INITIAL CONDITIONS, DISCRETENESS AND NON-LINEAR STRUCTURE FORMATION IN COSMOLOGY

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Abstract In this lecture we address three different but related aspects of the initial continuous fluctuation field in standard cosmological models. Firstly we discuss the properties of the so-called Harrison-Zeldovich like spectra. This power spectrum is a fundamental feature of all current standard cosmological models. In a simple classification of all stationary stochas-tic processes into three categories, we highlight with the name “super-homogeneous” the properties of the class to which models like this, with \(P(0) = 0\), belong. In statistical physics language they are well described as glass-like. Secondly, the initial continuous density field with such small amplitude correlated Gaussian fluctuations must be discretised
in order to set up the initial particle distribution used in gravitational N-body simulations. We discuss the main issues related to the effects of discretisation, particularly concerning the effect of particle induced fluctuations on the statistical properties of the initial conditions and on the dynamical evolution of gravitational clustering.

**Keywords:** Galaxy: correlation Cosmology: Large Scale Structures

In standard theories of structure formation in cosmology the matter density field in the early Universe is described as a homogeneous and isotropic distribution, with superimposed tiny fluctuations characterized by some particular correlation properties (e.g. [1]). These fluctuations are believed to be the initial seeds from which, through a complex dynamical evolution, galaxies and galaxy structures have emerged. In particular the initial fluctuations are taken to have Gaussian statistics and a spectrum which is, on large enough scales, the so-called *Harrison-Zeldovich* (hereafter H-Z) [2, 3] or “scale-invariant” power spectrum (hereafter PS). Because fluctuations are Gaussian, the knowledge of the PS, or its Fourier conjugate, the real-space correlation function, gives a complete statistical description of the fluctuations. The H-Z type spectrum was first given a special importance in cosmology with arguments for its “naturalness” as an initial condition for fluctuations in the framework of the expanding universe cosmology, and it is in this context that the use of the term “scale-invariant” to designate it can be understood. It subsequently gained in importance with the advent of inflationary models in the eighties, and the demonstration that such models quite generically predict a spectrum of fluctuations of this type. Since the early nineties, when the COBE experiment [4] measured for the first time fluctuations in the temperature in the Cosmic Microwave Background Radiation (hereafter CMBR) at large scales, and found results consistent with the predictions of models with a H-Z spectrum at such scales, the H-Z type spectra have become a central pillar of standard models of structure formation in the Universe.

The H-Z spectrum arises in cosmology through a particular condition applied to perturbations of Friedman-Robertson-Walker (FRW) models, which describe a homogeneous Universe in expansion. This condition - commonly referred to in cosmology as “scale invariance” of the perturbations - gives rise to a spectrum (the “scale-invariant” perturbation spectrum) with \( P(k) \sim k \) at small \( k \). All current standard cosmological models of structure formation in the Universe assume a spectrum exactly like this, or close to it, as initial condition (IC) for density perturbations in the Universe. In such models there is at any time only one characteristic scale corresponding to the causal horizon, which increases with time, and below which causal physics can act to modify the
spectrum. This causal physics depends, in general, on the details of the model, i.e. on the nature of its content in matter and radiation (or other forms of energy), until a characteristic time (the time when matter and radiation have comparable densities), after which purely gravitational evolution takes over. In this lecture we firstly try to clarify the statistical properties in real space of these distributions, which have been almost completely overlooked in the literature on the subject (for further readings we refer to [5, 6]).

A related context in which an understanding of the statistical properties in real space of the H-Z PS of the mass density field is important is represented by cosmological N-body simulations (hereafter NBS), the aim of which is to calculate the formation of structures under gravity in the Universe by a direct numerical calculation (see e.g. [7]). Since the time scale of evolution in these simulations is short compared to the dynamical time of the system (i.e. a particle moves a small distance relative to the size of the box representing a large volume of the Universe) the final configuration depends strongly on the IC at all but the smallest scales. Indeed a central idea is that from the final distribution - which should be closely related to the observed one of galaxies - one should be able to “reconstruct” some important features of the IC, which can be related to other observations such as those of the CMBR.

This leads to the problem of setting-up IC in standard cosmological NBS. The problem concerns the setting-up of the initial particle distribution, as standard theories of galaxy formation predict the properties of the initial continuous density field with small amplitude correlated Gaussian fluctuations. In general, most of the procedures which for the discretisation of a continuous field gives rise to Poisson noise which dominates the dynamics at small-scales. In order to avoid this and treat the dynamics due only to large scale (smooth) fluctuations, an ad-hoc method (lattice or glassy system plus correlated displacements) has been introduced and used in cosmological simulations. We discuss the fact that such a method gives rise to a particle distribution which does not have any of the correlation properties of the theoretical continuous density field [8]. This is because discreteness effects, different from Poisson noise but nevertheless very important, determine particle fluctuations at any scale so that the original continuous fluctuations become negligible.

The third and last point which we address in this lecture concerns the following question: which kind of fluctuations is the source of the gravitational dynamics which gives rise to non-linear and power-law correlated point structures? In a particle distribution with large-scale small-amplitude correlations there are two different sources of fluctuations which can drive the gravitational dynamics. On one hand the intrinsic
small-scale fluctuations (which are inherent to any particle distribution) having amplitude of order one at the smallest scales in the system (average distance between nearest-neighbors $<\Lambda_i>$). On the other hand there are, on large scales, some given small-amplitude (i.e. $\delta \rho/\rho \approx 10^{-3}$) correlations among density fluctuations. We find that in cosmological NBS small-scale particle fluctuations play an essential role in the development of power law correlations in the range $(0.03, 3) <\Lambda_i>$. This implies that the theoretical description should focus on the discrete nature of the particle distribution. We refer the reader to [9, 10] for further readings.

1. REAL SPACE PROPERTIES OF H-Z LIKE MODELS

Discussions of real space properties of the density fluctuations encountered in cosmology are puzzlingly sparse in the literature on the subject. Peebles briefly notes ([1] - see pg.523) that a very particular characteristic of H-Z models is that “on large scales the fluctuations have to be anti-correlated to suppress the root mean square mass contrast on the scale of the Hubble length”. Indeed, as we discuss below (and see [5]) these models are characterized at large scales by a correlation function $\xi(r)$ which has a negative power-law tail: detecting it would be the real space equivalent of finding the turnover to H-Z behavior to scales at which the PS goes as $P(k) \sim k$. A basic question we try to answer is the following: What “kind” of two-point correlation function is the one corresponding to the H-Z behaviour in cosmological models? We compare it to some different statistical homogeneous and isotropic systems: (i) Poisson-like distributions, (ii) systems with a power-law correlation function found in critical phenomena [11] and (iii) distributions characterized by long-range order (e.g. lattice or glass-like) [5, 6]. Through this comparison we can classify H-Z models in the third category. We introduce the term “super-homogeneous” to refer to these kinds of distributions, as their primary characteristic is that mass fluctuations decay at large scales faster than in a completely uncorrelated (Poisson) system. For critical systems one has instead a decay of the (normalized) mass variance which is slower than Poisson. Formally the definition of this class of “super-homogeneous” distributions is given by the condition that the PS has $P(0) = 0$, or equivalently in real space that the integral of the two point correlation function over all space is zero.

In the cosmological literature the latter property of cosmological models is often noted, but its meaning (as a strong non-local condition on a stochastic process) is not appreciated, or worse misunderstood as a trivial condition applying to any correlated system. In the textbook of
Padmanabhan[12], for example, it is “proved” on pg. 171 that the integral over all space of the correlation function always vanishes for any stationary stochastic process. The error is in an implicit assumption made that the number of particles in a large volume in a single realization converges exactly to the ensemble average. This is not true because, in general, extensive quantities such as particle number have fluctuations which are increasing functions of the volume (e.g. Poissonian, for which the integral of the correlation function over all space is not zero) even in homogeneous systems. A slightly different, but common, kind of misunderstanding of the meaning of the vanishing of the integral over the correlation function is evidenced in the book by Kolb & Turner [13]. There it is affirmed (after its statement in Eq.(9.39)) to be “...just a statement of mass conservation: if galaxies are clustered on small scales, then on large scale they must be “anti-clustered” to conserve the total amount of mass (number of galaxies)”. The source of this misconception seems to be a confusion with the so-called “integral constraint” in data analysis (e.g. [14, 15]), which imposes such a condition on the \( \hat{r} \) of the correlation function in a finite sample, due to the fact that the (unknown) average number of points in such a volume is estimated by the (exactly known) number of points in the actual sample. Despite their apparent similarity, these are different conditions: the first (infinite volume) integral constraint provides non-trivial physical information about the intrinsic probabilistic nature of fluctuations, while the second is just an artifact of the boundary conditions which holds in a finite sample independently of the nature of the underlying correlations.

\section{STATISTICAL PROPERTIES OF STATISTICALLY HOMOGENEOUS AND ISOTROPIC DISTRIBUTIONS}

We start by giving some basic definitions about the correlation properties of statistically homogeneous and isotropic (SHI) density fields. Inhomogeneities in cosmology are described using the general framework of stationary stochastic processes (hereafter SSP). Let us consider in general the description of a continuous or a discrete mass distribution \( \rho(\vec{r}) \) in terms of such a process. A stochastic process is completely characterized by its “probability density functional” \( \mathcal{P}[\rho(\vec{r})] \) which gives the probability that the result of the stochastic process is the density field \( \rho(\vec{r}) \) (e.g. see Gaussian functional distributions [16]). For a discrete mass distribution the space (e.g. infinite three dimensional space) is divided into sufficiently small cells and the stochastic process consists in occupying or not any cell with a point-particle, and \( \rho(\vec{r}) \) can be written
in general as:

$$\rho(\vec{r}) = \sum_{i=1}^{\infty} \delta(\vec{r} - \vec{r}_i),$$  \hspace{1cm} (1)

where $\vec{r}_i$ is the position vector of the particle $i$ of the distribution.

In a single realization of the mass distribution the existence of a well defined average density implies that \cite{17, 5}

$$\lim_{R \to \infty} \frac{1}{\|C(R; \vec{x}_0)\|} \int_{C(R; \vec{x}_0)} d^3r \rho(\vec{r}) = \rho_0 > 0$$  \hspace{1cm} (2)

where $\|C(R, \vec{x}_0)\| \equiv 4\pi R^3/3$ is the volume of a sphere $C(R, \vec{x}_0)$ of radius $R$, centered on the arbitrary point $\vec{x}_0$ of space $^2$. When Eq.2 is valid one can then define \cite{17, 5} a characteristic homogeneity scale as the scale $\lambda_0$ given by

$$\left| \frac{1}{C(R; \vec{x}_0)} \int_{C(R; \vec{x}_0)} d^3r \rho(\vec{r}) - \rho_0 \right| < \rho_0 \ \forall R > \lambda_0, \ \forall \vec{x}_0$$  \hspace{1cm} (3)

which depends on the nature of the fluctuations of the density in spheres (see \cite{5} for more details).

### 1.2. REAL SPACE CORRELATION FUNCTION

Let us analyze in further detail the auto-correlation properties of these systems. Due to the hypothesis of statistical homogeneity and isotropy, $\langle \rho(\vec{r}_1)\rho(\vec{r}_2) \rangle$ depends only on $r_{12} = |\vec{r}_1 - \vec{r}_2|$. The reduced two-point correlation function $\xi(r)$ is defined by:

$$\langle \rho(\vec{r}_1)\rho(\vec{r}_2) \rangle \equiv \rho_0^2 \left[ 1 + \xi(r_{12}) \right]$$  \hspace{1cm} (4)

The correlation function $\xi(r)$ is one way to measure the “persistence of memory” of spatial variations in the mass density \cite{18}.

Let us consider the paradigm of a stochastic homogeneous point-mass distribution: the Poisson case. For such a particle distribution the reduced two-point correlation function Eq. (4) can be written as (see \cite{17})

$$\xi(r) = \frac{\delta(\vec{r})}{\rho_0} \ \text{(i.e. } \xi(r) = 0).$$  \hspace{1cm} (5)

The previous relation is a direct consequence of the fact that there is no correlation between different spatial points. That is, the reduced correlation functions $\xi$ has only the (diagonal) part at $r = 0$. The
latter is present in the reduced correlation functions of any statistically homogeneous discrete distribution of particles with correlations.

In general [19, 15] for a SHI distribution of point-particles the reduced correlation function can be written as

$$\tilde{\xi}(r) = \frac{\delta(\vec{r})}{\rho_0} + \xi(r)$$  \hspace{1cm} (6)

where $\xi$ is the non-diagonal part which are meaningful only for $r > 0$. In general $\xi(r)$ is a smooth function of $r$ [19, 17].

**1.3. MASS VARIANCE**

Let us now consider the amplitude of the mass fluctuations in a generic sphere of radius $R$ with respect to the average mass. First let $M(R) = \int_{C(R)} \rho(\vec{r}) d^3r$ be the mass (for a discrete distribution the number of particles) inside the sphere $C(R)$ of radius $R$ (and then volume $\|C(R)\| = \frac{4\pi}{3} R^3$). The normalised mass variance is defined as

$$\sigma^2(R) = \frac{\langle M(R)^2 \rangle - \langle M(R) \rangle^2}{\langle M(R) \rangle^2},$$  \hspace{1cm} (7)

where

$$\langle M(R) \rangle = \int_{C(R)} d^3r \langle \rho(\vec{r}) \rangle = \rho_0 \|C(R)\|,$$  \hspace{1cm} (8)

and

$$\langle M(R)^2 \rangle = \int_{C(R)} d^3r_1 \int_{C(R)} d^3r_2 \langle \rho(\vec{r}_1)\rho(\vec{r}_2) \rangle.$$  \hspace{1cm} (9)

Note that there is no condition on the location of the center of the sphere, because of the assumed translational invariance of $\mathcal{P}[\rho(\vec{r})]$.

In the discrete Poisson case, using Eq. (5), we obtain that

$$\sigma^2(R) = \frac{1}{\rho_0 \|C(R)\|} \equiv \frac{1}{\langle M(R) \rangle}.$$  \hspace{1cm} (10)

In general, for a SHI mass density field with correlations, substituting Eq.(4) in Eq.(7), we obtain

$$\sigma^2(R) = \frac{1}{\|C(R)\|^2} \int_{C(R)} d^3r_1 \int_{C(R)} d^3r_2 \tilde{\xi}(|\vec{r}_1 - \vec{r}_2|).$$  \hspace{1cm} (11)

Using Eq. (6) in the discrete case we can write

$$\sigma^2(R) = \frac{1}{\rho_0 \|C(R)\|} + \frac{1}{\|C(R)\|^2} \int_{C(R)} d^3r_1 \int_{C(R)} d^3r_2 \xi(|\vec{r}_1 - \vec{r}_2|).$$  \hspace{1cm} (12)
Note that the sign of the second term of Eq.(12) is not uniquely determined. Equations (11) and (12) make evident the relation between fluctuations in one-point properties (as in this case the number of points in a sphere centered on a random point in space) and two-point correlations.

In the discrete case, to measure $\sigma^2(R)$ one has to take into account both terms in Eq.(12), not only the second one. From Eq.(12) the variance can, in general, be written as the sum of two contributions:

$$\sigma^2(R) = \sigma^2_{\text{poi}}(R) + \Xi(R),$$

where the first term $\sigma^2_{\text{poi}}(R) \sim R^{-3}$ represents the intrinsic Poisson noise of any stochastic particle distribution, and the second term $\Xi(r)$ (which, as noted above, does not have to be of a determined sign) is the additional contribution due to correlations (i.e. $\xi(r) \neq 0$). Note that the first term is specific of any point distribution, in which fluctuations are never absent, and have large amplitude (of order one) at small scale; in a continuous distribution, instead, only the second term is present in both expressions and fluctuations, which can be arbitrarily small at all scales, are uniquely associated with correlations between different points.

To end this section note that Equation (2) implies to the requirement that

$$\lim_{R \to \infty} \sigma^2(R) = 0,$$

which is therefore a condition satisfied by any SHI distribution. Let us return to further discussion of Eqs.(11) and (12). It is very important for our discussion to note that this condition (14) which holds for any mass distribution generated by a SSP, is very different from the requirement

$$\int d^3r \xi(r) = 0$$

which is a much stronger special condition which holds for certain distributions - those to which below we will ascribe the name “super-homogeneous”.

### 1.4. POWER SPECTRUM

The PS $P(\vec{k})$ is the main statistical tool used to describe cosmological models. It is defined as

$$P(\vec{k}) = \lim_{V \to \infty} \frac{|\delta_{\rho}(\vec{k})|^2}{V} = \langle |\delta_{\rho}(\vec{k})|^2 \rangle$$

(16)
where
\[ \delta_{\rho}(\vec{k}) = \int_V d^3r e^{-i\vec{k}\cdot\vec{r}} \frac{\rho(\vec{r}) - \rho_0}{\rho_0}, \]  
limited to a volume $V$. For a spatially stationary mass distribution $\rho(\vec{r})$ it is possible to demonstrate that it can be obtained by simply taking the FT of the correlation function $\tilde{\xi}(\vec{r})$ (up to a multiplicative constant) [24]:
\[ P(\vec{k}) = \frac{1}{(2\pi)^d} \int d^d r \exp(-i\vec{k}\cdot\vec{r}) \tilde{\xi}(\vec{r}). \]  
Let us analyze the relation between the PS and the mass-variance in real space. We first rewrite Eqs. (7-9), generalizing them to the case in which we calculate the mass variance in a topologically more complex volume $V$ of size $V$. To do this one introduces the window function $W_V(\vec{r})$ defined as
\[ W_V(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \in V \\ 0 & \text{otherwise} \end{cases}. \]  
Therefore we can rewrite Eq. (8) as
\[ \langle M(V) \rangle = \int W_V(\vec{r}) \langle \rho(\vec{r}) \rangle d^3r. \]  
and Eq. (9) as
\[ \langle M^2(V) \rangle = \int \int d^3r_1 d^3r_2 W_V(\vec{r}_1) W_V(\vec{r}_2) \langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle, \]  
where the integrals are over all space. The normalised variance is then given by
\[ \sigma^2(V) = \frac{1}{V^2} \int \int d^3r_1 d^3r_2 W_V(\vec{r}_1) W_V(\vec{r}_2) \tilde{\xi}(\vec{r}_1 - \vec{r}_2). \]  
On taking the FT one obtains
\[ \sigma^2(V) = \frac{1}{(2\pi)^d} \int d^3k P(\vec{k}) |\tilde{W}_V(\vec{k})|^2 \]  
which is explicitly positive, and $\tilde{W}_V(\vec{k})$ is the FT of $W_V(\vec{r})$, normalised by the volume defined by the window function itself,
\[ \tilde{W}_V(\vec{k}) = \frac{1}{V} \int d^3r e^{-i\vec{k}\cdot\vec{r}} W_V(\vec{r}) \]  
with $V = \int_\Omega W_V(\vec{r}) d^3r$. 

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Consider now again the real sphere of radius $R$ for which the FT of the window function (normalised as defined) is

$$\tilde{W}_R(\vec{k}) = \frac{3}{(kR)^3} \left( \sin kR - kR \cos kR \right). \tag{25}$$

One then has, assuming statistical isotropy so that $P(\vec{k}) = P(k)$, an expression for the variance in real spheres which is

$$\sigma^2(R) = \frac{1}{2\pi^2} \int_0^\infty dk \frac{9}{(kR)^6} (\sin kR - kR \cos kR)^2 k^2 P(k). \tag{26}$$

One may show [5] that, for power-law spectra $P(k) \sim k^n$ (for small $k$, $n > -3$) the integral in (26) has a very different behaviour for $n < 1$ and $n \geq 1$ [5]. To summarize clearly: For a power-law $P(k) \sim k^n$ (with an appropriate cut-off around the wavenumber $k_c$) the mass variance for real spheres with radius $R \gg k_c$ is given by

1. For $n < 1$, $\sigma^2(R) \sim 1/R^{3+n}$ and the dominant contribution comes from the PS modes at $k \sim R^{-1}$.
2. For $n > 1$, $\sigma^2(R) \sim 1/R^4$ and the dominant contribution comes from the PS modes at $k_c^{-1}$.
3. For $n = 1$, we have the limiting logarithmic divergence with $\sigma^2(R) \sim (\ln R)/R^4$.

In the cosmological literature [4] the divergences in the latter two cases are treated as a simple mathematical pathology due to the assumption of a perfect sphere (with a perfectly defined boundary). Replacing the real sphere with a smooth Gaussian filter $W_V(\vec{r}) \sim e^{-r^2/R^2}$ these integrals are also cut-off at the scale $k \sim R^{-1}$ and one recovers a behaviour $\sigma^2(R) \sim 1/R^{3+n}$ even for $n > 1$. While of course this is valid mathematically it misses an important point, which is that this limiting behaviour of the variance (as $1/R^4$) has a very real physical meaning which has to do with the nature of systems with such a rapidly decaying PS. They correspond to extremely homogeneous systems (i.e. extremely ordered systems) in which the variance really is dominated by the small scale fluctuations. Let us explain this point further.

An intuitive understanding of this fact can be gained by considering discrete distributions. One would reason that any continuous distribution can be arbitrarily well approximated at large scales by an appropriate discretisation process, and that therefore the same result may hold of discrete distributions. In fact such a result has been proved several
years ago [26]: In $d$-dimensions there exists no discrete distribution of points in which the variance in spheres decays faster than $1/R^{d+1}$. One can see roughly why this is so by considering the most ordered distribution of points one might think of: a simple cubic lattice. The variance in a sphere is given by averaging over spheres with center anywhere in the unit cell. As the sphere moves in the unit cell the variance, one would guess (correctly!), in the number of points is proportional to the difference in the volume of the spheres, which is proportional to the surface area of spheres, i.e. $\propto R^{d-1}$ in $d$-dimensions. Thus the normalised variance scales as $1/R^{d+1}$, a result proved rigorously in [27] (see also [26] for a more general discussion of the problem and the comments in [5]). They are distributions which are highly ordered (“glass-like”) in which the fluctuations in real space actually are at small scales (those at which the PS is cut-off). Because of this it is one of their characteristics, as we have seen, that there is no direct relation between the PS at scale $k$ and the physical variance in real space at the scale $R \sim k^{-1}$.

1.5. REAL SPACE CLASSIFICATION OF LONG RANGE FLUCTUATIONS

We now return to a the discussion of the nature of correlations in systems with H-Z like power spectra, with the aim of elucidating their properties by comparison with systems described in statistical physics. To this end, by following [5], we discuss a classification of all possible mass distributions in terms of the main features of the correlation function $\xi(r)$. Following from the above discussion concerning the behaviour of mass fluctuations, we define three distinct classes (for either the case of discrete particle distribution and of a continuous density field) [5]:

1. If

$$\int d^d r \, \xi(r) = \text{const.} > 0 \quad (27)$$

we can say that at large scale the system is substantially Poissonian. Indeed Eq. (27) implies that the PS goes to a constant non-zero value as $k$ goes to zero, and therefore that the large distance behavior of the mass fluctuations is

$$\langle M^2(R) \rangle - \langle M(R) \rangle^2 \sim R^d \sim \langle M(R) \rangle. \quad (28)$$

We write here the unnormalized form of the variance, as the result that the variance of an extensive quantity such as the mass is proportional to the volume on which it is measured is the most intuitive way of characterizing a Poisson type behaviour. In this class is, for example, a system with a finite range correlation $\xi(r) \sim$
\( e^{-r/r_c} \). Beyond the scale \( r_c \) (the correlation length - see below for a discussion about this length) the system is uncorrelated and effectively Poissonian.

If
\[
\int d^d r \, \tilde{\xi}(r) = +\infty \tag{29}
\]
then we are in a case similar to a system at the critical point of a second order phase transition (e.g. the liquid-gas critical point). Such systems have a positive correlation function which is asymptotically a positive power law, with \( \xi(r) \sim 1/r^\gamma \) and \( \gamma < d \), corresponding to a PS \( P(k) \sim k^{\gamma-d} \) as \( k \to 0 \). One then has at large scales the variance

\[
\langle M^2(R) \rangle - \langle M(R) \rangle^2 \sim R^\alpha \quad \text{with} \quad d - 1 < \alpha < d, \tag{30}
\]
or
\[
\langle M^2(R) \rangle - \langle M(R) \rangle^2 \sim \langle M(R) \rangle^\beta \quad \text{with} \quad \beta = \alpha/d > 1.
\]

It is in this context that the concept of self-similarity and scale-invariance has been introduced in statistical mechanics. These terms refer to the fact that in these systems the mass fluctuation field has well defined fractal properties \[20\].

If
\[
\int d^d r \, \tilde{\xi}(r) = 0 \tag{31}
\]
then, as we have discussed, we have for the behaviour of the mass fluctuations

\[
\langle M^2(R) \rangle - \langle M(R) \rangle^2 \sim R^\alpha \quad \text{with} \quad d - 1 < \alpha < d, \tag{32}
\]
i.e. \( \langle M^2(R) \rangle - \langle M(R) \rangle^2 \sim \langle M(R) \rangle^\beta \) with \( \beta = \alpha/d < 1 \), so that the mass fluctuations are always asymptotically smaller than in the uncorrelated Poisson case. This also corresponds to a strongly correlated, long-range ordered, system. We will refer to them with the term “super-homogeneous” to underline this feature that they are more homogeneous than a Poisson system. (Indeed, the Poisson particle distribution is considered as the paradigm of a stochastic homogeneous mass distribution \[24\]). In the context of statistical mechanics they can be described as glass-like, as they have the properties of glasses, which are highly ordered compact systems. That can be said to be typically lattice-like, with a long-range ordered packing, but without the discrete symmetries of an exact
lattice. Note again that, since $\tilde{\xi}(0) > 0$ (a Dirac delta function in the discrete case) by definition, $\tilde{\xi}(r)$ must change sign with $r$ at least once. They are systems with finely balanced positive and negative correlation.

The distinction between 1 and 2 is typical of the statistical physics of critical phenomena in order to distinguish a critical state (case 2) from a non-critical state (case 1). In this context the concept of correlation length is central. The correlation length is a measure of the distance up to which one has spatial memory of the spatial variations in the mass density [18]. There is no unique definition of this length scale, but from a phenomenological point of view it can be defined as the length scale up to which the effect of a small local perturbation in the system is felt. This is due to the fluctuation-dissipation theorem which links the response of the system to a local perturbation and the large scale behavior of the two-point correlation function (for the different precise definitions of the correlation length see for example [11]). A simple definition is

$$r_{\text{corr}}^2 = \frac{\int r^2 d^d r \mid \tilde{\xi}(r) \mid}{\int d^d r \mid \tilde{\xi}(r) \mid}. \quad (33)$$

In case 1 one can generally define a finite correlation length, while in case 2 it will generally diverge. In particular in the case $\xi(r) \sim \exp(-r/r_c)$, $r_c$ is indeed then the correlation length, while for a positive power-law $\xi(r) \sim 1/r^\gamma$ and $\gamma < d$ (case 2) $r_{\text{corr}} \to \infty$.

Case 3 is typical of ordered compact systems with small correlated perturbations. One can meet this kind of correlation function for example in the statistical physics of liquids, glasses, phonons in lattices. The concept of correlation length in this context is less central, and the extension of its use to this class of systems is not particularly useful. Instead it is appropriate to classify the correlation properties of these systems directly through the integral of the correlation function as we have done. It is this behaviour of their correlations which distinguishes them from the other two cases, just as these cases are typically distinguished from one another by the value (finite or infinite) of their correlation length. Certainly, as we have noted, the use of the term “correlation length” in the cosmological literature, which is defined [14] as a scale defining the amplitude of the correlation function, is in no way related to its use in statistical physics.
1.6. CDM AND HDM REAL-SPACE CORRELATION PROPERTIES

Let us firstly consider a simple and instructive example: a H-Z spectrum with a simple exponential cut-off:

\[ P(k) = A \times k \times e^{-\frac{k}{k_c}}, \quad (34) \]

where \( A \) is the amplitude and \( k_c^{-1} \) the cut-off scale. The correlation function is given by

\[ \tilde{\xi}(r) = \frac{A}{\pi^2} \left( \frac{3}{k_c^2} - \frac{r^2}{k_c^2 + r^2} \right). \quad (35) \]

For \( r < r_c \equiv k_c^{-1} \) we have \( \tilde{\xi}(r) \simeq \frac{A}{\pi^2} \frac{3}{k_c^2} > 0 \), changing at \( r \sim r_c \) to an asymptotic behaviour \( \xi(r) \sim -r^{-4} \). Note that the correlation does not oscillate, its only zero crossing being at scale \( r = \sqrt{3} r_c \). Simply because of the condition \( P(0) = 0 \), which implies that the integral of the correlation function must be zero, the correlation function must change sign and in this case it only does so once and thus remains negative at large scales.

In the normalised mass variance \( \sigma^2(R) \) shows a corresponding change in behaviour from being approximately constant at small scales \( R < r_c \) to a \( \ln R/R^4 \) decay at large scales, as was shown above. Note that, unlike for the variance in spheres, there is no limit to the rapidity of the decay of the correlation function (for the more general expression see [14]). Despite the weakness of this correlation at large scales, however, the variance in spheres does not behave like that of a Poisson system, because of the balance between positive correlations at small and negative at large scales imposed by the non-local condition \( P(0) = 0 \).

In cosmological HDM models the form of the PS is almost the same as we have just considered with an exponential cut-off [12]

\[ P(k) \sim k \exp(-k/k_c)^{3/2}. \quad (36) \]

A numerical integration verifies that the correlation function is essentially unchanged.

For CDM models, the class by far favored in the last few years, the form of the PS at scales below turn-over from H-Z behaviour is considerably more complicated. In a linear analysis the PS of CDM matter density field decays below the turn-over with a power-law \( \sim k^{-3/2} \) at large \( k \) until a smaller scale (larger \( k \)) at which it is cut-off with an exponential (in a manner similar to that in the HDM model). Numerical studies of these models designed to include the non-linear evolution bring
further modifications, roughly increasing the exponent in the negative power law regime (see discussion in [5]).

In conclusion two simple real space characteristics today in the distribution of matter coming from the primordial H-Z PS are a negative non-oscillating power-law tail in the two point correlation function $\xi(r) \sim -r^{-4}$, and a $(\ln R)/R^4$ decay in the variance of mass in spheres of radius $R$. These are the peculiar distinctive feature of H-Z type spectra which should possibly be detected in real space by the new galaxy catalogs.

1.7. DISCUSSION

The H-Z spectrum has the same behaviour characteristic of lattice-like order at large scales, while its small $k$ PS is $P(k) \sim k$ instead of $\sim k^2$ [5, 6], typical of a lattice with completely random short range distortions. This spectrum corresponds to more power at large scales and it can be associated with an appropriate correlated shuffling of a perfect lattice [5, 6]. We thus say that the distribution described by the H-Z spectrum has a lattice-like or, more appropriately because of the isotropy, glass-like long range order. More specifically it can be described as a glass characterised by an opportune coherent long-range perturbative waves of distortions [5, 6]. These mass distributions are so ordered at large scales that the mass variance at large scales really does come from small scales. The H-Z spectrum marks the transition to a pure lattice-like behaviour of the normalised variance in spheres $\sigma^2(R) \sim 1/R^4$, which has been shown to be the most rapid possible decay of this quantity for any stochastic distribution of points.

We now return to the use of the term “scale-invariance” in cosmology: it refers to the fact that the variance of the mass (or equivalently gravitational potential) has an amplitude at the horizon scale which does not depend on time. The PS associated with this behaviour is that of a correlated system which is of the super-homogeneous type. This use of the term “scale-invariance” therefore is not in any way analogous to its (original) use in statistical physics. In this context it is associated with a distinctly different class of distributions which have special properties with respect to scale transformations: typically critical systems, like a liquid-gas coexistence phase at the critical point, which have a well defined homogeneity scale and a reduced two-point correlation function which decays as a non-integrable power law: $\xi(r) \sim r^{-\gamma}$ with $0 < \gamma < 3$. In particular, in statistical physics, the term does not have anything to do with the amplitudes of fluctuations but with the persistence of the memory [18].
We have highlighted the fact that all current cosmological models will share at large scales the characteristic behaviour in real space of the H-Z spectrum. Specifically we note primarily the very characteristic lattice-like behaviour of the variance in spheres $\sigma^2(R) \sim R^{-4}$ (up to a small correction which is formally logarithmic for the case of exact H-Z), as well as the characteristic negative (non-oscillating) power-law tail in the two point correlation function $\xi(r) \sim -r^{-4}$. One would expect such behaviour to be seen in principle, if these models were correct, in the distribution of matter in the Universe at large scales, and in particular in the distribution of galaxies. So far such behaviour has not been observed. Rather the characteristic feature of galaxy clustering at small scales is that it shows fractal behaviour [20], which corresponds to a very different kind of distribution than that described by CDM type models. A central (and much debated [20, 23]) question is the determination of the scale marking the transition from this behaviour to homogeneity. In order to detect the correlations predicted by CDM in the distribution of galaxies, one should first find a clear crossover towards homogeneity i.e. a scale beyond which the average density becomes a well-defined (i.e. sample-independent) quantity [23, 20]. On much larger scales galaxy structures should then present the super-homogeneous character of the H-Z type PS. Indeed this should be a critical test of the interpretation of measurements of CMBR in terms of the H-Z picture on large spatial scales [4, 28]. Clearly the link between the observed fractal properties of the galaxy distribution and such super-homogeneous temperature fluctuations is a central problem for theoretical cosmology. Observationally a crucial question is the feasibility of measuring the transition between these regimes directly in galaxy distributions. With large forthcoming galaxy surveys it may be possible to do so, but this is a question which addresses exactly the statistics of these surveys and the exact nature of the signal in any given model.

2. SETTING-UP INITIAL CONDITIONS IN STANDARD COSMOLOGICAL SIMULATIONS

The purpose of cosmological NBS is to calculate the non-linear growth of structures in the universe by following individual particles trajectories under the action of their mutual gravity [7]. These particles are not galaxies but are meant to represent collisionless clouds of elementary dark matter particles. In order to make them move, one must calculate the force acting on each of them due to all the others. In general one may find several algorithms which speed up the $N^2$ sum necessary to
compute the force on each particle \[7\]. The force used is not a pure \( r^{-2} \) one: instead one smooths it at small \( r \) by choosing for instance a force proportional to \((r^2 + \epsilon^2)^{-1}\) in order to avoid “collisions” between close particles. This brings us to an important hypothesis sometimes made in cosmological NBS: with a softened force and a proper choice of the softening parameter \( \epsilon \), the evolution of the NBS should be the same as the evolution of a continuous density field (made of a huge number of particles behaving like a fluid) under its own gravity. In a series of papers Melott & collaborators \[29, 30, 31\] have discussed the effects of discretisation in NBS, showing serious discrepancies in the dynamical evolution described by different algorithms. In particular, and very importantly, they have questioned the capacity of high resolution NBS to describe correctly the evolution of a continuous density field. In our opinion this is still an open problem. For example an important parameter is the ratio \( \epsilon/\langle \Lambda \rangle \): in order to simulate a self-gravitating fluid one would consider that it should be larger than one, but this is not always the case. For instance in the Virgo project (nowadays the standard reference for simulations in the field) where \( \epsilon = 0.036\,\text{Mpc}/\text{h} \) \[33\] and \( \langle \Lambda \rangle \approx 1\,\text{Mpc}/\text{h} \) (see discussion in \[8\]) .

As already discussed, a continuous and smooth density field with correlated density fluctuations is given as IC. However, if one wants to study the time evolution of this field with NBS based on particle dynamics, it is then necessary to discretise the field. This means that one has to create a particle distribution which is representative of the continuous density field and to control any finite size effect. In cosmology a certain procedure has been used since twenty years: let us discuss it in more detail.

2.1. THE UNIFORM BACKGROUND

The standard ad-hoc procedure for setting up IC is described in \[34, 32, 33\]. For the problem of galaxy structure formation the IC generation splits into two parts. The first is to set up a “uniform” distribution of particles, which should represent the unperturbed universe. The second is to impose density fluctuations with the desired characteristics. There are different procedures (e.g. random sampling, threshold sampling, etc.) which can be chosen and they result in different point distributions. Clearly one should have some physical reasons to choose one or another since any procedure introduces some discreteness effects, like Poisson noise, which could play an important role in the non-linear dynamics of the system. For instance, in a Poisson distribution the dominant part of the gravitational force acting on an average particle is
due to its NN [35, 36]: this is because statistical isotropy and the trivial three-point correlation properties make the long-range component of the force cancel. If a simulation is run from a pure Poisson IC the intrinsic small scale fluctuations grow rapidly into non-linear objects at small scales. Instead, in cosmological NBS, one would like to simulate a system where the main contribution to non-linear structure formation, is due to the large scale distribution of the other particles and not to local NN interactions [32].

To overcome the fact that a Poisson distribution leads to unwanted structure formation even without perturbations, the most widely used solution has been to represent instead the unperturbed universe by a regular cubic grid of particles [34, 32]. An infinite lattice, or a lattice with periodic boundary conditions, is “gravitationally stable” because of symmetry. However a lattice is a distribution with fluctuations at all scales and non-trivial correlations. As discussed in the previous section, the unconditional variance in spheres of radius $R$ decays as $\sigma^2(R) \sim R^{-4}$. The two-point CF is such that $\xi(\vec{r}_1, \vec{r}_2) = \xi(\vec{r}_1 - \vec{r}_2) \neq \xi(|\vec{r}_1 - \vec{r}_2|)$ because it is not invariant for space rotation [5]: A lattice breaks space isotropy. Moreover, the grid-like system has the disadvantage to introduce a strong characteristic length on small scales - the grid spacing - and it leads to strongly preferred directions on all scales.

An alternative way to generate an “uniform background” is by means of the following procedure. One starts from a Poisson distribution and then the N-body integrator is used with a repulsive gravitational force in such a way that, after the simulation is evolved for a sufficiently long time, the particles settle down to a glass-like configuration in which the force on each particle is very close to zero [32]. The resulting distribution is very isotropic but it is still characterized by long-range order of the same kind as in a lattice (see Fig.1). As for the lattice the distribution is characterized by the presence of an excluded volume: two particles cannot lie at a distance smaller than a certain fixed length scale [5]. In the lattice this scale is the grid space, for a glass such a distance depends on the number of points one has distributed in a given volume. The fact that a lattice is ordered is due to the existence of the deterministic small scale distance. The unconditional variance scales as $\sigma^2(R) \sim R^{-\alpha}$ where $3 < \alpha \leq 4$, and it is again a strongly correlated system [5]. Its two-point correlation function depends on the detailed procedures used to generate the glass distribution. As already mentioned, glass-like systems belong to a wide family of distributions for which the common feature is that $P(k) \sim k^a$ with $a > 0$ for $k \to 0$ and hence $P(0) = 0$ [5, 6]. However such behaviors in the PS do not imply directly that $\xi(r)$ has a negative
power-law tail at large scales. In particular this is not true if the PS has a singularity for $P(0) \neq 0$, as happens in many systems.

### 2.2. IMPOSING CORRELATED DISPLACEMENTS...

Given a “suitably unperturbed” particle distribution, any desired linear fluctuation distribution can be in principle generated using the Zeldovich approximation [34]. Let us see how this method works for the ideal case of a continuous field. Let the uniform density field be $\rho_0(\vec{r}) = \rho_0$ and superimpose on it the stochastic displacement field $\vec{u}(\vec{r})$ (the infinitesimal volume $dV$ at $\vec{r}$ is displaced by $\vec{u}(\vec{r})$). Let us call $\rho(\vec{r})$ the resulting density field. We suppose that the stochastic field displacement is the realization of a stationary and isotropic stochastic process characterized

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{poisson_glass.png}
\caption{In the top panel it is shown a Poisson distribution, while in the bottom one a glass-like distribution (see [6] for details).}
\end{figure}
by the probability density functional \( P[\vec{u}(\vec{r})] \). In this way \( P[\vec{u}(\vec{r})] \) defines also an ensemble of density fields \( \rho(\vec{r}) \) which is stationary and isotropic, with \( \langle ... \rangle \) the ensemble average.

By applying the mass conservation (i.e. the continuity equation) we find

\[
\frac{\rho(\vec{r}) - \rho_0}{\rho_0} \approx -\vec{\nabla} \cdot \vec{u}(\vec{r}).
\]  \( \text{(37)} \)

If we call as usual \( \xi(r) \) the reduced two-point correlation function of the density field we can write

\[
\xi(r) = \langle \vec{\nabla} \cdot \vec{u}(\vec{r}) \vec{\nabla} \cdot \vec{u}(\vec{0}) \rangle.
\]  \( \text{(38)} \)

Then, taking the FT of both sides of Eq. (38), and making use of the statistical isotropy, we obtain

\[
P(k) \sim k^2 P_u(k),
\]  \( \text{(39)} \)

where \( P(k) \) is the usual PS of the mass density field and \( P_u(k) \) is the PS of the displacement field.

As already discussed the pre-initial configuration can be a lattice or a glass-like system and hence one should consider the intrinsic fluctuations and correlation inherent to such a particle distribution. This is not taken into account by the simple approximation of Eq.39. The standard procedure neglects precisely these “pre-initial” perturbations. In the continuous case, if the initial distribution has a PS \( P_i(k) \), what results from the infinitesimal displacements is a PS

\[
P(k) \sim k^2 P_u(k) + P_i(k).
\]  \( \text{(40)} \)

The effect of the second term could be important both for the determination of the statistical properties of IC, and for the dynamical evolution, as we discuss below. This is especially true for the finite sample problem (i.e. finite displacement and finite sample volume). The analytical calculation in the discrete case is far more complex than Eq.40. In such a situation one has to ensure that the correlations among density fluctuations implemented by the displacement field are larger than the intrinsic fluctuations of the original particle distribution, and that the large-scale fluctuations dominate the non-linear small-scale clustering instead of nearest-particle interactions. Only if one considers very large scales, and/or a displacements field which introduces correlations which are larger than the intrinsic one of the original distribution, one can recover the PS as in Eq.39. Otherwise in a certain range of small enough scales, the point distribution is dominated by discreteness effects, which
in this context can be seen as finite-size effects and which are important for what concerns the small-scale non-linear structure formation (see [8, 5, 6]).

In order to clarify the noise introduced by discretisation and perturbations introduced with respect to the case of the continuous field describe by Eq.39 we have checked numerically the statistical properties of the IC of some NBS. In [8] we have analysed a CDM simulation with $\Omega_0 = 1$ as an example, but our result are generally valid for any other particular model chosen, as they involve the same method for setting-up initial conditions. The result is that there is a very bad between the real-space theoretical correlation properties of the CDM continuous density field and the actual correlations of the particle distribution used as IC (see Fig.2).

2.3. DISCUSSION

We have addressed the problem whether discreteness effects due to the imprint of the (correlated) fluctuations on the pre-initial point distribution are strong enough to dominate the fluctuations of the continuous distribution: the standard ad-hoc procedure used to create NBS IC does not give rise to the desired CDM-like correlation of density fluctuations. This implies that small-scale non-linear dynamical evolution of the system is driven by fluctuations which arise from the particular ad-hoc procedure used to discretise the field. In this context, these fluctuations can be seen as finite size effects, and are completely different from CDM-like fluctuations.

3. DISCRETENESS AND FORMATION OF NON-LINEAR POWER-LAW STRUCTURES IN COSMOLOGICAL N-BODY SIMULATIONS

Non-linear structure formation represents a long-standing problem in cosmology. The central question concerns the organization of galaxies in the highly irregular distribution characterized by large voids, clusters and super-clusters which we observe today in redshift surveys, from some IC. There is a general agreement that the main feature of galaxy structures is their fractal nature [20], which is observed in the power-law decay of the two-point correlation function. The value of the fractal dimension and the possible detection of a well-defined crossover toward homogeneity are still matter of debate [20, 37]. As already mentioned, according to standard theories of galaxy formation, IC of the mass distribution are represented by a continuous field with average density $\rho_0$ and with very
Figure 2. Theoretical behaviour of the CDM correlation function $\xi(r)$ (absolute value) for the IC of a NBS. The vertical solid line correspond to the average distance between nearest neighbors: at smaller scales the correlation function is oscillating due to the imprint of the glass like properties. The horizontal dashed line corresponds to the intrinsic lower limit (Poisson distribution) in the case one uses all the $256^3$ to perform the statistical average. One may see that even in this case that particle induced noise fluctuations determine the behavior of the actual correlation function for most of its range of scale. Finally the solid line is the determination by [8].

Small amplitude correlated Gaussian fluctuations ($\delta \rho/\rho_0 \approx 10^{-5}$) [12, 1], NBS $^7$ permit then to study numerically the evolution of the discretised
density field by particle dynamics [32, 34, 33]. The final result of cosmological NBS, i.e. the particle distribution at the present time, should be similar to the observed galaxy structures and it should depend on the particular choice of the statistical properties of the IC and of the various cosmological parameters [12, 1]: this is the crucial test for the different models, like Cold Dark Matter (hereafter CDM) and its variants. NBS represent the primary tool which allows one to study gravitational many-body dynamics in the non-linear regime (which corresponds to the observed power-law correlated galaxy structures) and, possibly, the transition from linear fluid-like evolution to strongly non-linear clustering. In fact, the understanding of the linear growth of structures, when the density field has small-amplitude fluctuations, is based on linear perturbation theory of the equations of motion of a fluid under the influence of gravity in an expanding universe [12]. In this context, NBS are used as a guide for any analytical approach to the weakly non-linear gravitational regime.

Given the fluctuations present in the system at the beginning (particle fluctuations at small scales, small-amplitude correlated fluctuations at large scales - which, we as discussed before, are not of CDM type) which are the dominant ones for the generation of non-linear structures with power-law correlations (as found in the cosmological NBS [34, 33])? In order to study this question, we characterize here the formation of power-law correlated structures in cosmological NBS (based on particle dynamics with a small initial velocity dispersion [8]) by a detailed study of real space statistical properties of the particle distribution during time evolution. Power-law structures seem to arise from the (small-scale) particle fluctuations and large-scale small-amplitude fluctuations appears to play little role in this dynamics [9, 10]. Such a result implies a radical change of perspective about the problem of structure formation, as we discuss below.

We present here the analysis of a cosmological NBS performed by the Virgo consortium [33], and we refer the interested reader to [9, 10] for further discussions. This is a CDM model with \( N = 256^3 \) particles and gravitational force smoothing length \( \epsilon = 0.036 \text{ Mpc}/h \). The volume \( V \) of the simulation is a cube of side \( L = 239.5 \text{ Mpc}/h \). Each particle then represents a mass of \( m_p = 2.27 \times 10^{11} \) solar masses, and we find that the initial mean interparticle separation \( \langle \Lambda_i \rangle \approx 1 \text{ Mpc}/h \). The NBS are all run from a red-shift of \( z = 50 \) until today \( (z = 0) \), i.e. for a time which is essentially the age of the Universe \( \approx 15 \text{ Gyr} \). This means that the time of evolution is essentially one dynamical time \( \tau_{\text{dyn}} \approx 1/\sqrt{\Omega\rho} \), where the latter is simply the characteristic time scale associated with a mass.
density $\rho = N m_p / V$. A particle typically moves by a distance of order the lattice spacing in this time.

The first statistic we consider is the conditional average density [20, 5] given by $\Gamma(r) = \langle n(r) n(0) \rangle / \langle n \rangle$ where $n(r)$ is the microscopic number density. It is simply the mean density of points at a distance $r$ from an occupied point. It is plotted in Fig.2 for a sequence of time slices, beginning from the initial distribution at $z = 50$ until today. Also shown is the pre-initial ‘glass’ configuration. The behaviour of $\Gamma(r)$ for the glassy configuration and these IC at $z = 50$ are very similar. This is because the perturbations are very small in amplitude compared to the initial interparticle separation $\langle \Lambda_i \rangle$. The highly ordered lattice-like nature of the initial distribution is manifest: there is an excluded volume around each point so that $\Gamma(r)$ is negligible until very close to $\langle \Lambda_i \rangle$, where it shows a small peak, with some oscillation about the mean density evident corresponding to the long-range order [5]. When the evolution under self-gravity starts the excluded volume feature is rapidly diluted, having almost completely disappeared by $z = 3$ ($t = 0.125 t_\text{ad}$). This corresponds to the development of power-law clustering at very small scales where it was completely absent in the IC.

Let us consider what is driving this dynamics. When points move towards their NN the contribution of the force acting on particle due to its nearest neighbor grows as one over the distance squared while the latter changes, supposing that the validity of linear growth of fluctuations at larger scales, in proportion to the scale factor, i.e. as $1/(1 + z)$. Thus at $z = 5$ we would expect the latter to grow by a factor of 10 and thus the NN force to dominate below about $\langle \Lambda_i \rangle / 5$.

Between $z = 5$ and $z = 3$ we see - in this range of scales completely dominated by NN interactions - the appearance of an approximately power-law behaviour in $\Gamma(r)$. At $z = 3$ the amplitude of $\Gamma(r)$ over most of the range $r < \langle \Lambda_i \rangle$ is greater than the mean density so that this power law is also now seen in the normalised correlation function $\xi(r) = \Gamma(r) / \langle n \rangle - 1$ shown in the inset of Fig.3. At the next time slice, at $z = 1$, the form of $\Gamma(r)$ up to approximately $0.3 \langle \Lambda_i \rangle$ is almost precisely the same, being simply amplified by an order of magnitude. At the same time the power law extends slightly further before flattening from $\Gamma \approx 3 \langle n \rangle$ to a smooth interpolation towards its asymptotic value. The evolution in the remaining two time slices is well described over the range $\Gamma(r) > 3 \langle n \rangle$ as a simple amplification, and translation towards larger scales, of $\Gamma(r)$. At the final time the power law (with exponent $\gamma \approx -1.7$) extends to $\approx 2 \langle \Lambda_i \rangle$.

In [38] a very similar behaviour is observed in the evolution of clustering of particles by Newtonian forces, without expansion and with simple
Figure 3. Two-point conditional density for the simulation considered. The initial mean interparticle separation $\langle \Lambda_i \rangle$ and the softening length $\epsilon$ are shown, as well together with the best power-law fit at the end of the simulation. In the insert panel the evolution of the reduced correlation function $\xi(r)$ is shown.
Poissonian IC. The authors give a physical interpretation of this clustering in terms of the exportation of the initial “granularity” in the distribution to larger scales through clustering. The self-similarity in time of $\Gamma(r)$ is explained as due to a coarse-graining performed by the dynamics: one supposes a NN dynamics between particle-like discrete masses, with the mass and physical scale changing as a function of time as the clustering evolves (particles forming clusters, clusters forming clusters of clusters etc.). This would appear to provide a good explanation for the behaviour observed here as well, but further theoretical work is clearly needed to establish this and find a more quantitative description.

The primary conclusion we draw from our real space analysis of the Virgo NBS is therefore that the fluctuations at the smallest scales in these NBS - i.e. those associated with the discreteness of the particles - play a central role in the dynamics of clustering in the non-linear regime. In particular the power-law type correlations appear to be built up from the initial clustering at the smallest scales. The nature of the clustering (in particular the exponent of the power-law) seems to be independent of the IC, and its physical origin should be explained through the dynamics of discrete self-gravitating systems. The fluid-like statistical description and equation of motions, which is the framework used to describe a CDM universe, do not consider the non-analytical “particle” term of noise which is represented by NN interactions. The latter as we have seen are strongly present in the NBS we have analysed and appear to play a crucial role in the formation of the correlated structures observed to emerge. As already mentioned in the paradigmatic example of stochastic (homogeneous and isotropic) point processes, the Poisson distribution, the gravitational force on an average point is due very predominantly to its NN [35]. Large-scale small-amplitude density fluctuations do not give rise to a significant contribution to the force acting a point, because of symmetry: i.e. large-scale isotropy [35].

The dynamics we have described is essentially dependent on the gravitational forces at the smallest resolved scale in the NBS, and the small smooth component of the force added to this by the perturbations at larger scales appears to be irrelevant in the non-linear regime. In cosmology this supposed link between the IC and the “predictions” for structure formation at larger scales is very important, as it is through it that one tries to constrain models using observations of galaxy distributions. Our conclusions completely change the perspective on this problem. Power-law correlations of this type are the most striking and well established feature of such distributions [1, 20]. The theoretical problem of their origin therefore must deal with the apparently crucial role in their formation of an intrinsically highly fluctuating (and thus
non-analytic) density field. In particular the origin of the exponent in the correlations and the dependence of the extent of such correlations on the discretisation (physical or numerical) needs to be understood.

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Notes

1. Note that H-Z type spectra are the only ones compatible with FRW metric on large scales, because they do not give divergent fluctuations of the gravitational potential, as the purely Poisson distribution [5]
2. Because of the arbitrariness of the position of the center of the sphere, the average density is a one-point statistical property.
3. Note that this term can give a contribution to the variance which dominates over that due to the intrinsic correlations.
4. See, for example, the section entitled “Problems with filters” in the book by Lucchin and Coles [25].
5. For example these properties near the critical point of the liquid-gas transition gives place to opalescence phenomena.
6. To avoid any possible confusion for those somewhat familiar with these simulations, we note the description of the H-Z model as lattice-like or glass-like, has no direct relation to the use of lattices or glasses in setting up IC in current NBS. Their small $k$ behavior is in general different as well as the real space correlation function [5]
7. Note that there are basically two different families of N-body codes: $P^3M$ and $PM$ [32]. We refer hereafter to the first kind of NBS, whose aim is to study particle dynamics with high small-scale resolution, as discussed below. Instead, $PM$ codes avoid close particles interactions [29, 30, 31].
8. With this smoothing the force is $53.6\%$ of the true $1/r^2$ force at $\epsilon$ and more than $99\%$ at $2\epsilon$ [33].
9. In these matter dominated cosmologies the time is given by $t = t_o/(1 + z)^{3/2}$ where $t_o$ is the age of the Universe today.

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