The Hermitian Jacobi Process: A Simplified Formula for the Moments and Application to Optical Fiber MIMO Channels

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ABSTRACT. Using a change of basis in the algebra of symmetric functions, we compute the moments of the Hermitian Jacobi process. After a careful arrangement of terms and the evaluation of the determinant of an “almost upper-triangular” matrix, we end up with a moment formula which is considerably simpler than the one derived in [8]. As an application, we propose the Hermitian Jacobi process as a dynamical model for an optical fiber MIMO channel and compute its Shannon capacity in the case of a low-power transmitter. Moreover, when the size of the Hermitian Jacobi process is larger than the moment order, our moment formula can be written as a linear combination of balanced terminating \(_4F_3\)-series evaluated at unit argument.

Key words: unitary Brownian motion, orthogonal projection, Jacobi unitary ensemble, Schur polynomials, symmetric Jacobi polynomials, MIMO channels, Shannon capacity.

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1. Motivations

The Hermitian Jacobi process was introduced in [11] as a multidimensional analogue of the real Jacobi process. This is a stationary matrix-valued process whose distribution converges weakly in the large-time limit to the matrix-variate beta distribution describing the Jacobi unitary ensemble (JUE). The latter was used in [6] as a random matrix model for a multi-input-multi-output (MIMO) optical fiber channel. In the same paper, numerical data of Shannon capacity and outage probability were provided, which confirmed the efficiency of the matrix model. By using a general fact about unitarily-invariant matrix models, this capacity can be expressed through the Christoffel–Darboux kernel for Jacobi polynomials, which is the one-point correlation function of the underlying eigenvalue process [15]. Yet another expression for it was recently obtained in [17] on the basis of a remarkable formula for the moments of the unitary Selberg weight [3]. The strategy employed in [3] was partially adapted in [8] to the Hermitian Jacobi process and led to a quite complicated formula for its moments, which did not allow deriving their large-size limits. The main ingredients used in [8] were the expansion of Newton power sums in the basis of Schur functions, the determinantal form of the symmetric Jacobi polynomials, and the integral form of the Cauchy–Binet formula (known as Andréief’s identity).

In this paper, we follow another approach to compute the moments of the Hermitian Jacobi process, which is based on a change of basis in the algebra of symmetric functions (with a fixed number of indeterminates). More precisely, we express Newton power sums in the basis of symmetric Jacobi polynomials, which are mutually orthogonal with respect to the unitary Selberg weight. This leads to the determinant of an “almost triangular” matrix, which we express in a product form by using row operations. After a careful rearrangement of the terms, we end up with a considerably simpler moment formula compared to the one obtained in [8] (Theorem 1). Actually, the latter involves three nested alternating sums together with a determinant whose entries are beta functions. To the best of our knowledge, this determinant has no closed form, except in very few special cases. The moment formula obtained in this paper contains only two nested alternating sums, whose summands are ratios of gamma functions.
As a potential application of our formula, we propose the Hermitian Jacobi process as a dynamical analogue of the MIMO Jacobi channel studied in [6] and compute its Shannon capacity for a transmitter with low-power antennas. Being motivated by free probability theory, we also study the case when the size of the Hermitian Jacobi process is larger than the moment order. In this case, our moment formula can be written as a linear combination of terminating balanced $4F_3$-hypergeometric series evaluated at unit argument [1; Chap. 3].

The paper is organized as follows. In the next section, we briefly review the construction of the Hermitian Jacobi process and recall the definition of the semigroup density of its eigenvalue process (when it exists). In the third section, we state our main result (Theorem 1) and prove it. For ease of reading, we proceed in several steps until we obtain the sought moment formula. In the last section, we discuss the application of our main result to optical fiber MIMO channel and to the large-size limit of the moments of the Hermitian Jacobi process.

2. A Review of the Hermitian Jacobi Process

For the sake of completeness, we recall the construction of the Hermitian Jacobi process and the expression for the semigroup density of its eigenvalue process. We refer the reader to [11] and [8] for further details.

We denote the group of complex unitary matrices of order $d \geq 2$ by $U(d)$. Let $p, m \leq d$ be two integers, and let $Y = (Y_t)_{t \geq 0}$ be a $U(d)$-valued stochastic process. We set

$$X_t \oplus 0 := PY_tQ, \quad t \geq 0,$$

where

$$P := \begin{pmatrix}
\Id_{m \times m} & 0_{m \times (d-m)} \\
0_{(d-m) \times m} & 0_{(d-m) \times (d-m)}
\end{pmatrix}, \quad Q := \begin{pmatrix}
\Id_{p \times p} & 0_{p \times (d-p)} \\
0_{(d-p) \times p} & 0_{(d-p) \times (d-p)}
\end{pmatrix}$$

are orthogonal projections. In other words, $X$ is the upper left corner of $Y$. Assume now that $Y$ is the Brownian motion on $U(d)$ starting at the identity matrix. Then

$$J_t := X_t X_t^* = PY_t Q Y_t^* P, \quad t \geq 0,$$

is called the Hermitian Jacobi process of size $m \times m$ with parameters $(p, q)$, where $q = d - p$. As $t \to +\infty$, we have $Y_t \to Y_\infty$, where $Y_\infty$ is a Haar unitary matrix and the convergence is weak. Moreover, it was proved in [4] that the random matrix

$$J_\infty = X_\infty X_\infty^* = PY_\infty Q Y_\infty^* P$$

has distribution of the same form drawn from a JUE with suitable parameters.

For each $n \geq 1$, we define the $n$th moment of $J_t$ by

$$M_{n,p,m,d}(t) := \mathbb{E}(\text{tr}((J_t)^n))$$

for fixed time $t \geq 0$ and write simply $M_n(t)$. Since the matrix Jacobi process is Hermitian, it follows that

$$M_n(t) = \mathbb{E} \left( \sum_{k=1}^{m} (\lambda_k(t))^n \right),$$

where $(\lambda_k(t), t \geq 0)_{k=1}^{m}$ is the eigenvalue process of $(J_t)_{t \geq 0}$ and $\mathbb{E}$ stands for the expectation in the underlying probability space. If

$$r := p - m \geq 0, \quad s := d - p - m = q - m \geq 0,$$
then the distribution of the eigenvalue process is absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}^m\). Moreover, its semigroup density is given by a bilinear generating series of symmetric Jacobi polynomials with Jack parameter equal to 1. More precisely, let

\[
\tau = (\tau_1 \geq \cdots \geq \tau_m \geq 0)
\]

be a partition of length at most \(m\), and let \((P_k^{r,s})_{k \geq 0}\) be the sequence of orthonormal Jacobi polynomials with respect to the beta weight:

\[
u^r(1-u)^s1_{[0,1]}(u).
\]

These polynomials can be defined through the Gauss hypergeometric function as

\[
P_k^{r,s}(u) := \left[ \frac{(2k + r + s + 1)\Gamma(k + r + s + 1)k!}{\Gamma(r + k + 1)\Gamma(s + k + 1)k!} \right]^{1/2} \frac{(r + 1)_k}{k!} 2F_1(-k, k + r + s + 1, r + 1; x).
\]

Then the orthonormal symmetric Jacobi polynomial corresponding to \(\tau\) is defined by

\[
P_{\tau}^{r,s,m}(x_1, \ldots, x_m) := \frac{\det(P_{\tau_i - i + m}(x_j))_{1 \leq i, j \leq m}}{V(x_1, \ldots, x_m)}, \quad V(x_1, \ldots, x_m) := \prod_{1 \leq i < j \leq m} (x_i - x_j),
\]

if the coordinates \((x_1, \ldots, x_m)\) do not overlap and by l'Hôpital's rule otherwise. An expansion of these polynomials in the basis of Schur functions can be found in [14]. These polynomials are mutually orthonormal with respect to the unitary Selberg weight

\[
W_{r,s,m}(y_1, \ldots, y_m) := [V(y_1, \ldots, y_m)]^2 \prod_{i=1}^m y_i^r (1 - y_i)^s 1_{0 < y_m < \cdots < y_1 < 1},
\]

in the sense that two elements corresponding to different partitions are orthogonal and the norm of each \(P_{\tau}^{r,s,m}\) equals 1 (see, e.g., [2; Theorem 3.1]). Moreover, the semigroup density of the eigenvalue process of \(J_t\) admits the following absolutely convergent expansion [9]:

\[
G_{r,s,m}^t(1^m, y) := \sum_{\tau} e^{-\nu_\tau t} P_{\tau}^{r,s,m}(1^m) P_{\tau}^{r,s,m}(y) W_{r,s,m}(y_1, \ldots, y_m),
\]

where \(1^m := (1, \ldots, 1)\) and

\[
\nu_\tau := \sum_{i=1}^m \tau_i(\tau_i + r + s + 1 + 2(m - i)).
\]

If \((\tilde{P}_k^{r,s})_{k \geq 0}\) denotes the sequence of orthogonal Jacobi polynomials

\[
\tilde{P}_k^{r,s}(u) := \frac{(r + 1)_k}{k!} 2F_1(-k, k + r + s + 1, r + 1; x),
\]

then \(G_{r,s,m}^t(1^m, y)\) can be written as

\[
G_{r,s,m}^t(1^m, y) = \sum_{\tau} e^{-\nu_\tau t} \prod_{j=1}^m \frac{1}{(\|P_{\tau_j + m-j}^{r,s}\|)^2} \tilde{P}_{\tau}^{r,s,m}(1^m) \tilde{P}_{\tau}^{r,s,m}(y) W_{r,s,m}(y_1, \ldots, y_m),
\]

(2.1)
where \((\| \tilde{P}_{r,s}^r \|_2)^2\) is the squared \(L^2\)-norm of the one-variable Jacobi polynomial and

\[
\tilde{P}_{r,s,m}^r(x_1, \ldots, x_m) := \frac{\det(\tilde{P}_{r,s}^r(x_j))_{1 \leq i,j \leq m}}{V(x_1, \ldots, x_m)}.
\]

Indeed, Andreief’s identity [7; p. 37] shows that \((\tilde{P}_{r,s,m}^r)_{\tau}\) is an orthogonal set with respect to the unitary Selberg weight and that the squared \(L^2\)-norm of \(\tilde{P}_{r,s,m}^r\) with respect to \(W_{r,s,m}\) is nothing else but

\[
\prod_{j=1}^m (\| \tilde{P}_{r,s}^r \|_2)^2.
\]

On the other hand, the set \((\tilde{P}_{r,s,m}^r)_{\tau}\) of polynomials can be mapped to the set \((Q_{r,s,m}^r)_{\tau}\) of symmetric Jacobi polynomials considered in [19] by the affine transformation

\[
(x_1, \ldots, x_m) \in [0,1]^m \mapsto (1 - 2x_1, \ldots, 1 - 2x_m) \in [-1,1]^m.
\]

More precisely, one has

\[
P_{r,s,m}^r(x_1, \ldots, x_m) = (-2)^{m(m-1)/2}Q_{r,s,m}^r(1 - 2x_1, \ldots, 1 - 2x_m).
\]

Moreover, the polynomials \(Q_{r,s,m}^r\) have the mirror property

\[
Q_{r,s,m}^r(-x_1, \ldots, -x_m) = (-1)^{|\tau|}Q_{s,r,m}^r(x_1, \ldots, x_m),
\]

which is inherited from their one-variable analogues. Indeed, when \(x\) has distinct coordinates, this property is checked directly by using the determinantal form of \(Q_{r,s,m}^r\), and to the general case it is extended by continuity. In particular,

\[
P_{r,s,m}^r(1^m) = (-2)^{m(m-1)/2}Q_{r,s,m}^r(1, \ldots, -1) = (-1)^{|\tau|}(-2)^{m(m-1)/2}Q_{s,r,m}^s(1^m).
\]

But Proposition 7.1 in [19] gives

\[
Q_{s,r,m}^s(1^m) = \prod_{1 \leq i < j \leq m} (\tau_i + \tau_j + 2m - i - j + r + s + 1)(\tau_i - \tau_j + j - i)
\]

\[
\times \prod_{j=1}^m \frac{\Gamma(\tau_j + m - j + s + 1)2^{-m-j}}{\Gamma(\tau_j + m - j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}.
\]

As a result, we obtain the special value

\[
\tilde{P}_{r,s,m}^r(1^m) = (-1)^{|\tau|+m(m-1)/2} \prod_{1 \leq i < j \leq m} (\tau_i + \tau_j + 2m - i - j + r + s + 1)(\tau_i - \tau_j + j - i)
\]

\[
\times \prod_{j=1}^m \frac{\Gamma(\tau_j + m - j + s + 1)}{\Gamma(\tau_j + m - j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}, \quad (2.2)
\]

which will be used in our computations below.
3. Main Result: The Moment Formula

Let $n \geq 1$ and recall that a hook $\alpha$ of weight $|\alpha| = n$ is a partition of the form

$$\alpha = (n - k, 1^k).$$

Recall also that there is an ordering of partitions induced by the containment of their Young diagrams: given partitions $\alpha$ and $\tau$, $\tau \subseteq \alpha$ if and only if $\tau_i \leq \alpha_i$ for any $1 \leq i \leq l(\tau) \leq l(\alpha)$, where the length $l(\tau)$ is the number of nonzero components of $\tau$. On the other hand, the $n$th moment of the stationary distribution $J_\infty$ is given by the normalized integral\footnote{As in the case of fixed $t > 0$, we omit the dependence of the stationary moments on $(r, s, m)$ from the notation.}

$$M_n(\infty) := \frac{1}{Z^{r,s,m}} \int \left( \sum_{i=1}^{m} y_i^n \right) W^{r,s,m}(y) dy,$$

where

$$Z^{r,s,m} := \int W^{r,s,m}(y) dy$$

is the Selberg integral. An explicit expression for $M_n(\infty)$ is contained in Corollary 2.3 of [3]. In this notation, our main result is stated as follows.

**Theorem 1.** The $n$th moment of the Hermitian Jacobi process is given by

$$M_n(t) = M_n(\infty) + \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha} \sum_{\tau \subseteq \alpha} \frac{e^{-\nu t} \overline{V}_{\alpha, l(\tau)}^{r,s,m} U_{l(\alpha), l(\tau)}^{r,s,m}}{(r + s + \tau_1 + 2m - l(\tau))(\tau_1 + l(\tau) - 1)}, \quad (3.1)$$

where

$$\overline{V}_{\alpha, r_1}^{r,s,m} = \frac{(r + s + 2\tau_1 + 2m - 1) \Gamma(\tau_1 + 2m + r + s) \Gamma(\alpha_1 + m) \Gamma(r + \alpha_1 + m) \Gamma(\tau_1 + m + s)}{\Gamma(\alpha_1 - \tau_1 + 1) \Gamma(r + \alpha_1 + \tau_1 + 2m) \Gamma(r + \tau_1 + m)}$$

and

$$U_{l(\alpha), l(\tau)}^{r,s,m} := \frac{(2m + r + s + 1 - 2l(\tau)) \Gamma(r + m - l(\tau) + 1)}{(l(\alpha) - l(\tau))! \Gamma(r + m - l(\tau) + 1)} \times \frac{\Gamma(r + s + 2m - l(\alpha) - l(\tau) + 1)}{\Gamma(m + s - l(\tau) + 1) \Gamma(2m + r + s - l(\tau) + 1)}.$$

The rest of this section is devoted to the proof of this result. Because of lengthy computations, we shall proceed in several steps; at each step we simplify the moment expression obtained at the previous one.

### 3.1. The basis change.

We start with changing the basis from Schur polynomials to symmetric Jacobi polynomials. This leads to the following formula for $M_n(t)$.

**Proposition 1.** For any $m, n \geq 1$ and $t > 0$,

$$M_n(t) = \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha} \sum_{\tau \subseteq \alpha} e^{-\nu t} \overline{V}_{\tau}^{r,s,m}(1) \det \left( \frac{-(\alpha_i - i + m)_{\tau_j - j + m}}{\Gamma(r + s + \tau_j - j + \alpha_i - i + 2m + 2)} i,j = 1 \right)^m \times \prod_{j=1}^{m} \frac{(r + s + 2(\tau_j - j + m) + 1) \Gamma(r + \alpha_j + m - j + 1) \Gamma(r + s + \tau_j - j + m + 1)}{\Gamma(r + \tau_j - j + m + 1)}, \quad (3.2)$$
where
\[ (-\alpha_i - i + m)_{\tau_j - j + m} = (-1)^{\tau_j + m - j} \frac{\alpha_i + m - i!}{(\alpha_i - \tau_j + j - i)!} \tau_{\alpha_i - \tau_j - j}. \]

**Proof.** Recall that the $n$th Newton power sum is defined by (see [16])
\[ p_n(y) := \sum_{i=1}^{m} y_i^n \]
and the Schur polynomials associated to a partition $\tau$ of length $l(\tau) \leq m$ are [16]
\[ s_{\tau}(x) := \frac{\det(x_{i-j}^{\tau_i + \mu_j})_{1 \leq i,j \leq m}}{V(x_1, \ldots, x_m)}. \]

These symmetric functions are related by the representation-theoretic formula (see, e.g., [16; p. 48])
\[ p_n(y) = \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha_1} s_{\alpha}(y). \]

In order to integrate the Newton sum against the semigroup density (2.1), we will expand the Schur polynomials in the basis $(\tilde{P}_{r,s,m}^\tau)$ of symmetric Jacobi polynomials. To this end, we employ the inversion formula ([12])
\[ y^j = \Gamma(r + j + 1) \sum_{l=0}^{j} \frac{(-j)_l (r + s + 2l + 1) \Gamma(r + s + l + 1)^{r,s}(y)}{\Gamma(r + l + 1) \Gamma(r + s + l + j + 2)}, \]
\[ \text{together with Proposition 3.1 in [18] (the change of basis formula). We obtain} \]
\[ s_{\alpha}(y) = \prod_{i=1}^{m} \Gamma(r + \alpha_i + m - i + 1) \sum_{\mu \subset \alpha} \det(a(\alpha_i - i + m, \mu_j - j + m))_{1 \leq i,j \leq m} \tilde{P}_{\mu}^{r,s,m}(y), \]

where we set
\[ a(\alpha_i - i + m, \mu_j - j + m) := (-\alpha_i - i + m)_{\mu_j - j + m + 1} \frac{(r + s + 2(\mu_j - j + m) + 1) \Gamma(r + s + \mu_j - j + m + 1)}{\Gamma(r + \mu_j - j + m + 1) \Gamma(r + s + \mu_j - j + \alpha_i - i + 2m + 2)}. \]

Integrating $y \mapsto p_n(y)G_{r,s,m}^{1,m}(y)$ and applying Fubini’s theorem, we arrive at
\[
\int \left( \sum_{i=1}^{m} y_i^n \right) \tilde{P}_{r,s,m}^\tau(y) W_{r,s,m}^\tau(y) \, dy \\
= \frac{1}{m!} \int_{[0,1]^m} \left( \sum_{i=1}^{m} y_i^n \right) \tilde{P}_{r,s,m}^\tau(y) [V(y_1, \ldots, y_m)]^2 \prod_{i=1}^{m} y_i^i (1 - y_i)^s \, dy \\
= \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha_1} \sum_{\mu \subset \alpha} \det(a(\alpha_i - i + m, \mu_j - j + m))_{1 \leq i,j \leq m} \\
\times \frac{1}{m!} \int_{[0,1]^m} \tilde{P}_{\mu}^{r,s}(y) \tilde{P}_{r,s,m}^\tau(y) [V(y_1, \ldots, y_m)]^2 \prod_{i=1}^{m} y_i^i (1 - y_i)^s \, dy \\
= \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha_1} \prod_{\tau \subset \alpha, j=1}^{m} (\|P_{\tau_j + m - j}^{r,s}\|_2)^2 \det(a(\alpha_i - i + m, \tau_j - j + m))_{1 \leq i,j \leq m},
\]
where the last equality follows again from Andreev’s identity. In view of the series expansion (2.1), the stated moment formula follows. \qed
3.2. An almost upper-triangular matrix. For the sake of simplicity, we introduce the notation
\[ n_i = \alpha_i + m - i, \quad m_i = \tau_i + m - i. \]
Using (2.2), we write the moment formula (3.2) more explicitly as
\[
M_n(t) = \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha_1} \sum_{\tau \subseteq \alpha} \prod_{1 \leq i < j \leq m} (m_i + m_j + r + s + 1)(m_i - m_j) \\
\times (-1)^{|\tau|+m(m-1)/2} \prod_{j=1}^{m} \frac{(r + s + 2m_j + 1)\Gamma(r + n_j + 1)\Gamma(m_j + s + 1)\Gamma(r + s + m_j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}{\Gamma(r + m_j + 1)\Gamma(m + 1)\Gamma(m - j + 1)} \\
\times \det \left( \frac{(-n_i)_{m_j}}{\Gamma(r + s + n_i + m_j + 2)} \right)_{i,j=1}^{m}.
\] (3.3)
Since \( \alpha_i = \tau_j \) for \( i,j > l(\alpha), \) \( 2 \leq i,j \leq l(\tau), \) it follows that \( n_i < m_j \) provided that \( i > j. \) Similarly, \( 1 = \alpha_i > \tau_j = 0, l(\tau) + 1 \leq i,j \leq l(\alpha) - 1, \) implies the same inequality for \( i > j + 1. \) These elementary observations show that the matrix above is “almost upper-triangular.”

**Lemma 1.** For any hook \( \alpha \) of weight \( n \geq 1 \) and length \( l(\alpha) \leq m \) and any \( \tau \subset \alpha, \) we set
\[ b_{\alpha,\tau}(i,j) := \frac{(-n_i)_{m_j}}{\Gamma(r + s + n_i + m_j + 2)}, \quad B_{\alpha,\tau} := (b_{\alpha,\tau}(i,j))_{i,j=1}^{m}. \]
Then \( b_{\alpha,\tau}(i,j) = 0 \) for \( i \geq j + 2. \) Moreover,
- if \( l(\alpha) < l(\tau) + 2, \) then \( B_{\alpha,\tau} \) is upper-triangular;
- otherwise, \( b_{\alpha,\tau}(j + 1,j) = 0 \) for \( j \geq l(\alpha) \) or \( j \leq l(\tau), \) while
\[ b_{\alpha,\tau}(j + 1,j) = \frac{(-1)^{m-j}(m-j)!}{\Gamma(r + s + 2m - 2j + 2)}, \quad l(\tau) + 1 \leq j \leq l(\alpha) - 1, \tau \neq \emptyset. \]

**Proof.** Take \( i > j \geq 1. \) Then \( \alpha_i - i \leq 1 - i \) and \( 1 - i < \tau_j - j, \) except in the case when \( \tau_j = 0 \) and \( i = j + 1. \) Consequently, \( (-\alpha_i + i + m)_{\tau_j - j + m} = 0 \) in the following three cases:
1. \( \alpha_i = 0; \)
2. \( i \geq j + 2; \)
3. \( i = j + 1 \) and \( \tau_j \geq 1. \)
In particular, \( B_{\alpha,\tau} \) is upper-triangular if \( l(\alpha) = l(\tau) \) or \( l(\alpha) = l(\tau) + 1, \) since \( \alpha_i \leq \tau_j \) in this case. Otherwise, if \( l(\alpha) \geq l(\tau) + 2, \) then \( b_{\alpha,\tau}(j + 1,j) \) vanishes unless \( l(\tau) + 1 \leq j \leq l(\alpha) - 1, \) in which case
\[ \alpha_{j+1} = 1 > \tau_j = 0 \implies n_{j+1} = m_j = m - j. \]

3.3. Further simplifications. According to Lemma 1, (3.3) is expanded as
\[
M_n(t) = \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha_1} \sum_{\tau \subseteq \alpha} \prod_{1 \leq i < j \leq m} (m_i + m_j + r + s + 1)(m_i - m_j) \\
\times \prod_{j=1}^{l(\tau)} \frac{n_{j}!}{\Gamma(r + s + n_j + m_j + 2)} \det \left( \frac{n_i!}{\Gamma(r + s + n_i + m_j + 2)} \right)_{i,j=1}^{m} \\
\times \prod_{j=l(\alpha)+1}^{m} \frac{n_{j}!}{\Gamma(r + s + 2m_j + 2)} \prod_{j=1}^{l(\alpha)+1} \frac{(r + s + 2m_j + 1)\Gamma(r + n_j + 1)\Gamma(m_j + s + 1)\Gamma(r + s + m_j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}{\Gamma(r + m_j + 1)\Gamma(m + 1)\Gamma(m - j + 1)} \\
\] (3.4)
where the empty determinant and the empty product equal 1. This expression can be considerably simplified. Namely, it can be reduced to the formula given in the corollary below, where we prove that the factors corresponding to the indices \( l(\alpha) + 1 \leq i, j \leq m \) cancel. To this end, we find it convenient to single out the contribution of the empty partition, which corresponds to the stationary regime \( t \to +\infty \).

**Corollary 1.** The moment formula (3.4) reduces to

\[
M_n(t) = M_n(\infty) + \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha_1} \sum_{\tau \subseteq \alpha, \tau \neq \emptyset} (r + s + \tau_1 + 2m - l(\tau))(\tau_1 + l(\tau) - 1) \left( \prod_{i=2}^{l(\tau)} \frac{(m-i+1)(m-i+s+1)}{(2m-i-l(\tau) + r + s + 2)(1 + \tau - i)} \prod_{j=l(\tau)+1}^{\tau_1} (m-j+1)(r + m - j + 1) \right. \\
\left. \times \det \left( \frac{\Gamma(r + s + 2(m-j) + 2)}{(n_i-n_j)!\Gamma(r + s + n_i + m_j + 2)} \right)_{j=l(\tau)+1}^{l(\tau)+1, l(\alpha)} \right)_{i,j=1}^{\tau_1+1}.
\]

where, for a nonempty hook \( \tau \subseteq \alpha \),

\[
V_{\alpha_1, \tau_1}^{r,s,m} := \frac{(r + s + 2\tau_1 + 2m - 1)\Gamma(\tau_1 + 2m + r + s)\Gamma(\alpha_1 + m)\Gamma(r + \alpha_1 + m)\Gamma(\tau_1 + m + s)}{(\alpha_1 - \tau_1)!\Gamma(\tau_1 - 1)!\Gamma(r + s + \alpha_1 + \tau_1 + 2m)\Gamma(m)\Gamma(r + \tau_1 + m)\Gamma(m + s)}.
\]

**Proof.** We consider only hooks \( \tau \) such that \( l(\tau) \geq 1 \). We proceed in three steps. At the first one, we work out the product

\[
\prod_{l(\alpha)+1 \leq i < j \leq m} (m_i + m_j + r + s + 1)(m_i - m_j)
\]

\[
= \prod_{i=l(\alpha)+1}^{m-1} \prod_{j=i+1}^{m} (2m - i - j + r + s + 1)(j - i)
\]

\[
= \prod_{i=l(\alpha)+1}^{m-1} \Gamma(m - i + 1)(m - i + r + s + 1)_{m-i}
\]

\[
= \prod_{i=l(\alpha)+1}^{m} \frac{\Gamma(m - i + 1)\Gamma(2m - 2i + r + s + 1)}{\Gamma(m - i + r + s + 1)}.
\]

Since \( n_j = m_j = m - j \) for \( j \geq l(\alpha) + 1 \), it follows that

\[
\prod_{l(\alpha)+1 \leq i < j \leq m} (m_i + m_j + r + s + 1)(m_i - m_j)
\]

\[
\times \prod_{j=l(\alpha)+1}^{m} \frac{n_j!}{\Gamma(r + s + 2m_j + 2)} \frac{(r + s + 2m + 1)\Gamma(r + n_j + 1)\Gamma(m_j + s + 1)\Gamma(r + s + m_j + 1)}{\Gamma(r + m_j + 1)\Gamma(m_j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}
\]

equals 1. At the second step, we split the product

\[
\prod_{1 \leq i \leq l(\alpha)} (m_i + m_j + r + s + 1)(m_i - m_j)
\]

\[
\times \prod_{i=l(\tau)+1}^{l(\alpha)} \prod_{j=i+1}^{l(\alpha)} (m_i + m_j + r + s + 1)(m_i - m_j)
\]

into

\[
\prod_{1 \leq i \leq l(\alpha)} (m_i + m_j + r + s + 1)(m_i - m_j)
\]

\[
\prod_{i=l(\tau)+1}^{l(\alpha)} \prod_{j=i+1}^{m} (m_i + m_j + r + s + 1)(m_i - m_j)
\]

264
and

\[
\prod_{i=1}^{l(\tau)} \prod_{j=i+1}^{m} (m_i + m_j + r + s + 1)(m_i - m_j).
\]

The first product is expressed as

\[
\prod_{i=l(\tau)+1}^{l(\alpha)} \prod_{j=i+1}^{m} (2m - i - j + r + s + 1)(j - i)
= \prod_{i=l(\tau)+1}^{l(\alpha)} \frac{\Gamma(m - i + 1)\Gamma(2m - 2i + r + s + 1)}{\Gamma(m - i + r + s + 1)}.
\]

(3.7)

In turn, the second splits into

\[
\prod_{1 \leq i < j \leq l(\tau)} (m_i + m_j + r + s + 1)(m_i - m_j)
= \prod_{i=1}^{l(\tau)} \frac{\Gamma(\tau_i + 2m - 2i + r + s + 2)\Gamma(\tau_i + l(\tau) - i)}{\Gamma(\tau_i + 2m - i - l(\tau) + r + s + 1)\Gamma(\tau_i + l(\tau) - i + 1)}
\]

and

\[
\prod_{i=1}^{l(\tau)} \prod_{j=i+1}^{m} (m_i + m_j + r + s + 1)(m_i - m_j)
= \prod_{i=1}^{l(\tau)} \frac{\Gamma(\tau_i + 2m - i - l(\tau) + r + s + 1)\Gamma(\tau_i + m - i + 1)}{\Gamma(\tau_i + m - i + r + s + 1)\Gamma(\tau_i + l(\tau) - i + 1)}
\]

which yields

\[
\prod_{i=1}^{l(\tau)} \prod_{j=i+1}^{m} (m_i + m_j + r + s + 1)(m_i - m_j)
= \prod_{i=1}^{l(\tau)} \frac{\Gamma(\tau_i + 2m - 2i + r + s + 2)\Gamma(m_i + 1)}{(\tau_i + 2m - i - l(\tau) + r + s + 1)(\tau_i + l(\tau) - i)\Gamma(m_i + r + s + 1)}
\]

Since \(n_j = 1 + m - j = m_j + 1\) for \(l(\tau) + 1 \leq j \leq l(\alpha)\), it follows from (3.7) that

\[
\left[ \prod_{i=l(\tau)+1}^{l(\alpha)} \prod_{j=i+1}^{m} (2m - i - j + r + s + 1)(j - i) \right]
\times \prod_{j=l(\tau)+1}^{l(\alpha)} \frac{(r + s + 2m_j + 1)\Gamma(n_j + 1)\Gamma(r + n_j + 1)\Gamma(m_j + s + 1)\Gamma(r + s + m_j + 1)}{\Gamma(r + m_j + 1)\Gamma(m_j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}
\]

reduces to

\[
\prod_{j=l(\tau)+1}^{l(\alpha)} \frac{\Gamma(r + s + 2(m - j) + 2)(m - j + 1)(r + m - j + 1)}{\Gamma(r + m - j + 1)}
\]

(3.8)
Moreover, \( n_j = m_j = 1 + m - j \) when \( 2 \leq j \leq l(\tau) \), whence

\[
\left[ \prod_{i=2}^{l(\tau)} \prod_{j=i+1}^{m} (m_i + m_j + r + s + 1)(m_i - m_j) \right] \\
\times \prod_{i=2}^{l(\tau)} \frac{(r + s + 2m_j + 1)\Gamma(r + n_j + 1)\Gamma(n_j + 1)\Gamma(m_j + s + 1)\Gamma(r + s + m_j + 1)}{\Gamma(r + s + n_j + m_j + 2)\Gamma(r + m_j + 1)\Gamma(m_j + 1)\Gamma(m - j + s + 1)\Gamma(m - j + 1)}
\]

\[
= \prod_{i=2}^{l(\tau)} \frac{(m - i + 1)(m - i + s + 1)}{(2m - i - l(\tau) + r + s + 2)(1 + l(\tau) - i)}. \tag{3.9}
\]

Finally, the contribution of the terms corresponding to \( (\alpha_1, \tau_1) \) is given by

\[
\frac{r + s + 2\tau_1 + 2m - 1}{r + s + \tau_1 + 2m - l(\tau)} \times \frac{\Gamma(\alpha_1 + m)\Gamma(r + \alpha_1 + m)\Gamma(\tau_1 + m + s)\Gamma(\tau_1 + 2m + r + s)}{\Gamma(r + s + \alpha_1 + \tau_1 + 2m)\Gamma(m)\Gamma(r + \tau_1 + m)\Gamma(m + s)\Gamma(\tau_1(\tau_1 + l(\tau) - 1))}. \tag{3.10}
\]

Combining (3.8), (3.9), and (3.10) with (3.4) taken into account, we are done. \( \square \)

### 3.4. An auxiliary determinant: End of the proof. In this section, we complete the proof of Theorem 1 after expressing the determinant of the submatrix

\[
(b_{\alpha, \tau}(i, j))_{i,j=(\tau)+1}^{l(\alpha)},
\]

when it is not empty, in a product form. This expression is given in the following lemma.

**Lemma 2.** Let \( \tau \) be a hook of length \( l(\tau) \geq 1 \), and let \( \alpha \supset \tau \) be a hook such that \( l(\alpha) \geq l(\tau) + 1 \). Then

\[
\det \left( \frac{\Gamma(r + s + 2(m - j) + 2)}{(n_i - m_j)!\Gamma(r + s + n_i + m_j + 2)} \right)_{j=l(\tau)+1}^{i=l(\alpha)},
\]

\[
= \frac{1}{(l(\alpha) - l(\tau))!} \prod_{j=l(\tau)+1}^{l(\alpha) - 1} \frac{1}{r + s + 2m - l(\tau) + 1 - j}.
\]

**Proof.** When \( l(\alpha) \geq l(\tau) + 1 \geq 2 \), we have \( \alpha_i = 1 \) and \( n_i = m - i + 1 = m_i + 1 \), so that

\[
\det \left( \frac{\Gamma(r + s + 2(m - j) + 2)}{(n_i - m_j)!\Gamma(r + s + n_i + m_j + 2)} \right)_{j=l(\tau)+1}^{i=l(\alpha)},
\]

\[
= \det \left( \frac{\Gamma(r + s + 2(m - j) + 2)}{(j - i + 1)!\Gamma(r + s + 2m - i - j + 3)} \right)_{j=l(\tau)+1}^{i=l(\alpha)}.
\]

We set

\[
N := r + s + 2m + 2, \quad L := l(\alpha) - l(\tau),
\]

and, for every \( i, j \in \{l(\tau) + 1, \ldots, l(\alpha)\} \),

\[
a_{i-l(\tau), j-l(\tau)} := \frac{\Gamma(N - 2j)}{(j - i + 1)!\Gamma(N - i - j + 1)} 1_{i \leq j + 1}.
\]
The determinant which we need to compute is

\[
\det[a_{kl}]_{k,l=1}^L = \begin{vmatrix}
 a_{11} & a_{12} & \cdots & \cdots & a_{1L} \\
 1 & a_{22} & a_{23} & \cdots & a_{2L} \\
 0 & 1 & \ddots & \ddots & \vdots \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & \cdots & 1 & a_{LL}
\end{vmatrix},
\]

where

\[ a_{kk} = \frac{1}{N - 2k - 2l(\tau)}, \quad k \in \{1, \ldots, L\}. \]

Using the row operation

\[ R_2 \rightarrow R_2 - \frac{1}{a_{11}} R_1, \]

one obtains

\[
\det[a_{kl}]_{k,l=1}^L = \frac{1}{N - 2l(\tau) - 2} \begin{vmatrix}
 a_{22}' & a_{23}' & \cdots & \cdots & a_{2L}' \\
 1 & a_{33}' & \ddots & \ddots & \vdots \\
 0 & \ddots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 0 & \cdots & \cdots & 1 & a_{LL}'
\end{vmatrix},
\]

where

\[ a_{kl}' = \begin{cases} 
 a_{kl} - a_{(k-1)l}/a_{11} & \text{if } k = 2, \\
 a_{kl} & \text{otherwise}. 
\end{cases} \]

More explicitly,

\[
a_{2l}' = \frac{\Gamma(N - 2l(\tau) - 2l)}{l(N - 2l(\tau) - l - 1)(l - 2)!\Gamma(N - 2l(\tau) - l - 2)} = \frac{1}{l(N - 2l(\tau) - l - 1)^{a_{3l}'}, \quad l \geq 2.}
\]

In particular,

\[ a_{22}' = \frac{1}{2(N - 2l(\tau) - 3)}, \quad a_{23}' = \frac{1}{3(N - 2l(\tau) - 4)(N - 2l(\tau) - 6)}. \]

Now, the second row operation

\[ R_3 \rightarrow R_3 - \frac{1}{a_{11}} R_2 \]

transforms the third row of the matrix into

\[ \frac{\Gamma(N - 2l(\tau) - 2l)!}{l(N - 2l(\tau) - l - 1)(l - 3)!\Gamma(N - 2l(\tau) - l - 3)} = \frac{1}{l(N - 2l(\tau) - l - 1)^{a_{4l}'}, \quad l \geq 3.} \]

Iterating these row operations, one obtains

\[
\det[a_{kl}]_{k,l=1}^L = \prod_{k=1}^L \frac{1}{k(N - 2l(\tau) - k - 1)} = \frac{1}{(l(\alpha) - l(\tau))!} \prod_{k=l(\tau)+1}^{l(\alpha)} \frac{1}{r + s + 2m - l(\tau) + 1 - k},
\]

as claimed.
Remark. The determinant computed in the previous lemma can be written as

\[
\prod_{j=1}^{\ell(\alpha) - \ell(\tau)} \Gamma(r + s + 2m - 2l(\tau) - 2j + 2) \det \left( \frac{1}{(j - i + 1)! \Gamma(r + s + 2m - 2l(\tau) + 2 - i - j)!} \right)_{i,j=1}^{L(\alpha) - L(\tau)},
\]

where we use the convention that \((j - i + 1)! = \infty\) when \(j - i + 1 < 0\). On the other hand, if \(A\) and \(L_i, 1 \leq i \leq L\), are indeterminates, then the following identity holds (take \(B = 2\) in [13; Theorem 26, Eq. (3.13)]):

\[
\det \left( \frac{1}{(L_i + j)! (A + L_i - j)!} \right)_{i,j=1}^{L} = \prod_{1 \leq i < j \leq L} (L_i - L_j) \prod_{i=1}^{L} \frac{(A - 2i + 1)i-1}{(L_i + L)! (L_i + A - 1)!}.
\]

Choosing \(L_i = -i + 1\) and \(A = r + s + 2m - 2l(\tau) + 1\) and recalling that \(L = l(\alpha) - l(\tau)\), one obtains another proof of the previous lemma after some simplifications. The authors thank the referee for this comment.

End of the proof of the main result. With the help of Lemma 2, the formula (3.5) is written as

\[
M_n(t) = M_n(\infty) + \sum_{\alpha \text{ is a hook}} (-1)^{n-\alpha} \sum_{\tau \subseteq \alpha, \tau \neq \emptyset} \frac{e^{-\nu t} \tau^{r,s,m}}{(r + s + \tau_1 + 2m - l(\tau)) (\tau_1 + l(\tau) - 1) (l(\alpha) - l(\tau))! (l(\tau) - 1)!} \prod_{i=2}^{l(\tau)} \frac{(m - i + 1)(m - i + s + 1)}{(2m - i - l(\tau) + r + s + 2)} \prod_{j=l(\tau)+1}^{l(\alpha)} \frac{(r + m - j + 1)}{r + s + 2m - l(\tau) + 1 - j}.
\]

Now, the products appearing on the right-hand side can be expressed through the gamma function as

\[
\prod_{i=2}^{l(\alpha)} (m - i + 1) = \frac{\Gamma(m)}{\Gamma(m - l(\alpha) + 1)},
\]

\[
\prod_{i=2}^{l(\tau)} (m - i + s + 1) = \frac{\Gamma(m + s) \Gamma(2m + r + s + 2 - 2l(\tau))}{\Gamma(m + s + 1 - l(\tau)) \Gamma(2m + r + s + 1 - l(\tau))},
\]

\[
\prod_{j=l(\tau)+1}^{l(\alpha)} \frac{(r + m - j + 1)}{r + s + 2m - l(\tau) + 1 - j} = \frac{\Gamma(m + r + 1 - l(\tau)) \Gamma(2m + r + s + 1 - l(\tau) - l(\alpha))}{\Gamma(m + r + 1 - l(\tau)) \Gamma(2m + r + s + 1 - 2l(\tau))}.
\]

Finally, we apply (3.6) to obtain (3.1). This completes the proof of Theorem 1. \(\square\)

Remark. Recall the notation:

\[
r = p - m, \quad s = d - p - m = q - m.
\]

In this notation,

\[
\tilde{V}_{\alpha_1, \tau_1}^{r,s,m} = \frac{(d + 2\tau_1 - 1) \Gamma(d + \tau_1) \Gamma(\alpha_1 + m) \Gamma(p + \alpha_1) \Gamma(q + \tau_1)}{(\alpha_1 - \tau_1)! \Gamma(d + \alpha_1 + \tau_1) \Gamma(p + \tau_1) \Gamma(\tau_1)}
\]

and

\[
U_{l(\alpha), l(\tau)}^{r,s,m} := \frac{(d + 1 - 2l(\tau)) \Gamma(d - l(\alpha) - l(\tau) + 1) \Gamma(p - l(\tau) + 1)}{(l(\alpha) - l(\tau))! (l(\tau) - 1)! \Gamma(m - l(\alpha) + 1) \Gamma(p - l(\alpha) + 1) \Gamma(q - l(\tau) + 1) \Gamma(d - l(\tau) + 1)}.
\]
4. Further Perspectives

We have computed the moments of the Hermitian Jacobi process. In this section, we discuss two perspectives worth developing in future research. The first perspective is concerned with a possible application to optical fiber MIMO channels when the time variation of the communication system taken into account. The second one is motivated by the random-matrix approach to free probability theory. More precisely, the marginal of the Hermitian Jacobi process at any fixed time \( t > 0 \) converges strongly as \( m \to \infty \) to the so-called free Jacobi process [5]; the moments of the spectral measure of the latter were determined for equal projections in [10]. As a matter of fact, it would be quite interesting to determine the large-\( m \) limit of \( M_n(t) \) (after rescaling the parameters \( r = r(m), s = s(m), \) and \( d = d(m) \) so as to obtain a nontrivial limit) in order to generalize and give another proof of the expression derived in [10]. As a step toward this goal, we write the moment formula for \( m \geq n \) (or \( m \) large enough) as a linear combination of terminating balanced \( _4F_3 \)-series evaluated at unit argument. Though this hypergeometric series obeys Whipple’s transformation (see, e.g., [1; Theorem 3.3.3]), we have not succeeded in deriving a closed formula for it, which would certainly open the way to investigate the large \( m \)-limit of \( M_n(t) \).

4.1. Application to optical fiber MIMO channels. In [6], the authors used the JUE to model an optical fiber MIMO channel. Actually, the transfer matrix of this model is a truncation of a Haar unitary matrix and reflects the situation when only a part of the modes in the fiber is used. If we further take into account the time variation of the transfer matrix, then a natural dynamical candidate for modeling an optical fiber MIMO channel with \( m \) antennas at the receiver and \( p \) antennas at the transmitter would be an \( m \times p \) truncation of a \( d \times d \) unitary Brownian motion. In this case, the statistical behavior of the channel is governed by the eigenvalues of the Hermitian Jacobi process. In particular, the unitary invariance of \( J_t \) for fixed time \( t \) implies that the Shannon capacity of the channel is given by (we assume that the Gaussian noise is centered and has identity covariance matrix [20])

\[
C_t(m, p, d, \rho) := \mathbb{E} \left[ \log \det \left( \mathbb{I}_{m \times m} + \frac{\mathcal{P}}{p} J_t \right) \right],
\]

where \( \mathcal{P} \) is the total power at the transmitter and \( \rho := \frac{\mathcal{P}}{p} \). If \( \rho \leq 1 \), then the capacity is expanded as

\[
C_t(m, p, d, \rho) := \mathbb{E} \left[ \sum_{k=1}^{m} \log \left( 1 + \rho \lambda_k \right) \right] = \sum_{n=1}^{\infty} \frac{(-\rho)^n}{n} M_n(t)
\]

\[
= C_\infty(m, p, d, \rho) + \sum_{\alpha \text{ is a hook } \tau \subseteq \alpha \text{ with } 1 \leq |\alpha| \leq m, \tau \neq \emptyset} \sum_{1 \leq l(\alpha) \leq m} \frac{(-1)^{\alpha_1} |\alpha|}{(r + s + \tau_1 + 2m - l(\tau)(\tau_1 + l(\tau) - 1))}
\]

where \( C_\infty(m, p, d, \rho) \) is the channel capacity drawn from the JUE ([6], [17]). Of course, the condition \( \rho \leq 1 \) is redundant, since it is only needed to expand the logarithm in a power series. In future, we will work out the expression for \( C_t(m, p, d, \rho) \) and get rid of this condition.

4.2. The large-\( m \) limit. Reversing the summation order in (3.1), we fix a hook

\[
\tau = (h - j, 1^j), \quad 0 \leq j \leq h - 1,
\]

of weight \( 1 \leq h = |\tau| \leq n \) and then sum over the hooks

\[
\alpha = (n - k, 1^k), \quad j \leq k \leq j + n - h,
\]

269
of weight \( n \) containing \( \tau \). Extracting the terms depending only on \( \alpha \), we obtain the alternating sum

\[
\sum_{k=j}^{j+n-h} \frac{(-1)^k \Gamma(n-k+m)\Gamma(p+n-k)\Gamma(d-k-j-1)}{(n-h+j-k)!(k-j)!\Gamma(m-k)\Gamma(p-k)\Gamma(d+n-k+h-j)}.
\]

Making the index change \( k \mapsto n-h+j-k \), we transform this sum into

\[
(-1)^{n-h+j} \sum_{k=0}^{n-h} \frac{(-1)^k \Gamma(h-j+m+k)\Gamma(p+h-j+k)\Gamma(d-n+h-2j-1+k)}{k!(n-h-k)!\Gamma(m+h-n-j+k)\Gamma(p+h-n-j+k)\Gamma(d+k+2h-2j)}.
\]

Up to gamma factors which do not depend on \( k \), this sum can be expressed as a terminating \( 4F_3 \) hypergeometric series at unit argument:

\[
\frac{(-1)^{n-h+j} \Gamma(h-j+m)\Gamma(p+h-j)\Gamma(d-n+h-2j-1)}{(n-h)!\Gamma(d+2h-2j)\Gamma(m+h-n-j)\Gamma(p+h-n-j)} \times 4F_3\left(\begin{array}{c}
-(n-h), m+h-j, p+h-j, d-n+h-2j-1; \\
m-n+h-j, p-n+h-j, d+2h-2j;
\end{array} \right),
\]

which is balanced (1 plus the sum of the upper parameters is equal to the sum of the lower ones).

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