Performance analysis of control system tuning under inner disturbance influence by means of least square approximation to suboptimal algorithm and calculation using frequency optimality criteria

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Abstract. The performance of two known variants of automatic control system tuning has been analysed. The control systems contained PI and PID-regulators and disturbances influenced the process under consideration, the transfer function of the process was the second order with a transport delay typical for thermal processes. The methods used were the least square approximation to the suboptimal algorithm and the calculation method using indirect frequency optimality criteria. The PID-regulator used had a second order filter in the differentiator. The suboptimal algorithm had a second order filter with time constant $T_c$ and correction coefficient $k_t$, for the regulator step response argument. Dependencies and approximation formulae for unambiguous definition of parameters $T_c$ and $k_t$ using the process model parameters for the systems with PI and PID-regulators are obtained, some recommendations for determining approximation frequency range are given. When choosing these parameters the frequency oscillation index of the system is taken into account. When tuning for the indirect indexes approximating formulae determining the indirect indexes optimal values according to the process model parameters in the wide range are used. It is shown that for the system with a PID-algorithm when disturbances influence the process the tuning method using the indirect frequency indexes provides better performance of the control system. Moreover, it is more convenient for implementation in controllers, because it allows to make all the necessary calculations automatically during the one working cycle of the controller. For the system with a PI-algorithm both methods provide approximately equal performance of the control system.

1. Introduction
Microprocessor systems are highly developed now and it gives us a possibility to include the regulator tuning modules into industrial controller software. There are a lot of methods and formulae for PI and PID-regulators tuning, they may be found, for example, in [1]. It should be mentioned that a method can be implemented in the controller software only if it can provide an unambiguous algorithm of the regulator tuning on the basis of information about the process dynamical properties.

One of the autotuning problem solution trend is to use indirect frequency optimality indexes, based on the common properties of closed loop system frequency characteristics around the resonance frequency [2]. The indirect optimality conditions are equations in the frequency characteristics of an automatic control system (ACS) and it is not necessary to find the extremum of a goal function. All the necessary
calculations may be carried out during the one working cycle of the controller. The autotuning algorithms for PI and PID-regulators based on the above mentioned optimality criteria are implemented in some industrial controllers.

Paper [3] offers a tuning method based on the approximation of the frequency response of the regulator been tuned to the frequency response of the suboptimal regulator using the least square method and taking into consideration some conditions. Let us call this method the LS-approximation. The authors of the method notice the following advantages of it, the first is that it is universal and the second is that it does not need iterative calculations, and it is very important for implementation of the method in the industrial controllers.

Because of these reasons, it may be interesting to compare the efficiency of the methods mentioned for tuning PI and PID-regulators in control systems for some processes with typical dynamic characteristics. This paper describes the analysis for the processes described by the following transfer function

\[
W_{ob}(s) = K_{ob} \cdot \frac{\exp(-s \cdot \tau)}{(T_1 \cdot s + 1) \cdot (T_2 \cdot s + 1)} = K_{ob} \cdot \frac{\exp(-s \cdot T_i \cdot \beta)}{(T_1 \cdot s + 1) \cdot (T_1 \cdot n \cdot s + 1)},
\]

where \(\beta = \frac{\omega}{T_1}\) and \(n = \frac{T_2}{T_1}\) are relative parameters.

The paper considers some variants of the parameters values:

\[
\beta = [0.1; 0.2; 0.4; 0.8; 1.6]; n = [1; 2; 4; 8; 16],
\]

and it may be enough for thermal processes representation.

![Figure 1. The structure of the ACS.](image)

The regulator algorithm is PID with a second order filter for the differentiator, the transfer function is

\[
W_i(s) = K_i \left[1 + \frac{1}{T_i \cdot s} + \frac{T_0 s}{(T_i s + 1)^2}\right]
\]

where \(T_o = \alpha \cdot T_i\), \(T_o = T_o / C_i\), \(\alpha = T_o / T_i\); \(C_i = T_o / T_i\). The default value is \(C_i = 8\).

The filter for the differentiator is important for the industrial control systems because the process value signals usually contain random noises that can cause the system equipped with a differentiator without the filter to be inoperative.

2. Brief information about the calculation methods under consideration

2.1. Tuning parameters calculation using indirect frequency indexes

The detailed description of the method may be found, for example, in [2]. The indirect optimality criteria for the systems with the PI and PID-regulators may be represented by a system of three equations for the closed-loop system frequency response \(W_{cs}(j\omega) = R_{cs}(\omega) \cdot \exp[j \cdot F_{cs}(\omega)]\):

\[
R_{cs}(\omega, Kp, Ti, \alpha) = R_{cs,0}; \quad F_{cs}(\omega, Kp, Ti, \alpha) = F_{cs,0}; \quad T_o / T_i = b^3, \quad (1)
\]

where \(R_{cs,0}, F_{cs,0}, b^3\) are predetermined values (the indirect indexed optimal values); \(T_0\) is the oscillation period for frequency \(\omega_0\):

\[
T_0 = \frac{2\pi}{\omega_0}.
\]

For the system with the PID-regulator paper [2] gives formulae for determining optimal values of indirect indexes \(\alpha\) and \(b^3\) according to parameters of the plant model (2) \(n = T_2 / T_1\) and \(\beta = \omega / T_1\), the formulae are:

for parameter \(a_{opt} = \alpha(\beta, n):\)
y01(n) = b01 + c01 / (n - a01); \quad a01 = -4; b01 = 0.38; c01 = 0.2;
y16(n) = b16 + c16 / (n - a16); \quad a16 = -3.7143; b16 = 0.1817;
z4(\beta) = bz4 + cz4 / (\beta - az4); \quad c16 = 0.5576; az4 = -0.2613;
kz(\beta) = [z4(\beta) - y01(4)]/[y16(4) - y01(4)]; \quad b24 = 0.2176; cz4 = 0.0677;
\alpha(\beta, n) = y01(n) + [y16(n) - y01(n)] \cdot kz(\beta);
for parameter b3_opt = b3 (\beta, n):
\quad a01 = -0.1074; b01 = 1.252; c01 = 0.6065; a16 = -2.2022;
\quad b16 = 1.609; c16 = 8.9375; az4 = -1.40; bz4 = 4.70; cz4 = -4.95;
y01(n) = b01 + c01 / (n - a01); y16(n) = b16 + c16 / (n - a16);
z4(\beta) = bz4 + cz4 / (\beta - az4); kz(\beta) = [z4(\beta) - y01(4)]/[y16(4) - y01(4)];
b3(\beta, n) = y01(n) + [y16(n) - y01(n)] \cdot kz(\beta).

If it is better to use the error-squared performance index (I2 = min), it is recommended to make b3_opt approximately 1.4 times lower.

After calculating indirect indexes optimal values (\alpha(\beta, n) and b3(\beta, n)), the regulator parameters are calculated by means of solving system (4)-(6). The solution algorithm allows to make all the calculations through one working cycle of a controller. The tuning on the indirect indexes basis provides the following ACS quality indexes:
- desired free oscillations damping degree by mean of imposing limitations of the frequency oscillation index \(M \leq M_{geo}[4]\) that is provided by means of (4) and (5);
- Minimal integral quality index \(I\{ \epsilon(t) \} = \min\) (linear or squared), characterizing the control error that is provided by (6).

2.2. Tuning regulators by means of the LS-approximation

According to [2] the step response \(h_{xy}^{\text{opt}}(t)\) of the closed optimal control system may be represented by the difference of the following components:
\[h_{xy}^{\text{opt}}(t) = h_{xy}(t) \cdot 1(t) - h_{xy}(t) \cdot 1(t - \tau),\]
where: \(h_{xy}(t)\) is the process step response; \(\tau\) is the delay time.

In order to implement the system we introduce so-called suboptimal systems. The suboptimal system step response does not immediately decrease to zero (when \(t > 2\tau\)), it has a smooth part which is formed by the filter with transfer function \(W_c(s)\).

In order to provide the necessary stability margin we soften demands to the speed of the system by means of introduction of correction coefficient \(k_3\) for the delay time of the regulator step response. For plant (1) the transfer function of the suboptimal system \(W_{k_3, opt}(\omega)\) when the plant is influenced by disturbance \(\lambda\) may be written as follows:
\[
\alpha_1 = \frac{1}{T_1}; \quad \alpha_2 = \frac{1}{T_2}; \quad B_0 = C_0 = K_{ob}; \quad C_1 = \frac{\alpha_2 \cdot K_{ob}}{\alpha_1 \cdot K_{ob}}; \quad C_2 = \frac{-\alpha_1 \cdot K_{ob}}{\alpha_1 - \alpha_2}; \quad W_c(s) = \frac{1}{(s \cdot T_c + 1)};
\]
\[
B_1 = C_1 \cdot \exp[-\alpha_1 \cdot \tau \cdot (2k_3 \cdot \tau - 1)]; \quad B_2 = C_2 \cdot \exp[-\alpha_2 \cdot \tau \cdot (2k_3 \cdot \tau - 1)];
\]
\[
h_{xy}(t) = \Phi(t - \tau) \cdot [C_0 + C_1 \cdot \exp[-\alpha_1 \cdot (t - \tau)] + C_2 \cdot \exp[-\alpha_2 \cdot (t - \tau)]];
\]
\[
W_1(s) = h_{xy}(\tau 2k_3) \cdot [1 - W_c(s)]; \quad W_2(s) = B_0 + \frac{B_1 \cdot s}{s + \alpha_1} + \frac{B_2 \cdot s}{s + \alpha_2}; \quad W_3(s) = W_2(\omega) - W_1(\omega); \quad W_{k_3, opt}(s) = W_{xy}(s) - W_3(\omega) \cdot \exp(-s \cdot \omega \cdot 2 \cdot k_3).\]
Having taken into consideration process model (1), we use here a second order filter with transfer function \( W_c(s) \) and time constant \( T_c = r \cdot k_r \).

The transfer function of the suboptimal regulator is

\[
W_{sub}(s) = \frac{1}{W_{k_y,op}(s)} - \frac{1}{W_{ob}(s)}.
\]

After that parameters of the regulator been tuned can be calculated, the optimality condition is the approximation of its frequency response to the frequency response of the suboptimal regulator using the LS-method. The calculations have shown that the result of the regulator tuning using the LS-method depends considerably upon its parameters, namely, coefficients \( k_r, k_t \) and approximation frequency range \([\omega_1, \omega_2] \). Paper [3] written by the developers of the method does not give sufficient and complete recommendations on calculating these parameters. Because of this reason one of the purposes of this paper is to develop practical recommendations for some processes with transfer function (1) where parameters \( n \) and \( \beta \) change in the large range.

For each variant of the process parameters \( n \) and \( \beta \) values of coefficients \( k_r, k_t \) and frequency range \([\omega_1, \omega_2] \) were chosen by means of calculations in Mathcad. The criterion was the transient process close to the optimal one when the disturbance influenced the control plant. The necessary condition was the value of the closed loop resonance peak, i.e., the frequency oscillation index \( M \), and the optimality criterion was the minimal transient process duration.

It should be mentioned that the suboptimal transient process in theory should be nonperiodical (i.e., \( M=1 \)) while the transient process duration is minimal. But in many cases the nonperiodical process may result from the low speed of the system and in industry it is undesirable. So, we made an assumption that the process may be a little oscillatory with \( M_{op} = 1.3 \). On the basis of the calculations made some practical recommendations have been developed, they are given below.

3. Recommendations on determination of approximation frequency range \([\omega_1, \omega_2] \)

For the system with the PID-algorithm it is recommended to determine the upper frequency \( \omega_2 \) from the condition that the phase frequency characteristic of the process \( F_{ob}(\omega) = \text{arg}(W_{ob}(\omega)) \) is equal to

\[
F_{0, \text{PID}} = -2.7 \text{ rad}. \tag{10}
\]

For the process with transfer function (1) and parameters \( n=T_2/T_1 \) and \( \beta=r/T_1 \) this condition is as follows:

\[
F_0 - F_{ob}(x) = F_0 + \beta \cdot x + \arctg(x) + \arctg(x \cdot n) = 0, \quad x = \omega_2 \cdot T_1 \tag{11}
\]

The root of the equation (11) \( x_0 \) may be found using the following algorithm:

\[
x_{k+1} = x_k - \frac{G(x_k)}{G(x_k)} \tag{12},
\]

where: \( G(x) = \beta \cdot x + \arctg(x) + \arctg(x \cdot n) + F_0 \); \( GG(x) = \beta + 1/(x^2 + 1) + n/(x^2 \cdot n^2 + 1) \).

After that frequency \( \omega_2 \) may be calculated using the formula

\[
\omega_2 = x_0/T_1. \tag{13}
\]

For the system with the PI-algorithm it is also recommended to determine the upper frequency \( \omega_2 \) by means of (11), but the value of the phase frequency characteristic must be lower:

\[
F_{0, \text{PI}} = -2.0 \text{ rad}. \tag{14}
\]

The formula for determining the lower frequency is the same for both cases:

\[
\omega_1 = 0.1 \omega_2. \tag{15}
\]

4. Recommendations on determination of coefficients \( k_r \) and \( k_t \)

Functional connections between coefficients \( k_r, k_t \) of the least square algorithm and process parameters \( n \) and \( \beta \) have been obtained. The connections were derived from the calculation for an array of parameters (2) for plant model (1). The curves for the system with the PID-algorithm (3) are shown in figure 2.
Figure 2. The parameters of the least square algorithm for the PID-regulator: a) for \( k_T \); b) for \( k_r \).

These curves are approximated by the formulae given below in order to make it possible to use them in controller programs:

\[
al_1 = 0.04; \quad b = 0.299; \quad c_1 = 1.1; \quad a_{16} = -0.047; \quad b_{16} = 1.795; \quad c_{16} = 2.3; \\
a_2 = -0.455; \quad b_2 = 8.079; \quad c_2 = -65.642; \quad u = 1/\beta; \quad z(u) = u - 0.625; \\
y_1(u) = a_1 \cdot z(u)^2 + b_1 \cdot z(u) + c_1; \quad y_{16}(u) = a_{16} \cdot z(u)^2 + b_{16} \cdot z(u) + c_{16}; \\
v_2(n) = bx_2 + cx_2 / (n - ax_2); \quad k_{16}(n, u) = y_1(u) + [y_{16}(u) - y_1(u)] \frac{v_2(n) - 1.8}{3.7}. \]

The curves for the system with the PI-algorithm are given in figure 3.

Figure 3. The parameters of the least square algorithm for the PI-regulator: for \( k_T \); b) for \( k_r \).

These curves are approximated by the following formulae:

\[
al_1 = 0.01138; \quad b_1 = 0.9387; \quad c_1 = 1.2; \quad a_{16} = 0.04836; \quad b_{16} = 1.8827; \quad c_{16} = 3.1; \\
a_2 = -10.25; \quad b_2 = 9.65; \quad c_2 = -74.812; \quad u = 1/\beta; \quad z(u) = u - 0.625; \\
y_1(u) = a_1 \cdot z(u)^2 + b_1 \cdot z(u) + c_1; \quad y_{16}(u) = a_{16} \cdot z(u)^2 + b_{16} \cdot z(u) + c_{16}; \\
v_2(n) = bx_2 + cx_2 / (n - ax_2); \quad k_{16}(n, u) = y_1(u) + (y_{16}(u) - y_1(u)) \frac{v_2(n) - 3}{3.8}. \]
az1 = -0.05354; bz1 = 0.7087; cz1 = 0.8124; az16 = -0.04356; bz16 = 0.9280; 
cz16 = 1.7618;

\[
a_{04} = -7.0769; \quad b_{04} = 6.1923; \quad c_{04} = -29.822; \quad z_{1}(\beta) = bz1 + cz1 / (\beta - az1);
\]

\[
z_{16}(\beta) = bz16 + cz16 / (\beta - az16); \quad y_{04}(n) = b_{04} + c_{04} / (n - a_{04});
\]

\[
k_{z}(\beta, n) = z_{1}(\beta) + [z_{16}(\beta) - z_{1}(\beta)] - y_{04}(n) - 2.5 / 2.4.
\]  

5. Examples of the regulator tuning

5.1. Example 1. PID-regulator tuning by means of the LS-approximation

The calculation order is shown below. The calculation has been made in Mathcad for one variant of process (1):

\[
K_{ob} = 1; \quad T_{1} = 20; \quad T_{2} = 80; \quad \tau = 8; \quad n = T_{2}/T_{1} = 4; \quad \beta = \tau/T_{1} = 0.4.
\]  

The real PID-algorithm is considered. First upper frequency \(\omega_2\) and lower \(\omega_1\) are determined by means of (10)-(13). The result is: \(\omega_2 = 0.039; \quad \omega_1 = 0.0039\). After that parameters \(k_{Tc}\) and \(k_{\tau}\) are calculated by means of (16) and (17) for \(n = 4\) and \(\beta = 0.4\). The result is \(k_{Tc} = 3.2; \quad k_{\tau} = 2.85\). In this case: \(T_{c} = \tau \cdot k_{Tc} = 25.6\).

Figure 4 shows the step responses of the plant (curve 1) and for the system with the suboptimal regulator (curve 2).

\[\text{Figure 4. The step responses of the plant and the system with the suboptimal regulator.}\]

Figure 5a shows the frequency response of the process, the upper and the lower frequencies are marked; fig. 5b shows the frequency responses of the suboptimal regulator (curve 1) and the real PID-regulator (curve 2).

\[\text{Figure 5. Frequency responses of the process and regulators; 1 \text{ – suboptimal regulator, 2 \text{ – PID.}}}\]
After that we specify the point quantity in the frequency range for the LS-approximation: \( N = 100 \); and organize a calculation cycle with step \( \Delta \omega = (\omega_2 - \omega_1)/N \). We also specify the initial values of the parameters being calculated: \( c_1 = 4; \; c_2 = 0.05; \; c_3 = 10; \; T_f = 1 \)s (for the filter in \( D \)-part of the PID). Next we write the equation for the imaginary and real parts of the frequency response and use built-in function \( \text{Minerr}(c_1, c_2, c_3) \) in order to calculate the real PID-regulator parameters by means of the least square approximation \( (i = \sqrt{-1}) \):

\[
\begin{align*}
\sum_{j=1}^{N} \left[ \text{Re}(W_{sub}(\omega)) - \text{Re} \left[ c_1 + \frac{c_2}{i \cdot \omega_j} + \frac{c_3 \cdot i \cdot \omega}{(1 + T_f \cdot i \cdot \omega)^2} \right] \right]^2 &= 0, \\
\sum_{j=1}^{N} \left[ \text{Im}(W_{sub}(\omega)) - \text{Im} \left[ c_1 + \frac{c_2}{i \cdot \omega_j} + \frac{c_3 \cdot i \cdot \omega}{(1 + T_f \cdot i \cdot \omega)^2} \right] \right]^2 &= 0,
\end{align*}
\]

\[
\begin{bmatrix}
(c_1) \\
(c_2) \\
(c_3)
\end{bmatrix} = \text{Minerr}(c_1, c_2, c_3) \Rightarrow K_r := c_1; \; T_i := \frac{c_1}{c_2}; \; T_d := \frac{c_3}{c_1}.
\]

As a result the PID (3) parameters have been obtained:

\[
K_r = 4.673; \; T_i = 65.0; \; T_d = 8.37; \; T_f = 1 \text{ s}.
\]

Here we have \( T_d/T_f = 8.37 \), and it is quite common.

### 5.2. Example 2. PID-regulator tuning using indirect frequency indexes

The calculation is made for the same process (1) with parameters (20), the desired values of the indirect indexes are \( R_{zs.op} = 1.4; \; G_{zs.op} = 90^\circ \) (it is the control point on the closed loop system frequency response). The calculation is made using the method described in [1]. The control point for the open system is calculated using the point of the closed loop system: \( R_{zs.op} = 0.814; \; F_{zs.op} = 2.521 \) rad.

The optimal values of indirect indexes \( \alpha(\beta, n) \) and \( b3(\beta, n) \) are calculated by formulae (8) and (9) for plant parameters \( n = 4; \beta = 0.4 \); the result is \( b3. op = 1.95; \alpha. op = 0.355 \). The regulator frequency response vector for the control point when \( K_r = 1 \) is calculated by formulae

\[
\begin{align*}
\omega_{_T_i} &= 2\pi/b3. op; \; \omega_{_T_d} = \omega_{_T_i} \cdot \alpha. op; \; \omega_{_T_f} = \omega_{_T_d} \cdot 0.125; \\
W_{r.op} &= 1 + \frac{1}{i \cdot \omega_{_T_i}} + i \cdot \omega_{_T_d} \frac{1}{(1 + i \cdot \omega_{_T_f})^2}; \; R_{r.op} = |W_{r.op}|; \; F_{r.op} = \arg(W_{r.op});
\end{align*}
\]

and the result is \( R_{r.op} = 1.52; \; F_{r.op} = 0.527 \).

After that the argument of the process frequency response vector for the optimal regulator parameters is calculated:

\[
F_{ob.op} = F_{r.op} - F_{zs.op} = -3.049.
\]

The optimal value of the non-dimensional frequency \( (x_0) \) is calculated using the argument equation:

\[
\beta \cdot x + \arctg(x) + \arctg(x \cdot n) + F_{ob.op} = 0. \]

The root of the equation is obtained by means of algorithm (13), the value of the root is \( x_0 = 1.575 \).

Then the process frequency response modulus for non-dimensional frequency \( x_0 \) and \( K_{ob} = 1 \) is determined:

\[
\text{Rob.op} = 1/ \sqrt{(1 + x_0^2)(1 + (n \cdot x_0)^2)}; \; \text{Rob.op} = 0.084. \]

The absolute frequency in the control point is \( \omega_0 = x_0/ T_f = 0.07876 \). The optimal values of the regulator parameters are calculated by formulae:

\[
\begin{align*}
Kr.op &= R_{zs.op} / (R_{r.op} \cdot \text{Rob.op} \cdot Kob); \; Ti.op = 2\pi T_i / (b3. op \cdot x_0); \\
T_d.op &= \alpha \cdot Ti.op.
\end{align*}
\]

The result is:

\[
Kr.op = 6.37; \; Ti.op = 40.9; \; T_d.op = 14.5.
\]

### 5.3. Calculation result analysis for the PID-regulator tuning

Figure 6 gives the frequency characteristics of the open and closed systems with PID-regulator (3) with the obtained parameters and process (1) with parameters (20). Curve 1 is for the LS-approximation
method with parameters (21), curve 2 is for the calculation using the indirect frequency indexes for \( R_{zs,op} = 1.4 \) and \( G_{zs,op} = -70^\circ \) and regulator parameters (22).

**Figure 6.** Open loop frequency responses (a) and closed loop amplitude characteristics (b): 1 – the LS-method; 2 – the indirect frequency indexes method; 3 – the control point for the indirect frequency indexes method; 4 – frequency \( \omega_2 \) for the least square method; 5 – circle for \( M=1.4 \).

One can see in figure 6 that both methods provides the necessary stability margin. The comparison of the frequency properties shows that the closed loop amplitude characteristic \( R(\omega) \) is higher frequency for the indirect indexes than for the least square method.

Figure 7 shows the step responses of the control systems with the PID-regulator when the process is influenced by the disturbance; the parameters of the system and notation are given above. The quality indexes are the maximum deviation \( (h_{max,1}=0.182; \ h_{max,2}=0.13) \) and the error-squared performance index \( (I_{2.1}=1.69; \ I_{2.2}=0.639) \).

**Figure 7.** Step responses of the ACS with the PID-regulator.

Thus, in the case under consideration, the indirect frequency indexes method provides better quality.

5.4. Example 3. The PI-regulator tuning using the LS-approximation

The calculations are shown below, they are carried out in Mathcad for process (1) with parameters (20). The upper and the lower frequencies (\( \omega_2 \) and \( \omega_1 \)) are determined using (11)-(13) for \( F_{0,pi} = -2.0 \) rad (14). The result is \( \omega_2 = 0.0313; \ \omega_1 = 0.00313 \).

Parameters \( k_{TC} \) and \( k_c \) for the LS-algorithm for the system with the PI-regulator are calculated using (18) and (19), the process parameters are \( n=4; \ \beta=0.4 \). The result is \( k_{TC}=4.4; \ k_c=3.5 \) and \( T_c = \tau \ k_{TC}=35.2 \).

In order to calculate the PI-parameters by means of the LS-approximation method we can derive formulate from the system of the least square approximation equations:
\[ C_1 = \frac{1}{\omega_2^2 - \omega_1^2} \int_{\omega_1}^{\omega_2} \text{Re}(W_{sub}(\omega)) d\omega; \]
\[ C_{21} = \frac{\omega_2^3 - \omega_1^3}{3}; \quad C_{22} = \frac{1}{\omega_1} \int_{\omega_1}^{\omega_2} \text{Im}(W_{sub}(\omega)) d\omega; \]
\[ C_{23} = \frac{\omega_2^2}{\omega_1} \cdot \text{Im}(W_{sub}(\omega)) d\omega; \quad C_{24} = \frac{1}{\omega_1} - \frac{1}{\omega_2}; \quad C_2 = \frac{C_{21} \cdot C_{22} - C_{23} \cdot (\omega_2 - \omega_1)}{(\omega_2 - \omega_1)^2 - C_{21} \cdot C_{24}}; \]
\[ K_r = C_1; \quad T_i = C_1 / C_2. \]

As a result we have obtained the following parameters:
\[ K_r = 2.63; \quad T_i = 70.9. \] (23)

5.5. Example 4. The PI-stimulator tuning using the indirect frequency indexes

The calculations are made similarly to Example 2, condition \( \alpha = 0 \) was imposed. The optimal value of indirect index \( b3(\beta, n) \) is calculated by means of formulae (10) for the given process model parameters \( n = 4; \beta = 0.4 \) and the linear integral index. The result is \( b3_1 = 2.87 \). The calculation for the error-squared performance index is made using formula \( b3_2 = b3_1 / 1.4 \). It results in \( b3_2 = 2.05 \).

Then we obtain the frequency response in the control point: \( R_{rob} = 1.052; \quad F_{rob} = -0.315 \). The argument of the process frequency response when the regulator is optimally tuned is \( F_{rob} = -2.206 \).

The optimal value of non-dimensional frequency \( x_0 \) is obtained from the equation for arguments: \( \beta \cdot x + \arctg(x) + \arctg(x \cdot n) + F_{rob} = 0 \). The root of the equation is found using algorithm (12), the value of it is \( x_0 = 0.759 \). The process frequency response modulus for frequency \( x_0 \) and \( K_{rob} = 1 \) is equal to \( R_{rob} = 0.249 \). The absolute frequency in the control point is \( \omega_0 = x_0 / T_i = 0.03797 \).

The optimal PI-regulator parameters are calculated by means of the following formulae:
\[ K_{rob} = R_{rs} / (R_{rob} \cdot R_{rob} \cdot K_{rob}); \quad T_i = 2\pi T_i / (b3_2 \cdot x_0); \] and the result in this case is
\[ K_{rob} = 3.11; \quad T_i = 80.7. \] (24)

5.6. Calculation result analysis for the PI-regulator tuning

Figure 8 shows the frequency responses for the open systems with the PI and suboptimal regulators for process (1). Curve 1 is for the PI-regulator tuned by the LS-method with parameters (23), curve 2 is for suboptimal regulator \( W_{sub}(\omega) \), curve 3 is for a piece of the \( M \)-circle for \( M = 1.4 \).

As one can see from the figure, the system with the suboptimal regulator does not have the stability margin. It confirms that it is necessary to impose limitations and choose carefully the upper frequency of the approximation. Fig. 8 also shows that upper frequency \( \omega_2 \) determined using (15) is in the interval of the resonance frequency of the system with the PI-regulator and it confirms that the frequency is suitable.

Figure 9 shows step responses of the ACS with the PI-regulator when the disturbance influences the process. Curve 1 is for the LS-approximation method, curve 2 is for the calculation using the indirect indexes. The quality indexes are the maximum deviation \( (h_{max1} = 0.282; \ h_{max2} = 0.267) \) and the error-squared performance index \( (I_{2.1} = 5.155; \ I_{2.2} = 4.404) \). The transient process duration is more or less the same. So, we can draw the conclusion that for the system with the PI-regulator when the disturbance influences the process both methods provide approximately equal control system performance. But the LS-method needs some complicated calculations with complex variables that are not easy to implement in industrial controllers.
Figure 8. Open system frequency response for the PI (1) and suboptimal (2) regulators.

Figure 9. Transient processes in the ACS with the PI-regulator.

6. Conclusion

1. The curves and approximating formulae for determination of the LS-approximation method parameters have been obtained. The parameters may be calculated using the second-order process model with the time delay for the systems with the PID and PI-regulators when the disturbance influences the process.

2. It is shown that for the PID-regulator in the case under consideration the indirect frequency indexes method provides better performance of the ACS.

3. Moreover, the indirect frequency method is more convenient for implementation in the controllers because all calculations are carried out during one working cycle of the controller using standard functions.

4. For the system with the PI-regulator in the case under consideration both methods provide more or less the same ACS performance.

7. References

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