Entanglement generation between distant atoms by Lyapunov control

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We show how to apply Lyapunov control design to the problem of entanglement creation between two atoms in distant cavities connected by optical fibers. The Lyapunov control design is optimal in the sense that the distance from the target state decreases monotonically and exponentially, and the concurrence increases accordingly. This method is far more robust than simple geometric schemes.

I. INTRODUCTION

Atoms, or their artificial counterparts, quantum dots, in cavities or traps have great potential for applications in quantum communication, metrology and information processing. Since entanglement is a crucial resource in quantum computation and communication, the preparation of maximally entangled states is a crucial task. Nonlocal interactions between two physical qubits are required to generate entanglement and there have been numerous proposals to effect such interactions, especially for atoms trapped in distant cavities [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], and similar schemes are conceivable for artificial atoms such as quantum dots. Some of these proposals make use of continuous feedback in open quantum systems [14] but most are based on Hamiltonian systems, and in most cases only simple geometric control schemes are employed to create the maximally entangled state. These methods have the advantage of simplicity but unfortunately often suffer from robustness issues.

In this work we explore an alternative control design inspired by Lyapunov functions [15, 16, 17, 18, 19, 20, 21, 22, 23, 24] to find control designs for robust entanglement creation. Lyapunov control design has the advantage of being sufficient simple to be amenable to rigorous analysis, and much is known about their convergence properties, robustness and stability. In particular such design can be shown to be highly effective for systems that satisfy certain sufficient conditions, which are roughly equivalent to the controllability of the linearized system [20, 22]. Unfortunately, this appears to be a strong requirement not satisfied by many physical systems. However, in certain cases, in particular for systems like the two-atom model proposed by Mancini and Bose [13], we can circumvent these restrictions by considering the dynamics on a subspace and successfully apply Lyapunov control to create maximally entangled states from certain initial product states in robust fashion.

The paper is organized as follows. In Sec. 2 we briefly review the distant-atom model and the geometric control scheme proposed in [13] to generate entanglement. In Sec. 3 we briefly review Lyapunov control and show how to apply it to the problem of steering the system from certain product state to one of the four Bell state in a robust fashion. We will consider two control paradigms: one is to control the local Hamiltonian which is easier to implement experimentally; the other is to control the non-local interaction Hamiltonian, which might be possible for certain systems.

II. TWO-DISTANT-ATOM MODEL AND GEOMETRIC CONTROL

We consider a two-qubit model where the qubits are encoded in two atoms or two quantum dots in distant cavities connected into a closed loop by optical fibers, as illustrated in Fig. 1. It was shown in [13] that eliminating the interacting light field between the two atoms in the dispersive regime leads to an effective Hamiltonian for the two-atom system of form $H_{\text{tot}} = H_{\text{local}} + H_{\text{eff}}$, where the local Hamiltonian induced by interaction with resonant light and the effective interaction Hamiltonian are:

$$H_{\text{local}} = B(X \otimes I + I \otimes X)$$  \hspace{1cm} (1a)

$$H_{\text{eff}} = 2JZ \otimes Z$$  \hspace{1cm} (1b)

where $X$, $Y$, $Z$ are Pauli operators and $I$ is the identity operator, and the coupling constant $B = \eta J$ where $\eta$...
Concurrence

against imperfections of switching time. Entanglement generation is quite simple, it is not robust
time but the concurrence fluctuations increase by a factor
prepare a maximally entangled state in about 1 sec.

J is fixed coupling strength to small fluctuations. In the model we have assumed a
quires very precise switching as the concurrence is subject
fidelity with respect to the maximally entangled state re-
plotting the concurrence of the final state versus the in-
system state ends at the correct target state. However,
can choose the rotation angle such as to ensure that the
in the two-qubit space, and with the correct timing we
of 

Broadly speaking, by applying a constant Hamil-
entangled state from the initial ground state by turning
B
against imperfections of switching time.

In the previous section we have seen that the method
on for a fixed amount of time is highly sensitive to small
switching time errors. Ideally, we would like a control
scheme where the concurrence of the two qubits converges
to 1 asymptotically, and better without any fluctuations.
In that way, the control is robust against switching time
errors. A simple method that seems well suited to this
task is Lyapunov-based design. Roughly speaking, the
idea of Lyapunov control is to choose a suitable so-called
Lyapunov function V and then try to find a control that
ensures that V is monotonically decreasing along any dy-
namical evolution.

In the time scale where the Hamiltonian evolution is
still a good approximation, many physical systems satisfy
the quantum Liouville equation (with \( \hbar = 1 \))

\[
\dot{\rho} = -i[H_0 + f(t)H_1, \rho],
\]

where we have assumed the Hamiltonian has two parts:
\( H_0 \) is the system Hamiltonian and \( H_1 \) is the inter-
action Hamiltonian, with the interaction coupling constant
modulated by the function \( f(t) \). For example, for a two-
level structure of a single atom, \( H_0 = \frac{\Omega}{2}\sigma_z \) is the en-
ergy splitting, and \( H_1 = \sigma_x \) is the dipole interaction be-
tween the laser and the atom, with a varying \( f(t) \) by modu-
lating the laser amplitude. The fact that \( f(t) \) can be
varied is very crucial from control point of view, since
this degree of freedom allows us to design the dynamics
to derive the desired evolution.

We can define a general control task thus: for a given
target state \( \rho_d \), for example, a maximally entangled state,
we wish to find a control function \( f(t) \), such that the
system state \( \rho(t) \) will converge to \( \rho_d \), as \( t \to \infty \). In many
applications, we allow \( \rho_d(t) \) to evolve under \( H_0 \), and the
control requirement becomes \( \rho(t) \to \rho_d(t) \) as \( t \to \infty \),
which is generally known as tracking control [23]. In the following we assume:

\[
\dot{\rho}_d = -i[H_0, \rho_d]
\]

Motivated from the theory of Lyapunov function and
the Hilbert Schmidt distance \( \| \rho(t) - \rho_d(t) \|_2 \), we define

\[
V(\rho, \rho_d) = \frac{1}{2} \| \rho - \rho_d \|^2 = \frac{1}{2} \text{Tr}[(\rho - \rho_d)^2].
\]  

Assuming \( \kappa > 0 \), if we choose

\[
f(t) = f(\rho(t), \rho_d(t)) = \kappa \text{Tr}(\rho_d(t)[-iH_1, \rho(t)]),
\]

we find that for \( V(t) = V(\rho(t), \rho_d(t)) \),

\[
\dot{V}(t) = -f(t) \text{Tr}(\rho_d(t)[-iH_1, \rho(t)]) = -\kappa f(t)^2 \leq 0.
\]

Hence \( V \) is a Lyapunov function and the value of \( V \) monotonically decreases along any solution \((\rho(t), \rho_d(t))\). Moreover, every solution \((\rho(t), \rho_d(t))\) converges to an invariant
set $E$, called the LaSalle invariant set, on which $\dot{V}$ vanishes.

Discussions on Lyapunov-based design in terms of density operators have been analyzed in Refs. [21, 22, 23]. In particular, many target states can be shown to be almost globally asymptotically stable if the Hamiltonian satisfies certain demanding conditions: (i) $H_0$ be strongly regular and (ii) $H_1$ be fully connected [23]. The former condition translates into the requirement that $H_0$ have distinct transition frequencies between any pair of energy levels. This rules out systems with degenerate energy levels. This restricts the applicability of the method especially for higher-dimensional systems, including two-qubit models and spin chains.

However, for high-dimensional systems with Hamiltonian not satisfying the above conditions, it is still possible to make the target state asymptotically stable on a subspace, where the Lyapunov control can be applied effectively. In the following, for the two-distant-atom model (Fig. 1), we illustrate how the Lyapunov control design can be utilized to drive the system state from a product state and the Bell state $|\Psi^+\rangle$, the control design be-

and it is easy to see that the eigenvectors of $H_0$ are the Bell states

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |+\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|-\rangle - |+\rangle).$$

To generate maximally entangled state, we can choose $\rho_d = |\Phi^+\rangle \langle \Phi^+ |$, for instance, and the control $f(t) = \kappa \text{Tr}(\rho_d [|-iH_1, \rho(t)|])$, according to (3). Notice that $H_0$ and $H_1$ do not satisfy the strict condition in Section III. Thus this design cannot drive every state to the target state, but we can see that if the initial state of the system is $\rho(0) = |++\rangle \langle ++ |$ or $|--\rangle \langle -- |$ then the state will converge to the target state. In fact, in the Bell-state basis, the Hamiltonian can be written as

$$\tilde{H}_0 = 2J \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \tilde{H}_1 = 2\eta J \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

where

$$\rho_d = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \rho(0) = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \pm 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

FIG. 3: Local Control: control fields obtained from Lyapunov design for different values of $\kappa$ and distance between the system state and the Bell state $|\Psi^+\rangle$. The control design is robust in that the field amplitude gently decreases to zero, and the semilog distance plot shows that the convergence to the target state is not only monotonic but also exponential with the converging rate determined by $\kappa$.
For states initially prepared in the subspace \( S \) spanned by \(|++\rangle\) and \(|-\rangle\), we clearly see that the dynamics under the Hamiltonian \( H_0 + f(t)H_1 \) will be confined in that subspace, and thus we can consider the dynamics on this two-dimensional subspace \( S \) where the Hamiltonians and state take the form:

\[
H_0 = 2J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H_1 = 2\eta J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The results in [23] now guarantee that all solutions in \( S \) except for \(|\Phi^-\rangle\) will converge to the target state. The control field varies smoothly and steers the system gently to the target state as shown in Fig. 3. Moreover convergence is exponential and

\[
|f(t)| = \kappa \left| \text{Tr}(i[\rho(t), \rho_d(t)]H_1) \right| = \kappa |i[\rho(t), \rho_d(t)]H_1| \leq \kappa |i[\rho(t), \rho_d(t)]| \cdot \|H_1\|
\]

shows that \( f(t) \) is bounded and we can choose \( \kappa \) to ensure that \( |f(t)| \) is sufficiently small and the approximations inherent in the model remain valid.

The method can also be utilized to increase the entanglement in the initial state, i.e., to prepare a maximally entangled state starting with a partially entangled one. Specifically, if the system initial starts in the state \( |\psi_0\rangle = \lambda_1|++\rangle + \lambda_2|-\rangle \) then the control design produces a control field that steers the system from this state to the desired maximally entangled state \(|\Phi^+\rangle\). Choosing \( \rho_d = |\Phi^+\rangle\langle\Phi^+| \) instead, we can similarly prepare \(|\Phi^-\rangle\), and it can be verified that steering the state to \(|\Psi^-\rangle\) simply requires inverting the sign of the control field. Thus, not only can we prepare a maximally entangled state, but we can select which state we prepare.

If the coupling constants of the local Hamiltonian for the two atoms are not exactly identical, e.g., if \( H_{\text{local}} = \eta J (X \otimes I + I \otimes X) \) then changing to the \( X \)-basis gives \( \hat{H}_1 = \eta J \text{diag}(1+k, 1-k, -1+k, -1-k) \), which transforms to

\[
\tilde{H}_1 = \eta J \begin{pmatrix} 0 & 0 & 0 & 1-k \\ 0 & 0 & 1+k & 0 \\ 0 & 1+k & 0 & 0 \\ 1-k & 0 & 0 & 0 \end{pmatrix}.
\]

Thus for \( k \neq 1 \) we can also steer the system from the product states \(|-\rangle\rangle\) or \(|-\rangle\rangle\) to the Bell state \(|\Phi^\pm\rangle\), i.e., for this two-atom model Lyapunov control can be used to prepare any of the four Bell states.

One limitation of the scheme is that the initial state must be in the subspace \( S \), for example, \( S = \text{span}(|+\rangle, |-\rangle) \), for the control to be effective. This is not a shortcoming of the proposed control scheme, however, because we can see from the structure of \( \hat{H}_0 \) and \( \hat{H}_1 \) that the control system is decomposable, hence not controllable on the whole space [26]. More specifically, the dynamics on the orthogonal subspaces \( S \) and \( S^\perp \) are independent, and subspace populations are conserved quantities. Thus, for the above Hamiltonian, no control exists that steers population from subspace \( S \) to \( S^\perp \) and vice versa.

\[\text{B. Interaction control}\]

Instead of controlling the atoms locally, we can alternatively control the nonlocal Hamiltonian \( \hat{H}_{\text{eff}} \), if the underlying physical system allows. In this case we choose \( H_0 = \eta J (X \otimes I + I \otimes X) \) and \( H_1 = 2J (Z \otimes Z) \), or in the \( X \)-eigenbasis

\[
H_0 = 2\eta J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H_1 = 2J \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

The Bell states are no longer the eigenstates of \( H_0 \). Hence, for \( \rho_d(0) = |\Phi^+\rangle\langle\Phi^+| \), the target state is also evolving with time, but for \( \rho(0) = |++\rangle\rangle \) the dynamics is still confined to the subspace \( S \) spanned by \(|++\rangle\rangle \) and \(|-\rangle\rangle \). Therefore, the dynamics can again be reduced to a 2D subspace on which we have

\[
H_0 = 2\eta J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H_1 = 2J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

as well as

\[
\rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_d(0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

where the orbit of \( \rho_d(t) \) is the equator of the Bloch sphere.
From the analysis in [23] we can conclude that all solutions in $S$ will converge to the equator of the Bloch sphere, i.e., states of the form

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\alpha} \\ e^{i\alpha} & 1 \end{pmatrix}$$

which corresponds to the LaSalle invariant set $E$ of the original problem satisfying

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & e^{-i\alpha} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \\ e^{i\alpha} & 0 & 0 & 1 \end{pmatrix}.$$ 

Thus, we can no longer guarantee $\rho(t) \to \rho_d(t)$ as $t \to +\infty$, i.e., that the state converges to a particular Bell state. This is illustrated in Fig. 4, which shows that the distance from the target state still decreases monotonically and exponentially but the asymptotic value of $V(\rho(t), \rho_d(t))$ for $t \to \infty$ now depends on $\kappa$ and is generally larger than zero. However, since all the states in the set to which $\rho(t)$ converge are maximally entangled, we can still steer the system to a maximally entangled state, and the concurrence still increases monotonically to one (see Fig. 5) but the relative phase $\alpha$ of the state we converge to now depends on the exact initial state and the feedback strength $\kappa$.

Strictly speaking, as the control $f(t)$ reduces to zero, the norm of $H_{\text{local}}$ will cease to be significantly smaller than that of $H_{\text{eff}}$, rendering the approximations made in the derivation of $H_{\text{eff}}$ invalid, unless we reduce the strength of $H_{\text{local}}$ accordingly. However, in practice, the system should already have reached a state with significant entanglement before the model becomes invalid.

V. CONCLUSION

We have shown how to apply Lyapunov control to the problem of generating entanglement between two distant two-levels atoms in cavities connected by optical fibers. Given the Lyapunov control design, despite the fact that the sufficient condition for a target state to be asymptotically stable is not satisfied on the whole state space, we can still ensure it is almost globally asymptotically stable on certain subspace. Therefore, within that subspace we can drive the system from a product state to a maximally entangled state. The Lyapunov control design has the advantage of much greater robustness compared to simple geometric schemes, and optimality in the sense that the distance from the maximally entangled target state is monotonically decreasing, and the convergence speed is exponential. We have discussed two control paradigms: to control the local Hamiltonian, as well as to control the effective interaction Hamiltonian between the two atoms. In both cases we can generate a maximally entangled state from an initial product state: for the formal case the system state will converge to a stationary Bell state, while for the latter case the relative phase of the final state will keep varying under the Hamiltonian, since the target state is non-stationary. Moreover, in the latter case, the model becomes invalid when the control amplitude becomes sufficiently small. Therefore, the former control paradigm is preferable. The Lyapunov control design can be also used to steer partially entangled states to a maximally entangled state, however, the control is only effective for initial states in the subspace where the target state is asymptotically stable. This is not a limitation of the control design, however, but a consequence of the fact that the controlled system is decomposable into two orthogonal subspaces on each of which the dynamics is invariant. In this sense, the Lyapunov control design is as effective as is possible within the constraints of the model.

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