Generalized Unitarity Relation for Linear Scattering Systems in One Dimension

Ali Mostafazadeh*
Departments of Mathematics and Physics, Koç University, 34450 Sariyer, Istanbul, Turkey

Abstract

We derive a generalized unitarity relation for an arbitrary linear scattering system that may violate unitarity, time-reversal invariance, $\mathcal{PT}$-symmetry, and transmission reciprocity.

1 Introduction

The scattering phenomenon defined by a real scattering potential $v(x)$ through the time-independent Schrödinger equation,

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x),$$

(1)

satisfies the unitarity relation:

$$|R_l^r(k)|^2 + |T_l^r(k)|^2 = 1,$$

(2)

where $R_l^r(k)$ and $T_l^r(k)$ are respectively left/right reflection and transmission amplitudes. The latter determine the asymptotic behavior of the scattering solutions of (1) according to

$$\psi_l(k,x) \rightarrow \left\{ \begin{array}{ll} \mathcal{N}_+(k)\left[e^{ikx} + R_l^l(k)e^{-ikx}\right] & \text{for } x \rightarrow -\infty, \\ \mathcal{N}_+^*(k)T_l^l(k)e^{ikx} & \text{for } x \rightarrow +\infty, \end{array} \right.$$  

(3)

$$\psi_r(k,x) \rightarrow \left\{ \begin{array}{ll} \mathcal{N}_-(k)T_r^r(k)e^{-ikx} & \text{for } x \rightarrow -\infty, \\ \mathcal{N}_-(k)\left[e^{-ikx} + R_r^l(k)e^{ikx}\right] & \text{for } x \rightarrow +\infty. \end{array} \right.$$  

(4)

These respectively correspond to scattering setups where left-/right-incident waves of amplitude $\mathcal{N}_+/-(k)$ are scattered by the potential $v(x)$.

For a real scattering potential, one can show that

$$|R_l^l(k)| = |R_r^r(k)|,$$

(5)

$$T_l^l(k) = T_r^r(k).$$

(6)

*E-mail address: amostafazadeh@ku.edu.tr
Therefore the unitarity relation takes the form

\[ |R^{l/r}(k)|^2 + |T(k)|^2 = 1, \]  

(7)

where \( T(k) \) stands for the common value of \( T^l(k) \) and \( T^r(k) \).

Reciprocity in transmission (6) turns out to be a universal feature of all real and complex scattering potentials [2, 3]. To see this we first recall that the Wronskian of any pair of solutions \( \psi_{1,2}(x) \) of (1), i.e.,

\[ W[\psi_1(x), \psi_2(x)] := \psi_1(x)\psi'_2(x) - \psi'_1(x)\psi_2(x), \]

is independent of \( x \). If we compute \( W[\psi_l(x), \psi_r(x)] \) for \( x \to -\infty \) and \( x \to +\infty \), we respectively find \( 2ik/T^l(k) \) and \( 2ik/T^r(k) \). The fact that these must be equal to the same constant implies (6) for \( k \neq 0 \). This is actually the one-dimensional realization of the celebrated reciprocity theorem which is for example proven for real potentials in Ref. [4].

Unlike (6), (5) is violated by generic complex scattering potentials. A striking demonstration of this fact is the existence of unidirectionally reflectionless complex potentials [5]. These are potentials whose reflection amplitudes fulfil either \( R^l(k) = 0 \neq R^r(k) \) or \( R^r(k) = 0 \neq R^l(k) \) for some \( k \in \mathbb{R}^+ \). It turns out that these conditions are invariant under the combined action of parity and time-reversal transformation (PT), where \( T\psi(x) := \psi(x)^* \) and \( P\psi(x) := \psi(-x) \) respectively define the parity and time-reversal transformations [6]. This in turn makes PT-symmetric potentials [7] the principal examples of unidirectionally reflectionless potentials. This together with the interesting properties of their spectral singularities [7] have made PT-symmetric scattering potentials a focus of intensive research activity during the past decade [8].

Among the outcomes of the research done in the subject is the discovery of the following generalization of the unitarity relation (7) for PT-symmetric potentials [9]:

\[ |T(k)|^2 \pm |R^l(k)R^r(k)| = 1. \]  

(8)

Another curious observation is that reflection and transmission amplitudes of PT-symmetric scattering potentials satisfy

\[ |R^l(-k)| = |R^r(k)|, \quad \quad \quad |T(-k)| = |T(k)|. \]  

(9)

These were initially conjectured in [10] based on evidence provided by the study of a complexified Scarf II potential. They were subsequently proven as immediate consequences of the following identities that hold for PT-symmetric scattering potentials [11]:

\[ R^{l/r}(-k) = -e^{2i\tau(k)}R^{r/l}(k), \quad \quad \quad T(-k) = T(k)^*, \]  

(10)

where \( e^{i\tau(k)} := T(k)/|T(k)| \). In view of the second of these equations, we can write the first in the form

\[ R^{l/r}(-k)T(-k) + R^{r/l}(k)T(k) = 0. \]  

(11)

These are potentials that satisfy \( v(-x)^* = v(x) \).
The analysis leading to the proof of (10) also reveals that the reflection and transmission amplitudes of both real and $\mathcal{PT}$-symmetric scattering potentials fulfil
\[ R^{l/r}(k)R^{l/r}(-k) + |T(k)|^2 = 1. \] (12)

It is not difficult to see that this reduces to (7) and (8) for real and $\mathcal{PT}$-symmetric potentials, respectively.

The purpose of the present article is to establish a generalization of (12) that holds for every linear scattering system, even those that are not defined by a local potential [11, 12].

2 General scattering systems in one dimension

Consider a wave equation in $1+1$ dimensions that admits time-harmonic solutions: $e^{-i\omega t}\psi(x)$, where $\psi : \mathbb{R} \to \mathbb{C}$ solves a time-independent wave equation,
\[ \mathcal{H}[\psi, x] = 0. \] (13)

This equation, which may be nonlocal or even nonlinear, defines a meaningful scattering phenomenon if for $x \to \pm\infty$ its solutions tend to those of
\[ -\psi''(x) = k^2\psi(x). \] (14)

In other words, solutions of (13) satisfy the asymptotic boundary conditions:
\[
\begin{align*}
\psi(x) &\to A_-(k)e^{ikx} + B_-(k)e^{-ikx} \quad \text{for} \ x \to -\infty, \\
\psi(x) &\to A_+(k)e^{ikx} + B_+(k)e^{-ikx} \quad \text{for} \ x \to +\infty,
\end{align*}
\] (15)-(16)

where $A_\pm$ and $B_\pm$ are complex-valued coefficient functions. We call the $2 \times 2$ matrices $M(k)$ and $S(k)$ satisfying
\[
\begin{align*}
M(k) \begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix} &= \begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix}, \\
S(k) \begin{bmatrix} A_-(k) \\ B_+(k) \end{bmatrix} &= \begin{bmatrix} A_+(k) \\ B_-(k) \end{bmatrix},
\end{align*}
\] (17)-(18)

the transfer and scattering matrices of the scattering system. If (13) is nonlinear, their entries, $M_{ij}(k)$ and $S_{ij}(k)$, are respectively nonlinear functions of $(A_-, B_-)$ and $(A_-, B_+)$. In the following we focus our attention to scattering phenomena defined by linear wave equations.

Because $(A_-, B_-)$ and $(A_+, B_+)$ determine the behavior of the solutions $\psi(x)$ at $x = -\infty$ and $x = +\infty$, the global existence and uniqueness of the solution of the initial-value problem defined by (13) and (15) implies that $M(k)$ is an invertible matrix, i.e.,
\[ \det M(k) \neq 0. \] (19)

\[ \text{A linear wave equation is an equation of the form (13) such that the linear combinations of its solutions are also solutions of this equation.} \]
Under this condition the scattering problem for the wave equation (13) is well-posed. We therefore assume that it holds true. The inverse of \( M(k) \) allows us to specify the asymptotic expression for the solutions of (13) at \( x = -\infty \) in terms of their asymptotic expression at \( x = +\infty \).

Let \( \psi_\pm(k, x) \) be the solutions of (13) that satisfy
\[
\psi_\pm(k, x) = e^{\pm ikx} \quad \text{for} \quad x \to \pm \infty.
\] (20)

Then Eq. (17) implies
\[
\psi_-(k, x) \to M_{22} e^{ikx} + M_{12} e^{-ikx} \quad \text{for} \quad x \to +\infty, \quad (21)
\]
\[
\psi_+(k, x) \to \frac{-M_{21} e^{ikx} + M_{22} e^{-ikx}}{\det M(k)} \quad \text{for} \quad x \to -\infty. \quad (22)
\]

\( \psi_\pm \) are called the Jost solutions of the wave equation (13). Comparing (20) – (22) with (3) and (4) and using the linearity of (13), we can respectively identify \( \psi_l(k, x) \) and \( \psi_r(k, x) \) with \( N_+ T_l(k) \psi_+(k, x) \) and \( N_- T_r(k) \psi_-(k, x) \). Furthermore, this identification implies
\[
M_{11}(k) = \frac{D(k)}{T_r(k)}, \quad M_{12}(k) = \frac{R_r'(k)}{T_r(k)}, \quad M_{21}(k) = -\frac{R_l'(k)}{T_r(k)}, \quad M_{22}(k) = \frac{1}{T_r(k)}, \quad (23)
\]
\[
R_l'(k) = -\frac{M_{21}(k)}{M_{22}(k)}, \quad T_l'(k) = \frac{\det M(k)}{M_{22}(k)}, \quad R_r'(k) = \frac{M_{12}(k)}{M_{22}(k)}, \quad T_r'(k) = \frac{1}{M_{22}(k)}, \quad (24)
\]
where
\[
D(k) := T_l'(k) T_r'(k) - R_l'(k) R_r'(k) = \frac{M_{11}(k)}{M_{22}(k)}. \quad (25)
\]

We can similarly relate the entries of the scattering matrix to the reflection and transmission amplitudes by enforcing (18) for the coefficient functions of the Jost solutions \( \psi_\pm(k, x) \). In view of (20) – (22), this gives
\[
S_{11}(k) = T_l'(k), \quad S_{12}(k) = R_r'(k), \quad S_{21}(k) = R_l'(k), \quad S_{22}(k) = T_l'(k). \quad (26)
\]
In particular,
\[
\det S(k) = D(k). \quad (27)
\]

The above-mentioned requirements on the global existence of the solutions of (13) that satisfy asymptotic boundary conditions (15), (16), and (20) restrict the wave operator \( \mathcal{W} \). For example if \( \mathcal{W} \) is the Schrödinger operator \( -\partial_x^2 + v(x) \) for a potential \( v : \mathbb{R} \to \mathbb{C} \), we can satisfy these requirements provided that \( v(x) \) fulfills the Faddeev condition (13):
\[
\int_{-\infty}^{\infty} (1 + |x|) |v(x)| dx < \infty. \quad (28)
\]
3 Generalized unitarity relation

Let us make the $k$-dependence of the solutions of the wave equation (13) explicit by using $\psi(k, x)$ in place of $\psi(x)$ in (13) and (13). Consider the implications of the transformations:

$$\psi(k, x) \xrightarrow{R} \tilde{\psi}(k, x) := (R\psi)(k, x) := \psi(-k, x),$$  \hspace{1cm} (29)
$$\psi(k, x) \xrightarrow{P} \bar{\psi}(k, x) := (P\psi)(k, x) := \psi(k, -x),$$  \hspace{1cm} (30)
$$\psi(k, x) \xrightarrow{T} \tilde{T}(k, x) := (T\psi)(k, x) := \psi(k, x)^*,$$  \hspace{1cm} (31)
$$\psi(k, x) \xrightarrow{PT} \tilde{\psi}(k, x) := (PT\psi)(k, x) := \psi(k, -x)^*. \hspace{1cm} \text{(32)}$$

It is not difficult to see that the transformed wave functions, $\tilde{\psi}(k, x), \bar{\psi}(k, x), \tilde{\psi}(k, x)$, and $\tilde{\psi}(k, x)$ also tend to plane waves at spatial infinities. Therefore they determine scattering phenomena.

By analogy to the definition of the transfer matrix $M(k)$ for $\psi(k, x)$, i.e., (17), we can introduce the transfer matrices for $\tilde{\psi}(k, x), \bar{\psi}(k, x), \tilde{\psi}(k, x)$, and $\bar{\psi}(k, x)$. We respectively label them by $M(-k), M(k), M(k), \text{ and } \bar{M}(k)$. In view of (29) – (31), we can show that:

$$M(-k) = \sigma_1 M(k) \sigma_1, \hspace{1cm} \tilde{M}(k) = \sigma_1 M(k)^{-1} \sigma_1, \hspace{1cm} \text{(33)}$$
$$\bar{M}(k) = \sigma_1 M(k)^* \sigma_1, \hspace{1cm} \bar{\tilde{M}}(k) = M(k)^{-1*}, \hspace{1cm} \text{(34)}$$

where $\sigma_1$ is the first Pauli matrix:

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Similarly we can introduce the reflection and transmission amplitudes for $\tilde{\psi}(k, x), \bar{\psi}(k, x), \tilde{\psi}(k, x), \text{ and } \bar{\psi}(k, x)$, which by virtue of their relationship to $M(-k), \tilde{M}(k), \bar{M}(k), \text{ and } \bar{\tilde{M}}(k)$ and Eqs. (33) and (34), take the form:

$$R^l(-k) = -\frac{R^r(k)}{D(k)}, \hspace{1cm} T^l(-k) = \frac{T^l(k)}{D(k)}, \hspace{1cm} R^r(-k) = -\frac{R^l(k)}{D(k)}, \hspace{1cm} T^r(-k) = \frac{T^r(k)}{D(k)}, \hspace{1cm} \text{(35)}$$

$$\tilde{R}^l(k) = R^r(k), \hspace{1cm} \tilde{T}^l(k) = T^r(k), \hspace{1cm} \tilde{R}^r(k) = R^l(k), \hspace{1cm} \tilde{T}^r(k) = T^l(k), \hspace{1cm} \text{(36)}$$

$$\bar{R}^l(k) = -\frac{R^r(k)^*}{D(k)^*}, \hspace{1cm} \bar{T}^l(k) = \frac{T^l(k)^*}{D(k)^*}, \hspace{1cm} \bar{R}^r(k) = -\frac{R^l(k)^*}{D(k)^*}, \hspace{1cm} \bar{T}^r(k) = \frac{T^r(k)^*}{D(k)^*}, \hspace{1cm} \text{(37)}$$

$$\tilde{\bar{R}}^l(k) = -\frac{R^r(k)^*}{D(k)^*}, \hspace{1cm} \tilde{\bar{T}}^l(k) = \frac{T^l(k)^*}{D(k)^*}, \hspace{1cm} \tilde{\bar{R}}^r(k) = -\frac{R^l(k)^*}{D(k)^*}, \hspace{1cm} \tilde{\bar{T}}^r(k) = \frac{T^r(k)^*}{D(k)^*}, \hspace{1cm} \text{(38)}$$

respectively.

Next, we invert (35) to express $R^r(k)$ and $T^r(k)$ in terms of $R^l(-k), T^l(-k), \text{ and } D(k)$. Substituting the result in (25), we find

$$D(k) \left[ T^r(-k) T^l(k) + R^r(-k) R^l(k) - 1 \right] = 0. \hspace{1cm} \text{(39)}$$

Similarly, we can solve (35) for $R^l(k)$ and $T^l(k)$ in terms of $R^r(-k), T^r(-k), \text{ and } D(k)$, and use (25) to establish:

$$D(k) \left[ T^l(-k) T^r(k) + R^l(-k) R^r(k) - 1 \right] = 0. \hspace{1cm} \text{(40)}$$
Equations (39) and (39) imply that whenever $D(k) \neq 0$,

$$T^{l/r}(-k)T^{r/l}(k) + R^{l/r}(-k)R^{l/r}(k) = 1.$$  (41)

This is a generalized unitarity relation that reduces to (12) whenever the scattering system has reciprocal transmission and $D(k) \neq 0$ for all $k \in \mathbb{R}^+$. Both of these conditions are satisfied for scattering systems determined by the Schrödinger equation for a local time-reversal invariant (real) or $\mathcal{PT}$-symmetric potential. According to the reciprocity theorem they have reciprocal transmission, and as we show in the sequel they satisfy $|D(k)| = 1$. To see this, first we note that according to (34) the transfer matrix for time-reversal-invariant and $\mathcal{PT}$-symmetric systems respectively fulfil

$$M(k)^* = \sigma_1 M \sigma_1,$$  (42)

$$M(k)^* = M(k)^{-1}.$$  (43)

We can use these equations to show that

- $\mathcal{T}$-symmetry $\Rightarrow$ $M_{11}(k)^* = M_{22}(k),$  (44)

- $\mathcal{PT}$-symmetry $\Rightarrow$ $M_{11}(k)^* = \frac{M_{22}(k)}{\det M(k)}$.  (45)

For time-reversal-invariant systems, Eqs. (25) and (44) imply:

$$|D(k)| = \left| \frac{M_{11}(k)}{M_{22}(k)} \right| = \left| \frac{M_{11}(k)^*}{M_{22}(k)^*} \right| = 1.$$  (46)

In light of (25) and (45), we also find the following result for $\mathcal{PT}$-symmetric scattering systems.

$$|D(k)| = \left| \frac{M_{11}(k)}{M_{22}(k)} \right| = \left| \frac{\det M(k)}{\det M(k)} \right| = \left| \frac{M_{22}(k)^*}{M_{11}(k)^*} \right| = \left| \frac{M_{22}(k)}{M_{11}(k)} \right| = \frac{1}{|D(k)|},$$  (47)

which means $|D(k)| = 1$.

Note that the proof of the identity $|D(k)| = 1$ we have just presented does not make use of the transmission reciprocity. Therefore it holds for every scattering system possessing time-reversal invariance or $\mathcal{PT}$-symmetry. In view of (40), it implies that the reflection and transmission amplitudes of these systems fulfill (41) for all $k \in \mathbb{R}^+$.

For scattering systems that are neither time-reversal-invariant nor $\mathcal{PT}$-symmetric, there may exist values of $k$ for which $D(k) = 0$, in which case (41) may be violated for these values of $k$. According to (27), these are the real and positive zeros $k_0$ of $\det S(k)$. Clearly $\det S(k_0) = 0$ means that $S(k_0)$ has a vanishing eigenvalue, i.e., there are complex numbers $A_{0-}$ and $B_{0+}$ such that

$$S(k_0) \begin{bmatrix} A_{0-} \\ B_{0+} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  (48)

3By definition, time-reversal-invariance and $\mathcal{PT}$-symmetry of a scattering system respectively mean that its reflection and transmission amplitudes, and consequently its transfer and scattering matrices are invariant under time-reversal and $\mathcal{PT}$ transformations.
In light of (15), (16), and (18), this equation proves the existence of a solution $\psi_{in}(k, x)$ of the wave equation that satisfies purely incoming asymptotic boundary conditions for $k = k_0$, i.e.,

$$
\psi_{in}(k_0, x) \rightarrow \begin{cases} 
A_0 e^{ik_0x} & \text{for } x \to -\infty, \\
B_0 e^{-ik_0x} & \text{for } x \to +\infty.
\end{cases}
$$

This solution describes a rather remarkable situation where the system absorbs a pair of incident waves traveling towards it in opposite directions. This phenomenon is called coherent perfect absorption or antilasing [14, 15, 16, 17, 18].

The above analysis shows that for every scattering system and $k \in \mathbb{R}^+$, either $k$ is a wavenumber at which the system acts as a coherent perfect absorber or its reflection and transmission amplitudes satisfy the generalized unitarity relation (11).

Let us conclude by noting that the term ‘generalized unitarity relation’ refers to the fact that for a real scattering potential where the wave operator is a Hermitian Schrödinger operator, this relation reduces to the unitarity relation (7). This follows from the reciprocity theorem and Eqs. (35) and (37), which for time-reversal-invariant systems imply

$$
R^{l/r}(-k) = R^{l/r}(k)^*, \quad T^{l/r}(-k) = T^{l/r}(k)^*.
$$

Acknowledgments

This work has been supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) in the framework of the project no: 114F357 and by Turkish Academy of Sciences (TÜBA).

References

[1] A. Mostafazadeh, Generalized unitarity and reciprocity relations for $\mathcal{PT}$-symmetric scattering potentials, J. Phys. A: Math. Theor. 47, 505303 (2014).

[2] Z. Ahmed, Schrödinger transmission through one-dimensional complex potentials, Phys. Rev. A 64, 042716 (2001)

[3] A. Mostafazadeh and H. Mehri-Delmavi, Spectral singularities, biorthonormal systems and a two-parameter family of complex point interactions, J. Phys. A: Math. Theor. 42, 125303 (2009).

[4] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Pergamon, Oxford, 1977.

[5] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, D. N. Christodoulides, Unidirectional invisibility induced by PT-symmetric periodic structures, Phys. Rev. Lett. 106, 213901 (2011).

[6] A. Mostafazadeh, Invisibility and $\mathcal{PT}$-symmetry, Phys. Rev. A 87, 012103 (2013).
[7] A. Mostafazadeh, Physics of Spectral Singularities, in Proceedings of XXXIII Workshop on Geometric Methods in Physics, held in Bialowieza, Poland, June 29-July 5, 2014, Trends in Mathematics, pp. 145-165, Springer International Publishing, Switzerland, 2015; preprint arXiv:1412.0454.

[8] V. V. Konotop, J. Yang, and D. A. Zezyulin, Nonlinear waves in PT-symmetric systems, Rev. Mod. Phys. 88, 035002 (2016).

[9] L. Ge, Y. D. Chong, and A. D. Stone, Conservation relations and anisotropic transmission resonances in one-dimensional $\mathcal{PT}$-symmetric photonic heterostructures, Phys. Rev. A 85 023802 (2012).

[10] Z. Ahmed, New features of scattering from a one-dimensional non-Hermitian (complex) potential, J. Phys. A: Math. Theor. 45, 032004 (2012).

[11] J. G. Muga, J. P. Palao, B. Navarro, and I. L. Egusquiza, Complex absorbing potentials, Phys. Rep. 395, 357-426 (2004).

[12] A. Ruschhaupt, T. Dowdall, M. A. Simon, and J. G. Muga, Asymmetric scattering by non-hermitian potentials, preprint arXiv 1709:07027.

[13] R. R. D. Kemp, A singular boundary value problem for a non-self-adjoint differential operator, Canadian J. Math. 10, 447-462 (1958).

[14] Y. D. Chong, L. Ge, H. Cao, and A. D. Stone, Coherent perfect absorbers: Time-reversed lasers, Phys. Rev. Lett. 105, 053901 (2010).

[15] S. Longhi, Backward lasing yields a perfect absorber, Physics 3, 61 (2010).

[16] S. Longhi, $\mathcal{PT}$-symmetric laser absorber, Phys. Rev. A 82, 031801 (2010).

[17] W. Wan, Y. Chong, L. Ge, H. Noh, A. D. Stone, and H. Cao, Time-reversed lasing and interferometric control of absorption, Science 331, 889-892 (2011).

[18] A. Mostafazadeh, Self-dual spectral singularities and coherent perfect absorbing lasers without $\mathcal{PT}$-symmetry, J. Phys. A: Math. Gen. 45, 444024 (2012).