Algebraic Image Processing

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Abstract

We propose an approach to image processing related to algebraic operators acting in the space of images. In view of the interest in the applications in optics and computer science, mathematical aspects of the paper have been simplified as much as possible. Underlying theory, related to rigged Hilbert spaces and Lie algebras, is discussed elsewhere.

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1 Introduction

The fundamental problem in image analysis is that the information contained in an image is immense, too much in order that the human mind can manage it. So in science we have to disregard a large part of the contained information and isolate the required one, quite more restricted, in function of our specific interests.

We suggest that this work to clean the signal from forgettable elements and to highlight the relevant information could be improved by the mathematical theory of operators acting in the space of images.

Same application of these ideas, somehow similar to adaptive optics, are described in the following. Yet, while in adaptive optics the perturbations to be removed are related to the phases of a complex function $f(r, \theta)$ our approach, that we could call Algebraic Image Processing (AIP), acts on the module of the image $|f(r, \theta)|$. So, we suggest that
the image $|f(r, \theta)|^2$ registered by an experimental apparatus can be assumed not as the
final result of the optical measure but possibly as an intermediate step to be further
elaborated by a computer program based on operators acting in the space of images. AIP
is indeed a soft procedure and operates on a set of pixels to obtain another set of pixels
independently from the causes of the effects we wish to eliminate.

Moreover AIP is a general theory of the connections of images between them and thus
can be applied also outside the cleaning of images to any manipulation of images.

We restrict ourselves in this paper to practical aspects relevant in optics and computer
science. Underlying mathematics, related to rigged Hilbert spaces and Lie groups, is
considered in detail in [1] and, at higher theoretical level, in [2–4]. We recommend to the
reader interested in the theory to look there and references therein.

Features of the proposed approach are:
1. It can be used on-line and also off-line if, at later time, more accuracy is required.
2. It is a soft procedure, cheap and without mechanical moving parts.
3. It allows to consider together images of different origin and frequencies like optical
   and radio images.

The relevant operative points are:
• The vector space of images on the disk has as a basis the Zernike functions [5].
• The space of images can be integrated with the operators in the space of images i.e.
  the operators that transform Zernike functions into Zernike functions.
• This space of images and its operators define the unitary irreducible representation
  $D_{1/2} \otimes D_{1/2}$ of the Lie algebra $su(1, 1) \oplus su(1, 1)$.
• Every operator of AIP can be written as a polynomial in the generators of the
  algebra $su(1, 1) \oplus su(1, 1)$
• and can be computed applying this polynomial to the Zernike functions.

In sect.2 we summarize the vector space of images. In sect.3 we introduce the operators
acting on this space, their algebraic structure and their realization. In sect.4 we give a
description of a possible modus operandi that could be realized in automatic or semi-
automatic way by a computer program. Color images can also be introduced by means
of an additional factorization of a color code.

2 Vector Space of Images

Radial Zernike polynomials $R^m_n(r)$ can be found in [5] and are real polynomials defined
for $0 \leq r \leq 1$ such that $R^m_n(1) = 1$, where $n$ is a natural number and $m$ an integer, such
that $0 \leq m \leq n$ and $n - m$ is even.
As the interest in optics is focused on real functions defined on the disk, starting from $R^m_n(r)$ it is usual to introduce, with $0 \leq \theta < 2\pi$, the functions

$$Z^{-m}_n(r, \theta) := R^m_n(r) \sin(m\theta) \quad Z^m_n(r, \theta) := R^m_n(r) \cos(m\theta)$$

and, because only smooth functions are normally considered, to take into account only low values of $n$ and $m$, summarized in a unique sequential index [6].

However, in a general theory, functions are not necessarily smooth and formal properties are relevant. We thus came back to the classical form of Born and Wolf in the complex space [5]:

$$Z^m_n(r, \theta) := R^{|m|}_n(r) e^{im\theta}$$

with $n$ natural number, $m$ integer with $n - m$ even and $-n \leq m \leq n$. The symmetry can be improved writing $n$ and $m$ in function of two arbitrary independent natural numbers $k$ and $l$ [7]

$$n = k + l \quad m = k - l \quad (k = 0, 1, 2, \ldots; l = 0, 1, 2, \ldots)$$

and introducing a multiplicative factor. The Zernike functions we consider here are thus

$$V_{k,l}(r, \theta) := \sqrt{k + l + 1} R^{[k-l]}_{k+l}(r) e^{i(k-l)\theta} \quad (1)$$

and depend from two natural numbers $k$ and $l$ and two continuous variables $r$ and $\theta$. The functions $V_{k,l}(r, \theta)$ are an orthonormal basis in the space $L^2(D)$ of square integrable complex functions defined on the unit disk $D$ as:

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^1 dr^2 V_{k,l}(r, \theta)^* V_{k',l'}(r, \theta) = \delta_{k,k'} \delta_{l,l'}$$

$$\frac{1}{2\pi} \sum_{k=0}^\infty \sum_{l=0}^\infty V_{k,l}(r, \theta)^* V_{k,l}(s, \phi) = \delta(r^2 - s^2) \delta(\theta - \phi) \quad (2)$$

and have the symmetries

$$V_{l,k}(r, \theta) = V_{k,l}(r, \theta)^* = V_{k,l}(r, -\theta).$$

Every function $f(r, \theta) \in L^2(D)$ can thus be written

$$f(r, \theta) = \sum_{k=0}^\infty \sum_{l=0}^\infty f_{k,l} V_{k,l}(r, \theta) \quad (3)$$

where

$$f_{k,l} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^1 dr^2 V_{k,l}(r, \theta)^* f(r, \theta) \quad (4)$$

is the component along $V_{k,l}(r, \theta)$ of the function $f(r, \theta)$. In this paper we consider only normalized states, so that the Parseval identity gives for each state:
\[
\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^1 dr r^2 |f(r, \theta)|^2 = \sum_{k=0}^\infty \sum_{l=0}^\infty |f_{k,l}|^2 = 1. \tag{5}
\]

Restrictions on the values of \( k \) and \( l \) can be used as filters. \( k + l > h (h) \) is a high-pass (low-pass) filter in \( r \), while \( |k - l| > h (h) \) is high-pass (low-pass) filter in \( \theta \) and a combination of the restrictions can be also considered. Of course, using filters, the results must be multiplied for adequate factors to obtain normalized states.

3 Operators and Algebra in the disk space

Now let us consider the differential applications:

\[
A_+ := \frac{e^{+i\theta}}{2} \left[ -(1 - r^2) \frac{d}{dr} + r(k + l + 2) + \frac{1}{r}(k - l) \right] \sqrt{\frac{k + l + 2}{k + l + 1}} \\
A_- := \frac{e^{-i\theta}}{2} \left[ + (1 - r^2) \frac{d}{dr} + r(k + l) + \frac{1}{r}(k - l) \right] \sqrt{\frac{k + l}{k + l + 1}} \\
B_+ := \frac{e^{-i\theta}}{2} \left[ -(1 - r^2) \frac{d}{dr} + r(k + l + 2) - \frac{1}{r}(k - l) \right] \sqrt{\frac{k + l + 2}{k + l + 1}} \\
B_- := \frac{e^{+i\theta}}{2} \left[ + (1 - r^2) \frac{d}{dr} + r(k + l) - \frac{1}{r}(k - l) \right] \sqrt{\frac{k + l}{k + l + 1}} \tag{6}
\]

that, by inspection, are the rising and lowering recurrence applications on the Zernike functions \( V_{k,l}(r, \theta) \):

\[
A_+ V_{k,l}(r, \theta) = (k + 1) V_{k+1,l}(r, \theta) , \quad A_- V_{k,l}(r, \theta) = k V_{k-1,l}(r, \theta) , \\
B_+ V_{k,l}(r, \theta) = (l + 1) V_{k,l+1}(r, \theta) , \quad B_- V_{k,l}(r, \theta) = l V_{k,l-1}(r, \theta) . \tag{7}
\]

\( A_\pm \) and \( B_\pm \) establish the recurrence relations but are not operators because each application of \( A_\pm \) or \( B_\pm \) modifies the parameters \( k \) or \( l \) to be read by the following applications. To obtain the true rising and lowering operators, we need indeed to introduce the operators \( R, D_R, \Theta, D_\Theta, K, L \):

\[
R \ V_{k,l}(r, \theta) := r V_{k,l}(r, \theta) , \quad D_R \ V_{k,l}(r, \theta) := \frac{dV_{k,l}(r, \theta)}{dr} , \\
\Theta \ V_{k,l}(r, \theta) := \phi V_{k,l}(r, \theta) , \quad D_\Theta \ V_{k,l}(r, \theta) := \frac{dV_{k,l}(r, \theta)}{d\theta} , \\
K \ V_{k,l}(r, \theta) := k V_{k,l}(r, \theta) , \quad L \ V_{k,l}(r, \theta) := l V_{k,l}(r, \theta) ;
\]
Now, as a particular we can calculate the action of the commutators: rewrite eqs. (6) as operators:

\[ A_+ := \frac{e^{i\Theta}}{2} \left[ -(1 - R^2)D_R + R(K + L + 2) + \frac{1}{R}(K - L) \right] \sqrt{\frac{K + L + 2}{K + L + 1}}, \]

\[ A_- := \frac{e^{-i\Theta}}{2} \left[ +(1 - R^2)D_R + R(K + L) + \frac{1}{R}(K - L) \right] \sqrt{\frac{K + L}{K + L + 1}}, \]

\[ B_+ := \frac{e^{-i\Theta}}{2} \left[ -(1 - R^2)D_R + R(K + L + 2) - \frac{1}{R}(K - L) \right] \sqrt{\frac{K + L + 2}{K + L + 1}}, \]

\[ B_- := \frac{e^{i\Theta}}{2} \left[ +(1 - R^2)D_R + R(K + L) - \frac{1}{R}(K - L) \right] \sqrt{\frac{K + L}{K + L + 1}}. \]  

(8)

Now, as \( A_\pm \) and \( K \) are operators, we can apply them iteratively on \( V_{k,l}(r, \theta) \) and in particular we can calculate the action of the commutators:

\[ [A_+, A_-] V_{k,l}(r, \theta) = -2(k + 1/2) V_{k,l}(r, \theta), \quad [K, A_\pm] V_{k,l}(r, \theta) = \pm V_{k\pm,1}(r, \theta). \]

So, defining \( A_3 := K + 1/2 \), we find that \( \{A_+, A_3, A_-\} \) are on the \( V_{k,l}(r, \theta) \) the generators of one algebra \( su(1,1) \):

\[ [A_+, A_-] = -2A_3, \quad [A_3, A_\pm] = \pm A_\pm. \]  

(9)

Analogously

\[ [B_+, B_-] V_{k,l}(r, \theta) = -2(l + 1/2) V_{k,l}(r, \theta), \quad [L, B_\pm] V_{k,l}(r, \theta) = \pm B_{k,\pm,1}(r, \theta) \]

exhibit that \( \{B_+, B_3 := L + 1/2, B_-\} \) are on the \( V_{k,l}(r, \theta) \) the generators of another Lie algebra \( su(1,1) \):

\[ [B_+, B_-] = -2B_3, \quad [B_3, B_\pm] = \pm B_\pm. \]  

(10)

Finally, as \( A_i \) and \( B_j \) commute on the \( V_{k,l}(r, \theta) \), the algebra is completed by

\[ [A_i, B_j] = 0. \]  

(11)

Thus in the vector space of images on the disk, eqs. (9) (11) define a differential realization of the 6 dimensional Lie algebra \( su(1,1) \oplus su(1,1) \).

We can now calculate on \( V_{k,l}(r, \theta) \) the Casimir invariants of the two \( su(1,1) \):

\[ C_A V_{k,l}(r, \theta) = \frac{1}{2} (A_+ A_3 - A_-^2) V_{k,l}(r, \theta) = \frac{1}{4} V_{k,l}(r, \theta), \]

\[ C_B V_{k,l}(r, \theta) = \frac{1}{2} (B_+ B_3 - B_-^2) V_{k,l}(r, \theta) = \frac{1}{4} V_{k,l}(r, \theta). \]

As the Casimir of the discrete series \( D_+^j \) of \( su(1,1) \) is \( j(1 - j) \) with \( j = 1/2, 1, \ldots \), the space \( \{V_{k,l}(r, \theta)\} \) is isomorphic to \( \{|k,l|k,l = 0,1,2,\ldots\} \), the unitary irreducible
representation \( D_{1/2}^+ \otimes D_{1/2}^+ \) of the group \( SU(1, 1) \otimes SU(1, 1) \), where indeed the action of the generators of the algebra is:

\[
A_+ |k, l\rangle = (k + 1) |k + 1, l\rangle, \quad B_+ |k, l\rangle = (l + 1) |k, l + 1\rangle, \\
A_3 |k, l\rangle = k |k, l\rangle, \quad B_3 |k, l\rangle = l |k, l\rangle, \\
A_- |k, l\rangle = k |k - 1, l\rangle, \quad B_- |k, l\rangle = l |k, l - 1\rangle. \tag{12}
\]

Now we move from the generators of the algebra \( su(1, 1) \otimes su(1, 1) \) to their products that belong to the associated universal enveloping algebra (UEA). It is the algebra \( UEA[su(1, 1) \oplus su(1, 1)] \) constructed on the ordered monomials \( A_+^\alpha A_3^\beta A_-^\gamma B_+^\delta B_3^\mu B_-^\nu \) (where \( \alpha_i \) and \( \beta_j \) are natural numbers) submitted to the relations \((9),(11)\). So every operator \( \mathcal{O} \in UEA[su(1, 1) \oplus su(1, 1)] \) can be written as

\[
\mathcal{O} = \sum_{\bar{\alpha}, \bar{\beta}} c_{\bar{\alpha}, \bar{\beta}} A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3}. \tag{13}
\]

where \( c_{\bar{\alpha}, \bar{\beta}} \) are arbitrary complex functions of \( \bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \) and \( \bar{\beta} = (\beta_1, \beta_2, \beta_3) \). Since \( \{V_{k,l}(r, \theta)\} \) is a differential representation of the algebra \( su(1, 1) \oplus su(1, 1) \), it is also a differential representation of the \( UEA[su(1, 1) \oplus su(1, 1)] \).

Because the representation \( D_{1/2}^+ \otimes D_{1/2}^+ \) is unitary and irreducible the set of unitary operators acting on the space \( L^2(\mathbb{D}) \), \( \{\mathcal{O}[L^2(\mathbb{D})]\} \), is isomorphic to the set of operators acting on \( D_{1/2}^+ \otimes D_{1/2}^+ \). Therefore all invertible operators that transform disk images into disk images can be written in the form \((13)\) and belong to the \( UEA[su(1, 1) \oplus su(1, 1)] \).

In concrete: all transformations of an arbitrary image in whatsoever other image can be realized by means of iterated applications of the operators \((8)\).

## 4 Applications to image processing

In physics a fundamental point of image processing is that every image is the result of a measure and each measure has a measure error. This implies that all numbers in the preceding sections—that in mathematics are unlimited— in physics can be considered always finite, because the limited level of accuracy. All problems related to the rigged Hilbert space are thus irrelevant as, in finite dimensions, rigged Hilbert spaces and Hilbert spaces are equivalent (see \([1, 3]\) and \([4]\)).

Let us start depicting the procedure that, starting from one image \( |f(r, \theta)|^2 \), by means of an operator \( \mathcal{O} \) of the \( UEA \), gives us another image \( |g(r, \theta)|^2 \).

As each image \( |f(r, \theta)|^2 \) does not depend from the phases, it is completely determined by \( |f(r, \theta)| \). Thus we look for the components along \( V_{k,l}(r, \theta) \) of \( |f(r, \theta)| \):

\[
f_{k,l} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^1 dr^2 \ V_{k,l}(r, \theta)^* \ |f(r, \theta)| \tag{14}
\]
where, as \( |f(r, \theta)| \) is real, \( f_{l,k} = (f_{k,l})^* \).

Because of the measure errors, the relevant values of \( k \) and \( l \) are limited to the \( k_M \) and \( l_M \) such that the Parseval identity eq. (5) is satisfied in the approximation appropriate to the accuracy of the experimental result \( |f(r, \theta)|^2 \). We thus have:

\[
|f(r, \theta)| \approx \sum_{k=0}^{k_M} \sum_{l=0}^{l_M} f_{k,l} V_{k,l}(r, \theta). \tag{15}
\]

Let us describe now how a transformation \( \mathcal{O} \) allows to obtain from the image \( |f(r, \theta)|^2 \) a new image \( |g(r, \theta)|^2 \).

The operator \( \mathcal{O} \) will be

\[
\mathcal{O} = \sum_{\bar{\alpha}, \bar{\beta}} \mathcal{O}_{\bar{\alpha}, \bar{\beta}} = \sum_{\bar{\alpha}, \bar{\beta}} c_{\bar{\alpha}, \bar{\beta}} A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3}.
\]

where in physics the sums on \( \bar{\alpha} \) and \( \bar{\beta} \) can be assumed to be finite.

We thus write

\[
\mathcal{O}_{\bar{\alpha}, \bar{\beta}} |f(r, \theta)| = \sum_{k=0}^{k_M} \sum_{l=0}^{l_M} f_{k,l} c_{\bar{\alpha}, \bar{\beta}} A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3} V_{k,l}(r, \theta),
\]

then –by means of iterated applications of operators (8)– we calculate

\[
A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3} V_{k,l}(r, \theta)
\]

finding the coefficients \( g_{k,l} \) that satisfy

\[
A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3} V_{k,l}(r, \theta) = g_{k+\alpha_1-\alpha_3,l+\beta_1-\beta_3} V_{k+\alpha_1-\alpha_3,l+\beta_1-\beta_3}(r, \theta).
\]

Thus we have

\[
\mathcal{O}_{\bar{\alpha}, \bar{\beta}} |f(r, \theta)| = \sum_{k,l=0}^{k_M,l_M} f_{k,l} c_{\bar{\alpha}, \bar{\beta}} g_{k+\alpha_1-\alpha_3,l+\beta_1-\beta_3} V_{k+\alpha_1-\alpha_3,l+\beta_1-\beta_3}(r, \theta),
\]

that, combined with eq. (16), allows to obtain:

\[
g(r, \theta) = \mathcal{O} |f(r, \theta)|
\]

from which \( |g(r, \theta)|^2 \), the transformed image under \( \mathcal{O} \) of \( |f(r, \theta)|^2 \), is obtained.

Analogous procedure can be applied to obtain the operator \( \mathcal{O} \) from \( |g(r, \theta)| \) and \( |f(r, \theta)| \).

To conclude, let us consider now a possible application to improve, by means of AIP, an image obtained by a flawed instrument. We start observing a null signal that, with a
perfect tool, would give $|f(r, \theta)| = |V_{0,0}(r, \theta)|$. A defective instrument, on the contrary, will give a perturbed image $|f(r, \theta)|^2$ such that

$$|f(r, \theta)| = \sum_{k=0}^{k_M} \sum_{l=0}^{l_M} f_{k,l} V_{k,l}(r, \theta) = \sum_{k=0}^{k_M} \sum_{l=0}^{l_M} f_{k,l} \frac{A_k^l B_l^l}{k! l!} V_{0,0}(r, \theta)$$

(17)

where $f_{k,l} (= f^*_{l,k})$ are the parameters, obtained from eq.(14), that characterize the distortion of the null image made by the instrument.

So that the operator that eliminates the defects of the instrument is

$$\left( \sum_{k=0}^{k_M} \sum_{l=0}^{l_M} f_{k,l} \frac{A_k^l B_l^l}{k! l!} \right)^{-1}.$$

If the observation of the object we are interested in gives, with $g_{k,l} = (g_{l,k})^*$,

$$|g(r, \theta)| = \sum_{k=0}^{k_M'} \sum_{l=0}^{l_M'} g_{k,l} V_{k,l}(r, \theta) = \sum_{k=0}^{k_M'} \sum_{l=0}^{l_M'} g_{k,l} \frac{A_k^l B_l^l}{k! l!} V_{0,0}(r, \theta)$$

(18)

the final cleaned image will be

$$\left( \sum_{k=0}^{k_{M'}} \sum_{l=0}^{l_{M'}} g_{k,l} \frac{A_k^l B_l^l}{k! l!} \right) \left( \sum_{k=0}^{k_M} \sum_{l=0}^{l_M} f_{k,l} \frac{A_k^l B_l^l}{k! l!} \right)^{-1} V_{0,0}(r, \theta)^2,$$

(19)

formula that can be easily calculated in series as all operators commute.

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