Non-Gaussian statistics of pencil beam surveys

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Abstract

We study the effect of the non-Gaussian clustering of galaxies on the statistics of pencil beam surveys. We find that the higher order moments of the galaxy distribution play an important role in the probability distribution for the power spectrum peaks. Taking into account the observed values for the kurtosis of galaxy distribution we derive the general probability distribution for the power spectrum modes in non-Gaussian models and show that the probability to obtain the $128h^{-1}$ Mpc periodicity found in pencil beam surveys is raised by roughly one order of magnitude. The non-Gaussianity of the galaxy distribution is however still insufficient to explain the reported peak-to-noise ratio of the periodicity, so that extra power on large scales seems required.

Subject headings: cosmology: large-scale structure of the Universe; galaxies: clustering

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1 Introduction

The surprising discovery of a $128h^{-1}$ Mpc periodicity in the distribution of galaxies (Broadhurst et al. 1989) has raised an intense debate about the statistical significance of the signal detected. The main question is whether the periodicity is consistent with the local observations or is rather to be regarded as a new feature appearing only when very large scales ($\gg 100h^{-1}$ Mpc) are probed. In the original paper by Broadhurst et al. (1989; BEKS) the statistical significance of the peak in the one-dimensional power spectrum was assessed making use of an external estimator, i.e. adopting a model for the clustering of galaxies. The clustering was assumed to be described by the usual correlation function $\xi(r) = (r/r_0)^{-\gamma}$ up to the scale of $30h^{-1}$ Mpc, without any correlation beyond this scale, and without any higher order moment. As Szalay et al. (1991) pointed out, however, external estimators are very model dependent. Even slightly different assumptions, concerning e.g. selection functions or the parameters $r_0, \gamma$, can result in dramatic variations of statistical significances. Indeed, Kaiser & Peacock (1991), investigating essentially the same dataset as BEKS, found that the noise level was to be significatively raised, resulting in a much higher probability to find a peak as large as, or larger, the one at $128h^{-1}$ Mpc, so as to reconcile the standard model of galaxy clustering with the BEKS data. Similarly, Luo & Vishniac (1993) found that the redshift distortions can alter the estimate of the noise level.

This seems to force one to use internal estimators of the noise level. This has been done by Szalay et al. (1991), who showed then that the probability to find a peak as high as the one in the BEKS data, or higher, is $2.2 \cdot 10^{-4}$, matching their original estimate. Luo & Vishniac (1993) also confirmed the result that, while the rest of the power spectrum is consistent with the hypotheses of clustering and Gaussianity, the single prominent spike at $128h^{-1}$ Mpc is not.

They also showed that even a delta-like feature in the tridimensional power spectrum of the galaxy distribution can barely account for the BEKS spike. As we will show below, the probability estimate on which these conclusions are based relies essentially on two hypotheses: a) that the spatial bins of the BEKS survey are uncorrelated, i.e. that the clustering beyond $30h^{-1}$ Mpc is negligible, and b) that the components of the power spectrum can be assumed, by virtue of the central limit theorem, to be Gaussian distributed. The very fact that the probability estimate based on these two hypotheses is as low as $2.2 \cdot 10^{-4}$ points to the conclusion that one of the two, or both, are false. This implies either that there is some previously unknown, and theoretically unexpected, feature in the tridimensional power spectrum at large scale, or that it is the other hypothesis, the Gaussianity, to be abandoned.

The first possibility has been explored for instance in the Voronoi simulations (see e.g. Coles 1990, SubbaRao & Szalay 1992), or in truncated HDM models (Weiss & Buchert 1993). Unlike the precedent studies, in this paper we consider in detail the latter way out.

The scheme of this paper is as follows. First, we derive the probability distribution of the components of a one-dimensional power spectrum in presence of higher order moments of
the spatial distribution. Second, we ask ourselves which is the probability to find a spike as high as, or higher than, the one in the BEKS data in such non-Gaussian galaxy distribution. Finally, adopting the actual higher order moments found in local (≤ 100h⁻¹ Mpc) observations, we will show that the formal probability for the BEKS periodicity increases roughly by an order of magnitude. This, however, may still be insufficient to explain the data.

Let us note that we will not question in any way the reliability of the BEKS data or of their noise estimate. Rather, we derive our conclusion only taking into due account the already known level of non-Gaussianity in the galaxy distribution.

2 Non-Gaussian pencil beam statistics

The BEKS data consist in a set of counts along a survey geometry that approximates a long, thin cylinder directed towards the galactic poles. The galaxy positions are binned in N small cylinders of radius $R = 3h^{-1}$ Mpc and radial length $30h^{-1}$ Mpc, out to $L/2 \sim 1000h^{-1}$ Mpc in both directions. The details of the survey are given in the original paper (BEKS) and in Szalay et al. (1991). Let us denote the cell counts as $n_i$, with $i = 1, \ldots, N \approx 67$. The discrete Fourier transform of the dataset is

$$f_k = \frac{1}{P} \sum_{j=1}^{N} n_j \exp(i2\pi kr_j/L),$$

where $r_j = 30jh^{-1}$ Mpc is the radial distance to the $j-th$ bin, and $P = \sum n_j$ is the total number of galaxies (396 in BEKS). The counts $n_j$ have mean $\hat{n} = P/N$ and variance $\sigma^2 = \langle (n_j - \hat{n})^2 \rangle$ as well as higher order irreducible moments (or cumulants, or disconnected moments) $k_n$.

The power spectrum is defined as

$$A_k = |f_k|^2.$$ (2)

Let us define the quantity

$$a_k = \frac{\sum_j n_j \cos(2\pi kr_j/L)}{\sigma \sqrt{N/2}}.$$ (3)

Squaring $a_k$ we obtain $a^2_k = (\Re f_k)^2/[\sigma^2(N/2P^2)]$. Likewise, we can define the quantity $b_k = \sum_j n_j \sin(2\pi kr_j/L)/\sigma \sqrt{N/2}$ and form the modulus

$$z \equiv a^2_k + b^2_k = \frac{A_k}{\sigma^2(N/2P^2)}.$$ (4)

The problem is now to find out the probability distribution density (PDD) of $A_k$ when we know the one for $n_j$. First, however, we have to derive the PDD of $a_k$ and $b_k$. They are constructed as a linear sum of $N$ independent variables (as long as the various $n_j$ are uncorrelated), so by the central limit theorem $a_k, b_k$ should tend to be Gaussian distributed. However, since $N \sim 70$ is not really very large, one should check if the higher order terms
are significant. This is indeed what will be shown to happen. We make use of the so-called Edgeworth expansion (see e.g. Cramer 1966, Abramovitz & Stegun 1972, whose notation we will follow), according to which the variable
\[ X = \frac{\sum_i (Y_i - m_i)}{(\sum_i \sigma_i^2)^{1/2}} \] (5)
(the sums run over \( N \) terms) where the \( Y_i \) are independent random variables with mean \( m_i \), variance \( \sigma_i \) and \( n \)-th order cumulants \( k_{n,i} \), is distributed like a function \( f(X) \) that can be expanded in powers of \( N^{-1/2} \)
\[ f(X) \sim G(X) \left[ 1 + \frac{\gamma_1}{6N^{1/2}} H_{e3}(X) + \frac{\gamma_2}{24N} H_{e4}(X) + \frac{\gamma_2^2}{72N} H_{e6}(X) + O(N^{-3/2}) \right]. \] (6)

Here, \( G(X) \) is the normal distribution, \( H_{e_n} \) is the Hermite polynomial of order \( n \), and \( \gamma_1 = (\sum_i k_{3i}/N)/((\sum_i \sigma_i/N)^3) \), \( \gamma_2 = (\sum_i k_{4i}/N)/((\sum_i \sigma_i/N)^4) \). The Edgeworth expansion has been used recently in astrophysics by several authors to quantify slight deviations from Gaussianity (Juszkiewicz et al. 1993, and references therein). Now we can notice that, as long as the counts \( n_j \) are uncorrelated, the variables \( a_k \) and \( b_k \) are indeed in the form (5), where \( Y_i = n_i \cos(2\pi kr_i/L) \) [or \( Y_i = n_i \sin(2\pi kr_i/L) \)] so we are allowed to apply the Edgeworth expansion. To the order \( 1/N \), the expansion will include the skewness and the kurtosis of the counts \( n_i \).

However, the skewness sum \( \sum k_{3i} \) for the variables \( Y_i \) vanishes due to the oscillating term, so that \( \gamma_1 = 0 \).

Let us estimate then the expansion coefficient \( \gamma_2 \) in our case. The higher order moments in the galaxy counts have been calculated by several authors for different surveys (Saunders et al. 1991; Bouchet, Davis & Strauss 1992; Gaztaña 1992; Loveday et al. 1992). The general result is that, for scales which range from some megaparsecs to more than 50 \( h^{-1} \) Mpc, the dimensionless cumulants \( \mu_m = k_m/\bar{n}^m \) (for \( m = 2 \), \( \mu_2 = \sigma^2/\bar{n}^2 \))

obey the hierarchical scaling relation
\[ \mu_m = S_m \mu_2^{m-1}, \] (7)
where \( S_m \) are the scaling constants (we have checked that the shot-noise correction is negligible in our case). To the lowest order in the variance and for scales much larger than the correlation length of the fluctuation field, the scaling relation is in reality a direct consequence of the Edgeworth expansion (and can actually be derived by a much simpler argument, see Amendola & Borgani 1994). Then we see that
\[ \gamma_2 \approx (3/2)S_4 \mu_2 . \] (8)
where the numerical factor is due to the sum over the sines and cosines in \( \sum_i k_{4,i} \). The relations between the scaling constants and the physics of the clustering process have been
investigated in several works, from the book of Peebles (1980) to recent generalizations as in Bernardeau (1992).

The value of $\mu_2 = \sigma^2/\hat{n}^2$ can be expressed as a function of the correlation function (e.g. Peebles 1980), $\sigma^2 = (\hat{n}+\hat{n}^2\xi_0)$, where $\xi_0 = V^{-2} \int d^3r_1d^3r_2 W(r_1)W(r_2)\xi(\mathbf{r}_1 - \mathbf{r}_2)$ and where $W$ is the window function corresponding to the BEKS cylindrical cells of volume $V$. Since $\hat{n} \approx 6$ and $\xi_0$ is of order unity, we can approximate $\mu_2$ with $\xi_0$, so that

$$\gamma_2 \approx (3/2)S_4\xi_0.$$ The value of $\gamma_2$ will result to be crucial.

Several uncertainties, however, prevent its exact estimate. For $\xi_0$ we must rely on very local observations; we may assume the value given in Szalay et al. (1991).

$\xi_0 \approx 0.83$, or the one that we derive from Luo & Vishniac (1993), $\xi_0 \approx 1.24$, or similar values, depending on models of the correlation function. For $S_4$ one problem is that we need the scaling constants for quite elongated cylindrical cells, while the observations have been carried out mostly for large spherical or cubic cells. We can find observational values from near unity to 30 or 40 (see e.g. the table in Gaztañaga (1992)). Further, Lahav et al. (1993) find that $S_3$ and $S_4$, rather than being constants, sharply increase with the rms density contrast $\delta$, and thus decrease with the cell volume, when $\delta > 1$ in CDM simulations. We then absorb the uncertainties of $S_4$ and of $\xi_0$ in $\gamma_2$ and explore numerically the range $\gamma_2 \in (0 - 40)$.

Let us come back to the Edgeworth PDD $f(a_k)$ for $a_k$ (to the order $1/N$). The PDD for $y = a_k^2$ is then $P(y) = f(a_k)(da_k/dy) = f(y^{1/2})/2y^{1/2}$, that is

$$P(y = a_k^2) \sim g_1P_1 + \frac{\gamma_2}{24N} [g_5P_5 - 6g_3P_3 + 3g_1P_1] \equiv \sum_i c_iP_i(y),$$

where $g_n \equiv 2^{n/2}\Gamma(n/2)$ and $P_n(y) = g_n^{-1}y^{n/2-1}e^{-y/2}$ is the $\chi^2$ PDD with $n$ degrees of freedom.

Now that we have the PDD for $a_k^2, b_k^2$ we must find the distribution for $z = a_k^2 + b_k^2$. Let us denote with $\phi(t)$ the characteristic function (CF) of a generic probability distribution $P(x)$, where $\phi(t) = \int e^{itx}P(x)dx$. The general theorems about probability distributions say that the CF of the sum of two variables is the product of the CF of the variables. Furthermore, by linearity, we see that the CF of $P = P_1 + P_2$ is $\phi(P_1) + \phi(P_2)$. We are to use these two properties to derive the general distribution $P(A_k)$. First, we calculate the CF $\phi(P)$ for $P(y)$ given by (9),

$$P(y) = \sum_i c_iP_i.$$ Denoting the CF for the $\chi^2$ distribution $P_n$ as $\psi_n \equiv (1 - 2it)^{-n/2}$, we have

$$\phi[z] = \phi[a_k^2]\phi[b_k^2] = \phi^2[y] = (\sum_i c_i\psi_i)^2,$$

where the sum runs over all the $\chi^2$ PDD in the expansion (9), with the same $c_i$’s. Now, since $\psi_n\psi_m = \psi_{n+m}$, we can see that the CF for the unknown distribution $P(z)$ is a sum of $\chi^2$ CFs, so that the final result $P(z)$ is again a sum of $\chi^2$ PDDs.

Before writing down the result, we note that $z = A_k/[\sigma^2(N/2P^2)] = 2A_k/A_0$, where $A_0$ is the
noise level in the notation of BEKS. It follows $A_0 = \sigma^2 N/P^2 = (\xi_0/N + 1/P)$, which gives an external estimate of the noise level. However, as already remarked, the estimate of $A_0$ by Szalay et al. (1991) is internal in that is not based on a \textit{a priori} model for $\xi(r)$, but rather on fitting the observational distribution function for $A_k$ at small amplitudes with the exponential $P(A_k) = (1/A_0) \exp(A_k/A_0)$, as it should be in the purely Gaussian case (or for $N \to \infty$). The same internal estimate applies here, since as we will see the Gaussian and non-Gaussian PDD are equivalent at low amplitudes.

Finally, the normalized distribution function for $A_k$ to the order $1/N$ is

$$P(z = 2A_k/A_0) = P_2 + a(P_0 - 2P_4 + P_2)$$  \hspace{1cm} (11)

where $a = \gamma_2/4N$.

Eq. (11) gives then the general PDD for the power spectrum amplitudes relative to a set of pencil beam counts with scaling coefficient $S_4$.

When $S_4 = 0$ we return to the exponential distribution $P(z) = P_2$ on which the calculation of BEKS, and of all the other works on the subject, was based. We can see from $P(z)$ why the higher order terms are important. Since the peak-to-noise ratio $X \equiv A_k/A_0$ found by BEKS is very large, $X_{BEKS} = 11.8$, the terms containing higher order $\chi^2$ functions will dominate over the $P_2$ term when integrated to give the cumulative probability, even if the constant $a$ are small, i.e. even if $N$ is large. Actually, for any given $N$ there is a value $z_c$ such as the higher order terms dominate over the lower orders in the integral $\int_{z_c}^{+\infty} P(z)dz$. This is a consequence of the fact that, while the convergence of any distribution $f(X)$ to the normal one for $N \to \infty$ is ensured by the central limit theorem, the convergence itself need not be uniform. The fractional difference between the cumulative distribution of $f(X)$ and the one relative to a normal distribution can be arbitrarily large for large deviations from the mean.

We can now directly compare the PDD (11) with the power spectrum coefficients found by BEKS. We use the tabulated values provided by Luo & Vishniac (1993), binned in peak-to-noise intervals of 0.5. We plot in Fig. 1 the cumulative function of the BEKS coefficients versus peak-to-noise ratio (a point at abscissa $x$ represents the fraction of values of $A_k$ in the BEKS data with peak-to-noise ratio larger than $x$) and compare this with our theoretical cumulative function

$$F(X) = \int_{2X}^{+\infty} P(z)dz,$$  \hspace{1cm} (12)

where $X = A_k/A_0 = z/2$ is the peak-to-noise ratio. The functions plotted are for the Gaussian case ($S_4 = 0$), and for three possible values of the constant $\gamma_2$: from bottom to top,

$\gamma_2 = 5, 20, 40$. It is clear that as $\gamma_2$ increases, the observed distribution becomes more and more consistent with the non-Gaussian behavior, except for the last point, the $128h^{-1}$ Mpc spike, which appears still far away from its expected frequency value. However, we can estimate now the probability to have $X_{BEKS} = 11.8$ or higher in one of the $\sim 30$ $k$-bins to which BEKS assigned the data (see Szalay et al. 1991 for a detailed exposition) and compare with
the very unlikely value $2.2 \cdot 10^{-4}$ originally found for $\gamma_2 = 0$. The non-Gaussian result is

$$P(>11.8) \approx 30F(11.8) = 0.001 - 0.005,$$  

(13)

for the range $\gamma_2 = 10 - 40$.

The inclusion of non-Gaussianity pushed the probability to obtain the BEKS spike by about one order of magnitudes, without any need to invoke non-standard features in the galaxy distribution. The result (13) states that the BEKS spike should occur roughly in 0.1-0.5% of the cases if the very large scale galaxy distribution has to be consistent with the local observations of variance and kurtosis. If further data do not reduce the peak significativity, our result

indicate that is very difficult for the non-Gaussianity alone to explain the observations.

In Fig. 2 we display the behavior of $P(>11.8)$ vs. $\gamma_2$. Only for very large values of $\gamma_2$, and thence of $S_4$ or of $\xi_0$, the BEKS spike approaches the 3σ level. In other words, if the BEKS periodicity is strengthened by further data, we will be forced to assume values of $S_4$ or $\xi_0$ larger than local observations would require, and/or to discard the assumption that the spatial bins are uncorrelated. On the other hand, a value of $X_{BEKS}$ smaller by even a ten percent would result in quite higher values of $P(>11.8)$, as shown by the dot-dashed curve in Fig. 2.

3 Conclusions

We have shown that the higher order moments of the galaxy clustering play a not negligible role in assessing the significance level of peaks in one-dimensional power spectra. The scaling constant $S_4$ and the correlation average $\xi_0$ on the spatial bin in a pencil beam survey are combined in the crucial parameter $\gamma_2$. Assuming that the spatial bins are uncorrelated, the question raised by the remarkable periodicity discovered by Broadhurst et al. (1989) in the very large scale galaxy clustering can then be expressed in the following way: are the values of $S_4$ and of $\xi_0$ determined by local observations compatible with the clustering of galaxies at the very deep scales probed by the pencil beam? To give an answer, we have to determine the probability distribution for the power spectrum amplitudes of a non-Gaussian field sampled in spatial bins. We find by means of the Edgeworth expansion that the BEKS most prominent spike around $128h^{-1}$ Mpc has a probability of roughly $1 - 5 \cdot 10^{-3}$ for acceptable values of $\gamma_2$, to be compared with the value $2.2 \cdot 10^{-4}$ obtained by Szalay et al. (1991) neglecting the kurtosis correction. This result seems to show that non-Gaussianity alone cannot be responsible for the BEKS periodicity, unless further observations will allow very large values of $S_4$ or $\xi_0$ or will decrease the peak-to-noise ratio of the $128h^{-1}$ Mpc peak. As the data stand, we should conclude that the spatial bins cannot be assumed uncorrelated.

Indeed, this is what the large coherent structures reported in deep surveys seems to require. We also compared the peak occurrences of the full spectrum of BEKS and found it in a good agreement with our non-Gaussian probability distribution if large values for $\gamma_2$ are allowed. This raises the
interesting possibility that further pencil beam data can be employed to measure the parameter $\gamma_2$, i.e. the product $S_4\xi_0$, down to very deep distances.

Let us conclude by the remark that is not unlikely that further higher order terms in the Edgeworth expansion are significant. Further terms would require however the knowledge of new scaling coefficients, like $S_6$ and so on, which are not available. Here we confined ourselves to the order $1/N$ for simplicity, with the aim to show how the non-Gaussian properties of the galaxy clustering have a strong effect on the peak probability estimate. In this sense, our calculation gives only a lower bound on the probability estimate.

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References

Amendola L. & Borgani S. 1994, Mon. Not. R. Ast. Soc. 266, 203
Bernardeau F. 1992 Ap. J. 392, 1
Bouchet F., Davis M., & Strauss M. 1992, in Proc. of the 2nd DAEC Meeting on the Distribution of Matter in the Universe, eds. G.A. Mamon, & D. Gerbal, 287
Bouchet F. R., & Hernquist L. 1992, Ap. J. 25, 400
Broadhurst T.J., Ellis R.S., Koo D.C. & Szalay A. 1990 Nature 343, 726
Coles P. 1990 Nature 346, 446
Coles P. & Frenk C.S. 1991, Mon. Not. R. Ast. Soc. , 253, 727
Cramer H. 1966, Mathematical Methods of Statistics (Princeton: Princeton Univ. Press)
Gaztañaga E. 1992, Ap. J. , 398, L17
Juszkiewicz R., & Bouchet F.R. 1992, in Proc. of the 2nd DAEC Meeting on the Distribution of Matter in the Universe, eds. G.A. Mamon, & D. Gerbal, 301
Juszkiewicz R., Weinberg D. H., Amsterdamski P., Chodorowski M. & Bouchet F. 1993, preprint (IANSS-AST 93/50)
Kaiser N. & Peacock J.A. 1991, Ap. J. 379 482
Lahav O., Itoh M., Inagaki S. & Suto Y. 1993, Ap. J. 402, 387
Loveday J., Efstathiou G., Peterson B. A. & Maddox S.J. 1992, Ap. J. , 400, L43
Luo S. & Vishniac E.T. 1993, Ap. J. 415, 450.
Peebles P.J.E. 1980, The Large-Scale Structure of the Universe (Princeton: Princeton Univ. Press)
Saunders W., Frenk C., Rowan-Robinson M., Efstathiou G., Lawrence A., Kaiser N., Ellis R., Crawford J., Xia X.-Y., Parry I. 1991, Nature, 349, 32
SubbaRao M. & Szalay A. 1992, Ap. J. 391, 483
Szalay A.S., Ellis R.S., Koo D.C. & Broadhurst T.J. 1991, in Proc. After the First Three Minutes, ed. S. Holt, C. Bennett & V. Trimble (New York: AIP)
Weiss A. & Buchert T. 1993, Astron. Astrophys. 274,1
**Figure Caption**

**Fig. 1.**
Cumulative probability distribution for the Gaussian model (dashed straight line), for the non-Gaussian models with $\gamma_2 = 5, 20, 40$ (solid lines, bottom to top), and for the BEKS data (filled squares). The vertical long-dashed line marks the peak-to-noise ratio of BEKS, $X = 11.8$.

**Fig. 2.**
Probability to find a peak as high as, or higher than, the $128h^{-1}$ Mpc peak of BEKS as a function of the crucial parameter $\gamma_2$. The long-dashed curve is for a value of $X$ ten percent lesser than $X_{BEKS} = 11.8$. The horizontal dashed lines show the rejection levels of 99.7 %.