BOREL’S CONJECTURE AND MEAGER-ADDITIVE SETS

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Abstract. We prove that it is relatively consistent with ZFC that every strong measure zero subset of the real line is meager-additive, yet Borel’s conjecture fails. This answers a long-standing question due to Bartoszyński and Judah.

1. Introduction

In this paper, we continue the study of the structure of strong measure zero sets. Strong measure zero sets were introduced by Borel in [5], and have been studied from the beginning of the previous century. Borel conjectured that every strong measure zero set of real numbers must be countable. A few years later, Sierpiński proved in [19] that if the continuum hypothesis (CH) is assumed, then there exists an uncountable strong measure zero set of reals. Nevertheless, the question about the relative consistency of Borel’s conjecture remained open until 1976 when Laver, in his ground-breaking [14], constructed a model of set theory in which every strong measure zero set of reals is countable. In his construction, Laver used Cohen’s forcing technique.

A result of Galvin, Mycielski, and Solovay (see [7]) provides a characterization of Borel’s strong nullity in terms of an algebraic (or translation-like) property for subsets of the real line. By means of this characterization, a strengthening of strong nullity, meager-additivity, appeared on the scene. Meager-additivity, as well as other smallness notions on the real line have received considerable attention in recent years. A 1993’s question due to Bartoszyński and Judah (see [2], or [20, Problem 12.4]) asks whether strong nullity and meager-additivity have a very rigid relationship, in the following sense.

Problem (Bartoszyński–Judah, 1993). Suppose that every strong measure zero set of reals is meager-additive. Does Borel’s conjecture follow?

The main result of this paper is a negative answer to this question.

Theorem A. It is relatively consistent with ZFC that every strong measure zero set of reals is meager-additive, yet Borel’s conjecture fails.

For the proof of Theorem A we use the technique of iterated forcing with countable support to construct a model of set theory in which there are uncountable strong measure zero sets, and every strong measure zero set in the final extension appear in some intermediate stage of the iteration. This allows us to “catch the tail” in a way such that every strong measure zero set of reals in the final extension appears in some intermediate stage of the iteration.
is forced to satisfy certain selection principle in the sense of Scheepers (see \[17\])
that implies meager-additivity.

The work is organized as follows: In Section 2 we will introduce (in a self-
contained as possible way) the essential preliminaries to this paper, as well as the
notation that will be used hereby. In Section 3 we will introduce and analyze a
forcing notion similar to Silver’s forcing, and that will be crucial for our construc-
tion. In Section 4 we will offer a proof of Theorem A. Finally, in Section 5, we will
discuss some concluding remarks and open problems.

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2. Preliminaries

2.1. Notation. We will denote by \( \mathbb{N} \) the set of non-negative integers. Recall that
a subset \( a \) of \( \mathbb{N} \) is called an initial segment if for every \( n \in a \), and \( m < n \), we have
that \( m \in a \). Every initial segment of \( \mathbb{N} \) is either finite or equals \( \mathbb{N} \). If \( a \) is an initial
segment of \( \mathbb{N} \), and \( X \) is a countable set, a function \( \pi : X \to a \) is called a partition of \( X \).

2.2. Meager-additivity and strong nullity. In [5], Borel introduced the notion
of strong nullity for sets of real numbers. Recall that if \( Y \) is a metric space with
distance function \( d \), then the diameter of its subset \( X \) is defined as
\[
\text{diam } X := \sup_{x,y \in X} d(x,y).
\]
A metric space \( X \) has strong measure zero if for every sequence \( (\varepsilon_n : n \in \mathbb{N}) \) of
positive real numbers, there is an open cover \( \{U_n : n \in \mathbb{N}\} \) of \( X \) with \( \text{diam } U_n \leq \varepsilon_n 
\)
for every \( n \in \mathbb{N} \). If \( Y \) is a metric space, we will denote \( \mathcal{N}^+(Y) \) the class of subsets
of \( Y \) that are strong measure zero spaces.

A classical result of Galvin, Mycielski, and Solovay (see [7]) establishes a link
between Borel’s strong nullity and a translation-like property for subsets of the
deals. The Galvin–Mycielski–Solovay theorem asserts that a set \( X \) of real numbers
is a strong measure zero space if, and only if, \( X + M \neq \mathbb{R} \) for every meager set
\( M \subseteq \mathbb{R} \). The same result holds if subsets of the Cantor space are considered instead
of subsets of the real line. The Galvin–Mycielski–Solovay theorem motivates the
definition of meager-additivity. A set \( X \) of real numbers is meager-additive if \( X + M \)
is meager for every meager set \( M \subseteq \mathbb{R} \). The notion of meager-additivity can be
extended to any topological group. If \( Y \) is a topological group, we will denote by
\( \mathcal{M}^\ast(Y) \) the class of meager-additive subsets of \( Y \).

In [21], Zindulka offered a Borel-like characterization of meager-additivity for
subsets of the Cantor space \( C \). A metric space \( X \) has sharp measure zero if for
every sequence \( (\varepsilon_n : n \in \mathbb{N}) \) of positive real numbers, there exists an open cover
We will respectively denote by $P_p$ if, for every sequence $U$ some open set $A$ ambient space is clear from the context, we will avoid as much bombastic notation of pre-

Let $A = \{U_n : n \in \mathbb{N}\}$ of $X$ with $\text{diam} U_n \leq \varepsilon_n$, and there is a partition $\pi : A \to \mathbb{N}$ into finite sets such that every $x \in X$ is in all but finitely many elements of the set

$$\left\{ \bigcup [n]_\pi : n \in \mathbb{N} \right\}.$$ 

Zindulka’s results (see [21, Theorem 1.3]) implies that a subset of the Cantor space is meager-additive if, and only if, it has sharp measure zero. By [21, Theorem 6.3], the same characterization holds for subsets of the real line. Finally, let $c : C \to [0,1]$ be the usual map given by

$$c(x) := \frac{x(n)}{2^n+1}. $$

The following proposition, that is a combination of [3, Lemma 8.1.12] and [21, Proposition 6.2], will be useful later on.

**Proposition 2.1.** A subset $X$ of the unit interval $[0,1]$ has strong (sharp) measure zero if, and only if, $c^{-1}(X)$ has strong (sharp) measure zero. \hfill $\square$

### 2.3. Topological combinatorics

A relative open cover of a subset $X$ of the topological space $Y$ is a family $A$ of open subsets of $Y$ such that $X \subseteq \bigcup A$. Since our definition of a relative cover $A$ depends both in $X$, and its ambient space $Y$, we will often refer to $A$ as a cover of $X|Y$. If either $X$ equals $Y$, or always that the ambient space is clear from the context, we will avoid as much bombastic notation as possible. A cover $A$ of $X|Y$ is called:

- a pre-$\gamma$-cover if every $x \in X$ is in all but finitely many elements of $A$,
- a $\lambda$-cover if every $x \in X$ is in infinitely many elements of $A$,
- an $\omega$-cover if every finite subset of $X$ is included in a single element of $A$, and no element of $A$ covers $X$.

We will respectively denote by $PT[X|Y]$, $A[X|Y]$, $O[X|Y]$, and $\Omega[X|Y]$ the classes of pre-$\gamma$-covers, $\lambda$-covers, open covers, and $\omega$-covers of $X|Y$. In seek of completeness, we include a proof of the following well-known fact.

**Lemma 2.2** (Folklore). For every topological space $Y$, and every $A \in \Omega[X|Y]$, if $A$ is partitioned into finitely many pieces, then at least one of the pieces is an $\omega$-cover of $Y$. In particular, for every finite subset $F \subseteq A$, we have that $A \setminus F \in \Omega[X|Y]$.

**Proof.** Let $A \in \Omega[X|Y]$, and let $\pi : A \to \{0, \ldots, k-1\}$ be a partition. Clearly, no $[i]_\pi$ has $Y$ as an element. If no $[i]_\pi$, for $i < k$, is an $\omega$-cover of $Y$, choose finite subsets $F_i \subseteq Y$, for $i < k$, such that no element of $[i]_\pi$ includes $F_i$. Then no element of $A$ includes the finite set

$$\bigcup_{i<k} F_i \subseteq Y,$$

which is a contradiction. For the second part of the statement, let $U \in A$. Since $A = (A \setminus \{U\}) \cup \{U\}$, and $U \neq Y$, then $A \setminus \{U\}$ is an $\omega$-cover of $Y$. If $F \subseteq A$ is an arbitrary finite set, proceed inductively. \hfill $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be classes of relative covers (not necessarily of the same subset) on a space $Y$. We will say that the selection principle $S_1(\mathcal{A}, \mathcal{B})$ holds if, and only if, for every sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists, for each $n \in \mathbb{N}$, some open set $U_n \in A_n$ such that the set $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

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1This notation is inspired by the probabilistic $X$ given $Y$.

2A pre-$\gamma$-cover, in opposition to a $\gamma$-cover (see [17] for definitions), is not required to be infinite.
Following Kočinac and Scheepers in [12], a cover $A$ of $X|Y$ is called \textit{$\lambda$-groupable} if it is infinite and there exists a partition $\pi : A \to \mathbb{N}$ into finite sets such that every element of $X$ is in all but finitely many elements of the set 

$$\left\{ \bigcup[n]_{\pi} : n \in \mathbb{N} \right\}.$$ 

We will denote by $GA[X|Y]$ the class of $\lambda$-groupable covers of $X|Y$.

2.4. \textbf{Forcing.} A forcing notion is a partially ordered set $\mathbb{P}$. The elements of $\mathbb{P}$ are also called conditions, and if $p \leq q$ then $p$ is said to extend $q$. Two conditions $p$ and $q$ are compatible, denoted by $p \not\leq q$, if a single condition extends both of them.

A subset $D$ of $\mathbb{P}$ is called open if it contains all extensions of all of its elements. A subset $D$ of $\mathbb{P}$ is called dense if it contains some extension of every condition in $\mathbb{P}$.

A subset $G$ of $\mathbb{P}$ is a filter if it satisfies the following two conditions:

1. If $p \in G$, and $p \leq q$, then $q \in G$.
2. Every two elements of $G$ have a common extension in $G$.

If $\mathcal{D}$ is a family of dense open subsets of $\mathbb{P}$, then a filter $G$ is called $\mathcal{D}$-generic if it intersects every element of $\mathcal{D}$ non-trivially.

If a filter $G$ on $\mathbb{P}$ intersects all dense open subsets of $\mathbb{P}$ that belong to the transitive model $V$, then $G$ is said to be $V$-generic. In this situation, one can define the forcing (or generic) extension $V[G]$ which is a transitive model of set theory that includes $V$ and contains $G$ as an element. The model $V$ is usually referred to as the ground model. Our notation is standard and follows [9], [11], and [13]; which are also canonical references for the general theory concerning the forcing technique.

3. \textbf{The forcing notions $\mathbb{P}(t)$}

In this section, we will introduce (and analyze some of the properties of) a variation of Silver’s forcing of partial functions into $\{0, 1\}$, with their domain included in $\mathbb{N}$, and such that the complement of each of their domains is infinite (see [9] Chapter 22]. Before introducing this family of forcing notions, we will need some definitions and results from the general theory.

\textbf{Definition 3.1.} The forcing notion $\mathbb{P}$ satisfies Axiom B if there exists a sequence $(\leq_n : n \in \mathbb{N})$ of partial orders on $\mathbb{P}$ such that:

1. for every $n \in \mathbb{N}$, if $p \leq_n q$, then $p \leq q$.
2. for every $n \in \mathbb{N}$, if $p \leq_{n+1} q$, then $p \leq_n q$.
3. for every sequence $(p_n : n \in \mathbb{N})$ of elements of $\mathbb{P}$ such that $p_{n+1} \leq_n p_n$, there exists a condition $p \in \mathbb{P}$ such that $p \leq_n p_n$ for every $n \in \mathbb{N}$.
4. for every $q \in \mathbb{P}$, and every $n \in \mathbb{N}$, if $q \models \tau \in V$, then there exists a finite set $H \in V$, and $p \leq_n q$, such that $p \models \tau \in H$.

Recall that the forcing notion $\mathbb{P}$ is called $\omega^2$-bounding if for every $V$-generic filter $G$ on $\mathbb{P}$, and for every function $f : \mathbb{N} \to \mathbb{N}$ in $V[G]$, there exists a function $g : \mathbb{N} \to \mathbb{N}$ in $V$ such that $f(n) < g(n)$ for every $n \in \mathbb{N}$.

\textbf{Proposition 3.2.} If the forcing notion $\mathbb{P}$ satisfies Axiom B, then it is proper and $\omega^2$-bounding.

\textit{Proof.} It is clear that if $\mathbb{P}$ satisfies Axiom B, then it satisfies Baumgartner’s Axiom A (see [4] §7 for definitions), and therefore is proper.
To see that \( P \) is \( \omega^\omega \)-bounding, let \( G \) be a \( V \)-generic filter on \( P \), let \( f: N \to N \) be a function in \( V[G] \), and set \( p_0 := 1_P \). For each \( n \in N \), let \( H_n \in V \) be a finite set, and let \( p_{n+1} \leq_n p_n \) be such that \( p_n \Vdash \"f(n) \in H_n\". Let \( g: N \to N \) be the function in \( V \) defined by \( g(n) := \max H_n + 1 \). If \( p \leq_n p_n \) for every \( n \in N \), then
\[
p \Vdash \"(\forall n \in N)(f(n) < g(n))\",
and therefore, in \( V[G] \), \( f(n) < g(n) \) for all \( n \in N \).
\[\square\]

**Corollary 3.3.** If \( P_{\omega_2} \) is a countable support iteration of forcing notions, all of them satisfying Axiom \( B \), then \( P_{\omega_2} \) is proper and \( \omega^\omega \)-bounding.

**Proof.** Follows from Theorem 3.2 and Theorem 4.3 in [18]. \[\square\]

**Proposition 3.4.** If \( P_{\omega_2} \) is a countable support iteration of forcing notions, all of them satisfying Axiom \( B \), \( P_{\omega_2} \) has the \( \aleph_2 \)-chain condition, and \( \Vdash_\alpha \text{CH} \) for all \( \alpha < \omega_2 \), then \( \Vdash_{\omega_2} \"^{\omega^2}(C) \subseteq [C]^{\aleph_1}\".\)

**Proof.** Every forcing notion satisfying Axiom \( B \) is strongly \( \omega^\omega \)-bounding in the sense of Goldstern, Judah, and Shelah in [8] Definition 1.13]. Then the proposition follows from [8] Corollary 3.6] and Proposition 2.21. \[\square\]

If \( \pi: N \to N \) is a partition into finite sets such that every \( [n]_\pi \) is a finite, non-empty interval, and \( \text{max}[m]_\pi < \text{min}[n]_\pi \) always that \( m < n \), we will say that \( \pi \) is an interval partition of \( N \). We will denote by \( \zeta: N \to N \) the canonical interval partition of \( N \) determined by
\[
\text{min}[n]_\zeta = \sum_{i \leq n} i.
\]
A function \( f \) whose domain is some proper subset of \( N \) is called a partial function. In opposition, if the function \( g \) has \( N \) as its domain, we will say that \( g \) is total. If \( f \) is a partial function, we define the gap-counting function \( g_f: N \to N \) by
\[
g_f(n) := |[n]_\zeta \setminus \text{dom } f|.
\]
A total function \( g: N \to N \) is called staggered divergent if it is non-decreasing and divergent, i.e., \( \lim_n g(n) = \infty \). If \( g: N \to N \) is divergent, we define
\[
\mu_n(g) := \text{min } \{ m \in N : g(m) \geq n \}.
\]
An easy (but useful) observation is that if \( f \) is a partial function with divergent gap-counting \( g_f \), then for every \( n \in N \) we have that \( g_f(\mu_n(g_f)) \geq n \).

Let \( t = (t_n : n \in N) \) be some sequence of sets. A partial function
\[
f: \text{dom } f \to \bigcup_{n \in N} t_n
\]
is called a partial \( t \)-selector if \( f(n) \in t_n \) for all \( n \in \text{dom } f \).

For the duration of this section, fix a sequence \( t = (t_n : n \in N) \) of finite sets.

**Definition 3.5.** The forcing notion \( P(t) \) is the set of partial \( t \)-selectors \( p \) such that the gap-counting function \( g_p \) is staggered divergent.

We order \( P(t) \) by \( p \leq q \) if \( p \supseteq q \).

The aim of this section is to prove that the forcing notion \( P(t) \) satisfies Axiom \( B \). To do this, we will start by defining, for each \( n \in N \), a binary relation \( \leq_n \) on \( P(t) \) given by \( p \leq_n q \) if, and only if, \( p \leq q \) and for all \( i \leq \mu_n(q) \),
\[
[i]_\zeta \setminus q = [i]_\zeta \setminus \text{dom } p.
\]
The relation $\leq_n$ may be thought of in the following way: $p \leq_n q$ if $p$ is an extension of $q$, and these two partial functions are exactly the same until (and including) the first interval of the partition $\zeta$ in which the domain of $q$ avoids at least $n$ non-negative integers. Of course, the expression “exactly the same” means that they even have the same gaps in their domains.

**Lemma 3.6.** For every $n \in \mathbb{N}$, the relation $\leq_n$ is a partial order on $P(t)$.

**Proof.** It is enough to prove that the relation $\leq_n$ on $P(t)$ is transitive. Suppose that $p \leq_n q \leq_n r$. We want to conclude that $p \leq_n r$. Since $q \leq_n r$, then $\mu_n(g_q) = \mu_n(g_r)$. Since also $p \leq_n q$, then for every $i \leq \mu_n(g_q) = \mu_n(g_r)$, we have that

$$[i]_\zeta \setminus \text{dom } r = [i]_\zeta \setminus \text{dom } q = [i]_\zeta \setminus \text{dom } p.$$ 

Therefore, $p \leq_n r$. □

**Theorem 3.7.** The forcing notion $P(t)$ satisfies Axiom B.

**Proof.** The only items in Definition 3.1 that require a proof are (3) and (4):

(3) Let $(p_n : n \in \mathbb{N})$ be a sequence of conditions such that $p_{n+1} \leq_n p_n$, and let

$$p := \bigcup_{n \in \mathbb{N}} p_n.$$ 

If it turns out that $p \in P(t)$, then $p \leq_n p_n$ for all $n \in \mathbb{N}$. Thus, it is enough to prove that the gap-counting function $g_p$ is staggered divergent.

(3.1) Let us check first that $g_p$ is divergent. Fix a non-negative integer $n \in \mathbb{N}$. Note that for every $i \leq \mu_n(g_{p_n})$, $[i]_\zeta \setminus \text{dom } p_n = [i]_\zeta \setminus \text{dom } p$. In particular, we have that if $i \leq \mu_n(g_{p_n})$, then $g_p(i) = g_{p_n}(i)$. Thus, $g_p(\mu_n(g_{p_n})) = g_{p_n}(\mu_n(g_{p_n})) \geq n$. Since $n$ was arbitrary, then the function $g_p$ is divergent.

(3.2) To see that $g_p$ is non-decreasing, let $m \in \mathbb{N}$ be fixed. Since

$$\lim_{n \to \infty} \mu_n(g_{p_n}) = \infty,$$

let $n \in \mathbb{N}$ be such that $m \leq \mu_n(g_{p_n})$. As before, for every $i \leq \mu_n(g_{p_n})$, we have that $g_p(i) = g_{p_n}(i)$. This implies that $g_p$ is non-decreasing in the interval $[0, m]$. Since this holds for every $m \in \mathbb{N}$, then $g_p$ is non-decreasing.

(4) Let $q \in P(t)$ be such that $q \models \lnot \exists \tau \in V$, fix $n \in \mathbb{N}$, and set

$$F := \bigcup_{i \leq \mu_n(g_q)} [i]_\zeta \setminus \text{dom } q.$$ 

Let also $\{s_k : k < l\}$ be the finite set of partial $t$-selectors with domain $F$.

We will define $p \leq_n q$ recursively: Start by choosing a condition $q_0 \leq q$ such that $\text{dom } q_0 \cup F \subseteq \text{dom } q_0$ and $q_0 \upharpoonright F = s_0$. Then choose some $p_0 \leq q_0$ such that there exists some $x_0 \in V$ such that $\models p_0 \exists \tau = x_0$. Now, if $k \geq 1$, and we already defined $p_i$, for $i < k$, let $q_k \leq q$ be such that $\text{dom } q_k = \text{dom } p_{k-1}$, $q_k \upharpoonright F = s_k$, and $q_k \upharpoonright ( \text{dom } q_k \setminus F) = p_{k-1} \upharpoonright ( \text{dom } p_{k-1} \setminus F)$. Then choose some $p_k \leq q_k$ such that there exists some $x_k \in V$ such that $\models p_k \exists \tau = x_k$. Finally, let $p := p_{k-1} \upharpoonright ( \text{dom } p_{k-1} \setminus F)$. Clearly, $p \leq_n q$. Now, if $H := \{x_k : k < l\} \in V$, and $r \leq p$ is arbitrary, we can find a further extension $s \leq r$ such that $F \subseteq \text{dom } s$. If $s \upharpoonright F = s_k$, then $s \leq p_k$, and therefore $s \models \lnot \exists \tau = x_k$. Thus, $p \models \lnot \exists \tau \in H$, as required. □
4. A Proof of the Main Result

In this section we will offer a proof of Theorem A.

**Definition 4.1.** Let \( \pi : N \to N \) be a partition into finite sets. A subset \( X \) of the topological space \( Y \) is called \( \pi \)-supernull if for every sequence \( (A_n : n \in N) \) of \( \omega \)-covers of \( Y \), there exists an infinite collection of open sets \( \{U_n : n \in N\} \) such that \( U_n \in A_n \) for every \( n \in N \), and every element of \( X \) is an element of all but finitely many elements of the set

\[
\left\{ \bigcup \{U_k : k \in [n]_\pi \} : n \in N \right\}.
\]

**Proposition 4.2.** Every \( \pi \)-supernull subset of the topological space \( Y \) satisfies the selection principle \( S_1(\Omega[Y], GA[X|Y]) \). In particular, every \( \pi \)-supernull subset of the Cantor space has sharp measure zero.

**Proof.** That the selection principle \( S_1(\Omega[Y], GA[X|Y]) \) holds for every \( \pi \)-supernull subset \( X \) of the topological space \( Y \) follows from the definition. Thus, for the second part of the statement, it is enough to check that if the selection principle \( S_1(\Omega[C], GA[X|C]) \) holds, then \( X \) has sharp measure zero.

Let \( (\varepsilon_n : n \in N) \) be a sequence of positive real numbers. Without loss of generality, let us assume that \( \varepsilon_{n+1} < \varepsilon_n \) for all \( n \in N \), and that \( \varepsilon_n \to 0 \) as \( n \to \infty \). Let \( B_n \) be the collection of all open subsets \( U \) of \( C \) such that

\[
\varepsilon_{n+1} < \text{diam } U \leq \varepsilon_n.
\]

Let \( \rho : N \to N \) be a partition such that every \( [n]_\rho \) is infinite, and let \( A_n \) be the set of finite unions \( U_{n_0} \cup \cdots \cup U_{n_{k-1}} \) such that:

1. \( n_0 < n_1 < \cdots < n_{k-1} \) are all elements of \( [n]_\rho \).
2. \( U_{n_i} \in B_{n_i} \) for every \( i < k \).

Since the sequence \( \varepsilon_n \to 0 \) as \( n \to \infty \), we may assume that no \( A_n \) has \( C \) as an element, and therefore every \( A_n \) is an \( \omega \)-cover of \( C \). Using the selection principle \( S_1(\Omega[C], GA[X|C]) \), choose open sets \( V_n \in A_n \) such that \( \{V_n : n \in N\} \in GA[X|C] \). Now, for each \( n \in N \), let \( k(n) \in N \) be such that \( V_n = U_{n_0} \cup \cdots \cup U_{n_{k(n)-1}} \) in a way such that \( n_0 < \cdots < n_{k(n)-1} \) are all elements of \( [n]_\rho \), and \( U_{n_i} \in B_{n_i} \) for every \( i < k(n) \). Since all the \( U_{n_i} \)'s have different diameters (in particular they are different), then the set

\[
\{U_{n_i} : n \in N \& i < k(n)\} \in GA[X|C].
\]

The argument finishes by recalling that \( \text{diam } U_n \leq \varepsilon_n \). \( \square \)

Although the classes \( \mathcal{N}^+(\mathbb{R}) \) and \( \mathcal{M}^*(\mathbb{R}) \) may differ in many models of set theory (every model in which CH holds, for example), these classes cannot be extremely different. More precisely, it cannot be the case that every meager-additive set of reals in countable, while there exists an uncountable strong measure zero set. Recall that \( b \) is the minimal cardinality of a set \( \mathcal{F} \) of functions \( f : N \to N \) such that for every \( g : N \to N \) there exists some \( f \in \mathcal{F} \), and there are infinitely many \( n \in N \) such that \( g(n) < f(n) \).

**Corollary 4.3.** The following are equiveridical.

1. Every strong measure zero set of reals is countable.
2. Every meager-additive set of reals is countable.
Proof. It is enough to prove that \( \neg(1) \Rightarrow \neg(2) \). We proceed by cases:

(i) If \( b = \aleph_1 \), it is a result of Bartoszyński (see [1, Theorem 2 (1)]) that there exists an uncountable meager-additive subset of the real line.

(ii) If \( b > \aleph_1 \), let \( X \) be a strong measure zero set of reals with \( |X| = \aleph_1 \). Since \( b > \aleph_1 \), then \( X \) has the Hurewicz property (see [10]). By [15, Theorem 8], \( X \) is a set of reals that has the Hurewicz property and such that the selection principle \( S_1(O[X], O[X]) \) holds. By [16, Theorem 14], this implies that the selection principle \( S_1(\Omega[R], G\Lambda[X|R]) \) holds. Therefore, by Proposition 4.2, \( X \) has sharp measure zero, and is meager-additive.

This concludes the proof. \( \square \)

Corollary 4.3 implies that the classes \( M_{\prec}R_q \) and \( N_{\prec}R_q \) are provable in ZFC not extremely different in the sense that it cannot be case that \( M_{\prec}R_q \) coincides with the class of countable subsets of \( R \), while there exists an uncountable strong measure zero set. The Bartoszyński–Judah problem asks whether the classes of meager-additive sets and strong measure zero sets are intrinsically distinct in the sense that they can only be equal if they are trivially equal, i.e., if \( N_{\prec}R_q \) and \( M_{\prec}R_q \) coincide, is because both classes collapse to the class of countable subsets of the reals. As announced in the introduction, we will give a negative answer to this question by building a model of set theory in which the notions of meager-additivity and strong nullity coincide, yet these classes (this class) include uncountable sets.

**Theorem 4.4.** If ZFC has a model, then it has a model in which:

1. Every strong measure zero subset of \( C \) is \( \zeta \)-supernull.
2. There are uncountable strong measure zero subsets of \( C \).
3. There are no unbounded reals over \( L \).
4. \( 2^{\aleph_0} = \aleph_2 \).

Before getting into the details of Theorem 4.4, let us show how it can be used to prove Theorem A.

**Proof of Theorem A.** Let \( M \) be a model of set theory in which all the items in Theorem 4.4 hold. By Proposition 4.2, \( N^{\ast}(C) = M^{\ast}(C) \). Now, if \( X \) is a strong measure zero set of real numbers that is not meager-additive, since both \( N^{\ast}(R) \) and \( M^{\ast}(R) \) coincide, is because both classes collapse to the class of countable subsets of the reals. As announced in the introduction, we will give a negative answer to this question by building a model of set theory in which the notions of meager-additivity and strong nullity coincide, yet these classes (this class) include uncountable sets.

Let \( T = (A_n : n \in \mathbb{N}) \) be some sequence of \( \omega \)-covers of the Cantor space. We will construct a sequence of finite sets \( \hat{T} := (t_n : n \in \mathbb{N}) \) such that each \( t_n \subseteq A_n \) is a cover of \( C \), and all the \( t_n \)'s are pairwise disjoint. Since \( C \) is compact, let \( t_0 \subseteq A_0 \) be a finite cover of \( C \). For every \( n \geq 1 \), if we already defined \( t_k \) for all \( k < n \), let \( B_n := A_n \setminus \bigcup_{k<n} t_k \).

By Lemma 2.2, \( B_n \) is an \( \omega \)-cover of \( C \), so there exists a finite set \( t_n \subseteq B_n \) that covers \( C \). This finishes the construction.

Fix a sequence \( T \) of \( \omega \)-covers of \( C \), and set \( \hat{T} \) as described above.

**Definition 4.5.** The forcing notion \( Q(\hat{T}) \) is the set of pairs \( p = (C_p, f_p) \) such that

1. \( C_p \) is a finite subset of \( C \).
2. \( f_p \) is a partial \( T \)-selector.
(3) the gap-counting function $g_{f_p}$ is staggered divergent.
(4) $(\forall x \in C_p) (\exists N \in \mathbb{N}) (\forall n \geq N) (\exists k \in [n]_\omega \cap \text{dom } f_p) (x \in f_p(k)).$

We order $Q(\hat{T})$ by $p \leq q$ if $C_p \supseteq C_q$ and $f_p \supseteq f_q$.

**Lemma 4.6.** For every $n \in \mathbb{N}$, and every $x \in \mathcal{C}$, the sets
$$D_n := \left\{ p \in Q(\hat{T}) : n \in \text{dom } f_p \right\} \quad \text{and} \quad E_x := \left\{ p \in Q(\hat{T}) : x \in C_p \right\}$$
are dense open subsets of $Q(\hat{T})$.

**Proof.** That all the $D_n$’s and all the $E_x$’s are open is obvious, so we only need to check the density.

(1) Let $q \in Q(\hat{T})$ be a condition, and let $n \in \mathbb{N}$ be fixed. We are looking for a partial $\mathbb{T}$-selector $f_p$ extending $f_q$, with $n \in \text{dom } f_p$, and such that the gap-counting $g_{f_p}$ still monotone. If $n \in \text{dom } f_q$, there is nothing to prove. Otherwise, let $m \in \mathbb{N}$ be such that $n \in [m]_\omega$, and let $\{m - i : i < l\}$ be the set of all the non-negative integers such that $g_{f_q}(m) = g_{f_q}(m - i)$. For each $i < l$, choose $k_i \in [m - i]_\omega \setminus \text{dom } f_q$, with $k_0 = n$, and choose arbitrary open sets $U_{k_i} \in A_{k_i}$. Setting $f_p := f_q \cup \{(k_i, U_{k_i}) : i < l\}$, we obtain that $p := (C_q, f_p) \in Q(\hat{T})$ is the desired condition.

(2) Let $q \in Q(\hat{T})$, and let $x \in \mathcal{C}$ be fixed. For each $n \geq \mu_1(g_{f_q})$, choose some $k_n \in [n]_\omega \setminus \text{dom } f_q$. Since each $A_{k_n}$ is an open cover of $\mathcal{C}$, there must exist open sets $U_{k_n} \in A_{k_n}$ such that $x \in U_{k_n}$ for all $n \in \mathbb{N}$. If we define
$$f_p := f_q \cup \{(k_n, U_{k_n}) : n \geq \mu_1(g_{f_q})\},$$
then the condition $p := (C_q \cup \{x\}, f_p) \in Q(\hat{T})$ is as desired.

This concludes the argument. \qed

If $G$ is a $V$-generic filter on $Q(\hat{T})$, we define
$$f_G := \bigcup_{p \in G} f_p.$$

**Theorem 4.7.** If $G$ is a $V$-generic filter on $Q(\hat{T})$, then:

(1) $f_G$ is a total function, i.e., it has $\mathbb{N}$ as its domain.
(2) For every $n \in \mathbb{N}$, $f_G(n) \in A_n$. In particular, $f_G$ is one-to-one.
(3) Every element of $\mathcal{C}$ is in all but finitely many elements of the set
$$\left\{ \bigcup \{f_G(k) : k \in [n]_\omega \} : n \in \mathbb{N} \right\}.$$

**Proof.** Fix a $V$-generic filter $G$ on $Q(\hat{T})$.

(1) For each $n \in \mathbb{N}$, choose a condition $p \in G \cap D_n$. Then
$$n \in \text{dom } f_p \subseteq \text{dom } f_G.$$

(2) If $n \in \text{dom } f_p$, since $f_p$ is a partial $\mathbb{T}$-selector, $f_G(n) = f_p(n) \in A_n$.

(3) For each $x \in \mathcal{C}$, choose a condition $p \in G \cap E_x$. Then $x$ is in all but finitely many elements of the set
$$\left\{ \bigcup \{f_p(k) : k \in [n]_\omega \cap \text{dom } f_p \} : n \in \mathbb{N} \right\},$$
and therefore (3) follows. \qed
Lemma 4.8. The map \( i : \mathcal{Q}(\mathbb{T}) \to \mathcal{P}(\mathbb{T}) \) given by \( i(p) := f_p \) is a surjective forcing embedding.

Proof. All the items in the definition of a surjective forcing embedding (see [13] Definition III.3.65) are easily checked with the exception of: if \( i(p) \not\leq i(q) \), then \( p \not\leq q \). This is an easy application of density: if \( i(p) \) and \( i(q) \) have some common extension \( f \), find in \( \mathcal{Q}(\mathbb{T}) \) some condition \( r \leq (\emptyset, f) \) such that \( C_p \cup C_q \subseteq C_r \). Then \( r \) is a common extension of both \( p \) and \( q \). \( \square \)

We are now in shape to finish the proof of the main result.

Proof of Theorem 4.4. The model in which we are interested is obtained by doing a countable support iteration of \( \mathcal{P}(\mathbb{T}_\alpha) \), for \( \alpha < \omega_2 \), over a model of \( V = L \), where at each stage in the iteration \( V[G_\alpha] = \mathbb{T}_\alpha \) is a sequence of \( \omega \)-covers of \( \mathbb{C} \), and where we have dovetailed so as to ensure that for any \( \mathbb{T} \) such that

\[
V[G_\omega] = \mathbb{T} \text{ is a sequence of } \omega \text{-covers of } \mathbb{C},
\]

then for cofinally many \( \alpha < \omega_2 \) we have that \( \mathbb{T} = \mathbb{T}_\alpha \). This dovetailing can be done since there are only continuum many sequences of \( \omega \)-covers of \( \mathbb{C} \), and the intermediate models satisfy the continuum hypothesis.

1. By [11] Theorem 16.30, the iteration poset \( \mathcal{P}_{\omega_2} \) has the \( \kappa_2 \)-chain condition. By Proposition 3.3, every strong measure zero subset \( X \) of \( \mathbb{C} \) in \( V[G_{\omega_2}] \) has cardinality \( \kappa_1 \). This implies that there exists some \( \alpha < \omega_2 \) such that \( X \in V[G_\alpha] \). Now, if \( \mathbb{T} = (A_n : n \in \mathbb{N}) \) is such that

\[
V[G_{\omega_2}] = \mathbb{T} \text{ is a sequence of } \omega \text{-covers of } \mathbb{C},
\]

let \( \beta \geq \alpha \) be such that \( \mathbb{T} = \mathbb{T}_\beta \). By Lemma 4.8 and Theorem 4.7 there is in \( V[G_{\beta + 1}] \) an infinite cover \( \{U_n : n \in \mathbb{N}\} \) of \( X|\mathbb{C} \), for every \( n \in \mathbb{N} \), and such that every element of \( X \) is in all but finitely many elements of the set

\[
\left\{ \bigcup \{U_k : k \in [n]_{\zeta} \} : n \in \mathbb{N} \right\}.
\]

Therefore, \( V[G_{\omega_2}] = X \text{ is a } \zeta \text{-supernull subset of } \mathbb{C} \).

2. By (2) in Corollary 5.3, every function \( f : \mathbb{N} \to \mathbb{N} \) in \( V[G_{\omega_2}] \) is dominated by some function \( g : \mathbb{N} \to \mathbb{N} \) in the ground model.

3. The ground-model Cantor space \( \mathbb{C}^V \) is \( \zeta \)-supernull, thus has strong measure zero. Since \( \mathcal{P}_{\omega_2} \) is proper, then \( \mathbb{C}^V \) is uncountable in \( V[G_{\omega_2}] \).

4. Follows from the usual argument. \( \square \)

5. Concluding remarks

We will say that a subset \( X \) of the real numbers is a relative pre-\( \gamma \)-set if the selection principle \( S_1(\Omega[\mathbb{R}], PT[X|R]) \) holds.

Question 5.1. Suppose that every strong measure zero set of reals is a relative pre-\( \gamma \)-set. Does Borel’s conjecture follow?

At first sight, Question 5.1 could seem like just a random variation of the Bartoszyński–Judah problem. We will spend the remainder of this section to explain why this is not the case. There exists an obvious forcing notion analogous to \( \mathcal{Q}(\mathbb{T}) \) (see [15] Theorem 5), that adds a pre-\( \gamma \)-cover by selecting one open set in
each coordinate of a given sequence of \( \omega \)-covers. Fix a sequence \( T \) of \( \omega \)-covers of \( C \). A partial \( T \)-selector \( f \) is called initial if \( \text{dom } f \) is a finite initial segment of \( \mathbb{N} \).

**Definition 5.2.** The forcing notion \( S(T) \) is the set of pairs \( p = (C_p, f_p) \) such that

1. \( C_p \) is a finite subset of \( C \)
2. \( f_p \) is an initial \( T \)-selector.

We order \( S(T) \) by \( p \leq q \) if

1. \( C_p \subseteq C_q \)
2. \( f_p \subseteq f_q \)
3. \( (\forall k \in \text{dom } f_p \setminus \text{dom } f_q)(\forall x \in C_q)(x \in f_p(k)) \).

By [15, Theorem 6], the forcing notion \( S(T) \) has the countable chain condition. Clearly, \( S(T) \) forces that for the sequence of \( \omega \)-covers \( T \), we can select one open set in each coordinate of \( T \) to obtain a pre-\( \gamma \)-cover of \( C \). Nevertheless, it is easy to see that \( S(T) \) adds a Cohen real to the universe. This is problematic since if Cohen (in particular unbounded) reals are added to the universe, then it is not clear at all how to assure that strong measure zero sets in an extension given by a suitable iteration appeared in some intermediate stage. On the other hand, it is not clear how to force the existence of a pre-\( \gamma \)-cover by selecting one open set in each coordinate of \( T \) if countable conditions are used instead of finite ones; once a point avoids infinitely many open sets, it avoids them forever. In particular, one may ask the following.

**Question 5.3.** Let \( T = (A_n : n \in \mathbb{N}) \) be a sequence of \( \omega \)-covers of the Cantor space. Is it possible to generically add a cover \( \{U_n : n \in \mathbb{N}\} \in \mathcal{P}(\mathcal{T}[C]) \) such that \( U_n \in A_n \) for every \( n \in \mathbb{N} \) without adding unbounded (or even Cohen) reals?

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