Symmetry Algebras in Chern-Simons Theories with Boundary: Canonical Approach

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ABSTRACT

I consider the classical Kac-Moody algebra and Virasoro algebra in Chern-Simons theory with boundary within the Dirac’s canonical method and Noether procedure. It is shown that the usual (bulk) Gauss law constraint becomes a second-class constraint because of the boundary effect. From this fact, the Dirac bracket can be constructed explicitly without introducing additional gauge conditions and the classical Kac-Moody and Virasoro algebras are obtained within the usual Dirac method. The equivalence to the symplectic reduction method is presented and the connection to the Bañados’s work is clarified. It is also considered the generalization to the Yang-Mills-Chern-Simons theory where the diffeomorphism symmetry is broken by the (three-dimensional) Yang-Mills term. In this case, the same Kac-Moody algebras are obtained although the two theories are sharply different in the canonical structures. The both models realize the holography principle explicitly and the pure CS theory reveals the correspondence of the Chern-Simons theory with boundary/conformal field theory, which is more fundamental and generalizes the conjectured anti-de Sitter/conformal field theory correspondence.

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I. Introduction

Recently, there has been vast interest on the role of the space boundary in diverse areas of physics [1 - 3]. Though the complete understanding of the boundary physics has not been attained yet, this boundary theory opened new rich areas both in physics and mathematics. One of the interesting areas is what comes from the existence of the central terms in the Kac-Moody algebra and Virasoro algebra even at “classical” level. These unusual classical algebras were found first in the asymptotic isometry group $SO(2, 2)$ of the three-dimensional anti-de Sitter space ($AdS_{2+1}$) more than 10 years ago [4]. It is only in recent times that this algebra was applied to a practical physical problem of the statistical entropy calculation for BTZ black hole [5] by Strominger, which might provide important clues for understanding the mystery of black holes [6].

On the other hand, recently there was also an interesting report on the similar “classical” central terms in the Kac-Moody and Virasoro algebras in the Chern-Simons (CS) theory [7] with boundary [1,8] using the Regge-Teitelboim’s canonical method [4,9] by Bañados [10]. However, in this work he considered several hypothetical procedures which make it difficult to understand the work by the usual and familiar field theory methods. Motivated by this problem, the well-known symplectic reduction method [11] was considered more recently [12] and the Bañados’s Kac-Moody and Virasoro algebras were rigorously derived with the help of the Noether procedure for constructing the conserved charges [13]. Then, following the equivalence of the CS theory to the (2+1)-dimensional gravity theory with a cosmological constant [14, 15], it was straightforward to apply the Bañados’s algebras to the BTZ black hole (negative cosmological constant) entropy [16] and Kerr-de Sitter space (positive cosmological constant) entropy [17] a’ la Strominger. A merit of this approach was that the Virasoro algebra can be found for any radii, although the details of the central charge depend on the boundary diffeomorphism ($Diff$). However, even in this symplectic reduction method, where only the boundary degrees of freedom are treated by imposing the (Gauss law) constraint, the origin of the center was not understood at the more fundamental level as the result of interaction of bulk and boundary degrees of freedom. Moreover, the connection to the Bañados’s work [9] was not clear either.

In this paper, I clarify these issues within the usual Dirac method [19]. In this method, several remarkable implications of the classical center are manifest. In Sec. II, it is shown that the usual (bulk) Gauss law constraint of the (pure) CS theory becomes a second-class constraint because of the boundary effect; because of this fact, the Dirac bracket can be explicitly constructed without introducing additional gauge conditions contrast to the boundary-less case. Following
the Noether procedure, the conserved charge is calculated, which contains the surface integral term \( (Q_S) \) as well as bulk term \( (Q_B) \). Functional variations of \( Q_B \) and \( Q_S \) have the boundary contributions but their sum \( Q(= Q_B + Q_S) \) has no boundary contributions. However, \( Q_B \) and \( Q_S \) as well as \( Q \) are still differentiable contrast to recent claims of Bañados et al. [10, 16]. It is shown that the Poisson and Dirac bracket algebras of the Noether charges \( Q \) for the both gauge symmetry and \( Diff \) symmetry across the boundary are the same and become the Kac-Moody (Sec. II) and Virasoro (Sec. III) algebras with classical central terms, respectively. In Sec. II. C, the origin of the central terms is re-examined and it is found that the unusual delta-function formulas, which contain the full information of the boundary, are the essential source of the (classical) center. Furthermore, in Sec. III. B it is emphasized that the CS theory provides an concrete realization of the holography principle and a correspondence of the three-dimensional CS theory with boundary/ one-dimensional conformal field theory \( (CS_{2+1}/CFT_1) \), which is more fundamental and generalizes the conjectured correspondence of the three-dimensional anti-de Sitter space/two-dimensional conformal field theory \( (AdS_{2+1}/CFT_2) \). In Sec. IV, the equivalence to the symplectic reduction method is shown directly by projecting the Dirac bracket of base fields onto the boundary. The connection to Bañados’s work is clarified also. In Sec. V, the Yang-Mills-Chern-Simons (YMCS) theory is considered as a generalization. Even with the sharp differences in the symplectic structures and the Noether charges, the Kac-Moody algebra is exactly the same as that of the CS theory. There is no Virasoro algebra in this model because the \( Diff \) symmetry is broken by the three-dimensional Yang-Mills term explicitly. In Sec. VI, a summary and several applications and generalizations are discussed. In Appendix A, it is shown that the (smearing) Gauss law constraint, which becomes a second-class constraint when there is the boundary, satisfies the consistency condition of the Dirac’s Hamiltonian algorithm for both the CS and YMCS theories. In Appendix B, the symmetry algebras for the \( Diff \) along the boundary, which has only the quantum theoretical center, is considered.

II. Kac-Moody algebra of gauge transformation

A. Noether charge

I start with the Chern-Simons Lagrangian on a two-dimensional disc \( D_2 \)

\[
L_{CS} = \kappa \int_{D_2} d^2 x \epsilon^{\mu \nu \rho} \left\langle A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right\rangle,
\]

where \( \langle \cdots \rangle \) denotes trace. Up to a boundary term, (1) can be put into the canonical form with the Lagrangian

\[
L_{CS} = \frac{1}{2} \kappa \int_{D_2} d^2 x \epsilon^{ij} (-A_i^a \dot{A}_j^a + A_0^a F_0^{aj}).
\]
(Here, $\epsilon^{012} \equiv \epsilon^{12} \equiv 1$, $A_i = A_i^a t_a$, $F_{ij} = F_{ij}^a t_a$, $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f^{abc} A_b^a A_c^j$, and the group generators $t^a$ satisfy $[t^a, t^b] = f^{abc} t^c$, $\langle t^a t^b \rangle = \frac{1}{2} \delta^{ab}$.) I shall take (2) as my starting point [2, 10, 12, 16]. Variation of the Lagrangian (2) gives

$$\delta L_{CS} = \kappa \int_{D_2} d^2 x \langle \delta A_{\rho} \epsilon^{\rho \mu \nu} F_{\mu \nu} \rangle + 2\kappa \oint_{\partial D_2} d\varphi \langle A_0 \delta A_{\varphi} \rangle .$$

(3)

[ I neglect the total time derivative term which is removed by considering action variation $\delta I = \int_{t^1}^{t^2} dt \delta L_{CS}$ and the usual boundary conditions $\delta A_i|_{t^1} = \delta A_i|_{t^2} = 0$. However, the total space derivatives term, which becomes the boundary Lagrangian in (3), can not be removed by choosing $\delta A_{\varphi}|_{\partial D_2} = 0$: This boundary condition kills all the local boundary degrees of freedom which is a dangerous situation.] In order to get the usual equation of motion

$$F_{\mu \nu} = 0,$$

(4)
even when there is the boundary, I choose the boundary conditions [16]

$$A_0|_{\partial D_2} \propto A_{\varphi}|_{\partial D_2},$$

(5)

$$\oint_{\partial D_2} d\varphi \langle A_{\varphi} A_{\varphi} \rangle = \text{fixed}$$

(6)

for each time $t$; actually this boundary conditions have no role in the Kac-Moody algebra for the gauge transformation but important role in the Virasoro algebra for $\text{Diff}$ [11]. The spatial part of the equations of motion (4) gives the Gauss law constraint

$$G^a = \frac{1}{2} \kappa \epsilon^{ij} F_{ij}^a = 0$$

(7)

from the variation with respect to $A_0^a$, independently on the boundary conditions (5) and (6). In the symplectic reduction method, the analysis of symmetry algebras is carried out after the constraint (7) is explicitly solved [11-13]. But I am considering an alternative approach where the constraint is not solved but imposed only after all the processes of analysis are completed $a^prime la$ Dirac. It is widely believed that these two methods are equivalent, of course when they

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4The non-covariant Lagrangian form (2) is the simplest action which is gauge invariant and retains the usual (bulk) equations of motion (4): If the covariant Lagrangian is considered as the starting point, one must introduce the additional function $c(\varphi, t)$ which relates $A_0|_{\partial D_2} = c(\varphi, t) A_{\varphi}|_{\partial D_2}$ [see Ref. [20] for comparison] where the variation with respect to $c(\varphi, t)$ produces the boundary condition (6). But, from the gauge invariance requirement, which is necessary in the discussions of the Kac-Moody algebra, additional condition "$A_0|_{\partial D_2} = \varphi$ independent" should be introduced in order not to obtain the trivial result of zero center.

5Here, I define $A_r = \hat{r} A_i$, $A_{\varphi}/r = \hat{\varphi} A_i$ for the radial, (polar) angular coordinate $r$, $\varphi$ and their corresponding orthogonal unit vectors $\hat{r}$, $\hat{\varphi}$ on $\partial D_2$.

6Depending on the supplemented boundary Lagrangians and boundary conditions, the equations of motion can be modified by the boundary term [21]. But the consistency and equivalence to the theory with (4) is not clear.
both can be applied, although there is no general proof. However, the Dirac method will be unique one when the constraint can not be solved like as in the YMCS model which will be treated in Section V. Moreover, in the boundary theory the equivalence is not trivial matter as will be shown in this paper which involves some non-trivial facts.

Now, I consider the time-independent gauge transformation which is generated by

\begin{align}
\delta A_i^a &= D_i \lambda^a, \\
\delta A_0^a &= f^{abc} A_0^b \lambda^c.
\end{align}

(D_i is the covariant derivative $D_i^{ab} = \delta^{ab} \partial_i + f^{abc} A_i^c$.) Under this transformation, the Lagrangian (2) transforms as

\[
\delta L_{CS} = -\kappa \frac{d}{dt} \int_{D_2} d^2 x \epsilon^{ij} \langle \partial_i \lambda A_j \rangle \equiv \frac{d}{dt} X. \tag{9}
\]

Then, the Noether charge associated with this gauge transformation is given by

\[
Q(\lambda) = -\frac{\delta L_{CS}}{\delta \dot{A}_j^a} \delta A_j^a + X = \kappa \int_{D_2} d^2 x \epsilon^{ij} \langle F_{ij} \lambda \rangle - 2\kappa \int_{\partial D_2} d\varphi \langle A_\varphi \lambda \rangle \\
\equiv Q_B(\lambda) + Q_S(\lambda) \tag{10},
\]

where $Q_B(\lambda)$ and $Q_S(\lambda)$ are the bulk and surface integration terms, respectively.

**B. Poisson and Dirac bracket algebras of Noether charge**

The basic Poisson bracket which can be directly read off from the symplectic structure of the Lagrangian (2) is [11]

\[
\{ A_i^a(x), A_j^b(y) \} = \frac{1}{\kappa} \epsilon^{ij} \delta^{ab} \delta^2(x - y), \tag{11}
\]

and the Poisson bracket of any two function(al) $A, B$ is given by

\[
\{ A, B \} = \int_{D_2} d^2 z \frac{\delta A}{\delta A_j^a(z)} \epsilon^{ij} \frac{\delta B}{\delta A_j^a(z)}. \tag{12}
\]
For the Noether charge \( Q \) and its two constituents \( Q_B \) and \( Q_S \), the functional derivatives are calculated simply by considering the functional variations for the field \( A_i^a \):

\[
\delta Q = \frac{1}{2} \kappa \int_{D_2} d^2 x \epsilon^{ij} \delta F_i^a \lambda^a - \kappa \oint_{\partial D_2} d\varphi \delta A_i^a \lambda^a \\
= \kappa \int_{D_2} d^2 x \epsilon^{ij} \delta A_i^a (D_j \lambda)^a \\
= \kappa \int_{D_2} d^2 x (\epsilon^{ij} \delta A_i^a (D_j \lambda)^a + \delta (r-a) \delta A_i^a \dot{\varphi}^i \lambda^a), \tag{13}
\]

\[
\delta Q_B = \kappa \int_{D_2} d^2 x \epsilon^{ij} \delta A_i^a (D_j \lambda)^a + \kappa \oint_{\partial D_2} d\varphi \delta A_i^a \lambda^a \\
= \kappa \int_{D_2} d^2 x \left[ (\epsilon^{ij} \delta A_i^a (D_j \lambda)^a + \delta (r-a) \delta A_i^a \dot{\varphi}^i \lambda^a) \right], \tag{14}
\]

\[
\delta Q_S = -\kappa \oint_{\partial D_2} d\varphi \delta A_i^a \lambda^a \\
= -\kappa \int_{D_2} d^2 x \delta (r-a) \delta A_i^a \dot{\varphi}^i \lambda^a. \tag{15}
\]

I note that \( Q_B \) and \( Q_S \) as well as their sum \( Q \) have the well-defined functional variations contrast to recent claims \([10, 16]\). (‘\( a \)’ is the radius of the boundary circle \( \partial D_2 \)) Then, the functional derivatives become

\[
\frac{\delta Q}{\delta A_i^a} = \kappa \epsilon^{ij} (D_j \lambda)^a, \\
\frac{\delta Q_B}{\delta A_i^a} = \kappa \epsilon^{ij} (D_j \lambda)^a + \kappa \delta (r-a) \dot{\varphi}^i \lambda^a, \tag{16}
\]

\[
\frac{\delta Q_S}{\delta A_i^a} = -\kappa \delta (r-a) \dot{\varphi}^i \lambda^a.
\]

One notes that the derivatives of \( Q_B \) and \( Q_S \) have the boundary effect terms which appear only for the variation on the boundary. Then, using the formula (12) and the result (16), it is easy to show the Poisson algebras of the \( Q \)’s as follows

\[
\{ Q_B(\lambda), Q_B(\eta) \} = Q_B([\lambda, \eta]) - 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda D_\varphi \eta \rangle, \tag{17}
\]

\[
\{ Q_S(\lambda), Q_S(\eta) \} = 0,
\]

\[
\{ Q_B(\lambda), Q_S(\eta) \} = \{ Q_S(\lambda), Q_B(\eta) \} \\
= 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda D_\varphi \eta \rangle, \\
\{ Q(\lambda), Q(\eta) \} = Q([\lambda, \eta]) + 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda \partial_\varphi \eta \rangle, \tag{18}
\]

where \([\lambda, \eta]^a = f^{abc} \lambda^b \eta^c\). This results show the Kac-Moody algebra with the central term for the charge \( Q \) even at the Poisson bracket level; however, I note that, the Poisson algebra of \( Q_B \) as well as \( Q_S \) is not the Kac-Moody algebra. In general, the algebras will be modified by considering the Dirac bracket but except one case which has been studied in Refs. [22]; as will
be shown later this is actually the exceptional case but now let me first consider the Dirac bracket algebra explicitly following the usual Dirac’s procedures.

To this end, it is important to note that the bulk charge \( Q_B(\lambda) \), which is a smearing quantity of the Gauss law constraint \( G^a = 0 \) of (7) with the smearing function \( \lambda \), becomes a second-class constraint for the function \( \lambda \) whose (angular) derivative \( \partial_\varphi \lambda \) as well as \( \lambda \) itself does not vanish on the boundary \([2\kappa \oint_{\partial D_2} (\lambda D_\varphi \eta) = Q_S(\lambda, \eta)] = 2\kappa \oint_{\partial D_2} (\lambda \partial_\varphi \eta) \); when \( \lambda, \partial_\varphi \lambda \) vanish on the boundary, the theory becomes a trivial bulk one and this situation is not what I want to study. Now, it is found that the second-class constraint algebra comes from only the boundary effect and so additional gauge conditions are not needed contrast to recent claims [10, 16].

(See the Appendix A about the consistency with the Dirac’s algorithm, i.e., \{Q_B , H_c \} \approx 0 without introducing additional (secondary) constraints. Then, the Dirac bracket of any two function(al) \( A, B \) is defined by

\[
\{A, B\}^* = \{A, B\} - \int [du][dv] \{A, Q_B(u)\} \Delta^{-1}(u, v) \{Q_B(v), B\}, \tag{19}
\]

where \( \Delta^{-1} \) is defined as the functional inverse of \( \Delta(\lambda, \eta) : = \{Q_B(\lambda), Q_B(\eta)\} \approx -2\kappa \oint_{\partial D_2} d\varphi \langle \lambda D_\varphi \eta \rangle \) which depends, eventually, only on the functions \( \lambda, \eta, A_\varphi \) which live only on the boundary:

\[
\int [du] \Delta(\lambda, u) \Delta^{-1}(u, \eta) = \int [du] \Delta^{-1}(\eta, u) \Delta(u, \lambda) = \delta(\lambda - \eta). \tag{18}
\]

[Weak equality ‘\( \approx \)’ means the equality after implementation of the constraint \( Q_B = 0 \).] This bracket satisfies

\[
\{Q_B, B\}^* \approx 0 \tag{20}
\]

for any function(al) \( B \) and so the Gauss law constraint \( Q_B = 0 \) can be imposed consistently in the Hamiltonian dynamics. (Here, it is difficult to find the explicit solution of \( \Delta^{-1} \) although it can be argued that this actually exists. But I don’t need the explicit solution in the main issue of this paper.) With this Dirac bracket, it is easy to calculate the charge algebras as follows

\[
\{Q_S(\lambda), Q_S(\eta)\}^* \approx Q_S(\lambda, \eta) + 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda \partial_\varphi \eta \rangle, \tag{21}
\]

\[
\{Q(\lambda), Q(\eta)\}^* \approx Q(\lambda, \eta) + 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda \partial_\varphi \eta \rangle. \tag{22}
\]

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8This fact is related to that of the non-degeneracy of the symplectic structure of the boundary Lagrangian which is reduced from the original Lagrangian (2) by imposing the Gauss law constraint (7).

9I thank Prof. R. Jackiw who first asked about this problem.

10I thank Prof. P. Oh who motivated for me to consider the Dirac bracket explicitly.

11Here, it would be more correct to confine \( \lambda, \eta, \ldots \) etc. which live only on the boundary. However, since all the calculations involving \( \Delta, \Delta^{-1} \) are performed on the boundary \( \partial D_2 \), this rather formal definition also works as well. I thank Prof. S. Carlip for discussion about this matter.

12The matrix \( \Delta(u, v) \), which is defined in the space of \( u, v \), which live only on the boundary more correctly, has the non-vanishing determinant \( \det \Delta(u, v) = (\Delta(u, v))^2 \neq 0 \) (\( u, v \) are treated as the indices of the matrix and \( \Delta(u, u) = 0 \) is used) unless one considers a trivial (bulk) theory of \( \Delta(u, v) = 0 \).
Here, one can observe the Dirac bracket algebra (22) of $Q$ is the same as the corresponding Poisson algebra (18) but not for others $Q_B$ and $Q_S$: (21) can be considered as the result of implementation of $Q_B = 0$ in (22) but it is different from the corresponding Poisson algebra (17). This peculiar property of $Q$ can be also found in its generating gauge transformation: The gauge transformation generated by the charges in the Poisson brackets are given by

\[
\{Q_B(\lambda), A^{ai}(x)\} = (D^i\lambda)^a + \xi^{ai}(\lambda),
\]

\[
\{Q_S(\lambda), A^{ai}(x)\} = -\xi^{ai}(\lambda),
\]

\[
\{Q(\lambda), A^{ai}(x)\} = (D^i\lambda)^a,
\]

where $\xi^{ai}(\lambda) = \epsilon_{ij}\hat{\phi}^j\delta(|x| - a)\lambda^a$ is the gauge transformation only on the boundary \footnote{Temporal gauge transformation in (8) can be also obtained by including $-\frac{\hat{\phi}^j}{\hat{\phi}}\delta A^0_a = -\int \pi^0_a f^{abc} A^b_0 \lambda^c$ to the formula of the Noether charge (10). But, since this additional term is not important in my discussion, I will not consider this in this paper.}. $Q_S$ generates only the boundary gauge transformation $-\xi^{ai}$ which cancels that of $Q_B$, and $Q = Q_B + Q_S$ generates the usual bulk gauge transformation even with boundary. The corresponding ones in the Dirac bracket are

\[
\{Q_B(\lambda), A^{ai}(x)\}^* \cong 0,
\]

\[
\{Q_S(\lambda), A^{ai}(x)\}^* \cong \{Q(\lambda), A^{ai}(x)\}^* \cong (D^i\lambda)^a.
\]

So, one can find again the algebras involving $Q$ are the same for the Poisson and Dirac brackets: But here, the bulk charge $Q_B$ is frozen and does not generate the gauge transformation; instead, surface charge $Q_S$ acts like as the true generator of the full bulk gauge transformation (8). This result means that the full bulk gauge degrees of freedom for the system without boundary are transferred completely into the boundary gauge degrees of freedom: Hence, this CS theory with boundary can be considered as a concrete realization of the holography principle which states ‘bulk world is an image of data that can be stored on a boundary projection’ \footnote{Temporal gauge transformation in (8) can be also obtained by including $-\frac{\hat{\phi}^j}{\hat{\phi}}\delta A^0_a = -\int \pi^0_a f^{abc} A^b_0 \lambda^c$ to the formula of the Noether charge (10). But, since this additional term is not important in my discussion, I will not consider this in this paper.}. Because of the connection of the CS theory to the diverse areas of physics, the principle can be applied more widely than currently limited cases of anti-de Sitter space \footnote{Temporal gauge transformation in (8) can be also obtained by including $-\frac{\hat{\phi}^j}{\hat{\phi}}\delta A^0_a = -\int \pi^0_a f^{abc} A^b_0 \lambda^c$ to the formula of the Noether charge (10). But, since this additional term is not important in my discussion, I will not consider this in this paper.}, like as in (Kerr-) de Sitter space \footnote{Temporal gauge transformation in (8) can be also obtained by including $-\frac{\hat{\phi}^j}{\hat{\phi}}\delta A^0_a = -\int \pi^0_a f^{abc} A^b_0 \lambda^c$ to the formula of the Noether charge (10). But, since this additional term is not important in my discussion, I will not consider this in this paper.}.

I conclude this subsection by summarizing that both the Poisson algebra of $Q$ and Dirac algebra of $Q$ (or $Q_S$) show the Kac-Moody algebra with classical central term $2\kappa \int_\partial D^2 \langle \lambda \partial \varphi \eta \rangle = \kappa \int_\partial D^2 \lambda^a \partial \varphi \eta^a$ and noting that the existence of the central term is the purely Abelian effect with no mixing between different colors.

C. Re-examining the origin of the central terms
Up to now, the calculation has been straightforward and the appearance of the central term seems not to be so strange unless λ’s and their derivatives ∂xλ vanish on the boundary. But as will be shown in this Section these conditions imply the unusual formulas of delta-function which cannot be seen in the calculation of Section B. In Section B, one has observed only some remnant effects of these unusual formulas which contain the full informations about the boundary. Here, I only consider the Abelian case for simplicity because the non-Abelian properties have no important role. To this end, I first note that the Poisson algebra of \( Q_B \)’s, without using the formulas (13)-(16) but only the basic Poisson bracket (11), becomes

\[
\{ Q_B(\lambda), Q_B(\eta) \} = \frac{1}{4} \kappa \int_{D_2} d^2 x \int_{D_2'} d^2 x' \epsilon^{ij} \epsilon^{kl} \{ F_{ij}(x), F_{kl}(x') \} \lambda(x) \eta(x') \\
= \kappa \int_{D_2} d^2 x \int_{D_2'} d^2 x' \epsilon^{ij} \partial_i \delta^2(\lambda(x) \eta(x')).
\]

Here, if one uses the usual formula for the derivative of delta-function

\[
\partial^i \delta^2(x - x') = -\partial_i \delta^2(x - x'),
\]

(25) will vanish trivially. But actually this formula (26) is not true in this case and rather this depends on the smearing functions λ, η which are the test functions of the ∂δ2(x - x’) in (25); for example, the radial part of (26) is modified as

\[
\hat{r}^i \partial^i \delta^2(x - x') = -\hat{r}^i \partial_i \delta^2(x - x') + \delta^2(x - x') \delta(r - a)
\]

(27)
or in an integral form with the smearing (test) function η

\[
\int_{D_2} d^2 x' \hat{r}^i \partial^i \delta^2(x - x') \eta(x') = -\hat{r}^i \partial_i \eta(x) + \delta(r - a) \eta(a, \varphi),
\]

by carefully treating the boundary terms in the process of integration by parts for the smearing function η which does not vanish on the boundary r = a. The angular part is not modified if η is single-valued function η(r, \varphi = 2\pi) = η(r, \varphi = 0) : \hat{\varphi}^i \partial^i \delta^2(x - x') = -\hat{\varphi}^i \partial_i \delta^2(x - x').

Moreover, the quantity εij \partial_i \partial^i \delta^2(x - x’) does not vanish if its test function η is not constant on the boundary: In an integral form it becomes

\[
\int_{D_2} d^2 x'[\epsilon^{ij} \partial_i \partial^j \delta^2(x - x')] \eta(x') = -\delta(r - a) \hat{\varphi}^i \partial_i \eta(a, \varphi).
\]

(28)

Now then, with formula (28) or by carefully treating the boundary terms in the process of integration by part, one can find that

\[
\{ Q_B(\lambda), Q_B(\eta) \} = \kappa \int_{D_2} d^2 x \lambda(x) \partial_i \int_{D_2'} d^2 x' \epsilon^{ij} \partial^j \delta^2(x - x') \eta(x') \\
= -\kappa \int_{\partial D_2} d\varphi \lambda \partial_\varphi \eta
\]

(29)
which is an Abelian result of (17). So, one can find that the existence of classical central term implies that the usual delta-function formulas need the boundary corrections\footnote{The discrepancy between the usual formula (26) and the integration by parts of (25) was observed by Balachandran et al. \cite{8} but he did not provide the complete solution. For a related early work about the formula in other contexts, see Ref. \cite{25}; I thank Dr. K. Bering for informing this reference.}; actually the boundary terms in (13)-(16) are the ramnent effects of these corrections. All the other algebras (17)-(24) can be calculated in this way but after more tedious calculations than previous calculations of Sec. B.

III. Virasoro algebra of diffeomorphism

A. Noether charge

The CS Lagrangian (2) has not only the gauge symmetry but also the other class of symmetry, \textit{Diff} symmetry which involves the reparametrization of the geometrical coordinates. As is well-known \cite{13, 26} \textit{Diff} symmetry is an expected one because this corresponds to a \textit{field-dependent} gauge transformation when the equations of motion of the (pure) CS Lagrangian, \( F_{\mu\nu}^a = 0 \) is used. But it is not straightforward to obtain the \textit{classical} central term for the \textit{Diff} symmetry algebra (Virasoro central term) from the central term for the gauge transformation (Kac-Moody central term): In the derivation of the Kac-Moody algebras (18), (21), (22) the existence of the central term does not depend on what boundary conditions one chooses for \( \lambda \)'s only if the \( \lambda \)'s are non-constant and single-valued functions on the boundary. However, in the derivation of the Virasoro algebra, the existence of central term will depend crucially on the boundary condition; in other words, the (classical) Virasoro algebra cannot be anticipated simply from the Kac-Moody algebra. There are several possible boundary conditions which allow the \textit{Diff} symmetry but since I am only interested in the Virasoro algebra with the center, I will consider only one and unique boundary condition which allows the Virasoro central term. To this end, I start with the Lagrangian (2) and study the response of \( L_{CS} \) to a spatial and time-independent \textit{Diff}:

\[
\begin{align*}
\delta_f x^\mu &= -\delta^\mu_k f^k, \\
\delta_f A_i^a &= f^k \partial_k A_i^a + (\partial_i f^k) A_k^a, \\
\delta_f A_0^a &= f^k \partial_k A_0^a.
\end{align*}
\]

Under (30), one finds

\[
\delta_f L_{CS} = \kappa \int_{\partial D^2} d^2 x \epsilon^{ij} \partial_k \left\langle -f^k A_i \dot{A}_j + f^k A_0 F_{ij} \right\rangle \\
= -\kappa \oint_{\partial D^2} d\varphi f^r \left\langle A_r \dot{A}_\varphi - \dot{A}_r A_\varphi - A_0 \epsilon^{ij} F_{ij} \right\rangle.
\]
Now, one has two possible boundary conditions in order that there are Diff invariance, i.e., $\delta f_{LCS} = \frac{d}{dt}X$: (a). $f^r|_{\partial D_2} = 0$, (b). $A^a_{\partial D_2} =$constant. [Remember that one needs the boundary conditions (5), (6) already. Since the condition (6) is equivalent to $\partial_r A^a_{\partial D_2} = 0$, the condition (b) is exactly the same as that of the symplectic reduction method [12].]. But, the condition (a) does not produce the Virasoro algebra with center and I will concentrate only the more interesting case (b) which produces the Virasoro center. (See Appendix B for analysis of case (a)) From the condition (b), (31) becomes $dX/dt$ with $X = -\frac{1}{2}\kappa \oint_{\partial D_2} d\varphi f^r A^a_{\varphi} A^a_{\varphi}$. The Noether charge becomes

$$Q(f) = -\frac{\partial L_{CS}}{\partial A^a_i} \delta f^a_i + X$$

$$= \kappa \int_{D_2} d^2 x \left( f^k A^a_k \varepsilon^{ij} F^a_{ij} \right) - \kappa \oint_{\partial D_2} d\varphi \left( 2 f^r A_r A_\varphi + f^a A_\varphi A_\varphi \right) \quad (32)$$

$$\equiv Q_B(f) + Q_S(f),$$

where $Q_B(f)$ and $Q_S(f)$ are the bulk and surface integration terms, respectively as in (10). [The Noether charges for the Diff are distinguished from those of the gauge transformation by the Roman parameters $f, g, ...$ etc.]

**B. Dirac bracket algebra of Noether charge**

By noting that there is no functional variation of $A^a_{\partial D_2}$ because of the boundary condition $A^a_{\partial D_2} =$constant in the last Section, the functional variations of the Noether charge $Q$ and its bulk and surface constraints $Q_B, Q_S$ become

$$\delta Q(f) = \kappa \int_{D_2} d^2 x \left[ \varepsilon^{ij} \delta A^a_i D_j (A_k f^k)^a + \frac{1}{2} \varepsilon^{ij} F^a_{ij} \delta A^a_k f^k \right], \quad (33)$$

$$\delta Q_B(f) = \kappa \int_{D_2} d^2 x \left[ \varepsilon^{ij} \delta A^a_i D_j (A_k f^k)^a + \frac{1}{2} \varepsilon^{ij} F^a_{ij} \delta A^a_k f^k \right] + \kappa \oint_{\partial D_2} d\varphi \delta A^a_\varphi A^a_k f^k, \quad (34)$$

$$\delta Q_S(f) = -\kappa \oint_{\partial D_2} d\varphi \delta A^a_\varphi A^a_k f^k, \quad (35)$$

and their functional derivatives become

$$\frac{\delta Q(f)}{\delta A^a_i} = \kappa \varepsilon^{ij} \delta A^a_i D_j (A_k f^k)^a + \frac{1}{2} \kappa \varepsilon^{jk} F^a_{jk} f^i,$$

$$\frac{\delta Q_B(f)}{\delta A^a_i} = \kappa \varepsilon^{ij} \delta A^a_i D_j (A_k f^k)^a + \frac{1}{2} \kappa \varepsilon^{jk} F^a_{jk} f^i + \kappa \delta(r - a) \hat{\varphi}^i A^a_k f^k,$$

$$\frac{\delta Q_S(f)}{\delta A^a_i} = -\kappa \delta(r - a) \hat{\varphi}^i A^a_k f^k. \quad (36)$$

Here, I note that the functional variations (33)-(35) and derivatives (36) for Diff is exactly the same form as those of gauge transformations (13)-(15) and (16), respectively with the field-dependent gauge transformations with the transformation function $\lambda^a = A^a_k f^k$ although the
Using the fact \[9, 10, 22\] \(\delta_f A^a_i = D_i (f^k A_k)^a + f^k F_{ki}^a\) \(\ldots\)  
(37)

Using the formula (12) and the result (36), one finds the Poisson algebras of the \(Q(f)\)'s as follows

\[
\{Q_B(f), Q_B(g)\} = -2\kappa \int_{\partial D_2} d\varphi \left\langle A_k f^k D_{\varphi}(A_l g^l) \right\rangle \\
+2\kappa \int_{D_2} d^2 x \left\langle [A_k, A_l] f^k g^l G + (f \times g) G \cdot G - (A_k f^k D_j (G g^j) - (f \leftrightarrow g)) \right\rangle \\
\approx -2\kappa \int_{\partial D_2} d\varphi \left\langle A_k f^k D_{\varphi}(A_l g^l) \right\rangle, 
\]

\[
\{Q_S(f), Q_S(g)\} = 0, \\
\{Q_B(f), Q_S(g)\} = \{Q_S(f), Q_B(g)\} = 0 \\
\{Q(f), Q(g)\} = 2\kappa \int_{\partial D_2} d\varphi \left\langle A_k f^k D_{\varphi}(A_l g^l) \right\rangle \\
+2\kappa \int_{D_2} d^2 x \left\langle [A_k, A_l] f^k g^l G + (f \times g) G \cdot G - (D_j (A_k f^k) G g^j - (f \leftrightarrow g)) \right\rangle \\
\approx 2\kappa \int_{\partial D_2} d\varphi \left\langle A_k f^k D_{\varphi}(A_l g^l) \right\rangle. 
\]  
(38)

Using the fact \[9, 10, 22\] \([f, g]_{\text{Lie}} = f^n \partial_n g^k - g^n \partial_n f^k\) is Lie bracket\textsuperscript{15} on the bulk \((D_2)\)

\[
\int_{\partial D_2} d\varphi \left\langle A_k f^k \partial_{\varphi}(A_l g^l) \right\rangle = \int_{\partial D_2} d\varphi \left\langle [f, g]_{\text{Lie}} A_{\varphi} A_{\psi} + 2[f, g]_{\text{Lie}} A_{\varphi} + 2f^r \partial_{\varphi} g^r A_r A_r \right\rangle 
\]  
(41)

and \(\left\langle A_k f^k D_{\varphi}(A_l g^l) \right\rangle = \left\langle A_k f^k \partial_{\varphi}(A_l g^l) \right\rangle\), the algebras become

\[
\{Q_B(f), Q_B(g)\} \approx -Q_S([f, g]_{\text{Lie}} - 2\kappa A_r A_r) \int_{\partial D_2} d\varphi f^r \partial_{\varphi} g^r, 
\]

\[
\{Q_S(f), Q_S(g)\} = 0, \\
\{Q_B(f), Q_S(g)\} = \{Q_S(f), Q_B(g)\} = 0 \\
\{Q(f), Q(g)\} \approx Q_S([f, g]_{\text{Lie}} + 2\kappa A_r A_r) \int_{\partial D_2} d\varphi f^r \partial_{\varphi} g^r. 
\]  
(42)

\[
\{Q(f), Q(g)\} \approx Q_S([f, g]_{\text{Lie}} + 2\kappa A_r A_r) \int_{\partial D_2} d\varphi f^r \partial_{\varphi} g^r. 
\]  
(43)

\[
\{Q(f), Q(g)\} \approx Q_S([f, g]_{\text{Lie}} + 2\kappa A_r A_r) \int_{\partial D_2} d\varphi f^r \partial_{\varphi} g^r. 
\]  
(44)

\textsuperscript{15}By considering the \(\text{Diff}\) of \(A^a_i|_{\partial D_2} (= \text{constant})\) which reads \(0 = \delta_f A^a_i|_{\partial D_2} = [D_r (f^k A_k)^a + f^k F_{ki}^a]|_{\partial D_2} = [(\partial_r f^r) A^a_i + (\partial_r f^r) A^a_i]|_{\partial D_2}\), one deduces an additional condition \(\partial_r f^r|_{\partial D_2} = 0\). From this fact, it is found that this Lie bracket is equivalent to \([f, g]_{\text{Lie}} = f^r \partial_{\varphi} g^k - g^r \partial_{\varphi} f^k\) which is the Lie bracket on the circle \((\partial D_2)\) \[12, 16\]. This property will be useful in the analysis of the higher-dimensional algebra, if there is.
Here, there is no non-trivial non-Abelian effect which mixes the different colors for the center: Similar to the central term of (18), all the color degrees of freedom are simply added up.

Now, using the Dirac bracket (19) the Dirac bracket algebra of $Q(f)$’s, which has never been calculated explicitly due to the complications, becomes

$$\{Q(f), Q(g)\}^* \cong \{Q_S(f), Q_S(g)\}^* \cong Q_S([f, g]_{Lie}) + 2\kappa \left\langle A_r A_r \right\rangle \int_{\partial D_2} d\varphi f^\nu \partial \varphi g^\nu. \quad (45)$$

By considering a particular $\text{Diff}$ with $f^r|_{\partial D_2} \propto \partial \varphi f^\nu|_{\partial D_2}$, which is required from the Jacobi identity [12], and the additional constant term $\int_{\partial D} d\varphi f^\nu A_c^a A^a_r$, one finds that this becomes the standard form of the Virasoro algebra after a proper normalization. Similar to the gauge transformation of Sec. II, the Dirac bracket algebra of $Q(f)$ in (45) is the same as the corresponding Poisson algebra (44) although not the same for $Q_B(f)$ and $Q_S(f)$. The $\text{Diff}$ which is generated by the Noether charge of (34) becomes

$$\{Q_B(f), A^{ai}\} = f^k \partial_k A^{ai} + (\partial^i f^k) A^{ai}_{k},$$
$$\{Q_S(f), A^{ai}\} = -\xi_f^{ai},$$
$$\{Q(f), A^{ai}\} = f^k \partial_k A^{ai} + (\partial^i f^k) A^{ai}_{k}. \quad (46)$$

where $\xi_f^{ai} = \epsilon_{ij} \hat{\varphi}^j \delta(|\mathbf{x}| - a) A^k_i f^k$ is the $\text{Diff}$ on the boundary. The corresponding Dirac brackets are

$$\{Q_B(f), A^{ai}\}^* \cong 0,$$
$$\{Q(f), A^{ai}\}^* \cong \{Q_S(f), A^{ai}\}^* \cong f^k \partial_k A^{ai} + (\partial^i f^k) A^{ai}_{k}. \quad (47)$$

Hence in the $\text{Diff}$ case also, one finds the algebras involving $Q(f)$ are the same for the Poisson and Dirac brackets and all the information of $\text{Diff}$ are stored in the surface charge $Q_S$ and the bulk charge $Q_B$ is frozen in the Dirac bracket, coherently with the holography principle. Moreover, one finds that the CS theory shows a correspondence of the three-dimensional CS theory with boundary/one-dimensional conformal field theory ($CS_{2+1}/CFT_1$) which generalizes the conjectured $AdS_{2+1}/CFT_2$ correspondence [29]: The CS theory has one copy of the (real) Virasoro algebra and thus describes a one-dimensional (real) conformal field theory. On the other hand, the $AdS_{2+1}$, which can be constructed by the two copies of the $SL(2, \mathbb{R})$ CS theories has two copies of the (real) Virasoro algebras and thus describes two copies, i.e., two-dimensional (real) conformal field theory ($CFT_2$). Hence, the $CS_{2+1}/CFT_1$ correspondence

\[16\] In the quantized theory, there is a further positive contribution to the center owing to the normal ordering effect for the unitary highest weight representation of Kac-Moody algebra with $\kappa > 0$ [10, 27]. Thus, $Q(f)$ does not produce the closed algebra, i.e., Witt algebra even when quantized. This situation is similar to the linear dilaton CFT and Liouville theory [28].
is more fundamental than the conjectured AdS$_{2+1}$/CFT$_2$ correspondence; moreover the former correspondence derives also a correspondence of the three-dimensional (Kerr-) de Sitter space/one-dimensional complex conformal field theory (with the group $SL(2,\mathbb{C})$) [17] which has not been studied well.

IV. Relation to previous works

There are two closely related works which used different methods to obtain the anomalous symmetry algebras: One is the work done by Bañados [10] and the other is the work of Ref. [12]. In this Section, I discuss the relations between the calculation of this paper and those of two previous works.

A. Relation to Bañados’s work

Let me start first by recalling an interesting points which have been emphasized in two previous Sections. It is the fact that the Poisson bracket algebra of the Noether charge $Q = Q_B + Q_S$ itself is the same as the corresponding Dirac bracket algebra for both the gauge transformation and Diff. Actually, this is a peculiar situation in the Dirac method and it is known that there is a unique case where this happens, in a recent analyses [22, 30]: Translated into my case, the result says that

$$\{L_a, L_b\} \cong \{L_a, L_b\}^*$$ (48)

when $L_a$ is a quantity which commutes with the $Q_B$ in the Poisson bracket, $\{L_a, Q_B\} \cong 0$ (which can be called “gauge invariant quantity” if there is no boundary [20]). Now, since it is easy to observe that

$$\{Q, Q_B\} \cong 0$$ (49)

for both the gauge and Diff transformations in the previous Sections, it is now clear why the formula (48) works for $L \equiv Q$. In the literatures, a formula which is essentially the same as (48) was assumed, implicitly or explicitly, in several models with an asymptotic boundary [4, 9]; the method which has been used in these models was applied recently to the CS theory with the finite and/or infinite boundary by Bañados et al. [10, 16] but it was unclear how the method could be applied to the the finite boundary, which has an important meaning for

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17 In the previous case of Ref.[22, 30], $T(Q_B$ in this paper) was the first-class constraint and additional gauge fixing condition $\Gamma$ was considered such that the formula (called master formula) becomes $\{L_a, L_b\} \cong \{L_a, L_b\}^\Gamma$, which expressing the gauge independence of the equality. However, the important thing is that the validity of the formula is not limited to the first-class constraint $T$. 

18
understanding the nature of entropy of the horizon space-times [2, 6, 16, 17], as well as to the infinite boundary. Now, it is clear why their method, which use (48) essentially, can be applied to even the finite boundary systems with the help of (49) and so the Bañados’s calculation method for the boundary CS theory can be justified. However, unfortunately I haven’t been able to find any general argument for the validity of (49) for the general Noether charge $Q$ which produces the equality (48); if it is generally valid, it will be a powerful tool for the evaluation of the Dirac bracket when the straightforward calculation is difficult due to technical reasons.

On the other hand, the Noether charge $Q$ of (10) and (32) are the same as the Bañados’s smeared generator

$$H(\eta) = \int_{D_2} d^2x \eta^a G^a + J(\eta),$$

(50)

where the boundary term $J(\eta)$ is introduced (by hand) such that $H(\eta)$ has no boundary terms in the functional variation. However, it is not evident how the Noether charge and smeared generator are equivalent in general: When $G^a$ represents the first-class constraints and one restricts to the bulk symmetry transformation without the surface transformation, the equivalence can be considered as a form of the Dirac’s conjecture [19], which states all the first-class constraints (secondary as well as primary) become the symmetry generators, which has been widely believed without complete proof. However, my result implies that the equivalence may be valid even for the second-class constraint $G^a$ although it is not found any formal proof for the validity similar to the case of (49).

### B. Relation to symplectic reduction method

In the symplectic reduction method, the bulk Lagrangian (1) reduces to, with the pure gauge solution $A_i = g^{-1} \partial_i g$,

$$L_{CS} = -\kappa \int_{D_2} d^2x \epsilon^{ij} \left( \partial_i g^{-1} \partial_j gg^{-1} \frac{\partial}{\partial g} - \kappa \int_{\partial D_2} d\varphi \left( g^{-1} \partial_\varphi gg^{-1} \frac{\partial}{\partial \varphi} \right) \right)$$

(51)

which is essentially a boundary Lagrangian upon the local parameterization of $g$ [1, 13], and its corresponding Poisson bracket is

$$\{A_a^\varphi(\varphi), A_b^\varphi(\varphi')\} = \frac{1}{\kappa} (D_\varphi \delta(\varphi - \varphi'))^{ab}, \quad \text{others} = 0$$

(52)

---

18 This construction was first considered by Regge-Teitelboim [9] and this has been used later, sometimes in their name [4, 9, 10].

19 This has been called ‘differentiability’ but this will be miss-named one according to the differentiability even with the boundary terms in the functional variations.

20 Recently, some proofs of the conjecture have been known with several assumptions and explicit examples were also found, where the assumption were not valid [30, 31].
which are defined on the boundary $\partial D_2$. In this case, both the base manifold and the symplectic structure of the Lagrangian are drastically changed when the symplectic reduction is performed, but the equivalence to the Dirac method was not well known. However, the results of the previous Sections show the complete equivalence at least in the symmetry algebras of the boundary CS theory which can be considered as affirmative sign of the equivalence even for the boundary theories. In this subsection I present the more direct equivalence proof in the basic bracket of fields $A^a_i$ from which the equivalence of the charge algebras can be easily inferred. Following the definition of (19), the Dirac bracket between $A^a_i$’s is given by

$$\{A^a_i(x), A^b_j(x')\}^* = \frac{1}{\kappa} \epsilon^{ij} \delta^{ab} \delta^2(x - x') + \int [du][dv] [(D_i u)^a + \xi^a_i] \Delta^{-1}(u, v) [(D_j v)^b + \xi^b_j], \quad (53)$$

where the first equation of (23) is used; this is valid over all space including boundary by construction. Now, let me project this bracket onto the boundary $\partial D_2$ by multiplying $\hat{\phi}^i$:

$$\{A^a_i(x), A^b_j(x')\}^* = \{aA^a_i \phi^i(x), aA^b_j \phi^j(x')\}^* = \int [du][dv] D_i u^a \Delta^{-1}(u, v) D_j v^b$$

$$= \frac{1}{\kappa} (D_i \delta(\phi - \phi'))^a, \quad (54)$$

where I used the fact of $\xi^i = a \phi^i \xi_i = 0$. Moreover, from ‘$A^a_i|_{\partial D_2} = \hat{r}^i A^a_i|_{\partial D_2}=$constant’ the bracket for $A^a_i$ vanishes on the boundary $\partial D_2$, i.e.,

$$\{A^a_i(x), A^b_j(x')\}^* = \{A^a_i(x), A^b_j(x')\}^* = 0. \quad (55)$$

These basic bracket algebras (54) and (55) are the same as (52) of the symplectic reduction method which solves the Gauss law from the start and reduce the action (2) to the boundary action. Hence the Dirac bracket of the all space (boundary as well as bulk) is reduced to the symplectically reduced bracket on the boundary by projecting the Dirac bracket onto the boundary. This proves the equivalence of the Dirac method and symplectic reduction method at the fundamental level. This is the first time to derive the boundary brackets (54), (55) directly from the bulk bracket (11) as far as I know.

V. Inclusion of Yang-Mills term

So far, I have considered the pure CS term with the space boundary. In this Section, I consider a generalized model with the Yang-Mills term (Yang-Mills-Chern-Simons model (YMCS)) [7, 33, 34]. In this model, the Dirac method is unique one because the equations of motion can

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21There is a known proof of the equivalence when the base manifold is not changed upon the symplectic reduction [30, 32]. The generalization of the proof to the changeable manifold will be interesting.
not be identically solved by the pure-gauge type solution contrast to the pure CS theory such that the symplectic reduction method can not be applied.

I start with the YMCS Lagrangian on the disc $D_2$,

$$L_{YMCS} = \int_{D_2} d^2x \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{\kappa}{2} \epsilon_{ij} A^a_i \dot{A}^a_j + \frac{\kappa}{2} A^a_0 \epsilon_{ij} F^{ij}_a\right],$$

(56)

where the CS part is the pure CS Lagrangian (2). Here, I note that the added YM term break the $Diff$ symmetry of the CS part although the gauge symmetry is preserved. So, in this model, there is no Virasoro algebra but only the Kac-Moody algebra [13, 26]. Furthermore, because of the YM term, the symplectic structure of the total Lagrangian $L_{YMCS}$ is changed from the pure CS Lagrangian: The basic Poisson bracket is just the canonical one

$$\{A^i_a(x), \pi^{bj}_a(y)\} = \delta_{ij} \delta^{ab} \delta^2(x-y),$$

(57)

$$\text{others} = 0,$$

where $\pi^{ai} = \frac{\delta L}{\delta \dot{A}^a_i}$ contrast to the pure CS case (10): Although the Lagrangian $L_{YMCS}$ and $\pi^{ia}$ converge to those of pure CS theory by $\kappa \to \infty$ limit, the Poisson bracket does not; hence, it is not clear at the algebraic level whether the symmetry algebra of YMCS model converge into that of CS model in the large $\kappa$ limit or not.

Before considering the symmetry algebra, I first consider the variation principle for the Lagrangian (56): The variation of the YM part of (56) becomes (neglecting the total time derivative terms)

$$\delta L_{YM} = 2 \int_{D_2} d^2x \left\langle \delta A_\rho F^{\rho\mu} \right\rangle + 2 \oint_{\partial D_2} \, d\varphi \left\langle -\frac{1}{a} (\partial_\tau A_\varphi - \partial_\varphi A_\tau + [A_\tau, A_\varphi]) \right\rangle \delta A_\varphi$$

$$+ a \left( \partial_\tau A_0 - \partial_0 A_\tau + [A_\tau, A_0] \right) \delta A_0.$$

(58)

So, by considering the total variation of $L_{YMCS}$ with $\delta L_{CS}$ of (3), one can get the usual equations of motion

$$D_\mu F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\rho\mu\nu} F_{\mu\nu} = 0$$

(59)

if one chooses the boundary conditions

$$A_0|_{\partial D_2} = \mp \dot{\varphi} \cdot A|_{\partial D_2},$$

$$\left( \partial_0 \mp \dot{\varphi} \cdot \nabla \right) A_\tau|_{\partial D_2} = 0$$

(60)

as well as the condition (6). Here, I note that the condition (5) for the pure CS theory is more confined to the first condition of (60), which corresponds to the horizon space-times in pure CS

17
Using the result (65) and the Poisson bracket which is given by

The functionals become

Now, returning to the symmetry algebra, it is noted that the YMCS theory has the gauge symmetry with the same as the CS theory: Under the gauge transformation (7), the YMCS Lagrangian transforms as $\delta L_{YMCS} = \frac{d}{dt} X$ with $X = -\kappa \int d^2x \epsilon_{ij} (\partial_i A_j)$. Then, the associated Noether charge is given by, according to (9),

$$Q(\lambda) = \int_{D_2} d^2x \left( 2D_j \pi^j + \kappa \epsilon_{ij} \partial_i A_j \right) \lambda - \oint_{\partial D_2} d\varphi \left( 2a \pi^r \lambda + \kappa A_r \lambda \right)$$

$$\equiv Q_B(\lambda) + Q_S(\lambda),$$

(61)

where $Q_B(\lambda)$ and $Q_S(\lambda)$ are the bulk and surface terms, respectively. Here, I note that $Q_B(\lambda) = 2 \int_{D_2} d^2x \left( (D_j F^{j0} + \frac{\kappa}{2} \epsilon^{lij} F_{lij}) \lambda \right)$ which is a smearing form of the Gauss law (0'th component) of (59).

In parallel with the CS theory, the functional variations of the Noether charge $Q$ and its constituents $Q_B, Q_S$ are calculated as

$$\delta Q = \int_{D_2} d^2x \left[ \delta A^a_j \left( f^{abc} \pi^b_{ji} \lambda^c - \frac{\kappa}{2} \epsilon_{ij} \partial_i \lambda^a \right) - \delta \pi_{ij}^a \left( f^{abc} A^b_j \lambda^c + \partial_j \lambda^a \right) \right],$$

(62)

$$\delta Q_B = \int_{D_2} d^2x \left[ \delta A^a_j \left( f^{abc} \pi^b_{ji} \lambda^a - \frac{\kappa}{2} \epsilon_{ij} \partial_i \lambda^a \right) - \delta \pi_{ij}^a \left( f^{abc} A^b_j \lambda^c + \partial_j \lambda^a \right) \right] + \delta(r-a) \left( \delta \pi_{ai}^r + \frac{\kappa}{2} \delta A^a_i \lambda^r \right) \lambda^a,$$

(63)

$$\delta Q_S = -\int_{D_2} d^2x \delta(r-a) \left( \delta \pi_{ai}^r + \frac{\kappa}{2} \delta A^a_i \lambda^r \right) \lambda^a.$$  

(64)

These all have the well-defined functional variations. Then, the functional derivatives become

$$\frac{\delta Q}{\delta A^a_i} = f^{abc} \pi^b_{ai} \lambda^c + \frac{\kappa}{2} \epsilon_{ij} \partial_j \lambda^a,$$

$$\frac{\delta Q}{\delta \pi^a_{bi}} = -(D_i \lambda)^b,$$

$$\frac{\delta Q_B}{\delta A^a_i} = f^{abc} \pi^b_{ai} \lambda^c + \frac{\kappa}{2} \epsilon_{ij} \partial_j \lambda^a + \delta(r-a) \frac{\kappa}{2} \delta A^a_i \lambda^r,$$

$$\frac{\delta Q_B}{\delta \pi^a_{bi}} = -(D_i \lambda)^b + \delta(r-a) \lambda^b,$$

$$\frac{\delta Q_S}{\delta A^a_i} = -\delta(r-a) \frac{\kappa}{2} \lambda^b,$$

$$\frac{\delta Q_S}{\delta \pi^a_{bi}} = -\delta(r-a) \lambda^b.$$  

(65)

Using the result (65) and the Poisson bracket which is given by

$$\{A, B\} = \int_{D_2} d^2z \left( \frac{\delta A}{\delta A^a_i(z)} \frac{\delta B}{\delta \pi_{ai}(z)} - \frac{\delta A}{\delta \pi_{ai}(z)} \frac{\delta B}{\delta A^a_i(z)} \right),$$

(66)
the Poisson algebra of the $Q$’s becomes as follows

\[
\{Q_B(\lambda), Q_B(\eta)\} = Q_B([\lambda, \eta]) - Q_S([\lambda, \eta]) - 2\kappa \int_{\partial D_2} d\varphi \langle \lambda \partial_\varphi \eta \rangle, \quad (67)
\]

\[
\{Q_S(\lambda), Q_S(\eta)\} = 0,
\]

\[
\{Q_B(\lambda), Q_S(\eta)\} = \{Q_S(\lambda), Q_B(\eta)\} = Q_S([\lambda, \eta]) + 2\kappa \int_{\partial D_2} d\varphi \langle \lambda \partial_\varphi \eta \rangle,
\]

\[
\{Q(\lambda), Q(\eta)\} = Q([\lambda, \eta]) + 2\kappa \int_{\partial D_2} d\varphi \langle \lambda \partial_\varphi \eta \rangle. \quad (68)
\]

This algebra is exactly the same form as that of pure CS theory (17), (18) although the charges and the basic symplectic structures are sharply different for the YMCS and pure CS theories.\footnote{This result of $\{Q(\lambda), Q(\eta)\}$ agrees to the Dunne and Trugenberger’s one [34]. But, they didn’t find any good reason for including the surface term in $Q$. Mickelsson [33] only considered bulk part $Q_B$ and he found the correct central term for the first time but he missed the $Q_S$ term in (67).}

Moreover, the large $\kappa$ limit is singular for the commutation relation (65) although not for $Q$’s themselves; on the other hand, the small $\kappa$ limit (pure Yang-Mills phase) is well-defined one which has vanishing center Virasoro algebra (Wit algebra). The Dirac bracket is well-defined in the same as (19) from the second-class algebra of $Q_B$ also, and the Dirac bracket algebra is the same as the pure CS theory: Moreover, since $\{Q_B(\lambda), Q(\eta)\} \approx 0$ also, the Poisson bracket and Dirac bracket algebra are the same for the Noether charge $Q$’s. The realization of holography principle is the same as that of CS theory due to the same gauge transformations (23) and (24). (See Appendix A. (b) for the consistency with the Dirac algorithm.)

Before ending this Section, it seems to appropriate to note that the angular projection of the Dirac bracket between $A^a_{\varphi}$’s, $\{A^a_{\varphi}, A^b_{\varphi}\}^\ast$ is the same as that of pure CS theory. However, the Dirac brackets containing $A^a_{r}$ can not be calculated completely without knowing the explicit form of $\Delta^{-1}$:

\[
\{A^a_{\varphi}(x), A^b_{\varphi}(x')\}^\ast = D^b_{\varphi} \left[ du [dv] v^c D_\varphi u^a \Delta^{-1}(u, v) \right] + \delta(r' - a) \int [du [dv] \Delta^{-1}(u, v) v^b D_\varphi u^a],
\]

\[
\{A^a_{r}(x), A^b_{r}(x')\}^\ast = \int [du [dv] \left[ (D_r u)^a - \delta(r - a) u^a \right] \Delta^{-1}(u, v) \left[ (D_r v)^a - \delta(r' - a) v^a \right] \right]. \quad (69)
\]

This result implies that the symplectic structure involving $A^a_{\varphi}$’s for the boundary YMCS theory is the same as the boundary (pure) CS theory, if one finds and considers the symplectic reduction which solves the equations of motion (58), although not the same for other components.

VI. Summary and discussions

It has been studied how the space boundary modifies drastically the symmetry algebras in the pure CS theory and YMCS theory. The most drastic one is the fact that the Gauss law
constraint $Q_B$, which was the first-class constraint in the usual theory where the boundary effect was not considered, became the second-class one; due to this fact, the Dirac bracket was able to be constructed *explicitly*, which had never been done previously from the lack of complete understanding of the constraints structure, without introducing additional gauge conditions. Moreover, the symmetry algebras of the Noether charges, Kac-Moody and Virasoro algebras, which had been known recently, with the “classical” centers have their origin in the non-commutability of the Gauss law $Q_B$. In a mathematical terms, all these unusual things were simple results of the unusual delta-function formulas which had the boundary correction terms. Although it has been found that the Dirac method is equivalent to the symplectic reduction method by which the anomalous Bañados algebra was explicitly derived first, the previously noted peculiar properties were manifest only in the Dirac method.

The boundary modified also the conserved Noether charge by surface integral term $Q_S$ in addition to the usual bulk term $Q_B$ and only the combination $Q_B + Q_S (= Q)$ generated the correct (bulk) symmetry transformation. A peculiar result of this fact is that the physical states of the quantum theory are not annihilated by the symmetry generator $Q$ due to the non-vanishing part $Q_S$ which does not constitute a constraint. Thus, the central term in the algebra of $Q$ is not harmful in the quantization contrast to the usual (quantum) anomaly of symmetry constraints. Moreover, it is important to note that the second-class constraint algebra of $Q_B$ does not imply the breaking of some symmetries: The only candidate of the broken symmetry due to the boundary will be the time-dependent gauge (and Diff also for the pure CS theory) symmetry by which the Lagrangian (2) and (54) do not transforms as $\delta L = \frac{d}{dt}X$ and thus are not gauge invariant. But, the center of the second-class algebra of $Q_B$ are independent on the time-dependence of gauge transformation parameters $\lambda$ or $\eta$ and thus invalidate the connection of the non-commutability of $Q_B$ and gauge non-invariance.

The holography principle *i.e.*, bulk theory/boundary theory correspondence was also an interesting effect of the boundary. In this paper, the $CS_{2+1}/CFT_1$ correspondence occurred and this is more fundamental and can generalize the conjectured $AdS_{2+1}/CFT_2$ correspondence.

Besides of these things, several things remains unclear. One is about the explicit solution of $\Delta^{-1}$ although it was not needed in many physically interesting cases. Second is about the question of the general validity of the property $\{Q, Q_B\} \cong 0$ (49), which makes the Poisson and Dirac bracket algebras for $Q$ be the same, for the general Noether charge $Q$ and the bulk part $Q_B$. Third one is the question about the *formal* equivalence of the Noether procedure and the Regge-Teitelboim procedure for constructing the symmetry generators. Final one is the question about the origin for the same symmetry (Kac-Moody) algebras of both CS and YMCS theories despite of the sharp differences in the basic Poisson bracket and the Noether

\[23\] Similar situation is also occurred in the symplectic method. See the paper of Bak et al. [13].
charges.

As the final remarks, it would be interesting to extend to supersymmetric [35] and higher-dimensional CS theories [36] in relation to the supergravity theories, higher-dimensional black hole systems, M-theory and anticipated \( CS_{d+1}/CFT_{d-1} \) correspondence: Especially, in the higher-dimensional CS theories the use of Dirac method will be crucial in the manipulation of the symmetry algebras because the general pure gauge solution is not known in that case. Moreover, it is interesting to find a transformation which transforms the different-dimensions CS theories, which may reflect the \( U \)-duality of the D-brane configurations of the black holes [37].

**Appendix A**

In this Appendix, I show that the second-class Gauss law (smearing) constraint \( Q_B \cong 0 \) is consistent with the Dirac’s Hamiltonian algorithm, i.e., \( \{Q_B, H_c\} \cong 0 \) without introducing additional (secondary) constraints for both pure CS and YMCS theories.

(a) pure CS theory:

I start by noting that the canonical Hamiltonian of CS Lagrangian (2) becomes

\[
H_c = \frac{\kappa}{2} \int_{D_2} d^2x \left( -A_0^a \epsilon^{ij} F_{ij}^a \right) = Q_B(-A_0). \tag{70}
\]

Then, it is easy to show that

\[
\{Q_B, H_c\} = -Q_B(\lambda, A_0^0) - 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda D_\varphi A_0 \rangle = -Q_B(\lambda, A_0^0) - 2\kappa \oint_{\partial D_2} d\varphi \langle \lambda \partial_\varphi A_0 \rangle \approx 0 \tag{71}
\]

if \( \partial_\varphi A_0 |_{\partial D_2} \cong 0 \) (restriction of the Lagrange multiplier \( A_0 \)) is satisfied. [In the second line, the boundary condition (5) was used.] Hence, The consistency condition [19] is satisfied without introduction of additional (secondary) constraints.

(b) YMCS theory:

In this case, the canonical Hamiltonian becomes

\[
H_c = \int_{D_2} d^2x \left[ \frac{1}{2} E^{ai} E^{ai} + \frac{1}{2} B^a B^a - A_0^a \left( (D_i E^i)^a + \frac{\kappa}{2} \epsilon^{ij} F_{ij}^a \right) + \partial_i (A_0^a E^{ai}) \right]
\]

\[
H_0 \equiv \int_{D_2} d^2x \frac{1}{2} \left( E^{ai} E^{ai} + B^a B^a \right),
\]

\[
H_S \equiv \int_{D_2} d^2x \partial_i (A_0^a E^{ai}),
\]

21
where $B^a \equiv -\frac{1}{2} \epsilon^{ij} F^a_{ij}$, $E^a_i \equiv F^{0ia}$. Now, I need the functional derivatives of $H_0$ and $H_S$ in order to evaluate $\{Q_B, H_c\}$ in parallel with manipulation of the text. The functional variations and their derivatives are as follow:

\[
\delta H_0 = \int_{D^2} d^2 x \left[ E^a_i (\delta \pi^a_i - \frac{\kappa}{2} \delta \varphi^a_i) + (D_i B)^a \epsilon^{ij} \delta A^a_j - B^a \delta \varphi^i (r - a) \right],
\]

\[
\delta H_S = \int_{D^2} d^2 x (r - a) \left[ \delta A^a_i E^a_i \hat{r}^i + A^a_0 (\delta \pi^a_i - \frac{\kappa}{2} \epsilon^{ij} \delta A^a_j) \hat{r}^i \right],
\]

\[
\frac{\delta H_0}{\delta A^a_i} = \frac{\kappa}{2} \epsilon^{ij} E^a_j - \epsilon^{ij} (D_j B)^a - B^a \delta \varphi^i (r - a),
\]

\[
\frac{\delta H_0}{\delta \pi^a_i} = E^a_i,
\]

\[
\frac{\delta H_S}{\delta A^a_i} = \delta (r - a) \frac{\kappa}{2} \epsilon^{ij} \hat{r}^j A^a_0,
\]

\[
\frac{\delta H_S}{\delta \pi^a_i} = \delta (r - a) \hat{r}^i A^a_0.
\]

(73)

Then, one could show that

\[
\{Q_B(\lambda), H_c\} = A + B + C,
\]

where

\[
A = \{Q_B(\lambda), Q_B(-A_0)\}
\]

\[
= -Q_B([\lambda, A_0]) + \oint_{\partial D^2} d\varphi \left\langle -2a \pi^*[\lambda, A_0] + 2\kappa \lambda \partial_\varphi A^0 \right\rangle,
\]

(75)

\[
B = \int_{D^2} d^2 z \frac{\delta Q_B(\lambda)}{\delta A^a_i(z)} \frac{\delta (H_0 + H_S)}{\delta \pi^a_i(z)}
\]

\[
= \int_{D^2} d^2 z \kappa \epsilon^{ij} \left\langle \lambda(D_j E^j) \right\rangle + \oint_{\partial D^2} d\varphi \left\langle -\kappa \lambda \partial_\varphi A_0 + 2a E^*[\lambda, A_0] \right\rangle,
\]

(76)

\[
C = -\int_{D^2} d^2 z \frac{\delta Q_B(\lambda)}{\delta \pi^a_i(z)} \frac{\delta (H_0 + H_S)}{\delta A^a_i(z)}
\]

\[
= -\int_{D^2} d^2 z \kappa \epsilon^{ij} \left\langle \lambda(D_j E^j) \right\rangle + \oint_{\partial D^2} d\varphi \left\langle \kappa \lambda \partial_\varphi A_0 + 2\lambda(D_\varphi B) \right\rangle.
\]

(77)

Finally, one obtains

\[
\{Q_B(\lambda), H_c\} = -Q_B([\lambda, A_0]) + 2 \oint_{\partial D^2} d\varphi \left\langle \lambda(\kappa \partial_\varphi A_0 + D_\varphi B) \right\rangle 
\]

\[
\cong 0
\]

if $\kappa \partial_\varphi A^0 + D_\varphi B = D_\varphi (\kappa A^0 + B) \cong 0$ is satisfied. [Here, the boundary condition (5) was used several times.] Hence, in the YMCS case also, the consistency condition is satisfied without introduction of additional (secondary) constraints.
Appendix B

Here, I present the Virasoro algebra for the Diff symmetry of the CS theory with \( f^r|_{\partial D_2} = 0 \) (i.e., Diff along the boundary \((\partial D_2)\)) which does not produce the central term at classical level. The Noether charge for this Diff becomes \((X=0)\)

\[
Q(f) = \kappa \int_{D_2} d^2 x \left( f^k A_k \epsilon^{ij} F_{ij} \right) - \kappa \oint_{\partial D_2} d\varphi \left( f^\varphi A_\varphi \right),
\]

\[
\equiv Q_B(f) + Q_S(f)
\]

with the bulk and surface terms \(Q_B(f)\) and \(Q_S(f)\) respectively. Actually, these charges \(Q\)'s can be directly obtained by setting \(f^r|_{\partial D_2} = 0\) in (32). Moreover, all the other algebras can be simply obtained by this reduction procedure from the corresponding ones in Sec. III. B: Since \(f^r\) appears always together with \(A_r\) in the Noether charge and there is no fundamental variation of \(A_r|_{\partial D_2}\) already, this reduction is well-defined. So, the only modification is the surface term involving \(f^r|_{\partial D_2}\) in the results of Sec. III. B. The functional variation and their functional derivatives of the Noether charge become

\[
\delta Q(f) = \kappa \int_{D_2} d^2 x \left[ \epsilon^{ij} \delta A_i^a D_j(A_k f^k)^a + \frac{1}{2} \epsilon^{ij} F_{ij}^a \delta A_i^a f^k \right],
\]

\[
\delta Q_B(f) = \kappa \int_{D_2} d^2 x \left[ \epsilon^{ij} \delta A_i^a D_j(A_k f^k)^a + \frac{1}{2} \epsilon^{ij} F_{ij}^a \delta A_i^a f^k \right] + \kappa \oint_{\partial D_2} d\varphi \delta A_i^a A_\varphi f^\varphi,
\]

\[
\delta Q_S(f) = -\kappa \oint_{\partial D_2} d\varphi \delta A_i^a A_\varphi f^\varphi,
\]

\[
\frac{\delta Q(f)}{\delta A_i^a} = \kappa \epsilon^{ij} \delta A_i^a D_j(A_k f^k)^a + \frac{1}{2} \epsilon^{ij} F_{ij}^a f^i,
\]

\[
\frac{\delta Q_B(f)}{\delta A_i^a} = \kappa \epsilon^{ij} \delta A_i^a D_j(A_k f^k)^a + \frac{1}{2} \epsilon^{ij} F_{ij}^a f^i + \kappa \delta(r - a) \delta^i A_\varphi f^\varphi,
\]

\[
\frac{\delta Q_S(f)}{\delta A_i^a} = -\kappa \delta(r - a) \delta^i A_\varphi f^\varphi.
\]

The Poisson and Dirac bracket algebras of \(Q(f)\)'s are as follows

\[
\{Q_B(f), Q_B(g)\} \cong -2\kappa \oint_{\partial D_2} d\varphi \left( A_\varphi f^\varphi D_\varphi(A_\varphi g^\varphi) \right),
\]

\[
\{Q_S(f), Q_S(g)\} = 0,
\]

\[
\{Q_B(f), Q_S(g)\} = \{Q_S(f), Q_B(g)\} \cong 2\kappa \oint_{\partial D_2} \left( A_\varphi f^\varphi D_\varphi(A_\varphi g^\varphi) \right)
\]

\[
\{Q(f), Q(g)\} \cong 2\kappa \oint_{\partial D_2} \left( A_\varphi f^\varphi D_\varphi(A_\varphi g^\varphi) \right).
\]

23
Now, using the fact
\[
\oint_{\partial D} \left\langle A_\varphi f_\varphi D_\varphi (A_\varphi g_\varphi) \right\rangle = \oint_{\partial D} \left\langle A_\varphi f_\varphi \partial_\varphi (A_\varphi g_\varphi) \right\rangle \\
= \oint_{\partial D} \left\langle [f, g]_{\text{Lie}} A_\varphi A_\varphi \right\rangle,
\]
the algebras become
\[
\{Q_B(f), Q_B(g)\} \cong -Q_S([f, g]_{\text{Lie}}), \\
\{Q_S(f), Q_S(g)\} = 0, \\
\{Q_B(f), Q_S(g)\} = \{Q_S(f), Q_B(g)\}, \\
\cong Q_S([f, g]_{\text{Lie}}), \\
\{Q(f), Q(g)\} \cong Q_S([f, g]_{\text{Lie}}).
\]

Here, there is no central term classically contrast to the algebra for the corresponding charges of (32). The center arise only as a quantum mechanical effect of normal ordering [9, 11].

Now, the Dirac bracket algebra of \(Q(f)'s\) becomes
\[
\{Q(f), Q(g)\}^* \cong \{Q_S(f), Q_S(g)\}^* \cong Q_S([f, g]_{\text{Lie}})
\]
and this is the same algebra of the Poisson bracket which is inferred from the fact of \(\{Q_B(f), Q(g)\} \cong 0\). The gauge transformation which is generated by \(Q(f)\) and the realization of holography principle is the same as in Section III. B.

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