Quantum information with conserved quantities

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Conserved quantities are crucial in quantum physics. Here we discuss a general scenario of Hamiltonians. All the Hamiltonians within this scenario share a common conserved quantity form. For unitary parametrization processes, the characteristic operator of this scenario is analytically provided, as well as the corresponding quantum Fisher information (QFI). As the application of this scenario, we focus on two classes of Hamiltonians: su(2) category and canonical category. Several specific physical systems in these two categories are discussed in detail. Besides, we also calculate an alternative form of QFI in this scenario.

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I. INTRODUCTION

With the development of quantum physics, quantum mechanics nowadays is way beyond the phase of logical argument and mind experiments, but deep into the applied field and even our daily lives. Quantum metrology is one of the most successful applications of quantum mechanics. Starting from the pioneer work of Caves [1,2], people gradually realized that many quantum effects and quantum mechanics nowadays is way beyond the phase of logical argument and mind experiments, but deep into the applied field and even our daily lives. Quantum metrology is one of the most successful applications of quantum mechanics. Starting from the pioneer work of Caves [1,2], people gradually realized that many quantum effects and states [3,4] are available to enhance the measurement precision of physical parameters via various quantum systems.

Quantum Fisher information (QFI) is a central quantity in quantum metrology because it depicts the lower bound on the variance of the estimator $\hat{\theta}$ due to the Cramér-Rao theory: $\delta^2\hat{\theta} \geq (\nu F)^{-1}$ [16,17], where $\delta^2\hat{\theta}$ is the variance, $\nu$ is the number of repeated experiments, and $F$ is the QFI. Generally, the QFI is defined as $F = \text{Tr}(\rho L^2)$. Here $\rho$ is a parametrized state and $L$ is the symmetric logarithmic derivative (SLD), which is determined by the equation $\partial_\theta \rho = (\rho L + L \rho)/2$. $\theta$ is the parameter under estimation. Quantum Fisher information matrix (QFIM) $F$ is the counterpart of QFI in multi-parameter estimation. The element of QFIM is defined as $F_{mn} = \text{Tr}(\rho[L_m,L_n])/2$. $L_m$, $L_n$ are the SLD operators for $m$th and $n$th parameters under estimation, respectively.

Because of the importance of QFI, its calculation is always an interesting and attractive topic. Recently, several alternative formulas of QFI have been developed [20,21,23–27]. For unitary parametrization processes, new expressions of QFI and QFIM were given for both pure and mixed states [20,21]. The QFI and QFIM here are totally determined by a characteristic operator $\mathcal{H}$ and the initial states. In these expressions, all the information of parametrization is involved in $\mathcal{H}$. Furthermore, $\mathcal{H}$ can be written into an expanded form [21], which is particularly useful when the $n$th order commutation between the Hamiltonian and its partial derivative can be truncated or is periodic. Moreover, for a general parametrized exponential state, the SLD operator is also found to be expressed in an expanded form [23]. These facts prompt us to study various Hamiltonians owning above properties. This is the major motivation of this paper.

Conserved quantities are crucial in both classical and quantum physics. Since Emmy Noether connected them with differentiable symmetries in Noether’s theorem, searching for conserved quantities becomes a prior mission when facing a novel new system. Locating conserved quantities will not only help us to find hidden symmetries of these systems, but also give us an easy perspective to describe and classify them.

In this paper, we discuss a scenario, in which all systems share a common conserved quantity form. For the unitary parametrization processes, we provide the analytical expressions of the characteristic operator and the QFI. The maximum QFI and the corresponding optimal initial states are discussed. Moreover, we also study QFI for the parametrized thermal states in these systems. The scenario we discuss includes many systems, of which two typical classes, su(2) category and canonical category, are the main applications of this paper. In the su(2) category, ferromagnetic two-spin system, anisotropic two-spin system and a spin-one system are discussed in detail. In the canonical category, a cavity optomechanical system are provided as an example. At the end of this paper, we discuss an alternative form of QFI given by Luo et al. [28]. The formula of this alternative QFI for above scenario is analytically calculated and discussed.

The paper is organized as follows. In Sec. I, we review quantum metrology with unitary parametrization processes. In Sec. II, we propose a scenario in which all systems share a common conserved quantity form. For unitary parametrization processes, the characteristic operator and the QFI are analytically provided. We also
discussed the QFI for the parametrized thermal states in the scenario. In Sec. [4] the applications of this scenario, including two typical classes: \textit{su}(2) category and canonical category, are given and discussed. In Sec. [4] an alternative form of QFI is calculated for this scenario. Section [4] is the conclusion of this paper.

II. UNITARY PARAMETRIZATION

The unitary parametrization process is a widely used parametrization strategy in quantum metrology. Recently, it has been found that the QFI and QFIM for both pure and mixed states can be expressed via a characteristic function \(H\) \cite{20, 21}. Denoting the parameter under estimation as \(\theta\), \(H\) is defined as \cite{18, 21}

\[
H_{\theta} := i \left( \partial_{\theta} U \right)^{\dagger} U,
\]

where \(U\) is a unitary parametrization transformation, i.e., \(\rho_0 = U \rho_0 U^\dagger\) with \(\rho_0\) a \(\theta\)-independent density matrix. Now we consider the situation that the transformation is generated by a time-independent Hamiltonian, namely, \(U\) can be written in the form

\[
U = \exp[-i t H(\theta)],
\]

where \(\theta = (\theta_1, \theta_2, \ldots)^T\) is a vector of parameters under estimation and the parametrized Hamiltonian \(H(\theta)\) is time-independent. Based on a recent work \cite{21}, the characteristic operator \(H_{\theta}\) for parameter \(\theta\) can be expressed in an expanded form

\[
H_{\theta} = i \sum_{n=0}^{\infty} \frac{(i t)^{n+1}}{(n+1)!} \left( H^x \right)^n \partial_{\theta} H,
\]

where \(H^x = [H, \cdot]\) is a superoperator. With a known \(H\), the QFIM can be obtained easily. The element of QFIM can be expressed by \cite{21}

\[
\mathcal{F}_{mn} = \sum_{i=1}^{M} 4 p_i \text{cov}_i(H_m, H_n)
- \sum_{i \neq j} 8 p_i p_j \text{Re} \left( \langle \psi_i | H_m | \psi_j \rangle \langle \psi_j | H_n | \psi_i \rangle \right),
\]

where the covariance reads

\[
\text{cov}_i(H_m, H_n) = \frac{1}{2} \left( \langle H_m | H_n \rangle + \langle H_n | H_m \rangle \right).
\]

In above equations, \(H_m\) is short for \(H_{\theta_m}\); \(p_i\) and \(\langle \psi_i \rangle\) are the \(i\)th eigenvalue and eigenstate of initial state \(\rho_0\), which means \(p_i\) and \(\langle \psi_i \rangle\) are independent of \(\theta\). \(M\) is the dimension of the support of \(\rho_0\). \(\langle \cdot \rangle_i\) is the expected value on \(\langle \psi_i \rangle\), i.e., \(\langle \cdot \rangle_i = \langle \psi_i | \cdot | \psi_i \rangle\). It is known that the diagonal elements of QFIM are the QFIs for corresponding single-parameter estimations, thus, the QFI for a unitary parametrization process can be written into the form \cite{21}

\[
\mathcal{F}_{\theta} = \sum_{i=1}^{M} 4 p_i (\Delta^2 H_{\theta})_i
- \sum_{i \neq j} 8 p_i p_j (\langle \psi_i | H_{\theta} | \psi_j \rangle)^2,
\]

where \(\langle \Delta^2 H_{\theta} \rangle_i := \langle H_{\theta}^2 \rangle_i - \langle H_{\theta} \rangle_i^2\) is the variance of the characteristic operator on \(\langle \psi_i \rangle\). For a purely initial state, the element of QFIM reduces to \cite{21}

\[
\mathcal{F}_{mn} = 4 \text{cov}(H_m, H_n).
\]

The covariance is taken on the initial state. Based on this equation, the QFI for purely initial states is \cite{20, 21}

\[
F_{\theta} = 4 \langle \Delta^2 H_{\theta} \rangle,
\]

namely, the QFI for parameter \(\theta\) is actually the variance of the corresponding characteristic operator.

In the expression of QFIM in Eq. (4), for any \(m\), if we assume \(H_m = H_m' + c_m\), with \(c_m\) a complex number, it can be checked that \(\text{cov}_i(H_m, H_n) = \text{cov}_i(H_m', H_n')\). Meanwhile, in the second part of Eq. (4), the overlap \(\langle \psi_i | c_m | \psi_j \rangle\) always vanishes for any \(m\) when \(i \neq j\). Thus, \(H_m\) and \(H_m'\) share the same expression of QFIM. This fact indicates that the number terms of characteristic operator do not affect the value of QFIM, therefore they can be neglected in metrological problems.

III. CONSERVED QUANTITIES

Conserved quantities are important in theoretical physics. In the following we propose a scenario in which all systems share a common conserved quantity. At first, we introduce a \(\theta\)-dependent Hermitian operator \(V\), which is defined as

\[
V = \left[(H^x)^2 - \Omega^2\right] \partial_{\theta} H,
\]

where \(\Omega^2\) is a real number and \(\theta\) is a parameter in the Hamiltonian. This operator is generated by the Hamiltonian \(H\) and related to the parameter \(\theta\). The central quality of the scenario we discuss here is that \(V\) is a conserved quantity for all systems in this scenario, namely, the Hamiltonian \(H\) satisfies

\[
[V, H] = 0.
\]

Based on the definition of \(V\), above equation can be rewritten into

\[
\left[(H^x)^2 - \Omega^2\right] H^x \partial_{\theta} H = 0.
\]

This equation implies that \(H^x \partial_{\theta} H\) is the eigensuperoperator \((H^x)^2\), with \(\Omega^2\) the corresponding eigenvalue. From this aspect, one can check that \(H^x \partial_{\theta} H\) and \((H^x)^2 \partial_{\theta} H\) are also eigensuperoperators of \((H^x)^{2n}\) with \(\Omega^{2n}\) the eigenvalues for \(n \geq 0\), i.e.,

\[
\left[(H^x)^{2n} - \Omega^{2n}\right] (H^x)^{2n-1} \partial_{\theta} H = 0,
\]

where \(i = 1, 2\). In the following we take \(\theta\) as the parameter under estimation. Based on this equation, the characteristic operator \(H_{\theta}\) in Eq. (3) will be separated into
two parts via the parity of $n$. Through some straightforward calculations, the analytical expression of $\mathcal{H}_\theta$ can be obtained as below

$$\mathcal{H}_\theta = \left[-t - i(\partial_t f)H^\times + f(H^\times)^2\right]\partial_\theta H.$$  

(13)

Here $f$ is short for the function $f(\Omega, t)$, which is defined as

$$f(\Omega, t) := \frac{1}{\Omega^3} [\Omega t - \sin (\Omega t)].$$

(14)

If $\Omega > 0$, $f$ is positive and monotone increasing with the passage of time. Utilizing the conserved quantity $\mathcal{V}$, the expression of $\mathcal{H}_\theta$ can be rewritten into

$$\mathcal{H}_\theta = f\mathcal{V} + (f\Omega^2 - t) \partial_\theta H - i (\partial_t f) H^\times \partial_\theta H.$$  

(15)

Moreover, from the expression of $f$, the coefficients can be simplified as $f\Omega^2 - t = -\Omega^{-1} \sin (\Omega t)$ and $\partial_t f = 2\Omega^{-2} \sin^2 (\Omega/2)$, then $\mathcal{H}_\theta$ can be finally expressed by

$$\mathcal{H}_\theta = f\mathcal{V} - \frac{1}{\Omega} \sin(\Omega t) \partial_\theta H - i \frac{2}{\Omega^2} \sin^2 \left(\frac{\Omega}{2}t\right) H^\times \partial_\theta H.$$  

(16)

This is the general expression of the characteristic operator $\mathcal{H}_\theta$ for the scenario in which $\mathcal{V}$ is a conserved quantity. In this scenario, the characteristic operator is the linear combination of the conserved quantity $\mathcal{V}$, partial derivative $\partial_\theta H$ and commutation between $\partial_\theta H$ and $H$. With the expression of $\mathcal{H}_\theta$, the QFIM and QFI can be calculated through Eqs. (14), (15) for both mixed and pure states. For the long-time limit or the situations that the value of $\Omega$ is very large, the oscillating terms in Eq. (16) can be neglected and $\mathcal{H}_\theta$ reduces to

$$\mathcal{H}_\theta = \frac{t}{\Omega^2} \mathcal{V}.$$  

(17)

The characteristic operator is then proportional to $\mathcal{V}$. For purely initial states, the QFI is

$$F_\theta = \frac{t^2}{\Omega^2} (\Delta^2 \mathcal{V}),$$  

(18)

namely, it is actually determined by the fluctuation of the conserved quantity.

When $\mathcal{V}$ is a number (which is a trivial conserved quantity), it can be neglected according to the analysis in Sec. [11]. Thus, the characteristic operator in this case reduces to

$$\mathcal{H}_\theta = -\frac{1}{\Omega} \left[\sin (\Omega t) \partial_\theta H - i \frac{2}{\Omega} \sin^2 \left(\frac{\Omega}{2}t\right) H^\times \partial_\theta H\right].$$  

(19)

For purely initial states, the corresponding QFI is

$$F_\theta = \frac{1}{\Omega^2} \left[\sin^2 (\Omega t) \langle \partial_\theta H \rangle^2 - \frac{4}{\Omega^2} \sin^4 \left(\frac{\Omega}{2}t\right) \langle (H^\times \partial_\theta H)^2 \rangle - i \frac{4}{\Omega} \sin (\Omega t) \sin^2 \left(\frac{\Omega}{2}t\right) \langle (H^\times \partial_\theta H)^2 \rangle\right] - \frac{4}{\Omega^2} \sin^2 \left(\frac{\Omega}{2}t\right) \langle (H^\times \partial_\theta H)^2 \rangle.$$  

(20)

where the equality $\{\partial_\theta H, H^\times \partial_\theta H\} = H^\times (\partial_\theta H)^2$ has been applied. $\{\cdot, \cdot\}$ represents the anticommutation. A simple example of this scenario is that $H^\times \partial_\theta H$ is proportional to $\partial_\theta H$, i.e., $H^\times \partial_\theta H = \Omega \partial_\theta H$. In this example, $\mathcal{V} = 0$, then $\mathcal{H}$ reduces to $i\Omega^{\times -1} (e^{\Omega t} - 1) \partial_\theta H$ [21]. Especially, if $\Omega$ can be chosen as $\Omega = 0$ with a positive number in this example, the characteristic operator $\mathcal{H}_\theta$ will reduce to $-\partial_\theta H/\omega$ for the long-time limit. A realistic case of this scenario is a collective spin system in an external magnetic field, which has been discussed in Ref. [21] in detail.

Moreover, if the operator $\mathcal{V}_1 = (H^\times - \Omega) \partial_\theta H$ is a non-trivial conserved quantity, then $\mathcal{V} = \Omega \mathcal{V}_1$ is also a non-trivial conserved quantity. Therefore the characteristic operator in this case can also be expressed in the form of Eq. (16).

**Thermal states of the scenario**. Thermal states widely appear in realistic world. In quantum theory, a general thermal state can be written as

$$\rho = \frac{1}{Z} \exp (-\beta H),$$

(21)

where $\beta = 1/(k_B T)$ and the partition function $Z = \text{Tr}(e^{-\beta H})$. $T$ is the temperature and $k_B$ is the Boltzmann constant. In Plank unit, $k_B = 1$. Recently, Jiang [23] provides the expression of SLD for a general exponential state $\rho_\theta = \exp[G(\theta)]$, which is

$$L = \sum_{n=0}^\infty \frac{4 (4n+1) - 1}{(2n+2)!} B_{2n+2} (G^\times)^{2n} \partial_\theta G.$$  

(22)

Here $B_{2n+2}$ is the $2n$th Bernoulli number. For the thermal state expressed in (21), $G = -\beta H - \ln Z$. Then it is easy to check the equality $(G^\times)^{2n} \partial_\theta G = (-\beta)^{2n+1} (H^\times)^{2n} \partial_\theta H$. For any system in the scenario, $\mathcal{V}$ is a conserved quantity. The SLD operator can then be calculated as

$$L = \beta \left[r \mathcal{V} + (r \Omega^2 - 1) \partial_\theta H\right],$$  

(23)

where $r$ is short for $r(\beta, \Omega)$, which is defined as

$$r(\beta, \Omega) := \frac{1}{\Omega^2} \left[1 - \frac{1}{\beta \Omega} \tanh (\beta \Omega)\right].$$

(24)

The regime of $r$ is $[0, \Omega^{-2}]$. Meanwhile, one can see that $r\Omega^2 - 1 = -\tanh (\beta \Omega) / (\beta \Omega)$. Then the SLD can be alternatively written as

$$L = \beta r \mathcal{V} - \frac{1}{\Omega} \tanh (\beta \Omega) \partial_\theta H.$$  

(25)

With above equation, the QFI $F_T = \langle L^2 \rangle$ can be finally expressed by

$$F_T = \beta^2 r^2 \langle V^2 \rangle_T + \frac{1}{\Omega} \tanh (\beta \Omega) \langle (\partial_\theta H)^2 \rangle_T - \beta \Omega \tanh (\beta \Omega) \langle \{\mathcal{V}, \partial_\theta H\} \rangle_T.$$  

(26)
Here $\langle \gamma \rangle_T$ is the expected value on the thermal states. If the thermal states in Eq. (21) can be rewritten into the form $Ue^{-\beta H_0}U^\dagger$, with $H_0$ a parameter independent Hamiltonian, the QFI can also be calculated utilizing the function $\mathcal{H}_\theta$ in Eq. (16). At the zero-temperature limit, $\tanh(\beta \Omega) = 1$ and $\sigma = \Omega^{-2}$, the SLD reduces to $L = \beta \Omega^{-2}V$ and the QFI is $F_T = \beta^2 \Omega^{-4}(V^2)_{\gamma_T}$. Similarly, at the high-temperature limit, $\tanh(\beta \Omega) \simeq \beta \Omega$ and $r = 0$, the SLD is $L = -\beta \partial_0H$ and the QFI can be written as $F_T = \beta^2(\langle 0_bH \rangle^2)_{\gamma_T}$.

IV. APPLICATIONS

In the following we will solve the metrological problems in some realistic systems in the scenario where $\mathcal{V}$ is a conserved quantity. We mainly focus on two classes of Hamiltonians. The first one is mainly related to the generators of $\mathfrak{su}(2)$ algebra and is called $\mathfrak{su}(2)$ category; the second one is related to the canonical variables: the position operator $x$ and the momentum operator $p$, and it is called canonical category. We first discuss $\mathfrak{su}(2)$ category.

A. $\mathfrak{su}(2)$ category

Ferromagnetic two-spin system

As the first application of the $\mathfrak{su}(2)$ category, we now consider a ferromagnetic two-spin system in an external magnetic field. The Hamiltonian of this system reads

$$H_1 = -\sigma_1^x \sigma_2^x - B (\sigma_1^z + \sigma_2^z),$$

where $\sigma_i^x = \sigma_i \otimes 1$ and $\sigma_i^y = 1 \otimes \sigma_i$ for $i = x, y, z$. $\sigma_i$ is a Pauli matrix, 1 is the identity matrix, and $B$ is the strength of the external field. The optimization of QFI in a general Ising model with GHZ-type state has been discussed recently [22]. In this case, we take $B$ as the parameter under estimation. Before the main calculation, we introduce three operators

$$\begin{aligned}
J_x &= \frac{1}{4} (\sigma_1^x \sigma_2^x - \sigma_1^y \sigma_2^y), \\
J_y &= \frac{1}{4} (\sigma_1^x \sigma_2^y + \sigma_1^y \sigma_2^x), \\
J_z &= \frac{1}{4} (\sigma_1^z + \sigma_2^z).
\end{aligned}$$

(28)

It is worth to notice that $J_x, J_y$ and $J_z$ satisfy the $\mathfrak{su}(2)$ commutation $[J_i, J_j] = i\epsilon_{ijk} J_k$ with $\epsilon_{ijk}$ the Levi-Civita symbol. In addition, the anticommutation is $\{J_i, J_j\} = 2\delta_{ij} J_i$ with $\delta_{ij}$ the Kronecker delta function. Using these operators, the first and second order commutations between the Hamiltonian and its derivative are calculated as below

$$\begin{aligned}
(H_1^x)^2 \partial_B H_1 &= -8J_y, \\
(H_1^x)^2 \partial_B H_1 &= 16 (2BJ_x - J_z).
\end{aligned}$$

(29)

(30)

Utilizing above two equations, we can obtain

$$\left[(H_1^x)^2 - 4 (1 + 4B^2)^2\right] H_1^x \partial_B H_1 = 0.$$  \hspace{1cm} (31)

This equation implies that if we choose

$$\Omega = 2\sqrt{1 + 4B^2},$$

(32)

the operator $\mathcal{V}$ defined in Eq. (9) is a conserved quantity. Specifically, it is

$$\mathcal{V} = 32B\vec{v} \cdot \vec{J},$$

(33)

where $\vec{J} = (J_x, J_y, J_z)^T$ and $\vec{v} = (1, 0, 2B)^T$. Using this method, we find a non-trivial conserved quantity in this two-spin system. Moreover, based on the property of quantum conserved quantity, all the operators, for which the corresponding vectors share the same or opposite directions with $\vec{v}$, are conserved quantities. $\vec{v}$ is a vector in the $x-z$ plane. When $B$ is zero, $\vec{v}$ is along the $x$ axis. With the increase of $B$, $\vec{v}$ rotates around the $y$ axis from $x$ axis to $z$ axis. For a very large $B$, $\vec{v}$ is almost along the $z$ axis.

Compared with the expressions of $\Omega$ and $\vec{v}$, one can see that $\Omega = 2v$ with $v = |\vec{v}|$, the function $f$ can then be rewritten into $f = [2vt - \sin (2vt)]/8v^3$. The characteristic operator $\mathcal{H}_B$ can be calculated via Eq. (13). Its explicit expression is

$$\mathcal{H}_B = 4\vec{v} \cdot \vec{J},$$

(34)

where the vector $\vec{x}$ reads $\vec{x} = (8Bf, -2\partial_tf, t-4f)^T$. With above $\mathcal{H}_B$, the QFI for purely initial states in this case can be written as

$$F_B = \frac{16}{3} \left[|\vec{x}|^2 |\vec{J}|^2 - 3 \left(\vec{x} \cdot \langle \vec{J}\rangle\right)^2\right],$$

(35)
where \(|\vec{J}|^2 = 3(1 + \sigma_1^2\sigma_2^2)/8\). During the calculation of \(F_B\), the relation \(\{J_i, J_j\} = 2\delta_{ij}J_i\) has been used. Above equation of QFI implies that the its maximum is attained when \((|\vec{J}|^2)\) is maximum and \((\vec{J})\) is vertical to \(\vec{x}\). In a 4-dimensional Hilbert space, the maximum value of \((\sigma_1^2\sigma_2^2)\) is 1, which indicates the formula of the maximum QFI must be

\[
F_{B,\text{max}} = 4|x|^2,
\]

and the optimal initial state is required to be in the form

\[
|\psi_{\text{opt}}\rangle = a_1|00\rangle + a_2e^{i\phi}|11\rangle,
\]

where \(a_{1,2}\) is a real number. To make \(x \cdot \langle \vec{J} \rangle = 0\), the amplitudes need to satisfy the equation

\[
a_1a_2(x_x\cos\phi + x_y\sin\phi) + \frac{1}{2}x_z(a_1^2 - a_2^2) = 0.
\]

(38)

Here \(x_i (i = x, y, z)\) is a element of \(\vec{x}\). Since \(x_z = t - 4f\) is not always zero for \(t > 0\), then \(a_1a_2 = 0\) cannot be a solution of this equation, which means above equation can be further simplified into

\[
x_x\cos\phi + x_y\sin\phi + \frac{1}{2}x_z(a_1 - a_2)(a_1 + a_2) = 0.
\]

(39)

All states satisfying this equation are available to access \(F_{B,\text{max}}\). At the long-time limit, \(f \approx t/4v^2\), and \(\frac{\partial f}{\partial t} \ll f\), then Eq. (39) reduces to

\[
4B\left[\cos\phi + B\left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right)\right] = 0.
\]

(40)

Figure 1 gives the optimal points \((\theta, a_1/a_2)\) to access the maximum QFI for different \(B\) at the long-time limit. The solid blue, dashed red, dash-dot black and dotted pink lines in this figure represent the optimal points for \(B = 0.4, 0.8, 1.0\) and 10, respectively. From this figure, it can be found that with the increase of \(B\), the curve becomes more flat. This behavior indicates that for a strong external field, the maximum QFI is insensitive to the relative phase \(\phi\). This is actually due to the fact that when \(B\) is very large in above equation, \(\cos\phi\) can be neglected and the equation reduces to

\[
a_1 - \frac{a_2}{a_1} = 0.
\]

(41)

One solution of this equation is \(a_1 = a_2\). Thus, for a strong external field, the maximum QFI can be saturated at \(a_1 = a_2\) with any phase \(\phi\).

A widely studied special form is that \(a_1 = a_2 = 1/\sqrt{2}\), i.e., \(|\psi_{\text{opt}}\rangle = (|00\rangle + e^{i\phi}|11\rangle)/\sqrt{2}\), then the optimal relative phase \(\phi_{\text{opt}}\) reads

\[
\phi_{\text{opt}} = \arctan\left(\frac{4Bf}{\partial f}\right).
\]

(42)

At the long-time limit, this optimal phase reduces to a constant number \(\pi/2\), independent of the external field. This fact means at the long-time limit, the state \((|00\rangle + e^{i\phi}|11\rangle)/\sqrt{2}\) is always an optimal state for any strength of external field. Figure 2 shows the general variation of optimal relative phase as a function of \(B\) and \(t\). For a weak external field, \(\phi_{\text{opt}}\) is growing rapidly with the increase of \(B\) and \(t\). This is because when the external field is very weak, \(4Bf/\partial tf \approx 4Bt/3\), then \(\phi_{\text{opt}} \approx \arctan(4Bt/3) \approx 4Bt/3\). Thus, the optimal phase \(\phi_{\text{opt}}\) grows almost linearly with \(B\) and \(t\) in this regime. With the continue increase of \(B\) and \(t\), \(\phi_{\text{opt}}\) shows a oscillating behavior for intermediate strength of external field. The oscillation amplitude of \(\phi_{\text{opt}}\) trends to shrink with the passage of time as \(\phi_{\text{opt}} = \pi/2\) when \(t\) is infinite large.

Thermal state.-Here we consider the thermal state of this ferromagnetic two-spin system, of which the density matrix reads \(\rho_1 = \exp(-\beta H_1)/Z_1\), with \(Z_1 = \text{Tr} [\exp(-\beta H_1)]\). It is already known that \(V\) is a conserved quantity, then based on Eq. (23), the SLD operator for \(B\) of this thermal state can be written as

\[
L_B = \beta r V + \frac{2}{v} \tanh (2\beta v) J_z
\]

(43)

where \(r\) is defined in Eq. (24) and in this case, it has the form

\[
r = \frac{1}{4v^2} \left[ 1 - \frac{1}{2\beta v} \tanh(2\beta v) \right].
\]

(44)

Based on the expressions of \(L_B\), the QFI can be calculated as

\[
F_T = 2\beta^2 \left( 16v^2r^2 - 8r + 1 \right) (1 + \langle \sigma_1^2 \sigma_2^2 \rangle).
\]

(45)

\(F_T\) here is determined by the correlation function \(\langle \sigma_1^2 \sigma_2^2 \rangle\), which can be analytically solved as

\[
\langle \sigma_1^2 \sigma_2^2 \rangle = -1 + \frac{2 \cosh (v\beta)}{\cosh (v\beta) + \cosh \beta}.
\]

(46)
Thus, $F_T$ can be finally expressed in the form

$$F_T = \left(16\nu^2 r^2 - 8r + 1\right) \frac{4\beta^2 \cosh (\nu\beta)}{r^2 \cosh (\nu\beta) + \cosh \beta}. \quad (47)$$

Figure 3 shows the variation of $F_T$ as a function of $B$. The temperatures are set as $T = 0.5, 1.0$ and $1.5$ for the solid blue, dashed red and dash-dot black lines. In this figure, for the weak external field, the QFI grows greatly with the increase of $B$. However, for the strong external field, this growth is not significant. With respective to the temperature, the decrease of $T$ shows a positive effect on the QFI and is useful for the precision measure in this system.

For a very low temperature and nonzero external field, $\cosh \beta/k \cosh (\nu\beta) \simeq e^{\beta(1-\nu)} \approx 0$, then the QFI in Eq. (47) reduces to the form

$$F_T \simeq \frac{4}{T^2} \left(1 - \frac{1}{T^2}\right). \quad (48)$$

This equations shows that in the low-temperature regime, the QFI is inversely proportional to the square of $T$. Thus, the decrease of $T$ will dramatically improve the value of $F_T$. Meanwhile, the increase of $B$ will also enhance the value of QFI. The maximum value of $F_T$ in this regime is $F_{T,max} = 4/T^2$.

**Anisotropic two-spin system**

In the following we consider a more general case: the anisotropic two-spin ferromagnetic XY model with an inhomogeneous external magnetic field. The Hamiltonian of this system is

$$H_2 = -\frac{1 + \gamma}{2} \sigma_i^x \sigma_j^x - \frac{1 - \gamma}{2} \sigma_i^y \sigma_j^y - B_+ (\sigma_i^1 + \sigma_j^1) - B_- (\sigma_i^2 + \sigma_j^2), \quad (49)$$

where $\gamma$ is the anisotropic parameter. This Hamiltonian can reduces to the Hamiltonian $H_1$ in Eq. (27) with $\gamma = 1$ and $B_- = 0$. Before the main calculation, we introduce a new group of operators

$$\begin{align*}
S_x &= \frac{1}{4} (\sigma_1^x \sigma_2^y - \sigma_2^y \sigma_1^x), \\
S_y &= \frac{1}{4} (\sigma_1^x \sigma_2^y + \sigma_2^y \sigma_1^y), \\
S_z &= \frac{1}{4} (\sigma_1^z - \sigma_2^z).
\end{align*} \quad (50)$$

Similarly with $J_{x,y,z}, S_{x,y,z}$ also satisfies $su(2)$ commutation, i.e., $[S_i, S_j] = i\epsilon_{ijk} S_k$. Meanwhile, the anti-commutation relation is $\{S_i, S_j\} = 2\delta_{ij} S_i$. A very interesting property between these two groups of operators is that

$$J_i S_j = S_j J_i = 0, \quad \forall i, j = x, y, z. \quad (51)$$

$[J_i, S_j] = 0$ for any $i$ and $j$ is a natural result of this property.

Utilizing these two set of operators, Hamiltonian (49) can be written as the sum of two parts, i.e.,

$$H_2 = 2(H_+ + H_-), \quad (52)$$

where the sub-Hamiltonians $H_+$ is only related to $J_{x,y,z}$ and $H_-$ is only related to $S_{x,y,z}$. Their specific formulas are

$$\begin{align*}
H_+ &= -\gamma J_x - 2B_+ J_z, \\
H_- &= -S_y - 2B_- S_z.
\end{align*} \quad (53, 54)$$

Since $J_i$ and $S_j$ are commutative for any $i$ and $j$, $H_+$ and $H_-$ are also commutative. Here we take both $B_+$ and $B_-$ as the parameters under estimation. Through some algebra, we find that if one choose

$$\Omega_+ = 2\sqrt{\gamma^2 + 4B_+^2}, \quad \Omega_- = 2\sqrt{1 + 4B_-^2}, \quad (55)$$

the corresponding operator $\mathcal{V}_+$ and $\mathcal{V}_-$ defined in Eq. (9) are conserved quantities and have the form

$$\mathcal{V}_+ = 32B_+ \bar{v}_+ \cdot \vec{J}, \quad \mathcal{V}_- = 32B_- \bar{v}_- \cdot \vec{S}, \quad (56)$$

where $\vec{J} = (J_x, J_y, J_z)^T, \vec{S} = (S_x, S_y, S_z)^T$ and the vector $\bar{v}_+$ and $\bar{v}_-$ read

$$\begin{align*}
\bar{v}_+ &= (\gamma, 0, 2B_+)^T, \\
\bar{v}_- &= (0, 1, 2B_-)^T.
\end{align*} \quad (57)$$

Then there is $\Omega_{\pm} = 2v_{\pm}$ with $v_{\pm} = |\bar{v}_{\pm}|$.

Utilizing these conserved quantities, the characteristic operators $\mathcal{H}_+$ and $\mathcal{H}_-$ can be expressed by

$$\mathcal{H}_+ = 4\bar{x}_+ \cdot \vec{J}, \quad \mathcal{H}_- = 4\bar{x}_- \cdot \vec{S}. \quad (58)$$

The vector $\bar{x}_{\pm}$ in above equation reads

$$\begin{align*}
\bar{x}_+ &= (8\gamma B_+ f_+ - 2\gamma \partial_t f_+ t - 4\gamma^2 f_+^2)^T, \\
\bar{x}_- &= (2\partial_t f_-, 8B_- f_-, t - 4f_-)^T.
\end{align*} \quad (59, 60)$$
Here $f_{\pm} = [2v_{\pm} t - \sin(2v_{\pm} t)]/8\theta^2$. For purely initial states, the QFI is the variance of $\mathcal{H}$ on the initial state. For parameter $B_+$, the QFI $F_+$ shares the same form with that in Eq. (35), i.e.,

$$F_+ = \frac{16}{3} \left[ |\bar{x}_+|^2 \langle |\bar{S}|^2 \rangle - 3 \left( \bar{x}_+ \cdot \langle \bar{S} \rangle \right)^2 \right].$$  

(61)

For parameter $B_-$, the QFI $F_-$ has the similar form

$$F_- = \frac{16}{3} \left[ |\bar{x}_-|^2 \langle |\bar{S}|^2 \rangle - 3 \left( \bar{x}_- \cdot \langle \bar{S} \rangle \right)^2 \right].$$  

(62)

where $|\bar{S}|^2 = 3(1 - \sigma_1^2 \sigma_2^2)/8$. The maximum value of $F_+$ is $4|\bar{x}_+|^2$ and the corresponding optimal initial state has the same form as that in Eq. (37). For the long-time limit, the equation for $B_+$ that the optimal points satisfy is

$$4 \gamma B_+ \cos \phi + (v_+^2 - \gamma^2) \left( \frac{a_1}{a_2} - \frac{a_2}{a_1} \right) = 0.$$  

(63)

Figure 4 gives the optimal points to access the maximum QFI for the long-time limit. The solid blue, dashed red, and dash-dot black lines represent the optimal points for $\gamma = 0.3$, 0.6 and 0.9. It is found in this figure that with the increase of $\gamma$, the curve of the optimal points gets more sharp, indicating that the maximum QFI is more sensitive to the relative phase for a large $\gamma$.

Similarly, the maximum value of $F_-$ is $4|\bar{x}_-|^2$ with the optimal initial state

$$|\Phi_{\text{opt}}\rangle = b_1 |01\rangle + b_2 e^{i\varphi} |10\rangle,$$  

(64)

where $b_1$ and $b_2$ are real numbers. The equation for $b_1$, $b_2$ to satisfy to access the maximum QFI is

$$2 \partial_{f_-} \sin \varphi - 8 B_- f_- \cos \varphi = \frac{1}{2} (t - 4 f_-) \left( \frac{b_1}{b_2} - \frac{b_2}{b_1} \right).$$  

(65)

For the long-time limit, this equation reduces to the same form with Eq. [40]. Moreover, taking $b_1 = b_2 = 1/\sqrt{2}$, the optimal relative phase can be written as

$$\varphi_{\text{opt}} = \arctan \left( \frac{4 B_- f_-}{\partial_f f_-} \right).$$  

(66)

At the long-time limit, $\varphi_{\text{opt}}$ also equals to $\pi/2$.

Now we consider the situation that both $B_+$ and $B_-$ are unknown parameters simultaneously. It is known for the saturation of the multiparameter Cramér-Rao bound on unitary parametrization process is [21]

$$\langle \psi_{\text{in}} | [\mathcal{H}_+, \mathcal{H}_-] | \psi_{\text{in}} \rangle = 0.$$  

(67)

From the expressions of $\mathcal{H}_\pm$ in Eq. [58] and the property that $J_i$ and $S_j$ are commutative for all $i$ and $j$, it is easy to see that $[\mathcal{H}_+, \mathcal{H}_-] = 0$. Thus, $B_+$ and $B_-$ can be jointly measured for any purely initial state. Since $\mathcal{H}_+$ and $\mathcal{H}_-$ are commutative and based on Eq. (51), the off-diagonal element of QFIM is

$$F_{++} = F_{--} = -\langle \mathcal{H}_+ \rangle \langle \mathcal{H}_- \rangle.$$  

(68)

The expected value above is taken on the initial state. According to the Cramér-Rao theory, we have

$$\delta^2 B_\pm \geq \frac{F_{++} F_{--}}{F_{++} - F_{--}}.$$  

(69)

When the initial state is chosen as $|\psi_{\text{opt}}\rangle$ or $|\Phi_{\text{opt}}\rangle$, $F_{++}$ vanishes and the inequality above reduces to the form of the single-parameter cases, which indicates that the joint measurement of $B_+$ and $B_-$ can be performed by $|\psi_{\text{opt}}\rangle$ or $|\Phi_{\text{opt}}\rangle$. However, it should be noticed that $|\psi_{\text{opt}}\rangle$ and $|\Phi_{\text{opt}}\rangle$ are orthogonal, which means even $B_+$ and $B_-$ can be jointly measured, there does not exist an optimal state to access the maximum QFI for both $B_+$ and $B_-$ simultaneously. This fact implies that the joint measurement here is not as good as the single measurement.

**Thermal state.** Next we consider the thermal state of $H_2$, which is $\rho_2 = \exp(-\beta H_2)/Z_2$, with $Z_2$ the partition function. Based on Eq. (23), the SLD operators for $B_+$ and $B_-$ can be expressed by

$$L_\pm = \beta r_\pm V_\pm - \frac{1}{v_\pm} \tanh (2\beta v_\pm) \partial_{\pm} H_\pm.$$  

(70)

where $\partial_\pm$ is short for $\partial_{B_\pm}$ and the coefficient

$$r_\pm = \frac{1}{4v_\pm^2} \left[ 1 - \frac{1}{2\beta v_\pm} \tanh (2\beta v_\pm) \right].$$  

(71)

Since $\{S_i, S_j\} = \{J_i, J_j\} = 0$ for $i \neq j$, the thermal QFI for $B_+$ and $B_-$ can then be written as

$$F_{T+} = 2\beta^2 (16\gamma^2 v_+^2 r_+^2 - 8\gamma^2 r_- + 1) (1 + \langle \sigma_1^2 \sigma_2^2 \rangle),$$  

(72)

$$F_{T-} = 2\beta^2 (16\gamma^2 r_-^2 - 8r_- + 1) (1 - \langle \sigma_1^2 \sigma_2^2 \rangle).$$  

(73)
For the thermal state, the correlation function $\langle \sigma_1^z \sigma_2^z \rangle$ is
\[ \langle \sigma_1^z \sigma_2^z \rangle = -1 + \frac{2 \cosh(\beta v_\perp)}{\cosh(\beta v_\perp) + \cosh(\beta v_\parallel)}. \tag{74} \]

In the low-temperature regime, there is
\[ \frac{\cosh(\beta v_\parallel)}{\cosh(\beta v_\perp)} \approx e^{\beta (v_\perp - v_\parallel)}. \tag{75} \]

Therefore, when $v_\parallel$ equals to $v_\perp$, $e^{\beta (v_\perp - v_\parallel)}$ equals to 1, and $\langle \sigma_1^z \sigma_2^z \rangle = 0$. $F_{T,+}$ and $F_{T,-}$ reduce to
\[ F_{T,+} \approx \frac{2}{T^2} \left( 1 - \frac{\gamma^2}{v_\perp^2} \right), \quad F_{T,-} \approx \frac{2}{T^2} \left( 1 - \frac{1}{v_\parallel^2} \right). \tag{76} \]

When $v_\parallel$ is smaller than $v_\perp$, $e^{\beta (v_\perp - v_\parallel)}$ trends to infinity in the low-temperature regime, then $\langle \sigma_1^z \sigma_2^z \rangle \approx -1$ and $F_{T,+}$ and $F_{T,-}$ is in the form
\[ F_{T,+} \approx 0, \quad F_{T,-} \approx \frac{4}{T^2} \left( 1 - \frac{1}{v_\parallel^2} \right). \tag{77} \]

When $v_\parallel$ is larger than $v_\perp$, $e^{\beta (v_\perp - v_\parallel)}$ equals to 0, then $\langle \sigma_1^z \sigma_2^z \rangle \approx 1$, and the QFI $F_{T,+}$ and $F_{T,-}$ can be written as
\[ F_{T,+} \approx \frac{4}{T^2} \left( 1 - \frac{\gamma^2}{v_\perp^2} \right), \quad F_{T,-} \approx 0. \tag{78} \]

Above analysis shows that when $v_\parallel$ is smaller than $v_\perp$, the parameter $B_\perp$ can be barely estimated via the Cramér-Rao theory, so as $B_\parallel$ when $v_\parallel$ is larger than $v_\perp$. Thus, if either of $B_\perp$ and $B_\parallel$ is the parameter under estimation, we have to tune down the value of the other one to make sure that nonzero QFI exists.

**Spin-one model**

Not only the spin-half systems, but also some spin-one systems, can fit in the $\mathfrak{su}(2)$ category. Here we show such a spin-one system in the one-axis twisting model. The Hamiltonian of a one-axis twisting model with a transverse field can be written in the form $[29][32]
\[
H_3 = \chi J_x^2 + B J_z, \tag{79}
\]
where $J_x = (a b + a^\dagger b^\dagger)/2$ and $J_z = (a^\dagger a - b b^\dagger)/2i$. $\chi$ is the coupling constant and $B$ is the strength of the transverse field. This Hamiltonian can be realized in many physical systems including two-component Bose-Einstein condensates $[33][34]$. Now we consider one realization that a two-boson system in a double well. Since the particle number is a conserved quantity, this system can be expanded in the basis $\{|02\},|11\},|20\}$ in Fock space, which can be mapped as a spin-one system. In this basis, the Schwinger operators has the form
\[
J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{80}
\]

and
\[
J_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{81}
\]

Based on these matrices and taking the parameter $B$ as the one under estimation, one can easily check that the Hamiltonian $H_3$ satisfies the following equation
\[
[(H_3^2 - (\chi^2 + 4B^2))H_3 \partial_\phi H_3 = 0, \tag{82}
\]
which implies that if we choose $\Omega = \sqrt{\chi^2 + 4B^2}$, the operator $\mathcal{V}$ defined in Eq. (79) will be a conserved quantity. Specifically, $\mathcal{V}$ can be written in the form
\[
\mathcal{V} = (\chi^2 - \Omega^2) J_z - 2B \chi \mathcal{I}, \tag{83}
\]
where $\mathcal{I} = |02\rangle\langle 20| + |20\rangle\langle 02|$. Furthermore, the characteristic function $\mathcal{H}_B$ can be written as
\[
\mathcal{H}_B = (\chi^2 J - t) J_z - 2B \chi \mathcal{I} + \frac{12}{\Omega^2} \sin^2 \left( \frac{\Omega}{2} t \right) J_z \mathcal{I}. \tag{84}
\]

For the long-time limit, it reduces to
\[
\mathcal{H}_B = -4B^2 t \chi^2 J_z - 2B \chi \mathcal{I}. \tag{85}
\]

For the unitary parametrization processes with a purely initial state, the QFI for above $\mathcal{H}_B$ reads
\[
F_B = \left( \frac{4B^2 t}{\chi^2 + 4B^2} \right)^2 \langle \Delta^2 J_z \rangle + 4B^2 \chi^2 \langle \Delta^2 \mathcal{I} \rangle, \tag{86}
\]
where we have used the equality $\langle J_z \mathcal{I} \rangle = 0$. Furthermore, the QFI can be simplified into
\[
F_B = \left( \frac{4B^2 t}{\chi^2 + 4B^2} \right)^2 \langle J_z^2 \rangle + 4B^2 \chi^2 \langle \mathcal{I} \rangle^2 - \left( \frac{4B^2 t}{\chi^2 + 4B^2} \right)^2 \langle J_z^2 \rangle - 4B^2 \chi^2 \langle \mathcal{I} \rangle^2. \tag{87}
\]

During the calculation we used the equality $\langle J_z^2 \rangle = 2\chi^2$. The maximum value of QFI above is attained when $\langle J_z^2 \rangle$ reaches its maximum value and $\langle J_z \mathcal{I} \rangle$ vanish simultaneously. Denoting the initial state as $|\psi \rangle = c_1 |02\rangle + c_2 e^{i \phi_2} |11\rangle + c_3 e^{i \phi_3} |20\rangle$ with $c_{1,2,3}$ real numbers, there are $\langle J_z^2 \rangle = \langle \mathcal{I} \rangle^2 = c_1^2 + c_2^2 + c_3^2$, $\langle J_z \mathcal{I} \rangle = c_1^2 - c_2^2$ and $\langle \mathcal{I} \rangle = 2c_1c_3 \cos \phi_3$. Utilizing these expressions, it can be checked that when $c_1 = c_2 = 1/\sqrt{2}$ and $\phi_3 = \pi/2$, all the conditions can be satisfied simultaneously, which implies that one optimal initial state here is a NOON-type state, i.e.,
\[
|\psi_{opt} \rangle = \frac{1}{\sqrt{2}} (|02\rangle + |20\rangle). \tag{88}
\]

The corresponding maximum QFI is
\[
F_{B,\text{max}} = 4B^2 \left[ \frac{4B^2 t^2}{(\chi^2 + 4B^2)^2} + \chi^2 \right]. \tag{89}
\]
This expression shows that the maximum QFI will be square-enhanced with the passage of time. For the coupling constant $\chi$, $F_{B, \text{max}}$ does not change monotonously. For a small $\chi$, $F_{B, \text{max}}$ increases sharply with the decrease of $\chi$ and the trend is totally reverse for a large $\chi$. The minimum value of $F_{B, \text{max}}$ is attained around $\chi^2 = 4(Bl)^{2/3} - 4B^2$. Thus, the value of $\chi$ should be tuned carefully to avoid this regime during the measurement of the transverse field.

B. Canonical category

The most obvious property of Hamiltonians in canonical category is that they can be partly or entirely rewritten via the canonical variables: $x$, $p$ and the number operator. A typical case of canonical category is the optomechanical systems, which has been widely discussed as a novel artificial device [35]. A simple model for optomechanical systems is a single-mode cavity coupling with a movable mirror, of which the schematic is shown in Fig. 5. The total Hamiltonian can be written as [36, 37]

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b - \tilde{g} a^\dagger a (b + b^\dagger),$$

(90)

where $a$, $a^\dagger$, $b$, $b^\dagger$ are the annihilation and creation operators for the cavity and mirror, respectively. $\omega_a$, $\omega_b$ are the corresponding frequencies. The coupling strength $\tilde{g}$ has the form

$$\tilde{g} = \frac{\omega_a}{T} \sqrt{\frac{1}{2m\omega_b}},$$

(91)

with $l$, $m$ the length of the cavity and mass of the mirror. With the introduction of the quadratic operators $x_{a(b)}$, $p_{a(b)}$ defined as

$$x_a = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad p_a = \frac{1}{\sqrt{2}T} (a - a^\dagger),$$

$$x_b = \frac{1}{\sqrt{2}} (b + b^\dagger), \quad p_b = \frac{1}{\sqrt{2}l} (b - b^\dagger),$$

(92, 93)

and the number operators $N_a = a^\dagger a$, $N_b = b^\dagger b$, Hamiltonian (90) can be rewritten into the form

$$H = \omega_a N_a + \omega_b N_b - g N_a x_b,$$

(94)

where $g = \sqrt{2}\tilde{g}$. In this case we take $m$ or $l$ as the parameter under estimation. Utilizing the commutation relations $[N, x] = -i\hbar$, $[N, p] = i\hbar$ and $[x, p] = i$, one can see that

$$H^x \partial_{m(t)} H = i\omega_b g' N_a p_b,$$

$$\left(H^x\right)^2 \partial_{m(t)} H = -\omega_b^2 g' N_a x_b + \omega_b g g' N_a^2,$$

(95, 96)

where we denote $g' := \partial_{m(t)} g$. From these equations, one can check that if we choose $\Omega = \omega_b$, the operator $V$ defined in Eq. (9) is a conserved quantity. Its specific expression is

$$V = \omega_b g g' N_a^2.$$  

(97)

As a matter of fact, the photon number in the cavity, i.e., $N_a$ is a conserve quantity, thus, it is natural that any exponentiation of $N_a$ is also a conserved quantity. Based on above information, the characteristic function reads

$$\mathcal{H}_{m(t)} = \frac{n_a g'}{\omega_b} \{ \sin (\omega_b t) x_b + \omega_b (\partial_t f) p_b + g \omega_b f N_a \}.$$  

(98)

In the Fock space of the cavity, $N_a$ in above equation can be replaced by the average photon number $n_a$, which is a constant, therefore, the characteristic operator can be simplified into

$$\mathcal{H}_{m(t)} = \frac{n_a g'}{\omega_b} \{ \sin (\omega_b t) x_b + [1 - \cos (\omega_b t)] p_b \}.$$  

(99)

For a purely initial state, the QFI is in the form

$$F_{m(t)} = \left(\frac{n_a g'}{\omega_b}\right)^2 \left\{ \sin^2 (\omega_b t) \Delta^2 x_b + [1 - \cos (\omega_b t)]^2 \Delta^2 p_b + 2 \sin (\omega_b t) [1 - \cos (\omega_b t)] \text{cov} (x_b, p_b) \right\}.$$  

(100)

If the movable mirror is initially in the vacuum state, above expression of QFI reduces to

$$F_{m(t)} = \left(\frac{n_a g'}{\omega_b}\right)^2 [1 - \cos (\omega_b t)].$$  

(101)

When $t = (2k + 1)\pi/\omega_b$ with $k = 0, 1, 2, \ldots$, the QFI above reaches its maximum value with respect to time, which is $F_{m(t), \text{max}} = (n_a g'/\omega_b)^2$. Contrarily, when $t = 2k\pi/\omega_b$, the QFI vanishes, the parameter cannot be estimated via Cramér-Rao inequality. This fact shows that the measure should not be performed at these time points. Moreover, the increase of photon number in the cavity can squarely benefit the estimation of $m$ and $l$.

For the mass $m$ and the length $l$, the specific expressions of maximum QFI are

$$F_{m, \text{max}} = \frac{n_a^2 \omega_b^2}{4m^3 l^2}, \quad F_{l, \text{max}} = \frac{n_a^2 \omega_b^2}{ml^4 \omega_b}.$$  

(102)
In both expressions above, tuning down the frequency $\omega_b$ will help to improve the precision of $l$ and $m$. Especially for the estimation of mass $m$, the decrease of $\omega_b$ will show a dramatic enhancement of the precision.

There are several other systems in this category, including a quantum harmonic oscillator in a classical field. The corresponding Hamiltonian is $H = \omega_0 a^\dagger a + ga^\dagger + g^* a$. In this case, the characteristic operator $H$ is also the linear combination of operators $x$ and $p$. The $su(2)$ and canonical categories discussed above are representative. However, there are still systems out of these two categories in which $V$ is a conserved quantity. For instance, a two-level atom in a single-mode cavity with the Hamiltonian $H = \omega_0 a^\dagger a + \frac{1}{2} \omega_0 \sigma_z - g (a + a^\dagger) \sigma_z$ can also fit in the scenario discussed in this paper.

V. ALTERNATIVE FORM OF QFI

The classical Fisher information has more than one extensions in quantum mechanics. Besides the traditional one discussed above, an alternative definition of quantum Fisher information is [28]

$$I_\theta = 4 \text{Tr} (\partial_\theta \sqrt{\rho})^2. \quad (103)$$

For the unitary parametrization $\rho(\theta) = U(\theta) \rho_0 U^\dagger(\theta)$, this alternative form of QFI can be expressed by

$$I_\theta = 8 \text{Tr} \left[ H^2 \rho_0 - (H^\dagger \rho_0)^2 \right], \quad (104)$$

where $H$ is the corresponding characteristic operator. Similarly with the traditional expression, above formula is also determined by $\mathcal{H}$ and the initial state $\rho_0$. Recalling the spectral decomposition of $\rho_0$ as $\rho_0 = \sum_{i=1}^M p_i |\psi_i\rangle \langle \psi_i|$, with $M$ the dimension of the support of $\rho_0$, Eq. (104) can be written into

$$I_\theta = 8 \sum_{i=1}^M \left( p_i \langle \Delta^2 \mathcal{H} \rangle_i - 2 \sum_{j>i} \sqrt{p_i p_j} \langle \psi_i| \mathcal{H} |\psi_j\rangle \langle \psi_j| \mathcal{H} |\psi_i\rangle \right), \quad (105)$$

with $\langle \Delta^2 \mathcal{H} \rangle_i$ the variance of $\mathcal{H}$ on the $i$th eigenstate of $\rho_0$. Similarly with the traditional form of QFI [24, 27], $I_\theta$ is also determined by the support of $\rho_0$. For a purely initial state, $I_\theta = 8 \langle \Delta^2 \mathcal{H} \rangle$.

For a general exponential form state $\rho_\theta = \exp(G\theta)$, $I_\theta$ is actually a correlation function, namely,

$$I_\theta = \langle \Gamma_+(G, \theta) \Gamma_-(G, \theta) \rangle, \quad (106)$$

where

$$\Gamma_\pm(G, \theta) := \int_0^1 e^{\pm \frac{i}{2} G \beta} \partial_\beta G \ d\beta. \quad (107)$$

When $e^{\frac{i}{2} G \beta} \partial_\beta G$ is a real operator, $\Gamma_+ = \Gamma_-$. $I_\theta$ then reduces to $\langle \Gamma_+^2 \rangle$. Similarly with the SLD operator, above integrating form of $\Gamma_\pm$ can also be rewritten into an expended form

$$\Gamma_\pm(G, \theta) = \sum_{n=0}^{\infty} \left( \frac{\pm 1/2}{n+1} \right)^n \partial_\theta^n G. \quad (108)$$

For a thermal state expressed in Eq. [21], i.e., $G = -\beta H - \ln Z$, where $V$ is a conserved quantity, $\Gamma_\pm$ can be expressed by

$$\Gamma_\pm = f_1 V + \left( f_1 \Omega^2 - \beta \right) \partial_\beta H \mp 2 (\beta G f_1) H X \partial_\beta H. \quad (109)$$

where $f_1$ is defined as $f_1 = 2\Omega - 3 \left[ \frac{\Omega}{2} - \sinh \left( \frac{\Omega}{2} \right) \right]$. Similarly with $\mathcal{H}$ in unitary parametrization process, $\Gamma_\pm$ is also the linear combination of $V$, the partial derive $\partial_\beta H$ and its commutation between the Hamiltonian $H$. Thus, $I_\theta$ for thermal states of all systems discussed previously can be calculated analytically.

VI. CONCLUSION

In summary, we discuss a general scenario in which the Hermietian operator $V$ (defined in Eq. [9]) is a conserved quantity. For the unitary parametrization processes, we provide analytical expression of the characteristic operator $\mathcal{H}$, which is totally determined by the Hamiltonian, the commutation between the Hamiltonian and its partial derivative, and the conserved quantity $V$. With the expression of $\mathcal{H}$, we further give the expression of the QFI, calculate its maximum value and the corresponding optimal initial states. For the parametrized thermal states in this scenario, the SLD is the linear combination of $V$ and $\partial_\beta H$.

The scenario in this paper includes many other specific physical systems. As the application, we mainly focus on two categories: $su(2)$ category and canonical category. In the $su(2)$ category, we detailedly discuss the QFI in the ferromagnetic two-spin system, the anisotropic two-spin XY model and a spin-one model. The characteristic operator $\mathcal{H}$ in these systems can basically be expressed via $su(2)$ generators. With the expressions of QFI, we locate the optimal initial states in these systems to access the maximum QFI. Meanwhile, the QFI for the parametrized thermal states of two-spin systems are also discussed. In the canonical category, we provide the QFI for a cavity optomechanical system. Increasing the photon number in the cavity or tuning down the frequency of the movable mirror will enhance the QFI.

At the end of this paper, an alternative form of QFI is discussed in the scenario. Its formula for unitary parametrization processes is analytically given. For a general parametrized exponential state, we also provide the expression of this alternative QFI, which is a correlation function of $\Gamma_+$ and $\Gamma_-$. In the scenario where $V$ is a conserved quantity, $\Gamma_\pm$ is actually governed by the Hamiltonian, the parameter under estimation and the conserved quantity. We hope this paper could prompt
more and more researchers to study the connection between the conserved quantities and the quantum Fisher information, and search for various ways to enhance the parameter precision via conserved quantities.

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