ANALYSIS OF A LÉVY-DIFFUSION LESLIE-GOWER
PREDATOR-PREY MODEL WITH NONMONOTONIC
FUNCTIONAL RESPONSE

HONGWEI YIN
School of Sciences, Nanchang University Nanchang 330031, China
Numerical Simulation and High-Performance Computing Laboratory
Nanchang University, Nanchang 330031, China

XIAOYONG XIAO AND XIAOQING WEN∗
School of Sciences, Nanchang University Nanchang 330031, China
(Communicated by Björn Schmalfuß)

ABSTRACT. In this paper, a Lévy-diffusion Leslie-Gower predator-prey model with a nonmonotonic functional response is studied. We show the existence, uniqueness and attractiveness of the globally positive solution to this model. Moreover, to its corresponding steady-state model, we obtain the stability of the semi-trivial solutions, the existence and nonexistence of coexistence states by the method of topological degree, the uniqueness and stability of coexistence states by Grandall-Rabinowitz bifurcation theorem. In addition, to get these results, we study the property of the Lévy diffusion operator, and give out the comparison principle of the generalized parabolic Lévy-diffusion differential equation, as well as the existence and stability of the solution for the steady-state Logistic equation with Lévy diffusion. Furthermore, we obtain the comparison principle of the steady-state Lévy-diffusion equation. As far as we know, these results are new in the ecological model.

1. Introduction. From the pioneering works of Lotka and Volterra [23], a lot of authors studied dynamical features of ecosystem, such as the interaction of predator and prey. Therein, the well-known predator-prey model is considered as the following form:

\[
\begin{align*}
\frac{dx}{d\tau} &= x \left( s_1 - \frac{xk_1}{k_2} \right) - yp(x), \\
\frac{dy}{d\tau} &= y \left( s_2 - \frac{hy}{x+k_2} \right),
\end{align*}
\]

which is called as a modified Leslie-Gower predator-prey model, where \(x\) and \(y\) represent population densities of prey and predator at time \(\tau\), respectively. All parameters \(s_1, s_2, k_1, k_2\) and \(h\) are positive. \(s_1\) and \(s_2\) denote the intrinsic growth rates of the prey and predator, respectively. \(k\) is carrying capacity of the prey. \(p(x)\), which is called a functional response, is a consumption rate of prey by a predator.

2010 Mathematics Subject Classification. Primary: 92B05, 92D25; Secondary: 35S05.
Key words and phrases. Lévy diffusion, Leslie-Gower predator-prey model, bifurcation.
Hongwei Yin is supported by NSF of China (61563033 and 11461044) and NSF of Jiangxi Province (2016MBAB201010). Xiaoqing Wen is supported by NSF of China (11563005).
∗ Corresponding author: Xiaoqing Wen.
Figure 1. Response functions.

The concept of the functional response was first introduced by Solomon (1949). The carrying capacity of the predator’s environment is assumed to be proportional to the prey abundance, i.e., \((x + k_2)/h\), which is called as a modified Leslie-Gower term [2]. The functional response \(p(x)\) has been developed during various different processes of energy transfer (see [35]). In detail, there are three important forms of functional responses, that is, (I) \(p(x) = mx\), (II) \(p(x) = mx/(a + x)\) and (III) \(p(x) = mx^2/(a + x^2)\)(sigmoidal), where \(m\) is the maximum predation rate and \(a\) is a half-maximum rate. These three responses are monotone-increasing functions with respect to the prey \(x\). In addition, the form of \(p(x) = mx/(a + x^2)\) in [12, 25] is called Holling type IV functional response, which is a nonmonotonic function with respect to \(x\).

The generalized expression of Holling IV type is
\[ p(x) = \frac{mx}{ax^2 + bx + 1}, \]  
which is introduced in [26, 34]. In this functional response, the parameter \(a\) is a positive constant. If \(b > -2\sqrt{a}\), then \(ax^2 + bx + 1 > 0\) for \(x \geq 0\). This functional response seems reasonable if the prey and webbing densities are directly related [1]. When \(b\) is nonnegative, the functional response (2) resembles the Holling type II for small enough \(x\) while the effect of inhibition appears for large \(x\), see Figure 1(A). If \(-2\sqrt{a} < b < 0\), the function (2) remains nonnegative and the inhibition effect still works for large \(x\). This function is like Holling type III for small \(x\), see Figure 1(B) [34]. Introducing this functional response into the system (1) yields
\[
\begin{align*}
\frac{dx}{d\tau} &= x \left( s_1 - \frac{x}{k_1} \right) - \frac{mxy}{ax^2 + bx + 1}, \\
\frac{dy}{d\tau} &= y \left( s_2 - \frac{hy}{x + k_2} \right).
\end{align*}
\]  
(3)

For simplicity, we apply the following transformation to the variables and the parameters in the system (3)
\[
\begin{align*}
x &= k_1 u, \quad y = v/k_1, \quad ak_1^2 = \ddot{a}, \quad bk_1 = \ddot{b}, \quad \tau = k_1 t, \\
s_2k_1 &= \eta, \quad r = h/k_1^2, \quad k_2/k_1 = k,
\end{align*}
\]
and we still use \(a, b\) to stand for \(\ddot{a}, \ddot{b}\). Thus (3) becomes the following form:
\[
\begin{align*}
\frac{du}{dt} &= u (c - u) - \frac{mv}{au^2 + bu + 1}, \\
\frac{dv}{dt} &= v \left( \eta - \frac{ru}{u + k} \right).
\end{align*}
\]  
(4)

In real world, the spatial heterogeneity of the predator’s and prey’s distributions is obvious more or less. Thus, the predator or prey can spread from higher density regions to lower density ones. This phenomenon is called as Gaussian diffusion (or
normal diffusion). This process can be often described by the Laplacian operator. Then, (4) can be rewritten into a reaction-diffusion system as the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + u (c - u) - \frac{muv}{u^2 + vu + t}, \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v + v \left( \eta - \frac{rvu}{u + k} \right), \quad x \in \Omega, t > 0,
\end{align*}
\]

(5)

where the diffusion coefficients of the prey and predator are taken as 1. The region \( \Omega \subset \mathbb{R}^n \) is open, bounded and equipped with the Lipschitz boundary, where \( \mathbb{R}^n \) is an arbitrary positive-integer \( n \) dimension space. \( \Delta \) is the Laplacian operator and the infinitesimal generator of Brown motion (also called as Gaussian process) with the corresponding characteristic of \((0, I, 0)\). It is well known that Brown motion is a special rotationally invariant stable Lévy process. At present, many authors usually use the Laplacian operator to describe diffusion process of population and study mathematical properties of this kind of reaction-diffusion population models, such as stationary patterns [15, 22, 13, 14, 24, 29, 30, 31], globally asymptotic stability [16, 11, 10, 27, 7] and the existence of positive solutions for steady state equation [32, 33] and so on.

However, diffusion process of population is always complex and diverse, for example, population often face some obstacles, stay and jump etc. As a result, population can not obey the rule of Gaussian diffusion. Moreover, the recent researches [21, 9] have showed that predator (such as sharks, bony fishes, sea turtles and penguins) and prey exhibit Lévy-walk-like behaviours. Based on this case, we should consider the predator-prey system equipped with Lévy diffusion (also called as anomalous diffusion), instead of normal diffusion. A Lévy diffusion process is a stochastic process with independent and stationary increments, which extends the concept of Brownian motion. Essentially, Lévy processes are obtained when one relaxes the assumption of continuity of paths (which gives the Brownian motion) by the weaker assumption of stochastic continuity. By the Lévy-Khintchine formula, the infinitesimal generator of any Lévy processes is an operator in the following form

\[
L_{\nu}(u) = \sum_{i,j} \frac{1}{2} a_{ij} \partial_{ij} u + \sum_j b_j \partial_j u + \int_{\mathbb{R}^n} \{u(x + y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y)\} d\nu(y),
\]

(6)

where \( \nu \) is the Lévy measure, and satisfies \( \int_{\mathbb{R}^n} \min\{1, |y|^2\} d\nu(y) < \infty \). When the process has no diffusion and drift parts, (6) takes the following form

\[
L_{\nu}(u) = \int_{\mathbb{R}^n} \{u(x + y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y)\} d\nu(y).
\]

(7)

Further, if one assumes that the process is symmetric, and that the Lévy measure is absolutely continuous, then (7) is turned into

\[
L_K u = \int_{\mathbb{R}^n} [u(x + y) + u(x - y) - 2u(x)] K(y) dy.
\]

(8)

Here \( K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty) \) is a function, and

\[
mK \in L^1(\mathbb{R}^n), \ m(x) = \min\{|x|^2, 1\}.
\]

(9a)

We assume that the operator \( L_K \) satisfies the conditions: there exists a constant \( \theta > 0 \) such that

\[
K(x) \geq \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\},
\]

(9b)

\[
K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}.
\]

(9c)
The parameter $s \in (0, 1)$ is fixed and $n > 2s$. The conditions (9) are throughout this paper.

A typical example for $K$ is given by $K(x) = |x|^{-(n+2s)}$, $s \in (0, 1)$. For this case, the operator $L_K$ is the fractional Laplace operator $-(-\Delta)^s$, see [6], which is the infinitesimal generator of the rotationally invariant stable Lévy process with stable index $s \in (0, 1)$. The fractional Laplace operator is an important nonlocal diffusion one and has been successfully used in quantum physics [17, 20] and epidemic model [8]. For the case of $K(x) = |x|^{-(n+2)}$, the operator $L_K$ corresponds to the normal Laplace operator (or normal diffusion operator) $\Delta$. It is worth mentioning that the fractional Laplace operator $-(-\Delta)^s$ converges to the normal Laplace operator $\Delta$ as $s \to 1^-$. 

In this paper, we introduce the Lévy diffusion operator $L_K$ in the form of (8) with the conditions (9) into (4) as the following form

\[
\frac{\partial u}{\partial t} - L_K u = u(c - u) - \frac{muv}{au^2 + bu + 1}, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (10a)
\]

\[
\frac{\partial v}{\partial t} - L_K v = \left( \eta - \frac{rv}{u + k} \right), \quad (x, t) \in \Omega \times \mathbb{R}^+. \quad (10b)
\]

Here, we consider the initial-boundary condition

\[
u(x, t) = v(x, t) = 0, \quad (x, t) \in (\mathbb{R}^n \setminus \Omega) \times \mathbb{R}^+, \quad (10c)
\]

\[
u(x, 0) = u_0(x), \quad v(x, t) = v_0(x), \quad x \in \Omega. \quad (10d)
\]

The Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial \Omega$, consistently with the non-local character of the operator $L_K$.

The steady-state equations of (10) is

\[
-L_K u = u(c - u) - \frac{muv}{au^2 + bu + 1}, \quad x \in \Omega, \quad (11a)
\]

\[
-L_K v = \left( \eta - \frac{rv}{u + k} \right), \quad x \in \Omega, \quad (11b)
\]

\[
u(x, t) = v(x, t) = 0, \quad x \in \mathbb{R}^n \setminus \Omega. \quad (11c)
\]

As far as we know, the systems (10) and (11) have not been investigated. In this paper, we make the first attempt to fill this gap and study the mathematical properties for (10) and (11). To this end, we first examine the properties of the Lévy diffusion operator $L_K$, and then obtain the existence and uniqueness of the positive solution to (10). By the comparison principle of the parabolic Lévy-diffusion equation, asymptotic behavior of this solution is obtained. By the method of the topological degree, we show the existence, uniqueness and nonexistence of coexistence states to (11). Moreover, by applying Grandall-Rabinowitz bifurcation theorem and the property of the Lévy diffusion operator $L_K$, we get the multiplicity and stability of coexistence states to (11).

The remainder of the paper is organized as below. In Sec.2, some necessary results and notations are introduced. In Sec.3, the basic properties of the Lévy diffusion operator $L_K$ are given out. In Sec.4, the existence of the globally positive solution of (10) is showed. In Sec.5, we examine the stability of semi-trivial solutions of the steady state system to (11). In Sec.6, we give out the conditions of the existence and nonexistence of coexistence states to (11). In Sec.7, we prove the uniqueness and stability of coexistence state to (11). In Sec.8, we apply Grandall-Rabinowitz bifurcation theorem and the spectrum of the Lévy diffusion operator $L_K$ to get the...
multiplicity and stability of the coexistence states to (11). Finally, we conclude this paper with a discussion in Sec.9.

2. Preliminaries. In this section we state the general assumptions on the quantity we are dealing with. We keep these assumptions throughout the paper. Denote $X$ by the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $g \in X$ belongs to $L^2(\Omega)$ and the map $(x, y) \mapsto (g(x) - g(y))\sqrt{K(x - y)}$ is in $L^2(\mathbb{R}^{2n} \setminus (C\Omega \times C\Omega), dx dy)$, where $C\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_0 = \{ g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$ 

Note that

$$C_0^2(\Omega) \subseteq X_0,$$

so $X$ and $X_0$ are non-empty. In the sequel we set $Q = \mathbb{R}^{2n} \setminus C\Omega$, where

$$O = (C\Omega \times C\Omega) \subset \mathbb{R}^{2n}.$$

The space $X$ is endowed with the norm defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \left( \int_Q |g(x) - g(y)|^2 K(x - y) dx dy \right)^{1/2}. \tag{12}$$

From [18], it is well known that $\| \cdot \|_X$ is a norm on $X$. We denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dxdy \right)^{1/2}. \tag{13}$$

In [18], when $K(x) = |x|^{-(n + 2s)}$, the norms in (12) and (13) are not the same, because $\Omega \times \Omega$ is strictly contained in $Q$. Thus, the classical fractional Sobolev space approach is not sufficient for studying the problem. But, there is the connection between the spaces $X$ and $X_0$ with the fractional Sobolev spaces, see [18].

**Lemma 2.1.** [18] $(X_0, \| \cdot \|)$ is a Hilbert space, with scalar product

$$(u, v)_{X_0} = \int_Q (u(x) - u(y))(v(x) - v(y))K(x - y) dx dy. \tag{14}$$

**Lemma 2.2.** [19] Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ to satisfy assumptions (9) and let $v_j$ be a bounded sequence in $X_0$. Then, there exists $v_\infty \in L^\nu(\mathbb{R}^n)$ such that, up to a subsequence

$$v_j \to v_\infty \text{ in } L^\nu(\mathbb{R}^n)$$

as $j \to +\infty$ for any $\nu \in [1, 2^*)$ and $2^* = \frac{2n}{n - 2s}$.

**Definition 2.3.** [28] A linear operator $A$ is called as a sectorial operator in Banach space $X$, if it satisfies the following conditions

1. $A$ is closed and dense in $X$.
2. In the complex plane, there exists a section

$$S_{\alpha, \beta} = \{ \lambda | \lambda \in \mathbb{C}, \beta \leq |\arg(\lambda - \alpha)| \leq \pi, \lambda \neq \alpha \} \subset \rho(A),$$

where $\beta \in (0, \pi/2)$ and $\beta$ are some constants.
3. $\| (\lambda - A)^{-1} \| \leq \frac{M}{|\lambda - \alpha|}(\lambda \in S_{\alpha, \beta})$, $M \geq 1$ is some positive constant.
Theorem 2.4. [28] Assume that \( A \) is a sectional operator with the section \( S_{\alpha,\beta} \), and \( \Gamma \) is a path in \( \rho(-A) \) satisfying the condition that there exists a constant \( \theta \in (\pi/2, \pi) \) such that \( \arg \lambda \to \pm \theta \) as \( |\lambda| \to \infty \) and \( \lambda \in \Gamma \), then \( e^{-tA} \) is an analytical semigroup, whose infinitesimal generator is \(-A\), where

\[
e^{-tA} = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma} (\lambda + A)^{-1} e^{\lambda t} d\lambda, & t > 0 \\ I, & t = 0. \end{cases}
\]

Lemma 2.5. [28] An operator \( A \) is self-adjoint and dense on Hilbert space \( H \), and bounded below, that is, there exists a real constant \( \mu \) such that

\[\langle Ah, h \rangle \geq \mu \|h\|^2, \quad h \in H,\]

then \( A \) is a sectorial operator.

3. Basic properties of Lévy diffusion equation. To study the existence, uniqueness, stability and bifurcation of (10) and (11), we first discuss the comparison principle of Lévy diffusion equation and the existence and stability of the steady-state Logistic equation with Lévy diffusion. For convenience, we introduce some notations. For \( x, y \in \mathbb{R} \), we denote \( x \wedge y := \min \{x, y\} \), \( x \vee y := \max \{x, y\} \), \( x^+ := x \vee 0 \), \( x^- := (-x) \vee 0 \) and extend these notations to real-valued functions. If \((V, \geq)\) is partially ordered vector space, we denote its positive cone by \( V^+ := \{v \in V : v \geq 0\} \).

Next, we consider the following linear Poisson equation with the Lévy diffusion operator as the following form

\[
\begin{cases}
- L_K u = f(x, u), & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where \( f \in L^2(\Omega) \). The operator \(-L_K\) is an invertible operator. In view of Lemma 2.2, the operator \((-L_K)^{-1} : X_0(\Omega) \to X_0(\Omega)\) is compact. Then, from Proposition 2 in Appendix, it is known that the eigenfunction sequence \( \{e_k\}_{k \in \mathbb{N}} \) corresponding to the eigenvalue \( \lambda_k \) of \(-L_K\) is a complete orthonormal basis of \( X_0 \).

Lemma 3.1. The operator \(-L_K\) is a sectorial operator.

Proof. According to the definition of the operator \( L_K \), we know that the definition domain of \(-L_K\) is \( X_0 \). \( X_0 \) is dense in \( L^2(\Omega) \). From the above discussion, for any \( h \in X_0 \) there exist a constant sequence \( a_k, k \in \mathbb{N} \), such that \( h = \sum_{k=1}^{\infty} a_k e_k \), \( a_k = \langle h, e_k \rangle \). From Proposition 2 and by taking \( \mu = 0.5\lambda_1 \), we have

\[
\langle -L_K h, h \rangle = \sum_{k=1}^{\infty} \lambda_k a_k^2 \geq \mu \sum_{k=1}^{\infty} a_k^2 = \mu \|h\|_{X_0}.
\]

Thus, by Lemma 2.5, it is well known that the operator \(-L_K\) is a sectorial operator.

In view of Lemma 3.1, (d) in Proposition 2 and Theorem 2.4, one can get the following result.

Theorem 3.2. The operator \(-L_K\) is the infinitesimal generator of the analytically positive semigroup \( \{e^{L_K t}\}_{t \geq 0} \).

Remark 1. It is well known that the normal Laplace operator \(-\Delta\) as a special case of the operator \(-L_K\) is a sectorial operator, and that it also is the infinitesimal generator of the analytically positive semigroup, see [28].
Next, we introduce the comparison principle of the following equation,
\[
\begin{aligned}
& -L_K u = f(x,u), \quad x \in \Omega, \\
& u = g(x), \quad x \in \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]  
(16)

**Lemma 3.3.** Let \( u \) be a solution to (16) with \( f,g \geq 0 \) a.e. in \( \Omega \). Then \( u \geq 0 \) a.e. in \( \Omega \).

**Proof.** The integral form of (16) is
\[
\int_{\mathbb{R}^n} (u(x) - u(y)) (\phi(x) - \phi(y)) K(x - y) dxdy = \int_\Omega f(x,u(x)) \phi(x) dx, \quad \forall \phi \in X_0, u \in X_0.
\]
Assume that \( u^- \) is not identically zero and write \( u = u^+ - u^- \) in \( \Omega \). Let \( \phi = u^- \), and then
\[
\int_{\mathbb{R}^n} (u(x) - u(y)) (u^- (x) - u^- (y)) K(x - y) dxdy
= \int_{\Omega \times \Omega} (u(x) - u(y)) (u^- (x) - u^- (y)) K(x - y) dxdy
+ 2 \int_\Omega dx \int_{\Omega} (u(x) - g(y)) u^- (x) K(x - y) dy
= \int_\Omega fu^- dx.
\]
Noticing that \( (u^+ (x) - u^+ (y)) (u^- (x) - u^- (y)) = 0 \), one has
\[
\int_{\Omega \times \Omega} (u(x) - u(y)) (u^- (x) - u^- (y)) K(x - y) dxdy
= - \int_{\Omega \times \Omega} (u^- (x) - u^- (y))^2 K(x - y) dy < 0,
\]
and
\[
\int_\Omega dx \int_{\Omega} (u(x) - g(x)) u^- (x) K(x - y) dxdy < 0.
\]
On the other hand, we have known that
\[
\int_\Omega fu^- \geq 0.
\]
Thus, we obtain a contradiction and finish this proof.

**Remark 2.** For the case of the normal Laplace operator, the nonnegativity of the solution for Eq.(16) is directly obtained by the maximum principle of the elliptic equation.

Of course, we can get the following result immediately.

**Corollary 1.** Let \( L_K \) be any operator and \( u_1,u_2 \in X_0 \) satisfy
\[
\begin{aligned}
& -L_K u_1 - f(x,u_1) \geq -L_K u_2 - f(x,u_2), \quad x \in \Omega, \\
& u_1 \geq u_2, \quad x \in \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]
then
\[
u_1(x) \geq u_2(x), \text{ a.e. } x \in \Omega.
\]
According to the comparison principle, we can introduce the definition of upper and lower solutions of (15).

**Definition 3.4.**
(i) \( \bar{u} \in X_0 \) is called as a upper solution if
\[
\begin{cases}
-L_K \bar{u} \geq f(x, \bar{u}), & x \in \Omega, \\
\bar{u} \geq g(x), & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
(ii) \( \underline{u} \in X_0 \) is called as a lower solution if
\[
\begin{cases}
-L_K \underline{u} \leq f(x, \underline{u}), & x \in \Omega, \\
\underline{u} \leq g(x), & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

Next, we introduce the comparison principle for the parabolic Lévy-diffusion equation as the following form
\[
\begin{align*}
\frac{\partial u}{\partial t} &= L_K w + f(x, t, w), & (x, t) &\in \Omega \times (0, T], \\
w &= g(x, t), & (x, t) &\in \mathbb{R}^n \setminus \Omega \times (0, T], \\
w(x, 0) &= w_0(x), & x &\in \Omega.
\end{align*}
\] (18)

**Lemma 3.5.** Let \( w \) be a solution to (18), assume \( f \geq 0, w_0 \geq 0 \) a.e in \( \Omega \) and \( g \geq 0 \) a.e. \((x, t) \in \mathbb{R}^n \setminus \Omega \times (0, T] \). Then, \( w \geq 0 \) in \((x, t) \in \Omega \times [0, T] \).

**Proof.** Let \( w(x, t) \) is a solution of (18). Assume that \( w(x, t) \) is not identically zero. Thus, there exists a maximal subset \( \Omega' \times J \) with \( \Omega' \subset \Omega \) and \( J \subset [0, T] \) such that \( w(x, t) < 0 \) for \((x, t) \in \Omega' \times J \). By Theorem 3.2, the solution \( w(x, t) \) is continuous and derivable with respect to the time variable \( t \). Since the initial value \( w(x, 0) \geq 0, x \in \Omega \), there must exist a point \((x_0, t_0)\) on the boundary of \( \Omega' \times J \), such that \( w(x_0, t_0) = 0 \), \( \frac{\partial w(x_0, t_0)}{\partial t} < 0 \) and \( w(x, t_0) \geq 0, x \in \Omega \). By the definition of nonlocal operator, we check the sign of the first equation in (18) on the point \((x_0, t_0)\),
\[
L_K w(x_0, t_0) = \int_{\mathbb{R}^n} [w(x_0 + y, t_0) + w(x_0 - y, t_0) - 2w(x_0, t_0)]K(y)dy \\
= \int_{\mathbb{R}^n} [w(x_0 + y, t_0) + w(x_0 - y, t_0)]K(y)dy \\
\geq 0.
\]

But, the sign for the left hand side of the first equation in (18) on the point \((x_0, t_0)\) is the less-than sign \( (\langle \rangle \)\). This contradiction is obtained. Therefore, we confirm \( w(x, t) \geq 0 \) for \((x, t) \in \Omega \times (0, T] \), and complete the proof of this lemma. \( \square \)

**Remark 3.** If Eq.(18) is equipped with the normal Laplace operator, then Eq.(18) degenerates into the standard second-order partial differential equation of parabolic type. For this case, the conclusion of Lemma 3.5 is directly gotten by the maximum principle of second-order parabolic equation.

Hence, we can obtain the following Corollary.

**Corollary 2.** Assume that \( w_1, w_2 \in C([0, T], X_0) \cap C^1((0, T], X_0) \) satisfy
\[
\begin{align*}
\frac{\partial w_1}{\partial t} - L_K w_1 - f(x, t, w_1) &\geq \frac{\partial w_2}{\partial t} - L_K w_2 - f(x, t, w_2), & (x, t) &\in \Omega \times (0, T], \\
w_1(x, 0) &\geq w_2(x, 0), & x &\in \Omega,
\end{align*}
\] (19)
then

\[ w_1(x, t) \geq w_2(x, t) \]

for \((x, t) \in \Omega \times [0, T]\).

Let \(\lambda_1(-L_K + q)\) be the principal eigenvalue of the following equation

\[
\begin{aligned}
- L_K u + q(x)u &= \lambda u, \quad x \in \Omega, \\
u &= 0, \quad x \in \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

We denote \(\lambda_1(-L_K)\) by \(\lambda_1\) for simplicity and the eigenfunction corresponding to \(\lambda_1\) is denoted by \(\phi_1\).

First, we consider the steady-state Lévy diffusion equation as the following form

\[
\begin{aligned}
- L_K u &= u(a - b(x)u), \quad x \in \Omega, \\
u &= 0, \quad x \in \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \(a > 0, b(x) \geq \varepsilon, \ a.e., x \in \Omega\). The solution to (20) is denoted by \(\Theta_{(a,b(x))}\). Then, we have the following result.

**Lemma 3.6.** (i) If \(a \leq \lambda_1\), then (20) has no nontrivial positive solution. Moreover, the trivial solution \(u = 0\) is globally asymptotically stable.

(ii) If \(a > \lambda_1\), then (20) has a unique nontrivial positive solution which is asymptotically stable. For this case, the trivial solution \(u = 0\) is unstable.

**Proof.** (i) Assume that (20) has a nontrivial positive solution \(u(x)\). We define a mapping \(J : X_0 \rightarrow \mathbb{R}\) as follows

\[
J(u) := \int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x - y)dx dy + \int_{\Omega} b u^3 dx.
\]

By Corollary 1, we know that \(0 \leq u(x) \leq \frac{a}{2}, \ a.e., x \in \Omega\), and then the following inequation holds

\[
a \equiv J(u) \geq \frac{\int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x - y)dx dy}{\int_{\Omega} u^2 dx} = \lambda_1.
\]

We get a contradiction. Thus, when \(a < \lambda_1\), then (20) has no nontrivial positive solution. Next, we will check the stability of the trivial solution \(u \equiv 0\). Linearizing the equation (20) at \(u = 0\), we obtain an eigenvalue problem

\[
\begin{aligned}
- L_K \phi(x) - a \phi(x) &= \lambda \phi(x), \quad x \in \Omega \\
\phi(x) &= 0, \quad x \in \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

Since \(a < \lambda_1\), all eigenvalues of (21) are positive. Therefore, the trivial solution \(u = 0\) is stable.

(ii) Let \(e_1\) be an eigenfunction corresponding to the principle eigenvalue \(\lambda_1\) for the operator \(-L_K\), where \(e_1 \in X_0\) and \(\lambda_1\) have been defined in Proposition 2. We define two functions with respect to the parameter \(\theta \in \mathbb{R}^+\),

\[
\begin{aligned}
F_1(\theta) &= \frac{\int_{\mathbb{R}^n} (\theta \cdot e_1(x) - \theta \cdot e_1(y))^2 K(x - y)dx dy}{\int_{\Omega} (\theta \cdot e_1)^2 dx} \\
&= \frac{\int_{\mathbb{R}^n} (e_1(x) - e_1(y))^2 K(x - y)dx dy}{\int_{\Omega} e_1^2 dx} = \lambda_1,
\end{aligned}
\]

and

\[
\begin{aligned}
F_2(\theta) &= a - \frac{\int_{\Omega} b (\theta \cdot e_1)^3 dx}{\int_{\Omega} (\theta \cdot e_1)^2 dx} = a - \theta \int_{\Omega} be_1^3 dx.
\end{aligned}
\]
Clearly, $\int_{\Omega} b(x)e_1^3(x)dx$ is a positive constant. The function $F_2(\theta)$ is a monotone decreasing function with respect to the parameter $\theta$. Since $a > \lambda_1$, there exist a positive constant $\theta_0$ such that

$$F_1(\theta_0) = a - F(\theta_0),$$

that is, $u = \theta_0 \cdot e_1$ is a nontrivial positive solution to (20).

Next, we will prove the uniqueness of the nontrivial positive solution to (20). Assume that there exists two nontrivial positive solutions to (20), denoted by $u_1$ and $u_2$, respectively. Then, for any $\phi_1(x), \phi_2(x) \in X_0(\Omega)$ we have

$$\int_{\mathbb{R}^2} (u_1(x) - u_2(y))(\phi_1(x) - \phi_2(y))K(x-y)dx = \int_{\Omega} u_1(a - bu_1)\phi_1 dx,$$ (23)

and

$$\int_{\mathbb{R}^2} (u_2(x) - u_2(y))(\phi_2(x) - \phi_2(y))K(x-y)dx = \int_{\Omega} u_2(a - bu_2)\phi_2 dx.$$ (24)

Let $\phi_1 = u_2$ and $\phi_2 = u_1$. Then (23) subtracting (24) yields

$$0 = -\int_{\Omega} b(x)u_1(x)u_2(x)(u_1(x) - u_2(x))dx.$$ (25)

Thus, we have $u_1(x) = u_2(x)$, a.e., $x \in \Omega$. The unique nontrivial positive solution to (20) is $\bar{u} = \theta_0 \cdot e_1$, where $\theta_0$ is some positive constant.

Finally, we discuss the stability of the solutions in the case of $\lambda_1 < a$. It is obvious that the first eigenvalue of (21) is less than 0. Therefore, the solution $u = 0$ is unstable. Linearizing the equation (20) at $u = \bar{u}$, we obtain an eigenvalue problem

$$\begin{cases}
-L_K \phi(x) - a\phi(x) + 2b\bar{u}\phi(x) = \lambda \phi(x), & x \in \Omega \\
\phi(x) = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}$$ (26)

By virtue of (22), the minimum eigenvalue $\lambda^*_1$ of (26) is

$$\lambda^*_1 = \int_{\mathbb{R}^2} (e_1(x) - e_2(y))^2K(x-y)dx - a + 2\theta_0 \int_{\Omega} b\bar{u}^3dx = \theta_0 \int_{\Omega} b\bar{u}^3dx > 0.$$ (27)

Therefore, the solution $u = \bar{u}$ of (20) is stable in the case of $a > \lambda_1$. We complete this proof.

**Remark 4.** The condition of $b(x) \geq \epsilon$ in Lemma 3.6 can be relaxed into $b(x) \geq 0$ a.e. $x \in \Omega$. For this case, Lemma 3.6 is still available. This proof only needs to make small modification. In detail, let $\epsilon$ be a sufficient small positive constant, we consider the following equation

$$\begin{cases}
-L_K u_\epsilon = u_\epsilon[a - (b(x) + \epsilon)u_\epsilon], & x \in \Omega \\
u_\epsilon = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}$$ (27)

where $b(x) + \epsilon \geq \epsilon$ a.e. $x \in \Omega$. By the method in Lemma 3.6 and by letting $\epsilon \to 0$, we can obtain $\lim_{\epsilon \to 0} u_\epsilon = u$ in the sense of the $X_0$ norm, where $u$ satisfies (20).

**Remark 5.** The results about Eq.(20) with the special case of the normal Laplace operator are well known. But, for the general Lévy-diffusion form, few result is known. Eq.(20) is crucial to prove the stability and bifurcation of the steady-state solution for (11). Thus, here we give out the detailed proof of Lemma 3.6.
4. Existence and uniqueness of globally positive solution to (10). In this section, by using Theorem 3.2 we will prove the existence and uniqueness of the solution to the system (10). Let \( H = L^2(\Omega) \times L^2(\Omega) \). Assume \( w = (u, v) \in X_0 \times X_0 \) and its norm is defined as \( ||w|| = ||u||_{X_0} + ||v||_{X_0} \). The model (10) can be rewritten as an abstract differential equation:

\[
\frac{dz}{dt} = Az + F(z), t \in (0, +\infty),
\]

(28a)

\[
z(0) = z_0,
\]

(28b)

where \( z = (u(\cdot, t), v(\cdot, t))^T \), \( z_0 = (u_0, v_0)^T \),

\[
A = \begin{pmatrix} L_K & 0 \\ 0 & L_K \end{pmatrix}
\]

(29)

with

\[
D(A) = \{(u, v) \in X_0 \times X_0 \},
\]

and

\[
F(z) = (f_1, f_2)^T = \begin{pmatrix} u(c - u) - \frac{mu}{av^2 + bu + 1} \\ v(\eta - \frac{rv_i}{u_i + \xi}) \end{pmatrix}.
\]

(30)

First, we give out the following Lemma.

**Lemma 4.1.** For every \( z_0 \in H_+ \), the Cauchy problem (28) has a unique nonnegative maximal local solution

\[
z \in C([0, T_{\text{max}}); H) \cap C^1((0, T_{\text{max}}); H) \cap C((0, T_{\text{max}}); D(A)),
\]

(31)

which satisfies the following Duhamel formula for \( t \in [0, T_{\text{max}}) \):

\[
z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}F[z(s)]ds,
\]

(32)

Moreover, if \( T_{\text{max}} < \infty \) then \( \limsup_{T_{\text{max}}} ||z(t)|| = \infty \).

**Proof.** The operator \( A \) defined in \( H \) can generate an analytic, condensed, strong continuous operator semigroup \( e^{tA} \). Furthermore, from Proposition 2 and for \( t > 0 \) we have

\[
||e^{tA}w|| \leq e^{-Mt}||w||, \forall w \in X,
\]

(33)

where \( M \) is a positive constant and \( 0 < M < \lambda_1 \). Noting that \( F : D(A) \to H \) satisfies locally Lipschitz condition on a bounded set and by the contracting mapping principle and theorem 7.2.1 in [3], we know that there exists the unique solution for the evolution equation (28) defined on a maximal interval \((0, T_{\text{max}})\). Next, we shall show that this solution is nonnegative. To prove \( u(x, t), v(x, t) \geq 0 \) for \( t \in (0, T_{\text{max}}) \) and a.e. \( x \in \Omega \), we consider the following auxiliary system

\[
\begin{align*}
\frac{\partial u_i'}{\partial t} &= L_K u_i' + u_i' (c - u_i') - \frac{mu_i' v_i'}{a(u_i') x + bu_i' + 1}, & x &\in \Omega, \\
\frac{\partial v_i'}{\partial t} &= L_K v_i' + v_i' (\eta - \frac{rv_i'}{u_i' + \xi}), & x &\in \Omega
\end{align*}
\]

(34)

with boundary and initial conditions

\[
\begin{align*}
u_i' &= v_i' = 0, & x &\in \mathbb{R}^n \setminus \Omega, \\
u_i'(x, 0) &= u_0(x), & v_i'(x, 0) &= v_0(x).
\end{align*}
\]
Multiplying (34) by \( u'_- \) and \( v'_- \), respectively, integrating over the domain \( \Omega \) and noting
\[
\int_{\mathbb{R}^n} -L_K u'_- u_- dx = \int_{\mathbb{R}^n} (u'_+(x) - u'_-(y))(u'_-(x) - u'_-(y))K(x-y)dx dy = 0,
\]
we obtain
\[
- \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_-|^2 dx - \int_{\mathbb{R}^n} (u'_-(x) - u'_-(y))^2 K(x-y)dx dy = 0,
\]
and
\[
- \frac{1}{2} \frac{d}{dt} \int_{\Omega} |v_-|^2 dx - \int_{\mathbb{R}^n} (v'_-(x) - v'_-(y))^2 K(x-y)dx dy = 0.
\]

Hence, for \( t \in (0, T_{max}) \)
\[
\|u'_-(\cdot, t)\|_2^2 + \|v'_-(\cdot, t)\|_2^2 \leq \|u'_-(\cdot, 0)\|_2^2 + \|v'_-(\cdot, 0)\|_2^2 = 0.
\]

Consequently,
\[
u'(x, t) \geq 0, \quad v'(x, t) \geq 0, \quad (x, t) \in \Omega \times [0, T_{max}).
\]

Now, we can know that \( (u'(x, t), v'(x, t)) \) is a solution of (10), and according to the uniqueness of the solution in Lemma 4.1 we get that
\[
u(x, t) = u'(x, t) \geq 0, \quad v(x, t) = v'(x, t) \geq 0, \quad t \in (0, T_{max}).
\]

So, we get this lemma.

\[ \square \]

**Remark 6.** Of course, there are other ways to prove the nonnegativity of this solution. For example, we may apply the positive property of the operator semigroup \( \{e^{tA}\}_{t \geq 0} \) because the resolvent \( (\lambda - L_K)^{-1} \) is positive for \( \lambda \in \rho(L_K) \).

Next, we shall prove the existence of the globally positive solution for (10). Via Lemma 4.1 we only need to show that the solution of (10) is essentially bounded, i.e., dissipation.

**Lemma 4.2.** The solution of (10) satisfies the following inequations
\[
u(x, t) \leq \bar{u}, \quad v(x, t) \leq \bar{v}, \tag{35}
\]
for \( (x, t) \in \Omega \times (0, T], \quad 0 < T < T_{max}, \) and where \( \bar{u} = \max\{c, \|u_0\|_\infty\}, \quad \bar{v} = \max\left\{ \frac{(\bar{u} + k)\eta}{\tau}, \|v_0\|_\infty \right\} \).

**Proof.** By Lemma 4.1, the system (10) has a unique nonnegative solution \( (u, v) \) for \( (x, t) \in \Omega \times [0, T] \).

Then, the following inequations hold
\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - L_K \bar{u} - f_1(\bar{u}, v) \geq \frac{\partial u}{\partial t} - L_K u - f_1(u, v), \quad (x, t) \in \Omega \times (0, T], \\
\bar{u} > u = 0, \quad (x, t) \in x \in \mathbb{R} \setminus \Omega \times (0, T], \\
\bar{u} \geq u_0(x), \quad x \in \Omega.
\end{array} \right.
\]

By Corollary 2, we can know that \( u(x, t) \leq \bar{u} \) a.e. \( (x, t) \in \Omega \times (0, T] \).

Similarly, we have
\[
\left\{ \begin{array}{l}
\frac{\partial v}{\partial t} - L_K \bar{v} - f_2(u, \bar{v}) \geq \frac{\partial v}{\partial t} - L_K v - f_2(u, v), \quad (x, t) \in \Omega \times (0, T], \\
\bar{v} \geq v = 0, \quad (x, t) \in x \in \mathbb{R} \setminus \Omega \times (0, T], \\
\bar{v} \geq v_0(x), \quad x \in \Omega.
\end{array} \right.
\]

By the same method, we can obtain \( v(x, t) \leq \bar{v} \) a.e. \( (x, t) \in \Omega \times (0, T] \). As a result, we complete this proof.

\[ \square \]
By Lemmas 4.1 and 4.2, we can get the following theorem.

**Theorem 4.3.** The system (28) has a unique, nonnegative and bounded solution 
\[ z(x, t) = (u(x, t), v(x, t))^T, \]
such that
\[ z \in C((0, \infty); H) \cap C^1((0, \infty); H) \cap C((0, \infty); D(A)), \]

5. **Stability and asymptotic behavior.** In this section, the asymptotic behavior of (10) is considered. By means of Lemma 3.6, we will study the stability of semi-trivial solutions to (11). Obviously, (11) have two semi-trivial solutions, \((u_*, 0)\) with \(c > \lambda_1\) satisfying
\[
\begin{align*}
    -L_Ku &= u(c - u), & x \in \Omega, \\
    u &= 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{align*}
\]
and \((0, v_*)\) with \(\eta > \lambda_1\) satisfying
\[
\begin{align*}
    -L_Kv &= \eta \left( \gamma - \frac{r v}{\Omega} \right), & x \in \Omega, \\
    v &= 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{align*}
\]
Firstly, we give out the stability of trivial solution \((0, 0)\) and semi-trivial solutions \(((u_*, 0), (0, v_*))\).

**Theorem 5.1.**

(i) If \(c < \lambda_1\) and \(\eta < \lambda_1\), then the trivial solution \((0, 0)\) is stable; if \(c > \lambda_1\) or \(\eta > \lambda_1\), then trivial solution \((0, 0)\) is unstable;

(ii) Let \(c > \lambda_1\). If \(\eta < \lambda_1\), then semi-trivial solution \((u_*, 0)\) is stable; if \(\eta > \lambda_1\), then semi-trivial solution \((u_*, 0)\) is unstable;

(iii) Let \(\eta > \lambda_1\). If \(c < \lambda_1\), then semi-trivial solution \((0, v_*)\) is stable; if \(c > \lambda_1\), then semi-trivial solution \((0, v_*)\) is unstable.

**Proof.** (i) and (iii) can be shown similarly as in the proof of (ii), and so we only prove (ii). Let \(L_1\) be the linearized operator of (11) at \((u_*, 0)\) as follows
\[
L_1 = \begin{pmatrix}
-L_K - (c - 2u_*) & \frac{mu_*}{u_* + b u_* + 1} \\
0 & -L_K - \eta
\end{pmatrix}.
\]
The spectrum of \(L_1\) consists of real eigenvalues and \(\sigma(L_1) = \sigma(-L_K - (c - 2u_*)) \cup \sigma(-L_K - \eta).\) Obviously, \(\lambda_1(-L_K - (c - 2u_*)) > \lambda_1(-L_K - (c - u_*)) = 0.\) If \(\eta < \lambda_1\), then \(\lambda_1(-L_K - \eta) > 0.\) Hence \(\sigma(L_1) > 0.\) This implies that semi-trivial solution \((u_*, 0)\) is stable. Similarly, if \(\eta > \lambda_1\), then \(\lambda_1(-L_K - \eta) < 0.\) Hence, semi-trivial solution \((u_*, 0)\) is unstable.

Next, we will discuss the sufficient conditions for the globally asymptotic behavior of the semi-trivial solutions of (11).

**Theorem 5.2.** Let \((u, v)\) be a positive solution of (10).

(i) If \(c > \lambda_1, \eta \leq \lambda_1, \) then \((u, v) \to (u_*, 0)\) as \(t \to \infty;\)

(ii) If \(c \leq \lambda_1, \eta > \lambda_1, \) then \((u, v) \to (0, v_*)\) as \(t \to \infty.\)

**Proof.** (i) Since \(c > \lambda_1\) and \(-L_Ku \leq u(c - u)\) and by Corollaries 2 and 1, we get
\[
\limsup_{t \to \infty} u(x, t) \leq u_*.
\]
Let \(\epsilon\) be a sufficiently small positive constant with \(c - m\epsilon > \lambda_1.\) Then, there exists a constant \(T_\epsilon \geq 0\) such that \(u(x, t) \leq u_* + \epsilon\) for \(t > T_\epsilon.\) Then, we have
\[
\begin{align*}
\frac{\partial v}{\partial t} - L_Kv &\leq \left( \eta - \frac{r v}{(u_* + v*) + k} \right), & x \in \Omega, t > T_\epsilon, \\
v &= 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{align*}
\]
Since \( \eta < \lambda_1 \), by virtue of Lemmas 3.6, 3.5 and Corollary 1, we get \( v(x, t) \to 0 \) as \( t \to \infty \) for all \( x \in \Omega \). Therefore, there exists a constant \( T'_e \) such that \( v(x, t) < \epsilon \) for \( t > T'_e \). Hence, we have
\[
\begin{align*}
&\frac{\partial u}{\partial t} - L_K u \geq u \left( \epsilon - u - \frac{m \epsilon}{z} \right), \quad x \in \Omega, t > T'_e, \\
&u = 0, \quad x \in \mathbb{R}^n \setminus \Omega, t > T'_e,
\end{align*}
\]
where
\[
z = \begin{cases} 
1, & b \geq 0, \\
\frac{4a-b^2}{4a}, & -2\sqrt{a} < b < 0.
\end{cases}
\]
Similarly, we obtain
\[
\liminf_{t \to \infty} u(x, t) \geq \Theta_{(c-m \eta/z, 1)}.
\]
Using the continuity for \( \epsilon \to 0 \), (36) and (39), it follows that \( u(x, t) \to u_* \) as \( t \to \infty \).

(ii) It can be shown similarly as in the proof of (i).

Assume that \( c > \lambda_1 \), and by Lemma 3.6, the following problem
\[
\begin{align*}
&-L_K v = v \left( \eta - \frac{rv}{u_*+r} \right), \quad x \in \Omega, \\
v &= 0, \quad x \in \mathbb{R}^n \setminus \Omega
\end{align*}
\]
has a unique nontrivial positive solution if \( \eta > \lambda_1 \). We denote the unique nontrivial positive solution by \( v^* \). From Corollary 1 we know \( v_* \leq v^* \).

The following theorem provides sufficient conditions to ensure permanence to (10).

**Theorem 5.3.** If \( c - \frac{m(c+k)\eta}{z} > \lambda_1 \), \( \eta > \lambda_1 \), where \( z \) is defined by (38), then there exists a pair of functions \((\hat{u}, \hat{v})\) and \((\tilde{u}, \tilde{v})\) in \( X_0(\Omega) \times X_0(\Omega) \) such that they satisfy
\[
\Theta_{(c-m \eta/z, 1)} \leq \hat{u} \leq \tilde{u} \leq u_*, \quad v_* \leq \hat{v} \leq \tilde{v} \leq v^*
\]
and the equations
\[
\begin{align*}
-\hat{L}_K \hat{u} &= \hat{u}(c - \hat{u}) - \frac{m \hat{u}}{\alpha \hat{u}^2 + \beta \hat{u} + 1}, \\
-\hat{L}_K \hat{\tilde{v}} &= \hat{\tilde{v}}(c - \hat{\tilde{v}}) - \frac{m \tilde{v}}{\alpha \tilde{v}^2 + \beta \tilde{v} + 1}, \\
-\hat{L}_K \hat{\tilde{v}} &= \hat{\tilde{v}}(c - \hat{\tilde{v}}) - \frac{m \hat{v}}{\alpha \hat{v}^2 + \beta \hat{v} + 1}, \\
-\hat{L}_K \hat{v} &= \hat{v}(c - \hat{v}) - \frac{m \hat{\tilde{v}}}{\alpha \hat{\tilde{v}}^2 + \beta \hat{\tilde{v}} + 1}, \\
\hat{u} &= \tilde{u} = 0, \quad \hat{\tilde{v}} = \hat{\tilde{v}} = 0, \quad x \in \mathbb{R}^n \setminus \Omega.
\end{align*}
\]
Furthermore, \([\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}]\) is a positive global attractor of (10).

**Proof.** Let
\[
f(u, v) = u(c - u) - \frac{muv}{au^2 + bu + 1},
\]
\[
g(u, v) = v \left( \eta - \frac{rv}{u+k} \right).
\]
It is obvious that \( f \) is monotone decreasing with respect to \( v \), and that \( g \) is monotone increasing with respect to \( u \). According to the definition 3.4 of the upper and lower solutions, it follows that \((u_*, v^*)\) and \((\Theta_{(c-m \eta/z, 1)}, v_*)\) are a pair of ordered upper and lower solutions of (11). Furthermore, \( f, g \) satisfies the Lipschitz condition in a bounded set \([\Theta_{(c-m \eta/z, 1)}, u_*] \times [v_*, v^*]\). There exist two pairs of functions \((\hat{u}, \hat{v})\) and \((\tilde{u}, \tilde{v})\), which satisfy (41) and (42).
Next, we prove that $[\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}]$ is a positive global attractor of (10). Let $\epsilon$ be a sufficiently small positive constant. Using similar method in the proof of Theorem 5.2, there exists a $T_\epsilon \geq 0$ such that

$$u(x, t) \leq u_* + \epsilon$$

for all $t > T_\epsilon$, which satisfies (37). Thus, $\limsup_{t \to \infty} v(x, t) \leq v^*$ by Lemma 3.5 and Corollary 1. Hence, there exists a $T'_\epsilon \geq 0$ such that

$$v(x, t) \leq v^* + \epsilon$$

for all $t > T'_\epsilon$. On the other hand, the first and second equations of (10) indicate that

$$\frac{\partial u}{\partial t} - L_K u = u(c - u) - mv^* u / z, \quad x \in \Omega, t > 0,$$

and

$$\frac{\partial v}{\partial t} - L_K v = v(\eta - \frac{v}{K}), \quad x \in \Omega, t > 0,$$

where $u, v \in \Omega^n \setminus \Omega$. The condition of $c - \frac{m(c + k) \eta}{z} > \lambda_1$ ensures that the solution $\Theta_{(c - mv^*/z, 1)}$ exists. By Lemma 3.5, we have

$$\liminf_{t \to \infty} u(x, t) \geq \Theta_{(c - mv^*/z, 1)}$$

and

$$\liminf_{t \to \infty} v(x, t) \geq v_*.$$

Then, there exists a constant $T''_\epsilon \geq 0$ such that

$$u(x, t) \geq \Theta_{(c - mv^*/z, 1)} - \epsilon, \quad v(x, t) \geq v_* - \epsilon$$

for all $t > T''_\epsilon$. Finally, take $T^* = \max\{T_\epsilon, T'_\epsilon, T''_\epsilon\}$, and then for all $t > T^*$,

$$(u, v) \in \left[\Theta_{(c - mv^*/z, 1)} - \epsilon, [u_* + \epsilon, v^* + \epsilon], \forall \epsilon > 0.$$.

Letting $\epsilon \to 0$, we can obtain this result.

6. Existence and nonexistence of coexistence states. In this section, we will discuss the existence and nonexistence of coexistence states to (11).

6.1. Existence of coexistence states. In this subsection, we determine the sufficient conditions for the existence states to (11) by calculating the index of fixed points. First, we give out the priori estimates for coexistence states to (11).

**Lemma 6.1.** Suppose that $(u, v)$ is a coexistence state to (11), then

$$u(x) \leq u_* < c, \quad v(x) \leq v(x) \leq \Theta_{(\eta, r/(c+k))} < \frac{\eta(c + k)}{r}.$$

**Proof.** At first, from the first equation in (11), it follows that

$$0 = \lambda_1 \left( -L_K - c + u + \frac{mv}{au^2 + bu + 1} \right) > \lambda_1 (-L_K - c).$$

Thus, $c > \lambda_1$. Similarly, we can obtain $\eta > \lambda_1$. Since $-L_K u \leq u(c - u), u \leq u_*$ and by the uniqueness of $u_*$ and the method of the upper and lower solution, we have $u(x) \leq u_* < c$. Similarly, we can get $v_* \leq v(x) \leq \Theta_{(\eta, r/(c+k))} < \eta(c + k)/\eta$. □

Thus, from the proof of Lemma 6.1 we have the following result.

**Lemma 6.2.** If the system (11) has a coexistence state, then $c > \lambda_1$ and $\eta > \lambda_1$. 
Let $E = X_0(\Omega) \bigoplus X_0(\Omega)$, $P = \{u \in X_0(\Omega) : u \geq 0, \text{a.e. } x \in \Omega\}$, $W = P \bigoplus P$. $E = W - W = \{(u_1, v_1) - (u_2, v_2) : (u_i, v_i) \in W, i = 1, 2\}$. Let $\vartheta \subset W$ be a bounded open set with respect to the relative topology of $W$. Assume $F : \vartheta \to W$ is a compact and Fréchet differentiable operator. For $(z_1, z_2) \in \vartheta$ be a fixed point of $F$, define a wedge by

$$W_{(z_1, z_2)} = \{(u, v) \in E : (z_1, z_2) + s(u, v) \in W \text{ for some } s > 0\}.$$ 

Let $E_{(z_1, z_2)}$ be the maximal subspace of $E$ contained in $W_{(z_1, z_2)}$. If there exists a subspace $\tilde{E}_{(z_1, z_2)}$ such that $E = E_{(z_1, z_2)} \bigoplus \tilde{E}_{(z_1, z_2)}$, then $\text{index}_W(F, (z_1, z_2))$ of $(z_1, z_2)$ can be calculated by analyzing certain eigenvalue problems in $E_{(z_1, z_2)}$ and $\tilde{E}_{(z_1, z_2)}$. For details, we have the following theorem.

**Theorem 6.3.** [32] Let $T : E \to \tilde{E}_{(z_1, z_2)}$ be a projection operator of $\tilde{E}_{(z_1, z_2)}$ along $E_{(z_1, z_2)}$. If the Fréchet derivative $F'(z_1, z_2)$ of $F$ at $(z_1, z_2)$ has no nonzero fixed point in $W_{(z_1, z_2)}$, then $\text{index}_W(F, (z_1, z_2))$ exists. Moreover, we have

(i) If $T \circ F'(z_1, z_2)$ has an eigenvalue $\lambda > 1$ with eigenvector in $W_{(z_1, z_2)}$, then $\text{index}_W(F, (z_1, z_2)) = 0$.

(ii) If $T \circ F'(z_1, z_2)$ has no such eigenvalue, then

$$\text{index}_W(F, (z_1, z_2)) = \text{index}_{E_{(z_1, z_2)}}(F'(z_1, z_2), (0, 0)) = (−1)^s,$$

where, $\text{index}_{E_{(z_1, z_2)}}(F'(z_1, z_2), (0, 0))$ is the index of the linear operator $F'(z_1, z_2)$ at $(0, 0)$ in the space $E_{(z_1, z_2)}$ and $r$ is the sum of multiplicities of the eigenvalues $\lambda$ of $F'(z_1, z_2)$ restricted in $E_{(z_1, z_2)}$ such that $\lambda < 1$.

Let $\vartheta = \{(u, v) \in W : u \leq c + 1, v \leq \eta(c + k)/r + 1\}$. Take $q$ sufficiently large with $q > \max\{c + 1, \eta(c + k)/r + 1\}$. Define the operator $F : E \to E$ by

$$F(u, v) = (-L_K + q)^{-1} \left( \begin{array}{c} u(c - u) - \frac{mvw}{au + \eta} + qu \\ v(\eta - \frac{uw}{a + c}) +qv \end{array} \right).$$

By Lemma 2.2, we know that the mapping $F : \vartheta \to W$ is compact. Furthermore, finding a positive solution of (11) is equivalent to finding a positive fixed point of $F$. Now, we can show the indexes of $F$ at $(0, 0), (u_*, 0), (0, v_*)$ in $W$.

**Lemma 6.4.** If $c \neq \lambda_1$, $\eta \neq \lambda_1$ then

(i) $\text{index}_W(F, (0, 0)) = 0$ if $c > \lambda_1$ or $\eta > \lambda_1$, $\text{index}_W(F, (0, 0)) = 1$ if $c < \lambda_1$ and $\eta < \lambda_1$.

(ii) Assume $c > \lambda_1$, then $\text{index}_W(F, (u_*, 0)) = 0$ if $\eta > \lambda_1$, $\text{index}_W(F, (u_*, 0)) = 1$ if $\eta < \lambda_1$.

(iii) Assume $\eta > \lambda_1$ and $c \neq \lambda_1(-L_K + mv_*), \text{index}_W(F, (0, v_*)) = 0$ if $c > \lambda_1(-L_K + mv_*)$, $\text{index}_W(F, (0, v_*)) = 1$ if $c < \lambda_1(-L_K + mv_*)$.

**Proof.** (i) By the direct calculation we can get $W_{(0, 0)} = W$, $E_{(0, 0)} = \{(0, 0)\}$, $\tilde{E}_{(0, 0)} = E$, $T = Id$, where $Id$ is the indentity operator in $E$. It is easy to prove

$$F'(0, 0) \left( \begin{array}{c} u \\ v \end{array} \right) = (-L_K + q)^{-1} \left( \begin{array}{c} (c + q)u \\ (\eta + q)v \end{array} \right).$$

(45)

If $\exists(u, v) \in W_{(0, 0)}$ such that $F'(0, 0)(u, v) = (u, v)$, then $(u, v)$ satisfies

$$\begin{cases} -L_Ku = cu, & x \in \Omega, \\ -L_Kv = \eta v, & x \in \Omega, \\ u = v = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$
Since $c \neq \lambda_1$ and $\eta \neq \lambda_1$, we have $(u, v) = (0, 0)$. So $F'(0, 0)$ has no nonzero fixed point in $\mathbb{W}_{(0,0)}$. Note that $\mu$ is an eigenvalue of $Id \circ F'(0, 0) = F'(0, 0)$ and $(u, v) \in \mathbb{W}_{(0,0)}$ is the corresponding eigenvector if and only if $(u, v)$ is a solution of the problem

$$
\mu \begin{pmatrix} u \\ v \end{pmatrix} = (-L_K + q)^{-1} \begin{pmatrix} (c + q)u \\ (\eta + q)v \end{pmatrix}, \quad (u, v) \neq (0, 0). \quad (46)
$$

So, the first eigenvalue (maximal eigenvalue) $\mu_1$ of (46) is $\mu_1 = \mu_c = \frac{c^2 + q}{\lambda_1 + q}$ or $\mu_1 = \mu_\eta = \frac{\eta + q}{\lambda_1 + q}$. If $c > \lambda_1$ or $\eta > \lambda_1$, then $\mu_c > 1$ or $\mu_\eta > 1$, so $\text{index}_W(F, (0, 0)) = 0$. If $c < \lambda_1$ and $\eta < \lambda_1$, then $\mu_c < 1$ and $\mu_\eta < 1$, so $\text{index}_W(F, (0, 0)) = \text{index}_{E_{(0,0)}}(F'(0, 0), (0, 0)) = (-1)^r$. Since $E_{(0,0)} = \{(0, 0)\}$, it follows that $r = 0$. Hence, $\text{index}_W(F, (0, 0)) = 1$.

(2) By direct calculation, we get $\mathbb{W}_{(u,0)} = \{(u, v) \in E : v \geq 0\}$, $E_{(u,0)} = \{(u, v) \in E : v = 0\}$. Set $E_{(u,0)} = \{(u, v) \in E : u = 0\}$, we have $E = E_{(u,0)} \oplus E_{(u,0)}$, $T : (u, v) \rightarrow (0, v)$, and

$$
F'(u, 0) \begin{pmatrix} u \\ v \end{pmatrix} = (-L_K + q)^{-1} \begin{pmatrix} (c - 2u_*)u - \frac{mu_*v}{au_*^2 + bv_* + 1} + qu \\ \eta v + qv \end{pmatrix}. \quad (47)
$$

Here, we take $q$ large such that $c - 2u_* + q > 0$. If $\exists (u,v) \in \mathbb{W}_{(u,0)}$ such that $F'(u, 0)(u,v) = (u,v)$, then $(u,v)$ satisfies

$$
\begin{cases}
-L_K u = (c - 2u_*)u - \frac{mu_*v}{au_*^2 + bv_* + 1}, & x \in \Omega, \\
-L_K v = \eta v, & x \in \Omega, \\
u = v = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (48)
$$

Since $v \geq 0$ and $\eta \neq \lambda_1$, we have $v = 0$, and $u$ satisfies

$$
\begin{cases}
-L_K u = (c - 2u_*)u, & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (48)
$$

It is obvious that $\lambda_1(-L_K - c + 2u_*) > \lambda_1(-L_K - c + u_*) = 0$. Thus $u = (-c + 2u_*)^{-1}0 = 0$. We obtain that $F'(u, 0)$ has no nonzero fixed point in $\mathbb{W}_{(u,0)}$.

Next, we consider the eigenvalue of $T \circ F'(u, 0)$. The projection $T : (u, v) \rightarrow (0, v)$ gives that every eigenvector of $T \circ F'(u, 0)$ has the form $(0, v)$ with $v \neq 0$. Let $T \circ F'(u, 0)$ be the corresponding eigenvector if and only if $v$ is the solution of the problem

$$
\begin{cases}
\mu v = (-L_K + q)^{-1}(\eta + q)v, & x \in \Omega, \\
v = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (49)
$$

So the first eigenvalue $\mu_1$ of (49) is $\mu_1 = \mu_\eta = \frac{\eta + q}{\lambda_1 + q}$. If $\eta > \lambda_1$, then $\mu_1 > 1$, so $\text{index}_W(F, (u,0), 0) = 0$. If $\eta < \lambda_1$, then $\mu_1 < 1$, so $\text{index}_W(F, (u,0), 0) = \text{index}_{E_{(u,0)}}(F'(u, 0), (0, 0)) = (-1)^r$. Suppose $\lambda$ is an eigenvalue of $F'(u, 0)$ and $(\xi_1, \xi_2)$ is the corresponding eigenvalue in $E_{(u,0)}$, then $\xi_2 = 0$, and $\xi_1 \neq 0$ solves

$$
\lambda \xi_1 = (-L_K + q)^{-1}(c + q - 2u_*)\xi_1, \quad \xi_1 \neq 0. \quad (50)
$$

Thus, $\lambda \neq 0$ and

$$
\begin{cases}
-L_K \xi_1 + 2u_*\xi_1 + \frac{\lambda - 1}{\lambda}(c + q - 2u_*)\xi_1 = c\xi_1, & x \in \Omega, \\
\xi_1 = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (51)
$$
Hence, $c = \lambda_j(-L_K + 2u_\ast + \frac{\lambda - 1}{\lambda}(c + q - 2u_\ast))$ for some $j \geq 1$. Since $c < \lambda_1(-L_K + 2u_\ast)$ by Lemma 3.6 and $c + q - 2u_\ast > 0$, we have $\lambda < 1$. Thus $r = 0$ and $index_W(F,(u_\ast,0)) = 1$.

(3) By direct computation we get $W((u_\ast,0)) = \{(u,v) \in E : u \geq 0\}$, $E((u_\ast,0)) = \{(u,v) \in E : u = 0\}$. Set $E(u_\ast,0) = \{(u,v) \in E : v = 0\}$, we have $E = E((u_\ast,0)) \oplus E((u_\ast,0))$, $T : (u,v) \rightarrow (u,0)$, and

$$F'(0,v) = \begin{pmatrix} u \\ v \end{pmatrix} = (-L_K + q)^{-1} \begin{pmatrix} (-mv_\ast + c + q) u \\ \frac{u^2r}{k^2} + (\eta + q - \frac{2rv_\ast}{k}) v \end{pmatrix}. \quad (52)$$

Here, we take $q$ large such that $c + q - mv_\ast > 0$ and $\eta + q - \frac{2rv_\ast}{k} > 0$. If $\forall (u,v) \in W((u_\ast,0))$ such that $F'(0,v_\ast)(u,v) = (u,v)$, then $u,v$ satisfies

$$\begin{cases} -L_K u = (c - mv_\ast)u, & x \in \Omega, \\ -L_K v = \frac{u^2r}{k^2} + (\eta - \frac{2rv_\ast}{k}) v, & x \in \Omega, \\ u = v = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (53)$$

Since $u \geq 0$ and $c \neq \lambda_1(-L_K + mv_\ast)$, we have $u = 0$, and $v$ satisfies

$$\begin{cases} -L_K v = (\eta - \frac{2rv_\ast}{k}) v, & x \in \Omega, \\ v = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (54)$$

By the monotone property of $\lambda_1$, it follows that

$$\lambda_1\left(-L_K - \eta + \frac{2rv_\ast}{k}\right) > \lambda_1\left(-L_K - \eta + \frac{rv_\ast}{k}\right) = 0.$$ 

So, $F'(0,v_\ast)$ has no nonzero fixed point in $W((u_\ast,0))$.

Next, we consider the eigenvalue of $T \circ F'(0,v_\ast)$. The projection $T : (u,v) \rightarrow (u,0)$ gives that every eigenvector of $T \circ F'(u_\ast,0)$ has the form $(u,0)$ with $u \neq 0$. Note that $\mu$ is an eigenvalue of $T \circ F'(0,v_\ast)$ and $u$ is the corresponding eigenvector if and only if $u$ is the solution of the problem

$$\mu u = (-L_K + q)^{-1} (c + q - mv_\ast) u, \ u \neq 0. \quad (55)$$

Therefore $\mu \neq 0$ and

$$\begin{cases} -L_K u + mv_\ast u + \frac{\mu - 1}{\mu}(c + q - mv_\ast) u = cu, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (56)$$

Hence, $c = \lambda_j(-L_K + 2u_\ast + \frac{\lambda - 1}{\lambda}(c + q - 2u_\ast))$ for some $j \geq 1$. Since $c + q - 2u_\ast > 0$, we obtain $\mu > 1$ if $c > \lambda_1(-L_K + 2u_\ast)$ and $\mu < 1$ if $c < \lambda_1(-L_K + 2u_\ast)$. So, we get $index_W(F(0,v_\ast)) = 0$ if $c > \lambda_1(-L_K + 2u_\ast)$ and $index_W(F(0,v_\ast)) = index_E(F(0,v_\ast),(0,0)) = (-1)^r$ if $c < \lambda_1(-L_K + 2u_\ast)$. Suppose $\lambda$ is an eigenvalue of $F'(0,v_\ast)$ and $\xi_1, \xi_2$ is the corresponding eigenvalue in $E((u_\ast,0))$, then $\xi_1 = 0$, and $\xi_2 \neq 0$ solves

$$\lambda \xi_2 = (-L_K + q)^{-1}(\eta + q - \frac{2rv_\ast}{k}) \xi_2, \ \xi_2 \neq 0. \quad (57)$$

Therefore $\lambda \neq 0$ and

$$\begin{cases} -L_K \xi_2 + \frac{2rv_\ast}{k} \xi_2 + \frac{\lambda - 1}{\lambda}(\eta + q - \frac{2rv_\ast}{k}) \xi_2 = \eta \xi_2, & x \in \Omega, \\ \xi_2 = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (58)$$

Hence, $\eta = \lambda_j(-L_K + \frac{2rv_\ast}{k} + \frac{\lambda - 1}{\lambda}(\eta + q - \frac{2rv_\ast}{k}))$ for some $j \geq 1$. Since $\eta < \lambda_1(-L_K + 2u_\ast)$ and $\eta + q - \frac{2rv_\ast}{k} > 0$ and by Lemma 3.6, we have $\lambda < 1$. Thus $r = 0$ and $index_W(F,(0,v_\ast)) = 1$. \qed
Theorem 6.6. If anyone of the following five conditions holds, then the problem (11) has at least one positive solution.

Proof. For \( s \in [0,1] \), we define the compact homotopy operators \( F_s : E \to E \) by

\[
F_s(u, v) = F(u, v) = (-L_K + q)^{-1}
\left( \frac{su(c - u) - \frac{smv}{au^2 + bu + 1} + qu}{sv \left( \eta - \frac{rv}{u + k} \right) +qv} \right).
\]

The comparison principle guarantees that if \( F_s(u, v) = (u, v) \) for some \((u, v) \in W\) and \( s \in [0,1] \), then \((u, v) \in \vartheta\). Hence, \( F_s \) has no fixed point on \( \vartheta \) for all \( s \in [0,1] \). So the homotopy invariance of index shows that \( \text{index}_{\vartheta}(F) = \text{index}_{\vartheta}(F_s(0, \vartheta)) \). Observe that \( F_s \) has only the trivial fixed point \((0,0)\), then \( \text{index}_{\vartheta}(F_0) = \text{index}_{\vartheta}(F_0(0,0)) \). Suppose that \( F \) has no fixed point in \( \vartheta \). Similar to the proof of (i) in Lemma 6.4, we have \( \text{index}_{\vartheta}(F_0(0,0)) = 1 \), and hence \( \text{index}_{\vartheta}(F) = 1 \). Since \( c > \lambda_1(-L_K + mv_\ast) \) and \( \eta > \lambda_1 \), by the additivity of index and Lemma 6.4, we have \( 0 = \text{index}_{\vartheta}(F(0,0)) + \text{index}_{\vartheta}(F(u,0)) + \text{index}_{\vartheta}(F(0,v)) = \text{index}_{\vartheta}(F, \vartheta) = 1 \). Thus, this contradiction is gotten. Hence (11) has at least one positive solution. \( \square \)

6.2. Nonexistence of coexistence states. In this subsection, we give some conditions to ensure the nonexistence of coexistence states to (11).

Theorem 6.6. If anyone of the following five conditions holds,

(i) \( b > 0, \eta > c, km > r(ac^2 + bc + 1) \);
(ii) \( -2\sqrt{a} < b < 0, 0 < c < -\frac{b}{2\sqrt{a}}, \eta > c \) and \( km > (a + b + c) \);
(iii) \( 0 < a < 4, -2\sqrt{a} < b < 0, c > -\frac{b}{a}, \eta > c \) and \( km > r(ac^2 + bc + 1) \);
(iv) \( a \geq 4, -2\sqrt{a} < b < 0, c \geq -\frac{b}{2\sqrt{a}}, \eta > c \) and \( km > r(ac^2 + bc + 1) \);
(v) \( 0 < a < 4, -2\sqrt{a} < b < 0, -\frac{b}{2\sqrt{a}} < c < -\frac{b}{a}, \eta > c \) and \( km > r(a + b + c) \),

then the problem (11) has no positive solution.

Proof. Assume that problem (11) has a positive solution \((u, v)\), and by Lemma 3.3, we obtain \( 0 \leq u < c \) and \( 0 \leq v < \frac{\eta(u+k)}{r} \). Further, according to the property of principle eigenvalue, we have

\[
0 = \lambda_1 \left( -L_K + u - c + \frac{mv}{au^2 + bu + 1} \right)
= \lambda_1 \left( -L_K + \eta - c + \frac{mv}{au^2 + bu + 1} - \frac{rv}{u + k} \right) - \eta + \frac{rv}{u + k}.
\]

(i) Since \( b > 0, \eta > c, km > r(ac^2 + bc + 1) \) we get

\[
0 > \lambda_1 \left( -L_K + \eta - c + \frac{mv}{ac^2 + bc + 1} - \frac{rv}{k} \right) - \eta + \frac{rv}{u + k}
= \lambda_1 \left( -L_K + \eta - c + \frac{mk - r(ac^2 + bc + 1)v}{(ac^2 + bc + 1)(u + k)} \right) - \eta + \frac{rv}{u + k}
> \lambda_1 \left( -L_K - \eta + \frac{rv}{u + k} \right)
= 0.
\]

We get a contradiction.
(2) Since $-2\sqrt{a} < b < 0$, $0 < c < -\frac{b}{2\sqrt{a}}$, $\eta > c$ and $mk > r(a+b+c)$, we have

$$0 > \lambda_1 \left( -L_K + \eta - c + \frac{mv}{a + b + 1} - \frac{rv}{k} - \eta + \frac{rv}{u + k} \right)$$

$$= \lambda_1 \left( -L_K + \eta - c + \frac{[mk - r(a+b+c)]v}{(a + b + 1)k} - \eta + \frac{rv}{u + k} \right)$$

$$> \lambda_1 \left( -L_K - \eta + \frac{rv}{u + k} \right)$$

$$= 0.$$

We get a contradiction.

(iii) Since $0 < a < 4$, $-2\sqrt{a} < b < 0$, $c > -\frac{b}{a}$, $\eta > c$ and $km > r(ac^2 + bc + 1)$, we have

$$0 > \lambda_1 \left( -L_K + \eta - c + \frac{mv}{ac^2 + bc + 1} - \frac{rv}{k} - \eta + \frac{rv}{u + k} \right)$$

$$= \lambda_1 \left( -L_K + \eta - c + \frac{[mk - (ac^2 + bc + 1)]v}{(ac^2 + bc + 1)k} - \eta + \frac{rv}{u + k} \right)$$

$$> \lambda_1 \left( -L_K - \eta + \frac{rv}{u + k} \right)$$

$$= 0.$$

We obtain a contradiction.

(iv) Since $a \geq 4$, $-2\sqrt{a} < b < 0$, $c \geq -\frac{b}{2\sqrt{a}}$, $\eta > c$ and $km > r(ac^2 + bc + 1)$, we have

$$0 > \lambda_1 \left( -L_K + \eta - c + \frac{mv}{ac^2 + bc + 1} - \frac{rv}{k} - \eta + \frac{rv}{u + k} \right)$$

$$= \lambda_1 \left( -L_K + \eta - c + \frac{[mk - (ac^2 + bc + 1)]v}{(ac^2 + bc + 1)k} - \eta + \frac{rv}{u + k} \right)$$

$$> \lambda_1 \left( -L_K - \eta + \frac{rv}{u + k} \right)$$

$$= 0.$$

We obtain a contradiction.

(v) Since $0 < a < 4$, $-2\sqrt{a} < b < 0$, $-\frac{b}{2\sqrt{a}} < c < -\frac{b}{a}$, $\eta > c$ and $mk > r(a+b+c)$, we have

$$0 > \lambda_1 \left( -L_K + \eta - c + \frac{mv}{a + b + 1} - \frac{rv}{k} - \eta + \frac{rv}{u + k} \right)$$

$$= \lambda_1 \left( -L_K + \eta - c + \frac{[mk - (a+b+c)]v}{(a + b + 1)k} - \eta + \frac{rv}{u + k} \right)$$

$$> \lambda_1 \left( -L_K - \eta + \frac{rv}{u + k} \right)$$

$$= 0.$$

We get a contradiction. The proof of this result is completed.
7. Uniqueness of coexistence state. In this section, we will study the uniqueness of coexistence state. At first, we discuss the following equation

\[
\begin{align*}
-L_Ku &= u(c - u), & x &\in \Omega, \\
-L_Kv &= v \left( \eta - \frac{rv}{u+k} \right), & x &\in \Omega, \\
u = v &= 0, & x &\in \mathbb{R}^n \setminus \Omega.
\end{align*}
\] (61)

Lemma 7.1. If \( c > \lambda_1, \eta > \lambda_1 \), then (61) has a unique positive solution \( u_*, v^* \), where \( v^* \) is the unique positive solution of (40). Furthermore, the unique positive solution \( (u_*, v^*) \) is non-degenerate and linearly stable.

Proof. By virtue of Lemma 3.6, the existence, stability and uniqueness of positive solution \( (u_*, v^*) \) are obvious.

Let \( E \) be a real Banach space. \( W \) is called a wedge if \( W \) is a closed convex set and \( \beta W \subset W \) for all \( \beta \geq 0 \). For \( y \in W \), we define

\[ W_y = \{ x \in E : \exists r = r(x) > 0, s.t. y + rx \in W \}, \quad S_y = \{ x \in \overline{W}_y : -x \in \overline{W}_y \}. \]

We always assume that \( E = \overline{W} - \overline{W} \). Let \( T : W_y \to W_y \) be a compact linear operator on \( E \). We say that \( T \) has property \( \alpha \) on \( \overline{W}_y \) if there exists \( s \in (0, 1) \) and \( w \in \overline{W}_y \setminus S_y \) such that \( (1 - sT)w \in S_y \). Let \( A : W \to W \) be a compact operator with a fixed point \( y \in W \) and \( A \) is Fréchet differentiable at \( y \). Let \( L = A'(y) \) be the Fréchet derivative of \( A \) at \( y \). Then, \( L \) maps \( \overline{W}_y \) into itself. We denote by \( \text{index}_W(A, y) \) the fixed point index of \( A \) at \( y \) relative to \( W \).

Proposition 1. [5] Assume that \( I - L \) is invertible on \( \overline{W}_y \). Then

(i) If \( L \) has property \( \alpha \) on \( \overline{W}_y \), then \( \text{index}_W(A, y) = 0 \);

(ii) If \( L \) does not have property \( \alpha \) on \( \overline{W}_y \), then \( \text{index}_W(A, y) = (-1)^\sigma \), where \( \sigma \) is the sum of multiplicities of all eigenvalues of \( L \) which is greater than 1.

Theorem 7.2. Suppose \( \eta > \lambda_1 \) is fixed. If \( c > \lambda_1(-L_K + mv) \) and \( m \) is sufficient small, then (11) has a unique positive solution and it is linearly stable;

Proof. From Theorem 6.5, (11) has at least a positive solution. In the following, we show the uniqueness and stability. When \( m \) is sufficient small, (11) is a regular perturbation of (61). In view of Lemma 7.1 and a standard regular perturbation argument, we know that any positive solution of (11) is non-degenerate and linearly stable for small \( m \). On the other hand, since \( c > \lambda_1(-L_K + mv) \), \( c > \lambda_1 \), it can easily be checked that trivial solution and semi-trivial solution of (11) are bounded away from any positive solution. In help of a compactness argument and the assumption, the equation (11) has at most finitely positive solutions. Let them be \( \{(u_i, v_i)\} : 0 \leq i \leq n \). From the discussion above, we know that \( I - F'(u_i, v_i) \) is invertible on \( \overline{W}_{(u_i, v_i)} \) and \( F'(u_i, v_i) \) has no real eigenvalue being greater than 1. Since \( \overline{W}_{(u_i, v_i)} = S_{(u_i, v_i)} \), \( F'(u_i, v_i) \) does not have \( \alpha \) on \( \overline{W}_{(u_i, v_i)} \). Thus, from [5], we have \( \text{index}_W(F, (u_i, v_i)) = 1 \). Finally, it follows from Lemma 6.4 and the additivity property of the degree that

\[ 1 = \text{index}_W(F, \partial) = \sum_{1 \leq i \leq n} \text{index}_W(F, (u_i, v_i)) + 0 = n, \]

which implies the uniqueness. Next, we will study the stability of this solution. To the contrary, we assume that there exist sequences \( \{m_i\} \) such that \( m_i \to 0, \mu_i \to \infty \) with
$Re(\mu_i) \leq 0$ and $(\xi_1, \xi_2)$ with $\|\xi_1\|_{X_0(\Omega)}^2 + \|\xi_2\|_{X_0(\Omega)}^2 = 1$ satisfying

$$\begin{cases}
-\Delta - c - 2u_i + \frac{mv_i}{au_i + bu_i + 1} - \frac{m_iu_iv_i(2au_i + b)}{(au_i^2 + bu_i + 1)^2} \xi_1 = \mu \xi_1, & x \in \Omega \\
-\Delta - \eta \xi_2 - \frac{v_i^2 r}{(u_i + k)^2} \xi_1 = \mu \xi_2, & x \in \Omega \\
\xi_1 = \xi_2 = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases} \quad (62)$$

where $(u_i, v_i)$ is a positive solution of (11) with $c = c_i$. Multiplying the first and second equations of (62) by $\xi_1$ and $\xi_2$, respectively, integrating the results over $\Omega$, and then adding the results, we obtain

$$\mu_i = \|\xi_1\|_{X_0(\Omega)}^2 + \|\xi_2\|_{X_0(\Omega)}^2$$

$$+ \int_\Omega \left[-c + 2u_i - \frac{mv_i}{au_i^2 + bu_i + 1} - \frac{m_iu_iv_i(2au_i + b)}{(au_i^2 + bu_i + 1)^2} \right] |\xi_1|^2 \, dx$$

$$+ \int_\Omega \left[\frac{m_iu_i}{au_i^2 + bu_i + 1} \xi_1 \xi_2 \right] \, dx + \int_\Omega \left[-\eta + \frac{rv_i}{u_i + k} \right] |\xi_2|^2 \, dx$$

$$- \int_\Omega \left[\frac{v_i^2 r}{(u_i + k)^2} \right] \xi_1 \xi_2 \, dx.$$

Since $Re(\mu_i) \leq 0$, $\|\xi_1\|_{X_0(\Omega)}^2 + \|\xi_2\|_{X_0(\Omega)}^2 = 1$ and $\{(u_i, v_i)\}$ is uniformly bounded, then we obtain that both $Re(\mu_i)$ and $Im(\mu_i)$ are uniformly bounded. So $\{\mu_i\}_{i=1}^\infty$ are uniformly bounded and there exists a subsequence of $\{\mu_i\}_{i=1}^\infty$ denoted by itself, such that $\lim_{i \to \infty} \mu_i = \mu$ with $Re(\mu) < 0$. Using the boundedness of $\{\mu_i\}_{i=1}^\infty$, we get that $\{\xi_1\}_{i=1}^\infty$ and $\{\xi_2\}_{i=1}^\infty$ are uniformly bounded in $X_0(\Omega)$. Since $X_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exist subsequences of $\{\xi_1\}_{i=1}^\infty$ and $\{\xi_2\}_{i=1}^\infty$, denoted by themselves, such that $\lim_{i \to \infty} \xi_1 = \xi_1$ and $\lim_{i \to \infty} \xi_2 = \xi_2$ in $L^2(\Omega)$ and $\|\xi_1\|_{L^2(\Omega)}^2 + \|\xi_2\|_{L^2(\Omega)}^2 \leq 1$. Letting $n \to \infty$ in (63), we obtain that $(\mu, \xi_1, \xi_2)$ satisfies the following equation in the sense of distribution

$$\begin{cases}
-\Delta - (c - 2u_*) \xi_1 = \mu \xi_1, & x \in \Omega \\
-\Delta \xi_2 - \left(\eta - \frac{2v^*}{u_* + k}\right) \xi_2 = \mu \xi_1, & x \in \Omega \\
\xi_1 = \xi_2 = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases} \quad (63)$$

Since $\xi_1, \xi_2 \in L^2(\Omega)$, we get $(\xi_1, \xi_2) \in X_0(\Omega) \times X_0(\Omega)$. Furthermore $\mu$ is a real number with $\mu \leq 0$. If $\xi_1 \neq 0$, we see that $\mu$ is an eigenvalue of the following problem

$$\begin{cases}
-\Delta \phi - (c - 2u_*) \phi = \mu \phi, & x \in \Omega \\
\phi = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (64)$$

Thus, we have $0 \geq \mu \geq \lambda_1(-\Delta + 2u_* - c) > \lambda_1(-\Delta + u_* - c) = 0$, which is a contradiction. On the other hand, if $\xi_1 = 0, \xi_2 \neq 0$, then $\mu$ is an eigenvalue of the following problem

$$\begin{cases}
-\Delta \phi - \left(\eta - \frac{2v^*}{u_* + k}\right) \phi = \mu \phi, & x \in \Omega \\
\phi = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (65)$$
Since $v^*$ is the unique solution of (61), one can obtain
\[ 0 \geq \mu \geq \lambda_1 \left( -L_K + \frac{2r}{u_* + k} v^* - \eta \right) > \lambda_1 \left( -L_K + \frac{r}{u_* + k} v^* - \eta \right) = 0. \]
We get a contradiction again. The proof is finished.

8. Multiplicity and stability of coexistence states. In this section, we discuss the bifurcation of the parameter $c$ by using the Grandall-Rabinowitz bifurcation theorem. To simplify the notation we define
\[ \hat{c} = \lambda_1(-L_K + mv_*). \]

Our main result in this section is that there exists a positive constant $c^* \in (\lambda_1, \hat{c})$ such that the problem (11) has at least two positive solutions for $c \in (\hat{c} - \varepsilon, \hat{c})$ for some small $\varepsilon > 0$, and has at least one positive solution for $c \in [c^*, \hat{c}]$. We consider the bifurcation of positive solutions from the branch of semi-trivial solution: $\{(c,0,v_*) : c > \lambda_1 \}$. By linearizing (11) at $(c,0,v_*)$, we obtain the following eigenvalue problem:
\[
\begin{cases}
-L_K \xi_1 - (c - mv_*) \xi_1 = \mu \xi_1, & x \in \Omega, \\
-L_K \xi_2 = \frac{\nu r \xi_1}{k^2} + (\eta - \frac{2r v_*}{k}) \xi_2, & x \in \Omega, \\
\xi_1 = \xi_2 = 0, & x \in \mathbb{R} \setminus \Omega.
\end{cases}
\]
(66)

A necessary condition for bifurcation is that the principle eigenvalue $\mu$ of (66) is zero, which occurs if $c = \hat{c}$.

Let $\phi$ be the positive eigenfunction corresponding to $\hat{c}$, i.e., $(\hat{c}, \phi)$ satisfies
\[
\begin{cases}
-L_K \phi + mv_* \phi = \hat{c} \phi, & x \in \Omega, \\
\phi = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
(67)

We assume that $\phi$ is normalized so that $\int_{\Omega} \phi^2 dx = 1$. Since
\[ \lambda_1 \left( -L_K + \frac{2r}{k} v_* - \eta \right) > \lambda_1 \left( -L_K + \frac{r}{k} v_* - \eta \right) = 0, \]
then the operator $-L_K + \frac{2r}{k} v_* - \eta$ is invertible, and the operator $\left( -L_K + \frac{2r}{k} v_* - \eta \right)^{-1}$ maps positive functions to positive functions because of the compare principle. Define
\[ \varphi_1 = \left( -L_K - \eta + \frac{2r}{k} v_* \right)^{-1} \left( \frac{r}{k^2} v_*^2 \phi_1 \right) > 0. \]
(68)

Thus, we have the following result regarding the bifurcation of positive solution of (11) from $(c,0,v_*)$ at $c = \hat{c}$.

**Theorem 8.1.** Assume $c > \lambda_1$ and $\eta > \lambda_1$. Then $c = \hat{c}$ is a bifurcation value of (11) where positive solutions bifurcation from the line of semi-trivial solution $\Gamma_0 = \{(c,0,v_*) : c > 0 \}$ near $(\hat{c},0,v_*)$, all the positive solutions of (11) lie on a smooth curve $\Gamma_1 = \{(c(s),u(s),v(s)) : s \in (0,\delta)\}$ for some $\delta > 0$ such that
\[
\begin{cases}
c(s) = \hat{c} + sc_2 + sc_3(s), \\
u(s) = s\phi + su_1(s,x), \\
v(s) = v_* + s\varphi + sv_1(s,x),
\end{cases}
\]
(69)
where \( s \to (\tilde{c}_3(s), u_1(s, x), v_1(s, x)) \) is a smooth function from \((0, \delta)\) to \(\mathbb{R} \times X_0(\Omega) \times X_0(\Omega)\), such that \(\tilde{c}_3(0) = 0\), \(u_1(0, x) = v_1(0, x) = 0\), and
\[
\tilde{c}_2 = -\int_{\Omega} [(mv_* b - 1)\phi^3 - m\phi^2\varphi] \, dx.
\]
Moreover \( c = \tilde{c} \) is the unique bifurcation value for which positive solutions bifurcate from \(\Gamma_0\).

**Proof.** We define a nonlinear mapping \( F^* : \mathbb{R} \times X_0(\Omega) \times X_0(\Omega) \to L^2(\Omega) \times L^2(\Omega) \) by
\[
F^* (c, u, v) = \begin{pmatrix} L_K u + u(c - u) - \frac{mvu}{au^2 + bu + 1} - \frac{mv(2au + b)}{(au^2 + bu + 1)^3} \xi_1 - \frac{mu\xi_2}{au^2 + bu + 1} \xi_1 \\ L_K v + v (\eta - \frac{rv}{u + k}) \xi_2 \end{pmatrix}.
\]
We consider the bifurcation at \((c, u, v) = (\tilde{c}, 0, v_*)\). From straightforward calculations, we obtain that
\[
F^*_{(u,v)}(c, u, v) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} c - 2u - \frac{mv}{au^2 + bu + 1} + \frac{mv(2au + b)}{(au^2 + bu + 1)^3} \xi_1 - \frac{mu\xi_2}{au^2 + bu + 1} \\ \frac{mv^2r\xi_1}{(u + k)^3} + \left(\eta - 2\frac{rv}{u + k}\right) \xi_2 \end{pmatrix}.
\]
\[
F^*_{c}(c, u, v) = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad F^*_{c(u,v)}(c, u, v) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix},
\]
\[
F^*_{(u,v)(u,v)}(c, u, v) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (F_1, F_2)^T,
\]
where
\[
F_1 = \begin{pmatrix} \left(-2 + \frac{mv}{au^2 + bu + 1} \right)\xi_1 + \frac{mv(2au + b)}{(au^2 + bu + 1)^3} \xi_1 + \frac{mu\xi_2}{au^2 + bu + 1} \xi_1 \\ -\frac{m}{au^2 + bu + 1} + \frac{mu(2au + b)}{(au^2 + bu + 1)^3} \xi_1 \end{pmatrix}
\]
\[
F_2 = \begin{pmatrix} \left(-2\frac{v^2r\xi_1}{(u + k)^3} + \frac{rv\xi_2}{(u + k)^2} \right)\xi_1 + \frac{2}{(u + k)^2} \xi_1 - \frac{rv\xi_2}{(u + k)^2} - \frac{r\xi_2}{u + k} \xi_2 \\ \frac{2}{(u + k)^2} \xi_1 - \frac{rv\xi_2}{(u + k)^2} - \frac{r\xi_2}{u + k} \xi_2 \end{pmatrix}.
\]
At \((c, u, v) = (\tilde{c}, 0, v_*),\) it is easy to verify that the kernel \( \mathcal{N}(F_{u,v}(\tilde{c}, 0, v_*)) = \text{span}\{\phi, \varphi\}, \) the range space
\[
\mathcal{R}(F^*_{u,v}(\tilde{c}, 0, v_*)) = \left\{(f, g) \in L^2(\Omega) \times L^2(\Omega) : \int_{\Omega} f \phi dx = 0 \right\},
\]
and
\[
F^*_{c(u,v)}(\tilde{c}, 0, v_*)(\phi, \varphi)^T = (\phi, 0) \notin \mathcal{R}(F^*_{u,v}(\tilde{c}, 0, v_*)) \text{ since } \int_{\Omega} \phi^2 dx \neq 0.
\]
Thus, we can apply \([4]\) to conclude that the set of positive solution to (11) near \((\tilde{c}, 0, v_*)\) is a smooth curve
\[
\Gamma_1 = \{(c(s), u(s), v(s)) : s \in (0, \delta)\},
\]
such that \( c(0) = \tilde{c}, u(s) = \phi s + o(s), v(s) = v^* + \varphi s + o(s) \). Moreover, \( c'(0) \) can be calculated by
\[
c'(0) = \tilde{c}_2 = - \frac{(l, F^r_{(u,v)}(\tilde{c}, 0, v_\ast)) (\phi, \varphi)^2}{2(l, F^r_{c(u,v)}(\tilde{c}, 0, v_\ast)) (\phi, \varphi)}
\]
\[
= - \frac{\int_{\Omega} [(2mv_b - 2) \phi - mv\varphi] \phi - m\phi\varphi] \phi}{2 \int_{\Omega} \phi^2 dx}
\]
\[
= - \int_{\Omega} [(mv_b - 1)\phi^3 - m\phi^2\varphi] dx
\]
where \( l \) is a linear functional on \( L^2(\Omega) \times L^2(\Omega) \) defined as \( \langle l, (f, g) \rangle = \int_{\Omega} f(x)g(x) \) dx.

Finally, we prove that \( c = \tilde{c} \) is the unique bifurcation point where positive solutions of \( (11) \) bifurcate from \((0, v_\ast)\). Suppose that there is a sequence \( \{c_n, u_n, v_n\} \) of positive solutions of \( (11) \) with
\[
\lim_{n \to \infty} (c_n, u_n, v_n) = (\tilde{c}_1, 0, v_\ast) \in \mathbb{R} \times X_0(\Omega) \times X_0(\Omega).
\]
Then, we find from the first equation of \( (11) \) with \( c = c_n \), that is, for every \( n \geq 1 \),
\[
-L_K \left( \frac{u_n}{\|u_n\|_{L^2(\Omega)}} \right) = c_n \frac{u_n}{\|u_n\|_{L^2(\Omega)}} - \frac{u_n^2}{\|u_n\|_{L^2(\Omega)}} - \frac{mv_n}{au_n^2 + bu_n + 1} \frac{u_n}{\|u_n\|_{L^2(\Omega)}},
\]
that is,
\[
\frac{u_n}{\|u_n\|_{L^2(\Omega)}} = (c_n - \tilde{c}_1)(-L_K)^{-1} \left( \frac{u_n}{\|u_n\|_{L^2(\Omega)}} \right)
\]
\[
+ (-L_K)^{-1} \left( \tilde{c}_1 \frac{u_n}{\|u_n\|_{L^2(\Omega)}} - \frac{u_n^2}{\|u_n\|_{L^2(\Omega)}} - \frac{mv_n}{au_n^2 + bu_n + 1} \frac{u_n}{\|u_n\|_{L^2(\Omega)}} \right).
\]
By the compactness of \((-L_K)^{-1}\), it is easy to see that, by extracting subsequence, re-labeled by \( n \), we have that
\[
\lim_{n \to \infty} \frac{u_n}{\|u_n\|_{L^2(\Omega)}} = \phi > 0
\]
for some \( \phi \in X_0(\Omega) \) with \( \|\phi\|_{L^2(\Omega)} = 1 \). Thus, passing to the limit as \( n \to \infty \) in the previous identities, we find that
\[
\phi = (-L_K)^{-1} (\tilde{c}_1 \phi - mv\phi),
\]
that is,
\[
\begin{cases}
-L_K \phi + mv\phi = \tilde{c}_1 \phi, & x \in \Omega, \\
\Phi = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Thus, \( \tilde{c}_1 = \tilde{c} \). This proof is finished.

Next, the stability of the positive solutions bifurcating from the semi-trivial solutions will be studied.

**Theorem 8.2.** Assume that the conditions of Theorem 8.1 are satisfied, and let \( \tilde{c}_2 \) be defined as in \((70)\). If \( \tilde{c}_2 \neq 0 \), then \( s \in (0, \delta) \), the positive solution \((c(s), u(s), v(s))\) bifurcating from \((\tilde{c}, 0, v_\ast)\) is non-degenerate. Furthermore, \((u(s), v(s))\) is unstable if \( \tilde{c}_2 < 0 \), and stable if \( \tilde{c}_2 > 0 \).
Proof. Denote $c = c(s)$ and $(u, v) = (u(s), v(s))$. Then the corresponding linearized problem at $(u, v)$ can be written as

$$\mathfrak{L}(s)(\xi_1, \xi_2)^T = \mu(s)(\xi_1, \xi_2)^T,$$

$$\mathfrak{L}(s) = \begin{pmatrix} -L_K - c + 2u + \frac{mv}{au^2 + bu + 1} - \frac{mu(2au + b)}{(au^2 + bu + 1)^2} & \frac{mu}{au^2 + bu + 1} \\ -\frac{v^2}{(u + k)^2} - L_K - \eta + 2\frac{rv}{u + k} & -L_K - \eta + 2\frac{rv}{u + k} \end{pmatrix}.$$ 

Letting $s \to 0^+$, we get

$$\mathfrak{L}(s) \to \mathfrak{L}_0 = \begin{pmatrix} -L_K - c + mv_s & 0 \\ -\frac{v}{u}v_s^2 & -L_K - \eta + 2\frac{v}{u}v_s \end{pmatrix}, \quad (74)$$

It is obvious that 0 is the first eigenvalue of the operator $-L_K - \tilde{c} + mv_s$. Moreover, all other eigenvalues of $\mathfrak{L}_0$ are positive and apart from 0. By the perturbation theory of linear operator, we know that for the small $s > 0$, $\mathfrak{L}(s)$ has a unique eigenvalue $\mu(s)$ satisfying $\mu(s) \to 0$ as $s \to 0^+$ and all other eigenvalues of $\mathfrak{L}(s)$ have positive real parts and apart from 0. In the following, we denote $\mathfrak{L}(s) = \mathfrak{L}$ and $\mu(s) = \mu$.

Now we determine the sign of $Re(\mu)$ for small enough $s > 0$. Let $(\xi_1, \xi_2)$ be the corresponding eigenfunction to $\mu$ such that $(\xi_1, \xi_2) \to (\phi, \varphi)$ as $s \to 0^+$, then $(\xi_1, \xi_2)$ satisfies

$$\begin{cases} 
-L_K - c + 2u + \frac{mv}{au^2 + bu + 1} - \frac{mu(2au + b)}{(au^2 + bu + 1)^2} \xi_1 + \frac{mu}{au^2 + bu + 1} \xi_2 = \mu \xi_1, & x \in \Omega \\
-L_K - \eta + 2\frac{rv}{u + k} \xi_2 - \frac{v^2}{(u + k)^2} \xi_1 = \mu \xi_2, & x \in \Omega \\
\xi_1 = \xi_2 = 0, & x \in \mathbb{R}^n \setminus \Omega. 
\end{cases} \quad (75)$$

Multiplying the first equation of (75) by $u$ and integrating over $\Omega$, we get

$$\int_{\mathbb{R}^n} (u(x) - u(y))(\xi_1(x) - \xi_1(y))K(x - y)dxdy + \int_{\Omega} \left[ -c + 2u + \frac{mv}{au^2 + bu + 1} - \frac{mu(2au + b)}{(au^2 + bu + 1)^2} \right] \xi_1udx + \int_{\Omega} \frac{mu}{au^2 + bu + 1} \xi_2udx = \mu \int_{\Omega} u\xi_1dx. \quad (76)$$

By multiplying the first equation of (11) with $(u, v) = (u(s), v(s))$ and integrating over $\Omega$, we have

$$\int_{\mathbb{R}^n} (u(x) - u(y))(\xi_1(x) - \xi_1(y))K(x - y)dxdy = \int_{\Omega} \left[ u(c - u) - \frac{mu}{au^2 + bu + 1} \right] \xi_1dx. \quad (77)$$

Combining (76) and (77) yields

$$\mu \int_{\Omega} u\xi_1dx = \int_{\Omega} \left[ u - \frac{mu(2au + b)}{(au^2 + bu + 1)^2} \right] \xi_1dx + \int_{\Omega} \frac{mu^2\xi_2}{au^2 + bu + 1}dx. \quad (78)$$
Recall that \((u, v) = (\phi s + O(s^2), v_s + \phi s + O(s^2))\) and \((\xi_1, \xi_2) \rightarrow (\phi, \varphi)\) as \(s \rightarrow 0^+\). Taking the real part in (78), then dividing the results by \(s^2\) and letting \(s \rightarrow 0^+\), we have
\[
\lim_{s \rightarrow 0^+} \frac{\text{Re}(\mu)}{s} = -\int_{\Omega} [(mv_s b - 1)\phi^3 - m\phi^2 \varphi] dx = \hat{c}_2 \neq 0,
\]
where \(\text{Re}(\mu) \neq 0\) for \(s > 0\) small. Since all the other eigenvalues of \(\mathcal{L}\) has positive real parts and apart from 0, then the stability assertions follow from (79).

Next, we will discuss multiplicity of the positive solutions of (11).

**Theorem 8.3.** Assume that the conditions of Theorem 8.1 are satisfied, and let \(\hat{c}_2\) be defined as in (70). If \(\hat{c}_2 < 0\), there exists a positive constant \(c^* \in (\lambda_1, \hat{c})\) and \(\varepsilon \in (0, \hat{c} - c^*)\) such that (11) has at least two positive solutions for \(c \in [c^*, \hat{c})\), and has at least one positive solution for \(c \in [c^*, \hat{c}]\).

**Proof.** From Theorem 8.1, the equation (11) has a curve \(\Gamma = \{(c(s), u(s), v(s)) : s \in (0, \hat{c})\}\) of positive solutions near \((\hat{c}, 0, v_\ast)\). Since \(\hat{c}_2 < 0\), we get \(c(s) < \hat{c}\) for \(s > 0\) small. Assume that (11) has a unique positive solution \((\hat{u}, \hat{v})\) when \(c < \hat{c}\) but near \(\hat{c}\). By Theorem 8.1, we know that \((\hat{u}, \hat{v})\) must be the positive solution bifurcating from \((\hat{c}, 0, v_\ast)\). That is \((\hat{u}, \hat{v}) = (u(s), v(s))\), which is non-degenerate by Theorem 8.2. Therefore \((I - F(u,v))(\hat{u}, \hat{v}) : \mathcal{W}(\hat{u}, \hat{v}) \rightarrow \mathcal{W}(\hat{u}, \hat{v})\) is invertible. Since \(W(\hat{u}, \hat{v}) \setminus S(\hat{u}, \hat{v}) = \emptyset\), \(F(u,v)(\hat{u}, \hat{v})\) does not have property \(\alpha\) on \(W(\hat{u}, \hat{v})\). Consequently, \(\text{index}_W(F, (\hat{u}, \hat{v})) = 1\) or \(-1\). Note that \(\lambda_1 < c < \hat{c}\) and \(\eta > \lambda_1\). Applying Lemma 6.4, we have
\[
1 = \text{index}_W(F, \emptyset) = \text{index}_W(F, (0, 0)) + \text{index}_W(F, (u_\ast, 0)) + \text{index}_W(F, (0, v_\ast)) \\
+ \text{index}_W(F, (\hat{u}, \hat{v})) = 0 + 0 + 1 \pm 1.
\]
So, we get a contradiction. When \(c < \hat{c}\) and near to \(\hat{c}\), there exists at least two positive solutions of (11).

From a global bifurcation result of Rabinowitz [5], the curve \(\Gamma\) of the bifurcating positive solutions is contained in a connected component \(S_0\) of the set of positive solution of (11). Moreover, either the closure of \(S_0\) contains another trivial solution on \{(c, 0, v_\ast) : c > 0\} or \(S_0\) is unbounded. By Theorem 8.1 \(c = \hat{c}\) is the unique bifurcation value to positive solutions of (11) from the line of trivial solutions \{(c, 0, v_\ast) : c > 0\}, so the first alternative is not possible and \(S_0\) must be unbounded. Furthermore, \(0 < u < c\) for \(\lambda_1 \leq c \leq \hat{c}\). Finally, there is no positive solution when \(c \leq \lambda_1\) by Theorem 6.5. Thus the projection of \(S_0\) contains an interval \([c^*, \infty)\) for some \(c^*\) satisfying \(\lambda_1 < c^* < \hat{c}\). In particular, (11) has at least one nontrivial positive solution for \(c \in [c^*, \hat{c}]\).

9. **Conclusion.** The Lévy diffusion operator is a nonlocal diffusion operator, which can stand for particles’ or species’ spread in the form of jump from one position to another position. However, the Laplace operator is local. The Lévy diffusion operator is a generalized form of the Laplace operator and has been successfully introduced into quantum physics and biology and so on. Furthermore, there are some phenomena that some species exhibit Lévy-walk-like behaviours, see [21, 9]. Therefore, it is very significant to introduce the Lévy diffusion operator into the ecosystem. We are first to introduce this operator into the predator-prey model, which is a partial integro-differential equations. In this paper, we first argue the basic properties of the Lévy diffusion equation. In detail, we have shown that the Lévy
diffusion operator is a sectional operator, which can generate an analytically positive semigroup \( \{e^{tL_K}\}_{t \geq 0} \); we have given out the compare principles of the generalized Lévy-diffusion equation and the parabolic Lévy-diffusion differential equation. In help of these results above, we have succeeded to obtain the dynamical properties of the Lévy-diffusion Leslie-Gower predator-prey model with nonmonotonic functional response, such as the existence, uniqueness and attractiveness of the globally positive solution, the stability of the semi-trivial solutions, the existence and nonexistence of coexistence states, the multiplicity and stability of coexistence states. Our results enrich and extend the present research results of the predator-prey system. The Lévy diffusion of ecosystem is a new open problem. In our future work, we will show other dynamical properties of the ecosystem with Lévy diffusion, such as pattern formation.

Appendix. In this appendix, we discuss the spectrum of the Lévy-diffusion operator \( L_K \) with the condition (9) and consider the following eigenvalue problem

\[
\begin{aligned}
\begin{cases}
-L_K u = \lambda u, & \text{in } \Omega \\
u = 0, & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\end{aligned}
\] (A.80)

The weak formulation of (A.80) consists in the following eigenvalue problem

\[
\begin{aligned}
\begin{cases}
\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\phi(x) - \phi(y))K(x - y)dxdy = \lambda \int_{\Omega} u(x)\phi(x)dx, & \forall \phi \in X_0,
\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y)dxdy, & \forall u \in X_0.
\end{cases}
\end{aligned}
\] (A.81)

We recall that \( \lambda \in \mathbb{R} \) is an eigenvalue of \(-L_K\) provided there exists a non-trivial solution \( u \in X_0 \) to (A.80).

**Proposition 2.** [19] The function \( K \) satisfies the assumptions (9), and then

(a) the problem (A.80) admits an eigenvalue \( \lambda_1 \) which is positive and that can be characterized as follows

\[
\lambda_1 = \min_{u \in X_0, \|u\|_{L^2(\Omega)} = 1} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y)dxdy,
\] (A.82)

or, equivalently,

\[
\lambda_1 = \min_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x - y)dxdy}{\int_{\Omega} |u(x)|^2 dx};
\] (A.83)

(b) there exists a non-negative function \( e_1 \in X_0 \), which is an eigenfunction corresponding to \( \lambda_1 \), attaining the minimum in (A.82), that is \( \|e_1\|_{L^2(\Omega)} = 1 \) and

\[
\lambda_1 = \int_{\mathbb{R}^{2n}} |e_1(x) - e_1(y)|^2 K(x - y)dxdy;
\] (A.84)

(c) \( \lambda_1 \) is simple, that is if \( u \in X_0 \) is a solution of the following equation

\[
\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\phi(x) - \phi(y))K(x - y)dxdy = \lambda \int_{\Omega} u(x)\phi(x)dx, \forall \phi \in X_0,
\]

then \( u = \theta e_1 \), with \( \theta \in \mathbb{R} \);

(d) the set of the eigenvalues of the problem (A.80) consists of a sequence \( \{\lambda_k\}_{k \in \mathbb{N}} \) with

\[
0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots
\] (A.85)
and
\[ \lambda_k \to +\infty \text{ as } k \to +\infty. \] (A.86)

Moreover, for any \( k \in \mathbb{N} \) the eigenvalues can be characterized as follows:
\[ \lambda_{k+1} = \min_{u \in P_{k+1} \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) \, dx \, dy}{\int_{\Omega} |u(x)|^2 \, dx}, \] (A.87)
or, equivalently,
\[ \lambda_{k+1} = \min_{u \in P_{k+1} \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) \, dx \, dy}{\int_{\Omega} |u(x)|^2 \, dx}, \] (A.88)

where
\[ P_{k+1} := \{ u \in X_0 \ s.t. \langle u, e_j \rangle = 0 \ \forall j = 1, \ldots, k \}. \]

(e) for any \( k \in \mathbb{N} \) there exists a function \( e_{k+1} \in P_{k+1} \), which is an eigenfunction corresponding to \( \lambda_{k+1} \), attaining the minimum in (A.87), that is \( \|e_{k+1}\|_{L^2(\Omega)} = 1 \) and
\[ \lambda_{k+1} = \int_{\mathbb{R}^n} |e_{k+1}(x) - e_{k+1}(y)|^2 K(x - y) \, dx \, dy; \] (A.89)

(f) the sequence \( \{e_k\}_{k \in \mathbb{N}} \) of eigenfunctions corresponding to \( \lambda_k \) is an orthonormal basis of \( L^2(\Omega) \) and an orthogonal basis of \( X_0 \);

(g) each eigenvalue \( \lambda_k \) has finite multiplicity; more precisely, if \( \lambda_k \) is such that
\[ \lambda_k - 1 < \lambda_k = \cdots = \lambda_k + h < \lambda_k + h + 1 \] (A.90)
for some \( h \in \mathbb{N}_0 \), then the set of all the eigenfunctions corresponding to \( \lambda_k \) agrees with
\[ \text{span}\{e_k, \ldots, e_k+h\}. \]

Acknowledgments. The authors are grateful to the editor for his help and anonymous referees for the careful reading of the original manuscript and useful comments that helped to improve the manuscript.

REFERENCES
[1] P. Aguirre, E. González-Olivares and E. Sáez, Two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect, Nonlinear Analysis: Real World Applications, 10 (2009), 1401–1416.
[2] N. Ali and M. Jazar, Global dynamics of a modified Leslie-Gower predator-prey model with Crowley-Martin functional responses, Journal of Applied Mathematics and Computing, 43 (2013), 271–293.
[3] J. W. Cholewa and T. Dlotko, Global Attractors in Abstract Parabolic Problems, Cambridge University Press, 2000.
[4] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, Journal of Functional Analysis, 8 (1971), 321–340.
[5] E. N. Dancer, On the indices of fixed points of mappings in cones and applications, Journal of Mathematical Analysis and Applications, 91 (1983), 131–151.
[6] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bulletin Des Sciences Mathematiques, 136 (2012), 521–573.
[7] S. Fu, L. Zhang and P. Hu, Global behavior of solutions in a Lotka-Volterra predator-prey model with prey-stage structure, Nonlinear Analysis: Real World Applications, 14 (2013), 2027–2045.
[8] D. Hnaien, F. Kellil and R. Lassoued, Asymptotic behavior of global solutions of an anomalous diffusion system, Journal of Mathematical Analysis and Applications, 421 (2014), 1519–1530.
[9] N. E. Humphries, N. Queiroz, J. R. M. Dyer, N. G. Pade, M. K. Musyl, K. M. Schaefer, D. W. Fuller, J. M. Brunnschweiler, T. K. Doyle, J. D. R. Houghton and Others, Environmental context explains Lévy and brownian movement patterns of marine predators, Nature, 465 (2010), 1066–1069.
[10] Y.-J. Kim, O. Kwon and F. Li, Global asymptotic stability and the ideal free distribution in a starvation driven diffusion, Journal of Mathematical Biology, 68 (2014), 1341–1370.
[11] E. Latos and T. Suzuki, Global dynamics of a reaction-diffusion system with mass conservation, Journal of Mathematical Analysis and Applications, 411 (2014), 107–118.
[12] Y. Li and D. Xiao, Bifurcations of a predator–prey system of Holling and Leslie types, Chaos, Solitons & Fractals, 34 (2007), 606–620.
[13] J. Liu, H. Zhou and L. Zhang, Cross-diffusion induced Turing patterns in a sex-structured predator-prey model, International Journal of Biomathematics, 5 (2012), 1250016, 23PP.
[14] Y. F. Lv, R. Yuan and Y. Z. Pei, Turing pattern formation in a three species model with generalist predator and cross-diffusion, Nonlinear Analysis: Theory Methods & Applications, 85 (2013), 214–232.
[15] R. Peng, M. Wang and G. Yang, Stationary patterns of the Holling-Tanner prey-predator model with diffusion and cross-diffusion, Applied Mathematics and Computation, 196 (2008), 570–577.
[16] R. Peng and M. Wang, Global stability of the equilibrium of a diffusive Holling-Tanner prey-predator model, Applied Mathematics Letters, 20 (2007), 664–670.
[17] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^n$, Journal of Mathematical Physics, 54 (2013), 031501, 17PP.
[18] R. Servadei and E. Valdinoci, Mountain pass solutions for non-local elliptic operators, Journal of Mathematical Analysis and Applications, 389 (2012), 887–898.
[19] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, Discrete Contin. Dyn. Syst., 33 (2013), 2105–2137.
[20] X. Shang and J. Zhang, Ground states for fractional Schrödinger equations with critical growth, Nonlinearity, 27 (2014), 187–207.
[21] D. W. Sims, E. J. Southall, N. E. Humphries, G. C. Hays, C. J. A. Bradshaw, J. W. Pitchford, A. James, M. Z. Ahmed, A. S. Brierley, M. A. Hindell, D. Morritt, M. K. Musyl, D. Righton, E. L. C. Shepard, V. J. Wearmouth, R. P. Wilson, M. J. Witt and J. D. Metcalfe, Scaling laws of marine predator search behaviour, Nature, 451 (2008), 1098–1102.
[22] Y. Song and X. Zou, Spatiotemporal dynamics in a diffusive ratio-dependent predator-prey model near a Hopf-Turing bifurcation point, Computers & Mathematics with Applications, 67 (2014), 1978–1997.
[23] V. Volterra, Variazioni e fluttuazioni del numero D’individui in specie conviventi, Mem Acad Lincei Roma, 2 (1926), 31–113.
[24] M. Wang, Stationary patterns for a prey-predator model with prey-dependent and ratio-dependent functional responses and diffusion, Physica D: Nonlinear Phenomena, 196 (2004), 172–192.
[25] D. Xiao and S. Ruan, Codimension two bifurcations in a predator–prey system with group defense, International Journal of Bifurcation and Chaos, 11 (2001), 2123–2131.
[26] D. Xiao and H. Zhu, Multiple focus and Hopf bifurcations in a predator-prey system with nonmonotonic functional response, Siam Journal On Applied Mathematics, 66 (2006), 802–819.
[27] W. Yang, Global asymptotical stability and persistent property for a diffusive predator-prey system with modified Leslie-Gower functional response, Nonlinear Analysis: Real World Applications, 14 (2013), 1323–1330.
[28] Q. Ye, Z. Li, M. Wang and Y. Wu, Introduction to Reaction-Diffusion Equations, 2nd edition, Science Press, Beijing, 2011.
[29] H. Yin, X. Xiao and X. Wen, Turing patterns in a predator-prey system with self-diffusion, Abstract and Applied Analysis, 2013 (2013), Art. ID 891738, 10 pp.
[30] H. Yin, J. Zhou, X. Xiao and X. Wen, Analysis of a diffusive Leslie-Gower predator-prey model with nonmonotonic functional response, Chaos, Solitons & Fractals, 65 (2014), 51–61.
[31] H. Yin, X. Xiao, X. Wen and K. Liu, Pattern analysis of a modified Leslie-Gower predator-prey model with Crowley-Martin functional response and diffusion, Computers & Mathematics with Applications, 67 (2014), 1607–1621.
[32] J. Zhou, Positive solutions of a diffusive predator-prey model with modified Leslie-Gower and Holling-type II schemes, Journal of Mathematical Analysis and Applications, 389 (2012), 1380–1393.

[33] J. Zhou, Positive steady state solutions of a Leslie-Gower predator-prey model with Holling type II functional response and density-dependent diffusion, Nonlinear Analysis: Theory, Methods & Applications, 82 (2013), 47–65.

[34] H. Zhu, G. S. K. Wolkowicz and S. A. Campbell, Bifurcation analysis of a predator-prey system with nonmonotonic functional response, SIAM Journal On Applied Mathematics, 63 (2002), 636–682.

[35] B. Zimmermann, H. Sand, P. Wabakken, O. Liberg and H. P. Andreassen, Predator-dependent functional response in wolves: From food limitation to surplus killing, Journal of Animal Ecology, 84 (2015), 102–112.

Received June 2016; revised March 2017.

E-mail address: hongwei-yin@hotmail.com
E-mail address: xiaoy31@gmail.com
E-mail address: wendy.xiaoqing@sina.com