Generalized integral type Hilbert operator acting between weighted Bloch spaces

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1 INTRODUCTION

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk of the complex plane \( \mathbb{C} \) and \( H(\mathbb{D}) \) denote the space of all analytic functions in \( \mathbb{D} \).

A positive continuous function \( v \) on \( [0, 1) \) is called normal if there exist \( 0 < a \leq b \leq \infty \) and \( 0 \leq s_0 < 1 \) such that \( \frac{v(s)}{(1-s^2)^p} \) almost decreasing on \([s_0, 1) \) and \( \frac{v(s)}{(1-s^2)^p} \) almost increasing on \([s_0, 1) \) (see Shields and Williams [1]).

A function \( g \) is almost increasing if there exists \( C > 0 \) such that \( r_1 < r_2 \) implies \( g(r_1) \leq Cg(r_2) \). Almost decreasing functions are defined in an analogous manner.

Functions such as
\[ v(s) = (1-s^2)^{\delta} \log^6 \left( \frac{e}{1-s^2} \right) (t > 0, \delta \in \mathbb{R}) \text{ and } v(s) = \left( \sum_{k=1}^{\infty} \frac{ks^{2k-2}}{\log^3(k+1)} \right)^{-1} \]
are normal functions.

In this paper, we use \( \mathcal{N} \) to denote the set of all normal functions on \([0, 1) \) and let \( s_0 = 0 \). The letters \( a \) and \( b \) always represent the parameters in the definition of normal function.

Let \( v \in \mathcal{N} \), the normal weight Bloch space \( B_v \) consists of those functions \( f \in H(\mathbb{D}) \) for which
\[ \| f \|_{B_v} = |f(0)| + \sup_{z \in \mathbb{D}} v(|z|) |f'(z)| < \infty. \]
In particular, if \( \nu(|z|) = (1 - |z|^2)^\gamma (\gamma > 0) \), then \( B_\gamma \) is the Bloch type space \( B' \). If \( \nu(|z|) = (1 - |z|^2)^\log^{-\beta} \frac{e}{1 - |z|} (\beta \in \mathbb{R}) \), then \( B_\gamma \) is just the logarithmic Bloch space \( B_{\log^\beta} \). A more general logarithmic Bloch space was introduced in Stević [2].

Let \( I \subset \partial \mathbb{D} \) be an arc and \(|I|\) denote the length of \( I \). The Carleson square \( S(I) \) is defined as \( S(I) = \{ re^{i\theta} : e^{i\theta} \in I, 1 - \frac{|I|}{2\pi} \leq r < 1 \} \). Let \( t > 0 \) and \( \mu \) be a positive Borel measure on \( \mathbb{D} \). We say that \( \mu \) is a \( t \)-Carleson measure if there exists a positive constant \( M > 0 \) such that

\[
\mu(S(I)) \leq M|I|^t, \quad \text{for any interval} \quad I \subset \partial \mathbb{D}.
\]

If \( \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^t} = 0 \), we say that \( \mu \) is a vanishing \( t \)-Carleson measure.

Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). For \( 0 \leq \beta < \infty \) and \( 0 < t < \infty \), we say that \( \mu \) is a \( \beta \)-logarithmic \( t \)-Carleson measure (resp. a vanishing \( \beta \)-logarithmic \( t \)-Carleson measure) if

\[
\sup_{|I| \in \partial \mathbb{D}} \frac{\mu(S(I)) \left( \log \frac{2\pi}{|I|} \right)^\beta}{|I|^t} < \infty, \quad \text{resp.} \quad \lim_{|I| \to 0} \frac{\mu(S(I)) \left( \log \frac{2\pi}{|I|} \right)^\beta}{|I|^t} = 0.
\]

See Zhao [3] for \( \beta \)-logarithmic \( t \)-Carleson measure.

A positive Borel measure \( \mu \) on \([0, 1)\) can be seen as a Borel measure on \( \mathbb{D} \) by identifying it with the measure \( \overline{\mu} \) defined by

\[
\overline{\mu}(E) = \mu(E \cap [0, 1)), \quad \text{for any Borel subset} \ E \text{ of } \mathbb{D}.
\]

In this way, a positive Borel measure \( \mu \) on \([0, 1)\) is an \( t \)-Carleson measure if and only if there exists a constant \( M > 0 \) such that

\[
\mu([s, 1)) \leq M(1 - s)^t, \quad 0 \leq s < 1,
\]

and we have similar statements for vanishing \( t \)-Carleson measures and for \( \beta \)-logarithmic \( t \)-Carleson and vanishing \( \beta \)-logarithmic \( t \)-Carleson measures.

Let \( \mu \) be a finite Borel measure on \([0, 1)\) and \( n \in \mathbb{N} \). We use \( \mu_n \) to denote the moments of \( \mu \), that is, \( \mu_n = \int_{[0,1]} t^n d\mu(t) \). Let \( H_\mu \) be the Hankel matrix \( (\mu_{n,k})_{n,k \geq 0} \) with entries \( \mu_{n,k} = \mu_{n+k} \). The matrix \( H_\mu \) induces an operator on \( H(\mathbb{D}) \) by its action on the Taylor coefficients: \( (a_n)_{n \in \mathbb{N}} \to \sum_{k=0}^{\infty} \mu_{n,k} a_k, \ n = 0, 1, 2, \ldots \).

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \), the generalized Hilbert operator defined as follows:

\[
H_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,
\]

whenever the right hand side makes sense and defines an analytic function in \( \mathbb{D} \).

It is known that the generalized Hilbert operator \( H_\mu \) is closely related to the integral operator

\[
I_\mu(f)(z) = \int_0^1 \frac{f(t)}{1-tz} d\mu(t).
\]

If \( \mu \) is the Lebesgue measure on \([0, 1)\), then \( H_\mu \) and \( I_\mu \) reduce to the classical Hilbert operators \( H \) and \( I \).

The action of the operators \( I_\mu \) and \( H_\mu \) on distinct spaces of analytic functions have been studied in a number of articles (see, e.g., previous works [4–10]). In this paper, we consider the generalized integral type Hilbert operator

\[
I_{\mu_n+1}(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}} d\mu(t), \ (\alpha > -1).
\]
If $a = 0$, the operator $I_{\mu_{\alpha+1}}$ is just $I_{\mu}$. The integral type operator $I_{\mu_{\alpha+1}}$ is closely related to the Hilbert type operator

$$H_\alpha^n(f)(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + \alpha)}{\Gamma(n + 1)\Gamma(\alpha + 1)} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad (\alpha > -1),$$

whenever the right hand side makes sense and defines an analytic function in $D$. The operator $H_\alpha^n$ can be regarded as the fractional derivative of $H_\mu$. If $a = 1$, then $H_\alpha^n$ called the Derivative-Hilbert operator, which has been studied in the literature [11, 12].

The connection between $I_{\mu}$ (or $H_\mu$) and $I_{\mu_{\alpha+1}}$ (or $H_\alpha^n$) motivates us to consider the operator $I_{\mu_{\alpha+1}}$ in a unified manner. Girela and Merchán [13] (see also their other work [7]) have studied the boundedness of $I_{\mu}$ acting on $B$. Li and Zhou studied the operator $H_\mu$ from Bloch type spaces to the BMOA and the Bloch space in Li and Zhou [14]. Ye and Zhou investigated $I_{\mu}$ acting on $B$ in Ye and Zhou [11] and $I_{\mu_{\alpha+1}}$ acting between Bloch-type space in Ye and Zhou [15]. But only partial results were obtained for the boundedness of $I_{\mu_{\alpha+1}}$ acting between Bloch-type spaces. The aim of this article is to deal with the operator $I_{\mu_{\alpha+1}}$ acting from normal weight Bloch space into another of the same kind. As consequences of our study, we obtain complete results for the boundedness of $I_{\mu_{\alpha+1}}$ acting between Bloch type spaces, and between logarithmic Bloch spaces.

Throughout the paper, the letter $C$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $||P \leq Q||$ if there exists a constant $C = C(\cdot)$ such that $||P \leq CQ||$, and $||P \geq Q||$ is understood in an analogous manner. In particular, if $||P \leq Q||$ and $||P \geq Q||$, then we will write $||PQ||$.

## 2 | PRELIMINARY RESULTS

In Pavlović [16], a sequence $\{V_n\}$ was constructed in the following way: Let $\psi$ be a $C^\infty$-function on $\mathbb{R}$ such that (1) $\psi(s) = 1$ for $s \leq 1$, (2) $\psi(s) = 0$ for $s \geq 2$, (3) $\psi$ is decreasing and positive on the interval $(1, 2)$.

Let $\varphi(s) = \psi \left( \frac{s}{2} \right) - \psi(s)$, and let $v_0 = 1 + z$, for $n \geq 1$,

$$V_n(z) = \sum_{k=0}^{\infty} \varphi \left( \frac{k}{2^{n-1}} \right) z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi \left( \frac{k}{2^{n-1}} \right) z^k.$$

The polynomials $V_n$ have the properties:

1. $f(z) = \sum_{n=0}^{\infty} V_n * f(z)$, for $f \in H(D)$;
2. $||V_n * f||_p \leq ||f||_p$, for $f \in H^p, p > 0$;
3. $||V_n||_p \approx 2^n(1 - \frac{1}{p})$, for all $p > 0$, where $*$ denotes the Hadamard product and $|| \cdot ||_p$ denotes the norm of Hardy space $H^p$.

**Lemma 2.1.** Let $v \in \mathcal{N}$ and $f \in H(D)$, then $f \in B_v$ if and only if

$$\sup_{n \geq 0} v(1 - 2^{-n})2^n ||V_n * f||_\infty < \infty.$$

Moreover,

$$||f||_{B_v} \leq \sup_{n \geq 0} v(1 - 2^{-n})2^n ||V_n * f||_\infty.$$

The proof of this lemma is similar to theorem 3.1 in Pavlović [17], we leave it to the interested readers.
Lemma 2.2. Let $\nu \in \mathcal{N}$ and
\[
g(\zeta) = 1 + \sum_{s=1}^{\infty} 2^s \zeta^n, \quad (\zeta \in \mathbb{D}),
\]
where $n_s$ is the integer part of $(1 - r_s)^{-1}$, $r_0 = 0$, $\nu(r_s) = 2^{-s} (s = 1, 2, \ldots)$. Then $g(r)$ is strictly increasing on $[0, 1)$ and there exist two positive constants $N_1$ and $N_2$ such that
\[
\inf_{[0,1]} \nu(r)g(r) = N_1 > 0, \quad \sup_{\zeta \in \mathbb{D}} \nu(|\zeta|)|g(\zeta)| = N_2 < +\infty.
\]
This result is originated from theorem 1 in Hu [18].

Lemma 2.3. If $\nu \in \mathcal{N}$, then
\[
\frac{\nu(|z|)}{\nu(|w|)} \lesssim \left( \frac{1 - |z|^2}{1 - |w|^2} \right)^a + \left( \frac{1 - |z|^2}{1 - |w|^2} \right)^b \quad \text{for all } z, w \in \mathbb{D}.
\]
This result comes from lemma 2.2 in Zhang et al. [19].

Lemma 2.4. Let $\nu \in \mathcal{N}$, $0 < \delta < \frac{1}{e^2}$, then
\[
\int_{e}^{\infty} \frac{e^{-\delta t}}{tv\left(1 - \frac{1}{t}\right)} dt \leq \frac{1}{\nu(1 - \delta)}.
\]

Proof. Here, we use the method in previous works [20, 21], with some modifications. It is clear that
\[
\int_{e}^{\infty} \frac{e^{-\delta t}}{tv\left(1 - \frac{1}{t}\right)} dt = \int_{e}^{1} \frac{e^{-\delta t}}{tv\left(1 - \frac{1}{t}\right)} dt + \int_{1}^{\infty} \frac{e^{-\delta t}}{tv\left(1 - \frac{1}{t}\right)} dt = I_1 + I_2.
\]
By the definition of normal function, we have that
\[
I_1 \leq \int_{e}^{1} \frac{dt}{tv\left(1 - \frac{1}{t}\right)} \leq \frac{\delta^a}{\nu(1 - \delta)} \int_{e}^{\frac{1}{\delta}} t^{a-1} dt \leq \frac{1}{\nu(1 - \delta)}.
\]
If $t > \frac{1}{\delta}$, then $1 - \frac{1}{t} > 1 - \delta$. The definition of normal function shows that
\[
\frac{\nu(1 - \delta)}{[1 - (1 - \frac{1}{t})]^b} \lesssim \frac{\nu\left(1 - \frac{1}{t}\right)^b}{[1 - (1 - \frac{1}{t})]^b}.
\]
Hence, we have that
\[
I_2 = \int_{1}^{\infty} \frac{\nu(1 - \delta)}{tv\left(1 - \frac{1}{t}\right)} \frac{e^{-\delta t}}{tv(1 - \delta)} dt \lesssim \int_{1}^{\infty} \frac{\delta^b t^{b-1} e^{-\delta t}}{tv(1 - \delta)} dt = \frac{1}{\nu(1 - \delta)} \int_{1}^{\infty} e^{-\delta s^{b-1}} ds \lesssim \frac{1}{\nu(1 - \delta)}.
\]
The proof is complete. \qed
Lemma 2.5. Let $\mu$ be a positive Borel measure on $[0, 1)$, $\beta > 0$, $\gamma > 0$. Let $\tau$ be the Borel measure on $[0, 1)$ defined by

$$d\tau(t) = \frac{d\mu(t)}{(1 - t)^\gamma}.$$

Then, the following two conditions are equivalent.

(a) $\tau$ is a $\beta$-Carleson measure.
(b) $\mu$ is a $(\beta + \gamma)$-Carleson measure.

Proof.

(a) $\Rightarrow$ (b). By the definition of Carleson measure, there exists a positive constant $C > 0$ such that

$$\int_0^1 \frac{d\mu(r)}{(1 - r)^\gamma} \leq C(1 - t)^\beta, \ t \in [0, 1).$$

Using this and the fact that the function $x \to \frac{1}{(1 - x)^\gamma}$ is increasing in $[0, 1)$, we obtain

$$\frac{\mu([t, 1))}{(1 - t)^\gamma} \leq \int_t^1 \frac{d\mu(r)}{(1 - r)^\gamma} \leq C(1 - t)^\beta, \ t \in [0, 1).$$

This shows that $\mu$ is a $(\beta + \gamma)$-Carleson measure.

(b) $\Rightarrow$ (a). It is clear that there exists a positive constant $C > 0$ such that

$$\mu([t, 1)) \leq C(1 - t)^{\beta + \gamma}, \ t \in [0, 1).$$

For $0 < x < 1$, let $h(x) = \mu([0, x)) - \mu([0, 1)) = -\mu([x, 1))$. Integrating by parts and using the inequality above, we obtain that

$$\tau([t, 1)) = \int_t^1 \frac{d\mu(x)}{(1 - x)^\gamma} = \frac{1}{(1 - t)^\gamma} \mu([t, 1)) - \lim_{x \to 1^-} \frac{1}{(1 - x)^\gamma} \mu([x, 1)) + \gamma \int_t^1 \frac{\mu([x, 1))}{(1 - x)^{\gamma + 1}} dx$$

$$= \frac{1}{(1 - t)^\gamma} \mu([t, 1)) + \gamma \int_t^1 \frac{\mu([x, 1))}{(1 - x)^{\gamma + 1}} dx$$

$$\leq (1 - t)^\beta + \int_t^1 (1 - x)^{\beta - 1} dx \leq (1 - t)^\beta.$$

Thus, $\tau$ is a $\beta$-Carleson measure.

Lemma 2.6. Let $\omega, \nu \in \mathcal{N}$. If $T$ is a bounded operator from $B_\omega$ into $B_\nu$, then $T$ is compact operator from $B_\omega$ into $B_\nu$ if and only if for any bounded sequence $\{h_n\}$ in $B_\omega$, which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have $\lim_{n \to \infty} \|T(h_n)\|_{B_\nu} = 0$.

The proof is similar to that of proposition 3.11 in Cowen and MacCluer [22], so we omit the details.

3 | NONNEGATIVE COEFFICIENTS OF NORMAL WEIGHT BLOCH FUNCTIONS

First, we give a characterization of the functions $f \in H(\mathbb{D})$ whose sequence of Taylor coefficients is nonnegative, which belongs to $B_\nu$. 
Theorem 3.1. Let \( \nu \in \mathcal{N} \) and \( f \in H(\mathbb{D}) \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n \geq 0 \) for all \( n \geq 0 \). Then \( f \in B_\nu \) if and only if

\[
S(f) := \sup_{n \geq 1} \nu \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} ka_k < \infty.
\]

Moreover,

\[
\|f\|_{B_\nu} \approx S(f) + a_0.
\]

Proof. If \( f \in B_\nu \), then for each \( n \in \mathbb{N} \),

\[
\|f\|_{B_\nu} \geq \sup_{|z|= \frac{1}{n+1}} \nu(|z|) |f'(z)| \geq \nu \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} ka_k \left( 1 - \frac{1}{n} \right)^{k-1} \geq \nu \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} ka_k,
\]

and hence, \( S(f) \leq \|f\|_{B_\nu} \). Since \( a_0 = |f(0)| \leq \|f\|_{B_\nu} \), we obtain

\[
S(f) + a_0 \leq \|f\|_{B_\nu}.
\]

On the other hand, if \( S(f) < \infty \), then

\[
\nu \left( 1 - 2^{-j} \right) \sum_{k=2^j}^{2^{j+1}-1} ka_k \lesssim S(f), \quad j \in \mathbb{N}.
\]

For each \( z \in \mathbb{D} \) with \( \frac{1}{2} \leq |z| < 1 \), we have

\[
|f'(z)| = \left| \sum_{j=0}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} ka_k z^{k-1} \right| \leq \sum_{j=0}^{\infty} \left( \sum_{k=2^j}^{2^{j+1}-1} ka_k |z|^{k-1} \right) \lesssim \nu \left( 1 - 2^{-j} \right) \sum_{j=0}^{\infty} \left| \sum_{k=2^j}^{2^{j+1}-1} ka_k |z|^{k-1} \right| \lesssim S(f) \sum_{j=0}^{\infty} \frac{|z|^{2^j}}{\nu \left( 1 - 2^{-j} \right)}.
\]

To finish the proof, it suffices to prove that

\[
\sum_{j=0}^{\infty} \frac{|z|^{2^j}}{\nu \left( 1 - 2^{-j} \right)} \leq \frac{1}{\nu(|z|)} \quad \text{for all} \quad \frac{1}{2} \leq |z| < 1.
\] (3.1)

For each \( \frac{1}{2} \leq |z| = r < 1 \), by choosing \( m \geq 2 \) such that \( r_{m-1} \leq r \leq r_m \), where \( r_m = 1 - 2^{-m} \). Then

\[
\sum_{j=0}^{\infty} \nu^{-1} \left( 1 - 2^{-j} \right) r^{2^j} \leq \sum_{j=0}^{m} \nu^{-1} \left( 1 - 2^{-j} \right) + \sum_{j=m+1}^{\infty} \nu^{-1} \left( 1 - 2^{-j} \right) r^{2^j} = S_1 + S_2.
\]

\[
S_1 = \nu^{-1} \sum_{j=0}^{m} \frac{1}{\left( 1 - 2^{-j} \right)} r^{2^j} = \nu^{-1} \frac{1 - r^{2^{m+1}}}{2 - 1} = \nu^{-1} \frac{1 - r^{2^{m+1}}}{2}.
\]

\[
S_2 = \nu^{-1} \sum_{j=m+1}^{\infty} \frac{1}{\left( 1 - 2^{-j} \right)} r^{2^j} \leq \nu^{-1} \sum_{j=m+1}^{\infty} \frac{1}{\left( 1 - 2^{-j} \right)} r^{2^j} = \nu^{-1} \frac{1 - r^{2^{m+1}}}{2 - 1} = \nu^{-1} \frac{1 - r^{2^{m+1}}}{2}.
\]

Thus, \( S_1 + S_2 \leq \nu^{-1} \frac{1 - r^{2^{m+1}}}{2} \leq \frac{1}{\nu(|z|)} \) for all \( \frac{1}{2} \leq |z| < 1 \), which completes the proof.
Using Lemma 2.3, we have
\[ S_1 \lesssim \nu^{-1} (1 - 2^{-m}) \sum_{j=0}^{m} \left( \left( \frac{1}{2} \right)^{(m-j)a} + \left( \frac{1}{2} \right)^{(m-j)b} \right) \lesssim \nu^{-1} (1 - 2^{-m}). \]

On the other hand,
\[ S_2 = \sum_{j=m+1}^{\infty} \nu^{-1} (1 - 2^{-j}) r^{2j} \leq \sum_{j=m+1}^{\infty} \nu^{-1} (1 - 2^{-j}) r^{m,2j-m} \]
\[ \leq \sum_{j=m+1}^{\infty} \nu^{-1} (1 - 2^{-j}) e^{-2(j+m)} = \sum_{l=1}^{\infty} \nu^{-1} (1 - 2^{-l+m}) e^{-2l} \]
\[ \lesssim \nu^{-1} (1 - 2^{-m}) \sum_{l=1}^{\infty} e^{-2l} 2^b \lesssim \nu^{-1} (1 - 2^{-m}). \]

Since \( \nu^{-1} (1 - 2^{-m}) \asymp \nu^{-1}(r) \), it follows that (3.1) is valid for all \( \frac{1}{2} \leq |z| < 1 \).

Therefore,
\[ |f(0)| + \sup_{z \in \mathbb{D}} \nu(|z|) |f'(z)| \lesssim a_0 + S(f). \]

The proof is complete. \( \square \)

**Corollary 3.2.** Let \( \gamma > 0 \) and \( f \in H(\mathbb{D}), f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n \geq 0 \) for all \( n \geq 0 \). Then \( f \in B^\gamma \) if and only if
\[ \sup_{n \geq 1} n^{-\gamma} \sum_{k=1}^{n} k a_k < \infty. \]

If \( f \in B_c \) has nonnegative and nonincreasing coefficients, then the result of Theorem 3.1 can be stated as follows.

**Theorem 3.3.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \) with \( a_n \) nonnegative and nonincreasing. Then \( f \in B_c \) if and only if
\[ \sup_{n \geq 1} n^2 \nu \left( 1 - \frac{1}{n} \right) a_n < \infty. \]

Moreover,
\[ \|f\|_{B_c} \asymp a_0 + \sup_{n \geq 1} n^2 \nu \left( 1 - \frac{1}{n} \right) a_n. \]

**Proof.** If \( a_n \) nonnegative and nonincreasing, then \( \sum_{k=1}^{n} k a_k \geq n^2 a_n \). The proof of the necessity follows from Theorem 3.1 immediately.

On the other hand, if \( M := \sup_{n \geq 1} n^2 \nu \left( 1 - \frac{1}{n} \right) a_n < \infty \), then
\[ a_n \lesssim \frac{M}{n^2 \nu \left( 1 - \frac{1}{n} \right)} \text{ for all } n \geq 1. \]

For every \( z \in \mathbb{D} \) and \( \frac{1}{2} < |z| < 1 \),
\[ |f'(z)| \leq \sum_{n=1}^{\infty} n a_n |z|^{n-1} \lesssim M \sum_{n=1}^{\infty} \frac{|z|^n}{n \nu \left( 1 - \frac{1}{n} \right)}. \]
To complete the proof, we use some methods from the literature [2, 20, 21]. Let 
\[ h_x(t) = \frac{x'}{t \nu \left( 1 - \frac{1}{t} \right)} \quad x \in (0, 1), \]
then \( h_x \) is decreasing in \( t \), for sufficiently large \( t \) and each \( x \in (0, 1). \) So, by Lemma 2.4, we have
\[
\sum_{n=1}^{\infty} \frac{|z|^n}{n \nu \left( 1 - \frac{1}{n} \right)} \asymp \int_0^{\infty} \frac{e^{-\ln |z|}}{t \nu \left( 1 - \frac{1}{t} \right)} dt \lesssim \frac{1}{\nu \left( 1 - \ln \frac{1}{|z|} \right)} \lesssim \frac{1}{\nu(|z|)}. \]
This means that
\[
\|f\|_{\mathcal{B}_1} \lesssim a_0 + \sup_{n \geq 1} n^2 \nu \left( 1 - \frac{1}{n} \right) a_n.
\]
The proof is complete. \( \square \)

**Corollary 3.4.** Let \( \gamma > 0 \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \) with \( a_n \) nonnegative and nonincreasing. Then \( f \in B^\gamma \) if and only if
\[
\sup_{n \geq 1} n^{2-\gamma} a_n < \infty.
\]

### 4 | Generalized Integral Type Hilbert Operator Acting on Weighted Bloch Space

Let \( \omega \in \mathcal{N} \), we write \( \tilde{\omega}(t) = \int_0^t \frac{1}{\omega(s)} ds \). We begin with characterizing those measure \( \mu \) for which the operator \( I_{\mu_{x+1}} \) is well defined on \( B_w \).

**Proposition 4.1.** Let \( \mu \) be a positive Borel measure on \( [0, 1) \) and \( \alpha > -1 \). For any given \( f \in B_w \), \( I_{\mu_{x+1}}(f) \) uniformly converges on any compact subset of \( \mathbb{D} \) if and only if
\[
\int_0^{1} \left( \tilde{\omega}(t) + 1 \right) d\mu(t) < \infty. \tag{4.1}
\]

**Proof.** Let \( f \in B_w \), it is easy to verify that
\[
|f(z)| \lesssim (\tilde{\omega}(|z|) + 1) \|f\|_{\mathcal{B}_w} \quad \text{for all } z \in \mathbb{D}. \tag{4.2}
\]
If (4.1) holds, then for each \( 0 < r < 1 \) and \( z \in \mathbb{D} \) with \( |z| \leq r \), we have
\[
|I_{\mu_{x+1}}(f)(z)| \leq \int_0^{1} \frac{|f(t)|}{|1 - rz|^{\alpha+1}} d\mu(t)
\lesssim \frac{\|f\|_{\mathcal{B}_w}}{(1 - r)^{\alpha+1}} \int_0^{1} \left( \tilde{\omega}(t) + 1 \right) d\mu(t)
\lesssim \frac{\|f\|_{\mathcal{B}_w}}{(1 - r)^{\alpha+1}}.
\]
This implies that \( I_{\mu_{x+1}}(f) \) uniformly converges on any compact subset of \( \mathbb{D} \) and hence is analytic in \( \mathbb{D} \).

Suppose that the operator \( I_{\mu_{x+1}} \) is well defined in \( B_w \). Considering the function
\[
f(z) = \int_0^{z} g(s) ds + 1
\]
where $g$ is the function in Lemma 2.2 with respect to $\omega$. Then Lemma 2.2 implies that $f \in B_\omega$. Since $I_{\mu^+}$ is well defined for every $z \in \mathbb{D}$, we have

$$|I_{\mu^+}(f)(0)| = \left|\int_0^1 f(t)d\mu(t)\right| < \infty.$$ 

Since $\mu$ is a positive measure and $g(s) > 0$ for all $s \in [0, 1)$, it follows from Lemma 2.2 that

$$f(t) = \int_0^t g(s)ds + 1 = \omega(t) + 1. \quad (4.3)$$

Therefore,

$$\int_0^1 (\omega(t) + 1)d\mu(t) < \infty.$$

The proof is complete.

The sublinear generalized integral type Hilbert operator $\tilde{I}_{\mu^+}$ is defined by

$$\tilde{I}_{\mu^+}(f)(z) = \int_0^1 \frac{|f(t)|}{(1 - tz)^{\alpha+1}} d\mu(t), \quad (\alpha > -1).$$

It is obvious that Proposition 4.1 also holds if $I_{\mu^+}$ is replaced by $\tilde{I}_{\mu^+}$. By mean of Lemma 2.1, Theorem 3.1 and the sublinear integral type Hilbert operator $\tilde{I}_{\mu^+}$, we have the following results.

**Theorem 4.2.** Let $\omega, \nu \in \mathcal{N}$ and $\alpha > -1$. Suppose $\mu$ is a positive Borel measure on $[0, 1)$ and satisfies (4.1). Then the following statements are equivalent.

a. $I_{\mu^+} : B_\omega \to B_\nu$ is bounded;

b. $\tilde{I}_{\mu^+} : B_\omega \to B_\nu$ is bounded;

c. $\sup_{n \geq 1} n^{\alpha+2} \nu \left(1 - \frac{1}{n}\right) \int_0^1 t^n (\omega(t) + 1)d\mu(t) < \infty.$

**Proof.**

(a) $\Rightarrow$ (c) : If $I_{\mu^+} : B_\omega \to B_\nu$ is bounded. For each $f \in B_\omega$, Proposition 4.1 implies that $I_{\mu^+}(f)$ converges absolutely for every $z \in \mathbb{D}$ and

$$I_{\mu^+}(f)(z) = \sum_{n=0}^\infty \left(\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n f(t)d\mu(t)\right) z^n, \quad z \in \mathbb{D}.$$

Take

$$f(z) = \int_0^z g(s)ds + 1,$$

where $g$ is the function in Lemma 2.2 with respect to $\omega$. Then $f \in B_\omega$ and

$$I_{\mu^+}(f)(z) = \int_0^1 \frac{f(t)}{(1 - tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^\infty b_n z^n$$

where

$$b_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n \left(\int_0^t g(s)ds + 1\right) d\mu(t).$$
It is clear that \( \{b_n\}_{n=1}^{\infty} \) is a nonnegative sequence. Using Theorem 3.1, (4.3), and Stirling’s formula, we have

\[
\|I_{\mu_+}\|(f) \geq \sup_{n \geq 1} n^{\frac{1}{n+1}} \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} k \alpha_k
\]

\[
\geq \sup_{n \geq 1} n^{\frac{1}{n+1}} \left( 1 - \frac{1}{n} \right) \int_{0}^{1} t^n (\tilde{\phi}(t) + 1) \, d\mu(t) \sum_{k=1}^{n} k \alpha_k
\]

\[
= \sup_{n \geq 1} n^{\frac{2}{n+1}} \left( 1 - \frac{1}{n} \right) \int_{0}^{1} t^n (\tilde{\phi}(t) + 1) \, d\mu(t).
\]

Therefore,

\[
\sup_{n \geq 1} n^{\frac{2}{n+1}} \left( 1 - \frac{1}{n} \right) \int_{0}^{1} t^n (\tilde{\phi}(t) + 1) \, d\mu(t) < \infty.
\]

\((c) \Rightarrow (b)\) : For each \( n \in \mathbb{N} \), we have

\[
\int_{0}^{1} t^n (\tilde{\phi}(t) + 1) \, d\mu(t) \leq \frac{1}{n^{\frac{2}{n+1}} \left( 1 - \frac{1}{n} \right)}.
\]

For a given \( 0 \neq f \in B_{\omega_0} \),

\[
\hat{I}_{\mu_+} \circ f(z) = \int_{0}^{1} \frac{|f(t)|}{(1 - tz)^{\alpha_0+1}} \, d\mu(t) = \sum_{n=0}^{\infty} c_n z^n.
\]

where

\[
c_n = \frac{\Gamma(n + 1 + \alpha)}{\Gamma(n + 1) \Gamma(\alpha + 1)} \int_{0}^{1} t^n |f(t)| \, d\mu(t).
\]

Obviously, \( \{c_n\}_{n=1}^{\infty} \) is a nonnegative sequence. Using (4.2), (4.4), and the definition of normal weight, we deduce that

\[
|c_0| + \sup_{n \geq 1} n^{\frac{1}{n+1}} \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} k \alpha_k
\]

\[
\leq \|f\|_{B_\omega} + \|f\|_{B_\omega} \sup_{n \geq 1} n^{\frac{1}{n+1}} \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} (k + 1)^{\alpha_k+1} \int_{0}^{1} t^k (\tilde{\phi}(t) + 1) \, d\mu(t)
\]

\[
\leq \|f\|_{B_\omega} + \|f\|_{B_\omega} \sup_{n \geq 1} n^{\frac{1}{n+1}} \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} \frac{1}{k^n} \left( 1 - \frac{1}{n} \right)
\]

\[
\leq \|f\|_{B_\omega} + \|f\|_{B_\omega} \sup_{n \geq 1} \frac{1}{(n + 1)^\alpha} \sum_{k=1}^{n} (k + 1)^{\alpha_k-1}
\]

\[
\leq \|f\|_{B_\omega}.
\]

Hence, \( \hat{I}_{\mu_+} : B_{\omega_0} \to B_\omega \) is bounded by Theorem 3.1.

\((b) \Rightarrow (a)\) : If \( \hat{I}_{\mu_+} : B_{\omega_0} \to B_\omega \) is bounded, then for each \( f \in B_{\omega_0} \), by Lemma 2.1, we have

\[
\sup_{n \geq 1} \nu(1 - 2^{-n}) 2^n \|V_n \ast \hat{I}_{\mu_+}(f)\|_{\infty} \leq \|\hat{I}_{\mu_+}(f)\|_{B_\omega} \leq \|f\|_{B_\omega} \|\hat{I}_{\mu_+}\|.
\]

Since the coefficients of \( \hat{I}_{\mu_+}(f) \) are nonnegative, it is easy to check that

\[
M_\infty(r, V_n \ast \hat{I}_{\mu_+}(f)) \leq M_\infty \left( r, V_n \ast \hat{I}_{\mu_+}(f) \right) \quad \text{for all } 0 < r < 1.
\]
Therefore,
\[ \|V_n \ast I_{\mu_{\alpha+1}}(f)\|_{\infty} = \sup_{0 < r < 1} M_{\infty}(r, V_n \ast I_{\mu_{\alpha+1}}(f)) \leq \sup_{0 < r < 1} M_{\infty}(r, V_n \ast I_{\mu_{\alpha+1}}(f)) \leq \|f\|_{\infty} \|f\|_{\infty}. \]

Consequently,
\[ \|f\|_{\infty} \leq \sup_{0 < r < 1} M_{\infty}(r, V_n \ast I_{\mu_{\alpha+1}}(f)) \leq \|f\|_{\infty} \|f\|_{\infty}. \]

This implies that \( I_{\mu_{\alpha+1}} : B_\omega \to B_\nu \) is bounded.

\[ \square \]

**Theorem 4.3.** Let \( \omega, \nu \in \mathcal{N} \) and \( \alpha > -1 \). Suppose \( \mu \) is a finite positive Borel measure on \([0, 1]\) and \( \tilde{\omega}(1) < \infty \). Then the following statements are equivalent.

(a) \( I_{\mu_{\alpha+1}} : B_\omega \to B_\nu \) is bounded;
(b) \( I_{\mu_{\alpha+1}} : B_\omega \to B_\nu \) is bounded;
(c) \( I_{\mu_{\alpha+1}} : B_\omega \to B_\nu \) is compact;
(d) \( I_{\mu_{\alpha+1}} : B_\omega \to B_\nu \) is compact;
(e) \( \sup_{n \geq 1} n^{\alpha+2} \nu \left(1 - \frac{1}{n}\right) \mu_n < \infty \).

**Proof.** The equivalence of (a) \( \iff \) (b) \( \iff \) (e) follows from Theorem 4.2 immediately and the implications of (d) \( \implies \) (c) \( \implies \) (a) are obvious. Therefore, we only need to prove that (e) \( \implies \) (d).

Let \( \{f_k\}_{k=1}^{\infty} \) be a bounded sequence in \( B_\omega \), which converges to 0 uniformly on every compact subset of \( \mathbb{D} \). Since \( \tilde{\omega}(1) < \infty \), arguing as the proof of lemma 2.5 in Stević [23], we have that
\[ \lim_{k \to \infty} \sup_{z \in \mathbb{D}} |f_k(z)| = 0. \]

For each \( k \in \mathbb{N} \), we have
\[ I_{\mu_{\alpha+1}}(f_k)(z) = \int_0^1 \frac{|f_k(t)|}{(1 - tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} c_{n,k} z^n, \]
where
\[ c_{n,k} = \frac{\Gamma(n + 1 + \alpha)}{\Gamma(n + 1)\Gamma(\alpha + 1)} \int_0^1 t^n |f_k(t)| d\mu(t). \]

It is obvious that \( \{c_{n,k}\}_{n=1}^{\infty} \) is a nonnegative sequence for each \( k \in \mathbb{N} \). To prove that \( I_{\mu_{\alpha+1}} : B_\omega \to B_\nu \) is compact, it is sufficient to prove that
\[ \lim_{k \to \infty} \left( c_{0,k} + \sup_{n \geq 1} \nu \left(1 - \frac{1}{n}\right) \sum_{j=1}^{n} j c_{j,k} \right) = 0 \]
by using Theorem 3.1 and Lemma 2.6. If \( \sup_{n \geq 1} n^{\alpha+2} \nu \left(1 - \frac{1}{n}\right) \mu_n < \infty \), then
\[ \mu_n \leq \frac{1}{n^{\alpha+2} \nu \left(1 - \frac{1}{n}\right)} \text{ for all } n \in \mathbb{N}. \]
By Stirling’s formula and the above inequality, we have

\[ |c_{0,k}| + \sup_{n \geq 1} \nu \left( 1 - \frac{1}{n} \right) \sum_{j=1}^{n} j c_{j,k} \lesssim \int_{0}^{1} |f_k(t)| d\mu(t) + \sup_{n \geq 1} \nu \left( 1 - \frac{1}{n} \right) \sum_{j=1}^{n} j^{\gamma+1} \int_{0}^{1} t^j |f_k(t)| d\mu(t) \]

\[ \lesssim \sup_{t \in [0,1]} |f_k(t)| + \sup_{t \in [0,1]} |f_k(t)| \sup_{n \geq 1} \nu \left( 1 - \frac{1}{n} \right) \sum_{j=1}^{n} j^{\gamma+1} \mu_j \]

\[ \lesssim \sup_{t \in [0,1]} |f_k(t)| + \sup_{t \in [0,1]} |f_k(t)| \sup_{n \geq 1} \nu \left( 1 - \frac{1}{n} \right) \sum_{j=1}^{n} \frac{1}{j^{\gamma+1}} \]

\[ \lesssim \sup_{t \in [0,1]} |f_k(t)| \rightarrow 0, \ (k \rightarrow \infty). \]

Hence, (d) holds.

5 | SOME APPLICATIONS

As a direct application of the above results, we first consider the operator \( I_{\mu_+^1} \) acting from \( B^\beta \) to \( B^\gamma \). If \( \gamma \geq \alpha + 2 \), then it is easy to see that \( I_{\mu_+^1} : B^\beta \rightarrow B^\gamma \) is always a bounded operator under the condition (4.1). Therefore, we only need to consider the case \( 0 < \gamma < \alpha + 2 \).

**Corollary 5.1.** Let \( \mu \) be a positive Borel measure on \([0,1]\) and satisfies \( \int_{0}^{1} \log \frac{1}{1-t} d\mu(t) < \infty, \alpha > -1 \). If \( 0 < \gamma < \alpha + 2 \), then the following statements are equivalent.

(a) \( I_{\mu_+^1} : B \rightarrow B^\gamma \) is bounded;
(b) \( \mu \) is a 1-logarithmic \( \alpha + 2 - \gamma \)-Carleson measure;
(c) \( \int_{0}^{1} t^\alpha \log \frac{1}{1-t} d\mu(t) = O \left( \frac{1}{n^{\alpha+2-\gamma}} \right) \).

**Proof.** Let \( d\lambda(t) = \log \frac{1}{1-t} d\mu(t) \), then lemma 2.5 in Girela and Merchán [13] shows that \( \mu \) is a 1-logarithmic \( \alpha + 2 - \gamma \)-Carleson measure if and only if \( \lambda \) is an \( \alpha + 2 - \gamma \)-Carleson measure. By theorem 2.1 in Bao and Wulan [24], \( \lambda \) is an \( \alpha + 2 - \gamma \)-Carleson measure if and only if

\[ \int_{0}^{1} t^\alpha \lambda(t) = O \left( \frac{1}{n^{\alpha+2-\gamma}} \right). \]

The desired result follows from Theorem 4.2 immediately.

**Remark 5.2.** If \( \gamma = 1 \) and \( \alpha = 0 \), the result of Theorem 5.1 have been obtained in Girela and Merchán [13] (or [7]). In addition, if \( \gamma = 1 \) and \( \alpha = 1 \), the result have been given in Ye and Zhou [11].

**Corollary 5.3.** Let \( \mu \) be a positive Borel measure on \([0,1]\) and satisfies \( \int_{0}^{1} \frac{d\mu(t)}{(1-t)^{\beta+1}} < \infty, \alpha > -1 \). If \( 0 < \gamma < \alpha + 2 \) and \( \beta > 1 \), then \( I_{\mu_+^1} : B^\beta \rightarrow B^\gamma \) is bounded if and only if \( \mu \) is an \( \alpha + 1 + \beta - \gamma \)-Carleson measure.

**Proof.** It follows from Theorem 4.2 that \( I_{\mu_+^1} : B^\beta \rightarrow B^\gamma \) is bounded if and only if

\[ \int_{0}^{1} t^\alpha \frac{d\mu(t)}{(1-t)^{\beta+1}} = O \left( \frac{1}{n^{\alpha+2-\gamma}} \right). \]

This is equivalent to saying that \( \frac{d\mu(t)}{(1-t)^{\beta+1}} \) is an \( \alpha + 2 - \gamma \)-Carleson measure. The proof can be done by using Lemma 2.5.
Corollary 5.4. Let \( \mu \) be a finite positive Borel measure on \([0, 1]\) and \( \alpha > -1 \). If \( 0 < \gamma < \alpha + 2 \) and \( 0 < \beta < 1 \), then the following statements are equivalent.

(a) \( I_{\mu_{\alpha+1}} : B^\beta \to B^\gamma \) is bounded;
(b) \( I_{\mu_{\alpha+1}} : B^\beta \to B^\gamma \) is compact;
(c) \( \mu \) is an \( \alpha + 2 - \gamma \)-Carleson measure.

Proof. This is a direct consequence of Theorem 4.3. \( \square \)

Remark 5.5. It should be mentioned that Ye and Zhou [15] have obtained some results of Corollary 5.1–5.4 by using the duality theorem. In fact, they dealt with \( \gamma = \alpha \) and \( \alpha \geq 1 \).

In what follows, we consider the operator \( I_{\mu_{\alpha+1}} \) acting between logarithmic Bloch spaces.

Corollary 5.6. Let \( \alpha > -1, \beta > -1, \gamma \in \mathbb{R} \). Suppose \( \mu \) is a positive Borel measure on \([0, 1]\) and satisfies \( \int_0^1 \log^\beta t^{-1} d\mu(t) < \infty \). Then the following statements are equivalent.

(a) \( I_{\mu_{\alpha+1}} : B_{\log^\beta} \to B_{\log^\gamma} \) is bounded;
(b) \( \sup_{n \geq 1} n^{\alpha+1} \log^{-\gamma} \left( n + 1 \right) \int_0^1 t^n \log^\beta e \frac{t}{1-t} d\mu(t) < \infty \);
(c) \( \sup_{t \in [0, 1)} \frac{\mu([t, 1)) \left( \log \frac{e}{1-t} \right)^{\beta+\gamma}}{(1-t)^{\alpha+1}} < \infty \).

Proof. It follows from Theorem 4.2 that \( (a) \iff (b) \). We only need to show that \( (b) \iff (c) \). The implication \( (b) \Rightarrow (c) \) follows from the inequalities

\[
\mu \left( \left[ 1 - \frac{1}{n}, 1 \right) \right) \log^{\alpha+1} (n+1) \leq \int_0^1 t^n \log^{\beta+1} e \frac{t}{1-t} d\mu(t) \leq \frac{\log(n+1)}{n^{\alpha+1}}.
\]

(\( c \Rightarrow b \)). Assume (c). Then there exists a positive constant \( C \) such that

\[
\mu \left( [t, 1) \right) \left( \log \frac{e}{1-t} \right)^{\beta+\gamma} \leq C(1-t)^{\alpha+1}, 0 \leq t < 1.
\]

Integrating by parts, we obtain

\[
\int_0^1 t^n \log^{\beta+1} e \frac{t}{1-t} d\mu(t) = n \int_0^1 t^{n-1} \mu([t, 1)) \log^{\beta+1} e \frac{t}{1-t} dt + (\beta + 1) \int_0^1 t^n \mu([t, 1)) \log^\beta e \frac{t}{1-t} dt
\]

\[
\leq n \int_0^1 t^{n-1} (1-t)^{\alpha+1} \log^\gamma e \frac{t}{1-t} dt + \int_0^1 t^n (1-t)^{\alpha} \log^{\gamma-1} e \frac{t}{1-t} dt.
\]

Note that

\[
\phi_1(t) = (1-t)^{\alpha+1} \log^\gamma e \frac{t}{1-t}, \quad \phi_2(t) = (1-t)^{\alpha} \log^{\gamma-1} e \frac{t}{1-t}
\]

are regular in the sense of Peláez and Rättyä [25]. Then, using lemma 1.3 and (1.1) in Peláez and Rättyä [25], we have

\[
n \int_0^1 t^{n-1} (1-t)^{\alpha+1} \log^\gamma e \frac{t}{1-t} dt \asymp \frac{\log(n+1)}{n^{\alpha+1}}
\]

and

\[
\int_0^1 t^n (1-t)^{\alpha} \log^{\gamma-1} e \frac{t}{1-t} dt \asymp \frac{\log^{\gamma-1}(n+1)}{n^{\alpha+1}}.
\]
These two estimates imply that
\[
\int_0^1 t^n \log^{\beta+1} \frac{2}{1-t} \, d\mu(t) \lesssim \frac{\log^{\gamma}(n+1)}{n^{\beta+1}}.
\]

Thus, (b) holds.

Arguing as the proof of previous theorem, one can obtain the following theorems.

**Corollary 5.7.** Let \( \alpha > -1, \beta = -1, \gamma \in \mathbb{R} \). Suppose \( \mu \) is a positive Borel measure on \([0, 1]\) and satisfies
\[
\int_0^1 \log \log \frac{\epsilon}{1-t} \, d\mu(t) < \infty.
\]
Then the following statements are equivalent.

1. \( I_{\mu_{\gamma}} : B_{\log^{\gamma}} \to B_{\log^{\gamma}} \) is bounded;
2. \( \sup_{n \geq 1} n^\gamma \log^{\gamma} (n+1) \int_0^1 t^n \log \log \frac{\epsilon}{1-t} \, d\mu(t) < \infty; \)
3. \( \sup_{t \in [0,1]} \frac{\mu([t,1)) \log \log \frac{\epsilon}{1-t}}{1-t} < \infty. \)

**Corollary 5.8.** Let \( \alpha > -1, \beta < -1, \gamma \in \mathbb{R} \). Suppose \( \mu \) is a finite positive Borel measure on \([0, 1]\), then the following statements are equivalent.

1. \( I_{\mu_{\gamma}} : B_{\log^{\gamma}} \to B_{\log^{\gamma}} \) is bounded;
2. \( I_{\mu_{\gamma}} : B_{\log^{\gamma}} \to B_{\log^{\gamma}} \) is compact;
3. \( \sup_{n \geq 1} n^\gamma \log^{\gamma} (n+1) \mu_n < \infty; \)
4. \( \sup_{t \in [0,1]} \frac{\mu([t,1)) \log \log \frac{\epsilon}{1-t}}{1-t} < \infty. \)

It is known that \( H \) maps \( B_{\log^{\beta}} \) into \( B_{\log^{\beta+1}} \) for all \( \beta \in \mathbb{R} \) (see e.g., Karapetrović [26]). If \( \mu \) is Lebesgue measure on \([0, 1]\), then Corollary 5.6-5.8 show that the integral type Hilbert operator \( I : B_{\log^{\beta}} \to B_{\log^{\beta+1}} \) is bounded if and only if \( \beta > -1. \)

**CONFLICT OF INTEREST STATEMENT**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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