Time Dynamics of Probability Measure and Hedging of Derivatives

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Abstract

We analyse derivative securities whose value is \textit{not} a deterministic function of an underlying which means presence of a basis risk at any time. The key object of our analysis is conditional probability distribution at a given underlying value and moment of time.

We consider time evolution of this probability distribution for an arbitrary hedging strategy (dynamically changing position in the underlying asset). We assume log-brownian walk of the underlying and use convolution formula to relate conditional probability distribution at any two successive time moments. It leads to the simple PDE on the probability measure parametrized by a hedging strategy. For delta-like distributions and risk-neutral hedging this equation reduces to the Black-Scholes one. We further analyse the PDE and derive formulae for hedging strategies targeting various objectives, such as minimizing variance or optimizing quantile position.
1 Introduction

In the classical theory of derivatives a number of restrictions is conventionally imposed. Risk free rate of interest, growth rates and volatilities of log-normal processes are assumed to be constant, short selling of securities is permitted, transaction costs, taxes and dividends are absent, securities are perfectly divisible, trading is continuous, and there are no arbitrage opportunities [1]. Attempts to overcome these restrictions may involve some basis risk. In competitive markets the price is such that a participant may receive a profit or suffer a loss. We expect that market participants are prepared to price basis risk in a form of profit/loss probability distribution function (PDF) according to their portfolio needs. Our problem here is to develop a technique for computing PDFs of derivatives for any hedging strategy.

Risk-neutral valuation is a general approach to studying derivatives [1]. It basically says, that in order to compute a present value of a contract (European, for example), one must average the cash flow function at maturity over equivalent risk-neutral martingale measure \( \tilde{M}[S(t)] \) on a space of random walks of the underlying asset \([S(t)]\) and make an appropriate discounting

\[
F(S,t) = e^{r(t-T)} E_{\tilde{M}}[F(\cdot, T)]
\]

This prescription follows from the no-arbitrage argument. The effective risk-neutral measure \( \tilde{M} \) comes as a result of gauging of an objective measure \( M[S(t)] \) by the riskless hedging strategy \( \phi_0(S,t) \)

\[
(M, \phi_0) \rightarrow \tilde{M}
\]

Riskless hedging is merely a replication process used for correct pricing. In addition to replicating riskless portfolio an investor may choose to explore innovative hedging strategies. Then investors should be provided by a tool which would enable them to choose hedging strategies serving best their personal objectives.

So, what if a generic hedging strategy \( \phi \neq \phi_0 \) is chosen? This would lead to an effective measure \( \tilde{M}_\phi \) which is generally neither risk-neutral nor martingale. We shall solve the following problem.

Given the objective measure \( M \) (say, log-normal with drift \( \mu \) and volatility \( \sigma \)) and a hedging function \( \phi \), derive an effective measure \( \tilde{M}_\phi \).

It is worthwhile noting that this problem is also related to pricing in incomplete markets [2-4], when \( \phi_0 \) is not unique.

As long as effective measure \( \tilde{M}_\phi \) is not riskless a value of the contract \( F \) is clearly a random number. Therefore, we should describe \( \tilde{M}_\phi \) by a conditional probability distribution function (PDF) \( P^\phi(F|S,t) \) for a given \( S \) (underlying value) and \( t \) (time): 

\[
\int_{-\infty}^{\infty} dF P^\phi(F|S,t) = 1
\]
Then equation (1) takes more general form

\[ \mathcal{P}(F|S,t) = e^{r(T-t)} E_{\tilde{\mathcal{M}}_0}[\mathcal{P}(F e^{r(T-t)}| \cdot, T)] \] (4)

Time evolution of \( \mathcal{P}(F|S,t) \) can be described by the PDE which we derive in the next section. It can be viewed as *generalized Black-Scholes equation*. In fact, we show that it coincides with the Black-Scholes equation in the risk free limit. In the sequel we omit the \( \phi \) index. However, we always imply that a PDF depends on \( \phi \).

### 2 The PDE on a PDF

#### 2.1 One Time Step

Let us consider the time interval \((t, t')\). Assume that the underlying stock value at the beginning \( S(t) = S \) and conditional PDF \( \mathcal{P}(F|\cdot, t') \) at the end of the interval as well as a risk neutral discount rate \( r \) are known. Markovian transfer matrix \( \rho(S', t'|S, t) \) will be considered log-normal in the sequel. We should also assume that hedging position \( \phi \) is unchanged during the interval. We consider a position where the contract is sold and shares are purchased. Portfolio consisting of \( \phi \) shares and one short contract has a present value \( X = F - \phi S \). A PDF of the portfolio value \( X \) at the moment \( t \) is simply expressed by the following convolution

\[ \tilde{\mathcal{P}}(X|t) = \int dS' \rho(S', t'|S, t) \mathcal{P}(X + \phi S'| S', t') \] (5)

Now suppose that a value of the portfolio at the moment \( t' \) is \( X \) with some probability \( p \). Then clearly a contract value at the moment \( t \) is \( F = X e^{r(t-t')} + \phi S \) with the same probability. This simple reasoning leads to the useful relation between the contract value statistics at the moments \( t \) and \( t' \)

\[ \mathcal{P}(F|S, t) = e^{r(t'-t)} \int dS' \rho(S', t'|S, t) \mathcal{P}((F - \phi S)e^{r(t'-t)} + \phi S'| S', t') \] (6)

So we did a remarkable loop in time when deriving (6). First, we used forward transfer matrix on \( S \) to mix \( S \) and \( F \) statistics at \( t' \) and obtain statistics of \( X \), Eq(5). Then we went back in time discounting portfolio statistics and subtracting stock part of the portfolio at \( t \).

Another important lesson which we learn from this exercise it that statistics of a contract value is closely related to a hedging strategy. To find out how valuable a contract is one needs to examine it’s statistical behaviour under *optimal* hedging. We should return to this point in the next section.
2.2 Continuous Limit

We now take a continuous limit assuming that the time step \((t, t') = t + dt\) is small and therefore, the kernel

\[
\rho(S', t'|S, t) = \frac{1}{S'\sqrt{2\pi\sigma^2(t' - t)}} \exp\left\{ -\frac{[\ln(S'/S) - (\mu - \sigma^2/2)(t' - t)]^2}{2\sigma^2(t' - t)} \right\}
\]

(7)

is sharply peaked as compared to the typical support of \(P(F|S, t)\). Then expanding r.h.s. of (6) to the first order in \(dt\) one finds [5]

\[
\frac{\partial P}{\partial t} + r \frac{\partial}{\partial F} \left( (F - \phi S)P \right) + \mu S \left( \phi \frac{\partial P}{\partial F} + \frac{\partial P}{\partial S} \right) + \frac{1}{2} \sigma^2 S^2 \left( \phi^2 \frac{\partial^2 P}{\partial F^2} + 2\phi \frac{\partial^2 P}{\partial F \partial S} + \frac{\partial^2 P}{\partial S^2} \right) = 0 \quad (8)
\]

Second term in the l.h.s. corresponds to continuous discounting of the portfolio, third term is responsible for the drift of the underlying and fourth one is a diffusion term \(a la\ Fokker-Plank. It is instructive to derive equations for mean, conditional variance and skewness of the contract value. Let us define

\[
\bar{F}(S, t) = E_P[F], \quad V(S, t) = E_P[F^2] - \bar{F}^2, \quad Q(S, t) = E_P[F^3] - 3E_P[F^2]\bar{F} + 2\bar{F}^3. \quad (9)
\]

Differentiating (9) with respect to time and using (8) gives

\[
\frac{\partial \bar{F}}{\partial t} + \mu S \frac{\partial \bar{F}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \bar{F}}{\partial S^2} - r\bar{F} = (\mu - r)S\phi, \quad (10)
\]

\[
\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - 2rV = -\sigma^2 S^2 \left( \frac{\partial \bar{F}}{\partial S} - \phi \right)^2 \quad (11)
\]

and

\[
\frac{\partial Q}{\partial t} + \mu S \frac{\partial Q}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} - 3rQ = -3\sigma^2 S^2 \frac{\partial V}{\partial S} \left( \frac{\partial \bar{F}}{\partial S} - \phi \right) \quad (12)
\]

Pre-factor 2 (3) in the term \(2rV\) (3\(rQ\)) accounts for the fact that the variance (skewness) is measured in squared (cubed) units of currency. Equation (10) reduces to the Black-Scholes equation on the average \(\bar{F}\) if one chooses the hedge \(\phi = \partial \bar{F}/\partial S\). If one adds to it final condition on variance \(V(S, T) = 0\) then (11) gives \(V(S, t) = 0\), i.e. randomness disappears at any moment \(t\). Thus we obtain the classical European contract dynamics as a special case.

PDE’s for the momenta \(E_P[F^n]\) can be easily derived in the same fashion. Lower momenta \(E_P[F^{n-1}]\) and \(E_P[F^{n-2}]\) will enter the \(n\)th equation as long as we have got first- and second-order partial derivatives in (8).
2.3 Path Integral Solution and Effective Measure

Deriving equation (8) we had in mind that a PDF is a smooth function of its arguments (otherwise partial derivatives would not be well defined). Unfortunately, it is not always true. The PDE is not applicable if we have to deal with binary, delta-like or any other singular distribution at maturity. However, it is possible to derive closed form integral solution (evolution kernel) which would make sense for any reasonable final conditions \( P(F|S, T) \).

Suppose that we are interested in a PDF at initial moment \( t \). Let us divide the time interval \((t, T)\) in \( N \) little segments \((t_k, t_{k+1})\) such that \( t_0 = t \) and \( t_N = T \). We should move back in time - from maturity to the present moment - applying backward transfer matrix (6) at each step. Introducing notations \( S(t_k) = S_k \) and \( \phi(t_k) = \phi_k \) one can rewrite (6) as

\[
P(F|S_{k-1}, t_{k-1}) = \int dS_k T_\phi(S_{k-1}|S_k)P(F|S_k, t_k). \tag{13}
\]

The \( T \)-operator acts on both arguments of \( P \) - on \( S \) as a \( \rho \)-matrix and on \( F \) as a shift. After \( N \) successive backward steps we get

\[
P(F|S, t) = \int dS_1dS_2 \ldots dS_N T_\phi(S|S_1)T_\phi(S_1|S_2) \ldots T_\phi(S_{N-1}|S_N)P(F|S_N, T). \tag{14}
\]

In the \( N \to \infty \) limit the \( \rho \)-matrices accumulate to the log-brownian measure

\[
\int \mathcal{D}M|_{S(t)=S} = \int dS_1dS_2 \ldots dS_N \rho(S, t|S_1, t_1)\rho(S_1, t_1|S_2, t_2) \ldots \rho(S_{N-1}, t_{N-1}|S_N, t_N),
\]

whereas stepwise shifts of \( F \) produce an integral shift

\[
\Psi[S, \phi] = e^{-rT} \int_t^T \phi(dS(u) - rSdu)e^{ru} \tag{16}
\]

defined by the path \( S(u) \) and the hedging function \( \phi(S, u) \). So, we get a compact answer for the PDF

\[
P(F|S, t) = e^{r(T-t)} \int \mathcal{D}M[S] \ P(e^{r(T-t)}(F + \Psi[S, \phi]) | \cdot, T) \tag{17}
\]

It’s a right time now to recall the formula (4). Effective measure \( \tilde{M}_\phi \) on the space of random walks \([S]\) turns out to be a differential operator acting on \( P \)

\[
\int D\tilde{M}_\phi[S] = \int D\mathcal{M}[S] \ exp(\Psi[S, \phi] \partial F). \tag{18}
\]

It should be stressed that formula (17) does not require any special properties of the probability measure like smoothness or continuity. It perfectly works for any probability distribution. As an example let us consider a European call option with random strike. Say, the strikes are \( K_1 \) and \( K_2 \) with probabilities \( p \) and \( 1-p \) respectively (hybrids or dual
triggers are contracts of this type). Maturity distribution for such an option is a linear combination of two $\delta$-functions and equation (17) immediately gives

$$\mathcal{P}^\phi(F|S,t) = e^{r(T-t)} \int DM[S] \{ p \delta[e^{r(T-t)}(F + \Psi[S,\phi]) - (S(T) - K_1)] + (1 - p) \delta[e^{r(T-t)}(F + \Psi[S,\phi]) - (S(T) - K_2)] \}. \quad (19)$$

3 Optimal Hedges

In this section we show how to optimize different characteristics of a PDF. First, let us consider variance at the moment $t$: $V(S,t)$. Suppose that the goal of a hedger is to optimize this variance by using the best strategy [6]. One should minimize variance as a functional of $\phi(S',t')$, where $t' \in (t,T)$. A convenient Green function

$$\rho_\Delta(S',t'|S,t) = e^{\Delta r(t-t')} \rho(S',t'|S,t) \quad (20)$$

turns out to be useful to express evolution equation solutions whose dimensionality with respect to a numeraire is $\Delta$. For example, solution of the equation for variance (11) can be presented as

$$V^\phi(S,t) = \int dS'dt' \rho_2(S',t'|S,t) \{ V(S',t')\delta(t' - T) + \sigma^2 S'^2 (\phi(S',t') - \partial_{S'} \bar{F}(S',t'))^2 \} \quad (21)$$

Variance minimizing hedge $\phi^*$ solves the variational equation $\delta V^\phi / \delta \phi = 0$, which immediately gives $\phi^*(S,t) = \partial_S \bar{F}(S,t)$. Eq. (10) for the variance minimizing hedging strategy $\phi^*$ is nothing but the Black-Scholes equation on the mean.

$$\frac{\partial \bar{F}}{\partial t} + rS \frac{\partial \bar{F}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \bar{F}}{\partial S^2} - r\bar{F} = 0. \quad (22)$$

Final condition to be used here is $\bar{F}(S,T) = E\mathcal{P}[F]_{t=T}$. Thus variance minimizing hedge is given by

$$\phi^*(S,t) = \frac{\partial F_{BS}(S,t)}{\partial S}. \quad (23)$$

Minimal variance $V^{\phi^*}$ can be found then as a solution of the homogenous PDE

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - 2rV = 0. \quad (24)$$

Another important characteristic of the statistical distribution which we consider is a quantile position [7] $F_q(S,t)$ defined by

$$\int_{-\infty}^{F_q(S,t)} dF \mathcal{P}(F|S,t) = q \quad (25)$$
Optimal hedging $\phi^{**}$ now is such that $\max_{\phi}[F_\phi^q] = F^\phi_{q}^{**}$. Similar variational equation $\delta F^\phi_q/\delta \phi = 0$ can be solved for this case as well. It leads to

$$\phi^{**}(S, t) = -\frac{(\mu - r)/\sigma^2 S + \partial_S \ln \mathcal{P}^{\phi^{**}}|_{F=F_\phi(S,t)}}{\partial_F \ln \mathcal{P}^{\phi^{**}}|_{F=F_\phi(S,t)}}$$  \hspace{1cm} (26)$$

where $\mathcal{P}^{\phi^{**}}$ is the solution of (8) for $\phi = \phi^{**}$. Although the system of two equations (8) and (26) looks nonlocal in time, it makes perfect sense once we adopt the stepwise approach. Namely, we assume again that the time is discrete $(t, t_1, t_2, \ldots t_N = T)$. Then knowing $\mathcal{P}(F|S,t_k)$ and therefore $F_q(S,t_k)$ one can compute $\phi^{**}(S,t_k-1)$ using formula (26) and $\mathcal{P}(F|S,t_{k-1})$ using (13) (discrete version of (8)), etc. What we describe here is a dynamical programming procedure which would result in a hedging function maximizing a quantile position. Such a procedure will be implemented numerically and presented in a separate paper.

4 Conclusion

In this paper we have indicated an approach which allows one to study financial derivatives depending on the method of their hedging. We think that this approach can be useful for developing optimal trading strategies in the framework of portfolio management. This approach can also be helpful for quantitative analysis of contracts which value at maturity is random rather than deterministic. Randomness may come from various sources such as default, contingency on non-traded indices, or uncertainty of the statistics of tradable underlyings. This method also leads to analysis of incomplete markets, and helps to develop preferences regarding investment strategies (the case $\mathcal{P}(F|\cdot, T) = \delta(F)$). In the latter case, an interesting problem is to find the hedge which replicates a preferred probability distribution. One way to do this is optimize the Kullback-Leibler distance between the replicated and target distribution functions [8]. The work on application of objective measure analysis to American options and fixed-income instruments is in progress.

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