Comment on “Casimir energies with finite-width mirrors”

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We comment on a recent publication [1] on Casimir energies for material slabs (‘finite width mirrors’) and report a discrepancy between results obtained there for a single mirror and some previous calculations. We provide a simple consistency check which proves that the method used in [1] is not reliable when applied to approximations of piecewise constant profile of the mirror.

We also present an alternative method for calculation of the Casimir energy in such systems based on [2]. Our results coincide both with perturbation theory and with some older [3] and more recent [4] calculations, but differ from those of [1].

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In a recent publication on Casimir energies and interaction of material slabs (‘finite width mirrors’) [1] there was presented a thorough and rather general treatment of the problem at hand. To model the presence of matter in the system of massless scalar field the authors of [1] introduce into the action an additional term concentrated in a given domain of the space-time. In the framework of renormalizable quantum field theory this method was initially proposed by Symanzik [5] though his paper is not cited in [1]. Such models with position dependent mass terms (also called defects) were studied extensively in recent years, in particular in the case of delta-potentials which effectively describe thin films, see for instance [6], [7]. Surprisingly the problem of interaction of finite width mirrors hasn’t been investigated in full, despite several attempts e.g. [2], [3], [4].

The Casimir energy for massive scalar field interacting with a homogeneous plane slab of finite width has been calculated quite a long time ago by Bordag [3], and also recently rederived in [4] by Vassilevich and Konoplya [13]. A similar problem is considered in Section III.B of [1]. A smooth approximation of piecewise constant potential is investigated there and Casimir energy for the case when this limit is approached is obtained. However the outcome presented there (equation (68)) disagree with (well defined) massless limit of the above mentioned results, and in [1] there is no any discussion of this discrepancy.

In [6] we elaborated a calculation approach for singular potentials describing the interaction of thin material films with fields of quantum electrodynamics. In order to generalize it to the case of 3-dimensional defects we studied recently the model of massive scalar field interacting with a single plane slab [2]. We developed a simple calculation method that differs both from [3, 4] and also from one employed in [1]. We calculated the propagator and the Casimir energy [2]. In appropriate regularization the later result coincides with ones calculated in [3, 4] (but differs from that of [1]). It is also in agreement with usual perturbation theory both for the massive case and in the massless limit.

One could argue that the discrepancy between results of [2, 3, 4] and [1] originates from the smooth approximation used in [1] to describe singular (non-continuous) potential. Similar problems have been discussed in the framework of Dirac equation perturbed by a delta-function potential (e.g. see [10]). However, in our case it is not the reason of the above mentioned discrepancy. As we show below, the calculation method proposed in [1] is not reliable when applied to the case of (approximated) piecewise constant potential, and the final expression (68) [1] is to be reconsidered.

In Section I of this Comment we present a consistency check of the calculation method developed in [1], while in Section II we sketch our own approach to similar problem.

I. CONSISTENCY CHECK

The calculation approach of [1] is based on expressing the Casimir energy as a trace of a logarithm of an integral operator, and this step is widely used in Casimir calculations. The trace of the operator can be calculated directly as a sum of its eigenvalues. For their determination authors of [1] propose particular Sturm-Liouville problem (on a finite interval) which is obtained using a specific coordinate transformation, see Section II.A [1].
As a consistency check of this method we use it to calculate the trace of the integral operator $D(z,z')$, defined in (32) of [1], and compare it with result of a straightforward alternative calculation. Expressing the trace of $D(z,z')$ as a sum of its eigenvalues $\mu_l$ and using (33) [1] we obtain

$$\text{Tr} D(z,z') = \sum_l \mu_l = \sum_l \frac{\alpha_l - 1}{\lambda(\omega,k_l)}.$$  

(1)

For the limit of piecewise constant profile being approached, $\alpha_l$ is given in (66) [1]

$$\alpha_l = \alpha_l(\omega,k_l,\epsilon) = 1 + \frac{2\epsilon \lambda(\omega,k_l)}{l^2\pi^2 + (2\epsilon k)^2}.$$  

(2)

Then one easily finds

$$\text{Tr} D(z,z') = \sum_{l=1}^{\infty} \frac{2\epsilon}{l^2\pi^2 + (2\epsilon k)^2} = \frac{2\epsilon \text{coth}(2\epsilon k) - 1}{4\epsilon k^2}.$$  

(3)

On the other hand, it is straightforward to calculate the trace of $D$ without appealing to any eigenvalue problem

$$\text{Tr} D(z,z') = \int_{-1/2}^{1/2} dz D(z,z) = \int_{-\infty}^{\infty} dx D(x,x)\sigma_\epsilon(x) = \frac{1}{2\kappa}.$$  

(4)

Evidently, (4) contradicts to (5). At the same time it is straightforward to prove the equivalence of the trace definitions used in (3) and (4) taking into account that $\psi_\alpha$ (33) [1] constitute a complete set of eigenfunctions of Sturm-Liouville problem (39) [1]. Moreover, it must be emphasized here that (4) is valid independently of any particular profile of $\sigma_\epsilon(x)$ provided it satisfies the normalization condition (7)[1]. It is due to the fact that $D(z,z)$ (or equivalently $D(x,x)$) is position-independent.

We see that the spectral problem used for calculation of the eigenvalues of $D$ in the case of (approximated) piecewise constant profile is either ill posed, or not equivalent to the original problem. It gives a wrong answer (3) for the trace of $D$. In virtue of (31) [1] the same must also hold for the trace of the operator $\ln \hat{K}$ which defines the Casimir energy (30) [1]. Hence, it is not legitimate to use the eigenvalues $\alpha_l$ (2) for the (correct) calculation of the Casimir energy (43) [1].

Thus, we must conclude that the calculation method proposed in [1] contains an internal inconsistency when applied to (approximations of) piecewise constant profile of the mirror. One can also check that an expansion of (67) [1] in a power series in constant $\lambda$ (acknowledged in Section III.C [1]) does not coincide with the usual perturbation theory.

These arguments unambiguously show that the result for the Casimir energy of a single slab (68) [1] is presumably incorrect.

II. ALTERNATIVE APPROACH

In [2] we presented a detailed treatment of the similar system of piecewise constant profile without reference to more general cases. We restricted ourselves to consideration of homogenous and isotropic infinite plane slab of thickness $\epsilon$, placed in the $x_1x_2$ plane.

Casimir interaction of (multi) layered systems has been actively studied for dielectric materials (e.g., within surface modes formalism [11], or macroscopical field operators method [12]). However, the mathematical formulation of these problems differ from one considered here, and no direct comparison of the results is possible.

For modeling of the interaction of massive scalar field with volume defects we followed the Symanzik approach [3]. The complete action of the model is following

$$S = S_0 + S_I + J\phi,$$

$$S_0 = \frac{1}{2} \int d^4x (\partial_\mu \phi(x)\partial_\mu \phi(x) + m^2 \phi^2(x)),$$

$$S_I = \frac{\lambda}{2} \int d^4x \theta(\epsilon,x_3)\phi^2(x).$$

The distribution function $\theta(\epsilon,x_3)$ represents the piecewise constant profile being equal to $1/\epsilon$ when $|x_3| < \epsilon/2$, and zero otherwise. Then there is no need for introduction of implicit variables’ change defined by (21) [1], and we can proceed explicitly in Cartesian coordinates.

We consider the generating functional of Green’s functions and similarly to [3] introduce auxiliary fields defined on the support of the defect (this line is also followed in [1] with minor generalizations). These fields satisfy free boundary conditions on the edges of the layer (in other words there are no constrains imposed on the fields). Following this approach one introduces integral operators acting on the auxiliary fields with support in the finite interval, such as $\lambda^{-1}$ in (12) of [1]. In our case it reduces to a unity operator and there is no need for any special treatment of its boundary conditions.

Functional integral takes then explicitly gaussian form and we are able to derive both the Casimir energy and the propagator, arriving at

$$Z[J] = (\text{Det} Q)^{-1/2} \exp \left\{ \frac{1}{2} J\hat{S}J \right\},$$

$$\hat{S} = D - \eta(\Delta^2 + m^2)^{-1}Q^{-1}(\Delta^2), \quad Q = 1 + \eta(\Delta^2).$$  

(5)

here $\eta = \lambda \epsilon^{-1}$. The definitions of $\hat{S}$ and $Q$ must be understood in terms of integral operators. With $\Delta = (-\partial^2 + m^2)^{-1}$ we denote the standard free propagator of scalar field, and the projecting operator $O$ acts as

$$\psi O\phi \equiv \int d\bar{x} \int_{-\epsilon/2}^{\epsilon/2} dx_3 \psi \phi.$$  

The unity operator $1$ is also defined on the defect only. We shall note here that our operator $Q$ is the direct analog of the $K$ (24) [1]. However, the trace calculation which
defines the Casimir energy and is presented below, differs significantly in our approach.

The Casimir energy density per unit area of the layer is given by

\[ \mathcal{E} = \int \frac{d^3 \vec{p}}{2(2\pi)^3} \text{Tr} \ln[Q(\vec{p}; x,y_3)], \]  

(6)

where the Fourier transformation of the coordinates parallel to the defect (i.e. \( \vec{x} = (x_0, x_1, x_2) \)) was performed, \( \vec{p} = (p_0, p_1, p_2) \).

Introducing operator \( U \equiv Q^{-1} - 1 \) and using the definition of \( Q \) we can express the \( \eta \)-derivative of the integrand in the following form

\[ \partial_\eta \ln Q = -\frac{U}{\eta}. \]

For explicit calculation of \( U \) we note that it is proportional to a Green function of an ordinary differential operator:

\[ K_\rho U = -\eta \]
\[ K_V(x,y) \equiv \left( -\frac{\partial^2}{\partial x^2} + V^2 \right) \delta(x-y). \]

(7)

where \( \rho = \sqrt{\eta + E^2}, \quad E = \sqrt{\vec{p}^2 + m^2}. \)

Employing the symmetry conditions \( U(x,y) = U(y,x) = U(-x,-y) \) which follow from the definition of \( U \) we can write

\[ U = -\eta \frac{e^{-\rho|x-y|}}{2\rho} + 2a \cosh((x+y)\rho) + 2b \cosh((x-y)\rho). \]

(8)

From (8) and definition of \( U \) it follows that

\[ U + \eta \partial_\eta U(1 + U) = 0. \]

Then one can derive the coefficients \( a, b \) as

\[ a = -\frac{\xi \eta^2 e^{\epsilon \rho}}{2\rho}, \quad b = -\frac{\xi \eta (E - \rho)^2}{2\rho}, \]

\[ \xi = \left( e^{2\epsilon \rho}(E + \rho)^2 - (E - \rho)^2 \right)^{-1}. \]

(9)

For the energy density we have

\[ \mathcal{E} = -\mu^{3-d} \int \frac{d^d \vec{p}}{2(2\pi)^d} \int_0^\eta \frac{d\eta}{\eta} \text{Tr} U. \]

(10)

where we introduced dimensionization regularization to free oneself from ultraviolet divergencies (\( d = 3 \) corresponds to removing of regularization), and an auxiliary mass parameter \( \mu \). We have chosen the lower limit of integration over \( \eta \) to satisfy the energy normalization condition \( \mathcal{E}|_{\eta=0} = 0 \). It can be shown that the integral is convergent at \( \eta = 0 \).

The calculation of trace of the operator \( U \) is straightforward

\[ \text{Tr} U \equiv \int_{-\epsilon/2}^{\epsilon/2} dx U(x,x) = 2b \epsilon + 4a \sinh(\epsilon \rho) - \epsilon \eta. \]

(11)

Next, putting (9) into (11) we can prove directly that

\[ \text{Tr} U = -\eta \frac{\partial}{\partial \eta} \ln \left[ \frac{e^{-\epsilon(E+\rho)} }{4E\rho \xi} \right]. \]

(12)

Thus, from (11) and (12) we obtain the following expression for the Casimir energy

\[ \mathcal{E} = \mu^{3-d} \int \frac{d^d \vec{p}}{2(2\pi)^d-1} \ln \left[ \frac{e^{-\epsilon E} }{4E \rho} \left( e^{\epsilon \rho}(E + \rho)^2 - e^{-\epsilon \rho}(E - \rho)^2 \right) \right], \quad \rho = \sqrt{E^2 + \lambda \epsilon^{-1}}. \]

(13)

It can be shown that (13) is in full agreement with the usual perturbation theory.

It is easily to generalize (13) for non-local translation invariant \( \lambda \equiv \lambda(\vec{x} - \vec{y}) \) as considered in \([1]\) (see equation (4) there). In this case the final expression for \( \mathcal{E} \) remains the same provided that \( \lambda \) is replaced there with \( \lambda(\vec{p}) \) – Fourier image of \( \lambda(\vec{x}) \) defined in (26) \([1]\). In the massless model (with \( m = 0, \quad E = |\vec{p}| \)) the energy \( \mathcal{E} \) obtained in such a way can be compared with \( \mu^{3-d} \mathcal{E}_0 \) for \( \mathcal{E}_0 \) presented in (67) \([1]\) for plane slab. One can easily check that these results do not coincide.

To compare (13) with results of \([3]\) and \([4]\) one needs to replace the zeta-function regularization used there with dimensional one applied in our approach. Using (15) of \([3]\) one can obtain a generalization of (8) \([3]\) (written for \( d = 3 \)) to \( d \)-dimensional case in the form

\[ V_{\text{eff}} = \frac{\Omega_d \mu^{3-d}}{2(2\pi)^d} \int_0^\infty dk (k^2 - m^2)^{d/2} \frac{\partial}{\partial k} \ln s_{11}(ik), \]

(14)

where \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \) is the volume of the \((d-1)\)-dimensional sphere in the \( d \)-dimensional space, and \( s_{11}(ik) \) for considered case of plane slab is defined by (22) \([3]\). Integrating by part in (14) and changing integration variable \( k = \sqrt{|\vec{p}|^2 + m^2} \) one can check that the right hand sides of (13) and (14) coincide (up to change...
of notation $L = \epsilon$, $V_0 = \eta$). In a similar way one can verify that result obtained in [4] for the plane slab for $d = 2$ with help of zeta-function regularization agrees with [13].

We hope that the arguments given above are sufficient to conclude that the calculation methods proposed in [1] need to be thoroughly verified in order to expose and eliminate theirs defects. Then it may be possible to employ effectively the basic ideas of [1] in calculation of Casimir energy in the models of quantum field theory with nontrivial background.

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[1] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. D **77**, 085018 (2008), arXiv:0801.0760v1.
[2] I. V. Fialkovsky, V. N. Markov, Yu. M. Pis'mak, *On the Casimir energy for scalar fields with bulk inhomogeneities*, arXiv:0804.3603v2.
[3] M. Bordag, *Vacuum energy in smooth background fields*, J. Phys. A **28**, 755 (1995). Note that the author uses the term *effective potential* for the quantity which we identify as Casimir energy.
[4] R. A. Konoplya, D. V. Vassilevich, *Quantum corrections to the noncommutative kink*, JHEP 01 (2008) 068, arXiv:0712.0360.
[5] K. Symanzik, Nucl. Phys. B**190**, 1 (1981).
[6] I. V. Fialkovsky, V. N. Markov, Yu. M. Pis'mak, Int. J. Mod. Phys. A **21** (2006) 2601-2616, arXiv:hep-th/0311236; J. Phys. A: Math. Theor. **41** (2008) 075403.
[7] K. A. Milton, *Casimir Energies and Pressures for $\delta$-function Potentials*, J. Phys. A **37** (2004) 6391-6406, arXiv:hep-th/0401090.
[8] M. Bordag, K. Kirsten, D. V. Vassilevich J.Phys. A **31** (1998) 2381-2389, arXiv:hep-th/9709084v2.
[9] J. Feinberg, A. Mann, M. Revzen, Annals Phys. **288** (2001) 103-136, arXiv:hep-th/9908149v2.
[10] B.H.J. McKellar, G.J. Stephenson, Phys. Rev. C **35** (1987) 2262-2271.
[11] F. Zhou and L. Spruch, Phys. Rev. A **52**, 297, 1995. G.L. Klimchitskaya, U. Mohideen, and V.M. Mostepanenko, Phys. Rev. A **61**, 062107 (2000).
[12] R. Matloob, R. Loudon, S. M. Barnett, and J. Jeffers, Phys. Rev. A **52**, 4823, 1995; M.S. Tomas, Phys. Rev. A **66**, 052103 (2002).
[13] By their titles these papers are seemingly not connected with the problem at hand but still both of them contain explicit calculation of the Casimir energy of a plane slab.