ON THE ESSENTIAL SPECTRUM OF NADIRASHVILI-MARTIN-MORALES MINIMAL SURFACES

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ABSTRACT. We show that the spectrum of a complete submanifold properly immersed into a ball of a Riemannian manifold is discrete, provided the norm of the mean curvature vector is sufficiently small. In particular, the spectrum of a complete minimal surface properly immersed into a ball of $\mathbb{R}^3$ is discrete. This gives a positive answer to a question of Yau [22].

1. Introduction

An interesting problem in the Geometry of the Laplacian is to understand the relations of the geometry of a Riemannian manifold and its spectrum. For instance, to understand the restrictions on the geometry of a Riemannian manifold implying that its spectrum is purely continuous or discrete. There are several important work along these lines. See [5], [6], [8], [11], [19], [21] for geometric conditions implying that the spectrum is purely continuous and [2], [7], [9], [12], [13] for geometric conditions implying that the spectrum is discrete.

Since every complete Riemannian $m$-manifold can be realized as a complete submanifold embedded into a ball of radius $r$ of an $n$-dimensional Euclidean space, with $n$ depending only on $m$, see [18], it would be important to understand the relations between the spectrum and the extrinsic geometry of bounded embeddings of complete Riemannian manifolds in Euclidean spaces. A particularly interesting aspect of this problem is the spectrum related part of the so called Calabi-Yau conjectures on minimal surfaces.

Yau in his 2000 millennium lectures [22], [23], revisiting these conjectures, wrote: It is known [17] that there are complete minimal surfaces properly immersed into a [open] ball. . . . Are their spectrum discrete? It is worthwhile to point out that it is not clear that the Nadirashvili’s complete bounded minimal surface [17] is properly immersed. However, in [15], [16], F. Martin and S. Morales constructed, for any open convex subset $B$ of $\mathbb{R}^3$, a complete proper minimal immersions $\varphi : \mathbb{D} \hookrightarrow B$, where $\mathbb{D}$ is the standard disk on $\mathbb{R}^2$. The Martin-Morales’ method is a highly non-trivial refinement of Nadirashvili’s method, thus we name, (as we should), these complete properly immersed minimal surfaces into convex subsets $B$ of $\mathbb{R}^3$ as Nadirashvili-Martin-Morales minimal surfaces.

The purpose of this paper is to answer positively Yau’s question. In fact, we show as a particular case of our main result that the spectrum of any Nadirashvili-Martin-Morales minimal surface is discrete if the convex set $B$ is a ball $B_{\mathbb{R}^3}(r)$ of $\mathbb{R}^3$. We prove the following.

Theorem 1.1. Let $\varphi : M \hookrightarrow B_{\mathbb{R}^3}(r) \subset \mathbb{R}^3$ be a complete surface, properly immersed into a ball. If the norm of the mean curvature vector $H$ of $M$ satisfies

$$\sup_M |H| < 2/r$$

then $M$ has discrete spectrum.
Our main result Theorem 1.2 is a natural generalization of Theorem 1.1. It shows that the spectrum of a complete properly immersed submanifold $\varphi: M \hookrightarrow B_N(r)$ is discrete provided the norm of the mean curvature vector $H = \text{Tr}\alpha$ is sufficiently small. Here $B_N(r) \subset N$ is a normal geodesic ball of radius $r$ of a Riemannian manifold $N$ and $\alpha$ is the second fundamental form. In the following we denote
\begin{align}
C_b(t) = \begin{cases}
\sqrt{b}\cot(\sqrt{b}t) & \text{if } b > 0, \\
1/t & \text{if } b = 0, \\
\sqrt{-b}\coth(\sqrt{-b}t) & \text{if } b < 0.
\end{cases}
\end{align}

**Theorem 1.2.** Let $\varphi: M \hookrightarrow B_N(r)$ be a complete $m$-submanifold properly immersed into a geodesic ball, centered at $p$ with radius $r$, of a Riemannian $n$-manifold $N$. Let $b = \sup K^\text{rad}_N$ where $K^\text{rad}_N$ are the radial sectional curvatures along the geodesics issuing from $p$. Assume that $r < \min\{\text{inj}_N(p), \pi/2\sqrt{b}\}$, where $\pi/2\sqrt{b} = +\infty$ if $b \leq 0$. If the norm of the mean curvature vector $H$ satisfies,
\[
\sup_M |H| < m \cdot C_b(r),
\]
then $M$ has discrete spectrum.

The properness condition is a marginal technical hypothesis in Theorem 1.2. It is used only to choose a natural sequence of compact subsets of $M$ so that we can construct a sequence of positive smooth functions on their complements. The result should hold without it.

Isabel Salavessa in a beautiful paper [20], generalized Theorem 1.2 in the minimal case proving discreteness of the spectrum of $X$-bounded minimal submanifolds of Riemannian manifolds carrying strongly convex vector field $X$.

A Riemannian manifold $M$ is said to be **stochastically complete** if for some (and therefore, for any) $(x, t) \in M \times (0, +\infty)$ it holds that
\[
\int_M p(x, y, t) dy = 1,
\]
where $p(x, y, t)$ is the heat kernel of the Laplacian operator. Otherwise, the manifold $M$ is said to be **stochastically incomplete** (for further details about this see, for instance, [21]). It seems to have a close relation between discreteness of the spectrum of a complete noncompact Riemannian manifold and stochastic incompleteness. For instance, it was proved in [19] that submanifolds satisfying the hypotheses of Theorem 1.2, (without the properness condition) are stochastically incomplete. M. Harmer [9], shows that stochastic incompleteness implies discreteness of the spectrum in a certain class of Riemannian manifolds. Based on these evidences, we believe that the following conjecture should be true.

**Conjecture 1.3.** A complete noncompact Riemannian manifold has discrete spectrum if and only if it is stochastically incomplete.

### 2. Preliminaries.

Let $M$ be a complete noncompact Riemannian manifold. The Laplacian $\Delta$ acting on $C^\infty_0(M)$ has a unique self-adjoint extension to an unbounded operator acting on $L^2(M)$, also denoted by $\Delta$, whose domain are those functions $f \in L^2(M)$ such that $\Delta f \in L^2(M)$ and whose spectrum $\Sigma(M) \subset [0, \infty)$ decomposes as $\Sigma(M) = \Sigma_p(M) \cup \Sigma_{ess}(M)$ where $\Sigma_p(M)$ is formed by eigenvalues with finite multiplicity and $\Sigma_{ess}(M)$ is formed by accumulation points of the spectrum and by the eigenvalues with infinite multiplicity. It is said that $M$ has discrete spectrum if $\Sigma_{ess}(M) = \emptyset$ and that $M$ has purely continuous spectrum if $\Sigma_p(M) = \emptyset$.

If $K \subset M$ is a compact manifold with boundary, of the same dimension as $M$ then there is a self-adjoint extension $\Delta'$ of the Laplacian $\Delta$ of $M \setminus K$ by imposing Dirichlet conditions.
The *Decomposition Principle* \[7\] says that \(\triangle\) and \(\triangle'\) have the same essential spectrum \(\Sigma_{ess}(M) = \Sigma_{ess}(M \setminus K)\). On the other hand, the bottom of the spectrum of \(M \setminus K\) is equal to the fundamental tone of \(M \setminus K\), i.e. \(\inf \Sigma(M \setminus K) = \lambda^*(M \setminus K)\), where

\[
\lambda^*(M \setminus K) = \inf \left\{ \frac{\int_{M \setminus K} |\nabla f|^2}{\int_{M \setminus K} f^2}, f \in C_0^\infty(M \setminus K) \setminus \{0\} \right\}.
\]

To give lower estimates for \(\lambda^*(M \setminus K)\) we need of the following version of Barta’s Theorem.

**Theorem 2.1** (Barta, \[3\]). Let \(\Omega \subset M\) be an open subset of a Riemannian manifold \(M\) and let \(f \in C^2(\Omega), f|\Omega > 0.\) Then

\[
\lambda^*(\Omega) \geq \inf_{\Omega} (-\Delta f/f).
\]

**Proof:** Let \(X = -\nabla \log f\) be a \(C^1\) vector field in \(\Omega\). It was proved in \[4\] that

\[
\lambda^*(\Omega) \geq \inf_{\Omega} (\text{div } X - |X|^2) = \inf_{\Omega} (-\frac{\Delta f}{f}).
\]

The second main ingredient of our proof is the Hessian comparison theorem.

**Theorem 2.2.** Let \(M^m\) be a Riemannian manifold and \(x_0, x_1 \in M\) be such that there is a minimizing unit speed geodesic \(\gamma\) joining \(x_0\) and \(x_1\) and let \(p(x) = \text{dist}(x_0, x)\) be the distance function to \(x_0\). Let \(a \leq K, b \leq b\) be the radial sectional curvatures of \(M\) along \(\gamma\). If \(b > 0\) assume \(p(x_1) < \pi/2\sqrt{b}\). Then, we have \(\text{Hess}(\gamma') = 0\) and

\[
C_b(\rho(x))\|X\|^2 \geq \text{Hess}(\gamma)(X, X) \geq C_b(\rho(x))\|X\|^2,
\]

where \(X \in T_x M\) is perpendicular to \(\gamma'(\rho(x))\).

Let \(\varphi : M \hookrightarrow W\) be an isometric immersion of a complete Riemannian \(m\)-manifold \(M\) into a Riemannian \(n\)-manifold \(W\) with second fundamental form \(\alpha\). Consider a \(C^2\)-function \(g : W \to \mathbb{R}\) and the composition \(f = g \circ \varphi : M \to \mathbb{R}\). Identifying \(X\) with \(d\varphi(X)\) we have at \(q \in M\) that the Hessian of \(f\) is given by

\[
\text{Hess}(\varphi(q))(X,Y) = \text{Hess}(g(q))(X,Y) + \langle \nabla g, \alpha(X,Y) \rangle_{\varphi(q)}.
\]

Taking the trace in \(\text{(2.3)}\), with respect to an orthonormal basis \(\{e_1, \ldots, e_m\}\) for \(T_q M\), we have the Laplacian of \(f\),

\[
\Delta f(q) = \sum_{i=1}^m \text{Hess}(\varphi(q))(e_i, e_i) + \langle \nabla g, \sum_{i=1}^m \alpha(e_i, e_i) \rangle.
\]

The formulas \(\text{(2.3)}\) and \(\text{(2.4)}\) are well known in the literature, see \[10\].

3. **Proof of Theorem 1.2**

Let \(K_1 \subset K_2 \subset \cdots\) be an exhaustion sequence of \(M\) by compact sets. The *Decomposition Principle* states that \(M\) and \(M \setminus K_i\) have the same essential spectrum, \(\Sigma_{ess}(M) = \Sigma_{ess}(M \setminus K_i)\). Therefore, the **Theorem 1.2** is proved if we show that \(\lim_{i \to \infty} \lambda^*(M \setminus K_i) = \infty\) since \(\lambda^*(M \setminus K_i) \leq \inf \Sigma_{ess}(M \setminus K_i)\).

By hypothesis we have a complete \(m\)-submanifold \(\varphi : M \hookrightarrow B_N(r)\) properly immersed into a ball \(B_N(r) = B_N(p, r)\) with center at \(p\) and radius \(r\) in a Riemannian \(n\)-manifold \(N\) with radial sectional curvatures \(K_r\) along the radial geodesics issuing from \(p\) bounded as \(a = \inf K_r \leq K_r = b = \sup K_r\) in \(B_N(r)\), where \(r < \min\{\text{inf}_N(p), \pi/2\sqrt{b}\}\). Here we replace \(\pi/2\sqrt{b}\) by \(+\infty\) if \(b \leq 0\).

Define a function \(v : B_N(p, r) \to \mathbb{R}\) by \(v(y) = \phi_a(\rho(y))\), where \(\phi_a : [0, r] \to \mathbb{R}\) given by

\[
\phi_a(t) = \begin{cases} 
\cos(\sqrt{a} t) - \cos(\sqrt{a} r) & \text{if } a > 0, t < \pi/2\sqrt{a}, \\
\frac{1}{2}t^2 - \frac{1}{2}a^2r & \text{if } a = 0, \\
\cosh(\sqrt{-a} r) - \cosh(\sqrt{-a} r) & \text{if } a < 0.
\end{cases}
\]
Observe that \( \phi(t) > 0 \) in \([0, r], \phi_a(r) = 0, \phi_a'(r) < 0 \) and \( \phi_a''(t) - C_a(t)\phi_a'(t) = 0 \) in \([0, r] \).

This function \( \phi_a \) we learned from Markvorsen \([14]\). Let \( f: M \to \mathbb{R} \) defined by \( f = v \circ \varphi \) and consider an exhaustion sequence of \( M \) by compact sets \( K_i = \varphi^{-1}(B_N(p, r_i)) \), where \( r_i < r, r_i \to r \). By Barta’s Theorem we have that \( \lambda^*(M \setminus K_i) \geq \inf_{M \setminus K_i} (-\frac{\Delta f}{f}) \).

Now by \((2.4)\) we have

\[
\Delta f(x) = \sum_{i=1}^{m} \text{Hess}_N v(\varphi(x))(e_i, e_i) + \langle \text{grad} v, \sum_{i=1}^{m} a(e_i, e_i) \rangle.
\]

The metric of \( N \) inside the normal geodesic ball \( B_N(p, r) \) can be written in polar coordinates as \( ds^2 = dt^2 + |A(t, \xi)|^2 d\xi^2 \), \( A(t, \xi) \) satisfies the Jacobi equation \( A'' + RA = 0 \) with initial conditions \( A(0, \xi) = 0, A'(0, \xi) = 1 \). We have at the point \( \varphi(x) \) an orthonormal basis \( \{\partial/\partial t, \partial/\partial \xi_1, \ldots, \partial/\partial \xi_{n-1}\} \) for \( T_{\varphi(x)}N \). We may choose an orthonormal basis for \( T_{\varphi(x)}N \) as \( e_1 = \langle e_1, \partial/\partial t \rangle \cdot \partial/\partial t + e_1^\perp \), where \( e_1^\perp \perp \partial/\partial t \) and \( \{e_2, \ldots, e_m\} \subset \{\partial/\partial \xi_1, \ldots, \partial/\partial \xi_{n-1}\} \).

Computing \( \text{Hess}_N v(\varphi(x))(e_i, e_i) \) we have

\[
\text{Hess}_N v(\varphi(x))(e_1, e_1) = [\phi''(t) - \phi'(t) \cdot \text{Hess}_N \rho(e_1^\perp/|e_1^\perp|, e_1^\perp/|e_1^\perp|)] \langle e_1, \text{grad} \rho \rangle^2
+ \phi'(t) \cdot \text{Hess}_N \rho(e_1^\perp/|e_1^\perp|, e_1^\perp/|e_1^\perp|)
\]

(3.2)

and for \( i \geq 2 \)

\[
\text{Hess}_N v(\varphi(x))(e_i, e_i) = \phi'(t) \cdot \text{Hess}_N \rho(e_i, e_i)
\]

where \( t = \rho(\varphi(x)) \). Now,

\[
- \Delta f = -\phi'(t) \cdot [C_a(t) - \text{Hess}_N \rho(e_1^\perp/|e_1^\perp|, e_1^\perp/|e_1^\perp|)] \langle e_1, \text{grad} \rho \rangle^2
- \phi'(t) \cdot \left[ \text{Hess}_N \rho(e_1^\perp/|e_1^\perp|, e_1^\perp/|e_1^\perp|) + \sum_{i=2}^{m} \text{Hess}_N \rho(e_i, e_i) \right]
- \phi'(t) \langle \text{grad} \rho, H \rangle
\]

\[
\geq -\phi'(t) \cdot [m \cdot C_b(t) - \sup |H|]
\]

We used that \( C_a(t) \geq \text{Hess}_N \rho(e_1^\perp/|e_1^\perp|, e_1^\perp/|e_1^\perp|) \geq C_b(t), \text{Hess}_N \rho(e_i, e_i) \geq C_b(t) \) by the Hessian Comparison Theorem and that \( -\phi'(t) > 0 \).

Hence

\[
\lambda^*(M \setminus K_i) \geq \inf_{M \setminus K_i} \left(-\frac{\Delta f}{f}\right) \geq \inf_{M \setminus K_i} \frac{\phi'(t)}{\phi_a(t)} [m \cdot C_b(t) - \sup |H|]
\]

(3.4)

Thus \( \lambda^*(M \setminus K_i) \to \infty \) as \( r_i \to r \). This proves Theorem \([12]\).

4. CYLINDRICALLY BOUNDED SUBMANIFOLDS.

Let \( \varphi: M^m \to B_N(r) \times \mathbb{R}^\ell \subset N^{n-\ell} \times \mathbb{R}^\ell \), \( m \geq \ell + 1 \), be an isometric immersion of a complete Riemannian \( m \)-manifold \( M^m \) into the \( B_N(r) \times \mathbb{R}^\ell \), where \( B_N(r) \) is a geodesic ball in a Riemannian \((n-\ell)\)-manifold \( N^{n-\ell} \), centered at a point \( p \) with radius \( r \). Let \( b = \sup K_N^a \)
where $K_N^\text{rad}$ are the radial sectional curvatures along the geodesics issuing from $p$. Assume that $r < \min\{\text{inj}_N(p), \pi/2\sqrt{b}\}$, where $\pi/2\sqrt{b} = +\infty$ if $b \leq 0$.

**Theorem 4.1.** Suppose that $\varphi : M^m \to B_N(r) \times \mathbb{R}^\ell$ as above satisfies the following.

1. For every $s < r$, the set $\varphi^{-1}(B_N(s) \times \mathbb{R}^\ell)$ is compact in $M$.
2. $\sup_M |H| < (m - \ell)C_b(r)$ where $|H|(x)$ is norm of the mean curvature vector of $\varphi(M)$ at $\varphi(x)$.

Then $M$ has discrete spectrum.

Observe that the condition 1. is a stronger property than being a proper immersion except when $\ell = 0$.

**Proof:** As before, let $r_i \to r$ be a sequence of positive real numbers $r_i < r$ and the compacts sets $K_i = \varphi^{-1}(B_N(r_i) \times \mathbb{R}^\ell)$. We need only to show that $\lambda^*(M \setminus K_i) \to \infty$ as $r_i \to r$. Define $v$ on $B_N(r) \times \mathbb{R}^\ell$ by $v(x, y) = \rho_0(\rho(x))$, where $\rho(x) = \text{dist}_N(p, x)$, $\rho_0$ given in \textbf{3.1} and $a = \inf K_N^\text{rad}$. Let $f = v \circ \varphi : M \to \mathbb{R}$. We have by \textbf{2.4}

\[
\triangle f(x) = \sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}^\ell} v(\varphi(x))(e_i, e_i) + \langle \text{grad} v, H \rangle.
\]

(4.1)

\[
= \sum_{i=1}^m \text{Hess}_N(\rho_0 \circ \rho)(\varphi(x))(e_i, e_i) + \langle \text{grad} (\rho_0 \circ \rho), H \rangle.
\]

At $\varphi(x) = (y_1, y_2)$, consider the orthonormal basis

\[
\text{Polar basis} \quad \{\partial/\partial t, \partial/\partial \xi_1, \ldots, \partial/\partial \xi_{n-\ell-1}, \partial/\partial s_1, \ldots, \partial/\partial s_\ell\}
\]

\[
\text{Cartesian basis} \quad \{e_1, e_2, \ldots, e_m\}
\]

for $T_{(y_1, y_2)}N^{n-\ell} \times \mathbb{R}^\ell = T_{y_1}N^{n-\ell} \oplus T_{y_2}\mathbb{R}^\ell$. Choose an orthonormal basis $\{e_1, e_2, \ldots, e_m\}$ as follows

\[
e_i = a_1 \frac{\partial}{\partial t} + \sum_{j=1}^{n-\ell-1} b_{ij} \frac{\partial}{\partial \xi_j} + \sum_{j=1}^{\ell} c_{ij} \frac{\partial}{\partial s_j}.
\]

Using that $\phi''_a(t) < 0$, $\phi'''_a = C_a(t)\phi''_a(t)$ and $\text{Hess}_N(\rho_0)(\partial/\partial \xi_j, \partial/\partial \xi_j) \geq C_b(t)$ for all $j = 1, \ldots, n - \ell - 1$, we have that

\[
\text{Hess}_N(\rho_0 \circ \rho(y_1))(e_i, e_i) = \phi''_a(t)a^2_1 + \phi''_a(t) \sum_{j=2}^{n-\ell-1} b^2_{ij} \text{Hess}_N(\rho_1)(\partial/\partial \xi_j, \partial/\partial \xi_j)
\]

\[
\leq \phi''_a(t) a^2_1 + \phi''_a(t) \sum_{j=2}^{n-\ell-1} b^2_{ij} C_b(t)
\]

\[
= C_a(t) \phi''_a(t) a^2_1 + \phi''_a(t) \sum_{j=2}^{n-\ell-1} b^2_{ij} C_b(t)
\]

\[
= \phi''_a(t) a^2_1 (C_a(t) - C_b(t)) + \phi''_a(t) \sum_{k=1}^{\ell} c^2_{ik} C_b(t)
\]

\[
\leq \phi''_a(t) (1 - \sum_{k=1}^{\ell} c^2_{ik}) C_b(t)
\]
since $C_a(t) \geq C_b(t)$ and where $t = \rho(y_1)$. Therefore

$$- \sum_{i=1}^{n} \text{Hess} \phi_a \circ \rho(y_1)(e_i, e_i) \geq -\phi'_a(t)(m - \sum_{i=1}^{m} \sum_{k=1}^{\ell} C_{ik}^2)C_b(t) \geq -\phi'_a(t)(m - \ell)C_b(t)$$

From this we have that

$$\frac{\Delta f}{f}(x) \geq \frac{\phi'_a(t)}{\phi_a(t)} \left[ (m - \ell)C_b(t) - \sup_{M} |H| \right]$$

so that

$$\inf_{M \setminus K_i} \left( -\frac{\Delta f}{f} \right) \geq \frac{\phi'_a(r_i)}{\phi_a(r_i)} \left[ (m - \ell)C_b(r) - \sup_{M} |H| \right] .$$

Therefore $\inf_{M \setminus K_i} \left( -\frac{\Delta f}{f} \right) \to +\infty$ as $r_i \to r$ proving Theorem (4.1).

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