Energy analysis of Duffing oscillators with quadratic damping: exact solutions

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Abstract. Oscillators with a Duffing-type restoring force and quadratic damping are dealt with in this paper. Four characteristic cases of this restoring force are analysed: hardening, softening, bistable and a pure cubic one. Their energy-displacement relationships are considered, and the corresponding closed-form exact solutions are obtained in terms of the incomplete Gamma function, which represent new results. Such results provide insight into damped dynamics of the class of system, including finding the phase trajectories as well as the comparison between these cases from the viewpoint of the energy loss per cycle.

1. Introduction

This work deals with Duffing-type oscillators that are quadratically damped. A general form for the Duffing-type restoring force is given by:

$$F(x) = k_1 x + k_2 x^3,$$

where \(x\) is the displacement, while \(k_1\) and \(k_2\) are constant stiffness parameters. Depending on their sign and values, four characteristic cases can be distinguished, as follows:

- \(k_1 < 0\) and \(k_2 > 0\) : bistable case;
- \(k_1 > 0\) and \(k_2 < 0\) : softening restoring force;
- \(k_1 > 0\) and \(k_2 > 0\) : hardening restoring force;
- \(k_1 = 0\) and \(k_2 > 0\) : pure cubic restoring force.

The damping force is assumed to be dependent on the square of velocity

$$F_v(\dot{x}) = c \text{sgn}(\dot{x}) \dot{x}^2,$$

where \(c\) is constant and \(\text{sgn}\) stands for the signum function, while the dot stands for the differentiation with respect to time \(t\).

Thus, the equation of motion of the system under consideration reads as:
\[ m \ddot{x} + c \text{sgn} (\dot{x}) \dot{x}^2 + k_1 x + k_2 x^3 = 0, \quad (3) \]

where \( m \) is the mass. The initial conditions are assumed to correspond to the existence of an initial non-zero amplitude and a zero initial velocity:

\[ x(0) = x_0, \quad \dot{x}(0) = 0, \quad (4) \]

where \( x_0 \) is assumed to be a positive real constant.

There are various types of damping in mechanical systems, one of which is dependent on the square of the velocity. It can occur during vibration of body/particle in a gas or liquid medium. Besides this example, the pressure of acoustic flow through of number of different orifices when excited at very low excitation amplitudes, is known to be proportional to the square of the velocity [1], and then can be modeled as a motion of a point mass in a medium with a square law of drag [2]. This paper is a follow-up to the research published in [3], where a system with quadratic damping and a purely nonlinear restitution force was considered. The pendulum with quadratic damping was considered in [4], where the condition for a time-varying damping coefficient was examined for the origin to be asymptotically stable. A vibration isolation system with a non-polynomial form of the restoring force and a real-power exponent of damping was investigated in [5]. In [6], an optimal variational method was employed for the investigation of the oscillator with a linear and cubic restoring force and quadratic damping. An isolation performance of the system which has an arbitrary non-negative velocity exponent was considered in [7]. Based on Jacobi’s last multiplier and Chielmini’s condition, the system of the Lienard-type with quadratic damping was studied in [8]. In [9], the base excited oscillator with real-power exponents of the restoring force and damping was investigated. The problem of Liouvillian integrability for the oscillator with quadratic damping and a polynomial force was investigated in [10]. In [11], the delayed feedback control of a tristable isolation system with a real-power exponent of damping was studied. Klotter [12] investigated free vibrations of a system with quadratic damping and an arbitrary restoring force. By using the Lambert W-function, Cvetičanin [13] presented an exact analytical form of the energy-displacement relationship for the system with quadratic damping and a linear restoring force. Although oscillatory nonlinear systems with quadratic damping have captured the attention of researchers, there has been no study dedicated to the Duffing-type restoring force (1) in this respect so far. Given the fact that this type of the restoring force appears in a variety of engineering systems [14], it is of interest to carry out such considerations, and this study contributes to this shortcoming. Hence, this paper provides exact analytical expressions that cover all four cases listed above in a unique way.

2. Energy displacement relations

Introducing the non-dimensional variables

\[
\xi = \frac{x}{x_0}, \quad \tau = \sqrt{\frac{\text{sgn}(k_2)k_2}{m}} \frac{x_0}{\text{sgn}(k_2)}, \quad \mu = \frac{cx_0}{m}, \quad \epsilon = \frac{k_1}{\text{sgn}(k_2)k_2} x_0^2,
\]

equation (3) reads as

\[
\ddot{\xi} + \mu \text{sgn} (\dot{\xi}) \dot{\xi}^2 + \epsilon \dot{\xi} + \text{sgn}(k_2) \xi^3 = 0, \quad \xi(0) = 1, \quad \dot{\xi}(0) = 0, \quad (6)
\]

where the prime denotes differentiation with respect to non-dimensional time \( \tau \). While deriving equation (6), it has been taken into account that \( k_2 \neq 0 \), so \( 1 / \text{sgn}(k_2) = \text{sgn}(k_2) = \pm 1 \).
The total energy $E$, kinetic energy $T$ and potential energy $V$ of the system governed by equation (6) are respectively given by:

$$E = T + V, \quad T = \frac{1}{2} (\ddot{\xi})^2, \quad V = \frac{1}{2} \epsilon \dot{\xi}^2 + \frac{1}{4} \text{sgn}(k_z) \xi^4.$$  

(7)

Equation (6) can be written down in the form

$$\frac{d}{dt} \left[ \frac{1}{2} (\ddot{\xi})^2 + \frac{1}{2} \epsilon \dot{\xi}^2 + \frac{1}{4} \text{sgn}(k_z) \xi^4 \right] = -\mu \text{sgn}(\ddot{\xi})(\ddot{\xi})^3,$$

(8)

which, after using equations (7b), yields

$$\frac{dT}{d\xi} + 2\mu \text{sgn}(\ddot{\xi}) T = -\epsilon \dot{\xi} - \text{sgn}(k_z) \xi^3.$$  

(9)

So, the problem is transformed from a second-order differential equation (6) to a first-order differential equation (9). The former governs how the displacement changes in time, while the latter models how the kinetic energy changes with the displacement. To find the solution of equation (9), the motion will be assumed to begin from a positive initial displacement with respect to the equilibrium position, so that $\text{sgn}(\ddot{\xi}) = -1$. It should be noted that during one cycle, the analysis needs to be divided into two parts: the first part represents the motion from the initial position which is on the positive side of the $\xi$ axis to the stop position on the other side of the equilibrium. The kinetic and total energies corresponding to this part of the motion will be denoted by $T_\sigma$ and $E_\sigma$ respectively. Analogously, the kinetic and total energies corresponding to the second part of the whole cycle, will be denoted by $T_\rho$ and $E_\rho$. Using this notation, the general solution of equation (9) can be written down as

$$T(\xi) = C e^{2\mu i} + \text{sgn}(k_z) \left( \frac{1}{2\mu} \right)^4 e^{2\mu i} \Gamma[4, 2\mu \xi] + \frac{1}{4\mu^2} \epsilon (1 + 2\mu \xi),$$

(10)

where $C$ is an unknown constant, while $\Gamma$ stands for the upper incomplete Gamma function [15].

The total mechanical energy reads now as

$$E = C e^{2\mu i} + \text{sgn}(k_z) \left( \frac{1}{2\mu} \right)^4 e^{2\mu i} \Gamma[4, 2\mu \xi] + \frac{1}{4\mu^2} \epsilon (1 + 2\mu \xi) + \frac{1}{2} \epsilon \dot{\xi}^2 + \text{sgn}(k_z) \frac{1}{4} \xi^4.$$  

(11)

In equations (10) and (11), the constant $C$ can be obtained from the initial conditions (6)

$$C = -\text{sgn}(k_z) \left( \frac{1}{2\mu} \right)^4 \Gamma[4, 2\mu] - \frac{1}{4\mu^2} \epsilon (1 + 2\mu)e^{-2\mu},$$

(12)

so the total energy during the first interval is defined by the following equation:
\[ E = \frac{1}{4\mu} \varepsilon (1 + 2\mu \dot{\xi}) \]  
\[ + \frac{1}{2} \varepsilon \dot{\xi}^3 + \text{sgn} (k_z) \frac{1}{4} \xi^4 \]  
\[ + \left( \frac{1}{2\mu} \right)^2 \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^2 \Gamma [4, 2\mu \dot{\xi}] - \]  
\[ - \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^2 \Gamma [4, 2\mu] - \varepsilon (1 + 2\mu)e^{2\mu}. \]  

(13)

For the second interval, i.e. the motion from the left to the right-hand side, the corresponding energy-displacement relation can be obtained analogously as before. It should be noted that in this case \( \text{sgn} (\xi') = 1 \), so based on equation (9), the general solution is found to be:

\[ \Gamma (\xi) = C_1 e^{-2\mu \xi} + \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^4 e^{-2\mu \xi} \Gamma [4, -2\mu \dot{\xi}] - \frac{1}{4\mu^2} \varepsilon (-1 + 2\mu \dot{\xi}). \]  

(14)

The energy-displacement function is

\[ E = C_1 e^{-2\mu \xi} + \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^4 e^{-2\mu \xi} \Gamma [4, -2\mu \dot{\xi}] - \frac{1}{4\mu^2} \varepsilon (-1 + 2\mu \dot{\xi}) + \frac{1}{2} \varepsilon \dot{\xi}^2 + \text{sgn} (k_z) \frac{1}{4} \xi^4. \]  

(15)

The constant \( C_1 \) can be obtained having in mind that in the position \( \xi_i \) the velocity is zero, so that equation (14) yields

\[ C_1 = \frac{1}{4\mu} \varepsilon (-1 + 2\mu \dot{\xi}) e^{2\mu \xi} - \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^4 \Gamma [4, -2\mu \xi_i]. \]  

(16)

Finally, the expression for the energy-displacement function, for the motion to the right-hand side, reads as:

\[ E = -\frac{1}{4\mu^2} \varepsilon (-1 + 2\mu \dot{\xi}) + \frac{1}{2} \varepsilon \dot{\xi}^2 + \text{sgn} (k_z) \frac{1}{4} \xi^4 + \left( \frac{1}{2\mu} \right)^2 e^{-2\mu \xi} \left[ \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^2 \Gamma [4, -2\mu \dot{\xi}] - \right. \]  
\[ - \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^2 \Gamma [4, -2\mu \xi_i] + \left. \left( \frac{1}{2\mu} \right)^2 \varepsilon (-1 + 2\mu \dot{\xi}) \right]. \]  

(17)

Thus, using equations (13) and (17), the way how the energy changes during one full cycle is determined. The value of \( \xi_i \) is obtained by solving the following equation:

\[ \frac{1}{4\mu} \varepsilon (1 + 2\mu \dot{\xi}_i) + \left( \frac{1}{2\mu} \right)^2 e^{2\mu \xi}_i \left[ \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^2 \Gamma [4, 2\mu \dot{\xi}_i] - \right. \]  
\[ \text{sgn} (k_z) \left( \frac{1}{2\mu} \right)^2 \Gamma [4, 2\mu] - \varepsilon (1 + 2\mu)e^{-2\mu} \right] = 0. \]  

(18)

3. Phase Trajectories

Based on equation (9), it is possible to determine the relationship between the velocity and displacement. Knowing that \( \dot{\xi}^* = dT / d\xi \), equation (6) can be presented as
\[
\frac{dT}{d\xi} + e\xi + \text{sgn}(k_2)\xi^3 = -\mu \text{sgn}(\xi')\xi'^2.
\] (19)

From equation (19), the velocity can be expressed as
\[
\xi' = \pm \sqrt{-\frac{1}{\mu \text{sgn}(\xi')} \frac{e}{d\xi} - \frac{1}{\mu \text{sgn}(\xi')} \frac{\text{sgn}(k_2)}{\mu \text{sgn}(\xi')} \xi^3}.
\] (20)

The previously defined expressions \(T(\xi')\) can now be used to establish an explicit connection between \(\xi'\) and \(\xi\). Here, we again consider the motion in two subintervals as previously. For the motion to the left-hand side, one has
\[
\xi'' = -\frac{1}{\sqrt{2}\mu} \left[ e + 2\mu e\xi - e(1 + 2\mu)e^{2\mu(\xi - 1)} - 4\text{sgn}(k_2)e^{2\mu}\mu^2 \left( E_n[-3, 2\mu] - \xi^4 E_n[-3, 2\mu\xi]\right)^{1/2},
\] (21)

where \(E_n\) is the exponential integral [16]. For the part of the motion to the right-hand side, the relationship between \(\xi'\) and \(\xi\) is:
\[
\xi'' = \frac{1}{\sqrt{2}\mu} \left[ e^{2\mu} \left( 4\mu e^2 \left( e^{2\mu} (1 - 2\mu\xi) + e^{2\mu} (-1 + 2\mu\xi)\right)\right) + \text{sgn}(k_2)\Gamma[4, -2\mu\xi] - \text{sgn}(k_2)\Gamma[4, -2\mu\xi]\right]^{1/2}.
\] (22)

4. Analysis

Using the derived expressions, it is possible to perform an analysis of the change of the energy, as well as to get the trajectories in the phase plane. Note that as above, it is of interest to consider the behaviour of the system during one full cycle. This analysis is carried out and illustrated below for four characteristic cases defined in the Introduction: bistable, softening, hardening and pure cubic.

It should be noted that the derived expressions refer to a non-dimensional mathematical model represented by equation (6), in which two dimensionless parameters \(e\) and \(\mu\) appear. These non-dimensional parameters depend on the dimensional initial condition \(x_0\). Thus, the shape of the potential energy diagram varies depending on the choice of the parameter \(e\).

Figure 1 shows a bistable case for which \(e < 0\) and \(\text{sgn}(k_2)=1\). The potential energy diagram (represented by the thicker dashed line) has a characteristic shape with a hill at the origin and two wells that are situated symmetrically with respect to the hill. As it is known, these points correspond respectively to an unstable fixed point (labelled by FP3), and the stable fixed points (labelled by FP1 and FP2) occurring at \(\xi_{1/2}=\pm e^{1/2}\). In the case of a conservative system, two types of oscillatory motion can exist: one around each stable fixed points (the so-called ‘in-well oscillations’) or outside the homoclinic orbit (‘out-of-well oscillations’, which are characterised by larger amplitudes). However, here depending on the parameters \(e\) and \(\mu\), the phase trajectories can perform out-of-well motion in the first part of oscillations, but due to the existence of damping, there will be oscillations around one of the fixed points only. The analytical expressions (13) and (17) provide the conditions for determining the behaviour of the system. Using equation (13), a curve of the energy change \(E(\xi)\) is drawn during the first part of the motion to the left-hand side (represented by the thick red line). The intersection of this curve with the potential energy diagram is labelled by Point A, the coordinate \(\xi\) of this point is equal to unity and this is in accordance with the expression (5). The horizontal line through it represents the initial energy of the considered system which will decrease due to the existence of
damping. The curve $E(\xi)$ has a characteristic shape, where at the intersections with the diagram of the potential energy (Points A and C) there are a local maximum and a local minimum, respectively. Therefore, there are two inflection points of the curve $E(\xi)$ between Points A and C. The position of Point B, i.e. the sign of its ordinate $\eta=E(\xi=0)$ is also important for the way of how the oscillations are performed. Namely, if $\eta>0$, there will be a crossing over the potential barrier, directed towards the side of the fixed point FP2, as it is the case in the example shown. The intersection points of the curves $E(\xi)$ and $V(\xi)$ in which the tangents to curve $E(\xi)$ are horizontal (places of the local extremes of function $E(\xi)$) are marked with $t$. The abscissa of Point C is the location of the current stop position. Using equation (17), the curve $E(\xi)$ is drawn for the second part of the right-hand cycle (CDE curve). In this case, the ordinate of Point D is positive, so that the system returns to the side of the initial position. Since the ordinate of Point E is negative, the system will then oscillate around the stable fixed point FP1. In this way, equations (13) and (17) make it possible to estimate the influence of the parameters $\varepsilon$ and $\mu$ on the possibility of crossing the potential barrier. In Figure 1(a), $\Delta E_1=E_A-E_E$ shows the energy loss during the first cycle of motion. The energy loss during other parts of the oscillations would be assessed in an analogous way. The analytical expressions (13) and (17) enable one to estimate the energy loss as a function of the system parameters.

![Figure 1](image.png)

Figure 1. Bistable case; $\mu=0.2$; $\varepsilon=−0.4$; $\text{sgn}(k_2)=1$; (a) Energy diagrams; (b) Phase trajectories.

Figure 1(b) shows the solution in the phase plane. Using equation (20), a phase trajectory was drawn for the first part of the motion to the left-hand side (this is represented by the curve GH plotted by the red solid line). Similarly, using equation (21), a curve HI is drawn representing the phase trajectory for the second part of the motion to the right-hand side. These phase trajectories are in accordance with Figure 1(a) confirming that in this cycle, the system has enough energy to cross the potential barrier at the unstable fixed point FP3. After that, the phase trajectories indicate that there is an oscillation around the stable fixed point FP1.
Figure 2 is plotted for the case of the softening restoring force, where Figure 1(a) contains the energy diagrams and Figure 1(b) phase trajectories. In this case, there are three fixed points, but the stable fixed point is FP3 placed at the origin. The parameter $\varepsilon$ ($\varepsilon=1$) is adopted here so that the system is in an unstable fixed point FP1 at the initial moment. This is also the limit (minimum) value of the parameter $\varepsilon$ yielding the oscillatory motion around the stable fixed point FP3. Numerical integrations of differential equation (6) for these initial conditions corresponding to the boundary case could not lead to such a solution. Therefore, a value for $\varepsilon=1+\delta$ should be taken, where $\delta$ is positive. It should be emphasized that for different values of the parameter $\varepsilon$, different forms of the diagram $V(\xi)$ are obtained, where only the motion for $\varepsilon>1$ is physically feasible. The feature of the derived analytical expressions is that they give solutions even in this limit case when $\varepsilon=1$, as well as that the shape of the curve $E(\xi)$ clearly explains the physical meaning of a possible real solution, i.e. real physical motion.

Based on equation (13), the curve $E(\xi)$ is drawn for the first part of the cycle (motion to the left-hand side), containing Points A, B and C. Point A denotes the value of $E(\xi=0)$. The shape of the curve $E(\xi)$ at position A should be noted. Unlike in the bistable case, herein the curve $E(\xi)$ has no local extrema and it is valid only for this point. At all other Points, B, C, D, E and F, the curves $E(\xi)$ have local extrema at the intersections with $V(\xi)$. This shape of the curve $E(\xi)$ at Point A indicates that motion is possible only to the right-hand side, because otherwise there would be an increase of energy, which is not possible because there is no energy supply to the system, but only dissipation due to damping. The motion is only possible towards Point B, and this part of the line is represented by the red dash-dotted line. As already indicated in the previous example, at the intersection of the curve $E(\xi)$ and the diagram of the potential energy $V(\xi)$, the tangent to the curve $E(\xi)$ is horizontal and these places are marked with $t$ in Figure 2(a). At Point B, the curve $E(\xi)$ has a local minimum, and at Point C a local
maximum. The curve $E(\xi)$ is represented by the blue solid line, which reflects the other part of the cycle that is physically realized. Other parts of this curve are shown by the blue dash-dotted line, and these parts of curve $E(\xi)$ do not refer to real physical motion. At Points F and D, the curve $E(\xi)$ reflecting the motion to the right-hand side has local minima, and at Points B and E local maxima. Figure 2(b) shows the phase trajectories based on expressions (20) and (21). In addition to the phase trajectories HJ and JI, which refer to physically achievable motion, theoretical solutions that cannot be physically realized are also presented by the dash-dotted line.

Figure 3 shows the hardening type of the restoring force (the corresponding lines and points are consequently enriched with the index H and they are plotted in red) and pure cubic restoring force (the corresponding lines and points are consequently enriched with the index PC and plotted in blue). These two cases are shown in the same diagrams for the easy comparison between them, which enable one to determine the influence of the linear term that exist in the hardening case but not in the pure cubic case either.

![Figure 3](image)

**Figure 3.** (a) Hardening restoring force ($\epsilon=0.5$); (b) Pure cubic restoring force ($\epsilon=0$); For both cases: $\mu=0.5$.

Part of the curve $A_HB_H$ reflects the change in energy in the first part of the motion to the left-hand side in the hardening case, while the part $A_{PC}B_{PC}$ corresponds to the pure cubic case. It can be noticed that in the hardening case, the initial energy value is higher than in the pure cubic case. Also, from Figure 3(b), which shows the phase trajectories, it can be noticed that the velocities in the hardening case are
higher, which is why both the resistance and loss of energy are higher than in the pure cubic case. The derived analytical expressions allow the estimation of the energy loss $\Delta E$ during the cycles. Also, the derived expressions enable one to determine the value of parameters in the case of overdamped oscillations, which can take place when the abscissas of Points B are positive, or the abscissas of Points C are negative.

5. Conclusion

The exact analytical expressions for the relationship between the energy and displacement, as well as the expressions for phase trajectories for the Duffing-type of restoring force and for quadratic damping have been obtained. They have been found for four characteristic cases of the Duffing-type restoring force: hardening, softening, bistable and pure cubic. The derived expressions are non-dimensional, and the non-dimensional procedure has been introduced in a way that only two non-dimensional parameters appear in the expressions derived, as well as the sign of the coefficient of the cubic term of the restoring force. As far as the authors are aware, these expressions have not been known so far and represent new results given in terms of special functions: the upper incomplete Gamma function and the exponential integral. Besides being exact and not the approximated ones, these results give deep insight into the damped dynamics of these systems, enabling one to perform their mutual comparison and the analysis of the influence of the systems parameters, as illustrated by the energy change and phase trajectories in this paper.

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Acknowledgments

This research has been supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia through the project 451-03-9/2021-14/ 200156.