Abstract

A Jacobi structure $J$ on a line bundle $L \to M$ is weakly regular if the sharp map $J^\#: J^1L \to DL$ has constant rank. A generalized contact bundle with regular Jacobi structure possess a transverse complex structure. Paralleling the work of Bailey in generalized complex geometry, we find condition on a pair consisting of a regular Jacobi structure and an transverse complex structure to come from a generalized contact structure. In this way we are able to construct interesting examples of generalized contact bundles. As applications: 1) we prove that every 5-dimensional nilmanifold is equipped with an invariant generalized contact structure, 2) we show that, unlike the generalized complex case, all contact bundles over a complex manifold possess a compatible generalized contact structure. Finally we provide a counterexample presenting a locally conformal symplectic bundle over a generalized contact manifold of complex type that do not possess a compatible generalized contact structure.

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1 Introduction

Generalized geometry was set up in the early 2000’s by Hitchin [9] and further developed by Gualtieri [8], and the literature about them is now rather wide. Generalized complex manifolds encompass symplectic and complex manifolds, i.e. they provide a common framework for symplectic and complex manifolds. Generalized complex structure are even dimensional, so from the very beginning people searched for their odd dimensional counterpart. The odd dimensional counterparts of symplectic manifolds are contact manifolds and actually this gave a hint: the odd dimensional analogue of generalized complex structures should at least contain contact manifolds. First attempts to give meaningful definitions have been done by Iglesias-Ponte and Wade [18], Poon and Wade [11] and Aldi and Grandini [1]. Recently, Vitagliano and Wade proposed a definition [17] that seems to be well motivated and has the main advantage that all the previous attempts are included. This is the definition we will work with. Generalized contact structures (in the sense of Vitagliano and Wade) encompass non necessarily co-orientable contact structures and integrable complex structures on the Atiyah-algebroid of a line bundle. In [14] it was proven that, locally, generalized contact structures are products of either a contact and a homogeneous generalized complex or a symplectic and a generalized contact structure (of a simpler type). This local normal form theorem can be seen as a manual to build up examples which are not included in the extreme cases by taking products. On the other hand, it is interesting to find examples which do not belong to this class, i.e. are not globally products. This is the aim of this work: we want to describe a method, similar to the one described in [4] for generalized complex structures, to detect whenever a weakly regular Jacobi structure can come from a generalized contact bundle. We use then using our method to find examples, which do not come from a global product.

This note is organized as follows. First we recall the arena for generalized contact bundles, the omni-Lie algebroid, and the very definition of generalized contact structures. This is far from being complete and is more meant to fix notation than to give a proper introduction to the topic. A more detailed discussion of these notions can be found in [16]. The second step is to introduce the notion of weakly regular Jacobi structures, which are the equivalent notion of a regular Poisson structures in Poisson geometry, transversally complex subbundles, which are basically complex structures on a normal bundle, and their connection to generalized contact structures. The general idea is that every generalized contact structure with weakly regular Jacobi structure comes together with a transversally complex subbundle, but the converse is not true in general. We will see that we can find necessary and sufficient condition to find the converse statement, which will be a cohomological obstruction in some spectral sequence.

In the last part we construct examples and a counterexample, with the help of the previous part. This part is in turn divided in 3 parts. First of all, we prove that every five dimensional nilpotent Lie group possesses an invariant generalized contact structure. This can be seen as the odd dimensional analogue of the existence of generalized complex structures on six dimensional nilpotent Lie groups proven in [2]. Our second examples are contact fiber bundles over a complex base. It turns out that they posses generalized contact structures, since the obstructions of the previous parts become trivial.

Finally, we discuss a specific counterexample, i.e. an example where the obstructions do not vanish.

2 Preliminaries and Notation

This introductory section is divided into two parts: first we recall the Atiyah algebroid of a vector bundle and the corresponding Der-complex with applications to contact and Jacobi geometry. Afterwards, we introduce the arena for generalized geometry in odd dimensions, the omni-Lie algebroids, and give a quick reminder of generalized contact bundles together with the properties we will need.
2.1 The Atiyah algebroid and the Der-complex

The notions of Atiyah algebroid of a vector bundle and the associated Der-complex are known and are used in many other situations. This section is basically meant to fix notation. A more complete introduction to this can be found in [14], which also discusses the notion of Dirac structures on the omni-Lie algebroid of line bundles in more detail, nevertheless the notion of Omni-Lie algebroids was first defined in [3], in order to study Lie algebroids and local Lie algebra structures on vector bundles.

Definition 2.1 Let $E \to M$ be a vector bundle. A derivation is a map $\Delta : \Gamma^\infty(E) \to \Gamma^\infty(E)$, fulfilling

$$\Delta(fe) = X(f)e + f\Delta(e) \quad \forall f \in \mathcal{C}^\infty(M) \quad \text{and} \quad \forall e \in \Gamma^\infty(E),$$

for a necessarily unique $X \in \Gamma^\infty(TM)$.

Remark 2.2 Derivations of a vector bundle form a subspace of the first order differential operators from $E$ to itself. Moreover, if the vector bundle has rank one then each differential operator is a derivation.

Lemma 2.3 Let $E \to M$ be a vector bundle. The derivations of of $E \to M$ are the sections of a Lie algebroid $DE \to M$. The Lie bracket of $\Gamma^\infty(DE)$ is the commutator. An element in the fiber $D_pE$ is a map $\delta_p : \Gamma^\infty(E) \to E_p$, such that

$$\delta_p(fe) = v_p(f)e(p) + f(p)\delta(e) \quad \forall f \in \mathcal{C}^\infty(M) \quad \text{and} \quad \forall e \in \Gamma^\infty(E),$$

for a necessarily unique $v_p \in T_pM$. The anchor, or symbol, $\sigma : DE \to TM$ the assignment $\sigma(\delta_p) = v_p$. Moreover, $DE \to M$ is called Atiyah algebroid of $E \to M$.

The Atiyah algebroid fits into the following exact sequence of Lie algebroids, which is called the Spencer sequence,

$$0 \to \text{End}(E) \to DE \to TM \to 0,$$

where the Lie algebroid structure of $\text{End}(E)$ is the (pointwise) commutator and the trivial anchor. The first arrow is given by the inclusion. With this we see immediately that $\text{rank}(DE) = \text{rank}(E)^2 + \dim(M)$.

We assign a vector bundle $E \to M$ to a Lie algebroid $DE \to M$. A natural question is: is this assignment functorial? In fact it is not, unless we restrict the category of vector bundles by just taking regular vector bundle morphisms, i.e. vector bundle morphisms which are fiber-wise invertible. Note that a regular vector bundle morphism $\Phi : E \to E'$ covering a smooth map $\phi : M \to M'$ allows us to define the pull-back $\Phi^*(e')$ of a section $e' \in \Gamma^\infty(E')$ by putting

$$(\Phi^*e')(p) = \Phi_p^{-1}(e'(\phi(p))).$$

Lemma 2.4 Let $E \to M$ and $E' \to M'$ be vector bundles and let $\Phi : E \to E'$ be a regular vector bundle morphism covering a smooth map $\phi : M \to M'$. Then the map $D\Phi : DE \to DE'$ defined by

$$D\Phi(\delta_p)(e') = \Phi_p(\delta_p(\Phi^*e'))$$

is a Lie algebroid morphism covering $\phi$. Moreover, this assignment makes $D$ a functor from the category of vector bundles with regular vector bundle morphisms to the category of Lie algebroids.
Note that for a given vector bundle \( E \to M \), \( DE \) has a tautological representation on \( E \). Hence, we can define its de Rham complex with coefficients in \( E \).

**Definition 2.5** Let \( E \to M \) be a vector bundle. The complex \( (\Omega^*_E(M) := \Gamma^\infty(\Lambda^\bullet(DE)^* \otimes E), d_E) \), where we define

\[
d_E \alpha(\Delta_0, \ldots, \Delta_k) = \sum_{i=0}^{k} (-1)^i \Delta_i (\alpha(\Delta_0, \ldots, \widehat{\Delta_i}, \ldots, \Delta_k)) \\
+ \sum_{i<j} (-1)^{i+j} \alpha([\Delta_i, \Delta_j], \Delta_0, \ldots, \widehat{\Delta_i}, \ldots, \widehat{\Delta_j}, \ldots, \Delta_k)
\]

for \( \alpha \in \Omega^k_E(M) \) and \( \Delta_i \in \Gamma^\infty(DE) \), is called the Der-complex of \( E \). We call elements of this complex Atiyah forms.

**Lemma 2.6** Let \( E \to M \) be a vector bundle. The Der-complex \( (\Omega^*_E(M), d_E) \) is acyclic with contracting homotopy \( \iota_B \), the contraction with the identity operator \( 1 \in \Gamma^\infty(DE) \).

**Remark 2.7** For a regular vector bundle morphism \( \Phi: E \to E' \) covering the smooth map \( \phi: M \to M' \), we have a pull-back for Atiyah forms :

\[
(\Phi^* \alpha')(\Delta_1, \ldots, \Delta_k) := \Phi^{-1}_{p}(\alpha'_{\phi(p)}(D\Phi(\Delta_1), \ldots, D\Phi(\Delta_k)))
\]

for \( \Delta_i \in D_p E \), \( \alpha \in \Omega^k_{E'}(M') \) and \( p \in M \). This pull-back commutes with the differential \( d_E \).

**Remark 2.8** In view of Remark 2.2 we have \( J^1L = (DL)^* \otimes L \), where \( J^1L \) is first jet bundle, i.e. \( \Omega^1_L(M) = \Gamma^\infty(J^1L) \).

**Remark 2.9** (Contact Geometry reloaded) A contact manifold is a pair \( (M, \xi) \) where \( M \) is a manifold and \( \xi \subset TM \) is a co-dimension one maximally non-integrable distribution. Let us define the contact 1-form \( \Theta: TM \to L \) for \( L := TM/\xi \) simply as the projection. Now we consider

\[
\Theta \circ \sigma: DL \to L,
\]

which is Atiyah 1-form and \( \omega = d_L \sigma^* \Theta \). We call \( \omega \) a contact 2-form. The maximal non-integrability of \( \xi \) is equivalent to the invertibility of

\[
\omega^\flat: DL \to (DL)^* \otimes L.
\]

Since the Der-complex is acyclic the data of a line bundle and a non-degenerate and closed Atiyah 2-form in the Der-complex is actually equivalent to a contact structure. From now on, we will always take this point of view on contact structure. So, in the sequel, a contact structure will be just a contact 2-form i.e. a closed and non-degenerate Atiyah 2-form on a line bundle. Using Remark 2.7 we can define contactomorphisms: Let \( (L \to M, \omega) \) and \( (L' \to M', \omega') \) be two contact manifolds and let \( \Phi: L \to L' \) be a regular line bundle morphism. We call \( \Phi \) a contactomorphism, if \( \Phi^* \omega' = \omega \). This approach has several advantages, which will be clearer later on.

We discuss now briefly Jacobi brackets

**Definition 2.10** A Jacobi bracket on a line bundle \( L \to M \) is a Lie bracket \( \{-, -\}: \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L) \), such that the bracket is a derivation in each slot.
Remark 2.11 Let \{-,-\} be a Jacobi bracket on a line bundle \( L \to M \). Then there is a unique tensor, called the Jacobi tensor, \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \), such that
\[
\{\lambda, \mu\} = J(j^1\lambda, j^1\mu)
\]
for \( \lambda, \mu \in \Gamma^\infty(L) \). Conversely, every \( L \)-valued 2-form \( J \) on \( J^1L \) defines a skew-symmetric bilinear bracket \( \{-,-\} \), but the latter needs not to be a Jacobi bracket. Specifically, it does not need to fulfill the Jacobi identity. However, there is the notion of a Gerstenhaber-Jacobi bracket
\[
[-,-]: \Gamma^\infty(\Lambda^i(J^1L)^* \otimes L) \times \Gamma^\infty(\Lambda^j(J^1L)^* \otimes L) \to \Gamma^\infty(\Lambda^{i+j-1}(J^1L)^* \otimes L),
\]
such that the Jacobi identity of \( \{-,-\} \) is equivalent to \([J,J] = 0\) see [13] Chapter 1.3] for a detailed discussion. Finally, a Jacobi tensor defines a map \( J^2: J^1L \to (J^1L)^* \otimes L = DL \).

Lemma 2.12 Let \( L \to M \) be a line bundle and let \( \omega \in \Gamma^\infty(\Lambda^2(DL)^* \otimes L) \) be a contact 2-form. The inverse of the map
\[
\omega^\flat: DL \to (DL)^* \otimes L = J^1L
\]
is the sharp map of a tensor \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \), i.e.
\[
(\omega^\flat)^{-1} = J^2: J^1L \to (J^1L)^* \otimes L = DL,
\]
for some \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \). Moreover, \( J \) is a Jacobi tensor and we will refer to it as the Jacobi tensor corresponding to the contact 2-form \( \omega \).

When \( L \) is the trivial line bundle, than the notion of Jacobi bracket boils down to that of Jacobi pair.

Remark 2.13 (Trivial Line bundle) Let \( \mathbb{R}_M \to M \) be the trivial line bundle and let \( J \) be a Jacobi tensor on it. Let us denote by \( 1_M \in \Gamma^\infty(\mathbb{R}_M) \) the canonical global section. Using the canonical connection
\[
\nabla: TM \ni v \mapsto (f \cdot 1_M \mapsto v(f)1_M) \in D\mathbb{R}_M,
\]
we can see that \( DL \cong TM \oplus \mathbb{R}_M \) and hence
\[
J^1\mathbb{R}_M = (D\mathbb{R}_M)^* \otimes \mathbb{R}_M = T^*M \oplus \mathbb{R}_M.
\]

With this splitting, we see that
\[
J = \Lambda + 1 \wedge E
\]
for some \( (\Lambda, E) \in \Gamma^\infty(\Lambda^2TM \oplus TM) \). The Jacobi identity is equivalent to \([\Lambda, \Lambda] + E \wedge \Lambda = 0\) and \( \mathcal{L}_E \Lambda = 0 \). The pair \( (\Lambda, E) \) is often referred to as Jacobi pair. Moreover, if we denote by \( 1^* \in \Gamma^\infty(J^1\mathbb{R}_M) \) the canonical section then we can write any \( \psi \in J^1\mathbb{R}_M \) as \( \psi = \alpha + r1^* \in \Gamma^\infty(J^1\mathbb{R}_M) \), for some \( \alpha \in T^*M \) and \( r \in \mathbb{R} \). We obtain
\[
J^2(\alpha + r1^*) = \Lambda^2(\alpha) + rE - \alpha(E)1.
\]

A more detailed discussion about Jacobi structures on trivial line bundles can be found in [13] Chapter 2]
2.2 Generalized Geometry in odd dimensions

In this section we briefly discuss generalized geometry in odd dimensions. This introduction is far from being complete. For a more detailed outlook to the topic we refer the reader to [16] and [17]. The arena for generalized geometry in odd dimensions is the so-called omni-Lie algebroid.

Definition 2.14 Let \( L \to M \) be a line bundle and let \( H \in \Omega^3_L(M) \) be a closed Atiyah 2-form. The vector bundle \( \mathbb{D}L := DL \oplus J^1L \) together with

i.) the (Dorfman-like, H-twisted) bracket on sections
\[
[(\Delta_1, \psi_1), (\Delta_2, \psi_2)]_H = ([\Delta_1, \Delta_2], \mathcal{L}_{\Delta_1} \psi_2 - \imath_{\Delta_2} d_L \psi_1 + \imath_{\Delta_1} \imath_{\Delta_2} H)
\]

ii.) the non-degenerate \( L \)-valued pairing
\[
\langle (\Delta_1, \psi_1), (\Delta_2, \psi_2) \rangle := \psi_1(\Delta_2) + \psi_2(\Delta_1)
\]

iii.) the canonical projection \( \text{pr}_D: \mathbb{D}L \to DL \)

is called the (\( H \)-twisted) omni-Lie algebroid of \( L \to M \).

As in even dimensions, the objects of interest are certain subbundles of the Omni-Lie algebroid, or in our case its complexification.

Definition 2.15 Let \( L \to M \) be a line bundle and let \( H \in \Omega^3_L(M) \) be closed. A subbundle \( L \subseteq \mathbb{D}C \mathbb{L} \) is called almost generalized contact, if the following conditions are fulfilled:

i.) \( L \) is maximally isotropic with respect to (the \( \mathbb{C} \)-linear extension of) the pairing \( \langle -,- \rangle \).

ii.) \( L \cap \overline{L} = \{0\} \)

If \( L \) is additionally involutive with respect to (the \( \mathbb{C} \)-linear extension of) \([[-,-]]_H\), we call \( L \) an \( H \)-generalized contact structure, if \( H = 0 \) we say just generalized contact structure. Moreover, an \( H \)-generalized contact bundle is a line bundle together with a \( H \)-generalized contact structure.

Remark 2.16 Note that so far, in the literature only the case \( H = 0 \) considered. The main motivation of introducing a non-trivial \( H \) is, that if one considers just a short exact sequence of vector bundles
\[
0 \to J^1L \to E \to DL \to 0
\]

for a line bundle \( L \to M \) and a splitting \( \nabla^L: DL \to E \), then the closed Atiyah 3-form is the curvature of this splitting.

We can rephrase the definition in terms of an endomorphism of the omni-Lie algebroid, which shows the similarity to generalized complex geometry:

Proposition 2.17 Let \( L \to M \) be a line bundle and let \( H \in \Omega^3_L(M) \) be closed. For a \( K \in \Gamma^\infty(\text{End} \mathbb{D}L) \), fulfilling

i.) \( K^2 = -\text{id} \)

ii.) \( \langle K-,- \rangle = \langle -,- \rangle \)
the +i-Eigenbundle is an almost generalized contact structure. Moreover, every almost generalized
contact structure \( L \) arises as the +i-Eigenbundle of an endomorphism \( K \) satisfying i.)-ii.). If addi-
tionally
\[
N^H_K(A, B) := \left[ [K(A), K(B)]_H - K([K(A), B]_H) - K([A, K(B)]_H) - [A, B]_H \right] = 0
\]
holds for all \( A, B \in \Gamma^\infty(DL) \), then \( L \) is \( H \)-generalized contact.

Remark 2.18 Using Proposition 2.17, we will often call the endomorphism \( K \) itself an \( H \)-generalized
contact structure. If just the conditions i.) and ii.) are fulfilled, we will refer to \( K \) as an almost
generalized contact structure.

Lemma 2.19 Let \( L \rightarrow M \) be a line bundle, let \( H \in \Omega^3 L(M) \) be closed and let \( K \in \Gamma^\infty(\text{End} DL) \)
be an almost generalized contact structure. Then, using the decomposition \( DL = DL \oplus J^1 \), \( K \) can be
written as
\[
K = \begin{pmatrix}
\phi & J^2 \\
-\omega^b & -\phi^a
\end{pmatrix},
\]
where \( \phi \in \Gamma^\infty(\text{End} DL) \), \( J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L) \) and \( \sigma \in \Omega^3_L(M) \). If \( K \) is additionally \( H \)-generalized
contact, then \( J \) is a Jacobi tensor.

Proof: The proof is an easy verification, but can also be found in [14] or [17]. □

Example 2.20 Let \( L \rightarrow M \) be a line bundle and let \( \omega \in \Omega^2_L(M) \) be a contact 2-form. Then
\[
\mathcal{L} = \{(\Delta, i_\Delta \omega) \in DCL \mid \Delta \in DCL\}
\]
is a generalized contact structure. The corresponding endomorphism \( K \in \Gamma^\infty(\text{End} DL) \) is given by
\[
K = \begin{pmatrix}
0 & J^2 \\
-\omega^b & 0
\end{pmatrix},
\]
where \( J \) is the Jacobi tensor of \( \omega \).

Example 2.21 Let \( L \rightarrow M \) be a line bundle and let \( \phi \in \Gamma^\infty(\text{End} DL) \) be a complex structure, i.e.
an almost complex structure such that the Nijenhuis torsion with respect to the Lie algebroid bracket
vanishes. If we denote by \( DL^{(1,0)} \) its +i-Eigenbundle, then
\[
\mathcal{L} = DL^{(1,0)} \oplus \text{Ann}(DL^{(1,0)})
\]
is a generalized contact structure with corresponding endomorphism \( K \in \Gamma^\infty(\text{End} DL) \) given by
\[
K = \begin{pmatrix}
\phi & 0 \\
0 & -\phi^*
\end{pmatrix},
\]
where \( \phi^* \in \text{End}(J^1 L) \) is the adjoint of \( \phi \) with respect to the \( L \)-valued pairing of \( J^1 L \) and \( DL \). In the
following, we will refer to \( \phi \) as a Gauge complex or Atiyah complex structure on \( L \).

We will refer to the last two examples, i.e. contact structures and Atiyah complex structures, as the extreme
cases of generalized contact structures.

We will not discuss the automorphisms of \( DL \) in detail. A conceptual discussion can be found in [14]. Nevertheless, we will need a kind of action of Atiyah 2-forms on \( H \)-generalized contact structures:
Lemma 2.22 Let \( L \to M \) be a line bundle, let \( H \in \Omega^2_L(M) \) be closed and let \( \mathcal{L} \) be a \( H \)-generalized contact structure. For a real \( B \in \Omega^2_L(M) \), the subbundle

\[
\mathcal{L}^B = \{ (\Delta, \psi + \iota_\Delta B) \in \mathcal{D}L \mid (\Delta, \psi) \in \mathcal{L} \}
\]

is a \((H + d_L B)\)-generalized contact structure. If \( B \) is closed we will refer to it as a \( B \)-field.

Remark 2.23 Let \( L \to M \) be a line bundle, let \( H \in \Omega^2_L(M) \) be closed and let \( \mathcal{L} \) be a \( H \)-generalized contact structure. Since the Der-complex is acyclic with contracting homotopy \( \iota_1 \), we have that \( \mathcal{L}^{-\iota_1 H} \) is a generalized contact structure by the previous Lemma.

3 Tranversally Complex Jacobi Structures and Generalized Contact Bundles

Unlike in Poisson geometry, a Jacobi structure \( J \) may also have odd dimensional characteristic leaves. This comes from the fact that the leaves are the integral manifolds of the singular distribution \( \text{im}(\sigma \circ J^2) \), where the image of the Jacobi tensor \( J \) is surely an even dimensional subbundle of \( DL \), but composed with symbol we might lose one dimension. It seems therefore reasonable to distinguish between regular Jacobi structures, i.e. Jacobi structure with a regular distribution, and

Definition 3.1 Let \( L \to M \) be a line bundle and let \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \) be a Jacobi tensor. \( J \) is said to be weakly regular, if \( \text{im}(J^2) \subseteq DL \) is a regular subbundle.

Remark 3.2 A Jacobi structure which is weakly regular is not always regular. To illustrate this, we take for example the canonical Jacobi structure \((\Lambda_{can}, E_{can}) \in \Gamma^\infty(\Lambda^2 T^R 2k+1 \oplus T^R 2k+1)\) coming from the contact structure and consider \( Z \in \Gamma^\infty(T^R) \) given by \( Z = x \frac{\partial}{\partial x} \). Then \((\Lambda = \Lambda_{can} + E_{can} \wedge Z, E_{can})\) defines a weakly regular Jacobi structure on \( R^{2k+1} \times R^\infty \) where the set of contact points are \( \{ (x, 0) \in R^{2k+1} \times R \} \).

Remark 3.3 Let \( L \to M \) be a line bundle and let \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \) be weakly regular, then by definition \( \text{im}(J^2) \subseteq DL \) is a regular subbundle. Moreover, one can prove, that it is in fact a subalgebroid and there is a canonical form \( \omega \in \Gamma^\infty(\Lambda^2(\text{im}(J^2)) \otimes L) \) such that \( d_{\text{im}(J^2)} \omega = 0 \) and \( \omega(J^2(\alpha), J^2(\beta)) = \alpha(J^2(\beta)) \), where \( d_{\text{im}(J^2)} \) the de Rham differential with coefficients in the tautological representation \( \text{im}(J^2) \to DL \). We will refer to this form as the inverse of \( J \) and denote it by \( J^{-1} \).

A weakly regular Jacobi structure alone is not enough to construct an almost generalized contact structure out of, more precisely we need to consider transversal information to be seen in the following

Definition 3.4 A transversally complex subbundle on \( L \to M \) is a pair \((S, K)\) consisting of two involutive subbundles \( S \subseteq DL \) and \( K \subseteq D_{C}L \), such that

i.) \( K + T_{C} = D_{C}L \),

ii.) \( K \cap T_{C} = S_{C} \).

Remark 3.5 The name transversally complex involutive subbundle comes from the fact, that the decomposition

\[
(DL/S)_C = (K/S_C) \oplus (\overline{K}/S_C)
\]

defines an almost complex structure on \( DL/S \).
We are mainly interested in transversally complex structures with an additional Jacobi structure, let us therefore be precise in the following

**Definition 3.6** Let $L \to M$ be a line bundle. A transversally complex Jacobi structure is a pair $(J, K)$ consisting of a weakly regular Jacobi structure $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ and an involutive subbundle $K \subset DCL$, such that $(\im(J^1), K)$ is transversally complex subbundle.

This kind of structure appear naturally in generalized contact geometry if one assumes some regularity conditions, to be seen in the next

**Proposition 3.7** Let $L \to M$ be a line bundle and let $L \subseteq DCL$ be a generalized contact structure, such that the corresponding Jacobi structure $J$ is weakly regular. Then $(J, \pr_D L)$ is a transversally complex Jacobi structure.

**Proof:** Let us define $K := \pr_D(L)$. Having in mind that $\im(J^1) = \pr_D(L) \cap \pr_D(\overline{\mathbb{C}})$ and that $\pr_D(L)$ is involutive ([14]), we get the result.

With the previous proposition in mind, it is natural to ask which transversally complex Jacobi structure can be induced by a generalized contact structure. To formalize the term "induced by", we use the proof of the previous Lemma.

**Definition 3.8** Let $L \to M$ be a line bundle, let $L \subseteq DCL$ be a generalized contact structure, and $(J, K)$ be a transversally complex Jacobi structure. We say $L$ induces $(J, K)$, if $J$ is the Jacobi structure of $L$ and $K = \pr_D L$.

Let us give a first characterization of a transversally complex Jacobi structure induced by a generalized contact structure.

**Lemma 3.9** Let $L \to M$ be a line bundle and let $(J, K)$ be transversally complex Jacobi structure induced by the generalized contact structure $L \subseteq DCL$. Then

i.) for any real extension $\omega$ of the inverse of $J$ there exists a real $B \in \Omega^2(M)$, such that

$$d_L(i\omega + B)(\Delta_1, \Delta_2, \Delta_3) = 0 \ \forall \Delta_i \in K.$$

ii.) $L = (K \oplus \text{Ann}(K))^\omega + B$

**Proof:** Every generalized contact structure $L$ can be represented by a two form $\tilde{\epsilon}: \Lambda^2 pr_D L \to L_C$ by

$$L = \{(\Delta, \alpha) \in DCL \mid \alpha|_{pr_D L} = i\Delta \tilde{\epsilon}|_{pr_D L}\},$$

such that $\im(\tilde{\epsilon})|_{\Lambda^2_S} = (J|_S)^{-1}$ for the Jacobi structure $J$ of the generalized contact structure, so in our case with the given weakly regular one. The proof of this can be found in [14] Section 2.2.3. If the generalized contact structure induces the given transversally complex Jacobi structure, the we have $\pr_D(L) = K$. Since $K$ is regular, we extend $\tilde{\epsilon}$ to a (complex valued) 2-form $\varepsilon$ and get

$$L = (\pr_D(L) \oplus \text{Ann}(\pr_D(L)))^\varepsilon = (K \oplus \text{Ann}(K))^{\text{Re}(\varepsilon) + i\text{Im}(\varepsilon)},$$

which is the first statement, since $\im(\varepsilon)$ extends the inverse of $J$. The second statement follows directly from the integrability of $L$. \qed
Remark 3.10 It is easy to see, that if a generalized contact structure induces a given transversally complex Jacobi structure, then every $B$-field transform of it will induce the same structure. So in view of Remark 2.23, we can even say that a transversally complex Jacobi structure is induced by a $H$-generalized contact structure, if and only if it is induced by a generalized contact structure. Note that this is not the case in generalized complex geometry, since the third de Rham cohomology need not to be zero, while the Der-complex is always acyclic.

As an endpoint of this section we collect all the previous results in the following

Corollary 3.11 Let $L \to M$ be a line bundle and let $(J,K)$ be a transversally complex Jacobi structure. These data come from a ($H$-)generalized contact structure, if and only if there exists a real extension $\omega$ of the inverse of $J$ and a real $B \in \Omega^2_L(M)$, such that

$$d_L(\omega + B)(\Delta_1, \Delta_2, \Delta_3) = 0 \ \forall \Delta_i \in K.$$

This condition seems not to be very easy to handle and also involves an extension of the inverse of the Jacobi structure and the existence of a 2-form $B$. We will see in the following that their existence can be encoded completely in properties of $J$.

4 The Splitting of the Spectral Sequence of a Transversally Complex Subbundle

We have seen that every generalized contact structure with weakly regular Jacobi structure induces a transversally complex Jacobi structure. The latter special case of a transversally complex subbundles. We want to explore these subbundles by means of a canonical spectral sequence attached to them. It turns out that a transversally complex Jacobi structure comes from a generalized contact bundle, if and only if a certain cohomology class in the first page of this spectral sequence vanishes.

Throughout this subsection we will assume the following data: a line bundle $L \to M$ and the subalgebroids $S \subseteq DL$ and $K \subseteq DCL$, such that $K + K = DCL$ and $K \cap K = SC$, in other words we want to fix a transversally complex subbundle $(S,K)$. Moreover, if not stated otherwise, we see every Atiyah form as complex valued.

4.1 General Statements and Preliminaries

This part is not only meant to fix notation and give a quick reminder on spectral sequences, but also to give a splitting of the zeroth and first page of the spectral sequence induced by the transversally complex subbundle $(S,K)$. Let us begin by showing that $(S,K)$ induces two filtered complexes.

Lemma 4.1 The subspaces

$$F^m_n := \{ \alpha \in \Omega^m_L(M) \mid \iota_X \alpha = 0 \ \forall X \in \Lambda^{m-n+1}S \} \quad \text{and}$$

$$G^m_n := \{ \alpha \in \Omega^m_L(M) \mid \iota_X \alpha = 0 \ \forall X \in \Lambda^{m-n+1}K \}$$

fulfill the following properties

i.) $\Omega^m_L(M) = F^m_0 \supseteq F^m_1 \supseteq \ldots$ and $\Omega^m_L(M) = G^m_0 \supseteq G^m_1 \supseteq \ldots$

ii.) $d_L(F^m_n) \subseteq F^m_{n+1}$ and $d_L(G^m_n) \subseteq G^m_{n+1}$

Moreover, we have the following relations between them:

(1) $G^m_n \subseteq F^m_n$
A spectral sequence is a series of bigraded vector spaces
\[ \{E_r^{i,j}\} \]
\[ \text{The maps } d_r : E_r^{i,j} \to E_r^{i+r,j+1} \]
\[ \text{Proof: The proof is an easy verification exploiting the involutivity of } \]
\[ \text{spectral sequence and its differentials.} \]
\[ \text{Let } E_r^{i,j} = \{G_j^m \cap G^m_i\} \text{ together with the maps } \]
\[ d_r : E_r^{i,j} \to E_r^{i+r,j+1} \]
\[ \text{□} \]

The properties i.) and ii.) in the previous Lemma show that the subspaces \( F^m_n \) and \( G^m_n \) induce filtrations of the Der-complex. Note that we did not explicitly introduce the spaces \( G^m_i \), but from the notation it should be clear that we mean the complex conjugation of the spaces \( G^m_i \) or equivalently the filtered complex with respect to \( K \). Properties (I)-(V) will give us a canonical splitting of the spectral sequence and its differentials.

Let us therefore briefly recall the definition of a spectral sequence

**Definition 4.2** A spectral sequence is a series of bigraded vector spaces \( \{E_r^{i,j}\} \geq 0 \), called the pages, and a series of maps \( \{d_r : E_r^{i,j} \to E_r^{i+r,j+1}\} \geq 0 \), the differentials, such that

i.) \( (d_r)^2 = 0 \)

ii.) \( E_r^{i,j} = \operatorname{ker}(d_r : E_r^{i,j} \to E_r^{i+r,j+1}) / \operatorname{im}(d_r : E_r^{i-1,j} \to E_r^{i,j}) \)

There is a canonical way to associate a spectral sequence to a filtered complex. We will define it for the filtered complex \( \Omega^m_n(M) = F^m_0 \supseteq F^m_1 \supseteq \ldots \) We consider the quotients
\[ E_r^{p,q} = \{ \alpha \in F^p_r | d_L \alpha \in F^{p+1}_r \} \]
\[ \text{together with the maps} \]
\[ d_r : E_r^{p,q} \ni \alpha + F^{q+1}_r + d_L(F^{q+1}_r) \to \alpha + F^{q+1}_r + d_L(F^{q+1}_r) \in E_r^{p+1,q-1} \]

**Lemma 4.3** The maps \( \{d_r\} \geq 0 \text{ from Equation 4.1 are well-defined and } \{\{E_r^{i,j}, d_r\}\} \geq 0 \text{ is a spectral sequence.} \)

**Proof:** The proof is a easy exercise, but can be found in every book treating spectral sequences of filtered complexes, see e.g. [19].

In our case, we do not have only one filtered complex, but two more filtered complexes \( \Omega^m_L(M) = G^m_0 \supseteq G^m_1 \supseteq \ldots \) and its complex conjugate, which are in relation with \( \Omega^m_L(M) = F^m_0 \supseteq F^m_1 \supseteq \ldots \). Let us now consider the quotients
\[ E_r^{(i,j),q} = \{ \alpha \in G^{i+j}_j \cap G^{i+j}_i | d_L \alpha \in G^{i+j+1}_j \}
\[ \text{together with the maps} \]
\[ d_r : E_r^{(i,j),q} \ni \alpha + G^{q+1}_j + d_L(G^{q+1}_j) \to \alpha + G^{q+1}_j + d_L(G^{q+1}_j) \in E_r^{(i,j),q+1} \]

**Lemma 4.4** Let \( s = 0, 1 \), then the canonical maps
\[ E_r^{(i,j),q} \ni \alpha + (F^{q+1}_j + d_L(F^{q+1}_j)) \cap G^{i+j}_i \ni \alpha + (F^{q+1}_j + d_L(F^{q+1}_j)) \ni \alpha + (F^{q+1}_j + d_L(F^{q+1}_j)) \ni \alpha + (F^{q+1}_j + d_L(F^{q+1}_j)) \in E_r^{(i,j),q} \]
are injective and moreover \( E_r^{p,q} = \bigoplus_{i+j=p} E_r^{(i,j),q} \).
For the differentials

\[ E_0^{i,j,q} = \frac{G_j^{q+i+j} \cap G_i^{q+i+j}}{F_i^{q+i+j} \cap G_i^{q+i+j}}, \]

where we used property (ii.) of Lemma 4.1. By (III) of Lemma 4.1 we obtain that \( E_0^{p,q} = (E_0^{i,j,q})_{i+j=p} \). Let now \( \omega_{ij} \in G_j^{q+i+j} \cap G_i^{q+i+j} \) for \( i + j = p \), such that

\[ \sum_{i+j=p} \omega_{ij} \in F_i^{q+i+j}. \]

We have that

\[ \omega_{kl} \in G_l^{q+i+j} \cap G_k^{q+i+j} \cap ((G_j^{q+i+j} \cap G_i^{q+i+j}))_{k+l=n,(i,j)\neq(k,l)} + F_i^{q+i+j} \subseteq F_i^{q+i+j} \]

for every choice of \( k + l = i + j \) and hence, using (IV) of Lemma 4.1, \( \omega_{kl} \in F_i^{q+i+j} \). If we pass to the quotients, we get the result for \( s = 0 \). Let us pass to \( s = 1 \) and let \( \omega \in F_i^{q+i+j} \) such that

\[ d_L \omega \in F_i^{q+i+j+1}. \]

Since \( F_i^{q+i+j} = ((G_l^{q+i+j} \cap G_k^{q+i+j}))_{k+l=i+j} \), we can find \( \omega_{kl} \in G_l^{q+i+j} \cap G_k^{q+i+j} \) for \( k + l = i + j \), such that

\[ \omega = \sum_{k+l=i+j} \omega_{kl}. \]

We have that \( d_L \omega_{kl} \in G_l^{q+i+j+1} \cap G_k^{q+i+j+1} \) similarly as in case \( s = 0 \), we can prove that actually \( d_L \omega_{kl} \in F_i^{q+i+j+1} \), using \( d_L \omega \in F_i^{q+i+j+1} \). Thus

\[ \omega \in \{\alpha \in G_f^{q+i+j} \cap G_i^{q+i+j} \mid d_L \alpha \in F_i^{q+i+j+1}\}_{k+l=i+j} \]

and hence \( E_1^{i,j,q} = (E_1^{k,l,q})_{k+l=i+j} \). Let now \( \omega_{kl} \in G_l^{q+i+j} \cap G_k^{q+i+j} \) for \( k + l = i + j \), such that \( d_L \omega_{kl} \in F_i^{q+i+j+1} \) and \( \sum_{k+l=i+j} \omega_{kl} \in F_i^{q+i+j+1} + d_L(F_i^{q+i+j+1}). \) Therefore there exists \( \alpha \in F_i^{q+i+j+1} \), such that

\[ \sum_{k+l=i+j} \omega_{kl} + d_L \alpha \in F_i^{q+i+j+1}. \]

Splitting \( \alpha = \sum_{k+l=i+j} \alpha_{kl} \) for some \( \alpha_{kl} \in G_k^{q+i+j+1} \cap G_i^{q+i+j+1} \), we get that

\[ \sum_{k+l=i+j} \omega_{kl} + d_L \alpha_{kl} \in F_i^{q+i+j+1}. \]

Additionally we have \( \omega_{kl} + d_L \alpha_{kl} \in G_l^{q+i+j} \cap G_k^{q+i+j} \). Applying the same argument as in the case \( s = 0 \), we get \( \omega_{kl} + d_L \alpha_{kl} \in F_i^{q+i+j+1} \) for all \( k + l = i + j \). Passing to the quotient, we get the result for \( s = 1 \).

We consider the differentials on the zeroth and first page and use this splitting to decompose them.

**Proposition 4.5** For the differentials \( d^0: E_0^{p,q} \rightarrow E_0^{p,q+1} \) and \( d^1: E_0^{p,q} \rightarrow E_0^{p+1,q} \), the following hold

i.) \( d^0 (E_0^{i,j,q}) \subseteq E_0^{i,j,q+1} \)

ii.) \( d^1 (E_1^{i,j,q}) \subseteq E_1^{i+j+1,q} \oplus E_1^{i,j+1,q} \)

Hence there is a canonical splitting \( d^1 = \partial^1 + \overline{\partial}^1 \), where \( \partial^1 (E_0^{i,j,q}) \subseteq E_0^{i+1,j,q} \) and \( \overline{\partial}^1 (E_0^{i,j,q}) \subseteq E_0^{i+1,j,q} \). Finally, \( (\partial^1)^2 = (\overline{\partial}^1)^2 = \partial^1 \overline{\partial}^1 + \overline{\partial}^1 \partial^1 = 0. \)
In Section 3, we have seen that a transversally complex Jacobian structure

4.2 The Obstruction Class of transversally Complex Subalgebroids

alized contact structure, if and only if there exists an extension of the inverse of

a real 2-form

We have that

The condition

Corollary 4.7

We start with the zeroth page. Let \( \omega + F_{i,j+1}^{j+i} \in E_{(i,j),q} \), such that \( \omega \in G_j^{q+i+j} \cap G_i^{j+i+j} \), then

\[
d^0(\omega + F_{i,j+1}^{j+i+j}) = d_L \omega + F_{i,j+1}^{j+i+j+1}.
\]

We have that \( d_L \omega \in G_j^{q+i+j+1} \cap G_i^{j+i+j+1} \) and hence \( d^0(\omega + F_{i,j+1}^{j+i+j}) \in E_{(i,j),q+1} \). For the first page let us choose \( \omega + F_{i,j+1}^{j+i+j} + d_L(F_{i,j}^{j+i+j-1}) \), with \( \omega \in G_j^{j+i+j} \cap G_i^{j+i+j} \) and \( d_L \omega \in F_{i,j+1}^{j+i+j+1} \). Then

\[
d_L \omega \in G_j^{j+i+j+1} \cap G_i^{j+i+j+1} \cap F_{i,j+1}^{j+i+j+1} = G_j^{j+i+j+1} \cap G_i^{j+i+j+1} + G_j^{j+i+j+1} \cap G_i^{j+i+j+1}
\]

and the claim follows.

Remark 4.6 The whole spectral sequence (rather than just its zeroth and first pages) seem to be very interesting objects in themselves. However, we will not explore it here in more detail.

4.2 The Obstruction Class of transversally Complex Subalgebroids

In Section 3 we have seen that a transversally complex Jacobian structure \((J,K)\) comes from a generalized contact structure, if and only if there exists an extension of the inverse of \(J\), \(\omega \in \Omega^3_L(M)\), and a real 2-form \(B \in \Omega^2_L(M)\), such that

\[
d_L(\omega + B)(\Delta_1, \Delta_2, \Delta_3) = 0 \quad \forall \Delta_i \in \text{pr}_D\mathcal{L}.
\]

We want to apply the techniques from the previous subsection to obtain a cohomological obstruction on this condition. Using the formalism of Subsection [11] and using the notation \(\text{im}(J^2) = S\), we see that this is equivalent to

\[
d_L(\omega + B) \in G^3_1 = G^3_1 \cap G^3_0.
\]

Since \(\omega\) is non-degenerate on \(S\), we have that \(\omega \in F^3_0 \) and \(\omega \notin F^3_1\). Thus, \(\omega + B \in G^3_0 \cap G^3_0\) and \(d_L \omega, d_L B \in F^3_1\). Hence both forms define classes in \(E_0^{(0,0),2}\), denoted by \([\omega]_0\) and \([B]_0\). Moreover, the latter are both \(d^0\)-closed in \(E_0^{(0,0),2}\) and hence they define iterated classes in \(E_1^{(0,0),2}\), denoted by \([\omega]_0\) and \([B]_0\).

Corollary 4.7 The condition \(d_L(\omega + B) \in G^3_1 = G^3_1 \cap G^3_0\) is equivalent to \(\partial^1(i[\omega]_0) + [B]_0) = 0\).

Proof: We have that \(d_L(\omega + B) \in G^3_1 \cap G^3_0\), which implies for the class

\[
d^1(i[\omega]_0 + B + F_1^2 + d_L(F_0^1)) = d_L(\omega + B) + F_2^1 + d_L(F_1^2) \in G^3_1 \cap G^3_0 + F^3_2 + d_L(F^2_1)
\]

Hence \(d^1(i[\omega]_0 + [B]_0) \in E_1^{(0,1),2}\). Using the splitting of the differential \(d^1 = \partial^1 + \overline{\partial^1}\), we get that \(\partial^1(i[\omega]_0) + [B]_0) = 0\).

We want to go a step further and ask for which \(\omega\) can we find a \(B\), such that \(d_L(\omega + B) \in G^3_1 \cap G^3_0\). The answer gives the following

Lemma 4.8 Let \(\omega \in \Omega^2_L(M)\) be real, such that \(d_L \omega \in F^3_1\). Then there exists a \(B \in \Omega^2_L(M)\), such that \(d_L(\omega + B) \in G^3_1 = G^3_1 \cap G^3_0\) if and only if

i.) \(\partial^1 \overline{\partial^1} ([\omega]_0) = 0\)

ii.) \(\overline{\partial^1}([\omega]_0) - \partial^1([\omega]_0)\) is \(d^1\)-exact in \(E_3^{(0,0),3}\).
Before we prove the Lemma, we want to discuss i.) and ii.). Note that ii.) can only be fulfilled, if i.) is fulfilled, since i.) just ensures that $\partial^1[\omega]_1 - \partial^1[\omega]_1$ is $d^1$-closed. Let us now prove the Lemma.

**Proof (of Lemma 4.8):** Let us first assume, that $d_L(\iota([\omega] + B)) \in G^3_1 = G^3_1 \cap G^3_0$ for a real $B$, which is equivalent to $\partial^1(i[\omega]_1 + [B]_0) = 0$ by Proposition 4.7. Hence we have

$$2d^1[B]_0 = 2\text{Re}(d^1[\iota([\omega] + B)]_1),$$

and hence $\partial^1[\omega]_1$ is $d^1$-closed. Assuming, on the other hand, that $\iota(\partial^1[\omega]_1 - \partial^1[\omega]_1) = d^1[B]_1$, where we can choose a real representant $B$ of $[B]_0$, then it is easy to see that $\partial^1([\iota([\omega] + B)]_1 = 0$ and hence the claim follows. 

Let us conclude this section with the main theorem of this note, which is basically just a summary of the previous results. Afterwards we will discuss the connection to generalized complex structures.

**Theorem 4.9** Let $L \rightarrow M$ be a line bundle and let $(J, K)$ be a transversally complex Jacobi structure on $L$. These data are induced by a generalized contact structure, if and only if

i.) $\partial^1 \partial^1 J^{-1} = 0$

ii.) $\partial^1 J^{-1} = \partial^1 J^{-1}$ is $d^1$-exact in $E_3^{(\ast, \ast)}$.

where we interpret $J^{-1}$ as an element in $E_0^{(0,0,2)}$. Moreover, the generalized contact structure inducing the data is of the form

$${\mathcal L} = (K \oplus \text{Ann}(K))^{\omega + B}$$

for any choice of $\omega \in J^{-1}$ real and any real $B \in \Omega^2_L(M)$ such that $[B]_0$ is closed and $d^1[B]_0 = \iota(\partial^1 J^{-1} - \partial^1 J^{-1})_1$.

**Corollary 4.10** Let $L \rightarrow M$ be a line bundle and let $(J, K)$ be a transversally complex Jacobi structure. If $[J^{-1}]_0 = 0$, then the data comes from a generalized contact structure of the form

$${\mathcal L} = (K \oplus \text{Ann}(K))^{\omega},$$

where $\omega \in J^{-1}$ is a real extension of the inverse of $J^{-1}$.

**Remark 4.11 (Generalized Complex Geometry)** Let us recall the mirror result in generalized complex geometry. In [2] the author obtains similar results, given a regular Poisson structure $\pi \in \Gamma^\infty(\Lambda^2 TM)$ and a transversally complex involutive distribution $K \subseteq T_C M$. Using the same formalism introduced in Subsection 4.1, we find that the data comes from a generalized complex structure, if and only if

i.) $\partial^1 \partial^1 [\pi^{-1}]_1 = 0$

ii.) $\partial^1 [\pi^{-1}]_1 - \partial^1 [\pi^{-1}]_1$ is $d^1$-exact in $E_3^{(\ast, \ast)}$. 

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These obstructions differ quite a lot from those found in [2]. The reason for this is that in [2] the author searches for $H$-generalized complex structures, while we are mainly interested in the obstructions for the existence of honest generalized complex structures. In view of Remark 5.10 this is not a difference in the case of generalized contact geometry, in fact it is in generalized complex geometry: not every $H$-generalized complex structure is a transformation of a generalized complex structure via a 2-form in the sense of Lemma 2.22. It is an easy exercise to see that there is an $H$-generalized complex structure inducing $(\pi, K)$, if and only if

\begin{itemize}
  \item[i.)] $d^1(\mathcal{F}[\pi^{-1}]_1 - \partial^1[\pi^{-1}]_1) = 0$
  \item[ii.)] $d^2(\mathcal{F}[\pi^{-1}]_1 - \partial^1[\pi^{-1}]_1)_2 = 0$
  \item[iii.)] $d^3(\mathcal{F}[\pi^{-1}]_1 - \partial^1[\pi^{-1}]_1)_3 = 0$.
\end{itemize}

To be more precise ii.) is only well-defined, if i.) is fulfilled and iii.) is only well-defined, if ii.) is fulfilled. These obstructions are equivalent to the ones found in [2]. It is a bit of a computational effort to prove this, since the author used a transversal to search for effort to prove this, since the author used a transversal to obtain his results. We want to stress that, as we work completely within the spectral sequence, this is not really necessary and we prefer not to make this arbitrary choice.

5 Examples I: Five dimensional nilmanifolds

Jacobi structures appear, among many other situations, as invariant Jacobi structures on Lie groups, which are canonically regular, and hence weakly regular. We begin this section describing invariant Jacobi structures. Afterwards, we will formulate everything at the level of Lie algebras.

**Definition 5.1** Let $L \to G$ be a line bundle over a Lie group $G$ and let $\Phi: G \to \operatorname{Aut}(L)$ be a group action covering the left multiplication. A Jacobi bracket $\{-, -\}: \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$ is said to be invariant, if

$$\Phi^*_g\{\lambda, \mu\} = \{\Phi^*_g\lambda, \Phi^*_g\mu\} \ \forall \lambda, \mu \in \Gamma^\infty(L), \ \forall g \in G.$$ 

Our arena is the omni-Lie algebroid $DL \oplus J^1L$ for a line bundle $L \to G$ over a Lie group. For a Lie group action $\Phi: G \to \operatorname{Aut}(L)$, we have the the canonical action

$$\mathbb{D}\Phi_g : \mathbb{D}L \ni (\Delta, \psi) \mapsto (D\Phi_g(\Delta), (D\Phi_g^{-1})^*\psi) \in \mathbb{D}L.$$ 

**Definition 5.2** Let $L \to G$ be a line bundle over a Lie group $G$, let $H \in \Gamma^\infty(\Lambda^3(DL) \otimes L)$ be closed and let $\Phi: G \to \operatorname{Aut}(L)$ be a Lie group action covering the left multiplication. A $H$-generalized contact structure $\mathcal{L} \subseteq \mathbb{D}_CL$ is said to be $G$-invariant, if and only if $\mathbb{D}\Phi_g(\mathcal{L}) = \mathcal{L}$ for all $g \in G$ and $H = \Phi^*_gH$.

**Proposition 5.3** Let $L \to G$ be a line bundle over a Lie group $G$, let $H \in \Gamma^\infty(\Lambda^3(DL) \otimes L)$ be closed, let $\Phi: G \to \operatorname{Aut}(L)$ be a Lie group action covering the left multiplication and let $\mathcal{L} \subseteq \mathbb{D}_CL$ be a $G$-invariant $H$-generalized contact structure, then its Jacobi-structure is $G$-invariant.

Let us now trivialize, with the help of the Lie group action $\Phi: G \to \operatorname{Aut}(L)$, the line bundle itself, its derivations and its first jet. Similarly to the tangent bundle of a Lie group, we have

$$L \cong G \times \ell.$$
where \( \ell = \Gamma^\infty(L)^G \). Note that \( \ell \) is a 1-dimensional vector space over \( \mathbb{R} \). Moreover, the action of \( G \) looks in this trivialization like
\[
\Phi_g(h,l) = (gh,l) \quad \forall (h,l) \in L, \forall g \in G.
\]

Additionally the Atiyah algeroid is also a trivial vector bundle by
\[
DL \cong G \times \Gamma^\infty(DL)^G,
\]
where \( \Gamma^\infty(DL)^G \) is a \((\dim(G)+1)\)-dimensional vector space over \( \mathbb{R} \). Moreover, since the symbol maps invariant derivations to left-invariant vector fields, we have the \( G \)-invariant Spencer sequence
\[
0 \to \mathbb{R} \to \Gamma^\infty(DL)^G \to \mathfrak{g} \to 0,
\]
by using the fact that \( G \)-invariant Endomorphisms are just multiplications by constants. This sequence splits canonically, since \( \Gamma^\infty(DL)^G \) acts as derivations on \( \ell \), we have \( \Gamma^\infty(L) \cong \mathcal{C}^\infty(M) \otimes_\mathbb{R} \ell \). Thus we have
\[
\Gamma^\infty(DL)^G \cong \mathfrak{g} \oplus \mathbb{R},
\]
with bracket
\[
[(\xi,r), (\eta,k)] = [(\xi,\eta),0] \quad \forall (\xi,r), (\eta,k) \in \mathfrak{g} \oplus \mathbb{R}.
\]

Similarly, using \( J^1L = (DL)^* \otimes L \), plus the choice of a basis of \( \ell \), we get
\[
\Gamma^\infty(J^1L)^G \cong \mathfrak{g}^* \oplus \mathbb{R}.
\]

The differential \( d_L \) reduces two
\[
d_L(\alpha + k1^*) = \delta_{CF}\alpha + 1^* \wedge \alpha,
\]
where \( \alpha \in \mathfrak{g}^* \) and \( 1^* \) is the projection to the \( \mathbb{R} \)-component. These are all the ingredients, we need to describe \( G \)-invariant generalized contact structures via their infinitesimal data, i.e. in terms of the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \).

**Proposition 5.4** Let \( L \to G \) be a line bundle over a Lie group \( G \), let \( H \in \Gamma^\infty(\Lambda^3(DL) \otimes L) \) be closed and let \( \Phi: G \to \text{Aut}(L) \) be a Lie group action. A \( G \)-invariant \( H \)-generalized contact structure \( L \subseteq \mathcal{D}C \) is equivalently described by a subspace \( L^\mathfrak{g} \subseteq \{(\mathfrak{g} \oplus \mathbb{R}) \oplus (\mathfrak{g}^* \oplus \mathbb{R})\}|_{\mathbb{C}} \), which is \( H \)-involutive, maximally isotropic and fulfills \( L^\mathfrak{g} \cap L^\mathfrak{g}^* = \{0\} \), where all the operations on \( \{(\mathfrak{g} \oplus \mathbb{R}) \oplus (\mathfrak{g}^* \oplus \mathbb{R})\}|_{\mathbb{C}} \) are given by the identification \( \Gamma^\infty(DL \oplus J^1L)^G = \Gamma^\infty(DL)^G \oplus \Gamma^\infty(J^1L)^G \cong \{(\mathfrak{g} \oplus \mathbb{R}) \oplus (\mathfrak{g}^* \oplus \mathbb{R})\} |_{\mathbb{C}} \).

**Proof:** The proof is based on the fact that an invariant generalized contact structure is completely characterized by its invariant sections. \( \Box \)

The idea is now to forget about the Lie group and perform all the existence proofs directly on the Lie algebra, having in mind, of course, that we can reconstruct a generalized contact structure on the Lie group by translating. Being a bit more precise, we give the following

**Definition 5.5** Let \( \mathfrak{g} \) be a Lie algebra with the abelian extension \( \mathfrak{g}_R := \mathfrak{g} \oplus \mathbb{R} \), where we denote by \( 1 \) and \( 1^* \) the canonical elements in \( \mathfrak{g} \) and \( \mathfrak{g}^* \), respectively, and let \( H \in \Lambda^3(\mathfrak{g}_R^*) \) be \( d \)-closed. The omni-Lie algebra of \( \mathfrak{g} \) is the vector space \( \mathfrak{g}_R \oplus \mathfrak{g}_R^* \) together with

i.) the (Dorfman-like, \( H \)-twisted) bracket
\[
[(X_1,\psi_1),(X_2,\psi_2)]_H = ([X_1,X_2],\mathcal{L}_{X_1}\psi_2 - \iota_{X_2}d\psi_1 + \iota_{X_1}\iota_{X_2}H)
\]
ii.) the non-degenerate pairing
\[ \langle (X_1, \psi_1), (X_2, \psi_2) \rangle := \psi_1(X_2) + \psi_2(X_1) \]

iii.) the canonical projection \( \text{pr}_D: \mathfrak{g}_R \oplus \mathfrak{g}_R^* \to \mathfrak{g}_R \)

Here the differential is given by \( d = \delta_{CE} + \imath_X^* \wedge \) and \( \mathcal{L}_X = [\iota_X, d] \).

The obvious way to define now a generalized contact Lie algebra is the following

**Definition 5.6** Let \( \mathfrak{g} \) be a Lie algebra. A generalized contact structure on \( \mathfrak{g} \) is a subbundle \( \mathcal{L} \in (\mathfrak{g}_R \oplus \mathfrak{g}_R^*)_C \), which is involutive, maximally isotropic and fulfills \( \mathcal{L} \cap \mathcal{L} = \{0\} \).

From the above discussion, we can immediately obtain

**Lemma 5.7** Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). Then there is a 1:1-correspondence of generalized contact structures on \( \mathfrak{g} \) and left-invariant generalized contact structures on \( G \times \mathbb{R} \to G \) given by the left-translation.

As in the geometric setting we have extreme cases

**Example 5.8** Let \((\mathfrak{g}, \Theta)\) be a \((2n+1)\) contact Lie algebra, i.e. \( \Theta \in \mathfrak{g}^* \), such that \( \Theta \wedge (\delta_{CE} \Theta)^n \neq 0 \), then we denote by \( \Omega = p^* \Theta \) for \( p: \mathfrak{g}_R \to \mathfrak{g} \) and get that
\[
\mathcal{L} = \{(X, \iota_X d\Omega) \in (\mathfrak{g}_R \oplus \mathfrak{g}_R^*)_C \mid X \in (\mathfrak{g}_R)_C \}
\]
gives \( \mathfrak{g} \) the structure of a generalized contact Lie algebra.

**Example 5.9** Let \( \mathfrak{g} \) be a Lie algebra and \( \phi \in \text{End}(\mathfrak{g}_R) \) be a complex structure, then
\[
\mathcal{L} = \mathfrak{g}_R^{(1,0)} \oplus \text{Ann}(\mathfrak{g}_R^{(1,0)})
\]
gives \( \mathfrak{g} \) the structure of a generalized contact Lie algebra, where \( \mathfrak{g}_R^{(1,0)} \) is the \(+i\)-Eigenbundle of \( \phi \).

We shrink ourselves now to the case of 5-dimensional nilpotent Lie algebras, since we are able to use already existing classification results, which are not available in more general classes of Lie algebras. To be more precise, we want to prove the following

**Theorem 5.10** Every five dimensional nilpotent Lie algebra possesses a generalized contact structure.

From this theorem, we can immediately conclude

**Corollary 5.11** Every five dimensional nilmanifold possesses an invariant generalized contact structure.

To prove Theorem 5.10, we will use Section 4.2. In particular, we will find a generalized contact structure on a given Lie algebra by looking for a transversally complex Jacobi structure. Afterwards, we use Theorem 4.9 to prove the existence of a generalized contact structure. Note that we did not prove the invariant analogue of Theorem 4.9, but as the proof suggests this can be done with a bit of effort.

A big help in proving Theorem 5.10 is the classification of five dimensional nilpotent Lie algebras provided in [6]. In that work the author proved that there are exactly nine (isomorphism classes of) five dimensional nilpotent Lie algebras. Since we want to prove that there are generalized contact structures on all of them, it seems convenient to test first the extreme examples, i.e. integrable
complex structures on \( g_R \) (Example 5.9) on the one hand and contact structures on the other hand (Example 5.8). For the complex structures we can use the work of Salamon in [13], where he classified all the complex nilpotent Lie algebras of six dimensions. In the following, we denote by \( \{e_1, \ldots, e_5\} \) a given basis of a five dimensional vector space \( g \). The only 5-dimensional nilpotent Lie algebras, such that \( g_R \) admits a complex structure are (we use the notation of [6] for his description of 5-dimensional nilpotent Lie algebras):

1. \( \mathfrak{L}_{5,1} \) (abelian)
2. \( \mathfrak{L}_{5,2} : [e_1, e_2] = e_3 \)
3. \( \mathfrak{L}_{5,4} : [e_1, e_2] = e_5, \ [e_3, e_4] = e_5 \)
4. \( \mathfrak{L}_{5,5} : [e_1, e_2] = e_3, \ [e_1, e_3] = e_5, \ [e_2, e_4] = e_5 \)
5. \( \mathfrak{L}_{5,8} : [e_1, e_2] = e_4, \ [e_1, e_3] = e_5 \)
6. \( \mathfrak{L}_{5,9} : [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_2, e_3] = e_5 \)

We have to check that the remaining 5-dimensional nilpotent Lie algebras \( \mathfrak{L}_{5,3}, \mathfrak{L}_{5,6} \) and \( \mathfrak{L}_{5,7} \) are also generalized contact. Let us denote by \( \{e^1, \ldots, e^5\} \) the dual basis of \( \{e_1, \ldots, e_5\} \).

5.1 \( \mathfrak{L}_{5,3} : [e_1, e_2] = e_3, \ [e_1, e_3] = e_4 \)

It is easy to see that \( J = e_3 \wedge e_1 + \mathbb{1} \wedge e_4 \) is a Jacobi structure. Additionally
\[
K := \langle 1, e_1, e_3, e_4, e_2 - i e_5 \rangle
\]
is a subalgebra of \( (g_R)_\mathbb{C} \) and that \( (J, K) \) is a transversally complex Jacobi structure. Moreover, \( \omega = e^1 \wedge e^3 - \mathbb{1}^* \wedge e^4 \) is an extension of the inverse of \( J \) and we obtain that \( d\omega = \delta_{CE}\omega + \mathbb{1}^* \wedge \omega = 0 \), which implies that \( [J^{-1}]_1 = [\omega]_0 = 0 \). Using Corollary 4.10 we see that there is a generalized contact structure inducing this data, an explicit example is given by
\[
(K \oplus \text{Ann}(K))^{i\omega}.
\]

5.2 \( \mathfrak{L}_{5,6} : [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_2, e_3] = e_5 \)

This Lie algebra is actually a contact Lie algebra with contact 1-form \( \Theta = e^5 \).

5.3 \( \mathfrak{L}_{5,7} : [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5 \)

It is easy to see that \( J = e_1 \wedge e_3 + e_4 \wedge (1 + e_5) \) is a Jacobi structure. Let us define
\[
K := \langle 1 + e_5, e_1, e_3, e_4, 1 + ie_2 \rangle.
\]
We have \( [g_R, g_R] \subset \text{im}(J^2) \) and hence integrability of the corresponding \( K \) is canonically fulfilled. Moreover, we have that \( \omega = -(e^1 \wedge e^3 + e^4 \wedge e^5) + (1 - e^5) \wedge e^4 \) is an extension of \( J^{-1} \) and is closed with respect to \( d = \delta_{CE} + \mathbb{1} \wedge \). Using Corollary 4.10 we find a generalized contact structure given by
\[
(K \oplus \text{Ann}(K))^{i\omega}.
\]
We have already seen that the Lie algebras \( L_{5,3}, L_{5,6} \) and \( L_{5,7} \) do not admit a complex structure on their one dimensional abelian extension. Moreover, \( L_{5,6} \) is a contact Lie algebra. In the following we want to show that \( L_{5,3} \) and \( L_{5,7} \) do not admit a contact structure, so that there are generalized contact structures on them but not of the extreme types. Let us first collect some basic properties of contact Lie algebras

**Theorem 5.12** Let \( g \) be a nilpotent Lie algebra and \( \Theta \in g^* \) be a contact form. Then the center \( Z(g) \) has dimension one.

A reference of Theorem 5.12 and of its proof is [12]. As a first consequence we have

**Corollary 5.13** The Lie algebra \( L_{5,3} \) is not contact.

The only Lie algebra, which is left over is \( L_{5,7} \). Here we do not have a general statement about contact Lie algebras that we can use, nevertheless we have

**Lemma 5.14** The Lie algebra \( L_{5,7} \) is not contact.

**Proof:** From the commutation relation in Subsection 5.3 it is clear that we have \( \delta_{CE}(\Lambda^* g^*) \subseteq e^1 \Lambda^* g^* \). Hence we have that for all \( \alpha \in g^* \delta_{CE}\alpha = e^1 \wedge \beta \) for some \( \beta \in g^* \). As a consequence \( \alpha \wedge (\delta_{CE}\alpha)^2 = 0 \) for all \( \alpha \in g^* \), and hence the Lie algebra can not be contact. \( \square \)

**Remark 5.15** To prove that \( L_{5,1}, L_{5,2}, L_{5,8} \) and \( L_{5,9} \) are not contact it is enough to obtain that all of them can be excluded by Theorem 5.12. Moreover, for the remaining ones the contact structures are given by \( e^5 \).

As a summary we have the following table

|       | contact | \( g_\mathbb{R} \)-complex | generalized contact |
|-------|---------|---------------------------|---------------------|
| \( L_{5,1} \) | ×       | ✓                         | ✓                   |
| \( L_{5,2} \) | ×       | ✓                         | ✓                   |
| \( L_{5,3} \) | ×       | ×                         | ✓                   |
| \( L_{5,4} \) | ✓       | ✓                         | ✓                   |
| \( L_{5,5} \) | ✓       | ✓                         | ✓                   |
| \( L_{5,6} \) | ✓       | ×                         | ✓                   |
| \( L_{5,7} \) | ×       | ×                         | ✓                   |
| \( L_{5,8} \) | ×       | ✓                         | ✓                   |
| \( L_{5,9} \) | ×       | ✓                         | ✓                   |

### 6 Examples II: Contact Fiber Bundles

The next class of examples are contact fiber bundles over a complex base manifold. We begin explaining what we mean by contact fiber bundle. Similarly to symplectic fiber bundles, there is a contact structure on the vertical bundle

\[ \text{Ver}_L(P) = \sigma^{-1}(\text{Ver}(P)) \subseteq DL, \]

for a line bundle \( L \rightarrow P \), such that \( P \rightarrow M \) is a fiber bundle. More precisely:
Definition 6.1 Let $\pi : P \to M$ be a fiber bundle and let $L \to P$ by a line bundle. A smooth family of contact manifolds is the data of $L \to P$ together with a closed non-degenerate 2-form $\omega \in \Gamma^\infty(\Lambda^2(\text{Ver}_L(P))^* \otimes L)$. If additionally the contact structures $(L|_{P_m} \to P_m, \omega|_{D(L|P_m)})$ are contactomorphic, we say that $L \to P$ is a contact fiber bundle.

Before we come to examples, we want to make some general remarks on smooth families of contact structures and contact fiber bundles, which are more or less known.

Remark 6.2 Let $(L \to P, \pi : P \to M, \omega)$ be a smooth family of contact structures. If the fiber is compact and connected and the base is connected, then the data automatically is a contact fiber bundle. This follows from the stability theorem of Gray in [7], which states that two contact forms which are connected by a smooth path of contact structures are contactomorphic.

Remark 6.3 As in the setting of symplectic fiber bundles, we can express the data in local terms, namely: the datum of a contact fiber bundle over a manifold $M$ with typical fiber $F$ is equivalent to:

i.) a line bundle $L_F \to F$ and a contact 2-form $\omega \in \Gamma^\infty(\Lambda^2(DL)^* \otimes L)$

ii.) an open cover $\{U_i\}_{i \in I}$

iii.) smooth transition maps $T_{ij} : U_i \cap U_j \to \text{Aut}(L)$ which are point-wise contactomorphisms

Remark 6.4 Obviously, one can define smooth families of contact structures as a Jacobi structure of contact type, such that the characteristic distribution of it is the vertical bundle of a fiber bundle.

Using this remarks, we can show that under certain assumptions on the base, a smooth family of contact structures always induces a generalized contact structure on the total space.

Lemma 6.5 Let $\pi : P \to M$ be a fiber bundle with typical fiber $F$ over a complex base $M$, let $L \to M$ by a line bundle and let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi structure giving $P$ the structure of a smooth family of contact manifolds. Then $P$ possesses a generalized contact structure with Jacobi structure $J$.

Proof: First of all, we notice that that the Jacobi structure is weakly regular, since $\text{im}(J^2) = \text{Ver}_L(P)$. The only thing what we have to show is that the data induce a transversally complex Jacobi structure, since we have that the inverse of the Jacobi structure is leaf-wise exact with canonical primitive $\iota_1 \omega$, which implies that $|J^{-1}| = 0$ by Corollary 4.10. Let us denote by $T^{(1,0)}M \subseteq T^C_M$ the holomorphic tangent bundle induced by the complex structure on $M$. With this we define

$$K := (\sigma \circ T\pi)^{-1}(T^{(1,0)}M) \subseteq D^C_L.$$  

It is an easy consequence of the definitions of the bundles that $(J, K)$ is a transversally complex Jacobi structure, and hence there exists a generalized contact structure inducing it.  

We see that in this case the existence of a generalized contact structure is unobstructed. Thus we want to show that this kind of structures exist and give some classes of examples.

Example 6.6 (Projectivized Vertical Bundle) Let $\pi : P \to M$ fiber bundle with typical fiber $F$ over a complex base $M$. Given a set of local trivializations $(U_i, \tau_i)_{i \in I}$

![Diagram](image-url)
with transition functions $\tau_{ij} : U_i \cap U_j \rightarrow \text{Diffeo}(F)$. Let us denote by $T_\ast \tau_{ij} : U_i \cap U_j \rightarrow \text{Diffeo}(T^\ast F)$ there cotangent lifts, which fulfill also the cocycle condition for transition functions and hence belong to a fiber bundle $\tilde{V} \rightarrow M$ (actually this is $\text{Ver}^\ast(P)$) with local trivializations $(U_i, \phi_i)_{i \in I}$, such that the transition functions fulfill $\phi_{ij} = T_\ast \tau_{ij}$. We consider now the canonical symplectic form $\omega_{can} \in \Gamma^\infty(\Lambda^2 T^\ast (T^\ast F))$ on $T^\ast F$, note that the functions $T\tau_{ij}$ are symplectomorphisms, since they are point transformations. The next step is to consider the fiber bundle $V \rightarrow M$ with typical fiber $T^\ast F \setminus 0_F$, which we get by the obvious restrictions. Note that on $T^\ast F \setminus 0_F$ we have a canonical $\mathbb{R}^\times$-action which is free and proper and the for restricted symplectic form $\omega_{can}$ we have that $\mathcal{L}_{(1)}\tau_{ij} \omega_{can} = \omega_{can}$. Using the results from [3], we conclude that the associated line bundle $L \rightarrow \mathbb{R} T^\ast F := \frac{T^\ast F \setminus 0_F}{\mathbb{R}^\times}$ carries a contact structure and the transitions functions, which are obviously commuting with the $\mathbb{R}^\times$-action, act as line bundle automorphisms preserving the contact structure. This is exactly the data we need to cook up a contact fiber bundle \cite{6.3} and hence its total space possess a generalized contact structure, due to Lemma [6.5]. Note that here the input was a generic fiber bundle over a complex base and the output is a generalized contact bundle. Moreover, if both the base and the fiber are compact, then the output is also compact. We hence proved the existence of compact examples.

After this rather general construction of examples of contact fiber bundles, we want to give a more down-to-earth class which also naturally appears.

**Example 6.7 (Principal fiber Bundles)** Let $\mathfrak{g}$ be a Lie algebra with a contact 1-form $\Theta \in \mathfrak{g}^\ast$. Let us consider a Lie group $G$ integrating $\mathfrak{g}$ and a manifold $M$ with a complex structure. Additionally, let $P \rightarrow M$ be a $G$-principal fiber bundle and let $\mathbb{R}_P \rightarrow P$ be the trivial line bundle, where we denote by $1_P$ the generating section. Note that here the gauge algebroid splits canonically as $D\mathbb{R}_P = TP \oplus \mathbb{R}_P$. Moreover, we have that $\text{Ver}_{\mathbb{R}_P}(P) = \text{Ver}(P) \oplus \mathbb{R}_P$. Thus, a generic derivation $\Delta_p \in \text{Ver}_{\mathbb{R}_P}(P)$ is of the form $\Delta = ((\xi_p(p), k) \in \text{Ver}(P) \oplus \mathbb{R}_P$ for the fundamental vector field $\xi_p$ of a unique $\xi \in \mathfrak{g}$ and for $p \in P$. We define $\omega \in \Gamma^\infty(\Lambda^2(\text{Ver}_{\mathbb{R}_P}(P))^\ast \otimes \mathbb{R}_P)$ by

$$\omega(\xi, \eta) = \frac{1}{2} \left( \delta_{\xi} \Theta(\eta) + k \Theta(\eta) - r \Theta(\xi) \right) \cdot 1_P(p).$$

It is easy to check that $\omega$ gives $P \rightarrow M$ the structure of a contact fiber bundle. Since $M$ was assumed to be complex, we can apply Lemma [6.5] to obtain a generalized contact bundle on $P$. Note that this notion includes $S^1$-principal fiber bundles over a complex manifold. Moreover, contact Lie algebras are an active field of research and there are many examples around and even a classification of nilpotent contact Lie algebras in [10].

**Remark 6.8** There is also the notion of smooth families of locally conformal symplectic structures, which corresponds to a weakly regular Jacobi structures with just even dimensional leaves such that the projection to the leaf space is the projection map of fiber bundle. A transversally complex Jacobi structure can be induced, at least in some cases, by an Atiyah-complex structure on the base. In this specific case there are examples which do not come from generalized contact structure (one explicit counterexample is provided in Section [7]). Since the notion of Atiyah-complex structures is by far not as developed as the notion of complex structures, it is not very easy at the moment to construct bigger classes of counter examples.

### 7 A Counterexample

In this last section, we want to construct a transversally complex Jacobi structure which cannot be induced by a generalized contact structure. The remarkable feature of this counterexample is, that it is, as manifolds, a global product of a (locally conformal) symplectic manifold and an Atiyah-complex manifold. Note that in [14] it was proven that every generalized contact bundle is locally isomorphic to a product, however not all products arise in this way.
Let us consider the 2-sphere $S^2$ and its symplectic form $\omega \in \Gamma^\infty (\Lambda^2 T^* S^2)$. Its inverse $\pi \in \Gamma^\infty (\Lambda^2 TM)$ is a Poisson structure and hence $\pi + 1 \wedge 0 = \pi$ is a Jacobi structure on the trivial line bundle.

The second manifold which is involved is the circle $S^1$. Our counterexample will live on the trivial line bundle over the product

$$\mathbb{R}_M \to M := S^2 \times S^1.$$ Using Remark 2.13, we see that

$$D \mathbb{R}_M = TM \oplus \mathbb{R}_M = TS^2 \oplus TS^1 \oplus \mathbb{R}_M$$

and we can define a Jacobi structure $J = \pi + 1 \wedge 0 = \pi$ on it by "pulling back". We see that $\text{im}(J^2) = TS^2 \subseteq D \mathbb{R}_M$.

The next step is to choose an everywhere non-vanishing vector field $e \in \Gamma^\infty (TS^1)$ and define

$$K := T_C S^2 \oplus (1 - ie) \subseteq D_C \mathbb{R}_M.$$ Note that the derivation $1 - ie \in (TS^1 \oplus \mathbb{R}_S)_C \subseteq D_C \mathbb{R}_S$ is the $+i$-Eigenbundle of an Atiyah-complex structure on $\mathbb{R}_S \to S^1$. Moreover, we have

$$D_C \mathbb{R}_M = T_C S^2 \oplus (1 - ie) \oplus (1 + ie).$$

An easy computation shows that $(J, K)$ is a transversally complex Jacobi structure. Our claim is now that $(J, K)$ can not be induced by a generalized contact structure. To see this, let us examine the $\text{Der}$-complex a bit closer. We have that

$$\Lambda^k(D \mathbb{R}_M)^* \otimes \mathbb{R}_M = \Lambda^k(TM^* \oplus \mathbb{R}_M).$$

Using the notation of Remark 2.13, we obtain that a $\psi \in \Gamma^\infty (\Lambda^k(TM^* \oplus \mathbb{R}_M))$ can be uniquely written as $\psi = \alpha + 1^* \wedge \beta$ for some $(\alpha, \beta) \in \Gamma^\infty (\Lambda^k T^* M \oplus \Lambda^{k-1} T^* M)$. It is an easy verification to show that for the differential $d_{\mathbb{R}_M}$ we have

$$d_{\mathbb{R}_M}(\alpha + 1^* \wedge \beta) = d\alpha + 1^* \wedge (\alpha - d\beta)$$

where $d$ is the usual de Rham differential, [14 Remark 1.1.1]. Now we want to pass to the spectral sequence, therefore we split the $\text{Der}$-complex according to the splitting of Equation 7.1 we have

$$\Omega^{(i,j)}_{\mathbb{R}_M} = \Gamma^\infty(\Lambda^j T^* S^2 \otimes \Lambda^i (1^* + i\alpha) \otimes \Lambda^j (1^* - i\alpha)),$$

where $\alpha \in \Gamma^\infty(TS^1)$ such that $\alpha(e) = 1$.

Note that here we have the canonical identification $D \mathbb{R}_M \supseteq \text{Ann}(TS^2) = T^* S^1 \oplus \mathbb{R}_M$, which allows us to identify

$$E_0^{(i,j), q} = \Omega^{(i,j), q}_{\mathbb{R}_M}.$$ In case of a product, the differential $d_{\mathbb{R}_M}$ splits canonically with respect to the bi-grading into

$$d_L = d^0 + \partial^1 + \bar{\partial}^1,$$

where $d^0 : \Omega^{(i,j), q}_{\mathbb{R}_M} \to \Omega^{(i,j), q+1}_{\mathbb{R}_M}$, $\partial^1 : \Omega^{(i,j), q}_{\mathbb{R}_M} \to \Omega^{(i+1,j), q}_{\mathbb{R}_M}$ and $\bar{\partial}^1 : \Omega^{(i,j), q}_{\mathbb{R}_M} \to \Omega^{(i,j+1), q}_{\mathbb{R}_M}$. Additionally, all three maps are differentials themselves and anticommute pairwise. Now we want to consider the inverse of $J$, which is the pullback of $\omega$ with respect to the canonical projection $S^2 \times S^1 \to S^2$. With
a tiny abuse of notation we will see $\omega$ is an element of $\Gamma^\infty(\Lambda^2 T^*S^2) \subseteq F_0^{(0,0),2}$. A long and not very enlightening computation shows that

$$\partial^1 \overline{\partial}^1 \omega = \frac{1}{4}((\overline{1}^* - i\alpha) \wedge (1^* + i\alpha) \wedge \omega)$$

Hence, we have for the cohomology class

$$\partial^1 \overline{\partial}^1 [\omega]_0 \overline{1}_1 = [[(\overline{1}^* - i\alpha) \wedge (1^* + i\alpha) \wedge \omega]_0]_1.$$  

But this cannot vanish, since a $d^0$-primitive $\psi$ has to be of the form

$$\psi = (\overline{1}^* - i\alpha) \wedge (1^* + i\alpha) \wedge \beta$$

for $\beta \in \Gamma^\infty(T^*S^2)$, which implies that $d\beta = \omega$, which is an absurd because the symplectic form on the sphere is not exact.

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