Exact multilocal renormalization on the effective action: application to the random sine Gordon model statics and non-equilibrium dynamics

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We extend the exact multilocal renormalization group (RG) method to study the flow of the effective action functional. This important physical quantity satisfies an exact RG equation which is then expanded in multilocal components. Integrating the nonlocal parts yields a closed exact RG equation for the local part, to a given order in the local part. The method is illustrated on the $O(N)$ model by straightforwardly recovering the $\eta$ exponent and scaling functions. Then it is applied to study the glass phase of the Cardy-Ostlund, random phase sine Gordon model near the glass transition temperature. The static correlations and equilibrium dynamical exponent $z$ are recovered and several new results are obtained. The equilibrium two-point scaling functions are obtained. The nonequilibrium, finite momentum, two-time $t, t'$ response and correlations are computed. They are shown to exhibit scaling forms, characterized by novel exponents $\lambda_R \neq \lambda_C$, as well as universal scaling functions that we compute. The fluctuation dissipation ratio is found to be non trivial and of the form $X(q^*(t - t'), t/t')$. Analogies and differences with pure critical models are discussed.

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I. INTRODUCTION

Recently a method was devised, the exact multilocal renormalization group (EMRG) [1], to obtain perturbative renormalization group equations from first principles, in a controlled way to any order, and for arbitrary smooth cutoff function. It starts, as numerous previous exact RG studies [2, 3, 4, 6, 8, 9, 10, 11, 12], from the exact Polchinski-Wilson renormalization group equation [13, 14] for the action functional $S(\phi)$. The next step however consists in splitting it onto local and higher multilocal components [15], and integrating exactly all multilocal components in terms of the local part. This yields an exact and very general RG flow equation for the local part of the action, i.e. a function, expressed in an expansion in powers of the local part.

The aim of this paper is first to develop a similar method using instead the effective action functional $\Gamma(\phi)$, relevant for e.g. superconductors and density waves [20, 21, 22, 23, 24], and the EMRG has been applied to study that problem [1, 15]. Here, and this is the second aim of this paper, we will study another instance of a glass phase, arising in the random phase sine Gordon model excluding vortices, as discovered by Cardy and Ostlund [25]. This model has been studied extensively, in its statics [19, 26, 27, 28, 29, 30, 31] and its dynamics [20, 21, 22, 23, 24], and the EMRG has been applied to study that problem [20, 21, 22, 23, 24]. Here, and this is the second aim of this paper, we will study another instance of a glass phase, arising in the random phase sine Gordon model excluding vortices, as discovered by Cardy and Ostlund [25]. This model has been studied extensively, in its statics [19, 26, 27, 28, 29, 30, 31] and its dynamics [20, 21, 22, 23, 24], and the EMRG has been applied to study that problem [20, 21, 22, 23, 24].

The outline of the paper is as follows. First in Section II we derive the EMRG method for the effective action, and give the explicit general lowest order RG equations. In Section III we apply these RG equations to the pure $O(N)$ model, as a test of the method. In Section IV we consider the Cardy-Ostlund model statics. In Section V we study the CO model equilibrium dynamics. Section VI is devoted to the non-equilibrium dynamics of the CO model. All calculational details are contained in the appendices.
II. METHOD

A. Exact RG method

We want to study interacting bosonic degrees of freedom described by a set of fields denoted \( \phi \equiv \phi_x \) where \( x \) is the position in space and \( i \) a general label denoting any quantity which will not undergo the coarse-graining (e.g. fields indices, spin, replica indices, additional coordinate). The problem is defined by an action functional:

\[
S(\phi) = \frac{1}{2} : G^{-1} : \phi + \mathcal{V}(\phi) \tag{1}
\]

and by the functional integral (i.e. the partition function)

\[
Z = \int D\phi \ e^{-S(\phi)}. \tag{2}
\]

The action consists of a quadratic part \((G^{-1}_{ij} \equiv G_{ij}^x \) is a symmetric invertible matrix\) and \(\mathcal{V}(\phi)\) the interaction, a functional of \(\phi\). The notation : denotes full contractions over \(x, i\) (i.e. : \( \phi : G^{-1} : \phi \) = \( \sum_{i,j} \int \phi_i(x) (G^{-1})_{ij} \phi_j(x) \)). We will denote \( \int_x = \int d^d x \) where \( d \) is the space dimension, and \( \int_x = \int (d^d x) \) for integration in Fourier. Our aim is to compute the effective action \( \Gamma(\phi) \), i.e. the generating function of proper vertices, since once it is known, all correlation functions are known being simply obtained as sums of all tree diagrams drawn using \( \Gamma \). For all observables to be well defined one usually requires both an ultraviolet UV cutoff (e.g. \( \Lambda_0 \) in momentum space) and an infrared IR cutoff (noted here \( \Lambda_t = e^{-1}\Lambda_0 \)). For example, in a single scalar theory one chooses \( G \equiv G_t \) with:

\[
G_t^q = e^{-2c(q^2/2\Lambda_t^2)} \tag{3}
\]

in Fourier. Here \( c(z,s) \) is a cutoff function which decreases to zero as \( z \to 0 \) or \( s \to \infty \) and for convenience, see below, we choose \( c(z,s) = 0 \). To study finite momentum observables in a massless theory one is also interested in the zero IR cutoff limit, \( \Lambda_t = 0 \) with \( G \equiv G_t=\infty \) denoting \( c(z) = c(\infty, z) \).

In this paper we will use that \( \Gamma(\phi) \) satisfies the following exact functional equation when the quadratic part \( G \) is varied (for a fixed \( \mathcal{V}(\phi) \)):

\[
\partial \Gamma(\phi) = \frac{1}{2} Tr G^{-1} : \left[ \frac{\delta^2 \Gamma(\phi)}{\delta \phi \delta \phi} \right]^{-1} + \frac{1}{2} : \phi : G^{-1} : \phi \tag{4}
\]

Derivations and more details are given in Appendix A. This can be used to express how the effective action \( \Gamma(\phi) \equiv \Gamma_t(\phi) \) of the model \( \Gamma_t \) with \( G \equiv G_t \) depends on the IR cutoff \( \Lambda_t \). Indeed the following property holds:

\[
\Gamma_t(\phi) = -\frac{1}{2} Tr \ln G_t + \frac{1}{2} : G_t^{-1} : : \phi + \mathcal{U}_t(\phi) \tag{5}
\]

with \( \mathcal{U}_t(\phi) \equiv \mathcal{U}_t(G_t) \) satisfies the exact flow equation:

\[
\partial_l \mathcal{U}_t(\phi) = \frac{1}{2} Tr \partial_l G_t : (G_t^{-1} - G_t^{-1} (1 + G_t : \frac{\delta^2 \mathcal{U}_t}{\delta \phi \delta \phi} )^{-1} ) \tag{6}
\]

with the initial condition \( \mathcal{U}_t(0) = \mathcal{V}(\phi) \), simply reflecting that the effective action equals the action when all fluctuations are suppressed (at \( l = 0 \) where the running propagator satisfies \( G_t = 0 \) from the property \( c(z,s) = 0 \)). The above equation \( \Box \) simply expresses how \( \Gamma(\phi) \) in \( \Box \) depends on the final value \( G \equiv G_t \). The zero IR cutoff limit \( \Lambda_t = 0 \) can then studied by integrating the above equation up to \( l = \infty \).

For actual calculations, simpler and useful choices read, in momentum space:

\[
G_t^q = \frac{1}{q^2} (c(q^2/2\Lambda_t^2) - c(q^2/2\Lambda_t^2)) \tag{7}
\]

where the cutoff function \( c(x) \) satisfies \( c(0) = 1 \) and \( c(\infty) = 0 \). With the choice \( c(x) = 1/(1 + 2x) \) one finds the massive, Pauli-Villars like, propagator:

\[
G_t^q = \frac{1}{q^2 + m^2} - \frac{1}{q^2 + M^2} \tag{8}
\]

a different choice.

The full exact RG equation \( \Box \) can also be expanded in series of \( \mathcal{U}_t \) as:

\[
\partial_l \mathcal{U}_t(\phi) = \frac{1}{2} Tr \partial_l G_t : \frac{\delta^2 \mathcal{U}_t(\phi)}{\delta \phi \delta \phi} + \frac{1}{2} Tr \partial_l G_t : \frac{\delta^2 \mathcal{U}_t(\phi)}{\delta \phi \delta \phi} + O(\mathcal{U}_t^3) \tag{9}
\]

which admits the graphical representation given in Fig.1.

To summarize, the philosophy of the method is, in a sense the exact opposite of the Wilson one, since it amounts to start from the action with no fluctuations \( \Lambda_t = \Lambda_0 \), and then add modes and their fluctuations until one reaches the desired theory \( \Lambda_t \ll \Lambda_0 \). In that limit one expects that the effective action reaches a fixed point form, given by the asymptotic solution of \( \Box \) at large \( l \).

\[
\frac{d}{dl} = \bullet + \bullet + \bullet + \bullet + ..\tag{10}
\]

FIG. 1: Representation of the exact RG equation \( \Box \). The dot is the vertex \( \mathcal{U} \), the solid line a propagator \( G_t \) and the crossed solid line the on shell propagator \( \partial_l G_t \). The sum is over all one loop graphs with a factor \( (-1)^{p-1}/2 \) for each \( p \) vertex graph represented.
\[ \frac{d}{d\phi} \phi = \phi + \phi + \phi + \ldots \]

\[ \frac{d}{d\phi} \phi = (1-P) \cdot \phi + \phi \]

\[ \frac{d}{d\phi} \phi = \phi \]

\[ \frac{d}{d\phi} \phi = 0 \]

We recall here only its action on a bilocal operator \( F(\phi_1, \phi_2, x - y) \), namely \( (\bar{P}_1 F)(\phi) = \int_y F(\phi, \phi, y) \). It can be used to split an action depending only on two points into:

\[
\int_{xy} F(\phi_x, \phi_y, x - y) = \int_x (\bar{P}_1 F)(\phi_x) + \int_{xy} ((1 - P_1) F)(\phi_x, \phi_y, x - y)
\]

where, by definition, \( (P_1 F)(\phi, \psi, z) = \delta(z) \int_y F(\phi, \psi, y) \), in such a way that the second part is properly bi-local (i.e. \( (P_1(1 - P_1) F)(\phi) = 0 \)). A similar construction holds for higher multilocal operators.

The idea is then to project the functional equation so that the bilocal, trilocal etc., can be expressed exactly in terms of the local part \( U_l \) only. One notices that there is a simplest way to do it so that the bilocal part is \( V \sim O(U^2) \), trilocal \( W \sim O(U^3) \) etc., which determines one possible splitting of the higher multilocal components (e.g. bilocal vs trilocal) as is represented in the Fig. and further explained in [1]. This expansion is clearly suited to the situations where the flowing functional \( U_l \) becomes "small" and dominated by its local part (e.g. in the context of a dimensional expansion), but it has a more general validity, since in all cases it is an exact expansion in series of the local part of the full effective action functional.

We now pursue the analysis exactly to order \( O(U_l^2) \), sufficient to a number of one loop applications. Details are given in Appendix [4]. The bilocal part is exactly given by:

\[
V_l(\phi_1, \phi_2, x) = \frac{1}{2} F_l(\phi_1, \phi_2, x - \delta(x) \int_y F_l(\phi_1, \phi_2, y))
\]

with:

\[
F_l(\phi_1, \phi_2, x) = - \int_0^t \left( \delta^1 \partial^1 G_{l_{i_1}^1} \cdot \partial^2 \right) \partial^1 G_{l_{i_2}^1} \cdot \partial^2 \epsilon^{-\frac{1}{2} \partial^1 G_{l_{i_1}^1} \cdot \partial^1 - \frac{1}{2} \partial^2 G_{l_{i_2}^1} \cdot \partial^2} U_{l'}(\phi_1) U_{l'}(\phi_2)
\]

(13)

to all orders (by definition), and the resulting exact RG equation for the local part of the effective action (i.e. the exact \( \beta \)-function up to \( O(U_l^2) \)) is:

\[
\partial_1 U_l(\phi) = \frac{1}{2} \partial G_{l,i} \partial_1 U_l(\phi) - \frac{1}{2} \int \partial G_{l,i} \partial_1 U_l(\phi) (G_{l,k})_{km} \partial_m \partial_1 U_l(\phi)
\]

\[
- \frac{1}{2} \int_x \partial^1 (\partial G_{l} - \partial G_{0}) \partial^2 \int_0^t \left( \delta^1 \partial^1 G'_{l} \partial^2 \right) \partial^1 G'_{l} \partial^2 \epsilon^{-\frac{1}{2} \partial^1 G_{l_{i_1}^1} \cdot \partial^1 - \frac{1}{2} \partial^2 G_{l_{i_2}^1} \cdot \partial^2} U_{l'}(\phi_1) U_{l'}(\phi_2)|_{\phi_1 = \phi_2 = \phi}
\]

We use the following notations: \( \partial_i \equiv \partial_{\phi_i} \) (resp. \( \partial^i \)) denotes derivation with respect to the first argument.

FIG. 2: Schematic representation of the splitting of the functional \( U \) vertex into local, bilocal, trilocal etc., parts respectively (top line). Representation of the exact RG equation for the bilocal, trilocal, etc., as well as local vertex (last several lines). Note that by definition the ERG equation for the bilocal part contains only exactly two feeding terms, trilocal three etc... The solid lines represent a propagator \( G \) and the crossed solid lines the on shell propagator \( \partial G \). Combinatorial factors are not represented. \( P \) here denotes the projection operator on the local part (denoted \( \bar{P}_1 \) and \( P_1 \) in the text).

B. Multilocal expansion
for the RG equation of $\tilde{g}$ drop from now on these higher monomials and study only $\tilde{g}$.

From power counting, it is more convenient to introduce

$$G^q_{l,ij} = \delta_{ij} G^q_l$$

with $G^q_l$ as in (12). We study this model near the dimension 4, in $d = 4 - \epsilon$, and compute the effective action to order $O(\epsilon^2)$. For some explicit calculations, we will further use the form (13) with the following convenient parametrization and notation for the cut-off function $c(z)$:

$$c(z) = \int_0^{+\infty} da e^{az} = \int_0^1 e^{-ax}$$

The condition $c(0) = 1$ imposes $\int_a = 1$.

### A. Derivation of the $\beta$-functions and fixed points.

The local part of the running effective action admits the polynomial expansion:

$$U_1(\phi) = g_{0,l} + \frac{g_{2,l}}{2} \phi^2 + \frac{g_{4,l}}{4!} (\phi^2)^2 + \frac{g_{6,l}}{6!} (\phi^2)^3 + ..$$

(18)

From power counting, it is more convenient to introduce the dimensionless couplings $\tilde{g}_{2n,l}$ defined from:

$$g_{2,l} = \Lambda^2_l \tilde{g}_{2,l}$$

$$g_{4,l} = \Lambda^4_l \tilde{g}_{4,l}$$

and more generally $\tilde{g}_{2n,l} = g_{2n,l} \Lambda_l^{(d-2)n-d}$ which flows to some fixed point values $\tilde{g}_{2n}^*$ as discussed below. Since $\tilde{g}_{6}^* \sim O(\epsilon^3)$ and $\tilde{g}_{2n}^* \sim O(\epsilon^n)$ for $n \geq 3$ (see Appendix C for the RG equation of $\tilde{g}_{6,l}$ and the free energy $g_{0,l}$), we drop from now on these higher monomials and study only the coupled RG equation for $\tilde{g}_{4}$ and $\tilde{g}_{2}$ easily obtained by inserting (13) into (14) as detailed in Appendix C:

$$\partial_l \tilde{g}_{4,l} = \frac{N - 8}{3} \tilde{I}_{(1)}^{(1)} \tilde{g}_{4,l}^2 = -\beta(\tilde{g}_{4,l})$$

$$\partial_l \tilde{g}_{2,l} = 2 \tilde{g}_{2,l} + \frac{N + 2}{6} \tilde{I}_{(1)}^{(0)} \tilde{g}_{4,l} - \frac{N + 2}{3} \tilde{I}_{(1)}^{(1)} \tilde{g}_{2,l} \tilde{g}_{4,l},$$

(20)

(21)

with the integrals:

$$\tilde{I}_{(1)}^{(0)} = \Lambda^{2+\epsilon}_l \int d^d q G^1_l,$$

(22)

$$\tilde{I}_{(1)}^{(1)} = \Lambda^{\epsilon}_l \int d^d q G^1_l G^1_l$$

(23)

$\tilde{I}_{(1)}^{(2)}$ is given in (15) where we show that the coefficient of the term proportional to $\tilde{g}_{4,l}^2$ in (22) is well defined in the limit $l \rightarrow \infty$. One finds that $\tilde{I}_{(1)}^{(0)}$ is $l$-independent and that $\lim_{l \rightarrow \infty} \tilde{I}_{(1)}^{(1)} = I^{(1)}$ is universal (independent of $c(s)$) in dimension $d = 4$:

$$\tilde{I}_{(1)}^{(0)} = -\frac{1}{4\pi^2} \int_{0}^{\infty} d\omega \omega^{0} + O(\epsilon)$$

(24)

$$\tilde{I}_{(1)}^{(1)} = S_d \int_{s > 0} (2s)^{-1/2} c'(s)(c(s) - 1) = \frac{1}{16\pi^2} + O(\epsilon)$$

(25)

where $S_d$ is the unit sphere area divided by $(2\pi)^d$ and we recall $c'(s) < 0$. Finally eq. (20, 21) together with (22) lead to the fixed point values

$$\tilde{g}_{4}^* = \frac{48\pi^2}{N + 8} \epsilon + O(\epsilon^2)$$

(26)

$$\tilde{g}_{2}^* = -\frac{N + 2}{12} \tilde{I}_{(1)}^{(0)} \tilde{g}_{4}^* + O(\epsilon^2)$$

This fixed point describes the standard $O(N)$ critical system exactly at the critical temperature $T = T_c$. The initial conditions which end up for $l = \infty$ exactly at the fixed point describe the critical manifold.

Besides we obtain the correction of the critical exponent characterizing the divergence of the magnetic susceptibility near the critical temperature from the positive eigenvalue $\lambda_l$ (corresponding to the instable direction)

$$\lambda_l = 2(1 - \frac{N + 2}{2(N + 8)} \epsilon)$$

(27)

which gives correctly (33) the exponent $\gamma$ to order $\epsilon$

$$\gamma = 2 + \frac{N + 2}{2(N + 8)} \epsilon + O(\epsilon^2)$$

(28)

### B. Computation of the 2 and 4 points proper vertices.

We now compute the effective action on the critical manifold, up to order $O(\epsilon)$ for the local part, and $O(\epsilon^2)$ for the 

\[\boxed{\text{the function}}\]
for the bi-local part (i.e. the $q$ dependent part), in the limit of large $l$. Eq. (12) allows to construct the bilocal term in the effective action by inserting (13) in (12). As we restrict our analysis to order $\epsilon^2$, we do not consider monomials higher than $(\phi_2^2)^2$ in (13) and therefore we expand the exponential in (13) to order one. Using the combinatorics already explained for the local part in Appendix C one gets

$$V_l(\phi_1, \phi_2, q) = \frac{1}{2} \int_x (e^{iqx} - 1) F_l(\phi_1, \phi_2, x)$$

$$F_l(\phi_1, \phi_2, x) = \frac{N + 2}{3} \phi_1 \cdot \phi_2 \int_0^l ds \partial_s G_{\epsilon} G_{\epsilon} G_{\epsilon} \phi_s \phi_s \lambda_{\epsilon}^2$$

$$- \left( \frac{N + 4}{3!} \phi_1^2 \phi_2^2 + \frac{4}{3!} \phi_1 \phi_2 \right) \int_0^l ds \partial_s G_{\epsilon} G_{\epsilon} G_{\epsilon} \phi_s \phi_s \lambda_{\epsilon}^2$$

where we have not written terms of the form $f(\phi_1, x)$ (i.e. which depend only on one field argument) as they cancel out from the effective action. To this order in $\epsilon (O(\epsilon^2))$ there are no other contributions. The explicit expressions of $G_{\epsilon}$ and $\partial G_{\epsilon}$ using (12) are given in Appendix C (30). This bilocal term (29) allows to treat the renormalization of the wave function and compute the exponent $\eta$ to order $\epsilon^2$. A natural way to obtain it, within this method, is to compute directly the 1 particle irreducible (1PI) two-points function and then take the limit $l \to \infty$ (directly at $T_c$). Its local part comes from the quadratic contribution of (13) and the bi-local part is the sum of $G_{\epsilon}^{-1}(q)$ (31) and the quadratic contribution of (30)

$$\Gamma_l^{(2)}(q) = \frac{\delta^2 \Gamma_l}{\delta \phi^2} \bigg|_{\phi = 0} = \delta_{ij} \Gamma_l^{(2)}(q)$$

$$\Gamma_l^{(2)}(q) = G_l^{-1}(q) + \lambda_l^2 \partial_2 l - \frac{N + 2}{18} \partial_2 l \int_x (e^{iqx} - 1) (G_l^{-1})^3$$

In the appendix C we show that it has the form, up to terms of order $(\Lambda_l/\Lambda_0)^2$

$$\Gamma_l^{(2)}(q) = G_l^{-1}(q) + \lambda_l^2 \partial_2 l - q^2 \eta[\partial_4 l] (\ln \frac{\Lambda_l}{\Lambda_0} + \chi^{(2)}(\frac{q}{\Lambda_l}))$$

$$\eta[\partial_4 l] = \frac{N + 2}{18(4\pi)^4} g_4 l$$

with the following asymptotic behaviors

$$\chi^{(2)}(k) \sim ak^2 \quad k \ll 1$$

$$\chi^{(2)}(k) \sim \ln k \quad k \gg 1$$

with $a$ some non universal (i.e. dependent of the cutoff function (14)) coefficient. The two-point scaling function $\chi^{(2)}(k)$ which is computed here (see Appendix C) for an arbitrary infrared cutoff function $c(x)$, is up to an additive constant, independent of the UV cutoff $\Lambda_0$. For the particular choice (14) one recovers the result of (30).

The large argument behavior of $\chi^{(2)}(k)$ allows to take the limit $l \to \infty$, using the fixed point value $\eta_4$ (24), we have (for $q \ll \Lambda_0$):

$$\lim_{l \to \infty} \Gamma_l^{(2)}(q) = (q^2 - q^2 \frac{N + 2}{2(N + 8)^2} \ln \frac{q}{\Lambda_0})$$

which coincides with the expansion of $\lim_{l \to \infty} \Gamma_l^{(2)}(q) \sim q^2 (q/\Lambda_0)^{-\eta}$ to order $\epsilon^2$ with the universal value of the $\eta$ exponent to this order

$$\eta = \eta_4 = \frac{N + 2}{2(N + 8)^2}$$

in agreement with standard results (35).

Let us focus on the construction of the quartic term in $\Gamma_l(\phi)$, obtained from the quartic contribution of (13) and (29). After combinatorial manipulations, we obtain

$$\Gamma_l^{\text{quart}} = \frac{\partial_4 l}{4!} \int_{q_4} (\partial_4 l)(\phi_{q_4} \cdot \phi_{q_4})$$

$$+ \frac{1}{3!} \int_{q_3} (\phi_{q_3} \cdot \phi_{q_3} \cdot \phi_{q_4} \cdot \phi_{q_4}) \chi^{(4)}(q_3 + q_4)$$

with $\chi^{(4)}(q)$ defined by

$$\chi^{(4)}(q) = \int_x (e^{iqx} - 1)(G_l^{-1})^2 + O(\Lambda_0^{-2})$$

and where we used the notation $\int_{q_4} = \int_{q_1, q_2, q_3, q_4} (2\pi)^d \delta(4)(q_1 + q_2 + q_3 + q_4)$. The local term, i.e. the first line in (35), contains a contribution of order $\epsilon^2$ which is divergent in the limit $l \to \infty$. Indeed, expanding it to second order gives $\partial_4 l \partial_4 l = \partial_4 l (1 + \epsilon \ln \Lambda_l) + O(\epsilon^3)$ and at first sight this term would lead to a divergent contribution in the limit $l \to \infty$. However, the analysis of $\chi^{(4)}(q)$ shows the following asymptotic behaviors

$$\chi^{(4)}(k) \sim bk^2 \quad k \ll 1$$

$$\chi^{(4)}(k) \sim \frac{1}{16\pi^2} \ln(k^2) \quad k \gg 1$$

with $b$ a non universal constant. When considering the large $l$ limit of the effective action, we are interested in the large argument behavior of $\chi^{(4)}(k)$ (35). Using the fixed point value $\eta_4$ (24), one gets that this cancels exactly the divergence when $l \to \infty$ due to the local term. Thus we obtain:

$$\lim_{l \to \infty} \Gamma_l^{\text{quart}} = \frac{2\pi^2 \epsilon}{(N + 8)} \int_{q_4} (\phi_{q_4} \cdot \phi_{q_4})$$

$$+ \frac{4\epsilon}{(N + 8)} \int_{q_3} (\phi_{q_3} \cdot \phi_{q_3} \cdot \phi_{q_4} \cdot \phi_{q_4})$$

$$+ \frac{1}{\Lambda_0} (\phi_{q_4} \cdot \phi_{q_4} \cdot \phi_{q_4} \cdot \phi_{q_4}) \ln(\frac{q_3 + q_4}{\Lambda_0})$$

which is independent of $\Lambda_0$ to order $O(\epsilon^2)$. Note that in the large $N$ limit one recovers correctly the "screened"
four point renormalized vertex $\sim \epsilon q^3$ (where $q$ is the transfer momentum).

The result of this analysis is that we have constructed the large scale theory by obtaining directly a fixed point for the effective action, keeping the UV cut-off $\Lambda_0$ finite, which is the relevant object for statistical physics, and for an arbitrary cutoff function.

C. Relation with field-theoretical methods

It is interesting to make the connection with standard field-theoretical methods for critical phenomena. There one is usually interested in the limit $\Lambda_0 \to \infty$. Note that in this limit there diverges. It is however possible to define a “renormalized” effective action $\Gamma_R(\phi_R)$ which is well defined in that limit.

One can first check directly on the standard Callan-Symanzik (CS) ”bare” RG equation \eqref{eq:CS} for the physical correlation function of the massless theory at the fixed point

$$\left(\Lambda_0 \frac{\partial}{\partial \Lambda_0} - \eta \right) \left( \lim_{l \to \infty} \Gamma_l^{(2)}(q) = 0 + O(\epsilon^3) \right) \quad (40)$$

One can also connect to the CS equation for the renormalized theory. One defines:

$$\Gamma_R(\phi) = \Gamma_1(\sqrt{Z}\phi) \quad (41)$$

where $Z \equiv Z(\frac{\Lambda_0}{\Lambda}, \tilde{g}_{4.4})$ is the so called ”wave-function renormalization” factor such that

$$\Gamma_R^{(2)}(q) = m_R^2 + q^2 + O(q^4) \quad (42)$$

Using \eqref{eq:CS} and noting that $G^{-1}_l(q) = -\frac{2\Lambda^2}{\epsilon(q \Lambda)} + \epsilon \omega^2(0) + O(\epsilon^3)$, with $A = -\frac{\omega^2(0)}{\epsilon(q \Lambda)}$ one finds the renormalized mass $m_R^2 = \frac{1}{\epsilon(q \Lambda)} \eta(\tilde{g}_{4.4})$ and $Z = \frac{1}{\epsilon}(1 + \eta(\tilde{g}_{4.4})(\frac{\Lambda_0}{\Lambda}))$. One can see that up to higher order terms, $\Lambda_1$ plays the role of the renormalized mass. From \eqref{eq:CS} one finds, to order $O(\eta \tilde{g}_{4.4}^2)$

$$m_R \delta_{n,R} | \Lambda_0 \ln Z(\Lambda_1 \Lambda_0, \tilde{g}_{4.4}) = -\partial_1 \ln Z = \eta(\tilde{g}_{4.4}) \quad (43)$$

these derivatives being taken at fixed $\tilde{g}_{4.4}$. This is the standard definition for the $\eta(q)$ function. One can go further, define a renormalized coupling $g_R$, e.g. through $\Gamma_R^{(2)}(q = 0) = m_R^2 g_R$, with $g_R = \tilde{g}_{4.4}$ up to higher order terms, and derive the CS equations for the renormalized vertices. Here, we just mention one such equation \eqref{eq:CS} for the ”renormalized” two-point vertex function in the critical regime $\Lambda_1 \ll \Lambda_0$ but finite

$$\left( \partial_1 + \bar{\eta}(\tilde{g}_{4.4}) \right) \Gamma_1^{(2)}(q) \simeq 0 \quad q/\Lambda_1 \gg 1 \quad (44)$$

obtained using the large $k$ behavior of $\chi^{(2)}(k)$ \eqref{eq:large_k}. We get again the universal value of the $\eta$ exponent form $\eta = \eta(\tilde{g}_{4.4}) \quad (45)$.

The connection between the EMRG method and the standard field theoretical methods in the massless scheme (i.e. imposing $\Gamma^{(2)}(q = 0) = 0$) is more subtle here (since one should use $l = \infty$ strictly).

IV. CARDY OSTLUND MODEL: STATICS

In this section, we show how this EMRG method can be used to study perturbatively the Cardy Ostlund model \[32] near its glass transition.

A. Model, choice of propagator

This model is a random phase Sine-Gordon model which can represent an XY model in a random magnetic field where the vortices are excluded by hand. As mentioned in the introduction, the statics of this model has been extensively studied using various methods \[16, 27, 28, 29, 30, 31\]. The system at equilibrium is described by the partition function $Z = \int D\phi e^{-H^{CO}(\phi)/T}$, $T$ being the temperature with the hamiltonian

$$H^{CO}[\phi] = \frac{1}{2} \int d^2 x (\nabla \phi)^2 - \int d^2 x (h_x^1 \cos \phi_x + h_x^2 \sin \phi_x) \quad (45)$$

while $\phi_x \in ]-\infty, +\infty[ \text{ as there are no vortices, where } h_x = (h_x^1, h_x^2) \text{ is a } 2d \text{ random Gaussian vector of zero average with fluctuations decorrelated from site to site: }$

$$\langle h_x^1 h_x^2 \rangle = 2g_0 \Lambda_0^2 \delta_{i,j}(2)(x - x') \quad (46)$$

The quenched average over this random variable is performed by the means of replicas, which is used here as a simple trick to restore translational invariance and to organize perturbation theory. After averaging over the disorder, one obtains

$$\overline{\ln Z} = \lim_{n \to 0} \frac{Z^n - 1}{n} = \int D\phi e^{-\frac{H^{rep}[\phi]}{T}} = \frac{1}{27} \sum_{a,b} \int d^2 x d\phi_a^2 d\phi_b^2 \delta_{a,b}$$

$$-\frac{g_0 \Lambda_0^2}{T^2} \sum_{a,b} \int d\phi_a^2 d\phi_b^2 \cos(\phi_a - \phi_b) \quad (47)$$

where $a, b = 1, \ldots, n$ are replica indices. We use the same propagator as for the O(N) model, the Gaussian part of \[17\] being diagonal in replicas, one has

$$G^{(2)}_{ab} = \delta_{ab} \frac{T}{q^2} (c(q^2/2\Lambda_0^2) - c(q^2/2\Lambda_1^2)) \quad (48)$$

with the same decomposition of the cutoff function \(c(x)\). \[17\]

Notice that the hamiltonian $H^{rep}$ possesses the statistical tilt symmetry (STS) \[38\], the last term in \eqref{eq:hamiltonian} is invariant under the change of variable $\phi_x^a \to \phi_x^a + u_x$ which protects the diagonal (in replica space) quadratic term in the effective action to all orders in perturbation theory \[29, 34\].
B. \( \beta \)-functions and fixed point

For this model, the Fourier representation in the fields (143) is more natural. Although only one harmonic is present in the starting hamiltonian (144), higher harmonics are generated by perturbation theory and we write the local interacting part of the effective action (140) as

\[
U_l(\phi) = -\Lambda_l^2 \sum_{K \neq 0} \frac{g_l^K}{T^2} e^{iK \cdot \phi} \tag{49}
\]

where \( K = (K_1, \ldots, K_n) \), \( \phi = (\phi^1, \ldots, \phi^n) \) are \( n \)-components vectors and one defines \( K.K' = \sum_a K_a K'_a \). The sum is over all \( K \) such that \( K_a \) are integers not all zero with \( \sum_a K_a = 0 \). \( U_l(\phi) \) is real, imposing \( g_l^K = g_l^{-K} \), and the symmetry under replica indices permutation, which is assumed here, imposes \( g_l^K = g_l^{\sigma(K)} \), \( \sigma(K) \) being any vector obtained from \( K \) by a permutation of the \( K_a \).

By inserting (49) in (14) (see also (B6) in Appendix B) one obtains the RG equation for the local part to second order in \( g_l^K \):

\[
\partial_t g_l^K = (2 - \frac{T K^2}{4\pi}) g_l^K + \frac{J_l^{(1)}}{2T^2} \sum_{P,Q,P+Q=K} g_l^P g_l^Q (P,Q)^2 - \frac{1}{2T^2} \sum_{P,Q,P+Q=K} (P,Q)^3 \int_0^l dt' J_{l,t'}^P g_l^P g_l^Q (50)
\]

with the integrals

\[
J_l^1 = \Lambda_l^2 \int \partial G_l^T G_l^T \tag{51}
\]

\[
J_{l,t'}^2 = \Lambda_l^{-2} \int \partial \{ G_l^T - \partial G_l^{T=0} \} \partial G_l^T G_l^T \Lambda_l^4 \times e^{\frac{T^2}{4\pi} \int_0^l dt' \partial G_l^T G_l^T} \tag{52}
\]

The glass transition temperature \( T_c \) below which the charges of minimal modulus such that \( K_{1,-1} = (0, \ldots, 1, \ldots, -1, \ldots, 0) \), \( K_{1,-1} = 2 \) become relevant is

\[
T_c = \frac{8\pi}{K_{1,-1}^2} = 4\pi \tag{53}
\]

and a small parameter \( \tau = (T_c - T)/T_c > 0 \) can be defined, which allows to construct perturbatively the effective action of this model (147) in its glass phase. Indeed just below \( T_c \) the higher harmonics are irrelevant (the eigenvalues \( 2 - \frac{T K^2}{4\pi} \) are negative and of order one). Such irrelevant higher harmonics include for instance 3 replicas term (162) \( g_l^{1,-2,1} \sum_{a \neq \pm c} e^{i(a^2 - 2a^1 + \phi^0_a)} \), corresponding to \( K_{2,-2,1} = 4 \). We denote \( g_l = g_l^{1,-1} \) the coupling constant associated to \( K_{1,-1} \), and obtain its RG flow from (147) by taking into account the 2\((n-2)\) possible fusions such that \( P + Q = K_{1,-1} \), \( P, Q \) being themselves obtained by a permutation of the components of \( K_{1,-1} \)

\[
(g_l^P = g_l^Q = g_l) \text{ with } P,Q = -1 \tag{54}
\]

After some transformations detailed in the Appendix D one obtains

\[
\partial_t g_l = (2 - \frac{T}{2\pi}) g_l - B_l g_l^2 \tag{54}
\]

\[
B_l = 2\partial_\gamma_0(0) \int_\tilde{x} \gamma(\tilde{x}) + 2 \int_\tilde{x} (\partial_\gamma_0(\tilde{x}) - \partial_\gamma_0(0)) (e^{T_\gamma(\tilde{x})} - 1) \tag{55}
\]

where we used the dimensionless variable \( \tilde{x} = x\Lambda_l \) and defined:

\[
\partial G_l^T = T \partial_\gamma_\mu = l - l'(\tilde{x}) \tag{56}
\]

\[
G_l^{\tau} = -T_\gamma_\mu = l - l'(\tilde{x}) \tag{56}
\]

where the two functions \( \partial_\gamma_\mu(x) \) and \( \gamma_\mu(x) \) are given in (138).

As shown in the appendix, we can transform the integral over \( \tilde{x} \) in (151) and express its cutoff dependence in a simple way. One finds \( B_\infty = \frac{\pi^2}{12} e^{-(\gamma_E - f_a \ln 2\alpha)} \) yielding for \( T < T_c \), the stable fixed point of the RG flow is given by

\[
g^* = 8\pi e^{(\gamma_E - f_a \ln 2\alpha)} + O(\tau^2) \tag{57}
\]

with \( \tau = (T_c - T)/T_c \) and \( \gamma_E = 0.577216 \) the Euler constant.

C. Bilocal term and 2-point correlation function.

Eq. (151) allows to construct the bilocal term in the effective action to lowest order (i.e. \( O(\tau^2) \)) using a Fourier representation (152):

\[
\tilde{V}_l(\phi, \psi, x) = \sum_{K, P} \tilde{V}_l^{K, P_{\psi}} e^{iK \cdot \phi + iP \cdot \psi} \tag{58}
\]

Just below \( T_c \), only the charges of minimal modulus \( K_{2,-1}^2 = 2 \) are relevant, therefore to order the sums in (155) are restricted to such harmonics. By inserting (155) into (153) one has

\[
\tilde{V}_l^{K, P_{\psi}} = \frac{1}{2} \int_x (e^{q(x)} - 1) \tilde{F}_l^{K, P_{\psi}} \tag{59}
\]

\[
\tilde{F}_l^{K, P_{\psi}} = -\frac{(K.P)^2}{T^4} \int_0^l dt' \partial G_l^T G_l^T \tag{59}
\]

\[
e^{\frac{K^2 - P^2}{T^2} G_l^{T=0}} e^{K.P G_l^{T=0} \Lambda_l^4 g_l} g_l^P \tag{59}
\]

where \( K, P \) are of the form \( K_{1,-1} \), and thus \( g_l^K = g_l^P = g_l \). Performing the integral over \( l' \) as explained in Appendix D we have:

\[
\tilde{F}_l^{K, P_{\psi}} = -\frac{\Lambda_l^4}{T^2} g_l^2 \left( \frac{1}{2} e^{-T.K.P \gamma(\tilde{x})} - 1 \right) K.P \gamma(\tilde{x}) \tag{60}
\]
with $\hat{x} = \Lambda_t x$. For the charges $K, P$ we are considering here, there are a priori 5 different cases of $K, P = -2, -1, 0, 1, 2$ to consider. However, we see immediately on the previous expression that the charges such that $K, P = 0$ do not contribute to the bilocal part of the effective action, which only exists for $K = -P$ (Appendix D). We show in Appendix D that $\hat{V}_{t,K,P}^q$ takes the form, up to terms of order $(\Lambda_t/\Lambda_0)^2$

$$\hat{V}_{t,K,P}^q = -A q^2 (\delta_{K,-P} \ln \frac{\Lambda_t}{\Lambda_0} + \chi_{K,P}(\frac{q}{\Lambda_t})) \quad (61)$$

$$A_t = \frac{\pi \hat{g}_t^2}{4T_0} e^{-2\gamma_0 + 2\sum \ln 2a} \quad (62)$$

where $\chi_{K,P}(k)$ behaves asymptotically at small argument as

$$\chi_{K,P}(k) \sim \begin{cases} \frac{a_{K,P}}{a_{-2k^2}} K, P \neq -2 \\ -\frac{a_{-2k^2}}{K, P = -2} k \ll 1 \end{cases} \quad (63)$$

and at large argument (relevant for the limit $l \rightarrow \infty$) as

$$\chi_{K,P}(k) \sim \begin{cases} \frac{b_{K,P} T_0}{k} K, P = 1, 2 \\ -\frac{b_{-2k^2}}{k} K, P = -2 \quad k \gg 1 \end{cases} \quad (64)$$

The large argument behavior of $\chi_{K,P}(x)$ allows to take the limit $\Lambda_t \rightarrow 0$ of $\hat{V}_{t,K,P}^q$ as the logarithmic divergence (which only exists for $K = -P$) is cancelled. We notice also that only such terms with $K = -P$ survive in this limit: in particular, three replicas term as $g_t^2 \sum_{\alpha \neq \beta \neq \gamma} e^{i(x_{\alpha} - x_{\beta}) + i(x_{\gamma} - x_{\alpha})}$ do not exist in the effective action to order $\tau^2$ at the fixed point for $\Lambda_t = 0$. Besides, by inserting the fixed point value $g_0^2$ in $A_t$ (62), we see that the cut-off dependence (encoded in the factor $e^{\sum \ln 2a}$) disappears in $\lim_{l \rightarrow \infty} A_t$ leading to

$$\lim_{l \rightarrow \infty} \hat{V}_{t,K,P}^q = -\delta_{K,-P} \frac{\tau^2}{16\pi} q^2 \ln \frac{q}{\Lambda_0} \quad (65)$$

Eq. (21) together with (60) allow to construct the bilocal term as

$$\lim_{l \rightarrow \infty} \Gamma_{t,K,P}^{bilocal}(\phi) = \frac{1}{2T} \sum_\alpha \int q \frac{q^2}{c(2\pi)^2} \phi_\alpha^q \phi_{-q}^q \quad \begin{cases} a_{K,P} K, P \neq -2 \\ -a_{-2k^2} K, P = -2 \end{cases} \quad (66)$$

$$\quad + \sum_{\alpha, b} \int_{x, x'} \int_q \lim_{l \rightarrow \infty} \hat{V}_{t,K,-Q}^q e^{iq(x-x')} e^{i(\phi_{b,x}^q - \phi_{b}^q)} e^{-i(\phi_{a,b}^q - \phi_{a,b}^q)}$$

from which we extract the two-point 1 PI function

$$\lim_{l \rightarrow \infty} \Gamma_{t,ab}^{(2)}(q) = \frac{q^2}{Tc(q^2/2\Lambda_0^2)} \delta_{ab} + \frac{\tau^2}{4\pi} q^2 \ln \frac{q}{\Lambda_0} \quad (68)$$

V. CARDY OSTLUND: EQUILIBRIUM DYNAMICS

We now turn to dynamics, which, within the EMRG framework can be conveniently studied by introducing an infrared cutoff on space only, keeping the full time dependence.

A. Model and propagator.

Within this EMRG framework we want to study the dynamics of the model (15) (32), described by a Langevin type equation:

$$\eta \frac{\partial}{\partial t} u_t = -\frac{\delta H^{CO}}{\delta u_t} + \zeta(x, t) \quad (71)$$

where $\zeta(x, t) = 0$ and $\zeta(x, t) \zeta(x', t') = 2\eta T \delta(x - x')\delta(t - t')$ is the thermal noise and $\eta$ the friction coefficient. A convenient way to study the dynamics is to use the Martin-Siggia-Rose (39) generating functional, on which perturbation theory can be done. Moreover, using the Ito prescription, it can be readily averaged over the disorder. The disorder averaged generating functional reads

$$Z[j, \tilde{j}] = \int \mathcal{D}u \mathcal{D}i \tilde{u} e^{-S[u, i] + j: u + \tilde{j}: \hat{i}} \quad (72)$$

$$S[u, i] = S_0[u, i] + S_{int}[u, i] \quad (73)$$

$$S_0[u, i] = \int_{\tau t} i u_t - \eta \frac{\delta}{\partial t} + c_q u_{qt} - \eta T \int_{\tau t} i u_{xt} i u_{xt} \quad (74)$$

$$S_{int}[u, i] = -g_0 \Lambda_t \int_{\tau t} i u_{xt} i u_{xt} \cos(u_{xt} - u_{xt}) \quad (75)$$

where $\int_t^\infty ds\, dt$, where in this section the initial time $t_i$ is sent to $t_i = -\infty$ before taking $\Lambda_t/\Lambda_0$ large, in order to describe equilibrium dynamics.
In our formulation \(\Gamma\), the field \(\phi\) is now a two components vector

\[
\phi_{xt} = \begin{pmatrix} u_{xt} \\ i\bar{u}_{xt} \end{pmatrix}
\] (73)

and from \(S_0\) in (72), we compute the inverse bare propagator \(G_t^{-1}\)

\[
G_t^{-1}(q) = \begin{pmatrix} \delta(t-t') & \delta(t-t')(-\eta\partial_t + cq^2) \\ \frac{1}{c(q^2/2\Lambda_0^2) - c(q^2/2\Lambda_t^2)} \end{pmatrix}
\] (74)

By inverting this matrix we obtain the bare response and correlation functions

\[
C_{ltt'} = C^q_{ltt'} = \langle u_{qt}i\bar{u}_{qt'} \rangle = \frac{T}{q^2}e^{-q^2t-qt'}(c(q^2/2\Lambda_0^2) - c(q^2/2\Lambda_t^2))
\] (75)

\[
R_{ltt'}^{it} = \frac{\delta\langle u_{qt}\rangle}{\delta u_{qt'}} = \langle u_{qt}u_{qt'} \rangle
\] (76)

where we have set the bare \(\eta = 1\). As we consider here the equilibrium dynamics of the system, the time translation invariance (TTI) and the fluctuation dissipation theorem (FDT) hold. These properties hold to all orders in perturbation theory and, as we will see has strong consequences on the structure of the effective action \(\Gamma_t(u, i\bar{u})\). This means for the dressed (i.e. exact) response and correlation functions:

\[
C_{ltt'} = C_{ltt'}^q
\] (77)

\[
R_{ltt'}^{it} = R_{ltt'}^{it}
\] (78)

\[
R_{ltt'}^{it} = -\theta(t-t')\frac{1}{T}\partial_t C_{ltt'}^q
\] (79)

B. Response function and dynamical exponent.

We will study the dynamics near the transition temperature \(T_c\), below which the lowest harmonic of the disordered potential becomes relevant. Near \(T_c\), we showed previously that the higher harmonics, although generated by perturbation theory, are irrelevant.

As we considered here static disorder, the average over the disorder generates an effective interaction \(S_{\text{int}}[u, i\bar{u}]\) in (72) which is non local in time, so we expect the friction coefficient to be renormalized by the disorder. We therefore construct the effective action to order one in \(r = (T - T_c)/T_c\), and extract the dynamical exponent \(z\) from the response function. In the starting dynamical action (72) the interacting part is purely local in space, so to order one the interacting part of the associated effective action \(\Gamma_t(u, i\bar{u})\) will remain so. We therefore search a perturbative solution of the equation for \(\Gamma_t(u, i\bar{u})\) of the form (19):

\[
\mathcal{U}_t(u, i\bar{u}) = \int_x U_t(u_{xt}, i\bar{u}_{xt})
\]

\[
= \int_x i\bar{u}_{xt}F_t(u_{xt}) - \frac{1}{2} \int_{x,t'} i\bar{u}_{xt}\Delta_{ltt'}(u_{xt})i\bar{u}_{xt'}
\] (81)

where \(F_t(u_{xt})\) and \(\Delta_{ltt'}(u_{xt})\) are functionals only with respect to the time dependence, i.e. functions of the “vector” \(u_{xt} \equiv \{u_{xt}\}\) at a given point \(x\) in space. In addition these will acquire an explicit time dependence, indicated by their \(t\) and \(t'\) indices. One has the initial conditions

\[
\Delta_{t=0t'}(u) = 2g_0\Lambda_0^2 \cos(u_t - u_{t'})
\]

\[
F_t(u_{xt}) = 0
\] (82)

The \(F_t(u)\) term is indeed generated by perturbation theory and is related - in the case of equilibrium dynamics - to \(\Delta_{ltt'}(u)\) by a generalized FDT relation, namely a Ward Identity, which can be written to lowest order:

\[
\frac{\delta F_t(u)}{\delta u_{xt'}} = -\frac{1}{T}\partial_t \Delta_{ltt'}(u) \quad t > t'
\] (83)

where \(\partial_t\) acts only on the explicit time dependence (i.e. not on \(u_{xt}\)). Notice finally that terms containing higher powers of the field \(i\bar{u}\), i.e. \((i\bar{u})^{p+2}\) are of order \(g'u^{p+1}\). They correspond to higher cumulant of the disorder (i.e. higher number of replica terms in the statics). The exact RG equation to order one (19) then reads (see Appendix 3)

\[
\frac{\partial}{\partial t} \Delta_{ltt'}(u) = \int_{t_{t_1}} \delta_{lt_1}^{(1)} \Delta_{ltt'}(u)
\]

\[
\frac{\partial}{\partial t} F_t(u) = \int_{t_{t_1}} \delta_{lt_1}^{(1)} F_t(u) - \int_{t_{t_1}} \delta_{lt_1}^{(2)} \Delta_{ltt'}(u)
\]

with

\[
\delta_{lt_1}^{(1)} = \frac{1}{2} \partial_{u_{xt}} \partial_{u_{xt'}} \delta_{lt_1}^{x=0} \delta_{lt_1}^{x=0}
\]

\[
\delta_{lt_1}^{(2)} = \frac{1}{\partial_{u_{xt}}^2} \partial_{R_{lt_1}^{x=0}} \delta_{lt_1}^{x=0}
\] (85)

The solution of this coupled set of equations (81) together with (82) is given by

\[
\Delta_{ltt'}(u) = 2\Lambda_0^2 g_{i\bar{u}} C_{ltt'}(u)
\]

\[
\frac{\delta F_t(u)}{\delta u_{xt'}} = -2\Lambda_0^2 g_{i\bar{u}} C_{ltt'}(u) R_{ltt'}^{x=0} \cos(u_{xt} - u_{xt'}) \quad t > t'
\] (86)

where we can check explicitly the previously mentioned generalized FDT relation (83). Finally, as we consider here static disorder, the flow of \(g_{i\bar{u}}\) is given by the previous study, the fixed point value \(g^*\) being given by (84).

From \(\Gamma_t(u, i\bar{u})\), we obtain the response function in the following way

\[
\mathcal{R}_{ltt'}^{q} = \langle u_{qt}i\bar{u}_{qt'} \rangle = \left( \frac{\delta^2 \Gamma_t}{\delta i\bar{u}_{qt'} \delta u_{qt}} \bigg|_{u=i\bar{u}=0} \right)^{-1}
\] (87)
We define
\[ D_{ltt'} = \Delta_{ltt'}(u = 0) \]
\[ \Sigma_{ltt'} = \frac{\delta F_{lt}}{\delta u_{tt'}} \bigg|_{u=0} \] (88)
notice that in the case of equilibrium dynamics \( D_{ltt'} = D_{ltt'} \) and \( \Sigma_{ltt'} = \Sigma_{ltt'} \). One gets
\[ \frac{\delta^2 \Gamma_l}{\delta u_{lt} \delta u_{qt}} \bigg|_{u=0} = \delta(t - t')(q^2 + \partial_t) + \Sigma_{ltt'} \] (89)
When considering equilibrium dynamics, the use of Fourier transform allows to compute this matrix element in a simple way
\[ R^q_{l\omega} = \frac{1}{q^2 - i\omega + \Sigma_{l\omega}} \] (90)
where \( R^q_{l\omega} = \int \frac{d^4 x}{(2\pi)^4} e^{-i\omega t + i q \cdot x} \Sigma_{ltt' \omega} \) and \( \Sigma_{l\omega} \) is the Fourier transform, of \( \Sigma_{ltt'} \). In Appendix E we show that it has the following form (up to terms of order \( \Lambda_f^2/\Lambda_0^2 \))
\[ \Sigma_{l\omega} = i\omega B_l (\ln \frac{\Lambda_f^2}{\Lambda_0^2} + \chi^{(\text{dyn})}(\frac{\omega}{\Lambda_f^2})) \] (91)
\[ B_l = \frac{g_l e^{l/4} \ln 2a}{2 T_c} \] (92)
with the following asymptotic behaviors
\[ \chi^{(\text{dyn})}(\nu) \sim a_{\text{dyn}} \ln \nu \quad \nu \ll 1 \] (93)
\[ \chi^{(\text{dyn})}(\nu) \sim \ln \nu \quad \nu \gg 1 \] (94)
where \( a_{\text{dyn}} \) is a non universal constant. The large argument behavior of \( \chi^{(\text{dyn})} \) allows to take the large \( l \) limit in \( 91 \) as the logarithmic divergence is cancelled, which gives
\[ \lim_{l \to \infty} R^q_{l\omega} = \frac{1}{q^2 - i\omega + i\omega B^* \ln \frac{\Lambda_f^2}{\Lambda_0^2}} \] (95)
\[ B^* = \lim_{l \to \infty} B_l = e^{\gamma_E} \tau \] (96)
where we have used (97) to compute \( B^* \) which is universal : the cut-off dependence encoded in \( e^{\gamma_E} \tau \) has disappeared. On the other hand we expect that the scaling function in Fourier should read:
\[ \lim_{l \to \infty} R^q_{l\omega} = \frac{1}{q^2 - i\omega(\frac{\Lambda_f^2}{\Lambda_0^2})^{2/3}} \] (97)
from scaling. If the initial model possess STS then the coefficient of \( q^2 \) is fixed to unity. The \( q \)-independence of the self-energy is expected to hold only to the order in \( \tau \) that we are working at, and it should be corrected by higher loops. Expansion of the denominator of \( 95 \) coincides with the expansion to order \( \tau \) of the denominator of \( 97 \) and yields the universal value of the dynamical exponent \( z \)
\[ z - 2 = 2B^* = 2e^{\gamma_E} \tau + O(\tau^2) \] (98)
in agreement with previous studies. It is interesting in view of later applications to non-equilibrium dynamics, and a useful check, to compute this response function in the time domain. Indeed, writing simply the identity \( \Gamma_l^{(2)} \Gamma_l^{(2)\dagger} = 1 \), where \( \Gamma_l^{(2)} \) is the matrix of the second functional derivatives of the effective action with respect at the fields \( u_{tt} \) and \( i\bar{u}_{xt} \), we obtain a system of closed equations for the exact response and correlation functions \( R_{ltt'} \) and \( C_{ltt'} \) to order one (more generally, \( F_{lt}(u) \) and \( \Delta_{ltt'} \) can be bilocal in space)
\[ \partial_t R_{ttx'} - \nabla^2 R_{ttx'} + \int_{t_1}^{t} dt_1 \Sigma_{ltt}, R_{ttx'} = \delta(t - t')\delta(x - x') \] (99)
\[ \partial_t C_{ttx'} - \nabla^2 C_{ttx'} + \int_{t_1}^{t} dt_1 \Sigma_{ltt}, C_{ttx'} = 2\partial_l R_{ltx'} + \int_{t_1}^{t} dt_1 D_{ltt'}, R_{ttx'} \] (100)
We remind that we have chosen the Ito presription, which fixes the following initial condition for the response function
\[ \lim_{\epsilon \to 0} R_{ltt',t-\epsilon} = 1 \]
\[ R_{ltt,t} = 0 \] (101)
Before using these equations to study non-equilibrium dynamics, we show how the equation for the response (Eq. 99) function allows to recover the dynamical exponent \( z \). Using (88) together with (90) and TTI (which holds for equilibrium dynamics), the equation for the response function reads
\[ (\partial_l + q^2) R_{ltt'} = \partial_l + \frac{2g_l \Lambda_f^2}{\ln 2a} \int_{t_1}^{t} dt_1 R_{l-\tau} \exp(\frac{\partial l}{\partial l} (R_{l+\tau} - R_{l-\tau})) \] (102)
The limit \( l \to \infty \) is taken as explained in the Appendix E [20], and a way to solve this equation is simply to say that in the rhs, we may replace \( R_{l+\tau} \), by its bare
value, which is simply $\theta(t_1 - t') e^{-q^2(t_1 - t')}$ as this term is already of order $\tau$.

One expects that the response function can be written as:

$$R^q_{\ell - t'} = \lim_{t \to \infty} R^q_{\ell, t - t'} = \tilde{q}^{-2} F^q_R(\tilde{q}^z (\tilde{t} - \tilde{t}'))$$  \hfill (103)

where $\tilde{q} = q/\Lambda_0$, $\tilde{t} = t \Lambda_0^2$, $\tilde{t}' = t' \Lambda_0^2$, with $F^q_R$ a universal scaling function (up to an overall non universal scale) such that $F^q_R(v) \sim v^{(z-2)/z}$ for $v \to 0$. As a function it admits an expansion in powers of $\tau$, obtained as:

$$F^q_R(v) = F^q_R(v') + \tau F^{1eq}_R(v') + O(\tau^2)$$  \hfill (104)

$$F^{1eq}_R(v) = e^{-v} + \tau F^{2eq}_R(v') + O(\tau^2)$$  \hfill (105)

as shown in the appendix. This is established by identifying the direct expansion of in terms of the argument $v' = \tilde{q}^z (\tilde{t} - \tilde{t}')$:

$$R^q_{\tilde{t}} = F^q_R(v') + (z - 2) \ln \tilde{q} F^q_R(v') + \tau F^{1eq}_R(v') + O(\tau^2)$$  \hfill (106)

with the result of solving. Note that the term proportional to $\ln \tilde{q}$ has precisely the expected $v$ dependence, a check of the calculation. Since there is an overall non-universal scale $\tilde{q} \to \lambda \tilde{q}$, $F^q_R(v)$ is defined up to a change in the constant $\rho$ defined in the Appendix.

One can check explicitly that the scaling function in the time domain obtained by this second method coincides with the inverse Fourier transform of to the lowest order in $\tau$. The asymptotic behavior of the scaling function in the time domain is:

$$F^q_R(v) \approx e^{-v} \ln(1/(e^v v))$$, $v \to 0$ \hfill (107)

$$F^{1eq}_R(v) \approx e^{-v} v^{-2}$$, $v \to \infty$ \hfill (108)

the slow time decay $1/t^{1+z\frac{z}{2}}$, for $z > 2$, arises from the disorder. Notice that a similar power law tail for large $\tilde{q}^z t$ has already been obtained for the diluted Ising model.

Using the FDT we also obtain the equilibrium correlation function in the scaling regime as:

$$C^q_{\ell t'} = T \tilde{q}^{-2} F^q_C(\tilde{q}^z (\tilde{t} - \tilde{t}')) \hfill (109)

We conclude this section on equilibrium dynamics by noticing a few interesting properties. The first one is an exact consequence of the scaling form combined with the STS. Indeed, the STS imposes:

$$\lim_{t \to \infty} \int_{t'}^t dt' R^q_{\ell t'} = \frac{1}{\tilde{q}^2}$$  \hfill (110)

Using the scaling property we showed previously, this symmetry implies

$$\int_0^1 dt \tilde{q}^{-2} F^q_R(\tilde{q}^z t) = \frac{1}{\tilde{q}^2} \Rightarrow \int_0^1 du F^q_R(\tilde{u}) = 1$$  \hfill (111)

from which it follows that

$$\mathcal{R}^q_{\ell t'} = \int_0^1 \tilde{q}^{-2} F^q_R(\tilde{q}^z (\tilde{t} - \tilde{t}')) = \frac{1}{2\pi z (t - t')}$$  \hfill (112)

where we have used FDT in the last line. Note that the unrescaled time $t$ appears in these formulae. Although the scaling form is only valid for small $\tilde{q}$ we believe that the behaviors may actually be the exact leading ones in the large $t - t'$ limit, their coefficients being fixed (non-perturbatively) by the STS. This would be interesting to check numerically.

The second property is a comparison with the so-called Porod’s law. If the form were to hold to all orders, the scaling functions would decay at large arguments as $F^q_R(v) \sim 1/v^2$ and $F^q_C(v) \sim 1/v^{\frac{1}{2}}$. That yields

$$C^q_{\ell t'} \sim \frac{1}{(t - t')^{2/z} q^4}$$  \hfill (113)

as in the Porod’s law with $d = 2$ and $n = 2$. Here this property holds to the order of our calculation $O(\tau)$.

VI. NON-EQUILIBRIUM DYNAMICS OF THE CO MODEL

Applying standard scaling arguments, we expect $R^q_{\ell t'}$, and $C^q_{\ell t'}$ to be functions of the scaling variables $\tilde{q}^z \tilde{t}$ and $\tilde{q}^z \tilde{t}'$ where $\tilde{q} = q/\Lambda_0$ and $\tilde{t} = \Lambda_0^2 t$ and $z$ is the dynamical exponent. As is the case for pure systems at a critical point one can write from RG arguments with little restriction:

$$R^q_{\ell t'} = \tilde{q}^{-2 + z\frac{z}{2}} \int_{\tilde{t}'}^{\tilde{t}} F_R(\tilde{q}^z (\tilde{t} - \tilde{t}'), \tilde{t}/\tilde{t}')$$  \hfill (114)

$$C^q_{\ell t'} = T \tilde{q}^{-2 + \frac{z}{2}} \int_{\tilde{t}'}^{\tilde{t}} F_C(\tilde{q}^z (\tilde{t} - \tilde{t}'), \tilde{t}/\tilde{t}')$$  \hfill (115)

where the exponent $\theta$ is defined by imposing the following behavior of the response scaling function $F_R(v, u)$ when $u \to \infty$:

$$F_R(v, u) = F_R(\infty, v) + O(u^{-1})$$  \hfill (116)

This has been checked for pure systems and, partially for one case of a disordered system (only for the response function in and for the Fourier mode $q = 0$ for both functions in ). It was found in all the pure cases that one also has

$$F_C(v, u) = \frac{F_{C(\infty)}(v)}{u} + O(u^{-2})$$  \hfill (117)
These forms \([16, 17]\) yield a non trivial Fluctuation Dissipation Ratio (FDR) characterizing the violation of the Fluctuation Dissipation Theorem (FDT) \([17, 18]\). It has been computed exactly for the spherical model in \(d > 2\) \([13]\), using dynamical RG methods for the pure \(O(N)\) model at criticality up to two loops in an \(\epsilon = 4 - d\) expansion \([15]\), and up to one loop for the critical diluted Ising model in a \(\sqrt{\epsilon}\) expansion \([16]\).

Another standard definition for the autocorrelation exponent \(\lambda_C\) \([12, 50, 51]\) and for the autoresponse exponent \(\lambda_R\) \([52]\) is:

\[
C_{t\nu}^q = \tilde{t}^{d-\lambda_C} \phi_C(q^2 \tilde{t})
\]

\[
R_{t\nu}^q = \tilde{t}^{d-\lambda_R} \phi_R(q^2 \tilde{t})
\]

in the limit \(\tilde{t} \to \infty\), \(\tilde{q} \to 0\) with \(\tilde{t}\) fixed and \(\tilde{q}\) fixed, with \(\phi_{R,C}(0) = Cst\). Assuming the behaviors \([16, 17]\) one finds the connection:

\[(d - \lambda_C) z^{-1} = \theta - 1 + (2 - \eta) z^{-1}\]

\[
\lambda_R = \lambda_C
\]

\[
\phi_C(v) = T v^{\frac{d-2}{2}} F_{C,\infty}(v) (t')^{1-\theta}
\]

\[
\phi_R(v) = v^{\frac{d-2}{2}} F_{R,\infty}(v) (t')^{\theta}
\]

which seems to hold for pure models, together with the inequality \(d/2 \leq \lambda_C = \lambda_R\) \([49, 52]\).

For the nonequilibrium dynamics of the CM model, we obtain similar scalings \([13, 113]\), \((\eta = 0\) in this case because of STS) but with a different asymptotic behavior at large \(u\) of the scaling function \(F_C(v, u)\). As we will see, this has strong consequences on the FDR. Note that although \(C_{t\nu}^q\) is the full correlation function, to this order in the \(\tau\) expansion it coincides with the connected one (which is the correct one to consider e.g. to obey FDT in the equilibrium regime), the difference between the two being of order \(g^2 = O(\tau^2)\).

### A. General framework

We want to study the dynamics of the system described by \([71]\) which, at the initial time \(t_0 = 0\), is in a non equilibrium configuration \(u_{xt_0} = u_0^x\), whose statistical weight is given by \(e^{-H_0[u_0]}\) (where \(H_0[u_0] \neq H_{CO}[u_0]\)). The general framework to incorporate this feature in the MSR formalism has been developed in \([12]\) and it amounts to describe the system in terms of the generating functional \(S[u, i\dot{u}] \to S[u, i\dot{u}] + H_0[u_0]\). If the system is prepared in a high temperature state, with short range correlations \(\langle u_0^x u_0^{x'} \rangle = m_0^{-2} \delta^4(x - x')\), the corresponding \(H_0[u_0]\) is given by

\[
H_0[u_0] = \frac{m_0^2}{2} \int_x (u_0^x)^2
\]

any addition of anharmonic terms in \(H_0[u_0]\) is irrelevant as long \(m_0^2 \neq 0\). Moreover by power counting one has that \(m_0^2\) is irrelevant \([42]\), so that to study the leading scaling behavior it is sufficient to assume \(m_0^{-2} = 0\), i.e. \(u_0^x = 0\). The effect of this nonequilibrium initial condition is then completely encoded in the lower bound \(t_0 = 0\) on the time integrals in the MSR functional \([72]\). The running bare response and correlation functions are given by \([42]\):

\[
R_{tt'}^q = \theta (t - t') e^{-q^2 (t-t')} \left( c(\frac{q^2}{2\Lambda_0^2}) - c(\frac{q^2}{2\Lambda_1^2}) \right)
\]

\[
C_{tt'}^q = \frac{T}{q^2} \left( e^{-q^2 (t-t')} - e^{-q^2 (t+t')} \right) \left( c(\frac{q^2}{2\Lambda_0^2}) - c(\frac{q^2}{2\Lambda_1^2}) \right)
\]

### B. Nonequilibrium response function

In order to compute the response function, we solve perturbatively the equation for \(R_{tt'}^q, C_{tt'}^q\) using the trick explained above, i.e. replacing the exact \(R_{tt'}^q\) in the rhs of \([79]\) by its bare value. Doing this, we obtain a perturbative expansion of the exponents \(z\) (already obtained previously \([52]\)), \(\theta\) and of the scaling function \(F_R(v, u)\) in the same spirit as \([103]\). Indeed, as shown in Appendix \([14]\) one has the scaling \([112, 140]\), in terms of the scaling variables \(v = \tilde{q}^2 (\tilde{t} - \tilde{t}')\) and \(u = \tilde{t}/\tilde{t}'\) with

\[
F_R(v, u) = F_R^0(v) + \tau F_R^1(v, u) + \tau^2 F_R^{1\text{noneq}}(v, u)
\]

\[
\theta = e^{\gamma \tau} + O(\tau^2)
\]

which is established by comparison with the direct perturbative expansion of \([113]\) in powers of \(\tau\):

\[
R_{tt'}^q = F_R^0(v') + (z - 2) \ln \tilde{q} (F_R^0(v') + v' F_R^0(v')) + \theta \ln u F_R^0(v') + \tau F_R^1(v', u) + O(\tau^2)
\]

with \(v' = \tilde{q}^2 (\tilde{t} - \tilde{t}')\) and \(F_R^{1\text{noneq}}(v, u)\) given in \([110]\) has a complicated expression left in the Appendix \([118]\). However its asymptotic behaviors, which we now focus on have remarkably simple forms. First, in order to compare with the prediction for pure critical systems one is interested in the limit of large \(u\), keeping \(v\) fixed. This defines \(F_{R,\infty}(v)\) \([110]\) which, we find to be:

\[
F_{R,\infty}(v) = e^{-v} + e^{\gamma \tau} \tau \left\{ -\sqrt{\pi v} \operatorname{Erf} \sqrt{v} 
\right.
\]

\[
- e^{-v} \left( (1 - v) \ln (4\pi v^2) 
\right.
\]

\[
- 2v(v - \frac{1}{2}) z F_2\left(\{1, 1\}, \{\frac{3}{2}, 2\}, v\right) \right\} + O(\tau^2)(125)
\]

where \(\operatorname{Erf}(z)\) is the error function and \(z F_2\left(\{1, 1\}, \{\frac{3}{2}, 2\}, z\right)\) is a generalized hypergeometric series \([53, 55, 54]\). This shows that the response function has a scaling behavior as predicted for pure systems at a critical point \([110]\). The small \(v\) behavior of \(F_{R,\infty}(v) \sim 1 - e^{\gamma \tau} \tau \ln v\) shows that \(\phi_R(v)\) \([120]\) has a
good limit when \( v \to 0, \phi_R(0) = Cst, \) and this gives the autocorrelation exponent \( \lambda_R \).

\[
\lambda_R = 2 + O(\tau^2)
\]  

(126)

It is also interesting to analyze the asymptotic behavior in the limit of large \( v \) (and in particular \( v \gg u \)), keeping \( u \) fixed. This limit is relevant e.g. to study the behavior at fixed \( q \), large \( t, t' \) with \( u = \tilde{t}/\tilde{t}' \) fixed. It is obtained from (119) as explained in the Appendix. The behavior of the response function in this limit is then given by

\[
\lim_{v \to \infty, \tilde{u} \text{ fixed}} F_R(v, u) \sim e^{-v} + \frac{\tau}{v^2} P_R(u) + O(\tau v^{-3}, \tau^2)
\]

(127)

Notice that in the limit \( u \to 1 \), we recover the result of the equilibrium dynamics (107). This is more one can check from (119) that \( F_{\tilde{u}}(v, u) = O((u-1)^2) \) as \( u \to 1 \). Finally, one must keep in mind that the limits \( v \to \infty \) and \( u \to \infty \) do not commute, indeed one expects that a scaling function of \( v/u \sim q^2 t' \) interpolates between these limits, left for future investigation.

Another interesting behavior is the limit of vanishing momentum \( \tilde{q} = 0 \), the so-called diffusion mode. Although well defined, this limit is a bit peculiar due to the pre-factor \( \tilde{q}^{2-2} \) in the scaling function (118). However, the function \( F_R(v, u) \) behaves when \( v \to 0 \) in such a way to cancel this divergence as in (109) and leads to well a defined response function \( R_{\tilde{u} \tilde{v}}^{\tilde{q}=0} \) which has the scaling form

\[
\begin{align*}
R_{\tilde{u} \tilde{v}}^{\tilde{q}=0}(v, u) &= \frac{1}{(\tilde{t} - \tilde{t}')^{(z-2)/z}} \left( \frac{\tilde{t}}{\tilde{v}} \right)^{\theta} F_{\tilde{R}}^{\text{diff}} \left( \frac{\tilde{t}}{\tilde{v}} \right) \\
F_{\tilde{R}}^{\text{diff}}(u) &= F_{\tilde{R}}^{\text{diff}}(u) + \tau F_{\tilde{R}}^{\text{diff}}(u) \\
P_{\tilde{R}}^{\text{diff}}(u) &= 1 \\
P_{\tilde{R}}^{\text{diff}}(u) &= 2e^{\gamma_E} \ln \left( \frac{1 + \sqrt{u}}{2\sqrt{u}} \right)
\end{align*}
\]

which is identified with the perturbative expansion of \( R_{\tilde{u} \tilde{v}}^{\tilde{q}=0} \) straightforwardly obtained from the general expression (119).

C. Nonequilibrium correlation function.

To compute the correlation function, instead of solving the equation for \( C_{\tilde{u} \tilde{v}}^{\tilde{q}}(v, u) \), we obtain it using the following formal solution for \( \tilde{t} > \tilde{v} \)

\[
C_{\tilde{u} \tilde{v}}^{\tilde{q}} = \lim_{\tilde{t} \to \infty} C_{\tilde{u} \tilde{v}}^{\tilde{q}}(v, u)
\]

(129)

\[
= 2T \int_{0}^{\tilde{v}} dt_1 R_{\tilde{t}_1 \tilde{v}}^{\tilde{q}}(v, u) + \int_{0}^{\tilde{v}} dt_1 \int_{0}^{\tilde{v}} dt_2 R_{\tilde{t}_1 \tilde{t}_2}^{\tilde{q}}(v, u) + \int_{0}^{\tilde{v}} dt_1 D_{\tilde{t}_1 \tilde{t}_2} R_{\tilde{t}_1 \tilde{t}_2}^{\tilde{q}}(v, u)
\]

where \( D_{\tilde{t}_1 \tilde{t}_2} = \lim_{\tilde{u} \to \infty} D_{\tilde{t}_1 \tilde{t}_2} \) is defined in (118) and explicitly given in (119), that we expand perturbatively using the expression we obtained for \( R_{\tilde{u} \tilde{v}}^{\tilde{q}=0} \). In the Appendix, we show that \( C_{\tilde{u} \tilde{v}}^{\tilde{q}} \) has the following scaling form with

\[
F_C(v, u) = F_C^0(v, u) + \tau F_C^1(v, u)
\]

(130)

\[
F_C^0(v, u) = e^{-v} - e^{-v + \frac{\pi v}{2}}
\]

and \( F_C^1(v, u) \) given in Appendix. Again, this is established by identifying the direct perturbative expansion of (118):

\[
\begin{align*}
C_{\tilde{u} \tilde{v}}^{\tilde{q}}(v, u) &= \frac{T}{\tilde{q}^2} \left( F_C^0(v', u) + (z - 2) \ln(\tilde{q})v' \frac{\partial F_C^0(v', u)}{\partial v'} \right) \\
&\quad + \theta \ln u F_C^1(v', u) \tau F_C^1(v', u)
\end{align*}
\]

(131)

with \( v' = \tilde{q}^2(\tilde{t} - \tilde{v}) \), which is similar to the scaling form expected for pure systems at a critical point (117). However, the large \( u \) behavior is different, indeed one has in the large \( u \) limit, keeping \( v \) fixed

\[
\lim_{u \to \infty} F_C(v, u) \sim 2e^{-v} + \tau F_C^1(v)\sqrt{u} + O(u^{-2}, \tau u^{-1}, \tau^2)
\]

(132)

\[
F_C^1(v) = e^{\gamma_E} e^{-v} \sqrt{\pi u} \text{Erfi} \sqrt{v}
\]

which decays more slowly than the predicted scaling for pure system at a critical point (117). Besides, using (129)

\[
F_C^1(v) \sim v + O(v^2), \phi_C(0) = Cst
\]

(120)

this defines the autocorrelation exponent \( \lambda_C \):
with the asymptotic behaviors
\[ F_C^{\text{diff}}(u) \sim 1 + \tau e^{\gamma_E} \ln(u - 1) \quad u \to 1^+ \]
\[ F_C^{\text{diff}}(u) \sim 1 + \tau e^{\gamma_E} \sqrt{\frac{u}{v}} \quad u \gg 1 \]

These behaviors are such that the singularity as \( \tilde{t} - \tilde{t}' \to 0 \) cancels and one finds that the diffusion of the zero mode become anomalous at large time:
\[ C^{q=0}_{\tilde{t}'t} \sim A \tilde{t}^{2/z} \]
\[ A = 2T_c + O(\tau) \]

this formula being valid for \( \tilde{t} - \tilde{t}' \ll \tilde{t}' \), the random walk result being recovered when \( z = 2 \).

D. Fluctuation Dissipation Ratio.

We now give the results for The Fluctuation Dissipation Ratio (FDR) \( X_{\tilde{t}'t}^q \) defined by
\[ \frac{T}{X_{\tilde{t}'t}^q} = \frac{\partial_t C_{\tilde{t}'t}^q}{R_{\tilde{t}'t}^q} \]

Starting from the scaling laws that we established above, we can compute the FDR \( X_{\tilde{t}'t}^q \equiv X_{\tilde{t}'t}^q \) as a function of the scaling variables \( \tilde{q}^2(\tilde{t} - \tilde{t}') \) and \( \tilde{t}/\tilde{t}' \). As we saw previously, both the exponent \( z \) and the scaling function associated with the FDR will have an expansion in powers of \( \tau \), i.e.
\[ \frac{T}{X_{\tilde{t}'t}^q} = F_X(\tilde{q}^2(\tilde{t} - \tilde{t}'), \frac{\tilde{t}}{\tilde{t}'}, \tau) \]
\[ F_X(v, u) = F_X^0(v, u) + \tau F_X^1(v, u) + O(\tau^2) \]

the expansion of \( z \) to order \( \tau \) being given by [73].

\( F_X(v, u) \) corresponds to the Gaussian model and from the perturbative expansions that we obtained for \( R_{\tilde{t}'t}^q \) [124] and \( C_{\tilde{t}'t}^q \) [131] one can identify (perturbatively) this scaling form, with
\[ F_X^0(v, u) = 1 + e^{-2\frac{u}{v}} \]
\[ F_X^1(v, u) = -\frac{\theta u - 1}{\tau} \frac{v}{u} (1 - e^{-2\frac{u}{v}}) - e^v F_R^1(1 + e^{-2\frac{u}{v}}) \]
\[ -e^v (\frac{\partial F_C^0(v, u)}{\partial u} + \frac{u(u - 1)}{v} \frac{\partial F_C^0(v, u)}{\partial u}) \]

Inserting the formulae for \( F_R^0 \) and \( F_R^1 \) obtained in the Appendix yields the general result for \( F_X \) as a non trivial function of the two variables \( u, v \). Here we only give the behaviors of this scaling function in the different asymptotic limits studied previously. First, we note that this formula gives back the FDT result \( F_X = 1 \) for \( u = 1 \).

Second, focusing on the limit \( u \gg 1 \), keeping \( v \) fixed one has
\[ F_X(v, u) = 1 + e^{-2\frac{u}{v}} + \frac{\sqrt{\pi}}{2} e^{\gamma_E} \tau \sqrt{\frac{u}{v}} \text{Erfi} \sqrt{v} + O(\tau u^0, \tau^2) \]

Thus in this regime \( X \) decreases below its FDT value \( X_{\text{FDT}} = 1 \). Looking at this result one is tempted to conclude that \( X_{\tilde{t}'t}^q \) vanishes as \( t/t' \to \infty \) when \( q^2(t - t') \) is kept fixed. In particular for \( q = 0 \) (see below the direct calculation in this case) one finds the analogous quantity \( X_{\text{FDT}}^q \) computed in [15, 17] for several models. However one must keep in mind that [141] is perturbative in \( \tau \) and the divergence of the coefficient of \( \tau \) could also be a sign of a non analyticity in \( \gamma \) of the \( u = \infty \) result. Elucidation of this point is left for future study.

In the other limit that we studied previously, corresponding to \( v \gg 1 \), keeping \( u \) fixed, we obtain straightforwardly the following behavior
\[ F_X(v, u) = 1 + e^{-2\frac{u}{v}} - \frac{e^{\gamma_E} \tau v(u - 1)^2}{2\sqrt{u}} + O(\tau v^{-3}, \tau^2) \]

This limit is relevant to study fixed \( q \). It shows that there is still aging behavior in a given non zero mode, and appears to contradict some claims [15] that only the zero mode (diffusion) exhibits interesting aging behavior. Note also that in this regime one has \( X > X_{\text{FDT}} \), a feature found in other disordered models [51].

Finally in the limit of vanishing momentum \( \tilde{q} = 0 \), the FDR is a function of the scaling variable \( \tilde{t}/\tilde{t}' \) whose perturbative expansion is given by
\[ \frac{T}{X_{\tilde{t}'t}^q} = F_X^{\text{diff}}(\frac{\tilde{t}}{\tilde{t}'}) \]
\[ F_X^{\text{diff}}(u) = F_X^{\text{diff}0}(u) + \tau F_X^{\text{diff}1}(u) + O(\tau^2) \]
\[ F_X^{\text{diff}0}(u) = 2 \]
\[ F_X^{\text{diff}1}(u) = 2 F_R^{\text{diff}1}(u) - 2 u \frac{d F_C^{\text{diff}1}(u)}{du} - 2 F_R^{\text{diff}1}(u) \]
\[ + 2(z - 2) \frac{\theta}{\tau(u - 1)} - 2 \frac{\theta}{\tau} \]

Using the results of previous Sections we find:
\[ \frac{T}{X_{\tilde{t}'t}^q} = 2 + 2 \tau e^{\gamma_E} (\sqrt{\frac{u}{v}} + \ln(\frac{\sqrt{u} - 1}{\sqrt{v} + 1}) + \sigma) \]

where \( \sigma \) is a numerical constant. This constant depends on additive constants to respectively \( F_R^0 \) and \( F_C^0 \), each of them being nonuniversal as discussed above (see Appendix). However, a distinct possibility is that \( F_X^{\text{diff}}(u) \) is universal (i.e. that the non universal parts cancel). Checking this can be done with the present method, and is left for future study. The value obtained here, \( \sigma = 5 - 12 \ln 2 \), may only be indicative since we did not keep track of all additive constants. In particular, in the scaling regime \( \tilde{t} \gg \tilde{t}' \gg 1 \), one obtains
\[ \frac{T}{X_{\tilde{t}'t}^q} \sim 2 + 2 \tau e^{\gamma_E} \sqrt{\frac{u}{v}} + O(\tau u^0, \tau^2) \]

Notice that taking the limit \( v \to 0 \) (using [123]) on the asymptotic expression [111] where we have taken the
limit \( u \gg 1 \) before \( v \) small one recovers the same result \((146)\).

One way to understand the result \((145)\), i.e. the divergence of \( X^\tilde{q}_t=0 \) when \( t' \to t \) is to note that the same divergence occurs for a simple diffusion process with the same close times asymptotic behaviors:

\[
C^\tilde{q}_t=0 \sim t'^{2/z} \quad (147)
\]
\[
R^\tilde{q}_t=0 \sim (t-t')^{(2-z)/z} \quad (148)
\]

which yields straightforwardly \( X^\tilde{q}_t=0 \sim A(u-1)^{(2-z)/z} \) as \( u \to 1 \). Note however that to obtain the correct amplitude \( A \) one needs to take into account further corrections to \( C^\tilde{q}_t=0 \), specifically we note that one can rewrite \((146)\) as:

\[
C^\tilde{q}_t=0 = t'^{2/z} A(u) \quad (149)
\]

and that the detailed asymptotics of \( A(u) \) near \( u = 1 \) determines the amplitude of the divergence.

\section*{VII. CONCLUSION}

In this paper we have developed a novel EMRG method to perform first principle perturbative calculations based on exact RG. Contrarily to previous works it is based on a multilocal expansion of the effective action functional. It allows to perform conveniently calculations with an arbitrary cutoff function in a fully controlled way and check explicitly the universality of the observables.

We have tested the method on the standard \( O(N) \) model. We have shown that the exponent \( \eta \) to order \( O(e^2) \) can be simply recovered within the exact RG multilocal expansion. This is interesting since previous approaches relied on approximations such as polynomial and derivative expansions, which are not needed here. We have also obtained several two-point scaling functions and explicitly checked universality. Finally, we explained how the method compares with more standard field theoretical approaches. In a sense the present method directly yields the renormalized theory.

We have applied the EMRG method to study the glass phase of the two dimensional random Sine-Gordon model (Cardy-Ostlund) near the glass transition temperature. We have first recovered known results for the statics and for the equilibrium dynamical exponent \( z \) which we showed to be universal. The method of derivation however is quite different from previous ones, since it yields directly the self energy \( \Sigma_l(\omega) \) as a scaling function of \( \omega/\Lambda_l \) where \( \Lambda_l \) is the infrared cutoff. We have given for the first time the scaling functions associated to finite momentum equilibrium response and correlation.

Next we studied the out of equilibrium dynamics of the Cardy-Ostlund model. We obtained the two time response and correlations at finite momentum. These were found to take a scaling form and we computed analytically the corresponding scaling functions which depend on two arguments \( v = \tilde{q}^2(t - t') \) and \( u = t/t' \). We showed

that they exhibit aging behavior characterized by a non trivial fluctuation dissipation ratio \( X \), itself a universal function of \( u, v \) that we obtained. We also obtained the off equilibrium exponents \( \theta \) and \( \lambda \). Interestingly we found that, at variance with pure systems, one must introduce two distinct exponents \( \lambda_R \) and \( \lambda_C \) for response and correlation respectively. Our study raises the question of whether this could be a more general property of glassy dynamics in disordered systems.

Our method is promising for further RG studies of disordered systems, as it allows to attack the problem with few assumptions. Other situations where it can be applied are elastic manifolds in random media, where it can be used to put the so-called Functional RG on a more solid basis \((58, 59)\). Concerning the results of the present paper, a numerical simulation of the Cardy Ostlund glass phase can be performed \((60)\) and should provide an interesting test of the predictions of our RG calculation. In particular, some points require further examination, e.g. the asymptotic value \( X_\infty \) of the FDR. This would be interesting especially in the light of the present activity on FDR in mean field models, and interpretations in terms of effective temperatures. Indeed developing real space, RG type methods beyond mean field remains a challenge in the theory of glasses.
APPENDIX A: EXACT RG EQUATION FOR THE EFFECTIVE ACTION

Here we present a simple derivation of the exact RG equation satisfied by the effective action, denoted here \( \Gamma_G(\phi) \) (and \( \Gamma(\phi) \) in the text), for the theory of action given in (1), when the propagator \( G \) is varied, for a fixed interacting functional \( V(\phi) \). One first introduces the generating functional:

\[
Z_G(j) = \int D\phi \ e^{-\frac{1}{2} \phi G^{-1} : \phi - V(\phi) + j : \phi} \tag{A1}
\]

i.e. the partition function in presence of a set of sources denoted \( j \equiv j^x \). For any variation \( \partial G \) of \( G \), its variation \( \partial Z_G(j) \) satisfies:

\[
\partial Z_G(j) = \frac{1}{2} \text{Tr} \partial G^{-1} \int D\phi \phi \phi e^{-\frac{1}{2} \phi G^{-1} : \phi + V(\phi) + j : \phi} \tag{A2}
\]

\[
\partial G = \frac{1}{2} \text{Tr} \partial G^{-1} \frac{\delta^2 Z_G(j)}{\delta j \delta j}
\]

where \( \partial G^{-1} = -G^{-1} \partial G^{-1} \) and \( \text{Tr} \) denotes a trace over all spatial coordinates and indices \( x, i \). Next, one introduces the generating functional \( W_G(j) = \ln Z_G(j) \) of connected correlations, which varies as:

\[
\partial W_G(j) = \frac{1}{2} \text{Tr} \partial G^{-1} \left( \frac{\delta^2 W_G(j)}{\delta j \delta j} + \frac{\delta W_G(j)}{\delta j} \frac{\delta W_G(j)}{\delta j} \right) \tag{A3}
\]

an exact RG equation for this quantity. From there it is simple to obtain the RG equation obeyed by its Legendre transform \( \Gamma_G(\phi) = \min_j (\phi : j - W_G(j)) \). We will assume that no problem arise from the convexity condition and that \( \Gamma_G(\phi) \) can be obtained using only the saddle point conditions:

\[
\frac{\delta W_G(j_G(\phi))}{\delta j} = \phi \tag{A4}
\]

\[
\frac{\delta \Gamma_G(\phi)}{\delta \phi} = j_G(\phi) \tag{A5}
\]

For the variation of \( \Gamma_G(\phi) = \phi : j_G(\phi) - W_G(j_G(\phi)) \), this yields:

\[
\partial \Gamma_G(\phi) = -\partial W_G(j_G(\phi)) \tag{A6}
\]

\[
\partial \Gamma_G(\phi) = \frac{1}{2} \text{Tr} \partial G^{-1} \left( \frac{\delta^2 W_G(j_G(\phi))}{\delta j \delta j} + \frac{\delta W_G(j_G(\phi))}{\delta j} \frac{\delta W_G(j_G(\phi))}{\delta j} \right)
\]

since the term proportional to \( \partial^2 \Gamma \) cancels as usual from the saddle point conditions (A3). Using (A3) once more, as well as the standard relation \( \frac{\delta W_G(j_G(\phi))}{\delta j} = \left[ \frac{\delta \Gamma_G(\phi)}{\delta \phi} \right]^{-1} \), gives the equation (A) of the text.

Writing then

\[
\Gamma_G(\phi) = \frac{1}{2} \phi : G^{-1} : \phi + U_G(\phi) - \frac{1}{2} \text{Tr} \ln G \tag{A7}
\]

this is equivalent to the equation for \( U_G(\phi) \):

\[
\partial U_G(\phi) = \frac{1}{2} \text{Tr} \partial G : (G^{-1} - G^{-1} (1 + G : \frac{\delta^2 U_G(\phi)}{\delta \phi \delta \phi})) \tag{A8}
\]

or its equivalent form given in the text.

Now that we have an exact equation for \( \Gamma_G(\phi) \), we can relate the effective action in theories with the same \( V(\phi) \) but different \( G \). All we need to fully determine the effective action is an initial condition. It is provided by the action itself. Indeed one has the following perturbative loop expansion:

\[
\Gamma(\phi) = -\frac{1}{2} \text{Tr} \ln G + S(\phi) + \sum_{k \geq 1} \Gamma^k(\phi) \tag{A9}
\]

where \( \Gamma^k(\phi) \) is the sum of all \( k \) loop 1PI graphs using \( V(\phi) \) as interaction and \( G \) as propagator. Thus if the initial condition for the propagator \( G_{i=0} \) is such that all \( \Gamma^k(\phi) \) graphs vanish when computed with \( G_{i=0} \), then one can choose the initial condition as \( U_{i=0}(\phi) = V(\phi) \). This is the case for the choice (A8) made in the text (similarly the initial condition for \( W_G(j) \) in (A3) is the Legendre transform of the initial action \( S(\phi) \)).

Finally let us note that the RG equation can also be written as:

\[
\frac{dU_G(\phi)}{dG} = \frac{1}{2} \text{Tr} \ln (1 + G ; \frac{\delta^2 U_G(\phi)}{\delta \phi \delta \phi}) \tag{A10}
\]

\[
= \frac{1}{2} \frac{\delta^2 U_G(\phi)}{\delta \phi \delta \phi} : (1 + G ; \frac{\delta^2 U_G(\phi)}{\delta \phi \delta \phi})^{-1} \tag{A11}
\]

where the derivative \( \frac{d}{dG} \) in the r.h.s. of the first equation is restricted to the explicit \( G \) dependence (i.e. not the one implicit in \( U_G(\phi) \)).

APPENDIX B: MULTILOCAL EXPANSION TO \( O(U^2) \)

To \( O(U^2) \) one needs only \( U \) and \( V \sim O(U^2) \) in the expansion (10) of \( U \). The functional derivative reads:

\[
\frac{\delta U}{\delta \phi_i \delta \phi_j} = \delta_{ij} \partial_i \partial_j U(\phi_x) + \int \left( \partial_i^2 \partial_j^2 V(\phi_x, \phi_y, x - z) + \partial_i^4 \partial_j^4 V(\phi_x, \phi_y, x - y) + O(W) \right)
\]

Using parity \( V(\phi, \psi, -x) = V(\phi, \psi, x) \). Inserting in (A) and keeping only terms up to order \( O(U^2) \) one finds the resulting RG equation:
\[ \partial_t U_l(\phi) = \frac{1}{2} \partial G^{\tau=0}_{ij} \partial_i \partial_j U_l(\phi) + \int_x \partial G^{\tau}_{ij} \partial_i \partial_j V_l(\phi, x) - \frac{1}{2} \int_x \partial G^{\tau}_{ij} \partial_j \partial_k U_l(\phi)(G^\tau_{km}) \partial_m \partial_i U_l(\phi) \] (B2)

\[ \partial_t V_l(\phi, \psi, x) = -\frac{1}{2} \partial G^{\tau}_{ij} \partial_i \partial_j U_l(\phi)(G^\tau_{km}) \partial_m \partial_i U_l(\psi) \]

\[ + \frac{1}{2} \partial G^{\tau=0}_{ij} (\partial_i \partial_j + \partial^2 \partial^2) V_l(\phi, \psi, x) + \partial_l \partial^2 \partial G^{\tau}_{ij} V_l(\phi, \psi, x) \]

\[ - \delta(x) \int_y \partial G^{\tau}_{ij} V_l(\phi, \psi, y) + \frac{1}{2} \delta(x) \int_y \partial G^{\tau}_{ij} \partial_j \partial_k U_l(\phi)(G^\tau_{km}) \partial_m \partial_i U_l(\psi) \] (B3)

where the local projection $\mathcal{T}_l$ operator has been applied to obtain the first equation, and the operator $1 - P_l$ to obtain the second. This is illustrated in Fig. 2 (dropping all terms of order $O(U^3)$ and higher). Note that \( \int_x V_l(\phi, \psi, x) = 0 \). The differential equation for the bilocal part $V_l$ is linear, a general property which allows to solve all higher multilocal components (here $V_l$) as a function of the local part $U_l$ only. The equation for $V_l$ can be integrated in the forms \([12, 13]\) given in the text. The method is similar to \([1]\) to which we refer for further details. Inserting this solution in the equation for $U_l$ one obtains \([11]\) in the text. We have assumed that no bilocal term exists in the original action. Near the fixed point form at large $l$ these assumptions are not strictly necessary, a statement which can be checked using the present method.

It can be useful, in particular for the Cardy Ostlund model that we study in the text, to introduce a Fourier representation in the fields:

\[ U_l^K = \int d\phi e^{-iK.\phi} U_l(\phi) \]

\[ V_l^{KPx} = \int d\phi d\psi e^{-iKP.\phi - iKP.\psi} V_l(\phi, \psi, x) \] (B4)

Using this representation, we obtain the RG equations \([12, 14]\) in Fourier space

\[ V_l^{KPx} = \frac{1}{2} (F_l^{KPx} - \delta(x) \int_y F_l^{KPx}) \] (B5)

\[ F_l^{KPx} = -\int_0^l \langle L(K.\partial G^{\tau}_{ij}.P)(K.\partial G^{\tau}_{ij}.P) \rangle \delta(K - P) \]

\[ \partial_t U_l^K = -\frac{1}{2} K.\partial G^{\tau=0}_{ij}.K U_l^K - \frac{1}{2} \int_{P, Q, P + Q = K} \langle P(\partial G^{\tau}_{ij}.Q)(P.G^\tau_{ij}.Q)U_l^K U_l^Q \rangle \]

\[ + \frac{1}{2} \int_{P, Q, P + Q = K} \langle P(\partial G^{\tau}_{ij} - \partial G^\tau_{ij}).Q \rangle \int_0^l \langle L(P.\partial G^{\tau}_{ij}.Q)(P.G^\tau_{ij}.Q) \rangle \delta(K - P - Q) \] (B6)

where \( \int_{P, Q, P + Q = K} \equiv \int \frac{dN_P d^N \delta_0(Q) \delta(K - P - Q)}{(2\pi)^N} \delta_l(P - Q) \) where $N$ is the number of components of $\phi$. In the text we have used \( V_l^{KPx} \) to distinguish the Fourier series coefficients from the Fourier transform.

**APPENDIX C: DETAILED CALCULATIONS FOR THE $O(N)$ MODEL**

1. $\beta$-function

Let us insert \([18]\) into the ERG equation \([14]\), keeping only $g_0$, $g_2$ and $g_4$ for now, and first focus on the first line in \([14]\) which reads:

\[ \partial_t g_{0,1} + \frac{g_2 + 3g_4}{2!} \Delta^2 \partial^2 + \frac{g_4 + 3g_4}{4!} \Delta^4(\partial^2)^2 = \]

\[ \frac{1}{2} \int_q \partial G^\tau_{ij} \partial_i \partial_j U_l(\phi) - \frac{1}{2} \int_q \partial G^\tau_{ij} G^\tau_{ij}(\partial_i \partial_j U_l(\phi))^2 \] (C1)

with implicit sums on repeated indices. Using that:

\[ \partial_i \partial_j U_l(\phi) = g_{2,l} \delta_{ij} + \frac{g_{4,l}}{3!} (\delta_{ij} \phi^2 + 2\phi^i \phi^j) \] (C2)
which yields:
\[
\begin{align*}
\partial_i \partial_j U_l &= N g_{2,l} + \frac{N + 2}{3!} g_{4,i} \phi^2 \\
\partial_i \partial_j \partial_k U_l(\phi_1) \partial_l \partial_k \partial^2 U_l(\phi_2) &= N g_{2,l}^2 \\
&+ \frac{N + 2}{3!} g_{2,i} g_{4,l} (\phi^2 + \phi_2^2) \\
&+ \frac{(3!)^2}{2} \partial^2 g_{4,l} ((N + 4) \phi_1^2 \phi_2^2 + 4 (\phi_1 \cdot \phi_2)^2) 
\end{align*}
\]
Setting \( \phi_1 = \phi_2 \) in (C8) and identifying the coefficients of \( \phi^2 \) and \((\phi^2)^2\), one then easily obtains all terms in (21) apart from the last one, with the scaled integrals defined in (22).

Inserting now (18) into the second line of the ERG equation (14) one obtains only a correction to \( g_2 \) from the term with the lowest number of derivatives (six). Noting that:
\[
\partial_i \partial_j \partial_k U_l(\phi) = \frac{g_{4,l}}{3} (\delta_{ij} \phi^k + \delta_{ik} \phi^j + \delta_{jk} \phi^i) 
\]
one finds:
\[
\sum_i (\partial_i \partial_i \partial_j g_{4,l} U_l(\phi_1) U_l(\phi_2) = \frac{g^2_{4,l}}{3} (N + 2) \phi_1 \cdot \phi_2 
\]
yielding the last term in (21)
\[
- \frac{N + 2}{3} \int_0^l d l' \tilde{I}_{l, l'}^{(2)} g_{4,l'}^2 
\]
with
\[
\tilde{I}_{l, l'}^{(2)} = \Lambda_{l'}^{-2} \int x (\partial_i G^e_{l'} - \partial_i G_{l'}^{e=-0}) \partial_i G^e_{l'} \partial_i G^e_{l'} \Lambda_{l'}^{2e} 
\]
This term does not modify the fixed point value \( \tilde{g}_{2}^* \) to order \( \epsilon \), provided it remains finite in the limit \( l \to \infty \). A way to compute it is to make an integration by part to treat the integral over \( l' \):
\[
\int_0^l d l' \partial_i G^e_{l'} \partial_i G^e_{l'} \partial_i g_{4,l'} \Lambda_{l'}^{2e} 
\]
\[
= \frac{1}{2} (G^e_{l'})^2 g_{4,l'} \Lambda_{l'}^{2e} - \int_0^l d l'' (G^e_{l''})^2 g_{4,l''} \Lambda_{l''} \partial_i (\tilde{g}_{4,l''} \Lambda_{l''}) 
\]
\[
= \frac{1}{2} (G^e_{l'})^2 g_{4,l'}^{2e} + O(\epsilon) 
\]
as from Eq. (20) \( \partial_i (\tilde{g}_{4,l'} \Lambda_{l'}) \) is of order \( \tilde{g}_{4,l'}^{2e} \) and where we have used \( G^e_{l''} = 0^+ \). The terms we dropped are of order \( \epsilon^5 \) in the limit \( l \to \infty \). Finally, in the large \( l \) limit we are left with
\[
\int_0^l d l' \tilde{I}_{l, l'}^{(2)} g_{4,l'}^{2e} = \frac{g_{4,l'}}{2} \Lambda_{l'}^{-2} \int x (\partial_i G^e_{l'} - \partial_i G_{l'}^{e=-0}) (G^e_{l''})^2 + O(\epsilon) 
\]
which is already of order \( \epsilon^2 \), so that the integral over \( x \) can be performed in \( d = 4 \) exactly. Using the decomposition of the cutoff (17), we compute the following integrals exactly in \( d = 4 \)
\[
G^e_{l'} = \frac{1}{4 \pi^2} \int_0^l \frac{1}{a} \left( e^{-x^2 \Lambda_{l'}^{2e}/2a} - e^{-x^2 \Lambda_{l'}^{2e}/2a} \right) 
\]
\[\Lambda_{l'}^{-2} \partial_i G^e_{l'} = \frac{1}{4 \pi^2} \int_0^l \frac{1}{a^2} e^{-x^2 \Lambda_{l'}^{2e}/2a} \]
Eq. (C10) can finally be written as an integral over the rescaled variable \( \tilde{x} = \Lambda_l x \)
\[
\frac{N + 2}{3} \int_0^l d l' \tilde{I}_{l, l'}^{(2)} g_{4,l'}^{2e} \propto g_{4,l'}^{2e \tilde{G}_{l'}^{e=0}} \int_x \left( e^{-x^2/2a} - 1 \right) \frac{1}{a \tilde{x}^4} \left( \int e^{-\left(\Lambda_0/\Lambda_l\right)^2 \tilde{x}^2/(2a)} - e^{-\tilde{x}^2/2a} \right)^2 
\]
In the limit \( l \to \infty \), this integral is well defined. Indeed, there is no UV divergence (due to the term \( e^{-\tilde{x}^2/2a} \) which behaves as \( \tilde{x}^2 \)) nor IR divergence due to the term \( e^{-\tilde{x}^2/2a} \).

By the same calculation one obtains the flow of the free energy:
\[
\partial_l g_{0,l} = \frac{N}{2} \Lambda_{l'}^{-d} \left( \tilde{I}_{l, l'}^{(1)} g_{2,l} - \tilde{I}_{l, l'}^{(1)} g_{2,l}^{e=0} \right) \\
+ \frac{N + 2}{3} \Lambda_{l'}^{-d} \int_0^l \tilde{I}_{l, l'}^{(3)} g_{4,l'} 
\]
with:
\[
\tilde{I}_{l, l'}^{(3)} = \Lambda_{l'}^{-d} \int_x \left( \partial_i G^e_{l'} - \partial_i G_{l'}^{e=0} \right) \partial_i G^e_{l'} \partial_i G^e_{l'} \Lambda_{l'}^{2e} 
\]
We finally obtain the flow for \( \tilde{g}_{6,l} \) in (15) with the same kind of manipulations, and using furthermore
\[
\partial_i \partial_j ((\phi^2)^3) = 6 \delta_{ij} (\phi^2)^2 + 24 \phi^i \phi^j \phi^2 
\]
\[\partial_i \partial_j \partial_k ((\phi^2)^3) = 24 \phi^2 (\delta_{ij} \phi^k + \delta_{ik} \phi^j + \delta_{jk} \phi^i) + 48 \phi^i \phi^j \phi^k \]
one gets
\[
\partial_l \tilde{g}_{6,l} = (2 \epsilon - 2 \tilde{g}_{6,l} - (N + 14) \tilde{I}_{l, l'}^{(1)} \tilde{g}_{4,l} \tilde{g}_{6,l} \\
- \frac{8}{5} (3N + 16) \int_0^l d l'' \tilde{I}_{l', l''} \tilde{g}_{4,l''}^2 + O(\tilde{g}_{4,l}) 
\]
which shows that \( \tilde{g}_{6,l} \sim \epsilon^3 \). Similarly there is a term proportional to \( \tilde{I}_{l, l'}^{(1)} \tilde{g}_{4,l} \tilde{g}_{6,l} \) in the flow equation of \( \tilde{g}_{4,l} \) which affect the fixed point value \( \tilde{g}_{2}^* \) only to next order in \( \epsilon \).

2. Computation of the exponent \( \eta \)

The quadratic term in (20) is obtained by inserting (15) in (14) and expanding the exponential in (13) to order one. One gets, using (C8)
\[
F_l(\phi_1, \phi_2, x) = \int_0^l d l' \partial_i G^e_{l'} \partial_i G^e_{l'} \tilde{I}_{l, l'}^{(1)} g_{4,l'} (\partial_i \partial_i \partial_i g_{4,l'} (G^e_{l'})^2 + O(\epsilon)) 
\]
\[
= \frac{N + 2}{3} \phi_1 \cdot \phi_2 \int_0^l d l' \partial_i G^e_{l'} \partial_i G^e_{l'} \tilde{I}_{l, l'}^{(1)} g_{4,l'} \Lambda_{l'}^{2e} 
\]
(C15)
which is the second line of (29). We have dropped terms of the form $f(\phi_1, x)$ such as $g_2(\phi_1, \phi^2_2)$, $g_2(\phi_1, \phi^4_2)$ (resulting from the expansion of the exponential in (13) to order 0), or $\phi^2_1, \phi^4_2, \phi^6_2$ (resulting from the expansion of the exponential in (13) to order 1 but acting respectively with $\partial^2 G_{\eta \eta}^{(0)} \partial \phi^2$ or $\partial^1 G_{\eta \eta}^{(0)} \partial \phi^1$) because they do not give any contribution to the effective action. Indeed, the contribution of such terms to the interaction functional $U_\phi^{(2)}$ will be

$$V_l(\phi_x, \phi_y, x - y) = f(\phi_x, x - y) - \delta(x - y) \int_z f(\phi_x, z)$$

$$U_l(\phi) = \int \left( f(\phi_x, x - y) - \delta(x - y) \int_z f(\phi_x, z) \right)$$

where we assumed parity $f(\phi_1, x) = f(\phi_1, -x)$ (which is the case here) and translational invariance. To treat the integral over $l'$ in (C15), we use an integration by part as in (C8), one gets

$$\frac{N + 2}{3} \phi_1 \phi_2 \int_0^l d^4 \phi_i G^2 \Gamma^2 \Gamma^2 G^2 \Gamma^2 G^2 \Lambda^2 + N + 2 \frac{2}{18} \phi_1 \phi_2 (G^2 \Gamma^2) + O(\phi^2_4, \phi^4_4, \Lambda^2)$$

which leads to (30). Using (C10), the last term in (30) (for getting the discussion the numerical prefactor)

$$H(q, 1, 1) = \int x \left( e^{i q x} - 1 \right) (G^2 \Gamma^2)^3$$

where the integral over $x$ is evaluated in $d = 4$ (as this term is already of order $\frac{2}{18}$). For any $1, A_1$, this integral is well defined but in the limit $A_1 \to \infty$, the integrand is not any more regularized at small $x$ and there is a logarithmic divergence. We are interested in the limit $q, A_1 \ll A_0$. A simple way to isolate this divergence is to rewrite it as

$$H(q, A_0, 1) = \frac{1}{(4\pi^2)^3} \int_x \left( e^{i q x} - 1 \right) \frac{1}{x^6} \left( \int_a e^{-x^2 A_0^2/2a} - e^{-x^2 A_0^2/2a} \right)^3$$

where in the second line we performed the change of variable $x \to \lambda x^2$ and denoted $S_4 = 2\pi^2$ the unit sphere area in dimension $d = 4$. Interestingly, we have (using the variable $u = \lambda x^2$), up to terms of order $\lambda^{-2}$:

$$h'(\lambda) = \frac{-3S_4 q^2}{16\lambda} \int_0^\infty du \frac{1}{2a} e^{-u/2a} \left( \int_a e^{-u/2a} \right)^2$$

$$= \frac{-S_4 q^2}{16\lambda} \left( \int_a e^{-u/2a} \right)^3_{u=0} + O(\lambda^{-2}) = \frac{-S_4 q^2}{16\lambda} \frac{a}{\lambda}$$

where we have used $c(0) = \int_a = 1$, which leads to $h(\lambda) \sim -\frac{\lambda^2}{\lambda} \ln \lambda + O(\lambda^{-1})$. Finally one obtains

$$H(q, A_0, 1) = \frac{q^2}{(4\pi^2)^3} \left( \ln A_0 + \chi^{(2)}(q/\Lambda_1) \right) + O(\Lambda^2)$$

$$\chi^{(2)}(q) = \frac{1}{\Lambda^2} \int_0^\infty dx (\frac{1}{x} \ln x + \chi^{(2)}(q/\Lambda_1))$$

which gives (up to the factor $-\frac{2}{18}(N + 2)/18$), the last term in (30). Using $1/x^6 = 1/2 \int_0^\infty dt 2t e^{-tu^2}$, one can compute the integral over $x$ in $\chi^{(2)}(q)$:

$$\chi^{(2)}(q) \sim \frac{1}{\Lambda^2} \int_0^\infty dt (t + \alpha_3) \chi^{(2)}(q/\Lambda_1) \sim \frac{1}{\Lambda^2} \int_0^\infty dt (t + \alpha_3)$$

with $\alpha_3 = \frac{1}{4\pi^2} + \frac{1}{12} + \frac{1}{2\pi^2}$ from which we easily obtain the asymptotic behavior

$$\chi^{(2)}(q) \sim \frac{1}{\Lambda^2} \int_0^\infty dt (t + \alpha_3) \chi^{(2)}(q/\Lambda_1) \sim \frac{1}{\Lambda^2} \int_0^\infty dt (t + \alpha_3)$$

as announced in the text (32). This yields a universal result for the $\eta$ exponent. In addition $\chi^{(2)}(q)$ gives the scaling function of the two point correlator $\chi^{(2)}(q) = 2q^2 \chi(q^2)$ where $Q(y)$ was computed in (30) in the particular case of a IR "massive" cutoff function of the form (7). Although our expression is more general we have checked through series expansion that it coincides with the expression given in (30) for that choice of the cutoff.

Performing two integrations by part one can rewrite:

$$\chi^{(2)}(q) = \int_0^\infty dt \chi^{(2)}(q/\Lambda) \frac{C(a)}{(a + b)^2} \frac{C(c)}{a^2} \frac{4}{q^2} \left( 1 - \frac{1}{\alpha_3 + \frac{1}{8\alpha_3}} \right)$$

with $C(a) = \int_0^\infty da \chi^{(2)}(a)$.}

3. Quartic contribution to $\Gamma_\phi$

The quartic term in (29) is obtained by inserting (13) and expanding the exponential in (13) to order
zero. One gets, using (C3)

\[ F_l(\phi_1, \phi_2, x) = -\int_0^l dt' \partial_t G^r_l G^r_0 \left( \sum_i \partial_t^2 \chi_i^2 \right) U_l(\phi_1) U_l(\phi_2) \]

\[ = -\left( \frac{N + 4}{3!^2} \chi_1^2 + \frac{4}{3!^2} (\phi_1 \cdot \phi_2)^2 \right) \int_0^l dt' \partial_t G^r_l G^r_0 \chi_0^2 \Lambda^r_{l'} \]  

(C19)

which is the last line in (29) (here again we have dropped terms of the form (C3)). The integral over \( \Lambda \) in (C19) is then treated as previously (C8).

Then, when computing the Fourier transform, one obtains (C10) with, using (C10)

\[ \chi^{(4)}(\bar{q}) = \int_x (e^{iqx} - 1) (G^r_l)^2 \]

\[ = \frac{1}{16\pi^2} \int_x (e^{iqx} - 1) \frac{1}{x^4} \left( \int_a e^{-x^2/2a} - e^{-x^2/2b} \right)^2 \]

For any \( \Lambda_1, \Lambda_0 \) finite, this function is well defined, and we see that the limit \( \Lambda_0 \to \infty \) is also well defined, thus

\[ \chi^{(4)}_l(q) = \frac{1}{16\pi^4} \int_x (e^{iqx} - 1) \frac{1}{x^4} \left( \int_a e^{-x^2/2a} \right)^2 + O \left( \frac{\Lambda_1^2}{\Lambda_0^2} \right) \]

the integral over \( x \) can be computed using \( 1/x^4 = \int_0^\infty dt e^{-tx^2} \), one obtains

\[ \chi^{(4)}(\bar{q}) = \frac{1}{16\pi^2} \int_0^\infty dt \frac{t}{(t + \alpha_2)^2} \left( e^{-\alpha_2 t} - 1 \right) \]

with \( \alpha_2 = 1/2a + 1/2b \), from which we extract the following asymptotic behaviors

\[ \chi^{(4)}(\bar{q}) \sim -\bar{q}^2 \int_{abc} \frac{1}{128\alpha_2} \hat{q} \ll 1 \]

\[ \chi^{(4)}(\bar{q}) \sim -\frac{1}{16\pi^2} \ln \bar{q}^2 \hat{q} \gg 1 \]

as announced in the text (37, 38).

**APPENDIX D: DETAILED CALCULATIONS FOR THE CO MODEL - STATICS.**

1. \( \beta_{g_1} \)-function

The \( \beta \)-function for the coupling constant \( g^K \) is obtained by inserting (C19) in (B9). This gives straightforwardly (50) using \( \partial_t G^r_{l=0} = -\frac{l}{2\pi} \int_0^\infty du \partial_t' (u) = \frac{l}{2\pi} \).

One has also \( G^r_{l'=0} = \frac{l}{2\pi} (l' - l) \). Considering specifically \( g_1^{(1)} = g_1 \), we first consider the possible fusion rules such that \( P + Q = K_{-1,1} \):

\[ P + Q = K_{1,-1} \]

(D1)

\[ (.) (.) (.) \]

(D2)

\[ (.) (.) (.) \]

\[ (.) (.) (.) \]

\[ (.) (.) (.) \]

where \( . \equiv 0 \), and there are \( 2(n - 2) \) different ways to choose \( P, Q \) like that, notice \( PQ = -1 \). Other possible fusions rules involve charges of higher modulus, for instance we could consider

\[ P^2 = Q^2 = 6. \]

It is then useful to write the integrals \( j^{(1)}_l \) and \( j^{(2)}_l \) in (C3) in terms of the variables \( \tilde{x} = \Lambda_l x \) and \( \mu = l - l' \). Using (50), and specifying to \( g_1 \) one has:

\[ \frac{j^{(1)}}{2T^2} \sum_{P, Q} g_1^P g_1^Q (P, Q)^2 = (n - 2) g_1^2 \int_{\tilde{x}} \partial_\gamma_0(\tilde{x}) \gamma_1(\tilde{x}) \]  

(D5)

and

\[ \frac{j^{(1)}}{2T^2} \sum_{P, Q} g_1^P g_1^Q (P, Q)^2 = (n - 2) g_1^2 \int_{\tilde{x}} (\partial_\gamma_0(\tilde{x}) - \partial_\gamma_0(0)) \int \mu \partial_\gamma_1(\tilde{x}) (\gamma_1(\tilde{x}) - \gamma_1(\tilde{x})) e^{(1 - \frac{\mu}{2})} e^{T_\gamma_0(\tilde{x}) g_1^2} \]  

(D6)
with $\sum_{P,Q} g_{P,Q} = \sum_{P,Q,P+Q=K}$. We study the flow near $T_c = 4\pi$, and as $\mu \rightarrow -\mu$, we can evaluate the integral over $\mu$ exactly at $T_c$, in particular $e^{(4-\pi/\mu)} = 1 + O(\tau)$. Moreover, as the integral is convergent, it is dominated by the vicinity of the fixed point $\mu = 0$. We can then substitute in $g_\mu$ by $g_\mu$. The remaining integral over $\mu$ is then straightforwardly computed by integration by parts. Then, together with $\mu=0$, and using $\mu$ in the limit $n \rightarrow 0$ to $(D7)$:

$$\partial_\mu g_\mu = \frac{1}{2\pi} \int_a e^{-x^2 e^{2\mu}/(2a)} dy y e^{-y}$$

and the following identities:

$$\partial_\mu \mu = \partial_\mu (\mu)$$

$$2x^2 \partial_\mu \mu = \partial_\mu (\mu) - \partial_\mu (0)$$

To compute the integrals over $\hat x$ in $(D7)$, in the limit $l \rightarrow \infty$ at $T = T_c$ we first quote some useful relations. Using the decomposition of the cut-off function $(D7)$, we have

$$\partial_\gamma (0) \int_x \gamma (x) + \frac{1}{T_c} \int_x' (\partial_\gamma (x) - \partial_\gamma (0))(e^{T_c \gamma (x)} - 1)$$

$$= \frac{1}{T_c} \int_x' \partial_\gamma (x)(e^{T_c \gamma (x)} - 1) + \int_x \partial_\gamma (0) \gamma (x) - \frac{1}{T_c} \int_x \partial_\gamma (0)(e^{T_c \gamma (x)} - 1)$$

for $l \rightarrow \infty$, $2x^2 \partial_\gamma \gamma = -\partial_\gamma (0), x > 0$ since $\partial_\gamma (0) = 0$, and together with $\partial_\gamma (0) = 2/T_c$, we are left with performing the change of variable $u = x^2$, and denoting $\gamma (x) = \tilde \gamma (x^2)$

$$\partial_\gamma (0) \int_x \gamma (x) + \frac{1}{T_c} \int_x' (\partial_\gamma (x) - \partial_\gamma (0))(e^{T_c \gamma (x)} - 1)$$

$$= \frac{2\pi}{T_c} \int_x \int_x' du u T_c \partial_\gamma (u)(e^{T_c \gamma (u)} - 1)$$

and

$$+ \frac{2\pi}{T_c} \int_x \int_x' du (\gamma (u) - u \partial_\gamma (u))$$

whith the asymptotic behaviors of $E_1 (z)$ given in $(D13)$ and $(D14)$. For any $\Lambda_1, \Lambda_2$ the integral over $x$ in $(D16)$ is well defined, but we see that in the limit $\Lambda_0 \rightarrow \infty$ (i.e. $\Lambda_1, \Lambda_2 \ll \Lambda_0$), there is a logarithmic divergence (for small $\Lambda$) and only for charges such that $K \Lambda = -2$. Indeed, at small $x$, using $(D13)$, $-T_c K \Lambda \gamma (\Lambda_1 x) \sim K \Lambda \gamma (\Lambda_2 x^2)$, leading to $e^{-T_c K \Lambda \gamma (\Lambda_1 x)} \sim x^{2K \Lambda}$. This implies that the limit $\Lambda_0 \rightarrow \infty$ only diverges for $K \Lambda = -2$ (there is no problem with the large $x$ behavior as the integrand decay exponentially for any couple of charges we consider here).
a. The case of charges \( K.P = 1 \) or \( 2 \).

For these charges, the limit \( \Lambda_0 \to \infty \) can be taken directly on (D10). This leads to, performing the change of variable \( x \to \Lambda_0 x \) and the integral over the angular variable on (D16)

\[
\hat{V}_t^{K.P.q} = -q^2 g^2 \frac{\pi}{T^2} \int_0^\infty dr \frac{r}{\hat{q}^2} (J_0(\hat{q}r) - 1) \quad (D19)
\]

\[
(e^{-K.P \int_a E_1(r^2/2a)} - 1 + K.P \int_a E_1(r^2/2a))
\]

where \( \hat{q} = q/\Lambda_0 \) and \( J_0(z) \) is a Bessel function of the first kind. This defines the function \( \chi_{K.P}(k) \) in that case

\[
\chi_{K.P}(k) = 4e^{2(\gamma_{E} - \int_a \ln(2a))} \int_0^\infty dr \frac{r}{k^2} (J_0(|k|r) - 1)
\]

\[
(e^{-K.P \int_a E_1(r^2/2a)} - 1 + K.P \int_a E_1(r^2/2a)) \quad (D20)
\]

The small \( k \) behavior (the first line of (D3)) is straightforwardly obtained as

\[
\chi_{K.P}(k) \sim a_{K,P} + O(k^2)
\]

\[
a_{K,P} = -e^{2(\gamma_{E} - \int_a \ln(2a))} \int_{u>0} u(e^{-K.P \int_a E_1(u/2a)} - 1)
\]

\[
+ K.P \int_a E_1(u/2a))
\]

where we performed the change of variable \( u = r^2 \).

For \( K.P = 1 \) or \( 2 \), \( r e^{-K.P \int_a E_1(r^2/2a)} \sim e^{2K.P+1} \) when \( r \ll 1 \) is analytic in 0 and using \( J_0(k) \sim k^{-1/2} \cos(k - \frac{\pi}{4}) \) one finds for \( k \gg 1 \)

\[
\int_0^\infty dr \frac{r}{k^2} J_0(|k|r) (e^{-K.P \int_a E_1(r^2/2a)} - 1) \sim O(\frac{1}{k^{5/2}})
\]

(D22)

We have moreover

\[
\int_0^\infty dr \frac{r}{k^2} (J_0(|k|r) - 1) \int_a E_1(r^2/2a)
\]

\[
= \int_a 2 - 2e^{-ak^2/2} - ak^2
\]

(D23)

Using (D23) together with (D22) one obtains the leading behavior of \( \chi_{K,P}(k) \) in the large \( k \) limit, i.e. the first line of (D4)

\[
\chi_{K,P}(k) \sim b_{K,P} \frac{1}{k^2}
\]

\[
b_{K,P} = -e^{2(\gamma_{E} - \int_a \ln(2a))} \int_{u>0} (e^{-K.P \int_a E_1(u/2a)} - 1)
\]

\[
-2K.P c'(0)
\]

where we made the change of variable \( u = r^2 \) and used \( c'(0) = -\int_a r \).

b. The case of charges \( K.P = -1 \)

In that case, \( \chi_{K,P}(k) \) is formally obtained as previously (D21), the small \( k \) behavior being still given by (D24). However the large \( k \) behavior is dominated by the small \( r \) region and as noticed previously for \( r \ll 1 \), \( r(e^{-K.P \int_a E_1(r^2/2a)} - 1) \sim r^{2K.P+1} = r^{-1} \), which leads to a logarithmic divergence in the large \( k \) limit. It can be obtained by computing

\[
\int_0^\infty dr (J_0(kr) - 1)(\int_a E_1(r^2/2a) - 1)
\]

\[
\sim e^{-\gamma_E + \int_a \ln(2a)} \int_0^\infty \frac{dr}{r} (J_0(kr) - 1)
\]

\[
\sim -e^{-\gamma_E + \int_a \ln(2a)} \ln k \quad k \gg 1
\]

(D25)

The last term in (D20) has the same behavior (D25) independently of \( K.P \) and (D21) together with (D20) for \( K.P = -1 \) lead to the second line of (D4)

\[
\chi_{K,P}(k) \sim b_{-1} \ln k \quad \frac{k}{k^2}
\]

\[
b_{-1} = -4e^{-\gamma_E + \int_a \ln(2a)}
\]

(D26)

c. The case of charges \( K.P = -2 \)

As pointed out previously, there is in that case a logarithmic divergence when \( \Lambda_0 \to \infty \). We isolate this divergence by writing

\[
\hat{V}_t^{K.P.q} = -\frac{1}{4} \int_x (q) x^2 \hat{\mathcal{E}}_l^{K.P.x} + \frac{1}{2} \int_x (e^{q x} - 1 - \frac{1}{2}(q x)^2) \hat{\mathcal{E}}_l^{K.P.x}
\]

(D27)

the second term being well defined in the limit \( \Lambda_0 \to \infty \). We focus now on the first part, using the explicit expression of \( \gamma_l(\Lambda_0 x) \) (D3)

\[
-\frac{1}{4} \int_x (q) x^2 \hat{\mathcal{E}}_l^{K.-K.x} =
\]

\[
\frac{1}{8 \pi^2} g^2 \int_x x^2 \left\{ e^{2 \int_a E_1(\frac{x^2}{2a})} - E_1(x^2/2a) \right\} - 1
\]

\[
-2 \left\{ \int_a E_1 \left( \frac{x^2}{2a} \right) - E_1 \left( \frac{x^2}{2a} \frac{\Lambda_0}{\Lambda_0^2} \right) \right\} = \mathcal{H} \left( \frac{\Lambda_0^2}{\Lambda_0^2} \right)
\]

(D28)

where we made the change of variable \( x \to \Lambda_0 x \). To analyse the large argument behavior of \( \mathcal{H}(\lambda) \), we take the derivative w.r.t \( \lambda \)

\[
\mathcal{H}'(\lambda) = \frac{\pi}{4 \lambda} g^2 \int_0^\infty dx x^3
\]

\[
\int_0^\infty \frac{2}{\lambda} e^{-\frac{\lambda}{2a}} (e^{2 \int_a E_1(\frac{x^2}{2a})} - E_1(\frac{\Lambda_0^2}{\Lambda_0^2})) - 1
\]

(D29)
where we have used $E'_i(z) = -e^{-z}/z$. Making the change of variable $u = \lambda x^2$ in $\mathcal{H}'(\lambda)$ one obtains:

$$\mathcal{H}'(\lambda) = \frac{\pi}{8T_c} q q^2 \int_0^\infty \frac{du}{u} \lambda f_x \left( e^{2f_x E_i((-\gamma_E+2\ln 2a)} - 1 \right)$$

using the large $\lambda$ behavior $E_1(u/(2a\lambda)) \sim -\gamma E + f_x \ln (2a) - \ln (u/\lambda) + O(1)$ one gets

$$\mathcal{H}'(\lambda) = \frac{\pi}{8T_c} q q^2 e^{2f_x (-\gamma_E+2\ln 2a)} \frac{1}{\lambda} \lambda$$

$$\int_0^\infty \frac{du}{u} e^{2f_x} E_i((-\gamma_E+2\ln 2a)} + O(\lambda^{-2})$$

$$= \frac{\pi}{8T_c} q q^2 e^{2f_x} E_i((-\gamma_E+2\ln 2a)} \frac{1}{\lambda} + O(\lambda^{-2})$$

$$(D30)$$

where we have used the asymptotic behaviors $[D13, D14]$. This leads finally to

$$\frac{1}{4} \int_x (q x)^2 \hat{F}_{K,P} - \delta_{K,-P} A q^2 \ln \left( \frac{\Lambda_1}{\Lambda_0} \right) + O(\Lambda_0^2)$$

$$A_i = \frac{\pi}{4T_c} q q^2 e^{2f_x} E_i((-\gamma_E+2\ln 2a)}$$

$$(D31)$$

which is the first term in $[D27]$ with the amplitude $A_1$ given in $[D21]$.

In the second line of $[D27]$, we perform the change of variable $x \rightarrow \Lambda_1 x$ and the integral over the angular variable to get

$$\frac{1}{2} \int_x (e^{q x} - 1 + \frac{1}{2} (q x)^2) \hat{F}_{K,P}$$

$$= -q^2 \frac{\pi}{T_c} \int_0^\infty \frac{dr}{r} (J_0(|q|r) - 1 + 1 + q^2 r^2$$

$$(c^2 f_x E_i(r^2/2a}) - 1 - 2 \int_a E_1(r^2/2a))$$

$$(D32)$$

where $J_0(z)$ is a Bessel function of the first kind, from which we get the function $\chi^{K,P}(k)$ defined in the text for $K,P = -2$

$$\chi^{K,P}(k) = 4e^{2(-\gamma_E-f_x \ln 2a)} \int_0^\infty \frac{dr}{r^2} (J_0(|k|r) - 1 + 1 + k^2 r^2)$$

$$(c^2 f_x E_i(r^2/2a}) - 1 - 2 \int_a E_1(r^2/2a))$$

$$(D33)$$

the small $k$ behavior (i.e. the second line of $[D3]$ in the text) is easily obtained

$$\chi^{K,P}(k) \sim a_2 k^2 k \ll 1$$

$$a_2 = \frac{e^{2(-\gamma_E-f_x \ln 2a}}{32} \int_0^\infty du u^2$$

$$(D34)$$

where we made the change of variable $u = r^2$. The large $k$ behavior is governed by the small $r$ region in the integral $[D3]$, where $r(e^{2f_x E_i(r^2/2a}) - 1 - 2 \int_a E_1(r^2/2a)) \sim e^{2(-\gamma_E-f_x \ln 2a}-3}$, which implies for $k \gg 1$

$$\chi^{K,P}(k) \sim \frac{4}{k^2} \int_0^\infty \frac{dr}{r^4} (J_0(|k|r) - 1 + 1 + k^2 r^2) + O(1)$$

$$\sim \ln k + O(1)$$

$$(D35)$$

which is the last line of $[D31]$ in the text.

**APPENDIX E: DETAILED CALCULATIONS FOR THE CO MODEL : EQUILIBRIUM DYNAMICS.**

### 1. Derivation of the RG flow.

We restrict our analysis to order one $O(U_i)$, and at this order the RG flow reads $[14]$

$$\partial_t U_i(u, i\hat{u}) = \frac{1}{2} \partial\partial_{\hat{u}} G^x_{i,j} \partial_{\hat{u}} U_i(u, i\hat{u})$$

$$(E1)$$

where $U_i(u, i\hat{u})$ is given by $[89]$ and the indices $i, j$ formally refer to the components of the vector $\phi$ $[63]$ and the time dependence, i.e. $\partial_t \equiv \delta_{\hat{u}}$. From $[44]$, the matrix $G^x_{i,j}$ has the following expression

$$G^x_{i,j} = \left( \frac{C^x_{i}}{R^x_{i}} \frac{R^x_{j}}{0} \right)$$

$$(E2)$$

With these notations, we have

$$\frac{1}{2} \partial G^x_{i,j} \partial_{\hat{u}} \partial_{\hat{u}} = \frac{1}{2} \frac{\delta}{\delta u} \partial C^x_{i,j} \cdot \frac{\delta}{\delta u} + \frac{\delta}{\delta u} \partial R^x_{i,j} \cdot \frac{\delta}{\delta \hat{u} \hat{u}}$$

$$(E3)$$

where we will often use the matrix notation for time, i.e

$$u \cdot v = \int_t u_t v_t.$$ Acting with this operator on $U_i(u, i\hat{u})$, one gets

$$\frac{1}{2} \partial G^x_{i,j} \partial_{\hat{u}} \partial_{\hat{u}} \left( \int_t i u_t F_{it}(u) - \frac{1}{2} \int_{t'} i u_t i u_{t'} F_{it'}(u) \right)$$

$$= \frac{1}{2} \int_{t'} i u_{t'} F_{it'}(u)$$

$$+ \frac{1}{2} \int_t i u_t \int_{t'} \frac{\delta}{\delta u_{t'}} \partial C^x_{i,j} \partial R^x_{i,j} \frac{\delta}{\delta u_{t'}} F_{it}(u)$$

$$- \int_t i u_{t'} \int_{t'} \frac{\delta}{\delta u_{t'}} \partial R^x_{i,j} \frac{\delta}{\delta u_{t'}} F_{it}(u)$$

$$(E4)$$

The last term vanishes by causality since $F_{i}(u)$ depends on $u_t$, with $t_1 < t$ only. Identifying in $[E4]$ the coefficient of the powers of the field $i \hat{u}$ one gets $[E3]$ in the text.
The first equation of (E5) is easily solved, and it gives
\[ \Delta_{lt'}(u) = e^{\frac{1}{4} \int_{t'}^t \frac{1}{\pi ti} e^{C_{lt}^{\pi t} \frac{1}{\pi ti}} \Delta_{t=0} dt'} \] (E5)
\[ \Delta_{lt'}(u) = 2e^{-C_{lt}^{\pi t} \frac{1}{t} \Lambda_0^2 \cos(u_{zt} - u_{zt'})} \]
where we have used \( C_{lt}^{\pi t} = 0 \). From the previous study of the statics, one has that
\[ e^{-C_{lt}^{\pi t}} \Lambda_0^2 \cos(u_{zt} - u_{zt'}) = \Lambda_0^2 \gl + O(g_l^2) \] (E6)
which leads together with (E5) to the first line in (E6). By taking the functional derivative w.r.t. \( u_{zt'} \) in the second line of (E5) and using the same manipulations one gets the second line in (E6).

2. Computation of the dynamical exponent \( z \).

Here we compute the self energy \( \Sigma_{l0} \) given by
\[ \Sigma_{l0} = \int_0^\infty dt e^{i\omega t} \Sigma_{lt} \] (E7)
\[ \Sigma_{lt} = -2\Lambda_0^2 \gl \left( R_{lt}^{\pi t} e^{C_{lt}^{\pi t}} - \delta(t) \int_0^\infty dt' R_{lt'}^{\pi t} e^{C_{lt'}^{\pi t}} \right) \]
Notice the terms proportional to \( \delta(t) \) in \( \Sigma_{lt} \) (not given in the text for clarity) which guarantees that \( \Sigma_{l=0} = 0 \), and with the explicit expressions for the bare correlation and response functions computed with the cut-off decomposition (E7)
\[ C_{lt}^{\pi t} = \frac{T}{4\pi} \int_a^\infty \ln \left( \frac{|t| + \frac{\gl}{2\Lambda_0^2}}{|t| + \frac{\gl}{2\Lambda_0^2}} \right) \] (E8)
\[ R_{lt}^{\pi t} = \frac{\theta(t)}{4\pi} \int_a^\infty \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \] (E9)
After an integration by part in (E7) and using those explicit expressions one gets to order \( \tau \)
\[ \Sigma_{l0} = -2\Lambda_0^2 \gl \int_0^\infty dt e^{i\omega t} \left( e^{\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \right)} - 1 \right) \] (E10)
This expression is logarithmically divergent for \( \Lambda_0 \to \infty \)
the integrand behaves as \( 1/\tau \) at small \( t \) in this limit, and a way to isolate this divergence is to decompose this integral in the following way (and performing the change of variable \( \tau \to \tau/\Lambda_0^2 \))
\[ \Sigma_{l0} = -2\Lambda_0^2 \gl \int_1^{\infty} dt e^{i\omega t} \left( e^{\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \right)} - 1 \right) \]
\[ -2\Lambda_0^2 \gl \int_0^1 dt (e^{i\omega t} - 1)(e^{\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \right)} - 1) \]
\[ -2\Lambda_0^2 \gl \int_0^1 dt (e^{\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \right)} - 1) \] (E11)
where \( \omega = \omega/\Lambda_0^2 \) and \( \lambda = \Lambda_0^2/\Lambda_0^2 \). In the first two lines we can take safely the limit \( \lambda \to 0 \) and we focus now on the divergent part of the last term
\[ \mathbb{H}(\lambda) = -2g_l \frac{i\omega}{T_c} \int_0^1 dt e^{\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \right)} \] (E12)
Taking the derivative w.r.t. \( \lambda \), one has
\[ \mathbb{H}'(\lambda) = -2g_l \frac{i\omega}{T_c} \int_0^1 dt \int_a^\infty \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \] \[ -2g_l \frac{i\omega}{T_c} \int_0^1 dt \int_a^\infty \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \] (E13)
In the integral of the second line, we can take the limit \( \lambda \to 0 \), it gives
\[ -2g_l \frac{i\omega}{T_c} \int_0^1 dt \int_a^\infty \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \] \[ -2g_l \frac{i\omega}{T_c} \int_0^1 dt \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \] (E14)
The last term in (E13) can be integrated by parts, to get
\[ -2g_l \frac{i\omega}{T_c} \lambda \int_0^1 dt \int_a^\infty \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \] \[ = \mathbb{H}(\lambda) + 2g_l \frac{i\omega}{T_c} \lambda e^{\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \right)} + O(1) \] (E15)
Finally \( \mathbb{H}'(\lambda) \) in (E13) can be written as
\[ \mathbb{H}'(\lambda) \sim i\omega \frac{g_l}{2T_c} e^{\int_a^\infty e^{i\omega t} 2n_a \lambda + O(1)} \]
\[ \mathbb{H}(\lambda) \sim i\omega \frac{g_l}{2T_c} e^{\int_a^\infty e^{i\omega t} 2n_a \lambda + O(1)} \] (E15)
which gives together with the last line of (E11) the first term in (E11) with the amplitude \( B_l = \frac{g_l}{2T_c} e^{\int_a^\infty e^{i\omega t} 2n_a \lambda} \). The first two lines of (E11) where we take the limit \( \lambda \to 0 \) define the function \( \chi^{(\text{dyn})}(\nu) \) of (D4): \[ \chi^{(\text{dyn})}(\nu) = -4e^{-\int_a^\infty e^{i\omega t} \left( \frac{1}{t + \frac{\gl}{2\Lambda_0^2}} - \frac{1}{t - \frac{\gl}{2\Lambda_0^2}} \right)} + \int_0^1 dt \left( e^{i\omega t} - 1 \right) + \int_0^1 dt \left( e^{i\omega t} - 1 \right) \] (E16)
The small argument behavior of \( \chi^{(\text{dyn})}(\nu) \) is dominated by the large \( t \) region of the integrand (i.e. the first line of
Using that \( (\frac{1}{t} \epsilon_l \ln{(t + \frac{2}{q})}) - 1 \sim \int_a \frac{1}{t} \) for \( t \gg 1 \), one gets

\[
\chi^{\text{(dyn)}}(\nu) \sim -4e^{-j_0 \ln{2a}} \int_a \frac{1}{2} \int_1^\infty e^{i\nu t} \frac{1}{t} \nu \ll 1
\]

\[
\sim 4e^{-j_0 \ln{2a}} \int_a \frac{1}{2} \ln \nu \nu \ll 1
\]

which is the asymptotic behavior announced in the text [93] with the non universal amplitude \( a_{\text{dyn}} = 4e^{-j_0 \ln{2a}} \int_a \frac{1}{2} \). The large \( \nu \) behavior of \( \chi^{\text{(dyn)}}(\nu) \) is governed by the small \( t \) region of the integrand, i.e. the second line of [E10]:

\[
\chi^{\text{(dyn)}}(\nu) \sim -4e^{-j_0 \ln{2a}} e^{j_0 \ln{\frac{1}{t}}} \int_0^1 dt(e^{int} - 1) \frac{1}{t}
\]

\[
\sim \ln \nu \nu \gg 1
\]

which is the asymptotic behavior announced in the text [94].

We show here how to take directly, in a cruder way, the limit \( l \to \infty \) in \( \Sigma_{lt} \) [97]. Indeed, using the explicit expression of \( C_{lt} = 0 \) and \( R_{lt} = 0 \) [88], one has

\[
C_{lt}^{\text{pm}} = \frac{T}{2\pi} \int_a \ln{(4\Lambda^2 t^2 + 2a)} - \ln{(2a) - 2l} + O(e^{-2l})
\]

\[
R_{lt}^{\text{pm}} = \frac{1}{4\pi} \int_a \frac{1}{t + \frac{a}{2\Lambda^2}} + O(e^{-2l})
\]

This allows to take the large \( l \) limit in \( \Sigma_{lt} \) at \( T_e \) (as it is already of order \( r \))

\[
\lim_{l \to \infty} \Sigma_{lt} = -\frac{\Lambda^2_0}{2\pi} g^* e^{j_0 \ln{(2a)}} \int_a \frac{1}{t + \frac{a}{2\Lambda^2_0}} e^{-j_0 \ln{(4\Lambda^2 t^2 + 2a)}}
\]

for \( t > 0 \). We then obtain directly \( \Sigma_{l\omega} \) in the limit \( l \to \infty \) as

\[
\lim_{l \to \infty} \Sigma_{l\omega} = -\frac{g^* \Lambda_0^2 e^{j_0 \ln{(2a)}}}{2\pi} i\omega \int_0^\infty dt e^{i\omega t} e^{-j_0 \ln{4\Lambda^2 t^2 + 2a}}
\]

\[
= -\frac{g^* e^{j_0 \ln{(2a)}}}{2\pi} i\omega \int_0^\infty dt e^{i\omega t} e^{-j_0 \ln{4t^2 + 2a}}
\]

The small \( \omega/\Lambda^2_0 \) behavior is governed by the large \( t \) region of the integrand, which gives

\[
\lim_{l \to \infty} \Sigma_{l\omega} \sim \frac{-g^* e^{j_0 \ln{(2a)}}}{2\pi} i\omega \int_0^\infty e^{i\omega t} \frac{1}{4t}
\]

\[
\sim B^* i\omega \ln{(\frac{\omega}{\Lambda^2_0})} + O(\frac{i\omega}{\Lambda^2_0} \ll 1)
\]

which gives the same result obtained by the previous analysis [93].

3. Scaling function at equilibrium

In this section, we show how to solve the equation for the response function [102]. First, it is natural to search for a solution under the form \( R_{lt}^q = e^{-q^2 t} G_{lt}^q \). Then, performing the change of variable \( u = t - t_1 \) and using the explicit expression [E20], one gets the following equation for \( G_{lt}^q \):

\[
\partial_t G_{lt}^q = 4B^* \int_0^1 du \int_a \frac{1}{u + \frac{q}{2}} e^{-j_0 \ln{(4u + 2a)}} e^{q^2 u}
\]

\[
- 4B^* \int_a \frac{1}{u + \frac{q}{2}} e^{-j_0 \ln{(4u + 2a)}}
\]

where \( q = q/\Lambda_0 \) and \( t = \Lambda^2_0 t \), with the initial conditions:

\[
G_{lt}^{q_0+} = 1
\]

\[
G_{lt}^{q_0-} = 0
\]

The second term in the l.h.s. is a total derivative and can be integrated. Performing an integration by part on the first term one gets

\[
G_{lt}^q = 1 + 4B^* (\frac{q}{2} \int_0^1 dv \int_v^1 du e^{-j_0 \ln{(4v + 2a)}} e^{q^2 v})
\]

\[
- \int_0^1 dv e^{q^2 v} e^{-j_0 \ln{(4v + 2a)}}
\]

Performing an integration by part in the integral over \( v \) on the first integral and performing the change of variable \( u' = q^2 u \) in the remaining integrals one gets

\[
G_{lt}^q = 1 + 4B^* ((\frac{q^2 t}{2} - 1) \frac{1}{q^2} \int_0^{q^2 t} du e^{-j_0 \ln{(4u + 2a)}} e^{u'} - 1)
\]

\[
- \frac{1}{q^2} \int_0^{q^2 t} dv e^{u'} e^{-j_0 \ln{(4v + 2a)}}
\]

\[
+ ((\frac{q^2 t}{2} - 1) \frac{1}{q^2} \int_0^{q^2 t} du e^{-j_0 \ln{(4u + 2a)}} + O(q^2)
\]

We now want to find the scaling function, i.e. the asymptotic behavior when \( q \to 0 \) (\( \Lambda_0 \to \infty \)), keeping \( q^2 t = \gamma \) fixed. In the two first lines of the above expression, the limit \( q \to 0 \) can be taken safely, although the last term is divergent in this limit. Thus one has

\[
G_{lt}^q = 1 + 4B^* ((\frac{q^2 t}{2} - 1) \frac{1}{q^2} \int_0^{q^2 t} du e^{u'} - 4 \int_0^{q^2 t} du e^{u'})
\]

\[
+ ((\frac{q^2 t}{2} - 1) \frac{1}{q^2} \int_0^{q^2 t} du e^{-j_0 \ln{(4u + 2a)}} + O(q^2)
\]

To find the asymptotic behavior of the last term we write

\[
\frac{1}{q^2} \int_0^{q^2 t} du e^{u'} = \frac{1}{q^2} \int_0^{q^2 t} du e^{-j_0 \ln{(4u + 2a)}} - \int_a \frac{1}{q^2} \]

\[
+ \frac{1}{q^2} \int_0^{q^2 t} du \int_a \frac{1}{4u + 2a}
\]
In the integral on the second line, we can take the limit \( \hat{q} \to 0 \) by making the change of variable \( \lambda = u/\hat{q}^2 \), and the second can be done exactly. We thus have

\[
\frac{1}{\hat{q}^2} \int_0^{\hat{q}} dt \left( \rho - f_a \ln (\hat{q}^2 + 2a) \right) = \frac{1}{\hat{q}^2} \int_0^{\hat{q}} dt \left( \frac{1}{2} \ln \frac{y}{\hat{q}^2} - \int_a^\infty \frac{\ln u}{2} - \int_a^\infty \frac{\ln u}{2} + O(\hat{q}^2) \right)
\]

Finally, using

\[
\int_0^y du \frac{u^a - 1}{4u} = \frac{1}{4} (-\gamma_E + E\{y \} - \ln y) \quad (E33)
\]

one has, up to terms of order \( \hat{q}^2 \)

\[
C_t^q = 1 + B^*(((y - 1)E\{y \} + 1 - q^y + (1 - y)(\ln q^2 + \rho))
\]

\[
\rho = \gamma_E + \int_a^\infty \frac{a}{2} - 4 \int_0^\infty d\lambda e^{-f_a \ln (\lambda^2 + 2a)} + 4 \int_a^\infty \frac{1}{4\lambda + 2a} \quad (E34)
\]

which yields the scaling function given in the text.

**APPENDIX F: NONEQUILIBRIUM DYNAMICS OF THE CO MODEL.**

1. Some useful expressions

To begin with, we give the explicit expression of \( \Delta_{tt'} \) and \( \Sigma_{tt'} \) and their limiting expression when \( l \to \infty \) in the case of nonequilibrium dynamics. The general expression of \( \Delta_{tt'}(u) \), i.e. the first line of (E25) is still valid for nonequilibrium dynamics. To evaluate it, we only need the expression of \( C_{tt'}(u) \) that we compute from (142) using the same cutoff function \( c(z) \) as previously (17):

\[
C_{tt'}(u) = T \int_0^\infty \left( \ln (t + t' + a/2\Lambda_0^2) - \ln (|t - t'| + a/2\Lambda_0^2) \right) - \frac{T}{2\pi} \ln (t + t' + a/2\Lambda_0^2) + \ln (|t - t'| + a/2\Lambda_0^2) \right) \quad (F1)
\]

Notice that the response function \( R_{tt'}^q(0) \) has its equilibrium expression. From (E25) and (F1) one obtains

\[
\Delta_{tt'}(u) = e^{-\frac{C_{tt'}(0)}{2}} e^{-\frac{C_{tt'}(u)}{2}} C_{tt'}(u) \quad (F2)
\]

Using (F1) one has, using \( T = T_e = 4\pi \) to this order:

\[
D_{tt'} = \lim_{l \to \infty} D_{tt'} = \frac{C_{tt'}(0)}{2} \quad (F3)
\]

and using the definition (88), one obtains finally

\[
\Delta_{tt'} = \lim_{l \to \infty} \Delta_{tt'} = \frac{\Lambda_0^2 T_e B^*}{2} e^{f_a - \ln (\Lambda_0^2 + \frac{a}{2}) + \ln (\Lambda_0^2 + \frac{a}{2})} \quad (F4)
\]

where we have used the expression of \( B^* \) given in (F5). The expression for \( \Sigma_{tt'} \) can be obtained in a very similar way:

\[
\Sigma_{tt'} = \lim_{l \to \infty} \Sigma_{tt'} = \frac{-\Lambda_0^2 B^*}{2} \int_0^\infty \left\{ \frac{\theta(t - t')}{2\Lambda_0^2 (t - t')} + \frac{2}{\pi} e^{\frac{1}{2} \ln (\Lambda_0^2 + \frac{a}{2})} \right\} \quad (F5)
\]

These expressions (E4, E5) will be very useful to determine explicit expressions for \( R_{tt'}^q \) and \( \Delta_{tt'}^q \), thus solving perturbatively (94) (100).

2. Nonequilibrium response function: detailed calculations.

The starting point of our analysis is the equation (94) that we solve perturbatively by replacing, in the rhs of this equation, \( R_{tt'}^q \) by its bare value. One obtains, in the limit \( l \to \infty \), using (F5), and in terms of the rescaled variables \( t = \Lambda_0^2 t, \hat{q} = q/\Lambda_0, \)

\[
\partial_t R_{tt'}^q + \hat{q} \partial_{\hat{q}} R_{tt'}^q = -\frac{B^*}{2} \int_0^t dt e^{f_a - \ln (\Lambda_0^2 t - t + \frac{a}{2})} \quad (F6)
\]

\[
\times e^{f_a - \ln (\Lambda_0^2 (t + t) + \frac{a}{2})} \quad (F7)
\]

Let us first focus on the last term in the rhs of (F6) where we make the change of variable \( t_1 = ut \) and analysing the limit \( t \gg 1 \):

\[
-\frac{B^*}{2} e^{-\hat{q} t} \int_0^1 du \int_0^1 \frac{1}{u + \frac{a}{2}} e^{f_a - \ln (1 - u + \frac{a}{2})} \quad (F8)
\]

\[
\times e^{f_a - \ln (1 + u + \frac{a}{2})} - \frac{1}{2} \ln (1 + \frac{a}{2}) - \frac{1}{2} \ln (1 + \frac{a}{2}) - \frac{1}{2} \ln (1 + \frac{a}{2}) \quad (F7)
\]

In the integrand one can not directly take the limit \( l \to \infty \) because it generates a divergence of the integral when \( u \to 1 \). Therefore we substract the divergent term in the following way

\[
Q = -\frac{B^*}{2} e^{-\hat{q} t} \int_0^1 du \int_0^1 \frac{1}{u + \frac{a}{2}} e^{f_a - \ln (1 - u + \frac{a}{2})} \quad (F8)
\]

\[
\times \left( e^{f_a - \ln (1 + u + \frac{a}{2})} - \frac{1}{2} \ln (1 + u + \frac{a}{2}) - 2 \right) \quad (F8)
\]
Interestingly except in the first line, one can take directly the limit $\ell \to \infty$ in the integrand of the two last lines using
\[
\lim_{\ell \to \infty} \frac{1}{\ell} \frac{e^{f_a} - \ln(1 - u + \frac{u}{\ell})}{1 - u + \frac{u}{\ell}} = \frac{1}{(1 - u)^2} \left( \frac{1}{\sqrt{u}} - 2 \right) = \frac{1}{\sqrt{u}(1 + \sqrt{u})^2} \quad \text{(F9)}
\]
and the divergence for $u \to 1$ is cured. Then all the remaining integrals can be performed exactly, giving finally
\[
Q = B^* e^{-q^2(\ell - \ell')} \left( -e^{-f_a \ln \frac{\ell}{\ell} + \frac{1}{2\ell} + O(\ell^{-2})} \right) \quad \text{(F10)}
\]
We now perform exactly the same manipulations on the first term in the rhs of (F6). Performing the change of variable $t_1 = uf$ and considering the limit $\ell \gg 1$ (keeping $\tilde{q}^2(\ell - \ell')$ and $\tilde{q}/\ell'$ fixed) one obtains
\[
\frac{B^*}{2} \int_{\ell' \ell} \frac{dt}{t} \int_{a} \frac{e^{-q^2(t_1 - \ell')}}{1 - u + \frac{u}{\ell}} \times e^{f_a - \ln(1 + u + \frac{u}{\ell}) + \ln(1 + u + \frac{u}{\ell})} = B^* \int_{\ell' \ell} \frac{dt}{t} \left( \int_{a} \frac{e^{-q^2(t_1 - \ell')}}{1 - u + \frac{u}{\ell}} e^{f_a - \ln(1 - u + \frac{u}{\ell})} \right)
\]
\[
+ \frac{B^*}{2} \int_{\ell' \ell} \frac{dt}{t} \left( \frac{\sqrt{u}}{\sqrt{1 + \sqrt{u}}} \right) + O(\ell^{-2}) \quad \text{(F11)}
\]
where we have used the same trick (F9) as previously. Using (F11) together with (F10) on can write (F10) in a rather simple way
\[
\frac{\partial_t H^q_{\ell' \ell}}{2} q^2 R^q_{\ell' \ell} =
\]
\[
4B^* \int_{\ell' \ell} \frac{dt_1}{t_1} \left( \int_{a} \frac{1}{t_1 - t_1 + \frac{a}{2}} e^{f_a (\ln(1 + u + \frac{u}{\ell}) - \ln(1 - u + \frac{u}{\ell}))} \right)
\]
\[
- B^* e^{-q^2(\ell - \ell')} e^{-f_a \ln(1 + u + \frac{u}{\ell}) + \ln(1 + u + \frac{u}{\ell})}
\]
\[
+ \frac{B^*}{2} \left( \int_{\ell' \ell} \frac{dt_1}{t_1} \left( \frac{e^{-q^2(t_1 - \ell')}}{\sqrt{1 + \sqrt{u}}} + \frac{e^{-q^2(\ell - \ell')}}{t} \right) \right) \quad \text{(F12)}
\]

The two first lines correspond to equilibrium fluctuations (102) and their contribution to the response function has already been computed (105). The last term does not depend any more on the cutoff function and characterizes the contributions coming from nonequilibrium fluctuations. The linearity of this equation suggests then to look for a solution under the form
\[
H^q_{\ell' \ell} = R^q_{\ell' \ell} + R^q_{\ell' \ell}^{\text{noneq}},
\]
where $R^q_{\ell' \ell} = \tilde{q}^2 F^R_{\ell' \ell}(q^2(\ell - \ell'))$ and $R^q_{\ell' \ell}^{\text{noneq}} = e^{-q^2(\ell - \ell')} H^q_{\ell' \ell}$ with $H^q_{\ell' \ell}$ determined by (F6)
\[
\frac{\partial_t H^q_{\ell' \ell}}{2} = \frac{B^*}{2} \left( \int_{\ell' \ell} \frac{dt_1}{\sqrt{1 + \sqrt{u}}} e^{-q^2(t_1 - \ell')} + \frac{1}{t} \right)
\]
\[
H^q_{\ell' \ell} = H^q_{\ell' \ell}^{\text{noneq}} = 0 \quad \text{(F13)}
\]
This allows to write a close expression for the perturbative expansion of $R^q_{\ell' \ell}^{\text{noneq}}$ in terms of the scaling variables
\[
v' = q^2(\ell - \ell'), \quad u = \frac{1}{\ell'}
\]
\[
R^q_{\ell' \ell}^{\text{noneq}} = \frac{B^* e^{-v'}}{2} \left( \int_{\ell' \ell} \frac{dt_1}{\sqrt{1 + \sqrt{u}}} + \frac{1}{\sqrt{1 + \sqrt{1 + \sqrt{u}}}} \right)
\]
\[
+ \ln u \quad \text{(F14)}
\]
Unfortunately it is quite difficult to extract directly the asymptotic behaviors from this double integral. However one can perform straightforward (although tedious) manipulations to obtain a quasi-explicit expression for $R^q_{\ell' \ell}^{\text{noneq}}$.

Performing the natural change of variables $\alpha = \sqrt{t_1} - \sqrt{t_1}$, $\beta = \sqrt{t_1} + \sqrt{t_1}$ one is left with integrals over one variable
\[
\frac{B^*}{2} e^{-v} \int_{\sqrt{\alpha \beta}} \frac{dt_2}{\sqrt{\alpha \beta}} \int_{\sqrt{\alpha \beta}} \frac{dt_1}{\sqrt{1 + \sqrt{\beta + \beta^2}}} e^{t_1 - t_2} \left( \frac{u - 1}{8u} - \frac{u - 1}{8v} + Q\left(\frac{v}{u - 1}, u\right) + Q\left(\frac{v}{u - 1}, u\right) \right)
\]
\[
Q(x, y) = \int_{\sqrt{x}} \frac{dt}{(x + 1)^3} e^{t(x^2 - 1)} \quad \text{(F15)}
\]
Performing further manipulations we find that one can write:
\[
R^q_{\ell' \ell}^{\text{noneq}} = \theta \ln u e^{-v} + \tau F^R_{\ell' \ell}(v', u) + O(\tau^2)
\]
\[
\theta = B^* + O(\tau^2)
\]
\[
(F16) \quad (F17)
\]
where the logarithmic behavior determining \( \theta \) has been extracted such that the function \( F_R^{\text{loneq}}(v, u) \) has a good limit for \( u \to \infty \), as will be shown below. A useful expression for this function is found as:

\[
F_R^{\text{loneq}}(v, u) = e^{-v} \left( 1 - e^{-v} - \sqrt{\pi} v u + e^{-v} \left( \frac{v u}{u - 1} \right) + e^{-v} \left( 1 - (v - \gamma_E) \ln(v - \gamma_E) \right) + 2 e^{-v} \left( \frac{v u}{u - 1} \right) \left( \frac{v}{u - 1} + \frac{1}{2} \right) - 2 \sqrt{\pi} \left( (1 - v) e^{-v} \int_0^{\frac{v u}{u - 1}} s e^{-s^2} \text{Erfi}(s) \right) + 2(1 - v)e^{-v} \ln \left( 1 + \frac{1}{2} - \frac{v e^{-v}}{u - 1} \ln u \right) \right)
\]

Under this form, asymptotic behaviors are more easily checked that \( F_2^{(18)}(v, u) \) and the starting integral \( F_1^{(14)}(v, u) \) do indeed coincide.

\[
F_R^{\text{loneq}}(v, u) = e^{-v} \left( 1 - e^{-v} - \sqrt{\pi} v u + e^{-v} \left( \frac{v u}{u - 1} \right) + e^{-v} \left( 1 - (v - \gamma_E) \ln(v - \gamma_E) \right) + 2 e^{-v} \left( \frac{v u}{u - 1} \right) \left( \frac{v}{u - 1} + \frac{1}{2} \right) - 2 \sqrt{\pi} \left( (1 - v) e^{-v} \int_0^{\frac{v u}{u - 1}} s e^{-s^2} \text{Erfi}(s) \right) + 2(1 - v)e^{-v} \ln \left( 1 + \frac{1}{2} - \frac{v e^{-v}}{u - 1} \ln u \right) \right)
\]

with Erf \( z \) the error function, Erf \( iz \) the imaginary error function:

\[
\text{Erf} \pm \sqrt{\pi} z \ll 1 \quad \text{Erf} z \sim 1 - e^{-z^2} / (\sqrt{\pi} z) \quad z \gg 1
\]

and

\[
\text{Erfi} z \sim 2 z / \sqrt{\pi} \quad z \ll 1 \quad \text{Erfi} z \sim e^{z^2} / (\sqrt{\pi} z) \quad z \gg 1
\]

and \( 2 \text{F}_2(\{1, 1\}, \{\frac{3}{2}, 2\}, z) \) is a generalized hypergeometric series which has the following asymptotic behaviors

\[
2 \text{F}_2(\{1, 1\}, \{\frac{3}{2}, 2\}, z) \sim 1 + O(z) \quad (F25)
\]

\[
2 \text{F}_2(\{1, 1\}, \{\frac{3}{2}, 2\}, z) \sim z^{\rightarrow \infty} \sqrt{\pi} \frac{e^z}{2 \sqrt{2 \pi}} (1 + O(z^{-1}))
\]

\[
2 \text{F}_2(\{1, 1\}, \{\frac{3}{2}, 2\}, z) \sim z^{\rightarrow \infty} - \frac{\ln(-z)}{2z}
\]

Under this form, asymptotic behaviors are more easily obtained. Note that we have also performed numerical checks that \( F_2^{(18)} \) and the starting integral \( F_1^{(14)} \) do indeed coincide.

Note some simple formulae for the same point response:

\[
\mathcal{R}_{t^t}^{z=0} = \frac{1}{2 \pi z (t - t')} h(t/t') \quad (F27)
\]

\[
h(u) = u^\theta \int_0^\infty dF_R(v, u) \quad (F28)
\]

The asymptotic behavior of \( F_2^{(1)}(v, u) \) is easily obtained in this limit. From \( F_2^{(23)} \), one has \( \lim_{u \to \infty} F^{(1)}_R(v, u) = F_2^{(1)}(v) + \lim_{u \to \infty} F^{(1)}_R(v, u) \), where \( F_2^{(1)}(v) \) is given in \( F_2^{(10)} \). On the expressions \( F_2^{(16)} \) together with the asymptotic behaviors \( F_2^{(14)} \), we see that all terms vanish in this limit except the following ones

\[
\lim_{u \to \infty, u, v \text{fixed}} F^{(1)}_R(v, u) = -F^{(1)}_R(v) + e^{-v^2} \left[ \sqrt{\pi} \text{Erf} \sqrt{v} - e^{-v} \left( 1 - v \right) \ln(4ve^{-v}) \right]
\]

which leads to \( F_2^{(26)} \) in the text.

b. Expansion at large \( u, v \) fixed.

Although one can extract more rigorously the large \( v \) behavior at \( u \) fixed from the complete expression \( F_2^{(16)} \), it is easier to compute it from the starting integral \( F_1^{(14)} \). Indeed, in the large \( v \) limit, the integral will be dominated by the region \( t_2 - t_1 \sim v \), i.e. one can replace in the integrand (except of course in the term \( e^{t_2 - t_1} \) \( t_2 \) by \( vu/(u - 1) \) and \( t_1 \) by \( v/(u - 1) \):

\[
\frac{B^* e^{-v}}{2} \left( \int_{\frac{w_2}{v}}^{\frac{w_1}{v}} dt_2 \int_{t_2}^{\infty} dt_1 e^{t_2 - t_1} \right)
\]

\[
\sim B^* \frac{1}{2} \frac{1}{v^2} \frac{(\sqrt{u} - 1)^2}{\sqrt{u}} e^{-v} \int_{\frac{w_2}{v}}^{\frac{w_1}{v}} dt_2 e^{t_2} \int_{t_2}^{\infty} dt_1 e^{t_2 - t_1}
\]

which leads finally to

\[
F^{(1)}_R(v, u) \sim B^* \frac{1}{2 \pi} \left( \frac{\sqrt{u} - 1)^2}{\sqrt{u}} + O(v^{-3}) \right)
\]

We have checked that we obtain the same result by performing this expansion on \( F_2^{(10)} \). Finally, using the large
\[ F_R^1(v, u) \sim e^{\gamma_E} \frac{1}{2v^2} \frac{u + 1}{\sqrt{u}} + O(v^{-3}) \] (F31)

which gives (127) in the text.

c. The limit of vanishing momentum.

The limit \( \tilde{q} \to 0 \) is easily obtained by looking for the leading term in \( F_R(v, u) \) when \( v = \tilde{q}^2(t - t') \to 0 \). Using [128] together with (129) and (103) one has

\[ F_R^1(v) \sim -e^{\gamma_E} (\ln v + \gamma_E - 2 \ln \frac{1 + \frac{1}{\sqrt{u}}}{2}) \] (F32)

This logarithmic behavior together with (88) cancels the log \( \tilde{q} \) divergence in (123), and allows to take the limit of vanishing momentum. We also give here the expression of \( F_R^{1\text{noneq}}(0, u) \), obtained from (106)

\[ F_R^{1\text{noneq}}(0, u) = 2e^{\gamma_E} \ln \frac{1 + \frac{1}{\sqrt{u}}}{2} \] (F33)

this will be useful for further applications.

3. Nonequilibrium correlation function: detailed calculations.

The starting point of our analysis is the following expression given in the text [129], for \( t > t' \):

\[ C_{t't'}^{\tilde{q}} = \lim_{l \to \infty} C_{t't'}^l \] (F34)

\[ = 2T \int_0^1 dt_1 R_{t t_1}^\tilde{q} R_{t' t_1}^\tilde{q} + \int_0^1 dt_1 \int_0^{t'} dt_2 R_{t t_1}^\tilde{q} D_{t_1 t_2} R_{t' t_2}^\tilde{q} \] (F35)

where \( D_{t_1 t_2} = \lim_{l \to \infty} D_{t_1 t_2} \) is given in [124], that we expand perturbatively using the expression we obtained for \( R_{t t'}^{\tilde{q}l} \) [124]. As we did previously for the response function we could keep the complete cut-off dependence in (124). However given the complexity of these manipulations and the experience we acquired before, we know that the only cutoff dependence is contained in an overall nonuniversal scale \( \tilde{q} \to \lambda \tilde{q} \). For these reasons we will perform the computations using a simplified cutoff \( \hat{c}(a) = \delta(a - a_0) \) and we will choose \( a_0 = 2 \) for simplicity. \( D_{t_1 t_2} \) can then be written as (124)

\[ D_{t_1 t_2} = \frac{1}{2} e^{\gamma_E} T \hat{c} \tau \left( \frac{t_1 + t_2}{(|t_1 - t_2| + 1) \sqrt{|t_1 t_2|}} + O(\tau^2) \right) \] (F36)

where we have dropped the \( a_0 \) dependence where it turns out to be unimportant.

Performing the integrals that do not involve \( F_R^0(v, u) \) one has

\[ C_{t't'}^\theta = \frac{T}{q^2} F_C^0(v, u) + \frac{T}{q^2} \theta \ln u F_C^0(v, u) \] (F37)

\[ + \frac{T}{q^2} (z - 2) \ln q \left( \frac{\partial F_C^0(v, u)}{\partial v} + F_C^0(v, u) \right) \]

\[ + \frac{2T}{q^2} e^{-\frac{v}{u + 1}} \left( E_1 \left( \frac{2v}{u - 1} \right) - \ln \left( \frac{2v}{u - 1} \right) - \gamma_E \right) \]

\[ + \frac{2T}{q^2} e^{-\frac{v}{u - 1}} \int_0^1 ds F_R^0 \left( \frac{u - s}{u - 1} \right) \frac{F_R^0(1 - s, \frac{1}{u - 1})}{u} \]

\[ + \frac{2T}{q^2} e^{-\frac{v}{u - 1}} \int_0^1 dt_1 \int_0^{t_1} dt_2 e^{(t_1 + t_2)} \times \left( \frac{1}{|t_1 - t_2| + q^2} + \frac{1}{2} \frac{\sqrt{t_1 - t_2}}{\sqrt{t_1 + t_2}} \right) \]

where we have used the trick (159), and dropped the prime in \( v' = q^2(t - t') \) for simplicity. A natural way to perform this computation is to use for \( F_R^1(v, u) \) the decomposition in an equilibrium and a nonequilibrium contributions (128). Parts of (137) can then be computed analytically:

\[ 2T \tau \frac{v}{q^2} \int_0^1 ds F_R^0 \left( \frac{u - s}{u - 1} \right) F_R^{1\text{eq}} \left( \frac{1 - s}{u - 1} \right) \]

\[ + \frac{2T}{q^2} e^{-\frac{v}{u + 1}} \int_0^1 dt_1 \int_0^{t_1} dt_2 e^{(t_1 + t_2)} \times \left( \frac{1}{|t_1 - t_2| + q^2} + \frac{1}{2} \frac{\sqrt{t_1 - t_2}}{\sqrt{t_1 + t_2}} \right) \]

The expressions (F37) together with (F38) allows to iden-
tify the following perturbative scaling behavior \[ C^0_{uv} = \frac{T}{q^2} F^0_C(v, u) + (z - 2)v \ln q \frac{\partial F^0_C(v, u)}{\partial v} \]
\[ + \theta \ln u F^0_C(v, u) + \mu F^1_C(v, u) + O(\tau^2) \] \[ F^1_C(v, u) = 2 e^{v \gamma_E} e^{-v + \frac{1}{u + 1}} \left( E\left(\frac{2v}{u - 1}\right) - \ln \left(\frac{2v}{u - 1}\right) - \gamma_E \right) \]
\[ + \frac{1}{2} e^{v \gamma_E} \left( e^{-\frac{v(u+1)}{u-1}} - 4 - 2 \frac{uv}{u - 1} E\left(\frac{uv}{u - 1}\right) \right) \]
\[ - 2 \frac{v}{u - 1} E\left(\frac{v}{u - 1}\right) + 2(e^{-v} + e^{-\frac{v}{u - 1}}) \]
\[ + 2 \left( e^{-v} - 1 + ve^{-v} E\left(v\right) \right) \}
\[ \sim O(u^{-1}) \] \[ \text{with the exponents } z \text{ and } \theta \text{ given in } [\text{F38}] \text{ and } [\text{F32}]. \]

The scaling functions are universal up to a cutoff dependent additive constant. It was explicitly computed for the equilibrium response in \[ [\text{F33}] \]. Here, we do not determine it and thus we can drop some multiplicative factors of momentum in the log \( \bar{y} \) term.

\( a. \) Expansion at large \( u, v \) fixed.

First, one has
\[ F^0_C(v, u) = e^{-v} - e^{-v + \frac{1}{u + 1}} \sim \frac{2e^{-v}}{u} + O(u^{-2}) \] \[ \text{(F40)} \]

We now focus on the asymptotic behavior of \( F^0_C(v, u) \) for large \( u \), keeping \( v \) fixed. Using the small argument behavior of \( E\left(z\right) \sim \ln z + \gamma_E + O(z) \) one has for the first line of \[ [\text{F39}] \] in this limit
\[ 2 e^{v \gamma_E} e^{-v + \frac{1}{u + 1}} \left( E\left(\frac{2v}{u - 1}\right) - \ln \left(\frac{2v}{u - 1}\right) - \gamma_E \right) \]
\[ \sim O(u^{-1}) \] \[ \text{(F41)} \]

Again using the small argument behavior of \( E\left(z\right) \), one has
\[ + \frac{1}{2} e^{v \gamma_E} \left( e^{-\frac{v(u+1)}{u-1}} - 4 - 2 \frac{uv}{u - 1} E\left(\frac{uv}{u - 1}\right) \right) \]
\[ - 2 \frac{v}{u - 1} E\left(\frac{v}{u - 1}\right) + 2(e^{-v} + e^{-\frac{v}{u - 1}}) \]
\[ + 2 \left( e^{-v} - 1 + ve^{-v} E\left(v\right) \right) \}
\[ \sim e^{v \gamma_E} e^{-v} \frac{\ln u}{u} + O(u^{-1}) \] \[ \text{(F42)} \]

One then analyses the integrals involving \( F^1_{R \text{non}}(v, u) \):
\[ \frac{2T}{q^2} \left( \int_0^1 ds \left\{ F^0_R \left(\frac{u - s}{u - 1} v\right) F^1_{R \text{non}} \left(\frac{u - s}{u - 1} v, \frac{1}{s} \right) \right. \right. \]
\[ + F^0_R \left(\frac{1 - s}{u - 1} v\right) F^1_{R \text{non}} \left(\frac{1 - s}{u - 1} v, \frac{u}{s} \right) \}
\[ \sim \frac{2T}{q^2} \left( \int_0^1 ds F^1_{R \text{non}}(0, \frac{1}{s}) \right) \]
\[ + v F^0_R(v) \int_0^1 ds F^1_{R \text{non}}(0, \frac{1}{s}) + O(u^{-1}) \] \[ \text{(F43)} \]

and the remaining integral in \[ [\text{F39}] \] where we perform the natural change of variable \( \alpha = \sqrt{t_1}, \beta = \sqrt{t_2} \)
\[ e^{T_E T_C \tau} \frac{e^{-v + \frac{1}{u + 1}}}{q^2} \int_0^1 \int_0^{\frac{1}{u - 1}} dt_1 \int_0^{\frac{1 + v}{u - 1}} dt_2 (\sqrt{t_1 - t_2} e^{t_1 + t_2} \]
\[ = \frac{2e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \int_0^{\frac{1}{u - 1}} d\alpha \int_0^{\frac{1}{u - 1}} d\beta e^{\alpha^2 + \beta^2} \]
\[ - 4e^{T_E T_C \tau} \frac{e^{-v + \frac{1}{u + 1}}}{q^2} \int_0^{\frac{1}{u - 1}} \int_0^{\frac{1}{u - 1}} d\alpha \int_0^{\frac{1}{u - 1}} d\beta e^{\alpha^2 + \beta^2} \]
\[ \frac{\beta}{\alpha + \beta} \]

The first double integral can be performed exactly
\[ \frac{2e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \int_0^{\frac{1}{u - 1}} d\alpha \int_0^{\frac{1}{u - 1}} d\beta e^{\alpha^2 + \beta^2} \]
\[ = \frac{\pi e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \left( \text{Erfi} \left(\frac{uv}{u - 1}\right) - \text{Erfi} \left(\frac{v}{u - 1}\right) \right) \]
\[ \times \text{Erfi} \sqrt{\frac{v}{u - 1}} \]
\[ \sim \sqrt{\pi e^{T_E T_C \tau}} \frac{e^{-v}}{q^2} \text{Erfi} \sqrt{\frac{v}{u}} + O(u^{-1}) \] \[ \text{(F45)} \]

And we expand the second one in the following way
\[ - \frac{4e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \int_0^{\frac{1}{u - 1}} d\alpha \int_0^{\frac{1}{u - 1}} d\beta e^{\alpha^2 + \beta^2} \]
\[ = - \frac{2e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \int_0^{\frac{1}{u - 1}} d\alpha e^{\alpha^2} \left( \frac{1}{2} \frac{v}{u} \right) \]
\[ + \sum_{n \geq 2} \left( \frac{v}{u} \right)^{n/2} \frac{a_n}{\alpha^{n-1}} (1 + O(\alpha)) \]
\[ \sim - \frac{2e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \int_0^{\frac{1}{u - 1}} d\alpha e^{\alpha^2} + O(u^{-1}) \]
\[ - \frac{e^{T_E T_C \tau}}{q^2} e^{-v + \frac{1}{u + 1}} \ln u + O(u^{-1}) \] \[ \text{(F46)} \]

Finally, \[ [\text{F40} \text{ F42} \text{ F45} \text{ F46}] \] lead to the asymptotic following form for \( F_C(v, u) \) in the limit \( u \to \infty, v \) fixed,
\[ \lim_{v \to \infty} F_C(v, u) = \frac{2e^{-v}}{u} + \frac{F^1_{C,\text{non}}(v)}{\sqrt{u}} + O(u^{-2}, \tau u^{-1}, v^2) \]
\[ F^1_{C,\text{non}}(v) = e^{v \gamma_E} e^{-v} \sqrt{\pi} \text{Erfi} \sqrt{v} \] \[ \text{(F47)} \]
notice that the subdominant terms in \( \ln u/u \) cancel between (F42) and (F43), so that the leading corrections are of order \( u^{-1} \). (F44) gives the asymptotic behavior given in the text (F32).

b. Expansion at large \( v, u \) fixed.

In this limit, the terms in the four first lines of (F39) decay exponentially in this limit. The fifth line however (which corresponds to the equilibrium contribution) decays like a power law. Indeed, using the large \( v \) behavior of \( E_i(v) \) gives the asymptotic behavior given in the text (F32).

\[
e^{-\gamma_E} (e^{-v} - 1 + ve^{-v} E_i(v)) \sim \frac{e^{-\gamma_E}}{v} + O(v^{-2}) \quad \text{(F48)}
\]

We now analyse the behavior of the terms involving \( F_{\alpha}^{\text{noneq}}(v, u) \) (F33). Using the large \( v \) behavior of \( F_{\alpha}^{\text{noneq}}(v, u) \) (F30), one has

\[
2 v \int_0^1 \left[ \frac{u}{u-1} \frac{u}{u-v} \int_0^1 ds \left( \frac{1}{1-s} \right) \right]
\]

Notice first on this expression that we are left with convergent integrals over \( s \). Moreover, in the large \( v \) limit, due to the exponential prefactors the first term decays also exponentially (for \( u > 1 \)), and the second one is dominated by \( s = 1 \), which leads to a power law decay

\[
\sim \frac{1}{v(u-1)^2} \frac{1}{\sqrt{u}} \int_0^1 ds e^{-\frac{s-1}{1-s}} \sim O(v^{-2})
\]

We are now left with the double integral in (F39), which is dominated - also due to the exponential prefactor - by \( t_1 \sim v/u(u-1) \) and \( t_2 \sim v/(u-1) \). Therefore to get the leading behavior, we substitute \( t_1 \) and \( t_2 \) by these values in the integrand (except of course in the exponential \( e^{t_1+t_2} \)). This yields

\[
\frac{1}{2} e^{-\gamma_E} e^{-\frac{s-1}{1-s}} \int_0^1 dt_1 \int_0^1 dt_2 \frac{\sqrt{1-t_1-t_2 e^{t_1+t_2}}}{\sqrt{1-t_1 t_2}}
\]

\[
= \frac{1}{2} e^{-\gamma_E} \frac{1}{\sqrt{u}} \frac{1}{v} e^{-\frac{s-1}{1-s}} \int_0^1 dt_1 e^{t_1} \int_0^1 e^{-\frac{s-1}{1-s}} dt_2 e^{-t_2}
\]

\[
= O(v^2)
\]

which together with the other term in \( v^{-1} \) (F48) yields (F32) in the text.

c. The limit of vanishing momentum.

To obtain the limit of vanishing momentum \( \tilde{q} \to 0 \) of the correlation function, we look at the behavior of \( F_{\alpha}(v, u) \) when \( v \to 0 \), up to order \( O(v) \) terms (due to the \( q^{-2} \) prefactor in (F33)). This is done in the following way

\[
2 e^{-\gamma_E} e^{-\frac{s-1}{1-s}} \left( E_i \left( \frac{2v}{u-1} \right) - \ln \left( \frac{2v}{u-1} \right) - \gamma_E \right) \quad \text{(F51)}
\]

\[
+ \frac{1}{2} e^{-\gamma_E} e^{-\frac{s-1}{1-s}} \left( 4 - 2 \frac{uv}{u-1} E_i \left( \frac{uv}{u-1} \right) \right)
\]

\[
-2 \frac{uv}{u-1} E_i \left( \frac{v}{u-1} \right) + 2 e^{-\frac{s-1}{1-s}} + e^{-\frac{s-1}{1-s}}
\]

\[
+ 2 \left( e^{-v} - 1 + ve^{-v} E_i(v) \right)
\]

\[
= \frac{e^{-\gamma_E}}{v(u-1)} (6 - 2 \ln v e^{-v} - u \ln u + (u+1) \ln (u-1) + O(v^2)
\]

Then using the expression of \( F_{\alpha}^{\text{noneq}}(0, u) \) (F33), one has

\[
2 v \int_0^1 \left[ \frac{u}{u-1} \frac{u}{u-v} \int_0^1 ds \left( \frac{1}{1-s} \right) \right]
\]

Notice first on this expression that we are left with convergent integrals over \( s \). Moreover, in the large \( v \) limit, due to the exponential prefactors the first term decays also exponentially (for \( u > 1 \)), and the second one is dominated by \( s = 1 \), which leads to a power law decay

\[
\sim \frac{1}{v(u-1)^2} \frac{1}{\sqrt{u}} \int_0^1 ds e^{-\frac{s-1}{1-s}} \sim O(v^{-2})
\]

We are now left with the double integral in (F39), which is dominated - also due to the exponential prefactor - by \( t_1 \sim v/u(u-1) \) and \( t_2 \sim v/(u-1) \). Therefore to get the leading behavior, we substitute \( t_1 \) and \( t_2 \) by these values in the integrand (except of course in the exponential \( e^{t_1+t_2} \)). This yields

\[
\frac{1}{2} e^{-\gamma_E} e^{-\frac{s-1}{1-s}} \int_0^1 dt_1 \int_0^1 dt_2 \frac{\sqrt{1-t_1-t_2 e^{t_1+t_2}}}{\sqrt{1-t_1 t_2}}
\]

\[
= \frac{1}{2} e^{-\gamma_E} \frac{1}{\sqrt{u}} \frac{1}{v} e^{-\frac{s-1}{1-s}} \int_0^1 dt_1 e^{t_1} \int_0^1 e^{-\frac{s-1}{1-s}} dt_2 e^{-t_2}
\]

\[
= O(v^2)
\]

which together with the other term in \( v^{-1} \) (F48) yields (F32) in the text.

To treat the double integral in the last line of (F39) we come back to the variables \( \tilde{t}, \tilde{u}, \tilde{q} \)

\[
\int_0^1 dt_1 \int_0^1 dt_2 \frac{\sqrt{1-t_1-t_2 e^{t_1+t_2}}}{\sqrt{1-t_1 t_2}}
\]

\[
= \frac{1}{2} \int dt_1 \int_0^1 \frac{\sqrt{1-t_1-t_2 e^{t_1+t_2}}}{\sqrt{1-t_1 t_2}}
\]

Under this form, the limit \( \tilde{q} \to 0 \) is very simply obtained:

\[
\lim_{\tilde{q} \to 0} \frac{1}{2} \int dt_1 \int_0^1 \frac{\sqrt{1-t_1-t_2 e^{t_1+t_2}}}{\sqrt{1-t_1 t_2}}
\]

\[
= \frac{1}{2} \int dt_1 \int_0^1 \frac{\sqrt{1-t_1}}{\sqrt{1-t_1 t_2}}
\]

\[
= 2 \int dt_1 \left( \ln(1 + \sqrt{u}) - \frac{u}{2} \ln u \right)
\]
Finally, together with (F30) and the complete expression of the correlation function $C_{t\nu}^{\tilde{q}}$ lead to

$$C_{t\nu}^{\tilde{q}=0} = 2t\tilde{e} \left(1 + \tau - \frac{z - 2}{2} (\ln (\tilde{t} - \tilde{\nu}) + \gamma_E) + \theta \ln \frac{\tilde{t}}{\tilde{\nu}} \right) + \tau F^{\text{diff}}_C \left( \frac{\tilde{t}}{\tilde{\nu}} \right)$$

$$F^{\text{diff}}_C (u) = \frac{1}{2} e^{\gamma_E} \left( (4\sqrt{u} + (u + 1) \ln (u - 1) - 2(u - 1) \ln (1 + \sqrt{u}) - 2 \ln u + 6 - 8 \ln 4 \right)$$

where we have used $F_C^{(v,u)} = v/(u - 1) + O(u^2)$,

$$\partial_v F_C^{(v,u)} = 1/(u - 1) + O(v)$$

and $v/(u - 1) = q^2 \tilde{t}$: this gives the scaling form (F55) given in the text.

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[61] note that in the particular choice used here the same function $c(x)$ appears both as IR and UV cutoff

[62] The higher replica operators used here are defined in terms of excluded replica sums. They thus form a different basis than the one used in, e.g. FRG studies of zero temperature fixed point in higher dimension $\text{IR}$.

These are defined in terms of unrestricted sums directly related to the cumulant of the disorder, which is not the case here