Do Arnold Tongues really constitute a Fractal Set?

Anderson Luis Gama
Mario Sergio Teixeira de Freitas
Paraná Federal University of Technology: Av. Sete de Setembro, 3165 - Rebouças - CEP 80230-901 - Curitiba - Paraná - Brazil
E-mail: gamaanderson@uol.com.br
msergio58@yahoo.com.br

Abstract. We review and analyze the main features of fractal sets, as well as argue that the geometric property called fractality is not a well defined concept in the literature. As an example, we consider the mode locking phenomenon exhibited by the Sine Circle Map, concerning those sets in parameter space called Arnold Tongues and Devil’s Staircase (more exactly, their complements). It is shown that well-known results for the fat fractal exponent of the ergodic region turn out to be valid only in a tiny region of parameter space where $\rho \in [0, a]$, $a \to 0$, being $\rho$ the winding number. A careful geometric analysis shows that a misleading simplification has led to a loss of information that hindered relevant conclusions. We propose an alternative, broader and more rigorous approach in that it selects a generic interval from the whole domain which excludes the mentioned restricted region. Our results reveal that the measure shows a different dependence on tongue widths, so we argue that such a discrepancy is only possible if we assume that the set in question does not satisfy a scale-free property. Since there is no invariant exponent for the complementary set of Arnold Tongues (and consequently for that of Devil’s Staircase), we conclude that the exponent found by Ecke et al. [4] does not characterize those sets as true fractals. We also consider those sets as an example that statistical criteria are insufficient to characterize any set as being a fractal; they should always follow an analysis of the topological process responsible for generating the set.

Keywords: Arnold Tongues, Fat fractal, Exponent, Ergodic Region.

1. Introduction
Geometrical features of phase space are a powerful tool for representing the behavior of dynamical systems, mainly since the concept of fractal sets was introduced by Mandelbrot in 1975 [1]. Though fractal geometry is strongly connected to nonlinear sciences, and power laws are ubiquitously related to its statistical properties, up to now there is no criterion in the bibliography, that could be considered as a formal definition for categorizing a geometrical set as being or not a fractal and, consequently, that leads to the risk of abusive fractal characterization of every apparently irregular set [2]. In this article, we provide an example of such a critical situation through a careful analysis of the particular case of Arnold Tongues.

The name ‘Arnold Tongues’ is normally applied to the parameter space of the dynamical system concerning the mode locking phenomenon, which occurs when a nonlinear oscillator is forced by another one; as the nonlinear parameter increases, periodic behavior can also be generated by some incommensurable frequencies [3]. The most complete bibliographic source on quantitative relations of the forementioned system dates from 1989 [4], where the authors have concluded that the complement of Arnold Tongues that is present in the Sine Map, namely, the
ergodic region, is a fat fractal set. Such conclusion have also appeared later in some books [3], [5], and papers [6], [7], [8], [9].

Our main goal in this paper is to emphasize the need of a more careful analysis prior to labeling a set as a fractal, taking the Arnold Tongues as an example of a set which, we believe, is inappropriately characterized as a fractal. Falconer has already pointed out to such lack of criterion that concerns not only ‘fractals’, but also the term ‘fractal dimension’ [2]. It is clear that nonlinear sciences ought to have a more rigorous definition of the concept of fractal in order to properly develop. In our previous works, we have already investigated the recursive geometrical processes that are involved in the formation of some particular fractal sets [10], [11]. We suggest that a possible criterion for characterizing fractals could be based on the nature of such processes. However, such question must be extensively discussed, and it would be unrealistic to expect it to be solved in one single article.

In section 2, we review some fundamental concepts and reproduce some well-known statistical results, in order to single out some features of the Arnold Tongues that are unexpected for a typical fractal set. Then, in section 3, we reanalyze [4], and we find out that a subtle misleading analytical procedure made it possible to obtain the non-representative $2/3$ exponent. Our results and conclusions are discussed in section 4.

2. Brief review about fat fractal sets and Arnold Tongues

Instead of introducing general basic notions about fractals, in this section we discuss key concepts that normally are not paid enough attention in the bibliography, regarding whether some geometrical set must be considered or not as a fractal.

Fractal is a recent discovered class of geometrical sets that violates some axioms and theorems of Euclidean Geometry [1]. The lack of a formal definition of fractals lead them to be analyzed through a comparison to some classic sets that are postulated as fractals, based on their geometric structure and also on some statistical properties, most of them involving power laws.

One of the most traditional fractals is the Cantor set [1], which is a recursive process in which one starts from one unit length segment and, upon successive middle-interval subtractions, ends up with a limit set of null measure. The fat fractal version of the Cantor set [2] has a positive Lebesgue measure and an integer dimension. In order to generate a fat Cantor set, one should remove the same sequence of intervals from a set that is otherwise larger than unity; as a result, the length of the final set will be the difference between the two initial intervals [12]. The intervals size, relatively to the remaining length in each iteration, will vanish much faster than that of the standard Cantor Set. Figure 1a shows the Cantor set formation, and Figure 1b its fat fractal counterpart.

It also is appropriate to briefly review what defines the multifractal version of the Cantor set, which can be either thin or fat: the removed intervals are not necessarily selected from the middle of the former iterated set; so, the two remaining intervals are not equal, although their size still vanish exponentially with the number of iterations, just like a standard Cantor set (Figure 1c).

We stress here that, along the iterations, both the decrease of size and increase of number of intervals follow an exponential law, although the Capacity Dimension, that is used to characterize thin fractals, is based on a simple power law [2], [1]

$$N(\varepsilon) = C\varepsilon^{-d}. \quad (1)$$

In equation 1, $N(\varepsilon)$ is the smallest number of intervals smaller than $\varepsilon$ needed to cover the set,

\footnote{Unfortunately the names Arnold Tongues and Devil’s Staircase [1] may also refer to those fractals that represent the Arnold Tongues and Devil’s Staircase obtained from physical systems. However, their geometry is completely different, and for that reason, they are true fractals.}
Figure 1. The recursive formation processes of the Cantor set (a), of the fat Cantor set (b), and of the multifractal Cantor set (c). In all cases, we start from a straight segment.

The exponent $d$ is the Capacity Dimension (in the case of the Cantor set we have $d = \log_3 2$), and $C$ is a scale-dependent positive constant.

Concerning the Capacity Dimension, it provides little relevant information about the fractal itself, since it is equals 1 for all one-dimensional fat fractals. For that reason, other kinds of measure that are valid for fat fractals have been established, and in this paper we will choose the coarse-grained measure [4]

$$
\mu(\varepsilon) = \mu(0) + A\varepsilon^\beta,
$$

where $\mu(\varepsilon)$ is the length of the set in which all discontinuities smaller than $\varepsilon$ are neglected; $\beta$ is the fat fractal exponent, and $A$ is a scale-depending positive constant. For thin fractals as well as for the original Cantor set, $\mu(0) = 0$, and it is easily demonstrated that

$$
\mu(\varepsilon) = A\varepsilon^{D-d}.
$$

Next, we review the structure of Arnold Tongues, stressing that, up to now, there have been no exact criterion for characterizing a fractal set, and most characteristics are not universal [2]. Following this short review, we reproduce and discuss some results obtained by Ecke [4], that will be relevant in next section.

Circle maps, generated by the Poincaré section of two coupled oscillators through a nonlinear term [4], [3], can be encountered in several areas, ranging from simple fixed points to highly chaotic motion. The particular Sine Map is given by

$$
X_{t+1} = X_t - (k/2\pi) \sin(2\pi X_t) + \Omega,
$$

where $k$ is the nonlinearity parameter and $\Omega$ is the frequency of the first oscillator, being the second frequency unitary. For $0 \leq k \leq 1$, the ubiquitous non-chaotic motion is characterized by the winding number

$$
\rho = \lim_{t \to \infty} \frac{X_t - X_0}{t}.
$$

For periodic behavior $\rho = p/q$, where $p$ and $q$ are the smallest integers that respect

$$
X_{t+q} = X_t + p.
$$

Each Arnold tongue is the set of all pairs $(\Omega, k)$ sharing the same pair $(p, q)$. Figure 2 depicts the set of the largest tongues in $\Omega \leq 1/2$. The Devil’s Staircase may help to visualize the whole parameter space.
Figure 2. a) The set of the largest Arnold tongues of the Sine Map is shown in black and the Complement, that is the Ergodic Region, in white. In higher resolution there would appear tiny black tongues inside every white region, but otherwise no tiny white region inside black ones. On the top of each tongue, the corresponding winding number \( p/q \) is indicated. b) The complete Devil’s Staircase represents the graph of the winding number \( p/q \) in function of \( \Omega \) for a specific \( k \), in this case \( k = 1 \). What seems to be a continuous line is in fact several smaller and smaller steps [5].

Based on extensive numerical simulations, Ecke et al. have found some power laws concerning dependency of tongues’ widths (\( \Delta \Omega \)) on periods \( (p \text{ and } q) \) for several values of nonlinearity parameter \( (k) \), the most important being [4]

\[
\lim_{k \to 1} \lim_{q \to \infty} \Delta \Omega = C \cdot q^{-3},
\]

where \( C \) is a proportionality constant that varies as a rather complicated and non-monotonic decreasing way with \( p \) [4]. After some approximations, the authors have established the \textit{ad hoc} assumption, which is supposed to be valid for large values of \( p \) and \( q \)

\[
\Delta \Omega = Aq^{-3}p^{-\alpha},
\]

where \( A \) and \( \alpha \) are positive constants, and \( \alpha \gg 3 \), based on data evidence. However, Ecke [4] could not ensure that an invariant \( \alpha \) would be established for large values of \( q \). Nevertheless, the statistical tool \( p^{-\alpha} \) will be essential in determining the fat fractal exponent. It is also interesting to point out that the Arnold Tongues are ruled by power laws, differently from fractals, which display exponential laws [1].

In order to evidence some measuring problems that Ecke and his collaborators [4] have faced regarding geometrical features of the Arnold Tongues, we reproduce in Figures 3 and 4, with the authors’ caption, two of their numerical results, where data do exhibit strong fluctuations. The fitting to black circles, in the second figure, provides an exponent that has been estimated to be \( 0.68 \pm 0.05 \), contrasting with \( 2/3 \), that was analytically obtained for \( 0.4 < k < 0.9 \).

In next section, we develop a broader analytical method to investigate the validity of a fat fractal exponent to the complement of Arnold Tongues.

\[ \text{2 Originally, Ecke [4] has adopted the symbol } \bar{p} \text{ instead of } p, \text{ representing the sequence of largest } \Delta \Omega. \text{ But it is particularly important that the differences between } \bar{p} \text{ and } p \text{ should not be significant, which is the case for most values of } k \text{ (more precisely, for } k < 0.9); \text{ so we decided to keep only } p. \]
Figure 3. The distribution of interval sizes at fixed period $q$. At fixed $k = 0.8$ we compute all the periodic intervals with a given $q$, and sort them in order according to their size. The order is labeled by an index $\bar{p} = 1, 2, \ldots, \bar{P}_{MAX}$. $\Delta \Omega$ is plotted against $\bar{p}$ for several different values of $q$ (extracted from [4]).

Figure 4. An illustration of the convergence of the computed value of the fat-fractal scaling exponent $\beta$ with the scale of resolution $\varepsilon$. The numerical data of figure 3 (not shown in this paper) are numerically differentiated. The resulting slopes are plotted against the corresponding values of $\log \varepsilon$, to give an idea of the rate of convergence. We show two cases. At $k = 0.80(\bullet)$ the convergence is within experimental error at the smallest values of $\varepsilon$. At $k = 0.95(\circ)$, however, due to crossover phenomena there is no convergence, even at the smallest value of $\varepsilon$ that we were able to compute (extracted from [4]).

3. Algebraic Procedures for the Fat Fractal Exponent
It is convenient here, before presenting our results, to review how Ecke and his collaborators [4] have obtained the value $2/3$ for the fat fractal exponent. Though all approximate procedures seem to be reasonable, we have detected a paradox: while we were developing a broader and
more exact model, we obtained our unexpected result for a fractal. That was due to some characteristics of a very strict interval in parameter space, which prevail over those of the whole set and masks the true hierarchic structure of the tongues; we will show that, if any generic segment of the set is sampled, an exponent different from $2/3$ is obtained. As a result, we conclude that the authors’ approximate method has hindered a reliable diagnosis, in other words, we will demonstrate that the ergodic region does not scale as a fat fractal exponent.

Ecke et al. [4] started considering the length of the ergodic region. For a particular value of $k$, such length equals the total interval of parameter $\Omega$, minus the summation of the widths of all tongues; so, they conclude that

$$\mu(\varepsilon) - \mu(0) = \sum_{\Delta\Omega < \varepsilon} \Delta\Omega(q, p) = \sum_{\Delta\Omega < \varepsilon} Aq^{-3}p^{-\alpha}. \quad (9)$$

For methodological reasons, the authors separate out the sum of equation 9 in two parts: in Figure 5, values of $p$ inside region I hold for values of $q$ that are greater than the critical curve $Q(p)$ in which $\Delta\Omega = \varepsilon$; in region II, otherwise, values of $p$ hold for every $q$ that is greater than $p$. The boundary value of $p$ between regions I and II is called $p'$. The authors chose the value $\alpha = 0.1$.

**Figure 5.** Sketch of the pairs $(p, q)$ corresponding to the summation in equation 9 [4]. $Q(p)$ represents the critical curve where $\Delta\Omega = \varepsilon$, and $p'$ is the value for which $Q(p') = p'$. Just for a better qualitative visualization, the authors chose the value $\alpha = 0.1$.

Summation I is limited by lines $q = Q(p)$ and $p = p'$. The scalings of $Q(p)$ and of $p'$ are obtained from equation 8

$$\Delta\Omega(Q(p), p) = \varepsilon \Leftrightarrow Q(p) = (\varepsilon \cdot p^\alpha / A)^{-1/3}, \quad (10)$$

$$Q(p') = p' \Leftrightarrow p' = (\varepsilon / A)^{-1/(3+\alpha)}. \quad (11)$$

Summing over each region, equation 9 becomes like

$$\mu(\varepsilon) - \mu(0) = \sum_{1 \leq p < p'} Aq^{-3}p^{-\alpha} + \sum_{p' \leq p < q} Aq^{-3}p^{-\alpha}. \quad (12)$$
In order to solve the above equation, variables $p$ and $q$ are interpreted as being continuous, so the summations are written as integrals

$$\mu(\varepsilon) - \mu(0) = \int_1^{p'} \int_{Q(p)}^{\infty} Aq^{-3}p^{-\alpha} \, dq \, dp + \int_0^{p'} \int_p^{\infty} Aq^{-3}p^{-\alpha} \, dq \, dp.$$

Using equation 10 for $Q(p)$ and equation 11 for $p'$, the integrals are straightforward, resulting in

$$\mu(\varepsilon) - \mu(0) = C\varepsilon^{2/3} + \tilde{C}\varepsilon^{(\alpha+1)/(\alpha+3)},$$

where $C$ and $\tilde{C}$ are constants. In fact, the approximation of equation 12 by 13 represents a lower bound and the upper bound can be found replacing $p$ by $p-1$ and $q$ by $q-1$ in equation 13. Ecke shows that this implies the same equation 14 with only different constants [4]. As a result, he argues $\mu(\varepsilon) - \mu(0)$ follows the same scaling of equation 14.

The key point is that, since $\alpha \gg 3$, we have $\lim_{\varepsilon \to 0} \mu(\varepsilon) - \mu(0) = C\varepsilon^{2/3}$; from that, Ecke and his collaborators have concluded that the ergodic region scales as a power law with a fat fractal exponent that equals 2/3 for the subcritical regime [4].

We now present our approach. Although the methodology is similar, an important difference lies in the restriction of $p/q$ by the variables $a$ and $b$.

Although Ecke’s result seems a reasonable one, a more careful investigation reveals that the measure value is not scale invariant. In order to make evident the misleading step, we follow all of the steps that have been described previously, adopting a more general analysis.

Although Ecke’s result seems a reasonable one, a more careful investigation reveals that the measure value is not scale invariant. In order to make evident the misleading step, we follow all of the steps that have been described previously, adopting a more general analysis.

In what follows, we apply the methodological procedure of separating equation 15 into two summations, corresponding to regions III and IV in Figure 6, which is analogous to the qualitative graph of Figure 5, originally presented by Ecke et al. [4]. Figure 6 is our original result, and it will lead us to the main conclusion.

Summation III is limited by $p=p''$ and by $p=p'''$. According to equation 8, $p''$ and $p'''$ scale as

$$p'' = \frac{\varepsilon}{a^3 A},$$

$$p''' = \frac{\varepsilon}{b^3 A}.$$

So, analogously to equations 12 and 13 we have

$$\mu(\varepsilon) - \mu(0) = \sum_{1 \leq p' < p < p'' \atop Q(p) < q < p/a} Aq^{-3}p^{-\alpha} + \sum_{p'' \leq p < p''' \atop p/b < q < p/a} Aq^{-3}p^{-\alpha},$$

$$\mu(\varepsilon) - \mu(0) = \int_{p''}^{p''' \atop Q(p)} \int_{Q(p)}^{\infty} Aq^{-3}p^{-\alpha} \, dq \, dp + \int_{p''}^{p''' \atop p/b} \int_p^{\infty} Aq^{-3}p^{-\alpha} \, dq \, dp.$$

Solving the above integrals, we obtain

$$\mu(\varepsilon) - \mu(0) = Ce^{(\alpha+1)/(\alpha+3)},$$
Figure 6. Representation of the pairs \((p, q)\) that are summed in equation 15. \(Q(p)\) represents the critical curve for which \(\Delta\Omega = \varepsilon\), and \(p\), \(p'\) and \(p''\) represent the values for which \(Q(p') = p\), \(Q(p'') = p''/a\) and \(Q(p''') = p'''/b\). Differently from the other figures, this one is original from our work.

where \(C\) is some positive constant. Also here the approximation of equation 18 by 19 is a lower bound and since \(\Delta\Omega\) is monotonic an upper bound can be found replacing \(p\) by \(p - 1\) and \(q\) by \(q - 1\) in equation 18. By solving the resulting integral the same equation 20 is found, up to a constant. As a result we conclude that \(\mu(\varepsilon) - \mu(0)\) scales as \(\varepsilon^{(\alpha+1)/(\alpha+3)}\) for \(a < (p/q) < b\). Note that in our modified method the unitary lower limit does not appear in the first integral of the right hand side of equation 13. Solving the integral allows us to conclude that this particular change led the first term of the right hand side of equation 14 to vanish. As a result we find a different fat fractal exponent \(\beta = (\alpha + 1)/(\alpha + 3)\).

Furthermore, equations 14 and 20 do evidence a paradox: the complement of the Arnold Tongues in the interval \(\rho \in [a, b] \subset [0, 1)\) scales as the fat fractal exponent \((\alpha + 1)/(\alpha + 3) > 2/3\), while in the particular case of \(\rho \in [0, 1]\) the fat fractal exponent equals 2/3. We make clear that our method is somewhat limited: as \(a\) approaches zero, the method fails for \(p'' < 1\). As a result, we conclude that, close enough to zero, the intervals \([a, b]\) also have a fat fractal exponent equal to 2/3. Therefore, we also conclude that the 2/3 fat fractal exponent obtained by Ecke et al. [4] results from the tongues only with small winding numbers, being the remaining tongues too narrow, so that they were not considered. But, by removing the tongues with small winding numbers, our method shows a different fat fractal exponent.

This kind of behavior is not only unexpected for a fractal, it is not desirable either. Nevertheless, that is not accidental but rather a direct consequence from the rules that describe the system, mainly equation 8. Ecke himself stress that as the exponent 2/3 is not dependent on \(\alpha\), only the tongues with small \(p\) are relevant for the geometry of the set, for the other ones are just too narrow to affect the exponent [4]. But we remark that such argument involves a geometric consequence: all tongues with small \(p\) are located on the left side of the set, having \(\rho \to 0\). Though we agree that multifractals also have a concentration of large discontinuities on one of the set’s edge, they also have smaller concentration on other points, unlike the ergodic region we are analyzing. Equation 8 predicts that the size of discontinuities generally decreases

\(^3\) here small has an absolute meaning, not relative to \(q\)
from left to right, with no smaller concentration, and such a structure resembles a logarithmic accumulation, rather than a fractal [13], [14].

A possible solution for this situation would be to consider that the ergodic region excluding the neighborhood of \( \rho = 0 \) can be a fractal set with exponent equal to \( (\alpha + 1)/(\alpha + 3) \). However, Ecke and his collaborators themselves stress that the obtained exponent shows no dependence on \( \alpha \) [4]. On the other hand, our result is strongly influenced by the \( \Delta \Omega(p) \) dependence. Moreover, numerical data that we reproduce in Figure 3 show that not only \( \alpha \) increases with \( q \), but also that such increase is uniform. As a result, we argue that it would not be prudent to suppose \( (\alpha + 1)/(\alpha + 3) \) as the exponent’s value, before proving its own validity.

In the next section, we discuss the previous results, and we conclude that the complement of Arnold Tongues, and consequently the complement of the Devil’s Staircase, have no reliable relationship to any traditional fractal set, unless a new procedure for such an analysis leads to an invariant value of its exponent.

4. Discussion

In this article, we developed a careful algebraic analysis of properties of the parameter space of the Sine Map and showed that some current approximate approaches have led to a misinterpretation of the set known as Arnold Tongues as being a true fractal. We have shown that, although some well-known results are already widely mentioned in the literature concerning Arnold Tongues, such results are, unfortunately, not reliable.

Our main contribution is that, according to the analytical results that we have obtained through a broader algebraic method, the value of 2/3 for the fat fractal exponent corresponds strictly to the region where the winding number \( \rho \to 0 \), not being representative for the whole set. Furthermore, such exponent is not an invariant one.

We have also established, through geometrical arguments, that the accumulated structure near the edge that corresponds to small values of \( p/q \) in parameter space does not display the behavior that would be expected from a fractal set.

Consequently, we do not wish to replace the exponent of 2/3 by the remaining exponent, that is \( \beta = (\alpha + 1)/(\alpha + 3) \) in equation 20, as some kind of correction since such exponent turns out to be based on a heuristic assumption present in the literature, namely the \( \alpha \) in \( \Delta \Omega = Aq^{-3}p^{-\alpha} \). Such assumption was supposedly proved through some doubtful statistical evidences. We argue that, since it has not been possible to determine the value of \( \alpha \) and the only fact we know it that \( \alpha > 3 \).

In summary, since the complement of Arnold Tongues persists up to now without a valid exponent that characterizes it, we do not have a clear proof that it has any relationship to traditional fractals and multifractals, despite its apparent irregularity.

A thorough study of intricate geometrical sets deserves more rigorous criteria than arbitrarily comparing them to some so-called irregular objects or applying some indicative statistical measurements. The conclusion we have reached at this writing, concerning the lack of criterion for characterizing the Arnold Tongues as a fractal, refers to just one of the examples we are investigating [10], [11]. We expect that our methodology and results may contribute to stimulate more intensive research on fractal geometry, aiming at the establishment of a formal, rigorous definition of what a fractal is. We argue that such eventual definition should take into account the topological process that generates the set, rather than indicative statistical properties, such as the several different exponents called Fractal Dimensions, including, among others, Box-Counting Dimension, Correlation Dimension, Lyapunov Dimension, and Coarse-Grained Measure [2], [3], [5], [4].
Acknowledgments
The authors are grateful to Dr. Pedro Zambianchi from Department of Physics of UTFPR for his careful revision of our text.

References
[1] Mandelbrot B B 1983 *The Fractal Geometry of Nature* (New York: W.H. Freeman and Company)
[2] Falconer K 1990 *Fractal Geometry: Mathematical Foundations and Applications* (Chichester: John Wiley)
[3] Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Dynamics* (Berlin: Springer)
[4] Ecke R E, Farmer J D and Umbler D K Scaling of the Arnold tongues 1989 *Nonlinearity* 2 175-196
[5] Ott E 1993 *Chaos in Dynamical Systems* (Cambridge: Cambridge University Press)
[6] Paar V and Pavin N Intermingled fractal Arnold tongues 1998 *Phys. Rev. E* 57 1544-1549
[7] Paar V, Pavin N and Rosandi M Link between Truncated Fractals and Coupled Oscillators in Biological Systems 2001 *Journal of Theoretical Biology* 218 47-56
[8] Kennedy M P, Krieg K R and Chua L O The Devil’s Staircase: The Electrical Engineer’s Fractal 1989 *Circuits and Systems, IEEE Transactions on* 36 1113-1139
[9] Wenzel W and Biham O Periodic orbits in the dissipative standard map *Phys. Rev. A* 43 6550-6557
[10] Gama A L and de Freitas M S T Fractal Formation in the Circular Chaos Game: a Topological Investigation 2010 *International Journal of Bifurcation and Chaos* 20 703-713
[11] Gama A L and de Freitas M S T Fractal Formation Process in the Mandelbrot Set, submitted for the Journal Fractals
[12] Vicsek T 1992 *Fractal Growth Phenomena* (Singapore: World Scientific)
[13] McDonald S W, Grebogi C, Ott E and Yorke J A Fractal basin boundaries 1985 *Phys. D* 17 125-153
[14] de Freitas M S T, Viana R L and Grebogi C Multistability, basin boundary structure, and chaotic behavior in a suspension bridge model 2004 *International Journal of Bifurcation and Chaos* 14 927-950