Matrix Theory in Curved Space

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Abstract

According to the Matrix theory proposal of Banks, Fischler, Shenker and Susskind M-theory in the infinite momentum frame is the large N limit of super Yang-Mills theory in a flat background. To address some physical issues of classical gravity such as gravitational collapse and cosmological expansion we consider an extension of the BFSS proposal by defining M-theory in curved space as the large N limit of super Yang-Mills theory in a curved background. Motivations and possible implications of this extension are discussed.
It was suggested that there is a consistent quantum theory underlying superstring theories and eleven dimensional supergravity called M-Theory \[1, 2\]. According to the Matrix theory proposal of Banks, Fischler, Shenker and Susskind (BFSS) \[3\] M-theory in the infinite momentum frame is the large N limit of the supersymmetric Yang-Mills theory in a flat background. It seems that there are various M-theories depending on the curved classical supergravity background. The most ambitious project would be a derivation of all possible classical backgrounds just from the BFSS Matrix theory, for recent reviews see \[4, 5, 6\].

In this talk we discuss a less ambitious proposal. Instead of the derivation of the classical background from Matrix theory we consider Matrix theory in a given classical background. This proposal was previously considered in \[7\].

Such a consideration seems natural if we want to address some physical issues of classical gravity such as gravitational collapse and cosmological expansion. Scattering of branes with the large impact parameter was successfully considered in Matrix theory in flat space \[3, 8, 9, 10, 11, 12\]. However it is not clear how to deal in Matrix theory in flat space with scattering of 0-brane off black hole with small impact parameter which is a characteristic process for black holes.

We discuss an extension of the BFSS proposal by defining M-theory in curved space as the large N limit of super Yang-Mills theory in a curved background. Another approach to Matrix theory in curved space is considered in \[13, 14, 15\].

It is well known that by using the simple string Lagrangian

\[
S = \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu}
\]

where \(\alpha, \beta = 0, 1, \mu, \nu = 0, ..., D-1\) and \(\eta_{\mu\nu}\) is a flat metric in Minkowski spacetime one can reproduce tree scattering amplitudes in (super)gravity \[16\]. So, in string theory we can compute small corrections to the flat Minkowski background. However this remarkable fact does not mean that we are able to derive a nontrivial curved background such as a black hole from superstring theory in flat spacetime. To compute amplitude for black hole creation we have to start with an appropriate curved background, for a discussion of this point see \[17, 18, 19\]. If we want to deal with string theory in a curved background with nontrivial metric \(g_{\mu\nu}\) we have to use a Lagrangian containing this metric

\[
S = \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\nu g_{\mu\nu}
\]

For the theory to be self consistent, i.e. conformal invariant, the metric \(g_{\mu\nu}\) should be Ricci flat, \(R_{\mu\nu} = 0\).

The Lagrangian \[1\] is very simple but the procedure of derivation of supergravity amplitude is not so simple, it includes string perturbation theory.

Banks, Fischler, Shenker and Susskind \[3\] have suggested that the \(U(N)\)-invariant
Lagrangian

\[ L = \frac{1}{2} tr[\dot{Y}^i \dot{Y}^i] + \frac{1}{2} [Y^i, Y^j]^2 + 2\theta^T \dot{\theta} + 2\theta^T \gamma_i[\theta, Y^i] \] (3)

can be used to describe eleven dimensional supergravity in the infinite momentum frame if one takes the large \( N \) limit. Here \( Y^i \) are Hermitian \( N \times N \) matrices while \( \theta \) is a 16-component fermionic spinor each component of which is an Hermitian \( N \times N \) matrix and \( i, j = 1, ..., 9 \).

Although the Lagrangian (3) looks more complicated than the string Lagrangian (1) but there is a hope that it has an advantage leading to a non-perturbative formulation of quantum gravity. This remarkable proposal does capture the essential degrees of freedom of quantum gravity. It has passed several nontrivial tests including the derivation of the effective action for the long-distance and low-energy scattering.

However this remarkable fact does not mean that we are already able to derive a nontrivial background such as a black hole with its nontrivial topology from the Matrix theory Lagrangian (3). There are skilful constructions of branes in Matrix theory as operator matrices of special form but they are lacking the crucial global property of black hole, i.e. its horizon.

We treat the Matrix theory Lagrangian (3) as an analogue of the String Lagrangian (1) in flat Minkowski spacetime. Indeed the Lagrangian (3) can be regarded as \( U(N) \) supersymmetric Yang-Mills theory in ten-dimensional flat Minkowski spacetime dimensionally reduced to \( (0 + 1) \) space-time dimensions [20, 21, 22]. Therefore if we want to deal with Matrix theory in a nontrivial curved background with metric \( g_{\mu\nu} \) then by analogy with (2) we have to consider instead of (3) another Lagrangian which is obtained from \( U(N) \) supersymmetric Yang-Mills theory in ten-dimensional curved spacetime dimensionally reduced to \( (0 + 1) \) space-time dimensions. The bosonic part of the obtained Lagrangian reads [7]

\[ L = -\frac{1}{2} tr[\dot{Y}^i \dot{Y}^j g^{00} g^{ij} - \frac{1}{2} [Y^i, Y^j][Y^m, Y^n]g^{in}g^{jm}] \] (4)

Here \( g^{\mu\nu} = g^{\mu\nu}(t) \) are functions of time \( t \). The Lagrangian (3) is reduced to the bosonic part of (3) if one takes \( g^{00} = -1 \), \( g^{ij} = \delta^{ij} \).

Notice that we would obtain the dependence on time even if we consider a static metric. This is because one can take the dimensional reduction of the Yang-Mills theory to a geodesic in curved space. In this case functions \( g^{\mu\nu} \) depend from the parameter on the geodesic.

If we want to treat (3) as an extension of the Matrix theory Lagrangian (3) then we should interpret \( g^{\mu\nu}(t) \) not as a spacetime metric but just as functions of parameter \( t \) because spacetime is dynamically generated as a collective mode being constructed from matrices \( Y^i \).
Classical dynamical system (3) has been discussed in [23]. Properties of the dynamical system (4) are different. For example for the Kasner metric

\[ ds^2 = -dt^2 + \sum_i t^{2p_i} dx_i^2 \]  

(5)

for the ansatz \( Y_1 = y_1\sigma_1, Y_2 = y_1\sigma_2 \) one has equations

\[ \frac{d}{dt}(t^{-2p_1+1}y_1) + t^{-2p_1-2p_2+1}y_1y_2^2 = 0, \quad \frac{d}{dt}(t^{-2p_2+1}y_2) + t^{-2p_1-2p_2+1}y_2y_1^2 = 0 \]  

(6)

The low energy effective theory of D-branes in Minkowski spacetime is given by the dimensional reduction of the supersymmetric gauge theory in ten dimensional Minkowski spacetime [24]. If one has D-branes in flat Minkowski spacetime but in curved (non-Cartesian) coordinates we have to start from the supersymmetric Yang-Mills theory in curved coordinates. Then we get a version of the M(atrix) theory Lagrangian (4) in the curved coordinates. If one has D-branes in a curved spacetime, for instance D-branes in the presence of black hole then it is natural to expect that the low energy effective theory will be given by the dimensional reduction of the supersymmetric gauge theory coupled with supergravity in the ten dimensional curved spacetime.

Let us consider the supersymmetric Yang-Mills theory in the \( D \)-dimensional space-time with metric \( g_{MN} \). The action is

\[ I = \int d^Dx \sqrt{g} tr \{-\frac{1}{4} F_{MN} F_{PQ} g^{MP} g^{NQ} - \frac{1}{2} \bar{\theta} \Gamma^M D_M \theta \} \]  

(7)

where \( F_{MN} = i[D_M, D_N], \ D_M = \nabla_M - iA_M \). Let \( \gamma : x^M = x^M(\sigma), \ \sigma = (\sigma_0, ..., \sigma_p) \) be a \( p+1 \)-dimensional submanifold and let us consider the dimensional reduction to \( \gamma \). One has \( A_M = (A_{\alpha}, Y_i), \alpha = 0, ..., p; \ i = p+1, ..., D-1 \) and \( F_{MN} = (F_{\alpha\beta}, F_{\alpha i}, F_{ij}), \ g^{MN} = (g^{\alpha\beta}, g^{\alpha i}, g^{ij}) \). The bosonic Lagrangian is

\[ L = -\frac{1}{2} tr[D_\alpha Y_i D_\beta Y_j g^{\alpha\beta} g^{ij} - \frac{1}{2} [Y_i, Y_j][Y_m, Y_n] g^{im} g^{jn} + ...] \]  

(8)

Here \( Y_i = Y_i(x^P(\sigma)), \ g_{MN} = g_{MN}(x^P(\sigma)) \). For a 0-brane \( x^M = x^M(\tau) \) in the gauge \( A_0 = 0, \ g^{0i} = 0 \) one gets the Lagrangian (8) describing the bosonic part of M(atrix) theory in curved space. If one takes \( p = 1 \) then the Lagrangian (8) describes the matrix string [32] in curved background.

The corresponding Hamiltonian is

\[ H = \frac{1}{2} tr[P^i P^j g_{ij} - \frac{1}{2} [Y_i, Y_j][Y_m, Y_n] g^{im} g^{jn}] \]  

(9)
One deals with quantum mechanics in the dependent on time background \( g^{ij}(\tau) = g^{ij}(x(\tau)) \). Now the properties of the matrix quantum mechanics depend on the choice of the curve \( x(\tau) \). The one-loop effective action for the theory (4) with the metric \( g_{MN} \) can be computed using the background field method by the standard procedure. One gets corrections to the phase shift \( \delta \) obtained in [3, 8]. If one takes geodesics near the singularity then generically one gets the creation of particles (D0-branes) and there is back reaction of the gas of D-branes to the metric which generically is described by the equations:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = < T_{\mu\nu} >
\]

where \( T_{\mu\nu} \) is the energy-momentum tensor of D-branes. One hopes that one can study the singular regime [27] by using the Matrix theory framework [28, 27, 28].

Matrix theory Lagrangian (3) is known to be closely related to the matrix regularization of supermembrane [29, 30]. Kappa-symmetry of the supermembrane in the curved background requires the background be a solution of eleven dimensional supergravity [31]. We have considered here the metric \( g_{\mu\nu} \) in Matrix theory and in matrix string theory as an arbitrary phenomenological background. This is different from the compactification prescription discussed in [33, 34]. Perhaps the metric should be fixed to admit the large N limit. One expects here a relation with recent considerations in [35, 36, 37].

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