Perfect divisibility and 2-divisibility

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Abstract
A graph $G$ is said to be 2-divisible if for all (nonempty) induced subgraphs $H$ of $G$, $V(H)$ can be partitioned into two sets $A, B$ such that $o(A) < o(H)$ and $o(B) < o(H)$. (Here $o(G)$ denotes the clique number of $G$, the number of vertices in a largest clique of $G$). A graph $G$ is said to be perfectly divisible if for all induced subgraphs $H$ of $G$, $V(H)$ can be partitioned into two sets $A, B$ such that $H[A]$ is perfect and $o(B) < o(H)$. We prove that if a graph is $(P_3, C_5)$-free, then it is 2-divisible. We also prove that if a graph is bull-free and either odd-hole-free or $P_5$-free, then it is perfectly divisible.

KEYWORDS
2-divisibility, graph coloring, perfect divisibility

1 | INTRODUCTION

All graphs considered in this article are finite and simple. Let $G$ be a graph. The complement $G^c$ of $G$ is the graph with vertex set $V(G)$ and such that two vertices are adjacent in $G^c$ if and only if they are nonadjacent in $G$. For two graphs $H$ and $G$, $H$ is an induced subgraph of $G$ if $V(H) \subseteq V(G)$, and a pair of vertices $u, v \in V(H)$ is adjacent if and only if it is adjacent in $G$. We say that $G$ contains $H$ if $G$ has an induced subgraph isomorphic to $H$. If $G$ does not contain $H$, we say that $G$ is $H$-free. For a set $X \subseteq V(G)$ we denote by $G[X]$ the induced subgraph of $G$ with vertex set $X$. For an integer $k > 0$, we denote by $P_k$ the path on $k$ vertices, and by $C_k$ the cycle on $k$ vertices. A path in a graph is a sequence $p_1 \cdots p_k$ (with $k \geq 1$) of distinct vertices such that $p_i$ is adjacent to $p_j$ if and only if $|i - j| = 1$. Sometimes we say that $p_1 \cdots p_k$ is a $P_k$. A hole in a graph is an induced subgraph that is isomorphic to the cycle $C_k$ with $k \geq 4$, and $k$ is the length of the hole. A hole is odd if $k$ is odd, and even otherwise. The vertices of a hole can be numbered $c_1, \ldots, c_k$ so that $c_i$ is adjacent to $c_j$ if and only if $|i - j| \in \{1, k - 1\}$; sometimes we write $C = c_1 \cdots c_k$. An antihole in a graph is an induced subgraph that is isomorphic to $C^c_k$ with $k \geq 4$, and again $k$ is the length of the antihole. Similarly, an antihole is odd if $k$ is odd, and even otherwise. The bull is the graph consisting of a triangle with two disjoint pendant edges. A graph is bull-free if no induced subgraph of it is isomorphic to the bull. The
chromatic number of a graph $G$ is denoted by $\chi(G)$ and the clique number by $\omega(G)$. A graph $G$ is called \textit{perfect} if for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. For a set $X$ of vertices, we will usually write $\chi(X)$ instead of $\chi(G[X])$, and $\omega(X)$ instead of $\omega(G[X])$. If $X$ is a set of vertices and $x$ is a vertex, we will write $X + x$ for $X \cup \{x\}$.

A graph $G$ is said to be \textit{2-divisible} if for all (nonempty) induced subgraphs $H$ of $G$, $V(H)$ can be partitioned into two sets $A, B$ such that $\omega(A) < \omega(H)$ and $\omega(B) < \omega(H)$. Hoàng and McDiarmid \cite{8} defined the notion of 2-divisibility. They actually conjecture that a graph is 2-divisible if and only if it is odd-hole-free. A graph is said to be \textit{perfectly divisible} if for all induced subgraphs $H$ of $G$, $V(H)$ can be partitioned into two sets $A, B$ such that $H[A]$ is perfect and $\omega(B) < \omega(H)$. Hoàng \cite{7} introduced the notion of perfect divisibility and proved (\cite{7}) that (banner, odd hole)-free graphs are perfectly divisible. A nice feature of proving that a graph is perfectly divisible is that we get a quadratic upper bound for the chromatic number in terms of the clique number. More precisely:

\textbf{Lemma 1.1.} Let $G$ be a perfectly divisible graph. Then $\chi(G) \leq \binom{\omega(G) + 1}{2}$.

\textit{Proof.} Induction on $\omega(G)$. Let $\omega(G) = t$. Let $X \subseteq V(G)$ such that $G[X]$ is perfect and $\chi(G \setminus X) < t$. Since $G \setminus X$ is perfectly divisible, $\chi(G \setminus X) \leq \binom{\omega(G) - 1}{2} = \binom{t}{2}$. Since $G[X]$ is perfect, $\chi(X) \leq t$. Consequently, $\chi(G) \leq \chi(G \setminus X) + \chi(X) \leq t + \binom{t}{2} = \binom{t + 1}{2}$. ■

Analogously, 2-divisibility gives an exponential $\chi$-bounding function.

\textbf{Lemma 1.2.} Let $G$ be a 2-divisible graph. Then $\chi(G) \leq 2^{\omega(G) - 1}$.

\textit{Proof.} Induction on $\omega(G)$. Let $\omega(G) = t$. Let $(A, B)$ be a partition of $V(G)$ such that $\omega(A) < t$ and $\omega(B) < t$. Now $\chi(A) \leq 2^{t-2}$ and $\chi(B) \leq 2^{t-2}$. Consequently, $\chi(G) \leq \chi(A) + \chi(B) \leq 2^{t-2} + 2^{t-2} = 2^{t-1}$. ■

We end the introduction by setting up the notation that we will be using. For a vertex $v$ of a graph $G$, $N(v)$ will denote the set of neighbors of $v$ (we write $N_G(v)$ if there is a risk of confusion). The closed neighborhood of $v$, denoted $N[v]$, is defined to be $N(v) + v$. We define $M(v)$ (or $M_G(v)$) to be $V(G) \setminus N[v]$. Let $X$ and $Y$ be disjoint subsets of $V(G)$. We say $X$ is complete to $Y$ if every vertex in $X$ is adjacent to every vertex in $Y$. We say $X$ is anticomplete to $Y$ if every vertex in $X$ is nonadjacent to every vertex in $Y$. A set $X \subseteq V(G)$ is a \textit{homogeneous set} if $1 < |X| < |V(G)|$ and every vertex of $V(G) \setminus X$ is either complete or anticomplete to $X$.

This article is organized as follows. In Section 2, we prove that if a graph contains neither a $P_5$ nor a $C_5$, then it is 2-divisible. In Section 3, we prove that if a graph is bull-free and either odd-hole-free or $P_5$-free, then it is perfectly divisible.

\section{\textbf{$(P_5, C_5)$-Free Graphs are 2-Divisible}}

We start with some definitions. Let $G$ be a graph. $X \subseteq V(G)$ is said to be \textit{connected} if $G[X]$ is connected, and \textit{anticonnected} if $G'[X]$ is connected. For $X \subseteq V(G)$, a \textit{component} of $X$ is a maximal connected subset of $X$, and an \textit{anticomponent} of $X$ is a maximal anticonnected subset of $X$.

The following lemma is used several times in the sequel.

\textbf{Lemma 2.1.} Let $G$ be a graph. Let $C \subseteq V(G)$ be connected, and let $v \in V(G) \setminus C$ such that $v$ is neither complete nor anticomplete to $C$. Then there exist $a, b \in C$ such that $v - a - b$ is a path.
Proof. Since $v$ is neither complete nor anticomplete to $C$, it follows that both the sets $N(v) \cap C$ and $M(v) \cap C$ are nonempty. Since $C$ is connected, there exist $a \in N(v) \cap C$ and $b \in M(v) \cap C$ such that $ab \in E(G)$. But now $v - a - b$ is the desired path. This completes the proof. 

We are ready to prove the main result of this section.

**Theorem 2.1.** Every $(P_5, C_5)$-free graph is 2-divisible.

**Proof.** Let $G$ be a $(P_5, C_5)$-free graph. We may assume that $G$ is connected. Let $v \in V(G)$, let $N = N(v)$, $M = M(v)$. Let $Z_1, \ldots, Z_t$ be the components of $M$.

(1) We may assume that there is $i$ such that no vertex of $N$ is complete to $Z_i$.

For, otherwise, $X_1 = M + v$, $X_2 = N$ is the desired partition. This proves (1). Let $i$ be as in (1), we may assume that $i = 1$.

(2) There do not exist $n_1, n_2$ in $N$ and $m_1, m_2$ in $M$ such that $n_1$ is adjacent to $m_1$ and not to $m_2$, and $n_2$ is adjacent to $m_2$ and not to $m_1$, and $n_1$ is nonadjacent to $n_2$.

For, otherwise, $G[[n_1, n_2, m_1, m_2, v]]$ is a $P_5$ or a $C_5$. This proves (2).

(3) For every $i > 1$ there exists $n \in N$ complete to $Z_i$.

For suppose that there does not exist $n \in N$ that is complete to $Z_2$. For $i = 1, 2$ let $n_i \in N$ have a neighbor in $Z_i$. Since $Z_1, Z_2$ are connected, by Lemma 2.1, there exist $a_i, b_i \in Z_i$ such that $a_i - n_i - b_i$ is a path. Since $b_i - a_i - n_i - a_2 - b_2$ is not a $P_5$, we deduce that $n_1 \neq n_2$, and therefore $n_1$ is complete or anticomplete to $Z_2$, and $n_2$ is complete or anticomplete to $Z_1$. By the choice of $Z_1$ and the assumption, $n_1$ is anticomplete to $Z_2$, and $n_2$ to $Z_1$. By (2) $n_1$ is adjacent to $n_2$. But now $b_2 - a_2 - n_2 - n_1 - a_1$ is a $P_3$, a contradiction. This proves (3).

From the set of vertices in $N$ that have a neighbor in $Z_1$, choose one that has the maximum number of neighbors in $M$; call it $n$. (Such a vertex exists because $G$ is connected.) Let $X_1 = N(n)$, and let $X_2 = V(G) \setminus X_1$. Clearly $X_1$ does not contain a clique of size $\omega(G)$. We claim that $\omega(X_2) < \omega(G)$, thus proving that $(X_1, X_2)$ is a partition certifying 2-divisibility.

Suppose that there is a clique $K$ of size $\omega(G)$ in $X_2$. Then $n \notin K$.

(4) $K \notin Z_j$ for $j = 1, \ldots, t$.

By (3), $K \setminus (Z_2 \cup \cdots \cup Z_t) \neq \emptyset$. Suppose that $K \subseteq Z_1$. Then $K \subseteq C_1 \setminus N(n)$. Let $D$ be the component of $C_1 \setminus N(n)$ containing $K$. Then some vertex $p \in N(n) \cap Z_1$ has a neighbor in $D$. Since $D$ contains a clique of size $\omega(G)$, $p$ is not complete to $D$. Since $D$ is connected, by Lemma 2.1, there exist $d_1, d_2 \in D$ such that $p - d_1 - d_2$ is a path. But now $d_2 - d_1 - p - n - v$ is a $P_5$, a contradiction. This shows that $K \not\subseteq Z_1$. By (3), $K \setminus (Z_2 \cup \cdots \cup Z_t) \neq \emptyset$. This proves (4).

It follows from (4) that $K$ has a vertex $k_1 \in N \setminus X_1$, and a vertex $k_2 \in M \setminus X_1$. Then $k_1$ is nonadjacent to $n$, and $k_2$ is nonadjacent to $n$. But now by (2) $N(k_1) \cap M$ strictly contains $N(n) \cap M$, and in particular $k_1$ has a neighbor in $Z_1$, contrary to the choice of $n$. This completes the proof. 

An easy consequence of this is

**Corollary 2.1.** Let $G$ be a $(P_5, C_5)$-free graph. Then $\chi(G) \leq 2^{\omega(G)-1}$.

**Proof.** Follows from Theorem 2.1 and Lemma 1.2.

It is worth recalling Gyárfás’ result [6] that a $P_5$-free graph $G$ satisfies $\chi(G) \leq 4^{\omega(G)-1}$. 


3 | PERFECT DIVISIBILITY IN BULL-FREE GRAPHS

For an induced subgraph $H$ of a graph $G$, a vertex $c \in V(G) \setminus V(H)$ that is complete to $V(H)$ is called a center for $H$. Similarly, a vertex $a \in V(G) \setminus V(H)$ that is anticomplete to $V(H)$ is called an anticenter for $H$. For a hole $C = c_1 - c_2 - c_3 - c_4 - c_5 - c_1$, an $i$-clone is a vertex adjacent to $c_{i+1}$ and $c_{i-1}$, and not to $c_{i+2}, c_{i-2}$ (in particular $c_i$ is an $i$-clone). An $i$-star is a vertex complete to $V(C) \setminus c_i$, and nonadjacent to $c_i$. A clone is a vertex that is an $i$-clone for some $i$, and a star is a vertex that is an $i$-star for some $i$. We will need the following results from [3,4], and [5].

Theorem 3.1 (from [4,5]). If $G$ is bull-free, and $G$ has a $P_4$ with a center and an anticenter, then $G$ contains a homogeneous set, or $G$ contains $C_5$.

Theorem 3.2 (from [3]). If $G$ is bull-free and contains an odd hole or an odd antihole with a center and an anticenter, then $G$ contains a homogeneous set.

Theorem 3.3 (from [3]). If $G$ is bull-free, then either $G$ contains a homogeneous set, or for every $v \in V(G)$, either $G[N(v)]$ or $G[M(v)]$ is perfect.

The next two theorems refine Theorem 3.3 in the special cases we are dealing with in this article.

Theorem 3.4. If $G$ is bull-free and odd-hole-free, then either $G$ contains a homogeneous set, or for every $v \in V(G)$ the graph $G[M(v)]$ is perfect.

Proof. We may assume that $G$ does not contain a homogeneous set. Let $v \in V(G)$ such that $G[M(v)]$ is not perfect. Since $G$ is odd-hole-free, by the strong perfect graph theorem [2], $G[M(v)]$ contains an odd antihole of length at least seven, and therefore a three-edge-path $P$ with a center. Now $v$ is an anticenter for $P$, and so by Theorem 3.1, $G$ contains a homogeneous set, a contradiction. This proves the theorem.

One of the referees and T. Karthick (private communication) pointed out that 3.4 was already proved by Brandstadt and Mosca [1]. Actually the result also follows from Reed and Sbihi’s Wheel Lemma [10] stating that if a bull-free graph contains a wheel, then it has a homogeneous set. (Here a wheel consists of a hole of length at least seven with an additional vertex complete to the hole.)

Theorem 3.5. If $G$ is bull-free and $P_5$-free, then either $G$ contains a homogeneous set, or for some $v \in V(G)$, $G[M(v)]$ is perfect.

Proof. By Theorem 3.4 we may assume that $G$ contains a $C_5$, say $C = c_1 - c_2 - c_3 - c_4 - c_5 - c_1$. We may assume that $G$ does not contains a homogeneous set.

(1) Let $D$ be a hole of length 5, and let $v \notin V(D)$. Then $v$ is a clone, a star, a center or an anticenter for $D$.

Since $G$ has no $P_5$, $v$ cannot have exactly one neighbor in $D$. Suppose that $v$ has exactly two neighbors in $D$. Since $G$ is bull-free, the neighbors are nonadjacent, so $v$ is a clone. Suppose that $v$ has exactly two nonneighbors in $D$. Since $G$ is bull-free, the nonneighbors are adjacent, and $v$ is a clone. The cases when $v$ has 0, 4, 5 neighbors in $D$ result in $v$ being an anticenter, star, and a center for $D$, respectively. This proves (1).

(2) Let $D$ be a hole of length 5 in $G$. Then there is no anticenter for $D$.

Suppose that $v$ is an anticenter for $D$, we may assume that $D = C$. By Theorem 3.3 there is no center for $D$. Since $G$ is connected, we may assume that $v$ has a neighbor $u$ such that $u$ has a neighbor in $V(D)$. Let $P$ be a path starting at $u$ and with $V(P) \setminus u \subseteq V(D)$ with $|V(P)|$ maximum.
Since \( v - u - P \) is not a \( P_5 \), and \( u \) is not a center for \( P \setminus u \), it follows that for some \( i, u \) is adjacent to \( c_i \) and to \( c_{i+1} \), but not to \( c_{i+2} \). But now \( G[\{c_i, c_{i+1}, c_{i+2}, u, v\}] \) is a bull, a contradiction. This proves (2).

(3) Let \( d_i \) and \( d'_i \) be \( i \)-clones nonadjacent to each other. Let \( v \) be adjacent to \( d_i \) and not to \( d'_i \). Then \( v \) is a center for \( C \), or \( v \) is an \( i \)-star for \( C \), or \( v \) is an \( i \)-clone for \( C \). Moreover, let \( D \) be the hole obtained from \( C \) by replacing \( c_i \) with \( d_i \), and let \( D' \) be the hole obtained from \( C \) by replacing \( c_i \) with \( d'_i \). It follows that either:

- \( v \) is an \( i \)-clone for both \( D \) and \( D' \), or
- \( v \) is a center for \( D \), and an \( i \)-star for \( D' \).

To prove (3), we may assume that \( i = 1 \). If \( v \) is anticomplete to \( \{c_2, c_4\} \), then we get a contradiction to (1) or (2) applied to \( v \) and \( D' \). Thus we may assume that \( v \) is adjacent to \( c_2 \). Suppose that \( v \) is nonadjacent to \( c_5 \). By (1) applied to \( D, v \) is adjacent to \( c_5 \). But now \( d'_1 - c_5 - d_1 - v - c_4 \) is a \( P_5 \), a contradiction. Thus \( v \) is adjacent to \( c_5 \). By (1) applied to \( D' \), \( v \) is either complete or anticomplete to \( \{c_3, c_4\} \). Now if \( v \) is anticomplete to \( \{c_3, c_4\} \), then \( v \) is an \( i \)-clone; if \( v \) is complete to \( \{c_3, c_4\} \) then \( v \) is a center or an \( i \)-star for \( C \). This proves (3).

(4) There do not exist \( d_1, d'_1, d_3, d'_3, v_1, v_3 \) such that:

- \( \{d_1, d'_1\} \) is not complete to \( \{d_3, d'_3\} \), and
- for \( i = 1, 3 \)
  - \( d_i \) and \( d'_i \) are \( i \)-clones nonadjacent to each other, and
  - \( v_i \) is adjacent to \( d_i \) and nonadjacent to \( d'_i \), and
  - \( v_i \) is not an \( i \)-clone.

Observe that by (3), no vertex of \( \{d_1, d'_1\} \) is neither complete nor anticomplete on \( \{d_3, d'_3\} \) and the same with the roles of 1, 3 exchanged. It follows that \( \{d_1, d'_1\} \) is anticomplete to \( \{d_3, d'_3\} \), and in particular \( v_1, v_3 \notin \{d_1, d'_1, d_3, d'_3\} \). By (3) applied to the hole \( d'_1 - c_2 - c_3 - c_4 - c_5 - d'_1 \) and \( d_3, d'_3 \), it follows that \( v_3 \) is complete to \( \{d_1, d'_1\} \). Similarly \( v_1 \) is complete to \( \{d_3, d'_3\} \). In particular \( v_1 \neq v_3 \). But now \( G[\{d'_1, v_3, d_1, v_1, d'_3\}] \) is either a bull or a \( P_5 \), in both cases a contradiction. This proves (4).

(5) There is not both a 1-clone nonadjacent to \( c_1 \), and a 3-clone non-adjacent to \( c_3 \).

For suppose that such clones exist. For \( i = 1, 3 \) let \( X_i \) be a maximal anticonnected set of \( i \)-clones with \( c_i \) in \( X_i \). Then \( |X_i| > 1 \) for \( i = 1, 3 \). Since \( X_i \) is anticonnected, it follows from (3) that \( X_i \) is anticonnected to \( X_3 \). Since \( |X_1|, |X_3| > 1 \), and \( G \) does not admit a homogeneous set decomposition, it follows that neither \( X_1 \) nor \( X_3 \) is a homogeneous set in \( G \). Therefore for \( i = 1, 3 \) there exists \( v_i \notin X_i \) with a neighbor and a nonneighbor in \( X_i \). Then \( v_i \notin X_1 \cup X_3 \). Note that \( X_i + v_i \) is anticonnected, and hence by the maximality of \( X_i \), it follows that \( v_i \) is not an \( i \)-clone. By applying Lemma 2.1 in \( G^c \) with \( v_i \) and \( X_i \) for \( i = 1, 3 \), it follows that there exist \( d_i, d'_i \in X_i \) such that \( d_i \) is non-adjacent to \( d'_i \), \( v_i \) is adjacent to \( d_i \), and \( v_i \) is nonadjacent to \( d'_i \). But now we get a contradiction to (4). This proves (5).

(6) For some \( i \), \( V(G) = N[c_i] \cup N[c_{i+2}] \) (here addition is modulo 5).

Suppose that (6) is false. Since (6) does not hold with \( i = 1, 3 \) and (2) and symmetry imply that we may assume that there is a 1-clone \( c'_i \) nonadjacent to \( c_1 \). Since (6) does not hold with \( i = 5 \), again by (1), (2) and symmetry we may assume that there is a 2-clone \( c'_i \) non-adjacent to \( c_2 \). Finally, since (6) does not hold with \( i = 3 \), by (1), (2) and symmetry we get a 3-clone \( c'_i \) nonadjacent to \( c_3 \). But this is a contradiction to (5). This proves (6).
Let $i$ be as in (6); we may assume that $i = 1$. Suppose that $G[M(c_1)]$ is not perfect. Then, by the strong perfect graph theorem [2], $G[M(c_1)]$ contains an odd hole or an odd antihole $H$. But now $c_1$ is a center for $H$, and $c_1$ is an anticenter for $H$, contrary to Theorem 3.2. This proves the theorem. $lacksquare$

A graph $G$ is perfectly weight divisible if for every nonnegative integer weight function $w$ on $V(G)$, there is a partition of $V(G)$ into two sets $P, W$ such that $G[P]$ is perfect and the maximum weight of a clique in $G[W]$ is smaller than the maximum weight of a clique in $G$.

**Theorem 3.6.** A minimal nonperfectly weight divisible graph does not admit a homogeneous set decomposition.

**Proof.** Let $G$ be such that all proper induced subgraphs of $G$ are perfectly weight divisible. Let $w$ be a weight function on $V(G)$. Let $X$ be a homogeneous set in $G$, with common neighbors $N$ and let $M = V(G) \setminus (X \cup N)$. Let $G'$ be obtained from $G$ by replacing $X$ with a single vertex $x$ of $X$ with weight $w(x)$ equal to the maximum weight of a clique in $G[X]$. Let $T$ be the maximum weight of a clique in $G$.

Let $(P', W')$ be a partition of $V(G')$ corresponding to the weight $w$. Let $(X_p, X_w)$ be a partition of $X$ where $G[X_p]$ is perfect and the maximum weight of a clique in $G[X_w]$ is smaller than the maximum weight of a clique in $G$. We construct a partition of $V(G)$.

Suppose first that $x \in W'$. Then let $P = P'$ and $W = W' \cup X$. Clearly this is a good partition. Now suppose that $x \in P'$. Let $P = (P' \setminus x) \cup X_p$ and let $W = W' \cup X_w$. By a theorem of [9], $G[P]$ is perfect. Suppose that $W$ contains a clique $K$ with weight $T$. Then $K \cap X_w$ is nonempty. Let $K'$ be a clique of maximum weight in $X$. Now $(K \setminus X_w) \cup K'$ is a clique in $G$ with weight greater than $T$, a contradiction. This proves the theorem. $lacksquare$

We can now prove our main result:

**Theorem 3.7.** Let $G$ be a bull-free graph that is either odd-hole-free or $P_5$-free. Then $G$ is perfectly weight divisible, and hence perfectly divisible.

**Proof.** Let $G$ be a minimal counterexample to the theorem. Then there is a nonnegative integer weight function $w$ on $V(G)$ for which there is no partition of $V(G)$ as in the definition of being perfectly weight divisible. Let $U$ be the set of vertices of $G$ with $w(v) > 0$, and let $G' = G[U]$. By theorems 3.4, 3.5, 3.6, $G'$ has a vertex $v$ such that $G'[M_{G'}(v)]$ is perfect. But now, since $w(v) > 0$, setting $P = M_{G'}(v) + v$ and $W = N_{G'}(v) \cup (V(G) \setminus U)$ we get a partition of $V(G)$ as in the definition of being perfectly weight divisible, a contradiction. This proves the theorem. $lacksquare$

**Corollary 3.1.** Let $G$ be a bull-free graph that is either odd-hole-free or $P_5$-free. Then $\chi(G) \leq \binom{\omega(G) + 1}{2}$.

**Proof.** Follows from Theorem 3.7 and Lemma 1.1. $lacksquare$

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