PROPERTIES OF GENERALIZED DERANGEMENT GRAPHS

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ABSTRACT. A permutation $\sigma \in S_n$ is a $k$-derangement if for any subset $X = \{a_1, \ldots, a_k\} \subseteq [n]$, $\{\sigma(a_1), \ldots, \sigma(a_k)\} \neq X$. One can form the $k$-derangement graph on the set of permutations of $S_n$ by connecting two permutations $\sigma$ and $\tau$ if $\sigma^{-1} \tau$ is a $k$-derangement. We characterize when such a graph is connected or Eulerian. For $n$ an odd prime power, we determine the independence, clique and chromatic number of the 2-derangement graph.

1. Introduction

Permutations which leave no element fixed, known as derangements, were first considered by Pierre Raymond de Montmort in 1708 and have been extensively studied since. A derangement graph is a graph whose vertices are the elements of the symmetric group $S_n$ and whose edges connect two permutations that differ by a derangement. Derangement graphs have been shown to be connected (for $n > 3$), Hamiltonian, and their independence number, clique number, and chromatic number have been calculated [2].

The concept of a derangement can be generalized to a $k$-derangement, a permutation in $S_n$ such that the induced permutation on the set of all unordered $k$-tuples leaves no $k$-tuple fixed. A $k$-derangement graph is defined in an analogous manner to a derangement graph. In this paper, we investigate some of the graph-theoretical properties of $k$-derangement graphs.

2. Preliminaries

Let $S_n$ be the group of permutations on the set $[n] = \{1, 2, \ldots, n\}$, and denote by $[n]^{(k)}$ the set of unordered $k$-tuples with entries from $[n]$. Note that a permutation $\sigma \in S_n$ induces a permutation $\sigma^{(k)}$ of unordered $k$-tuples by $\sigma^{(k)}(\{a_1, \ldots, a_k\}) = \{\sigma(a_1), \ldots, \sigma(a_k)\}$. For example, with $n = 4$, $k = 2$, and $\sigma = (1234)$ in cycle notation, we have

$$(1234)_{(2)}(\{1,2\}) = \{(1234)(1), (1234)(2)\} = \{2,3\}$$

$$(1234)_{(2)}(\{1,3\}) = \{(1234)(1), (1234)(3)\} = \{2,4\}$$

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Let $D_n := \{\sigma \in S_n | \sigma(x) \neq x, \forall x \in [n] \}$ denote the ordinary derangements on $[n]$. Extending this concept, we say that a permutation $\sigma \in S_n$ is a $k$-derangement if $\sigma^{(k)}(x) \neq x$ for all $x \in [n]^{(k)}$. In other words, a $k$-derangement in $S_n$ is a permutation (of $[n]$) which induces a permutation (of $[n]^{(k)}$) which leaves no $k$-tuple fixed. The set of $k$-derangements in $S_n$ is denoted $D_{k,n}$, and the number of $k$-derangements in $S_n$ is denoted $D_k(n)$ ($D_k(n) = |D_{k,n}|$). The example above shows that $(1234)$ is in $D_2$. Specifically, $D_{2,4} = \{(1234), (1243), (1324), (1342), (1423), (1432), (123)(4), (124)(3)(132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1)\}$, and thus $D_2(4) = 14$. Note that $D_n = D_{1,n}$, and $D_1(n)$ is the ordinary derangement number.

The cycle structure of a permutation $\sigma$, denoted $C_\sigma$, is the multiset of the lengths of the cycles in its cycle decomposition (e.g., $C_{(12)(3)(45)} = \{2, 2, 1\}$). Note that the cycle structure of $\sigma \in S_n$ is a partition of $n$. Given a partition $r \vdash n$, let $P_r$ be the set of all permutations in $S_n$ whose cycle structure is $r$. For example, $P_{[2,1,1]} = \{(12), (13), (14), (23), (24), (34)\}$.

We first note that if the cycle structure of a permutation $\sigma$ contains a multiset which partitions $k$, then $\sigma$ is not a $k$-derangement. For example, $(12)(3)(4)$ will not be, because $\{2,1\} \subseteq C_{(12)(3)(4)} = \{2,1,1\}$ is a partition of $3$. Indeed, if $\{q, r, \ldots, s\}$ is a partition of $k$, and $(a_1 \ldots a_q)(b_1 \ldots b_r)\ldots(c_1 \ldots c_s)$ are cycles of $\sigma$, then for $x = \{a_1, \ldots, a_q, b_1, \ldots, b_r, c_1, \ldots, c_s\}$, $\sigma^{(k)}(x) = x$. Conversely, if $\sigma$ has no set of cycles whose lengths partition $k$, then given any $x \in [n]^{(k)}$, there is a cycle in $\sigma$ which contains at least one element in $x$ and contains some element not in $x$. Hence $\sigma$ sends an element in $x$ to an element not in $x$ and so $\sigma^{(k)}(x) \neq x$.

Thus we observe that the cycle structure of a permutation determines whether or not it is a $k$-derangement, and we have the following.

**Proposition 1.** A permutation $\sigma \in S_n$ is a $k$-derangement if and only if the cycle decomposition of $\sigma$ does not contain a set of cycles whose lengths partition $k$.

Let $CD_{k,n}$ be the set of cycle structures corresponding to $k$-derangements in $S_n$ [e.g., $CD_{2,4} = \{\{4\}, \{3,1\}\}$], Note that $D_{k,n} = D_{n-k,n}$. This follows from the fact that if a cycle structure $C_\sigma$ is in $CD_{k,n}$, then $C_\sigma$ is in $CD_{n-k,n}$ as well.

Let $G$ be a group, and let $S \subseteq G$ such that if $s$ is in $S$, then $s^{-1}$ is in $S$. The Cayley graph $\Gamma(G,S)$ is the graph whose vertices are the elements of $G$ such that an edge connects two vertices $u, v \in G$ if $su = v$ for some $s \in S$. A $k$-derangement graph is a Cayley graph defined by $\Gamma_k := \Gamma(S,D_{k,n})$. (Note that $D_{k,n}$ is symmetric, as the inverse of a $k$-derangement is a $k$-derangement, and thus satisfies the requirements for a Cayley graph.) It is worth noting that $\Gamma_k$ is, by construction, $D_k(n)$-regular, and that since $D_{k,n} = D_{n-k,n}$, $\Gamma_k = \Gamma(n-k,n)$. Figure 1 illustrates the $2$-derangement graph on $6$ vertices, $\Gamma_{2,3}$.

It is possible to consider $k$-derangements in $S_n$ for any positive $k$ and $n$. However, if $k = n$, there will be no $k$-derangements in $S_n$, since every partition in $S_n$ will have a cycle structure such that the cycle length partition $k$. As such, $\Gamma_{k,n}$ will be the empty graph on $n$ vertices. If $k > n$, then every permutation in $S_n$ is a $k$-derangement vacuously, and thus
Figure 1. $\Gamma_{2,3}$

$\Gamma_{k,n}$ will be the complete graph on $|S_n|$ vertices. As neither of these cases is particularly interesting, henceforth we will only consider $k$-derangements where $k < n$.

3. Properties of derangement graphs

Figure 1 shows that $\Gamma_{2,3}$ is not a connected graph, and since $\Gamma_{2,3} = \Gamma_{1,3}$, we see that $\Gamma_{k,3}$ is disconnected, for all $k < n$. But this is an exception rather than the rule, as the following theorem demonstrates.

**Theorem 2.** For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is connected.

**Proof.** Every permutation in $S_n$ can be written as the product of adjacent transpositions $(h \, (h + 1))$. These, in turn, can be expressed as the product of two $k$-derangements, so long as $n > 3$, as we will demonstrate. As a result, for $n > 3$, the elements of $\mathcal{D}_{k,n}$ generate $S_n$, which means that every vertex of $\Gamma_{k,n}$ can be reached by a path from the identity.

We show that the permutation $(1 \, 2)$ can be written as the product of two $k$-derangements and then note that since it is the form and not the individual labels that are important, any adjacent transposition can be written as the product of two $k$-derangements. We consider two cases, the case where $k = 1$, and the case where $k \geq 2$.

Case 1: If $k = 1$, then $(1 \, 2) = (1 \, 2 \ldots n)^2 \cdot (n \, (n-1) \ldots 1)^2(1 \, 2)$. We claim that $(1 \, 2 \ldots n)^2$ and $(n \, (n-1) \ldots 1)^2(1 \, 2)$ are each 1-derangements in $S_n$ for all $n > 3$. If $n$ is even, then $(1 \, 2 \ldots n)^2 = (1 \, 3 \ldots (n-3) \, (n-1))(2 \, 4 \ldots (n-2) \, n)$, which is a 1-derangement in $S_n$, for all $n$. Additionally, $(n \, (n-1) \ldots 1)^2(1 \, 2) = (1 \, n \, (n-2) \, (n-4) \ldots 2 \, (n-1) \, (n-3) \ldots 3)$, which is also a 1-derangement in $S_n$, for any $n$.

On the other hand, if $n$ is odd, then $(1 \, 2 \ldots n)^2 = (1 \, 3 \ldots (n-2) \, n \, 2 \ldots 4 \ldots (n-3) \, (n-1))$, which is a 1-derangement in $S_n$ for all $n$. And $(n \, (n-1) \ldots 1)^2(1 \, 2) = (n \, (n-2) \, (n-4) \ldots 3 \, 1 \, (n-1) \, (n-3) \ldots 4 \, 2)(1 \, 2) = (1 \, n \, (n-2) \, (n-4) \ldots 3)(2 \, (n-1) \, (n-3) \ldots 4)$, which is a 1-derangement in $S_n$ so long as $n > 3$. (If $n = 3$, $(3 \, 1 \, 2)(1 \, 2) = (13)(2)$, which is not a 1-derangement.)
Thus for \( n > 3 \), we have shown that \((1 2)\) can be written as the product of two 1-derangements, and, by extension, every adjacent transposition can be written as the product of two 1-derangements.

Case 2: For \( k \geq 2 \), \((1 2) = (1 2 \ldots n)^{-1}(1 3 4 \ldots n)\). We know \((1 2 \ldots n)^{-1}\) is a \( k\)-derangement for all \( k \) since the inverse of a \( k\)-derangement is a \( k\)-derangement. And, by the cycle structure, we see that \((1 3 4 \ldots n) = (1 3 4 \ldots n)(2)\) is a \( k\)-derangement for all \( k \), except \( k = 1 \) and \( k = (n - 1)\). (However, since \( \Gamma_{1,n} = \Gamma_{(n-1),n} \), Case 1 addresses \((n - 1)\)-derangements as well as 1-derangements).

So we have shown that for \( k \geq 2 \), \((1 2)\) can be written as the product of two \( k\)-derangements, and again, by extension, we can write any adjacent transposition as the product of two \( k\)-derangements. Thus every vertex is connected by a path to the identity, and \( \Gamma_{k,n} \) is connected.

It is worth noting that Theorem 1 holds for \( n = 2 \) as well. Since we are only interested in \( k\)-derangements in \( S_n \) such that \( k < n \), when \( n = 2 \), \( k \) must equal 1, and so \( \Gamma_{1,2} \) is the connected graph on two vertices.

Next, we give a characterization in terms of \( n \) and \( k \) for when a derangement graph is Eulerian. We will require the following result.

**Lemma 3.** If a cycle structure includes a cycle of length greater than 2, then there are an even number of permutations with that cycle structure.

**Proof.** Consider \( P_r \), the set of permutations with a given cycle structure, \( r \). We can pair each \( \sigma \in P_r \) with its inverse \( \sigma^{-1} \in P_r \), and so long as \( \sigma \neq \sigma^{-1} \) for any \( \sigma \in P_r \), \(|P_r|\) will be even. Suppose there exists a \( \sigma \in P_r \) such that \( \sigma = \sigma^{-1} \). Then \( \sigma^2 = e \), and so the order of \( \sigma \) is at most 2. The order of a permutation is the least common multiple of the orders of the elements of its cycle structure, so \( \sigma \) must not include a cycle of length greater than 2. This is a contradiction; thus \(|P_r|\) is even. \( \Box \)

**Theorem 4.** For \( n > 3 \) and \( k < n \), \( \Gamma_{k,n} \) is Eulerian if and only if \( k \) is even or \( k \) and \( n \) are both odd.

**Proof.** A graph is Eulerian if and only if it is connected and each vertex has an even degree. In light of Theorem 2 and the previously noted fact that \( \Gamma_{k,n} \) is \( D_k(n)\)-regular, in order to ascertain if \( \Gamma_{k,n} \) is Eulerian, we must determine whether \( D_k(n) \) is even or odd.

If \( k \) is even, we claim that \( D_k(n) \) is the sum of even numbers. Any cycle structure composed entirely of 2- or 1- cycles will partition an even \( k \), and thus any permutation which is in \( D_k \) for an even \( k \) will contain a cycle of length 3 or greater in its cycle decomposition. Now, \( D_{k,n} = P_{r_1} \cup P_{r_2} \cdots \cup P_{r_m} \) such that no \( r_i \) partitions \( k \), and by Lemma 3 \(|P_{r_i}|\) is even for all \( i \in \{1, \ldots, m\} \). Thus, when \( k \) is even, \( D_k(n) \) is even.

If \( k \) and \( n \) are both odd, again we see that every permutation in \( D_{k,n} \) will contain a cycle of length 3 or greater in its cycle decomposition, since an odd \( k \) can be partitioned by a set of cycles of lengths 1 or 2 if there is at least one 1-cycle. Furthermore, since \( n \) is...
odd, there are no permutations whose cycle structure is composed only of length-2 cycles. Thus, \( D_k(n) \) is even.

Finally, we show that if \( k \) is odd and \( n \) is even, then \( \Gamma_{k,n} \) is not Eulerian. In this case, \( P_{\{2,2,\ldots,2\}} \) is in \( CD_{k,n} \). By choosing pairs of elements for the cycles and dividing by the number of ways to order the cycles, we see that the number of permutations in \( P_{\{2,2,\ldots,2\}} \) is given by

\[
\frac{n(n-1)(n-2)\cdots(\frac{n}{2})!}{(\frac{n}{2})!} = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{(2 \cdot \frac{n}{2})(2 \cdot (\frac{n}{2} - 1)) \cdots (6)(4)(2)} = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{n(n-2)\cdots(6)(4)(2)} = (n-1)(n-3)\cdots(5)(3)(1).
\]

Since \( n \) is even, the product \((n-1)(n-3)\cdots(5)(3)(1)\) is odd. Every other \( k \)-derangement in \( S_n \) will contain a cycle with length greater than 2, since any combination of 1-cycles or 1- and 2-cycles will partition \( k \). So \( D_k(n) \) is the sum of one odd number and even numbers, and so is odd. \( \Box \)

4. Chromatic, Independence and Clique Numbers for \( k = 2 \) and \( n \) an Odd Prime Power

For the majority of this section, we will think of permutations in terms of the result of their application to the ordering \( \{1,2,3,\ldots,n\} \). Thus, \( \{2,3,1,4,5\} \) represents the permutation which has moved 2 to the first position, 3 to the second, 1 to the third, and left 4 and 5 fixed; that is, the permutation \((132)(4)(5)\) in cycle notation, or the inverse of the permutation \((12345)\), in two line notation.

We note that in order for \( vu^{-1} \) (or equivalently, \( v^{-1}u \)) to be a \( k \)-derangement, it is necessary and sufficient that no unordered \( k \)-tuple of elements be sent to the same unordered \( k \)-tuple of positions by both \( u \) and \( v \). For example, the permutation \( u = \{2,3,1,4,5\} \) and \( v = \{4,1,3,5,2\} \) both send the pair \( \{1,3\} \) to the second and third positions. Thus \( (vu^{-1})_{\{2\}}(\{2,3\}) = \{2,3\} \), and so \( vu^{-1} \) is not a 2-derangement and there is no edge between \( u \) and \( v \) in the 2-derangement graph. More formally, suppose \( u \) and \( v \) both send the \( k \)-tuple \( M' = \{a_1', a_2', \ldots, a_k'\} \) to positions \( M = \{a_1, a_2, \ldots, a_k\} \). Then, \((vu^{-1})_{(k)}(M) = v_{(k)}(M') = M \). Thus, \( vu^{-1} \) is not a \( k \)-derangement.

On the other hand, if \( u \) and \( v \) send no \( k \)-tuple to the same positions we claim \( vu^{-1} \) is a \( k \)-derangement. Consider an arbitrary \( k \)-tuple, \( M = \{a_1, a_2, \ldots, a_k\} \), and suppose \( u \) maps the \( k \)-tuple \( M' = \{a_1', a_2', \ldots, a_k'\} \) to the positions given in \( M \). Then \((vu^{-1})_{(k)}(M) = v_{(k)}(M') \neq M \) since \( v \) cannot send the \( k \)-tuple \( M' \) to the same positions as \( u \) does. Thus, \( vu^{-1} \) is a \( k \)-derangement.

In Theorem \[1\], we find the clique number of the 2-derangement graph, \( \omega(\Gamma_{2,n}) \), for \( n \) an odd prime power, by constructing a clique of maximal size. Before establishing this clique number, we note an upper bound on the clique number of a general \( k \)-derangement graph.

**Lemma 5.** For \( k < n \), \( \omega(\Gamma_{k,n}) \leq \binom{n}{k} \).
Proof. The clique number of the $k$-derangement graph, $\omega(\Gamma_{k,n})$ cannot be greater than $\binom{n}{k}$, since there are only $\binom{n}{k}$ subsets of size $k$ and hence at most $\binom{n}{k}$ different unordered $k$-tuples of positions for an arbitrary $k$-tuple of elements to be sent under a permutation. 

Theorem 6. If $n$ is an odd prime power, then $\omega(\Gamma_{2,n}) = \binom{n}{2}$.

Proof. We will explicitly construct a clique with $\binom{n}{2}$ elements. Let $n = p^k$ with $p$ a prime greater than 2, and let $\mathbb{F}_{p^k}$ denote the field with $p^k$ elements. Rather than letting $S_n$ act on $[n]$, we will let it act on $\mathbb{F}_{p^k}$ and construct $\Gamma_{2,n}$ accordingly. Let $v = (x_1, \ldots, x_n)$ be an ordered $n$-tuple whose entries are the elements of $\mathbb{F}_{p^k}$ in some order. Given any function $\phi : \mathbb{F}_{p^k} \to \mathbb{F}_{p^k}$, we define $\phi(v) = (\phi(x_1), \ldots, \phi(x_n))$. Partition the non-zero elements of $\mathbb{F}_{p^k}$ by pairing each element with its (additive) inverse, and let $T$ be a set obtained by choosing exactly one element from each pair, giving $|T| = (p^k - 1)/2$.

Define $f_{s,\alpha}(x) = sx + \alpha$, and consider the set $X = \{f_{s,\alpha}(v) | s \in T \text{ and } \alpha \in \mathbb{F}_{p^k}\}$. Since $s \neq 0$, $f_{s,\alpha}$ is a bijection and $f_{s,\alpha}(v)$ is a permutation of the elements of $\mathbb{F}_{p^k}$. We claim that $X$ is a clique in $\Gamma_{2,n}$. Suppose not; that is, suppose there are $s, t \in T$ and $\alpha, \beta \in \mathbb{F}_{p^k}$, $(s, \alpha) \neq (t, \beta)$, such that $f_{s,\alpha}(v)$ is not a 2-derangement of $f_{s,\beta}(v)$. In that case there exist $x, y \in \mathbb{F}_{p^k}$, $x \neq y$, such that either $f_{s,\alpha}(x) = f_{t,\beta}(x)$ and $f_{s,\alpha}(y) = f_{t,\beta}(y)$ or $f_{s,\alpha}(x) = f_{t,\beta}(y)$ and $f_{s,\alpha}(y) = f_{t,\beta}(x)$. In the first case, subtracting the two equations and rewriting yields $(s - t)(x - y) = 0$. If $s = t$, then $\alpha = \beta$ giving a contradiction. If $s \neq t$, then $x = y$ and again we have a contradiction. In the second case, subtracting and rewriting yields $(s + t)(x - y) = 0$ and since $s + t \neq 0$ for $s, t \in T$, $x = y$ and this also give a contradiction. Thus, $X$ is a clique of size $p^k((p^k - 1)/2) = \binom{n}{2}$. 

Example 7. We build a clique of size \(\binom{7}{2}\) in the derangement graph $\Gamma_{2,7}$ consisting of \(\frac{7-1}{2}\) blocks, each of which contains 7 permutations. We let $v = (1, 2, 3, 4, 5, 6, 7)$ (writing 7 instead of 0) and take $T = \{1, 4, 5\}$. Then $f_{1,0}(v) = (1, 2, 3, 4, 5, 6, 7)$, $f_{1,0}(v) = (1, 2, 3, 4, 5, 6, 7)$, $f_{0,0}(v) = (5, 3, 1, 6, 4, 2, 7)$. Increasing $\alpha$ from 0 cyclically permutes the 7-tuples. Block 1 consists of the arrangements $\{f_{1,\alpha}(v) | \alpha \in \mathbb{F}_7\}$, that is the arrangement $(1, 2, 3, 4, 5, 6, 7)$ and the remaining 6 rotations of this arrangement (e.g., $(2, 3, 4, 5, 6, 7, 1)$, $(3, 4, 5, 6, 7, 1, 2)$, etc.). Block 2 consists of the arrangement $f_{0,0}(v)$ along with all of its rotations. Finally, Block 3 consists of $f_{5,0}(v)$ and its rotations. To see that these permutations form a clique, consider, for example, the pair $\{1, 2\}$. These elements are one position apart in block 1, two positions apart in block 2 and three positions apart in block 3 (counting the shortest distance between them either forwards or backwards). So the pair $\{1, 2\}$ cannot occupy the same positions in two permutations which appear in different blocks. Furthermore, within a block, the rotations insure that the pair never occupies the same positions.

Next we turn to the independence number $\alpha(\Gamma_{k,n})$ and the chromatic number $\chi(\Gamma_{2,n})$ of the $k$-derangement graph. We will require the following lemma which has been adapted from Frankl and Deza’s lemma \[ and applied to $k$-tuples of elements.

Lemma 8. For $k < n$, $\alpha(\Gamma_{k,n}) \omega(\Gamma_{k,n}) \leq n$!
Proof. Let \( \mathcal{P} \) be a set of permutations in \( S_n \), every pair of which has at least one unordered \( k \)-tuple of elements in the same unordered \( k \)-tuple of positions. That is, for any \( u, v \in \mathcal{P} \), there exists a set \( M = \{a_1, \ldots, a_k\} \subseteq [n] \) such that \( (v^{-1}u)_k(M) = M \). Note that \( \mathcal{P} \) is an independent set in the \( k \)-derangement graph. Let \( Q \) be a set of permutations in \( S_n \) such that each pair of permutations has no \( k \)-tuple of elements in the same positions; that is, \( Q \) is a clique in the \( k \)-derangement graph. We claim that products of the form \( PQ \) with \( P \in \mathcal{P} \) and \( Q \in Q \) give distinct permutations of \( n \). Suppose, for the sake of contradiction, that \( P_1 Q_1 = P_2 Q_2 \) for \( P_1, P_2 \in \mathcal{P} \) and \( Q_1, Q_2 \in Q \) with \( P_1 \neq P_2 \) and \( Q_1 \neq Q_2 \). This implies that \( P_1^{-1} P_2 = Q_1 Q_2^{-1} \). Now, since \( P_1 \) and \( P_2 \) are in \( \mathcal{P} \), there is a \( k \)-tuple of elements \( M = \{a_1, \ldots, a_k\} \) such that \( (P_1^{-1} P_2)_k(M) = M \). However, this implies \( (Q_1 Q_2^{-1})_k(M) = M \). But we know that the permutations in \( Q \) agree on no \( k \)-tuples, and so we must have \( Q_1 = Q_2 \) and hence, \( P_1 = P_2 \). Finally, since each product gives a unique permutation of \( n \), there can be no more than \( n! \) such products. \( \square \)

**Theorem 9.** For \( k < n \), \( \alpha(\Gamma_{k,n}) \geq k!(n-k)! \) and \( \chi(\Gamma_{k,n}) \leq \binom{n}{k} \).

**Proof.** Consider \( H \), the set of all permutations in \( S_n \) that send \( \{1, 2, \ldots, k\} \) to itself (and hence \( \{k+1, \ldots, n\} \) to itself). It is clear that \( H \) is a subgroup of \( S_n \) isomorphic to \( S_k \times S_{n-k} \) and that \( |H| = k!(n-k)! \). Since the unordered \( k \)-tuple \( \{1, 2, \ldots, k\} \) is fixed, none of these are \( k \)-derangements of each other, so \( H \) is an independent set and \( \alpha(\Gamma_{k,n}) \geq k!(n-k)! \).

The cosets of \( H \) partition \( S_n \), and each forms an independent set, since \( \tau_1, \tau_2 \in \sigma H \) implies that \( \tau_1^{-1} \tau_2 \in H \) is not a \( k \)-derangement and hence the vertices associated to \( \tau_1 \) and \( \tau_2 \) are not connected by an edge. Giving each of the \( n! \) cosets a different color results in a valid coloring of \( \Gamma_{k,n} \), so \( \chi(\Gamma_{k,n}) \leq \binom{n}{k} \). \( \square \)

**Corollary 10.** For \( n \) an odd prime power, \( \alpha(\Gamma_{2,n}) = 2(n-k)! \) and \( \chi(\Gamma_{2,n}) = \binom{n}{2} \).

**Proof.** By Lemma 8 and Theorem 6 we have \( \binom{n}{2} \cdot \alpha(\Gamma_{2,n}) \leq n! \). Thus \( \alpha(\Gamma_{2,n}) \leq n! \frac{2(n-2)!}{n!} = 2(n-2)! \) and Theorem 9 gives the reverse inequality. For any graph \( G \), \( \chi(G) \geq \omega(G) \), so by Theorem 6, \( \chi(\Gamma_{2,n}) \geq \binom{n}{2} \) and again Theorem 9 gives the reverse inequality. \( \square \)

5. **Further Questions**

While the last section focused on properties of the 2-derangement graphs for \( n \) an odd prime power, we are interested in finding formulas for \( \omega(\Gamma_{k,n}) \), \( \alpha(\Gamma_{k,n}) \) and \( \chi(\Gamma_{k,n}) \) for arbitrary \( k \) and \( n \). We have some faint hope that the bounds given in Theorem 9 are actually equalities, but those of Theorem 6 cannot be, since \( \omega(\Gamma_{2,4}) = 5 < \binom{4}{2} \) (via a computer search). When \( n \) is not an odd prime power, the clique construction of Theorem 6 fails to work. If \( n \) is not a prime power, then there is no field of that cardinality, and if \( n = 2^k \), then the condition that \( \alpha + \beta \neq 0 \) fails since \( \alpha + \alpha = 0 \) for all \( \alpha \in \mathbb{F}_{2^k} \). In fact, we believe the clique number for \( n \) not an odd prime power is strictly smaller than \( \binom{n}{k} \). We found a clique of size 9 for \( n = 6 \) and \( k = 2 \), and so \( 9 \leq \omega(\Gamma_{2,6}) \leq 15 \).

In another direction, the numerical evidence is overwhelming that the derangement graphs are Hamiltonian. We hope to explore this and other questions in future work.
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References

[1] P. Frankl and M. Deza, On the Maximum Number of Permutations with Given Maximal or Minimal Distance, *J. Comb. Theory (A)* 22 (1977), 352–360.
[2] P. Renteln, On the Spectrum of the Derangement Graph, *Elec. J. of Comb.* 14 (2007), #R82.

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