The general spin triangle

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\textbf{Abstract.} We consider the Heisenberg spin triangle with general coupling coefficients and general spin quantum number $s$. The corresponding classical system is completely integrable. In the quantum case the eigenvalue problem can be reduced to that of tridiagonal matrices in at most $2s+1$ dimensions. The corresponding energy spectrum exhibits what we will call spectral symmetries due to the underlying permutational symmetry of the considered class of Hamiltonians. As an application we explicitly calculate six classes of universal polynomials that occur in the high temperature expansion of spin triangles and more general spin systems. Aspects of quantum integrability are also discussed in this context.

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1. Introduction

Theoretical investigations of spin systems are often motivated by the applications to real systems as magnetic molecules or spin lattices. In the case of a system of three spins with realistic spin quantum numbers varying from \( s = \frac{1}{2} \) to, say, \( s = \frac{7}{2} \), the underlying Hilbert space has a maximal dimension of \((2s + 1)^3 = 8^3 = 512\). Hence, even without using any dimensional reduction due to rotational symmetry, it is an easy exercise to numerically calculate physically relevant quantities, e.g., specific heat or magnetization curves, by using standard codes. Other aspects of spin triangles have been treated in recent publications [1], [2], and [3]. Thus a detailed theoretical investigation of the general spin triangle requires additional justification. In my opinion, the reasons to study this special class of spin systems are threefold.

First, the spin triangle is interesting in its own right since it exhibits salient features which might evade the purely numerical treatment. It is a straightforward task to reduce the pertinent Hamiltonian \( H \) to that of a two-chain with coupling constants \( 1 \) and \( y \) and to restrict it to invariant subspaces of maximal dimension \( 2s + 1 \), see section 2. Here the reduced Hamiltonian has a tridiagonal matrix representation, see section 3, which can be understood by virtue of the Wigner-Eckhard theorem. Some properties of tridiagonal matrices that are relevant for our purpose are collected in Appendix A. The corresponding characteristic polynomial and the spectral curves \( e_\nu(y) \) exhibit a number of symmetries that I have dubbed “spectral symmetries” in section 4. Usually, symmetries in quantum mechanics lead to operators commuting with \( H \). Spectral symmetries are of a different kind and can be expressed by functional equations for the \( e_\nu(y) \). They can be traced back to permutational invariance, not of a particular Hamiltonian, but of the whole class of triangle Hamiltonians. Despite its elementary origin, spectral symmetry allows a qualitative discussion of the spectral curves and, e.g., the calculation of the moments of the restricted Hamiltonian up to fifth order without using its explicit matrix representation.

A second reason to study spin triangles is that their properties might help to understand more complex systems. One example, not considered in this article, is the \( S = 0 \) ground state (for integer \( s \) and approximately equal coupling constants) that has been used to build ground states of larger spin systems including spin lattices, see [4] and [5]. Another example is the observation that the first few terms of the perturbation series for the eigenvalues of \( H \) yield certain classes of universal polynomials that occur in the graph expansion of the \( k \)-th moments of \( H \), see section 5 and 6. These polynomials also occur in the high temperature expansion for general Heisenberg systems, not only triangles, see [8] and [9]. Appendix B contains explicit expressions for the \( k \)-th moments of spin dimers that are used in the formulas for the above-mentioned polynomials.

Finally, spin triangles may serve as simple but non-trivial examples to study
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Theoretical problems of general interest. Here we will concentrate on one aspect, namely integrability. Classical systems are completely integrable in the sense of Arnold-Liouville iff they possess $N$ Poisson-commuting integrals of motion, where $2N$ is the phase space dimension. Sometimes the corresponding quantum systems are also analytically solvable (harmonic oscillator, hydrogen atom) but not always. One counter-example is the one-dimensional particle in a potential $V(x)$, another one is the general spin triangle. In the latter case we have classically three unit spin vectors, i.e., a six-dimensional phase space, and three constants of motion, $H, S^2$ and $S_3$. The corresponding quantum system, notwithstanding the partial results of this paper, cannot be solved in closed form. However, there are weaker notions of “quantum integrability” that justify, e.g., to consider the $s = \frac{1}{2}$ chain as integrable by means of the Bethe ansatz, see [6] or [7] for generalizations. In this context one could see the results of the present paper as another example within the wide range of the not yet fully understood notion of quantum integrability. We will come back to this question in the section 7 containing the summary and an outlook.

2. Basic definitions and elementary results

We consider the general Heisenberg spin triangle with one and the same individual spin quantum number $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ and the Hamiltonian

$$H = J_1 \mathbf{s}_2 \cdot \mathbf{s}_3 + J_2 \mathbf{s}_3 \cdot \mathbf{s}_1 + J_3 \mathbf{s}_1 \cdot \mathbf{s}_2$$

(1)

where the $J_1, J_2, J_3$ are three arbitrary real parameters and the $\mathbf{s}_i$ denote the three individual spin vector operators, $i = 1, 2, 3$. The Hamiltonian (1) can be analytically diagonalized if two of the $J_i$'s coincide (i.e., for the isosceles triangle) or if $s$ is small in a sense to be made more precise subsequently. In the first case of, say, $J_1 = J_2 \equiv J$ we have

$$\widetilde{H} = H_J = J (\mathbf{s}_1 + \mathbf{s}_2) \cdot \mathbf{s}_3 + J_3 \mathbf{s}_1 \cdot \mathbf{s}_2$$

(2)

$$= \frac{J}{2} \mathbf{S}^2 + \frac{J_3 - J}{2} \mathbf{S}_{12}^2 - \left(\frac{J}{2} + J_3\right) s(s + 1),$$

(3)

where

$$\mathbf{S} \equiv \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3,$$

(4)

$$\mathbf{S}_{12} \equiv \mathbf{s}_1 + \mathbf{s}_2.$$  

(5)

It is well-known that $\mathbf{S}^2$ and $\mathbf{S}_{12}^2$ commute and have simultaneous eigenvalues of the form $S(S + 1), S_{12}(S_{12} + 1)$ where

$$S = 3s, 3s - 1, \ldots, S_{\min} \geq 0$$

(6)

$$S_{12} = 2s, 2s - 1, \ldots, 0,$$

(7)

and the triangle inequality

$$|S_{12} - s| \leq S \leq S_{12} + s$$

(8)
holds. From this the eigenvalues of $\tilde{H}_J$ immediately follow from (3). The corresponding eigenbasis of $\tilde{H}_J$ will be used to split the matrix representation of the general $\tilde{H}$ into smaller block matrices.

In the general case we write

$$H = \frac{J_2}{2} \left( S^2 - 3s(s + 1) \right) + (J_3 - J_2) \left( \mathbf{s}_1 \cdot \mathbf{s}_2 + \frac{J_1 - J_2}{J_3 - J_2} \mathbf{s}_2 \cdot \mathbf{s}_3 \right),$$

and, since $H$ and $S^2$ commute, the problem of diagonalizing $H$ can be reduced to the task of finding the eigenvalues and eigenvectors of

$$\tilde{H}_2(y) \equiv \mathbf{s}_1 \cdot \mathbf{s}_2 + y \mathbf{s}_2 \cdot \mathbf{s}_3,$$

where

$$y \equiv \frac{J_1 - J_2}{J_3 - J_2}.$$

in the invariant subspace $\mathcal{H}(S, S_3)$. The latter denotes the common eigenspace of $S^2$ and $S^3$ corresponding to the eigenvalues $S(S + 1)$ and, say,

$$S_3 = S_{\text{min}} \equiv \begin{cases} 0 & : s \text{ integer} \\ \frac{1}{2} & : s \text{ half-integer} \end{cases}$$

It will be simply denoted by $\mathcal{H}(S)$, since the eigenvalues of $\tilde{H}_2(y)$ do not depend on the chosen eigenspace of $S_3$. The dimension of $\mathcal{H}(S)$ can be determined by counting the possible values of the quantum number $S_{12}$. We obtain

$$s - S \leq S_{12} \leq S + s \quad \text{for} \quad 0 \leq S \leq s,$$

$$S - s \leq S_{12} \leq 2s \quad \text{for} \quad s \leq S \leq 3s,$$

according to the triangle inequality (8). In order to avoid the case distinction (13),(14) in what follows, we will write the above condition in a uniform way as

$$g_1 \leq g \leq g_2,$$

where $g \equiv S_{12}$ and $g_1(s, S), g_2(s, S)$ are defined according to (13) and (14).

The dimension $N$ of $\mathcal{H}(S)$ increases in steps of two in the range $0 \leq S \leq s$ and decreases in steps of one in the range $s \leq S \leq 3s$. The maximal dimension $N_{\text{max}} = 2s + 1$ is assumed for $S = s$. The figures 1 and 2 show typical examples. Especially, it follows that for $s \leq 3/2$ the dimension of $\mathcal{H}(S)$ will be at most four and hence $\tilde{H}$ can be analytically diagonalized. At this point one should mention that for moderate $s$, also exceeding $s = 3/2$, the computer-algebra system MATHEMATICA renders the eigenvalues of $H_2(y)$ in the form of root objects which have many properties of special functions. For example, the $n$-th eigenvalue of $H_2(y)$ can be expanded into an exact power series w.r.t. $y$ up to some finite order.

The total dimension $d_{\text{tot}}$ of the eigenspace of $S_3$ with eigenvalue $S_{\text{min}}$ amounts to

$$d_{\text{tot}} = 3s(s + 1) + \begin{cases} 1 & : s \text{ integer} \\ \frac{3}{4} & : s \text{ half-integer} \end{cases}$$

in accordance with the general formula in [10].
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Figure 1. The range of the quantum numbers $S_{12}$ and $S$ for $s = 2$. It is typical for integer $s$ that $S$ has a minimal value of $S_{\text{min}} = 0$ and the corresponding eigenspace $\mathcal{H}(S)$ will be one-dimensional. Note that dimensions of the various spaces $\mathcal{H}(S)$ are $1 + 3 + 5 + 4 + 3 + 2 + 1 = 19 = 3s(s + 1) + 1$ in accordance with (16).

Figure 2. The range of the quantum numbers $S_{12}$ and $S$ for $s = 5/2$. It is typical for half-integer $s$ that $S$ has a minimal value of $S_{\text{min}} = 1/2$ and the corresponding eigenspace $\mathcal{H}(S)$ will be two-dimensional. Note that dimensions of the various spaces $\mathcal{H}(S)$ are $2 + 4 + 6 + 5 + 4 + 3 + 2 + 1 = 27 = 3s(s + 1) + 3/4$ in accordance with (16).

3. Matrix representation of the reduced Hamiltonian

In order to find the matrix representation of $H_2(y)$ within the invariant subspace $\mathcal{H}(S)$ one has to choose an appropriate basis. An obvious choice is the eigenbasis of $H_2(0) = s_1 \cdot s_2 = \frac{1}{2} \left( S_{12}^2 - 2s(s + 1) \right)$. The transformation to the eigenbasis of $s_2 \cdot s_3$ within $\mathcal{H}(S)$ is achieved by the unitary, real, symmetric matrix $U$ with the entries

$$U_{gg'} = (-1)^{3s + S} \sqrt{(2g + 1)(2g' + 1)} \begin{pmatrix} s & s & g \\ s & S & g' \end{pmatrix}, \quad g_1 \leq g, g' \leq g_2,$$

where (...) denotes the $6J$–symbol, see, e. g., [11]. For the sake of simplicity we will not distinguish between operators and their matrix representation in $\mathcal{H}(S)$, if no
misunderstanding is likely to arise. Hence we may write
\[ \mathbf{s}_1 \cdot \mathbf{s}_2 = \text{diag}(\frac{1}{2}(g(g+1)-2s(s+1))|g=g_1, \ldots, g_2) \] (18)
and
\[ \mathbf{s}_2 \cdot \mathbf{s}_3 = U^* \mathbf{s}_1 \cdot \mathbf{s}_2 U. \] (19)

Usually, the matrix \( U \) is fully occupied and contains only few zeroes. Thus it is somewhat
surprising that \( \mathbf{s}_2 \cdot \mathbf{s}_3 \) will always be tridiagonal. On the other hand it is a straight forward
task to directly calculate \( \mathbf{s}_2 \cdot \mathbf{s}_3 \) in the eigenbasis of \( \mathbf{s}_1 \cdot \mathbf{s}_2 \) if one adopts the theory of
irreducible tensor operators (ITO), see, e. g. [12] or, for a recent account, [13]. In this
theory the tridiagonal form of \( \mathbf{s}_2 \cdot \mathbf{s}_3 \) is a direct consequence of the Wigner-Eckhard
theorem, since \( \mathbf{s}_2 \) is an ITO of rank one. Moreover, it follows by the same theorem that,
for \( S = s, \ (\mathbf{s}_2 \cdot \mathbf{s}_3)_{00} = 0 \) which we will derive later by different arguments. We will not
dwell upon the details of the ITO calculations and simply present the result. In order to
avoid the case distinction (13),(14) in the formula for the secondary diagonals of \( H_2(y) \)
we will express \( s \) and \( S \) through \( g_1 \) and \( g_2 \), the bounds of \( g \), see (15):
\[ s = \frac{1}{2}(g_1 + g_2), \quad S = \frac{1}{2}(g_2 - g_1), \quad \text{for} \quad 0 \leq S \leq s, \] (20)
\[ s = \frac{g_2}{2}, \quad S = \frac{1}{2}(2g_1 + g_2), \quad \text{for} \quad s \leq S \leq 3s. \] (21)

Then
\[ \begin{aligned}
\left[ H_2(y) \right]_{g,g} &= \frac{1}{2}(g(g+1) - 2s(s+1)) + \frac{y}{4}(-g(1+g) - s(1+s) + S(S+1)), \\
\left[ H_2(y) \right]_{g,g+1} &= \left[ H_2(y) \right]_{g+1,g}, \\
&= \sqrt{(1 + g - g_1)(1 + g + g_1)(g - g_2)(g - g_1 - g_2)(2 + g + g_2)(2 + g + g_1 + g_2)}.
\end{aligned} \] (22)

The other matrix elements of \( H_2(y) \) vanish. For example, if \( s = S = 3 \) then \( H_2(y) \) has the
matrix form
\[ \begin{pmatrix}
-12 & 4\sqrt{3}y & 0 & 0 & 0 & 0 \\
4\sqrt{3}y & -11 - \frac{y}{2} & \frac{3\sqrt{15}y}{2} & 0 & 0 & 0 \\
0 & \frac{3\sqrt{15}y}{2} & -9 - \frac{3y}{2} & 6\sqrt{\frac{5}{7}}y & 0 & 0 \\
0 & 0 & 6\sqrt{\frac{5}{7}}y & -6 - 3y & \frac{11y}{\sqrt{7}} & 0 \\
0 & 0 & 0 & \frac{11y}{\sqrt{7}} & -2 - 5y & \frac{10y}{\sqrt{11}} \\
0 & 0 & 0 & 0 & \frac{10y}{\sqrt{11}} & 3 - \frac{15y}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{3}{2} \sqrt{\frac{13}{11}y} \\
0 & 0 & 0 & 0 & 0 & \frac{3}{2} \sqrt{\frac{13}{11}y} \\
0 & 0 & 0 & 0 & 0 & \frac{9}{2} - \frac{21y}{2} \\
\end{pmatrix}. \] (25)

We note in passing that a negative sign of the square root in (24) gives the matrix repre
sentation of \( \tilde{H}_2(y) \equiv \mathbf{s}_1 \cdot \mathbf{s}_2 + y \mathbf{s}_1 \cdot \mathbf{s}_3 \). This will be later used in section 6 to derive (90).

The diagonalization of tridiagonal matrices is considerably simpler than for general
matrices and has been extensively studied also from a numerical point of view, see,
e. g. [14]. For the related problem of the inversion of tridiagonal matrices see [15]. We have collected some results which will be of relevance for our purpose in the Appendix A. For example, in our problem the entries (24) of the secondary diagonals do not vanish. In this case the eigenvalues of the tridiagonal matrix will never be degenerate. Of course, there is the usual degeneracy due to rotational invariance and some accidental degeneracy between the energy eigenvalues of \( \mathcal{H}(S) \), not to mention the special case of \( J_1 = J_2 = J_3 \).

Another interesting property is the following: If we perturb a diagonal matrix by a tridiagonal perturbation the resulting perturbation series will be given by relatively simple formulas, at least for the first few terms, see (114)-(116). These formulas seem to be of little use at first sight since, at least for small \( s \), the eigenvalues can be numerically calculated very fast, but we will present different applications in the sections 5 and 6.

4. Spectral symmetries

For given quantum numbers \( s \) and \( S \) let \( p(x, y) \) denote the characteristic polynomial of \( \mathcal{H}_2(y) \) restricted to \( \mathcal{H}(S) \), that is,

\[
p(x, y) \equiv \det \left( \mathcal{H}_2(y) - x \ 1_{\mathcal{H}(S)} \right).
\]

It is a polynomial in the variables \( x \) and \( y \) of total degree \( N \equiv \dim(\mathcal{H}(S)) \). For example, if \( s = S = 3/2 \) we have \( N = 4 \) and

\[
p(x, y) = -\frac{135}{256}(y - 1)^2(33 + 94y + 33y^2) - \frac{9}{16}(y + 1)(47 - 110y + 47y^2)x - \frac{9}{8}(1 - 26y + y^2)x^2 + 5(1 + y)x^3 + x^4.
\]

(27)

Obviously, the above coefficients of \( y^\nu x^\mu \) are symmetric under the reflection \( \nu \mapsto N - \mu - \nu \) or

\[
yN p \left( \frac{x}{y}, \frac{1}{y} \right) = p(x, y) \text{ for } y \neq 0.
\]

(28)

This can be proven generally as follows.

The two Hamiltonians \( \mathcal{H}_2(y) = \mathbf{s}_1 \cdot \mathbf{s}_2 + y \mathbf{s}_2 \cdot \mathbf{s}_3 \) and \( y \mathcal{H}_2 \left( \frac{1}{y} \right) = y \left( \mathbf{s}_1 \cdot \mathbf{s}_2 + \frac{1}{y} \mathbf{s}_2 \cdot \mathbf{s}_3 \right) \) are unitarily equivalent under the permutation (13) and hence have the same eigenvalues, say, \( e_\nu(y) \), \( \nu = 1, \ldots, N \). This means that, upon an appropriate ordering of eigenvalues, \( e_\nu(y) = ye_\nu \left( \frac{1}{y} \right) \).

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from which (28) follows by

\[ p(x, y) = \prod_{\nu=1}^{N} (e_\nu(y) - x) \]

\[ = \prod_{\nu=1}^{N} \left( y e_\nu \left( \frac{1}{y} \right) - x \right) \]

\[ = y^N \prod_{\nu=1}^{N} (e_\nu \left( \frac{1}{y} \right) - \frac{x}{y}) \]

\[ = y^N p \left( \frac{x}{y}, \frac{1}{y} \right) . \]

The coefficients of \( y^0 x^\mu \) are determined by the well-known eigenvalues of \( H_2(0) \)

\[ e_g(0) = \frac{1}{2} \left( g(g+1) - 2s(s+1) \right), \quad g = g_1, \ldots, g_2 . \]

For example, the factors in front of the lines of (27) result from

\[ \left( -\frac{15}{4} - x \right) \left( -\frac{11}{4} - x \right) \left( -\frac{3}{4} - x \right) \left( \frac{9}{4} - x \right) \]

\[ = - \frac{135 \cdot 33}{256} \cdot \frac{9 \cdot 47}{16} \cdot x - \frac{9}{8} x^2 + 5 x^3 + x^4 . \]

There exist further symmetries of \( p(x, y) \) which are not so obvious as the reflection symmetry. They are revealed by employing arbitrary permutations of the spin triangle. These permutations can be composed of transpositions \((i \, j)\) that induce rational transformations of \( y = J_1 - J_2 J_3 - J_2 \) in the following way:

\[ y \xrightarrow{(13)} \frac{1}{y} \xrightarrow{(12)} 1 - \frac{1}{y} = \frac{y - 1}{y} \xrightarrow{(23)} y \xrightarrow{(13)} 1 - \frac{y}{y - 1} = \frac{1}{1 - y} \xrightarrow{(12)} 1 - y \xrightarrow{(23)} y . \]

For almost all real numbers \( y \) the permutation group \( S_3 \) thus produces cycles of six different numbers. The exceptions are 1 \( \rightarrow \) 1 \( \rightarrow \) 0 \( \rightarrow \) \( \pm \infty \) \( \rightarrow \) 0 \( \rightarrow \) 1 and \( \frac{1}{2} \rightarrow \) 2 \( \rightarrow \) -1 \( \rightarrow \) -1 \( \rightarrow \) 2 \( \rightarrow \) \( \frac{1}{2} \rightarrow \) \( \frac{1}{2} \), where only three different numbers are involved. In general, if \( y \) runs through such a 6-cycle the spectrum of the corresponding Hamiltonians \( H_2(y) \) will be related by affine transformations. (29) is a first example. A second example which, together with (29) generates all remaining ones, is

\[ e_\nu(1 - y) = E_0 - e_\nu(y) , \]

where

\[ E_0 \equiv \frac{1}{2} \left( S(S+1) - 3s(s+1) \right) , \]

and a suitable ordering of the eigenvalues is assumed.

Geometrically, (37) means that the system of spectral curves \( y \mapsto e_\nu(y) \), \( \nu = 1, \ldots, N \) in the \( y - E \)-plane is point-symmetric w. r. t. the point \( P \equiv (y = \frac{1}{2}, E = \frac{1}{2}E_0) \). For odd \( N \), the curve \( y \mapsto e_{N+1}(y) \) passes through \( P \) and is point-symmetric in itself. Especially, \( E = \frac{1}{2}E_0 \) is always an eigenvalue of \( H_2(\frac{1}{2}) \) for odd \( N \) and the other eigenvalues of \( H_2(\frac{1}{2}) \) lie symmetric to the value \( \frac{1}{2}E_0 \) for all \( N \). These properties are illustrated in figure 3.
Figure 3. Spectral curves $e_\nu(y)$, $\nu = 0, \ldots, 8$ of a 2-chain with coupling coefficients $J_1 = 1$ and $J_2 = y$ and $s = S = 4$. The system of spectral curves is point-symmetric w. r. t. the red point $P = (1/2, -10)$. For pairs of $y$-values connected by the transformation (36) the corresponding eigenvalues $e_\nu(y)$ are related by an affine transformation. This is illustrated here for the pair $y_1 = 1/2$, $y_2 = 1 - \frac{1}{y_1} = -1$ by demonstrating that the (dashed) lines joining the corresponding eigenvalues $e_\nu(y)$ meet at the point $(2, -20)$.

Figure 4. The same system of spectral curves as in figure 3, but with a larger $y$-domain. For large $|y|$ the spectral curves approach certain (dashed) lines which are determined by the first two terms of the perturbation series of $e_\nu(y)$ via (43). For $y \to +\infty$ the slope of the asymptotic line equals the unperturbed eigenvalue $e_\nu(0)$; for $y \to -\infty$ the slope is $e_{N-\nu}(0)$. For $y > 0$ the convergence of the spectral curves to their asymptotic lines becomes slower for the lower curves due to larger values of $|x_2^{(\nu)}|$. The lowest spectral curve with a large (negative) curvature at $y = 0$ shows the slowest convergence. For $y < 0$ analogous remarks applies to the upper curves. The asymptotic lines meet at the red point $P = (1/2, -10)$. 
A related consequence of (37) is that the shifted characteristic polynomial
\[ q(x, y) = p \left( x + \frac{E_0}{2}, y + \frac{1}{2} \right) = \sum_{\nu=0}^{N} q_\nu(y) x^\nu \]  \tag{39}
will contain only even or odd sub-polynomials \( q_\nu(y) \). For example, the shifted polynomial pertaining to (27) reads:
\[ q(x, y) = \frac{1}{256} (945 - 5130 y^2 - 4455 y^4) - \frac{x}{16} (27 y + 423 y^3) \]
\[- \frac{x^2}{8} (63 + 9 y^2) + 5 x^3 y + x^4. \]  \tag{40}

Unfortunately, these and the above-mentioned symmetries (29) are not sufficient to determine the coefficients of \( p(x, y) \) completely.

We will explore some further consequences of (29) and (37). For \(|y| \to \infty\) the spectral curves \( y \mapsto e_\nu(y) \) will approach certain lines. This can be easily understood by inserting the perturbation series
\[ e_\nu(z) = x_0^{(\nu)} + x_1^{(\nu)} z + x_2^{(\nu)} z^2 + \ldots \]  \tag{41}
into the r. h. s. of (29) and setting \( z = \frac{1}{y} \)
\[ e_\nu(y) = y \left( x_0^{(\nu)} + x_1^{(\nu)} \frac{1}{y} + x_2^{(\nu)} \frac{1}{y^2} + \ldots \right) \]
\[ = x_0^{(\nu)} y + x_1^{(\nu)} + x_2^{(\nu)} \frac{1}{y} + \ldots, \quad \text{for } y \to +\infty. \]  \tag{42}

For \( y \to -\infty \) one obtains
\[ e_\nu(y) = x_0^{(N-\nu)} y + x_1^{(N-\nu)} + x_2^{(N-\nu)} \frac{1}{y} + \ldots. \]  \tag{43}

Since the spectral curves do not intersect (see Appendix A), it follows that for \( y \to +\infty \) the spectral curve \( e_\nu(y) \) approaches the line with slope \( x_0^{(\nu)} \) and for \( y \to +\infty \) it approaches the line with slope \( x_0^{(N-\nu)} \). This effect is illustrated in figure 4.

Anticipating the result (50) it can be shown that the \( N \) asymptotic lines meet at the point \( P = (y = \frac{1}{2}, E = \frac{1}{2} E_0) \):
\[ x_0^{(\nu)} \frac{1}{2} + x_1^{(\nu)} \overset{(50)}{=} x_0^{(\nu)} \frac{1}{2} + \frac{1}{2} \left( E_0 - x_0^{(\nu)} \right) = \frac{1}{2} E_0. \]  \tag{45}

By the sequence of rational maps (36) the closed interval \([0, \frac{1}{2}]\) is mapped as follows:
\[ [0, \frac{1}{2}] \to [2, +\infty] \to [-\infty, -1] \to [-1, 0] \to [1, 2] \to [\frac{1}{2}, 1] \to [0, \frac{1}{2}]. \]  \tag{46}

This set of six closed intervals covers the whole real axis, hence the values of \( e_\nu(y) \) for \( y \in [0, \frac{1}{2}] \) completely determine the full spectral curve \( e_\nu(y), y \in \mathbb{R} \). We have already exploited this principle for the discussion of the asymptotic behavior of \( e_\nu(y) \) for \(|y| \to \infty\).

Conversely, if we choose some analytic function, restrict it to \( y \in [0, \frac{1}{2}] \) and extend it
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to the full real line by means of the functional equations (29) and (37), the resulting
function will not automatically be analytic at the borders of the intervals, e. g. at \( y = 0 \).
This makes it plausible that (29) and (37) impose conditions on the perturbation series
(41). Let us consider some small \( y \). After the first three steps of the sequence (36)
\( y \) is mapped onto \(-y\) which is again small. Simultaneously, \( e_{\nu}(y) \) is mapped onto
\[
E_{0} - \frac{1}{1-y}(E_{0} - e_{\nu}(y)) = -E_{0} \frac{y}{1-y} + \frac{1}{1-y} \sum_{i=0}^{\infty} x_{i}^{(\nu)} y^{i}
\]
(47)
\[
e_{\nu}\left(\frac{-y}{1-y}\right) = \sum_{i=0}^{\infty} x_{i}^{(\nu)} \left(\frac{-y}{1-y}\right)^{i}.
\]
(48)
Equating the coefficients of the power series (47) and (48) leads to the following system
of equations:
\[
0 = -E_{0} + \sum_{n=0}^{m} \left(1 + (-1)^{n+1}\left(\frac{m-1}{n-1}\right)\right) x_{n}^{(\nu)}, \quad m = 1, 2, \ldots
\]
(49)
Consequently, for all \( \nu = 1, \ldots, N \) the perturbation coefficients \( x_{n}^{(\nu)} \), \( n \) odd, can be
expressed in terms of the perturbation coefficients \( x_{i}^{(\nu)} \), \( i \) even and \( i < n \). The first few
of the corresponding equations read:
\[
x_{1}^{(\nu)} = \frac{1}{2} \left(E_{0} - x_{0}^{(\nu)}\right)
\]
(50)
\[
x_{3}^{(\nu)} = \frac{1}{2} x_{2}^{(\nu)}
\]
(51)
\[
x_{5}^{(\nu)} = \frac{1}{4} \left(-x_{2}^{(\nu)} + 6x_{4}^{(\nu)}\right)
\]
(52)
\[
x_{7}^{(\nu)} = \frac{1}{2} \left(x_{2}^{(\nu)} - 5x_{4}^{(\nu)} + 5x_{6}^{(\nu)}\right)
\]
(53)
\[
x_{9}^{(\nu)} = -\frac{17}{8} x_{2}^{(\nu)} + \frac{21}{2} x_{4}^{(\nu)} - \frac{35}{4} x_{6}^{(\nu)} + \frac{7}{2} x_{8}^{(\nu)}.
\]
(54)
Sometimes we will switch from the index notation of eigenvalues using \( \nu = 1, \ldots, N \)
to the notation using \( g = g_1, \ldots, g_2 \) introduced before. In this notation we have
\[
x_{0}^{(g)} = \frac{1}{2}(g(g+1) - 2s(s+1)).
\]In the special case of \( S = s \) we conclude \( E_{0} = \frac{1}{2}(S(S+1) - 3s(s+1)) = -s(s+1) \) and, by (50), \( x_{1}^{(g)} = -\frac{1}{2}g(g+1) \). Especially
it follows that \( x_{1}^{(0)} = 0 \), which was previously, in section 3, mentioned as a consequence
of the Wigner-Eckhard theorem. This fact will simplify some calculations in section 5.
There are two other special cases that should be mentioned: One case is \( S = 0 \) with
\( N = 1 \) for integer \( s \). The corresponding eigenstate does not depend on \( y \), see [4], hence
the spectral curve coincides with its asymptotic line.
The other case is \( s = 1/2 \) and the vector \( \mathbf{J} = (J_1, J_2, J_3) \) running through a circle in
the plane \( J_1 + J_2 + J_3 = 3j = \text{const} \) with center \( (j, j, j) \). In this case the eigenvalues of
\( \mathbf{H} \) will be constant, i. e., we have the rare case of isospectrality for spin systems, see [16].
As a further application of the spectral symmetry we calculate the first five moments of $H_2(y)$. Let

$$M_n(y) \equiv \sum_{\nu=1}^{N} (e_{\nu}(y))^n, \quad n \in \mathbb{N}.$$  \hspace{1cm} (55)

Since the moments can be expressed through the coefficients of the characteristic polynomial, see, e. g., [17] (14.35), the $M_n(y)$ will be polynomials in the variable $y$ of maximal degree $n$ and their coefficients will exhibit the same reflection symmetry as the coefficients of $p(x, y)$, see (29). For example,

$$M_1(y) = M_{10}(1 + y).$$  \hspace{1cm} (56)

The coefficient $M_{10}$ can be calculated directly or by using

$$M_1(1 - y) = \sum_{\nu=1}^{N} (E_0 - e_{\nu}(y)) = N E_0 - M_1(y) = N E_0 - M_{10}(1 + y) = M_{10}(1 + 1 - y),$$  \hspace{1cm} (57)

hence

$$M_{10} = \frac{1}{3} N E_0.$$  \hspace{1cm} (58)

Similarly,

$$M_2(y) = M_{20}(1 + y^2) + M_{21}y,$$

$$M_2(1 - y) = (2M_{20} + M_{21})(1 - y) + M_{20}y^2$$

$$= \sum_{\nu=1}^{N} (E_0 - e_{\nu}(y))^2$$

$$= N E_0^2 - 2E_0 M_1(y) + M_2(y)$$

$$= N E_0^2 - 2E_0 M_{10}(1 + y) + M_{20}(1 + y^2) + M_{21}y,$$

hence

$$M_2(y) = M_{20}(1 + y^2) + (E_0 M_{10} - M_{20})y.$$  \hspace{1cm} (59)

Here and in the following equations we assume the $M_{n0}$ as given since they can be expressed through the well-known eigenvalues of $H_2(0)$. By the same method we obtain

$$M_3(y) = M_{30}(1 + y^3) + \frac{3}{2}(E_0 M_{20} - M_{30})y(1 + y),$$  \hspace{1cm} (60)

$$M_4(y) = M_{40}(1 + y^4) + 2(E_0 M_{30} - M_{40})y(1 + y^2) + \left(-E_0^3 M_{10} + 6E_0^2 M_{20} - 8E_0 M_{30} + 3M_{40}\right)y^2,$$

and

$$M_5(y) = M_{50}(1 + y^5) + \frac{5}{2}(E_0 M_{40} - M_{50})y^2(1 + y^2) + \left(-E_0^4 M_{10} + 5E_0^3 M_{20} - 5E_0^2 M_{30} + M_{50}\right)y^3.$$  \hspace{1cm} (61)
For $n \geq 6$ this method fails to give $M_{n2}$ in terms of $M_{k0}$, $k \leq n$ and simple formulas like those above seem to be no longer available. Note, however, that the first five moments have been derived solely by utilizing $y = 0$ results and spectral symmetry. The knowledge about the explicit matrix representation of $H_2(y)$ has not yet been used. The first five moments of $H_2(y)$ depend on $S$ only implicitly through $E_0$ and $M_{n0}$. If these arguments are explicitly evaluated it is possible to calculate the corresponding total moments of the 2-chain by summation over $S$ and to compare the results with those in the literature, see [8] and [9]. This has been done successfully but will not be detailed here. Rather, in the next two sections, we will use an alternate method to calculate moments which leads to results not already reported in the literature.

5. Moments of the two-chain

The total $k$-th moment of $H_2(y)$ will be defined as

$$t_k(y) \equiv \frac{\text{Tr} \, H_2(y)^k}{(2s+1)^3}.$$  

(71)

Obviously, it is a polynomial in the variable $y$ with the same reflection symmetry as the local moments $M_k(y)$ defined in (55). The knowledge of the first terms in the perturbation series of the eigenvalues $e_{g,S}(y)$ of $H_2(y)$ will give the corresponding coefficients of $t_k(y)$ via summation over the quantum numbers $S$ and $g \equiv S_{12}$. Moreover, we will see that there is a close connection between these coefficients and the terms of the so-called graph expansion of $t_k(y)$. In the case of the two-chain $H_2(y)$ this connection is even a 1 : 1-correspondence.

Let

$$e_{g,S}(y) = x_0^{(g)} + x_1^{(g)} y + x_2^{(g)} y^2 + x_3^{(g)} y^3 + \ldots$$  

(72)

denote the perturbation series of the eigenvalues $e_{g,S}(y)$. Upon multinomial expansion we obtain

$$e_{g,S}(y)^k = x_0^{(g)k} + k x_0^{(g)(k-1)} x_1^{(g)} y + \left( \binom{k}{2} x_0^{(g)(k-2)} x_1^{(g)2} + k x_0^{(g)(k-1)} x_1^{(g)} \right) y^2 + \left( \binom{k}{3} x_0^{(g)(k-3)} x_1^{(g)3} + k(k-1) x_0^{(g)(k-2)} x_1^{(g)} x_2^{(g)} + k x_0^{(g)(k-1)} x_3^{(g)} \right) y^3 + \ldots$$  

(73)

Into this series we insert the known equations (50) and (51):

$$x_0^{(g)} = \frac{1}{2} (g(g+1) - 2s(s+1)),$$  

(74)

$$x_1^{(g)} = \frac{1}{2} (E_0 - x_0^{(g)}) = \frac{1}{2} (S(S+1) - g(g+1) - s(s+1)),$$  

(75)

$$x_3^{(g)} = \frac{1}{2} x_2^{(g)}.$$  

(76)
For $x_2^{(g)}$ we will use the general result (114) for tridiagonal matrices and the explicit form (24) of the secondary diagonal entries of $H_2(y)$. This yields, after some calculations, for $g > 0$:

$$x_2^{(g)} = \frac{1}{16(2g + 1)} \left( \frac{(g - 2s - 1)(g + 2s + 1)(g - s - S - 1)(g + s - S)(g - s + S)(g + s + S + 1)}{g(2g - 1)} - \frac{(g - 2s)(g + 2s + 2)(g - s - S)(g + s - S + 1)(g - s + S + 1)(g + s + S + 2)}{(g + 1)(2g + 3)} \right) \right)$$

$$= -\frac{3}{32}x_0^{(g)} + \frac{E_0}{16} + \frac{3}{256}(1 + 8s(s + 1)) + \frac{A_1}{g(g + 1)} - \frac{4A_2}{4g(g + 1) - 3},$$

where

$$A_1 = \frac{1}{16}(1 + 2s)^2(s - S)^2(1 + s + S)^2,$$

and

$$A_2 = \frac{1}{1024}(1 + 4s)(3 + 4s)(1 + 2s - 2S)(1 + 2s - 2S)(1 + 2s + 2S)(3 + 2s + 2S).$$

For $g = 0$ the first term in the brackets in (77) or, equivalently, the term $\frac{A_1}{g(g+1)}$ in (78) has to be set to 0. These results have to be inserted into

$$t_k(y) = \sum_{g=0}^{s+g} \sum_{S=|s-g|}^{2s} (2S + 1)e_{g,S}(y)^k.$$  

The double sum in (81) can be evaluated up to third order in $y$ by adopting the following strategy: First, expand all terms depending on $S(S + 1)$ into polynomials in $E_0$ and express the resulting sums of the form

$$\sum_{S=|s-g|}^{s+g} (2S + 1)E_0^m,$$

up to a factor $2g + 1$, as polynomials in $x_0^{(g)}$. This makes it possible to write the final sums $\sum_{g=0}^{2s} x_0^{(g)}$ in terms of the known moments $t_\lambda(0)$ (dimer polynomials).

Before writing down the final results of the above-sketched calculations we will shortly recapitulate the general theory of moments of Heisenberg Hamiltonians $H$, following [8] and [9], that is also the starting point of the high temperature expansion of the corresponding thermodynamic quantities. The $k$-th moment of $H$ has the following “graph expansion”

$$t_k = \frac{\text{Tr } H_k}{(2s + 1)^N} = \sum_{\mathcal{G}} \mathcal{P}p(\mathcal{G}).$$

The latter summation extends over a finite set of (multi-)graphs with $k$ edges (bonds). $\mathcal{G}$ denotes a certain polynomial of the coupling constants of $H$ that is obtained via
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summing over all embeddings of $G$ into the spin system, see [9]. $p(G)$ denotes certain other “universal” polynomials of maximal degree $k$ in the variable $r \equiv s(s + 1)$ that are independent of the coupling constants of the spin system under consideration.

In our case of the two-chain $H_2(y)$ the graphs needed in the graph expansion of $t_k(y)$ are either of the form $\vec{k}$ (two vertices and $k$ edges) or $\vec{kk}'$ (three vertices and a total number of $k = k_1 + k_2$ edges). The cases $G = \vec{k}$ or $G = 1/k$ can be disregarded since then $p(G) = 0$, see [8]. Moreover, we have $\overline{G} = 1 + y^k$ in the first case of $G = \vec{k}$ and $\overline{G} = y^{k_1} + y^{k_2}$ in the second case of $G = \vec{kk}'$. Hence the coefficients of the $y$-expansion of $t_k(y)$ are exactly the universal polynomials $p(G)$ of the relevant graphs and will be written as $p(k)$ in the first case and $p(k_1, k - k_1)$, $k_1 \leq k - k_1$, in the second one. The dimer polynomials $p(k)$ are the well-known moments of the one-chain and will be given in explicit form in the Appendix B. The polynomials $p(k_1, k - k_1)$, $k_1 = 2, 3, 4, 5$ can be expressed in terms of dimer polynomials $p(k')$ as the result of the above-sketched calculations. The results are

$$p(2, k - 2) = \frac{k}{12} \left[ (-r)^k + (k - 1) r^2 p(k - 2) + k r p(k - 1) \right],$$

and

$$p(3, k - 3) = \frac{1}{144} \left[ -3k(2r + k - 1)(-r)^{k-1} + 6k^2 r p(k - 1) - 2r(6r + k + 1) \binom{k}{2} p(k - 2) - 6r^2 \binom{k}{3} p(k - 3) \right].$$

If $k_1 = 4, 5$ the expressions for $p(k_1, k - k_1)$ are more complex and will involve sums over $O(k)$ dimer polynomials. It will be economic to write the results in terms of the modified dimer functions

$$P(\ell) \equiv p(\ell) - \frac{(-r)^\ell}{4r + 1}, \quad \ell = 0, 1, 2, \ldots,$$

although the expressions for $p(k_1, k - k_1)$ will then no longer be manifestly polynomial. The final results are:

$$p(4, k - 4) =$$

$$kr \left( -32r(4r - 3)(80r + 21) \left( \frac{4}{3} - r \right)^{k-1} - 128(-r)^k(r(-30k + 80r - 9) - 72) \right)$$

$$+ \frac{5k^2r}{192} P(k - 1) + \frac{(k - 1)kr(16k^2(3r - 1) - 16k(3r + 4) - 3(284r + 57))}{23040} P(k - 2)$$

$$+ \frac{k(k^2 - 3k + 2) r (k (12r^2 - 5r + 2) - 36r^2 + 39r + 6)}{2880} P(k - 3)$$

$$+ \frac{(k - 3)(k - 2)(k - 1)k r^2(6r^2 - 3r + 2)}{2880} P(k - 4)$$

$$+ \frac{k}{960} \sum_{i=0}^{k-3} (i + 1) \left( \frac{1}{64} r(4r - 3)(16r + 3)^2 \left( \frac{3}{8} - r \right)^{-i+k-3} + (r - 2)(4r + 1)^2(-r)^{-i+k-2} \right) P(i),$$

(87)
and
\[ p(5, k - 5) = \]
\[
4kr^2(4r - 3) \left( \frac{3}{8} - r \right)^{k-2} \frac{(80r + 21)(4k + 8r - 7)}{46080(4r + 1)} \\
- 64k(-r)^k \frac{20kr^2 - 45kr - 15k + 160r^3 - 98r^2 - 39r + 15}{46080(4r + 1)} \\
+ \frac{7}{384} k^2rP(k - 1) + (k - 1)kr \frac{k^2(96r - 32) + k(-96r - 68) - 3(532r + 111)}{46080} P(k - 2) \\
- (k - 2)(k - 1)k \frac{2k(8r - 3)r(k(2r - 1) - 1) - 108r^3 + 372r^2 - 141r - 45}{11520(8r - 3)} P(k - 3) \\
- (k - 3)(k - 2)(k - 1)kr \frac{8kr^2 - 5kr + k + 24r^3 - 52r^2 + 34r + 6}{11520} P(k - 4) \\
- (k - 4)(k - 3)(k - 2)(k - 1)kr^2 \frac{4r^2 - 3r + 1}{11520} P(k - 5) \\
+ \frac{k}{3840} \sum_{i=0}^{k-3} (i + 1) \left( - \frac{3}{256} r(4r - 3)(16r + 3)^2(4k + 8r - 7) \left( \frac{3}{8} - r \right)^{-i+k-4} \\
- (4r + 1)^2(5k(r - 1) + r(6r - 17) + 5)(-r)^{-i+k-3} \right) P(i). \]
\[ (88) \]

The formulas (84), (85), (87) and (88) have been checked with the known results for \( k = 2, \ldots, 8 \) in [8] and unpublished results for \( k = 9, 10 \) that have been obtained by different methods.

6. Moments of the general triangle

The graph expansion of \( t_k = \frac{1}{(2s+1)^2} \text{Tr} H^k \) for the general triangle Hamiltonian (1) involves additional triangular graphs with \( k_1, k_2, k_3 \) bonds such that \( k_1 + k_2 + k_3 = k \). We will write the corresponding universal polynomials as \( p(k_1, k_2, k_3) \), \( k_1 \leq k_2 \leq k_3 \).

Following the ideas of the last section, a possible strategy to determine these polynomials would consist of expanding \( t_k(z) \) into an even power series w. r. t. \( z \), if the Hamiltonian (1) is written in the form
\[ H(z) = J_3 \mathbf{s}_1 \cdot \mathbf{s}_2 + J \left( \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s}_1 \cdot \mathbf{s}_3 \right) + (J_3 - J) z \left( \mathbf{s}_2 \cdot \mathbf{s}_3 - \mathbf{s}_1 \cdot \mathbf{s}_3 \right). \]
\[ (89) \]

In this paper we will take only the first step of this expansion and consider the integrable case \( z = 0 \). Even this case is sufficient to obtain explicit expressions for some classes of universal polynomials.

Consider the eigenvalues \( e_{g,s} \) of \( H(0) \) that, according to (50) and the remark following (25), can be written as
\[ e_{g,s} = J_3 x_0^{(g)} + 2J x_1^{(g)} \]
\[ = J_3 x_0^{(g)} + J (E_0 - x_0^{(g)}). \]
\[ (90) \]
\[ (91) \]
Hence
\[
e^{k}_{g,S} = \sum_{\ell=0}^{k} \binom{k}{\ell} J^\ell_{3} x^{(g)}_{0} J^{k-\ell} (E_{0} - x^{(g)}_{0})^{k-\ell}
\]
\[
= \sum_{\ell=0}^{k} \sum_{\mu=0}^{k-\ell} \binom{k}{\ell} \binom{k-\ell}{\mu} J^\ell_{3} J^{k-\ell} (E_{0}^{\mu} (-1)^{k-\ell-\mu} x^{(g)(k-\mu)}_{0}.
\]

The moments \(t_{k}(0)\) are obtained by summing (93) over \(S\) and \(g\). This leads to sums of the form \(\sum_{S=|s-g|}^{s+g} (2S+1) E_{0}^{\mu}\). It can be shown that these are of the form \((2s+1)(2g+1)\) times a polynomial in the variables \(r = s(s+1)\) and \(g(g+1)\). Hence we obtain the following expansion
\[
\sum_{S=|s-g|}^{s+g} (2S+1) E_{0}^{\mu} = (2s+1)(2g+1) \sum_{\lambda=0}^{\mu} P_{\mu,\lambda} x^{(g)}_{0}^{\lambda},
\]
which implicitly defines the polynomials \(P_{\mu,\lambda}(r)\). For our applications we will only use the following ones:

\[P_{nn} = 1, \quad n = 0, 1, 2, \ldots,\]
\[P_{10} = 0,\]
\[P_{20} = \frac{2}{3} r^{2}, \quad P_{21} = \frac{2}{3} r,\]
\[P_{30} = -\frac{1}{3} r^{2}, \quad P_{31} = \frac{1}{3} r(6r - 1), \quad P_{32} = 2r.\]

Note that the factor \((2s+1)(2g+1)\) is just the degeneracy of the eigenvalue \(x^{(g)}_{0}\) of \(g_{1} \cdot g_{2}\) in the total Hilbert space. Hence the sums of the form \(\sum_{g=0}^{2s} (2s+1)(2g+1) x^{(g)\lambda}_{0}\) are equal to \((2s+1)^{3} p(\lambda)\) and we obtain
\[
t_{k}(0) = \sum_{\ell=0}^{k} \sum_{\mu=0}^{k-\ell} \sum_{\lambda=0}^{\mu} \binom{k}{\ell} \binom{k-\ell}{\mu} J^\ell_{3} J^{k-\ell} (E_{0}^{\mu} (-1)^{k-\ell-\mu}) P_{\mu,\lambda} p(k + \lambda - \mu).
\]

We will only consider two terms of the expansion (99), namely those with \(\ell = k - 2\) and \(\ell = k - 3\). Let \([A(x)]_{x}\) denote the coefficient of \(x\) in the polynomial \(A(x)\). Then the first term is
\[
[t_{k}(0)]_{J^{k-2}J^{2}} = \binom{k}{2} \sum_{\mu=0}^{2} \binom{2}{\mu} (-1)^{2-\mu} P_{\mu,\lambda} p(k + \lambda - \mu)
\]
\[
= p(1, 1, k - 2) + 2p(2, k).
\]

After some calculations, using (84) and (85), we finally obtain
\[
p(1, 1, k - 2) = \frac{k}{6} \left[ (-r)^{k} + (k - 1)r^{2}p(k - 2) + (k - 2)p(k - 1) \right],
\]
for \(k = 3, 4, \ldots\).

For \(\ell = k - 3\) the analogous results read
\[
[t_{k}(0)]_{J^{k-3}J^{3}} = \binom{k}{3} \sum_{\mu=0}^{3} \binom{3}{\mu} (-1)^{3-\mu} P_{\mu,\lambda} p(k + \lambda - \mu)
\]
\[
= 2 \left( p(1, 2, k - 3) + 2p(3, k) \right).
\]
and

\[ p(1, 2, k-3) = \frac{k}{48} \left\{ \left( k+2r-1 \right) (-r)^{k-1} \\
+ r \left[ \left( k-1 \right) \left( (2-k)r p(k-3) + (-k+2r+3) p(k-2) \right) \\
- 2 k p(k-1) \right] \right\}, \quad \text{for } k = 4, 5, \ldots . \] (105)

The formulas (102) and (105) agree with the values of the polynomials published in [8] for \( k = 4, \ldots , 8 \) and with unpublished results for \( k = 9, 10 \) obtained by different methods. For an example see Appendix B.

It should be mentioned that the universal polynomials (84), (85), (87), (88), (102), and (105) do not only occur in the graph expansion of the moments of the two-chain or the general triangle but for arbitrary Heisenberg spin systems. As a rule, general formulas for these polynomials are rare. The only ones previously known to the author are, besides the explicit expressions for the dimer polynomials \( p(k) \), see Appendix B, those which result from certain rules given in [8]. For example, \( p(\mathcal{G}) = 0 \) if \( \mathcal{G} \) is a single-bond connection of two disjoint (possibly empty) graphs. Another rule in [8] refers to the case where \( \mathcal{G} \) contains two vertices connected by a single-bond chain of length \( n > 2 \). Let \( \tilde{\mathcal{G}} \) be the graph resulting by shortening the single-bond chain by one vertex. Then \( p(\mathcal{G}) = \ell r^{3} p(\tilde{\mathcal{G}}) \), where \( \ell \) is the total number of bonds of \( \mathcal{G} \). (Note that the authors of [8] use a definition of the graph polynomials \( p(\mathcal{G}) \) that differs from our definition by a multinomial factor). In particular, the last rule implies that the polynomial \( q_{n} \) of a single-bond polygon with \( n > 2 \) vertices will be \( q_{n}(r) = \frac{n!}{3^{n-1}} r^{n} \). Of course, this rule can also be applied to graphs that can ultimately be reduced to those for which (84), (85), (87), (88), (102), and (105) apply.

7. Summary and outlook

The general quantum spin triangle is not integrable in the strict sense that its eigenvalues and eigenvectors can be given in closed form, as, e.g., for the isosceles triangle. However, our results show that we are rather close to integrability. The reduced Hamiltonian has a triangular matrix representation the entries of which are explicitly given. Hence also the characteristic polynomial can be expressed either recursively or explicitly in terms of finite sums, see Appendix A. Similarly, the perturbation series for the eigenvalues could, in principle, be calculated up to any finite order, although with rapidly increasing complexity. Therefore we have contented ourselves with third order perturbation results. Comparing this situation with, e.g., the Bethe ansatz for the \( s = \frac{1}{2} \) chain with \( N \) spins shows that the difference is not huge. Also in the Bethe ansatz the eigenvalues have to be calculated as the numerical solutions of certain equations which become more and more complex for large \( N \). Perhaps the general triangle should also be regarded as “quantum integrable” in some weaker sense.

One virtue of integrability in whatever sense is the possibility to calculate certain limits,
e. g., the thermodynamic limit $N \to \infty$ in the case of the $s = \frac{1}{2}$ chain. In the present case a possible candidate is the classical limit $s \to \infty$, see [18] for the equilateral triangle case. This seems to be a sensible project for future work. Another aspect that has only be touched in the present paper is the closer investigation of physical properties of the general spin triangle, as, for instance, ground state diagrams or magnetization curves. As already pointed out in the Introduction, these properties could be of paradigmatic character in order to understand more complex systems.

**Acknowledgement**

To check the equations (84), (85), (87), (88), (102), and (105) I have used unpublished results for $k = 9, 10$ that have been obtained in collaboration with Andre Lohmann and Johannes Richter. I thank Johannes Richter and Jürgen Schnack for critical reading of the first version of the manuscript and Thomas Bröcker for encouraging discussions.

**Appendix A: Properties of tridiagonal matrices**

We will collect some properties of tridiagonal matrices relevant for this article without claiming originality. For sake of simplicity we assume $T$ to be an $N \times N$-symmetric real tridiagonal matrix with positive entries in the secondary diagonals. Let $T^{(n)}$ denote the submatrix of $T$ obtained by deleting its first $n$ rows and columns and $D_n \equiv \det T^{(n)}$. Then $D_0 = \det T$ satisfies the obvious recursion relation

$$D_0 = T_{11} D_1 - T_{12}^2 D_2 .$$

It follows that the explicit formula for $\det T$ does not contain $N!$ terms as for general $T$, but only $F_{N+1}$ terms, where $F_N$ denotes the $N$-th Fibonacci number. For example, if $N = 3$ we have $\det T = T_{11} T_{22} T_{33} - T_{11} T_{23}^2 - T_{12} T_{23}^2$, which consists of $F_4 = 3$ terms instead of $3! = 6$ ones. The number of terms still grows exponentially with $N$ but not super-exponential like $N!$. Consequently, the computer-algebraic calculation of the characteristic polynomial of $T$ is markedly faster than for a general $N \times N$-matrix.

We will write down the explicit formula for $\det T$ in order to substantiate our claim that the characteristic polynomial of the general triangle can be given in closed form. To this end let $(d_1, \ldots, d_N)$ denote the diagonal of $T$ and $(c_2, \ldots, c_N)$ its secondary diagonals. Further let $B(m, k)$ denote the set of $(0, 1)$-sequences of length $m + k$ containing exactly $m$ zeroes and $k$ ones. There are $\binom{m+k}{k}$ such sequences. For any $\sigma \in B(m, k)$ we define

$$n(i, \sigma) \equiv i + \sum_{j=1}^{i} \sigma(j), \quad i = 1, \ldots, m + k .$$

(107)
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Then

\[
\det T = \sum_{m,k \in \mathbb{N}} (-1)^k \sum_{\sigma \in B(m,k)} \prod_{i=1}^{m+k} \begin{cases} d_{n(i,\sigma)} : \sigma(i) = 0, \\ c_{n(i,\sigma)}^2 : \sigma(i) = 1. \end{cases}
\]  

(108)

Another simplification occurs for the perturbation series of, say, a diagonal matrix \(A\) with pairwise different eigenvalues perturbed by a symmetric tridiagonal matrix \(yB\). We will write the tridiagonal matrix \(A + yB\) in the form

\[
\begin{pmatrix}
* & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & y\sqrt{B_{n-2}} & 0 & 0 & 0 \\
0 & 0 & y\sqrt{B_{n-2}} & A_{n-1} + y\alpha_{n-1} & y\sqrt{B_{n-1}} & 0 & 0 \\
0 & 0 & 0 & y\sqrt{B_{n-1}} & A_n + y\alpha_n & y\sqrt{B_{n+1}} & 0 \\
0 & 0 & 0 & 0 & y\sqrt{B_{n+1}} & A_{n+1} + y\alpha_{n+1} & * \\
0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & *
\end{pmatrix},
\]

(109)

which is symmetric w. r. t. the \(n\)-th diagonal entry (note that there is no \(B_n\)). This symmetric form facilitates the notation in what follows. We introduce, for fixed \(n\), the abbreviations

\[
a_i \equiv A_i - A_n,
\]

(110)

\[
b_i \equiv \frac{B_i}{A_i - A_n}, \quad (i \neq n)
\]

(111)

\[
\beta_i \equiv \frac{\alpha_i - \alpha_n}{A_i - A_n}, \quad (i \neq n).
\]

(112)

Let

\[
x^{(n)} = A_n + \alpha_n y + \sum_{i=2}^{\infty} x_i^{(n)} y^i
\]

(113)

denote the perturbation series for the \(n\)-th eigenvalue \(x^{(n)}\) of \(A + yB\), where we have anticipated the elementary result for the first two terms. Then the next terms are of the form

\[
x_2^{(n)} = -b_{n-1} - b_{n+1} = -b_{n-1} + \text{RT},
\]

(114)

\[
x_3^{(n)} = b_{n-1}\beta_{n-1} + b_{n+1}\beta_{n+1} = b_{n-1}\beta_{n-1} + \text{RT},
\]

(115)

\[
x_4^{(n)} = -\frac{b_{n-2} b_{n-1}}{a_{n-1}} + \frac{b_n^2}{a_{n-1}} + \frac{b_{n-1} b_{n+1}}{a_{n-1}} - b_{n-1} \beta_{n-1}^2 + \text{RT}.
\]

(116)

Here “RT” (reflected terms) stands for the same terms as before but with reflected indices \(n - i \leftrightarrow n + i\). The above formulas also hold for small \(n = 1, 2, 3, \ldots\) or large \(n = N, N - 1, N - 2, \ldots\) if the corresponding \(A_i\) and \(B_i\) are set to zero. It is remarkable that the perturbation has a “local” character in so far as \(x_{2k}^{(n)}\) and \(x_{2k+1}^{(n)}\) only depend on matrix elements with indices \(n - k, \ldots, n + k\). Therefore it is possible to obtain the
above results by computer-algebraic calculations with at most $5 \times 5$-matrices. We note that there are further simplifications for our case of $A + yB = s_1 \cdot s_2 + y s_2 \cdot s_3$, see (50)-(54), which do not hold in the case of general tridiagonal matrices.

A further property of tridiagonal matrices $T$ with non-vanishing entries in the secondary diagonals is that their eigenvalues are never degenerate. For the proof, let $\varphi$ be an eigenvector of $T$ with eigenvalue $x$, that is

$$A \varphi \equiv (T - xI) \varphi = 0.$$  

(117)

It follows that $A_{11} \varphi_1 + A_{12} \varphi_2 = 0$, or $\varphi_2 = -\frac{A_{11}}{A_{12}} \varphi_1$ since $A_{12} = T_{12} \neq 0$. Similarly, $A_{21} \varphi_1 + A_{22} \varphi_2 + A_{23} \varphi_3 = 0$ implies

$$\varphi_3 = \frac{1}{A_{12}A_{23}}(A_{11}A_{22} - A_{12}A_{21}) \varphi_1,$$  

(118)

since $A_{23} \neq 0$. By induction, one easily shows that

$$\varphi_n = \left(\prod_{i=1}^{n-1} A_{i,i+1}\right)^{-1} (-1)^{n-1} D^{(n-1)} \varphi_1, \quad n = 2, \ldots, N,$$  

(119)

where $D^{(n)}$ denotes the $n$-th principal minor of $A$, e.g., $D^{(1)} = A_{11}$, $D^{(2)} = A_{11}A_{22} - A_{12}A_{21}$, and so on. (119) implies that the eigenspace of $T$ corresponding to the eigenvalue $x$ will be one-dimensional, which completes the proof. As a by-product we have obtained an explicit formula for the eigenvector $\varphi$ corresponding to the eigenvalue $x$ of a tridiagonal matrix $T$.

Appendix B: Moments of the one-chain

Since the eigenvalues of the one-chain (or “spin dimer”) are $e_S = \frac{1}{2}(S(S + 1) - 2s(s + 1))$ with degeneracy $2S + 1$, its $k$-th moment is given by

$$p(k) = \frac{1}{(2s + 1)^2} \sum_{S=0}^{2s} (2S + 1) \left[\frac{1}{2}(S(S + 1) - 2s(s + 1))\right]^k.$$  

(120)

This sum can be conveniently evaluated by computer-algebra software but it is not an explicit formula for the polynomials $p(k)(r)$, $r \equiv s(s + 1)$ in the strict sense. However, such an explicit formula exists since (120) can be reduced to power sums, which in turn are expressible by means of the Bernoulli numbers $B_n$:

$$p(k)(r) = \frac{2^{-3k-2}}{k+1} \sum_{\nu=0}^{k} (-1)^\nu (8r + 1)^\nu \binom{k+1}{\nu} \sum_{\mu=1}^{k-\nu+1} (4r + 1)^{\mu-1} \sum_{j=2\mu}^{2(k-\nu+1)} 2^j B_j \binom{j}{2\mu} \binom{2(k-\nu+1)}{j}.$$  

(121)

For example,

$$p(2) = \frac{1}{3} r^2.$$  

(122)
The general spin triangle

\[ p(3) = -\frac{1}{6}r^2 \]
\[ p(4) = \frac{1}{15}r^2(2 - 2r + 3r^2) \tag{124} \]
\[ p(5) = -\frac{1}{6}r^2(2 - 2r + 2r^2) . \tag{125} \]

From these polynomials we can, for example, calculate \( p(1, 2, 3) \) by using (105):

\[ p(1, 2, 3) = \frac{1}{8} \left\{ -r^5(5 + 2r) + r \left[ 5(-4rp(3) + (-3 + 2r)p(4)) - 12p(5) \right] \right\} \tag{126} \]
\[ = \frac{1}{3}r^4(1 - 2r) , \tag{127} \]
in accordance with [8] and [9].

Let us write \( p_k(r) = \sum_{n=2}^{k} a_n^{(k)} r^n \). Then the first few leading coefficients of the polynomials \( p_k(r) \) assume the following form:

\[ a_k^{(k)} = \begin{cases} \frac{1}{k+1} & : k \text{ even}, \\ 0 & : k \text{ odd}, \end{cases} \tag{128} \]
\[ a_{k-1}^{(k)} = \begin{cases} \frac{k(2-k)}{12(k+1)} & : k \text{ even}, \\ \frac{1-k}{12} & : k \text{ odd}, \end{cases} \tag{129} \]
\[ a_{k-2}^{(k)} = \begin{cases} \frac{(k-2)^2k(7k+2)}{720(k+1)} & : k \text{ even}, \\ \frac{k-2}{720} & : k \text{ odd}, \end{cases} \tag{130} \]
\[ a_{k-3}^{(k)} = \begin{cases} \frac{(4-k)(k-2)k(62k^3-174k^2+9k+174)}{60480(k+1)^2} & : k \text{ even}, \\ \frac{60480(k+1)^2}{60480} & : k \text{ odd}. \end{cases} \tag{131} \]

Another representation of the polynomials \( p_k(r) \) can be obtained from the Euler-MacLaurin summation formula, see [19] 23.1.30. After some simplifications we arrive at the following expression involving only a single summation:

\begin{align*}
  p(k) &= \frac{1}{(2s+1)^2} \frac{1}{k+1} \\
  &\quad \sum_{m=1}^{k} 2^{m} \frac{B_{2m} \left( \frac{k+1}{m} \right)}{m} \left( s^2 \right)^{k-m+1} F\left( -m, -k + m - 1; \frac{1}{2}; \frac{16s^2 + 8s + 1}{8s^2} \right) \\
  &\quad - \left( -s(s+1) \right)^{k-m+1} F\left( -m, -k + m - 1; \frac{1}{2}; -\frac{1}{8s(s+1)} \right) \\
  &\quad + \frac{1}{2} \left( s^{2k} \left( 4ks + k + (2s + 1)^2 \right) + (-s(s+1))^{k} \left( k + (2s + 1)^2 \right) \right) . \tag{132} \end{align*}

Note that the hypergeometric function \( F(a, b; c; z) \) reduces to a polynomial in \( z \) if \( a \) or \( b \) are negative integers, see [19] 15.4.1. Hence (132) is a priori a rational function of \( s \). Only after the summation it turns out that the factor \((2s+1)^2\) cancels and the remaining sum is a polynomial in \( r = s(s+1) \).

The \( p(k) \) can be viewed as polynomials in the complex variable \( r \). Interestingly, the
Figure 5. The sequence of complex zeroes of the dimer polynomials $p(k), k = 2, \ldots, 50$, where different colors correspond to different values of $k$. Obviously, besides the trivial value of $r = 0$, there are accumulation points for the physical values $r = s(s+1), s = 1/2, 1, 3/2, 2, 5/2, 3$.

physical values of $r = s(s+1), s = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ can be recovered as accumulation points of the sequence of complex zeroes of all $p(k), k = 2, 3, 4, \ldots$. This is not rigorously proven but demonstrated by numerical evidence, see figure 5.

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