On the derivation of the neutrino oscillation length formula.

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Abstract

Forsaking the traditionnal hand-waving in the treatment of the motion, we show that the ultra-relativistic approximation and the equality of kinematical variables are unnecessary ingredients in the derivation of the oscillation length using plane waves, at least in a two flavor world. It ensues that the formula is valid as it is in the non relativistic regime, provided one uses the correct variable which is found to be momentum, not energy, and that the precise production kinematics is irrelevant. Consequences for the more complete treatments are briefly evoked.

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The expression of the neutrino oscillation length is usually derived in the ultra-relativistic regime using a superposition of plane waves supposed to have either equal momenta or equal energies. In the course of the derivation, the approximation $t \approx x$ is often made (see e.g. [1]) to identify the oscillating pattern as a function of the distance from the production point. It is readily seen, however, that the correct result obtains in such a treatment because the equal $p$’s or equal $E$’s hypothesis reduces the time dependence or the space dependence of the oscillation amplitude to an overall phase factor which disappears upon calculating probabilities. We will show that making the $t \approx x$ ‘approximation’ is unnecessary and can lead to senseless results if one uses the equal velocity hypothesis, which is admittedly neither better nor worse than the above mentioned two other possibilities.

The more correct treatment given below hinges on a definition of the (pseudo) ‘center’ of the would-be wave packet and shows that the usual oscillation length also obtains in the non relativistic case. In the end, it allows to get rid of any hypothesis about the kinematics of the production process.

The meaning of these findings is, however, particularly clear in a simplified two-flavor world, where there is but one oscillation length. The relevance of all this to a more realistic situation is briefly discussed.

1 Three derivations

Let us represent the neutrino born at space-time point $(0,0)$ in some charged current reaction involving charged lepton $l$ by

$$|0,0> = |\nu_l> = \sum_h U_{lh}|h> \tag{1}$$

where $|h>$ is a mass eigenstate with eigenvalue $m_h$ and definite energy and momentum. The fate of this neutrino is governed by the space-time translation operator

$$\mathcal{U} = e^{-i(Ht-\vec{P}\cdot\vec{r})}$$

and the problem amounts to correctly evaluate its action on (1):

$$\mathcal{U}|0,0> = |x,t> = \sum_h U_{lh}e^{-i(E_h t - p_h x)}|h> \tag{2}$$

where we have assumed that the propagation is along the $x$ axis. The precise values of $E_h$ and $p_h$ in this formula depend on the production kinematics. In the case of a $\pi l_2$ decay for example, they are fixed by the masses in the $\pi$ rest-frame and from there in the lab, once the $\pi$’s decay angle and velocity are given.
Projection of $|x, t>$ onto $|\nu_l>$ yields then

$$A(l \rightarrow l')(x, t) = \sum_h U^*_{lh} U_{hl} e^{-i(E_h t - p_h x)}$$

(3)

for the amplitude to detect a neutrino of flavor $l'$ at point $x$ and time $t$ relative to the production point at $(0, 0)$, granted that the neutrino interacts. 

Henceforth, we shall assume a two flavor-two mass world, which simplifies the matter greatly. 

The all-important object is then the one phase difference which appears upon squaring (3):

$$P(l \rightarrow l' \neq l)(x, t) = \sin^2(2\theta) \sin^2 \left( \frac{\delta\phi}{2} \right)$$

(4)

where we have reverted to the simplified notation in use in the two flavor world and abbreviated: $\delta\phi = \delta E t - \delta p x$

In order to make contact with experiments which register the coordinates but not the time and because the object of study should be more properly described by some more or less localized wave function, a connection between $x$ and $t$ must be made at some point to describe what shall be considered as the motion of the center of the wave packet. Also, the phase difference should be expressed in terms of the quantities which are really at stake, viz. the masses and a single kinematical value representing the average energy or momentum of the beam. To this effect, various supplementary hypotheses are added to the basic ingredient represented by formula (2)

None of these is necessary in the case at hand as we shall see now, provided the first problem is properly treated.

### 1.1 Equal energies

It is assumed here that the two massive components have the same energy and different momenta; then $\delta\phi = -\delta p x$ and the oscillation pattern is described by:

$$O(x) = \sin^2 \left( \frac{\delta p x}{2} \right)$$

Invoking a relativistic situation, people usually expand $p \approx E - \frac{m^2}{2E}$ hence $\delta\phi \approx \frac{\delta m^2}{2E} x$ and

$$O(x) \approx \sin^2 \left( \frac{\Delta m^2}{2E} x \right)$$

(5)

However, this is unnecessary since the exact relation:

$$\delta p^2 = \delta E^2 - \delta m^2$$

(6)

\footnote{\(\delta\) is a true, signed, difference, and \(\Delta\) is an absolute value}
yields in this case:
\[ \delta p = -\frac{\delta m^2}{\Sigma p} = -\frac{\delta m^2}{2\bar{p}} \]
with an obvious definition for \( \bar{p} \).
Consequently:
\[ O(x) = \sin^2\left(\frac{\Delta m^2}{4\bar{p}} x\right) \quad \text{and} \quad L_{osc} = \frac{4\pi\bar{p}}{\Delta m^2} \quad (7) \]

Simple as is it, the meaning of this procedure is completely clear only in the case of two masses, since in the more general situation we could not introduce the third momentum into the definition of \( \bar{p} \).

1.2 Equal momenta

Here, \( \delta \phi = \delta Et \) and using again a first order expansion, this becomes: \( \delta \phi \approx \frac{\delta m^2}{2p} t \) after which the further ‘relativistic approximations’ \( t \to x \) and \( p \to E \) allow to find consistency with the approximate result (7).

This again is unnecessary because (7) yields here \( \delta E = \frac{\delta m^2}{\Sigma E} \) and, upon defining the velocity \( v \) of the center of the would-be wave packet, one finds:
\[ \delta \phi = \frac{\delta m^2}{\Sigma E} \frac{x}{v} = \frac{\delta m^2}{2p} x \]
provided
\[ v = \frac{2p}{E_1 + E_2} \]
which shall be justified presently, but is seen to agree with the arithmetic mean up to and including first degree terms in the small quantity \( \frac{\Delta E}{\Sigma E} \).

Hence
\[ O(x) = \sin^2\left(\frac{\Delta m^2}{4\bar{p}} x\right) \quad \text{and} \quad L_{osc} = \frac{4\pi\bar{p}}{\Delta m^2} \quad (8) \]

exactly as in (1.1), granted the definition used for \( v \), and with an analogous restriction since only two energies must be considered in defining \( v \).

1.3 Equal velocities

In this case, \( \delta \phi \) does not reduce to a single term and it is very important not to approximate \( t \) by \( x \), for in so doing one would arrive at:
\[ \delta \phi = (\delta E - \delta p)x = \delta m e^{-\eta} x \]
upon introducing \( \eta = \tanh^{-1}(v) \)
The oscillating pattern would be described by:

\[ \sin^2 \left( \frac{\Delta m}{2} e^{-\eta x} \right) \]

and the oscillation length:

\[ L'_{\text{osc}} = \frac{2\pi \epsilon \eta}{\Delta m} \] (9)

However, since

\[ e^{\eta} = \frac{E_1 + p_1}{m_1} = \frac{E_2 + p_2}{m_2} \approx \frac{4p}{m_1 + m_2} \]

this yields finally

\[ L'_{\text{osc}} = \frac{8p\pi}{\Delta m^2} \] (10)

viz. twice the usual value.

Confronted with this result, people have been tempted to think that the standard formula is either false or does not apply in the case at hand. A more careful treatment of the motion of the would-be wave packet shows that this is not correct. Indeed, if the hypothesis of equal velocities has any meaning, then the center of the wave packet moves with \textit{that} velocity, not with velocity \( v \). Therefore, defining its position by \( x = vt \) yields:

\[ \delta \phi = \delta E(t - vx) = \delta m \gamma (1/v - v) x = \frac{\delta m}{v \gamma} x \]

Now

\[ \frac{1}{v \gamma} = \frac{m_1}{p_1} = \frac{m_2}{p_2} = \frac{\Sigma m}{2\bar{p}} \]

Hence \( \delta \phi = \frac{\delta m^2}{2\bar{p}} x \) and the correct formula found in (1.1) results. The same restriction as before applies, since the third momentum cannot enter the definition of \( \bar{p} \).

Clearly, replacing \( 1/v - v \) by \( 1 - v \) (equivalent to \( t \to x \)) cannot be harmless; in (1.2), \( v \) is only an overall factor, and replacing it by 1 induces a relative error on the phase shift which goes to 0 with \( 1 - v \). Not so in the present case where the relative error is \( 1/(1 + v) \) - hence the factor 2 found above. Stated differently, \( v \) and \( E \) were treated separately in (1.2) but here the connection between \( m \), \( v \) and \( E \) (or \( p \)) must be used.

2 ..and a fourth one.

First observe that all three derivations above use exact relativistic kinematics but that none uses any sort of ‘ultra-relativistic approximation’, especially not \( ^{2\gamma} = 1/\sqrt{1-v^2} \) as usual.
the ubiquitous but very unreasonable \( x \approx t \). A moment of reflexion reveals their common feature: in all three cases, the center of the would-be wave packet is endowed with the average momentum and energy of the components: \( p^c = \bar{p}, \ E^c = \bar{E} \), and it is assumed to have velocity \( v = \frac{\bar{p}}{\bar{E}} \). This is the real justification behind the definition of \( v \) in (1.2).

One is thus led to think that this is all that is needed to yield the well-known \( L_{osc} \); indeed, barring any ad hoc hypothesis on the production kinematics:

\[
v = \frac{p_1 + p_2}{E_1 + E_2} \Rightarrow \delta \phi = \delta px - \delta Et = (\delta p - \delta E \frac{\Sigma E}{\Sigma p})x = \frac{(\delta p^2 - \delta E^2)x}{\Sigma p}
\]

\[
= -\frac{\delta m^2x}{2\bar{p}}
\]

which proves our point.

3 Lessons

The usefulness of all this is of course lessened by the well known shortcomings of the use of plane waves for the purpose of describing neutrino oscillations (see e.g. [5] for a list of these); the necessity of using wave packets (see e.g. [2]) or field theory ([3],[4]) has been the subject of a long and still ongoing debate and many sophisticated treatments have appeared over the years. However, these more elaborate methods together with the inclusion of the neutrino production and/or detection processes in the description of the phenomenon ([3],[4]) all result in formulae which are subtended by the basic oscillation pattern described by (7), provided there exists a middle-zone where coherence is not lost but finite source length and momentum spread effects are negligible. In all cases, the same (vacuum) oscillation length obtains when and where resolution or decoherence do not blur the oscillations.

The above demonstration sheds some light on this robustness of the classical formula, by showing that none of the extra hypotheses usually made is necessary, at least in the two flavor world where the oscillation length has its clearest meaning. Buried at the heart of the more sophisticated treatments is always some definition of the \( x \leftrightarrow t \) relationship which avoids the hand-waving \( x \approx t \).

It is also seen that the relevant variable is the momentum, not the energy, when the distinction applies; provided one uses this variable, the standard result seems to follow also in the non relativistic regime. This might be usefull if slow, Karmen-anomaly-like objects [8] are confirmed; consideration of non-relativistic effects in oscillations due to such states have already appeared in the litterature (see [9])
A note of caution is, however, in order: it is not entirely clear that an oscillation probability - and therefore, an oscillation length- has a meaning in itself in the non-relativistic regime; the wave packet treatment (except in its most primitive form) and the field theoretical approach both include the production and/or detection processes in an overall probability calculation which generally factorizes in the relativistic regime, where all masses are small with respect to the kinetic energy scale. It should be determined under what conditions this also applies in the non relativistic regime. Since oscillations can only occur in the case of nearly degenerate mass states, the production phase-space mass dependence should not be of concern, but the detection reaction is a potential source of problem in that, e.g. the $\nu_e$ and possibly the $\nu_\mu$ component can be inhibited for lack of energy; one can really appreciate at this point how much ill-defined are the so called ‘weak eigenstates’ (besides the fact that they are not eigenstates of any operator which distinguishes them from the mass eigenstates!)

Moreover, one must expect a larger yield of ‘wrong’ helicity when $\gamma \to 1$ and therefore an additional entanglement of ‘flavor’ with the other variables which might further preclude the definition of an ‘oscillation probability’ disconnected from the rest of the process [6].

4 Summary and conclusion

Simple calculations show that none of the hypotheses usually employed in deriving the neutrino oscillation length in the plane wave formalism is necessary and that the only requirement is a proper treatment of the motion of the would-be wave packet, at least in a two-mass world. This should apply to real life whenever the number of active mass states is reduced to two, in particular when production phase-space is restricted or in case of degeneracy.

Although a plane wave treatment of neutrino oscillations is, admittedly, an over-simplification, we believe that what has been done here has the merit of giving some clues in answering questions as to what are the conditions that should be met to allow observation of oscillations or that should be hypothetized in a sensible theoretical treatment. In particular, it casts some shadow on the relevance of the equal energies versus equal momenta arguments that have appeared in the litterature, given that the one important ingredient seems to be a suitably defined velocity for the center of the wave packet representing the object under study.
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