Differential Sandwich-Type Results for Symmetric Functions Connected with a $Q$-Analog Integral Operator

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Abstract: In this paper, we obtain some applications of the theory of differential subordination, differential superordination, and sandwich-type results for some subclasses of symmetric functions connected with a $q$-analog integral operator.

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1. Introduction

The theory of $q$-analysis has an important role in many areas of mathematics and physics. Jackson [1,2] was the first that gave some application of $q$-calculus and introduced the $q$-analog of derivative and integral operator (see also [3]). Let $\mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{H}[a,m]$ denote the subclass of functions $f \in \mathcal{H}(U)$ of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \ldots, \quad z \in U,$$

with $a \in \mathbb{C}$ and $m \in \mathbb{N} := \{1,2,\ldots\}$.

In addition, let $\mathcal{A}(m)$ denote the subclass of functions $f \in \mathcal{H}(U)$ of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \quad z \in U, \quad (1)$$

with $m \in \mathbb{N}$, and let $\mathcal{A} := \mathcal{A}(1)$.

We define the integral operator $K_{n,m}^\alpha : \mathcal{A}(m) \to \mathcal{A}(m)$, with $\alpha > 0$ and $n \geq 0$, as follows:

$$K_{n,m}^0 f := f,$$

and

$$K_{n,m}^\alpha f(z) := \frac{(n+1)\alpha}{\Gamma(\alpha)z^n} \int_0^z t^{n-1} \left(\log^\alpha \frac{z}{t}\right)^{\alpha-1} f(t) dt,$$
where all the powers are the principal ones, and log 1 = 0.

If \( f \in \mathcal{A}(m) \) has the power expansion of the form in Equation (1), it can be easily verified that
\[
\mathcal{K}^\alpha_{n,m} f(z) = z + \sum_{k=m+1}^{\infty} \left( \frac{n+1}{n+k} \right)^a a_k z^k, \quad z \in \mathbb{U}.
\]

For \( 0 < q < 1 \), the \( q \)-derivative of the operator \( \mathcal{K}^\alpha_{n,m} \) is defined by
\[
\partial_q \mathcal{K}^\alpha_{n,m} f(z) := \frac{\mathcal{K}^\alpha_{n,m} f(qz) - \mathcal{K}^\alpha_{n,m} f(z)}{z(q-1)}, \quad z \in \mathbb{U},
\]
that is
\[
\partial_q \left[ z + \sum_{k=m+1}^{\infty} \left( \frac{n+1}{n+k} \right)^a a_k z^k \right] = 1 + \sum_{k=m+1}^{\infty} \left( \frac{n+1}{n+k} \right)^a [k,q] a_k z^{k-1}, \quad z \in \mathbb{U}, \tag{2}
\]
where
\[
[k,q] = \frac{1-q^k}{1-q} = 1 + \sum_{i=1}^{k-1} q^i, \quad [0,q] = 0,
\]
It is easily to verify from Equation (2) that
\[
z \partial_q \mathcal{K}^\alpha_{n,m} f(z) = z + \sum_{k=m+1}^{\infty} \left( \frac{n+1}{n+k} \right)^a [k,q] a_k z^k, \quad z \in \mathbb{U}.
\]

For any non negative integer \( k \), the \( q \)-number shift factorial is given by
\[
[k,q]! = \begin{cases} 1, & \text{if } k = 0, \\ [1,q][2,q][3,q] \ldots [k,q], & \text{if } k \in \mathbb{N}, \end{cases}
\]
while the \( q \)-generalized Pochhammer symbol for \( r > 0 \) is defined by
\[
[r,q]_k = \begin{cases} 1, & \text{if } k = 0, \\ [r,q][r+1,q] \ldots [r+k-1,q], & \text{if } k \in \mathbb{N}. \end{cases}
\]

For \( \lambda > -1 \), we define the operator \( \mathcal{N}^{\lambda,a}_{n,m,q} : \mathcal{A}(m) \to \mathcal{A}(m) \) by
\[
\mathcal{N}^{\lambda,a}_{n,m,q} f(z) + \mathcal{M}_{q,\lambda+1}(z) = z \partial_q \mathcal{K}^\alpha_{n,m} f(z),
\]
where
\[
\mathcal{M}_{q,\lambda+1}(z) := z + \sum_{k=m+1}^{\infty} \frac{[\lambda + 1,q][k-1,q-1]}{[k-1,q]!} a_k z^k, \quad z \in \mathbb{U}.
\]

From the above definition, we obtain
\[
\mathcal{N}^{\lambda,a}_{n,m,q} f(z) = z + \sum_{k=m+1}^{\infty} \left( \frac{n+1}{n+k} \right)^a \frac{[k,q][k-1,q]!}{[\lambda + 1,q]_{k-1}} a_k z^k
\]
\[
= z + \sum_{k=m+1}^{\infty} \frac{[k,q]!}{[\lambda + 1,q]_{k-1}} \left( \frac{n+1}{n+k} \right)^a a_k z^k, \quad z \in \mathbb{U}, \tag{3}
\]
\((a > 0, \lambda > -1, m \geq 0, 0 < q < 1)\)
and from Equation (3) we can easily verify that
\[
[q + \lambda, q] \mathcal{N}^{\lambda,a}_{n,m,q} f(z) = [\lambda, q] \mathcal{N}^{\lambda+1,a}_{n,m,q} f(z) + q^\lambda z \partial_q \mathcal{N}^{\lambda+1,a}_{n,m,q} f(z), \quad z \in \mathbb{U}.
We note that
\[
\lim_{q \to 1} \lambda_{n,m}\alpha_n^\alpha f(z) = L_{n,m}\alpha_n^\alpha f(z) = z + \sum_{k=m+1}^{\infty} \frac{k!}{(\lambda + 1)_{k-1}} \frac{(n + 1 + \lambda)}{n + k} a_k z^k, \quad z \in \mathbb{U}.
\] (4)

**Definition 1.** For \( f, g \in \mathcal{H}(\mathbb{U}) \), we say that \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \), if there exists a Schwarz function \( w \), which is analytic in \( \mathbb{U} \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in \mathbb{U} \), such that \( f(z) = g(w(z)), \ z \in \mathbb{U} \). Furthermore, if the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence (see [4,5]):
\[
f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

Let \( k, h \in \mathcal{H}(\mathbb{U}) \), and let \( \varphi(r,s;z) : \mathbb{C} \times \mathbb{U} \to \mathbb{C} \).

(i) If \( k \) satisfies the first order differential subordination
\[
\varphi(k(z), z^k(z); z) \prec h(z),
\] (5)
then \( k \) is said to be a solution of the differential subordination in Equation (5). The function \( q \) is called a dominant of the solutions of the differential subordination in Equation (5) if \( k(z) \prec q(z) \) for all the functions \( k \) satisfying Equation (5). A dominant \( \tilde{q} \) is said to be the best dominant of Equation (5) if \( \tilde{q}(z) \prec q(z) \) for all the dominants \( q \).

(ii) If \( k \) satisfies the first order differential superordination
\[
h(z) \prec \varphi(k(z), z^k(z); z),
\] (6)
then \( k \) is called to be a solution of the differential superordination in Equation (6). The function \( q \) is called a subordinant of the solutions of the differential superordination in Equation (6) if \( q(z) \prec k(z) \) for all the functions \( k \) satisfying Equation (6). A subordinant \( \tilde{q} \) is said to be the best subordinant of Equation (6) if \( q(z) \prec \tilde{q}(z) \) for all the subordinants \( q \).

Miller and Mocanu [6] obtained conditions on the functions \( h, q \) and \( \varphi \) for which the following implication holds:

\[
h(z) \prec \varphi(k(z), z^k(z); z) \Rightarrow q(z) \prec k(z).
\]

Applying these methods, in [7,8], the author studied general classes of first order differential superordinations and superordination-preserving integral operators. Using the results of Bulboacă [4] (see also [9,10]), the authors of [11] obtained sufficient conditions for functions \( f \in \mathcal{A} \) to satisfy the double subordination
\[
q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),
\]
where \( q_1 \) and \( q_2 \) are univalent functions in \( \mathbb{U} \), normalized with \( q_1(0) = q_2(0) = 1 \).

Sakaguchi [12] introduced a class \( S^*_n \) of functions starlike with respect to symmetric points, which consists of functions \( f \in \mathcal{A} \) satisfying the inequality
\[
\Re \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in \mathbb{U},
\]
that represents a subclass of close-to-convex functions, and hence univalent in \( \mathbb{U} \). Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin (see [12,13]).

In addition, Aouf et al. [14] introduced and studied the class \( S^*_n T(1,1) \) of functions \( n \)-starlike with respect to symmetric points, which consists of functions \( f \in \mathcal{A} \), with \( a_k \leq 0 \) for \( k \geq 2 \), and satisfying the inequality
\[
\Re \frac{D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} > 0, \quad z \in \mathbb{U},
\]
where $D^n$ is the 
\textit{Sandwich operator} \cite{15}.

The classes defined in \cite{12,13} could be generalized by introducing the next class of functions, defined with the aid of the $N^\lambda_{n,m,q}$ operator defined as follows:

**Definition 2.** A function $f \in \mathcal{A}(m)$ with

$$N^\lambda_{n,m,q}f(z) - N^\lambda_{n,m,q}f(-z) \neq 0, \quad z \in \mathbb{U} \setminus \{0\}, \tag{7}$$

is said to be in the class $\mathcal{M}^\lambda_{n,m,q}(\gamma, \mu, A, B)$ if it satisfies the subordination condition

$$\left(1 + \gamma \right) \left( \frac{2z}{\mathcal{N}^\lambda_{n,m,q}(f(z)) - \mathcal{N}^\lambda_{n,m,q}(f(-z))} \right)^\mu \left(1 + \frac{Az}{1 + Bz}\right), \tag{8}$$

where $\gamma \in \mathbb{C}$, $0 < \mu < 1$, $-1 < B < A \leq 1$, $m \in \mathbb{N}$, $\alpha > 0$, $n \geq 0$, $0 < q < 1$, $\lambda > -1$.

By specializing the parameters $a$, $\lambda$ and $q$, we obtain the following subclasses:

(i) For $q \to 1^-$, we define the class $\mathcal{W}^\lambda_{n,m}(\gamma, \mu, A, B)$ as follows:

$$\mathcal{W}^\lambda_{n,m}(\gamma, \mu, A, B) := \left\{ f \in \mathcal{A}(m) : (1 + \gamma) \left( \frac{2z}{I^\lambda_{n,m}(f(z)) - I^\lambda_{n,m}(f(-z))} \right)^\mu \left(1 + \frac{Az}{1 + Bz}\right) \right\},$$

where the operator $I^\lambda_{n,m}$ is defined by Equation (4);

(ii) For $q \to 1^-$, $a = 0$ and $\lambda = 1$, we define the class $\mathcal{N}^\gamma_{\gamma,0}(m, A, B)$ that corrects the class defined by Muhammad and Marwan \cite{16} as follows:

$$\mathcal{N}^\gamma_{\gamma,0}(m, A, B) := \left\{ f \in \mathcal{A}(m) : (1 + \gamma) \left( \frac{2z}{f(z) - f(-z)} \right)^\mu \left(1 + \frac{Az}{1 + Bz}\right) \right\}.$$

In this paper, we obtain some sharp differential subordination and superordination results for the functions belonging to the class $\mathcal{M}^\lambda_{n,m,q}(\gamma, \mu, A, B)$ to try to make a connection between a special subclass of analytic functions whose coefficients are given by the $q$-analog of integral operator and the differential subordination theory.

2. Preliminaries

To prove our results, we need the following definition and lemmas.

**Definition 3** \cite{15}. (Definition 2.2b., p. 21) Let $Q$ be the set of all functions $f$ that are analytic and injective on $\mathbb{U} \setminus E(f)$, where $E(f) := \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$.

**Lemma 1** \cite{15}. (Theorem 3.1b., p. 71) Let the function $H$ be convex in $\mathbb{U}$, with $H(0) = a$, and $\zeta \neq 0$ with $\text{Re} \zeta \geq 0$. If $\Phi \in \mathcal{H}[a, m]$ and

$$\Phi(z) + \frac{2\Phi'(z)}{\zeta} < H(z), \tag{9}$$
Theorem 1. If \( f \) is univalent in \( U \), then \( \Psi \in \mathcal{H}[a,m] \), and is the best dominant of Equation (9).

**Lemma 2** ([17]). (Lemma 2.2., p. 3) Let \( q \) be univalent in \( U \), with \( q(0) = 1 \). Let \( \xi, \varphi \in \mathbb{C} \) with \( \varphi \neq 0 \), and assume that

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0; -\frac{\xi}{\varphi} \right\}, z \in U.
\]

If \( k \) is analytic in \( U \) and

\[
\xi k(z) + \varphi k'(z) \prec \xi q(z) + \varphi q'(z),
\]

then \( k(z) \prec q(z) \), and \( q \) is the best dominant of Equation (10).

From [6] (Theorem 6, p. 820), we could easily obtain the following lemma:

**Lemma 3.** Let \( q \) be convex in \( U \), and \( k \neq 0 \) with \( \text{Re} k \geq 0 \). If \( g \in \mathcal{H}[q(0), 1] \cap \mathcal{Q} \), such that \( g(z) + kzg'(z) \) is univalent in \( U \), then

\[
q(z) + kq'(z) \prec g(z) + kzg'(z).
\]

implies that \( q(z) \prec g(z) \), and \( q \) is the best subdominant of Equation (11).

**Lemma 4** ([18]). Let \( F \) be analytic and convex in \( U \), and \( 0 \leq \lambda \leq 1 \). If \( f, g \in A \), such that \( f(z) \prec F(z) \) and \( g(z) \prec F(z) \), then

\[
\lambda f(z) + (1 - \lambda)g(z) \prec F(z).
\]

3. Main Results

Unless otherwise mentioned, we assume in the remainder of this paper that \( \gamma \in \mathbb{C}, 0 < \mu < 1, -1 \leq B < A \leq 1, m \in \mathbb{N}, \alpha > 0, n \geq 0, 0 < q < 1, \lambda > -1 \), and all the powers are understood as principle values.

**Theorem 1.** If \( f \in \mathcal{M}_{\alpha}^{\lambda, \alpha}(\gamma, \mu, A, B) \) and \( \gamma \in \mathbb{C}^*: = \mathbb{C} \setminus \{0\} \) with \( \text{Re} \gamma \geq 0 \), then

\[
\left( \frac{2z}{N_{\alpha}^{\lambda, \alpha}(m, A, B)(z) - N_{\alpha}^{\lambda, \alpha}(m, A, B)(-z)} \right)^{\mu} \prec \Psi(z) := \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 + Azu}{1 + Bzu} u^{\gamma - 1} du \prec \frac{1 + Az}{1 + Bz}
\]

and \( \Psi \) is convex, \( \Psi \in \mathcal{H}[1, m] \), and is the best dominant.

**Proof.** If we define the function \( h \) by

\[
h(z) := \left( \frac{2z}{N_{\alpha}^{\lambda, \alpha}(m, A, B)(z) - N_{\alpha}^{\lambda, \alpha}(m, A, B)(-z)} \right)^{\mu}, z \in U,
\]

from Equation (7), it follows that \( h \) is an analytic function in \( U \), with \( h(0) = 1 \). Differentiating Equation (12) with respect to \( z \), we obtain that
with unknowns \( \gamma \) where we have hence and equating the corresponding coefficients it follows that

\[
(1 + \gamma) \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu \\
- \gamma \left( \frac{z \left( N_{n,m,q}^\lambda f(z) \right) - z \left( N_{n,m,q}^\lambda f(-z) \right)}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu 
= h(z) + \frac{2}{\beta} z^\mu'(z) < \frac{1 + \alpha \zeta}{1 + \beta \zeta}.
\]

Since

\[
N_{n,m,q}^\lambda f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \quad \text{and} \quad N_{n,m,q}^\lambda f(-z) = -z + \sum_{k=m+1}^{\infty} a_k (-1)^k z^k,
\]

where

\[
a_k = \frac{[k,q]!}{[\lambda + 1,q]_{k-1}} \left( \frac{n + 1}{n + k} \right)^a a_k, \quad k \geq m + 1,
\]

we have

\[
U(z) := \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} = \frac{2z}{2z + \sum_{k=m+1}^{\infty} a_k \left[1 + (-1)^{k+1}\right] z^k} = \frac{1}{1 + \sum_{s=m}^{\infty} \beta_s z^s},
\]

with

\[
\beta_s = \frac{a_{s+1} \left[1 + (-1)^s\right]}{2}, \quad s \geq m.
\]

Moreover,

\[
U(z) = \frac{1}{1 + \sum_{s=m}^{\infty} \beta_s z^s} = 1 + \sum_{j=1}^{\infty} \gamma_j z^j, \quad z \in \mathbb{U},
\]

with unknowns \( \gamma_j, j \geq 1 \), we have

\[
1 = \left(1 + \beta_m z^m + \beta_{m+1} z^{m+1} + \ldots\right) \left(1 + \gamma_1 z + \gamma_2 z^2 + \ldots + \gamma_m z^m + \gamma_{m+1} z^{m+1} + \ldots\right),
\]

and equating the corresponding coefficients it follows that

\[
\gamma_1 = \gamma_2 = \ldots = \gamma_{m-1} = 0, \quad \gamma_m = -\beta_m, \quad \gamma_{m+1} = -\beta_{m+1}, \ldots,
\]

hence

\[
U(z) = 1 + \sum_{j=m}^{\infty} \gamma_j z^j \in \mathbb{H}[1,m].
\]

According to Equation (12), we have

\[
h = U^\mu, \quad \text{with} \quad U \in \mathbb{H}[1,m],
\]

and using the binomial power expansion formula, we get

\[
h = U^\mu \in \mathbb{H}[1,m].
\]

Now, from the subordination in Equation (13), using Lemma 1 for \( \zeta = \frac{\mu}{\gamma} \), we obtain our result. \( \square \)

Taking \( q \to 1^- \) in Theorem 1, we obtain the following corollary:
Corollary 1. If \( f \in \mathcal{W}_{n,m}^{\lambda,\alpha}(\gamma, \mu, A, B) \) and \( \gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) with \( \text{Re} \gamma \geq 0 \), then
\[
\left( \frac{2z}{I_{n,m}^{\lambda,\alpha}f(z) - I_{n,m}^{\lambda,\alpha}f(-z)} \right)^\mu < \Psi(z) := \frac{\mu}{\gamma m} \int_0^{\frac{1+Az}{1+Bz}} \frac{1}{u^m} \frac{1}{1+Bz} \, du < \frac{1+Az}{1+Bz},
\]
and \( \Psi \) is convex, \( \Psi \in \mathcal{H}[1,m] \), and is the best dominant.

Remark 1. The above theorem shows that
\[
\mathcal{M}_{n,m,d}^{\lambda,\alpha}(\gamma, \mu, A, B) \subset \mathcal{M}_{n,m,d}^{\lambda,\alpha}(0, \mu, A, B),
\]
for all \( \gamma \in \mathbb{C} \) with \( \text{Re} \gamma \geq 0 \).

Moreover, the next inclusion result for the classes \( \mathcal{M}_{n,m,d}^{\lambda,\alpha}(\gamma, \mu, A, B) \) holds:

Theorem 2. If \( \gamma_1, \gamma_2 \in \mathbb{R} \) such that \( 0 \leq \gamma_1 \leq \gamma_2 \), and \( -1 \leq B_1 \leq B_2 \), then \( A_2 \leq A_1 \leq 1 \), then
\[
\mathcal{M}_{n,m,d}^{\lambda,\alpha}(\gamma_2, \mu, A_2, B_2) \subset \mathcal{M}_{n,m,d}^{\lambda,\alpha}(\gamma_1, \mu, A_1, B_1).
\]

Proof. If \( f \in \mathcal{M}_{n,m,d}^{\lambda,\alpha}(\gamma_2, \mu, A_2, B_2) \), since \(-1 \leq B_1 \leq B_2 \), \( A_2 \leq A_1 \leq 1 \), it is easy to check that
\[
(1 + \gamma_2) \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu - \gamma_2 \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu < \frac{1+A_2z}{1+B_2z},
\]
that is \( f \in \mathcal{M}_{n,m,d}^{\lambda,\alpha}(\gamma_1, \mu, A_1, B_1) \), hence the assertion in Equation (14) holds for \( \gamma_1 = \gamma_2 \).

If \( 0 \leq \gamma_1 < \gamma_2 \), from Remark 1 and Equation (15), it follows \( f \in \mathcal{M}_{n,m,d}^{\lambda,\alpha}(0, \mu, A_1, B_1) \), that is
\[
\left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu < \frac{1+A_1z}{1+B_1z}.
\]
A simple computation shows that
\[
(1 + \gamma_1) \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu - \gamma_1 \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu = \left( 1 - \frac{\gamma_1}{\gamma_2} \right) \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu + \frac{\gamma_1}{\gamma_2} \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu - \gamma_2 \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^\mu, \quad z \in \mathbb{U}.
\]
Moreover,
\[
0 \leq \frac{\gamma_1}{\gamma_2} < 1,
\]
and the function \( \frac{1 + A_1 z}{1 + B_1 z} \) with \(-1 \leq B_1 < A_1 \leq 1\) is analytic and convex in \(U\). According to Equation (17), using the subordinations in Equations (15) and (16), from Lemma 4, we deduce that

\[
(1 + \gamma_1) \left( \frac{2z}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu \tag{18}
\]

\[
-\gamma_1 \left( \frac{z \left( \mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z) \right)^\prime}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu \left( \frac{2z}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu < 1 + A_1 z
\]

that is \( f \in \mathcal{M}_{n,m,q}^\alpha (\gamma_1, \mu, A_1, B_1) \). \( \square \)

Taking \( q \to 1^- \) in Theorem 2, we obtain the following corollary:

**Corollary 2.** If \( \gamma_1, \gamma_2 \in \mathbb{R} \) such that \( 0 \leq \gamma_1 \leq \gamma_2 \) and \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1\), then

\[
\mathcal{W}_{n,m,q}^\alpha (\gamma_2, \mu, A_2, B_2) \subset \mathcal{W}_{n,m,q}^\alpha (\gamma_1, \mu, A_1, B_1).
\]

**Example 1.** For the special case \( A_1 = 1 \) and \( B_1 = -1 \), Theorem 2 and Corollary 2 reduce to the next examples, respectively:

Suppose that \( \gamma_1, \gamma_2 \in \mathbb{R} \) such that \( 0 \leq \gamma_1 \leq \gamma_2 \) and \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \).

1. If \( f \in \mathcal{M}_{n,m,q}^\alpha (\gamma_2, \mu, A_2, B_2) \), then

\[
\text{Re} \left\{ (1 + \gamma_1) \left( \frac{2z}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu \right. \text{Re} \left( \frac{z \left( \mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z) \right)^\prime}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu > 0, \; z \in U; \right. \tag{19}
\]

2. If \( f \in \mathcal{W}_{n,m,q}^\alpha (\gamma_2, \mu, A_2, B_2) \), then

\[
\text{Re} \left\{ (1 + \gamma_1) \left( \frac{2z}{\mathcal{I}_{n,m,q}^\alpha f(z) - \mathcal{I}_{n,m,q}^\alpha f(-z)} \right)^\mu \right. \text{Re} \left( \frac{z \left( \mathcal{I}_{n,m,q}^\alpha f(z) - \mathcal{I}_{n,m,q}^\alpha f(-z) \right)^\prime}{\mathcal{I}_{n,m,q}^\alpha f(z) - \mathcal{I}_{n,m,q}^\alpha f(-z)} \right)^\mu > 0, \; z \in U; \right. \tag{20}
\]

**Theorem 3.** Suppose that \( q \) is univalent in \( U \), with \( q(0) = 1 \), and let \( \gamma \in \mathbb{C}^* \) such that

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0; -\text{Re} \frac{\mu}{\gamma} \right\}, \; z \in U. \tag{21}
\]

If \( f \in \mathcal{A}(m) \) such that Equation (7) holds, and satisfies the subordination

\[
(1 + \gamma) \left( \frac{2z}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu \tag{22}
\]

\[
-\gamma \left( \frac{z \left( \mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z) \right)^\prime}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu \left( \frac{2z}{\mathcal{A}_{n,m,q}^\alpha f(z) - \mathcal{A}_{n,m,q}^\alpha f(-z)} \right)^\mu < q(z) + \frac{1}{\mu} z q'(z), \quad \tag{23}
\]
whenever we easily check that Equation (22) holds if and only if the assumption in Equation (20) is satisfied, and $q$ is the best dominant of Equation (19).

**Proof.** Since $f \in A(m)$ such that Equation (7) holds, it follows that the function $h$ defined by Equation (12) is analytic in $U$, and $h(0) = 1$. As in the proof of Theorem 1, differentiating Equation (12) with respect to $z$, we obtain that Equation (19) is equivalent to

$$h(z) + \frac{\gamma}{\mu} z h'(z) < q(z) + \frac{\gamma}{\mu} q'(z).$$

Using Lemma 2 for $\zeta := 1$ and $\varphi := \frac{\gamma}{\mu}$, we get that the above subordination implies $h(z) < q(z)$, and $q$ is the best dominant of Equation (19). \qed

For the special case $q(z) = \frac{1 + Az}{1 + Bz}$, with $-1 \leq B < A \leq 1$, Theorem 3 reduces to the following corollary:

**Corollary 3.** Let $\gamma \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$, such that

$$\max \left\{ -1; \frac{1 + \Re \frac{\mu}{\gamma}}{1 - \Re \frac{\mu}{\gamma}} \right\} \leq B \leq 0, \quad \text{or} \quad 0 \leq B \leq \min \left\{ 1; \frac{1 + \Re \frac{\mu}{\gamma}}{1 - \Re \frac{\mu}{\gamma}} \right\}. \quad (20)$$

If $f \in A(m)$ such that Equation (7) holds, and satisfies the subordination

$$\left( 1 + \gamma \right) \left( \frac{2z}{N_{n,m,q}^1(z) - N_{n,m,q}^1(-z)} \right)^\mu = \frac{1 + Az}{1 + Bz},$$

then

$$\left( \frac{2z}{N_{n,m,q}^1(z) - N_{n,m,q}^1(-z)} \right)^\mu < \frac{1 + Az}{1 + Bz},$$

and $\frac{1 + Az}{1 + Bz}$ is the best dominant of Equation (21).

**Proof.** For $q(z) = \frac{1 + Az}{1 + Bz}$, the condition in Equation (18) reduces to

$$\Re \frac{1 - Bz}{1 + Bz} > \max \left\{ 0; - \Re \frac{\mu}{\gamma} \right\}, \quad z \in U. \quad (22)$$

Since

$$\inf \left\{ \Re \frac{1 - Bz}{1 + Bz} : z \in U \right\} = \begin{cases} \frac{1 + B}{1 - B}, & \text{if} \quad -1 \leq B \leq 0, \\ \frac{1 - B}{1 + B}, & \text{if} \quad 0 \leq B < 1, \end{cases}$$

we easily check that Equation (22) holds if and only if the assumption in Equation (20) is satisfied, whenever $-1 \leq B < 1$. \qed

Taking $q \to 1^{-}$ in Theorem 3, we obtain the following corollary:
Corollary 4. Suppose that \( q \) is univalent in \( \mathbb{U} \), with \( q(0) = 1 \), and let \( \gamma \in \mathbb{C}^* \) such that
\[
\text{Re} \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \max \left\{ 0; \text{Re} \frac{\mu}{\gamma} \right\}, \quad z \in \mathbb{U}.
\]

If \( f \in \mathcal{A}(m) \) such that Equation (7) holds, and satisfies the subordination
\[
(1 + \gamma) \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu 
- \gamma \left( \frac{z(\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z))}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right) \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu < q(z) + \frac{\gamma}{\mu} z q'(z),
\]
then
\[
\left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu < q(z),
\]
and \( q \) is the best dominant of Equation (19).

Theorem 4. Let \( q \) be convex in \( \mathbb{U} \), with \( q(0) = 1 \), and \( \gamma \in \mathbb{C}^* \), with \( \text{Re} \gamma \geq 0 \). In addition, let \( f \in \mathcal{A}(m) \) such that
\[
\left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu \in \mathcal{H}[q(0),1] \cap \mathcal{Q}, \tag{23}
\]
and assume that the function
\[
(1 + \gamma) \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu 
- \gamma \left( \frac{z(\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z))}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right) \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu,
\]
is univalent in \( \mathbb{U} \).

If
\[
q(z) + \frac{\gamma}{\mu} z q'(z) < (1 + \gamma) \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu,
\]
\[
- \gamma \left( \frac{z(\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z))}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right) \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu,
\]
then
\[
q(z) \prec \left( \frac{2z}{\mathcal{N}_{n,m} f(z) - \mathcal{N}_{n,m} f(-z)} \right)^\mu,
\]
and \( q \) is the best subordinant of Equation (25).

Proof. Letting the function \( h \) be defined by Equation (12), then \( h \in \mathcal{H}[q(0),m] \), and from Equation (23) we have that \( h \in \mathcal{H}[q(0),1] \cap \mathcal{Q} \). As in the proof of Theorem 1, differentiating Equation (12) with respect to \( z \), we obtain that
\[
q(z) + \frac{\gamma}{\mu} z q'(z) < h(z) + \frac{\gamma}{\mu} z h'(z).
\]
Now, according to Lemma 3 for \( k := \frac{\gamma}{\mu} \) we obtain the desired result. \( \square \)

Taking \( q(z) = \frac{1 + A z}{1 + B z} \), with \(-1 \leq B < A \leq 1\), in Theorem 4, we obtain the following corollary:
Corollary 5. Let \( \gamma \in \mathbb{C}^* \), with \( \text{Re}\gamma \geq 0 \), and \(-1 \leq B < A \leq 1\). If \( f \in \mathcal{A}(m) \) such that the assumptions in Equations (23) and (24) hold, and satisfies the subordination

\[
\frac{1 + Az}{1 + Bz} + \frac{\gamma (A - B)z}{\mu (1 + Bz)^{\mu}} < (1 + \gamma) \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^{\mu} 
\]

\[
-\gamma \left( \frac{z \left( N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z) \right)}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right) \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^{\mu},
\]

then

\[
\frac{1 + Az}{1 + Bz} < \left( \frac{2z}{N_{n,m,d}^{\lambda,\alpha}f(z) - N_{n,m,d}^{\lambda,\alpha}f(-z)} \right)^{\mu},
\]

and \( \frac{1 + Az}{1 + Bz} \) is the best subordinant of Equation (26).

Taking \( q \to 1^- \) in Theorem 4, we obtain the following corollary:

Corollary 6. Let \( q \) be convex in \( \mathbb{U} \), with \( q(0) = 1 \), and \( \gamma \in \mathbb{C}^* \), with \( \text{Re}\gamma \geq 0 \). In addition, let \( f \in \mathcal{A}(m) \) such that

\[
\left( \frac{2z}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right)^{\mu} \in \mathcal{H}[q(0),1] \cap \mathcal{Q},
\]

and assume that the function

\[
(1 + \gamma) \left( \frac{2z}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right)^{\mu}
\]

\[
-\gamma \left( \frac{z \left( T_{n,m,f}(z) - T_{n,m,f}(-z) \right)}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right) \left( \frac{2z}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right)^{\mu}
\]

is univalent in \( \mathbb{U} \).

If

\[
q(z) + \frac{\gamma}{\mu} zq'(z) < (1 + \gamma) \left( \frac{2z}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right)^{\mu}
\]

\[
-\gamma \left( \frac{z \left( T_{n,m,f}(z) - T_{n,m,f}(-z) \right)}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right) \left( \frac{2z}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right)^{\mu},
\]

then

\[
q(z) \left( \frac{2z}{T_{n,m,f}(z) - T_{n,m,f}(-z)} \right)^{\mu},
\]

and \( q \) is the best subordinant of Equation (25).

Combining Theorems 3 and 4, we obtain the following sandwich-type theorem:
**Theorem 5.** Let $q_1$ and $q_2$ be two convex functions in $\mathbb{U}$, with $q_1(0) = q_2(0) = 1$, and let $\gamma \in \mathbb{C}^*$, with $\text{Re} \gamma \geq 0$. If $f \in \mathbb{A}(m)$ such that the assumptions in Equations (23) and (24) hold, then

$$
q_1(z) + \frac{2}{\mu} \frac{q_1'(z)}{q_1(z)} \prec \Theta(z) := (1 + \gamma) \left( \frac{2z}{\hat{N}_{n,m,q}^{\lambda,\alpha}(z) - \hat{N}_{n,m,q}^{\lambda,\alpha}(-z)} \right) \mu
$$

implies that

$$
q_1(z) \prec \Phi(z) := \left( \frac{2z}{\hat{N}_{n,m,q}^{\lambda,\alpha}(z) - \hat{N}_{n,m,q}^{\lambda,\alpha}(-z)} \right)^\mu < q_2(z),
$$

and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant of Equation (27).

Combining Corollaries 4 and 6, we obtain the following sandwich-type theorem:

**Corollary 7.** Let $q_1$ and $q_2$ be two convex functions in $\mathbb{U}$, with $q_1(0) = q_2(0) = 1$, and let $\gamma \in \mathbb{C}^*$, with $\text{Re} \gamma \geq 0$. If $f \in \mathbb{A}(m)$ such that the assumptions in Equations (23) and (24) hold for the operator $\hat{N}_{n,m,q}^{\lambda,\alpha}$ replaced by $\hat{T}_{n,m,q}^{\lambda,\alpha}$, then

$$
q_1(z) + \frac{2}{\mu} \frac{q_1'(z)}{q_1(z)} \prec \hat{\Theta}(z) := (1 + \gamma) \left( \frac{2z}{\hat{T}_{n,m,q}^{\lambda,\alpha}(z) - \hat{T}_{n,m,q}^{\lambda,\alpha}(-z)} \right) \mu
$$

implies that

$$
q_1(z) \prec \hat{\Phi}(z) := \left( \frac{2z}{\hat{T}_{n,m,q}^{\lambda,\alpha}(z) - \hat{T}_{n,m,q}^{\lambda,\alpha}(-z)} \right)^\mu < q_2(z),
$$

and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant of Equation (27).

**Example 2.** Taking $q_j = 1 + rz$, with $0 < r_1 < r_2$, $j = 1, 2$ in Theorem 5 and Corollary 7, we obtain the next examples, respectively:

1. If $f \in \mathbb{A}(m)$ such that the assumptions in Equations (23) and (24) hold, then

$$
\left| \frac{1}{r_2} + \frac{\gamma}{\mu} \right| < |\Theta(z) - 1| < \left| 1 + \frac{\gamma}{\mu} \right|, \quad z \in \mathbb{U} \Rightarrow r_1 < |\Theta(z) - 1| < r_2, \quad z \in \mathbb{U}, \quad (0 < r_1 < r_2)
$$

where $\Theta$ and $\Phi$ are given in Theorem 5, and the obtained bounds $r_1$ and $r_2$ are the best possible.

2. If $f \in \mathbb{A}(m)$ such that the assumptions in Equations (23) and (24) hold for the operator $\hat{N}_{n,m,q}^{\lambda,\alpha}$ replaced by $\hat{T}_{n,m,q}^{\lambda,\alpha}$, then

$$
\left| \frac{1}{r_2} + \frac{\gamma}{\mu} \right| < |\hat{\Theta}(z) - 1| < \left| 1 + \frac{\gamma}{\mu} \right|, \quad z \in \mathbb{U} \Rightarrow r_1 < |\hat{\Theta}(z) - 1| < r_2, \quad z \in \mathbb{U}, \quad (0 < r_1 < r_2)
$$

where $\hat{\Theta}$ and $\hat{\Phi}$ are given in Corollary 7, and the obtained bounds $r_1$ and $r_2$ are the best possible.

**Example 3.** Putting $q_j = e^{|z|^2}$, with $0 < r_1 < r_2 \leq 1$, $j = 1, 2$ in Theorem 5 and Corollary 7, we obtain the next examples, respectively:

Let $\gamma \in \mathbb{C}^*$, with $\text{Re} \gamma \geq 0$. 

1. If \( f \in A(m) \) such that the assumptions in Equations (23) and (24) hold, then
\[
\left(1 + \frac{\gamma}{\mu} z\right) e^{\alpha z} < \Theta(z) < \left(1 + \frac{\gamma}{\mu} z\right) e^{\alpha z} \Rightarrow e^{\alpha z} < \Phi(z) < e^{\alpha z}, \quad (0 < r_1 < r_2 \leq 1)
\]
where \( \Theta \) and \( \Phi \) are given in Theorem 5, and \( e^{\alpha z} \) and \( e^{\alpha z} \) are, respectively, the best subordinant and the best dominant.

2. If \( f \in A(m) \) such that the assumptions in Equations (23) and (24) hold for the operator \( N_{n,m,q}^{\lambda,\alpha} \) replaced by \( T_{n,m,q}^{\lambda,\alpha} \), then
\[
\left(1 + \frac{\gamma}{\mu} z\right) e^{\alpha z} < \hat{\Theta}(z) < \left(1 + \frac{\gamma}{\mu} z\right) e^{\alpha z} \Rightarrow e^{\alpha z} < \hat{\Phi}(z) < e^{\alpha z}, \quad (0 < r_1 < r_2 \leq 1)
\]
where \( \hat{\Theta} \) and \( \hat{\Phi} \) are given in Corollary 7, and \( e^{\alpha z} \) and \( e^{\alpha z} \) are, respectively, the best subordinant and the best dominant.

**Theorem 6.** If \( f \in M_{n,m,q}^{\lambda,\alpha}(0, \mu, 1 - 2\rho, -1) \), with \( 0 \leq \rho < 1 \), then \( f \in M_{n,m,q}^{\lambda,\alpha}(0, \mu, 1 - 2\rho, -1) \) for \( |z| < R \), where
\[
R = \left(\sqrt{\frac{|\gamma|^2 m^2}{\mu^2}} + 1 - \frac{|\gamma| m}{\mu}\right)^{\frac{1}{2}}. \tag{29}
\]

**Proof.** For \( f \in M_{n,m,q}^{\lambda,\alpha}(0, \mu, 1 - 2\rho, -1) \), with \( 0 \leq \rho < 1 \), let the function \( h \) be defined by
\[
\left(\frac{2z}{N_{n,m,q}^{\lambda,\alpha} f(z) - N_{n,m,q}^{\lambda,\alpha} f(-z)}\right)^{\mu} = (1 - \rho) h(z) + \rho, \quad z \in \mathbb{U}. \tag{30}
\]

Hence, the function \( h \) is analytic in \( \mathbb{U} \), with \( h(0) = 1 \), and since \( f \in M_{n,m,q}^{\lambda,\alpha}(0, \mu, 1 - 2\rho, -1) \) is equivalent to,
\[
\left(\frac{2z}{N_{n,m,q}^{\lambda,\alpha} f(z) - N_{n,m,q}^{\lambda,\alpha} f(-z)}\right)^{\mu} < \frac{1 + (1 - 2\rho)z}{1 - z},
\]
it follows that \( \text{Re} \ h(z) > 0, \ z \in \mathbb{U} \).

As in the proof of Theorem 1, since \( f \in M_{n,m,q}^{\lambda,\alpha}(0, \mu, 1 - 2\rho, -1) \), with \( 0 \leq \rho < 1 \), we deduce that
\[
\left(\frac{2z}{N_{n,m,q}^{\lambda,\alpha} f(z) - N_{n,m,q}^{\lambda,\alpha} f(-z)}\right)^{\mu} \in \mathcal{H}[1, m],
\]
and from the relation in Equation (30), we get \( h \in \mathcal{H}[1, m] \). Therefore, the following estimate holds
\[
|zh'(z)| \leq \frac{2mr^m \text{Re} h(z)}{1 - r^m}, \quad |z| = r < 1,
\]
that represents the result of Shah [19] (the inequality (6), p. 240, for \( \alpha = 0 \)), which generalize Lemma 2 of [20].
A simple computation shows that

\[
\frac{1}{1 - \rho} \left\{ (1 + \gamma) \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu \right. \\
- \gamma \left( \frac{z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right) \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu - \rho \left. \right\} = h(z) + \frac{2zh'(z)}{\mu}, \quad z \in \mathbb{U},
\]

hence, we obtain

\[
\text{Re} \left\{ \frac{1}{1 - \rho} \left\{ (1 + \gamma) \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu \right. \right. \\
- \gamma \left( \frac{z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right) \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu - \rho \left. \right\} \geq \text{Re} h(z) \left[ 1 - \frac{2|\gamma|m}{\mu(1 - \rho \gamma m)} \right], \quad |z| = r < 1,
\]

and the right-hand side of Equation (31) is positive provided that \( r < R \), where \( R \) is given by Equation (29). □

**Theorem 7.** Let \( f \in M_{n,m,q}^{\lambda,a}(\gamma, \mu, A, B) \), let \( \gamma \in \mathbb{C}^* \) with \( \text{Re} \gamma \geq 0 \), and \(-1 \leq B < A \leq 1\).

1. Then,

\[
\frac{\mu}{\gamma m} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{\mu}{m} - 1} du < \text{Re} \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu < \frac{\mu}{\gamma m} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{\mu}{m} - 1} du, \quad z \in \mathbb{U}.
\]

2. For \(|z| \leq r < 1\), we have

\[
2r \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{\mu}{m} - 1} du \right)^{-\frac{1}{2}} < \left| N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z) \right| < 2r \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{\mu}{m} - 1} du \right)^{-\frac{1}{2}}.
\]

All these inequalities are the best possible.

**Proof.** From the assumptions, using Theorem 1, we obtain that

\[
\left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu \prec \Psi(z) := \frac{\mu}{\gamma m} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{\mu}{m} - 1} du,
\]

and the convex function \( \Psi \in \mathcal{H}[1, m] \) is the best dominant. Therefore,

\[
\text{Re} \left( \frac{2z}{N_{n,m,q}^\lambda f(z) - N_{n,m,q}^\lambda f(-z)} \right)^\mu < \sup_{z \in \mathbb{U}} \text{Re} \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{\mu}{m} - 1} du \right)
\]

\[
= \frac{\mu}{\gamma m} \sup_{z \in \mathbb{U}} \left( \frac{1 + Au}{1 + Bu} \right) u^{\frac{\mu}{m} - 1} du = \frac{\mu}{\gamma m} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{\mu}{m} - 1} du, \quad z \in \mathbb{U},
\]

and
\[
\text{Re} \left( \frac{2z}{N_{n,m,q}^{\lambda,a}(z) - N_{n,m,q}^{\lambda,a}(-z)} \right)^{\mu} > \inf_{z \in \mathbb{U}} \text{Re} \left( \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 - Au}{1 - Bu} u^{\frac{\mu - 1}{m}} \, du \right)
\]

\[
= \frac{\mu}{\gamma m} \int_{0}^{1} \inf_{z \in \mathbb{U}} \left( \frac{1 - Azu}{1 - Bzu} \right) u^{\frac{\mu - 1}{m}} \, du = \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 - Au}{1 - Bu} u^{\frac{\mu - 1}{m}} \, du, \quad z \in \mathbb{U}.
\]

In addition, since
\[
\left| \frac{2z}{N_{n,m,q}^{\lambda,a}(z) - N_{n,m,q}^{\lambda,a}(-z)} \right|^{\mu} < \sup_{z \in \mathbb{U}} \left| \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 + Azu}{1 + Bzu} u^{\frac{\mu - 1}{m}} \, du \right|
\]

we get
\[
\left| N_{n,m,q}^{\lambda,a}(z) - N_{n,m,q}^{\lambda,a}(-z) \right| > 2r \left( \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 + Aur}{1 + Bur} u^{\frac{\mu - 1}{m}} \, du \right)^{-\frac{1}{\mu}},
\]

while
\[
\left| \frac{2z}{N_{n,m,q}^{\lambda,a}(z) - N_{n,m,q}^{\lambda,a}(-z)} \right|^{\mu} > \inf_{z \in \mathbb{U}} \left| \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 - Azu}{1 - Bzu} u^{\frac{\mu - 1}{m}} \, du \right|
\]

implies
\[
\left| N_{n,m,q}^{\lambda,a}(z) - N_{n,m,q}^{\lambda,a}(-z) \right| < 2r \left( \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 - Aur}{1 - Bur} u^{\frac{\mu - 1}{m}} \, du \right)^{\frac{1}{\mu}}.
\]

The inequalities of Equations (32) and (33) are the best possible because the subordination in Equation (34) is sharp. \(\square\)

Taking \(q \to 1^-\) in Theorem 7, we obtain the following corollary:

**Corollary 8.** Let \(f \in W_{n,m}^{\lambda,a}(\gamma, \mu, A, B)\), let \(\gamma \in \mathbb{C}^*\) with \(\text{Re} \gamma \geq 0\), and \(-1 \leq B < A \leq 1\).

1. Then,
\[
\frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 - Au}{1 - Bu} u^{\frac{\mu - 1}{m}} \, du < \text{Re} \left( \frac{2z}{T_{n,m,q}^{\lambda,a}(z) - T_{n,m,q}^{\lambda,a}(-z)} \right)^{\mu}
\]
\[
< \frac{\mu}{\gamma m} \int_{0}^{1} \frac{1 + Au}{1 + Bu} u^{\frac{\mu - 1}{m}} \, du, \quad z \in \mathbb{U}.
\]
2. For $|z| = r < 1$, we have
\[
2r \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 + A u z_{\gamma^{-1}}}{1 + B u z_{\gamma^{-1}}} \, du \right)^{-\frac{1}{\beta}} < |T_{n,m}^{\lambda,\alpha} f(z) - T_{n,m}^{\lambda,\alpha} f(-z)|
< 2r \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 - A u z_{\gamma^{-1}}}{1 - B u z_{\gamma^{-1}}} \, du \right)^{-\frac{1}{\beta}}.
\]

All these inequalities are the best possible.

Taking $q \to 1^-$, $\alpha = 0$ and $\lambda = 1$ in Theorem 7, we obtain the following corollary:

**Corollary 9.** Let $f \in N^{\gamma,\mu}(m, A, B)$, let $\gamma \in \mathbb{C}^*$ with $\text{Re } \gamma \geq 0$, and $-1 \leq B < A \leq 1$.

1. Then,
\[
\frac{\mu}{\gamma m} \int_0^1 \frac{1 - A u z_{\gamma^{-1}}}{1 - B u z_{\gamma^{-1}}} \, du < \text{Re} \left( \frac{2z}{f(z) - f(-z)} \right)^\mu < \frac{\mu}{\gamma m} \int_0^1 \frac{1 + A u z_{\gamma^{-1}}}{1 + B u z_{\gamma^{-1}}} \, du, \quad z \in \mathbb{U}.
\]

2. For $|z| = r < 1$, we have
\[
2r \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 + A u z_{\gamma^{-1}}}{1 + B u z_{\gamma^{-1}}} \, du \right)^{-\frac{1}{\beta}} < |f(z) - f(-z)|
< 2r \left( \frac{\mu}{\gamma m} \int_0^1 \frac{1 - A u z_{\gamma^{-1}}}{1 - B u z_{\gamma^{-1}}} \, du \right)^{-\frac{1}{\beta}}.
\]

All these inequalities are the best possible.

**Example 4.** Putting $\mu = \gamma = m = 1$, $A = 1 - 2\beta$ ($0 \leq \beta < 1$), and $B = -1$ in Corollary 9, we get the next special case.

If $f \in N^{1,1}(1, 1 - 2\beta, -1)$ with $0 \leq \beta < 1$, then:

1. The next inequality holds:
\[
\text{Re} \frac{2z}{f(z) - f(-z)} > 2\beta - 1 + 2(1 - \beta) \ln 2, \quad z \in \mathbb{U}.
\]

2. For $|z| = r := 0.9$, we have
\[
\frac{1.8}{1 + 3.116855762\beta} < |f(z) - f(-z)| < \frac{1.8}{1 - 0.5736580307\beta}.
\]

**Remark 2.** Part (ii) of Corollary 9 corrects the Corollary (3.10) studied by Muhammad and Marwan [16].

Concluding, all the above results give us information about subordination and superordination properties, inclusion results, radius problem, and sharp estimations for the classes $N_{n,m,q}^{l,\alpha}(\gamma, \mu, A, B)$ together general sharp subordination and superordination for the operator $N_{n,m,q}^{l,\alpha}$. For special choices of the parameters $\gamma \in \mathbb{C}, 0 < \mu < 1, -1 \leq B < A \leq 1, \mu \in \mathbb{N}, \alpha > 0, n \geq 0, 0 < q < 1$, and $\lambda > -1$, we may obtain several simple applications connected with the above-mentioned classes and operator.

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