Abstract. We consider partial theta series associated with periodic sequences of coefficients, of the form \( \Theta(\tau) := \sum_{n>0} n^{\nu} f(n) e^{i\pi n^2 \tau / M} \), with \( \nu \in \mathbb{Z}_{\geq 0} \) and an \( M \)-periodic function \( f : \mathbb{Z} \to \mathbb{C} \). Such a function is analytic in the half-plane \( \{ \Im \tau > 0 \} \) and as \( \tau \) tends non-tangentially to any \( \alpha \in \mathbb{Q} \), a formal power series appears in the asymptotic behaviour of \( \Theta(\tau) \), depending on the parity of \( \nu \) and \( f \). We discuss the summability and resurgence properties of these series by means of explicit formulas for their formal Borel transforms, and the consequences for the modularity properties of \( \Theta \), or its “quantum modularity” properties in the sense of Zagier’s recent theory. The Discrete Fourier Transform of \( f \) plays an unexpected role and leads to a number-theoretic analogue of Écalle’s “Bridge Equations”.

The motto is: (quantum) modularity = Stokes phenomenon + Discrete Fourier Transform.

This text gathers selected results from our forthcoming paper [8]. Here, we aim at giving the global picture, omitting the most technical details of the proofs for the sake of clarity. We are interested in analytic functions defined in \( \mathbb{H} := \{ \tau \in \mathbb{C} \mid 3m \tau > 0 \} \) as partial theta series, i.e. of the form

\[
\tau \in \mathbb{H} \mapsto \Theta(\tau) := \sum_{n \geq 1} a_n e^{i\pi n^2 \tau / M},
\]

mostly in the case when the coefficients \( a_n \) are of the form \( n^{\nu} f(n) \) and \( f : \mathbb{Z} \to \mathbb{C} \) is \( M \)-periodic. We will show that their boundary behaviour is better understood when viewed from the perspective of Écalle’s Resurgence Theory [6, 18].

Resurgence Theory originates in local analytic dynamics (Écalle-Voronin theory of parabolic points and Stokes phenomena for nonlinear ODEs [6, 5, 4, 19, 13]) and has recently gained recognition in topology and mathematical physics—see [1] and [14] for the most recent examples. It defines certain subspaces of the space of all formal series in one indeterminate and shows how to endow them with a Fréchet algebra structure [20]; it then relies on a systematic use of the Laplace transform to relate certain analytic functions and their divergent asymptotic expansions, while providing us with the apparatus of Alien calculus to analyse Stokes phenomena (linear or nonlinear).

Keywords and phrases. Resurgence, Modularity, Partial Theta Series, Analytic Combinatorics.

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In our case, the Stokes phenomenon will prove to be related to the action of the Discrete Fourier Transform on \( f \) and, according to the parity of \( \nu \) and \( f \), to classical modularity or quantum modularity properties (in the sense of Zagier [22]). We shall thus recover and deepen the analytic aspects of various results found in the literature [1, 3, 10, 12, 15, 21, 24].

1. From partial theta series to generating series

Suppose for the moment that \( M \in \mathbb{R}_{>0} \), \((a_n)_{n \geq 1}\) is a sequence of complex numbers and there exists \( K > 0 \) such that \( |a_n| \leq K^n \) for all \( n \geq 1 \). Since \( |e^{i\pi n^2/\tau}/M| = e^{-\pi n^2(\Im \tau)/M} \), the function \( \Theta \) defined by (0.1) is analytic on \( \mathbb{H} \) and exponentially small as \( \Im \tau \to +\infty \).

**Theorem 1.** Take any \( c \in \mathbb{R}_{>0} \) large enough so that the sum of the generating series
\[
F(t) := \sum_{n \geq 1} a_n e^{-nt}
\] (1.1)
is bounded on \( \{\Re t \geq c\} \) and analytic in a neighbourhood of this half-plane. Then the function
\[
\hat{\phi}(\xi) := \pi^{-1/2} \xi^{-1/2} F(C\xi^{1/2}) \quad \text{with} \quad C := \left(\frac{4\pi}{M}\right)^{1/2} e^{i\pi/4},
\] (1.2)
where the branch of \( \xi^{1/2} \) is specified by \(-\frac{3\pi}{2} < \arg \xi < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < \arg(\xi^{1/2}) < \frac{\pi}{4} \), is bounded on \( P_c := \{\Re(C\xi^{1/2}) \geq c\} \) and analytic in a neighbourhood of this set, and we have
\[
\Theta(\tau) = \frac{1}{2} \tau^{-1/2} \int_{\partial P_c} e^{-\xi/\tau} \hat{\phi}(\xi) \, d\xi \quad \text{for all} \quad \tau \in \mathbb{H},
\] (1.3)
where the parabola \( \partial P_c \) is oriented anticlockwise—see Figure 1—and we choose the branch of the square root so that \( 0 < \arg(\tau^{1/2}) < \frac{\pi}{2} \).

Notice that when \( \xi \) is on the parabola \( \partial P_c \) and \( |\xi| \to \infty \), \( \arg \xi \) tends to \(-\frac{3\pi}{2} \) or \( \frac{\pi}{2} \) so that \( |e^{-\xi/\tau}| \) decays exponentially as soon as \( 0 < \arg \tau < \pi \). If \( 0 < \varepsilon < \frac{\pi}{2} \) then, for \( \arg \tau \in (\varepsilon, \pi - \varepsilon) \), the Cauchy theorem allows us to replace the integration contour \( \partial P_c \) with a contour \( \Gamma_\varepsilon \subset P_c \)
beginning like a half-line of direction $-\frac{3\pi}{2} + \varepsilon$ and finishing like a half-line of direction $\frac{\pi}{2} - \varepsilon$, as on Figure 1:

$$
\varepsilon < \arg \tau < \pi - \varepsilon \implies \Theta(\tau) = \frac{1}{2} \tau^{-1/2} \int_{\Gamma_\varepsilon} e^{-\xi/\tau} \phi(\xi) \, d\xi.
$$  

(1.4)

Theorem 1 is implicit in [12] and [1]. A direct proof is obtained from the identity

$$
\tau^{1/2} e^{\sigma^2 \tau} = \frac{1}{2} \tau^{-1/2} \left( - \int_0^\infty e^{i(\frac{1}{2} + \varepsilon)\xi} + \int_0^{i(\frac{1}{2} - \varepsilon)\xi} e^{-\xi/\tau} \xi^{-1/2} e^{-2\sigma \xi^{1/2}} \, d\xi \right) 
$$

(1.5)

(for any $\sigma \in \mathbb{C}^*$), which follows from Borel-Laplace summation of the convergent Puiseux expansion $\sum \frac{\sigma^n \tau^{p+\frac{1}{2}}}{n!}$ of the left-hand side at 0 (indeed: the Borel transform of this series happens to be the odd part of $\Psi(\xi^{1/2}) := \pi^{-1/2} \xi^{-1/2} e^{-2\sigma \xi^{1/2}}$, we thus recover the left-hand side by applying the Laplace transform (3.8) to $[\Psi(\xi^{1/2}) - \Psi(-\xi^{1/2})]/2$ in any direction, e.g. $\frac{\pi}{2} \pm \varepsilon$, but we also have $\mathcal{L}^{\frac{\pi}{2} + \varepsilon}[\Psi(-\xi^{1/2})] = \mathcal{L}^{\frac{\pi}{2} + \varepsilon}[\Psi(\xi^{1/2})]$). Alternatively, one can use Lemma 1 of [9].

## 2. Partial theta series associated with periodic sequences

From now on, in (0.1) we consider sequences $(a_n)$ of the form $a_n = n^\nu f(n)$ with

$$
\nu \in \mathbb{Z}_{\geq 0}, \quad M \in \mathbb{Z}_{>0}, \quad f: \mathbb{Z} \to \mathbb{C}, \quad M\text{-periodic function.}
$$

(2.1)

For each $(\nu, f, M)$, the corresponding partial theta series is

$$
\tau \in \mathbb{H} \mapsto \Theta(\tau; \nu, f, M) := \sum_{n \geq 1} n^\nu f(n) e^{i\pi n^2 \tau/M}
$$

(2.2)

and we are particularly interested in the local behaviour of this analytic function as $\tau$ approaches a point $\alpha \in \mathbb{Q}$ of the boundary of $\mathbb{H}$.

**Remark 2.1.** For any $\alpha \in \mathbb{Q}$, we can write

$$
\Theta(\alpha + \tau; \nu, f, M) = \Theta(M_{\alpha} \tau; \nu, f_{\alpha}, M_{\alpha}) 
$$

with notation $f_{\beta}(n) := f(n) e^{i\pi n^2 \beta}$ for $\beta \in \mathbb{Q}$,

(2.3)

where $M_{\alpha}$ is a period of $\alpha$, for instance $\text{lcm}(M, 2 \text{ denominator}(\alpha/M))$. Therefore, the local study near $\tau = 0$ of $\Theta(\tau; \nu, f, M)$ with arbitrary $M$ and $f$ will give us access to the local behaviour of $\tau \mapsto \Theta(\tau; \nu, f, M)$ near any rational.

Instead of $\Theta(\tau; \nu, f, M)$, we will often write $\Theta(\tau; \nu, f)$ or $\Theta(\tau)$, omitting some of the parameters if they are clear from the context. The cases $\nu = 0$ and $\nu = 1$ will be the most important ones.

**Remark 2.2.** Note that, for $\mu \geq 0$ and $\nu = 2\mu$ or $2\mu + 1$,

$$
\Theta(\tau; 2\mu, f, M) = \left( \frac{M}{\tau \partial \tau} \right)^\mu \Theta(\tau; 0, f, M), \quad \Theta(\tau; 2\mu + 1, f, M) = \left( \frac{M}{\tau \partial \tau} \right)^\mu \Theta(\tau; 1, f, M).
$$

(2.4)

Consequently, even if we state some results with $\nu = 0$ or $\nu = 1$ as $\tau \to 0$ only, they entail results for arbitrary $\nu$ with $\tau$ approaching any $\alpha \in \mathbb{Q}$, simply by putting together (2.3) and (2.4).
A few examples.
- The classical Jacobi theta function appears as $\theta_3 = 1 + 2\Theta(\tau; 0, 1, 1)$.
- The even primitive quadratic Dirichlet character $f(n) = (\frac{12}{n})$ (Kronecker symbol), listed as $f = \chi_{12}(11, \cdot)$ in [16], gives rise to the Dedekind eta-function $\eta(\tau) = \Theta(\tau; 0, f, 12)$, and its Eichler integral $\tilde{\eta}(\tau) = \Theta(\tau; 1, f, 12)$ has been investigated particularly in [24] and [3].
- The Rogers-Ramanujan identity [23] involves $\theta_{5, j}(\tau) = \Theta(\tau; 0, f_{5, j}, 20)$, where $f_{5, 1}$ and $f_{5, 2}$ respectively are the real and imaginary part of $\chi_20(7, \cdot)$, an even primitive Dirichlet character [16].
- In Chern-Simons theory with gauge group SU(2) or in the context of Witten-Reshetikhin-Turaev invariants for the Poincaré homology sphere [15, 12], there is a partition function involving $\Theta(\tau; 0, f_+, 60)$ with $f_+ := \Re(\chi_{20}(23, \cdot))$, where $\chi_{60}(23, \cdot)$ is an odd primitive Dirichlet character [16].
- More generally, for any oriented Seifert fibered integral homology sphere with $r \geq 3$ exceptional fibers, Andersen and Mistegård show in [1] that the “Gukov-Pei-Putrov-Vafa invariant” is essentially a partial theta series as in (0.1), whose limit as $\tau \to 1/k$ is related to the level $k$ Witten-Reshetikhin-Turaev invariant, at least for $k > 0$. One can check that, in this case, $a_n$ is a finite sum of odd sequences of the form $n^\nu f^{[\nu]}(n)$, with periodic functions $f^{[\nu]}$ of the same period and $\nu$ ranging from 0 to $r - 3$.

3. A resurgent divergent series that has three canonical sums

Let $\nu, M, f$ be as in (2.1). With a view to applying Theorem 1, we observe that the generating function has an analytic continuation as a rational function of $e^{-t}$:

$$F(t) = \sum_{n \geq 1} n^\nu f(n) e^{-nt} = \left(-\frac{d}{dt}\right)^\nu F_0(t), \quad F_0(t) = \sum_{n \geq 1} f(n) e^{-nt} = \frac{1}{1 - e^{-Mt}} \sum_{1 \leq \ell \leq M} f(\ell) e^{-\ell t},$$

hence we can take $c > 0$ arbitrarily small and $F(t)$ extends from $\{\Re t \geq c\}$ to a meromorphic function on $\mathbb{C}$, regular on $\mathbb{C} - \frac{2n}{M} \mathbb{Z}$. The mean value $\langle f \rangle = \frac{1}{M} \sum_{\ell \in \mathbb{Z}/M} f(\ell)$ controls the regularity at $t = 0$: if $\langle f \rangle \neq 0$, then 0 is a simple pole for $F_0(t)$ with residue $\langle f \rangle$; all other possible poles are of the form $t_n := 2\pi n / M$, $n \in \mathbb{Z}^*$.

To compute the function $\hat{\phi}(\xi) = \pi^{-1/2} \xi^{-1/2} F(C \xi^{1/2})$ of (1.2), we decompose $F$ into

$$F(t) = \frac{\nu! (f)}{\nu + 1} + F^+(t) + F^-(t), \quad F^\pm(t) \in \mathbb{C}\{t\}, \quad F^+ \text{ even, } F^- \text{ odd.} \quad (3.1)$$

Defining two functions $\hat{\phi}^+$ and $\hat{\phi}^-$ by

$$F^+(t) = \pi^{1/2} \hat{\phi}^+ \left( \frac{t^2}{C^2} \right), \quad F^-(t) = \pi^{1/2} \frac{t}{C} \hat{\phi}^- \left( \frac{t^2}{C^2} \right), \quad (3.2)$$

we get

$$\hat{\phi}(\xi) = \frac{\nu! (f)}{\pi^{1/2} C^{\nu + 1}} \xi^{-\nu - 1} + \xi^{-1/2} \hat{\phi}^+ (\xi) + \hat{\phi}^- (\xi), \quad \hat{\phi}^\pm (\xi) \in \mathbb{C}\{\xi\}. \quad (3.3)$$
Notice that $\hat{\phi}^+$ and $\hat{\phi}^-$ are meromorphic on $\mathbb{C}$ and their possible poles are of the form

$$\xi_n := \frac{i^n}{n^2} = \frac{i\pi n^2}{M} \in i\mathbb{R}_{>0}, \quad n \in \mathbb{Z}_{>0}. \quad (3.4)$$

The Laplace-like formula (1.4) thus gives rise to a sum of three terms:

$$\Theta(\tau; \nu, f, M) = \left\{ \frac{1}{2} \Gamma\left(\frac{\nu+1}{2}\right) \langle f \rangle \left(\frac{\pi}{M} \xi \right)^{-\frac{\nu+1}{2}} + \Theta^+(\tau; \nu, f, M) + \Theta^-(\tau; \nu, f, M) \right\} \quad (3.5)$$

with

$$\Theta^+ := \frac{\tau^{-1/2}}{2} \int_{\Gamma_{\varepsilon}} e^{-\varepsilon/\tau} \xi^{-1/2} \hat{\phi}^+(\xi) d\xi, \quad \Theta^- := \frac{\tau^{-1/2}}{2} \int_{\Gamma_{\varepsilon}} e^{-\varepsilon/\tau} \hat{\phi}^-(\xi) d\xi,$$

where the first term in (3.5) has been obtained from the known value of the Hankel-type Laplace integral of $\xi^{-\frac{\nu}{2} - 1}$, which is $2\pi i^{\nu+1} \tau^{-\nu/2} / \Gamma(1 + \nu/2)$, and the Legendre duplication formula.

Now, the key point is that the Cauchy theorem allows us to raise the integration contour $\Gamma_{\varepsilon}$ until it touches 0 and to replace it with the two half-lines $e^{i(\frac{\pi}{2} + \varepsilon)} \mathbb{R}_{>0}$ properly oriented, taking into account the change of branch for $\xi^{-1/2}$ from $e^{i(-\frac{\pi}{2} + \varepsilon)} \mathbb{R}_{>0}$ to $e^{i(\frac{\pi}{2} + \varepsilon)} \mathbb{R}_{>0}$ in the case of $\Theta^+$, whence

• $\Theta^+ = \tau^{-1/2} \int_{\frac{\pi}{2} \pm \varepsilon} e^{-\varepsilon/\tau} \xi^{-1/2} \hat{\phi}^+(\xi) d\xi,$

$$\Theta^+ = \tau^{-1/2} \times \frac{1}{2} \left( \mathcal{L}^{\frac{\pi}{2} - \varepsilon} + \mathcal{L}^{\frac{\pi}{2} + \varepsilon} \right) \left[ \xi^{-1/2} \hat{\phi}^+(\xi) \right], \quad (3.6)$$

• $\Theta^- = \tau^{-1/2} \times \left( \mathcal{L}^{\frac{\pi}{2} - \varepsilon} - \mathcal{L}^{\frac{\pi}{2} + \varepsilon} \right) \left[ \frac{1}{2} \hat{\phi}^- \right], \quad (3.7)$

with the notation

$$\mathcal{L}^\theta \hat{\varphi}(\tau) = \int e^{i\theta \tau} e^{-\varepsilon/\tau} \hat{\varphi}(\xi) d\xi \quad (3.8)$$

in the half-plane $\{ \arg \tau \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}) \}$.

In the case of $\Theta^-(\tau)$, since $\hat{\phi}^-(\xi)$ is regular at 0 and meromorphic, we can go on raising the integration contour as long as we do not hit the first pole on $i\mathbb{R}_{>0}$, which results in an exponentially small contribution as $\tau \to 0$.

In the case of $\Theta^+(\tau)$, it is the Puiseux expansion at 0 of

$$\hat{\varphi}^+(\xi) := \xi^{-1/2} \hat{\phi}^+(\xi) \in \xi^{-1/2} \mathbb{C}\{\xi\} \quad (3.9)$$

that governs the asymptotics: the standard properties of the Laplace transform $\mathcal{L}^{\frac{\pi}{2} \pm \varepsilon}$ give the asymptotic behaviour as $\tau \to 0$ non-tangentially, i.e. uniformly in any sector $\varepsilon < \arg \tau < \pi - \varepsilon$, which we denote by $\tau \xrightarrow{\mathbb{N}} 0$. A few computations lead to formulas involving the analytic continuation in $s$ of the $L$-function $L(s, f) := \sum_{n \geq 1} f(n)n^{-s}$ for the Taylor coefficients of $\hat{\phi}^+(\xi)$ (see [8] for the details). The upshot is

Theorem 2. • If the mean value $\langle f \rangle$ is non-zero, then the dominant term in (3.5) is the first one and $|\Theta(\tau; \nu, f, M)| \xrightarrow{\tau \xrightarrow{\mathbb{N}} 0} \infty$. 
• The term $\Theta^+(\tau; \nu, f, M)$ is the **median sum** in the direction $\frac{\pi}{2}$ of the formal series

$$\Theta(\tau; \nu, f, M) := \sum_{p \geq 0} \frac{1}{p!} L(-2p - \nu, f) \left(\frac{\pi 1}{M}\right)^p \tau^p. \quad (3.10)$$

The formal series $\Theta$ is **resurgent** in $1/\tau$ and its Borel transform has multivalued analytic continuation in $C - \{\xi_n\}_{n \geq 1}$. In particular $\Theta^+ \sim_1 \Theta$ as $\tau \downarrow 0$.

• The term $\Theta^-(\tau; \nu, f, M)$ is exponentially small as $\tau \downarrow 0$: it is $O(e^{-cM(-1/\tau)})$ with any $0 < c < \frac{\pi}{M}$ (much more will be said on it in Section 5).

A few words on the terminology. The concept of resurgence [6] is based on the Borel transform

$$\mathcal{B}: \hat{\psi}(\tau) = \sum_{p \in \mathcal{N}} c_p \tau^p \mapsto \check{\psi}(\xi) = \sum_{p \in \mathcal{N}} \frac{\xi^{p-1}}{\Gamma(p)} \quad \text{for } \mathcal{N} = a + b\mathbb{Z}_{\geq 0} \text{ with } a, b \in \mathbb{R}_{>0}. \quad (3.11)$$

The series $\hat{\psi}(\tau)$ is said to be **resurgent** in $1/\tau$ if its Borel transform $\check{\psi}(\xi)$ has positive radius of convergence and has “endless analytic continuation”—see [6], or Chapters 5 and 6 of [18]. If $|\hat{\psi}(\xi)|$ is bounded by an exponential along a singularity-free sector arg $\xi \in [\theta_1, \theta_2]$, then its Laplace transforms $\mathcal{L}^\theta \hat{\psi}$ in the directions $\theta \in [\theta_1, \theta_2]$ are the analytic continuation of one another and can be glued into one function:

$$\mathcal{L}^{[\theta_1, \theta_2]} \hat{\psi}(\tau) = \int_0^{e^{\theta_2}} e^{-\xi/\tau} \hat{\psi}(\xi) \, d\xi, \quad \theta \in [\theta_1, \theta_2] \text{ such that } \arg \tau \in \left(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right), \quad (3.12)$$

thus yielding the Borel sum $\mathcal{J}^{[\theta_1, \theta_2]} \hat{\psi} := \mathcal{L}^{[\theta_1, \theta_2]} \mathcal{B} \hat{\psi}$, that is analytic in a sectorial neighbouhood of $\tau = 0$ with arguments ranging from $\theta_1 - \frac{\pi}{2}$ to $\theta_2 + \frac{\pi}{2}$, and has 1-Gevrey asymptotic expansion $\check{\psi}(\tau)$ in that domain—this is the meaning of the symbol “$\sim_1$” in Theorem 2.

If there are singularities along $e^{\theta} \mathbb{R}_{>0}$ but not in the sectors arg $\xi \in \left[\theta - \varepsilon, \theta\right)$ and $(\theta, \theta + \varepsilon]$, then one may resort to median summation $\mathcal{J}^\theta \hat{\psi} = \mathcal{L}^\theta \mathcal{B} \hat{\psi}$, where $\mathcal{L}^\theta$ is a variant of the Laplace operator introduced by J. Écalle; in general, $\mathcal{L}_\text{med} \hat{\psi}$ is not the mere arithmetic average $\frac{1}{2}(\mathcal{L}^{\theta - \varepsilon} + \mathcal{L}^{\theta + \varepsilon}) \hat{\psi}$ but something more elaborate that involves, in a symmetric way, with well-chosen weights, the various branches of the analytic continuation of $\hat{\psi}$ along $e^{\theta} \mathbb{R}_{>0} - \{\text{singular points}\}$; see §1.4 of [7] or [17]. These weights are such that $\mathcal{J}^\theta_{\text{med}}(\hat{\psi}_1 \hat{\psi}_2) = (\mathcal{J}^\theta_{\text{med}} \hat{\psi}_1)(\mathcal{J}^\theta_{\text{med}} \hat{\psi}_2)$, which means that

$$\mathcal{L}^\theta_{\text{med}}(\hat{\psi}_1 \ast \hat{\psi}_2) = (\mathcal{L}^\theta_{\text{med}} \hat{\psi}_1)(\mathcal{L}^\theta_{\text{med}} \hat{\psi}_2), \quad (3.13)$$

where $\hat{\psi}_1 \ast \hat{\psi}_2(\xi) := \mathcal{B}(\hat{\psi}_1 \hat{\psi}_2)$ is the convolution.

A case in which the median sum coincides with the arithmetic average is when $\hat{\psi}(\xi)$ is single-valued for arg $\xi \in [\theta - \varepsilon, \theta + \varepsilon]$, e.g. a meromorphic function, as is $\hat{\phi}^+ = \xi^{-1/2} \hat{\phi}^+(\xi)$.

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1 The convolution product $\ast$ is given by the formula $\hat{\psi}_1 \ast \hat{\psi}_2(\xi) = \int_0^\xi \hat{\psi}_1(\xi_1) \hat{\psi}_2(\xi - \xi_1) \, d\xi_1$ for $|\xi|$ small enough and is known to have endless analytic continuation when both factors have—see [6], or Chapter 6 of [18]. Of course, $\mathcal{L}^\theta_{\text{med}}(\hat{\psi}_1 \ast \hat{\psi}_2) = (\mathcal{L}^\theta \hat{\psi}_1)(\mathcal{L}^\theta \hat{\psi}_2)$ for a singularity-free direction $\theta'$, and $\mathcal{J}^\theta_{\text{med}}(\hat{\psi}_1 \hat{\psi}_2) = (\mathcal{J}^\theta \hat{\psi}_1)(\mathcal{J}^\theta \hat{\psi}_2)$ in that case.
Another case is when $\hat{\psi} = \hat{A} \ast \hat{\phi}$ with $\hat{\phi}$ meromorphic and $\hat{A}$ holomorphic in the sector $\arg \xi \in [\theta - \varepsilon, \theta + \varepsilon]$; indeed, the identity $L^\theta_{\text{med}} \hat{\psi} = \frac{1}{2} (L^{\theta-\varepsilon} + L^{\theta+\varepsilon}) \hat{\phi}$ follows from (3.13) because $L^{\theta-\varepsilon} \hat{A} = L^{\theta+\varepsilon} \hat{A}$ and $L^\theta_{\text{med}} \hat{\phi} = \frac{1}{2} (L^{\theta-\varepsilon} + L^{\theta+\varepsilon}) \hat{\phi}$ in that case.

What happens with Theorem 2 is that (3.9) defines a function $\hat{\phi}^+(\xi) \in \xi^{-1/2} \mathbb{C}\{\xi\}$ that is meromorphic with singularities in direction $\frac{\pi}{2}$ only (in particular it is endlessly continuiable), so (3.6) can be rewritten $\Theta^+(\tau) = \tau^{-1/2} L^\tau_{\text{med}} \hat{\phi}^+$, which implies that the function $\psi^+(\xi) := \tau \Theta^+(\tau)$ can be written $\hat{\psi}^+(\tau) = \tau^{1/2} L^\tau_{\text{med}} \hat{\phi}^+$. Using $\tau^{1/2} = L^0 \hat{A}$ with $\hat{A}(\xi) := \frac{\xi^{1/2}}{\Gamma(1/2)}$ for any $\theta$ close to $\frac{\pi}{2}$, we get $\hat{\psi}^+(\tau) = L^\tau_{\text{med}} \hat{\phi}^+$ with $\hat{\phi}^+(\xi) := \hat{A} \ast \hat{\phi}^+(\xi) \in \mathbb{C}\{\xi\}$. Multiplying $L^\tau_{\text{med}} \hat{\phi}^+$ by $\tau^{-1}$, we finally get

$$\Theta^+(\tau) = \tau^{-1} \psi^+(\tau) = \hat{\psi}^+(0) + L^\tau_{\text{med}}(\frac{d \hat{\psi}^+}{d \xi})(\tau) \sim_1 \hat{\psi}^+(0) \ast \mathcal{B}_1 =: \hat{\Theta}(\tau).$$

We end up with

$$\Theta^+(\tau) = L^\tau_{\text{med}} \hat{\Theta} = \mathcal{S}^\tau_{\text{med}} \hat{\Theta}, \quad \hat{\Theta} = \hat{\psi}^+(0) \delta + \frac{d \hat{\psi}^+}{d \xi}, \quad \hat{\psi}^+(\xi) = \frac{\xi^{-1/2}}{\Gamma(1/2)} \ast \left[ \xi^{-1/2} \hat{\phi}^+(\xi) \right]$$

by extending (3.11) and setting $\mathcal{B}1 = \delta$ (“Dirac mass at 0”).

In the Borel plane, $\hat{\phi}^+(\xi)$ and $\xi^{-1/2} \hat{\phi}^+(\xi)$ are meromorphic and hence endlessly continuiable. Now, convolution with $\xi^{-1/2}$ does not preserve meromorphy, but it preserves endless continuability, and so does $\frac{d}{d \xi}$, therefore $\hat{\Theta}$ is endlessly continuable and the formal series $\hat{\Theta}$ is thus proved to be resurgent. To check that $\hat{\Theta}$ defined by (3.14) is given by (3.10), one just needs to compute the Taylor expansion at $\xi = 0$ of $\hat{\psi}^+$—we omit the details here and refer the reader to [8].

**Remark 3.1.** Notice that the resurgent formal series $\hat{\Theta}(\tau)$ has, for $\tau \in \mathbb{H}$, three “canonical” sums: $L^\tau_{\text{med}} \hat{\Theta}(\tau)$, $L^\tau_{\text{med}} \hat{\Theta}(\tau)$ and $L^\tau_{\text{med}} \hat{\Theta}(\tau)$, the third of which is the function $\Theta^+(\tau)$. Interesting phenomena are observed when one compares these three functions—see Section 5. Notice also that if $f$ is real-valued, then the coefficients of $\hat{\Theta}(\tau)$ with respect to $\tau/i$ are real; the realness of $\Theta^+(\tau) = L^\tau_{\text{med}} \hat{\Theta}(\tau)$ for $\tau \in i \mathbb{R}_{>0}$ in that case can be viewed as a consequence of a general property of Écalle’s median summation (anyway, the decomposition (3.5) respects realness).

**Remark 3.2.** Using Remark 2.2, we get from Theorem 2 the asymptotic behaviour of the function $\Theta(\tau; \nu, f, M)$ as $\tau \xrightarrow{n.t.} \alpha \in \mathbb{Q}$:

$$\Theta(\alpha + \tau; \nu, f, M) = \frac{1}{2} \Gamma \left( \frac{\nu + 1}{2} \right) \langle f_M \rangle \left( \frac{\pi i}{M} \right)^{-\nu + 1} + \mathcal{S}^\tau_{\text{med}} \hat{\Theta}_\alpha(\tau) \ast 0(e^{-c M(-1)\tau}) \quad \text{as } \tau \xrightarrow{n.t.} 0 \quad (3.16)$$

with arbitrary $0 < c < \frac{\pi M}{M^2}$, and with

$$\hat{\Theta}_\alpha(\tau) := \sum_{p \geq 0} \frac{1}{p!} L(-2p - \nu, f_M) \left( \frac{\pi i}{M} \right)^p \tau^p. \quad (3.17)$$
We are thus led to define
\[ \overline{\mathcal{D}}_{f,M} := \{ \alpha \in \mathbb{Q} \mid \langle f_{\alpha} \rangle = 0 \}, \]
so that \( \alpha \notin \overline{\mathcal{D}}_{f,M} \Rightarrow |\Theta(\tau; \nu, f, M)| \to \infty \) as \( \tau \nrightarrow \alpha \),
\[ \alpha \in \overline{\mathcal{D}}_{f,M} \Rightarrow \Theta^\text{nt}(\alpha; \nu, f, M) := \lim_{\tau \nrightarrow \alpha} \Theta(\tau; \nu, f, M) = L(-\nu, f_{\alpha}). \]

Note that \( f \) and \( f_{\alpha} \) have same parity, in particular
\[ f \text{ odd } \Rightarrow \overline{\mathcal{D}}_{f,M} = \mathbb{Q}. \]  

Note also that the full domain of definition of the boundary function \( \Theta^\text{nt} \) is \( \overline{\mathcal{D}}_{f,M} \cup \{i\infty\} \), since \( \Theta(\tau; \nu, f, M) \to 0 \) as \( \Im \tau \to \infty \). We will see in the next section that \( L(-\nu, f_{\alpha}) = 0 \) if \( \nu \) and \( f \) are odd, and also if \( \nu \geq 2 \) and \( f \) are even, while \( L(0, f_{\alpha}) = -f(0)/2 \) if \( f \) is even.

4. THE ROLE OF PARITY

It so happens that the decomposition of \( f \) into even and odd parts,
\[ f = f^\text{ev} + f^\text{od}, \quad f^\text{ev}(n) := \frac{f(n) + f(-n)}{2}, \quad f^\text{od}(n) := \frac{f(n) - f(-n)}{2}, \]
relates to the decomposition (3.1): for \( \nu = 0 \) one finds \( F_0 = F_0^\text{ev} + F_0^\text{od} \) with
\[ F_0^\text{ev}(t) = -\frac{1}{2} f^\text{ev}(0) + \frac{1}{1 - e^{-\pi t}} \sum_{\ell=0}^{M-1} f^\text{od}(\ell) e^{-\ell t}, \quad F_0^\text{od}(t) = \frac{1}{1 - e^{-\pi t}} \sum_{\ell=0}^{M-1} f^\text{ev}(\ell) e^{-\ell t}, \]
from which one gets \( F^\pm \) by applying \( (-\frac{d}{dt})^\nu \) and removing \( \frac{\nu! (f)}{\nu! \pi \tau} \). Therefore, when \( f \) is even or odd, only one of the two functions \( \Theta^+ \) and \( \Theta^- \) can be nonzero, with the sole exception of \( \{\nu = 0, f \text{ even}\} \), in which case \( \Theta^+(\tau) = -\frac{1}{2} f(0) \). In general,

- \( \tilde{\Theta}(\tau; 0, f) = -\frac{1}{2} f^\text{ev}(0) + \sum_{p \geq 0} \frac{1}{p!} L(-2p, f^\text{od})(\frac{\pi}{M})^p \tau^p \) only depends on \( f^\text{od} \) and \( f^\text{ev}(0) \) and gives rise to \( \Theta^+ (\tau; 0, f) \) by median summation, while \( \Theta^- (\tau; 0, f) \) only depends on \( f^\text{ev} \),
- \( \tilde{\Theta}(\tau; 1, f) = \sum_{p \geq 0} \frac{1}{p!} L(-2p - 1, f^\text{ev})(\frac{\pi}{M})^p \tau^p \) only depends on \( f^\text{ev} \) and its median sum is \( \Theta^+(\tau; 1, f) \), while \( \Theta^- (\tau; 1, f) \) only depends on \( f^\text{od} \),

and so on.

**Remark 4.1.** If \( \nu \) is even (resp. odd), unless \( f \) is even (resp. odd), the Borel transform of \( \tilde{\Theta}(\tau; \nu, f) \) has singularities in the direction \( \frac{\pi}{2} \) and \( \tilde{\Theta}(\tau; \nu, f) \) is not Borel summable in that direction, median summation is needed. Moreover, the median sum coincides with \( \Theta(\tau; \nu, f) = -\frac{1}{2} \Gamma(\frac{\nu+1}{2}) \langle f \rangle (\frac{\pi}{M})^{\nu} \tau^{\nu+1} \) only if \( f \) is odd (resp. even).

**Discrete Fourier Transform.** The DFT operator \( U_M \) is defined on the space of \( M \)-periodic functions \( \mathbb{Z} \to \mathbb{C} \) by
\[ U_M : f \mapsto \hat{f}, \quad \hat{f}(n) := \frac{1}{\sqrt{M}} \sum_{\ell \mod M} f(\ell) e^{-2\pi i \ell n / M} \text{ for all } n \in \mathbb{Z}. \]
Theorem 3. In the sake of clarity: (4.4) gives rise to particularly nice formulas, which we spell out only when \( \nu \) is even.

Such a difference usually goes under the name of “Stokes phenomenon”. In the present case, (4.2) that leads us to consider the DFT, since for all \( n \in \mathbb{Z} \),

\[
\text{Res}(\hat{F}_0^{ev}(t), t = \frac{2\pi i n}{M}) = M^{-\frac{1}{2}} \hat{f}^{ev}(n), \quad \text{Res}(\hat{F}_0^{od}(t), t = \frac{2\pi i n}{M}) = M^{-\frac{1}{2}} \hat{f}^{od}(n). \tag{4.4}
\]

5. Number-theoretic Stokes phenomena

According to Section 3, the third term of the decomposition (3.5) is the difference of two Laplace transforms:

\[
\Theta^{-}(\tau, \nu, f, M) = \tau^{-\frac{1}{2}} \left( \mathcal{L}^{\frac{\pi}{2} - \epsilon} - \mathcal{L}^{\frac{\pi}{2} + \epsilon} \right) \left[ \frac{\hat{\phi}^{-}(\xi)}{2} \right] = \left( \mathcal{L}^{\frac{\pi}{2} - \epsilon} - \mathcal{L}^{\frac{\pi}{2} + \epsilon} \right) \left[ \frac{d\hat{\psi}^{-}}{d\xi} \right],
\]

with \( \hat{\psi}^{-}(\xi) := \frac{\xi - 1}{\Gamma(1/2)} \ast \frac{\hat{\phi}^{-}(\xi)}{2} \in \mathbb{C} \{\xi\} \). \tag{5.1}

Such a difference usually goes under the name of “Stokes phenomenon”. In the present case, (4.4) gives rise to particularly nice formulas, which we spell out only when \( \nu = 0 \) or \( 1 \) for the sake of clarity:

**Theorem 3.** We have

\[
\Theta(\tau; 0, f) = \frac{1}{2} \hat{f}^{ev}(0) \left( \tau^{-1/2} - \frac{1}{2} \hat{f}^{ev}(0) + \mathcal{S}_{\text{med}} \Theta(\tau; 0, f^{od}) + (\tau^{-1}) \hat{\Theta}(-\tau^{-1}; 0, \hat{f}^{ev}) \right) \tag{5.2}
\]

\[
\Theta(\tau; 1, f) = -\frac{M^{1/2}}{2\pi i} \hat{f}^{ev}(0) \tau^{-1} + \mathcal{S}_{\text{med}} \hat{\Theta}(\tau; 1, f^{ev}) + i(\tau^{-1})^{3/2} \hat{\Theta}(-\tau^{-1}; 1, \hat{f}^{od}). \tag{5.3}
\]

The proof consists in analysing the singularity at \( \xi_n = \frac{i\pi n^2}{M} \) of the function \( \hat{\phi}^{-} \) of (3.2) so as to write \( \Theta^{-} \) as the sum over \( n \geq 1 \) of one exponentially small contribution for each singularity. From (4.4) we get

\[
\frac{1}{2} \hat{\phi}^{-}(\xi) = \frac{\epsilon_{\nu} f^{ev}(n)}{2\pi i(\xi - \xi_n)} + \text{reg} \quad \text{if } \nu = 0, \quad \frac{1}{2} \hat{\phi}^{-}(\xi) = \frac{\epsilon_{\nu} f^{od}(n)}{2\pi i(\xi - \xi_n)} + \text{reg} \quad \text{if } \nu = 1, \tag{5.4}
\]

where “reg” denotes a function that is regular at \( \xi_n \). The last term in (5.2)/(5.3) thus results from

\[
(\mathcal{L}^{\frac{\pi}{2} - \epsilon} - \mathcal{L}^{\frac{\pi}{2} + \epsilon}) \left[ \frac{e^{-\xi_n/\tau}}{2\pi i(\xi - \xi_n)^2} \right] = e^{-\xi_n/\tau}, \quad \left( \mathcal{L}^{\frac{\pi}{2} - \epsilon} - \mathcal{L}^{\frac{\pi}{2} + \epsilon} \right) \left[ \frac{-1}{2\pi i(\xi - \xi_n)^2} \right] = \tau^{-1} e^{-\xi_n/\tau}. \tag{5.5}
\]

The other terms in (5.2)/(5.3) are just a rewriting of the first two terms of (3.5), using \( \langle f \rangle = \langle f^{ev} \rangle = \frac{1}{M^{1/2}} \hat{f}^{ev}(0) \) and incorporating the parity information on \( \Theta^{+} \). By (2.4), differentiation with respect to \( \tau \) yields a statement for \( \Theta(\tau; \nu, f) \) with \( \nu \geq 2 \) as well.

We now arrive at the first instance of our motto “(quantum) modularity = Stokes phenomenon + DFT” (according to parity):
Toward Modularity. The median sum term of (5.2), resp. (5.3), is absent if \( f \) is even, resp. odd. If \( f \) is even, we can rewrite (5.2) as a weight \( \frac{1}{2} \) modularity relation for a full theta series as far as the modular transformation under consideration is the negative inversion \( S: \tau \mapsto -\tau^{-1} \):

\[
f \text{ even, } \quad \theta(\tau; f) := f(0) + 2\Theta(\tau; 0, f) \Rightarrow \theta(\tau; f) = (\frac{\tau}{1})^{-\frac{1}{2}} \theta(-\tau^{-1}; f),
\]

while if \( f \) is odd (whence \( \hat{f}^{\text{ev}}(0) = 0 \)), we can rewrite (5.3) as a weight \( \frac{3}{2} \) modularity relation:

\[
f \text{ odd } \Rightarrow \quad \Theta(\tau; 1, f) = i(\frac{\tau}{1})^{-\frac{3}{2}} \Theta(-\tau^{-1}; 1, f) \quad (\exp. \text{ small as } \tau \xrightarrow{n.t.} 0),
\]

with a caveat: a priori we have different partial theta series on the two sides of the relation, one associated with \( f \) and the other associated with \( \hat{f} \). However, if \( f \) is an eigenvector of \( U_M \), we do end up with modularity relations with half-integral weight. The classical Jacobi function \( \theta_3 \) and the Dedekind \( \eta \) function fall in that category.

If \( f \) is a primitive Dirichlet character modulo \( M \), then \( f(-1) = (-1)^\nu \) with \( \nu = 0 \) or 1, and \( f \) is even (\( \nu = 0 \)) or odd (\( \nu = 1 \)). It is then known that \( \hat{f} = \hat{f}(1)\hat{f} \), and (5.6)–(5.7) are related to the famous results by Shimura on modular forms of half-integral weight [21].

Notice that if \( f \) is even or odd but not an eigenvector of \( U_M \), we still can write

\[
F := \left( \frac{f}{f} \right), \quad U_M F := \left( \frac{U_M f}{U_M f} \right) = JF, \quad J := \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad \nu := \text{parity of } f,
\]

which allows one to construct 2-dimensional vector-valued modular forms of half-integral weight:

\[
\theta(\tau; F) = \left( \begin{array}{c} \theta(\tau; f) \\ \theta(\tau; j) \end{array} \right), \quad \Theta(\tau; 1, F) = \left( \begin{array}{c} \Theta(\tau; 1, f) \\ \Theta(\tau; 1, j) \end{array} \right),
\]

\[
f \text{ even } \Rightarrow \quad \theta(\tau; F) = (\frac{\tau}{1})^{-\frac{1}{2}} J \theta(-\tau^{-1}; F), \quad f \text{ odd } \Rightarrow \quad \Theta(\tau; 1, F) = i(\frac{\tau}{1})^{-\frac{3}{2}} J \Theta(-\tau^{-1}; 1, F).
\]

With \( \theta_{5,1}(\tau) \) or \( \theta_{5,2}(\tau) \), since the real 20-periodic functions \( f_{5,j} \) satisfy \( f_{5,1} + if_{5,2} = \chi \) with \( \chi = \chi_{20}(7, \cdot) \) even primitive Dirichlet character, it is better to use \( F^* := \left( \frac{f_{5,1}}{f_{5,2}} \right) \)

instead of the previous \( F \): one finds \( U_{20} \chi = \lambda \chi, \lambda = (-\frac{3+4i}{5})^{1/4} = s_2 + is_1 \) with \( s_k := \sqrt{\frac{5+(-1)^k\sqrt{5}}{10}}, \sin \frac{k\pi}{5} \), hence \( U_{20} F^* = SF^* \) with reflection matrix \( S := (\begin{array}{cc} s_2 & s_1 \\ s_1 & -s_2 \end{array}) \).

Since \( f_{5,1}(0) = f_{5,2}(0) = 0, \) (5.6) yields

\[
\left( \begin{array}{c} \theta_{5,1}(\tau) \\ \theta_{5,2}(\tau) \end{array} \right) = (\frac{\tau}{1})^{-\frac{1}{2}} S \left( \begin{array}{c} \theta_{5,1}(\tau^{-1}) \\ \theta_{5,2}(\tau^{-1}) \end{array} \right).
\]

Remark 5.1. Let \( \alpha \in \mathbb{Q}^* \). For \( f \) even, replacing \( \tau \) by \( \alpha + \tau \) in (5.6), we can use (3.16) and equate the leading order terms in the asymptotic behaviour of both sides. We get

\[
\langle f_{20} \rangle = (i\alpha)^{1/2} \langle (U_M f)_{\frac{1}{\alpha M}} \rangle,
\]

which stays true even if \( f \) is not even. This is an identity on quadratic Gauss sums, since

\[
\langle f_{20} \rangle = \frac{1}{M_\alpha} \sum_{n \mod M_\alpha} f(n)e^{i\pi n^2\alpha/M}, \quad \langle (U_M f)_{\frac{1}{\alpha M}} \rangle = \frac{1}{M_{-1}/\alpha} \sum_{n \mod M_{-1}} (U_M f)(n)e^{-i\pi n^2/\alpha M},
\]

which is related to the quadratic reciprocity law. In particular, we see that the negative inversion \( \alpha \mapsto -1/\alpha \) maps \( \mathcal{D}_{f,M} \cap \mathbb{Q}^* \) to \( \mathcal{D}_{U_M f,M} \cap \mathbb{Q}^* \).
Toward Quantum Modularity. In (3.5), the median sum term $\Theta^+(\tau; \nu, f, M)$ itself has an interesting Stokes phenomenon. In view of (5.2)–(5.3), we restrict our attention to $\{\nu = 0$ and $f$ odd$\}$ or $\{\nu = 1$ and $f$ even$\}$, so $\Theta^{-} \equiv 0$, and we consider the exponentially small difference

$$D(\tau) := \left(\mathcal{S}^{\frac{\pi}{2}} - \mathcal{S}^{\frac{\pi}{2} + \varepsilon}\right)\tilde{\Theta}(\tau; \nu, f) = \tau^{-\frac{1}{2}}(\mathcal{L}^{\frac{\pi}{2} - \varepsilon} - \mathcal{L}^{\frac{\pi}{2} + \varepsilon})\tilde{\phi}^+$$

with $\tilde{\phi}^+ = \xi^{-\frac{1}{2}}\tilde{\phi}^+(\xi)$ (cf. Section 3), so that

$$\Theta^+ = \mathcal{L}^{\frac{\pi}{2}}B\tilde{\Theta} = \mathcal{L}^{\frac{\pi}{2} - \varepsilon}B\tilde{\Theta}(\tau; \nu, f) - \frac{D(\tau)}{2} = \mathcal{L}^{\frac{\pi}{2} + \varepsilon}B\tilde{\Theta}(\tau; \nu, f) + \frac{D(\tau)}{2}. \quad (5.12)$$

A residue computation similar to the one that yielded Theorem 3 gives

$$\tilde{\phi}^+(\xi) = \frac{2\pi i \hat{\theta}(\pi, n)}{2\pi i (\xi - \xi_n)^2} + \text{reg} \quad \text{if} \quad \nu = 0, \quad \tilde{\phi}^+(\xi) = \frac{2\pi i \hat{\theta}(\pi, n)}{2\pi i (\xi - \xi_n)^2} + \text{reg} \quad \text{if} \quad \nu = 1, \quad (5.13)$$

whence, again by (5.5),

$$\{\nu = 0 \text{ and } f \text{ odd}\} \Rightarrow D(\tau) = 2(\frac{\pi}{1})^{-\frac{3}{2}}\Theta(-\tau^{-1}; 0, \hat{f}) \quad (5.14)$$

$$\{\nu = 1 \text{ and } f \text{ even}\} \Rightarrow D(\tau) = 2i(\frac{\pi}{1})^{-\frac{3}{2}}\Theta(-\tau^{-1}; 1, \hat{f}).$$

Therefore, in view of (5.12), we can rephrase (5.2)–(5.3) as

$$f \text{ odd } \Rightarrow \Theta(\tau; 0, f) \pm (\frac{\pi}{1})^{-\frac{3}{2}}\Theta(-\tau^{-1}; 0, \hat{f}) = \mathcal{S}^{\frac{\pi}{2} + \varepsilon}\tilde{\Theta}(\tau; 0, f) \quad (5.15)$$

$$f \text{ even } \Rightarrow \Theta(\tau; 1, f) \pm i(\frac{\pi}{1})^{-\frac{3}{2}}\Theta(-\tau^{-1}; 1, \hat{f}) = -\frac{M(f)}{2\pi i}\tau^{-1} + \mathcal{S}^{\frac{\pi}{2} + \varepsilon}\tilde{\Theta}(\tau; 1, f), \quad (5.16)$$

where the right-hand sides involve the Borel-Laplace sums of $\tilde{\Theta}(\tau; \nu, f)$ in directions close to $\frac{\pi}{2}$ instead of the median sum in the direction $\frac{\pi}{2}$. We thus end up with

**Theorem 4.** Suppose $\{\nu = 0, f \text{ odd}\}$ or $\{\nu = 1, f \text{ even}\}$. If $f$ is an eigenvector of $U_M$, put $F := f$ and $J := \frac{f}{\tau} = \pm \nu^{1+1}$, else $F := \left(\frac{f}{1}\right)$ and $J := \left(\frac{0}{(-1)^{\nu+1}}\right)$. Then the “modular obstruction”

$$G_{\pm}(\tau) := \Theta(\tau; \nu, F) \pm i^{\nu}\left(\frac{\pi}{1}\right)^{-\frac{3}{2}} J \Theta(-\tau^{-1}; \nu, F) + \frac{M(F)}{2\pi i}\tau^{-1} \quad (5.17)$$

can be written $\mathcal{S}^\theta \tilde{\Theta}(\tau; \nu, F)$ with $\theta = \frac{\pi}{2} - \varepsilon$ for $G_+$ and $\theta = \frac{\pi}{2} + \varepsilon$ for $G_-$, and thus extends analytically (by varying $\theta$ as in (3.12)) from $\mathbb{H}$ through the real semi-axis $\mathbb{R}_{>0}$ to the sector $-2\pi < \arg \tau < \pi$ for $G_+$, and through $\varepsilon i\mathbb{R}_{>0}$ to the sector $0 < \arg \tau < 3\pi$ for $G_-$, with 1-Gevrey asymptotic $\tilde{\Theta}(\tau; \nu, F)$.

This unexpected regularity property of the modular obstruction $G_{\pm}$ was first observed in the case of the Eichler integral $\tilde{\eta}$ by D. Zagier and in the aforementioned case of $\Theta(\tau; 0, f_+, 60)$ in [15], and gave rise to his theory of quantum modular forms [22]. Theorem 4 is related to the results of [10]—see Section 7.

Note the “built-in modularity”: $G_{\pm}(\tau) = \pm\left(\frac{\pi}{1}\right)^{-\frac{3}{2}} J G_{\mp}(\tau^{-1})$ if $\nu = 0$, and $G_{\pm}(\tau) = \pm i^{\nu}\left(\frac{\pi}{1}\right)^{-\frac{3}{2}} J G_{\mp}(\tau)$ if $\nu = 1$ and $\{
u, f\} = 0$. In particular, out of the asymptotics of $G_{\pm}(\tau)$ as $\tau \to 0$ we get asymptotic expansions for $G_{\pm}(\tau)$ as $\tau \to \infty$ in terms of $\tilde{\Theta}(-\tau^{-1}; \nu, F)$.

A function of interest is the non-tangential limit $\alpha \in \mathcal{D}_{f,M} \mapsto \Theta^{at}(\alpha; \nu, F, M)$, well-defined (according to Remark 3.2) on $\mathcal{D}_{f,M} := \mathcal{D}_{f,M} \cap \mathcal{D}_{U_M f, M}$, which is $2M$-periodic. In particular,
for $k$ integer, $\Theta^\dagger(k; \nu, F, M)$ depends on $k \mod 2M$ only and, by taking non-tangential limits in (5.17), we get

**Corollary 5.** If $\alpha \in \mathbb{Q}^*$, then $\alpha \in \mathcal{D}_{f,M} \Leftrightarrow -\alpha^{-1} \in \mathcal{D}_{f,M}$. For $\alpha = \pm \frac{1}{k}$ with $k \in \mathcal{D}_{f,M} \cap \mathbb{Z}_{>0}$,

\begin{align*}
\text{if odd} & \quad \Rightarrow \quad \Theta^\dagger\left(\pm \frac{1}{k}; 0, F\right) = -i e^{\frac{-\pi i}{4}} k^{1/2} J \Theta^\dagger\left(\mp k; 0, F\right) + G^\pm\left(\pm \frac{1}{k}\right), \\
\text{if even} & \quad \Rightarrow \quad \Theta^\dagger\left(\pm \frac{1}{k}; 1, F\right) = e^{\frac{-\pi i}{4}} k^{3/2} J \Theta^\dagger\left(\mp k; 1, F\right) \mp \frac{M(F)}{2k^4} k + G^\pm\left(\pm \frac{1}{k}\right),
\end{align*}

with $G^\pm\left(\pm \frac{1}{k}\right) \sim_1 \tilde{\Theta}\left(\pm \frac{1}{k}; \nu, F\right) = \sum_{p \geq 0} (\pm 1)^p \frac{L(-2p - \nu, F) (\pi i)^p}{p!} \frac{1}{k^p}$ as $k \to +\infty$.

So, the divergent resurgent series $\tilde{\Theta}$ reappears in the asymptotics of $\Theta^\dagger\left(\pm \frac{1}{k}; \nu, F\right)$. Note that in general the dominant terms in (5.18)–(5.19) are the first terms of the right-hand sides, which are of the form $k^{\frac{1}{2}+\nu} \times \{2M\text{-periodic function of } k\}$. There are similar formulas for $\Theta^\dagger(\alpha \pm \frac{1}{k}; \nu, F)$. See Section 7 for more on the sets $\mathcal{D}_{f,M}$.

Being primitive and real, the Dirichlet character $f(n) = \left(\frac{12}{n}\right) = \chi_{12}(11, n)$ is an eigenvector of $U_{12}$; the eigenvalue is 1, we thus recover weight $\frac{1}{2}$ modularity for the Dedekind $\eta$-function ($\nu = 0$) and quantum modularity for its Eichler integral $\tilde{\eta} (\nu = 1)$; in the latter case, our results shed a new light on the resurgence properties already investigated in [3]. For the Chern-Simons partition function of the Poincaré homology sphere $\Theta(\tau; 0, f_+, 60)$, recalling that $f_+ = \operatorname{Re} \left(\chi_{60}(23, \cdot)\right)$ and introducing $f_- = 3m \left(\chi_{60}(23, \cdot)\right)$, we get

\[ F^* := \left( f_+ \right) \Rightarrow \quad U_{60} F^* = SF^*, \quad S := -i \left( s_1 \frac{s_2}{s_2 - s_1} \right) \]

(5.20)

with $s_1, s_2$ as before (because $\Phi := f_+ + i f_-$ is an odd primitive Dirichlet character and one finds $\Phi_6 \Phi = (s_2 - is_1) \Phi$, whence a 2D quantum modular form $\Theta(\tau; 0, F^*, 60)$.

### 6. Alien Derivatives and “Bridge Equation”

In this work, the two resurgent series $\tilde{\Theta}(\tau) = \psi^+ (0) + \mathcal{B}^{-1} \left[ \frac{d \psi^+}{d \xi} \right] \in \mathbb{C}[[\tau]]$ and $\tau^{-1} \psi^- = \mathcal{B}^{-1} \left[ \frac{d \psi^-}{d \xi} \right] \in \tau^{\frac{1}{2}} \mathbb{C}[[\tau]]$ have appeared in the decomposition (3.5) of $\Theta(\tau; \nu, f, M)$:

\[ \Theta^+ = \mathcal{L}_{\text{med}}^\frac{\tau}{2} \tilde{\Theta} = \psi^+ (0) + \mathcal{L}_{\text{med}}^\frac{\tau}{2} \left[ \frac{d \psi^+}{d \xi} \right], \quad \Theta^- = \left( \mathcal{L}^\frac{\tau}{2} + \epsilon - \mathcal{L}^\frac{\tau}{2} - \epsilon \right) \left[ \frac{d \psi^-}{d \xi} \right] \]

(cf. Theorem 2, (3.14) and (5.1)). This has led to two interesting Stokes phenomena, via singularity computations in the Borel plane.

Écalle’s Resurgence Theory provides us with an efficient tool to handle such singularity computations: with each point $\omega$ of the Riemann surface of the logarithm is associated the so-called “alien derivation” $\Delta_\omega$. This is an operator which maps a resurgent series $\tilde{\varphi}$ to a resurgent series whose Borel transform somehow measures the singularities of certain branches of the analytic continuation of $\mathcal{B} \tilde{\varphi}$ at $\omega$; in particular, $\Delta_\omega \tilde{\varphi} = 0$ if all branches of $\mathcal{B} \tilde{\varphi}$ are regular at $\omega$, which is the case if $\tilde{\varphi}$ is a convergent power series. The reader is
referred to [6, 7, 18] for more details. In the case of meromorphic singularities, the recipe is motivated by (5.5) and can be illustrated with $\xi_n = \frac{\pi n^2}{M} e^{i\pi}$. For $n \geq 1$: the formulas

$$
\nu = 0 \quad \Rightarrow \quad \Delta_{\xi_n} \tilde{\varphi}^+ = 2 e^{i\pi} \hat{f}^{\text{od}}(n) \quad \Delta_{\xi_n} \tilde{\varphi}^- = 2 e^{i\pi} \hat{f}^{\text{ev}}(n) 
$$

$$
\nu = 1 \quad \Rightarrow \quad \Delta_{\xi_n} \tilde{\varphi}^+ = 2 i e^{\frac{3i\pi}{4}} n \hat{f}^{\text{ev}}(n) \tau^{-1} \quad \Delta_{\xi_n} \tilde{\varphi}^- = 2 i e^{\frac{3i\pi}{4}} n \hat{f}^{\text{od}}(n) \tau^{-1}
$$

must be understood as a mere rephrasing of (5.13) and (5.4). For the two resurgent series of our problem, since $\tilde{\Theta} = \tau^{-1} \tilde{\psi}^+ = \tau^{-\frac{3}{2}} \tilde{\varphi}^+$ and $\tau^{-1} \tilde{\psi}^- = \tau^{-\frac{3}{2}} \tilde{\phi}^-/2$, the operator $\Delta_{\xi_n}$ being a derivation annihilating $\tau^{-\frac{3}{2}}$, we thus get

$$
\nu = 0 \quad \Rightarrow \quad \Delta_{\xi_n} \tilde{\Theta} = 2 e^{i\pi} \hat{f}^{\text{od}}(n) \tau^{-\frac{1}{2}} \quad \Delta_{\xi_n} (\tau^{-1} \tilde{\psi}^-) = e^{i\pi} \hat{f}^{\text{ev}}(n) \tau^{-\frac{1}{2}} 
$$

$$
\nu = 1 \quad \Rightarrow \quad \Delta_{\xi_n} \tilde{\Theta} = 2 i e^{\frac{3i\pi}{4}} n \hat{f}^{\text{ev}}(n) \tau^{-\frac{3}{2}} \quad \Delta_{\xi_n} (\tau^{-1} \tilde{\psi}^-) = i e^{\frac{3i\pi}{4}} n \hat{f}^{\text{od}}(n) \tau^{-\frac{3}{2}}
$$

which means that their Borel transforms have singularities proportional to $(\xi - \xi_n)^{-\frac{3}{2}}$ if $\nu = 0$ and $(\xi - \xi_n)^{-\frac{5}{2}}$ if $\nu = 1$.

The Stokes phenomena that we have observed just reflect the action of the directional alien derivation

$$
\Delta_{\tilde{\xi}} = \sum_{\arg \omega = \frac{\pi}{2}} e^{-\omega/\tau} \Delta_{\omega}. 
$$

Indeed, for $\nu = 0$:

$$
\Delta_{\tilde{\xi}} \tilde{\Theta}(\tau; 0, f) = 2 \left( \frac{\tau}{1} \right)^{-\frac{3}{2}} \Theta(-\tau^{-1}; 0, U_M f^{\text{od}}), \quad \Delta_{\tilde{\xi}} (\tau^{-1} \tilde{\psi}^-) = \left( \frac{\tau}{1} \right)^{-\frac{1}{2}} \Theta(-\tau^{-1}; 0, U_M f^{\text{ev}}),
$$

and for $\nu = 1$:

$$
\Delta_{\tilde{\xi}} \tilde{\Theta}(\tau; 1, f) = 2 i \left( \frac{\tau}{1} \right)^{-\frac{3}{2}} \Theta(-\tau^{-1}; 1, U_M f^{\text{ev}}), \quad \Delta_{\tilde{\xi}} (\tau^{-1} \tilde{\psi}^-) = i \left( \frac{\tau}{1} \right)^{-\frac{1}{2}} \Theta(-\tau^{-1}; 1, U_M f^{\text{od}}).
$$

In the context of dynamical systems, J. Écalle has observed that the action of the alien derivations on divergent series of natural origin often coincide with the action of a classical operator, typically a differential operator, giving rise to a “bridge” between alien calculus and classical differential calculus. Here, the equations just obtained can be called Bridge Equations for the problem at hand inasmuch as they show that the action of the directional alien derivation on our two resurgent series amount to the action of the DFT operator $U_M$ on the even or odd parts of $f$, together with the modular transformation $S$: $\tau \mapsto -\tau^{-1}$.

**Remark 6.1.** Écalle’s Alien Calculus gives a formula for the commutator of an alien derivation $\Delta_{\omega}$ and the natural derivation $\frac{d}{d\tau}$, which amounts to the fact that each homogeneous component $e^{-\omega/\tau} \Delta_{\omega}$ in (6.5) commutes with the natural derivation $\frac{d}{d\tau}$, namely

$$
\Delta_{\omega} \frac{d}{d\tau} = \left( \frac{d}{d\tau} + \omega \tau^{-2} \right) \Delta_{\omega}. 
$$

(6.6)
We can thus differentiate (6.3)–(6.4) with respect to \( \tau \) and, together with (2.4), this implies
\[
\nu = 2 \Rightarrow \Delta_{\xi_n} \tilde{\Theta}(\tau; 2, f, M) = -2 n^2 \hat{f}^{\text{od}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{7}{2}} + \frac{M}{\pi} \hat{f}^{\text{od}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{3}{2}},
\]
\[
\nu = 3 \Rightarrow \Delta_{\xi_n} \tilde{\Theta}(\tau; 3, f, M) = -2i n^3 \hat{f}^{\text{ev}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{7}{2}} + 3i \frac{M}{\pi} n \hat{f}^{\text{ev}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{5}{2}},
\]
\[
\nu = 4 \Rightarrow \Delta_{\xi_n} \tilde{\Theta}(\tau; 4, f, M) = 2 n^4 \hat{f}^{\text{od}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{9}{2}} - 6 \frac{M}{\pi} n^2 \hat{f}^{\text{od}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{7}{2}} + 3 \left( \frac{M}{\pi} \right)^2 \hat{f}^{\text{od}}(n) \left( \frac{\tau}{\pi} \right)^{-\frac{5}{2}},
\]
etc.

**Remark 6.2.** The growth as \( p \to \infty \) of \( \left[ \tau^p \right] \tilde{\Theta} = \frac{1}{p!} L(-2p - \nu, f)(\frac{\pi i}{M})^p \frac{d^p \hat{\psi}^+}{dp}(0) \) is governed by the nearest-to-origin singularity of \( \hat{\psi}^+ = \mathcal{B}(\tau \tilde{\Theta}) \in \mathbb{C}\{\xi\} \); taking \( n^* \geq 1 \) minimum such that \( \hat{f}^{\text{od}}(n^*) \) or \( f^{\text{ev}}(n^*) \neq 0 \), we thus have, by virtue of (6.3)–(6.4), \( \Delta_{\xi_n^*}(\tau \tilde{\Theta}) = S_{\nu} \tau^{\frac{3}{2} - \nu} \) with \( S_0 = 2 e^{\frac{i\pi}{4}} \hat{f}^{\text{od}}(n^*), S_1 = 2i e^{\frac{3i\pi}{4}} n^* \hat{f}^{\text{ev}}(n^*) \), which means
\[
\hat{\psi}^+(\xi) = \frac{S_0}{2i(\frac{1}{2})}(\xi - \xi_n^*)^{-\frac{1}{2}} + \hat{R}_0(\xi) \text{ or } -\frac{S_1}{4i(\frac{1}{2})}(\xi - \xi_n^*)^{-\frac{3}{2}} + \hat{R}_1(\xi)
\]
with \( \hat{R}_\nu(\xi) \in \mathbb{C}\{\xi\} \) convergent in a disc of radius larger than \( |\xi_n^*| \). The coefficients of the Taylor expansion of \( (\xi - \xi_n^*)^{-\frac{1}{2} - \nu} \) thus give an asymptotic equivalent of \( \left[ \tau^p \right] \tilde{\Theta} \) up to exponential precision.

Another commonly used variable is \( g = e^{2i\pi \tau} \), and the resulting asymptotic expansions in powers of \( q - 1 \) as \( q \to 1 \) (non-tangentially from within the unit disc) can be handled as follows: put \( Q = \frac{q - 1}{2\pi i} = g(\tau) \), so \( \tau = g^{-1}(Q) \in Q + Q^2 \mathbb{C}\{Q\} \), then the new asymptotic expansion is \( \tilde{\Theta}^* = \tilde{\Theta} \circ g^{-1} \), hence it must be resurgent too and the **alien chain rule** gives \( \Delta_{\xi_n} \tilde{\Theta}^*(Q) = (e^{-\xi_n \left( \frac{1}{2} - \frac{1}{Q} \right)} \Delta_{\xi_n} \tilde{\Theta}(\tau))_{|\tau = g^{-1}(Q)} \), thus providing access to the asymptotics of \( \left[ Q^p \right] \tilde{\Theta}^* \) as \( p \to \infty \). Applying this with the quadratic character \( f(n) = \left( \frac{12}{n} \right) \), one recovers the asymptotic of Glaisher’s T-numbers, as well as that of Stoimenov’s numbers, which count regular linearized chord diagrams of degree \( p \) [24].

### 7. Action of the Modular Group and (Quantum) Modularity

The standard action of a general modular transformation \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{H} \),
\[
\gamma(\tau) = \frac{a \tau + b}{c \tau + d},
\]
induces an action on our partial theta series.

**Parabolic Case.** The case \( \gamma = \left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \) is straightforward; actually, it is a particular case of (2.3):
\[
\Theta(\tau + b; \nu, f, M) = \Theta(M^{-1} \tau; \nu, f, M') \quad \text{where} \quad f_{\frac{b}{M}} := \Lambda_{M}^b f, \quad \Lambda_{M}(n) := e^{in^2/M} \quad (7.1)
\]
and \( M' \) is a period of \( f_{\frac{b}{M}} \). We note that \( \Lambda_M(n + M) = (-1)^M \Lambda_M(n) \), hence
\[
M \text{ or } b \text{ even } \Rightarrow \Theta(\tau + b; \nu, f, M) = \Theta(\tau; \nu, f_{\frac{b}{M}}, M) \quad (7.2)
\]
(because one can then take \( M' = M \)). Notice that if the support of \( f \) is such that
\[
\exists n_0 \in \mathbb{Z} \text{ such that, } \forall n \in \mathbb{Z}, \quad f(n) \neq 0 \Rightarrow n^2 = n_0^2 \mod 2M, \quad (7.3)
\]
then $f_{\frac{1}{2M}} = \Lambda_M(n_0)f$ and $f_{\frac{b}{M}} = \Lambda_M^b(n_0)f$. Notice also that, since $\Lambda_M^{2M} = 1$, in all cases
\[ b = 0 \mod 2M \implies f_{\frac{b}{M}} = f \tag{7.4} \]

(indeed: $\Theta(\tau; \nu, f, M)$ is $2M$-periodic in $\tau$).

**Non-Parabolic Case.** If $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ with $c \neq 0$, then one can assume $c > 0$ without loss of generality (replacing $\gamma$ with $-\gamma$ if necessary). Using what is known from Section 5 on the action of $S = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and writing
\[ \gamma(\tau) = c^{-1}(a + S(c\tau + d)), \tag{7.5} \]

one can compute $\Theta(\gamma(\tau); \nu, f, M)$. Here is what one finds, assuming for the sake of simplicity $M$ even, $\nu \in \{0, 1\}$, $f$ even or odd (7.6)
till the end of this section (otherwise some adjustments are required):

**Theorem 6.** Assume (7.6) and $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \operatorname{SL}(2, \mathbb{Z})$ with $c > 0$. Then the formula
\[ h(n) := (cM)^{-1/2}\Lambda_{M}^{bd}(n) \sum_{r \mod M} f(r + dn)g(r)e^{2\pi i bmr/M}, \quad g(r) := \sum_{\ell \mod cM \text{ s.t. } \ell \equiv r \mod M} \Lambda_{cM}^{\ell}(\ell) \tag{7.7} \]
defines an $M$-periodic function $h$ of the same parity as $f$ and, using the notation $\theta(\cdot; \cdot, \cdot)$ introduced in (5.6),

\[ f \text{ even } \implies \quad \theta(\gamma(\tau); f) = \left( \frac{cr+d}{i} \right)^{1/2} \theta(\tau; h) \tag{7.8} \]
\[ f \text{ odd } \implies \quad \Theta(\gamma(\tau); 1, f, M) = i\left( \frac{cr+d}{i} \right)^{3/2} \Theta(\tau; 1, h, M) \tag{7.9} \]
\[ f \text{ even } \implies \quad \Theta(\tau; 1, h, M) \pm i\left( \frac{cr+d}{i} \right)^{-1/2} \Theta(\gamma(\tau); 1, f, M) \]
\[ = \frac{(cM)^{1/2}}{2\pi} f(0) \frac{1}{cr+d} + \mathcal{S}^{2\mp \varepsilon} \tilde{\Theta}(c\tau + d; 1, \Lambda_{cM}^{-d}h, cM) \tag{7.10} \]
\[ f \text{ odd } \implies \quad \Theta(\tau; 0, h, M) \mp \left( \frac{cr+d}{i} \right)^{-1/2} \Theta(\gamma(\tau); 0, f, M) \]
\[ = \mathcal{S}^{2\mp \varepsilon} \tilde{\Theta}(c\tau + d; 0, \Lambda_{cM}^{-d}h, cM). \tag{7.11} \]

Notice that, in (7.10) and (7.11), the right-hand side has analytic continuation through the real axis to the right or to the left of $\gamma^{-1}(i\infty) = -d/c$ (according to the $\pm$ sign) by the standard properties of Borel-Laplace summation.

Imposing to $\gamma$ to belong to a certain congruence subgroup and making use of classical properties of quadratic Gauss sums, one obtains a relation between $h$ and $f$ simple enough to allow us to interpret the left-hand side of (7.10) and (7.11) as a genuine modularity obstruction, with appropriate automorphy factor. We thus get classical modularity from (7.8)–(7.9) or quantum modularity from (7.10)–(7.11) according to the parity of $\nu$ and $f$.

Specifically, if we assume $\gamma \in \Gamma_0(2M) := \{ c = 0 \mod 2M \}$, then $g(r) = 0$ in (7.7) unless $r = 0 \mod M$ and
\[ g(0) = (cM)^{1/2}e^{\pi i/4}\varepsilon_d^{-1}(2Mc|d|) \quad \text{with} \quad \varepsilon_d := \begin{cases} 1 & \text{if } d = 1 \mod 4 \\ i & \text{if } d = 3 \mod 4 \end{cases} \tag{7.12} \]
and \( \left( \frac{m}{d} \right) \) := Jacobi symbol (notice that \( d \) is necessarily odd since \( ad = 1 \mod 2M \)), whence
\[
\gamma \in \Gamma_0(2M) \quad \Rightarrow \quad h(n) = e^{i\pi/4} \varepsilon_d^{-1} \left( \frac{2Mc}{|d|} \right) \Lambda_M^b(n)f(dn).
\]
Since here we assume \( M \) even, it follows that
\[
\gamma \in \Gamma(2M) \quad \Rightarrow \quad h = e^{i\pi/4} \left( \frac{2Mc}{|d|} \right) f,
\] (7.13)
where \( \Gamma(2M) := \{ a = d = 1 \mod 2M \text{ and } b = c = 0 \mod 2M \} \).

If we use the larger subgroup \( \Gamma_1(2M) := \{ a = d = 1 \mod 2M \text{ and } c = 0 \mod 2M \} \) instead of \( \Gamma(2M) \), then we still have a relation similar to one of the results of [10] under the support assumption (7.3):
\[
\gamma \in \Gamma_1(2M) \text{ and Assumption (7.3) holds} \quad \Rightarrow \quad h = \Lambda_M^b(n_0) e^{i\pi/4} \left( \frac{2Mc}{|d|} \right) f.
\] (7.14)

Another option available when \( f \) is a Dirichlet character modulo \( M \) is to use \( \Gamma_0^b(2M) := \{ b = 0 \mod 2M \text{ and } c = 0 \mod 2M \} \):
\[
\gamma \in \Gamma_0^b(2M) \text{ and } f \text{ Dirichlet character mod } M \quad \Rightarrow \quad h = e^{i\pi/4} \varepsilon_d^{-1} \left( \frac{2Mc}{|d|} \right) f(d)f.
\] (7.15)

The relations (7.8)–(7.11) thus give rise to explicit transformation formulas for the function \( \Theta(\cdot; \nu, f, M) \) under the action of the congruence subgroup \( \Gamma \) defined by
\[
\Gamma := \Gamma(2M) \text{ or, if Assumption (7.3) holds, } \Gamma := \Gamma_1(2M)
\]

or, if \( f \) is a Dirichlet character mod \( M \), \( \Gamma := \Gamma_0^b(2M) \). (7.16)

**The Boundary Function \( \Theta^{nt} \).** Let us give more details on the cases
\[
\{ \nu = 0 \text{ and } f \text{ odd} \} \text{ or } \{ \nu = 1 \text{ and } f \text{ even} \}
\] (7.17)
(still with \( M \) even). Recall that the boundary function \( \Theta^{nt}(\alpha) = \Theta^{nt}(\alpha; \nu, f, M) \) is defined for \( \alpha \in \mathcal{D}_{f,M} \cup \{ i\infty \} \).

If \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) is parabolic, we have \( \Theta(\gamma(\tau); \nu, f, M) = \lambda(\gamma) \Theta(\tau; \nu, f, M) \) with \( \lambda(\gamma) := 1 \) or \( \lambda(\gamma) := \Lambda_M^b(n_0) \) according as which option holds in (7.16); in particular, \( \mathcal{D}_{f,M} \) is stable under \( \gamma \).

If \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) is non-parabolic, then we assume \( c > 0 \) and (7.13)–(7.15) say that the function \( h \) of (7.7) is \( h = \lambda(\gamma) f \) with \( \lambda(\gamma) := e^{i\pi/4} \left( \frac{2Mc}{|d|} \right) \) or \( \lambda(\gamma) := \Lambda_M^b(n_0) e^{i\pi/4} \left( \frac{2Mc}{|d|} \right) \) or \( \lambda(\gamma) := e^{i\pi/4} \varepsilon_d^{-1} \left( \frac{2Mc}{|d|} \right) f(d) \). The relations (7.10)–(7.11) thus yield
\[
\Theta(\gamma(\tau); \nu, f, M) \equiv \left( \frac{(c\tau+d)}{|d|} \right)^{-\nu - \frac{1}{2}} \Theta(\gamma(\tau); \nu, f, M)
\]
\[
= \left( \frac{(cM)^{1/2}}{2\pi \lambda(\gamma)} \right) f(0) \frac{i}{c\tau+d} + \mathcal{S}_{M; \tau} \Theta(c\tau+d; \nu, \Lambda_M^d f, cM). \tag{7.18}
\]

Assuming \( f(0) = 0 \) and taking the limit as \( \tau \) tends non-tangentially to \( \gamma^{-1}(i\infty) \) (in which case the second term tends to 0 exponentially fast), or to a rational \( \alpha > \gamma^{-1}(i\infty) \) (in which case we use ‘−’), or to a rational \( \alpha < \gamma^{-1}(i\infty) \) (in which case we use ‘+’), we get
Theorem 7. Suppose $M$ is even and $f(0) = 0$. If $f$ is even then the domain of definition $\mathcal{Q}_{f,M} \cup \{i\infty\}$ of $\Theta^{nt}$ is stable under $\Gamma$: it is a union of cusps, which contains at least the cusp at infinity, and also the cusp at $0$ if $\langle f \rangle = 0$. If $f$ is odd then $\mathcal{Q}_{f,M} = \mathbb{Q}$. In all cases, the function $\Theta^{nt}: \mathcal{Q}_{f,M} \cup \{i\infty\} \to \mathbb{C}$ is a quantum modular form with respect to $\Gamma$, of weight $\frac{1}{2}$ if $\{\nu = 0$ and $f$ is odd\}, of weight $\frac{3}{2}$ if $\{\nu = 1$ and $f$ is even\}.

Notice that, as a consequence, $\mathcal{Q}_{f,M}$ is dense in $\mathbb{Q}$ if $f(0) = 0$. Indeed, each cusp of $\Gamma$ is dense in $\mathbb{Q}$ because $\Gamma$ contains $\Gamma(2M)$, which is a normal subgroup of finite index of $\text{SL}(2, \mathbb{Z})$.

Remark 7.1. The number $\Theta^{nt}(\alpha)$ is just the limit value of $\Theta(\tau; \nu, f, M)$ as $\tau \to \alpha$, but we can also consider the whole asymptotic expansion of (7.18) as $\tau \to \alpha$: the formal series $(\tilde{\Theta})_{\alpha \in \mathcal{Q}_{f,M}}$ introduced in Remark 3.2 make up what is called a strong quantum modular form in [22].

The case when Assumption (7.3) holds has already been considered (with $M$ even or odd) in [10], and we get the same congruence subgroup as them in that case. A notable example where this assumption is fulfilled is, following [11] or [1], the partial theta series giving the Gukov-Pei-Putrov-Vafa invariant for an oriented Seifert fibered integral homology sphere with 3 exceptional fibers (a Brieskorn 3-sphere).

Remark 7.2. For arbitrary $\nu \geq 2$, since Borel-Laplace (median) summation commutes with $\frac{d}{d\tau}$, we can use (2.4) and derive transformation formulas for $\Theta(\tau; \nu, f, M)$, not as simple as when $\nu = 0$ or 1, but rather pertaining to the theory of higher depth quantum modular forms [2].

Remark 7.3. Examples with $f$ even and $\mathcal{Q}_{f,M} = \mathbb{Q}$. Suppose that $\nu = 1$ and $f$ is even. If Assumption (7.3) holds for $f$, then we get $f_{\alpha+1} = \Lambda_M(n_0) f_{\alpha}$ for every $\alpha \in \mathbb{Q}$, hence

$$\text{Assumption (7.3) holds } \implies \mathcal{Q}_{f,M} \text{ is invariant under } T: \tau \mapsto \tau + 1.$$  \hspace{1cm} (7.19)

Using (5.16), we also have

$$f \text{ eigenvector of } U_M \text{ and } \langle f \rangle = 0 \implies \mathcal{Q}_{f,M} \cup \{i\infty\} \text{ is invariant under } S: \tau \mapsto -1/\tau.$$  \hspace{1cm} (7.20)

Since $\langle S, T \rangle = \text{SL}(2, \mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{i\infty\}$, we deduce that $\mathcal{Q}_{f,M} = \mathbb{Q}$ whenever $f$ is a zero mean value even eigenvector of $U_M$ such that (7.3) holds. This is what happens with the Eichler integral $\tilde{\eta}(\tau) = \Theta(\tau; 1, \chi_{12}(11, \cdot), 12)$ of the Dedekind $\eta$ function.

There are cases where $f$ is not an eigenvector of $U_M$, but $U_M f$ is a linear combination of $f$ and $\tilde{f}$ with $f$ satisfying (7.3) with a certain $n_0$ and $\tilde{f}$ satisfying (7.3) with a certain $\tilde{n}_0$, so that $\mathcal{Q}_{f,M} \cap \mathcal{Q}_{U_M f,M}$ is invariant under $T$ by (7.19). If moreover $\langle f \rangle = f(0) = 0$, then (5.16) shows that $(\mathcal{Q}_{f,M} \cup \{i\infty\}) \cap (\mathcal{Q}_{U_M f,M} \cup \{i\infty\})$ is invariant under $S$, whence $\mathcal{Q}_{f,M} = \mathcal{Q}_{U_M f,M} = \mathbb{Q}$ in those cases. This is what happens with the Eichler integrals $\Theta(\tau; 1, f_{\alpha, j}, 20)$ of the two modular forms involved in the Rogers-Ramanujan identity.
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