Recurrence of random walks with long-range steps generated by fractional Laplacian matrices on regular networks and simple cubic lattices

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Abstract

We analyze a Markovian random walk strategy on undirected regular networks involving power matrix functions of the type \( L^\alpha \) where \( L \) indicates a ‘simple’ Laplacian matrix. We refer to such walks as ‘fractional random walks’ with admissible interval \( 0 < \alpha \leq 2 \). We deduce probability-generating functions (network Green’s functions) for the fractional random walk. From these analytical results we establish a generalization of Polya’s recurrence theorem for fractional random walks on \( d \)-dimensional infinite lattices: The fractional random walk is transient for dimensions \( d > \alpha \) (recurrent for \( d \leq \alpha \)) of the lattice. As a consequence, for \( 0 < \alpha < 1 \) the fractional random walk is transient for all lattice dimensions \( d = 1, 2, .. \) and in the range \( 1 \leq \alpha < 2 \) for dimensions \( d \geq 2 \). Finally, for \( \alpha = 2 \), Polya’s classical recurrence theorem is recovered, namely the walk is transient only for lattice dimensions \( d \geq 3 \). The generalization of Polya’s recurrence theorem remains valid for the class of random walks with Lévy flight asymptotics for long-range steps. We also analyze the mean first passage probabilities, mean residence times, mean first passage times and global mean first passage times (Kemeny constant) for the fractional random walk. For an infinite 1D lattice (infinite ring) we obtain for the transient regime \( 0 < \alpha < 1 \) closed form expressions for the fractional lattice Green’s function matrix containing the escape and ever passage probabilities. The ever passage probabilities (fractional lattice...
Green’s functions) in the transient regime fulfil Riesz potential power law decay asymptotic behavior for nodes far from the departure node. The non-locality of the fractional random walk is generated by the non-diagonality of the fractional Laplacian matrix with Lévy-type heavy tailed inverse power law decay for the probability of long-range moves. This non-local and asymptotic behavior of the fractional random walk introduces small-world properties with the emergence of Lévy flights on large (infinite) lattices.

Keywords: fractional random walk, Polya walk, recurrence theorem, first passage probabilities, Kemeny constant, mean first passage times (MFPT), Lévy flight

(Some figures may appear in colour only in the online journal)

1. Introduction

Due to the rapid growth of online networks and search engines such as Google, there is an increasing interest in improved and faster search and navigation strategies for complex networks [1–6]. The number of systems which can be thought of as complex networks is huge. They include biological, social and friendship networks, human cities, electricity networks, water supply networks, transport networks (rivers, streets), computer networks such as the world wide web, and the crystalline structures of solids. There are numerous other examples. Due to this huge variety of different fields and contexts, the study of the dynamic processes in networks has become a vast interdisciplinary field. Many of these processes, such as internet searches, the spread of rumors, news headlines, propagation of pandemic deceases, foraging, the transitions in chemical reactions and many other examples in very different contexts, can be considered as random walks on abstract sets of points (nodes) on networks (graphs).

In 1921, Polya was probably one of the first to provide a thorough analysis of Markovian time-discrete random walks on periodic $d$-dimensional lattices. In these ‘Polya walks’, the walker is allowed to step with equal probability only to any of its neighbor nodes [20–23]. Polya proved for this kind of random walk that the walker is sure to return to its starting node for dimensions $d = 1, 2$ of the lattice whereas for dimensions $d > 2$ a finite escape probability (probability of never returning) exists. This celebrated result has become known as Polya’s theorem or the recurrence theorem [20, 23]. As one of the subjects of the present paper, we will generalize this theorem to ‘fractional random walks’ (FRWs).

Polya considered recurrence for random walks performing steps only to next-neighbor nodes. Hudges and Shlesinger analyzed random walks on simple cubic lattices with an asymptotic power-law behavior of the vibrational dispersion relation and derived conditions for recurrence of these walks [11]. In that seminal paper, it was demonstrated that the Polya recurrence theorem has to be modified for such walks with power-law asymptotics. Later, we will see that this scaling dimension (index of the dispersion relation) can be identified with the Lévy index of a random walk performing asymptotically long-range steps which are drawn from a Lévy distribution (see also appendix B).

Noh and Rieger [3] considered important characteristics of ‘normal random walks’ (NRWs) in complex networks which are a generalization of the Polya-type walk to networks of variable degree of the nodes. In that paper, characteristics such as mean first passage times (MFPTs) and first passage probabilities were deduced. Watts and Strogatz [24] showed that, in many ‘real-world’ networks, features like a small world emerge that are not captured by
the previously mentioned classical network models [25]. Indeed, in the meantime, numerous further models were developed to generate new types of sophisticated small-world networks, including randomly generated features such as the Erős–Rényi network [26] and scale-free (self-similar) and fractal networks [2, 25].

A generalization of the NRW concept [3] to Lévy type random walks on complex undirected networks was presented by Riascos and Mateos [6]. They demonstrated that if a Lévy type navigation strategy is performed on a large-world network, small-world properties emerge with increasing efficiency as compared to a NRW.

The fractional calculus approach has turned out to be a powerful analytical tool to describe a large variety of phenomena such as anomalous diffusion and fractional transport [9, 12] (and the references therein). Usually the fractional approach is often applied to problems in continuous spaces. A fractional lattice approach has been suggested by Tarasov [17].

In many practical applications, in the development of search strategies the simple question arises as to how the average number of necessary steps can be reduced until a result is found. In the random walk picture this question corresponds to finding random walk strategies that reduce first passage times. One goal of this paper is to demonstrate that this can be achieved for random walk strategies based on fractional Laplacian matrices (FRWs).

The present paper is arranged as follows. In the subsequent section 2 we invoke some basic general features of Markovian random walks on undirected regular networks. We describe some important spectral properties of the transition matrix for the FRW and determine probability-generating functions (which we refer alternatively to as Green’s functions) for the node occupation probabilities and first passage probabilities, which are highly powerful analytical tools in the determination of these probabilities. To keep our demonstration as simple as possible, we especially focus on regular networks, i.e. networks having constant degree for all nodes such as \(d\)-dimensional cubic primitive periodic and infinite lattices \((d = 1, 2, 3, 4, ...).\) Nevertheless, many of the obtained results can be generalized to more complex network types.

In section 3, we deduce the Green’s functions for the FRW in \(d\)-dimensional lattices and analyze infinite lattice limits for the probabilities that a node is ever visited including probabilities of (n)ever returning. In this way we establish a generalization of Polya’s recurrence theorem for FRWs holding in the limit of infinite lattices. Further, we also obtain for \(d\)-dimensional periodic and infinite lattices exact expressions for the mean first passage times (MFPT) and global MFPT (Kemeny constant) in terms of spectral properties of the fractional Laplacian matrix.

In section 4 we develop, for the transient regime of the FRW, explicit expressions for the 1D infinite lattice (infinite ring) for the ever passage probabilities and escape probabilities and obtain power law evanescent asymptotic behavior of Riesz potential forms for the ever passage probabilities for nodes far from the departure node.

The main dynamic effect is that an FRW performed on a large-world network appears as a walk in a small-world network. The long-range moves appearing in the FRW make the dynamics of the FRW remarkably rich. One aim of the present paper is to analyze the FRW (i.e. a random walk generated by \(L^\alpha (0 < \alpha \leqslant 2)\)) with special emphasis on the emergence of universal features such as Lévy flight behavior with inverse power-law heavy tailed distribution for large steps for \(0 < \alpha < 2\) (see also appendix B), and universal limit of extreme transience \(\alpha \to 0\). Among these universal features, we analyze in section 3 the recurrence features of the FRW.

The present study aims to demonstrate some of these dynamic effects and complement some previous studies on the subject [6, 30–35].
2. Basic notions on Markovian time discrete random walks

2.1. Fractional Laplacian matrix and the FRW on regular undirected networks

We analyze random walks on regular undirected connected networks (graphs) of \( N \) nodes which we denote by \( p = 0, \ldots, N - 1 \). In the regular networks considered in the present paper, all nodes (vertices) \( p \) have constant degree \( K_p = K \) \( \forall p = 0, \ldots, N - 1 \). This is the case for the lattice structures considered in the present paper. Whether or not a pair of nodes \( p, q \) is connected (by an edge) is described by the \( N \times N \) adjacency matrix \( A \) with elements \( A_{pq} = 1 \) if the nodes \( p, q \) are connected and \( A_{pq} = 0 \) otherwise. Further, we assume \( A_{pp} = 0 \). In an undirected network, the connections (edges) between nodes have no direction and as a consequence the adjacency matrix is symmetric \( A_{pq} = A_{qp} \). The properties of the network are characterized by the Laplacian matrix which we can write in its spectral representation \([1, 27]\)

\[
L_{pq} = \delta_{pq}K_p - A_{pq} = \sum_{j=1}^{N} \mu_j \langle p|\Psi_j \rangle \langle \Psi_j|q \rangle,
\]

(1)

where we adopt in this paper the common Dirac bra-ket notation and we have the symmetry \( L_{pq} = L_{qp} \) of the Laplacian matrix of undirected networks. In the present paper we assume constant degree \( K_p = K \) for any node \( p \) thus the Laplacian matrix takes the form \( L = K I - A \) where \( I \) denotes the \( N \times N \) unity matrix. Due to the symmetry of the Laplacian matrix, the set of eigenvectors constitutes a complete \( N \)-dimensional ortho-normal canonic basis. The degree \( K_p \) of a node \( p \) counts the number of connections of node \( p \) with other nodes. This is expressed by the relationship \( K_p = \sum_{q=0}^{N-1} A_{pq} \). It follows that the constant vector which we denote as \( |\Psi_1 \rangle = \frac{1}{\sqrt{N}} (1, \ldots, 1) \) is the eigenvector of the Laplacian matrix \( L \) to the zero eigenvalue \( \mu_1 = 0 \). Generally the Laplacian matrix is positive-semidefinite and in connected networks the vanishing eigenvalue \( \mu_1 \) appears uniquely together with \( N - 1 \) positive eigenvalues \( 0 < \mu_2 \leq \ldots \leq \mu_N \) \([27]\).

To analyze random walks on networks, we introduce the probability vector \( \vec{P}_t = \sum_{p=0}^{N-1} P_t(p)|p \rangle \) having the occupation probabilities \( P_t(p) \) of the nodes \( p \) as Cartesian components where variable \( t \) denotes the time. As the random walker is definitely somewhere on the network, the normalization condition \( \sum_{p=0}^{N-1} P_t(p) = 1 \) is fulfilled for the entire time of observation \( 0 \leq t < \infty \) where we define \( t = 0 \) as the time of departure of the random walker. We consider time-discrete random walks at integer times \( t = 0, 1, 2, \ldots \) where the walk is assumed to start at \( t = 0 \) and during one time increment \( \Delta t = 1 \) the random walker is allowed to move from one node to another where only steps between connected nodes \((A_{pq} \neq 0)\) are allowed.

The time evolution of the occupation probabilities for a Markovian walk is governed by a discrete master equation where we utilize alternatively matrix and index notations \([23]\)

\[
P_{t+1}(p) = \sum_{q=0}^{N-1} W_{pq} P_t(q), \quad \vec{P}_{t+1} = \mathcal{W} \cdot \vec{P}_t.
\]

(2)

The \( N \times N \) matrix \( \mathcal{W} = : W(\Delta t = 1) \) is referred to as the transition matrix connecting the probabilities \( \vec{P}_{t+1} \) with \( \vec{P}_t \). We analyze in the present paper time-discrete random walks, a transition to time continuous walks was considered in \([32]\)^2. For random walks taking place as Markovian processes, the transition matrix of one time step \( \mathcal{W} = W(\Delta t) \) is constant depending

\(^4\langle \Psi_j|\Psi_i \rangle = \delta_{ij} \) and \( \sum_{n=0}^{N} \langle \Psi_n|l \rangle \langle l|\Psi_m \rangle = \delta_{nm} \).

\(^5\)Letting \( \Delta t \to 0 \) infinitesimal yields the transition matrix of time-continuous random walks.
only on the time step \( \delta t \), but not on the history of the walk. The time-evolution (2) of the transition matrix iterating \( t = n \) time steps is then

\[
P_n(p) = \sum_{q=0}^{N-1} \langle p|\mathcal{W}^n|q \rangle P_0(q),
\]

where \( \mathcal{W}^n = \hat{1} \) denotes the \( N \times N \) unity matrix and where subsequently we utilize synonymously \( \langle p|\mathcal{W}^n|q \rangle = W_{pq}(t) \) for the elements of the transition matrix \( \mathcal{W}^t = W(t) \). The transition matrix fulfills the normalization condition

\[
\sum_{p=0}^{N-1} W_{pq}(t) = 1, \quad 0 \leq W_{pq}(t) \leq 1,
\]

reflecting the normalization of the occupation probabilities. We also have the restriction \( 0 \leq W_{pq}(t) \leq 1 \) allowing the probability interpretation to be maintained for the entire observation time \( 0 \leq t = n < \infty \). As mentioned above, we confine ourselves here to undirected regular networks which are characterized by the symmetry property \( W_{pq}(t) = W_{qp}(t) \), reflecting the fact that there is no preference for moves between \( p \) to \( q \) and vice versa. For the analysis to follow it is worthwhile considering the spectral properties of the transition matrix. As the transition matrix is symmetric (self-adjoint), it can be expressed by its (purely real) eigenvalues \( \lambda_m \) and eigenvectors \( |\Psi_m\rangle \) as

\[
\mathcal{W} = W(t = 1) = \sum_{m=1}^{N} \lambda_m |\Psi_m\rangle \langle \Psi_m|, \quad W(t = n)_{pq} = (\mathcal{W}^n)_{pq} = \sum_{m=1}^{N} (\lambda_m)^n \langle p|\Psi_m\rangle \langle \Psi_m|q \rangle.
\]

In connected ergodic networks with constant degree, the stationary distribution is constituted by the equal distribution \( W_{pq}(t \to \infty) = \langle p|\Psi_1\rangle \langle \Psi_1|q \rangle = \frac{1}{N} \) (\( \forall p, q = 0, \ldots, N - 1 \)) [6, 27].

Now we relate network properties with the random walk dynamics by means of the Laplacian matrix. For the NRW on a regular network the transition matrix of one time step takes the following form [3, 6, 32]:

\[
\mathcal{W}_{pq} = \delta_{pq} - \frac{1}{K_p} L_{pq} = \frac{1}{K} A_{pq}, K = \frac{1}{N} \text{tr}(L),
\]

where \( \text{tr}(..) \) denotes the trace \( \sum_{p=0}^{N-1} (..)_{pp} \) of a matrix and \( \frac{1}{K} L_{pq} = \frac{1}{K} A_{pq} \) the (for regular networks considered in the present paper symmetric) generator matrix of the random walk. We notice that \( \mathcal{W}_{pp} = 0 \), that is the walker is forced to change its node at each time step, moving with equal probability \( \frac{1}{K} \) to any next-neighbor node\(^7\). The transition matrix (6) is invariant by rescaling \( L \to \zeta L \) of the Laplacian matrix by any non-zero scaling factor \( \zeta \). We utilize this simple observation later for the physical interpretation of the dynamics of the fractional random walk.

It follows from (6) that the eigenvalues of \( \mathcal{W} \) and of the Laplacian matrix \( L \) are related by \( \lambda_m = 1 - \frac{m}{K} (m = 1, 2, \ldots, N) \), where both the transition matrix and the Laplacian matrix as

\(^6\)From \( \lim_{n \to \infty} \mathcal{W}^n = W(\infty) = |\Psi_1\rangle \langle \Psi_1| \) it follows that \( \lambda_1 = 1 \) is unique where the remaining \( N - 1 \) eigenvalues \( |\lambda_m| < 1 \) \( m = 2, \ldots, N \) where all eigenvalues are real due to \( W_{pq} = W_{qp} \).

\(^7\)We identify ‘next-neighbor node’ with ‘connected node’.

\(^8\)This scaling invariance is absent in the time continuous random walks when the transition matrix is defined as in [32].
well as any matrix functions of these matrices have an identical space of eigenvectors with the canonical set $|\Psi_j\rangle$ $(j = 1, 2, \ldots, N)$.

The transition matrix (6) defines an NRW on regular networks with constant degree. After $n = 0, 1, 2, \ldots$ time steps the transition matrix elements take the form

$$W_{pq}(n) = P_n(p, q) = [W^\alpha]_{pq} = \frac{1}{K^n} Z_n(p, q),$$

indicating the probability that node $p$ is occupied by the walker at the $n$th-time step for a walk departing at node $q$. In this relation we have introduced

$$Z_1(p, q) = A_{pq},$$

$$Z_n(p, q) = (A^n)_{pq} = \sum_{j_1, j_2, \ldots, j_{n-1}} A_{p j_1} A_{j_1 j_2} \cdots A_{j_{n-1} q}, \quad n = 2, 3, \ldots$$

indicating the number of possible paths the walker can take when performing a walk of $n$ time steps starting at node $q$ ending at node $p$. Each of these paths occurs in (7) with equal probability $\frac{1}{K^n}$ where $K^n$ indicates the number of paths a random walker can choose performing $n$ time steps where all paths depart from the same node.

We refer to random walks of the type (7) and (8), with equal probability of any possible path, as Polya walks [20, 23] where for regular networks considered in the present paper the notions ‘Polya walk’ and ‘normal random walk (NRW)’ can be used synonymously. The equal probability distribution of paths is characteristic of Polya-type walks and is no more true for the FRWs analyzed subsequently. Summarizing (8) over all possible paths starting at node $q$ of $n$ time steps, we obtain the total number of $n$-time step paths

$$\sum_{q=0}^{N-1} Z_n(p, q) = \sum_{q=0}^{N-1} K^n W_{pq}(n) = K^n$$

starting at $q$ reflecting the normalization condition of the transition matrix $W_{pq}(n)$ at any time step.

The main subject of the present analysis is to study a generalization of the NRW which we refer to as FRW where the Laplacian matrix (1) which was employed in (6) is replaced by a fractional non-integer power matrix function of the form

$$L_\alpha^2 = \sum_{m=2}^{N} (\mu_m)^\alpha |\Psi_m\rangle \langle \Psi_m|, \quad 0 < \alpha \leq 2,$$

which we refer to as the fractional Laplacian matrix. It is important to notice that the admissible interval for $\alpha$ in order to maintain the good properties of a random walk generator matrix is $0 < \alpha < 2$ [32]. For $\alpha = 2$ (9) recovers the Laplacian matrix (1) where the FRW then recovers the NRW (Polya walk). We define the transition matrix of one time step for the FRW corresponding to (6) [30, 31]

$$W^{(\alpha)}_{pq} = W_{pq}^{(\alpha)}(t = 1) = \delta_{pq} - \frac{1}{K^{(\alpha)}} (L_\alpha^2)_{pq} =: \frac{1}{K^{(\alpha)}} (A^{(\alpha)})_{pq}, \quad 0 < \alpha \leq 2. \quad (10)$$

Here we introduced the fractional degree $K^{(\alpha)}$ which is a constant in regular networks and given by the constant diagonal element of the fractional Laplacian matrix

$$K^{(\alpha)} = [L_\alpha^2]_{pp} = \frac{1}{N} \text{tr}(L_\alpha^2) = \frac{1}{N} \sum_{m=1}^{N} (\mu_m)^\alpha, \quad 0 < \alpha \leq 2. \quad (11)$$

So we observe that the diagonal element of the transition matrix $W_{pp}^{(\alpha)} = \frac{1}{N} \text{tr}(W^{(\alpha)}) = 0$ is vanishing as for the NRW and further due to $\mu_1 = \mu_2^\alpha = 0$ the stationary distribution

$^9$Where the summation over all paths indeed is a discrete network version of Feynman’s path integral.
(ergodicity) for the FRW $\langle p | \Psi_1 \rangle \langle \Psi_1 | q \rangle = \frac{1}{N}$ together with $\lambda_1 = 1$ of the fractional transition matrix is maintained. In contrast to the FRW on finite networks, it has been demonstrated that Lévy flights in infinite spaces exhibit non-ergodic features in the sense of diverging time and ensemble averages [10].

We introduced in relation (10) the fractional adjacency matrix

$$A_{pq}^{(\alpha)} = \delta_{pq} K^{(\alpha)} - (L^2)^{\alpha}_{pq} \geq 0,$$  \hspace{1cm} 0 < \alpha \leq 2, \tag{12}

where we have analogous properties as in the non-fractional case $K^{(\alpha)} = \sum_{q=0}^{N-1} A_{pq}^{(\alpha)}$ reflecting conservation of eigenvalue zero and corresponding eigenvector $|\Psi_1\rangle$ of the fractional Laplacian matrix. The fractional adjacency matrix $A_{pq}^{(\alpha)}$ has uniquely non-negative off diagonal elements $A_{pq}^{(\alpha)} = -(L^2)^{\alpha}_{pq} \geq 0$ ($p \neq q$)\footnote{Where it was demonstrated [32] that $(L^2)^{\alpha}_{pq}$ has non-positive off-diagonal elements for $0 < \alpha \leq 2.$} and zero diagonal elements $A_{pp}^{(\alpha)} = 0$ allowing the probability interpretation of (10) fulfilling the necessary conditions (i) $0 \leq W_{pq}^{(\alpha)}(t) \leq 1$ and (ii) $\sum_{t=0}^{N-1} W_{pq}^{(\alpha)}(t) = 1.$ As already mentioned for regular undirected networks, these good properties of the fractional Laplacian matrix allowing definitions (11) and (12) with (i) and (ii) are fulfilled within $0 < \alpha \leq 2.$ We notice that this range of exponent is exactly the range of definition of the Lévy index occurring in the context of Lévy flights and Lévy ($\alpha$-stable) distributions. Moreover, the emergence of Lévy flights for FRWs on $d$-dimensional lattices has been demonstrated recently [30–33]. A short demonstration is given in appendix B.

2.2. Some general remarks on first passage probabilities and mean first passage times

In this section we evoke some basic relations between occupation probabilities and first passage probabilities and their statistical interpretations as far as is required for the present analysis of the FRW. The efficiency of a random walk strategy to explore the network can be measured by first passage quantities such as mean first passage probability and mean first passage times (MFPTs) for a node. In the deductions to follow we first consider finite networks, i.e. with a finite number $N$ of nodes and analyze the limit $N \to \infty$ of infinite networks. It turns out that new features such as transience of the random walk may emerge in the limiting cases of infinite networks. For a thorough analysis and further discussions of general properties we refer to [23, 27–29, 37].

In the subsequent analysis, when a quantity $B$ refers to the FRW, we employ the notation $B^{(\alpha)}$ with a superscript $(\alpha).$ Otherwise for general relations as well as for NRWs we utilize $B.$ The following definitions and notions will be used.

1. $P_n(p, q)$ denotes the occupation probability of node $p$ by a random walker starting at node $q$ undertaking a walk of $n$ time steps. This probability coincides with the ratio of the number $Z_n(p, q)$ of paths starting at $q$ ending at $p$ of $n$ time steps and the total number $K^n$ of paths of $n$ time steps with the same departure node. The occupation probabilities $P_n(p, q)$ were already defined in above relation (7).

2. As a generalization of (8) we interpret $(K^{(\alpha)})^p$ (where $K^{(\alpha)}$ denotes the fractional degree (11)) as the ‘fractional’ number of allowed paths for an FRW of $t = n$ time steps with the same starting node.

3. The quantity $Z_n^{(\alpha)}(p, q) = ( (A^{(\alpha)})^n )_{pq} = \sum_{j_1, \ldots, j_n} A^{(\alpha)}_{j_n, j_{n-1}} A^{(\alpha)}_{j_{n-1}, \ldots, j_1} A^{(\alpha)}_{j_1, p}$ indicates the ‘fractional’ number of paths of $n$ time steps starting at node $q$ ending at node $p$ where it...
turns out that the equal probability distribution of paths of the Polya walk is not true for the FRW when \(0 < \alpha < 2\). The occupation probability of the FRW is \(P_n^{(\alpha)}(p, q) = W_{pq}^{(\alpha)}(n) = \frac{Z_k^{(\alpha)}(p, q)}{W_p^{(\alpha)}(n)}\) constituting the matrix elements of the fractional transition matrix \(W_{pq}^{(\alpha)}(n) = \langle p(W^{(\alpha)})^n | q \rangle\). In the fractional case \(0 < \alpha < 2\) the \(Z_k^{(\alpha)}(p, q)\) and \((K^{(\alpha)})^n\) are generally non-negative non-integers. For \(\alpha = 2\), all characteristics of the Polya walk are recovered.

(4) \(F_n(p, q)\) denominates the first passage probability, that is the probability that a random walker starting from node \(q\) visits node \(p\) at the \(n^{th}\) time step for the first time. For interpretation purposes we introduce the number \(f_n(p, q)\) of first passage paths of \(n\) time steps. A first passage path is a path starting at node \(q\) containing node \(p\) only once as the end node. These are the paths of \(n\) time step walks departing at node \(q\) passing at node \(p\) for the first time. When \(p = q\), these paths constitute closed cycles representing paths of first return to the starting node \(q\). It follows that the first passage probabilities can be represented as \(F_n(p, q) = \frac{f_n(p, q)}{K_n}\), where \(K_n\) denotes the total number of possible paths of \(n\) time steps starting all from node \(q\) where \(K\) indicates the constant degree. In a regular undirected network the probabilities of first return \(F_n(q, q) = F_n(0, 0)\) are constant for all nodes and the probabilities of first passage \(F_n(p, q) = F_n(q, p)\) as well as the occupation probabilities (transition matrices) represent symmetric matrices with respect of interchanging starting and end nodes.

With these definitions we can establish a relationship between the first passage probabilities \(F_i\) and the occupation probabilities \(P_i\) which holds for Markovian walks \([3, 6, 23]\)

\[
P_i(p, q) = \delta_{i0}\delta_{pq} + \sum_{k=0}^{t} F_{t-k}(p, q)P_k(0, 0)
\]

with \(P_0(p, q) = \delta_{pq}\) where \(F_0(p, q) = 0\) and \(F_1(p, q) = P_1(p, q)\) as at \(t = 1\) only next-neighbor nodes can be visited for the first time. Relation (13) can be interpreted as follows by multiplying relation (13) by the total number of possible paths \(K_n\) of \(n\) time steps. In Markovian random walks the number of possible paths \(Z_n(p, q)\) connecting the nodes \(q\) and \(p\) of \(n\) time steps can be decomposed into \(Z_n(p, q) = \sum_{k=0}^{n} f_{n-k}(p, q)Z_k(0, 0)\) \((f_0 = 0)\). That is the number \(f_{n-k}(p, q)\) of first passage paths of \(n - k\) time steps multiplied with the number \(Z_k(0, 0)\) of return paths of \(k\) time steps whereas all combinations \(k = 0, \ldots, n\) occur as a sum reflecting the property that first passage events at different times are exclusive events representing different paths. Whereas the occupation probabilities are determined by (3), the first passage probabilities are uniquely determined by (13).

For the determination of the probabilities \(Q_n = (P_n, F_n)\) of further characteristics it is convenient to employ the method of probability-generating functions. The probability-generating function of the probabilities \(\{Q_n\}\) is defined as a power series having these probabilities as non-negative coefficients

\[
Q(p, q, \xi) = \sum_{n=0}^{\infty} Q_n(p, q)\xi^n \quad |\xi| < 1
\]

with, according to Abel’s theorem, (at least) the radius of convergence \(\xi = 1\). The \(Q(p, q, \xi) = \langle F(p, q, \xi), P(p, q, \xi) \rangle\) stand in the following analysis for the generating functions of the first passage and occupation probabilities, respectively. The occupation probability-generating function \(P(p, q, \xi)\) is also referred to as the network Green’s function \([21–23]\).

The matrix elements of the network Green’s function (at \(\xi = 1\)) \(P(p, q, \xi = 1) = \sum_{i=q}^{\infty} P_i(p, q)\) indicate the mean residence times (MRTs), i.e. the average number of time
steps the walker occupies a node \( p \) (when starting the walk at node \( q \)) for an infinite time of observation \( t \to \infty \). For discrete time random walks defined as in (6), where the walker at any time step moves to another node (as \( W_{pp} = 0 \)), the MRT \( P(p, q, 1) \) counts the average number of visits of a node during an infinite observation time [42]. It follows that on an infinite network a walk is transient if the average number of visits of a node is finite, i.e. \( P(p,q,\xi = 1) < \infty \), and further a walk is recurrent if a node \( p \) in the average infinitely often is visited where \( P(p,q,\xi = 1) \to \infty \). An instructive study of the MRT and MFPT among other features of Lévy flights versus Lévy walks in the 1D continuous space is given in [13], and a further analysis of first escapes and arrivals in finite domain is performed in [14].

The zero-order \( F_0(p,q) \) in the series for \( F(p,q,\xi) \) is vanishing whereas \( P_0(p,q) = \delta_{pq} \). The probability-generating function (14) can be read as a discrete Laplace transform by \( \xi = e^{-s} \) converging for \( \Re(s) > 0 \). We mention this point as technically for the determination of the moments the Laplace transform is more convenient to use \( \langle (r^n)_{pq} \rangle = (-1)^n \frac{d^n}{ds^n} \mathcal{L}(p,q,e^{-s}) \big|_{s=0} \). (13) can be identified with the \( n \)th orders \( \sim \xi^n (n = 1, 2, ...) \) of the functional identity

\[
P(p,q,\xi) - \delta_{pq} = F(p,q,\xi)P(0,0,\xi). \tag{15}
\]

Thus the generating function for the first passage probabilities is obtained as [22]

\[
F(p,q,\xi) = \frac{P(p,q,\xi) - \delta_{pq}}{P(0,0,\xi)}, \quad F(\xi) = \frac{1}{P(0,0,\xi)} \left( P(\xi) - \hat{1} \right). \tag{16}
\]

where in the second equation we write this relation in matrix form. For our subsequent analysis of the FRW it is useful to relate (16) with the spectral properties of transition matrix and (fractional) Laplacian matrix. To this end, we evaluate the generating matrix for a finite network of \( N \) nodes

\[
P(\xi) = \sum_{n=0}^{\infty} \mathcal{W}^n \xi^n = \left[ \hat{1} - \xi \mathcal{W} \right]^{-1} = \frac{\langle \Psi_1 \rangle \langle \Psi_1 \rangle}{(1 - \xi)} + \sum_{m=2}^{N} \frac{\langle \Psi_m \rangle \langle \Psi_m \rangle}{(1 - \lambda_m \xi)}, |\xi| < 1. \tag{17}
\]

The diagonal element of (17), which is identical for all nodes, is obtained as

\[
P(p,p,\xi) = P(0,0,\xi) = \frac{1}{N} \text{tr}(P(\xi)) = \frac{1}{N} \left( \frac{1}{1 - \xi} + \sum_{m=2}^{N} \frac{1}{1 - \lambda_m \xi} \right). \tag{18}
\]

For \( p = q \) (16) contains the probabilities of first return to the starting node \( F(0,0,\xi) = 1 - \frac{\delta_{pp}}{P(0,0,\xi)} \) being identical for all departure nodes.

We mention the important property that \( F(p,q,\xi \to 1) = \sum_{n=1}^{\infty} F_n(p,q) \) yields the probability that the random walker starting at node \( q \) ever visits node \( p \), or equivalently that the random walker visits node \( p \) at least once during the infinite observation time \( t \to \infty \) [23]. This information is hence contained in (16) in the limiting case \( \xi \to 1 - 0 \). This quantity is of great importance in many contexts, such as survival time models and the subsequently analyzed question of recurrence (transience) of a random walk. For an infinite network \( N \to \infty \), the contribution of the stationary distribution \( \langle p|\Psi_1 \rangle \langle \Psi_1 |q \rangle = \frac{1}{N} \to 0 \) is suppressed reflecting the property that the Green’s function (17) in the limit \( N \to \infty \) becomes a ‘generalized function’ in the distributional sense [39]. The matrix elements of the Green’s function of the infinite network are defined by the distributional identity \( r_{pq}(\xi) = P_{N \to \infty}(p,q,\xi) \) in the sense of below given limiting integral (19). The suppressed stationary distribution becomes important only when performing infinite sums such as \( \sum_{p=0}^{\infty} (P_{\infty}(p,q,\xi) - r_{pq}(\xi)) = 1 \) reflecting the normalization of the probabilities.
To evaluate the infinite network Green’s function which we denote subsequently as \( P_{N \to \infty}(p, q, \xi) = r_{pq}(\xi) \), its spectral representation in infinite networks is determined by the spectral sum (17) accounting only for the relaxing eigenvalues \(|\lambda_n| < 1\), namely
\[
P_{\infty}(\xi) = r(\xi) = \sum_{m=2}^{\infty} |\Psi_m\rangle \langle \Psi_m| \frac{1}{(1 - \lambda_m \xi)} = \int_{\lambda_{\text{min}}}^{1} \frac{|\tilde{\Psi}(\lambda)|^2 \langle \tilde{\Psi}(\lambda) \rangle D(\lambda)}{(1 - \xi \lambda)} \, d\lambda,
\]
(19)
where, as mentioned for \( N \to \infty \), the stationary contribution corresponding to \( \lambda = 1 \) is suppressed. Whether or not (19) converges for \( \xi = 1 \) depends on the properties of the infinite network. On the right-hand side of (19) we accounted for the property that the spectrum \( \lambda_m \to \lambda \) becomes continuous when \( N \to \infty \) and the eigenvalue density \( D(\lambda) \, d\lambda \) counts the number of eigenvalues within \([\lambda, \lambda + d\lambda]\) and can be represented\(^{11}\)
\[
D(\lambda) = \lim_{N \to \infty} \sum_{m=2}^{N} \delta(\lambda - \lambda_m)
= \frac{1}{\pi} \Im \left( \frac{1}{\lambda \tilde{W} - \tilde{W} - i0} \right)^{-1},
\]
(20)
where \( \delta(\cdot) \) denotes Dirac’s \( \delta \)-function and \( |\tilde{\Psi}(\lambda)| \) indicate (appropriately renormalized) eigenfunctions. In (20) we further introduced \( \tilde{W} = W - |\Psi_1\rangle \langle \Psi_1| \) and \( \tilde{1} = \tilde{1} - |\Psi_1\rangle \langle \Psi_1| \). For the general derivations to follow in this section, however, it is sufficient to write (19) for the sake of simplicity as an infinite sum. Before we analyze the FRW let us carefully consider how properties change when passing from finite to infinite networks.

For finite and infinite networks, the geometrical series in (17) and (19) always converge for \(|\xi| < 1\). However, on finite networks the series in (17) is convergent only for \(|\xi| < 1\) but always divergent for \(\xi = 1\) due to the existence of the largest eigenvalue \(\lambda_1 = 1\) of the transition matrix \(W\). It follows then from (16) that in a finite network the probability of ever returning to the departure node is \( F(0, 0, \xi = 1) = 1 - \frac{1}{p_{pp}} \) with divergent \( P(0, 0, \xi = 1) = 1 \). It follows recurrence of the walk on finite networks. To prove the recurrence of the random walk, the divergence of \( P(\xi \to 1) \) is a sufficient criteria (due to the presence of eigenvalue \( \lambda_1 = 1 \) corresponding to the non-zero stationary distribution \( \frac{1}{N} \) on finite networks). Hence random walks on finite networks are always recurrent [23]. Further, we observe in view of (17) with (18) that in finite connected networks for all nodes \( p \) independent of the departure node, \( F(p, q, \xi \to 1) = 1 \) (due to the existence of \( \lambda_1 = 1 \)). In finite networks any node \( p \) (including the departure node) is visited. As a consequence, a search strategy based on the random walk in finite connected networks is always successful.

Depending on the properties of the network, this property may change in the case of infinite networks, which we will analyze more closely in what follows. Before we do so, let us further analyze the above-introduced \( F \)-matrix (16) (generating matrix of the first passage probabilities) taking the representation
\[
F(p, q, \xi) = \sum_{a=0}^{\infty} F_a(p, q) \xi^a = \frac{N^{-1} + (1 - \xi)(r_{pq}(\xi) - \delta_{pq})}{N^{-1} + (1 - \xi)r_{pp}(\xi)},
\]
(21)
for a finite network of \( N \) nodes where we have introduced
\(^{11}\) Where \( \lim_{a \to \infty} \frac{1}{\pi} \Im \langle \cdot \rangle_{\text{re}} = \frac{1}{\pi} \Im \langle \cdot \rangle_{\text{im}} = \delta(\cdot) \) and \( \Im(\cdot) \) indicates the imaginary part of (\( \cdot \)) with \( \tilde{W} = \sum_{m=2}^{\infty} \lambda_m |\Psi_m\rangle \langle \Psi_m| \) and \( 1 = \sum_{m=2}^{\infty} |\Psi_m\rangle \langle \Psi_m| \); see also below relation (22).
\( r(\xi) = \sum_{n=0}^{\infty} \xi^n \hat{W}^n = [I - \xi \hat{W}]^{-1}, \hat{W} = W - |\Psi_1 \rangle \langle \Psi_1 | \)

\[ r_{pq}(\xi) = P(p,q,\xi) - \frac{1}{(1-\xi)} \langle p | \Psi_1 \rangle \langle \Psi_1 | q \rangle = \sum_{m=2}^{N} \langle p | \Psi_m \rangle \langle \Psi_m | q \rangle \frac{1}{(1-\lambda_m \xi)} \]  \hspace{1cm} (22)

with \( \hat{I} = I - |\Psi_1 \rangle \langle \Psi_1 | \) indicating the unity in the \( N-1 \)-dimensional subspace of relaxing modes. Then we have

\[ r_{pp}(\xi) = \frac{1}{N} \sum_{q=0}^{N-1} r_{pq}(\xi) = \frac{1}{N} \sum_{m=2}^{N} \frac{1}{(1-\lambda_m \xi)} \]  \hspace{1cm} (23)

Since \( r_{pq}(\xi = 0) = \delta_{pq} = \frac{1}{N} \), we have \( F(p,q,\xi = 0) = 0 \), i.e. all matrix elements of the zero order in \( \xi \) are vanishing due to the fact that first passage probabilities at \( t = 0 \) are vanishing for all nodes. Thus series (21) \( F(p,q,\xi) = \sum_{m=1}^{\infty} F_m(p,q)\xi^m \) starts with the first order in \( \xi \) and with \( F_1(p,q) = \frac{1}{N} F(p,q,\xi)|_{\xi=0} = P_1(p,q) = \mathcal{W}_{pq} \) recovers the transition matrix as occupation probabilities coinciding with first passage probabilities at \( t = 1 \). For infinite networks \( N \to \infty \) relation (21) takes the form

\[ F_{\infty}(p,q,\xi) = \frac{(r_{pq}(\xi) - \delta_{pq})}{r_{pp}(\xi)} \]  \hspace{1cm} (24)

with

\[ r_{pq}(\xi) = P_{N \to \infty}(p,q,\xi) = \sum_{m=2}^{\infty} \langle p | \Psi_m \rangle \langle \Psi_m | q \rangle \frac{1}{(1-\lambda_m \xi)} \]  \hspace{1cm} (25)

which has to be read in the sense of an asymptotic integral (19). It follows that the matrices of first passage and occupation probability-generating functions \( F_{N \to \infty}(\xi) \), \( P_{N \to \infty}(\xi) \) are fully determined by the spectral properties of the infinite network Laplacian matrix. We notice that the matrix \( r(\xi) \) of (25) at \( \xi = 1 \) is also referred to as the fundamental matrix of the walk [19, 32].

On infinite networks, a walk is recurrent only if \( r_{pp}(\xi \to 1) \) is diverging. Otherwise the infinite spectral sum \( r(\xi = 1) \) of (25) is converging with \( F_{\infty}(p,p,1) = 1 - \frac{1}{r_{pp}(1)} < 1 \) where the escape probability is constant for all nodes in a regular network \( 1 - F_{\infty}(p,p,1) = \frac{1}{r_{pp}(1)} > 0 \) is non-zero. Such a walk is transient.

We will analyze in the next section the question of recurrence for the FRW on infinite \( d \)-dimensional simple cubic lattices to establish a generalization of Polya’s recurrence theorem holding for the entire class of random walks with the same asymptotic power law behavior as the FRW.

Another important characteristic is the mean first passage time (MFPT) indicating the average number of time steps \( \langle T_{pq} \rangle \) that a random walker needs starting at \( q \) to reach node \( p \). The MFPT with (21) is obtained as

\[ \langle T_{pq} \rangle = \sum_{n=1}^{\infty} n F_n(p,q) = \frac{d}{d\xi} F(p,q,\xi)|_{\xi=1} \]

\[ = \lim_{\xi \to 1-0} N^{-1} (\delta_{pq} - r_{pq}(\xi)) + r_{pp}(\xi) + N^{-1}(1 - \xi) (r'_{pq} - r''_{pp}) + (1 - \xi)^2 a(\xi) \]

\hspace{1cm} (26)
where \(a(\xi) = r_{pp}(\xi) r'_{pq}(\xi) - (r_{pq}(\xi) - \delta_{pq}) r'_{pp}(\xi)\) and \(\langle..\rangle' = \frac{d}{d\xi} \langle..\rangle\). For finite networks (26)\(^{12}\)

\[
\langle T_{pq} \rangle = \sum_{n=1}^{\infty} nF_n(p, q) = \frac{d}{d\xi} F(p, q, \xi)_{|\xi=1} = N(\delta_{pq} - r_{pq}(1) + r_{pp}(1)) = N \left( \delta_{pq} + \sum_{m=2}^{N} \frac{\langle p|\Psi_{m}\rangle\langle\Psi_{m}|p\rangle - \langle p|\Psi_{m}\rangle\langle\Psi_{m}|q\rangle}{(1 - \lambda_{m})} \right)
\]

(27)

which was also obtained earlier \([6]\)\(^{13}\). For \(p = q\) (27) yields the average number of steps for first return as \(\langle T_{pp} \rangle = N\) being constant for all nodes on a regular network increasing linearly with the number \(N\) of nodes and independent of the spectral properties of the Laplacian matrix. We emphasize that (27) holds for finite connected regular networks \((N < \infty)\) with a constant degree of the nodes. For infinite networks, (26) take the asymptotic form

\[
\langle T_{pq} \rangle_{\infty} = \lim_{\xi \to 1^+} a(\xi) (r_{pp}(\xi))^{-2}
\]

(28)

where \(a(\xi) = r_{pp}(\xi) r'_{pq}(\xi) - (r_{pq}(\xi) - \delta_{pq}) r'_{pp}(\xi)\) and \(r_{pq}(\xi) = \mathcal{P}_{N \to \infty}(p, q, \xi)\).

A further interesting quantity is the global mean first passage time which is defined as the average value of (27) averaged over all nodes of the network

\[
\langle T \rangle = \frac{1}{N} \sum_{p=0}^{N-1} \langle T_{pq} \rangle = 1 + Nr_{pp}(1) = 1 + \sum_{m=2}^{N} \frac{1}{(1 - \lambda_{m})} = 1 + K \sum_{m=2}^{N} \mu_{m}^{-1}
\]

(29)

where in the last relation we have used \(\sum_{p=0}^{N-1} r_{pq} = 0\). The global MFPT \(\langle T \rangle\) can be interpreted as the average number of time steps to reach any node of the network when starting at a node \(q\). We observe the remarkable property that (29) does not depend on \(q\), for a further general discussion see also \([19, 28]\).

When we exclude in the average (29) the contribution of return walks \(\langle T_{pp} \rangle\) then we arrive at a global mean first passage time which indicates the average number of steps to reach a randomly chosen destination node (different from the departure node) for the first time. Without counting the contribution of recurrent walks, the global MFPT yields

\[
K_{c} = \langle T \rangle - 1 = \sum_{m=2}^{N} \frac{1}{(1 - \lambda_{m})} = Nr_{pp}(\xi = 1)
\]

(30)

and is referred to as the Kemeny constant \([6, 19, 28, 29]\). In the picture of diffusive transport phenomena described by the random walk, the inverse Kemeny constant (inverse global MFPT) \(K_{c}^{-1}\) measures the speed of the random walk. The smaller the Kemeny constant \(K_{c}\), the faster the random walker moves through the network.

### 2.3. Some general features and useful formulas for the FRW

Before analyzing lattices, let us briefly deduce in this subsection some useful formulas to analyze the FRW. The good properties of the fractional Laplacian matrix \(\mathcal{L}^{\alpha}\) are maintained in the interval \(0 < \alpha \leq 2\)\(^{14}\). These properties are the following: the zero eigenvalue \(\mu_1 = 0\)

---

\(^{12}\)Since on finite networks \(r_{pq}(\xi = 1), r'_{pq}(\xi = 1), N^{-1}\) are finite.

\(^{13}\)(Where \(r_{pq}(\xi) = (r_{pq}(\sigma^{-})|_{\sigma = 0} = R_{pq}^{(0)}\) and \(R_{pq}^{(0)}\) and \(\frac{1}{\alpha} = F_{pq}\) is the notation used in \([6]\), see equation (10)).

\(^{14}\)A discussion and detailed proof can be found in \([32]\).
and the remaining \( N - 1 \) positive eigenvalues \( \mu^2_m > 0 \) of the fractional Laplacian matrix are maintained. The off-diagonal elements \( \langle p|L^2|q \rangle \leq 0 \) remain non-positive as in (1). These properties guarantee the existence of the representation \( \langle p|L^2|q \rangle = K^{(\alpha)} \delta_{pq} - A^{(\alpha)}_{pq} \) with the positive fractional degree \( K^{(\alpha)} \) and non-negative elements of the fractional adjacency matrix (12). The fundamental matrix (Green’s function) (22) at \( \xi = 1 \) takes the form

\[
r^{(\alpha)}(\xi = 1) = K^{(\alpha)} \sum_{m=2}^{N} |\Psi_m\rangle\langle\psi_m|\left(\mu_m\right)^{-\frac{\alpha}{2}}
\]

(31)

where \( K^{(\alpha)} = \frac{1}{N} \sum_{m=2}^{N} (\mu_m)^{\alpha} \) indicates the fractional degree introduced in (11). An interesting representation of the fractional fundamental matrix (31) useful in the analysis of FRWs on lattices is obtained in terms of Mellin transforms. Let \( f(\tau) \) be an arbitrary bounded function \( |f(\tau)| < \infty \) with the existing Mellin transform\(^{15} \) which is defined by [38]

\[
\mathcal{M}_f \left( \frac{\alpha}{2} \right) = \int_0^\infty f(\tau) \tau^{\frac{\alpha}{2} - 1} d\tau < \infty
\]

(32)

where we always confine ourselves to the admissible range of the fractional Laplacian matrix \( 0 < \alpha \leq 2 \). The fractional fundamental matrix (31) can then be represented by Mellin transform of the \( N \times N \) matrix function \( f(L\tau) \) as

\[
\frac{r^{(\alpha)}(1)}{K^{(\alpha)}} = \frac{1}{\mathcal{M}_f \left( \frac{\alpha}{2} \right)} \int_0^\infty \left( f(L\tau) - f(0) \right) |\Psi_1\rangle\langle\psi_1| \tau^{\frac{\alpha}{2} - 1} d\tau
\]

(33)

which is well defined by scalar integrals by employing the spectral representation of the matrix function \( f(L\tau) = f(0) |\Psi_1\rangle\langle\psi_1| + \sum_{m=2}^{N} f(\mu_m\tau) |\Psi_m\rangle\langle\psi_m| \). The large choice of functions \( f \) provides a convenient tool for generating useful integral representations for the fundamental matrix of the FRW. By choosing \( f(\tau) = e^{-\tau} \), which refers to this class of functions \( f \) and \( \mathcal{M}_\exp(-\cdot) = \int_0^\infty e^{-\tau} \tau^{\frac{\alpha}{2} - 1} d\tau = \Gamma \left( \frac{\alpha}{2} \right) \), we get

\[
r^{(\alpha)}(1) = \frac{K^{(\alpha)}}{\Gamma \left( \frac{\alpha}{2} \right)} \int_0^\infty \left( e^{-\tau L} - |\Psi_1\rangle\langle\psi_1| \right) \tau^{-\frac{\alpha}{2} - 1} d\tau
\]

(34)

where \( \Gamma(\cdot) \) denotes the \( \Gamma \)-function and \( e^{-\tau L} = |\Psi_1\rangle\langle\psi_1| + \sum_{m=2}^{N} |\Psi_m\rangle\langle\psi_m|e^{-\mu_m\tau} \) the matrix exponential of Laplacian matrix \( L \). Relation (34) is especially useful as it links the fundamental matrix of the time-discrete FRW with the matrix exponential \( e^{-L\tau} \), which can be interpreted as the transition matrix of a time-continuous NRW having a probability distribution evolving as \( \tilde{P}(t) = e^{-\tau\tilde{L}}\tilde{P}(0) \) with the master equation \( \frac{d}{dt}\tilde{P}(t) = -\tilde{L}\tilde{P}(t) \) [32].

Consider now the important limit \( \alpha \to 0^+ \). In this limiting case the \( N - 1 \) non-zero eigenvalues take asymptotically \( (\mu_m)^{\alpha} \to 1 \) and hence we get for the fractional Laplacian matrix the limiting expression

\[
\lim_{\alpha \to 0^+} L^2 = \sum_{m=2}^{N} |\Psi_m\rangle\langle\psi_m| = \tilde{1} - |\Psi_1\rangle\langle\psi_1|,
\]

\[
\lim_{\alpha \to 0^+} \langle p|L^2|q \rangle = \delta_{pq} \frac{N - 1}{N} - \frac{1}{N}(1 - \delta_{pq}).
\]

(35)

\(^{15}\) Defined on \( 0 \leq \tau < \infty \) and decaying as \( \tau \to \infty \) at least as \( f(\tau) \leq \text{const} \tau^{-\beta} \) with \( \beta > \frac{\alpha}{2} \) as \( \tau \to \infty \).
This limiting expression for the fractional Laplacian matrix is universal for finite ergodic regular networks in the sense that it only requires one vanishing and \(N - 1\) positive eigenvalues \(\mu_m\) independent of their values. Especially, we obtain for the limit of the fractional adjacency matrix \(A_{pq}^{(\alpha \to 0)} = \frac{1}{N}(1 - \delta_{pq})\) and for fractional degree \(K^{(\alpha \to 0)} = \frac{N-1}{N}\). This leads for \(\alpha \to 0\) to the transition matrix

\[
\lim_{\alpha \to 0^+} W^{(\alpha)} = \frac{1}{K^{(\alpha \to 0)}} A_{pq}^{(\alpha \to 0)} = \frac{1}{N-1} (1 - \delta_{pq}) \tag{36}
\]

coinciding with the transition matrix of a Polya walk on a fully connected network where the walker can reach in one time step any node with equal probability \(\frac{1}{N-1}\). This observation can be confirmed when we return to above ‘trivial’ scaling invariance of transition matrices of the form (6).

Namely that any rescaled Laplacian matrix generates an identical random walk. Choosing a scaling factor \(\zeta = N\) gives a rescaled Laplacian corresponding to the same random walk as the FRW for \(\alpha \to 0^+\) (36) generated by the Laplacian matrix \(N \lim_{\alpha \to 0^+} [L^{(\alpha)}]_{pq} = \delta_{pq}(N-1) - (1 - \delta_{pq})\) of a fully connected regular network where the degree of any node is \(K = N - 1\). In other words, in the limit of vanishing \(\alpha \to 0^+\) the FRW exhibits the extremely fast small-world dynamics of a Polya-type NRW performed on a fully connected network.

This universal extreme small-world property of the FRW for \(\alpha \to 0^+\) on finite regular networks is independent of the spectral properties of the Laplacian. This observation underlines the main effect of the FRW dynamics becoming especially pronounced at small \(\alpha\) where any network appears as a small-world network and (for \(\alpha \to 0^+)\) any node can be reached in one time step with equal probability \(\frac{1}{N-1}\) as in a completely connected network.

In the subsequent section, we consider FRWs on lattices with a more profound analysis of some features of the FRW demonstrating the remarkable richness of the FRW. A key role in this dynamics is played by the fractional scaling index \(\alpha\) appearing as a controlling parameter which switches between large world (for \(\alpha = 2\) corresponding to the NRW) to a small world \(0 < \alpha < 2\) where the small world becomes extremely pronounced for small \(\alpha\).

An important quantity in this analysis is, as already mentioned, the Kemeny constant of the FRW \(K_\alpha^{(\alpha)}\) being invariant towards a rescaling with any nonzero scaling factor of the fractional Laplacian. Using relation (30), the Kemeny constant (global MFPT) of the FRW can be represented by the eigenvalues of the fractional Laplacian as

\[
K_\alpha^{(\alpha)} = \text{tr}(r^{(\alpha)}(\xi = 1)) = \frac{1}{N} \sum_{m=2}^{N} (\mu_m)^{\frac{1}{\alpha}} \frac{1}{N} \sum_{m=2}^{N} (\mu_m)^{-\frac{1}{\alpha}} \tag{37}
\]

taking in the limit for \(\alpha \to 0^+\) the value \(K_\alpha^{(\alpha \to 0^+)} = (N-1)K^{(\alpha \to 0^+)} = \frac{(N-1)^2}{N}\), coinciding with the limiting value obtained for long-range Lévy type walks when assuming regular networks [6].

Further, let us revisit the inverse of the escape probability in the infinite network limit being defined as the diagonal element \(r_{pp}^{(\alpha)}(\xi = 1) = \lim_{N \to \infty} \frac{1}{N} K_\alpha^{(\alpha)}\) and consider the limit \(\alpha \to 0^+\)

\[
\lim_{\alpha \to 0^+} r_{pp}^{(\alpha)}(\xi = 1) = \lim_{N \to \infty} \lim_{\alpha \to 0^+} \frac{1}{N} \sum_{m=2}^{N} (\mu_m)^{\frac{1}{\alpha}} \frac{1}{N} \sum_{m=2}^{N} (\mu_m)^{-\frac{1}{\alpha}} = \lim_{N \to \infty} \left(\frac{N-1}{N}\right)^2 = 1, \tag{38}
\]

i.e. for infinite networks \(\lim_{\alpha \to 0^+} (r_{pp}^{(\alpha \to 0^+)}(\xi = 1))^{-1} = 1\), i.e. the walker is sure to escape. This property of extreme transience at \(\alpha = 0^+\) of the FRW is independent of the spectral
details of the Laplacian matrix and therefore universal. It remains true for an FRW in an infinite \( d \)-dimensional lattice for any lattice dimension \( d \). For any infinite network, the limit \( \alpha \to 0^+ \) represents a limit of extreme transience.

This picture is consistent with the above-mentioned interpretation of the components of the Green’s function \( r_{pq}^{(\alpha)}(\xi = 1) \) (see below equation (14)), namely as the mean residence times (MRTs) of the walker in a node: in the limit of extreme transience \( \alpha \to 0^+ \) with \( r_{pp}^{(\alpha \to 0^+)} = 1 \) indicates that the walker on average is present at the departure node only once, namely at the time of its departure \( t = 0 \), and never returning to the departure node. We will return to the important property of extreme transience in the subsequent section when analyzing recurrence of FRWs on \( d \)-dimensional infinite lattices.

3. FRWs on simple \( d \)-dimensional cubic lattices

We now consider periodic \( d \)-dimensional periodic lattices (\( d \)-tori) where \( d = 1, 2, 3, 4, ... \) can take any integer dimension. We assume that the lattice points represent the nodes of the network and denote them by \( \vec{p} = (p_1, ..., p_d) \). In each dimension \( j = 1, ..., d \) we denote the nodes by \( p_j = 0, ..., N_j - 1 \) where the total number of nodes of the network is \( N = \prod_{j=1}^d N_j \). We assume that the lattice is \( N_j \)-periodic in each spatial dimension \( j = 1, ..., d \). All quantities defined on the nodes such as occupation and first passage probabilities fulfill the periodicity conditions \( Q(\vec{p}, ..., \vec{p}_j, ..., \vec{p}_n) = Q(\vec{p}, ..., \vec{p}_j + N_j, ..., \vec{p}_n) \).

It follows that the matrices defined above all have the same canonic basis of eigenvectors \( |\Psi_{\vec{\ell}}\rangle = \prod_{\ell_{j=1}}^d |\Psi_{\ell_j}\rangle(\vec{\ell} = (\ell_1, ..., \ell_d), \ell_j = 0, ..., N_j - 1) \) represented by the Bloch eigenvectors with the notation \( \langle \vec{p} |\Psi_{\vec{\ell}}\rangle = \prod_{j=1}^d e^{i \kappa_{\ell_j} p_j} \sqrt{N_j} = \frac{e^{i \vec{p} \cdot \vec{\kappa}_{\ell}}}{\sqrt{N}} \). In order to fulfill \( N_j \)-periodicity, it follows that the components of wave vectors \( \vec{\kappa}_{\ell} = (\kappa_{\ell_j}) \) of the Bloch eigenvectors can only take the values \( \kappa_{\ell_j} = \frac{2 \pi}{N_j} \ell_j, \ell_j = 0, ..., N_j - 1 \) \( (j = 1, ..., d) \).

We plotted \( d \)-dimensional periodic lattices for dimensions \( d = 1, 2 \) in figure 1. Topologically such a lattice can be conceived as a \( d \)-dimensional hypertorus (’\( d \)-torus’), for instance in 1D this is a cyclic ring, in 2D a conventional torus, and so forth.

The spectral decomposition \( B_{pq} = b_1 \langle p |\Psi_1\rangle \langle \Psi_1 |q\rangle + \sum_{m=2}^{N} b_m \langle p |\Psi_m\rangle \langle \Psi_m |q\rangle \) of a matrix \( B \) on the lattice has the representation

![Figure 1. Finite lattices with periodic boundary conditions. (a) 1D lattice with length \( N_1 = 20 \), the resulting structure is a ring. (b) 2D lattice with dimensions \( N_1 = N_2 = 20 \), in this case the nodes define a torus obtained from the Cartesian product of two rings with dimensions \( N_1, N_2 \).](image)
where we use alternatively the notations \( B_\bar{\gamma} = B(p_1 - q_1, ..., p_d - q_d) \) = \( \sum_{\bar{\ell}} b_{\bar{\ell}} e^{i(\bar{\ell} - \bar{\bar{\ell}}) \cdot \bar{\kappa}} = \sum_{\bar{\ell}} \prod_{j=1}^{d} \frac{\mu_j}{N_j} \) (39)

\( B(p_1 - q_1, ..., p_d - q_d) \) = \( \sum_{\bar{\ell}} b_{\bar{\ell}} e^{i(\bar{\ell} - \bar{\bar{\ell}}) \cdot \bar{\kappa}} \)

\( \delta_b \) indicates the fractional degree. We notice further that the diagonal elements of the fractional transition matrix are vanishing due to \( \prod_{j=1}^{d} N_j \rightarrow \infty \), which we write compactly as:

\[
B(\bar{\rho}) = \frac{1}{(2\pi)^d} \int b(\bar{\kappa}) e^{i\bar{\rho} \cdot \bar{\kappa}} d\kappa =: \frac{1}{(2\pi)^d} \int d\kappa_1 ... d\kappa_d b(\kappa_1, ..., \kappa_d) e^{i\bar{\rho} \cdot \bar{\kappa}}.
\]

Especially we identify in the context of lattices the general representation of the components of the unity matrix \( \delta_{p\bar{\gamma}} \rightarrow \delta_{\bar{\gamma}} = \prod_{j=1}^{d} \delta_{p_jq_j} \).

Let us now introduce the fractional Laplacian matrix on the \( d \)-dimensional lattice, which we generate from an \( N \times N \)-matrix \( L \) defined on the lattice with next-neighbor connections [32]

\[
L(\bar{\rho} - \bar{\gamma}) = L(p_1 - p_0, q_1 - q_0, ..., p_d - q_d) = 2d \sum_{j=1}^{d} \prod_{j=1}^{d} \delta_{p_j q_j} - \sum_{j=1}^{d} \left( \delta_{p_{j+1} q_j} + \delta_{p_{j-1} q_j} \right) \prod_{s \neq j}^{d} \delta_{p_j q_s}.
\]

(41)

where the constant degree of the \( d \)-dimensional lattice is the number of next-neighbor nodes \( K = 2d \). The spectral representation of the fractional Laplacian on the finite lattice \( L^\frac{\alpha}{2} \) is

\[
[L^\frac{\alpha}{2}]_{\bar{p}\bar{q}} = [L^\frac{\alpha}{2}]_{(\bar{p} - \bar{\gamma})} = \frac{1}{N} \sum_{\bar{\ell}} e^{i\bar{\ell} \cdot (\bar{p} - \bar{\gamma})} \mu_{\bar{\ell}}, \mu(\bar{\kappa}) = 2d - 2 \sum_{j=1}^{d} \cos(\kappa_j), 0 < \alpha \leq 2
\]

(42)

where throughout this analysis we confine ourselves to the admissible range \( 0 < \alpha \leq 2 \) where \( \alpha = 2 \) recovers (41). We identify the vanishing eigenvalue \( \mu_1 = \mu_{\bar{\gamma}} = 0 \) of the stationary eigenvector \( \langle \bar{\gamma} | \Psi_{\bar{\gamma}} \rangle = \frac{1}{\sqrt{N}} \) corresponding to the zero (Bloch-wave) vector \( \bar{\kappa}_\ell = (0, ..., 0) \).

The fractional transition matrix on the lattice which we define as in (10) for the finite lattice is then written as

\[
\mathcal{W}^{(\alpha)}(\bar{p} - \bar{\gamma}) = \sum_{\bar{\ell}} \lambda^{(\alpha)}_{\bar{\ell}} e^{i(\bar{\ell} - \bar{\gamma}) \cdot \bar{\kappa}_{\bar{\ell}}} = 1 - \frac{\mu^2}{K^{(\alpha)}}
\]

(43)

where \( K^{(\alpha)} = \frac{1}{N} \text{tr}(L^\frac{\alpha}{2}) = \frac{1}{N} \sum_{\bar{\ell}} \mu^2_{\bar{\ell}} \) indicates the fractional degree. We notice further that the diagonal elements of the fractional transition matrix are vanishing due to \( \sum_{\ell} \lambda^{(\alpha)}_{\ell} = 0 \) forcing the fractional random walker to change node at each step.

We analyze now infinite lattices \( N_j \rightarrow \infty \) \((\forall j = 1, ..., d)\) to consider the question of recurrence of the FRW where we employ the limiting formula (40) for the spectral representations. Let us first analyze the probability \( F^{(\alpha)}(p, q, \xi = 1) \) of ever passage which is determined by (see equation (24))

\( 16 \kappa_{\ell} = \frac{2\pi}{L} j \rightarrow \xi_j \) thus the eigenvalues become continuous functions \( b(\kappa_1, ..., \kappa_d) = b(\bar{\gamma}) \) where

\( 0 \leq \kappa_j \leq 2\pi \) and \( \frac{1}{\kappa} = \frac{4\pi}{2\pi} \)

\( (N_j \rightarrow \infty) \rightarrow \bar{p} - \bar{\gamma}. \)
In order to analyze recurrence, it is sufficient to consider the diagonal element of (44) which indicates the probability that the walker ever returns to the departure node

$$F_\beta^0(\xi = 1) = r_\beta^0(\xi = 1) - \delta_{\beta,q} \frac{\alpha}{r_0^\alpha(\xi = 1)}.$$  \hspace{1cm} (44)

It follows that $1 - F_\beta^0(\xi = 1) = \frac{1}{r_\beta^0(\xi = 1)}$ indicates the (escape) probability that the walker never returns to its departure node. In (44) and (45) we may use general relation (25) which is written as, for the infinite $d$-dimensional lattice,

$$r_\beta^0(\xi = 1) = K_\beta^0 \left( 2\pi \right)^d \int e^{i(\beta-q) \cdot \kappa} \mu^{-\frac{\alpha}{2}}(\kappa) d^d \kappa$$  \hspace{1cm} (46)

which contains the fractional degree

$$K_\beta^0 = \frac{1}{(2\pi)^d} \int \mu^\frac{\alpha}{2}(\kappa) d^d \kappa, \mu(\kappa) = 2d - 2 \sum_{j=1}^d \cos(\kappa_j).$$  \hspace{1cm} (47)

and where the eigenvalues of the fractional Laplacian $\mu^\frac{\alpha}{2}(\kappa)$ become a continuous function defined on the $d$-dimensional cube (Brillouin zone) of volume $(2\pi)^d$. We refer (46) to as lattice Green’s function of the FRW and return to the interpretation of its entries as MRT [13] indicating the average number of visits of a node $\beta$ for an infinite time of observation when the walker starts at a node $q$.

We now analyze recurrence (transience) of the FRW: if a node in the average infinitely is visited often, i.e. if Green’s function (46) diverges, then the walk is recurrent. If a node in the average is visited only a finite number of times, i.e. when (46) converges, then the walk is transient. To analyze this behavior, it is sufficient to consider the existence of identical diagonal elements of (46)

$$r_\beta^0(\xi = 1) = \frac{1}{(2\pi)^d} \int \mu^\frac{\alpha}{2}(\kappa) d^d \kappa \int \mu^{-\frac{\alpha}{2}}(\kappa') d^d \kappa'.$$  \hspace{1cm} (48)

We notice that (48) is related with the Kemeny constant (37) (global MFPT) by $r_\beta^0(\xi = 1) = \lim_{N \to \infty} \frac{K_\beta^0}{N}$ and hence the probability of never returning for the walker performing the FRW $(r_\beta^0(\xi = 1))^{-1} \sim \frac{N}{K_\beta^0}$ is the infinite network limit of the inverse of the global MFPT.

In the above considerations we have seen that $\frac{1}{r_\beta^0(\xi = 1)}$ indicates the escape probability, i.e. probability of the walker never returning to the departure node. As mentioned, the FRW is hence recurrent only if $r_\beta^0(\xi = 1) \to \infty$ is divergent and transient otherwise. Since the integral (47) for the fractional degree in the admissible $\alpha$ range always exists, the question of recurrence depends uniquely on the divergence (or convergence) of the second integral in (48) depending crucially on the features of the $\mu^{-\frac{\alpha}{2}}$ for small $|\kappa| \to 0$ around the origin.
Taking into account that the eigenvalues (42) of the fractional Laplacian matrix around the origin are behaving as \( \mu^\frac{\alpha}{2}(\vec{r}) \sim \kappa^\alpha (\kappa = |\vec{r}|) = \sqrt{\sum_{j=1}^{d} \kappa_j^2} \), then \( r_0^{(\alpha)} (\xi = 1) \) of (48) and the matrix elements (46) are finite if\(^18\)

\[
\frac{r_0^{(\alpha)} (\xi = 1)}{K^{(\alpha)}} = \frac{1}{(2\pi)^d} \int \mu^{-\frac{\alpha}{2}}(\vec{r}')d\kappa' \nonumber
\]

\[
= \frac{1}{(2\pi)^d} \left\{ \frac{2\pi^d}{\Gamma(d)} \lim_{\epsilon \to 0} \int_{\epsilon}^{\kappa_0} \kappa^{d-1-\alpha}d\kappa + \int_{\kappa_0}^{\rho_0} \mu^{-\frac{\alpha}{2}}(\kappa')d\kappa' \right\} \sim \lim_{\epsilon \to 0} a(\epsilon) + C(\kappa_0) \tag{49}
\]

exists. In (49), \( 0 < \kappa_0 << 1 \) is sufficiently small that \( (\mu(\kappa_0))^{-\frac{\alpha}{2}} \approx \kappa_0^{-\alpha} \) and \( C(\kappa_0) \) is the contribution of the integral of \( \mu^{-\frac{\alpha}{2}}(\kappa') \) over \( V_c \), which is the cube \( -\pi < \kappa_j < \pi \) without the \( d \)-sphere of radius \( \kappa = \rho_0 \).

The first integral in (49)\(^2\) is crucial for the divergence or convergence of \( r_0^{(\alpha)} (1) \): It behaves as \( a(\epsilon) \sim -\frac{\alpha}{d-1} \) for \( d \neq \alpha \) and \( a(\epsilon) \sim -\log(\epsilon) \) when \( d = \alpha \) where \( \epsilon \to 0+ \). Hence (49) diverges for \( d \leq \alpha \) and as a consequence the FRW then is recurrent. On the other hand, integral (49) is finite for \( d > \alpha \) and as a consequence the FRW is then transient where the walker has a finite escape probability (probability of never returning to the departure node) \( \frac{1}{r_0^{(\alpha)}(\xi=1)} \).

The generalized recurrence theorem for FRWs can hence be formulated as follows. The FRW is recurrent for lattice dimensions \( d \leq \alpha \) and transient for \( d > \alpha \) where always \( 0 < \alpha \leq 2 \). We emphasize that the recurrence behavior of the FRW is a universal feature, that is, it does not depend on the spectral details of the generating Laplacian \( L \). This behavior is represented in figure 2: lattice dimensions of transient FRWs are indicated by bullet points. The recurrence theorem remains true for the entire class of random walks on infinite networks with the same power law asymptotics as the FRW leading to the emergence of Lévy flights.

The recurrence theorem holds also for Lévy flights in the continuous \( d \)-dimensional infinite space [16] (see chapter 7, pages 261, 262) and also see the analysis in [7, 8] where symmetric stable processes are considered. For Lévy flights on lattices, transience has been demonstrated for lattice dimensions \( d \geq 2 \) (\( 0 < \alpha < 2 \)) in [15], and we refer also to the analysis in [11] where several recurrence features of random walks with power-law asymptotics where derived (relation (34) in that paper).

Generally, for Lévy flights it is sufficient to consider the infinite space Green’s function which yields a Riesz potential \( \lim_{\epsilon \to 0+} \frac{1}{(2\pi)^d} \int e^{i\vec{r}\cdot\vec{k}}|\vec{k}|^{-\alpha}e^{-\epsilon|\vec{k}|}d\vec{k} = \frac{C_{d-\alpha}}{r_0^{d-\alpha}} \) for \( d > \alpha \) (see equation (B.4)) and is divergent for \( d \leq \alpha \).

As \( 0 < \alpha \leq 2 \) only the following cases exist:

1. \( 0 < \alpha < 1 \): \( d - \alpha > 0 \) \( \forall d \) where \( (r_0^{(\alpha)} (\xi \to 1))^{-1} > 0 \) is a nonzero escape probability. In this range the FRW is transient for all lattice dimensions \( d \). We therefore refer the interval \( 0 < \alpha < 1 \) to as a ‘strongly transient regime’ (see figure 2). The transience becomes more and more pronounced with smaller \( \alpha \). This includes the above-discussed limiting case of extreme transience \( \alpha \to 0+ \) (38) with \( r_0^{(\alpha \to 0+)} (1) = 1 \) where the departure node in the average is visited only once, namely at \( t = 0 \).

\(^{18}\) The appearance of the additional factor \( \kappa^{d-1} \) in the integrand is due to scaling of the volume element \( d^d\kappa = \kappa^{d-1}d\kappa d\Omega_d \) and \( \int_{\kappa=1}^{\infty} d\kappa d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \) is the surface of the \( d \)-dimensional unit ball.
\(1 \leq \alpha < 2\): \(d - \alpha > 0\) \(r_0^{(\alpha)}(\xi \to 1))^{-1} > 0\), i.e. we have nonzero escape probability with transience for \(d = 2, 3, \ldots\), recurrence for \(d = 1\) (figure 2).

\((\text{iii})\) \(\alpha = 2\) (Polya walk): (49) diverges for dimensions \(d = 1, 2\), whereas it converges for dimensions \(d = 3, 4, \ldots\). This recovers Polya’s classical recurrence theorem [20]: the Polya walk is recurrent for dimensions \(d = 1, 2\) and transient for dimensions \(d > 2\) (figure 2).

Statements (i)–(iii) generalize Polya’s recurrence theorem to FRWs. In appendix B we give a brief demonstration of the emergence of Lévy flights for FRWs within \(0 < \alpha < 2\) (and Brownian motion for the Polya case \(\alpha = 2\)) on infinite lattices due to the power law asymptotics of the eigenvalues. The same asymptotic behavior is also responsible for the convergence or divergence of (49) determining transience or recurrence of the FRW. The recurrence theorem for the FRW therefore remains true for the whole class of random walks with asymptotic emergence of Lévy flights. These are walks generated by Laplacian matrices where the eigenvalues behave asymptotically as a power law \(\sim \kappa^\alpha\) when \(\kappa \to 0\) with asymptotic behavior of the transition matrix elements as for \(|\vec{p} - \vec{q}| \gg 1\) as the kernel of the fractional Laplacian operator \(\sim |\vec{p} - \vec{q}|^{-(d+\alpha)}\), leading in the transient regime \(d > \alpha\) to the ever passage probabilities (and lattice Green’s functions) decaying as Riesz potentials \(\sim |\vec{p} - \vec{q}|^{-(d-\alpha)}\) (see appendix B and [18]).

A physical interpretation is as follows. (i) In the interval \(0 < \alpha < 1\) the smaller \(\alpha\), the ‘faster’ the FRW: Due to the slower decay of the transition matrix elements as \(|\vec{p} - \vec{q}|^{-(d+\alpha)}\), long-range jumps are more frequent leading to strong transience in case (i). With increasing \(\alpha\) in case (ii), i.e. for \(1 \leq \alpha < 2\), the FRW becomes slower than in case (i) (the stronger decay of the transition matrix elements makes long-range jumps less likely to take place). This relative
slowness of the FRW can only be compensated when $d$ increases (thus transience only for dimensions $d = 2, 3, \ldots$ in case (iii)). This tendency is even more pronounced in case (iii) of the Polya walk where no long-range jumps occur, thus transience occurs only for dimensions $d = 3, 4, \ldots$. The slower the FRW (the larger $\alpha$), the higher the minimum dimension $d$ must be that the walk becomes transient, reflecting the effect that higher dimensions offer to the random walker more escape paths.

In conclusion the transience of an FRW at $d - \alpha > 0$ is the more pronounced by higher spatial dimensions $d$ (more escape paths) and lower exponents $\alpha$ increasing the speed of the FRW, that is the escape probability (in the infinite lattice limit $N \to \infty$) $(r_{q}(1))^{-1} \sim \kappa^{-\alpha}$ increases when the (renormalized) global MFPT (Kemeny constant) decreases.

### 4. Transient regime $0 < \alpha < 1$ for the infinite ring

We saw above that, in case (i), i.e. for $0 < \alpha < 1$ the FRW is transient for all dimensions $d$ of the lattice. Let us now analyze for the transient regime the limiting case $N \to \infty$ of a cyclic ring ($d = 1$) which allows explicit evaluations for the fractional lattice Green’s function. The elements of the fractional Laplacian matrix have been obtained in closed form [33, 34, 36]. For a cyclic ring, the Laplacian matrix (41) takes the simple representation

$$L_{pq} = 2\delta_{pq} - \delta_{p,q+1} - \delta_{p,q-1}$$

(50)
of a symmetric second-difference operator where the eigenvalues of the Laplacian for the infinite ring are

$$\mu(\kappa) = 2(1 - \cos(\kappa)) = 4 \sin^{2} \left( \frac{\kappa}{2} \right), \quad -\pi \leq \kappa \leq \pi.$$  

(51)
The fractional Laplacian matrix $L_{pq}^\alpha$ for the infinite ring has the spectral representation where we account for $(L_{pq}^\alpha)_{p,q} = (L_{pq}^\alpha)_{|p|=|q|}$

$$(L_{pq}^\alpha)_{|p|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\kappa p} \left( 4 \sin^{2} \frac{\kappa}{2} \right)^{\frac{\alpha}{2}} d\kappa.$$  

(52)
The matrix elements of (52) have been obtained in the following representation [33, 34] where the Töplitz structure of this matrix allows $|p - q| \to |p| = p$

$$(L_{pq}^\alpha)_{|p|} = \frac{2^{\alpha}}{\sqrt{\pi(p - \frac{1}{2})!}} \int_{0}^{1} \xi^\frac{\alpha}{2} \frac{d\xi}{d\xi^{p}} (\xi(1 - \xi))^{p - \frac{1}{2}} d\xi$$

$$= (-1)^{p} \frac{\alpha^{2}}{2^{\alpha} (\frac{\alpha}{2} + p)! (\frac{\alpha}{2} - p)!} = \frac{\alpha!}{\pi} \sin \left( \frac{\alpha \pi}{2} \right) \left( \frac{1 + \frac{\alpha}{2} - \frac{1}{2}}{\frac{\alpha}{2} + |p|} \right) !,$$  

(53)
where we utilized the notation with generalized binomial coefficients [33] and generalized factorials holding for (non-negative) integers and non-integers $\zeta! = \Gamma(\zeta + 1)$ where $\Gamma(\zeta)$ denotes the $\Gamma$-function. Detailed derivations and discussions of the properties of the
explicit form of 1D fractional Laplacian are performed in [33]. In view of expression (53), let us consider the sign of the matrix elements (53) first for \( p \neq 0 \) the sign is determined by 
\[
sgn[(-1)^p \frac{1}{2(p-q)}] = sgn[(-1)^p \prod_{s=0}^{p-1} (\frac{\alpha}{2} - s)] = (-1) \text{ whereas } +1 \text{ for } p = 0.\]
Thus, we confirm what we mentioned above that the diagonal element \( (p = 0) \) of the fractional Laplacian (the fractional degree) is positive whereas the off-diagonal elements are all negative within the range \( 0 < \alpha < 2 \) thus the fractional Laplacian matrix constitutes a good generating matrix for a random walk.

As outlined in the previous section, the integral of the fundamental matrix \( r^{(\alpha)}(\xi = 1) \) converges in the transient regime \( 0 < \alpha < 1 \) for all lattice dimensions \( d \): we now evaluate (46) in the transient regime for \( d = 1 \) in explicit form. \( L^{-\frac{\alpha}{2}} \) is obtained by formally replacing \( \alpha \rightarrow -\alpha \) in (53), however, we verify this necessary property by a brief explicit calculation.

For the 1D infinite ring this integral is determined by the inverse fractional Laplacian matrix (fractional lattice Green’s function) \( r^{(\alpha)}(\xi = 1) = K^{(\alpha)}L^{-\frac{\alpha}{2}} \) defined in (46). In contrast to finite networks, the fractional Laplacian matrix \( L^{-\frac{\alpha}{2}} \) becomes invertible in the transient regime in the limit of infinite lattices \( N \rightarrow \infty \) since \( 1 - |\Psi_1| \langle \Psi_1 | \rightarrow 1 \) due to suppression of the stationary distribution \( |\Psi_1| \langle \Psi_1 | = \frac{1}{N} \rightarrow 0 \). For the 1D infinite ring \( r^{(\alpha)}_{pq} (\xi = 1) = r^{(\alpha)}_{|p|} (\xi = 1) \) exists and is obtained as (where we denote always \( r_{pq} = r_{|p|,|q|} \) and \(|p-q| \rightarrow p\))
\[
r^{(\alpha)}_{|p|} (\xi = 1) = K^{(\alpha)} L^{-\frac{\alpha}{2}} = \frac{2}{\pi} \int_{-\pi}^{\pi} e^{ips} \left( 4 \sin^2 \left( \frac{\kappa}{2} \right) \right)^{-\frac{\alpha}{2}} d\kappa
= K^{(\alpha)} \frac{2^{-\alpha}}{\sqrt{\pi(p-\frac{1}{2})!}} \int_{0}^{1} \xi^{-\frac{\alpha}{2}} \frac{d\rho}{d\xi^p} (\xi(1-\xi))^{p-\frac{1}{2}} d\xi, \quad 0 < \alpha < 1
\]
where we emphasize again that this integral converges only in the transient regime \( 0 < \alpha < 1 \). In (54) \( K^{(\alpha)} \) indicates the fractional degree being the diagonal element of (53) which yields
\[
K^{(\alpha)} = (L^{-\frac{\alpha}{2}})_{0} = \frac{2^{\alpha} \left( \frac{\alpha+1}{2} \right)! \left( \frac{1}{2} \right)!}{\frac{1}{2}!} = \frac{\alpha!}{\left( \frac{\alpha}{2} \right)!}, \quad 0 < \alpha < 1
\]

The matrix elements (54) can be evaluated in the same way as (53): upon \( p = |p| \) partial integrations (54) yields
\[
r^{(\alpha)}_{|p|} (\xi = 1) = K^{(\alpha)} \frac{2^{-\alpha}}{\sqrt{\pi(p-\frac{1}{2})!}} \left( (-1)^p \prod_{s=0}^{p-1} \left( \frac{\alpha}{2} - s \right) \right) \int_{0}^{1} \xi^{-\frac{\alpha+1}{2}} (1-\xi)^{p-\frac{1}{2}} d\xi
= K^{(\alpha)} \frac{2^{-\alpha} \left( \frac{\alpha-1}{2} \right)!}{\sqrt{\pi(-\frac{\alpha}{2} + p)!}} \left( -1 \right)^p \frac{\left( \frac{-\alpha}{2} \right)!}{\left( \frac{-\alpha}{2} + p \right)!}, \quad 0 < \alpha < 1
\]
where \( (-1)^p \prod_{s=0}^{p-1} (-\frac{\alpha}{2} - s) = \prod_{s=0}^{p-1} (\frac{\alpha}{2} + s) = (-1)^p \frac{\left( \frac{-\alpha}{2} \right)!}{\left( \frac{-\alpha}{2} + p \right)!} > 0 \) and hence (56) is uniquely positive. The first relation (56) is written for \( p \neq 0 \). For \( p = 0 \), the product \( (-1)^p \prod_{s=0}^{p-1} (...) \) has to be replaced by 1 whereas the second equation (56) holds for all components \( |p| = 1, 2, ... \) including \( p = 0 \) where in all expressions we write \( p = |p| \). Using the identity \( 2^{-\alpha} \frac{\left( \frac{-\alpha+1}{2} \right)!}{\sqrt{\pi\left( \frac{-\alpha}{2} \right)!}} = \frac{(-\alpha)!}{\sqrt{\pi\left( \frac{-\alpha}{2} \right)!}} \) finally yields for (56) the handier expression
\[
2^{-\alpha} \frac{\left( \frac{-\alpha+1}{2} \right)!}{\sqrt{\pi\left( \frac{-\alpha}{2} \right)!}} = \frac{(-\alpha)!}{\sqrt{\pi\left( \frac{-\alpha}{2} \right)!}}.
\]

For details, see again [33].
\[ r^{(\alpha)}_{|p|}(\xi = 1) = K^{(\alpha)}(\mathcal{L}^{-\frac{\alpha}{2}})_{|p|} = K^{(\alpha)}(-1)^p \frac{(-\alpha)!}{(\frac{\alpha}{2} + p)!(\frac{\alpha}{2} - p)!} \]

\[ = \frac{\alpha!}{\pi^{\frac{\alpha}{2}}} \frac{(-\alpha)!}{(\frac{\alpha}{2} + p)!(\frac{\alpha}{2} - p)!} > 0, \quad 0 < \alpha < 1 \quad (57) \]

which is indeed consistent with (53) when replacing there \( \alpha \to -\alpha \). For numerical evaluations and to obtain the asymptotic behavior, the following equivalent representation of (57) is useful\(^{21}\)

\[ r^{(\alpha)}_{|p|}(\xi = 1) = K^{(\alpha)} \left( -\alpha \right) \sin \left( \frac{\pi \alpha}{2} \right) \frac{(|p| + \frac{\alpha}{2} - 1)!}{(|p| - \frac{\alpha}{2})!}, \quad 0 < \alpha < 1. \quad (58) \]

Let us consider the asymptotic behavior for \(|p| \gg 1\). Since for \( \beta \gg 1 \) we have the asymptotics \( \frac{\delta_{|p|}}{\pi^{\frac{3}{2}}} \sim \beta^{a-b} \). So (58) yields for \(|p| \gg 1\) the asymptotic behavior

\[ \frac{r^{(\alpha)}_{|p|}}{K^{(\alpha)}}(\xi = 1) \approx \frac{(-\alpha)!}{\pi} \sin \left( \frac{\pi \alpha}{2} \right) \frac{1}{|p|^{1-\alpha}}, \quad 0 < \alpha < 1 \quad (59) \]

which is uniquely positive and evanescent at infinity. Expression (59) coincides with the inverse kernel of the fractional Laplacian, the Riesz potential \((-\frac{d^2}{dp^2})^{-\frac{\alpha}{2}} \delta(p)\) (where \(\delta(\cdot)\) denotes Dirac’s \(\delta\)-function) which is in the present case of the 1D infinite space \([40]\). We notice that (57) \( r^{(\alpha)}_{|p|=q}(\xi = 1) \) is also a Töplitz matrix.

With these results we obtain the probability \( F^{(\alpha)}_{|p-q|} \) of ever passage (44) for the infinite ring in closed form. Assuming that the walker starts at node 0, the probability that the walker ever reaches a node \( p \), i.e. a node with distance \( |p| \) from the departure node, is obtained as

\[ F^{(\alpha)}_{|p|} = \frac{r^{(\alpha)}_{|p|}(\xi = 1) - \delta_{0p}}{r^{(\alpha)}_{|0|}(\xi = 1)} \quad (60) \]

which becomes with (57) an explicit expression.

Let us now analyze probabilities of ever return to the departure node and escape probabilities. Accounting for (57) for \( p = 0 \) yields\(^{22}\)

\[ r^{(\alpha)}_0(1) = K^{(\alpha)} K^{(-\alpha)} = \frac{\alpha!}{\pi^{\frac{\alpha}{2}}} \frac{(-\alpha)!}{(\frac{\alpha}{2} + 1)!(\frac{\alpha}{2} - 1)!} = \frac{\Gamma(1 + \alpha) \Gamma(1 - \alpha)}{\Gamma^2(1 + \frac{\alpha}{2}) \Gamma^2(1 - \frac{\alpha}{2})} 0 < \alpha < 1 \quad (61) \]

which are well-defined expressions within the transient regime \( 0 < \alpha < 1 \) with the escape probability (probability of never returning to the departure node) \( \frac{1}{r^{(\alpha)}_0(1)} \). From above general expression (48) we observe that for \( \alpha \to 0 \) the escape probability \( \frac{1}{r^{(\alpha)}_0(1)} \to 1 \) which is recovered by (61): This limit of extreme transience (sure escape of the walker at \( \alpha \to 0 \)) is in accordance with above-obtained general relation (38).

In figure 3 the escape probability \( r^{(\alpha)}(1)^{-1} \) of the explicit expression (61) for the transient regime \( 0 < \alpha < 1 \) for 1D \((d = 1)\) is plotted. We further notice that in the limit \( \alpha \to 1 \) the relation (61), due to \( \Gamma(1 - \alpha) \to \infty \), tends to infinity and, as a consequence, the escape probability is vanishing. This is consistent with the above recurrence theorem as \( \alpha = 1 \) constitutes

\(^{21}\) Which is obtained by applying Euler’s reflection formula in (57) and is also obtained from (53) by replacing \( \alpha \to -\alpha \).

\(^{22}\) Where we denote \( K^{(-\alpha)} = \frac{(-\alpha)!}{(\frac{\alpha}{2})!(\frac{\alpha}{2})!} \).
for \( d = 1 \) the limit of recurrence. We emphasize that expression (61) exists only in the transient interval \( 0 < \alpha < 1 \) of case (i).

Let us return to the interpretation below relation (30): figure 3 shows that the escape probability \( (r_\alpha(0))^{-1} \sim N/K^{\alpha} \) is the infinite lattice limit of the renormalized inverse Kemeny constant \( (K^{\alpha}/N)^{-1} \) being a measure how fast the walker visits a randomly selected node different from the departure node. The more transient the FRW is for small \( \alpha \), the faster in the infinite network the walk necessarily is.

In view of our above-mentioned interpretation of the Green’s function, namely that \( r_\alpha(0) \) counts the average number of time steps (the MRT) the walker is present in the departure node \( p = 0 \) (i.e. \( r_\alpha(0) - 1 \) indicates the average number of returns to the departure node), we observe in (61) and figure 3 that in the extreme transient limit \( \alpha \to 0^+ \) as \( r_{\alpha \to 0^+}(0) = 1 \) the average number of returns is vanishing corresponding to sure escape of the walker.

Now, let us discuss the ever passage probabilities (60) for \( p \neq 0 \), i.e. for nodes different as the departure node. Then (60) assumes the form

\[
F^{(\alpha)}_{|p|} = \frac{r^{(\alpha)}_{|p|}(\xi = 1)}{r^{(\alpha)}_{|0|}(\xi = 1)} = (L^{-\frac{\alpha}{2}})_{|p|}/(L^{-\frac{\alpha}{2}})_{|0|}, \quad p \neq 0
\]

which takes with (57)

\[
F^{(\alpha)}_{|p|} = (-1)^p \frac{(-\frac{\alpha}{2})!(-\frac{\alpha}{2}!)!}{(-\frac{\alpha}{2} + p)!(-\frac{\alpha}{2} - p)!} = (-1)^p \frac{\Gamma(1 - \frac{\alpha}{2}) \Gamma(1 + \frac{\alpha}{2} - p) \Gamma(1 - \frac{\alpha}{2} + p)}{\Gamma(1 - \frac{\alpha}{2}) \Gamma(1 + \frac{\alpha}{2} - p)}, \quad 0 < \alpha < 1
\]

with \( p \neq 0 \) where \( |p| \) indicates the distance of the departure node with \( 0 < \alpha < 1 \). In view of the initial representation (54) the property \( 0 < F^{(\alpha)}_{|p|} < 1 \) reflecting the probability interpretation is verified in appendix A.

In figures 4 and 5, the ever passage probabilities \( F^{(\alpha)}_{|p|} \) for the transient regime \( 0 < \alpha < 1 \) are drawn. In figure 4 we depict \( F^{(\alpha)}_{|p|} \) as a function of \( \alpha \) for different nodes \( |p| = 1, \ldots, 100 \). We observe what we see analytically for \( \alpha \to 0 \) in (58), that \( \lim_{\alpha \to 0^+} F^{(\alpha)}_{|p|} = 0 \). This holds
for all $p$ including $p = 0$ (see figure 3) where $\lim_{\alpha \to 0} F^{(\alpha)}_{|p|} = \lim_{\alpha \to 0} (1 - (r^{(\alpha)}(1))^{-1}) = 0$ as $r^{(\alpha \to 0+)}(1) = 1$ as demonstrated by above general relation (38). On the other hand, we observe in figure 4 and also in figure 3, that as we approach the limit $\alpha = 1 - 0$ of recurrence, that the ever passage probability for all nodes approaches asymptotically the value $\lim_{\alpha \to 1-0} F^{(\alpha)}_{|p|} = 1$ of sure ever passage which also follows directly from relations (58) together with (62) and (A.2). Finally, in figure 5 we plot $F^{(\alpha)}_{|p|}$ as a function of $p$ for the values of $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$. We notice that $F^{(\alpha)}_{|p|}$ for a fix $\alpha$ is monotonously decreasing with increasing $|p|$ (the result is a power-law relation approaching to zero as a Riesz potential $\sim |p|^{\alpha-1}$ for $|p| \to \infty$, see asymptotic relation (59) and appendix B). In appendix A we give a short proof that $F^{(\alpha)}_{|p+1|} < F^{(\alpha)}_{|p|}$ which also can be seen in figures 4 and 5.
5. Conclusions

In this paper we have analyzed FRWs on regular networks, especially $d$-dimensional simple cubic lattices in the framework of Markovian processes. The FRW generalizes the Polya walk by replacing the Laplacian matrix $L$ by a fractional power $L^\alpha$ with $0 < \alpha \leq 2$ allowing within $0 < \alpha < 2$ long-range moves where in sufficiently large networks the probability of occurrence of long-range steps decays as an inverse power law heavy tailed (Lévy-) distribution 

$$W^{(\alpha)}_{p \rightarrow q} \sim |p - q|^{-\alpha - d}$$

of the form of the fractional Laplacian operator kernel (Riesz fractional derivative) of the $d$-dimensional infinite space. This property is a landmark of the emergence of Lévy flights in sufficiently ‘large’ lattices \([30–32]\). (For a brief demonstration, see appendix B, relation (B.2)). The tendency in the FRW to perform long-range steps increases as $\alpha$ decreases. As a consequence the speed of the walk, and finally – in the infinite lattice – the escape probability increases the smaller $\alpha$ [measured by the renormalized inverse Kemeny constant $\mathcal{N}(K^{(\alpha)}_0)$] which includes the limiting cases of extreme transience at $\alpha = 0+$ with sure escape (Depicted in figure 3 for the infinite ring).

We established for $d$-dimensional infinite lattices a generalization of Polya’s recurrence theorem to FRWs and random walks having the same (heavy tailed) inverse power law characteristics which include Lévy flights: FRWs are transient for lattice dimensions $d > \alpha$ and recurrent for $d \leq \alpha$ where $\alpha$ is always restricted to $0 < \alpha \leq 2$. For the strongly transient regime $0 < \alpha < 1$ where the FRW is transient for all lattice dimensions we have obtained for the infinite ring ($d = 1$) closed form expressions for the fractional lattice Green’s function containing complete statistical information such as the average number of visits to nodes and the ever passage and escape probabilities. In the limiting case of extreme transience ($\alpha \rightarrow 0+$) the walker is sure to escape. In contrast in the recurrent limit $\alpha \rightarrow 1 – 0$ the escape probability approaches zero (figure 3). In the extreme transient case $\alpha \rightarrow 0+$, the ever passage probability for any fixed node $p$ approaches zero. This can be understood by overleaping of nodes (especially close to the departure node) due to frequent long-range steps. The opposite effect takes place when approaching the recurrent limit $\alpha \rightarrow 1 - 0$: the walker is sufficiently slow with fewer long-range steps thus nodes located ‘not too far away from the departure node’ become sure to be visited. As a consequence, a searched target on such nodes becomes sure to be found (figures 4 and 5). Important future applications of FRWs on networks are the development of search strategies with improved search efficiencies. For Lévy flights and Lévy walks several models tackling these issues exist, especially in the context of foraging and related problems [43–45].

The overall significance of the FRW is the interplay of discrete random motion on a well-defined set (network) and Lévy motions with the emergence of an asymptotically scale-free nonlocality $\sim |p - q|^{-\alpha - d}$ for the probabilities of large steps on sufficiently large networks. This scale-free nonlocality of the FRW is independent of the details of the generating Laplacian $L$ and therefore is a universal small-world feature depending only on the dimension $d$ of the lattice and the scaling index $\alpha$.

In view of its universal properties, the FRW deserves further investigation, especially in the context of dynamical processes in complex networks (information exchange in complex systems). Especially, there is a need for experimental studies of recurrence properties in real-world networks. In this way the applicability of our approach to real-world situations could be examined. Further, detailed analysis of recurrence features of Lévy walks on networks, i.e. when the velocity of the walker is finite [12, 13], would be desirable.

The remarkably rich dynamics behind FRWs should be further analyzed, for instance when passing from random walks to random flights where the latter take place in the limit of a
continuous distribution of nodes in continuous spaces. Especially, the first passage events of FRWs deserve further attention since they play a key role in the understanding of chemical reactions and chaotic turbulent motions, population dynamics and in a vast number of interdisciplinary problems.

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Appendix A

We analyze some necessary properties of expression (63). First we have to prove that

$$0 \leq F^{(\alpha)}_{|p|} \leq 1$$

allowing probability interpretation:

$$F^{(\alpha)}_{|p|} = (-1)^p \frac{(-\frac{\alpha}{2})!}{(-\frac{\alpha}{2} + p)!(-\frac{\alpha}{2} - p)!} = (-1)^p \frac{\Gamma^2(1 - \frac{\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2} + p)\Gamma(1 - \frac{\alpha}{2} - p)}, \quad (p \neq 0), \quad 0 < \alpha < 1. \quad (A.1)$$

Using the property $$(\zeta + p)! = \zeta! \prod_{s=1}^{p} (\zeta + s)$$ and by setting $$\zeta = -\frac{\alpha}{2} - p$$ gives

$$\frac{(-\frac{\alpha}{2})!}{(-\frac{\alpha}{2} + p)!} = \prod_{s=1}^{p} (-\frac{\alpha}{2} - p + s) = (-1)^p \prod_{s=0}^{p-1} \left(1 + \frac{\alpha}{2} + s\right)$$

and further

$$\frac{(-\frac{\alpha}{2})!}{(-\frac{\alpha}{2} + p)!} = \prod_{s=0}^{p-1} \left(1 + \frac{\alpha}{2} + s\right)$$

thus we can write for (A.1) the product representation

$$F^{(\alpha)}_{|p|} = \prod_{s=0}^{p-1} \frac{\left(1 + \frac{\alpha}{2} + s\right)}{1 - \frac{\alpha}{2} + s}, \quad (p \neq 0), \quad 0 < \alpha < 1. \quad (A.2)$$

The observation is that as $$1 - \frac{\alpha}{2} > 0$$ all factors are positive thus $$F^{(\alpha)}_{|p|} > 0$$. Now because of $$0 < \alpha < 1$$ we can put $$\frac{\alpha}{2} = \frac{1}{2} - \epsilon (\epsilon > 0)$$ thus

$$0 < F^{(\alpha)}_{|p|} = \prod_{s=0}^{p-1} \frac{(\frac{1}{2} + s - \epsilon)}{(\frac{1}{2} + s + \epsilon)} < 1, \quad (p \neq 0), \quad 0 < \alpha < 1. \quad (A.3)$$

It follows that $$0 < F^{(\alpha)}_{|p|} < 1$$ as each factor fulfills $$0 < \frac{(\frac{1}{2} + s - \epsilon)}{(\frac{1}{2} + s + \epsilon)} < 1$$ ($$\epsilon = \frac{1}{2} - \frac{\alpha}{2} > 0$$).

Especially, we observe that

$$F^{(\alpha)}_{|p+1|} = \frac{\left(1 + p - \epsilon\right)}{(\frac{1}{2} + p + \epsilon)} F^{(\alpha)}_{|p|} \quad (A.4)$$

and hence $$F^{(\alpha)}_{|p+1|} < F^{(\alpha)}_{|p|}$$ that is the ever passage probability decays monotonously when the distance $$|p|$$ from the departure node increases. We hence have proved that $$0 < F^{(\alpha)}_{|p|} < 1$$ for $$0 < \alpha < 1$$ as a necessary condition allowing (ever passage) probability interpretation. We further observe in $$F^{(\alpha=0+)}_{|p|} = 0$$ (extreme transience) and $$F^{(\alpha=1-0)}_{|p|} = 1$$ (recurrence), see figures 3 and 4.

Appendix B

Here our goal is to briefly demonstrate the asymptotic behavior for the transition matrix (43) for $$|\vec{p} - \vec{q}| \gg 1$$ and $$N_j \to \infty$$ as $$j = 1, \ldots, d$$ in the fractional interval $$0 < \alpha < 2$$ as a landmark for
the emergence of Lévy flights in sufficiently large $d$-dimensional lattices. The spectral representation of the transition matrix is

$$\mathcal{W}^{(\alpha)}(\vec{p} - \vec{q}) = \sum_{\ell} \lambda_{\ell}^{(\alpha)} \frac{e^{i(\vec{p} - \vec{q}) \cdot \vec{r}_{\ell}}}{N}, \quad \lambda_{\ell}^{(\alpha)} = 1 - \frac{\mu_{\ell}}{K^{(\alpha)}}, \quad 0 < \alpha < 2. \tag{B.1}$$

Consider now the probability that the walker makes a long-range move of $|\vec{p} - \vec{q}| \gg 1$. Using $(\mu(\vec{r}))^{\frac{1}{\alpha}} \sim |\vec{r}|^\alpha$ for $|\vec{r}| \to 0$ the principal contribution to the fractional adjacency matrix elements is written as

$$A^{(\alpha)}(\vec{p} - \vec{q}) = K^{(\alpha)} \mathcal{W}^{(\alpha)}(\vec{p} - \vec{q}) \approx -\frac{1}{(2\pi)^d} \int |\vec{r}|^\alpha e^{i(\vec{p} - \vec{q}) \cdot \vec{r}} = -(-\Delta |\vec{p} - \vec{q}|)^{\frac{d}{\alpha}} \delta^d(\vec{p} - \vec{q})$$

with the positive constant $C_{\alpha,d} = \frac{2^{\alpha-1} \alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}} \Gamma(1 - \frac{\alpha}{2})}$ [41] where the latter inverse power law kernel holds for $0 < \alpha < 2$ ($\alpha \neq 2$). The transition matrix (probability of a long-range jump distance $|\vec{p} - \vec{q}| \gg 1$) thus scales in the fractional interval $0 < \alpha < 2$ as an inverse power law $\mathcal{W}^{(\alpha)}(\vec{p} - \vec{q}) \sim |\vec{p} - \vec{q}|^{-d-\alpha}$ having the form of the kernel of the fractional Laplacian operator (Riesz fractional derivative) in the $d$-dimensional infinite space [32, 41]. Returning to the master equation (2), the time evolution of the occupation probabilities with (B.2) is for $|\vec{p} - \vec{q}| \gg 1$ asymptotically described by (with $P_{t+1}(\vec{p} - \vec{q}) - P_t(\vec{p} - \vec{q}) \approx \frac{d}{dt} P_t(\vec{p} - \vec{q})$)

$$\frac{d}{dt} P_t(\vec{p} - \vec{q}) \big|_{t=0} \approx \frac{(L^{\frac{d}{\alpha}})_{\vec{p} - \vec{q}}}{K^{(\alpha)}} \approx -\frac{1}{K^{(\alpha)}} (-\Delta |\vec{p} - \vec{q}|)^{\frac{d}{\alpha}} \delta^d(\vec{p} - \vec{q}) \tag{B.3}$$

which is the evolution equation of a (time-continuous) Lévy flight in $d$-dimensional infinite space with Lévy index $0 < \alpha < 2$ where $\delta^d(\vec{p} - \vec{q})$ denotes the $d$-dimensional Dirac $\delta$-function [32]. Asymptotic relation (B.3) recovers for $\alpha = 2$ the conventional diffusion equation indicating the Brownian nature of the Polya walk.

By a similar consideration the asymptotic representation of the fractional lattice Green’s function (46) of the $d$-dimensional infinite lattice for $|\vec{p} - \vec{q}| \gg 1$ for the transient regime, $d - \alpha > 0$ is obtained in Riesz potential form (see also [16], pp. 261)

$$\frac{1}{K^{(\alpha)}} f^{(\alpha)}(\vec{p} - \vec{q}) \approx (-\Delta |\vec{p} - \vec{q}|)^{-\frac{d}{\alpha}} \delta^d(\vec{p} - \vec{q}) = -\frac{C_{-\alpha,d}}{|\vec{p} - \vec{q}|^{d+\alpha}} > 0 \tag{B.4}$$

where this expression formally is obtained when replacing $\alpha \to -\alpha$ (and adding multiplier $(-1)$) in (B.2). It is worth noticing that the constant $-C_{-\alpha,d} = 2^{-\alpha} \pi^{-\frac{d}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2}\right)} > 0$ occurring in (B.4) is positive. It is further worth mentioning that asymptotic relation (B.4) also holds for the transient Polya walks $d > \alpha = 2$. For a Polya walk on a 3D lattice relation (B.4) takes the representation of a Newtonian potential $\frac{1}{4\pi |\vec{p} - \vec{q}|} = (-\Delta |\vec{p} - \vec{q}|)^{-1} \delta^3(\vec{p} - \vec{q})$ as a landmark of the Brownian nature of the Polya walk.

We now return to the interpretation of the components of the lattice Green’s function (46) indicating the average number of visits of nodes (the MRT). Let us briefly consider the MRT of the walker in a large sphere in the $d$-dimensional space of radius $R \gg 1$ (sufficiently large that the Lévy flight characteristics of the FRW emerge). The walker is assumed to depart in the origin of the sphere; integrating (B.4) over this sphere yields for the MRT a behavior $\sim R^\alpha$
independent of the lattice dimension \( d \) for the transient regime \( d > \alpha \) whereas the Green’s function (46) and hence the MRT diverges in the recurrent regime \( d \leq \alpha \). This observation coincides with the results obtained in [13] (equation (32) in that paper) for the MRT in the transient regime \( 0 < \alpha < 1 \) for a Lévy flyer in the 1D space \( d = 1 \) (and for \( d = 1 \leq \alpha \leq 2 \) the MRT diverges in the recurrent regime [13]).

(B.4) remains uniquely positive allowing probability interpretation of the ever passage probabilities. The probability that the walker in the transient regime \((d > \alpha, 0 < \alpha \leq 2)\) ever reaches a far distant node \(|\mathbf{p} - \mathbf{q}| \gg 1\) from the departure node is with (B.4) and (44) given by the inverse power law

\[
F_{\mathbf{p} - \mathbf{q}}^{\alpha}(\xi = 1) = \frac{r_{\mathbf{p} - \mathbf{q}}^{\alpha}(1)}{r_{\mathbf{0}}^{\alpha}(1)} \approx \frac{D_{-\alpha,d}}{|\mathbf{p} - \mathbf{q}|^{d-\alpha}}, \quad d > \alpha, \quad 0 < \alpha \leq 2 \tag{B.5}
\]

of a Riesz potential type [40] where the constant \( D_{-\alpha,d} = \frac{\zeta_\alpha(-C_{-\alpha,d})}{\zeta_\alpha(1)} > 0 \) is uniquely positive. We further mention that also the asymptotic representations (B.4) and (B.5) are universal as they do not depend on the details of the generating Laplacian matrix \( L \). For the infinite ring \( d = 1 \), (B.5) takes the Riesz potential power law asymptotics \( \sim r^{\alpha-1} \) of explicit expression (62).

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