Fully Coupled Quasi-Metric Gravity

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Abstract

The original theory of quasi-metric gravity, allowing only a partial coupling between space-time geometry and the active stress-energy tensor, is too restricted to allow the existence of gravitational waves. Therefore, said theory can at best be regarded as a waveless approximation theory. In this paper, it is shown how to relax the restrictions on quasi-metric space-time geometry such that a full coupling between space-time geometry and the active stress-energy tensor becomes possible. The extended field equations are consistent with the general quasi-metric geometric framework, and their dynamical structure is somewhat similar to that of canonical general relativity (GR). Moreover, results from the original quasi-metric gravitational theory are recovered for metrically static vacua and for isotropic cosmology. This means that the current experimental status of the new quasi-metric gravitational theory is the same as for the original theory, except for the prediction of weak GR-like gravitational waves in vacuum.

1 Introduction

The so-called quasi-metric framework (QMF), an alternative geometric framework for formulating relativistic gravitation, was invented a number of years ago [1, 2]. Currently, the QMF turns out to have a non-viable status due to its predicted properties of the cosmic relic neutrino background (assuming standard neutrino physics) [3]. This status could possibly change if future experiments show evidence of the necessary non-standard neutrino physics to resolve the apparent conflict with experiment. Now said predicted properties of the cosmic neutrino background depends crucially on the neutrino physics, but not on the gravitational sector of the QMF. Therefore, the currently non-viable status of the QMF is valid independently of any particular theory of quasi-metric gravity.

However, the original theory of quasi-metric gravity (OQG) has a much more serious problem. That is, due to the very restricted form postulated for quasi-metric space-time geometry, only a partial coupling is possible between the active stress-energy tensor

*Dedicated to the memory of my mother*
and space-time curvature. Unfortunately, this property makes the theory essentially waveless since said restricted form of the quasi-metric space-time geometry makes it fully determinable by the matter sources alone. Unlike general relativity (GR), where the field equations directly determine the Ricci tensor only, leaving the Weyl curvature free, the OQG leaves free no aspect of quasi-metric space-time geometry. Thus no GR-like gravitational waves can exist according to the OQG. With the recent direct experimental evidence for GR-like gravitational waves, this means that the OQG must be abandoned and at best be treated as a waveless approximation theory.

Fortunately, it turns out that it is possible to relax the postulated form of the quasi-metric space-time geometry sufficiently to allow a full coupling to the active stress-energy tensor. In this paper, we show how this can be done such that the resulting gravitational theory is still consistent with the QMF. Moreover, due to the somewhat (constructed) similarity to the dynamical structure of canonical GR, the extended field equations predict weak GR-like gravitational waves in vacuum, implying that the field equations leave free some aspect of quasi-metric geometry. This means that there is some hope that fully coupled quasi-metric gravity may eventually turn out to be viable.

2 Quasi-metric gravity with full coupling

The QMF has been published in detail in [1] (see also [2]). Here we include only the minimum basics and the required adoptions made to accommodate the extended quasi-metric field equations.

In short, the basic motivation for introducing the QMF is the idea that the cosmic expansion should be described as a general phenomenon not depending on the causal structure associated with any pseudo-Riemannian manifold. And as we will see in what follows, certain properties intrinsic to quasi-metric space-time ensure that this alternative way of describing the cosmic expansion is mathematically consistent and fundamentally different from its counterpart in GR. In what follows it is shown how said motivation is realized geometrically.

The geometrical basis of the QMF consists of a 5-dimensional differentiable manifold with topology $\mathcal{M} \times \mathbb{R}_1$, where $\mathcal{M} = S \times \mathbb{R}_2$ is a Lorentzian space-time manifold, $\mathbb{R}_1$ and $\mathbb{R}_2$ both denote the real line and $S$ is a compact 3-dimensional manifold (without boundaries). That is, in addition to the usual time dimension and 3 space dimensions, there is an extra time dimension represented by the global time function $t$. The reason for introducing this extra time dimension is that by definition, $t$ parametrizes any change in the space-time geometry that has to do with the cosmic expansion. By construction, the
extra time dimension is degenerate to ensure that such changes will have nothing to do with causality. Mathematically, to fulfil this property, the manifold $\mathcal{M} \times \mathbb{R}_1$ is equipped with two degenerate 5-dimensional metrics $\bar{g}_t$ and $g_t$. The metric $\bar{g}_t$ is found from field equations as a solution, whereas the “physical” metric $g_t$ can be constructed from $\bar{g}_t$ in a way described in refs. [1, 2].

The global time function is unique in the sense that it splits quasi-metric space-time into a unique set of 3-dimensional spatial hypersurfaces called fundamental hypersurfaces (FHSs). Observers always moving orthogonally to the FHSs are called fundamental observers (FOs). The topology of $\mathcal{M}$ indicates that there also exists a unique “preferred” ordinary global time coordinate $x^0$. We use this fact to construct the 4-dimensional quasi-metric space-time manifold $\mathcal{N}$ by slicing the submanifold determined by the equation $x^0 = ct$ out of the 5-dimensional differentiable manifold. (It is essential that this slicing is unique since the two global time coordinates should be physically equivalent; the only reason to separate between them is that they are designed to parametrize fundamentally different physical phenomena.) Thus the 5-dimensional degenerate metric fields $\bar{g}_t$ and $g_t$ may be regarded as one-parameter families of Lorentzian 4-metrics on $\mathcal{N}$. Note that there exists a set of particular coordinate systems especially well adapted to the geometrical structure of quasi-metric space-time, the global time coordinate systems (GTCSs). A coordinate system is a GTCS iff the time coordinate $x^0$ is related to $t$ via $x^0 = ct$ in $\mathcal{N}$.

Expressed in an isotropic GTCS, the most general form allowed for the family $\bar{g}_t$ is represented by the family of line elements valid on the FHSs (this may be taken as a definition)

$$ds^2_t = \bar{N}_t^2 \left\{ [N^k_{(t)} N^s_{(t)} \bar{h}_{(t)ks} - 1] (dx^0)^2 + 2 t_0 t \bar{N}^k_{(t)} \bar{h}_{(t)ks} dx^s dx^0 + \frac{t^2 t_0^2}{t_0} \bar{h}_{(t)ks} dx^k dx^s \right\}. \quad (1)$$

Here $t_0$ is some arbitrary reference epoch (usually chosen to be the present epoch) setting the scale of the spatial coordinates, $\bar{N}_t$ is the family of lapse functions of the FOs and $\frac{\partial}{\partial t} \bar{N}^k_{(t)}$ are the components of the shift vector family of the FOs in $(\mathcal{N}, \bar{g}_t)$. Also, $\bar{h}_{(t)ks} dx^k dx^s = \bar{N}_t^2 \bar{h}_{(t)ks} dx^k dx^s$ is the spatial metric family intrinsic to the FHSs.

In the OQG, the form of the metric family (1) was severely restricted by postulating that $\bar{h}_{(t)ik} dx^i dx^k$ must be set equal to the metric $S_{ik} dx^i dx^k$ of the 3-sphere (with radius $ct_0$). The reason for this restriction was to ensure the uniqueness of $t$ by requiring the FHSs to be compact [1]. However, this requirement inevitably leads to some form of prior 3-geometry. Then said restriction was also thought to prevent the possibility that the prior 3-geometry might interfere with the dynamics of $\bar{g}_t$. On the other hand, except for the dependence on $t$, the form of (1) may seem completely general. But this is not really so since the FHSs are still required to be compact, so there still may be prior 3-geometry.
The difference from the old theory is that in the revised theory, the prior 3-geometry will be indirectly implemented via certain terms in the extended field equations rather than as an explicit restriction of (1).

The families $\bar{g}_t$ and $g_t$ are related by the transformation $\bar{g}_t \to g_t$ as described in [1, 2]. A general form for the family $g_t$ is given by the family of line elements (using a GTCS)

$$ds_t^2 = [N^k_t N^s_t h_{tk} - N^2](dx^0)^2 + 2\frac{t}{t^0} N^k_t h_{tk} ds dx^0 + \frac{t^2}{t^0} h_{tk} ds dx^s,$$

where the symbols have similar meanings to their (barred) counterparts in equation (1). Note that the propagation of sources (and test particles) is calculated by using the equations of motion in $(N, g_t)$ (see below). Moreover, since the proper time as measured along a world line of a FO should not directly depend on the cosmic expansion, the lapse function $N$ should not depend explicitly on $t$. Therefore, any potential $t$-dependence of $N$ must be eliminated by substituting $t$ with $x^0/c$ (using a GTCS) whenever it occurs before using the equations of motion. In the same way, any extra $t$-dependence of $g_t$ coming from the transformation $\bar{g}_t \to g_t$ must be eliminated. Consequently, any $t$-dependence of $h_{tk}$ will stem from that of $\tilde{h}_{tk}$. Also notice that, if for some reason one wants to use the equations of motion in $(N, \bar{g}_t)$, any explicit dependence of $\bar{N}_t$ on $t$ must be eliminated as well.

Next, $(N, \bar{g}_t)$ and $(N, g_t)$ are equipped with linear and symmetric connections $\hat{\nabla}$ and $\tilde{\nabla}$, respectively. These connections are identified with the usual Levi-Civita connection for constant $t$, yielding the standard form of the connection coefficients not containing $t$. The rest of the connection coefficients are determined by the condition that the connections $\hat{\nabla}$ and $\tilde{\nabla}$, should be compatible with the non-degenerate part of $\bar{g}_t$ and $g_t$, respectively. That is, we have the conditions

$$\hat{\nabla}_\alpha \bar{g}_t = 0, \quad \hat{\nabla}_\alpha \bar{\bar{\mathbf{n}}}_t = 0, \quad \hat{\nabla}_\alpha g_t = 0, \quad \hat{\nabla}_\alpha \mathbf{n}_t = 0,$$

where $\bar{\bar{\mathbf{n}}}_t$ and $\mathbf{n}_t$ are families of unit normal vector fields to the FHSs in $(N, \bar{g}_t)$ and $(N, g_t)$, respectively. The conditions shown in equation (3) will hold if we make the requirements (where a comma denotes taking a partial derivative)

$$\frac{\partial}{\partial t} \left[ N^k_t \bar{N}^s_t \tilde{h}_{tk} \right] = 0, \quad \Rightarrow \quad \bar{N}^s_t, t = -\frac{1}{2} \tilde{h}_{tik} \bar{h}_{tik}, \quad (4)$$

and

$$\frac{\partial}{\partial t} \left[ N^k_t N^s_t h_{tk} \right] = 0, \quad \Rightarrow \quad N^s_t, t = -\frac{1}{2} N^k_t h_{tik} h_{tik}. \quad (5)$$
Given the requirements (4) and (5), the conditions shown in equation (3) now yield the nonzero extra connection coefficients (using a GTCS)

\[ \dot{\Gamma}^0_{00} = \frac{\bar{N}_{t,0}}{N_t}, \quad \dot{\Gamma}^i_j = \left( \frac{1}{t} + \frac{\bar{N}_{t,t}}{N_t} \right) \delta^i_j + \frac{1}{2} \bar{h}^{is}_{(t)} \bar{h}(s)_{sj,t}, \quad \dot{\Gamma}^i_j = \frac{1}{t} \delta^i_j + \frac{1}{2} \bar{h}^{is}_{(t)} h(s)_{sj,t}. \quad (6) \]

Note that all connection coefficients are symmetric in the lower indices. The equations of motion in \((\bar{N}, \bar{g}_t)\) are given by \([1, 2]\)

\[ \frac{d^2 x^\mu}{d\lambda^2} + \left( \dot{\Gamma}^\mu_{\nu\tau} \frac{dt}{d\lambda} + \dot{\Gamma}^\mu_{\beta\nu} \frac{dx^\beta}{d\lambda} \right) \frac{dx^\nu}{d\lambda} = \left( \frac{dt}{d\lambda} \right)^2 a^\mu(t). \quad (7) \]

Here, \(d\tau_t\) is the proper time interval as measured along the curve, \(\lambda\) is some general affine parameter, and \(a_t\) is the 4-acceleration measured along the curve.

One important postulate of the OQG is that gravitational quantities should be “formally” variable when measured in atomic units. This formal variability is also a postulate of revised quasi-metric gravity and applies to all dimensionful gravitational quantities. Said formal variability may be viewed as an interpretation of equation (1) and is directly connected to the spatial scale factor \(\bar{F}_t \equiv \bar{N}_t \frac{a}{a_0}\) of the FHSs \([1, 2]\). In particular, the formal variability applies to any potential gravitational coupling parameter \(G_t\). It is convenient to transfer the formal variability of \(G_t\) to mass (and charge, if any) so that all formal variability is taken into account of in the active stress energy tensor \(T_t\), which is the object that couples to space-time geometry via field equations. However, dimensional analysis yields that the gravitational coupling must be non-universal, i.e., that the electromagnetic active stress-energy tensor \(T_t^{(EM)}\) and the active stress-energy tensor for material particles \(T_t^{\text{mat}}\) couple to space-time curvature via two different (constant) coupling parameters \(G^B\) and \(G^S\), respectively. This non-universality of the gravitational coupling is required for consistency reasons. As a consequence, compared to GR, the non-universal gravitational coupling yields a modification of the right hand side of any quasi-metric gravitational field equations. (Said modification was missed in the original formulation of quasi-metric gravity.) The quantities \(G^B\) and \(G^S\) play the roles as gravitational constants measured in some local gravitational measurements at some chosen event at the arbitrary reference epoch \(t_0\).

As mentioned above, the form (1) of \(\bar{g}_t\) in the OQG was too restricted to allow the existence of a full coupling between space-time curvature and the active stress-energy tensor \(T_t\). Rather, a subset of the (modified) Einstein field equations was tailored to \(\bar{g}_t\), yielding partial couplings to space-time curvature of \(T_t^{(EM)}\) and \(T_t^{\text{mat}}\) \([1, 2]\). Unfortunately, using this approach, no aspects of \(\bar{g}_t\) were left free, meaning that \(\bar{g}_t\) would be fully determinable by the matter sources alone. Thus the OQG is essentially a waveless ap-
proximation theory, and it must therefore be discarded as a potentially viable candidate for quasi-metric gravity.

Now the problem is to find an extended set of field equations representing a full coupling between space-time curvature and $T_t$, in addition to being compatible with equation (1). To do that, we first define the correspondence with the OQG (and with GR) by requiring the validity of the old field equations (valid on the FHSs)

$$2\bar{\mathcal{R}}_t = 2(c^{-2}a^{ij}_F + c^{-4}a^i_F a^j_F - \bar{K}_{(t)ik}\bar{K}^{ij}_{(t)} + \mathcal{L}_{\bar{n}_t}\bar{K}_t)$$

$$= \kappa^B(T^{(EM)}_{(t)\perp} + \hat{T}^{(EM)}_{(t)i}) + \kappa^S(T^{\text{mat}}_{(t)\perp} + \hat{T}^{\text{mati}}_{(t)}),$$

$$c^{-2}a^i_F = \frac{\bar{N}_{t,i}}{\bar{N}_t},$$

(8)

$$\bar{R}_{(t)j\perp} = \bar{K}^{ij}_{(t)j|i} - \bar{K}_{t,j} = \kappa^B T^{(EM)}_{(t)j\perp} + \kappa^S T^{\text{mat}}_{(t)j\perp}.$$  

(9)

Here $\bar{R}_t$ is the Ricci tensor family corresponding to the metric family $\bar{g}_t$ and the symbol $'\perp'$ denotes a scalar product with $-\bar{n}_t$. Moreover, $\mathcal{L}_{\bar{n}_t}$ denotes a projected Lie derivative in the direction normal to the FHSs, $\hat{\bar{K}}_t$ denotes the extrinsic curvature tensor family (with trace $\bar{K}_t$) of the FHSs, a “hat” denotes an object projected into the FHSs and the symbol $'|$’ denotes spatial covariant derivation. (Note that $\mathcal{L}_{\bar{n}_t}$ operates on spatial objects only.) Finally $\kappa^B \equiv 8\pi G^B/c^4$ and $\kappa^S \equiv 8\pi G^S/c^4$, where the values of $G^B$ and $G^S$ are by convention chosen as those measured in some local gravitational measurements at some chosen event at the arbitrary reference epoch $t_0$.

Next, we will find a new spacetime tensor family $\bar{Q}_t$ defined from its projections $\bar{Q}_{(t)\perp}, \bar{Q}_{(t)j\perp} = \bar{Q}_{(t)i\perp}$ and $\bar{Q}_{(t)ik}$ with respect to the FHSs. These projections will play almost the same role as do the projections of the Einstein tensor in canonical GR. It is important to notice that unlike $\bar{R}_t$ and the Einstein tensor family $\bar{G}_t$, the definition of $\bar{Q}_t$ depends directly on the geometry of the FHSs and their extrinsic curvature. This means that the obtained expressions for the projections of $\bar{Q}_t$ will not be valid for any hypersurfaces other than the FHSs. In contrast, in canonical GR, the projections of the Einstein tensor $\bar{G}$ on a Lorentzian manifold with metric $\bar{g}$ is valid for any foliation of $\bar{g}$ into spatial hypersurfaces. These projections are given by (see, e.g., [4] for a derivation)

$$\bar{G}_{\perp \perp} = \frac{1}{2}(P + K^2 - K_{mn}K^{mn}),$$

(10)

$$\bar{G}_{i\perp j} = (K^k_j - \bar{K}\bar{a}^k_j)_{ij},$$

(11)

$$\bar{G}_{i\perp k} = -\frac{1}{N}\mathcal{L}_{\bar{n}_t}(\bar{K}_{ik} - \bar{K}\bar{h}_{ik}) + 3\bar{K}\bar{K}_{ik} - 2\bar{K}^s_i\bar{K}_{sk} - \frac{1}{2}(\bar{K}^2 + \bar{K}_{mn}\bar{K}^{mn})\bar{h}_{ik}$$

$$-c^{-2}\bar{a}_{ik} + c^{-4}\bar{a}_i\bar{a}_k + (c^{-2}\bar{a}^s_i + c^{-4}\bar{a}^s\bar{a}_i)\bar{h}_{ik} + \bar{H}_{ik},$$

$$c^{-2}\bar{a}_i \equiv \bar{N}^i_j/N,$$

(12)
where \( h_{ik} \) are the components of the spatial metric. Moreover, \( \bar{P} \) and \( \bar{H}_{ik} \) are the spatial Ricci scalar and the components of the spatial Einstein tensor, respectively.

We will now require that \( \bar{Q}_t \) and \( \bar{G}_t \) should have somewhat similar dynamical structures. That is, \( \bar{Q}_{(t)ik} \) and \( \bar{G}_{ik} \) should both predict weak GR-like gravitational waves in vacuum via having common (up to signs) second order terms \(-\frac{1}{8N}\mathcal{L}_{\bar{N}_t} \bar{K}_{ik} \) and \( \bar{H}_{ik} \) in equation (12). Furthermore, we must have that \( \bar{Q}_{(t)\perp \perp} + \bar{Q}_{(t)s} = 2\bar{R}_{(t)\perp \perp} \) to fulfil equation (8), and \( \bar{Q}_{(t)\perp j} = \bar{R}_{(t)\perp j} \) to fulfil equation (9). Besides, the extended field equations should also yield the same solutions as the OQG for the metrically static vacuum cases (for which the extrinsic curvature vanishes identically). Thus for these cases, the equation \( \bar{Q}_{(t)ik} = 0 \) should yield the relationship \( \bar{H}_{(t)ik} + c^{-2}\bar{a}_{F[i}k + c^{-4}\bar{a}_{F}a_{Fk} - (c^{-2}\bar{a}_{F}|s - \frac{1}{(ctN_t)^2})\bar{h}_{(t)ik} = 0 \), which follows directly from the OQG [2]. These requirements still leave some leeway regarding terms quadratic in extrinsic curvature, so one may expect that some amount of guesswork will be necessary. However, requiring that the extended field equations should also yield the same solution as the OQG for isotropic cosmology (see the next section) leaves almost no ambiguity, and we can thus postulate

\[
\bar{Q}_{(t)\perp \perp} \equiv \mathcal{L}_{\bar{N}_t} \bar{K}_t - \frac{1}{2}(\bar{P}_t + \bar{K}_t^2 - \bar{K}_{(t)mn}\bar{K}_{(t)}^{mn}) + 3c^{-4}\bar{a}_{F}s\bar{a}_{F} + \frac{3}{(ctN_t)^2},
\]

(13)

\[
\bar{Q}_{(t)\perp j} = \bar{Q}_{(t)\perp j} \equiv \bar{G}_{(t)\perp j} = (\bar{K}_{(t)j} - \bar{K}_{t}\bar{s}^{k})|_{k},
\]

(14)

\[
\bar{Q}_{(t)ik} \equiv \frac{1}{N_t}\mathcal{L}_{\bar{N}_t,\bar{n}_t} \bar{K}_{(t)ik} - \bar{K}_t\bar{K}_{(t)ik} + \bar{K}_{(t)is}\bar{K}^{s}_{(t)k} + \frac{1}{2}(\bar{K}_t^2 - \bar{K}_{(t)mn}\bar{K}_{(t)}^{mn})\bar{h}_{(t)ik}
- c^{-2}\bar{a}_{F[i}k - c^{-4}\bar{a}_{F}a_{Fk} + (c^{-2}\bar{a}_{F}|s - \frac{1}{(ctN_t)^2})\bar{h}_{(t)ik} - \bar{H}_{(t)ik}.
\]

(15)

Projections of \( \bar{Q}_t \) are related to projections of \( \bar{G}_t \) and \( \bar{R}_t \) via the formulae

\[
\bar{Q}_{(t)\perp \perp} = -\bar{G}_{(t)\perp \perp} + \bar{R}_{(t)\perp \perp} + \bar{K}_{(t)mn}\bar{K}_{(t)}^{mn} - c^{-2}\bar{a}_{F}|s + 2c^{-4}\bar{a}_{F}s\bar{a}_{F} + \frac{3}{(ctN_t)^2},
\]

(16)

\[
\bar{Q}_{(t)ik} = -\bar{G}_{(t)ik} - \bar{K}_{(t)is}\bar{K}^{s}_{(t)k} - 2c^{-2}\bar{a}_{F[i}k - 2c^{-4}\bar{a}_{F}a_{Fk}
+ (\bar{R}_{(t)\perp \perp} + c^{-2}\bar{a}_{F}|s - \frac{1}{(ctN_t)^2})\bar{h}_{(t)ik}.
\]

(17)

Equations (13)-(15) may be thought of as projections of the space-time tensor family \( \bar{Q}_t \) with components \( \bar{Q}_{(t)\mu \nu} \) (in a general coordinate system). The extended quasi-metric field equations representing a full coupling to the active stress-energy tensor may then be written in the form

\[
\bar{Q}_{(t)\mu \nu} = \kappa B T_{(t)\mu \nu}^{(EM)} + \kappa S T_{(t)\mu \nu}^{\text{mat}}.
\]

(18)
but as mentioned earlier, $\bar{Q}_t$ is not a “genuine” space-time object since it is defined from its projections with respect to a particular space-time foliation.

Equation sets (10)-(12) and (13)-(15), considered as dynamical systems for initial value problems, share some similarities. Both sets consist of dynamical equations plus constraints. The constraint equations are determined by the initial data on an initial FHS (or, for the set (10)-(12), initial data on an arbitrary spatial hypersurface), whereas the dynamical equations are not. Now we see that the quantities $\bar{G}_{\perp\perp}$, $\bar{G}_{ik}$ and $\bar{Q}(t)_{\perp\perp}$ are all determined by the initial data, while the quantities $\bar{Q}(t)_{\perp\perp}$, $\bar{G}_{ik}$ and $\bar{Q}(t)_{ik}$ are not. So it would seem that the number of constraints/dynamical equations is different for the two systems. However, the dynamical equation (13) is not independent of equation (15) since its trace yields the combination

$$\bar{Q}(t)_{\perp\perp} - \bar{Q}(t)_{i} = -\bar{P}_{t} + 3\bar{K}(t)_{mn}\bar{K}(t)^{mn} - \bar{K}_{t}^{2} - 2c^{-2}\bar{a}_{F}^{i} + 4c^{-4}\bar{a}_{F}^{i}\bar{a}_{F}^{i} + \frac{6}{(ctN_{t})^{2}}, \quad (19)$$

which is a new scalar equation that is indeed determined by the initial data. Equation (19) may then be regarded as a constraint counterpart to equation (10). Thus both dynamical systems have the same basic form.

Due to some concern that the global cosmic expansion might unduly interfere with the gravitational dynamics of $\bar{g}_t$, via the global scale changes of the FHSs, it was argued in [1, 2] that in quasi-metric gravity, matter sources should not couple explicitly to the intrinsic geometry of the FHSs. But from equations (13), (15) and (18) we see that there is such a coupling in the extended field equations. However, the terms in equations (13) and (15) representing the prior 3-geometry makes it likely that said coupling should not have any serious consequences.

As we have seen, relaxing the original restrictions on the form of equation (1) leads to the requirements (4) and (5) and their consequences for the extra connection coefficients in equation (6) and ultimately to the extended field equations (13) and (15) (implying equations (18) and (19)). However, there will be no further change in the basic equations of quasi-metric gravity when switching from the OQG to the revised theory. In particular, the transformation $\bar{g}_t \rightarrow g_t$ will be defined as before [1, 2]. Also, the local conservation laws in $\bar{g}_t$ of $\bar{T}_t$ are unchanged, i.e.,

$$T_{(t)\mu;\nu} = 2\bar{N}_{t}^{\nu}T_{(t)\mu} = 2c^{-2}\bar{a}_{F}\bar{T}_{s} = 2\bar{N}_{t,\perp}T_{(t)\perp\mu}. \quad (20)$$

Said conservation laws must take the form (20) to be consistent with classical electrodynamics coupled to quasi-metric gravity [5]. However, equation (20) applies to both $T_{t}^{(EM)}$ and to $T_{t}^{\text{mat}}$ alike. Notice that, unlike its counterpart in GR, equation (20) does
not automatically follow from the field equations. That is, equation (20) represents real restrictions on what kind of sources can be admitted in the field equations for a given gravitational system. We shall see an example of this in the next section.

3 Some example solutions

In this section, we find some solutions of the field equations (18) for simple systems. Corresponding solutions may have been found previously for the OQG. Said solutions do not cover metrically static vacua since, by construction, for such systems the solutions of the extended field equations and of the OQG coincide. See [6] for the spherically symmetric case.

3.1 Isotropic cosmology

Isotropic cosmology in the OQG has been treated in [3]. Here, we will show that the solution found there is the unique solution also of the revised theory. We start by introducing a spherical GTCS \( \{x^0, \chi, \theta, \phi\} \) in which equation (1) takes the form

\[
\text{d}s^2_t = \bar{N}_t^2 \left\{ - (dx^0)^2 + (ct)^2 \bar{S}^2(t) \left( d\chi^2 + \sin^2 \chi d\Omega^2 \right) \right\}, \tag{21}
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and \( \bar{S}(t) \) is a global scale factor presumably depending on the matter density of the quasi-metric universe. The extrinsic curvature tensor and the intrinsic curvature of the FHSs obtained from equation (21) are given by

\[
\bar{K}_{(t)ik} = \frac{\bar{N}_{t,i}}{\bar{N}_t} \bar{h}_{(t)ik}, \quad \bar{K}_t = 3 \frac{\bar{N}_{t,i}}{\bar{N}_t}, \quad \bar{H}_{(t)ik} = -\frac{1}{(ct\bar{N}_t \bar{S}(t))^2} \bar{h}_{(t)ik}, \quad \bar{P}_t = \frac{6}{(ct\bar{N}_t \bar{S}(t))^2}, \tag{22}
\]

We assume that the quasi-metric universe is filled with a perfect fluid with active mass density \( \tilde{\varrho}_m \) and corresponding pressure \( \tilde{p} \), so that

\[
T_{(t)\perp\perp} = \tilde{\varrho}_mc^2 = \left(\frac{t_0}{\bar{N}_t t}\right)^2 \tilde{\varrho}_m(t)c^2, \quad T_{(t)\chi}^\chi = T_{(t)\theta}^\theta = T_{(t)\phi}^\phi = \tilde{p} = \left(\frac{t_0}{\bar{N}_t t}\right)^2 \bar{p}(t), \tag{23}
\]

where we have set the arbitrary boundary condition \( \bar{N}_t(t_0) = 1 \) for the present reference epoch \( t_0 \). Furthermore we have the relationship

\[
\tilde{\varrho}_m = \begin{cases} \frac{t_0^3 \bar{N}_t^3 \varrho_m}{t_0 t} & \text{for a fluid of material particles}, \\ \frac{t_0^4 \bar{N}_t^4 \varrho_m}{t_0 t} & \text{for the electromagnetic field}, \end{cases} \tag{24}
\]

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between the quantity $\bar{\rho}_m$ and the directly measurable passive (inertial) mass density $\rho_m$. Now, by taking the normal projection of equation (20) with respect to the FHSs we find (see, e.g., [4] for explicit projection formulae)

$$\mathcal{L}_{\bar{n}_t} T_{(t)\perp\perp} = \frac{N_{t,\perp}}{N_t} \left( T_{(t)\perp\perp} + \dot{T}_{(t)\perp} \right) = \frac{t^2_0}{t^2 N_t^3} \left( \bar{\rho}_m c^2 + 3\bar{\rho} \right),$$

while taking the Lie derivative directly of equation (23) we find

$$\mathcal{L}_{\bar{n}_t} T_{(t)\perp\perp} = 2\frac{N_{t,\perp}}{N_t} T_{(t)\perp\perp} = 2\frac{t^2_0}{t^2 N_t^3} \bar{\rho}_m c^2.$$

But then, to be consistent equations (25) and (26) imply that the perfect fluid must satisfy the equation of state $\rho_m = 3\rho/c^2$, i.e., it must be a null fluid. That is, any material component of the fluid must be ultrarelativistic, so that any deviation from said equation of state is negligible. This is a good approximation for a hot plasma mainly consisting of photons and neutrinos. The above result was arrived at also for the OQG, see [3] for a further discussion.

The null fluid condition implies that the left hand side of equation (19) must vanish. But then equations (19) and (22) yield $\dot{S}(t) = 1$, so that equation (21) has the same form as for the OQG. Moreover, with $\dot{S}(t) = 1$, it turns out that for the metric family (21) it is sufficient to solve equation (8) in order to find a solution $\bar{N}_t$. Such a solution with a correct vacuum limit was found in [3]. The extended field equations (13)-(15) do not yield anything further besides the solution found from equation (8), and it is given by [3] (expressed by the “critical” density $\bar{\rho}_m^{ct}(t)=\frac{3}{8\pi G \bar{\rho}_m^{ct}(EM)}$)

$$\bar{N}_t = \exp \left[ -\frac{1}{2} \frac{(x^0)^2}{(ct)^2} \frac{\dot{\bar{\rho}}_m(t)}{\bar{\rho}_m^{ct}} + \frac{1}{2} \frac{\ddot{\bar{\rho}}_m(t)}{\bar{\rho}_m^{ct}} \right], \quad \dot{\bar{\rho}}_m^{ct} = \frac{G^B}{G^S} \bar{\rho}_m^{EM} + \bar{\rho}_m^{mat}. \quad (27)$$

Here, the epoch $t_1$ is interpreted as the epoch when matter creation ceases, so that essentially $\frac{\partial}{\partial t} \bar{\rho}_m = 0$ for $t \geq t_1$. For a further discussion of the solution (27), see [3].

### 3.2 Spherically symmetric electrovacuum

Classical electrodynamics formulated within the QMF has been published in [5]. There will be no change in the basic formulae and definitions of this theory when considering the more general form (1) of the metric family. However, compared to the OQG, there will be some extra features when solving the extended field equations (18) with electromagnetic field sources. In this paper, we find the metric family for the electrovacuum outside a spherically symmetric, metrically static, charged body according to equations (18). The counterpart solution for the OQG was found in [5].
We start by writing down the chosen form of the spherically symmetric line element family $ds_t^2$ using a spherical GTCS $\{x^0, \rho, \theta, \phi\}$, where $\rho$ is a radial coordinate. It would be very convenient if one could choose a form of said family in a way such that equation (8) becomes identical to its counterpart for the OQG. It turns out that this is possible, and that this choice simplifies the calculations considerably. We thus consider the line element family

$$ds_t^2 = \bar{B}(\rho) \left[ -(dx^0)^2 + \frac{t^2}{t_0^2} \left( \frac{\bar{A}^2(\rho)d\rho^2}{1 - \frac{\rho^2}{\Xi_0^2}} + \bar{A}(\rho)\rho^2 d\Omega^2 \right) \right],$$  \hspace{1cm} (28)

where $\bar{B} \equiv \bar{N}_t^2$ and $\Xi_0 \equiv ct_0$. What enters as a source in the field equations is the active electromagnetic stress-energy tensor family $T^{(EM)}_{\alpha\beta}$, given by the familiar formula

$$T^{(EM)}_{\alpha\beta} = \frac{1}{4\pi} \left( \bar{F}_{(t)\alpha} \gamma^{\alpha} \bar{F}_{(t)\beta} - \frac{1}{4} \bar{F}_{(t)\rho\sigma} \bar{F}_{(t)\rho\sigma} \bar{g}_{(t)\alpha\beta} \right),$$ \hspace{1cm} (29)

where $\bar{F}_t$ and $\tilde{F}_t \equiv \frac{1}{t_0} \bar{N}_t \bar{F}_t$ are respectively the passive and the active electromagnetic field tensor in $(N, \bar{g})$, see [5]. Now the active electric field $\tilde{E}_t$ can be defined from its components in the above defined GTCS. Due to the spherical symmetry its only nonvanishing component is $\tilde{E}_{(t)\rho} \equiv \bar{F}_{(t)\perp}$. Using Maxwell’s equations for $\bar{F}_t$ [5] and performing a standard calculation, integrating the (passive) charge density $\tilde{\rho}_c$ over the source, we then find (see [5] for the details of a similar calculation)

$$\tilde{E}_{(t)\rho} = \frac{Q}{\rho^2 \sqrt{1 - \frac{\rho^2}{\Xi_0^2}}}, \hspace{1cm} Q \equiv \int \int \int \tilde{\rho}_c d\bar{V}_t. \hspace{1cm} (30)$$

Next we put these expressions into the field equations. It is convenient to use equation (8) since it only depends on the function $\bar{B}(\rho)$ and its derivatives. We find the same equation as for the OQG, i.e., [5]

$$(1 - \frac{\rho^2}{\Xi_0^2}) \frac{\bar{B}_{\rho\rho}}{\bar{B}} + \frac{2}{\rho} (1 - \frac{3\rho^2}{2\Xi_0^2}) \frac{\bar{B}_{\rho}}{\bar{B}} = \frac{r_{Q0}^2}{\rho^2}, \hspace{1cm} (32)$$

where $r_{Q0} \equiv \sqrt{2G\bar{B}|Q|/c^2}$. An (unique) exact solution of equation (32) is [5]

$$\bar{B}(\rho) = \cosh \left[ \frac{r_{Q0}}{\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right] - \frac{r_{a0}}{r_{Q0}} \sinh \left[ \frac{r_{Q0}}{\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right]. \hspace{1cm} (33)$$

Here $Q$ is the (constant) passive charge of the source. From equations (29) and (30) we then have (it is straightforward to check that these expressions satisfy equation (20))

$$T^{(EM)}_{(t)\perp\perp} = \frac{t_0^2}{t^2} \frac{Q^2}{8\pi B A^2 \rho^2}, \hspace{1cm} \tilde{T}^{(EM)}_{(t)\rho} = -T^{(EM)}_{(t)\perp\perp}, \hspace{1cm} \tilde{T}^{(EM)}_{(t)\theta} = \tilde{T}^{(EM)}_{(t)\phi} = T^{(EM)}_{(t)\perp\perp}. \hspace{1cm} (31)$$
Here $r_{s0}$ is the modified (generalized) Schwarzschild radius of the central source,
\[
r_{s0} \equiv \left( \frac{2M_{0}^{\text{mat}}G^S}{c^2} + \frac{2M_{0}^{\text{(EM)}}G^B}{c^2} \right) \text{sech} \left[ \frac{r_{Q0}}{\rho_s} \sqrt{1 - \frac{\rho_s^2}{\Xi_0}} \right] + r_{Q0} \tanh \left[ \frac{r_{Q0}}{\rho_s} \sqrt{1 - \frac{\rho_s^2}{\Xi_0}} \right],
\]
(34)

\[
M_{t0}^{\text{mat}} \equiv c^{-2} \int \int \int \sqrt{B} \left[ T_{(t0)\perp\perp}^{\text{mat}} + \tilde{T}_{(t0)i}^{\text{mat}i} \right] d\tilde{V}_{t0},
\]
\[
M_{t0}^{\text{(EM)}} \equiv c^{-2} \int \int \int \sqrt{B} \left[ T_{(t0)\perp\perp}^{\text{(EM)}} + \tilde{T}_{(t0)i}^{\text{(EM)}i} \right] d\tilde{V}_{t0},
\]
(35)

where the integration is taken over the source and $\rho_s$ is its coordinate radius (i.e., $\rho \leq \rho_s$).

The quantity $r_{s0}$ is thus the generalized Schwarzschild radius of the source at epoch $t_0$, modified to include the electrostatic field energy.

To determine the function $\bar{A}(\rho)$, it is convenient to use the $\rho\rho$- or the $\theta\theta$-components of equation (18), since these equations only depend on $\bar{A}(\rho)$ and its derivatives. After some calculations we find the equation
\[
(1 - \frac{\rho^2}{\Xi_0}) \left[ - \frac{\bar{A}_{\rho\rho}}{A} + \frac{3}{2} \left( \frac{\bar{A}_{\rho}}{A} \right)^2 \right] + \frac{\rho}{\Xi_0} \frac{\bar{A}_{\rho}}{A} + \frac{2}{\Xi_0} \left( 1 - \bar{A}^2 \right) = \frac{r_{Q0}^2}{\rho^4},
\]
(36)

from the $\theta\theta$-component (or equivalently, from the $\phi\phi$-component), and the equation
\[
(1 - \frac{\rho^2}{\Xi_0}) \left[ \frac{1}{2} \left( \frac{\bar{A}_{\rho}}{A} \right)^2 + \frac{2}{\rho} \frac{\bar{A}_{\rho}}{A} \right] + \frac{2}{\rho^2} \left( 1 - \bar{A} \right) - \frac{2}{\Xi_0} \left( 1 - \bar{A}^2 \right) = \frac{r_{Q0}^2}{\rho^4},
\]
(37)

from the $\rho\rho$-component. As they should, these equations have an unique, common solution with the correct limit $\bar{A} \to 1$ for $r_{Q0} \to 0$. Said solution is exact and can be found by computer (MAPLE). We get
\[
\bar{A}(\rho) = \frac{r_{Q0}^2}{2\rho^2} \left( \sinh \left[ \frac{r_{Q0}}{\sqrt{2}\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0}} \right] + \frac{r_{Q0}^2}{2\Xi_0} \exp \left[ - \frac{\sqrt{2}r_{Q0}}{\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0}} \right] \right)^{-1}.
\]
(38)

To construct the “physical” metric family $g_t$ via the transformation $\tilde{g}_t \to g_t$ (see, [1, 2] for the general method), we need the quantity
\[
v(\rho)/c = \frac{\bar{B}_{\rho}}{\bar{B}} \left[ \frac{\bar{B}_{\rho}}{\bar{B}} + \frac{\bar{A}_{\rho}}{A} + \frac{2}{\rho} \right]^{-1}.
\]
(39)

The resulting exact expression for $v(\rho)/c$ is rather complicated, so we do not include it here (see [5] for the counterpart expression for the OQG, where $\bar{A}_{\rho} \equiv 0$). We then have [1, 2]
\[
\dot{s}_t^2 = \bar{B}(\rho) \left\{ - \left( 1 - \frac{v^2(\rho)}{c^2} \right)^2 (dx^0)^2 + \left( \frac{t}{t_0} \right)^2 \left[ \left( \frac{1 + \frac{v(\rho)}{c}}{1 - \frac{v(\rho)}{c}} \right)^2 \bar{A}^2(\rho) d\rho^2 + \bar{A}(\rho) \rho^2 d\Omega^2 \right] \right\}.
\]
(40)

Substituting the exact expressions for $\bar{B}(\rho)$, $\bar{A}(\rho)$ and $v(\rho)/c$ into equation (40) now yields an exact expression for the metric family $g_t$. 

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3.3 Weak gravitational waves in vacuum

If we ignore the global curvature of space, the weak field (linear) approximations of the field equations (18) for vacuum are the same as for GR provided that \( \bar{a}_F \) vanishes identically. (This can be easily seen directly from equations (16) and (17).) Thus the counterpart weak-field GR-solution of locally plane-fronted waves with two independent polarizations will also be an approximate solution of equation (18). The only difference from the GR-solution is that the global cosmic expansion is included via the scale factor. Thus the family of line elements takes the form

\[
\bar{ds}_t^2 = - (dx^0)^2 + \frac{t^2}{t_0^2} \left[ (E_{ks} + \bar{\varepsilon}_{(t)ks}) dx^k dx^s \right],
\]

(41)

where \( E_{ks}dx^k dx^s \) is the metric of Euclidean space and where the terms

\[
\bar{\varepsilon}_{(t)ks} = \Re \left[ \bar{A}_{(t)ks} \exp(i\bar{\vartheta}_t) \right],
\]

(42)

describe the plane wave perturbation from the Euclidean background. Moreover, we have

\[
\bar{\vartheta}_t \equiv \bar{k}_{(t)0} (x^0 - x^0_0) + \bar{k}_{(t)i} x^i, \quad \bar{k}_{(t)\mu} = \bar{\vartheta}_{t,\mu}, \quad \bar{k}_{(t)0,t} = -\frac{1}{t} \bar{k}_{(t)0}, \quad \bar{k}_{(t)j,t} = 0,
\]

(43)

where \( \bar{\vartheta}_t \) is the phase factor and where \( \bar{k}_{(t)\mu} \) denotes the components of the wave 4-vector family. (Also, \( x^0_0 \) is an arbitrary reference epoch.) Finally, \( \bar{A}_{(t)ks} = \frac{\mu}{t} \bar{A}_{(t0)ks} \) is the (possibly complex) polarization tensor. As for the counterpart GR case, equation (18) for vacuum (ignoring global space curvature) yields that the plane wave is null, transverse and traceless. That is, choosing Cartesian coordinates \((x, y, z)\) with the wave travelling in the \(z\)-direction, equation (41) takes the form

\[
\bar{ds}_t^2 = - (dx^0)^2 + \frac{t^2}{t_0^2} \left[ (1 + \bar{\varepsilon}_{(t)xx}) dx^2 + (1 - \bar{\varepsilon}_{(t)xx}) dy^2 + 2\bar{\varepsilon}_{(t)xy} dx dy + dz^2 \right].
\]

(44)

Since equations (41) and (44) only describe approximative solutions of equation (18), to further investigate the nature of gravitational radiation in quasi-metric gravity some exact solutions should be found. Such solutions are expected to differ from their GR counterparts. However, finding exact wave-like solutions of equation (18) may turn out to be difficult, and is beyond the scope of the present paper.

4 Conclusion

In this paper, we have relaxed the original restrictions on the quasi-metric space-time geometry \((\bar{N}, \bar{g})\) so that its most general form is now given by equation (1). The reason for this revision was to make possible the prediction of (weak) GR-like gravitational
waves since such have now been directly detected. Moreover, the original quasi-metric gravitational field equations have been extended with the extra equations (13) and (15) (together with the field equations (18)), which by construction are designed to predict GR-like gravitational waves in the weak-field approximation for vacuum. On the other hand, exact gravitational wave solutions are expected to differ from their GR counterparts.

Besides the prediction of gravitational waves, the differences between the predictions of the extended quasi-metric gravitational theory and the OQG are small. This means that, disregarding gravitational waves and systems emitting gravitational waves (e.g., binary pulsars), the observational status of the extended gravitational theory is the same as for the OQG (i.e., currently nonviable [3]).

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