A GEOMETRIC FORMULA FOR THE
WITTEN-RESHETIKHIN-TUAEV QUANTUM INVARIANTS
AND SOME APPLICATIONS

JØRGEN ELLEGAARD ANDERSEN

Abstract. We provide a geometric construction of the boundary states for
handlebodies which we in turn use to give a geometric formula for the Witten-
Reshetikhin-Turaev quantum invariants. We then analyze the asymptotics of
this invariant in the special case of a three manifold given by 1-surgery on
a knot and we show that if the knot has an irreducible representation of its
fundamental group into SU(2), then its quantum invariant cannot equal those
of the three sphere. From this we conclude that if a knot has the same colored
Jones polynomials as the unknot, it must be the unknot.

1. Introduction

Witten constructed, via path integral techniques, a quantization of Chern-Simons
theory in 2 + 1 dimensions, and he argued in [Wi] that this produced a TQFT ac-
cording to the Atiyah-Segal-Witten axioms [At], [Se], indexed by a compact simple
Lie group and an integer level k. For the group SU(n) and level k, let us denote this
TQFT by $Z_k^{(n)}$. Combinatorially, this theory was first constructed by Reshetikhin
and Turaev, using representation theory of $U_q(sl(n, \mathbb{C}))$ at $q = e^{2\pi i/(k+n)}$, in [RT1]
and [RT2] (see also [T]). Subsequently, the TQFT's $Z_k^{(n)}$ were constructed using
skein theory by Blanchet, Habegger, Masbaum and Vogel in [BHMV1], [BHMV2]
and [B1].

Let us first review the geometric construction of the Witten-Reshetikhin-Turaev
TQFT. Let $\Sigma$ be a closed surface of genus $g > 1$. Let $\Gamma$ be the mapping class group
of $\Sigma$. We will denote the moduli space of flat SU(2)-connections on $\Sigma$ by $M$ and its
smooth locus for $M'$. It is well known that $M'$ carries the Goldmann symplectic
structure $\omega$, which is determined by choosing an invariant inner product on the Lie
algebra of SU(2). For the appropriate choice of scaling of this inner product we get
that the class of $\omega$ generates $H^2(M', \mathbb{Z})$. Let now $(\mathcal{L}, \nabla, \langle \cdot, \cdot \rangle)$ be a prequantum line
bundle over $(M', \omega)$, e. g. the curvature of $\nabla$ is the symplectic form

$$F_{\nabla} = -i\omega.$$ 

It is well known that $\Gamma$ acts by symplectomorphisms on $(M', \omega)$ and this action
can be lifted to an action of $\Gamma$ on $\mathcal{L}$ which preserves $\nabla$ and $\langle \cdot, \cdot \rangle$. There is a
very natural $\Gamma$-equivariant family of complex structures on $M$ parametrized by
Teichmüller space $\mathcal{T}$ of $\Sigma$. Suppose $\sigma \in \mathcal{T}$ is a complex structure on $\Sigma$, then we
can consider the moduli space of stable holomorphic bundles of rank 2 and trivial
determinant on the Riemann surface $\Sigma_\sigma$. This moduli space is naturally a complex

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algebraic variety \( M_\sigma \) and by the theorem of Narasimhan and Seshadri \([\text{NS1}],\ [\text{NS2}]\), we get a natural homeomorphism, which is a diffeomorphism on the smooth locus, from the algebraic geometric moduli space \( M_\sigma \) to gauge theory moduli space \( \mathcal{M} \). We get this way induced a complex structure on \( \mathcal{M} \) and we denote \( \mathcal{M} \) with this complex structure \( \mathcal{M}_{\sigma} \). This structure in fact depends holomorphically on \( \sigma \in \mathcal{T} \).

The complex structure on \( \mathcal{M} \) combines with the connection \( \nabla \) to produce the structure of a holomorphic line bundle on \( \mathcal{L} \) over \( \mathcal{M}_{\sigma}' \) and one gets a vector bundle \( H^{(k)}_{\sigma} = H^0(\mathcal{M}_{\sigma}', \mathcal{L}^k) \).

We remark that we can extend \( \mathcal{L}^k \) with its holomorphic complex structure to a rank one invertible locally free sheaf (which we also denote \( \mathcal{L}^k \)) over all of \( \mathcal{M}_{\sigma} \) and in fact it is well known that \( H^0(\mathcal{M}_{\sigma}', \mathcal{L}^k) \approx H^0(\mathcal{M}_{\sigma}, \mathcal{L}^k) \).

Please \([\text{H}]\), where this is used extensively to construct the connection. See also references to the literature for this fact in \([\text{A10}]\). We note that this gives us the freedom to work with either model, precisely as is being used in \([\text{H}]\). There is a natural holomorphic structure on the bundle \( H^{(k)} \) over \( \mathcal{T} \). The main result pertaining to this bundle is the following theorem.

**Theorem 1** (Axelrod, Della Pietra and Witten; Hitchin). The bundle \( H^{(k)} \) supports a natural projectively flat \( \Gamma \)-invariant connection \( \nabla \).

This is a result proved independently by Axelrod, Della Pietra and Witten \([\text{ADW}]\) and by Hitchin \([\text{H}]\). In section 2, we review our differential geometric construction of the connection \( \nabla \) in the general setting discussed in \([\text{A9}]\).

**Definition 1.** Let \( \mathbb{P}^{(k)}(\Sigma) \) be the space of covariant constant sections of \( \mathbb{P}(H^{(k)}) \).

We observe that there is a representation of the mapping class group \( \Gamma \) on \( \mathbb{P}^{(k)}(\Sigma) \).

By a theorem of Laszlo \([\text{La1}]\), we know that there is an isomorphism between the bundle \( H^{(k)} \) and then the bundle of conformal block constructed by Tsuchiya, Ueno and Yamada in \([\text{TUY}]\), which takes the Hitchin connection to the TUY connection. In the paper \([\text{AU1}]\), it is explain how one twist the bundle of conformal block with a fractional power of the Quillen determinant line bundle (in a version described in \([\text{AU2}]\)) over Teichmüller space, so as to obtain a vector bundle one which a central extension of the mapping class group naturally acts and which supports an invariant flat connection. Further it is shown in \([\text{AU1}]\), that the resulting representations of a central extension of the mapping class group is part of a modular functor, which we in \([\text{AU1}]\) show is isomorphism to the modular functor underlying the Reshetikhin-Turaev TQFT for \( U_q(sl(2, \mathbb{C})) \) at \( q = e^{(2\pi i)/(k+2)} \). Combining these results we get the following theorem

**Theorem 2** (Andersen and Ueno). There is a natural \( \Gamma \)-equivariant isomorphism

\[ I_\Sigma : \mathbb{P}Z^{(k)}(\Sigma) \rightarrow \mathbb{P}^{(k)}(\Sigma). \]

This allows us to use the Hitchin connection in the bundle \( H^{(k)} \) to study the Reshetikhin-Turaev TQFT, as we do in this paper.

Suppose now \( H \) is a handlebody whose boundary is identified with \( \Sigma \). We will in this paper give a geometric definition of the boundary state, which the Reshetikhin-Turaev TQFT associated to \( H \), i. e. we will associate a covariant constant section
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To do this, suppose $P$ is a pair of pants decomposition of $\Sigma$, which is compatible with $H$, in the sense that all curves in $P$ bounds discs in $H$. By mapping a flat SU(2)-connection to the traces of its holonomy around each of the curves in $P$, we get a smooth map $h_P: M \to [-2, 2]^{3g-3}$. The fibers of this map is the so-called Jeffrey-Weitsman real polarization $F_P$ on the moduli space $M$. The fibers over the part of the image which is contained in $(-2, 2)^{3g-3}$ are Lagrangian sub-tori of $M$. Fibers which map to the boundary of the image $h_P(M) \subset [-2, 2]^{3g-3}$ are singular. We will give a precise description of them in section ??.

We observe that, if $L$ is a leaf of $F_P$, then $L$ is a Bohr-Sommerfeld fiber if and only if $(L, \nabla)|_L$ admits a covariant constant section defined on all of $L$. By choosing a covariant constant section of $(L, \nabla)|_L$ for each $L \in B_k(P)$ and considering them as distributional sections of $L$ over $M$, we obtain a basis for $H^{(k)}_P$. The main result of [20] is that

$$\dim H^{(k)}_\sigma = \dim H^{(k)}_P$$

for all $\sigma \in T$ and every pair of pants decomposition of $\Sigma$. Consider a sequence $\sigma_t$, $t \in \mathbb{R}_+ \cup \{0\}$ obtained from some arbitrary starting point $\sigma_0 \in T$, such that $\sigma_t$ is obtained from $\sigma_0$ by insertion of a flat cylinder of length $t$ into the cut of $\Sigma$ along each of the curves in $P$. We have the following theorem from [A3].

**Theorem 3.** The complex polarizations of $M$ induced from $\sigma_t$ converges to $F_P$ as $t$ goes to infinity.

Let

$$P_t(\sigma_0, P) : H^{(k)}_{\sigma_0} \to H^{(k)}_{\sigma_t}$$

be the parallel transport with respect to the Hitchin connection in $H^{(k)}$ over $T$ along the curve $(\sigma_s)$, $s \in [0, t]$. In Section 4 we show that there exist a limiting linear map

$$P_\infty(\sigma_0, P) : H^{(k)}_{\sigma_0} \to H^{(k)}_P.$$
leaves of $F_P$ is identified with a subset of the set of maps from the set of edges $E_{\Gamma_P}$ of $\Gamma_P$ to $[-2, 2]$. By identifying $[0, k]$ with $[-2, 2]$ using the bijection
\[ t \mapsto 2\cos(\pi t/k) \]
we can consider the set of leaves of $F_P$ as a subset of the set of maps from $E_{\Gamma_P}$ to $[0, k]$. For each vertex $v$ in the set of vertices $V_{\Gamma_P}$ in $\Gamma_P$, we let $e_1(v)$, $e_2(v)$ and $e_3(v)$ be the three edges emanating from $v$ in some ordering of the edges around the vertex $v$.

**Definition 3.** For each pair of pants decomposition $P$ of $\Sigma$ we have that
\[
L_k(P) = \left\{ l : E_{\Gamma_P} \to [0, k] \right\} \begin{cases}
    & l(e) \in 2\mathbb{Z} \text{ if } e \in E_{\Gamma_P} \text{ is separating} \\
    & (l(e_1(v)), l(e_2(v)), l(e_3(v))) \text{ is admissible } \forall v \in V_{\Gamma_P}
\end{cases},
\]
where a triple of integers $(l_1, l_2, l_3)$ is said to be admissible if the following three conditions are satisfied.
\[
\begin{align*}
    |l_1 - l_2| &\leq l_3 \leq l_1 + l + 2 \\
    l_1 + l_2 + l_3 &\leq 2k \\
    l_1 + l_2 + l_3 &\in 2\mathbb{Z}
\end{align*}
\]

Theorem 8.1 in [JW] states that:

**Theorem 5 (Jeffrey-Weitsman).** Under the above identification we have that
\[ B_k(P) = L_k(P). \]

We recall that the Reshetikhin–Turaev TQFT assigns a Hermitian vector space to $\Sigma$, which given the pair of pants decomposition $P$ of $\Sigma$ is provided with a basis indexed exactly by $L_k(P)$, see [RT1], [RT2], [H]. We also refer to the skein theory model of Blanchet, Habegger, Masbaum and Vogel, [BHMV1], [BHMV2], [H]. We let the vector corresponding to $l \in L_k(P)$ be denoted by $v_l$. Let $0 \in L_k(P)$ be the labeling corresponding to the zero map from $E_{\Gamma_P}$ to $[0, k]$. We recall that in the Reshetikhin-Turaev TQFT the boundary vector associated to $H$ is exactly $v_0$. We observe that the fiber of $h_P$ corresponding to $0 \in L_k(P)$ is exactly the same as the space of connections in $M$ that extends over $H$ which is the same as $h_{p,1}^{-1}(2)$, where $2 \in B_k(P)$ refers to $(2, 2, \ldots, 2) \in [-2, 2]^{g-3}$. Moreover, as it is explained in [Fr], the Chern-Simons functional defines a section of $\mathcal{L}^k|_{h_{p,1}^{-1}(2)}$, whose $k$’th tensor power we denote $\exp(2\pi ikCS)$.

**Definition 4.** We let $s^{(k)}_{H,P}(P) \in H^{(k)}_P$ be a covariant constant section of $\mathcal{L}^k$ over the Bohr-Sommerfeld fiber of $P$ corresponding to $0 \in L_k(P)$ which is given by $\exp(2\pi ikCS)$. Let $s^{(k)}_{H,P}(\sigma_0) \in H^{(k)}_{\sigma_0}$ be given by
\[
s^{(k)}_{H,P}(\sigma_0) = P_\infty(\sigma_0, P)^{-1}(s^{(k)}_{H,P}(P)),
\]
for $\sigma_0 \in \mathcal{T}$.

We prove the following theorem in Section 7.

**Theorem 6.** Suppose that $P_1$ and $P_2$ are pair of pants decomposition of $\Sigma$, which are compatible with $H$. Then $s^{(k)}_{H,P_1}(\sigma_0)$ agrees with $s^{(k)}_{H,P_2}(\sigma_0)$ up to multiplication by a root of one.
We do this by just analyzing the parallel transport of this state using the Hitchin connection. If we where to also take into account the the fractional power of the Quillen determinant bundle, one would be able to normalize the boundary vector, so as to eliminate this root of one. This will however not be important for us in this paper.

Now recall the definition of the Hermitian structure from [BHMV]. By Theorem 4.11 in [BHMV], we have that the basis is orthogonal and the norms are given by the following formula

\[ [v_l, v_l] = \eta^{1-g} \frac{\prod_{l(e) \in \Gamma_P} \langle l(e) \rangle}{\prod_{l(e) \in \Gamma_P} \langle l(e) \rangle}, \]

where

\[ \eta = \sqrt{\frac{r}{2}} \sin(\pi/r) \]

with \( \langle j \rangle = (-1)^j \sqrt{j+1} \) for any integer \( j \) and for any triple of integers \((a, b, c)\)

\[ \langle a, b, c \rangle = (-1)^{a+b+c} \frac{\Gamma(a+b+\gamma+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(a)\Gamma(b)\Gamma(c)} \]

with

\[ a = \beta + \gamma, \ b = \alpha + \gamma, \ c = \alpha + \beta. \]

Furthermore \( r = k + 2 \). We observe that (1) is positive for all \( l \in L_k(P) \). We now introduce an orthonormal basis \( \tilde{v}_l, l \in L_k(P) \), given by

\[ \tilde{v}_l = \frac{v_l}{[v_l, v_l]^{1/2}}. \]

As will demonstrated in this paper, the basis vector \( \tilde{v}_l \) correspond to a covariant constant section of \( L^k \) of unit norm over the leaf of \( F_P \) corresponding to \( l \). We therefore define a Hermitian structure \( \langle \cdot, \cdot \rangle_P \) on \( H^k_P \) as follows. Suppose \( s_1, s_2 \in H^k_P \), then for each \( L \in B_k(P) \) we have that \( s_1, s_2, i = 1, 2 \), are covariant constant sections of \( L^k |_L \). Hence we see that \( \langle s_1, s_2 \rangle \) is constant along the leaves of \( P \) in \( B_k(P) \) and thus \( \langle s_1, s_2 \rangle \) becomes a function on \( B_k(P) \). Under the above identification of \( B_k(P) \) with \( L_k(P) \), we can thus interpret \( \langle s_1, s_2 \rangle \) as a function defined on \( L_k(P) \).

**Definition 1.** For any \( s_1, s_2 \in H^k_P \), we define

\[ \langle s_1, s_2 \rangle_P^{(k)} = \sum_{l \in L_k(P)} \langle s_1, s_2 \rangle(l). \]

We observe that \( \langle \cdot, \cdot \rangle_P^{(k)} \) is positive definite. Similarly to [A12], we define the Hermitian structure \( \langle \cdot, \cdot \rangle_P \) determined by \( P \) by the formula

\[ [s_1, s_2]_P^{(k)} = (P_\infty(\sigma_0, P)(s_1), P_\infty(\sigma_0, P)(s_2))_P^{(k)}, \]

for all \( s_1, s_2 \in H^k_P \). We have the following Theorem analogous to the main result of [A12].

**Theorem 7.** The Hermitian structure \( \langle \cdot, \cdot \rangle_P \) is projectively preserved by the Hitchin connection and it is projectively invariant under the mapping class group action.

This Theorem is proved in Section 7. In complete analogy with the case studied in [A12], we can then similarly conclude that
Theorem 8. There exist functions $G^{(k)} \in C^\infty(T, C^\infty(M))$, such that
$$\langle s_1, s_2 \rangle^{(k)} = \int_M \langle s_1, s_2 \rangle G^{(k)} \omega^m \frac{\omega^m}{m!}$$
for $s_1, s_2 \in H^0(M, \mathcal{L}^k)$, which has the asymptotic expansion
$$G^{(k)}_\sigma = \exp(-F_\sigma + O(1/k))$$
for all $\sigma \in T$, where $F_\sigma \in C^\infty(M)$ is the Ricci potential for $(M'_\sigma, \omega)$.

Suppose now that we have a Heegaard decomposition of a compact 3-manifold $X$, i.e.
$$X = H_1 \cup_{\Sigma} H_2.$$ 
In Section 7 we also prove the following theorem.

Theorem 9. The Reshetikhin-Turaev invariant $Z^{(k)}(X)$ of $X$ is given by
$$Z^{(k)}(X) = c_g^{(k)} \langle s_{H_1, P_1}(\sigma), s_{H_2, P_2}(\sigma) \rangle^{(k)}_{P_1, \sigma}$$
for any $\sigma \in T$ and $P_i$ any pair of pants decomposition of $\Sigma$ which is compatible with $H_i$, $i = 1, 2$ and where $c_g^{(k)}$ is a constant that only depends on the genus of $\Sigma$ and the level $k$.

In Section 9 we use Theorem 9 to establish the following application.

Theorem 10. Suppose $X$ is obtained by 1 surgery on a knot $K$. If $K$ is not the unknot, then there exist a $k$ such that
$$|Z^{(k)}(X)| \neq |Z^{(k)}(S^3)|.$$
This theorem uses the Theorem by Kronheimer and Mrowka [1], which states that 1 surgery on a knot yields a three-manifold, which has an irreducible representation of its fundamental group to SU(2), in case the knot is not the unknot. As we argue in Section 9 this has the following corollary.

Corollary 1. Suppose that $K$ is a knot. If the colored Jones polynomial of $K$ are the same as those of the unknot, then $K$ is the unknot.

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2. The Hitchin connection

In this section, we review our construction of the Hitchin connection using the global differential geometric setting of [A9]. This approach is close in spirit to Axelrod, Della Pietra and Witten’s in [ADW], however we do not use any infinite dimensional gauge theory. In fact, the setting is more general than the gauge theory setting in which Hitchin in [H] constructed his original connection. But when applied to the gauge theory situation, we get the corollary that Hitchin’s connection agrees with Axelrod, Della Pietra and Witten’s.

Hence, we start in the general setting and let $(M, \omega)$ be any compact symplectic manifold.
Definition 5. A prequantum line bundle \((L,\langle\cdot,\cdot\rangle,\nabla)\) over the symplectic manifold \((M,\omega)\) consists of a complex line bundle \(L\) with a Hermitian structure \(\langle\cdot,\cdot\rangle\) and a compatible connection \(\nabla\) whose curvature is

\[
F_\nabla(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]} = -i\omega(X,Y).
\]

We say that the symplectic manifold \((M,\omega)\) is prequantizable if there exist a prequantum line bundle over it.

Recall that the condition for the existence of a prequantum line bundle is that

\[
\left[\frac{\omega}{2\pi}\right] \in \text{Im}(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})).
\]

Furthermore, the inequivalent choices of prequantum line bundles (if they exist) are parametrized by \(H^1(M,U(1))\) (see e.g. [Wo]).

We shall assume that \((M,\omega)\) is prequantizable and fix a prequantum line bundle \((L,\langle\cdot,\cdot\rangle,\nabla)\).

Assume that \(T\) is a smooth manifold which smoothly parametrizes Kähler structures on \((M,\omega)\). This means that we have a smooth map \(I : T \to C^\infty(M,\text{End}(TM))\) such that \((M,\omega,I_\sigma)\) is a Kähler manifold for each \(\sigma \in T\).

We will use the notation \(M_\sigma\) for the complex manifold \((M,I_\sigma)\). For each \(\sigma \in T\), we use \(I_\sigma\) to split the complexified tangent bundle \(TM_\mathbb{C}\) into the holomorphic and the anti-holomorphic parts. These we denote by

\[
T_\sigma = E(I_\sigma,i) = \text{Im}(\text{Id} - iI_\sigma)
\]

and

\[
\overline{T}_\sigma = E(I_\sigma,-i) = \text{Im}(\text{Id} + iI_\sigma)
\]

respectively.

The real Kähler-metric \(g_\sigma\) on \((M_\sigma,\omega)\), extended complex linearly to \(TM_\mathbb{C}\), is by definition

\[
g_\sigma(X,Y) = \omega(X,I_\sigma Y),
\]

where \(X,Y \in C^\infty(M,TM_\mathbb{C})\).

The divergence of a vector field \(X\) is the unique function \(\delta(X)\) determined by

\[
\mathcal{L}_X\omega^m = \delta(X)\omega^m,
\]

with \(m = \text{dim } M\). It can be calculated by the formula \(\delta(X) = \Lambda d(i_X\omega)\), where \(\Lambda\) denotes contraction with the Kähler form. Even though the divergence only depend on the volume, which is independent of the of the particular Kähler structure, it can be expressed in terms of the Levi-Civita connection on \(M_\sigma\) by \(\delta(X) = \text{Tr} \nabla_\sigma X\).

Inspired by this expression, we define the divergence of a symmetric bivector field

\[
B \in C^\infty(M,\Lambda^2(TM_\mathbb{C}))
\]

by

\[
\delta_\sigma(B) = \text{Tr} \nabla_\sigma B.
\]

\footnote{Here a smooth map from \(T\) to \(C^\infty(M,W)\), for any smooth vector bundle \(W\) over \(M\), means a smooth section of \(\pi_M^* W\) over \(T \times M\), where \(\pi_M\) is the projection onto \(M\). Likewise, a smooth \(p\)-form on \(T\) with values in \(C^\infty(M,W)\) is, by definition, a smooth section of \(\pi_T^* \Lambda^p(T) \otimes \pi_M^* W\) over \(T \times M\). We will also encounter the situation where we have a bundle \(W\) over \(T \times M\) and then we will talk about a smooth \(p\)-form on \(T\) with values in \(C^\infty(M,\pi_\sigma^* W)\) and mean a smooth section of \(\pi_T^* \Lambda^p(T) \otimes \pi_M^* W\) over \(T \times M\).}
Notice that the divergence of bivector fields does depend on the point $\sigma \in \mathcal{T}$.

Suppose $V$ is a vector field on $\mathcal{T}$. Then we can differentiate $I$ along $V$ and we denote this derivative by $V[I]: \mathcal{T} \to C^\infty(M, \text{End}(TM_\mathbb{C}))$. Differentiating the equation $I^2 = -\text{Id}$, we see that $V[I]$ anti-commutes with $I$. Hence, we get that

$$V[I]_\sigma \in C^\infty(M, (\bar{T}_\sigma^* \otimes T_\sigma) \oplus (T_\sigma^* \otimes \bar{T}_\sigma))$$

for each $\sigma \in \mathcal{T}$. Let

$$V[I] = V[I]'_\sigma + V[I]''_\sigma$$

be the corresponding decomposition such that $V[I]'_\sigma \in C^\infty(M, \bar{T}_\sigma^* \otimes T_\sigma)$ and $V[I]''_\sigma \in C^\infty(M, T_\sigma^* \otimes \bar{T}_\sigma)$.

Now we will further assume that $\mathcal{T}$ is a complex manifold and that $I$ is a holomorphic map from $\mathcal{T}$ to the space of all complex structures on $M$. Concretely, this means that

$$V'[I]_\sigma = V[I]'_\sigma$$

and

$$V''[I]_\sigma = V[I]''_\sigma$$

for all $\sigma \in \mathcal{T}$, where $V'$ means the $(1,0)$-part of $V$ and $V''$ means the $(0,1)$-part of $V$ over $\mathcal{T}$.

Let us define $\tilde{G}(V) \in C^\infty(M, TM_\mathbb{C} \otimes TM_\mathbb{C})$ by

$$V[I] = \tilde{G}(V)$$

and define $G(V) \in C^\infty(M, T_\sigma \otimes T_\sigma)$ such that

$$\tilde{G}(V) = G(V) + \overline{G}(V)$$

for all real vector fields $V$ on $\mathcal{T}$.

We see that $\tilde{G}$ and $G$ are one-forms on $\mathcal{T}$ with values in $C^\infty(M, TM_\mathbb{C} \otimes TM_\mathbb{C})$ and $C^\infty(M, T_\sigma \otimes T_\sigma)$, respectively. We observe that

$$V'[I] = G(V)$$

and $G(V) = G(V')$.

Using the relation (2), one checks that

$$\tilde{G}(V) = -V[g^{-1}]$$

where $g^{-1} \in C^\infty(M, S^2(TM))$ is the symmetric bivector field obtained by raising both indices on the metric tensor. Clearly, this implies that $\tilde{G}$ takes values in $C^\infty(M, S^2(TM_\mathbb{C}))$ and thus $G$ takes values in $C^\infty(M, S^2(T_\sigma))$.

On $\mathcal{L}^k$, we have the smooth family of $\bar{\partial}$-operators $\nabla^{0,1}$ defined at $\sigma \in \mathcal{T}$ by

$$\nabla_{\sigma}^{0,1} = \frac{1}{2}(1 + i\mathcal{L}_\mathcal{E})\nabla.$$

For every $\sigma \in \mathcal{T}$, we consider the finite-dimensional subspace of $C^\infty(M, \mathcal{L}^k)$ given by

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k) = \{ s \in C^\infty(M, \mathcal{L}^k) \mid \nabla_{\sigma}^{0,1}s = 0 \}.$$

Let $\hat{\nabla}^i$ denote the trivial connection in the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$, and let $\mathcal{D}(M, \mathcal{L}^k)$ denote the vector space of differential operators on $C^\infty(M, \mathcal{L}^k)$. For any smooth one-form $u$ on $\mathcal{T}$ with values in $\mathcal{D}(M, \mathcal{L}^k)$, we have a connection $\nabla$ in $\mathcal{H}^{(k)}$ given by

$$\nabla_V = \hat{\nabla}_V^i - u(V)$$
Lemma 1. The connection $\nabla$ in $\mathcal{H}^{(k)}$ preserves the subspaces $H^{(k)}_\sigma \subset \mathcal{C}^\infty(M, L^k)$, for all $\sigma \in T$, if and only if
\begin{equation}
\frac{i}{2} V[I] \nabla^{1,0} s + \nabla^{0,1} u(V)s = 0
\end{equation}
for all vector fields $V$ on $T$ and all smooth sections $s$ of $H^{(k)}$.

This result is not surprising. See [A9] for a proof this lemma. Observe that if this condition holds, we can conclude that the collection of subspaces $H^{(k)}_\sigma \subset \mathcal{C}^\infty(M, L^k)$, for all $\sigma \in T$, form a subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$.

We observe that $u(V'') = 0$ solves (4) along the anti-holomorphic directions on $T$ since $V''[I] \nabla^{1,0} s = 0$.

In other words, the $(0,1)$-part of the trivial connection $\hat{\nabla}^t$ induces a $\bar{\partial}$-operator on $H^{(k)}$ and hence makes it a holomorphic vector bundle over $T$.

This is of course not in general the situation in the $(1,0)$-direction. Let us now consider a particular $u$ and prove that it solves (4) under certain conditions.

On the Kähler manifold $(M_\sigma, \omega)$, we have the Kähler metric and we have the Levi-Civita connection $\nabla$ in $T_\sigma$. We also have the Ricci potential $F_\sigma \in \mathcal{C}^\infty_0(M, \mathbb{R})$.

The Ricci potential is the element of $F_\sigma \in \mathcal{C}^\infty_0(M, \mathbb{R})$ which satisfies
\begin{equation}
\text{Ric}_\sigma = \text{Ric}^H_\sigma + 2i \partial_\sigma \bar{\partial}_\sigma F_\sigma,
\end{equation}
where $\text{Ric}_\sigma \in \Omega^{1,1}(M_\sigma)$ is the Ricci form and $\text{Ric}^H_\sigma$ is its harmonic part. In this way we get a smooth function $F : T \rightarrow \mathcal{C}^\infty_0(M, \mathbb{R})$.

For any symmetric bivector field $B \in \mathcal{C}^\infty(M, S^2(TM))$ we get a linear bundle map
\begin{equation}
B : TM^* \rightarrow TM
\end{equation}
given by contraction. In particular, for a smooth function $f$ on $M$, we get a vector field
\begin{equation}
Bdf \in \mathcal{C}^\infty(M, TM).
\end{equation}

We define the operator
\begin{equation}
\Delta_B : \mathcal{C}^\infty(M, L^k) \xrightarrow{\nabla} \mathcal{C}^\infty(M, TM^* \otimes L^k) \xrightarrow{B \otimes \text{Id}} \mathcal{C}^\infty(M, TM \otimes L^k) \xrightarrow{\text{Tr}_\alpha} \mathcal{C}^\infty(M, L^k).
\end{equation}

Let’s give a more concise formula for this operator. Define the operator
\begin{equation}
\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y},
\end{equation}
which is tensorial and symmetric in the vector fields $X$ and $Y$. Thus, it can be evaluated on a symmetric bivector field and we have
\begin{equation}
\Delta_B = \nabla^2_B + \delta(B).
\end{equation}
Putting these constructions together, we consider, for some $n \in \mathbb{Z}$ such that $2k + n \neq 0$, the following operator
\begin{equation}
    u(V) = \frac{1}{k + n/2} o(V) - V'[F],
\end{equation}
where
\begin{equation}
    o(V) = -\frac{1}{4} (\Delta_G(V) + 2\nabla_G(V) dF - 2n V'[F]).
\end{equation}

The connection associated to this $u$ is denoted $\nabla$, and we call it the Hitchin connection in $\mathcal{H}^{(k)}$. Following \cite{A9}, we now introduce the notion of a rigid family of Kähler structures.

**Definition 6.** We say that the complex family $I$ of Kähler structures on $(M, \omega)$ is rigid if
\begin{equation}
    \bar{\partial}_{G}(G(V)_{\sigma}) = 0
\end{equation}
for all vector fields $V$ on $\mathcal{T}$ and all points $\sigma \in \mathcal{T}$. We will assume our holomorphic family $I$ is rigid. There are plenty of examples of rigid holomorphic families of complex structures, see e.g. \cite{AGL}.

**Theorem 11.** Suppose that $I$ is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold $(M, \omega)$ which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of $(M, \omega)$ is $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R}) = 0$. Then $u$ given by \eqref{eq:5} and \eqref{eq:6} satisfies \eqref{eq:1} for all $k$ such that $2k + n \neq 0$.

Hence, the Hitchin connection $\nabla$ preserves the subbundle $\mathcal{H}^{(k)}$ under the stated conditions. Theorem 11 is established in \cite{A9} through the following three lemmas.

**Lemma 2.** Assume that the first Chern class of $(M, \omega)$ is $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$. For any $\sigma \in \mathcal{T}$ and for any $G \in H^0(M_{\sigma}, S^2(T_{\sigma}))$, we have the following formula
\begin{equation}
    \nabla_{\sigma}^{G}(\Delta_{G}(s) + 2\nabla_{GdF_{\sigma}}(s)) = -i(2k + n)\omega G\nabla(s) + 2ik\omega(GdF_{\sigma})s + ik\omega_{\sigma}(G)s,
\end{equation}
for all $s \in H^0(M_{\sigma}, \mathcal{L}^{k})$.

**Lemma 3.** We have the following relation
\begin{equation}
    4i\bar{\partial}_{\sigma}(V'[F]_{\sigma}) = 2(G(V)dF)_{\sigma} + \delta_{\sigma}(G(V))_{\sigma} \omega,
\end{equation}
provided that $H^{1}(M, \mathbb{R}) = 0$.

**Lemma 4.** For any smooth vector field $V$ on $\mathcal{T}$, we have that
\begin{equation}
    2(V'[\text{Ric}])^{1,1} = \partial(\delta(G(V))\omega).
\end{equation}

Let us here recall how Lemma 3 is derived from Lemma 4. By the definition of the Ricci potential
\begin{equation}
    \text{Ric} = \text{Ric}^{H} + 2i\partial \bar{\partial} F,
\end{equation}
where $\text{Ric}^{H} = n \omega$ by the assumption $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$. Hence
\begin{equation}
    V'[\text{Ric}] = -dV'[I] dF + 2i d\bar{\partial} V'[F],
\end{equation}
and therefore
\begin{equation}
    4i\partial \bar{\partial} V'[F] = 2(V'[\text{Ric}])^{1,1} + 2\partial V'[I] dF.
\end{equation}
From the above, we conclude that
\begin{equation}
    (2(G(V)dF)\omega + \delta(G(V))\omega - 4i\partial \bar{\partial} V'[F])_{\sigma} \in \Omega^{0,1}_{\sigma}(M)
\end{equation}
is a $\partial_\sigma$-closed one-form on $M$. From Lemma 2 it follows that it is also $\overline{\partial}_\sigma$-closed, hence it must be a closed one-form. Since we assume that $H^1(M, \mathbb{R}) = 0$, we see that it must be exact. But then it in fact vanishes since it is of type $(0,1)$ on $M_\sigma$.

From the above we conclude that

$$u(V) = \frac{1}{k + n/2} \sigma(V) - V'[F] = -\frac{1}{4k + 2n}(\Delta_{G(V)} + 2\nabla_{G(V)}dF + 4kV'[F])$$

solves (4). Thus we have established Theorem 11 and hence also provided an alternative proof of Theorem ??.

In [AGL] we use half-forms and the metaplectic correction to prove the existence of a Hitchin connection in the context of half-form quantization. The assumption that the first Chern class of $(M, \omega)$ is $n[\omega/2\pi] \in H^2(M, \mathbb{Z})$ is then replaced by the vanishing of the second Stiefel-Whitney class of $M$ (see [AGL] for more details).

Suppose $\Gamma$ is a group which acts by bundle automorphisms of $L$ over $M$ preserving both the Hermitian structure and the connection in $L$. Then there is an induced action of $\Gamma$ on $(M, \omega)$. We will further assume that $\Gamma$ acts on $T$ and that $I$ is $\Gamma$-equivariant. In this case we immediately get the following invariance.

**Lemma 5.** The natural induced action of $\Gamma$ on $H^{(k)}$ preserves the subbundle $H^{(k)}$ and the Hitchin connection.

**Remark 1.** We remark that if $M$ is not compact, but we know there exist a family of functions $F : T \to C^\infty(M)$ which solves

$$\text{Ric} = n\omega + 2i\partial\overline{\partial}F,$$

then all the rest of the proof of Theorem 11 is local and thus it applies in the noncompact case as well, and the theorem remains valid in this more general case. We further observe from the above argument that if the family $I$ satisfies $F_\sigma = 0$ for all $\sigma \in T$, then the above construction also gives a Hitchin connection, which in that case is simply given by

$$u(V) = -\frac{1}{4k}\Delta_{G(V)}.$$

An example of this is if $M$ is a torus and $I$ is a family of linear complex structures on $M$, see e.g. [AB].

### 3. Non-negative polarizations on moduli spaces

In this section we review the setting and results from [A3] and we discuss the immediate generalizations to surfaces with marked points.

Let $\Sigma$ be a closed oriented surface and let $R$ be a finite set of points on $\Sigma$ and set $\Sigma = \Sigma - R$.

**Definition 2.** A system $\tilde{P}$ of $q$ disjoint closed curves on $\hat{\Sigma}$ is called admissible if no two curves from the system are homotopic on $\hat{\Sigma}$ and none of the curves are null-homotopic on $\hat{\Sigma}$ nor homotopic on $\hat{\Sigma}$ to a curve which is contained in a disc-neighborhood of one of the points in $R$.

For an admissible system of curves $\tilde{P}$ on $\hat{\Sigma}$, let $\hat{\Sigma}$ be the complement in $\hat{\Sigma}$ of the curves in $\tilde{P}$. Suppose $c_0$ is any assignment of conjugacy classes of SU(2) to each of the points in $R$. Let $N$ be the moduli space of flat SU(2)-connections on $\hat{\Sigma}$ with holonomy around each of the points in $R$ contained in the conjugacy classes determined by $c_0$. 
We let \( h_\hat{P} : N \to [-2, 2] \hat{P} \) be the map which maps a connection to the trace of the holonomy around the curves in \( \hat{P} \). We let \( N_c = h_\hat{P}^{-1}(c) \) for all \( c \in [-2, 2] \hat{P} \).

Consider the moduli space \( \tilde{N} \) consisting of flat connections on \( \tilde{\Sigma} \) with holonomy around each of the points in \( R \) contained in the conjugacy class determined by \( c_0 \). Let \( \tilde{N}^c \) be the subspace of \( \tilde{N} \) consisting of the connections, which also has holonomy around each of the two boundary components corresponding to any curve \( \gamma \in \hat{P} \) given by \( c(\gamma) \). The projection map

\[ \pi : N \to \tilde{N} \]

induces projection maps

\[ \pi_c : N_c \to \tilde{N}^c. \]

For each of the conjugacy classes \( c(\gamma) \in [-2, 2], \gamma \in \hat{P} \), we choose an element in the conjugacy class and let \( Z_{c(\gamma)} \) be the centralizer of this element in \( c(\gamma) \). Hence we see that for \( c(\gamma) \in (-2, 2) \), we have that \( Z_{c(\gamma)} \cong U(1) \), and for \( c(\gamma) = \pm 2 \), we have that \( Z_{c(\gamma)} \cong SU(2) \). Furthermore, if we have a flat connection \( A \) on \( \tilde{\Sigma} \), representing a point in \( \tilde{N}^c \), we define \( \bar{Z}_A \) to be the automorphism group of \( \bar{A} \). Now fix a flat connection \( A \) on \( \tilde{\Sigma} \) such that \( [A] \in N_c \) and \( \pi_c([A]) = [\bar{A}] \).

If we fix parametrizations of each of the components of a tubular neighborhood of \( \hat{P} \) by \( S^1 \times (-1, 1) \), which for each \( \gamma \in \hat{P} \) maps \( S^1 \times \{0\} \) to \( \gamma \), we can assume that \( A \) restricted to each component of this tubular neighborhood is of the form \( A = \xi_\gamma d\theta \), where \( \theta \) is a coordinate on \( S^1 \) and \( \xi_\gamma \in su(2) \) such that \( \exp(\xi_\gamma) \in c(\gamma) \) is the chosen element in the conjugacy class for all \( \gamma \in \hat{P} \). We can now associate to any element in the Lie group

\[ z \in Z_c = \prod_{\gamma \in \hat{P}} Z_{c(\gamma)} \]

a broken gauge transformation \( g_z \) with support in the chosen tubular neighborhood of \( \hat{P} \), such that the restriction of \( g_z \) to the connected component around \( \gamma \) is given by \( g = \exp(\psi(t)\eta(\gamma)) \), where \( z(\gamma) = \exp(\eta(\gamma)) \) and \( \psi : (-1, 1) \to [0, 1] \) is identically zero on \( (0, 1) \) and near \(-1\), and it is identically 1 on \( (-\varepsilon, 0] \), for some small positive \( \varepsilon \).

From this, it is clear that the Lie group \( Z_{\bar{A}} \) acts on \( Z_c \) and we have the following Lemma from [AA3].

**Lemma 1.** We have a smooth \( Z_{\bar{A}} \)-invariant surjective map

\[ \Phi_A : Z_c \to \pi_c^{-1}([\bar{A}]), \]

given by mapping \( g \in Z_c \) to \( g^*A \). This map induces an isomorphism

\[ \Phi_A : Z_c/Z_{\bar{A}} \to \pi_c^{-1}([\bar{A}]). \]

We observe that \( Z_{\bar{A}} \) is isomorphic to a product of Lie groups. The product is index the components of \( \tilde{\Sigma} \) and the Lie groups are of sub-groups of \( SU(2) \) from the following list: \( SU(2), Z_{SU(2)} = \{ \pm \text{Id} \} \), or a conjugate of \( U(1) \subset SU(2) \).

We denote by \( \mathcal{L}_N \) the Chern–Simons line bundle over \( N \) constructed in [Fr]. \( \mathcal{L}_N \) is a topological complex line bundle over \( N \). Moreover, there is a well-defined notion of parallel transport in this bundle along any curve in \( N \) which can be lifted to a piecewise \( C^1 \) curve of connections. Over the dense smooth part \( N' \) of \( N \), \( \mathcal{L}_N \) is equipped with a preferred Chern–Simons connection, whose parallel transport induces this parallel transport.
Since $\mathcal{L}_N$ is constructed in \cite{Fr} on the space of connections on $\hat{\Sigma}$ with holonomy contained in $c_0$, we see in fact that we get a well-defined line bundle $\mathcal{L}_{N,A}$ over $\mathbb{Z}_c \cong \mathbb{Z}_c$, with an induced action of $\mathbb{Z}_A$, with the property that there is a natural $\mathbb{Z}_A$-equivariant isomorphism from $\mathcal{L}_{N,A}$ to $\hat{\Phi}_A^*(\mathcal{L}_N)$. From this we see that the restriction of smooth sections of $\mathcal{L}_N^k$ to $\pi^{-1}([A])$ gets pulled back by $\hat{\Phi}_A$ to $\mathbb{Z}_A$-invariant smooth sections of the smooth bundle $\mathcal{L}_{N,A}^k$ over $\mathbb{Z}_c$. This gives us a means to use differential geometric techniques to study these restrictions, even though these fibers sometimes are singular.

**Definition 3.** The Bohr–Sommerfeld set $B_k(\hat{P})$ associated to $\hat{P}$ on $\Sigma'$ is by definition the subset of $c$’s in $h_{\hat{P}}(N) \subset [-2,2]^{\hat{P}}$, for which the holonomy in $\mathcal{L}_N^k|N_c$ along the fibers of $\pi_c$ is trivial.

We remark that if $c \in B_k(\hat{P})$, there is a unique complex line $\mathcal{L}_{c,k}$ over $\hat{N}_c$ and a preferred isomorphism

$$\pi_c^*(\mathcal{L}_{c,k}) \cong \mathcal{L}_N^k|N_c.$$ 

Let $\bar{s}$ be a complex structure on $\hat{\Sigma}$ with the following property:

1. The complex structure $\bar{s}$ restricted to each of the components of a tubular neighborhood of the curves in $\hat{P}$ are conformally equivalent to semi-infinite cylinders.

2. The complex structure $\bar{s}$ extends over the points in $R$.

The following theorem is an immediate generalization of Theorem 5.1 in \cite{A3}.

**Theorem 12.** The structure $(\hat{P}, \bar{s})$ induces a non-negative polarization $F_{\hat{P}, \bar{s}}$ on $N$, with the following properties:

- The coisotropic leaves of $F_{\hat{P}, \bar{s}}$ are given by the fibers $N_c$, $c \in [-2,2]^{\hat{P}}$.
- The isotropic leaves of $F_{\hat{P}, \bar{s}}$ in $N_c$ are fibers of $\pi_c : N_c \to \hat{N}_c$, for all $c \in [-2,2]^{\hat{P}}$.

**Definition 4.** Let $H_{\hat{P}, \bar{s}}^{(k)}$ denote the vector space of distributional sections of $\mathcal{L}_N^k$ over $N$, which are covariant constant along the directions of $F_{\hat{P}, \bar{s}}$.

We have the following factorization theorem, which is an analogue of the factorization theorem in \cite{A2}.

**Theorem 13.** We have the following natural isomorphism:

$$H_{\hat{P}, \bar{s}}^{(k)} \cong \bigoplus_{c \in B_k(\hat{P})} H^0(\hat{N}_c, \mathcal{L}_{c,k}).$$

**Proof.** The theorem follows directly from the arguments presented in \cite{A2}. First one observes that for any $c \in h_{\hat{P}}(N)$, the holonomy is trivial along some generic fiber of $\pi_c$ if and only if it is trivial along all the generic fibers of $\pi_c$. This follows since the symplectic annihilator of $TN_c$ is $\ker(\pi_c)_*$ at a generic point of $N_c$. From this one concludes that the support of any distribution in $H_{\hat{P}, \bar{s}}^{(k)}$ must be contained in $h_{\hat{P}}^{-1}(B_k(\hat{P}))$. For each $c \in B_k(\hat{P})$ one then observes that a distribution in $H_{\hat{P}, \bar{s}}^{(k)}$ can be restricted to $N_c$, and here it must be covariant constant along the fibers of $\pi_c$ and hence induces a section in $\mathcal{L}_{c,k}$ over $\hat{N}_c$. By analyzing the distributional section restricted to $N_c$ in the transverse directions to the fibers of $\pi_c$, one finds
that the induced section of $L_{c,k}$ over $\tilde{N}^c$ must be holomorphic with respect to the complex structure induced on $\tilde{N}^c$ by $\tilde{\sigma}$.

Suppose we now have a complex structure $\tilde{\sigma}_0$ on $\tilde{\Sigma}$, which extends over $\Sigma$. We now construct a family of complex structures $\tilde{\sigma}_t$ on $\tilde{\Sigma}$, obtained from $\tilde{\sigma}_0$ by cutting $\tilde{\Sigma}$ along each of the curves in $\tilde{P}$ and gluing in flat cylinders of length $t$ to each of the two copies of each curve in $\tilde{P}$, for all non-negative $t$. The complex structures $\tilde{\sigma}_t$ on $\tilde{\Sigma}$ induce complex structures on $\tilde{N}$. When identifying the surface we obtain by cutting $\Sigma$ along $\tilde{P}$ and then attaching semi-infinite flat cylinders to all boundary components, with $\tilde{\Sigma}$, we obtain a complex structure on $\tilde{\Sigma}$, which we denote $\tilde{\sigma}$.

The following theorem is an immediate generalization of Theorem 6.2 of [A2].

**Theorem 14.** The complex structures on $\tilde{N}$ induced from the complex structures $\tilde{\sigma}_t$ converge to the non-negative polarization $F_{\tilde{P},\sigma}$ as $t$ goes to infinity.

4. **The asymptotics of the Hitchin connection under degenerations**

In this section we prove Theorem 14. We consider the more general setting discussed in Theorem 14 from the previous section. However, we only need the following special cases:

1. The surface $\Sigma$ is of genus $g > 1$ and $R$ consists of one point,
2. The surface $\Sigma$ is a torus and $R$ consists of one point,
3. The surface $\Sigma$ is a sphere and $R$ consists of four points.

We recall that the moduli space $N$ of interest is the moduli space of flat connections on $\tilde{\Sigma}$ with holonomy around each of the points in $R$ determined by $c_0$.

In the case (1) we will only be interested in the moduli space $N = M$ of flat connections on $\tilde{\Sigma}$ with holonomy $c_0 = \{-\text{Id}\}$ around the one point $p$ in $R$. Consider a point $\sigma \in T$.

A holomorphic vector bundle $E \to \tilde{\Sigma}_0$ is semi-stable if for every proper holomorphic subbundle $F \subset E$ we have the following conditions on the slope $\mu$ of $E$, and $F$

$$\frac{\deg(F)}{\text{rk}(F)} = \mu(F) \leq \mu(E) = \frac{\deg(E)}{\text{rk}(E)}.$$

A holomorphic vector bundle is called stable if the inequality is strict.

To each semi-stable vector bundle there exists a unique (up to isomorphism) filtration called the Jordan-Hölder filtration

$$0 = E_0 \subset \cdots \subset E_m = E,$$

with the property that the slopes of each of the quotients is the same as the slope of $E$, i.e.

$$\mu(E_{i+1}/E_i) = \mu(E),$$

and each quotient $E_{i+1}/E_i$ is a stable vector bundle. We then define the associated graded vector bundle

$$\text{Gr}(E) = \bigoplus_i (E_{i+1}/E_i).$$

Two holomorphic vector bundles $E$, $E'$ are $S$-equivalent if and only if their associated graded vector bundles are isomorphic, i.e.

$$E \sim S E' \text{ if and only if } \text{Gr}(E) \simeq \text{Gr}(E').$$
Theorem 15 (Narasimhan & Seshadri). The moduli space of S-equivalence classes of semi-stable bundles of rank \( n \) and determinant \( O_\sigma([p]) \) is a smooth complex algebraic projective variety isomorphic as a Kähler manifold to \( M_\sigma \).

This theorem is proven by using Mumford’s Geometric Invariant Theory.

Hence we see that \( \mathcal{T} \) parametrizes complex structures which are all Kähler with respect to the symplectic structure \( \omega \) on \( M \). To get uniform notation we will in this case (1) also use the notation \( \tilde{T} \) for \( T \).

In the cases (2) and (3) we are interested in arbitrary rational holonomies around the points in \( R \), hence we need on the algebraic side to consider moduli space of parabolic vector bundles on \( \Sigma \) with the parabolic structures located at the points \( R \) with respect to some point \( \tilde{\sigma} \) in the Teichmüller space \( \tilde{T} \) of \( \tilde{\Sigma} \).

Definition 5. Let \( \tilde{\Sigma}_\sigma \) be a compact Riemann surface with distinct marked points \( R \subset \tilde{\Sigma}_\sigma \), and \( E \rightarrow \tilde{\Sigma}_\sigma \) a holomorphic vector bundle of rank \( r \). A parabolic structure on \( E \rightarrow \tilde{\Sigma}_\sigma \) at \( p \in S \) is a choice of partial flag

\[
E_p = E^1_p \supset E^2_p \supset \cdots \supset E^r_p \supset 0
\]

with a set of parabolic weights

\[
w_1(p) < \cdots < w_r(p), \quad \text{with} \quad w_r(p) - w_1(p) < 1.
\]

Multiplicities are denoted by \( m_i(p) = \dim E^j_p - \dim E^{j+1}_p \).

A parabolic vector bundle on \( \tilde{\Sigma}_\sigma \) is a holomorphic rank \( r \) vector bundle \( E \rightarrow \tilde{\Sigma}_\sigma \) with a choice of parabolic structure at each marked point.

In order for the moduli space of parabolic vector bundles to have nice geometric structure we need to impose stability conditions on the parabolic vector bundles – just as in the case of ordinary vector bundles.

The parabolic degree of a parabolic vector bundle \( E \rightarrow \tilde{\Sigma}_\sigma \) is defined by

\[
pdeg(E) = \deg(E) + \sum_{p \in R} \sum_i m_i(p)w_i(p).
\]

The parabolic slope of \( E \) is

\[
\mu(E) = \frac{pdeg(E)}{\text{rk}(E)}.
\]

Every holomorphic subbundle \( F \) of \( E \) naturally has a parabolic structure at each of the marked points \( p \in R \) by defining

\[
F_p \cap E^1_p \supset F_p \cap E^2_p \supset \cdots \supset F_p \cap E^r_p \supset 0,
\]

and removing repeated terms. The weights are the largest of the corresponding parabolic weights from \( E \), i.e \( w^F_i(p) = \max_j \{ w^j_i | F_p \cap E^j_p = F^j_p \} \).

As with vector bundles we now define stable parabolic vector bundles as those where for each proper subbundle \( F \subset E \) we have

\[
\mu(F) = \frac{\text{pdeg}(F)}{\text{rk}(F)} < \frac{\text{pdeg}(E)}{\text{rk}(E)} = \mu(E).
\]

The weights give the connection between the moduli space of parabolic vector bundles to the moduli space of flat unitary connections with holonomy around the punctured marked points being these weights. This is the Mehta–Seshadri theorem [MeSe].
Theorem 16 (Mehta–Seshadri). Let \( \tilde{\Sigma} \) be a surface as above and \( R \subset \tilde{\Sigma} \) a set of marked points of \( \tilde{\Sigma} \). Then there is a one-to-one correspondence between the moduli space of irreducible unitary connections on \( \tilde{\Sigma} - R \) with holonomy around \( p \in R \) having eigenvalues
\[
\{e^{2\pi i w_1(p)}, e^{2\pi i w_2(p)}, \ldots, e^{2\pi i w_r(p)}\},
\]
each \( e^{2\pi i w_i(p)} \) with multiplicity \( m_i(p) \), and the moduli space of parabolic vector bundles with parabolic degree zero on \( \tilde{\Sigma} \) with weights and multiplicities specified by the above data.

We will in the following only be interested in the case of SU(2)-connections corresponding to rank-2 degree 0 parabolic vector bundles. Furthermore we will only be interested in the cases where the Riemann surface is a torus with a single marked point and the sphere with four marked points.

At the marked points for a rank-2 parabolic vector bundle there is only a two step filtration,
\[
E = E^1_p \supset E^2_p \supset 0,
\]
for \( p \in R \). The weights must satisfy \( w_2(p) - w_1(p) < 1 \) and \( w_1(p) < w_2(p) \). If the parabolic vector bundle should correspond to a flat unitary connection the parabolic degree of \( E \) must be zero, so
\[
0 = \text{pdeg}(E) = \deg(E) + \sum_{p \in S} w_1(p) + w_2(p).
\]
At each marked point \( p \in R \) the holonomy of the connection around that point is conjugate to \( \text{diag}(e^{2\pi i w_1(p)}, e^{2\pi i w_2(p)}) \). Since this matrix must be an SU(2) matrix \( w_1(p) + w_2(p) \) must be an integer. Since \( \deg(E) = 0 \) we all in all have \( w_1(p) \in (-\frac{1}{2}, \frac{1}{2}) \). The consequence is that \( w_1(p) + w_2(p) = 0 \) and finally that \( w_1(p) = -w_2(p) \). Since \( w_1(p) < w_2(p) \) we get that \( w_2(p) = s_p \in [0, \frac{1}{2}) \) and \( w_1(p) = -s_p \in (-\frac{1}{2}, 0] \).

Let \( L \) be a proper line subbundle of \( E \rightarrow \tilde{\Sigma} \). If we assume \( E \) to be parabolically stable then \( \text{pdeg}(L) < 0 \). For a marked point \( p \) the filtration of \( L_p \) has only one step, and is
\[
L_p = L_p \cap E^1_p \supset L_p \cap E^2_p = \begin{cases} 
0 & L \neq E^2_p \\
L_p & L = E^2_p
\end{cases}
\]
In the case \( L_p \neq E^2_p \) the weight is \( w_1(p) = -s_p \) while if \( L_p = E^2_p \) the weight jumps to \( w_1(p) = s_p \).

In all of the three cases (1) - (3) above, we get a family of complex structures \( I \) on \( N \) parametrized by \( \mathcal{T} \). We denote \( N \) with the complex structure \( I(\tilde{\sigma}) \) by \( N_{\tilde{\sigma}} \) for \( \tilde{\sigma} \in \mathcal{T} \). We let \( H^{(k)} \) denote the vector bundle over \( \mathcal{T} \), whose fiber over \( \tilde{\sigma} \in \mathcal{T} \) is \( H^0(N_{\tilde{\sigma}}, \mathcal{L}^k_N) \).

Lemma 2. In the cases (1) — (3) above, we have that \( N \) and \( I \) satisfy either the assumptions of Theorem 11 or those of Remark 1, hence in all cases we have a Hitchin connection which is projectively flat.

Proof. In the case (1) this was demonstrated by Hitchin in [11]. The cases (2) and (3) follow from the special considerations in Sections 5 and 6. \( \square \)
Consider the family $\tilde{\sigma}_t$ constructed in the previous section from the starting data $(\tilde{\sigma}_0, \tilde{P})$. Let

$$P_t(\tilde{\sigma}_0, \tilde{P}) : H^{(k)}_{\tilde{\sigma}_0} \to H^{(k)}_{\tilde{\sigma}_t}$$

be the parallel transport with respect to the Hitchin connection in $H^{(k)}$ over $\tilde{T}$ along the curve $(\tilde{\sigma}_s), s \in [0, t]$.

Let $c \in [-2, 2]^{\tilde{P}}$ and consider the subspace $N_c \subset N$. Consider a point $x$ in $N'$ ($N'$ being the manifold of smooth points of $N$), which is also a smooth point of $N_c$. For each $t$, let $I_t$ be the corresponding complex structure on $N$. A covariant constant section $s_t \in H^{(k)}_{\tilde{\sigma}_t}, t \in [0, \infty)$, of the Hitchin connection along the curve $\tilde{\sigma}_t$ satisfies the following equations:

$$s'_t = u_c(\tilde{\sigma}'_t)(s_t),$$

and

$$\nabla_X s_t = -i \nabla I_t X s_t$$

for all vector fields $X$ and all $t$. Since the curves in $\tilde{P}$ are non-intersecting, the corresponding holonomy functions Poisson commute, hence we have that $TN_c$ is coisotropic, thus $TN_c^0 \subset TN_c$, where $(\cdot)^0$ refers to the symplectic complement. We observe that $TN_c^\perp = I_t(TN_c^0)$, where $(\cdot)^\perp$ refers to the orthogonal complement with respect to the metric induced by $\omega$ and $I_t$. From this we get the following decomposition:

$$TN|_{N_c} = TN_c \oplus I_t(TN_c^0).$$

For any section $X$ of $TN|_{N_c}$, we define $X'$ a section of $TN_c^0$ and $X''$ a section of $I_t(TN_c^0)$ such that $X = X' + X''$.

**Theorem 17.** Suppose $s_0 \in H^{(k)}_{\tilde{\sigma}_0}$. Then $s_t|_{N_c}$ only depends on $s_0|_{N_c}$ and we have that

$$(s_t|_{N_c})' = \tilde{u}_c(\tilde{\sigma}'_t)(s_t|_{N_c}),$$

where $\tilde{u}_c(\tilde{\sigma}'_t)$ is a second order differential operator acting on $C^\infty(N_c, L^k_N|_{N_c})$ depending linearly on $\tilde{\sigma}'_t$. Moreover, the limit

$$\tilde{u}_{c,\infty} = \lim_{t \to \infty} \tilde{u}_c(\tilde{\sigma}'_t)$$

exists, and the operator $\tilde{u}_{c,\infty}$ is a second order differential operator acting on sections of $L^k_N|_{N_c}$, whose kernel consists of sections of $L^k_N|_{N_c}$ that are covariant constant along the directions of $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}}$.

We will use the following notation

$$\tilde{u}_{c,t} = \tilde{u}_c(\tilde{\sigma}'_t).$$

**Proof.** For $X$ a smooth section of $TN|_{N_c}$, we have that

$$\nabla_X = \nabla_X' - i \nabla I_t(X''),$$

where $\nabla_X'$ is the usual covariant derivative along the curve $\tilde{\sigma}_t$, and $\nabla I_t(X'')$ is the covariant derivative in the direction of $I_t$ along $\tilde{\sigma}_t$. The operator $\tilde{u}_c(\tilde{\sigma}'_t)$ is a second order differential operator acting on $C^\infty(N_c, L^k_N|_{N_c})$ depending linearly on $\tilde{\sigma}'_t$. The limit $\tilde{u}_{c,\infty}$ exists, and the operator $\tilde{u}_{c,\infty}$ is a second order differential operator acting on sections of $L^k_N|_{N_c}$, whose kernel consists of sections of $L^k_N|_{N_c}$ that are covariant constant along the directions of $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}}$. 

We will use the following notation

$$\tilde{u}_{c,t} = \tilde{u}_c(\tilde{\sigma}'_t).$$
and if \( Y \) is a further smooth section of \( TN|_{N'_{c}} \), then
\[
\nabla_X \nabla_Y = \nabla_X \nabla_{Y'} - iL_i(Y'') - i\nabla_Y \nabla_{L_i(X'')} + i\nabla_{L_i(X'')} \nabla_{L_i(X''')}
\]
\[
+ \nabla_{X'''} \nabla_{Y'} - iL_i([X''', L_i(Y'')])
\]
\[
- \nabla_{[X'''', L_i(Y'')]} - iL_i([X'''', L_i(Y'')])
\]
\[
- k(\omega(X'', Y'') - \omega(X''', L_i(Y''))).
\]

From these formulae we immediately get the first part of the proposition, since we can use the above two formulae to rewrite \( u(\bar{\sigma}_t^i) \) on \( N_{c} \), to obtain an operator \( \tilde{u}_t(\bar{\sigma}_t^i) \), such that the evolution of \( s_t|_{N_{c}} \) is determined by \( \tilde{u}_t(\bar{\sigma}_t^i) \).

Let us now use the notation \( G_t = G(\bar{\sigma}_t^i) \).

**Claim 1.** There exists a unique section \( G_{\infty} \in C^\infty(N'_{c}, S^2(F_{\bar{P}, \bar{\sigma}} \cap \tilde{F}_{\bar{P}, \bar{\sigma}})) \) such that
\[\lim_{t \to \infty} G_t = G_{\infty}.\]

In order to establish the claim, we consider a point \( x_0 \in N'_{c} \) and a local symplectic frame \((w, v)\) of \( TN' \) around \( x_0 \) with the following properties: The bundles \( O = \text{Span } p \) and \( Q = \text{Span } q \) are complementary Lagrangian subbundles of \( TN' \) and further that \( p = (p', p'') \), such that
\[\text{Span } p' = F_{\bar{P}, \bar{\sigma}} \cap \tilde{F}_{\bar{P}, \bar{\sigma}} \cap TN'.\]

We now observe that there is a unique complex symmetric matrix \( Z_t(x) \) depending smoothly on \( x \) near \( x_0 \), such that
\[u^{(t)}(x) = p(x) + Z_t(x)q(x)\]
spans \( P_t(x) \), the fiber of the holomorphic tangent bundle of \( N' \) at \( x \) with respect to the complex structure induced from \( \bar{\sigma}_t \). If we write \( Z_t = X_t + iY_t \), where \( X_t \) and \( Y_t \) real, then from its definition we conclude that \( X_t \) and \( Y_t \) are symmetric and \( Y_t > 0 \). The decomposition \( p = (p', p'') \) gives a corresponding decomposition of \( q = (q', q'') \). This decomposition gives the following block-decomposition of \( Z_t \):
\[Z_t = \begin{pmatrix} Z_t^{(11)} & Z_t^{(12)} \\ Z_t^{(21)} & Z_t^{(22)} \end{pmatrix}.\]

By Theorem \[A3\] we have the following asymptotics:
\[Z_t \to \begin{pmatrix} 0 & 0 \\ 0 & Z_{\infty} \end{pmatrix}\]
as \( t \) goes to infinity, where \( Z_{\infty} = X_{\infty} + iY_{\infty} \) and \( Y_{\infty} > 0 \). By examining the proofs of Theorem 10 in \[A3\], one sees immediately that the convergence of \( P_t \) to \( F_{\bar{P}, \bar{\sigma}} \) is a convergence in the \( C^\infty \)-topology on \( N' \). In particular, we have that
\[Z_t = Z_{\infty} + Z_{\infty}' t^{-1} + R(t).\]

Let us now analyse the case where
\[O = F_{\bar{P}, \bar{\sigma}} = F_{\bar{P}, \bar{\sigma}} \cap \tilde{F}_{\bar{P}, \bar{\sigma}} \cap TN'.\]
The other cases are treated completely analogously.

Let \( L_t \) be a symplectic local bundle transformations of \( TN' \otimes \mathbb{C} \) such that \( L_t(O) = P_t \) and \( L_{\infty} = \text{Id} \). In this basis we have:
\[L_t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}\]
as $t \to \infty$. Since $A \to \text{Id}$ as $t \to \infty$, we may assume that $A$ is invertible. The symplectic transform
\[
\begin{pmatrix} A^{-1} & 0 \\ -C^t & A^t \end{pmatrix}
\]
preserves $O$ so we consider
\[
\begin{pmatrix} A^{-1} & 0 \\ -C^t & A^t \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \text{Id} & A^{-1}B \\ A^tC - C^tA & A^tD - C^tB \end{pmatrix} = \begin{pmatrix} \text{Id} & A^{-1}B \\ 0 & \text{Id} \end{pmatrix}
\]
which must map $O$ onto $P_t$. Hence, $Z = A^{-1}B$ and
\[
w_i = p_i \sum Z_{ij}q_j = p_i + \sum X_{ij}q_j + i \sum Y_{ij}q_j
\]
and
\[
w_i = p_i \sum \bar{Z}_{ij}q_j = p_i + \sum X_{ij}q_j - i \sum \bar{Y}_{ij}q_j
\]
is a basis of $\bar{P}_t$ (we have here suppressed the $t$-dependence of the $w_i$’s). Since $P_t \cap \bar{P}_t = \{0\}$ we that $O \cap P_t \neq \{0\}$. This follows since $P_t$ corresponds to $I_t$ and $I_t(O) \cap O = \{0\}$ since $O$ is Lagrangian.

Claim 2. $P \cap P_t = \{0\} \iff \det(Z) \neq 0$.

Proof. Assume $\det Z \neq 0$. Then there exists a non zero vector $c$ such that
\[
\sum_i c_i Z_{ij} = 0, \quad j = 1, \ldots, n.
\]
Hence $\sum c_i w_i = \sum c_i p_i$ thus $O \cap P_t \neq \{0\}$. Conversely, if $O \cap P_t \neq \{0\}$, let $c$ be such that
\[
\sum c_i w_i \in O \cap P_t - \{0\}.
\]
But then
\[
\sum c_i w_i = \sum c_i p_i + \sum (\sum c_i Z_{ij})q_j \in P \cap P_t - \{0\}
\]
which implies that $\sum_i c_i Z_{ij} = 0$ for $j = 1, \ldots, n$, thus $\det Z = 0$. \hfill \Box

Claim 3. $P_t \cap \bar{P}_t = \{0\}$ if and only if $\det Y \neq 0$.

Proof. Assume $\det Y = 0$ then there exist $(x_1, \ldots, x_n) \in \mathbb{R}^n - \{0\}$ s.t. $\sum x_i Y_{ij} = 0$. Now, $\sum x_i w_i = \sum x_i \bar{w}_i \neq 0$ and hence $P_t \cap \bar{P}_t \neq \{0\}$.

Conversely, assume $P_t \cap \bar{P}_t \neq \{0\}$. Let $(c_1, \ldots, c_n) \in \mathbb{C}^n - \{0\}$, such that $\sum c_i w_i \in P_t \cap \bar{P}_t \cap TM$. But then
\[
\sum (c_i w_i - \bar{c}_i \bar{w}_i) = 0,
\]
if and only if
\[
\sum (c_i^l w_i^l - c_i^r w_i^r + i(c_i^l w_i^r + c_i^r w_i^l) - c_i^l w_i^r + c_i^r w_i^l + i(c_i^l w_i^r + c_i^r w_i^l)) = 0
\]
which is equivalent to
\[
2 \sum (c_i^l Y_{ij}q_j + c_i^r p_i + c_i^r X_{ij}q_j) = 0,
\]
happening if and only if
\[
c_i^r = 0 \text{ and } c_i^l = 0
\]
which is the case if and only if $\det Y = 0$. \hfill \Box
Notice that $TM/O \simeq Q$ and so
\[
\hat{L}_\infty \in C^\infty(O, TM/O)
\]
is represented by $\tilde{Z}_\infty$ in the basis $(p_1, \ldots, p_n)$ of $O$ and $(q_1, \ldots, q_n)$ of $Q \simeq TM/O$. By Proposition 2.3 p.118 in [GS], we can identify the space of lagrangian subspaces transverse to a given one $O$ as an affine space associated to the vector space $S^2(TM/P)$, which we can identify with $S^2(Q)$. The quadratic form associated with $P_t$ becomes
\[
H_t(q_i, q_j) = (\pi_t y_1, y_2),
\]
where $\pi_t$ is the projection from $TN' \otimes \mathbb{C}$ onto $P_t$ along $O$. Now $w_i = \pi_t(w_i) = \sum Z_i^{-1}w_j$, and so
\[
H_t(q_i, q_j) = Z_i^{-1}.
\]
Now $I_t$ is determined from $P_t$ by the condition that
\[
P_t = E(I_t, i),
\]
and
\[
P_t = E(I_t, -i).
\]
Hence
\[
I(p_i) + \sum X_{ij}I(q_j) + i \sum Y_{ij}I(q_j) = ip_i + i \sum X_{ij}q_j - \sum Y_{ij}q_j
\]
and
\[
I(p_i) + \sum X_{ij}I(q_j) - i \sum Y_{ij}I(q_j) = -iq_i - i \sum X_{ij}q_j - \sum Y_{ij}q_j,
\]
which implies
\[
I(q_j) = \sum_k Y_{jk}^{-1} \left( p_k + \sum X_{ki}q_i \right) = \sum_k Y_{jk}^{-1}p_k + \sum_{k,i} Y_{jk}^{-1}X_{ki}q_i,
\]
and
\[
I(p_i) = - \sum Y_{ij}q_j - \sum X_{ij}I(q_j)
\]
\[
= - \sum Y_{ij}q_j - \sum X_{ij}Y_{jk}^{-1}p_k - \sum X_{ij}Y_{jk}^{-1}X_{kl}q_l
\]
\[
= - \sum X_{ij}Y_{jk}^{-1}p_k - \left( Yd + \sum X_{ij}Y_{jk}^{-1}X_{kl} \right) q
\]
This gives us the following matrix presentation
\[
\begin{pmatrix}
I(p) \\
I(q)
\end{pmatrix} = \begin{pmatrix}
-X Y^{-1} & -(Y + XY^{-1} X) \\
Y^{-1} & Y^{-1} X
\end{pmatrix}
\begin{pmatrix}
p \\
q
\end{pmatrix}.
\]
A simple computational check shows that this this matrix indeed squares to $-\text{Id}$. Let us now compute the derivative of $I_t$.
\[
\begin{pmatrix}
\dot{I}(p) \\
\dot{I}(q)
\end{pmatrix} = \begin{pmatrix}
-X Y^{-1} - X Y^{-1} & -(X Y^{-1} - (XY^{-1} X)) \\
Y^{-1} & Y^{-1} X + Y^{-1} X
\end{pmatrix}
\begin{pmatrix}
p \\
q
\end{pmatrix}
\]
Using that $YY^{-1} = \text{Id}$ we find that
\[
\dot{Y}^{-1} = Y^{-1} \dot{Y} Y^{-1}.
\]
On the other hand, using
\[
I(w_i) = i w_i
\]
we compute that
\[
\dot{I}(w_i) = i \dot{w}_i - I(\dot{w}_i) = (i \text{Id} - I)\dot{w}_i
\]
and that
\[ \dot{w}_i = \dot{X}_{ij} q_j + i \dot{Y}_{ij} q_j = (\dot{X}_{ij} + i \dot{Y}_{ij}) q_j = (\dot{X} + i \dot{Y}) Y^{-1} Y q_j \]
which gives
\[
\dot{I}(\dot{w}_i) = (\dot{X}_{ij} + i \dot{Y}_{ij}) I(q_j) = \dot{X}_{ij} Y^{-1}_k p_k + i \dot{Y}_{ij} Y^{-1}_k p_k = \dot{X}_{ij} Y^{-1}_k X_{kl} q_l + i \dot{Y}_{ij} Y^{-1}_k X_{kl} q_l.
\]
This gives us the following formula
\[
\dot{I}(w) = - (\dot{X} + i \dot{Y}) Y^{-1} (p + X q - Y q) = - (\dot{X} + i \dot{Y}) Y^{-1} \bar{w}
\]
and hence
\[
\dot{I}(w) = \dot{Z} Y^{-1} (p + X q) - \dot{Z} Y^{-1} \bar{w}.
\]
But \( I(w_i) = iw_i \), so \( \dot{I}(w) = i \dot{w} - I(\dot{w}) \). Now
\[
\dot{w} = Z q = \dot{Z} Y^{-1} Y q,
\]
so we conclude
\[
I(\dot{w}) = \dot{Z} Y^{-1} (p + X q).
\]
Which implies that
\[
\dot{I}(w) = - \dot{Z} Y^{-1} (p + X q) - iY q = - \dot{Z} Y^{-1} \bar{w}.
\]
Hence, with respect to the local frames we have the local matrix presentations
\[
\dot{I} = - \dot{Z} Y^{-1} \in C^\infty(P^*_t \otimes \bar{P}_t) \simeq C^\infty(\text{Hom}(P_t, \bar{P}_t))
\]
and
\[
\dot{I} = - \dot{Z} Y^{-1} \in C^\infty(P^*_t \otimes P_t) \simeq C^\infty(\text{Hom}(\bar{P}_t, P_t)).
\]
So we have the following formula for the derivative of the complex structure
\[
\dot{I} = - \sum_{i,j,k} \dot{Z}_{jk} Y^{-1}_{ki} w_i \otimes \bar{w}_j^* = \sum_{i,j} a_{ij} w_i \otimes \bar{w}_j^*.
\]
So if \( w = \sum \omega_{ij} w_i^* \wedge \bar{w}_j^* \) then
\[
a_{ij} = \sum G_{ik} w_{kj}
\]
so
\[
\sum_k a_{ik} w_{kj}^{-1} = - \sum_{l,k} Y^{-1}_{lk} \dot{Z}_{kl} \omega_{lj}^{-1}.
\]
Define
\[
Z^{-1} = V + i W.
\]
Then
\[
X V - Y W = \text{Id} \quad Y V + X W = 0
\]
and hence
\[
V = - Y^{-1} X W = - W X Y^{-1}.
\]
\[
X Y^{-1} X W + Y W = (X Y^{-1} X + Y) W = \text{Id}.
\]
Now
\[
(X Y^{-1} X + Y) v = 0
\]
will imply that $Yv = -XY^{-1}Xv$ which gives
$$0 \leq (Yv, v) = -(XY^{-1}Xv, v) = -(Y^{-1}Xv, Xv) \leq 0.$$  
Thus $(Yv, v) = 0$ and therefore $v = 0$.

$$W = -(XY^{-1}X + Y)^{-1},$$

$$V = Y^{-1}X(XY^{-1}X + Y)^{-1}.$$  

Let $(p_i^*, q_i^*)$ be a basis of $T^*N'$ dual to the basis $(p_i, q_i)$ of $TN'$. So $\omega = \sum p_i^* \wedge q_i^*$. Let $(w_i^*, \bar{w}_i^*)$ be a basis of $T^*N'$ dual to the basis $(w_i, \bar{w}_i)$ of $TN$. Then

$$O^* = \text{span}\{p_1^*, \ldots, p_n^*\}$$

and

$$P_t^* = \text{span}\{\bar{w}_1^*, \ldots, \bar{w}_n^*\}.$$  

A short computation gives that

$$\left(\begin{array}{c} w^* \\ \bar{w}^* \end{array}\right) = \frac{i}{2} \left(\begin{array}{cc} Y^{-1} & 0 \\ 0 & Y^{-1} \end{array}\right) \left(\begin{array}{c} \bar{Z} \\ \text{Id} \end{array}\right) \left(\begin{array}{c} p^* \\ q^* \end{array}\right).$$

Let us now compute the symplectic form on the $(w^*, \bar{w}^*)$ basis.

$$\omega = \sum_{i=1}^n p_i^* \wedge q_i^*$$

$$= \sum_{j=1}^{n} (w_i^* \bar{w}_i^*) \wedge (Z_{ij} w_j^* + \bar{Z}_{ij} \bar{w}_j^*)$$

$$= \sum_{j=1}^{n} w_i^* \wedge Z_{ij} w_j^* + \sum_{j=1}^{n} \bar{w}_i^* \wedge \bar{Z}_{ij} \bar{w}_j^* + \sum_{j=1}^{n} w_i^* \wedge \bar{Z}_{ij} \bar{w}_j^* + \sum_{j=1}^{n} \bar{w}_i^* \wedge Z_{ij} w_j^*$$

$$= \sum_{i<j} (Z_{ij} (w_i^* \wedge w_j^* + w_j^* \wedge w_i^*) + \bar{Z}_{ij} (\bar{w}_i^* \wedge \bar{w}_j^* + \bar{w}_j^* \wedge \bar{w}_i^*)) - 2i \sum_{i,j} w_i^* \wedge Y_{ij} \bar{w}_j^*$$

$$= -2i \sum_{i,j} w_i^* \wedge Y_{ij} \bar{w}_j^* = -2i w^* \wedge Y \bar{w}^*$$

$$= -2i \sum_{i,j} Y_{ij} w_i^* \wedge \bar{w}_j^*,$$

hence $\omega_{ij} = -2i Y_{ij}$. From this we see that

$$G = \sum_{i,j} G_{ij} w_i \otimes w_j = -\frac{i}{2} \sum_{i,j,k,l} Y^{-1}_{ik} \bar{Z} Y^{-1}_{lj} w_i \otimes w_j$$

Let $\pi_t : T^*M \rightarrow P_t^*$ be the projection onto $P_t^*$, whose kernel is $\bar{P}_t^*$, i.e. compatible with $T^*M = P_t^* \oplus \bar{P}_t^*$, and let $\pi'_t : T^*M \rightarrow O^*$ be the projection onto $O^*$, whose kernel is $\bar{P}_t^*$, i.e. compatible with $T^*M = O^* \oplus \bar{P}_t^*$. Since $\text{Im}(1 - \pi_t) = \text{Ker}\pi_t = \text{Ker}\pi'_t = \text{Im}(1 - \pi'_t)$ we see that

$$\pi_t \circ \pi'_t = \pi_t (\pi_t + (1 - \pi'_t)) = \pi_t$$

and

$$\pi'_t \circ \pi_t = \pi'_t (\pi_t + (1 - \pi_t)) = \pi'_t.$$
Let us now compute $\pi_t$ and $\pi'_t$ in the respective bases. We have that

$$-2iY(w^* + \bar{w}^*) = (\bar{Z} - Z)p^* = -2iYp^*$$

which implies

(11) 
$$p^* = w^* + \bar{w}^*$$

and further that

$$-2iY(w^* - \bar{w}^*) = -2q^* + 2Xp^*.$$ 

This implies

$$q^* = iY(w^* - \bar{w}^*) + X(w^* + \bar{w}^*) = Zw^* + \bar{Z}\bar{w}^*.$$ 

So

$$\pi_t(p^*) = w^* = \frac{i}{2}Y^{-1}(-q^* + \bar{Z}p^*), \quad \pi_t(q^*) = Zw^* = \frac{i}{2}ZY^{-1}(-q^* + \bar{Z}p^*).$$ 

Because of (11) we see that

$$\pi'_t(w^*) = p^*.$$ 

and

$$\pi'_t(\bar{w}^*) = 0$$

so

$$\pi'_t(q^*) = \pi'_t(Zw^*) + \pi'_t(\bar{Z}\bar{w}^*) = Zp^*.$$ 

Let us define the following operators

$$D' = \pi'_t \circ \nabla^{1,0} : C^\infty(\mathcal{L}^k) \to C^\infty(O^* \otimes \mathcal{L}^k)$$

$$G' = \pi'_t \circ G \circ \pi_t : C^\infty(O^* \otimes \mathcal{L}^k) \to C^\infty(O \otimes \mathcal{L}^k)$$

$$D'' = \pi'_t \circ (\nabla^{1,0} \otimes \text{Id} \otimes \text{Id} \otimes \nabla^{1,0}) \circ \pi_t : C^\infty(O \otimes \mathcal{L}^k) \to C^\infty(O \otimes O \otimes \mathcal{L}^k).$$

On ker $\nabla^{0,1}$, we shall now compute Tr($D'GD'$). Hence if we have a section $s$ of $\mathcal{L}^k$ over $N'$, which is holomorphic, we have that

$$\nabla_{p^i,s} = -\nabla_{\bar{Z}_i,q_j} s,$$

which we will use a number of times below. From the above we have that

$$G(w^*_i) = \frac{i}{2} \sum Y_{ik}^{-1}\bar{Z}_{ij}Y_{ij}^{-1}w_j$$

Now

$$w_i + \bar{w}_i = 2\left(p_i \sum X_{ij}q_j\right)$$

and

$$w_i - \bar{w}_i = 2i \sum Y_{ij}q_j$$

so

$$q_i = -\frac{i}{2} \sum Y_{ij}^{-1}(w_j - \bar{w}_j)$$

and therefore we have that

$$w_i + \bar{w}_i = 2p_i - i \sum X_{ij}Y_{jk}^{-1}(w_k - \bar{w}_k)$$

which implies

$$\sum_k \left(\delta_{ik} + i \sum X_{ij}Y_{jk}^{-1}\right)w_k = 2p_i + \sum_k \left(i \sum X_{ij}Y_{jk}^{-1} - \delta_{ik}\right).$$

Now

$$i\bar{Z}Y^{-1}w = 2p + iZY^{-1}\bar{w}$$
which gives
\[ w = -2iYZ^{-1}p + YZ^{-1}ZY^{-1}\bar{w} \]
hence
\[ \pi'_i(w) = -2iYZ^{-1}p, \quad \pi_i(p) = \frac{i}{2}YZ^{-1}w. \]

So
\[ \nabla^{1,0} = \sum_{i=1}^{n} w_i^* \otimes \nabla w_i = \sum_{i=1}^{n} w_i^* \otimes (\nabla p_i s + \nabla z_j t_j s). \]
and then if we use \( \nabla^{0,1} = 0 \) then
\[ \nabla^{1,0} = \sum_{i=1}^{n} w_i^* \otimes \nabla w_i = -2i \sum_{i,k,l=1}^{n} Y_{ik} \bar{Z}_{kl}^{-1} \nabla pl s \otimes w_i^*, \]
and hence
\[ D's = \pi'_i \nabla^{1,0} = -2i \sum_{i,k,l=1}^{n} Y_{il} \bar{Z}_{ik}^{-1} p_i^* \otimes \nabla p_k s. \]

Now
\[ G'(p_i^*) = \pi'_i \circ G \circ \pi_i(p_i^*) \]
\[ = \pi'_i \circ G(w_i^*) \]
\[ = -\frac{i}{2} \pi'_i \left( \sum_{k,l} Y_{ik}^{-1} \bar{Z}_{kl}^{-1} Y_{lj}^{-1} w_j \right) \]
\[ = - \sum_{k,l} Y_{ik}^{-1} \bar{Z}_{kl} Y_{lj}^{-1} Y_{jr} \bar{Z}_{rs}^{-1} p_s \]
\[ = - \sum_{k,l} Y_{ik}^{-1} \bar{Z}_{kl} \bar{Z}_{sr}^{-1} p_r. \]
so
\[ G' \circ D's = 2i \sum_{i,k,l=1}^{n} Y_{il} \bar{Z}_{ik}^{-1} Y_{ir}^{-1} \bar{Z}_{rs} \bar{Z}_{st}^{-1} p_l \otimes \nabla p_k s \]
\[ = 2i \sum_{i,k,l=1}^{n} \bar{Z}_{ik} \bar{Z}_{sl} \bar{Z}_{st}^{-1} p_l \otimes \nabla p_k s \]
\[ = 2i \sum_{l,k,i} \bar{Z}_{il} \bar{Z}_{lk} \bar{Z}_{ki} p_l \otimes \nabla p_s \]
giving
\[ \pi_i \circ G' \circ D's = - \sum_{j,l} \bar{Z}_{j,l} \bar{Z}_{lk} \bar{Z}_{ki}^{-1} Y_{rs}^{-1} w_i \otimes \nabla p_j s \]
\[ = - \sum_{j,l} \bar{Z}_{j,l} \bar{Z}_{lk} Y_{ki}^{-1} w_i \otimes \nabla p_j s \]
\[ = G \circ \nabla^{1,0} s. \]

and thus
\[ (\nabla^{1,0} \otimes 1 + 1 \otimes \nabla^{1,0}) \circ G \circ \nabla^{1,0} s = - \sum_{j,r} \bar{Z}_{jr} \bar{Z}_{rk} Y_{ki}^{-1} w_i^* \otimes \nabla w_i (w_i) \otimes \nabla p_j s \]
\[ - \sum_{j,r} \bar{Z}_{jr} \bar{Z}_{rk} Y_{ki}^{-1} w_i^* \otimes w_i \nabla w_i \nabla p_j s \]
\[ - \sum_{j,l} d(\bar{Z}_{jr} \bar{Z}_{rk} Y_{ki}^{-1}) (w_i) w_i^* \otimes w_i \otimes \nabla p_j s. \]

Write
\[ \nabla w_i (w_i) = \sum C_{i,j}^l w_j \]
Also, we rewrite
\[ \nabla_{w_i} \nabla_{p_j} s = -2i \sum_{k,m} Y_{ik} Z_{km} \nabla_{p_m} \nabla_{p_j} s + \sum_{k,m,l,r} Y_{ik} Z_{km} Y_{lr}^{-1} \nabla \bar{w}_r \nabla_{p_j} s \]
\[ = -2i \sum_{k,m} Y_{ik} Z_{km} \nabla_{p_m} \nabla_{p_j} s + k \sum_{k,m,l,r} Y_{ik} Z_{km} Y_{sr}^{-1} \omega(\bar{w}_r, p_j) s. \]
which allows us to conclude
\[ (\nabla_{1,0}^1 \otimes 1 + 1 \otimes \nabla_{1,0}^1) \circ G \circ \nabla_{1,0}^1 = 2i \sum \bar{Z}_{jk}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1} \zeta_{sm}^{-1} w_i^s \otimes w_l \otimes \nabla_{p_m} \nabla_{p_j} s \]
\[ \quad - \sum \bar{Z}_{jk}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1} C_{m,i}^j w_i^s \otimes w_s \otimes \nabla_{p_j} s \]
\[ \quad - k \sum \bar{Z}_{jk}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1} Y_{sr}^{-1} Z_{tm} Y_{mn}^{-1} \omega(\bar{w}_s, p_j) w_i^s \otimes w_l \otimes s \]
\[ \quad - d \left( \bar{Z}_{jk}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1} \right) (w_l) w_i^s \otimes w_l \otimes \nabla_{p_j} s. \]
So
\[ \Delta_G s = \text{Tr}(D^0 \circ G' \circ D') s \]
\[ = -2i \sum \bar{Z}_{ij}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1} C_{m,j} \nabla_{p_m} \nabla_{p_j} s \]
\[ \quad - \sum d(\bar{Z}_{ik}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1})(w_j) \nabla_{p_j} s \]
\[ \quad - k \sum \bar{Z}_{jk}^{-1} \hat{\zeta}_{kl} Y_{ij}^{-1} Z_{tm} Y_{mn}^{-1} \omega(\bar{w}_s, p_j) s, \]
where \( \omega(\bar{w}_s, p_j) = -\bar{Z}_{sj} \).

Since we have that
\[ Z = Z_{\infty}^{-1} \frac{1}{t} + R(t), \quad \det Z_{\infty} \neq 0. \]

where
\[ t \cdot R(t) \to 0 \quad \text{as} \quad t \to \infty \]
and
\[ t^2 R'(t) \to 0 \quad \text{as} \quad t \to \infty, \]
we get that
\[ Z^{-1} = t \cdot Z_{\infty}^{-1} (\text{Id} + t R(t) \cdot Z_{\infty}^{-1})^{-1} = t \cdot Z_{\infty}^{-1} + G(t) \]
such that
\[ \frac{1}{t} G(t) \to 0 \quad \text{as} \quad t \to \infty. \]
From this we see that
\[ \bar{Z}^{-1} \hat{\zeta} \bar{Z}^{-1} = (t \cdot Z_{\infty}^{-1} + G(t)) \cdot (-\bar{Z}_{\infty}^{-1} \frac{1}{t^2} + R'(t)) \cdot (t \bar{Z}_{\infty}^{-1} + G(t)) = -\bar{Z}_{\infty}^{-1} + H(t) \]
where \( H(t) \to 0 \) as \( t \to \infty \). Hence we have obtained the formula
\[ \lim_{t \to \infty} \Delta_G s = 2i \sum_{i,j} (\bar{Z}_{\infty}^{-1})_{i,j} \nabla_{p_i} \nabla_{p_j} s \]
\[ \sum_{i,j} 2G^{i,j} \frac{\partial F}{\partial z_i} \nabla_{j} s = 2G \cdot \partial F \otimes \nabla_{1,0}^1 s = -i \sum_{i,j} Y_{ik}^{-1} \hat{\zeta}_{kl} Y_{lj}^{-1} dF(w_i) \otimes \nabla_{w_j} s. \]
So using $\nabla^{1,0}s = 0$ we obtain
\[
2G\partial F \otimes \nabla^{1,0}s = -2 \sum_{i,j} Y_{ik}^{-1} \hat{Z}_{kl} Y_{lj}^{-1} dF(w_i) \otimes Y_{jr}^{-1} \nabla_{p_j}s.
\]

From this we get the following formula for
\[
\tilde{u}_{c,t}(s) = -\frac{1}{4k+2n} \left( -2i \sum Z_{ik}^{-1} \hat{Z}_{kl} Y_{lj}^{-1} C_{m,j}^{m,n} \nabla_{p_m} \nabla_{p_j}s 
\right.
\]
\[
- \sum d(Z_{ik}^{-1} \hat{Z}_{kl} Y_{lj}^{-1})(w_j) \nabla_{p_j}s 
\]
\[
- k \sum Z_{ik}^{-1} \hat{Z}_{kl} Y_{ij}^{-1} Z_{rm} Y_{ms} \omega(\bar{w}_s, p_j)s 
\]
\[
- 2 \sum_{i,j} Y_{ik}^{-1} \hat{Z}_{kl} Y_{lj}^{-1} dF(w_i) \otimes \nabla_{p_j}s 
\]
\[
+ 4k \dot{F}(s).
\]

where $\dot{F}_t$ refers to the derivative of $F_t$ with respect to the holomorphic part of $\tilde{\sigma}'_t$.

**Claim 4.** We have that

- The derivative along the directions of $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}}$ of $F_t$ converges to zero.
- The derivative of $F_t$ with respect to the holomorphic part of $\tilde{\sigma}'_t$ goes to zero as $t$ goes to infinity.
- The function $\tilde{F}_t$ converges to zero, as $t$ goes to infinity.

**Proof.** The claim follows directly from the equations which defines $F_t$ when combined with Theorem 14.

From these two claims it follows immediately that $\tilde{u}_{c}(\tilde{\sigma}'_t)$ has a limit, say $\tilde{u}_{c,\infty}$ as $t$ goes to infinity, and in fact
\[
\tilde{u}_{c,\infty} = \Delta G_{\infty}.
\]

**Claim 5.** We have that the kernel of $\tilde{u}_{c,\infty}$ consists of sections that are covariant constant along $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}}$.

**Proof.** We observe that $G_{\infty}$ induces a Hermitian structure on the leaves of $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}} \cap T\Sigma'$ and that $\Delta G_{\infty}$ is the corresponding Laplace–Beltrami operator associated to the restriction of $\nabla$ to the directions of $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}} \cap T\Sigma'$. But then it follows immediately that the kernel of $\Delta G_{\infty}$ are exactly the covariant constant sections of $\nabla$ along the directions of $F_{\tilde{P},\tilde{\sigma}} \cap \tilde{F}_{\tilde{P},\tilde{\sigma}}$.

Theorem 17 now follows directly from Claim 1, 4 and 5 together with the above derived formulae.

**Theorem 18.** In the cases (1)—(3) above and for $\tilde{P}$ any admissible system of curves on $\tilde{\Sigma}$, there exists a limiting linear map
\[
P_{\infty}(\tilde{\sigma}_0, \tilde{P}) : H^{(k)}_{\tilde{\sigma}_0} \to H^{(k)}_{\tilde{P},\tilde{\sigma}_0}.
\]
Proof. Assume $E(t)$ is a solution to

$$E'(t) = -P(t)E(t)$$

where $P(t) = [\tilde{u}_{c,t}, \cdot]$ and $E(t_0) = \text{Id}$, where $t_0$ is some starttime. We further let $P_\infty = P(\infty) = [\tilde{u}_{c,\infty}, \cdot]$. Let now $Q(t) = e^{(t-t_0)P_\infty}E(t)$. Then

$$Q'(t) = e^{(t-t_0)P_\infty}(P_\infty - P(t))E(t),$$

so

$$Q(t) = \text{Id} + \int_{t_0}^t e^{(s_0-t_0)P_\infty}(P_\infty - P(s_0))E(s_0)\,ds_0,$$

$$E(t) = e^{-(t-t_0)P_\infty} + \int_{t_0}^t e^{-(t-s_0)P_\infty}(P_\infty - P(s_0))E(s_0)\,ds_0$$

$$= e^{-(t-t_0)P_\infty} + \int_{t_0}^t e^{-(t-s_0)P_\infty}(P_\infty - P(s_0))e^{-(s_0-t_0)P}\,ds_0$$

$$+ \int_{t_0}^t e^{-(t-s_0)P_\infty}(P_\infty - P(s_0))\int_{t_0}^{s_0} e^{-(s_0-s_1)P_\infty}(P_\infty - P(s_1))E(s_1)\,ds_1\,ds_0.$$ Iterating this construction we arrive at the following formula

$$E(t) = \sum_{n=0}^\infty \int_{\Delta_n(t,t_0)} e^{-(t-s_0)P_\infty}(P_\infty - P(s_0))e^{-(s_0-s_1)P_\infty}(P_\infty - P(s_1))$$

$$\cdots (P_\infty - P(s_{n-1}))e^{-(s_{n-1}-t_0)P_\infty} ds_{n-1} \cdots ds_0,$$

where

$$\Delta_n(t,t_0) = \{(s_0, \ldots, s_{n-1}) \in \mathbb{R}^n \mid t_0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq t\}.$$ We need to justify the convergence of the series (15). First we observe that

$$\text{Vol}(\Delta_n(t,t_0)) = \frac{(t-t_0)^n}{n!}.$$ From the above we have that

$$|P_\infty - P(t)| \leq ct^\alpha$$

for all $t \in [t_0, \infty)$, where $\alpha < -1$. This allows us to show that (15) is absolutely summable. For large enough $t_0$ we will get that $|e^{-tP_\infty}| = 1$ for all $t \leq t_0$. So then

$$\int_{\Delta_n(t,t_0)} e^{-(t-s_0)P_\infty}(P_\infty - P(s_0))\cdots (P_\infty - P(s_{n-1}))e^{-(s_{n-1}-t_0)P_\infty} ds_{n-1} \cdots ds_0$$

$$\leq e^{cn} \int_{\Delta_n(t,t_0)} s_0^\alpha \cdots s_{n-1}^\alpha ds_{n-1} \cdots ds_0$$

$$= \frac{e^{cn}}{n!} \left( \frac{t_0^{\alpha+1}}{\alpha+1} + \frac{t^{\alpha+1}}{\alpha+1} \right)^n.$$ Hence, we see that (15) is summable and

$$|E(t)| \leq e^{-\frac{ct^\alpha+1}{\alpha+1}}.$$

Note that the estimate converges to $e^{-\frac{ct^{\alpha+1}}{\alpha+1}}$ as $t \to \infty$. 

Proposition 1. We have the following asymptotics in Teichmüller space: the large $k$ degenerating the complex structure is after a local coordinate change equivalent to

\begin{equation}
\lim_{k \to \infty} (O_{\infty} - P(s_0)) \cdots
\end{equation}

Proof. This Theorem follows by the same arguments as in [A11], since the effect of

\begin{equation}
|E(t_1) - E(t_2)| \leq |\sum_{n=0}^{\infty} \int_{T_2(t_2, t_2)} \langle e^{-t_1 P_{\infty} - e^{-t_2 P_{\infty}}} e^{s_0 P}, (O_{\infty} - P(s_0)) \cdots |
\end{equation}

\begin{equation}
+ |\sum_{n=0}^{\infty} \int_{T_2(t_2, t_2)} e^{-(t_1 - s_0) P_{\infty}} (P - P(s_0)) \cdots |
\end{equation}

\begin{equation}
\leq |e^{-t_1 P_{\infty}} - e^{-t_2 P_{\infty}}| e^{-\frac{s_0 + 1}{t_1}} e^{-\frac{s_0 + 1}{t_2}}
\end{equation}

which can be made arbitrary small provided $t_1$ and $t_2$ are large enough giving the

Cauchy condition. Hence $E(\infty)$ exists. Moreover, by dividing by $|t_1 - t_2|$ and

letting $t_2 \to t_1$. We see that $|E'(t)|$ can be made arbitrarily small, provided $t$ is

large enough, hence $E'(t) \to 0$ as $t \to \infty$. But then we get that

\begin{equation}
P_{\infty} E(\infty) = 0,
\end{equation}

proving $\text{Im} E(\infty) \subseteq \ker P_{\infty}$. It is clear that $E(t)$ defined this satisfies the required

equation. The theorem now follows from Claim [5].

Suppose we now have $s_P \in H^{(k)}_{P}$. Then we get an induced linear functional on

$H^{(k)}_{\sigma_i}$ given by

\begin{equation}
s_P(s) = \sum_{b \in B^{(k)}_{P}} \int_{x \in h^{-1}_P(b)} \langle s(x), s_P(x) \rangle \text{Vol}_{\sigma_i, b}(x),
\end{equation}

where $\text{Vol}_{\sigma_i, b}$ is the volume form on $h^{-1}_P(b)$ induced by the metric on $N$ associated to $\sigma_i$. Now let $s_{P, \sigma_i} \in H^{(k)}_{\sigma_i}$ be the state associated to this functional,

\begin{equation}
(s, s_{P, \sigma_i}) = S_P(s),
\end{equation}

for all $s \in H^{(k)}_{\sigma_i}$.

Proposition 1. We have the following asymptotics in Teichmüller space:

\begin{equation}
\lim_{t \to \infty} P_{\infty}(\sigma_i, P)(s_{P, \sigma_i}) = s_P.
\end{equation}

Proof. This Theorem follows by the same arguments as in [A11], since the effect of

degenerating the complex structure is after a local coordinate change equivalent to

the large $k$ limit considered in [A11].

Corollary 1. In the cases (1)—(3) above and for $\tilde{P}$ any admissible system of curves on $\Sigma$, the map (14) is an isomorphism.

Theorem [18] and this Corollary [1] implies Theorem [4].

5. The four punctured sphere case

Suppose $\Sigma$ is a 2-sphere, and that $R$ consists of four points on $\Sigma$. Let $\tilde{\Sigma} = \Sigma - R$. Assume that we have a labeling $c : R \to [-2, 2]$. Suppose we are given two transverse pair of pants decompositions $P_1$ and $P_2$ of $\Sigma$. Then $P_i = \{\gamma_i\}$, where $\gamma_1$ and $\gamma_2$ are two transverse simple closed curves on $\tilde{\Sigma}$. We will use the notation $h_i = h_{P_i}$.
Choose an ordered subset $R'$ of $R$ of cardinality three. In this case we have the identity
\[ \tilde{T} \cong \mathbb{C} - \{0,1\} \]
on obtained as follows. For each $\tilde{\sigma}$, there is a unique $z \in \mathbb{C} - \{0,1\}$ and a unique biholomorphism from $(\tilde{\Sigma}_\sigma, R)$ to $(\mathbb{CP}^1, \{0,1,\infty, z\})$ and which maps the ordered set $R'$ to the points $\{0,1,\infty\}$ on $\mathbb{CP}^1$.

In the following we determine the moduli space of flat connections on a four punctured sphere.

In stead of calculating the moduli spaces purely gauge theoretic we will make heavy use of the identification of the moduli space of flat connections on $\tilde{\Sigma}$ with the character variety $\mathcal{M}(\Sigma_\sigma) = \text{Hom}(\pi_1(\Sigma_{g,n}), \text{SU}(2))/ \text{SU}(2)$.

There are many ways of calculating these moduli spaces. We could use the Morse theoretic approach as [Th], or we could use pair of pants decomposition of $\tilde{\Sigma}$ into two pair of pants glued along a circle, and calculate the fundamental group as an amalgamation of fundamental groups of two fundamental groups of a pair of pants. We will however calculate it by specifying specific curves, and use them to define coordinates in $\mathcal{M}(\Sigma_\sigma)$ by using trace.

Let $A, B, C, D$ be four curves on $\tilde{\Sigma}$ each of which encircles a puncture. Then
\[ \pi_1(\tilde{\Sigma}_\sigma) = \langle A, B, C, D \mid ABCD = 1 \rangle. \]

We define seven coordinates on the moduli space, each for one of the trace of holonomies around the punctures $a = \text{Tr}(\rho(A)), b = \text{Tr}(\rho(B)), c = \text{Tr}(\rho(C)), d = \text{Tr}(\rho(D))$ and one for each of the belts dividing $\tilde{\Sigma}$ into two pair of pants $x = \text{Tr}(\rho(AB)), y = \text{Tr}(\rho(BC))$ and a last for the diagonal $z = \text{Tr}(\rho(AC))$, where $\rho$ is a $\text{SU}(2)$-representation of $\pi_1(\tilde{\Sigma}_\sigma)$. It can be shown ([?]) that these functions satisfy the equation
\[ x^2 + y^2 + z^2 + xyz = (ab+cd)x + (ad+bc)y + (ac+bd)z - (a^2 + b^2 + c^2 + d^2 + abcd - 4). \]

If the holonomies, $(\rho(A), \rho(B), \rho(C), \rho(D))$, around $A, B, C, D$ are fixed subject to $\rho(ABCD) = \text{Id}$, the moduli space $N_{(\rho(A), \rho(B), \rho(C), \rho(D))}(\tilde{\Sigma}_\sigma)$ is the zero-set of the polynomial (17) in $[-2,2]^3$. For the permitted $(a, b, c, d) \in (-2,2)^4$ all moduli spaces are topologically spheres. In the six boundary cases
\[ (a, b, c, d) \in \{(2, 2, t, t), (2, t, t, 2), (2, t, 2, t), (t, t, 2, 2), (t, 2, t, 2), (t, 2, 2, t), t \in [-2,2]\}, \]
the moduli spaces are just points – this corresponds to the case where two of the punctures has been filled in, and we consider the space of flat connections on a circle with specified holonomy $t \in [-2,2]$ – which is exactly a point.

**Remark 2.** For a more detailed study of the moduli spaces mentioned in the above examples see e.g. [Go1].

Let us now consider the moduli space of parabolic vector bundles on $\tilde{\Sigma}_\sigma$. By the Mehta–Seshadri Theorem and the calculations above this moduli space is generically a 2-sphere.

Let $E \to \tilde{\Sigma}_\sigma$ be a stable parabolic vector bundle of parabolic degree 0 on $\tilde{\Sigma}_\sigma$ and let $L \subset E$ be a proper subbundle.
From above we have
\[ p\deg L = \deg L + \sum_{p \in R} w_1(p) \]
\[ = \deg L + \sum_{p \in R} s_p - \sum_{p \in R} s_p \]
\[ = \deg L + 2 \sum_{p \in R} s_p - \sum_{p \in R} s_p \]
= \deg L + \sum_{p \in R} s_p - \sum_{p \in R} s_p

For \( E \) to be parabolically stable \( p\deg L < 0 \) so we get the following bound on the degree of \( L \):
\[ \deg L = p\deg L + \sum_{p \in R} s_p - 2 \sum_{p \in R} s_p \leq \sum_{p \in R} s_p. \]
Since \( \deg E = 0 \) the Grothendieck classification of vector bundles on \( \mathbb{P}^1 \) give that \( E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k) \) for an integer \( k \in \mathbb{N} \).
If \( L = \mathcal{O}(k) \) the restriction on degree gives \( k \leq \sum_{p \in R} s_p \). Now since there are four marked points and each of the \( s_p \) are less than \( \frac{1}{2} \) we get that \( k < 2 \). Thus there are only two options
\[ E \simeq \mathcal{O} \oplus \mathcal{O} \quad \text{or} \quad E \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1). \]

Having analyzed this moduli space, we now turn to its quantization and the associated Hitchin connection. In particular, we will below identify the Hitchin connection explicitly with the TUY connection in the bundle of conformal blocks in this case of a four holed sphere. Hence let us first recall the sheaf of vacua construction from [TUY].
Suppose \( \mathfrak{g} \) is a Lie algebra with a invariant inner product, which we will normalize such that the longest root have length \( \sqrt{2} \). Let
\[ B = \mathbb{C} - \{-1, 0, 1\} \]
and let \( C = B \times \mathbb{P}^1 \), which the canonical sections \( s_i : B \to C, i = 1, 2, 3, 4 \) determined by
\[ s_1(\tau) = -1, \ s_2(\tau) = 0, \ s_3(\tau) = 1 \text{ and } s_4(\tau) = \tau, \]
for \( \tau \in B \). Let \( \mathcal{F} = (C, B, s_1, s_2, s_3, s_4) \) with the natural formal neighbourhoods induced from the canonical identification \( \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \).
Let
\[ \hat{\mathfrak{g}}(\mathcal{F}) = \mathfrak{g} \otimes \mathbb{C} H^0(C, \mathcal{O}_C(\star \sum_{j=1}^N x_j)) \]
and recall from [TUY] that the sheaf of conformal blocks over \( B \) are given as follows
\[ \mathcal{V}_\lambda^l(\mathcal{F}) = \{ |\Psi\rangle \in \mathcal{O}_B \otimes \mathcal{H}_\lambda^l | \langle \Psi | \hat{\mathfrak{g}}(\mathcal{F}) = 0 \} \]
where \( \mathcal{H}_\lambda \) is the highest weight integrable \( \hat{\mathfrak{g}} \)-module and
\[ \mathcal{H}_\lambda^l = \mathcal{H}_\lambda^l \otimes \mathbb{C} \mathcal{H}_\lambda^{l-1} \cdots \otimes \mathbb{C} \mathcal{H}_\lambda^1. \]
As it is proved in [TUY], we get that the restriction map from $H^{\lambda}_{\bar{\lambda}}$ to $H^{(0)}_{\bar{\lambda}} = V_{\bar{\lambda}}$ induces an embedding of the sheaf of conformal block in genus 0 into trivial $V_{\bar{\lambda}}^*$-bundle:

$$V^{\lambda}_{\bar{\lambda}}(F) \hookrightarrow B \times (V_{\bar{\lambda}}^*)^g.$$  

Under this identification, the TUY-connection in the sheaf of conformal blocks gets identified with the KZ-connection in $B \times (V_{\bar{\lambda}}^*)^g$, which we now recall. Let $\Omega_{ij}$ is the quadratic Casimir acting in the $i$'th and $j$'th factor. Suppose that $(J_1, J_2, J_3)$ is an orthonormal basis of $g$, then

$$\Omega = \sum_{i=1}^{3} J_i \otimes J_i$$

So if $\rho_i : SU(2) \rightarrow Aut(V_{\bar{\lambda}})$ and $\dot{\rho}_i : g \rightarrow End(V_{\bar{\lambda}})$ are the representations of $SU(2)$ and $g$, and we embed them into $Aut(V_{\bar{\lambda}})$ and $End(V_{\bar{\lambda}})$ in the usual way, then

$$\Omega_{ij} = \dot{\rho}_i \otimes \dot{\rho}_j(\Omega).$$

The KZ-connection is then given by

$$\nabla^{KZ} = \nabla^t - \alpha\left(\frac{\partial}{\partial \tau}\right).$$

where

$$\alpha\left(\frac{\partial}{\partial \tau}\right) = \frac{\Omega_{41}}{\tau} + \frac{\Omega_{42}}{\tau-1} + \frac{\Omega_{43}}{\tau+1}.$$

We will now produce a geometric version of the KZ-connection.

The invariant inner product on the Lie algebra $g$ induces a natural symplectic structure on the coadjoint orbits. Let $h \subseteq g$ denote the Cartan subalgebra and denote by $X_{\lambda}$ the coadjoint orbit through $\lambda \in h^*$. If we use the right normalization of the inner product, we have that $X_{\lambda}$ is quantizable if and only if $\lambda$ is in the weight lattice. Let $G = SU(2)$ and assume that $\lambda$ is a dominant weight. We get a prequantum line bundle $L_\lambda \rightarrow X_{\lambda}$, and the action of $SU(2)$ lifts to this line bundle. Furthermore there exists a $SU(2)$-invariant complex structure on $X_{\lambda}$. It follows from the Bott–Borel–Weil Theorem that the representation of $SU(2)$ on $H^0(X_{\lambda}, L_\lambda)$ are the one determined by $\lambda$:

$$V_{\lambda} \cong H^0(X_{\lambda}, L_\lambda).$$

The action of $g$ on $V_{\lambda}$ can be described explicitly: we have an infinitesimal action of $g$ on $X_{\lambda}$ given by

$$g \rightarrow \mathcal{X}(X_{\lambda}) \quad \text{given by} \quad \xi \mapsto Z_{\xi}$$

We then have that the action of $g$ on $V_{\lambda}$ is described by

$$\xi(s) = \nabla_{x_{\xi}} s + 2\pi i \mu(\xi) s$$

where $s \in H^0(X_{\lambda}, L_\lambda)$ and $\mu(\xi)$ is the moment map evaluated on $\xi$. We remark that the action of $g$ is given by first order differential operators.

Let us now consider the situation where we have four dominant weights $\bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and consider the exterior tensor product

$$\mathcal{L}_{\bar{\lambda}} = p^1_*(\mathcal{L}_{\lambda_1}) \otimes p^2_*(\mathcal{L}_{\lambda_2}) \otimes p^3_*(\mathcal{L}_{\lambda_3}) \otimes p^4_*(\mathcal{L}_{\lambda_4})$$

which is a line bundle over

$$X = X_{\lambda_1} \times X_{\lambda_2} \times X_{\lambda_3} \times X_{\lambda_4}.$$
Thus we get a representation of SU(2) on
\[ H^0(X, \mathcal{L}_\lambda) \cong H^0(X_{\lambda_1}, \mathcal{L}_{\lambda_1}) \otimes H^0(X_{\lambda_2}, \mathcal{L}_{\lambda_2}) \otimes H^0(X_{\lambda_3}, \mathcal{L}_{\lambda_3}) \otimes H^0(X_{\lambda_4}, \mathcal{L}_{\lambda_4}). \]
We are interested in the invariant part
\[ V^G_X = H^0(X, \mathcal{L}_\lambda)^{SU(2)}. \]
We can provide an alternative description of \( V^G_X \) by applying the idea that quantization commutes with reduction: Consider the moment map for the diagonal action
\[ \mu : X_{\lambda_1} \times X_{\lambda_2} \times X_{\lambda_3} \times X_{\lambda_4} \to \mathfrak{g}^* \]
given by
\[ \mu(\xi_1, \xi_2, \xi_3, \xi_4) = \sum_{i=1}^{4} \xi_i. \]
Now we consider the symplectic reduction
\[ \mathcal{M} = \mu^{-1}(0)/SU(2), \]
which have an induced complex structure from \( X \). Furthermore there exists a unique line bundle \( \mathcal{L}_\mathcal{M} \to \mathcal{M} \) s.t.
\[ p^*(\mathcal{L}_\mathcal{M}) \cong \mathcal{L}_\lambda|_{\mu^{-1}(0)} \]
where \( p : \mu^{-1}(0) \to \mathcal{M} \) is the projection map.

**Theorem 19** (Guillemin & Sternberg). *Quantization commutes with reduction, i.e.*
\[ V^G \cong H^0(\mathcal{M}, \mathcal{L}_\mathcal{M}). \]
Now we consider the genus 0 surface \( \Sigma \) with 4 marked points \( x_1, \ldots, x_4 \). We assume that we are provided with an identification \( \Sigma \cong \mathbb{P}^1 \), s.t. \( (x_1, x_2, x_3) \) are mapped to \((−1, 0, 1)\) and \( x_4 \) to \( \tau \in \mathbb{P}^1 - \{-1, 0, 1, \infty\} \). We assume that we have dominant weights \( \lambda_1, \ldots, \lambda_4 \) attached to \( x_1, \ldots, x_4 \). The KZ-connection is defined as a connection in the trivial bundle
\[ V^G_X = (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_4})^G. \]
As stated above, the KZ-connection is described by the specific 1-form:
\[ \nabla^{KZ} = \nabla^{t} - \alpha(\frac{\partial}{\partial \tau}) \]
where
\[ \alpha(\frac{\partial}{\partial \tau}) = \frac{\Omega_{41}}{\tau} + \frac{\Omega_{42}}{\tau - 1} + \frac{\Omega_{43}}{\tau + 1} \]
From this we see that each of the operators \( \Omega_{ij} \) become second order differential operators on \( X \) acting on \( \mathcal{L}_\lambda \) such that they globally preserve \( V^G_X \). Let \( u^{KZ} = u^{KZ}(\frac{\partial}{\partial \tau}) \).
We now describe the resulting connection \( \hat{\nabla} \) acting on the trivial \( H^0(\mathcal{M}, \mathcal{L}_\mathcal{M}) \)-bundle over \( \mathbb{P}^1 - \{0, 1, \infty\} \):
\[ \hat{\nabla} = \nabla^{t} - \hat{u} \]
where \( \hat{u} \) is a 1-form on \( \mathbb{P}^1 - \{0, 1, \infty\} \) with values in differential operators on \( \mathcal{M} \) acting on \( \mathcal{L}_\mathcal{M} \). Explicitly we get a formula for \( \hat{u}(\frac{\partial}{\partial \tau}) \) by considering
\[ X \supset \mu^{-1}(0) \to \mathcal{M} \]
and the splitting:
\[ T_x \mu^{-1}(0) = T_x(Gx) \oplus (T_x(Gx))^\perp \cong T_x(Gx) \oplus p^*(T_x\mathcal{M}). \]
of the tangent space of \( \mu^{-1}(0) \) into a 3-dimensional and a 2-dimensional subspace. Furthermore, we have that
\[ T_x X = I(T_x(Gx)) \oplus T_x \mu^{-1}(0) \]
where \( I \) is the complex structure on \( X \). On \( G \)-invariant section of \( \mathcal{L}_X \) which are also holomorphic, i.e. \( V^G \), we see that the derivatives in the direction of \( T_x(Gx) \) and \( I(T_x(Gx)) \) vanishes, hence we can rewrite the action of \( u^{KZ} \) as a second order differential operator which only differentiates in the direction of \( (T(Gx))^\perp \). Since we have \( G \)-invariance, we get this way an expression for \( \hat{u}(\frac{\partial}{\partial \tau}) \) as a second order differential operator.

**Proposition 2.** The symbol of the second order differential operator \( \hat{u}(\frac{\partial}{\partial \tau}) \) is holomorphic, i.e.
\[ \sigma(\hat{u}(\frac{\partial}{\partial \tau})) \in H^0(\mathcal{M}, S^2(T)) \]

**Proof.** We observe that
\[ S^2(T) \cong \mathcal{O}(4) \]
der under the identification of \( \mathcal{M} \cong \mathbb{P}^1 \). Next we observe that \( u^{KZ} \in H^0(X, S^2(T)) \) which then gives the stated result by reduction. \( \square \)

We now compare this geometric version of the KZ-connection with the Hitchin connection. Since \( (\mathcal{M}, \omega, I) \) is isomorphic to \( \mathbb{P}^1 \) as a complex manifold, we know there exists a smooth family of complex isomorphisms
\[ \Phi_\tau : (\mathcal{M}, I) \to \mathbb{P}^1 \]
varying smoothly with \( \tau \in \mathcal{T} \). By comparing Chern-classes, we see that
\[ \Phi_\tau^*(\mathcal{L}_\mathcal{M}) \cong \mathcal{O}(k_{\mathcal{M}}) \]
as holomorphic line bundles, for some \( k_{\mathcal{M}} \in \mathbb{Z} \) independent of \( \tau \in \mathcal{T} \). From this we also get
\[ G_\tau = \Phi_\tau^* \left( G \left( \frac{\partial}{\partial \tau} \right)_\tau \right) \in H^0(\mathbb{P}^1, S^2(TT\mathbb{P}^1)) \cong H^0(\mathbb{P}^1, \mathcal{O}(4)). \]
We observe that
\[ S_0^2(H^0(\mathbb{P}^1, \mathcal{O}(2))) \cong H^0(\mathbb{P}^1, \mathcal{O}(4)) \]
as representations of \( \text{SL}(2, \mathbb{C}) \). Here we think of \( S^2(H^0(\mathbb{P}^1, \mathcal{O}(2))) \) as quadratic forms on \( H^0(\mathbb{P}^1, \mathcal{O}(2)) \) and \( S_0^2 \) mean trace zero such.

**Theorem 20.** There exists
\[ \Psi : \mathcal{T} \to \text{SL}(2, \mathbb{C}), \]
such that if we define \( \tilde{\Phi}_\tau = \Psi(\tau) \circ \Phi_\tau \) and let
\[ \tilde{G}_\tau = \tilde{\Phi}_\tau^*(G \left( \frac{\partial}{\partial \tau} \right)_\tau) \in H^0(\mathbb{P}^1, \mathcal{O}(4)) \]
then
\[ \tilde{G}_\tau = \sigma(\hat{\mu}(\frac{\partial}{\partial \tau})). \]
Proof. We consider $S^2_1(H^0(\mathbb{P}^1, \mathcal{O}(2)))$ as a representation of $SL(2, \mathbb{C})$, where we think of $S^2_1(H^0(\mathbb{P}^1, \mathcal{O}(2)))$ as quadratic forms on $H^0(\mathbb{P}^1, \mathcal{O}(2)) = H^0(\mathbb{P}^1, T^\mathbb{P}^1)$, hence we consider elements of $S^2_0(H^0(\mathbb{P}^1, \mathcal{O}(2)))$ as symmetric symmetric traceless $3 \times 3$ complex matrices on which $SL(2, \mathbb{C})$ acts by conjugation. We have that two symmetric traceless $3 \times 3$ complex matrices are conjugate if and only if they have the same eigenvalues. An explicit computation shows that $\tilde{G}_\tau$ and $\sigma(\hat{u}(\frac{\tau}{2}))$ has the same eigenvalues, hence we can find the required map $\Psi$. \hfill □

Since $\Phi$ is such that the two symbols of the two second order differential operators defining the Hitchin connection and the geometric KZ-connection have been aligned, it follows from the form the Hitchin connection has, in order to preserve the subbundle of holomorphic sections that $\Phi$ must take the Hitchin connection to the KZ-connection. We further see that the Bohr-Sommerfeld decomposition corresponding to the limiting real polarizations, when $\tau$ approaches $-1$ and 1, corresponds to the factorization decomposition for the covariant constant sections of the sheaf of vacua constructed in [TUY].

Theorem 21. If $P_1$ and $P_2$ are pair of pants decompositions related by an elementary flip on a four-punctured sphere, then $\langle \cdot, \cdot \rangle_{\rho, \sigma_0}$ and $\langle \cdot, \cdot \rangle_{\rho, \sigma_0}$ are projectively equivalent.

Proof. The projective equivalence is obtained by the tensor product of the parallel transport discussed above on the four-punctured sphere in question with the identity on the complementary part in the factorization. The fact that this is a projective equivalence follows from the above arguments identifying the parallel transport of the Hitchin connection with the KZ-connection, which by the results of [AU1] [AU2] [AU3] [AU4] know is an isometry, since the corresponding flip transformation in the Reshetikhin-Turaev TQFT is an isometry. \hfill □

6. THE ONCE PUNCTURED GENUS ONE CASE

Consider the specific case of a torus with a single puncture, $\hat{\Sigma}$ (in the notation above). Let $N_{c_0}$ be the moduli space of flat connections on $\hat{\Sigma}$ with $c_0 \in [-2, 2]$ the holonomy around the puncture. The generators of the fundamental group are the curves $a, b, c$ being the longitude, meridian and a small curve around the puncture. The fundamental group of $\hat{\Sigma}$ is $\pi_1(\hat{\Sigma}) = \langle a, b, c \mid aba^{-1}b^{-1}c \rangle$.

Let $\rho : \pi_1(\hat{\Sigma}) \to SU(2)$ be a SU(2)-representation of $\pi_1(\hat{\Sigma})$. Define $A = \rho(a)$, $B = \rho(b)$ and $C = \rho(c)$. We describe the moduli space by determining each of the fibers of the trace $\text{Tr} : N \to [-2, 2]$.

The case where $C$ corresponds to minus the identity (i.e. $\text{Tr}(C) = -2$) is the same as removing the puncture. Now since $a, b$ commute in $\pi_1(\hat{\Sigma})$ we have $AB = BA$. Every element of SU(2) can be diagonalized, so as SU(2) acts on the representation variety by diagonal conjugation we assume $A$ to be diagonal. Assume also that $A$ has distinct eigenvalues. Then the only element $B$ that commutes with $A$ are diagonal matrices. Hence $A$ and $B$ can be simultaneously diagonalised to be elements of $S^1$. We can however still conjugate $A$ and $B$ by elements of the Weyl group and still stay within $S^1 \subset SU(2)$ (this amounts to changing the order of the eigenvalues), so $N_1(\hat{\Sigma}) = S^1 \times S^1 / \mathbb{Z}_2$. In the case of $A$ or $B$ not having two distinct eigenvalues the above description is still valid; generally however these non-generic cases correspond to singular points of the moduli space.
Let $a, b, c$ be curves as above. The trace provides coordinates on the moduli space, so let $\rho$ be a SU(2)-representation of the fundamental group, and define $x = \text{Tr}(\rho(a))$, $y = \text{Tr}(\rho(b))$ and $z = \text{Tr}(\rho(ab))$. The moduli space is a subset of $[-2, 2]^3$ carved out by the relation from the presentation of the fundamental group. Now fix the holonomy around $c$ to be $C \in SU(2)$. By the relation $ABA^{-1}B^{-1} = C$, and it is a simple check that the following identity is satisfied for any $A, B \in SU(2)$:
\begin{equation}
\text{Tr}(ABA^{-1}B^{-1}) = \text{Tr}(A)^2 + \text{Tr}(B)^2 + \text{Tr}(AB)^2 - \text{Tr}(A)\text{Tr}(B)\text{Tr}(AB) - 2.
\end{equation}

In other words the moduli space with fixed holonomy around $c$ is
\[ N_{c_0}(\hat{\Sigma}_\sigma) = \{(x, y, z) \in [-2, 2]^3 \mid x^2 + y^2 + z^2 - xyz - 2 = c_0\}, \]
which is topologically a sphere, for all values of $c_0 \in (-2, 2]$.

We expect that we can find an argument completely parallel to the one given above in the genus zero case, since the moduli space is again a sphere. However we do not strictly need this, since by [AU3], we know that the genus zero part of a modular functor determines $S$-matrix, which is the need equivalence in this case. By the result of the previous section, we know that the quantization of the moduli spaces of does indeed give a modular functor which in genus zero is isomorphic to the one constructed in [AU2] for the Lie algebra of $SU(2)$. Hence we have the following theorem.

**Theorem 22.** If $P_1$ and $P_2$ are pair of pants decompositions related by an elementary flip on a once punctured torus, then $[\cdot, \cdot]_{P_1, \sigma_0}$ and $[\cdot, \cdot]_{P_2, \sigma_0}$ are projectively equivalent.

7. **The Hermitian structure and and the Handlebody vectors**

We recall the setting from the introduction, where $\Sigma$ is a closed oriented surface of genus $g > 1$ and $P$ is a pair of pants decomposition of $\Sigma$. Recalling the map $(??)$, we define the representative $[\cdot, \cdot]_{P}^{(k)}$ of $(\cdot, \cdot)^{(k)}$ determined by $P$ by the formula
\[ [s_1, s_2]_{P, \sigma_0}^{(k)} = (P_{\infty}(\sigma_0, P)(s_1), P_{\infty}(\sigma_0, P)(s_2))_{P}^{(k)}, \]
for all $s_1, s_2 \in H_{\sigma_0}^{(k)}$.

**Theorem 23.** The Hermitian structure $[\cdot, \cdot]_{P}^{(k)}$ is projectively preserved by the Hitchin connection.

**Proof.** We consider two arbitrary complex structures $\sigma_1$ and $\sigma_2$. Parallel transport along any curve from $\sigma_1$ to $\sigma_2$ is invariant up to scale under perturbation of the curve, hence the curve can be deformed to the canonical curve from $\sigma_1$ to $P$ and composed with the reverse of the canonical curve from $\sigma_2$ to $P$ without changing the projective class of the parallel transport. But by the definition of $[\cdot, \cdot]_{P}^{(k)}$, the result now follows. \qed

**Theorem 24.** For any two pair of pants decompositions $P_1$ and $P_2$ on $\Sigma$, any complex structure on $\sigma_0$ on $\Sigma$ and any level $k$, we have that $[\cdot, \cdot]_{P_1}^{(k)}$ and $[\cdot, \cdot]_{P_2}^{(k)}$ induce the same projective unitary structure on $H^{(k)}$.

**Proof.** This is an immediate consequence of Theorem 21 and 22. \qed
Theorem 6 now follows completely similarly, the isomorphism given by parallel transport, discussed above in the proof of Theorem 22, is identified with the corresponding isomorphism in the Reshetikhin-Turaev TQFT, hence they take the vector corresponding to the zero label of the graph for $P_1$ to the same for $P_2$. But then we can conclude Theorem 6. More generally, we see that the handlebody vector defined in Definition 4 under the isomorphism $I_\Sigma$ is taken to the corresponding vector in the Reshetikhin-Turaev TQFT.

But then, by combining the above, we have established Theorem 7. Having established geometric constructions for the Handlebody boundary vectors and the unitary structure as explained above, we can combine this to conclude Theorem 9, where we will determine the constants $c_{g}(k)$ in the following section.

Let now define a first order approximation to the boundary states $s^{(k)}_{H,P}$, where $H$ is a Handlebody whose boundary is identified with $\Sigma$ and $P$ is a pair of pants decomposition of $\Sigma$.

Consider the linear functional on $H^{(k)}_{\sigma}$ given by

$$s_{H}(s) = \sum_{b \in B^{(k)}_{\sigma}} \int_{x \in h^{-1}P(b)} \langle s(x), s^{(k)}_{H,P}(P)(x) \rangle \text{Vol}_{\sigma,b}(x).$$

Now let $s^{(k)}_{H,\sigma} \in H^{(k)}_{\sigma}$ be the state associated to this functional,

$$(s, s^{(k)}_{H,\sigma}) = s_{H}(s),$$

for all $s \in H^{(k)}_{\sigma}$.

**Theorem 25.** We have the following norm estimate

$$\left| \frac{s^{(k)}_{H,P}(\sigma)}{|s^{(k)}_{H,\sigma}|} - \frac{s^{(k)}_{H,\sigma}}{|s^{(k)}_{H,\sigma}|} \right| = O(1/k).$$

**Proof.** This theorem follows by the same arguments as presented in [A11], since the asymptotics in Teichmüller space is analytically equivalent to the large $k$ asymptotics. \hfill \Box

Suppose now that $e^{(k)}_{\alpha,\sigma} \in H^{(k)}_{\sigma}$ is the coherent state associated to $\alpha \in \mathcal{L}$. Now define $e^{(k)}_{H,\sigma} \in H^{(k)}_{\sigma}$ as follows

$$e^{(k)}_{H,\sigma} = \int_{x \in h^{-1}_{\sigma}(b)} e^{(k)}_{s^{(k)}_{H,P}(P)(x),\sigma} \text{Vol}_{\sigma,b}(x).$$

**Theorem 26.** We have the following norm estimate

$$\left| \frac{e^{(k)}_{H,\sigma}}{|e^{(k)}_{H,\sigma}|} - \frac{s^{(k)}_{H,\sigma}}{|s^{(k)}_{H,\sigma}|} \right| = O(1/k).$$

**Proof.** This theorem is a simple calculation and it is proved using the same techniques as presented in [A11], where we analyzed the asymptotic $k$-behavior of coherent states following on from the techniques used in [A6]. \hfill \Box
8. The geometric formula for the Reshetikhin-Turaev quantum invariants

Recall that the SU(2)-quantum invariant $Z_k(M)$ is defined via surgery. We briefly recall the construction and its relation with the coloured Jones polynomials.

Let $L$ be an $m$-component banded link in $S^3$. Choose a regular closed neighbourhood $U$ of $L$, consisting of $m$ disjoint solid tori $U_1, \ldots, U_m$. Each of these are homeomorphic to $S^1 \times D^2$ with boundary homeomorphic to $S^1 \times S^1$. Choose homeomorphisms $h_i : S^1 \times S^1 \to S^1 \times S^1$ and form the space

$$M_L = (S^3 \setminus U) \cup h_i \left( \bigcup_{i=1}^m D^2 \times S^1 \right)$$

which is the disjoint union of $S^3 \setminus U$ and $m$ copies of solid tori $D^2 \times S^1$, these two spaces being identified along their common boundary $\bigcup_{i=1}^m S^1 \times S^1$ using the homeomorphisms $h_i$. The resulting topological space $M_L$ is a closed orientable manifold.

The space $M_L$ constructed as such depends of course on the homeomorphisms involved in the gluing. Using the banded structure of $L$, one canonically obtains particular homeomorphisms $h_i$ depending only on $L$, and the resulting surgery is referred to as integral surgery; in particular, one could talk about surgery along a non-banded link together with an integer called the framing, of which there is a canonical 0-framing. Using this, we say that $M_L$ is obtained by surgery on $S^3$ along $L$.

**Theorem 27** (Lickorish, Wallace). Any closed connected oriented 3-manifold can be obtained by (integral) surgery on $S^3$ along a banded link.

Now, Kirby showed that two links give rise to the same 3-manifold if and only if they are related by certain Kirby moves. Thus, any invariant of banded links, invariant under Kirby moves, gives rise to a 3-manifold invariant, and this is where the coloured Jones polynomials enter the picture. Let $k \in \mathbb{N}$ and let $0 \leq n_1, \ldots, n_m \leq k$ be integers (corresponding to simple modules of dimensions $n_1, \ldots, n_m$ of the quantum group $U_q(\mathfrak{sl}_2)$ for $q = \exp(2\pi i/k)$). Then the coloured Jones polynomial of an $m$-component link $L$, as defined e.g. using the modular category construction of $\mathbf{H}$, is a complex number $J_L(n_1, \ldots, n_m) \in \mathbb{C}$, depending on $n_1, \ldots, n_m$ and $k$. For the special case where $L = U$ is the unknot with framing 0, $J_U(n) = [n]$, the $n$th quantum integer. Now, the coloured Jones invariant of a link is not invariant under the Kirby moves, but the average of the first $k$ coloured Jones polynomials turn out to be. Specifically, define

$$A_k(L) = \sum_{0 \leq n_1, \ldots, n_m \leq k} \left( \prod_{l=1}^m n_l \right) J_L(n_1, \ldots, n_m).$$

Then

$$Z_k(L) = \Delta^{\sigma(L)} D^{-\sigma(L) - m - 1} A_k(L)$$

is an invariant of banded links, invariant under the Kirby moves, thus giving rise to a 3-manifold invariant $Z_k(M_L)$, called the quantum SU(2)-invariant of $M_L$. Here, $U_+$ and $U_-$ denotes the banded unknot with a positive, respectively negative, twist, and for an oriented banded link $L$, we write $\sigma(L) = \sigma_+(L) - \sigma_-(L)$, where $\sigma_+(L)$, $\sigma_-(L)$ are the numbers of positive, respectively negative, eigenvalues of the linking matrix $\text{lk}(L_i, L_j)$ consisting of linking numbers of the components; the linking number of a
banded knot $L_i$ with itself is defined to be the linking number of its boundary knots. Furthermore, $D$ is the rank of the underlying modular category, given explicitly by

$$D = \sqrt{\sum_{n=0}^{k} (n+1)^2} \frac{1}{\sin(\pi/(k+2))}.$$ 

and $\Delta$ is defined as in [T, Ch. II, 1.6],

$$\Delta = \sum_{n=0}^{k} \exp\left(\frac{(n+1)^2}{4k+8}\right) [n+1]^2.$$ 

Note that this invariant has framing anomalies, that is, it is only well-defined up to a choice of 2-framing of the 3-manifold (of which there always is a particular canonical choice, see [AI]). Changing this 2-framing will change the invariant by a factor of the root of unity $\Delta D^{-1} = \exp(-3\pi i/(2r)) \exp(3\pi i/4)$ to some power.

Assume now that $X$ is given as a Heegaard splitting $X = H \cup H'$, where $H, H'$ are genus $g$ handlebodies. Then the TQFT gluing axioms imply (see [T, Thm. IV.4.3]) that

$$Z_k(X) = (D\Delta^{-1})^m [Z_k(H'), Z_k(H)]$$

where here $m \in \mathbb{Z}$ is an integer depending on the decomposition $M = H \cup H'$ and which has a description in terms of Maslov indices (cf. [T, Thm. IV.4.3]). The interest of this lies in the fact that the vectors associated to the handlebodies correspond in the picture of trivalent graphs to those graphs having all edges coloured 0.

From the discussion in the previous section we conclude that

$$Z_k(X) = (D\Delta^{-1})^m \langle s_{H_1,H_2}^{(k)}(\sigma), s_{H_1,H_2}^{(k)}(\sigma) \rangle_{F_0,\sigma}.$$ 

But then by combining the results of theorem 25 and 26 together with Theorem 8

$$Z_k(X) \sim (D\Delta^{-1})^m \int_{x_1 \in h_{P_1}^{-1}(b_1)} \int_{x_2 \in h_{P_2}^{-1}(b_2)} \int_M \langle e^{(k)}_{H_1,P_1}(x_1), e^{(k)}_{H_2,P_2}(x_2), \sigma \rangle \operatorname{Vol}_{\sigma,b}(x) \operatorname{Vol}_{\sigma,b}(x) e^{-F_0} \omega^n \frac{n!}{n!}$$

where $\sim$ mean leading order asymptotics. But by using the usual asymptotic expansion of the coherent states on the smooth part $M_0'$, we see that we can isolate the contributions from the irreducible flat connections as integrals of the form discussed in the following section.

9. Asymptotics of quantum invariants for 1 surgeries on nontrivial knots

By the above formula, we see that we need to consider the asymptotic expansion of integrals of the form

$$\int_M e^{i\lambda f(x)} \varphi(x) \omega,$$
where $M$ is a Riemannian manifold, $\omega$ a volume form, $\varphi$ a smooth real valued function of compact support, $f$ a real polynomial and $\lambda$ a real parameter. We will consider asymptotics as $\lambda \to \infty$. In our case the support of $\varphi$ are confined to very small neighbourhoods around the critical points of $f$. Thus the integral is a sum of the contributions from each of these patches. All in all are we led to consider integrals of the form

$$\int_{\mathbb{R}^k} e^{i\lambda f(x)} \varphi(x) dx.$$ 

In the situations we consider $f$ will for Gauge theoretic reasons always be a polynomial. We cannot be certain that the critical point will be non-degenerate, so we need to use a different method than stationary phase approximation. We will use a method given in e.g. [Va] to describe the asymptotics of these oscillatory integrals with degenerate critical points.

Let $f$ be an analytic function of $k$ variables, $f : \mathbb{R}^k \to \mathbb{R}$, that is $f(x) = \sum c_k x^k$, and let $\varphi : \mathbb{R}^k \to \mathbb{R}$ be a smooth compactly supported function. Then under some mild non-degeneracy conditions on $f$ (described below) we have the following asymptotic expansion

$$\int_{\mathbb{R}^k} e^{i\lambda f(x)} \varphi(x) dx \sim e^{i\lambda f(o)} \sum_{p} \sum_{n=0}^{k-1} a_{p,n}(\varphi) \lambda^p (\ln \lambda)^n.$$ 

Here the $p$-sum is taken over a finite number of arithmetic progressions not depending on $\varphi$, and these progressions all consists only of negative rationals.

As we are only interested in the asymptotics of the integral, we are mainly concerned with the largest $p$ occurring in the arithmetic progressions and the corresponding coefficient. As we will see the largest $p$ can easily be read off from a Taylor expansion of $f$.

Since $f$ is assumed to be analytic, expand it in a Taylor series around 0, $f(x) = \sum a_n x^n$. The Newton polyhedron $\Gamma(f)$ is the convex hull of $\cup_{n \in \mathbb{N}^k \setminus \{0\}} \{n+\mathbb{R}_+^k\}$. Along with the polyhedron we consider the union of all the compact faces of the Newton polyhedron, the Newton diagram $\Gamma(f)$. The principal part of $f$ is $\sum_{n \in \Gamma(f)} a_n x^n$, and a principal part is said to be non-singular with respect to $\Gamma(f)$ if for any closed face of the Newton diagram $\gamma \in \Gamma(f)$, $f_\gamma = \sum_{n \in \gamma} a_n x^n$ has no critical points, i.e. $x^i \frac{\partial f_\gamma}{\partial x^i}, i = 1, \ldots, k$ do not vanish simultaneously on $\{x \in \mathbb{R} | x_1 \cdots x_k \neq 0\}$. We will from now assume that the phase functions have non-singular principal part with respect to their Newton diagram.

Let $(t_0, \ldots, t_0)$ be the intersection point of the Newton diagram $\Gamma(f)$ and the diagonal $x_1 = \cdots = x_k$. Let furthermore $\tau_0$ be the smallest face of $\Gamma(f)$ that contains the above intersection point. Lastly we define $s_0 = -1/t_0$ and $\rho$ to be the codimension of $\tau_0$ in $\mathbb{R}^k$.

As mentioned above (19) is the main theorem in [Va]. The theorem is proven by examining a related integral (mentioned below), which is constructed as a meromorphic extension and studying the singularities of this integral. By using the Newton polyhedron they are able to determine the arithmetic progressions and there by also the leading order term.
\[ \sum_{\rho} \sum_{n=0}^{k-1} a_{p,n}(\rho) \lambda^n (\ln t)^n = \mu(\rho) t^{n_0} (\ln \lambda)^{\rho - 1} + O(\lambda^{n_0} (\ln \lambda)^{\rho - 2}). \]

Note that the values \( s_0 \) and \( \rho \) directly can be read off from the Newton polyhedron.

Determining the coefficient \( \mu(\rho) \) is more complicated, and the following is a result from [DNS]. If \( s_0 \not\in \mathbb{Z} \) the coefficient \( \mu(\rho) \) can be calculated by the following formula:
\[
\mu(\rho) = \frac{1}{(\rho - 1)!} \Gamma(-s_0) \left( \mu_+(\rho) e^{\frac{\pi i s_0}{2}} + \mu_-(\rho) e^{\frac{\pi i s_0}{2}} \right).
\]

Note that since \( s_0 \not\in \mathbb{Z} \) \( \mu(\rho) = 0 \) if and only if \( \mu_\pm(\rho) = 0 \). Here \( \mu_\pm(\rho) \) are defined by the positive and negative part of \( f \), as follows. On \( D = \{ s \in \mathbb{C} \mid \text{Re}(s) > 0 \} \) we can define the functions
\[
I_\pm(s) = \int_{\mathbb{R}_+} f_\pm(x)^s \varphi(x) dx,
\]
where \( f_+ = \max(f, 0) \) and \( f_- = \max(-f, 0) \). \( I_\pm \) can be extended meromorphically to \( \mathbb{C} \) and are also denoted \( I_\pm \). It turns out that in order to understand the asymptotics of \( I(\lambda) \) we must understand the poles of \( I_\pm \). Varchenko’s approach showed that the arithmetic progressions were related to understanding the singular part of the Laurent expansion of \( I_\pm \) at its poles. \( \mu_\pm(\rho) \) is defined to be the coefficient the degree \(-\rho\) term in the Laurent expansion of \( I_\pm \) about \( s_0 \):
\[
I_\pm(s) = \frac{\mu_\pm(\rho)}{(s - s_0)^\rho} + O\left( \frac{1}{(s - s_0)^{\rho-1}} \right) \quad \text{for } s \to s_0.
\]

In [DNS] we find the following residue formula

**Theorem 28 (DNS).** Assume that \( f \) is non-singular with respect to its Newton polyhedron, and furthermore that \( \tau_0 \) is compact. When the support of \( \varphi \) is sufficiently small, then

\[
\mu_\pm(\rho) = k! \text{Vol}(C) \varphi(0) \text{PV} \int_{\mathbb{R}_+} f_{\tau_0}(1, \ldots, 1, y_{\rho+1}, \ldots, y_k)^n dy,
\]

where the principal value integral is defined as the value of the analytic continuation at \( t = 1 \) of the function
\[
K_\pm(t) = \int_{\mathbb{R}_+} f_{\tau_0}(1, \ldots, 1, y_{\rho+1}, \ldots, y_k)^s t^y dy,
\]
where \( K_\pm(t) \) is defined for \( t \in \mathbb{R}_+ \setminus \{0\} \), \( ts_0 > -1 \) and \( t \) sufficiently small. Here \( y = \prod_{i=\rho+1} y_i \) and \( dy = dy_{\rho+1} \wedge \cdots \wedge dy_k \).

\( C \) is a convex hull independent of \( \varphi \). Hence the only way \( \mu(\rho) \) depends on \( \varphi \) is by its value at 0.

From the above discussion, we conclude that if we consider a Heedgaard decomposition of a three manifold \( X = H_1 \cup \Sigma H_2 \), which is obtained by doing 1 surgery on a non-trivial knot, then the result of Kronheimer and Mrowka [1], gives a non-trivial representation of fundamental group of \( X \) to \( SU(2) \), showing that the two corresponding Lagrangians for \( H_1 \) and \( H_2 \) must intersect and therefore give a non-vanishing contribution to the asymptotics of the quantum invariant, hence proving Theorem [10].
A GEOMETRIC FORMULA FOR THE WITTEN-RESHETIKHIN-TURAEV INVARIANTS

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Center for Quantum Geometry of Moduli Spaces, University of Aarhus, DK-8000, Denmark

E-mail address: andersen@qgm.au.dk