Soft topology redefined

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Abstract

In this paper we give a new definition of soft topology using elementary union and elementary intersection although these operations are not distributive. Also we have shown that this soft topology is different from Naz’s soft topology and studied some basic properties of this new type of soft topology. Here we use elementary complement of soft sets, though law of excluded middle is not valid in general for this type of complementation.

Keywords: Soft sets, soft elements, soft topological spaces, soft base, soft functions, soft continuous functions, soft separation axioms.

1 Introduction

The concept of soft set was introduced by Molodtsov [22] as a new mathematical tool for dealing with uncertainties. He has shown several applications of soft set in solving many practical problems in economics, engineering, social sciences, medical sciences, etc. Later, Maji et al. defined several operations over soft sets and applied soft set in decision making problems in [18, 19]. Soft sets are convenient to be applied in practice and this theory has potential application in many different fields [22] such as smoothness of functions, game theory, Riemann integration, Perron integration, probability theory and measure theory. Several authors are trying to develope different mathematical structures in soft set settings (for references please see [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 20, 23, 24, 25, 26, 27, 28, 30]). In [28], Shabir and Naz gave a definition of soft topology. Very recently Shi et al. in [29] has commented that soft topology in the sense of Shabir & Naz [28] can be interpreted as a crisp topology. In this paper, we define soft topology in different perspective using elementary union and elementary intersection which is different from that of Naz’s topology and study some basic properties of this new type soft topological space. The importance of this study lies on the fact that the operations of elementary union and elementary intersection are not
distributive. Also if we take a soft set and its elementary complement then law of excluded-middle is not valid in general. This has been mentioned in Note 2.24.

2 Preliminaries

Definition 2.1. [22] Let $X$ be a universal set and $E$ be a set of parameters. Let $P(X)$ denote the power set of $X$ and $A$ be a subset of $E$. A pair $(F, A)$ is called a soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over $X$ is a parametrized family of subsets of the universe $X$. For $\alpha \in A, F(\alpha)$ may be considered as the set of $\alpha$-approximate elements of the soft set $(F, A)$.

In [21] the soft sets are redefined as follows:

Let $E$ be the set of parameters and $A \subseteq E$. Then for each soft set $(F, A)$ over $X$ a soft set $(H, E)$ is constructed over $X$, where $\forall \alpha \in E, H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A \\ \phi & \text{if } \alpha \in E \setminus A. \end{cases}$

Thus the soft sets $(F, A)$ and $(H, E)$ are equivalent to each other and the usual set operations of the soft sets $(F_i, A_i), i \in \Delta$ is the same as those of the soft sets $(H_i, E), i \in \Delta$. For this reason, in this paper, we have considered our soft sets over the same parameter set $A$.

Following Molodtsov and Maji et al. [18, 19, 22] definitions of soft subset, absolute soft set, null soft set, arbitrary union, arbitrary intersection of soft sets etc. are presented in [25] considering the same parameter set.

Definition 2.2. [25] Let $(F, A)$ and $(G, A)$ be two soft sets over a common universe $X$.

(a) $(F, A)$ is said to be a soft subset of $(G, A)$ if $F(\alpha)$ is a subset of $G(\alpha), \forall \alpha \in A$.

(b) $(F, A)$ and $(G, A)$ are said to be soft equal if $(F, A)$ is a soft subset of $(G, A)$ and $(G, A)$ is a soft subset of $(F, A)$.

(c) The complement or relative complement of a soft set $(F, A)$ is denoted by $(F, A)^C$ and is defined by $(F, A)^C = (F^C, A), \forall \alpha \in A$.

(d) (Null soft set) $(F, A)$ over $X$ is said to be null soft set denoted by $(\tilde{\Phi}, A)$, where $F(\alpha) = \phi, \forall \alpha \in A$.

(e) (Absolute soft set) $(F, A)$ over $X$ is said to be absolute soft set denoted by $(\tilde{X}, A)$, if $F(\alpha) = X, \forall \alpha \in A$.

(f) Union of $(F, A)$ and $(G, A)$ is denoted by $(F, A)\tilde{\cup}(G, A)$ and defined by $[(F, A)\tilde{\cup}(G, A)](\alpha) = F(\alpha) \cup G(\alpha), \forall \alpha \in A$. 

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(g) Intersection of \((F, A)\) and \((G, A)\) is denoted by \((F, A) \cap (G, A)\) and defined by \([[(F, A) \cap (G, A)](\alpha) = F(\alpha) \cap G(\alpha), \forall \alpha \in A\].

**Definition 2.3.** \[[28]\] Let \(\tau\) be a collection of soft sets over \(X\). Then \(\tau\) is said to be a soft topology on \(X\) if

(i) \((\emptyset, A), (X, A) \in \tau\).

(ii) the intersection of any two soft sets in \(\tau\) belongs to \(\tau\).

(iii) the union of any number of soft sets in \(\tau\) belongs to \(\tau\). The triplet \((X, A, \tau)\) is called a soft topological space over \(X\).

**Definition 2.4.** \[[13]\] Let \(\tau\) be a collection of soft sets over \(X\). Define \(\tau(\alpha) = \{F(\alpha) : (F, A) \in \tau\}\) for \(\alpha \in A\). Then \(\tau\) is said to be a topology of soft subsets over \((X, A)\) if \(\tau(\alpha)\) is a crisp topology on \(X, \forall \alpha \in A\). In this case \(((X, A), \tau)\) is said to be a topological space of soft subsets. If \(\tau\) is a topology of soft subsets over \((X, A)\), then the members of \(\tau\) are called open soft sets and a soft set \((F, A)\) over \(X\) is said to be soft closed if \((F, A)^C \in \tau\).

**Definition 2.5.** \[[7]\] Let \(X\) be a non-empty set and \(A\) be a non-empty parameter set. Then a function \(\tilde{x} : A \to X\) is said to be a soft element of \(X\). A soft element \(\tilde{x}\) of \(X\) is said to belong to a soft set \((F, A)\) of \(X\), which is denoted by \(\tilde{x} \in (F, A)\), if \(\tilde{x}(\lambda) \in F(\lambda), \forall \lambda \in A\). Thus for a soft set \((F, A)\) over \(X\) with respect to the index set \(A\) with \(F(\lambda) \neq \phi, \forall \lambda \in A\), we have \(F(\lambda) = \{\tilde{x}(\lambda) : \tilde{x} \in (F, A)\}\), for all \(\lambda \in A\).

By the notation \(\tilde{x}\) we will denote a particular type of soft element such that \(\tilde{x}(\lambda) = x, \forall \lambda \in A\).

Let \(X\) be an initial universal set and \(A\) be a non-empty parameter set. Throughout the paper we consider the null soft set \((\emptyset, A)\) and those soft sets \((F, A)\) over \(X\) for which \(F(\alpha) \neq \phi, \forall \alpha \in A\). We denote this collection by \(S(\tilde{X})\). Thus for \((F, A)[\neq (\emptyset, A)] \in S(\tilde{X}), F(\alpha) \neq \phi, \forall \alpha \in A\).

**Proposition 2.6.** \[[9]\] Any collection of soft elements of a soft set can generate a soft subset of that soft set.

The soft set constructed from a collection of soft elements \(\mathcal{B}\) will be denoted by \(SS(\mathcal{B})\). For any soft set \((F, A) \in S(\tilde{X})\), the collection of all soft elements of \((F, A)\) is denoted by \(SE(F, A)\).

**Proposition 2.7.** \[[9]\] For any soft sets \((F, A), (G, A) \in S(\tilde{X}), (F, A) \subseteq (G, A)\) iff every soft element of \((F, A)\) is also a soft element of \((G, A)\).

**Definition 2.8.** \[[9]\] For any two soft sets \((F, A), (G, A) \in S(\tilde{X}),\)

(i) elementary union of \((F, A)\) and \((G, A)\) is denoted by \((F, A) \cup (G, A)\) and is defined by \((F, A) \cup (G, A) = SS(\mathcal{B})\), where \(\mathcal{B} = \{\tilde{x} \in (X, A) : \tilde{x} \in (F, A) \text{ or } \tilde{x} \in (G, A)\}\);

i.e. \((F, A) \cup (G, A) = SS(SE(F, A) \cup SE(G, A))\).

(ii) elementary intersection of \((F, A)\) and \((G, A)\) is denoted by \((F, A) \cap (G, A)\)
and is defined by \((F, A) \cap (G, A) = SS(\mathcal{B})\), where 
\(\mathcal{B} = \{\tilde{x} \in (X, A): \tilde{x} \in (F, A) \text{ and } \tilde{x} \in (G, A)\}\); i.e. \((F, A) \cap (G, A) = SS(SE(F, A) \cap SE(G, A))\).

**Definition 2.9.** \[9\] For any soft set \((F, A) \in S(X)\), the elementary complement of \((F, A)\) is denoted by \((F, A)^c\) and is defined by \((F, A)^c = SS(\mathcal{B})\), where \(\mathcal{B} = \{\tilde{x} \in (X, A): \tilde{x} \in (F, A)^c\}\) and \((F, A)^c\) is the complement of \((F, A)\).

**Remark 2.10.** \[9\] It can be easily verified that if \((F, A), (G, A) \in S(X)\), then 
\((F, A) \cup (G, A), (F, A) \cap (G, A)\) and \((F, A)^c\) are members of \(S(X)\).

**Proposition 2.11.** \[9\] For any two soft sets \((F, A), (G, A) \in S(X)\),
(i) \((F, A) \cup (G, A) = (F, A) \cup (G, A)\).
(ii) \((F, A) \cap (G, A) = (F, A) \cap (G, A)\), if \((F, A) \cap (G, A) \neq (\Phi, A)\).

**Remark 2.12.** The above results can be extended easily to arbitrary union and arbitrary intersection.

**Proposition 2.13.** \[9\] For any soft set \((F, A) \in S(X)\), in general, \((F, A)^c \subseteq (F, A)^c\) and \((F, A)^c = (F, A)^c\) if \((F, A)^c \neq (\Phi, A)\) i.e. if \((F, A)^c \in S(X)\).

**Proposition 2.14.** \[9\] For any soft set \((F, A) \in S(X)\),
(i) \((F, A) \cap (F, A)^c = (\Phi, A)\).
(ii) In general, \((F, A) \cup (F, A)^c \subseteq (X, A)\) but if \((F, A)^c \neq (\Phi, A)\), then \((F, A) \cup (F, A)^c = (X, A)\).

**Proposition 2.15.** \[9\] Let \(\{(Y_i, A), i \in A\}\) be a family of soft sets and \(\{\mathcal{B}_i, i \in A\}\) be a family of collection of soft elements such that \((Y_i, A) = SS(\mathcal{B}_i), \forall i \in A\). Then \(\bigcup_{i \in A} (Y_i, A) = SS(\bigcup_{i \in A} \mathcal{B}_i)\).

**Proposition 2.16.** Let \(\{(Y_i, A), i \in A\}\) be a family of soft sets. Then \(\bigcap_{i \in A} (Y_i, A) = \bigcap_{i \in A} (Y_i, A)\) if \(\bigcap_{i \in A} (Y_i, A) \in S(X)\).

**Proposition 2.17.** Let \(\{(Y_i, A), i \in A\}\) be a family of soft sets and \(\{\mathcal{B}_i, i \in A\}\) be a family of collection of soft elements such that \((Y_i, A) = SS(\mathcal{B}_i), \forall i \in A\). Then \(\bigcap_{i \in A} (Y_i, A) \subseteq SS(\bigcap_{i \in A} \mathcal{B}_i)\).

**Proof.** We have \(\mathcal{B}_i \subset SE(Y_i, A), \forall i \in A\)
\(\Rightarrow \bigcap_{i \in A} \mathcal{B}_i \subset \bigcap_{i \in A} SE(Y_i, A)\)
\(\Rightarrow SS(\bigcap_{i \in A} \mathcal{B}_i) \subseteq SS(\bigcup_{i \in A} SE(Y_i, A))\)
\(\Rightarrow SS(\bigcap_{i \in A} \mathcal{B}_i) \subseteq SS(\bigcup_{i \in A} (Y_i, A))\).

**Example 2.18.** In general, \(\bigcap_{i \in A} (Y_i, A) \neq SS(\bigcap_{i \in A} \mathcal{B}_i)\).
For example, let $X = \{x, y\}, A = \{\lambda, \mu\}, \mathcal{B}_1 = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}, \mathcal{B}_2 = \{\tilde{x}_4\}$; where $\tilde{x}_1(\lambda) = \{x\}, \tilde{x}_1(\mu) = \{y\}; \tilde{x}_2(\lambda) = \{x\}, \tilde{x}_2(\mu) = \{y\}; \tilde{x}_3(\lambda) = \{x\}, \tilde{x}_3(\mu) = \{y\}; \tilde{x}_4(\lambda) = \{y\}, \tilde{x}_4(\mu) = \{x\}$. Also, let $(Y_1, A) = SS\{\mathcal{B}_1\}, (Y_2, A) = SS\{\mathcal{B}_2\}$. Then $\mathcal{B}_1 \cap \mathcal{B}_2 = \phi$ i.e. $SS(\mathcal{B}_1 \cap \mathcal{B}_2) = (\tilde{\Phi}, A)$. But $(Y_1, A) \cap (Y_2, A) = (Y_2, A)$.

**Proposition 2.19.** Let $(Y, A), (Z, A) \in S(\tilde{X})$. Then 
(i) $SE[(Y, A) \cap (Z, A)] = SE(Y, A) \cap SE(Z, A)$.
(ii) $SE[(Y, A) \cup (Z, A)] \supset SE(Y, A) \cup SE(Z, A)$.

**Example 2.20.** In general, $SE[(Y, A) \cup (Z, A)] \neq SE(Y, A) \cup SE(Z, A)$.

Consider $X = \{x, y, z\}, A = \{\lambda, \mu\}, (Y, A), (Z, A) \in S(\tilde{X})$, where $Y(\lambda) = \{y\}, Y(\mu) = \{z\}; Z(\lambda) = \{z\}, Z(\mu) = \{x\}$. Also consider the soft elements $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ where $\tilde{x}_1(\lambda) = \{y\}, \tilde{x}_1(\mu) = \{z\}; \tilde{x}_2(\lambda) = \{z\}, \tilde{x}_2(\mu) = \{x\}; \tilde{x}_3(\lambda) = \{y\}, \tilde{x}_3(\mu) = \{x\}; \tilde{x}_4(\lambda) = \{z\}, \tilde{x}_4(\mu) = \{z\}$. Then $SE(Y, A) \cup SE(Z, A) = \{\tilde{x}_1, \tilde{x}_2\}$, but $SE[(Y, A) \cup (Z, A)] = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$.

**Remark 2.21.** The operations of elementary union and elementary intersection are not distributive over $S(\tilde{X})$. This is shown in the following example.

**Example 2.22.** Let $X = \{x, y, z\}$ and $A = \{\alpha, \beta\}$. Now consider three soft sets $(F, A), (G, A)$ and $(H, A) \in S(\tilde{X})$, where $F(\alpha) = \{x\}, F(\beta) = \{y\}; G(\alpha) = \{y\}, G(\beta) = \{z\}$ and $H(\alpha) = \{y\}, H(\beta) = \{y\}$.

Then $[(F, A) \cup (G, A)] \cap (H, A) = [(F, A) \cap (H, A)] \cup [(G, A) \cap (H, A)]$ as $(F, A) \cap (H, A) = (\tilde{\Phi}, A), (G, A) \cap (H, A) = (\tilde{\Phi}, A)$ i.e. $[(F, A) \cap (H, A)] \cup [(G, A) \cap (H, A)] = (\tilde{\Phi}, A)$ but $[(F, A) \cup (G, A)] \cap (H, A) = (\tilde{\Phi}, A)$.

Also $[(F, A) \cap (H, A)] \cup (G, A) = [(F, A) \cup (G, A)] \cap [(H, A) \cup (G, A)]$ as $[(F, A) \cup (G, A)](\alpha) = \{x, y\}, [(F, A) \cup (G, A)](\beta) = \{y, z\}$ and $[(H, A) \cup (G, A)](\alpha) = \{y\}, [(H, A) \cup (G, A)](\beta) = \{y, z\}$ i.e. $[(F, A) \cup (G, A)] \cap [(H, A) \cup (G, A)](\alpha) = \{y\}, [(F, A) \cup (G, A)] \cap [(H, A) \cup (G, A)](\beta) = \{y, z\}$ but $[(F, A) \cup (G, A)](\alpha) = \{y\}, [(F, A) \cup (G, A)](\beta) = \{y, z\} = (\tilde{\Phi}, A)$.

**Proposition 2.23.** (i) If $(F, A), (F, A)^C \in S(\tilde{X})$, then $(F, A)^C = (F, A)^C$ i.e. then $[(F, A)^C]^C = (F, A)$.
(ii) Let $\{(F_i, A) : i \in \Delta\}$ be any collection of soft sets in $S(\tilde{X})$, then

(a) $[\cup\{(F_i, A) : i \in \Delta\}]^C = \cap\{(F_i, A)^C : i \in \Delta\}$.

(b) $[\cap\{(F_i, A) : i \in \Delta\}]^C = \cup\{(F_i, A)^C : i \in \Delta\}$.

**Note 2.24.** Following observations are to be noted:

(i) The operations of elementary union and elementary intersection are not distributive over $S(\tilde{X})$.

(ii) In general, $(F, A) \cup (F, A)^C \subseteq (\tilde{\Phi}, A), (F, A) \cup (F, A)^C \neq (\tilde{\Phi}, A)$. However if $(F, A)^C \neq (\tilde{\Phi}, A)$, then $(F, A) \cup (F, A)^C = (\tilde{\Phi}, A)$. 

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(iii) In general, \((F, A)^C)^C \neq (F, A)\). However if \((F, A)^C \neq (\Phi, A)\), then \((F, A)^C)^C = (F, A)\).

3 Soft topology

**Definition 3.1.** Let \(\tau\) be the collection of soft sets of \(S(\hat{X})\). Then \(\tau\) is said to be a soft topology on \((\hat{X}, A)\) if

(i) \((\Phi, A)\) and \((\hat{X}, A)\) belong to \(\tau\).

(ii) the elementary union of any number of soft sets in \(\tau\) belongs to \(\tau\).

(iii) the elementary intersection of two soft sets in \(\tau\) belongs to \(\tau\). The triplet \((\hat{X}, \tau, A)\) is called a soft topological space.

**Definition 3.2.** In a soft topological space \((\hat{X}, \tau, A)\), the members of \(\tau\) are called soft open sets in \((\hat{X}, \tau, A)\).

**Definition 3.3.** Let \((\hat{X}, \tau, A)\) be a soft topological space. A soft set \((F, A) \in S(\hat{X})\) is said to be a soft closed set in \((\hat{X}, \tau, A)\) if its relative complement \((F, A)^C \in S(\hat{X})\) and \((F, A)^C \in \tau\).

**Proposition 3.4.** Let \((\hat{X}, \tau, A)\) be a soft topological space. Then

(i) \((\Phi, A)\) and \((\hat{X}, A)\) are soft closed soft sets in \((\hat{X}, \tau, A)\).

(ii) arbitrary elementary intersection of soft sets is soft closed.

**Proof.** (i) Since \((\Phi, A)^C = (\hat{X}, A) \in S(\hat{X})\) and \((\Phi, A)^C = (\hat{X}, A) \in \tau\) it follows that \((\Phi, A)\) is soft closed.

Similarly, \((\hat{X}, A)\) is soft closed.

(ii) Let \(\{Y_i, A\} : i \in \Delta\) be a family of soft closed sets in \((\hat{X}, \tau, A)\). We have to show that \(\cap \{Y_i, A\} : i \in \Delta\} = (Y, A)\) is soft closed in \((\hat{X}, \tau, A)\).

From Remark 2.10, \((Y, A) \in S(\hat{X})\)

If \((Y, A) = (\Phi, A)\), then \((Y, A)\) is soft closed in \((\hat{X}, \tau, A)\).

If \((Y, A) \neq (\Phi, A)\), then from Remark 2.12, \(\cap \{Y_i, A\} : i \in \Delta\} = \cap \{(Y_i, A) : i \in \Delta\}\)

and hence \((Y, A)^C = \cap \{(Y_i, A) : i \in \Delta\}\)^C = \(\cup \{Y_i, A\} : i \in \Delta\}\)

= \(\Psi \{Y_i, A\} : i \in \Delta\}\) \(\in S(\hat{X}), \forall i \in \Delta\).

Now \((Y, A)^C = \cap \{Y_i, A\} : i \in \Delta\}\) \(\in S(\hat{X}), \forall i \in \Delta\) and \(\tau\) is closed under arbitrary elementary union.

Therefore \((Y, A)\) is soft closed in \((\hat{X}, \tau, A)\).

**Remark 3.5.** Elementary union of two soft closed sets is not soft closed in general. This is shown in the following example.

**Example 3.6.** Let \(X = \{x, y, z\}\) and \(A = \{\alpha, \beta\}\). Then \(\tau = \{(\Phi, A), (\hat{X}, A), (F, A), (G, A)\}\) where \(F(\alpha) = \{y, z\}\), \(F(\beta) = \{x, z\}\) and \(G(\alpha) = \{x\}, G(\beta) = \{y, z\}\). Then \(\tau\) is a soft topology as per Definition 3.1. The soft closed sets
are \((\bar{\Phi}, A), (\bar{X}, A), (F, A)^C, (G, A)^C\), where \(F^C(\alpha) = \{x\}\), \(F^C(\beta) = \{y\}\) and \(G^C(\alpha) = \{y, z\}\), \(G^C(\beta) = \{x\}\). But \((F, A)^C \cup (G, A)^C\) is not soft closed.

**Proposition 3.7.** Let \(\{(Y_i, A) : i = 1, 2, ..., n\}\) be a finite family of soft closed sets in \((\bar{X}, \tau, A)\). Then \(\bigcup_{i=1}^n (Y_i, A)\) is soft closed in \((\bar{X}, \tau, A)\) if \(\bigcap_{i=1}^n (Y_i, A)^C(\neq (\bar{\Phi}, A)) \in S(\bar{X})\).

**Proof.** Let \(\{(Y_i, A) : i = 1, 2, ..., n\}\) be a finite family of soft closed sets in \((\bar{X}, \tau, A)\). We have to show that \(\bigcup_{i=1}^n (Y_i, A)\) is soft closed in \((\bar{X}, \tau, A)\).

Since for each \(i = 1, 2, ..., n\), \((Y_i, A)\) is closed in \((\bar{X}, \tau, A)\), \((Y_i, A), (Y_i, A)^C \in S(\bar{X})\) and \((Y_i, A)^C \in \tau\).

Now \((Y, A)^C = [\bigcup\{(Y_i, A) : i = 1, 2, ..., n\}]^C = [\bigcap\{(Y_i, A) : i = 1, 2, ..., n\}]\) = \(\bigcap_{i=1}^n (Y_i, A)^C : i = 1, 2, ..., n\) = \(S(\bar{X})\), as \(\bigcap_{i=1}^n (Y_i, A)^C(\neq (\bar{\Phi}, A))\).

Also \((Y, A)^C = [\bigcup\{(Y_i, A) : i = 1, 2, ..., n\}]^C = \bigcap\{(Y_i, A)^C : i = 1, 2, ..., n\}\). Then \((Y, A)^C \in \tau\), as \((Y, A)^C \in \tau\), \(\forall i = 1, 2, ..., n\) and \(\tau\) is closed under finite elementary intersection. Hence \((Y, A)^C\) is soft closed in \((\bar{X}, \tau, A)\).

**Remark 3.8.** (a) Soft topology due to Shabir and Naz and the soft topology as given by us in **Definition 3.1** are different.

Let \(X = \{x, y, z\}\) and \(A = \{\alpha, \beta\}\). Then \(\tau = \{(\bar{\Phi}, A), (\bar{X}, A), (F, A), (G, A)\}\) where \(F(\alpha) = \{x, y\}\), \(F(\beta) = \{x, z\}\) and \(G(\alpha) = \{z\}\), \(G(\beta) = \{y, z\}\). Then \((F, A) \cup (G, A) = (X, A)\) and \((F, A) \cap (G, A) = (\bar{\Phi}, A)\). Thus \(\tau\) is a soft topology as per **Definition 3.1**. Now as \((F, A) \cap (G, A) = \{\phi, \{z\}\}\), \{\phi, \{z\}\} does not belong to \(\tau\), it is not a soft topology as per Shabir and Naz. Now if \(\tau' = \{(\bar{\Phi}, A), (\bar{X}, A), (F, A), (G, A), (H, A)\}\) where \(H(\alpha) = \phi, H(\beta) = \{z\}\). Then \(\tau'\) is a soft topology as per Shabir and Naz but not a soft topology as per **Definition 3.1**, since \((H, A) \notin S(\bar{X})\).

(b) Soft topology due to Hazra et al. and the soft topology as per **Definition 3.1** are different.

Consider the universal set \(X\), the parameter set \(A\) and \(\tau, \tau'\) are similar as above. Then \(\tau\) is a soft topology as per **Definition 3.1**. Here \(\tau(\alpha) = \{\phi, X, \{x, y\}, \{z\}\}\) and \(\tau(\beta) = \{\phi, X, \{x, z\}, \{y, z\}\}\). So, \(\tau(\beta)\) is not a crisp topology on \(X\). Thus \(\tau\) is not a topology of soft subsets over \((X, A)\) as per Hazra et al. Since \(\tau'(\alpha) = \{\phi, X, \{x, y\}, \{z\}\}\) and \(\tau'(\beta) = \{\phi, X, \{x, z\}, \{y, z\}, \{z\}\}\) are crisp topologies on \(X\). Thus \(\tau'\) is a topology of soft subsets over \((X, A)\) as per Hazra et al. but not a soft topology as per **Definition 3.1**, as we have seen before.

However we have the following result:

**Proposition 3.9.** Let \((\bar{X}, \tau, A)\) be a soft topological space as per **Definition 3.1** and if \((F, A), (G, A) \in \tau \Rightarrow (F, A) \cap (G, A) \in S(\bar{X})\), then \(\tau\) defines a soft topology as per Hazra et al. Further if \(\tau\) is a soft topology as per Hazra et al., then
\( \tau' = \{(F, A) \in S(\tilde{X}) : F(\alpha) \in \tau(\alpha), \forall \alpha \in A\} \) is a soft topology on \((\tilde{X}, A)\) as per Definition 3.1.

**Proof.** Let \((F, A), (G, A) \in \tau\) and \((F, A) \cap (G, A) \in S(\tilde{X})\). Then we have to show that \(\tau(\alpha) = \{F(\alpha) : (F, A) \in S(\tilde{X})\}\) is a crisp topology on \(X\). We only show the intersection property as rest are obvious. Let \(F(\alpha), G(\alpha) \in \tau(\alpha)\) for some \((F, A), (G, A) \in \tau\).

Now \(F(\alpha) \cap G(\alpha) = [(F, A) \cap (G, A)](\alpha) = [(F, A) \cup (G, A)](\alpha) \cap (F, A) \cap (G, A) = (F, A) \cap (G, A) \in S(\tilde{X}) \Rightarrow (F, A) \cap (G, A) \in \tau\)

\(\tau(\alpha)[as (F, A) \cap (G, A) \in \tau]\)

Thus \(\tau\) defines a soft topology as per Hazra et al.

Next, Obviously \((\tilde{F}, A), (\tilde{X}, A) \in \tau'\).

Now let \((F_1, A), (F_2, A) \in \tau'.\) If \((F_1, A) \cap (F_2, A) = (\tilde{F}, A)\). Then obviously \((F_1, A) \cap (F_2, A) \in \tau'\).

Again if \((F_1, A) \cap (F_2, A) \neq (\tilde{F}, A)\). Then \((F_1, A) \cap (F_2, A) = (F_1, A) \cap (F_2, A)\).

Also \((F_1, A), (F_2, A) \in \tau(\alpha), \forall \alpha \in A\).

Thus \([(F_1, A) \cap (F_2, A)](\alpha) = [(F_1, A) \cap (F_2, A)](\alpha) = F_1(\alpha) \cap F_2(\alpha) \in \tau(\alpha), \forall \alpha \in A\).

Hence, \((F_1, A) \cap (F_2, A) \in \tau'\).

Similarly if we take \((F_i, A) \in \tau', \forall i \in \Delta,\) then \(\psi\{(F_i, A), i \in \Delta\} \in \tau'\).

Therefore, \(\tau\) is a soft topology on \((\tilde{X}, A)\).

**Proposition 3.10.** Let \((\tilde{X}, \tau_1, A)\) and \((\tilde{X}, \tau_2, A)\) be two soft topological spaces. Then \((\tilde{X}, \tau_1 \cap \tau_2, A)\) is a soft topological space.

**Remark 3.11:** The union of two soft topologies on \((\tilde{X}, A)\) may not be a soft topology on \((\tilde{X}, A)\).

**Example 3.12:** Let \(X = \{x, y, z\}\) and \(A = \{\alpha, \beta\}\).

Let \(\tau_1 = \{(\tilde{F}, A), (\tilde{X}, A), (F_1, A), (G_1, A)\}\) where \(F_1(\alpha) = \{x, y\}, F_1(\beta) = \{x, z\}\)

and \(G_1(\alpha) = \{z\}, G_1(\beta) = \{y, z\}\) and \(\tau_2 = \{(\tilde{F}, A), (\tilde{X}, A), (F_2, A), (G_2, A)\}\) where \(F_2(\alpha) = \{y\}, F_2(\beta) = \{y\}\) and \(G_2(\alpha) = \{x, z\}, G_2(\beta) = \{x, z\}\).

Now, we define \(\tau = \tau_1 \cup \tau_2 = \{(\tilde{F}, A), (\tilde{X}, A), (F_1, A), (G_1, A), (F_2, A), (G_2, A)\}\).

If we take \((F_1, A) \cup (F_2, A) = (H, A)\) then \(H(\alpha) = \{x, y\}, H(\beta) = \{x, y, z\}\). But \((H, A) \notin \tau\).

Thus \(\tau\) is not a soft topology as in Definition 3.1.

**Definition 3.13.** Let \((\tilde{X}, \tau, A)\) be a soft topological space and \((F, A) \in S(\tilde{X})\).

Then the soft closure of \((F, A)\), denoted by \(\overline{(F, A)}\), is defined as the elementary intersection of all soft closed super sets of \((F, A)\). Clearly \(\overline{(F, A)}\) is the smallest soft closed set in \((\tilde{X}, \tau, A)\) which contains \((F, A)\) by Proposition 3.4.

**Proposition 3.14.** Let \((\tilde{X}, \tau, A)\) be a soft topological space and \((F, A), (G, A) \in S(\tilde{X})\). Then
(i) \((\Phi, A) = (\Phi, A)\), \((X, A) = (X, A)\).
(ii) \((F, A) \subseteq (F, A)\).
(iii) \((F, A)\) is soft closed if and only if \((F, A) = (F, A)\).
(iv) \((F, A) = (F, A)\).
(v) \((F, A) \subseteq (G, A) \Rightarrow (F, A) \subseteq (G, A)\).
(vi) \((F, A) \cup (G, A) \subseteq (F, A) \cup (G, A)\). Equality holds if, \((F, A) \cup (G, A) \subseteq (F, A) \cup (G, A)\).

Proof. Proofs of (i)-(v) are obvious.

(vii) Since \((F, A) \subseteq (F, A) \cup (G, A)\) and \((G, A) \subseteq (F, A) \cup (G, A)\), so by (v), \((F, A) \subseteq (F, A) \cup (G, A) \subseteq (F, A) \cup (G, A)\).

Let, \((F, A) \cup (G, A) \subseteq (F, A) \cup (G, A)\). Also, \((F, A) \cup (G, A) \subseteq (F, A) \cup (G, A)\).

[From Proposition 2.23] and this implies that \((F, A) \cup (G, A)\) is soft closed set.

Therefore, \((F, A) \cup (G, A) \subseteq (F, A) \cup (G, A)\). Thus \((F, A) \cup (G, A) = (F, A) \cup (G, A)\).

Example 3.15. Here we give an example where \((F, A) \cup (G, A) \neq (F, A) \cup (G, A)\).

Consider the soft topological space \((X, \tau, A)\) of Example 3.12. Where the non-null soft closed sets are \((F, A)\) and \((Q, A)\) such that \(P(\alpha) = \{z\}, P(\beta) = \{y\}\) and \(Q(\alpha) = \{x, y\}, Q(\alpha) = \{x\}\). Then \((P, A) = (P, A)\) and \((Q, A) = (Q, A)\).

Let \((V, A) = (P, A) \cup (Q, A)\). Then \((V, A) = (P, A) \cup (Q, A)\). But \((P, A) \cup (Q, A) = (X, A)\). Therefore, \((P, A) \cup (Q, A) \neq (P, A) \cup (Q, A)\).

Definition 3.16. Let \((X, \tau, A)\) be a soft topological space. A soft element \(\hat{x} \in (X, A)\) is said to be a limiting soft element of a soft set \((F, A) \in S(X)\) if \(\forall (G, A) \in \tau\) and for any \(\alpha \in A, \hat{x}(\alpha) \in G(\alpha)\) implies \(F(\alpha) \cap G(\alpha) \neq \emptyset\).

The soft set formed out of all limiting soft elements of \((F, A) \in S(X)\) is called the derived soft set of \((F, A)\) and is denoted by \((F, A)\).

The weak soft closure of a soft set \((F, A) \in S(X)\), denoted by \((F, A)\), is defined by \((F, A) = (F, A) \cup (F, A)\).

Proposition 3.17 If a soft set \((F, A) \in S(X)\) is soft closed in a soft topological space \((X, \tau, A)\), then \((F, A)\) contains all its limiting soft elements as in Definition 3.16.
Proof. Let \((F, A) \in S(\tilde{X})\) be soft closed and \(\tilde{x} \tilde{\varepsilon}(F, A)\). Since \((F, A)\) is soft closed \((F, A)^C \in S(\tilde{X})\) and \((F, A)^C\) is soft open.

Now, \(\tilde{x}\) either belongs to \((F, A)^C\) or does not belongs to \((F, A)^C\). If \(\tilde{x} \in (F, A)^C\), then \(\tilde{x}(\lambda) \in [(F, A)^C](\lambda) = X \setminus F(\lambda), \forall \lambda \in A\). Since \(F(\lambda) \cap X \setminus F(\lambda) = \phi, \forall \lambda \in A\), \(\tilde{x}\) can not be limiting soft element of \((F, A)\). If \(\tilde{x} \notin (F, A)^C\), then \(\tilde{x}(\lambda) \in F(\lambda)\) for some \(\lambda \in A\) and \(\tilde{x}(\mu) \notin F(\mu)\) for some \(\mu(\neq \lambda) \in A\). Then \(\tilde{x}(\mu) \in [(F, A)^C](\mu)\) but \(F(\mu) \cap [(F, A)^C](\mu) = \phi\). So, \(\tilde{x}\) can not be limiting soft element of \((F, A)\) as per Definition 3.16.

Therefore \((F, A)\) contains all its limiting soft element.

Remark 3.18. Converse of Proposition 3.17 is not true.

Let \(X = \{x, y, z\}, A = \{\alpha, \beta\}\) and \(\tau = \{\{\tilde{X}, A\}, \{\tilde{X}, (F, A), (G, A), (H, A)\}\}\) where \(F(\alpha) = \{y, z\}, F(\beta) = \{x\}; G(\alpha) = \{y\}, G(\beta) = \{y, z\}; H(\alpha) = \{y, z\}, H(\beta) = \{x, y, z\}\). Then \(\tau\) is a soft topology as per Definition 3.1. Now, let \((C, A)\) be any soft set where \(C(\alpha) = \{x, z\}, C(\beta) = \{y, z\}\). Let \(\xi_1(\alpha) = x, \xi_1(\beta) = y; \xi_2(\alpha) = x, \xi_2(\beta) = z\). Then \(\xi_1, \xi_2\) are the only limiting soft elements of \((C, A)\) as per Definition 3.16 and \(\tilde{X} \in (C, A), i = 1, 2\). But \((C, A)\) is not closed in \((\tilde{X}, \tau, A)\), as \((C, A)^C\) is not soft open.

Definition 3.19. Let \((\tilde{X}, \tau, A)\) be a soft topological space and \((F, A) \in S(\tilde{X})\).

A soft element \(x \in S(\tilde{X})\) is said to be an interior soft element of \((F, A)\) if \(\exists (G, A) \in \tau\) such that \(\tilde{x} \in (G, A) \subseteq (F, A)\).

Definition 3.20. The interior of a soft set \((F, A)\) is defined to be the set consisting of all interior soft elements of \((F, A)\). The interior of the soft set \((F, A)\) denoted by \(\text{Int}(F, A)\). Thus,

\[\text{Int}(F, A) = \{x \in (F, A) : \exists (G, A) \in \tau, \tilde{x} \in (G, A) \subseteq (F, A)\}\]

SS\[\text{Int}(F, A)\] is said to be soft interior of \((F, A)\) and denoted by \((F, A)^{\circ}\).

Proposition 3.21. Let \((\tilde{X}, \tau, A)\) be a soft topological space and \((F, A), (G, A) \in S(\tilde{X})\).

Then

(i) \((F, A)^{\circ} \subseteq (G, A)^{\circ}\).

(ii) \((F, A)^{\circ} \subseteq (G, A)^{\circ} \Rightarrow (F, A)^{\circ} \subseteq (G, A)^{\circ}\).

(iii) \(\text{Int}([F, A] \cup (G, A)]^C) \subseteq \text{Int}(F, A) \cup \text{Int}(G, A)\). Also, \([([F, A] \cup (G, A)]^C)^{\circ} \subseteq (F, A)^{\circ} \cup (G, A)^{\circ}\).

(iv) \(\text{Int}(F, A) \cap \text{Int}(G, A) \subseteq \text{Int}((F, A) \cap (G, A))\).

Proposition 3.22. Let \((\tilde{X}, \tau, A)\) be a soft topological space. Then \((F, A)(\neq (\tilde{F}, A)) \in S(\tilde{X})\) is soft open if and only if every \(x \in (F, A)\) is an interior soft element of \((F, A)\).

Proposition 3.23. Let \((\tilde{X}, \tau, A)\) be a soft topological space and \((F, A) \in S(\tilde{X})\).

Then \((F, A)^{\circ}\) is the elementary union of all soft open sets contained in \((F, A)\). It is the largest soft open set in \((\tilde{X}, \tau, A)\) contained in \((F, A)\).
Definition 3.24. Let \((\tilde{X}, \tau, A)\) be a soft topological space. Then \((F, A)(\neq (\Phi, A)) \in S(X)\) is a soft neighbourhood (soft nbd) of the soft element \(\tilde{x}\) if there exists a soft set \((G, A) \in \tau\) such that \(\tilde{x} \in \mathcal{C}(G, A) \subseteq (F, A)\). The soft nbd system at a soft element \(\tilde{x}\), denoted by \(\mathcal{N}_\tau(\tilde{x})\), is the family of all its soft nbds.

Proposition 3.25. Let \((\tilde{X}, \tau, A)\) be a soft topological space. Then \((F, A)(\neq (\Phi, A)) \in S(X)\) is soft open if and only if \((F, A)\) is soft nbd of all of its soft elements.

Proposition 3.26. Let \((\tilde{X}, \tau, A)\) be a soft topological space and for \(\tilde{x} \in SE(\tilde{X})\), let \(\mathcal{N}_\tau(\tilde{x})\) be the soft nbd system at the soft element \(\tilde{x}\). Then

(i) \(\mathcal{N}_\tau(\tilde{x}) \neq \emptyset\), \(\forall \tilde{x} \in SE(\tilde{X})\)
(ii) \(\tilde{x} \in (F, A)\) \(\forall (F, A) \in \mathcal{N}_\tau(\tilde{x})\)
(iii) \((F, A) \in \mathcal{N}_\tau(\tilde{x})\), \((F, A) \subseteq (G, A) \Rightarrow (G, A) \in \mathcal{N}_\tau(\tilde{x})\)
(iv) \((F, A), (G, A) \in \mathcal{N}_\tau(\tilde{x}) \Rightarrow (F, A) \cap (G, A) \in \mathcal{N}_\tau(\tilde{x})\)
(v) \((F, A) \in \mathcal{N}_\tau(\tilde{x}) \Rightarrow \exists (G, A) \in \mathcal{N}_\tau(\tilde{x})\) such that \((G, A) \subseteq (F, A)\) and \((G, A) \in \mathcal{N}_\tau(y), \forall y \in \tilde{G}(G, A)\).

Definition 3.27. A mapping \(\nu : SE(\tilde{X}) \rightarrow P(S(\tilde{X}))\) is said to be a soft nbd operator on \(SE(\tilde{X})\) if the following conditions hold:

\(N1\) \(\nu(\tilde{x}) \neq \emptyset, \forall \tilde{x} \in SE(\tilde{X})\)
\(N2\) \(\tilde{x} \in \mathcal{C}(F, A), \forall (F, A) \in \nu(\tilde{x})\)
\(N3\) \((F, A) \in \nu(\tilde{x}), (F, A) \subseteq (G, A) \Rightarrow (G, A) \in \nu(\tilde{x})\)
\(N4\) \((F, A), (G, A) \in \nu(\tilde{x}) \Rightarrow (F, A) \cap (G, A) \in \nu(\tilde{x})\)
\(N5\) \((F, A) \in \nu(\tilde{x}) \Rightarrow \exists (G, A) \in \nu(\tilde{x})\) such that \((G, A) \subseteq (F, A)\) and \((G, A) \in \nu(y), \forall y \in \tilde{G}(G, A)\).

Example 3.28. If \((\tilde{X}, \tau, A)\) is a soft topological space, then the mapping \(\nu : SE(\tilde{X}) \rightarrow P(S(\tilde{X}))\) defined by \(\nu(\tilde{x}) = \mathcal{N}_\tau(\tilde{x})\), where \(\mathcal{N}_\tau(\tilde{x})\) is the soft nbd system at the soft element \(\tilde{x}\), is a soft nbd operator on \(SE(\tilde{X})\).

4 Soft base of a soft topology

Definition 4.1. Let \((\tilde{X}, \tau, A)\) be a soft topological space. Then a subcollection \(\mathcal{B}\) of \(\tau\), containing \((\Phi, A)\), is said to be an open base \(\tau\) iff \(\forall \tilde{x} \in (X, A)\) and for any soft open set \((F, A)\) containing the soft element \(\tilde{x}\), there exists \((G, A) \in \mathcal{B}\) such that \(\tilde{x} \in \mathcal{C}(G, A) \subseteq (F, A)\).

Examples 4.2. Let \(X = \{x, y, z, t\}, A = \{\alpha, \beta\}\) and \(\tau = \{(\Phi, A), (\tilde{X}, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A), (F_7, A)\}\) where
\(F_1(\alpha) = \{x\}, F_1(\beta) = \{t\}; F_2(\alpha) = \{y\}, F_2(\beta) = \{z\}; F_3(\alpha) = \{t\}, F_3(\beta) = \{x\}; F_4(\alpha) = \{x, y\}, F_4(\beta) = \{z, t\};\)
Then
\[ F_5(\alpha) = \{y, t\}, \quad F_5(\beta) = \{z, x\}; \quad F_6(\alpha) = \{x, t\}, \quad F_6(\beta) = \{x, t\}; \]
\[ F_7(\alpha) = \{x, y, t\}, \quad F_7(\beta) = \{x, z, t\}. \]

Then \( \tau \) is a soft topology on \((X, A)\).

Let \( \mathcal{B} = \{ (\Phi, A), (X, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A) \} \).

Then \( \mathcal{B} \) forms an open base for \( \tau \).

**Proposition 4.3.** Let \((X, \tau, A)\) be a soft topological space and \( \mathcal{B} \) is an open base for \( \tau \). Then every member of \( \tau \) can be expressed as the elementary union of some members of \( \mathcal{B} \).

**Remark 4.4.** Converse of Proposition 4.3 is not true. Consider the soft topological space \((X, \tau, A)\) of Example 4.2 and let \( \mathcal{B} = \{ (\Phi, A), (X, A), (F_1, A), (F_2, A), (F_3, A) \} \).

Then \( \mathcal{B} \) satisfies the condition of Proposition 4.3 but \( \mathcal{B} \) is not an open base for \( \tau \) as the soft element \( x \in (F_6, A) \) but there is no soft set in \( \mathcal{B} \) containing \( x \) and contained in \((F_6, A)\).

**Proposition 4.5.** If a collection \( \mathcal{B} \) of soft sets of \( S(X) \) forms an open base of a soft topological space \((X, \tau, A)\), then the following conditions are satisfied:

(i) \((\Phi, A) \in \mathcal{B}\).

(ii) \((X, A)\) is elementary union of some members of \( \mathcal{B} \).

(iii) If \((F_1, A), (F_2, A) \in \mathcal{B}\) and \( x \in (F_1, A) \cap (F_2, A) \), then there exists \((F_3, A) \in \mathcal{B}\) such that \( x \in (F_3, A) \subseteq (F_1, A) \cap (F_2, A) \).

**Remark 4.6.** Converse of Proposition 4.5 is not true. Consider the soft topological space \((X, \tau, A)\) of Example 4.2 and let \( \mathcal{B} \) as in Remark 4.4. Then \( \mathcal{B} \) satisfies all the condition of Proposition 4.5 but \( \mathcal{B} \) is not base for the soft topology \( \tau \).

**Definition 4.7.** A family \( S \) of subsets of \( X \) is said to be a sub-base for a soft topological space \((X, \tau, A)\), if the family of all finite elementary intersection of members of \( S \) is a soft base for \( \tau \).

## 5 Soft function and soft continuous function

Proceeding as in [23], where definition of soft mapping has been given using 'soft point' concept, we introduce here a definition of soft function using the concept of 'soft element'.

**Definition 5.1.** Let \( X \) and \( Y \) be two non-empty sets and \( \{ f_\lambda : X \to Y : \lambda \in A \} \)

be a collection of functions. Then a function \( f : SE(X) \to SE(Y) \) defined by

\[ [f(\tilde{x})](\lambda) = f_\lambda(\tilde{x}(\lambda)), \forall \lambda \in A \]

is called a soft function.

**Definition 5.2.** Let \( f : SE(X) \to SE(Y) \) be a soft function. Then
(i) image of a soft set \((F, A)\) over \(X\) under the soft function \(f\), denoted by 
\(f[(F, A)]\), is defined by 
\[f[(F, A)] = SS\{f(SE(F, A))\}\] i.e. 
\[f[(F, A)](\lambda) = f_\lambda(F(\lambda)), \forall \lambda \in A.\]

(ii) inverse image of a soft set \((G, A)\) over \(Y\) under the soft function \(f\), denoted by 
\(f^{-1}[(G, A)]\), is defined by 
\[f^{-1}[(G, A)] = SS\{f^{-1}(SE(G, A))\}\] i.e. 
\[f^{-1}[(G, A)](\lambda) = f_\lambda^{-1}(G(\lambda)), \forall \lambda \in A.\]

**Definition 5.3.** Let \(f: SE(\tilde{X}) \rightarrow SE(\tilde{Y})\) be a soft function associated with the family of functions \(\{f_\lambda: X \rightarrow Y, \lambda \in A\}\). Then \(f\) is said to be (i) injective if \(\tilde{x} \neq \tilde{y}\) implies \(f(\tilde{x}) \neq f(\tilde{y})\).

(ii) surjective if \(f(X, A) = (\tilde{Y}, A)\).

(iii) bijective if both injective and surjective.

**Proposition 5.4.** Let \(X\) and \(Y\) be two non-empty sets and \(A\) be the parameter set. Also let \(f: SE(\tilde{X}) \rightarrow SE(\tilde{Y})\) be a soft function associated with the family of functions \(\{f_\lambda: X \rightarrow Y, \lambda \in A\}\). If \((F, A) \in S(\tilde{X})\) then

(i) \(ff^{-1}(F, A) \subseteq (F, A)\).

(ii) \((F, A) \subseteq f^{-1}(F, A)\).

**Proposition 5.5.** Let \(X\) and \(Y\) be two non-empty sets and \(A\) be the parameter set. Also let \(f: SE(\tilde{X}) \rightarrow SE(\tilde{Y})\) be a soft function associated with the family of functions \(\{f_\lambda: X \rightarrow Y, \lambda \in A\}\). If \((F_1, A), (F_2, A) \in S(\tilde{X})\) then

(i) \((F_1, A) \subseteq (F_2, A) \Rightarrow f[(F_1, A)] \subseteq f[(F_2, A)]\).

(ii) \(f[(F_1, A) \uplus (F_2, A)] = f[(F_1, A)] \uplus f[(F_2, A)]\).

(iii) \(f[(F_1, A) \sqcap (F_2, A)] \subseteq f[(F_1, A)] \sqcap f[(F_2, A)]\).

(iv) \(f[(F_1, A) \sqcup (F_2, A)] = f[(F_1, A)] \sqcup f[(F_2, A)]\), if \(f\) is one-one.

**Proposition 5.6.** Let \(X\) and \(Y\) be two non-empty sets and \(A\) be the parameter set. Also let \(f: SE(\tilde{X}) \rightarrow SE(\tilde{Y})\) be a soft function associated with the family of functions \(\{f_\lambda: X \rightarrow Y, \lambda \in A\}\). If \((F_1, A), (F_2, A) \in S(\tilde{X})\) then

(i) \((F_1, A) \subseteq (F_2, A) \Rightarrow f^{-1}[(F_1, A)] \subseteq f^{-1}[(F_2, A)]\).

(ii) \(f^{-1}[(F_1, A) \uplus (F_2, A)] = f^{-1}[(F_1, A)] \uplus f^{-1}[(F_2, A)]\).

(iii) \(f^{-1}[(F_1, A) \sqcap (F_2, A)] = f^{-1}[(F_1, A)] \sqcap f^{-1}[(F_2, A)]\).

**Definition 5.7.** Let \((\tilde{X}, \tau, A)\) and \((\tilde{Y}, \nu, A)\) be two soft topological spaces and \(f: SE(\tilde{X}) \rightarrow SE(\tilde{Y})\) be a soft function associated with the family of functions \(\{f_\lambda: X \rightarrow Y, \lambda \in A\}\). Then we denote this soft function as \(f: (\tilde{X}, \tau, A) \rightarrow (\tilde{Y}, \nu, A)\).

Now \(f: (\tilde{X}, \tau, A) \rightarrow (\tilde{Y}, \nu, A)\) is said to be soft continuous at \(\tilde{x}_0 \in (\tilde{X}, A)\), if for every \((V, A) \in \nu\) such that \(f(\tilde{x}_0) \in (V, A)\), there exists \((U, A) \in \tau\) such that \(\tilde{x}_0 \in (U, A)\) and \(f(U, A) \subseteq (V, A)\).

\(f\) is said to be soft continuous on \((\tilde{X}, \tau, A)\) if it is soft continuous at each soft element \(\tilde{x}_0 \in (X, A)\).

**Proposition 5.8.** Let \((\tilde{X}, \tau, A)\) and \((\tilde{Y}, \nu, A)\) be two soft topological spaces and \(f: (\tilde{X}, \tau, A) \rightarrow (\tilde{Y}, \nu, A)\) be a soft function. Then the followings are related as
follows:

(i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

(i) $f$ is soft continuous.

(ii) For all $(V, A) \in \nu$, $f^{-1}(V, A) \in \tau$.

(iii) There exists a subbase $\varphi$ for $\nu$ such that $f^{-1}(V, A) \in \tau$ for all $(V, A) \in \varphi$.

(iv) for any closed soft set $(F, A) \in S(Y)$ in $(\tilde{Y}, \nu, A)$, $f^{-1}(F, A)$ is soft closed in $(\tilde{X}, \tau, A)$.

**Remark 5.9.** In Proposition 5.8 (iv) $\Rightarrow$ (i) does not hold.

Let $X = \{x, y, z\}$ and $A = \{\alpha, \beta\}$. Let $\tau_1 = \{(\tilde{F}, A), (\tilde{X}, (F, A))\}$ and $\tau_2 = \{(\tilde{F}, A), (\tilde{X}, A)\}$, where $F(\alpha) = \{x, y, z\}, F(\beta) = \{x, z\}$. Then $\tau_1$ and $\tau_2$ are soft topologies on $(\tilde{X}, A)$. Consider the soft function $i : \tilde{X}, \tau, A \rightarrow (\tilde{X}, \tau_1, A)$ corresponding to the identity function $i : X \rightarrow X$. Then $i^{-1} : (\tilde{X}, \tau_1, A) \rightarrow (\tilde{X}, \tau_2, A)$ maps all soft closed sets of $\tau_1$ to soft closed sets of $\tau_2$. But $i^{-1}(F) = F$ does not hold. Therefore, $i : \tilde{X}, \tau_2, A \rightarrow (\tilde{X}, \tau_1, A)$ is not soft continuous function.

**Definition 5.10.** A soft function $f : (\tilde{X}, \tau, A) \rightarrow (\tilde{Y}, \nu, A)$ is said to be

(i) soft open if $f$ maps soft open sets of $\tau$ to soft open sets of $\nu$.

(ii) soft closed if $f$ maps soft closed sets of $\tau$ to soft closed sets of $\nu$.

**Definition 5.11.** A soft function $f : (\tilde{X}, \tau, A) \rightarrow (\tilde{Y}, \nu, A)$ is said to be soft homeomorphism if

(i) $f$ is bijective

(ii) $f$ is soft continuous

(iii) $f^{-1}$ is soft continuous.

**Proposition 5.12.** Let $f : (\tilde{X}, \tau, A) \rightarrow (\tilde{Y}, \nu, A)$ be a soft function. Then the followings are equivalent:

(i) $f$ is a soft homeomorphism.

(ii) $f$ is bijective, $f$, $f^{-1}$ are soft continuous.

(iii) $f$ is bijective, soft open and soft continuous.

(iv) $f^{-1}$ is a soft homeomorphism.

6 Soft separation axioms

**Definition 6.1.** Let $(\tilde{X}, \tau, A)$ be a soft topological space. If for $\tilde{x}, \tilde{y} \in SE(\tilde{X})$ with $\tilde{x}(\lambda) \neq \tilde{y}(\lambda), \forall \lambda \in A$,

(i) there exists $(F, A) \in \tau$ such that $[\tilde{x}(\lambda) \in (F, A)(\lambda)$ and $\tilde{y}(\lambda) \notin (F, A)(\lambda)]$ or $[\tilde{y}(\lambda) \in (F, A)(\lambda)$ and $\tilde{x}(\lambda) \notin (F, A)(\lambda)]$, $\forall \lambda \in A$, then $(\tilde{X}, \tau, A)$ is called a soft $T_0$ space.

(ii) there exist $(F, A), (G, A) \in \tau$ such that $[\tilde{x}(\lambda) \in (F, A)(\lambda), \tilde{y}(\lambda) \notin (F, A)(\lambda)]$ and $[\tilde{y}(\lambda) \in (G, A)(\lambda), \tilde{x} \notin (G, A)(\lambda)], \forall \lambda \in A$ then $(\tilde{X}, \tau, A)$ is called a soft $T_1$
(iii) there exist \((F, A), (G, A) \in \tau\) such that \(\tilde{x} \in \tilde{F}(F, A), \tilde{y} \in \tilde{G}(G, A)\) and \((F, A) \cap (G, A) = (\emptyset, A)\), then \((\tilde{X}, \tau, A)\) is called a soft \(T_2\) space.

**Remark 6.2.** (i) Every soft \(T_1\) space is a soft \(T_0\) space.

(ii) Every soft \(T_2\) space is a soft \(T_1\) space.

**Example 6.3.** Let \(\mathbb{R}\) be the real number space and \(A\) be a non-empty parameter set. Let for each \(\alpha \in A\), \(\tau(\alpha)\) be the usual topology on \(\mathbb{R}\). Then \((\mathbb{R}, \tau, A)\) where \(\tau\) be the soft topology generated by \(\tau(\alpha)\) as in Proposition 3.9, is a soft \(T_2\) space.

**Definition 6.4.** If a soft set of \(S(\tilde{X})\) contains exactly one soft element \(\tilde{x}\), then we denote this soft set by \((\tilde{x}, A)\) i.e. \((\tilde{x}, A)(\lambda) = \tilde{x}(\lambda), \, \forall \lambda \in A\).

**Proposition 6.5.** Let \((\tilde{X}, \tau, A)\) be a soft \(T_1\) topological space. Then for any \(\tilde{x} \in SE(\tilde{X}), (\tilde{x}, A)\) is soft closed.

**Proof.** Let \(\tilde{x} \in SE(\tilde{X})\). We claim that \((\tilde{x}, A)^C = (\tilde{x}, A)^C\) is a soft nbd of each of its soft element. For, let \(\tilde{y} \in (\tilde{x}, A)^C\). Then \(\tilde{x}(\lambda) \neq \tilde{y}(\lambda), \, \forall \lambda \in A\) and since \((\tilde{X}, \tau, A)\) is soft \(T_1\), there exists \((G, A) \in \tau\) such that \([\tilde{y}(\lambda) \in (G, A)(\lambda), \tilde{x} \notin (G, A)(\lambda)], \, \forall \lambda \in A\). But this means \((G, A) \subseteq (\tilde{x}, A)^C\) and hence \((\tilde{x}, A)^C\) is a soft nbd of \(\tilde{y}\). So, \((\tilde{x}, A)^C = (\tilde{x}, A)^C\) is open. Hence \((\tilde{x}, A)\) is soft closed.

**Definition 6.6.** A soft topological space \((\tilde{X}, \tau, A)\) is said to be a soft regular space if for any soft closed set \((F, A)\) and any soft element \(\tilde{x}\) such that \(\tilde{x}(\lambda) \notin (F, A)(\lambda), \forall \lambda \in A\), \(\exists (G, A), (H, A) \in \tau\) such that \((F, A) \subseteq (G, A), \tilde{x} \in (H, A)\) and \((F, A) \cap \subseteq (G, A) = (\emptyset, A)\).

If in addition, \((\tilde{X}, \tau, A)\) is soft \(T_1\), then \((\tilde{X}, \tau, A)\) is called soft \(T_3\) space.

**Example 6.7.** Let \(X = \{x, y, z\}, A = \{\alpha, \beta\}\) and \(\tau = \{\emptyset, A\}, (\tilde{X}, A), (F, A), (G, A)\}, where \(F(\alpha) = \{x, z\}, F(\beta) = \{y\}; G(\alpha) = \{y\}, G(\beta) = \{x, z\}\). Then \((\tilde{X}, \tau, A)\) is a soft regular space.

**Proposition 6.8.** Let \((\tilde{X}, \tau, A)\) be a soft topological space. If for all \(\tilde{x} \in SE(\tilde{X})\) and for all soft open sets \((U, A)\) such that \(\tilde{x} \in (U, A), \exists (V, A) \in \tau\) such that \(\tilde{x} \in (V, A) \subseteq (\tilde{X}, A) \subseteq (U, A)\), then \((\tilde{X}, \tau, A)\) is soft regular, where closure is taken as per Definition 3.13.

**Proof.** Let \(\tilde{x} \in SE(\tilde{X})\) and \((F, A)\) be any soft closed set such that \(\tilde{x}(\lambda) \notin (F, A)(\lambda), \forall \lambda \in A\). Consider \((U, A) = (F, A)^C\). Then \((U, A) \in \tau\), as \((U, A) \in S(\tilde{X})\) and \((U, A) = (F, A)^C = (F, A)^C\) and \(\tilde{x} \in (U, A)\). So by given condition \(\exists (V, A) \in \tau\) such that \(\tilde{x} \in (V, A) \subseteq (\tilde{X}, A) \subseteq (U, A)\). Let \((W, A) = (V, A)^C\). Then \((W, A) \in \tau\), as \((W, A) \in S(\tilde{X})\) and \((W, A) = (V, A)^C = (V, A)^C\). Thus \((F, A) = (U, A)^C \subseteq (\tilde{X}, A) \setminus (V, A) = (W, A)\). Also, \((V, A) \cap (W, A) = (\emptyset, A)\) i.e. \((V, A) \cap \)
\((W, A) = (\Phi, A)\). Therefore, \((\tilde{X}, \tau, A)\) is a soft regular space.

The following example shows that the converse of Proposition 6.8 is not true.

**Example 6.9.** Let \(X = \{x, y\}, A = \{\alpha, \beta\}\) and \(\tau = \{(\Phi, A), (\tilde{X}, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}\), where \(F_1(\alpha) = \{x, y\}, F_1(\beta) = \{x\}; F_2(\alpha) = \{y\}, F_2(\beta) = \{x, y\}; F_3(\alpha) = \{y\}, F_3(\beta) = \{x\}; F_4(\alpha) = \{x\}, F_4(\beta) = \{y\}\). Then \((\tilde{X}, \tau, A)\) is a soft regular space. But \((F_1, A)\) is a soft open set containing the soft element \(\Phi\) for which the given condition of Proposition 6.8 is not satisfied.

**Definition 6.10.** A soft topological space \((\tilde{X}, \tau, A)\) is said to be a soft normal space if for any two soft closed sets \((F, A)\) and \((G, A)\) such that \((F, A) \cap (G, A) = (\Phi, A), \exists (U, A), (V, A) \in \tau\) such that \((F, A) \subseteq (U, A), (G, A) \subseteq (V, A)\) and \((U, A) \cap (V, A) = (\Phi, A)\). If in addition \((\tilde{X}, \tau, A)\) is soft \(T_1\), then \((\tilde{X}, \tau, A)\) is called soft \(T_4\) space.

**Proposition 6.11.** Let \((\tilde{X}, \tau, A)\) be a soft topological space. If for all soft closed sets \((F, A)\) and for all soft open sets \((U, A)\) such that \((F, A) \subseteq (U, A)\), \(\exists (V, A) \in \tau\) such that \((F, A) \subseteq (V, A) \subseteq (U, A)\), then \((\tilde{X}, \tau, A)\) is soft normal, where closure is taken as per Definition 3.13.

**Proof.** Let the given condition be satisfied.

Let \((F_1, A), (F_2, A)\) be two soft closed sets such that \((F_1, A) \cap (F_2, A) = (\Phi, A)\). Consider \((U, A) = (F_2, A)^c\). Then \((U, A) \in \tau\), as \((U, A) \in S(\tilde{X})\) and \((U, A) = (F_2, A)^c\). Also, \((F_1, A) \subseteq (U, A)\). So, by given condition \(\exists (V, A) \in \tau\) such that \((F_1, A) \subseteq (V, A) \subseteq (U, A)\). Let \((W, A) = (V, A)^c\). Then \((W, A) \in \tau\), as \((W, A) \in S(\tilde{X})\) and \((W, A) = (V, A)^c\). Thus \((F_2, A) = (\tilde{X}, A) \setminus (U, A) \subseteq (V, A) \subseteq (W, A)\). Also, \((V, A) \cap (W, A) = (\Phi, A)\) i.e. \((V, A) \cap (W, A) = (\Phi, A)\). Therefore, \((\tilde{X}, \tau, A)\) is a soft normal space.

The following example shows that the converse of Proposition 6.11 is not true.

**Example 6.12.** Let \(X = \{x, y\}, A = \{\alpha, \beta\}\) and \(\tau = \{(\Phi, A), (\tilde{X}, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}\), where \(F_1(\alpha) = \{x, y\}, F_1(\beta) = \{x\}; F_2(\alpha) = \{x\}, F_2(\beta) = \{y\}\). Then \((\tilde{X}, \tau, A)\) is a soft normal space. But \((F_1, A)\) is a soft open set containing the soft closed set \((F_3, A)\), (say), where \(F_3(\alpha) = \{y\}, F_3(\beta) = \{x\}\) for which the given condition of Proposition 6.11 is not satisfied.

**References**

[1] H. Aktas, N.Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726-2735.
[2] K.V. Babitha, J. J. Sunil, Soft set relations and functions, Comput. Math. Appl. 60 (2010) 1840-1849.

[3] K.V. Babitha, J. J. Sunil, Studies on soft topological spaces, Journal of Intelligent & Fuzzy Systems 28 (2015) 1713-1722.

[4] S. Bayramov, C. Gunduz, Soft locally compact spaces and soft paracompact spaces, Journal of Mathematics and System Science 3 (2013) 122-130.

[5] N. Cagman, S. Karatas and S. Enginoğlu, Soft topology, Comput. Math. Appl. 62 (1) (2011) 351-358.

[6] M. Chiney, S. K. Samanta, Vector soft topology, Ann. Fuzzy Math. Inform. 10 (1) (2015) 45-64.

[7] S. Das, S. K. Samanta, Soft real sets, soft real numbers and their properties, The Journal of Fuzzy Mathematics 20 (3) (2012) 551-576.

[8] S. Das, S. K. Samanta, On soft complex sets and soft complex numbers, The Journal of Fuzzy Mathematics 21 (1) (2013) 195-216.

[9] S. Das, S. K. Samanta, On soft metric spaces, The Journal of Fuzzy Mathematics 21 (3) (2013) 707-734.

[10] S. Das, P. Majumdar, S. K. Samanta, On soft linear spaces and soft normed linear spaces, Ann. Fuzzy Math. Inform. 9 (1) (2015) 91-109.

[11] F. Feng, Y. B. Jun, X. Zhao, Soft semirings, Comput. Math. Appl. 56 (10) (2008) 2621-2628.

[12] D. N. Georgiou, A. C. Megaritis, V. I. Petropoulos, On soft topological spaces, Appl. Math. Inf. Sci. 7 (5) (2013) 1889-1901.

[13] H. Hazra, P. Majumdar and S. K. Samanta, Soft topology, Fuzzy Information and Engineering 4 (1) (2012) 105-115.

[14] S. Hussain, B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62 (11) (2011) 4058-4067.

[15] Y. B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (5) (2008) 1408-1413.

[16] Y. B. Jun, C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Information Sciences 178 (11) (2008) 2466-2475.

[17] A. Kharal, B. Ahmad, Mappings on soft classes, New Math. Nat. Comput. 7 (3) (2011) 471-481.

[18] P. K. Maji, R. Biswas, A. R. Roy, An Application of soft sets in a decision making problem, Comput. Math. Appl. 44 (8-9) (2002) 1077-1083.
[19] P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (4-5) (2003) 555-562.

[20] P. Majumdar, S. K. Samanta, On soft mappings, Comput. Math. Appl. 60 (9) (2010) 2666-2672.

[21] Z. Ma, W. Yang, B. Hu, Soft set theory based on its extension, Fuzzy Information and Engineering 2 (4) (2010) 423-432.

[22] D. Molodtsov, Soft set theory first results, Comput. Math. Appl. 37 (4-5) (1999) 19-31.

[23] S. Mondal, M. Chiney, S. K. Samanta, Urysohn’s lemma and Tietze’s extension theorem in soft topological spaces, Ann. Fuzzy Math. Inform. 10 (6) (2015) 883-894.

[24] Sk. Nazmul, S. K. Samanta, Group soft topology, The Journal of Fuzzy Mathematics 22 (2) (2014) 435-450.

[25] Sk. Nazmul, S. K. Samanta, Neighbourhood properties of soft topological spaces, Ann. Fuzzy Math. Inform. 6 (1) (2013) 1-15.

[26] Sk. Nazmul, S. K. Samanta, Some properties of soft topologies and group soft topologies, Ann. Fuzzy Math. Inform. 8 (4) (2014) 645-661.

[27] M. Shabir, M. Ifran ali, Soft ideals and generalized fuzzy ideals in semigroups, New Math. Nat. Comput. 5 (3) (2009) 599-615.

[28] M. Shabir, M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (7) (2011) 1786-1799.

[29] F. G. Shi, B. Pang, A note on soft topological spaces, Iranian Journal of Fuzzy Systems 12 (5) (2015) 149-155.

[30] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2) (2012) 171-185.