Anisotropic viscoacoustic wave modelling in VTI media using frequency-dependent complex velocity

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Abstract
Under the conditions of acoustic approximation and isotropic attenuation, we derive the pseudo- and pure-viscoacoustic wave equations from the complex constitutive equation and the decoupled P-wave dispersion relation, respectively. Based on the equations, we investigate the viscoacoustic wave propagation in vertical transversely isotropic media. The favourable advantage of these formulas is that the phase dispersion and the amplitude dissipation terms are inherently separated. As a result, we can conveniently perform the decoupled viscoacoustic wavefield simulations by choosing different coefficients. In the computational process, a generalised pseudo-spectral method and a low-rank decomposition scheme are adopted to calculate the wavenumber-domain and mixed-domain propagators, respectively. Because low-rank decomposition plays an important role in the simulated procedure, we evaluate the approximation accuracy for different operators using a linear velocity model. To demonstrate the effectiveness and the accuracy of our method, several numerical examples are carried out based on the new pseudo- and pure-viscoacoustic wave equations. Both equations can effectively describe the viscoacoustic wave propagation characteristics in vertical transversely isotropic media. Unlike the pseudo-viscoacoustic wave equation, the pure-viscoacoustic wave equation can produce stable viscoacoustic wavefields without any SV-wave artefacts.

Keywords: complex velocity, pseudo-viscoacoustic, pure-viscoacoustic, generalised pseudo-spectral method, low-rank decomposition

1. Introduction
The ubiquitous seismic anisotropy is deemed as an important property that has an effect on wave propagation and reverse-time migration. Generally, there are several numerical algorithms to achieve the wavefield simulation in anisotropic media. A typical approach is to directly solve the full elastic wave equations and then decouple the wavefield into P- and S-wave components. However, the wavefield separation in heterogeneous anisotropic media is very complicated and extremely computational inefficient (Yan & Sava 2009; Xu & Zhou 2014; Wang et al. 2016).

Given this problem, some scholars propose to simulate the P-wave propagation using a pseudo-acoustic (PSA) wave equation, which is derived from the acoustic approximation (Alkhalifah 1998, 2000; Du et al. 2007; Fowler et al. 2010). Rao & Wang (2019) explicitly derive the numerical dispersion relation and stability condition for the PSA wave equation. Compared
with the elastic case, this approach can effectively implement P-wave propagation and largely reduce the computational cost. Unfortunately, setting SV-wave velocity to zero along the axis cannot completely eliminate the SV-wave artefacts in all directions (Grechka & Tsvankin 2004; Chu et al. 2011; Xu & Liu 2018). On the contrary, the pure-acoustic (PUA) wave equation, derived from the decoupled P-wave dispersion relation, provides a stable and SV-wave free solution for the acoustic wavefield simulation in anisotropic media (Chu et al. 2011; Xu & Zhou 2014; Yan & Liu 2016).

The currently available modelling approaches for anisotropic acoustic wave usually ignore the attenuating effects of the real Earth materials. However, seismic wave propagation in attenuating media usually suffers from amplitude dissipation and phase dispersion (Wang & Guo 2004; Wang 2008; Guo et al. 2016; Chen et al. 2019). Without properly considering the attenuating effects, poor illuminations and incorrect structural positions may occur in migrated image (Sun et al. 2016; Guo & McMechan 2018; Sun & Zhu 2018).

So far, there are mainly three kinds of theory to describe the attenuating behaviours in the numerical simulation. The first method, based on the constant-Q model, gives a convenient description about seismic attenuation (Kjartansson 1979). Under this theory, the relation between stress and strain is expressed as a convolution operator, which can be computed by fractional time derivatives (Caputo & Mainardi 1971; Podlubny 1999; Carcione et al. 2002; Carcione 2009). It is notable that solving the fractional time derivative requires storing whole previous wavefields, which is unacceptable for practical calculation. To overcome this limitation, Carcione (2010) adopts fractional Laplacians to replace fractional time derivatives. Following Carcione’s work, Zhu & Harris (2014) further derive a viscoacoustic wave equation with decoupled phase dispersion and amplitude dissipation terms. This novel formulation makes it convenient to compensate for energy loss during wavefield backward propagation (Zhu 2014; Sun et al. 2015; Sun & Zhu 2018). On the other hand, the fractional Laplacians associated with phase dispersion and amplitude dissipation are spatially varied. Commonly, they can be calculated by locally homogeneous approximations (Zhu & Harris 2014), constant-order fractional derivative polynomials (Chen et al. 2016; Li et al. 2016; Wang et al. 2018) or low-rank decompositions (Chen et al. 2016; Sun et al. 2016).

The second approach to simulate Q effects is based on the generalised standard linear solid theory. Using a series of standard linear solid elements can closely approximate the constant-Q model over a specific frequency range (Day & Minster 1984; Carcione et al. 1988; Blanch et al. 1995). However, the related equation does not explicitly contain a quality factor, instead, it is replaced with a set of optimised relaxation times (Emmerich & Korn 1987; Liao & McMechan 1996; Zhu et al. 2013). Moreover, it will increase the complexity of the viscoacoustic wave equations when multiple elements are adopted to approximate the constant-Q model in a certain frequency band. Simultaneously, the separation of phase dispersion and amplitude dissipation in viscoacoustic wave equation becomes much harder (Zhu et al. 2014; Yang & Zhu 2018a).

The third effective methodology depicts the attenuating property by using a complex velocity (Aki & Richards 2002). Based on this theory, Yang & Zhu (2018a) derive a novel viscoacoustic wave equation to simulate the wavefield propagation in the time domain. Analogous to the viscoacoustic wave equation derived by Zhu & Harris (2014), this new expression also contains a real part and an imaginary part, which correspond to the phase dispersion and amplitude dissipation, respectively. However, the equation derived by Zhu & Harris (2014) includes two variable-order fractional Laplacian operators, which are determined by wavenumbers and quality factors. Calculating the fractional operators with a low-rank decomposition method would consume many computational resources. Based on the approximation of angular frequency, the wavenumber and the velocity are separated from each other. Thus, the calculation of the new complex-valued time-domain viscoelastic wave equation is more flexible and more efficient. Based on the new equation, Yang & Zhu (2018b) further develop a viscoacoustic reverse-time migration approach to correct the phase dispersion and the amplitude dissipation caused by seismic attenuation, and thus, obtain the subsurface migration profiles with accurate spatial locations as well as amplitudes.

To simulate the attenuating mechanism in anisotropic media, Bai & Tsvankin (2016) present a 2D time-domain finite-difference algorithm for viscoelastic wavefield simulation in vertical transversely isotropic (VTI) attenuating media. Zhu (2017) derives the viscoelastic wave equations based on the fractional time derivatives. By transforming fractional time derivatives into fractional Laplacians, Zhu & Bai (2019) further implement wavefield simulation in VTI attenuating media. Moreover, Silva et al. (2019) also develop a system of viscoacoustic wave equations to simulate the viscoacoustic wave propagation in VTI media. However, all of these approaches use either standard linear solid theory or a fractional Laplacian method.

In this paper, we first introduce a complex velocity to describe the attenuating property. Second, we derive the novel pseudo-viscoacoustic (PSVA) and pure-viscoacoustic (PUVA) wave equations in VTI media from the complex constitutive equation and decoupled P-wave dispersion relation, respectively. Third, a generalised pseudo-spectral method and a low-rank decomposition scheme are introduced to solve the wavenumber-domain and mixed-domain operators, respectively. Fourth, we carry out accuracy analyses in terms of several operators with a linear velocity model based on low-rank approximation.
Fifth, three numerical examples are adopted to prove the applicability and the validity of our new approach during viscoacoustic wavefield simulation. Finally, conclusions are summarised based on analyses and experiments.

2. Theory and method

2.1. Pseudo-viscoacoustic wave equations

Based on acoustic approximation, the stress–strain relationship for VTI elastic media can be symbolically written as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{bmatrix} = \rho \begin{bmatrix}
V_p^2 (1 + 2\varepsilon) & V_p^2 (1 + 2\varepsilon) & V_p^2 \sqrt{(1 + 2\delta)} \\
V_p^2 (1 + 2\varepsilon) & V_p^2 (1 + 2\varepsilon) & V_p^2 \sqrt{(1 + 2\delta)} \\
V_p^2 \sqrt{(1 + 2\delta)} & V_p^2 \sqrt{(1 + 2\delta)} & V_p^2
\end{bmatrix} \begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33}
\end{bmatrix},
\]

(1)

where \( \rho \) is the density, \( \sigma_{ij} \) and \( e_{ij} \) are the components of stress and strain tensor, \( V_p \) is the velocity along symmetry axis and \( \varepsilon \) and \( \delta \) are the Thomsen (1986) anisotropy parameters. To simulate the attenuating property of viscoacoustic wave propagation, a frequency-dependent complex velocity is introduced in the frequency domain (Aki & Richards 2002; Yang & Zhu 2018a):

\[
V(x, \omega) = V_0(x) \left[ 1 + \frac{1}{\pi Q(x)} \ln \left| \frac{\omega}{\omega_0} \right| - \frac{\text{sgn}(\omega)}{Q(x)} \right],
\]

(2)

where \( x \) is the spatial coordinate, \( \omega \) is the angular frequency, \( \omega_0 \) denotes the reference angular frequency, \( V_0 \) is a reference velocity at \( \omega_0 \) and \( i = \sqrt{-1} \) is the imaginary unit. The conspicuous advantage of equation (2) is that the phase dispersion and the amplitude dissipation terms are inherently decoupled. Here, \( (1 + \frac{1}{\pi Q} \ln |\omega/\omega_0|) \) is related to the phase dispersion, and \( \frac{\text{sgn}(\omega)}{Q} \) associates with amplitude dissipation.

In an elastic medium, the bulk modulus is defined as

\[
M_0 = \rho V_0^2.
\]

(3)

Combining equations (2) and (3), and neglecting high-order terms with respect to \( Q \), we can formulate the complex bulk modulus as follow

\[
M_c = \rho V_0^2 \left[ 1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right| - \frac{\text{sgn}(\omega)}{Q} \right].
\]

(4)

Taking the isotropic quality factor into account, we get the following relation in attenuating media based on equations (1–4):

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{bmatrix} = \begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{12} & D_{11} & D_{13} \\
D_{13} & D_{13} & D_{33}
\end{bmatrix} \begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33}
\end{bmatrix},
\]

(5)

in which

\[
D_{11} = \rho V_p^2 \left[ 1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right| - \frac{\text{sgn}(\omega)}{Q} \right] (1 + 2\varepsilon),
\]

(6)

\[
D_{13} = \rho V_p^2 \left[ 1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right| - \frac{\text{sgn}(\omega)}{Q} \right] \sqrt{(1 + 2\delta)},
\]

(7)

\[
D_{33} = \rho V_p^2 \left[ 1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right| - \frac{\text{sgn}(\omega)}{Q} \right].
\]

(8)
Figure 1. Error variations for phase dispersion (a) and amplitude dissipation (b) with reference velocities, phase angles and anisotropy parameters. Note that the model is an elliptically anisotropic medium.

Equation (5) has a similar structure to that of PSA wave equation in transversely isotropic (TI) media. After some transformations, we can obtain the equivalent 2D PSVA wave equations as follows:

\[
\frac{\partial^2 \sigma_H}{\partial t^2} = D_{11} \frac{\partial^2 \sigma_H}{\partial x^2} + D_{13} \frac{\partial^2 \sigma_V}{\partial z^2},
\]

(9)

\[
\frac{\partial^2 \sigma_V}{\partial t^2} = D_{13} \frac{\partial^2 \sigma_H}{\partial x^2} + D_{33} \frac{\partial^2 \sigma_V}{\partial z^2},
\]

(10)

where \(\sigma_H\) and \(\sigma_V\) are the stresses in the horizontal and vertical directions, respectively. Based on equations (9) and (10), we can resolve the PSVA wavefield extrapolation in attenuating media. However, the modelling results of PSVA wave still contain some undesired SV-wave components. In addition, the stability of these two equations is seriously restricted by the condition of \(\epsilon \geq \delta\) (Chu et al. 2011; Xu & Liu 2018). When ignoring the phase dispersion and the amplitude dissipation terms of the complex velocity, equations (9) and (10) then degenerate into the PSA wave equations.

2.2. Pure-viscoacoustic wave equation

Another effective scheme to implement the acoustic wave propagation is solving the decoupled P-wave dispersion relation (Tsankin 1996). For the attenuating media, the complex-valued P-wave phase velocity can be briefly written as follows (Appendix A):

\[
2\rho V^2(\theta) = D_{11} \sin^2 \theta + D_{33} \cos^2 \theta + [(D_{11} \sin^2 \theta - D_{33} \cos^2 \theta)^2 + 4D_{13}^2 \sin^2 \theta \cos^2 \theta]^{1/2},
\]

(11)

where \(\theta\) is the phase angle. After some simplifications, equation (11) can be reformulated as follows:

\[
\omega^2 = \frac{D_{33}}{2\rho} \left\{ \frac{(1 + 2\epsilon)k_x^2 + k_z^2}{(1 + 2\epsilon)k_x^2 + k_z^2} \right\} \left\{ 1 - \frac{4(\epsilon - \delta)k_x^2 k_z^2}{(1 + 2\epsilon)k_x^2 + k_z^2} \right\},
\]

(12)

where \(k_x = k \sin \theta\) is the wavenumber in the \(x\) direction and \(k_z = k \cos \theta\) is the wavenumber in the \(z\) direction. When applying the first-order Taylor series expansion to approximate the square-root in equation (12) and ignoring the influence of \(\epsilon\) on denominator, we attain the following PUVA wave equation (Chu et al. 2011; Zhan et al. 2012; Yan & Liu 2016)

\[
\omega^2 = V_p^2 \left[ 1 + \frac{2}{\pi Q} \ln \frac{\omega}{\omega_0} \right] \left[ 1 + 2\epsilon \right] k_x^2 + k_z^2 - \frac{2(\epsilon - \delta)k_x^2 k_z^2}{k_x^2 + k_z^2} \right\},
\]

(13)

After that, the PUVA wavefield extrapolation in VTI media can be effectively implemented by solving equation (13).
Table 1. Some abbreviations of different numerical methods.

| Abbreviation | Meaning                      |
|--------------|------------------------------|
| PSA          | Pseudo-acoustic wave         |
| PSVA         | Pseudo-viscoacoustic wave    |
| PUA          | Pure-acoustic wave           |
| PUVA         | Pure-viscoacoustic wave      |

2.3. Numerical implementation

With respect to the amplitude dissipation term \( \frac{\text{sgn}(\omega)}{Q} \), we adopt the following approximation (Yang & Zhu 2018a)

\[
\frac{\text{sgn}(\omega)}{Q} = \frac{i\omega}{|\omega|} \approx \frac{i\omega}{|k|V_P},
\]

(14)

where \( k = (k_x, k_z) \) is the wavenumber vector. Equation (14) can be conveniently computed by using a generalised pseudo-spectral method. With respect to the dispersion term \( \frac{2}{\pi Q} \ln |\omega_{00}| \), the wavenumber and the velocity are coupled in the natural logarithm, which cannot be straightforwardly computed by Fourier transform (FT). Given this issue, a low-rank decomposition method is introduced to approximate this operator. After that, the mixed-domain propagator can be decomposed into a separable representation (Engquist & Ying 2009; Fomel et al. 2013):

\[
W(x, k) \approx M \sum_{m=1}^{M} \sum_{n=1}^{N} W(x, k_m) a_{mn} W(x_n, k),
\]

(15)

where \( W(x, k_m) \) is a submatrix of \( W(x, k) \) that is composed of selected columns related to wavenumbers, \( W(x_n, k) \) is another submatrix that comprises selected rows associated with spatial positions and \( a_{mn} \) is the middle matrix coefficients. The low-rank approximation is highly accurate. Generally, the parameters \( M = N = 3 \) can satisfy the requirement of accuracy well.

To illustrate the mechanism of low-rank decomposition, we take \( D_1 \frac{\partial^2 \sigma_H}{\partial x^2} \) in equation (9) as an example:

\[
D_1 \frac{\partial^2 \sigma_H}{\partial x^2} = \rho V_P^2 (1 + 2\varepsilon) \left\{ F^{-1} \left[ -k_x^2 F(\sigma_H) \right] + \frac{2}{\pi Q} F^{-1} \left[ -k_x^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_H) \right] + \frac{1}{Q V_P} F^{-1} \left[ \frac{-k_x^2}{|k|} F(\partial \sigma_H) \right] \right\},
\]

(16)

where \( \partial \) denotes the first-order time derivative, \( F \) and \( F^{-1} \) denote forward and inverse FTs, respectively. Using a low-rank decomposition method to approximate \( -k_x^2 \ln \left| \frac{\omega}{\omega_0} \right| \), we have

\[
F^{-1} \left[ -k_x^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_H) \right] \approx \sum_{m=1}^{M} W(x, k_m) \left\{ \sum_{n=1}^{N} a_{mn} F^{-1} \left[ W(x_n, k) F(\sigma_H) \right] \right\}.
\]

(17)

Analogously, the other terms in equations (9), (10) and (13) also can be calculated by using a generalized pseudo-spectral method and a low-rank decomposition scheme (Appendix B). For each time circulation, solving equations (9) and (10) requires four forward FTs and \( (4 + 2N) \) inverse FTs, and solving equation (13) requires two forward FTs and \( (6 + 3N) \) inverse FTs. Note that \( N \) is a parameter of low-rank decomposition.

2.4. Accuracy analyses

(1) Accuracy analyses for the approximation of \( \omega \approx |k|V_P \)

During the numerical implementation, \( \omega \approx |k|V_P \) is adopted to solve PSVA and PUVA wave equations. However, the approximation has some limitations in anisotropic media. Therefore, the accuracy analyses for this approximation are necessary.

In an anisotropic medium, the relationship between angular frequency and wavenumber is written as:

\[
\omega = |k(\theta)| \psi(\theta),
\]

(18)
where \( v(\theta) \) is the phase velocity. Under the acoustic assumption, the phase velocity in VTI media can be expressed as (Thomsen 1986):

\[
v^2(\theta) \approx V_p^2(1 + \varepsilon \sin^2 \theta + E(\theta)), \tag{19}
\]

and

\[
E(\theta) = \frac{1}{2} \left[ \sqrt{1 + (8\delta - 4\varepsilon)\sin^2 \theta \cos^2 \theta + 4\varepsilon(1 + \varepsilon)\sin^4 \theta} - 1 \right]. \tag{20}
\]
Figure 3. Wavefield snapshots computed by PSA (a), PSVA (b and c), PUA (e) and PUVA (f and g) wave equations using the generalised pseudo-spectral method (a, b, e and f) and the low-rank decomposition scheme (c and g). (d) is the wavefield difference between (b) and (c) and (h) is the wavefield difference between (f) and (g). Note that all the wavefield snapshots are plotted using the same colorbar.
Figure 4. Wavefield snapshots of PSVA and PUVA at 0.7 s in a homogeneous medium. (a) and (b) are obtained by using the anisotropic acoustic, dispersion-effect, dissipation-effect and viscoacoustic wave equations, respectively, and (c) and (d) are obtained by using the viscoacoustic wave equation with different Q values. Note that all the wavefield snapshots are plotted using the same colorbar.

In the complex stiffness coefficients $D_{11}$, $D_{13}$ and $D_{33}$, $(1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right|)$ is related to the phase dispersion, and $\frac{\text{sgn}(\omega)}{Q}$ associates with the amplitude dissipation. For the phase dispersion term, we have

$$
\left( 1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right| \right) = \left( 1 + \frac{2}{\pi Q} \ln \left| \frac{\mathbf{k} \cdot \mathbf{v}(\theta)}{\omega_0} \right| \right) \approx \left( 1 + \frac{2}{\pi Q} \ln \left| \frac{\mathbf{k} \cdot V_p}{\omega_0} \right| \right). \quad (21)
$$

Using reference velocity $V_p$ to replace phase velocity $v(\theta)$ in anisotropic media would produce some errors, which can be defined as follows:

$$
\text{error} = \frac{2}{\pi Q} \left( \ln |v(\theta)| - \ln |V_p| \right). \quad (22)
$$

Based on equation (22), we calculate the errors with different reference velocities, phase angles and anisotropy parameters, shown in figure 1a (for convenience, the medium is elliptically anisotropic). It can be seen that the approximation error is very small.
Figure 5. Extracted single traces of PSVA and PUVA at the distance of 750 m in a homogeneous medium. (a) and (b) are calculated by using the anisotropic acoustic, dispersion-effect, dissipation-effect and viscoacoustic wave equations, respectively, and (c) and (d) are calculated by using the viscoacoustic wave equation with different Q values.

Figure 6. Two-layered model.

Without considering the dispersion effect of complex velocity, we have

$$\left[1 - \frac{\text{isgn}(\omega)}{Q}\right] \approx \left[1 - \frac{i\omega}{Q |k| V_P}\right]. \quad (23)$$

After importing the approximation $\omega \approx |k| V_P$, the error of equation (23) can be defined as:

$$\text{error} = \frac{i\omega}{Q |k| v(\theta)} - \frac{i\omega}{Q |k| V_P} = \frac{i}{Q} \left(1 - \frac{v(\theta)}{V_P}\right). \quad (24)$$
Ignoring the effect of imaginary unit, the error becomes $\frac{1}{Q} \left(1 - \frac{v(\theta)}{V_P}\right)$. Figure 1b displays the corresponding error variations with reference velocities, phase angles and anisotropy parameters in an elliptically anisotropic medium. Compared with the value of equation (23), the error is negligible.

Based on these analyses, we can see that the approximation of $\omega \approx |k| V_P$ is limited in anisotropic media, but the error is relatively small and thus acceptable under the assumption of weak anisotropy and attenuation.

(2) Accuracy analyses for low-rank decomposition

To reduce the numerical error caused by computational algorithms, we jointly adopt the pseudo-spectral method and the low-rank decomposition scheme to solve equations (9), (10) and (13), respectively. For the sake of evaluating the accuracy of low-rank approximation, we design a simple model with linear velocity, which varies from $1500$ to $3000 \text{ m} \cdot \text{s}^{-1}$. Moreover, the wavenumbers of $k_x$ and $k_z$ are between $0$ and $0.314 \text{ m}^{-1}$, and the reference frequency is set to $\omega_0 = 1000 \text{ rad} \cdot \text{s}^{-1}$. Figure 2 shows the approximations of $k_x^2 \ln |\omega_0|$, $k_z^2 \ln |\omega_0|$ and $\frac{k_x^2}{k_x^2 + k_z^2} \ln |\omega_0|$, and their corresponding errors using low-rank decomposition with $M = N = 2$. For simplicity, the exact values are not displayed here. From figure 2, we can see that the values of these three operators are between $-0.074$ and $0.0278$, while their errors are as small as $10^{-8}$. It illustrates that the low-rank decomposition has very high accuracy.

3. Numerical examples

In this section, three numerical examples are carried out using our proposed PSVA and PUVA wave equations. In each example, we set $\omega_0 = 1000 \text{ rad} \cdot \text{s}^{-1}$. A Ricker signal with main frequency of $20$ Hz is used to produce waves. The parameters
Figure 8. Seismic records computed by using the (a) PSA, (b) PSVA, (c) PUA and (d) PUVA wave equations in a two-layered model. (e) and (f) are the extracted traces at the distance of 1000 m as indicated by the black dotted lines in (a–d). (g) and (h) are the extracted traces at the time of 500 ms as indicated by the white dotted lines in (a–d), respectively.
\( M = N = 3 \) are adopted in the low-rank approximation. To eliminate the artefact boundary reflections, a modified hybrid absorbing boundary condition is applied during the wavefields simulation (Liu & Sen 2010, 2018). For convenience, we first give some abbreviations of different numerical methods in Table 1.

### 3.1. A homogeneous model

Our first example is for a 2D homogeneous model with grids of 301 \( \times \) 301. The grid intervals in the \( x \)- and \( z \)-directions are 10 m. The velocity along the symmetry axis is 1500 m s\(^{-1}\). The Thomsen anisotropy parameters are \( \varepsilon = 0.20 \) and \( \delta = 0.15 \), respectively. The quality factor is \( Q = 20 \). The record length and the time step are 1.2 and 0.001 s, respectively. The source wavelet is placed at the centre of the model. The reference solution in the homogeneous model is directly computed by using the generalised pseudo-spectral method. Figure 3 shows several wavefield snapshots and their differences compared with the reference solutions. Unlike the acoustic modelling wavefields (figure 3a and e), the viscoacoustic modelling wavefields (figure 3b, c, f and g) have a large amount of energy loss. Besides, the wavefields computed by PSA (figure 3a) and PSVA (figure 3b and c) wave equations contain some strong SV-wave artefacts around the source, while they are effectively eliminated in the wavefields of PUA (figure 3e) and PUVA (figure 3f and g). In addition, the differences (figure 3d and h) between the reference solutions (figure 3b and f) and our new modelling results (figure 3c and g) are very small, which illustrates that our proposed algorithm is highly accurate.

Because the phase dispersion and the amplitude dissipation are naturally decoupled in the complex velocity, we can conveniently implement different viscoacoustic behaviours by choosing different coefficients in equations (9), (10) and (13). For example, figure 4a and b show four wavefields at 0.7 s, calculated by the anisotropic acoustic, dispersion-effect, dissipation-effect and viscoacoustic wave equations, respectively. Compared with the acoustic wavefield, the dispersion-effect wavefield has little amplitude variation, but presents a delayed phase. In contrast, the dissipation-effect wavefield only occurs amplitude variation.
dissipation without phase dispersion. When the dispersion and dissipation effects are considered, the viscoacoustic wavefield exhibits amplitude dissipation and phase dispersion. To compare the attenuating effects caused by different quality factors, we adopt the parameter of $Q = 500, 100, 50$ and $20$ to perform the viscoacoustic wavefield modelling. The corresponding wavefields are shown in figure 4c and d, which indicate that the effects of amplitude dissipation and phase dispersion increase with the decreasing $Q$. Some comparisons among different traces at the distance of 750 m (figure 5a–d) also give such conclusions.

### 3.2. A two-layered model

The second example is a two-layered model shown in figure 6. The computational area is divided into $301 \times 301$ grids with a grid spacing of 10 m. The simulated length is 1.2 s with the time step 0.001 s. The interface is located at the depth of 1600 m. Figure 7 displays several snapshots at 0.6 s calculated by using the PSA, PSVA, PUA and PUVA wave equations, respectively. Meanwhile, figure 8 plots the corresponding seismic records and extracted traces at the distance of 1000 m or at the time of 500 ms, respectively. Compared with acoustic modelling wavefields (figure 7a and c) and records (figure 8a and c), viscoacoustic simulated wavefields (figure 7b and d) and records (figure 8b and d) present obvious amplitude dissipation. Besides, seismograms from PSA and PSVA wave equations contain a large amount of SV-wave noise, while those from PUA and PUVA wave equations are completely free of SV-wave noise.

### 3.3. A heterogeneous model

To verify the accuracy and the robustness of our new schemes, a part of BP 2007 tilted TI (TTI) model with a constant dip angle $\theta = 0^\circ$ is used to perform the viscoacoustic wavefield simulation. The model is discretised by $501 \times 361$ with a grid spacing of 15 m. The record length is 1.5 s. The time step of 0.001 s is applied in the wave propagation. Figure 9 displays
Figure 11. Seismic records computed by using the (a) PSA, (b) PSVA, (c) PUA and (d) PUVA wave equations in the heterogeneous model. (e) and (f) are the extracted traces at the distance of 6000 m as indicated by the black dotted lines in (a–d), respectively. (g) and (h) are the magnification of (e) and (f).
some model parameters, including the velocity $v$ along the symmetry axis, the quality factor $Q = 14 \times (v/1000)^{2.2}$, and the Thomsen anisotropy parameters $\varepsilon$ and $\delta$, respectively. The source signal, represented by a Ricker wavelet, located at (3750, 2175 m), is used to generate the waves.

Figures 10 and 11 show several snapshots and seismograms for the modified BP 2007 TTI model computed by using the PSA, PSVA, PUA and PUVA wave equations, respectively. Compared with the acoustic modelling wavefield, the viscoacoustic modelling wavefield contains some amplitude dissipation effects along the wavefront. This phenomenon can be obviously observed from the extracted traces at the distance of 6000 m (figure 11e–h). In addition, we also compare the computational efficiency between PSVA and PUVA wave equations in a personal computer (Intel Core i7-8850H 2.60 GHz). The elapsed times of PSVA and PUVA wave equations are 511.66 and 713.11 s, respectively.

4. Conclusions

We have derived the new PSVA and PUVA wave equations with isotropic quality factor from the complex constitutive wave equation and the decoupled P-wave dispersion relation. Since phase dispersion and amplitude dissipation terms are inherently separated in these formulas, we can conveniently realise the decoupled viscoacoustic wavefield extrapolation. During the numerical implementation, the generalised pseudo-spectral method and the low-rank decomposition scheme were adopted to solve the wavenumber-domain and mixed-domain propagators, respectively. Meanwhile, we also examined the accuracy of low-rank approximations for different propagators. Several numerical experiments were performed to prove the validity and the robustness of our approach. Moreover, we compared the modelling results between PSVA and PUVA wave equations. Unlike the PSVA wave equation, our PUVA wave equation could obtain stable viscoacoustic wavefield without any SV-wave artefacts.

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Appendix A. Decoupled P-wave dispersion relation

In attenuating media, the Christoffel equation is generally written as follows:

$$\mathbf{G} - \rho \tilde{V}^2 \mathbf{I} = 0,$$  \hspace{1cm} \text{(A1)}

where $\mathbf{G}$ is the complex Christoffel matrix that depends on the complex stiffness and the phase direction, $\tilde{V} = \omega \tilde{k}$ is the complex velocity, $\tilde{k} = k - \alpha i$ is the complex wavenumber, $\alpha$ is the attenuation coefficient and $\mathbf{U}$ is the displacement vector of the viscoacoustic plane wave.

Setting the characteristic determinant of equation (A1) to zero yields the following dispersion relation:

$$\det (\mathbf{G} - \rho \tilde{V}^2 \mathbf{I}) = 0.$$ \hspace{1cm} \text{(A2)}

Under the acoustic assumption, the decoupled P-wave velocity in 2D attenuating media is obtained by solving equation (A2):

$$2 \rho V^2(\theta) = D_{11} \sin^2 \theta + D_{33} \cos^2 \theta + \left[(D_{11} \sin^2 \theta - D_{33} \cos^2 \theta)^2 + 4D_{13}^2 \sin^2 \theta \cos^2 \theta\right]^{1/2}.$$ \hspace{1cm} \text{(A3)}

Appendix B. Specific implementations for PSVA and PUVA wave equations

Under the acoustic assumption, we derive the PSVA and PUVA wave equations, respectively. In the implementation, a generalised pseudo-spectral method and a low-rank decomposition scheme are adopted to solve the wavenumber-domain and
mixed-domain propagators. As for equations (9) and (10), each term can be solved by the following equations:

\[
D_{11} \frac{\partial^2 \sigma_{H}}{\partial x^2} = \rho V_p^2 (1 + 2\epsilon) \left\{ F^{-1} \left[ -k_x^2 F(\sigma_{H}) \right] + \frac{2}{\pi Q} F^{-1} \left[ -k_{2z}^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_{H}) \right] \right\} \tag{B1}
\]

\[- \rho V_p^2 (1 + 2\epsilon) \frac{1}{Q V_p} F^{-1} \left[ \frac{-i k_z}{|k|} F(\sigma_{H}) \right], \]

\[
D_{13} \frac{\partial^2 \sigma_{V}}{\partial z^2} = \rho V_p^2 \sqrt{1 + 2\delta} \left\{ F^{-1} \left[ -k_z^2 F(\sigma_{V}) \right] + \frac{2}{\pi Q} F^{-1} \left[ -k_{2z}^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_{V}) \right] \right\} \tag{B2}
\]

\[- \rho V_p^2 \sqrt{1 + 2\delta} \frac{1}{Q V_p} F^{-1} \left[ \frac{-i k_z}{|k|} F(\sigma_{V}) \right], \]

\[
D_{13} \frac{\partial^2 \sigma_{H}}{\partial z^2} = \rho V_p^2 \sqrt{1 + 2\delta} \left\{ F^{-1} \left[ -k_z^2 F(\sigma_{H}) \right] + \frac{2}{\pi Q} F^{-1} \left[ -k_{2z}^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_{H}) \right] \right\} \tag{B3}
\]

\[- \rho V_p^2 \sqrt{1 + 2\delta} \frac{1}{Q V_p} F^{-1} \left[ \frac{-i k_z}{|k|} F(\sigma_{H}) \right], \]

\[
D_{33} \frac{\partial^2 \sigma_{V}}{\partial z^2} = \rho V_p^2 \left\{ F^{-1} \left[ -k_z^2 F(\sigma_{V}) \right] + \frac{2}{\pi Q} F^{-1} \left[ -k_{2z}^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_{V}) \right] \right\} \tag{B4}
\]

\[- \rho V_p^2 \frac{1}{Q V_p} F^{-1} \left[ \frac{-i k_z}{|k|} F(\sigma_{V}) \right]. \]

In equations (B1)–(B4), \( F^{-1} \left[ -k_x^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_{H}) \right] \) and \( F^{-1} \left[ -k_z^2 \ln \left| \frac{\omega}{\omega_0} \right| F(\sigma_{V}) \right] \) are calculated by using the low-rank decompositions, and the other terms are computed with a pseudo-spectral method. Therefore, solving PSVA wave equation would spend twice the low-rank decomposition and four times the pseudo-spectral method.

To solve equation (13), we have

\[
\frac{\partial^2 p}{\partial t^2} = V_p^2 \left\{ (1 + 2\epsilon) F^{-1} \left[ -k_{2x}^2 F(P) \right] + F^{-1} \left[ -k_{2z}^2 F(P) \right] - 2(\epsilon - \delta) F^{-1} \left[ \frac{-k_{2z}^2}{|k|^2} F(P) \right] \right\} \tag{B5}
\]

\[+ \frac{2V_p}{\pi Q} \left\{ (1 + 2\epsilon) F^{-1} \left[ -k_{2x}^2 \ln \left| \frac{\omega}{\omega_0} \right| F(P) \right] + F^{-1} \left[ -k_{2z}^2 \ln \left| \frac{\omega}{\omega_0} \right| F(P) \right] \right\} \]

\[+ \frac{V_p}{Q} \left\{ (1 + 2\epsilon) F^{-1} \left[ -k_z^2 F(\partial_P) \right] + F^{-1} \left[ -k_z^2 F(\partial_P) \right] \right\}, \]

where \( P \) is the pressure. Similarly, \( F^{-1} \left[ -k_x^2 \ln \left| \frac{\omega}{\omega_0} \right| F(P) \right], F^{-1} \left[ -k_x^2 \ln \left| \frac{\omega}{\omega_0} \right| F(P) \right] \) and \( F^{-1} \left[ \frac{-k_{2z}^2}{|k|^2} \ln \left| \frac{\omega}{\omega_0} \right| F(P) \right] \) are calculated by using the low-rank decompositions and the other terms are computed with pseudo-spectral method. So, solving PUVA wave equation would spend three times the low-rank decomposition and six times the pseudo-spectral method. Based on these analyses, we can see that the computational cost of PUVA is a little larger than that of the PSVA wave equation.

**Appendix C. Theoretical quality factor investigation**

For a P-wave, the Christoffel equation can be written as:

\[
(D_{11} k_z^2 \sin^2 \theta - \rho \omega^2) (D_{33} k_x^2 \cos^2 \theta - \rho \omega^2) - (D_{13} k_z^2 \sin \theta \cos \theta)^2 = 0. \tag{C1}
\]
Based on the complex stiffness coefficients and the definition of quality factor, we get

$$Q_\alpha = \left| \frac{\text{Re} \left( D_{ij} \right)}{\text{Im} \left( D_{ij} \right)} \right| \approx \left| \left( 1 + \frac{2}{\pi Q} \ln \left| \frac{\omega}{\omega_0} \right| \right) \right| \approx \left( Q + \frac{2}{\pi} \ln \left| \frac{\omega}{\omega_0} \right| \right). \quad (C2)$$

For weak attenuation, $\frac{2}{\pi} \ln \left| \frac{\omega}{\omega_0} \right| \ll Q$ and $\left| \frac{\text{Re} \left( D_{ij} \right)}{\text{Im} \left( D_{ij} \right)} \right| \approx Q$. Then, equation (C1) can be rewritten as follows (Zhu & Tsvankin 2006):

$$\left( c_{11} \sin^2 \theta \kappa_1 - \rho \omega^2 + ic_{11} \sin^2 \theta \kappa_2 \right) \left( c_{33} \cos^2 \theta \kappa_1 - \rho \omega^2 + ic_{33} \cos^2 \theta \kappa_2 \right) - \left[ c_{13} \sin \theta \cos \theta \left( \kappa_1 + i \kappa_2 \right) \right]^2 = 0, \quad (C3)$$

where $c_{11}$, $c_{13}$ and $c_{33}$ are the corresponding real parts of complex stiffness coefficients $D_{11}$, $D_{13}$ and $D_{33}$. Solving equation (C3) yields

$$\kappa_1 = k^2 - \alpha^2 + \frac{2k\alpha}{Q}, \quad (C4)$$

$$\kappa_2 = \frac{1}{Q} \left[ k^2 - \alpha^2 \right] - 2k\alpha. \quad (C5)$$

The only physically meaningful solution of the imaginary part of equation (C3) is $\kappa_2 = 0$, which yields the isotropic expression for attenuation coefficient $\alpha$ as follows:

$$\alpha = k \left( \sqrt{1 + Q^2} - Q \right). \quad (C6)$$

The normalised attenuation coefficient $\chi$ that defines the rate of amplitude decay per wavelength can be expressed as:

$$\chi \equiv \frac{\alpha}{k} = \left( \sqrt{1 + Q^2} - Q \right). \quad (C7)$$

Based on equation (C7), we can see that the normalised attenuation coefficient for P-wave in velocity anisotropic media with isotropic $Q$ is independent of the phase angle. When attenuation is weak ($1/Q \ll 1$), equation (C7) becomes

$$\chi \approx \frac{1}{2Q}. \quad (C8)$$

Therefore, we deem that the theoretical quality factor is in accordance with isotropic attenuation.

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