THE SCHEME OF MONOGENIC GENERATORS AND ITS TWISTS

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Abstract. Given an extension of algebras $B/A$, when is $B$ generated by a single element $\theta \in B$ over $A$? We show there is a scheme $M_{B/A}$ parameterizing the choice of a generator $\theta \in B$, a “moduli space” of generators. This scheme relates naturally to Hilbert schemes and configuration spaces. We give explicit equations and ample examples.

A choice of a generator $\theta$ is a point of the scheme $M_{B/A}$. This inspires a local-to-global study of monogeneity, piecing together monogenerators over points, completions, open sets, and so on. Local generators may not come from global ones, but they often glue to twisted monogenerators that we define. We show a number ring has class number one if and only if each twisted monogenerator is in fact a global generator $\theta$. The moduli spaces of various twisted monogenerators are either a Proj or stack quotient of $M_{B/A}$ by natural symmetries. The various moduli spaces defined can be used to apply cohomological tools and other geometric methods for finding rational points to the classical problem of monogenic algebra extensions.

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1. Introduction

The theorem of the primitive element states that given a finite separable field extension \( L/K \), there is an element \( \theta \) of \( L \) such that \( L = K(\theta) \). This holds for any extension of number fields \( L/K \). By contrast, the extension of their rings of integers \( \mathbb{Z}_L/\mathbb{Z}_K \) may require up to \( \lceil \log_2([L:K]) \rceil \) elements of \( \mathbb{Z}_L \) to generate \( \mathbb{Z}_L \) as a \( \mathbb{Z}_K \)-algebra [Ple74].

**Question 1.1.** Which extensions \( \mathbb{Z}_L/\mathbb{Z}_K \) are generated by a single element over \( \mathbb{Z}_K \)? More generally, which finite locally free algebras \( B/A \) are generated by a single element over \( A \)? How many elements does it take to generate \( B \) over \( A \) otherwise?

**Definition 1.2.** A finite locally free \( A \)-algebra \( B \) is monogenic\(^1\) if there is an element \( \theta \in B \) such that \( B = A[\theta] \). The element \( \theta \) is called a monogenic generator or monogenerator of \( B \) over \( A \).

If there are elements \( \theta_1, \ldots, \theta_k \in B \) such that \( B = A[\theta_1, \ldots, \theta_k] \), then \( B/A \) is \( k \)-genic and \((\theta_1, \ldots, \theta_k)\) is a generating \( k \)-tuple.

Monogeneity of an algebra can be restated geometrically:

**Remark 1.3.** Let \( A \subseteq B \) be an inclusion of rings. A choice of element \( \theta \in B \) is equivalent to a commutative triangle

\[
\begin{array}{ccc}
\text{Spec } B & \xrightarrow{s_{\theta}} & \mathbb{A}^1_A, \\
\downarrow & & \downarrow s_{\theta} \\
\text{Spec } A & \end{array}
\]

where \( s_{\theta} \) is the map induced by the \( A \)-algebra homomorphism \( A[t] \to B \) taking \( t \) to \( \theta \).

The element \( \theta \) is a monogenerator if and only if the map \( s_{\theta} \) is a closed immersion.

Likewise, a \( k \)-tuple \( \vec{\theta} = (\theta_1, \ldots, \theta_k) \in B^k \) determines a corresponding map \( s_{\vec{\theta}} : \text{Spec } B \to \mathbb{A}^k_A \) induced by the \( A \)-algebra homomorphism \( A[t_1, \ldots, t_k] \to B \) taking \( t_i \mapsto \theta_i \). The tuple \((\theta_1, \ldots, \theta_k)\) generates \( B \) over \( A \) if and only if \( s_{\vec{\theta}} \) is a closed immersion.

\(^1\)The literature often uses the phrase “\( L \) is monogenic over \( K \)” to mean \( \mathbb{Z}_L/\mathbb{Z}_K \) is monogenic as above. We prefer “\( \mathbb{Z}_L \) is monogenic over \( \mathbb{Z}_K \)” in order to treat fields and more exotic rings uniformly. If \( B \) is monogenic of degree \( n \) over \( A \) with monogenerator \( \theta \), the elements \( \{1, \theta, \theta^2, \ldots, \theta^{n-1}\} \) are elsewhere referred to as a “power \( A \)-integral basis.”
We ask if there is a scheme that represents such commutative triangles as in moduli theory [Fan+05, Chapter 1]. For extensions of number rings or any other algebras fitting into Situation 2.1 below, there is a representing scheme $M_{B/A}$ over Spec $A$ called the scheme of monogenerators. There is an analogous scheme of generating $k$-tuples or scheme of $k$-generators denoted by $M_{k,B/A}$. When the extension $B/A$ is implied, we simply write $M$ or $M_k$ as appropriate.

**Theorem 1.4.** Let $B/A$ be an extension of rings such that Spec $B \to$ Spec $A$ is in Situation 2.1. There is an affine $A$-scheme $M_{B/A}$ and finite type quasiaffine $A$-schemes $M_{k,B/A}$, for $k \geq 1$, with natural bijections

\[
\text{Hom}_{(Sch/A)}(\text{Spec } A, M_{B/A}) \simeq \{ \theta \in B \mid \theta \text{ is a monogenerator for } B \text{ over } A \}
\]

and

\[
\text{Hom}_{(Sch/A)}(\text{Spec } A, M_{k,B/A}) \simeq \{ \vec{\theta} \in B^k \mid \vec{\theta} \text{ is a generating } k\text{-tuple for } B \text{ over } A \}.
\]

We compute explicit affine charts for $M_{k,B/A}$ and deduce affineness for $M_{B/A}$.

The problem of finding generating $k$-tuples of $\mathbb{Z}_L/\mathbb{Z}_K$ is thereby identified with that of finding $\mathbb{Z}_K$-points of the schemes $M_{k,\mathbb{Z}_L/\mathbb{Z}_K}$. The functors $\text{Hom}(-, M_k)$ automatically form sheaves in the fpqc, fppf, étale, and Zariski topologies, permitting monogeneity to be studied locally.

These spaces are already interesting for trivial covers of a point:

**Example 1.5.** Let $A = \mathbb{C}$ and $B = \mathbb{C}^n$. The complex points of the monogeneity space are naturally homeomorphic to the configuration space of $n$ distinct points in $\mathbb{C}$:

\[
M_{1,B/A}(\mathbb{C}) \simeq \text{Conf}_n(\mathbb{C}) := \{ (x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ for } i \neq j \}.
\]

Monogeneity generalizes configuration spaces by conceiving of $B/A$ as “families of points” to be configured in $\mathbb{A}^1$. We spell out the relation to both $\text{Conf}_n$ and the Hilbert scheme. When $B/A$ is étale (unramified), this connection relates $M$ to log geometry and Example 5.3 shows the étale fundamental group of $M_1$ yields a well-known action of the Grothendieck-Teichmüller group $\hat{GT}$ on the profinite completion of the braid group. Similarly, Example 5.6 gives an isomorphism of a quotient of $M$ with the moduli space of $n$-pointed genus zero curves $M_{0,n}$.

1.1. **Twists.** The space of monogenerators admits several natural group actions, such as affine transformations $\text{Aff}^1$ or scaling by units $\mathbb{G}_m$. Two monogenerators $\theta_1, \theta_2 \in \mathbb{Z}_L$ are “affine equivalent” if there are $a \in \mathbb{Z}_K^*, b \in \mathbb{Z}_K$ such that

\[
a \theta_1 + b = \theta_2.
\]

Section 6 studies the twists of $M_{B/A}$ by this action using the stack quotient. The points of one of the twisted spaces are then commutative triangles like Diagram (1), but with $\mathbb{A}^1_\mathbb{A}$ replaced by a line bundle $L$. The line bundle $L \to \text{Spec } A$ has a class $\alpha \in \text{Pic}(A)$.

We say $B/A$ is twisted monogenic of class $\alpha$ if such a triangle exists. Twisted monogeneity of an extension of algebras can also be characterized without the language of schemes, see Definition 6.1. Likewise, we construct a space of twisted monogenerators that is itself a twisted form of the space of ordinary monogenerators $M_{B/A}$. For a precise study, see Section 6. To see a practical example, consult Example 6.22. Twisted monogenic extensions are a strong invariant of a number field:
Theorem 1.6 (=Theorem 6.20). A number field $K$ has class number one if and only if all twisted monogenic extensions of $\mathbb{Z}_K$ are in fact monogenic.

Warning 1.7. The reader with a background in classical monogeneity or number theory is advised to turn directly to Sections 3 and 7 for concrete worked examples. Appendix A is similarly directed at those unfamiliar with torsors and stack quotients. It contains a minimum for the purposes of Sections 5 and 6. More thorough expository accounts of stacks include [Fan+05].

1.2. Summary of the paper. Section 2 defines our main object of study $M_{S'/S}$ for a finite flat map $S' \to S$. This scheme parametrizes choices of monogenerators for the algebra extension $\mathcal{O}_{S'}/\mathcal{O}_S$. We offer basic properties, functoriality, examples, and a relation with the classical Hilbert scheme and the work of Poonen [Poo06].

In Section 3, we obtain explicit equations for $M_{S'/S}$ using index forms. The index form tells whether a given section $\theta \in \mathcal{O}_{S'}$ is a monogenerator for $\mathcal{O}_{S'}/\mathcal{O}_S$ or not. We generalize to $k$-generators.

Section 4 concerns local monogeneity. If every fiber of the map $S' \to S$ is monogenic, does this imply the map $S' \to S$ is monogenic? Zariski-locally on $S$, it does. The classical notion of common index divisors addresses the same question. One gets a different notion if only the fibers of $S' \to S$ over geometric points are monogenic. We conclude with concrete deformation theory for monogenerators, applicable in the case where $S$ is a geometric point.

In Example 1.5, the scheme $M_{S'/S}$ coincides with the classical configuration space of $n$ points in $\mathbb{C}$. Section 5 generalizes this phenomenon to any étale map $S' \to S$. Theorem 5.2 establishes the correspondence in general and the section culminates in a variety of sample applications in other areas of mathematics.

Finding monogenerators for $S' \to S$ is equivalent to finding rational points on the scheme $M_{S'/S}$. Section 6 applies techniques for finding rational points to $M$, specifically local-to-global methods. To glue local monogenerators to a global monogenerator, one must control how two monogenerators can differ. We consider symmetries of monogenerators given by actions of $\mathbb{G}_m$ or $\text{Aff}^1$ and define twisted monogenerators analogous to Cartier divisors. Twisted monogenerators relate to “affine equivalence classes” of monogenerators the same way Cartier divisors relate to principal Cartier divisors. There are cohomological obstructions to lifting a given twisted monogenerator to a global monogenerator.

For any group $G$, we define $G$-twisted monogenerators and show there is a moduli space parameterizing them. The Picard group, Brauer group, and Shafarevich-Tate groups can all obstruct gluing $G$-twisted monogenerators to a global monogenerator. Theorem 6.20 states that a number field $K$ has class number one if and only if all twisted monogenic extensions of $\mathbb{Z}_K$ are monogenic.

Section 7 gives a variety of concrete examples of the scheme of monogenerators $M$, including: separable and inseparable field extensions, a variety of orders in number fields including Dedekind’s non-monogenic cubic, jet spaces, curves, and completions. These explicit equations put into practice classical theory in addition to the theory we have built. We encourage the reader to consult these examples to complement the earlier sections.

Appendix A reviews torsors and stack quotients for the reader’s convenience. This establishes conventions and contains requisite material for Section 6.

1.3. Guide to notions of “Monogeneity”.
1.4. Summary of Previous Results. The question of which rings of integers are monogenic was posed to the London Mathematical Society in the 1960’s by Helmut Hasse. Hence the study of monogeneity is sometimes known as Hasse’s problem. For an in-depth look at monogeneity with a focus on algorithms for solving index form equations, see Gaál’s book [Gaá19]. Evertse and Győry’s book [EG17] provides background with a special focus on the relevant Diophantine equations. For another bibliography of monogeneity, see Narkiewicz’s texts [Nar04, pages 79-81] and [Nar18, pages 75-77].

The prototypical examples of number rings are monogenic over $\mathbb{Z}$, such as quadratic and cyclotomic rings of integers. Dedekind [Ded78] produced the first example of a non-monogenic number ring (see Example 7.6). Dedekind used the splitting of (2) to show that the field obtained by adjoining a root of $x^3 - x^2 - 2x - 8$ to $\mathbb{Q}$ is not monogenic over $\mathbb{Z}$. Hensel [Hen94] showed that local obstructions to monogeneity come from primes whose splitting cannot be accommodated by the local factorization of polynomials. See [Ser79, Proposition III.12] and also [Ple74] which is discussed briefly below.

Global obstructions also preclude monogeneity. For a number field $L/\mathbb{Q}$, the field index is the greatest common divisor $\gcd_{\alpha \in \mathbb{Z}_L} [\mathbb{Z}_L : \mathbb{Z}[\alpha]]$. A number field $L$ can have field index 1 and $\mathbb{Z}_L$ may still not be monogenic, for example the ring of integers of $\mathbb{Q}(\sqrt[3]{25} \cdot 7)$; see [Nar04, page 65] or Example 7.9. Define the minimal index to be $\min_{\alpha \in \mathbb{Z}_L} [\mathbb{Z}_L : \mathbb{Z}[\alpha]]$. The monogeneity of $\mathbb{Z}_L/\mathbb{Z}$ is equivalent to having minimal index equal to 1. An early result of Hall [Hal37] shows that there exist cubic fields with arbitrarily large minimal indices. In [SYY16], this is generalized to show that every cube-free integer occurs as the minimal index of infinitely many radical cubic fields.

The monogeneity of a given extension of $\mathbb{Z}$ is encoded by a Diophantine equation called the index form equation. Győry made the initial breakthrough regarding the resolution of index form equations and related equations in the series of papers [Gyö73], [Gyö74], [Gyö76], [Gyö78a], and [Gyö78b]. These papers investigate monogeneity and prove effective finiteness results for affine inequivalent monogenerators in a variety of number theoretic contexts. For inequivalent monogenic generators
one should also consult [EG85], [BEG13], and the survey [Eve11]. Specializing families of polynomials to obtain monogenic extensions is investigated in [Kön18]. In large part due to the group in Debrecen, there is a vast literature involving relative monogeneity: [Gyö80], [Gyö81], [Gaá01], [GP00], [GS13], [GRS16], [GR19b], and [GR19a].

Pleasants [Ple74] bounds the number of generators needed for a field of degree $n$ by $\lceil \log_2(n) \rceil$, with equality if 2 splits completely. This upper bound is a consequence of a precise description of exactly when an extension is locally $k$-genic. For number rings, Pleasants answers the question of what the minimal positive integer $k > 1$ is such that $\mathbb{Z}_L$ is $k$-genic. Global obstructions only appear in the case of monogeneity: for $k > 1$, Pleasants shows that local $k$-genesity is equivalent to global $k$-genesity.

A brief context of the above is detailed in [EG17, Chapter 11], where it is shown that given an order $\mathcal{O}$ of a finite étale $\mathbb{Q}$-algebra, one can effectively compute the smallest $k$ such that $\mathcal{O}$ is $k$-genic. In the spirit of the previous work of Györy, one can also effectively compute the $k$ generators of $\mathcal{O}$ over $\mathbb{Z}$.

Monogeneity has recently been viewed from the perspective of arithmetic statistics: Bhargava, Shankar, and Wang [BSW16] have shown that the density of monic, irreducible polynomials in $\mathbb{Z}[x]$ such that a root is a monogenerator is $\frac{6}{\pi^2} = \zeta(2) - 1 \approx 60.79\%$. That is, about 61% of monic, integer polynomials correspond to monogenerators. They also show the density of monic integer polynomials with square-free discriminants (a sufficient condition for a root to be a monogenerator) is

$$\prod_p \left(1 - \frac{1}{p} + \frac{(p-1)^2}{p^2(p+1)}\right) \approx 35.82\%.$$ 

Thus these polynomials only account for slightly more than half of the polynomials yielding a monogenerator. Using elliptic curves, [ABS20] shows that a positive proportion of cubic number fields are not monogenic despite having no local obstructions (common index divisors). In a pair of papers that investigate a variety of questions ([Sia20a] and [Sia20b]), Siad shows that monogeneity appears to increase the average amount of 2-torsion in the class group of number fields. In particular, monogeneity has a doubling effect on the average amount of 2-torsion in the class group of odd degree number fields. Previously, [BHS20] had established this result in the case of cubic fields.

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For numerous computations throughout, we were very thankful to be able to employ Magma [BCP97] and SageMath [Sage]. For a number of examples the [LMFDB] was invaluable.
2. The Scheme of Monogenic Generators

We now use Remark 1.3 to construct the scheme of monogenic generators \( M_{S'/S} \), our geometric reinterpretation of the classical question of monogeneity. We arrive at a scheme \( M_{k,Z_L/Z_K} \) over \( \text{Spec} Z_K \) whose \( Z_K \)-points are in bijection with generating \( k \)-tuples for \( Z_L \) over \( Z_K \). When \( k = 1 \), \( M_{Z_L/Z_K} := M_{1,Z_L/Z_K} \) is the scheme of monogenerators.

The map \( S' := \text{Spec} Z_L \to S := \text{Spec} Z_K \) is finite locally free and \( X = A^k_{\mathbb{Z}_K} \to \text{Spec} Z_K \) is quasi-projective; these properties suffice for our purposes. This permits analogues of monogeneity such as embeddings into \( \mathbb{P}^k \), which are more natural when \( S' \to S \) is a map of proper varieties. We invite the reader to picture the well-known index form equations.

**Situation 2.1.** Let \( S' \to S \) be a finite locally free morphism of constant degree \( n \geq 1 \) with \( S \) locally noetherian. Consider a quasiprojective morphism \( f : X \to S \) and write \( X' := X \times_S S' \).

The constant degree assumption is for simplicity; the reader may remove it by working separately on each connected component of \( S \). Leave \( n = 0 \) to scholars of the empty set. We allow \( S \) to be an algebraic stack in Sections 5 and 6, though the morphism \( S' \to S \) is always representable. Write \((\text{Sch}/S)\) for the big étale site, the category of \( S \)-schemes with the étale topology. We now define \( M_{X,S'/S} \) by describing the functor of \( S \)-maps into it and then showing it is representable.

**Definition 2.2.** In Situation 2.1, consider the presheaf on \((\text{Sch}/S)\) which sends an \( S \)-scheme \( T \) to the set of morphisms \( s \) fitting into the diagram:

\[
T \mapsto \begin{cases} 
S' \times_S T \to X \times_S T \\ T \times 
\end{cases}
\]

with restriction given by pullback. Refer to this presheaf either as the relative hom presheaf \( \text{Hom}_S(S', X') \) [Stacks, p. 0D19] or the Weil Restriction \( R_{X', S'/S} \) [BLR12, §7.6].

To see the collection of monogenerators within this presheaf, define a subpresheaf

\[
M_{X,S'/S} \subseteq R_{X', S'/S}
\]

whose sections are diagrams exactly as above, but with \( s \) a closed immersion. We write \( M_{k,S'/S}, R_{k,S'/S} \) for \( X = A^k_{\mathbb{Z}_K} \) and \( M_{S'/S} = M_{1,S'/S}, R_{S'/S} = R_{1,S'/S} \). When \( S' \to S \) or \( X \) is understood, we may drop “\( S'/S \)” or “\( X \)” from the notation. If \( S' = \text{Spec} B \) and \( S = \text{Spec} A \) are affine, we write \( M_{k,B/A} \) or \( M_{B/A} \) instead.

We offer a few examples from [grg] for those unfamiliar with Weil Restrictions:

**Example 2.3.** One can always view a complex \( n \)-dimensional manifold \( X' \) as a real \( 2n \)-dimensional manifold. If \( S' = \text{Spec} \mathbb{C}, S = \text{Spec} \mathbb{R} \), the Weil Restriction \( R_{X', S'/S} \) is simply \( X' \) considered as a variety over \( \mathbb{R} \) instead of \( \mathbb{C} \). For example, \( R_{\text{Spec} \mathbb{C}, S'/S} = \mathbb{R}^2 = A^2_{\mathbb{R}} \).
Take \( X' = \mathbb{G}_m = \text{Spec} \mathbb{C}[z_1, z_2]/(z_1z_2 - 1) \). One replaces the complex variables with pairs of real variables:

\[
\begin{align*}
  z_1 &= x_1 + iy_1; \\
  z_2 &= x_2 + iy_2.
\end{align*}
\]

The equation \( z_1z_2 = 1 \) becomes

\[
\begin{align*}
  x_1x_2 - y_1y_2 &= 1; \\
  x_1y_2 + y_1x_2 &= 0.
\end{align*}
\]

The Weil Restriction \( R_{X'/L} \) is defined by

\[
\text{Spec} \mathbb{R}[x_1, x_2, y_1, y_2]/(x_1x_2 - y_1y_2 - 1, x_1y_2 + y_1x_2).
\]

If \( S' = \text{Spec} L, S = \text{Spec} K \) for a degree-\( n \) finite extension \( L/K \) and \( X' = \text{Spec} L[z_1, \ldots, z_n]/(f_1, \ldots, f_r) \), one obtains \( R_{X'/L} \) the same way. Choose a basis \( L = \bigoplus \mathbb{K}e_i \) and split the variables \( z_i \) into \( n \) variables over \( K \):

\[
z_i = x_1e_1 + \cdots + x_ne_n.
\]

Substitute into the \( f_j \)'s and split them accordingly

\[
f_j(\overline{T}) = g_{j1}e_1 + \cdots + g_{jn}e_n
\]

to get

\[
R_{X'/L} = \text{Spec} \mathbb{K}[\overline{T}]/(\overline{g}).
\]

Suppose \( L/K \) is finite separable of degree \( d \) and \( X \) an \( L \)-scheme. Let \( \bar{L} \) be the normal closure of \( L \) in a separable closure \( K^{sep} \) with \( d \) embeddings \( \sigma_i : L \subseteq \bar{L} \). This gives \( d \) different base changes of \( X' \) to \( \bar{L} \), denoted \( X'_{\sigma_i} \). The Galois group \( G = G(L/K) \) acts freely on \( X'_{\sigma_1} \times \cdots \times X'_{\sigma_d} \) and \( R_{X'/L} \) is the quotient.

Remark 1.3 expresses the presheaf \( M_{k,S'/S} \) as

\[
M_{k,S'/S}(T) = \{ (\theta_1, \ldots, \theta_k) \in \Gamma(T \times SS', O_{T \times SS'})^\oplus_k \mid O_{T \times SS'} = O_T[\theta_1, \ldots, \theta_k] \}
\]

An \( S \)-point of \( M_1 \) is said to be a monogenerator of \( S'/S \) and we say \( S'/S \) is monogenic if such a point exists. This recovers the definition of monogeneity of algebras when \( S \) is affine. These presheaves are representable:

**Proposition 2.4.** The presheaves \( M_X \subseteq R_{X'/S'} \) are both representable by quasiprojective \( S \)-schemes and the inclusion is a quasicompact open immersion. If \( f : X \to S \) is smooth, unramified, or étale, the same is true for \( R_{X'/S'} \) and \( M_X \to S \).

**Proof.** The Weil restriction \( R_{X'/S'} \) is representable by a scheme quasiprojective over \( S \) [Ji+17, Theorem 1.3, Proposition 2.10]. If \( X' \to S' \) is a finite-type affine morphism, the same is true for \( R_{X'/S'} \to S \) by locally applying [Ji+17, Proposition 2.2(1), (2)]. The inclusion \( M_X \subseteq R_{X'/S'} \) is open by [Stacks, 05XX] and automatically quasicompact because \( R_{X'/S'} \) is locally noetherian. The second statement is immediate.

**Corollary 2.5.** The presheaves \( M_X, R_{X'/S'} \) are sheaves in the Zariski, Nisnevich, étale, fppf, and fpqc topologies on \( (\text{Sch}/S) \).

**Definition 2.6.** Let \( \tau \) be a subcanonical Grothendieck topology on schemes, for example the Zariski, Nisnevich, étale, fppf, or fpqc topologies. We say that \( S'/S \) is \( \tau \)-locally \( k \)-generic if the sheaf \( M_{k,S'/S} \) is locally non-empty in the topology \( \tau \). I.e., there is a \( \tau \)-cover \( \{ U_i \to S \}_{i \in I} \) of \( S \) such that \( M_{k,S'/S}(U_i) \) is non-empty for all \( i \in I \). By default, we use the étale topology.
A $k$-genic extension $S'/S$ is $\tau$-locally $k$-genic. If $\tau_1$ is a finer topology than $\tau_2$, then $\tau_2$-locally monogenic implies $\tau_1$-locally monogenic.

**Remark 2.7.** We pose a related moduli problem $\mathcal{F}$ in § 2.2.1 that parameterizes a choice of finite flat map $S' \to S$ together with a monogenerator. It is also representable by a scheme. The mere choice of a finite flat map $S' \to S$ is representable by an algebraic stack, as shown in [Poo06] and recalled in Section 2.2.

Our main example of $S' \to S$ comes from rings of integers in number fields $\mathbb{Z}_L/\mathbb{Z}_K$, but here is another:

**Example 2.8.** Let $S = \text{Spec } \mathbb{Z}$ and $S'_n = \text{Spec } \mathbb{Z}[\varepsilon]/\varepsilon^n$. The Weil Restriction $\mathcal{R}_{X', S'_n/S}$ is better known as the jet space $J_{X, n-1}$ [Voj04]. For any ring $A$, $(n-1)$-jets are maps

$$\text{Spec } A[\varepsilon]/\varepsilon^n \to X.$$ 

Jet spaces are usually considered over a field $k$ by base changing from $S$. The monogeneity space $M_{S'_n/S, X} \subseteq J_{X, n-1}$ parametrizes embedded $(n-1)$-jets, whose map $\text{Spec } A[\varepsilon]/\varepsilon^n \subseteq X$ is a closed embedding. If $n = 2$, $J_{X, 1}$ is the Zariski tangent bundle and $M_{X, S'_2/S}$ is the complement of the zero section.

The truncation maps $J_{X, n} \to J_{X, n-1}$ restrict to maps $M_{X, S'_n/S} \to M_{X, S'_n/S}$. The inverse limit $\lim_n J_{X, n}$ sends rings $A$ to maps called arcs

$$\text{Spec } A[[t]] \to X$$

according to [Bha16, Remark 4.6, Theorem 4.1]. Under this identification, the limit $\lim_n M_{X, S'_n/S}$ parametrizes those arcs that are closed immersions into $X \times \text{Spec } A$.

Compare embedded $(n - 1)$-jets to “regular” ones. An $(n - 1)$-jet $f : \text{Spec } k[\varepsilon]/\varepsilon^n \to X$ over a field $k$ is called regular [Dem91, §5] if $f'(0) \neq 0$. I.e., the truncation of higher order terms $\overline{f} : \text{Spec } k[\varepsilon]/\varepsilon^2 \to X$ is a closed immersion. Regular $(n - 1)$-jets are precisely the pullback $M_{X, S'_n/S} \times_{J_{X, n}} J_{X, n}$.

Extensions of number rings are generically étale, with a divisor of ramification. The finite flat map $S'_n \to S$ in jet spaces is the opposite, ramified everywhere.

**Remark 2.9.** If $\mathcal{R}_{X', S'/S}$ is affine, then $M_X$ is quasi-affine. Even for number fields, $M_k$ need not be affine; see Theorem 3.8.

A sanity check for our new definition:

**Example 2.10.** Suppose $\pi : S' \to S$ is the identity, so $n = 1$. Then $\mathcal{R}_{X', S'/S} = X' = X$ and all sections of the separated $X \to S$ are closed: $M_X = \mathcal{R}_{X'}$.

**Remark 2.11** (Steinitz Classes). We have assumed that $\pi : S' \to S$ is finite locally free of rank $n$, so $\pi_* \mathcal{O}_{S'}$ is a locally free $\mathcal{O}_S$-module of rank $n$. By taking an $n$th exterior power, one obtains a locally free $\mathcal{O}_S$-module

$$\det \pi_* \mathcal{O}_{S'} := \bigwedge^n \pi_* \mathcal{O}_{S'}.$$ 

of rank 1 [Har77, Chapter II, Exercise 6.11]. The Steinitz class of $\pi : S' \to S$ is the isomorphism class of $\det \pi_* \mathcal{O}_{S'}$ in $\text{Pic}(S)$.

The Weil Restriction $\mathcal{R}_{k, S'/S}$ is precisely the rank $kn$ vector bundle with sheaf of sections $\pi_* \mathcal{O}_{S'}$. To see this, recall that the set of $T$-points of the $S$-scheme $\mathcal{R}_{k, S'/S}$ is by definition the set of $T$-morphisms $\{S' \times_S T \to k_T^k\}$. By the universal property
defines a map 

It follows that \( \text{Hom}_S(\cdot, \mathcal{R}_{S'/S}) \simeq (\pi_*\mathcal{O}_S)^k \) as quasicohherent sheaves on \( S \). When \( \pi_*\mathcal{O}_S \simeq \bigoplus^n \mathcal{O}_S \cdot e_i \) is trivial, so is \( \mathcal{R}_{S'/S} \).

Example 2.12. The Steinitz class of the jet space of \( \mathbb{A}^k \) is the trivial vector bundle: 

\[
J_{n, \mathbb{A}^k} = \mathbb{A}^{k(n+1)} \quad \text{[Voj04, Corollary 5.2]}
\]

Remark 2.13. Although \( \mathbb{A}^1 \) and hence \( \mathcal{R}_{S'/S} \) are ring objects, \( M_1 \) is neither closed under addition nor multiplication. For multiplication, note that a power of a generator is typically a non-generator. For addition, note that if \( \theta \in \mathcal{O}_{S'} \) is a generator, then so is \(-\theta\), but \( \theta + (-\theta) = 0 \) is not for \( n \neq 1 \).

Lemma 2.14. Let \( \theta \in \Gamma(S', \mathcal{O}_{S'}) \) be any element. There is a canonical monic polynomial \( m_\theta(t) \in \Gamma(S, \mathcal{O}_S)[t] \) of degree \( n \) such that \( m_\theta(\theta) = 0 \).

Proof. We begin by constructing \( m_\theta(t) \) locally, following [AM69, Proposition 2.4].

Assume first that \( \pi : S' \to S \) is such that \( \pi_*\mathcal{O}_{S'} \simeq \mathcal{O}_S[n] \) as an \( \mathcal{O}_S \)-module. Choose a \( \Gamma(S, \mathcal{O}_S) \)-basis \( \{x_1, \ldots, x_n\} \) of \( \Gamma(S', \mathcal{O}_{S'}) \). For each \( i = 1, \ldots, n \), write \( \theta x_i = \sum_{j=1}^n a_{ij} x_j \) where \( a_{ij} \in \Gamma(S', \mathcal{O}_{S'}) \). Now we let

\[
m_\theta(t) = \det(\delta_{ij} t - a_{ij}).
\]

As in the proof of [AM69, Proposition 2.4], \( m_\theta(t) \) has coefficients in \( \Gamma(S, \mathcal{O}_S) \), is monic of degree \( n \), and \( m_\theta(\theta) = 0 \). Moreover, \( m_\theta(t) \) does not depend on the basis chosen since \( m_\theta(\theta) \) is computed by a determinant.

Now for general \( \pi : S' \to S \), choose an open cover \( \{U_i\} \) of \( S \) on which \( \pi_*\mathcal{O}_{S'} \) trivializes. On each open set, the construction of the previous paragraph yields a monic polynomial \( m_i(t) \in \mathcal{O}_S(U_i)[t] \) of degree \( n \) vanishing on \( \theta|_{U_i} \). Since the construction of the polynomials commutes with restriction and is independent of choice of basis, we have

\[
m_i(t)|_{U_i \cap U_j} = m_j(t)|_{U_i \cap U_j}.
\]

We conclude by the sheaf property. \( \square \)

Remark 2.15. Lemma 2.14 defines a map \( m : \mathcal{R}_{S'/S} \to \mathbb{A}_S^n \) sending an element \( \theta \in \Gamma(S', \mathcal{O}_{S'}) \) to the coefficients \( (b_{n-1}, \ldots, b_0) \) of the universal canonical monic minimal polynomial

\[
m_\theta(t) = t^n + b_{n-1}t^{n-1} + \cdots + b_0.
\]

The preimage of a point of \( \mathbb{A}_S^n \) is the set of roots of the corresponding polynomial in \( \mathcal{O}_{S'} \). Locally in \( S' \), \( \mathcal{R}_{S'/S} = \mathbb{A}_S^n \) and \( m_\theta(t) \) is given by elementary symmetric polynomials, up to sign.

Trivializing \( \mathcal{R}_{S'/S} \) locally, this map becomes the map \( \mathbb{A}_S^n \to \mathbb{A}_S^n \) parametrizing the \( n \) elementary symmetric functions, the coarse quotient space for the natural action of the symmetric group \( \Sigma_n \circlearrowright \mathbb{A}_S^n \). This action need not globalize. For example, take the 3-power map \( [3] : \mathbb{G}_m \to \mathbb{G}_m \) for \( S' \to S \). We have more to say in Section 5.

2.1. Functoriality of \( M_X \). In Situation 2.1, let \( T \to S \) be a map from a locally noetherian scheme. Write \( X_T := X \times_T T \), \( T' := S' \times_T \), leading to a pullback
In this way, $m_{X,S'/S}$ is functorial in $S$. What about $X$? A closed immersion $X_1 \subseteq X_2$ gives

$$m_{X_1,S'/S} \rightarrow m_{X_2,S'/S}.$$ 

**Warning 2.16.** Beware that $m_{X,S'/S}$ is not functorial for all commutative squares $S'$ even though the Weil Restriction $R_{X',S'/S}$ is. Similarly, there’s no natural map $m_{X,S'/S} \rightarrow m_{X,S''/S}$ except for closed immersions.

If $g : S'' \rightarrow S'$ and $f : S' \rightarrow S$ are as in Situation 2.1, consider a commutative diagram

$$
\begin{array}{ccc}
S'' & \longrightarrow & X \\
\downarrow & \searrow & \downarrow \\
S' & \longrightarrow & S \\
\end{array}
\quad
\begin{array}{ccc}
X \times_S S' & \longrightarrow & X \\
\downarrow & \searrow & \downarrow \\
S' & \longrightarrow & S \\
\end{array}
$$

If $i$ is a closed immersion, so is $i'$ because $S' \rightarrow S$ is separated [Stacks, 07RK]. This yields a map

$$f^* : m_{X,S''/S} \times_S S' \rightarrow m_{X \times_S S',S''/S}.$$ 

**Corollary 2.17** (Relative Extensions of $k$-genic Extensions are $k$-genic). As above, if $S'' \rightarrow S$ is $k$-genic, then $S' \rightarrow S$ is $k$-genic.

**Proof.** Since $S'' \rightarrow S$ is $k$-genic, $m_{k,S''/S}$ has an $S$-point. Apply $f^*$ to the corresponding $S'$-point of $m_{k,S''/S} \times_S S'$ to get an $S'$-point of $m_{k,S'/S}$. 

In particular, if $M/L/K$ is a tower of number fields and $\mathbb{Z}_M/\mathbb{Z}_K$ is $k$-genic, then $\mathbb{Z}_M/\mathbb{Z}_L$ is $k$-genic. More prosaically, if $A \subseteq B \subseteq C$ and $C = A[\theta_1, \ldots, \theta_k]$, then $C = B[\theta_1, \ldots, \theta_k]$.

**Remark 2.18.** If $S' = \bigsqcup S'_i$ is a finite disjoint union of finite locally free maps $S'_i \rightarrow S$, the pullback $T \times_S S'$ is the disjoint union $\bigsqcup T \times_S S'_i$. Write $X'_i = X' \times_S S'_i$, it follows from the universal property of coproducts and the above that the Weil Restriction decomposes as

$$R_{X',S'/S} = \prod_i R_{X'_i,S'_i/S}.$$
The monogeneity space is not the product $M_{S'/S,X} \neq \prod M_{S_i'/S,X}$. Rather, we claim a map
\[
\bigsqcup S'_i \to X
\]
is a closed immersion if and only if each map
\[
S'_i \to X
\]
is a closed immersion and the closed immersions are disjoint:
\[
S'_i \times_X S'_j = \emptyset
\]
for all $i \neq j$. To see the claim, we may check affine locally on $X$, where it reduces to the statement that $A \to \prod B_i$ is surjective if and only if $A \to B_i$ is surjective for each $i$ and $B_i \otimes_A B_j = 0$ whenever $i \neq j$. This follows quickly in turn from the Chinese remainder theorem.

**Remark 2.19.** Not only is $M_{X,S}/S$ functorial in $S$, but we show its normalization and reduction can be performed on $S$.

If $X \to S$ is smooth and $T \to S$ is the normalization of $S$, one uses [Stacks, 03GV] and properties of $M_{X,S'/S}$ in Proposition 2.4 to see that $\mu$ in Diagram (3) is also normalization.

If a map $Y \to Z$ is smooth, the square is cartesian
\[
\begin{array}{ccc}
Y_{red} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z_{red} & \longrightarrow & Z.
\end{array}
\]
We need only check $Z_{red} \times_Z Y$ is reduced using [Stacks, 034E], since a surjective closed immersion of a reduced scheme must be $Y_{red}$. For smooth $X \to S$ and $T := S_{red}$, the pullback $M_{X,S'/S} \times_S S_{red} \simeq M_{X_{red},T'/S_{red}}$ is the reduction of $M_{X,S'/S}$.

### 2.2. Relation to the Hilbert Scheme.

With this section we pause our development of $M$ to relate our construction to other well-known objects. Though the rest of the paper does not use this section, it behooves us to situate our work in the existing literature.

At the possible cost of representability of $M_{X,S'/S}$, let $X \to S$ be any morphism of schemes in this section. Recall the Hilbert scheme of points [Stacks, 0B94]
\[\text{Hilb}_{X/S}^n(T) := \{\text{closed embeddings } Z \subseteq X \times_T T \mid Z \to T \text{ finite locally free, deg. } n\} \ldots\]

There is an algebraic moduli stack $\mathfrak{A}_n$ of finite locally free maps of degree $n$ [Poo06, Definition 3.2], with universal finite flat map $\mathfrak{F}_n \to \mathfrak{A}_n$. Any map $S' \to S$ in Situation 2.1 is pulled back from $\mathfrak{F}_n \to \mathfrak{A}_n$. We restrict to $S$-schemes without further mention: $\mathfrak{A}_n = \mathfrak{A}_n \times S$.

Recall Poonen’s description of $\mathfrak{A}_n$: A finite locally free map $\pi : Z \to T$ is equivalent to the data of a finite locally free $O_T$-algebra $\mathcal{Q}$ given by $\pi_* O_Z$. Suppose for the sake of exposition that a locally free algebra $\mathcal{Q}$ has a global basis $\mathcal{Q} \simeq O_T^\oplus n$. The algebra structure is a multiplication map
\[\mathcal{Q} \otimes^L \mathcal{Q} \to \mathcal{Q}\]
that can be written as a matrix using the basis. Conditions of associativity and commutativity are polynomial on the entries of this matrix. We get an affine scheme
of finite type \( \mathfrak{B}_n \) parametrizing matrices satisfying the polynomial conditions, or equivalently multiplication laws on globally free finite modules [Poo06, Proposition 1.1]. Two different choices of global basis \( \Theta_T^m \cong \Theta_T^\oplus \) differ by an element of \( \text{GL}_n(\Theta_T) \). Taking the stack quotient by this action \( \text{GL}_n \circ \mathfrak{B}_n \) erases the need for a global basis and gives \( \mathfrak{A}_n \).

There is a map \( \text{Hilb}^n_X/S \to \mathfrak{A}_n \) sending a closed embedding \( Z \subseteq X|_T \) to the finite flat map \( Z \to T \). The fibers of this map are exactly monogeneity spaces:

\[
\begin{array}{ccc}
M_{X,S'/S} & \longrightarrow & S \\
\downarrow & & \downarrow \phi_{S'/S} \\
\text{Hilb}^n_{X/S} & \longrightarrow & \mathfrak{A}_n.
\end{array}
\]

Conversely, the monogeneity space of the universal finite flat map \( Z \to \mathfrak{A}_n \) is isomorphic to the Hilbert Scheme

\[
M_{X,3_n/\mathfrak{A}_n} \cong \text{Hilb}^n_{X/S}
\]

over \( \mathfrak{A} \). The space \( \mathfrak{b}_n(A^k) \) of [Poo06, §4] is \( M_k \) for the universal finite flat map to \( \mathfrak{B}_n \).

Proposition 2.4 shows \( M_{S'/S}(X) \to S \) is smooth for \( X \) smooth, and likewise for unramified or étale. This means the map \( \text{Hilb}^n_{X/S} \to \mathfrak{A}_n \) is smooth, unramified, or étale if \( X \to S \) is.

Suppose \( X \to S \) flat to identify the Chow variety of dimension 0, degree \( n \) subvarieties of \( X \) with \( \text{Sym}^n X \) [Ryd08] and take \( S \) equidimensional. The Hilbert-Chow morphism \( \text{Hilb}^n_{X/S} \to \text{Sym}^n X \) sending a finite, flat, equidimensional \( Z \to S \) to the pushforward of its fundamental class \( [Z] \) in Chow \( A_*(X) \) restricts to \( M_{X,S'/S} \). When \( S'/S \) is étale, we will see the restriction of the Hilbert-Chow morphism \( M_{X,S'/S} \to \text{Sym}^n X \) is an open embedding by hand in Section 5.

**Question 2.20.** Can known cohomology computations of \( \text{Hilb}^n_{X/S} \) offer obstructions to monogeneity under this relationship?

### 2.2.1. Finite flat algebras with monogenerators

We mention a related moduli problem and how it fits into the present schema. We encourage the reader to skip to Section 3 on a first reading; this moduli problem will not arise in the sequel. We rely on a classical representability result:

**Theorem 2.21** ([Fan+05, Theorem 5.23]). If \( X \to S \) is flat and projective and \( Y \to S \) quasiprojective over a locally noetherian base \( S \), the functor \( \text{Hom}_S(X,Y) \) is representable by an \( S \)-scheme.

The scheme \( \text{Hom}_S(X,Y) \) is a potentially infinite disjoint union of quasiprojective \( S \)-schemes.

Fix a flat, projective map \( C \to S \) and quasiprojective \( X \to S \). Assign to any \( S \)-scheme \( T \) the groupoid of finite flat maps \( Y \to C \times_S T \) of degree \( n \). One can think of this as a \( T \)-indexed family of finite flat maps \( Y_t \to C \). This problem is represented by

\[
\mathfrak{R}_{\mathfrak{A}_n,C/S} \coloneqq \text{Hom}_S(C,\mathfrak{A}_n).
\]

We study moduli of finite flat maps \( Y \to C \times_S T \) **together with** a choice of monogenerator:
Definition 2.22. The moduli problem $\mathcal{F}$ on $S$-schemes $(\text{Sch}/S)$ has $T$-points given by:

- A finite, flat family $Y \to C \times_S T$ of degree $n$,
- A closed embedding $Y \subseteq X \times_S C \times_S T$ over $C \times_S T$.

These data form a fibered category via pullback. Define a variant $\mathcal{F}'$ parameterizing the data above together with a global basis $Q \simeq \mathcal{O}_T^{\oplus n}$ for the finite, flat algebra $\mathcal{O}$ corresponding to $Y \to C \times_S T$.

The map $\mathcal{F}' \to \mathcal{F}$ forgetting the basis is a torsor for the smooth group scheme $\text{Hom}_S(C, \text{GL}_n)$. Let $X_C$ denote the pullback $X \times_S C$. The stack $\mathcal{F}$ is the Weil Restriction $\mathcal{R}_{\text{Hilb}^n_{X_C/C}/S}$ of the Hilbert Scheme $\text{Hilb}^n_{X_C/C}$ for $X_C \to C$ along the map $C \to S$. Both are therefore representable by schemes using the theorem. One must use caution: $\text{Hilb}^n_{X_C/C}$ is an infinite disjoint union of projective schemes indexed by Hilbert polynomials and not itself projective, but this suffices for representability.

There are universal finite flat maps

$$
\begin{array}{c}
\tilde{Z} \\
\downarrow \\
C \times_S \text{Hom}_S(C, \mathfrak{B}_n)
\end{array}
\quad
\begin{array}{c}
\tilde{Y} \\
\downarrow \\
C \times_S \text{Hom}_S(C, \mathfrak{A}_n)
\end{array}
$$

with and without a global basis $Q \simeq \mathcal{O}_T^{\oplus n}$. The sheaf $\mathcal{F}$ may also be obtained by the Weil Restriction along $C \times_S \text{Hom}_S(C, \mathfrak{A}_n) \to \text{Hom}_S(C, \mathfrak{A}_n)$ of the monogeneity space $M_{X, Y/C \times_S \text{Hom}_S(C, \mathfrak{A}_n)}$. The same construction of $\mathcal{F}'$ can be obtained with $\mathfrak{B}_n$ in place of $\mathfrak{A}_n$.

We argue $\text{Hom}_S(C, \mathfrak{A}_n)$ is also representable by an algebraic stack. Olsson’s result [Ols06, Theorem 1.1] does not apply here because $\mathfrak{A}_n$ is not separated. This means the diagonal $\Delta_{\mathfrak{A}_n}$ is not proper, and this diagonal is a pseudotorsor for automorphisms of the universal finite flat algebra. The automorphism sheaf $\text{Aut}(\mathcal{O})$ of some finite flat algebras is not proper: take the 2-adic integers $\mathbb{Z}_2$, $\mathcal{O} = \mathbb{Z}_2[x]/x^2$, and the map $\mathcal{O} \to \mathcal{O}$ sending $x \mapsto 2x$. This is an automorphism over the generic point $Q_2 = \mathbb{Z}_2[1/2]$ and the zero map over the special point $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2/2\mathbb{Z}_2$.

The scheme $\mathfrak{B}_n$ on the other hand is a closed subscheme of an affine space, hence separated. The Weil Restriction $\text{Hom}_S(C, \mathfrak{B}_n)$ is a scheme by the above theorem and the map

$$
\text{Hom}_S(C, \mathfrak{B}_n) \to \text{Hom}_S(C, \mathfrak{A}_n)
$$

is again a torsor for the smooth group scheme $\text{Hom}_S(C, \text{GL}_n)$. Therefore $\text{Hom}_S(C, \mathfrak{A}_n)$ is algebraic.

The diagonal $\Delta_{\mathfrak{A}_n}$ even fails to be quasifinite because some finite flat algebras have infinitely many automorphisms:

Example 2.23 (Infinite automorphisms). The dual numbers $k[x, y]/(x^3, y^3 - 1)$ have an action of $\mathbb{G}_m$ by $\varepsilon \mapsto u \cdot \varepsilon$ for a unit $u \in \mathbb{G}_m(k)$.

For another example, let $k$ be an infinite field of characteristic three and consider $\mathcal{O} = k[x, y]/(x^3, y^3 - 1)$. Because $(y + x)^3 = y^3$, there are automorphisms $y \mapsto y + ux$ for any $u \in k$.

The reader may define stable algebras $\mathcal{O}$ as those with unramified automorphism group [Stacks, 0DSN]. There is a universal open, Deligne-Mumford substack $\mathfrak{A}_n \subseteq$
\( \mathfrak{A}_n \) of stable algebras [Stacks, 0DSL]. This locus consists of points where the action \( \text{GL}_n \circ \mathfrak{B}_n \) has unramified stabilizers [Poo06, §2].

3. The Local Index Form and Construction of \( M \)

This section describes equations for the monogeneity space \( M_{1,S'/S} \) inside \( \mathcal{R} = \text{Hom}_S(S', \mathbb{A}^1) \) by working with the universal homomorphism over \( \mathcal{R} \). In the classical case \( \mathbb{Z}_L/\mathbb{Z}_K \) we recover the well-known index form equation. Section 3.2 gives examples, while 3.3 generalizes the equations to \( k \)-geneity \( M_{k,S'/S} \).

Remark 3.1 (Representable functors). Recall that if a functor \( F : C^{op} \to \text{Set} \) is represented by an object \( X \), then there is an element \( \xi \) of \( F(X) \) corresponding to the identity morphism \( X \overset{id}{\to} X \), called the universal element of \( F \). The proof of the Yoneda lemma shows that for all objects \( Y \) and elements \( y \in F(Y) \), there is a morphism \( f_y : Y \to X \) such that \( y \) is obtained by applying \( F(f_y) \) to \( \xi \).

For a map \( f : X \to Y \) of schemes, we write \( f^* : \mathcal{O}_Y \to \mathcal{O}_X|_Y \) for the map of sheaves and its kin.

3.1. Explicit Equations for the Scheme \( M \). The scheme \( \mathcal{R} = \mathcal{R}_{S'/S} = \text{Hom}_S(S', \mathbb{A}^1) \) is a “moduli space” of maps \( S' \to \mathbb{A}^1 \). For any \( T \to S \), every morphism \( S' \times_S T \to \mathbb{A}^1_T \) is pulled back along some \( T \to \mathcal{R} \) from the universal homomorphism

\[
\begin{array}{ccc}
S' \times_S \mathcal{R} & \xrightarrow{u} & \mathbb{A}^1_{\mathcal{R}} \\
\downarrow & & \downarrow \\
\mathcal{R} & & \\
\end{array}
\]

We want explicit equations for \( M_{S'/S} \subseteq \mathcal{R} \).

Let \( t \) be the coordinate function on \( \mathbb{A}^1 \). The map \( u \) corresponds to an element \( \theta = u^\dagger(t) \in \Gamma(\mathcal{O}_{S' \times_S \mathcal{R}}) \). Let \( m(t) \) be the polynomial of Lemma 2.14 for \( \theta \), i.e. \( m(t) \) is a monic polynomial in \( \Gamma(\mathcal{O}_{\mathcal{R}})[t] \) of degree \( n \) such that \( m(\theta) = 0 \).

Definition 3.2. We call the polynomial \( m(t) \) the universal minimal polynomial of \( \theta \). Let \( V(m(t)) \) be the closed subscheme of \( \mathbb{A}^1_{\mathcal{R}} \) cut out by \( m(t) \).

The universal map \( u \) factors through this closed subscheme:

\[
\begin{array}{ccc}
S' \times_S \mathcal{R} & \xrightarrow{v} & V(m(t)) \xrightarrow{\pi} \mathbb{A}^1_{\mathcal{R}} \\
\downarrow & & \downarrow \\
\mathcal{R} & & \\
\end{array}
\]

Since \( V(m(t)) \to \mathbb{A}^1_{\mathcal{R}} \) is a closed immersion, the locus in \( \mathcal{R} \) over which \( u \) restricts to a closed immersion agrees with the locus over which \( v \) is a closed immersion.

Remark 3.3. The map \( V(m(t)) \to \mathcal{R} \) is finite globally free \( \tau_* \mathcal{O}_{V(m(t))} \simeq \bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathcal{R}} \cdot t^i \). The map \( v : S' \times_S \mathcal{R} \to V(m(t)) \) comes from a map

\[
v^* : \tau_* \mathcal{O}_{V(m(t))} \to \pi_* \mathcal{O}_{S' \times_S \mathcal{R}}
\]
of finite locally free \( \mathcal{O}_{\mathcal{R}} \)-modules. Locally, it is an \( n \times n \) matrix. The determinant of this matrix is a unit when it is full rank, i.e. when \( v \) is a closed immersion. The \( i \)th column is \( \theta^i \), written out in terms of the local basis of \( \pi_* \mathcal{O}_{S' \times_S \mathcal{R}} \). We work this out explicitly to get equations for \( M \subseteq \mathcal{R} \).
Remark 2.19 lets us find equations locally. Suppose $S' = \text{Spec } B$ and $S = \text{Spec } A$, where $B = \bigoplus_{i=1}^{n} A \cdot e_i$ is a finite free $A$-algebra of rank $n$ with basis $e_1, \ldots, e_n$. Let $I = \{1, \ldots, n\}$. Write $t$ for the coordinate function of $\mathbb{A}^1$ and write $x_I$ as shorthand for $n$ variables $x_i$ indexed by $i \in I$.

The scheme $\mathcal{R} = \text{Hom}_A(S', \mathbb{A}^1)$ is the affine scheme $\mathbb{A}^n_S = \text{Spec } A[x_I]$ and Diagram (4) becomes

$$
\begin{array}{cccc}
B[x_I] & \xrightarrow{\pi^t} & A[x_I]/(m(t)) & \xrightarrow{\pi^t} & A[x_I] \\
\downarrow & & \downarrow & & \downarrow \\
\pi^t & & \pi^t & & \pi^t \\
A[x_I] & & A[x_I] & & A[x_I]
\end{array}
$$

The $A[x_I]$-homomorphism $\pi^t$ sends

$$
t \mapsto \theta := x_1e_1 + \cdots + x_ne_n.
$$

Note that $A[x_I]/(m(t))$ has an $A[x_I]$ basis given by the equivalence classes of $1, t, \ldots, t^{n-1}$ and $B[x_I]$ has an $A[x_I]$-basis given by $e_1, \ldots, e_n$. With respect to this basis, $\pi^t$ is represented by the matrix of coefficients

$$
M(e_1, \ldots, e_n) = [a_{ij}]_{1 \leq i,j \leq n}.
$$

where $a_{ij} \in A[x_I]$ are the unique coefficients such that $\theta^j = \sum_{i=1}^{n} a_{ij}e_i$ for each $j = 1, \ldots, n$.

**Example 3.4.** Let $S' := \text{Spec } \mathbb{C}^n \rightarrow S := \text{Spec } \mathbb{C}$ as in Example 1.5. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors of $\mathbb{C}^n$. The monogenerators of $S'$ over $S$ are precisely the closed immersions:

$$
\begin{array}{ccc}
S' & \xrightarrow{} & \mathbb{A}^1_S \\
\downarrow & & \downarrow \\
S & & \mathbb{A}^1_S
\end{array}
$$

Identify $\mathcal{R} \simeq \mathbb{A}^n_S$. Let $t$ denote the coordinate of $\mathbb{A}^1_{\mathbb{A}^n_S}$, and let $x_1, x_2, \ldots, x_n$ denote the coordinates of $\mathbb{A}^n_S$. The analogue of Diagram (4) for this case is:

$$
\begin{array}{cccc}
S' \times_S \mathbb{A}^n_S & \xrightarrow{V(m(t))} & \mathbb{A}^1_{\mathbb{A}^n_S} \\
\downarrow & & \downarrow \\
\mathbb{A}^n_S & & \mathbb{A}^1_{\mathbb{A}^n_S}
\end{array}
$$

We write down $m_\theta(t)$, as in Lemma 2.14. The coordinate $t$ of $\mathbb{A}^1_{\mathbb{A}^n_S}$ maps to the universal element $\theta = x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Since $e_i = (\delta_{ij})_{j=1}^{n} \in \mathbb{C}^n$, we have $\theta e_i = x_ie_i$. Computing the minimal polynomial, $m_\theta(t) = \det(\delta_{ij}t - a_{ij}) = \prod_{i=1}^{n}(t - a_i)$.

Notice that $e_i e_j = \delta_{ij} e_i$. It follows that $\theta^i = \sum_{j=0}^{n-1} x_i e_j$. Therefore $M(e_1, \ldots, e_n)$ is the Vandermonde matrix with $i$th row given by $[1 \ x_i \ x_i^2 \ \cdots \ x_i^{n-1}]$.

**Definition 3.5.** With notation as above, let $i(e_1, \ldots, e_n) = \det(M(e_1, \ldots, e_n)) \in A[x_I]$. We call this element a local index form for $S'$ over $S$. When the basis is clear from context, we may omit the basis elements from the notation.
**Proposition 3.6.** Suppose $S' \to S$ is finite free and $S$ is affine. With notation as above, $M$ is the distinguished affine subscheme $D(i(e_1, \ldots, e_n))$ inside $\mathcal{R} \simeq \text{Spec } A[x_1]$.

**Proof.** By Proposition 2.4, $M$ is an open subscheme of $\mathcal{R}$. Therefore it suffices to check that $D(i(e_1, \ldots, e_n))$ and $M$ have the same points. Let $j : y \to \mathcal{R}$ be the inclusion of a point with residue field $k(y)$. Then

$j$ factors through $M \iff j^* u$ is a closed immersion, where $u$ is the univ. hom.

$\iff j^* v$ is a closed immersion, for $v$ as in (4)

$\iff v^\# \otimes_A k(y)$ is surjective

$\iff j^2(M(e_1, \ldots, e_n))$ is full rank

$\iff j^2(i(e_1, \ldots, e_n))$ is nonzero.

This establishes the claim. \hfill \Box

**Remark 3.7.** If $B$ is a free $A$-algebra with basis $\{e_1, \ldots, e_n\}$, it follows that $B$ is monogenic over $A$ if and only if there is a solution $(x_1, \ldots, x_n) \in A^n$ to one of the equations

$$i(e_1, \ldots, e_n)(x_1, \ldots, x_n) = a$$

as $a$ varies over the units of $A$. These are the well-known index form equations. In the case that $A$ is a number ring there are only finitely many units, so only finitely many equations need be considered. This perspective gives the set of global monogenerators the flavor of a closed subscheme of $\mathcal{R}$ even though $M_1$ is an open subscheme.

**Corollary 3.8.** The map $M_{1,S'/S} \to S$ classifying monogenerators is affine.

**Proof.** Proposition 3.6 shows that $S$ possesses an affine cover on which $M_{1,S'/S}$ restricts to a single distinguished affine subset of the affine scheme $\mathcal{R} = \text{Spec } A[x_1]$. \hfill \Box

**Remark 3.9.** Consider a local index form $i(e_1, \ldots, e_n) = \det(M(e_1, \ldots, e_n))$ defined by a basis $B \simeq \bigoplus_{i=0}^{n-1} A \cdot e_i$ as above. Suppose $\tilde{e}_1, \ldots, \tilde{e}_n$ is a second basis of $B$ over $A$ and $\tilde{M}$ is the matrix representation of $v^\#$ with respect to the bases $\{1, \ldots, t^{n-1}\}$ and $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. Then $\det(\tilde{M}) = u \det(M)$ for some unit $u$ of $A$. Although the determinant of $M$ may not glue to a global datum $S$, this shows the ideal that it generates does.

**Definition 3.10 (Non-monogenerators $\mathcal{N}_{S'/S}$).** Let $\mathcal{G}_{S'/S}$ be the locally principal ideal sheaf on $\mathcal{R}$ generated locally by local index forms. We call this the index form ideal. Let $\mathcal{N}_{S'/S}$ be the closed subscheme of $\mathcal{R}$ cut out by the vanishing of $\mathcal{G}_{S'/S}$. We call this the scheme of non-monogenerators, since its support is the complement of $M_{S'/S}$ inside of $\mathcal{R}$.

**Remark 3.11.** The local index forms $i(e_1, \ldots, e_n)$ are homogeneous with respect to the grading on $A[x_1]$. To see this, note that since the $i$th column of $M(e_1, \ldots, e_n)$ represents $\theta^{i-1}$, its entries are of degree $i - 1$ in $x_1, \ldots, x_n$. The Leibnitz formula for the determinant

$$\det(a_{ij}) = \sum_{\sigma \in \Sigma_n} (-1)^{\sigma} \prod a_{\sigma(i)j}$$
shows that $i(e_1, \ldots, e_n)$ is homogeneous of degree $\sum_{i=1}^n (i - 1) = \frac{(i(i-1)}{2}$. The transition functions induced by change of basis respect this grading, so the index form ideal $\mathfrak{d}_{S'/S}$ is a sheaf of homogeneous ideals.

3.2. First examples. To illustrate our machinery and gain some intuition, we compute the set and the scheme of monogenerators for quadratic extensions of number fields.

Example 3.12 (Quadratic Number Fields). Let $K = \mathbb{Q}(\sqrt{d})$, for any square-free integer $d$. It is well-known that the ring of integers $\mathbb{Z}_K$ is monogenic: $\mathbb{Z}_L \simeq \mathbb{Z}[\sqrt{d}]$ or $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$, depending on $d \mod 4$. We will confirm this using our framework, and determine the scheme $M_1$ of monogenic generators.

Let $\alpha$ denote the known generator of $\mathcal{O}_L$, either $\sqrt{d}$ or $1+\sqrt{d}$. Let us take $\{1, \alpha\}$ as the basis $e_1, \ldots, e_n$. The universal map diagram (4) becomes:

$$
\begin{array}{c}
\mathbb{Z}[a, b, \alpha] \leftarrow \mathbb{Z}[a, b, t]/(m(t)) \leftarrow \mathbb{Z}[a, b, t] \\
\mathbb{Z}[a, b] \\
\end{array}
$$

where the map $\mathbb{Z}[a, b, t]/(m(t)) \rightarrow \mathbb{Z}[a, b, \alpha]$ is given by $t \mapsto a + b\alpha$. The universal minimal polynomial $m(t)$ is given by $t^2 - \text{Tr}(a + b\alpha)t + N(a + b\alpha)$.

This diagram encapsulates all choices of generators as follows. The elements of $\mathbb{Z}[\alpha]$ are all of the form $a_0 + b_0\alpha$ for $a_0, b_0 \in \mathbb{Z}$. Integers $a_0, b_0 \in \mathbb{Z}$ are in bijection with maps $\phi : \mathbb{Z}[a, b] \rightarrow \mathbb{Z}$. Applying the functor $\mathbb{Z} \otimes_{\phi, \mathbb{Z}[a, b]} -$ to the diagram above yields a diagram

$$
\begin{array}{c}
\mathbb{Z}[\alpha] \leftarrow \mathbb{Z}[t]/(m(t)) \leftarrow \mathbb{Z}[t] \\
\mathbb{Z} \\
\end{array}
$$

where the map $\mathbb{Z}[t]/(m(t)) \rightarrow \mathbb{Z}[\alpha]$ takes $t \mapsto a_0 + b_0\alpha$. The image is precisely $\mathbb{Z}[a_0 + b_0\alpha]$, and the index form that we are about to compute detects whether this is all of $\mathbb{Z}[\alpha]$.

Returning to the universal situation, the matrix representation of the map $\mathbb{Z}[a, b, t]/(m(t)) \rightarrow \mathbb{Z}[a, b, \alpha]$ (what we have been calling the matrix of coefficients (5)) is given by

$$
\begin{bmatrix}
1 & a \\
0 & b
\end{bmatrix}.
$$

Notice that we did not need to compute $m(t)$ to get this matrix. The determinant, $b$, is the local index form associated to the basis $\{1, \alpha\}$. Therefore $M_1 \simeq \mathbb{Z}[a, b, b^{-1}]$. Taking $\mathbb{Z}$-points of $M_1$, we learn that $a + b\alpha$ $(a, b \in \mathbb{Z})$ is a monogenic generator precisely when $b$ is a unit, i.e. $b = \pm 1$.

Proposition 3.16 generalizes this example to any degree-two $S' \rightarrow S$.

Example 3.13 (Jets in $\mathbb{A}^1$). Let $S = \text{Spec} \mathbb{Z}$ and $S'_n = \text{Spec} \mathbb{Z}[\varepsilon]/\varepsilon^n$, as in Example 2.8. We explicitly describe $M_{1,S'_n/S} \subseteq \mathcal{R} = J_{n-1, \mathbb{A}^1}$.
Choose the basis $1, \varepsilon, \ldots, \varepsilon^{n-1}$ for $\mathbb{Z}[\varepsilon]/\varepsilon^n$. With respect to this basis, we may write the universal map diagram as

\[
\begin{array}{c}
\mathbb{Z}[x_1, \ldots, x_n, \varepsilon]/\varepsilon^n \\
\mathbb{Z}[x_1, \ldots, x_n, t]/(m(t)) \\
\mathbb{Z}[x_1, \ldots, x_n, t] \\
\mathbb{Z}[x_1, \ldots, x_n]
\end{array}
\]

where $t \mapsto x_1 + x_2 \varepsilon + \cdots + x_n \varepsilon^{n-1}$.

Change coordinates by $t \mapsto t - x_1$ so that the image of $t$ is

\[
t \mapsto \theta = x_2 \varepsilon + \cdots + x_n \varepsilon^{n-1}.
\]

Update $m(t)$ accordingly: $m(t) = t^n$. Our next task is to compute the representation of $\theta^j$ in $\{1, \varepsilon, \ldots, \varepsilon^{n-1}\}$-coordinates for $j = 0, \ldots, n - 1$. The multinomial theorem yields

\[
(x_2 \varepsilon + x_3 \varepsilon^2 + \cdots + x_n \varepsilon^{n-1})^j = \sum_{i_2+i_3+\cdots+i_n=j} \left( \begin{array}{c} j \\ i_2, \ldots, i_n \end{array} \right) \prod_{t=2}^n x_t^{i_t} \varepsilon^{(t-1)i_t}.
\]

The coefficient of $\varepsilon^p$ is

\[
\sum_{i_2+i_3+\cdots+i_n=j \atop i_2+2i_3+\cdots+(n-1)i_n=p} \left( \begin{array}{c} j \\ i_2, \ldots, i_n \end{array} \right) \prod_{t=2}^n x_t^{i_t}.
\]

The matrix of coefficients in Figure 3.1 represents the $\mathbb{Z}[x_1, \ldots, x_n]$-linear map from $\mathbb{Z}[x_1, \ldots, x_n, t]/(m(t))$ to $\mathbb{Z}[x_1, \ldots, x_n, \varepsilon]/\varepsilon^n = \bigoplus_{t=0}^{n-1} \mathbb{Z}[x_1, \ldots, x_n] \cdot \varepsilon^t$. The coefficient of $\varepsilon^p$ above appears in the $(j+1)$st column and $(p+1)$st row.

\[
M_n = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & x_2 & 0 & 0 & 0 & \cdots & 0 \\
0 & x_3 & x_2^2 & 0 & 0 & \cdots & 0 \\
0 & x_4 & 2x_2x_3 & x_2^3 & 0 & \cdots & 0 \\
0 & x_5 & 2x_2x_4 + x_3^2 & 3x_2^2x_3 & x_2^4 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & x_n & \cdots & \cdots & \cdots & \cdots & x_2^{n-1}
\end{bmatrix}
\]

**Figure 3.1.** The matrix determined by an $(n-1)$-jet.

Since $M_n$ is lower triangular, it has determinant $x_2^{\binom{n(n-1)}{2}}$. An $(n-1)$-jet thereby belongs to $M_{A^1, S_n/S}$ if and only if the coefficient $x_2$ is a unit. The $x_i$ are naturally coordinates of the jet space, yielding $M_{A^1, S_n/S} = \text{Spec} \mathbb{Z}[x_1, \ldots, x_n, x_2^{-1}] \subseteq \mathcal{R}_{S_n/S} = J_{n-1, A^1}$. 
3.3. Explicit Equations for Polygenerators $M_k$. The work above readily generalizes to describe $M_k$.

Fix a number $k \in \mathbb{N}$. We now construct explicit equations for $M_k$ as a subscheme of $\mathcal{O}_{k} = \text{Hom}_S(S^?, \mathbb{A}^k)$ when $S^?$ is free and $S$ is affine. These hypotheses hold Zariski locally on $S$, so by Remark 2.19, this gives a construction for $M_k$ locally on $S$ in the general case.

Let $S^? = \text{Spec } B$ and $S = \text{Spec } A$, where $B = \bigoplus_{i=1}^n A \cdot e_i$ is a finite free $A$-algebra of rank $n$ with basis $e_1, \ldots, e_n$. Let $J = \{1, \ldots, k\}$ and $I = \{1, \ldots, n\}$. Write $t_J$ for the $|J|$ coordinate functions $t_1, \ldots, t_k$ of $\mathbb{A}^k$ and write $x_{I \times J}$ as shorthand for $|I \times J|$ variables $x_{ij}$ indexed by $(i, j) \in I \times J$.

The scheme $\mathcal{O}_k$ is represented by the affine scheme $\text{Spec } A[x_{I \times J}]$ and the universal map for $\mathcal{O}_k$ is the commutative triangle

$$
\begin{array}{ccc}
S^\times_S \mathcal{O}_k & \xrightarrow{u} & \mathbb{A}^k_{S_k} \\
\downarrow & & \downarrow \\
\mathcal{O}_k & \xrightarrow{v} & B_{\mathcal{O}_k}
\end{array}
$$

where the horizontal arrow $u$ is induced by the ring map $A[x_{I \times J}, t_J] \to B[x_{I \times J}]$ sending $t_J \mapsto \theta_J := \sum_{i \in I} x_{ij} e_i$.

Notice that $S^\times_S \mathcal{O}_k \to \mathcal{O}_k$ is in Situation 2.1. Apply Lemma 2.14 to find monic degree $n$ polynomials $m_j(t_J) \in A[x_{I \times J}, t_J]$ such that $m_j(\theta_J) = 0$ in $B[x_{I \times J}]$. Write $v^S$ for the unique map $v^S : A[x_{I \times J}, t_J]/(m_j(t_J) : j \in J) \to B[x_{I \times J}]$ factoring $v^S : A[x_{I \times J}, t_J] \to B[x_{I \times J}]$.

Now, $A[x_{I \times J}, t_J]/(m_j(t_J) : j \in J)$ is a free $A[x_{I \times J}]$-module of rank $n^k$ with basis given by the equivalence classes of the products $t_1^{r_1} \cdots t_k^{r_k}$ as the powers $r_j$ vary between 0 and $n - 1$. Since $B[x_{I \times J}]$ is also a free $A[x_{I \times J}]$-module with basis $e_1, \ldots, e_n$, we may choose an ordering of the powers $t_1^{r_1} \cdots t_k^{r_k}$ and represent the map $v^S$ by an $n^k \times n$ matrix $M$. For each subset $C \subseteq \{1, \ldots, kn\}$ of size $n$, let $M_C$ be the submatrix of $M$ whose columns are indexed by $C$ and let $\det(M_C)$ be the determinant.

**Proposition 3.14.** Suppose $S^? \to S$ is finite free and $S$ is affine. Then with notation as above, $M_k$ is the union of the distinguished affines $D(\det(M_C))$ inside $Y$.

**Proof.** Check on points as in Proposition 3.6. □

The scheme of $k$-generators $M_k$ need not be affine. Even for the Gaussian integers $\mathbb{Z}[i]/\mathbb{Z}$, we have that $M_k = \mathbb{A}^k \times (\mathbb{A}^k \setminus \{0\})$. We prove this in Proposition 3.16, after a small lemma. The second factor begs to be quotiented by group actions of $\mathbb{G}_m, \Sigma_n$, or $\text{GL}_n$, which is the topic of Sections 5 and 6.

**Lemma 3.15.** Locally on $S$, the ring $\mathcal{O}_S^?$ has an $\mathcal{O}_S$-basis in which one basis element is 1.

**Proof.** This is part of the proof of [Poo06, Proposition 3.1]. □
Proposition 3.16. Suppose $S' \to S$ has degree 2 and let $\bar{0} \in \mathbb{A}^k_S$ be the zero section. Then affine locally on $S$ we have an isomorphism

$$M_{k,S'/S} \cong \mathbb{A}^k_S \times \left( \mathbb{A}^k_S \setminus \bar{0} \right).$$

Proof. Working affine locally and applying Lemma 3.15, we may take $S' = \text{Spec } B$ and $S = \text{Spec } A$, where $B$ has an $A$-basis of the form $\{1, e\}$. Write $b_1, \ldots, b_k$ for the coordinates on the second $\mathbb{A}^k_S$, so $\bar{0} = V(b_1, \ldots, b_k) \subseteq \mathbb{A}^k_S$.

In the notation preceding Proposition 3.14, we may take $x_{i,1} = a_i$, $x_{i,2} = b_i$, and $t_i \mapsto a_i + b_i e$. The matrix of coefficients will have columns given by the $\{1, e\}$-basis representation of the images of $1$, $t_i$, and $t_it_j$ as $i, j$ vary over distinct integers in $\{1, \ldots, k\}$. Write $e^2 = c + de$ where $c, d \in A$. Then the column of the matrix representing $1$ is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

the columns representing the images of the $t_i$ are

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix},$$

and the columns representing the images of $t_it_j$ are

$$\begin{bmatrix} a_i a_j + b_i b_j c \\ a_i b_j + a_j b_i + b_i b_j d \end{bmatrix}.$$

Among the determinants of the $2 \times 2$ minors of this matrix are $b_1, \ldots, b_k$, coming from the submatrices

$$\begin{bmatrix} 1 & a_i \\ 0 & b_i \end{bmatrix}.$$ 

The remaining determinants all lie in the ideal $(b_1, \ldots, b_k)$ since all elements of the second row of the matrix lie in this ideal. We conclude by Proposition 3.14 that $M_{k,S'/S}$ is the union of the open subsets $D(b_i)$ of $\text{Spec } A[a_1, b_1, \ldots, a_k, b_k]$. This gives us the result. \qed

As an alternative to taking the union of $k$-determinants, we can use a generalization of the determinant first introduced by Cayley, later rediscovered and generalized by Gel'fand, Kapranov and Zelevinsky:

Question 3.17. The map $v^\sharp$ above is a multilinear map from the tensor product of $k$ free modules $A[x_{I \times J}, t_{j}/m_j(t_j)]$ of rank $n$ over $A[x_{I \times J}]$ to rank-$n$ free $B[x_{I \times J}]$. For $k = 1$, $M_1$ is the complement of the determinant of $v^\sharp$. In general, the map $v^\sharp$ is locally given by a hypermatrix of format $(n - 1, \ldots, n - 1)$ [Ott13]. This $n \times n \times \cdots \times n$-hypercube of elements of $A[x_{I \times J}]$ describes a multilinear map the same way ordinary $n \times n$ matrices describe a linear map. What locus does the hyperdeterminant cut out in $R_k$?

Example 7.14 addresses the case $k = 2$ for jet spaces.

Remark 3.18. We compare our local index form to the classical number theoretic situation. For references see [EG17] and [Gaö19]. If $K$ is a number field and $L$ is an extension of finite degree $n$, then there are $n$ distinct embeddings of $L$ into an algebraic closure that fix $K$. Denote them $\sigma_1, \ldots, \sigma_n$. Let $\text{Tr}$ denote the trace
from $L$ to $K$. The discriminant of $L$ over $K$ is defined to be the ideal $\text{Disc}(L/K)$ generated by the set of elements of the form
\[
(\det[\sigma_i(\omega_j)])_{1 \leq i,j \leq n}^2 = \det[\text{Tr}(\omega_i\omega_j)]_{1 \leq i,j \leq n},
\]
where we vary over all $K$-bases for $L$, $\{\omega_1, \ldots, \omega_n\}$, with each $\omega_i \in \mathbb{Z}_L$. If $\alpha$ is any element of $L$, then the discriminant of $\alpha$ over $K$ is defined to be
\[
\text{Disc}_{L/K}(\alpha) = \left(\det(\sigma_i(\alpha^{j-1}))_{1 \leq i,j \leq n}\right)^2 = \det(\text{Tr}(\alpha^{i-1}\alpha^{j-1}))_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} (\sigma_i(\alpha) - \sigma_j(\alpha))^2,
\]
where the last equality comes from Vandermonde’s identity. Note that $\text{Disc}_{L/K}(\alpha)$ is a power of the discriminant of the minimal polynomial of $\alpha$. For every $\alpha$ generating $L$ over $K$ one has
\[
\text{Disc}_{L/K}(\alpha) = [\mathbb{Z}_L : \mathbb{Z}_K[\alpha]]^2 \text{Disc}(L/K).
\]
One defines the index form of $\mathbb{Z}_L$ over $\mathbb{Z}_K$ be to
\[
g_{\mathbb{Z}_L/\mathbb{Z}_K} = [\mathbb{Z}_L : \mathbb{Z}_K[\alpha]] = \sqrt{\frac{\text{Disc}_{L/K}(\alpha)}{\text{Disc}(L/K)}}.
\]
Confer [EG17, Equation 5.2.2]. In the case where $\{\omega_1, \ldots, \omega_n\}$ is a $\mathbb{Z}_K$-basis for $\mathbb{Z}_L$, employing some linear algebra [EG17, Equation (1.5.3)], one finds $g_{\mathbb{Z}_L/\mathbb{Z}_K}$ is, up to an element of $\mathbb{Z}_K^\times$, the determinant of the change of basis matrix from $\{\omega_1, \ldots, \omega_n\}$ to $\{1, \alpha, \ldots, \alpha^{n-1}\}$.

The matrix in Equation (5) is just such a matrix and its determinant coincides up to a unit with the index form in situations where the index form is typically defined. The generality of our setup affords us some flexibility that is not immediate from the definition of the classical index form equation.

**Proposition 3.19.** When the scheme of non-monogenerators $\mathcal{H}_{S'/S}$ is an effective Cartier divisor (equivalently, when none of the local index forms are zero divisors), the divisor class of $\mathcal{H}_{S'/S}$ in $\mathcal{R} = \underline{\text{Hom}}_S(S', \mathbb{A}^1)$ is the same as the pullback of the Steinitz class of $S'/S$ from $S$.

**Proof.** Recall that $V(m(t))$ is the vanishing of $m(t)$ in $\mathbb{A}_R^1$, where $m(t)$ is the generic minimal polynomial for $S'/S$. Let $\tau$ be the natural map $\tau : V(m(t)) \to \mathcal{R}$. Consider the morphism
\[
v^t : \tau^*\mathcal{O}_{V(m(t))} \to \pi_*\mathcal{O}_{S' \times_S \mathcal{R}}
\]
of sheaves on $\mathcal{R}$. The first sheaf is free of rank $n$ since $m(t)$ is a monic polynomial: there is a basis given by the images of $1, t, \ldots, t^{n-1}$. Therefore, taking $n$th wedge products in the previous equation, we have a map
\[
\mathcal{O}_{\mathcal{R}} \simeq \det(\tau^*\mathcal{O}_{V(m(t))}) \to \det(\pi_*\mathcal{O}_{S' \times_S \mathcal{R}}).
\]
By construction, this map is locally given by a local index form, $i(e_1, \ldots, e_n)$. Since we have assumed that $\mathcal{H}_{S'/S}$ is an effective Cartier divisor, the determinant is locally a nonzero-divisor. Therefore, the determinant identifies a non-zero section of $\pi_*\mathcal{O}_{S' \times_S \mathcal{R}}$. By definition of $\mathcal{H}_{S'/S}$, we may identify $\det(\pi_*\mathcal{O}_{S' \times_S \mathcal{R}})$ with $\mathcal{O}(\mathcal{H}_{S'/S})$. 


Writing $\psi : R \to S$ for the structure map, we also have that
$$\det(\pi_* O_{S'^* \times S/R}) \simeq \psi^* \det(\pi_* O_{S'}).$$

since taking a determinental line bundle commutes with arbitrary base change and $\pi_*$ commutes with base change for flat maps. The class of the line bundle $\det(\pi_* O_{S'})$ in Pic$(S)$ is by definition the Steinitz class.

$\square$

4. Local monogeneity

4.1. Zariski-local monogeneity. This section shows Zariski-local monogeneity can be detected over points and completions, as spelled out in Remark 4.3.

**Theorem 4.1.** The following are equivalent:

1. $\pi : S' \to S$ is Zariski-locally monogenic.
2. There exists a family of maps $\{f_i : U_i \to S\}$ such that
   a. the $f_i$ are jointly surjective;
   b. for each point $p \in S$, there is an index $i$ and point $q_p \in f_i^{-1}(p)$ so that $f_i$ induces an isomorphism $k(p) \to k(q_p)$;
   c. $S' \times_S U_i \to U_i$ is monogenic for all $i$.
3. $\pi : S' \to S$ is monogenic over points, i.e., $S' \times_S \text{Spec } k(p) \to \text{Spec } k(p)$ is monogenic for each point $p \in S$.

**Proof.** $(1) \implies (2)$: Choose the $U_i$ to be a Zariski cover on which $S' \to S$ is monogenic.

$(2) \implies (3)$: Suppose such a cover $\{f_i : U_i \to S\}$ is given. For each $i$, let $\sigma_p : \text{Spec } k(p) \to U_i$ be the section through $q_p$. Monogeneity is preserved by pullback on the base, so pulling back $S' \times_S U_i \to U_i$ along $\sigma_p$ implies $(3)$.

$(3) \implies (1)$: Let $p \in S$ be a point with residue field $k(p)$ and let $\theta$ be a monogenerator for $S' \times_S \text{Spec } k(p) \to \text{Spec } k(p)$. We claim that $\theta$ extends to a monogenerator over an open subset $U \subseteq S$ containing $p$, from which $(1)$ follows. The claim is Zariski local, so assume $S = \text{Spec } A$ and $\pi_* O_{S'} \simeq \bigoplus_{i=1}^n O_{S' \times S}$ is globally free. The Weil Restriction $R_{S'/S}$ is then isomorphic to affine space $A^n_S$.

We first extend $\theta$ to a section of $R_{S'/S}$. The monogenerator entails a point $\theta : \text{Spec } k(p) \to M_{S'/S} \subseteq R_{S'/S} \simeq A^n_S$, i.e. $n$ elements $\pi_1, \ldots, \pi_n \in k(p)$. Choose arbitrary lifts $x_i \in A(p)$ of $\pi_i$. The $n$ elements $x_i$ must have a common denominator, so we have $x_1, \ldots, x_n \in A[f^{-1}]$ for some $f$. Thus our point $\tilde{\theta} : \text{Spec } k(p) \to A^n_S$ extends to $\tilde{\theta} : D(f) \to R_{S'/S}$ for some distinguished open neighborhood $D(f) \subseteq S$ containing $p$.

Finally, we restrict $\tilde{\theta}$ to a section of $M_1$. The monogeneity space $M_1$ is an open subscheme of the Weil Restriction $R_{S'/S}$, so $\tilde{\theta} : D(f) \to R_{S'/S}$ restricts to a monogenerator $\tilde{\theta}|_U : U \to M_1$ where $U = \tilde{\theta}^{-1}(M_1) \subset D(f)$. By hypothesis, $p \in U$, so $\tilde{\theta}|_U$ is the desired extension of $\theta$.

$\square$

**Remark 4.2.** The same proof shows that $S' \to S$ is Zariski-locally $k$-genic if and only if its fibers $S' \times_S \text{Spec } k(p) \to \text{Spec } k(p)$ are $k$-genic.

**Remark 4.3.** Item (2) of Theorem 4.1 implies the following are also equivalent to Zariski-local monogeneity:

1. $S' \to S$ is “monogenic at local rings,” i.e., for each point $p$ of $S$, $S' \times_S \text{Spec } O_{S,p} \to \text{Spec } O_{S,p}$ is monogenic.
(2) $S' \to S$ is “monogenic at completions,” i.e., for each point $p$ of $S$, $S' \times_S \text{Spec } \mathcal{O}_{S,p} \to \text{Spec } \hat{\mathcal{O}}_{S,p}$ is monogenic, where $\mathcal{O}_{S,p}$ denotes the completion of $\mathcal{O}_{S,p}$ with respect to its maximal ideal.

(3) $S' \to S$ is locally monogenic in the Nisnevich topology as in Definition 2.6.

Corollary 4.4 ([Ser79, Proposition III.6.12]). Let $S' \to S$ be an extension of local rings inducing a separable extension of residue fields. Then $S'$ is monogenic over $S$.

Proof. Use the equivalence of item (1) in Remark 4.3 and (3) in Theorem 4.1. □

Definition 4.5. Let $\mathbb{Z}_L/\mathbb{Z}_K$ be an extension of number rings. A non-zero prime of $p \subseteq \mathbb{Z}_K$ is a common index divisor for the extension $\mathbb{Z}_L/\mathbb{Z}_K$ if $q_{\mathbb{Z}_L/\mathbb{Z}_K}(\theta) \in p$ for every $\theta \in \mathbb{Z}_L$ generating $L/K$.

As mentioned in Section 1, common index divisors are exactly the primes $p$ whose splitting in $\mathbb{Z}_L$ cannot be mirrored by irreducible polynomials in $k(p)[x]$; see [Hen94] and [Ple74].

Restating the property of being monogenic at points in terms of the index form, we obtain a generalization of the notion of having no common index divisors:

Proposition 4.6. $S' \to S$ is monogenic over points if and only if for each point $p$ of $S$ and local index form $i$ around $p$, there is a tuple $(x_1, \ldots, x_n) \in k(p)^n$ such that $i(x_1, \ldots, x_n)$ is nonzero in $k(p)$.

Proof. By Proposition 3.6 and functoriality of $M_1$, $S' \times \text{Spec } k(p) \to \text{Spec } k(p)$ is monogenic precisely when $\text{Spec } k(p)[x_1, \ldots, x_n, i(x_1, \ldots, x_n)^{-1}]$ has a $k(p)$ point. □

Immediately we recover an explicit corollary validating the generalization:

Corollary 4.7. Suppose $S' \to S$ is induced by an extension of number rings $\mathbb{Z}_L/\mathbb{Z}_K$. Then $S' \to S$ is Zariski-locally monogenic if and only if there are no common index divisors for $\mathbb{Z}_L/\mathbb{Z}_K$.

Example 4.8. There are extensions of number rings that are locally monogenic but not monogenic. Narkiewicz [Nar04, Page 65] gives the following family. Let $L = \mathbb{Q}(\sqrt[3]{m})$ with $m = ab^2$, $ab$ square-free, $3 \nmid m$, and $m \not\equiv \pm 1 \pmod{9}$. The number ring $\mathbb{Z}_L$ is not monogenic over $\mathbb{Q}$ despite having no common index divisors. To be even more explicit in Example 7.9 we consider the case $ab^2 = 7 \cdot 5^2$.

4.2. Monogeneity over Geometric Points.

Definition 4.9. Say that $S'$ over $S$ is monogenic over geometric points if, for each morphism $\text{Spec } k \to S$ where $k$ is an algebraically closed field, $S' \times_S \text{Spec } k \to \text{Spec } k$ is monogenic.

While it is a weaker condition than monogeneity over points in general, it is equivalent to some conditions that might seem more natural.

Theorem 4.10. The following are equivalent:

---

2Common index divisors are also called essential discriminant divisors and inessential or nonessential discriminant divisors. The shortcomings of the English nomenclature likely come from what Neukirch [Neu99, page 207] calls “the untranslatable German catch phrase [...] ‘außer-wesentliche Diskriminatoranteile.’” Our nomenclature is closer to Fricke’s ‘ständiger Indexteiler.’
(1) the local index forms for $S'/S$ are nonzero on each fiber of $R \to S$;  
(2) for each point $p \in S$ with residue field $k(p)$, there is a finite Galois extension $L/k(p)$ such that $S' \times_S \text{Spec } L \to \text{Spec } L$ is monogenic, and this extension may be chosen to be trivial if $k(p)$ is an infinite field;  
(3) $S' \to S$ is monogenic over geometric points;  
(4) there is a jointly surjective collection of maps $\{U_i \to S\}$ so that $S' \times_S U_i \to U_i$ is monogenic for each $i$;  
(5) $M_1 \to S$ is surjective;  

If, in addition, $N$ is a Cartier divisor in $R$ (i.e., the local index forms are non-zero divisors), the above are also equivalent to:  
(6) $N \to S$ is flat.

To see some of the subtleties one can compare item (1) above with Lemma 4.6 and item (4) above with item (2) of Theorem 4.1.

**Proof.** The assertions are Zariski-local, so we may choose local coordinates as usual $(S = \text{Spec } A, S' = \text{Spec } B, \text{coordinates } x_1)$.  

(1) $\implies$ (2): Suppose first that $p \in S$ is a point with $k(p)$ an infinite field. Let $p$ be the corresponding prime of $A$ and write $\bar{t}$ for the restriction of the local index form modulo $p$. Recall that since $k(p)$ is infinite, polynomials in $k(p)[x_1, \ldots, x_n]$ are determined by their values on $(x_1) \in k(p)^n$. Since $\bar{t}$ is nonzero, $\bar{t}(a_1, \ldots, a_n)$ must be nonzero for some $(a_1, \ldots, a_n) \in k(p)^n$. This shows that $S' \times_S \text{Spec } k(p) \to \text{Spec } k(p)$ is monogenic, so we have (2) again.

(2) $\implies$ (3): Let $k$ be an algebraically closed field and $\text{Spec } k \to S$ a map. Let $p$ be the image of $\text{Spec } k$, and let $L$ be the field extension given by (2). Then pullback along $\text{Spec } k \to \text{Spec } L$ implies that $S' \times_S \text{Spec } k \to \text{Spec } k$ is monogenic.

(3) $\implies$ (4): Take $\{U_i \to S\}$ to be $\{\text{Spec } k(p) \to S\}_{p \in S}$.

(4) $\implies$ (1): For each point $p \in S$, choose an index $i$ and a point $q_p \in U_i$ mapping to $p$. Let $i$ be an index form around $p$. By pullback, $S' \times_S q_p \to q_p$ is monogenic, so $i$ pulls back to a nonzero function over $k(q_p)$. Therefore $i$ is nonzero over $k(p)$ as well.

(5) $\implies$ (4): Take $M \to S$ as the cover.

(2) $\implies$ (5): For each point $p \in S$, the $\text{Spec } L$ point of $M_1$ witnessing monogenicity of $S' \times_S \text{Spec } L \to \text{Spec } L$ is a preimage of $p$.

(1) $\iff$ (6): The sequence

$$0 \to J_{S'/S} \to \mathcal{O}_S \to \mathcal{O}_{R_{S'/S}} \to 0$$

may be written as

$$0 \to A[x_1] \xrightarrow{i} A[x_1] \to A[x_1]/i \to 0$$

where $i$ is a local index form for $S'/S$.

Recall that an $A[x_1]$-module $M$ is flat if and only if for each prime $p$ of $A$ and ideal $q$ of $A[x_1]$ lying over $p$, $M_q$ is flat over $A_p$. Therefore, by the local criterion for flatness [Stacks, 00MK], $J_{S'/S}$ is flat over $S$ if and only if

$$\text{Tor}_1^{A_p} (A_p/pA_p, (A[x_1]/i)_q) = 0$$
for all such ideals $p$ and $q$. Therefore, $\mathfrak{n}_{S'/S}$ is flat if and only if

$$A[x_1]/pA[x_1]_q \cong^{(\text{mod } p)} A[x_1]/pA[x_1]_q$$

is injective for all $p$ and $q$ as above. All of these maps are injective if and only if the maps of $A[x_1]$-modules

$$A_p[x_1]/pA_p[x_1] \cong^{(\text{mod } p)} A_p[x_1]/pA_p[x_1]$$

are all injective as $p$ varies over the prime ideals of $A$. Since $A_p[x_1]/pA_p[x_1] \cong (A_p/pA_p)[x_1]$ is an integral domain for each $p$, injectivity fails if and only if $i$ reduces to 0 in the fiber over some $p$. We conclude that (1) holds if and only if (6) holds. \hfill \Box

Corollary 4.11. If all of the points of $S$ have infinite residue fields, then the following are equivalent:

1. $S' \to S$ is monogenic over geometric points;
2. $S'/S$ is Zariski-locally monogenic.

Remark 4.12. This conclusion fails dramatically if $S$ has finite residue fields. For $S' \to S$ coming from an extension of number rings condition (1) always holds (see Corollary 4.18 below), yet there are extensions that are not locally monogenic. In this sense, monogeneity is more subtle in the arithmetic context than the geometric one. For an example of an extension that is monogenic over geometric points but is not monogenic over points see Example 7.6.

4.3. Monogeneity Over Points. In light of Theorem 4.1, it is particularly interesting to work out the problem of monogeneity in the case that $S$ is the spectrum of a field $k$. In this case $S'$ is the spectrum of an $n$-dimensional $k$-algebra $B$. Such algebras are Artinian rings, and a well-known structure theorem implies that $B$ is a direct product of local Artinian rings $B_i$. We will exploit this to give a complete characterization of Zariski-local monogeneity.

The result in the case that both $S'$ and $S$ are spectra of fields is well-known.

Theorem 4.13 (Theorem of the primitive element). Let $\ell/k$ be a finite field extension. Then $\text{Spec } \ell \to \text{Spec } k$ is monogenic if and only if there are finitely many intermediate subfields $\ell/\ell'/k$.

In particular, a finite separable extension of fields is monogenic.

We next consider the monogeneity of $S' \to S$ when $S'$ is a nilpotent thickening of $\text{Spec } \ell$, leaving $S = \text{Spec } k$ fixed. A key ingredient is a study of the square zero extensions of $\text{Spec } \ell$.

We remark that the proof below does not consider whether monogenators extend to a deformation of $S' \to S$ along a nilpotent thickening of the base $S \to \bar{S}$. In fact, if $S \to \bar{S}$ is a nilpotent closed immersion with $S' = S \times_{\bar{S}} \bar{S}'$, any monogenator $\theta : S \to M_{S'/S}$ extends to $\bar{S}$ locally in the étale topology. This results from the smoothness of $M_{\bar{S}'/\bar{S}} \to \bar{S}$.

Theorem 4.14. Let $S' \to S$ be induced by $k \to B$ where $k$ is a field and $B$ is a local Artinian $k$-algebra with residue field $\ell$ and maximal ideal $m$. Then $S' \to S$ is monogenic if and only if

1. $\text{Spec } \ell \to \text{Spec } k$ is monogenic
(2) \( \dim \kappa / \kappa^2 \leq 1. \)

(3) If \( \dim \kappa / \kappa^2 = 1 \) and \( \ell / k \) is inseparable, then

\[ 0 \to \kappa / \kappa^2 \to \mathcal{B}/\mathcal{B}^2 \to \ell \to 0 \]

is a non-split extension.

**Proof.** If the tangent space \( (\kappa / \kappa^2)^\vee \) has dimension greater than 1, then no map \( S' \to \mathbb{A}^1_k \) can be injective on tangent vectors as is required for a closed immersion.

Conversely, if the tangent space of \( S' \) has dimension 0, we have \( B = \ell \), and the result is true by hypothesis.

Now suppose the tangent space of \( S' \) has dimension 1. A morphism \( S' \to \mathbb{A}^1_k \) is a closed immersion if and only if it is universally closed, universally injective, and universally closed from \( \text{Spec} \mathcal{B} \). Hence, we will show that \( S' \to \mathbb{A}^1_k \) is a closed immersion if and only if its restriction to the vanishing of \( \mathcal{B} \) is. On the other hand, any map \( V(\mathcal{B}^2) \to \mathbb{A}^1_k \) extends to \( S' \to \mathbb{A}^1_k \) (choose a lift of the image of \( t \) arbitrarily). Therefore, it suffices to consider the case that \( S' \) is a square zero extension of \( \text{Spec} \mathcal{B} \).

By hypothesis, we have a presentation of \( \ell \) as \( k[t]/(f(t)) \). We conclude with some elementary deformation theory, see for example [Ser06, §1.1]. We have a square zero extension of \( \ell \)

\[ 0 \to (f(t))/((f(t))^2) \to k[t]/(f(t))^2 \to \ell \to 0. \]

By assumption, \( B \) is also a square zero extension of \( \ell \):

\[ 0 \to \kappa \to B \to \ell \to 0. \]

By [Ser06, Proposition 1.1.7], there is a morphism of \( k \)-algebras \( \phi : k[t]/(f(t))^2 \to B \) inducing the identity on \( \ell \). Since \( (f(t))/((f(t))^2) \simeq \ell/\ell^2 \) as a \( k[t]/(f(t))^2 \) module, \( \phi \) either restricts to an isomorphism \( f(t)/(f(t))^2 \to \kappa \) or else the zero map. In the former case, the composite \( k[t] \to k[t]/(f(t))^2 \to B \) is a surjection, and we are done. In the latter case, \( B \) is the pushout of the extension \( k[t]/(f(t))^2 \) along \( (f(t))/((f(t))^2) \to \kappa \), so \( B \) is the split extension \( \ell/\ell^2 \).

If \( \ell / k \) is separable, then the extension \( k[t]/(f(t))^2 \) is itself split [Ser06, Proposition B.1, Theorem 1.1.10], i.e. there is an isomorphism \( k[t]/(f(t))^2 \simeq \ell/\ell^2 \). Composing with \( k[t] \to k[t]/(f(t))^2 \) gives the required monogenerator.

If \( \ell / k \) is inseparable and \( B \simeq \ell/\ell^2 \), we will show that \( S' \to \mathbb{A}^1_k \) is not monogenic. Any generator for \( \ell/\ell^2 \) over \( k \) must also be a generator for \( \ell/\ell^2 \) over the maximal separable subextension \( k' \) of \( \ell / k \), so we may assume that \( \ell / k \) is purely inseparable. Moreover, any generator \( \theta \) for \( \ell/\ell^2 \) over \( k' \) must reduce modulo \( \ell \) to a generator \( \overline{\theta} \) of \( \ell / k \). Since \( \ell / k \) is purely inseparable, the minimal polynomial \( f(t) \) of \( \overline{\theta} \) satisfies \( f'(t) = 0 \). Note \( \theta = \overline{\theta} + c \ell \) for some \( c \in \ell \). Since \( \ell \) is assumed to be a monogenerator, there is a polynomial \( g(t) \in k[t] \) such that \( \ell = g(\theta) \). Reducing,
\(g(\theta) = g(\overline{\theta}) = 0\), so \(g(t) = q(t)f(t)\) for some \(q(t) \in k[t]\). Then
\[
g(\theta) = g(\overline{\theta}) + g'(\theta)\varepsilon = 0 + q'(\overline{\theta})f(\overline{\theta})\varepsilon + q(\overline{\theta})f'(\overline{\theta})\varepsilon = 0,
\]
a contradiction. We conclude that in this case \(S' \to S\) is not monogenic. \(\square\)

**Remark 4.15.** In the case that \(k\) is perfect, the first and third conditions hold automatically. If \(S'\) is regular of dimension 1 the second condition is trivial.

**Theorem 4.16.** Suppose \(S' \to S\) is induced by \(k \to A\) where \(k\) is a field and \(B\) is an Artinian \(k\)-algebra. Write \(B = \prod_i B_i\) where the \(B_i\) are local artinian \(k\)-algebras with respective residue fields \(k_i\). Then \(S' \to S\) is monogenic if and only if

1. \(\text{Spec } B_i \to S\) is monogenic for each \(i\);
2. for each finite extension \(k'\) of \(k\), \(S'\) has fewer points with residue field isomorphic to \(k'\) than \(k\) has elements that generate \(k'\) over \(k\).

**Proof.** Remark 2.18 tells us that a map from a disjoint union is a closed immersion when it restricts to a closed immersion on each connected component and the components have non-overlapping images. The proof of Theorem 4.14 shows that a closed immersion \(\text{Spec } B_i \to \mathbb{A}^1\) can be chosen with image any of the points of \(\mathbb{A}^1_k\) with residue field \(k_i\). The condition on numbers of points is exactly what we need for the images of the Spec \(B_i\)s not to overlap without running out of points. (Since topologically, the components are single points.) \(\square\)

**Remark 4.17.** Condition (2) is trivial in the case that the residue fields of \(S\) are infinite, highlighting the relative simplicity of monogeneity in the geometric context.

If \(S' \to S\) is instead induced by an extension of number rings, then Remark 4.15 implies condition (1) is trivial. In particular, an extension of \(\mathbb{Z}\) has common index divisors if and only if there is “too much prime splitting” in the sense of condition (2). This recovers the theorem of [Hen94] (also see [Ple74, Cor. to Thm. 3]) that \(p\) is a common index divisor if and only if there are more primes in \(\mathbb{Z}_L\) above \(p\) of residue class degree \(f\) than there are irreducible polynomials of degree \(f\) in \(k(p)[x]\) for some positive integer \(f\).

**Corollary 4.18.** If \(S' \to S\) is induced by an extension of number rings \(\mathbb{Z}_L/\mathbb{Z}_K\), then \(S' \to S\) is monogenic over geometric points.

**Proof.** Let \(p\) be a point of \(S\). Let \(A = \mathbb{Z}_L \otimes_{\mathbb{Z}_K} k(p)\) be the ring for the fiber of \(S'\) over \(\text{Spec } k(p)\). Note that \(k(p)\) is either of characteristic 0 or finite, so \(k(p)\) is perfect. Decompose \(A\) into a direct product of local Artinian \(k(p)\) algebras \(A_i\). Since \(k(p)\) is perfect, conditions (1) and (3) of Theorem 4.14 hold for \(\text{Spec } A_i \to \text{Spec } k(p)\). Condition (2) holds as well since \(S'\) is regular of dimension 1. Therefore \(\text{Spec } A_i \to \text{Spec } k(p)\) is monogenic for each \(i\).

Now consider the base change of \(S'\) to the algebraic closure \(\overline{k(p)}\) of \(k(p)\). Write \(B\) for the ring of functions \(\mathbb{Z}_L \otimes_{\mathbb{Z}_K} \overline{k(p)}\) of this base change, and write \(B\) as a product of local Artinian algebras \(B_j\). For each \(i\) we have that \(\text{Spec } A_i \otimes_{\overline{k(p)}} \overline{k(p)} \to \text{Spec } \overline{k(p)}\) is monogenic. Each \(\text{Spec } B_j\) is a closed subscheme of exactly one of the \(\text{Spec } A_i \otimes_{k(p)} \overline{k(p)}\) \(\overline{k(p)}\)s, so by composition, \(\text{Spec } B_j \to \text{Spec } \overline{k(p)}\) is monogenic for each \(j\). This gives
us condition (1) of Theorem 4.16 for $S' \times_S \text{Spec } k(p) \to \text{Spec } k(p)$. Since $k(p)$ is infinite, condition (2) holds trivially. We conclude that $S' \times_S \text{Spec } k(p) \to \text{Spec } k(p)$ is monogenic, as required. \hfill \Box

5. Étale Maps, Configuration Spaces, and Monogeneity

This section concerns maps $\pi : S' \to S$ that are étale, or unramified. The monogeneity space becomes a “configuration space” classifying arrangements of $n$ points on a given topological space.

Example 1.5 showed monogeneity is a configuration space for the trivial cover of $C$:

$$M_{1,\mathbb{C}^n/C}(C) = \text{Conf}_n(C) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ for } i \neq j\}. $$

Configuration spaces arise for finite étale covers $S' \to S$ more generally. There is a universal finite étale cover whose space of monogenerators is a configuration space. Philosophically, $M_{S'/S}$ regards $S' \to S$ as a twisted family of points to be configured in $\mathbb{A}^1$ or any other target. In other words, we may interpret $M_{S'/S}$ as an arithmetic refinement of the configuration space. Example 5.3 shows the corresponding action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is known in anabelian geometry. This indicates the power of the topological invariants of $M$. We end Subsection 5.1 with a handful of exotic applications in other areas.

All extensions $S' \to S$ sit somewhere between the étale case and the jet spaces of Example 3.13, between being totally unramified and totally ramified. The trace pairing gives equations which cut out the locus of ramification, as described in Section 5.2. Specifically, [Poo06, §6] says that our description in the étale case holds precisely away from the vanishing of the discriminant. The discriminant plays a similar role in the classical case when investigating the monogeneity of a polynomial. We end with some remarks on using stacks to promote a ramified cover of curves to an étale cover of stacky curves as in [Cos06].

5.1. The case of étale $S' \to S$. Write $(n) = \{1, 2, \ldots, n\}$ and $(n)_S = \bigsqcup_{1 \leq i \leq n} S \to S$ for the trivial degree $n$ finite étale cover. Consider the category $(\text{Sch}/*)_S$ of schemes over a final scheme $*$ equipped with the étale topology. For example, take $* = \text{Spec } \mathbb{Z} \text{ or Spec } \mathbb{C}$. Write $\Sigma_n$ for the symmetric group on $n$ letters and $B\Sigma_n$ for the stack on $(\text{Sch}/*)_S$ of étale $\Sigma_n$-torsors.

Étale-locally in $S$, all finite étale morphisms $S' \to S$ are isomorphic to $(n)_S$ [Stacks, 04HN]. The sheaf of isomorphisms

$$P = \text{Isom}_{(S', (n)_S)}$$

is a left $\Sigma_n$-torsor, classified by some map $S \to B\Sigma_d$.

Conversely, any such torsor $\Sigma_n \times P$ begets a finite étale $n$-sheeted cover via

$$(n) \wedge^\Sigma_n P := ((n) \times P)/(\Sigma_n, \Delta) \cong S \times_{B\Sigma_n} B\Sigma_{n-1},$$

where we are quotienting by the diagonal action $\Sigma_n \times (n) \times P$. The quotient is a “contracted product” discussed in Appendix A.

We therefore have an equivalence between finite étale $n$-sheeted covers and $\Sigma_n$-torsors [Cos06, Lemma 2.2.1], [HW21, Lemma 3.2]. I.e., the map $B\Sigma_{n-1} \to B\Sigma_n$ is the universal $n$-sheeted cover. The map $B\Sigma_{n-1} \to B\Sigma_n$ comes from the inclusion $\Sigma_{n-1} \subseteq \Sigma_n$ obtained by omitting any one of the $n$ letters and does not depend on which letter is omitted up to isomorphism as in Remark A.10.
Pullback squares
\[ T' \to S' \]
\[ T \to S \]
induce identifications
\[ R_{T'/T} \cong R_{S'/S} \times_S T \quad \text{and} \quad m_{1,T'/T} \cong m_{1,S'/S} \times_S T. \]
Reduce thereby to the universal \( n \)-sheeted finite étale cover
\[ S' = B \Sigma_n^{-1}, S = B \Sigma_n. \]
Each has an affine line \( \mathbb{A}_B \Sigma_n^1 = [\mathbb{A}^1/\Sigma_n] \) obtained via quotienting by the trivial action.
We claim the Weil Restriction is the vector bundle over \( B \Sigma_n \) coming from the usual permutation representation \( \Sigma_n \circ \mathbb{A}^n \):
\[ R_{B \Sigma_n^{-1}/B \Sigma_n} = [\mathbb{A}^n/\Sigma_n]. \]
Diagram (2) becomes:
\[
\begin{cases}
|\mathbb{A}^n/\Sigma_n| \\
S \to S' \to B \Sigma_n
\end{cases}
\to
\begin{cases}
S' \to \mathbb{A}_S^1 \\
S \leftarrow
\end{cases}
\]
If we pull back the diagram along \( * \to B \Sigma_n \), we get
\[
\begin{cases}
\langle n \rangle \to \mathbb{A}_K^1 \\
* \leftarrow
\end{cases}
\]
Because \( \langle n \rangle \to * \) is the trivial cover, we have \( n \) sections of \( \mathbb{A}^1 \) over the base \( * \), represented by \( \mathbb{A}^n \). An equivalent description is as \( n \) ordered points in \( \mathbb{A}^1 \).

The group \( \Sigma_n \) acts on \( n \)-tuples of sections by acting on the source:
\[ \Sigma_n \circ \text{Hom}(\langle n \rangle, \mathbb{A}^1) \cong \mathbb{A}^n. \]
This translates to the usual action on the coordinates of \( \mathbb{A}^n \).

Identifying the Weil Restriction with \( [\mathbb{A}^n/\Sigma_n] \) in general requires working over an arbitrary base, arguing on the total space of the torsor by pulling back along \( * \to B \Sigma_n \), and checking things are equivariant.

The case of a point \( S = \text{Spec} K \) is more classical:

**Example 5.1 (\( S = \text{Spec} K \)).** Let \( L/K \) be a finite separable extension, i.e., an \( n \)-sheeted finite étale cover \( \text{Spec} L \to \text{Spec} K \) which is classified by a map \( \text{Spec} K \to B \Sigma_n \). The corresponding torsor is the sheaf of isomorphisms
\[ \text{Isom}_K(L, K^n) = \text{Isom}_{\text{Spec} K}(\langle n \rangle_K, \text{Spec} L) \]
of \( K \)-algebras. After base changing to a separable closure \( K^{\text{sep}} \) of \( K \) (or \( L \) if one wishes), we get \( L \otimes_K K^{\text{sep}} = (K^{\text{sep}})^n \) as algebras and \( \text{Spec} L \otimes_K K^{\text{sep}} = \langle n \rangle_{K^{\text{sep}}} \to \text{Spec} K^{\text{sep}} \) as schemes. Then \( L/K \) is étale locally an \( n \)-sheeted cover, so we see the Weil Restriction \( R_{L/K} \) is the pullback of the universal Weil Restriction \( R_{B \Sigma_n^{-1}/B \Sigma_n} = [\mathbb{A}^n/\Sigma_n] \). Such a pullback is simply \( \mathbb{A}^n_K \), just like Example 2.3.
Returning to the general discussion, the closed substack of non-monogenerators
\[ N_{1, B\Sigma_n - 1/B\Sigma_n} \subseteq R_{B\Sigma_n - 1/B\Sigma_n} = [\mathbb{A}^n/\Sigma_n] \]
is the quotient by \( \Sigma_n \) of the “fat diagonal” \( \hat{\Delta} \subseteq \mathbb{A}^n \) given by the union of all loci \( V(x_i - x_j) \subseteq \mathbb{A}^n \) where two coordinates are equal. This is checked after pulling back along \( * \to B\Sigma_n \), where the universal diagram is

\[
\begin{array}{ccc}
\langle n \rangle & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
* & \longrightarrow & B\Sigma_n.
\end{array}
\]

Identifying the diagram with \( n \)-tuples of sections of \( \mathbb{A}^1 \) over the base, some of the \( n \) sections coincide if and only if some coordinates are equal \( x_i = x_j \). Remark 2.18 affirms our intuition that such a map \( s \) is a closed immersion if and only if the images of two distinct points in \( \langle n \rangle \) are not the same point. In other words, \( M_{B\Sigma_n - 1/B\Sigma_n} \) is the space of ordered configurations of \( n \) points

\[ \text{Conf}_n (\mathbb{A}^1) = \{(x_1, \ldots, x_n) \mid x_i \neq x_j \text{ for } i \neq j \} \]
The stack quotient is likewise the \( n \)th unordered configuration space of points in \( \mathbb{A}^1 \):

\[ \text{UConf}_n (\mathbb{A}^1) = \{(x_1, \ldots, x_n) \mid x_i \neq x_j \text{ for } i \neq j \}/\Sigma_n. \]
The quotient by \( \Sigma_n \) can be seen as forgetting the order on \( n \) points in \( \mathbb{A}^1 \). Moreover, we have a pullback square

\[
\begin{array}{ccc}
\text{Conf}_n(\mathbb{A}^1) & \longrightarrow & \text{UConf}_n(\mathbb{A}^1) \\
\downarrow & & \downarrow \\
* & \longrightarrow & B\Sigma_n.
\end{array}
\]

The action of \( \Sigma_n \circ \mathbb{A}^n \) restricts to an action on \( \hat{\Delta} \), and

\[ N_{1, B\Sigma_n - 1/B\Sigma_n} = [\hat{\Delta}/\Sigma_n] \subseteq [\mathbb{A}^n/\Sigma_n]. \]
The fat diagonal \( \hat{\Delta} \) is exactly the locus of \( \mathbb{A}^n \) where \( \Sigma_n \) has stabilizers. The coarse moduli space of \( [\mathbb{A}^n/\Sigma_n] \) is precisely \( \mathbb{A}^n \) by the fundamental theorem of symmetric functions, with the composite

\[ \mathbb{A}^n \to [\mathbb{A}^n/\Sigma_n] \to \mathbb{A}^n; \quad \varphi = (x_1, \ldots, x_n) \mapsto (s_1(\varphi), s_2(\varphi), \ldots, s_n(\varphi)) \]
given by the elementary symmetric polynomials \( s_i(x_1, \ldots, x_n) \) [Art11, §16.1-2]. The composite sends a list of \( n \) roots to the coefficients of the monic polynomial of degree \( n \) vanishing at those roots, up to sign:

\[ (t - x_1) \cdots (t - x_n) = t^n - s_1(\varphi)t^{n-1} + s_2(\varphi)t^{n-2} - \cdots \pm s_n(\varphi). \]
The assignment is plainly equivariant in \( \Sigma_n \) relabeling the \( x_i \).

The map to the coarse moduli space \( [\mathbb{A}^n/\Sigma_n] \to \mathbb{A}^n \) is an isomorphism precisely over \( M_{B\Sigma_n - 1/B\Sigma_n} \). The image of \( N_{1, B\Sigma_n - 1/B\Sigma_n} \) in \( \mathbb{A}^n \) is the closed subscheme cut
out by the discriminant of the above polynomial
\[
\text{Disc} \left( \prod_{i=1}^{n} (t - x_i) \right) = D(s_1(\overline{x}), s_2(\overline{x}), \ldots, s_n(\overline{x})) = \prod_{i<j} (x_i - x_j)^2,
\]
the square of the Vandermonde determinant. The resulting divisor is the pushforward of \( N \) to the coarse moduli space \( \mathbb{A}^n \).

Consider for example, the case where \( n = 2, \Sigma_2 = \mu_2 \) and the map
\[
[A^2/\Sigma_2] \to \mathbb{A}^2
\]
is the 2nd root stack at the discriminant \( \text{Disc} = s_1^2 - 4s_2 \) that cuts out the fat diagonal. The discriminant matches the one that shows up in the quadratic formula:
\[
t^2 + s_1t + s_2 = 0 \leadsto t = \frac{-s_1 \pm \sqrt{s_1^2 - 4s_2}}{2}.
\]

The general stack quotient \([A^n/\Sigma_n] \to \mathbb{A}^n\) is a “2nd root stack” adding a \( \mu_2 \)-stabilizer along a single \( V(x_i - x_j) \), away from the intersection with the other vanishing loci [Cad07]. At the intersections, more stack structure is needed.

We summarize the above discussion for general targets \( X \) in the place of \( \mathbb{A}^1 \):

**Theorem 5.2.** Let \( X_{B\Sigma_n} = X \times B\Sigma_n \to B\Sigma_n \) be the stack quotient of a quasiprojective scheme by the trivial \( \Sigma_n \) action.

- The Weil restriction is the stacky symmetric product:
  \[
  \mathcal{R}_{X_{B\Sigma_n}, B\Sigma_n/B\Sigma_n} := [\text{Sym}^n X] = [X^n/\Sigma_n].
  \]

- The space of monogenerators for \( S' = B\Sigma_{n-1}, S = B\Sigma_n \) is precisely the \( n \)th unordered configuration space:
  \[
  m_{X, B\Sigma_{n-1}/B\Sigma_n} = \text{UConf}_n X := \{(x_1, \ldots, x_n) \mid x_i \neq x_j \text{ for } i \neq j\}/\Sigma_n.
  \]

- The complementary space of non-monogenerators is the stack quotient by \( \Sigma_n \) of the “fat diagonal” of \( n \) points in \( X \) which are not pairwise distinct:
  \[
  n_{X, B\Sigma_{n-1}/B\Sigma_n} = [\Delta_X/\Sigma_n] = \{(x_1, \ldots, x_n) \mid \text{some } x_i = x_j, i \neq j\}/\Sigma_n.
  \]

The rest of the section gives sample applications, exotic examples, and directions based on the correspondence with configuration spaces.

**Example 5.3.** Braid groups are extensions of symmetric groups that show up naturally in situations involving monodromy. They have representations everywhere from exceptional collections in derived categories to solutions of the Yang-Baxter equations in physics.

The étale fundamental groups of monogeneity spaces give extensions of the profinite completions of braid groups as in [Fur12, Remark 1.14]. Take \( \ast = \text{Spec } \mathbb{Q} \), \( X = \mathbb{A}^1 \), and \( S' \to S \) the universal étale \( n \)-sheeted cover \( B\Sigma_{n-1} \to B\Sigma_n \).

We claim \( M_{\mathbb{A}^k} \) is connected for \( k \geq 1 \). It is the complement of a union of \( \mathbb{C} \)-hyperplanes \( V(x_i - x_j) \subset \mathbb{A}^nk \) which are \( \mathbb{R} \)-codimension 2, hence connected [Con]. In particular, \( M_X \) is geometrically connected. We have a short exact sequence of étale fundamental groups [Stacks, 0BTX]:
\[
1 \to \pi_1(M_{\mathbb{A}^k}) \to \pi_1(M_X) \to \pi_1(\text{Spec } \mathbb{Q}) \to 1.
\]
Extending along an embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) does not affect the fundamental group, and the étale fundamental group over \( \mathbb{C} \) is the profinite completion of the ordinary topological fundamental group:

\[
\pi_1(M_{X_{\overline{\mathbb{Q}}}}) = \pi_1(M_{X_\mathbb{C}}) = \pi_1^\text{top}(M_{X}(\mathbb{C})^{an})^\wedge.
\]

The topological fundamental group of \( M_{X}(\mathbb{C}) = \text{Conf}_n(\mathbb{A}^1) \) is famously [Knu18, §1.2] the braid group \( B_n \) on \( n \) strings generated by \( \sigma_1, \ldots, \sigma_n \) with relations

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| > 1.
\]

Write \( \hat{B}_n \) for its profinite completion. The fundamental group \( \pi_1(\text{Spec } \mathbb{Q}) = G_\mathbb{Q} \) is the absolute Galois group of \( \mathbb{Q} \).

Choose a section \( \sigma \) of \( X_{\mathbb{Q}} \to \text{Spec } \mathbb{Q} \). The section induces a splitting of the sequence (6) and a continuous conjugation action

\[
G_\mathbb{Q} \rtimes \hat{B}_n; \quad g.b := \sigma(g)b\sigma(g)^{-1}.
\]

As discussed in [Fur12], the action of \( G_\mathbb{Q} \) extends to an action of the Grothendieck-Teichmüller group \( \hat{GT} \). Conjecturally, \( G_\mathbb{Q} = \hat{GT} \).

**Question 5.4.** Fulton and MacPherson defined a natural compactification of configuration spaces by blowing up the locus in \( \mathbb{A}^n \) where the points come together in a specific way [FM94]. Their compactification has a natural \( \Sigma_n \) action and descends to a blowup of \( R_{\Sigma_n-1}/\Sigma_n \). One could make the same definitions and apply their computations of rational cohomology to monogeneity. Can their computations tell us whether specific extensions are monogenic? Is there a version when \( S' \to S \) is not étale? Can it capture the operadic nature of the usual configuration spaces over \( \mathbb{R} \) in an algebraic setting?

The stacks \( R_{1,\Sigma_n-1}/\Sigma_n \) we study arise naturally in log geometry as “Artin fans” [Abr+14].

**Example 5.5 (Thanks to Sam Molcho).** Consider the stack quotient

\[
[(\mathbb{A}^n/\mathbb{G}_m^n)/\Sigma_n]
\]

of \( R \) by the image of the dense torus, an Artin fan [Abr+14]. The quotient is the Artin fan of the stack \( [\mathbb{A}^3/\Sigma_3] \), but we can describe a log scheme \( Y \) for \( n = 3 \) with the same Artin fan.

Let \( Z = \mathbb{G}_m^3 \times \mathbb{A}^3 = \text{Spec } k[t^{\pm 1}, u^{\pm 1}, v^{\pm 1}, x, y, z] \) over a field \( k \) not of characteristic 2 and containing a primitive 3rd root of unity \( \omega \in k \). Write \( G = \Sigma_3 \) and define an action \( G \circ Z \) by

\[
(12).(t, u, v, x, y, z) := (u, t, -v, y, x, z)
\]

\[
(123).(t, u, v, x, y, z) := (\omega \cdot t, \omega^2 \cdot u, v, z, x, y).
\]

Check the action of (12) has order 2, (123) has order 3, and they satisfy the relation

\[
(12). (123). (12). (123). (t, u, v, x, y, z) = (t, u, v, x, y, z)
\]

to induce a free action of \( \Sigma_3 \). One can replace \( \mathbb{G}_m^3 \) with any free action of \( \Sigma_3 \) instead.

Endow \( Z \) with the natural log structure on \( \mathbb{A}^3 \) and none on \( \mathbb{G}_m^3 \), i.e., the divisorial log structure from \( x, y, z \). Define \( Y := Z/G \), with descended log structure. To
compute its Artin fan, endow \( \mathbb{A}^3 \) with the natural \( \Sigma_3 \) action and note the projection \( Z \to \mathbb{A}^3 \) is \( G \)-equivariant. The Artin fan of \( Y \) is that of \( [\mathbb{A}^3/G] \), our \( \mathcal{R} \) above.

For larger \( n \), use a similar presentation of \( \Sigma_n \) and the natural action on \( \mathbb{A}^n \), entwined with a free action on some \( \mathbb{G}_m^k \). The Artin cones for \( n = 3 \) show up naturally as the top-dimensional cone of the moduli space of tropical genus-one 3-pointed curves \( \overline{M}_{1,3}^{\text{top}} \) corresponding to the triangle with marked points on each vertex.

**Example 5.6** (Moduli spaces of curves in genus 0). Let \( M_{0,n} \) be the moduli stack of smooth curves of genus 0 (i.e. \( \mathbb{P}^1 \)) with \( n \) marked points. The evident [Wes11] isomorphism with a quotient of configuration space gives:

\[
[M_{g,n}/\mathbb{C}/\text{PGL}_2] \simeq [\text{Conf}_n(\mathbb{P}^1)/\text{PGL}_2] \simeq M_{0,n}.
\]

One can always put the first point at \( \infty \) and get equivalent descriptions:

\[
M_{0,n} \simeq [M_{g,1}/\mathbb{C}/\text{Aff}^1] \simeq [\text{Conf}_n(\mathbb{C})/\text{Aff}^1],
\]

where \( \text{Aff}^1 \) is the group of affine transformations \( \mathbb{G}_m \ltimes \mathbb{A}^1 \). The stack quotient classifies local affine equivalence classes of monogenerators, as detailed in Section 6.

One can likewise obtain the other moduli spaces of curves by an ad hoc construction. Consider \( \mathfrak{U} \to \mathfrak{M} \) the universal connected, proper, genus-\( g \) nodal curve, its relative smooth locus \( \mathfrak{U}^{sm} \subseteq \mathfrak{U} \), and the monogeneity stack

\[
M_{\mathfrak{U}^{sm},(n)_{\geq 2}/\mathfrak{M}}
\]

of the trivial cover \( (n) \) over the moduli space \( \mathfrak{M} \). The monogeneity stack is naturally isomorphic to the space of nodal, \( n \)-marked curves \( \mathfrak{M}_{g,n} \). One can also obtain the open substack of stable curves as the universal Deligne-Mumford locus \( \overline{\mathfrak{M}}_{g,n} \subseteq \mathfrak{M}_{g,n} \).

**Example 5.7** (Torsors for finite groups). Let \( k \) be a separably closed field and \( G \) a finite group, yielding a group scheme \( \bigsqcup_{g \in G} \text{Spec} k \) over \( k \). A \( G \)-torsor \( S' \to S \) is, in particular, a finite \( \text{étale} \) map of degree \( n = \#G \) admitting the above description. Notice that the action of \( G \) on \( S' \) induces an action of \( G \) on \( M_{S'/S} \). The map \( S' \to S \) is classified by a map \( S \to BG \), the stack of \( G \)-torsors \( \mathbf{A}.4 \). The left regular representation \( G \circ G \) gives an inclusion \( G \subseteq \Sigma_n \) upon ordering the set \( G \). The induced representable map \( BG \to B\Sigma_n \) is essentially independent of the ordering as in Proposition \( \mathbf{A}.9 \). The classifying map \( S \to B\Sigma_n \) is the composite \( S \to BG \to B\Sigma_n \) with the left regular representation. The universal family is \( * \to BG \), pulled back from \( B\Sigma_{n-1} \to B\Sigma_n \). Using \( BG \) instead of \( B\Sigma_n \) refines the transition functions and is closer to the original approach of E. Galois, since “groups” were then only defined as subsets of \( \Sigma_n \).

A similar description locally holds for other finite \( \text{étale} \) group schemes. For merely finite flat group schemes \( G \) such as \( \alpha_p, \mu_p \) in characteristic \( p \), the group action on the monogeneity space of \( G \)-torsors \( S' \to S \) still holds but the local decomposition \( S' = \bigsqcup S \) and \( \Sigma_n \) action do not.

**Example 5.8** (Explicit Elliptic Curve Monogeneity). Example 5.7 applies to isogenies of elliptic curves over a field \( K \). Let \( \varphi : E_1 \to E_2 \) be a separable isogeny of degree \( N \), with \( S' = E_1, S = E_2 \). The map \( \varphi \) is a torsor for \( \text{ker}(\varphi) \). We want
monogenerators

\[ E_1 \xrightarrow{\theta} \mathbb{A}^1_{E_2}, \]

\[ \varphi \quad \xrightarrow{\cdot} \quad E_2 \]

but all global maps \( E_1 \to \mathbb{A}^1_{E_2} \) are constant on the fibers of \( \mathbb{A}^1_{E_2} \to E_2 \). We will have to look for local monogenerators instead.

A good source of such monogenerators are the rational functions on \( E_1 \). Recall that \( \varphi \) induces an Galois extension \( K(E_2) \subseteq K(E_1) \) with Galois group \( \ker \varphi \). An element \( s \in K(E_1) = \mathcal{R} \text{Spec}(K(E_2)) \) is a monogenerator over \( K(E_2) \) if and only if it lies in no proper subextension, i.e., if it is not invariant under any non-trivial subgroup or element of \( \ker \varphi \). Such an \( s \) extends to a regular function \( s \in \mathcal{O}_{E_1}(U) \) on some open subset \( U \subseteq E_1 \). We can assume \( U = \varphi^{-1}V \) is pulled back from some \( V \subseteq E_2 \). Since \( M_{1,E_1/E_2} \) is open in \( \mathcal{R} \) and \( s \mid \text{Spec}(K(E_2)) \in M_1 \), we may shrink \( V \) to ensure \( s \in M_1(V) \).

Apply our results to a concrete situation. Consider the elliptic curves over \( \mathbb{Q} \) defined by the affine equations

\[ E_1: y^2 = x^3 - 9x + 9 \]

\[ E_2: y^2 = x^3 - 189x - 999. \]

The rational map

\[ \varphi(x, y) = \left( \frac{x^3 - 6x^2 + 45x - 72}{x^2 - 6x + 9}, \frac{x^3y - 9x^2y - 9xy + 9y}{x^3 - 9x^2 + 27x - 27} \right) \]

gives an isogeny \( \varphi : E_1 \to E_2 \) with \( \deg(\varphi) = 3 \) and \( \ker(\varphi) = \{ \infty_{E_1}, (3, 3), (3, -3) \} \).

The isogeny gives a torsor in the sense of Example 5.7 via the group action given by adding points of the kernel. Writing

\[ A(x, y) := \frac{x^3y - y^3 - 27x^2y + 9y^2 + 162x + 27y - 216}{(x - 3)^3}, \]

the kernel acts via

\[ (3, 3)_{E_1} (x, y) = \left( \frac{(y - 3)^2 - (x + 3)(x - 3)^2}{(x - 3)^2}, A(x, y) \right) \]

\[ (3, -3)_{E_1} (x, y) = \left( \frac{(y - 3)^2 - (x + 3)(x - 3)^2}{(x - 3)^2}, -A(x, y) \right). \]

We claim that the rational function \( y \) on \( E_1 \) defines a monogenerator for \( E_1 \to E_2 \) over an open subset \( V \subseteq E_2 \). To see that \( y \) works on some open subset, all we have to note is that \( y \) is not fixed by the action of \( \ker \varphi \) on \( K(E_1) \), since \( y \neq A(x, y) \).

Better, we can compute \( V \) explicitly. Which points of \( E_2 \) need to be thrown out? The function \( y \) fails to define a closed immersion where

1. \( y \) has a pole,
2. there is a pair of points \((x_0, y_0), (x_1, y_1) \in E_1 \) with the same \( y \)-value \( y_0 = y_1 \) and the same \( \varphi \)-value \( \varphi(x_0, y_0) = \varphi(x_1, y_1) \).

The points of \( E_2 \) which it is necessary to remove are therefore

1. the point at infinity of \( E_2 \) and
the images of the points \((x, y) \in E_1\) satisfying one of the following pairs of equations:

\[
y = A(x, y), \quad y^2 = x^3 - 9x + 9,
\]

or

\[
y = -A(x, y), \quad y^2 = x^3 - 9x + 9.
\]

Using a computer algebra system, we solve the first pair of equations: the solutions to \(y = A(x, y)\) are \(y = 3\), \(y = 12 - 3x\), \(y = 3x - 6\), with \(x \neq 3\), and plugging these possibilities into the equation for \(E_1: y^2 = x^3 - 9x + 9\), we obtain the points:

\((0, 3), (-3, 3), (3 - 6i, 3 + 18i), (3 + 6i, 3 - 18i)\).

The solutions to the second pair of equations are less tidy; they are approximately

\((2.43, 1.22), (10.48, -32.66), (-2.36 - 1.41i, -5.62 + 0.72i),
\)

\((-2.36 + 1.41i, -5.62 - 0.72i), (0.40 - 1.78i, 3.35 + 3.11i), (0.40 + 1.78i, 3.35 - 3.11i)\).

We have a complete finite set of points (defined over a finite extension of \(\mathbb{Q}\)) to remove from \(E_1\) in order to obtain a closed immersion. To obtain \(V\), we take the complement of the images of these points in \(E_2\).

5.2. **When is a map étale?** We recall from [Poo06, §6] that a map \(S' \to S\) is étale precisely when the discriminant of the algebra does not vanish. Below we employ the algebraic moduli stack \(\mathfrak{A}_n\) of finite locally free algebras and the affine scheme of finite type \(\mathfrak{B}_n\) parametrizing such algebras together with a choice of global basis \(\mathcal{O} \simeq \bigoplus \mathcal{O}_S \cdot e_i\), both defined in § 2.2.

Suppose \(\pi: S' \to S\) comes from a finite flat algebra \(\mathcal{O}\) with a global basis \(\varphi: \mathcal{O} \simeq \bigoplus_{i=1}^n \mathcal{O}_S \cdot e_i\), corresponding to a map \(S \to \mathfrak{B}_n\). There is a trace pairing \(\text{Tr}: \mathcal{O} \to \mathcal{O}_S\) which we can use to define the discriminant:

\[
\text{Disc}(\mathcal{O}, \varphi) := \det [\text{Tr}(e_i e_j)] \in \Gamma(\mathcal{O}_S)
\]

Changing \(\varphi\) changes the function \(\text{Disc}\) by a unit. The function \(\text{Disc}\) does not descend to \(\mathfrak{A}_n\), but the vanishing locus \(V(\text{Disc}) \subseteq \mathfrak{B}_n\) does. Writing \(\mathfrak{B}_n^{et}, \mathfrak{A}_n^{et}\) for the open complements of the vanishing locus \(V(\text{Disc})\), a map \(\pi: S' \to S\) is étale if and only if \(S \to \mathfrak{A}_n^{et}\) factors through the open substack \(\mathfrak{A}_n^{et} \subseteq \mathfrak{A}_n\) [Poo06, Proposition 6.1].

**Remark 5.9.** Most finite flat algebras are *not* étale, nor are they degenerations of étale algebras. B. Poonen shows the moduli of étale algebras inside of all finite flat algebras \(\mathfrak{B}_n^{et} \subseteq \mathfrak{A}_n\) cannot be dense by computing dimensions [Poo06, Remark 6.11]. The closure \(\overline{\mathfrak{A}_n^{et}}\) is nevertheless an irreducible component.

What if \(S' \to S\) is not étale? Readers familiar with [Cos06] know one can sometimes endow \(S\) and \(S'\) with stack structure \(\tilde{S}\) and \(\tilde{S}'\) at the ramification to make \(\tilde{S}' \to \tilde{S}\) étale in the ramified case. Then all \(\tilde{S}' \to \tilde{S}\) are \(\Sigma_n\)-torsors, and not just unramified covers \(S' \to S\). We sketch these ideas over \(\mathbb{C}\). The ideas in Section 5.1 apply in this level of generality.

Consider

\[
y^2 = x(x - 1)(x - \lambda),
\]

for some \(\lambda \in \mathbb{C}\). If \(C := \mathbb{P}^1\) and \(C'\) is the projective closure of the above affine equation, the projection \((x, y) \to x\) extends to a finite locally free map \(\pi: C' \to C\). This is in Situation 2.1 so our definitions make sense for it. However \(\pi\) is ramified.
at four points, preventing us from interpreting its monogeneity space using the perspective of this section. Nevertheless, we may observe that the function $y$ gives a section of $M_{1,C'/C}$ over $C \setminus \infty$. The section naturally extends to a section of $\mathcal{R}_{P^1,C'/C}$ over all of $C$.

Let $X := \mathbb{P}^1$. If we work over $\mathbb{C}$ and endow $C'$ and $C$ with stack structure to obtain a finite étale cover of stacky curves $\tilde{C}' \to \tilde{C}$ as in [Cos06], the stacky finite étale cover together with the map $\tilde{C}' \to X$ is parameterized by a representable map $\tilde{C} \to [\text{Sym}^n X]$ to the stack quotient $[\text{Sym}^n X] := [X^n/\Sigma_n]$.

We can similarly allow $C'$ and $C$ to be nodal families of curves over some base $S$. Maps from nodal curves $\tilde{C}$ over $S$ entail an $S$-point of the moduli stack $\mathcal{M}(\text{Sym}^n X)$ of prestable maps to the symmetric product. As in Proposition 2.4, there is an open substack for which the map from the coarse space $C' \to X$ is a closed immersion.

The stack $\mathcal{M}(\text{Sym}^n X)$ splits into components indexed by the ramification profiles of the cover of coarse spaces $C' \to C$.

There are some subtleties in characteristic $p$ – one cannot treat all ramification as a $\mu_n$ torsor because some ramification is a $\mathbb{Z}/p\mathbb{Z}$-torsor in characteristic $p$. The formalism of tuning stacks [ESZ21] is a substitute in arbitrary characteristic.

6. Twisted Monogeneity

The Hasse local-to-global principle is the idea that “local” solutions to a polynomial equation over all the $p$-adic fields $\mathbb{Q}_p$ and the real field $\mathbb{R}$ can piece together to a single “global” solution over $\mathbb{Q}$. We ask the same for monogeneity: given local monogenerators, say over completions or local in the Zariski or étale topologies, do they piece together to a single global monogenerator?

The Hasse principle fails for elliptic curves. Let $E$ be an elliptic curve over a number field $K$ and consider all its places $\nu$. The Shafarevich-Tate group $\Sha(E/K)$ of an elliptic curve sits in an exact sequence

$$0 \to \Sha(E/K) \to H^1_{et}(K, E) \to \prod_{\nu} H^1_{et}(K_{\nu}, E).$$

Elements of $\Sha$ are genus-one curves with rational points over each completion $K_{\nu}$ that do not have a point over $K$. Similarly, we want sequences of cohomology groups to control when local monogenerators do or do not come from a global monogenerator.

For such a sequence, one needs to know how a pair of local monogenerators can differ. One would like a group $G$ or sheaf of groups transitively acting on the set of local monogenerators so that cohomology groups can record the struggle to patch local monogenerators together into a global monogenerator.

Suppose $B/A$ is an algebra extension inducing $S' \to S$ and $\theta_1, \theta_2 \in B$ are both monogenerators. Then

$$\theta_1 \in B = A[\theta_2], \quad \theta_2 \in B = A[\theta_1],$$

so each monogenerator is a polynomial in the other:

$$\theta_1 = p_1(\theta_2) \quad \text{and} \quad \theta_2 = p_2(\theta_1), \quad \text{with} \quad p_1(x), p_2(x) \in A[x].$$
We can think of the $p_i(x)$ as transition functions or endomorphisms of the affine line $\mathbb{A}^1$. Even though $p_i(p_2(\theta_1)) = \theta_1$, it is doubtful that $p_1 \circ p_2 = \operatorname{id}_{\mathbb{A}^1}$ or even that $p_i(x)$ are automorphisms of $\mathbb{A}^1$.

One remedy would be to take $G$ to be the group completion of the noncommutative monoid $\operatorname{End}^g(\mathbb{A}^1)$ of nonzero endomorphisms of $\mathbb{A}^1$ under composition. Each polynomial $p_i(x)$ has image in the group completion. Unfortunately, this group is terribly complicated. It is generally undecidable whether the group completion of a noncommutative monoid is nonzero. The degree homomorphism shows polynomial transition functions that $p_i(x)$, but we are not able to say anything meaningful about the kernel. Moreover, the map from $\operatorname{End}^g(\mathbb{A}^1)$ to the group completion is non-injective: for example, the images of $x$ and $-x$ must coincide, since the composite of $x^2$ with both polynomials is $x^2$.

Instead of working with the most general group as above, we require our transition functions $p_i(x)$ to lie in a group $G \circlearrowright \mathbb{A}^1$ acting on $\mathbb{A}^1$. We focus on the two specific group sheaves with group laws

$$
\mathbb{G}_m(A) = A^*, \quad u \cdot u' := uu'
$$

$$
\operatorname{Aff}^1(A) = A^* \rtimes A, \quad (u, v) \cdot (u', v') := (uu', uu' + v).
$$

Affine transformations $\operatorname{Aff}^1$ are essentially polynomials $ux + v$ of degree one under composition. These act on monogenerators:

$$
\mathbb{G}_m \circlearrowright \mathbb{A}^1 : a \in A^*, \theta \in M(A), \quad a, \theta := a \cdot \theta,
$$

$$
\operatorname{Aff}^1 \circlearrowright \mathbb{A}^1 : a \in A^*, b \in A, \theta \in M(A), \quad (a, b) \cdot \theta := a\theta + b.
$$

**Definition 6.1** (Twisted Monogenerators). A $(\mathbb{G}_m)$-twisted monogenerator for $B/A$ is:

1. a Zariski open cover $\operatorname{Spec} A = \bigcup_i D(f_i)$ for elements $f_i \in A$,
2. a system of “local” monogenerators $\theta_i \in B[\frac{1}{f_i-1}]$ for $B[\frac{1}{f_i}]$ over $A[\frac{1}{f_i}]$,
3. units $a_{ij} \in A[\frac{1}{f_i-1}, \frac{1}{f_j-1}]^*$

such that

- for all $i, j$, we have $a_{ij} \theta_j = \theta_i$,
- for all $i, j, k$, the “cocycle condition” holds:

$$
a_{ij} a_{jk} = a_{ik}.
$$

Two such systems $\{(a_{ij}), (\theta_i)\}, \{(a'_{ij}), (\theta'_i)\}$ are equivalent if they differ by further refining the cover $\operatorname{Spec} A = \bigcup D(f_i)$ or global units $u \in A^*$: $u \cdot a_{ij} = a'_{ij}, u \cdot \theta_i = \theta'_i$.

Likewise $B/A$ is $\operatorname{Aff}^1$-twisted monogenic if there is a cover with $\theta_i$’s as above, but with units (3) replaced by pairs $a_{ij}, b_{ij} \in A[\frac{1}{f_i}, \frac{1}{f_j}]$ such that each $a_{ij}$ is a unit and $a_{ij} \theta_j + b_{ij} = \theta_i$.

The elements $\theta_i$ may or may not come from a single global monogenerator $\theta \in A$. Nevertheless, the transition functions $(a_{ij})$ or $(a_{ij}, b_{ij})$ define an affine bundle $L$ on $\operatorname{Spec} A$ with global section $\theta$ induced by the $\theta_i$’s. We say $S'/S$ is “twisted monogenic” to mean there exists a $\mathbb{G}_m$-twisted monogenerator and similarly say “$\operatorname{Aff}^1$-twisted” monogenic. Both are clearly Zariski-locally monogenic.

Compare with Cartier divisors:
Proposition 6.15: \( B/A \) is \( \mathbb{G}_m \)-twisted monogenic if and only if it is \( \text{Aff}^1 \)-twisted monogenic.

Theorem 6.20: The class number of a number ring \( \mathbb{Z}_K \) is one if and only if all twisted monogenic extensions of number rings \( \mathbb{Z}_L/\mathbb{Z}_K \) are in fact monogenic.

Remark 6.13: There is a local-to-global sequence relating affine equivalence classes of monogenerators with global monogenerators as above.

Theorem 6.2: There are moduli spaces of \( \mathbb{G}_m \) and \( \text{Aff}^1 \)-twisted monogenerators.

Theorem 6.17: There are finitely many twisted monogenerators up to equivalence.

We warm up with a classical approach to \( \mathbb{G}_m \)-quotients, namely taking Proj. Then we study \( \text{Aff}^1 \)-twisted monogenerators before finally introducing \( G \)-twisted monogenerators for arbitrary groups \( G \).

There is a moduli space for each notion of twisted monogeneity. We use these moduli spaces now and defer the proof until Theorem 6.27:

\[ \text{Theorem } 6.2 \quad (=\text{Theorem } 6.27): \] Let \( \mathbb{G}_m, \text{Aff}^1 \) act on \( \mathbb{A}^1 \) on the left in the natural way, inducing a left action on \( \mathcal{M} \). The stack quotients \( [\mathcal{M}/\mathbb{G}_m] \) and \( [\mathcal{M}/\text{Aff}^1] \) represent \( \mathbb{G}_m \)- and \( \text{Aff}^1 \)-twisted monogenerators up to equivalence, respectively.

### 6.1. \( \mathbb{G}_m \)-Twisted Monogenerators and Proj of the Weil Restriction.

Writing \( S' = \text{Spec } B \) and \( S = \text{Spec } A \), a twisted monogenerator amounts to a Zariski cover \( S = \bigcup U_i \), a system of closed embeddings \( \theta_i : S_{U_i} \subseteq \mathbb{A}^1_{U_i} \), over \( U_i \), and elements \( a_{ij} \in \mathbb{G}_m(U_i) \) such that

\[ a_{ij} \cdot \theta_j = \theta_i : S'_{U_{ij}} \to \mathbb{A}^1_{U_{ij}}. \]

Equivalently, a twisted monogenerator is a line bundle \( L \) on \( S \) defined by the above cocycle \( a_{ij} \) and a global embedding \( \theta : S' \subseteq L \) over \( S \). Twisted monogenerators are identified if they differ by global units \( u \in \mathbb{G}_m(S) \) or refinements of the cover \( U_i \), i.e., if the corresponding line bundles \( L, L' \) are isomorphic in a way that identifies the closed embeddings \( \theta, \theta' \).

For number fields \( L/K \) with \( \theta \in \mathbb{Z}_L \) and \( a \in \mathbb{Z}_K \), one has \( a\theta \in \mathbb{Z}_L \). If \( a \in \mathbb{Z}_K^* \), then \( \theta \) is a monogenerator if and only if \( a\theta \) is. The multiplication action \( \mathbb{G}_m(\mathbb{Z}_K) \circ \mathcal{R}(\mathbb{Z}_K) \) corresponds to the global \( \mathbb{G}_m \) action on the vector bundle \( \mathcal{R} \) over \( S \).

An action of \( \mathbb{G}_m \) corresponds to a \( \mathbb{Z} \)-grading on the sheaf of algebras [Stacks, 0EKJ]. Locally in \( S \), \( \pi_* \mathcal{O}_{S'} \simeq \bigoplus^n \mathcal{O}_S \cdot e_i \) and \( \mathcal{R} \simeq \mathbb{A}^n \). The \( \mathbb{G}_m \) action
is the diagonal action and corresponds to the total degree of polynomials in \( \mathcal{O}_{\Delta_S^g} = \mathcal{O}_S[x_1, \ldots, x_n] \).

The associated projective bundle to the vector bundle \( \mathcal{R} \) is given by the relative \( \text{Proj} \) \cite{stacks-project, 01NS}

\[
\mathbb{P}\mathcal{R} := \text{Proj}_S \mathcal{O}_\mathcal{R},
\]

with the total-degree grading. The ideal \( \mathcal{I}_{\mathcal{S}'/\mathcal{S}} \) cutting out the complement \( \mathcal{N}_{1,\mathcal{S}'/\mathcal{S}} \subseteq \mathcal{R} \) is graded by Remark 3.11, defining a closed subscheme \( \mathbb{P}\mathcal{N} \subseteq \mathbb{P}\mathcal{R} \).

**Definition 6.3.** Define the scheme of projective monogenerators

\[
\mathbb{P}\mathcal{M}_{\mathcal{S}'/\mathcal{S}} \subseteq \mathbb{P}\mathcal{R}_{\mathcal{S}'/\mathcal{S}} := \text{Proj}_{\mathcal{O}_{\mathcal{R}_{\mathcal{S}'/\mathcal{S}}}}
\]

to be the open complement of the closed subscheme \( \mathbb{P}\mathcal{N}_{\mathcal{S}'/\mathcal{S}} \) cut out by the graded homogeneous ideal \( \mathcal{I}_{\mathcal{S}'/\mathcal{S}} \).

The reader may define projective polygenerators in the same fashion.

**Lemma 6.4.** The vanishing of the irrelevant ideal \( V(\mathcal{O}_{\mathcal{R},+}) \) of \( \mathcal{R}_{\mathcal{S}'/\mathcal{S}} \) is contained inside of the non-monogeneity locus \( \mathcal{N}_{\mathcal{S}'/\mathcal{S}} \) for \( \mathcal{S}' \neq \mathcal{S} \).

**Proof.** Locally, the lemma states that \( \theta = 0 \) is not a monogenerator. \( \square \)

**Remark 6.5.** We relate the \( \text{Proj} \) construction to stack quotients by \( \mathbb{G}_m \) according to \cite[Example 10.2.8]{stacks-project}. The ring \( \mathcal{O}_{\mathcal{R}_{\mathcal{S}'/\mathcal{S}}} \) is generated by elements of degree one.

Locally, \( \mathcal{R}_{\mathcal{S}'/\mathcal{S}} \simeq \mathcal{A}_S^n \) and \( \mathcal{O}_{\mathcal{R}_{\mathcal{S}'/\mathcal{S}}} \simeq \mathcal{O}_S[x_1, \ldots, x_n] \) is generated by the degree one elements \( x_i \). Write \( \text{Spec}_S \mathcal{O}_{\mathcal{R}} \) for the relative spectrum \cite{stacks-project, 01LQ}. The map

\[
\text{Spec}_S \mathcal{O}_{\mathcal{R}} \setminus V(\mathcal{O}_{\mathcal{R},+}) \to \text{Proj}_S \mathcal{O}_{\mathcal{R}}
\]

is therefore a stack quotient or \( \mathbb{G}_m \)-torsor.

We have a pullback square

\[
\begin{array}{ccc}
M_{\mathcal{S}'/\mathcal{S}} & \rightarrow & \text{Spec}_S \mathcal{O}_{\mathcal{R}} \setminus V(\mathcal{O}_{\mathcal{R},+}) \\
\downarrow & & \downarrow \\
\mathbb{P}M_{\mathcal{S}'/\mathcal{S}} & \rightarrow & \text{Proj}_S \mathcal{O}_{\mathcal{R}}
\end{array}
\]

of \( \mathbb{G}_m \)-torsors and a stack quotient \( \mathbb{P}M_{\mathcal{S}'/\mathcal{S}} = [M_{\mathcal{S}'/\mathcal{S}}/\mathbb{G}_m] \).

Theorem 6.2 states that \( [M/\mathbb{G}_m] \) represents twisted monogenerators, and now we know the quotient stack is wondrously a scheme:

**Corollary 6.6.** The scheme \( \mathbb{P}M_{\mathcal{S}'/\mathcal{S}} = [M/\mathbb{G}_m] \) represents the \( \mathbb{G}_m \)-twisted monogenerators of Definition 6.1. That is, \( \mathbb{P}M_{\mathcal{S}'/\mathcal{S}} \) is a moduli space for twisted monogenerators. The action \( \mathbb{G}_m \circ M \) is free.

**Warning 6.7.** Given a monogenerator \( \theta \in \mathbb{Z}_L \) and a pair \( a \in \mathbb{Z}_K^*, \beta \in \mathbb{Z}_L \), write

\[
\beta = b_0 + b_1 \theta + \cdots + b_{n-1} \theta^{n-1}.
\]

One may try to define a second action

\[
a.\beta := b_0 + b_1 a \theta + b_2 a^2 \theta^2 + \cdots + b_{n-1} a^{n-1} \theta^{n-1}
\]
encoding the degree with respect to \( \theta \), but this action does not define a grading as it is almost never multiplicative. For example, take \( \mathbb{Z}_L = \mathbb{Z}[\sqrt{2}] \) with monogenerator \( \sqrt{2} \) over \( \mathbb{Z}_K = \mathbb{Z} \). Then
\[
a.2 = 2 \neq a.\sqrt{2} \cdot a.\sqrt{2}.
\]
Jet spaces are the only example where \( a.x \) is multiplicative and induces a second action \( \mathbb{G}_m \circ \mathcal{R}_{S'/S} = J_{A^1,m} \). The two actions of \( \lambda \in \mathbb{G}_m \) on a jet
\[
f(\varepsilon) = a_0 + a_1 \varepsilon + \cdots + a_m \varepsilon^m
\]
on \( \mathbb{A}^1 \) are \( \lambda f(\varepsilon) = \lambda \cdot f(\varepsilon) \) and \( \lambda f(\varepsilon) = f(\lambda \varepsilon) \). The Proj of \( J_{X,m} \) with respect to this second \( \mathbb{G}_m \) action is known as a “Demailly-Semple jet” or a “Green-Griffiths jet” in the literature [Voj04, Definition 6.1]. For certain \( S' \to S \), there may be a distinguished one-parameter subgroup, i.e., the image of \( \mathbb{G}_m \to \text{Aut}_S(S') \), that results in a second action \( \mathbb{G}_m \circ \mathcal{R}_{S'/S} \) and allows an analogous construction.

6.2. Aff\(^1\)-Twisted Monogenerators and Affine Equivalence. We enlarge our study to Aff\(^1\)-twisted monogenerators and the related study of affine equivalence classes of ordinary monogenerators. We delay twisting by general sheaves of groups other than \( \mathbb{G}_m \) and Aff\(^1\) until the next section. For an \( S\)-scheme \( X \), the automorphism sheaf \( \text{Aut}(X) \) is the subsheaf of automorphisms in \( \text{Hom}_S(X,X) \).

**Remark 6.8.** The automorphism sheaf \( \text{Aut}_S(\mathbb{A}^1) \) has a subgroup of affine transformations Aff\(^1\) under composition. These are identified in turn with \( \mathbb{A}^1 \rtimes \mathbb{G}_m \) via
\[
(a,b) \mapsto (x \mapsto bx + a).
\]
The automorphism sheaf can be much larger for other \( \mathbb{A}^k \). For example,
\[
(x,y) \mapsto (x + y^3, y)
\]
is an automorphism of \( \mathbb{A}^2 \).

The automorphism sheaf \( \text{Aut}(\mathbb{A}^1) \) is not the same as Aff\(^1\), though they have the same points over reduced rings. See [Dup13] for some discussion over nonreduced rings.

**Definition 6.9.** The group sheaf \( \text{Aff}^k \subseteq \text{Aut}(\mathbb{A}^k) \) of affine transformations is the set of functions
\[
\vec{x} \mapsto M\vec{x} + \vec{b}
\]
where \( M \in \text{GL}_k \) and \( \vec{b} \in \mathbb{A}^k \), under composition. Note \( \text{Aff}^k \simeq \mathbb{A}^k \times \text{GL}_k \).

There is a sub-group scheme \( \mathbb{A}^k \times \mathbb{G}_m \subseteq \text{Aff}^k \) cut out by the condition that \( M \) is a unit multiple of the identity matrix. The sub-group scheme sits in a short exact sequence
\[
1 \to \mathbb{A}^k \times \mathbb{G}_m \to \text{Aff}^k \to \text{PGL}_k \to 1.
\]

Recall that two monogenerators \( \theta_1, \theta_2 \) of an \( A \)-algebra \( B \) are said to be equivalent if
\[
\theta_1 = u\theta_2 + v,
\]
where \( u \in A^* \) and \( v \in A \). Likewise, say that two embeddings \( \theta_1, \theta_2 : S' \to L \) of \( S' \) into an Aff\(^1\) bundle \( L \) over \( S \) are equivalent if there is in \( f \in \text{Aff}^1(S) \) such that \( \theta_1 = f.\theta_2 \). The set of monogenerators up to equivalence is then
\[
\Gamma(S,M)/\Gamma(S,\text{Aff}^1).
\]

If \( S' \xrightarrow{\sim} S \) is an isomorphism and \( n = 1 \), the action of Aff\(^1\) is trivial. Otherwise, the Aff\(^1\)-action is often free:
Theorem 6.10. The action $\text{Aut}(S') \circ M$ has trivial stabilizers. If $S$ is normal and $S' \to S$ is not an isomorphism, the action $\text{Aff}^1 \circ M$ has trivial stabilizers as well.

Proof. A stabilizer of the action $\text{Aut}_S(S') \circ M_X$ entails a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow & & \downarrow \\
S' & \cong & S'.
\end{array}
$$

The fact that $S' \subseteq X$ is a monomorphism forces $S' \cong S'$ to be the identity.

Normality of $S$ means $S$ is a finite disjoint union of integral schemes [Stacks, 033N]; we assume $S$ is integral without loss of generality.

Computing stabilizers of $\text{Aff}^1 \circ M$ is local, so we may assume $S' \to S$ is induced by a non-identity finite map $A \to B$ of rings where $A$ is an integrally closed domain with field of fractions $K$. A stabilizer $a + b\theta = \theta$; $a \in A, b \in A^*$ implies $(1 - b)\theta = a$. If $b = 1$, then $a = 0$ and the stabilizing affine transformation is trivial. Otherwise, $1 - b \in K^*$ and $\theta = \frac{a}{1 - b} \in K$. Elements $\theta \in B$ are all integral over $A$. Since $A$ is integrally closed, $\theta \in A$. Hence $B = A[\theta] = A$, a contradiction. 

Remark 6.11. Suppose given transition functions $(a_{ij}, b_{ij})$ and local monogenerators $(\theta_i)$ as in an $\text{Aff}^1$-twisted monogenerator that may not satisfy the cocycle condition a priori. For normal $S$ with $n > 1$ as in the lemma, the cocycle condition holds automatically, since $\text{Aff}^1$ acts without stabilizers.

Corollary 6.12. If $S$ is normal, the stack quotient $[M_{1,S'/S}/\text{Aff}^1]$ is represented by the ordinary sheaf quotient $M/\text{Aff}^1$.

Proof. If $G \circ X$ is a free action, the stack quotient $[X/G]$ coincides with the sheaf quotient $X/G$. 

Remark 6.13. If $S$ is normal, Corollary 6.12 tells us that an $\text{Aff}^1$-twisted monogenerator is the same as a global section $\Gamma(S, M/\text{Aff}^1)$. Equivalence classes of monogenerators are given by the presheaf quotient $\Gamma(S, M)/\Gamma(S, \text{Aff}^1)$.

Affine equivalence classes of monogenerators thereby relate to twisted monogenerators in something like an exact sequence:

$$\Gamma(S, M)/\Gamma(S, \text{Aff}^1) \to \Gamma(S, M/\text{Aff}^1) \to H^1(S, \text{Aff}^1).$$

As in sheaf cohomology, the second map takes $\overline{\theta}$ to its torsor of lifts $\delta(\overline{\theta})$ in $M$:

$$\delta(\overline{\theta})(U) = \{ f \in M(U) : f + \text{Aff}^1(U) = \overline{\theta}|_U \}.$$

A section of the sheaf quotient $\overline{\theta} \in \Gamma(S, M/\text{Aff}^1)$ lifts to an affine equivalence class in the presheaf quotient $\theta \in \Gamma(S, M)/\Gamma(S, \text{Aff}^1)$ if and only if the induced $\text{Aff}^1$-torsor is trivial.

The exact sequence is analogous to Cartier divisors. If $X$ is an integral scheme with rational function field $K(X)$, the long exact sequence associated to

$$1 \to \mathcal{O}_X^* \to K(X)^* \to K(X)^*/\mathcal{O}_X^* \to 1$$
Remark 6.14. One can do the same with $\mathbb{G}_m$, or any other group. Compare twisted monogenerators $\mathbb{P}M = [M/\mathbb{G}_m]$ with ordinary monogenerators $M$ up to $\mathbb{G}_m$-equivalence to obtain a sequence

$$\Gamma(S, M)/\Gamma(S, \mathbb{G}_m) \to \Gamma(S, M/\mathbb{G}_m) \to H^1(S, \mathbb{G}_m).$$

We preferred the case of affine equivalence already prevalent in the literature, and the reader can make the necessary modifications in general. Freeness of the action is necessary to identify the stack quotient with the ordinary sheaf quotient.

Sometimes, being $\mathbb{G}_m$-twisted monogenic is the same as being $\text{Aff}^1$-twisted monogenic:

Proposition 6.15. If $S = \text{Spec } A$ is affine, all $\text{Aff}^1$-torsors on $S$ are induced by $\mathbb{G}_m$-torsors:

$$H^1(\text{Spec } A, \mathbb{G}_m) \simeq H^1(\text{Spec } A, \text{Aff}^1).$$

The corresponding twisted forms of $\mathbb{A}^1$ are the same, so we can furthermore identify $\mathbb{G}_m$-twisted monogenerators with $\text{Aff}^1$-twisted monogenerators.

Proof. The inclusion

$$\mathbb{A}^1 \to \text{Aff}^1; \quad a \mapsto (a, 1)$$

and the projection $\text{Aff}^1 \to \mathbb{G}_m$ fit into a short exact sequence

$$0 \to \mathbb{A}^1 \to \text{Aff}^1 \to \mathbb{G}_m \to 1.$$ 

The sheaf $\text{Aff}^1$ is not commutative. Cohomology sets $H^i(S, \text{Aff}^1)$ are nevertheless defined for $i = 0, 1, 2$. Serre Vanishing [Stacks, 01XB] says $H^i(\text{Spec } A, \mathbb{A}^1) = 0$ for $i \neq 0$ and $\Gamma(\text{Aff}^1) \to \Gamma(\mathbb{G}_m)$ is surjective, yielding an identification in all nonzero degrees:

$$H^i(\text{Spec } A, \text{Aff}^1) \simeq H^i(\text{Spec } A, \mathbb{G}_m), \quad i = 1, 2.$$

The action $\mathbb{G}_m \circ \mathbb{A}^1$ is the restriction of that of $\text{Aff}^1$, factoring

$$\mathbb{G}_m \subseteq \text{Aff}^1 \to \text{Aut}(\mathbb{A}^1).$$

The corresponding twisted forms of $\mathbb{A}^1$ are the same. \qed

The literature abounds with finiteness results on equivalence classes of monogenerators, for example:

Theorem 6.16 ([EG17, Theorem 5.4.4]). Let $A$ be an integrally closed integral domain of characteristic zero and finitely generated over $\mathbb{Z}$. Let $K$ be the quotient field of $A$, $\Omega$ a finite étale $K$-algebra with $\Omega \neq K$, and $B$ the integral closure of $A$ in $\Omega$. Then there are finitely many equivalence classes of monogenic generators of $B$ over $A$.

Assume in addition that $\text{Pic}(A)$ is finite, as when $A$ is of the form $\mathbb{Z}_L[x_1, \ldots, x_n, 1/p]$. We have an analogous finiteness result for equivalence classes of $\text{Aff}^1$-twisted monogenerators:

Corollary 6.17. Let $A, K, \Omega, B$ be as in Theorem 6.16, with $S' \to S$ induced from $A \to B$. Assume $\text{Pic}(A)$ is finite. Then there are finitely many equivalence classes of $\text{Aff}^1$-twisted monogenerators for $S' \to S$.

is analogous to the above.
Proof. We essentially use the sequence
\[ \Gamma(S, \mathcal{M})/\Gamma(S, \mathcal{A}ff^3) \to \Gamma(S, \mathcal{M}/\mathcal{A}ff^3) \to H^1(S, \mathcal{A}ff^3) \]
of Remark 6.13. If this were a short exact sequence of groups, the outer terms being finite would force the middle term to be; our proof is similar in spirit.

Since \( S \) is quasicompact and there are finitely many elements of the Picard group
\[ \text{Pic}(S) := H^1(S, \mathbb{G}_m) = H^1(S, \mathcal{A}ff^1), \]
we can find an affine open cover \( S = \bigcup U_i \) by finitely many open sets of \( S \) that simultaneously trivializes all line bundles on \( S \).

The above sequence of presheaves restricts to the \( U_i \)'s in a commutative diagram
\[
\begin{array}{ccc}
\Gamma(S, \mathcal{M})/\Gamma(S, \mathcal{A}ff^3) & \longrightarrow & \Gamma(S, \mathcal{M}/\mathcal{A}ff^3) & \longrightarrow & H^1(S, \mathcal{A}ff^3) \\
\downarrow & & \downarrow & & \downarrow 0 \\
\prod \Gamma(U_i, \mathcal{M})/\Gamma(U_i, \mathcal{A}ff^3) & \longrightarrow & \prod \Gamma(U_i, \mathcal{M}/\mathcal{A}ff^3) & \longrightarrow & \prod H^1(U_i, \mathcal{A}ff^3).
\end{array}
\]
The restriction \( H^1(S, \mathcal{A}ff^1) \to \prod H^1(U_i, \mathcal{A}ff^1) \) is zero by construction of the \( U_i \)'s. The restriction \( \rho : \Gamma(S, \mathcal{M}/\mathcal{A}ff^3) \to \prod \Gamma(U_i, \mathcal{M}/\mathcal{A}ff^3) \) is injective by the sheaf condition. A diagram chase reveals that the restriction \( \rho(\bar{\varrho}) \) of any section \( \bar{\varrho} \in \Gamma(S, \mathcal{M}/\mathcal{A}ff^3) \) is in the image of \( \prod \Gamma(U_i, \mathcal{M}/\mathcal{A}ff^3) \). Theorem 6.16 asserts that each set \( \Gamma(U_i, \mathcal{M})/\Gamma(U_i, \mathcal{A}ff^3) \) is finite. \( \square \)

We conclude with several consequences of twisted monogeneity and our Theorem 6.20 that shows twisted monogenerators detect class number-one number rings.

**Theorem 6.18.** Suppose \( S' \to S \) is twisted monogenic, with an embedding into a line bundle \( E \). Let \( \mathcal{E} \) be the sheaf of sections of \( E \). Then
\[ \det(\pi_* \mathcal{O}_{S'}) := \wedge^{top} \pi_* \mathcal{O}_{S'} \simeq \mathcal{E}^{\frac{\alpha(n-1)}{2}} \]
in \( \text{Pic}(S) \).

In particular, if an extension of number rings \( \mathbb{Z}_L/\mathbb{Z}_K \) is twisted monogenic, then its Steinitz class is a triangular number power in the class group.

**Proof.** Write \( \text{Sym}^* \mathcal{E} := \bigoplus_d \text{Sym}^d \mathcal{E} \) for the symmetric algebra. Recall that \( E \simeq \mathcal{V}(\mathcal{E}^\vee) := \text{Spec}_S \text{Sym}^* (\mathcal{E}^\vee) \). We have a surjection of \( \mathcal{O}_S \)-modules
\[ \text{Sym}^* \mathcal{E}^\vee \twoheadrightarrow \pi_* \mathcal{O}_{S'}, \]
which we claim factors through the projection of \( \mathcal{O}_S \)-modules \( \text{Sym}^* \mathcal{E}^\vee \to \bigoplus_{i=0}^{n-1} \text{Sym}^i \mathcal{E}^\vee \). Such a factorization is a local question and local factorizations automatically glue because there is at most one. Locally, we may assume \( S' \to S \) is induced by a ring homorphism \( A \to B \) and \( \mathcal{E}^\vee \) is trivialized. We have a factorization of \( A \)-modules
\[
\begin{aligned}
\text{Sym}^* \mathcal{E}^\vee &\simeq A[t] \\
\bigoplus_{i=0}^{n-1} \text{Sym}^i \mathcal{E}^\vee &\simeq \bigoplus_{i=0}^{n-1} A \cdot t^i
\end{aligned}
\]
due to the existence of a monic polynomial \( m_\theta(t) \) of degree \( n \) for the image \( \theta \) of \( t \) in \( \mathcal{O}_{S'} \) (Lemma 2.14). The \( A \)-modules \( \bigoplus_{i=0}^{n-1} A \cdot t^i \) and \( B \) are abstractly isomorphic,
and any surjective endomorphism of a finitely generated module is an isomorphism [MRB89, Theorem 2.4].

We conclude that globally
\[
\pi_* \mathcal{O}_{S'} \simeq \bigoplus_{i=0}^{n-1} \text{Sym}^{i} \mathcal{E}^\vee.
\]

Since \( \mathcal{E} \) is invertible, \( \text{Sym}^{i} \mathcal{E}^\vee = (\mathcal{E}^\vee)^i \). Taking the determinant,
\[
\det(\pi_* \mathcal{O}_{S'}) = \det \left( \bigoplus_{i=0}^{n-1} (\mathcal{E}^\vee)^i \right) = \mathcal{E}^{- \sum_{i=0}^{n-1} i} = \mathcal{E}^{- \frac{n(n-1)}{2}}.
\]

\[\square\]

**Lemma 6.19.** Degree-two extensions are all \( \text{Aff}^1 \)-twisted monogenic. If \( S \) is affine, they are also \( \mathbb{G}_m \)-twisted monogenic.

**Proof.** Localize and use Lemma 3.15 to write \( \pi_* \mathcal{O}_{S'} \simeq \mathcal{O}_S \oplus \mathcal{O}_S \theta \) for some \( \theta \in \Gamma(\mathcal{O}_{S'}) \). Given two such generators \( \theta_1, \theta_2 \), we may write
\[
\theta_1 = a + b \theta_2, \quad \theta_2 = c + d \theta_1, \quad a, b, c, d \in \mathcal{O}_S.
\]

Hence \( bd = 1 \) are units, and the transition functions come from \( \text{Aff}^1 = \mathbb{A}^1 \times \mathbb{G}_m \). By choosing such generators on a cover of \( S \), one obtains a twisted monogenerator. Proposition 6.15 further refines our affine bundle to a line bundle. \[\square\]

**Theorem 6.20.** A number field \( K \) has class number one if and only if all twisted monogenic extensions of \( \mathbb{Z}_K \) are in fact monogenic.

**Proof.** If the class number of \( K \) is one, then all line bundles on \( \text{Spec} \mathbb{Z}_K \) are trivial and the equivalence is clear. Mann [Man58] has shown that \( K \) has quadratic extensions without an integral basis if and only if the class number of \( K \) is not one: adjoin the square root of \( \alpha \), where \( (\alpha) = b^2 \mathfrak{c} \) with \( b \) non-principal and \( \mathfrak{c} \) square-free. By Lemma 6.19, such an extension is necessarily \( \mathbb{G}_m \)-twisted monogenic. As the monogeneity of quadratic extensions is equivalent to the existence of an integral basis, the result follows. \[\square\]

**Remark 6.21.** Theorem 6.20 implies that the ring of integers of a number field is twisted monogenic over \( \mathbb{Z} \) if and only if it is monogenic over \( \mathbb{Z} \). Example 7.9 thus provides an example of a number field which is not twisted monogenic.

**Example 6.22 (Properly Twisted Monogenic, Not Quadratic).** Let \( K = \mathbb{Q}(\sqrt[3]{5 \cdot 23}) \) and let \( p_3, p_5, \) and \( p_{23} \) be the unique primes of \( K \) above 3, 5, and 23, respectively. One can compute \( p_3 = (\rho_3) = 1970(\sqrt[3]{5 \cdot 23})^2 + 9580(\sqrt[3]{5 \cdot 23}) + 46587 \).

Consider \( \mathbb{Z}_L/\mathbb{Z}_K \), where \( L = K(\sqrt[3]{23^{17} p_3}) \). On \( D(p_{23}) \), the local index form with respect to the local basis \( \{ 1, \sqrt[3]{23^{17} p_3}, (\sqrt[3]{23^{17} p_3})^2 \} \) is \( b^3 - 23 \rho_3 c^3 \). On \( D(p_5) \), we have the local index form \( B^3 - 5^2 \rho_3 C^3 \) with respect to the local basis \( \{ 1, \sqrt[3]{5^2 p_3}, (\sqrt[3]{5^2 p_3})^2 \} \).

We transition via \( \sqrt[3]{23^{22} \cdot 5^2 23} \), which is not a global unit, so the extension \( \mathbb{Z}_L/\mathbb{Z}_K \) is twisted monogenic.

To see what is going on more explicitly, we investigate how the transitions affect the local index forms. We have
\[
b^3 - 23 \rho_3 c^3 = \frac{5^2}{23} B^3 - \frac{5^4}{23^2} \cdot 23 \rho_3 C^3 = \frac{5^2}{23} B^3 - \frac{5^4}{23^2} \rho_3 C^3 = \text{ a unit in } \mathcal{O}_{D(p_{23})}.
\]
If $B$ and $C$ could be chosen to be $\mathbb{Z}_K$-integral so that local index form represented a unit of $\mathbb{Z}_K$, then $\sqrt{5^2 p_3}$ would be a global monogenerator. However, $p_5$-adic valuations tell us $\sqrt{5^2 p_3}$ is not a monogenerator. One can also apply Dedekind’s index criterion to $x^3 = 5^2 p_3$. Similarly, we have

$$B^3 - 5^2 p_3 C^3 = \frac{23}{52} b^3 - \frac{23^2}{5} \cdot 5^2 p_3 c^3 = \frac{23}{52} b^3 - \frac{23^2}{5} p_3 c^3 = \text{ a unit in } \mathcal{O}_{\mathcal{D}(p_5)}.$$ 

If $b$ and $c$ could be chosen to be $\mathbb{Z}_K$-integral so that local index form represented a unit of $\mathbb{Z}_K$, then $\sqrt{23 p_3}$ would be a global monogenerator. As above, the $p_{23}$-adic valuations tell us this cannot be the case. Again, we could also use polynomial-specific methods.

We have shown that $\mathbb{Z}_L/\mathbb{Z}_K$ is twisted monogenic, but it remains to show that the twisting is non-trivial. We need to show the ideal $p_5 = (5, \sqrt{5 \cdot 23})$ is not principal. On $\mathcal{D}(p_{23})$ it can be generated by $\sqrt{5 \cdot 23}$ and on $\mathcal{D}(p_5)$ it can be generated by 5. We transition between these two generators via $\sqrt{5^2 \cdot 23^2/23}$, exactly as above. Thus our twisted monogenerators correspond to a non-trivial ideal class.

A computer algebra system can compute a $K$-integral basis for $\mathbb{Z}_L$:

$$\begin{cases} 1, \left(-2 \sqrt{23 \cdot 5} + \frac{5}{23} \left(\sqrt{23 \cdot 5}\right)^2\right) \sqrt[3]{23 p_3} + \left(-3 + \frac{3}{23} \sqrt{23 \cdot 5}\right) \left(\sqrt{23 p_3}\right)^2, \\ 120589 + 5243 \sqrt{23 \cdot 5} + \frac{5243}{23} \left(\sqrt{23 \cdot 5}\right)^2 \sqrt{23 p_3} + \left(22850 + \frac{57125}{23} \sqrt{23 \cdot 5} + 1828 \left(\sqrt{23 \cdot 5}\right)^2\right) \left(\sqrt{23 p_3}\right)^2 \end{cases},$$

with index form:

$$g_{\mathbb{Z}_L/\mathbb{Z}_K} = 13796817 \left(\sqrt{5 \cdot 23}\right)^2 b^3 - 1367479703949 \left(\sqrt{5 \cdot 23}\right)^2 b^2 c + 45179341009193328 \left(\sqrt{5 \cdot 23}\right)^2 b c^2 + 67103709 \sqrt{5 \cdot 23} b^4 - 497537273719431009077 \left(\sqrt{5 \cdot 23}\right)^2 c^3 - 6650125342740 \sqrt{5 \cdot 23} b c^2 + 219702478196413227 \sqrt{5 \cdot 23} b c^2 - 2419492830176044166763 \sqrt{5 \cdot 23} c^3 + 326269891 b^3 - 3233992309080b^2 c + 1068411032584717260bc^2 - 11765841517121285321908c^3.$$ 

Because $\mathbb{Z}_L/\mathbb{Z}_K$ is twisted monogenic, there are no common index divisors. Thus we will always find solutions to $g_{\mathbb{Z}_L/\mathbb{Z}_K}$ when we reduce modulo a prime of $\mathbb{Z}_K$. We do not expect $\mathbb{Z}_L$ to be monogenic over $\mathbb{Z}_K$; however, showing that there are no values of $b, c \in \mathbb{Z}_K$ such that $g_{\mathbb{Z}_L/\mathbb{Z}_K}(b, c) \in \mathbb{Z}_K$ appears to be rather difficult.

**Remark 6.23.** One can perform the same construction of Example 6.22 with radical cubic number rings other than $\mathbb{Q}(\sqrt{5 \cdot 23})$. Specifically, take any radical cubic where $(3) = p_3^3 = (\alpha)^3$, $\ell$, and $q$ are distinct primes with $(\ell) = \ell^3$, $(q) = q^3$, and neither $\ell$ nor $q$ principal. The ideas behind this construction can be taken further by making appropriate modifications.

### 6.3. Twisting in general

Definition 6.1 readily generalizes. Replace $\mathbb{G}_m$ by any sheaf of groups $G$ acting on $\mathbb{A}^1$ or any left action $G \times X$ and work in the étale topology. A $G$-twisted monogenerator for $S' \to S$ (into $X$) is an étale cover $U_i \to S$, ...
closed embeddings $\theta_i : S'_{ij} \subseteq X_{U_i}$, and elements $g_{ij} \in G(U_{ij})$ such that

$$g_{ij, \theta_j} = \theta_i : S'_{U_{ij}} \to X_{U_{ij}}.$$  

Equivalently, the $\theta_i$'s glue to a global closed embedding $S' \subseteq \hat{X}$ into a twisted form $\hat{X}$ of $X$ the same way the $\mathbb{G}_m$-twisted monogenerators give embeddings into a line bundle. By “twisted form,” we mean a scheme $\hat{X} \to S$ which becomes isomorphic to $X \to S$ locally in the étale topology.

The twisted form $\hat{X}$ arises from transition functions in $G$, meaning there is a $G$-torsor $P$ such that $\hat{X}$ is the contracted product of Definition A.4:

$$\hat{X} = X \wedge^G P := X \times P/(G, \Delta)$$

We have already seen the variant $G = \text{Aff}^1$, $X = \mathbb{A}^1$. Other interesting cases include $G = \text{PGL}_2 \circ \mathbb{P}^1$, $\text{GL}_n \circ \mathbb{A}^n$, an elliptic curve $E$ acting on itself $E \circ E$, etc.

**Example 6.24.** Usually, contracted products are defined for a left action $G \circ P$ and a right action $G \circ X$ by quotienting by the antidiagonal action $X \wedge^G P := X \times P/(-\Delta, G)$ defined by

$$G \circ X \times P; \quad g.(x, p) := (x.g^{-1}, g.p).$$

We instead take two left actions and quotient by the diagonal action of $G$. Remark A.5 equates the two. The literature often turns left actions $\mathbb{G}_m \circ \mathbb{A}^1$ into right actions anyway, as in [Bre06, Remark 1.7].

Throughout this section, fix notation as in Situation 2.1 and work in the category $(\text{Sch}/S)$ of schemes over $S$ equipped with the étale topology. Group sheaves $G$ beget stacks $BG = BG_S$ classifying $G$-torsors on $S$-schemes with universal $G$-torsor $S \to BG$; these are reviewed in Appendix A.

Twisted forms $\hat{X}$ of $X$ are equivalent to torsors for $\text{Aut}(X)$, as in [Poo17, Theorem 4.5.2]. Given a twisted form $\hat{X} \to S$, we obtain the torsor $\text{Isom}(\hat{X}, X)$ of local isomorphisms. Given a $\text{Aut}(X)$-torsor $P$, we define a twisted form via contracted product (Definition A.4):

$$\hat{X}_P := X \wedge^{\text{Aut}(X)} P.$$  

The stack $B\text{Aut}(X)$ is thereby a moduli space for twisted forms of $X$ with universal family $X \wedge_{B\text{Aut}(X)}^ {\text{Aut}(X)} S = [X/\text{Aut}(X)]$. An action $G \to \text{Aut}(X)$ lets one turn a $G$-torsor $P$ into a twisted form

$$\hat{X}_P := X \wedge^G P$$

classified by the map $BG \to B\text{Aut}(X)$.

The automorphism sheaf $\text{Aut}(X)$ acts on the scheme $\mathcal{M}_X$ via postcomposition with the embeddings $S' \to X$, yielding a map of sheaves $\gamma : \text{Aut}(X) \to \text{Aut}(\mathcal{M}_X)^3$.

Similarly, the automorphism sheaf $\text{Aut}(S')$ acts on $\mathcal{M}_X$ on the right via precomposition:

$$\xi : \text{Aut}(S')^{op} \to \text{Aut}(\mathcal{M}_X).$$

\footnote{The map $\gamma$ need not be injective. Consider $S = \text{Spec} k$ a geometric point, $S' = \bigsqcup^3 S$ the trivial 3-sheeted cover, and $X = \bigsqcup^2 S$ only 2-sheeted. There are no closed immersions $S' \subseteq X$, so $\mathcal{M}_X = \emptyset$ has global automorphisms $\text{Aut}(\emptyset) \cong \{ id \}$, but $\text{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$ is nontrivial. Similar examples abound for non-monogenic $S' \to S$.}
The induced map $H^1(S, \text{Aut}(X)) \to H^1(S, \text{Aut}(M))$ sends a twisted form $\hat{X}$ of $X$ to the twisted form $\hat{M}_X$ of $M_X$ given by looking at closed embeddings into $\hat{X}$. We package these twisted forms $\hat{M}_X$ into a universal version: $\text{Twist} M$.

**Definition 6.25 (Twist $M$).** Let $\text{Twist} M \to B \text{Aut}(X)$ be the $B \text{Aut}(X)$-stack whose $T$-points are given by

$$\left\{ \begin{array}{c} \text{Twist} M \\ \xymatrix{T \ar[r]^-{\hat{X}} & B \text{Aut}(X)} \end{array} \right\} := \left\{ S' \times_S T \ar@{..>}[rr]^s \ar[d] & & \hat{X} \ar[d] \\ T & & \end{array} \right\} s \text{ is a closed immersion}.$$

The fibers of $\text{Twist} M \to B \text{Aut}(X)$ are representable because they are twisted forms $\hat{M}_X$ of $M_X$ itself:

$$\xymatrix{ m_{\hat{X}} \ar[r]^s \ar[d] & S \ar[d] \\ \text{Twist} M \ar[r] & B \text{Aut}(X).}$$

The universal torsor over $B \text{Aut}(X)$ is $S$, but the universal twisted form is obtained by the contracted product of Definition A.4 with $X$ over $B \text{Aut}(X)$:

$$X \hat{\wedge}_{B \text{Aut}(X)} S \simeq [X/\text{Aut}(X)].$$

One can exhibit $\text{Twist} M$ as an open substack of the Weil Restriction of $[X/\text{Aut}(X)] \to B \text{Aut}(X)$ as in Proposition 2.4. There is a universal closed embedding over $\text{Twist} M$ into the universal twisted form of $X$ as in the definition of $M$ in Section 3.1:

$$\xymatrix{ \text{Twist} M \times_S S' \ar@{..>}[rr]^{\nu} \ar[dr] & & [X/\text{Aut}(X)] \ar[dl] \\ & \text{Twist} M &}$$

The universal case is concise to describe but unwieldy because $\text{Aut}(X)$ need not be finite, smooth, or well-behaved in any sense. We simplify by specifying our twisted form $\hat{X} \to T$ to get a scheme $\text{Twist} M_{\hat{X}} = T \times_{\hat{X}, B \text{Aut}(X)} \text{Twist} M$ or by specifying the structure group $G$.

Fixing the structure group $G$ requires $\hat{X} = X \hat{\wedge}^G P$ for the specified sheaf of groups $G$ and some $G$-torsor $P$. These $G$-twisted forms are parameterized by the pullback

$$\xymatrix{ \text{Twist} M^G \ar[r]^-{\nu} \ar[d] & \text{Twist} M \ar[d] \\ B G \ar[r] & B \text{Aut}(X).}$$

The fibers of $\text{Twist} M^G \to B G$ over maps $T \to B G$ are again twisted forms of $M_X$. If $X = \mathbb{A}^k$ and $\hat{X}$ is a $G$-twisted form, we may refer to the existence of sections of $\text{Twist} M_{\hat{X}}$ by saying $S'/S$ is $G$-twisted $k$-genic, etc.
Remark 6.26. Trivializing \( \hat{X} \) is not the same as trivializing the torsor \( P \) that induces \( \hat{X} \) unless the group \( G \) is \( \text{Aut}(X) \) itself. For example, take the trivial action \( G \circ X \).

Theorem 6.27. The stack of twisted monogenerators \( \text{Twist}M \) is isomorphic to \( [M_{X/\text{Aut}(X)}] \) over \( B \text{Aut}(X) \). More generally, for any sheaf of groups \( G \circ X \), we have an isomorphism \( \text{Twist}M^G \simeq [M_{X/G}] \) over \( BG \).

Proof. Address the second, more general assertion and let \( T \) be an \( S \)-scheme. Write \( X_T := X \times_S T, T' := T \times_S S', \) etc. A \( T \)-point of \( \text{Twist}M^G \) is a \( G \)-torsor \( P \rightarrow T \) and a solid diagram

\[
\begin{array}{ccc}
P' & \rightarrow & X_T \times_T P \\
\downarrow & & \downarrow_{\Delta^G} \\
T' & \rightarrow & X_T \wedge^G_T P, \\
& & \rightarrow T
\end{array}
\]

with \( T' \rightarrow X_T \wedge^G_T P \) a closed immersion. Form \( P' \) by pullback: \( P' \) is a left \( G \)-torsor with an equivariant map to \( X_T \times_T P \) with the diagonal action. The map \( P' \rightarrow P \) over \( T \) forces \( P' \simeq P \times_T T' \). These data form an equivariant map \( P \rightarrow M_X \) over \( T \), or \( T \rightarrow [M_{X/G}] \). Reverse the process to finish the proof. \( \square \)

Example 6.28. Consider \( S' = \text{Spec} \mathbb{Z}[i], S = \text{Spec} \mathbb{Z} \). The space of \( k \)-generators is \( M_k = \mathbb{A}^k \times (\mathbb{A}^k \setminus 0) \) according to Proposition 3.16. Take the quotient by the groups of affine transformations:

\[
[M_k/\mathbb{A}^k \times \mathbb{G}_m] = \mathbb{P}^{k-1}, \quad [M_k/\text{Aff}^k] = [\mathbb{A}^k/\text{GL}_k] = [\mathbb{P}^{k-1}/\text{PGL}_k].
\]

These quotients represent twisted monogenerators according to Theorem 6.27. The corresponding \( \text{PGL}_k \)-torsors were classically identified with Azumaya algebras and Severi-Brauer varieties (see the exposition in [Kol16]), or twisted forms of \( \mathbb{P}^{k-1} \). These yield classes in the Brauer group \( H^2(\mathbb{G}_m) \) via the connecting homomorphism from

\[
1 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_k \rightarrow \text{PGL}_k \rightarrow 1.
\]

The same holds locally for any degree-two extension \( S' \rightarrow S \) with \( S \) integral using Proposition 3.16.

If \( G \) is an abelian variety over a number field \( S = \text{Spec} K \), let \( P \rightarrow S \) be a \( G \)-torsor inducing a twisted form \( X \) of \( X \). Given a twisted monogenerator \( \theta : S' \subseteq \hat{X} \), one can try to promote \( \theta \) to a bona fide monogenerator by trivializing \( P \) and thus \( X \).

Suppose one is given trivializations of \( P \) over the completions \( K_\nu \) at each place. Whether these glue to a global trivialization of \( P \) over \( K \) and thus a monogenerator \( S' \subseteq X \) is governed by the Shafarevich-Tate group \( \text{III}(G/K) \).

Given a \( G \)-twisted monogenerator with local trivializations, the Shafarevich-Tate group obstructs lifts of \( \theta \) to a global monogenerator the same way classes of line bundles in \( \text{Pic} \) obstruct \( \mathbb{G}_m \)-twisted monogenerators from being global monogenerators. Theorem 6.20 showed a converse – nontrivial elements of \( \text{Pic} \) imply twisted monogenerators that are not global monogenerators.
Question 6.29. Is the same true for III? Does every element of the Shafarevich-Tate group arise this way?

The Shafarevich-Tate group approach is useless for $G = \mathbb{G}_m$ or $\text{GL}_n$ because of Hilbert’s Theorem 90 [Stacks, 03P8]:
$$H^1_{\text{ét}}(\text{Spec } K, G) = H^1_{\text{Zar}}(\text{Spec } K, G) = 0.$$ 

The same goes for any “special” group with étale and Zariski cohomology identified. The strategy may work better for $\text{PGL}_n$ or elliptic curves $E$.

Remark 6.30. This section defined $G$-twisted monogenerators using covers in the étale topology, whereas Definition 6.1 used the Zariski topology. For $G = \mathbb{G}_m$ or $\text{Aff}^1$, either topology gives the same notion of twisted monogenerators. Observe that $\mathbb{G}_m$ has the same Zariski and étale cohomology by Hilbert’s Theorem 90. The same is true for $\mathbb{A}^1$ by [Stacks, 03P2] and so also $\text{Aff}^1 = \mathbb{G}_m \ltimes \mathbb{A}^1$.

7. Examples of the scheme of monogenerators

We conclude with several examples to further illustrate the nature and variety of the scheme of monogenerators. We will consider situations in which the classical index form of Remark 3.18 is well-studied, such as field extensions and extensions of number rings, as well as more exotic situations, such as covers of curves and jet spaces. We suggest starting with Subsection 3.2, which contains what the authors consider the most approachable examples. We will make frequent reference to computation of the index form using the techniques of Section 3.1.

7.1. Field extensions. When $S' = \text{Spec } L \to S = \text{Spec } K$ is induced by a field extension $L/K$, we know that the monogenic generators of $L$ over $K$ are precisely the elements of $L$ that do not belong to any proper subfield of $L$. Therefore, on the level of $K$-points of $M_1$, we can expect to see that the index form vanishes on precisely the proper subfields of $L$. However, it has further structure that is better seen after extension to a larger field.

Example 7.1 (A $\mathbb{Z}/2 \times \mathbb{Z}/2$ field extension). Let $S = \text{Spec } \mathbb{Q}$ and $S' = \text{Spec } \mathbb{Q}(\sqrt{2}, \sqrt{3})$. The isomorphism of groups $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \simeq \mathbb{Q} \oplus \mathbb{Q}\sqrt{2} \oplus \mathbb{Q}\sqrt{3} \oplus \mathbb{Q}\sqrt{6}$ identifies the Weil Restriction $R_{\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}}$ and its universal maps with Spec of

$$\mathbb{Q}[a, b, c, d][\sqrt{2}, \sqrt{3}] \overset{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}-t}{\leftarrow} \mathbb{Q}[a, b, c, d][t] \overset{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}-t}{\rightarrow} \mathbb{Q}[a, b, c, d].$$

Hence $R_{\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}} = \mathbb{Q}[a, b, c, d]$ and the universal morphism

$$u : S' \times_S R_{\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}}$$

is induced by

$$t \mapsto a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}.$$ 

We expand the images of the powers $1, t, t^2, t^3$ to find the matrix of coefficients

$$\begin{bmatrix}
1 & a & a^2 + 2b^2 + 3c^2 + 6d^2 & a^2 + 6ab^2 + 9ac^2 + 36bcd + 18a^2d^2 \\
0 & b & 2ab + 6cd & 3a^2b + 2b^3 + 9bc^2 + 18a^2cd + 18bd^2 \\
0 & c & 2ac + 4bd & 3a^2c + 6b^2c + 3c^3 + 12abcd + 18c^2d^2 \\
0 & d & 2bc + 2ad & 6abc + 3a^2d + 6b^2d + 9c^2d + 6d^3
\end{bmatrix}.$$
We compute the local index form associated to our chosen basis by taking the determinant:
\[ i(a, b, c, d) = -8b^4c^2 + 12b^2c^4 + 16b^4d^2 - 36c^4d^2 - 48b^2d^4 + 72c^2d^4 \]
\[ = -4(2b^2 - 3c^2)(b^2 - 3d^2)(c^2 - 2d^2). \]

Note that this determinant has degree 6. Dropping subscripts, the factorization implies that the closed subscheme of non-generators \( \mathcal{N} \) inside \( \mathcal{R} \cong A^3_\mathbb{Q} \) has three components of degree 2.

Consider the \( \mathbb{Q} \)-points of \( M_{1, S'/S} = \mathcal{R} - \mathcal{N} \). These are in bijection with the elements \( a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) where \( a, b, c, d \) are in \( \mathbb{Q} \) and the index form does not vanish. Equivalently,
\[ 2b^2 - 3c^2 \neq 0, \quad b^2 - 3d^2 \neq 0, \quad \text{and} \quad c^2 - 2d^2 \neq 0. \]

Let \( \theta = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \) for some \( a, b, c, d \in \mathbb{Q} \) and consider what it would mean to fail one of these conditions. If \( 2b^2 - 3c^2 = 0 \) for \( b, c \in \mathbb{Q} \), it must be that \( b = c = 0 \). Then \( \theta \in \mathbb{Q}(\sqrt{6}) \), a proper subfield of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \). Similarly, if \( b^2 - 3d^2 = 0 \) then \( \theta \in \mathbb{Q}(\sqrt{3}) \), and if \( c^2 - 2d^2 = 0 \) then \( \theta \in \mathbb{Q}(\sqrt{2}) \). It follows that the \( \mathbb{Q} \)-points of \( M_{1, S'/S} \) are in bijection with the elements of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) that do not lie in a proper subfield, as we expect from field theory.

See Example 7.8 for an analysis of the monogeneity of some orders contained in the field considered above.

**Example 7.2** (A \( \mathbb{Z}/4\mathbb{Z} \)-extension). Let \( S = \text{Spec} \mathbb{Q}(i) \) and \( S' = \text{Spec} \mathbb{Q}(i, \sqrt{2}) \).

We have a global \( \mathbb{Q}(i) \)-basis \( \{1, \sqrt{2}, \sqrt{3}, \sqrt{2}\sqrt{3} \} \) for \( \mathbb{Q}(i, \sqrt{2}) \) over \( \mathbb{Q}(i) \). We may use this basis to write \( \mathcal{R} \cong \text{Spec} \mathbb{Q}(i)[a, b, c, d] \) where the universal map from \( A^1 \) is \( t \mapsto a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \). The matrix of coefficients is
\[
\begin{bmatrix}
1 & a & a^2 + 2c^2 + 4bd & a^3 + 6b^2c + 6ac^2 + 12abd + 12cd^2 \\
0 & b & 2ab + 4cd & 3a^2b + 6bc^2 + 6b^2d + 12acd + 4d^3 \\
0 & c & b^2 + 2ac + 2d^2 & 3ab^2 + 3a^2c + 2c^3 + 12bcd + 6d^2 \\
0 & d & 2bc + 2ad & b^3 + 6abc + 3a^2d + 6c^2d + 6d^2 \\
\end{bmatrix}.
\]

The determinant yields the local index form with respect to this basis:
\[ (b^2 - 2d^2)(b^4 + 8c^4 - 16bc^2d + 4b^2d^2 + 4d^4). \]

We note that the first factor vanishes for \( a, b, c, d \in \mathbb{Q}(i) \) when \( a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \in \mathbb{Q}(i, \sqrt{2}) \). At first glance the second factor is more mysterious, but after adjoining enough elements, the entire index form factors into distinct linear terms:
\[
(b - \sqrt{2}d)(b + \sqrt{2}d)(ib - (1 + i)\sqrt{2}c + \sqrt{2}d)(-ib - (1 - i)\sqrt{2}c + \sqrt{2}d) \cdot (-ib + (1 - i)\sqrt{2}c + \sqrt{2}d)(ib + (1 + i)\sqrt{2}c + \sqrt{2}d).
\]

This behavior of factorization into distinct linear factors occurs in general:

**Proposition 7.3.** Let \( S' \to S \) be induced by a finite separable extension of fields \( L/K \). Let \( e_1, \ldots, e_n \) be a \( K \)-basis for \( L \), and let \( x_1, \ldots, x_n \) be the corresponding coordinates for \( \mathcal{R} \). Then the local index form \( i(e_1, \ldots, e_n) \) factors completely into distinct linear factors in \( x_1, \ldots, x_n \) over the normal closure \( \overline{L} \) of \( L/K \).
Compare this with Example 5.1. Monogenerators correspond to configurations of \( n \) points in \( \mathbb{A}^1 \) and the distinct linear factors of \( \hat{t}(e_1, \ldots, e_n) \) correspond to when pairs of points collide. This is the image of the “fat diagonal” \( \Delta \).

Some interesting and useful specifics in the case of number fields are investigated in more depth in chapter 7 of [Gaâ19].

**Proof.** We may consider \( \hat{t}(e_1, \ldots, e_n) \) as an element of \( \overline{L}[x_1, \ldots, x_n] \) by pulling back to \( \mathcal{R}_{L/K} \times_S \text{Spec} \overline{L} \simeq \mathcal{R}_{L \otimes_K L/\overline{L}} \). Our strategy is to compute a second generator of the pullback of \( \mathcal{I}_{S'/S} \) with respect to a more convenient basis.

By the Chinese remainder theorem, \( L \otimes_K \overline{L} \simeq \prod_{i=1}^n \overline{L} \). Let \( \overline{e}_1, \ldots, \overline{e}_n \) be the standard basis of \( (\overline{L})^n \), let \( \overline{x}_1, \ldots, \overline{x}_n \) be the corresponding coordinates on \( \mathcal{R}_{L \otimes_K L/\overline{L}} \simeq \mathcal{R}_{L'/L} \), and let \( \theta = \overline{x}_1 \overline{e}_1 + \cdots + \overline{x}_n \overline{e}_n \). Computing a matrix \( M \) for the map \( L[\overline{x}_1, \ldots, \overline{x}_n, t]/m(t) \to \overline{L}[\overline{x}_1, \ldots, \overline{x}_n] \) sending \( t \to \theta \), we see that it is a Vandermonde matrix with factors \( \overline{x}_1, \ldots, \overline{x}_n \), since

\[
(\overline{x}_1 \overline{e}_1 + \cdots + \overline{x}_n \overline{e}_n)^k = \overline{x}_1^k \overline{e}_1 + \cdots + \overline{x}_n^k \overline{e}_n,
\]

when computed in the product ring \( (\overline{L})^n \). Therefore \( |\hat{t}(\overline{e}_1, \ldots, \overline{e}_n)| = |\det(M)| = |\prod_{i<j} (\overline{x}_i - \overline{x}_j)| \). Applying the \( \overline{L} \)-linear change of basis from \( \{\overline{x}_i\} \) to \( \{x_i\} \), we see that \( \hat{t}(e_1, \ldots, e_n) \) is a product of distinct linear factors in \( x_1, \ldots, x_n \).

The lemma above does not consider inseparable extensions. To see what can happen then, we begin with an example.

**Example 7.4** (A purely inseparable extension). For \( \mathbb{F}_3(\alpha)[\beta]/(\beta^3 - \alpha) \) over \( \mathbb{F}_3(\alpha) \), write \( a, b, c \) for the universal coefficients of the basis \( 1, \beta, \beta^2 \). In other words, \( \theta = a + b\beta + c\beta^2 \). One computes that the index form is then

\[
b^3 - c^3 \alpha.
\]

To find the monogenic generators of this extension, we look for \( a, b, c \in \mathbb{F}_3(\alpha) \) so that \( b^3 - c^3 \alpha \neq 0 \). Clearly, at least one of \( b, c \) must be nonzero. Choose \( b, c \) arbitrarily so that one is nonzero. Is this enough to ensure we have a monogenerator?

Suppose first that \( b \neq 0 \). Then \( b^3 - c^3 \alpha = 0 \) implies \( c \in \mathbb{F}_3(\alpha^{1/3}) \setminus \mathbb{F}_3(\alpha) \), a contradiction. Symmetrically, if \( c \neq 0 \) and \( b^3 - c^3 \alpha = 0 \), then \( b \in \mathbb{F}_3(\alpha^{1/3}) \setminus \mathbb{F}_3(\alpha) \), a contradiction again. We conclude that the set of monogenerators is

\[
M_{1, S'/S}(\mathbb{F}_3(\alpha)) = \{a + b\beta + c\beta^2 \mid a, b, c \in \mathbb{F}_3(\alpha) \text{ and } (b, c) \neq (0, 0)\}.
\]

as one expects from field theory.

The polynomial \( b^3 - c^3 \alpha \) is irreducible in \( \mathbb{F}_3(\alpha)[a, b, c] \), so the scheme of non-generators \( \mathcal{N} \) is an irreducible subscheme of \( \mathcal{R} \simeq \mathbb{A}^3 \). However, \( \mathcal{N} \) is not geometrically reduced: after base extension to \( \mathbb{F}_3(\alpha, \sqrt[3]{\alpha}) \), the index form factors as \( (b - c\beta)^3 \).

The factorization noted above is not an isolated phenomenon:

**Proposition 7.5.** Let \( S' \to S \) be induced by a degree \( n := p^m \) completely inseparable extension of fields \( K(\alpha^{1/p^m})/K \). Then over \( K(\alpha^{1/p^m}) \), the local index form factors into a repeated linear factor of multiplicity \( p^m \).
Proof. Consider the local index form $i(e_1, \ldots, e_n)$ as an element of $K(\alpha^{1/p^m})[x_1, \ldots, x_n]$ by pulling back to $\mathcal{R}_{K(\alpha^{1/p^m})/K} \times S^t \simeq \mathcal{R}_{K(\alpha^{1/p^m})} \otimes_K K(\alpha^{1/p^m})/K(\alpha^{1/p^m})$.

Once again, to arrive at the result, we will compute a second generator of the pull back of $\mathcal{G}_{S}/S$ with respect to another basis.

By the Chinese remainder theorem, $K(\alpha^{1/p^m}) \otimes_K K(\alpha^{1/p^m}) \simeq K[t]/(t^{p^m} - \alpha) \otimes_K K(\alpha^{1/p^m}) 
\quad \simeq K(\alpha^{1/p^m})[t]/((t - \alpha^{1/p^m})^{p^m}) 
\quad \simeq K(\alpha^{1/p^m})[\varepsilon]/\varepsilon^{p^m},$

where $\varepsilon = t - \alpha^{1/p^m}$.

Let $b_1 = 1, b_2 = \varepsilon, \ldots, b_{p^m} = \varepsilon^{p^m-1}$ be a basis for $K(\alpha^{1/p^m})[\varepsilon]/\varepsilon^{p^m}$ over $K(\alpha^{1/p^m})$, and let $y_1, \ldots, y_n$ be corresponding coordinates on $\mathcal{R}_{K(\alpha^{1/p^m})}/K \times S^t$. We are now in the situation of Example 3.13. Following the calculation there, we do a second change of coordinates to the basis $c_1 = 1, c_2 = \varepsilon - y_1, \ldots, c_n = (\varepsilon - y_1)^{n-1}$ and let $z_1, \ldots, z_n$ be the corresponding coordinates on $\mathcal{R}_{K(\alpha^{1/p^m})}/K \times S^t$. Taking the determinant of the matrix $M$ of the map $K(\alpha^{1/p^m})[z_1, \ldots, z_n, t]/m(t) \rightarrow K(\alpha^{1/p^m})[\varepsilon][z_1, \ldots, z_n]/\varepsilon^{p^m}$ sending $t \mapsto z_1 c_1 + \cdots + z_n c_n$ with respect to the bases $\{1, \ldots, t^{n-1}\}$ and $\{c_1, \ldots, c_n\}$, we obtain $i(c_1, \ldots, c_n) = z_2 \left(\frac{z^{(p^m-1)}}{2}\right)$.

Applying the change of basis from $\{z_1, \ldots, z_n\}$ to $\{x_1, \ldots, x_n\}$, we see that $i(e_1, \ldots, e_n)$ is a power of a linear term. \qed

7.2. Orders in number rings.

Example 7.6 (Dedekind’s Non-Monogenic Cubic Field). Let $\eta$ denote a root of the polynomial $X^3 - X^2 - 2X - 8$ and consider the field extension $L := \mathbb{Q}(\eta)$ over $K := \mathbb{Q}$. When Dedekind constructed this example [Ded78] it was the first example of a non-monogenic extension of number rings. Indeed two generators are necessary to generate $\mathbb{Z}_L/\mathbb{Z}_K$: take $\eta^2$ and $\frac{\eta^2 + 2}{2}$, for example. In fact, $\{1, \frac{\eta^2 + 2}{2}, \eta^2\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}_K$. The matrix of coefficients with respect to the basis $\{1, \frac{\eta^2 + 2}{2}, \eta^2\}$ is

$$
\begin{bmatrix}
1 & a & a^2 + 6b^2 + 16bc + 8c^2 \\
0 & b & 2ab + 7b^2 + 24bc + 20c^2 \\
0 & c & -2b^2 + 2ac - 8bc - 7c^2
\end{bmatrix}.
$$

Taking its determinant, the index form associated to this basis is $-2b^3 - 15b^2 c - 31bc^2 - 20c^3$.

Were the extension monogenic, we would be able to find $a, b, c \in \mathbb{Z}$ so that the index form above is equal to $\pm 1$.

In fact, $\mathbb{Z}_{\mathbb{Q}(\eta)}/\mathbb{Z}$ is not even locally monogenic. By Lemma 4.6, we may check by reducing at primes. Over the prime 2 the index form reduces to $b^2 c + bc^2$,
and iterating through the four possible values of \((b, c) \in (\mathbb{Z}/2\mathbb{Z})^2\) shows that the index form always reduces to 0. That is, 2 is a common index divisor.

Dedekind showed that \(\mathbb{Z}_{\mathbb{Q}(\eta)}/\mathbb{Z}\) is non-monogenic, not by using an index form, but by deriving a contradiction from the \(\mathbb{Z}_L\)-factorization of the ideal 2, which splits into three primes. In our terms, Spec \(\mathbb{Z}_{\mathbb{Q}(\eta)}\) → Spec \(\mathbb{Z}\) has three points over Spec \(\mathbb{F}_2\), all with residue field \(\mathbb{F}_2\). Therefore, condition (2) of Theorem 4.16 for monogeneity at the prime (2) fails, so \(S' \to S\) is not monogenic.

The example above is minimal in the sense that the smallest a common index divisor can be is 2, and to fail to be monogenic over the integral prime (2) requires at least 3 points over Spec \(\mathbb{F}_2\) by Theorem 4.16, hence an extension of degree 3. The next smallest that a common index divisor can be is 3. By Theorem 4.16, this requires at least 4 points over the Spec \(\mathbb{F}_3\), so we will need an extension of at least degree 4. Our next example highlights a number field with common index divisor 3.

**Example 7.7** (Non-monogenic with common index divisor 3). Let \(\eta\) denote a root of the polynomial \(X^4 - X^3 + 2X^2 + 4X + 3\) and consider the field extension \(L := \mathbb{Q}(\eta)\) over \(K := \mathbb{Q}\). The integral prime 3 splits completely in this quartic field, making 3 a common index divisor. Let us check directly that the index form for \(\mathbb{Z}_L/\mathbb{Z}_K\) admits no solutions modulo 3.

One can compute a \((\mathbb{Z}_K = \mathbb{Z})\)-module basis for \(\mathbb{Z}_L\): \(\{1, \eta^2, \eta^3, (2\eta^3 + \eta)/3\}\). The matrix of coefficients is:

\[
\begin{pmatrix}
1 & a & a^2 - 4bc - 3ac^2 + 2bd - 6cd + 3d^2 & A \\
0 & b & 2ab - 4b^2 - 28bc - 12c^2 - 4bd - 42cd + 3d^2 & B \\
0 & c & -3b^2 + 2ac - 8bc - 2c^2 - 8bd - 12cd - 5d^2 & C \\
0 & d & 18bc + 9c^2 + 2ad - 6bd + 26cd - 9d^2 & D
\end{pmatrix}
\]

where

\[
A = a^3 + 6b^3 - 12abc + 51b^2c - 9ac^2 + 39bc^2 + 3c^3 + 6abd + 15b^2d - 18acd + 150bcd + 63c^2d + 9ad^2 + 3bd^2 + 108cd^2 - 9d^3;
\]

\[
B = 3a^2b - 12ab^2 + 58b^3 - 84abc + 276b^2c - 36ac^2 + 165bc^2 + 3c^3 - 12abd + 213b^2d - 126acd + 834bcd + 279c^2d + 9ad^2 + 237bd^2 + 621cd^2 + 72d^3;
\]

\[
C = -9ab^2 + 24b^3 + 3a^2c - 24abc + 66b^2c - 6ac^2 + 24bc^2 - 5c^3 - 24abd + 102b^2d - 36acd + 210bcd + 45c^2d - 15ad^2 + 141bd^2 + 165cd^2 + 63d^3;
\]

\[
D = -27b^3 + 54abc - 189b^2c + 27ac^2 - 135bc^2 - 9c^3 + 3a^2d - 18abd - 81b^2d + 78acd - 558bcd - 222c^2d - 27ad^2 - 54bd^2 - 405cd^2 + 10d^3.
\]

The index form equation corresponding to monogeneity is obtained by setting the determinant of this matrix equal to a unit of \(\mathbb{Z}\), so ±1. Since 3 is a known common index divisor, we consider this equation modulo 3:

\[-b^4cd + b^3c^2d - b^2c^3d - b^3d^2 - bc^2d^2 - b^3d^3 - b^4cd^3 - bc^2d^3 + b^2d^4 + bcd^4 + bd^5 = \pm 1.\]

Running through the possible values of \(b, c, d \in \mathbb{Z}/3\mathbb{Z}\), we see that there are no solutions to this equation, and thus no monogenerators of \(\mathbb{Z}_L\) over \(\mathbb{Z}\).

**Example 7.8** (A non-monogenic order and monogenic maximal order). Consider the extension \(\mathbb{Z}[\sqrt{2}, \sqrt{3}]\) of \(\mathbb{Z}\). Note that \(\mathbb{Z}[\sqrt{2}, \sqrt{3}]\) is not the maximal order of
\( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \). As we will see below, the maximal order is \( \mathbb{Z}[\sqrt{\sqrt{3} + 2}] \). The isomorphism of groups \( \mathbb{Z}[\sqrt{2}, \sqrt{3}] \cong \mathbb{Z} \oplus \mathbb{Z}[\sqrt{2}] \oplus \mathbb{Z}[\sqrt{3}] \oplus \mathbb{Z}[\sqrt{6}] \) identifies the Weil Restriction \( \mathcal{R}_{\mathbb{Z}[\sqrt{2}, \sqrt{3}]/\mathbb{Z}} \) and its universal maps with Spec of

\[
\begin{array}{cccc}
\mathbb{Z}[a, b, c, d][\sqrt{2}, \sqrt{3}] & \overset{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}}{\rightarrow} & \mathbb{Z}[a, b, c, d][t].
\end{array}
\]

Now,

\[
\begin{align*}
1 & \mapsto 1 \\
t & \mapsto a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \\
t^2 & \mapsto (a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6})^2 \\
t^3 & \mapsto (a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6})^3
\end{align*}
\]

is given by

\[
\begin{bmatrix}
a & a^2 + 2b^2 + 3c^2 + 6d^2 \\
b & 2ab + 6cd \\
c & 2ac + 4bd \\
d & 2bc + 2ad
\end{bmatrix}
\begin{bmatrix}
a^3 + 6ab^2 + 9ac^2 + 36bcd + 18ad^2 \\
a^2b + 2b^3 + 9bc^2 + 18acd + 18bd^2 \\
a^2c + 6b^2c + 3c^3 + 12abd + 18cd^2 \\
a^2d + 6bd^2 + 9c^2d + 6d^3
\end{bmatrix}.
\]

Taking the determinant, the index form with respect to our chosen basis is

\[
-8b^4c^2 + 12b^2c^3 + 16b^4d^2 - 36c^4d^2 - 48b^2d^4 + 72c^2d^4 = -4(2b^2 - 3c^2)(b^2 - 3d^2)(c^2 - 2d^2).
\]

The \( \mathbb{Z} \)-points of \( M_{1, \mathbb{Z}[\sqrt{2}, \sqrt{3}]/\mathbb{Z}} \) are in bijection with the tuples \( (a, b, c, d) \in \mathbb{Z}^4 \) such that the determinant is a unit. Since the determinant is divisible by 2, this never happens. We conclude that \( \mathbb{Z}[\sqrt{2}, \sqrt{3}] \) is not monogenic over \( \mathbb{Z} \). In light of Proposition 4.10, \( S' \to S \) fails to be monogenic over even geometric points, since the index form reduces to 0 in the fiber over 2. In terms of Theorem 4.14, \( S' \to S \) fails to be monogenic since the fiber over 2 consists of a single point with a two-dimensional tangent space.

The non-monogeneity of the order \( \mathbb{Z}[\sqrt{2}, \sqrt{3}] \) is in marked contrast to the maximal order of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \), which is monogenic. A computation shows that a power integral basis for the maximal order is given by \( \{1, \alpha, \alpha^2, \alpha^3\} \), where \( \alpha \) is a root of \( t^4 - 4t^2 + 1 \). One could take \( \alpha = \sqrt{\sqrt{3} + 2} \). Here the Weil Restriction \( \mathcal{R}_{\mathbb{Z}[\alpha]/\mathbb{Z}} \) and its universal maps are identified with Spec of

\[
\begin{array}{cccc}
\mathbb{Z}[a, b, c, d][\alpha] & \overset{a+b\alpha+ca^2+d\alpha^3}{\rightarrow} & \mathbb{Z}[a, b, c, d][t].
\end{array}
\]

The element-wise computation

\[
\begin{align*}
1 & \mapsto 1 \\
t & \mapsto a + b\alpha + ca^2 + d\alpha^3 \\
t^2 & \mapsto (a + b\alpha + ca^2 + d\alpha^3)^2 \\
t^3 & \mapsto (a + b\alpha + ca^2 + d\alpha^3)^3
\end{align*}
\]
yields the matrix of coefficients
\[
\begin{bmatrix}
1 & a & a^2 - c^2 - 2bd - 4d^2 & A \\
0 & b & 2ab - 2cd & B \\
0 & c & b^2 + 2ac + 4c^2 + 8bd + 15d^2 & C \\
0 & d & 2bc + 2ad + 8cd & D
\end{bmatrix},
\]
where
\[
A = a^3 - 3b^2c - 3ac^2 - 4c^3 - 6abd - 24bcd - 12ad^2 - 45cd^2,
\]
\[
B = 3a^2b - 3bc^2 - 3b^2d - 6acd - 12c^2d - 12bd^2 - 15d^3,
\]
\[
C = 3ab^2 + 3a^2c + 12b^2c + 12ac^2 + 15c^3 + 24abd + 90bcd + 45ad^2 + 168cd^2,
\]
\[
D = b^3 + 6abc + 12bc^2 + 3a^2d + 12d^2 + 24acd + 45c^2d + 45bd^2 + 56d^3.
\]
The determinant of this matrix yields the index form
\[
(b^2 - 2c^2 + 6bd + 9d^2)(b^2 - 6c^2 + 10bd + 25d^2)(b^2 + 4bd + d^2).
\]

One can compute that the index of \( \mathbb{Z}[\sqrt{2}, \sqrt{3}] \) inside of \( \mathbb{Z}[\sqrt{3} + 2] \) is 2. Therefore the index forms are equivalent away from the prime 2.

**Example 7.9** (A locally monogenic, but not twisted-monogenic extension). Here we take a closer look at one member of the family in Example 4.8. Let \( K = \mathbb{Q}, L = K(\sqrt[5]{2}, \sqrt[5]{7}). \) The ring of integers \( \mathbb{Z}_L = \mathbb{Z}[\sqrt[5]{2}, \sqrt[5]{7}] \) is not monogenic over \( \mathbb{Z}. \) Let \( \alpha = \sqrt[5]{2}, \beta = \sqrt[5]{7}. \) It turns out that \( \{1, \alpha, \beta\} \) is a \( \mathbb{Z} \)-basis for \( \mathbb{Z}_L, \) so the universal map may be identified with

\[
\mathbb{Z}_L[a, b, c] \xleftarrow{a + b\alpha + c\beta - d} \mathbb{Z}[a, b, c][t].
\]

Expanding
\[
1 \mapsto 1 \\
t \mapsto a + b\alpha + c\beta,
\]
\[
t^2 \mapsto (a + b\alpha + c\beta)^2
\]
we find that the matrix of coefficients is
\[
\begin{bmatrix}
1 & a & a^2 + 70bc \\
0 & b & 2ab + 7c^2 \\
0 & c & 2ac + 5\beta^2
\end{bmatrix}.
\]

Computing the determinant, we get the index form \( 5b^3 - 7c^3. \) Thus, for a given choice of \( a, b, c \in \mathbb{Z}, \) the primes which divide the value \( 5b^3 - 7c^3 \) are precisely the primes at which \( a + b\alpha + c\beta \) will fail to generate the extension.

The values obtained by this index form are \( \{0, \pm 5\} \) modulo 7. Since 5 is a unit in \( \mathbb{Z}/7\mathbb{Z}, \) we do have local monogenerators over \( D(7). \) Similarly, we have local monogenerators over \( D(5). \) Together, \( D(5) \) and \( D(7) \) form an open cover, and we see that this extension is Zariski-locally monogenic. However, as we can see by reducing modulo 7, the index form cannot be equal to \( \pm 1. \) Therefore there are no global monogenerators and \( \mathbb{Z}_L/\mathbb{Z} \) is not monogenic.

This is not a \( \mathbb{G}_m \)-twisted monogenic extension by Theorem 6.20, because \( h(\mathbb{Z}) = 1 \) and this extension is not globally monogenic. See Example 6.22 for a properly
\[\mathbb{G}_m\text{-twisted monogenic extension and an interesting comparison to the example presented here.}\]

### 7.3. Other examples.

**Example 7.10.** We investigate the analog of the integers in Example 7.4. We keep the same notation. The base ring is \(\mathbb{F}_3[\alpha]\) and the extension ring is \(\mathbb{F}_3[\alpha][x]/(x^3 - \alpha) = \mathbb{F}_3[\beta]\), where \(\beta^3 = \alpha\).

\[
\begin{align*}
\mathbb{F}_3[a, b, c][\beta] & \xrightarrow{a + b\beta + c\beta^2 \mapsto x} \mathbb{F}_3[\alpha][a, b, c][x]. \\
\mathbb{F}_3[\alpha][a, b, c] & \xrightarrow{1 \mapsto 1} \\
x & \mapsto a + b\beta + c\beta^2 \\
x^2 & \mapsto (a + b\beta + c\beta^2)^2
\end{align*}
\]

is given by

\[
\begin{bmatrix}
1 & a & a^2 + 2b\alpha \\
0 & b & c^2\alpha + 2ab \\
0 & c & b^2 + 2ac
\end{bmatrix}.
\]

The determinant is \(b^3 - c^3 \alpha\), which is not geometrically reduced: it factors as \((b - c\beta)^3\). To find the monogenerators of this extension, we set this expression equal to the units of \(\mathbb{F}_3[\alpha]\). Since \((\mathbb{F}_3[\alpha])^* = \pm 1\), the only solutions are \(b = \pm 1\), \(c = 0\). Thus

\[
M_{1,\mathbb{F}_3[\beta]/\mathbb{F}_3[\alpha]}(\mathbb{F}_3[\alpha]) = \{a \pm \beta : a \in \mathbb{F}_3[\alpha]\}.
\]

We can see that, much like number rings, monogeneity imposes a stronger restriction here than it does for the extension of fraction fields.

Our next examples concern the case that \(S' \to S\) is a finite map of algebraic curves, which are essentially never monogenic. On the other hand, we find explicit examples of \(\mathbb{G}_m\text{-twisted monogenic} S' \to S\). Theorem 6.18 constrains the possible line bundles that we may use to show \(\mathbb{G}_m\text{-twisted monogeneity}\). We make this precise in the lemma below.

**Lemma 7.11.** Let \(\pi : C \to D\) be a finite map of smooth projective curves of degree \(n\) and let \(g\) denote the genus.

1. \(\pi\) is only monogenic if it is the identity map;
2. If \(\pi\) is \(\mathbb{G}_m\text{-twisted monogenic},\) then \(1 - g(C) - n(1 - g(D))\) is divisible by \(\frac{1}{2}n(n - 1)\) in \(\mathbb{Z}\). Moreover, if \(\pi\) factors through a closed embedding into a line bundle \(E\) with sheaf of sections \(\mathcal{E}\), then

\[
\deg(\mathcal{E}) = \frac{1 - g(C) - n(1 - g(D))}{\frac{1}{2}n(n - 1)}.
\]

**Proof.** To see (1), note that a map \(f : C \to \mathbb{A}^1_D\) is determined by a global section of \(\mathcal{O}_C\). Since \(C\) is a proper variety, the global sections of \(\mathcal{O}_C\) are constant functions. It follows that a map \(f : C \to \mathbb{A}^1_D\) is constant on fibers of \(\pi\). Therefore \(f\) cannot be an immersion unless \(\pi\) has degree 1, i.e., is the identity.
Suppose \( \pi : C \to D \) is \( \mathbb{G}_m \)-twisted monogenic with an embedding into a line bundle \( E \). By [Stacks, 0AYQ] and Riemann-Roch,
\[
\deg(\det(\pi_\ast \mathcal{O}_C)) = 1 - g(C) - n(1 - g(D)).
\]

By Theorem 6.18,
\[
\det(\pi_\ast \mathcal{O}_{S'}) \simeq \mathcal{E} \cdot \frac{n(n-1)}{2}
\]
where \( \mathcal{E} \) is the sheaf of sections of \( E \). Taking degrees of both sides,
\[
1 - g(C) - n(1 - g(D)) = -\deg(\mathcal{E}) \cdot \frac{n(n-1)}{2}.
\]
This shows (2). \( \square \)

**Example 7.12** (Maps \( \mathbb{P}^1 \to \mathbb{P}^1 \)). Let \( k \) be an algebraically closed field and let \( \pi : \mathbb{P}^1_k \to \mathbb{P}^1_k \) be a finite map of degree \( n \). If \( n = 1 \), then \( \pi \) is trivially monogenic. When \( n = 2 \), Lemma 7.11 tells us that \( \pi \) cannot be monogenic, while Lemma 6.19 tells us that \( \pi \) is \( \text{Aff}^1 \)-twisted monogenic. Lemma 7.11 tells us that for degrees \( n > 2 \) the map \( \pi \) is neither monogenic nor \( \mathbb{G}_m \)-twisted monogenic, although Theorems 4.1 and 4.16 tell us that \( \pi \) is Zariski-locally monogenic.

Working with \( \mathbb{Z} \) instead of an algebraically closed field, consider the map \( \pi : S' = \mathbb{P}^1_{\mathbb{Z}} \to S = \mathbb{P}^1_{\mathbb{Z}} \) given by \( [a : b] \mapsto [a^2 : b^2] \). We will show by direct computation that this map is \( \mathbb{G}_m \)-twisted monogenic. Write \( U = \text{Spec} \mathbb{Z}[x] \) and \( V = \text{Spec} \mathbb{Z}[y] \) for the standard affine charts of the target \( \mathbb{P}^1 \). The map \( \pi \) is then given on charts by
\[
\mathbb{Z}[x] \to \mathbb{Z}[a], \quad x \mapsto a^2
\]
and
\[
\mathbb{Z}[y] \to \mathbb{Z}[b], \quad y \mapsto b^2.
\]

Let us compute \( M_{1,S'/S} \). Over \( U \), \( \pi_\ast \mathcal{O}_U \) has \( \mathbb{Z}[x] \)-basis \( \{1, a\} \). Let \( c_1, c_2 \) be the coordinates of \( \mathcal{R}|_U = \mathbb{A}^2 \), with universal map
\[
\mathbb{Z}[c_1, c_2, t] \to \mathbb{Z}[c_1, c_2, a], \quad t \mapsto c_1 + c_2 a.
\]
The index form associated to this basis is
\[
i(c_1, c_2) = c_2.
\]
Similarly, \( \pi_\ast \mathcal{O}_V \) has \( \mathbb{Z}[y] \)-basis \( \{1, b\} \), \( \mathcal{R}|_V \) analogous coordinates \( d_1, d_2 \), and the index form associated to this basis is
\[
i(d_1, d_2) = d_2.
\]
An element of \( \mathbb{Z}[x] \) (resp. \( \mathbb{Z}[y] \)) is a unit if and only if it is \( \pm 1 \), so
\[
M_1(U) = \{c_1 \pm a \mid c_1 \in \mathbb{Z}[x]\}
\]
\[
M_1(V) = \{d_1 \pm b \mid d_1 \in \mathbb{Z}[y]\}.
\]
We can see directly that \( \pi \) is not monogenic: the condition that a monogenerator \( c_1 \pm a \) on \( U \) glue with a monogenerator \( d_1 \pm b \) on \( V \) is that
\[
(c_1 \pm a)|_{U \cap V} = (d_1 \pm b)|_{U \cap V}.
\]
But this is impossible to satisfy since \( a|_{U \cap V} = b|_{U \cap V}^{-1} \).
Lemma 7.11 tells us that if $S' \to S$ is twisted monogenic, the line bundle into which $S'$ embeds must have degree 1. Let us therefore attempt to embed $S'$ into the line bundle with sheaf of sections $\mathcal{O}_{\mathbb{P}^1}(1)$. The sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$ restricts to the trivial line bundle on both $U$ and $V$, and a section $f \in \mathcal{O}_U$ glues to a section $g \in \mathcal{O}_V$ if
\[ y \cdot f|_{U \cap V} = g|_{U \cap V}. \]

Embedding $S'$ into this line bundle is therefore equivalent to finding a monogenerator $c_1 \pm a$ on $U$, and a monogenerator $d_1 \pm b$ on $V$ such that
\[ y((c_1 \pm a)|_{U \cap V}) = (d_1 \pm b)|_{U \cap V}. \]

Bearing in mind that $y = b^2 = a^{-2}$ on $U \cap V$, we find a solution by taking positive signs, $c_1 = 0$, and $d_1 = 0$. Therefore $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ is twisted monogenic.

Lemma 7.11 tells us that we must pass to higher genus to find a $\mathbb{G}_m$-twisted monogenic cover of $\mathbb{P}^1$ of degree greater than 2. Here is an example where the source is an elliptic curve.

**Example 7.13** (Twisted monogenic cover of degree 3). Let $E$ be the Fermat elliptic curve $V(x^3 + y^3 - z^3) \subset \mathbb{P}^2_k$. Consider the projection from $[0 : 0 : 1]$, i.e., the map $\pi : E \to \mathbb{P}^1$ defined by $[x : y : z] \mapsto [x : y]$. Write $U = \text{Spec } \mathbb{Z}[x]$ and $V = \text{Spec } \mathbb{Z}[y]$ for the standard affine charts of $\mathbb{P}^1$. The map is given on charts by $\mathbb{Z}[x] \to \mathbb{Z}[x, z]/(x^3 + 1 - z^3)$ and $\mathbb{Z}[y] \to \mathbb{Z}[y, z]/(1 + y^3 - (yz)^3)$. The gluing on overlaps is given by $x \mapsto y^{-1}$ on $\mathbb{P}^1$ and by $x \mapsto y^{-1}, z \mapsto z$ on $E$.

We now compute $M_{E/\mathbb{P}^1}$. Note that over $U$, $\mathcal{O}_E$ has the $\mathbb{Z}[x]$-basis $1, z, z^2$. The index form associated to $1, z, z^2$ is
\[ i(c_1, c_2, c_3) = c_2^3 - c_3^3(1 + z). \]

Over $V$, $\mathcal{O}_E$ has the $\mathbb{Z}[y]$-basis $1, yz, y^2z^2$. The index form associated to this basis is
\[ i(d_1, d_2, d_3) = d_2^3 - d_3^3(y^3 + 1). \]

An element of $\mathbb{Z}[x]$ or $\mathbb{Z}[y]$ is a unit if and only if it is $\pm 1$. This implies that
\[ M_1(U) = \{ c_1 \pm z \mid c_1 \in \mathbb{Z}[x] \} \]
\[ M_1(V) = \{ d_1 \pm yz \mid d_1 \in \mathbb{Z}[y] \}. \]

We see that there are no global sections of $M_1$, since coefficients of $z$ cannot match on overlaps.

However, if we twist so that we are considering embeddings of $E$ into $\mathcal{O}_{\mathbb{P}^1}(1)$, then the condition for a monogenerator $c_1 \pm z$ on $U$ to glue with a monogenerator $d_1 \pm yz$ on $V$ is that
\[ y((c_1 \pm z)|_{U \cap V}) = (d_1 \pm yz)|_{U \cap V}. \]

This is satisfiable, for example by taking the positive sign for both generators and $c_1 = d_1 = 0$. Therefore $E \to \mathbb{P}^1$ is twisted monogenic with class $1 \in \text{Pic}(\mathbb{P}^1_k)$.

**Example 7.14** (Jet spaces of $\mathbb{A}^2$). Consider an $m$-jet of $\mathbb{A}^2 = \text{Spec } k[t, u]$ determined as in Example 3.13 by
\[ t = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \cdots + a_m \varepsilon^m \]
\[ u = b_0 + b_1 \varepsilon + b_2 \varepsilon^2 + \cdots + b_m \varepsilon^m. \]
Linear changes of coordinates ensure \( a_0 = b_0 = 0 \) and that our jets satisfy \( t^{m+1} = u^{m+1} = 0 \) in \( k[\varepsilon]/\varepsilon^{m+1} \). To find the matrix for the induced \( k \)-linear map from

\[
k[t, u]/(t^{m+1}, u^{m+1}) = \bigoplus k \cdot t^e u^f
\]

to the jets \( k[\varepsilon]/\varepsilon^{m+1} = \bigoplus k \cdot \varepsilon^i \), we need the coefficient of each \( \varepsilon^p \) in the expression:

\[
t^e u^f = (a_1 \varepsilon + a_2 \varepsilon^2 + \cdots + a_m \varepsilon^m)(b_1 \varepsilon + b_2 \varepsilon^2 + \cdots + b_m \varepsilon^m)
\]

\[
= \left( \sum_{1 \leq r \leq m} \varepsilon^r \cdot \sum_{i_1 + i_2 + \cdots + i_m = m \atop i_1 + 2i_2 + \cdots + mi_m = r} \left( e \atop i_1, \ldots, i_m \right) \prod_{t=0}^{m} a_t^{i_t} \right)
\cdot \left( \sum_{1 \leq s \leq m} \varepsilon^s \cdot \sum_{j_1 + j_2 + \cdots + j_m = m \atop j_1 + 2j_2 + \cdots + mj_m = s} \left( f \atop j_1, \ldots, j_m \right) \prod_{t=0}^{m} b_t^{j_t} \right)
\]

\[
= \sum_{1 \leq p \leq m} \varepsilon^p \left( \sum_{i_1 + i_2 + \cdots + i_m = m \atop j_1 + j_2 + \cdots + j_m = m \atop (i_1 + j_1) + 2(i_2 + j_2) + \cdots + m(i_m + j_m) = p} \left( e \atop i_1, \ldots, i_m \right) \left( f \atop j_1, \ldots, j_m \right) \prod_{t=0}^{m} a_t^{i_t} b_t^{j_t} \right)
\]

If \( e + f > p \), the coefficient of \( \varepsilon^p \) in \( t^e u^f \) is again zero. If \( e + f > m \), all the coefficients are zero. The corresponding \( m^2 \times m \) matrix is “lower triangular” in this sense.

Take \( m = 1 \) to reduce to \( A \). Cayley’s original situation of a \( 2 \times 2 \times 2 \) hypermatrix; compute his second hyperdeterminant \( \text{Det} \) to be \( a_1^2 b_1^2 \). In this case, \( \mathbb{R}^2 \) is the tangent space of \( \mathbb{A}^2 \), the index forms cut out the locus where both \( a_1 \) and \( b_1 \) are zero, and the hyperdeterminant cuts out the locus where either \( a_1 \) or \( b_1 \) are zero.

Computability is a serious constraint for even simple cases. Taking \( \mathbb{A}^3 \) and \( m = 1 \) yields a \( 2 \times 2 \times 2 \times 2 \) hypermatrix. The formula for such a hyperdeterminant is degree 24 and has 2,894,276 terms [Ott13, Remark 5.7].

**Example 7.15** (Limits and Colimits). Let \( B \) be an \( A \)-algebra which is complete with respect to \( I \subseteq B \). If each \( B_m := B/I^m \) is finite locally free over \( A \) and \( X \to \text{Spec} \ A \) is quasiprojective, there are affine restriction maps \( M_{X, B_m+1}/A \to M_{X, B_m}/A \). One can define

\[
M_{X, B}/A := \lim_m M_{X, B_m}/A
\]

which is a scheme [Stacks, 01YX]. By [Bha16, Remark 4.6, Theorem 4.1], this limit parametrizes closed embeddings \( s : \text{Spec} \ B \to X \) over \( \text{Spec} \ A \) as in Definition 2.2. The arc space examples \( k[t]/k, k[x, y]/k \) were mentioned in Example 2.8.

We cannot make a similar statement for colimits of algebras. Suppose \( \{B_i\} \) is a diagram of \( A \)-algebras indexed by \( N \). Then for each \( i < j \) there is a natural map \( \mathcal{R}_{B_i}/A \to \mathcal{R}_{B_j}/A \). Notice that if the image of some \( \theta \in B_i \) is a monogener of \( B_j \), then \( B_i \to B_j \) is surjective. It follows that \( \mathcal{R}_{B_i}/A \to \mathcal{R}_{B_j}/A \) only takes \( M_{B_i}/A \) into \( M_{B_j}/A \) if \( \text{Spec} \ B_j \to \text{Spec} \ B_i \) is a closed immersion over each open set \( U \subseteq \text{Spec} A \) over which \( M_{B_i}/A \) is non-empty. Assuming \( M_{B_i}/A \) is locally non-empty, the only diagrams \( \{B_i\} \) for which the colimit \( \text{colim}_i M_{B_i}/A \) can even be formed are those for
which each $B_i \to B_j$ is surjective. Since $B_0$ is Noetherian, all such diagrams are eventually constant and uninteresting.

Appendix A. Basics of Torsors and Stack Quotients

With this appendix our aim is to introduce the minimum prerequisites for torsors and stack quotients. The concepts covered here will be employed throughout the paper, especially in Sections 5 and 6. The reader interested in a more thorough treatment should consult [Fan+05].

Let $C$ be the category $(\text{Sch}/\ast)$ of schemes over a fixed final scheme $\ast$ such as $\ast = \text{Spec} \mathbb{Z}$ or $\text{Spec} \mathbb{C}$, equipped with the étale topology. One can also take the Zariski, fppf, or fpqc topologies, or any other site for $C$. Let $G$ be a sheaf of groups on $C$, which one may take to be $\mathbb{G}_m$, $\text{Aff}^1$, $\text{GL}_n$ for concreteness.

**Definition A.1.** A (left) $G$-pseudotorsor $P$ over an object $X \in C$ is a sheaf on $C/X$ such that

$$G \times P \xrightarrow{\sim} P \times_X P; \quad (g, p) \mapsto (gp, p)$$

is an isomorphism. It is called a $G$-torsor if, for some cover $\{U_i \to X\}$, $P(U_i)$ is nonempty for each $i$.

**Slogan:** for any two sections $p, p'$ of $P$, there is a unique section $g$ of $G$ such that $gp = p'$. If one locally chooses sections $p_i \in P(U_i)$, the multiplication map

$$G(U_i) \xrightarrow{\sim} P(U_i); \quad g \mapsto g.p_i$$

is a bijection. Locally, but not necessarily globally, $P$ is isomorphic to $G$ – a torsor is called trivial if $P$ is actually globally isomorphic to $G$.

**Example A.2.** Let $L \to X$ be a line bundle and $P := L \setminus X$ be $L$ without the zero section. Locally in $X$, $L \simeq \mathbb{A}^1_X$ and $P \simeq \mathbb{G}_m \times X$. But two such trivializations $L \simeq \mathbb{A}^1_X$ differ by units in $\mathbb{G}_m \times X$ and the same holds for $P$. Regardless, $P$ is a $\mathbb{G}_m$-torsor, and so is the sheaf of isomorphisms $\text{Isom}(L, \mathbb{A}^1)$.

**Remark A.3.** Define right $G$-torsors the same way but with a right $G$-action. Right $G$-torsors $P$ become left $G$-torsors $P^L$ via

$$g.p := p.g^{-1}$$

in the usual way. For abelian groups, these two actions and left and right $G$-torsors coincide.

We exclusively use left torsors, even with contracted products:

**Definition A.4.** Let $G$ be a sheaf of groups on $X$. Let $BG$ be the $X$-stack whose $T$ points are

$$BG(T) := \{\text{left } G|_T\text{-torsors on } T\},$$

where maps are given by $G$-equivariant isomorphisms. Remark that $H^1(T, G|_T) \simeq BG(T)/\text{isomorphism}$.

If $G \times X$ is a left action and $G \circ P$ a left torsor, form the contracted product

$$X \times^G P := X \times P / (\Delta, G)$$

as the quotient of $X \times P$ by the diagonal action:

$$g.(x, p) := (g.x, g.p).$$
The same can be done for a fiber product over a base $Z$:

$$X \wedge_Z^G P := X \times_Z P/(\Delta, G).$$

For example, let $\varphi : G \to H$ be a morphism of sheaves of groups and $P$ a $G$-torsor. Obtain an action $G \triangleleft H$ and form the contracted products $H \wedge^G P$ to get a map

$$BG \to BH; \quad P \mapsto P \wedge^G H,$$

which is well-defined. If $G$ is abelian, the multiplication map $m : G \times G \to G$ is a group homomorphism and we have a group structure on $BG$ via contracted product along $m$.

If $G \triangleleft X$ and $H \triangleleft Y$ are two actions and $f : X \to Y$ is “equivariant along” $\varphi : G \to H$, i.e. $f(g.x) = \varphi(g).f(x)$, then there is a unique $H$-equivariant map $H \wedge^G X \to Y$ such that $X \to H \wedge^G X \to Y$ is $f$.

**Remark A.5.** Contracted products are usually defined for a left action $G \triangleleft P$ and a right action $G \triangleleft X$ by taking the quotient

$$X \wedge^G P := X \times P/(\Delta, G)$$

by the antidiagonal action $\Delta$:

$$G \times X \times P; \quad g.(x, p) := (x, g^{-1}, g.p).$$

Turning the right action $G \triangleleft X$ into a left action $X^L$ as in Remark A.3, this is the same as quotienting by the diagonal action

$$X \wedge^G P = X^L \times P/(\Delta, G) = X \times P/(-\Delta, G).$$

**Definition A.6.** Let $G \triangleleft X$ be an action on a scheme or a sheaf $X$. Let $T \to BG$ parametrize a $G|_T$-torsor $P \to T$ and define the quotient stack by

$$[X/G] := \{ f : P \to X \mid f \text{ is } G\text{-equivariant} \}.$$ 

In other words, a map $T \to [X/G]$ from a scheme to the quotient stack parametrizes a $G|_T$-torsor $P \to T$ and an equivariant morphism $P \to X$. If $\ast$ is the final sheaf with trivial $G$ action, $[\ast/G] = BG$. If $f : X \to Y$ is equivariant along $\varphi : G \to H$ as in A.4, there is a map

$$[X/G] \to [Y/H]$$

similarly induced by contracted product.

**Example A.7.** If $\varphi : G \to M$ is a group homomorphism, the nonempty preimages $\varphi^{-1}(m)$ for $m \in M$ become torsors (“cosets”) for $\ker \varphi$. This applies to sheaves of groups as well, giving possibly nontrivial torsors. From a short exact sequence of sheaves of groups

$$0 \to K \to G \xrightarrow{\varphi} M \to 0$$

and a section $s \in \Gamma(M)$, define the $K$-torsor $P_s$ by

$$P_s(U) := \{ t \in G(U) \mid s = \varphi(t) \}.$$ 

The connecting map $\Gamma(M) \to H^1(K)$ sends $s \in \Gamma(M)$ to the isomorphism class of $P_s \in H^1(K)$. This leads to the usual long exact sequence

$$1 \to \Gamma(K) \to \Gamma(G) \to \Gamma(M) \to H^1(K) \to H^1(G) \to H^1(M) \to \cdots$$
Remark A.8 ([Stacks, 003O]). Suppose given functors $f : \mathcal{F} \to \mathcal{H}$, $g : \mathcal{G} \to \mathcal{H}$ between stacks over $C$. The $T$-points of the fiber product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ are:

- a pair of points $x \in \mathcal{F}(T)$, $y \in \mathcal{G}(T)$ and
- an isomorphism $\beta : f(x) \cong g(y)$ between their images in $\mathcal{H}$.

Proposition A.9. Let $G, H$ be finite groups, and suppose $\mathcal{H} = \text{Spec } k$ where $k$ is separably closed. Then the category $\text{Hom}(BG, BH)$ is equivalent to the category whose objects are homomorphisms $\phi : G \to H$ of groups, and where a morphism $f : (\phi : G \to H) \to (\psi : G \to H)$ is an element $h_f \in H$ such that $\psi(g) = h_f^{-1}\phi(g)h_f$ for all $g \in G$.

Proof. By definition, a morphism $BG \to BH$ is equivalent to an $H$-torsor $P \to BG$. The map $\mathcal{H} \to BG$ is an étale cover because $G$ is finite. Since $BH$ is a stack, an $H$-torsor $P$ on $BG$ is uniquely determined by its associated descent datum on the cover $\mathcal{H} \to BG$. Conversely, every such $H$-torsor arises from such a descent datum.

Consider a descent datum for $BH$ with respect to the cover $\mathcal{H} \to BG$. In the notation of [Fan+05, Section 4.1.2], we have

1. an $H$-torsor $P_{\mathcal{H}} \to \mathcal{H}$;
2. an isomorphism of $H$-torsors $\alpha : pr_{12}^*P_{\mathcal{H}} \to pr_{23}^*P_{\mathcal{H}}$ on $\mathcal{H} \times_{BG} \mathcal{H}$ such that the cocycle condition $pr_{23}^*\alpha \circ pr_{12}^*\alpha = pr_{13}^*\alpha$ holds on $\mathcal{H} \times_{BG} \mathcal{H} \times_{BG} \mathcal{H}$.

We argue that the fiber products of the cover $\mathcal{H} \to BG$ are:

\begin{align*}
\begin{array}{ccc}
G \times G & \xrightarrow{pr_{23}} & G \\
pr_{13} \downarrow & & \downarrow pr_{13} \\
G & \xrightarrow{pr_{12}} & \mathcal{H} \\
pr_{1} \downarrow & & \downarrow pr_{1} \\
\mathcal{H} & \xrightarrow{pr_{2}} & BG \\
pr_{1} \downarrow & & \downarrow pr_{1} \\
\mathcal{H} & \xrightarrow{pr_{2}} & BG \\
& \downarrow & \\
& BG & \\
\end{array}
\end{align*}

An $H$-torsor $P_{\mathcal{H}} \to \mathcal{H}$ over a point can only be the trivial torsor $H \to \mathcal{H}$ since we have assumed our base separably closed: $\mathcal{H}$ has no non-trivial étale covers, so for $P_{\mathcal{H}}$ to be étale-locally a trivial torsor, $P_{\mathcal{H}}$ must itself be a trivial torsor.

We next consider the pullbacks of $H \to \mathcal{H}$ to $\mathcal{H} \times_{BG} \mathcal{H}$ as in Remark A.8. In our case, $\mathcal{H} = \text{Spec } k(T)$ is the one-element set and its lonely element maps to the trivial torsor $G \times T \to T$ in $BG(T)$. Therefore $(\mathcal{H} \times_{BG} \mathcal{H})(T)$ may be identified with the set of automorphisms $\beta$ of $G \times T \to T$. Such automorphisms are in bijection with sections of $G \times T \to T$: one only has to choose an element of $G$ by which to translate in each fiber.

Thus the two pullbacks $pr_{12}^*P_{\mathcal{H}}$ and $pr_{23}^*P_{\mathcal{H}}$ may both be identified with the trivial $H$-torsor $H \times G \to G$. To complete a descent datum, we ask what isomorphisms $\alpha : H \times G \to H \times G$ satisfy the cocycle condition. Suppose that $\alpha$ is such a morphism. Then since $\alpha$ is an isomorphism of torsors, its action on points takes the form

$$\alpha : (h, g) \mapsto (h \cdot \phi(g), g),$$
where \( \phi : G \to H \) is the function taking \( g \in G \) to the \( H \)-coordinate of \( \alpha(1, g) \).

It remains to check the implications of the cocycle condition. Arguing similarly to \( * \times BG * \), we may identify the \( T \)-points of \( * \times BG * \times BG * \) with pairs of isomorphisms \( (\beta, \gamma) \) of the trivial torsor, whose three projections to \( * \times BG * \) are given by \( \text{pr}_{12}(\beta, \gamma) = \beta, \text{pr}_{23}(\beta, \gamma) = \gamma, \) and \( \text{pr}_{13}(\beta, \gamma) = \gamma \circ \beta \). Since such isomorphisms can be identified with elements of \( G \), we may identify \( * \times BG * \times BG * \) with \( G \times G \). The projections then become

\[
\begin{align*}
\text{pr}_{12} & : G \times G \to G \\
(g_{12}, g_{23}) & \mapsto g_{12} \\
\text{pr}_{23} & : G \times G \to G \\
(g_{12}, g_{23}) & \mapsto g_{23} \\
\text{pr}_{12} & : G \times G \to G \\
(g_{12}, g_{23}) & \mapsto g_{12}g_{23}
\end{align*}
\]

Therefore, the pullbacks of \( \alpha \) to \( * \times BG * \times BG * \) may be identified with

\[
\begin{align*}
\text{pr}_{12}^{*} \alpha & : H \times G \times G \to G \\
(h, g_{1}, g_{2}) & \mapsto (h \cdot \phi(g_{1}), g_{1}, g_{2}) \\
\text{pr}_{23}^{*} \alpha & : H \times G \times G \to G \\
(h, g_{1}, g_{2}) & \mapsto (h \cdot \phi(g_{2}), g_{1}, g_{2}) \\
\text{pr}_{13}^{*} \alpha & : H \times G \times G \to G \\
(h, g_{1}, g_{2}) & \mapsto (h \cdot \phi(g_{1}g_{2}), g_{1}, g_{2}).
\end{align*}
\]

We conclude that an isomorphism \( \alpha \) satisfies the cocycle condition if and only if

\[
\phi(g_{1})\phi(g_{2}) = \phi(g_{1}g_{2}), \quad \text{for all } g_{1}, g_{2} \in G.
\]

That is, if and only if \( \phi \) is a group homomorphism. This establishes our claim on the elements of \( \text{Hom}(BG, BH) \).

Now suppose that \( P_{\phi} \to BG \) and \( P_{\psi} \to BG \) are \( H \)-torsors with associated group homomorphisms \( \phi : G \to H, \psi : G \to H \). A morphism \( f : P_{\phi} \to P_{\psi} \) is equivalent to a morphism of the associated descent data, i.e., a morphism between the restrictions of \( P_{\phi} \) and \( P_{\psi} \) to \( * \) that commutes with the gluing morphisms. Now, since \( P_{\phi}|_{*} \) and \( P_{\psi}|_{*} \) are both trivial torsors over a point, the morphisms \( P_{\phi}|_{*} \to P_{\psi}|_{*} \) are in bijection with elements of \( H \): one just has to choose what element of \( H \) to translate by. Write \( h_{f} \) for the element associated to \( f : P_{\phi} \to P_{\psi} \). Compatibility with gluing morphisms translates to

\[
(h_{f} \phi(g)h_{f}, g) = (h \cdot h_{f} \psi(g), g) \quad \text{for all } h \in H, g \in G.
\]

Equivalently,

\[
\psi(g) = h_{f}^{-1} \phi(g)h_{f} \quad \text{for all } g \in G.
\]

Remark A.10. When \( H = \Sigma_{n} \) and \( G \) is a finite group, the proposition describes the category \( \text{Hom}(BG, B\Sigma_{n}) \) as:

- Ob: Objects are group homomorphisms \( G \to \Sigma_{n} \), or actions \( G \circ \langle n \rangle \).
- Mor: Morphisms are reorderings of the set \( \langle n \rangle \) on which \( G \) acts.
A functor \( BG \to B\Sigma_n \) up to isomorphism is an action of \( G \) on an unordered \( n \)-element set. In particular, all the inclusions \( \Sigma_{n-1} \subseteq \Sigma_n \) given by omitting one of the \( n \)-letters induce the same map \( B\Sigma_{n-1} \to B\Sigma_n \) up to isomorphism.

**Remark A.11.** The assumption \( \ast = \text{Spec} \ k \) separably closed was used for one purpose in the proof: to show \( \ast \to BG \to BH \) parametrized the trivial torsor. For general final schemes \( \ast \) there could be many \( H \)-torsors on \( \ast \) and no guarantee that \( \ast \to BG \to BH \) parametrizes the trivial one.

Suppose \( \ast \to BG \to BH \) parametrizes a torsor \( P \). Restriction along the map \( \ast \to BG \) gives a map

\[
\hom(BG,BH) \xrightarrow{P} BH.
\]

The fiber of this map over the trivial torsor admits the above description exactly: the objects are group homomorphisms \( G \to H \), and the morphisms are given by elements \( h_f \in H \) conjugating one homomorphism to the other. This fiber is also the full subcategory of functors \( BG \to BH \) which are monoidal, preserving the contracted product of torsors.

What about the other fibers? The fiber of \( \hom(BG,BH) \to BH \) over an \( H \)-torsor \( P \) consists of functors \( BG \to BH \) such that \( \ast \to BG \to BH \) parametrizes \( P \). The composite

\[
\ast \to BG \to BH \xrightarrow{\cdot - P} BH
\]

with contracting by \(-P\) then parametrizes the trivial \( H \) torsor on \( \ast \) and admits the above description. The other fibers are “torsors” for the fiber over the trivial torsor.

This is reminiscent of the fact that any map of abelian varieties as schemes is a group homomorphism plus a translation – any functor \( BG \to BH \) is a monoidal functor composed with contraction by \( P \).

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