GREEN RINGS OF POINTED RANK ONE HOPF ALGEBRAS OF NON-NILPOTENT TYPE

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Abstract. In this paper, we continue our study of the Green rings of finite dimensional pointed Hopf algebras of rank one initiated in [22], but focus on those Hopf algebras of non-nilpotent type. Let $H$ be a finite dimensional pointed rank one Hopf algebra of non-nilpotent type. We first determine all non-isomorphic indecomposable $H$-modules and describe the Clebsch-Gordan formulas for them. We then study the structures of both the Green ring $r(H)$ and the Grothendieck ring $G_0(H)$ of $H$ and establish the precise relation between the two rings. We use the Cartan map of $H$ to study the Jacobson radical and the idempotents of $r(H)$. It turns out that the Jacobson radical of $r(H)$ is exactly the kernel of the Cartan map, a principal ideal of $r(H)$, and $r(H)$ has no non-trivial idempotents. Besides, we show that the stable Green ring of $H$ is a transitive fusion ring. This enables us to calculate Frobenius-Perron dimensions of objects in the stable category of $H$. Finally, as an example, we present both the Green ring and the Grothendieck ring of the Radford Hopf algebra in terms of generators and relations.

1. Introduction

In [22], we studied the Green rings of finite dimensional pointed rank one Hopf algebras of nilpotent type. The Green ring of a finite dimensional pointed rank one Hopf algebra $H$ of nilpotent type is isomorphic to a polynomial ring in one variable over the Grothendieck ring of $H$ modulo one relation. This Green ring is a symmetric algebra over $Z$ with the Jacobson radical being a principal ideal. The stable Green ring (i.e., the Green ring of the stable category) of $H$ is isomorphic to the quotient ring of the Green ring of $H$ modulo the ideal generated by isomorphism classes of indecomposable projective $H$-modules. In particular, the complexified stable Green algebra is both a group-like algebra and a bi-Frobenius algebra introduced by Doi and Takeuchi (cf. [6, 7, 8, 9]).

In this paper, we continue the study of the Green rings of finite dimensional pointed Hopf algebras of rank one, but concentrate us on pointed rank one Hopf algebras of non-nilpotent type. The representation theory of a pointed

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rank one Hopf algebra of non-nilpotent type is closely related to that of some pointed rank one Hopf algebra of nilpotent type. However, as we shall see in the non-nilpotent case, the Green ring is much more complicated. The main reason is due to the lack of the Chevalley property in the non-nilpotent case. This leads to the problem that the free abelian group generated by simple modules does not form a subring of the Green ring. Because of this, it is hard to describe the Green ring in terms of generators and relations as we did in the nilpotent case [22]. However, we can still study the ring-theoretical properties of the Green ring, including the Frobenius property, the Jacobson radical and idempotents of the Green ring.

The paper is organized as follows. In Section 2 we first recall the Hopf algebra structure of a finite dimensional pointed rank one Hopf algebra $H$ of non-nilpotent type. We construct a central idempotent element $e$ of $H$ such that the subalgebra $H(1 - e)$ is a Hopf ideal of $H$. It turns out that the quotient Hopf algebra $\overline{H} := H / H(1 - e)$ is a pointed rank one Hopf algebra of nilpotent type. Thus, the indecomposable $\overline{H}$-modules are also indecomposable modules over $H$. On the other hand, every finite dimensional simple module over the subalgebra $H(1 - e)$ is projective. Consequently, $H(1 - e)$ is a semisimple subalgebra of $H$. As a result, we obtain all indecomposable $H$-modules from both the indecomposable modules over $\overline{H}$ and the simple modules over $H(1 - e)$.

In Section 3 and Section 4, we study the Clebsch-Gordan formulas for the decompositions of tensor products of indecomposable $H$-modules. This includes the decompositions of the tensor products of simple $H(1-e)$-modules. We work out the relation between the Green rings $r(H)$ of $H$ and $r(\overline{H})$ of $\overline{H}$. Let $\mathcal{P}$ be the ideal of $r(H)$ generated by isomorphism classes of indecomposable projective $H$-modules and the direct sum $r(\overline{H}) \oplus \mathcal{P}$ a ring with a new multiplication rule. Instead of describing the Green ring $r(H)$ in terms of generators and relations, we show that $r(H)$ is isomorphic to a quotient ring of $r(\overline{H}) \oplus \mathcal{P}$ modulo an ideal described in detail.

In Section 5, we investigate the Grothendieck ring $G_0(H)$ of $H$. We first show that the Green ring $r(H)$ admits an associative, symmetric and nonsingular $\mathbb{Z}$-bilinear form. The Grothendieck ring $G_0(H)$ is isomorphic to the quotient ring $r(H) / \mathcal{P}^\perp$, where $\mathcal{P}^\perp$ is the orthogonal complement of $\mathcal{P}$ with respect to the bilinear form. In addition, $G_0(H)$ can also be regarded as a subring of the Green ring $r(kG)$ of the group algebra $kG$, where $G$ is the group of group-like elements of $H$. Besides, we describe the Cartan matrix of the Cartan map from $\mathcal{P}$ to $G_0(H)$ with respect to the given bases. It turns out that this Cartan matrix is of block-diagonal form with each block described explicitly.

In Section 6, we use the Cartan map to study the Jacobson radical and idempotents of $r(H)$. Our approach to the Jacobson radical of $r(H)$ is
different from the nilpotent case carried out in [22, Section 5], where we had to study the irreducible representations of the complexified Green ring $r(H) \otimes_{\mathbb{Z}} \mathbb{C}$ in order to get the rank of the Jacobson radical of the Green ring $r(H)$. With the new approach by using the Cartan map, we can show that the Jacobson radical of $r(H)$ is precisely the kernel of the Cartan map, which is exactly the intersection $\mathcal{P} \cap \mathcal{P}^\perp$. This implies that the Jacobson radical of $r(H)$ is a principal ideal and that the Green ring $r(H)$ has no non-trivial idempotents, a property that holds as well in the nilpotent case.

In Section 7, we consider the stable Green ring of $H$. Based on the representations of $H$, we see that the stable Green rings $r_{st}(H)$ of $H$ and $r_{st}(\overline{H})$ of $\overline{H}$ coincide, where the latter has been studied in [22, Section 6]. Here we go further to show that the stable Green ring $r_{st}(H)$ is a transitive fusion ring. This allows us to study the Frobenius-Perron dimensions of objects in the stable category of $H$ in the framework of $r_{st}(H)$. Using the Dickson polynomials (of the second type), we calculate all Frobenius-Perron dimensions of indecomposable objects of the stable category of $H$.

In Section 8, as an example, we compute the Green ring $r(H)$ of the Radford Hopf algebra $H$. Although the Green ring of a pointed rank one Hopf algebra of non-nilpotent type is difficult to describe in terms of generators and relations, we can do so for the Radford Hopf algebra $H$ because the quotient Hopf algebra $\overline{H} = H/(1 - e)$ is a Taft Hopf algebra. According to [2, Theorem 3.10] or [22, Theorem 4.3], we are able to show that the Green ring $r(H)$ is isomorphic to a quotient ring of a polynomial ring in several variables over $\mathbb{Z}$ modulo certain relations described explicitly. As a quotient of the Green ring $r(H)$, the Grothendieck ring $G_0(H)$ of $H$ can be presented as well in terms of generators and relations.

Throughout, we work over an algebraically closed field $k$ of characteristic zero. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over $k$; all modules are left modules and finite dimensional; $\dim$ and $\otimes$ stand for $\dim_k$ and $\otimes_k$ respectively. For a finite set $B$, $|B|$ is the cardinality of $B$ and $\text{sp}B$ is the $k$-vector space spanned by $B$. We refer to [14, 19] for the theory of Hopf algebras.

2. INDECOMPOSABLE REPRESENTATIONS

A quadruple $D = (G, \chi, g, \mu)$ is called a group datum if $G$ is a finite group, $\chi$ a $k$-linear character of $G$, $g$ an element in the center of $G$, and $\mu \in k$ subject to $\chi^n = 1$ or $\mu(g^n - 1) = 0$, where $n$ is the order of $\chi(g)$. The group datum $D$ is said to be of nilpotent type if $\mu(g^n - 1) = 0$, and it is of non-nilpotent type if $\mu(g^n - 1) \neq 0$ and $\chi^n = 1$ (cf. [4, 15]).

Let $H_D$ be the Hopf algebra associated to the group datum $D = (G, \chi, g, \mu)$. Namely, as an associative algebra, $H_D$ is generated by $y$ and all elements $h$
in $G$ such that $kG$ is a subalgebra of $H_D$ and
\begin{equation}
(2.1) \quad y^n = \mu(g^n - 1), \; yh = \chi(h)hy, \; \text{for} \; h \in G.
\end{equation}
The comultiplication $\Delta$, the counit $\varepsilon$, and the antipode $S$ of $H_D$ are given respectively by
\begin{equation}
(2.2) \quad \Delta(y) = y \otimes g + 1 \otimes y, \; \varepsilon(y) = 0, \; S(y) = -yg^{-1},
\end{equation}
and
\begin{equation}
(2.3) \quad \Delta(h) = h \otimes h, \; \varepsilon(h) = 1, \; S(h) = h^{-1},
\end{equation}
for all $h \in G$.

The Hopf algebra $H_D$ is a finite dimensional pointed Hopf algebra of rank one [15] with a canonical $k$-basis $\{ y^ih \mid h \in G, \; 0 \leq i \leq n - 1 \}$. The group of group-like elements of $H_D$ is $G$ and $\dim H_D = n|G|$. The Hopf algebra $H_D$ is said to be of nilpotent type (resp. non-nilpotent type) if the associated group datum $D$ is nilpotent (resp. non-nilpotent), see [15, 22].

Throughout this paper, $D = \langle G, \chi, g, \mu \rangle$ is a fixed group datum of non-nilpotent type with $n$ the order of $q := \chi(g)$. Without loss of generality we may assume that $\mu = 1$ (see [15, Corollary 1]). For the simplicity, we shall drop $D$ from $H_D$, and denote by $H$ the Hopf algebra $H_D$ associated to the group datum $D = \langle G, \chi, g, 1 \rangle$. In this case, the relations in (2.1) become
\begin{equation}
(2.1') \quad y^n = g^n - 1, \; yh = \chi(h)hy, \; \text{for} \; h \in G.
\end{equation}
Note that $\chi^n = 1$ and the order of $q = \chi(g)$ is $n$. The order of $\chi$ is also $n$. If $n = 1$, the Hopf algebra $H$ is nothing but $kG$. To avoid this, we always assume that $n \geq 2$. In this case, $\chi(g) \neq 1$, implying that $g \neq 1$ and $\chi \neq \varepsilon$.

Note that the order of $q = \chi(g)$ is $n$. The order of $g$ in the group $G$ is $nr$, where $r > 1$ since the group datum $D$ is of non-nilpotent type. Denote by
\begin{equation}
N = \{1, g^n, g^{2n}, \ldots, g^{(r-1)n}\}
\end{equation}
and
\begin{equation}
(2.3') \quad e = \frac{1}{r}(1 + g^n + g^{2n} + \cdots + g^{(r-1)n}).
\end{equation}
Then $N$ is a normal subgroup of $G$ and $e$ is a central idempotent of $H$ satisfying $eg^n = e$. Let $G = G/N$. The following lemma is obvious.

**Lemma 2.1.** Let $kG(g^n - 1)$ be the ideal of $kG$ generated by $g^n - 1$. Then $kG/kG(g^n - 1) \cong kG$.

**Remark 2.2.** Note that $1 - e$ and $1 - g^n$ are both central elements of $H$ and $1 - e = (1 - g^n)\frac{1}{r} \sum_{k=1}^{r-1}(1 + g^n + \cdots + g^{(k-1)n})$ whereas $1 - g^n = (1 - e)(1 - g^n)$.

1. The ideals of $kG$ generated respectively by $1 - e$ and $1 - g^n$ coincide. It follows from Lemma 2.1 that $kG \cong kGe$.
2. The ideals of $H$ generated respectively by $1 - e$ and $1 - g^n$ coincide. Thus, $H(1 - e)$ is a Hopf ideal of $H$ because $H(1 - g^n)$ is.
Let $\pi$ be the natural projection from $G$ to $\overline{G}$ and $\overline{h} := \pi(h)$ for $h \in G$. The character $\chi$ of $G$ induces a character $\overline{\chi}$ of $\overline{G}$ such that $\overline{\chi} \circ \pi = \chi$. Denote by $\overline{\mathcal{D}} = (\overline{G}, \overline{\chi}, \overline{g}, 0)$. Then the group datum $\overline{\mathcal{D}}$ is of nilpotent type since $\gamma^n - 1 = 0$, where $n$ is the order of $\chi(g) = \overline{\chi}(g)$. Let $\overline{H}$ be the Hopf algebra associated to the group datum $\overline{\mathcal{D}}$. More precisely, $\overline{\mathcal{H}}$ is generated as an algebra by $z$ and all elements $\overline{h} \in \overline{G}$ such that $k\overline{G}$ is a subalgebra of $\overline{H}$ and the relations (2.1), (2.2) and (2.3) are satisfied when we replace $y$ and elements $h \in G$ with $z$ and $\overline{h} \in \overline{G}$ respectively. So $\overline{H}$ is a finite dimensional pointed rank one Hopf algebra of nilpotent type with a $k$-basis \[\{z^i\overline{h} \mid \overline{h} \in \overline{G}, 0 \leq i \leq n - 1\}.\]

The relation between Hopf algebras $H$ of non-nilpotent type and $\overline{\mathcal{H}}$ of nilpotent type is described as follows:

**Proposition 2.3.**

1. $\overline{H}$ is isomorphic to $H/H(1-e)$ as a Hopf algebra.
2. $\overline{\mathcal{H}}$ is isomorphic to $He$ as an algebra.

**Proof.**

(1) It is easy to check that the algebra epimorphism $\rho : H \to \overline{H}$ given by $\rho(y) = z$ and $\rho(h) = \overline{h}$, for any $h \in G$ respects the Hopf algebra structure. The inclusion $H(1-e) \subseteq \ker \rho$ is obvious since $\rho(e) = 1$. To verify that $\ker \rho = H(1-e)$, we only need to verify that the restriction of $\rho$ to the summand $He$ of $H$ is injective. In fact, if $\rho(\sum_{i=0}^{n-1} y' a_i e) = 0$ for $a_i \in kG$, then $\sum_{i=0}^{n-1} z^i \overline{a_i} \overline{e} = 0$, and hence each $\overline{a_i} \overline{e} = 0$. By Lemma 2.1, $\overline{a_i} \overline{e} = 0$ if and only if $a_i e \in kG(g^n - 1)$ if and only if $a_i e \in kG(1-e)$. It follows that $a_i e = 0$, as desired.

(2) Follows from the algebra decomposition $H = He \oplus H(1-e)$. \qed

Let $V$ be a $kG$-module and $x$ a formal variable. For any $k \in \mathbb{N}$, the set $x^kV = \{x^k v \mid v \in V\}$ is a $k$-vector space defined by $x^k u + x^k v = x^k (u + v)$ and $\lambda (x^k v) = x^k (\lambda v)$ for $u, v \in V$ and $\lambda \in k$. Obviously, $\dim x^kV = \dim V$. Moreover, $x^kV$ is a $kG$-module with the $G$-action given by

$$h(x^k v) = \chi^{-k}(h)x^k hv,$$

for any $h \in G$ and $v \in V$.

Let $\{V_i \mid i \in \Omega\}$ be a complete set of simple $kG$-modules up to isomorphism, where $V_0$ is the trivial module $k$. Since $g^n$ is a central element of $G$, the action of $g^n$ on each $V_i$ is a scalar multiple by a non-zero element of $k$, say $\lambda_i$. Let $\Omega_0 = \{i \in \Omega \mid \lambda_i = 1\}$ and $\Omega_1 = \{i \in \Omega \mid \lambda_i \neq 1\}$. In particular, $0 \in \Omega_0$. It follows from Lemma 2.1 that $\{V_i \mid i \in \Omega_0\}$ is a complete set of simple $kG$-modules up to isomorphism.

Let $M(j, i) := V_i \oplus xV_i \oplus \cdots \oplus x^{j-1}V_i$ for $i \in \Omega_0$ and $1 \leq j \leq n$. Then $M(j, i)$ is an $H$-module, where the action of $h \in G$ on $M(j, i)$ is given by
(2.4) and the action of $y$ on $M(j, i)$ is
\[
y(x^k v) = \begin{cases} 
x^{k+1} v, & 0 \leq k \leq j - 2, \\
0, & k = j - 1,
\end{cases}
\]
for any $v \in V_i$. Note that $\lambda_i = 1$ for $i \in \Omega_0$. We have the following action of the idempotent element $e$:
\[
e(x^k v) = \frac{1}{r} \sum_{s=0}^{r-1} g^m(x^k v) = x^k (\frac{1}{r} \sum_{s=0}^{r-1} g^m v) = x^k v,
\]
for any $x^k v \in M(j, i)$. It follows from Proposition 2.3 that each $M(j, i)$ is an $\mathcal{H}$-module. In particular, by [22, Theorem 2.5],
\[
\{M(j, i) \mid i \in \Omega_0, 1 \leq j \leq n\}
\]
forms a complete set of finite dimensional indecomposable $\mathcal{H}$-modules up to isomorphism, where $M(1, 0) \cong k$, $M(1, i) \cong V_i$ and $M(n, i)$ is the projective cover of $V_i$.

So far we have obtained all finite dimensional indecomposable modules over $\mathcal{H} \cong He$. Now we determine all finite dimensional indecomposable modules over $H(1-e)$. It turns out that the indecomposable modules over $H(1-e)$ are induced from the simple $kG$-modules $V_j$, for $j \in \Omega_1$.

Let $P_j := V_j \oplus xV_j \oplus \cdots \oplus x^{n-1}V_j$ for $j \in \Omega_1$. Then $P_j$ is an $H$-module, where the action of $h \in G$ on $P_j$ is given by (2.4) and the element $y$ acts on $P_j$ as follows:
\[
y(x^k v) = \begin{cases} 
x^{k+1} v, & 0 \leq k \leq n - 2, \\
(\lambda_j - 1) v, & k = n - 1,
\end{cases}
\]
for any $v \in V_j$. Note that $\lambda_j \neq 1$ and $\lambda_j^r = 1$ for $j \in \Omega_1$. Now the action of $e$ on $P_j$ is zero:
\[
e(x^k v) = \frac{1}{r} \sum_{s=0}^{r-1} g^m(x^k v) = x^k (\frac{1}{r} \sum_{s=0}^{r-1} g^m v) = x^k (\frac{1}{r} \sum_{s=0}^{r-1} \lambda_j^s v) = 0,
\]
for any $x^k v \in P_j$. This means that each $P_j$ is in fact an $H(1-e)$-module.

For any $H$-module $V$, the subspace $V_y = \{v \in V \mid yv = 0\}$ is a submodule of $V$. If $V_y = V$, then $V$ is called $y$-torsion. If $V_y = 0$, then $V$ is called $y$-torsionfree. Obviously, if $V$ is a simple $H$-module, then $V$ is either $y$-torsion or $y$-torsionfree. It follows from Lemma 2.1 that an $H$-module $V$ is simple $y$-torsion if and only if $V$ is a simple $kG$-module. We have the following description of the simple $y$-torsionfree $H$-modules.

**Proposition 2.4.**

1. Every simple $y$-torsionfree $H$-module is isomorphic to $P_j$ for some $j \in \Omega_1$.

2. Every $H$-module $P_j$, $j \in \Omega_1$, is simple, projective and $y$-torsionfree.
Proof. (1) If $V$ is a simple $y$-torsionfree $H$-module, then $V$ is semisimple as a $kG$-module since $kG$ is a semisimple subalgebra of $H$. Hence there is some $j \in \Omega$ such that $V_j$ is a direct summand of $V$. By the simplicity of $V$, we have $V = V_j + yV_j + \cdots + y^{n-1}V_j$. We claim that $j \in \Omega_1$ (i.e., $\lambda_j \neq 1$) and $V_j + yV_j + \cdots + y^{n-1}V_j$ is a direct sum. In fact, if $\lambda_j = 1$, then for any $v \in V_j$, $y^nv = (g^n - 1)v = (\lambda_j - 1)v = 0$. It follows that $v = 0$ since $V$ is $y$-torsionfree, a contradiction. Note that $g$ is a central element of $G$. There is a non-zero scalar $\omega_j$ such that $g\omega_jv$, for any $v \in V_j$. If $v_0 + yv_1 + \cdots + y^{n-1}v_{n-1} = 0$ for $v_i \in V_j$, then
\[0 = g^i(v_0 + yv_1 + \cdots + y^{n-1}v_{n-1}) = \omega_j^i(v_0 + y^{-i}v_1 + \cdots + y^{-(n-1)i}v_{n-1}).\]
Thus,
\[v_0 + y^{-i}v_1 + \cdots + y^{-(n-1)i}v_{n-1} = 0, \text{ for } 0 \leq i \leq n - 1.\]
This implies that $v_0 = yv_1 = \cdots = y^{n-1}v_{n-1} = 0$ since the coefficient matrix determined by Equations (2.7) is a Vandermonde matrix (which is invertible). Now we have $V = V_j \oplus yV_j \oplus \cdots \oplus y^{n-1}V_j$ for some $j \in \Omega_1$, and hence $V$ is isomorphic to $P_j$ with the isomorphism given by
\[V \rightarrow P_j, \quad \sum_{i=0}^{n-1} y^i v_i \mapsto \sum_{i=0}^{n-1} x^i v_i,\]
as desired.

(2) We first claim that $P_j = V_j \oplus xV_j \oplus \cdots \oplus x^{n-1}V_j$ is $y$-torsionfree, for any $j \in \Omega_1$. Indeed, if $y(v_0 + xv_1 + \cdots + x^{n-1}v_{n-1}) = 0$, then
\[(g^n - 1)(v_0 + xv_1 + \cdots + x^{n-1}v_{n-1}) = y^n(v_0 + xv_1 + \cdots + x^{n-1}v_{n-1}) = 0.\]
It follows that $(\lambda_j - 1)(v_0 + xv_1 + \cdots + x^{n-1}v_{n-1}) = 0$. We obtain that $v_0 + xv_1 + \cdots + x^{n-1}v_{n-1} = 0$ since $\lambda_j \neq 1$.

Now we check that $P_j$ is simple for any $j \in \Omega_1$. If $V$ is a non-zero simple submodule of $P_j$, then $V$ is simple $y$-torsionfree. By Part (1), $V \cong P_j$ for some $j' \in \Omega_1$. Moreover, $V_j \subseteq P_j \cong V \subseteq P_j$. Note that $V_{j'}$ is a simple $kG$-module and $P_j$ is a direct sum with the summands of simple $kG$-modules. There exists some $k$ such that $V_{j'} \cong x^kV_j$. Thus, dim $V_{j'} = \dim x^kV_j = \dim V_j$. This implies that dim $V = \dim P_{j'} = \dim P_j$. As a result, $V = P_j$.

To see that $P_j$ is projective for any $j \in \Omega_1$, we denote by $e_j$ the primitive idempotent of $kG$ such that $V_j \cong kGe_j$ as $kG$-modules. Then $e_j$ is an idempotent of $H$ as well. Let $He_j$ be the left ideal of $H$ generated by $e_j$. We have the following decomposition of $He_j$ into direct sum of simple $kG$-modules:
\[He_j = kGe_j \oplus ykGe_j \oplus \cdots \oplus y^{n-1}kGe_j.\]
Denote by $\zeta_j$ the isomorphism from $V_j$ to $kGe_j$ and consider the $k$-linear map as follows:

$$P_j \rightarrow He_j, \quad \sum_{k=0}^{n-1} x^k v_k \mapsto \sum_{k=0}^{n-1} y^k \zeta_j(v_k).$$

It is straightforward to verify that this map is an $H$-module isomorphism. $\square$

We are now ready to determine all finite dimensional indecomposable modules over the subalgebra $H(1-e)$ of $H$.

**Theorem 2.5.** The following hold:

1. For any $j \in \Omega_1$, $P_j$ is a simple projective $H(1-e)$-module.
2. Every simple $H(1-e)$-module is isomorphic to some $P_j$, $j \in \Omega_1$.
3. The subalgebra $H(1-e)$ of $H$ is semisimple.
4. The Jacobson radical of $H$ is a principal ideal generated by the element $ye$.

**Proof.** (1) Follows from Proposition 2.4 (2) and Equation (2.6).

(2) If $V$ is a simple $H(1-e)$-module, then $V$ is a simple $H$-module via the projection from $H$ to $H(1-e)$. Moreover, $V$ is $y$-torsionfree. Indeed, if $yv = 0$ for some $v \in V$, then $(g^n - 1)v = y^n v = 0$ and hence $g^n v = v$. This implies that $0 = ev = \frac{1}{r} \sum_{k=0}^{r-1} g^k v = v$. Thus, $V$ is a simple $y$-torsionfree $H$-module. It follows from Proposition 2.4 (1) that $V \cong P_j$ for some $j \in \Omega_1$.

(3) Follows from Part (1) and Part (2).

(4) Follows from the fact that the ideal $(ye)$ of $H$ generated by $ye$ is nilpotent and the quotient algebra $H/(ye) \cong H(1-e) \bigoplus kGe$ is semisimple. $\square$

**Remark 2.6.** Recall that a finite dimensional Hopf algebra over $k$ is said to have the Chevalley property if the tensor product of any two simple modules is semisimple [11, Definition 7.2.1]. One of the equivalent conditions is that the Jacobson radical of the Hopf algebra is a Hopf ideal [11, Proposition 7.2.2]. Accordingly, the finite dimensional pointed rank one Hopf algebra $H$ of non-nilpotent type does not have the Chevalley property since the Jacobson radical of $H$ is not a Hopf ideal. As we shall see that the tensor product of any two simple $H$-modules is not necessary semisimple.

In order to determine all simple $H(1-e)$-modules up to isomorphism, we need a permutation on the index set $\Omega$ of simple $kG$-modules. Let $V_\chi$ and $V_{\chi^{-1}}$ be the two 1-dimensional simple $kG$-modules corresponding to the $k$-linear character $\chi$ and $\chi^{-1}$ respectively. Similar to the nilpotent case in [22], we have a unique permutation $\tau$ of the index set $\Omega$ determined by

$$V_{\chi^{-1}} \otimes V_s \cong V_s \otimes V_{\chi^{-1}} \cong V_{\tau(s)}.$$
as $kG$-modules, for some $\tau(s) \in \Omega$. Moreover, it is easy to see that $s \in \Omega_0$ (resp. $s \in \Omega_1$) if and only if $\tau(s) \in \Omega_0$ (resp. $\tau(s) \in \Omega_1$), i.e., $\tau$ permutes the index set $\Omega_0$ and $\Omega_1$ respectively.

**Lemma 2.7.** For any $s \in \Omega$ and $t \in \mathbb{Z}$, the following hold for the $kG$-modules:

1. $V_s \otimes V_s \cong V_s \otimes V_s \cong V_{\tau^{-1}(s)}$.
2. $V_s \otimes V_{s^{-1}} \cong V_{\tau(s)}$.
3. There is a bijection $\sigma_{s,t}$ from $V_s$ to $V_{\tau(s)}$ such that $\sigma_{s,t}(v) = \chi(t)h\sigma_{s,t}(v)$, for any $h \in G$ and $v \in V_s$.
4. $xV_s \cong V_{\tau(s)}$. Moreover, $V_i \cong V_j$ if and only if $xV_i \cong xV_j$, for $i, j \in \Omega$.
5. The order of the permutation $\tau$ is $n$. Moreover, for any $s \in \Omega$, $x^iV_s \cong V_s$ if and only if $it$ is divisible by $n$.

**Proof.** Part (1) and Part (2) are obvious. Part (3) is the same as [22, Lemma 2.3].

4. The $k$-linear map $xv \mapsto u \otimes v$ gives an isomorphism from $xV_s$ to $V_{\chi^{-1}} \otimes V_s$, where $0 \neq u \in V_{\chi^{-1}}$. Moreover, $V_i \cong V_j$ if and only if $V_{\tau(i)} \cong V_{\tau(j)}$ if and only if $xV_i \cong xV_j$, for $i, j \in \Omega$.

5. Note that the order of $\chi$ is $n$. It follows from Part (2) that the order of $\tau$ is $n$ as well. Suppose the action of $g$ on $V_s$ is a scalar multiple by an element $\omega_s$. Then the action of $g$ on $x^iV_s$ is a scalar multiple by $\omega_s q^{-i}$. If $x^iV_s \cong V_s$, then $\omega_s = \omega_s q^{-i}$, and hence $t$ is divisible by $n$ since the order of $q$ is $n$. Conversely, if $t$ is divisible by $n$, it is obvious that $x^iV_s \cong V_s$ since the order of $\tau$ is $n$.

Let $\langle \tau \rangle$ be the group generated by the permutation $\tau$. Then $\langle \tau \rangle$ acts on the index sets $\Omega_k, k \in \{0, 1\}$, respectively. Let $\sim$ be the equivalence relation on $\Omega_k$: $i \sim j$ if and only if $i$ and $j$ belong to the same $\langle \tau \rangle$-orbit. Denote by $\langle i \rangle$ the equivalence class (or the $\langle \tau \rangle$-orbit) of $i$ and $\Omega_k = \{ \langle i \rangle \mid i \in \Omega_k \}$ the set of all equivalence classes. By Lemma 2.7 (5), we have $|\langle i \rangle| = n$ for any $i \in \Omega_k$, and hence $|\Omega_k|$ is divisible by $n$.

With the equivalence relation above, we have the following result.

**Proposition 2.8.** For any $i, j \in \Omega_1$, $P_i \cong P_j$ as $H$-modules (or equivalently, as $H(1-e)$-modules) if and only if $|i| = |j|$.

**Proof.** Suppose $P_i \cong P_j$ as $H$-modules, then the two are isomorphic as $kG$-modules. By Krull-Schmidt theorem, the direct summand $V_i$ of $P_i$ is isomorphic to a direct summand $x^kV_j$ of $P_j$. It follows from Lemma 2.7 (4) that $|i| = |j|$. Conversely, if $|i| = |j|$, then $i = \tau^k(j)$ for some $k$. This implies that $V_i \cong V_{\tau^k(j)} \cong x^kV_j$ as $kG$-modules. Thus, $\dim P_i = \dim P_j$ since $\dim V_i = \dim x^kV_j = \dim V_j$. Let $\zeta$ denote the isomorphism from $V_i$ to
$x^k V_j$. Then the $k$-linear map

$$P_i \rightarrow P_j, \quad \sum_{s=0}^{n-1} x^s v_s \mapsto \sum_{s=0}^{n-1} y^s \zeta(v_s),$$

is an injective $H$-module morphism, where the term $y^s \zeta(v_s)$ means the action of $y^s$ on $\zeta(v_s)$. Since the dimensions of the two modules are equal, we conclude that $P_i \cong P_j$. \qed

Following Proposition 2.8, we obtain a partition on the set $\{P_j \mid j \in \Omega_1\}$. Let $P_{[j]}$ stand for a representative of the isomorphism class $[P_j]$ of $P_j$. As a direct consequence of Theorem 2.5 and Proposition 2.8, we obtain that the set $\{P_{[j]} \mid j \in \Omega_1\}$ forms a complete set of simple $H(1-e)$-modules up to isomorphism. We summarize the main result of this section as follows:

**Theorem 2.9.** The set $\{M(k, i), P_{[j]} \mid i \in \Omega_0, 1 \leq k \leq n, j \in \Omega_1\}$ forms a complete set of finite dimensional indecomposable $H$-modules up to isomorphism.

To end this section, we describe the dual of $H$-modules, which will be used later. For any finite dimensional $H$-module $M$, the dual $M^* := \text{Hom}_H(M, k)$ is an $H$-module given by $(hf)(v) = f(S(h)v)$, for $h \in H$, $f \in M^*$ and $v \in M$.

**Proposition 2.10.** For any $i \in \Omega_0$, $j \in \Omega_1$ and $1 \leq k \leq n$, the following hold:

1. $M(k, i)^* \cong M(k, \tau^1 - k(i^*))$, where $i^* \in \Omega_0$ such that $(V_i)^* \cong V_i^{**}$.
2. $P_{[j]}^* \cong V_{\chi^{-1}} \otimes P_{[j^*]}$, where $j^* \in \Omega_1$ such that $(V_j)^* \cong V_j^{**}$.
3. $M(k, i)^{**} \cong M(k, i)$ and $P_{[j]}^{**} \cong P_{[j]}$.

**Proof.** The proof of Part (2) is similar to the proof of Part (1), whereas the proof of Part (1) follows from [22, Proposition 3.7 (1)]. Part (3) is obvious because the square of the antipode of $H$ is inner. \qed

### 3. Clebsch-Gordan Formulas

In this section, we compute the Clebsch-Gordan formulas for the decompositions of tensor products of indecomposable $H$-modules. For any two simple $kG$-modules $V_i$ and $V_j$, if $V_s$ is a direct summand of the tensor product $V_i \otimes V_j$, we let $\pi_s$ be the projection from $V_i \otimes V_j$ to $V_s$.

**Lemma 3.1.** Let $V_s$ be a direct summand of $kG$-module $V_i \otimes V_j$.

1. If $i, j \in \Omega_0$, then $s \in \Omega_0$.
2. If $i \in \Omega_0$, $j \in \Omega_1$ or $i \in \Omega_1$, $j \in \Omega_0$, then $s \in \Omega_1$.
3. If $i, j \in \Omega_1$, and $\lambda_i \lambda_j = 1$, then $s \in \Omega_0$.
4. If $i, j \in \Omega_1$, and $\lambda_i \lambda_j \neq 1$, then $s \in \Omega_1$. 
Proof. Follow the fact that $\lambda_i \lambda_j = \lambda_s$ because the projection $\pi_s : V_i \otimes V_j \to V_s$ exists.

Lemma 3.2. Let $B_i$ be a basis of $V_i$ for any $i \in \Omega$.

(1) The set
\[
\{ y^s(x^t u \otimes v) \mid 0 \leq s, t \leq n-1, u \in B_i, v \in B_j \}
\]
forms a basis of $P_i \otimes P_j$, for any $i, j \in \Omega_1$.

(2) The set
\[
\{ y^s(x^t u \otimes v) \mid 0 \leq s \leq n-1, 0 \leq t \leq k-1, u \in B_i, v \in B_j \}
\]
forms a basis of $M(k, i) \otimes P_j$, for any $i \in \Omega_0$, $j \in \Omega_1$ and $1 \leq k \leq n$.

(3) The set
\[
\{ y^s(v \otimes x^t u) \mid 0 \leq s \leq n-1, 0 \leq t \leq k-1, u \in B_i, v \in B_j \}
\]
forms a basis of $P_j \otimes M(k, i)$, for any $i \in \Omega_0$, $j \in \Omega_1$ and $1 \leq k \leq n$.

Proof. We only prove Part (1), and the proofs of Part (2) and Part (3) are similar. Note that \{ $x^t u \otimes x^s v \mid 0 \leq t, s \leq n-1, u \in B_i, v \in B_j$ \} forms a basis of $P_i \otimes P_j$. It is sufficient to show that the following is true:

(3.1) $x^t u \otimes x^s v \in \text{sp}\{ y^s(x^t u \otimes v) \mid 0 \leq s, t \leq n-1, u \in B_i, v \in B_j \}$,

for any $0 \leq s, t \leq n-1$, $u \in B_i$ and $v \in B_j$. We proceed by induction on $s$. It is obvious that (3.1) holds when $s = 0$, for all $0 \leq t \leq n-1$, $u \in B_i$ and $v \in B_j$. For a fixed $1 \leq d \leq n-2$, suppose that (3.1) holds for $1 \leq s \leq d$. We consider the case $s = d + 1$. Note that

\[
\Delta(y^{d+1}) = \sum_{p=0}^{d+1} \binom{d+1}{p} y^{d+1-p} \otimes g^{d+1-p} y^p,
\]

see e.g. [15, Eq.(1)]. Then

\[
y^{d+1}(x^t u \otimes v) \\
= \sum_{p=0}^{d+1} \binom{d+1}{p} y^{d+1-p} \otimes g^{d+1-p} y^p (x^t u \otimes v) \\
= \sum_{p=0}^{d} \binom{d+1}{p} y^{d+1-p} \otimes g^{d+1-p} y^p (x^t u \otimes v) + (1 \otimes y^{d+1})(x^t u \otimes v) \\
= \sum_{p=0}^{d} \mu_p (x^{n_p} u \otimes x^p v) + (x^t u \otimes x^{d+1} v),
\]

where $\mu_p \in k$, and $n_p$ is the remainder after dividing $d + 1 - p + t$ by $n$. By induction assumption, we have that the element $\sum_{p=0}^{d} \mu_p (x^{n_p} u \otimes x^p v)$
belongs to $\text{sp}\{y^s(x^t u \otimes v) \mid 0 \leq s, t \leq n - 1, u \in B_i, v \in B_j\}$. This implies that

$$x^t u \otimes x^{d+1} v = y^{d+1}(x^t u \otimes v) - \sum_{p=0}^{d} \mu_p(x^{p} u \otimes x^{p} v)$$

belongs to $\text{sp}\{y^s(x^t u \otimes v) \mid 0 \leq s, t \leq n - 1, u \in B_i, v \in B_j\}$, for any $0 \leq t \leq n - 1, u \in B_i$ and $v \in B_j$, as desired.

With the bases of the tensor products of indecomposable modules given in Lemma 3.2, we are able to describe the decomposition of the tensor product of any two indecomposable modules. We start with two indecomposable modules of form $P_i$, for $i \in \Omega_1$.

**Proposition 3.3.** For any $i, j \in \Omega_1$, we have the following decompositions:

1. If $\lambda_i \lambda_j = 1$, then $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_0} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$. In this case,
   $$P_i \otimes P_j \cong \bigoplus_{t=0}^{n-1} \bigoplus_{s \in \Omega_0} \gamma_{ij}^s M(n, \tau^t(s)).$$

2. If $\lambda_i \lambda_j \neq 1$, then $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_1} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$. In this case,
   $$P_i \otimes P_j \cong \bigoplus_{t=0}^{n-1} \bigoplus_{s \in \Omega_1} \gamma_{ij}^s P_{\tau^t(s)}.$$

**Proof.** For any $0 \leq t \leq n - 1$, we let $\Gamma_t$ be the subset

$$\{x^t u \otimes v, y(x^t u \otimes v), \ldots, y^{n-1}(x^t u \otimes v) \mid u \in B_i, v \in B_j\}$$

of the basis $\{y^s(x^t u \otimes v) \mid 0 \leq s, t \leq n - 1, u \in B_i, v \in B_j\}$ of $P_i \otimes P_j$. It is easy to see that the subspace $\text{sp}\Gamma_t$ is a submodule of $P_i \otimes P_j$ with dimension $n \dim V_i \dim V_j$. Moreover, we have $P_i \otimes P_j \cong \bigoplus_{t=0}^{n-1} \text{sp}\Gamma_t$.

1. If $\lambda_i \lambda_j = 1$, by Lemma 3.1, $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_0} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$. For any $0 \leq t \leq n - 1$, we consider the following $k$-linear map:
   $$\varphi_t : \text{sp}\Gamma_t \rightarrow \bigoplus_{s \in \Omega_0} \gamma_{ij}^s M(n, \tau^t(s)), \ y^k(x^t u \otimes v) \mapsto \sum_{s \in \Omega_0} \gamma_{ij}^s x^k \sigma_{s,t} \pi_s (u \otimes v),$$

for $0 \leq k \leq n - 1, u \in B_i$ and $v \in B_j$, where the map $\sigma_{s,t}$ is given in Lemma 2.7 (3). It is tedious but straightforward to check that the map $\varphi_t$ is an $H$-module isomorphism. It follows that

$$P_i \otimes P_j = \bigoplus_{t=0}^{n-1} \text{sp}\Gamma_t \cong \bigoplus_{t=0}^{n-1} \bigoplus_{s \in \Omega_0} \gamma_{ij}^s M(n, \tau^t(s)).$$
(2) Suppose that $\lambda_i \lambda_j \neq 1$. By Lemma 3.1, we have $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_1} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$. Now for any $0 \leq t \leq n - 1$, similar to Part (1), we have the following $k$-linear map:

$$
\psi_t: \text{sp} \Gamma_t \to \bigoplus_{s \in \Omega_1} \gamma_{ij}^s P_{t'}(s), \quad y^k(x^t u \otimes v) \mapsto \sum_{s \in \Omega_1} \gamma_{ij}^s x^k \sigma_s t \pi_s (u \otimes v),
$$

for $0 \leq k \leq n - 1$, $u \in B_i$ and $v \in B_j$, which is in fact an $H$-module isomorphism. Thus,

$$
P_i \otimes P_j \cong \bigoplus_{t=0}^{n-1} \bigoplus_{s \in \Omega_1} \gamma_{ij}^s P_{t'}(s),$$

We have completed the proof. □

Now we consider the decomposition of the tensor product of $M(k, i)$ with $P_j$.

**Proposition 3.4.** If $i \in \Omega_0$ and $j \in \Omega_1$, then $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_0} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$. In this case,

$$M(k, i) \otimes P_j \cong P_j \otimes M(k, i) \cong \bigoplus_{t=0}^{k-1} \bigoplus_{s \in \Omega_1} \gamma_{ij}^s P_{t'}(s),$$

for any $1 \leq k \leq n$.

**Proof.** Similar to the proof of Proposition 3.3. □

Note that the left $H$-module category is a monoidal full subcategory of left $H$-module category by Proposition 2.3. The $H$-module decomposition of the tensor product $M(k, i) \otimes M(l, j)$ is the same as the $\overline{H}$-module decomposition. We present this decomposition directly as follows (see [22, Proposition 4.2]).

**Proposition 3.5.** Let $i, j \in \Omega_0$. Then $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_0} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$.

1. If $k + l - 1 \leq n$, then

$$M(k, i) \otimes M(l, j) \cong \bigoplus_{s \in \Omega_0} \bigoplus_{t=0}^{\min\{k, l\}-1} \gamma_{ij}^s M(k + l - 1 - 2t, \tau^l(s)).$$

2. If $k + l - 1 \geq n$, then

$$M(k, i) \otimes M(l, j) \cong \bigoplus_{s \in \Omega_0} \bigoplus_{t=0}^{r} \gamma_{ij}^s \bigoplus_{t=\tau^l(s)}^{\min\{k, l\}-1} M(k + l - 1 - 2t, \tau^l(s)),$$

where $r = k + l - 1 - n$.

As an immediate consequence of Proposition 3.3, Proposition 3.4 and Proposition 3.5, we have the following.
Corollary 3.6. We have that $M \otimes N \cong N \otimes M$, for any two finite dimensional $H$-modules $M$ and $N$.

4. The structure of Green rings

Let $A$ be a Hopf algebra over a field $k$ and $F(A)$ the free abelian group generated by the isomorphism classes $[M]$ of finite dimensional $A$-modules $M$. The abelian group $F(A)$ becomes a ring if we endow $F(A)$ with a multiplication given by the tensor product $[M][N] = [M \otimes N]$. The Green ring (or representation ring) $r(A)$ of the Hopf algebra $A$ is defined to be the quotient ring of $F(A)$ modulo the relations $[M \oplus N] = [M] + [N]$. The identity of the associative ring $r(A)$ is represented by the trivial $A$-module $k$. Note that $r(A)$ has a $\mathbb{Z}$-basis consisting of isomorphism classes of finite dimensional indecomposable $A$-modules (see e.g., [3, 5, 16, 17]).

The Grothendieck ring $G_0(A)$ of Hopf algebra $A$ is the quotient ring of $F(A)$ modulo short exact sequences of $A$-modules, i.e., $[Y] = [X] + [Z]$ if $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is exact. The Grothendieck ring $G_0(A)$ possesses a basis given by isomorphism classes of simple $A$-modules. Both $r(A)$ and $G_0(A)$ are augmented $\mathbb{Z}$-algebras with the dimension augmentation. Moreover, there is a natural ring epimorphism from $r(A)$ to $G_0(A)$ given by

$$
\phi : r(A) \rightarrow G_0(A), [M] \mapsto [M] = \sum [M : V][V],
$$

where the sum runs over all finite dimensional simple $A$-modules and $[M : V]$ is the multiplicity of $V$ in the composition series of $M$.

Let $r(\overline{H})$ and $r(H)$ be the Green rings of Hopf algebras $\overline{H}$ and $H$ respectively. Then $r(H)$ is a commutative ring (Corollary 3.6) with the subring $r(\overline{H})$ (Proposition 2.3). Denote by $M[k,i]$ and $P[j]$ the isomorphism classes of indecomposable modules $M(k,i)$ and $P_j$ respectively. We write $1$ for $[k]$ and $a$ for $[V_{\chi^{-1}}]$ respectively. Since the order of $\chi$ is $n$, we have $a^n = 1$.

Proposition 4.1. For any $i, j \in \Omega_1$, the following hold in $r(H)$:

1. If $\lambda_i \lambda_j = 1$, then $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_0} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$.

   In this case,

   $$
P_{[i]} P_{[j]} = (1 + a + \cdots + a^{n-1}) \sum_{s \in \Omega_0} \gamma_{ij}^s M[n, s].
$$

2. If $\lambda_i \lambda_j \neq 1$, then $V_i \otimes V_j \cong \bigoplus_{s \in \Omega_1} \gamma_{ij}^s V_s$ for some integers $\gamma_{ij}^s \geq 0$.

   In this case,

   $$
P_{[i]} P_{[j]} = n \sum_{s \in \Omega_1} \gamma_{ij}^s P[s].
$$
Proof. (1) It follows from Proposition 3.3 (1) that
\[
\begin{align*}
P_{[i]}P_{[j]} &= \sum_{t=0}^{n-1} \sum_{s \in \Omega_0} \gamma_{ij}^s M[n, \tau^t(s)] = \sum_{t=0}^{n-1} \sum_{s \in \Omega_0} \gamma_{ij}^s a^t M[n, s] \\
&= (1 + a + \cdots + a^{n-1}) \sum_{s \in \Omega_0} \gamma_{ij}^s M[n, s].
\end{align*}
\]
In addition, the foregoing expression is well-defined. Indeed, if \(i_1 = \tau^k(i)\) and \(j_1 = \tau^p(j)\), for some integers \(k\) and \(p\), then \(V_{i_1} \cong V_i \otimes V_{\chi^{-k}}\) and \(V_{j_1} \cong V_j \otimes V_{\chi^{-p}}\). These yield that \(\lambda_{i_1} \lambda_{j_1} = (g^n) \lambda_i \chi^{-k}(g^n) = \lambda_i \lambda_j = 1\) and
\[
V_{i_1} \otimes V_{j_1} \cong V_i \otimes V_{\chi^{-k}} \otimes V_j \otimes V_{\chi^{-p}} \cong V_i \otimes V_j \otimes V_{\chi^{-(k+p)}}
\]
\[
\cong \bigoplus_{s \in \Omega_0} \gamma_{ij}^s V_{\tau^{k+p}(s)}.
\]
Note that \(a^n = 1\). By Proposition 3.3 (1), we have
\[
P_{[i]}P_{[j]} = \sum_{t=0}^{n-1} \sum_{s \in \Omega_0} \gamma_{ij}^s M[n, \tau^t(\tau^{k+p}(s))] = \sum_{t=0}^{n-1} \sum_{s \in \Omega_0} \gamma_{ij}^s a^{t+k+p} M[n, s]
\]
\[
= (1 + a + \cdots + a^{n-1}) \sum_{s \in \Omega_0} \gamma_{ij}^s M[n, s] = P_{[i]}P_{[j]}.
\]
(2) Note that \(P_{[\tau^t(s)]} = P_{[s]}\) since \(\tau^t(s) \sim s\). By Proposition 3.3 (2), we have
\[
P_{[i]}P_{[j]} = \sum_{t=0}^{n-1} \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[\tau^t(s)]} = \sum_{t=0}^{n-1} \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[s]} = n \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[s]}.
\]
To show that this expression is well-defined, we assume that \(i_1 = \tau^k(i)\) and \(j_1 = \tau^p(j)\), for some integers \(k\) and \(p\). Then \(\lambda_{i_1} \lambda_{j_1} \neq 1\) since \(\lambda_i \lambda_j \neq 1\), and it is similar that \(V_{i_1} \otimes V_{j_1} \cong \bigoplus_{s \in \Omega_1} \gamma_{ij}^s V_{\tau^{k+p}(s)}\). It follows from Proposition 3.3 (2) that
\[
P_{[i]}P_{[j]} = \sum_{t=0}^{n-1} \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[\tau^t(\tau^{k+p}(s))]}) = \sum_{t=0}^{n-1} \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[s]} = n \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[s]} = P_{[i]}P_{[j]}.
\]
The proof is completed. \(\square\)

Proposition 4.2. If \(i \in \Omega_0\) and \(j \in \Omega_1\), then \(V_i \otimes V_j \cong \bigoplus_{s \in \Omega_1} \gamma_{ij}^s V_s\) for some integers \(\gamma_{ij}^s \geq 0\). In this case,
\[
M[k, i]P_{[j]} = k \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[s]}
\]
for any \(1 \leq k \leq n\). In particular, \([V_i]P_{[j]} = \sum_{s \in \Omega_1} \gamma_{ij}^s P_{[s]}\), \(M[2, 0]P_{[j]} = 2P_{[j]}\) and \((1 + a - M[2, 0])P_{[j]} = 0\).
Proof. By Proposition 3.4, we have

\[ M[k, i]P_{[j]} = \sum_{t=0}^{k-1} \sum_{s \in \Omega_1} \gamma^t_{ij} P_{[\tau_t(s)]} = k \sum_{s \in \Omega_1} \gamma^s_{ij} P_{[s]} = k \sum_{s \in \Omega_1} \gamma^s_{ij} P_{[s]}. \]

The proof that the above expression is well-defined is similar to those in Proposition 4.1.

In view of the relations given in Proposition 4.1 and Proposition 4.2, it is difficult to present the Green ring \( r(H) \) of \( H \) in terms of (minimal) generators and relations as done in [2, 13, 17, 21]. However, we are still able to describe the structure of \( r(H) \) in terms of the Green ring \( r(H) \).

Let \( P \) be the free abelian group generated by isomorphism classes of finite dimensional indecomposable projective \( H \)-modules. That is,

\[ P = \mathbb{Z}\{M[n, i], P_{[j]} \mid i \in \Omega_0, j \in \Omega_1\}. \]

Then \( P \) is an ideal of \( r(H) \) since any \( H \)-module tensors with a projective module is also projective. The direct sum \( r(H) \oplus P \) is a commutative ring with the multiplication given by

\[ (b_1, c_1)(b_2, c_2) = (b_1b_2, b_1c_2 + c_1b_2 + c_1c_2), \]

for any \( b_1, b_2 \in r(H) \) and \( c_1, c_2 \in P \). Obviously, \( r(H) \oplus P \) can be regarded as a certain trivial extension of \( r(H) \) with respect to \( P \).

**Theorem 4.3.** Let \( I \) be the submodule of \( \mathbb{Z} \)-module \( r(H) \oplus P \) generated by the elements \( -(M[n, i], M[n, i]) \) for \( i \in \Omega_0 \). Then \( I \) is an ideal of \( r(H) \oplus P \) and the quotient ring \( (r(H) \oplus P)/I \) is isomorphic to \( r(H) \).

**Proof.** For any \( b \in r(H) \) and \( c \in P \), we have

\[ (b, c)(-M[n, i], M[n, i]) = (-bM[n, i], bM[n, i]) \in I \]

since \( bM[n, i] \) is always a \( \mathbb{Z} \)-linear combination of elements of the form \( M[n, j] \) for \( j \in \Omega_0 \). Thus, \( I \) is a two-sided ideal of \( r(H) \oplus P \) since \( r(H) \oplus P \) is commutative. Note that the \( \mathbb{Z} \)-linear map from \( r(H) \oplus P \) to \( r(H) \) given by \( (b, c) \mapsto b + c \) is a ring epimorphism with the kernel equal to \( I \). We conclude that \( (r(H) \oplus P)/I \cong r(H) \), as desired.

5. Grothendieck rings and Cartan matrices

In this section, we study the Grothendieck ring \( G_0(H) \) of \( H \) in terms of the Green ring \( r(H) \). We then describe the Cartan map from \( P \) to the Grothendieck ring \( G_0(H) \) and obtain the Cartan matrix with respect to given bases.

Let \( M \) be a finite dimensional \( H \)-module. Recall that the \( k \)-linear dual \( M^* \) is an \( H \)-module with the \( H \)-module structure given by Proposition 2.10. This
leads to an anti-automorphism $\ast$ of the Green ring $r(H)$: $[M]^\ast := [M^\ast]$. Since $M^{**} \cong M$, the $\ast$-operator is an involution of $r(H)$.

Now for any indecomposable $H$-module $Z$, if $Z$ is not projective, there exists a unique almost split sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ ending at $Z$. We take the same notation introduced in [1, Section 4, ChVI] (see also [22]) and denote by $\delta_Z$ the element $[X] - [Y] + [Z]$ in $r(H)$. If $Z$ is projective, then we write $\delta_Z$ for $[Z] - [\text{rad } Z]$.

For any two indecomposable $H$-modules $X$ and $Y$, similar to the one in [22], we define:

$$([X], [Y]) = \dim \operatorname{Hom}_H(X, Y^\ast),$$

which extends to an associative, symmetric and non-degenerate $\mathbb{Z}$-bilinear form on $r(H)$. This bilinear form satisfies the following:

\begin{equation}
(\delta^\ast_{[X]}, [Y]) = 1 \text{ if } X \cong Y, \text{ and } 0 \text{ otherwise.}
\end{equation}

It follows that $r(H)$ is a symmetric algebra over $\mathbb{Z}$ with a pair of dual bases

$$\{M[k, i] \mid i \in \Omega_0, \ j \in \Omega_1, \ 1 \leq k \leq n\}$$

and

$$\{\delta^\ast_{M[k, i]}, P[j] \mid i \in \Omega_0, \ j \in \Omega_1, \ 1 \leq k \leq n\}.$$

With the above pair of dual bases, we can express any element $x$ in $r(H)$ as follows:

\begin{equation}
x = \sum_{i \in \Omega_0} \sum_{k=1}^n (\delta^\ast_{M[k, i]}, x) M[k, i] + \sum_{[j] \in \Omega_1} (P[j], x) P[j],
\end{equation}

or equivalently,

\begin{equation}
x = \sum_{i \in \Omega_0} \sum_{k=1}^n (x, M[k, i]) \delta^\ast_{M[k, i]} + \sum_{[j] \in \Omega_1} (x, P[j]) P^\ast_{[j]}.
\end{equation}

**Lemma 5.1.** The following equations hold in $r(H)$:

1. $\delta_{M[k, i]} = \delta_{[k]} M[k, i]$ and $\delta_{P[j]} = P[j]$, where $\delta_{[k]} = 1 + a - M[2, 0]$, $i \in \Omega_0$, $j \in \Omega_1$, and $1 \leq k \leq n - 1$.
2. $\delta_{[k]} M[n, i] = \delta_{[k]} P[j] = 0$, for $i \in \Omega_0$ and $j \in \Omega_1$.

**Proof.** (1) Note that an almost split sequence of $\mathcal{H}$-modules is also an almost split sequence of $H$-modules since the $\mathcal{H}$-module category is a monoidal full subcategory of the $H$-module category. It follows that the almost split sequence of $\mathcal{H}$-modules (see [22, Proposition 3.5])

$$0 \rightarrow V_{-1} \otimes M(k, i) \rightarrow M(2, 0) \otimes M(k, i) \rightarrow M(k, i) \rightarrow 0$$
is almost split as $H$-modules, for any $i ∈ Ω_0$ and $1 ≤ k ≤ n - 1$. In particular, the almost split sequence ending at $M(1, 0) = k$ is
\[ 0 → V_{r - 1} → M(2, 0) → k → 0. \]
Consequently, we have $δ_{[k]} = 1 + a − M[2, 0]$ and $δ_{M[k, i]} = δ_{[k]}M[k, i]$, for $i ∈ Ω_0$ and $1 ≤ k ≤ n − 1$. It is obvious that $δ_{P_j} = [P_j] − [radP_j] = P_j$ since $P_j$ is simple projective for any $j ∈ Ω_1$.

(2) Follows from the fact that an almost split sequence tensoring with a projective module is split. \hfill \Box

The natural ring epimorphism from the Green ring $r(H)$ to the Grothendieck ring $G_0(H)$ of $H$ is as follows:
\[ φ : r(H) → G_0(H), M[k, i] → [V_i](1 + a + \cdots + a^{k-1}), P_j → P_j \]
for $i ∈ Ω_0$, $j ∈ Ω_1$ and $1 ≤ k ≤ n$. The kernel of $φ$ is exactly the free abelian group generated by almost split sequences ending at indecomposable non-projective modules (cf. [1, Theorem 4.4, ChVI]). That is,
\[ ker φ = \mathbb{Z}δ_{M[k, i]} | i ∈ Ω_0, \ 1 ≤ k ≤ n - 1. \]
By Lemma 5.1, $ker φ$ is a principal ideal of $r(H)$ generated by $δ_{[k]}$. Moreover, $(ker φ)^* = ker φ$ since $δ_{[k]}^* = \alpha^{-1}δ_{[k]}$.

Thanks to the map $φ$ given above, one is able to describe the multiplication rule in $G_0(H)$. Since $\{[V_i], P_j | i ∈ Ω_0, j ∈ Ω_1\}$ is a basis of $G_0(H)$, it is enough to look at the products of basis elements. The products $[V_i][V_j]$ and $[V_i]P_k$ in $G_0(H)$ are the same as in $r(H)$. However, the product of $P_i$ and $P_j$ in $G_0(H)$ becomes
\[ P_iP_j = \begin{cases} n(1 + a + \cdots + a^{n-1})\sum_{s∈Ω_0}\gamma_{ij}^s[V_i], & λ_i λ_j = 1, \\ n\sum_{s∈Ω_1}\gamma_{ij}^sP_i, & λ_i λ_j \neq 1. \end{cases} \]
We refer to Proposition 4.1 for their multiplication in $r(H)$.

Denote by
\[ \mathcal{P}^⊥ = \{ x ∈ r(H) \mid (x, y) = 0, \ ∀ y ∈ \mathcal{P} \}, \]
the orthogonal ideal of $\mathcal{P}$ with respect to the form $(-, -)$. We are ready to describe the structure of the Grothendieck ring $G_0(H)$ as follows.

**Proposition 5.2.** Let $r(kG)$ be the Green ring of the group algebra $kG$.

1. We have $ker φ = \mathcal{P}^⊥$, hence $G_0(H) ∼= r(H)/\mathcal{P}^⊥$.
2. The Grothendieck ring $G_0(H)$ is isomorphic to the subring of $r(kG)$ generated by $[V_i]$ and $(1 + a + \cdots + a^{n-1})[V_j]$, for any $i ∈ Ω_0$ and $j ∈ Ω_1$. 


Proof. (1) By Lemma 5.1 (2), $(\delta_{[k]}, P_{[j]}) = ([k], \delta_{[k]}P_{[j]}) = ([k], 0) = 0.$ Similarly, we have $(\delta_{[k]}, M[n, i]) = 0.$ These imply that $\delta_{[k]} \in \mathcal{P}^\perp,$ and hence $\ker \phi \subseteq \mathcal{P}^\perp.$ Conversely, for any $x \in \mathcal{P}^\perp,$ by (5.3),

$$x = \sum_{i \in \Omega_0} \sum_{k=1}^n (x, M[k, i])\delta^*_{M[k, i]} + \sum_{[j] \in \mathcal{P}_1} (x, P_{[j]})P_{[j]}^*$$

$$= \sum_{i \in \Omega_0} \sum_{k=1}^{n-1} (x, M[k, i])\delta^*_{M[k, i]} \quad \text{(as } x \in \mathcal{P}^\perp).$$

It follows that $x^* = \sum_{i \in \Omega_0} \sum_{k=1}^{n-1} (x, M[k, i])\delta^*_{M[k, i]} \in \ker \phi.$ Hence, $x = x^{**} \in (\ker \phi)^* = \ker \phi,$ and $\mathcal{P}^\perp \subseteq \ker \phi.$

(2) Consider the $\mathbb{Z}$-linear map $\varphi$ from $G_0(H)$ to $r(kG)$ given by

$$\varphi([V_i]) = [V_i] \text{ and } \varphi(P_{[j]}) = (1 + a + \cdots + a^{n-1})[V_j],$$

for any $i \in \Omega_0$ and $j \in \Omega_1.$ The map $\varphi$ is well-defined since the sum $(1 + a + \cdots + a^{n-1})[V_j]$ is a $\langle \tau \rangle$-orbit sum and $(1 + a + \cdots + a^{n-1})[V_j] = (1 + a + \cdots + a^{n-1})[V_i]$ if and only if $[j] = [k].$ It is straightforward to verify that the map $\varphi$ is a ring homomorphism. We claim that the map is injective. Indeed, if $\sum_{i \in \Omega_0} \alpha_i [V_i] + \sum_{[j] \in \mathcal{P}_1} \beta_{[j]} P_{[j]}$ belongs to the kernel of $\varphi,$ then $\sum_{i \in \Omega_0} \alpha_i [V_i] + \sum_{[j] \in \mathcal{P}_1} \beta_{[j]} (1 + a + \cdots + a^{n-1})[V_j] = 0.$ This yields that each $\alpha_i = 0$ and $\beta_{[j]} = 0$ since the elements of the set $\{[V_i], (1+a+\cdots+a^{n-1})[V_j] \mid i \in \Omega_0, [j] \in \mathcal{P}_1\}$ are $\mathbb{Z}$-linear independent. \hfill $\square$

Remark 5.3. It was shown in [18] that if the Jacobson radical of a finite-dimensional Hopf algebra is a Hopf ideal, then its Grothendieck ring is semiprime. The Proposition 5.2 (2) means that the Grothendieck ring $G_0(H)$ of $H$ is semiprime although the Jacobson radical of $H$ is not a Hopf ideal.

The map $\phi$ restricting to the the ideal $\mathcal{P}$ of $r(H)$ gives rise to the Cartan map as follows:

$$\phi|_{\mathcal{P}} : \mathcal{P} \rightarrow G_0(H), \quad M[n, i] \mapsto [V_i](1 + a + \cdots + a^{n-1}), \quad P_{[j]} \mapsto P_{[j]}$$

for $i \in \Omega_0$ and $j \in \Omega_1.$ Note that, for any $i \in \Omega_0,$ the $\langle \tau \rangle$-orbit containing $i$ is $[i] = \{i, \tau(i), \cdots, \tau^{n-1}(i)\}.$ Accordingly, we write $V_{[i]}$ for the set $\{[V_i], [V_{\tau(i)}], \cdots, [V_{\tau^{n-1}(i)}]\}.$ The disjoint union

$$(\cup_{[i] \in \mathcal{P}_0} V_{[i]}) \cup \{P_{[j]} \mid [j] \in \mathcal{P}_1\}$$

forms a basis of $G_0(H)$ and the set of their projective covers

$$(\cup_{[i] \in \mathcal{P}_0} P_{V_{[i]}}) \cup \{P_{[j]} \mid [j] \in \mathcal{P}_1\}$$

forms a basis of $\mathcal{P},$ where $P_{V_{[i]}} = \{M[n, i], M[n, \tau(i)], \cdots, M[n, \tau^{n-1}(i)]\}.$ Note that the map $\phi|_{\mathcal{P}}$ maps any element in $P_{V_{[i]}}$ into the same element
It follows that the Cartan matrix of the map $\phi|_P$ with respect to the above bases is of a block-diagonal form with $|\Omega_0| + 1$ blocks:

\[
\begin{pmatrix}
E & & \\
& \ddots & \\
& & E \\
& & & I
\end{pmatrix},
\]

where $E$ is a square matrix of order $n$ with each entry 1 and $I$ is the identity matrix of order $|\Omega_1|$. 

6. Jacobson radicals and idempotents of Green rings

In this section, we shall use the Cartan map of $H$ to describe the Jacobson radical and the idempotents of $r(H)$. We show that the Jacobson radical of $r(H)$ is exactly the kernel of the Cartan map generated by one element, and $r(H)$ has no non-trivial idempotents. We first need the following lemma.

**Lemma 6.1.** The following hold in $r(H)$:

1. For any two indecomposable $H$-modules $X$ and $Y$, the coefficient of $[k]$ in the expression of $[Y][X]^*$ is 1 if $X \cong Y$ and neither of them is projective. Otherwise, the coefficient is 0.

   \[
   (1) \text{ If } x \in r(H), \text{ then } xx^* = 0. 
   \]

   \[
   \text{Proof.} \quad (1) \text{ If } X \text{ is projective, then the coefficient of } [k] \text{ in } [Y][X]^* \text{ is 0 since } k \text{ itself is not projective. If } X \text{ is not projective, by (5.1) the coefficient of } [k] \text{ in } [Y][X]^* \text{ is } (\delta_{[k]}^*, [Y][X]^*) = (\delta_{[X]}^*, [Y]), \text{ which is equal to 1 if } X \cong Y, \text{ and to 0 otherwise.} 
   \]

   (2) Suppose $x = \sum_{i \in \Omega_0} \sum_{1 \leq k \leq n-1} \alpha_{ki} M\{k, i\} + x_0$, for $\alpha_{ki} \in \mathbb{Z}$ and $x_0 \in \mathcal{P}$. By Part (1), the coefficient of $[k]$ in $xx^*$ is $\sum_{i \in \Omega_0} \sum_{1 \leq k \leq n-1} \alpha_{ki}^2$. Thus, if $xx^* = 0$, then $\alpha_{ki} = 0$ for any $i \in \Omega_0$ and $1 \leq k \leq n - 1$. This implies that $x = x_0 \in \mathcal{P}$. \qed

Note that the Green ring $r(H)$ is commutative and finitely generated as an algebra over $\mathbb{Z}$. This means that $r(H)$ is a commutative Jacobson ring. In this case, the Jacobson radical of $r(H)$ is the set consisting of all nilpotent elements of the ring.

**Theorem 6.2.** The Jacobson radical of $r(H)$ is $J(r(H)) = \mathcal{P} \cap \mathcal{P}^\perp = \ker(\phi|_\mathcal{P})$.

**Proof.** The equality $\mathcal{P} \cap \mathcal{P}^\perp = \ker(\phi|_{\mathcal{P}})$ is obvious since $\mathcal{P}^\perp = \ker \phi$ by Proposition 5.2. In the following, we verify that $J(r(H)) = \mathcal{P} \cap \mathcal{P}^\perp$. 

For any \( x \in P \cap P^\perp \) and \( y \in r(H) \), we have \( xy \in P \cap P^\perp \), implying that \( (x^2, y) = (x, xy) = 0 \). This means that \( x^2 = 0 \) since the bilinear form \((-,-)\) on \( r(H) \) is non-degenerate. Consequently, \( x \in J(r(H)) \) and \( P \cap P^\perp \subseteq J(r(H)) \).

Conversely, for any \( x \in J(r(H)) \), we have \( x \in \ker \phi \) since \( \phi(x) \) is a nilpotent element of \( G_0(H) \), whereas \( G_0(H) \) has no non-trivial nilpotent elements since it is semiprime (Remark 5.3). On the other hand, we claim that \( x \in P \). In fact, we let

\[
x = \sum_{i \in \Omega_0} \sum_{1 \leq j \leq n-1} \alpha_{ji} M[j, i] + x_0,
\]

for \( \alpha_{ji} \in \mathbb{Z} \) and \( x_0 \in P \). Then

\[
xx^* = \sum_{i, k \in \Omega_0} \sum_{1 \leq l \leq n-1} \alpha_{ji} \alpha_{lk} M[j, i] M[l, k]^* + x_1
\]

for some \( x_1 \in P \) since \( P \) is an ideal of \( r(H) \) satisfying \( P = P^* \). Denote by

\[
y := xx^* = \sum_{i \in \Omega_0} \sum_{1 \leq j \leq n-1} \beta_{ji} M[j, i] + x_2,
\]

for some \( x_2 \in P \). By Lemma 6.1 (1), the coefficient of \([k]\) in \( y \) is \( \beta_{10} = \sum_{i \in \Omega_0} \sum_{1 \leq j \leq n-1} \alpha_{ji}^2 \). Consider

\[
y^2 = yy^* = \sum_{i, k \in \Omega_0} \sum_{1 \leq l \leq n-1} \beta_{ji} \beta_{lk} M[j, i] M[l, k]^* + x_3
\]

for some \( x_3 \in P \). Again by Lemma 6.1 (1), the coefficient of \([k]\) in \( y^2 \) is \( \sum_{i \in \Omega_0} \sum_{1 \leq j \leq n-1} \beta_{ji}^2 \). If \( \beta_{10} \neq 0 \), then \( \sum_{i \in \Omega_0} \sum_{1 \leq j \leq n-1} \beta_{ji}^2 \neq 0 \), and hence \( y^2 \neq 0 \). Repeating this process, we obtain that if \( \beta_{10} \neq 0 \), then \( y^{2n} \neq 0 \) for any \( n > 0 \). This contradicts to the fact that \( y \in J(r(H)) \). Thus, \( \beta_{10} = 0 \) and \( x = x_0 \in P \). We obtain that \( J(r(H)) \subseteq \ker \phi \cap P = P \cap P^\perp \). The proof is completed.

In the following, we use Theorem 6.2 to describe the Jacobson radical of \( r(H) \) in terms of generators.

**Theorem 6.3.** The Jacobson radical of \( r(H) \) is a principal ideal generated by \((1 - a)M[n, 0]\).

**Proof.** Note that \((1 - a)M[n, 0] \in \ker(\phi|_P)\). The ideal of \( r(H) \) generated by \((1 - a)M[n, 0] \) is contained in \( \ker(\phi|_P) \). Conversely, for any \( \sum_{i \in \Omega_0} \alpha_i M[n, i] + \sum_{|j| \in \Omega_1} \beta_j P_{|j|} \) in \( \ker(\phi|_P) \), we have

\[
\sum_{i \in \Omega_0} \alpha_i (1 + a + \cdots + a^{n-1}) [V_i] + \sum_{|j| \in \Omega_1} \beta_j P_{|j|} = 0.
\]
It follows that $\beta_{[j]} = 0$ for any $[j] \in \overline{\Omega}$, and

\[(6.1) \quad \sum_{i \in \Omega_0} \alpha_i(1 + a + \cdots + a^{n-1})[V_i] = 0.\]

Note that $(1 + a + \cdots + a^{n-1})[V_i] = (1 + a + \cdots + a^{n-1})[V_j]$ if and only if $[i] = [j]$. Thus, the equality (6.1) can be written as

\[
\sum_{[i] \in \Omega_0} (\alpha_i + \alpha_{r(i)} + \cdots + \alpha_{r^{n-1}(i)})(1 + a + \cdots + a^{n-1})[V_i] = 0.
\]

This implies that $\alpha_i + \alpha_{r(i)} + \cdots + \alpha_{r^{n-1}(i)} = 0$ since the $\langle \tau \rangle$-orbit sum $(1 + a + \cdots + a^{n-1})[V_i]$ for $[i] \in \Omega_0$ are $\mathbb{Z}$-linear independent. So far we have verified that any element of $\ker(\phi|_P)$ is of the form $\sum_{i \in \Omega_0} \alpha_i M[n, i]$, where the coefficients satisfy $\alpha_i + \alpha_{r(i)} + \cdots + \alpha_{r^{n-1}(i)} = 0$, for $i \in \Omega_0$. The equality $\alpha_i + \alpha_{r(i)} + \cdots + \alpha_{r^{n-1}(i)} = 0$ implies the following equality:

\[
\alpha_i + \alpha_{r(i)} a + \cdots + \alpha_{r^{n-1}(i)} a^{n-1} = \alpha_{r(i)}(a - 1) + \cdots + \alpha_{r^{n-1}(i)}(a^{n-1} - 1),
\]

where the right hand side is in the ideal of $r(H)$ generated by $1 - a$. As a result, we obtain that the element

\[
\sum_{i \in \Omega_0} \alpha_i M[n, i] = \left(\sum_{i \in \Omega_0} \alpha_i[M_i]\right)M[n, 0]
= \sum_{[i] \in \Omega_0} (\alpha_i[M_i] + \alpha_{r(i)}[V_{r(i)}] + \cdots + \alpha_{r^{n-1}(i)}[V_{r^{n-1}(i)}])M[n, 0]
= \sum_{[i] \in \Omega_0} (\alpha_i + \alpha_{r(i)} a + \cdots + \alpha_{r^{n-1}(i)} a^{n-1})[V_i]M[n, 0],
\]

sits in the ideal of $r(H)$ generated by $(1 - a)M[n, 0]$. The proof is completed.

We have shown that the Jacobson radical $J(r(H))$ of $r(H)$ is precisely $P \cap P^\perp$. So the generator $M[n, 0](1 - a)$ of $J(r(H))$ belongs to $P \cap P^\perp$. In particular, $M[n, 0](1 - a)$ is in $P^\perp$, a principal ideal of $r(H)$ generated by $\delta_{[k]}$. In other words, the element $M[n, 0](1 - a)$ can be written as the product of $\delta_{[k]}$ with some element of $r(H)$. Indeed, by induction on $k$, we have $\delta_{[k]}(M[1, 0] + \cdots + M[k, 0]) = 1 + aM[k, 0] - M[k + 1, 0]$, for $1 \leq k \leq n - 1$. This implies that

\[
\delta_{[k]} \sum_{k=1}^{n-1} \left( M[1, 0] + \cdots + M[k, 0] \right) a^{n-1-k} = \sum_{k=1}^{n-1} \left( 1 + aM[k, 0] - M[k + 1, 0] \right) a^{n-1-k} = (1 + a + \cdots + a^{n-1}) - M[n, 0].
\]
Thus, \( M[n,0] \) can be written as

\[
M[n,0] = (1 + a + \cdots + a^{n-1}) - \delta_{[k]} \sum_{k=1}^{n-1} (M[1,0] + \cdots + M[k,0]) a^{n-1-k}.
\]

Multiplying both sides of the equation by \((1-a)\), we obtain:

\[
M[n,0](1-a) = -\delta_{[k]} \sum_{k=1}^{n-1} (M[1,0] + \cdots + M[k,0]) a^{n-1-k}(1-a).
\]

To end this section, we show that the Green ring \( r(H) \) has no non-trivial idempotents. First, observe that the Green ring of a group algebra possesses such a property. This fact might be found in other literature which we don’t have at hands. So we include it in the following remark.

**Remark 6.4.** For any finite group \( G \), the Green ring (i.e., Grothendieck ring) \( r(kG) \) of the group algebra \( kG \) has no non-trivial idempotents. Indeed, suppose that \( \{ V_i \mid i \in \Omega \} \) is a complete set of simple \( kG \)-modules up to isomorphism. For any \( x \in r(kG) \), if we write \( x = \sum_{i \in \Omega} \alpha_i [V_i] \) for \( \alpha_i \in \mathbb{Z} \), then the coefficient of \([k]\) in \( xx^* \) is \( \sum_{i \in \Omega} \alpha_i^2 \). Thus, \( x = 0 \) if and only if \( xx^* = 0 \). If \( E \) is a primitive idempotent of \( r(kG) \), so is \( E^* \). Since \( r(kG) \) is commutative and the duality operator \( \ast \) is an anti-automorphism of \( r(kG) \), either \( E = E^\ast \) or \( EE^* = 0 \). If \( EE^* = 0 \), it follows that \( E = 0 \). If \( E = E^\ast \), comparing the coefficient of \([k]\) in both sides of the equation \( EE^* = E \), we obtain that \( E = 0 \) or \( E = 1 \).

**Theorem 6.5.** The Green ring \( r(H) \) has no non-trivial idempotents.

**Proof.** Let \( E \) be a primitive idempotent of \( r(H) \). We first prove that \( E \in \mathcal{P} \) or \( 1 - E \in \mathcal{P} \). Note that \( E \) is primitive. So is \( E^* \). Then \( E = E^* \) or \( EE^* = 0 \). If \( EE^* = 0 \), by Lemma 6.1 (2), \( E \in \mathcal{P} \). If \( E = E^* \), let \( E = \sum_{i \in \Omega_0} \alpha_i M[k,i] + E_0 \), for \( \alpha_i \in \mathbb{Z} \) and \( E_0 \in \mathcal{P} \). Comparing the coefficient of \([k]\) in both sides of the equation \( EE^* = E \), we obtain that \( E = \alpha_{10}[k] + E_0 \), where \( \alpha_{10} = 0 \) or 1. Therefore \( E \in \mathcal{P} \) or \( 1 - E \in \mathcal{P} \). If \( E \in \mathcal{P} \), let \( E = \sum_{i \in \Omega_0} \alpha_{ni} M[n,i] + \sum_{[j] \in \Pi_1} \alpha_{[j]} P_{[j]} \). Then

\[
\phi(E) = \left( \sum_{i \in \Omega_0} \alpha_{ni}[V_i] \right) (1 + a + \cdots + a^{n-1}) + \sum_{[j] \in \Pi_1} \alpha_{[j]} P_{[j]}.
\]

Note that \( \phi(E) \) is an idempotent of \( G_0(H) \subseteq r(kG) \) since \( E \) is an idempotent of \( r(H) \). By Remark 6.4, \( \phi(E) \) is equal to 0 or 1. However, \( \phi(E) \neq 1 \) because \( \phi(E)(1 - a) = 0 \) and \( a \neq 1 \). Thus, \( \phi(E) = 0 \), and hence \( E \in \ker(\phi|_{\mathcal{P}}) = J(r(H)) \). It follows that \( E = 0 \). If \( 1 - E \in \mathcal{P} \), one can show in a similar way that \( 1 - E = 0 \), and therefore \( E = 1 \).

Note that Theorem 6.2 and Theorem 6.5 hold as well for the Green rings of Hopf algebras of nilpotent type. In [22, Theorem 5.4] we showed that the Jacobson radical of the Green ring of a finite dimensional pointed rank one
Hopf algebra of nilpotent type is a principal ideal. The method we applied there was to compute the dimension of the Jacobson radical of the complexified Green algebra $R(H) = r(H) \otimes \mathbb{C}$ by determining the irreducible representations of $R(H)$. With the help of the Cartan map, the study of the Jacobson radical of the Green ring becomes more handy.

7. Stable Green rings and Frobenius-Perron dimensions

Let $A$ be a finite dimensional non-semisimple Hopf algebra over the complex field $k := \mathbb{C}$. Denote by $A\text{-mod}$ the category of finite dimensional left $A$-module. The stable category $A\text{-mod}^s$ has the same objects as $A\text{-mod}$ does, and the space of morphisms from $X$ to $Y$ in $A\text{-mod}$ is the quotient space

$$\text{Hom}_A(X, Y) := \text{Hom}_A(X, Y) / P(X, Y)$$

where $P(X, Y)$ is the subspace of $\text{Hom}_A(X, Y)$ consisting of morphisms factoring through projective modules.

The stable category $A\text{-mod}^s$ is a triangulated [12] monoidal category with the monoidal structure stemming from that of $A\text{-mod}$. The Green ring $r_{st}(A)$ of the stable category $A\text{-mod}^s$ is called the stable Green ring of $A$. Obviously, the stable Green ring $r_{st}(A)$ admits a $\mathbb{Z}$-basis consisting of all isomorphism classes of finite dimensional indecomposable non-projective $A$-modules.

Let $R$ be a unital associative ring which is free as a $\mathbb{Z}$-module with a finite basis $B = \{b_i \mid i \in I\}$ containing 1. Then $R$ is a fusion ring if the following conditions hold (cf. [10, Definition 1.42.2]):

1. For any $i, j \in I$, $b_ib_j = \sum_k p^{k}_{ij} b_k$, where $p^{k}_{ij} \in \mathbb{Z}_+$.  
2. There exists a subset $I_0 \subset I$ such that $\sum_{i \in I_0} b_i = 1$.  
3. Let $\psi : R \to \mathbb{Z}$ be the group homomorphism defined by $\psi(b_i) = 1$ if $i \in I_0$, and 0 otherwise. There exists an involution $i \mapsto i^*$ of $I$ such that the induced map $b = \sum_{i \in I} p_i b_i \mapsto b^* = \sum_{i \in I} p_i b_{i^*}$ for $p_i \in \mathbb{Z}$ is an anti-involution of ring $R$ and such that $\psi(b_ib_j) = 1$ if $i = j^*$, and 0 otherwise.

The fusion ring $R$ is transitive if for any $b_i$ and $b_j$, there exist $b_k$ and $b_l$ such that $b_ib_k$ and $b_l$ involve $b_j$ with a nonzero coefficient, for $i, j, k, l \in I$ (cf. [10, Definition 1.45.1]).

For the case when $H$ is a finite dimensional pointed rank one Hopf algebra of non-nilpotent type, the stable Green ring $r_{st}(H)$ of $H$ coincides with the stable Green ring $r_{st}(\overline{H})$ of $\overline{H}$, where the latter has been studied in [22, Proposition 6.1]. Accordingly, the stable Green ring $r_{st}(H)$ is isomorphic to the quotient ring $r(H) / P$. More precisely, it is isomorphic to the quotient of the polynomial ring $r(kG)[z]$ modulo the ideal generated by $F_n(a, z)$, where $F_n$ is the Dickson polynomial (of the second type) defined recursively as
The fact that Proposition 1.45.2 using Lemma 6.1 (1).

\[ F_1(Y, Z) = 1, F_2(Y, Z) = Z \text{ and } F_j(Y, Z) = ZF_{j-1}(Y, Z) - YF_{j-2}(Y, Z), \text{ for } j \geq 3. \]

**Proposition 7.1.** The stable Green ring \( r_{st}(H) \) is a transitive fusion ring.

**Proof.** Let \( I = \{(j, i) \mid i \in \Omega_0, 1 \leq j \leq n - 1 \} \). Then the set \( \{M[j, i] \mid (j, i) \in I\} \) consists of all non-projective indecomposable \( H \)-modules. It follows that \( \{M[j, i] \mid (j, i) \in I\} \) forms a basis of \( r_{st}(H) = r(H)/\mathcal{P} \). Note that the duality \( \ast \) operator on \( r(H) \) is given by \( M[j, i]^\ast = M[j, \tau^{-1}(i)^\ast] \) (cf. Proposition 2.10 (1)). This induces an involution on the index set \( I \) as \( (j, i)^\ast := (j, \tau^{-1}(i)^\ast) \), for any \( (j, i) \in I \). Define the group homomorphism \( \psi : r_{st}(H) \to \mathbb{Z} \) by \( \psi(\pi) = (\delta^\ast_{[k]}, x) \), for any \( \pi \in r_{st}(H) \). This map is well-defined since \( (\delta^\ast_{[k]}, x) = 0 \) if \( x \in \mathcal{P} \). It is straightforward to verify that \( r_{st}(H) \) satisfies the definition of a fusion ring given above. The stable Green ring \( r_{st}(H) \) is transitive. We omit the proof because it is similar to the proof of [10, Proposition 1.45.2] using Lemma 6.1 (1).

The fact that \( r_{st}(H) \) is a transitive fusion ring allows us to calculate the Frobenius-Perron dimensions of objects of the stable category \( H\text{-mod} \) in the framework of the stable Green ring \( r_{st}(H) \). Let \( \text{FPdim}(M[j, i]) \) be the maximal nonnegative eigenvalue of the matrix of left multiplication by \( M[j, i] \) with respect to the basis \( \{M[j, i] \mid (j, i) \in I\} \) of \( r_{st}(H) \). Then \( \text{FPdim}(M[j, i]) \) is the Frobenius-Perron dimension of the object \( M(j, i) \) in \( H\text{-mod} \). The function \( \text{FPdim} \) has the following properties, which can be seen directly from [10, Section 1.45].

**Proposition 7.2.** For any \((j, i) \in I\), the following hold:

1. The function \( \text{FPdim} : r_{st}(H) \to \mathbb{C} \) is a ring homomorphism.
2. \( \text{FPdim} \) is a unique nonzero character of \( r_{st}(H) \) which takes positive values at each \( M[j, i] \).
3. \( \text{FPdim}(M[j, i]^\ast) = \text{FPdim}(M[j, i]) \).
4. \( \text{FPdim}(M[j, i]) \geq 1 \). If \( \text{FPdim}(M[j, i]) < 2 \), then \( \text{FPdim}(M[j, i]) = 2 \cos \frac{2}{n} \), for some integer \( n > 2 \).

Remark that the function \( \text{FPdim} \) restricting to the subring \( r(kG) \) of \( r_{st}(H) \) gives the usual dimensions of \( kG \)-modules, namely, \( \text{FPdim}(M) = \dim(M) \), for any \( kG \)-module \( M \), see [10, Example 1.45.6].

**Theorem 7.3.** \( \text{FPdim}(M[j, i]) = \dim(V_i)F_j(1, 2 \cos \frac{2}{n}) \), for any \((j, i) \in I\), where \( F_j \) is the \( j \)-th Dickson polynomial.

**Proof.** By [22, Proposition 4.1], one is able to verify by induction on \( j \) that

\[ F_j(a, M[2, 0]) = M[j, 0], \text{ for } 1 \leq j \leq n. \]
Note that \( M[n, 0] = 0 \) holds in \( r_{st}(H) \) since \( M(n, 0) \) is a projective \( H \)-module. Consequently, we have:

\[
0 = \text{FPdim}(M[n, 0]) \\
= \text{FPdim}(F_n(\pi, M[2, 0])) \\
= F_n(\text{FPdim}(\pi), \text{FPdim}(M[2, 0])) \\
= F_n(1, \text{FPdim}(M[2, 0])).
\]

Note that the equality \( F_n(1, Z) = 0 \) has \( n - 1 \) distinct roots: \( 2 \cos \frac{k\pi}{n} \), for \( 1 \leq k \leq n - 1 \) (cf. [22, Lemma 5.1]). It follows that \( \text{FPdim}(M[j, 0]) \) is equal to \( 2 \cos \frac{k\pi}{n} \) for some \( 1 \leq k \leq n - 1 \) satisfying \( k \mid n \) by Proposition 7.2 (4). We claim that \( k = 1 \). Otherwise, \( n_1 := \frac{n}{k} < n \) and \( \text{FPdim}(M[n_1, 0]) = F_{n_1}(1, 2 \cos \frac{\pi}{n_1}) = 0 \), a contradiction to Proposition 7.2 (4). Now for any \((j, i) \in I\), we have

\[
\text{FPdim}(M[j, i]) = \text{FPdim}(\overline{V}_i) \text{FPdim}(\overline{M[j, 0]}) \\
= \dim(\overline{V}_i) \text{FPdim}(F_j(\pi, M[2, 0])) \\
= \dim(\overline{V}_i) F_j(\text{FPdim}(\pi), \text{FPdim}(M[2, 0])) \\
= \dim(\overline{V}_i) F_j(1, 2 \cos \frac{\pi}{n}).
\]

The proof is completed. \( \square \)

Extending \( \text{FPdim} \) from the basis of \( r_{st}(H) \) to \( R_{st}(H) := \mathbb{C} \otimes_{\mathbb{Z}} r_{st}(H) \) by linearity, we obtain a function \( \text{FPdim} \) from \( R_{st}(H) \) to \( \mathbb{C} \). Denote by \( b_{(j,i)} := \text{FPdim}(M[j,i])M[j,i] \), for any \((j, i) \in I\). Then the set \( B = \{ b_{(j,i)} \mid (j, i) \in I \} \) is a basis of \( R_{st}(H) \). The stable Green algebra \( R_{st}(H) \) admits some special properties, e.g., the quadruple \((R_{st}(H), \text{FPdim}, B, \ast)\) is a group-like algebra, and hence a bi-Frobenius algebra, see [22, Section 6] for details.

### 8. The Green rings of the Radford Hopf algebras

In this section, we apply the results obtained in the previous sections to a family of Hopf algebras, known as the Radford Hopf algebras. The Radford Hopf algebras was introduced by Radford in [20] so as to give an example of Hopf algebras whose Jacobson radical is not a Hopf ideal. As we shall see that this family of Hopf algebras are pointed of rank one and can be derived from group data of non-nilpotent type.

Let \( G \) be a cyclic group of order \( mn \) \((m > 1)\) generated by \( g \). Suppose \( V_i \) is a one dimensional vector space such that the action of \( g \) on \( V_i \) is the scalar multiply by \( \omega^i \), where \( \omega \) is a primitive \( mn \)-th root of unity. Then \( \{ V_i \mid i \in \mathbb{Z}_{mn} \} \) forms a complete set of simple \( kG \)-modules up to isomorphism. Let \( \chi \) be the \( k \)-linear character of \( V_{m(n-1)} \). Namely, \( \chi(g) = \omega^{m(n-1)} = \omega^{-m} \),
where \( \omega^{-m} \) is a primitive \( n \)-th root of unity. The order of \( \chi \) is \( n \) and the \( k \)-linear character of \( V_m \) is \( \chi^{-1} \).

Let \( D = (G, \chi, g, 1) \). Then the group datum \( D \) is of non-nilpotent type since \( g^n - 1 \neq 0 \) and \( \chi^n = 1 \). Let \( H \) be the Hopf algebra associated to the group datum \( D \). Then \( H \) is generated as an algebra by \( g \) and \( y \) subject to

\[
g^{mn} = 1, \quad yg = \chi(g)gy = \omega^{-m}gy, \quad y^n = g^n - 1.
\]

The comultiplication \( \Delta \), counit \( \varepsilon \), and antipode \( S \) are given respectively by (2.2) and (2.3). \( H \) is a finite dimensional pointed Hopf algebra of rank one with a \( k \)-basis \( \{ y^i g^j \mid 0 \leq i \leq n - 1, \ 0 \leq j \leq mn - 1 \} \) and \( \dim H = mn^2 \).

Let \( N = \{ 1, g^n, g^{2n}, \ldots, g^{(m-1)n} \} \), \( \overline{G} = G/N \) and \( \chi \) the \( k \)-linear character of \( G/N \) such that \( \overline{\chi}(g^n) = \chi(g^n) \), for \( 0 \leq i \leq mn - 1 \). Then the Hopf algebra associated to the group datum \( \overline{D} = (\overline{G}, \overline{\chi}, \overline{g}, 0) \) of nilpotent type is nothing but the Taft Hopf algebra \( T_n \). Let \( e = \frac{1}{m} \sum_{k=0}^{m-1} g^k n \). Then \( e \) is a central idempotent of \( H \) and \( H \) has the decomposition \( H = He \oplus H(1 - e) \).

By Proposition 2.3, the quotient \( H/H(1 - e) \) is a Hopf algebra isomorphic to \( T_n \) and the subalgebra \( H(1 - e) \) is semisimple.

Denote by \( \Omega_0 \) the subset of \( \mathbb{Z}_{mn} \) consisting of elements divisible by \( m \) and \( \Omega_1 \) the complementary subset of \( \Omega_0 \). Let \( \tau \) be the permutation of \( \mathbb{Z}_{mn} \) determined by \( V_{\chi^{-1}} \otimes V_i \cong V_{\tau(i)} \), where \( V_{\chi^{-1}} \) is exactly the simple \( kG \)-module \( V_m \) with the character \( \chi^{-1} \). It is easy to see that \( \tau(i) = m + i \), for any \( i \in \mathbb{Z}_{mn} \). Let \( \langle \tau \rangle \) be the subgroup of the symmetric group \( S_{mn} \) generated by the permutation \( \tau \). Then \( \langle \tau \rangle \) acts on the index set \( \mathbb{Z}_{mn} \). With this action, the index set \( \mathbb{Z}_{mn} \) is divided into \( m \) distinct \( \langle \tau \rangle \)-orbits \([0], [1], [2], \ldots, [m-1]\), where \([i]\) = \{ \( i, m+i, 2m+i, \ldots, (n-1)m+i \} \), for \( 0 \leq i \leq m - 1 \). Moreover, \( \Omega_0 = [0] \) and \( \Omega_1 = [1] \cup [2] \cup \ldots \cup [m-1] \).

It follows from Theorem 2.9 that

\[
\{ M(k, i), P_{[j]} \mid i \in \Omega_0, 0 \leq k \leq n, 1 \leq j \leq m - 1 \}
\]

forms a complete set of finite dimensional indecomposable \( H \)-modules up to isomorphism. Observe that \( V_i \otimes V_j \cong V_{i+j} \) and \( (\omega^a \omega^b)^n = 1 \) if and only if \( m \mid i + j \) for any \( i, j \in \mathbb{Z}_{mn} \). By Proposition 4.1 and Proposition 4.2, we have the following.

**Proposition 8.1.** Let \( i, j \in \Omega_1, s \in \Omega_0, 1 \leq k \leq n \) and \( a = [V_{\chi^{-1}}] = [V_m] \).

1. \( P_{[j]} P_{[j]} = \begin{cases} (1 + a + \cdots + a^{n-1})M[n, 0], & m \mid i + j, \\ nP_{[i+j]}, & m \nmid i + j. \end{cases} \)

2. \( M[k, s]P_{[j]} = kP_{[j]} \). Moreover \([V_s]P_{[j]} = P_{[j]} \) and \( M[2, 0]P_{[j]} = 2P_{[j]} \).
Let $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]$ be the polynomial ring over $\mathbb{Z}$ in the variables $Y,Z,X_1,X_2,\ldots,X_{m-1}$ and $I$ the ideal of the polynomial ring generated by the following elements:

\begin{align}
(8.1) & \quad Y^n - 1, \ (1 + Y - Z)F_n(Y,Z), \ YX_1 - X_1, \ ZX_1 - 2X_1, \\
(8.2) & \quad X_j^I - n^{j-1}X_j, \text{ for } 1 \leq j \leq m - 1, \\
(8.3) & \quad X_j^m - n^{m-2}(1 + Y + \cdots + Y^{n-1})F_n(Y,Z).
\end{align}

**Theorem 8.2.** The Green ring $r(H)$ of the Radford Hopf algebra $H$ is isomorphic to the quotient ring $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]/I$. The Jacobson radical of the quotient ring is a principal ideal generated by $(1 - Y)F_n(Y,Z)$.

**Proof.** Let $r(T_n)$ be the Green ring of the Taft algebra $T_n$. Then $r(T_n)$ is isomorphic to the quotient ring of the polynomial ring $\mathbb{Z}[Y,Z]$ modulo the relations $Y^n = 1$ and $(1 + Y - Z)F_n(Y,Z) = 0$, see [2, Theorem 3.10] or [22, Theorem 4.3]. Note that $r(H)$ is commutative and is generated as a ring by $P_j$, for $1 \leq j \leq m - 1$, over the subring $r(T_n)$. There is a unique ring epimorphism $\Phi$ from $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]$ to $r(H)$ such that

$$
\Phi(Y) = a, \ \Phi(Z) = M[2,0], \ \Phi(X_j) = P_j, \text{ for } 1 \leq j \leq m - 1.
$$

It follows from Proposition 8.1 that the map $\Phi$ vanishes at the generators of the ideal $I$ given by (8.1)-(8.3). Hence $\Phi$ induces a unique ring epimorphism $\overline{\Phi}$ from $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]/I$ to $r(H)$ such that $\overline{\Phi}(z + I) = \Phi(z)$, for any $z$ in $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]/I$. Observe that $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]/I$ as a $\mathbb{Z}$-module has a $\mathbb{Z}$-basis \{\(Y^iZ^kX_j\mid 0 \leq i, k \leq n - 1, 1 \leq j \leq m - 1\)\}. Thus, as free $\mathbb{Z}$-modules, $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]/I$ and $r(H)$ both have the same rank $n^2 + m - 1$. As a result, the map $\overline{\Phi}$ is an isomorphism. Thanks to Theorem 6.3, we have that the Jacobson radical of the quotient ring $\mathbb{Z}[Y,Z,X_1,X_2,\ldots,X_{m-1}]/I$ is a principal ideal generated by $(1 - Y)F_n(Y,Z)$.

By Proposition 5.2 and Theorem 8.2, we have the following corollary.

**Corollary 8.3.** The Grothendieck ring $G_0(H)$ of the Radford Hopf algebra $H$ is isomorphic to the quotient ring $\mathbb{Z}[Y,X_1,X_2,\ldots,X_{m-1}]/I_0$, where $I_0$ is the ideal of $\mathbb{Z}[Y,X_1,X_2,\ldots,X_{m-1}]$ generated by $Y^n - 1, YX_1 - X_1, X_j^I - n^{j-1}X_j$ for $1 \leq j \leq m - 1$ and $X_j^m - n^{m-1}(1 + Y + \cdots + Y^{n-1})$.

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