DISTANCES IN DENSE SUBSETS OF $\mathbb{Z}^d$

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Abstract. In [2] Katznelson and Weiss establish that all sufficiently large distances can always be attained between pairs of points from any given measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density. A second proof of this result, as well as a stronger “pinned variant”, was given by Bourgain in [1] using Fourier analytic methods. In [5] the second author adapted Bourgain’s Fourier analytic approach to establish a result analogous to that of Katznelson and Weiss for subsets $\mathbb{Z}^d$ provided $d \geq 5$. In this article we establish an optimal strengthening of this discrete distance set result as well as the natural “pinned variant”.

1. Introduction

Recall that upper Banach density $\delta^*$ is defined for $A \subseteq \mathbb{Z}^d$ by

$$\delta^*(A) = \limsup_{N \to \infty} \sup_{x \in \mathbb{Z}^d} \frac{|A \cap (x + \{1, \ldots, N\}^d)|}{N^d}.$$ 

1.1. Distance sets and existing results. A result of Katznelson and Weiss [2] states that all sufficiently large distances can always be attained between pairs of points from any given measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density. Specifically, if $A$ is a measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density, then there exists $\lambda_0 = \lambda_0(A)$ such that the distance set

$$\text{dist}(A) = \{|x-y| : x, y \in A\} \supseteq [\lambda_0, \infty).$$

This result was later established using Fourier analytic methods by Bourgain in [1]. Bourgain in fact also established a “pinned variant”, namely that for any $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that

$$\text{dist}(A; x) = \{|x-y| : y \in A\} \supseteq [\lambda_0, \lambda_1].$$

In [5] the second author adapted Bourgain’s Fourier analytic approach to establish a result analogous to that of Katznelson and Weiss for subsets $\mathbb{Z}^d$, namely that if $A \subseteq \mathbb{Z}^d$ of positive upper (Banach) density, then there exists $\lambda_0 = \lambda_0(A)$ and an integer $q$, depending only on the density of $A$, such that

$$\text{dist}^2(A) = \{|x-y|^2 : x, y \in A\} \supseteq [\lambda_0, \infty) \cap q\mathbb{Z}.$$ 

One should note that the fact that $A$ could fall entirely into a fixed congruence class of some integer $1 \leq r \leq \delta^*(A)^{-1/d}$ ensures that $q$ must be divisible by the least common multiple of all integers $1 \leq r \leq \delta^*(A)^{-1/d}$.

1.2. New results. In what follows we will denote the discrete sphere of radius $\sqrt{\lambda}$ by $S_\lambda$, namely

$$S_\lambda := \{x \in \mathbb{R}^d : |x|^2 = \lambda\} \cap \mathbb{Z}^d.$$ 

Our first result is the following optimal strengthening of the discrete distance set result from [5].

**Theorem 1** (Optimal Unpinned Distances). Let $\varepsilon > 0$ and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.

There exist $q = q(\varepsilon)$ and $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any $\lambda \geq \lambda_0$ there exist $x \in A$ for which

$$\frac{|A \cap (x + qS_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon.$$

While the main result of this paper is the following (optimal) “pinned variant” of Theorem 1 above, in other words the (optimal) discrete analogue of Bourgain’s pinned distances theorem.

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Theorem 2 (Optimal Pinned Distances). Let $\varepsilon > 0$ and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.

There exist $q = q(\varepsilon)$ and $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there exists a fixed $x \in A$ such that

$$\frac{|A \cap (x + qS_A)|}{|S_A|} > \delta^*(A) - \varepsilon \quad \text{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1.$$  

2. Uniformly Distributed Sets

Definition 1 (Definition of $q_\eta$ and $\eta$-uniform distribution). For any $\eta > 0$ we define

$$q_\eta := \text{lcm}\{1 \leq q \leq C\eta^{-2}\}$$

with $C > 0$ a (sufficiently) large absolute constant and $A \subseteq \mathbb{Z}^d$ to be $\eta$-uniformly distributed (modulo $q_\eta$) if its relative upper Banach density on any “residue class” modulo $q_\eta$ never exceeds $(1 + \eta^2)$ times its density on $\mathbb{Z}^d$, namely if

$$\delta^*(A \cap (x + q_\eta\mathbb{Z})^d) \leq (1 + \eta^2) \delta^*(A)$$

holds for all $s \in \{1, \ldots, q_\eta\}$.

Theorems 1 and 2 are immediate consequences, via an easy density increment argument, of the following analogous results for uniformly distributed sets.

Theorem 3 (Theorem 1 for Uniformly Distributed Sets). Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^2$, and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.

If $A$ is $\eta$-uniformly distributed, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any $\lambda \geq \lambda_0$ one has

$$\frac{|A \cap (x + S_A)|}{|S_A|} > \delta^*(A) - \varepsilon \quad \text{for some} \quad x \in A$$

Theorem 4 (Theorem 2 for Uniformly Distributed Sets). Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^3$, and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.

If $A$ is $\eta$-uniformly distributed, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there exists a fixed $x \in A$ such that

$$\frac{|A \cap (x + S_A)|}{|S_A|} > \delta^*(A) - \varepsilon \quad \text{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1.$$  

3. Preliminaries

3.1. Fourier analysis on $\mathbb{Z}^d$. If $f : \mathbb{Z}^d \to \mathbb{C}$ is a function for which

$$\sum_{x \in \mathbb{Z}^d} |f(x)| < \infty$$

we will say that $f \in \ell^1(\mathbb{Z}^d)$ and define

$$\|f\|_1 = \sum_{x \in \mathbb{Z}^d} |f(x)|.$$  

For $f \in \ell^1$ we define its Fourier transform $\hat{f} : \mathbb{T}^d \to \mathbb{C}$ by

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x)e^{-2\pi ix \cdot \xi}$$

noting that the summability assumption on $f$ ensures that the series defining $\hat{f}$ converges uniformly to a continuous function on the torus $\mathbb{T}^d$, which we will freely identify with the unit cube $[0,1)^d$ in $\mathbb{R}^d$.

Furthermore, Parseval’s identity, namely that if $f, g \in \ell^1$ then

$$\langle f, g \rangle := \sum_{x \in \mathbb{Z}^d} f(x)\overline{g(x)} = \int_{\mathbb{T}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)} \, d\xi$$

is a simply and immediate consequence of the familiar orthogonality relation

$$\int_{\mathbb{T}^d} e^{2\pi ix \cdot \xi} \, d\xi = \begin{cases} 1 & \text{if} \ x = 0 \\ 0 & \text{if} \ x \neq 0 \end{cases}.$$
Defining the convolution of $f$ and $g$ to be
\[ f * g(x) = \sum_{y \in \mathbb{Z}^d} f(x - y)g(y) \]
it follows that if $f, g \in \ell^1$ then $f * g \in \ell^1$ with
\[ \|f * g\|_1 \leq \|f\|_1 \|g\|_1 \quad \text{and} \quad \widehat{f * g} = \widehat{f} \widehat{g}. \]

Finally, we recall following consequence of the Poisson Summation Formula, namely that if $\psi$ is a Schwartz function on $\mathbb{R}^d$, then
\[ \hat{\psi}(\xi) = \sum_{y \in \mathbb{Z}^d} \hat{\psi}(\xi - y) \]
where
\[ \hat{\psi}(\xi) = \int_{\mathbb{R}^d} \psi(x)e^{-2\pi i x \cdot \xi} \, dx \]
denotes the Fourier transform on $\mathbb{R}^d$ of $\psi$.

3.2. Counting differences in $S_\lambda$. Let $A \subseteq B_N$, where $B_N \subseteq \mathbb{Z}^d$ denotes some arbitrary translate of the cube $\{1, \ldots, N\}^d$, and recall that we are denoting the discrete sphere of radius $\sqrt{\lambda}$ by $S_\lambda$, namely
\[ S_\lambda := \{x \in \mathbb{R}^d : |x|^2 = \lambda\} \cap \mathbb{Z}^d. \]

It is easy to verify, using the properties of the Fourier transform discussed above, that
\[ \sum_{x \in A} \frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} = \langle 1_A, A_\lambda(1_A) \rangle = \int |\hat{1}_A(\xi)|^2 \sigma_\lambda(\xi) \, d\xi \]
where $A_\lambda(f)(x)$ denotes the spherical average
\[ A_\lambda(f)(x) := f * \sigma_\lambda(x) = \frac{1}{|S_\lambda|} \sum_{y \in S_\lambda} f(x - y). \]

3.3. Exponential sum estimates. In light of (8) we will naturally be interested estimates for the Fourier transform of the surface measure $\sigma_\lambda$, namely
\[ \sigma_\lambda(\xi) := \frac{1}{|S_\lambda|} \sum_{x \in S_\lambda} e^{-2\pi i x \cdot \xi}. \]

It is clear that whenever $|\xi|^2 \ll \lambda^{-1}$ there can be no cancellation in the exponential sum (10), in fact it is easy to verify that the same is also true whenever $\xi$ is close to a rational point with small denominator. The following Proposition is a precise formulation of the fact that this is the only obstruction to cancellation.

**Proposition 1** (Key exponential sum estimates, Proposition 1 in [5]). Let $\eta > 0$. If $\lambda \geq C \eta^{-4}$ and
\[ \xi \notin (\eta^{-1} \mathbb{Z})^d + \{\xi \in \mathbb{R}^d : |\xi|^2 \leq \eta^{-1} \lambda^{-1}\}, \]
then
\[ \left| \frac{1}{|S_\lambda|} \sum_{x \in S_\lambda} e^{-2\pi i x \cdot \xi} \right| \leq \eta. \]
3.4. **Smooth cutoff functions.** It will be convenient to introduce a smooth function \( \psi_{q,L} \) whose Fourier transform (on \( \mathbb{Z}^d \)) will serve as a substitute for the characteristic function of the set

\[
\mathcal{M}_{q,L} = \left( q^{-1} \mathbb{Z} \right)^d + \{ \xi \in \mathbb{R}^d : |\xi| \leq L^{-1} \}.
\]

Towards this end, let \( \psi : \mathbb{R}^d \to (0, \infty) \) be a Schwartz function satisfying

\[
1 = \hat{\psi}(0) \geq \hat{\psi}(\xi) \geq 0 \quad \text{and} \quad \hat{\psi}(\xi) = 0 \quad \text{for} \quad |\xi| > 1
\]

where \( \hat{\psi} \) denotes the Fourier transform (on \( \mathbb{R}^d \)) of \( \psi \). For a given \( q \in \mathbb{N} \) and \( L \geq q \) we define

\[
\psi_{q,L}(x) = \begin{cases} (\hat{\psi})^d \psi \left( \frac{\xi}{q} \right) & \text{if} \quad x = (q\mathbb{Z})^d \\ 0 & \text{otherwise} \end{cases}
\]

(11)

It follows from the Poisson summation formula that the Fourier transform (on \( \mathbb{Z}^d \)) of \( \psi_{q,L} \) takes the form

\[
\hat{\psi}_{q,L}(\xi) = \sum_{\ell \in \mathbb{Z}^d} \hat{\psi} \left( L \left( \xi - \frac{\ell}{q} \right) \right)
\]

(12)

and is supported on \( \mathcal{M}_{q,L} \).

3.5. **Properties of \( \psi_{q,L} \) and \( \hat{\psi}_{q,L} \).** We first note that since \( \hat{\psi} \) is compactly supported and \( q \leq L \), it follows from (12) that

\[
\sum_{x \in \mathbb{Z}^d} \psi_{q,L}(x) = \hat{\psi}_{q,L}(0) = \sum_{\ell \in \mathbb{Z}^d} \hat{\psi}(\ell/q) = \hat{\psi}(0) = 1.
\]

We next make the simple but important observation that \( \psi \) may be chosen so that for any \( \eta > 0 \), the function \( 1 - \hat{\psi}_{q,L} \) will be essentially supported on the complement of \( \mathcal{M}_{q,\eta^{-1}L} \) in the sense that

\[
|1 - \hat{\psi}_{q,L}(\xi)| \ll \eta \quad \text{whenever} \quad \xi \in \mathcal{M}_{q,\eta^{-1}L}.
\]

Finally we record a precise formulation of the fact that \( \psi_{q,L} \) is essentially supported on a box of size \( \eta^{-1}L \) and is approximately constant on smaller scales.

**Lemma 1.** Let \( \eta > 0 \) and \( 1 \leq q \leq L \), then

\[
\sum_{|x| \geq \eta^{-1}L} \psi_{q,L}(x) \ll \eta.
\]

(14)

and

\[
\| \chi_{q,L} \ast \psi_{q,L_1} - \psi_{q,L_1} \|_1 \ll \eta
\]

(15)

whenever \( L_1 \geq \eta^{-1}L \), where

\[
\chi_{q,L}(x) = \begin{cases} (\hat{\psi})^d \psi \left( \frac{\xi}{q} \right) & \text{if} \quad x \in (q\mathbb{Z})^d \cap \left[-\frac{q}{2}, \frac{q}{2}\right]^d \\ 0 & \text{otherwise} \end{cases}.
\]

(16)

**Proof.** Estimate (14) is easily verified using the fact that \( \psi \) is a Schwartz function on \( \mathbb{R}^d \) as

\[
\sum_{|x| \geq \eta^{-1}L} \psi_{q,L}(x) = \left( \frac{q}{L} \right)^d \sum_{|\ell| \geq \eta^{-1}L/q} \psi(\ell/q) \ll \left( \frac{q}{L} \right)^d \sum_{|\ell| \geq \eta^{-1}L/q} \left( 1 + \frac{|\ell q|}{L} \right)^{-d-1} \ll \eta.
\]

To verify estimate (15) we make use of the fact that both \( \psi \) and its derivative are rapidly decreasing, specifically

\[
\| \chi_{q,L} \ast \psi_{q,L_1} - \psi_{q,L_1} \|_1 \leq \left( \frac{q}{L_1} \right)^d \sum_{x \in (q\mathbb{Z})^d} \sum_{y \in (q\mathbb{Z})^d} \left| \psi \left( \frac{x - y}{L_1} \right) - \psi \left( \frac{x}{L_1} \right) \right|
\]

\[
\leq \frac{L}{L_1} \left( \frac{q}{L_1} \right)^d \sum_{x \in (q\mathbb{Z})^d} \left( 1 + \frac{|x|}{L_1} \right)^{-d-1} \leq \frac{L}{L_1}.
\]

\[\square\]
4. Reducing Theorems 3 and 4 to Key Dichotomy Propositions

First a definition.

**Definition 2** (Definition of \((\eta, L)\)-uniform distribution). Let \(N\) be a large positive integer and \(B_N \subseteq \mathbb{Z}^d\) denote some arbitrary translate of the cube \(\{1, \ldots, N\}^d\). For any \(\eta > 0\) and positive integer \(L\) with the property that \(q_\eta|L|N\) we define \(A \subseteq B_N\) to be \((\eta, L)\)-uniformly distributed if

\[
\frac{|A \cap B_L \cap (s + (q_\eta \mathbb{Z})^d)|}{(L/q_\eta)^d} \leq (1 + \eta^2) \frac{|A|}{N^d}
\]

holds for all \(s \in \{1, \ldots, q_\eta\}^d\) and each sub-cube \(B_L\) in the partition of the original cube \(B_N\) into \((N/L)^d\) sub-cubes each of “ sidelength” \(L\).

4.1. Dichotomy Propositions. As with the second author’s approach in [5], itself adapted from [1], we will deduce Theorems 3 and 4 as consequences of the following quantitative finite versions.

**Proposition 2** (Dichotomy for Theorem 3). Let \(\varepsilon > 0\), \(0 < \eta \ll \varepsilon^2\), and \((L, N)\) be a pair of integers such that \(q_\eta|L|N\). If \(A \subseteq B_N \subseteq \mathbb{Z}^d\) with \(d \geq 5\) is \((\eta, L)\)-uniformly distributed, then for all integers \(\lambda\) satisfying \(\eta^{-4}L^2 \leq \lambda \leq \eta^{-1}N^2\) one of the following statements must hold:

(i) there exists \(x \in A\) such that

\[
\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \frac{|A|}{N^d} - \varepsilon
\]

(ii)

\[
\frac{1}{|A|} \left| \int_{\Omega_\lambda} |1_A(\xi)|^2 d\xi \right| \gg \varepsilon
\]

where \(\Omega_\lambda = \Omega_\lambda(\eta, q_\eta)\) denotes the set theoretic sum \((q_\eta^{-1}\mathbb{Z})^d + \{\xi \in \mathbb{R}^d : \eta^2 \lambda^{-1} \leq |\xi|^2 \leq \eta^{-2} \lambda^{-1}\}\).

**Proposition 3** (Dichotomy for Theorem 4). Let \(\varepsilon > 0\), \(0 < \eta \ll \varepsilon^3\), and \(A \subseteq B_N \subseteq \mathbb{Z}^d\) with \(d \geq 5\).

If \(A\) is \((\eta, L)\)-uniformly distributed (this implicitly assumes that \(q_\eta|L|N\)), then for all integer pairs \((\lambda_0, \lambda_1)\) that satisfy \(\eta^{-4}L^2 \leq \lambda_0 \leq \lambda_1 \leq \eta^{-1}N^2\) one of the following statements must hold:

(i) there exists \(x \in A\) with the property that one has

\[
\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \frac{|A|}{N^d} - \varepsilon \quad \text{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1
\]

(ii)

\[
\frac{1}{|A|} \left| \int_{\Omega_{\lambda_0, \lambda_1}} |1_A(\xi)|^2 d\xi \right| \gg \varepsilon^2
\]

where \(\Omega_{\lambda_0, \lambda_1} = \Omega_{\lambda_0, \lambda_1}(\eta, q_\eta) = (q_\eta^{-1}\mathbb{Z})^d + \{\xi \in \mathbb{R}^d : \eta^2 \lambda_0^{-1} \leq |\xi|^2 \leq \eta^{-2} \lambda_0^{-1}\}\).

4.2. The Proof of Theorems 3 and 4. We naturally start with a short Lemma relating our two notions of uniform distribution.

**Lemma 2.** If \(A \subseteq \mathbb{Z}^d\) is \(\eta\)-uniformly distributed. Then there exists a constant \(L(A, \eta)\) such that for every positive integer \(L \geq L(A, \eta)\) satisfying \(q_\eta|L|N\) the following holds: There exist arbitrarily large positive integers \(N\) satisfying \(L|N|\) such that

(i) \(\frac{|A \cap B_N|}{N^d} \geq \delta^*(A) - \varepsilon/2\) and (ii) \(A \cap B_N\) is \((2\eta, L)\)-uniformly distributed hold simultaneously for some cube \(B_N\).

**Proof.** By our assumption there exists a positive integer \(L(A, \eta)\) such that if \(L' = L(A, \eta)\), then

\[
\delta(A|(s + (q_\eta \mathbb{Z})^d) \cap B_{L'}) := \frac{|A \cap (s + (q_\eta \mathbb{Z})^d) \cap B_{L'}|}{|(s + (q_\eta \mathbb{Z})^d) \cap B_{L'}|} \leq (1 + 2\eta^2) \delta^*(A)
\]

for any \(s \in \{1, \ldots, q_\eta\}^d\) and any cube \(B_{L'}\) of size \(L'\). Let \(L \geq L'\) such that \(q_\eta|L|\).
Now choose any cube $B_N$ of size $N' \gg \varepsilon^{-1} \eta^{-2} L$ such that $\delta(A|B_N') \geq \delta^*(A)(1 - \varepsilon \eta^2/20)$. Choosing $N' \leq N \leq N' + L$ one can ensure $L|N$ and $\delta(A|B_N') \geq \delta^*(A)(1 - \varepsilon \eta^2/10)$, thus (i) holds (easily). To see (ii) note that $|(s + (q\eta)k) \cap B_L| = (L/q\eta)^d$ and for $A' := A \cap B_N$ and any cube $B_L \subset B_N$ of size $L$ we have

$$\frac{|A' \cap (s + (q\eta)k) \cap B_L|}{(L/q\eta)^d} \leq (1 + 3\eta^2/2) \delta^*(A) \leq (1 + 2\eta^2)(1 - \varepsilon \eta^2/10)^{-1} |A'| _{N^d} \leq (1 + 4\eta^2) |A'| _{N^d}. \quad \square$$

4.2.1. Proof that Proposition 2 implies Theorem 3. Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^2$. Suppose that $A \subseteq \mathbb{Z}^d$ with $d \geq 5$ is an $\eta$-uniformly distributed set for which the conclusion of Theorem 3 fails to hold, namely that there exists arbitrarily large integers $\lambda$ for which

$$\frac{|A \cap (x + S_{\lambda})|}{|S_{\lambda}|} \leq \delta^*(A) - \varepsilon$$

for all $x \in A$. For a fixed integer $J \gg \varepsilon^{-1}$ we choose a sequence $\{\lambda^{(j)}\}_{j=1}^J$ of such $\lambda$’s with the property that $\lambda^{(1)} \geq \eta^{-4} L^2$, $\lambda^{(j)} \leq \eta^4 \lambda^{(j+1)}$ for $1 \leq j < J$, and $\lambda^{(J)} \leq \eta^{11} N^2$ with $L$ and $N$ satisfying the conclusion of Lemma 2. From Lemma 2 we obtain a set $A \cap B_N$, which we will abuse notation and denote by $A$.

An application Proposition 2 thus allows us to conclude that for this set one must have

$$\sum_{j=1}^J \frac{1}{|A|} \int_{\Omega_{\lambda^{(j)}}} |\widehat{1_A}(\xi)|^2 d\xi \gg J \varepsilon > 1. \quad (17)$$

On the other hand it follows from the disjointness property of the sets $\Omega_{\lambda^{(j)}}$, which we guaranteed by our initial choice of sequence $\{\lambda^{(j)}\}$, and Plancherel that

$$\sum_{j=1}^J \frac{1}{|A|} \int_{\Omega_{\lambda^{(j)}}} |\widehat{1_A}(\xi)|^2 d\xi \leq \frac{1}{|A|} \int_{\mathbb{T}^d} |\widehat{1_A}(\xi)|^2 d\xi = 1 \quad (18)$$

giving a contradiction. \quad \square

4.2.2. Proof that Proposition 3 implies Theorem 4. Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^3$. Suppose that $A \subseteq \mathbb{Z}^d$ with $d \geq 5$ is an $\eta$-uniformly distributed set for which the conclusion of Theorem 4 fails to hold, namely that there exists arbitrarily large integer pairs $(\lambda_0, \lambda_1)$ such that for all $x \in A$

$$\frac{|A \cap (x + S_{\lambda_0})|}{|S_{\lambda}|} \leq \delta^*(A) - \varepsilon$$

for some $\lambda_0 \leq \lambda \leq \lambda_1$.

For a fixed integer $J \gg \varepsilon^{-2}$ we choose a sequence of such pairs $\{(\lambda_0^{(j)}, \lambda_1^{(j)})\}_{j=1}^J$ with the property that $\lambda_0^{(1)} \geq \eta^{-4} L^2$, $\lambda_0^{(j)} \leq \eta^4 \lambda_0^{(j+1)}$ for $1 \leq j < J$, and $\lambda_1^{(J)} \leq \eta^{11} N^2$ with $L$ and $N$ satisfying the conclusion of Lemma 2. From Lemma 2 we obtain a set $A \cap B_N$, which we will abuse notation and denote by $A$.

An application Proposition 3 thus allows us to conclude that for this set one must have

$$\sum_{j=1}^J \frac{1}{|A|} \int_{\Omega_{\lambda_0^{(j)}, \lambda_1^{(j)}}} |\widehat{1_A}(\xi)|^2 d\xi \gg J \varepsilon > 1. \quad (19)$$

On the other hand it follows from the disjointness property of the sets $\Omega_{\lambda_0^{(j)}, \lambda_1^{(j)}}$, which we guaranteed by our initial choice of pair sequence $\{(\lambda_0^{(j)}, \lambda_1^{(j)})\}$, and Plancherel that

$$\sum_{j=1}^J \frac{1}{|A|} \int_{\Omega_{\lambda_0^{(j)}, \lambda_1^{(j)}}} |\widehat{1_A}(\xi)|^2 d\xi \leq \frac{1}{|A|} \int_{\mathbb{T}^d} |\widehat{1_A}(\xi)|^2 d\xi = 1 \quad (20)$$

giving a contradiction. \quad \square
5. Proof of Proposition \[2\]

Let \( f = 1_A \) and \( \delta = |A|/N^d \). Suppose that \( \eta^{-4}L^2 \leq \lambda \leq \eta^{11}N^2 \) and that (i) does not hold, then
\[
\langle f, \mathcal{A}_\lambda(f) \rangle \leq \langle f, \delta - \varepsilon \rangle = (\delta - \varepsilon)|A|.
\]
We now define
\[
f_1 = f * \psi_{q_L} \quad \text{ and } \quad f_2 = f * \psi_{q_L}
\]
with \( L_1 = \eta^{-1/2}L^{1/2} \) and \( L_2 = \eta^{1/2}L^{1/2} \). Since
\[
|\langle \psi_{q_L} - \hat{\psi}_{q_L} \rangle| \ll \eta^{1/2}
\]
whenever \( \xi \not\in \Omega_\lambda = \mathcal{M}_{q_L} \setminus \mathcal{M}_{q_L, \eta} \), the proof of Proposition \[2\] is therefore reduced (via Parseval) to showing that if (21) holds, then
\[
\langle f, \mathcal{A}_\lambda(f_2 - f_1) \rangle \gg \varepsilon|A|.
\]
The observation that
\[
\langle f, \mathcal{A}_\lambda(f_2 - f_1) \rangle \geq \langle f, \mathcal{A}_\lambda(f_1) \rangle - \langle f, \mathcal{A}_\lambda(f) \rangle - \langle f, \mathcal{A}_\lambda(f - f_2) \rangle
\]
further reduces the entire argument to

**Lemma 3** (Main term). If \( f_1 := f * \psi_{q_L} \) with \( L_1 = \eta^{-1/2}L^{1/2} \), then \( \langle f, \mathcal{A}_\lambda(f_1) \rangle \geq (\delta - C\eta^{1/2})|A| \).

**Lemma 4** (Error term). If \( f_2 := f * \psi_{q_L} \) with \( L_2 = \eta^{1/2}L^{1/2} \), then \( \langle f, \mathcal{A}_\lambda(f - f_2) \rangle \ll \eta^{1/2}|A| \).

**Proof of Lemma 3** Since \( A \) is \((\eta, L)\)-uniformly distributed it follows that \( f * \chi_{\eta,L}(x) \leq \delta(1 + \eta^2) \) for all \( x \in \mathbb{Z}^d \).

As \( L_1 \geq \eta^{-5/2}L \) and \( \eta^{1/2} \ll \delta \) it further follows from the properties of \( \psi_{q,L} \) discussed in Section 3.5 that
\[
f_1(x) = f * \psi_{q_L}(1) \leq f * \chi_{\eta,L} \psi_{q_L}(1) + |f * (\psi_{q_L} - \chi_{\eta,L} \psi_{q_L})(x)| \leq \delta(1 + \eta^2) + C\eta^{5/2} \leq \delta(1 + C\eta^2).
\]

Let \( N' = N + \eta^{-5/2}L_1 \) and let \( B_{N'} \) be a cube of size \( N' \) centered at the same point as \( B_N \). As \( f \) is supported on \( B_N \) and \( \eta^{1/2} \ll \delta \) we have
\[
\sum_{x \in B_N} f_1(x) = \sum_{x \in \mathbb{Z}^d} f_1(x) - \sum_{x \not\in B_{N'}} f_1(x) - \sum_{x \in B_{N'} \backslash B_N} f_1(x) \geq (1 - C\eta^2)|B_N|.
\]
Indeed, since \( N \gg \eta^{-5}L_1 \) we have
\[
\frac{|B_{N'} \backslash B_N|}{|B_N|} \ll \left( \frac{N'}{N} - 1 \right) \ll \eta^{-5/2}L_1/N \ll \eta^{5/2}
\]
while from (14) we have
\[
\sum_{x \not\in B_{N'}} f_1(x) \leq \sum_{|y| \gg \eta^{-5/2}L_1} \psi_{q_L}(y) \sum_{x} f(x - y) \leq C\eta^{5/2}|B_N|.
\]

We now define the set
\[
E := \{ x \in B_N : f_1(x) \leq \delta - C\eta \}.
\]
From estimate (25) it follows that
\[
\delta(1 - C\eta^2)|B_N| \leq \sum_{x \in E} f_1(x) + \sum_{x \in B_N \setminus E} f_1(x) \leq |E|((\delta - C\eta) + (|B_N| - |E|)\delta(1 + C\eta^2))
\]
and hence that \(|E| \leq C\eta \delta |B_N| = C\eta |A| \). Using the bound
\[
f_1(x) \geq \delta - C\eta - E(x)
\]
for \( x \in B_N \) it follows that
\[
\langle f, \mathcal{A}_\lambda(f_1) \rangle \geq \langle f, \delta - C\eta \rangle - \langle f, \mathcal{A}_\lambda(E) \rangle \geq (\delta - C\eta)|A| - \langle f, \mathcal{A}_\lambda(1_E) \rangle.
\]
The result follows via an application of Cauchy-Schwarz and the $\ell^2$ boundedness of the operator $A_\lambda$, namely that
\[
\sum_{x \in \mathbb{Z}^d} |A_\lambda(g)(x)|^2 \leq C \sum_{x \in \mathbb{Z}^d} |g(x)|^2
\]
for any $g \in L^2$, which is an immediate consequence of Plancherel and the fact that $|\hat{\sigma}_\lambda(\xi)| \leq 1$ for all $\xi \in \mathbb{T}^d$.

Indeed, with $g = 1_E$, we thus obtain
\[
|\langle f, A_\lambda(1_E) \rangle| \leq \left( \sum_{x \in B_N} f(x)^2 \right)^{1/2} \left( \sum_{x \in B_N} 1_E(x) \right)^{1/2} \leq |A|^1/2 |E|^{1/2} \leq \eta^{1/2} |A|.
\]

**Proof of Lemma 4**. Note that
\[
|\langle f, A_\lambda(f - f_2) \rangle| \leq \int |\hat{f}(\xi)|^2 \left| 1 - \hat{\psi}_{q_n, L_2}(\xi) \right| |\hat{\sigma}_\lambda(\xi)| \, d\xi.
\]

Now Proposition 4 ensures that $|\hat{\sigma}_\lambda(\xi)| \leq \eta$ for all $\xi \notin M_{q_n, \eta^{-1/2}L_2}$ and $\psi$ was constructed so that
\[
1 - \hat{\psi}_{q_n, L_2}(\xi) \ll \eta^{1/2}
\]
whenever $\xi \in M_{q_n, \eta^{-1/2}L_2}$. The result follows via Plancherel as $|\hat{\sigma}_\lambda(\xi)| \leq 1$ for all $\xi \in \mathbb{T}^d$.

**6. Proof of Proposition 3**

Suppose that we have a pair $(\lambda_0, \lambda_1)$ satisfying $\eta^{-4}L^2 \leq \lambda_0 \leq \lambda_1 \leq \eta^{11}N^2$, but for which (i) does not hold. It follows that there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that
\[
\langle f, A_\lambda(f) \rangle \leq (\delta - \varepsilon)|A|
\]
and hence that
\[
\langle f, A_\lambda(1 - f) \rangle \geq (1 - \delta + \varepsilon/2)|A|
\]
where $1 = 1_{B_N}$ and for any function $g : \mathbb{Z}^d \to \mathbb{C}$, $A_\lambda(g)$ denotes the discrete spherical maximal function defined by
\[
A_\lambda(g)(x) := \sup_{\lambda_0 \leq \lambda \leq \lambda_1} |A_\lambda(g)(x)|.
\]

**Proposition 4** ($\ell^2$-Boundedness of the Discrete Spherical Maximal Function). If $d \geq 5$, then
\[
\sum_{x \in \mathbb{Z}^d} |A_\lambda(g)(x)|^2 \leq C \sum_{x \in \mathbb{Z}^d} |g(x)|^2.
\]

In light of Proposition 4 the proof of Proposition 3 reduces (via Cauchy-Schwarz and Plancherel) to showing that if (27) holds, then
\[
|\langle f, A_\lambda(f_2 - f_1) \rangle| \gg \varepsilon |A|
\]
with $f_1 = f \ast \psi_{q_n, L_1}$ and $f_2 = f \ast \psi_{q_n, L_2}$, where now $L_1 = \eta^{-1/2} \lambda_1^{1/2}$ and $L_2 = \eta \lambda_0^{1/2}$.

Since
\[
|\langle f, A_\lambda(f_2 - f_1) \rangle| \geq |\langle f, A_\lambda(1 - f) \rangle| - |\langle f, A_\lambda(f - f_1) \rangle| - |\langle f, A_\lambda(f - f_2) \rangle|,
\]
the whole argument reduces to

**Lemma 5** (Main term). If $f_1 := f \ast \psi_{q_n, L_1}$ with $L_1 = \eta^{-1/2} \lambda_1^{1/2}$, then
\[
|\langle f, A_\lambda(1 - f_1) \rangle| \leq (1 - \delta + C\eta^{1/2})|A|.
\]

**Lemma 6** (Error term). If $f_2 := f \ast \psi_{q_n, L_2}$ with $L_2 = \eta \lambda_0^{1/2}$, then $|\langle f, A_\lambda(f - f_2) \rangle| \leq C\eta^{1/3}|A|$. 

6.1. Proof of Lemma 5. We use the lower bound
\[ f_1(x) \geq \delta - C\eta - 1_E(x) \]
for \( x \in B_N \) together with the bound \(|E| \leq C\eta|B_N|\) proved in Lemma 4. Then, as in the proof of Lemma 5 we obtain
\[
|\langle f, \mathcal{A}_*(1 - f) \rangle| \leq (1 - \delta + C\eta)|A| + |\langle f, \mathcal{A}_*(1_E) \rangle|.
\]
The result follows via an application of Cauchy-Schwarz and Proposition 4.1 since
\[
|\langle f, \mathcal{A}_*(1 - f) \rangle| \leq \left( \sum_{x \in B_N} f(x)^2 \right)^{1/2} \left( \sum_{x \in B_N} 1_E(x) \right)^{1/2} \leq |A|^{1/2}|E|^{1/2} \leq \eta^{1/2}|A|.
\]

\[ \square \]

6.2. Proof of Lemma 6. Note that
\[
\mathcal{A}_*(f - f_2) = \sup_{\lambda_0 \leq \lambda \leq \lambda_1} |(f - \psi_{\eta,L_2} \ast f) \ast \sigma| = \sup_{\lambda_0 \leq \lambda \leq \lambda_1} |f \ast (\sigma_\lambda - \sigma_\Lambda \ast \psi_{\eta,L_2})| =: \mathcal{A}_{\ast,\eta}(f)
\]
where the maximal operator \( \mathcal{A}_{\ast,\eta} \) corresponds to the “mollified” multiplier \( \sigma_{\lambda,\eta} := \sigma_\lambda(1 - \psi_{\eta,L_2}) \). Thus in order to prove the Lemma 6 it is suffices establish the following proposition.

Proposition 5 (\( \ell^2 \)-Decay of the “Mollified” Discrete Spherical Maximal Function). Let \( f \in \ell^2 \), then for any \( \eta > 0 \) we have
\[
\sum_{x \in \mathbb{Z}^d} |\mathcal{A}_{\ast,\eta}(f)(x)|^2 \leq C\eta^{2/3} \sum_{x \in \mathbb{Z}^d} |f(x)|^2.
\]

Proof of Proposition 6. We follow the proof of Proposition 4 given in [6]. For each \( x \in \mathbb{Z}^d \) we now define
\[
\tilde{A}_\lambda f(x) = \mathcal{A}_{\lambda^2} f(x)
\]
noting that \( \mathcal{A}_* f(x) = \sup_{\lambda_0^{1/2} \leq \lambda \leq \lambda_1^{1/2}} \tilde{A}_\lambda f(x) =: \tilde{A}_* f(x) \) and \( \tilde{A}_{\ast,\eta} f(x) = \tilde{A}_*(f - f_2)(x) \).

We now recall the approximation to \( \tilde{A}_\lambda \) given in Section 3 of [6] as a convolution operator \( \mathcal{M}_\lambda \) acting on functions on \( \mathbb{Z}^d \) of the form
\[
\mathcal{M}_\lambda = c_d \sum_{\substack{a = 1 \ldots \infty \\in \mathbb{Z}^d \\in \mathbb{Z}^d}} e^{-2\pi i a/q} \mathcal{M}_{\lambda a/q}^{a/q}
\]
where for each reduced fraction \( a/q \) the corresponding convolution operator \( \mathcal{M}_{\lambda a/q}^{a/q} \) has Fourier multiplier
\[
m_{\lambda a/q}^{a/q}(\xi) := \sum_{\ell \in \mathbb{Z}^d} G(a/q, \ell) \varphi_q(\xi - \ell/q) \tilde{\sigma}_\lambda(\xi - \ell/q)
\]
with \( \varphi_q(\xi) = \varphi(q\xi) \) a standard smooth cut-off function, \( G(a/q, l) \) a normalized Gauss sum, and \( \tilde{\sigma}_\lambda(\xi) = \tilde{\sigma}(\lambda\xi) \) where \( \tilde{\sigma}(\xi) \) is the Fourier transform (on \( \mathbb{R}^d \)) of the measure on the unit sphere in \( \mathbb{R}^d \) induced by Lebesgue measure and normalized to have total mass 1. By Proposition 4.1 in [6] we have
\[
\left\| \sup_{\lambda_0 \leq \lambda \leq 2\Lambda} |\tilde{A}_\lambda(f) - \mathcal{M}_\lambda(f)| \right\|_{L^2(\mathbb{Z}^d)} \leq C\lambda^{-1/2} \|f\|_{L^2(\mathbb{Z}^d)}
\]
provided \( d \geq 5 \). Writing \( \mathcal{M}_\ast(f) := \sup_{\lambda_0^{1/2} \leq \lambda \leq \lambda_1^{1/2}} |\mathcal{M}_\lambda(f)| \) and \( \mathcal{M}_{\ast,\eta}(f) := \mathcal{M}_*(f - f_2) \), this implies
\[
\|\tilde{A}_{\ast,\eta}(f) - \mathcal{M}_{\ast,\eta}(f)\|_{L^2} = \|\tilde{A}_*(f - f_2) - \mathcal{M}_*(f - f_2)\|_{L^2} \leq C\lambda_0^{-1/4} \|f - f_2\|_{L^2} \leq C\lambda_0^{-1/4} \|f\|_{L^2}.
\]
Thus by choosing \( \lambda_0 \gg \eta^{-4} \) matters reduce to showing (29) for the operator \( \mathcal{M}_{\ast,\eta} \).

For a given reduced fraction \( a/q \) define the maximal operator
\[
\mathcal{M}_{\ast}^{a/q}(f) := \sup_{\lambda_0^{1/2} \leq \lambda \leq \lambda_1^{1/2}} |\mathcal{M}_{\lambda a/q}^{a/q}(f)|,
\]
where \( \mathcal{M}_{\lambda a/q}^{a/q} \) is the convolution operator with multiplier \( m_{\lambda a/q}^{a/q}(\xi) \). It is proved in Lemma 3.1 of [6] that
\[
\|\mathcal{M}_{\ast}^{a/q}(f)\|_{L^2} \leq Cq^{-d/2} \|f\|_{L^2}.
\]
We will show here that if $q \leq C\eta^{-2/3}$, then
\begin{equation}
\|M_{\lambda}^{\lambda/q}(f - f_2)\|_{L^2} \leq C\eta^{1/3}q^{-d/2}\|f\|_{L^2}.
\end{equation}
Taking estimates (36) and (37) for granted, one obtains
\begin{equation}
\|M_*(f - f_2)\|_{L^2} \ll \left( \eta^{1/3} \sum_{1 \leq q \leq C\eta^{-2/3}} q^{-d/2+1} + \sum_{q \geq C\eta^{-2/3}} q^{-d/2+1} \right)\|f\|_{L^2} \ll \eta^{1/3}\|f\|_{L^2}
\end{equation}
as required. It thus remains to prove (37).

Writing $\varphi_q(\xi) = \varphi'_{q/\ell}(\xi)\varphi(\xi)$, with a suitable smooth cut-off function $\varphi'$, we can introduce the decomposition
\begin{equation}
m^{\lambda/q}_\lambda(\xi) = \left( \sum_{\ell \in \mathbb{Z}} G(a/q, \ell)\varphi'_{q/\ell}(\xi - \ell/q) \right) \left( \sum_{\ell \in \mathbb{Z}} \varphi_q(\xi - \ell/q)\tilde{\sigma}(\xi - \ell/q) \right) =: g^{a/q}(\xi)\eta^{\lambda}(\xi),
\end{equation}
since for each $\xi$ at most one term in each of the above sums is non-vanishing. Accordingly
\begin{equation}
M_{\lambda}^{\lambda/q}(f - f_2) = G^{a/q} N_\lambda(\xi - \ell/q)
\end{equation}
where the maximal operator $N_\lambda$ and the convolution operator $G_{a/q}$ correspond to the multipliers $n^{\lambda}_\lambda$ and $g^{a/q}$ respectively. Now by the standard Gauss sum estimate we have $|g^{a/q}(\xi)| \ll q^{-d/2}$ uniformly in $\xi$, hence
\begin{equation}
\|G^{a/q} N_\lambda(\xi - \ell/q)\|_{L^2} \ll q^{-d/2}\|N_\lambda(\xi - \ell/q)\|_{L^2}.
\end{equation}
thus by our choice $q_0 := \text{lcm}\{1 \leq q \leq C\eta^{-2}\}$ it remains to show that if $q$ divides $q_0$ then
\begin{equation}
\|N_\lambda^{\lambda/q}(f - f_2)\|_{L^2} \ll \eta^{1/3}\|f\|_{L^2}.
\end{equation}

As before we may write $N_\lambda^{\lambda/q}(f) = N_\lambda^{\lambda/q}(f - f_2)$, and note that this is a maximal operator with multiplier
\begin{equation}
n^{\lambda}_\lambda(\xi)(1 - \tilde{\psi}_{q_0, L_2}(\xi)) = \sum_{\ell \in \mathbb{Z}^d} \varphi_q(\xi - \ell/q)(1 - \tilde{\psi}_{q_0, L_2}(\xi - \ell/q))\tilde{\sigma}_\lambda(\xi - \ell/q).
\end{equation}
For a fixed $q$, the multiplier $\varphi_q(1 - \tilde{\psi}_{q_0, L_2})\tilde{\sigma}_\lambda$ is supported on the cube $[-\frac{1}{2q} \frac{1}{2q}]^d$ thus by Corollary 2.1 in [10]$$\|N_\lambda^{\lambda/q}\|_{L^2 \rightarrow L^2} \leq C \|N_\lambda^{\lambda/q}\|_{L^2 \rightarrow L^2}$$where the latter is the maximal operator corresponding to the multipliers $\varphi_q(1 - \tilde{\psi}_{q_0, L_2})\tilde{\sigma}_{\lambda}$, for $\lambda_0^{1/2} \leq \lambda \leq \lambda_1^{1/2}$, acting on $L^2(\mathbb{R}^d)$. By the definition of the function $\psi_{q,L}$
$$|1 - \tilde{\psi}_{q_0, L_2}(\xi)| \ll \min\{1, L_2|\xi|\},$$
thus from Theorem 6.1 (with $j = 1$) in [3] we obtain
$$\|N_\lambda^{\lambda/q}\|_{L^2 \rightarrow L^2} \ll \left( \frac{L_2}{\lambda_0^{1/2}} \right)^{1/3} \ll \eta^{1/3}$$
which establishes (42) and completes the proof. \hfill \Box

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