On Free Pseudo-Product
Fundamental Graded Lie Algebras

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Abstract. In this paper we first state the classification of the prolongations of complex free fundamental graded Lie algebras. Next we introduce the notion of free pseudo-product fundamental graded Lie algebras and study the prolongations of complex free pseudo-product fundamental graded Lie algebras. Furthermore we investigate the automorphism group of the prolongation of complex free pseudo-product fundamental graded Lie algebras.

Key words: fundamental graded Lie algebra; prolongation; pseudo-product graded Lie algebra

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1 Introduction

Let \( m = \bigoplus_{p<0} g_p \) be a graded Lie algebra over the field \( \mathbb{R} \) of real numbers or the field \( \mathbb{C} \) of complex numbers, and let \( \mu \) be a positive integer. The graded Lie algebra \( m = \bigoplus_{p<0} g_p \) is called a fundamental graded Lie algebra if the following conditions hold: (i) \( m \) is finite-dimensional; (ii) \( g_{-1} \neq \{0\} \), and \( m \) is generated by \( g_{-1} \). Moreover a fundamental graded Lie algebra \( m = \bigoplus_{p<0} g_p \) is said to be of the \( \mu \)-th kind if \( g_{-\mu} \neq \{0\} \), and \( g_p = \{0\} \) for all \( p < -\mu \). It is shown that every fundamental graded algebra \( m = \bigoplus_{p<0} g_p \) is prolonged to a graded Lie algebra \( g(m) = \bigoplus_{p < 0} g(m)_p \) satisfying the following conditions: (i) \( g(m)_p = g_p \) for all \( p < 0 \); (ii) for \( X \in g(m)_p \) \( (p \geq 0) \), \( [X, m] = \{0\} \) implies \( X = 0 \); (iii) \( g(m) \) is maximum among graded Lie algebras satisfying conditions (i) and (ii) above. The graded Lie algebra \( g(m) \) is called the prolongation of \( m \). Note that \( g(m)_0 \) is the Lie algebra of all the derivations of \( m \) as a graded Lie algebra.

Let \( m = \bigoplus_{p<0} g_p \) be a fundamental graded Lie algebra of the \( \mu \)-th kind, where \( \mu \geq 2 \). The fundamental graded Lie algebra \( m \) is called a free fundamental graded Lie algebra of type \((n, \mu)\) if the following universal properties hold:

(i) \( \dim g_{-1} = n \);

(ii) Let \( m' = \bigoplus_{p<0} g'_p \) be a fundamental graded Lie algebra of the \( \mu \)-th kind and let \( \varphi \) be a surjective linear mapping of \( g_{-1} \) onto \( g'_{-1} \). Then \( \varphi \) can be extended uniquely to a graded Lie algebra epimorphism of \( m \) onto \( m' \).

In Section 3 we see that a universal fundamental graded Lie algebra \( b(V, \mu) \) of the \( \mu \)-th kind introduced by N. Tanaka [11] becomes a free fundamental graded Lie algebra of type \((n, \mu)\), where \( \mu \geq 2 \), and \( V \) is a vector space such that \( \dim V = n \geq 2 \).
In [13], B. Warhurst gave the complete list of the prolongations of real free fundamental graded Lie algebras by using a Hall basis of a free Lie algebra. The complex version of his theorem has the completely same form except for the ground number field as follows:

**Theorem I.** Let \( m = \bigoplus_{p<0} g_p \) be a free fundamental graded Lie algebra of type \((n,\mu)\) over \( \mathbb{C} \). Then the prolongation \( g(m) = \bigoplus_{p \in \mathbb{Z}} g(m)_p \) of \( m \) is one of the following types:

(a) \((n,\mu) \neq (n,2)\) \((n \geq 2), (2,3)\). In this case, \( g(m)_1 = \{0\} \).

(b) \((n,\mu) = (n,2)\) \((n \geq 3), (2,3)\). In this case, \( \dim g(m) < \infty \) and \( g(m)_1 \neq \{0\} \). Furthermore, \( g(m) \) is isomorphic to a finite-dimensional simple graded Lie algebra of type \((B_n,\{\alpha_n\})\) \((n \geq 3)\) or \((G_2,\{\alpha_1\})\) \((n = 2)\) (see [15] or Section 5 for the gradations of finite-dimensional simple graded Lie algebras over \( \mathbb{C} \)).

(c) \((n,\mu) = (2,2)\). In this case, \( \dim g(m) = \infty \). Furthermore, \( g(m) \) is isomorphic to the contact algebra \( K(1) \) as a graded Lie algebra.

The first purpose of this paper is to give a proof of Theorem I by using the classification of complex irreducible transitive graded Lie algebras of finite depth (cf. [6]). Note that Warhurst’s methods in [13] are available to the proof of Theorem I.

Next we introduce the notion of free pseudo-product fundamental graded Lie algebras. Let \( m = \bigoplus_{p<0} g_p \) be a fundamental graded Lie algebra, and let \( \mathfrak{e} \) and \( \mathfrak{f} \) be nonzero subspaces of \( g_{-1} \).

Then \( m \) is called a pseudo-product fundamental graded Lie algebra with pseudo-product structure \((\mathfrak{e},\mathfrak{f})\) if the following conditions hold: (i) \( g_{-1} = \mathfrak{e} \oplus \mathfrak{f}; \) (ii) \( [\mathfrak{e},\mathfrak{e}] = [\mathfrak{f},\mathfrak{f}] = \{0\} \) (cf. [10]).

Let \( m = \bigoplus_{p<0} g_p \) be a pseudo-product fundamental graded Lie algebra with a pseudo-product structure \((\mathfrak{e},\mathfrak{f})\), and let \( g(m) = \bigoplus_{p \in \mathbb{Z}} g(m)_p \) be the prolongation of \( m \). Moreover, let \( g_0 \) be the Lie algebra of all the derivations of \( m \) as a graded Lie algebra preserving \( \mathfrak{e} \) and \( \mathfrak{f} \). Also for \( p \geq 1 \) we set \( g_p = \{X \in g(m)_p : [X,g_k] \subset g_{p+k} \text{ for all } k < 0\} \) inductively. Then the direct sum \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) becomes a graded subalgebra of \( g(m) \), which is called the prolongation of \((m;\mathfrak{e},\mathfrak{f})\).

Let \( m = \bigoplus_{p<0} g_p \) be a pseudo-product fundamental graded Lie algebra of the \( \mu \)-th kind with pseudo-product structure \((\mathfrak{e},\mathfrak{f})\), where \( \mu \geq 2 \). The pseudo-product fundamental graded Lie algebra \( m = \bigoplus_{p<0} g_p \) is called a free pseudo-product fundamental graded Lie algebra of type \((m,n,\mu)\) if the following conditions hold:

(i) \( \dim \mathfrak{e} = m \) and \( \dim \mathfrak{f} = n; \)

(ii) Let \( m' = \bigoplus_{p<0} g'_p \) be a pseudo-product fundamental graded Lie algebra of the \( \mu \)-th kind with pseudo-product structure \((\mathfrak{e}',\mathfrak{f}')\) and let \( \varphi \) be a surjective linear mapping of \( g_{-1} \) onto \( g'_{-1} \) such that \( \varphi(\mathfrak{e}) \subset \mathfrak{e}' \) and \( \varphi(\mathfrak{f}) \subset \mathfrak{f}' \). Then \( \varphi \) can be extended uniquely to a graded Lie algebra epimorphism of \( m \) onto \( m' \).

The main purpose of this paper is to prove the following theorem.

**Theorem II.** Let \( m = \bigoplus_{p<0} g_p \) be a free pseudo-product fundamental graded Lie algebra of type \((m,n,\mu)\) with pseudo-product structure \((\mathfrak{e},\mathfrak{f})\) over \( \mathbb{C} \), and let \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) be the prolongation of \((m;\mathfrak{e},\mathfrak{f})\). If \( g_1 \neq \{0\} \), then \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is a finite-dimensional simple graded Lie algebra of type \((A_{m+n},\{\alpha_m,\alpha_{m+1}\})\).
Let \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) be the prolongation of a free pseudo-product fundamental graded Lie algebra \( m = \bigoplus_{p < 0} g_p \) with pseudo-product structure \((\epsilon, f)\) over \( \mathbb{C} \). We denote by \( \text{Aut}(g; \epsilon, f)_0 \) the group of all the automorphisms as a graded Lie algebra preserving \( \epsilon \) and \( f \), which is called the automorphism group of the pseudo-product graded Lie algebra \( g = \bigoplus_{p \in \mathbb{Z}} g_p \). In Section 9, we show that \( \text{Aut}(g; \epsilon, f)_0 \) is isomorphic to \( GL(\epsilon) \times GL(f) \).

**Notation and conventions**

1. From Section 2 to the last section, all vector spaces are considered over the field \( \mathbb{C} \) of complex numbers.
2. Let \( V \) be a vector space and let \( W_1 \) and \( W_2 \) be subspaces of \( V \). We denote by \( W_1 \wedge W_2 \) the subspace of \( \Lambda^2 V \) spanned by all the elements of the form \( w_1 \wedge w_2 \) \((w_1 \in W_1, w_2 \in W_2)\).
3. Graded vector spaces are always \( \mathbb{Z} \)-graded. If we write \( V = \bigoplus_{p < 0} V_p \), then it is understood that \( V_p = \{0\} \) for all \( p \geq 0 \). Let \( V = \bigoplus_{p \in \mathbb{Z}} V_p \) be a graded vector space. We denote by \( V_- \) the subspace \( V = \bigoplus_{p < 0} V_p \). Also for \( k \in \mathbb{Z} \) we denote by \( V_{\leq k} \) the subspace \( \bigoplus_{p \leq k} V_p \).

Let \( V = \bigoplus_{p \in \mathbb{Z}} V_p \) and \( W = \bigoplus_{p \in \mathbb{Z}} W_p \) be graded vector spaces. For \( r \in \mathbb{Z} \), we set

\[
\text{Hom}(V, W)_r = \{ \varphi \in \text{Hom}(V, W) : \varphi(V_p) \subset W_{p+r} \text{ for all } p \in \mathbb{Z} \}.
\]

**2 Free fundamental graded Lie algebras**

First of all we give several definitions about graded Lie algebras. Let \( g \) be a Lie algebra. Assume that there is given a family of subspaces \( (g_p)_{p \in \mathbb{Z}} \) of \( g \) satisfying the following conditions:

(i) \( g = \bigoplus_{p \in \mathbb{Z}} g_p \);

(ii) \( \dim g_p < \infty \) for all \( p \in \mathbb{Z} \);

(iii) \( [g_p, g_q] \subset g_{p+q} \) for all \( p, q \in \mathbb{Z} \).

Under these conditions, we say that \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is a graded Lie algebra (GLA). Moreover we define the notion of homomorphism, isomorphism, monomorphism, epimorphism, subalgebra and ideal for GLAs in an obvious manner.

A GLA \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is called transitive if for \( X \in g_p \) \((p \geq 0)\), \([X, g_-] = \{0\}\) implies \( X = 0 \), where \( g_- \) is the negative part \( \bigoplus_{p < 0} g_p \) of \( g \). Furthermore a GLA \( g = \bigoplus_{p < 0} g_p \) is called irreducible if the \( g_0 \)-module \( g_-1 \) is irreducible.

Let \( \mu \) be a positive integer. A GLA \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is said to be of depth \( \mu \) if \( g_{-\mu} \neq \{0\} \) and \( g_p = \{0\} \) for all \( p < -\mu \).

Next we define fundamental GLAs. A GLA \( m = \bigoplus_{p \leq 0} g_p \) is called a fundamental graded Lie algebra (FGLA) if the following conditions hold:

(i) \( \dim m < \infty \);

(ii) \( g_- \neq \{0\} \), and \( m \) is generated by \( g_- \), or more precisely \( g_{p-1} = [g_p, g_-] \) for all \( p < 0 \).
If an FGLA $m = \bigoplus_{p<0} g_p$ is of depth $\mu$, then $m$ is also said to be of the $\mu$-th kind. Moreover an FGLA $m = \bigoplus_{p<0} g_p$ is called non-degenerate if for $X \in g_{-1}$, $[X, g_{-1}] = \{0\}$ implies $X = 0$.

Let $m = \bigoplus_{p<0} g_p$ be an FGLA of the $\mu$-th kind, where $\mu \geq 2$. $m$ is called a free fundamental graded Lie algebra of type $(n, \mu)$ if the following conditions hold:

(i) $\dim g_{-1} = n$;
(ii) Let $m' = \bigoplus_{p<0} g'_p$ be an FGLA of the $\mu$-th kind and let $\varphi$ be a surjective linear mapping of $g_{-1}$ onto $g'_{-1}$. Then $\varphi$ can be extended uniquely to a GLA epimorphism of $m$ onto $m'$.

**Proposition 2.1.** Let $n$ and $\mu$ be positive integers such that $n, \mu \geq 2$.

(1) There exists a unique free FGLA of type $(n, \mu)$ up to isomorphism.

(2) Let $m = \bigoplus_{p<0} g_p$ be a free FGLA of type $(n, \mu)$. We denote by $\text{Der}(m)_0$ the Lie algebra of all the derivations of $m$ preserving the gradation of $m$. Then the mapping $\Phi : \text{Der}(m)_0 \ni D \mapsto |g_{-1} \in \text{gl}(g_{-1})$ is a Lie algebra isomorphism.

**Proof.** (1) The uniqueness of a free FGLA of type $(n, \mu)$ follows from the definition. We set $X = \{1, \ldots, n\}$. Let $L(X)$ be the free Lie algebra on $X$ (see [1, Chapter II, § 2]) and let $i : X \to L(X)$ be the canonical injection. We define a mapping $\phi$ of $X$ into $\mathbb{Z}$ by $\phi(k) = -1$ ($k \in X$). The mapping $\phi$ defines the natural gradation $(L(X)_p)_{p<0}$ on $L(X)$ such that: (i) $L(X)$ is generated by $L(X)_{-1}$; (ii) $\{i(1), \ldots, i(n)\}$ is a basis of $L(X)_{-1}$ (see [1, Chapter II, § 2, no. 6]). Note that if $n > 1$, then $L(X)_p \neq 0$ for all $p < 0$. We set $a = \bigoplus_{p<0} L(X)_p$; then $a$ is a graded ideal of $L(X)$ and the factor GLA $m = L(X)/a$ becomes an FGLA of the $\mu$-th kind. We put $a_p = a \cap L(X)_p$ and $g_p = L(X)_p/a_p$.

Now we prove that $m = \bigoplus_{p<0} g_p$ is a free FGLA of type $(n, \mu)$. Let $m' = \bigoplus_{p<0} g'_p$ be an FGLA of the $\mu$-th kind and let $\varphi$ be a surjective linear mapping of $g_{-1}$ onto $g'_{-1}$. Let $h$ be a mapping of $X$ into $m'$ defined by $h(k) = \varphi(i(k))$ ($k \in X$). Then there exists an algebra homomorphism $\tilde{h}$ of $L(X)$ into $m'$ such that $\tilde{h} \circ i = h$. Since $L(X)$ (resp. $m'$) is generated by $L(X)_{-1}$ (resp. $g'_{-1}$), $\tilde{h}$ is surjective. Since $m' = \bigoplus_{p<0} g'_p$ is of the $\mu$-th kind, $\tilde{h}(a) = 0$, so $\tilde{h}$ induces a GLA epimorphism $L(\varphi)$ of $m$ onto $m'$ such that $L(\varphi)|g_{-1} = \varphi$. The homomorphism $L(\varphi)$ is unique, because $m = \bigoplus_{p<0} g_p$ is generated by $g_{-1}$. Thus $m$ is a free FGLA of type $(n, \mu)$.

(2) Assume that $m$ is a free FGLA constructed in (1). Let $\phi$ be an endomorphism of $g_{-1}$. By Corollary to Proposition 8 of [1, Chapter II, § 2, no. 8], $\phi$ can be extended uniquely to a unique derivation $D$ of $L(X)$. Since $D(L(X)_{-1}) = \phi(L(X)_{-1}) = \phi(g_{-1}) \subset L(X)_{-1}$, and since $L(X)$ is generated by $L(X)_{-1}$, we see that $D(L(X)_p) \subset L(X)_p$ and $D(a) \subset a$. Thus there is a derivation of $D_\phi$ of $m$ such that $\pi \circ D = D_\phi \circ \pi$, where $\pi$ is the natural projection of $L(X)$ onto $m$. The correspondence $\text{gl}(g_{-1}) \ni \phi \mapsto D_\phi \in \text{Der}(m)_0$ is an injective linear mapping. Hence $\dim \text{gl}(g_{-1}) \leq \dim \text{Der}(m)_0$. On the other hand, since $m$ is generated by $g_{-1}$, the mapping $\Phi$ is a Lie algebra monomorphism. Therefore $\Phi$ is a Lie algebra isomorphism. ■

**Remark 2.1.** Let $n$ and $\mu$ be positive integers with $n, \mu \geq 2$, and let $m = \bigoplus_{p<0} g_p$ be a free FGLA of type $(n, \mu)$. Furthermore let $m' = \bigoplus_{p<0} g'_p$ be an FGLA of the $\mu$-th kind, and let $\varphi$ be a linear mapping of $g_{-1}$ into $g'_{-1}$.

(1) From the proof of Proposition 2.1, there exists a unique GLA homomorphism $L(\varphi)$ of $m$ into $m'$ such that $L(\varphi)|g_{-1} = \varphi$. 

(2) Let \( m'' = \bigoplus_{p<0} g''_p \) be an FGLA of the \( \mu \)-th kind, and let \( \varphi' \) be a linear mapping of \( g'_{-1} \) into \( g''_{-1} \). Assume that \( m' = \bigoplus_{p<0} g'_p \) is a free FGLA. By the uniqueness of \( L(\varphi' \circ \varphi) \), we see that \( L(\varphi' \circ \varphi) = L(\varphi') \circ L(\varphi) \).

(3) Assume that \( m' = \bigoplus_{p<0} g'_p \) is a free FGLA and \( \varphi \) is injective. By the result of (2), \( L(\varphi(\cdot)) \) is a monomorphism.

(4) Let \( W \) be an \( m \)-dimensional subspace of \( g_{-1} \) with \( m \geq 2 \). By the result of (3), the subalgebra of \( m \) generated by \( W \) is a free FGLA of type \( (m, \mu) \).

By Remark 2.1 (4) and [1, Chapter II, § 2, Theorem 1], we get the following lemma.

**Lemma 2.1.** Let \( m = \bigoplus_{p<0} g_p \) be a free FGLA of type \( (n, \mu) \) with \( \mu \geq 3 \). If \( X, Y \) are linearly independent elements of \( g_{-1} \), then

\[
\text{ad}(X)^\mu(Y) = 0, \quad \text{ad}(X)^{\mu-1}(Y) \neq 0, \\
\text{ad}(Y) \text{ad}(X)^{\mu-1}(Y) = 0, \quad \text{ad}(Y) \text{ad}(X)^{\mu-2}(Y) \neq 0.
\]

### 3 Universal fundamental graded Lie algebras

Following N. Tanaka [11], we introduce universal FGLAs of the \( \mu \)-th kind.

Let \( V \) be an \( n \)-dimensional vector space. We define vector spaces \( b(V)_p \) (\( p < 0 \)) and linear mappings \( B_p \) of \( \sum_{r+s=p} b(V)_r \otimes b(V)_s \) into \( b(V)_p \) (\( p \leq -2 \)) as follows: First of all, we put \( b(V)_{-1} = V \) and \( b(V)_{-2} = \Lambda^2 V \). Further we define a mapping \( B_{-2} : b(V)_{-1} \otimes b(V)_{-1} \rightarrow b(V)_{-2} \) to be the identity mapping. For \( k \leq -3 \), we define \( b(V)_k \) and \( B_k \) inductively as follows: We set \( b(V)^{(k+1)} = \bigoplus_{p=-1}^{k+1} b(V)_p \) and we define a subspace \( c(V)_k \) of \( \Lambda^2 (b(V)^{(k+1)}) \) to be \( \sum_{r+s=k} b(V)_r \otimes b(V)_s \). We denote by \( A(V)_k \) the subspace of \( c(V)_k \) spanned by the elements

\[
\mathcal{S}_\bullet \sum_{r+s=k} \sum_{u+v=r} B_r(X_u \wedge Y_v) \wedge Z_s, \quad X, Y, Z \in b(V)^{(k+1)},
\]

where \( \mathcal{S}_\bullet \) stands for the cyclic sum with respect to \( X, Y, Z \), and \( X_u \) denotes the \( b(V)_u \)-component in the decomposition \( b(V)^{(k+1)} = \bigoplus_{p=-1}^{k+1} b(V)_p \). Now we define \( b(V)_k \) to be the factor space \( c(V)_k/A(V)_k \), and \( B_k \) to be the projection of \( c(V)_k \) onto \( b(V)_k \). We put \( b(V) = \bigoplus_{p<0} b(V)_p \) and define a bracket operation \( \{ , \} \) on \( b(V) \) by

\[
[X, Y] = \sum_{p\leq -2} \sum_{r+s=p} B_p(X_r \wedge Y_s)
\]

for all \( X, Y \in b(V) \). Then \( b(V) = \bigoplus_{p<0} b(V)_p \) becomes a GLA generated by \( b(V)_{-1} \), and \( b(V)_p \neq 0 \) for all \( p < 0 \) if \( \dim V > 1 \).

Note that \( b(V)_{-3} \) is isomorphic to \( \Lambda^2(V) \otimes V/\Lambda^3 V \). Let \( \mu \) be a positive integer. Assume that \( \mu \geq 2 \) and \( \dim V = n \geq 2 \). Since \( \bigoplus_{p<\mu} b(V)_p \) is a graded ideal of \( b(V) \), we see that the factor space \( b(V, \mu) = b(V)/\bigoplus_{p<\mu} b(V)_p \) becomes an FGLA of \( \mu \)-th kind, which is called a universal fundamental graded Lie algebra of the \( \mu \)-th kind. By [11, Proposition 3.2], \( b(V, \mu) \) is a free FGLA of type \( (n, \mu) \).
4 The prolongations of fundamental graded Lie algebras

Following N. Tanaka [11], we introduce the prolongations of FGLAs. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA. A GLA $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ is called the prolongation of $\mathfrak{m}$ if the following conditions hold:

(i) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$;
(ii) $\mathfrak{g}(\mathfrak{m})$ is a transitive GLA;
(iii) If $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ is a GLA satisfying conditions (i) and (ii) above, then $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ can be embedded in $\mathfrak{g}(\mathfrak{m})$ as a GLA.

We construct the prolongation $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ of $\mathfrak{m}$. We set $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ ($p < 0$). We define subspaces $\mathfrak{g}(\mathfrak{m})_k$ ($k \geq 0$) of $\text{Hom}(\mathfrak{m}, \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p)$ and a bracket operation on $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ inductively. First $\mathfrak{g}(\mathfrak{m})_0$ is defined to be $\text{Der}(\mathfrak{m})_0$ and a bracket operation $[ , ] : \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p \times \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p \rightarrow \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p$ is defined by

$[X,Y] = -[Y,X] = X(Y), \quad X \in \mathfrak{g}(\mathfrak{m})_0, \quad Y \in \mathfrak{m}$,

$[X,Y] = XY - YX, \quad X, Y \in \mathfrak{g}(\mathfrak{m})_0$.

Next for $k > 0$ we define $\mathfrak{g}(\mathfrak{m})_k$ ($k \geq 1$) inductively as follows:

$\mathfrak{g}(\mathfrak{m})_k = \left\{ X \in \text{Hom}(\mathfrak{m}, \bigoplus_{p \leq k-1} \mathfrak{g}(\mathfrak{m})_p) : X([u,v]) = [X(u), v] + [u, X(v)] \text{ for all } u, v \in \mathfrak{m} \right\}$,

where for $X \in \mathfrak{g}(\mathfrak{m})_p$, $u \in \mathfrak{m}$, we set $[X,u] = -[u,X] = X(u)$. Further for $X \in \mathfrak{g}(\mathfrak{m})_k$, $Y \in \mathfrak{g}(\mathfrak{m})_l$ ($k, l \geq 0$), by induction on $k + l \geq 0$, we define $[X,Y] \in \text{Hom}(\mathfrak{m}, \mathfrak{g}(\mathfrak{m}))_{k+l}$ by

$[X,Y](u) = [X, [Y,u]] - [Y, [X,u]], \quad u \in \mathfrak{m}$.

It follows easily that $[X,Y] \in \mathfrak{g}(\mathfrak{m})_{k+l}$. With this bracket operation, $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ becomes a graded Lie algebra satisfying conditions (i), (ii) and (iii) above.

Let $\mathfrak{m}$ and $\mathfrak{g}(\mathfrak{m})$ be as above. Assume that we are given a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}(\mathfrak{m})_0$. We define subspaces $\mathfrak{g}_k$ ($k \geq 1$) of $\mathfrak{g}(\mathfrak{m})_k$ inductively as follows:

$\mathfrak{g}_k = \{ X \in \mathfrak{g}(\mathfrak{m})_k : [X, \mathfrak{g}_p] \subset \mathfrak{g}_{p+k} \text{ for all } p < 0 \}$.

If we put $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ becomes a transitive graded Lie subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

By Proposition 2.1 (2) we get the following proposition.

**Proposition 4.1.** Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free FGLA and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of $\mathfrak{m}$. Then the mapping $\mathfrak{g}(\mathfrak{m})_0 \ni D \mapsto D|_{\mathfrak{g}_{-1}} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ is an isomorphism.

Conversely we obtain the following proposition.

**Proposition 4.2.** Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA of the $\mu$-th kind and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of $\mathfrak{m}$. Assume that $\mathfrak{g}(\mathfrak{m})_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{g}_{-1})$. If $\mu = 2$ or $\mu = 3$, then $\mathfrak{m}$ is a free FGLA.
Proof. We put \( n = \dim \mathfrak{g}_{-1} \). We consider a universal FGLA \( b(\mathfrak{g}_{-1}, \mu) = \bigoplus_{p<0} b(\mathfrak{g}_{-1}, \mu)_p \) of the \( \mu \)-th kind. Since \( b(\mathfrak{g}_{-1}, \mu) \) is a free FGLA of type \((n, \mu)\), there exists a GLA epimorphism \( \varphi \) of \( b(\mathfrak{g}_{-1}, \mu) \) onto \( \mathfrak{m} \) such that the restriction \( \varphi | b(\mathfrak{g}_{-1}, \mu)_{-1} \) is the identity mapping. Let \( \tilde{b}(\mathfrak{g}_{-1}, \mu) = \bigoplus_{p \in \mathbb{Z}} b(\mathfrak{g}_{-1}, \mu)_p \) be the prolongation of \( b(\mathfrak{g}_{-1}, \mu) \). Since the mapping \( \mathfrak{g}(\mathfrak{m})_0 \ni D \mapsto D| \mathfrak{g}_{-1} \in \mathfrak{gl}(\mathfrak{g}_{-1}) \) is an isomorphism, \( \varphi \) can be extended to be a homomorphism \( \tilde{\varphi} \) of \( \bigoplus_{p \leq 0} \tilde{b}(\mathfrak{g}_{-1}, \mu)_p \) onto \( \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p \).

Let \( \mathfrak{a} \) be the kernel of \( \tilde{\varphi} \); then \( \mathfrak{a} \) is a graded ideal of \( \bigoplus_{p \leq 0} \tilde{b}(\mathfrak{g}_{-1}, \mu)_p \). We set \( \mathfrak{a}_p = \mathfrak{a} \cap \tilde{b}(\mathfrak{g}_{-1}, \mu)_p \); then \( \mathfrak{a} = \bigoplus_{p \leq 0} \mathfrak{a}_p \). Since the restriction of \( \tilde{\varphi} \) to \( \tilde{b}(\mathfrak{g}_{-1}, \mu)_{-1} \) is injective, \( \mathfrak{a}_p = \{0\} \) for \( p \geq -1 \).

Also each \( \mathfrak{a}_p \) is a \( \tilde{b}(\mathfrak{g}_{-1}, \mu)_0 \)-submodule of \( \tilde{b}(\mathfrak{g}_{-1}, \mu)_p \). From the construction of \( b(\mathfrak{g}_{-1}, \mu) \), we see that \( b(\mathfrak{g}_{-1}, \mu)_{-2} \) (resp. \( b(\mathfrak{g}_{-1}, \mu)_{-3} \)) is isomorphic to \( \Lambda^2(\mathfrak{g}_{-1}) \) (resp. \( \Lambda^2(\mathfrak{g}_{-1}) \otimes \mathfrak{g}_{-1}/\Lambda^3(\mathfrak{g}_{-1}) \)) as a \( \tilde{b}(\mathfrak{g}_{-1}, \mu)_0 \)-module. By the table of [8], \( \Lambda^2(\mathfrak{g}_{-1}) \) and \( \Lambda^2(\mathfrak{g}_{-1}) \otimes \mathfrak{g}_{-1}/\Lambda^3(\mathfrak{g}_{-1}) \) are irreducible \( \mathfrak{gl}(\mathfrak{g}_{-1}) \)-modules. Thus we see that \( \mathfrak{a}_{-2} = \mathfrak{a}_{-3} = \{0\} \). From \( \mu \leq 3 \) it follows that \( \varphi \) is an isomorphism.

\section{Finite-dimensional simple graded Lie algebras}

Following [15], we first state the classification of finite-dimensional simple GLAs.

Let \( \mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \) be a finite-dimensional simple GLA of the \( \mu \)-th kind over \( \mathbb{C} \) such that the negative part \( \mathfrak{g}_{-} \) is an FGLA. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g}_0 \); then \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \) such that \( E \in \mathfrak{h} \), where \( E \) is the element of \( \mathfrak{g}_0 \) such that \( [E, x] = px \) for all \( x \in \mathfrak{g}_p \) and \( p \).

Let \( \Delta \) be a root system of \( (\mathfrak{g}, \mathfrak{h}) \). For \( \alpha \in \Delta \), we denote by \( \mathfrak{g}^\alpha \) the root space corresponding to \( \alpha \). We set \( \mathfrak{h}_R = \{ h \in \mathfrak{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in \Delta \} \) and let \( (h_1, \ldots, h_l) \) be a basis of \( \mathfrak{h}_R \) such that \( h_1 = E \). We define the set of positive roots \( \Delta^+ \) as the set of roots which are positive with respect to the lexicographical ordering in \( \mathfrak{h}_R^* \) determined by the basis \( (h_1, \ldots, h_l) \) of \( \mathfrak{h}_R \). Let \( \Pi \subset \Delta^+ \) be the corresponding simple root system. We denote by \( \{m_1, \ldots, m_l\} \) the coordinate functions corresponding to \( \Pi \), i.e., for \( \alpha \in \Delta \), we can write \( \alpha = \sum_{i=1}^l m_i(\alpha)\alpha_i \).

We set \( \alpha_i(E) = s_i \) and \( s = (s_1, \ldots, s_l) \); then each \( s_i \) is a non-negative integer. For \( \alpha \in \Delta \), we call the integer \( \ell_s(\alpha) = \sum_{i=1}^l m_i(\alpha)s_i \) the \( s \)-length of \( \alpha \). We put \( \Delta_p = \{ \alpha \in \Delta : \ell_s(\alpha) = p \} \) and \( I = \{ i \in \{1, \ldots, l\} : s_i = 1 \} \). Let \( \theta \) be the highest root of \( \mathfrak{g} \); then \( \ell_s(\theta) = \mu \).

Also since the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_{-\mu} \) is irreducible, \( \dim \mathfrak{g}_{-\mu} = 1 \) if and only if \( (\theta, \alpha_i^\vee) = 0 \) for all \( i \in \{1, \ldots, l\} \setminus I \), where \( \alpha_i^\vee \) is the simple root system of the dual root system \( \Delta^\vee \) of \( \Delta \) corresponding to \( \{\alpha_i\} \). In our situation, since \( \mathfrak{g}_{-} \) is generated by \( \mathfrak{g}_{-1} \), we have \( s_i = 0 \) or 1 for all \( i \). The \( l \)-tuple \( s = (s_1, \ldots, s_l) \) of non-negative integers is determined only by the ordering of \( (\alpha_1, \ldots, \alpha_l) \). In what follows, we assume that the ordering of \( (\alpha_1, \ldots, \alpha_l) \) is as in the table of [2].

If \( \mathfrak{g} \) has the Dynkin diagram of type \( X_l \) (\( X = A, \ldots, G \)), then the simple GLA \( \mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \) is said to be of type \( (X_l, \Pi_1) \). Here we remark that for an automorphism \( \tilde{\mu} \) of the Dynkin diagram, a simple GLA of type \( (X_l, \Pi_1) \) is isomorphic to that of type \( (X_l, \tilde{\mu}(\Pi_1)) \). We will identify a simple GLA of type \( (X_l, \Pi_1) \) with that of type \( (X_l, \tilde{\mu}(\Pi_1)) \).

For \( i \in I \), we put \( \Delta_p^{(i)} = \{ \alpha \in \Delta : m_i(\alpha) = p \text{ and } m_j(\alpha) = 0 \text{ for } j \in I \setminus \{i\} \} \) and \( \mathfrak{g}_p^{(i)} = \sum_{\alpha \in \Delta_p^{(i)}} \mathfrak{g}_\alpha \); then \( \mathfrak{g}_0^{(i)} \) is an irreducible \( \mathfrak{g}_0 \)-submodule of \( \mathfrak{g}_{-1} \) with highest weight \(-\alpha_i \). In particular, if the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_{-1} \) is irreducible, then \( \#(I) = 1 \).
For $i \in I$, we denote by $g^{(i)}$ the subalgebra of $g$ generated by $g^{(i)}_{-1} \oplus g^{(i)}_1$; then $g^{(i)}$ is a simple GLA whose Dynkin diagram is the connected component containing the vertex $i$ of the subdiagram of $X_l$ corresponding to vertices $(\{1, \ldots, l\} \setminus I) \cup \{i\}$. We denote by $\theta^{(i)}$ the highest root of $g^{(i)}$. Then $[g^{(i)}_{-1}, g^{(i)}_1] = \{0\}$ if and only if $m_i(\theta^{(i)}) = 1$.

From Theorem 5.2 of [15], we obtain the following theorem:

**Theorem 5.1.** Let $g = \bigoplus_{p \in \mathbb{Z}} g_p$ be a finite-dimensional simple GLA over $\mathbb{C}$ such that $g_-$ is an FGLA and the $g_0$-module $g_{-1}$ is irreducible. Then $g = \bigoplus_{p \in \mathbb{Z}} g_p$ is the prolongation of $g_-$ except for the following cases:

(a) $g_-$ is of the first kind;
(b) $g_-$ is of the second kind and $\dim g_{-2} = 1$.

Let $g = \bigoplus_{p \in \mathbb{Z}} g_p$ be a finite-dimensional simple GLA. Now we assume that $g_0$ is isomorphic to $gl(g_{-1})$; then the $g_0$-module $g_{-1}$ is irreducible. The derived subalgebra $[g_0, g_0]$ of $g_0$ is a semisimple Lie algebra whose Dynkin diagram is the subdiagram of $X_l$ consisting of the vertices $\{1, \ldots, l\} \setminus I$. Since $[g_0, g_0]$ is of type $A_{l-1}$ and since the $g_0$-module $g_{-1}$ is elementary, $(X_l, \Delta_1)$ is one of the following cases:

$$(A_l, \{\alpha_l\}), \quad (B_l, \{\alpha_l\}), \quad l \geq 2, \quad (G_2, \{\alpha_1\}).$$

From this result and Propositions 4.1 and 4.2, we get the following theorem:

**Theorem 5.2.** Let $g = \bigoplus_{p \in \mathbb{Z}} g_p$ be a finite-dimensional simple GLA of type $(X_l, \Pi_1)$ over $\mathbb{C}$ satisfying the following conditions:

(i) $g_-$ is an FGLA of the $\mu$-th kind;
(ii) The $g_0$-module $g_{-1}$ is irreducible;
(iii) $g_0$ is isomorphic to $gl(g_{-1})$;
(iv) $g$ is the prolongation of $g_-$.

Then $g_-$ is a free FGLA of type $(l, \mu)$, and $g = \bigoplus_{p \in \mathbb{Z}} g_p$ is one of the following types:

(a) $l \geq 3$, $\mu = 2$, $(X_l, \Pi_1) = (B_l, \{\alpha_l\})$.
(b) $l = 2$, $\mu = 3$, $(X_l, \Pi_1) = (G_2, \{\alpha_1\})$.

6 Graded Lie algebras $W(n)$, $K(n)$ of Cartan type

In this section, following V.G. Kac [3], we describe Lie algebras $W(n)$, $K(n)$ of Cartan type and their standard gradations.

Let $A(m)$ denote the monoid (under addition) of all $m$-tuples of non-negative integers. For an $m$-tuple $s = (s_1, \ldots, s_m)$ of positive integers and $\alpha = (\alpha_1, \ldots, \alpha_m) \in A(m)$ we set $\|\alpha\|_s = \sum_{i=1}^m s_i \alpha_i$. Also we denote the $m$-tuple $(1, \ldots, 1)$ by $1_m$ and we denote the $(m+1)$-tuple $(1, \ldots, 1, 2)$ by $(1_m, 2)$. Let $A(m) = \mathbb{C}[x_1, \ldots, x_m]$. For any $m$-tuple $s$ of positive integers, we denote by $A(m; s)_p$ the subspace of $A(m)$ spanned by polynomials

$$x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}, \quad \alpha = (\alpha_1, \ldots, \alpha_m) \in A(m), \quad \|\alpha\|_s = p.$$
Let $W(m)$ be the Lie algebra consisting of all the polynomial vector fields

$$
\sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i}, \quad P_i \in \mathfrak{A}(m).
$$

(6.1)

For an $m$-tuple $s = (s_1, \ldots, s_m)$ of positive integers, we denote by $W(m; s)_p$ the subspaces of $W(m)$ consisting of those polynomial vector fields (6.1) such that the polynomials $P_i$ are contained in $\mathfrak{A}(m; s)_{p+s_i}$; then $W(m; s) = \bigoplus_{p \in \mathbb{Z}} W(m; s)_p$ is a transitive GLA. In particular, $W(m; 1_m) = \bigoplus_{p \geq 0} W(m; 1_m)_p$ is a transitive irreducible GLA such that: (i) $W(m; 1_m)_0$ is isomorphic to $\mathfrak{gl}(m, \mathbb{C})$; (ii) the $W(m; 1_m)_0$-module $W(m; 1_m)_{-1}$ is elementary; (iii) $W(m; 1_m)$ is the prolongation of $W(m; 1_m)_{-1}$.

We now consider the following differential form

$$\omega_K = dx_{2n+1} - \sum_{i=1}^{n} x_{i+n} dx_i.$$

Define

$$K(n) = \{ D \in W(2n+1) : D\omega_K \in \mathfrak{A}(2n+1)\omega_K \}.$$

(Here the action of $D$ on the differential forms is extended from its action $\mathfrak{A}(2n+1)$ by requiring that $D$ be derivation of the exterior algebra satisfying $D(df) = d(Df)$, where $df = \sum \frac{\partial f}{\partial x_i} dx_i$, $f \in \mathfrak{A}(m)$. We set $K(n)_p = W(2n+1; (1_{2n+2})_p \cap K(n)$. Then $K(n) = \bigoplus_{p \geq 0} K(n)_p$ is a transitive irreducible GLA such that: (i) $K(n)_0$ is isomorphic to $\mathfrak{osp}(n, \mathbb{C})$; (ii) the $K(n)_0$-module $K(n)_{-1}$ is elementary; (iii) $K(n)$ is the prolongation of $K(n)_{-1}$ (cf. [3, 5]).

From Proposition 2.2 of [6], we get

**Theorem 6.1.** Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a transitive GLA over $\mathbb{C}$ satisfying the following conditions:

(i) $\mathfrak{g}_-$ is an FGLA of the $\mu$-th kind;

(ii) $\mathfrak{g}$ is infinite-dimensional;

(iii) $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible;

(iv) $\mathfrak{g}$ is the prolongation of $\mathfrak{g}_-$.

Then $\mu \leq 2$ and $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is isomorphic to $W(m; 1_m)$ or $K(n)$.

### 7 Classification of the prolongations

of free fundamental graded Lie algebras

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA of type $(n, \mu)$ over $\mathbb{C}$, and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of $\mathfrak{m}$. First of all, we assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. By Theorem 6.1, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(m)$ as a GLA, where $n = 2m$. Since $K(m)_0$ is isomorphic to $\mathfrak{osp}(m, \mathbb{C})$ and since $\mathfrak{g}(\mathfrak{m})_0$ is isomorphic to $\mathfrak{gl}(n, \mathbb{C})$, we see that $m = 1$. Therefore $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(1)$ as a GLA.

Next we assume that $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ and $\mathfrak{g}(\mathfrak{m})_1 \neq 0$. Since the $\mathfrak{g}(\mathfrak{m})_0$-module $\mathfrak{g}(\mathfrak{m})_{-1}$ is irreducible, $\mathfrak{g}(\mathfrak{m})$ is a finite-dimensional simple GLA (see [4, 7]). By Theorem 5.2, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to one of the following types:

$$(B_l, \{a_l\}) \quad l \geq 3, \quad (G_2, \{a_1\}).$$

Thus we get a proof of the following theorem:
Theorem 7.1. Let \( m = \bigoplus_{p<0} g_p \) be a free FGLA of type \((n, \mu)\) over \(\mathbb{C}\), and let \( g(m) = \bigoplus_{\mu \in \mathbb{Z}} g(m)_\mu \) be the prolongation of \( m \). Then one of the following cases occurs:

(a) \((n, \mu) \neq (n, 2) (n \geq 2), (2, 3)\). In this case, \( g(m)_1 = \{0\} \).

(b) \((n, \mu) = (n, 2) (n \geq 3), (2, 3)\). In this case, \( \dim g(m) < \infty \) and \( g(m)_1 \neq \{0\} \). Furthermore \( g(m) \) is isomorphic to a finite-dimensional simple GLA of type \((B_n, \{\alpha_n\}) (n \geq 3)\) or \((G_2, \{\alpha_1\}) (n = 2)\).

(c) \((n, \mu) = (2, 2)\). In this case, \( \dim g(m) = \infty \). Furthermore, \( g(m) \) is isomorphic to \( K(1) \) as a GLA.

8 Free pseudo-product fundamental graded Lie algebras

An FGLA \( m = \bigoplus_{p<0} g_p \) equipped with nonzero subspaces \( \epsilon, f \) of \( g_{-1} \) is called a pseudo-product FGLA if the following conditions hold:

(i) \( g_{-1} = \epsilon \oplus f \);
(ii) \( [\epsilon, \epsilon] = [f, f] = \{0\} \).

The pair \((\epsilon, f)\) is called the pseudo-product structure of the pseudo-product FGLA \( m = \bigoplus_{p<0} g_p \).

We will also denote by the triplet \((m; \epsilon, f)\) the pseudo-product FGLA \( m = \bigoplus_{p<0} g_p \) with pseudo-product structure \((\epsilon, f)\). Let \( m = \bigoplus_{p<0} g_p \) (resp. \( m' = \bigoplus_{p<0} g'_p \)) be a pseudo-product FGLA with pseudo-product structure \((\epsilon, f)\) (resp. \((\epsilon', f')\)). We say that two pseudo-product FGLAs \((m; \epsilon, f)\) and \((m'; \epsilon', f')\) are isomorphic if there exists a GLA isomorphism \( \varphi \) of \( m \) onto \( m' \) such that \( \varphi(\epsilon) = \epsilon' \) and \( \varphi(f) = f' \).

Proposition 8.1. Let \( m = \bigoplus_{p<0} g_p \) be a pseudo-product FGLA of the \( \mu \)-th kind with pseudo-product structure \((\epsilon, f)\). If \( m \) is a free FGLA of type \((n, \mu)\), then \( n = 2 \).

Proof. Let \((e_1, \ldots, e_m)\) (resp. \((f_1, \ldots, f_l)\)) be a basis of \( \epsilon \) (resp. \( f \)). Since \([\epsilon, f] = g_{-2}\), the space \( g_{-2} \) is generated by \( \{[e_i, f_j] : i = 1, \ldots, m, j = 1, \ldots, l\} \) as a vector space, so \( \dim g_{-2} \leq ml \).

On the other hand, since \( m \) is a free FGLA,

\[
\dim g_{-2} = \dim b(g_{-1}, \mu)_{-2} = \dim A^2(g_{-1}) = \frac{(m+l)(m+l-1)}{2},
\]

so \( ml \geq \frac{(m+l)(m+l-1)}{2} \). From this fact it follows that \( m = l = 1 \).

Let \( m = \bigoplus_{p<0} g_p \) be a pseudo-product FGLA of the \( \mu \)-th kind with pseudo-product structure \((\epsilon, f)\), where \( \mu \geq 2 \). \( m \) is called a free pseudo-product FGLA of type \((m, n, \mu)\) if the following conditions hold:

(i) \( \dim \epsilon = m \) and \( \dim f = n \);

(ii) Let \( m' = \bigoplus_{p<0} g'_p \) be a pseudo-product FGLA of the \( \mu \)-th kind with pseudo-product structure \((\epsilon', f')\) and let \( \varphi \) be a surjective linear mapping of \( g_{-1} \) onto \( g'_{-1} \) such that \( \varphi(\epsilon) \subset \epsilon' \) and \( \varphi(f) \subset f' \). Then \( \varphi \) can be extended uniquely to a GLA epimorphism of \( m \) onto \( m' \).

Proposition 8.2. Let \( m, n \) and \( \mu \) be positive integers such that \( \mu \geq 2 \).
(1) There exists a unique free pseudo-product FGLA of type \((m,n,\mu)\) up to isomorphism.

(2) Let \(m = \bigoplus_{p<0} g_p\) be a free pseudo-product FGLA of type \((m,n,\mu)\) with pseudo-product structure \((\epsilon, f)\). We denote by \(\text{Der}(m; \epsilon, f)_0\) the Lie algebra of all the derivations of \(m\) preserving the gradation of \(m\), \(\epsilon\) and \(f\). Then the mapping \(\Phi : \text{Der}(m; \epsilon, f)_0 \ni D \mapsto (D|\epsilon, D|f) \in \mathfrak{gl}(\epsilon) \times \mathfrak{gl}(f)\) is a Lie algebra isomorphism.

**Proof.** (1) The uniqueness of a free pseudo-product FGLA of type \((m,n,\mu)\) follows from the definition. Let \(V\) be an \((m+n)\)-dimensional vector space and let \(\epsilon, f\) be subspaces of \(V\) such that \(V = \epsilon \oplus f\), \(\dim \epsilon = m\) and \(\dim f = n\). Let \(a = \bigoplus_{p<0} a_p\) be the graded ideal of \(b(V,\mu)\) generated by \([\epsilon, \epsilon] + [f, f]\). We set \(m = b(V,\mu)/a\), \(g_p = b(V,\mu)_p/a_p\). Clearly \(m = \bigoplus_{p<0} g_p\) is a pseudo-product FGLA. We show that the factor algebra \(m\) is a free pseudo-product FGLA of type \((m,n,\mu)\). First we prove that \(m\) is of the \(\mu\)-th kind. Let \(n = \bigoplus_{p<0} g''_p\) be a free FGLA of type \((2,\mu)\) and let \(\epsilon''\) and \(f''\) be one-dimensional subspaces of \(g''_{-1}\) such that \(g''_{-1} = \epsilon'' \oplus f''\). Let \(\varphi_1\) be an injective linear mapping of \(g''_{-1}\) into \(V\) such that \(\varphi_1(\epsilon'') \subset \epsilon\) and \(\varphi_1(f'') \subset f\). Let \(\varphi_2\) be a linear mapping of \(V\) into \(g''_{-1}\) such that \(\varphi_2 \circ \varphi_1 = 1_{g''_{-1}}, \varphi_2(\epsilon) = \epsilon''\) and \(\varphi_2(f) = f''\).

There exists a homomorphism \(L(\varphi_1)\) (resp. \(L(\varphi_2)\)) of \(n\) (resp. \(b(V,\mu)\)) into \(b(V,\mu)\) (resp. \(n\)) such that \(L(\varphi_1)(\epsilon''_{-1}) = \varphi_1(\epsilon'')\) (resp. \(L(\varphi_2)(V) = \varphi_2\)). Since \(L(\varphi_2)([\epsilon, f] + [f, f]) = \{0\}\), \(L(\varphi_2)\) induces a homomorphism \(\hat{L}(\varphi_2)\) of \(m\) into \(n\) such that \(\varphi_2 = \hat{L}(\varphi_2) \circ \pi\), where \(\pi\) is the natural projection of \(b(V,\mu)\) onto \(m\). Since \(1_n = L(\varphi_2) \circ L(\varphi_1) = \hat{L}(\varphi_2) \circ \pi \circ L(\varphi_1),\)

\(\pi \circ L(\varphi_1)\) is a monomorphism of \(n\) into \(m\), so \(g_{-\mu} \neq \{0\}\). Thus \(m\) is of the \(\mu\)-th kind. Let \(m' = \bigoplus_{p<0} g'_p\) be a pseudo-product FGLA of the \(\mu\)-th kind with pseudo-product structure \((\epsilon', f')\) and let \(\phi\) be a surjective linear mapping of \(b(V,\mu)_{-1}\) onto \(g'_{-1}\) such that \(\phi(\epsilon) \subset \epsilon'\) and \(\phi(f) \subset f'\).

By the definition of a free FGLA, there exists a GLA epimorphism \(L(\phi)\) of \(b(V,\mu)\) onto \(m'\) such that \(L(\phi)b(V,\mu)_{-1} = \phi\). Since \(L(\phi)([\epsilon, f] + [f, f]) \subset [\epsilon', f'] + [f', f'] = \{0\}\), we see that \(L(\phi)(a) = \{0\}\), so the epimorphism \(L(\phi)\) induces a GLA epimorphism \(\hat{L}(\phi)\) of \(m\) onto \(m'\) such that \(\hat{L}(\phi)|g_{-1} = \phi\).

(2) We may prove the fact that the mapping \(\Phi\) is surjective. Let \(\phi\) be an endomorphism of \(g_{-1}\) such that \(\phi(\epsilon) \subset \epsilon\) and \(\phi(f) \subset f\). By Proposition 2.1 (2), there exists a \(D \in \text{Der}(b(V,\mu))_0\) such that \(D(b(V,\mu)_{-1}) = \phi\). Since \(D([\epsilon, f] + [f, f]) \subset [\epsilon, f] + [f, f], D\) induces a derivation \(\hat{D}\) of \(m\) such that \(\hat{D}|g_{-1} = \phi\). ■

**Remark 8.1.** Let \(m, n, m', n'\) and \(\mu\) be positive integers with \(\mu \geq 2\), and let \(m = \bigoplus_{p<0} g_p\) (resp. \(m' = \bigoplus_{p<0} g'_p\)) be a free pseudo-product FGLA of type \((m,n,\mu)\) (resp. \((m',n',\mu)\)) with pseudo-product structure \((\epsilon, f)\) (resp. \((\epsilon', f')\)). Furthermore let \(\varphi\) be a linear mapping of \(g_{-1}\) into \(g'_{-1}\) such that \(\varphi(\epsilon) \subset \epsilon'\) and \(\varphi(f) \subset f'\).

(1) From the proof of Proposition 8.2, there exists a unique GLA homomorphism \(\hat{L}(\varphi)\) of \(m\) into \(m'\) such that \(\hat{L}(\varphi)|g_{-1} = \varphi\). If \(\varphi\) is injective, then \(\hat{L}(\varphi)\) is a monomorphism.

(2) Assume that \(m = n = 1\) and \(\varphi\) is injective. Then \(\hat{L}(\varphi)(m)\) is a graded subalgebra of \(m'\) isomorphic to a free FGLA of type \((2,\mu)\). From this result, the subalgebra of \(m'\) generated by a nonzero element \(X\) of \(\epsilon'\) and a nonzero element \(Y\) of \(f'\) is a free FGLA of type \((2,\mu)\).

Let \(m = \bigoplus_{p<0} g_p\) be a pseudo-product FGLA of the \(\mu\)-th kind with pseudo-product structure \((\epsilon, f)\). We denote by \(g_0\) the Lie algebra of all the derivations of \(m\) preserving the gradation
of \( m, c \) and \( f \):

\[
g_0 = \{ D \in \text{Der}(g)_0 : D(c) \subset c, D(f) \subset f \}.
\]

The prolongation \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) of \((m, g_0)\) is called the prolongation of \((m; c, f)\).

A transitive GLA \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is called a pseudo-product GLA if there are given nonzero subspaces \( c \) and \( f \) of \( g_{-1} \) satisfying the following conditions:

(i) The negative part \( g_- \) is a pseudo-product FGLA with pseudo-product structure \((c, f)\);

(ii) \([g_0, c] \subset c \) and \([g_0, f] \subset f \).

The pair \((c, f)\) is called the pseudo-product structure of the pseudo-product GLA \( g = \bigoplus_{p \in \mathbb{Z}} g_p \). If the \( g_0\)-modules \( c \) and \( f \) are irreducible, then the pseudo-product GLA \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is said to be of irreducible type.

The following lemma is due to N. Tanaka (cf. \[9\]). Here we give a proof for the convenience of the readers.

**Lemma 8.1.** Let \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) be a pseudo-product GLA of depth \( \mu \) with pseudo-product structure \((c, f)\).

1. If \( g_- \) is non-degenerate, then \( g \) is finite-dimensional.
2. If \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) is of irreducible type and \( \mu \geq 2 \), then \( g \) is finite-dimensional.

**Proof.** (1) The proof is essentially due to the proof of [11, Corollary 3 to Theorem 11.1]. For \( p \in \mathbb{Z} \), we set \( h_p = \{ X \in g_p : [X, g_{-2}] = \{0\} \} \). We define \( I \in \text{gl}(g_{-1}) \) as follows: \( I(x) = -\sqrt{-1}x \) for \( x \in c \), \( I(x) = \sqrt{-1}x \) for \( x \in f \). Then \( I^2 = -1 \), \( I([a, x]) = [a, I(x)] \) and \([I(x), I(y)] = [x, y] \) for \( a \in g_0 \) and \( x, y \in g_{-1} \). We put \( \langle x, y \rangle = \langle I(x), y \rangle \) for \( x, y \in g_{-1} \). Then \( \langle x, y \rangle = \langle y, x \rangle \), and for \( x \in g_{-1}, \langle x, g_{-1} \rangle = \{0\} \) implies \( x = 0 \), since \( g_- \) is non-degenerate. Also \( \langle [a, x], y \rangle + \langle x, [a, y] \rangle = 0 \) and \([b, x], y] = [b, y], x] \) for \( a \in h_0 \), \( b \in h_1 \) and \( x, y \in g_{-1} \). Then, for \( b \in h_1 \), \( x, y, z \in g_{-1} \), we have \( \langle [b, x], y], z \rangle = -\langle y, [b, x], z] \rangle = -\langle y, [b, z], x] \rangle = \langle [b, y], z], x \rangle = -\langle b, y], x]], z \rangle = -\langle [b, x], y], z \rangle \) for \( \langle [b, x], y], z \rangle = 0 \). By transitivity of \( g \), \( h_1 = \{0\} \). Therefore by [11, Corollary 1 to Theorem 11.1], \( g \) is finite-dimensional.

(2) We may assume that \( g_1 \neq \{0\} \). By [16, Lemma 2.4], the \( g_0\)-modules \( c, f \) are not isomorphic to each other. We put \( d = \{ X \in g_{-1} : [X, g_{-1}] = \{0\} \} \); then \( d \) is a \( g_0\)-submodule of \( g_{-1} \). Hence \( d = \{0\} \), \( d = e \), \( d = f \) or \( d = g_{-1} \). If \( d = \{0\} \), then \( g_{-2} = \{c, f \} = \{0\} \), which is a contradiction. Thus \( g_- \) is non-degenerate. By (1), \( g \) is finite-dimensional. \( \blacksquare \)

The prolongation of a pseudo-product FGLA becomes a pseudo-product GLA. By Proposition 8.2 (2), the prolongation of a free pseudo-product FGLA is a pseudo-product GLA of irreducible type. By Lemma 8.1 (2), the prolongation of a free pseudo-product FGLA is finite-dimensional.

**Proposition 8.3.** Let \( m = \bigoplus_{p < 0} g_p \) be a free pseudo-product FGLA of type \((m, n, \mu)\) with pseudo-product structure \((c, f)\) and let \( g = \bigoplus_{p \in \mathbb{Z}} g_p \) be the prolongation of \((m; c, f)\).

1. \( g_0 \) is isomorphic to \( gl(c) \oplus gl(f) \) as a Lie algebra.
2. \( g_{-2} \) is isomorphic to \( c \otimes f \) as a \( g_0\)-module. In particular, \( \dim g_{-2} = mn \).
Proof. (1) This follows from Proposition 8.2 (2).

(2) Let $a = \bigoplus_{p<0} a_p$ be the graded ideal of $b(g_{-1}, \mu)$ generated by $[e, e] + [f, f]$. By the construction of $b(g_{-1}, \mu) = -2$, $a_{-2}$ is isomorphic to $\Lambda^2(e) \oplus \Lambda^2(f)$, so $g_{-2} = b(g_{-1}, \mu)/a_{-2}$ is isomorphic to $e \otimes f$.

(3) By the construction of $b(g_{-1}, \mu) = -3$, $b(g_{-1}, \mu)/a_{-3}$ is isomorphic to

$$(e \otimes f) \otimes \Lambda^2(e) \otimes f \otimes f / \Lambda^2(e) \otimes f \otimes f \cong S^2(e) \otimes f \otimes S^2(f) \otimes f.$$ 

Moreover, $a_{-3}$ is isomorphic to

$$(e \otimes f) \otimes \Lambda^2(e) \otimes f \otimes f / \Lambda^2(e) \otimes f \otimes f \cong e \otimes \Lambda^2(e) \otimes f \otimes \Lambda^2(f).$$

Hence $g_{-3} = b(g_{-1}, \mu)/a_{-3}$ is isomorphic to

$$(e \otimes e \otimes f) / \Lambda^2(e) \otimes f \otimes f / \Lambda^2(e) \otimes f \otimes f \cong S^2(e) \otimes f \otimes S^2(f) \otimes e.$$ 

This completes the proof.

Proposition 8.4. Let $m = \bigoplus_{p<0} g_p$ be a pseudo-product FGLA of the $\mu$-th kind with pseudo-product structure $(e, f)$, where $\mu \geq 2$. We denote by $c$ the centralizer of $g_{-2}$ in $g_{-1}$. Let $g = \bigoplus_{p \in \mathbb{Z}} g_p$ be the prolongation of $(m; e, f)$. Assume that $g_0$ is isomorphic to $gl(e) \oplus gl(f)$ as a Lie algebra.

(1) If $\mu = 2$, then $m = \bigoplus_{p<0} g_p$ be a free pseudo-product FGLA.

(2) If $\mu \geq 3$ and $c \neq \{0\}$, then $(m; e, f)$ is not a free pseudo-product FGLA.

(3) If $\mu = 3$ and $c = \{0\}$, then $(m; e, f)$ is a free pseudo-product FGLA.

Proof. Let $\tilde{m} = \bigoplus_{p<0} \tilde{g}_p$ be the free pseudo-product FGLA of type $(m, n, \mu)$ with pseudo-product structure $(\tilde{e}, \tilde{f})$ such that $\tilde{g}_{-1} = g_{-1}, \tilde{e} = e$ and $\tilde{f} = f$. Let $\tilde{g} = \bigoplus_{p \in \mathbb{Z}} \tilde{g}_p$ be the prolongation of $(\tilde{m}; \tilde{e}, \tilde{f})$. There exists a GLA epimorphism $\varphi$ of $\tilde{m}$ onto $m$ such that the restriction $\varphi|\tilde{g}_{-1}$ is the identity mapping. Since the mapping $g_0 \ni D \mapsto (D|e, D|f) \in gl(e) \otimes gl(f)$ is an isomorphism, $\varphi$ can be extended to be a homomorphism $\tilde{\varphi}$ of $\bigoplus_{p \leq 0} \tilde{g}_p$ onto $\bigoplus_{p \leq 0} g_p$. Let $a$ be the kernel of $\tilde{\varphi}$; then $a$ is a graded ideal of $\bigoplus_{p \leq 0} \tilde{g}_p$. We set $a_p = a \cap \tilde{g}_p$; then $a = \bigoplus_{p \leq 0} a_p$. Since the restriction of $\tilde{\varphi}$ to $\tilde{g}_{-1} \oplus \tilde{g}_0$ is injective, $a_p = \{0\}$ for $p \geq -1$. Also each $a_p$ is a $\tilde{g}_0$-submodule of $\tilde{g}_p$. Since the $\tilde{g}_0$-module $\tilde{g}_{-2}$ is irreducible (Proposition 8.3 (2)), $\varphi|\tilde{g}_{-2}$ is injective. If $\mu = 2$, then $\varphi$ is an isomorphism. This proves the assertion (1). Now we assume that $\mu \geq 3$. Then

$$\tilde{g}_{-3} = [[e, f], f] \oplus [[e, f], e].$$

Since $\tilde{g}_0$-modules $[[e, f], f]$ and $[[e, f], e]$ are irreducible and not isomorphic to each other (Proposition 8.3 (3)), one of the following cases occurs: (i) $a_{-3} = [[e, f], f]$; (ii) $a_{-3} = [[e, f], e]$; (iii) $a_{-3} = \{0\}$. If $a_{-3} = [[e, f], f]$ (resp. $a_{-3} = [[e, f], e]$), then $e = f$ (resp. $e = e$). Also since $g_0$-modules $e, f$ are irreducible and not isomorphic to each other, one of the following cases occurs: (i) $e = e$; (ii) $e = f$; (iii) $e = \{0\}$. If $e = e$ (resp. $e = f$), then $a_{-3} = [[e, f], e]$ (resp. $a_{-3} = [[e, f], f]$). In this case, $\varphi$ is not injective. Hence $(m; e, f)$ is not free. If $e = \{0\}$, then $a_{-3} = \{0\}$. Hence $\varphi|\tilde{g}_{-3}$ is an isomorphism. In particular, if $\mu = 3$, then $(m; e, f)$ is free. 

$\blacksquare$
Example 8.1. Let $V$ and $W$ be finite-dimensional vector spaces and $k \geq 1$. We set

$$\mathfrak{c}^k(V,W) = \bigoplus_{p=-k-1}^{-1} \mathfrak{c}^k(V,W)_p,$$

$$\mathfrak{c}^k(V,W)_p = W \otimes S^{k+p+1}(V^*), \quad -k-1 \leq p \leq -2,$$

$$\mathfrak{c}^k(V,W)_{-1} = V \oplus (W \otimes S^k(V^*)).$$

The bracket operation of $\mathfrak{c}^k(V,W)$ is defined as follows:

$$[W,V] = \{0\}, \quad [V,V] = \{0\}, \quad [W \otimes S^r(V^*), W \otimes S^s(V^*)] = \{0\},$$

$$[w \otimes s_r, v] = w \otimes (v \circ s_r) \quad \text{for} \ v \in V, \ w \in W, \ s_r \in S^r(V^*).$$

Equipped with this bracket operation, $\mathfrak{c}^k(V,W)$ becomes a pseudo-product FGLA of the $(k+1)$-th kind with pseudo-product structure $(V,W \otimes S^k(V^*))$, which is called the contact algebra of order $k$ of bidegree $(n,m)$, where $n = \dim V$ and $m = \dim W$ (cf. [14, p. 133]). We assume that $\mathfrak{c}^k(V,W)$ is a free pseudo-product FGLA. Since

$$\dim \mathfrak{c}^k(V,W)_{-2} = m \binom{n+k-2}{k-1}, \quad \dim V \dim(W \otimes S^k(V^*)) = mn \binom{n+k-1}{k},$$

we get $n = 1$. Since $W \otimes S^k(V^*)$ is contained in the centralizer of $\mathfrak{c}^k(V,W)_{-2}$ in $\mathfrak{c}^k(V,W)_{-1}$, we get $k = 1$. Thus we obtain that $\mathfrak{c}^k(V,W)$ is a free pseudo-product FGLA if and only if $k = 1, n = 1$.

Example 8.2. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.

We set $\mathfrak{c} = \mathfrak{g}_{-1}^{(m)}, \mathfrak{f} = \mathfrak{g}_{-1}^{(m+1)}$. Then $(\mathfrak{g}_-; \mathfrak{c}, \mathfrak{f})$ is a pseudo-product FGLA. Since $\dim \mathfrak{c} = m$, $\dim \mathfrak{f} = n$ and $\dim \mathfrak{g}_{-2} = mn$, the pseudo-product FGLA $(\mathfrak{g}_-; \mathfrak{c}, \mathfrak{f})$ is a free pseudo-product FGLA of type $(m,n,2)$ (Proposition 8.3 (2)). Also $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{g}_-$ except for the following cases (see [15]):

1. $m = n = 1$. In this case, the prolongation of $\mathfrak{g}_-$ is isomorphic to $K(1)$.
2. $m = 1$ or $n = 1$ and $l = \max\{m,n\} \geq 2$. In this case, the prolongation of $\mathfrak{g}_-$ is isomorphic to $W(l+1; s)$, where $s = (1,2,\ldots,2)$.

Example 8.3. Let $V$ and $W$ be finite-dimensional vector spaces such that $\dim V = m \geq 1$ and $\dim W = n \geq 1$. We set

$$\mathfrak{g}_{-1} = V \oplus W, \quad \mathfrak{g}_{-2} = V \otimes W,$$

$$\mathfrak{g}_{-3} = V \otimes S^2(W) \oplus S^2(V) \otimes W, \quad \mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}.$$

The bracket operation of $\mathfrak{m}$ is defined as follows:

$$[\mathfrak{g}_{-3}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}] = [\mathfrak{g}_{-2}, \mathfrak{g}_{-2}] = \{0\}, \quad [V, V] = [W, W] = \{0\},$$

$$[v, w] = -[w, v] = v \otimes w, \quad [v, v' \otimes w] = -[v' \otimes w, v] = v \otimes v' \otimes w,$$

where $v, v' \in V$ and $w, w' \in W$. Equipped with this bracket operation, $\mathfrak{m}$ becomes a free pseudo-product FGLA of type $(m,n,3)$ with pseudo-product structure $(V, W)$ (Proposition 8.3).
Theorem 8.1. Let $\mathfrak{m} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a free pseudo-product FGLA of type $(m, n, \mu)$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ over $\mathbb{C}$. Furthermore let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ (resp. $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$) be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ (resp. $\mathfrak{m}$).

(1) Assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. Then $m = 1$ or $n = 1$, and $\mu = 2$. Furthermore $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is isomorphic to a finite-dimensional simple GLA of type $(A_{l+1}, \{\alpha_1, \alpha_2\})$, where $l = \max\{m, n\}$. If $l = 1$, then $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(1)$. If $l \geq 2$, then $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $W(l + 1; s)$, where $s = (1, 2, \ldots, 2)$.

(2) If $\mathfrak{g}_1 \neq \{0\}$, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a finite-dimensional simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.

Proof. (1) For $p \geq -1$, we put $\mathfrak{h}_p = \{X \in \mathfrak{g}(\mathfrak{m})_p : [X, \mathfrak{g}_{\leq -2}] = \{0\}\}$. Assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$ and $\mu \geq 3$. By Proposition 8.4 (2), $\mathfrak{h}_{-1} = \{0\}$. Since $[\mathfrak{h}_0, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{-1} = \{0\}$, we see that $\mathfrak{h}_0 = \{0\}$. By [11, Corollary 1 to Theorem 11.1], we obtain that $\dim \mathfrak{g}(\mathfrak{m}) < \infty$, which is a contradiction. Thus we see that $\mu = 2$ if $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. The remaining assertion follows from Example 8.2.

(2) Assume that $\mathfrak{g}_1 \neq \{0\}$ and $\mu \geq 3$. By transitivity of $\mathfrak{g}$, $[\mathfrak{g}_1, \mathfrak{e}] \neq \{0\}$ or $[\mathfrak{g}_1, \mathfrak{f}] \neq \{0\}$. We may assume that $[\mathfrak{g}_1, \mathfrak{e}] \neq \{0\}$. Then there exists an irreducible component $\mathfrak{g}'_1$ of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ such that $[\mathfrak{g}'_1, \mathfrak{e}] \neq \{0\}$ and $[\mathfrak{g}'_1, \mathfrak{f}] = \{0\}$. The subalgebra $\mathfrak{e} \oplus [\mathfrak{e}, \mathfrak{g}'_1] \oplus \mathfrak{g}'_1$ is a simple GLA of the first kind. Since $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$, $\mathfrak{e} \oplus [\mathfrak{e}, \mathfrak{g}'_1] \oplus \mathfrak{g}'_1$ is of type $(A_m, \{\alpha_1\})$.

Let $D$ be a nonzero element of $\mathfrak{g}'_1$. There exist $\lambda \in \mathfrak{e}^*$ and $\eta \in \mathfrak{f}^*$ such that

$$[[[D, Z], U] = \lambda(U)Z + \lambda(Z)U, \quad [[D, Z], W] = \eta(Z)W,$$

where $Z, U \in \mathfrak{e}$ and $W \in \mathfrak{f}$ (cf. [12, p. 4]). Let $X$ (resp. $Y$) be a nonzero element of $\mathfrak{e}$ (resp. $\mathfrak{f}$).

Since the subalgebra generated by $X, Y$ is a free FGLA of type $(2, \mu)$ (Remark 8.1 (2)),

$$\text{ad}(X)^\mu(Y) = 0, \quad \text{ad}(X)^\mu - 1(Y) \neq 0,$$

$$\text{ad}(Y) \text{ad}(X)^\mu - 1(Y) = 0, \quad \text{ad}(Y) \text{ad}(X)^\mu - 2(Y) \neq 0$$

(Lemma 2.1). By induction on $\mu$, we see that

$$0 = \text{ad}(D) \text{ad}(X)^\mu(Y) = (\mu(\mu - 1)\lambda(X) + \mu\eta(X)) \text{ad}(X)^\mu - 1(Y),$$

$$0 = \text{ad}(D) \text{ad}(Y) \text{ad}(X)^\mu - 1(Y)$$

$$= ((\mu - 1)(\mu - 2)\lambda(X) + (\mu - 1)\eta(X)) \text{ad}(Y) \text{ad}(X)^\mu - 2(Y).$$

Since

$$\det \begin{bmatrix} \mu(\mu - 1) & \mu \\ (\mu - 1)(\mu - 2) & \mu - 1 \end{bmatrix} = \mu(\mu - 1) \neq 0,$$

we see that $\lambda(X) = \eta(X) = 0$, which is a contradiction. Thus we obtain that $\mu = 2$ if $\dim \mathfrak{g}_1 \neq \{0\}$. From Example 8.2, it follows that $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$ if $\dim \mathfrak{g}_1 \neq \{0\}$. $$\blacksquare$$

9 Automorphism groups of the prolongations of free pseudo-product fundamental graded Lie algebras

For a GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ we denote by $\text{Aut}(\mathfrak{g})_0$ the group of all the automorphisms of $\mathfrak{g}$ preserving the gradation of $\mathfrak{g}$:

$$\text{Aut}(\mathfrak{g})_0 = \{\varphi \in \text{Aut}(\mathfrak{g}) : \varphi(\mathfrak{g}_p) = \mathfrak{g}_p \text{ for all } p \in \mathbb{Z}\}.$$
Proposition 9.1. Let $m = \bigoplus_{p < 0} g_p$ be an FGLA and let $g(m) = \bigoplus_{p \in \mathbb{Z}} g(m)_p$ be the prolongation of $m$. The mapping $\Phi : \text{Aut}(g(m))\,|_0 \ni \phi \mapsto \phi|m \in \text{Aut}(m)\,|_0$ is an isomorphism.

Proof. It is clear that $\Phi$ is a group homomorphism. We prove that $\Phi$ is injective. Let $\phi$ be an element of $\ker \Phi$. Assume that $\phi(X) = X$ for all $X \in g(m)_p$ ($p < k$). For $X \in g(m)_k$, $Y \in g_{-1}$, we have $[\phi(X) - X, Y] = \phi([X, Y]) - [X, Y]$.

Since $[X, Y] \in g(m)_{k-1}$, we have $[\phi(X) - X, Y] = 0$. By transitivity, $\phi(X) = X$. By induction, we prove $\phi$ to be the identity mapping. Hence $\Phi$ is a monomorphism.

We prove that $\Phi$ is surjective. Let $\phi \in \text{Aut}(m)_0$. We construct the mapping $\varphi_p : g(m)_p \rightarrow g(m)_p$ inductively as follows: First for $p < k$, we set $\varphi_0(X) = \varphi(X)^{-1}$. Then for $Y, Z \in m$

$$\varphi_0(X)([Y, Z]) = \varphi([X, \varphi^{-1}(Y)], Z) + [Y, \varphi(X, \varphi^{-1}(Z))],$$

so $\varphi_0(X) \in g(m)_0$. Furthermore we can prove easily that $\varphi_0(X), \varphi_p(Y) = \varphi_p([X, Y])$ for $X \in g_0$ and $Y \in g_p$ ($p \leq 0$). Here for $p < 0$ we set $\varphi_p = \varphi|g(m)_p$. Assume that we have defined linear isomorphisms $\varphi_p$ of $g(m)_p$ onto itself ($0 \leq p < k$) such that

$$\varphi_{r+s}([X, Y]) = [\varphi_r(X), \varphi_s(Y)]$$

for $X \in g(m)_r$, $Y \in g(m)_s$ ($r + s < k$, $r < k$, $s < k$). For $X \in g(m)_k$ we define $\varphi_k(X) \in \text{Hom}(m, \bigoplus_{p \leq k-1} g(m)_p)$ as follows:

$$\varphi_k(X)(Y) = \varphi_{k+s}([X, \varphi^{-1}(Y)]), \quad Y \in g_s, s < 0.$$ 

For $Y \in g_s$, $Z \in g_t$ ($s, t < 0$),

$$\varphi_k(X)([Y, Z]) = \varphi_{k+t+s}([[X, \varphi^{-1}(Y)], \varphi^{-1}(Z)] + [\varphi^{-1}(Y), [X, \varphi^{-1}(Z)]])$$

$$= [\varphi_{k+s}([X, \varphi^{-1}(Y)]), Z] + [Y, \varphi_{k+t}([X, \varphi^{-1}(Z)])]$$

$$= [\varphi_k(X)(Y), Z] + [Y, \varphi_k(X)(Z)],$$

so $\varphi_k(X) \in g(m)_k$. Next we prove that for $X \in g_p$, $Y \in g_q$ ($p + q = k$, $0 \leq p \leq k$, $0 \leq q \leq k$),

$$\varphi_k([X, Y]) = [\varphi_p(X), \varphi_q(Y)].$$

For $Z \in g_s$ ($s < 0$),

$$[\varphi_p(X), \varphi_q(Y)], Z] = [\varphi_p(X), [\varphi_q(Y), Z]] - [\varphi_q(Y), [\varphi_p(X), Z]]$$

$$= \varphi_{p+q+s}([[X, [Y, \varphi^{-1}(Z)]], [Y, [X, \varphi^{-1}(Z)]])$$

$$= \varphi_{p+q+s}([[X, Y], \varphi^{-1}(Z)]) = [\varphi_k([X, Y]), Z].$$

By transitivity, we see that $\varphi_k([X, Y]) = [\varphi_p(X), \varphi_q(Y)]$. We define a mapping $\bar{\varphi}$ of $g(m)$ into itself as follows:

$$\bar{\varphi}(X) = \begin{cases} \varphi(X), & X \in m, \\ \varphi_k(X), & k \geq 0, \ X \in g(m)_k. \end{cases}$$

From the above results and the definition of $\varphi_k$ ($k \geq 0$), we see that $\bar{\varphi}$ is a GLA homomorphism. Assume that $\varphi_{k-1}$ ($k \geq 0$) is a linear isomorphism. For $X \in g(m)_k$, if $\varphi_k(X) = 0$, then $0 = [\varphi_k(X), Y] \in g_{-1}$ for all $Y \in g_{-1}$. By transitivity, we see that $X = 0$, so $\varphi_k$ is a linear isomorphism. Therefore $\bar{\varphi}$ is an automorphism of $g(m)$. ■
Theorem 9.1. Let $m = \bigoplus_{p<0} g_p$ be a free FGLA over $\mathbb{C}$, and let $g(m) = \bigoplus_{p \in \mathbb{Z}} g(m)_p$ be the prolongation of $m$. The mapping $\Phi : \text{Aut}(g(m))_0 \ni \phi \mapsto \phi|_{g_{-1}} \in GL(g_{-1})$ is an isomorphism.

Proof. We may assume that $m$ is a universal FGLA $b(g_{-1}, \mu)$ of the $\mu$-th kind. By Corollary 1 to Proposition 3.2 of [11], the mapping $\text{Aut}(m)_0 \ni a \mapsto a|_{g_{-1}} \in GL(g_{-1})$ is an isomorphism. By Proposition 9.1, we see that the mapping $\Phi : \text{Aut}(g(m))_0 \ni \phi \mapsto \phi|_{g_{-1}} \in GL(g_{-1})$ is an isomorphism.

For a pseudo-product GLA $g = \bigoplus_{p \in \mathbb{Z}} g_p$ with pseudo-product structure $(e, f)$, we denote by $\text{Aut}(g; e, f)_0$ the group of all the automorphisms of $g$ preserving the gradation of $g$, $e$ and $f$:

$$\text{Aut}(g; e, f)_0 = \{ \varphi \in \text{Aut}(g)_0 : \varphi(e) = e, \varphi(f) = f \}.$$

Theorem 9.2. Let $m = \bigoplus_{p<0} g_p$ be a free pseudo-product FGLA of type $(m, n, \mu)$ with pseudo-product structure $(e, f)$ over $\mathbb{C}$, and let $g = \bigoplus_{p \in \mathbb{Z}} g_p$ be the prolongation of $(m; e, f)$. The mapping $\Phi : \text{Aut}(g; e, f)_0 \ni \phi \mapsto (\phi|_{e}, \phi|_{f}) \in GL(e) \times GL(f)$ is an isomorphism. Furthermore if $\dim e \neq \dim f$, then $\text{Aut}(g; e, f)_0 = \text{Aut}(g)_0$.

Proof. Clearly $\Phi$ is a monomorphism. We show that $\Phi$ is surjective. Let $(\phi_1, \phi_2)$ be an element of $GL(e) \times GL(f)$. We set $\phi = \phi_1 \oplus \phi_2 \in GL(g_{-1})$. By Corollary 1 to Proposition 3.2 of [11], there exists an element $\varphi_1 \in \text{Aut}(b(g_{-1}, \mu))_0$ such that $\varphi_1|_{g_{-1}} = \phi$. Since $\varphi_1([e, f] + [f, f]) = [e, f] + [f, f]$, $\varphi_1$ induces an element $\varphi_2 \in \text{Aut}(m; e, f)_0$ such that $\varphi_2|_{g_{-1}} = \phi$. By Proposition 9.1, there exists $\varphi_3 \in \text{Aut}(g(m))_0$ such that $\varphi_3|_{g} = \varphi_2$. We prove that $\varphi_3|_{g} = g$. For $X_0 \in g_0$ and $Y \in e$, we see that $[\varphi_3(X_0), Y] = \varphi_3([X_0, \varphi_3^{-1}(Y)]) \in \varphi_3(e) = e$, so $\varphi_3(X_0)(e) \subset e$. Similarly we get $\varphi_3(X_0)(f) \subset f$. Thus we obtain that $\varphi_3(g_0) = g_0$. Now we assume that $\varphi_3(g_i) = g_i$ for $0 \leq i \leq k$. Then for $X_{p+1} \in g_{k+1}$ and $Y \in g_p$ ($p < 0$), we see that $[\varphi_3(X_{p+1}), Y] = \varphi_3([X_{p+1}, \varphi_3^{-1}(Y)]) \in \varphi_3(g_{p+k+1}) = g_{p+k+1}$, so $\varphi_3(g_{k+1}) \subset g_{k+1}$. Hence $\varphi_3(g) = g$ and $\Phi$ is surjective. Now we assume that $\dim e \neq \dim f$. Let $\varphi \in \text{Aut}(g)_0$. Since $g_0$-modules $e$ and $f$ are not isomorphic to each other, we see that (i) $\varphi(e) = e$, $\varphi(f) = f$ or (ii) $\varphi(e) = f$, $\varphi(f) = e$. According to the assumption $\dim e \neq \dim f$, we get $\varphi(e) = e$, $\varphi(f) = f$, so $\varphi \in \text{Aut}(g; e, f)_0$.

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