Invertible completions of $n \times n$ upper triangular operator matrices via normed spaces enclosures

Nikola Sarajlija

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Abstract

Let $T^d_n(A)$ be an upper triangular operator matrix of dimension $n$ with given diagonal entries. We assume $T^d_n(A)$ acts on a direct topological sum of Banach spaces, where tuple $A = (A_{ij})_{1 \leq i < j \leq n}$ consists of unknown operators defined on appropriate domains. We prove that regularity of $T^d_n(A)$ can be characterized in terms of diagonal operators solely, and we extend the well known result from [11] and a recent result from [23] from $n = 2, 3$ to arbitrary dimension $n$. Thus, invertibility of upper triangular operator matrices is completely characterized, and the answer to a question raised by the present author and D. S. Djordjević in [23] is given. We obtain some perturbation and filling in holes results as consequences.

Keywords: invertibility, enclosure, $n \times n$, upper triangular, Banach spaces,
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1 e-mail address: nikola.sarajlija@dmi.uns.ac.rs
University of Novi Sad, Faculty of Sciences, Novi Sad, 21000, Serbia
1 Introduction

Let $D_1 \in B(X_1)$, $D_2 \in B(X_2)$, ..., $D_n \in B(X_n)$ be given. We denote by $T_n^d(A)$ an $n \times n$ partial upper triangular operator matrix of the form

$$T_n^d(A) = \begin{bmatrix} D_1 & A_{12} & A_{13} & \cdots & A_{1,n-1} & A_{1n} \\ 0 & D_2 & A_{23} & \cdots & A_{2,n-1} & A_{2n} \\ 0 & 0 & D_3 & \cdots & A_{3,n-1} & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{n-1} & A_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & D_n \end{bmatrix} \in B(X_1 \oplus X_2 \oplus \cdots \oplus X_n),$$

(1.1)

where $A := (A_{12}, A_{13}, \ldots, A_{ij}, \ldots, A_{n-1,n})$ is an operator tuple consisting of unknown variables $A_{ij} \in B(X_j, X_i)$, $1 \leq i < j \leq n$, $n \geq 2$. For convenience, we denote by $B_n$ the collection of all such tuples $A$. This notation is due to J. Huang et al. [12], and the present author has used it in some of his articles [21], [22], etc.

Main result of this article is the following:

**Theorem 1.1** Let $D_1 \in B(X_1)$, ..., $D_n \in B(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$ are inner regular.

Then there exists $A \in B_n$ such that $T_n^d(A)$ is invertible if and only if

(a) $D_1$ is left invertible;
(b) $D_n$ is right invertible;
(c) $\bigoplus_{s=1}^{n-1} X_s/R(D_s) \cong \bigoplus_{s=2}^n N(D_s)$.

This Theorem is a generalization of [11, Theorem 2] and [23, Theorem 4] to arbitrary dimension $n$. Explanation of the terminology appearing in its statement is given in the sequel. Its proof is conducted below as well.

In the last few decades considerable attention has been devoted to the study of spectral properties of operator matrices, having in mind their governing importance in various areas of mathematics (see for example [9, Chapter VIII]). One has soon realized that one way for successful work on problems arising in spectral theory, is to see operator matrices as entries of simpler blocks. Block operator matrices, and especially upper triangular operator matrices, have been extensively studied by numerous authors (see [3], [4], [6], [8], [11], [14], [15], [25], [26] and many others...). The reason for this lies in the fact that if an operator $T$ is acting on a direct sum of Banach spaces, it takes the upper triangular form under condition that certain number of those spaces is invariant for $T$.

Development of this topic began in the last century, and is of great importance
ever since. In the beginning, authors have only considered the case of $2 \times 2$ operator matrices. Pioneering work in that direction was the article of Du and Jin from 1994 ([8]) treating the usual spectrum. Han et al. have generalized their result to Banach spaces ([11]), and Lee has proved some facts concerning the Weyl spectrum ([14]). Afterwards, Djordjević in 2002 gave some characterizations for $2 \times 2$ upper triangular operators to be Fredholm, Weyl, and Browder ([6]). After that, many authors have explored various properties of $2 \times 2$ block operators in a connection with intersection of spectra, Weyl and Browder type theorems, etc. (see for example [15], [3]).

Investigation of spectral properties of general $n \times n$ operators began no sooner than 2013, when Zguitti published the article dealing with some properties of the Drazin spectrum ([27]). Huang et al. continued his work in 2016 by investigating properties of the point, residual, and continuous spectrum of $n \times n$ matrix operators ([12]). Fredholm and the Weyl spectrum of such operators have been studied by Wu and Huang in [25], [26] only a few years ago, and this article is concerned with generalizing their results from separable Hilbert to arbitrary Banach spaces.

Partial operator matrix is an operator matrix with some entries specified and the others unknown. Most commonly, those entries are linear and bounded operators with appropriate domains. We mention that there has been some interest in block operators with unbounded entries lately, see [2], [18], [20], but we shall not pursue this point any further. Usually, it is required that unknown entries satisfy some special properties, and finding those entries is the task known as the completion of the matrix. It is sometimes a rather difficult task, but there are some ways to make this task easier. Perturbation of an upper triangular operator matrix is an intersection of its certain spectrum, some of the entries being fixed, for example the diagonal ones, and the others chosen arbitrarily with appropriate domains. Finding perturbation is a very useful pre-step to the completion of a matrix. This article is ought to be just a such pre-step.

Article is organized as follows. In Section 2 we give some basic concepts and terminology that we use in the rest of this work. We introduce notion of enclosed spaces. In Section 3 we prove our main result. Section 4 is devoted to perturbation and filling in holes results.

## 2 Preliminaries

We always assume that $X_1, \ldots, X_n$, $X$, $Y$ are complex Banach spaces, unless different is said. Collection of all linear and bounded functions from $X_i$ to $X_j$ is denoted by $B(X_i, X_j)$, where $B(X_i) := B(X_i, X_i)$. For element $T \in B(X_i, X_j)$ we use a term operator. If $T$ is an operator, we define its kernel $\mathcal{N}(T) = \{x \in$
\( X_i : Tx = 0 \) and range \( R(T) = \{Tx : x \in X_i\} \). Kernel is always closed. 

\( X' = B(X, \mathbb{C}) \) is the dual space of \( X \). It is a Banach space as well.

Now we list some elementary notions from Fredholm theory (see [29]). Let \( T \in B(X) \), \( \alpha(T) = \dim N(T) \) and \( \beta(T) = \dim X/R(T) \). Quantities \( \alpha \) and \( \beta \) are called the nullity and deficiency of \( T \), respectively, and in the case where at least one of them is finite we define \( \text{ind}(T) = \alpha(T) - \beta(T) \) to be the index of \( T \). Notice that \( \text{ind}(T) \) may be \( \pm \infty \) or integer. Families of left and right Fredholm operators, respectively, are defined as

\[
\Phi_l(X) = \{ T \in B(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed} \}
\]

and

\[
\Phi_r(X) = \{ T \in B(X) : \beta(T) < \infty \}.
\]

The set of Fredholm operators is

\[
\Phi(X) = \Phi_l(X) \cap \Phi_r(X) = \{ T \in B(X) : \alpha(T) < \infty \text{ and } \beta(T) < \infty \}.
\]

Families of left and right Weyl operators, respectively, are defined as

\[
\Phi_l^-(X) = \{ T \in \Phi_l(X) : \text{ind}(T) \leq 0 \}
\]

and

\[
\Phi_r^+(X) = \{ T \in \Phi_r(X) : \text{ind}(T) \geq 0 \}.
\]

The set of Weyl operators is

\[
\Phi_0(X) = \Phi_l^- (X) \cap \Phi_r^+(X) = \{ T \in \Phi(X) : \text{ind}(T) = 0 \}.
\]

Let \( T \) be an operator from \( X_i \) to \( X_j \). Then \( T \) is left invertible if there is operator \( S \) from \( X_j \) to \( X_i \) such that \( ST \) is the identity operator on \( X_i \). \( T \) is right invertible if there is operator \( R \) from \( X_j \) to \( X_i \) such that \( TR \) is the identity operator on \( X_j \). It is well known that \( T \) is left invertible if and only if \( N(T) = 0 \) and \( R(T) \) is closed and topologically complemented. Dually, \( T \) is right invertible if and only if \( R(T) = X_j \) and \( N(T) \) is closed and complemented. We say that \( T \) is invertible if it is both left and right invertible, that is if \( N(T) = 0 \) and \( R(T) = X_j \), or in other words if \( T \) is injective and surjective. For our purposes, we introduce the notion of almost invertible operators: \( T \) is almost invertible if it is injective with dense range.

In view of the concepts of the previous paragraph, we consider spectrum \( \sigma(T) \) of \( T \in B(X_i, X_j) \) as the following collection: \( \sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not invertible} \} \). Next, we consider different parts of the spectrum: \( \sigma_p(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not injective} \} \) is the point spectrum, \( \sigma_d(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not surjective} \} \) is the defect spectrum, \( \sigma_l(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not left invertible} \} \) is the left defect spectrum, \( \sigma_r(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not right invertible} \} \) is the right defect spectrum, and \( \sigma_0(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not invertible} \} \) is the total defect spectrum.
invertible} is the left spectrum, \( \sigma_r(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not right invertible} \} \) is the right spectrum, \( \sigma_{re}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not left Fredholm} \} \) is the left essential spectrum, \( \sigma_{le}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not left Fredholm} \} \) is the right Fredholm spectrum, \( \sigma_{lw}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not left Weyl} \} \) is the left Weyl spectrum, \( \sigma_{rw}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not right Weyl} \} \) is the right Weyl spectrum of \( T \).

For \( U \subseteq X \) we define set \( U^\circ \subseteq X' \), and for \( V \subseteq X' \) we define set \( \circ V \subseteq X \) as

\[
U^\circ := \{ f \in X' : f |_U = 0 \}.
\]

\[
\circ V := \{ x \in X : f(x) = 0 \text{ for every } f \in V \}.
\]

\( U^\circ \) and \( \circ V \) are called the right and the left annihilator of \( U \) and \( V \), respectively.

Above all interesting features that hold for annihilators, we point out only a few of them, needed in duality arguments.

**Lemma 2.1** ([24]) If \( A \in \mathcal{B}(X, Y) \), then

(a) \( \mathcal{R}(A)^\circ = \mathcal{N}(A') \);
(b) \( \mathcal{R}(A)^\circ = [\mathcal{N}(A')] \);
(c) \( \mathcal{R}(A') = \mathcal{N}(A)^\circ \).

The following lemma imposes a connection between \( T \) and its dual operator \( T' \) in terms of nullity and deficiency of \( T \). This claim will be crucial at some points.

**Lemma 2.2** ([5, p. 7-8]) For \( T \in \mathcal{B}(X) \) with closed range the following holds:

(a) \( \alpha(T) = \beta(T') \), \( \beta(T) = \alpha(T') \);
(b) \( T \in \Phi_l(X) \text{ if and only if } T' \in \Phi_r(X') \);
(c) \( T \in \Phi_r(X) \text{ if and only if } T' \in \Phi_l(X') \);
(d) \( \text{ind}(T') = -\text{ind}(T) \).

Consequence of Lemma 2.1 is the following:

**Corollary 2.3** If \( A \in \mathcal{B}(X, Y) \), then \( \mathcal{R}(A) \) is dense if and only if \( A' \) is injective. Furthermore, \( A \) is injective if and only if \( \mathcal{R}(A') \) is dense.

Next fact is also elementary.

**Lemma 2.4** If \( A \in \mathcal{B}(X, Y) \), then \( \mathcal{R}(A) \) is closed if and only if \( \mathcal{R}(A') \) is closed.

Combining the previous two claims, one can get.
**Lemma 2.5** If \( A \in \mathcal{B}(X, Y) \), then \( A \) is surjective if and only if \( A' \) is injective with closed range. \( A \) is injective with closed range if and only if \( A' \) is surjective.

Next Theorem is well known in the literature.

**Theorem 2.6** Let \( Z \) be a proper subspace of \( Y \). Then:
(a) \( (Y/Z)' \) is isometrically isomorphic to \( Z^0 \);
(b) \( Z' \) is isometrically isomorphic to \( Y'/Z^0 \).

We introduce a notion that will have a major role in our work.

**Definition 2.7** If there exists an injective bounded linear mapping from \( X \) to \( Y \), we say that \( X \) can be enclosed into \( Y \) and write \( X \hookrightarrow Y \). If such a mapping is with closed range, we say that \( X \) can be completely enclosed into \( Y \) and write \( X \hookrightarrow_c Y \).

**Remark 2.8** One can notice a similarity with the concept of Banach space embeddings introduced by Djordjević in [6]. Namely, the correct link between the two is the following: if \( X \) can be embedded into \( Y \), then \( X \) can be (completely) enclosed into \( Y \). In other words:

\[
(X \preceq Y) \Rightarrow (X \hookrightarrow Y) \Rightarrow (X \hookrightarrow Y).
\]

It is also clear that the notion of enclosed Banach spaces extends to normed spaces as well.

We introduce the following variant of Djordjević’s notion:

**Definition 2.9** If there exists a left Fredholm bounded linear mapping from \( X \) to \( Y \), we say that \( X \) can be \( F \)-embedded into \( Y \) and write \( X \preceq_{le} Y \).

Following Lemma is a consequence of the Han-Banach Theorem.

**Lemma 2.10** \( X \hookrightarrow Y \) if and only if \( X' \hookrightarrow Y' \).

It is known that a closed subspace of a Banach space need not be complemented. We overcome this obstacle by summoning a concept from [22].
**Definition 2.11** We say that \( T \in B(X_i, X_j) \) satisfies the complements condition if:

(i) \( \mathcal{N}(T) \) is complemented;

(ii) \( \mathcal{R}(T) \) is complemented.

One important class of operators satisfying the complements condition are inner regular operators. Operator \( T \) is inner regular if and only if an equality

\[ T = \tilde{T}T \]

holds for some operator \( \tilde{T} \) with appropriate domain. In this case \( \tilde{T} \) is an inner regular inverse for \( T \). Notice that such an operator \( \tilde{T} \), if exists, need not be unique. One can prove that operator \( T \) is inner regular if and only if its kernel and range are closed and complemented subspaces [7, 1.1.5 Corollary]. Hence, inner regular operators satisfy the complements condition of Definition 2.11. Observe also that every operator between Hilbert spaces satisfies the complements condition. We will use inner regular operators in the statement of our main theorem.

We will also consider a partial lower triangular operator matrix \( \widehat{T}_n(A) \) of the form

\[
\widehat{T}_n(A) = \begin{bmatrix}
D_1 & 0 & 0 & \ldots & 0 & 0 \\
A_{21} & D_2 & 0 & \ldots & 0 & 0 \\
A_{31} & A_{32} & D_3 & \ldots & 0 & 0 \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{n-1,1} & A_{n-1,2} & A_{n-1,3} & \ldots & D_{n-1} & 0 \\
A_{n1} & A_{n2} & A_{n3} & \ldots & A_{n,n-1} & D_n
\end{bmatrix} \in B(X_1 \oplus X_2 \oplus \cdots \oplus X_n),
\]

(2.1)

where \( A := (A_{21}, A_{31}, \ldots, A_{ij}, \ldots, A_{n,n-1}) \) consists of unknown variables \( A_{ij} \in B(X_j, X_i) \), \( n \geq i > j \geq 1 \), \( n \geq 2 \). It is sometimes necessary to consider (1.1) and (2.1) together, since it is clear that an upper triangular operator matrix has the dual operator matrix of the lower triangular form.

Proofs of the next two lemmas are not difficult, and so we omit them.

**Lemma 2.12** If \( T_n(A) \) is injective, then \( D_1 \) is injective. If \( T_n(A) \) has dense range, then \( D_n \) has dense range.

**Lemma 2.13** If \( T_n(A) \) is left invertible, then \( D_1 \) is left invertible and \( \mathcal{R}(D_n) \) is closed and complemented. If \( T_n(A) \) is right invertible, then \( D_n \) is right invertible.

**Lemma 2.14** If \( T_n(A) \) is left Fredholm invertible, then \( D_1 \) is left Fredholm invertible and \( \mathcal{R}(D_n) \) is closed and complemented. If \( T_n(A) \) is right Fredholm invertible, then \( D_n \) is right Fredholm invertible.
Following characterizations will be useful as well.

**Lemma 2.15** $T \in B(X)$ is left invertible if and only if $T$ is injective with closed and complemented range. $T$ is right invertible if and only if $T$ is surjective with complemented kernel.

### 3 Main results

In this section we will prove Theorem 1.1 stated in the Introductory section. We start with a characterization of left Fredholm invertibility of $T_n(A)$. After that we get a characterization of injectivity as a consequence. We use the concept of enclosed Banach spaces introduced in Definition 2.7.

**Theorem 3.1** Let $D_1 \in B(X_1), \ldots, D_n \in B(X_n)$. Assume that $D_s, 2 \leq s \leq n-1$, satisfy the complements condition. Assume further that $R(D_1)$ and $N(D_n)$ are (topologically) complemented.

Then there exists $A \in B_n$ so that $T_n^d(A)$ is left Fredholm invertible with nullity $n \in \mathbb{N} \cup \{0\}$ if and only if

(a) $D_1$ is left Fredholm invertible;

(b) there exists an embedding $\bigoplus_{s=2}^n N(D_s) \lesssim_{le} \bigoplus_{s=1}^{n-1} X_s/R(D_s)$ with nullity $n$.

**Proof.** Let $D_1$ and $T : \bigoplus_{s=2}^n N(D_s) \to \bigoplus_{s=1}^{n-1} X_s/R(D_s)$ be left Fredholm invertible operators. Consider matrix representation of $T$, $T = [T_{ij}]$, where $T_{ij} \in B(N(D_j), X_i/R(D_i))$. Then operator matrix $T$ is equivalent to a left Fredholm upper triangular operator matrix which we again denote by $T$

$$T = \begin{bmatrix} A_{12} & A_{13} & A_{14} & \ldots & A_{1n} \\ 0 & A_{23} & A_{24} & \ldots & A_{2n} \\ 0 & 0 & A_{34} & \ldots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_{n-1,n} \end{bmatrix} \begin{bmatrix} N(D_2) \\ N(D_3) \\ \vdots \\ N(D_n) \end{bmatrix} \to \begin{bmatrix} X_1/R(D_1) \\ X_2/R(D_2) \\ X_3/R(D_3) \\ \vdots \\ X_{n-1}/R(D_{n-1}) \end{bmatrix} \quad (3.1)$$

Consider $T_n^d(A)$ as an operator from $X_1 \oplus X_2/N(D_2) \oplus N(D_3) \oplus X_3/N(D_3) \oplus \cdots \oplus X_n/N(D_n) \oplus N(D_n)$ into $R(D_1) \oplus X_1/R(D_1) \oplus R(D_2) \oplus$
Evidently, $D^{(1)}_2, \ldots, D^{(1)}_n$ as in (3.3) are all injective and $D^{(1)}_1$ is left Fredhom. Thus,
there exist invertible operator matrices $U$ and $V$ such that

$$U \mathcal{T}_n^d(A)V = \begin{bmatrix}
D_1^{(1)} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & A_2^{(4)} & 0 & A_3^{(4)} & \ldots & 0 & A_{4n}^{(4)} \\
0 & D_2^{(1)} & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & A_2^{(4)} & \ldots & 0 & A_{4n}^{(4)} \\
0 & 0 & 0 & D_3^{(1)} & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & A_{4n}^{(4)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & A_{4n-1,n}^{(4)} \\
0 & 0 & 0 & 0 & 0 & \ldots & D_n^{(1)} & 0
\end{bmatrix}$$

and $D_s^{(1)}$, $2 \leq s \leq n$ are injective, $D_1^{(1)}$ is left Fredholm. Note that $A_{ij}^{(4)}$ and $D_s^{(1)}$ in (3.4) are not the original ones in (3.3) in general, but we still use them for convenience. Now, since $U$ and $V$ are regular, $U \mathcal{T}_n^d(A)V$ is left Fredholm. This, together with injectivity of $D_s^{(1)}$, $2 \leq s \leq n$ and left Fredholmness of $D_1^{(1)}$ implies left Fredholmness of

$$T = \begin{bmatrix}
A_{12}^{(4)} & A_{13}^{(4)} & A_{14}^{(4)} & \ldots & A_{1n}^{(4)} \\
0 & A_{23}^{(4)} & A_{24}^{(4)} & \ldots & A_{2n}^{(4)} \\
0 & 0 & A_{34}^{(4)} & \ldots & A_{3n}^{(4)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_{n-1,n}^{(4)}
\end{bmatrix}; \quad \mathcal{N}(D_2) \rightarrow \mathcal{N}(D_3) \rightarrow \mathcal{N}(D_4) \rightarrow \mathcal{N}(D_n) \rightarrow X_n/\mathcal{R}(D_n).

\text{We conclude that } \bigoplus_{s=2}^{n} \mathcal{N}(D_s) \leq \bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s). \square

Now, we get as a direct consequence.

**Theorem 3.2** Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s$, $2 \leq s \leq n-1$, satisfy the complements condition. Assume further that $\mathcal{R}(D_1)$ and $\mathcal{N}(D_n)$ are (topologically) complemented.

Then there exists $A \in \mathcal{B}_n$ so that $\mathcal{T}_n^d(A)$ is injective if and only if

(a) $D_1$ is injective;

(b) $\bigoplus_{s=2}^{n} \mathcal{N}(D_s) \hookrightarrow \bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s)$.

Lower triangular operator matrices have an analogous characterization of injectivity.

**Theorem 3.3** Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s$, $2 \leq s \leq n-1$, satisfy the complements condition. Assume further that $\mathcal{R}(D_n)$ and $\mathcal{N}(D_1)$ are...
(topologically) complemented.
Then there exists $A \in B_n$ so that $\tilde{T}_d^n(A)$ is injective if and only if:
(a) $D_n$ is injective;
(b) $\bigoplus_{s=1}^{n-1} \mathcal{N}(D_s) \hookrightarrow \bigoplus_{s=2}^n X_s/\mathcal{R}(D_s)$.

Now, the following perturbation results immediately follow.

**Corollary 3.4** Let $D_1 \in B(X_1), \ldots, D_n \in B(X_n)$. Assume that $D_s - \lambda$, $2 \leq s \leq n - 1$, $\lambda \in \mathbb{C}$ satisfy the complements condition. Assume further that $\mathcal{R}(D_1 - \lambda)$, $\lambda \in \mathbb{C}$ and $\mathcal{N}(D_n - \lambda)$, $\lambda \in \mathbb{C}$ are complemented. Then

$$\bigcap_{A \in B_n} \sigma_p(\tilde{T}_d^n(A)) = \sigma_p(D_1) \cup \{\lambda \in \mathbb{C} : \bigoplus_{s=1}^n \mathcal{N}(D_s - \lambda) \not\hookrightarrow \bigoplus_{s=2}^{n-1} X_s/\mathcal{R}(D_s - \lambda)\}.$$  

**Corollary 3.5** Let $D_1 \in B(X_1), \ldots, D_n \in B(X_n)$. Assume that $D_s - \lambda$, $2 \leq s \leq n - 1$, $\lambda \in \mathbb{C}$ satisfy the complements condition. Assume further that $\mathcal{R}(D_n - \lambda)$, $\lambda \in \mathbb{C}$ and $\mathcal{N}(D_1 - \lambda)$, $\lambda \in \mathbb{C}$ are complemented. Then

$$\bigcap_{A \in B_n} \sigma_p(\tilde{T}_d^n(A)) = \sigma_p(D_n) \cup \{\lambda \in \mathbb{C} : \bigoplus_{s=1}^{n-1} \mathcal{N}(D_s - \lambda) \not\hookrightarrow \bigoplus_{s=2}^n X_s/\mathcal{R}(D_s - \lambda)\}.$$  

**Remark 3.6** Notice that Corollary [3.4] is a far-reaching improvement and generalization to arbitrary Banach spaces of [12, Theorem 2.4].

If we observe the proof of Theorem [3.2] with the help of Lemma [2.13] one can easily prove the characterization of left invertibility of $T_d^n(A)$ in two steps.

**Theorem 3.7** Let $D_1 \in B(X_1), \ldots, D_n \in B(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$, are inner regular. Assume further that $\mathcal{N}(D_n)$ is (topologically) complemented.
Then there exists $A \in B_n$ so that $T_d^n(A)$ is injective with closed range if and only if:
(a) $D_1$ is injective;
(b) $\mathcal{R}(D_n)$ is closed;
(c) $\bigoplus_{s=2}^n \mathcal{N}(D_s) \hookrightarrow_c \bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s)$.

**Theorem 3.8** Let $D_1 \in B(X_1), \ldots, D_n \in B(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$, are inner regular. Assume further that $\mathcal{N}(D_n)$ is (topologically) complemented.
Then there exists $A \in B_n$ so that $T_n^d(A)$ is left invertible if and only if
(a) $D_1$ is left invertible;
(b) $\mathcal{R}(D_n)$ is closed and complemented;
(c) $\bigoplus_{s=2}^n \mathcal{N}(D_s) \preceq \bigoplus_{s=1}^{n-1} X_s / \mathcal{R}(D_s)$.

**Corollary 3.9** Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s - \lambda, 2 \leq s \leq n-1, \lambda \in \mathbb{C}$ are inner regular. Assume further that $\mathcal{N}(D_n - \lambda), \lambda \in \mathbb{C}$ is complemented. Then
\[
\bigcap_{A \in B_n} \sigma_l(T_n^d(A)) = \sigma_l(D_1) \cup \{\lambda \in \mathbb{C} : \mathcal{R}(D_n - \lambda) is not closed and complemented\} \cup \{\lambda \in \mathbb{C} : \bigoplus_{s=2}^n \mathcal{N}(D_s - \lambda) \not\preceq \bigoplus_{s=1}^{n-1} X_s / \mathcal{R}(D_s - \lambda)\}.
\]

**Remark 3.10** Notice that Theorem 3.8 and Corollary 3.9 are generalizations and improvements of [6, Theorem 5.2] and [6, Corollary 5.3].

Now, duality argument is employed to get a characterization of denseness of $\mathcal{R}(T_n^d(A))$.

**Theorem 3.11** Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s - \lambda, 2 \leq s \leq n-1$ satisfy the complements condition. Assume further that $\mathcal{R}(D_n)$ and $\mathcal{N}(D_1)$ are (topologically) complemented. Then there exists $A \in B_n$ so that $T_n^d(A)$ has dense range if and only if
(a) $D_n$ has dense range;
(b) $\bigoplus_{s=1}^{n-1} X_s / \mathcal{R}(D_s) \hookrightarrow \bigoplus_{s=2}^n \mathcal{N}(D_s)$.

**Proof.** By Corollary 2.3 we know that $T_n^d(A)$ is with dense range if and only if $T_n^d(A)'$ is injective. However, since $T_n^d(A)'$ takes lower triangular form, the former is equivalent to injectivity of $D_n'$ and enclosure $\bigoplus_{s=1}^{n-1} \mathcal{N}(D_s') \hookrightarrow \bigoplus_{s=2}^n X_s' / \mathcal{R}(D_s')$ by Theorem 3.3 that is to denseness of $\mathcal{R}(D_n)$ and enclosure $\bigoplus_{s=1}^{n-1} \mathcal{N}(D_s) \hookrightarrow \bigoplus_{s=2}^n X_s' / \mathcal{N}(D_s)'$ by Lemma 2.1 and Corollary 2.3. Now there is only one more appeal to Theorem 2.6 and Lemma 2.10 to conclude the desired. □

By examining the proof of Theorem 3.11 and with regards to Lemma 2.5 we see that the following two are valid.
Theorem 3.12 Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$, are inner regular. Assume further that $N(D_1)$ is (topologically) complemented. Then there exists $A \in \mathcal{B}_n$ so that $T^d_n(A)$ is surjective if and only if
(a) $D_n$ is surjective;
(b) $\mathcal{R}(D_1)$ is closed;
(c) $\bigoplus_{s=2}^{n} N(D_s) \hookrightarrow_c \bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s)$.

Theorem 3.13 Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$, are inner regular. Assume further that $N(D_1)$ is (topologically) complemented. Then there exists $A \in \mathcal{B}_n$ so that $T^d_n(A)$ is right invertible if and only if
(a) $D_n$ is right invertible;
(b) $\mathcal{R}(D_1)$ is closed and complemented;
(c) $\bigoplus_{s=2}^{n} N(D_s) \preceq \bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s)$.

Statements of Theorems 3.2 and 3.11 lead us to a characterization of almost invertible upper triangular operators.

Theorem 3.14 Let $D_1 \in \mathcal{B}(X_1), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$ satisfy the complements condition. Then there exists $A \in \mathcal{B}_n$ such that $T^d_n(A)$ is almost invertible if and only if
(a) $D_1$ is injective;
(b) $D_n$ has dense range;
(c) $\bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s) \cong \bigoplus_{s=2}^{n} N(D_s)$.

Proof. Necessity is immediate from Theorems 3.2 and 3.11. Now we prove sufficiency. Assume that $D_1$ is injective, $D_n$ has dense range, and $\bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s) \cong \bigoplus_{s=2}^{n} N(D_s)$. There exists invertible operator matrix $T$ as in (3.1). Put $T^d_n(A)$ as in (3.2). Injectivity of $T^d_n(A)$ is proved in the same way as in Theorem 3.2. To prove that $T^d_n(A)'$ has dense range, it is enough, by Corollary 2.3, to prove injectivity of $T^d_n(A)'$. Dual operator $T^d_n(A)'$ consists of operator $T'$ in the bottom left corner, and of operators $D_s^{(1)}$, $1 \leq s \leq n$ on the diagonal. However, surjectivity of $T'$ implies injectivity of $T'$, and an obvious denseness of $\mathcal{R}(D_s^{(1)})$, $1 \leq s \leq n - 1$ together with denseness of $\mathcal{R}(D_n)$ imply injectivity of $D_s^{(1)}$, $1 \leq s \leq n$. Therefore, $T^d_n(A)'$ must be injective. □

Now we are able to prove the main result of this paper.
Theorem 3.15  Let $D_1 \in \mathcal{B}(X_1), ..., D_n \in \mathcal{B}(X_n)$. Assume that $D_s$, $2 \leq s \leq n - 1$ are inner regular.

Then there exists $A \in \mathcal{B}_n$ such that $T_n^d(A)$ is invertible if and only if

(a) $D_1$ is left invertible;
(b) $D_n$ is right invertible;
(c) $\bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s) \cong \bigoplus_{s=2}^n \mathcal{N}(D_s)$.

Proof. Necessity part is obvious from Theorem 3.14 and Lemma 2.13. Assume that $D_1$ is left invertible, $D_n$ is right invertible, and $\bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s) \cong \bigoplus_{s=2}^n \mathcal{N}(D_s)$.

Then, according to Theorem 3.14 there exists $A \in \mathcal{B}_n$ such that $T_n^d(A)$ (again given by (3.2)) is almost invertible. To end this proof, it is enough to observe that in our situation all $D_s^{(1)}$, $1 \leq s \leq n$, as in (3.2) have closed range. Thus, $T_n^d(A)$ has closed range as well and so $\mathcal{R}(T_n^d(A)) = \bigoplus_{s=1}^n X_s$. □

Corollary 3.16  Let $D_1 \in \mathcal{B}(X_1), ..., D_n \in \mathcal{B}(X_n)$. Assume that $D_s - \lambda$, $2 \leq s \leq n - 1$, $\lambda \in \mathbb{C}$ are inner regular. Then

$$\bigcap_{A \in \mathcal{B}_n} \sigma(T_n^d(A)) = \sigma_l(D_1) \cup \sigma_r(D_n) \cup \{\lambda \in \mathbb{C} : \bigoplus_{s=2}^n \mathcal{N}(D_s - \lambda) \not\cong \bigoplus_{s=1}^{n-1} X_s/\mathcal{R}(D_s - \lambda)\}.$$

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