Spinors, superalgebras and the signature of space-time

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ABSTRACT

Superconformal algebras embedding space-time in any dimension and signature are considered. Different real forms of the $R$-symmetries arise both for usual space-time signature (one time) and for Euclidean or exotic signatures (more than one times). Application of these superalgebras are found in the context of supergravities with $32$ supersymmetries, in any dimension $D \leq 11$. These theories are related to $D = 11$, $M, M^*$ and $M'$ theories or $D = 10$, IIB, IIB* theories when compactified on Lorentzian tori. All dimensionally reduced theories fall in three distinct phases specified by the number of (128 bosonic) positive and negative norm states: $(n^+, n^-) = (128, 0), (64, 64), (72, 56)$.

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1 Introduction

We describe superconformal algebras embedding space-time $V_{s,t}$ with arbitrary dimension $D = s + t$ and signature $\rho = s - t$ [1]. The relation between $R$-symmetries and space-time signatures is elucidated [2]. For supergravities with 32 supersymmetries, there exist theories in three distinct phases specified by the number of (128 bosonic) positive and negative metric “states”:

$$(n^+, n^-) = (128, 0), (64, 64), (72, 56).$$

This fact is closely related to the compactification of $M, M^*$ and $M'$ theories on Lorentzian tori [3].

When dimensionally reduced to $D = 3$, two of these phases are Minkowskian with $\sigma$-model $E_{8(8)}/SO(16)$ and $E_{8(8)}/SO(8, 8)$; one is Euclidean with $\sigma$-model $E_{8(8)}/SO^*(16)$. The three real forms of $D_8$ correspond to the superconformal algebras $Osp(16/4, R), Osp(8, 8/4, R)$ and $Osp(16^*/2, 2)$ associated to real and quaternionic spinors respectively.

The paper is organized as follows:

In Sections 2 and 3, we describe symmetry and reality properties of spinors in arbitrary space-time signatures and dimensions. In Sections 4 and 5, we consider real forms of classical Lie algebras and superalgebras. In Section 6, we study extended superalgebras. Supersymmetry algebras with non-compact $R$-symmetries arise in the context of Euclidean theories as well as in theories with exotic (more than one times) signature. In Section 7, we show that non-compact $R$-symmetries arise when $D = 11$ and $M, M^*, M'$ theories introduced by C. Hull [3] are compactified to lower dimensions on Lorentzian tori.

Euclidean supergravity from time-reduction [1] of $M$-theory was considered by Hull and Julia [3] and BPS branes in theories with more than one times were studied by Hull and Khuri [3]. Hull’s observation is that theories on exotic space-time signatures arise by $T$-duality on time-direction from conventional $M$ theory on string theory so this can perhaps be regarded as different phases of the same non-perturbative $M$-theory [3]. Irrespectively of whether it is sensible to use time-like $T$-duality in string theory [7], supergravity theories in exotic space-time dimensions certainly exist, on the basis of invariance principles due to the existence of appropriate superalgebras [1,3]. $U$-duality also implies the existence of two new kinds of type II string theories, IIA*, IIB in $V_{9,1}$ space-time where all bosonic $RR$ fields have reversed sign for the metric [8]. A strong evidence of this reasoning is that the $U$-duality groups in all lower dimensions are the same, independently of the space-time signature. BPS states are classified by orbits of the $U$-duality group. These orbits were given in Refs. [9,10] and [11]. Since they depend only on the $U$-duality group and not on the $R$-symmetry, it follows that the orbit classification is insensitive to the signature of space-time and is still valid for $M^*, M'$ theories.

Finally, in Section 8 we describe state counting of supergravities in exotic space-time. This is given in $D = 4$ and $D = 3$ dimensions, as well as in the $D = 11, 10$ original dimensions. This counting shows that there are essentially three distinct phases, in any dimension and signature where the (128) bosonic degrees of freedom fall in positive and negative norm classes

$$(n^+, n^-) = (128, 0), (64, 64), (72, 56).$$

The first two are Minkowskian phases (one time or more than one times), the latter is the Euclidean phase (no time). State counting in theories with
16 supersymmetries is also given, and again three different phases emerge.

2 Properties of spinors of SO(V)

Let $V$ be a real vector space of dimension $D = s + t$ and $\{v_\mu\}$ a basis of it. On $V$ there is a non degenerate symmetric bilinear form, which in the basis is given by the matrix

$$\eta_{\mu\nu} = \text{diag}(+\ldots(s\text{ times})\ldots,+,\ldots,+\ldots(t\text{ times})\ldots,-).$$

We consider the group $\text{Spin}(V)$, the unique double covering of the connected component of $\text{SO}(s,t)$ and its spinor representations. A spinor representation of $\text{Spin}(V)^C$ is an irreducible complex representation whose highest weights are the fundamental weights corresponding to the right extreme nodes in the Dynkin diagram. These do not descend to representations of $\text{SO}(V)$. A spinor-type representation is any irreducible representation that does not descend to $\text{SO}(V)$. A spinor representation of $\text{Spin}(V)$ over the reals is an irreducible representation over the reals whose complexification is a direct sum of spin representations $[12] - [15]$.

Two parameters, the signature $\rho$ mod(8) and the dimension $D$ mod(8) classify the properties of the spinor representation. Through this paper we will use the following notation:

$$\rho = s - t = \rho_0 + 8n, \quad D = s + t = D_0 + 8p,$$

where $\rho_0, D_0 = 0, \ldots, 7$. We set $m = p - n$, so

$$s = \frac{1}{2}(D + \rho) = \frac{1}{2}(\rho_0 + D_0) + 8n + 4m,$$

$$t = \frac{1}{2}(D - \rho) = \frac{1}{2}(D_0 - \rho_0) + 4m.$$

The signature $\rho$ mod(8) determines if the spinor representations are of the real($\mathbb{R}$), quaternionic ($\mathbb{H}$) or complex ($\mathbb{C}$) type. Also note that reality properties depend only on $|\rho|$ since $\text{Spin}(s,t) = \text{Spin}(t,s)$.

The dimension-$D$ mod(8) determines the nature of the $\text{Spin}(V)$-morphisms of the spinor representation $S$. Let $g \in \text{Spin}(V)$ and let $\Sigma(g) : S \rightarrow S$ and $L(g) : V \rightarrow V$ the spinor and vector representations of $l \in \text{Spin}(V)$ respectively. Then a map $A$

$$A : S \otimes S \rightarrow \Lambda^k,$$

where $\Lambda^k = \Lambda^k(V)$ are the $k$-forms on $V$, is a $\text{Spin}(V)$-morphism if

$$A(\Sigma(g)s_1 \otimes \Sigma(g)s_2) = L^k(g)A(s_1 \otimes s_2).$$

In Tables 1 and 2, reality and symmetry properties of spinors are reported.
ρ_0(odd) | Real dim(S) | Reality | ρ_0(even) | Real dim(S^±) | Reality
---|---|---|---|---|---
1 | 2^{(D-1)/2} | ℝ | 0 | 2^{D/2-1} | ℝ
3 | 2^{(D+1)/2} | ℍ | 2 | 2^{D/2} | ℂ
5 | 2^{(D+1)/2} | ℍ | 4 | 2^{D/2} | ℍ
7 | 2^{(D-1)/2} | ℝ | 6 | 2^{D/2} | ℂ

Table 1: Reality properties of spinors

| D | k even | k odd |
|---|---|---|
| Morphism | Symmetry | Morphism | Symmetry |
| 0 | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} | S^± ⊗ S^± → Λ^k |
| 1 | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} |
| 2 | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} |
| 3 | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} |
| 4 | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} |
| 5 | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} |
| 6 | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} | S^± ⊗ S^± → Λ^k | (-1)^{k(k-1)/2} |
| 7 | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} | S ⊗ S → Λ^k | (-1)^{k(k-1)/2} |

Table 2: Properties of morphisms

3 Orthogonal, symplectic and linear spinors

We now consider morphisms

\[ S \otimes S \longrightarrow Λ^0 \approx ℂ. \]

If a morphism of this kind exists, it is unique up to a multiplicative factor. The vector space of the spinor representation then has a bilinear form invariant under Spin(V). Looking at Table 2, one can see that this morphism exists except for D_0 = 2, 6, where instead a morphism

\[ S^± \otimes S^± \longrightarrow ℂ \]

occurs.

We shall call a spinor representation orthogonal if it has a symmetric, invariant bilinear form. This happens for D_0 = 0, 1, 7 and Spin(V)^ℂ (complexification of Spin(V)) is then a subgroup of the complex orthogonal group SO(n, ℂ), where n is the dimension of the spinor representation (Weyl spinors for D even). The generators of SO(n, ℂ) are n × n antisymmetric matrices. These are obtained in terms of the morphisms

\[ S \otimes S \longrightarrow Λ^k, \]
which are antisymmetric. This gives the decomposition of the adjoint representation of $\text{SO}(n, \mathbb{C})$ under the subgroup $\text{Spin}(V)^{\mathbb{C}}$. In particular, for $k = 2$ one obtains the generators of $\text{Spin}(V)^{\mathbb{C}}$.

A spinor representation is called symplectic if it has an antisymmetric, invariant bilinear form. This is the case for $D_0 = 3, 4, 5$. $\text{Spin}(V)^{\mathbb{C}}$ is a subgroup of the symplectic group $\text{Sp}(2p, \mathbb{C})$, where $2p$ is the dimension of the spinor representation. The Lie algebra $\text{sp}(2p, \mathbb{C})$ is formed by all the symmetric matrices, so it is given in terms of the morphisms $S \otimes S \to \Lambda^k$, which are symmetric. The generators of $\text{Spin}(V)^{\mathbb{C}}$ correspond to $k = 2$ and are symmetric matrices.

For $D_0 = 2, 6$ one has an invariant morphism

$$B : S^+ \otimes S^- \to \mathbb{C}.$$ 

One of the representations $S^+$ and $S^-$ is one the contragradient (or dual) of the other. The spin representations extend to representations of the linear group $\text{GL}(n, \mathbb{C})$, which leaves the pairing $B$ invariant. These spinors are called linear. $\text{Spin}(V)^{\mathbb{C}}$ is a subgroup of the simple factor $\text{SL}(n, \mathbb{C})$.

These properties depend exclusively on the dimension. When combined with the reality properties, which depend on $\rho$, one obtains real groups embedded in $\text{SO}(n, \mathbb{C})$, $\text{Sp}(2p, \mathbb{C})$ and $\text{GL}(n, \mathbb{C})$, which have an action on the space of the real spinor representation $S^\sigma$. The real groups contain $\text{Spin}(V)$ as a subgroup.

We first need some general facts about real forms of simple Lie algebras. Let $S$ be a complex vector space of dimension $n$, which carries an irreducible representation of a complex Lie algebra $\mathcal{G}$. Let $\mathcal{G}$ be the complex Lie group associated to $\mathcal{G}$. Let $\sigma$ be a conjugation or a pseudoconjugation on $S$ such that $\sigma X \sigma^{-1} \in \mathcal{G}$ for all $X \in \mathcal{G}$. Then the map

$$X \mapsto X^\sigma = \sigma X \sigma^{-1}$$

is a conjugation of $\mathcal{G}$. We shall write

$$\mathcal{G}^\sigma = \{ X \in \mathcal{G} | X^\sigma = X \}.$$ 

$\mathcal{G}^\sigma$ is a real form of $\mathcal{G}$; if $\tau = h \sigma h^{-1}$, with $h \in \mathcal{G}$, $\mathcal{G}^\tau = h \mathcal{G}^\sigma h^{-1}$. We have $\mathcal{G}^\sigma = \mathcal{G}^{\sigma'}$ if and only if $\sigma' = \epsilon \sigma$, for $\epsilon$ a scalar with $|\epsilon| = 1$; in particular, if $\mathcal{G}^\sigma$ and $\mathcal{G}^\tau$ are conjugate by $\mathcal{G}$, $\sigma$ and $\tau$ are both conjugations or both pseudoconjugations. The conjugation can also be defined on the group $\mathcal{G}$, $g \mapsto \sigma g \sigma^{-1}$.

4 Real forms of the classical Lie algebras

We describe the real forms of the classical Lie algebras from this point of view. (See also Ref. [16].)
Linear algebra, $\text{sl}(S)$.

(a) If $\sigma$ is a conjugation of $S$, then there is an isomorphism $S \to \mathbb{C}^n$ such that $\sigma$ goes over to the standard conjugation of $\mathbb{C}^n$. Then $\mathcal{G}^\sigma \simeq \text{sl}(n, \mathbb{R})$. (The conjugation acting on $\text{gl}(n, \mathbb{C})$ gives the real form $\text{gl}(n, \mathbb{R})$).

(b) If $\sigma$ is a pseudoconjugation and $\mathcal{G}$ does not leave invariant a non-degenerate bilinear form, then there is an isomorphism of $S$ with $\mathbb{C}^n$, $n = 2p$ such that $\sigma$ goes over to

$$(z_1, \ldots, z_p, z_{p+1}, \ldots, z_{2p}) \mapsto (z^*_{p+1}, \ldots, z^*_{2p}, -z^*_1, \ldots, -z^*_p).$$

Then $\mathcal{G}^\sigma \simeq \text{su}^*(2p)$. (The pseudoconjugation acting in on $\text{gl}(2p, \mathbb{C})$ gives the real form $\text{su}^*(2p) \oplus \text{so}(1,1)$.)

To see this, it is sufficient to prove that $\mathcal{G}^\sigma$ does not leave invariant any non-degenerate hermitian form, so it cannot be of the type $\text{su}(p,q)$. Suppose that $\langle \cdot, \cdot \rangle$ is a $\mathcal{G}^\sigma$-invariant, non-degenerate hermitian form. Define $(s_1, s_2) := \langle \sigma(s_1), s_2 \rangle$. Then $(\cdot, \cdot)$ is bilinear and $\mathcal{G}^\sigma$-invariant, so it is also $\mathcal{G}$-invariant.

(c) The remaining cases, following E. Cartan’s classification of real forms of simple Lie algebras, are $\text{su}(p,q)$, where a non-degenerate hermitian bilinear form is left invariant. They do not correspond to a conjugation or pseudoconjugation on $S$, the space of the fundamental representation. (The real form of $\text{gl}(n, \mathbb{C})$ is in this case $\text{u}(p,q)$).

Orthogonal algebra, $\text{so}(S)$. $\mathcal{G}$ leaves invariant a non-degenerate, symmetric bilinear form. We will denote it by $(\cdot, \cdot)$.

(a) If $\sigma$ is a conjugation preserving $\mathcal{G}$, one can prove that there is an isomorphism of $S$ with $\mathbb{C}^n$ such that $(\cdot, \cdot)$ goes to the standard form and $\mathcal{G}^\sigma$ to $\text{so}(p,q)$, $p + q = n$. Moreover, all $\text{so}(p,q)$ are obtained in this form.

(b) If $\sigma$ is a pseudoconjugation preserving $\mathcal{G}$, $\mathcal{G}^\sigma$ cannot be any of the $\text{so}(p,q)$. By Cartan’s classification, the only other possibility is that $\mathcal{G}^\sigma \simeq \text{so}^*(2p)$. There is an isomorphism of $S$ with $\mathbb{C}^{2p}$ such that $\sigma$ goes to

$$(z_1, \ldots, z_p, z_{p+1}, \ldots, z_{2p}) \mapsto (z^*_{p+1}, \ldots, z^*_{2p}, -z^*_1, \ldots, -z^*_p).$$

Symplectic algebra, $\text{sp}(S)$. We denote by $(\cdot, \cdot)$ the symplectic form on $S$. 

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(a) If $\sigma$ is a conjugation preserving $G$, it is clear that there is an isomorphism of $S$ with $\mathbb{C}^{2p}$, such that $G^\sigma \simeq \text{sp}(2p, \mathbb{R})$.

(b) If $\sigma$ is a pseudoconjugation preserving $G$, then $G^\sigma \simeq \text{usp}(p, q)$, $p + q = n = 2m$, $p = 2p'$, $q = 2q'$. All the real forms $\text{usp}(p, q)$ arise in this way. There is an isomorphism of $S$ with $\mathbb{C}^{2p}$ such that $\sigma$ goes to

$$(z_1, \ldots, z_m, z_{m+1}, \ldots z_n) \mapsto J_m K_{p',q'}(z_1^*, \ldots, z_m^*, z_{m+1}^*, \ldots z_n^*),$$

where

$$J_m = \begin{pmatrix} 0 & I_{m \times m} \\ -I_{m \times m} & 0 \end{pmatrix}, \quad K_{p',q'} = \begin{pmatrix} -I_{p' \times p'} & 0 & 0 & 0 \\ 0 & I_{q' \times q'} & 0 & 0 \\ 0 & 0 & -I_{p' \times p'} & 0 \\ 0 & 0 & 0 & I_{q' \times q'} \end{pmatrix}.$$ 

In Section 2 we saw that there is a conjugation on $S$ when the spinors are real and a pseudoconjugation when they are quaternionic [1] (both denoted by $\sigma$). We have a group, $\text{SO}(n, \mathbb{C})$, $\text{Sp}(2p, \mathbb{C})$ or $\text{GL}(n, \mathbb{C})$ acting on $S$ and containing $\text{Spin}(V)^\mathbb{C}$. We note that this group is minimal in the classical group series. If the Lie algebra $G$ of this group is stable under the conjugation

$X \mapsto \sigma X \sigma^{-1}$,

then the real Lie algebra $G^\sigma$ acts on $S^\sigma$ and contains the Lie algebra of $\text{Spin}(V)$. We shall call it the $\text{Spin}(V)$ algebra.

Let $B$ be the space of $\text{Spin}(V)^\mathbb{C}$-invariant bilinear forms on $S$. Since the representation on $S$ is irreducible, this space is at most one-dimensional. Let it be one-dimensional, let $\sigma$ be a conjugation or a pseudoconjugation, and let $\psi \in B$. We define a conjugation in the space $B$ as

$$B \longrightarrow \psi \longrightarrow B$$

$$\psi \mapsto \psi^\sigma$$

$$\psi^\sigma(v, u) = \psi(\sigma(v), \sigma(u))^*.$$ 

It is then immediate that we can choose $\psi \in B$ such that $\psi^\sigma = \psi$. Then if $X$ belongs to the Lie algebra preserving $\psi$, so does $\sigma X \sigma^{-1}$.

One can determine the real Lie algebras in each case [1]. All the possible cases must be studied separately. All dimension and signature relations are mod(8). In the following, a relation like $\text{Spin}(V) \subseteq G$ for a group $G$ will mean that the image of $\text{Spin}(V)$ under the spinor representation is in the connected component of $G$. The same applies for the relation $\text{Spin}(V) \simeq G$. For $\rho = 0, 1, 7$ spin algebras commute with a conjugation, for $\rho = 3, 4, 5$ they commute with a pseudoconjugation. For $\rho = 2, 6$ they are complex. The complete classification is reported in Table 3.
Table 3: Spin($s, t$) algebras

| Orthogonal | Real, $\rho_0 = 1, 7$ | so(2$^{(D - 1)/2}$, $\mathbb{R}$) if $D = \rho$
|            | Quaternion, $\rho_0 = 3, 5$ | so*(2$^{(D - 1)/2}$) if $D = \rho$
|            | Quaternion, $\rho_0 = 3, 5$ | so*(2$^{(D - 1)/2}$) if $D \neq \rho$
| Symplectic  | Real, $\rho_0 = 1, 7$ | sp(2$^{(D - 1)/2}$, $\mathbb{R}$)
|            | Quaternion, $\rho_0 = 3, 5$ | usp(2$^{(D - 1)/2}$, $\mathbb{R}$) if $D = \rho$
|            | Quaternion, $\rho_0 = 3, 5$ | usp(2$^{(D - 1)/2}$, $\mathbb{R}$) if $D \neq \rho$
| Orthogonal | Real, $\rho_0 = 0$ | so(2$^{(D - 1)/2}$, $\mathbb{R}$) if $D = \rho$
|            | Quaternion, $\rho_0 = 4$ | so*(2$^{(D - 1)/2}$) if $D = \rho$
|            | Complex, $\rho_0 = 2, 6$ | so(2$^{(D - 1)/2}$, $\mathbb{C}$)$_\mathbb{R}$
| Symplectic  | Real, $\rho_0 = 0$ | sp(2$^{(D - 1)/2}$, $\mathbb{R}$)
|            | Quaternion, $\rho_0 = 4$ | usp(2$^{(D - 1)/2}$, $\mathbb{R}$) if $D = \rho$
|            | Quaternion, $\rho_0 = 4$ | usp(2$^{(D - 1)/2}$, $\mathbb{R}$) if $D \neq \rho$
|            | Complex, $\rho_0 = 2, 6$ | sp(2$^{(D - 1)/2}$, $\mathbb{C}$)$_\mathbb{R}$
| Linear     | Real, $\rho_0 = 0$ | sl(2$^{(D + 1)/2}$, $\mathbb{R}$)
|            | Quaternion, $\rho_0 = 4$ | su*(2$^{(D + 1)/2}$)
|            | Complex, $\rho_0 = 2, 6$ | su(2$^{(D + 1)/2}$) if $D = \rho$
|            | Quaternion, $\rho_0 = 4$ | su(2$^{(D + 1)/2}$) if $D \neq \rho$

5 Spin(V) superalgebras

We now consider the embedding of Spin(V) in simple real superalgebras. We require in general that the odd generators be in a real spinor representation of Spin(V). In the cases $D_0 = 2, 6$, $\rho_0 = 0, 4$ we have to allow the two independent irreducible representations $S^+$ and $S^-$ in the superalgebra, since the relevant morphism is

$$S^+ \otimes S^- \rightarrow \Lambda^2.$$

The algebra is then non-chiral.

We first consider minimal superalgebras [17, 18], i.e. those with the minimal even subalgebra. From the classification of simple superalgebras [19–21] one obtains the results listed in Table 4.

We note that the even part of the minimal superalgebra contains the Spin(V) algebra obtained in Section 4 as a simple factor. For all quaternionic cases, $\rho_0 = 3, 4, 5$, a second simple factor SU(2) is present. For the linear cases there is an additional non-simple factor, SO(1,1) or U(1), as discussed in Section 4.
Table 4: Minimal Spin($V$) superalgebras.

For $D = 7$ and $\rho = 3$ there is actually a smaller superalgebra, the exceptional superalgebra $f(4)$ with bosonic part spin(5,2) \times su(2). The superalgebra appearing in Table 4 belongs to the classical series and its even part is $so^*(8) \times su(2)$, $so^*(8)$ being the Spin(5,2)-algebra.

Keeping the same number of odd generators, the maximal simple superalgebra containing Spin($V$) is an orthosymplectic algebra with Spin($V$) \subseteq Sp(2n, \mathbb{R})$, $2n$ being the real dimension of $S$. The various cases are displayed in Table 5. We note that the minimal superalgebra is not a subalgebra of the maximal one, although it is so for the bosonic parts.
Table 6: \(N\)-extended Spin\((s, t)\) superalgebras

| \(D_0\) | \(\rho_0\) | R-symmetry | Spin\((s, t)\) superalgebra |
|---|---|---|---|
| 1,7 | 1,7 | \(\text{sp}(2N, \mathbb{R})\) | \(\text{osp}(2^{\frac{N-1}{2}}, 2^{\frac{N-1}{2}}|2N, \mathbb{R})\) |
| 1,7 | 3,5 | \(\text{usp}(2N - 2q, 2q)\) | \(\text{osp}(2\frac{N-1}{2}|2N - 2q, 2q)\) |
| 3,5 | 1,7 | \(\text{so}(N - q, q)\) | \(\text{osp}(N - q, q^{\frac{N-1}{2}})\) |
| 3,5 | 3,5 | \(\text{so}^*(2N)\) | \(\text{osp}(2N^*|2^{\frac{N-1}{2}}, 2^{\frac{N-1}{2}})\) |
| 0 | 0 | \(\text{sp}(2N, \mathbb{R})\) | \(\text{osp}(2^{\frac{N-1}{2}}, 2^{\frac{N-1}{2}}|2N)\) |
| 0 | 2,6 | \(\text{sp}(2N, \mathbb{C})_\mathbb{R}\) | \(\text{osp}(2\frac{N-1}{2}|2N, \mathbb{C})_\mathbb{R}\) |
| 0 | 4 | \(\text{usp}(2N - 2q, 2q)\) | \(\text{osp}(2\frac{N-1}{2}|2N - 2q, 2q)\) |
| 2,6 | 0 | \(\text{sl}(N, \mathbb{R})\) | \(\text{sl}(2^{\frac{N-1}{2}}|N, \mathbb{R})\) |
| 2,6 | 2,6 | \(\text{su}(N - q, q)\) | \(\text{su}(2^{\frac{N-1}{2}}, 2^{\frac{N-1}{2}}|N - q, q)\) |
| 2,6 | 4 | \(\text{su}^*(2N, \mathbb{R})\) | \(\text{su}^*(2\frac{N-1}{2}|2N)\) |
| 4 | 0 | \(\text{so}(N - q, q)\) | \(\text{osp}(N - q, q^{\frac{N-1}{2}})\) |
| 4 | 2,6 | \(\text{so}(N, \mathbb{C})_\mathbb{R}\) | \(\text{osp}(N|2^{\frac{N-1}{2}}, \mathbb{C})_\mathbb{R}\) |
| 4 | 4 | \(\text{so}^*(2N)\) | \(\text{osp}(2N^*|2^{\frac{N-1}{2}}, 2^{\frac{N-1}{2}})\) |

6 Extended Superalgebras

The present analysis can be generalized to the case of \(N\) copies of the spinor representation of spin\((s, t)\) algebras [3]. By looking at the classification of classical simple superalgebras [17]–[21], we find extensions for all \(N\), where the number of supersymmetries is always even if spinors are quaternionic (because of reality properties) or orthogonal (because of symmetry properties).

In Table 6 the classification is analogous to the one given in Table 4. Super-Poincaré algebras can be obtained from the simple superalgebras either by contraction Spin\((s, t)\) \(\rightarrow\) InSpin\((s, t - 1)\) or as subalgebras Spin\((s, t)\) \(\rightarrow\) InSpin\((s - 1, t - 1)\). It is important to observe that the \(R\)-symmetry may be non-compact for different signatures of space-time.

All these superalgebras have a space-time symmetry that commutes with the \(R\)-symmetry group. We can further extend these superalgebras to orthosymplectic superalgebras where these symmetries no longer commute. The number of fermionic generators remains unchanged. The bosonic part is simply a real symplectic algebra, which contains as a subalgebra the direct sum of the space-time and \(R\)-symmetry algebra [24]–[27]. These extensions are reported in Table 7. These superalgebras contain by contraction or as subalgebras, all Poincaré superalgebras in any dimension with all possible “central charges”.
Table 7: Orthosymplectic superalgebras

| $D_0$ | $\rho_0$ | $osp(1/2n, R) \supset sp(2n, R)$ |
|-------|---------|----------------------------------|
| 1,7   | 1,7     | $sp(2N \times 2^{\frac{D-1}{2}})$ |
| 1,7   | 3,5     | $sp(2N \times 2^{\frac{D-1}{2}})$ |
| 3,5   | 1,7     | $sp(N \times 2^{\frac{D-1}{2}})$  |
| 3,5   | 3,5     | $sp(2N \times 2^{\frac{D-1}{2}})$ |
| 0     | 0       | $sp(2n \times 2^{\frac{D-1}{2}})$ |
| 0     | 2,6     | $sp(2N \times 2^{\frac{D}{2}})$   |
| 0     | 4       | $sp(2N \times 2^{\frac{D-1}{2}})$ |
| 2,6   | 0       | $sp(N \times 2^{\frac{D}{2}})$    |
| 2,6   | 2,6     | $sp(N \times 2^{\frac{D}{2}})$    |
| 2,6   | 4       | $sp(2N \times 2^{\frac{D}{2}})$   |
| 4     | 0       | $sp(N \times 2^{\frac{D}{2}})$    |
| 4     | 2,6     | $sp(N \times 2^{\frac{D}{2}})$    |
| 4     | 4       | $sp(2N \times 2^{\frac{D-2}{2}})$ |

7 11D and 10D on $V_{s,t}$ supergravities revisited

Theories with $N$ super-Poincaré supersymmetries can be obtained as subalgebras of superconformal algebras with $2N$ spinorial generators. This is so because a conformal spinor $S_{s,t}$ of the conformal group $SO(s,t)$ always decomposes into two Poincaré spinors of opposite dimensions:

$$S_{s,t} \rightarrow s_{s-1,t-1}^{1/2} + s_{s-1,t-1}^{-1/2}.$$  

The $R$-symmetry is inherited from the simple superalgebra whose odd generators contain the spin group of the conformal group. Poincaré superalgebras with at most 32 supersymmetries are obtained as subalgebras of superconformal algebras with at most 64 charges. Then the $R$-symmetries of these theories are read from their superconformal extension. In particular, although a super-Poincaré graviton exists with at most 32 supersymmetries, a conformal graviton multiplet exists with at most 64 supercharges [29]–[31].

From Table 6, we can read the $R$-symmetries of super-Poincaré algebras by replacing $(D, \rho)$ with $(D-2, \rho)$.

Let us now consider maximal supergravity theories in $D = 11$ ($N = 1$) and $D = 10$, IIA, IIB. For $D = 11$, we know that a real spinor exists for $\rho = \pm 1 \mod 8$. This implies the existence of two more theories other than $M$ theory with signature $(9,2)$ and $(6,5)$.

These theories were introduced by C. Hull and called $M^*$ and $M'$ theories [3]. They were discovered on the basis of time-like $T$ duality of type IIA and IIB string theory, which requires the existence of new kinds of type II strings called IIA* and IIB* string theories [3]. The strong coupling limit of IIA* theory is $M^*$ theory.
By dimensional reduction of $M, M^*$ and $M'$ theories to $D = 10$, new type IIA theories with signatures (10,0), (9,1), (8,2), (6,4) and (5,5) are found \[3\]. Type IIA (10,0) is IIA Euclidean supergravity, (9,1)* supergravity is type IIA*. In the (8,2) (6,4) supergravities, there is a single (16) dimensional complex Weyl spinor such that $\psi_L^* = \psi_R$.

In type IIB, the story is rather different: (2,0) supergravity has an $R$-symmetry that can be either SO(2) or SO(1,1) for $\rho = 0$ or SO*$(2) = SO(2)$ for $\rho = 4$ (see Table 6 for $d = 4$). This leads to the existence of two types of IIB theories for $\rho = 0$, (9,1) and (5,5) signature called type IIB (SO(2)) and IIB* (SO(1,1)) by C. Hull \[8\] and one theory for (7,3) signature with SO(2) $R$-symmetry. Their respective 10D $\sigma$-model is $\frac{SL(2,R)}{SO(2)}$ for type IIB and $\frac{SL(2,R)}{SO(1,1)}$ for IIB* theories \[8, 4\].

Most interestingly, in type II* theories, the sign of the kinetic term in the RR bosonic fields is reversed with respect to the NS fields case. Lifting string theory to $D = 11 M$ theory, the above implies that in $M^*$ theory the three-form part of the action has a reversed sign with respect to the $M$ and $M'$ theories case \[3\].

We have just seen that a non-compact $R$-symmetry arises in IIB* theory in $D = 10$. To find other non-compact $R$-symmetries, it is sufficient to compactify $M, M^*$ and $M'$ theories on Lorentzian tori $T^{(p,q)}$. A property of $R$-symmetries is that they must contain, as a subgroup, the orthogonal group $SO(p,q)$ related to the classical moduli space of a Lorentzian torus

$$\mathcal{M}_{T^{(p,q)}} = \frac{GL(p + q, R)}{SO(p,q)}.$$ 

Moreover, the $R$-symmetries must also be a subgroup of the $U$-duality group $E_{11-D(11-D)}$ and be related to reality properties and dimension of space-time spinors from Table 6. This uniquely fixes the non-compact form of the $R$-symmetry $H_D$.

As illustrative examples, let us consider the $D = 5, 4$ and 3 cases for space-time of all possible signatures. For $D = 5, 4, 3$, $M, M^*$ and $M'$ theories we get

$$V_{(s,11-s)} \rightarrow V_{(s',t')} \times T^{(p,q)} \quad (s = 10, 9, 6) \quad (s' + p = s, \ t' + q = 11 - s).$$

For $D = 5, 4, 3$, the $R$-symmetries must be different real forms of the $C_4, A_7$ and $D_8$ Lie algebras, appropriate to spinors of $V_{s',t'}$, and they must contain $SO(p,q)$ as subgroup. The appropriate real forms can be read from Table 8 and are a consequence of Table 6.

Note that all real forms of $C_4, A_7$ and $D_8$ are a maximal subgroup of the $U$-duality groups $E_{6(6)}, E_{7(7)}, E_{8(8)}$. The associated moduli spaces of these theories are therefore $E_{11-D(11-D)}/H_D$, when $H_D$ are the appropriate real forms of the $R$-symmetries in Table 8.
| $D = 5$ | | | | $R$-symmetry |
|---|---|---|---|
| $(s, t)$ | $(s', t')$ | $(p, q)$ | |
| (10,1) | (4,1) | (6,0) | usp(8) |
| | (5,0) | (5,1) | usp(4,4) |
| (9,2) | (4,1) | (5,1) | usp(4,4) |
| | (3,2) | (6,0) | sp(8, $R$) |
| | (5,0) | (4,2) | usp(4,4) |
| (6,5) | (4,1) | (2,4) | usp(4,4) |
| | (1,4) | (5,1) | usp(4,4) |
| | (5,0) | (1,5) | usp(4,4) |
| | (0,5) | (6,0) | usp(8) |
| | (3,2) | (3,3) | sp(8, $R$) |
| | (2,3) | (4,2) | sp(8, $R$) |

| $D = 4$ | | | | $R$-symmetry |
|---|---|---|---|
| $(s, t)$ | $(s', t')$ | $(p, q)$ | |
| (10,1) | (3,1) | (7,0) | su(8) |
| | (4,0) | (6,1) | su*($8$) |
| (9,2) | (3,1) | (6,1) | su(4,4) |
| | (4,0) | (5,2) | su*($8$) |
| | (2,2) | (7,0) | sl($8$, $R$) |
| (6,5) | (3,1) | (3,4) | su(4,4) |
| | (1,3) | (5,2) | su(4,4) |
| | (4,0) | (2,5) | su*($8$) |
| | (0,4) | (6,1) | su*($8$) |
| | (2,2) | (4,3) | sl($8$, $R$) |

| $D = 3$ | | | | $R$-symmetry |
|---|---|---|---|
| $(s, t)$ | $(s', t')$ | $(p, q)$ | |
| (10,1) | (2,1) | (8,0) | so(16) |
| | (3,0) | (7,1) | so*($16$) |
| (9,2) | (2,1) | (7,1) | so(8,8) |
| | (1,2) | (8,0) | so(8,8) |
| | (3,0) | (6,2) | so*($16$) |
| (6,5) | (2,1) | (4,4) | so(8,8) |
| | (1,2) | (5,3) | so(8,8) |
| | (3,0) | (3,5) | so*($16$) |
| | (0,3) | (6,2) | so*($16$) |

Table 8: $R$-symmetries of 11$D$ supergravity compactified on Lorentzian tori
8 State counting and ghosts in $M, M^*, M'$ theories and IIB* theories

In this section, we count the degrees of freedom and show that in any theory, with any signature, supersymmetry implies that there are at most three distinct phases, two Minkowskian (at least one time), where all 128 bosons either are no-ghost or they split in $64^+$ positive norm and $64^-$ negative norm states. The other phase is Euclidean (no time) and the states arrange in a $72^+$ positive norm and $56^-$ negative norm sector. For the Euclidean theories, care is needed to give a meaning to a state and its norm since there are no truly massless particle in this case.

Our analysis is made as counting scalar degrees of freedom with their kinetic term factor after all these theories are dimensionally reduced to three dimensions. In this extreme situation, there are only three possibilities, as shown in Table 8, since the 128 bosons are coordinates of the $\sigma$-models $E_{8(8)}/SO(16)$ or $E_{8(8)}/SO(8,8)$ for Minkowskian phases or $E_{8(8)}/SO^*(16)$ for the Euclidean phase. Note that no other possibility is allowed since there are no other real forms of $D_8$ contained in $E_{8(8)}$.

The first case has no negative norm states since $SO(16)$ is the maximal compact subgroup of $E_{8(8)}$. For the other two cases, we precisely get $n^- = 64, 56$.

The rest of this section is devoted to showing how these states are arranged with the spin degrees of freedom when these theories are lifted to higher dimensions.

To do state counting in Minkowskian spaces with arbitrary signatures, one must use some rules, which are dictated by the underlying gauge invariance of the higher-dimensional supergravity with respect to coordinate transformations and gauge transformations of the $p$-form gauge potentials. For the metric part, one uses the fact that a massless graviton, in a Minkowskian space $V_{(s,t)}$, is associated to the coset $SL(s + t - 2)/SO(s - 1, t - 1)$. The number of positive and negative norm states is

\[
\begin{align*}
 n^+ &= \frac{(s + t - 2)(s + t - 1)}{2} - (s - 1)(t - 1) - 1 \\
n^- &= \frac{(s + t - 2)(s + t - 3)}{2} - (s - 1)(s - 2) + (t - 1)(t - 2) \frac{2}{2}.
\end{align*}
\]

Similar formulae exist for $p$-forms gauge fields. The above analysis does not apply for the Euclidean case $(t = 0)$. However, in this case, one can use the rule that a positive norm 3d Euclidean vector is dual to a negative norm Euclidean scalar. By lifting this rule, using the $KK$ ansatz, one is led to conclude that a Euclidean graviton in $D$ dimensions parametrizes the coset $SL(d - 2, R)/SO(D - 3, 1)$. Curiously, this is identical to the coset of a graviton in $V_{D-2,2}$ Minkowski space. The reason for this is that, after compactification on $T_{D-4}$, one gets 4d gravity in $(4,0)$ and $(2,2)$ space-time and they both give, upon $S_1$ reduction, the $\sigma$-model $SL(2,R)/SO(1,1)$, where the role of the two scalars has just been interchanged. Note that for $(3,1)$ gravity, the $S^1$ reduction gives the standard $SL(2,R)/SO(2)$ coset. Another explanation of this rule, related to $M^*$ theory, is given later. The above rules are also consistent with the
fact that upon reduction on a time-like $S^1$, gravity gives a $KK$ vector with a reversed sign of the metric (and a scalar with the correct sign), while a $p$-form gives a $(p-1)$ form with a wrong sign of the metric (in addition to a $p$-form with the original sign) [5].

Let us now consider $M, M^*$ and $M'$ theories. On an 11D Minkowskian background, the state counting goes as follows:

\[
\begin{align*}
M \text{ theory} & \quad 128 = 44^+ + 84^+ \\
M^* \text{ theory} & \quad 128 = 64^+ + 64^- \\
 & \quad 44 = 36^+ + 8^- \\
 & \quad 84 = 28^+ + 56^- \\
M' \text{ theory} & \quad 128 = 64^+ + 64^- \\
 & \quad 44 = 24^+ + 20^- \\
 & \quad 84 = 40^+ + 44^- \\
\end{align*}
\]

If we reduce $M^*$ theory on $S^1$, we get IIA* theory and the negative norm states correspond to the $RR$ vector and three forms. In a similar way, if we consider IIB* theory, the $NS$ and $RR$ states have reversed signs of the metric and since they are equal in number, they give $128 = 64^+_{NS} + 64^-_{RR}$ on a $(9,1)$ background. For the Euclidean gravity, IIA$_E$, the counting can be obtained by time-reduction on $S^1$ from $M$ theory and using the previous rules, the ghost counting goes as follows:

\[
\begin{align*}
128 & = 72^+ + 56^- \\
44 & = 30^+ + 14^- \\
84 & = 42 + 42^- \\
\end{align*}
\]

Note that in type IIA$_E$, the $RR$ vector and the $NS$ antisymmetric tensor have reversed sign, not the $RR$ three-form. If we instead consider $M^*$ theory as a space-like $S^1$, we get an $(8,2)$ theory where the $RR$ vector and $RR$ three-forms have reversed sign with respect to the IIA$_E$ theory. In the $(8,2)$ theory, these states contribute as $(36^+, 28^+)$ to the total $(64^+, 64^-)$ states. If we flip the signs of these states, we get $(36^+, 28^-)$ added to $(36^+, 28^-)$, explaining the $(72^+, 56^-)$ signs in the Euclidean case. In three dimensions, each phase corresponds to a different $\sigma$-model and there is a one-to-one correspondence. For higher dimensions, many different theories can be in the same phase. For instance in $D = 4$, $(64^+, 64^-)$ correspond to two $N = 8$ supergravities with $\sigma$-models $E_7(7)/SU(4,4)$ and $E_7(7)/SL(8,R)$, corresponding to $(3,1)$ signature and $(2,2)$ signature, respectively. The $(72^+, 56^-)$ phase corresponds to $N = 8$ Euclidean supergravity with $\sigma$-model $E_7(7)/SU^*(8)$ [8]. The higher the dimension, the more theories correspond to the same phase.

### 8.1 Theories with 16 supersymmetries

The previous analysis can be extended to any theory with a lower number of supersymmetries. Here we just consider theories with 16 supersymmetries. In $D = 10$ the $(1,0)$ algebra can
be constructed for signatures (9,1) and (5,5). This algebra admits both a supergravity and a Yang–Mills theory whose Lagrangians are identical to the standard (9,1) Lagrangians since their space-time is related by mod 8 periodicity. No other signatures are possible for real Weyl fermions.

By compactification on Lorentzian tori, one can get several theories in lower dimensions. In particular, descending to $D = 4$, one gets $N = 4$ supergravities and $N = 4$ superconformal Yang–Mills theories with space-time, with signature (3,1) (2,2) and (4,0) [33]–[35]. The relevant superalgebras are $SU(2,2/4)$, $SU(2,2/2,2)$ for a (3,1) signature (corresponding to super Yang–Mills on $T_6$ and $T_{2,4}$ respectively), $SL(4/4)$ for a (2,2) signature (corresponding to super Yang–Mills on $T_{(3,3)}$) and $SU^*(4/4)$ for a (4,0) signature (corresponding to super-Yang–Mills on $T_{5,1}$). In $D = 4$ the moduli space of (1,0) supergravity on the corresponding tori are:

\[
\begin{align*}
(3,1) \times (6,0) & \quad \frac{SO(6,6)}{SU(4) \times SU(4)} \times \frac{SU(1,1)}{U(1)} \\
(3,1) \times (2,4) & \quad \frac{SO(6,6)}{SU(2,2) \times SU(2,2)} \times \frac{SU(1,1)}{U(1)} \\
(4,0) \times (5,1) & \quad \frac{SO(6,6)}{SU^*(4) \times SU^*(4)} \times \frac{SU(1,1)}{SO(1,1)} \\
(2,2) \times (3,3) & \quad \frac{SO(6,6)}{SL(4,R) \times SL(4,R)} \times \frac{SU(1,1)}{SO(1,1)}
\end{align*}
\]

Further reducing to $D = 3$, we get three $\sigma$-models:

\[
\begin{align*}
\frac{SO(8,8)}{SO(8) \times SO(8)} \quad \frac{SO(8,8)}{SO(4,4) \times SO(4,4)} \quad \frac{SO(8,8)}{SO^*(8) \times SO^*(8)}
\end{align*}
\]

This again leads to three phases for the 64 (bosonic) degrees of freedom. In the Minkowskian phases, either $n^- = 0$ ($n^+ = 64$) or $n^- = 32$ ($n^+ = 32$). In the Euclidean phase ,$n^- = 24$ ($n^+ = 40$). Lifting the theory to $D = 10$, the graviton has $35 = 23^+ + 12^-$ degrees of freedom, the antisymmetric tensor $28 = 16^+ + 12^-$, and the dilaton $1^+$. The graviton states span the $\sigma$-model $SL(8,R)/SO(6,2)$. The positive and negative norm states fall, as usual, in the representation of the maximal compact subgroup $U(4)$.

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