BENIGN OVERFITTING IN TIME SERIES LINEAR MODEL WITH OVER-PARAMETERIZATION

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ABSTRACT. The success of large-scale models in recent years has increased the importance of statistical models with numerous parameters. Several studies have analyzed over-parameterized linear models with high-dimensional data that may not be sparse; however, existing results depend on the independent setting of samples. In this study, we analyze a linear regression model with dependent time series data under over-parameterization settings. We consider an estimator via interpolation and developed a theory for the excess risk of the estimator. Then, we derive bounds of risks by the estimator for the cases where the temporal correlation of each coordinate of dependent data is homogeneous and heterogeneous, respectively. The derived bounds reveal that a temporal covariance of the data plays a key role; its strength affects the bias of the risk, and its nondegeneracy affects the variance of the risk. Moreover, for the heterogeneous correlation case, we show that the convergence rate of risks with short-memory processes is identical to that of cases with independent data, and the risk can converge to zero even with long-memory processes. Our theory can be extended to infinite-dimensional data in a unified manner. We also present several examples of specific dependent processes that can be applied to our setting.

1. Introduction

In this study, we analyze the stochastic regression problem with an over-parameterized setting. Let us suppose \( n \) covariates, \( x_1, ..., x_n \in \mathbb{H} \), with Hilbert space \( \mathbb{H} \) and \( n \) responses, \( y_1, ..., y_n \in \mathbb{R} \), observed from the following linear model with an unknown true parameter, \( \beta^* \in \mathbb{H} \):

\[
y_t = \langle \beta^*, x_t \rangle + \epsilon_t, \ t = 1, ..., n.
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product for \( \mathbb{H} \). Here, covariates \( \{x_t : t = 1, ..., n\} \) and independent noise \( \{\epsilon_t : t = 1, ..., n\} \) are centered stationary Gaussian processes independent of each other. It should be noted that we allow dependence within \( x_t \) and \( \epsilon_t \). We focus

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on the over-parameterized case, wherein the dimension of \( H \) is significantly larger than \( n \) (sometimes infinite). In this study, we derive sufficient conditions for a risk of an estimator for a model to converge to zero as \( n \to \infty \), i.e., benign overfitting, under several settings for the dependence of \( \{x_t : t = 1, ..., n\} \).

The statistics for high-dimensional and large-scale data analyses have received significant attention over the decades. The most representative approach involves the utilization of sparsity via \( \ell_1 \)-norm regularization and its variants (Candes and Tao, 2007; Van de Geer, 2008; Bühlmann and Van De Geer, 2011; Hastie et al., 2019), which is effective when a signal to be estimated has many zero elements. Another study investigated the high-dimensional limit of risks of estimators (Dobriban and Wager, 2018; Belkin et al., 2019; Hastie et al., 2022; Bartlett et al., 2020) and revealed the risk limit when \( n \) data instances and \( p \) parameters diverged infinitely, while their ratio \( p/n \) converged to a positive constant value. Interpolators are typical estimators that perfectly fit the observed data under the \( p \gg n \) setting; moreover, it has been demonstrated that the risk or variance of interpolators converges to zero (Liang and Rakhlin, 2020; Hastie et al., 2022; Ba et al., 2019). It should be noted that these studies did not use the sparsity of data or signals; however, they are compatible with recent large-scale data analyses.

We focus on the risk of interpolators for high-dimensional data with temporal dependence but no sparsity. Although various have been conducted on high-dimensional dependent data with sparsity (Wang et al., 2007; Basu and Michailidis, 2015; Han et al., 2015; Wong et al., 2020), the analysis of interpolators without sparsity is still an emerging problem (Daskalakis et al., 2019; Kandiros et al., 2021). This is because the fundamental tools for studying interpolators, obtained from the random matrix theory (Bai and Silverstein, 2010; Vershynin, 2018), rely on the independence assumption of data.

In this study, we investigate the risk of interpolators in stochastic linear regression (1) with high-dimensional dependent processes. Specifically, we derive non-asymptotic upper and lower bounds on the prediction risk (9) of the interpolation estimator (7), for two situations where the temporal correlation structure of the dependent data is homogeneous or heterogeneous. In describing the bounds with dependent data, we specify several notations related to the covariance structure of the data in their time direction. Furthermore, we demonstrate that the bounds converge to zero as \( n \to \infty \) under tail assumptions on eigenvalues of the covariance matrix of the data. These results are valid when the data dimension was significantly large with a condition on autocovariance, and it is even possible for \( p = \infty \).

We demonstrate the validity of the results using the following example: a stationary Gaussian process, \( \{x_t\}_{t=1}^\infty \), is (i) an autoregressive moving-average (ARMA) process with homogeneous correlation structure; (ii) an ARMA process with heterogeneous correlation structure; and (ii) an autoregressive fractionally integrated moving-average (ARFIMA) process.
We have identified several effects of the dependent data on a risk of over-parameterized models, based on our derived bounds. The main contributions of our theory are summarized below. (i) We identify an effect of the dependence of data on bias and variance of risks. If dependent data has a homogeneous correlation, the bias is unaffected by the dependence, but the variance is increased by a degree of degeneracy of a temporal covariance of the data. The result on the variance is remarkable because the similarity of temporal structures of covariates and noises is not important in high-dimensional time series analysis in general (Basu and Michailidis 2015; Wong et al. 2020). We also derive the upper bounds for the bias and the variance under a heterogeneous correlation, then show that the bias is also affected by the strength of the temporal covariance. Especially, for the dependent data that has long-memory, the risk increase as its convergence rate slows, while short-term memory does not increase as the rate is slowed. (ii) Our results can handle various correlation structures and short/long-memory processes in a unified manner and remain valid without sparsity. Treating long-memory processes in high-dimensional time series studies is a challenging task; however, our non-asymptotic analysis facilitates it. Therefore, we find that the convergence rate of risk can remain the same as that in independent cases even under temporal dependence, and the risk can be convergent even under long-term dependence.

On the technical side, our core results depend on the following two contributions. First, we derive inequalities on empirical covariance matrices for the homogeneous/heterogeneous correlation cases. Especially, in the heterogeneous case, the empirical eigenvalues differ for each instance of data. We resolve the heterogeneity by deriving the inequalities for individual eigenvalues. Second, we derive the moment inequality for empirical covariance generated from possibly infinite-dimensional Gaussian processes under dependence. This is an extension of moment bounds on covariance matrices by Han and Li (2020) based on Sudakov’s inequality. The developed inequality can be used to perform systematic investigations on dependent data, while the work by Bartlett et al. (2020) on independent data used matrix concentration inequalities based on the generic chaining technique, which is not efficient for dependent data.

1.1. Related Studies. We discuss the following two types of related studies in this paper. 

High-dimensional time series: Time series analysis of high-dimensional data has been performed for the stochastic regression and vector autoregression (VAR) estimation problems with several sparsity-induced regularizations. These studies used regularization under the sparsity assumption. Wang et al. (2007) proposed a flexible parameter tuning method for the lasso, which can be applied to VAR. Alquier and Doukhan (2011) studied a general scheme of \( \ell_1 \)-norm regularization for dependent noise with a wide class of loss functions. Song and Bickel (2011) focused on a large-scale VAR and its sparse estimation. Basu and Michailidis (2015) developed a spectrum-based characterization of dependent data and utilized
a restricted eigenvalue condition for the \( \ell_1 \)-norm regularization of dependent data. Kock and Callot (2015) studied lasso for time series data and derived an oracle inequality and model selection consistency. Han et al. (2015) used the Danzig selector for high-dimensional VAR and studied its efficiency by the spectral norm of a transition matrix. Wu and Wu (2016) developed a widely applicable error analysis method on time series lasso with various tail probabilities of data. Guo et al. (2016) focused on lasso for VAR with banded transition matrices. Davis et al. (2016) proposed a two-step algorithm for VAR with sparse coefficients. Medeiros and Mendes (2016) studied time series lasso with non-Gaussian and heteroskedastic covariates. Masini et al. (2019) derived an oracle inequality for sparse VAR with fat probability tail and various dependent settings. Wong et al. (2020) studied time series lasso with general tail probability and dependence by relaxing a restriction in Basu and Michailidis (2015). These studies imposed sparsity on data; thus, they can handle non-sparse high-dimensional data in our setting. Bunea et al. (2022) studied a factor model with over-parameterization by the framework of benign overfitting.

**Large-scale model with dependent data:** In contrast to sparse-based studies, not many studies have analyzed dependent data in the context of large-scale models. The learning theory involves advancements in the predictive error analysis of dependent data using uniform convergence and complexity, wherein sparsity is not used, and it can be easily (relatively) applied to large-scale models. Yu (1994) demonstrated uniform convergence in empirical processes with mixing observations and derived its convergence rate. Mohri and Rostamizadeh (2008) developed the Rademacher complexity to study the predictive errors in stationary mixing processes. Berti et al. (2009) derived uniform convergence in empirical processes with an exchangeable condition. Mohri and Rostamizadeh (2010) proved the data-dependent Rademacher complexity for several types of mixing processes. Agarwal and Duchi (2012) developed an upper bound for generalization errors with general loss functions and mixing processes. Kuznetsov and Mohri (2015) derived an upper bound for predictive errors in stochastic processes without stationary or mixing assumptions. Dagan et al. (2019) derived a generalization error bound with dependent data that satisfied the Dobrushin condition. Daskalakis et al. (2019) and Kandiros et al. (2021) applied the Ising model, studied several linear models with dependent data, and derived novel estimation error bounds. These studies yielded general results, but they did not assume situations where the number of parameters diverges to infinity, such as over-parameterization.

**1.2. Organization.** Section 2 presents the stochastic regression problem and the definition of estimators with interpolation. Section 3 discusses the homogeneous autocorrelation case, followed by the assumptions and main results, including the risk bounds and convergence rates. Section 6 presents an overview of the proofs of the main result as well as the moment inequality as our technical contribution. Section 8 provides several examples of the Gaussian
and Σ negative of Λ and an orthogonal matrix {X}
w\in \mathbb{R}^d \otimes \mathbb{R}^d, A^{(i_1,i_2)} \in \mathbb{R} denotes an element at the i_1-th row and i_2-th column of A for i_1, i_2 = 1, ..., d. ||A|| := sup_{z \in \mathbb{R}^d:||z||=1} ||Tz|| is an operator norm of A, and \mu_i(A) is the i-th largest eigenvalue of A, and tr(A) = \sum_i \mu_i(A) denotes a trace of A. For positive sequences \{a_n\}_n and \{b_n\}_n, a_n \lesssim b_n and a_n = O(b_n) indicate that there exists C > 0 such that a_n \leq Cb_n for any n \geq \tilde{n} with some \tilde{n} \in \mathbb{N}. a_n \ll b_n and a_n = o(b_n) denote that for any C > 0, a_n \leq Cb_n holds true for any n \geq \tilde{n} with some \tilde{n} \in \mathbb{N}. a_n \asymp b_n denotes that both a_n \lesssim b_n and b_n \lesssim a_n hold true. For a \in \mathbb{R}, \log^a n denotes (\log n)^a. For an event E, 1\{E\} is an indicator function such that 1\{E\} = 1 if E is true and 1\{E\} = 0 otherwise. For arbitrary random variables r_1, ..., r_i, let E_{r_1, ..., r_i} indicates the conditional expectation given all random variables other than r_1, ..., r_i. \mathbb{N}_0 indicates the union of \mathbb{N} and {0}. For a positive semi-definite matrix with a form \Sigma = U\Lambda U^\top with a diagonal matrix \Lambda and an orthogonal matrix U, we define \Sigma_{0:k} := [e_1, ..., e_k]\text{diag}\{\lambda_1, ..., \lambda_k\}[e_1, ..., e_k]^\top and \Sigma_{k:\infty} := [e_{k+1}, ..., e_p]\text{diag}\{\lambda_{k+1}, ..., \lambda_p\}[e_{k+1}, ..., e_p]^\top where e_i are the column vectors of U = [e_1, ..., e_p]. For any positive semi-definite matrix A \in \mathbb{R}^d \otimes \mathbb{R}^d with A = VDV^\top where V is an orthogonal matrix and \Lambda = \text{diag}\{\delta_1, ..., \delta_d\} with non-increasing and non-negative \delta_i \geq 0, we define A_{0:k} := \sum_{i=1}^k \delta_i v_i v_i^\top for k = 1, ..., d, A_{k:\infty} := \sum_{i=k+1}^d \delta_i v_i v_i^\top for all k = 0, ..., d - 1, and A_{0:0} = A_{d:\infty} = O \in \mathbb{R}^d \otimes \mathbb{R}^d, where v_i is the i-th column vector of V.

2. Setting and Assumption

2.1. Gaussian Process with Spatio-Temporal Covariance. We firstly define a p-dimensional covariate process \{x_i\}_{i=1}^n and the corresponding design matrix \(X = [x_1, ..., x_n]^\top \in \mathbb{R}^n \otimes \mathbb{R}^p\). We define the covariate process by multiplying matrices of temporal and spatial covariance to a random matrix with i.i.d. coordinates, which is a generalization to separable models in spatial/spatio-temporal statistics (for details, see \cite{cressie1993, kyriakidis1999, chen2021}).

We consider positive definite matrices \(\Xi_{i,n} \in \mathbb{R}^n \otimes \mathbb{R}^n, i = 1, ..., p\) and \(\Sigma \in \mathbb{R}^p \otimes \mathbb{R}^p\), that represent the temporal and the spatial covariance, respectively. We also introduce \(Z = [z_1, ..., z_p], z_i \sim N(0, I_n)\) as a matrix with i.i.d. standard Gaussian entries. Let \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0\) be eigenvalues of \(\Sigma\), \(\Lambda = \text{diag}\{\lambda_1, ..., \lambda_p\}\) be a diagonal matrix, and 

\(U \in \mathbb{R}^p \otimes \mathbb{R}^p\) is an orthogonal matrix such that \(\Sigma = U\Lambda U^\top\). Then, we set the design matrix \(X = [x_1, ..., x_n]^\top \in \mathbb{R}^n \otimes \mathbb{R}^p\) for covariates as

\[
X = \begin{bmatrix} \Xi_{1,n}^{1/2} z_1, \ldots, \Xi_{p,n}^{1/2} z_p \end{bmatrix} \Lambda^{1/2} U^\top.
\]
If there exists $\Xi_n \in \mathbb{R}^n \otimes \mathbb{R}^n$ with $\Xi_{i,n} = \Xi_n$ for all $i = 1, \ldots, n$, which will be referred to a homo-correlation setting, the definition (2) is simplified to

$$X = \Xi_n^{1/2} Z A^{1/2} U^\top. \quad (3)$$

The covariate process $\{x_t\}_{t=1}^n$ has the spectral decomposition, that is, $\{x_t^\top u_i\}_t$ and $\{x_t^\top u_j\}_t$ with the column vectors $\{u_i\}$ of $U$ and $i \neq j$ are independent of each other. We set this independence to yield the concentration inequalities for some random matrices, which play important roles in our analysis (see Appendix C). This property appears in some physical models with stochastic partial differentiable equations; see Lototsky (2009); Lototsky and Rozovskii (2017).

We reveal a covariance matrix of the covariate process $\{x_t\}_{t=1}^n$. The process has simple cross-covariance matrices as follows: for all $t_1, t_2 \in \{1, 2, \ldots, n\}$, the covariance matrix is written as

$$\mathbb{E}[x_{t_1}x_{t_2}^\top] = U \text{diag}\{\Xi_{1,n}(t_1, t_2) \lambda_1, \ldots, \Xi_{p,n}(t_1, t_2) \lambda_p\} U^\top. \quad (4)$$

If all $\Xi_{i,n}$ are Toeplitz matrices, i.e. their $(t_1, t_2)$-th elements dependent only on $|t_1 - t_2|$, the covariance (4) is further simplified. Note that the covariance matrix of $x_t$, i.e., $\mathbb{E}[x_t x_t^\top]$, is not necessarily equal to $\Sigma$. This fact does not matter in our study.

2.1.1. Covariance Settings. We consider two different settings for the temporal covariance matrices $\Xi_{i,n}, i = 1, \ldots, p$. Rigorously, we consider two separate situations in which the temporal covariance $\Xi_{i,n}$ varies or does not vary for each coordinate of the covariate. To describe the situations, we define the following processes:

**Definition 1** (homo/hetero-correlation). The process $X = [x_1, \ldots, x_n]^\top$ is called

(1) a **homo-correlated process**: there exists a matrix $\Xi_n$ such that $\Xi_n = \Xi_{i,n}$ for all $i = 1, \ldots, p$.

(2) a **hetero-correlated process**: we cannot assume the existence of $\Xi_n$ as above.

The difference between these two settings has a significant impact on approaches and results of our analysis, as will be shown later. We present auto-regressive moving-average (ARMA) processes as a simple example of homo/hetero-correlated processes. Let $\{w_t\}_{t \in \mathbb{N}}$ be a sequence of independent $p$-dimensional Gaussian vectors whose mean is zero and covariance matrix is $Q$ which is a trace class and $\text{rank}(Q) > n$.

**Example 1** (homo-correlated ARMA process). A homo-correlated ARMA process, $x_t$, is defined as

$$x_t = \sum_{i=1}^{\ell_1} (a_i I_p) x_{t-i} + w_t + \sum_{i=1}^{\ell_2} (b_i I_p) w_{t-i}, \quad (5)$$
where \( \ell_1, \ell_2 \in \mathbb{N} \) is a number of lag, and \( a_i, b_i \in \mathbb{R} \) is a coefficient. Let us assume that the characteristic polynomial of the autoregressive (AR) part, \( \varpi_1(z) := 1 - \sum_{i=1}^{\ell_1} a_i z^i \), for \( z \in \mathbb{C} \) does not have roots for any \( z \) with \( |z| \leq 1 \), nor common roots with the characteristic polynomial for the moving-average (MA) part, \( \varpi_2(z) := 1 + \sum_{i=1}^{\ell_2} b_i z^i \) in \( \mathbb{C} \). There exists a causal MA representation \( x_t = \sum_{j=0}^{\ell_2} \phi_j w_{t-j} \) with a sequence \( \{ \phi_j \in \mathbb{R} \} \), which is square-summable and satisfies \( (1 + \sum_i b_i z^i) / (1 - \sum_i a_i z^i) = \sum_{j=0}^{\infty} \phi_j z^j \) (e.g., Brockwell and Davis [1991] Theorem 3.1.1). We can see that \( x_t \) is a homo-correlated process such that \( \Xi^{(t, t+h)} = \sum_{j=0}^{\infty} \phi_j \phi_{j+|h|} \) for all \( t = 1, \ldots, n \) and \( h \) with \( t + h \in \{1, \ldots, n\} \) and \( \Sigma = Q \). If those assumptions are not satisfied, stationarity may not hold; see Section 3.1 of Brockwell and Davis [1991].

**Example 2** (hetero-correlated ARMA process). We present a hetero-correlated ARMA process:

\[
x_t = \sum_{j=1}^{\ell_1} \rho_j x_{t-j} + w_t + \sum_{j=1}^{\ell_2} \varphi_j w_{t-j},
\]

where \( \rho_j, \varphi_j \in \mathbb{R}^p \otimes \mathbb{R}^p \) are coefficient matrices with spectral decompositions with an orthogonal basis \( \{ e_k \} \) such that \( \rho_j = \sum_{k=1}^{\ell_1} \rho_{j,k} e_k e_k^\top \), and \( \varphi_j = \sum_{k=1}^{\ell_2} \varphi_{j,k} e_k e_k^\top \). We also assume that \( Q = \sum_{k=1}^{p} q_k e_k e_k^\top \) and represent \( e_k w_t = \sqrt{q_k} z_{t,k} \) with i.i.d. \( z_{t,k} \sim N(0, 1) \). Here, we define \( \varpi_1,k(z) = 1 - \sum_{j=1}^{\ell_1} \rho_{j,k} z^j \) and \( \varpi_2,k(z) = 1 + \sum_{j=1}^{\ell_2} \varphi_{j,k} z^j \) as characteristic polynomials of the AR and MA parts, respectively, and set that \( \varpi_1,k(z) \neq 0 \) for all \( z \) with \( |z| \leq 1 \) and they do not have roots on the unit circle. There exist causal MA representations \( e_k x_t = \sum_{j=0}^{\infty} \phi_{j,k} e_k e_k^\top w_{t-j} \) with \( \{ \phi_{j,k} \in \mathbb{R} \} \) such that \( \varpi_{2,k}(z)/\varpi_{1,k}(z) = \sum_{j=0}^{\infty} \phi_{j,k} z^j \) (Brockwell and Davis [1991] Theorem 3.1.1). It holds that \( \Xi^{(t, t+h)}_{k,n} = \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+|h|,k} / \sum_{j=0}^{\infty} \phi_{j,k}^2 \) and \( \Sigma = \sum_{k=1}^{p} q_k (\sum_{j=0}^{\infty} \phi_{j,k}^2) e_k e_k^\top \).

**Remark 1.** Note that the representation of \( \Xi_{k,n} \) and \( \Sigma \) are not unique. We see that \( \Xi_{k,n}^{\text{new}} := c \Xi_{k,n} \) and \( \Sigma^{\text{new}} := c^{-1} \Sigma \) with a scaling factor \( c > 0 \) give the same covariance structure for \( x_t \) as \( \Xi_{k,n} \) and \( \Sigma \). The representation of \( \Xi_{k,n} \) and \( \Sigma \) in Example 2 uses this fact to scale each coordinate process so that all the diagonal elements of \( \Xi_{k,n} \) equal 1, which is assumed in the analysis for hetero-correlated processes. One of the reasons to introduce the MA representation with \( \{ \phi_{j,k} \} \) is to let these representations in hetero-correlated cases be explicit.

2.2. Stochastic Regression Problem and Interpolation Estimator. We consider the stochastic regression problem with the covariate process \( \{ x_t \}_{t=1}^n \) defined above. Suppose that we have \( n \) observations, \( (x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \mathbb{R} \) with dimension \( p \), which follow the stochastic linear regression model (1) for \( t = 1, \ldots, n \) with an unknown true parameter, \( \beta^* \in \mathbb{R}^p \). \( \{ \varepsilon_t : t = 1, \ldots, n \} \) is a stationary Gaussian process such that \( \mathcal{E} := (\varepsilon_1, \ldots, \varepsilon_n)^\top \sim N(0, \Upsilon_n) \), where \( \Upsilon_n \in \mathbb{R}^n \otimes \mathbb{R}^n \) is a positive definite matrix denoting the autocovariance of
Note that $x_t$ and $\varepsilon_t$ are independent of each other. We define $Y = (y_1, \ldots, y_n) \top \in \mathbb{R}^n$ as a design vector.

We consider an interpolation estimator with the minimum norm:

$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \| \beta \|_2 \text{ s.t. } \sum_{t=1}^n (y_t - \langle \beta, x_t \rangle)^2 = 0,$$

with norm $\| \cdot \|$ for $\mathbb{R}^p$. A solution satisfying this constraint in (7) is guaranteed to exist if $\{x_t\}_{t=1}^n$ spans an $n$-dimensional linear space, i.e., if $\mathbb{R}^p$ has a larger dimensionality than $n$ (for example, $\mathbb{R}^p = \mathbb{R}^p$ and $p \geq n$), then multiple solutions satisfy the linear equation $Y = \langle X, \beta \rangle$. The interpolation estimator (7) can be rewritten as:

$$\hat{\beta} = (X^\top X)^+ X^\top Y = X^\top (XX^\top)^{-1} Y,$$

where $^+$ denotes the pseudo-inverse of matrices.

We study an excess prediction risk of the interpolation estimator $\hat{\beta}$ as

$$R(\hat{\beta}) := \mathbb{E}^*[\langle y^* - \langle \hat{\beta}, x^* \rangle, y^* - \langle \hat{\beta}, x^* \rangle \rangle]$$

where $(x^*, y^*)$ is an independent random element such that $x^* \sim N(0, \Sigma)$, $\varepsilon^* \sim N(0, \sigma_\varepsilon^2)$ with arbitrary $\sigma_\varepsilon > 0$, and $y^* = (x^*)^\top \beta^* + \varepsilon^*$. The setting of $(x^*, y^*)$ can be justified as an approximation of $(x_T, y_T)$ with large $T$ when $x_t$ is mixing, or $x_t$ and $x_{t+h}$ with large $h$ are asymptotically independent.

**Remark 2** (motivation to interpolation). We study the interpolation estimator, because it has been shown to have interesting properties in the setting of over-parameterization with the independent data setting. For example, Bartlett et al. (2020) and Liang and Rakhlin (2020) showed that the interpolation estimator reduce its risk to zero in their setting on models and data; Hastie et al. (2022) found a phase transition of the risk of interpolation estimator, called the double descent. Interpolation estimators have attractive properties under independent data, and the results motivate us to study the interpolation under dependent data.

2.3. Notion for Matrix. In preparation for our analysis, we define several notions about matrices. The first is an effective rank of matrices, which is a measure of a matrix complexity using eigenvalues of the matrix.

**Definition 2** (effective rank). For a matrix $T \in \mathbb{R}^p \otimes \mathbb{R}^p$ and $k \in \mathbb{N}$, we define two types of effective ranks of $T$:

$$r_k(T) = \sum_{i=k}^\infty \frac{\mu_i(T)}{\mu_{k+1}(T)}, \text{ and } R_k(T) = \frac{(\sum_{i=k}^\infty \mu_i(T))^2}{(\sum_{i=k}^\infty \mu_i(T))^2}.$$  

Furthermore, we define an effective number of bases with constant $b > 0$ and $n \in \mathbb{N}$:

$$k^*(b) = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}.$$

If the set is empty, we set $k^*(b) = \infty$.  

This notion has been used in studies related to the matrix concentration inequality (Koltchinskii and Lounici, 2017), and its application to the high-dimensional linear regression (Bartlett et al., 2020). This measure is based on the degree of decay of the eigenvalues, rather than a number of nonzero eigenvalues.

We also define the notion of $\epsilon$-neighboring of sequences of matrices by using their eigenvalues. Recall that for a matrix $A \in \mathbb{R}^p \otimes \mathbb{R}^p$, $\mu_j(A)$ denotes the $j$-th largest eigenvalue of $A$.

**Definition 3** (neighboring of matrix sequences). For sequences of invertible matrices $\{A_n \in \mathbb{R}^n \otimes \mathbb{R}^n; n \in \mathbb{N}\}$ and $\{B_n \in \mathbb{R}^n \otimes \mathbb{R}^n; n \in \mathbb{N}\}$, $A_n$ and $B_n$ are $\epsilon$-neighboring with $\epsilon \in (0, 1]$, if

$$
\epsilon \leq \inf_{n \in \mathbb{N}} \mu_n (A_n^{-1} B_n) \leq \sup_{n \in \mathbb{N}} \mu_1 (A_n^{-1} B_n) \leq \frac{1}{\epsilon}.
$$

This notion ensures that the sequences are sufficiently close. If the sequences are $(1 - \delta)$-neighboring with $\delta \in (0, 1)$, we have $\max_{n \in \mathbb{N}} \|A_n^{-1} B_n - I\| \leq \max \{\delta, \delta/(1 - \delta)\}$. The neighboring can present the closeness of the sequences in a more flexible way. Note that $\epsilon$ for this notion is independent of $n$, which is necessary for our analysis. In Section 8, we show that several stochastic processes have covariance matrices that are neighboring under regularity conditions.

3. Error Analysis with Homo-Correlated Case

We present an analysis of excess risk of the interpolation estimator. Subsequently, we provide a sufficient condition on the covariance matrices of the covariate for the excess risk to converge to zero.

3.1. Main Result: General Bound on Excess Risk.

3.1.1. Upper Bound. We first provide the upper bound of the risk. Here, we utilize the effective rank of $\Sigma$ and weighted norm of $\beta^*$, which is projected onto a subspace spanned by $\Sigma_{k^*:\infty}$ and $\Sigma_{0:k}^\dagger$. Note that $\Sigma_{0:k}^\dagger$ has the representation $\Sigma_{0:k}^\dagger = \sum_{i=1}^{k^*} \lambda_i^{-1} e_i e_i^\top$. In addition, we use the notation $A_k := \sum_{j>k} \lambda_j z_j z_j^\top$ for $k = 1, \ldots, p$.

**Theorem 1** (homo-correlated case). Consider the interpolation estimator $\hat{\beta}$ in (7) for the stochastic regression problem (1) with a homo-correlated covariate process. Suppose that $\lambda_{n+1} > 0$ holds and a constant $L > 0$ is fixed. Then, there exist $b, c > 1$ such that for all $\delta \in (0, 1)$ with $\delta < 1 - c \exp(-n/c)$: (i) if $k^* \geq n/c$ with $k^* := k^*(b)$, then $\mathbb{E}_c R(\hat{\beta}) \geq 1/c \|\Sigma^{-1}_n\|$ with probability at least $1 - c \exp(-n/c)$; (ii) otherwise,

$$
R(\hat{\beta}) \leq c \left( \|\beta^*\|_{\Sigma_{k^*:\infty}}^2 + \|\beta^*\|_{\Sigma_{0:k}^\dagger}^2 \left( \sum_{i>k^*} \frac{\lambda_i}{n} \right)^2 \right) + c \|\Sigma^{-1}_n\| \left( \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \log(\delta^{-1}).
$$

with probability at least $1 - \delta - c \exp(-n/c)$. 

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The upper bound is composed of two elements: the first term bounds the bias part of the risk, and the second term represents the variance. We will describe a way to derive this bound in Section 6.1.

An important implication of this theorem is that the temporal dependence affects a variance part of the upper bound. Let us compare our results with the bound for i.i.d. sub-Gaussian settings: a combination of Bartlett et al. (2020) and Tsigler and Bartlett (2020) gives an upper bound

$$R(\hat{\beta}) \leq c \left( \frac{\beta^*}{\sum_{k=1}^{\infty}} \left( \frac{\sum_{i > k} \lambda_i}{n} \right)^2 \right) + c \sigma^2 \left( \frac{k^*}{n} + \frac{n}{R_k^*(\Sigma)} \right) \log(\delta^{-1})$$

with probability at least $1 - \delta - c \exp(-n/c)$, where $\sigma > 0$ is the upper bound of the sub-Gaussian norms of $\varepsilon$. We find that the bias part is identical to that of (10) for the i.i.d. case, on the other hand, the second term of (10) depends on $\sigma^2$ while our bound is dependent on $\|\Xi_n - 1_n \Upsilon_n\|$. Setting $\Xi_n = I_n$ and $\Upsilon_n = \sigma^2 I_n$ coincides the two bounds.

We provide additional insights by comparing Theorem 1 with other high-dimensional time-series studies: (i) Theorem 1 depends on the closeness of the covariances of $x_t$ and $\epsilon_t$, i.e. $\|\Xi_n - 1_n \Upsilon_n\|$, rather than the dependence of $x_t$ and $\epsilon_t$. This is an unusual property in time series analysis; for example, a sparsity-based high-dimensional analysis (Basu and Michailidis, 2015) utilizes the long-term dependence for characterizing the risk with high-dimensional time-series models. (ii) The upper bound by Theorem 1 depends on the $\ell^2$ norm of $\beta^*$, which is contrast to Basu and Michailidis (2015) which depends on the $\ell^1$-norm (or the number of nonzero coordinates) of $\beta^*$. (iii) The variance part increases in an inverse of the smallest eigenvalue of $\Xi_n$ when we assume $\Upsilon_n = I_n$, which indicates that the covariate process should not degenerate in the time direction. We can regard this point as a time series version of the restricted eigenvalue condition in high-dimension analysis (Bickel et al., 2009; Van de Geer, 2008).

Remark 3 (implicit temporal decorrelation). Theorem 1 shows an implicit temporal decorrelation effect by an interpolation estimator (7) under homo-correlation If there exists $\sigma > 0$ such that $\Xi_n^{-1} \Upsilon_n = \sigma^2 I_n$, then the bound in Theorem 1 are indifferent to those in i.i.d. cases (10). This observation indicates that, when $\Xi_n^{-1} \Upsilon_n = \sigma^2 I_n$, the temporal dependence is implicitly decorrelated by the interpolation estimator, even if we do not know the true $\Xi_n$ or $\Upsilon_n$. We can interpret Theorem 1 as an approximation of this implicit decorrelation.

3.1.2. Lower Bound. We show a lower bound of the risk. To the aim, we consider a random version of $\beta^*$ by a prior distribution. We utilize this setting only in this section for a lower bound. We also assume that $U = I_p$ without loss of generality. We set the random matrices $A_{-j} := \sum_{i \neq j} \lambda_i z_i z_i^T$ for all $j = 1, \ldots, p$. 

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**Assumption 1** (prior of $\beta$). Let $\bar{\beta} \in \mathbb{R}^p$ be an arbitrary nonrandom vector and $\beta^*$ is an $\mathbb{R}^p$-valued random vector such that\[
(\beta^*)^{(i)} = R_i \bar{\beta}^{(i)}, \quad i = 1, \ldots, p,
\]where $\{R_i\}_{i=1}^p$ is a sequence of i.i.d. random variables with $P(R_i = 1) = P(R_i = -1) = 1/2$ and independent of other random variables.

This setting is employed in Tsigler and Bartlett (2020) for the i.i.d. data setting. The response variable $\{y_i\}_{i=1}^n$ is assumed to be generated by the regression model (1) with conditioning $\beta^*$. With this setting, we derive the following lower bound, which considers an expectation of the risk with respect to $\beta^*$ in a part of the results.

**Theorem 2.** The following statements hold:

(i) Under the same assumptions as Theorem 1, there exists $b, c \geq 1$ such that if $k^* \leq n/c$ with $k^* := k^*(b)$, then\[
\mathbb{E}[R(\hat{\beta})] \geq \frac{1}{c} \left\| \Sigma^{-1/2} \Xi_n \right\| \left( \frac{k^*}{n} + \frac{n}{R_{k^*} (\Sigma)} \right)
\]with probability at least $1 - c \exp(-n/c)$.

(ii) Let $a, b > 0$ be a fixed constant with $0 < a < b$ and $L > 0$. Set $k^* := k^*(b)$ and assume that there exists $k^# \in \{1, \ldots, k^*\}$ such that $\mu_1(A_{k^#})/\mu_n(A_{k^#}) \leq L$ with probability at least $1 - \delta$ and $r_{k^#}(\Sigma) \geq an$. Under Assumption 1 and the assumptions in Theorem 1, there exists a constant $c$ depending only on $a, b, L$ such that\[
\mathbb{E}_{\bar{\beta}^*}[R(\hat{\beta})] \geq \frac{1}{c} \left( \left\| \bar{\beta} \right\|^2_{\Sigma_{k^#}^{-1} \Xi_n} + \left\| \bar{\beta} \right\|^2_{\Sigma_{a, k^#}^{-1} \Xi_n} \left( \sum_{i > k^#} \lambda_i / n \right)^2 \right)
\]with probability at least $1 - \delta - c \exp(-t/c)$.

These lower bounds show the tightness of each term in the upper bound in Theorem 1. That is, the first lower bound (i) corresponds to the variance part of Theorem 1 and the second bound coincides with the bias part, up to some constants.

As for the comparison with the i.i.d. setting, the lower bounds coincide with those of the i.i.d. setting (Theorem 5 of Tsigler and Bartlett (2020)) up to a constant factor. Rigorously, the result implies several intuitions: (i) the dependence of data does not affect the bias term of the risk as shown in Theorem 1 showed, and (ii) the dependence affects the variance part of the bounds.

4. **Error Analysis with Hetero-correlated Case**

We provide the excess risk bound in the case of the hetero-correlated process with matrices $\Xi_{i,n}$ separately. We provide result for the hetero-correlated setting independently, because its proof strategy and implications are different from the homo-correlated case.
4.1. Assumption and Notion. First, we impose a constraint on the sequence of temporal covariance matrices \( \{\Xi_{i,n}\}_{i=1}^p \). Unlike the homo-correlated case, the sequence has a large degree of freedom, hence it is essential to introduce the following assumptions.

**Assumption 2.** The temporal covariance matrices, \( \Xi_{i,n} \), satisfy the following:

(i) All the \( \Xi_{i,n} \) are positive definite Toeplitz matrices, which lead to the stationarity of \( x_t \).

(ii) All the diagonal elements of \( \Xi_{i,n} \) equal 1.

(iii) There exists a family of reference matrices \( \Xi_{0,n} \in \mathbb{R}^n \otimes \mathbb{R}^n \) and \( \epsilon \in (0, 1] \) such that all \( \Xi_{i,n}, i = 1, \ldots, p \) and \( \Xi_{0,n} \) are \( \epsilon \)-neighboring, that is,

\[
\epsilon \leq \inf_{i,n} \mu_n(\Xi_{0,n}^{-1} \Xi_{i,n}) \leq \sup_{i,n} \mu_1(\Xi_{0,n}^{-1} \Xi_{i,n}) \leq 1/\epsilon.
\]

The condition (ii) leads to \( \mathbb{E}[x_t x_t^\top] = \Sigma \) for all \( t \). Under the condition (iii), we can expect that the properties of the interpolator for \( \Xi_{0,n}^{-1/2} X \) and \( \Xi_{0,n}^{-1/2} Y \) should be similar to those under i.i.d. settings. In Section 8, we provide several examples to satisfy Assumption 2.

Second, we introduce a notion of integrated covariance, which is a generalized version of the spatial covariance matrix \( \Sigma \), which also reflects the temporal dependence. We define a positive version of a cross-covariance matrix as

\[
\tilde{\Sigma}_h := U \text{diag} \left\{ \left| \lambda_1 \Xi_{1,n}^{(1,h+1)} \right|, \ldots, \left| \lambda_i \Xi_{i,n}^{(1,h+1)} \right|, \ldots, \left| \lambda_p \Xi_{p,n}^{(1,h+1)} \right| \right\} U^\top, \ h = 1, \ldots, n - 1.
\]

Then, for \( n \in \mathbb{N} \), we define the integrated covariance:

**Definition 4** (integrated covariance). For \( n \in \mathbb{N} \), we define

\[
\tilde{\Sigma}_n := \Sigma + 2 \sum_{h=1}^{n-1} \tilde{\Sigma}_h,
\]

This matrix describes a unified covariance of the entire process, \( \{x_t\}_{t=1}^n \). Note that the time width of \( \tilde{\Sigma}_n \) depends on the number of data instances, \( n \). We discuss a limit of \( \tilde{\Sigma}_n \) as \( n \to \infty \) in Section 5.

A similar notion is used in another high-dimensional time-series studies. For example, Basu and Michailidis (2015) considered a spectral density of integrated covariance with infinite time width \( \sum_{h=-\infty}^{\infty} \Sigma_h \), where \( \Sigma_h := \mathbb{E}[x_1 x_{1+h}^\top] \) for \( h \in \mathbb{N} \cup \{0\} \). While Basu and Michailidis (2015) required the largest eigenvalues of \( \sum_{h=-\infty}^{\infty} \Sigma_h \) to be bounded, our analysis only used integrated covariance with finite time width.

4.2. Main Result: General Bound on Excess Risk. Using the assumption and the notion above, we develop an upper bound of a risk in the hetero-correlated setting.

**Theorem 3** (hetero-correlated case). Consider an interpolation estimator, \( \hat{\beta} \), in (7) for the stochastic regression problem (1) with a hetero-correlated covariate process. Suppose that
\(\lambda_{n+1} > 0\) and Assumption \(^2\) hold true. Then, there exist \(b, c > 1\) such that for all \(\delta \in (0, 1/2)\) with \(2\delta < 1 - c \exp(-n/c)\): (i) if \(k^* \geq n/c\) with \(k^* := k^*(b)\) then \(E[R(\hat{\beta})] \geq 1/c \|\Upsilon_{1}^{-1}\Xi_{0,n}\|\); (ii) otherwise,

\[
R(\hat{\beta}) \leq c\delta^{-1} \|\beta^*\|^2 \left( \sqrt{\|\Sigma\|} \max \left\{ \frac{r_0(\Sigma)}{n}, \frac{r_0(\Sigma_n)}{n} \right\} + \frac{\|\Sigma_n\| r_0(\Sigma_n)}{n} \right) \\
+ c\|\Xi_{0,n}^{-1}\| \left( \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \log(\delta^{-1}).
\]

with probability at least \(1 - 2\delta - c \exp(-n/c)\), and

\[
E_{\varepsilon}[R(\hat{\beta})] \geq \frac{1}{c \|\Upsilon_{1}^{-1}\Xi_{0,n}\|} \left( \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right)
\]

with probability at least \(1 - c \exp(-n/c)\).

The first term of the bound is a bias part, and the second term is a variance part, as same to the homo-correlated case in Theorem \(^1\). However, the bias part is quite different from that of Theorem \(^1\) due to the hetero-correlation setting. This difference appears, because it is difficult to decorrelate the temporal correlation of the covariate process \(\{x_t\}_{t=1}^{n}\) effectively in the hetero-correlation setting. On the other hand, the variance part is indifferent to that of Theorem \(^1\) owing to Assumption \(^2\) introducing \(\Xi_{0,n}\).

A main difference between the bounds in Theorem \(^3\) and Theorem \(^1\) appears in the bias part of the bounds. The bias part in Theorem \(^3\) is based on a moment inequality of random matrices as in \(\text{Bartlett et al. (2020)}\), although the bias part in Theorem \(^3\) is based on \(\text{Tsigler and Bartlett (2020)}\). This difference comes from the fact that there is the implicit decorrelation of the interpolation estimator in the homo-correlated case (see Remark \(^3\)), it does not appear in the hetero-correlated case. Due to the reason, the hetero-correlated case does not allow applying the analysis by \(\text{Tsigler and Bartlett (2020)}\).

We compare Theorem \(^3\) with Theorem 1 of \(\text{Bartlett et al. (2020)}\), which gives the upper bound in this i.i.d. setting such that

\[
R(\hat{\beta}) \leq c \|\beta^*\|^2 \|\Sigma\| \left( \sqrt{\frac{r_0(\Sigma)}{n}} + \frac{r_0(\Sigma)}{n} + \frac{\log(\delta^{-1})}{n} \right) + c\sigma^2 \left( \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \log(\delta^{-1}).
\]

with probability \(1 - \delta\). For fixed \(\delta\), the bound for the bias part by Theorem \(^3\) and that by \(\text{Bartlett et al. (2020)}\) coincide up to a constant factor. The bound for the variance by Theorem \(^3\) and that by \(\text{Bartlett et al. (2020)}\) also coincide with setting \(\Xi_{i,n} = I_n\) and \(\Upsilon_n = \sigma^2 I_n\). Hence, our result on the upper bound is a generalization for fixed \(\delta\) in terms of reflecting the temporal dependence by \(\Sigma_n\) and \(\|\Xi_{0,n}^{-1}\Upsilon_n\|\).
5. BENIGN COVARIANCE: CONVERGENCE RATE ANALYSIS

We present sufficient conditions under which the upper bound converges to zero as \( n \to \infty \). Let us recall that both \( \Sigma \) and its eigenvalues can depend on the sample size, \( n \) (Theorem 31 in Bartlett et al., 2020).

5.1. Preparation. Further, we introduce a characterization on the autocovariance, \( \Sigma \), developed by Bartlett et al. (2020).

**Definition 5** (benign autocovariance). An autocovariance matrix \( \Sigma \) is 
**benign**, if there exist sequences, \( \{\zeta_n : n \in \mathbb{N}\}, \{\tau_n : n \in \mathbb{N}\}, \{\eta_n : n \in \mathbb{N}\} \subset \mathbb{R}_+ \), such that

\[
\zeta_n := \text{tr}(\Sigma) = o(n), \quad \tau_n := \lambda_{k^*} = o(1), \quad \text{and} \quad \eta_n := \max\left\{ \frac{k^*}{n}, \frac{n}{R_{k^*}(\Sigma)} \right\} = o(1), \quad \text{as} \quad n \to \infty.
\]

The study provides specific examples of benign autocovariance. Although the analysis of autocovariance does not have a significant direct relationship with dependent data, we have included it for completeness.

**Example 3** (Theorem 31 in Bartlett et al., 2020). Autocovariance \( \Sigma \) with eigenvalues \( \{\lambda_i : i \in \mathbb{N}\} \) is benign, as stated in Definition 5, with corresponding \( \{\zeta_n : n \in \mathbb{N}\}, \{\tau_n : n \in \mathbb{N}\}, \) and \( \{\eta_n : n \in \mathbb{N}\} \):

(i) Eigenvalues are \( \lambda_i = i^{-1} \log^{-\gamma} i \) for \( \gamma > 0 \), then the sequences satisfy \( \zeta_n = O(1), \tau_n = o(1), \) and \( \eta_n = o(1) \).

(ii) Eigenvalues are \( \lambda_i = i^{-(1+\gamma_n)} \) for \( \gamma_n = o(1) \), then the sequences satisfy \( \tau_n = O(n^{-1+\gamma_n}), \) and \( \zeta_n = O(1), \) and \( \eta_n = O(\min\{\gamma_n^{-1} + \gamma_n, 1\}) \).

(iii) Eigenvalues are \( \lambda_i = i^{-\gamma} \mathbb{1}\{i \leq p_n\} \) for \( \gamma \in (0,1) \) and the dimension \( p = p_n \) is \( n \ll p_n \ll n^{1/(1-\alpha)} \), then the sequences satisfy \( \zeta_n = O(1), \tau_n = O(n^{-1}p_n^{1-\gamma}), \) and \( \eta_n = o(1) \).

(iv) Eigenvalues are \( \lambda_i = (\gamma_i + \epsilon_n) \mathbb{1}\{i \leq p_n\} \), for \( \gamma_i = \Theta(\exp(-i)) \) and the dimension \( p = p_n \) is \( n \ll p_n \), then the sequences satisfy \( \epsilon_n = p_n^{-1}ne^{-o(n)}, \) and \( \zeta_n = O(ne^{-o(n)}), \tau_n = O(\epsilon_n(1 + p_n/n)), \) and \( \eta_n = O(n^{-1}(1 + \log(n/(\epsilon_n p_n))) + np^{-1}) \).

In case (i), \( \Sigma \) is independent of \( n \); otherwise, it depends on \( n \). Cases (i) and (ii) yield infinite nonzero eigenvalues, while cases (iii) and (iv) yield \( p_n \) nonzero eigenvalues, which represent finite dimensionality increasing in \( n \). Note that all the sequences \( \{\zeta_n\}_n \), \( \{\tau_n\}_n \) and \( \{\eta_n\}_n \) are simply calculated from the proof of Theorem 31 in Bartlett et al. (2020).

5.2. Convergence Rate.

5.2.1. Homo-Correlated Case. We study the convergence rate of the upper bound by Theorems in homo-correlated cases. To discuss the influence of them, we introduce a notation
for the inverse of the spectral norm of \( \Sigma_n^{-1} \Gamma_n \):

\[
\nu_n := \|\Sigma_n^{-1} \Gamma_n\|.
\]

This value describes the relative degeneracy of \( \Sigma_n \) with respect to \( \Gamma_n \), which is specific to the dependent data setting. If \( \Gamma_n = I_n \), this quantity equals the smallest eigenvalue of \( \Sigma_n \).

We provide examples of \( \nu_n \) in Section [8]. In the study by Basu and Michailidis (2015), a similar condition was required, i.e., the lower bound on the spectral density of the sums of cross-covariance matrices should be positive. By contrast, we used the non-asymptotic approach of studying the eigenvalues of matrices under finite time width.

We derive a convergence rate of risk by using the notations for the benign autocovariance. In the case of homo-correlated processes, we obtain the following result.

**Proposition 4** (convergence rate in homo-correlated case). Consider the setting and assumption for Theorem [7]. Suppose that \( \Sigma \) is a benign covariance, as stated in Definition [3]. Also suppose that \( \|\beta^*\| = O(1) \) as \( n \to \infty \). Then, we have

\[
R(\hat{\beta}) = O_P(\tau_n + \nu_n \eta_n), \quad \text{as } n \to \infty.
\]

This rate consists of two terms; the first corresponds to the bias part and the second part for the variance. The bias part is determined by the volume of \( k^* \) and the decay speed eigenvalues \( \lambda_i \), and the variance part is determined by \( \eta_n \) and the strength of temporal correlation \( \nu_n \). Given that the rate in the i.i.d. case by Tsigler and Bartlett (2020) is given as

\[
R(\hat{\beta}) = O_P(\tau_n + \eta_n),
\]

we can see that \( \nu_n^{-1} \) reveals an effect of temporal dependence.

### 5.2.2. Hetero-Correlated Case

We also study the convergence rate in the hetero-correlated case, which is slightly complicated than the homo-correlated case.

First, we define a notation for the temporal correlation, which is an extended version of \( \nu_n \) in (11).

\[
\nu_{0,n} := \|\Sigma_{0,n}^{-1} \Gamma_n\|.
\]

The difference from \( \nu_n \) is that we use a reference matrix \( \Sigma_{0,n} \) instead of \( \Sigma_n \) to represent temporal correlations.

Next, to study the dependency of data, we characterize the eigenvalues of cross-covariance matrices \( \Xi^{(t,t+h)}_{i,n} \) for \( h \in \mathbb{Z} \setminus \{0\} \). We consider the following form:

\[
|\Xi^{(t,t+h)}_{i,n}| \leq c|h|^{-\alpha}, \quad \forall t, t + h \in \{1, \ldots, n\}, \forall i \in \mathbb{N}, \forall n \in \mathbb{N}.
\]

with \( \alpha > 0 \) and some constant, \( c > 0 \) independent of \( n \). For \( \alpha > 1 \), the covariate process is considered a short-memory process, i.e., \( \|\sum_{h=1}^{\infty} \Sigma_h\| < \infty \) and \( \|\sum_{h=1}^{\infty} \Sigma_h\| \neq 0 \) hold
true by the decay of $\Xi_{i,n}^{(t,t+h)}$ in $h$. For $\alpha \leq 1$, the process is possibly a long-memory process, which has $\| \sum_{h=\infty}^{\infty} \Sigma_h \| = \infty$.

For both cases of short- and long-memory processes, we obtain the following result:

**Proposition 5** (convergence rate in hetero-correlated case). Consider the setting and assumptions for Theorem $\mathcal{B}$. Suppose that $\Sigma$ is a benign covariance, as stated in Definition $\mathcal{A}$, and $\Xi_{i,n}^{(t,t+h)}$ satisfies (13) $\alpha > 0$. Also suppose that $\| \beta^* \| = O(1)$ as $n \to \infty$. Then, we have

$$R(\hat{\beta}) = O_P \left( \zeta_n^{1/2} \kappa_n^{(\alpha)} + \nu_0 n \eta_n \right), \text{ as } n \to \infty,$$

where

$$\kappa_n^{(\alpha)} = \begin{cases} O(n^{-\alpha/2}) & (\alpha \in (0, 1)) \\ O(n^{-1/2} \log^{1/2} n) & (\alpha = 1) \\ O(n^{-1/2}) & (\alpha > 1). \end{cases}$$

This rate consists of two items, $\zeta_n^{1/2} n^{-1/2}$ and $\nu_n^{-1} \eta_n$, corresponding to the bias and variance in Theorem $\mathcal{I}$, respectively. Importantly, the results based on $\kappa$ can handle both long- and short-memory processes simultaneously. Because the existing dependent data studies have been limited to short-memory processes, such as Basu and Michailidis (2015) where $\| \sum_{h=-\infty}^{\infty} \Sigma_h \| < \infty$ was required, this result is more general.

We compare the convergence rate with that of the i.i.d. case by Bartlett et al. (2020), which is more suitable in the hetero-correlated case. If $\Sigma$ is benign covariance in the i.i.d. case, the bound by Bartlett et al. (2020) is written as

$$R(\hat{\beta}) = O_P \left( \zeta_n^{1/2} n^{-1/2} + \eta_n \right). \quad (14)$$

A comparison with the rate in the independent case (14) yielded several implications: (i) In the short-memory case ($\alpha > 1$), the bound for bias parts exhibits the same rate, and only the variance increases by the deteriorated convergence rate owing to the autocorrelation matrix, $\Xi_n$. (ii) The rate of the bound for bias deteriorated only when the covariates involved a long-memory process. (iii) There was no guarantee that the risk would converge to zero with a part of benign autocovariances in Example $\mathcal{H}$ according to the worsening rate of dependency. Figure $\mathcal{J}$ demonstrates the comparison.

6. **Proof Outline of Upper Bounds**

6.1. **Homo-Correlated Case.** Theorem $\mathcal{I}$ is mainly based on the upper and lower bounds for the bias term by Tsingler and Bartlett (2020). The proof is based on the bias–variance decomposition of the excess risk $R(\hat{\beta})$ into a bias term $T_B$ and a variance term $T_V$, then we bound each of the terms. Note that this decomposition holds regardless of the homo- and hetero-correlated setting.
Figure 1. Comparison of the convergence rates of the upper bound for risk. \( \alpha \) (decay rate of eigenvalues of cross-covariance) and \( \nu_{0,n} \) (inverse of the smallest eigenvalue of the autocovariance matrix) affect the convergence rate in the dependent case.

Lemma 6. For any \( \delta \in (0, 1) \), we obtain the following with probability at least \( 1 - \delta \):

\[
R(\hat{\beta}) \leq 2\beta^*\top T_B\beta^* + 2C_\delta \text{tr}(T_V),
\]

where \( C_\delta = 2 \log(1/\delta) + 1 \), where

\[
T_B := (I - X^\top (XX^\top)^{-1}X)\Sigma(I - X^\top (XX^\top)^{-1}X)
\]

\[
T_V := \Upsilon^{-1/2}_n (XX^\top)^{-1} X\Sigma X^\top (XX^\top)^{-1} \Upsilon^{-1/2}_n.
\]

Furthermore, we have \( \mathbb{E} R(\hat{\beta}) \geq \beta^*\top T_B\beta^* + \text{tr}(T_V) \).

Under the homo-correlation setting, we can easily observe that the bias term \( T_B \) is invariant for any positive definite \( \Xi_n \) and thus we directly apply the result of i.i.d. by Tsigler and Bartlett (2020).

In contrast, the bound of the variance term \( T_V \) is not straightforward as that of the bias term. The term \( |1/\|\Upsilon^{-1}_n\Xi_n\|, \|\Xi^{-1}_n\Upsilon_n\| | \) appears at the upper bound of the variances part, because we handle the dependency of the data.

6.2. Hetero-Correlated Case. In this setting, we derive the upper bound based on the decomposition of Lemma 6, however, the derivations of the bounds of them are quite different.

For the bias term \( T_B \), we bound it by a deviation inequality on an empirical covariance matrix, adapted from Bartlett et al. (2020). That is, we use the fact that the bias term is bounded by

\[
T_B \leq \|\beta^*\|^2 \|\hat{\Sigma} - \Sigma\|,
\]
where $\hat{\Sigma} = (1/n) \sum_{t=1}^{n} x_t x_t^\top$ is the empirical covariance matrix. In our setting with the temporal dependence, we use the moment inequality for $E\|\hat{\Sigma} - \Sigma\|$ with dependent data (Han and Li 2020, Proposition 4.6) and Markov’s inequality, which suffice to derive the desired result.

The proof for the variance term $T_V$ is based on the spectral decomposition of $X$ by the eigenvectors $e_i$ of $\Sigma$, which is also adapted from Bartlett et al. (2020). Rigorously, we consider the decomposed vector $Xe_i/\sqrt{\lambda_i}$ and analyze its summation. Since $Xe_i/\sqrt{\lambda_i} \sim N(0, \Xi_{i,n})$ is not standard Gaussian unlike the i.i.d. setting, we utilize the reference covariance matrices $\Xi_{0,n}$ and the $\epsilon$-neighboring, then approximately decorrelate the variance terms by $\Xi_{0,n}$. Rigorously, we obtain the properties such that each $\Xi_{0,n}^{-1/2} Xe_i/\sqrt{\lambda_i}$ is $a\epsilon$-sub-Gaussian with a universal constant $a$ and $n \leq \text{tr}(E[(\Xi_{0,n}^{-1/2} Xe_i/\sqrt{\lambda_i})(\Xi_{0,n}^{-1/2} Xe_i/\sqrt{\lambda_i})^\top]) \leq \epsilon^{-1}n$. These properties yield results similar to those in i.i.d. settings.

7. Extension to Infinite-Dimensional Case

We extend the result of the hetero-correlated case (Theorem 3) to a setup with $p = \infty$. In this setting, each data point $x_t$ is a function-valued, for example, and its covariance is represented as an operator. For the explicit benign covariance given in Example 3, only (i) and (ii) are feasible in this setting.

To handle the infinite dimension, we give additional notations and also change the definitions of some notions, but they are used only in this section. Let $H$ be a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ induced by the inner product. $I : H \to H$ is an identity operator. For a linear operator $T : H \to H$, $T^\top$ denotes an adjoint operator of $T$. $\|T\| := \sup_{x \in H : \|x\| = 1} \|Tz\|$ is an operator norm. $\mu_i(T)$ is the $i$-th largest eigenvalue of $T$, and $\text{tr}(T) = \sum_i \mu_i(T)$ denotes a trace of $T$. For $z \in H$, $z^\top$ denotes a linear functional $z^\top : H \to \mathbb{R}$, $z' \mapsto \langle z, z' \rangle$; moreover, $z = (z^\top)^\top$ is considered an adjoint operator of $z^\top$. We prepare some matrix-type notations. Let $X$ be a linear map:

$$X : \mathbb{R}^p \to \mathbb{R}^n, z \mapsto (x_1^\top z, \ldots, x_n^\top z)^\top,$$

which is also a bounded operator. Similarly, we define $X^\top : \mathbb{R}^n \to \mathbb{R}^p$ as an adjoint of $X$.

We still consider the interpolation estimator

$$\hat{\beta} = (X^\top X)^\dagger X^\top Y,$$

which uses the changed notation of $X$ and $Y$. Because the operator $X^\top X$ is bounded and linear, its pseudo-inverse is guaranteed to exist (Desoer and Whalen 1963). Then, we obtain the following result:

**Theorem 7** (infinite-dimensional hetero-correlated case). Consider an interpolation estimator, $\hat{\beta}$, in (15) with a hetero-correlated covariate process. Suppose that $\lambda_{n+1} > 0$ and Assumption 3 hold true. Then, there exist $b, c > 1$ such that for all $\delta \in (0, 1/2)$ with
\[ 2\delta < 1 - c \exp(-n/c): \text{ (i) if } k^* \geq n/c \text{ with } k^* := k^*(b) \text{ then } \mathbb{E}_C[R(\hat{\beta})] \geq 1/c \| T_n^{-1} \Xi_{0,n} \| \text{ with probability at least } 1 - c \exp(-n/c); \text{ (ii) otherwise,} \]

\[ R(\hat{\beta}) \leq c\delta^{-1} \| \hat{\beta}^* \|^2 \left( \sqrt{\| \Sigma \|^2 \| \Sigma_n \|^2} \max\{r_0(\Sigma), r_0(\Sigma_n)\} \right) + \frac{\| \Sigma_n \|^2 r_0(\Sigma)}{n} + c\| \Xi_{0,n} T_n \| \left( \frac{k^* n}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \log(\delta^{-1}). \]

with probability at least \( 1 - 2\delta - c \exp(-n/c) \).

Except for the setting of \( p = \infty \), the upper bound in Theorem 7 is exactly the same as in the finite-dimensional case in Theorem 3. Since the upper bound with the finite dimension does not depend on \( p \), we can obtain the corresponding bound.

In proof of this result, the most technically important part is to bound the bias term \( T_B \) in Lemma 6. Rigorously, we need a moment inequality for \( \mathbb{H} \)-valued dependent processes. Hence, we developed an inequality to bound the empirical deviation of covariance operators by (possibly) infinite-dimensional dependent processes as follows:

**Proposition 8.** Let us assume that \( \mathbb{H} \) is a separable Hilbert space and \( x_t \) is an \( \mathbb{H} \)-valued centered stationary Gaussian process whose cross-covariance operator \( \Sigma_h \) such that \( \Sigma_h u := \mathbb{E}[x_{t+h}^\top u] \) for all \( u \in \mathbb{H} \). Suppose that \( \Sigma_h \) is a trace class for all \( h \in \mathbb{Z} \). Then, we have

\[
\mathbb{E} \left[ \| \hat{\Sigma}_0 - \Sigma_0 \| \right] \leq \frac{2\sqrt{2}}{n} \left( \sqrt{2\text{tr} (\Sigma_n)} + \sqrt{2n \| \Sigma_0 \| \text{tr} (\Sigma_n)} + \sqrt{n\text{tr} (\Sigma_0) \| \Sigma_n \|} \right),
\]

where for all \( u \in \mathbb{H} \),

\[
\hat{\Sigma}_0 u := \frac{1}{n} \sum_{t=1}^n (x_t^\top u) x_t, \quad \hat{\Sigma}_n u := \Sigma_0 u + 2 \sum_{h=1}^{n-1} \hat{\Sigma}_h u, \quad \tilde{\Sigma}_h u := \frac{1}{2} \sum_{i=1}^{\text{r}_h(\Sigma)} (f_{h,i}^\top u) f_{h,i} + (g_{h,i}^\top u) g_{h,i} \quad \text{ for all } u \in \mathbb{H},
\]

\( \sigma_i(\Sigma_h) \) are singular values of \( \Sigma_h \), \( \{f_{h,i}\} \) are left singular vectors and \( \{g_{h,i}\} \) are right ones of \( \Sigma_h \).

We can obtain the application of the monotone convergence theorem and Proposition 4.6 of Han and Li (2020) (e.g., see Giulini, 2018, Section 3, or Nakakita et al., 2022, Section 3.2). To make the discussion self-contained, we give a direct proof of Proposition 8 in Appendix D.

8. **Example**

As an example, we analyze multiple stochastic processes and risks of the stochastic regression problem with the processes. Specifically, we deal with the two types of ARMA processes listed in Section 2.1, and the autoregressive fractionally integrated moving-average (ARFIMA) process. The ARMA processes are known as short-memory processes, and the
ARFIMA process can be both short- and long-memory processes depending on its hyperparameter.

8.1. Homo-Correlated ARMA Process. We consider homo-correlated ARMA processes displayed in (5) in Example 1.

Recall that $\Sigma = Q$ holds and $\varpi_1(z)$ and $\varpi_2(z)$ are characteristic polynomials of the process. With the conditions on $\varpi_1(z)$ and $\varpi_2(z)$ in Example 1 and Proposition 4.5.3 of Brockwell and Davis (1991) yield the following inequality for the neighbouring condition:

$$\min_{z \in C: |z| = 1} \left| \frac{\varpi_2(z)}{\varpi_1(z)} \right| \leq \inf_{n} \mu_n(\Xi_n) \leq \sup_{n} \mu_1(\Xi_n) \leq \max_{z \in C: |z| = 1} \left| \frac{\varpi_2(z)}{\varpi_1(z)} \right|.$$ 

This property gives the following convergence of excess risk without proof:

**Corollary 9.** Consider the setting and assumptions for Theorem 1 and that the covariate follows process (6). Suppose that $\Sigma = Q$ is a benign covariance with sequences $\{\tau_n\}_{n \in N}$ and $\{\eta_n\}_{n \in N}$ as stated in Definition 5, and $\sup_n \mu_1(\Upsilon_n) < \infty$. Then, we have $\nu_n \leq \sup_n \mu_1(\Upsilon_n)/\min_{z: |z| = 1}(|\varpi_2(z)| - \epsilon)$.

This result shows that the case of ARMA has the same convergence rate as that of independent data with the same condition on $\Sigma$.

8.2. Hetero-Correlated ARMA Process. We present an example of hetero-correlated processes in (6) in Example 2. For the characteristic polynomials $\varpi_{1,k}(z)$ and $\varpi_{2,k}(z)$ defined in Example 2, we additionally assume there exists $\epsilon > 0$ such that $|\varpi_{1,k}(z)|$ and $|\varpi_{2,k}(z)|$ are in $[\epsilon, \epsilon^{-1}]$ for all $z \in C$ such that $|z| = 1$. Recall that we have $\Sigma = \sum_{k=1}^{p} \lambda_k e_k e_k^\top$ with $\lambda_k := q_k(\sum_{j=0}^{\infty} \phi_{j,k}^2)$ where $\{\phi_{j,k} \in R\}$ satisfies $\Xi_{k,n}^{(t,\ell)} = \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+k+|\ell|,k}/\sum_{j=0}^{\infty} \phi_{j,k}^2$.

The following proposition verifies that the spatial covariance $\Sigma$ is benign covariance, which holds by the assumptions on the characteristic polynomials.

**Proposition 10.** If rank $(Q) > n$, the following properties hold true for (6):

(i) $1 \leq \sum_{j=0}^{\infty} \phi_{j,k}^2 \leq \epsilon^{-4}$.
(ii) $\epsilon^8 \leq \inf_{k,n} \mu_n(\Xi_{k,n}) \leq \sup_{k,n} \mu_1(\Xi_{k,n}) \leq \epsilon^{-4}$.
(iii) For some $c > 0$ dependent only on $\epsilon$, $\ell_1$, and $\ell_2$, $|\Xi_{k,n}^{(1,1+h)}| \leq c (1 + \epsilon)^{-|h|/2}$.
(iv) $\|Q\| \leq \|\Sigma\| \leq \epsilon^{-4} \|Q\|$, $\text{tr}(Q) \leq \text{tr}(\Sigma) \leq \epsilon^{-4} \text{tr}(Q)$.
(v) There exists $c > 0$ dependent only on $\epsilon$, $\ell_1$, and $\ell_2$ such that for all $n \in N$, $\|\Sigma_n\| \leq c \|\Sigma\|$, $\text{tr}(\Sigma_n) \leq c \text{tr}(\Sigma)$.

Assumption 2 can be satisfied as follows: (i) $\Xi_{k,n}$ is clearly positive definite by (ii) of Proposition 10 and Toeplitz by stationarity; (ii) follows from the definition of $\Xi_{k,n}$; and (iii)
holds by (ii) of Proposition 10 and setting $\Xi_{0,n} = I_n$. Therefore, Theorem 3 immediately gives the following result without proof:

**Corollary 11.** Consider the setting in Theorem 3 with covariate $\{x_t\}_t$ that follows the process (6). Suppose that $\Sigma = \sum_{k=1}^p q_k (\sum_{j=0}^{\infty} a_{j,k}^2) c_k c_k^\top$ is a benign covariance with sequences $\{\zeta_n\}_{n \in \mathbb{N}}$ and $\{\eta_n\}_{n \in \mathbb{N}}$, as stated in Definition 5. Then, we have

$$R(\hat{\beta}) = O_P \left( \zeta_n n^{-1} + \eta_n \right).$$  

(16)

This upper bound does not depend on dependence-based terms, such as $\bar{\Sigma}_n$; therefore, it can be observed that the upper bound is equivalent to the independent data case in Bartlett et al. (2020). Additionally, the derived convergence rate was identical to that of the independent case, as displayed in (14). In other words, the same rate is achieved for excess risk as that in the independent case, because it is short-memory.

8.3. ARFIMA Process. We also consider homo-correlated ARFIMA processes. We define the ARFIMA process using the lag operator $L$ for stochastic processes on $\mathbb{Z}$, i.e., for an arbitrary process $\{p_t : t \in \mathbb{Z}\}$, $L^d p_t = p_{t-d}$ for all $t, j \in \mathbb{Z}$. We also define the difference operator, $\nabla^d$, for $d > -1$ such that $\nabla^d = (1 - L)^d$. Using the coefficients $a_j, b_j \in \mathbb{R}$, we consider a homo-correlated ARFIMA($\ell_1, d, \ell_2$) process $x_t$ with $d \in (-1/2, 1/2)$ such that

$$\left(1 - \sum_{j=1}^{\ell_1} a_j L^j\right) \nabla^d x_t = \left(1 + \sum_{j=1}^{\ell_2} b_j L^j\right) w_t.$$  

(17)

where $w_t$ is a white noise with covariance $Q$ as the previous examples. We assume that the characteristic polynomials $\varpi_1(z) := 1 - \sum_{j=1}^{\ell_1} a_j z^j$ and $\varpi_2(z) := 1 + \sum_{j=1}^{\ell_2} b_j z^j$ neither have common roots in $\mathbb{C}$ nor roots in the unit circle, i.e., $z$ with $|z| \leq 1$. The process (17) has the spatial covariance $\Sigma = Q$ as same as homo-correlated ARMA processes. About the temporal covariance $\Xi_n$, we obtain the following proposition under the assumptions on the characteristic polynomials (see Lu and Hurvich 2005).

**Proposition 12.** The homo-correlated ARFIMA($\ell_1, d, \ell_2$) process (17) with $d \in (-1/2, 1/2) \setminus \{0\}$ has the following properties:

1. If $d \in (0, 1/2)$, $\mu_1(\Xi_n) \asymp n^{2d}$ and $\inf_n \mu_n(\Xi_n) > 0$.
2. If $d \in (-1/2, 0)$, $\sup_n \mu_1(\Xi_n) < \infty$ and $\mu_n(\Xi_n) \asymp n^{2d}$.

Using the result, we achieve the following convergence rate.

**Corollary 13.** Consider the setting and assumptions for Theorem 4 and that the covariate follows the ARFIMA process (17). Suppose that $\Sigma = Q$ is a benign covariance with sequences $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{\eta_n\}_{n \in \mathbb{N}}$, as stated in Definition 5, and $\sup_n \mu_1(\Upsilon_n) < \infty$. Then, we have $\nu_n \leq \sup_n \mu_1(\Upsilon_n)n^{-2\min\{0,d\}}$ and thus

$$R(\hat{\beta}) = O_P \left( \tau_n + n^{-2\min\{0,d\}} \eta_n \right).$$

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This result shows that the convergence rate reduces owing to the effect of \( d \) caused by the long-memory ARFIMA with \( d \in (0, 1/2) \). In this case, the error converges to zero with \( \zeta_n = O(1) \), i.e., \( \Sigma = Q \) is a trace class. For \( d \in (-1/2, 0) \), if \( \eta_n = O(n^{-1/2}) \) holds true, i.e., for property (ii) in Example 3 with \( \gamma_n = n^{-1/2} \), the convergence of the risk to zero is guaranteed.

9. Conclusion and Discussion

In this study, we investigated the excess risk of an estimator of an over-parameterized linear model with dependent time series data. We constructed an estimator using an interpolator that fits the data perfectly and measured the excess risk of prediction. In addition to the notion of the effective ranks of the autocovariance matrices used for independent data, we developed several notions to handle the temporal covariance and its heterogeneity; subsequently, we derived the upper and lower bounds of the excess risk in the cases of homo/hetero-correlation. This result is valid in the high-dimensional case \( p \gg n \) regardless of sparsity and in long-memory processes. Our theory shows that only the variance is affected by the dependence in the homo-correlated setting, and both the bias and the variance are affected in the hetero-correlated setting. We also showed that the result can be extended to the infinite-dimensional setting \( p = \infty \).

A limitation of this study is that the dependent data are assumed to be Gaussian. While it is common to assume Gaussianity in the dependent data analysis, there are several ways to relax this assumption. However, in our study, it is non-trivial to relax it because Sudakov’s inequality for the derived moment inequality heavily depends on Gaussianity. Further, Gaussianity also plays a critical role in reducing the correlation of data to handle the bias term. In the over-parameterized setting, avoiding Gaussianity is important for future work.

An important direction of extension is the application to more complicated models, such as latent variable models. For example, Bunea et al. (2022) considered a factor model in an independent data setting and showed that the effect of latent noise has a significant impact on the conditions for the establishment of benign overfitting. How this would be affected under dependent data is an interesting direction of extension.
Supplementary Material of “Benign Overfitting in Time Series Linear Model with Over-Parameterization”

Additional Notation

For the matrix $X = [x_1, ..., x_n]^\top \in \mathbb{R}^n \otimes \mathbb{R}^p$ by the observations, we define its Hilbert-Schmidt norm as $\|X\|_{HS} = \sqrt{\sum_{t=1}^{n} \|x_t\|^2}$.

Appendix A. Proofs of Main Theorems

A.1. Proof of Theorems 1 and 2.

Proof of Theorem 1. The proof relies on the bias–variance decomposition given by Lemma 6 and the fact that temporal structures with homo-correlation are almost negligible. By Lemma 6, upper and lower bounds for $T_B$ and $T_V$ give the bounds for the excess risk. Note that Lemma 5 of Bartlett et al. (2020) and Gaussianity yield the existence of constants $b, c_1 \geq 1$ such that for all $k$ with $r_k(\Sigma) \geq bn$, $\mu_1(A_k)/\mu_n(A_k) \leq c_2^2$ with probability at least $1 - 2 \exp(-n/c_1)$. We only give the proof for the statement (ii) because the derivation of (i) is quite parallel by using Lemma 23.

(Step 1: invariance of $T_B$). We see that $T_B$ is invariant in $\Xi_n$ and thus the bounds for $T_B$ are the same as i.i.d. case. For any invertible matrix $A \in \mathbb{R}^n \otimes \mathbb{R}^n$,

$$
\left( I - X^\top (XX^\top)^{-1} X \right) \Sigma \left( I - X^\top (XX^\top)^{-1} X \right) = \left( I - (AX)^\top \left( (AX)(AX)^\top \right)^{-1} (AX) \right) \Sigma \left( I - (AX)^\top \left( (AX)(AX)^\top \right)^{-1} (AX) \right). \quad (18)
$$

Therefore, without loss of generality, we can regard $\Xi_n = I_n$. Hence, we consider the upper and lower bounds of $T_B$ by Theorem 1 of Tsigler and Bartlett (2020) and obtain

$$
2(\beta^*)^\top T_B \beta^* \leq c_3 \left( \|\beta^*\|_{\Sigma_{k^*}^{\infty}}^2 + \|\beta^*\|^2_{\Sigma_{0,k^*}} \frac{\sum_{i>k^*} \lambda_i}{n} \right)^2
$$

with probability at least $1 - 2 \exp(-n/c_1) - c_2 \exp(-n/c_2)$ for some $c_2$ dependent only on $b, c_1$.

(Step 2: decorrelation of $\text{tr}(T_V)$). Let $X_{\text{deco}} := \Xi_n^{-1/2}X$, a temporally decorrelated version of $X$. It holds that

$$
\text{tr}(T_V) = \text{tr} \left( \Sigma_n^{1/2} (XX^\top)^{-1} X \Sigma X^\top (XX^\top)^{-1} \Sigma_n^{1/2} \right) = \text{tr} \left( \Xi_n^{-1/2} \Sigma_n^{-1/2} (X_{\text{deco}} X_{\text{deco}}^\top)^{-1} X_{\text{deco}} \Sigma X_{\text{deco}}^\top (X_{\text{deco}} X_{\text{deco}}^\top)^{-1} \right) \leq \|\Xi_n^{-1/2} \Sigma_n^{-1/2}\| \text{tr} \left( (X_{\text{deco}} X_{\text{deco}}^\top)^{-1} X_{\text{deco}} \Sigma X_{\text{deco}}^\top (X_{\text{deco}} X_{\text{deco}}^\top)^{-1} \right)
$$
and
\[
\text{tr} \left( \left( X_{\text{deco}}X_{\text{deco}}^\top \right)^{-1} X_{\text{deco}} \Sigma X_{\text{deco}}^\top \left( X_{\text{deco}}X_{\text{deco}}^\top \right)^{-1} \right) \\
= \text{tr} \left( \left( \Xi_n^{-1/2} \Upsilon_n \Xi_n^{-1/2} \right)^{-1} \Xi_n^{-1/2} \Upsilon_n \Xi_n^{-1/2} \left( X_{\text{deco}}X_{\text{deco}}^\top \right)^{-1} X_{\text{deco}} \Sigma X_{\text{deco}}^\top \left( X_{\text{deco}}X_{\text{deco}}^\top \right)^{-1} \right) \\
\leq \| \Upsilon_n^{-1} \Xi_n \| \text{tr} \left( \Xi_n^{-1/2} \Upsilon_n \Xi_n^{-1/2} \left( X_{\text{deco}}X_{\text{deco}}^\top \right)^{-1} X_{\text{deco}} \Sigma X_{\text{deco}}^\top \left( X_{\text{deco}}X_{\text{deco}}^\top \right)^{-1} \right) \\
= \| \Upsilon_n^{-1} \Xi_n \| \text{tr}(T_V).
\]
Hence Lemma 6 of [Bartlett et al. (2020)] yields the existence of \( c_3 \geq 1 \) such that for all \( k \) with \( 0 \leq k \leq n/c_3 \), \( r_k(\Sigma) \geq bn \),

\[
\text{tr}(T_V) \leq c_3 \| \Xi_n^{-1} \Upsilon_n \| \left( \frac{k}{n} + \frac{n}{R_k(\Sigma)} \right) (2 \log(\delta^{-1}) + 1).
\]
with probability at least \( 1 - \delta - 7 \exp(-n/c_3) \). Hence we yield the desired conclusion for sufficiently large \( c > 0 \).

\[ \square \]

**Proof of Theorem 2.** The identical discussion to the previous proof along with Theorem 2 of [Tsigler and Bartlett (2020)] and Lemma 8 of [Bartlett et al. (2020)] yields the conclusion. \[ \square \]

A.2. **Proof of Theorem 3.** These lemmas are adapted from Lemmas 6 and 8 of [Bartlett et al. (2020)]. The most significant difference is that the constants are dependent on \( \epsilon \), the parameter controlling neighboring between \( \Xi_{i,n} \) and \( \Xi_{0,n} \). Because \( \epsilon = 1 \) for \( \Xi_{0,n} = I_n \) in i.i.d. settings, we can see that relatively short-range dependent \( x_t \) with respect to some sequence \( \Xi_{0,n} \) lead to similar results in i.i.d. (note that \( x_t \) is short-range dependent if \( \Xi_{i,n} \) and \( I_n \) are \( \epsilon \)-neighboring; we can regard that \( x_t \) with \( \epsilon \)-neighboring \( \Xi_{i,n} \) and \( \Xi_{0,n} \) are “relatively” short-range dependent by introducing the reference covariance \( \Xi_{0,n} \) instead of \( I_n \)).

**Proof of Theorem 3.** As the proof of Theorem 1, the proof is based on the bias–variance decomposition of the excess risk shown by Lemma 6.

(Step 1: \( T_B \) and moment inequality). Firstly, we consider the upper bound of \( \beta^\top T_B \beta \) and adapt the proof of Lemma S.18 of [Bartlett et al. (2020)]. We see

\[
\left( I - X^\top (XX^\top)^{-1} X \right) X^\top = X^\top - X^\top (XX^\top)^{-1} XX^\top = 0
\]
and for any \( v \) in the orthogonal complement to the span of the columns of \( X^\top \),

\[
\left( I - X^\top (XX^\top)^{-1} X \right) v = v.
\]
Hence

\[
\left\| I - X^\top (XX^\top)^{-1} X \right\| \leq 1.
\]
Combining it with (8), we see
\[
(\beta^*)^\top T_B \beta^* = (\beta^*)^\top \left( I - X^\top (XX^\top)^{-1} X \right) \Sigma \left( I - X^\top (XX^\top)^{-1} X \right) \beta^*
\]
\[
= (\beta^*)^\top \left( I - X^\top (XX^\top)^{-1} X \right) \left( \Sigma - \frac{1}{n} X^\top X \right) \left( I - X^\top (XX^\top)^{-1} X \right) \beta^*
\]
\[
\leq \left\| \Sigma - \frac{1}{n} X^\top X \right\| \|\beta^*\|^2.
\]

Proposition 4.6 of Han and Li (2020) along with Assumption 2 (ii) leads to the bound
\[
E \left[ \left\| \frac{1}{n} X^\top X - \Sigma \right\| \right] \leq \frac{2\sqrt{2}}{n} \left( \sqrt{2 \text{tr} (\Sigma_n)} + \sqrt{2n \|\Sigma\| \text{tr} (\Sigma_n)} + \sqrt{n \text{tr} (\Sigma)} \|\Sigma\| \right).
\]

Markov’s inequality yields that for all \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \),
\[
\left\| \Sigma - \frac{1}{n} X^\top X \right\| \leq \delta^{-1} \frac{2\sqrt{2}}{n} \left( \sqrt{2 \text{tr} (\Sigma_n)} + \sqrt{2n \|\Sigma\| \text{tr} (\Sigma_n)} + \sqrt{n \text{tr} (\Sigma)} \|\Sigma\| \right),
\]
and on the same event,
\[
(\beta^*)^\top T_B \beta^* \leq \delta^{-1} \|\beta^*\|^2 \frac{2\sqrt{2}}{n} \left( \sqrt{2 \text{tr} (\Sigma_n)} + \sqrt{2n \|\Sigma\| \text{tr} (\Sigma_n)} + \sqrt{n \text{tr} (\Sigma)} \|\Sigma\| \right).
\]

**Step 2:** \( T_V \) and \( \epsilon \)-neighboring. For any positive definite matrix \( A \in \mathbb{R}^n \otimes \mathbb{R}^n \), it holds that
\[
\text{tr} (T_V) = \text{tr} \left( \Upsilon_n (XX^\top)^{-1} X \Sigma X^\top (XX^\top)^{-1} \right)
\]
\[
= \text{tr} \left( A^{-1/2} \Upsilon_n A^{-1/2} \left( A^{-1/2} XX^\top A^{-1/2} \right)^{-1} A^{-1/2} XX^\top A^{-1/2} \left( A^{-1/2} XX^\top A^{-1/2} \right)^{-1} \right)
\]
By letting \( A = \Xi_{0,n} \), we obtain
\[
\left\| \Upsilon_n^{-1} \Xi_{0,n} \right\|^{-1} \text{tr} \left( \left( \Xi_{0,n}^{-1/2} XX^\top \Xi_{0,n}^{-1/2} \right)^{-1} \Xi_{0,n}^{-1/2} XX^\top \Xi_{0,n}^{-1/2} \right)
\]
\[
\leq \text{tr} (T_V) \leq \left\| \Xi_{0,n}^{-1} \Upsilon_n \right\| \text{tr} \left( \left( \Xi_{0,n}^{-1/2} XX^\top \Xi_{0,n}^{-1/2} \right)^{-1} \Xi_{0,n}^{-1/2} XX^\top \Xi_{0,n}^{-1/2} \right)
\]
and thus we only need to give bounds of the trace appearing in both the lower and upper bounds of \( \text{tr} (T_V) \). Lemmas 21 and 23 along with Assumption 2 (uniform \( \epsilon \)-neighboring of \( \Xi_{i,n} \) and \( \Xi_{0,n} \) over all \( i = 1, \ldots, p \)) lead to the conclusion.

**Appendix B. Bias-Variance Decomposition of the Excess Risk**

We give the bias-variance decomposition of the excess risk shown in Lemma 6. The following lemma is adapted from Lemma S.1 of Bartlett et al. (2020).

**Lemma 14.** The excess risk of the minimum norm estimator satisfies
\[
R(\hat{\beta}) = E_{x^*} \left[ \left( x^* \right)^\top \left( \beta^* - \hat{\beta} \right) \right] \leq 2 (\beta^*)^\top T_B \beta^* + 2 \left( \Upsilon_n^{-1/2} \mathcal{E} \right)^\top T_V \left( \Upsilon_n^{-1/2} \mathcal{E} \right)
\]
and

\[ E_{\mathcal{E}} R(\hat{\beta}) = (\beta^*)^T T_B \beta^* + \text{tr} (T_V) \]

where

\[ T_B := \left( I - X^T (XX^T)^{-1} X \right) \Sigma \left( I - X^T (XX^T)^{-1} X \right), \]
\[ T_V := \Upsilon_{n}^{1/2} (XX^T)^{-1} X \Sigma X^T (XX^T)^{-1} \Upsilon_{n}^{1/2}. \]

**Proof of Lemma 14.** Because \( y^* - (x^*)^T \beta^* \) is zero-mean conditionally on \( x^* \),

\[
R(\hat{\beta}) = E_{x^*, y^*} \left[ (y^* - (x^*)^T \hat{\beta})^2 \right] - E_{x^*, y^*} \left[ (y^* - (x^*)^T \beta^*)^2 \right] \\
= E_{x^*, y^*} \left[ (y^* - (x^*)^T \beta^* + (x^*)^T (\beta^* - \hat{\beta}))^2 \right] - E_{x^*, y^*} \left[ (y^* - (x^*)^T \beta^*)^2 \right] \\
= E_{x^*} \left[ (x^*)^T (\beta^* - \hat{\beta})^2 \right].
\]

The equality (8), the definition of \( \Sigma \) and \( Y = X \beta^* + \mathcal{E} \) lead to

\[
R(\hat{\beta}) = E_{x^*} \left[ (x^*)^T \left( \beta^* - X (XX^T)^{-1} (X \beta^* + \mathcal{E}) \right) \right]^2 \\
= E_{x^*} \left[ (x^*)^T \left( I - X (XX^T)^{-1} X \right) \beta^* - (x^*)^T X (XX^T)^{-1} \mathcal{E} \right]^2 \\
\leq 2 E_{x^*} \left[ (x^*)^T \left( I - X (XX^T)^{-1} X \right) \beta^* \right]^2 + 2 E_{x^*} \left[ (x^*)^T X (XX^T)^{-1} \mathcal{E} \right]^2 \\
= 2 (\beta^*)^T \left( I - X^T (XX^T)^{-1} X \right) \Sigma \left( I - X^T (XX^T)^{-1} X \right) \beta^* \\
+ 2 \mathcal{E}^T (XX^T)^{-1} X \Sigma X^T (XX^T)^{-1} \mathcal{E} \\
= 2 (\beta^*)^T T_B \beta^* + 2 \mathcal{E}^T \Upsilon_{n}^{-1/2} \Upsilon_{n}^{1/2} (XX^T)^{-1} X \Sigma X^T (XX^T)^{-1} \Upsilon_{n}^{1/2} \Upsilon_{n}^{-1/2} \mathcal{E} \\
= 2 (\beta^*)^T T_B \beta^* + 2 \left( \Upsilon_{n}^{-1/2} \mathcal{E} \right)^T T_V \left( \Upsilon_{n}^{-1/2} \mathcal{E} \right).
\]

Because \( \mathcal{E} \) is centered and independent of \( x^* \) and \( X \),

\[
E_{\mathcal{E}} R(\hat{\beta}) = E_{x^*, \mathcal{E}} \left[ (x^*)^T \left( I - X (XX^T)^{-1} X \right) \beta^* - (x^*)^T X (XX^T)^{-1} \mathcal{E} \right]^2 \\
= (\beta^*)^T \left( I - X^T (XX^T)^{-1} X \right) \Sigma \left( I - X^T (XX^T)^{T} X \right) \beta^* \\
+ \text{tr} \left( (XX^T)^{-1} X \Sigma X^T (XX^T)^{-1} \mathcal{E} \mathcal{E}^{T} \right) \\
= (\beta^*)^T T_B \beta^* + \text{tr} \left( \Upsilon_{n}^{1/2} (XX^T)^{-1} X \Sigma X^T (XX^T)^{-1} \Upsilon_{n}^{1/2} \right) \\
= (\beta^*)^T T_B \beta^* + \text{tr} (T_V).
\]

Hence we obtain the equality. \( \square \)
To bound the tail probability of the term $T_V$, we give a corollary of Lemma S.2 of Bartlett et al. (2020).

**Lemma 15.** For all $M$, $n \times n$-dimensional a.s. positive semi-definite random matrices, with probability at least $1 - e^{-t}$,

$$(\Upsilon_n^{-1/2} \mathcal{E})^\top M \Upsilon_n^{-1/2} \mathcal{E} \leq \text{tr} (M) + 2 \|M\| t + 2 \sqrt{\|M\|^2 t^2 + \text{tr} (M^2) t}.$$

**Proof of Lemma 15.** It follows from Lemma S.2 of Bartlett et al. (2020) and the facts such that $E$ is independent of $X$ and $\Upsilon_n^{-1/2} \mathcal{E} \sim N (0, I_n)$. □

As Bartlett et al. (2020), we see that with probability at least $1 - e^{-t}$,

$$(\Upsilon_n^{-1/2} \mathcal{E})^\top T_V (\Upsilon_n^{-1/2} \mathcal{E}) \leq (4t + 2) \text{tr} (T_V).$$

By combining these results, we obtain Lemma 6.

**Appendix C. Upper and Lower Bounds on the Variance Term $T_V$ with (Possibly) Hetero-Correlated Process**

In this section, we describe a $p$-dimensional Euclidean space as a Hilbert space $\mathbb{H}$, because we can use this result directly to the extension to infinite dimensions in Section 7. With this notation, the terms are regarded as $\Sigma : \mathbb{H} \to \mathbb{H}$, $X : \mathbb{H} \to \mathbb{R}^n$, and $T_V \in \mathbb{H} \otimes \mathbb{H}$ defined below are generalizations of the matrices appearing in the papers with the same notations.

Let $n \in \mathbb{N}$ be a finite number with $n < p$, $\{e_i \in \mathbb{H}; i = 1, \ldots, p\}$ be an orthonormal system in $\mathbb{H}$, and $\{\lambda_i; i = 1, \ldots, n\}$ be a nonincreasing sequence of nonnegative numbers with $\lambda_{n+1} > 0$ and $\sum_i \lambda_i < \infty$. We also define positive definite matrices $\Upsilon_n$, $\Xi_{i,n}$, $i = 1, \ldots, p$, and $\Xi_{0,n}$. We set the following assumptions on $\Xi_{i,n}$ and $\Xi_{0,n}$.

**Assumption 3.** For some $\epsilon \in (0, 1]$, for all $i = 1, \ldots, p$ and $n \in \mathbb{N}$ with $n < p$,

$$\epsilon \leq \mu_n (\Xi_{0,n}^{-1} \Xi_{i,n}^{-1}) \leq \mu_1 (\Xi_{0,n}^{-1} \Xi_{i,n}^{-1}) \leq \epsilon^{-1}.$$

We define a non-random positive semi-definite operator $\Sigma : \mathbb{H} \to \mathbb{H}$ and a random operator $X : \mathbb{H} \to \mathbb{R}^n$ such that for all $u \in \mathbb{H}$,

$$\Sigma u = \sum_{i: \lambda_i > 0} \lambda_i (e_i^\top u) e_i,$$

$$X u = \sum_{i: \lambda_i > 0} \sqrt{\lambda_i} \Xi_{i,n}^{1/2} z_i (e_i^\top u),$$

where $(e_i^\top u)$ is the inner product between $e_i$ and $u$, and $z_i$ are i.i.d. $n$-dimensional standard Gaussian random variables. The infinite series in $X$ converges almost surely (and in $L^p$ for
all $p \geq 1$) by Assumption 3 along with the Itô–Nisio theorem because $\Xi_{0,n}^{-1/2}Xu$ converges. Furthermore, we define a random matrix $T_V$ such that

$$T_V := \Upsilon_n^{1/2} (XX^\top)^{-1} XX^\top (XX^\top)^{-1} \Upsilon_n^{1/2}$$

(the invertibility of $XX^\top$ follows from $\lambda_{n+1} > 0$, independence of $z_i$, and positive definiteness of $\Xi_{i,n}$).

Furthermore, we set several notations. Let $\tilde{\Xi}_{k,n} := \Xi_{0,n}^{-1/2} \Xi_{k,n}^{-1/2}$, $\tilde{\Upsilon}_n := \Xi_{0,n}^{-1/2} \Upsilon_n^{-1/2}$, and $\tilde{z}_k = [\tilde{z}_{1,k}, \ldots, \tilde{z}_{n,k}]^\top$ be a random vector defined as

$$\tilde{z}_k := \Xi_{0,n}^{-1/2} \Xi_{k,n}^{-1/2} z_k = \Xi_{0,n}^{-1/2} Xe_k / \sqrt{\lambda_k} \sim N \left(0, \tilde{\Xi}_{k,n} \right).$$

We also define some random matrices for $i, k \in \mathbb{N}$ satisfying $\lambda_i > 0$ and $\lambda_k > 0$ such that

$$A = \sum_{i : \lambda_i > 0} \lambda_i \tilde{z}_i \tilde{z}_i^\top, \quad A_{-i} = \sum_{j \neq i : \lambda_j > 0} \lambda_j \tilde{z}_j \tilde{z}_j^\top, \quad \text{and} \quad A_k = \sum_{i < k : \lambda_i > 0} \lambda_i \tilde{z}_i \tilde{z}_i^\top.$$

C.1. **Supportive Result.** Using the notation above, we give another variation of Lemma 3 of [Bartlett et al. (2020)] by introducing the effect of the noise covariance.

**Lemma 16.** Let us assume $\lambda_{n+1} > 0$. We have

$$\text{tr} \left( T_V \right) = \sum_{i : \lambda_i > 0} \left[ \lambda_i^2 \tilde{z}_i^\top \left( \sum_{j : \lambda_j > 0} \lambda_j \tilde{z}_j \tilde{z}_j^\top \right)^{-1} \tilde{\Upsilon}_n \left( \sum_{j : \lambda_j > 0} \lambda_j \tilde{z}_j \tilde{z}_j^\top \right)^{-1} \tilde{z}_i \right],$$

where $\tilde{z}_i \sim N \left(0, \tilde{\Xi}_{i,n} \right)$'s are independent $\mathbb{R}^n$-valued random variables. In addition,

$$\text{tr} \left( T_V \right) \leq \mu_1 \left( \tilde{\Upsilon}_n \right) \sum_{i : \lambda_i > 0} \left[ \lambda_i^2 \tilde{z}_i^\top \left( \sum_{j : \lambda_j > 0} \lambda_j \tilde{z}_j \tilde{z}_j^\top \right)^{-2} \tilde{z}_i \right],$$

$$\text{tr} \left( T_V \right) \geq \mu_n \left( \tilde{\Upsilon}_n \right) \sum_{i : \lambda_i > 0} \left[ \lambda_i^2 \tilde{z}_i^\top \left( \sum_{j : \lambda_j > 0} \lambda_j \tilde{z}_j \tilde{z}_j^\top \right)^{-2} \tilde{z}_i \right],$$

and for all $i$ with $\lambda_i > 0$,

$$\lambda_i^2 \tilde{z}_i^\top \left( \sum_{j : \lambda_j > 0} \lambda_j \tilde{z}_j \tilde{z}_j^\top \right)^{-2} \tilde{z}_i = \frac{\lambda_i^2 \tilde{z}_i^\top A_{-i}^{-1} \tilde{z}_i}{\left(1 + \lambda_i \tilde{z}_i^\top A_{-i}^{-1} \tilde{z}_i \right)^2}.$$

**Proof of Lemma 16.** The linear map $X : \mathbb{H} \to \mathbb{R}^n$ has the decomposition with respect to the orthonormal system $\{e_k\}$

$$X = \sum_{i : \lambda_i > 0} \sqrt{\lambda_i} \Xi_{i,n}^{1/2} z_i e_i^\top.$$
It leads to
\[ XX^\top = \left( \sum_{i: \lambda_i > 0} \sqrt{\lambda_i} \Xi_{i,n}^{1/2} z_i e_i^\top \right) \left( \sum_{i: \lambda_i > 0} \sqrt{\lambda_i} \Xi_{i,n}^{1/2} z_i e_i^\top \right)^\top = \left( \sum_{i: \lambda_i > 0} \lambda_i \Xi_{i,n}^{1/2} z_i z_i^\top \Xi_{i,n}^{1/2} \right) \]
and
\[ X\Sigma X^\top = \left( \sum_{i: \lambda_i > 0} \sqrt{\lambda_i} \Xi_{i,n}^{1/2} z_i e_i^\top \right) \left( \sum_{i: \lambda_i > 0} \lambda_i e_i e_i^\top \right) \left( \sum_{i: \lambda_i > 0} \sqrt{\lambda_i} \Xi_{i,n}^{1/2} z_i e_i^\top \right)^\top = \left( \sum_{i: \lambda_i > 0} \lambda_i^2 \Xi_{i,n}^{1/2} z_i z_i^\top \Xi_{i,n}^{1/2} \right). \]

We obtain
\[
\text{tr} (T_V) = \text{tr} \left( \Upsilon_n^{1/2} (XX^\top)^{-1} X\Sigma X^\top (XX^\top)^{-1} \Upsilon_n^{1/2} \right) \\
= \text{tr} \left( \Xi_{0,n}^{-1/2} \Upsilon_n^{1/2} \Xi_{0,n}^{-1/2} (\Xi_{0,n}^{-1/2} XX^\top \Xi_{0,n}^{-1/2})^{-1} \Xi_{0,n}^{-1/2} X\Sigma X^\top \Xi_{0,n}^{-1/2} (\Xi_{0,n}^{-1/2} XX^\top \Xi_{0,n}^{-1/2})^{-1} \right) \\
= \sum_{i: \lambda_i > 0} \left( \lambda_i^2 \Xi_i \left( \sum_{j: \lambda_j > 0} \lambda_j \Xi_j \Xi_j^\top \Xi_i \right)^{-1} \Xi_i \left( \sum_{j: \lambda_j > 0} \lambda_j \Xi_j \Xi_j^\top \Xi_i \right)^{-1} \Xi_i \right).
\]

The following statements immediately follow by Lemma S.3 of Bartlett et al. (2020).

The following lemma is a variation of Corollary 1 of Bartlett et al. (2020).

**Lemma 17.** There is a universal constant \( a > 0 \) such that for any mean-zero Gaussian random variable \( \bar{z} \in \mathbb{R}^n \) with the covariance matrix \( \Xi \), any random subspace \( \mathcal{L} \) of \( \mathbb{R}^n \) of codimension \( d \) that is independent of \( z \), and any \( u > 0 \), with probability at least \( 1 - 3e^{-u} \),
\[
\|\bar{z}\|^2 \leq \text{tr} (\Xi) + a \mu_1 (\Xi) (u + \sqrt{nu}), \\
\|\Pi_{\mathcal{L}} \bar{z}\|^2 \geq \text{tr} (\Xi) - a \mu_1 (\Xi) (d + u + \sqrt{nu}),
\]
where \( \Pi_{\mathcal{L}} \) is the orthogonal projection on \( \mathcal{L} \).

**Proof of Lemma 17.** We use the notation \( z := \Xi^{-1/2} \bar{z} \), whose distribution is \( N (0, I_n) \). Theorem 1.1 of Rudelson and Vershynin (2013) verifies that there is a universal constant \( c > 0 \) such that at least probability \( 1 - 2e^{-u} \),
\[
|z^\top \Xi z - \text{tr} (\Xi)| \leq c \left( \mu_1 (\Xi) u + \|\Xi\|_{\text{HS}} \sqrt{u} \right) \leq c \mu_1 (\Xi) (u + \sqrt{nu})
\]
We can easily obtain the first inequality of the statement, because we have
\[
\|\bar{z}\|^2 = z^\top \Xi z \leq \text{tr} (\Xi) + c \mu_1 (\Xi) (u + \sqrt{nu}).
\]
With respect to the second inequality of the statement, we have the decomposition as [Bartlett et al. (2020)]:

$$\|\Pi_{\mathcal{L}}\tilde{z}\|^2 = \|\tilde{z}\|^2 - \|\Pi_{\mathcal{L}^\perp}\tilde{z}\|^2.$$  

Note that

$$\|\tilde{z}\|^2 = z^T\Xi z \geq \text{tr} (\Xi) - c\mu_1 (\Xi) (u + \sqrt{nu}) .$$

Using the transpose $\Pi_{\mathcal{L}^\perp}^T$ of the projection $\Pi_{\mathcal{L}^\perp}$, we define $M := \Pi_{\mathcal{L}^\perp}^T \Pi_{\mathcal{L}^\perp}$, $\|M\| = 1$ and $\text{tr} (M) = \text{tr} (M^2) = d$. Then, Lemma S.2 of [Bartlett et al. (2020)] leads to the bound that at least probability $1 - e^{-u}$,

$$\|\Pi_{\mathcal{L}^\perp}\tilde{z}\|^2 \leq z^T\Xi^{1/2}M\Xi^{1/2}z$$

$$\leq \text{tr} (\Xi M) + 2 \|\Xi^{1/2}M\Xi^{1/2}\| u + 2\sqrt{\|\Xi^{1/2}M\Xi^{1/2}\|^2 u^2 + \text{tr} ((\Xi M)^2) u}$$

$$\leq \mu_1 (\Xi) \text{tr} (M) + 2\mu_1 (\Xi) u + 2\sqrt{\mu_1^2 (\Xi) u^2 + \mu_1^2 (\Xi) \text{tr} (M^2) u}$$

$$\leq \mu_1 (\Xi) d + 2\mu_1 (\Xi) u + 2\mu_1^2 (\Xi) u^2 + \mu_1^2 (\Xi) du$$

$$= \mu_1 (\Xi) (d + 2u + 2\sqrt{u(d + u)})$$

$$\leq \mu_1 (\Xi) (2d + 4u)$$

by some trace inequalities (e.g., immediately obtained by von Neumann trace inequalities)

$$\text{tr} (\Xi M) \leq \mu_1 (\Xi) \text{tr} (M),$$

$$\text{tr} (\Xi M\Xi M) \leq \mu_1 (\Xi) \text{tr} (M\Xi M) = \mu_1 (\Xi) \text{tr} (\Xi M^2) \leq \mu_1^2 (\Xi) \text{tr} (M^2).$$

Hence

$$\|\Pi_{\mathcal{L}^\perp}\tilde{z}\|^2 \geq \text{tr} (\Xi) - c\mu_1 (\Xi) (u + \sqrt{nu}) - \mu_1 (\Xi) (2d + 4u)$$

$$\geq \text{tr} (\Xi) - \mu_1 (\Xi) (c (u + \sqrt{nu}) + 2d + 4u).$$

We obtain the second inequality. \hfill \Box

**Lemma 18.** Under Assumption \(\mathbf{3}\), there is a universal constant $c > 0$ such that with probability at least $1 - 2e^{-n/c}$,

$$\frac{c}{c} \sum_i \lambda_i - c\epsilon^{-2}\lambda_1 n \leq \mu_n (A) \leq \mu_1 (A) \leq c \left( \epsilon^{-1} \sum_i \lambda_i + \epsilon^{-2}\lambda_1 n \right).$$  

(19)

**Proof of Lemma 18.** For any $v \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $\lambda_k > 0$, $v^T\tilde{z}_k$ is clearly $c_0\epsilon^{-1} \|v\|^2$-sub-Gaussian with a universal constant $c_0$. Because for all fixed unit vector $v \in \mathbb{R}^n$, $(v^T\tilde{z}_k)^2 - v^T\Xi_k v$ is a centered $c_1\epsilon^{-1}$-sub-exponential random variable with a universal constant $c_1 > 0$, Proposition 2.5.2 and Lemma 2.7.6 of [Vershynin (2018)] yield that there exists a universal
constant $c_2 = c_2 (c_1) > 0$ such that for any fixed unit vector $v \in \mathbb{R}^n$, with probability at least $1 - 2e^{-t}$,

$$
\left| v^T A v - v^T \left( \sum_i \lambda_i \tilde{\Xi}_{i,n} \right) v \right| \leq c_2 \epsilon^{-1} \max \left( \sup_{k: \lambda_k > 0} \left( v^T \tilde{\Xi}_{k,n} v \right) \lambda_k t, \sqrt{t \sum_i \lambda_i \left( v^T \tilde{\Xi}_{i,n} v \right)} \right)
$$

$$
\leq c_2 \epsilon^{-1} \max \left( \sup_{k: \lambda_k > 0} \left( v^T \tilde{\Xi}_{k,n} v \right) \lambda_1 t, \sqrt{\sup_{k: \lambda_k > 0} \left( v^T \tilde{\Xi}_{k,n} v \right) t \sum_i \lambda_i} \right)
$$

$$
\leq c_2 \epsilon^{-1} \left( t^{-1} \lambda_1, \sqrt{t^{-1} \sum_i \lambda_i} \right)
$$

$$
\leq c_2 \epsilon^{-2} \left( t \lambda_1, \sqrt{t \sum_i \lambda_i} \right).
$$

(see also as Lemma S.9 of [Bartlett et al., 2020]). Let $\mathcal{N}$ be a $1/4$-net on $S^{n-1} := \{v \in \mathbb{R}^n : \|v\| = 1\}$ such that $|\mathcal{N}| \leq 9^n$. The union bound argument leads to that with probability at least $1 - 2e^{-t}$, for all $v \in \mathcal{N}$,

$$
\left| v^T Av - v^T \left( \sum_i \lambda_i \tilde{\Xi}_{i,n} \right) v \right| \leq c_2 \epsilon^{-2} \max \left( (t + n \log 9) \lambda_1, \sqrt{(t + n \log 9) \sum_i \lambda_i} \right)
$$

By Lemma S.8 of [Bartlett et al. (2020)], there exists a universal constant $c_3 = c_3 (c_2) > 0$ such that with probability at least $1 - 2e^{-t}$,

$$
\left\| A - \sum_i \lambda_i \tilde{\Xi}_{i,n} \right\| \leq c_3 \epsilon^{-2} \left( (t + n \log 9) \lambda_1 + \sqrt{(t + n \log 9) \sum_i \lambda_i} \right).
$$

Note that

$$
\left\| A - \sum_i \lambda_i \tilde{\Xi}_{i,n} \right\| = \max_{v \in S^{n-1}} \left| v^T \left( A - \sum_i \lambda_i \tilde{\Xi}_{i,n} \right) v \right| \geq \max_{v \in S^{n-1}} v^T Av - \epsilon^{-1} \sum_i \lambda_i,
$$

and

$$
\left\| A - \sum_i \lambda_i \tilde{\Xi}_{i,n} \right\| = \max_{v \in S^{n-1}} \left| v^T \left( \sum_i \lambda_i \tilde{\Xi}_{i,n} - A \right) v \right| \geq \epsilon \sum_i \lambda_i - \min_{v \in S^{n-1}} v^T Av.
$$

The remaining discussion is parallel to [Bartlett et al. (2020)].

The following corollary is a version of Lemma 4 of [Bartlett et al. (2020)].
Corollary 19. Under Assumption 3, there is a constant \( c = c(\epsilon) > 0 \) such that for any \( k \geq 0 \), with probability at least \( 1 - 2e^{-n/c} \),

\[
\frac{1}{c} \sum_{i > k} \lambda_i - c \lambda_1 n \leq \mu_n(A_k) \leq \mu_1(A_k) \leq c \left( \sum_{i > k} \lambda_i + \lambda_1 n \right).
\]

We give a variation of Lemma 5 of Bartlett et al. (2020).

Lemma 20. Under Assumption 3, there are constants \( b = b(\epsilon) \geq 1 \) and \( c = c(\epsilon) \geq 1 \) such that for any \( k \geq 0 \), with probability at least \( 1 - 2e^{-n/c} \),

(1) for all \( i \geq 1 \),

\[
\mu_{k+1}(A_{-i}) \leq \mu_{k+1}(A) \leq \mu_1(A_k) \leq c \left( \sum_{j > k} \lambda_j + n \lambda_{k+1} \right);
\]

(2) for all \( 1 \leq i \leq k \),

\[
\mu_n(A) \geq \mu_n(A_{-i}) \geq \mu_n(A_k) \geq \frac{1}{c} \sum_{j > k} \lambda_j - cn \lambda_{k+1};
\]

(3) if \( r_k(\Sigma) \geq bn \), then

\[
\frac{1}{c} \lambda_{k+1} r_k(\Sigma) \leq \mu_n(A_k) \leq \mu_1(A_k) \leq c \lambda_{k+1} r_k(\Sigma).
\]

Proof of Lemma 20. Let us recall Corollary 19: there exists a constant \( c_1 = c_1(\epsilon) > 0 \) such that for all \( k \geq 0 \), with probability at least \( 1 - 2e^{-n/c_1} \),

\[
\frac{1}{c_1} \sum_{i > k} \lambda_i - c_1 \lambda_1 n \leq \mu_n(A_k) \leq \mu_1(A_k) \leq c_1 \left( \sum_{i > k} \lambda_i + \lambda_1 n \right).
\]

Firstly, notice that the matrix \( A - A_k \) has its rank at most \( k \). Thus, there is a linear space \( \mathcal{L} \) of dimension \( n - k \) such that for all \( v \in \mathcal{L} \), \( v^T Av = v^T A_k v \leq \mu_1(A_k) \|v\|^2 \) and therefore \( \mu_{k+1}(A) \leq \mu_1(A_k) \) (Lemma S.10 of Bartlett et al., 2020).

In the second place, Lemma S.11 of Bartlett et al. (2020) yields that for all \( i \) and \( j \), \( \mu_j(A_{-i}) \leq \mu_j(A) \). On the other hand, for all \( i \leq k \), \( \mu_n(A_{-i}) \geq \mu_n(A_k) \) by \( A_k \preceq A_{-i} \) and Lemma S.11 of Bartlett et al. (2020) too.
Finally, if \( r_k(\Sigma) \geq bn \) for some \( b \geq 1 \),
\[
c_1 \left( \sum_{j>k} \lambda_j + n\lambda_{k+1} \right) = c_1 \left( \lambda_{k+1}r_k(\Sigma) + n\lambda_{k+1} \right) \\
\leq \left( c_1 + \frac{c_1}{b} \right) \lambda_{k+1}r_k(\Sigma),
\]
\[
\frac{1}{c_1} \sum_{j>k} \lambda_j - c_1n\lambda_{k+1} = \frac{1}{c_1} \lambda_{k+1}r_k(\Sigma) - c_1n\lambda_{k+1} \\
\geq \left( \frac{1}{c_1} - \frac{c_1}{b} \right) \lambda_{k+1}r_k(\Sigma).
\]
Let \( b > c_1^2 \) and \( c > \max \{ c_1 + 1/c_1, 1/ (1/c_1 - c_1/b) \} \), then the third statement holds. \( \square \)

C.2. Derivation of Upper Bound. We provide the following lemma for an upper bound by developing a variation of Lemma 6 of [Bartlett et al. (2020)].

**Lemma 21.** Under Assumption 3, there are constants \( b = b(\epsilon) \geq 1 \) and \( c = c(\epsilon) \geq 1 \) such that if \( 0 \leq k \leq n/c, r_k(\Sigma) \geq bn, \) and \( l \leq k, \) then with probability at least \( 1 - 7e^{-n/c}, \)
\[
\text{tr}(T_V) \leq c\mu_1 \left( \tilde{Y}_n \right) \left( \frac{l}{n} + n \sum_{i=l}^k \frac{\lambda_i^2}{(\sum_{k+1}^\infty \lambda_{i+1})^2} \right).
\]

**Proof of Lemma 21.** Fix \( b = b(\epsilon) \) to its value in Lemma 20. By Lemma 16
\[
\text{tr}(T_V) \leq \mu_1 \left( \tilde{Y}_n \right) \sum_i \lambda_i^2 z_i^\top A^{-2} z_i \\
\leq \mu_1 \left( \tilde{Y}_n \right) \left( \sum_{i=1}^l \frac{\lambda_i^2 z_i^\top A^{-2} z_i}{(1 + \lambda_i z_i^\top A^{-1} z_i)^2} + \sum_{i=l}^k \lambda_i^2 z_i^\top A^{-2} z_i \right).
\]
Firstly, let us consider the sum up to \( l \). Lemma 20 shows that there exists a constant \( c_1 = c_1(\epsilon) \geq 1 \) such that on the event \( E_1 \) with \( P(E_1) \geq 1 - 2e^{-n/c_1} \), if \( k \) satisfies \( r_k(\Sigma) \geq bn \), then for all \( i \leq k \) \( \mu_i(\Sigma-i) \geq \lambda_{k+1}r_k(\Sigma)/c_1 \), and for all \( i \geq 1 \) \( \mu_{k+1}(\Sigma-i) \leq c_1\lambda_{k+1}r_k(\Sigma) \). Hence on \( E_1 \), for all \( z \in \mathbb{R}^n \) and \( 1 \leq i \leq l, \)
\[
\bar{z}^\top A^{-2} \bar{z} \leq \frac{c_1^4 \|z\|^2}{(\lambda_{k+1}r_k(\Sigma))^2},
\]
\[
\bar{z}^\top A^{-1} \bar{z} \geq (\Pi_{L_i} \bar{z})^\top A^{-1}_{L_i} \Pi_{L_i} \bar{z} \geq \frac{\|\Pi_{L_i} \bar{z}\|^2}{c_1 \lambda_{k+1}r_k(\Sigma)},
\]
where \( L_i \) is the span of the \( n-k \) eigenvectors of \( A_{-i} \) corresponding to its smallest \( n-k \) eigenvalues. Therefore, on \( E_1 \), for all \( i \leq l, \)
\[
\frac{\lambda_i^2 z_i^\top A^{-2} z_i}{(1 + \lambda_i z_i^\top A^{-1} z_i)^2} \leq \frac{\bar{z}_i^\top A^{-2} \bar{z}_i}{(\bar{z}_i^\top A^{-1} \bar{z}_i)^2} \leq c_1^4 \frac{\|z_i\|^2}{\|\Pi_{L_i} \bar{z}_i\|^2}.
\]
We apply Lemma 17 $l$ times together with a union bound to show that, there exist constant $c_0 = c_0 (a, \epsilon) > 0$, $c_2 = c_2 (a, c_0, \epsilon) > 0$ and $c_3 = c_3 (a, c_0, \epsilon) > 0$ such that on the event $E_2$ with $P (E_2) \geq 1 - 3e^{-t}$, for all $1 \leq i \leq l$,

$$\|\tilde{z}_i\|^2 \leq \epsilon^{-1}n + a \left( t + \log k + \sqrt{n \left( t + \log k \right)} \right) \leq c_2 n,$$

$$\|\Pi_{\mathcal{F}} \tilde{z}_i\|^2 \geq \epsilon n - a \left( k + t + \log k + \sqrt{n \left( t + \log k \right)} \right) \geq n / c_3,$$

provided that $t < n / c_0$ and $c > c_0$ owing to $\text{tr}(\Xi_{i,n}^{-1} \Xi_{i,n}) \in [\epsilon n, \epsilon^{-1} n]$ and $\log k \leq n / c \leq n / c_0$.

Then, there exists a constant $c_4 = c_4 (c_1, c_2, c_3) > 0$ on the event $E_1 \cap E_2$ with $P (E_1 \cap E_2) \geq 1 - 5e^{-n / c_0}$,

$$\sum_{i=1}^{l} \frac{\lambda_i^2 z_i^\top A^{-2} z_i}{(1 + \lambda_i z_i^\top A^{-1} z_i)^2} \leq c_4 \frac{1}{n}.$$

In the second place, we consider the sum $\sum_{i>l} \lambda_i^2 z_i^\top A^{-2} z_i$. Lemma 20 shows that on $E_1$, $\mu_n (A) \geq \lambda_{k+1}r_k (\Sigma) / c_1$, and thus

$$\sum_{i>l} \lambda_i^2 z_i^\top A^{-2} z_i \leq c_2^2 \sum_{i>l} \lambda_i^2 \left( \frac{\|z_i\|^2}{\lambda_{k+1}r_k (\Sigma)} \right)^2.$$

Note that

$$\sum_{i>l} \lambda_i^2 \|z_i\|^2 = \sum_{i>l} \lambda_i^2 z_i^\top \Xi_{i,n} z_i \leq \epsilon^{-1} \sum_{i>l} \lambda_i^2 \|z_i\|^2 = \epsilon^{-1} \sum_{i>l} \lambda_i^2 \sum_{t=1}^{n} \left( z_i^{(t)} \right)^2,$$

where $z_i = \Xi_{i,n}^{-1/2} z_i$, and $z_i^{(t)} \sim N (0, 1)$ are i.i.d. random variables. Lemma 2.7.6 of Vershynin (2018) yields that there exists a constant $c_5 = c_5 (a, c_0, \epsilon) > 0$ on the event $E_3$ with $P (E_3) \geq 1 - 2e^{-t}$,

$$\sum_{i>l} \lambda_i^2 \|z_i\|^2 \leq \epsilon^{-1} \sum_{i>l} \lambda_i^2 \sum_{t=1}^{n} \left( z_i^{(t)} \right)^2$$

$$\leq \epsilon^{-1} \left( n \sum_{i>l} \lambda_i^2 + a \max \left\{ \lambda_i^2 t_i, \sqrt{tn} \sum_{i>l} \lambda_i^4 \right\} \right)$$

$$\leq \epsilon^{-1} \left( n \sum_{i>l} \lambda_i^2 + a \max \left\{ \sum_{i>l} \lambda_i^2 t_i, \sqrt{tn} \sum_{i>l} \lambda_i^2 \right\} \right)$$

$$\leq c_5 n \sum_{i>l} \lambda_i^2.$$

because $t < n / c_0$. Then there exists a constant $c_6 = c_6 (c_1, c_5) > 0$ such that on $E_1 \cap E_2 \cap E_3$ with $P (E_1 \cap E_2 \cap E_3) \geq 1 - 7e^{-n / c_0}$,

$$\sum_{i>l} \lambda_i^2 z_i^\top A^{-2} z_i \leq c_6 n \frac{\sum_{i>l} \lambda_i^2}{\left( \lambda_{k+1}r_k (\Sigma) \right)^2}.$$
Choosing $c > \max \{c_0, c_4, c_6\}$ gives the lemma. \hfill \Box

C.3. Derivation of Lower Bound. We start with the following lemma as a variation of Lemma 8 of [Bartlett et al. (2020)].

**Lemma 22.** Under Assumption 3 there is a constant $c = c(\epsilon) > 0$ such that for any $i \geq 1$ with $\lambda_i > 0$ and any $0 \leq k \leq n/c$, with probability at least $1 - 5e^{-n/c}$,

$$
\frac{\lambda_i^2 z_i^\top A_{-i}^{-2} z_i}{(1 + \lambda_i z_i^\top A_{-i}^{-1} z_i)^2} \geq \frac{1}{cn} \left(1 + \frac{\sum_{j>k} \lambda_j + n\lambda_{k+1}}{\lambda_i n}\right)^{-2}.
$$

**Proof of Lemma 22.** Fix $i \geq 1$ with $\lambda_i > 0$ and $k$ with $0 \leq k \leq n/c_0$, where $c_0 = c_0(a, \epsilon) > 0$ is a sufficiently large constant such that there exists $c_2 = c_2(a, c_0, \epsilon)$ with

$$
en - ac^{-1} (2n/c_0 + n/\sqrt{c_0}) \geq n/c_2.
$$

By Lemma 20 there exists a constant $c_1 = c_1(\epsilon) \geq 1$ such that on the event $E_1$ with $P(E_1) \geq 1 - 2e^{-n/c_1}$,

$$
\mu_{k+1}(A_{-i}) \leq c_1 \left(\sum_{j>k} \lambda_j + n\lambda_{k+1}\right),
$$

and hence

$$
z_i^\top A_{-i}^{-1} \tilde{z}_i \geq \frac{\|\Pi_{\mathcal{L}_i} \tilde{z}_i\|^2}{c_1 \left(\sum_{j>k} \lambda_j + n\lambda_{k+1}\right)},
$$

where $\mathcal{L}_i$ is the span of the $n-k$ eigenvectors of $A_{-i}$ corresponding to its smallest $n-k$ eigenvalues.Lemma 17 and the definitions of $c_0$ and $c_2$ give that on the event $E_2$ with $P(E_2) \geq 1 - 3e^{-t}$,

$$
\|\Pi_{\mathcal{L}_i} \tilde{z}_i\|^2 \geq en - ae^{-1} \left(k + t + \sqrt{tn}\right) \geq en - ae^{-1} (2n/c_0 + n/\sqrt{c_0}) \geq n/c_2,
$$

provided that $t < n/c_0$. Hence, there exists a constant $c_3 = c_3(c_0, c_1, c_2) > 0$ such that on the event $E_1 \cap E_2$ with $P(E_1 \cap E_2) \geq 1 - 5e^{-n/c_3}$,

$$
z_i^\top A_{-i}^{-1} \tilde{z}_i \geq \frac{n}{c_3 \left(\sum_{j>k} \lambda_j + n\lambda_{k+1}\right)},
$$

and

$$
1 + \lambda_i z_i^\top A_{-i}^{-1} \tilde{z}_i \leq \left(\frac{c_3 \left(\sum_{j>k} \lambda_j + n\lambda_{k+1}\right)}{\lambda_i n}\right) + 1 \lambda_i z_i^\top A_{-i}^{-1} \tilde{z}_i.
$$

Then we have

$$
\frac{\lambda_i^2 z_i^\top A_{-i}^{-2} z_i}{(1 + \lambda_i z_i^\top A_{-i}^{-1} \tilde{z}_i)^2} \geq \left(\frac{c_3 \left(\sum_{j>k} \lambda_j + n\lambda_{k+1}\right)}{\lambda_i n}\right)^{-2} \frac{\tilde{z}_i^\top A_{-i}^{-2} \tilde{z}_i}{(z_i^\top A_{-i}^{-1} \tilde{z}_i)^2}.
$$
The Cauchy–Schwarz inequality and Lemma 17 yield that there exists $c_4 = (a, c_0, \epsilon) > 0$ such that on $E_1$,

$$\frac{\tilde{z}_i^T A^{-2} \tilde{z}_i}{(\tilde{z}_i^T A^{-1} \tilde{z}_i)^2} \geq \frac{\tilde{z}_i^T A^{-2} \tilde{z}_i}{\|A^{-1} \tilde{z}_i\| \|\tilde{z}_i\|^2} = \frac{1}{\epsilon^{-1} n + a \epsilon^{-1} (t + \sqrt{nt})} \geq \frac{1}{c_4 n}.$$  

Then there exists a constant $c_5 = c_5 (c_3, c_4)$ such that for all $i \geq 1$ with $\lambda_i > 0$ and $0 \leq k \leq n/c_0$, with probability at least $1 - 5 \epsilon^{-n/c_3}$,

$$\frac{\lambda_i^2 \tilde{z}_i^T A^{-2} \tilde{z}_i}{(1 + \lambda_i \tilde{z}_i^T A^{-1} \tilde{z}_i)^2} \geq \frac{c_3 \left( \sum_{j > k} \lambda_j + n \lambda_{k+1} \right) + 1}{c_4 n} \geq \frac{1}{c_5 n} \left( 1 + \frac{\sum_{j > k} \lambda_j + n \lambda_{k+1}}{n \lambda_i} \right)^{-2}.$$  

Choosing $c > \max \{c_0, c_3, c_5\}$ gives the lemma. \hfill \Box

Using the above result, we develop a lower bound of $\text{tr} (T V)$, by a variation of Lemma 10 of Bartlett et al. (2020).

**Lemma 23.** Under Assumption 3, there is a constant $c = c(\epsilon) > 0$ such that for any $0 \leq k \leq n/c$ and any $b > 1$ with probability at least $1 - 10 \epsilon^{-n/c}$,

1. if $r_k (\Sigma) < bn$, then $\text{tr} (T V) \geq \mu_n \left( \tilde{\Upsilon}_n \right) (k + 1) / (c b^2 n)$;
2. if $r_k (\Sigma) \geq bn$, then

$$\text{tr} (T V) \geq \frac{\mu_n \left( \tilde{\Upsilon}_n \right)}{c b^2} \min_{l \leq k} \left( \frac{l}{n} + \frac{b^2 n \sum_{i > l} \lambda_i^2}{(\lambda_{k+1} r_k (\Sigma))^2} \right).$$

In particular, if all choices of $k \leq n/c$ give $r_k (\Sigma) < bn$, then $r_{n/c} (\Sigma) < bn$ implies that, with probability at least $1 - 10 \epsilon^{-n/c}$, $\text{tr} (T V) \geq \mu_n \left( \tilde{\Upsilon}_n \right)$.

**Proof of Lemma 23.** By Lemma 16,

$$\text{tr} (T V) \geq \mu_n \left( \tilde{\Upsilon}_n \right) \sum_i \frac{\lambda_i^2 \tilde{z}_i^T A^{-2} \tilde{z}_i}{(1 + \lambda_i \tilde{z}_i^T A^{-1} \tilde{z}_i)^2}.$$
and then Lemma 9 of [Bartlett et al., 2020] and Lemma 22 yield that there exist constants $c_1 = c_1(\epsilon) > 0$ and $c_2 = c_2(\epsilon) > 0$ such that with probability at least $1 - 10e^{-n/c_1}$,
\[
\sum_i \frac{\lambda_i^2 z_i^\top A_i^{-2} z_i}{(1 + \lambda_i^2 z_i^\top A_i^{-1} z_i)^2} \geq \frac{1}{c_1 n} \sum_i \left(1 + \frac{\sum_{j>k} \lambda_j + n\lambda_{k+1}}{\lambda_i n}\right)^{-2}
\]
\[
= \frac{1}{c_1 n} \sum_i \left(1^{-1} + \left(\frac{\lambda_i n}{\sum_{j>k} \lambda_j}\right)^{-1} + \left(\frac{\lambda_i}{\lambda_{k+1}}\right)^{-1}\right)^{-2}
\]
\[
\geq \frac{1}{c_2 n} \sum_i \left(\max \left\{1^{-1}, \left(\frac{\lambda_i n}{\sum_{j>k} \lambda_j}\right)^{-1}, \left(\frac{\lambda_i}{\lambda_{k+1}}\right)^{-1}\right\}\right)^{-2}
\]
\[
= \frac{1}{c_2 b^2 n} \sum_i \min \left\{1, \frac{\lambda_i^2}{(\sum_{j>k} \lambda_j)^2}, \frac{\lambda_i^2}{\lambda_{k+1}^2}\right\}
\]
\[
\geq \frac{1}{c_2 b^2 n} \sum_i \min \left\{1, \left(\frac{bn}{r_k(\Sigma)}\right)^{2}, \frac{\lambda_i^2}{\lambda_{k+1}^2}\right\}.
\]

If $r_k(\Sigma) < bn$, then
\[
\text{tr} (T_V) \geq \frac{\mu_n (\widetilde{\Upsilon}_n)}{c_2 b^2 n} \sum_i \min \left\{1, \frac{\lambda_i^2}{\lambda_{k+1}^2}\right\} = \frac{(k + 1)\mu_n (\widetilde{\Upsilon}_n)}{c_2 b^2 n}
\]
because $\lambda_i \geq \lambda_{k+1}$ for $i \leq k + 1$. If $r_k(\Sigma) \geq bn$, then
\[
\text{tr} (T_V) \geq \frac{\mu_n (\widetilde{\Upsilon}_n)}{c_2 b^2 n} \sum_i \min \left\{\frac{1}{n}, \frac{b^2 n \lambda_i^2}{(\lambda_{k+1} r_k(\Sigma))^2}\right\} = \frac{\mu_n (\widetilde{\Upsilon}_n)}{c_2 b^2 n} \min_{i \leq k} \left(\frac{l}{n} + \frac{b^2 n \sum_{i > k} \lambda_i^2}{(\lambda_{k+1} r_k(\Sigma))^2}\right)
\]
since $\{\lambda_i\}$ is non-increasing and then the minimizer $l$ gets restricted in $1 \leq l \leq k$. \hfill \Box

**Lemma 24** (Lemma 11 of [Bartlett et al., 2020]). For any $b \geq 1$ and $k^* := \min \{k : r_k(\Sigma) \geq bn\}$, if $k^* < \infty$, then
\[
\min_{i \leq k^*} \left(\frac{l}{n} + \frac{b^2 n \sum_{i > k} \lambda_i^2}{(\lambda_{k+1} r_k(\Sigma))^2}\right) = \left(\frac{k^*}{n} + \frac{b^2 n \sum_{i > k^*} \lambda_i^2}{(\lambda_{k+1} r_k(\Sigma))^2}\right) = \frac{k^*}{bn} + \frac{bn}{R_{k^*}(\Sigma)}.
\]

**APPENDIX D. MOMENT BOUND FOR HILBERT-VALUED GAUSSIAN PROCESSES UNDER DEPENDENCE**

We develop a moment inequality of empirical covariance operators with (possibly) infinite-dimensional dependent processes. This inequality will be used to bound the bias term $T_B$ in the proof of Theorem 3 under the settings where $x_t$ is an $\mathbb{H}$-valued process for a separable Hilbert space $\mathbb{H}$.
We extend Theorem 2.2 of Han and Li (2020), a moment bound for sample autocovariance matrices of finite-dimensional centered stationary Gaussian processes whose dependence decays sufficiently fast such that
\[
\mathbb{E} \left[ \| \hat{\Sigma}_0 - \Sigma_0 \| \right] \leq c \| \Sigma_0 \| \left( \sqrt{\frac{r_0(\Sigma_0)}{n}} + \frac{r_0(\Sigma_0)}{n} \right),
\]
where
\[
\hat{\Sigma}_0 u := \frac{1}{n} \sum_{t=1}^{n} (x_t^t u) x_t, \quad \Sigma_0 u = \mathbb{E} \left[ (x_1^t u) x_1 \right]
\]
for all \( u \in \mathbb{H} \). Note that the bound is independent of the dimension of the state space, and hence we can expect that it is possible to generalize the result to infinite-dimensional separable Hilbert spaces. If this result holds even in infinite-dimensional settings, then we can obtain a simple concentration inequality by Markov’s inequality: for all \( \delta > 0 \),
\[
P \left( \| \hat{\Sigma}_0 - \Sigma_0 \| \geq \delta \right) \leq \frac{1}{\delta} \mathbb{E} \left[ \| \hat{\Sigma}_0 - \Sigma_0 \| \right] \leq \frac{1}{\delta} c \| \Sigma_0 \| \left( \sqrt{\frac{r_0(\Sigma_0)}{n}} + \frac{r_0(\Sigma_0)}{n} \right).
\]

The following statement is a complete version of Proposition 8 in the main body, which is an extension of Proposition 4.6 of Han and Li (2020). We recall the integrated covariance operator \( \bar{\Sigma}_n \) such that for all \( u \in \mathbb{H} \),
\[
\bar{\Sigma}_n u = \Sigma_0 u + 2 \sum_{h=1}^{n-1} \bar{\Sigma}_h u
\]
where
\[
\bar{\Sigma}_h u = \sum_{i=1}^{\infty} |\lambda_{h,i}| (e_i^t u) e_i.
\]

**Proposition 25.** Assume \( \mathbb{H} \) is a separable Hilbert space and \( x_t \) is an \( \mathbb{H} \)-valued centered stationary Gaussian process whose cross-covariance operator \( \Sigma_h \) of trace class for all \( h \in \mathbb{Z} \) defined as
\[
\Sigma_h u := \mathbb{E} \left[ (x_{t+h}^t u) x_t \right]
\]
for all \( u \in \mathbb{H} \). We have
\[
\mathbb{E} \left[ \| \bar{\Sigma}_0 - \Sigma_0 \| \right] \leq \frac{2\sqrt{2}}{n} \left( \sqrt{2\text{tr} (\Sigma_n)} + \sqrt{2n \| \Sigma_0 \| \text{tr} (\Sigma_n)} + \sqrt{n\text{tr} (\Sigma_0) \| \Sigma_n \|} \right),
\]
where for all \( u \in \mathbb{H} \),
\[
\bar{\Sigma}_n u := \Sigma_0 u + 2 \sum_{h=1}^{n-1} \bar{\Sigma}_h u, \quad \bar{\Sigma}_h u := \frac{1}{2} \sum_{i} \sigma_i (\Sigma_h) ( (f_{h,i}^t u) f_{h,i} + (g_{h,i}^t u) g_{h,i} ),
\]
\( \sigma_i (\Sigma_h) \) are singular values of \( \Sigma_h \), \( \{ f_{h,i} \} \) are left singular vectors and \( \{ g_{h,i} \} \) are right ones of \( \Sigma_h \).
The following lemma is an extension of Lemma 4.7 of Han and Li (2020).

**Lemma 26.** Under the assumptions same as Proposition 25, we have

$$\mathbb{E} \left[ \| \tilde{\Sigma}_n - \Sigma_n \| \right] \leq \frac{2\sqrt{2}}{n} \mathbb{E} \left[ \| X \|_{HS} \right] \left( \sqrt{\text{tr} (\Sigma_n)} + \sqrt{\| \Sigma_n \|} \right).$$

**Proof of Lemma 26.** Let us consider a decoupling seen in Lemma 5.2 of van Handel (2017). \{\tilde{x}_t\} denotes an independent copy of \{x_t\}: then

$$\mathbb{E} \left[ \| \tilde{\Sigma}_0 - \Sigma_0 \| \right] = \mathbb{E} \left[ \sup_{u,v:|u|,|v| \leq 1} u^\top (\tilde{\Sigma}_0 - \Sigma_0) v \right]$$

$$= \mathbb{E} \left[ \sup_{u,v:|u|,|v| \leq 1} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^{n} u^\top (x_t x_t^\top - \tilde{x}_t \tilde{x}_t^\top) v \mid \{x_t\} \right] \right]$$

$$= \mathbb{E} \left[ \sup_{u,v:|u|,|v| \leq 1} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^{n} ((x_t^\top u)(x_t^\top v) - (\tilde{x}_t^\top u)(\tilde{x}_t^\top v)) \mid \{x_t\} \right] \right]$$

$$\leq \mathbb{E} \left[ \sup_{u,v:|u|,|v| \leq 1} \frac{1}{n} \sum_{t=1}^{n} (x_t + \tilde{x}_t)^\top u \mid \{x_t\} \right] \left( (x_t - \tilde{x}_t)^\top v \right)$$

$$\leq 2 \mathbb{E} \left[ \sup_{u,v:|u|,|v| \leq 1} \frac{1}{n} \sum_{t=1}^{n} (x_t^\top u) (\tilde{x}_t^\top v) \right]$$

for \(u, v \in \mathbb{H}\), because

$$\mathbb{E} \left[ (\tilde{x}_t^\top u)(x_t^\top v) - (x_t^\top u)(\tilde{x}_t^\top v) \mid \{x_t\} \right] = \mathbb{E} \left[ (\tilde{x}_t^\top u) \{x_t\} (x_t^\top v) - \mathbb{E} \left[ (\tilde{x}_t^\top v) \{x_t\} (x_t^\top u) \right] = 0$$

and thus

$$\mathbb{E} \left[ ((x_t + \tilde{x}_t)^\top u)((x_t - \tilde{x}_t)^\top v) \mid \{x_t\} \right] = \mathbb{E} \left[ (x_t^\top u)(x_t^\top v) - (\tilde{x}_t^\top u)(\tilde{x}_t^\top v) \mid \{x_t\} \right]$$

$$+ \mathbb{E} \left[ (\tilde{x}_t^\top u)(x_t^\top v) - (x_t^\top u)(\tilde{x}_t^\top v) \mid \{x_t\} \right]$$

$$= \mathbb{E} \left[ (x_t^\top u)(x_t^\top v) - (\tilde{x}_t^\top u)(\tilde{x}_t^\top v) \mid \{x_t\} \right].$$

Note that \(\left( ((x_t + \tilde{x}_t)^\top u), (x_t - \tilde{x}_t)^\top v \right) \) has the same law as \(\sqrt{2} \left( (x_t^\top u), (\tilde{x}_t^\top v) \right) \) because \(\{\tilde{x}_t\} \) is an independent copy of \(\{x_t\}\), which is a sequence of centered Gaussian random variables. Let us define the following random process for \(u, v \in \mathbb{H}\) with \(|u| \leq 1\) and \(|v| \leq 1\)
as follows:

\[ W_{u,v} := \sum_{t=1}^{n} (x_t^T u) (\tilde{x}_t^T v). \]

Then

\[
(W_{u,v} - W_{u',v'})^2 = \left( \sum_{t=1}^{n} (x_t^T (u - u')) (\tilde{x}_t^T v) - \sum_{t=1}^{n} (x_t^T u') (\tilde{x}_t^T v') \right)^2
\]

\[
= \left( \sum_{t=1}^{n} (x_t^T (u - u')) (\tilde{x}_t^T v) + \sum_{t=1}^{n} (x_t^T u') (\tilde{x}_t^T (v - v')) \right)^2
\]

\[
\leq 2 \left( \sum_{t=1}^{n} (x_t^T (u - u')) (\tilde{x}_t^T v) \right)^2 + 2 \left( \sum_{t=1}^{n} (x_t^T u') (\tilde{x}_t^T (v - v')) \right)^2
\]

\[
= 2 \sum_{h=0}^{n-1} \sum_{t_1, t_2=1, \ldots, n \atop |t_1 - t_2| = h} (x_{t_1}^T u - u') (x_{t_2}^T (u - u')) (\tilde{x}_{t_1}^T v (\tilde{x}_{t_2}^T v) + \sum_{t_1, t_2=1, \ldots, n \atop |t_1 - t_2| = h} (x_{t_1}^T u') (\tilde{x}_{t_2}^T (v - v')) (\tilde{x}_{t_2}^T (v - v'))
\]

By the Cauchy–Schwarz inequality,

\[
\mathbb{E} \left[ (W_{u,v} - W_{u',v'})^2 \right | \{\tilde{x}_t\}] \leq 2 (u - u')^T \Sigma_0 (u - u') \sum_{t=1}^{n} (\tilde{x}_t^T v)^2
\]

\[
+ 2 \sum_{h=1}^{n-1} (u - u')^T (\Sigma_h + \Sigma_h^T) (u - u') \sum_{t_1, t_2=1, \ldots, n \atop |t_1 - t_2| = h} (\tilde{x}_{t_1}^T v (\tilde{x}_{t_2}^T v)
\]

\[
+ 2 (u')^T \Sigma_0 u' \sum_{t=1}^{n} (\tilde{x}_t^T (v - v'))^2
\]

\[
+ 2 \sum_{h=1}^{n-1} (u')^T (\Sigma_h + \Sigma_h^T) u' \sum_{t_1, t_2=1, \ldots, n \atop |t_1 - t_2| = h} (\tilde{x}_{t_1}^T (v - v')) (\tilde{x}_{t_2}^T (v - v'))
\]

\[
\leq 2 (u - u')^T (\Sigma_n) (u - u') \sum_{t=1}^{n} \|\tilde{x}_t\|^2
\]

\[
+ 2 \|\Sigma_n\| \sum_{t=1}^{n} (\tilde{x}_t^T (v - v'))^2.
\]
We define the following Gaussian process on \( u, v \in \mathbb{H} \) with \( \|u\| \leq 1, \|v\| \leq 1 \):

\[
Y_{u,v} := \sqrt{2} \sqrt{\sum_{t=1}^{n} \|x_t\|^2 (g^\top u) + \sqrt{2} \sqrt{\|\Sigma_n\| \sum_{t=1}^{n} (\tilde{x}_t^\top v) g'_t}}
\]

where \( g \) is an \( \mathbb{H} \)-valued centered Gaussian variable with the independence of other random variables and the covariance operator \( \Sigma_n \) and \( \{g'_t : t = 1, \ldots, n\} \) is an i.i.d. sequence of \( \mathbb{R} \)-valued standard Gaussian random variables independent of other random variables. Obviously

\[
\mathbb{E} \left[ (W_{u,v} - W_{u',v'})^2 \right] \leq \mathbb{E} \left[ (Y_{u,v} - Y_{u',v'})^2 \right].
\]

It yields that by the Sudakov–Fernique inequality (Adler and Taylor, 2007),

\[
\mathbb{E} \left[ \sup_{u,v: \|u\| \leq 1, \|v\| \leq 1} W_{u,v} \right] \leq \mathbb{E} \left[ \sup_{u,v: \|u\| \leq 1, \|v\| \leq 1} Y_{u,v} \right]
\]

\[
\leq \sqrt{2} \sqrt{\sum_{t=1}^{n} \|x_t\|^2 \mathbb{E} \left[ \|g\|^2 \right] + \sqrt{2} \sqrt{\|\Sigma_n\| \sum_{t=1}^{n} \|\tilde{x}_t g'_t\|}}
\]

\[
\leq \sqrt{2} \sqrt{\sum_{t=1}^{n} \|x_t\|^2 \sqrt{\text{tr} (\Sigma_n)} + \sqrt{2} \sqrt{\|\Sigma_n\| \sum_{t=1}^{n} \|\tilde{x}_t\|^2}}.
\]

and by taking the expectation with respect to \( \{\tilde{x}_t\} \),

\[
\mathbb{E} \left[ \|\hat{\Sigma}_0 - \Sigma_0\| \right] \leq \frac{2\sqrt{2}}{n} \mathbb{E} \left[ \|X\|_{\text{HS}} \right] \left( \sqrt{\text{tr} (\Sigma_n)} + \sqrt{\|\Sigma_n\|} \right).
\]

We now see that the statement holds. \( \square \)

The following lemma is a variation of Lemma 4.8 of Han and Li (2020).

**Lemma 27.** Under the assumptions same as Proposition 25, we have

\[
\mathbb{E} \left[ \|X\|_{\text{HS}} \right] \leq \sqrt{2 \text{tr} (\Sigma_n)} + \sqrt{2n \|\Sigma_0\|}.
\]

**Proof of Lemma 27.** Let us define the following Gaussian process such that for \( u \in \mathbb{H} \) with \( \|u\| \leq 1 \) and \( v = [v^{(1)}, \ldots, v^{(n)}] \in \mathbb{R}^n \) with \( \|v\| \leq 1 \),

\[
\widetilde{W}_{u,v} := \sum_{t=1}^{n} (x_t^\top u) v^{(t)}.
\]

Then
\[
\mathbb{E} \left[ (\tilde{W}_{u,v} - \tilde{W}_{u',v'})^2 \right] = \mathbb{E} \left[ (\tilde{W}_{u,v} - \tilde{W}_{u',v} + \tilde{W}_{u',v} - \tilde{W}_{u',v'})^2 \right]
\]
\[
\leq 2 \mathbb{E} \left[ (\tilde{W}_{u,v} - \tilde{W}_{u',v})^2 \right] + 2 \mathbb{E} \left[ (\tilde{W}_{u',v} - \tilde{W}_{u',v'})^2 \right]
\]
\[
\leq 2 \mathbb{E} \left[ \left( \sum_{t=1}^{n} (x_t^\top (u - u')) v(t) \right)^2 \right] + 2 \mathbb{E} \left[ \left( \sum_{t=1}^{n} (x_t^\top u') (v - v')^t \right)^2 \right]
\]
\[
\leq 2 \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} (u - u')^\top \Sigma_{t_1 - t_2} (u - u') v(t_1) v'(t_2)
\]
\[
+ 2 \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} (u')^\top \Sigma_{t_1 - t_2} u' (v - v')^{(t_1)} (v - v')^{(t_2)}.
\]

As Han and Li (2020), we define the \( n \times n \)-matrix
\[
\Sigma_{L,u} := \begin{bmatrix}
u^\top \Sigma_0 u & u^\top \Sigma_1 u & \cdots & u^\top \Sigma_{n-1} u \\
u^\top \Sigma_0 u & u^\top \Sigma_1 u & \cdots & u^\top \Sigma_{n-1} u \\
\vdots & \vdots & \ddots & \vdots \\
u^\top \Sigma_{n-1} u & u^\top \Sigma_{n-1} u & \cdots & u^\top \Sigma_0 u
\end{bmatrix}
\]
\[
\Sigma^\circ := ||\Sigma_0|| \mathbf{1}_n \mathbf{1}_n^\top,
\]
where \( \mathbf{1}_n \in \mathbb{R}^n \) whose all the elements are 1. Because for all \( u \in \mathbb{H} \) and \( v \in \mathbb{R}^n \) such that \( ||u|| \leq 1 \) and \( ||v|| \leq 1 \),
\[
v^\top \Sigma_{L,u} v = \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} (u^\top \Sigma_{t_1 - t_2} u) v(t_1) v(t_2) \leq \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} ||\Sigma_0|| v(t_1) v(t_2) = ||\Sigma_0|| \left( \sum_{t=1}^{n} v(t) \right)^2 = v^\top \Sigma^\circ v,
\]
we obtain that for all \( u \in \mathbb{H} \) such that \( ||u|| \leq 1 \),
\[
\Sigma_{L,u} \leq \Sigma^\circ
\]
in the meaning of the Loewner partial order. Hence, as Lemma 26,
\[
\mathbb{E} \left[ (\tilde{W}_{u,v} - \tilde{W}_{u',v'})^2 \right] \leq 2 (u - u')^\top (\Sigma_n) (u - u')
\]
\[
+ 2 ||\Sigma_0|| (v - v')^\top \mathbf{1}_n \mathbf{1}_n^\top (v - v').
\]
Let us define a Gaussian process such that
\[
\tilde{Y}_{u,v} := \sqrt{2} (g^\top u) + \sqrt{2} ||\Sigma_0||^{1/2} \left( (g')^\top v \right),
\]
where \( g \) is an \( \mathbb{H} \)-valued centered Gaussian random variable with independence of other random variables and the covariance operator \( \Sigma_n \) and \( g' \) is an \( \mathbb{R}^n \)-valued centered Gaussian.
random variable with independence of other random variables and the degenerate covariance matrix $1_n 1_n^\top$. Then we obtain
\[ E \left[ (\tilde{W}_{u,v} - \tilde{W}_{u',v'})^2 \right] \leq E \left[ (\tilde{Y}_{u,v} - \tilde{Y}_{u',v'})^2 \right]. \]

The Sudakov–Fernique inequality verifies that
\[
E \left[ \sup_{u,v: \|u\| \leq 1, \|v\| \leq 1} \tilde{W}_{u,v} \right] \leq E \left[ \sup_{u,v: \|u\| \leq 1, \|v\| \leq 1} \tilde{Y}_{u,v} \right] 
\leq \sqrt{2 \text{tr} (\bar{\Sigma}_n)} + \sqrt{2} \|\Sigma_0\|^{1/2} \sqrt{n}.
\]

Here we obtain the statement. \[ \square \]

**Proof of Proposition 25.** It holds immediately for Lemmas 26 and 27, and the fact
\[
E [\|X\|_{\text{HS}}] \leq E [\|X\|^2_{\text{HS}}]^{1/2} = \sqrt{\sum_{t=1}^n E [\|x_t\|^2]} = \sqrt{n \text{tr} (\Sigma_0)}
\]
by the Cauchy–Schwarz inequality. \[ \square \]

**Remark 4.** Let us give some remarks on $\tilde{\Sigma}_h$. The operator $\tilde{\Sigma}_h$ has the property such that for all $u \in \mathbb{H}$,
\[
u^\top (\Sigma_h + \Sigma_h^\top) u = \sum_i 2 \sigma_i (\Sigma_h) (f_{h,i}^\top u) (g_{h,i}^\top u) \leq \sum_i \sigma_i (\Sigma_h) \left( (f_{h,i}^\top u)^2 + (g_{h,i}^\top u)^2 \right) = 2u^\top \tilde{\Sigma}_h u.
\]

When $\Sigma_h$ is self-adjoint, for all $u \in \mathbb{H}$, we can set
\[
\tilde{\Sigma}_h u = \sum_i |\mu_i (\Sigma_h)| (e_{h,i}^\top u) e_{h,i},
\]
where $\mu_i (\Sigma_h)$ and $e_{h,i}$ are the $i$-th eigenvalue and the corresponding eigenvector of $\Sigma_h$. 

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Appendix E. Proof for Convergence Rate Analysis

Proof of Proposition 4. In this proof, we study the terms in the bias part of the bound in Theorem 1. By Hölder’s inequality, we obtain

\[
\|\hat{\beta}^*\|^2 \lesssim \frac{1}{\sum_{i,k^n} \lambda_i} \left( \frac{\sum_{i>k^*} \lambda_i}{n} \right)^2 \leq \|\hat{\beta}^*\|^2 \left( \lambda_{k^*+1} + \lambda_{k^*}^{-1} \left( \frac{\sum_{i>k^*} \lambda_i - \lambda_{k^*}}{n} \right)^2 \right) \\
= \|\hat{\beta}^*\|^2 \left( \lambda_{k^*+1} + \lambda_{k^*}^{-1} \left( \frac{\sum_{i>k^*} \lambda_i}{\lambda_{k^*} n} \right)^2 \right) \\
= \|\hat{\beta}^*\|^2 \left( \lambda_{k^*+1} + \lambda_{k^*}^{2} \left( \frac{\lambda_{k^*-1}(\Sigma) - 1}{n} \right)^2 \right)
\]

which follows the definition of \( r_k(\Sigma) \) in Definition 2. With the loss of generality, we set \( r_{k^*-1}(\Sigma) \geq 1 \). With the specified constant \( b \geq 1 \) with \( k^* = k^*(b) \geq 1 \), the relation \( r_{k^*-1}(\Sigma) \leq bn \) by Definition 2 yields

\[
\|\hat{\beta}^*\|^2 \left( \lambda_{k^*+1} + \lambda_{k^*} \left( \frac{bn}{n} - 1 \right)^2 \right) \leq \|\hat{\beta}^*\|^2 \left( \lambda_{k^*+1} + \lambda_{k^*}^2 b^2 \right) \\
\leq c\|\hat{\beta}^*\|^2\lambda_{k^*} \\
\leq c\|\hat{\beta}^*\|^2 \frac{\sum_{i \geq k^*} \lambda_i}{bn} \\
\leq c\|\hat{\beta}^*\|^2 \frac{\text{tr}(\Sigma)}{bn},
\]

where \( c := (1 + b^2) \) is a constant. The last inequality follows \( \lambda_{k^*+1} \leq \lambda_{k^*} \).

About the variance part of the upper bound in Theorem 1, the result is obvious. \( \square \)

Proof of Proposition 5. Without loss of generality, we set \( \lambda_1 = 1 \). By Theorem 1, it is sufficient to study the term \( n^{-1/2} \sqrt{\|\Sigma_n\|r_0(\Sigma)} \). We obtain

\[
\sqrt{\|\Sigma_n\|r_0(\Sigma)} = \left( 2 \sum_{h=0}^{n} \lambda_1 \sup_i |\xi_{i,n}^{(1+h)}| \right)^{1/2} \zeta_n^{1/2} \leq \left( 2 + 2c \sum_{h=1}^{n} h^{-\alpha} \right)^{1/2} \zeta_n^{1/2}.
\]

For \( \alpha > 1/2 \), we have

\[
1 + c \sum_{h=1}^{n} h^{-\alpha} = \begin{cases} 
O(n^{1-\alpha}) & (\alpha \in (1/2, 1)) \\
O(\log n) & (\alpha = 1) \\
O(1) & (\alpha > 1).
\end{cases}
\]
Hence, we obtain

\[
\sqrt{\|\Sigma_n\| r_0(\Sigma_0)} = \begin{cases} 
O(\zeta_n^{1/2} n^{-\alpha/2}) & (\alpha \in (1/2, 1)) \\
O(\zeta_n^{1/2} n^{-1/2} \log^{1/2} n) & (\alpha = 1) \\
O(\zeta_n^{1/2} n^{-1/2}) & (\alpha > 1).
\end{cases}
\]

Then, we obtain the statement. \qed

\section*{Appendix F. Proofs for the Examples}

We give proofs for lemmas and propositions for the example of processes presented in Section 8.

\textit{Proof for Proposition 10.} Let us begin with showing (i). Because \(\phi_{0,k} = 1\) for all \(k\) by the definition, the lower bound is obvious. By the assumptions of the unit variance of white noises and spectral densities, Theorem 4.4.2 and Proposition 4.5.3 of Brockwell and Davis (1991) lead to

\[
\epsilon^4 \leq \inf_{z \in \mathbb{C} : |z| = 1} \left| \frac{1 + \sum_{j=1}^{\ell_1} \varphi_{j,k} z^j}{1 - \sum_{j=1}^{\ell_1} \rho_{j,k} z^j} \right|^2 \leq \sum_{j=0}^{\infty} \phi_{j,k}^2 \leq \sup_{z \in \mathbb{C} : |z| = 1} \left| \frac{1 + \sum_{j=1}^{\ell_1} \varphi_{j,k} z^j}{1 - \sum_{j=1}^{\ell_1} \rho_{j,k} z^j} \right|^2 \leq \epsilon^{-4}
\]

because \(\sum_{j=0}^{\infty} \phi_{j,k}^2\) equals to all the diagonal elements of the corresponding autocovariance matrix.

(ii) follows from the same argument as (i) and the fact that \(\sum_{j=0}^{\infty} \phi_{j,k}^2\) is the inverse variance of the corresponding noise terms in each coordinate process \(\{ (x_t^\top e_k) / \sqrt{\lambda_k} : t = 1, \ldots, n \} \).
(iii) uses a discussion of near-epoch dependence. It is sufficient to consider $h \geq 0$; for all $h \geq 2 (\max \{ \ell_1, \ell_2 \} + 2)$,

\[
\left| \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+h,k} \right| \leq \left| \sum_{j=0}^{[h/2]} \phi_{j,k} \phi_{j+h,k} \right| + \left| \sum_{j=[h/2]+1}^{\infty} \phi_{j,k} \phi_{j+h,k} \right|
\leq \left( \sum_{j=0}^{[h/2]} \phi_{j,k}^2 \right)^{1/2} \left( \sum_{j=0}^{[h/2]} \phi_{j+h,k}^2 \right)^{1/2} + \left( \sum_{j=[h/2]+1}^{\infty} \phi_{j,k}^2 \right)^{1/2} \left( \sum_{j=[h/2]+1}^{\infty} \phi_{j+h,k}^2 \right)^{1/2}
\leq 2 \left( \sum_{j=0}^{\infty} \phi_{j,k}^2 \right)^{1/2} \left( \sum_{j=[h/2]}^{\infty} \phi_{j+k}^2 \right)^{1/2}
\leq 4 \left( \sum_{j=0}^{\infty} \phi_{j,k}^2 \right)^{1/2} \left( \sum_{j=[h/2]}^{\infty} \phi_{j+k}^2 \right)^{1/2} (1 + \epsilon)^{-[h/2]+\max\{\ell_1,\ell_2\}+1}
\leq 4 \left( \sum_{j=0}^{\infty} \phi_{j,k}^2 \right) (1 + \epsilon)^{\max\{\ell_1,\ell_2\}+2} (1 + \epsilon)^{-h/2}
\]

because of the conditions on spectral density functions and the result (i), and a discussion of near-epoch dependence (see (2.6) and Proposition 2.1 of Davidson [2002] and Propositions 1.1 and 10.1 of Hamilton [1994]). Clearly for all $h = 0, \ldots, 2 (\max \{ \ell_1, \ell_2 \} + 2)$,

\[
\left| \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+h,k} \right| \leq \sum_{j=0}^{\infty} \phi_{j,k}^2 \leq 4 \left( \sum_{j=0}^{\infty} \phi_{j,k}^2 \right) (1 + \epsilon)^{\max\{\ell_1,\ell_2\}+2} (1 + \epsilon)^{-h/2}.
\]

Hence for all $h \geq 0$,

\[
\left| \Xi^{(1,1+h)}_{0,n} \right| = \left| \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+h,k} \right| \leq \left( \sum_{j=0}^{\infty} \phi_{j,k}^2 \right) (1 + \epsilon)^{\max\{\ell_1,\ell_2\}+2} (1 + \epsilon)^{-h/2}.
\]

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(iv) is another consequence of (i). (v) holds because

\[
\text{tr} \left( \Sigma + 2 \sum_{h=1}^{n-1} \tilde{\Sigma}_h \right) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \phi_{j,k}^2 \vartheta_k + 2 \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+h,k} \vartheta_k \\
\leq \left( \sup_k \sum_{j=0}^{\infty} \phi_{j,k}^2 + 2 \sum_{h=1}^{\infty} \sup_k \sum_{j=0}^{\infty} \phi_{j,k} \phi_{j+h,k} \right) \text{tr} (Q) \\
\leq \epsilon^{-4} \left( 1 + 8 (1 + \epsilon)^{\max \{ \ell_1, \ell_2 \}} + 2 \sum_{h=1}^{n-1} (1 + \epsilon)^{-h/2} \right) \text{tr} (Q) \\
\leq \epsilon^{-4} \left( 1 + 8 (1 + \epsilon)^{\max \{ \ell_1, \ell_2 \}} + 2 \frac{(1 + \epsilon)^{-1/2}}{1 - (1 + \epsilon)^{-1/2}} \right) \text{tr} (Q) \\
\leq \epsilon^{-4} \left( 1 + 8 (1 + \epsilon)^{\max \{ \ell_1, \ell_2 \}} + 2 \frac{(1 + \epsilon)^{-1/2}}{1 - (1 + \epsilon)^{-1/2}} \right) \text{tr} (\Sigma_0)
\]

by (i) and (iii). A similar argument holds for \( \| \tilde{\Sigma}_n \| \). □

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