Reduced bandwidth: a qualitative strengthening of twin-width in minor-closed classes (and beyond)

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“Graph Product Structure Theory”
Identifying vertices

- We consider graphs where some edges are colored red.

- When we identify two vertices \( v \) and \( w \) to \( z \) in a graph \( G \),
  - all edges between \( z \) and \( N(v) \triangle N(w) \) become red,
  - for \( x \in N(v) \cap N(w) \),
    > if at least one of \( vx \) and \( wx \) was red, then \( zx \) becomes red,
    > otherwise, it becomes black.
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  - all edges between $z$ and $N(v) \triangle N(w)$ become red,
  - for $x \in N(v) \cap N(w)$,
    > if at least one of $vx$ and $wx$ was red, then $zx$ becomes red,
    > otherwise, it becomes black.

• **General question:** Can we recursively identify a given graph into one vertex without creating a vertex of large red degree?
A trigraph is a graph whose edges are colored black or red.

For a graph $G$, a sequence $G = G_n, G_{n-1}, \ldots, G_1$ of trigraphs is a reduction sequence if $G_1$ is a singleton graph.

Twin-width of a graph $G$ is the minimum $k$ such that there is a reduction sequence $G = G_n, G_{n-1}, \ldots, G_1$ of $G$ for which the maximum red degree of $G_i$ is at most $k$.

Cographs have twin-width 0 (Cographs on $\geq 2$ vertices always have twins).
Reduced-\(f\) of a graph

- We consider any natural graph parameter \(f\) (maximum degree, tree-width, band-width, component size, …)

- Reduced-\(f\) of a graph \(G\) is the minimum \(k\) such that there is a reduction sequence \(G = G_n, G_{n-1}, \ldots, G_1\) of \(G\) for which \(\max_{1 \leq i \leq k} f(G_i) \leq k\).

- (Bonnet et al. 2020 TWW I) Reduced-maximum degree = twin-width
  (Bonnet et al. 2021 TWW VI) Reduced-component size \sim rank-width
  Reduced-number of edges \sim linear rank-width
Reduced-\(f\) of a graph

- If \(f\) is bounded on all stars, then reduced-\(f\) is bounded for all graphs.

- We may consider \(\max\{f, \Delta\}\) as a function, where \(\Delta(G)\) denotes max degree.

- **Question**: Are there differences between following classes?
  - graphs of bounded reduced-\(\Delta\)
  - graphs of bounded reduced-max\{treewidth,\(\Delta\}\}
  - graphs of bounded reduced-max\{pathwidth,\(\Delta\)\}
  - graphs of bounded reduced-bandwidth
  - graphs of bounded reduced-component size

- **Question**: Do some known classes of bounded twin-width have actually bounded reduced-bandwidth?
Reduced-bandwidth of a graph

- Band-width of a graph $G$: minimum $k$ such that there is a permutation $L : V(G) \rightarrow [n]$ where $|L(u) - L(v)| \leq k$ for every edge $uv$.

- If band-width is at most $k$, then maximum degree is at most $2k$. 
Main results (product theorem + neighborhood complexity)

- Theorem (Bonnet, K, Wood 2021)
  Proper minor-closed classes have bounded reduced-bandwidth.
  Their $r$-powers also have bounded reduced-bandwidth.

- This strengthens the results in TWW I that proper minor-closed classes have bounded twin-width.
Main results (product theorem + neighborhood complexity)

- Theorem (Bonnet, K, Wood 2021)
  Proper minor-closed classes have bounded reduced-bandwidth. Their r-powers also have bounded reduced-bandwidth.

- This strengthens the results in TWW I that proper minor-closed classes have bounded twin-width.

- Theorem (Bonnet, K, Wood 2021)
  Planar graphs have reduced-bandwidth at most 466 and twin-width at most 583. By the result of (Morin 2021), we can produce in polynomial time.

  Graphs of Euler genus $g$ have reduced-bandwidth at most $164g + 468$. Planar map graphs have reduced-bandwidth at most 10000.

- Previous bounds for planar graphs in TWW I/TWW VI papers were $\geq 2^{1000}$. 
• Theorem
  Planar graphs have reduced-bandwidth at most 466 and twin-width at most 583.

• (Product theorem (Ueckerdt, Wood, Yi 2021)) Every planar graph is a subgraph of $H \boxtimes P$ for some graph $H$ of treewidth at most 6 and a path $P$.

• (Neighborhood complexity) For every vertex set $S$ in a planar graph $G$, $|\{|N(v) \cap S : v \in V(G) \setminus S\}| \leq 6|S| - 9$. 
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• (Product theorem (Ueckerdt, Wood, Yi 2021)) Every planar graph is a subgraph of $H \boxtimes P$ for some graph $H$ of treewidth at most 6 and a path $P = w_1w_2\ldots w_t$.

• (Neighborhood complexity) For every vertex set $S$ in a planar graph $G$,
  $$|\{N(v) \cap S : v \in V(G) \setminus S\}| \leq 6|S| - 9.$$

• **Difficulty**: when you identify two vertices, planarity may be destroyed, and it is hard to find a natural sequence preserving planarity.

• **Idea**: we will not use planarity when constructing a reduction sequence.

• We can slightly **improve** bounds by looking at neighborhood complexity in the product structure carefully. But we do not know whether we can improve to $\leq 100$. 
Theorem
Planar graphs have reduced-bandwidth at most 466 and twin-width at most 583.

- Look at a vertex \( v \) in \( (V(Q_1) \setminus V(B)) \times \{w_2\} \).
- Neighbors are contained in \( (V(Q_1) \cup V(B)) \times \{w_1, w_2, w_3\} \).
Theorem

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- Look at a vertex \( v \) in \( (V(Q_1) \setminus V(B)) \times \{w_2\} \).
- Neighbors are contained in \( (V(Q_1) \cup V(B)) \times \{w_1, w_2, w_3\} \).
- We want to identify \( ((V(Q_1) \cup V(Q_2)) \setminus V(B)) \times V(P) \) so that no red edges incident with \( V(B) \times V(P) \) are created.

Idea: pick two vertices in the same slice that are twins to \( V(B) \times V(P) \).
• $S_{x,q}$ (each circle is a clique)

• $\Delta(S_{x,q}) \leq 5q - 2$ and $\text{bandw}(S_{x,q}) \leq 4q - 2$
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- $(k, q)$-rooted decomposition
  - internal bags have size $\leq k + 1$
  - leaf bags have size $\leq q$

- rooted separation is a separation $(C, D)$ as in the picture
- $S_{x,q}$ (each circle is a clique)
- $\Delta(S_{x,q}) \leq 5q - 2$ and $\text{bandw}(S_{x,q}) \leq 4q - 2$
- $(k, q)$-rooted decomposition
  - internal bags have size $\leq k + 1$,
  - leaf bags have size $\leq q$
  - rooted separation is a separation $(C, D)$ as in the picture
- A parameter $f$ is good if it is closed under subgraph / disjoint union
Let $f$ be a good parameter and $g : \mathbb{N} \rightarrow \mathbb{R}$ be a function where $f(S_{x,q}) \leq g(q)$ for all $q$. Let $(\mathcal{T}, \mathcal{B})$ be a $(k, q)$-rooted tree-decomposition of $H$ and let $F$ be a trigraph with $V(F) \subseteq V(H \boxtimes P)$ such that

1) (red edge condition) for every red edge $vw$, there is a leaf bag $B$ with parent $B'$ so that $v, w \in (B \setminus B') \times V(P),$

2) (separation condition)
for every rooted separation $(C, D)$ of $H$ and $z \in V(P)$,
$$\left| \left\{ N_F(v) \cap (D \times V(P)) : v \in ((C \setminus D) \times \{z\}) \cap V(F) \right\} \right| \leq q$$

3) (neighborhood condition)
for every vertex $v \in (V(H) \times \{z\}) \cap V(F)$ for some $z \in V(P)$,
$$N_F[v] \subseteq V(H) \times N_P[z].$$

Then reduced-$f$ of $F$ is at most $g(q)$. 
• Let \( f \) be a good parameter and \( g : \mathbb{N} \to \mathbb{R} \) be a function where \( f(S_{x,q}) \leq g(q) \) for all \( q \). Let \((\mathcal{T}, \mathcal{B})\) be a \((k, q)\)-rooted tree-decomposition of \( H \) and let \( F \) be a trigraph with \( V(F) \subseteq V(H \Box P) \) such that

1) (red edge condition) for every red edge \( vw \), there is a leaf bag \( B \) with parent \( B' \) so that \( v, w \in (B \setminus B') \times V(P), \)

2) (separation condition) for every rooted separation \((C, D)\) of \( H \) and \( z \in V(P), \)[\[
\left| \{N_F(v) \cap (D \times V(P)) : v \in ((C \setminus D) \times \{z\}) \cap V(F)\} \right| \leq q
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3) (neighborhood condition) for every vertex \( v \in (V(H) \times \{z\}) \cap V(F) \) for some \( z \in V(P), \) \( N_F[v] \subseteq V(H) \times N_P[z] \).

Then reduced-\( f \) of \( F \) is at most \( g(q) \).

• For planar graphs, we can take \( k=6 \) and \( q=6 \times (7 \times 3) - 9 = 117. \)

\begin{itemize}
  \item \( \text{bandw}(S_{x,q}) \leq 4q - 2 \) and \( \Delta(S_{x,q}) \leq 5q - 2. \)
  \item So, reduced-bandwidth \( \leq 466 \) and twin-width \( \leq 583. \)
\end{itemize}
• Let $f$ be a good parameter and $g : \mathbb{N} \rightarrow \mathbb{R}$ be a function where $f(S_{x,q}) \leq g(q)$ for all $q$. Let $(\mathcal{T}, \mathcal{B})$ be a $(k, q)$-rooted tree-decomposition of $H$ and let $F$ be a trigraph with $V(F) \subseteq V(H \Box P)$ such that

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• For planar graphs, we can take $k=6$ and $q=6*(7*3)-9=117.$

• bandw$(S_{x,q}) \leq 4q - 2$ and $\Delta(S_{x,q}) \leq 5q - 2.$

• So, reduced-bandw $\leq 466$ and twin-width $\leq 583.$

• Can be applied to any class with product structure!
Let $f$ be a good parameter and $g : \mathbb{N} \to \mathbb{R}$ be a function where $f(S_{x,q}) \leq g(q)$ for all $q$. Let $(\mathcal{T},\mathcal{B})$ be a $(k,q)$-rooted tree-decomposition of $H$ and let $F$ be a trigraph with $V(F) \subseteq V(H \boxtimes P)$ such that

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3) (neighborhood condition) for every vertex $v \in (V(H) \times \{z\}) \cap V(F)$ for some $z \in V(P)$, $N_F[v] \subseteq V(H) \times N_P[z]$. Then reduced-$f$ of $F$ is at most $g(q)$. 

\[ \text{Diagram of rooted tree decomposition and trigraph}\]
Let $f$ be a good parameter and $g : \mathbb{N} \rightarrow \mathbb{R}$ be a function where $f(S_{x,q}) \leq g(q)$ for all $q$. Let $(\mathcal{T}, \mathcal{B})$ be a $(k, q)$-rooted tree-decomposition of $H$ and let $F$ be a trigraph with $V(F) \subseteq V(H \boxtimes P)$ such that

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2) (separation condition) for every rooted separation $((C, D) \times \{z\}) \cap V(F)$, $H$ and $z \in V(P)$,

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\[
\begin{align*}
\text{R} & \quad \text{R} & \quad \text{R} & \quad \text{R} \\
\text{D} & \quad \text{R} & \quad \text{R} & \quad \text{R} \\
\text{B} & \quad \text{R} & \quad \text{R} & \quad \text{R} \\
H & \quad V(H) \times \{w_1\} & \quad V(H) \times \{w_2\} & \quad V(H) \times \{w_3\} & \quad V(H) \times \{w_4\}
\end{align*}
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Let $f$ be a good parameter and $g : \mathbb{N} \to \mathbb{R}$ be a function where $f(S_{x,q}) \leq g(q)$ for all $q$. Let $(\mathcal{T}, \mathcal{B})$ be a $(k, q)$-rooted tree-decomposition of $H$ and let $F$ be a trigraph with $V(F) \subseteq V(H \boxtimes P)$ such that

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\begin{center}
\begin{tabular}{c|c|c|c|c}
$H$ & $V(H) \times \{w_1\}$ & $V(H) \times \{w_2\}$ & $V(H) \times \{w_3\}$ & $V(H) \times \{w_4\}$ \\
\end{tabular}
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3. (neighborhood condition) for every vertex $v \in (V(H) \times \{z\}) \cap V(F)$ for some $z \in V(P)$, $N_F[v] \subseteq V(H) \times N_P[z]$.

Then reduced-$f$ of $F$ is at most $g(q)$.

One can see that each red component is always a subgraph of $S_{[V(P)],q}$.

Since $f$ is closed under subgraph / disjoint union, red graph has $f$-value $\leq g(q)$. So, reduced-$f$ of $F$ is at most $g(q)$. 
• For r-th powers:
  1) we consider the r-th power of $S_{x,q}$ instead of $S_{x,q}$
  2) (neighbourhood condition) $N_P[z]$ is replaced with $N_P^r[z]$  
  3) (separation condition) We can use linear bounds on distance-r profiles
    by Eickmeyer et al. (2017) (or simply $(r + 1)^{|S|}$)
  4) For a map graph $G$ and vertex set $S$, we prove that
    \[
    | \{N(v) \cap S : v \in V(G) \setminus S \} | \leq \max \{2^{10}, 37|S| - 81\} \]  
    (where we apply $|S| = 35$)
X-minor free graphs

- (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood 2020)
  For every graph $X$, there exists $k, a \in \mathbb{N}$ such that every $X$-minor-free graph $G$ has a tree-decomposition in which every torso is a subgraph of $(H \boxtimes P) + K_a$ for some graph $H$ of treewidth at most $k$ and some path $P$.

- We consider the neighborhood complexity to bags in $H \boxtimes P$ together with $K_a$.
- We obtain a reduction sequence from bottom to top in the tree-decomposition, so that during the sequence, we do not create red edge to above bags.
- We need to extend the lemma to deal with information from below subtrees.
Conclusion

- Proper minor-closed classes and their r-powers have bounded reduced-bandwidth.

Question:

- Is it true that planar graphs have reduced-bandwidth / twin-width at most 10?

- We write $f_1 < f_2$ if there is a function $\phi$ such that for every graph $G$, $f_1(G) \leq \phi(f_2(G))$. Is there a parameter $f$ such that
  - planar graphs have bounded reduced-$f$ and $f < \text{bandwidth}$ but bandwidth $\not\prec f$?

- Is there a natural parameter tied to reduced-bandwidth?

- Is there an interesting application of reduced-bandwidth?