Bloch-type spaces and extended Cesàro operators in the unit ball of a complex Banach space

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Abstract

Let $B$ be the unit ball of a complex Banach space $X$. In this paper, we will generalize the Bloch-type spaces and the little Bloch-type spaces to the open unit ball $B$ by using the radial derivative. Next, we define an extended Cesàro operator $T_\varphi$ with holomorphic symbol $\varphi$ and characterize those $\varphi$ for which $T_\varphi$ is bounded between the Bloch-type spaces and the little Bloch-type spaces. We also characterize those $\varphi$ for which $T_\varphi$ is compact between the Bloch-type spaces and the little Bloch-type spaces under some additional assumption on the symbol $\varphi$. When $B$ is the open unit ball of a finite dimensional complex Banach space $X$, this additional assumption is automatically satisfied.

Keywords Bloch-type space, complex Banach space, extended Cesàro operator, little Bloch-type space.

MSC(2000) Primary 47B38; Secondary 32A37, 32A70, 46E15

1 Introduction

Let $D$ denote the unit disc in $C$. For a holomorphic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $D$, the Cesàro operator is defined by

$$C(f)(z) = \sum_{j=0}^{\infty} \left( \frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.$$  

The boundedness of this operator on some spaces of holomorphic functions was considered by many authors (see [11], [13], [14], [15], [20]). The integral form of the Cesàro operator is

$$C(f)(z) = \frac{1}{z} \int_{0}^{z} f(\zeta) \frac{1}{1-\zeta} d\zeta = \int_{0}^{1} f(tz) \left( \log \frac{1}{1-\zeta} \right)' |_{\zeta=tx} dt.$$
As a natural extension of the Cesàro operator, the extended Cesàro operator $T_\varphi$ with holomorphic symbol $\varphi$ is defined by

$$T_\varphi f(z) = \int_0^z f(\zeta)\varphi'(\zeta)d\zeta.$$ 

The boundedness and compactness of this operator on the Hardy space, the Bergman space and the Bloch type spaces have been studied in [1], [2], [18].

Let $B_n$ be the Euclidean unit ball in $\mathbb{C}^n$ and $H(B_n)$ be the family of holomorphic functions on $B_n$. Given $\varphi \in H(B_n)$, the extended Cesàro operator $T_\varphi$ with holomorphic symbol $\varphi$ is defined by

$$T_\varphi f(z) = \int_0^1 f(tz)R_\varphi(tz)\frac{1}{t}dt,$$

where

$$R_\varphi(z) = \sum_{j=1}^n \frac{\partial \varphi}{\partial z_j}(z)z_j$$

is the radial derivative of $\varphi$. The boundedness and the compactness of this operator on $\alpha$-Bloch spaces, little $\alpha$-Bloch spaces and the Bergman space have been studied in [8], [9], [16], [21]. Tang [17] characterized those holomorphic symbols $\varphi$ in the Euclidean unit ball of $\mathbb{C}^n$ for which the induced extended Cesàro operator $T_\varphi$ is bounded or compact on the Bloch-type spaces and the little Bloch-type spaces.

On the other hand, Wicker [19] and Blasco, Galindo and Miralles [4] generalized the Bloch space to the unit ball of an infinite dimensional complex Hilbert space. Deng and Ouyang [6] and Chu, Hamada, Honda and Kohr [5] independently generalized the Bloch space to an infinite dimensional bounded symmetric domain realized as the open unit ball of a JB*-triple $X$ and studied the boundedness and the compactness of composition operators between the Bloch spaces on bounded symmetric domains. Blasco, Galindo, Lindström and Miralles [3] provided necessary and sufficient conditions for compactness of composition operators on the space of Bloch functions on the unit ball of a complex Hilbert space with additional compactness assumptions on the set related to the composition symbol. Further, Hamada [7] studied the weighted composition operators from the Hardy space $H^\infty$ to the Bloch space on bounded symmetric domains.

In this paper, we will generalize the Bloch-type spaces and the little Bloch-type spaces to the open unit ball $\mathbb{B}$ of a general infinite dimensional complex Banach space $X$ by using the radial derivative. Our definition is new, but if $X$ is a complex Hilbert space, it is equivalent to the definition which is an extension of that in the finite dimensional case. Next, we define an extended Cesàro operator $T_\varphi$ with holomorphic symbol $\varphi$ and characterize those $\varphi$ for which $T_\varphi$ is bounded between the Bloch-type spaces and the little Bloch-type spaces. As in [3], under some additional assumption on the symbol $\varphi$, we also characterize those $\varphi$ for which $T_\varphi$ is compact between the Bloch-type spaces and the little
Bloch-type spaces. When $B$ is the open unit ball of a finite dimensional complex Banach space $X$, this additional assumption is automatically satisfied. There are some gaps in [17]. We overcome these gaps and give a complete proof in this paper in the setting of the unit ball of a general infinite dimensional complex Banach space.

2 Bloch-type spaces and little Bloch-type spaces

A positive continuous function $\omega$ on $[0,1)$ is said to be normal if there are constants $\delta \in [0,1)$ and $0 < a < b < \infty$ such that

$$\frac{\omega(r)}{(1-r)^a} \text{ is decreasing and } \frac{\omega(r)}{(1-r)^b} \text{ is increasing on } [\delta,1). \quad (2.1)$$

Then a normal function $\omega$ is strictly decreasing on $[\delta,1)$ and $\omega(r) \to 0$ as $r \to 1$.

Let $D$ be the unit disc in $C$.

**Lemma 2.1.** Let $\omega$ be a normal function. Denote $k_0 = \max(0, \lfloor \log_2 \frac{1}{\omega(\delta)} \rfloor)$, $r_k = (\omega|_{[\delta,1)})^{-1}(\frac{1}{k})$ and $n_k = \lfloor \frac{1}{1-r_k} \rfloor$ for $k > k_0$, where the symbol $[x]$ means the greatest integer not more than $x$. Let

$$g(\zeta) = 1 + \sum_{k>k_0}^{\infty} 2^k \zeta^{n_k}, \quad \zeta \in D.$$

Then

(i) $g$ is a holomorphic function on $D$ such that $g(r)$ is increasing on $[0,1)$ and

$$0 < C_1 = \inf_{r \in [0,1)} \omega(r) g(r) \leq \sup_{r \in [0,1)} \omega(r) g(r) = C_2 < \infty;$$

(ii) there exists a positive constant $C_3$ such that the inequality

$$\int_{r_1}^{r} g(t) dt \leq C_3 \int_{r_1}^{r} g(t) dt$$

holds for all $r \in [r_1,1)$, where $r_1 \in (0,1)$ is a constant such that

$$\int_{0}^{r_1} g(t) dt = 1.$$

**Proof.** (i) was proved in [10, Theorem 2.3]. We give a proof for (ii). Let $\delta$ be the constant in (2.1). We may assume that $r_1 < \delta^{1/4}$. First, we consider the case $r \in [r_1,\delta^{1/4}]$. Then $\int_{0}^{r} g(t) dt$ is bounded above and $\int_{0}^{r_1} g(t) dt$ is bounded below by a positive constant. So, there exists a constant $C > 0$ such that

$$\int_{0}^{r} g(t) dt \leq C \int_{0}^{r_1} g(t) dt, \quad r \in [r_1,\delta^{1/4}].$$

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Next, we consider the case \( r \in (\delta^{1/4}, 1) \). In this case, by (i) and (2.1), we have

\[
\int_{r^2}^{r} g(t) dt \leq C_2 \int_{r^2}^{r} \frac{1}{\omega(t)} dt
\]

\[
= C_2 \int_{r^2}^{r} \frac{(1 - t)^b}{\omega(t)} \frac{1}{(1 - t)^b} dt
\]

\[
\leq C_2 \frac{(1 - r^2)^b}{\omega(r^2)} \frac{r - r^2}{(1 - r)^b}
\]

\[
= C_2 \frac{(1 - r^2)^b}{\omega(r^2)} (1 + r)^b (1 + r^2)^b
\]

\[
\leq C_2 \frac{(1 - r^2)^b (1 + r)^b (1 + r^2)^b}{r^2 - r^4} \int_{r^4}^{r^2} \frac{(1 - t)^b}{\omega(t)} \frac{1}{(1 - t)^b} dt
\]

\[
\leq \frac{C_2 (1 + r)^b (1 + r^2)^b}{C_1 (r + r^2)} \int_{r^4}^{r^2} g(t) dt.
\]

Therefore, there exists a constant \( C' > 0 \) such that

\[
\int_{0}^{r} g(t) dt = \int_{0}^{r^2} g(t) dt + \int_{r^2}^{r} g(t) dt \leq C' \int_{0}^{r^2} g(t) dt, \quad r \in (\delta^{1/4}, 1).
\]

This completes the proof. \( \square \)

Remark 2.2. In [17, eq.(3.5)], it is claimed that there exists a constant \( C > 0 \) such that

\[
\int_{0}^{\|w\|} g(t) dt \leq C \int_{0}^{\|w\|^2} g(t) dt, \quad w \in \mathbb{B}.
\]

However, this is impossible for small \( \|w\| \), because \( g(0) = 1 \).

Let \( \mathbb{B} \) be the unit ball of a complex Banach space \( X \) with norm \( \| \cdot \| \). A normal function \( \omega \) will be extended to a function on \( \mathbb{B} \) by \( \omega(z) = \omega(\|z\|) \). Let \( H(\mathbb{B}) \) denote the set of holomorphic mappings from \( \mathbb{B} \) into \( \mathbb{C} \).

Definition 2.3. Let \( \mathbb{B} \) be the open unit ball of a complex Banach space \( X \) and let \( \omega \) be a normal function on \( \mathbb{B} \). A function \( f \in H(\mathbb{B}) \) is called a Bloch-type function with respect to \( \omega \) if

\[
\|f\|_{\mathcal{B}_{\mathbb{R}}(\mathbb{B}), \omega} = \sup \{ \omega(z) | \mathcal{R}f(z) | : z \in \mathbb{B} \} < +\infty,
\]

where \( \mathcal{R}f(z) = Df(z)z \) and \( Df(z) \) is the Fréchet derivative of \( f \) at \( z \).

The class of all Bloch-type functions with respect to \( \omega \) on \( \mathbb{B} \) is called a Bloch-type space on \( \mathbb{B} \) and is denoted by \( \mathcal{B}_{\mathbb{R}}(\mathbb{B}), \omega \). Then

\[
\|f\|_{\mathbb{R}, \omega} = |f(0)| + \|f\|_{\mathcal{B}_{\mathbb{R}}(\mathbb{B}), \omega}
\]

is a norm on \( \mathcal{B}_{\mathbb{R}}(\mathbb{B}), \omega \).

The following proposition is a generalization of the result on the Euclidean unit ball in \( \mathbb{C}^n \) [17, Lemma 3.1] to the unit ball of a complex Banach space.
Proposition 2.4. Let $\omega$ be a normal function. Then there exists a constant $C_4 > 0$ such that

$$|f(z)| \leq C_4 \left( 1 + \int_0^1 \|z\| \frac{1}{\omega(t)} \, dt \right) \|f\|_{\mathcal{R}, \omega}$$

for $f \in \mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega$ and $z \in \mathcal{B}$.

Proof. First we consider the case $\|z\| < 1/2$. Since $\mathcal{R} f(0) = 0$ and

$$|\mathcal{R} f(z)| \leq \frac{\|f\|_{\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega}}{\min_{t \in [0,1/2]} \omega(t)} \|z\|, \quad \|z\| < \frac{1}{2},$$

we have

$$|\mathcal{R} f(z)| \leq \frac{2 \|f\|_{\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega}}{\min_{t \in [0,1/2]} \omega(t)} \|z\|, \quad \|z\| < \frac{1}{2}$$

by the Schwarz lemma. Note that $\min_{t \in [0,1/2]} \omega(t) > 0$, since $\omega$ is a positive continuous function on $[0,1)$. Therefore, we have

$$|f(z)| \leq |f(0)| + |f(z) - f(0)|$$

$$\leq |f(0)| + \int_0^1 \left| \frac{\mathcal{R} f(tz)}{t} \right| \, dt$$

$$\leq |f(0)| + \frac{2 \|z\|}{\min_{t \in [0,1/2]} \omega(t)} \|f\|_{\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega}. \quad (2.3)$$

Next, let $z \in \mathcal{B}$ with $\|z\| \geq 1/2$. Then, applying (2.3) at the point $\frac{z}{2}$, we have

$$|f(z)| \leq \left| f\left(\frac{z}{2}\right)\right| + \left| f(z) - f\left(\frac{z}{2}\right)\right|$$

$$\leq \left| f\left(\frac{z}{2}\right)\right| + \int_{1/2}^1 \left| \frac{\mathcal{R} f(tz)}{t} \right| \, dt$$

$$\leq \left| f\left(\frac{z}{2}\right)\right| + 4 \|f\|_{\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega} \int_{1/2}^1 \frac{\|z\|}{\omega(t)|z|} \, dt$$

$$\leq |f(0)| + \frac{\|z\|}{\min_{t \in [0,1/2]} \omega(t)} \|f\|_{\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega} + 4 \|f\|_{\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega} \int_0^{1/2} \frac{1}{\omega(t)} \, dt.$$

Proposition 2.5. The Bloch-type space $\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega$ is a complex Banach space with the norm $\|f\|_{\mathcal{R}, \omega}$.

Proof. Let $(f^k)$ be a Cauchy sequence in $\mathcal{B}_{\mathcal{R}(\mathcal{B})}, \omega$. By Proposition 2.4, it follows that $(f^k)$ is a Cauchy sequence in the space $H(\mathcal{B})$, where $H(\mathcal{B})$ is equipped with the locally uniform topology. Hence $(f^k)$ converges locally uniformly to some function $f \in H(\mathcal{B})$. 

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To complete the proof, we show $\|f^k - f\|_{R, \omega} \to 0$ as $k \to \infty$. For this, fix $\varepsilon > 0$. Since $(f^k)$ is a Cauchy sequence in $B_{R}(\mathbb{B})$, there exists $k_0 \in \mathbb{N}$ such that

$$\|f^k - f^p\|_{R, \omega} < \varepsilon \quad \text{for} \quad k, p \geq k_0$$

which gives

$$|f^k(0) - f^p(0)| + \omega(z)\|Df^k(z) - Df^p(z)\|z| < \varepsilon \quad (z \in \mathbb{B}, k, p \geq k_0).$$

On the other hand, given $p \in \mathbb{N}$ and $z \in \mathbb{B}$, the locally uniform convergence of the sequence $(f^k)$ to $f$ implies that

$$|f(0) - f^p(0)| + \omega(z)\|Df(z) - Df^p(z)\|z| \leq \varepsilon$$

for $p \geq k_0$ and $z \in \mathbb{B}$. Consequently,

$$\|f^p - f\|_{R, \omega} \leq \varepsilon \quad \text{for} \quad p \geq k_0.$$

Therefore $f = (f - f^p) + f^p \in B_{R}(\mathbb{B})$ and $\lim_{p \to \infty} \|f^p - f\|_{R, \omega} = 0$. This proves that $B_{R}(\mathbb{B})$ is complete.

A function $f \in H(\mathbb{B})$ is said to belong to the little Bloch-type space $B_{R}(\mathbb{B}), 0$ if

$$\lim_{\|z\| \to 1} \omega(z)|Rf(z)| = 0$$

holds. Since for each $R \in (0, 1)$, there exists a constant $C(R) > 1$ such that

$$\sup_{0 \leq r \leq R} \omega(r) \leq C(R)\omega(R), \quad (2.4)$$

$B_{R}(\mathbb{B}), 0$ is a closed subspace of $B_{R}(\mathbb{B})$.

For $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* : l_x(x) = \|x\|, \|l_x\| = 1\}.$$  

Then $T(x) \neq \emptyset$ in view of the Hahn-Banach theorem.

Now, we generalize the test functions defined in [17] on the Euclidean unit ball of $\mathbb{C}^n$ to the unit ball of a complex Banach space. These test functions will be useful in the next sections.

Lemma 2.6. Let $g \in H(\mathbb{D})$ be the function defined in Lemma 2.1. For each $v \in \mathbb{B} \setminus \{0\}$ and $l_v \in T(v)$, let

$$f_v(z) = \int_0^{\|v\|l_v(z)} g(\zeta)d\zeta, \quad z \in \mathbb{B}. $$

Then $f_v \in B_{R}(\mathbb{B}), 0$ and $\|f_v\|_{R, \omega} \leq C_2$, where $C_2$ is the constant defined in Lemma 2.1.
Proof. By Lemma 2.1 (i), we have
\[
\omega(z) |\mathcal{R} f_v(z)| = \omega(z) g(||v|| l_v(z)) ||v|| l_v(z) \leq \omega(||z||) g(||z||) \leq C_2.
\]
Therefore, \( f_v \in \mathcal{B}_{\mathcal{R}(\mathbb{B})}_\omega \) and \( ||f_v||_{\mathcal{R},\omega} \leq C_2 \). Moreover, since \( \mathcal{R} f_v \) is bounded on \( \mathbb{B} \) and \( \omega(z) \to 0 \) as \( ||z|| \to 1 \), we have \( f_v \in \mathcal{B}_{\mathcal{R}(\mathbb{B})}_{\omega,0} \).

Lemma 2.7. For each \( v \in \mathbb{B} \) with \( ||v|| \geq r_1 \), let
\[
F_v(z) = \frac{1}{f_v(v)} (f_v(z))^2, \quad z \in \mathbb{B},
\]
where \( r_1 \) is the constant in Lemma 2.1 and \( f_v \) is the function defined in Lemma 2.6. Then \( F_v \in \mathcal{B}_{\mathcal{R}(\mathbb{B})}_{\omega,0} \) and \( ||F_v||_{\mathcal{R},\omega} \leq 2C_2C_3 \), where \( C_2, C_3 \) are the constants defined in Lemma 2.1. Moreover, if \( \int_0^1 \frac{1}{\omega(t)} dt = \infty \), then \( F_v \to 0 \) uniformly on any closed ball strictly inside \( \mathbb{B} \) as \( ||v|| \to 1 \).

Proof. By Lemma 2.1 we have
\[
\omega(z) |\mathcal{R} F_v(z)| = \omega(z) \frac{2}{f_v(v)} |f_v(z) g(||v|| l_v(z))||v|| l_v(z) |
\leq 2 \frac{\int_0^{||v||} g(t) dt}{\int_0^{||v||^2} g(t) dt} \omega(||z||) g(||z||)
\leq 2C_2C_3.
\]
Therefore, \( F_v \in \mathcal{B}_{\mathcal{R}(\mathbb{B})}_\omega \) and \( ||F_v||_{\mathcal{R},\omega} \leq 2C_2C_3 \). Moreover, since \( \mathcal{R} F_v \) is bounded on \( \mathbb{B} \) and \( \omega(z) \to 0 \) as \( ||z|| \to 1 \), we have \( F_v \in \mathcal{B}_{\mathcal{R}(\mathbb{B})}_{\omega,0} \).

Next, assume that \( \int_0^1 \frac{1}{\omega(t)} dt = \infty \). Fix \( r \in (0, 1) \). Since
\[
f_v(v) = \int_0^{||v||^2} g(t) dt \geq \int_0^{||v||^2} \frac{C_3}{\omega(t)} dt \to \infty \quad \text{as} \quad ||v|| \to 1
\]
and
\[
|f_v(z)| \leq \int_0^r g(t) dt, \quad ||z|| \leq r,
\]
\( F_v(z) \to 0 \) uniformly for \( ||z|| \leq r \) as \( ||v|| \to 1 \).

Let \( f \in H(\mathbb{B}) \). Then the relation \( |\mathcal{R} f(z)| \leq ||D f(z)|| \) holds. So, if
\[
\sup_{z \in \mathbb{B}} \omega(z) ||D f(z)|| < \infty
\]
holds, then \( f \in \mathcal{B}_{\mathcal{R}(\mathbb{B})}_\omega \) and
\[
||f||_{\mathcal{R},\omega} \leq ||f(0)|| + \sup_{z \in \mathbb{B}} \omega(z) ||D f(z)||
\]
holds. In the case \( \mathbb{B} = B_H \) is the unit ball of a complex Hilbert space \( H \), we have the following theorem, which is a generalization of the result on the Euclidean
unit ball in $\mathbb{C}^n$ [17, Theorem 2.1] to the unit ball of a complex Hilbert space. Note that there is a gap in the proof of [17, Theorem 2.1], because [17, eq.(2.7)] cannot be obtained from [17, eq.(2.2)]. To overcome this gap, we will change the path of integration of Cauchy’s integral formula.

**Theorem 2.8.** Let $B_H$ be the unit ball of a complex Hilbert space $H$ and let $\omega$ be a normal function. Let $f \in H(B_H)$. Then

(i) $f \in B_R(B_H, \omega)$ if and only if \( \sup_{z \in B_H} \omega(z)\|Df(z)\| < \infty \). Moreover,

\[
\|f\|_{R, \omega} \simeq |f(0)| + \sup_{z \in B_H} \omega(z)\|Df(z)\|;
\]

(ii) $f \in B_R(B_H, \omega, 0)$ if and only if $\lim_{\|z\| \to 1} \omega(z)\|Df(z)\| = 0$.

**Proof.** (i) We may assume that $\dim H \geq 2$. It suffices to show that there exists a constant $C > 0$ such that

\[
\sup_{z \in B_H} \omega(z)\|Df(z)v\| \leq C \sup_{z \in B_H} \omega(z)\|\mathcal{R}f(z)\|, \quad f \in B_R(B_H, \omega), \|v\| = 1. \tag{2.5}
\]

Let $z \in B_H$ and $v \in H$ with $\|v\| = 1$ be fixed. Then there exist orthonormal unit vectors $e_1, e_2 \in H$ and $\alpha, \beta_1, \beta_2 \in \mathbb{C}$ with $|\alpha| < 1$ and $|\beta_1|^2 + |\beta_2|^2 = 1$ such that $z = \alpha e_1$, $v = \beta_1 e_1 + \beta_2 e_2$. For $f \in B_R(B_H, \omega)$, let

\[
F(z_1, z_2) = f(z_1 e_1 + z_2 e_2), \quad (z_1, z_2) \in B_2,
\]

where $B_2$ is the Euclidean unit ball in $\mathbb{C}^2$. Then $F \in H(B_2)$ and $\mathcal{R}F(z_1, z_2) = \mathcal{R}f(z_1 e_1 + z_2 e_2) = \mathcal{R}f(z_1 e_1), \mathcal{R}f(z_1, 0) = \mathcal{R}f(z_1 e_1) e_1, \mathcal{R}f(z_1, 0) = \mathcal{R}f(z_1 e_1) e_2$ hold. Let $R \in (\delta, 1)$ be fixed. We assume that $|z_1| \geq R$ and let $r = |z_1|$. Since $\delta < R \leq \sqrt{t^2 + R^2(1 - r^{-2}t^2)} \leq r$ for $0 \leq t \leq r$ and $\omega$ is strictly decreasing on $[\delta, 1)$, we have

\[
\omega \left( \sqrt{t^2 + R^2(1 - r^{-2}t^2)} \right) \geq \omega(r), \quad 0 \leq t \leq r.
\]

Then, for $0 \leq t < r$, by Cauchy’s integral formula, we have

\[
\frac{\partial (\mathcal{R}F)}{\partial z_2}(t, 0) = \left. \frac{1}{2\pi i} \int_{|z_2| = R \sqrt{1 - r^{-2}t^2}} \frac{\mathcal{R}F(t, z_2)}{z_2^2} \, dz_2 \right| \leq \frac{1}{2\pi} \int_{|z_2| = R \sqrt{1 - r^{-2}t^2}} \frac{\mathcal{R}F(te_1 + z_2 e_2)}{z_2^2} \, dz_2 \leq \frac{\max_{|z_2| = R \sqrt{1 - r^{-2}t^2}} |\mathcal{R}f(te_1 + z_2 e_2)|}{R \sqrt{1 - r^{-2}t^2}} \leq \frac{\sup_{R \leq |z_2| < 1} \omega(z)\|\mathcal{R}f(z)\|}{\omega(r)R \sqrt{1 - r^{-2}t^2}}. \tag{2.6}
\]
Therefore, for $|z_1| = r \geq R$, by \[12\] Lemma 6.4.5(2) and (2.6), we have

$$|z_1| \left| \frac{\partial F}{\partial z_2}(z_1, 0) \right| = \left| \int_0^r \frac{\partial (RF)}{\partial z_2}(t, 0) dt \right| \leq \int_0^r \sup_{R \leq \|z\| < 1} \omega(z) |RF(z)| dt$$

$$= \frac{\sup_{R \leq \|z\| < 1} \omega(z) |RF(z)|}{\omega(r) R} \int_0^r \frac{1}{\sqrt{1 - r^{-2} t^2}} dt$$

$$= \frac{\pi}{2 \omega(|z_1|) R} |z_1| \sup_{R \leq \|z\| < 1} \omega(z) |RF(z)|.$$  

Thus, we have

$$\left| \frac{\partial F}{\partial z_2}(z_1, 0) \right| \leq \frac{\pi}{2 \omega(|z_1|) R} \sup_{R \leq \|z\| < 1} \omega(z) |RF(z)|, \quad |z_1| \geq R. \quad (2.7)$$

Also, we have

$$\left| \frac{\partial F}{\partial z_1}(z_1, 0) \right| = \frac{|RF(z_1 e_1)|}{|z_1|} \leq \frac{1}{\delta \omega(|z_1|)} \sup_{R \leq \|z\| < 1} \omega(z) |RF(z)|, \quad |z_1| \geq R. \quad (2.8)$$

From (2.7) and (2.8), we have

$$\omega(z) |Df(z)v| = \omega(\alpha) |Df(\alpha e_1)(\beta_1 e_1 + \beta_2 e_2)|$$

$$= \omega(\alpha) \left| \beta_1 \frac{\partial F}{\partial z_1}(\alpha, 0) + \beta_2 \frac{\partial F}{\partial z_2}(\alpha, 0) \right|$$

$$\leq \omega(\alpha) \left( \left| \frac{\partial F}{\partial z_1}(\alpha, 0) \right|^2 + \left| \frac{\partial F}{\partial z_2}(\alpha, 0) \right|^2 \right)^{1/2}$$

$$\leq \frac{\pi}{\sqrt{2} \delta} \sup_{R \leq \|z\| < 1} \omega(z) |RF(z)|, \quad \|z\| \geq R, \|v\| = 1. \quad (2.9)$$

Since $Df(z)v$ is a holomorphic function in $z \in \mathbb{B}$, by (2.4), (2.9) and the maximum principle for holomorphic functions, we have

$$\omega(z) |Df(z)v| \leq \frac{\pi}{\sqrt{2} \delta} C(R) \sup_{z \in \mathbb{B}} \omega(z)|RF(z)|, \quad z \in \mathbb{B}, \|v\| = 1.$$

This implies (2.5).

(ii) It suffices to show that $f \in \mathcal{B}_R(\mathbb{B}_H)_{\omega,0}$ implies

$$\lim_{\|z\| \to 1} \omega(z) \|Df(z)\| = 0. \quad (2.10)$$

Assume that the condition $\lim_{\|z\| \to 1} \omega(z)|RF(z)| = 0$ holds. Then for any $\varepsilon > 0$, there exists $R \in (\delta, 1)$ such that

$$\omega(z)|RF(z)| < \varepsilon, \quad \|z\| > R.$$  

Therefore, by using (2.9), we obtain (2.10). This completes the proof.
3 Boundedness of extended Cesàro operators

Given \( \varphi \in H(B) \), the extended Cesàro operator \( T_\varphi \) is defined by
\[
T_\varphi f(z) = \int_0^1 f(tz) R\varphi(tz) \frac{1}{t} \, dt, \quad f \in H(B), \, z \in B.
\]

The following lemma is a generalization of the result on the Euclidean unit ball in \( C^n \) [16, Lemma 2.1] to the unit ball of a complex Banach space.

**Lemma 3.1.** For every \( f, \varphi \in H(B) \), it holds that
\[
R[T_\varphi f](z) = f(z) R\varphi(z).
\]

**Proof.** \( f R\varphi \in H(B) \) has the Taylor series
\[
f(z) R\varphi(z) = \sum_{n=1}^{\infty} P_n(z),
\]
where \( P_n \) is a homogeneous polynomial of degree \( n \). Then we have
\[
R[T_\varphi f](z) = R \int_0^1 \sum_{n=1}^{\infty} P_n(z) t^n \frac{1}{t} \, dt
= R \sum_{n=1}^{\infty} \frac{P_n(z)}{n}
= \sum_{n=1}^{\infty} P_n(z)
= f(z) R\varphi(z).
\]

\[\square\]

Tang [17, Theorems 3.1 and 3.2] obtained the following theorems when \( B \) is the Euclidean unit ball of \( C^n \). The following theorems are generalization to the unit ball of a complex Banach space. Note that the proof in [17, Theorem 3.1] has a gap, because (2.2) is used in it. For non-negative constants \( A_\lambda \) and \( B_\lambda \) with a parameter \( \lambda \), the expression \( A_\lambda \sim B_\lambda \) means that there exists a constant \( C > 0 \) which is independent of \( \lambda \) such that
\[
C^{-1} A_\lambda \leq B_\lambda \leq C A_\lambda.
\]

**Theorem 3.2.** Let \( \omega \) and \( \mu \) be normal functions. Let \( \varphi \in H(B) \). Then \( T_\varphi : B_R(B)_\omega \rightarrow B_R(B)_\mu \) is bounded if and only if
\[
\| T_\varphi \| \sim \sup_{z \in B} \mu(z) |R\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)} \, dt < \infty. \tag{3.1}
\]

Moreover, if \( T_\varphi : B_R(B)_\omega \rightarrow B_R(B)_\mu \) is bounded, then
\[
\| T_\varphi \| \sim \sup_{z \in B} \mu(z) |R\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)} \, dt. \tag{3.2}
\]
Proof. Assume that (3.1) holds. Let \( \eta \in (0, 1) \) be such that \( |\mathcal{R}\varphi(z)| \leq 1 \) for \( \|z\| \leq \eta \). There exists \( C_5 > 0 \) such that

\[
1 \leq C_5 \int_0^\eta \frac{1}{\omega(t)} dt.
\]

(3.3)

Then, there exists \( C_6 > 0 \) such that

\[
\sup_{\|z\| \leq \eta} \mu(z)|\mathcal{R}\varphi(z)| \left( 1 + \int_0^\|z\| \frac{1}{\omega(t)} dt \right) \leq C_6 \sup_{\|z\| \geq \eta} \mu(z)|\mathcal{R}\varphi(z)| \int_0^\|z\| \frac{1}{\omega(t)} dt.
\]

(3.4)

Let \( C_7 = \max\{C_5 + 1, C_6\} \). Then, by Proposition 2.4, Lemma 3.1, (3.3) and (3.4), we have

\[
\mu(z)|\mathcal{R}(T_\varphi f)(z)| = \mu(z)|f(z)||\mathcal{R}\varphi(z)| \leq C_4 \mu(z)|\mathcal{R}\varphi(z)| \left( 1 + \int_0^\|z\| \frac{1}{\omega(t)} dt \right) \|f\|_{\mathcal{R}, \omega}
\]

\[
\leq C_4 C_7 \|f\|_{\mathcal{R}, \omega} \sup_{\|z\| \geq \eta} \mu(z)|\mathcal{R}\varphi(z)| \int_0^\|z\| \frac{1}{\omega(t)} dt
\]

\[
\leq C_4 C_7 \|f\|_{\mathcal{R}, \omega} \sup_{z \in \mathcal{B}} \mu(z)|\mathcal{R}\varphi(z)| \int_0^\|z\| \frac{1}{\omega(t)} dt
\]

for \( f \in \mathcal{B}_{\mathcal{R}(\mathcal{B})}, z \in \mathcal{B} \). Since \( (T_\varphi f)(0) = 0 \), we obtain that \( T_\varphi : \mathcal{B}_{\mathcal{R}(\mathcal{B})}, \mathcal{B}_{\mathcal{R}(\mathcal{B})} \mu \) is bounded and

\[
\|T_\varphi\| \leq C_4 C_7 \sup_{z \in \mathcal{B}} \mu(z)|\mathcal{R}\varphi(z)| \int_0^\|z\| \frac{1}{\omega(t)} dt.
\]

(3.5)

Conversely, assume that \( T_\varphi : \mathcal{B}_{\mathcal{R}(\mathcal{B})}, \mathcal{B}_{\mathcal{R}(\mathcal{B})} \mu \) is bounded. Then \( \varphi(z) = \varphi(0) + \int_0^1 \mathcal{R}\varphi(tz)\frac{1}{t} dt = \varphi(0) + (T_\varphi 1)(z) \in \mathcal{B}_{\mathcal{R}(\mathcal{B})} \mu \). Let \( v \in \mathcal{B} \setminus \{0\} \) be fixed and let \( f_v \in \mathcal{B}_{\mathcal{R}(\mathcal{B})},0 \) be the function defined in Lemma 2.6. Let \( r_1 \) be the constant in Lemma 2.1. If \( \|v\| \geq r_1 \), by Lemmas 2.4 and 2.6 we have

\[
\mu(v)|\mathcal{R}\varphi(v)| \int_0^\|v\| \frac{1}{\omega(t)} dt \leq \mu(v)|\mathcal{R}\varphi(v)| \int_0^\|v\| \frac{g(t)}{C_1} dt
\]

\[
\leq \frac{C_3}{C_1} \mu(v)|\mathcal{R}\varphi(v)| \int_0^\|v\|^2 g(t) dt
\]

\[
\leq \frac{C_3}{C_1} \|T_\varphi f_v\|_{\mathcal{R}, \mu}
\]

\[
\leq \frac{C_2 C_3}{C_1} \|T_\varphi\| < \infty.
\]

(3.6)
If $\|v\| < r_1$, then by Lemma 2.1 we have

$$
\mu(v)|\mathcal{R}\varphi(v)| \int_0^{\|v\|} \frac{1}{\omega(t)} dt \leq \mu(v)|\mathcal{R}\varphi(v)| \int_0^{\|v\|} \frac{g(t)}{C_1} dt \\
\leq \frac{1}{C_1} \mu(v)|\mathcal{R}\varphi(v)| \\
\leq \frac{1}{C_1} \|T_{\varphi}\|_{\mathcal{R}, \mu} \\
\leq \frac{1}{C_1} \|T_{\varphi}\| < \infty.
$$

The inequalities (3.6) and (3.7) yield (3.1), as desired. Moreover, from (3.5), (3.6) and (3.7), we obtain (3.2). This completes the proof.

**Theorem 3.3.** Let $\omega$ and $\mu$ be normal functions. Let $\varphi \in H(B)$. Then $T_{\varphi} : \mathcal{B}_R(B)_{\omega, 0} \to \mathcal{B}_R(B)_{\mu, 0}$ is bounded if and only if $\varphi \in \mathcal{B}_R(B)_{\mu, 0}$ and

$$
\sup_{z \in B} \mu(z)|\mathcal{R}\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)} dt < \infty.
$$

**Proof.** Assume that $\varphi \in \mathcal{B}_R(B)_{\mu, 0}$ and

$$
M = \sup_{z \in B} \mu(z)|\mathcal{R}\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)} dt < \infty.
$$

Then $T_{\varphi} : \mathcal{B}_R(B)_{\omega} \to \mathcal{B}_R(B)_{\mu}$ is bounded by Theorem 3.2. Therefore, it suffices to show that $T_{\varphi}(f) \in \mathcal{B}_R(B)_{\mu, 0}$ for any $f \in \mathcal{B}_R(B)_{\omega, 0}$. To this end, let $f \in \mathcal{B}_R(B)_{\omega, 0}$ be arbitrarily fixed. Let $\varepsilon > 0$ be fixed. Then there exists $r_0 \in (1/2, 1)$ such that

$$
\omega(z)|\mathcal{R}f(z)| < \frac{\varepsilon}{4M}, \quad r_0 \leq \|z\| < 1.
$$

(3.9)

For any $z \in B$ with $r_0 < \|z\| < 1$, let $\hat{z} = r_0 z/\|z\|$. Then, by (3.9), we have

$$
|f(z) - f(\hat{z})| = \left| \int_{r_0/\|z\|}^{r_1/\|z\|} \frac{\mathcal{R}f(tz)}{t} dt \right| \\
\leq \frac{\|z\|}{r_0} \int_{r_0/\|z\|}^{1} |\mathcal{R}f(tz)| dt \\
\leq \varepsilon \|z\| \int_{r_0/\|z\|}^{1} \frac{1}{4Mr_0} \int_{r_0/\|z\|}^{\|t\|} \omega(t) dt dt \\
\leq \frac{\varepsilon}{2M} \int_{r_0}^{\|z\|} \frac{1}{\omega(t)} dt.
$$

Set $K = \sup_{\|z\| \leq r_0} |f(z)|$. By Proposition 2.3, $K < \infty$. Then as in the proof of [17, Theorem 3.2], we have $T_{\varphi} f \in \mathcal{B}_R(B)_{\mu, 0}$. 

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Conversely, assume that $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0}$ is bounded. Since
\[
\varphi(z) = \varphi(0) + \int_0^1 \mathcal{R}_\varphi(tz) \frac{1}{t} \, dt = \varphi(0) + (T_\varphi 1)(z),
\]
$\varphi \in \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0}$. Since the function $f_\varphi$ defined in Lemma 2.6 belongs to $\mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0}$, we obtain (3.8) by the proof of Theorem 3.2. \hfill \square

From Theorems 3.2 and 3.3, we obtain the following corollary which is a generalization of the result on the Euclidean unit ball in $\mathbb{C}^n$ [17, Corollary 3.1] to the unit ball of a complex Banach space.

**Corollary 3.4.** Let $\omega$ and $\mu$ be normal functions and let $\varphi \in H(\mathbb{B})$. Then $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0}$ is bounded if and only if $\varphi \in \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0}$ and $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu}$ is bounded.

### 4 Compactness of extended Cesàro operators

In this section, we study the compactness of the extended Cesàro operator $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu}$ and $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0}$.

The following lemma is a generalization of the result on the Euclidean unit ball in $\mathbb{C}^n$ [17, Lemma 4.1] to the unit ball of a complex Banach space. It can be proved by a well-known argument which uses Montel’s theorem (cf. [3, Lemma 4.4]). We omit the proof.

**Lemma 4.1.** Let $\mathbb{B}$ be the unit ball of a complex Banach space. Let $\omega$ and $\mu$ be normal functions and let $\varphi \in H(\mathbb{B})$. Then $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu}$ is compact if and only if for any bounded sequence $\{f_j\}$ in $\mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega}$ which converges to 0 uniformly on any compact subset of $\mathbb{B}$, we have $\lim_{j \to \infty} \|T_\varphi f_j\|_{\mathcal{R},\mu} = 0$.

The following theorem is a generalization of the result on the Euclidean unit ball in $\mathbb{C}^n$ [17, Theorem 4.1] to the unit ball of a complex Banach space. Blasco, Galindo, Lindström and Miralles [3] provided necessary and sufficient conditions for compactness of composition operators on the space of Bloch functions on the unit ball of a complex Hilbert space under additional relatively compactness assumptions on the set related to the composition symbol. For $\varphi \in H(\mathbb{B})$, we consider the set
\[
E_{\varepsilon,\rho} = \{ z \in \mathbb{B} : \|z\| \leq \rho, \exists s \in [1, \rho^{-1}] \text{ s.t. } \mu(sz)|\mathcal{R}_\varphi(sz)| \geq \varepsilon \}
\]
and give the following compactness results of $T_\varphi$ under the assumption that $E_{\varepsilon,\rho}$ is relatively compact in $\mathbb{B}$ for any $\varepsilon > 0$ and $\rho \in (0, 1)$.

**Theorem 4.2.** Let $\mathbb{B}$ be the unit ball of a complex Banach space. Let $\omega$ and $\mu$ be normal functions and let $\varphi \in H(\mathbb{B})$ be such that the set $E_{\varepsilon,\rho}$ is relatively compact in $\mathbb{B}$ for any $\varepsilon > 0$ and $\rho \in (0, 1)$. Then

(i) Assume that $\int_0^1 \frac{1}{\mathcal{R}_\varphi(tz)} \, dt < \infty$. Then $T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu}$ is compact if and only if $\varphi \in \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu}$. 

(ii) Assume that \( \int_0^1 \frac{1}{\omega(t)} \, dt = \infty \). Then \( T_\varphi : B_{\mathcal{R}(B)_\omega} \to B_{\mathcal{R}(B)_\mu} \) is compact if and only if

\[
\lim_{\|z\| \to 1} \mu(z)|R\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)} \, dt = 0. \tag{4.1}
\]

**Proof.** (i) First, assume that \( T_\varphi : B_{\mathcal{R}(B)_\omega} \to B_{\mathcal{R}(B)_\mu} \) is compact. Then it is bounded and therefore, \( \varphi \in B_{\mathcal{R}(B)_\mu} \) by the proof of Theorem 3.3.

Conversely, assume that \( \varphi \in B_{\mathcal{R}(B)_\mu} \). Since \( \int_0^1 \frac{1}{\omega(t)} \, dt < \infty \), \((4.1)\) holds and therefore \( T_\varphi : B_{\mathcal{R}(B)_\omega} \to B_{\mathcal{R}(B)_\mu} \) is bounded by Theorem 3.2. For any \( \varepsilon > 0 \), there exists \( \rho \in (1/2, 1) \) such that

\[
\mu(z)|R\varphi(z)| \int_\rho^{\|z\|} \frac{1}{\omega(t)} \, dt < \frac{\varepsilon}{3}, \quad \rho < \|z\| < 1 \tag{4.2}
\]

holds. Let \( \{f_j\} \) be a bounded sequence in \( B_{\mathcal{R}(B)_\omega} \) which converges to 0 uniformly on any compact subset of \( B \). We may assume that \( \|f_j\|_{R, \omega} \leq 1 \). Then \( |f_j| \leq C_\rho \) for all \( j \) and \( \|z\| \leq \rho \) by Proposition 2.4 where

\[
C_\rho = C_4 \left( 1 + \int_0^\rho \frac{1}{\omega(t)} \, dt \right).
\]

There exists a positive integer \( N \) such that

\[
|f_j(w)| \leq \frac{\varepsilon}{3\|\varphi\|_{R, \mu} + 1}, \quad j > N, w \in E_{\varepsilon/(3C_\rho) \cdot \rho}.
\]

Therefore, for \( \|z\| \leq \rho \) and \( t = 1 \) or for \( \rho < \|z\| < 1 \) and \( t = \rho/\|z\| \), we have

\[
\mu(z)|R\varphi(z)||f_j(tz)| < \frac{\varepsilon}{3}, \quad j > N. \tag{4.3}
\]

For \( j \geq N \) and \( \rho < \|z\| < 1 \), by \( \|f_j\|_{R, \omega} \leq 1 \), \((4.2)\) and \((4.3)\), we have

\[
\mu(z)|R\varphi(z)||f_j(z)| \\
\leq \mu(z)|R\varphi(z)| \left( |f_j(z) - f_j(\frac{z}{\|z\|}) + \mu(z)|R\varphi(z)| f_j\left(\frac{z}{\|z\|}\right)\right) \\
\leq \mu(z)|R\varphi(z)| \int_{\|z\|}^{1} |\mathcal{R}f_j(tz)| \frac{dt}{t} + \frac{\varepsilon}{3} \\
\leq \mu(z)|R\varphi(z)| \int_{\rho/\|z\|}^{1} \frac{1}{\omega(t \|z\|)} \, dt + \frac{\varepsilon}{3} \\
\leq \frac{\mu(z)|R\varphi(z)|}{\rho} \int_{\rho/\|z\|}^{1} \frac{1}{\omega(t \|z\|)} \, dt + \frac{\varepsilon}{3} \\
\leq \frac{\mu(z)|R\varphi(z)|}{\rho} \int_{\rho}^{\|z\|} \frac{1}{\omega(t \|z\|)} \, dt + \frac{\varepsilon}{3} \\
< \varepsilon. \tag{4.4}
\]

From \((4.3)\) and \((4.4)\), we obtain \( \|T_\varphi f_j\|_{R, \mu} < \varepsilon \) for \( j > N \). By Lemma 4.1, \( T_\varphi : B_{\mathcal{R}(B)_\omega} \to B_{\mathcal{R}(B)_\mu} \) is compact.
(ii) Assume that $T_\varphi : \mathcal{B}_R(\mathbb{B}) \rightarrow \mathcal{B}_R(\mathbb{B})_\mu$ is compact. If $\varphi$ does not satisfy (4.1), then there exist $\varepsilon > 0$ and a sequence $\{z_j\} \subset \mathbb{B}$ such that $\lim_{j \to \infty} \|z_j\| = 1$ and

$$\mu(z_j)|\mathcal{R}\varphi(z_j)| \int_0^{\|z_j\|} \frac{1}{\omega(t)}dt \geq \varepsilon, \quad j = 1, 2, 3, \ldots \quad (4.5)$$

We may assume that $\|z_j\| > r_1$, where $r_1$ is the constant in Lemma 2.1. Let $f_j(z) = F_{z_j}(z)$ for $z \in \mathbb{B}$, where $F_{z_j}$ is the function defined in Lemma 2.7. From Lemma 2.7, $\{f_j\}$ is a bounded sequence in $\mathcal{B}_R(\mathbb{B})_\omega,0$ and $f_j \to 0$ uniformly on any compact subset of $\mathbb{B}$. Then $\lim_{j \to \infty} \|T_\varphi f_j\|_{\mathcal{R},\mu} = 0$ by Lemma 4.1. On the other hand, by Lemmas 2.1, 3.1 and (4.5), we have

$$\|T_\varphi f_j\|_{\mathcal{R},\mu} = \sup_{z \in \mathbb{B}} \mu(z)|\mathcal{R}\varphi(z)||f_j(z)|$$

$$\geq \mu(z_j)|\mathcal{R}\varphi(z_j)||f_j(z_j)|$$

$$= \mu(z_j)|\mathcal{R}\varphi(z_j)| \int_0^{\|z_j\|^2} g(t)dt$$

$$\geq C_1 C_3 \mu(z_j)|\mathcal{R}\varphi(z_j)| \int_0^{\|z_j\|} \frac{1}{\omega(t)}dt$$

$$\geq C_1 C_3 \varepsilon.$$

This is a contradiction. Thus, we obtain (4.1).

Conversely, assume that (4.1) holds. Then $\varphi \in \mathcal{B}_R(\mathbb{B})_\mu,0$ and for any $\varepsilon > 0$, there exists $\rho \in (1/2, 1)$ such that

$$\mu(z)|\mathcal{R}\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)}dt < \frac{\varepsilon}{3}, \quad \rho < \|z\| < 1$$

holds. The rest of the proof is similar to the case (i). This completes the proof.

The following theorem is a generalization of the result on the Euclidean unit ball in $\mathbb{C}^n$ [17, Theorem 4.2] to the unit ball of a complex Banach space.

**Theorem 4.3.** Let $\mathbb{B}$ be the unit ball of a complex Banach space. Let $\omega$ and $\mu$ be normal functions and let $\varphi \in H(\mathbb{B})$ be such that the set $E_{\omega,\rho}$ is relatively compact in $\mathbb{B}$ for any $\varepsilon > 0$ and $\rho \in (0, 1)$. Then $T_\varphi : \mathcal{B}_R(\mathbb{B})_\omega,0 \rightarrow \mathcal{B}_R(\mathbb{B})_\mu,0$ is compact if and only if

$$\lim_{\|z\| \to 1} \mu(z)|\mathcal{R}\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)}dt = 0. \quad (4.6)$$

**Proof.** Assume that (4.6) holds. Then, by Theorems 3.3 and 4.2, we obtain that $T_\varphi : \mathcal{B}_R(\mathbb{B})_\omega,0 \rightarrow \mathcal{B}_R(\mathbb{B})_\mu,0$ is compact.

Conversely, assume that $T_\varphi : \mathcal{B}_R(\mathbb{B})_\omega,0 \rightarrow \mathcal{B}_R(\mathbb{B})_\mu,0$ is compact. Then $\varphi \in \mathcal{B}_R(\mathbb{B})_\mu,0$ by Theorem 3.3. Therefore, if $\int_0^1 \frac{1}{\omega(t)}dt < \infty$, then (4.6) holds. We
Thus, we have \[ \| \text{contradicts with } (4.7). \] Thus, (4.6) holds. This completes the proof.

We may assume that \( \|z_j\| > r_1 \), where \( r_1 \) is the constant in Lemma 2.7. Let \( f_j(z) = F_j(z) \) for \( z \in \mathbb{B} \), where \( F_j \) is the function defined in Lemma 2.7. From Lemma 2.7 \( \{f_j\} \) is a bounded sequence in \( \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \) and \( f_j \to 0 \) uniformly on any compact subset of \( \mathbb{B} \). Since \( T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0} \) is compact, we may assume that there exists some \( g \in \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0} \) such that \( \|T_\varphi f_j - g\|_{\mathcal{R},\mu} \to 0 \) as \( j \to \infty \). Then for each \( z \in \mathbb{B} \), we have
\[
g(z) = \lim_{j \to \infty} T_\varphi f_j(z) = T_\varphi(\lim_{j \to \infty} f_j)(z) = T_\varphi 0(z) = 0.
\]
Thus, we have \( \|T_\varphi f_j\|_{\mathcal{R},\mu} \to 0 \) as \( j \to \infty \). By the proof of Theorem 4.2, this contradicts with (4.7). Hence, (4.6) holds. This completes the proof.

From Theorems 4.2 and 17 we obtain the following corollaries which are generalization of the results on the Euclidean unit ball in \( \mathbb{C}^n \) [17 Corollaries 4.1 and 4.2] to the unit ball of a complex Banach space.

**Corollary 4.4.** Let \( \mathbb{B} \) be the unit ball of a complex Banach space. Let \( \omega \) and \( \mu \) be normal functions and let \( \varphi \in H(\mathbb{B}) \) be such that the set \( E_{\varepsilon, \rho} \) is relatively compact in \( \mathbb{B} \) for any \( \varepsilon > 0 \) and \( \rho \in (0,1) \). Assume that \( \int_0^1 \frac{1}{\omega(t)} dt = \infty \). Then the following statements are equivalent:

(i) \( T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu} \) is compact;

(ii) \( T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0} \) is compact;

(iii) \[
\lim_{\|z\| \to 1} \mu(z)|\mathcal{R}\varphi(z)| \int_0^{\|z\|} \frac{1}{\omega(t)} dt = 0.
\]

**Corollary 4.5.** Let \( \mathbb{B} \) be the unit ball of a complex Banach space. Let \( \omega \) and \( \mu \) be normal functions and let \( \varphi \in H(\mathbb{B}) \) be such that the set \( E_{\varepsilon, \rho} \) is relatively compact in \( \mathbb{B} \) for any \( \varepsilon > 0 \) and \( \rho \in (0,1) \). Assume that \( \int_0^1 \frac{1}{\omega(t)} dt < \infty \). Then \( T_\varphi : \mathcal{B}_\mathcal{R}(\mathbb{B})_{\omega,0} \to \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0} \) is compact if and only if \( \varphi \in \mathcal{B}_\mathcal{R}(\mathbb{B})_{\mu,0} \).

**Remark 4.6.** Let \( \mathbb{B} \) be the unit ball of a finite dimensional complex Banach space. Then \( E_{\varepsilon, \rho} \) is relatively compact in \( \mathbb{B} \) for any \( \varepsilon > 0 \) and \( \rho \in (0,1) \). Thus, in the finite dimensional case, Theorems 4.2, 4.3 and Corollaries 4.4, 4.5 hold without the assumption that \( E_{\varepsilon, \rho} \) is relatively compact in \( \mathbb{B} \).

**Acknowledgments.** Hidetaka Hamada was partially supported by JSPS KAKENHI Grant Number JP16K05217.
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