INTRODUCTION OF A QUANTUM OF TIME ("CHRONON")
AND ITS CONSEQUENCES FOR QUANTUM MECHANICS

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Abstract

We discuss the consequences of the introduction of a quantum of time $\tau_0$ in the formalism of non-relativistic quantum mechanics, by referring ourselves, in particular, to the theory of the chronon as proposed by P. Caldirola. Such an interesting “finite difference” theory, forwards—at the classical level—a solution for the motion of a particle endowed with a non-negligible charge in an external electromagnetic field, overcoming all the known difficulties met by Abraham–Lorentz’s and Dirac’s approaches (and even allowing a clear answer to the question whether a free falling charged particle does or does not emit radiation), and—at the quantum level—yields a remarkable mass spectrum for leptons.

After having briefly reviewed Caldirola’s approach, our first aim is to work out, discuss, and compare to one another the new representations of Quantum Mechanics (QM) resulting from it, in the Schrödinger, Heisenberg and density–operator (Liouville–von Neumann) pictures, respectively.

Moreover, for each representation, three (retarded, symmetric and advanced) formulations are possible, which refer either to times $t$ and $t - \tau_0$, or to times $t - \tau_0/2$ and $t + \tau_0/2$, or to times $t$ and $t + \tau_0$, respectively. It is interesting to notice that, when the chronon tends to zero, the ordinary QM is obtained as the limiting case of the “symmetric” formulation only; while the “retarded” one does naturally appear to describe QM with friction, i.e., to describe dissipative quantum systems (like a particle moving in an absorbing medium). In this sense, discretized QM is much richer than the ordinary one.

We also obtain the (retarded) finite-difference Schrödinger equation within the Feynman path integral approach, and study some of its relevant solutions. We then derive the time–evolution operators of this discrete theory, and use them to get the finite-difference Heisenberg equations.

When discussing the mutual compatibility of the various pictures listed above, we find that they can be written down in a form such that they result to be equivalent (as it happens in the “continuous” case of ordinary QM), even if the Heisenberg picture cannot be derived by “discretizing” directly the ordinary Heisenberg representation.

Afterwards, some typical applications and examples are studied, as the free particle, the harmonic oscillator and the hydrogen atom; and various cases are pointed out, for which the predictions of discrete QM differ from those expected from “continuous” QM.

At last, the density matrix formalism is applied to the solution of the measurement problem in QM, with very interesting results, as for instance a natural explication of “decoherence”, which reveal the power of discretized (in particular, retarded) QM.
1 Introduction

The idea of a discrete temporal evolution is not a new one and, as almost all the physical ideas, has from time to time been recovered from oblivion. For instance, in classical Greece this idea came to light as part of the atomistic thought. In the Middle Age, belief in the discontinuous character of time was at the basis of the “theistic atomism” held by the Arabic thinkers of the Kalam. In Europe, discussions about the discreteness of space and time can be found for example in the writings of Isidore of Sevilla, Nicolaus Boneti and Henry of Harclay, who discussed the nature of continuum. In more recent times, the idea of the existence of a fundamental interval of time was rejected by Leibniz, since it was incompatible with his rationalistic philosophy. Within modern physics, however, Planck’s famous work on black body radiation inspired a new view of the subject. In fact, the introduction of the quanta opened a wide range of new scientific possibilities regarding the way the physical world can be conceived. Including considerations, like those in the present paper, on the “discretization” of time within the framework of Quantum Mechanics (QM).

In the early years of our century, Mach regarded the concept of continuum to be a consequence of our physiological limitations: <<...le temps et l’espace ne représentent, au point de vue physiologique, qu’un continue apparent, qu’ils se composent très vraisemblablement d’éléments discontinus, mais qu’on ne peut distinguer nettement les uns des autres.>> Also Poincaré took into consideration the possible existence of what he called an “atom of time”: the minimum amount of time which allows to distinguish between two states of a system. Finally, in the twenties, J.J.Thomson suggested the electric force to act in a discontinuous way, producing finite increments of momentum separated by finite intervals of time. Such a seminal work has ever since inspired a series of papers on the existence of a fundamental interval of time, named chronon; even if the repercussions of all that work was small, at that time. A further seminal article was the one by Ambarzumian and Ivanenko, appeared in 1930, which assumed space-time as being discrete and also stimulated a large number of subsequent papers.

It is important to stress that, in principle, time discretization can be introduced in two distinct (and completely different) ways:

(1) by attributing to time a discrete structure, i.e., by regarding time not as a continuum, but as a one-dimensional “lattice”;

(2) by considering time as a continuum, in which events can take place (discontinuously) only at discrete instants of time.

Almost all the attempts to introduce a discretization of time followed the first way, generally

\footnote{Historical aspects related to the introduction of a fundamental interval of time in Physics can be found in F.Casagrande, ref.[?].}
as part of a more extended procedure in which the space-time as a whole is considered as intrinsically discrete (four-dimensional lattice). Recently, also T.D. Lee introduced a time discretization on the basis of the finite number of experimental measurements performable in any finite interval of time. For an early approach in this direction, see e.g. T. Tati and references therein.

The second approach was first adopted in the twenties (e.g., by Levi and by Pokrowski), after Thomson's work, and resulted in the first real example of a theory based on the existence of a fundamental interval of time: the one set forth by Caldirola, in the fifties. Namely, Caldirola formulated a theory for the classical electron, with the aim of providing a consistent (classical) theory for its motion in an electromagnetic field. In the late seventies, Caldirola extended its procedure to non-relativistic quantum mechanics.

It is known that the classical theory of the electron in an electromagnetic field (despite the efforts by Abraham, Lorentz, Poincaré and Dirac, among others) actually presents many serious problems; except — of course — when the field of the particle is neglected. By replacing Dirac’s differential equation by two finite-difference equations, Caldirola developed a theory in which the main difficulties of Dirac’s theory were overcome. As we shall see, in his relativistically invariant formalism the chronon characterizes the changes suffered by the dynamical state of the electron when it is submitted to external forces. So that the electron will be regarded as an (extended-like) object, which is point-like only at discrete positions $x_n$ (along its trajectory) such that the electron takes a quantum of proper time to travel from one position to the following one [or, rather, two chronons: see the following]. It is tempting to examine extensively the generalization of such a theory to the quantum domain; and this will be performed in the present work. Let us recall that one of the most interesting aspects of the discretized Schrödinger equations is that the mass of the muon and of the tau lepton followed as corresponding to the two levels of the first (degenerate) excited state of the electron.

In conventional QM there is a perfect equivalence among its various pictures: Schrödinger’s, Heisenberg’s, density matrix’s. When discretizing the evolution equations, we shall succeed in writing down those pictures in a form such that they result to be still equivalent. However, in order to be compatible with the Schrödinger representation, our Heisenberg equations cannot, in general, be obtained by a direct discretization of the continuous Heisenberg equation.

The plan of this work is the following. In Chapt. we present a brief review of the main classical theories of the electron, including Caldirola’s. In Chapt. we introduce the three discretized forms (“retarded”, “advanced”, “symmetrical”) of the Schrödinger equation, analyze the main characteristics of such formulations, and derive the retarded one from Feynman’s path integral approach. In Chapt., our discrete theory is applied to some simple quantum

\footnote{It is interesting to note that all those problems have been — necessarily — tackled by Yaghjian in his recent book when he faced the question of the relativistic motion of a charged, macroscopic sphere in an external electromagnetic field.}
systems, such as the harmonic oscillator, the free particle and the hydrogen atom. The possible experimental deviations from the predictions of ordinary QM are investigated. In Chapt.??, a new derivation of the discretized Liouville–von Neumann equation, starting from the coarse grained hypothesis, is presented. Such a representation is then adopted for tackling the measurement problem in QM, with rather interesting results. Finally, a discussion on the possible interpretation of our discretized equations can be found in Chapt.??.

2 The Introduction of the Chronon in the Classical Theory of the Electron

Almost a century after its discovery, the electron continues to be an object waiting for a convincing description (cf., e.g., [?]), both in classical and quantum electrodynamics. As Schrödinger put it, the electron is —still— a stranger in electrodynamics. Maxwell’s electromagnetism is a field theoretical approach in which no reference is made to the existence of material corpuscles. Thus, one may say that one of the most controversial questions of the twentieth century physics, the wave–particle paradox, is not characteristic of QM only. In the electron classical theory, matching the description of the electromagnetic fields (obeying Maxwell equations) with the existence of charge carriers like the electron is still a challenging task.

The hypothesis that electric currents could be associated with charge carriers was already present in the early “particle electrodynamics” formulated in 1846 by Fechner and Weber. But such an idea was taken into consideration again only a few decades later, in 1881, by Helmholtz. Up to that time, electrodynamics had been developed on the hypothesis of an electromagnetic continuum[?] and of an ether. In that same year, J.J. Thomson wrote his seminal paper in which the electron mass was regarded as purely electromagnetic in nature. Namely, the energy and momentum associated with the (electromagnetic) fields produced by an electron were held entirely responsible for the energy and momentum of the electron itself.[?]

Lorentz’s electrodynamics, which described the particle–particle interaction via electromagnetic fields by the famous force law

\[ f = \rho \left( E + \frac{1}{c} v \times B \right), \]

\( \rho \) being the charge density of the particle on which the fields act, dates back to the beginning of the 1890 decade. The electron was finally discovered by Thomson in 1897, and in the following years various theories appeared. The famous (pre-relativistic) theories by Abraham, Lorentz and Poincaré regarded it as an extended–type object, endowed again with a purely electromagnetic mass. As well-known, in 1903 Abraham proposed the simple-minded (and questionable) model of a rigid sphere, with a uniform electric charge density on its surface. The theory of Lorentz (1904) was quite similar, trying to improve the situation with the mere introduction of the effects resulting from the Lorentz–Fitzgerald contraction.
2.1 The theory of the electron by Abraham–Lorentz

A main difficulty for an accurate description of the electron motion was the inclusion of the radiation reaction, i.e., of the effect produced on such a motion by the fields radiated by the particle itself. In the model proposed by Abraham–Lorentz the assumption of a purely electromagnetic structure for the electron implied that

$$\mathbf{F}_p + \mathbf{F}_{\text{ext}} = 0 \ ,$$

where $\mathbf{F}_p$ is the self-force due to the self-fields of the particle, and $\mathbf{F}_{\text{ext}}$ is the external force. According to Lorentz’s law, the self-force was given by

$$\mathbf{F}_p = \int \rho \left( \mathbf{E}_p + \frac{1}{c} \mathbf{v} \wedge \mathbf{B}_p \right) d^3r \ ,$$

where $\mathbf{E}_p$ and $\mathbf{B}_p$ are the fields produced by the charge density $\rho$ itself, according to the Maxwell–Lorentz equations. For the radiation reaction force, Lorentz obtained the following expression:

$$\mathbf{F}_4 = \frac{4 e^2}{3 c^3} W_{el} \mathbf{a} + \frac{2 k e^2}{3 c^3} \mathbf{a} - \frac{2 e^2}{3 c^3} \sum_{n=2}^{\infty} \frac{(-1)^n}{n! c^n} \frac{1}{d^n a} O(R^n) \ ,$$

where $k \equiv (4\pi\varepsilon_0)^{-1}$ [in the following, whenever convenient, we shall assume units such that numerically $k = 1$], and where

$$W_{el} = \frac{1}{2} \int \int \frac{\rho(r) \rho(r')}{|r - r'|} d^3r d^3r'$$

is the electrostatic self-energy of the considered charge distribution, and $R$ is the radius of the electron. All the terms in the sum are structure dependent; i.e., they depend on $R$ and on the charge distribution. By identifying the electromagnetic mass of the particle with its electrostatic self-energy

$$m_{el} = \frac{W_{el}}{c^2} \ ,$$

it was possible to write eq.(??) as

$$\frac{4}{3} m_{el} \mathbf{v} - \Gamma = \mathbf{F}_{\text{ext}} \ ,$$

so that one got:

$$\Gamma = \frac{2 e^2}{3 c^3} \mathbf{a} (1 + O(R)) \ ,$$

which was the equation of motion in the Abraham–Lorentz model. Quantity $\Gamma$ is the radiation reaction force, namely, the reaction force acting on the electron. A problem with equation (4) was constituted by the factor $\frac{4}{3}$. In fact, if the mass is supposed to be of electromagnetic origin only, then the total momentum of the electron would be given by

$$p = \frac{4 W_{el}}{3 c^2} \mathbf{v} \ .$$
which is not invariant under Lorentz transformations. That model, therefore, was non-relativistic. Finally, we can observe from equation (??) that the structure dependent terms are functions of higher derivatives of the acceleration; even more, the resulting differential equation is of the third order, so that initial position and initial velocity are not enough to single out a solution. In order to suppress the structure terms, one ought to reduce the electron to a point, \((R \to 0)\), but in this case the self-energy \(W_e\) and mass \(m_e\) would diverge!

After the emergence of the special theory of relativity or, rather, after the publication by Lorentz in 1904 of his famous transformations, some attempts were made to adapt the model to the new requirements.[?, ?, ?] Abraham himself (1905) succeeded in deriving the following generalization of the radiation reaction term (??):

\[
\Gamma_\mu = \frac{2 e^2}{3 c} \left( \frac{d^2 u_\mu}{ds^2} + \frac{u_\mu u^\nu}{c^2} \frac{d^2 u_\nu}{ds^2} \right).
\] (7)

A solution for the problem of the electron momentum non-covariance was proposed by Poincaré in 1905, by the addition of cohesive forces of non-electromagnetic character, which —however— made the electron no longer purely electromagnetic in nature.

On the other hand, electrons could not be considered pointlike, due to the obvious divergence of their energy when \(R \to 0\); thus, a description of the electron motion could not forget about the structure terms. Only Fermi succeeded to show, later, that the correct relation for the momentum of a purely electromagnetic electron could be obtained without Poincaré’s cohesive forces.[?]  

### 2.2 Dirac’s theory of the classical electron

Notwithstanding its inconsistencies, the theory by Abraham–Lorentz was the most accepted theory of the electron, until the publication of Dirac’s theory in 1938. During the long period in between, as well as afterwards, however, various further attempts to solve the problem were set forth, either by means of extended-type models (Mie, Page, Schott, etc.[?]), or by trying again to treat the electron as a pointlike particle (Fokker, Wentzel, etc.[?]).

Dirac’s approach[?] is the best known attempt to describe the classical electron. It by-passed the critical problem of the previous theories of Abraham and Lorentz by working out for the pointlike electron a trick which avoided divergences! By using the conservation laws of energy and momentum, and Maxwell equations, Dirac calculated the flux of the energy–momentum 4-vector through a tube of radius \(\epsilon \ll R\) (quantity \(R\) being the radius of the electron at rest) surrounding the world line of the particle, and obtained:

\[
m \frac{du_\mu}{ds} = F_\mu + \Gamma_\mu,
\] (8)

where \(\Gamma_\mu\) is the Abraham 4-vector (??), that is, the reaction force acting on the electron itself; and \(F_\mu\) is the 4-vector that represents the external field acting on the particle:

\[
F_\mu = \frac{e}{c} F_{\mu\nu} u^\nu.
\] (9)
According to such a model, the rest mass \( m_0 \) of the electron is the limiting, finite value obtained as the difference of two quantities tending to infinity when \( R \to 0 \):

\[
m_0 = \lim_{\varepsilon \to 0} \left( \frac{1}{2} \frac{e^2}{c^2 \varepsilon} - k(\varepsilon) \right),
\]

the procedure followed by Dirac being an early example of elimination of divergences by means of a subtractive method.

At the non-relativistic limit, Dirac’s equation goes into the one previously obtained by Abraham–Lorentz:

\[
\text{except for the fact that in Abraham–Lorentz’s approach } m_0 \text{ diverged. The last equation shows that the reaction force equals } 3 \frac{e^2}{c^3} \frac{d^2 \mathbf{v}}{dt^2}.
\]

Dirac’s dynamical equation (19) was later reobtained also from different, improved models.[?] Wheeler and Feynman, for example, rederived eq.(19) by basing electromagnetism on an action principle applied to particles only, via their absorber hypothesis.[?] However, eq.(19) also presents many troubles, related to the infinite many non-physical solutions that it possesses. Actually, as already mentioned, it is a third–order differential equation, requiring three initial conditions for singling out one of its solutions. In the description of a free electron, e.g., it even yields “self-accelerating” solutions (runaway solutions),[?] for which velocity and acceleration increase spontaneously and indefinitely.[?] Moreover, for an electron submitted to an electromagnetic pulse, further non-physical solutions appear, related this time to pre-accelerations[?].

If the electron comes from infinity with a uniform velocity \( V_0 \) and, at a certain instant of time \( t_0 \), is submitted to an electromagnetic pulse, then it starts accelerating before \( t_0 \). Drawbacks like these motivate further attempts to find out a coherent model for the classical electron.

### 2.3 Caldirola’s theory for the classical electron

Among the various attempts to formulate a more satisfactory theory, we want to focus our attention on the one proposed by P.Caldirola. Like Dirac’s, Caldirola’s theory is also Lorentz invariant. Continuity, in fact, is not an assumption required by Lorentz invariance.[?]

The theory postulates the existence of a universal interval \( \tau_0 \) of proper time, even if time flows continuously as in the ordinary theory. When an external force acts on the electron, however, the reaction of the particle to the applied force is not continuous: The value of the electron velocity \( u_\mu \) is supposed to jump from \( u_\mu(\tau - \tau_0) \) to \( u_\mu(\tau) \) only at certain positions \( s_n \) along its world line; these discrete positions being such that the electron takes a time \( \tau_0 \) for travelling from one position \( s_{n-1} \) to the next one \( s_n \).

In this theory[?] the electron, in principle, is still considered as pointlike, but the Dirac relativistic equations for the classical radiating electron are replaced: (i) by a corresponding finite–difference (retarded) equation in the velocity \( u^\mu(\tau) \).
which reduces to the Dirac equation (10) when \( \tau_0 \rightarrow 0 \), but cannot be derived from it (in the sense that it cannot be obtained by a simple discretization of the time derivatives appearing in Dirac’s original equation); and (ii) by a second equation, connecting this time the “discrete positions” \( x^\mu(\tau) \) along the world line of the particle. In fact, the dynamical law above is by itself unable to specify univocally the variables \( u_\mu(\tau) \) and \( x_\mu(\tau) \) which describe the motion of the particle. Caldirola named it the transmission law:

\[
x_\mu((n-1)\tau_0) - x_\mu(n\tau_0) = \frac{\tau_0}{2} \{ u_\mu((n-1)\tau_0) - u_\mu((n-1)\tau_0) \},
\]

which is valid inside each discrete interval \( \tau_0 \), and describes the internal or microscopic motion of the electron.

In these equations, \( u^\mu(\tau) \) is the ordinary four-vector velocity satisfying the condition

\[
u.(\tau))u.(\tau) = -c^2 \quad \text{for} \quad \tau = n\tau_0
\]

where \( n = 0, 1, 2, \ldots \) and \( \mu, \nu = 0, 1, 2, 3 \); \( F^{\mu\nu} \) is the external (retarded) electromagnetic field tensor, and the quantity

\[
\frac{\tau_0}{2} \equiv \theta_0 = \frac{2}{3} \frac{k e^2}{m_0 c^3} \simeq 6.266 \times 10^{-24} \text{ s}
\]

is defined as the chronon associated with the electron (as it will be justified below). The chronon \( \theta_0 = \tau_0/2 \) depends on the particle (internal) properties: namely, on its charge \( e \) and rest mass \( m_0 \).

As a result, the electron happens to appear eventually as an extended-like\[?] particle, with internal structure, rather than as a pointlike object (as initially assumed). For instance, one may imagine that the particle does not react instantaneously to the action of an external force because of its finite extension (the numerical value of the chronon is of the same order as the time spent by light to travel along an electron classical diameter). As already said, eq.(10) describes the motion of an object that happens to be pointlike only at discrete positions \( s_n \) along its trajectory\[?]; even if both position and velocity are still continuous and well-behaved functions of the parameter \( \tau \), since they are differentiable functions of \( \tau \).

It is essential to notice that a discrete character is given to the electron merely by the introduction of the fundamental quantum of time, without any need of a “model” for the electron. Actually it is well-known that many difficulties are met not only by the strictly pointlike models, but also by the extended-type particle models (“spheres”, “tops”, “gyroscopes”, etc.). In A.O.Barut’s words, “If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be?”.

We deem the answer stays with a third type of models,
the “extended-like” ones, as the present theory; or as the (related) theoretical approach\(^{13, 14}\) in which the center of the pointlike charge is spatially distinct from the particle center-of-mass. Anyway, it is not necessary to recall that the worst troubles met in quantum field theory (e.g., in quantum electrodynamics), like the presence of divergencies, are due to the pointlike character still attributed to (spinning) particles; since the problem of a suitable model for elementary particles was transported, unsolved, from classical to quantum physics. One could say that problem to be the most important in modern particle physics.

Equations (10) and (11) together provide a full description of the motion of the electron. Notice that the global, “macroscopic” motion can be the same for different solutions of the transmission law. The behaviour of the electron under the action of external electromagnetic fields is completely described by its macroscopic motion.

As in Dirac’s case, the equations above are invariant under Lorentz transformations; but, as we are going to see, are free from pre-accelerations, self-accelerating solutions, and problems (that had raised great debates in the first half of the century) with the hyperbolic motion.

In the non-relativistic limit the previous (retarded) equations reduced to the form

\[
\frac{m_0}{\tau_0} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)] = e \left[ \mathbf{E}(t) + \frac{1}{c} \mathbf{v}(t) \wedge \mathbf{B}(t) \right], \tag{14}
\]

\[
\mathbf{r}(t) - \mathbf{r}(t - \tau_0) = \frac{\tau_0}{2} [\mathbf{v}(t) - \mathbf{v}(t - \tau_0)], \tag{15}
\]

which can be obtained —this time— from eq.(11), by directly replacing the time derivatives by the corresponding finite-difference expressions. The macroscopic equation (12) had already been obtained also by other authors\(^ {15, 16, 17, 18}\) for the dynamics of extended-type electrons.

The important point is that eqs.(10),(11), or eqs.(12),(13), allow to overcome the difficulties met with the Dirac classical equation (10). In fact, the electron macroscopic motion is completely determined once velocity and initial position are given. Solutions of the relativistic equations (10),(11) for the radiating electron —or of the corresponding non-relativistic equations (12),(13)— were obtained for several problems, the resulting motions never presenting unphysical behaviour; so that the following questions can be regarded as having been solved\(^ {19}\): A) exact relativistic solutions: 1) free electron motion; 2) electron under the action of an electromagnetic pulse;\(^ {20}\) 3) hyperbolic motion\(^ {21}\); B) non-relativistic approximate solutions: 1) electron under the action of time-dependent forces; 2) electron in a constant, uniform magnetic field\(^ {22}\); 3) electron moving along a straight line under the action of an elastic restoring force\(^ {23}\).

Before going on, let us briefly study the electron radiation properties as deduced from the finite-difference relativistic equations (12), (13), with the aim of showing the advantages of the present formalism with respect to the Abraham–Lorentz–Dirac one. Such equations can be written\(^ {24, 25}\)

\[
\frac{\Delta Q_\mu(\tau)}{\tau_0} + R_\mu(\tau) + S_\mu(\tau) = \frac{\epsilon}{c} F_\mu\nu(\tau) u^\nu(\tau), \tag{16}
\]
where:

\[
\Delta Q_\mu = m_0 [u_\mu(\tau) - u_\mu(\tau - \tau_0)] ; \tag{17}
\]

\[
R_\mu(\tau) = -\frac{m_0}{2\tau_0} \left\{ \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} \left[ u''(\tau + \tau_0) + u''(\tau - \tau_0) - 2u''(\tau) \right] \right\} ; \tag{18}
\]

\[
S_\mu(\tau) = -\frac{m_0}{2\tau_0} \left\{ \frac{u_\mu(\tau)u_\nu(\tau)}{c^2} \left[ u''(\tau + \tau_0) - u''(\tau - \tau_0) \right] \right\} . \tag{19}
\]

In eq.(17), the first term \(\Delta Q_0/\tau_0\) represents the variation per unit of proper time (in the interval \(\tau - \tau_0\) to \(\tau\)) of the particle energy–momentum vector. The second one, \(R_\mu(\tau)\), is a dissipative term, since it contains only even derivatives of the velocity as one can prove by expanding \(u''(\tau + \tau_0)\) and \(u''(\tau - \tau_0)\) in terms of \(\tau_0\); furthermore, it is never negative,\[?, \, ?\] and can therefore represent the energy–momentum radiated by the electron in the unit of proper time. The third term, \(S_\mu(\tau)\), is conservative and represents the rate of change in proper time of the electron reaction energy–momentum.

The time component (\(\mu = 0\)) of eq.(17) writes:

\[
\frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} + R_0(\tau) + S_0(\tau) = P^{\text{ext}}(\tau) , \tag{20}
\]

quantity \(T(\tau)\) being the kinetic energy

\[
T(\tau) = m_0 c^2 \left( \frac{1}{\sqrt{1 - \beta^2}} - 1 \right) ; \tag{21}
\]

so that in eq.(20) the first term replaces the proper-time derivative of the kinetic energy, the second one is the energy radiated by the electron in the unit of proper time, \(S_0(\tau)\) is the variation rate in proper time of the electron reaction energy (radiative correction), and \(P^{\text{ext}}(\tau)\) is the work done by the external forces in the unit of proper time.

We are now ready to show, e.g., that eq.(17) yields a clear explanation for the origin of the so-called “acceleration energy” (Schott energy), appearing in the energy-conservation relation for the Dirac equation. In fact, by expanding in power series with respect to \(\tau_0\) the l.h. sides of eqs.(18),(19), (20),(21) for \(\mu = 0\), and keeping only the first-order terms, one gets

\[
\frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} \approx \frac{dT}{d\tau} - \frac{2}{3} \frac{e^2}{c^2} \frac{d\alpha_0}{d\tau} ; \tag{22}
\]

\[
R_0(\tau) \approx \frac{1}{\sqrt{1 - \beta^2}} \frac{2}{3} \frac{e^2}{c^3} \alpha_\mu a^\mu ; \tag{23}
\]

\[
S_0(\tau) \approx 0 ; \tag{24}
\]

where \(a^\mu\) is the four-acceleration

\[
a^\mu \equiv \frac{d\mu u^\mu}{drm^\mu} = \gamma \frac{d\mu u^\mu}{drm^\mu} .
\]
quantity $\gamma$ being the Lorentz factor. Therefore eq.(??), to the first order in $\tau_0$, becomes:

$$\frac{dT}{d\tau} = \frac{2}{3} \frac{e^2}{c^2} a_0 \frac{d}{d\tau} + \frac{2}{3} \frac{e^2}{c^3} \sqrt{1 - \beta^2} \approx F^\text{ext}(\tau) \tag{25}$$

or, passing from the proper time $\tau$ to the observer’s time $t$:

$$\frac{dT}{dt} = \frac{2}{3} \frac{e^2}{c^2} a_0 \frac{d}{dt} + \frac{2}{3} \frac{e^2}{c} a_\mu a^\mu \approx F^\text{ext}(\tau) \frac{d\tau}{dt} . \tag{26}$$

The last relation is identical with the energy conservation law found by Fulton and Rohrlich[?] for the Dirac equation. In eq.(26) the derivative of $(2e^2/3c^2)a_0$ appears, which is just the acceleration energy. Our approach clearly shows that it arises only by expanding in a power series of $\tau_0$ the kinetic energy increment suffered by the electron during the fundamental proper-time interval $\tau_0$; whilst such a Schott energy (as well as higher-order energy terms) does not show up when adopting the full formalism of finite–difference equations. We shall come back to this important point in Subsection 2.4.

Let us finally observe[?] that, when setting

$$\frac{m_0}{c\tau_0} \left[ u_\mu(\tau) - u_\mu(\tau - \tau_0) - u_\mu(\tau - \tau_0) u_\nu(\tau) \right] \equiv F_{\mu\nu}^\text{self} , \tag{27}$$

the relativistic equation of motion (??) reads:

$$\frac{e}{c} \left( F_{\mu\nu}^\text{self} + F_{\mu\nu}^\text{ext} \right) u^\nu = 0 , \tag{28}$$

confirming that $F_{\mu\nu}^\text{self}$ represents the (retarded) self-field associated with the moving electron.

### 2.4 The three alternative formulations of Caldirola’s theory

Two more (alternative) formulations are possible of Caldirola’s equations, based on different discretization procedures. In fact, equations (??) and (??) describe an intrinsically radiating particle. And, by expanding equation (??) in terms of $\tau_0$, a radiation reaction term appears. Caldirola called those equations the *retarded* form of the electron equations of motion.

By rewriting the finite–difference equations, on the contrary, in the form:

$$\frac{m_0}{\tau_0} \left\{ u_\mu(\tau + \tau_0) - u_\mu(\tau) + \frac{u_\mu(\tau) u_\nu(\tau)}{c^2} \left[ u_\nu(\tau + \tau_0) - u_\nu(\tau) \right] \right\} = \frac{e}{c} F_{\mu\nu}(\tau) u^\nu(\tau) , \tag{29}$$

$$x_\mu \left[ (n + 1) \tau_0 \right] - x_\mu (n\tau_0) = \tau_0 u_\mu (n\tau_0) , \tag{30}$$

one gets the *advanced* formulation of the electron theory, since the motion —according to eqs.(??),(??)— is now determined by advanced actions. In contrast with the retarded formulation, the advanced one describes an electron which absorbs energy from the external world.
Finally, by adding together retarded and advanced actions, Caldirola wrote down the symmetric formulation of the electron theory:

\[
\frac{m_0}{2\tau_0} \left\{ u_\mu (\tau + \tau_0) - u_\mu (\tau - \tau_0) + \frac{u_\mu (\tau) u_\nu (\tau)}{c^2} [u_\nu (\tau + \tau_0) - u_\nu (\tau - \tau_0)] \right\} = \frac{e}{c} F_{\mu\nu}(\tau) u_\nu (\tau),
\]

which does not include any radiation reactions, and describes a non radiating electron.

Before closing this brief introduction to Caldirola’s theory, it is worthwhile to present two more relevant results derived from it, one of them following below, in the next Subsection. If we consider a free particle and look for the “internal solutions” of the equation (??), we get —for a periodical solution of the type

\[
\begin{align*}
\dot{x} &= -\beta_0 c \sin \left( \frac{2\pi \tau}{\tau_0} \right) \\
\dot{y} &= -\beta_0 c \cos \left( \frac{2\pi \tau}{\tau_0} \right) \\
\dot{z} &= 0
\end{align*}
\]

(which describes a uniform circular motion) and by imposing the kinetic energy of the internal rotational motion to equal the intrinsic energy \(m_0c^2\) of the particle— that the amplitude of the oscillations is given by \(\beta_0^2 = \frac{3}{4}\). Thus, the magnetic moment corresponding to this motion is exactly the anomalous magnetic moment of the electron,\([\?]\) obtained here in a purely classical context:

\[
\mu_a = \frac{1}{4\pi} \frac{e^3}{m_0c^2}.
\]

This shows that the anomalous magnetic moment is an intrinsically classical, and not quantum, result; and the absence of \(\hbar\) in the last expression is a remarkable confirmation of this fact.

2.5 Hyperbolic motions

In a review paper on the theories of electron including radiation–reaction effects, Erber[?] criticized Caldirola’s theory for its results in the case of hyperbolic motion. Let us recall that the opinion of Pauli and von Laue (among others) was that a charge performing uniformly accelerated motions —e.g., an electron in free fall— could not emit radiation. That opinion was strengthened by the invariance of Maxwell equations under the group of conformal transformations,\([\?]\) which in particular includes transformations from rest to uniformly accelerated motions. Since the first decades of the twentieth century, this had been
—however— an open question, as the works by Born and Schott had on the contrary suggested a radiation emission in such a case. In 1960, Fulton and Rohrlich demonstrated that from Dirac’s equation for the classical electron the emission of radiation during the hyperbolic motion follows.

A solution of this paradox is possible within Caldirola’s theory, and it was worked out in ref.[?]. By analysing the energy conservation law for an electron submitted to an external force and following a procedure similar to that of Fulton and Rohrlich, Lanz obtained the expression above. By expanding it in terms of \( \tau \) and keeping only the first order terms, he arrived at expression (??), identical to the one obtained by Fulton and Rohrlich, in which —let us repeat—the Schott energy appears: a term that Fulton and Rohrlich (having obtained it from Dirac’s expression for the radiation reaction) interpreted as a part of the internal energy of the charged particle.

For the particular case of hyperbolic motion, it is

$$ a_\mu a^\mu = \frac{\mathrm{d}a_0}{\mathrm{d}\tau} $$

so that there is no radiation reaction [cf. eqs.(??) or (??)]. However, neither the acceleration energy, nor the energy radiated by the charge per unit of proper time, \( \frac{2}{3} e^2 a_\mu a^\mu \), are zero!

The difference is that in the discrete case this acceleration energy does not exist as such, being a term proceeding from the discretized expression for the charged particle kinetic energy variation. As we have seen in eq.(??), the Schott term appears when the variation of the kinetic energy during the fundamental interval of proper time is expanded in powers of \( \tau_0 \):

$$ \frac{T(\tau) - T(\tau - \tau_0)}{\tau_0} \sim \frac{\mathrm{d}T}{\mathrm{d}\tau} - \frac{2}{3} \frac{e^2}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} a_0. $$

This is an interesting result, since it was not easy to understand the physical meaning of the Schott acceleration energy. With the introduction of the fundamental interval of time, as we know, the changes in the kinetic energy are no longer continuous, and the Schott term merely expresses, to first order, the variation of the kinetic energy when passing from one discrete instant of time to the subsequent one.

In eqs.(??) and (??), the derivative \( \mathrm{d}T/\mathrm{d}\tau \) is a point function, forwarding the kinetic energy slope at the instant \( \tau \). And the dissipative term \( \frac{2}{3} e^2 a_\mu a^\mu \) is just a relativistic generalization of the Larmor radiation law: then, if there is acceleration, there is also radiation emission.

For the hyperbolic motion, however, the energy dissipated (because of the acceleration) has just the same magnitude as the energy gain due to the kinetic energy increase. We are not forced to resort to ‘pre-accelerations’ in order to justify the origin of such energies...[?] Thus, the present theory provides a clear picture of the physical processes involved in the uniformly accelerated motion of a charged particle.
3 The Hypothesis of the Chronon in Quantum Mechanics

Let us pass to the main topic of the present paper: the chronon in Quantum Mechanics. The speculations about the discreteness of time (on the basis of possible physical evidences) in QM go back to the first decades of this century, and various theories have been proposed developing QM on a space-time lattice (cf., e.g., refs.[?]). This is not the case with the hypothesis of the chronon, where we do not actually have a discretization of the time coordinate. In the twenties, for example, Pokrowski[?] suggested the introduction of a fundamental interval of time, starting from an analysis of the shortest wavelengths detected at that time in cosmic radiation. More recently, for instance, Ehrlich[?] proposed a quantization of the elementary particle lifetimes, suggesting the value $4.4 \times 10^{-24}$ s for the quantum of time. But a time discretization is suggested by the very foundations of QM. There are physical limits that prevent the distinction of arbitrarily close successive states in the time evolution of a quantum system. Basically, such limitations result from the Heisenberg relations: in such a way that, if a discretization is to be introduced in the description of a quantum system, it cannot possess a universal value, since those limitations depend on the characteristics of the particular system under consideration. In other words, the value of the fundamental interval of time has to change a priori from system to system. All these points make the extension of Caldirola’s procedure to QM justifiable.

In the seventies, Caldirola extended the introduction of the chronon to QM, following the same guidelines that had led him to his theory of the electron. So, time is still a continuous variable, but the evolution of the system along its world line is discontinuous. As for the electron theory in the non-relativistic limit, one has to substitute the corresponding finite-difference expression for the time derivatives; e.g.:

$$\frac{df(t)}{dt} \approx \frac{f(t) - f(t - \Delta t)}{\Delta t} \quad (33)$$

where proper time is now replaced by the local time $t$. Such a procedure was then applied to obtain the finite-difference form of the Schrödinger equation. As for the electron case, there are three different ways to perform the discretization, and three “Schrödinger equations” can be obtained[?]:

$$i\hbar \left[ \Psi(x, t) - \Psi(x, t - \tau) \right] = \hat{H} \Psi(x, t), \quad (34)$$

$$i\frac{\hbar}{2\tau} \left[ \Psi(x, t + \tau) - \Psi(x, t - \tau) \right] = \hat{H} \Psi(x, t), \quad (35)$$

$$i\frac{\hbar}{\tau} \left[ \Psi(x, t + \tau) - \Psi(x, t) \right] = \hat{H} \Psi(x, t), \quad (36)$$

which are, respectively, the retarded, symmetric and advanced Schrödinger equations, all of them transforming into the (same) continuous equation when the fundamental interval of time
(that can now be called just $\tau$) goes to zero. It can be immediately observed that the symmetric equation is of the second order, while the other two are first order equations. As in the continuous case, for a finite-difference equation of order $n$ a single and complete solution requires $n$ initial conditions to be specified.

As the equations are different, the solutions they provide are also fundamentally different. In order to study the properties of such equations there are two basic procedures. For some special cases, they can be solved by one of the various existing methods of solving finite-difference equations, or by means of an attempt solution, an ansatz. The other way is to find a new Hamiltonian $\tilde{H}$ such that a new continuous Schrödinger equation

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \tilde{H}\Psi(x,t)$$ \hspace{1cm} (37)

reproduces, at the points $t = n\tau$, the same results obtained from the discretized equations. As has been shown by Casagrande and Montaldi[?], it is always possible to find a continuous generating function which makes it possible to obtain a differential equation equivalent to the original finite-difference one, such that in every point of interest their solutions are identical. This procedure is useful since it is generally very difficult to work with the finite-difference equations on a qualitative basis. Except for some very special cases, they can be only numerically solved. This equivalent Hamiltonian $\tilde{H}$ is, however, non-hermitian and it is frequently very difficult to be obtained. For the special case where the Hamiltonian is time independent, the equivalent Hamiltonian is quite easy to calculate. For the symmetric equation, e.g., it is given by:

$$\tilde{H} = \frac{\hbar}{\tau} \sin^{-1}\left(\frac{\tau}{\hbar}\right).$$ \hspace{1cm} (38)

As expected, $\tilde{H} \rightarrow H$ when $\tau \rightarrow 0$. Caldirola[?] used the symmetric equation to describe the non-radiating electron (bound electron) since for Hamiltonians explicitly independent of time its solutions are always of oscillating character:

$$\Psi(x,t) = \exp\left(-i\frac{t}{\tau} \sin^{-1}\left(\frac{\tau}{\hbar}\right)\right) f(x).$$

In the classical theory of electron, the symmetric equation also represents a non-radiating motion. It provides only an approximate description of the motion without taking into account the effects due to the self fields of the electron. However, in the quantum theory it plays a fundamental role. In the discrete formalism, it is the only way to describe a bound non-radiating particle.

The solutions of the advanced and retarded equations show completely different behaviour. For a hamiltonian explicitly independent of time the solutions have a general form given by

$$\Psi(x,t) = \left[1 + i\frac{\tau}{\hbar} \tilde{H}\right]^{-t/\tau} f(x)$$

and, expanding $f(x)$ in terms of the eigenfunctions of $\hat{H}$,

$$\hat{H}u_n(x) = W_n u_n(x).$$
with

\[ \sum_n |c_n|^2 = 1, \]

it can be obtained that

\[ \Psi(x, t) = \sum_n c_n \left[ 1 + i \frac{\tau}{\hbar} W_n \right]^{-t/\tau} u_n(x). \]

In particular, the norm of this solution is given by

\[ |\Psi(x, t)|^2 = \sum_n |c_n|^2 \exp(-\gamma_n t) \]

with

\[ \gamma_n = \frac{1}{\tau} \ln \left( 1 + \frac{\tau^2}{\hbar^2} W_n^2 \right) = \frac{W_n^2}{\hbar^2 \tau} + O(\tau^3). \]

The presence of a damping factor depending critically on the value \( \tau \) of the chronon must be remarked.

This dissipative behaviour originates from the retarded character of the equation. The analogy with the electron theory also holds and the retarded equation possesses intrinsically dissipative solutions representing a radiating system. The Hamiltonian has the same status as in the continuous case: it is an observable since it is a hermitian operator and its eigenvectors form a basis of the state space. However, due to the damping term, the norm of the state vector is not constant anymore. An opposite behaviour is observed for the solutions of the advanced equation, in the sense that they increase exponentially.

Before going on, let us at least mention that the discretized QM (as well as Caldirola et al.’s approach to “QM with friction”) can find room within the theories based on the so-called Lie-admissible algebras. A lot of related work (not covered in the present review) can be found e.g. in refs.; see also.

For a different approach to decaying states, see e.g.

### 3.1 The mass of the muon

The most impressive achievement due to the introduction of the chronon hypothesis in the realm of QM is obtained in the description of a bound electron using the new formalism. Bound states are described by the symmetric Schrödinger equation and a Hamiltonian that does not depend explicitly on time. A general solution can be obtained by using a convenient ansatz:

\[ \Psi(x, t) = \sum_n u_n(x) \exp(-i\alpha_n t), \]
where $\hat{H} u_n (x) = E_n u_n (x)$ gives the spectrum of eigenvalues of the Hamiltonian. If the fundamental interval of time $\tau$ corresponds to the chronon $\theta_0$ associated with the classical electron, it can be straightforwardly obtained that

$$\alpha_n = \frac{1}{\theta_0} \sin^{-1} \left( \frac{E_n \theta_0}{\hbar} \right).$$

This solution gives rise to an upper limit for the eigenvalues of the Hamiltonian due to the condition

$$\left| \frac{E_n \theta_0}{\hbar} \right| \leq 1.$$

Since $\theta_0$ is finite, there is a maximum value for the energy of the electron given by

$$E_{\text{max}} = \frac{\hbar}{\theta_0} = \frac{2 \hbar m_0 c^3}{3 e^2} \approx 105.04 \text{ MeV}.$$

Now, including the rest energy of the electron, we finally get

$$E = E_{\text{max}} + E_{\text{electron}} \approx 105.55 \text{ MeV},$$

which is very close (an error of 0.1%) to the measured value of the rest mass of the muon. Using the equivalent Hamiltonian method it is possible to extend the basis of eigenstates out of the critical limit. However, for the eigenvalues above the critical limit the corresponding eigenstates are unstable and decay in time:

$$\Psi (x, t) = \sum_n c_n u_n (x) \exp (-i \gamma_n t) \exp (-k_n t),$$

As for the retarded equation, the norm of the state vector is not constant and decays exponentially with time for those eigenstates out of the stability range. Caldirola[?] interpreted this norm as giving the probability of the existence of the particle in its original Hilbert space and associated a mean lifetime with these states.

The considerations regarding the muon as an excited state of the electron can be traced back to the very days of its discovery. Particularly, it has already been observed that the ratio between the masses of the two particles is almost exactly $3/(2\alpha)$, where $\alpha$ is the fine structure constant.[?] It has already been remarked that $\frac{3}{2} \alpha$ is just the coefficient of the radiative reaction term in Dirac’s equation for the classical electron.[?] Bohm and Weinstein[?] put forward the hypothesis that various kinds of “mesons” could be excited states of the electron. Dirac[?] even proposed a specific model for an extended electron so as to interpret the muon as an excited state of the electron (on this point, cf. also refs.[?]).

Caldirola[?] observed that by means of the Heisenberg uncertainty relations it is possible to associate the existence of the muon as an excited state of the electron with the introduction of the chronon in the theory of electron. The relation

$$\Delta \tau \Delta E \geq \hbar/2$$

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 imposes limitations in the determination, at a certain instant \( \tau \), of the energy \( E \) associated with the internal motion of the electron. If excited states of the particle corresponding to larger values of mass exist, then it is possible only to speak of an “electron with rest mass \( m_0 \)” when \( \Delta E \leq (\mu_0 - m_0)c^2 \), where \( \mu_0 \) is the rest mass of the internal excited state. Such internal states could only be excited in the presence of sufficiently strong interactions. From the uncertainty relation we have that

\[
\Delta \tau \geq \frac{\hbar}{2(\mu_0 - m_0)c^2},
\]

and, supposing the muon as an excited state, we get

\[
(\mu_0 - m_0)c^2 \simeq \frac{3\hbar c}{2e^2m_0c^2}.
\]

Thus, it can be finally obtained that

\[
\Delta \tau \geq \frac{1}{3m_0c^2} \frac{e^2}{\tau_0} = \frac{\tau_0}{2},
\]

i.e., that the value of the rest mass of an interacting electron can be taken only inside an interval of the proper time larger than half a chronon. So, when we take into account two successive states, each one endowed with the same uncertainty \( \Delta \tau \), they must then be separated by a time interval of at least \( 2 \Delta \tau \), which corresponds exactly to the chronon \( \tau_0 \).

### 3.2 The mass spectrum of leptons

In order to obtain the mass of another particle, a possibility to be considered is to take the symmetric equation as describing the muon. According to this naïve argumentation, the equation also foresees a maximum limit for the energy of the eigenstates of the muon. By assuming the equation as successively describing the particles corresponding to these maxima, an expression can be set up for the various limit values, given by

\[
E_{\alpha}^{(n)} = m_0c^2 \left[ \frac{3\hbar c}{2e^2} + 1 \right]^n = m_0c^2 \left[ \frac{3\hbar c}{2e^2} + 1 \right]^n,
\]  

such that, for

\[
\begin{align*}
  n = 0 & \rightarrow \ E^{(0)} = 0.511 \text{ MeV} & \text{ (electron)} \\
  n = 1 & \rightarrow \ E^{(1)} = 105.55 \text{ MeV} & \text{ (muon)} \\
  n = 2 & \rightarrow \ E^{(2)} = 21801.54 \text{ MeV} & \text{ (heavy lepton?)}
\end{align*}
\]

the masses for the first excited states can be obtained, including a possible heavy lepton which, according to the experimental results up to now, does not seem to exist.

Following a suggestion of Barut[?], according to which it should be possible to obtain the excited states of the electron from the coupling of its intrinsic magnetic moment with its self field, Caldirola[?, ?], considering a model of the extended electron as a micro-universe, succeeded in evaluating also the mass of the lepton \( \tau \).
Caldirola took into account, for the electron, a model of a point-object moving around in a 4-dimensional de Sitter micro-universe characterized by

\[ c^2 t^2 - x^2 - y^2 - z^2 = c^2 \tau_0^2, \]

where \( \tau_0 \) is the chronon associated with the electron and the radius of the micro-universe is given by \( a = c\tau_0 \). Considering the spectrum of excited states obtained from the naïve argumentation above, we find that each excited state determines a characteristic radius for the micro-universe. Thus, for each particle, the trajectory of the point-object is confined to a spherical shell defined by its characteristic radius and by the characteristic radius of its excited state. For the electron, e.g., the point-object moves around, inside the spherical shell defined by its corresponding radius and by the one associated with its excited state: the muon. Such radii are given by

\[ a^{(n)} = \tau_0 c \left[ \frac{3}{2} \alpha + 1 \right]^{-n}. \tag{40} \]

According to the model — supposing that the intrinsic energy of the lepton \( e^{(n)} \) is given by \( m^{(n)} c^2 \) — the lepton moves in its associated micro-universe along a circular trajectory with a velocity \( \beta = \frac{\sqrt{3}}{2} \), to which corresponds an intrinsic magnetic moment

\[ \mu^{(n)} = \frac{1}{4\pi} \frac{e^2}{m^{(n)} c^2}. \tag{41} \]

Starting from Barut’s suggestion, Caldirola obtained for the lepton \( e^{(n)} \) an extra self-energy given by

\[ E^{(n,p)} = (2p)^4 m^{(n)} c^2. \]

The condition set down on the trajectory of the point-object, so that it remains confined to its corresponding spherical shell, is given by

\[ E^{(n,p)} \leq \left[ \frac{3}{2} \alpha + 1 \right] m_0 c^2, \]

and the values attainable by \( p \) are: \( p = 0 \) for \( n = 0 \), and \( p = 0,1 \) for \( n \neq 0 \). The spectrum of mass is then finally given by

\[ m^{(n,p)} = \left[ 1 + (2p)^4 \right] m^{(n)} = m_0 \left[ 1 + (2p)^4 \right] \left[ \frac{3}{2} \alpha + 1 \right]^{-n}. \tag{42} \]

Thus, for different values of \( n \) and \( p \) we have:

| \( n \) | \( p \) | \( m^{(n)} \)     |
|-------|-------|-----------------|
| 0     | 0     | 0.511 MeV       | electron          |
| 1     | 0     | 105.55 MeV      | muon              |
| 1     | 1     | 1794.33 MeV     | tau               |
It must be remarked that the \( \tau \) appears as an internal excited state of the muon and its mass is in fair agreement with the experimental values: \( m_\tau \approx 1784 \text{ MeV} \). The difference between these values is less than 1%. This is quite amazing if we consider the simplicity of the model. The model foresees the existence of other excited states which do not seem to exist. This is to some extent justifiable once the muon is obtained as an excited electron and the description of the electron does not anticipate the existence of any other state. In order to obtain the lepton \( \tau \) it was necessary to introduce in the formalism the coupling of the intrinsic magnetic moment with the self-field of the electron.

3.3 Feynman path integrals

The discretized Schrödinger equations can easily be obtained using Feynman’s path integral approach. This is particularly interesting since it gives a clearer idea of the meaning of these equations. According to the hypothesis of a chronon, time is still a continuous variable and the introduction of the fundamental interval of time is connected —let us recall— only with the reaction of the system to the action of a force. It is convenient to restrict the derivation to the one-dimensional case, considering a particle under the action of a potential \( V(x, t) \). We take the time coordinate as continuous and take into account a discretization at instants separated by steps given by \( \tau \).

The transition amplitude for a particle going from an initial point \((x_1, t_1)\) of the space-time to a final point \((x_n, t_n)\) is given by the propagator

\[
K(x_n, t_n; x_1, t_1) = \langle x_n, t_n | x_1, t_1 \rangle .
\] (43)

In Feynman’s approach this transition amplitude is associated with a path integral, where the classical action plays a fundamental role. It is convenient to introduce the notation

\[
S(n, n - 1) \equiv \int_{t_{n-1}}^{t_n} dt \mathcal{L}(x, \dot{x})
\] (44)
such that \( \mathcal{L}(x, \dot{x}) \) is the classical Lagrangian and \( S(n, n_1) \) is the classical action. Thus, for two consecutive instants of time, the propagator is given by

\[
K(x_n, t_n; x_{n-1}, t_{n-1}) = \frac{1}{A} \exp \left( \frac{i}{\hbar} S(x_n, t_n; x_{n-1}, t_{n-1}) \right).
\]  

(45)

The path integral has the meaning of a sum over all the possible paths traversed by the particle and can be written as

\[
\langle x_n, t_n | x_1, t_1 \rangle = \lim_{N \to \infty} A^{-N} \int dx_{N-1} \int dx_{N-2} \cdots \int dx_2 \prod_{n=2}^{N} \exp \left( \frac{i}{\hbar} S(n, n-1) \right),
\]

(46)

where \( A \) is a normalization factor. In order to obtain the discretized Schrödinger equations we have to consider the evolution of a quantum state between two consecutive configurations \((x_{n-1}, t_{n-1}) \in (x_n, t_n)\). The state of the system at \( t_n \) is:

\[
\Psi(x_n, t_n) = \int \Psi(x_{n-1}, t_{n-1}) \exp \left( \frac{i}{\hbar} S(n, n-1) \right) dx_{n-1}.
\]

(47)

On the other side, it follows from the definition of the classical action (eq. ??) that

\[
S(x_n, t_n; x_{n-1}, t_{n-1}) = \frac{m}{2\tau} (x_n - x_{n-1})^2 - \tau V \left( \frac{x_n + x_{n-1}}{2}, t_{n-1} \right).
\]

(48)

Thus, the state at \( t_n \) is given by

\[
\Psi(x_n, t_n) = \int_{-\infty}^{+\infty} \exp \left\{ \frac{im}{2\tau} (x_n - x_{n-1})^2 - \frac{\tau}{\hbar} V \left( \frac{x_n + x_{n-1}}{2}, t_{n-1} \right) \right\} \Psi(x_{n-1}, t_{n-1}) dx_{n-1}.
\]

(49)

When \( \tau \approx 0 \), for \( x_n \) slightly different from \( x_{n-1} \), the integral due to the quadratic term is rather small. The contributions are considerable only for \( x_n \approx x_{n-1} \). Thus, we can make the following approximation

\[
x_{n-1} = x_n + \eta \quad \Rightarrow \quad dx_{n-1} \equiv d\eta,
\]

such that

\[
\Psi(x_{n-1}, t_{n-1}) \cong \Psi(x_n, t_{n-1}) + \left( \frac{\partial \Psi(x_n, t_{n-1})}{\partial x} \right) \eta + \left( \frac{\partial^2 \Psi(x_n, t_{n-1})}{\partial x^2} \right) \eta^2.
\]

By inserting this expression into equation (??), suppose that\(^\S\)

\[
V \left( x + \frac{\eta}{2} \right) \approx V(x),
\]

\(^\S\)The potential is supposed to vary slowly with \( x \).
and taking into account only the terms to the first order in $\tau$, we get

$$
\Psi(x_n, t_n) = \frac{1}{A} \exp \left( -\frac{i}{\hbar} \tau V(x_n, t_{n-1}) \right) \left( \frac{2i\hbar\tau}{m} \right)^{1/2} \left( \Psi(x_n, t_{n-1}) + \frac{i\hbar \tau}{2m} \frac{\partial^2 \Psi}{\partial x^2} \right)
$$

Notwithstanding the fact that $\exp(-i\tau V(x_n, t_n)/\hbar)$ is a function defined only for certain well-determined values, it can be expanded in powers of $\tau$, around an arbitrary position $(x_n, t_n)$. Choosing $A = \left( \frac{2i\hbar\pi\tau}{m} \right)^{-1/2}$, such that $\tau \to 0$ in the continuous limit, we obtain

$$
\Psi(x_n, t_{n-1} + \tau) - \Psi(x_n, t_{n-1}) = -\frac{i}{\hbar} \tau V(x_n, t_{n-1}) \Psi(x_n, t_{n-1}) + \frac{i\hbar \tau}{2m} \frac{\partial^2 \Psi}{\partial x^2} + O(\tau^2)
$$

By a simple reordering of terms we finally get

$$
\frac{i}{\tau} \left[ \Psi(x_n, t_{n-1} + \tau) - \Psi(x_n, t_{n-1}) \right] = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x_n, t_{n-1}) \right\} \Psi(x_n, t_{n-1})
$$

Following this procedure we obtain the **advanced** finite-difference Schrödinger equation which describes a particle performing a one-dimensional motion under the potential $V(x, t)$.

The solutions of the advanced equation show an amplification factor which may suggest that the particle absorbs energy from the field described by the Hamiltonian in order to evolve in time. In the classical domain the advanced equation is interpreted as describing a positron. However, in the realm of the non-relativistic QM, it is more naturally interpreted as representing a system which takes in energy from the environment.

In order to obtain the discrete Schrödinger equation only the terms to the first order in $\tau$ have been taken into account. Since the limit $\tau \to 0$ has not been accomplished, the equation thus obtained is only an approximation. The meaning of this fact may be related to another one we are going to face below in this paper, when considering the measurement problem in QM.

It is rather interesting to remark that in order to obtain the **retarded** equation we have to regard the propagator as acting backward in time. The conventional procedure used in the continuous case always provides us with the advanced equation. Therefore, the potential describes a mechanism for transferring energy from a field to the system. The retarded equation can be obtained only by assuming an inversion of the time order, considering the expression

$$
\Psi(x_{n-1}, t_{n-1}) = \frac{1}{A} \int_{-\infty}^{+\infty} \exp \left\{ \frac{i}{\hbar} \int_{t_n}^{t_{n-1}} \mathcal{L} dt \right\} \Psi(x_n, t_n) dx_n,
$$

which can be rigorously obtained by merely using the closure relation for the eigenstates of the position operator and then redefining the propagator in the inverse time order. With this expression, it is possible to obtain the retarded Schrödinger equation. The symmetric equation can easily be obtained by a similar procedure.

An interesting characteristic related to these apparently opposed equations is the impossibility of obtaining one from the other by a simple time inversion. The time order in the propagators
must be related to the inclusion, in these propagators, of something like the advanced and retarded potentials. Thus, in order to obtain the retarded equation we have to take into account effects that act backward in time. Considerations like these that led to the derivation of the three discretized equations can supply useful guidelines for the comprehension of their meaning.

3.4 The Schrödinger and Heisenberg pictures

In discrete QM, as well as in the “continuous” one, the utilization of discretized Heisenberg equations is expected to be preferable for certain types of problems. As it happens for the continuous case, the discretized versions of the Schrödinger and Heisenberg pictures are also equivalent. However, we show below that the Heisenberg equations cannot, in general, be obtained by a direct discretization of the continuous equations.

First of all, it is convenient to introduce the discrete time evolution operator for the symmetric

$$\hat{U} (t, t_0) = \exp \left[ -\frac{i}{\tau} \frac{(t-t_0)}{\sin^{-1} \left( \frac{\tau \hat{H}}{\hbar} \right)} \right]$$

and for the retarded equation,

$$\hat{U} (t, t_0) = \left[ 1 + \frac{i}{\hbar} \tau \hat{H} \right]^{-(t-t_0)/\tau}$$

In order to simplify the equations, the following notation will be used throughout this Section

$$\Delta f(t) \longleftrightarrow \frac{f(t + \tau) - f(t - \tau)}{2\tau}$$

$$\Delta_R f(t) \longleftrightarrow \frac{f(t) - f(t - \tau)}{\tau}$$

For both the operators above it can easily be demonstrated that, if the Hamiltonian $\hat{H}$ is a hermitian operator, the following equations are valid:

$$\Delta \hat{U} (t, t_0) = \frac{1}{i\hbar} \hat{U} (t, t_0) \hat{H},$$

$$\Delta \hat{U}^\dagger (t, t_0) = -\frac{1}{i\hbar} \hat{U}^\dagger (t, t_0) \hat{H}.$$  

In the Heisenberg picture the time evolution is transferred from the state vector to the operator representing the observable according to the definition

$$\hat{A}^H \equiv \hat{U}^\dagger (t, t_0 = 0) \hat{A} \hat{U} (t, t_0 = 0)$$

For the symmetric case, for a given operator $\hat{A}^S$, the time evolution of the operator $\hat{A}^H(t)$ is given by

$$\Delta \hat{A}^H (t) = \Delta \left[ \hat{U}^\dagger (t, t_0 = 0) \hat{A} \hat{U} (t, t_0 = 0) \right]$$
\[
\Delta \hat{A}^H(t) = \frac{1}{i\hbar} \left[ \hat{A}^H, \hat{H} \right]
\]
which has exactly the same form as the equivalent equation for the continuous case. The important feature of the time evolution operator which is used to derive the expression above is that it is an *unitary* operator. This is true for the symmetric case. For the retarded case, however, this property is not satisfied anymore. Differently from the symmetric and continuous cases, the state of the system is also time dependent in the retarded Heisenberg picture:

\[
\left| \Psi^H(t) \right> = \left[ 1 + \frac{\tau^2 \hat{H}^2}{\hbar^2} \right]^{-(t-t_0)/\tau} \left| \Psi^S(t_0) \right>
\]

By using the property \( [\hat{A}, f(\hat{A})] = 0 \), it is possible to show that the evolution law for the operators in the retarded case is given by:

\[
\Delta \hat{A}^H(t) = \left\{ \frac{1}{i\hbar} \left[ \hat{A}^S(t), \hat{H}^S(t) \right] + \Delta \hat{A}^S(t) \right\}^H
\]

In short, we can conclude that the discrete symmetric case and the continuous case are formally very similar and the Heisenberg equation can be obtained by a direct discretization of the continuous equation. For the retarded and advanced cases, however, this does not hold. In the Appendices we analyse the compatibility between the Heisenberg and Schrödinger pictures.

Let us here mention that a lot of parallel work has been done by Jannussis et al.; they, e.g., studied the retarded, dissipative case in the Heisenberg representation, passing then to study in that picture the (normal or damped) harmonic oscillator: see[?] (cf. also[?, ?]).

### 3.5 Time-dependent Hamiltonians

When we restricted the analysis of the discretized equations to the time independent Hamiltonians this was made aiming at simplicity. When the Hamiltonian is explicitly time dependent the situation is very similar to the continuous case. It is always difficult to work with such Hamiltonians but, as in the continuous case, the theory of small perturbations can also be applied. For the symmetric equation, when the Hamiltonian is of the form

\[
\hat{H} = \hat{H}_0 + \hat{V}(t),
\]

such that \( \hat{V} \) is a small perturbation related to \( \hat{H}_0 \), the resolution method turns out to be very similar to the usual one. The solutions are equivalent to the continuous solutions followed by an exponentially varying term. It is always possible to solve this kind of problem using an appropriate ansatz.

However, another factor must be considered, related to the existence of a limit beyond which \( \hat{H} \) does not have stable eigenstates. For the symmetric equation, the equivalent Hamiltonian is given by

\[
\hat{H} = \frac{\hbar}{\tau} \sin^{-1} \left( \frac{\tau}{\hbar} \hat{H} \right).
\]
Thus, as previously stressed, beyond the critical value the eigenvalues are not real and the operator $\hat{H}$ is not hermitian anymore. Below that limit, $\hat{H}$ is a densely defined and a self-adjoint operator in the $L \subset L^2$ subspace defined by the eigenfunctions of $\hat{H}$. When the limit value is exceeded the system changes to an excited state and the previous state loses physical meaning. In this way, it is convenient to restrict the observables to self-adjoint operators that keep invariant the subspace $L$. The perturbation $\hat{V}$ is assumed to satisfy this requirement.

In the usual QM it is convenient, in order to deal with time dependent perturbations, to work with the interaction representation (Dirac’s picture). In this representation, the evolution of the state is determined by the time dependent potential $\hat{V}(t)$, while the evolution of the observable is determined by the stationary part of the Hamiltonian $\hat{H}_0$. In the discrete formalism, the time evolution operator defined for $\hat{H}_0$, in the symmetric case, is given by

$$\hat{U}_0(t, t_0) = \exp \left[ -\frac{i}{\hbar} \frac{(t - t_0)}{\tau} \sin^{-1} \left( \frac{\tau \hat{H}_0}{\hbar} \right) \right]$$

(64)

In the interaction picture the vector state is defined, from the state in the Schrödinger picture, as

$$\Psi^I(t) = \hat{U}_0^\dagger(t) \Psi^S(t_0)$$

(65)

where $\hat{U}_0^\dagger(t) \equiv \hat{U}_0(t, t_0 = 0)$. On the other hand, the operators are defined as

$$\hat{A}^I = \hat{U}_0^\dagger(t) \hat{A} \hat{U}_0(t)$$

(66)

So, it is possible to show that, in the interaction picture, the evolution of the vector state is determined by the equation

$$i\hbar \Delta \Psi^I(x, t) = \frac{i\hbar}{2\tau} \left[ \Psi^I(x, t + \tau) - \Psi^I(x, t - \tau) \right] = \hat{V}^I \Psi^I(x, t),$$

(67)

which is equivalent to a direct discretization of the continuous equation. For the operators we get that

$$\Delta \hat{A}^I(t) = \frac{\hat{A}^I(t + \tau) - \hat{A}^I(t - \tau)}{2\tau} = \frac{1}{i\hbar} \left[ \hat{A}^I, \hat{H}_0 \right],$$

(68)

which is also equivalent to the continuous equation.

Thus, for the symmetric case, the discrete interaction picture keeps the same characteristics of the continuous one for the evolution of the operators and state vectors once, obviously, the eigenstates of $\hat{H}$ remain below the stability limit. We can adopt, for the discrete case, a procedure similar to that one commonly used in QM to deal with small time dependent perturbations.

We consider, in the interaction picture, the same basis of eigenstates associated with the stationary Hamiltonian $\hat{H}_0$, given by $|n\rangle$. Then,

$$|\Psi(t)\rangle^I = \sum_n \Psi(t) \langle n | \Psi(t)\rangle^I |n\rangle = \sum_n c_n(t) |n\rangle$$
is the expansion, over this basis, of the state of the system at a certain instant $t$. It must be observed that the evolution of the state of the system is determined once the coefficients $c_n(t)$ are known. Using the evolution equation (??) it can be obtained that

$$i\hbar \Delta \langle n | \Psi(t) \rangle^\dagger = \sum_m \langle n | \hat{V}^\dagger | m \rangle \langle m | \Psi(t) \rangle^\dagger.$$

Using the evolution operator to rewrite the perturbation $\hat{V}$ in the Schrödinger picture we get that

$$i\hbar \Delta c_n(t) = \sum_m c_m(t) V_{nm}(t) \exp(i\omega_{nm}t), \quad (69)$$

such that

$$\omega_{nm} = \frac{1}{\tau} \left[ \sin^{-1} \left( \frac{\tau E_n}{\hbar} \right) \sin^{-1} \left( \frac{\tau E_m}{\hbar} \right) \right],$$

and we obtain the evolution equation for the coefficients $c_n(t)$, the solution of which gives the time evolution of the system.

As in the usual QM, it is also possible to work with the interaction picture evolution operator, $\hat{U}^I(t, t_0)$, which is defined as

$$[\Psi(t)]^\dagger = \hat{U}^I(t, t_0) [\Psi(t_0)]^\dagger,$$

such that (??) can be written as

$$i\hbar \Delta \hat{U}^I(t, t_0) = \hat{V}^I(t) \hat{U}^I(t, t_0). \quad (70)$$

The operator $\hat{U}^I(t, t_0)$ has to satisfy the initial condition $\hat{U}^I(t, t_0) = 0$. Given this condition, we have for the finite-difference equation above the solution

$$\hat{U}^I(t, t_0) = \exp \left[ -\frac{i(t - t_0)}{\tau} \sin^{-1} \left( \frac{\tau \hat{V}^I(t)}{\hbar} \right) \right].$$

Differently from the continuous case, where the approximate evolution operator turns out to be an infinite Dyson series, a well determined expression is obtained. The solution of the problem is obtained by correlating the elements of the matrix associated with such operator to the evolution coefficients $c_n(t)$.

In general, the finite-difference equations are harder to be analytically solved than the equivalent differential equations. In particular, such difficulty is far more stressed for the system of equations obtained from the formalism above.

An alternative approach is to use the equivalent Hamiltonians. Once the equivalent Hamiltonian is found the procedure is exactly the same as for the continuous theory. If the perturbation term $\hat{V}$ is small the equivalent Hamiltonian can be written as

$$\hat{H} = \frac{\hbar}{\tau} \sin^{-1} \left( \frac{\tau \hat{H}_0}{\hbar} \right) + \hat{V}(t) = \hat{H}_0 + \hat{V}(t).$$
In the interaction picture, the state of the system is now defined as

$$\left| \psi^I(t) \right> = \exp \frac{i \hat{H}_0 t}{\hbar} \left| \psi^S(t) \right>,$$

and the operators are given by

$$\hat{A}^I = \exp \left( \frac{i \hat{H}_0 t}{\hbar} \right) \hat{A}^S \exp \left( -\frac{i \hat{H}_0 t}{\hbar} \right).$$

The states (??) evolve according to the equation

$$i\hbar \frac{\partial}{\partial t} \left| \psi^I(t) \right> = \hat{V}^I \left| \psi^I(t) \right>,$$

where \( \hat{V}^I \) is obtained according to definition (??).

Now, small time dependent perturbations can be dealt with by taking into account the time evolution operator defined by

$$\left| \psi^I(t) \right> = \hat{U}^I(t, t_0) \left| \psi^I(t_0) \right>.$$  \hspace{1cm} (74)

According to the evolution law (??) we have

$$i\hbar \frac{d}{dt} \hat{U}^I(t, t_0) = \hat{V}^I(t) \hat{U}^I(t, t_0).$$ \hspace{1cm} (75)

Thus, once given that \( \hat{U}^I(t_0, t_0) = 1 \) it turns out that the time evolution operator is given by

$$\hat{U}^I(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} \hat{V}^I(t') \hat{U}^I(t', t_0) dt'.$$

or

$$\hat{U}^I(t, t_0) = 1 + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{V}^I(t_1) \hat{V}^I(t_2) \cdots \hat{V}^I(t_n),$$

where the evolution operator is obtained in terms of a Dyson series.

In order to draw a parallel between the elements of the matrix of the evolution operator and the evolution coefficients \( c_n(t) \) obtained from the continuous equation equivalent to (??), we have to use the basis of eigenstates of the stationary Hamiltonian \( H_0 \). If the initial state of the system is an eigenstate \( |m> \) of that operator then, at a subsequent time, we have

$$c_n(t) = \left< n | \hat{U}^I(t, t_0) | m > \right>.$$  

The method of the equivalent Hamiltonian is simpler to use, since it takes full advantage of the continuous formalism.
4 Some Applications of the Discretized Quantum Equations

Turning back to more general questions, it is interesting to analyse the physical consequences resulting from the introduction of the fundamental interval of time in QM. In this Section we apply the discretized equations to some typical problems.

4.1 The simple harmonic oscillator

The Hamiltonian that describes a simple harmonic oscillator does not depend explicitly on time. The introduction of the discretization in the time coordinate does not affect the outputs obtained from the continuous equation for the spatial branch of the solution. This is always the case when the potential does not have an explicit time dependence. For potentials like this, the solutions of the discrete equations are always formally identical, with changes in the numerical values which depend on the eigenvalues of the Hamiltonian considered and on the value of the chronon associated with the system described. We have the same spectrum of eigenvalues and the same basis of eigenstates but with the time evolution given by a different expression.

For the simple harmonic oscillator, the Hamiltonian is given by

\[
\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{m\omega^2}{2} \hat{x}^2,
\]

(76)

to which the eigenvalue equation corresponds:

\[
\hat{H} |u_n\rangle = E_n |u_n\rangle,
\]

(77)

so that \( E_n \) gives the energy eigenvalue spectrum of the oscillator.

As mentioned previously, since this Hamiltonian does not depend explicitly on time, there is always an upper limit for the possible values of its energy eigenvalues. In the basis of eigenfunctions of \( \hat{H} \) a general state of the oscillator can be written as

\[
|\Psi(t)\rangle = \sum_n c_n(0) |u_n\rangle \exp \left[ -i \frac{t}{\tau} \sin^{-1} \left( \frac{E_n}{\hbar} \right) \right],
\]

with \( c_n(0) = \langle u_n | \Psi(t = 0) \rangle \). Naturally, when \( \tau \to 0 \), the solution above recovers the continuous expression with its time dependency given by \( \exp \left( -\frac{tE_n}{\hbar} \right) \). Therefore, there is only a small phase difference between the two expressions. For the mean value of an arbitrary observable,

\[
\langle \Psi(t)|\hat{A}|\Psi(t)\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m^*(0) c_n(0) A_{mn} \exp \left[ i \frac{t}{\hbar} (E_m - E_n) \right] \cdot
\]

\[
\cdot \exp \left[ i \left( E_m^3 - E_n^3 \right) \frac{t\tau^2}{3\hbar^3} \right] + O(\tau^4),
\]

with \( A_{mn} = \langle u_m | \hat{A} | u_n \rangle \), we obtain an additional phase term which implies a small deviation of the resulting frequencies when compared to the Bohr frequencies of the harmonic oscillator.
To first approximation, this deviation is given by the term depending on $\tau^2$ in the expression above. It must be emphasized that the restrictions imposed on the spectrum of eigenvalues of $\dot{H}$ break the basis of eigenvectors: the number of eigenvectors becomes finite and does not constitute a complete set and, therefore, does not form a basis anymore. For eigenstates beyond the upper limit the states are unstable and decay exponentially with time.

For a time independent Hamiltonian, the retarded equation always furnishes damped solutions characteristic of radiating systems. In this case there is neither stationary solutions nor upper limit for the energy eigenvalues. The larger the eigenvalue the larger the damping factor and more quickly its contribution to the state of the system tends to zero. If we write the state of the oscillator as

$$|\Psi(t)\rangle = \sum_n c_n(0) |u_n\rangle \left[ 1 + \frac{i}{\hbar} \tau E_n \right]^{\frac{1}{\tau}} ,$$

which has a norm decaying according to

$$\langle \Psi(t)|\Psi(t)\rangle = \sum_n c_n(0) |u_n\rangle \left[ 1 + \frac{\tau^2 E_n^2}{\hbar^2} \right]^{\frac{1}{\tau}} ,$$

we have for an arbitrary observable that [with $\langle A(t) \rangle \equiv \langle A \rangle (t)$]:

$$\langle A(t) \rangle = \sum_m \sum_n c_m^*(0)c_n(0) A_{mn} \exp \left[ -\frac{t}{\tau} \ln \left[ 1 + \frac{\tau^2}{\hbar^2} E_n E_m - \frac{i\tau}{\hbar} (E_m - E_n) \right] \right]$$

or, to the first order in $\tau$,

$$\langle A(t) \rangle = \sum_m \sum_n c_m^*(0)c_n(0) A_{mn} \exp \left[ \frac{i}{\hbar} (E_m - E_n) t \right] \exp \left[ -i \left( \frac{E_m^2 - E_n^2}{2\hbar^2} \right) \frac{\tau}{2} \right] ,$$

so that, beside the Bohr frequencies defining the emission and absorption frequencies of the oscillator, we obtain a damping term which causes the average value of the observable—which is explicitly independent of time—to tend to zero with time. A cursory analysis shows that even for very small eigenvalues, smaller than 1.0 eV, the damping factor is large, so that the decay of the average values is very fast. The damping factor of the norm in equation (78) can be evaluated, and its behaviour can be seen in figure ??.

### 4.2 Free particle

For a free particle, an electron for example, the general solution of the symmetric equation (??) can be obtained, in the coordinate representation, using as an ansatz the solution for the continuous case. Thus, a spectrum of eigenfunctions (plane waves) is obtained by

$$\Psi_p(x, t) = (2\pi\hbar)^{-3/2} \exp \left( -i\alpha(|p|) t + i\frac{p \cdot x}{\hbar} \right) .$$

Inserting this expression into the symmetric equation, we obtain for the frequency $\alpha(|p|)$ that
When $\tau \rightarrow 0$, $\alpha(|p|)\hbar$ coincides with the energy of the particle. As has been observed for the bound particle, here we also have an upper limit for the spectrum of eigenvalues. Thus the upper limit for the possible values of momentum is given by

$$p \leq p_{\text{Max}} = \sqrt{\frac{2m_{0}\hbar}{\tau}} = 10 \text{ MeV}/c$$

for the electron. In other words, there is a limit beyond which the frequencies cease to be real.

As in the continuous case, the state of the particle is described by a superposition of the eigenstates and can be written as

$$\Psi(x, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p c(p) \exp \left(-i\alpha(|p|)t + i\frac{(p \cdot x)}{\hbar}\right).$$

The coefficients $c(p)$ are determined from the initial condition $\Psi(x, 0) = \Psi_0(x)$. From the expression for $\alpha$, it can be observed that beyond a certain value of $p$ the expression loses meaning. When $p \geq \sqrt{\frac{2m_{0}}{\tau}}$, the complete solution will be defined only if $c(p)$. From the stationary phase condition we have that
The position of the particle is given by
\[ x = \frac{p}{m_0} \frac{t}{\sqrt{1 - (\tau^2 \left( \frac{p^4}{4m_0^2} \right))^2}}, \]
and, supposing that \( c(p) \) corresponds to a distribution of probabilities with a peak at \( p = p_0 \), then the wave packet will move in the direction \( p_0 \) with uniform velocity
\[ v = \frac{p_0}{m_0} \left[ 1 - \left( \frac{\tau^2 \left( \frac{p_0^4}{4m_0^2} \right)}{1 - \left( \frac{p_0^4}{4m_0^2} \right)^2} \right)^{-1/2} \right] \]
which coincides with the group velocity of the packet. It can be promptly observed that when \( p \) reaches its maximum value permitted, the velocity diverges: \( v \to \infty \). Thus, the introduction of a fundamental interval of time does not bring in any restriction to the velocity of the particle, although it results in a limit for the canonical momentum of the eigenfunctions. Starting from the condition of stationary phase it is possible to redefine the momentum associated with the particle, so that this new momentum does not suffer any restriction at all. So, one can conclude that the existence of free electrons with any energy is possible, differently from what happens for the bound electron.

For \( p > p_{\text{Max}} \) the frequency \( \alpha(|p|) \) fails to be real and its dependence on \( p \) is shown in figure ???. An analysis of equation (??) shows that if \( \alpha(|p|) \) is complex then, for \( p \leq p_{\text{Max}} \), the imaginary component is null and the real part is given by expression (??). When \( p \geq p_{\text{Max}} \), then
\[ \text{Re}(\alpha(p)) = \frac{\pi}{2\tau}, \]
\[ \text{Im}(\alpha(p)) = -\frac{1}{\tau} \ln \left[ \frac{\tau p^2}{2m_0\hbar} + \sqrt{\left( \frac{\tau p^2}{2m_0\hbar} \right)^2 - 1} \right], \]
with the real part being a constant and the imaginary one tending logarithmically to \(-\infty\). Using the expressions above we can observe that, for \( p > p_{\text{Max}} \), the eigenstates become unstable, with a time dependent decay term. When we look for an equivalent Hamiltonian \( \tilde{H} \) that, for the continuous Schrödinger equation, supplies equivalent outputs, we have that this is possible only if \( \tilde{H} \) is a non-hermitian operator. It is straightforward to see that this is the case for \( \tilde{H} = H_1 + iH_2 \), with \( H_1 \) and \( H_2 \) hermitian and such that \( H_1 |p\rangle = \hbar \text{Re}(\alpha(p)) |p\rangle \) and \( H_2 |p\rangle = \hbar \text{Im}(\alpha(p)) |p\rangle \).

For the retarded equation, using the same ansatz of the symmetric case, the damping factor appears for every value of \( p \). There is no limitation on the values of \( p \) but, when \( p \to \infty \), the real part of \( \alpha(|p|) \) tends to the same limit value observed for the symmetric case. Figure ?? illustrates the behavior of the components of \( \alpha(|p|) \). The general expression for an eigenfunction is found to be
\[ \Psi_p(x, t) \propto \exp \left[ \frac{ipx}{\hbar} - \frac{it}{\tau} \tan^{-1} \frac{p^2 \tau}{2m\hbar} \right] \exp \left[ -\frac{t}{2\tau} \ln \left[ 1 + \left( \frac{p^2 \tau}{2m\hbar} \right)^2 \right] \right]. \]

Performing a Taylor expansion and keeping only the terms to the first order in \( \tau \) we obtain the continuous solution multiplied by a damping factor:
Figure 3: Real and imaginary components of the $\alpha(|p|)$ obtained for the symmetric equation compared to continuous case.

Figure 4: Real and imaginary components of the $\alpha(|p|)$ obtained for the retarded equation compared to continuous case.
where $\omega = p^2 / 2mh$ is the frequency obtained for the continuous case.

It must be remarked that the damping term depends only on the Hamiltonian, through the frequency $\omega$, and on the chronon associated with the particle. As the latter is constant for a given particle, that term shows that for very high frequencies the solutions decay quite fast and, as the system evolves, the decay for smaller frequencies also comes true.

In figure ?? it can be observed that the inflection point, delimiting the region of the spectrum where the decay is faster, moves in the direction of smaller frequencies as time goes by. The consequence of this decay is the narrowing of the frequency bandwidth which is relevant for the wave packet describing the particle. This is an echo of the continuous decrease of the energy. As in the symmetric case, obtaining an equivalent Hamiltonian is possible only if non-hermitian operators are considered.

At this point, it is worthwhile to reconsider the question of the physical meaning of the three discretized Schrödinger equations. Apparently, the choice of the equation to be used in a particular situation is determined by the boundary conditions, by the restrictions they impose on the system. The symmetric equation is used for special situations for which the system neither emits nor absorbs radiation, or does it in a perfectly “balanced” way: This is the case for the
electrons in their ‘atomic orbits’. So, the particle is stable until a certain energy limit, beyond which the behaviour of the states is similar to that of the retarded solutions. For energies far below that limit, the particle behaves almost identically to the continuous case, only that the new frequencies associated with each wave function differ from the continuous frequencies by a factor of order $\tau^2$. The probability that a particle is found with energy larger than the limit value decreases exponentially with time. For the bound electron, the limit is that one equivalent to the rest mass of the muon. If a parallel with the classical approach is valid, the symmetric equation describes an isolated system, which does not exchange energy with the surrounding environment; or a situation of perfect thermodynamic equilibrium, in which a perfect balance between absorbed and dissipated energies is verified. For the classical theory of the electron the symmetric equation is only an approximation which ignores the radiation reaction effects. In QM, however, the existence of non-radiating states are related to the very essence of the theory. The symmetric equation shows that, below the critical limit, the states are physically identical to the outputs from the continuous theory: they are non-radiating states.

The retarded equation represents a system which somehow loses energy for the environment. The mechanism of such energy dissipation is related to the hamiltonian of the system but also to some property of the environment —even the vacuum— as it can be inferred from the description of the free particle. From the solutions obtained it is now observed that time has a well-defined direction of flux and that the frequency composition of the wave packet associated with the particle depends on the instant of time considered. It is clear that it is always possible to normalize the state in a certain instant and consider it as being an initial state. This is permitted by the formalism. However, in a strictly rigorous description, the frequency spectrum corresponds to a specific instant of time which took place after the emission. This is an aspect that can be interesting from the point of view of possible experimental verifications.

4.3 The discretized Klein–Gordon equation

Another interesting application is the description of a free scalar particle —a scalar or zero spin photon,— using a finite-difference form of the Klein–Gordon equation for massless particles.

In the symmetric form the equation is written as

$$\Box^2 A_\mu = 0 \quad \implies \quad \frac{\Psi(t + 2\tau) - 2\Psi(t) + \Psi(t - 2\tau)}{4\tau^2} - \nabla^2 \Psi(t) = 0. \quad (82)$$

Using a convenient ansatz we obtain, for this equation, in the coordinate representation, that

$$\Psi_k(x, t) = A \exp \left(-i\frac{t}{2\tau} \cos^{-1}(1 - 2c^2\tau^2k^2)\right) \exp(ikx),$$

which can be written as

$$\Psi_p(x, t) = A \exp \left(-i\frac{t}{2\tau} \cos^{-1}(1 - 2c^2\tau^2E^2/h^2)\right) \exp(ipx/h),$$

since $E = p^2c^2$ and $p = \hbar k$. Expanding the time exponential in powers of $\tau$, we find that, to the second order in $\tau$, a solution which is very similar to the continuous expression:
A difference of the order of $\tau^2$ is observed between the energy values of the photons in the continuous and discrete approaches. The general solution is given by a linear combination of the eigenfunctions found. \textit{A priori}, the value of the chronon for the particle is not known. The time dependent exponential term in the expressions above leads to an upper limit for the allowed energy, which is given by $E < \frac{\hbar}{\tau}$. We could suppose that the value of the chronon for this ’photon’ is of about the fundamental time interval of the electromagnetic interactions, around $10^{-9}$ s, resulting in a critical value of approximately 6.6 keV, which is a very low limit. A smaller chronon should increase this limit but, if there is any generality in the classical expression obtained for the electron, we should expect a larger value for this massless particle.

Whether, instead of a ’photon’ we consider a scalar ’neutrino’, taking for the value of the chronon $\tau \sim 10^{-13}$ s —typical time for the weak decay—, the limit for the energy associated with the eigenfunctions is now approximately 0.007 eV. This means that in the composition of the wave packet describing this particle the only contribution comes from eigenfunctions, the energy of which is below that limit.

The eigenfunctions obtained for the Hamiltonian considered are “plane waves” solutions. The dependence of these solutions on energy and time is displayed in figures ?? and ?? . For smaller values of $\tau$ the decay of the modes with energy above the maximum is faster.

Apparently, it seems to be possible to determine a limiting value for the chronon starting from the uncertainty relations. This could be obtained, when describing particles, using the expression

$$\tau < \frac{\hbar}{2m_e c^2}$$

that provides for the electron a maximum limit given by $6.4 \times 10^{-22}$ s. However, this value is two degrees of magnitude larger than the classical value of the chronon for the electron, which is a considerable difference. For a complex system it is also possible to make use of this relation, as we are going to mention ahead.

We also have to consider the conditions a ’photon’ must be supplied with, in order to be described by the symmetric equation. [For the electron, it seems clear that not irradiating in a bound state —which is imposed by QM— implies the adoption of the symmetrical equation]. For the photon, as for a free particle, when using the retarded form of the Klein–Gordon equation, a solution is also obtained wherein the highest frequencies decay faster than the lowest ones. There is always a tendency in the sense that the lowest frequencies prevail. If we are allowed to assign a physical meaning to such a discretized Klein–Gordon equation, we are also allowed
Figure 6: Solution of the discretized Klein–Gordon equation when the energy is smaller than
the critical limit, depicted for different values of energy and time.
Figure 7: Solution of the discretized Klein–Gordon equation when the energy is larger than the critical limit, depicted for different values of energy and time. In this case the amplitude decays quite fast.
to think that, the farther the light source, the more the spectrum of the emitted light will be shifted for the largest wavelengths, even if the source is at rest with respect to the observer. Thus, we could obtain a red shift effect as a consequence of the introduction of the chronon which could be used in the construction of a ‘tired-light’ theory.

Finally, we have to keep in mind that the discretization considered for the Klein–Gordon equation does not follow exactly the same procedure which led to the discretized Schrödinger equation, since it is a relativistic invariant equation. We did not change the proper time but the time coordinate itself into the discretized form. We considered a discretized version of the hamiltonian operator by applying the transformations:

\[ p \rightarrow \frac{\hbar}{i} \nabla, \]

\[ H \rightarrow i\hbar \Delta, \]

with \( \Delta \) as defined in Subsection ??, on the hamiltonian of a relativistic free particle,

\[ H = \sqrt{p^2 c^2 + m^2 c^4} \]

as usual in the continuous case.

### 4.4 Time evolution of the position and momentum Operators: The harmonic oscillator

It is possible to apply the discretized equations to determine the time evolution of the position and momentum operators, which is rather interesting for the description of the simple harmonic oscillator. In order to do that, we use the discretized form of the Heisenberg equations which, in the symmetric case, can be obtained by a direct discretization of the continuous equation. Starting from this equation we determine the coupled Heisenberg equations for the two operators:

\[ \frac{\hat{p}(t + \tau) - \hat{p}(t - \tau)}{2\tau} = -m\omega^2 \hat{x}(t), \]  
\[ \frac{\hat{x}(t + \tau) - \hat{x}(t - \tau)}{2\tau} = \frac{1}{m} \hat{p}(t). \]

Such coupled equations yield two finite-difference equations of second order, the general solutions of which are easily obtained. The most immediate way to determine the evolution of these operators is to use the creation and annihilation operators. Keeping the Heisenberg equation and remembering that for the harmonic oscillator we have \( \hat{H} = \omega \left( \hat{A}^\dagger \hat{A} + \frac{1}{2} \right) \), we get for the symmetric case:

\[ \frac{\hat{A}(t + \tau) - \hat{A}(t - \tau)}{2\tau} = -i\omega \hat{A}(t), \]  
\[ \frac{\hat{A}^\dagger(t + \tau) - \hat{A}^\dagger(t - \tau)}{2\tau} = i\omega \hat{A}^\dagger(t), \]
such that

\[ \dot{A}(t) = A(0) \exp \left( -i \frac{t}{\tau} \sin^{-1} (\omega \tau) \right), \]  

(87)

\[ \dot{A}^\dagger(t) = A^\dagger(0) \exp \left( i \frac{t}{\tau} \sin^{-1} (\omega \tau) \right), \]  

(88)

where we used the fact that, for \( t = 0 \), the Heisenberg and Schrödinger pictures are equivalent: \( \dot{A}(t = 0) = \dot{A} = \dot{A}(0) \) and \( \dot{A}^\dagger(t = 0) = \dot{A}^\dagger = \dot{A}^\dagger(0) \), with \( \dot{A} \) and \( \dot{A}^\dagger \) independent of time. In order to obtain these equations we considered that, for the non-relativistic case, there is neither creation nor annihilation of particles, such that we can impose restrictions on the frequencies in the phase term of the operators. For the creation operators, e.g., the terms with negative frequencies —associated with antiparticles— are discarded.

We can observe that the Number, as well as the hamiltonian operator are not altered:

\[ \dot{N} = \dot{A}^\dagger(t) \dot{A}(t) = \dot{A}^\dagger(0) \dot{A}(0), \]

\[ \dot{H} = \hbar \omega \left( \dot{N} + \frac{1}{2} \right) = \hbar \omega \left( \dot{A}^\dagger(0) \dot{A}(0) + \frac{1}{2} \right). \]

Thus, starting from these operators, we obtain for the symmetric case:

\[ \dot{x}(t) = \dot{x}(0) \cos \left[ \frac{t}{\tau} \sin^{-1} (\omega \tau) \right] + \frac{\dot{p}(0)}{m \omega} \sin \left[ \frac{t}{\tau} \sin^{-1} (\omega \tau) \right], \]

\[ \dot{p}(t) = \dot{p}(0) \cos \left[ \frac{t}{\tau} \sin^{-1} (\omega \tau) \right] - m \omega \dot{x}(0) \sin \left[ \frac{t}{\tau} \sin^{-1} (\omega \tau) \right], \]

which differ from the continuous case since the frequency \( \omega \) here is replaced by a new frequency \( \frac{t}{\tau} \sin^{-1} (\omega \tau) \) that, for \( \tau \to 0 \), tends to the continuous one. Also, there is now an upper limit for the possible oscillation frequencies given by \( \omega \leq 1/\tau \). Above this frequency the motion becomes unstable, as it can be observed in figure ??.

The existence of a maximum limit for the frequency is equivalent to an upper limit for the energy eigenvalues given by \( E_n = (n + \frac{1}{2}) \hbar \omega \leq \hbar / \tau \), which is equal to the upper limit obtained using Schrödinger’s picture. Since

\[ \frac{t}{\tau} \sin^{-1} (\omega \tau) \approx \omega + \frac{1}{3!} \omega^3 \tau^2 + O(\tau^4), \]

the difference expected in the behaviour of the oscillator with respect to the continuous solution is quite small. If we take, for example, the vibration frequency of the hydrogen molecule (\( H_2 \)), we have that \( \omega \sim 10^{14} \) Hz, while the term of the second order in \( \tau \) is smaller than \( 10^{-3} \) Hz (if the analogy with the classical theory is valid, the chronon is expected to be smaller for more massive systems). In terms of average values we have that, for the position operator

\[ \langle \dot{x}(t) \rangle = \langle \dot{x}(t) \rangle_{\text{cont}} + \frac{\omega^2 \tau^2}{3! m} t \langle \dot{p}(t) \rangle, \]
Figure 8: Phase space of the harmonic oscillator when $\omega > \frac{1}{\tau}$ and in the discrete case, with time intervals multiples of $\tau$: in case (a) time is regarded as intrinsically discrete, so that in the picture only the points where the lines touch one another are meaningful, while in case (b) time is regarded as intrinsically continuous. In the continuous case no modification is expected with respect to the ordinary case, (c), under the present hypotheses.
in which the term of order $\tau^2$ is expected to be considerably smaller than the mean value for the continuous case. At this point, naturally, the mean values are determined taking for the system a state made up of a superposition of stationary states. For the stationary states $|u_n\rangle$ themselves the mean values of $\hat{x}$ and $\hat{p}$ are zero.

For the retarded case the solutions can be obtained using the time evolution operators for the Heisenberg equation (Appendix ??). As expected, decaying terms come out. The creation and annihilation operators obtained for this case are then given by

$$\hat{A}(t) = \hat{A}(0) \left[ 1 + i \omega \tau + \tau^2 \omega^2 \xi \right]^{-\frac{1}{2}} \approx \hat{A}(0) \exp \left[ -\left( \xi + \frac{1}{2} \right) \omega^2 \tau t \right],$$

$$\hat{A}^\dagger(t) = \hat{A}^\dagger(0) \left[ 1 - i \omega \tau + \tau^2 \omega^2 \xi \right]^{-\frac{1}{2}} \approx \hat{A}^\dagger(0) \exp \left[ -\left( \xi + \frac{1}{2} \right) \omega^2 \tau t \right],$$

with $\xi$ being a real positive factor. The relation $(\hat{A}^\dagger)^\dagger = \hat{A}$ continues to be valid but the Number operator and, consequently, the Hamiltonian, is now not constant anymore:

$$\hat{H}(t) = \hat{A}^\dagger(0) \hat{A}(0) \left[ \left( 1 + \omega^2 \tau^2 \xi \right)^2 + \omega^2 \tau^2 \right]^{-\frac{1}{2}} \approx \hat{H}(0) \exp \left[ -2 \left( \xi + \frac{1}{2} \right) \omega^2 \tau t \right],$$

$$\hat{H}(t) = \hbar \omega \left\{ \hat{A}^\dagger(0) \hat{A}(0) \left[ \left( 1 + \omega^2 \tau^2 \xi \right)^2 + \omega^2 \tau^2 \right]^{-\frac{1}{2}} + \frac{1}{2} \right\}.$$

Taking into account the terms to the second order in $\tau$, we get that the oscillation frequencies also decay with time. These results are consistent with the fact that the system is emitting radiation, with the consequent reduction of its total energy. However, it is remarkable that the energy of the quanta associated with the creation and annihilation operators is not constant, even with a very tiny variation rate. In the same way, when we calculate the position and momentum operators a damping factor is obtained. Figure ??-a shows the strange damping factor associated with the Number operator. It can be observed that this damping occurs within a period of time which is characteristic for each frequency, being slower and postponed for lower frequencies. Figure ??-b shows the dampening of the oscillations as described by the retarded equation. Once the expressions for the position and momentum operators are determined, we obtain that, to first order in $\tau$,

$$\langle \hat{x}(t) \rangle = \left\{ \hat{x}(0) \cos (\omega \tau) + \frac{\hat{p}(0)}{m \omega} \sin (\omega \tau) \right\} \exp \left[ -\left( \xi + \frac{1}{2} \right) \omega^2 \tau t \right],$$

$$\langle \hat{p}(t) \rangle = \langle \hat{x}(t) \rangle_{\text{cont}} \exp \left[ -\left( \xi + \frac{1}{2} \right) \omega^2 \tau t \right].$$

Taking into account the higher order terms, we can observe a small variation in the oscillation frequency just as observed in the symmetrical case. The introduction of time independent perturbations does not cause any additional variations apart from those found even in the continuous case. It must be pointed out that the results obtained with the procedure above are in agreement with those obtained following Schrödinger’s picture.
4.5 Hydrogen atom

The hydrogen atom is basically a system made up of two particles attracting each other through coulombian force, which is therefore inversely proportional to the square of the distance between them. The basic Hamiltonian is given by

\[ \hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{R}, \]  

(89)

and is composed of the kinetic energy of the atom in the centre-of-mass frame, and of the coulombian electrostatic potential (\(\mu\) is the reduced mass of the system electron-proton). A more complete description is obtained by adding correction terms (fine structure) to the Hamiltonian, including relativistic effects such as the variation of the electron mass with velocity and the coupling of the intrinsic magnetic moment of the electron with the magnetic field due to its orbit (spin-orbit coupling). Besides, there are also the hyperfine (hf) corrections which appear due to the interaction of the electron with the intrinsic magnetic moment of the proton and, finally, the Lamb shift, due to the interaction of the electron with the fluctuations of the quantized electromagnetic field. The hamiltonian can finally be written as[?]

\[ \hat{H}_1 = m_e c^2 + \hat{H}_0 - \frac{P^4}{m_e^2 c^2 R^3} \mathbf{L} \cdot \mathbf{S} + \hat{H}_{\text{hf}} + \hat{H}_{\text{Lamb}}. \]  

(90)

The introduction of the magnetic moment of the nucleus through the hyperfine correction
causes the total angular momentum to be $\mathbf{F} = \mathbf{J} + \mathbf{I}$. The Hamiltonian does not depend explicitly on time such that, for the symmetric Schrödinger equation

$$i\frac{\hbar}{2\tau} [\Psi(x, t + \tau) - \Psi(x, t - \tau)] = \dot{H}_1 \Psi(x, t),$$  

we obtain, using the separation of variables, the following uncoupled equations:

$$\dot{H}_1 \Phi(x) = E\Phi(x)$$

$$i\frac{\hbar}{2\tau} [T(t + \tau) - T(t - \tau)] = \dot{H}_1 T(t),$$

with the general solution

$$\Psi(x, t) = \Phi(x) \exp \left[ -i \frac{t}{\tau} \sin^{-1} \left( \frac{\tau E}{\hbar} \right) \right].$$

The difference related to the continuous case appears only in those aspects involving the time evolution of the states. Since the Hamiltonian is time independent, its eigenvalues are exactly the same as those obtained in the continuous case.

$$E_{(n, j)} \approx m_0 c^2 - \frac{N_0 e^2}{2}\alpha^2 - \frac{m_0 c^2}{2n^4} \left( \frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \alpha^4 + E_{\text{hf}} + E_{\text{Lamb}}.$$  

A situation where a difference between the two cases can appear is when taking into account the probabilities of transition between the eigenstates for an atom submitted to a time dependent potential. In the discrete approach it is possible to use the method of the equivalent Hamiltonian in order to obtain the transition probabilities. As mentioned previously (Subsection ??), the problem is treated using the conventional approximate methods for time dependent perturbations.

If we consider, for example, the non-relativistic interaction of an atom with an electromagnetic field described by the vector potential $\mathbf{A}(x, t)$, we have for the low intensity limit, in the Coulomb gauge, the Hamiltonian

$$\dot{H}(t) = \dot{H}_1 - \dot{V}(t) = \dot{H}_1 - \frac{e}{m_e c} \dot{\mathbf{A}}(\mathbf{R}, t) \cdot \mathbf{P},$$

where the potential term is taken as being the perturbation. If we consider that the potential describes a monochromatic field of a plane wave, then

$$\mathbf{A}(x, t) = A_0 \hat{e} \left[ \exp \left( i\omega \frac{\hat{n} \cdot \mathbf{x}}{c} - i\omega t \right) + \exp \left( -i\omega \frac{\hat{n} \cdot \mathbf{x}}{c} + i\omega t \right) \right]$$

where $\hat{e}$ is the linear polarization of the field and $\hat{n}$ is the propagation direction. The term depending on $(-i\omega t)$ corresponds to the absorption of a quantum of radiation $\hbar \omega$ and the $(i\omega t)$ term to stimulated emission. Let us assume that the system is initially in an eigenstate $|\Phi_1\rangle$ of the time independent Hamiltonian. Keeping only the perturbations to the first order in $\dot{V}(t)$, we obtain that

44
\[ c_n(t) = \frac{1}{\hbar} \int_0^t \exp(i\omega_{n\ell} t')V_n(t')dt' \]

where \( \omega_{n\ell} \) in the discrete case is given by

\[ \omega_{n\ell} = \frac{1}{\tau} \left[ \sin^{-1} \left( \frac{\tau E_n}{\hbar} \right) - \sin^{-1} \left( \frac{\tau E_i}{\hbar} \right) \right] \]

Working with the absorption term, we get by contrast that

Thus, the probability of transition from the initial state \(|\psi_i\rangle\) to the final state \(|\psi_f\rangle\) is given by

\[ P_{fi}(t) = \left| \langle \psi_f | e^{i\omega_{n\ell} \cdot \mathbf{p}} | \psi_i \rangle \right|^2 \exp \left[ \frac{i(\omega_{n\ell} - \omega)t}{\tau} \right] \]

or

\[ P_{fi}(t) = \frac{4e^2 |A_0|^2}{m^2 c^2 \hbar^2} \left| \langle \psi_f | e^{i\omega_{n\ell} \cdot \mathbf{p}} | \psi_i \rangle \right|^2 \frac{\sin^2 \left[ (\omega_{fi} - \omega)t/2 \right]}{(\omega_{fi} - \omega)^2} \]

so that the determination of the matrix elements of the spacial term, using the electric dipole approximation, provides the selection rules for the transitions. What is remarkable in this expression is the presence of a resonance showing a larger probability for the transition when

\[ \omega = \omega_{fi} = \frac{1}{\tau} \left[ \sin^{-1} \left( \frac{\tau E_i}{\hbar} \right) - \sin^{-1} \left( \frac{\tau E_f}{\hbar} \right) \right] \]  

(95)

This expression is formally different from the one obtained for the continuous approach. When we expand this expression in powers of \( \tau \), we get that

\[ \omega \approx \frac{E_f - E_i}{\hbar} + \frac{1}{6} \frac{E_f^3 - E_i^3}{\hbar^3} \tau^2 \]  

(96)

The first term supplies the Bohr frequencies as in the continuous case; the second, the deviation in the frequencies caused by the introduction of the time discretization:

\[ \Delta \omega_{fi} = \frac{1}{6} \frac{E_f^3 - E_i^3}{\hbar^3} \tau^2. \]

If we consider the chronon of the classical electron, \( \tau \approx 6.26 \times 10^{-24} \) s, it is possible to estimate the deviation in the frequency due to the time discretization. Then, for the hydrogen atom,

\[ \Delta \omega_R \approx 2.289 \times 10^{-2} \left( E_f^3 - E_i^3 \right). \]

If we take into account, for example, the transitions corresponding to the first lines of the series of Lyman and Balmer, i.e., of the non-disturbed states \( n = n_i \rightarrow n = n_f \), we have
\[ \Delta E \text{ is the difference of energy between the states, } \nu \text{ is the frequency of the photon emitted in the transition and } \Delta \nu_D \text{ is the frequency deviation due to the discretization. Such deviation is always very tiny. We must remember that the hyperfine corrections and those due to the Lamb shift are of order of a Gigahertz. For the transition } n = 1 \rightarrow n = 2, \text{ e.g., the correction due to the Lamb shift is approximately } 1.06 \text{ GHz.} \]

Larger deviations caused by the discretization occurs for mono-electronic atoms with larger atomic numbers. For the first transition the deviation is of approximately 90 Hz for the \(^2\text{He}, 1.1\text{ kHz for the } ^3\text{Li, and 420 kHz for the } ^6\text{C. However, these deviations are still quite smaller than that one due to the Lamb shift. That is also the case for the muonic atoms. For a muonic atom with a proton as nucleus, using for the chronon a value derived from the classical expression for the electron } (\tau_\mu = 3.03 \times 10^{-26} \text{ s}) \text{ the deviation is of about } 1.4 \text{ kHz for the transition } n = 1 \rightarrow n = 2. \text{ For that transition the frequency of the emitted radiation is approximately } 4.58 \times 10^{17} \text{ Hz.} \]

For the \textit{retarded} equation, a difference with respect to the symmetrical case is present in the time evolution of the states. The procedure is identical to the one used above and the general solution is now given by

\[ \Psi(x, t) = \Phi(x) \left[ 1 + i \frac{\tau E_i}{\hbar} \right]^{-t/\tau}, \]

so that the transitions now occur with frequencies given by

\[ \omega = \omega_{fi} = -\frac{i}{\tau} \left[ \ln \left( 1 + \frac{i \tau E_i}{\hbar} \right) - \ln \left( 1 + \frac{i \tau E_i}{\hbar} \right) \right]. \quad (97) \]

As results from the characteristics of the retarded equation, this is a complex frequency. The real component of such frequency can be approximated by

\[ \text{Re}(\omega_{fi}) \approx \frac{E_f - E_i}{\hbar} + \frac{1}{3} \frac{E_i^3}{\hbar^3} - \frac{E_i^3}{\hbar^3} \tau^2, \]

where the first term is the expression for the continuous case. For the particular transition \( n = 1 \rightarrow n = 2 \) we have that the deviation due to the discretization is of about 18 Hz.

The imaginary component, on the other hand, can be approximated by

\[ \text{Im}(\omega_{fi}) \approx \frac{i}{2} \frac{E_i^2}{\hbar^2} - \frac{E_i^2}{\hbar^2} \tau. \]

In the expression for the probability of transition we have the module of an integral involving the time dependency of the general solution. In this case, the characteristic damping causes the probability to tend to a fixed, non zero value. An example of such behaviour is shown in figure ??, which shows the variation of the time dependent term between an initial instant \( t_0 = 0 \) and

| \( n_i \) | \( n_f \) | \( \Delta E \) (eV) | \( \nu \) (Hz) | \( \Delta \nu_D \) (Hz) |
|---|---|---|---|---|
| 1 | 2 | 10.2 | \( 2.465 \times 10^{15} \) | \( \sim 10 \) |
| 1 | 3 | 12.1 | \( 2.922 \times 10^{14} \) | \( \sim 10 \) |
| 1 | 4 | 12.75 | \( 3.082 \times 10^{14} \) | \( \sim 10 \) |
| 2 | 3 | 1.89 | \( 4.566 \times 10^{14} \) | \(< 1 \) |
Figure 10: Behaviour of the time dependent component of the probability of transition

some hundred chronons later. In order to observe the decay of the amplitude factor we have used a larger value for the chronon, of about $10^{-18}$ s. When the chronon is of the order of the one we have been considering for the electron the decay is slower.

As it can be observed, the effect of the time discretization on the emission spectrum of the hydrogen is extremely small. Using the expressions obtained above we can estimate that, in order that the effect of the time discretization is of the same order of the Lamb shift, the chronon associated with the electron should be of about $10^{-18}$ s, far above the classical value (but close to the typical interval of the electromagnetic interactions). In any case, it should be remembered that the Lamb shift measurements do not seem to be in full agreement[?] with quantum electrodynamics.

Concluding this Subsection, it is worthwhile to remark that, for a time independent Hamiltonian, the outputs obtained in the discrete formalism using the symmetric equation are very similar to those from the continuous case. For such Hamiltonians, the effect of the discretization appears basically in the frequencies associated with the time dependent term of the wave function. As already observed, the difference in the time dependency is of the kind
The discretization causes a change in the phase of the eigenstate, which can be quite large. The eigenfunctions individually describe stationary states, so that the time evolution appears when we have a linear combination of such functions, this way describing the state of the system. This state evolves according to

\[ |\Psi(t)\rangle = \sum_n c_n(0) \exp\left[-i \sin^{-1}\left(\frac{\tau E_n}{\hbar}\right)\frac{(t - t_0)}{\tau}\right]|\phi_n\rangle, \]

considering that \( H|\phi_n\rangle = E_n|\phi_n\rangle \) is the eigenvalue equation associated with the Hamiltonian.

When the stationary states of a particle under, e.g., one-dimensional squared potentials are studied, the same reflection and transmission coefficients and the same tunnel effect are obtained, since they are calculated starting from the stationary states. When we consider a linear superposition of these stationary states, building a wave packet, the time dependent terms have to be taken into account, resulting in some differences with respect to the continuous case. Some attempts have been carried out in order to find out significant measurable differences between the two formalisms\[?, \, ?\] but no encouraging case has been found yet.

5 Density Operators and the “Coarse Graining” Hypothesis

5.1 The “coarse graining” hypothesis

First of all, it is convenient to present a brief review of some topics related to the introduction of the coarse grained description of a physical system. This hypothesis is then going to be used to obtain a discretized form of the Liouville equation, which represents the evolution law of the density operators in the usual QM.

An important point to be remarked is that the introduction of a fundamental interval of time is perfectly compatible with a coarse grained description. The basic premise of such description, in statistical physics, is the impossibility of a precise determination of the position and momentum of each particle forming the system, in a certain instant of time. Let us consider, for the sake of simplicity, a system composed by \( N \) similar pointlike particles, each of them with three degrees of freedom described by the coordinates \((q_1, q_2, q_3)\). We can associate with this ensemble of particles an individual phase space (named \( \mu \)-space) defined by the six coordinates \((q_1, q_2, q_3; p_1, p_2, p_3)\) so that the system as a whole is represented by a crowd of points in this space.

Since the macroscopic observation is unable to precisely determine the six coordinates for each particle, let us assume that it is possible only to know if a given particle has its coordinates inside the intervals \((q_i + \delta q_i)\) and \((p_i + \delta p_i)\), with \( i = 1, 2, 3 \). In order to describe the state of the system in the \( \mu \)-space, we divide it into cells corresponding to the macroscopic uncertainties \( \delta q_i \) and \( \delta p_i \), each one occupying in the \( \mu \)-space a volume...
These cells must be sufficiently small related to the macroscopically measurable dimensions but also sufficiently large to contain a great number of particles.

When considering the system as a whole, its macroscopic state is given by a collection of points \( n_i \) corresponding to the number of particles inside each cell. Now, if we take into account the 6N-dimensional phase space \( \Gamma \), in which each of the states assumed by the system is represented by a point, to each configuration \( n_i \) corresponds in \( \Gamma \) a cell with volume given by

\[
(\delta V)_\Gamma = \prod_{n=1}^{N} (w_i)^{n_i}.
\]

Considering that the permutation of the particles inside the cells of the \( \Gamma \) space does not change the macroscopic state of the system, then to each collection of numbers \( n_i \) corresponds a volume \( \Omega_n \) in the \( \Gamma \) space \( ^4 \) given by

\[
W(\Omega_n) = \frac{N!}{\prod_i n_i!} \prod_i (w_i)^{n_i} \left( \sum_i n_i = N \right).
\]

The state of the system is determined by the star occupied by the representative point of the system in the \( \Gamma \) space. This way, macroscopically, it is only possible to distinguish in which star the system is, such that any point in this star corresponds to a same macroscopic state. When we consider a system which is not in equilibrium, a change in its macroscopic state can only be observed when the point describing the system changes star. The crossing time is small but finite. During this period of time the macroscopic state of the system does not change notwithstanding its microscopic state is continuously changing.

Thus, from the point of view of statistical physics, the introduction of a fundamental interval of time appears in a very natural way. That is still more significant when we remember that the predictions of QM are always obtained as mean values of observables. The uncertainty relations, according to the usual interpretation of QM —the Copenhagen interpretation—, are independent of the arguments above. If we accept that they play a fundamental role in the microscopic world —and this is postulated by Copenhagen—, then the concept of chronon, as a fundamental interval of time, must be related to them.

5.2 Discretized Liouville equation and the time–energy uncertainty relation

An attempt to set up a relationship between the chronon and the time–energy uncertainty relation has been put forward by Bonifacio (1983)\cite{Bonifacio1983}, extended the coarse graining hypothesis to the time coordinate. In the conventional QM the density operator evolves according to the Liouville–von Neumann equation

\[
w_i = \delta q_1 \delta q_2 \delta q_3 \delta p_1 \delta p_2 \delta p_3.
\]
\[ \frac{\partial \hat{\rho}}{\partial t} = -i \mathcal{L} \hat{\rho}(t) = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right], \]  

(99)

where \( \mathcal{L} \) is the Liouville operator. One can immediately observe that, if \( H \) is time independent, the solution is given by

\[ \hat{\rho}(T) = \exp \left( -\frac{i}{\hbar} H \right) \hat{\rho}(0) \exp \left( \frac{i}{\hbar} H \right), \]  

(100)

which gives the time evolution of the density operator starting from an initial time \( t_0 \), such that \( T = t - t_0 \) is the evolution time.

When we build a coarse grained description of the time evolution, by introducing a graining of value \( \tau \) such that the evolution time is now given by \( T = k\tau \) \((k = 1, 2, \ldots, \infty)\), we have that the resulting density operator \( \rho \) does not satisfy the continuous equation (99) but a discretized form of it given by

\[ \frac{\hat{\rho}(t) - \hat{\rho}(t - \tau)}{\tau} = -i \mathcal{L} \hat{\rho}(t), \]  

(101)

with \( t = k\tau \), which reduces to the Liouville–von Neumann equation when \( \tau \to 0 \). In the energy representation \( |n\rangle \), once satisfied certain conditions which ensure that \( \rho(k) \) is a density operator, we have that (101) rules for \( \rho \) an evolution which preserves trace, obeys the semigroup law and is an irreversible evolution towards a stationary diagonal form. In other words, we observe a reduction of state in the same sense as in the measurement problem of QM. This reduction is not instantaneous and depends on the characteristic value \( \tau \):

\[ \rho(t) \xrightarrow{t \to 0} \sum_n \rho_{nn}(0) |n\rangle \langle n| . \]

It is important to observe that the non diagonal terms tend exponentially to zero according to a factor which, to the first order, is given by

\[ \exp \left| -\frac{\omega_{nm}^2 \tau}{2} \right| . \]

Thus, the reduction to the diagonal form occurs provided we have a finite value for \( \tau \), no matter how small, and provided we do not have \( \omega_{nm} \tau \ll 1 \) for every \( n,m \), where \( \omega_{nm} = (E_n - E_m)/\hbar \) are the transition frequencies between the different energy eigenstates. This latter condition is always satisfied for systems not bounded.

These results, together with an analysis of the discrete Heisenberg equation defined in terms of the average values of observables

\[ \hat{A}(t) = \text{Tr} \left( \hat{\rho}(t) \hat{A} \right), \]

50
in the coarse grained description, suggest an interpretation of \( \tau \) in terms of the uncertainty relation \( \Delta E \Delta t \geq \hbar/2 \) such that \( \tau \) is a characteristic interval of time satisfying the inequality

\[
\tau \geq \tau_E = \frac{\hbar}{2\Delta E}, \quad \text{with} \quad \Delta E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2},
\]

so that the mathematical meaning of the time-energy uncertainty relation is that of fixing a lower limit for the time interval within which the time evolution can be described. Thus, “... the coarse grained irreversibility would become a necessary consequence of an intrinsic impossibility to give an instantaneous description of time evolution due to the time–energy uncertainty relation”.

Since the density operator, in the energy representation, tends to a diagonal form, it seems to be tempting to apply it to the measurement problem. We can also observe that, even without assuming any coarse graining of time, namely, without using the statistical approach adopted by Bonifacio, the reduction to a diagonal form results straightforwardly from the discrete Liouville equation and some asymptotic conditions regarding the behaviour of the solution, once satisfied\[?\] the inequality \( \omega_{nm} \tau \ll 1 \). See also \[?\].

The crucial point, from which derives both the decay of the non-diagonal terms of the density operator and the very discrete Liouville equation, is that the time evolution operator obtained from the coarse grained description is not a unitary operator. This way, the operator

\[
\dot{\hat{V}}(t = k\tau, t = 0) = \frac{1}{\left(1 + \frac{it\hat{E}}{\hbar}\right)^k},
\]

as all the non-unitary operators, does not preserve the probabilities associated with each of the energy eigenstates that make up the expansion of the initial state in that basis of eigenstates. We must recall that the appearance of non-unitary time evolution operators is not associated with the coarse grained approach only, since they also result from the discrete Schrödinger equations.

### 5.3 The measurement problem in quantum mechanics

Let us apply the discrete formalism introduced in the previous Subsection to the measurement problem. Using a quite general formalization, we can describe the measurement process taking advantage of the properties observed for the evolution of the density operator as determined by the discrete Liouville–von Neumann equation.\[\dagger\]

When speaking of measurement, we have to keep in mind that, in the process, an object \( \mathcal{O} \), of which we want to measure a dynamic variable \( R \), and an apparatus \( \mathcal{A} \), which is used to perform such measurement, are involved. Let us suppose that \( \hat{R} \) is the operator associated with the observable \( R \), with an eigenvalue equation given by \( \hat{R}|r\rangle = r|r\rangle \) and defines a complete basis of eigenstates. Thus, considered by itself, any possible state of the object can be expanded in this basis:

---

\[^{\dagger}\]We follow closely the description exhibited in Ballantine.\[?\]
(104) \[ |\Psi\rangle_0 = \sum_r c_r |r\rangle_0. \]

As regards the apparatus \( A \), we are interested only in its observable \( A \), whose eigenvalues \( \alpha \) represent the possible values indicated by a pointer. Besides, let its various internal quantum numbers be labelled by an index \( n \). These internal quantum numbers are useful to specify a complete basis of eigenvectors associated with the apparatus:

\[ \hat{A} |\alpha, n\rangle_A = \alpha |\alpha, n\rangle_A. \] (105)

Now, let us suppose that the apparatus is prepared in an initial state given by \( |0, n\rangle_A \), i.e., in the initial state the value displayed is zero. The interaction between the two systems is introduced by means of the time evolution operator and is such that there is a correlation between the value of \( r \) and the measure \( \alpha_r \). In order to deal with the measurement process itself we consider a quite general situation. First of all, let us consider the following pure state of the system object + apparatus \((O + A)\):

\[ |\psi_n\rangle = |\psi_0, n\rangle_A. \] (106)

The evolution of this state, in the continuous description, using the evolution operator, is given by

\[ \hat{U}(t, t_0) |\psi_0, n\rangle_A = \sum_r c_r |\alpha_r, r, n\rangle = |\psi_n\rangle \] (107)

which is a coherent superposition of macroscopically distinct eigenstates, each one corresponding to a different measure \( \alpha_r \). The great problem for the Copenhagen interpretation results from the fact that it considers the state \( |\psi_n\rangle \) as associated with a single system: a pure state provides a complete and exhaustive description of an individual system. Thus, the coherent superposition above describes a single system so that, at the end of the interaction which settles the measurement, the display should not show a well-defined output since (107) describes a system which is a superposition of all its possible states.

However, we know from experience that the apparatus always displays a single value as the output of the measurement. It is this disagreement between observation and the description provided by the formalism, when interpreted according to Copenhagen, which results in the necessity of introducing the postulate of the reduction of the vector state

\[ |\psi_n\rangle \rightarrow |\alpha_{r_0}, r_0, n\rangle \]

where \( r_0 \) is the value displayed by the apparatus.

This fact has been considered by many as being a problem for the usual interpretation of quantum mechanics.\([?, \ ?]\) The attempts to find a solution, in the context of different interpretations, have been numerous, from the Many-Worlds interpretation, proposed by Everett and
Wheeler\cite{Wheeler} to the measurement theory by Daneri, Loinger and Prosperi\cite{Daneri}, in which the reduction of the quantum state is described as a process triggered by the appearance of aleatory phases in the state of the apparatus, just because of its interaction with the elementary object. The approach introduced here is — by contrast — rather simpler.

As an initial state in the measurement process, let us consider a mixed state for the composite system \( \mathcal{O} + \mathcal{A} \),

\[
\rho^i = \sum_n C_n |\Psi^i_n\rangle \langle \Psi^i_n|,
\]

where \( C_n \) is the probability associated with each of the states \( |\Psi^i_n\rangle \). Such probability is, as in the classical physics, an ‘ignorance’ probability, i.e., it is not intrinsic to the system. In the continuous case, when we apply the time evolution operator to that density operator we get a final state given by

\[
\rho^f = \dot{U} \rho \dot{U}^\dagger \sum_n C_n |\Psi^f_n\rangle \langle \Psi^f_n| = \sum_{r_1, r_2} c_{r_1}^* c_{r_2} \sum_n C_n \{|\alpha_{r_1}; r_1, n\rangle \langle \alpha_{r_1}; r_1, n|\},
\]

so that the presence of non-diagonal terms corresponds to a coherent superposition of states. In this case, the postulate of the reduction of the quantum state is connected with the non-diagonal terms of the density operator. It is postulated that when a measurement is carried out on the system, the non-diagonal terms tend instantaneously to zero. Since in the continuous case the time evolution of the state results from the application of a unitary operator, which preserves the pure state condition \( \hat{\rho}^2 = \hat{\rho} \), it is impossible to obtain the collapse of the pure state from the action of such operator. In the diagonal form the density operator describes an incoherent mixture of the eigenstates of \( \hat{A} \), and the indetermination regarding the output of the measurement is a sole consequence of our ignorance about the initial state of the system.

In the discrete case, which has the time evolution operator given by \( \hat{U}(t = k\tau, t = 0) \), with the interaction between apparatus and object embedded in the Hamiltonian \( \hat{H} \), the situation is quite different. The main cause of such difference lays in the fact that the time evolution operator is not unitary. Let us consider the energy representation, describing the eigenvalue equation of the Hamiltonian as \( \hat{H}|n\rangle = E_n|j\rangle \) so that the eigenstates \( |n\rangle \) are the states with defined energy. From the formalism of the density matrices we know that when the operator \( \hat{H} \) is diagonal in the energy representation then, when calculating the expected value of the observable, we do not obtain the interference terms describing the quantum beats typical of a coherent superposition of the states \( |n\rangle \).

As the time evolution operator is a function of the Hamiltonian and, therefore, commutes with it, the basis of the energy eigenstates is also a basis for this operator. We can now use a procedure identical to the one applied by Bonifacio, and consider the evolution of the system in this representation. Thus, we have that the operator \( \hat{V}(t = k\tau, t = 0) \) takes the initial density operator \( \rho^i \) to a final state for which the non-diagonal terms decay exponentially with time:
\[ \rho_{rs}(t) = \rho_{rs}(0)e^{-\gamma_{rs}t}e^{-i\omega_{rs}t}, \] (113)

such that,

\[ \gamma_{rs} = \frac{1}{2\tau} \ln \left( 1 + \omega_{rs}^2 \tau^2 \right), \] (114)

\[ \nu_{rs} = \frac{1}{\tau} \tan^{-1}(\omega_{rs} \tau). \] (115)

We can observe right away that the non-diagonal terms tend to zero with time and the decay is faster the larger the value of \( \tau \), which here is an interval of time related to the whole system \( \mathcal{O} + \mathcal{A} \). If we keep in mind that in the coarse grained description the value of the time interval \( \tau \) originates from the impossibility to distinguish between two different states of the system, we must remember that the system \( \mathcal{O} + \mathcal{A} \) is not an absolutely quantum system. That means that \( \tau \) could be significantly larger, implying an extremely faster damping of the non-diagonal terms of the density operator (figure ??). We then arrived at a process like the one of the reduction of the quantum state, even if in a very elementary formalization. This result seems to be very encouraging regarding future researches on such important and controversial subject.

Some points must be pointed out from this brief approach of the measurement problem. First of all, we must emphasize that this result does not occur when we use the time evolution operators obtained directly from the retarded Schrödinger equation. The dissipative character of that equation causes the norm of the state vector to decay with time, leading also to a non-unitary evolution operator. However, this operator is such that, in the definition of the density operator we get damping terms which are effective even for the diagonal terms. This point, as well as the question of the compatibility between Schrödinger’s picture and the formalism of the density matrix are going to be analysed in Appendix ??.

As the composite system \( \mathcal{O} + \mathcal{A} \) is a complex system, it is suitably described by the coarse grained description, so that the understanding of the relationship between the two pictures is necessary in order to have a deeper insight on the processes involved.

Notwithstanding the simplicity of the approach we could also observe the intrinsic relation between measurement process and irreversibility. The time evolution operator \( \hat{V} \) meets the properties of a semigroup, so that it does not necessarily possess an inverse: and non-invertible operators are related to irreversible processes. In a measurement process, in which the object...
Figure 11: Damping of the non-diagonal terms of the density operator for two different values of $\tau$. For both cases we have used $\Delta E = 4 \text{ eV}$. (a) Slower damping for $\tau = 6.26 \times 10^{-24} \text{ s}$; (b) $\tau = 1 \times 10^{-19} \text{ s}$.

is lost just after the detection, we have an irreversible process that could very well be described by an operator such as $\dot{V}$.

Finally, it is worth mentioning that the measurement problem is controversial even regarding its mathematical approach. In the simplified formalization introduced above, we did not include any consideration beyond those common to the quantum formalism, allowing an as clear as possible individualization of the effects of the introduction of a fundamental interval of time in the approach to the problem.

The introduction of a fundamental interval of time in the description of the measurement problem makes possible a simple but effective formalization of the state reduction process. Such behaviour is only observed for the retarded case. When we take into account a symmetric version of the Liouville-von Neumann equation the solution is given by

$$\rho_{nm}(t) = \rho_{nm}(0) \exp \left\{ -\frac{it}{\tau} \sin^{-1} \left[ \frac{\tau}{\hbar} (E_n - E_m) \right] \right\},$$

where the diagonal elements do not change with time and the non-diagonal elements have an oscillatory behaviour. This means that the symmetric equation is not suitable to describe a measurement process, and this is an important distinction between the two descriptions.

It is important to stress that the retarded case of direct discretization of the Liouville-von Neumann equation results in the same equation obtained via the coarse grained description. This lead us to the consideration of this equation as the basic equation to describe complex systems, which is always the case when a measurement process is involved.
6 Conclusions

In this review we tried to get a better insight into the applicability of the various distinct formalisms obtained when performing a discretization of the continuous equations. For example, what kind of physical description is provided by the retarded, advanced and symmetric versions of the Schrödinger equation? This can be achieved by observing the typical behaviour of the solutions obtained for each case and, particularly, attempting to the derivation of these equations from Feynman’s approach. Then we get that the advanced equation describes a system that absorbs energy from the environment. We can imagine that, in order to evolve from one instant to another, the system must absorb energy, and this could justify the fact that, by using Feynman’s approach with the usual direction of time, we can only obtain the advanced equation. The propagator depends only on the Hamiltonian, being independent of the wave function which describes the initial state. So, it describes a transfer of energy to the system.

The retarded equation is obtained by a time reversion, by an inversion of the direction of the propagator, i.e., by inverting the flux of energy. The damping factor characteristic of the retarded solutions refers to a system continuously releasing energy into the environment. Thus, both the retarded and the advanced equations describe open systems.

Finally, the symmetric equation describes a system in an energy equilibrium with the environment. Thus, the only way to obtain stationary states is by using the symmetric equation.

Regarding the nature of such an energy, it can be related to the very evolution of the system. It can be argued that a macroscopic time evolution is possible only if there is some energy flux between system and environment. The states described by the symmetric equation are basically equilibrium states, without nett dissipation or absorption of energy by the system as a whole. We can also conceive the symmetric equation as describing a closed system, which does not exchange energy with the external world.

On the other hand, when a comparison is made with the classical approach, we can speculate that the symmetric equation ceases to be valid when the interaction with the environment changes fastly within a chronon of time. Thus, phenomena such as the collision of highly energetic particles require the application of the advanced or retarded equations. The decay of the norm associated with the vector states described by the retarded equation would indicate the very decay of the system, i.e., of a system abandoning its initial “equilibrium state”. The behaviour of the advanced equation would indicate the transition of the system to its final state. This speculation suggests another interpretation, closer to the quantum spirit. We could consider the possible behaviour of the system as being described by all the three equations. However, the ordinary QM works with averages over ensembles, which is a description of an ideal, purely mathematical reality. The question is that, if we accept the ergodic hypothesis, such averages over ensembles are equivalent to averages over time. Anyway, the quantum formalism always deals with average values when tackling with the real world. When the potentials involved vary slowly with respect to the value of the chronon of the system, which means a long interaction time, we have that the contributions due to the transient factors coming from the retarded and
advanced equations compensate each other, and cancel out. Then, in the average, the system behaves according to the symmetric equation. On the contrary, when the potentials vary strongly within intervals of time of the order of the chronon, we do not have stationary solutions. The discrete formalism describes such a situation by making recourse, during the interaction, to the transient solutions, which will yield the state of the system after the interaction. Afterwards, the system will be described again by a symmetric solution.

The most conservative quantum interpretation would be that of believing that only the symmetric equation describes a quantum system. During the interaction process the theory does not provide any description of the system, pointing only to the possible states of the system after the transient period. The description of the interaction would demand one more ingredient: the knowledge of the interaction process (which would imply an additional theoretical development, as, for example, the working out of an interaction model).

Besides the question of the physical meaning of the discretized equations, i.e., of the type of physical description underlying it, there is also the question of the time evolution of the quantum states. The Schrödinger equations describe the evolution of a wave function, with which an amplitude of probability is associated. An analogy with the electron theory makes us suppose that this wave function does not react instantaneously to the external action, but reacts after an interval of time which is characteristic of the described system. In discrete QM, the justification of the non-instantaneous reaction comes from the fact that the uncertainty principle prevents a reaction arbitrarily close to the action application instant.[?,?] Such uncertainty could be related to the very perturbation caused by the Hamiltonian on the state of the system, resulting in an uncertainty relation like the Mandelstam and Tamm[?] time-energy correlation. What we meet is a time evolution in which the 'macroscopic' state of the system leaps discontinuously from one instant to the other. Therefore, the quantum jumps appear not only in the measurement process, but are an intrinsic aspect of the time evolution of the quantum system. The difference, in our case, is that the jump does not take the system suddenly out of the quantum state it was endowed with, but only determines the evolution of that state.

Another aspect characteristic of the discrete approach is the existence of an upper limit for the eigenvalues of the Hamiltonian of a bounded system. In the description of a free particle it has been observed the existence of an upper limit for the energy of the eigenfunctions composing the wave packet which describes the particle, but this limit does not imply an upper value for the energy of the particle. The existence of this limiting value determines the Hamiltonian eigenvalue spectrum within which a normalization condition can hold. Once exceeded that value, a transition to the internal excited states of the system takes place. This allowed us, e.g., to obtain the muon as an excited internal state of the electron.

It must be noticed the non-linear character of the relation between energy and oscillation frequency of a state, and the fact that the theory is intrinsically non-local, as can be confirmed by looking at the discretized equations. It must also be stressed that the theory described in this paper is non-relativistic.

Finally, it must be remembered that the symmetric form of the discrete formalism repro-
duces grosso modo the results of the continuous theory. The effects of the introduction of a fundamental interval of time are evident in the evolution of the quantum systems, but they are—in general—extremely tiny. There have been some attempts to find physical situations in which measurable differences between the two formalisms can be observed, but till now with little success. [...] Maybe this could be afforded by exploiting the consequences of the phase shifts caused by the discretization, that we saw in Subsections (?) and (?). Regarding the justifications for introducing a fundamental interval of time, let us for instance recall what Bohr replied to the famous 1935 paper by Einstein, Podolski and Rosen: “The extent to which an unambiguous meaning can be attributed to such an expression as physical reality cannot of course be deduced from a priori philosophical conceptions, but ... must be founded on a direct appeal to experiments and measurements”: Considering time as continuous may be regarded as a criticizable philosophical position since, at the level of experiments and measurements, nature seems to be discrete.

More important is to recall that, as already mentioned, the new formalism allows not only the description of the stationary states, but also a space-time description of transient states: The Retarded Formulation yields a natural quantum theory for dissipative systems. It is not without meaning that it leads to a simple solution of the measurement problem in QM. Since the present review is still in a preliminary form, we shall come back to such interesting problems also elsewhere.
A Evolution Operators in the Schrödinger and Liouville–von Neumann Discrete Pictures

When we think of applying the formalism introduced in the previous Sections to the measurement problem, the requirement of the existence of a well-defined evolution operator comes out. By well-defined we mean, as in the continuous case, a unitary operator satisfying the properties of a group.

In the continuous case, when the Hamiltonian is independent of time, the time evolution operator has the form

$$U(t, t_0) = \exp \left( -i (t - t_0) \hat{H} / \hbar \right)$$

and is a unitary operator which satisfies the condition that $\hat{H}$ be hermitian. In the continuous case, by definition, every observable is represented by a hermitian operator. An operator is unitary when its hermitian conjugate is equal to its inverse, such that

$$\hat{A}^\dagger \hat{A} = \hat{A} \hat{A}^\dagger = 1.$$

Another important aspect regarding a unitary operator is related to the probability conservation. In other words, if the initial state is normalized to one, it will keep its norm for all subsequent times. The evolution operator does not change the norm of the states it operates on. Thus, we know beforehand that the evolution operators associated with the retarded and advanced discretized Schrödinger equations are not unitary operators.

A.1 Evolution operators in the Schrödinger picture

For the discretized Schrödinger equation the discrete analogue of the time evolution operator can easily be obtained. Let us initially consider the symmetric equation that, as already remarked, is the closest to the continuous description. After some algebraical handling we get the evolution operator as being

$$\hat{U}(t, t_0) = \exp \left[ - \frac{i(t - t_0)}{\tau} \sin^{-1} \left( \frac{\tau \hat{H}}{\hbar} \right) \right],$$

so that

$$|\Psi(x, t)\rangle = \hat{U}(t, t_0) |\Psi(x, t_0)\rangle = \exp \left[ - \frac{i(t - t_0)}{\tau} \sin^{-1} \left( \frac{\tau \hat{H}}{\hbar} \right) \right] |\Psi(x, t_0)\rangle.$$

Thus, if the eigenvalue equation of the Hamiltonian is given by
\[ \hat{H} |\Psi(x, t_0)\rangle = E |\Psi(x, t_0)\rangle \]

we have that
\[ |\Psi(x, t)\rangle = \exp \left[ -\frac{i(t - t_0)}{\tau} \sin^{-1} \left( \frac{\tau E}{\hbar} \right) \right] |\Psi(x, t_0)\rangle. \]

As \( \hat{H} \) is a hermitian operator, the evolution operator for the symmetric equation is also hermitian. However, the existence of a limit for the possible values of the eigenvalues of \( \hat{H} \) implies that, beyond such threshold, the evolution operator is not hermitian anymore. In fact, if we consider that beyond the threshold the operator \( \hat{H} \) has the form
\[ \hat{H} = \hat{\nu} + i\hat{\kappa}, \]
where \( \hat{\nu} \) and \( \hat{\kappa} \) are hermitian operators, we obtain in the continuous approach the same results obtained in the discrete case. One of the characteristics of a non-hermitian operator is just the fact that it does not conserve the norm of the state it acts on.

For the retarded equation, the evolution operator is given by
\[ \hat{U}(t, t_0) = \left[ 1 + \frac{i}{\hbar} \tau \hat{H} \right]^-(t-t_0)/\tau, \] (116)
such that, in the limit \( \tau \to 0 \),
\[ \lim_{\tau \to 0} \left[ 1 + \frac{i}{\hbar} \tau \hat{H} \right]^{-\tau(t-t_0)/\tau} = e^{-\hat{\kappa}(t-t_0)} \hat{H}, \]
which is an expression known as the Trotter equality. Taking the conjugate hermitian operator \( \hat{U}^\dagger \) we can verify that this operator is not unitary. In the basis of eigenstates of \( \hat{H} \) we can verify that
\[ \langle n | \hat{U}^\dagger \hat{U} | n \rangle = \left[ 1 + \frac{\tau^2 E_n^2}{\hbar^2} \right]^{-\tau(t-t_0)/\tau}, \]
is not equal to 1. Thus, that is the reason why the probabilities are not conserved for the solutions of the retarded equation. Besides, as the evolution operator for the advanced equation is given by
\[ \hat{U}(t, t_0) = \left[ 1 - \frac{i}{\hbar} \tau \hat{H} \right]^{(t-t_0)/\tau}, \]
it can be verified that the formal equivalence between the two equations is obtained by the inversion of the time direction and of the sign of the energy. In the relativistic case, this is understandable if we remember that, if a transformation changes the sign of the time component of a coordinate 4-vector, then it also changes the sign of the energy, which is the corresponding element of the energy-momentum 4-vector. Then the retarded equation describes a particle endowed with positive energy travelling forward in time, and the advanced equation describes an object with negative energy travelling backwards in time, i.e., an antiparticle.\[?, ?, ?, ?\]
A.2 Evolution Operator in the density matrix picture

For the sake of simplicity let $|\psi(t)\rangle$ be a pure state. The density of states operator is defined as

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|.$$ 

It can be shown that such operator evolves according to the following dynamic laws. For the retarded case,

$$\Delta_R \hat{\rho}(t) = \frac{1}{i\hbar} \left[ \hat{H}(t), \hat{\rho}(t) \right] - \frac{i}{\hbar \tau^2} \hat{H}(t) \hat{\rho}(t) \hat{H}(t);$$

for the advanced case,

$$\Delta_A \hat{\rho}(t) = \frac{1}{i\hbar} \left[ \hat{H}(t), \hat{\rho}(t) \right] + \frac{i}{\hbar \tau^2} \hat{H}(t) \hat{\rho}(t) \hat{H}(t);$$

and, finally, for the symmetric case,

$$\Delta \hat{\rho}(t) = \frac{1}{i\hbar} \left[ \hat{H}(t), \hat{\rho}(t) \right].$$

We can thus observe that the retarded and the advanced equations cannot be obtained by a direct discretization of the continuous Liouville–von Neumann equation. Such formal equivalence occurs only for the symmetric case. Taking into account the retarded case, we can obtain the equivalent time evolution operator as being

$$\hat{V}(t, t_0) = \frac{1}{1 + \frac{i}{\hbar} \hat{L} + \frac{\tau^2}{\hbar^2} \hat{H} \ldots \hat{H}}^{(t-t_0)\tau}. \quad (117)$$

We must remark that this operator is different from the one obtained from the coarse grained approach,

$$\hat{V}_{CG}(t, t_0) = \frac{1}{1 + \frac{i}{\hbar} \hat{L}}^{(t-t_0)\tau}. \quad (118)$$

and that it is not unitary as well. Quantity $\hat{V}_{CG}$ is so defined to have the properties of a semigroup: without having necessarily an inverse, but possessing the other group properties such as commutativity and existence of an identity (besides the translational invariance of the initial condition).

What we can conclude from such a difference between the two operators is that, apparently, the descriptions clash with each other. In the coarse grained approach the starting point was the continuous Liouville–von Neumann equation and, by introducing the graining of the time coordinate, an evolution operator was obtained satisfying the retarded equation

$$\Delta_R \hat{\rho}(t) = \frac{1}{i\hbar} \left[ \hat{H}(t), \hat{\rho}(t) \right].$$
The second path consisted in starting from the definition of the density operator, in order to determine the dynamical equation it satisfies, and then obtaining the evolution operator.

For the symmetric case the evolution operator is given by

$$
\hat{V}(t, t_0) = \exp \left[ -\frac{i(t - t_0)}{\tau} \sin^{-1} \left( \frac{\tau \mathcal{L}}{\hbar} \right) \right],
$$

which is similar to the operator obtained for the continuous case.

### A.3 Compatibility between the previous pictures

We thus have two distinct evolution operators for the retarded Schrödinger and Liouville equations so that, once established a connection between them, we arrive at the compatibility of the two descriptions. We try to set up a relation between those operators by observing their action on the density operator. So, we expect that both operators satisfy the expression

$$
\hat{V}(t, t - \tau) \hat{\rho}_0 = \hat{U}(t, t - \tau) \hat{\rho}_0 \hat{U}^\dagger(t, t - \tau),
$$

where the different action of the operators is basically due to the bilinearity of the operator \( \hat{V} \) given by (??), while \( \hat{U} \), given by (??), is linear. This relation is valid in the continuous case, where the evolution operators act on the density operator according to

$$
\hat{\rho}(t) = \exp[-i\mathcal{L}(t - t_0)/\hbar] \hat{\rho}_0 = \exp[-iH(t - t_0)/\hbar] \hat{\rho}_0 \exp[iH(t - t_0)/\hbar].
$$

Considering the basis of hamiltonian eigenstates \( |n\rangle \), we have

$$
\langle n|\hat{\mathcal{L}}\hat{\rho}(0)|m\rangle = (E_n - E_m)\rho_{nm}(0),
$$

so that

$$
\exp(-i\hat{\mathcal{L}}t)\hat{\rho}(0) = \exp[-it(E_n - E_m)]\rho_{nm}(0),
$$

$$
\exp(-i\hat{H}t)\hat{\rho}(0) \exp(i\hat{H}t) = \exp[-it(E_n - E_m)]\rho_{nm}(0).
$$

The question is to know if the same is valid for the discrete case. For the retarded approach we must check whether the relation

$$
\frac{1}{1 + \frac{i\tau}{\hbar} \mathcal{L} + \frac{\tau^2}{\hbar^2} \hat{H} \ldots \hat{H}}^{(t-t_0)/\tau} \hat{\rho}_0 = \frac{1}{1 + \frac{i\tau}{\hbar} \hat{H}^{(t-t_0)/\tau} \rho_0} \frac{1}{1 - \frac{i\tau}{\hbar} \hat{H}^{(t-t_0)/\tau}}
$$

is valid. What we get is that, if we consider equations such as (??) and (??) to continue to be valid in the discrete case, then the above relation is valid. For a generic element of the operator we then get

$$
\frac{1}{1 + \frac{i\tau}{\hbar}(E_n - E_m) + \frac{\tau^2}{\hbar^2} E_n E_m}^{t/\tau} \rho_{nm}(0) = \frac{1}{1 + \frac{i\tau}{\hbar}E_n}^{t/\tau} \rho_{nm}(0) \frac{1}{1 - \frac{i\tau}{\hbar}E_m}^{t/\tau}.
$$
Such equivalence can be also observed for the other cases. However, when we consider the evolution operator obtained from the coarse grained approach we find an incompatibility with the operator deriving from the Schrödinger one. For the operator (??) we have

\[ \langle n \| \frac{1}{1 + i \hbar / \tau} L^{1/\tau} \hat{\rho}(0) \| m \rangle = \frac{1}{1 + i \hbar / \tau(E_n - E_m)} L^{1/\tau} \rho_{nm}(0). \]

The question is to know what is the fundamental difference between the two descriptions: if both are valid and under what conditions. Some points must be emphasized. First of all, we must remember that the coarse grained description is a semi-classical approach which assumes a system with a certain degree of complexity, while the vector state description is a fundamentally quantum approach without any imposition, in principle, on the number of degree of freedom of the system described. There is a basic difference even in the way of conceiving the chronon. In the coarse grained approach it is understood as a magnitude inwardly connected to the experimental limitations or, for an ideal measurement device, to the limitations imposed by the uncertainty relations. For the Schrödinger equation, the value of the chronon is taken as a fundamental interval of time associated with interaction processes among the components of the system, and of the system as a whole with some external potential; i.e., it is associated with the internal processes of the system (as it has been conceived for the classical electron). In this way, the absence of the mixed term in the evolution operator obtained with the semi-classical procedure is comprehensible, as well as its incompatibility with the pure quantum description provided by the Schrödinger equations. As a semi-classical approach, the range of applicability of the coarse grained formalism extends to the cases where the system to be studied is not purely microscopic, particularly in the measurement processes. We have to stress the fact that, in this formalization, only the retarded equation was obtained. Thus, the system described dissipates energy, i.e., it is an open system. This is just the characteristic that allows us to have access to the output of a measurement.

In connection with the operator obtained directly in the Schrödinger picture for the retarded case, we have that all the elements of the density matrix, even the diagonal ones, are damped with time. Besides, there is also the controversy linked to the non-existence, in QM, of an applicability limit of the theory due to the number of degrees of freedom involved. The formalism does not distinguish between a microscopic and a macroscopic system, so that it should reproduce what is obtained with the coarse grained formalism. This means that the measurement problem also appears in the discrete formalism through the non-equivalence of the evolution operators (??) and (??).

B Non-Hermitian Operators in the Discrete Formalism

One of the features we have been stressing through this work is the non-hermitian character of the discrete formalism. In the Schrödinger representation, for example, the continuous
equation can reproduce the outputs obtained with the discretized equations once we replace the conventional Hamiltonian by a suitable non-hermitian Hamiltonian we have called ‘equivalent Hamiltonian’. One of the characteristics of a non-hermitian operator is that its eigenvalues are defined over the complex number field. A linear non-hermitian operator can always be considered as being composed by a hermitian part, which supplies the real component of the eigenvalues, and by an anti-hermitian part, which gives the complex component.

In the continuous case, let us take the Hamiltonian as being a non-hermitian operator given by

$$\hat{H} = \hat{\nu} + i\hat{\kappa},$$

where $\hat{\nu}$ and $\hat{\kappa}$ are hermitian. Then we have, in the Schrödinger picture, that the time evolution operator is given by

$$U_{\text{cont}}(t, t_0) = \exp \left[ \frac{1}{\hbar} \left( \hat{\kappa} - i \cdot \hat{\nu} \right)(t - t_0) \right]. \quad (122)$$

For the discrete case, we get from Appendix ?? that the evolution operator for the retarded states is given by (??)

$$\hat{U}(t, t_0) = \left[ 1 + \frac{i}{\hbar} \frac{\tau}{\hbar} \right]^{-(t-t_0)/\tau}, \quad (123)$$

where $\hat{H}$ is the hermitian operator associated with the conventional hamiltonian. This evolution operator can be written as

$$\hat{U}_{\text{ret}}(t, t_0) = \exp \left[ -\frac{(t-t_0)}{2\tau} \ln \left( 1 + \frac{\tau^2\hat{H}^2}{\hbar^2} \right) \right] \exp \left[ -\frac{i(t-t_0)}{\tau} \tan^{-1} \left( \frac{\tau\hat{H}}{\hbar} \right) \right]. \quad (124)$$

Comparing (??) and (??) we obtain the equivalence of the hamiltonians once $\hat{\nu}$ and $\hat{\kappa}$ are given by

$$\hat{\nu} = \frac{\hbar}{\tau} \tan^{-1} \left( \frac{\tau\hat{H}}{\hbar} \right),$$

$$\hat{\kappa} = -\frac{\hbar}{2\tau} \ln \left( 1 + \frac{\tau^2\hat{H}^2}{\hbar^2} \right).$$

For the advanced case we obtain the same expressions except by a minus sign for $\hat{\kappa}$. For the symmetric case, below the critical limit, we have

$$\hat{\nu} = \frac{\hbar}{\tau} \sin^{-1} \left( \frac{\tau\hat{H}}{\hbar} \right),$$

$$\hat{\kappa} = 0.$$
Above that limit \( \tilde{\nu} \) ceases to be hermitian and, in this case, the evolution operator can be written as

\[
\hat{U}_{\text{sym}}(t, t_0) = \exp \left[ -\frac{i\pi}{2\tau}(t - t_0) \right] \exp \left\{ -\frac{(t - t_0)}{\tau} \ln \left[ \frac{\tau \hat{H}}{\hbar} + \sqrt{\left( \frac{\tau \hat{H}}{\hbar} \right)^2 - 1} \right] \right\}
\]

so that

\[
\tilde{\nu} = \frac{\hbar \pi}{2\tau},
\]

\[
\kappa = -\frac{\hbar}{\tau} \ln \left[ \frac{\tau \hat{H}}{\hbar} + \sqrt{\left( \frac{\tau \hat{H}}{\hbar} \right)^2 - 1} \right],
\]

with \( \tilde{\nu} \) being now independent of the Hamiltonian and \( \kappa \) ceases to be zero.

The expressions obtained above show the characteristics that \( \tilde{\nu} \) and \( \kappa \) must fulfil in order that the continuous equation reproduces the outputs of the discretized equations. By observing the continuous evolution operator we have that the anti-hermitian part of \( \hat{H} \) shows a non-stationary behaviour, resulting in a damping or amplifying term associated with the evolution of the quantum state. Thus, the stationary solutions appear only for the symmetric case below the critical limit. In all the other cases the transient term always appears.

In QM, the non-hermitian operators have been used only as mathematical shortcuts, as in the case of the Lippmann–Schwinger equation in the scattering theory. It has already been observed that the introduction of such operators could make possible the description of unstable states, by phenomenologically linking the transient factor to the lifetime of the considered state.\([?]\) If in a certain instant \( t_0 = 0 \) the system is in one of the eigenstates \( |n\rangle \) of the Hamiltonian \( \hat{H} \) then, if such state is unstable, the probability of the system to be found in the same state in a later instant \( t \) is

\[
P_n(t) = \langle n | \hat{U} \hat{U} | n \rangle = \exp(-t/\tau_L),
\]

and that allows us to specify a lifetime \( \tau_L \), for the retarded case, as being

\[
\tau_L = \frac{\tau}{\ln \left( 1 + \frac{\tau^2 E_n^2}{\hbar^2} \right)}, \tag{125}
\]

and for the symmetric case, above the critical energy,

\[
\tau_L = \frac{\tau}{2 \ln \left( \frac{\tau E_n}{\hbar} + \sqrt{\left( \frac{\tau^2 E_n^2}{\hbar^2} \right) - 1} \right)}.
\]

Such lifetimes are connected with states that, in the discretized formalism, are intrinsically unstable. Only the retarded equation seems to be associated with quantum states which decay with time. If that is really valid, we have an expression which could be used for phenomenologically determining the value of the chronon. Finally, we can conclude that the time discretization
brings forth a formalism which, even if only hermitian Hamiltonians are involved, is equivalent to the introduction of non-hermitian operators in the continuous QM.

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\begin{align*}
\text{(a)} & & p(t) \quad x(t) \\
\text{(b)} & & p(t) \\
\text{(c)} & & p(t) \\
\end{align*}

\begin{align*}
\text{(a)} & & -2 \times 10^{-1} \quad 0.0 \quad 3 \times 10^{-1} \\
\text{(b)} & & -1.0 \times 10^{-1} \quad 5 \times 10^{-1} \\
\text{(c)} & & -5.0 \times 10^{-1} \quad 0.0 \quad 5.0 \times 10^{-1} \\
\end{align*}
(a) $E > E_M : E = E_M (1 + 1 \times 10^7) \text{ eV}; t = 1 \times 10^8 \text{s}$. Discrete and continuous solutions differ in phase and amplitude.

(b) $E > E_M : E = E_M (1 + 1 \times 10^7) \text{ eV}$. Left: $t = 1 \times 10^8 \text{s}$. Right: Damping of the amplitude with time.
(a) $E < E_M: E = 0.001 \text{eV}; t = 0$. Discrete and continuous solutions are identical.

(b) $E < E_M: E = 0.001 \text{eV}; t = 1 \times 10^{-8}$ s. Discrete and continuous solutions differ in phase.
$e^{-\frac{\omega^2}{2}\frac{1}{t\tau}}$

$ponto de inflexão$
\[ t_n - t_{n-1} = \tau \]