On Solutions to the Nonlinear Phase Modification of the Schrödinger Equation

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Abstract

We present some physically interesting, in general non-stationary, one-dimensional solutions to the nonlinear phase modification of the Schrödinger equation proposed recently. The solutions include a coherent state, a phase-modified Gaussian wave packet in the potential of harmonic oscillator whose strength varies in time, a free Gaussian soliton, and a similar soliton in the potential of harmonic oscillator comoving with the soliton. The last of these solutions implies that there exist an energy level in the spectrum of harmonic oscillator which is not predicted by the linear theory. The free solitonic solution can be considered a model for a particle aspect of the wave-particle duality embodied in the quantum theory. The physical size of this particle is naturally rendered equal to its Compton wavelength in the subrelativistic framework in which the self-energy of the soliton is assumed to be equal to its rest-mass energy. The solitonic solutions exist only for the negative coupling constant for which the Gaussian wave packets must be larger than some critical finite size if their energy is to be bounded, i.e., they cannot be point-like objects.

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1 Introduction

Recently we have presented the nonlinear phase modification of the Schrödinger equation \[1\]. From the general scheme of the modification we selected the two simplest models which guarantee that the departure from the linear Schrödinger equation is minimal in some reasonable manner. One of the models turned out to have the same continuity equation as the continuity equation of the Doebner-Goldin modification \[3\] and we demonstrated that its Lagrangian leads to a particular variant of this modification. The other model though constitutes a novel proposal not investigated in the literature before. It is the purpose of this report to present some physically interesting one-particle solutions \[1\] to this proposal that we called the simplest minimal phase extension (SMPE) of the Schrödinger equation. Before doing so, let us briefly recall it.

In what follows, \(R\) and \(S\) denote the amplitude and the phase of the wave function \(\Psi = R \exp(i S)\), \(V\) stands for a potential, and \(C\) is the only constant of the modification that does not appear in linear quantum mechanics. The discussed extension, similarly as the Schrödinger equation, is invariant under the Galilean group of transformations and the space and time reflections. The Lagrangian density for the modification,

\[
- L(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m} \left[ (\vec{\nabla} R)^2 + R^2 (\vec{\nabla} S)^2 \right] + CR^2 (\Delta S)^2 + R^2 V, \tag{1}
\]

enables one to derive the energy functional,

\[
E = \int d^3x \left\{ \frac{\hbar^2}{2m} \left[ (\vec{\nabla} R)^2 + R^2 (\vec{\nabla} S)^2 \right] + CR^2 (\Delta S)^2 + VR^2 \right\}, \tag{2}
\]

a conserved quantity for the potentials that do not depend explicitly on time which coincides with the quantum-mechanical energy defined as the expectation value of the Hamiltonian for this modification \[1, 4\]. The equations of motion for the modification read,

\[
\hbar \frac{\partial R^2}{\partial t} + \frac{\hbar^2}{m} \vec{\nabla} \cdot (R^2 \vec{\nabla} S) - 2C \Delta \left( R^2 \Delta S \right) = 0, \tag{3}
\]

\[
\frac{\hbar^2}{m} \Delta R - 2R \hbar \frac{\partial S}{\partial t} - 2RV - \frac{\hbar^2}{m} R \left( \vec{\nabla} S \right)^2 - 2CR (\Delta S)^2 = 0. \tag{4}
\]

As argued in \[1\], the most natural way to represent the nonlinear coupling constant \(C\) is as a product \(\pm \hbar^2 L^2 / m\), where \(L\) is some characteristic length to be thought of as the size of an extended particle of mass \(m\). This leads us to a nonlinear quantum theory of the particle of mass \(m\) and finite size \(L\), but still leaves open the question of sign of \(C\). However, an alternative more traditional interpretation of \(C\) as a universal coupling constant is also possible. In this approach \(L\) is just a proxy for \(C\) deprived of any additional physical meaning of its own.

In general, the solutions to the modification do not possess the classical limit in the sense of the Ehrenfest theorem. It is so because the Ehrenfest relations for this modification contain some additional terms,

\[
m \frac{d}{dt} \langle \vec{r} \rangle = \langle \vec{p} \rangle + \frac{2Cm}{\hbar} \int d^3x \vec{r} \Delta \left( \Delta S R^2 \right), \tag{5}
\]

\(^{1}\)Multi-particle solutions are discussed in \[3\].

\(^{2}\)We follow here the convention of \[1\] that treats the phase as the angle. In a more common convention \(S\) has the dimensions of action and \(\Psi = R \exp(i S/\hbar)\).
\[
\frac{d}{dt} (\hat{p}) = -\langle \nabla V \rangle + C \int d^3x \left[ 2\nabla S \Delta (\Delta S R^2) - R^2 \nabla (\Delta S)^2 \right].
\]

(6)

However, in the one-dimensional case, as long as \( \Delta S = f(t) \), where \( f(t) \) is an arbitrary function of time, these relations reduce to the standard Ehrenfest relations.

For a system described by some Gaussian wave function an appropriate measure of its physical size seems to be the width of its probability density. We will take this measure as the definition of the physical size of the system. As we will see, \( L \) is related to the physical size of the system in some situations that we will consider. Moreover, we will show that it is possible to determine \( L \) in in the so-called subrelativistic approach discussed in connection with the solitonic solution. It is in this approach that \( L \) can be given a physical meaning only as a truly particle-dependent parameter of the theory, the particle’s attribute similarly as its mass or spin. We will find that the physical size of the particle in this framework is that of its Compton wavelength.

The stationary solutions to the linear Schrödinger equation for which

\[
S = -\frac{Et}{\hbar} + \text{const},
\]

where \( E \) is the energy of a system, are also stationary solutions to this modification. There may however exist other stationary solutions as well. The purpose of the next two sections is to present some non-stationary solutions that describe single systems in one dimension, but, as we will see, the solitonic solution reduces to a nontrivial stationary solution in the zero velocity limit.

2 Wave Packet Solutions

We will assume in this section that \( \hbar = 1 \). Let us start from the simplest solution that is also a solution to the linear Schrödinger equation. It is a coherent state for which

\[
R^2 = \frac{1}{\sqrt{\pi x_0}} \exp \left[ -\frac{(x - x_0 \sqrt{2} \cos(\omega t - \delta))^2}{x_0^2} \right]
\]

and

\[
S = -\left( \frac{\omega t}{2} - \frac{|\alpha|^2}{2} \sin 2(\omega t - \delta) + \frac{\sqrt{2}|\alpha|x}{x_0} \sin (\omega t - \delta) \right).
\]

(8)

Since \( \Delta S = 0 \), the coherent state being a solution to the linear Schrödinger equation in the potential of a simple harmonic oscillator, \( V = \frac{m\omega^2 x^2}{2} \), represents a solution to equations (3) and (4) in the very same potential. Here \( x_0 = 1/\sqrt{m\omega} \), while \( \alpha \) and \( \delta \) are arbitrary constants, complex and real, respectively. The physical size of this system is \( x_0 \), but since \( \Delta S = 0 \), no relation between the size in question and the characteristic size \( L \) introduced by the theory can be established.

The coherent state is an example of a wave packet. Another member of this class, the ordinary Gaussian wave packet is not a solution to this modification. Unlike the Gaussian packet, the coherent state does not spread in time, but requires the potential of harmonic oscillator to support it. Nevertheless, one can find another solution in this class that represents a modified Gaussian wave packet whose amplitude is the same as that of the ordinary Gaussian wave packet in the linear theory,

\[
R = R_L = \left[ \frac{mt_0}{\pi (t^2 + t_0^2)} \right]^{1/4} \exp \left[ -\frac{mt_0x^2}{2 (t^2 + t_0^2)} \right].
\]

(9)

However, its phase is different from the phase of the “linear” packet,

\[
S_L = \frac{mtx^2}{2 (t^2 + t_0^2)} - \frac{1}{2} \arctan \frac{t}{t_0}.
\]

(10)
To ensure that this difference is minimal, we assume that the phase of the modified packet has the form

$$S = S_L + \frac{1}{2} f(t) x^2 + h(t). \quad (11)$$

The parameter $t_0$ is related to the average of the square of the momentum of this system via $\langle p^2 \rangle = m/2t_0$, so $t_0$ has to be positive. This is also necessary for the normalization of the packet’s wave function. Moreover, this constant determines the minimal physical size of the system $L^2_{ph}(t = 0) = 4t_0/m$. In principle, this size could be arbitrarily small as the momentum of the packet can be arbitrarily large. We will see however that for negative values of $C$ the parameter $t_0$ must be larger than some finite value that depends on $C$.

Such a packet differs in the most minimal way from the Gaussian wave packet of linear theory, but unlike the latter it may not exist without the support of some external potential. We will now find the functions $f(t)$ and $h(t)$ and the potential $V(x, t)$ which is required to support this configuration. Denoting for simplicity $\Delta S$ by $g(t)$, we find that the first equation of the modification reduces to

$$\frac{1}{m} \nabla \left( x f(t) R^2 - 2 C m g(t) \nabla R^2 \right) = 0. \quad (12)$$

The solution of this equation is possible only if the expression in the brackets is constant, but in order for the ratio $f(t)/g(t)$ to be a function of time only this constant has to be zero. Consequently, one obtains that

$$f(t) = -\frac{4Cm^3t_0}{(t_0^2 + t^2) (t_0^2 + t^2 + 4Cm^2t_0)}. \quad (13)$$

and since $g(t) = \Delta S_L + f(t)$,

$$f(t) = -\frac{4Cm^3t_0}{(t_0^2 + t^2) (t_0^2 + t^2 + 4Cm^2t_0)}. \quad (14)$$

The other equation of the modification will determine $h(t)$ and $V(x, t)$. It boils down to

$$\frac{1}{2} f'(t) x^2 + \frac{f^2(t)x^2}{2m} + \frac{f(t)S'_L x}{m} + C g^2(t) + \dot{h}(t) + V(x, t) = 0, \quad (15)$$

where overdot s denote differentiation with respect to time and the prime denotes differentiation with respect to $x$. Its solution requires that $V(x, t) = A(t)x^2$. One finds then that

$$A(t) = \frac{2Cm^3t_0 \left[ t^4 - t_0^4(t_0 + 4Cm^2) \right]}{(t^2 + t_0^2)^2 (t^2 + t_0^2 + 4Cm^2t_0)^2} \quad (16)$$

and

$$h(t) = -C \int dt g^2(t) = -Cm^2 \int \frac{dt t^2}{(t^2 + t_0^2 + 4Cm^2t_0)^2}. \quad (17)$$

Calculating this integral gives

$$h(t) = -\frac{Cm^2}{2} \left\{ \frac{1}{\sqrt{B^2(t_0)}} \arctan \left( \frac{t}{\sqrt{B^2(t_0)}} \right) + \frac{t}{t^2 + B^2(t_0)} \right\} + \text{const,} \quad (18)$$
when \( B^2(t_0) = t_0^2 + 4Cm^2t_0 \) is non-negative, and

\[
h(t) = -\frac{Cm^2}{2} \left\{ \frac{t}{t^2 - B^2(t_0)} + \frac{1}{2a} \ln \left| \frac{t + \sqrt{B^2(t_0)}}{t - \sqrt{B^2(t_0)}} \right| \right\} + \text{const}
\]  \hspace{1cm} (19)

otherwise. The energy of this configuration is time-dependent,

\[
E = \frac{t^6 + 3t_0^2t^4 + t_0^2(t_0^2 + 2t_0(t_0 + 8Cm^2))t^2 + \left( t_0 + 4Cm^2 \right)^2 + 4Cm^2(t_0 + 4Cm^2) }{4t_0 \left( t^2 + t_0^2 \right) \left( t^2 + t_0^2 + 4Cm^2t_0 \right)^2},
\]  \hspace{1cm} (20)

and, as seen from this expression, it is asymptotically bounded by \( E_{\text{asympt}} \equiv E(|t| \to \infty) = 1/4t_0 \). Therefore, \( E_{\text{asympt}} = \langle p^2 \rangle / 2m \). The energy of the packet is asymptotically conserved, but it changes locally in time due to the time-dependent potential. Moreover, one observes that the energy of the Gaussian scales as \( 1/|C|m^2 \), which is precisely as anticipated in \([1]\) based exclusively on dimensional arguments. A particularly simple form of the formula for energy is obtained for the negative coupling constant, \( C = -|C| \), and \( t_0 = 8|C|m^2 \),

\[
E = \frac{t^6 + 3 \left( 8Cm^2 \right)^2 t^4 + \left( 8Cm^2 \right)^4 t^2}{16|C|m^2 \left( t^2 + \left( 8Cm^2 \right)^2 \right) \left( 2t^2 + \left( 8Cm^2 \right)^2 \right)^2}.
\]  \hspace{1cm} (21)

We see that in this case, \( E(t = 0) = 0 \) and \( E(t \neq 0) > 0 \).

What is the most interesting here is that the energy can become infinite for negative values of \( C \) unless \( t_0 > t_0^{cr,1} = 4|C|m^2 \). This critical value of \( t_0 \) determines the lower bound on the minimal size of the packet in question as discussed earlier. This bound cannot be attained. Consequently, the lower bound for the minimal physical size of the packet is related to the characteristic size as \( L_{\text{ph}}^{lb,1}(t = 0) = 4L \). It is through this relationship that \( L \) could be, in principle, established experimentally if the bound on the minimal physical size of the packet proved to be somehow measurable. In the subrelativistic framework to be discussed in the next section, \( L = \lambda_c / 4 \), which leads to \( L_{\text{ph}}^{lb,1}(t = 0) = \lambda_c \). Let us also note that for the energy to be non-negative, \( t_0 \geq t_0^{cr,2} = 8|C|m^2 \). Using \( t_0^{cr,2} \) would yield the higher lower bound on the minimal physical size of the packet under study. In particular, in the subrelativistic framework, \( L_{\text{ph}}^{lb,2}(t = 0) = \sqrt{2} \lambda_c \). This bound is attainable. The Gaussian wave packet under discussion does not alter the standard Ehrenfest relations.

## 3 Solitonic Solution

We will now demonstrate that the modification discussed possesses a solitonic solution. By the soliton we mean an object whose amplitude is well localized and does not spread in time unlike that of ordinary Gaussian wave packets. It should also be a solution to the nonlinear equations of motion, i.e., we exclude the case of \( \Delta S = 0 \). We will seek a solution that resembles that of the Gaussian, but is not dispersive. Therefore, as an Ansatz for the amplitude we take

\[
R(x, t) = N \exp \left[ - \frac{(x - vt)^2}{s^2} \right],
\]  \hspace{1cm} (22)
where $v$ is the speed and $s$ is the half-width of the Gaussian amplitude to be determined through the coupling constant $C$ and other fundamental constants of the modification. The normalization constant $N = (2/\pi s^2)^{1/4}$. We will seek the phase in the form

$$S(x, t) = a(x - vt)^2 + bvx + c(t),$$

where $a$ and $b$ are certain constants and $c(t)$ is a function of time, all of which need to be found from the equations of motion. Assuming that $V(x, t) = 0$ and substituting (22) and (23) into (3) and (4) reveals that the latter are satisfied provided

$$b = m/\hbar, s^2 = -8mC/\hbar^2, s^4a^2 = 1,$$

and

$$2\hbar s^4m \frac{dc(t)}{dt} + 2\hbar^2 s^2 + \hbar^2 s^4b^2v^2 + 8Ca^2s^4m = 0.$$  

We see that the coupling constant $C$ has to be negative, $C = -|C|$. From (24) we now obtain that

$$s^2 = 8m|C|/\hbar^2 = 8L^2 = 8q\lambda_c^2,$$

$$a = \pm\hbar^2/8m|C| = \pm \frac{1}{8L^2} = \pm \frac{1}{8q\lambda_c^2},$$

where $q$ is the Compton quotient equal to $L^2/\lambda_c^2$ and $\lambda_c = \hbar/mc$ is the Compton wavelength of particle of mass $m$. Combining (25-27) leads to

$$c(t) = -\frac{1}{16} \left( \frac{\hbar^3}{m^2|C|} + \frac{8mv^2}{\hbar} \right) t + const.$$  

The energy of the soliton is a function of its speed $v$,

$$E(v) = E_{st}(L) + \frac{mv^2}{2},$$

where

$$E_{st}(L) = \frac{\hbar^2}{16mL^2} = \frac{mc^2}{16q}$$

is the stationary part of it. This part can become of the order of the rest energy of the particle and even bigger for appropriately small $q$’s. Nevertheless, as long as one remains outside the realm of special relativity, the decay of particles due to energetic reasons is not an issue and it is only the difference in the kinetic energy that matters and is actually observed. This difference can be observed in the process of changing the energy of the particle by slowing it down in some detector, in particular by stopping it. In the latter case one detects that the change in the particle’s energy is $\Delta E = mv^2/2$. We also note the characteristic scaling of energy being proportional to $\hbar^2/mL^2$, in agreement with what we anticipated in [1].

It is tempting to assume that the stationary energy term represents the rest mass-energy of free particle, i.e., $E_{st}(L) = \hbar^2/16mL^2 = mc^2$. This determines the characteristic size $L$ of the particle to be a quarter of its Compton wavelength. However, as seen from (26), its physical size $L_{ph} = \sqrt{2s} = 4L$ turns out to be equal to the Compton wavelength itself. Implicit in this assumption is the fact that the
constant rest mass-energy term that one would obtain in the nonrelativistic approximation is dropped from the scheme and replaced by the self-energy term. The energy of this term gives rise to the rest mass-energy of the particle. We call this approach subrelativistic. It leads to a model type of particles whose physical size is precisely that of their Compton wavelength, but in no way can it describe particles of any other size. It seems that the most appropriate way to interpret these solitons is as the fundamental particular constituents of quantum realm complementary to waves. Quontons, as we choose to call them, would then be the unique realization of the particle aspect of the wave-particle duality of quantum mechanics.

This all seems to be too easy so one can suspect some trick here. The trick is that out of three constants $\hbar$, $m$, and $L$ the last two having dimensions of kg and meter, respectively, it is always possible to form a quantity of the dimensions of energy, $\hbar^2/mL^2$, and if this quantity is to be of the order of the rest mass-energy of the particle then $L$ should be of the order of its Compton wavelength $\lambda_c$. However, it is not necessarily as easy as this simple reasoning may suggest. First of all, this dimensional trick does not imply that the physical size of the quonton is to be precisely equal to its Compton wavelength. The fact that it is so is thus rather remarkable. Secondly, and even more importantly, if a good joke is not to be repeated too often ours is a good one indeed for it cannot be repeated neither in the Doebner-Goldin [3] nor in the Białynicki-Birula modification [5], although for two different reasons. In the former, the dimensions of its nonlinear parameters do not allow to make any new dimensional quantities beyond those that can be made up of $\hbar$ and $m$ and those two constants are not enough for our task. In the latter, the nonlinear parameter $\varepsilon$ has the dimensions of energy and the dimensional analysis of the problem implies that the characteristic size of an object of such energy is inversely proportional to the square root of it. Indeed, the soliton of this modification has the radius $\hbar/\sqrt{2m\varepsilon}$. Experimentally established [17], the upper bound on the value of this parameter is so small that it implies the existence of objects of macroscopic size and thus easy, in principle, to observe. Nevertheless, they have not been empirically confirmed.

One can however consider the self-energy independent of the rest-mass energy. The rest mass-energy would then constitute a separate part of the total energy or it could be eliminated from the considerations in a completely non-relativistic framework. None of these approaches is more advantageous than the other, both are just models. The first of them attempts to model the rest mass-energy of the quonton and thus its inertia by means of the self-interaction term, the other approach is devoid of such a goal and treats the rest mass-energy as given and inconsequential.

Let us now present the subrelativistic formulation in a more mathematical manner. As a subrelativistic Hamiltonian $H_{sub}$ of the free Schrödinger equation we define the Hamiltonian whose expectation value on its solutions is

$$< H_{sub} > = E = \frac{mv^2}{2} + mc^2. \quad (31)$$

It is through this equation that the quantum and classical world make contact. However, this equation by no means fixes the form of subrelativistic Hamiltonian. It is easy to find such a Hamiltonian in linear quantum mechanics. In fact, it is unique and it differs from the Hamiltonian of the free Schrödinger equation $H_{Sch}$ by the rest energy term. In other words, it is

$$H_{sub}^L = H_{Sch} + H_{rest}^L = H_{Sch} + mc^2. \quad (32)$$

3The requirement that the size of the quantum particle should be that of its Compton wavelength has recently been used as a postulate to build a model of nonlinear quantum mechanics of extended objects from first principles [3].
The solution to the free linear subrelativistic Schrödinger equation is, similarly as in the nonrelativistic case, a plane wave \( \Psi = \exp(iS_{\text{plane}}) \), but with a slightly different phase,

\[
S_{\text{plane}} = \frac{mvx}{\hbar} - (E - mc^2)t. \tag{33}
\]

In general, the rest energy term is of no relevance in the linear formulation of the subrelativistic Schrödinger equation for it can be absorbed in the phase of the quantum system without any further consequences. In nonlinear quantum mechanics, things can be very different. Again, we expect that the following decomposition

\[
H_{\sub}^{NL} = H_{\text{Sch}} + H_{\text{rest}}^{NL} \tag{34}
\]

will lead to the dispersion relation (31). Now, however, the choice of the rest energy Hamiltonian \( H_{\text{rest}}^{NL} \) is not unique. It seems that the most reasonable and minimal in some sense way to enhance this uniqueness is to stipulate that a free nonlinear subrelativistic Schrödinger equation has a solitonic solution which satisfies (31). As argued above, the Doebner-Goldin modification is unable to produce (31) for its nonlinear solutions due to dimensional reasons. The Białynicki-Birula and Mycielski modification can be thought of as a subrelativistic nonlinear extension of the Schrödinger equation only for a sufficiently light particle due to the experimental smallness of \( \varepsilon \). It is not out of the question that the SMPE is the only nonlinear modification of the Schrödinger equation that can be considered a nonlinear subrelativistic Schrödinger equation which has a solitonic solution fulfilling (31) and which, in addition, entails the unique value for the physical size of the soliton. Let us note that the plane wave that is the solution of the subrelativistic linear Schrödinger equation is also a solution to the modification in question. Nevertheless, it should be noted that the coupling constant of this modification is no longer universal in this approach for it is determined by other parameters of the theory to the effect that \( C_{\sub} = -\hbar^4/16m^3c^2 \). Consequently, if \( C \) is ever experimentally found to be independent of the mass of the particle, i.e., \( C \) is indeed a truly universal constant and not a product of \( \hbar^2, m, \) and \( c \), then the discussed approach is viable only for one mass and thus is much less appealing. For this reason, it is more appropriate in this case to use \( L \) rather than \( C_{\sub} \), for the former, being the characteristic size of the particle represents its attribute and therefore cannot be thought of as a universal constant.

A similar solitonic solution exists also in the following time-dependent potential of harmonic oscillator,

\[
V(x, t) = k(x - vt)^2, \tag{35}
\]

for any negative value of the coupling constant \( C \). The amplitude and phase of the soliton are assumed to be the same as before, i.e., given by (22) and (23). The parameter \( b \) is determined by (24) and the half-width of the soliton by (26), and so none of them is affected by the potential. Moreover \( c(t) \) is also determined by (25), except that now \( a \) satisfies the equation

\[
a^2 = \frac{1}{s^4} - \frac{km}{2\hbar^2} = \frac{1}{64L^4} - \frac{km}{2h^2}, \tag{36}
\]

which implies that the strength of the potential cannot be greater than \( k_{\text{crit}} = \hbar^2/32mL^4 = mc^2/32qL^2 \). Choosing the standard form of \( k \), \( k = mw^2/2 \), we obtain that for a fixed \( L, \omega \leq \omega_{\text{crit}} = \hbar/4mL^2. \) For a given \( \omega, L \leq L_{\text{max}} = \sqrt{\hbar/4m\omega}. \) The energy of this configuration is

\[
E(v, L; d) = E_{\text{st}}(L; k) + \frac{mv^2}{2}, \tag{37}
\]
where

\[ E_{st}(L; d) = \frac{\hbar^2}{16mL^2} + 2kL^2 = \frac{mc^2}{16q} + 2qk\lambda^2_c \]  

represents the stationary part of it. We note that it is only the phase and the energy of the particle that depend on the potential. The average position \( \langle x \rangle = vt \) and momentum \( \langle p \rangle = mv \) are the same for both of these solitonic solutions. In the case when \( v = 0 \), each of these solutions reduces to a stationary solution of energy \( E_{st}(L) \) and \( E_{st}(L; k) \), respectively.

For a given \( L \), the maximum stationary energy (38) equals \( E_{st}(L; k_{\text{crit}}) = \frac{\hbar^2}{8mL^2} = \frac{mc^2}{8q} \). However, as a function of \( L \), \( E_{st}(L; k) \) does not have a maximum, but a minimum. This minimum is attained for \( L = L_{\text{max}} \) and it is equal to the ground state energy of the harmonic oscillator, \( E = \hbar\omega/2 \).

Since for \( L_{\text{max}} \), \( a = 0 \) and \( \Delta S = 0 \), we see that this state corresponds to the ground state of linear theory.

For \( L < L_{\text{max}} \), there exist two nodeless wave functions of which one corresponds to the ground state of linear theory \( (\Delta S = 0) \). The state described by the other wave function has the energy,

\[ E_{st}(\omega) = \frac{\hbar\omega}{4} \left( 1 + \frac{Q_h^2}{Q_h^2} \right) = \frac{\hbar\omega_{\text{crit}}}{4} \left( 1 + \frac{\omega^2_{\text{crit}}}{\omega^2} \right) = \frac{mL^2}{16m^2L^4 + \omega^2}, \]  

where \( Q_h = (L/L_{\text{max}})^2 = 4L^2m\omega/\hbar = 4q\hbar\omega/mc^2 = \omega/\omega_{\text{crit}} \). Therefore,

\[ \Delta E_{\text{new}} = E_{st} - E_g = \frac{\hbar\omega}{4} \left( 1 + Q_h - 2 \right) = \frac{\hbar\omega_{\text{crit}}}{4} \left( 1 - \frac{\omega}{\omega_{\text{crit}}} \right)^2, \]  

represents a new line in the spectrum of harmonic oscillator not predictable by the linear theory. In terms of the separation between consecutive energy levels \( E_{\text{con}} \) in the spectrum of linear theory and the frequency ratio \( \eta = \omega/\omega_{\text{crit}}, (\eta \leq 1) \)

\[ \frac{\Delta E_{\text{new}}}{E_{\text{con}}} = \frac{1}{4} \left( 1 - \frac{\omega}{\omega_{\text{crit}}} \right) = \frac{1}{4\eta} (1 - \eta)^2. \]  

In principle, it is easy to verify the existence of the new line. One should start observing the spectrum of harmonic oscillator right below \( \omega_{\text{crit}} \). It is at this critical frequency that the new line splits off of the ground state and as we keep lowering the frequency, it moves towards the first excited state of linear theory. At \( \eta = 1/4 \) it is approximately half-way there. The critical frequency \( \omega_{\text{crit}} \) expressed in Hz is approximately

\[ \omega_{\text{crit}} = 3 \times 10^{19} \frac{m}{q_em_e}, \]  

where \( m_e \) is the mass of electron and \( q_e \) the Compton quotient of a particle with respect to the Compton wavelength of the electron. Even for the lightest stable particle, the electron, this is well above the top range of frequency of gamma rays of the order of \( 10^7 \) Hz, which seems to make impossible to carry out this type of experiment. We should note that if \( \omega \ll \omega_c \), which is a much more accessible regime, the new level has to be sought among highly excited states of harmonic oscillator as seen from (41). This may not necessarily be feasible either.

It is reasonable to expect that \( L \) is of the order of \( \lambda_c \). In the subrelativistic framework, it is \( L = \lambda_c/4 \) that should be chosen. This yields the formula

\[ E_{st} = mc^2 + \frac{m\lambda^2_c\omega^2}{8}. \]
valid for $\omega \leq \omega_{\text{crit}} = 4\hbar/m\lambda_c^2 = 4m/\hbar$. However, it seems more appropriate to impose the condition $m\lambda_c^2\omega^2 < 8$, so that the particle creation-annihilation does not occur. This defines the subrelativistic frequency regime to be $\omega < \omega_{\text{creat}} = 2\sqrt{2}m/\hbar$. What we obtained is a hard core particle regime with the physical size of the oscillator depending only on universal constants and equal to its Compton wavelength, in contrast to a “soft” core type of oscillator of linear theory whose size can be modified by changing its frequency. If this modification describes reality then we should be able to observe that one of the energy levels in the spectrum of the harmonic oscillator depends quadratically on $\omega$.

Solitonic solutions occur also in other nonlinear modifications of the Schrödinger equation, as, for instance, in the modification of Bialynicki-Birula and Mycielski [5, 7, 8, 9] and in the Doebner-Goldin type of modifications [10, 11, 12, 13]. It should be pointed out that the solitons presented in this paper exist for arbitrary values of the (negative) coupling constant which is not always the case in other nonlinear modifications where for this to happen some threshold value of nonlinear parameter(s) must be exceeded.

4 Conclusions

We have presented four non-stationary one-dimensional solutions to the simplest minimal phase extension of the Schrödinger equation introduced in [1]. The simplest of them, being also a solution to the linear Schrödinger equation, represents a coherent state which is a particular form of a wave packet. Its existence requires the potential of harmonic oscillator. Similar in nature is the second solution, the modified Gaussian packet whose amplitude is identical with the amplitude of the “linear” Gaussian wave packet, its phase being slightly different but having the same spatial shape as the phase of the ordinary Gaussian packet. This solution exists in the potential of harmonic oscillator with a time-dependent strength. The wave packet in question is dispersive, which is not the case for the coherent state and the other two solutions, the free Gaussian soliton and a similar soliton in the potential of harmonic oscillator travelling with the velocity of the soliton. These two objects are characteristic of nonlinear structures. All of these solutions have the standard Ehrenfest limit.

For the existence of the solitons it is necessary that $C < 0$. The other solutions exist for any value of the coupling constant, but it is only for the negative $C$ that the Gaussian packet seems to corroborate our hypothesis that the theory discussed describes extended particles. Indeed, if the coupling constant is negative, the minimal physical size of the packets must be larger than some finite value for otherwise they would develop infinite energy at some point. This squares quite nicely with the idea of extended, i.e., not point-like particles.

The most physically interesting of the solutions presented is the free solitonic solution. It is conceivable that this solution can serve as a particle representation of the wave-particle duality embodied in quantum mechanics. The standard quantum theory despite many successful years of development has not been able to provide an acceptable physical realization of this duality as only the wave aspect of the duality in question has been incorporated in the mathematical structure of the theory. The wave packets cannot serve as good models of particles for they spread in time, suggesting that there exist macroscopically extended quantum objects contrary to the empirical evidence in this matter. The fact that these packets are not free solutions to the SMPE can thus be viewed as a partial boon to the theory, even if the theory implies that it is possible to create similar wave packets if an appropriate time-dependent potential is applied. Other notable modifications of the Schrödinger equation also contain wave packet solutions for time-dependent potentials.
A good mathematical model of the particle should represent an object that is well localized and non-dispersive. The free soliton presented in this paper meets these requirements. What is specially attractive about it is that it is a particle solution to the modification that does not alter well verified properties of the quantum world established by pure wave mechanics such as, for instance, the atomic structure. This solution seems to be particularly relevant in the context of de Broglie-Bohm formulation of quantum mechanics. It is this formulation that puts a considerable emphasis on the particle aspect of the wave-particle duality. Whereas in the Copenhagen interpretation of this theory it is either the wave or the particle, and the particle can be viewed as the result of interference of waves, in the approach pioneered by de Broglie it is both the wave and the particle. In this picture, the waves are always associated with particles and serve as guides for them according to the original de Broglie idea of pilot waves. Needless to say that without a particle solution to the equations of motion, this picture is rather incomplete. The free particle solution of our modification can coexist with any solution of linear wave mechanics in the sense that they can be part of a bigger system described by a factorizable wave function without violating separability. This is indeed a perfect marriage of wave and particle in that they always remain separated.

Other nonlinear modifications also contain particle-like solutions that might fulfill the dream of de Broglie. In a model specifically designed for this purpose the existence of a class of possible solutions of particle-like properties is demonstrated. However, these are solutions to approximate nonlinear equations. In the Bialynicki-Birula and Mycielski modification, the width of free Gaussian soliton is \( \frac{\hbar}{\sqrt{2m\varepsilon}} \), where \( \varepsilon \) is the only nonlinear physically significant parameter of the theory. Since the current upper bound on this parameter is \( 3.3 \times 10^{-15} \text{eV} \), it implies that the size of gaussson of the electron mass is of the order of 3 mm which is a macroscopic value! Such solitons would be easy to observe, but so far they have somehow managed to escape our attention. It is thus likely that they simply do not exist. A remarkable class of new type of solitons, finite-length solitons, have been recently discovered in the Doebner-Goldin type of modifications. As observed in, “they realize the “dream of De Broglie,” in the sense that they permit to identify a quantum particle with a non-spreading wave-packet of finite length travelling with a constant velocity in the free space.” However, the length in question depends on the speed and the frequency of the soliton, and in some cases the smaller these are the bigger the length of the soliton. In particular circumstances nothing can prevent this length from becoming arbitrarily large, and so if these objects are to resemble microscopic quantum particles some additional physically justifiable assumptions are necessary. The Doebner-Goldin modification itself does not seem to provide any insight on how to handle this problem, in part because the physical meaning of its parameters is not well elucidated.

On the other hand, the width of the solitons found in this paper which is a measure of their localization is of the order of the characteristic length of the modification, the length of the extended particle-system which this theory can be thought of describing. It seems rather unlikely that one can find a soliton of reasonably small size for an arbitrary value of a nonlinear coupling constant that would be a physically sound model of quantum particle in a theory which does not involve implicitly or explicitly a parameter proportional to some characteristic length or its power. The examples presented in the preceding paragraph were intended to illustrate precisely this point.

The stationary soliton solution in the potential of harmonic oscillator implies that there exist an energy level in the spectrum of harmonic oscillator not predictable by the linear theory. The energy of this level depends on the characteristic size of the oscillator that is limited by a certain critical value

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4This type of solitonic solutions are known as compactons in the broader literature.
$L_{\text{max}}$ which corresponds to the linear theory. It is for this value that the level in question coincides with the ground state of the harmonic oscillator in linear quantum mechanics and attains its minimum. The solution that corresponds to the nonlinear theory must thus be more compact than the solution of linear theory. Indeed, unlike the discussed solution of linear quantum mechanics with a “soft” size of the oscillator that can be modified by changing its frequency, the nonlinear solution describes a hard core particle regime with the physical size of the oscillator depending only on universal constants. If this modification describes reality then we should be able to observe that one of the energy levels in the spectrum of the harmonic oscillator depends quadratically on $\omega$.

Finally, let us note that a particularly fitting approach to the SMPE as the theory of extended particles is the approach that we term subrelativistic. This approach introduces the speed of light $c$, as in the rest mass-energy of a system, but the framework of special relativity is not needed; the Galilean transformation is the symmetry of the theory. Being nonrelativistic, the Schrödinger equation provides only a limited description of physical phenomena. One can derive it from the Klein-Gordon equation in the limit in which the Compton wavelength is much smaller than de Broglie’s wavelength of quantum particle. Yet, the Klein-Gordon equation cannot be used as an equation for a generic relativistic spinless quantum system due to the problem of negative probabilities. It is tempting to extend the limits of the Schrödinger equation to the domain between the completely nonrelativistic and relativistic world, to the subrelativistic realm. Subrelativistic phenomena are not necessarily of only speculative character. They may arise due to certain peculiarities of nonrelativistic quantum mechanics that, as argued in [13], does not in all respects behave as a fully Galilean invariant theory as one would expect it in the nonrelativistic limit. The difference is empirically significant, as illustrated by the Sagnac effect [19], and is due to the fact that the “quantum” Galilei group is not identical with its classical counterpart [20] for the former bears the remnants of its relativistic origin. Therefore, the nonrelativistic quantum-mechanical description is sometimes inevitably subrelativistic as ultimately based on a broader group of symmetry than the classical Galilei group. Other effects of this kind that justify the subrelativistic approach may, in principle, be possible too. A particularly important instance of such an effect is provided by the spin-orbit coupling.

It is within the subrelativistic approach that one can uniquely determine the physical size of the free particle which turns out to be equal to its Compton wavelength, a most reasonable size for a quantum particle. However, the approach in question makes sense only if the nonlinear parameter of the theory is particle-dependent, i.e., this parameter, such as $L$, has to be the particle’s attribute in the same manner as its mass and not a universal constant.

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References

[1] W. Puszkarz, *Nonlinear Phase Modification of the Schrödinger Equation*, preprint quant-ph/9710010.

[2] W. Puszkarz, *Non-separability without Non-separability in Nonlinear Quantum Mechanics*, preprint quant-ph/9905040.

[3] H.-D. Doebner and G. A. Goldin, Phys. Lett. **162A**, 397 (1992); J. Phys. A **27**, 1771 (1994).

[4] W. Puszkarz, *Energy Ambiguity in Nonlinear Quantum Mechanics*, preprint quant-ph/9802001.

[5] I. Białynicki-Birula and J. Mycielski, Ann. Phys. (N. Y.) **100**, 62 (1976).

[6] R. R. Sastry, *Quantum Mechanics of Extended Objects*, preprint quant-ph/9903025.

[7] I. Białynicki-Birula and J. Mycielski, Phys. Scr. **20**, 539 (1979).

[8] J. Oficjalski and I. Białynicki-Birula, Acta Phys. Pol. **B9**, 759 (1978).

[9] T. A. Minelli and A. Pascolini, Lett. Nuovo Cim. **27**, 413 (1980).

[10] P. Nattermann and W. Scherer, in *Nonlinear, Deformed and Irreversible Quantum Systems*, edited by H.-D. Doebner, V. K. Dobrev, and P. Nattermann (World Scientific, Singapore, 1995); preprint quant-ph/9506033.

[11] P. Nattermann and R. Zhdanov, J. Phys. A **29**, 2869 (1996).

[12] E. C. Caparelli, S. S. Mizrahi, and V.V. Dodonov, Mod. Phys. Lett. B **12**, 519 (1998).

[13] E. C. Caparelli, S. S. Mizrahi, and V.V. Dodonov, Phys. Scr. **58**, 417 (1998).

[14] D. Bohm, Phys. Rev. **85**, 166 (1952); 188 (1952).

[15] L. de Broglie, *Nonlinear Wave Mechanics*, (Elsevier, Amsterdam, 1960).

[16] J.-P. Vigier, Phys. Lett. **A135**, 99 (1989).

[17] R. Gähler, A. G. Klein, and A. Zeilinger, Phys. Rev. A. **23**, 1611 (1981).

[18] D. Dieks and G. Nienhuis, Am. J. Phys. **58**, 650 (1990).

[19] J. Anandan, Phys. Rev. D **24**, 338 (1981).

[20] J-M. Levy-Leblond, in *Group Theory and Its Applications*, edited by E. M. Loebl (Academic, New York, 1971), Vol. 2, p. 221.