SPLITTING OF RESONANT FREQUENCIES OF
ACOUSTIC WAVES IN ROTATING
COMPRESSIBLE FLUID

A. N. Tarasenko, A. A. Sokolsky
Belarusian State University, 4 Nezalejnosti Ave., 220030 Minsk, Belarus,
E-mail: sokolan@tut.by

It is shown that in a rotating compressible fluid the resonant frequencies
(measured in a system of reference rotating together with the medium)
for the azimuthally running acoustic waves are split into two compo-
nents. The received results can be of practical significance as a basis of
a method of measurements of angular speed of medium and for acoustics
of rotating technical devices.

As is known [1], in a compressible fluid, localized between two rigid coax-
ial cylindrical walls, free oscillations of the character of both standing and
running in an azimuthal direction waves are possible. With the help of the
equations

\[
\rho_0 \ddot{V} = -\nabla \tilde{p}, \quad \dot{\tilde{p}} = -\rho_0 c_0^2 \text{div} \tilde{V}
\]

(1)
it is easy to show that for the standing waves, in the absence of axial
component of oscillation speed \( \tilde{V} \) (i.e., for \( V_z = 0 \)), the acoustic
pressure \( \tilde{p} = p - p_0 \) looks like \( \tilde{p}(r, \varphi, t) = P_m(r) \cos m \varphi \cos \omega_{mn} t \), and for
the azimuthally running waves \( \tilde{p}(r, \varphi, t) = P_m(r) \cos (m \varphi - \omega_{mn} t) \). Here,
\( P_m(r) = A \ J_m(\omega_{mn} r / c_0) + B \ Y_m(\omega_{mn} r / c_0) \), \( A \) and \( B \) are the constants,
\( r, \varphi, z \) are the cylindrical coordinates, \( J_m \) and \( Y_m \) are the Bessel functions
[2], \( m \ n \) are the integers, \( p_0, \rho_0 \) and \( c_0 \) are the equilibrium pressure, density
and sound speed. The sets of resonant frequencies \( \omega_{mn} \) for both cases are
identical and can be found from the equation

\[
J'_m (\omega R_1 / c_0) Y'_m (\omega R_2 / c_0) = J'_m (\omega R_2 / c_0) Y'_m (\omega R_1 / c_0),
\]

(2)

where \( R_1 \) and \( R_2 \) are the radiuses of the walls, \( J'_m (x) = dJ_m / dx, \ Y'_m (x) =
dY_m / dx \).

Let us consider the behaviour of such waves on the background of the
equilibrium state of the medium rotating with a constant angular speed \( \bar{\Omega} \).
Nondissipative hydrodynamic processes in the rotating compressed liquid
(gas), considered from the point of view of the system of reference S rotating together with the fluid, are described by system of equations

\[ \dot{\vec{V}} + (\vec{V} \cdot \vec{\nabla}) \vec{V} = -\rho^{-1} \vec{\nabla} p + \Omega^2 \vec{r} + 2 [\vec{V} \times \vec{\Omega}], \]

\[ \dot{\rho} + \text{div}(\rho \vec{V}) = 0, \quad \dot{p} + \vec{V} \cdot \vec{\nabla} p = c^2 (\dot{\rho} + \vec{V} \cdot \vec{\nabla} \rho), \]  

where \( \vec{V} \) is the velocity of a medium element relative to S. Within the limits of linear acoustics and for relatively low speeds of rotation (i.e., for small value of \( M = \Omega R/c \)), system (3), in view of a condition of equilibrium \( \vec{\nabla} p_0 = \rho_0 \Omega^2 \vec{r} \), leads (instead of (1)) to the following equations:

\[ \dot{\vec{V}} = -\rho_0^{-1} \vec{\nabla} \tilde{p} + 2[\vec{V} \times \vec{\Omega}], \]  

\[ \dot{\tilde{p}} = -\rho_0 c_0^2 \text{div} \vec{V}. \]  

System (4), (5) describes adequately the corrections of the first order in \( M \), caused by the influence of the medium rotation on oscillations, if the conditions \( M \ll 1 \) and \( V_A/c \sim M^2 \) are satisfied (\( V_A \) is the amplitude of the velocity of oscillations). Thus the ratio \( \Omega/\omega \) for the resonant values of \( \omega \) will be of the order \( M \), and \( p_0, \rho_0, \) and \( c_0 \) in equations (4), (5) can be considered constant.

This allows to use system (4), (5), for example, for the frequencies of rotation of the order 1 – 10 r.p.s. and the radial sizes 0.2 – 0.5 m for the sound intensity corresponding to the values \( V_A/c < 10^{-5} \) for both gases and liquids.

It is easy to check that, in the approximation considered, elimination of \( \vec{V} \) from (4), (5) leads to the same differential equation for the acoustic pressure \( \tilde{p} \) as in the case of non-rotating medium:

\[ \ddot{\tilde{p}} = c_0^2 \nabla^2 \tilde{p}, \quad \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \]  

At the same time, the boundary condition for \( \tilde{p}(r, \varphi, t) \) on a rigid cylindrical wall of the radius \( R \) (that follows from the requirement \( V_r(r = R) = 0 \)) gets, by virtue of (4), an additional term, which have the first order in \( \Omega/\omega \):

\[ \left( \frac{\partial^2 \tilde{p}}{\partial r^2} + \frac{2\Omega}{r} \frac{\partial \tilde{p}}{\partial \varphi} \right)_{r=R} = 0. \]
As for the azimuthally running wave (i.e., for the harmonic of number $m = \pm 1, \pm 2, \pm 3\ldots$) the acoustic pressure $\tilde{p}(r, \varphi, t)$ looks like $P_m(r) \cos(m \varphi - \omega t)$, it follows from condition (7):

$$\left(\frac{dP_m}{dr} - \frac{2\Omega m}{\omega r} P_m\right)_{r=R_{1,2}} = 0,$$

where $R_1$ and $R_2$ are the radiiuses of rigid coaxial cylindrical walls between which the wave is localized.

According to equation (6), when $\Omega \neq 0$, $P_m(r)$ holds the form $A J_m(\omega r/c_0) + B Y_m(\omega r/c_0)$. Substituting $P_m(r)$ in (8), we receive, instead of (2), the condition

$$J'_m\left(\frac{\omega R_1}{c_0}\right) - \frac{2mc_0\Omega}{\omega^2 R_1} J_m\left(\frac{\omega R_1}{c_0}\right) = Y'_m\left(\frac{\omega R_1}{c_0}\right) - \frac{2mc_0\Omega}{\omega^2 R_1} Y_m\left(\frac{\omega R_1}{c_0}\right).$$

Values of $\omega$ satisfying (9) form a spectrum of the resonant frequencies of the azimuthally running waves: $\omega_{mn}$ at $\Omega = 0$, $\omega^+_{mn}$, if $m\Omega > 0$ (i.e., for a wave running in a direction of medium rotation) and $\omega^-_{mn}$, if $m\Omega < 0$.

Frequencies $\nu_{mn} = 2\pi \omega_{mn}$ and frequency deviations $\nu^+_{mn} - \nu_{mn}$ and $\nu^-_{mn} - \nu_{mn}$, where $\nu^\pm_{mn} = 2\pi \omega^\pm_{mn}$ obtained by a numerical solution of equation (9) at $R_1 = 0.2$ m, $R_2 = 0.5$ m, $\Omega = \pm2\pi N$, $N = 10$ r.p.s, and $c_0 = 330$ m/s (for $m=1, 2, 3; n=1, 2, 3, 4, 5$) are presented in Table 1.

| $m$ | 1  | 2   | 3   | 4   | 5   |
|-----|----|-----|-----|-----|-----|
| 1   | 153.55 | 594.45 | 1122.19 | 1664.72 | 2211.01 |
|     | - 3.91 | 1.32 | 0.43 - 0.43 | 0.20 - 0.20 | 0.11 - 0.11 |
| 2   | 298.57 | 673.95 | 1161.34 | 1690.30 | 2230.04 |
|     | - 5.24 | 4.06 | 2.29 - 2.39 | 0.86 - 0.87 | 0.40 - 0.40 | 0.23 - 0.23 |
| 3   | 431.53 | 791.52 | 1225.43 | 1732.52 | 2261.55 |
|     | - 6.12 | 5.43 | 2.25 - 2.40 | 1.27 - 1.29 | 0.60 - 0.61 | 0.34 - 0.34 |

Table 1: Frequencies $\nu_{mn}$ (Hz) and deviations: $\nu^+_{mn} - \nu_{mn}$, $\nu^-_{mn} - \nu_{mn}$ (Hz).

For a cylinder of the radius $R$ (i.e., for $R_1 = 0$, $R_2 = R$) the function $P_m(r)$ is reduced to $A J_m(\omega r/c_0)$, and we can receive the approximated expression for the resonant frequencies in an explicit form:
\[
\omega_{mn} + \Delta \omega_{mn} = \frac{c_0}{R} X_{mn} + \frac{2m \Omega J_m (X_{mn})}{X_{mn}^2 J''_m (X_{mn})},
\]

(10)

where \( X_{mn} \) are the roots of the equation \( J'_m (x) = 0 \).

According to (10), ratio of splitting of frequencies to angular speed of medium in this case is of a universal character:

\[
f_{mn} = \frac{\omega_{mn}^+ - \omega_{mn}^-}{\Omega} = \frac{4m J_m (X_{mn})}{X_{mn}^2 J''_m (X_{mn})}.
\]

(11)

Table 2 gives the values of the roots \( X_{mn} \) and of the “splitting coefficients” \( f_{mn} \).

| m\n | 1     | 2     | 3     | 4     | 5     |
|-----|-------|-------|-------|-------|-------|
| 1   | 1.841 | 5.331 | 8.536 | 11.706| 14.864|
|     | -1.674| -0.146| -0.056| -0.029| -0.018|
| 2   | 3.054 | 6.706 | 9.969 | 13.170| 16.348|
|     | -1.501| -0.195| -0.084| -0.047| -0.030|
| 3   | 4.201 | 8.015 | 11.346| 14.586| 17.789|
|     | -1.387| -0.217| -0.100| -0.059| -0.039|

Table 2: Values of \( X_{mn} \) and \( f_{mn} \) for \( m = 1, 2, 3; \ n = 1, 2, 3, 4, 5 \).

As the frequencies of the running waves following and opposing a direction of rotation are not equal, it is impossible to construct a standing wave as some their superposition. The absence of standing waves for \( m \Omega \neq 0 \) follows directly from impossibility to satisfy boundary conditions (7) by the pressure \( \tilde{p}(r, \varphi, t) \) of the form \( P_m (r) \cos m \varphi \cos \omega t \). For \( m \Omega \neq 0 \), a pulse of waves moving in an azimuthal direction with the angular speed \( \Omega_{\text{puls}} \), which is equal to \( (\omega_{mn}^+ - \omega_{mn}^-)/(2m) \) relative to rotating medium, will occur instead of a standing wave.

The splitting of sound frequencies in a rotating medium can be, in some sense, considered as an acoustic analog of the Sagnac effect in optics [3].

For values of parameters \( M = \Omega R/c \) and \( V_{A}/c \), which are rather greater than considered above, taking into account of nonlinear members in equations and corresponding correction of the received results are necessary. We expect, however, that, at least for weak nonlinearity, the effect of splitting of frequencies will be of similar character, and that the resonant response of medium will begin to be also revealed on the frequencies of pulses.
References

[1] Isakovich M. A. General Acoustic. Moscow. 1973. P. 271.

[2] Handbook of mathematical functions. Edited by M. Abramovitz, I. Stegun, NBSAMS, 1964.

[3] Landau L. D., Lifshits E. M. Theory of Field. Moscow. 1988. P. 326.