Generation of strong magnetic fields in old neutron stars accounting for continuous chiral magnetic effect

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Abstract. We suggest a new mean field dynamo model in anomalous MagnetoHydroDynamics (AMHD) accounting for the mean spin (polarization) of the magnetized chiral (ultrarelativistic) plasma of a neutron star (NS). For simplicity we consider a non-superfluid NS with its rigid rotation neglecting also any matter turbulence (convection) within a star. On this way, we recover the Chiral Magnetic Effect (CME) as a possible source for the amplification of a seed, sufficiently strong magnetic field, \( B \sim 10^{13} \, \text{G} \), up to values \( B \gtrsim 10^{18} \, \text{G} \) in old NS's, having ages \( t \gtrsim 10^6 \, \text{yr} \). The important issue in AMHD model suggested is the continuous evolution of the chiral imbalance providing the CME for these ages, \( \partial_t \mu_5(t) \neq 0 \), in spite of the fast spin-flip in Coulomb collisions in the dense NS plasma that leads to vanishing \( \mu_5 \rightarrow 0 \) at an earlier epoch in the corresponding protoneutron star. In contrast to the conventional mean-field dynamos, the dynamo drivers in the model are produced due to magnetic field generated at the previous stages of stellar evolution. It makes our model basically nonlinear.

Keywords: Magnetohydrodynamics, neutron stars, magnetic fields
1 Introduction

The generation of strongest magnetic fields observed in magnetars $\sim 10^{15}$ G [1] is still an open problem in astrophysics. The observation of X-rays and soft $\gamma$-emission from such objects means the penetration of these fields to the magnetosphere of a neutron star (NS) filled by the magnetized plasma, which is a source of electromagnetic waves.

In the present work, we develop a model for the generation of such superstrong magnetic fields, $B \gg B_{\text{Schwinger}} = m_e^2/e = 4.41 \times 10^{13}$ G, inside NS. There were many attempts to understand that generation at different stages of the NS evolution. In the most quoted paper on the subject [2], the authors give rough estimates using $\alpha - \Omega$ dynamo at the early stage of a nascent NS. The magnetic fields as strong as $3 \times 10^{15}$ G can be generated as a random superposition of many small dipoles with the size $l \sim 1$ km through the convection in such nascent NS originated by a huge neutrino emission in the protoneutron star (PNS) during first seconds after the core bounce.

The magnetic field origin in various celestial bodies is mainly associated with a dynamo action, i.e. the transformation of the kinetic energy into magnetic one in an electrically conducting media, see, e.g., refs. [3, 4]. The drivers of the dynamo action are thought to be a differential rotation and various forms of mirror asymmetric flows in convective or turbulent shells of the body of interest. The dynamo action in neutron stars was considered, in particular, in ref. [2] (see also, e.g., ref. [5] for PNS in the context of pulsar magnetic field origin). However, the point is that a pulsar magnetic field can be thought to be amplified from a relatively weak magnetic field in a normal star during its compression into NS. Perhaps, it is why the NS dynamo does not attract the main attention of experts in dynamo studies, who more prefer to explore solar or galactic dynamos. One more point is that NSs are believed to be solid body rotators, see, e.g., section 6.12 in ref. [6], while differential rotation is a very popular (however, not obligatory) dynamo driver for dynamo modelers.

According to contemporary knowledge, dynamo driven stellar magnetic fields can reach the equipartition between magnetic and kinetic energy what looks sufficient for the explanation of pulsar magnetic fields even if details of the magnetic field evolution from a relatively
weak magnetic field in a normal star to NS remain debatable. The problem arises for superstrong magnetar magnetic fields where an additional amplification in NS looks inevitable even if various above discussed options are exploited. Here we face severe problems with the concept of the solid body rotation of neutron stars because it is unable to amplify quasi-stationary magnetic fields.

Some new attempts to explain the generation of strong magnetic fields in PNS, while in a different way, with the use of Anomalous Magnetohydro-Dynamics (AMHD), were undertaken, e.g., in refs. [7, 8]. The following development of AMHD as laminar and turbulent dynamos, applying such approaches both to the hot plasma in early Universe and for the degenerate ultrarelativistic electron plasma in PNS, was done in refs. [9, 10]. Notice also ref. [11], where the authors took into account the mean spin term in a hot plasma as an additional source for the evolution of the chiral anomaly in AMHD. However, in the hot universe plasma, the electron chemical potential \( \mu_e \) is negligible, \( \mu_e \ll T \). Therefore, the lepton asymmetry and the corresponding mean spin contribution cease. Oppositely to such a medium, in the degenerate electron gas within NS, for which \( \mu_e \) prevails over temperature, \( \mu_e \gg T \), and, moreover, its derivative differs from zero for a non-uniform density profile, \( dn_e/dr \sim d\mu_e/dr \neq 0 \), the mean spin contribution emerges as the permanent source for the chiral anomaly evolution [12].

Novelty of our approach is that the dynamo action suggested requires neither a differential rotation nor a stellar convection. Mirror asymmetric magnetic field driver, responsible for the dynamo action, is connected with a mirror asymmetry of particle physics characteristics. In this sense, the dynamo suggested here can be considered as a specific kind of so-called \( \alpha^2 \)-dynamo based on mirror asymmetry of convective or turbulent flows in standard MHD, see, e.g., ref. [13]. A substantial difference of the process under discussion from the conventional forms of \( \alpha^2 \)-dynamo is that our dynamo driver is connected with a magnetic field somehow created at a previous stage of the magnetar evolution. Correspondingly, the process occurs to be very nonlinear just from its starting point. It is why the usage of conventional methods of dynamo studies to separate the investigation of the evolution of the dynamo driven magnetic field from the evolution of dynamo drivers, in order to simplify the problem, occurs here even more limited than in stellar or galactic dynamos. Of course, contemporary computers can run very complicated equations, and it is not a big deal, in principle, to include various effects in a single numerical model. Of course, it would require to consider various details of the dynamo driven magnetic field configuration.

The point, however, is that our knowledge of the structure and physical properties of magnetars is very limited. Thus, it would be highly desirable to avoid detailed description of the magnetar magnetic field configuration and deal with a limited number of its first Fourier modes. In order to resolve these contradictory intentions, we consider a general set of governing equations in a suitable Fourier basis and truncate it as strongly as possible to keep a few Fourier modes to allow a dynamo action. A simple analysis, e.g., in ref. [14], shows that we need here two Fourier modes for the poloidal magnetic field component, as well as two Fourier modes for the toroidal magnetic field component. Correspondingly, we keep two Fourier modes for \( \alpha \)-effect, as well. Of course, our dynamo model is very crude and illustrative. However, we believe that it allows to understand the physics of the problem under discussion until one learns more about the magnetar structure.

Thus, in the present work we develop the mean field dynamo approach in AMHD for the generation of strong axisymmetric magnetic fields in NS. For this purpose, we derive in section 2 the master evolution equation for the helicity parameter \( \alpha \sim \mu_5 \). Then our work
is organized as follows. In the main section 3, we complete the set of AMHD equations for the mean field dynamo in NS with its rigid rotation, $\Omega = \text{const}$, by the evolution equations for the azimuthal potential $A_\varphi = A$ and the toroidal magnetic field $B_\varphi = B$. Using the low Fourier mode approximation in a thin layer under the NS crust, we derive in section 3.1 the kinetic equations for the helicity amplitudes $\alpha_{1,2}(t)$ and formulate the initial conditions for all six required functions: $\alpha_{1,2}(t)$, $a_{1,2}(t)$, and $b_{1,2}(t)$. The solution of the self-consistent non-linear differential equations for these functions is presented in section 3.2 for the helicity parameters $\alpha_{1,2}(t)$, being illustrated in figures 1 and 2.

The growth of magnetic fields due to the chiral magnetic effect (CME) with continuous helicity parameter $\alpha$ is discussed in section 4 and in figure 3, where we show the sharp amplification of amplitudes, $a_{1,2}(t)$ and $b_{1,2}(t)$, correlated with the spikes for the helicity amplitudes $\alpha_{1,2}$, shown in figure 2. Then, in section 5, we discuss both the results and the shortcomings of our AMHD approach for the generation of superstrong magnetic fields in an old NS. We add also some comments on unsolved problems in AMHD applied, in particular, to the NS crust. Finally, in appendix A, we present the full set of cumbersome non-linear differential AMHD equations for the amplitudes $\alpha_{1,2}(t)$, $a_{1,2}(t)$, and $b_{1,2}(t)$ using the low Fourier mode approximation.

2 Anomalous magnetic helicity parameter $\alpha$ in NS

In this section, we derive the evolution equation for the $\alpha$-helicity parameter in AMHD starting from the axial Ward anomaly in QED plasma.

The magnetic helicity parameter $\alpha$ plays a crucial role for the generation of magnetic fields in astrophysical plasmas. In the mean-field dynamo theory [15], this parameter is given by the kinetic helicity $\alpha \simeq \tau_{\text{corr}} (\mathbf{u} \cdot (\nabla \times \mathbf{u})) / 3$, where $\tau_{\text{corr}}$ is the correlation time, that enters the growth term in the induction equation $\nabla \times (\alpha \mathbf{B})$.

In this work, we develop the mean field dynamo model in AMHD applying it to the generation of magnetic fields in NS using of the CME. The CME is driven by the anomalous current $J_{\text{anom}} = e^2 \mu_5 \mathbf{B} / 2 \pi^2$ [16]. The evolution of the chiral imbalance $\mu_5 = (\mu_R - \mu_L) / 2$, where $\mu_{R,L}$ are the chemical potentials of right and left electrons, is based on the axial Ward anomaly [17] valid in the ultra-relativistic QED plasma:

$$\frac{\partial}{\partial x^\mu} \bar{\psi} \gamma^\mu \gamma^5 \psi = 2 i m_e \bar{\psi} \gamma_5 \psi + \frac{e^2}{8 \pi^2} F_{\mu \nu} F^{\mu \nu}. \quad (2.1)$$

This anomaly, being statistically averaged in a chiral equilibrium plasma (in the massless limit $m_e = 0$), reduces to the evolution equation for the chiral anomaly density $n_5(x,t) = \langle \bar{\psi} \gamma^5 \psi \rangle = n_R(x,t) - n_L(x,t)$ completed by losses for $n_5(x,t)$ because of the spin-flip through collisions in NS plasma ($\sim \Gamma_f$),

$$\frac{\partial n_5(x,t)}{\partial t} = - \nabla \cdot \mathbf{S}(x,t) + \frac{2 \alpha_{\text{em}}}{\pi} (\mathbf{E} \cdot \mathbf{B}) - \Gamma_f n_5(x,t), \quad (2.2)$$

where $\alpha_{\text{em}} = e^2 / 4 \pi \approx (137)^{-1}$ is the fine structure constant in units $\hbar = c = 1$, $n_{R,L}(x,t) = \mu_{R,L}^3(x,t) / 6 \pi^2$ are the densities of right and left-handed electrons related to the corresponding chemical potentials. For a small chiral imbalance $\mu_5 = (\mu_R - \mu_L) / 2$, one obtains that $n_5(x,t) = \mu_5^2(r) \mu_5(x,t) / \pi^2$ where $\mu_5(r)$ is the non-uniform chemical poten-
tial for the stationary electron density $n_e(r) = n_R + n_L = \mu_3^2(r)/3\pi^2$ in the NS degenerate electron gas.

The term $-\nabla \cdot S(x, t)$ in the evolution eq. (2.2) is given by the mean spin in magnetized plasma. It is also known as the chiral separation effect (CSE) (see, e.g., ref. [18]),

$$S(x, t) = \langle \bar{\psi} e^{i\gamma_5 \psi} e \rangle_0 \equiv \langle \psi^\dagger \Sigma \psi \rangle_0 = -\left(\frac{e\mu_e(r)}{2\pi^2}\right) \mathbf{B}(x, t). \quad (2.3)$$

We consider below the evolution of a mean axisymmetric magnetic field in NS with spherical coordinates, $x \rightarrow (r, \theta)$, corresponding to argument changes $\mathbf{B}(x, t) \rightarrow \mathbf{B}(r, \theta, t)$ and similarly for the magnetic helicity parameter $\alpha$ in AMHD, $\alpha(r, \theta, t) = 2\alpha_{em}\mu_5(r, \theta, t)/\pi\sigma$.

The evolution of the chiral imbalance $\mu_5(r, \theta, t)$ stems from eq. (2.2) and, being multiplied by the factor $2\alpha_{em}/\pi\sigma$ where $\sigma$ is the electric conductivity, takes the following form:

$$\frac{\partial \alpha}{\partial \tau} = \frac{2\alpha_{em} \pi}{\mu_e^2 \sigma} \left[ \frac{eB_r}{2\mu_e^2} \frac{dn_e}{dr} + \frac{2\alpha_{em}}{\pi\sigma} \mathbf{B} \cdot (\nabla \times \mathbf{B}) \right] - \alpha \left[ \Gamma_f \left(1 + \frac{\mathbf{B}^2}{B_0^2}\right) + \frac{F}{F} \right]. \quad (2.4)$$

Here the first term in the right hand side stems from the mean spin term $-\nabla \cdot S$ in eq. (2.2). The electric conductivity $\sigma$ increases over time during neutrino epoch (1 yr = $t_9 < t < 10^6$ yr), $\sigma = \sigma(t) = \sigma_0 F(t)$, where $\sigma_0 = 10^7$ MeV is the initial conductivity at the temperature $T = 10^8$ K, while the time depending factor $F(t) = (t/t_9)^{0.28}$ is 1 comes from the NS cooling due to the neutrino emission, $T/T_9 = (t/t_9)^{1/6}$ [19], and the conductivity growth owing to such cooling, $\sigma(T) = \sigma_0 (T/T_9)^{-5/3}$ [20].

Then in eq. (2.4) we use notations: $B_0^2 = \Gamma_f \sigma_0 F \mu_e^2/(2\alpha_{em})^2 = 1.94 \times 10^5$ MeV$^4 \times F(\tau)$, or $B_0 = (2.2 \times 10^{16} \text{G}) \times \sqrt{F(\tau)}$, where $F(\tau) = (1 + 5 \times 10^{11} \tau)^{0.28}$ and $t_9 = 1$ yr. Note that the last term in eq. (2.4), $F/F = \dot{\sigma}/\sigma = 0.28/t$, is essential for a very short time only, $t < \Gamma_f^{-1} = 10^{-11}$ s. In our studies (see, e.g., section 4), we consider significantly greater times. Thus, we neglect such a term. We omit also the small diffusion term $\sim \mathbf{B} \cdot (\nabla \times \mathbf{B})/\sigma$ in comparison with the mean spin term.

Multiplying eq. (2.4) by the diffusion time, $t_{diff} = \sigma_0 R_{NS}^2 = 5 \times 10^{11}$ yr, and introducing the dimensionless time $\tau = t/t_{diff}$, one obtains

$$\frac{\partial \alpha}{\partial \tau} = \frac{2\alpha_{em} \pi R_{NS}^2}{\mu_e^2 F} \left[ \frac{eB_r}{2\mu_e^2} \frac{dn_e}{dr} \right] - \alpha \left[ \Gamma_f \sigma_0 R_{NS}^2 \left(1 + \frac{\mathbf{B}^2}{B_0^2}\right) \right]. \quad (2.5)$$

The huge factor in the last term, $\Gamma_f \sigma_0 R_{NS}^2 = (6.6/4) \times 10^{30}$, is dimensionless. To get this expression one substitutes $\Gamma_f = 6.6 \times 10^{-11}$ MeV and $R_{NS} = 10^6$ cm = $(10^{17}/2)$ MeV$^{-1}$.

The growth for $\alpha$ arises due to the mean spin given by the first term in the right hand side of eq. (2.5). For NS with mass $M = 1.73M_\odot$, the central density $n_e = 2 \times 10^{39}$ cm$^{-3}$, the chemical potential $\mu_e = 250$ MeV, and the crust width $(\Delta R)_{crust} = 470$ m, this term depends on the azimuthal potential $A_\varphi \equiv A$,

$$-\frac{5.6 \times 10^{14}}{F[(\Delta R)_{crust}/m]} \left(\frac{B_r}{\text{MeV}^2}\right) = -\frac{1.2 \times 10^{12}}{\text{MeV}^2 R_{NS}} \left(\frac{\partial A}{\partial \theta} + \cot \theta A\right), \quad (2.6)$$

\textsuperscript{1}For late epochs when neutrinos and antineutrinos are leaving NS by pairs, e.g. through the nucleon-nucleon bremsstrahlung processes, $N + N \rightarrow N + N + \nu + \bar{\nu}$, the lepton number and the electron abundance $Y_e$ remain fixed in NS, $Y_e = 0.04$. Thus $n_e = Y_e n_\nu$, is the stationary electron density where $n_\nu = n_\nu$ is the central (neutron) density.

\textsuperscript{2}These quantities correspond to table 1 in ref. [21].
where getting the term in eq. (2.6) we substituted in numerator of the first term in eq. (2.5): \( R_{\text{NS}}^2 = (10^{34}/4) \text{MeV}^{-2} \), \( m^{-1} = 2 \times 10^{-13} \text{MeV} \), \( \dot{n}_c/\dot{r} = -n_c Y_e/(\Delta R)_{\text{crust}} \) with \( n_c Y_e = 0.64 \times 10^6 \text{MeV}^3 \), \( Y_e = 0.04 \), and in denominator for the same term in eq. (2.5): \( \mu^2 = 39 \times 10^8 \text{MeV}^4 \).

Thus, substituting eq. (2.6) into eq. (2.5) we obtain finally the master evolution equation for the \( \alpha \)-helicity parameter,

\[
\frac{\partial \alpha}{\partial r} = -\frac{1.2 \times 10^{12}}{F} \left( \frac{\partial A}{\partial \theta} + \cot \theta A \right) - \alpha \left[ 1.65 \times 10^{30} \left( 1 + \frac{B^2}{B_0^2} \right) \right].
\]  

(2.7)

In eq. (2.7), we use the AMHD helicity parameter \( \alpha = 2 \alpha_{\text{em}} \mu_3(r, \theta, t)/\pi \sigma(t) \) instead of \( \alpha_{\text{MHD}} \sim (\tau_{\text{corr}}/3) \langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle \) in standard MHD.\(^3\) Such a new equation should be completed by the AMHD evolution equations for the azimuthal potential \( A = A(r, \theta, t) \) and the toroidal magnetic field \( B = B(r, \theta, t) \).

### 3 AMHD dynamo model for rigid NS rotation

We consider the rigid NS rotation \( \Omega = \text{const} \), for which the differential rotation is absent, \( \partial_\theta \Omega = \partial_t \Omega = 0 \). Hence the dynamo term \( \nabla \times (\mathbf{v} \times \mathbf{B}) \) vanishes in the Faraday equation. Therefore, we should not involve the Navier-Stokes equation neglecting also a small vorticity contribution \( \sim \omega = \nabla \times \mathbf{v} \).

The complete system of the dynamic equations for 3-D dynamo model with an axisymmetric magnetic field includes the following three AMHD equations:

1. For \( \alpha(r, \theta, t) \)-parameter originated by the CME and given by eq. (2.7).

2. For the azimuthal potential \( A_\varphi = A(r, \theta, t) \).

3. For the toroidal magnetic field \( B_\varphi = B(r, \theta, t) \).

From standard equation for the axisymmetric azimuthal potential \( A_\varphi = A(r, \theta, t) \),

\[
\frac{\partial A}{\partial t} = \frac{1}{\sigma} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A) \right) + \alpha B,
\]

multiplied consistently by the diffusion time \( t_{\text{diff}} = \sigma R_{\text{NS}}^2 \), and then by \( F^{-1} \), one obtains

\[
\frac{\partial A}{\partial r} = F^{-1} \left[ -\mu^2 A + \frac{\partial^2 A}{\partial \theta^2} + \cot \theta \frac{\partial A}{\partial \theta} - \frac{A}{\sin^2 \theta} \right] + \frac{10^{24}}{2} \alpha B,
\]

(3.2)

where the dimensionless toroidal magnetic field is normalized on \( \text{MeV}^2 = 5 \times 10^{13} \text{G} \), i.e. \( B = B_\varphi/\text{MeV}^2 \). The potential \( A_\varphi \) is normalized on \( R_{\text{NS}}^2 \text{MeV}^2 \), giving one the dimensionless \( A = A_\varphi/R_{\text{NS}}^2 \text{MeV}^2 \). The factor \( 10^{24}/2 \) stems from the product \( \sigma_0 R_{\text{NS}} = 10^{24}/2 \).

The induction (Faraday) equation in a thin layer under NS crust,\(^4\) \( r \lesssim R_{\text{NS}} \),

\[
\frac{\partial B}{\partial t} = \frac{1}{\sigma} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial B}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta B) \right] - \frac{1}{\sigma} \frac{\partial}{\partial r} \left( \frac{\alpha}{\sigma} \frac{\partial}{\partial r} (\sigma A) \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{\alpha}{\sigma r \sin \theta} \frac{\partial}{\partial \theta} (\sigma A) \right],
\]

(3.3)

\(^3\)At late epochs of NS evolution, we neglect any turbulent (random) velocity component \( \mathbf{u} \), for which \( \langle \mathbf{u} \rangle = 0 \), in the total fluid velocity \( \mathbf{v} = \mathbf{V} + \mathbf{u} \). However we retain the rotation velocity \( \mathbf{v} = \mathbf{V} \equiv \Omega \times \mathbf{r} \) of NS as a whole.

\(^4\)We substitute \( r = R_{\text{NS}} \) for \( \alpha(R_{\text{NS}}, \theta, t) \). Thus \( \partial_r \alpha = 0 \) while \( \partial_\theta \alpha \neq 0 \). The other radial derivatives are parametrized in a thin layer as \( R_{\text{NS}} \times \partial_r (A, B) \rightarrow i \mu(A, B) \), following Parker’s suggestion in the standard MHD dynamo [22].
being multiplied by $t_{\text{diff}} = \sigma R_{\text{NS}}^2$, and then by $F^{-1}$, takes the following form,

$$
\frac{\partial B}{d\tau} = F^{-1} \left[ -\mu^2 B + \frac{\partial^2 B}{\partial \theta^2} + \cot \theta \frac{\partial B}{\partial \theta} - \frac{B}{\sin^2 \theta} \right] - \left( \frac{10^{24}}{2} \right)
\times \left[ \frac{\partial \alpha}{\partial \theta} \left( \frac{\partial A}{\partial \theta} + \cot \theta A \right) + \alpha \left( -\mu^2 A + \frac{\partial^2 A}{\partial \theta^2} + \cot \theta \frac{\partial A}{\partial \theta} - \frac{A}{\sin^2 \theta} \right) \right],
$$

(3.4)

where we account for the coordinates dependence of $\alpha$.

### 3.1 Low Fourier mode approximation

In a thin layer under the crust, $r \lesssim R_{\text{NS}}$, we use the low mode approximation for all functions above [14]:

$$
\alpha(t, \theta) = \alpha_1(t) \sin 2\theta + \alpha_2(t) \sin 4\theta + \ldots,
$$

$$
A(t, \theta) = A_1(t) \sin \theta + A_2(t) \sin 3\theta + \ldots,
$$

$$
B(t, \theta) = B_1(t) \sin 2\theta + B_2(t) \sin 4\theta + \ldots
$$

(3.5)

Using the expansions in eq. (3.5) and integrating over the colatitudes $\theta$, one gets from eq. (2.7) that

$$
\dot{\alpha}_1 = - \left( \frac{19.2 \times 10^{12}}{15\pi F} \right) (5a_1 - a_2) - 1.65 \times 10^{30} \left\{ \alpha_1 + \alpha_2 \frac{2}{\pi} \int_0^\pi d\theta \sin^2 \theta \left( \frac{B^2}{B_0^2} \right) \right\}
$$

$$
+ \alpha_2 \frac{2}{\pi} \int_0^\pi d\theta \sin 2\theta \sin 4\theta \left( \frac{B^2}{B_0^2} \right) \right\},
$$

$$
\dot{\alpha}_2 = - \left( \frac{38.4 \times 10^{12}}{105\pi F} \right) (7a_1 + 37a_2) - 1.65 \times 10^{30} \left\{ \alpha_2 + \alpha_2 \frac{2}{\pi} \int_0^\pi d\theta \sin^2 4\theta \left( \frac{B^2}{B_0^2} \right) \right\}
$$

$$
+ \alpha_1 \frac{2}{\pi} \int_0^\pi d\theta \sin 2\theta \sin 4\theta \left( \frac{B^2}{B_0^2} \right) \right\},
$$

(3.6)

where the ratio $B^2/B_0^2$ for $B^2 = B^2_\theta + B^2_r + B^2_\varphi$ is given by

$$
\frac{B^2}{B_0^2} = \left( \frac{1.94 \times 10^6 F(\tau)}{1.94 \times 10^6} \right) \left\{ \beta_1 \sin^2 2\theta + 2\beta_1 b_2 \sin 2\theta \sin 4\theta + \beta_2 \sin^2 4\theta
$$

$$
+ \mu^2 \left[ a_1^2 \sin^2 \theta + 2a_1 a_2 \sin \theta \sin 3\theta + a_2^2 \sin^2 3\theta \right]
$$

$$
+ 4(a_1 + a_2)^2 \cos^2 \theta + 16a_2^2 \cos^2 3\theta + 16(a_1 + a_2) a_2 \cos \theta \cos 3\theta \right\}.
$$

(3.7)

Here in the last line we substituted $(\partial_\theta A + \cot \theta A)^2$ for $B^2_\theta$ using corresponding linear function from eq. (3.5), $\partial_\theta A + \cot \theta A = 2(a_1 + a_2) \cos \theta + 4a_2 \cos 3\theta$, and used the expression for $|B_\theta|^2$ in the second line. For this purpose we use the poloidal components $B_{r, \theta}$ for the axisymmetric field given by the azimuthal potential $A \equiv A_\varphi$,

$$
B_\theta = - \frac{1}{r} \frac{\partial}{\partial r} (r A), \quad B_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A),
$$

(3.8)

where $A = A(r, \theta, t)$.

The evolution equations for all six functions, $a_{1,2}(t)$, $b_{1,2}(t)$, and for $\alpha_{1,2}(t)$, have cumbersome form because of the presence of non-linear terms. Therefore, they are given explicitly in appendix A. Thus, there are six self-consistent ordinary differential equations: for azimuthal potentials $\alpha_{1,2}$ given by eq. (A.1), the toroidal field amplitudes $b_{1,2}$, see eq. (A.2), and for the helicity parameters $\alpha_{1,2}$ given by eq. (A.3).
3.1.1 Initial conditions

The initial condition for the AMHD magnetic helicity parameter \( \alpha(t) = 2\alpha_{em}(t)/\pi \sigma \), which evolves as given in eq. (2.4), is chosen relying on a small value of the chiral imbalance, \( \mu_5 \ll \mu_e \simeq 250 \text{ MeV} \). We take the initial \( \mu_5(0) \simeq 1\text{ MeV} \) at the level of the neutron-proton mass difference. Thus \( \alpha_1(0) = 10^{-3} \), accounting for the initial conductivity \( \sigma_0 = 10^7 \text{ MeV} \) at the NS temperature \( T = 10^9 \text{ K} \). We choose also \( \alpha_2(0) = \alpha_1(0)/10 \) meaning for that a decrease of helicity amplitudes in the low mode approximation.

To get the initial conditions for eqs. (A.1) and (A.2), we equate the initial normalized components

\[
\frac{B_0(R_{NS}, \theta, t = 0)}{\text{MeV}^2} = -\left[ a_1(0) \sin \theta + a_2(0) \sin 3\theta \right],
\]

\[
\frac{B_r(R_{NS}, \theta, t = 0)}{\text{MeV}^2} = 2\cos \theta \left[ a_1(0) + a_2(0)(4 \cos 2\theta - 1) \right],
\]

that result from eq. (3.8) in the low mode approximation in eq. (3.5), \( B_0(t = 0) / \text{MeV}^2 = B_r(t = 0) / \text{MeV}^2 = 0.2 \). Then one can find at the same force line of the poloidal field \( B_p = \sqrt{B_0^2 + B_r^2} \), while at different latitudes when substituting corresponding \( \theta = 0 \) for \( B_r(t = 0) \) where \( B_0(t) = 0 \) and \( \theta = \pi/2 \) for \( B_0(t = 0) \) where \( B_r(t) = 0 \), the following algebraic system:

\[
a_2(0) - a_1(0) = 0.2, \quad a_1 + 3a_2(0) = 0.1. \tag{3.10}
\]

The initial amplitudes, resulting from eq. (3.10), have opposite signs, \( a_1(0) = -0.125, a_2(0) = +0.075 \). We choose the same initial condition for the azimuthal components at \( r \simeq R_{NS} \), \( b_{1,2}(0) = 0.2 \), corresponding, at the beginning, to the toroidal field \( B(R_{NS}, \theta, t = 0) = (\sin 2\theta + \sin 4\theta) \times 10^{13} \text{ G} \) in the low mode approximation in eq. (3.5).

3.2 Continuous CME helicity parameter \( \alpha(t) \) in an old neutron star

In this section, we present the results of the numerical solution of eqs. (A.1), (A.2), and (A.3) accounting for the initial conditions in section 3.1.1.

The changing helicity parameter \( \alpha(t) \), shown in figures 1 and 2, is the most important issue in the present work. In figures 1(a) and 1(b), one can see how the initial helicity parameters \( \alpha_1(0) = 10^{-9}, \alpha_2(0) = 10^{-10} \) decrease down to the values \( \sim 10^{-18} \) shown then in figure 2 as an initial value for \( \alpha_{1,2}(t) \). Such a decrease happens during the very short time \( \tau \sim 10^{-29} \) corresponding to \( t = \tau \times t_{\text{diff}} = 1.5 \times 10^{-11} \text{ s} \), exactly as it is expected due to the spin-flip, \( \Gamma_f^{-1} = 10^{-11} \text{ s} \). Then, the first terms in eq. (3.6), generated by the mean spin in the NS electron plasma, \( \sim \nabla \cdot \mathbf{S} \), that are proportional to potential components \( \sim a_{1,2} \), become greater than the second ones, \( \sim a_{1,2} \), which depend on the spin-flip rate \( \Gamma_f^{-1} \), resulting in a slow growth of \( \alpha(t) \) from a time \( \tau' \sim 2 \times 10^{-29} \) to \( \tau \lesssim 10^{-7} \div 10^{-6} \), or \( t \sim 10^4 \div 10^5 \text{ yr} \), see figure 2.

\footnote{One can easily estimate in the right hand side in eq. (3.6) that for small values \( |a_{1,2}| \sim 10^{-18} \) the both terms: the first ones given by the mean spin and the second ones given by the spin-flip occur of the same order of magnitude.}

Note that, at the beginning, both helicity components are positive, while \( \alpha_1(0) > 0 \) remains positive forever, \( \alpha_1(t) > 0 \), and \( \alpha_2(0) = \alpha_1(0)/10 > 0 \) changes sign somewhere at \( \tau \sim 10^{-29} \) (see figure 1(d)) thereby, in figure 2, we show \( |\alpha_2(t)| \). The change of sign for \( \alpha_2 \) happens owing to the fast change of the sign for the initial factor \( 7a_1(0) + 37a_2(0) = +1.9 > 0 \).
in eq. (3.6) on the opposite one, \((7a_1(t) + 37a_2(t)) < 0\). It takes place when the mean spin contribution becomes a leading term for the derivative \(\dot{\alpha}_2\) after the preceding spin-flip for the growing potentials \(a_{1,2}(t)\), where \(|a_1| > a_2\), see figure 3. Note that, at small times \(\tau \sim 10^{-20}\), non-linear terms \(\sim B^2/B_0^2\) do not influence the \(\alpha\)-helicity evolution. At later times, at \(\tau \sim 10^{-6}\), an interplay between them for the attached factors \(a_{1,2}\) in eq. (3.6) plays a crucial role for the appearance of the spikes shown in figure 2 (see section 5).

4 The growth of magnetic fields due to CME with continuous \(\alpha\)

In figure 3, we show how the magnetic field is amplified in an old NS exactly due to spikes in behavior of continuous \(\alpha(t)\) at times \(\tau_{\text{crit}} \sim 1.75 \times 10^{-6}\), or near \(t_{\text{crit}} \sim 8.75 \times 10^5\) yr; cf. figures 2 and 3. The dimensionless fields \(B/\text{MeV}^2\) and the potentials \(A/\text{MeV}^2R_{\text{NS}}\) are plotted there versus the dimensionless time \(\tau = t/t_{\text{diff}}\), where \(t_{\text{diff}} = 5 \times 10^{11}\) yr.

The poloidal components \(B_{r,\theta} \sim a_{1,2}\) and the toroidal ones \(B \sim b_{1,2}\), are growing below the crust in such NS on \(\sim 6\) orders of magnitude as shown in figure 3, and can reach a danger value \(B \gtrsim 10^{18}\) G where the dynamical instability of NS can appear. The results depend slightly on the free parameter \(\mu\) used for radial derivatives in a thin layer, \(d/dr \sim i\mu\). Thus
Figure 2. The behavior of the coefficients $\alpha_{1,2}$ at $\tau' < \tau < 4 \times 10^{-6}$, where $\tau' = 2 \times 10^{-29}$. The values of $\alpha_{1,2}$ at $\tau = \tau'$ are $\alpha_1(\tau') = 1.73 \times 10^{-18}$ and $\alpha_2(\tau') = -1.34 \times 10^{-18}$, as results from figure 1.

we present them in figures 2 and 3 for $\mu = 0$ only (to be compared with varying $\mu$ in plots in ref. [12]).

The evolution of magnetic fields in the mean field dynamo model applied here for old NSs depends crucially on the initial conditions. Indeed, we take all the initial 3D components of the axisymmetric field as strong as $B_{r,\theta,\phi}(t = 0) \simeq 10^{13}$ G, or $B_{r,\theta,\phi}(t = 0) / \text{MeV}^2 = 0.2$ in dimensionless units (see section 3.1.1 and figure 3). If we assume an one order less initial amplitude, $B(t = 0) \sim 10^{12}$ G, one obtains rather big critical time that exceeds the Universe age, $t_{\text{crit}} > t_{\text{Universe}} = 14 \times 10^9 \text{yr}$. This means that in our AMHD scenario the initial moderately strong field $B(t = 0) = 10^{13}$ G should arise somehow through a different mechanism at earlier epochs.

5 Discussion

In the present work we recover the CME as a possible source for the generation of strong magnetic fields in NSs. Previously the common opinion on this issue was negative because of the rather fast disappearance of the chiral imbalance $\mu_5$ that drives the CME in a chiral plasma, $\mu_5 = (\mu_R - \mu_L)/2 \to 0$, during a very short time $\Gamma_f^{-1} \sim 10^{-11}$ s.

For simplicity we consider here the case of a non-superfluid NS with its rigid rotation that allowed us to avoid the involvement of the Navier-Stokes equation for the fluid velocity.
Figure 3. The behavior of the amplitudes: for azimuthal potential \(a_{1,2}(\tau)\) and for toroidal magnetic field \(b_{1,2}(\tau)\) at \(\tau' < \tau < 4 \times 10^{-6}\), where \(\tau' = 2 \times 10^{-29}\).

\[\begin{align*}
|a_1| & \quad a_2 \\
b_1 & \quad b_2
\end{align*}\]

\(\tau\) \(\times 10^{-6}\)

\(0\) \(1\) \(2\) \(3\) \(4\)

\(10^{-2}\) \(10^0\) \(10^2\) \(10^4\) \(10^6\)

\(v\) when applying the total set of MHD equations. Moreover, since we neglect at late epochs any turbulence within NS matter, or a random small-scale velocity component is absent, \(u = 0\), such velocity field \(v(x, t) = u(x, t) + V(r, \theta)\) coincides with the NS rotation velocity as a whole, \(v = V(r, \theta) = \Omega \times r\), where \(\Omega = \text{const}\).

For an axisymmetric 3D magnetic field under simplifying assumptions above the total set of evolution equations includes the three ones: eq. (3.2) for the azimuthal potential \(A(r, \theta, t) = A_{\phi}\), then eq. (3.4) for the toroidal magnetic field \(B(r, \theta, t) = B_{\phi}\), and eq. (2.7) for the magnetic helicity parameter \(\alpha(r, \theta, t) = \alpha \sim \mu_5\) that is the single source of the magnetic field instability given by the term \(\nabla \times (\alpha \mathbf{B})\) in the induction (Faraday) equation in the case of AMHD. Contrary to ref. [12], where the saturation regime with the stationary helicity parameter \(\partial_t \alpha = 0\) was a priori assumed, in the present work, we solve the set of AMHD equations in a self-consistent way and find the continuous CME in NS, \(j_{\text{anom}} \sim \mu_5 \mathbf{B} \neq 0\) due to \(\alpha(t) \sim \mu_5(t) \neq 0\), existing in NS for long times \(t \gtrsim 10^6\) yr; cf. figure 2.

Let us stress again that the continuous CME is originated by the mean spin in eq. (2.3) given by the electron plasma polarization in the growing magnetic field, i.e., by electrons populating the main non-degenerate Landau level \(n=0\). It means that the stronger magnetic field is the greater that polarization is.

The most interesting effect in the \(\alpha(t)\) behavior is the jump (spike) at the time \(\tau_{\text{crit}} \simeq 1.75 \times 10^{-6}\), equaled to \(t_{\text{crit}} \simeq 8.75 \times 10^5\) yr, which coincides with the moment for a sharp growth of the magnetic field in figure 3 on \(\sim 6\) orders of magnitude. The jump for \(\alpha\) is
caused by a simultaneous growth of the non-linear terms $B^2/B_0^2$ in eq. (3.6) and potentials $a_{1,2}$ in the first terms there originated by the mean spin in eq. (2.3): positive for $a_2$ and negative for $a_1$. On the one hand, these non-linear terms $\sim B^2/B_0^2$ compensate each other due to different signs of $a_1 > 0$ and $a_2 < 0$ ahead corresponding integrals in eq. (3.6) ($a_2$ becomes negative after $\tau'$, see figure 1(d)). On the other hand, the ratio $|a_2|/a_1$ is less than unity at times $\tau \lesssim 0.1\tau_{\text{crit}}, |a_2|/a_1 < 1$, and becomes bigger at $\tau > 0.1\tau_{\text{crit}}, |a_2|/a_1 > 1$, see figure 2. This fact inevitably leads to the sharp change of signs for temporal derivatives $\dot{a}_{1,2}$ and causes the appearance of spikes for $a$.

The general opinion on the appearance of superstrong magnetic fields in NS, $B \gtrsim 10^{18}$ G, is negative since a dynamical instability of NS could arise. This could happen since the magnetic energy can exceed the gravitational binding energy for such fields.

Nevertheless, some interesting problems concerning superstrong magnetic fields in NS could concern also the dynamo mechanism suggested in the present work. First, for fields exceeding the level $B \gtrsim 2 \times 10^{18}$ G the neutron decay $n \to p + e^- + \nu_e$ is forbidden, while for a bigger strength $B \gtrsim 5 \times 10^{18}$ G the proton becomes heavier than neutron resulting in the proton decay, $p \to n + e^+ + \nu_e$ [23, 24]. Secondly, such strong magnetic fields could change the width of the NS crust $(\Delta R)_{\text{crust}}$ due to Lorentz force influence there [25] that, surely, is important for our simplest electron density profile within the crust (see the comment after eq. (2.6)). Both these effects could influence the amplitude of the mean spin contribution in our master eq. (2.7). First, an additional decrease of the electron abundance $Y_e \to 0$ instead of $Y_e = 0.04 = \text{const assumed above}$, proceeds owing to vanishing charge fermion abundances in electroneutral plasma, $Y_e = Y_\gamma$, that happens due to the proton decay and following fast positron annihilation $e^+e^- \to 2\gamma$ within NS in the case [23]. Oppositely to such issue, the same amplitude of the mean spin source, maintaining the $\alpha(t)$ evolution in eq. (2.7), increases when the crust width $(\Delta R)_{\text{crust}}$ decreases in strong magnetic fields, as found in ref. [25].

There remain other interesting problems for AMHD applications to NS with strong magnetic fields. In particular, this concerns the magnetic helicity evolution near the surface of the NS crust leading to an intertwining of the thin magnetic tubes and then to the reconnection of magnetic field lines with the following X-ray bursts from magnetars. First, in outer layers within the crust, all degenerate non-relativistic electrons populate the main Landau level only ($n = 0$). It happens because, in strong magnetic fields, the condition implying that case, $2eB \gg m_e^2 \gg p_{F,e}^2$, is fulfilled at small matter densities $\rho \lesssim 10^8$ g·cm$^{-3}$. In addition, for these densities it is still unclear how the mean pseudoscalar term $2m_n(\bar{\psi}\gamma_\gamma\psi)$ resulting from the axial Ward anomaly in eq. (2.1) and omitted above for ultrarelativistic (chiral) plasma within NS core, $p_{F,e} \gg m_e$, could rearrange for non-relativistic electrons within crust both the helicity $\alpha(t)$ evolution and the evolution of the magnetic helicity density $h(t) = V^{-1} \int d^3x (A \cdot B)$ studied in ref. [26] for the chiral plasma only.\footnote{For early times $\tau \sim 10^{-29}$ the terms $\sim B^2/B_0^2$ were negligible in eq. (3.6), see section 3.2.}

To resume we solved a problem why the CME survives for a long time and feeds the amplification of strong magnetic fields in old NSs. For this purpose we built the new self-consistent mean field dynamo model in AMHD, which is similar to $\alpha^2$-dynamo for rigid

\[ F = \frac{eBp_{F,e}}{2\pi^2}, \]

\[ \tau_{\text{crit}} = 3.6 \times 10^{12} \text{s}, \]

\[ n_{\text{c}} = \frac{p_{F,e}^3}{3\pi^2}, \]

\[ n_{\text{c}} \gg 1 \] is appropriate.\footnote{The mean pseudoscalar $2m_n(\bar{\psi}\gamma_\gamma\psi) = -V^{-1} \int d^3x \nabla \cdot S_0(x, t)$ was calculated in ref. [26] using the quasiclassical approach at large Landau numbers, $n \gg 1$. Such an assumption is valid for the NS chiral plasma obeying the opposite condition in strong magnetic fields, $p_{F,e}^3 \gg 2eB \gg m_e^2$. Indeed, under such a condition the electron density $n_e = (p_{F,e}^3/3\pi^2)[1+3eB/2p_{F,e}]$ includes the contribution of the main Landau level $n = 0$, $n_{\alpha\beta} = eBp_{F,e}/2\pi^2$, as a small correction to the quasi-classical density, $n_e = p_{F,e}^3/3\pi^2$ [27], hence the approximation $n \gg 1$ is appropriate.}
rotation in standard MHD. The $\alpha(t)$-parameter in our model is however continuously changing in time.\footnote{We refer to the dynamo under discussion as $\alpha^2$-one because the instantaneous magnetic field growth rate is proportional to $\alpha^2$.}

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### A Complete set of AMHD evolution equations in the mean field dynamo for rigid NS rotation

In this appendix, we present the full set of differential equations for the expansion coefficients in eq. (3.5).

First, we substitute the low mode series eq. (3.5) in eq. (3.2). Then, we multiply it by $\sin \theta$ and $\sin 3\theta$, correspondingly, and integrate over the colatitude angle, $(2/\pi) \int_0^\pi d\theta(\ldots)$. Finally, we obtain the system of the ordinary non-linear differential equations for the azimuthal potential components $a_{1,2}$,

\[
\dot{a}_1 = -\frac{1}{F} [(\mu^2 + 2)a_1 + 2a_2] + \frac{16 \times 10^{24}}{315\pi} \left( 21\alpha_1 b_1 - 6(\alpha_1 b_2 + \alpha_2 b_1) + 20\alpha_2 b_2 \right),
\]

\[
\dot{a}_2 = -\frac{1}{F} (\mu^2 + 12) a_2 + \frac{16 \times 10^{24}}{\pi} \left( \frac{4}{165} \alpha_2 b_2 + \frac{2}{45} (\alpha_1 b_2 + \alpha_2 b_1) + \frac{\alpha_1 b_1}{21} \right). \quad (A.1)
\]

Then, we again use the low mode approximation in the Faraday eq. (3.4) and multiply it by $\sin 2\theta$ and $\sin 4\theta$, correspondingly. After the integration it over the colatitude angle $(2/\pi) \int_0^\pi d\theta(\ldots)$, we obtain the system of the differential equations for the toroidal magnetic field amplitudes $b_{1,2}$,

\[
\dot{b}_1 = -\frac{1}{F} [ (\mu^2 + 6)b_1 + 4b_2 ]
\]

\[
+ \frac{16 \times 10^{24}}{315\pi} \left[ \mu^2 \left( 21a_1 \alpha_1 - 6a_1 \alpha_2 + 15a_2 \alpha_1 + 14a_2 \alpha_2 \right) + 21a_1 \alpha_1 + 66a_1 \alpha_2 + 111a_2 \alpha_1 + 38a_2 \alpha_2 \right],
\]

\[
\dot{b}_2 = -\frac{1}{F} (\mu^2 + 20)b_2 - \frac{32 \times 10^{24}}{3465\pi} \left[ \mu^2 \left( 33a_1 \alpha_1 - 110a_1 \alpha_2 - 77a_2 \alpha_1 - 42a_2 \alpha_2 \right) + 429a_1 \alpha_1 - 110a_1 \alpha_2 - 649a_2 \alpha_1 - 362a_2 \alpha_2 \right]. \quad (A.2)
\]

Finally, the full set of AMHD evolution equations is completed by the two evolution equations for $\alpha_{1,2}(t)$ helicity amplitudes resulting from eq. (3.6),

\[
\dot{\alpha}_1 = -\frac{19.2 \times 10^{12}}{15\pi F} (5a_1 - a_2) - 1.65 \times 10^{30} \alpha_1
\]

\[
- \frac{8.58 \times 10^{24}}{4F} \left( \mu^2 \left[ 2a_1^2 \alpha_1 - a_1^2 \alpha_2 + 2a_1 a_2 \alpha_1 + 2a_1 a_2 \alpha_2 
\right. 
\left. + 2a_2^2 \alpha_1 + a_2^2 \alpha_2 \right] + 8a_1^2 \alpha_1 + 4a_1^2 \alpha_2 + 24a_1 a_2 \alpha_2 + 24a_2^2 \alpha_1 \right),
\]
\begin{equation}
\dot{\alpha}_2 = - \frac{3.84 \times 10^{12}}{105\pi F} (7a_1 + 37a_2) - 1.65 \times 10^{30} \alpha_2 \\
+ \frac{8.58 \times 10^{24}}{4F} \left( \mu^2 \left[ a_1^2 \alpha_1 - 2a_1^2 \alpha_2 - 2a_1 a_2 \alpha_1 - a_2^2 \alpha_1 - 2a_2^2 \alpha_2 \right] \\
- 4a_1^2 \alpha_1 - 8a_1^2 \alpha_2 - 24a_1 a_2 \alpha_1 - 16a_1 a_2 \alpha_2 - 4a_2^2 \alpha_1 - 40a_2^2 \alpha_2 \\
- 2\alpha_2 b_1^2 - 3\alpha_2 b_2^2 - 4\alpha_1 b_1 b_2 \right),
\end{equation}

where we use the factor $B^2/B_0^2$ given in eq. (3.7). Note that for old NSs at the age $t \gtrsim 10^6$ yr, the NS cooling proceeds via photon emission, which prevails over cooling owing to the neutrino emission. At such epochs, using figure 3, we can take that $F = 1$ in our calculations. Thus, electric conductivity is constant, $\sigma = \text{const} = \sigma_0$.

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