{ 0 \leq \text{Im} \zeta \leq 1 \} \) and we let \( \Omega \) to be simply connected domains in \( \Omega \) whose boundaries, except \( \xi_{\Omega, \alpha} \), are moved to lines \( \{ \text{Im} \zeta = t \} \), \( 0 < t \leq 1 \), by this equivalence so that \( \partial \Omega \) goes to \( \{ \text{Im} \zeta = 0 \} \). Clearly we get a Radó continuous family of simply connected domains \( \Omega_t \). Let \( K_t \) be compact sets consisting of \( \Omega_t \), \( K_2 \) and the curve \( \beta \). Let \( \phi_t \) be a continuous family of conformal mappings of \( \Omega_t \) onto \( \Omega \) such that \( \phi_t(\Omega_t) = \Omega_t \). Define \( f_t \in \mathcal{S}_{a, w_0}(K_t, W, M) \) as \( f_t(\zeta) = h(\phi_t(\zeta)) \) on \( \Omega_t \) and \( f_t \) on \( K_2 \) and \( \beta \). Then by Corollary 3.5 \([f_t, \gamma] = [h, \gamma], [h]|_{\beta, D, \gamma} = [f]|_{\beta, D, \gamma} \) and \([f]|_{\beta, D, \gamma} \xi_{\Omega, \alpha} = [h]|_{\beta, D, \gamma} \xi_{\Omega, \alpha} \) for all \( t \in [0, 1] \).

The pair of the set \( L = K^1 \) consisting of \( \Omega, K_2 \) and \( \beta \) and the mapping \( g = f^1 \) satisfies all requirements of the Lemma.

**Remark:** If \( \alpha \) is a smooth curve then by Corollary 3.5 we can shift \( D \) along \( \alpha \) so it becomes attached to \( b \).

## 4. Holomorphic fundamental semigroup of Riemann domains

If \( f \in \mathcal{S}_{1, w_0}(\mathbb{C}, W, M) \) we will denote \([f]_{1, w_0} \) by \([f] \). To introduce on \( \eta_1(W, M, w_0) \) a semigroup structure compatible with \( \iota_1 \) we need additional construction since in the standard definition the sum of two loops cannot be realized as a boundary of an analytic disk.

Suppose that \( f_1, f_2 \in \mathcal{S}_{1, w_0}(\mathbb{C}, W, M) \) are representatives of equivalence classes \([f_1] \) and \([f_2] \) respectively in \( \eta_1(W, M, w_0) \). Let \( K \subset \mathbb{C} \) be the union of \( K_1 = \{ |t - 1| \leq 1 \} \) and \( K_2 = \{ |t + 1| \leq 1 \} \) and let \( \gamma(t) = -it \), \( 0 \leq t \leq 1 \). Then \( \gamma \) is an
access curve for $K$ to 0. We define the mapping

$$h_{f_1,f_2}(\zeta) = \begin{cases} f_1(1 - \zeta), & \zeta \in \partial K_1, \\ f_2(1 + \zeta), & \zeta \in \partial K_2 \end{cases}$$

of $\partial K$ into $W$. The mapping $\hat{h}_{f_1,f_2}$ maps $K$ into $M$ so $h_{f_1,f_2} \in S_{0,w_0}(K,W,M)$.

We let $[f_1] \ast [f_2] = I_\gamma(h_{f_1,f_2})$. If $f_1$ and $f_2$ are $h$-homotopic to $g_1$ and $g_2$ respectively in $S_{1,w_0}(\mathbb{D},W,M)$, then evidently $h_{f_1,f_2}$ is $h$-homotopic to $h_{g_1,g_2}$ in $S_{0,w_0}(K,W,M)$. Hence the class $[f_1] \ast [f_2]$ is well defined.

One of the advantages of the $\ast$ operation is its help to calculate the homotopic type of holomorphic mappings of compact sets. Let $\alpha_1$ and $\alpha_2$ be two simple curves connecting the origin and points $\zeta_1$ and $\zeta_2$ in $\mathbb{C}$ and meeting only at the origin. We attach to these points two disjoint compact sets $K_1$ and $K_2$, which are connected and have connected complements, so that $K_j \cap \alpha_j = \{\zeta_j\}$, $j = 1, 2$, and $K_j \cap \alpha_k = \emptyset$, $j \neq k$. Let $L_1 = \alpha_1 \cup K_1$ and $L_2 = \alpha_2 \cup K_2$ and $L = L_1 \cup L_2$. Let $\gamma$ be an access curve to $L$ at the origin such that if we move by $\gamma$ and then by $\partial L$ counterclockwise, then we meet $L_1$ first and then $L_2$.

**Proposition 4.1.** Suppose that $f \in S_{0,w_0}(L,W,M)$. Let $f_j$ be the restriction of $f$ to $L_j$, $j = 1, 2$. Then $I_\gamma(f) = I_\gamma(f_1) \ast I_\gamma(f_2)$.

**Proof.** Deforming curves $\alpha_1$ and $\alpha_2$ and the mapping $f$ near the origin we may assume that there is $t_0 > 0$ such that $\alpha_1(t) = t$ and $\alpha_2(t) = -t$ and $f(\alpha_1(t)) = f(\alpha_2(t)) = w_0$ when $0 \leq t \leq t_0$. To use Lemma 3.9 we split $L$ into compact sets $K'_1$ which consists of the restriction of the curve $\alpha_1$ to $[0,1]$ and $K_1$ while $K'_2$ consists of $\alpha_2$ and $K_2$. The role of the connecting curve $\alpha$ is played by the restriction of the curve $\alpha_1$ to $[0,t_0]$. The same splitting is applied to $L_1$ but in this case $K'_2$ is just the origin.

By Lemma 3.9 we can replace in both cases $K'_1$ by a closed disk attached to $\alpha_1(t_0)$ and the mapping $f$ by a mapping $g$ so that $I_\gamma(f) = I_\gamma(g)$ and $I_\gamma(f_1) = I_\gamma(g|_{K'_1\cup \partial})$.

We repeat the same trick for $K_2$ and obtain a compact set $A$ consisting of the intervals $I_1 = [0, t_0]$ and $I_2 = [-t_0, 0]$ and closed disks $D_1$ and $D_2$ attached to $t_0$ and $-t_0$ respectively. Then we construct a continuous path in $S^*(C,W,M)$. At the first step we rotate the disks $D_1$ and $D_2$ around $t_0$ and $-t_0$ respectively so that the intervals $I_1$ and $I_2$ become perpendicular to the boundary of the disks. The mappings $g_t$ are defined by compositions with these rotations. Since this is a continuous path in $S^*(C,W,M)$ by Corollary 3.9 homotopic types do not change.

Then we shrink intervals $I_1$ and $I_2$ to the origin applying dilations $d_t(\zeta) = t\zeta$, $0 \leq t \leq 1$, with the simultaneous parallel translations of the disks $D_1$ and $D_2$ along the real line so they stay connected with the intervals. Again the mappings $g_t$ are defined on the disks by compositions with these translations and stay equal to $w_0$ on the interval.

Finally, we dilate the disks using the mappings $t\zeta$ to make them of radius 1. In this way we obtained a compact set $K$ consisting of two disks exactly as in the definition of the $\ast$ operation with mappings $g_1$ and $g_2$ on the disks. Let $g = h_{g_1,g_2}$. By construction $I_\gamma(g_1) = I_\gamma(f_1)$, $I_\gamma(g_2) = I_\gamma(f_2)$ and $I_\gamma(g) = I_\gamma(f)$. Since $I_\gamma(g_1) = [g_1]$, $I_\gamma(g_2) = [g_2]$ and $I_\gamma(g) = [g_1] \ast [g_2]$ the result follows.

This construction allows us to prove that $\eta_1(W,M,w_0)$ with the operation $\ast$ is a semigroup.
Theorem 4.2. The operation \(*\) induces on \(\eta_1(W, M, w_0)\) the structure of a semi-
group with unity.

Proof. The unity is the class of the constant mapping equal to \(w_0\) on \(T\). If, say, \(f_1 \equiv w_0\) then continuously shrinking \(K_1\) to the origin leaving the functions equal to \(w_0\) we will get a continuous path in \(S^* (C, W, M)\) which ends at \((K_2, f_2(1 + \zeta))\).

By Corollary \(4.1\) \(I_\gamma (h_{f_1, f_2}) = [f_2]\).

To prove that the operation \(*\) is associative we consider a compact set \(L\) consisting of three intervals \(I_1 = [0, 1], I_2 = [0, i], I_3 = [-1, 0]\) and three closed disks \(D_1 = \{|\zeta - 2| \leq 1\}, D_2 = \{|\zeta - 2i| \leq 1\}\) and \(D_3 = \{|\zeta + 2i| \leq 1\}\). The access curve \(\gamma = [-i, 0]\). Given \(f_1, f_2, f_3 \in \mathcal{S}_{1,w_0}(\overline{D}, W, M)\) we define the mapping \(f\) on \(L\) to be equal to \(w_0\) on intervals \(I_1, I_2, I_3\) and \(f_1(2 - \zeta)\) on \(D_1\), \(f_2(2 + i\zeta)\) on \(D_2\) and \(f_3(2 + \zeta)\) on \(D_3\).

Let \(f'_j\) be the restriction of \(f\) to \(I_j \cup D_j\), \(j = 1, 2, 3\). Shifting continuously \(D_j\) to the origin we see that \(I_\gamma (f'_j) = [f_j]\). Let \(f''_j\) be the restriction of \(f\) to \(\cup_k \neq j I_k \cup D_k\).

By Proposition \(4.1\) \(I_\gamma (f''_j) = I_\gamma (f'_j) \ast I_\gamma (f'_2) = [f_1] \ast [f_2]\). Also by Proposition \(4.1\) \(I_\gamma (f) = I_\gamma (f''_j) \ast I_\gamma (f'_2) = ([f_1] \ast [f_2]) \ast [f_3]\). Now if repeat this argument taking instead \(f''_j\) then we get \(I_\gamma (f) = I_\gamma (f'_j) \ast I_\gamma (f'_2) = [f_1] \ast ([f_2] \ast [f_3])\). □

Another useful tool to calculate homotopic types using the \(*\) operation is the content of the following proposition. Suppose that \(D\) is a Jordan domain in \(C\) and \(\zeta_0 \in \partial D\). Suppose that there are \(k\) simple curves \(\alpha_1, \ldots, \alpha_k\) in \(\overline{D}\) such that the curves meet \(\partial D\) only at their endpoints, endpoints of each curve are not equal and the curves may meet each other only at endpoints. The curves are numbered in such a way that if we move counterclockwise from \(\zeta_0\) along \(D\) until we reach an endpoint of one of these curves such that the domain between this curve and \(\partial D\) contains no other curves, then this curve is \(\alpha_1\). We denote the domain between \(\alpha_1\) and \(\partial D\) by \(D_1\). We also denote by \(\zeta_1\) the endpoint of \(\alpha_1\) we encountered first. Let \(K_1\) be the compact set consisting of \(\overline{D}_1\) and the arc in \(\partial D\) connecting \(\zeta_0\) and \(\zeta_1\). The complement of \(D_1\) in \(D\) is also a Jordan domain and if we repeat this process in \(D_1\) then we get \(\alpha_2, D_2\) and \(K_2\) and so on. Thus we obtain Jordan domains \(D_j\) and compact sets \(K_j, 1 \leq j \leq k + 1\), and we will say that the curves \(\alpha_1, \ldots, \alpha_k\) divide \(D\).

Let \(B = \partial D \cup \partial \alpha_1 \cup \cdots \cup \partial \alpha_k\) and \(f \in \mathcal{S}_{\zeta_0, w_0}(B, \overline{D}, W, M)\). Let \(f_j\) be the restriction of \(f\) to \(\partial K_j\). Then \(f_j \in \mathcal{S}_{\zeta_0, w_0}(K_j, W, M)\).

Proposition 4.3. Suppose that \(D\) is a Jordan domain in \(C\), \(\zeta_0 \in \partial D\) and \(\gamma\) is an access curve to \(D\) at \(\zeta_0\). Let \(\alpha_1, \ldots, \alpha_k\) be simple continuous curves in \(\overline{D}\) dividing \(D\) into domains \(D_j, 1 \leq j \leq k + 1\). If \(f \in \mathcal{S}_{\zeta_0, w_0}(B, \overline{D}, W, M)\) then

\[ I_\gamma (f) = I_\gamma (f_1) \ast \cdots \ast I_\gamma (f_{k+1}). \]

Proof. Since the mappings \(f_j\) are defined inductively it suffices to prove this proposition for \(k = 1\). The rest follows by induction.

At the first step we will separate \(D_1\) and \(D_2\) along \(\alpha_1\). Let \(\alpha_1 : [0, 1] \rightarrow \overline{D}, \alpha_1(0) = \zeta_1\) and \(\alpha_1(1) = \xi_1\). We fix conformal mappings \(\Psi_1\) and \(\Psi_2\) of \(\overline{D}\) onto \(\overline{D}_1\) and \(\overline{D}_2\) respectively such \(\Psi_1(1) = \Psi_2(1) = \zeta_1\) and \(\Psi_1(-1) = \Psi_2(-1) = \xi_1\). We may extend \(f\) holomorphically into a neighborhood \(V\) of \(\alpha_1\) as a mapping into \(W\). There is \(\ell_0 > 0\) such that all arcs \(\beta_t\) of unit circles centered at \(t, 0 \leq t \leq \ell_0\), lying in \(\overline{D}\) belong to \(\Psi_1^{-1}(V)\). Let \(E_t\) be the closed set in \(\overline{D}\) bounded by \(\beta_t\) and \(T\) and containing \(0\). We let \(K_1^t\) be the union of \(\Psi_1(E_t)\) and the arc in \(\partial D\) connecting \(\zeta_0\)
and \( \zeta_t \), where \( \zeta_t = \Psi_1(\xi_t) \) and \( \xi_t \) is the point where \( \beta_t \) meets \( \mathbb{T} \) and \( \text{Re} \xi_t > 0 \). Set \( f'_t \) to be the restriction of \( f \) to \( \partial K^t_1 \). Then \( (K^t_1, f'_t) \) is a continuous path in \( S^*(\mathbb{C}, W, M) \) and \( f_1(\zeta_0) = w_0 \). By Corollary \( \text{II} \), \( I_\gamma(f'_t) = I_\gamma(f_t) \) when \( 0 \leq t < t_0 \).

We do the same with \( D_2 \) getting a continuous path \( (K^t_2, f'_2) \) in \( S^*(\mathbb{C}, W, M) \) with \( f_2(\zeta_0) = w_0 \) such that \( I_\gamma(f'_2) = I_\gamma(f_2) \) when \( 0 \leq t < t_0 \).

Let \( K^t \) be the union of \( K^t_1 \) and \( K^t_2 \) and let \( f^t \) be the restriction of \( f \) to \( \partial K^t \). If \( \eta \) is taken from Theorem \( \text{III} \), then the triple \( (\partial K^t, K^t, f^t) \) lies in the \( \Phi, \eta \)-neighborhood of \( (B, K, f) \) when \( t \) is small and by this theorem \( I_\gamma(f^t) = I_\gamma(f) \).

In the last step we separate in the same way \( K^t_2 \) from \( K^t_1 \) along the arc in \( \partial D \) connecting \( \zeta_0 \) and \( \zeta_t \). Again homotopic types will not change. We apply Proposition \( \text{III} \) to show that \( I_\gamma(f) = I_\gamma(f_1) \ast I_\gamma(f_2) \).

5. Examples of holomorphic fundamental semigroups

In this section \( W = A_{s,r} = \{ s < |z| < r \} \), where \( 0 < s < 1 < r \), and \( M = \mathbb{C}P^1 \) or \( M = \mathbb{D}(0, x) = \{ |\zeta| < x \} \), where \( r \leq x \leq \infty \). We fix \( \Pi(z) = z \) and \( w_0 = 1 \).

The examples below show that the mapping \( \iota_1: \eta_1(W, M, w_0) \to \pi_1(W, w_0) \) need not to be injective or surjective.

**Theorem 5.1.** The semigroup \( \eta_1(A_{s,r}, \mathbb{C}P^1, w_0) \) is isomorphic to \( \mathbb{N}_0 \oplus \mathbb{N}_0 \), where \( \mathbb{N}_0 \) is the semigroup by addition of non-negative integers. Under this isomorphism the class of \( f \in S_{1,w_0}(\mathbb{D}, A_{s,r}, \mathbb{C}P^1) \) is mapped into \( (m, n) \), where \( m \) and \( n \) are the numbers of zeros and poles of \( f \) respectively counted with multiplicity.

The semigroup \( \eta_1(A_{s,r}, \mathbb{D}(0, x), w_0) \) is isomorphic to \( \mathbb{N}_0 \). Under this isomorphism the class of \( f \in S_{1,w_0}(\mathbb{D}, A_{s,r}, \mathbb{D}(0, x)) \) is mapped into \( m \), where \( m \) is the number of zeros of \( f \) counted with multiplicity.

**Proof.** Firstly, we show that if \( f, g \in S_{1,w_0}(\mathbb{D}, A_{s,r}, \mathbb{C}P^1) \) and \( \hat{f} \) and \( \hat{g} \) have the same numbers of zeros and poles, then \( [f] = [g] \). For this we define \( F(z, \zeta) = \alpha(z)(\zeta - z)(1 - \bar{\zeta})^{-1} \), where \( \alpha(z) = (1 - z)^{-1}(1 - \bar{z}) \), so \( F(z, 1) = 1 \). If \( a_k \), \( 1 \leq k \leq m \), are zeros of \( \hat{f} \) and \( b_j \), \( 1 \leq j \leq n \), are poles of \( \hat{f} \), then we can write \( \hat{f}(\zeta) = Z(\zeta)(P(\zeta))^{-1} h(\zeta) \), where \( Z(\zeta) = \prod_{j=1}^{m} F(a_j, \zeta) \), \( P(\zeta) = \prod_{j=1}^{n} F(b_j, \zeta) \) and \( h \in S_{1,w_0}(\mathbb{D}, A_{s,r}, \mathbb{C}P^1) \) has no zeros or poles. Since \( s < |h(\zeta)| < r \) on \( \mathbb{T} \), we see that \( s < |h(\zeta)| < r \) on \( \overline{\mathbb{D}} \).

Take two distinct points \( a, b \in \mathbb{D} \). Let \( \alpha_k(t), 1 \leq k \leq m \), and \( \beta_j(t), 1 \leq j \leq n \), be families of continuous curves on \( [0, 1] \) in \( \mathbb{D} \) such that \( \alpha_k(0) = a_k \), \( \alpha_k(1) = a \), \( \beta_j(0) = b_j \), \( \beta_j(1) = b \), and no two curves intersects each other except at end points. For \( 0 \leq t \leq 1 \) let \( Z_t(\zeta) = \prod_{j=1}^{m} F(\alpha_j(t), \zeta) \) and \( P_t(\zeta) = \prod_{j=1}^{n} F(\beta_j(t), \zeta) \). Define \( h_t(\zeta) = h((1 - t)\zeta + t) \) and \( f_t = Z_t P_t^{-1} h_t \). Then \( f_t(1) = 1 = w_0 \), \( f_0 = f \) and \( f_1(\zeta) = F^m(a, \zeta) F^{-n}(b, \zeta) \). Thus we see that mappings with the same numbers of zeros and poles are \( h \)-homotopic to the same mapping and, hence, their homotopic types are the same.

If \( f_t \) is a continuous path in \( S(\mathbb{D}, A_{s,r}, \mathbb{C}P^1) \), then the mappings \( f_t \) form a continuous path in \( A(\mathbb{D}, \mathbb{C}P^1) \). Hence the number of zeros and poles counted with multiplicity stays constant. Therefore the mapping \( R([f]) = (m, n) \) of \( \eta_1(\mathbb{W}, \mathbb{C}P^1, w_0) \) into \( \mathbb{N}_0 \oplus \mathbb{N}_0 \), where \( m \) and \( n \) are the numbers of zeros and poles of \( f \) respectively counted with multiplicity, is well defined. By the argument above the mapping \( R \) is injective and, evidently, it is surjective. Let us show that if \( f_1 \) has \( m_1 \) zeros and \( n_1 \) poles and \( f_2 \) has \( m_2 \) zeros and \( n_2 \) poles, then \( R([f_1] \ast [f_2]) = (m_1 + m_2, n_1 + n_2) \).
This follows immediately from the definition of the \( \star \) operation because a sufficiently good \( \Phi, \varepsilon \)-approximations of the mapping \( h_{f_1,f_2} \) in the definition has exactly \( m_1 + m_2 \) zeros and \( n_1 + n_2 \) poles.

The case when \( M = \mathbb{D}(0,x) \) follows from this argument if we take into account that the number of poles is equal to 0. \( \square \)

Another example when \( t_1 \) is not be injective is based on an example by Wermer (13 [16]). He constructed a strongly pseudoconvex domain \( \Omega \in \mathbb{C}^3 \) diffeomorphic to a ball and a holomorphic imbedding \( F(z, w, t) = (z, z w + t, z w^2 - w + 2 t w) \) of \( \Omega \) into \( \mathbb{C}^3 \) such that the mapping \( f(\zeta) = (\zeta, 1, 0) \in \mathcal{S}(\mathbb{D}, F(\Omega), \mathbb{C}^3) \) but \( f(0) \not\in F(\Omega) \). While \( t_1([f]) = 0 \) the mapping \( f \) is not \( h \)-homotopic to a constant mapping because it will contradict the continuity principle.

6. Properties of Holomorphic Fundamental Semigroups

Let \((W_1, \Pi_1)\) and \((W_2, \Pi_2)\) be two Riemann domains over two complex manifolds \( M_1 \) and \( M_2 \) respectively. Suppose \( w_1 \in W_1, w_2 \in W_2 \) and there are holomorphic mappings \( \phi : W_1 \to W_2 \) such that \( \phi(w_1) = w_2 \) and \( \psi : M_1 \to M_2 \) which satisfy \( \psi \circ \Pi_1 = \Pi_2 \circ \phi \). Then for any \( f \in \mathcal{S}(K, W_1, M_1) \) we have \( \Pi_2 \circ \phi \circ f = \psi \circ \Pi_1 \circ f = \psi \circ \hat{f} \). So \( \phi \circ \hat{f} = \psi \circ \hat{f} \) and we get a continuous mapping from \( \mathcal{S}^*(N, W_1, M_1) \) to \( \mathcal{S}^*(N, W_2, M_2) \) which maps a pair \((K, f)\) to \((K, \phi \circ f)\). Hence, firstly, the mapping from \( S_{1,w_1}(\mathbb{D}, W_1, M_1) \) to \( S_{1,w_2}(\mathbb{D}, W_2, M_2) \) induces a well defined mapping \( \phi_* \) from \( \eta_1(W_1, M_1, w_1) \) to \( \eta_2(W_2, M_2, w_2) \) given by \( \phi_*([f]) = [\phi \circ f] \). Secondly, if \( \gamma \) is an access curve to \( K \), then \( \phi_*([f, \gamma]) = [\phi \circ f, \gamma] \). In particular, if \((K, h_{f_1,f_2})\) is the pair in the definition of the \( \star \) operation, then

\[
\phi_*([f_1] \star [f_2]) = \phi_*([h_{f_1,f_2}, \gamma]) = [\phi \circ h_{f_1,f_2}, \gamma] = [\phi \circ f_1, \phi \circ f_2] \in [\phi \circ f_1] \star [\phi \circ f_2] = \phi_*[f_1] \star \phi_*[f_2].
\]

This leads us to the following proposition.

**Proposition 6.1.** The induced mapping \( \phi_* : \eta_1(W_1, M_1, w_1) \to \eta_2(W_2, M_2, w_2) \) is a homomorphism.

This proposition has the following corollary.

**Corollary 6.2.** Let \( f \in S_{1,w_2}(\mathbb{D}, W_2, M_2) \) and let \( [f]^k \) be the product of \( k \) classes \([f] \). Then \([f]^k = [f(\zeta^k)] \).

**Proof.** We may assume that \( \hat{f} \) is defined on \( \mathbb{D}(0,r), r > 1 \), and \( f \) maps \( A_{r-1} \) into \( W_2 \). Set \( W_1 = A_{r-1}, M_1 = \mathbb{D}(0,r) \) and \( w_1 = 1 \). Let \( \phi = f \) and \( \psi = f \). By Proposition 6.1 and Theorem 5.1 we have

\[
|f(\zeta^k)| = \phi_*([\zeta^k]) = \phi_*([\zeta]^k) = \phi_*([\zeta])^k = [f]^k.
\]

\( \square \)

Let \((W_1, \Pi_1)\) and \((W_2, \Pi_2)\) be two Riemann domains over two complex manifolds \( M_1 \) and \( M_2 \) respectively. Then clearly \((W_1 \times W_2, (\Pi_1, \Pi_2))\) is a Riemann domain over \( M_1 \times M_2 \).

**Theorem 6.3.** If \((W_1, \Pi_1)\) and \((W_2, \Pi_2)\) are two Riemann domains over two complex manifolds \( M_1 \) and \( M_2 \) respectively, then

\[
\eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2)) \cong \eta_1(W_1, M_1, w_1) \times \eta_1(W_2, M_2, w_2).
\]

17
Proof. Let $p_i : W_1 \times W_2 \to W_i$ and $q_i : M_1 \times M_2 \to M_i$ be projection maps for $i = 1, 2$. Then by Proposition 6.1 induced mappings $p_{i*}$ from $\eta(W_1 \times W_2, M_1 \times M_2, (w_1, w_2))$ into $\eta(W_1, M_1, w_1)$ given by $p_{i*}([(f_1, f_2)]) = [f_1]$ are homomorphisms. Now define a mapping

$$\phi : \eta(W_1 \times W_2, M_1 \times M_2, (w_1, w_2)) \to \eta(W_1, M_1, w_1) \times \eta(W_2, M_2, w_2)$$

by taking $\phi = (p_{1*}, p_{2*})$, i.e.

$$\phi([(f_1, f_2)]) = (p_{1*}([(f_1, f_2)]), p_{2*}([(f_1, f_2)])) = ([f_1], [f_2]).$$

Since $p_{1*}$ and $p_{2*}$ are homomorphisms clearly $\phi$ is a homomorphism.

To show that $\phi$ is an isomorphism we construct its inverse. To do that define the mapping

$$\psi : \eta(W_1, M_1, w_1) \times \eta(W_2, M_2, w_2) \to \eta(W_1 \times W_2, M_1 \times M_2, (w_1, w_2))$$

by taking $\psi([f_1], [f_2]) = ([f_1], [f_2])$. This mapping $\psi$ is well defined since for any two continuous paths $f_1$ in $\mathcal{S}_{1,w_1}([\overline{D}, W_1, M_1])$ and $g_1$ in $\mathcal{S}_{1,w_2}([\overline{D}, W_2, M_2])$, $(f_1, g_1)$ is a continuous path in $\mathcal{S}_{1,(w_1, w_2)}([\overline{D}, W_1 \times W_2, M_1 \times M_2])$. It is easy to see that $\phi$ and $\psi$ are inverses of each other. Hence $\phi$ is an isomorphism. $\square$

We want to show that the homomorphic fundamental semigroup does not depend on the choice of base points. Let $w_0$ and $w_1$ be two points in $W$. Let $\alpha(t), t \in [0, 1]$, be a continuous curve in $W$ with $\alpha(0) = w_0$ and $\alpha(1) = w_1$. Let $L$ be a compact set on the plane consisting of the interval $I = [0, 1]$ and the closed disk $D = \{|\zeta - 2| \leq 1\}$. Given a mapping $f \in \mathcal{S}_{1,w_1}([\overline{D}, W, M])$ we define a mapping $\tilde{f}$ on $L$ to be equal to $\alpha$ on $I$ and to $f(2 - \zeta)$ on $\partial D$. Clearly, $\tilde{f} \in \mathcal{S}_{1,w_0}(L, W, M)$.

We take the access curve $\gamma(t) = -it, 0 \leq t \leq 1$, to $L$ at the origin. Clearly, if $[\tilde{f}] = [g]$, then $[\tilde{f}, \gamma] = [g, \gamma]$. Hence we have a well-defined mapping $F_\alpha$ from $\eta(W, M, w_1)$ into $\eta(W, M, w_0)$.

First of all, by Corollary 3.9 any curve connecting $w_0$ to $w_1$ which is homotopic to $\alpha$ will give us the same mapping $F_\alpha$. Thus $F_\alpha$ depends only on the homotopy class $\{\alpha\}$ of $\alpha$ in $\pi_1(W, w_0, w_1)$. Secondly, we let $\alpha^{-1}$ to be the curve $\alpha^{-1}(t) = \alpha(1-t)$ for $0 \leq t \leq 1$ and denote by $\beta$ the curve on $[0, 1]$ defined as $\alpha(2t)$ when $0 \leq t \leq 1/2$ and as $\beta(2t - 1)$ when $1/2 \leq t \leq 1$. Then $\iota_1(F_\alpha([\tilde{f}]))$ is equal to the homotopy class of $\alpha\beta\alpha^{-1}$ in $\pi_1(W, w_0)$. Slightly abusing the notation we will denote also by $F_\alpha$ the homomorphism $\alpha\beta\alpha^{-1}$ mapping $\pi_1(W, w_1)$ into $\pi_1(W, w_0)$. And, thirdly, if $g \in \mathcal{S}_{1,w_0}(\overline{D}, W, M)$ then the mapping $F_g = F_\alpha$, where $\alpha(t) = g(e^{2\pi i t})$.

Proposition 6.4. Let $\{w_0, w_1, w_2\} \subset W$, $\alpha$ be a curve which connects $w_0$ and $w_1$, $\beta$ be a curve which connects $w_1$ and $w_2$, $f_1, f_2 \in \mathcal{S}_{1,w_1}(\overline{D}, W, M)$ and $f, g \in \mathcal{S}_{1,w_0}(\overline{D}, W, M)$. Then,

1. $F_{\alpha\beta} = F_\alpha \circ F_\beta$.
2. $F_{\alpha^{-1}} \circ F_\alpha([f_1]) = [f_1]$.
3. $F_\alpha([f_1] \ast [f_2]) = F_\alpha([f_1]) \ast F_\alpha([f_2])$.
4. $F_g([f]) \ast [g] = [g] \ast [f]$.
5. $F_g([g]) = [g]$.

Proof. (1) Given $f \in \mathcal{S}_{1,w_2}(\overline{D}, W, M)$ we consider the pair $(L, \tilde{f})$ as above for the curve $\alpha\beta$. By Corollary 3.3 we may dilate $[0, 1]$ so that $L$ consists of intervals $I_1 = [0, 1], I_2 = [1, 2]$ and the closed disk $D$ of radius 1 attached to 2 and $\tilde{f}|_{I_1} = \alpha$ and $\tilde{f}|_{I_2} = \beta$. By Lemma 3.9 we can replace $L$ with a compact set $L'$ which consists
of $I_1$ and a disk $D'$ attached to 1 and replace $\bar{f}$ with $g$ on $L'$ so that $[g, \gamma] = [f, \gamma]$ and \(\{g|_{D'}, 1\} = \{\bar{f}I_2\cup D), I_1\} = F_\beta([f])\). Now $F_{\alpha\beta}([f]) = F_\alpha(F_\beta([f]))$ by the definition of $F_\alpha$.

(2) follows from (1) because $\alpha^{-1}\alpha$ and $\alpha\alpha^{-1}$ are homotopic to constant curves.

(3) Consider a compact set $K$ consisting of disks $K_1 = \{\zeta - 1 \leq 1\}$ and $K_2 = \{\zeta + 1 \leq 1\}$ and the interval $I = [-i, 0]$. We define a mapping $f$ on $K_1 \cup K_2$ as in the definition of the * operation and let $f(-it) = \alpha(t)$ when $0 \leq t \leq 1$. Let $\gamma = [-2i, -i]$ be an access curve to $K$ at $-i$.

By Lemma 3.9 we can replace $K_1 \cup K_2$ with a closed disk $D$ attached to the origin and $f_1$ and $f_2$ with $g \in S_{0,w_0}(\mathcal{D}, W, M)$ so that $\{g, 0\} = \{f_1\} \ast \{f_2\}$. Thus $I_1(f) = F_\alpha(\{f_1\} \ast \{f_2\})$.

On the other hand let $K^\prime$ be the compact sets consisting of disks $K^\prime_1 = \{\zeta - 1 - t| \leq 1\}$ and $K^\prime_2 = \{\zeta + 1 + t| \leq 1\}$, intervals $I^\prime_1 = [t, -i + t]$, $I^\prime_2 = [-t, -t - i]$ and the interval $I^\prime = [-t - i, -i + t]$. We define a mapping $f^\prime$ on $K^\prime_1$ as $f^\prime_1((1 + t - \zeta)$ and on $K^\prime_2$ as $f^\prime_2(\zeta + 1 - t)$. We let $f^\prime(t - si) = \alpha(s)$ and $f^\prime(-t - si) = \alpha(s)$ when $0 \leq s \leq 1$ and let $f \equiv w_0$ on $I^\prime$. The path $(K^\prime, f^\prime)$ is continuous when $0 \leq t \leq 1$ and by Corollary 3.5 $I^\prime(f^\prime) = I_1(f)$. Since $K^\prime_0 = K$ and $f^\prime_0 = f$ we see that $I_1(f^\prime) = I_1(f)$. Now we apply Proposition 3.14 where $K_1 := K^\prime_1 \cup I^\prime_1$, $K_2 := K^\prime_2 \cup I^\prime_2$ and $\alpha := I^\prime_1$ to see that $I_1(f) = F_\alpha(\{f_1\} \ast \{f_2\})$.

(4) Consider a compact set $K$ consisting of the disks $D^1 = \{\zeta \leq 1\}$ and $D^2 = \{\zeta - 3 \leq 1\}$ and the interval $I = [1, 2]$. We define a mapping $h(\zeta)$ on $D^1$ as $g(\zeta)$ and on $D^2$ as $f(3 - \zeta)$. Let $h(t) = g(e^{2\pi t(i - 1)})$ on $I$. Let $\gamma = [-i + 1, 1]$ be an access curve to $K$ at 1. Then $[h, \gamma] = F_\beta([f]) \ast [g]$.

Consider the compact family of compact sets $K_s$, $0 \leq s \leq 1/2$, consisting of the disk $D^1$, an interval $I_s = [e^{2\pi si}, (2 - s)e^{2\pi si}]$ and the closed disk $D^2_s$ of radius $e^{2\pi si}$. The mapping $h_s$ on $K_s$ is defined as $h$ on $D^1$ and as $h((s + |\zeta| - 1)$ when $\zeta \in I_s$. The mapping $h_s$ on $D^2_s$ is defined as a composition of $h$ on $D^2$ and a conformal mapping that maps $D^2_s$ onto $D^2$ moving $(2 - s)e^{2\pi si}$. Simply speaking we rotate $I_s \cup D^2$ around $D^1$ leaving one end of $I_s$ attached normally to $D^1$. Clearly, the pairs $(K_s, h_s)$ form a continuous path and $[h_s, \gamma] = F_\beta([f]) \ast [g]$.

When $s = 1/2$ the set $K_{1/2}$ consists of $D^1$, $I_{1/2} = [-1, -3/2]$ and the disk $D^2_{1/2}$. Since all access curves to $K_{1/2}$ at 1 are equivalent we replace $\gamma$ with $\gamma' = [i + 1, 1]$. Still $[h_{1/2}, \gamma'] = F_\beta([f]) \ast [g]$. This is done to be sure that the access curve lies outside of the sets in the family $K_s$ as it is required by Corollary 3.5.

Then we continue the process described above for $1/2 \leq s \leq 1$. Finally, $K_1$ will consists of $D^1$ and $D^2 = \{\zeta - 2 \leq 1\}$. The mapping $h_1$ is equal to $g$ on $D^1$ and to $f(2 - \zeta)$. Now it is clear that $[h_1, \gamma'] = [g] \ast [f]$.

(5) We start with the compact set $K_1$ consisting of the interval $I = [0, 1]$ and the unit disk $D_1 = \{\zeta - 2 \leq 1\}$. The mapping $f_1$ on $K_1$ is defined as $g(e^{2\pi it})$ on $I$ and as $g(2 - \zeta)$ on $D_1$. If the access curve $\gamma = [-i, 0]$, then $[f_1, \gamma] = F_\beta([g])$.

For $0 \leq s \leq 1$ we define compact sets $K_s$ consisting of the intervals $I_s = [0, s]$ and the disks $D_s = \{\zeta - (1 + s) \leq 1\}$. The mapping $f_s$ is defined as $g(e^{2\pi it})$ on $I_s$ and as $g(e^{2\pi is}(1 + s - \zeta))$ on $D_s$. The pairs $(K_s, f_s)$ form a continuous path and $[f_s, \gamma] = F_\beta([g])$. Since $K_0$ consists of the disk $\{\zeta - 1 \leq 1\}$ and the mapping $f_0(\zeta) = g(1 - \zeta)$ we see that $[f_1, \gamma] = [g]$.

As a consequence we have the following corollary.
Corollary 6.5. If \( w_0 \) and \( w_1 \) are two points of \( W \) then \( \eta_1(W, M, w_0) \) is isomorphic to \( \eta_1(W, M, w_1) \).

7. Finitely connected domains

The main goal of this section is to study \( \eta_1(W, \mathbb{C}, w_0) \) when \( W \) is a finitely connected domain in \( \mathbb{C}, M = \mathbb{C} \) and \( \Pi(z) = z \). First we consider the case where our domain \( W \) is a disk with \( n \) punctures, i.e. \( W = W' \setminus E \), where \( W' \) is a connected and simply connected domain and \( E = \{w_1, \ldots, w_m\} \) is a set of distinct points in \( W' \). Let \( w_0 \in W \) be a base point.

The fundamental group of \( W \) is a free group on \( m \) generators. We will fix the set of generators by choosing simple continuous curves \( \alpha_j : [0, 1] \to W \), \( 1 \leq j \leq m \), such that \( \alpha_j \) connects \( w_0 \) with \( w_j \), never meets \( E \), when \( 0 \leq t < 1 \), and these curves meet each other only at \( w_0 \). If we take sufficiently small disjoint disks \( d_j \) centered at \( w_j \) and a point \( \zeta_j \in \partial d_j \), where \( \alpha_j \) meets \( \partial d_j \), then the union of these curves and disks \( d_j \) with deleted centers is a homotopy retract of \( W \). Therefore, the set of homotopy classes of equivalence of curves \( \{\lambda_j, 1 \leq j \leq m\} \), where \( \lambda_j \) is the curve which starts at \( w_0 \) follows \( \alpha_j \) up to \( \zeta_j \), then goes counterclockwise by \( \partial d_j \) until \( \zeta_j \) and then returns to \( w_0 \) by \( \alpha_j \), will be the set of generators of \( \pi_1(W, w_0) \) which will be denoted by \( \{e_1, \ldots, e_m\} \). Clearly, the homotopy classes of \( \lambda_j \) do not depend on the radii of the disks \( d_j \) provided that they are sufficiently small.

Let \( \beta_j \) be the curve defined as \( \lambda_j \) from \( w_0 \) to \( \zeta_j \). For each \( j \) we consider the mapping \( f_j \) of \( K = [0, 1] \cup \{\zeta - 2 \leq 1\} \) equal to \( \beta_j \) on \( [0, 1] \) and a conformal mapping onto \( d_j \) on \( \{\zeta - 2 \leq 1\} \) moving 1 to \( \zeta_j \). We will define \( [g_j] \in \eta_1(W, \mathbb{C}, w_0) \) as \( I_*(f_j) \), where the access curve \( \gamma = [-i, 0] \). Then \( \iota_1([g_j]) = e_j \).

If \( f \in S_{1, w_0}(\overline{D}, W, \mathbb{C}) \) then \( \hat{f} \) takes values \( w_1, \ldots, w_m \) at finitely many points. The number of these points counted with multiplicity of \( f \) at these points will be called the index of \( f \). If \( f, g \in S_{1, w_0}(\overline{D}, W, \mathbb{C}) \) and \( [f] = [g] \), then their indexes coincide. We will need the following lemma.

Lemma 7.1. If \( f \in S_{1, w_0}(\overline{D}, W, \mathbb{C}) \) then \( [f] = [f_1] \star \cdots \star [f_k] \), where \( k \) is equal to the index of \( f \), and the index of each of \( f_j \), \( 1 \leq j \leq k \), is 1.

Proof. Shift \( f \) slightly to ensure that any value \( w_1, \ldots, w_m \) is taken with multiplicity 1. Choose \( k - 1 \) curves \( \alpha_j \) dividing \( \overline{D} \) into Jordan domains containing exactly one preimage of the set \( \{w_1, \ldots, w_m\} \) and apply Proposition 4.3.

Now we want to describe the \( h \)-homotopy classes of mappings with index 1.

Lemma 7.2. Let \( f \in S_{1, w_0}(\overline{D}, W, \mathbb{C}) \) has index 1 and \( \hat{f} \) takes the value \( w_j \). Then there is a loop \( \lambda \) at \( w_0 \) such that \( [f] = F_\lambda([g_j]) \) and \( \iota_1([f]) = F_\lambda(e_j) \).

Proof. By the continuity of the homotopic type we may assume that \( \hat{f} \) is defined on \( D_r = \{\zeta < r\} \), \( r > 1 \), maps \( D_r \) into \( W' \) and is of index 1 on \( D_r \). We take a small disk \( d_j \) centered at \( w_j \) so that \( \hat{f}^{-1} \) is defined on \( d_j \). Let \( d_j' = \hat{f}^{-1}(d_j), \xi_j = \hat{f}^{-1}(w_j) \) and let \( \zeta_j' = \hat{f}^{-1}(\zeta_j) \). We connect 1 and \( \zeta_j' \) by a curve \( \mu \) in \( D \setminus d_j' \).

Let \( K \) be the union of \( [0, 1] \) and \( D = \{\zeta - 2 \leq 1\} \) and the access curve \( \gamma \) to \( K \) at 0 is \( [-i, 0] \). We define \( \phi \in \mathcal{A}(K, D_r) \) as \( \mu \) on \( [0, 1] \) and a conformal mapping of \( D \) onto \( d_j' \) such that \( \phi(1) = \zeta_j' \).

We want to show that \( [f] = I_*(f \circ \phi) \). By Mergelyan Theorem we can approximate \( \phi \) by holomorphic mappings on neighborhoods of \( K \) as well as we want.
we can find a smooth Jordan domain $\Omega$ containing $K$ and meeting $\gamma$ and a holomorphic mapping $\psi \in A(\overline{\Omega}, \mathbb{D}_r)$ such that $\psi(\partial \Omega) \subset \mathbb{D}_r \setminus \{\xi_j\}$. $\psi$ takes the value $\xi_j$ only once, $\psi(\zeta_{j,1}) = 1$ and $\{f \circ \psi, \zeta_{j,1}\} = I_2(f \circ \phi)$. Let $e$ be a conformal mapping of $\mathbb{D}$ onto $\mathbb{C}$ such that $e(1) = \zeta_{j,1}$. Then $\{\psi \circ e\} = \{\psi, \zeta_{j,1}\}$ in $\eta_1(\mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C}, 1)$.

Considering a conformal mapping of $\mathbb{D}_r$ onto itself which maps $\xi_j$ to 0 we see that by Proposition 6.4 and Theorem 5.1 the semigroup $\mathcal{S}$ with both left and right cancelation properties is called a cancelative semigroup.

Proposition 6.1. Let $\{\psi, \zeta\}$ be a smooth Jordan domain $\Omega$ containing $K$ and connects $\psi$ only once, $\{\psi, \zeta\}$ in $\eta_1(\mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C}, 1)$. Then $\{\psi \circ e\}$ is $\{\psi, \zeta\}$ in $\eta_1(\mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C}, 1)$. Consequently, the path $f \circ \eta_1$ is also continuous and connects $f \circ \psi$ and $f$ in $\mathcal{S}_{1,w_o}(\mathbb{D}, \mathbb{W}, \mathbb{C})$. Thus $[f] = I_2(f \circ \phi)$.

Let $h_j \in \mathcal{S}_{1,\xi_j}(\mathbb{D}, \mathbb{W}, \mathbb{C})$ be the conformal mapping of $\mathbb{D}$ onto $d_j$ with $h_j(1) = \zeta_j$. Let $\nu = \dot{f} \circ \mu$. Then $[g_j] = F_{\beta_j}([h_j])$ while $[f] = I_2(f \circ \phi) = F_0([h_j])$. Hence $[f] = F_0(F_{\beta_j}([g_j]))$. The concatenation of curves $\nu$ and $\beta_j^{-1}$ is a loop $\lambda$ at $w_0$. By Proposition 6.4 $F_\lambda([g_j]) = F_\nu(F_{\beta_j}([g_j])) = [f]$.

The equality $\ell_1([f]) = F_\lambda(e_j)$ is evident.

Combining this lemma with Lemma 7.1 we obtain the following corollary.

Corollary 7.3. If $f \in \mathcal{S}_{1,w_o}(\mathbb{D}, \mathbb{W}, \mathbb{C})$ has index $k$ then there are loops $\mu_j$, $1 \leq j \leq k$, at $w_0$ such that $\ell([f]) = \prod_{j=1}^k F_{\mu_j}(e_{n_j})$ and $[f] = \prod_{j=1}^k F_{\mu_j}(g_{n_j})$.

The following lemma is rather crucial.

Lemma 7.4. Let $f_0, f_1 \in \mathcal{S}_{1,w_o}(\mathbb{D}, \mathbb{W}, \mathbb{C})$ have index 1 and $\ell_1([f_0]) = \ell_1([f_1])$. Then $[f_0] = [f_1]$.

Proof. Firstly we note that $\dot{f}_0$ and $\dot{f}_1$ take the same value $w_j$ in $W'$. Otherwise, $\ell_1([f_0]) \neq \ell_1([f_1])$. By Lemma 7.2 there are loops $\mu_0$ and $\mu_1$ at $w_0$ such that $[f_0] = F_{\mu_0}([g_j])$ and $[f_1] = F_{\mu_1}([g_j])$. Since $\ell_1([f_0]) = \ell_1([f_1])$ we see that $F_{\mu_0}(e_j) = F_{\mu_1}(e_j)$ or $e_j = F_{\mu_0}(e_j)$. If $\mu = \mu_1^{-1} \mu_0$ in the group $\pi_1(W, w_0)$ then $e_j = e_{j,0} \mu^{-1}$ or $e_j \mu = e_{j,0}$. But the group $\pi_1(W, w_0)$ is free and therefore, $\mu = e_{j,0}^\infty$.

Since the operator $F_\mu$ does not depend on the homotopy class of $\nu$ we can write that $F_\mu([g_j]) = F_{\mu_1}([g_j])$. By Proposition 6.4 $F_{\mu_1}([g_j]) = F_{\mu_1}([g_j]) = F_{\mu_1}^{-1} F_{\mu_0}([g_j]) = F_{\mu_1}^{-1}([f_0])$ or $[f_0] = F_{\mu_1}([g_j]) = [f_1]$.

A semigroup $S$ has the left cancelation property (see [3]) if $ab = ac$ implies $b = c$. Similarly $S$ has the right cancelation property if $ac = bc$ implies $a = b$. A semigroup with both left and right cancelation properties is called a cancelative semigroup. We will show that $\eta_1(W, \mathbb{C}, w_0)$ is a cancelative semigroup. Cancelation property plays a crucial role in proving the injectivity of $\ell_1$. First we need the following two lemmas.

Lemma 7.5. Let $f_t \in \mathcal{S}_{1,w_o}(\mathbb{D}, \mathbb{W}, \mathbb{C})$, $0 \leq t \leq 1$, be a continuous path. Then for each $\varepsilon > 0$ there is a continuous path $g_t$ in $\mathcal{S}_{1,w_o}(\mathbb{D}, \mathbb{W}, \mathbb{C})$ such that $|g_t(\zeta) - f_t(\zeta)| < \varepsilon$ for $\zeta \in \mathbb{D}$, all roots $\zeta_k(t)$ of the equations $g_t(\zeta) = w_j$, $1 \leq j \leq m$, are simple, functions $\zeta_k(t)$ are smooth and $[f_t] = [g_t]$ for all $0 \leq t \leq 1$.

Proof. There is $\varepsilon > 0$ such that the distance between $f_t(T)$ to $\partial W$ is greater than $\varepsilon$ for all $t$. By considering $f(t, \zeta) = f_t(\zeta)$ as a function from the unit interval to
the Banach space $\mathcal{A}(\mathbb{D}, \mathbb{C})$ we can use the Weierstrass theorem to approximate it by polynomials of the form $\sum_{j=0}^{k} h_j t^j$, where $h_j \in \mathcal{A}(\mathbb{D}, \mathbb{C})$. Then by approximating each $h_j$ by holomorphic polynomials on a neighborhood of $\mathcal{A}(\mathbb{D}, \mathbb{C})$, we get a polynomial $h(t, \zeta) = \sum_{k=0}^{N} a_k(t) \zeta^k$ in $t$ and $\zeta$ such that $|f(t, \zeta) - h(t, \zeta)| < \varepsilon/2$ on $[0, 1] \times \mathbb{D}$. Moreover, we may assume that $h(t, 1) = w_0$ for all $t$.

For each $c = (c_0, \cdots, c_N) \in \mathbb{C}^{N+1}$ let

$$P_c(\zeta) = \prod_{j=1}^{m} \left( \sum_{k=0}^{N} c_k \zeta^k - w_j \right).$$

Let $\zeta_1(c), \cdots, \zeta_m(c)$ be the roots of the polynomial $P_c(\zeta)$. Since all elementary symmetric polynomials of $\zeta_1(c), \cdots, \zeta_m(c)$ are holomorphic, the discriminant $\Delta(c) = \prod_{i<j} (z_i(c) - z_j(c))^2$ of $P_c(\zeta)$ is holomorphic on $\mathbb{C}^{N+1}$. The set $D = \{ \Delta = 0 \}$ is analytic of dimension $N$ in $\mathbb{C}^{N+1}$ and for all $c \in \mathbb{C}^{N+1} \setminus D$ the roots of $P_c(\zeta)$ are distinct.

Consider the hypersurface $L = \{(c_0, \cdots, c_N) | \sum_{k=0}^{N} c_k = w_0\}$, where $w_0$ is the base point in $W$. Since $\Delta(c) \neq 0$ for $c = (0, w_0, 0, \cdots, 0) \in L$, $D \cap L$ is an analytic subset of dimension $N - 1$ in $L$.

The curve $c(t) = (c_0(t), \cdots, c_N(t))$ is analytic and either intersects $D$ at finite number of points or completely lies in the $D \cap L$. In the second case we can find a vector $a$ with $\|a\| < \varepsilon/2$, $\sum_{j=0}^{N} a_j = 0$ and $\Delta(c(0) + a) \neq 0$. So, by replacing $c(t)$ by $c(t) + a$ we can assume that $c(t)$ intersects $D$ at a finite number of points.

Since the Hausdorff measure $H_{2N-1}(D \cap L) = 0$, for each intersection point $z$ there is a neighborhood $U$ of $z$ such that $U \setminus D$ is connected. So, in a sufficiently small neighborhood of each intersection point we can smoothly modify the curve $c(t)$ so that it will lie in $L \setminus D$ and if $g(t, \zeta) = P_{c(t)}(\zeta)$ then $|f(t, \zeta) - g(t, \zeta)| < \varepsilon$ on $[0, 1] \times \mathbb{D}$.

Now $g(t, \zeta) = g_t(\zeta)$ gives us a homotopy with simple roots $\zeta_k(t)$ of the equations $g_t(\zeta) = w_j, 1 \leq j \leq m$. Since the roots are simple by the implicit function theorem the functions $\zeta_k(t)$ are smooth.

Since the homotopy $s f_t(\zeta) + (1 - s) g_t(\zeta), 0 \leq s \leq 1$, connects $f_t$ and $g_t$ in $S_{1, w_0}(\mathbb{D}, W, \mathbb{C})$ we see that $|f_t| = |g_t|$ for all $0 \leq t \leq 1$.

The proof of the lemma below follows the same line of argument as that in the proof of Assertion 2 in the proof of [11] Lemma 2.1.

**Lemma 7.6.** Let $\zeta_k(t), 1 \leq k \leq n$, are smooth mappings of $[0, 1]$ into $\mathbb{D}$ such that $\zeta_i(t) \neq \zeta_j(t)$ when $i \neq j$ and $0 \leq t \leq 1$. Then there is a $C^\infty$ mapping $\Phi : \mathbb{D} \times [0, 1] \to \mathbb{D}$ such that $\Phi_t(\zeta) = \Phi(\zeta, t)$ is a diffeomorphism from $\mathbb{D}$ onto itself for each $t$, $\Phi_t(\zeta) = \zeta$ when $|\zeta| = 1$ and $\Phi_t(\zeta_j(0)) = \zeta_j(t)$ for $j = 1, \ldots, n$.

**Proof.** By Whitney extension theorem (see [11] Theorem 1.5.6) we can find a complex valued $C^\infty$-function $F(t, \zeta)$ on $[0, 1] \times \mathbb{C}$ such that $F(t, \zeta_j(t)) = \partial \zeta_j(t)/\partial t$ for $0 \leq t \leq 1$, $j = 1, \ldots, n$. Replacing $F$ with the product $F \phi$, where $\phi$ is a $C^\infty$-function with $\phi = 1$ on $\cup_{j=1}^{n} \{(t, \zeta_j(t)) : 0 \leq t \leq 1\}$ and $\phi = 0$ for $|\zeta| \geq 1$, we can make $F(t, \zeta) = 0$ for $|\zeta| \geq 1$. Then by standard existence and uniqueness theorems for ordinary differential equations, the initial value problem $\partial x/\partial t(t) = F(t, x(t))$, $x(0) = \zeta$, $0 \leq t \leq 1$, has a unique solution $x(t, \zeta)$. Since $F(t, \zeta)$ is smooth, this solution is smooth on $[0, 1] \times \mathbb{C}$.

Now define a mapping $\Phi : \mathbb{C} \times [0, 1] \to \mathbb{C}$ by $\Phi(\zeta, t) = x(t, \zeta)$. Then for each $0 \leq t \leq 1$, $\Phi_t$ is a diffeomorphism and since the related initial value problem has
a unique solution we have $\Phi(\zeta_j(0), t) = \zeta_j(t)$ for $j = 1, \cdots, n$. Also note that $\Phi(\zeta, t) = \zeta$ for all $0 \leq t \leq 1$ when $|\zeta| \geq 1$. So, the restriction of $\Phi$ to $\overline{D} \times [0, 1]$ has desired properties. \hfill $\square$

Now we have all the necessary tools to prove the left and right cancelation properties.

**Proposition 7.7.** Let $f, g_0, g_1 \in S_{1, w_0}(\overline{D}, W, C)$. Then $[f] \ast [g_0] = [f] \ast [g_1]$ implies $[g_0] = [g_1]$ and $[g_0] \ast [f] = [g_1] \ast [f]$ implies $[g_0] = [g_1]$.

**Proof.** By Corollary 7.3 it suffices to prove the left cancelation property only when the index of $f$ is 1 and we assume that $\tilde{f}$ takes value $w_1$. Consider the compact set $K \subset \mathbb{C}$ from the definition the $\ast$ operation which is the union of $K_1 = \{|\zeta - 1| \leq 1\}$ and $K_2 = \{|\zeta + 1| \leq 1\}$ and let the access curve be $\gamma(t) = -it$, $0 \leq t \leq 1$. Pick up a smooth simply connected domain $D$ symmetric with respect to the axes with $K \subset D$ and intersecting $\gamma$ at $\zeta_0 = -it_0$ and and $\Phi_1, \varepsilon$-approximations $(\overline{D}, h_0)$ and $(\overline{D}, h_1)$ with respect to $\gamma$ of $(K, h_{f,G_0})$ and $(K, h_{f,G_1})$ respectively such that $\{h_0, \zeta_0\} = [f] \ast [g_0]$ and $\{h_1, \zeta_0\} = [f] \ast [g_1]$. If $D_1 = D \cap \{\text{Re} \zeta > 0\}$ and $D_2 = D \cap \{\text{Re} \zeta < 0\}$ we may assume that $\{h_0|_{D_1}, \zeta_0\} = [f]$, $\{h_0|_{D_2}, \zeta_0\} = [g_0]$, $\{h_1|_{D_1}, \zeta_0\} = [f]$ and $\{h_1|_{D_2}, \zeta_0\} = [g_1]$. We also may assume that the roots of all equation $\tilde{h}_0 = w_j$ and $\tilde{h}_1 = w_j$ are simple, none of them lies on $[\zeta_0, -\zeta_0]$, the index of $\tilde{h}_0$ and $\tilde{h}_1$ on $D_1$ is 1 and $h_0(\zeta_1) = h_1(\zeta_1) = w_1$ at some $\zeta_1 \in D_1$.

Let $\Psi$ be a conformal mapping of $\mathbb{D}$ onto $D$ such that $\Psi(0) = 0$, $\Psi(1) = \zeta_0 \ast \zeta_0$. By the symmetry $\Psi$ maps $[1, -1]$ onto $[\zeta_0, -\zeta_0]$, $\mathbb{D}^+ = \mathbb{D} \cap \{\text{Im} \zeta > 0\}$ onto $D_1$ and $\mathbb{D}^- = \mathbb{D} \cap \{\text{Im} \zeta < 0\}$ onto $D_2$. We set $h_0 = h_0 \circ \Psi$ and $h_1 = h_1 \circ \Psi$. By the definition of the homotopic type there is an $h$-homotopy $h_t \in S_{1, w_0}(\overline{D}, W, C)$ connecting $h_0$ and $h_1$. By Lemma 6.6 we may assume that the roots $\zeta_k(t), 1 \leq k \leq n$, of all equation $h_t = w_j, 1 \leq j \leq m$, are simple for all $t \in [0, 1]$ and the curves $\zeta_t$ are smooth. Hence there is a mapping $\Phi : \overline{D} \times [0, 1] \to \overline{D}$ satisfying the conclusion of Lemma 6.6.

The curve $\alpha_t = \Phi_t([-1, 1])$, connecting 1 and $-1$ in $\overline{D}$, divides $\mathbb{D}$ into Jordan domains $G_1^1 = \Phi_t(\mathbb{D}^+)$ and $G_1^2 = \Phi_t(\mathbb{D}^-)$. Note that if $\zeta \in \alpha_t$ then $h_t(\zeta) \neq w_j$ for any $1 \leq j \leq m$. Since $\Phi_t$ is a smooth family of diffeomorphisms the families $G_1^1$ and $G_1^2$ are Radó continuous. Hence $(G_t^1, h_t|_{G_t^1})$ and $(G_t^2, h_t|_{G_t^2})$ are continuous paths in $S^+(\mathbb{C}, W, C)$ and, therefore, $\{h_t|_{G_t^1}, 1\} = [f]$ and $\{h_t|_{G_t^2}, 1\} = [g_0]$ for all $t \in [0, 1]$.

We can slightly shift the curve $\alpha_0$ so that the homotopic types will not change and the intersection of $G_1^1$ with $\mathbb{D}^-$ consists of finitely many domains $\Omega_j^1$ bounded by parts of $\alpha_1$ and $[-1, 1]$. Analogously the intersection of $G_1^2$ with $\mathbb{D}^+$ also consists of finitely many domains $\Omega_j^2$ bounded by parts of $\alpha_1$ and $[-1, 1]$. Each of them is a Jordan domain being bounded by simple curves. The restriction of $h_1$ to $\mathbb{D}^+$ is $h$-homotopic to $f$ and, therefore, contains exactly one preimage of the set $E$ under the mapping $h_1$, namely, of $w_1$. By construction the domain $G_1^1$ also contains exactly one preimage of the set $E$ under the mapping $h_1$ and it can be only $w_1$. Hence only at most one of the domains $\Omega_j^2$ may contain a preimage of $w_j$ under the mapping $h_1$ and if this happens then exactly one of $\Omega_j^1$ contains a preimage of $w_j$ and vice versa. Moreover, this is a preimage of $w_1$. 23
If domains $\Omega_1^-$ do not contain a preimage of $w_1$ under the mapping $h_1$, then the restrictions of $h_1$ to domains $\Omega_j^-$ and $\Omega_j^+$ are homotopic to a constant mapping. Let $G = G_1^2 \cap \mathbb{D}^-$ and let $q$ be the restriction of $h_1$ to $G$. By Proposition 7.3 we can erase in $G_1^2$ domains $\Omega_j^+$ to get $\{h_1|_{G_1^2}, 1\} = \{q, 1\} = \{g_0\}$. Also in $\mathbb{D}^-$ we can erase $\Omega_1^-$ to get $\{g_1\} = \{h_1|_{\mathbb{D}^-}, 1\} = \{q, 1\}$. Hence $\{g_0\} = \{g_1\}$.

If, say, $\Omega_1^+$ contains a preimage of $w_1$ under the mapping $h_1$, then only one domain $\Omega_j^+$, say, $\Omega_1^+$ also contains exactly one preimage of $w_1$ under the mapping $h_1$. Let us denote by $L^+$ the compact set consisting of the path starting at 1 and following the boundary of $G_1^1$ until it reaches $\Omega_1^+$ and the set $\Omega_1^+$. Let $p^+$ will be restriction of $h_1$ to $L$. By Proposition 4.3 $\{g_0\} = \{h_1|_{G_1^2}, 1\} = I_\gamma(p^+) \ast \{q, 1\}$, where $\gamma = [1, 2]$.

Let $L^-$ be the compact set consisting of the path starting at 1 and following $[1, -1]$ until it reaches $\Omega_1^-$ and the set $\Omega_1^-$. Let $p^-$ will be restriction of $h_1$ to $L^-$. By Proposition 7.3 $\{g_1\} = \{h_1|_{L^-}, 1\} = I_\gamma(p^-) \ast \{q, 1\}$. Since $\iota_1([g_0]) = \iota_1([g_1])$ we see that $\iota_1(I_\gamma(p^+)) = \iota_1(I_\gamma(p^-))$. By Lemma 7.4 $I_\gamma(p^+) = I_\gamma(p^-)$ and, consequently, $\{g_0\} = \{g_1\}$.

Similarly we can prove the right cancelation property. \qed

We need the notion of a nested word in $\eta_1(W, C, w_0)$. It is defined by induction. A nested word of level 0 is the word of the form $g_{j_1}^{k_1} \ast \cdots \ast g_{j_n}^{k_n}$, where $g_{j_{i+1}} \neq g_{j_i}$ and all $k_i > 0$. A nested word of level $n$ is a word

\[(1) \quad F_{\lambda_1}(B_1) \ast \cdots \ast F_{\lambda_n}(B_m),\]

where the words $B_1, \ldots, B_m$ are nested words of level at most $n - 1$. By Corollary 7.3 every word in $\eta_1(W, C, w_0)$ can be written as a nested word of level 1.

Given a nested word $g_{j_{11}}^{k_{11}} \ast \cdots \ast g_{j_{1n}}^{k_{1n}}$ of level 0 its precise copy in $\pi_1(W, w_0)$ is the word $e_{j_{11}} \ast \cdots \ast e_{j_{1n}}$. Clearly, this word is reduced, i.e., no cancelations are possible. If a nested word is of level $n$ and has the form as in (1) then its precise copy is

$\lambda_1B_1^{\lambda_1^{-1}} \cdots \lambda_nB_n^{\lambda_n^{-1}}$,

where the words $\bar{B}_1, \ldots, \bar{B}_m$ are precise copies of words $B_1, \ldots, B_m$ respectively.

For nested words we can prove the following lemma.

**Lemma 7.8.** Every word in $\eta_1(W, C, w_0)$ can be written as a nested word whose precise copy is reduced.

**Proof.** To prove the lemma we will show that if a precise copy of a nested word admits cancelations then we can rewrite it as a nested word such that the length of the precise copy is decreased at least by 2. We will do it considering four possible cases.

1) Suppose that a cancelation is possible at level 1, i.e., somewhere in the word we have $\lambda g_{j_{11}}^{k_{11}} \ast \cdots \ast g_{j_{1n}}^{k_{1n}} \lambda^{-1}$ and $\lambda = \mu g_{j_{11}}^{-1}$. Then using in turn properties (1), (4) and (5) from Proposition 6.4 the latter word can be rewritten as

\[
\begin{align*}
F_\mu(F_{\mu^{-1}}(g_{j_{11}}^{k_{11}}) \ast F_{\mu^{-1}}(g_{j_{21}}^{k_{21}} \ast \cdots \ast g_{j_{1n}}^{k_{1n}})) = F_\mu(g_{j_{11}}^{k_{11}} \ast F_{\mu^{-1}}(g_{j_{21}}^{k_{21}} \ast \cdots \ast g_{j_{1n}}^{k_{1n}})) = F_\mu(g_{j_{11}}^{k_{11}^{-1}} \ast g_{j_{21}}^{k_{21}} \ast \cdots \ast g_{j_{1n}}^{k_{1n}} \ast g_{j_{11}}) \end{align*}
\]

and we see that the length of the precise copy decreases at least by 2.
2) Suppose the cancelation is possible in operators $F_\lambda$, i.e. $\lambda$ has adjacent $g_j$ and $g_j^{-1}$ as factors. Then they can be canceled by Proposition 6.4(2) and again the length of the precise copy decreases by 2.

3) Suppose the cancelation is possible between adjacent factors, i.e., somewhere in the word we have $F_\lambda(B_1) \ast F_\mu(B_2)$ and $\lambda = g_j \lambda_1$ while $\mu = g_j \mu_1$. Then by Proposition 6.4(1)

$$F_\lambda(B_1) \ast F_\mu(B_2) = F_{g_j}(F_\lambda(B_1)) \ast F_{g_j}(F_\mu(B_2)) = F_{g_j}(F_{\lambda_1}(B_1) \ast F_{\mu_1}(B_2))$$

and we see that the length of the precise copy decreases by 2. The case when $\lambda = g_j^{-1} \lambda_1$ while $\mu = g_j^{-1} \mu_1$ can be considered analogously.

4) Suppose the cancelation occurs at adjacent levels, i.e., somewhere in the word we have $F_\lambda(F_{\mu_1}(B_1) \ast \cdots \ast F_{\mu_n}(B_n))$ and $\lambda = \lambda_1 g_j$ while $\mu_1 = g_j^{-1} \mu_2$. Then by Proposition 6.4(1)

$$F_\lambda(F_{\mu_1}(B_1) \ast \cdots \ast F_{\mu_n}(B_n))$$

and again the length of the precise copy decreases by 2. The cases when $\lambda = \lambda_1 g_j^{-1}$ while $\mu_1 = g_j \mu_2$ or cancelation occurs between $\lambda^{-1}$ and $\mu_n$ can be considered analogously.

Now we can prove the following proposition.

**Proposition 7.9.** If $f_0, f_1 \in S_{w_0}(\mathbb{D}, W, C)$ and $\iota_1([f_0]) = \iota_1([f_1])$, then $[f_0] = [f_1]$.

**Proof.** We denote by $l([f])$ the length of the reduced word $\iota_1([f])$ and call it the length of $f$. The proof will be the induction by $l([f])$. If $l([f_0]) = 1$ then $l([f_1]) = 1$ and this means that $f_0$ and $f_1$ have index 1. By Lemma 7.8 $[f_0] = [f_1]$.

Now suppose that we proved the theorem for the length less or equal to $k - 1$ and let $l([f_0]) = l([f_1]) = k$. Suppose also that the reduced word for $\iota_1([f_0])$ contains $e_j^{-1}$ for some $1 \leq j \leq m$, i.e., $\iota_1([f_0]) = \lambda e_j^{-1} \mu$. Consider $[h_0] = F_{\lambda^{-1}}([f_0])$ and $[h_1] = F_{\lambda^{-1}}([f_1])$. Then $\iota_1([h_0]) = \iota_1([h_1]) = e_j^{-1} \mu \lambda$. Hence $l([h_0]) \leq k$ and $l([g_j] \ast [h_0]) \leq k - 1$. By the induction assumption $[g_j] \ast [h_0] = [g_j] \ast [h_1]$ and by the cancelation property from Proposition 7.7 $[h_0] = [h_1]$. Thus $[f_0] = [f_1]$.

If $\iota_1([f_0])$ does not contain $g_j^{-1}$ for any $1 \leq j \leq m$, i.e., $\iota_1([f_0]) = e_{j_1} \cdots e_{j_n}$, where $e_{j_{k+1}} \neq e_{j_k}$ and all $k > 0$. Then by Lemma 7.8 we can rewrite $[f_0]$ as a nested word whose precise copy is reduced and is equal to $\iota_1([f_0])$. Then all operators $F_\lambda$ in this word are identities and we see that $[f_0] = g_{j_1}^{k_1} \cdots g_{j_n}^{k_n}$. The same is true for $[f_1]$ and the proposition is proved.

Now we can describe the semigroup $\eta_1(W, C, w_0)$ when $W$ is a general finitely connected domain. Let $\tilde{W} = W' \setminus H_1 \cup \cdots \cup H_m$, where $W'$ is a connected and simply connected domain and $H_1, \ldots, H_m$ are disjoint connected compact sets in $W'$. We will fix the set of generators in $\pi_1(\tilde{W}, w_0)$ by choosing points $w_j \in \partial H_j$ and simple continuous curves $\tilde{a}_j : [0, 1] \to W$, $1 \leq j \leq m$, such that $\tilde{a}_j$ connects $w_0$ with $w_j$, never meets $H = H_1 \cup \cdots \cup H_m$ when $0 \leq t < 1$ and these curves meet each other only at $w_0$. We take smooth disjoint Jordan curves $C_j \subset \tilde{W}$, whose
interiors contain only $H_j$, and the points $ζ_j$, where $α_j$ meets $C_j$ last time. Let $λ_j$ be a curve which starts at $w_0$, follows $α_j$ up to $ζ_j$, then goes counterclockwise by $C_j$ until $ζ_j$ and then returns to $w_0$ by $α_j$. Then the set of homotopy equivalence classes of curves $\{λ_j, 1 \leq j \leq m\}$ will be the set of generators in $π_1(\tilde{W}, w_0)$ which will be denoted by $\{\tilde{e}_1, \ldots, \tilde{e}_m\}$. Clearly, the homotopy classes of $\tilde{λ}_j$ do not depend on the choice of $C_j$ provided they are chosen sufficiently close to $H_j$.

For each $j$ we consider the mapping $f_j$ of $K = [0, 1] \cup \{|ζ - 2| \leq 1\}$ defined as $α_j$ from $w_0$ to $ζ_j$ on $[0, 1]$ and a conformal mapping of $\{|ζ - 2| \leq 1\}$ onto the bounded domain $C_j$ which has $C_j$ as its boundary such that $f_j(1) = ζ_j$. We will define $[\tilde{g}_j] ∈ η_1(W, C, w_0)$ as $I_*{f_j}$, where the access curve $γ = [-i, 0]$. Then $η_1([\tilde{g}_j]) = \tilde{e}_j$.

Let $ψ$ be a homeomorphism of $\tilde{W}$ onto $W = W' \setminus \{w_1, \ldots, w_m\}$. We assume that $ψ$ is a continuous mapping of $W'$ onto itself collapsing each $H_j$ to $w_j$. Let $α_j = ψ ◦ α_j$ and let $\{e_j\}$ and $\{[\tilde{g}_j]\}$, $1 \leq j \leq m$, be the generators of $π_1(\tilde{W}, w_0)$ and $η_1(W, C, w_0)$ respectively defined at the beginning of this section for $W$ using the curves $α_j$. Then it is easy to see that the isomorphism $ψ_*$ between $π_1(\tilde{W}, w_0)$ and $π_1(W, w_0)$ generated by $ψ$ sends $\tilde{e}_j$ to $e_j$.

**Theorem 7.10.** If $\tilde{W}$ is a finitely connected domain in $C$, $w_0 ∈ \tilde{W}$, then the mapping $η_1 : η_1(\tilde{W}, C, w_0) → π_1(\tilde{W}, w_0)$ is an imbedding and the semigroup $η_1(\tilde{W}, C, w_0)$ is isomorphic to the minimal subsemigroup of $π_1(\tilde{W}, w_0)$ containing $\{\tilde{e}_j, 1 \leq j \leq m\}$ and invariant with respect to the inner automorphisms.

**Proof.** If $f_0, f_1 ∈ S_{1,w_0}(\overline{W}, C)$ then $f_0, f_1 ∈ S_{1,w_0}(\overline{W}, W, C)$ and if $[f_0] = [f_1]$ in $η_1(\tilde{W}, C, w_0)$ then $[f_0] = [f_1]$ in $η_1(W, C, w_0)$. Hence we have a mapping $Λ : η_1(W, C, w_0) → η_1(\tilde{W}, C, w_0)$.

Let us show that $Λ$ is injective. Suppose that $[f_0] = [f_1]$ in $η_1(W, C, w_0)$, i.e., there is an $h$-homotopy $f_t ∈ S_{1,w_0}(\overline{W}, W, C)$, $0 \leq t \leq 1$, connecting $f_0$ and $f_1$. There are some closed disjoint disks $d_j$ centered at $w_j$, $1 \leq j \leq m$, such that $f_t ∈ S_{1,w_0}(\overline{W}, W, C)$, $0 \leq t \leq 1$, where $W = W' \setminus d_1 ∪ \cdots ∪ d_m$. We choose them so small that $d_j ⊂ C_j$ and there is a conformal mapping $q_j$ of $C_j \setminus d_j$ onto an annulus $A = \{r < |ζ| < 1\}$.

We may assume that the curves $C_j$ has been chosen so that the mappings $f_0, f_1 ∈ S_{1,w_0}(\overline{U}, U, C)$, where $U = W' \setminus C$ and $C$ is the closure of $C_1 ∪ \cdots ∪ C_m$. We denote by $B_{js}$ the preimages of the circles $\{|ζ| = s\}$ under the mapping $q_j$. Let $U_s = W' \setminus B_{1s} ∪ \cdots ∪ B_{ms}$. There are numbers $r < s_0 < s_1 < 1$ such that for all $t ∈ [0, 1]$ the mappings $f_t$ map $T$ into $U_{s_0}$ and $U_{s_1} ⊂ \tilde{W}$.

Let $s(t)$ be the maximal number of those $s$ for which $f_t$ maps $T$ into $U_s$ and $f_t(T)$ meets $W' \setminus U_{s_1}$. If $f_t(T)$ does not meet $W' \setminus U_{s_1}$ we set $s(t) = s_1$. We define a homeomorphism $Q_t$ of $W'$ onto itself in the following way. If $z ∈ U$ or $z ∈ d_j$ or $s(t) = s_1$ then $Q_t(z) = z$. If $s(t) < s_1$ and $z ∈ C_j \setminus d_j$ then $Q_t(z) = q_j^{-1}(p_j(q_j(z)))$, where $p_j(xe^{iθ}) = a_j(x)e^{iθ}$ and $a_j$ is an increasing function made from two linear functions so that $a_j(r) = r$, $a_j(s(t)) = s_1$ and $a(1) = 1$. Since $s(t) > s_0 > r$ this definition is correct.
A simple geometric argument shows that $Q_t$ is quasiconformal (see [1]) and there is $k < 1$ such that its Beltrami coefficient

$\mu_t(z) = \frac{\partial Q_t}{\partial \bar{z}}(z)/\frac{\partial Q_t}{\partial z}(z)$

is less than $k$ by the absolute value for all $t \in [0, 1]$. Moreover, $Q_t(z)$ is continuous in $t$ and if $t \to t_0$ then $\mu_t \to \mu_{t_0}$ almost everywhere in $W'$.

Let $h_t = Q_t \circ f_t$. Then

$$\frac{\partial h_t}{\partial \zeta}(\zeta) = \frac{\partial Q_t}{\partial \bar{z}}(f_t(\zeta))\frac{\partial f_t}{\partial \zeta}(\zeta)$$

and

$$\frac{\partial h_t}{\partial \zeta}(\zeta) = \frac{\partial Q_t}{\partial z}(f_t(\zeta))\frac{\partial f_t}{\partial \zeta}(\zeta).$$

By (2) we get

$$\frac{\partial h_t}{\partial \zeta}(\zeta) = \mu_t(f_t(\zeta))\frac{\partial Q_t}{\partial \bar{z}}(f_t(\zeta))\frac{\partial f_t}{\partial \zeta}(\zeta) = \mu_t(f_t(\zeta))(\frac{\bar{f_t}}{f_t})\frac{\partial h_t}{\partial \zeta}(\zeta).$$

So, $h_t$ is a continuous family of quasiregular mappings of $\mathbb{D}$ into $W'$ with Beltrami coefficients $\nu_t(\zeta) = \mu_t(f_t(\zeta))(\frac{\bar{f_t}}{f_t})$. Moreover, $h_t(\mathbb{T}) \subset U_{s_1}$ and if $t \to t_0$ then $\nu_t \to \nu_{t_0}$ almost everywhere in $W'$.

By [1, Theorem 9.0.3] there is a homeomorphism $\psi_t$ of $\mathbb{D}$ onto itself satisfying the equation

$$\frac{\partial \psi_t}{\partial \eta} = \nu_t(\eta)\frac{\partial \psi_t}{\partial \eta}$$

and such that $\psi_t(0) = 0$ and $\psi_t(1) = 1$. If $\phi_t = \psi_t^{-1}$ then by formula (2.51) in [1]

$$\frac{\partial \phi_t}{\partial \xi} = -\nu_t(\phi_t)\frac{\partial \phi_t}{\partial \xi}.$$

Hence

$$\frac{\partial h_t \circ \phi_t}{\partial \xi} = \frac{\partial h_t}{\partial \zeta}(\phi_t(\xi))\frac{\partial \phi_t}{\partial \zeta}(\xi) + \frac{\partial h_t}{\partial \zeta}(\phi_t(\xi))\frac{\partial \phi_t}{\partial \zeta}(\xi) = 0$$

and this means that the mappings $h_t \circ \phi_t$ are holomorphic and by the lemma below the path $h_t \circ \phi_t$ is continuous in $S_{1, u_{00}}(\mathbb{D}, \mathbb{W}, \mathbb{C})$.

Note that $Q_0$ and $Q_1$ are identities. Hence $\mu_0 \equiv \mu_1 \equiv \nu_0 \equiv \nu_1 \equiv 0$. Hence $\psi_0$, $\psi_1$, $\phi_0$ and $\phi_1$ are identities and since by the definition $h_0 \circ \phi_0 = Q_0 \circ f_0 \circ \phi_0$ we see that $h_0 = h_0 \circ \phi_0 = f_0$. By the same reason $h_1 = h_1 \circ \phi_1 = f_1$. Hence $[f_0] = [f_1]$ in $\eta_1(\mathbb{W}, \mathbb{C}, w_0)$ and we see that $\Lambda$ is injective. Clearly, $\Lambda([g]) = [g]$, where $[g]$ were defined at the beginning of this section and, therefore, $\Lambda$ is an isomorphism.

If $f_0, f_1 \in S_{1, u_{00}}(\mathbb{D}, \mathbb{W}, \mathbb{C})$ and $\iota_1([f_0]) = \iota_1([f_1])$ in $\pi_1(\mathbb{W}, w_0)$, then $\iota_1([f_0]) = \iota_1([f_1])$ in $\pi_1(\mathbb{W}, w_0)$. By Proposition 7.9 $[f_0] = [f_1]$ in $\eta_1(\mathbb{W}, \mathbb{C}, w_0)$ and by the previous result $[f_0] = [f_1]$ in $\eta_1(\mathbb{W}, \mathbb{C}, w_0)$. Hence the mapping $\iota_1 : \eta_1(\mathbb{W}, \mathbb{C}, w_0) \to \pi_1(\mathbb{W}, w_0)$ is an isomorphism of $\eta_1(\mathbb{W}, \mathbb{C}, w_0)$ onto a subsemigroup $G \subset \pi_1(\mathbb{W}, w_0)$.

If $\mu$ is a loop in $\mathbb{W}$ starting at $w_0$ and $[f] \in \eta_1(\mathbb{W}, \mathbb{C}, w_0)$, then $F_\mu([f]) \in \eta_1(\mathbb{W}, \mathbb{C}, w_0)$ and since $\iota_1(F_\mu([f])) = \iota_1([f])\mu^{-1}$ we see that $G$ is invariant with respect to the inner automorphisms. On the other hand, given $f \in S_{1, u_{00}}(\mathbb{D}, \mathbb{W}, \mathbb{C})$ by Corollary 7.8 $\Lambda([f]) = \Pi_{j=1}^k F_{\lambda_j}([g_j])$. Since we can find loops $\lambda_j \in \mathbb{W}$ homotopic to $\lambda_j$ in $\mathbb{W}$ we can write $\Lambda([f]) = \Pi_{j=1}^k F_{\mu_j}([g_j])$. But then $[f] = \Pi_{j=1}^k F_{\mu_j}([g_j])$ and we see that $G$ is the minimal subsemigroup of $\pi_1(\mathbb{W}, w_0)$ containing $\{\varepsilon_j, 1 \leq j \leq m\}$ and invariant with respect to the inner automorphisms. □
Lemma 7.11. Suppose we have a sequence of Beltrami coefficients \{\mu_n\} such that \(|\mu_n| \leq k < 1\) for all \(n\) and almost every \(z \in \mathbb{D}\) and the pointwise limit \(\mu(z) = \lim_{n \to \infty} \mu_n(z)\) exists almost everywhere in \(\mathbb{D}\). Let \(\phi^n\) be the normalized \((\phi^n(0) = 0\) and \(\phi^n(1) = 1)\) solution to \(\phi^n_z = \mu^n(z)\phi^n\) which is a homeomorphism of \(\overline{\mathbb{D}}\). Then the limit \(\phi(z) = \lim_{n \to \infty} \phi^n(z)\) exists, the convergence is uniform on \(\overline{\mathbb{D}}\) and \(\phi\) solves the Beltrami equation \(\phi_z = \mu(z)\phi\).

Proof. Each \(\phi^n : \overline{\mathbb{D}} \to \overline{\mathbb{D}}\) extends to a \(K\)-quasiconformal homeomorphism of \(\mathbb{C}\) onto \(\mathbb{C}\) defined by \(\Phi^n(z) = \phi^n(z)\) when \(|z| \leq 1\) and \(\Phi^n(z) = 1/z\) when \(|z| > 1\). Note that \(\Phi^n_z = \mu_n(z)\Phi^n\), where \(\mu_n(z) = \mu(z)\) when \(|z| < 1\) and \(\mu_n(z) = \Phi^n(1/z)\Phi^n\) when \(|z| > 1\). Thus \(\mu_n \to \mu\) almost everywhere in \(\mathbb{C}\).

By [1, Lemma 5.3.5] the limit \(\Phi(z) = \lim_{n \to \infty} \Phi^n(z)\) exists, the convergence is uniform on compact sets in \(\mathbb{C}\) and \(\Phi\) solve the equation \(\Phi_z = \mu(z)\Phi\). By [1, Theorem 3.9.4] \(\Phi\) is a non-constant \(K\)-quasiconformal homeomorphism of \(\mathbb{C}\) onto itself. Since \(\Phi(\overline{\mathbb{D}}) = \overline{\mathbb{D}}\) the lemma is proved.

Finally, we will prove the Oka principle for \(S_{1,w_0}(\overline{\mathbb{D}},W,\mathbb{C})\) when \(W\) is a domain in \(\mathbb{C}\).

Theorem 7.12. Let \(W \subset \mathbb{C}\) be a domain. Then the mapping \(\iota_1\) is an imbedding and the semigroup \(\eta_1(W, \mathbb{C}, w_0)\) is cancelative.

Proof. Suppose that \(f_0, f_1 \in S_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})\) and \(\iota_1([f_0]) = \iota_1([f_1])\). hence there is a continuous mapping \(F : [0,1] \times T \to W\) such that \(F(t, 1) = w_0, F(0, \zeta) = f_0(\zeta)\) and \(F(1, \zeta) = f_1(\zeta)\). There is \(\delta > 0\) such that the distance in the spherical metric from \(F(t, \zeta)\) to \(L = \mathbb{CP}^1 \setminus W\) is always greater than \(\delta\). If \(K\) is the closed \(\delta/2\)-neighborhood of \(L\) the \(\mathbb{CP}^1 \setminus K\) is the union of bounded finitely connected subdomains in \(W\). Let us denote by \(W'\) one of them which contains \(F([0,1] \times T)\). By Theorem 7.10 \([f_0] = [f_1]\) in \(S_{1,w_0}(\overline{\mathbb{D}}, W', \mathbb{C})\). Hence \([f_0] = [f_1]\) in \(S_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})\) and we see that \(\iota_1\) is an imbedding.

If \([f] \ast [g_0] = [f] \ast [g_1]\) then \(\iota_1([f] \ast [g_0]) = \iota_1([f] \ast [g_1]) = \iota_1([f]) \ast \iota_1([g_0]) = \iota_1([f]) \ast \iota_1([g_1])\) and we see that \(\iota_1([g_0]) = \iota_1([g_1])\). By the argument above \([g_0] = [g_1]\).

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28
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Department of Mathematics, Syracuse University,
215 Carnegie Hall, Syracuse, NY 13244

E-mail address: dbdharma@syr.edu, eapolet@syr.edu