A simple approach to the calculation of retarded dispersion forces

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Abstract

We propose a phenomenological Hamiltonian for the interaction of neutral macroscopic bodies with the electromagnetic field. Subsequently we revisit the assumption according to which the retarded interactions between neutral macroscopic bodies can be obtained through an additive principle i.e. summing the volume elements defined by the macroscopic bodies.

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I. INTRODUCTION

As it is very well known the discovery and the first calculation of the retarded forces between macroscopic bodies, due to the fluctuations of the electromagnetic field, can be found in Casimir’s 1948 seminal paper[1]. A few years later the Lifshitz theory appeared. Within the frame of this theory a powerful expression describing the force at finite temperature between two semiinfinite dispersive media separated by a slab was obtained[2]. However the fundamental method of calculating these interactions at zero temperature is still due to Casimir. One has to single-out the electromagnetic field normal modes consistent with the boundary conditions imposed by the presence of the macroscopic bodies. The shift of the zero point energy of the field with respect to vacuum is then considered as the sought interaction energy. In the case of complicated geometry the evaluation of the Casimir force is very difficult, therefore various approximation methods were developed such as the so called proximity-force approximation (PFA)[3], the semiclassical[4, 5] and optical approximations[6] and path-integral methods[7]. We are interested here in an approach used in reference[8] and discussed by Mostepanenko and Sokolov[9]. The interested reader can find a review of the subject in the comprehensive report by Bordag et al.[10]. It is assumed that the dependence of the interaction potential of macroscopic bodies on distance can be determined by summing the interaction potential between single atoms (molecules) belonging to different bodies. The coefficient of the expression so obtained, which depends on the kind of material involved, is renormalized in order to be in agreement with the interaction between flat plates of the same material as derived by the Lifshitz theory. The aforementioned additivity principle works very well in the case of low density media though it is not said that the principle could be extended in general and particularly in the case of metals. However because of the simplicity and usefulness of this approach we are here revisiting it with the idea according to which the retarded forces between two macroscopic bodies might be understood as being due to the attraction in pairs between their volume elements. Evidently such volume elements must be considered small as far as the differential calculus is concerned, yet rather large compared to interatomic distances. We will delineate our idea proposing a phenomenological Hamiltonian for the interaction of the electromagnetic field with a macroscopic body whose properties are phenomenologically described.

In the following sections we will discuss the proposed Hamiltonian showing where it
leads. Subsequently we will apply our results to a few geometrical dispositions of simple macroscopic objects.

II. A PHENOMENOLOGICAL PERTURBATION HAMILTONIAN

Consider two neutral, dielectric and homogeneous bodies in fixed positions. The presence of such bodies will modify the Hamiltonian of the electromagnetic field which will be written:

$$H = H_0 + H'$$

where $H_0$ is the Hamiltonian of the free electromagnetic field and where $H'$ indicates the perturbation due to the presence of the two bodies. Assuming as unperturbed field state the vacuum we will calculate the energy shift due to the extra term of the Hamiltonian. The part of the shift which depends on the configuration of the two bodies will be interpreted as their potential energy at zero temperature. We consider and propose now a phenomenological perturbation as follows:

$$H' = -\frac{1}{2} \int_V \mathbf{P} \cdot \mathbf{E} dV - \frac{1}{2} \int_V \mathbf{M} \cdot \mathbf{H} dV$$

where $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{P}$ and $\mathbf{M}$ are the polarization and magnetization vectors. The integral is extended to the volume occupied by the bodies. Always within a phenomenological approach the Fourier components $\mathbf{P}_k$ and $\mathbf{M}_k$ may be considered connected to those of the electric field and magnetic field by a linear relation.

We set:

$$\begin{aligned}
\hat{\mathbf{P}} &= i\sqrt{2\pi\hbar c} \sum_{k,\lambda} \sqrt{k} \beta(k) \left( a_{k,\lambda} e^{ik \cdot r} - a_{k,\lambda}^\dagger e^{-ik \cdot r} \right) e_\lambda(k) \\
\hat{\mathbf{M}} &= i\sqrt{2\pi\hbar c} \sum_{k,\lambda} \sqrt{k} \gamma(k) \left( a_{k,\lambda} e^{ik \cdot r} - a_{k,\lambda}^\dagger e^{-ik \cdot r} \right) \frac{k \times e_\lambda(k)}{k}
\end{aligned}$$

where $a_k$, $a_k^\dagger$ are the annihilation and creation operators and $e_\lambda(k)$ indicates the polarization.

For the field we have used only transversal waves and we set to unity the side of the cube which defines the periodic conditions. The $\beta(k)$ and $\gamma(k)$ coefficients are defined as:

$$\begin{aligned}
\beta(k) &= \frac{3\varepsilon(k) - 1}{4\pi\varepsilon(k) + 2} \\
\gamma(k) &= \frac{3\mu(k) - 1}{4\pi\mu(k) + 2}
\end{aligned}$$
where $\varepsilon$ and $\mu$ are the electric and magnetic permittivities which express the usual Clausius-Mossotti formula. We then obtain:

$$
\mathcal{H}' = \pi \hbar c \sum_{k,k'} \sum_{\lambda,\nu} \sum_i \sqrt{kk'} \left[ \mathbf{e}_\lambda(k) \cdot \mathbf{e}_\nu(k') \beta(k) + \frac{k \times \mathbf{e}_\lambda(k)}{k} \cdot \frac{k' \times \mathbf{e}_\nu(k')}{k'} \gamma(k) \right]
$$

$$
\left[ a_{k,\lambda} a_{k',\nu} v_i(k + k') - a_{k,\lambda}^\dagger a_{k',\nu}^\dagger v_i(k' - k) - a_{k,\lambda} a_{k',\nu}^\dagger v_i(k - k') + a_{k,\lambda}^\dagger a_{k',\nu} v_i(-k - k') \right] \tag{5}
$$

The index $i = 1,2$ indicates the summation over the two bodies. We furthermore defined:

$$
v_i(\eta) = \int_{V_i} e^{i\eta r_i} dV_i \tag{6}
$$

III. THE INTERACTION ENERGY AT LARGE SEPARATIONS

We stress that we consider the perturbation $\mathcal{H}'$ above to hold only for the calculation of the interaction energy of neutral macroscopic bodies at large separations. Let $d$ be the shortest distance between the surfaces of the two bodies. The expression “large separation” usually means $d \gg \lambda_i$ where $\lambda_i$ are the absorption wavelengths of the material under consideration. The mutual energy of the two bodies at zero temperature is obtained by perturbation at first order. In the case of large separations the calculation simplifies greatly because $\beta(k)$ and $\gamma(k)$ can be safely substituted by $\beta(0)$ and $\gamma(0)$. In fact the major contribution to the integral below comes from the $k \lesssim 1/d$ region where $d$, as previously mentioned, is the shortest distance between the surfaces of the bodies[11].

Thus we asymptotically obtain:

$$
U_{1,2} = -\frac{\hbar c}{2(2\pi)^4} [\beta_1(0)\beta_2(0) + \gamma_1(0)\gamma_2(0)] \times \int d\mathbf{k} \int d\mathbf{k'} \frac{k k'}{k + k'} \left( 1 + \frac{(\mathbf{k} \cdot \mathbf{k'})^2}{k^2 k'^2} \right) (v_1(k + k') v_2(-k + k') + c.c.) \tag{7}
$$

Setting $\mathbf{q} = \mathbf{k} + \mathbf{k'}$ the expression $(7)$ becomes:

$$
U_{1,2} = -\frac{\hbar c}{2(2\pi)^4} [\beta_1(0)\beta_2(0) + \gamma_1(0)\gamma_2(0)] \times \int d\mathbf{q} (v_1(\mathbf{q}) v_2(-\mathbf{q}) + c.c.) \int d\mathbf{k} \frac{k |\mathbf{q} - \mathbf{k}|}{k + |\mathbf{q} - \mathbf{k}|} \left( 1 + \frac{(\mathbf{k} \cdot (\mathbf{q} - \mathbf{k}))^2}{k^2 |\mathbf{q} - \mathbf{k}|^2} \right) \tag{8}
$$

The integral over $\mathbf{k}$ because of a lack of a quenching factor obviously diverges. Yet within the scope of our calculation we can neglect the infinities and consider only the singular part.
in the origin \((q = 0)\) which dominates the asymptotic behaviour. The present procedure is analogous to the usual renormalization techniques which make use of a cut-off. In our case the divergent part of the integral is already separated without ambiguities. Furthermore this part can be ignored because it does not depend on the system configuration and therefore it can not give origin to forces among the considered bodies. The final result, indicated in expression (9), can be obtained rather laboriously following the indications of reference \[12\]:

\[
U_{1,2} = \hbar c A_{1,2} \int dq (v_1(q)v_2(-q) + c.c.) q^4 \ln q
\]

where:

\[
A_{1,2} = \frac{23}{240(2\pi)^3} [\beta_1(0)\beta_2(0) + \gamma_1(0)\gamma_2(0)]
\]

For two perfect metals \([\varepsilon(0) = \infty, \mu(0) = 0]\) the constant \(A_{m,m}\) is:

\[
A_{m,m} = \frac{1035}{3840} \frac{1}{(2\pi)^5}
\]

Now using the tables of the Fourier transforms of the generalized functions \[13\] we have:

\[
\int dq q^4 \ln q e^{ik\cdot r} = -\frac{30(2\pi)^3}{\pi r^7}
\]

hence recalling (6) we obtain:

\[
U_{1,2} = -\hbar c B_{1,2} \int V_1 dr_1 \int V_2 dr_2 \frac{1}{|r_1 - r_2|^7}
\]

where:

\[
B_{1,2} = \frac{23}{4\pi} [\beta_1(0)\beta_2(0) + \gamma_1(0)\gamma_2(0)]
\]

In the case of two perfect metals the constant is:

\[
B_{m,m} = \frac{1035}{256} \frac{1}{\pi^3}
\]

As it can be seen from expression (13) our results confirm Mostepanenko’s assumptions \[9\] yielding in addition the coefficient as a function of the characteristics of the material.

**IV. A FEW EXAMPLES**

In the case of two alike homogeneous dielectrics of very low molecular density and where the molecules are assumed of the same kind \(\gamma(0) = 0\) and \(\beta(0)\) becomes \(N\alpha(0)\) where \(N\) is the molecular density and \(\alpha(0)\) the molecular polarization. From \[13\] we then have:

\[
U = -\frac{23}{4\pi} \hbar c \alpha_1(0)\alpha_2(0) \int N dr_1 \int N dr_2 \frac{1}{|r_1 - r_2|^7}
\]
Which means that the interaction potential of the retarded Van der Waals forces between two molecules is:

$$-\frac{23}{4\pi}\frac{\hbar c\alpha_1(0)\alpha_2(0)}{|\mathbf{r}_1 - \mathbf{r}_2|^7}$$

which is a well known result.\[11\], \[14\]

Shall we move on now to the classic case of two parallelepipeds made of perfect metal whose volumes are so defined:

$$-\frac{L}{2} \leq x_1 \leq \frac{L}{2} \quad -\frac{L}{2} \leq y_1 \leq \frac{L}{2} \quad -D \leq z_1 \leq 0 \quad (18)$$

$$-\frac{L}{2} \leq x_2 \leq \frac{L}{2} \quad -\frac{L}{2} \leq y_2 \leq \frac{L}{2} \quad a \leq z_2 \leq a + D \quad (19)$$

The volume Fourier transforms are:

$$v_1(q)v_2(-q) = 16 \left( \frac{\sin \frac{q_x L}{2}}{q_x} \right)^2 \times \left( \frac{\sin \frac{q_y L}{2}}{q_y} \right)^2 \times \left( 1 - e^{-iq_z D} \right) \left( e^{-iq_z (D+a)} - e^{-iq_z a} \right) \frac{q_z^2}{q_x^2} \quad (20)$$

To integrate (20) we find it convenient, not to mix up the calculations, to write:

$$\left( \frac{\sin \frac{q_x L}{2}}{q_x} \right)^2 = \frac{\sin \frac{q_x L}{2}}{q_x} \quad \frac{\sin \frac{q_y L}{2}}{q_y} \quad (21)$$

regarding the first of the factors on the right (assuming \(L\) large) as

$$\frac{\sin \frac{q_x L}{2}}{q_x} = \pi \delta(q_x) \quad (22)$$

we then calculate the limit \(L' \to L\) once the integration has been carried out. We follow the same route with the factor depending on \(q_y\). We are then led to the integral:

$$(2\pi)^2 L^2 \int dq_x q_x^2 \ln(q_x) \left[ (2e^{-iq_x(a+D)} - e^{-iq_x(a+2D)} - e^{-iq_x a}) + c.c. \right] \quad (23)$$

which can be evaluated using again reference [13]. One then finds:

$$-2(2\pi)^3 L^2 \left( \frac{1}{a^3} + \frac{1}{(a+D)^3} - \frac{2}{(a+D)^3} \right) \quad (24)$$

Inserting the previous result into expression (9) we obtain the interaction energy for two metallic parallelepipeds at zero temperature:

$$-\frac{1035}{7680} \frac{\hbar c}{\pi^2} L^2 \left( \frac{1}{a^3} + \frac{1}{(a+2D)^3} - \frac{2}{(a+D)^3} \right) \quad (25)$$
which at the limit $D \gg a$ becomes:

$$- \frac{1035 \hbar cL^2}{7680 \ a^3} = -0.01365 \frac{\hbar cL^2}{a^3}$$

(26)

Expression (26) has to be compared with Casimir’s more elegant result [1]:

$$- \frac{\pi^2 \ h c L^2}{720 \ a^3} = -0.01370 \frac{\hbar cL^3}{a^3}$$

(27)

The small discrepancy between the two results deserves a brief comment. Though its origin is not clearly evident, we think that it comes from the fact that our Hamiltonian, referring to the bulk of the surrounding bodies, does not produce an exact description of the field in the proximity of the surface where the Clausiu-Mossotti formula may be inadequate. In the Casimir’s approach the appropriate boundary conditions are generally referred tho the field, thus the description of the situation on the surface is certaininly more exact.

Consider now the interaction $u$ of a metallic corpuscle of infinitesimal volume $dV$ at distance $y$ from a metallic hemispace. From expression (13) with $B_{1,2} = B_{m,m}$ and using cylindrical coordinates we have:

$$u = -\hbar c B_{m,m} dV \int_0^{2\pi} d\varphi \int_0^\infty \rho d\rho \int_0^\infty dz \frac{1}{[\rho^2 + (z + y)^2]^{-7/2}}$$

(28)

which is:

$$u = -\hbar c B_{m,m} \frac{\pi}{10} \frac{1}{y^4} dV$$

(29)

Using this last expression one can calculate the interaction between a very thick and infinitely extended metallic plate and a generic body. Let this last be a metallic sphere of radius $R$ at a distance $r = d + R$. We obtain as interaction energy:

$$V_{ps} = -\hbar c B_{m,m} \frac{\pi}{10} \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos \theta \int_0^R \frac{\rho^2 d\rho}{(r + \rho \cos \theta)^4}$$

(30)

$$= -\hbar c B_{m,m} \frac{\pi^2}{30} \left( \frac{R}{d^2} - \frac{1}{d} + \frac{1}{d + 2R} + \frac{R}{(d + 2R)^2} \right)$$

(31)

with:

$$B_{m,m} \frac{\pi^2}{30} = 0.04289$$

(32)

This result should be compared with that of Balian and Duplantier [15] which in two scattering approximation for $d \ll R$ is:

$$V_{ps} \simeq -\frac{\hbar c}{8\pi} \left( \frac{R}{d^2} - \frac{1}{d} \right)$$

(33)
where:

\[
\left( \frac{1}{8\pi} = 0.03978 \right)
\]

We think that, apart from a few slight differences in the numerical coefficients, our expression can lead to a good approximation with respect to the exact calculation.

Although the dependence of the Casimir effect on the temperature is not within the scope of the present brief note, we think appropriate to mention at least the consequences that our Hamiltonian implies on the subject. Our Hamiltonian includes diagonal terms which, for each body which may be present, generate a field energy equivalent to the body volume times the black body radiation energy density as shown by Stefan-Boltzmann’s expression. But this is still a well known result. It is more interesting to investigate on the effect of temperature on the dispersion forces. To this end it is sufficient to insert as a factor of the integrand in equation (7) the quantities \(\bar{n}(k', T) + \bar{n}(k, T)\) where \(\bar{n}(k, T)\) is the average number of photons of wave number \(k\) at temperature \(T\) obtained from Planck distribution. The expression which comes out is not analytically manageable, however the insertion of the above mentioned factor introduces a natural cut-off which implies that the quantity \(U_{1,2}(T)\) can be easily evaluated using numerical methods. However an analytical result can be obtained in the case of high temperatures developing to the first order \(\bar{n}(k, T)\) and setting it equal to \(KT/(\hbar ck)\). This procedure introduces a \(k\) factor at the denominator of equation (7). Then from a power counting one can see that in this limit the interaction \(U_{1,2}(T)\) between two volume elements at distance \(r\) will behave as \(1/r^6\). As a consequence in the case of two perfect metals in Casimir’s configuration we will obtain, instead of expression (26), an interaction energy proportional to \(-KT(L^2/a^2)\) which is a well known result [16, 17].
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