Stability of peakons for the generalized modified Camassa-Holm equation

Xingxing Liu*

Department of mathematics, China University of Mining and Technology,
Xuzhou 221116, Jiangsu, P. R. China

Abstract

In this paper, we consider the orbital stability of peakons for the generalized modified Camassa-Holm (CH) equation, which admits a single peaked soliton and multi-peakon solutions. We firstly apply the definition of weak solution and some combination formulas to investigate the existence of single peakon. Then we prove two useful conservation laws which play a key role in our proof of stability of peakons. Finally by introducing a new auxiliary functional \( h \) and establishing a crucial polynomial inequality, we successfully extend the result of stability of peakons for the modified CH equation (Comm. Math. Phys., 322:967-997, 2013) to the generalized case when \( n \) is any positive odd integer.

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1 Introduction

In the past two decades, the Camassa-Holm (CH) equation

\[
y_t + u y_x + 2 u_x y = 0, \quad y = u - u_{xx},
\]

attracted a great deal of attention among the nonlinear integrable equations and the communities of the PDEs. In 1993, Camassa and Holm [4] obtained the CH equation by approximating directly in the Hamiltonian for Euler’s equations in the shallow water regime. It can model the unidirectional propagation of shallow water waves over a flat bottom [4, 14, 20], with \( u(t, x) \) standing for the fluid velocity at time \( t \geq 0 \) in the spatial \( x \in \mathbb{R} \) direction. Actually, the CH

*E-mail: liuxxmaths@cumt.edu.cn.
equation was initially introduced in 1981 by Fuchssteiner and Fokas [20] as a bi-Hamiltonian generalization of the KdV equation. Dai [16] derived it as an equation describing the propagation of axially symmetric waves in hyperelastic rods. The CH equation shares with the classical KdV equation the properties that it has bi-Hamiltonian structure [20] and is completely integrable [1, 8]. However, while all smooth solutions of the KdV equation are global, the CH equation admits global strong solutions [7, 10, 11] as well as breaking waves [7, 10, 11, 12], i.e., the wave profile remains bounded, but its slope becomes unbounded in finite time [38].

Another remarkable property of the CH equation is the presence of peaked solitary wave solutions [5], called peakons. They are given by

\[ u(t, x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \]

which are solitons, retaining their shape and speed after interacting with other peakons [2]. It is worth pointing out that the feature of peakons that their profile is smooth except for a peak at its crest, is analogous to that of the waves of greatest height, i.e., traveling waves of largest possible amplitude which are solutions to be governing equations for water waves [9, 13, 37]. It is well-known that the stability of solitary waves is one of the fundamental qualitative properties of the solutions for nonlinear wave equations. However, the stability of peakons to the CH equation seems not to enter the general framework developed for instance in [3, 23]. In 2000, Constantin and Strauss [15] applied the conservation laws to give an impressive proof of stability by a direct approach. In 2009, El Dika and Molinet [17] derived the stability of multi-peakons by combining the proof of stability of single peakon with a property of almost monotonicity of the local energy norm. Then they [18] also considered the stability of ordered trains of anti-peakons and peakons.

Recently, the great interest in the CH equation has inspired the search for various CH-type equations with cubic or higher-order nonlinearity. One of the most concerned is the following modified CH equation

\[ y_t + ((u^2 - u_{xx}^2)y)_x = 0, \quad y = u - u_{xx}, \]

which was derived by Fuchssteiner [22] and Olver and Rosenau [30] using the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation. Subsequently, it was shown to arise from the asymptotic theory of surface water waves by Fokas [19]. Moreover, Qiao [31] deduced it from the two-dimensional Euler equations, where \( u(t, x) \) denotes the fluid velocity and hence \( y(t, x) \) represents its potential density. Hence the modified CH equation is also called FORQ equation in some literature [25, 39].

The modified CH equation is completely integrable [30]. It has a bi-Hamiltonian structure and also admits a Lax pair [33], and hence can be solved by the inverse scattering transform method. Compared with the CH equation, the modified CH equation admits not only peaked
solitary waves (peakons), but also possesses cusp solitons (cuspons) and weak kink solutions
\((u, u_x, u_t)\) are continuous, but \(u_{xx}\) has a jump at its peak point) \([33, 34]\). It has also significant
differences from the CH equation on the dynamics of the two-peakons and peakon-kink solutions
\([32, 34]\). Fu et al. \([21]\) studied the Cauchy problem of the modified CH equation in Besov spaces
and the blow-up scenario. The non-uniform dependence on the initial data was established in
\([25]\). Gui et al. \([24]\) considered the formulation of singularities of solutions and showed that some
solutions with certain initial data would blow up in finite time. Then the blow-up phenomena
were systematically investigated in \([6, 29]\). The modified CH equation admits single peakon of
the form \([24]\)
\[
\phi_c(t, x) = \sqrt{\frac{3c^2}{2}} e^{-|x-ct|}, \quad c > 0.
\]
Later, inspired by \([15]\) and \([17]\), the orbital stability of single peakon and the train of peakons
for the modified CH equation were proven in \([35]\) and \([28]\), respectively.

In this paper, we consider the following generalized modified CH equation proposed in \([36]\):
\[
y_t + \left( (u^2 - u_x^2)^n y \right)_x = 0, \quad y = u - u_{xx}, \tag{1.1}
\]
where \(n\) is positive integer. When \(n = 1\), Eq. (1.1) becomes the modified CH equation. Motivated
by the fact that the generalized KdV equation \(v_t - v^p v_x - v_{xxx} = 0, \ (p \neq 0)\), shared one of the
KdV’s two Hamiltonian structure and reduced to the KdV equation when \(p = 1\), Recio and
Anco \([36]\) introduced Eq. (1.1) in the study of the family of nonlinear dispersive wave equations
involving two arbitrary functions. Very recently, the local well-posedness in Besov spaces for the
Cauchy problem associated to Eq. (1.1), and blow-up mechanism have been discussed in \([39]\).
On the other hand, one of the main interesting features of the CH equation (with quadratic
nonlinearity) and the modified CH equation (with cubic nonlinearity) is the existence of stable
peakons. Notice that, Eq. (1.1) with higher-order nonlinearity can also represent single peakon
and multi-peakons \([36]\). Thus it is of great interest to identify whether the single peakon of Eq.
(1.1) has the similar stable result as the CH and modified CH equations.

To pursue the above goal, our approach to prove the stability of peakons for Eq. (1.1) is
adapted from the excellent idea of Constantin and Strauss developed in \([15]\). However, due to
the generalized form with higher-order nonlinearity, the analysis of stability result for Eq. (1.1)
is more challenging than for the CH and modified CH equations. On one hand, owing to the
same conservation law \(E(u)\) as the CH and modified CH equations, we naturally expect the
orbital stability of peakons for Eq. (1.1) in the sense of the energy space \(H^1(\mathbb{R})\)-norm. While
the other conservation law
\[
F(u) = \int_{\mathbb{R}} u(u^2 - u_x^2)^n y dx, \quad y = u - u_{xx},
\]
of Eq. (1.1) given in [36], involving the terms of $u, u_x$ and the momentum density $y$, is much more complicated than the one of the modified CH equations. For our purpose, we first need to derive an equivalent form of $F(u)$ only involving the terms of $u$ and $u_x$. Then we verify that the form of $F(u)$ is conserved (see Lemma 2.3). The two useful conservation laws $E(u)$ and $F(u)$ play a crucial role in our approach.

On the other hand, in view of [15, 35], we see that the most difficult part of proof of stability is to introduce two functionals $g$ and $h$, so that two integral relations between two conservation laws $E(u)$ and $F(u)$ can be connected by a polynomial inequality (see Lemma 3.2). There are two difficulties encountered by Eq. (1.1) in establishing the polynomial inequality. First, although we can define the same functional $g$ as in the CH and modified CH equations because of the same conserved quantity $E(u)$, here we require more subtle analysis on constructing the other functional $h$. Fortunately, based on the observation that the functionals $g$ and $h$ are required to vanish at the peakons, we derive a polynomial of degree 2 as $h$ from the equivalent form of $F(u)$. The second difficulty, arising from the derivation of the required polynomial inequality, is to show $h \leq \frac{2}{5} M^2$, $M \triangleq \max_{x \in \mathbb{R}} u(x)$ (see (3.8)), which is substantially different from the cases of the CH equation ($h = u^2$) and modified CH equation ($h = u^2 + \frac{2}{5} uu_x - \frac{1}{3} u_x^2 \leq \frac{4}{5} M^2$).

Of course this new difficulty is caused by the complicated nonlinear structure and higher-order conservation laws. To overcome it, we here treat the estimate of $h$ by using a recursive argument and the Cauchy-Schwarz inequality in a delicate way. Note that, we also need the assumption $y_0 = u_0 - \partial_x^2 u_0 \geq 0$ as the case of $n = 1$ of Eq. (1.1) in [35] to guarantee that the solution $u$ is positive and $(1 \pm \partial_x) u \geq 0$. Indeed, after overcoming the above difficulties, the strategy of proof of stability is rather routine. By expanding $E(u)$ around the peakon $\varphi_c$, the $H^1$-norm between the solution $u$ and the peakon $\varphi_c(\cdot - \xi(t))$ is precisely controlled by the error term $|M - \max_{x \in \mathbb{R}} \varphi_c|$, while the error term can just be estimated by analyzing the root structure of the obtained polynomial inequality.

Moreover, the important point to note here is that we restrict our discussion of the stability of peakons of Eq. (1.1) when $n$ is positive odd integer, even though the existence of peakons and conservation laws $E(u), F(u)$ hold for all positive integer. As mentioned above, the current approach requires principally one ingredient: the precise polynomial inequality related to the conservation laws. Even if we can construct the functionals $g$ and $h$ for the even integer case, it seems to be lack of estimation of the functional $h$. Take Eq. (1.1) with $n = 2$ for example, the function $h(x)$ can be defined as

$$h(x) \triangleq \begin{cases} u^4 - \frac{6}{5} u^3 u_x - \frac{2}{5} u^2 u_x^2 + \frac{2}{3} uu_x^3 + \frac{1}{5} u_x^4(x), & x < \xi, \\ u^4 + \frac{6}{5} u^3 u_x - \frac{2}{5} u^2 u_x^2 - \frac{2}{3} uu_x^3 + \frac{1}{5} u_x^4(x), & x > \xi, \end{cases}$$

which cannot generally be bounded by $\frac{8}{5} u^4$ due to the positive term $\frac{1}{5} u_x^4$. Thus, when $n$ is positive even integer in Eq. (1.1), the approach applied here may break down.
The remainder of this paper is organized as follows. In Section 2, we mainly demonstrate that the existence of peakons which can be understood as weak solutions for Eq. (1.1), and then prove two conserved quantities which are crucial in the proof of stability. In Section 3, the orbital stability of peakons for Eq. (1.1) is proved based on several useful lemmas. We end the paper with an appendix devoted to the proofs of the last third equality (3.6) and last inequality of (3.27) in Section 4.

Notation. In the following, the symbols \( \lesssim \) is used to express the corresponding inequality that includes a universal constant. For example, \( f \lesssim g \) denotes that there exists a constant \( C > 0 \) such that \( f \leq C g \).

2 Preliminaries

In this section, we first recall the local well-posedness result regarding the Cauchy problem associated to Eq. (1.1) (up to a slight modification), and properties for strong solutions. Then we show the existence of single peakon and prove two useful conservation law, which will be frequently used in the rest of the paper.

Lemma 2.1. [39] Let \( u_0(x) = u(0, x) \in H^s(\mathbb{R}) \) with \( s > \frac{5}{2} \). Then there exists a time \( T > 0 \) such that the Cauchy problem (1.1) has a unique strong solution \( u(t, x) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})) \) and the map \( u_0 \to u \) is continuous from a neighborhood of \( u_0 \) in \( H^s(\mathbb{R}) \) into \( C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})) \).

The proof of the following lemma is quite similar to Lemmas 2.8-2.9 in [27], thus we omit it here.

Lemma 2.2. Assume \( u_0(x) \in H^s(\mathbb{R}), \ s > \frac{5}{2}. \) If \( y_0(x) = (1 - \partial^2_x)u_0(x) \) does not change sign, then \( y(t, x) \) will not change sign for all \( t \in [0, T) \). It follows that if \( y_0 \geq 0 \), then the corresponding solution \( u(t, x) \) of Eq. (1.1) is positive for \( (t, x) \in [0, T) \times \mathbb{R} \). Furthermore, if \( y_0 \geq 0 \), then the corresponding solution \( u(t, x) \) of Eq. (1.1) satisfies
\[
(1 \pm \partial_x)u(t, x) \geq 0, \ \text{for} \ \forall (t, x) \in [0, T) \times \mathbb{R}.
\]

Now, with \( y = u - u_{xx} \), we can rewrite Eq. (1.1) in a more convenient form.

\[
\begin{align*}
\frac{u_t}{u_{txx}} &+ \sum_{k=0}^{n-1} (-1)^k \left( C_n^k + 2nC_{n-1}^k \right) u^{2n-2k} u_x^{2k+1} + (-1)^n u_x^{2n+1} \\
- \sum_{k=0}^{n-1} (-1)^k 4nC_n^k u^{2n-2k-1} u_x^{2k+1} u_{xx} &+ \sum_{k=0}^{n-1} (-1)^k 2nC_n^k u^{2n-2k-2} u_x^{2k+1} u_{xx}^2 \\
- \sum_{k=0}^{n} (-1)^k C_{n-k}^{2k} u^{2n-2k} u_x^{2k} u_{xxx} & = 0. \quad (2.1)
\end{align*}
\]
Applying the operator \((1 - \partial_x^2)^{-1}\) to the both sides of the above equation, one can get

\[
\begin{align*}
    u_t + \left( u^{2n} - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k} u_x^{2k} \right) u_x
    &+ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k u^{2n-2k+1} u_x^{2k+1}
    + \left( 1 - \partial_x^2 \right)^{-1} \partial_x \left( \frac{2n}{2n+1} u^{2n+1} \right)
    + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k} u_x^{2k+1} \right) = 0. \quad (2.2)
\end{align*}
\]

Recall that if \(p(x) \triangleq \frac{1}{2} e^{-|x|}, x \in \mathbb{R}\), then \((1 - \partial_x^2)^{-1} f = p * f\) for all \(f \in L^2\). In order to provide a mathematical framework for the study of peakons, we apply (2.2) to define the notion of weak solutions for Eq. (1.1).

**Definition 2.1.** Let \(u_0 \in W^{1,2n+1}(\mathbb{R})\) be given. If \(u(t, x) \in L^\infty_{\text{loc}}([0, T); W^{1,2n+1}_{\text{loc}}(\mathbb{R}))\) and satisfies

\[
\begin{align*}
    &\int_0^T \int_{\mathbb{R}} \left[ u \phi_t + \frac{1}{2n+1} u^{2n+1} \phi_x + \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k} u_x^{2k+1} \right) \phi \right. \\
    &+ p * \left( \frac{2n}{2n+1} u^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k u^{2n-2k+1} u_x^{2k} \right) \cdot \phi_x \phi \\
    &- p * \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k} u_x^{2k+1} \right) \cdot \phi \right] dx dt + \int_{\mathbb{R}} u_0(x) \phi(0, x) dx = 0,
\end{align*}
\]

for all functions \(\phi(t, x) \in C^\infty([0, T] \times \mathbb{R})\), then \(u(t, x)\) is called a weak solution to Eq. (1.1). If \(u\) is a weak solution on \([0, T)\) for every \(T > 0\), then it is called a global weak solution.

**Remark 2.1.** Since the Sobolev space \(W^{1,2n+1}_{\text{loc}}(\mathbb{R})\) can be embedded in the Hölder space \(C^\alpha(\mathbb{R})\) with \(0 \leq \alpha \leq \frac{2n}{2n+1}\), Definition 2.1 precludes the admissibility of discontinuous shock waves as weak solutions.

According to Definition 2.1, we discuss the existence result of single peakon.

**Theorem 2.1.** For any \(a \neq 0\), the peaked functions of the form

\[
\varphi_c(t, x) = ae^{-|x-ct|}, \quad \text{where} \quad c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k) u^{2n}, \quad (2.3)
\]

is a global weak solution to Eq. (1.1) in the sense of Definition 2.1.

**Proof.** For any test function \(\phi(\cdot) \in C^\infty_\text{c}(\mathbb{R})\), integrating by parts, we infer

\[
\begin{align*}
    \int_{\mathbb{R}} e^{-|x|} \phi'(x) dx &= \int_{-\infty}^{0} e^x \phi(x) dx + \int_{0}^{\infty} e^{-x} \phi(x) dx \\
    &= e^x \phi(x) \big|_{-\infty}^{0} - \int_{-\infty}^{0} e^x \phi(x) dx + e^{-x} \phi(x) \big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x} \phi(x) dx \\
    &= - \int_{-\infty}^{0} e^x \phi(x) dx + \int_{0}^{\infty} e^{-x} \phi(x) dx = \int_{\mathbb{R}} \text{sign}(x) e^{-|x|} \phi(x) dx.
\end{align*}
\]
Thus, for all \( t \geq 0 \), we have
\[
\partial_x \varphi_c(t, x) = -\text{sign}(x - ct) \varphi_c(t, x),
\]
(2.4)
in the sense of distribution \( \mathcal{S}'(\mathbb{R}) \). Letting \( \varphi_{0,c}(x) \triangleq \varphi_c(0, x) \), then we get
\[
\lim_{t \to 0^+} \| \varphi_c(t, \cdot) - \varphi_{0,c}(x) \|_{W^{1,\infty}} = 0.
\]
(2.5)
The same computation as in (2.4), for all \( t \geq 0 \), yields,
\[
\partial_t \varphi_c(t, x) = c \text{ sign}(x - ct) \varphi_c(t, x) \in L^\infty.
\]
(2.6)
Combining (2.4)-(2.6) with integration by parts, for any test function \( \phi(t, x) \in C^\infty_c([0, \infty) \times \mathbb{R}) \), we obtain
\[
\begin{align*}
\int_0^{+\infty} \int_{\mathbb{R}} \left[ \varphi_c \partial_t \phi + \frac{1}{2n+1} \varphi_c^{2n+1} \partial_x \phi + \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_c^{2n-2k} (\partial_x \varphi_c)^{2k+1} \right) \phi \right] dx dt \\
+ \int_{\mathbb{R}} \varphi_{0,c}(x) \phi(0, x) dx \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \left[ \partial_t \varphi_c + \varphi_c^{2n} \partial_x \varphi_c - \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_c^{2n-2k} (\partial_x \varphi_c)^{2k+1} \right) \phi \right] dx dt \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \phi \text{ sign}(x - ct) \varphi_c \left[ c - \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_c^{2n} \right) \right] dx dt.
\end{align*}
\]
(2.7)
Using the definition of \( \varphi_c \) and the relation \( c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k) a^{2n} \), we deduce that for \( x > ct \),
\[
\begin{align*}
\text{sign}(x - ct) \varphi_c \left[ c - \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_c^{2n} \right) \right] \\
= \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) a^{2n+1} \left( e^{ct-x} - e^{(2n+1)(ct-x)} \right),
\end{align*}
\]
(2.8)
and for \( x \leq ct \),
\[
\begin{align*}
\text{sign}(x - ct) \varphi_c \left[ c - \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_c^{2n} \right) \right] \\
= - \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) a^{2n+1} \left( e^{x-ct} - e^{(2n+1)(x-ct)} \right).
\end{align*}
\]
(2.9)
On the other hand, by Definition 2.1, we derive
\[
\int_0^{+\infty} \int_{\mathbb{R}} \left[ (1 - \partial_x^2)^{-1} \left( \frac{2n}{2n+1} \varphi_c^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k-1}C^k_{n} \varphi_c^{2n-2k+1}(\partial_x \varphi_c)^{2k}}{2k-1} \right) \right] dxdt \\
- (1 - \partial_x^2)^{-1} \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}C^k_{n} \varphi_c^{2n-2k}(\partial_x \varphi_c)^{2k+1}}{2k+1} \right) dxdt
\]

\[
= \int_0^{+\infty} \int_{\mathbb{R}} \left[ \phi \cdot \partial_x p \ast \left( \sum_{k=1}^{n} \frac{(-1)^{k-1}C^k_{n} \varphi_c^{2n-2k+1}(\partial_x \varphi_c)^{2k}}{2k-1} \right) \right] dxdt
\]

We calculate from (2.10) that,

\[
n \varphi_c^{2n} \partial_x \varphi_c + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C^k_{n} \varphi_c^{2n-2k}(\partial_x \varphi_c)^{2k+1}
\]

\[
= n \varphi_c^{2n} \left( -\text{sign} (x - ct) \varphi_c \right) + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C^k_{n} \varphi_c^{2n-2k} \left( -\text{sign} (x - ct) \varphi_c \right)^{2k+1}
\]

\[
= \left[ 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C^k_{n} \right] (\text{sign} (x - ct) \varphi_c)^{2n+1}
\]

\[
= \frac{1}{2n+1} \left[ 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C^k_{n} \right] \partial_x (\varphi_c^{2n+1}),
\]

which together with (2.10) leads to

\[
\int_0^{+\infty} \int_{\mathbb{R}} \left[ (1 - \partial_x^2)^{-1} \left( \frac{2n}{2n+1} \varphi_c^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k-1}C^k_{n} \varphi_c^{2n-2k+1}(\partial_x \varphi_c)^{2k}}{2k-1} \right) \right] dxdt \\
- (1 - \partial_x^2)^{-1} \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}C^k_{n} \varphi_c^{2n-2k}(\partial_x \varphi_c)^{2k+1}}{2k+1} \right) dxdt
\]

\[
= \int_0^{+\infty} \int_{\mathbb{R}} \left[ \phi \cdot \partial_x p \ast \left( \sum_{k=1}^{n} \frac{(-1)^{k-1}C^k_{n} \varphi_c^{2n-2k+1}(\partial_x \varphi_c)^{2k}}{2k-1} \right) \right] dxdt \\
+ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C^k_{n} \varphi_c^{2n+1} \right) dxdt.
\]

Note that \( \partial_x p(x) = -\frac{1}{2} \text{sign}(x)e^{-|x|} \) for \( x \in \mathbb{R} \), we deduce

\[
\partial_x p \ast \left[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C^k_{n} \varphi_c^{2n-2k+1}(\partial_x \varphi_c)^{2k} + \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C^k_{n} \varphi_c^{2n+1} \right) \right]
\]
For $x > ct$, we can split the right hand side of (2.12) into the following three parts,

$$I ≜ I_1 + I_2 + I_3.$$  

A direct calculation for each one of the terms $I_i, 1 \leq i \leq 3$, yields

$$I_1 = -\frac{1}{2} \int_{-\infty}^{+\infty} \text{sign}(x - y) e^{-|x-y|} \left[ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1} C^k_n}{2k+1} \right) + \sum_{k=1}^{n} \frac{(-1)^{k-1} C^k_n}{2k-1} \right] a^{2n+1} dy.$$  

$$I_2 = \frac{1}{2} \int_{ct}^{x} e^{-(x-y)} e^{-(2n+1)(y-ct)} dy.$$  

$$I_3 = \frac{1}{2} \int_{ct}^{+\infty} e^{-(x-y)} e^{-(2n+1)|y-ct|} dy.$$  

For $x > ct$, we can split the right hand side of (2.12) into the following three parts,

$$\partial_x p \left[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C^k_n e^{2n+1} (\partial_x \varphi^2 e^{2k}) + \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1} C^k_n}{2k+1} \right) e^{2n+1} \right]$$

$$\times \left[ \int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty} \right] \text{sign}(x - y) e^{-|x-y|} e^{-(2n+1)|y-ct|} dy.$$
\[
\begin{gathered}
= -\frac{1}{4n} \left[ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] \\
\times a^{2n+1} (e^{\sigma - x} - e^{(2n+1)(\sigma - x)}),
\end{gathered}
\]

and
\[
I_3 = \frac{1}{2} \left[ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] a^{2n+1} \\
\times \int_x^{+\infty} e^{x-y} e^{-(2n+1)(y-\sigma)} dy \\
= \frac{1}{2} \left[ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] a^{2n+1} \\
\times e^{x+(2n+1)\sigma} \int_x^{+\infty} e^{-(2n+2)y} dy \\
= \frac{1}{4n+4} \left[ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] a^{2n+1} e^{(2n+1)(\sigma - x)}.
\]

Plugging the above equalities \(I_1-I_3\) into (2.13), we find that for \(x > \sigma\),
\[
\begin{gathered}
\partial_x p \left[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} c_n^k \varphi_{c}^{2n-2k+1} (\partial_x \varphi_{c})^{2k} + \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) \varphi_{c}^{2n+1} \right] \\
= -\frac{1}{4n \cdot (n+1)} \left[ \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + (2n+1) \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] \\
\times a^{2n+1} (e^{\sigma - x} - e^{(2n+1)(\sigma - x)}),
\end{gathered}
\]

While for \(x \leq \sigma\), we also split the right hand side of (2.12) into three parts,
\[
\begin{gathered}
\partial_x p \left[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} c_n^k \varphi_{c}^{2n-2k+1} (\partial_x \varphi_{c})^{2k} + \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) \varphi_{c}^{2n+1} \right] \\
= -\frac{1}{2} \left[ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] a^{2n+1} \\
\times \left[ \int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{+\infty} \right] \text{sign}(x-y) e^{-|x-y|} e^{-(2n+1)(y-\sigma)} dy \\
\triangleq II_1 + II_2 + II_3.
\end{gathered}
\]

Applying a similar computation as \(I_i, 1 \leq i \leq 3\), to the terms \(II_1-II_3\) on the right hand side of
(2.15), we find that for $x \leq c t$,
\[
\partial_x p = \left[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k n^{2n-2k+1} (\partial_x \varphi) \varphi^{2n+1} \frac{1}{2k-1} \left(2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) \right]^{2k} + 1 + \frac{1}{2n+1} \left(2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) \varphi^{2n+1}.
\]
\[
= \frac{1}{4n \cdot (n+1)} \left[ \left(2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) + (2n+1) \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] \phi^{2n+1} e^{x-ct} - e^{(2n+1)(x-ct)}.
\]

Now we claim that the following identity holds,
\[
\frac{1}{4n \cdot (n+1)} \left[ (2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k ) + (2n+1) \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \right] = 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k.
\]

Indeed, to prove (2.17), we need only to show that the following equivalent identity
\[
(2n + 1) \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_n^k + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k = 2n + (-1)^n,
\]

is true. According to the combination formula: $\sum_{k=0}^{n} \frac{(-1)^{k}}{2k+1} C_n^k = \frac{(2n)!}{(2n+1)!}$, we have
\[
\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k = -\left( \frac{(2n)!}{(2n+1)!} - 1 - \frac{(-1)^{n}}{2n+1} \right).
\]

Substituting (2.19) into (2.18), we deduce that it suffices to show that
\[
\sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k = \frac{(2n)!}{(2n-1)!} - 1.
\]

(2.20) is obviously right for $n = 1$. By induction, assuming (2.20) to hold for $n$, we will prove it for $n + 1$,
\[
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{2k-1} C_n^k + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k + \frac{(-1)^n}{2n+1} +\frac{(-1)^n}{2n+1} = \frac{(2n)!}{(2n-1)!} - 1 + \sum_{k=0}^{n} \frac{(-1)^{k}}{2k+1} C_n^k = \frac{(2n)!}{(2n-1)!} - 1 + \frac{(2n)!}{(2n+1)!}.
\]
which implies (2.20). Note that the identity (2.20) seems to be of independent interest. Hence, by (2.8), (2.9), (2.14), (2.16) and (2.17), we deduce that for all \((t, x) \in (0, +\infty) \times \mathbb{R}\),

\[
\text{sign}(x - ct) \varphi_c \left[ c - \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_{2k}^n \right) \right] + \partial_x p \ast \left[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \varphi_{2n-2k+1}^n (\partial_x \varphi_c)^{2k} \right] \\
+ \frac{1}{2n+1} \left( 2n + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_{2n}^n \right) (t, x) = 0.
\]  

(2.21)

Thanks to (2.7), (2.11) and (2.21), we conclude that

\[
\int_0^{+\infty} \int_{\mathbb{R}} \left[ \varphi_c \partial_t \phi + \frac{1}{2n+1} \varphi_{2n+1}^n \partial_x \phi + \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_{2n-2k}^n (\partial_x \varphi_c)^{2k} \right) \partial_x \phi \right] \, dx \, dt \\
+ (1 - \partial_x^2)^{-1} \left( \frac{2n}{2n+1} \varphi_{2n+1}^n + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_n^k \varphi_{2n-2k+1}^n (\partial_x \varphi_c)^{2k} \right) \partial_x \phi \\
- (1 - \partial_x^2)^{-1} \left( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \varphi_{2n-2k}^n (\partial_x \varphi_c)^{2k+1} \right) \phi \right] \, dx \, dt + \int_{\mathbb{R}} \varphi_0(x) \phi(0, x) \, dx = 0,
\]

for every test function \(\phi(t, x) \in C_c^\infty([0, \infty) \times \mathbb{R})\). This completes the proof of Theorem 2.1. 

In the following lemma, we give the proof of two useful conservation laws which are crucial in our development.

**Lemma 2.3.** If the initial data \(u_0 \in H^s(\mathbb{R})\) with \(s > \frac{5}{2}\), then the functionals

\[
E(u) \triangleq \int_{\mathbb{R}} (u^2 + u_x^2) \, dx,
\]

and

\[
F(u) \triangleq \int_{\mathbb{R}} \left( u^{2n+2} + \sum_{k=1}^{n} (\frac{(-1)^{k+1}}{2k-1} C_n^k u^{2n-2k+2} u_x^{2k} + (-1)^n \frac{u_x^{2n+2}}{2n+1} ) \right) \, dx,
\]

are invariants for Eq. (1.1).

**Proof.** By Eq. (1.1), we have

\[
u_t - u_{txx} + (u^2 - u_x^2)^n \partial_x y = -2n(u^2 - u_x^2)^{n-1} u_x y^2.
\]

Multiplying the above equation by \(u\) and integrating over \(\mathbb{R}\) with respect to \(x\), and then inte-
gration by parts yields

\[
\frac{1}{2} \frac{dE(u)}{dt} = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) dx = - \int_{\mathbb{R}} u(u^2 - u_x^2)u_y^2 dx - 2n \int_{\mathbb{R}} (u^2 - u_x^2)^{n-1} u u_x y^2 dx
\]

\[
= \int_{\mathbb{R}} u_x (u^2 - u_x^2)^n y dx + n(u^2 - u_x^2)^{n-1} u (2u u_x - 2u_x u_{xx}) y dx
\]

\[
- 2n \int_{\mathbb{R}} (u^2 - u_x^2)^{n-1} u u_x y^2 dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2)^n (u^2 - u_x^2) = 0,
\]

which implies (2.22).

Setting \( v(x, t) = \int_{-\infty}^x u(z, t) dz \), we consider the following equalities

\[
\frac{d}{dt} \int_{\mathbb{R}} u^{2n+2} dx = \int_{\mathbb{R}} (2n+2) u^{2n+1} u_t dx = (2n+2) \int_{\mathbb{R}} u^{2n+1} u_x dx
\]

\[
= -(2n+2) \cdot \int_{\mathbb{R}} (2n+1) u^{2n} u_x v dx,
\]

(2.24)

\[
\frac{d}{dt} \int_{\mathbb{R}} (-1)^n \frac{1}{2n+1} u_x^{2n+2} dx
\]

\[
= (-1)^n \frac{2n+2}{2n+1} \int_{\mathbb{R}} u_x^{2n+1} u_{tx} dx = (-1)^n \frac{2n+2}{2n+1} \int_{\mathbb{R}} u_x^{2n+1} u_{xx} dx
\]

\[
= (2n+2) \int_{\mathbb{R}} (-1)^n (2n u_x^{2n-1} u_{xx}^2 + u_x^{2n} u_{xxx}) v dx,
\]

(2.25)

and

\[
\frac{d}{dt} \int_{\mathbb{R}} (n+1) u^{2n} u_x^2 dx
\]

\[
= (n+1) \int_{\mathbb{R}} 2n u^{2n-1} u_x^2 u_t + 2n u^{2n} u_x u_{xx} dx = (2n+2) \int_{\mathbb{R}} (n u^{2n-1} u_x^2 v + u^{2n} u_x v_{xx}) dx
\]

\[
= (2n+2) \int_{\mathbb{R}} \left[ - (n(2n-1)) u^{2n-2} u_x^3 + 2n u^{2n-1} u_x u_{xx} v + (2n(2n-1)) u^{2n-2} u_x^3
\]

\[
+ 6n u^{2n-1} u_x u_{xx} + u^{2n} u_{xxx} v \right] dx
\]

\[
= (2n+2) \int_{\mathbb{R}} (n(2n-1)) u^{2n-2} u_x^3 + 4n u^{2n-1} u_x u_{xx} + u^{2n} u_{xxx} v dx.
\]

(2.26)
In a similar manner,
\[
\frac{d}{dt} \int_{\mathbb{R}} \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^k u^{2n-2k+2} u_x^{2k} dx
\]
\[
= \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} \int_{\mathbb{R}} (2(n-k+1)u^{2n-2k+1} u_x^{2k} u_t + 2ku^{2(n-k+1)} u_x^{2k-1} u_{xx}) dx
\]
\[
= \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^k \int_{\mathbb{R}} (2(n-k+1)u^{2n-2k+1} u_x^{2k} v_x + 2ku^{2(n-k+1)} u_x^{2k-1} v_{xx}) dx
\]
\[
= \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^k \int_{\mathbb{R}} \left[ -2(n-k+1)(2n-2k+1) u^{2n-2k} u_x^{2k+1} + 2ku^{2n-2k+1} u_x^{2k-1} u_{xx} + 2k(2k-1) u^{2n-2k+2} u_x^{2k-3} u_{xx}^{2} + 2k(2k-1) u^{2n-2k+2} u_x^{2k-2} u_{xxxx} \right] dx
\]
\[
= \sum_{k=2}^{n} (-1)^{k+1} C_{n+1}^k \int_{\mathbb{R}} (2(n-k+1)(2n-2k+1) u^{2n-2k} u_x^{2k+1} + 2(n-k+1) 4ku^{2n-2k+1} u_x^{2k-1} u_{xx} + 2k(2k-1) u^{2n-2k+2} u_x^{2k-3} u_{xx}^{2} + 2ku^{2n-2k+2} u_x^{2k-2} u_{xxxx}) dx \equiv J_1 + J_2 + J_3 + J_4. \tag{2.27}
\]

For $J_1-J_4$, a direct computation yields
\[
J_1 = \sum_{k=2}^{n} (-1)^{k+1} C_{n+1}^k \int_{\mathbb{R}} 2(n-k+1)(2n-2k+1) u^{2n-2k} u_x^{2k+1} dx
\]
\[
= \sum_{k=2}^{n} \frac{(-1)^{k+1}}{k!(n-k+1)!} 2(n-k+1)(2n-2k+1) \int_{\mathbb{R}} u^{2n-2k} u_x^{2k+1} dx
\]
\[
= (2n+2) \sum_{k=2}^{n} (-1)^{k+1} \frac{n!}{k!(n-k)!} (2n-2k+1) \int_{\mathbb{R}} u^{2n-2k} u_x^{2k+1} dx
\]
\[
= (-1)^{n+1}(2n+2) \int_{\mathbb{R}} u_x^{2n+1} dx + (2n+2) \sum_{k=2}^{n-1} (-1)^{k+1} C_{n+1}^k (2n-2k+1) \int_{\mathbb{R}} u^{2n-2k} u_x^{2k+1} dx
\]
\[
= (-1)^{n+1}(2n+2) \int_{\mathbb{R}} u_x^{2n+1} dx + (2n+2) \sum_{k=2}^{n-1} (-1)^{k+1} (C_{n+1}^k (2n+2) C_{n+1}^k) \int_{\mathbb{R}} u^{2n-2k} u_x^{2k+1} dx,
\]

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\[ J_2 = \sum_{k=1}^{n-1} (-1)^k C_{n+1}^{k+1} \int_{\mathbb{R}} 2(n-k)4(k+1)u^{2n-2k-1}u^{2k+1}x dx \]
\[ = \sum_{k=1}^{n-1} (-1)^k \frac{(n+1)!}{(k+1)!(n-k)!} 2(n-k)4(k+1) \int_{\mathbb{R}} u^{2n-2k-1}u^{2k+1}x dx \]
\[ = (2n+2) \sum_{k=1}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-k-1)!} 4n \int_{\mathbb{R}} u^{2n-2k-1}u^{2k+1}x dx \]
\[ = (2n+2) \sum_{k=1}^{n-1} (-1)^k 4n C_{n-1}^k \int_{\mathbb{R}} u^{2n-2k-1}u^{2k+1}x dx, \]

\[ J_3 = \sum_{k=0}^{n-2} (-1)^{k+1} C_{n+1}^{k+2} \int_{\mathbb{R}} 4(k+2)(k+1)u^{2n-2k-2}u^{2k+1}x^2 dx \]
\[ = \sum_{k=0}^{n-2} (-1)^{k+1} \frac{(n+1)!}{(k+2)!(n-k-1)!} 4(k+2)(k+1) \int_{\mathbb{R}} u^{2n-2k-2}u^{2k+1}x^2 dx \]
\[ = (2n+2) \sum_{k=0}^{n-2} (-1)^{k+1} \frac{(n-1)!}{k!(n-k-1)!} 2n \int_{\mathbb{R}} u^{2n-2k-2}u^{2k+1}x^2 dx \]
\[ = (2n+2) \sum_{k=0}^{n-2} (-1)^{k+1} 2n C_{n-1}^k \int_{\mathbb{R}} u^{2n-2k-2}u^{2k+1}x^2 dx, \]

and

\[ J_4 = \sum_{k=1}^{n-1} (-1)^k C_{n+1}^{k+1} \int_{\mathbb{R}} 2(k+1)u^{2n-2k}u^k u_x x dx \]
\[ = \sum_{k=1}^{n-1} (-1)^k \frac{(n+1)!}{(k+1)!(n-k)!} 2(k+1) \int_{\mathbb{R}} u^{2n-2k}u^k u_x x dx \]
\[ = (2n+2) \sum_{k=1}^{n-1} (-1)^k \frac{n!}{k!(n-k)!} \int_{\mathbb{R}} u^{2n-2k}u^k u_x x dx \]
\[ = (2n+2) \sum_{k=1}^{n-1} (-1)^k C_n^k \int_{\mathbb{R}} u^{2n-2k}u^k u_x x dx. \]
Plugging $J_1-J_4$ into (2.27), we deduce that
\[
\frac{d}{dt} \int \sum_{k=2}^{n} (-1)^{k+1} \frac{C_k^{n+1} u^{2n-2k+2} u_x^{2k}}{2k-1} dx
\]
\[
= (2n + 2) \int \left[ (-1)^{n+1} u_x^{2n+1} + \sum_{k=2}^{n-1} (-1)^{k+1} (C_k^{n} + 2nC_{n-k}^{k}) u^{2n-2k-2} u_x^{2k+1} \right] dx
\]
\[
+ \sum_{k=1}^{n} (-1)^{k} 4nC_{n-k}^{k} u^{2n-2k-1} u_x^{2k+1} u_{xx} + \sum_{k=0}^{n-2} (-1)^{k+1} 2nC_{n-k}^{k} u^{2n-2k-2} u_x^{2k+1} u_{xx}^{2}
\]
\[
+ \sum_{k=1}^{n} (-1)^{k} C_{n-k}^{k} u^{2n-2k-2} u_x^{2k} u_{xxx}^{2}] dx.
\]
Combining (2.24)-(2.26) with (2.28), we have
\[
\frac{dF(u)}{dt} = \frac{d}{dt} \int (u^{2n+2} + \sum_{k=1}^{n} (-1)^{k+1} \frac{C_k^{n+1} u^{2n-2k+2} u_x^{2k}}{2k-1} + (1)^{n} \frac{u^{2n+2}}{2n+1}) dx
\]
\[
= -(2n + 2) \int \left[ \sum_{k=0}^{n-1} (-1)^{k} (C_k^{n} + 2nC_{n-k}^{k}) u^{2n-2k-2} u_x^{2k+1} + (-1)^{n} u^{2n+1} \right]
\]
\[
- \sum_{k=0}^{n-1} (-1)^{k} 4nC_{n-k}^{k} u^{2n-2k-1} u_x^{2k+1} u_{xx} + \sum_{k=0}^{n-1} (-1)^{k} 2nC_{n-k}^{k} u^{2n-2k-2} u_x^{2k+1} u_{xx}^{2}
\]
\[
- \sum_{k=0}^{n} (-1)^{k} C_{n-k}^{k} u^{2n-2k-2} u_x^{2k} u_{xxx}^{2} \right] dx,
\]
which along with (2.24) yields,
\[
\frac{dF(u)}{dt} = (2n + 2) \int (u_t - u_{txx}) dx = \int (2n + 2)(v v_{xxx} - v v_{xx}) dx = 0,
\]
which gives (2.23). This completes the proof of Lemma 2.3.

3 Stability of peakons

In this section, our main theorem of the orbital stability of peakons for Eq. (1.1) is stated and proved. We now present the stability result.

**Theorem 3.1.** The peaked soliton $\varphi_c(t,x)$ defined in (2.3) traveling with the speed $c > 0$ is orbitally stable in the following sense. If $u_0(x) \in H^s(\mathbb{R})$, for some $s > \frac{3}{2}$, $y_0(x) = (1 - \partial_x^2)u_0 \neq 0$ is a nonnegative Radon measure of finite total mass, and
\[
\|u(0,\cdot) - \varphi_c\|_{H^1(\mathbb{R})} < \varepsilon, \quad \text{for} \quad 0 < \varepsilon < (3 - 2\sqrt{2})a,
\]

 illegally entered into the orbital stability of peakons for Eq. (1.1) is stated and proved. We now present the stability result.

**Theorem 3.1.** The peaked soliton $\varphi_c(t,x)$ defined in (2.3) traveling with the speed $c > 0$ is orbitally stable in the following sense. If $u_0(x) \in H^s(\mathbb{R})$, for some $s > \frac{3}{2}$, $y_0(x) = (1 - \partial_x^2)u_0 \neq 0$ is a nonnegative Radon measure of finite total mass, and
\[
\|u(0,\cdot) - \varphi_c\|_{H^1(\mathbb{R})} < \varepsilon, \quad \text{for} \quad 0 < \varepsilon < (3 - 2\sqrt{2})a,
\]
where $a > 0$ is defined by $c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n}^{k}) a^{2n}$ according to Theorem 2.1. Then the corresponding solution $u(t,x)$ of Eq. (1.1) when $n$ is positive odd integer satisfies

$$
\sup_{t \in [0,T)} \| u(t, \cdot) - \varphi_{c}(\cdot - \xi(t)) \|_{H^{1}(\mathbb{R})} \lesssim \sqrt{3a\varepsilon + 4a A(c, \| u_{0} \|_{H^{s}}) \varepsilon},
$$

where $T > 0$ is the maximal existence time, $\xi(t) \in \mathbb{R}$ is the maximum point of function $u(t,\cdot)$ and the constant $A(n,c,\| u_{0} \|_{H^{s}}) > 0$ depends only on wave speed $c > 0$, integer $n > 0$ and the norm $\| u_{0} \|_{H^{s}}$.

**Remark 3.1.** Note that by the above stability theorem we mean that even if a solution $u$ which is initially close to a peakon $\varphi_{c}$ blows up in a finite time, it stays close to some translate of the peakon up to the breaking time.

We break the proof of Theorem 3.1 into several lemmas. Note that the assumptions on the initial profile in Theorem 3.1 guarantee the existence of the unique local positive solution of Eq. (1.1) by Lemmas 2.1-2.2. It is obvious that $\varphi_{c}(x) = a\varphi(x) = ae^{-|x|} \in H^{1}(\mathbb{R})$ has the peak at $x = 0$, and hence

$$
\max_{x \in \mathbb{R}} \{ \varphi_{c}(x) \} = \varphi_{c}(0) = a.
$$

By a simple computation, we have

$$
E(\varphi_{c}) = \| \varphi_{c} \|_{H^{1}}^{2} = a^{2} \int_{\mathbb{R}} (\varphi^{2} + \varphi_{x}^{2}) dx = 2a^{2},
$$

(3.1)

and

$$
F(\varphi_{c}) = a^{2n+2} \int_{\mathbb{R}} \left( \varphi^{2n+2} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^{k} \varphi^{2n-2k+2} \varphi_{x}^{2k} + (-1)^{n} \frac{\varphi_{x}^{2n+2}}{2n+1} \right) dx
$$

$$
= \ a^{2n+2} \left( 1 + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^{k} \right) \cdot \frac{1}{n + 1}.
$$

Here $a$ is given by $c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n}^{k}) a^{2n}$ according to Theorem 2.1.

**Lemma 3.1.** For every $u \in H^{1}(\mathbb{R})$ and $\xi \in \mathbb{R}$, we have

$$
E(u) - E(\varphi_{c}(\cdot - \xi)) = \| u - \varphi_{c}(\cdot - \xi) \|_{H^{1}}^{2} + 4a (u(\xi) - a),
$$

(3.2)

where $a$ is defined by $c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n}^{k}) a^{2n}$.
Proof. Note the relation \( \varphi - \partial_x^2 \varphi = 2\delta \). Here \( \delta \) denotes the Dirac distribution. For simplicity, we abuse notation by writing integrals instead of the \( H^{-1}/H^1 \) duality pairing. Hence we have

\[
\|u - \varphi_c(\cdot - \xi)\|_{H^1}^2 = \int_\mathbb{R} (u^2 + u_x^2)\,dx + \int_\mathbb{R} (\varphi_c^2 + (\partial_x \varphi_c)^2)\,dx - 2a \int_\mathbb{R} u(x)\varphi(x - \xi)\,dx
\]

\[
\quad - 2a \int_\mathbb{R} u(x)\varphi(x - \xi)\,dx = E(u) + E(\varphi_c(\cdot - \xi)) - 2a \int_\mathbb{R} (1 - \partial_x^2)\varphi(x - \xi)\,u(x)\,dx
\]

\[
= E(u) + E(\varphi_c(\cdot - \xi)) - 4a \int_\mathbb{R} \delta(x - \xi)\,u(x)\,dx
\]

\[
= E(u) + E(\varphi_c(\cdot - \xi)) - 4au(\xi)
\]

\[
= E(u) - E(\varphi_c(\cdot - \xi)) - 4au(\xi) - a,
\]

where we used integration by parts and (3.1). This completes the proof of Lemma 3.1.

Next, we derive a crucial polynomial inequality between the two conserved quantities \( E(u) \) and \( F(u) \).

Lemma 3.2. Assume \( u_0 \in H^s(\mathbb{R}), s > \frac{5}{2}, \) and \( y_0 \geq 0 \). Let \( u(t, x) \) be the solution of Eq. (1.1) with initial data \( u_0 \). Denoting \( M \triangleq \max_{x \in \mathbb{R}} \{u(x)\} \), then

\[
\frac{n(2 - c_1)}{n + 1} M^{2n+2} - \frac{2 - c_1}{2} M^{2n} E(u) + F(u) \leq 0,
\]

where \( n \) is positive odd integer.

Proof. Let \( M \) be taken at \( x = \xi \), and define the same function \( g \) as in (15)

\[
g(x) \triangleq \begin{cases} 
    u(x) - u_x(x), & x < \xi, \\
    u(x) + u_x(x), & x > \xi.
\end{cases}
\]

Then we have

\[
\int_\mathbb{R} g^2(x)\,dx = E(u) - 2M^2.
\]

Next we introduce the following function

\[
h(x) \triangleq \begin{cases} 
    (u^{2n} + \sum_{k=1}^{2n-1} c_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n})(x), & x < \xi, \\
    (u^{2n} + \sum_{k=1}^{2n-1} d_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n})(x), & x > \xi.
\end{cases}
\]
where \( c_k, d_k, k = 1, 2, \cdots, 2n - 1 \) are given as follows:

\[
\begin{align*}
\begin{cases}
  c_1 = -d_1 = \frac{1}{2} + \sum_{j=1}^{n+1} (-1)^{j+1} \frac{2j-3}{2(2j-1)} C_{n+1}^j, \\
  c_{2m} = d_{2m} = \sum_{j=m+1}^{n+1} (-1)^{j+1} \frac{2j-(2m+1)}{2j-1} C_{n+1}^j, & m = 1, 2, \cdots, n - 1, \\
  c_{2m-1} = -d_{2m-1} = \sum_{j=m+1}^{n+1} (-1)^{j+1} \frac{2j-2m}{2j-1} C_{n+1}^j, & m = 2, 3, \cdots, n.
\end{cases}
\end{align*}
\] (3.5)

Integrating by parts, we calculate

\[
\int_R h(x) g^2(x) dx
\]

\[
= \int_{-\infty}^\xi (u - u_x)^2 (u^{2n} + \sum_{k=1}^{2n-1} c_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n}) dx
+ \int_\xi^\infty (u + u_x)^2 (u^{2n} + \sum_{k=1}^{2n-1} d_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n}) dx
\]

\[
= \int_{-\infty}^\xi (u^{2n+2} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^k u^{2n-2k+2} u_x^{2k} + (-1)^n \frac{u_x^{2n+2}}{2n+1}) dx
\]

\[
+ (c_1 - 2) \int_{-\infty}^\xi u^{2n+1} u_x dx + \int_\xi^\infty (u^{2n+2} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^k u^{2n-2k+2} u_x^{2k}
\]

\[
+ (-1)^n \frac{u_x^{2n+2}}{2n+1}) dx + (d_1 + 2) \int_\xi^\infty u_x^{2n+1} u_x dx
\]

\[
= F(u) \left. \frac{c_1 - 2}{2n + 2} u^{2n+2}(x) \right|_{-\infty}^\xi - \frac{d_1 - 2}{2n + 2} u_x^{2n+2}(x) \left|_{\xi}^\infty \right.
\]

\[
= F(u) + \frac{c_1 - 2}{n + 1} u_x^{2n+2}(\xi) = F(u) - \frac{2 - c_1}{n + 1} M^{2n+2},
\] (3.6)

where the last third equality of (3.6) is shown in Appendix A.1 for simplicity.

On the other hand, by Lemma 2.2, the solution \( u \) satisfies

\[
u(t, x) \geq 0, \quad (u \pm u_x)(t, x) \geq 0 \quad \text{and} \quad (u^2 - u_x^2)(t, x) \geq 0 \quad \text{for} \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\] (3.7)

We now claim that

\[
h(t, x) \leq \frac{2 - c_1}{2} u^{2n}(t, x) \quad \text{for} \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\] (3.8)

To see this, it suffices to show that

\[
\sum_{k=1}^{2n-1} c_k u^{2n-k} u_x^k \leq -\frac{c_1}{2} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n},
\] (3.9)
and
\[
\sum_{k=1}^{2n-1} d_k u^{2n-k} u_x^k \leq -\frac{c_1}{2} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n}.
\] (3.10)

We shall prove (3.9) and (3.10) by the recursive method, which can be achieved as follows.

**Step 1.** By the Cauchy-Schwarz inequality and (3.7), we deduce
\[
\pm u^{2n-1} u_x \pm u^{2n-3} u_x^3 = (\mp uu_x) u^{2n-4} (u^2 - u_x^2) \leq \frac{u^2 + u_x^2}{2} \cdot u^{2n-4} (u^2 - u_x^2) = \frac{1}{2} (u^{2n} - u_x^{2n-4} u_x^4).
\] (3.11)

Thanks to (3.11), to prove (3.9), we need only to show that
\[
(c_1 + c_3) u^{2n-1} u_x + c_2 u^{2n-2} u_x^2 + (c_4 - \frac{c_3}{2}) u^{2n-4} u_x^4 + \sum_{k=5}^{2n-1} c_k u^{2n-k} u_x^k \leq \frac{(-1)^n}{2n+1} u_x^{2n}.
\] (3.12)

In a similar manner,
\[
\pm u^{2n-1} u_x - u^{2n-2} u_x^2 = (\mp uu_x) u^{2n-3} (u \pm u_x) \leq \frac{u^2 + u_x^2}{2} \cdot (u^{2n-2} \pm u^{2n-3} u_x) = \frac{1}{2} (u^{2n} \pm u^{2n-1} u_x + u^{2n-2} u_x^2 \pm u^{2n-3} u_x^3).
\] (3.13)

Owing to (3.13), to prove (3.12), it suffices to show that
\[
(c_1 + c_3 - \frac{3c_2}{2}) u^{2n-1} u_x - \frac{c_2}{2} u^{2n-2} u_x^2 - \frac{c_2}{2} u^{2n-3} u_x^3 + (c_4 - \frac{c_3}{2}) u^{2n-4} u_x^4 + \sum_{k=5}^{2n-1} c_k u^{2n-k} u_x^k \leq \frac{(-1)^n}{2n+1} u_x^{2n}.
\] (3.14)

Using (3.11) again, to prove (3.14), we need only to show that
\[
(c_1 + c_3 - 2c_2) u^{2n-1} u_x - \frac{c_2}{2} u^{2n-2} u_x^2 + (c_4 - \frac{c_3}{2} + \frac{c_2}{4}) u^{2n-4} u_x^4 + \sum_{k=5}^{2n-1} c_k u^{2n-k} u_x^k \leq \frac{(-1)^n}{2n+1} u_x^{2n}.
\] (3.15)

Since from (3.3), we have \(c_3 - 2c_2 + c_1 = 0\). Thus, it follows from (3.15) that
\[
-\frac{c_2}{2} u^{2n-2} u_x^2 + (c_4 - \frac{c_3}{2} + \frac{c_2}{4}) u^{2n-4} u_x^4 + \sum_{k=5}^{2n-1} c_k u^{2n-k} u_x^k \leq \frac{c_2}{4} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n}.
\] (3.16)
Step 2. Using the similar argument as in Step 1, we have
\[ \pm u^{2n-5} u_x^5 \pm u^{2n-7} u_x^7 = (\mp uu_x) u^{2n-8} u_x^4 (u^2 - u_x^2) \]
\[ \leq \frac{u^2 + u_x^2}{2} \cdot u^{2n-8} u_x^4 (u^2 - u_x^2) = \frac{1}{2} (u^{2n} - u^{2n-4} u_x^4), \quad (3.17) \]
and
\[ \pm u^{2n-5} u_x^5 - u^{2n-6} u_x^6 = (\mp uu_x) u^{2n-7} u_x^4 (u \pm u_x) \leq \frac{u^2 + u_x^2}{2} \cdot (u^{2n-6} u_x^4 \pm u^{2n-7} u_x^5) \]
\[ = \frac{1}{2} (u^{2n-4} u_x^4 \pm u^{2n-5} u_x^5 + u^{2n-6} u_x^6 \pm u^{2n-7} u_x^7). \quad (3.18) \]

Combining (3.16) with (3.17)-(3.18), one can deduce that
\[ -\frac{c_2}{2} u^{2n-2} u_x^2 + (c_4 - \frac{c_3}{2} + \frac{c_2}{4} + \frac{c_7}{2} - \frac{3c_6}{4}) u^{2n-4} u_x^4 + (c_5 + c_7 - 2c_6) u^{2n-5} u_x^5 - \frac{c_6}{2} u^{2n-6} u_x^6 + (c_8 - \frac{c_7}{2} + \frac{c_6}{4}) u^{2n-8} u_x^8 \]
\[ + \sum_{k=9}^{2n-1} c_k u^{2n-k} u_x^k \leq -\frac{c_2}{4} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n}. \quad (3.19) \]

Applying the fact from (3.16) that \( c_5 - 2c_4 + c_3 = 0 \) and \( c_7 - 2c_6 + c_5 = 0 \) to (3.19) yields
\[ -\frac{c_2}{2} u^{2n-2} u_x^2 + \frac{c_2 + c_6}{4} u^{2n-4} u_x^4 - \frac{c_6}{2} u^{2n-6} u_x^6 + (c_8 - \frac{c_7}{2} + \frac{c_6}{4}) u^{2n-8} u_x^8 \]
\[ + \sum_{k=9}^{2n-1} c_k u^{2n-k} u_x^k \leq -\frac{c_2}{4} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n}. \quad (3.20) \]

In the same manner one find that
\[ -\frac{c_2}{2} u^{2n-2} u_x^2 + \frac{c_2 + c_6}{4} u^{2n-4} u_x^4 - \frac{c_6}{2} u^{2n-6} u_x^6 + \frac{c_2 + c_6}{4} u^{2n-8} u_x^8 - \frac{c_{10}}{2} u^{2n-10} u_x^{10} \]
\[ + (c_{12} - \frac{c_{11}}{2} + \frac{c_{10}}{4}) u^{2n-12} u_x^{12} + \sum_{k=13}^{2n-1} c_k u^{2n-k} u_x^k \leq -\frac{c_2}{4} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n}. \quad (3.21) \]

Step 3. Using the recursive rule shown in (3.16), (3.20) and (3.21), we obtain
\[ \begin{aligned}
-\frac{c_2}{2} u^{2n-2} u_x^2 &+ c_2 + c_6 u^{2n-4} u_x^4 + \cdots + \left( -\frac{c_{4k-2}}{2} \right) u^{2n-(4k-2)} u_x^{4k-2} \\
&+ \frac{c_{4k+2}}{4} u^{2n-4k+2} u_x^{4k} + \cdots + \left( -\frac{c_{2n-4}}{2} \right) u_x^{2n-4n} \\
&+ \left( c_{2n-2} - \frac{c_{2n-3}}{2} + \frac{c_{2n-4}}{4} \right) u_x^{2n-2} + c_{2n-1} u_x^{2n-1} \\
&\leq -\frac{c_2}{4} u^{2n} - \frac{(-1)^n}{2n+1} u_x^{2n} = -\frac{c_2}{4} u^{2n} - \frac{c_{2n-1}}{2} u_x^{2n},
\end{aligned} \quad (3.22) \]

where we used the fact that \( c_{2n-1} = 2 \cdot \frac{(-1)^n}{2n+1} \). Note that
\[ c_{2n-1} u_x^{2n-1} \leq -\frac{c_{2n-1}}{2} u_x^{2n-2} = -\frac{c_{2n-1}}{2} u_x^{2n}. \quad (3.23) \]
Thus, by (3.22) and (3.23), we have

\[
0 \geq \frac{c_2}{4} u^{2n} + \frac{-c_2}{2} u^{2n-2} u_x^2 + \frac{c_2 + c_6}{4} u^{2n-4} u_x^4 + \cdots + (\frac{-c_4 k - 2}{2}) u^{2n-(4k-2)} u_x^{4k-2} + \frac{c_{4 k - 2} + c_{4 k + 2}}{4} u^{2n-4k} u_x^{4k} + \cdots + (\frac{-c_{2 n - 4}}{2}) u^{2n-2} u_x^{2n-2} - \frac{c_{2 n - 3}}{2} + \frac{c_{2 n - 4}}{2}) u^{2n-2} u_x^{2n-2}.
\] (3.24)

According to Steps 1 to 3, to prove (3.9), by the relation \(c_{2 n - 1} - 2 c_{2 n - 2} + c_{2 n - 3} = 0\) and (3.24), we need only to show that for \(k = 2, \ldots, n\),

\[
0 \geq c_2 u^{2n-2} - 2 c_2 u^{2n-4} u_x^2 + (c_2 + c_6) u^{2n-6} u_x^4 + \cdots + (c_{4 k - 6} + c_{4 k - 2}) u^{2n-4k+2} u_x^{4k-4} + \frac{-2 c_{4 k - 2} u^{2n-4k} u_x^{4k-2} + (c_{4 k - 2} + c_{4 k + 2}) u^{2n-4k-2} u_x^{4k-2} + \cdots + (c_{2 n - 8} + c_{2 n - 4}) u^{2n-6} u_x^{2n-6}}{-2 c_{2 n - 4} u^{2n-2} u_x^{2n-2}}.
\] (3.25)

Indeed, due to \(c_2, c_6, \ldots, c_{2n-4} < 0\), (3.25) is clearly true by the following equivalent form

\[
0 \geq c_2 u^{2n-6} (u^2 - u_x^2)^2 + \cdots + c_{4 k - 2} u^{2n-4k-2} u_x^{4k-4} (u^2 - u_x^2)^2 + \cdots + c_{2 n - 4} u^{2n-6} (u^2 - u_x^2)^2,
\]

which implies the desired result (3.9). A similar argument as the proof of (3.9) leads to (3.10). Hence we prove our claim (3.8). Therefore, in view of (3.4), (3.6) and (3.8), we conclude that

\[
F(u) - \frac{2 - c_1}{n + 1} M^{2n+2} = \int_R h(x) g^2(x) dx \leq \frac{2 - c_1}{2} M^{2n} (E(u) - 2 M^2),
\]

which implies (3.3). This completes the proof of Lemma 3.2.

**Lemma 3.3.** For \(u \in H^s(\mathbb{R}), s > \frac{3}{2},\) if \(\|u - \varphi_c\|_{H^1(\mathbb{R})} < \varepsilon,\) with \(0 < \varepsilon < (3 - 2\sqrt{2}) a,\) then

\[
|E(u) - E(\varphi_c)| \leq 3 a \varepsilon \quad \text{and} \quad |F(u) - F(\varphi_c)| \leq A(n, c, \|u\|_{H^s}) \cdot \varepsilon,
\]

where the constant \(A(n, c, \|u\|_{H^s}) > 0\) depends only on wave speed \(c > 0,\) integer \(n > 0\) and the norm \(\|u\|_{H^s}.

**Proof.** Under the assumption \(0 < \varepsilon < (3 - 2\sqrt{2}) a,\) a direct computation yields

\[
|E(u) - E(\varphi_c)| = \left| (\|u\|_{H^1} - \|\varphi_c\|_{H^1}) \left( \|u\|_{H^1} + \|\varphi_c\|_{H^1} \right) \right| \leq \varepsilon (\varepsilon + 2\sqrt{2} a) \leq 3 a \varepsilon.
\] (3.26)
On the other hand, we have
\[ |F(u) - F(\varphi_c)| \]
\[ = \left| \int_{\mathbb{R}} (u^{2n+2} + \sum_{k=1}^{n} \frac{(-1)^{k+1} C_{n+1}^k u^{2n-2k+2} u_x^{2k}}{2k-1} + (-1)^n \frac{u_x^{2n+2}}{2n+1}) dx \right| \]
\[ - \int_{\mathbb{R}} (\varphi_c^{2n+2} + \sum_{k=1}^{n} \frac{(-1)^{k+1} C_{n+1}^k \varphi_c^{2n-2k+2} (\partial_x \varphi_c)^{2k}}{2k-1} + (-1)^n \frac{\varphi_c^{2n+2}}{2n+1}) dx \right| \]
\[ \leq \left| \int_{\mathbb{R}} (u^{2n+2} + (n+1)u_x^{2n} - \varphi_c^{2n+2} - (n+1)\varphi_c^{2n} (\partial_x \varphi_c)^2) dx \right| + \frac{1}{2n+1} \]
\[ \times \left| \int_{\mathbb{R}} (u_x^{2n+2} - (\partial_x \varphi_c)^{2n+2}) dx \right| + \sum_{k=2}^{n} \frac{C_{n+1}^k}{2k-1} \left| \int_{\mathbb{R}} (u^{2n-2k+2} + u_x^{2n} - \varphi_c^{2n-2k+2} (\partial_x \varphi_c)^{2k}) dx \right| \]
\[ \leq \left| \int_{\mathbb{R}} (u^{2n} - \varphi_c^{2n}) (u^n + (n+1)u_x^n) dx \right| + \left| \int_{\mathbb{R}} \varphi_c^{2n} ((u^n - \varphi_c^n) + (n+1)(u_x^n - (\partial_x \varphi_c)^n)) dx \right| \]
\[ + \frac{1}{2n+1} \left| \int_{\mathbb{R}} (u_x^{n+1} + (\partial_x \varphi_c)^{n+1}) (u_x^{n+1} - (\partial_x \varphi_c)^{n+1}) dx \right| + \sum_{k=2}^{n} \frac{C_{n+1}^k}{2k-1} \left| \int_{\mathbb{R}} (\partial_x \varphi_c)^{2k} (u^{2n-2k+2} - \varphi_c^{2n-2k+2}) dx \right| \]
\[ \triangleq K_1 + K_2 + \cdots + K_5 \leq A(n, c, \|u\|_{H^s}) \cdot \varepsilon, \quad (3.27) \]
where we prove the last inequality of (3.27) in Appendix A.2. This completes the proof of Lemma 3.3.

**Lemma 3.4.** For \(0 < u \in H^s(\mathbb{R}), s > \frac{5}{2},\) let \(M = \max_{x \in \mathbb{R}} \{\varphi_c\}.\) If
\[ |E(u) - E(\varphi_c)| \leq 3a \varepsilon \quad \text{and} \quad |F(u) - F(\varphi_c)| \leq A(n, c, \|u\|_{H^s}) \cdot \varepsilon, \]
with \(0 < \varepsilon < (3 - 2\sqrt{2})a\) and the constant \(A(n, c, \|u\|_{H^s}) > 0\) depends only on wave speed \(c > 0,\) integer \(n > 0\) and the norm \(\|u\|_{H^s},\) then
\[ |M - a| \leq \sqrt{A(n, c, \|u\|_{H^s}) \varepsilon}. \]

**Proof.** In view of (3.3) in Lemma 3.2, we have
\[ nM^{2n+2} - \frac{n+1}{2} M^{2n} E(u) + \frac{n+1}{2} \frac{1}{c_1} F(u) \leq 0. \quad (3.28) \]
Consider the \((2n + 2)\)-th order polynomial \(P(y) = ny^{2n+2} - \frac{n+1}{2} y^{2n} E(u) + \frac{n+1}{2} - c_1 F(u).\) In case \(E(u) = E(\varphi_c) = 2a^2\) and \(F(u) = F(\varphi_c) = a^{n+2} (1 + \sum_{k=1}^{n+1} \frac{(-1)^{k+1} C_{n+1}^k}{2k-1} C_{n+1}^k) \cdot \frac{1}{n+1},\) it takes the form
\[ P_0(y) = ny^{2n+2} - \frac{n+1}{2} y^{2n} E(\varphi_c) + \frac{n+1}{2} \frac{1}{c_1} F(\varphi_c) \]
\[ = ny^{2n+2} - (n+1)(a^2 y^{2n} + \frac{1}{2 - c_1} a^{n+2} (1 + \sum_{k=1}^{n+1} \frac{(-1)^{k+1} C_{n+1}^k}{2k-1} C_{n+1}^k)). \quad (3.29) \]
We now claim that the following identity holds,

\[ 2 - c_1 = 1 + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{2k - 1} C^k_{n+1}. \]  

(3.30)

Indeed, to prove (3.30), it is inferred from (3.5) that we need only to show

\[ \sum_{k=1}^{n+1} (-1)^{k+1} C^k_{n+1} = 1. \]  

(3.31)

Obviously (3.31) is right for \( n = 1 \). By induction, we assume that (3.31) holds for \( n \), and prove it for \( n + 1 \),

\[
\begin{align*}
\sum_{k=1}^{n+2} (-1)^{k+1} C^k_{n+2} &= \sum_{k=1}^{n+1} (-1)^{k+1} (C^k_{n+1} + C^k_{n+1}) + (-1)^{n+3} \\
&= \sum_{k=1}^{n+1} (-1)^{k+1} C^k_{n+1} + \sum_{k=0}^{n} (-1)^{k} C^k_{n+1} + (-1)^{n+1} \\
&= 1 + (-1)^{0} C^0_{n+1} - \sum_{k=1}^{n+1} (-1)^{k+1} C^k_{n+1} = 1,
\end{align*}
\]

which proves (3.31). Thus, the combination of (3.29) and (3.30) yields

\[
P_0(y) = ny^{2n+2} - (n + 1)a^2 y^{2n} + a^{2n+2}
= (y - a)^2 (ny^{2n} + \sum_{k=1}^{2n-1} (2n+1-k)a^k y^{2n-k} + a^{2n}).
\]  

(3.32)

By (3.28) and (3.29), we obtain

\[
P_0(M) \leq P_0(M) - P(M) = \frac{n + 1}{2} M^{2n} (E(u) - E(\varphi_c)) - \frac{n + 1}{2 - c_1} (F(u) - F(\varphi_c)),
\]

which together with (3.32) leads to

\[
a^{2n}(M - a)^2 \leq \frac{n + 1}{2} M^{2n} (E(u) - E(\varphi_c)) - \frac{n + 1}{2 - c_1} (F(u) - F(\varphi_c)).
\]  

(3.33)

By (3.4) and the assumption of this lemma, we deduce that for \( 0 < \varepsilon < (3 - 2\sqrt{2})a \),

\[
0 < M^2 \leq \frac{E(u)}{2} \leq \frac{2a^2 + 3a\varepsilon}{2} < \frac{3a^2}{2}.
\]  

(3.34)

Hence, combining (3.33) with (3.34), we find that

\[
a^n |M - a| \lesssim \sqrt{\frac{(n + 1)3^{n+1}}{2^n+1}a^{2n+1} \varepsilon + \frac{n + 1}{2} A(n, c, \|u\|_{H^s})\varepsilon}.
\]
Thus, it follows from the above inequality and the relation \( c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} c_n^k) a^{2n} \) that there is a constant, still denoted by \( A(n, c, \| u \|_{H^s}) \) for simplicity, such that

\[
|M - a| \lesssim \sqrt{A(n, c, \| u \|_{H^s})\varepsilon}.
\]

This completes the proof of Lemma 3.4.

**Proof of Theorem 3.1.** The above four lemmas actually constitute sufficient preparation for the proof of stability. Let \( u \in C([0, T); H^s(\mathbb{R})) \), \( s > \frac{3}{2} \) be the solution of Eq. (1.1) with the initial data \( u_0 \in H^s(\mathbb{R}) \). Since \( E(u) \) and \( F(u) \) are both conserved by the generalized modified CH equation (1.1), we have

\[
E(u(t, \cdot)) = E(u_0) \quad \text{and} \quad F(u(t, \cdot)) = F(u_0), \quad \forall \, t \in [0, T).
\]  

(3.35)

Since \( \| u(0, \cdot) - \varphi_c \|_{H^s} < \varepsilon \), with \( 0 < \varepsilon < (3 - 2\sqrt{2})a \), and \( 0 \neq (1 - \partial_x^2)u_0 \geq 0 \), in view of (3.35) and Lemma 3.3, the hypotheses of Lemma 3.4 are satisfied for \( u(t, \cdot) \) with a positive constant \( A(n, c, \| u_0 \|_{H^s}) \) to be chosen depending only on wave speed \( c > 0 \), integer \( n > 0 \) and the norm \( \| u_0 \|_{H^s} \). Hence

\[
|u(t, \xi(t)) - a| \lesssim \sqrt{A(n, c, \| u_0 \|_{H^s})\varepsilon}, \quad \forall \, t \in [0, T),
\]  

(3.36)

where \( \xi(t) \in \mathbb{R} \) is the maximum point of function \( u(t, \cdot) \). Combining (3.2) with (3.35), we find

\[
\| u - \varphi_c(\cdot - \xi(t)) \|_{H^1}^2 = E(u_0) - E(\varphi_c) - 4a(u(t, \xi(t)) - a).
\]

Therefore, it follows (3.36) and Lemma 3.3 that for \( t \in [0, T) \),

\[
\| u - \varphi_c(\cdot - \xi(t)) \|_{H^1} \leq \sqrt{\left| E(u_0) - E(\varphi_c) \right| + 4a|u(t, \xi(t)) - a|} \lesssim \sqrt{3a\varepsilon + 4a\sqrt{A(c, \| u_0 \|_{H^s})\varepsilon}}.
\]

This completes the proof of Theorem 3.1.

4 Appendix

In this section, we provide a detailed exposition of the proofs of the last third equality (3.6) and last inequality of (3.27) for convenience.

**A.1. Proof of the last third equality of (3.6).** In Lemma 3.2, we define the function \( h(x) \) as

\[
h(x) = \begin{cases} 
  (u^{2n} + \sum_{k=1}^{2n-1} c_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n})(x), & x < \xi, \\
  (u^{2n} + \sum_{k=1}^{2n-1} d_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n})(x), & x > \xi.
\end{cases}
\]
Why do we require the coefficients \( c_k, d_k, k = 1, 2, \ldots, 2n - 1 \) satisfy (3.5)? The reason lies in the fact that \( h(x) \) have to ensure the last third equality of (3.6) holds. Hence, we can calculate as follows

\[
(u^2 - 2uu_x + u_x^2)(u^{2n} + \sum_{k=1}^{2n-1} c_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n}) = u^{2n+2} + (c_1 - 2)u^{2n+1} u_x + (c_2 - 2c_1 + 1)u^{2n} u_x^2 + \sum_{k=3}^{2n-1} (c_k - 2c_{k-1} + c_{k-2})u^{2n-k+2} u_x^k
\]

\[
+ \left( \frac{(-1)^n}{2n+1} - 2c_{2n-1} + c_{2n-2} \right) u^2 u_x^2 + \left( \frac{2 \cdot (-1)^n}{2n+1} + c_{2n-1} \right) u u_x^{2n+1} + \left( \frac{(-1)^n}{2n+1} u_x^{2n+2} \right),
\]

(4.1)

and

\[
(u^2 + 2uu_x + u_x^2)(u^{2n} + \sum_{k=1}^{2n-1} d_k u^{2n-k} u_x^k + (-1)^n \frac{1}{2n+1} u_x^{2n}) = u^{2n+2} + (d_1 + 2)u^{2n+1} u_x + (d_2 + 2d_1 + 1)u^{2n} u_x^2 + \sum_{k=3}^{2n-1} (d_k + 2d_{k-1} + d_{k-2})u^{2n-k+2} u_x^k
\]

\[
+ \left( \frac{(-1)^n}{2n+1} + 2d_{2n-1} + d_{2n-2} \right) u^2 u_x^2 + \left( \frac{2 \cdot (-1)^n}{2n+1} + d_{2n-1} \right) u u_x^{2n+1} + \left( \frac{(-1)^n}{2n+1} u_x^{2n+2} \right).
\]

(4.2)

For our purpose, combining the conservation law \( F(u) \) with (4.1)-(4.2), we thus obtain

\[
\begin{align*}
-\frac{2(-1)^n}{2n+1} + c_{2n-1} &= 0, \\
\frac{(-1)^n}{2n+1} - 2c_{2n-1} + c_{2n-2} &= \frac{(-1)^{n+1}}{2n-1} C_n^{n+1}, \\
c_k - 2c_{k+1} + c_{k+2} &= 0, \quad k = 2j + 1, \quad (j = 1, 2, \ldots, n-1), \\
c_k - 2c_{k-1} + c_{k-2} &= \frac{(-1)^{k+1}}{2k-1} C_n^k, \quad k = 2j, \quad (j = 1, 2, \ldots, n-1), \\
c_2 - 2c_1 + 1 &= C_n^1 + C_n^0,
\end{align*}
\]

(4.3)

and

\[
\begin{align*}
\frac{2(-1)^n}{2n+1} + d_{2n-1} &= 0, \\
\frac{(-1)^n}{2n+1} + 2d_{2n-1} + d_{2n-2} &= \frac{(-1)^{n+1}}{2n-1} C_n^{n+1}, \\
d_k + 2d_{k-1} + d_{k-2} &= 0, \quad k = 2j + 1, \quad (j = 1, 2, \ldots, n-1), \\
d_k + 2d_{k-1} + d_{k-2} &= \frac{(-1)^{k+1}}{2k-1} C_n^k, \quad k = 2j, \quad (j = 1, 2, \ldots, n-1), \\
d_2 + 2d_1 + 1 &= C_n^1 + C_n^0.
\end{align*}
\]

(4.4)

By induction, it is inferred from the relations (4.3)-(4.4) that (3.5) is true. Hence, the last third equality of (3.6) also holds.
A.2. Proof of the last inequality of (3.27). To complete the proof of Lemma 3.3, it remains to prove the following inequality:

\[ K_1 + K_2 + \cdots + K_5 \lesssim A(n, c, \|u\|_{H^n}) \cdot \varepsilon. \]  

Using the relation (3.24), we have for all \( v \in H^1(\mathbb{R}) \),

\[ \sup_{x \in \mathbb{R}} |v(x)| \leq \frac{\sqrt{E(v)}}{\sqrt{2}} \leq \frac{\|v\|_{H^1}}{\sqrt{2}}. \]  

(4.6)

For the term \( K_1 \), by (3.26) and (4.6), we have

\[
K_1 \leq (n+1) \int_{\mathbb{R}} |u - \varphi_c| \cdot |u^{2n-1} + u^{2n-2} \varphi_c + \cdots + u \varphi_c^{2n-2} + \varphi_c^{2n-1}| \cdot (u^2 + u_x^2) \, dx \\
\leq (n+1)E(u)\|u - \varphi_c\|_{L^\infty} \big(||u||_{L^\infty}^{2n-1} + ||u||_{L^\infty}^{2n-2} ||\varphi_c||_{L^\infty} + \cdots + ||u||_{L^\infty} ||\varphi_c||_{L^\infty}^{2n-2} + ||\varphi_c||_{L^\infty}^{2n-1} \big) \\
\leq \frac{n+1}{2n} (E(\varphi_c) + 3a\varepsilon)\|u - \varphi_c\|_{H^1} \big(||u||_{H^1}^{2n-1} + ||u||_{H^1}^{2n-2} ||\varphi_c||_{H^1} + \cdots + ||u||_{H^1} ||\varphi_c||_{H^1}^{2n-2} + ||\varphi_c||_{H^1}^{2n-1} \big) \\
\leq \frac{n+1}{2n} (E(\varphi_c) + 3a\varepsilon) \big(||u - \varphi_c||_{H^1} + 2||\varphi_c||_{H^1} \big)^{2n-1} \cdot ||u - \varphi_c||_{H^1} \\
\leq \frac{n+1}{2n} (2a^2 + 3a\varepsilon)(2\sqrt{2}\varepsilon + \varepsilon)^{2n-1} \cdot \varepsilon.
\]

Similarly, for \( K_2 \), we have

\[
K_2 \leq \|\varphi_c\|_{L^\infty}^{2n} \int_{\mathbb{R}} \left| (u - \varphi_c)^2 + (n+1)(u_x - \partial_x \varphi_c)^2 + 2\varphi_c(u - \varphi_c) \\
+ 2(n+1)\partial_x \varphi_c(u_x - \partial_x \varphi_c) \right| \, dx \\
\leq \left( \frac{\|\varphi_c\|_{H^1}}{\sqrt{2}} \right)^2 2n \left( (n+1)1||u - \varphi_c||_{H^1}^2 + 2(n+1)||\varphi_c||_{H^1} ||u - \varphi_c||_{H^1} \right) \\
\leq (n+1)a^{2n}(2\sqrt{2}\varepsilon + \varepsilon) \cdot \varepsilon.
\]

For the term \( K_3 \), by the H"older inequality, we obtain

\[
K_3 = \frac{1}{2n+1} \left| \int_{\mathbb{R}} (u_x^{n+1} + (\partial_x \varphi_c)^{n+1})(u_x^n + u_x^{n-1}(\partial_x \varphi_c) + \cdots + u_x(\partial_x \varphi_c)^{n-1} \\
+ (\partial_x \varphi_c)^n) \cdot (u_x - \partial_x \varphi_c) \, dx \right| \\
\leq \frac{1}{2n+1} \left( \int_{\mathbb{R}} (u_x^{n+1} + (\partial_x \varphi_c)^{n+1})^2(u_x^n + u_x^{n-1}(\partial_x \varphi_c) + \cdots + u_x(\partial_x \varphi_c)^{n-1} \\
+ (\partial_x \varphi_c)^n)^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} (u_x - \partial_x \varphi_c)^2 \, dx \right)^{1/2} \\
\leq \frac{1}{2n+1} \sqrt{K_3^2} ||u - \varphi_c||_{H^1}.
\]

27
For $K'_3$, a direct use of Young’s inequality yields,

$$K'_3 = \int_{\mathbb{R}} \left( u_x^{4n+2} + 2u_x^{4n+1} \partial_x \varphi_c + 3u_x^{4n} (\partial_x \varphi_c)^2 + \cdots + (2n+1)u_x^{2n+2} (\partial_x \varphi_c)^2n \right.\
+ (2n+2)u_x^{2n+1} (\partial_x \varphi_c)^{2n+1} + (2n+1)u_x^{2n} (\partial_x \varphi_c)^{2n+2} + 2nu_x^{2n-1} (\partial_x \varphi_c)^{2n+3} + \cdots \\
+ 3u_x^2 (\partial_x \varphi_c)^{4n} + 2u_x (\partial_x \varphi_c)^{4n+1} + (\partial_x \varphi_c)^{4n+2} \right) dx\
\leq (n+1)^2 \left( \int_{\mathbb{R}} u_x^{4n+2} dx + \int_{\mathbb{R}} (\partial_x \varphi_c)^{4n+2} dx \right).$$

Since $u \in H^s(\mathbb{R}) \subset H^2(\mathbb{R}), s \geq \frac{5}{2}$, by the Gagliardo-Nirenberg inequality, we have

$$\|u_x\|_{L^{4n+2}} \leq C \|u\|_{L^0}^{n+1} \|u_{xx}\|_{L^2}^{3n+1},$$

with the constant $C > 0$ independent of $u$. Hence it follows from $\|\partial_x \varphi_c\|_{L^{4n+2}} = \frac{a^{4n+2}}{2n+1}$ that

$$K'_3 \lesssim A^2(n, a, \|u\|_{H^s}),$$

where the constant $A(n, a, \|u\|_{H^s}) > 0$ depends only on $a > 0$, integer $n > 0$, and the norm $\|u\|_{H^s}$. Since $c = (1 - \sum_{k=1}^{n} \frac{(-1)^{k+1} C_n}{2k+1}) a^{2n}$, then we have

$$K_3 \lesssim A(n, c, \|u\|_{H^s}) \cdot \varepsilon.$$

Applying the similar method to treat $K_4$, by the fact that $\|\partial_x \varphi_c\|_{L^{4k-2}} = \frac{a^{4k-2}}{2k-1}$, we obtain

$$K_4 \leq \sum_{k=2}^{n} \frac{C_{n+1}^{2k}}{2k-1} \|u\|_{L^2}^{2n-2k+2} \left( \int_{\mathbb{R}} \left( u_x^k + (\partial_x \varphi_c)^k \right) \left( u_x^{k-1} + u_x^{k-2} \partial_x \varphi_c + \cdots + u_x (\partial_x \varphi_c)^{k-2} \right) dx \right)^\frac{1}{2}\
+ (\partial_x \varphi_c)^{k-1} \right) dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2} dx \right)^\frac{1}{2}$$

$$\leq \sum_{k=2}^{n} \frac{C_{n+1}^{2k}}{2k-1} \|u\|_{L^2}^{2n-2k+2} \cdot \sqrt{2k} \left( \int_{\mathbb{R}} u_x^{4k-2} dx + \int_{\mathbb{R}} (\partial_x \varphi_c)^{4k-2} dx \right)^\frac{1}{2} \cdot \|u - \varphi_c\|_{H^1}$$

$$\lesssim \|u\|_{H^1}^{2n-2k+2} \left( \|u\|_{L^2}^{3k-2} + \|\partial_x \varphi_c\|_{L^{4k-2}}^{4k-2} \right)^\frac{1}{2} \cdot \|u - \varphi_c\|_{H^1}$$

$$\lesssim A(n, c, \|u\|_{H^s}) \cdot \varepsilon.$$
For $K_5$, a direct use of the Hölder inequality gives rise to,

\[
K_5 \leq \sum_{k=2}^{n} \frac{C_{n+1}^k}{2k-1} \left( \|u\|_{L^\infty}^{n-k+1} + \|\varphi_c\|_{L^\infty}^{n-k+1} \right) \left( \|u\|_{L^\infty}^{n-k} + \|u\|_{L^\infty}^{n-k-1} \|\varphi_c\|_{L^\infty} + \cdots \right)
+ \|u\|_{L^\infty} \|\varphi_c\|_{L^\infty}^{n-k-1} \left( \int_{\mathbb{R}} (\partial_x \varphi_c)^{4k} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} (u - \varphi_c)^2 \, dx \right)^{\frac{1}{2}}
\leq \sum_{k=2}^{n} \frac{C_{n+1}^k}{2k-1} \left( \|u - \varphi_c\|_{H^1} + 2 \|\varphi_c\|_{H^1} \right)^{2n-2k+1} \|\partial_x \varphi_c\|_{L^{4k}}^{2k} \cdot \|u - \varphi_c\|_{H^1}
\leq \sum_{k=2}^{n} \frac{C_{n+1}^k}{2k-1} \frac{a^{4k}}{2k} (2\sqrt{2a} + \varepsilon)^{2n-2k+1} \cdot \varepsilon,
\]

where we used the fact that $\|\partial_x \varphi_c\|_{L^{4k}}^{4k} = \frac{4k}{2k}$. Therefore, gathering the estimations $K_1-K_5$, we get the desired result (4.5).

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