AN EXTENSION THEOREM OF HOLOMORPHIC FUNCTIONS
ON HYPERCONVEX DOMAINS

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Abstract. Let \( n \geq 3 \) and \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) with a smooth negative plurisubharmonic exhaustion function \( \varphi \). As a generalization of Y. Tiba’s result, we prove that any holomorphic function on a connected open neighborhood of the support of \((i\partial\bar{\partial}\varphi)^{n-2}\) in \( \Omega \) can be extended to the whole domain \( \Omega \). To prove it, we combine an \( L^2 \) version of Serre duality and Donnelly-Fefferman type estimates on \((n,n-1)\)- and \((n,n)\)-forms.

1. Introduction

In this article, we study a kind of the Hartogs extension theorem, which appears in Y. Tiba’s paper [9]. The Hartogs extension theorem states that any holomorphic function on \( \Omega \setminus K \), where \( \Omega \) is a domain in \( \mathbb{C}^n \), \( n > 1 \), \( K \) is a compact set in \( \Omega \) and \( \Omega \setminus K \) is connected, extends holomorphically on the whole domain \( \Omega \).

This phenomenon is different from the case of the function theory of one complex variable, and have become a starting point of the function theory of several complex variables. For the several complex variables, the notion of the (strict) pseudoconvexity for the boundary of a given domain have become crucial. Let \( \Omega \) be a smoothly bounded pseudoconvex domain. Denote by \( A(\Omega) \) the uniform algebra of functions that are holomorphic on \( \Omega \) and continuous on \( \overline{\Omega} \). The Shilov boundary of \( A(\Omega) \) is the smallest closed subset \( S(\Omega) \) in \( \partial \Omega \) on which the maximum value of \(|f|\) coincides with that on \( \overline{\Omega} \) for every function \( f \) in \( A(\Omega) \). In fact, the Shilov boundary of \( A(\Omega) \) is the closure of the set of strictly pseudoconvex boundary points of \( \Omega \) (see [1]). By [6], it is known that any holomorphic function \( f \) on \( \overline{\Omega} \) can be represented as \( f(x) = \int f(z) d\mu_x(z) \) where \( d\mu_x \) is a measure supported on the Shilov boundary \( S(\Omega) \).

Assume further that \( \Omega \) has a negative smooth plurisubharmonic function \( \varphi \) on \( \Omega \) such that \( \varphi \to 0 \) when \( z \to \partial \Omega \). Denote by \( \text{Supp}(i\partial\bar{\partial}\varphi)^k \) the support of \((i\partial\bar{\partial}\varphi)^k\). By [2], it can be shown that, for small \( \epsilon > 0 \), the Shilov boundary \( S(\Omega_\epsilon) \) of \( \Omega_\epsilon = \{ \varphi < -\epsilon \} \) is a subset of \( \text{Supp}(i\partial\bar{\partial}\varphi)^{n-1} \).

In this context, it is natural to ask whether any holomorphic function on the support of \((i\partial\bar{\partial}\varphi)^{n-1}\) can be extended to the whole domain \( \Omega \). With this motivation, Y. Tiba proved the following theorem in [9].

Theorem 1.1. Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \), \( n \geq 4 \). Suppose that \( \varphi \in C^\infty(\Omega) \) is a negative plurisubharmonic function which satisfies \( \varphi(z) \to 0 \) as \( z \to \partial \Omega \). Let \( V \) be an open connected neighborhood of \( \text{Supp}(i\partial\bar{\partial}\varphi)^{n-3} \) in \( \Omega \). Then any holomorphic function on \( V \) can be extended to \( \Omega \).

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In [9], Y. Tiba proved Donnelly-Fefferman type estimates for \((0,1)\)- and \((0,2)\)-forms and used them for establishing suitable \(L^2\) estimates of \(\bar{\partial}\)-equations. In this process, the Donnelly-Fefferman type estimate of \((0,2)\)-form and an integrability condition contribute to appear the restriction of the power \(n-3\) in Theorem 1.1.

In this article, we use the Donnelly-Fefferman type estimates for \((n,n)\)- and \((n,n-1)\)-forms rather than \((0,1)\)- and \((0,2)\)-forms. In this case, the restriction of \(n-3\) is changed by \(n-2\), and it improves Theorem 1.1. Finally, using dualities between \(L^2\)-Dolbeault cohomologies, in the same way as [7, 8], we can simplify Y. Tiba’s proof and obtain the generalized result:

**Theorem 1.2.** Let \(\Omega\) be a bounded domain in \(\mathbb{C}^n\), \(n \geq 3\). Suppose that \(\varphi \in C^\infty(\Omega)\) is a negative plurisubharmonic function which satisfies \(\varphi(z) \to 0\) as \(z \to \partial\Omega\). Let \(V\) be an open connected neighborhood of \(\text{Supp}(i\bar{\partial}\varphi)^{n-2}\) in \(\Omega\). Then any holomorphic function on \(V\) can be extended to \(\Omega\).

If the boundary is smooth, then Lemma 4.3 and the proof of Theorem 1.2 show the following.

**Corollary 1.3.** Let \(\Omega\) be a smoothly bounded domain in \(\mathbb{C}^n\), \(n \geq 3\), with a smooth plurisubharmonic defining function \(\varphi\) on a neighborhood of \(\overline{\Omega}\). Let \(S\) be the closure of the subset \(\{z \in \partial\Omega : \text{the Levi form of } \varphi \text{ at } z \text{ is of rank at least } n-2\}\) in \(\partial\Omega\). Then for any connected open neighborhood \(V\) of \(S\) in \(\overline{\Omega}\), any holomorphic function on \(V \cap \Omega\) can be extended to \(\Omega\).

By using a convergence sequence of smooth plurisubharmonic functions to \(\varphi\), we also obtain the following corollary which is the improvement of Corollary 1 of [9].

**Corollary 1.4.** Let \(\Omega\) be a bounded domain in \(\mathbb{C}^n\), \(n \geq 3\). Suppose that \(\varphi \in C^0(\Omega)\) is a negative plurisubharmonic function which satisfies \(\varphi(z) \to 0\) as \(z \to \partial\Omega\). Let \(V\) be an open connected neighborhood of \(\text{Supp}(i\bar{\partial}\varphi)\) in \(\Omega\). Then any holomorphic function on \(V\) can be extended to \(\Omega\).

## 2. Preliminaries

In this section, we review \(L^2\) estimates of \(\bar{\partial}\)-operators and introduce some notations which are used in this paper. Let \(\Omega \subset \mathbb{C}^n\) be a domain, and let \(\omega\) be a Kähler metric on \(\Omega\). We denote by \(|\cdot|_\omega\) the norm of \((p,q)\)-forms induced by \(\omega\) and by \(dV_\omega\) the associated volume form of \(\omega\). Then, we denote by \(L^2_{p,q}(\Omega, e^{-\varphi}, \omega)\) the Hilbert space of measurable \((p,q)\)-forms \(u\) which satisfy

\[
||u||^2_{L^2_{p,q}(\Omega, e^{-\varphi}, \omega)} = \int_\Omega |u|^2 e^{-\varphi} dV_\omega < \infty.
\]

Let \(\bar{\partial} : L^2_{p,q}(\Omega, e^{-\varphi}, \omega) \to L^2_{p,q+1}(\Omega, e^{-\varphi}, \omega)\) be the closed densely defined linear operator, and \(\bar{\partial}^*_\varphi\) be the Hilbert space adjoint of the \(\bar{\partial}\)-operator. We denote the \(L^2\)-Dolbeault cohomology group as \(H^2_{p,q}(\Omega, e^{-\varphi}, \omega)\) and the space of Harmonic forms as

\[
H^2_{p,q}(\Omega, e^{-\varphi}, \omega) = L^2_{p,q}(\Omega, e^{-\varphi}, \omega) \cap \text{Ker} \bar{\partial} \cap \text{Ker} \bar{\partial}^*_\varphi.
\]

It is known that, if the image of the \(\bar{\partial}\)-operator is closed, then these two spaces are isomorphic:

\[
H^2_{p,q}(\Omega, e^{-\varphi}, \omega) \cong H^2_{p,q}(\Omega, e^{-\varphi}, \omega).
\]
Thus, we have

\[ |[i\partial\bar{\partial}\varphi, A_u]u, u|_\omega \geq (\lambda_1 + \cdots + \lambda_q - \lambda_{p+1} \cdots - \lambda_n) |u, u|_\omega \]

for any smooth \((p, q)\)-form. Here, \(A_u\) is the adjoint of left multiplication by \(\omega\).

Suppose that \(A_{\omega, \varphi} = [i\partial\bar{\partial}\varphi, A_u]\) is positive definite and \(\varphi\) is a Kähler metric. By [3], if \(\Omega\) is a pseudoconvex domain, then for any \(\partial\bar{\partial}\)-closed form \(f \in L^2_{p,q}(\Omega, e^{-\varphi}, \omega)\), there exists a \(u \in L^2_{n,q-1}(\Omega, e^{-\varphi}, \omega)\) such that \(\bar{\partial}u = f\) and

\[
\int_\Omega |u|^2 e^{-\varphi} dV_\omega \leq \int_\Omega (A^{-1}_{\omega, \varphi} f, f) e^{-\varphi} dV_\omega.
\]

### 3. Donnelly-Fefferman type estimates for \((n, q)\)-forms

Let \(\Omega \subset \mathbb{C}^n\) be a bounded domain with a negative plurisubharmonic function \(\varphi \in C^\infty(\Omega)\) such that \(\varphi \to 0\) as \(z \to \partial\Omega\). Consider a smooth strictly plurisubharmonic function \(\psi\) on \(\overline{\Omega}\). Since \(\phi = -\log(-\varphi)\) is a plurisubharmonic exhaustion function on \(\Omega\) and \(|\partial\phi|^2 \leq 1, \omega = i\partial\bar{\partial}(\frac{1}{2\pi} \psi + \phi)\) is a complete Kähler metric on \(\Omega\). Let \(A_{\omega, \delta}\) be \([i\partial\bar{\partial}(\psi + \delta\phi), A_u]\) with \(\delta' > 0\).

**Lemma 3.1.** Suppose that \(0 < \delta < q, 1 \leq q \leq n\). Then for any \(\partial\bar{\partial}\)-closed form \(f \in L^2_{n,q}(\Omega, e^{-\psi+(\delta-\delta')\phi}, \omega)\), there exists a solution \(u \in L^2_{n,q-1}(\Omega, e^{-\psi+(\delta-\delta')\phi}, \omega)\) such that \(\bar{\partial}u = f\) and

\[
\int_\Omega |u|^2 e^{-\psi+(\delta-\delta')\phi} dV_\omega \leq C_{q, \delta} \int_\Omega (A^{-1}_{\omega, \delta', \delta} f, f) e^{-\psi+(\delta-\delta')\phi} dV_\omega
\]

where \(C_{q, \delta}\) is a constant which depends on \(q, \delta\).

By Lemma 3.1, the \(L^2\)-Dolbeault cohomology group \(H^2_{n,q}(\Omega, e^{-\psi+(\delta-\delta')\phi}, \omega)\) vanishes.

**Corollary 3.2.** Under the same condition as Lemma 3.1, \(H^2_{n,q}(\Omega, e^{-\psi+(\delta-\delta')\phi}, \omega) = \{0\}\).

To prove Lemma 3.1, we use the idea of Berndtsson’s proof of the Donnelly-Fefferman type estimate in [3].

**Proof.** Since \(\Omega\) can be exhausted by pseudoconvex domains \(\Omega_k \subset \subset \Omega\), for any \(\partial\bar{\partial}\)-closed form \(f \in L^2_{n,q}(\Omega, e^{-\psi-(\delta+\delta')\phi}, \omega)\), there exists the minimal solution \(u_k \in L^2_{n,q-1}(\Omega_k, e^{-\psi-(\delta+\delta')\phi}, \omega)\) of \(\bar{\partial}u_k = f\) such that

\[
\int_{\Omega_k} |u_k|^2 e^{-\psi-\delta\phi} dV_\omega \leq \int_{\Omega_k} (A^{-1}_{\omega, \delta', \delta} f, f) e^{-\psi-\delta\phi} dV_\omega.
\]

We consider \(u_k e^{\delta\phi}\). Since \(\phi\) is bounded on \(\Omega_k\), \(u_k e^{\delta\phi} \in L^2_{n,q-1}(\Omega_k, e^{-\psi-(\delta+\delta')\phi}, \omega)\) and it is orthogonal to \(N_{n,q-1}\) where \(N_{n,q-1}\) is the kernel of

\[
\bar{\partial} : L^2_{n,q-1}(\Omega_k, e^{-\psi-(\delta+\delta')\phi}, \omega) \to L^2_{n,q}(\Omega_k, e^{-\psi-(\delta+\delta')\phi}, \omega).
\]

By \(|\partial\phi|^2 \leq 1\), we have \((f + \delta u_k \wedge \bar{\partial}\phi) e^{\delta\phi} \in L^2_{n,q}(\Omega_k, e^{-\psi-(\delta+\delta')\phi}, \omega)\). Therefore, \(u_k e^{\delta\phi}\) is the minimal solution of

\[
\bar{\partial}(u_k e^{\delta\phi}) = (f + \delta u_k \wedge \bar{\partial}\phi) e^{\delta\phi} \in L^2_{n,q}(\Omega_k, e^{-\psi-(\delta+\delta')\phi}, \omega).
\]

Thus, we have

\[
\int_{\Omega_k} |u_k|^2 e^{-\psi+(\delta-\delta')\phi} dV_\omega \leq \int_{\Omega_k} (A^{-1}_{\omega, \delta', \delta} (f + \delta u_k \wedge \bar{\partial}\phi), f + \delta u_k \wedge \bar{\partial}\phi) e^{-\psi+(\delta-\delta')\phi} dV_\omega.
\]
Let \(\phi\) be a plurisubharmonic defining function. By the Cauchy-Schwarz inequality, for any \(t > 0\),
\[
\int_{\Omega_k} \langle A_{-1}^{\omega, \delta + \delta'} (f + \delta u_k \wedge \partial \phi), f + \delta u_k \wedge \partial \phi \rangle \omega e^{-\psi + (\delta - \delta') \phi} dV_\omega \\
\leq \left(1 + \frac{1}{t}\right) \int_{\Omega_k} \langle A_{-1}^{\omega, \delta + \delta'} f, f \rangle \omega e^{-\psi + (\delta - \delta') \phi} dV_\omega \\
+ (1 + t) \delta^2 \int_{\Omega_k} \langle A_{-1}^{\omega, \delta + \delta'} (u_k \wedge \partial \phi), u_k \wedge \partial \phi \rangle \omega e^{-\psi + (\delta - \delta') \phi} dV_\omega.
\]
Since \(\omega = i \partial \overline{\partial}(\frac{1}{t} \psi + \phi)\) and \(\delta < 2n\), we have \(i \partial \overline{\partial}(\psi + (\delta' + \delta) \phi) \geq \delta i \partial \overline{\partial}(\frac{1}{t} \psi + \phi)\). Hence, \(\langle A_{\omega, \delta + \delta'}, f, f \rangle \omega \geq q \delta^2 f^2 \omega\) if \(f\) is an \((n, q)\)-form. If we take \(t\) sufficiently close to 0, then \(C_{q, \delta} := (1 + \frac{t}{q})/(1 - (1 + t)^{\frac{1}{q}})\) is positive since \(\delta < q\). Note that \(C_{q, \delta}\) does not depend on \(k\). For such a \(t\), we have
\[
\int_{\Omega_k} |u_k|^2 e^{-\psi + (\delta - \delta') \phi} dV_\omega \leq C_{q, \delta} \int_{\Omega} \langle A_{-1}^{\omega, \delta + \delta'} f, f \rangle \omega e^{-\psi + (\delta - \delta') \phi} dV_\omega.
\]
Note that \(L^2(\Omega, e^{-\psi + (\delta - \delta') \phi})\) is a subset of \(L^2(\Omega, e^{-\psi - \delta' \phi})\). Hence, for any \(\overline{\partial}\)-closed form \(f \in L^2(\Omega, e^{-\psi + (\delta - \delta') \phi})\), the right-hand side of (3.2) is finite. Since \(\{\Omega_k\}\) is an exhaustion of \(\Omega\) and \(u_k\) is uniformly bounded by (3.2), we obtain a weak limit \(u \in L^2_{\text{loc}}(\Omega, e^{-\psi + (\delta - \delta') \phi})\) of \(u_k\), and it satisfies \(\overline{\partial} u = f\) and, for each compact set \(K\) in \(\Omega\),
\[
\int_K |u|^2 e^{-\psi + (\delta - \delta') \phi} dV_\omega \leq C_{q, \delta} \int_{\Omega} \langle A_{-1}^{\omega, \delta + \delta'} f, f \rangle \omega e^{-\psi + (\delta - \delta') \phi} dV_\omega.
\]
Using the monotone convergence theorem for \(K\), we obtain the desired result. \(\square\)

4. PROOF OF THE MAIN THEOREM 1.2

The key proposition of this section is the following:

**Proposition 4.1.** Under the same condition as Theorem 1.2, if \(1 \leq q < n\), \(\delta' > 0\), and \(0 < \delta < n - q\) then \(H^2_{\omega, q}(\Omega, e^{\psi - (\delta - \delta') \phi}) = \{0\}\).

**Proof.** Corollary 3.2 implies that two cohomologies \(H^2_{n, n-q}(\Omega, e^{-\psi + (\delta - \delta') \phi})\) and \(H^2_{n, n-q+1}(\Omega, e^{-\psi + (\delta - \delta') \phi})\) are \(\{0\}\). Therefore, the Serre duality in [3] implies that
\[
\overline{\partial} : L^2_{0, q-1}(\Omega, e^{\psi - (\delta - \delta') \phi}) \rightarrow L^2_{0, q}(\Omega, e^{\psi - (\delta - \delta') \phi})
\]
has a closed range and
\[
H^2_{n, n-q}(\Omega, e^{-\psi + (\delta - \delta') \phi}) \cong H^2_{0, q}(\Omega, e^{\psi - (\delta - \delta') \phi}) = \{0\}
\]
since \(\omega\) is a complete Kähler metric on \(\Omega\). Hence, \(H^2_{0, q}(\Omega, e^{\psi - (\delta - \delta') \phi}) = \{0\}\). \(\square\)

For the convenience of readers, we repeat Lemma 5 in [2].

**Lemma 4.2.** Let \(\Omega\) be a smoothly bounded pseudoconvex domain in \(\mathbb{C}^n\) with a plurisubharmonic defining function \(\varphi \in C^\infty(\Omega)\), i.e., \(\Omega = \{\varphi < 0\}\) and \(d\varphi(z) \neq 0\) on \(\partial \Omega\). Let \(p \in \partial \Omega\) and let \(1 \leq k \leq n\) be an integer. Assume that \((i \partial \overline{\partial} \varphi)^k = 0\) in a neighborhood of \(p\). If \(\delta > k\), then \(\exp(\psi - \delta \varphi) dV_\omega\) is integrable around \(p\). Here, \(\varphi = \log(-\varphi)\) and \(\omega = i \partial \overline{\partial}(\frac{1}{\delta} \psi + \varphi)\).

The following lemma is a variant of Lemma 5 in [2]. It is used to prove Corollary 1.3.
Lemma 4.3. Let $\Omega$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with a smooth plurisubharmonic defining function $\varphi$. Let $p \in \partial \Omega$ and let $1 \leq k \leq n-1$ be an integer. Assume that

\[(i\partial \bar{\partial} \varphi)^k \wedge \partial \varphi \wedge \bar{\partial} \varphi = 0\]

in a neighborhood of $p$ in $\partial \Omega$, i.e. the Levi form of $\varphi$ is of rank less than $k$. If $\delta > k$, then $\exp(-\delta \varphi) dV_\omega$ is integrable around $p$ in $\Omega$. Here, $\phi = -\log(-\varphi)$ and $\omega = i\partial \bar{\partial}(1/2n\psi + \phi)$.

Proof. Denote by $\bar{\nu}$ the unit outward normal vector at $p$. For a point $q \in \partial \Omega$ near $p$, (4.3) implies that the Levi form of $\varphi$ at $q$ has at least $n-k$ zero eigenvalues. Now consider a holomorphic coordinate system $(z_1, \ldots, z_n)$ such that $i\partial \bar{\partial} \varphi = \sum_{ij} a_{ij} dz_i \wedge d\bar{z}_j$ and $i\partial \bar{\partial} \varphi|_q = \sum_i a_i(q) dz_i \wedge d\bar{z}_i$, where $a_{ij}$ is a smooth function on $\Omega$ and $a_i(q) = 0$ when $1 \leq i \leq n-k$.

By smoothness of $a_{ij}$, it follows that $a_{ij}(q - t\bar{\nu}) = O(t)$ for $i \neq j$ and $1 \leq i = j \leq n-k$. Hence,

\[(i\partial \bar{\partial} \varphi)^{k+1} \wedge \partial \varphi \wedge \bar{\partial} \varphi \wedge (i\partial \bar{\partial} \varphi|_q)^n = O(t^{k+1})(i\partial \bar{\partial} \varphi|_q)^n\]

and

\[(i\partial \bar{\partial} \varphi)^{k+1} \wedge (i\partial \varphi|_q)^n - k - l = O(t^l)(i\partial \bar{\partial} \varphi|_q)^n\]

on the real half line $q - t\bar{\nu}$, $t > 0$. Since

\[
\exp(-\delta \varphi) \omega^n \approx (-\varphi)\delta \left( \frac{(i\partial \bar{\partial} \varphi|_q)^n \wedge \partial \varphi \wedge \bar{\partial} \varphi}{(-\varphi)^{n+1}} + \frac{(i\partial \bar{\partial} \varphi|_q)^n}{(-\varphi)^n} \right),
\]

$\exp(-\delta \varphi) dV_\omega$ is integrable near $p$ by the Fubini theorem. \qed

We also need the following lemma. It is similar to the Lemma 5.1 of [7].

Lemma 4.4. Let $\Omega$ be a smoothly bounded domain in $\mathbb{C}^n$ with defining function $\varphi \in C^\infty(\Omega)$. Let $U$ be an open set in $\mathbb{C}^n$ such that $\partial \Omega \cap U \neq \emptyset$ and $\Omega \cap U$ is connected. If $u$ is a holomorphic function on $\Omega \cap U$ such that

\[(4.2) \quad \int_{\Omega \cap U} |u|^2(\varphi)^\alpha dV < \infty\]

for some $\alpha \leq -1$, then $u = 0$ on $\Omega \cap U$.

Proof. Take a point $p \in \partial \Omega \cap U$. Consider a holomorphic coordinate such that $p = 0$ and the unit outward normal vector $\nu$ to $\partial \Omega$ at $p$ is $(0, \ldots, 0, 1)$. Take an open ball $B(p, r)$ which is centered at $p$ with sufficiently small radius $r > 0$ such that $B(p, r) \subset U$.

Denote by $Z_q$ the complex line $\{q + \lambda \bar{\nu}: \lambda \in \mathbb{C}\}$ for each $q \in \partial \Omega \cap B(p, r)$. By the Fubini theorem and (4.2),

\[
\int_{q \in E} \left( \int_{Z_q \cap \Omega \cap B(p, r)} |u|^2(-\varphi)^\alpha d\lambda \right) d\sigma \leq \int_{\Omega \cap B(p, r)} |u|^2(-\varphi)^\alpha dV,
\]

where $E \subset \partial \Omega \cap B(p, r)$ is a local parameter space for $Z_q$ of finite measure. Note that $E$ can be chosen as $(2n-2)$-dimensional smooth surface in $\partial \Omega$ and any fiber $(p + \mathbb{C}\bar{\nu}) \cap \partial \Omega$ for $p \in E$ is transversal to $E$. Then, we have

\[(4.3) \quad \int_{Z_q \cap \Omega \cap B(p, r)} |u|^2(-\varphi)^\alpha d\lambda < \infty\]
for $q \in E$ almost everywhere. Since $Z_q' = Z_q$ if $q' \in Z_q \cap \partial \Omega$, there exists an connected open set $V$ in $\partial \Omega$ such that (4.4) holds for $q \in \partial \Omega \cap V$ almost everywhere. For such $q$, since $\alpha \leq -1$ and $u$ is holomorphic on $\Omega \cap \Omega$, by Lemma 5.1 of [7], $u = 0$ on $Z_q \cap \Omega \cap B(p, r)$. Therefore, $u = 0$ on $\Omega \cap U$.

Proof of the Theorem 1.2. First, we assume that $\partial \Omega$ is smooth, $\varphi$ is smooth plurisubharmonic on $\Omega$, $d \varphi \neq 0$ on $\partial \Omega$ and the distance between $\partial V \cap \Omega$ and Supp($i \partial \partial \varphi$)$^{n-2}$ is positive. Take a neighborhood $W$ of Supp($i \partial \partial \varphi$)$^{n-2}$ such that $W$ is contained in $V$, the distance between $\partial W \cap \Omega$ and Supp($i \partial \partial \varphi$)$^{n-2}$ is positive, and $\partial V \cap \partial W \cap \Omega = \emptyset$. Consider a real-valued smooth function $\chi$ on $\Omega$ which is equal to one on Supp($i \partial \partial \varphi$)$^{n-2}$ and equal to zero on $\Omega - W$.

Choose $0 < \delta < n - 1$ and $\delta' > 0$ so that $n - 2 < \delta - \delta' < n - 1$. By Lemma 1.7, $\partial \chi(f) \in L^{2,1}_2(\Omega, e^{\psi - (\delta - \delta')\varphi}, \omega)$. Applying Proposition 4.1, we can find a function $u \in L^2(\Omega, e^{\psi - (\delta - \delta')\varphi}, \omega)$ such that $\partial u = \partial \chi(f)$.

Take a strictly pseudoconvex point $p \in \partial \Omega$ and a connected open set $U$ such that $p \in U \cap \partial \Omega \subset \Omega \cap \text{Supp}(i \partial \partial \varphi)^{n-1}$. Since $\omega = i(\partial \bar{\partial} \psi + \frac{\partial \bar{\partial} \varphi}{\varphi} + \frac{\partial \varphi \wedge \bar{\partial} \varphi}{\psi^2})$, we have

$$\int_{\Omega \cap U} |u|^2 e^{\psi - (\delta - \delta')\varphi} (\varphi)^{(n+1)} dV \lesssim \int_{\Omega} |u|^2 e^{\psi - (\delta - \delta')\varphi} dV < \infty.$$  

Since $\partial u = 0$ on $\Omega \cap U$, by Lemma 4.4 $u = 0$ on $\Omega \cap U$. Therefore, the holomorphic function $\chi f - u$ on $\Omega$ coincides with $f$ on $V$ by the uniqueness of analytic continuation.

To prove the general case, we consider the subdomain $\Omega_\epsilon = \{ \varphi < -\epsilon \}$ of $\Omega$ with smooth boundary, where $\epsilon > 0$. If $\epsilon$ is sufficiently small, $f$ has the holomorphic extension for each $\Omega_\epsilon$ by the previous argument. Due to analytic continuation, we have the desired holomorphic extension of $f$ on $\Omega$.

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