Uniformly accelerated traveller in FLRW universe

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This paper provides an analytical treatment of accelerated and geodesic motion within the framework of the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. By employing conformal time transformations we manage to convert second order differential equations of motion in FLRW spacetime to first order equations in the conformally transformed spacetime. This allows us to derive a general analytical solution in closed-form for accelerated motion in spatially curved FLRW spacetime. We provide few examples of this general solution for the spatially flat cases. The last part of our work focuses on the return journey for a traveler exploring a FLRW universe. We derive certain condition for de Sitter universe that must be satisfied in order to have an actual return journey.

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1. INTRODUCTION

The paradigm of a homogeneous expanding isotropic universe in the framework of General Relativity is realised via the Friedmann-Lemaître-Robertson-Walker (FLRW) model\textsuperscript{1–4}.

In this work we are going to investigate the accelerated motion of a test particle in FLRW. Such a test particle corresponds to a rocketeer traveling in an FRW universe. The derivation and the interpretation of accelerated motion have suffered from ambiguous treatments which will be discussed later\textsuperscript{1}.

The motion of a uniformly accelerated traveller in an expanding universe is described by a set of differential equations which are in general non-trivial coupled. This set of equations does not become less complicated even if a specific cosmological model is applied. Thus, an analytical derivation of the path of a rocketeer is highly challenging. Actually a goal of this work is to present a general formulation which allows an analytical treatment. In particular, this work has been inspired by the paper by W. Rindler\textsuperscript{10} in which to solve the corresponding set of equations a generalization of the hyperbolic motion in Minkowski spacetime was proposed. However, Rindler solved it only for the de Sitter spacetime\textsuperscript{10}.

Studying the accelerated motion of a rocketeer is useful for the future accelerated space probe. For our universe (with $\Omega_\Lambda \approx 0.27$, $\Omega_m \approx 0.73$ and nearly spatially flat) a space traveler could visit a galaxy which is observed today at a redshift of 1.7 on a one-way journey with proper acceleration equal to the terrestrial gravitational acceleration, in almost 100 years\textsuperscript{11}. However, for galaxies at redshift less than 1.7, e.g. 0.65, it is not clear whether the traveller would succeed to return back home. Therefore, it might be appropriate to consider a traveller of intermittent accelerations to explore the universe\textsuperscript{9}. In this study, we are going to address this issue from different point of view, i.e. a rocketeer travelling with uniform decelerated motion in order to achieve a return trip.

The formalism presented in this work reduces to the geodesic motion in a spatially curved FLRW spacetime in the limit of zero proper acceleration. Since geodesic in an expanding universe has vast applications in cosmology, astrophysics and quantum gravity, many attempts have been undertaken to solve the geodesic equations of motion (for more details see references [12–39] of \textsuperscript{12}). The first attempt to tackle this issue was initiated by Whiting\textsuperscript{13}. Whiting derived the equations of motion for a free particle with Newtonian background and its relativistic generalization. In \textsuperscript{14} geodesic in low-velocity regime has been studied. These efforts by \textsuperscript{13, 14} for solving geodesic motion was not sufficient due to the number of shortcomings in calculation and interpretation. Latter on, Grøn & Elgarøy\textsuperscript{15} derived a general solution for geodesics in the full general relativity framework. Moreover, Ref. \textsuperscript{13} claimed that particle moving uniformly in an expanding universe will join the Hubble flow. This claim has been refuted in \textsuperscript{16}, in which it has been formally proven that particles following the geodesic motion in an eternally expanding universe do not asymptotically rejoin the Hubble flow. Recently, the method for deriving both timelike and spacelike geodesic distances in spatially flat FLRW spacetime with given initial-value or boundary-value constraints was presented in \textsuperscript{12}.

In this work, we use the conformal time transformation in order to get a general analytical formulation. Thus, it is useful to provide a brief overview of what has been already done in FLRW with conformal transformations. Conformal transformation and its symmetries help us to grasp the notion of the causal structure of spacetime\textsuperscript{17}. FLRW metric has vanishing Weyl tensor, therefore, all Friedmann cosmological models are conformally flat and their systematic description has been studied in detail in Ref. \textsuperscript{18–20}. The nature of FLRW models in conformal coordinates has been studied in \textsuperscript{21}. It has been demonstrated in \textsuperscript{22} that transformation into conformal coordinates do not eliminate superluminal recession velocities for open or flat matter dominated FLRW cosmologies, and all of them possess superluminal expansion. Ref. \textsuperscript{23} de-
rived the scalar field and the electromagnetic field of a moving charged particle in de Sitter spacetime, when the particle is following geodesic trajectories or it is uniformly accelerated. In order to achieve this, conformal transformation between de Sitter and Minkowski spacetime was applied.

The layout of this work is as follows; Sec. 2 provides the essential mathematical background in which the conformal time transformation is applied. In Sec. 3 a novel general formalism is presented. Namely, using the transformed FLRW spacetime enables us to solve the equations of motion of accelerated particle. In this way, second order differential equations reduce to first order differential equations which allow us to solve the trajectories for accelerated particle and free motion. In addition, this formalism specifies the four-velocities of particles. This extends the previous results provided in [12] covering only geodesic motion. We prove that accelerated and geodesic motions in FLRW universe depend on the expansion factor and its integral for any specific FLRW model. In Sec. 4 we give some examples for known FLRW models that have an analytic solution. In the other cases where there is no analytical solutions, we use numerical integration to solve it. In Sec. 5 we analyze the return journey. We show that, in order to obtain an actual return journey having uniformly deceleration is not sufficient for every spacetime. For de Sitter spacetime we derive the boundary condition that must be satisfied to fulfill the return journey. Concluding remarks are driven in Sec. 6.

2. MATHEMATICAL BACKGROUND

We begin by introducing the line element of the FLRW spacetime, which describes the metric of an expanding, homogeneous and isotropic universe

\[ ds^2 = -c^2 dt^2 + R^2(t)[d\chi^2 + S_k(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)], \]

where \( c \) is light speed (hereafter \( c = 1 \)),

\[ S_k(\chi) = \begin{cases} \sin \chi, & k = +1, \text{ closed}, \\ \chi, & k = 0, \text{ flat}, \\ \sinh \chi, & k = -1, \text{ open}, \end{cases} \]

expresses the space curvature and \( R(t) \) is the scale factor which describes the expansion of the universe. \( t \) is the coordinate time \( t \in [0, \infty) \); \( \chi \) lies in the range \( \chi \in [0, \infty) \) for \( k = 0, -1 \) and \( \chi \in [0, \pi] \) for \( k = 1 \); while the angles \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \) independently of the curvature.

Let us assume a cosmological model with a cosmological constant \( \Lambda \) and a fluid with equation of state given by

\[ P = P(\rho) = (\gamma - 1)\rho, \]

where \( P \) is the pressure and \( \rho \) is the energy density. Then, the Friedmann equation reads

\[ \frac{R^2(t)}{\dot{R}^2(t)} = \frac{\Lambda}{3} - \frac{k}{R^2(t)} + \frac{C}{R^3(t)}, \]

where dot means derivation with respect to \( t \) and \( C \) is a constant proportional to the matter density (see e.g. [24]).

The four-acceleration of a particle is given by

\[ a^\mu = u^\nu u_\nu \frac{dx^\mu}{dt} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{dt} \frac{dx^\sigma}{dt}, \]

where \( u^\mu \) is the four-velocity and \( \lambda \) is the proper time. \( a^\mu \) and \( \dot{a}_\mu \) satisfy the following constraints

\[ u^\mu u_\mu = -1, \]

\[ a^\mu a_\mu = \Lambda^2, \]

\[ a^\mu u_\mu = 0, \]

where \( \Lambda \) is the norm of the acceleration. Having uniform acceleration means that \( \Lambda = \text{const} \).

Solving Eq. (3) for a given acceleration (say for \( \Lambda = \text{const} \)) is almost analytically intractable (see e.g. [10]). Here we introduce the conformal time transformation in order to tackle this problem. In particular, the conformal time \( \eta \) is such that

\[ \eta = \int \frac{dt}{R(t)}. \]

Additionally, by putting \( \bar{\chi} = \chi \), the FLRW metric reads

\[ ds^2 = \bar{R}^2(\eta)[-d\eta^2 + d\bar{\chi}^2 + S_k(\bar{\chi})(d\theta^2 + \sin^2 \theta d\phi^2)], \]

Notice that, \( \bar{R}(\eta) = R(t) \).

When we have cosmological models with \( \Lambda = 0 \) or \( \Lambda \neq 0 \) but without matter, it holds that (see e.g. [24])

\[ \bar{R}(\eta) = \begin{cases} \bar{R}_c \sin^b(\frac{\eta}{b}), & k = +1, \\ \bar{R}_c \eta^b, & k = 0, \\ \bar{R}_c \sinh^b(\frac{\eta}{b}), & k = -1, \end{cases} \]

where \( \bar{R}_c \) is a constant length which determines the scale of the universe. The power coefficient \( b \) for \( \Lambda = 0 \) is \( b = \frac{2}{\gamma - 2} \). The value of \( b \) distinguishes between different cosmological models. For example, if \( b = 2 \) then the universe is filled with dust; for stiff matter \( b = \frac{1}{2} \); while \( b = 1 \) describes the radiation case. Moreover, for non-negative curvature when \( \Lambda \neq 0 \) and without matter, which is actually a de Sitter cosmological model, then \( b = -1 \).

3. PATH OF PARTICLES IN FLRW SPACETIME

We would like first to present the general formulation for the motion of particles in the transformed FLRW spacetime by considering only the radial motion. To do that, we shall define the four-velocity as follows

\[ u^\eta = \frac{d\eta}{d\lambda} = \frac{\cosh \zeta(\lambda)}{\bar{R}(\eta)}, \quad u^\lambda = \frac{d\bar{\chi}}{d\lambda} = \frac{\sinh \zeta(\lambda)}{\bar{R}(\eta)}, \]

where \( \zeta(\lambda) \) is the rapidity, which will be determined later. Note that equations (9) automatically satisfy constraint (4).
The only needed non-vanishing Christoffel symbols for this case are
\[ \Gamma^\eta_{\eta\eta} = \Gamma^\eta_{\eta \xi} = \Gamma^\xi_{\eta \eta} = \frac{1}{\hat{R}(\eta)} \frac{d\hat{R}(\eta)}{d\eta}. \] (10)

The four-acceleration in the set of coordinates \( (\xi) \) can be written in the following way
\[ a^\eta = \frac{du^\eta}{d\lambda} + \Gamma^\eta_{\eta\eta}(u^\eta)^2 + \Gamma^\eta_{\eta \xi}(u^\xi)^2, \] (11)
\[ a^\xi = \frac{du^\xi}{d\lambda} + 2\xi \eta u^\eta u^\xi \hat{\lambda}. \] (12)

From now on, since all used Christoffel symbols have equal value, we shall denote them by \( \Gamma \).

By differentiating the first term in the right-hand side of Eq. (11), and by using Eq. (9) we obtain
\[ \frac{du^\eta}{d\lambda} = \frac{\sinh \zeta (\lambda)}{\hat{R}(\eta)} \frac{\frac{d\zeta (\lambda)}{d\lambda}}{\hat{R}(\eta)^2} - \frac{\cosh \zeta (\lambda)}{\hat{R}(\eta)^2} \frac{d\hat{R}(\eta)}{d\lambda}. \] (13)

Using Eq. (13) together with Eqs. (9) and (10) we arrive to
\[ \frac{du^\xi}{d\lambda} = u^\eta \frac{d\zeta (\lambda)}{d\lambda} - (u^\eta)^2 \Gamma. \] (15)

Thus
\[ a^\eta = u^\eta \frac{d\zeta (\lambda)}{d\lambda} + (u^\xi)^2 \Gamma = u^\eta \left( \frac{d\zeta (\lambda)}{d\lambda} + \Gamma u^\xi \right). \] (16)

Similar calculation can be undertaken for \( a^\xi \) where
\[ \frac{du^\xi}{d\lambda} = u^\eta \frac{d\zeta (\lambda)}{d\lambda} - u^\eta u^\xi \Gamma, \] (17)
which finally gives
\[ a^\xi = u^\eta \frac{d\zeta (\lambda)}{d\lambda} + u^\eta u^\xi \Gamma = u^\eta \left( \frac{d\zeta (\lambda)}{d\lambda} + \Gamma u^\xi \right). \] (18)

We denote
\[ A = \frac{d\zeta (\lambda)}{d\lambda} + \Gamma u^\xi, \] (19)
where \( A \) is the norm of acceleration as mentioned earlier in Eq. (5). As a result, the four-acceleration becomes
\[ a^\eta = Au^\xi, \quad a^\xi = Au^\eta. \] (20)

Note that Eq. (20) satisfies also the constraints \( a \) and \( \xi \).

In the transformed FLRW metric \( (\xi) \) the coordinates \( \eta \) and \( \xi \) share a common coefficient, i.e. the \( R(\eta) \). If we constrain the motion only on the radial direction through this transformation we get a solvable set of equations from Eq. (2). This allows us to analyze the radial motion of the rocketeer in the FLRW spacetime.

### 3.1. Accelerated Radial Motion

It is convenient to express the equation of motion of the rocketeer in terms of \( \eta \). Therefore, from the four-velocity \( u^\eta \), we get
\[ \frac{d\zeta }{d\eta} = \frac{d\zeta }{d\lambda} = \tanh \zeta (\lambda). \] (21)

To find the unknown rapidity \( \zeta (\lambda) \) we need to use Eq. (20) and reparametrize it in terms of \( \eta \)
\[ A - u^\eta \frac{d\zeta (\eta)}{d\eta} - \Gamma u^\xi = 0, \] (22)
where \( \zeta (\eta) = \text{arcsinh}(A\hat{R}(\eta) + \frac{v}{R(\eta)}) \),

\[ \hat{R}(\eta) = \int \frac{R(\eta) d\eta}{\sqrt{(A\hat{R}(\eta) + \frac{v}{R(\eta)})^2 + 1}}, \] (24)

and \( v \) is an integration constant which is related to the initial velocity of particle. Consequently,
\[ \hat{\chi} = A \int \frac{\hat{R}(\eta)}{\sqrt{(A\hat{R}(\eta) + \frac{v}{R(\eta)})^2 + 1}} d\eta \]
\[ + v \int \frac{1}{\sqrt{(A\hat{R}(\eta) + \frac{v}{R(\eta)})^2 + 1}} d\eta. \] (25)

Now, we go back to the coordinates of the original FLRW metric \( (\xi) \). This is achieved by using the inverse transformation of Eq. (7), i.e. \( \hat{R}(\eta) d\eta = dt \), and by recalling that \( R(t) = \hat{R}(\eta) \). Thus, we obtain for Eq. (23)
\[ \hat{\chi} = \text{arcsinh}(A\hat{R}(t) + \frac{v}{R(t)}), \] (26)

where
\[ \hat{R}(t) = \int \frac{R(t') dt'}{R(t)}. \] (27)

Using Eq. (26) enables us to derive the four-velocity in standard FLRW spacetime as follows
\[ u' = \frac{dt}{d\lambda} = \cosh \hat{\chi} (t), \quad u^\xi = \frac{d\chi}{d\lambda} = \frac{\sinh \hat{\chi} (t)}{R(t)}. \] (28)

Finally, the trajectory of uniform acceleration motion is given by
\[ \chi = A \int \frac{\hat{R}(t)}{\sqrt{(A\hat{R}(t) + \frac{v}{R(t)})^2 + 1}} dt \]
\[ + v \int \frac{1}{R(t)^2} \sqrt{\frac{1}{(A\hat{R}(t) + \frac{v}{R(t)})^2 + 1}} dt. \] (29)

By specifying the evolution of the scale factor \( R(t) \) Eq. (29) provides the accelerated radial path of the rocketeer in the standard FLRW coordinate.
3.2. Some characteristic types of motion

a. Purely accelerated motion. The trajectories (25) and (29) will remain uniformly accelerated motion if one ignores the integration constant \( \nu \) (i.e. \( \nu = 0 \)). We call this type of motion purely accelerated. In the conformally transformed coordinates the trajectory is given by

\[
\hat{\chi}_a = A \int \frac{\mathcal{A}(\eta)}{A^2 \mathcal{A}(\eta)^2 + 1} d\eta, \tag{30}
\]

and in the original FLRW coordinates we get

\[
\chi_a = A \int \frac{1}{R(t)} \frac{\mathcal{A}(t)}{A^2 \mathcal{A}(t)^2 + 1} dt, \tag{31}
\]

where index \( a \) in both Eqs. (30) and (31) refers to the purely accelerated motion.

b. Geodesic motion. To get the trajectory for the geodesic motion one has to substitute \( A = 0 \) into the Eqs. (23), (25) and (29). Thus, the rapidity function \( \tilde{\chi}(\eta) \) becomes

\[
\tilde{\chi}(\eta) = \text{arcsinh} \left( \frac{\nu}{\bar{R}(\eta)} \right). \tag{32}
\]

Consequently, Eq. (25) will be

\[
\tilde{\chi}_a = \int \frac{\nu}{\sqrt{R(\eta)^2 + \nu^2}} d\eta, \tag{33}
\]

and for Eq. (29) we obtain

\[
\tilde{\chi}_v = \int \frac{\nu}{\sqrt{R(\eta)^2 + \nu^2}} d\eta. \tag{34}
\]

Here index \( v \) in Eqs. (33) and (34) denotes geodesic motion. For all \( \nu \) values, \( \nu^2 > 0 \), which guarantees that Eq. (34) is a timelike geodesic (12).

Eqs. (33) and (34) are geodesics in any conformal time FLRW spacetime and FLRW spacetime respectively. Eq. (34) is the same as equation derived in (15) and recently in (12).

c. Null geodesics. We can see from Eq. (29) that for large acceleration \( A \) the particle’s trajectory asymptotically reaches the null geodesic, that means

\[
\lim_{A \to > \nu} \chi_a = \pm \int \frac{1}{R(t)} dt. \tag{35}
\]

Moreover, this statement holds for large \( \nu \) value, i.e.

\[
\lim_{\nu \to > \nu} \chi_a = \pm \int \frac{1}{R(t)} dt. \tag{36}
\]

4. SOME EXAMPLES

In this section our formalism of section 3 will be applied to specific FLRW universe models. The motion of a particle both in the original and in the transformed coordinates depend only on the scale factor and its integral (Eq. (27) and Eq. (24) respectively). Thus, specifying the expansion factor for each cosmological model enables us to determine the particles worldlines. In this section we will study the behaviour of the trajectories presented in paragraphs a and b of Sec. 3.2.

Recently, the solution of Friedmann Eq. (2) was presented for various FLRW models with \( k = 0 \) (25). Namely, Chavanis has derived an analytical solution for \( R(t) \) in a universe undergoing a various combination of eras, e.g. stiff matter era, dark matter era, and dark energy era due to the cosmological constant. From this study we use the form of the scale factor in the cosmological examples of Sec.4 and Sec 5.

Keep in mind that, although transformation (7) is not in general conformally flat transformation (CFT) for specially curved FLRW models, it is CFT for all flat FLRW models. In the coming examples all cases have zero spatial curvature which guarantee that transformation (7), apart from Milne universe, is CFT, that means

\[
ds^2_{\text{FLRW}} = \Omega^2 ds^2_{\text{flat}}, \tag{37}
\]

where \( \Omega = \bar{R}(\eta) \).

Furthermore, it is clear that this formalism is able to re-derive the known hyperbolic motion for the Minkowski spacetime (26). Additionally, uniformly accelerated motion in the Einstein static universe can be derived in a similar manner as Minkowski spacetime since both spacetimes have scale factor \( R(t) = 1 \).

To provide a visualization of our examples we are going to plot the some trajectories in Penrose diagrams with coordinates \( \eta \) and \( \chi \) given by metric (8).

4.1. Dust-filled universe, \( \gamma = 1 \)

We consider a universe which is dominated by dust or dark matter, i.e. in a case in which stiff matter, radiation and dark energy are vanishing, which is known as EdS solution. In this case the scale factor is

\[
R(t) = R_0 t^\gamma \tag{38}
\]

where \( R_d = R_0 \left( \frac{9}{4} \Omega_{m0} H_0^2 \right)^{\frac{1}{2}} \) and \( R_0 \) is the scale factor value at present time (25). Using scale factor Eq. (38) into the equations Eqs. (31) and (34) we obtain

\[
\chi_a = \frac{9}{20 R_d} A t^2 F_1 \left( \frac{1}{2}, \frac{5}{3} ; 1 ; -\frac{9}{25} A^2 t^2 \right), \tag{39}
\]

and

\[
\chi_v = \frac{3E_F \left( \frac{1}{2}, \frac{1}{2}, 3 ; \frac{1}{25} A^2 t^2 \right)}{R_d t^{\frac{2}{3}}}, \tag{40}
\]

where \( a \) and \( v \) denote uniform acceleration and geodesic motion respectively. Here, \( F_1(a,b;c;z) \) is the Gauss hypergeometric function and \( F_E(z,k) \) is the incomplete Elliptic integral of the first kind which is defined by

\[
E_F = \text{Elliptic} F(z,k) = \int_0^z \frac{1}{\sqrt{-\sigma^2 + 1}} \frac{1}{\sqrt{-\sigma^2 k^2 + 1}} d\sigma, \tag{41}
\]
with no restrictions on \((z,k)\).

In conformal representation, where \(\tilde{R}(\eta) = \tilde{R}_d \eta^2\) for this case, we have

\[
\begin{align*}
\tilde{\chi}_a &= \frac{A\tilde{R}_d \eta^4}{20} - \frac{2\sqrt{F_1(1/2, 1/3; 1/3; -1/25 A^2 \tilde{R}_d \eta^6)},}{(42)} \\
\tilde{\chi}_r &= E_F\left(\eta \sqrt{\frac{i \tilde{R}_d}{\nu}}, i\right) \sqrt{i \tilde{R}_d / \nu}.
\end{align*}
\]

(43)

FIG. 1: Penrose diagram for the dust-filled universe. The thick black solid line denotes the photon trajectory. Solid lines have constant acceleration \(A\) and \(\nu = 0\), while the dotted dashed lines have constant acceleration \(A\) and \(\nu \neq 0\). Dashed curves are geodetics, i.e. \(\mathcal{I}^+\).

4.2. Radiation, \(\gamma = \frac{1}{2}\)

We consider a universe made of radiation. In the absence of stiff matter, dark matter, dust and dark energy. This FLRW model has scale factor

\[
R(t) = R_{rad} \sqrt{t},
\]

(44)

where \(R_{rad} = R_0 \sqrt{2H_0 \Omega_{rad,0}^{1/2}}\) [25]. Thus, \(\chi_a\) becomes

\[
\begin{align*}
\chi_a &= \frac{2\sqrt{3iA}}{AR_{rad}} E_F\left(1/3 \sqrt{3}/3 - 2iAR, 1/2 \sqrt{2}\right) \\
&- \frac{2\sqrt{3iA}}{AR_{rad}} E_F\left(1/3 \sqrt{3}/3 - 2iAR, 1/2 \sqrt{2}\right).
\end{align*}
\]

(45)

where \(E_{\mathcal{E}}(z,k)\) is the incomplete elliptic integral which is define by

\[
E_{\mathcal{E}} = \text{EllipticE}(z,k) = \int_0^\infty \sqrt{-\delta^2 k^2 + 1} \, d\delta,
\]

(46)

with no restrictions on \((z,k)\).

And for geodesic motion we get

\[
\begin{align*}
\tilde{\chi}_r &= E_F\left(\eta \sqrt{\frac{i \tilde{R}_d}{\nu}}, i\right) \sqrt{i \tilde{R}_d / \nu} \\
\tilde{\chi}_a &= \frac{\nu v}{R_{rad}^2} \ln \left(\frac{2\left(\frac{R_{rad}}{\nu}\right)^2 + 2 \sqrt{2\left(\frac{R_{rad}}{\nu}\right)^2 + 1}}{2\left(\frac{R_{rad}}{\nu}\right)^2 + 1}\right),
\end{align*}
\]

(47)

In conformal FLRW spacetime where \(\tilde{R}(\eta) = \tilde{R}_d \eta\) we get

\[
\begin{align*}
\tilde{\chi}_a &= \frac{i}{3AR_{rad}} E_F\left(1/3 \eta \sqrt{3}/iAR_{rad}, i\right) \\
&- \frac{i}{3AR_{rad}} E_F\left(1/3 \eta \sqrt{3}/iAR_{rad}, i\right),
\end{align*}
\]

(48)

and

\[
\begin{align*}
\tilde{\chi}_r &= \frac{\nu v}{R_{rad}^2} \ln \left(\frac{R_{rad}^2 \eta^2 + v^2}{R_{rad}^2}\right),
\end{align*}
\]

(49)

for uniformly accelerated trajectory and free motion.

Fig. 2 illustrates the motions of accelerated and geodesic rocketeer in the flat FLRW universe made of radiation. These trajectories have similar behaviour as the dust filled universe.

4.3. Stiff Matter, \(\gamma = 2\)

Assuming the universe which is filled only by stiff matter. This universe has the scale factor

\[
R(t) = R_{sm} t^{1/2},
\]

(50)

where \(R_{sm} = R(0)(3\sqrt{\Omega_{sm,0} H_0})^{1/2}\) [25]. In this spacetime we get the following trajectories

\[
\begin{align*}
\chi_a &= \frac{9A t^2}{20R_{sm}^5} F_1\left(1/2, 1/3, 1/6, -1/9 A^2 t^2\right),
\end{align*}
\]

(51)

\[
\begin{align*}
\chi_r &= \frac{3v \sqrt{R_{sm}^2 t^2 + v^2}}{R_{sm}^2} + C_4.
\end{align*}
\]

(52)
In conformal frame $\bar{x}_a$ does not have a closed-form solution.

Trajectories in the FLRW universe filled by stiff matter is shown in Fig. 3. Similar to the previous examples, behaviour of the accelerate trajectories and geodesics are the same as Fig. 1.

### 4.4. Milne Universe

Vacuum FLRW model with $\Lambda = 0$ and $k = -1$ is known as Milne universe [27], where $R(t) = t$. For this spacetime particle’s paths are

$$\chi_a = \ln \left( At + \sqrt{A^2 t^2 + 4} \right) + \ln (C_5),$$

where $C_5$ is integration constant. We shall see that for particle starting from origin, $C_5 = \frac{1}{2}$, $\chi_a$ reduces to

$$\chi_a = \arcsinh \left( \frac{A}{2} t \right),$$

and $\chi_v$ becomes

$$\chi_v = - \text{arctanh} \left( \frac{u}{\sqrt{t^2 + u^2}} \right) + \text{arctanh} (\beta),$$

where $\text{arctanh} (\beta)$ is an integration constant.

For Milne universe we shall not use Eqs. (25) and (33) since transformation (7) is not CFT here. Using the transformation between Milne Universe and Minkowski spacetime [26] where

$$t = \sqrt{T^2 - R^2}, \quad \chi = \text{arctanh} \left( \frac{R}{T} \right),$$

we get

$$(R_0 + \frac{1}{A})^2 - T^2 = \frac{1}{A^2},$$

and

$$R_v = - \beta T + C_6,$$

for transformed trajectories (54) and (55) in Minkowski spacetime respectively where $C_6 = \frac{\sqrt{1 - \beta^2}}{\alpha}$. 

Note that, Eqs. (57) and (58) are known hyperbolic motion and geodesic motion in Minkowski spacetime. In Fig. 4 these motions are depicted in the same manner as Fig. 1. The shaded region in this figure does not cover the Milne universe.

### 4.5. de-Sitter Universe, $\gamma = 0$

Considering the dark energy dominated universe in the absence of radiation, stiff fluid and dark matter. In this case scale factor is

$$R(t) = R_0 e^{\sqrt{\frac{2}{\Lambda} t}},$$

where $\Lambda$ is a cosmological constant [23]. This solution is known as the de Sitter solution.
Accelerated and geodesic motion in this particular spacetime describe by

$$\chi_a = -\frac{A}{\sqrt{\lambda R_0 e^{\sqrt{\lambda} t}/2} \sqrt{A^2 + \frac{2}{3}}} + C_1, \quad (60)$$

$$\chi_v = -\frac{3}{\lambda} \sqrt{\left(\frac{R_0 e^{\sqrt{\lambda} t}}{2} + v^2\right)}.$$ \quad (61)

And in conformal coordinate, where scale factor is \(R(\eta) = -\sqrt{\frac{2}{\lambda} \frac{1}{\eta}}\), we get

$$\tilde{\chi}_a = \frac{A}{\sqrt{\lambda^2 + \frac{2}{3}}} \eta, \quad (62)$$

$$\tilde{\chi}_v = -\sqrt{\frac{3}{\lambda \eta^2} + \eta^2 + C_3}. \quad (63)$$

It is clear from Eqs. (62) and (63) that the geodesic equation in conformally flat coordinate, Minkowski spacetime, get transformed to uniformly accelerated worldline in de Sitter spacetime whereas trajectory of fixed accelerated particle in Minkowski spacetime get transformed to geodesic in de Sitter spacetime (see Fig. 5). Same result was presented by W.Rindler\(^2\) and J. Bicák & P. Krtouš. \cite{10, 23}.

Fig. 5 shows the trajectories in the de Sitter spacetime. In this spacetime, all trajectories have the same description as in Fig [1] but some of them have different initial conditions. Namely, some of them do not pass through the origin \(t = 0\). Another difference is that de Sitter spacetime covers only the lower part of the Penrose diagram, since all trajectories end up at the \(\mathcal{I}^\pm\). We continue plotting the trajectories even to the upper shaded region in order to have a global view of the behaviour of these trajectories.

5. RETURN JOURNEY

In this section we are going to focus our study on analyzing the return journey of the rocketeer in spatially flat FLRW universe. In particular, we are going to study the behaviour of the \(\text{Eq. (29)}\) or \(\text{Eq. (25)}\) in the spacetimes studied in the previous section. Actually, in order to fulfill the return journey, our rocketeer must begin to decelerate, i.e. \(A_d < 0\), long enough time before it reaches the designated proper distance.

Assuming that, the spaceship is travelling with non-zero positive value \(v\), at \(t = \lambda = \chi = 0\) the rocketeer applies a deceleration \(A_d\). Thus, the rocketeer reaches the maximum comoving distance from the origin at the return point with coordinates \(\{t_1, x_1\} > 0\) when \(\hat{\chi}(t_1) = 0\) or equivalently \(u^\Lambda(t_1) = 0\).

Therefore, depending on the form of scale factor, one can analyze the return journey of the rocketeer.

5.1. \(R(t) = t^n\) spacetimes

In this section we analyze the return journey in the spacetime studied in sections 4.1-4.4, namely spacetimes having the scale factor like \(R(t) = t^3\). In these particular spacetimes the rocketeer reaches the maximum comoving distance from the origin at

$$t_1 = \left(-\frac{v(n+1)}{A_d}\right)^{\frac{1}{n+1}}. \quad (64)$$

 Afterwards, the rocketeer returns towards the origin. As \(t \to \infty\) the trajectory of the rocketeer asymptotically becomes

$$\lim_{t \to \infty} \chi = -\int \frac{1}{t^n} dt,$$ \quad (65)

which means that there is a finite \(t_2 > t_1\), when the rocketeer arrives back to the origin, \(\chi = 0\). In Figs. 1-4, the dotted dash trajectories represent the return journeys in each spacetimes.

5.2. de Sitter case

Return journey in de Sitter spacetime has different behavior with respect to previous examples since the scale factor \(A_\Lambda\) has different properties. In this specific spacetime, the rocketeer reaches the maximum comoving distance from the origin at

$$t_1 = \sqrt{\frac{3}{\Lambda}} \ln \left(-\sqrt{\frac{\Lambda}{3}} \frac{v}{A_d}\right). \quad (66)$$

Moreover, the total cosmic time \(t_2\) needed to cover the return journey for a rocketeer that leaves the origin at \(t = \lambda = 0\), is derived from Eq. (29) and it is given by

$$t_2 = \sqrt{\frac{3}{\Lambda}} \ln \left(\frac{\sqrt{3\Lambda v}}{\sqrt{3\Lambda v + 6A_d}}\right). \quad (67)$$

Thus, from Eqs. (66) and (67) one can see that the return journey does not happen for all values of \(A_d\) and \(v\) (see Fig. 6). Actually, to attain an actual return journey, the following relation

$$2A_d < -\sqrt{\frac{\Lambda}{3}} v < A_d, \quad (68)$$

has to be satisfied. In Fig. 6 we show several cases of return journeys in de Sitter spacetime for \(\Lambda = 3\) and \(\Lambda = -2\). The negative values of \(\chi\) represents the opposite direction from the one that the rocketeer is supposed to explore.

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\(^2\) Note that, W.Rindler obtained only one special case of accelerated motion in de Sitter spacetime, namely at \(t = \lambda = 0\) particle leaves the origin from the rest, i.e. \(u' = 1\) and \(u^\Lambda = 0\). One can rederive Rindelr’s trajectory by putting \(v = -\sqrt{\frac{3}{\Lambda} A}\) into the Eq. (29) together with \(\chi(t_a) = 0\).
FIG. 5: Penrose diagram for de Sitter universe. The shaded part is not covered by the flat de Sitter universe.

**Line 1.** For \(-\sqrt{\frac{\Lambda}{3}} \geq A_d\), there isn’t any return point for the particle and rocketeer will move toward the \(-\chi\) direction.

**Line 2.** For \(2A_d < -\sqrt{\frac{\Lambda}{3}} < A_d\), there is a return point and the rocketeer is able to come back to the origin.

**Line 3.** For \(-\sqrt{\frac{\Lambda}{3}} = 2A_d\), there is a return point but the rocketeer will return back to origin in an infinite cosmic time.

**Line 4.** For \(-\sqrt{\frac{\Lambda}{3}} > 2A_d\), there is a return point but the rocketeer will never go back to the origin.

Thus, we have seen that in de Sitter spacetime, having the uniform deceleration motion is not sufficient for the rocketeer to come back to the origin. One has to apply the deceleration which satisfies the Eq. (68).

FIG. 6: Penrose diagram for the worldlines of return journeys in de Sitter spacetime. Here \(\Lambda = 3\) and \(A = -2\) and the spacetime is depicted only for \(t \geq 0\).

6. CONCLUSIONS

In 1960, W.Rindler [10] proposed the problem of a "Hyperbolic Motion in Curved Spacetime" to study the accelerated
motion in curved spacetime.

In particular, it was suggested that the accelerated motion is the best way of exploring our universe in a reasonably short time [9]. Taking the above suggestion into account, the motion of an accelerated traveller in an expanding universe has been studied in this work. This involves solving the non-trivial Eq. (3) for a given acceleration. To achieve this, we applied a conformal time transformation [1] to the generic FLRW universe. Using the method introduced in Sec. 3 has helped us to determine a generalized form of rapidity function (26), which leads us to derive the trajectory (29) of an accelerated traveler.

We have shown that the accelerated and the geodesic motion in an expanding universe are solely determined by the expansion factor and its integral (27). The scale factor is the solution of the Friedmann equation (2), which depends on the spatial curvature $k$, the cosmological constant $\Lambda$, and the equation of state $P = P(\rho)$.

In the last part of our work we have focused on the return journey of the rocketeer. It had been suggested that having uniform deceleration would be enough in order to have an actual return journey [10]. Here we have proved that even if this condition is necessary, it is not sufficient for all spacetimes. In particular, among the cosmological models analyzed here, in the de Sitter case Eq. (38) must be satisfied for a return journey to be possible.

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