Bases of Minimal Vectors in Lagrangian Lattices.

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Abstract

Motivated by the ring of integers of cyclic number fields of prime degree, we introduce the notion of Lagrangian lattices. Furthermore, given an arbitrary non-trivial lattice \( L \) we construct a family of full-rank sub-lattices \( \{L_\alpha\} \) of \( L \) such that whenever \( L \) is Lagrangian it can be easily checked whether or not \( L_\alpha \) has a basis of minimal vectors. In this case, a basis of minimal vectors of \( L_\alpha \) is given.

1 Introduction and Background

In [8] Conway and Sloane constructed the first example of an 11-dimensional lattice that is generated by its minimal vectors but in which no set of \( N \) minimal vectors forms a basis. This construction implies that such lattices exist in every dimension \( N \geq 11 \). In [22] and [23], Martinet showed that a lattice of dimension \( N \leq 8 \), which is generated by its minimal vectors also has a basis of minimal vectors. This study was completed in [24] by Martinet and Schrmann, where the authors showed a similar result for 9-dimensional lattices and provided a counter-example in dimension 10.

Thus, the \( \mathbb{Z} \)-span of \( S(L) \) the set of minimal vectors in a lattice \( L \) can span \( L \) without containing a subset of \( N \) \( \mathbb{R} \)-linearly independent vectors. If \( \mathbb{R}^N = \text{span}_\mathbb{R}(S(L)) \) then \( L \) is called well-rounded. A stronger condition is when \( L = \text{span}_\mathbb{Z}(S(L)) \), in this case \( L \) is called strongly well-rounded. From the previous, a stronger condition is when \( L \) has a basis of minimal vectors.

Computationally, checking the weakest of these conditions, i.e., well-roundedness, for an arbitrary lattice is an \( \text{NP} \)-hard problem [19]. Note that this is equivalent to finding \( S(L) \) the set of minimal vectors in \( L \). Up to dimension 4, the three conditions are equivalent. Thus, we will first give a short overview on the weakest of the previously mentioned properties, namely, the well-roundedness property. Then we will proceed on the study of Lagrangian lattices and their full-rank sub-lattices with a minimal basis.

Well-rounded lattices appear in various arithmetic and geometric problems. In particular, in discrete geometry, number theory and topology. For example, a classical theorem due to Voronoi [15] implies that the local maxima of the sphere packing function are all realized at well-rounded lattices. In [25] well-rounded lattices has been investigated in the context of Minkowski conjecture. Furthermore, topological properties of the set of well-rounded lattices has also been of interest. To name just a few examples, in [1], Ash proved that the space of all (unimodular) lattices retracts to the space of well-rounded lattices. More recently, a result by Solan [29] state that for any lattice \( L \) there exists a unimodular diagonal real matrix \( a \) with positive entries such that \( a \cdot L \) is a well-rounded lattice.

\*M. T. Damir’s work was supported in part by the Academy of Finland, under Grant No. 318937 (Aalto University PROFI funding to C. Hollanti).

\†G. Mantilla Soler’s work was supported in part by the Aalto Science Institute.
In addition to their arithmetic and geometric appeal, well-rounded lattices are studies in communication theory. In particular, in physical layer communication reliability [16] and security [18]. It is also worth mentioning that most of the lattice-based cryptographic protocols lays on the hardness of the shortest vector problem. On the other hand, the problem of determining all the successive minima of an arbitrary lattice is believed to be strictly harder [26]. However, if the lattice is well-rounded, these two problems are equivalent.

Hence, from a theoretical point of view, as well as a practical, it is of interest to explicitly construct well-rounded lattices and to study when a given lattice has a well-rounded sub-lattice or a sub-lattice generated by its minimal vectors. It turns out, see §2.1 for details, that among all lattices well-rounded lattices are scarce. So in a probabilistic sense, such lattices are difficult to find. Studying the geometric structure of well-rounded sub-lattices, strongly well-rounded and lattices with a minimal basis is a non-trivial question that has been investigated by several authors; for instance, in dimension 2 work in this direction has been done in [14], [2] and [21].

One example of an infinite family of well-rounded lattices, in fact lattices with a minimal basis, comes from the study of the ring of integers of Galois number fields of prime degree. Suppose that $p$ is an odd prime and that $K$ is a degree $p$ tame Galois number field. In [11] and [12] the authors give a set of conditions on positive integers $m \equiv 1 \pmod{p}$ so that the sub-lattice of $O_K$ given by $\{x \in O_K : \text{Tr}_{K/Q}(x) \equiv 0 \pmod{m}\}$ has a minimal basis. One of the key properties of $O_K$ behind such result is a Theorem of Conner and Perlis that shows that $O_K$ has a Lagrangian basis. Motivated by the results of Conner and Perlis, see [7, IV.8], we define the notion of Lagrangian lattice (see Definition 3.12). Moreover, given an arbitrary non-trivial lattice $\mathcal{L}$ we construct a family of sub-lattices of it such that whenever $\mathcal{L}$ is Lagrangian we can give easy conditions to decide if a given sub-lattice in the family has a minimal basis. More explicitly,

**Theorem** (cf. Theorem 4.9). Let $\mathcal{L} \subseteq \mathbb{R}^N$ be a Lagrangian lattice with Lagrangian basis $\{e_1, ..., e_N\}$. Let $a := (e_1, e_1)$ and $h = -(e_1, e_2)$. Let $r, s$ be integers such that $0 \neq |r| < N$ and let $m := r + sN$. Suppose that

$$\frac{Na - 1}{N^2 - 1} \leq \left(\frac{m}{r}\right)^2 \leq \frac{(aN - 1)(N + 1)}{N - 1}.$$ 

Then the lattice $\mathcal{L}_{v_1}^{(r,s)}$ is a sub-lattice of $\mathcal{L}$ of index $m|r|^{N-1}$, minimum

$$\lambda_1(\mathcal{L}_{v_1}^{(r,s)}) = ar^2 + \frac{m^2 - r^2}{N}$$

and basis of minimal vectors

$$\{re_1 + sv_1, re_2 + sv_1, ..., re_N + sv_1\}.$$ 

**Remark 1.1.** Suppose that $K$ is a Galois number field of prime degree $p$, unramified at $p$. If one uses the theorem above with $r = 1$, over the Lagrangian lattice $O_K$, then the results of [11] and [12] are recovered.

2 General background on Lattices

A lattice $\mathcal{L}$ in $\mathbb{R}^N$ is a discrete additive subgroup of $\mathbb{R}^N$. If $t$ is the dimension of the sub-space generated by $\mathcal{L}$, it can be shown that $\mathcal{L}$ is a free $\mathbb{Z}$-module of rank $t$. In other words, a rank $t$ lattice

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1 According to Conner and Perlis see [7, IV.8.1] this name is due to Hilbert.
in $\mathbb{R}^N$ is a set of the form

$$\mathcal{L} = \left\{ \sum_{i=1}^{t} a_i e_i \mid a_i \in \mathbb{Z} \right\}. \quad (1)$$

where $\{e_1, \ldots, e_t\} \subseteq \mathbb{R}^N$ is a set independent vectors. We call $M := [e_1 \cdots e_t]$ a generator matrix of $\mathcal{L}$, where the vectors $e_1, \ldots, e_t$ are considered as column vectors, i.e., $\mathcal{L} = M \mathbb{Z}^N$. The matrix $G = M^T M$ is called a gram matrix of $\mathcal{L}$. The lattice $\mathcal{L}$ is said to be integral if the matrix $G$ has all its entries in $\mathbb{Z}$. We say that $\mathcal{L}$ has full rank if $t = n$. If $\mathcal{L}$ is a full lattice its volume, or more precisely co-volume, is defined as $\text{vol}(\mathcal{L}) := \sqrt{\det(G)}$. It can be shown that this definition is independent on the choice of Gram matrix. If $\mathcal{L}'$ is a full sub-lattice of $\mathcal{L}$ it can be shown that $[\mathcal{L} : \mathcal{L}'] = \frac{\text{vol}(\mathcal{L}')}{\text{vol}(\mathcal{L})}$.

Remark 2.1. In this paper we will deal mostly with full lattices. Thus, from this point whenever we say lattice we mean full unless we explicitly say the contrary.

An important invarian in the study of the sphere packing problem is the center density of $\mathcal{L}$ defined by

$$\delta(\mathcal{L}) := \frac{\lambda_1(\mathcal{L})^{N/2}}{2^N \text{vol}(\mathcal{L})}.$$

Given a lattice $\mathcal{L}$, the quantity $\lambda_1(\mathcal{L}) := \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|^2$ is called the minimum of $\mathcal{L}$. The set of minimal vectors in $\mathcal{L}$ is the set of vectors of minimum norm, i.e.,

$$S(\mathcal{L}) := \{x \in \mathcal{L} : \|x\|^2 = \lambda_1(\mathcal{L})\}. \quad (2)$$

The cardinality of $S(\mathcal{L})$ is known as kissing number of $L$. For a detailed exposition on lattices we refer the reader to [9].

2.1 Well-rounded lattices

Now we are ready to proceed on the study of well-rounded lattices and the lattices with minimal basis.

Definition 2.2. Let $n$ be a positive integer and let $\mathcal{L} \subset \mathbb{R}^n$ be a lattice.

- The lattice $\mathcal{L}$ is well-rounded (abbreviated WR) if $\text{span}_\mathbb{R}(S(\mathcal{L})) = \mathbb{R}^n$.
- We say that $\mathcal{L}$ is strongly well-rounded (abbreviated SWR) if $\mathcal{L} = \text{span}_\mathbb{Z}(S(\mathcal{L}))$.
- The lattice $\mathcal{L}$ is said to have a minimal basis if $L = \text{span}_\mathbb{Z}\{v_1, \ldots, v_k\}$, for some $\mathbb{R}$-linearly independent vectors $v_1, \ldots, v_k$ in $S(\mathcal{L})$.

Every SWR lattice is WR. The following example illustrates that the converse does not always hold.

Example 2.3. Let $n, k$ be positive integers with $n > k^2 > 1$. Let $\mathcal{L}$ be the lattice generated by $\mathbb{Z}^n$ and the vector $v = (1/k, \ldots, 1/k)$. By the hypothesis on $n$ and $k$, $S(\mathcal{L})$ consists of the standard basis vectors, i.e., $\pm e_i = (0, \ldots, \pm 1, \ldots, 0)$ and $\mathbb{Z}^n \neq \mathcal{L}$. In particular, $\mathcal{L}$ is WR. On the other hand $\text{span}_\mathbb{Z}(S(\mathcal{L})) = \mathbb{Z}^n \neq \mathcal{L}$ so $\mathcal{L}$ is not SWR.
We define two lattices \( L \) and \( L' \) to be similar if we can obtain \( L \) from \( L' \) using a rotation and a real dilation of \( L' \). It is clear that the well-roundedness property is invariant under similarity, so it is natural to consider the space of lattices of fixed volume 1, namely,

\[
S_N = \text{SO}_N(\mathbb{R}) \backslash \text{SL}_N(\mathbb{R}) / \text{SL}_N(\mathbb{Z}).
\]

It is well-known that \( S_N \) has a unique measure \( \mu_n \) (Haar measure) that is right \( \text{SL}_N(\mathbb{R}) \)-invariant. Unfortunately, the set of WR lattices in \( S_N \) has measure zero with respect to \( \mu_n \). Given a lattice \( \mathcal{L} \), the \( i \)th successive minimum is

\[
\lambda_i(\mathcal{L}) = \inf \{ r \mid \dim(\text{span}(\mathcal{L} \cap \overline{B}(0, r))) = r \},
\]

where \( B(0, r) \) is the closed ball of radius \( r \) around 0.

It follows from definitions that a lattice \( \mathcal{L} \) is WR if and only if all its successive minima are equal. With this in mind, we can see the set of WR lattices of dimension \( N \) as a set defined by \( N - 1 \) (successive minima) equalities, this shows that the space of WR is not a full-dimensional space in \( S_N \), hence the vanishing measure. The following figure illustrates \( S_2 \) and \( W_2 \) the sets of similarity classes and well-rounded planar lattices.

![Figure 1: Similarity classes of WR planar lattices.](image)

### 2.2 Classical lattices from algebraic number theory

One of the most common source of lattices coming from number theory are the so called ideal lattices. We briefly recall their construction since it is precisely a subset of such lattices that motivated the definitions of this paper.

Let \( N \) be a positive integer and let \( K \) be a degree \( N \) number field. Let \( \text{Hom}_{\mathbb{Q}-\text{alg}}(K, \mathbb{C}) = \{ \sigma_1, \ldots, \sigma_{r_1}, \tau_1, \overline{\tau_1}, \ldots, \tau_{r_2}, \overline{\tau_{r_2}} \} \) be the set of complex embeddings of \( K \), where the \( \sigma_i \) are the embeddings such that \( \sigma_i(K) \subseteq \mathbb{R} \) and the \( \{ \tau_j, \overline{\tau_j} \} \) are the pairs with images outside \( \mathbb{R} \). Let \( \Re \) and \( \Im \) be the real part function resp. imaginary part function on complex numbers. An element \( \alpha \in K \) is called totally positive if \( \sigma(\alpha) > 0 \) for all \( \sigma \in \text{Hom}_{\mathbb{Q}-\text{alg}}(K, \mathbb{C}) \).
Definition 2.4. Let $\alpha \in K$ be a totally positive element. The $\alpha$-twisted Minkowski embedding from $K$ into $\mathbb{R}^n$ is the $\mathbb{Q}$-linear map $j_{K,\alpha} : K \to \mathbb{R}^n$ defined by mapping $x$ to
\[
\left(\sqrt{\sigma_1(\alpha)\sigma_1(x)}, \ldots, \sqrt{\sigma_r(\alpha)\sigma_r(x)}, \sqrt{\tau_1(\alpha)\mathcal{R}(\tau_1(x))}, \sqrt{\tau_1(\alpha)\mathcal{Q}(\tau_1(x))}, \ldots, \sqrt{\tau_2(\alpha)\mathcal{R}\tau_2}, \sqrt{\tau_2(\alpha)\mathcal{Q}\tau_2}\right).
\]

In the case that $\alpha = 1$ the map $j_{K,\alpha}$ is denoted by $j_K$ and it is called the Minkowski embedding.

It is a classical fact from the geometry of numbers that $j_{K,\alpha}(O_K)$ is a full lattice in $\mathbb{R}^n$, hence the same is true for $j_{K,\alpha}(I)$ for any abelian sub-group $I$ of $O_K$ of finite index. In particular, this is the case for $I$ a non-zero ideal of $O_K$. If the field $K$ is totally real we have, as a simple calculation shows, that for all $x, y \in K$
\[
\text{Tr}_{K/\mathbb{Q}}(\alpha xy) = \langle j_{K,\alpha}(x), j_{K,\alpha}(y) \rangle.
\]

Hence, in this case, for any non-zero ideal $I < O_K$ we can identify the lattice $j_{K,\alpha}(I) \subseteq \mathbb{R}^n$ with the ideal $I$ together with the bilinear form on it given by $\text{Tr}_{K/\mathbb{Q}}(\alpha xy)$. The latter is sometimes denoted by $\langle I_\alpha, \text{Tr}_{K/\mathbb{Q}} \rangle$ and it is called an ideal lattice.

3 Sub-lattices from co-dimension one linear maps

Let $N$ be a positive integer and let $\mathcal{L} \subseteq \mathbb{R}^N$ be a full lattice. Suppose that $T : \mathcal{L} \to \mathbb{Z}$ is a non-trivial linear map. Let $v_1 \in \mathcal{L} \setminus \ker T$.

Proposition 3.1. Let $r, s$ be integers and let $m := r + sT(v_1)$. The map
\[
\Phi_{(r,s)} : \mathcal{L} \to \mathcal{L}
\]
\[
x \mapsto rx + sT(x)v_1
\]
is a linear. Moreover, $T \circ \Phi_{(r,s)} = [m] \circ T$ where $[m] : \mathbb{Z} \to \mathbb{Z}$ is the usual multiplication by $m$ map. In particular, if $r$ and $m$ are different from zero the map $\Phi_{(r,s)}$ is an injection.

Proof. Since $T$ is linear $\Phi_{(r,s)}$ is a composition of linear maps hence it is linear as well. To verify the second claim let $x \in \mathcal{L}$. Then,
\[
T(\Phi_{(r,s)}(x)) = T(rx + sT(x)v_1) = rT(x) + sT(v_1)T(x) = mT(x).
\]
For the last claim let $x \in \ker(T)$. Since $m \neq 0$ it follows from the second claim that $T(x) = 0$ hence, by the definition, $\Phi_{(r,s)}(x) = rx$ and since $\Phi_{(r,s)}(x) = 0$ and $r \neq 0$ it follows that $x = 0$.

\[
\Box
\]

Corollary 3.2. Let $L \subseteq \mathbb{R}^N$ be a full lattice, let $T \in \text{Hom}_\mathbb{Z}(\mathcal{L}, \mathbb{Z}) \setminus \{0\}$ and let $v_1 \in \mathcal{L} \setminus \ker T$. Let $r, s$ be integers, and define $m := r + sT(v_1)$. Suppose that $r$ satisfies
\[
\bullet \quad r \neq 0
\]
\[
\bullet \quad |r| < |T(v_1)|.
\]
Then $\Phi_{(r,s)}(L)$ a full sub-lattice of $\mathcal{L}$. Furthermore,
\[
\Phi_{(r,s)}(L) \subseteq \mathcal{L}_{(m)}^T := \{x \in \mathcal{L} : T(x) \equiv 0 \pmod{mn_T}\}
\]
where $n_T$ is the size of the co-kernel of $T$, i.e., $[\mathbb{Z} : T(\mathcal{L})]$.
Proof. This is just a reformulation of Proposition 3.1. By the hypothesis we have that there is an integer \( r \) and \( m \) are non-zero. Hence from the proposition we see that \( \Phi_{(r,s)}(L) \) and \( L \) have the same \( \mathbb{Z} \)-rank. The claimed inclusion follows from \( T \circ \Phi_{(r,s)} = [m] \circ T \).

\[ \square \]

Remark 3.3. By definition \( m \) is defined in terms of \( r, s \) and \( T(v_1) \). The conditions of \( r \) defined in the corollary are there to guarantee the reverse; in this case we have that the lattice \( \Phi_{(r,s)}(L) \) is determined by \( m \) and and \( T(v_1) \) since \( r \) is congruent to \( m \) modulo \( T(v_1) \).

Lemma 3.4. Let \( L \subseteq \mathbb{R}^N \) be a full lattice, let \( T \in \text{Hom}_\mathbb{Z}(L, \mathbb{Z}) \setminus \{0\} \) and let \( n_T \) be the order of the co-kernel of \( T \). For a given positive integer \( m \) let \( L_{(m)} \) := \{x \in L : T(x) \equiv 0 \pmod{mn_T}\}. Then, \( L_{(m)} \) is a full sub-lattice of \( L \) of index \( m \).

Proof. We claim that \( L/L_{(m)} \cong \mathbb{Z}/m\mathbb{Z} \), from which the result follows. Composing \( T \) with the reduction map \( \mathbb{Z} \to \mathbb{Z}/mn_T\mathbb{Z} \) we get a linear map \( \mathcal{L} \to \mathbb{Z}/mn_T\mathbb{Z} \) with image \( n_T\mathbb{Z}/mn_T\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \), and with kernel equal to \( L_{(m)} \), hence proving our claim.

\[ \square \]

Definition 3.5. Let \( L, T \) and \( v_1 \) as above. Let \( r, s \) be integers. The lattice \( L_{(r,s)} \) is defined as the sub-lattice of \( L \) given by the image of \( \Phi_{(r,s)} \) i.e.,

\[ L_{(r,s)}^{T,v_1} := \Phi_{(r,s)}(L). \]

It follows from Corollary 3.2 and Lemma 3.4 that, for \( m = r + sT(v_1) \), if \( r \neq 0, |r| < |T(v_1)| \) then we have a sequence of of rank \( N \) lattices

\[ L_{(r,s)}^{T,v_1} \subseteq L^{(m)} \subseteq L \]

such that \([ L : L_{(r,s)}^{T,v_1} ]\) is a multiple of \( m \). If \( L \) has a rigid basis with respect to \( T \), meaning that \( T \) takes the same value in all the elements of the basis, it turns out that such multiple is \(|r|^{N-1}\).

Example 3.6. Suppose \( N > 2 \) and let \( L = \mathbb{Z}^N \) the standard cubic lattice. Let \( T : L \to \mathbb{Z} \) the map that sends a vector to the sum of its coordinates, i.e., the trace map. Notice that the standard basis of \( \mathbb{Z}^n \) is rigid with respect to \( T \); all of the canonical vectors have trace equal to 1. In this case the lattice \( L_{(r,s)}^{(2)} \) is the lattice \( \mathbb{D}^N \).

1. Choosing as \( v_1 \) any vector with \( T(v_1) = 3 \), for instance \( v_1 = [1,1,1,0,...,0]^t \) if \( N > 2 \) and \( v_1 = [1,2]^t \) for \( N = 2 \), we obtain that \( L_{(r,s)}^{(1,1)} = L_{(2)} = \mathbb{D}^N \). This can be seen by calculating the image of the standard basis of \( \mathbb{Z}^N \) under \( \Phi_{(r,s)} \). Such image is the set

\[ \{[0,1,1,0,...,0]^t, [1,0,1,0,...,0]^t, [1,1,1,0,...,0]^t, [1,1,1,0,...0,-1]^t\} \]

for \( N > 2 \) and \{\([0,2]^t, [1,1]^t\}\} for \( N = 2 \). In either case such set is a basis of \( \mathbb{D}^N \).

2. Suppose that \( N > 2 \). Choosing as \( v_1 \) the vector \([1,...,1]^t\), which has trace equal to \( N \), we see that for \( m = 2 = r + sN \) the tuple \((r,s)\) is either \((2,0)\) or \((2-N,1)\) which define the lattices \( L_{(r,s)}^{(2,0)} \) and \( L_{(r,s)}^{(2-N,1)} \) respectively. In the former case the lattice \( L_{(r,s)}^{(2,0)} \) is \((2\mathbb{Z})^N \). In the latter case the structure of the lattice \( L_{(r,s)}^{(2-N,1)} \) varies with \( N \). Similar to the previous item a basis for \( L_{(r,s)}^{(2-N,1)} \) can be calculated as the image of the standard basis of \( \mathbb{Z}^N \) under \( \Phi_{(r,s)} \) to obtain the basis

\[ \{\[3-N,1,1,...,1]^t, [1,3-N,1,...,1]^t, ..., [1,1,...,3-N]^t\}. \]

For instance:
• If $N = 3$ the lattice $L_{T, v_1}^{(−1, 1)}$ is isomorphic to the root lattice $A_3$,
• If $N = 4$ the lattice $L_{T, v_1}^{(−2, 1)}$ is isomorphic to the lattice $(2\mathbb{Z})^4$
• If $N = 5$ the lattice $L_{T, v_1}^{(−3, 1)}$ is isomorphic to the lattice $L_{9, 5}$ defined in [3 Theorem 4.1].

3. For $N = 2$ and $v_1 = [1, 1]^T$ the lattices $L_{T, v_1}^{(−1, 2)}, L_{T, v_1}^{(1, 1)}$ and $L_{T}^{(3)}$ are all isomorphic to $A_2$.

**Proposition 3.7.** Let $L, T$ and $v_1$ as above. Let $r, s$ be integers with $r \neq 0$ and $|r| < |T(v_1)|$ and let $m := r + sT(v_1)$. Suppose that there exists a basis of $L$ that is rigid with respect to $T$. Then,

$$[L : L_{T, v_1}^{(r,s)}] = m|r|^{N-1}.$$ 

In particular, $L_{T, v_1}^{(r,s)} = L_T^{(m)}$ if and only if $r = \pm 1$.

**Proof.** Let $B = \{e_1, \ldots, e_N\}$ be a basis of $L$ that is rigid with respect to $T$ i.e., $T(e_i) = T(e_j)$ for all $1 \leq i, j \leq N$. Thanks to Proposition 3.1 and Corollary 3.2 the set $B_{\Phi(r,s)} = \{\Phi(r,s)(e_1), \ldots, \Phi(r,s)(e_N)\}$ is a basis for $L_{T, v_1}^{(r,s)}$. Hence, the index $[L : L_{T, v_1}^{(r,s)}]$ is equal to $|\det(A)|$ where $A$ is the matrix representation of the basis $B_{\Phi(r,s)}$ in terms of the basis $B_{\Phi(r,s)}$. If $v_1 = \sum_{j=1}^{N} a_j e_j$ then, for all $1 \leq i \leq N$, we have that

$$\Phi(r,s)(e_i) = re_i + sT(e_i)\sum_{j=1}^{N} a_j e_j$$

hence

$$A = \begin{bmatrix} r + sa_1 T(e_1) & sa_1 T(e_2) & \ldots & sa_1 T(e_N) \\ sa_2 T(e_1) & r + sa_2 T(e_2) & \ldots & sa_2 T(e_N) \\ \vdots & \ddots & \ddots & \vdots \\ sa_N T(e_1) & sa_N T(e_2) & \ldots & r + sa_N T(e_N) \end{bmatrix}.$$ 

Since $B$ is rigid with respect to $T$ the matrix $A$ is equal to

$$A = \begin{bmatrix} r + sa_1 T(e_1) & sa_1 T(e_1) & \ldots & sa_1 T(e_1) \\ sa_2 T(e_2) & r + sa_2 T(e_2) & \ldots & sa_2 T(e_2) \\ \vdots & \ddots & \ddots & \vdots \\ sa_N T(e_N) & sa_N T(e_N) & \ldots & r + sa_N T(e_N) \end{bmatrix}.$$ 

Notice that the sum of the elements of an arbitrary column of $A$ is equal to

$$r + s \sum_{i=1}^{N} a_i T(e_i) = r + sT(v_1) = m.$$ 

Therefore after adding all the rows of $A$, factorizing $m$ and then substracting to the $i$-th row $sa_i T(e_i)$ times the first we see that $\det(A) = m r^{N-1}$. 

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Lemma 3.8. Let \( N \geq 2 \) be an integer and let \( K \) be a degree \( N \) number field. Let \( m \) be an integer such that \( m \equiv \pm 1 \pmod{N} \). By the Minkowski embedding we can view \( L := O_K \) as a lattice in \( \mathbb{R}^N \), so for this lattice let \( T : L \to \mathbb{Z} \) be the trace map. Suppose that \( K \) is tame, i.e., that there is no prime that ramifies wildly in \( K \). Then,

\[
\{ x \in O_K : \text{Tr}_{K/\mathbb{Q}}(x) \equiv 0 \pmod{m} \} = L_{T,1}^{\pm,1,s}
\]

where \( s = \frac{m+1}{N} \).

Proof. Since \( K \) is tame the trace map \( T : L \to \mathbb{Z} \) is surjective, see [27, Corollary 5 to Theorem 4.24]. Hence the result follows from Proposition 3.7 and Lemma 3.4.

\[ \square \]

3.1 Construction

In this section we specialize the above construction to lattices that have properties that are motivated by the structure of the ring of integers of certain tame abelian totally real number fields. Suppose that \( L \) is a rank \( N \) lattice such that \( L \cap L^* \neq 0 \). This for instance can be achieved if \( L \) is integral. By definition any non-zero \( v_1 \in L \cap L^* \) defines an element \( T_{v_1} \in \text{Hom}_\mathbb{Z}(L, \mathbb{Z}) \setminus \{0\} \); namely

\[
T_{v_1}(x) = \langle x, v_1 \rangle.
\]

Definition 3.9. Let \( L \) and \( v_1 \) as above. Let \( r, s \) be integers with \( 0 \neq |r| < T(v_1) = \|v_1\|^2 \). The lattice \( L_{v_1}^{(r,s)} \) is defined as

\[
L_{v_1}^{(r,s)} := L_{T_{v_1}, v_1}^{(r,s)}.
\]

From now on \( v_1 \) will be a fixed non-zero element in \( L \cap L^* \) and, unless clarification is necessary, we will denote the map \( T_{v_1} \) by \( T \).

Proposition 3.10. Let \( L, v_1, r \) and \( s \) as above. Let \( m := r + sT(v_1) = r + s\|v_1\|^2 \). For all \( \alpha \in L \) we have

\[
\| \Phi_{(r,s)}(\alpha) \|^2 = A\|\alpha\|^2 + BT^2(\alpha),
\]

where \( A = r^2 \) and \( B = \frac{m^2 - r^2}{\|v_1\|^2} \).

Proof. Let \( \alpha \in L \) and recall that by definition \( \Phi_{(r,s)}(\alpha) = r\alpha + sT(\alpha)v_1 \). Then,

\[
\| \Phi_{(r,s)}(\alpha) \|^2 = \langle \Phi_{(r,s)}(\alpha), \Phi_{(r,s)}(\alpha) \rangle
= \langle r\alpha + sT(\alpha)v_1, r\alpha + sT(\alpha)v_1 \rangle
= r^2\|\alpha\|^2 + 2rsT(\alpha)\langle \alpha, v_1 \rangle + s^2T(\alpha)^2\|v_1\|^2
= r^2\|\alpha\|^2 + 2rsT(\alpha)^2 + s^2T(\alpha)^2\|v_1\|^2
= r^2\|\alpha\|^2 + (2rs + s^2\|v_1\|^2)T(\alpha)^2
= r^2\|\alpha\|^2 + s(r + m)T(\alpha)^2
= r^2\|\alpha\|^2 + \frac{(m^2 - r^2)}{\|v_1\|^2}T(\alpha)^2
\]

\[ \square \]
3.2 Lagrangian lattices

In this section we define the notion of Lagrangian lattice and we see how from the we can obtain strongly well rounded lattices. Lagrangian lattices appear naturally in several context, e.g., ring of integers of certain number fields (see [3]).

Example 3.11. Suppose \( N \geq 2 \) and let \( L = \mathbb{Z}^N \) the standard cubic lattice. Let \( \mathbf{v}_1 := [1, ..., 1]^t \). For this choice of \( \mathbf{v}_1 \) the trace map \( T \) is equal to the sum of entries of a vector in \( \mathbb{Z}^N \). As we seen in example 3.6 for any pair of integers \( (r, s) \) with \( 0 < |r| < N \) the lattice \( L^{(r,s)}_{\mathbf{v}_1} \) can be very interesting and diverse. For instance, for \( N > 2 \), by just picking \( r \equiv 2 \pmod{N} \) we obtained \( \mathbb{D}_N, \mathbb{A}_2, (2\mathbb{Z})^4, L_{9,5} \).

Another important feature of these examples is that all of them are strongly well rounded. We isolated some characteristics of \( \mathbb{Z}^N \) and \( \mathbf{v}_1 = [1, ..., 1]^t \) that we think lead to the nice behaviour of the above examples.

Definition 3.12. Let \( N \) be a positive integer and let \( L \) be a rank \( N \) lattice. We say that \( L \) is Lagrangian if there is a basis \( \{e_1, ..., e_N\} \) of \( L \) and a non-zero \( \mathbf{v}_1 \in L \cap L^* \) such that

1. \( e_1 + ... + e_N = \mathbf{v}_1 \).
2. \( T_{\mathbf{v}_1}(e_i) = \langle e_1, \mathbf{v}_1 \rangle = 1 \) for all \( 1 \leq i \leq N \).
3. \( \langle e_i, e_i \rangle = \langle e_j, e_j \rangle \) for all \( 1 \leq i, j \leq N \).
4. \( \langle e_i, e_j \rangle = \langle e_k, e_l \rangle \) for all \( 1 \leq i, j, k, l \leq N \) with \( i \neq j \) and \( k \neq l \).

The above definition is an attempt to axiomatize the following result:

Theorem 3.13. Let \( N \geq 2 \) and let \( K \) be a totally real tame \( \mathbb{Z}/N\mathbb{Z} \)-number field. Let \( n \) be the conductor of \( K \). Suppose that either \( n \) or \( N \) is prime. Then, \( O_K \) via the Minkowski embedding is a Lagrangian lattice in \( \mathbb{R}^N \).

The case \( N \) prime is done in [7, IV. 8]. The case \( n \) prime is done in [6, Lemma 3.3]; there, such basis is constructed for \( N \) a prime power however the same exact proof works for any \( N \). We sketch the general idea of how the Lagrangian basis is found. By definition of conductor \( K \subseteq \mathbb{Q}(\zeta_n) \), and in both cases the hypotheses imply that \( O_K \) is a free \( \mathbb{Z}[\text{Gal}(K/\mathbb{Q})] \)-module of rank 1. Moreover, a generator for \( O_K \) as a \( \mathbb{Z}[\text{Gal}(K/\mathbb{Q})] \)-module is given by \( e_1 := \text{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n) \). Hence if \( \sigma \) is a generator of the Galois group of \( K \), and if \( e_i = \sigma(e_1)^{i-1} \), then the set \( \{e_1, ..., e_N\} \) is an integral basis of \( O_K \). Since \( K \) is tame its conductor \( n \) is square free, thus \( \text{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n) = \pm 1 \). Therefore replacing \( e_1 \) by \( -e_1 \) if necessary we may assume \( \text{Tr}_{K/\mathbb{Q}}(e_1) = 1 \), in other words \( \mathbf{v}_1 = e_1 + e_2 + ... + e_n = 1 \). This shows that (1) and (2) for the basis \( \{e_1, ..., e_N\} \), in fact since 1 goes under the Minkowski embedding into the vector with 1’s in all its entries we also have shown that \( T_{\mathbf{v}_1}(x) = \text{Tr}_{K/\mathbb{Q}}(x) \) for all \( x \in O_K \). Conditions (3) and (4) require more work but the Lagrangian basis constructed in the references mentioned above is the one we just described.

Remark 3.14. Sets of vectors \( v_1, ..., v_k \) in \( \mathbb{R}^N \) satisfying \( \langle v_i, v_i \rangle = 1 \) and \( \alpha = |\langle v_i, v_j \rangle| \) for \( i \neq j \) and some \( \alpha \in \mathbb{R} \) are sometimes called equiangular unit frames (see [30]).

Lemma 3.15. Let \( L \) be a rank \( N \) Lagrangian lattice and let \( \mathbf{v}_1 \) be the vector defining the map \( T \). Then

\[ \|\mathbf{v}_1\|^2 = T(\mathbf{v}_1) = N. \]
Proof. By definition of $T$ we have that $\|v_1\|^2 = \langle v_1, v_1 \rangle = T(v_1)$. On the other hand let $\{e_1, ..., e_N\}$ be a basis of $L$ satisfying the conditions of definition \ref{def:lagrangian}. Then $T(v_1) = T(e_1 + ... + e_N) = T(e_1) + ... + T(e_N) = 1 + ... + 1 = N$. \qed

Proposition 3.16. Let $L$ be a rank $N$ Lagrangian lattice with $v_1 \in L$ and $\{e_1, ..., e_N\}$ satisfying the conditions of the definition. Let $a := \langle e_1, e_1 \rangle$ and $h = -\langle e_1, e_2 \rangle$. Let $r, s$ be integers such that $0 \neq |r| < N$ and let $m := r + sN$.

- $L^{(r,s)}$ is a sub-lattice of $L$ of index $m|r|^{N-1}$.
- $||\Phi_{(r,s)}(e_i)||^2 = ar^2 + \frac{m^2 - r^2}{N}$ for all $1 \leq i \leq N$.
- $\langle \Phi_{(r,s)}(e_i), \Phi_{(r,s)}(e_j) \rangle = -r^2h + \frac{m^2 - r^2}{N}$ for all $1 \leq i \neq j \leq N$.

Proof. The first two conditions follow from Propositions \ref{prop:lagrangian} \ref{prop:lagrangian2} and Lemma \ref{lem:lagrangian3}. To finish the proof take $i \neq j$. Then,

$$\langle \Phi_{(r,s)}(e_i), \Phi_{(r,s)}(e_j) \rangle = \langle re_i + sv_1, re_j + sv_1 \rangle = r^2\langle e_i, e_j \rangle + 2rs + s^2N = -r^2h + \frac{m^2 - r^2}{N}.$$ \qed

Definition 3.17. Let $L$ be a rank $N$ lattice and let $T : L \to \mathbb{Z}$ be a non-trivial linear map. We denote by $L^0_T$ the rank $N - 1$ lattice given by de Kernel of $T$; $L^0_T := \ker(T)$. We will denote this sub-lattice by $L^0$ since in general $T$ will be clear from the context.

Example 3.18. Taking $L = \mathbb{Z}^{n+1}$ the usual cubic lattice and $T$ the sum of coordinates function, then $L^0$ is the root lattice $A_n$.

The above example is just a particular case of what happens in general Lagrangian lattices.

Theorem 3.19. Let $L$ be a rank $N$ Lagrangian lattice with $v_1 \in L$ and $\{e_1, ..., e_N\}$ satisfying the conditions of the definition. Let $a := \langle e_1, e_1 \rangle$ and $h = -\langle e_1, e_2 \rangle$. Then,

$$L^0 \cong (a + h)A_{N-1}.$$ 

Proof. Let $w_1 := e_1 - e_2, w_1 := e_2 - e_3, ..., w_{N-1} := e_{N-1} - e_N$. Since $T$ is constant on the $e_i$'s then $w_i \in L^0$ for all $i$. We claim that $\{w_1, ..., w_{N-1}\}$ is a basis of $L^0$. They are clearly linearly independent, so it’s enough to show that $\text{span}_{\mathbb{Z}}\{w_1, ..., w_{N-1}\} = L^0$. Let $v = a_1e_1 + ... + a_N e_N \in L^0$. Then $a_1 + ... + a_N = 0$. Showing that $v \in \text{span}_{\mathbb{Z}}\{w_1, ..., w_{N-1}\}$ is equivalent to show that the following linear system is solvable in $\mathbb{Z}$

\[
\begin{align*}
b_1 &= a_1 \\
b_2 - b_1 &= a_2 \\
b_3 - b_2 &= a_3 \\
&\vdots \\
b_{N-1} - b_{N-2} &= a_{N-1} \\
-b_{N-1} &= a_N
\end{align*}
\]
Since \( b_1 + (b_2 - b_1) + (b_3 - b_2) + \ldots + (b_{N-1} - b_{N-2}) = b_{N-1} \) and \( a_1 + \ldots + a_{N-1} = -a_{N-1} \) the system above has a solution if and only if the system given by the first \( N-1 \) equations has a solution. The \( N-1 \times N-1 \) clearly has a unique solution. Returning to the main proof notice that for all \( 1 \leq i \leq N-1 \)

\[
\langle w_i, w_i \rangle = \langle e_i, e_i \rangle - 2\langle e_i, e_{i+1} \rangle + \langle e_{i+1}, e_{i+1} \rangle = 2(a + h).
\]

Suppose that \( 1 \leq i < j \leq N-1 \). If \( j \neq i + 1 \) then

\[
\langle w_i, w_j \rangle = \langle e_i, e_j \rangle - \langle e_i, e_{j+1} \rangle - \langle e_j, e_{i+1} \rangle + \langle e_{j+1}, e_{j+1} \rangle = -h + h + h - h = 0.
\]

If \( j = i + 1 \) then

\[
\langle w_i, w_{i+1} \rangle = \langle e_i, e_{i+1} \rangle - \langle e_{i+1}, e_{i+1} \rangle + \langle e_{i+1}, e_{i+2} \rangle = -h + h - a - h = -(a + h).
\]

Therefore the Gram matrix of \( L^0 \) in the basis \( \{w_1, \ldots, w_{N-1}\} \) is \((a + h)M\) where \( M \) is one of know Gram matrices of \( \mathbb{A}_{N-1} \) (See [10]).

**Corollary 3.20.** Let \( \mathcal{L} \) be a rank \( N \) Lagrangian lattice with \( v_1 \in \mathcal{L} \) and \( \{e_1, \ldots, e_N\} \) satisfying the conditions of the definition. Let \( a := \langle e_1, e_1 \rangle \) and \( h = -\langle e_1, e_2 \rangle \). Then

\[
\min_{v \in \mathcal{L} \setminus \{0\}} \|v\|^2 = 2(a + h).
\]

Moreover, such minimum is obtained at any of the vectors \( e_i - e_j \) for \( i \neq j \).

**Proof.** This follows from Theorem 3.19 and from the fact that for any integer \( n > 1 \) the root lattice \( A_n \) has minimal distance equal to 2. \( \square \)

## 4 Conditions

As we seen in previous examples by considering the construction \( \mathcal{L}^{(r,s)}_{v_1} \) applied to the Lagrangian lattice \( \mathbb{Z}^N \), for some values \((r, s)\), we obtained several well rounded lattices, moreover they are strongly well rounded. Here we show how to do this for an arbitrary Lagrange lattice \( \mathcal{L} \) provided we have some restrictions on the values \((r, s)\) with respect to \( \mathcal{L} \).

Throughout this section \( \mathcal{L} \subseteq \mathbb{R}^N \) will denote a Lagrangian lattice with \( v_1 \) and \( \{e_1, \ldots, e_N\} \) satisfying the conditions of Definition 3.12. Let \( a := \langle e_1, e_1 \rangle \) and \( h = -\langle e_1, e_2 \rangle \). Let \( r, s \) be integers such that \( 0 \neq |r| < N \) and let \( m := r + sN \). Also recall the definitions \( A = r^2 \) and \( B = \frac{m^2 - r^2}{N} \).

### 4.1 Shortest vector problem for \( \mathcal{L}^{(r,s)}_{v_1} \)

The shortest non-zero norm in \( \mathcal{L}^{(r,s)}_{v_1} \) is by definition

\[
\lambda_1(\mathcal{L}^{(r,s)}_{v_1}) := \min_{v \in \mathcal{L}^{(r,s)}_{v_1} \setminus \{0\}} \|v\|^2.
\]

Thanks to Proposition 3.10 and Lemma 3.15 we have that

\[
\lambda_1(\mathcal{L}^{(r,s)}_{v_1}) := \min_{\alpha \in \mathcal{L} \setminus \{0\}} (A\|\alpha\|^2 + BT^2(\alpha)).
\]

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Therefore we are left with the task of minimising the function
\[ f(\alpha) := A\|\alpha\|^2 + BT^2(\alpha) \]
on \( \mathcal{L} \setminus \{0\} \). To do this we use the natural partition of \( \mathcal{L} \) given by taking the quotient with the sub-lattice \( \mathcal{L}^0 \). Since \( \mathcal{L}/\mathcal{L}^0 \cong \mathbb{Z} \) via the map \( T \) such partition is
\[ \mathcal{L} = \bigcup_{d \in \mathbb{Z}} S_d \]
where \( S_d := \{ x \in \mathcal{L} \mid T(x) = d \} \). Hence we have
\[ \lambda_1(\mathcal{L}^{(r,s)}) := \min_{d \geq 0} \left( \min_{\alpha \in S_d \setminus \{0\}} f(\alpha) \right). \]

**Remark 4.1.** We included only non-negative values of \( d \) above since \( f \) is even and \( S_{-d} = -S_d \).

**Lemma 4.2.** Let \( d \) be an integer. Let \( \alpha = a_1e_1 + ... + a_N e_N \) be an element of \( S_d \). Suppose there are \( a_i, a_j \) such that \( a_i - a_j > 1 \). Then there is \( \beta \in S_d \) such that \( \|\beta\| < \|\alpha\| \).

**Proof.** Let \( \beta = \alpha + e_j - e_i \). Since \( e_j - e_i \in \ker(T) \) we have that \( \beta \in S_d \). Taking square norms we get
\[ \|\beta\|^2 = \|\alpha\|^2 + 2(\langle \alpha, e_j \rangle - \langle \alpha, e_i \rangle) + \|e_i\|^2 - 2\langle e_i, e_j \rangle + \|e_j\|^2 = \|\alpha\|^2 + 2(\langle \alpha, e_j \rangle - \langle \alpha, e_i \rangle + a + h). \]

On the other hand for all \( 1 \leq k \leq N \)
\[ \langle \alpha, e_k \rangle = \sum_{i \neq k} a_i \langle e_i, e_k \rangle + a_k a = -h \sum_{i \neq k} a_i + a_k a = -h(T(\alpha) - a_k) + a_k a = -hT(\alpha) + a_k (a + h). \]

In particular, \( \langle \alpha, e_j \rangle - \langle \alpha, e_i \rangle = (a_j - a_i)(a + h) \) and thus
\[ \|\beta\|^2 = \|\alpha\|^2 + 2(a + h)(a_j - a_i + 1) = \|\alpha\|^2 - 2(a + h)(a_i - a_j - 1) > \|\alpha\|^2. \]

\[ \square \]

For a subset \( I \subset \{1, ..., N\} \) we denote by \( E_I := \sum_{i \in I} e_i \), for instance \( E_\emptyset = 0 \) and \( E_{\{1, ..., N\}} = v_1 \).

Notice that \( T(E_I) = \#I \) for all subset \( I \).

**Corollary 4.3.** Let \( d \) be a positive integer. Then \( \min_{\alpha \in S_d \setminus \{0\}} \|x\|^2 \) is achieved at some
\[ \alpha = E_I + C v_1 \]
where \( C \) is a non-negative integer and \( \#I < N \). Moreover, \( \#I \) is the residue class of \( d \) modulo \( N \) and \( C = (d - \#I)/N \).

**Proof.** Let \( \alpha = \sum_{i=1}^N a_i e_i \in S_d \) a non-zero element having minimal norm. Let
\[ a_k = \max\{a_i : 1 \leq i \leq N\}. \]

Replacing \( \alpha \) by \( -\alpha \) if necessary we may assume that \( a_k \geq 1 \). Suppose that there is some \( a_j \leq 0 \). Then, since the \( a_j \)'s can not be more than one integer apart, by Lemma 4.2 we must have that all
non positive coefficients are equal to 0 and all positives are equal to 1. Thus in such case $\alpha$ is of the form $E_I$ where $\#I < N$ since there are coefficients equal to 0. Now, if all $a_j$ are positive let

$$c = \min\{a_i : 1 \leq i \leq N\}.$$  

If $c = a_k$ then $\alpha = cv_1$, otherwise $a_k = c + 1$ and $\alpha = E_I + cv_1$ where $I$ is the subset of elements $i$ such that $a_i = a_k$.

**Theorem 4.4.** Let $d$ be a positive integer. Let $k$ be the residue class of $d$ modulo $N$ and let $c = (d-k)/N$. Then,

$$\min_{\alpha \in S_d \setminus \{0\}} f(\alpha) = f(E_I) + c^2 f(v_1) + 2ck(A + NB)$$

where $I$ is a subset of size $k$.

**Proof.** Since in $S_d$ the function $T$ is constant the minimum value of $f(\alpha) = A\|\alpha\|^2 + BT(\alpha)^2$ is attained whenever $\|\alpha\|^2$ is minimal. Thanks to corollary 4.3 such minimum is attained at $\alpha = E_I + cv_1$, hence the minimum value of $f$ over $S_d$ is $f(E_I + cv_1) = f(E_I) + c^2 f(v_1) + 2ck(A + NB)$.

**Lemma 4.5.** Let $k$ be an integer $0 \leq k \leq N$, and let $I \subset \{1, \ldots, N\}$ such that $|I| = k$, then

$$\|E_I\|^2 = k(a - (k-1)h) = k(1 + (N-k)h).$$

In particular,

$$f(E_I) = Ak(1 + (N-k)h) + Bk^2.$$  

**Proof.** We may assume that $k \neq 0$.

$$\|E_I\|^2 = \langle E_I, E_I \rangle = \sum_{i \in I} (e_i, e_i) + \sum_{i,j \in I, i \neq j} (e_i, e_j) = \sum_{i \in I} a - \sum_{i,j \in I, i \neq j} h = ak - (k^2 - k)h = k(a - (k-1)h).$$

Using the case $k = N$, i.e., $E_I = v_1$ we see that $N = \|E_I\|^2 = N(a - (N-1)h)$ hence $a = 1 + (N-1)h$. Replacing this in the first equality the second follows, and so it does the claim about $f(E_I)$.

### 4.2 Conditions on minimal basis.

To see whether or not $L_{d,r,s}^{(r,s)}$ is strongly well rounded, firstly we should find a basis in which all the vectors have the same norm. Since we have calculated the min values of $f$ over each $S_d$, we then should compare the value of the norms of the proposed basis vs the minimal values of $f$. Thanks to Proposition 3.10 we have that all the elements of the basis $\{\Phi_{r,s}(e_1), \ldots, \Phi_{r,s}(e_N)\}$ have norm $aA + B$.

Hence, if $aA + B$ were to be equal to $\lambda_d(L_{d,r,s}^{(r,s)})$ we should have at least $aA + B \leq f(v_1) = N(A + NB)$. As it turns out this is already a pretty strong condition as the next theorem shows.

**Theorem 4.6.** Suppose that $aA + B \leq f(v_1)$. Then,

$$aA + B = \min_{d > 0} \min_{\alpha \in S_d \setminus \{0\}} f(\alpha).$$
Proof. Since $e_1 \in S_1$ and $aA + b = f(e_1)$ we have that \(\min_{d > 0} \min_{\alpha \in S_d \setminus \{0\}} f(\alpha) \leq aA + B\). To show the opposite inequality let \(d\) be a positive integer. Let \(k\) be the residue class of \(d\) modulo \(N\) and \(c = (d - k)/N\). Thanks to Theorem 4.4 we have that
\[
\min_{\alpha \in S_d \setminus \{0\}} f(\alpha) = f(E_1) + c^2f(v_1) + 2ck(A + NB)
\]
where \(I\) is subset of \(\{1, \ldots, N\}\) of size \(k\). If \(c \neq 0\) we have that \(f(E_1) + c^2f(v_1) + 2ck(A + NB) \geq f(v_1) \geq aA + B\). If \(c = 0\) then \(k \neq 0\) and thanks to the next proposition \(f(E_1) \geq aA + B\). Thus, in either case
\[
\min_{\alpha \in S_d \setminus \{0\}} f(\alpha) \geq aA + B
\]
from which the result follows. \(\square\)

**Proposition 4.7.** Let \(1 \leq k \leq N\) be an integer, and \(I \subset \{1, \ldots, N\}\) a subset of size \(k\). Suppose that \(aA + B \leq f(v_1)\). Then,
\[
aA + B \leq f(E_1).
\]

**Proof.** Consider the parabola
\[
g(t) := At(1 + (N - t)h) + Bt^2.
\]
By Lemma 4.5 we have that \(aA + B = g(1), f(v_1) = g(N)\) and \(f(E_1) = g(k)\). Also, notice that \(g(0) = 0\) hence \(g(0) < g(1) \leq g(N)\). Since \(g\) is a parabola and \(0 < 1 < N\) the function \(g\) is increasing in the interval \([1, N]\) hence \(aA + B = g(1) \leq g(k) = f(E_1)\). \(\square\)

**Corollary 4.8.** Suppose that \(aA + B \leq N(A + NB)\). Then,
\[
\lambda_1(L_{\mathcal{L}_{v_1}^{(r,s)}}) = \min\{2A(a + h), aA + B\}.
\]

**Proof.** Recall that by doing the quotient partition of \(L/L^0\) we have that
\[
\lambda_1(L_{\mathcal{L}_{v_1}^{(r,s)}}) = \min_{d \geq 0}(\min_{\alpha \in S_d \setminus \{0\}} f(\alpha))
\]
Therefore, thanks to Theorem 4.6 \(\lambda_1(L_{\mathcal{L}_{v_1}^{(r,s)}}) = \min\{\min_{\alpha \in S_0 \setminus \{0\}} f(\alpha), aA + B\}\). On the other hand \(f(\alpha) = A||\alpha||^2\) for \(\alpha \in S_0 = L^0\). The result follows from Corollary 3.20. \(\square\)

We are ready to summarize our results in the main theorem of the paper:

**Theorem 4.9.** Let \(\mathcal{L} \subseteq \mathbb{R}^N\) be a Lagrangian lattice with \(v_1\) and \(\{e_1, \ldots, e_N\}\) satisfying the conditions of Definition 3.12. Let \(a := \langle e_1, e_1\rangle\) and \(h = -\langle e_1, e_2\rangle\). Let \(r, s\) be integers such that \(0 \neq |r| < N\) and let \(m := r + sN\). Suppose that
\[
\frac{Na - 1}{N^2 - 1} \leq \left(\frac{m}{r}\right)^2 \leq \frac{(aN - 1)(N + 1)}{N - 1}.
\]
Then the lattice \(L_{\mathcal{L}_{v_1}^{(r,s)}}\) is a sub-lattice of \(\mathcal{L}\) of index \(m|r|^{N-1}\), with minimum
\[
\lambda_1(L_{\mathcal{L}_{v_1}^{(r,s)}}) = ar^2 + \frac{m^2 - r^2}{N}
\]
and basis of minimal vectors
\[
\{re_1 + sv_1, re_2 + sv_1, \ldots, re_N + sv_1\}.
\]
Proof. By Corollary \[4.8\] \( L_{1}^{(r,s)} \) is a strongly well-rounded with minimal norm \( aA + B \) if and only if

- \( Aa + B \leq N(A + NB) \)
- \( Aa + B \leq 2A(a + h) \).

Recall that \( A = r^2 \) and \( B = \frac{m^2 - r^2}{N} \). Thus, using that \( \frac{B}{A} = \frac{1}{N} \left( (\frac{m}{r})^2 - 1 \right) \) the inequality

\[
Aa + B \leq N(A + NB)
\]

turns into

\[
\frac{Na - 1}{N^2 - 1} \leq \left( \frac{m}{r} \right)^2
\]

and using that \( a = 1 + (N - 1)h \), see the proof of Lemma \[4.5\]

\[
Aa + B \leq 2A(a + h) \text{ turns into } \left( \frac{m}{r} \right)^2 \leq \frac{(aN - 1)(N + 1)}{N - 1}.
\]

\[\Box\]

**Corollary 4.10.** Let \( N \geq 2 \) be an integer. Let \( K \) be a tame totally real degree \( N \) number field. Let \( m \) be an integer such that \( m \equiv \pm 1 \pmod{N} \). Suppose that \( O_K \) has an integral Lagrangian basis \( \{e_1, ..., e_N\} \) such that \( 1 = e_1 + e_2 + ... + e_N \). Let \( a := \langle e_1 \rangle \) and \( h = -\langle e_1, e_2 \rangle \) and suppose that

\[
\frac{Na - 1}{N^2 - 1} \leq m^2 \leq \frac{(aN - 1)(N + 1)}{N - 1}.
\]

Then, the lattice

\[
\{x \in O_K : \text{Tr}_{K/Q}(x) \equiv 0 \pmod{m}\}
\]

is a sub-lattice of \( O_K \) that has a minimal basis with minimum \( \lambda_1 = a + \frac{m^2 - 1}{N} \).

Proof. The result follows immediately from Lemma \[3.3\] and Theorem \[4.9\] \[\Box\]

**Remark 4.11.** Note that for simplicity have only mentioned here the case \( m \equiv \pm 1 \pmod{N} \) however, the more general values of \( m = r + sN \) yield to lattices with minimal basis inside of \( \{x \in O_K : \text{Tr}_{K/Q}(x) \equiv 0 \pmod{m}\} \).

There are several examples of real number fields containing an integral Lagrangian basis. One of the first families of such number fields was found in the mid 80’s by Conner and Perlis while studying integral traces over tame Galois number fields of prime degree:

**Example 4.12.** Let \( p \) be a prime and let \( K \) be a Galois number field of degree \( p \). If \( K \) is tame, which in this case is equivalent to say that \( p \) does not ramify, then \( O_K \) has an integral Lagrangian basis \( \{e_1, ..., e_p\} \) with \( a = \frac{n(p-1)+1}{p} \) and \( h = \frac{a-1}{p} \) where \( n \) is the conductor of \( K \) (see \[7\]). Note that such values of \( a \) and \( h \) for for \( N = p \) we have that \( \frac{Na - 1}{N^2 - 1} = \frac{n}{p+1} \) and \( \frac{(aN - 1)(N + 1)}{N - 1} = n(p + 1) \). Therefore, if \( m \equiv \pm 1 \pmod{p} \) is such that \( \frac{n}{p+1} \leq m^2 \leq n(p + 1) \) then, thanks to Corollary \[4.10\]

\[
\{x \in O_K : \text{Tr}_{K/Q}(x) \equiv 0 \pmod{m}\}
\]

is a sub-lattice of \( O_K \) with a minimal basis and minimum \( \lambda_1 = \frac{n(p-1)+m^2}{p} \). When restricting this example to the case \( m \equiv 1 \pmod{p} \) the results [Theorem 4.1,\[11\]] and [Theorem 3.3,\[12\]] are recovered.
Recently the results of Conner and Perlis about Trace forms over cyclic number fields, see [6], have been generalized. Using these ideas the example above can be extended to number fields of not necessarily prime degree:

**Example 4.13.** Let $K$ be a tame totally real abelian number field of degree $N$. If the conductor $n$ of $K$ is prime, then $O_K$ has an integral Lagrangian basis $\{e_1, \ldots, e_N\}$ with $a = \frac{n(N-1)+1}{N}$ and $h = \frac{a+1}{N}$. (The construction of such basis can be done as in [6, Lemma 3.3]; there such basis is constructed for $N$ a prime power however the same exact proof works for any $N$.) Thus, as in the previous example our general construction can also be applied to such fields. For instance, the field $K = \mathbb{Q}(\zeta_{13} + \zeta_{13}^{-1})$ is a tame real Galois extension of $\mathbb{Q}$ with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and of conductor 13. The lattices $\{x \in O_K : \text{Tr}_{K/\mathbb{Q}}(x) \equiv 0 \pmod{5}\}$ and $\{x \in O_K : \text{Tr}_{K/\mathbb{Q}}(x) \equiv 0 \pmod{7}\}$ are sub-lattices of $O_K$ with minimal basis and with respective minima equal to $15 = \frac{13(6-1)+5^2}{6}$ and $19 = \frac{13(6-1)+7^2}{6}$.

The two families in the above examples are not the only examples of number fields with Lagrangian integral basis; the following example shows that there are abelian fields that are neither of prime degree or prime conductor having a Lagrangian integral basis.

**Example 4.14.** Let $K$ be the number field defined by the polynomial $f := x^4 - x^3 - 24x^2 + 4x + 16$. The field $K$ is $\mathbb{Z}/4\mathbb{Z}$-extension of $\mathbb{Q}$, has discriminant $5^3 \cdot 13^3$ and conductor $n = 65$. If $\{a_1, a_2, a_3, a_4\}$ is the set of roots of $f$ then they form an integral basis of $O_K$, $a_1 + a_2 + a_3 + a_4 = 1$, and the Gram matrix of the trace in such basis is

$$
\begin{pmatrix}
49 & -16 & -16 & -16 \\
-16 & 49 & -16 & -16 \\
-16 & -16 & 49 & -16 \\
-16 & -16 & -16 & 49
\end{pmatrix}.
$$

Hence, $O_K$ is a Lagrangian lattice with $a = 49, h = 16$ and $N = 4$. Applying the bounds of Corollary 4.10 here, we obtain $13 \leq m^2 \leq \frac{(aN-1)(N+1)}{N} = 325$ for $m \equiv 1 \pmod{4}$. This is equivalent to $m \in \{5, 7, 9, 11, 13, 15, 17\}$. Hence, by applying Corollary 4.10 to $K$ a non-degree prime with no prime conductor number field, we have constructed seven non-isometric sub-lattices of $O_K$ with minimal basis.

The above constructions are obtained from Galois number fields. In the next section, we will present a generic construction of Lagrangian integral lattices, and with it we will show how to construct Lagrangian lattices arising from ideal lattices over number fields that are non-Galois over $\mathbb{Q}$.

## 5 Integral Lagrangian lattices

Recall that $\mathcal{L}$ is a Lagrangian lattice if there is a basis $\{e_1, \ldots, e_N\}$ of $\mathcal{L}$ and a non-zero $v_1 \in \mathcal{L} \cap \mathcal{L}^*$ such that:

1. $e_1 + \ldots + e_N = v_1$.
2. $T(e_i) = \langle e_1, v_1 \rangle = 1$ for all $1 \leq i \leq N$.
3. $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle = a$ for all $1 \leq i, j \leq N$.
4. $\langle e_i, e_j \rangle = \langle e_k, e_l \rangle = -h$ for all $1 \leq i, j, k, l \leq N$ with $i \neq j$ and $k \neq l$.
Many of the lattices of interest in arithmetic are integral, hence it is natural to see when a Lagrangian lattice $L$ is integral. Since conditions (1) and (2) are equivalent to $a - (N-1)h = 1$ the integrality of $L$ is equivalent to have that $a$ and $h$ are integers. More precisely:

**Lemma 5.1.** Let $N > 1$ be an integer and let $L$ be an integral lattice in $\mathbb{R}^N$. The lattice $L$ is Lagrangian if and only if there is a basis $B$ of $L$ such that the Gram matrix of $L$ in the basis $B$ is of the form

$$G_B = \begin{bmatrix} a & -h & \cdots & -h \\ -h & a & \cdots & -h \\ \vdots & \ddots & \ddots & \vdots \\ -h & -h & \cdots & a \end{bmatrix}$$

where $a, h$ are non-negative integers, $a \neq 0$, such that $a - (N-1)h = 1$.

**Proof.** Suppose that $L$ is Lagrangian and let $B := \{e_1, \ldots, e_N\}$ be a basis satisfying the conditions of (1)-(4) of the definition. Since $L$ is integral, $a = \langle e_1, e_1 \rangle \in \mathbb{Z}^+$ and $h = -\langle e_1, e_2 \rangle \in \mathbb{Z}$. In the final part of the proof of [15] we showed that $a - (N-1)h = 1$, so $h$ can’t be negative since $a$ and $N-1$ are positive, thus the Gram matrix of $L$ in the basis $B$ satisfies the claim. Conversely suppose that $B := \{e_1, \ldots, e_N\}$ is a basis of $L$ with Gram matrix as claimed. By definition of Gram matrix the basis $B$ satisfies conditions (3) and (4) above. Since $L$ is integral $L$ is a subset of its dual in particular, $v = e_1 + \cdots + e_N \in L \cap L^*$ and (1) is satisfied. Finally we have condition (2) since $\langle v, e_i \rangle = \langle e_1 + \cdots + e_N, e_i \rangle = a - (N-1)h = 1$. \hfill $\Box$

### 5.1 Constructing integral Lagrangian lattices

So far most of the examples of Lagrangian lattices we have encountered, see [31] and [20], are of the form $\langle O_K, \text{Tr}(x^2) \rangle$ where $K$ is a cyclic number field. It is natural to ask if there are other type of Lagrangian lattices. Since lattices of the form $\langle O_K, \text{Tr}(x^2) \rangle$ are integral this question is interesting only for integral Lagrangian lattices. As it turns out, thanks to results due to Taussky [31] and Kräusckemper [20], every integral lattice is isomorphic to an ideal lattice over some totally real number field. In particular, any integral Lagrangian lattice is an ideal lattice over a totally real number field. Here we will show examples of integral Lagrangian lattices that can be described as ideal lattices over non-Galois number fields. For the sake of completeness, we first sketch Taussky-Kräusckemper method.

#### 5.1.1 Taussky-Kräusckemper method

A key result linking matrices with ideal bases is Theorem 1 in [31]. Let $N$ be a positive integer. The mentioned theorem states that any matrix $M \in M_{N \times N}(\mathbb{Z})$ with an irreducible characteristic polynomial $f$, has an eigenvector $w_\gamma = (w_1, \ldots, w_N)$ associated to a root $\gamma$ of $f$, such that $\{w_1, \ldots, w_N\}$ form a basis of an ideal in the order $\mathbb{Z}[\gamma]$ of $\mathbb{Q}(\gamma)$. In fact, and explicit expression for a choice of basis is

$$w_j = (-1)^{1+j} \Delta_{1j} (M - \gamma I_N), \quad (3)$$

where $\Delta_{1j} (M - \gamma I_N)$ is the $j$th minor of the first row of $M - \gamma I_N$, and $I_N$ is the identity matrix of dimension $N$.\hfill$\Box$
Theorem 5.2 (Taussky, [32]). Let \( N \) be a positive integer and let \( G \in M_{N \times N}(\mathbb{Z}) \) be a symmetric matrix with non-zero determinant. Let \( M \in M_{N \times N}(\mathbb{Z}) \) be its characteristic polynomial \( f \) irreducible and such that \( MG = GMT \). Then, there exists a degree \( N \) number field \( K \), \( \alpha \in K \), \( O \) an order in \( K \) and \( I \) an ideal in \( O \) such that \( G \) is the gram matrix of the bilinear pairing on \( I \) given by the twisted trace; \((x, y) \mapsto \text{Tr}_{K/Q}(\alpha xy)\).

The idea behind the theorem is as follows. Let \( \gamma \) be a root of \( f \) and \( K \) be the number field generated by \( \gamma \). Let \( w_\gamma = (w_1, \ldots, w_N) \) as in [3], generating an ideal \( I \) in \( \mathbb{Z}[\gamma] \). Taussky showed that there exists \( \alpha \in \mathbb{Q}(\gamma) \) such that for \( 1 \leq i, j \leq N \)

\[
G_{i,j} = \text{Tr}_{K/Q}(\alpha w_i w_j)
\]  

(4)

Moreover, the element \( \alpha \) satisfies the relation

\[
Gw_\gamma = \alpha w_\gamma,
\]  

(5)

where \( w_\gamma' = (w_\gamma', \ldots, w_N') \) is an eigenvector of \( M^T \) associated to the eigenvalue \( \gamma \). The ordered set \( \{w_\gamma', \ldots, w_N'\} \) is the dual basis of \( \{w_1, \ldots, w_N\} \) with respect the trace pairing, see [3], hence for all \( 1 \leq j \leq N \)

\[
w_j' = \sum_{i=1}^{N} M_{ij} \gamma^{j-1},
\]  

(6)

where \( M = A^{-1}(P^T)^{-1} \), \( P \) is the matrix of \( \{w_1, \ldots, w_i\} \) in the basis \( \{1, \ldots, \gamma^{N-1}\} \) and \( A \) is the gram matrix of the trace pairing in the basis \( \{1, \gamma, \ldots, \gamma^{N-1}\} \). Therefore, \( G \) is the Gram matrix, in the basis \( \{w_1, \ldots, w_i\} \) of \( I \), of the twisted trace pairing \( \text{Tr}_{K/Q}(\alpha xy) \). If \( G \) is positive definite, the polynomial \( f \) can be chosen to have only real roots so \( K \) is totally real and hence \( \alpha \) is totally positive. In particular, \( G \) represents a gram matrix of the ideal lattice \( J_{K,\alpha}(I) \).

Using Taussky’s theorem Kr"uskemper [20] showed that every integral lattice is an ideal lattice. His idea was to show that given \( G \) a gram matrix of an integral lattice, there is \( S \) a symmetric matrix such that \( GS \) has an irreducible characteristic polynomial with only real roots. Taking \( M = GS \), since clearly \( MG = GMT \) the result follows from Taussky’s theorem.

Theorem 5.3. [20] Any integral lattice is isometric to an ideal lattice \( J_{K,\alpha}(I) \), where \( I \) is an ideal in \( \mathbb{Z}[\gamma] \subset K \) for some algebraic integer \( \gamma \). Furthermore, \( K \) can be assumed to be totally real.

The proof of Theorem 5.3 uses the fact that if \( G \) is a symmetric integer matrix, then by seeing it as the gram matrix of a rational quadratic form in some basis, we can find an integral matrix \( E \), with non-zero determinant, and a diagonal integer matrix \( D \) such that

\[
E^T GE = D.
\]  

(7)

Now, for a diagonal matrix \( D \), positive definite, Kr"uskemper shows, using Hilbert’s irreducibility theorem, that there exists a symmetric matrix \( S' \) such that \( DS' \) has an irreducible characteristic polynomial \( f \) with only real roots. Letting \( S = ES'E^T \) and \( M = GS \) we see that \( M \) and \( DS' \) are conjugate hence have the same characteristic polynomial.

We illustrate Taussky-Kr"uskemper’s method in the cubic case in the case of integral Lagrangian lattices of dimension \( N = 3 \). Let \( a, h \) be non-negative integers with \( a \neq 0 \) and such that \( a - 2h = 1 \)
and let
\[ G_{a,h} = \begin{bmatrix} a & -h & -h \\ -h & a & -h \\ -h & -h & a \end{bmatrix} \]

By doing row/column reduction we find that if
\[ E = \begin{bmatrix} 6 & -2 & -3 \\ 6 & 4 & 0 \\ 6 & -2 & 3 \end{bmatrix} \]

then,
\[ E^T G_{a,h} E = D, \tag{8} \]

where
\[ D = 6 \begin{bmatrix} 36(a - 2h) & 0 & 0 \\ 0 & 4(a + h) & 0 \\ 0 & 0 & 3(a + h) \end{bmatrix} = 6 \begin{bmatrix} 36 & 0 & 0 \\ 0 & 4(a + h) & 0 \\ 0 & 0 & 3(a + h) \end{bmatrix}. \]

Assume that \( S' \) is a symmetric integral matrix such that \( DS' \) has an irreducible characteristic polynomial. We fix a root \( \gamma \) of \( f \), and we consider \( w_\gamma \) and \( w'_\gamma \) as in (3) and (6) respectively. Using equation (5), we obtain
\[ \begin{bmatrix} a & -h & -h \\ -h & a & -h \\ -h & -h & a \end{bmatrix} w'_\gamma = \alpha w_\gamma. \]

Multiplying this on the left by the row vector \([1, 1, 1]\), and using that \( a - 2h = 1 \), yields
\[ \alpha = \frac{\sum_{i=1}^{3} w'_i}{\sum_{i=1}^{3} w_i}. \tag{9} \]

Remark 5.4. Let \( N \geq 2 \), \( w_\gamma = (w_1, \ldots, w_N) \) and \( w'_\gamma = (w'_1, \ldots, w'_N) \) as in (3) and (6) respectively. Then, multiplying by \([1, \ldots, 1]\) and using that \( a - (N - 1)h = 1 \), we see that \( \alpha \) is
\[ \alpha = \frac{\sum_{i=1}^{N} w'_i}{\sum_{i=1}^{N} w_i}. \]

Remark 5.5. The above method is presented in [28], the authors proposed this technique to construct full-diversity rotations of the orthogonal lattice \( \mathbb{Z}^n \), i.e., \( a = 1 \) and \( h = 0 \).

Example 5.6. Using the same above notation, we consider \( a = 5 \) and \( h = 2 \). Thus
\[ D = \begin{bmatrix} 108 & 0 & 0 \\ 0 & 168 & 0 \\ 0 & 0 & 126 \end{bmatrix}. \]

We take
\[ S' = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}. \]

The matrix \( DS' \) has characteristic polynomial \( f(x) = x^3 - 444x^2 - 71064x - 2286144 \) which is irreducible over \( \mathbb{Q} \), and with discriminant \( 2^{10} \cdot 3^8 \cdot 7^2 \cdot 580639 \). Furthermore, fixing a real root \( \gamma \)
of \( f \), the number field \( K = \mathbb{Q}(\gamma) \) is cubic number field of discriminant \( 2^2 \cdot 580639 \). In particular, \( K \) is totally real, is not Galois over \( \mathbb{Q} \) and the order \( \mathbb{Z}[\gamma] \) is a sub-ring of index 9072 in the ring of integers of \( K \).

Now we proceed to calculate the ideal basis. We have

\[
M = G_{5,2} E S' E^T = \begin{bmatrix} 512 & 86 & 392 \\ -454 & -124 & -322 \\ 176 & 2 & 56 \end{bmatrix}.
\]

Using (3), we obtain that \( G_{5,2} \) is a Gram of an ideal lattice obtained from an ideal \( I \) with basis \( \{ w_1, w_2, w_3 \} \), where

\[
\begin{align*}
  w_1 &= \gamma^2 + 68\gamma - 6300. \\
  w_2 &= -454\gamma - 31248. \\
  w_3 &= 176\gamma + 20916.
\end{align*}
\]  

In order to calculate \( \alpha \), we first need to compute \( w'_i \) as in (6). Using (10), we get that the matrix of the basis \( \{ w_1, w_2, w_3 \} \) in terms of the basis \( \{ 1, \gamma, \gamma^2 \} \) is

\[
P = \begin{bmatrix} -6300 & -31248 & 20916 \\ 68 & -454 & 176 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Furthermore, the gram matrix of the trace pairing in the basis \( \{ 1, \gamma, \gamma^2 \} \) is

\[
A = \begin{bmatrix} 3 & 444 & 339264 \\ 444 & 339264 & 189044064 \\ 339264 & 189044064 & 109060069248 \end{bmatrix}.
\]

Thus

\[
\begin{bmatrix} w'_1 \\ w'_2 \\ w'_3 \end{bmatrix} = A^{-1} (P^T)^{-1} \begin{bmatrix} 1 \\ \gamma \\ \gamma^2 \end{bmatrix}.
\]

Hence, \( \alpha \) is given by

\[
\alpha = \frac{\sum_{i=1}^{3} w'_i}{\sum_{i=1}^{3} w_i} = \frac{65191931\gamma^2 - 39226301250\gamma + 1018601862840}{143063823242500989696}.
\]

Summarizing; inside the order \( \mathbb{Z}[\gamma] \) the ideal \( I \subseteq \mathbb{Z}[\gamma] \) with basis \( \{ w_1, w_2, w_3 \} = \{ \gamma^2 + 68\gamma - 6300, -454\gamma - 31248, 176\gamma + 20916 \} \) together with \( \alpha \) define an ideal lattice \( \langle I, \text{Tr}_{K/Q} \rangle \) which gram matrix in the basis \( \{ w_1, w_2, w_3 \} \) is equal to

\[
\begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}.
\]

**Remark 5.7.** We should observe that this particular Lagrangian lattice is also the ideal lattice of the full ring of integers of the maximal real sub-field of \( \mathbb{Q}(\zeta_7) \) and \( \alpha = 1 \).
6 Integral Lagrangian lattices and units in real quadratic fields

Let $N > 2$ be an integer such that $N - 1$ is square-free. The field $K = \mathbb{Q}(\sqrt{N-1})$ is a real quadratic field with ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$ where

$$\omega = \begin{cases} \sqrt{N-1} & \text{if } N \equiv 0, 3 \pmod{4} \\ \frac{1+\sqrt{N-1}}{2} & \text{if } N \equiv 2 \pmod{4} \end{cases}.$$

Let $\eta \in \mathcal{O}_K$. We denote by $\eta_x$ and $\eta_y$ the unique integers such that $\eta = \eta_x + \eta_y\omega$.

We define the map

$$\pi_K : \mathcal{O}_K \rightarrow \mathbb{Z}^N$$

$$\eta \mapsto (\eta_x^2, -\eta_y^2, \ldots, -\eta_y^2).$$

Clearly, the map $\pi_K$ is well-defined. Let $\tau$ be an $N$-cycle in the permutation group $S_N$. Then, $\tau(\pi_K(\eta)) \neq \pi_K(\eta)$ for any $0 \neq \eta \in \mathcal{O}_K$. Furthermore, if $\eta$ is a non-zero element in $\mathcal{O}_K$, the set

$$\mathcal{B}_\eta = \{\tau^i(\pi_K(\eta)) : 0 \leq i \leq N - 1\}$$

consists of $N$ $\mathbb{R}$-linearly independent vectors $e_i = \tau^i(\pi_K(\eta))$. Thus, $\mathcal{B}_\eta$ generates a full-rank sub-lattice of $\mathbb{Z}^N$. We denote by $\mathcal{L}_\eta$ the lattice with basis $\mathcal{B}_\eta$.

A straightforward calculation yields

$$\text{vol}(\mathcal{L}_\eta) = (\eta_x^2 + \eta_y^2)^{2(N-1)}.$$

Let $0 \neq \eta \in \mathcal{O}_K$. Then the basis $\mathcal{B}_\eta = \{e_0, \ldots, e_{N-1}\}$ satisfies the following properties:

- $v_1 = \sum_{i=0}^{N-1} e_i = (\eta_x^2 - (N-1)\eta_y^2)(1, \ldots, 1)^T$.
- $\langle v_1, e_i \rangle = (\eta_x^2 - (N-1)\eta_y^2)^2$ for all $0 \leq i \leq N - 1$.
- $\langle e_i, e_i \rangle = \eta_x^4 + (N-1)\eta_y^4$ for all $0 \leq i \leq N - 1$.
- $\langle e_i, e_j \rangle = -2\eta_x^2\eta_y^2 + (N-2)\eta_y^4$ for all $0 \leq i, j \leq N - 1$ with $i \neq j$.

Observe that the basis $\mathcal{B}_\eta$ is Lagrangian if and only if $\eta_x^2 - (N-1)\eta_y^2 = 1$, we get:

**Proposition 6.1.** Let $K = \mathbb{Q}(\sqrt{N-1})$ such that $N - 1$ is a positive square-free integer and $N \equiv 0, 3 \pmod{4}$. Then the basis $\mathcal{B}_\eta$ is Lagrangian if and only if $\eta \in \mathcal{O}_K^\ast$.

Using the same notation as in [13], we denote by $a = \eta_x^4 + (N-1)\eta_y^4$ and $h = 2\eta_x^2\eta_y^2 + (N-2)\eta_y^4$.

Hereon, we consider $N \equiv 0, 3 \pmod{4}$ and recall that $N - 1$ square-free.

Let $r, s$ be integers such that $0 \neq |r| < N$, let $\eta \in \mathcal{O}_K^\ast$ and let $m = r + sN$. Denoting by

$$L_{\eta}^{(r,s)} := \Phi_{(r,s)}(\mathcal{L}_\eta),$$

we have by Proposition 3.7 that $L_{\eta}^{(r,s)}$ is a sub-lattice of $\mathcal{L}_\eta$ of index $m|\tau|^N$.

Consequently,

$$\text{vol}(L_{\eta}^{(r,s)}) = m(|r|(|\tau|)^2)^{N-1}.$$
Theorem 6.2. Let $N$ be a positive integer such that $N \equiv 0, 3 \pmod{4}$ and $N-1$ square-free. Then the $N$-dimensional lattice $L_{\eta}^{(r, \eta_y^2)}$ has a minimal basis and a minimum

$$\lambda_1(L_{\eta}^{(r, \eta_y^2)}) = \frac{r^2(N-1)(N\eta_y^2 + 1)^2 + (N\eta_y^2 + r)^2}{N},$$

for any norm 1 unit $\eta$ in $\Q(\sqrt{N-1})$ and $1 \leq r \leq \sqrt{N}$.

Moreover, the lattice $L_{\eta}^{(r, \eta_y^2)}$ is orthogonal if and only if $r = 1$.

Proof. Assume that $N_{K/\Q}(\eta) = 1$. Remarking that

$$\eta_x^4 + (N-1)^2\eta_y^4 = 1 + 2\eta_x^2\eta_y^2(N-1),$$

we get

$$Na - 1 = N\eta_x^4 + N(N-1)\eta_y^4 - 1 = (N-1)(\eta_x^2 + \eta_y^2)^2 = (N-1)(N\eta_y^2 + 1)^2. $$

Similarly, we obtain $Nh + 1 = (\eta_x^2 + \eta_y^2)^2$.

Finally, by Theorem 4.9, the lattice $L_{\eta}^{(r, \eta_y^2)}$ has a minimal basis whenever

$$\frac{(N\eta_y^2 + 1)^2}{N+1} \leq \left(\frac{m}{r}\right)^2 \leq (N\eta_y^2 + 1)^2(N+1). \quad (12)$$

Moreover,

$$\lambda_1(L_{\eta}^{(r, \eta_y^2)}) = \frac{r^2(N-1)(N\eta_y^2 + 1)^2 + (N\eta_y^2 + r)^2}{N}.$$

Let $s = \eta_y^2$ and $1 \leq r \leq \sqrt{N}$. Then the trivial bound

$$\frac{(N\eta_y^2 + 1)^2}{N} \leq (\frac{m}{r})^2 \leq N(\sqrt{N}\eta_y^2 + 1)^2 \quad (13)$$

holds. Thus, the lattice $L_{\eta}^{(\eta_y^2, r)}$ has a minimal basis for all $1 \leq r \leq \sqrt{N}$.

Let $1 \leq i, j \leq N$ with $i \neq j$. Then, Proposition 3.16 yields

$$\langle \Phi_{(r, \eta_y^2)}(e_i), \Phi_{(r, \eta_y^2)}(e_j) \rangle = \frac{m^2 - r^2(Nh + 1)}{N} = \frac{(N\eta_y^2 + r)^2 - r^2(N\eta_y^2 + 1)^2}{N}. \quad (14)$$

Hence, the lattice $L_{\eta}^{(\eta_y^2, r)}$ is the orthogonal lattice if and only if $r = 1$.

Now we assume that $N_{K/\Q}(\eta) = -1$. Again the lattice $L_{\eta}^{(r, s)}$ has volume

$$\text{vol}(L_{\eta}) = m|r|^{N-1}(N\eta_y^2 - 1)^{2(N-1)}.$$ 

A similar manipulation of the condition $N_{K/\Q}(\eta)^2 = 1$ yields

$$Na - 1 = (N-1)(N\eta_y^2 - 1)^2.$$
Let \( m = r + sN \). By virtue of Theorem 4.9, the lattice \( \mathcal{L}^{(r,s)}_\eta \) has a minimal basis whenever

\[
\frac{(N\eta_y^2 - 1)^2}{N+1} \leq \left( \frac{m}{r} \right)^2 \leq (N\eta_y^2 - 1)^2(N+1). \tag{15}
\]

Taking \( r = -1 \) and \( s = \eta_y^2 \), then \( \mathcal{L}^{(-1,\eta_y^2)}_\eta \) is generated by its minimal vectors and has a minimum

\[
\lambda_1(\mathcal{L}^{(-1,\eta_y^2)}_\eta) = \frac{N\alpha - 1 + (N\eta_y^2 - 1)^2}{N} = (N\eta_y^2 - 1)^2.
\]

Moreover,

\[
\text{vol}(\mathcal{L}^{(-1,\eta_y^2)}_\eta) = (N\eta_y^2 - 1)^{2N-1}.
\]

Recalling that the center density of a lattice \( L \) is given by

\[
\delta(L) = \frac{\lambda_1(L)^{N/2}}{2^N \text{vol}(L)},
\]

we conclude that

\[
\delta(\mathcal{L}^{(-1,\eta_y^2)}_\eta) = \frac{1}{2^N (N\eta_y^2 - 1)^{N-1}}.
\]

These are different for different values of \( \eta_y \). Hence, the lattices \( \mathcal{L}^{(-1,\eta_y^2)}_\eta \) are non-similar for every \( \eta_y \in \mathcal{O}_K^\ast \) of norm \(-1\). Note that if \( K \) has a unite of norm \(-1\), then the lattices \( \mathcal{L}^{(-1,\eta_y^2)}_\eta \) are completely determined by the odd powers of the fundamental unit in \( \mathcal{O}_K \).

**Example 6.3.** Let \( s \) be a positive odd integer such that \( s^2 + 1 \) is square-free. Let \( N = s^2 + 2 \). Then \( N \equiv 3 \mod 4 \) and \( \varepsilon = s + \sqrt{s^2 + 1} \) is the fundamental unit in \( \mathcal{O}_K \), where \( K = \mathbb{Q}(\sqrt{N-1}) \). Moreover, we have \( N_{K/Q}(\varepsilon) = -1 \). It follows from the above analysis, that the \( N \)-dimensional lattices \( \mathcal{L}^{(-1,\eta_y^2)}_\eta \) are non-similar and generated by their minimal bases for every \( \eta = \varepsilon^{2k+1} \) with \( k \geq 1 \).

It is a classic result due to Estermann, see \([13]\), that there are infinitely many square free values of the form \( s^2 + 1 \), in fact a non-zero positive proportion of integers. (See also \([17]\) for details). Hence the following theorem, which summarizes the analysis of this section, gives us a recipe to construct infinitely many non-isomorphic lattices with minimal basis.

**Theorem 6.4.** Let \( N = s^2 + 2 \), where \( s \) is an odd integer such that \( s^2 + 1 \) is square-free. Then the \( N \)-dimensional lattice \( \mathcal{L}^{(-1,\eta_y^2)}_\eta \) has a minimal basis, whenever \( \eta = (s + \sqrt{s^2 + 1})^{2k+1} \) and \( k \geq 0 \).

Furthermore, the lattices \( \mathcal{L}^{(-1,\eta_y^2)}_\eta \) are non-similar for different values of \( k \).

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