Jaynes–Cummings Model and a Non–Commutative “Geometry” : A Few Problems Noted

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Abstract

In this paper we point out that the Jaynes–Cummings model without taking a renonance conditton gives a non–commutative version of the simple spin model (including the parameters $x$, $y$ and $z$) treated by M. V. Berry. This model is different from usual non–commutative ones because the x–y coordinates are quantized, while the z coordinate is not.

One of new and interesting points in our non–commutative model is that the strings corresponding to Dirac ones in the Berry model exist only in states containing the ground state $(F \times \{|0\} \cup \{|0\} \times F)$, while for other excited states $(F \times F \setminus F \times \{|0\} \cup \{|0\} \times F)$ they don’t exist.

It is probable that a non–commutative model makes singular objects (singular points or singular lines or etc) in the corresponding classical model mild or removes them partly.

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1 Introduction

The Hopf bundles (which are famous examples of fiber bundles) over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (the field of quaternion numbers), $\mathbb{O}$ (the field of octanion numbers) are classical objects and they are never written down in a local manner. If we write them locally then we are forced to encounter singular lines called the Dirac strings, see [1].

It is very interesting to comment that the Hopf bundles correspond to topological solitons called Kink, Monopole, Instanton, Generalized Instanton respectively, see for example [1], [2], [3]. Therefore they are very important objects to study in detail.

Berry has given the Hopf bundle and Dirac strings by making use of a Hamiltonian (a simple spin model including the parameters $x$, $y$ and $z$), see the paper in [4]. We call this the Berry model for simplicity in the following.

We would like to make the Hopf bundles non-commutative. Whether such a generalization is meaningful or not is not clear at the current time, however it may be worth trying, see for example [5] or more recently [6] and its references.

By the way, we are studying a quantum computation based on cavity QED and one of the basic tools is the Jaynes–Cummings model (or more generally the Tavis–Cummings one), [7], [8], [9], [10]. This is given as a “half” of the Dicke model under the resonance condition and rotating wave approximation associated to it. If the resonance condition is not taken, then this model gives a non-commutative version of the Berry model. However, this new one is different from usual one because $x$ and $y$ coordinates are quantized, while $z$ coordinate is not.

If we study the non-commutative Berry model by making use of so-called Quantum Diagonalization Method (QDM) developed in [11], then we see that the Dirac strings appear in only states containing the ground one $(\mathcal{F} \times \{|0\}\} \cup \{|0\}\} \times \mathcal{F})$ where $\mathcal{F}$ is the Fock space generated by $\{a, a^\dagger\}$, while in excited states $(\mathcal{F} \times \mathcal{F} \setminus \mathcal{F} \times \{|0\}\} \cup \{|0\}\} \times \mathcal{F})$ they don’t appear. That is, this means that classical singularities are not universal in the process of non-commutativization, which is a very interesting phenomenon.

Why do we consider non-commutative versions of classical field models? What is an advantage to consider such a generalization? Researchers in this subject should answer such
natural questions. This note may give one of answers.

2 Berry Model and Dirac Strings : Review

First of all we explain the Dirac strings and Hopf bundle which Berry constructed in [4]. The Hamiltonian considered by Berry is a simple spin model

\[ H_B = x\sigma_1 + y\sigma_2 + z\sigma_3 = (x - iy)\sigma_+ + (x + iy)\sigma_- + z\sigma_3 = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \]  \hspace{1cm} (1)

where \( \sigma_j \) \((j = 1 \sim 3)\) is the Pauli matrices, \( \sigma_{\pm} \equiv (1/2)(\sigma_1 \pm i\sigma_2) \) and \( x, y \) and \( z \) are parameters.

We would like to diagonalize \( H \) above. The eigenvalues are

\[ \lambda = \pm r \equiv \pm \sqrt{x^2 + y^2 + z^2} \]

and corresponding orthonormal eigenvectors are

\[ |r\rangle = \frac{1}{\sqrt{2r(r + z)}} \begin{pmatrix} r + z \\ x + iy \end{pmatrix}, \quad |-r\rangle = \frac{1}{\sqrt{2r(r + z)}} \begin{pmatrix} -x + iy \\ r + z \end{pmatrix}. \]

Here we assume \((x, y, z) \in \mathbb{R}^3 - \{(0,0,0)\} \equiv \mathbb{R}^3 \setminus \{0\}\) to avoid a degenerate case. Therefore a unitary matrix defined by

\[ U_I = (|r\rangle, |-r\rangle) = \frac{1}{\sqrt{2r(r + z)}} \begin{pmatrix} r + z & -x + iy \\ x + iy & r + z \end{pmatrix} \]  \hspace{1cm} (2)

makes \( H_B \) diagonal like

\[ H_B = U_I \begin{pmatrix} r & \phantom{-} \\ -r & \end{pmatrix} U_I^\dagger \equiv U_I D_B U_I^\dagger. \]  \hspace{1cm} (3)

We note that the unitary matrix \( U_I \) is not defined on the whole space \( \mathbb{R}^3 \setminus \{0\}\). The defining region of \( U_I \) is

\[ D_I = \mathbb{R}^3 \setminus \{0\} - \{(0,0,z) \in \mathbb{R}^3 \mid z < 0\}. \]  \hspace{1cm} (4)

The removed line \( \{(0,0,z) \in \mathbb{R}^3 \mid z < 0\}\) is just the (lower) Dirac string, which is impossible to add to \( D_I \).
Next, we have another diagonal form of $H_B$ like

$$H_B = U_{II} D_B U_{II}^\dagger$$  

(5)

with the unitary matrix $U_{II}$ defined by

$$U_{II} = \frac{1}{\sqrt{2r(r-z)}} \begin{pmatrix} x - iy & -r + z \\ r - z & x + iy \end{pmatrix}.$$  

(6)

The defining region of $U_I$ is

$$D_{II} = \mathbb{R}^3 \setminus \{0\} - \{(0, 0, z) \in \mathbb{R}^3 | z > 0\}.$$  

(7)

The removed line $\{(0, 0, z) \in \mathbb{R}^3 | z > 0\}$ is just the (upper) Dirac string, which is also impossible to add to $D_{II}$.

Here we have diagonalizations of two types for $H$, so a natural question comes about. What is a relation between $U_I$ and $U_{II}$? If we define

$$\Phi = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x - iy \\ x + iy \end{pmatrix}.$$  

(8)

then it is easy to see

$$U_{II} = U_I \Phi.$$
We note that $\Phi$ (which is called a transition function) is not defined on the whole $z$–axis.

What we would like to emphasize here is that the diagonalization of $H$ is not given globally (on $\mathbb{R}^3 \setminus \{0\}$). However, the dynamics is perfectly controlled by the system

$$\{(U_I, D_I), (U_{II}, D_{II}), \Phi, \mathbb{R}^3 \setminus \{0\} = D_I \cup D_{II}\}, \quad (9)$$

which defines a famous fiber bundle called the Hopf bundle associated to the complex numbers $\mathbb{C}$, see [1].

$$S^1 \to S^3 \to S^2,$$

see [2].

### 3 All Hopf Bundles and Dirac Strings

In this section we show that the contents in the preceding section are easily generalized to the all Hopf bundles ($n = 1, 2, 4, 8$). Then as a by-product Dirac strings associated to them are shown clearly.

Let $\mathbf{K}$ be the field of real numbers $\mathbb{R}$, of complex numbers $\mathbb{C}$, of quaternion numbers $\mathbb{H}$, of octanion numbers $\mathbb{O}$ respectively. We write an element of $\mathbf{K}$ by

$$w = \sum_{j=0}^{n-1} x_j k_j, \quad x_j \in \mathbb{R},$$

where $k_j$ are generators of $\mathbf{K}$. Explicitly

$$w = x_0 \quad \text{for} \quad \mathbf{K} = \mathbb{R}, \quad (10)$$

$$w = x_0 + x_1 i \quad \text{for} \quad \mathbf{K} = \mathbb{C}, \quad (11)$$

$$w = x_0 + x_1 i + x_2 j + x_3 k \quad \text{for} \quad \mathbf{K} = \mathbb{H}, \quad (12)$$

$$w = x_0 + \sum_{j=1}^{7} x_j e_j \quad \text{for} \quad \mathbf{K} = \mathbb{O} \quad (13)$$

and $\bar{w}$ is a conjugate of $w$ in $\mathbf{K}$. Then it is well–known that $\bar{w}w = w\bar{w} = ||w||^2 = \sum_{j=0}^{n-1} x_j^2$. We note that $n = \dim_{\mathbb{R}}\mathbf{K} = 1, 2, 4, 8$ respectively.

\footnote{The base space $\mathbb{R}^3 \setminus \{0\}$ is homotopic to the two–dimensional sphere $S^2$}
As a “unified” Hamiltonian whose model space is $K \times \mathbb{R}$ we consider

$$H_K = w\sigma_+ + \bar{w}\sigma_- + z\sigma_3 = \begin{pmatrix} z & \bar{w} \\ w & -z \end{pmatrix}$$

(14)

where $z \in \mathbb{R}$. Of course, $H_C = H_B$ in (1).

Then we have a decomposition of $H_K$ like

$$H_K = \begin{cases} U_I D_K U_I^\dagger & \text{on } D_I \\ U_{II} D_K U_{II}^\dagger & \text{on } D_{II} \end{cases}$$

(15)

where

$$U_I = \frac{1}{\sqrt{2r(r+z)}} \begin{pmatrix} r + z & -\bar{w} \\ w & r + z \end{pmatrix}, \quad U_{II} = \frac{1}{\sqrt{2r(r-z)}} \begin{pmatrix} \bar{w} & -r + z \\ r - z & w \end{pmatrix}$$

(16)

for $r = \sqrt{||w||^2 + z^2}$ and

$$D_K = \begin{pmatrix} r \\ -r \end{pmatrix}, \quad \Phi_K = U_I^\dagger U_{II} = \frac{1}{||w||} \begin{pmatrix} \bar{w} \\ w \end{pmatrix}.$$

Since we are not interested in the degenerate case, we assume that $r \neq 0$ in the following $((w, z) \in K \times \mathbb{R} \setminus (0, 0) \equiv K \times \mathbb{R} \setminus \{0\})^2$).

What we would like to emphasize here is that the diagonalization of $H_K$ is not given globally, so there are Dirac strings. However, the dynamics is perfectly controlled by the system

$$\{(U_I, D_I), (U_{II}, D_{II}), \Phi_K, K \times \mathbb{R} \setminus \{0\} = D_I \cup D_{II}\},$$

(17)

which defines famous fiber bundles called the Hopf bundles

$$\mathbb{Z}_2 \to S^1 \to S^1 \quad \text{for } K = \mathbb{R},$$

$$U(1) \to S^3 \to S^2 \quad \text{for } K = \mathbb{C},$$

$$Sp(1) \to S^7 \to S^4 \quad \text{for } K = \mathbb{H},$$

$$So(1) \to S^{15} \to S^8 \quad \text{for } K = \mathbb{O},$$

(18)–(21)

$K \times \mathbb{R} \setminus \{0\}$ is homotopic to the $n$–dimensional sphere $S^n$. 
where $U(1) \cong S^1$, $Sp(1) \cong S^3$ and $So(1) \cong S^7$ are well-known, \[1\].

The projectors corresponding to the Hopf bundles are given as

$$P(w, z) = U_1 P_0 U_1^\dagger = U_{II} P_0 U_{II}^\dagger = \frac{1}{2r} \begin{pmatrix} r + z & \bar{w} \\ w & r - z \end{pmatrix}, \quad r = \sqrt{||w||^2 + z^2}$$

(22)

where $P_0$ is a basic one

$$P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in M(2, K).$$

We note that in (22) Dirac strings don’t appear because the projectors $P$ are expressed globally.

It is interesting to note that the Hopf bundles correspond to topological solitons called Kink, Monopole, Instanton, Generalized Instanton respectively, see for example \[2\], \[3\].

4 Two Steps Decomposition

With the decomposition \[15\] it is not easy to see where the Dirac strings come from. To see this point we give in this section two steps decomposition to the Hamiltonian \[14\], which makes the Dirac strings of Hopf bundles clear. It is easy to see

$$H_K = \begin{pmatrix} z & \bar{w} \\ w & -z \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{w}{||w||} \end{pmatrix} \begin{pmatrix} z & ||w|| \\ ||w|| & -z \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\bar{w}}{||w||} \end{pmatrix},$$

(23)

so the middle matrix

$$\begin{pmatrix} z & ||w|| \\ ||w|| & -z \end{pmatrix}$$

which is common to all $R$, $C$, $H$ and $O$, play a central role in the Dirac strings. Now let us diagonalize it to be

$$\begin{pmatrix} z & ||w|| \\ ||w|| & -z \end{pmatrix} = \begin{pmatrix} U_1 D_K U_1^\dagger \\ U_{II} D_K U_{II}^\dagger \end{pmatrix}$$

(24)

where $r = \sqrt{||w||^2 + z^2}$ and

$$U_1 = \frac{1}{\sqrt{2r(r + z)}} \begin{pmatrix} r + z & -||w|| \\ ||w|| & r + z \end{pmatrix}, \quad U_{II} = \frac{1}{\sqrt{2r(r - z)}} \begin{pmatrix} ||w|| & -r + z \\ r - z & ||w|| \end{pmatrix}.$$
We in this stage encounter the Dirac strings. It is just this matrix \[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \] that bears the Dirac strings on its shoulders.

5 A Non–Commutative Berry Model from the Jaynes–Cummings Model

First, let us explain the Jaynes–Cummings model which is well–known in quantum optics, \[7\]. The Hamiltonian of Jaynes–Cummings model can be written as follows (we set \( \hbar = 1 \) for simplicity)

\[
H = \omega_1 \sigma_3 \otimes 1 + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1} + g \left( \sigma_+ \otimes a + \sigma_- \otimes a^\dagger \right),
\]

where \( \omega \) is the frequency of single radiation field, \( \Delta \) the energy difference of two level atom, \( a \) and \( a^\dagger \) are annihilation and creation operators of the field, and \( g \) a coupling constant. We assume that \( g \) is small enough (a weak coupling regime). Here \( \sigma_+ \), \( \sigma_- \) and \( \sigma_3 \) are given as

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

See the figure 2 as an image of the Jaynes–Cummings model.

Now we consider the evolution operator of the model. We rewrite the Hamiltonian \[25\] as follows.

\[
H = \omega_1 \sigma_3 \otimes \mathbf{1} + \frac{\Delta - \omega}{2} \sigma_3 \otimes \mathbf{1} + g \left( \sigma_+ \otimes a + \sigma_- \otimes a^\dagger \right) \equiv H_1 + H_2.
\]

Then it is easy to see \([H_1, H_2] = 0\), which leads to \( e^{-itH} = e^{-itH_1} e^{-itH_2} \).
In the following we consider $e^{-itH_2}$ in which the resonance condition $\Delta - \omega = 0$ is not taken. For simplicity we set $\theta = \frac{\Delta - \omega}{2g}(\neq 0)$ \(^3\) then

$$H_2 = g \left( \sigma_+ \otimes a + \sigma_- \otimes a^\dagger + \frac{\Delta - \omega}{2g} \sigma_3 \otimes 1 \right) = g \left( \sigma_+ \otimes a + \sigma_- \otimes a^\dagger + \theta \sigma_3 \otimes 1 \right).$$

For further simplicity we set

$$H_{JC} = \sigma_+ \otimes a + \sigma_- \otimes a^\dagger + \theta \sigma_3 \otimes 1 = \begin{pmatrix} \theta & a \\ a^\dagger & -\theta \end{pmatrix}, \quad [a, a^\dagger] = 1 \quad (28)$$

where we have written $\theta$ in place of $\theta 1$ for simplicity.

$H_{JC}$ can be considered as a non-commutative version of $H_B$ under the correspondence $a \leftrightarrow x - iy, \quad a^\dagger \leftrightarrow x + iy$ and $\theta \leftrightarrow z$. That is, $x$ and $y$ coordinates are quantized, while $z$ coordinate is not, which is different from usual one, see for example [5]. It may be possible for us to call this a non–commutative Berry model. We note that this model is derived not “by hand” but by the model in quantum optics itself.

We usually analyze (28) by reducing it to each component contained in $H(2, C)$, which is a typical analytic method. However, we don’t adopt such a method. That is, we treat (28) as a kind of bundle, which means a “geometric” method in the title.

To study Dirac strings in this quantized model let us decompose the Hamiltonian (28) like in Section 4. It is easy to see

$$H_{JC} = \begin{pmatrix} \theta & a \\ a^\dagger & -\theta \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\sqrt{N+1}} \\ \frac{1}{\sqrt{N+1}} \sqrt{N+1} & -\theta \end{pmatrix} \begin{pmatrix} \theta & \sqrt{N+1} \\ \sqrt{N+1} & -\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{N+1}} \end{pmatrix} \quad (29)$$

from (28) and [11], where $N$ is the number operator $N = a^\dagger a$. Then the middle matrix in the right hand side can be considered as a classical one, so we can diagonalize it by making use of [24].

\(^3\)Since the Jaynes–Cummings model is obtained by the Dicke model under some resonance condition on parameters included, it is nothing but an approximate one in the neighborhood of the point, so we must assume that $|\theta|$ is small enough.
\[
\begin{pmatrix}
\theta & \sqrt{N+1} \\
\sqrt{N+1} & -\theta
\end{pmatrix}
= \begin{pmatrix}
U_I & U_I^+ \\
U_{II} & U_{II}^+
\end{pmatrix}
\]

where \( R(N) = \sqrt{N+\theta^2} \) and \( U_I, U_{II} \) are defined by

\[
U_I = \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}} \begin{pmatrix}
R(N+1) + \theta & -\sqrt{N+1} \\
\sqrt{N+1} & R(N+1) + \theta
\end{pmatrix}, \quad (31)
\]

\[
U_{II} = \frac{1}{\sqrt{2R(N+1)(R(N+1)-\theta)}} \begin{pmatrix}
\sqrt{N+1} & -R(N+1) + \theta \\
R(N+1) - \theta & \sqrt{N+1}
\end{pmatrix}. \quad (32)
\]

Now let us rewrite (29) by making use of (30) with (31). Inserting the identity

\[
\begin{pmatrix} 1 \\ a \end{pmatrix} \begin{pmatrix} 1 \\ a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{N+1}} \end{pmatrix}
\]

gives

\[
H_{JC} = \begin{pmatrix} 1 \\ a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} U_I \begin{pmatrix} R(N+1) \\ -R(N+1) \end{pmatrix} U_I^+ \begin{pmatrix} 1 \\ \frac{1}{\sqrt{N+1}} a \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 \\ a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} U_I \begin{pmatrix} 1 \\ \frac{1}{\sqrt{N+1}} a \end{pmatrix} \begin{pmatrix} 1 \\ a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} \begin{pmatrix} R(N+1) \\ -R(N+1) \end{pmatrix} \times
\]

\[
\begin{pmatrix} 1 \\ a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} \begin{pmatrix} 1 \\ a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} U_I^+ \begin{pmatrix} 1 \\ \frac{1}{\sqrt{N+1}} a \end{pmatrix}
\]

\[
= V_I \begin{pmatrix} R(N+1) \\ -R(N) \end{pmatrix} V_I^+, \quad (33)
\]

where

\[
V_I = \begin{pmatrix}
\frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}} & \frac{1}{\sqrt{2R(N)(R(N)+\theta)}} \\
\frac{1}{\sqrt{2R(N)(R(N)+\theta)}} & \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}}
\end{pmatrix}
\begin{pmatrix}
R(N+1) + \theta & -a \\
\frac{1}{\sqrt{2R(N)(R(N)+\theta)}} & \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
R(N+1) + \theta & -a \\
\frac{1}{\sqrt{2R(N)(R(N)+\theta)}} & R(N) + \theta
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}} & \frac{1}{\sqrt{2R(N)(R(N)+\theta)}} \\
\frac{1}{\sqrt{2R(N)(R(N)+\theta)}} & \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}}
\end{pmatrix}.
\]
Similarly, we can rewrite (29) by making use of (30) with (32). By inserting the identity
\[
\begin{pmatrix}
\frac{1}{\sqrt{N+1}}a \\
1
\end{pmatrix}
\begin{pmatrix}
a^\dagger \\
\frac{1}{\sqrt{N+1}}
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]
we obtain
\[
H_{JC} = V_{II} \begin{pmatrix}
R(N) \\
-R(N+1)
\end{pmatrix} V_{II}^\dagger,
\]
where
\[
V_{II} = \begin{pmatrix}
\frac{1}{\sqrt{2R(N+1)(R(N+1) - \theta)}} \\
\frac{1}{\sqrt{2R(N)(R(N) - \theta)}}
\end{pmatrix}
\begin{pmatrix}
a \\
-R(N+1) + \theta
\end{pmatrix}
= 
\begin{pmatrix}
a \\
-R(N+1) + \theta
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2R(N)(R(N) - \theta)}} \\
\frac{1}{\sqrt{2R(N+1)(R(N+1) - \theta)}}
\end{pmatrix}.
\]

Tidying up these we have
\[
H_{JC} = \begin{cases}
V_{I} \begin{pmatrix}
R(N+1) \\
-R(N)
\end{pmatrix} V_{I}^\dagger \\
V_{II} \begin{pmatrix}
R(N) \\
-R(N+1)
\end{pmatrix} V_{II}^\dagger
\end{cases}
\]
with \(V_{I}\) and \(V_{II}\) above. From the equations
\[
R(N+1)|0\rangle = \sqrt{1 + \theta^2} > \theta, \quad R(N)|0\rangle = \sqrt{\theta^2} = |\theta| \geq \theta
\]
we know that the strings corresponding to Dirac ones exist in only states \(\mathcal{F} \times |0\rangle \cup |0\rangle \times \mathcal{F}\) where \(\mathcal{F}\) is the Fock space generated by \(\{a, a^\dagger\}\), while in other excited states \(\mathcal{F} \times \mathcal{F} \setminus \mathcal{F} \times |0\rangle \cup |0\rangle \times \mathcal{F}\) they don’t exist \(^4\), see the figure 3. The phenomenon is very interesting. For simplicity we again call these strings Dirac ones in the following.

Then the transition “function” (operator) is given by
\[
\Phi_{JC} = \begin{pmatrix}
a \frac{1}{\sqrt{N}} \\
\frac{1}{\sqrt{N+1}} a^\dagger
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{N+1}} a \\
\frac{1}{\sqrt{N+1}} a^\dagger
\end{pmatrix}.
\]

\(^4\)We have identified \(\mathcal{F} \times \mathcal{F}\) with the space of 2–component vectors over \(\mathcal{F}\)
Figure 3: The bases of $\mathcal{F} \times \mathcal{F}$. The black circle means bases giving Dirac strings, while the white one don't.

The projector in this case is

$$P_{JC} = V_I \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V_I^\dagger = V_{II} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V_{II}^\dagger$$

$$= \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} R(N + 1) + \theta & a \\ a^\dagger & R(N) - \theta \end{pmatrix} \\
\begin{pmatrix} R(N + 1) + \theta & a \\ a^\dagger & R(N) - \theta \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\end{cases}$$

(36)

, so we obtain a quantum version of (classical) spectral decomposition (a “quantum spectral decomposition” by Suzuki [12])

$$H_{JC} = \begin{pmatrix} R(N + 1) \\ R(N) \end{pmatrix} P_{JC} - \begin{pmatrix} R(N + 1) \\ R(N) \end{pmatrix} (\mathbf{1}_2 - P_{JC}).$$

(37)

As a bonus of the decomposition let us rederive the calculation of $e^{-igtH_{JC}}$ which has been given in [8]. The result is

$$e^{-igtH_{JC}} = \begin{pmatrix} \cos(tgR(N + 1)) - i\theta \frac{\sin(tgR(N + 1))}{R(N + 1)} & -i \frac{\sin(tgR(N + 1))}{R(N + 1)} a \\ -i \frac{\sin(tgR(N))}{R(N)} a^\dagger & \cos(tgR(N)) + i\theta \frac{\sin(tgR(N))}{R(N)} \end{pmatrix}$$

(38)
by making use of (35) (or (37)). We leave it to the readers.

Lastly in this section we make a comment on the book [8]. It is very interesting from a not only quantum optical but also geometric point of view. We believe strongly that crucial results in [8] must be reobtained from a “geometric” method developed in this paper.

6 Discussion

In this paper we showed that a non–commutative version of the Berry model derived from the Jaynes–Cummings model (in quantum optics) had not Dirac strings in excited states. They appear in only states containing the ground one ($\mathcal{F} \times |0\rangle \cup |0\rangle \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F}$).

In general, a non-commutative version of classical field theory is of course not unique. If our model is a “correct” one, then this paper give an example that classical singularities like Dirac strings are not universal in some non–commutative model. As to general case with higher spins which are not easy see [12].

More generally, it is probable that a singularity (singularities) in some classical model is (are) removed in the process of non–commutativization. Further study on both finding many examples and constructing a general theory will be required.

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Appendix

A Local Coordinate of the Projector

In this appendix we consider a meaning of the projector (36) from the viewpoint of (infinite dimensional) Grassmann manifold. As a general introduction to this topic [13] is recommended.
For that let us look for a “local coordinate” \( Z \) giving the global expression (36). By making use of the expression by Oike in [13] (we don’t repeat it here)

\[
\mathcal{P}(Z) = \begin{pmatrix} 1 & -Z^\dagger \\ Z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -Z^\dagger \\ Z & 1 \end{pmatrix}^{-1}
\]

(39)

where \( Z \) is some operator on the Fock space \( \mathcal{F} \). Let us rewrite this into more useful form. From the simple relation

\[
\begin{pmatrix} 1 & Z^\dagger \\ -Z & 1 \end{pmatrix} \begin{pmatrix} 1 & -Z^\dagger \\ Z & 1 \end{pmatrix} = \begin{pmatrix} 1 + Z^\dagger Z & 1 + ZZ^\dagger \\ 1 & Z^\dagger \end{pmatrix}
\]

we have

\[
\begin{pmatrix} 1 & -Z^\dagger \\ Z & 1 \end{pmatrix}^{-1} = \begin{pmatrix} (1 + Z^\dagger Z)^{-1} & 1 \\ (1 + ZZ^\dagger)^{-1} & -Z \end{pmatrix} \begin{pmatrix} 1 & Z^\dagger \\ Z^\dagger & 1 \end{pmatrix}.
\]

Inserting this into (39) and some calculation leads to

\[
\mathcal{P}(Z) = \begin{pmatrix} (1 + Z^\dagger Z)^{-1} & (1 + Z^\dagger Z)^{-1} Z^\dagger \\ Z(1 + Z^\dagger Z)^{-1} & Z(1 + Z^\dagger Z)^{-1} Z^\dagger \end{pmatrix}.
\]

(40)

Comparing (40) with (36) we finally obtain the “local coordinate”

\[
Z = \frac{1}{R(N) + \theta} a^\dagger = \frac{1}{R(N + 1) + \theta}
\]

(41)

where \( R(N) = \sqrt{N + \theta^2} \). This is relatively simple and beautiful.

Now if we take a classical limit \( a \rightarrow x - iy, \ a^\dagger \rightarrow x + iy \) and \( \theta = z \) then

\[
Z_c = \frac{x + iy}{r + z}
\]

(42)

where \( r = \sqrt{x^2 + y^2 + z^2} \). This is nothing but a well–known one for (22) with \( w = x + iy \).

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