AN ADDITIVE VERSION OF HIGHER CHOW GROUPS

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Abstract. The cosimplicial scheme
\[ \Delta^\bullet = \Delta^0 \rightarrow \Delta^1 \rightarrow \ldots \rightarrow \Delta^n := \text{Spec} \left( k[t_0, \ldots, t_n]/(\sum t_i - t) \right) \]
was used in [2] to define higher Chow groups. In this note, we let \( t \) tend to 0 and replace \( \Delta^\bullet \) by a degenerate version
\[ \mathcal{Q}^\bullet = \mathcal{Q}^0 \rightarrow \mathcal{Q}^1 \rightarrow \ldots \rightarrow \mathcal{Q}^n := \text{Spec} \left( k[t_0, \ldots, t_n]/(\sum t_i) \right) \]
to define an additive version of the higher Chow groups. For a field \( k \), we show the Chow group of 0-cycles on \( \mathcal{Q}^n \) in this theory is isomorphic to the absolute \((n-1)\)-Kähler forms \( \Omega^{n-1}_k \).

An analogous degeneration on the level of de Rham cohomology associated to “constant modulus” degenerations of varieties in various contexts is discussed.

1. Introduction

The purpose of this note is to study a common sort of limiting phenomenon which occurs in the study of motives. Here is a simple example. Let \( k \) be a field. Let \( S = \mathbb{A}^1_k = \text{Spec} \left( k[t] \right) \) and let \( T = \text{Spec} \left( k[x, t]/(x(x-t)) \right) \hookrightarrow \mathbb{A}^2_{t,x} \). Over \( S[1/t] \) the Picard scheme \( \text{Pic}(\mathbb{A}^2_{t,x}, T)/S \) is represented by \( \mathbb{G}_{m,S[1/t]} \). On the other hand, when \( t = 0 \) one gets \( \text{Pic}(\mathbb{A}^1_{x, \{x^2 = 0\}}) \cong \mathbb{G}_{a,k} \). In some sense, \( \mathbb{G}_{m} \) has “jumped” to \( \mathbb{G}_{a} \).

In higher dimension, for \( t \neq 0 \), the Chow groups of the cosimplicial scheme
\[ \Delta^\bullet = \Delta^0 \rightarrow \Delta^1 \rightarrow \ldots \rightarrow \Delta^n := \text{Spec} \left( k[t_0, \ldots, t_n]/(\sum t_i - t) \right) \]
are known to give motivic cohomology ([3]). What can one say about the Chow groups of the degenerate cosimplicial complex:
\[ \mathcal{Q}^\bullet = \mathcal{Q}^0 \rightarrow \mathcal{Q}^1 \rightarrow \ldots \rightarrow \mathcal{Q}^n := \text{Spec} \left( k[t_0, \ldots, t_n]/(\sum t_i) \right) \]
Our main result is a calculation of the Chow groups of 0-cycles on $Q^*$. Let $Z^n(Q^r)$ be the free abelian group on codimension $n$ algebraic cycles on $Q^r$ satisfying a suitable general position condition with respect to the face maps. Let $SH^n(k, r)$ be the cohomology groups of the complex

$$
\ldots Z^n(Q^{r+1}) \to Z^n(Q^r) \to Z^n(Q^{r-1}) \to \ldots
$$

where the boundary maps are alternating sums of pullbacks along face maps. Write $\Omega^*_k$ for the absolute Kähler differentials.

**Theorem 1.1.** $SH^n(k, n) \cong \Omega^{n-1}_k$.

Note for $n = 1$, this is the above $G_a$.

For another example of this limiting phenomenon, consider $P^{n+1}$ with coordinates $U_0, \ldots, U_{n+1}$ over a field $k$. Let $X : f(U_0, \ldots, U_{n+1}) = 0$ be homogeneous of degree $n + 2$. Let $V_i = t^{r_i}U_i$, $0 \leq i \leq n + 1$, where $t$ is a variable, $r_0 = 0$, and $r_i \geq 0$. Let $N$ be the degree in $t$ of $F(t, U) := f(U_0, t^{r_1}U_1, \ldots, t^{r_{n+1}}U_{n+1})$, and assume $s := N - \sum r_i > 0$. Write $u_i = U_i/U_0, v_i = V_i/V_0$. Define a log $(n + 1)$-form on $P^{n+1} \setminus X$

$$\omega_f := \frac{du_1 \wedge \ldots \wedge du_{n+1}}{f(1, u_1, \ldots, u_{n+1})}.$$  

We view $\omega_f$ (more precisely $pr_1^*\omega_f$) as a closed $(n + 1)$-form on $(P^{n+1} \setminus X) \times G_{m,t}$. Making the substitution $u_i = t^{-r_i}v_i$ and clearing denominators, the corresponding form relative to $G_{m,t}$ can be written

$$\omega_{f,rel} = \frac{t^{-\sum r_i}dv_1 \wedge \ldots \wedge dv_{n+1}}{f(1, t^{-r_1}v_1, \ldots, t^{-r_{n+1}}v_{n+1})} = \frac{t^s dv_1 \wedge \ldots \wedge dv_{n+1}}{t^s f(1, t^{-r_1}v_1, \ldots, t^{-r_{n+1}}v_{n+1})} =: t^s \nu_{f,rel}$$

Write $\nu_f$ for the evident absolute form lifting the relative form $\nu_{f,rel}$. Define an $n$-form $\gamma_f$ by the equation

$$\omega_f = t^s \nu_f + st^{s-1}dt \wedge \gamma_f$$

Then, assuming $s^{-1} \in k$

$$\nu_f|_{t=0} = d\gamma_f|_{t=0}.$$  

To combine cycles and forms, let

$$\Delta_{n+1}(1, u_1, \ldots, u_{n+1}) := u_1 \cdot u_2 \cdots u_{n+1}(1 - u_1 - \cdots - u_{n+1}).$$
Then, writing \( u_0 = 1 - u_1 - \ldots - u_{n+1} \)

\[
\begin{align*}
\omega_{\Delta_{n+1}} &= \frac{du_1 \wedge \ldots \wedge du_{n+1}}{u_0 \cdot u_1 \cdot u_2 \cdots u_{n+1}} = \\
&= \sum_{i=0}^{n+1} (-1)^i d\log(u_0) \wedge \ldots \wedge d\log(u_i) \wedge \ldots \wedge d\log(u_{n+1}).
\end{align*}
\]

(1.10)

Substitute \( v_i = tu_i, \quad 1 \leq i \leq n+1 \). Then \( s = 1 \), and a calculation yields

\[
\begin{align*}
\gamma_{\Delta_{n+1}} &= \frac{1}{v_0} \sum_{i=1}^{n+1} (-1)^i d\log(v_1) \wedge \ldots \wedge d\log(v_i) \wedge \ldots \wedge d\log(v_{n+1}); \\
\nu_{\Delta_{n+1}} &= \sum_{i=0}^{n+1} (-1)^i d\log(v_0) \wedge \ldots \wedge d\log(v_i) \wedge \ldots \wedge d\log(v_{n+1}).
\end{align*}
\]

(1.11)

For convenience we write

\[
\begin{align*}
\nu_{n+1} := \nu_{\Delta_{n+1}}|_{t=0}; \quad \gamma_n := \gamma_{\Delta_{n+1}}|_{t=0}
\end{align*}
\]

(1.12)

The differential form \( \nu_{n+1} \) plays an important rôle in the computation of de Rham cohomology of the complement of hyperplane configurations. Let \( \mathcal{A}_t \) be the configuration in \( \mathbb{P}^n \) of \( (n+1) \) hyperplanes in general position with with affine equation \( u_0u_1 \cdots u_n = 0, u_0 + u_1 + \ldots + u_n = t \neq 0 \). Then \( H^n(\mathbb{P}^n \setminus \mathcal{A}_t) = H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(\log \mathcal{A}_t)) \) is a pure Tate structure generated by \( \nu_{\Delta_{n+1}} \). Let us now make \( t \) tend to 0 and consider the degenerate configuration \( \mathcal{A}_0 \) with affine equation \( u_0u_1 \cdots u_n = 0, u_0 + u_1 + \ldots + u_n = 0 \). Exactness of \( \nu_{n+1} \) in the case (1.8) follows from Aomoto’s theory ([1]).

We view \( \nu_{\Delta_{n+1}} \) (resp. \( \gamma_n \)) as a map

\[
\begin{align*}
\mathcal{Z}_0(\mathbb{A}^{n+1}_k \setminus \{(1 - u_1 + \ldots + u_{n+1})u_1 \cdots u_{n+1} = 0\}) &\to \Omega^{n+1}_k \\
\mathcal{Z}_0(\mathbb{A}^{n+1}_k \setminus \{u_1u_2 \cdots u_{n+1}(u_1 + \ldots + u_{n+1}) = 0\}) &\to \Omega^n_k.
\end{align*}
\]

Here \( \mathcal{Z}_0 \) denotes the free abelian group on closed points (0-cycles) and the maps are respectively

\[
\begin{align*}
x &\mapsto \text{Tr}_{k(x)/k}\nu|_{\{x\}}; \\
x &\mapsto \text{Tr}_{k(x)/k}\gamma|_{\{x\}}.
\end{align*}
\]

(1.13)

In the first case, the Nesterenko-Suslin-Totaro theorem ([3], [4]) identifies a quotient of the zero cycles modulo relations coming from curves in \( \mathbb{A}^{n+2} \) with the Milnor \( K \)-group \( K^M_{n+1}(k) \). The evaluation map (1.13)
passes to the quotient, and the resulting map $K_{n+1}^M(k) \to \Omega_k^{n+1}$ is given
on symbols by the $d \log$-map

\begin{equation}
\{x_1, \ldots, x_{n+1}\} \mapsto d \log(x_1) \wedge \cdots \wedge d \log(x_{n+1}).
\end{equation}

In the second case, factoring out by curves on $\mathbb{A}^{n+2}$ as in (1.3) yields
the Chow group of 0-cycles $SH^{n+1}(k, n + 1)$, and our main result is
that evaluation on $\gamma$ gives an isomorphism

\begin{equation}
SH^{n+1}(k, n + 1) \cong \Omega_k^n.
\end{equation}

If the hypersurface $X$ in (1.4) is smooth, we can also take the residues
of these forms. For a non-zero value of $t$, $\text{Res}(\nu_f)$ is a holomorphic $n$-
form on $X$. Let $Y^0 \subset Y : F(0, U) = 0$ be the open set of smooth
points on the special fibre. We have $\text{Res}(\nu_f|_{t=0}) = d\text{Res}(\gamma_f|_{t=0})$ on $Y^0$.
Evaluation $\mathbf{(1.13)}$ gives a map

\begin{equation}
CH_0(X) \to \Omega_k^n; \quad x \mapsto \text{Tr}_{k(x)/k}(\omega|_x).
\end{equation}

What about the special fibre? Consider the special case

\begin{equation}
f(1, u_1, u_2) = u_1^2 - u_2^3 - au_2 - b; \quad v_1 = t^3 u_1, \ v_2 = t^2 u_2
\end{equation}

One computes

\begin{equation}
\nu = \frac{dv_1 \wedge dv_2}{v_1^2 - v_2^3 - at^4 v_2 - bt^6}; \quad \gamma = \frac{2v_2dv_1 - 3v_1dv_2}{v_1^2 - v_2^3 - at^4 v_2 - bt^6}.
\end{equation}

One checks that $\text{Res}(\gamma|_{t=0}) = v_2/v_1$ and

\begin{equation}
d(v_2/v_1) = -dv_2/2 = d\text{Res}(\nu|_{t=0}) \text{ on } v_1^2 - v_2^3 = 0
\end{equation}

The assignment $x \mapsto \text{Tr}_{k(x)/k}(v_2/v_1)(x)$ identifies the jacobian of the
special fibre $v_1^2 - v_2^3 = 0$ with $G_a(k) = k$.

A final example of specialization, which we understand less well,
though it was an inspiration for this article, concerns the hyperbolic
motives of Goncharov [5]. The matrix coefficients of his theory (in
the sense of [5]) are the objects $H^{2n-1}(\mathbb{P}^{2n-1} \setminus Q, M \setminus Q \cap M)$. Here
$Q \subset \mathbb{P}^{2n-1}$ is a smooth quadric and $M$ is a simplex (union of $2n$
hyperplanes in general position). The subschemes $Q$ and $M$ are taken
in general position with respect to each other. The notation is intended
to suggest a sort of abstract relative cohomology group. In de Rham
cohomology, the non-trivial class in $H^{2n-1}_{\text{DR}}(\mathbb{P}^{2n-1} \setminus \bar{Q})$ is represented by

\begin{equation}
\omega = \frac{du_1 \wedge \ldots \wedge du_{2n-1}}{(u_1^2 + \ldots + u_{2n-1}^2 - 1)^n}
\end{equation}
Substituting \( u_i = v_i t \), we get

\[
(1.21) \quad \omega = \frac{t^{2n-1}dv_1 \wedge \ldots \wedge dv_{2n-1} + t^{2n-2}dt \wedge \sum (-1)^i v_i dv_1 \wedge \ldots \wedge \hat{dv}_i \wedge \ldots \wedge dv_{2n-1}}{(t^2(v_1^2 + \ldots + v_{2n-1}^2) - 1)^n}
\]

We get \( \nu|_{t=0} = \pm dv_1 \wedge \ldots \wedge dv_{2n-1} \) and \( \gamma|_{t=0} = \frac{1}{2n-1} \sum (-1)^i v_i dv_1 \wedge \ldots \wedge \hat{dv}_i \wedge \ldots \wedge dv_{2n-1} \). Let \( \Delta_M \in H_{2n-1}(\mathbb{P}^{2n-1}, M; \mathbb{Z}) \) be a generator. The hyperbolic volume

\[
(1.22) \quad \int_{\Delta_M} \omega
\]

is the real period \((\boxed{5} \text{ section 4.1})\) of the Hodge structure associated to \( H_{2n-1}(\mathbb{P}^{2n-1} \setminus Q) \). Goncharov remarks (op. cit., Question 6.4 and Theorem 6.5) that this volume degenerates to the euclidean volume as \( t \to 0 \). (More precisely, from \((1.22)\), we see that as a relative form, \( dv_1 \wedge \ldots \wedge dv_{2n-1} = \lim_{t \to 0} t^{1-2n} \omega \).) He asks for an interpretation of the degenerated volume in terms of some sort of motive over \( k[t]/(t^2) \).

In the Goncharov picture we can view \( Q \) as fixed and degenerate \( M \). Suppose \( M : L_0L_1 \cdots L_{2n-1} = 0 \) where

\[
L_i = L_i(v_1, \ldots, v_{2n-1}) = L_i(u_1/t, \ldots, u_{2n-1}/t)
\]

Assuming the \( L_i \) are general, clearing denominators and passing to the limit \( t \to 0 \) yields a degenerate simplex \( M_0 \) consisting of \( 2n \) hyperplanes meeting at the point \( v = 0 \). This limiting configuration leads to the Chow groups \( SH^*(k, 2n-1) \), but we do not see how to relate \( \gamma|_{t=0} \) to cycles.

The authors wish to thank A. Goncharov for several very inspiring conversations.

2. The additive Chow groups

In this section, we consider a field \( k \), and a \( k \)-scheme \( X \) of finite type.

We will throughout use the following notations.

**Notations 2.1.** We set \( Q^n = \text{Spec} \ k[t_0, \ldots, t_n]/(\sum_{i=0}^n t_i) \), together with the faces

\[
\partial_j : Q^{n-1} \to Q^n; \quad \partial_j^*(t_i) = \begin{cases} t_i & i < j \\ 0 & i = j \\ t_{i-1} & i > j \end{cases}
\]
One also has degeneracies
\[ \pi_j : Q^n \to Q^{n-1}; \quad \pi_j^*(t_i) = \begin{cases} 
  t_i & \text{if } i < j \\
  t_i + t_{i+1} & \text{if } i = j \\
  t_{i+1} & \text{if } i > j
\end{cases} \]

We denote by \{0\} \in Q^n the vertex defined by \( t_i = 0 \). We write \( Q^n_X = Q^n \times_{\text{Spec}(k)} X \). The above face and degeneracy maps make \( Q^n_X \) a cosimplicial scheme.

**Definition 2.2.** Let \( SZ^p(X, n) \) be the free abelian group on codimension \( p \) algebraic cycles on \( Q^n_X \) with the property:

(i) They don’t meet \( \{0\} \times X \).

(ii) They meet all the faces properly, that is in codimension \( \geq p \).

Thus the face maps induce restriction maps
\[ \partial_i : SZ^p(X, n) \to SZ^p(X, n - 1); \quad i = 0, \ldots, n; \quad \partial = \sum_{i=0}^{n} (-1)^i \partial_i; \]
yielding complexes \( SZ^p(X, \bullet) \):
\[ \ldots \to SZ^p(X, n + 1) \xrightarrow{\partial} SZ^p(X, n) \xrightarrow{\partial} SZ^p(X, n - 1) \xrightarrow{\partial} \ldots \]

**Definition 2.3.** The additive Chow groups are given by
\[ SH^p(X, n) = H_n(SZ^p(X, \bullet)). \]

**Remarks 2.4.**

(i) The above should be compared with the higher Chow groups \( CH^p(X, n) \) defined as above with \( Q^n_X \) replaced by \( \Delta^n := \text{Spec}(k[t_0, \ldots, t_n]/(\sum t_i - 1)) \).

(ii) The cosimplicial scheme \( Q^n_X \) admits an action of \( \mathbb{G}_m \), which we define by
\[ x \star (t_0, \ldots, t_n) := (t_0/x, \ldots, t_n/x). \]
(The reason for the inverse will be clear below.) By functoriality, we obtain a \( \mathbb{G}_m \)-action on the \( SH^p(X, n) \).

(iii) Let \( f : X' \to X \) be a proper map with \( n = \dim X' - \dim X \). Then one has a push-forward map \( f_* : SZ^p(X', \bullet) \to SZ^{p-n}(X, \bullet) \). On homology this yields \( SH^p(X', n) \to SH^{p-n}(X, n) \). We will be particularly interested in the case \( X = \text{Spec}(k), X' = \text{Spec}(k') \) with \( [k' : k] < \infty \). We write \( \text{tr}_{k'/k} : SH^p(k', n) \to SH^p(k, n) \) for the resulting map. This trace map is compatible with the action of \( k^\times \) from (ii) in the sense that for \( x \in k^\times \) and \( a' \in SH^p(k', n) \) we have \( x \star \text{tr}_{k'/k}(a') = \text{tr}_{k'/k}(x \star a') \).
Lemma 2.5. The $\star$ action of $k$ on $SH_n(k, n)$ is linear, i.e. this action comes from a $k$-vector space structure on $SH_n(k, n)$.

Proof. We extend the action of $k^\times$ on $SH_n(k, n)$ to an action of the multiplicative monoid $k$ by setting $0 \star x = 0$. We have to show that for a closed point $x = (u_0, \ldots, u_n) \in Q^n \setminus \bigcup_{i=0}^{n-1} \partial_i(Q^{n-1})$, and $a, b \in k$, one has $(a + b) \star x = a \star x + b \star x$. For either $a$ or $b = 0$, this is trivial. Thus we assume $ab \neq 0$. Let $k' = k(x)$. Then the class in $SH_n(k, n)$ of $x$ is the trace from $k'$ to $k$ of a $k'$-rational point $x' \in SH_n(k', n)$. Using the compatibility of $\star$ and trace from Remarks 2.4(iii) above, we reduce to the case $x k$-rational.

Let $m = (t_0 - u_0, \ldots, t_n - u_n)$ be the sheaf of ideals of $x$. Define $\ell(t) = -\frac{ab}{u_0}t + a + b$. Consider the curve $W \subset Q^{n+1}$ defined parametrically by

$$W = \{(t, -t + \frac{u_0}{\ell(t)}, \frac{u_1}{\ell(t)}, \ldots, \frac{u_n}{\ell(t)})\}.$$ 

To check that this parametrized locus is Zariski-closed, we consider the ideal:

$$I_W = \left( (t_1 + t_0)\ell(t_0) - u_0, t_2\ell(t_0) - u_1, \ldots, t_n\ell(t_0) - u_n \right).$$

If $y = (y_0, \ldots, y_{n+1})$ is a geometric point in the zero locus of $I_W$, then since the $u_i \neq 0$ we see that $\ell(y_0) \neq 0$. Substituting $t = y_0$, we see that $y$ lies on the parametrized locus $W$.

The equation $-t + \frac{u_0}{\ell(t)} = 0$ leads to a quadratic equation in $t$ with solutions $t = \frac{u_0}{a}$, $t = \frac{u_0}{b}$. If $a + b \neq 0$ we have

$$\partial_0(W) = \left( \frac{u_0}{a + b}, \ldots, \frac{u_n}{a + b} \right) = (a + b) \star (u_0, \ldots, u_n);$$ 

$$\partial_1(W) = \left( \frac{u_0}{a}, \ldots, \frac{u_n}{a} \right) + \left( \frac{u_0}{b}, \ldots, \frac{u_n}{b} \right)$$ 

$$= a \star (u_0, \ldots, u_n) + b \star (u_0, \ldots, u_n)$$ 

$$\partial_i(W) = 0; \quad i \geq 2,$$

so the lemma follows in this case. If $a + b = 0$, then $\partial_0 W = 0$ as well, and again the assertion is clear. \qed

3. Additive Chow groups and Milnor $K$-theory

We consider the map of complexes $Z^p(k, \bullet) \to SZ^p(k, \bullet + 1)$ defined on $k$-rational points by $\iota : (u_0, \ldots, u_n) \mapsto (-1, u_0, \ldots, u_n)$. This map induces then a map

$$\iota : CH^p(k, n) \to SH^{p+1}(k, n + 1).$$
By [3] and [4], one has an isomorphism
\begin{equation}
K^M_n(k) \cong CH^n(k, n)
\end{equation}
of the higher Chow groups of 0-cycles with Milnor $K$-theory. It is defined by:
\begin{equation}
(u_0, \ldots, u_n) \mapsto \{- u_0/u_n, \ldots, -u_{n-1}/u_n\}
\end{equation}
\begin{equation}
\{b_1, \ldots, b_n\} \mapsto \left(\frac{b_1}{c}, \ldots, \frac{b_n}{c}, -\frac{1}{c}\right); \quad c = -1 + \sum_{i=1}^n b_i.
\end{equation}
Note that if $\sum_i^u b_i = 1$, then the symbol $\beta := \{b_1, \ldots, b_n\}$ is trivial in Milnor $K$-theory, and one maps $\beta$ to 0.

In this way, one obtains a map
\begin{equation}
K^M_{n-1}(k) \to SH^n(k, n); \quad \{x_1, \ldots, x_{n-1}\} \mapsto
\left(-1, \frac{x_1}{-1 + \sum_{i=1}^{n-1} x_i}, \ldots, \frac{x_{n-1}}{-1 + \sum_{i=1}^{n-1} x_i}, -\frac{1}{-1 + \sum_{i=1}^{n-1} x_i}\right).
\end{equation}

4. Differential forms

In this section we construct a $k$-linear map $\Omega^{n-1}_k \to SH^n(k, n)$. (Here $\Omega^i_k$ are the absolute Kähler differential $i$-forms.)

The following lemma is closely related to calculations in [4].

**Lemma 4.1.** As a $k$-vector space, the differential forms $\Omega^{n-1}_k$ are isomorphic to $(k \otimes_{\mathbb{Z}} \Lambda^{n-1} k^\times)/\mathcal{R}$, where the $k$-structure on $k \otimes_{\mathbb{Z}} \Lambda^{n-1} k^\times$ is via multiplication on the first argument, and where the relations $\mathcal{R}$, for $n \geq 2$, are the $k$-subspace spanned by $a \otimes (a \wedge b_1 \wedge \ldots \wedge b_{n-2}) + (1 - a) \otimes ((1 - a) \wedge b_1 \wedge \ldots \wedge b_{n-2})$, for $b_i \in k^\times$, $a \in k$, and where $0 \wedge b_2 \wedge \wedge b_{n-2} = 0$. For $n = 1$, one has $\mathcal{R} = \{0\}$.

**Proof.** For $n = 1$, there is nothing to prove. We assume $n \geq 2$. It will be convenient to change the relations slightly. Replacing $a$ by $-a$, the relations become
\[
a \otimes (-a \wedge b_1 \wedge \ldots \wedge b_{n-2}) - (1 + a) \otimes ((1 + a) \wedge b_1 \wedge \ldots \wedge b_{n-2})
= a \otimes (a \wedge b_1 \wedge \ldots \wedge b_{n-2}) - (1 + a) \otimes ((1 + a) \wedge b_1 \wedge \ldots \wedge b_{n-2}).
\]
(To justify this, the $-1$ which appears multiplicatively can be dropped if $k$ has characteristic $\neq 2$ because the additive group $k$ is 2-divisible. If $k$ has characteristic 2, of course, $-a = a$.) One has a surjective $k$-linear map $(k \otimes_{\mathbb{Z}} \Lambda^{n-1} k^\times) \to \Omega^{n-1}_{k/2}$ defined by $a \otimes (b_1 \wedge \ldots \wedge b_{n-1}) \mapsto ad \log b_1 \wedge \ldots d \log b_{n-1}$. It factors through $\mathcal{R}$ as $ad \log a = da = (a + 1)d \log(a + 1)$ for $a \in k$, with $da = 0$ if $a \in \mathbb{Z}$. 
Let us first assume that \( n = 2 \). We define \( D : k \to (k \otimes \mathbb{Z} k^\times)/\mathcal{R} \)
by \( D(a) = a \otimes a \) for \( a \in k^\times \), else \( D(0) = 0 \). One has for \( a \in k^\times \) the relation
\( D(a + b) = (a + b) \otimes (a + b) = a(1 + \frac{b}{a}) \otimes a(1 + \frac{b}{a}) \).
By definition of the \( k \)-structure, this expression is
\[
= (1 + \frac{b}{a})(a \otimes a) + a((1 + \frac{b}{a}) \otimes (1 + \frac{b}{a})).
\]
Modulo \( \mathcal{R} \), this is
\[
= (1 + \frac{b}{a})(a \otimes a) + a\left(\frac{b}{a} \otimes \frac{b}{a}\right) = a \otimes a + \frac{b}{a} (a \otimes a) + a\left(\frac{b}{a} \otimes \frac{b}{a}\right).
\]
Applying again the \( k \)-structure, one obtains \( \frac{b}{a} (a \otimes a) = b \otimes a \) and
\[
a\left(\frac{b}{a} \otimes \frac{b}{a}\right) = b \otimes \frac{b}{a} = b \otimes b - b \otimes a.
\]
Summing up, one obtains \( D(a + b) = D(a) + D(b) \). Now if \( ab = 0 \), \( D(ab) = 0 \)
and since either \( a = 0 \) or \( b = 0 \) one has \( aD(b) + bD(a) = 0 \).
Else one has
\[
-D(ab) + aD(b) + bD(a) = -(ab) \otimes (ab) + a(b \otimes b) + b(a \otimes a) =
(ab)(1 \otimes (ab)^{-1}ab) = (ab)(1 \otimes 1) \equiv 0.
\]
Thus \( D \) is a derivation and factors through \( \Omega^1_k \to (k \otimes k^\times)/\mathcal{R} \) to yield
the inverse to the surjection defined above. If \( n \geq 2 \), we extend the
map \( D \) as follows:
\[
D : k^{n-1} \to (k \otimes \mathbb{Z} ^{n-1} k^\times)/\mathcal{R}
\]
\[
D(b_1, \ldots , b_{n-1}) := D(b_1) \wedge \ldots \wedge D(b_{n-1}) := b_1 \cdot \ldots \cdot b_{n-1} \otimes b_1 \wedge \ldots \wedge b_{n-1}.
\]
This symbol is immediately seen to be alternating. Furthermore, the
computations above yields
\[
D(ab) \wedge D(b_2) \wedge \ldots \wedge D(b_{n-1}) =
aD(b) \wedge D(b_2) \wedge \ldots \wedge D(b_{n-1}) + bD(a) \wedge D(b_2) \wedge \ldots \wedge D(b_{n-1}),
\]
and
\[
D(a + b) \wedge D(b_2) \ldots \wedge D(b_{n-1}) =
D(a) \wedge D(b_2) \ldots \wedge D(b_{n-1}) + D(b) \wedge D(b_2) \ldots \wedge D(b_{n-1}).
\]
This defines the inversed map to the surjection \( (k \otimes \mathbb{Z} ^{n-1} k^\times)/\mathcal{R} \to \Omega^{n-1}_k \)
defined above and proves the lemma. \( \square \)
Proposition 4.2. One has a well-defined \( k \)-linear map
\[
\phi : \Omega_{k}^{n-1} \rightarrow SH^{n}(k, n)
\]
\[
\alpha := ad \log b_{1} \wedge \ldots \wedge d \log b_{n-1} \mapsto a \ast (-1, \frac{b_{1}}{c}, \ldots, \frac{b_{n-1}}{c}, -\frac{1}{c})
\]
where \( c = -1 + \sum_{i=1}^{n-1} b_{i} \). The diagram
\[
\begin{array}{cccc}
K_{n-1}^{M}(k) & \xrightarrow{d \log} & \Omega_{k}^{n-1} & \\
\downarrow & & \downarrow \phi & \\
SH^{n}(k, n) & & \underline{\rightarrow} & \underline{SH^{n}(k, n)}
\end{array}
\]
is commutative.

Proof. If \( c = -1 + \sum_{i=1}^{n-1} b_{i} = 0 \), then \( \alpha = 0 \), else \( a \ast (-1, \frac{b_{1}}{c}, \ldots, \frac{b_{n-1}}{c}, -\frac{1}{c}) \) is defined. We now change notation and write \( c = -1 + a + \sum_{i=2}^{n-1} b_{i} \).

By Lemma 4.1, we have to show
\[
0 = \rho := a \ast (-1, \frac{a}{c}, \ldots, \frac{b_{n-1}}{c}, -\frac{1}{c}) - (a+1) \ast (-1, \frac{a+1}{c+1}, \ldots, \frac{b_{n-1}}{c+1}, -\frac{1}{c+1}).
\]

If \( a = 0 \), then one has
\[
\rho = -(-1, \frac{1}{c}, \ldots, \frac{b_{n-1}}{c}, -\frac{1}{c}) = -\iota\{1, b_{2}, \ldots, b_{n-1}\} = 0.
\]
Similarly, \( \rho = 0 \) if \( a = -1 \). Assume now \( a \neq 0, -1 \). Set \( b = -a \in k \setminus \{0, 1\} \). One defines, for \( n \geq 3 \) and \((u_{1}, \ldots, u_{n-1}) \in (\Delta^{n-2} \setminus \cup_{i=1}^{n-1} \Delta^{n-3})(k)\), the parametrized curve
\[
\Gamma(b, u) := \{(\frac{-1}{b} + t, \frac{1}{b-1} - t, \frac{-u_{1}}{b(b-1)}, \ldots, \frac{-u_{n-1}}{b(b-1)}) \} \subset Q^{n+1}
\]
and for \( n = 2 \)
\[
(4.1) \quad \Gamma(b) := \{(\frac{-1}{b} + t, \frac{1}{b-1} - t, \frac{-1}{b(b-1)}) \} \subset Q^{3}.
\]
(See [4] for the origin of this definition). This curve is indeed in good position, so it lies in \( SZ^{n}(Q^{n+1}) \). One computes
\[
(4.2) \quad \partial \Gamma(b, u) = (1 - b) \ast (-1, 1 - \frac{1}{b}, \frac{u_{1}}{b}, \ldots, \frac{u_{n-1}}{b})
\]
\[
+ b \ast (-1, \frac{b}{b-1}, \frac{-u_{1}}{b-1}, \ldots, \frac{-u_{n-1}}{b-1}).
\]
(Resp. in the case \( n = 2 \))
\[
\partial \Gamma(b) = (1 - b) \ast (-1, 1 - \frac{1}{b}, \frac{1}{b}) + b \ast (-1, \frac{b}{b-1}, \frac{-1}{b-1}).
\]
Now one has
\[
(−1, 1 − \frac{1}{b}, \frac{u_1}{b}, \ldots, \frac{u_{n-1}}{b}) = \iota\{1 − b, \frac{1}{u_{n-1}}, \ldots, \frac{u_{n-2}}{u_{n-1}}\}
\]
\[
= \iota\{(1 − b, −\frac{u_1}{u_{n-1}}, \ldots, −\frac{u_{n-2}}{u_{n-1}}) − \{u_{n-1}, u_1, \ldots, u_{n-2}\}\},
\]
as the rest of the multilinear expansion contains only symbols of the shape \{\ldots, u_{n-1}, \ldots, −u_{n-1}, \ldots\}. On the other hand, since \(\sum_{i=1}^{n-1} u_i = 1\), one has \(\{u_{n-1}, u_1, \ldots, u_{n-2}\} = 0\). Similarly, one has
\[
(−1, \frac{b}{b − 1}, \frac{1}{b − 1}, \ldots, \frac{u_{n-1}}{b − 1}) = \iota\{\frac{b}{u_{n-1}}, \frac{1}{u_{n-1}}, \ldots, \frac{u_{n-2}}{u_{n-1}}\}.
\]
The same argument yields that this is
\[
\iota\{b, −\frac{u_1}{u_{n-1}}, \ldots, −\frac{u_{n-2}}{u_{n-1}}\}.
\]
It follows now from (4.2) that for \(n ≥ 3\) we have the relation in \(SH^n(k, n)\)
\[
(1 − b) \ast \iota\{1 − b, −\frac{u_1}{u_{n-1}}, \ldots, −\frac{u_{n-2}}{u_{n-1}}\} +
\]
\[
b \ast \iota\{b, −\frac{u_1}{u_{n-1}}, \ldots, −\frac{u_{n-2}}{u_{n-1}}\} = 0
\]
(The analogous relation for \(n = 2\) is similar.) The assertion of the proposition now follows from Lemma 4.1. \(\square\)

**Remark 4.3.** The coordinates \(u_i\) of \(Q^n\) respecting the boundary \(\partial(Q^{n+1})\) are defined up to scalar \(\lambda ∈ k^\times\), thus \(φ_{\lambda u_i}(α) = λ \ast φ_{u_i}(α)\), and the map \(φ\) is coordinate dependent.

**Proposition 4.4.** With notation as above, the map
\[
φ : Ω^{n-1}_k \rightarrow SH^n(k, n)
\]
is surjective. In particular, \(SH^n(k, n)\) is generated by the classes of \(k\)-rational points in \(Q^n\).

**Proof.** It is easy to check that the image of \(φ\) coincides with the subgroup of \(SH^n(k, n)\) generated by \(k\)-points. Clearly, \(SH^n(k, n)\) is generated by closed points, and any closed point is the trace of a \(k'\)-rational point for some finite extension \(k'/k\). We first reduce to the case \(k'/k\) separable. If \(x ∈ Q^n_k\) is a closed point in good position (i.e. not lying on any face) such that \(k(x)/k\) is not separable, then a simple Bertini argument shows there exists a curve \(C\) in good position on \(Q^n+1\) such that \(∂C = x + y\) where \(y\) is a zero cycle supported on points with separable residue field extensions over \(k\). Indeed, let \(W ⊂ Q^{n+1}\) be...
the union of the faces. View \( x \in W \). Since \( x \) is in good position, it is a smooth point of \( W \). Bertini will say that a non-empty open set in the parameter space of \( n \)-fold intersections of hypersurfaces of large degree containing \( x \) will meet \( W \) in \( x \) plus a smooth residual scheme. Since \( k \) is necessarily infinite, there will be such an \( n \)-fold intersection defined over \( k \). Since the residual scheme is smooth, it cannot contain inseparable points. Then \( x \equiv -y \) which is supported on separable points.

We assume now \( k'/k \) finite separable, and we must show that the trace of a \( k' \)-point is equivalent to a zero cycle supported on \( k \)-points. Since the image of \( \phi \) is precisely the subgroup generated by \( k \)-rational points, it suffices to check that the diagram

\[
\begin{array}{ccc}
\Omega_{k'}^{n-1} & \xrightarrow{\phi} & SH^n(k', n) \\
\downarrow \text{Tr}_{k'/k} & & \downarrow \text{Tr}_{k'/k} \\
\Omega_k^{n-1} & \xrightarrow{\phi} & SH^n(k, n)
\end{array}
\]

commutes. Because \( k'/k \) is separable, one has \( \Omega_{k'}^{n-1} \hookrightarrow \Omega_k^{n-1} \), and \( \Omega_k^{n-1} = k' \cdot \Omega_k^{n-1} \). One reduces to showing, for \( \alpha = (-1, \alpha_1, \ldots, \alpha_n) \in Q^n(k) \) and \( t \in k' \), that \( \text{Tr}(t \ast \alpha) = (\text{Tr}(t)) \ast \alpha \).

Let \( P(V) = V^N + a_{N-1}V^{N-1} + \ldots + a_1V + a_0 \in k[V] \) be the minimal polynomial of \(-\frac{1}{t}\). We set \( b_N = \frac{1}{\alpha_0}; \) \( b_i = \frac{a_i}{\alpha_i}, i = N - 1, \ldots, 2 \) and \( b_i = a_i, i = 1, 0 \). We define the polynomial \( Q(V, u) = b_NV^{N-1}u + \ldots b_2Vu + b_1V + b_0 \in k[V, u] \), which by definition fulfills \( Q(V, -\alpha_nV) = P(V) \). We define the ideal

\[ \mathcal{I} = (Q(V_0, u), V_i + \alpha_1V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0) \subset k[V_0, \ldots, V_{n-1}, u]. \]

It defines a curve \( W \subset \mathbb{A}^{n+1} \). We think of \( \mathbb{A}^{n+1} \) as being \( Q^{n+1} \) with the faces \( V_0 = 0, \ldots, V_{n-1} = 0, u = 0, u + \sum_{i=0}^{n-1} V_i = 0 \). Then this curve is in general position and defines a cycle in \( \mathcal{S}Z^1(k, n+1) \).

Since \( b_0 \neq 0 \), and \( \alpha_i \neq 0 \), one has

\[
\partial_i W = 0, i = 0, 1, \ldots, n-1.
\]

One has

\( \partial_n W \) defined by \( (a_1V_0 + a_0V_0, V_1 + \alpha_1V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0). \)

To compute the last face, we observe that the ideal

\[
(u + \sum_{i=0}^{n-1} V_i = 0, V_1 + \alpha_1V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0),
\]
contains \( u + \alpha_n V_0 \). Consequently \( \partial_{u+\sum_{i=0}^{n-1} v_i} W \) is defined by
\[
\left( Q(V_0, -\alpha_n V_0) = P(V_0), V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1} V_0 \right).
\]
Thus one obtains
\[
0 \equiv (-1)^n \partial W = \frac{a_1}{a_0} \ast (-1, \alpha) - t \ast (-1, \alpha).
\]
Since \( P \) is the minimal polynomial of \(-\frac{1}{t}, \frac{a_1}{a_0} \) is the trace of \( t \).

5. The main theorem

Recall (1.12) we have a logarithmic \((n-1)\) form \( \gamma_{n-1} \) on \( Q^n = \text{Spec}(k[v_0, \ldots, v_n]/(\sum v_i)) \)
\[
(5.1) \quad \gamma_{n-1} = \frac{1}{v_0} \sum_{i=1}^{n} (-1)^i d \log(v_1) \wedge \ldots \wedge d \log(v_i) \wedge \ldots \wedge d \log(v_n)
\]
\[
d\gamma_{n-1} = \nu_n = \sum_{i=0}^{n} (-1)^i d \log(v_0) \wedge \ldots \wedge d \log(v_i) \wedge \ldots \wedge d \log(v_n)
\]
Writing \( v_i = V_i/V_{n+1} \), we can view \( \gamma_{n-1} \) as a meromorphic form on \( \mathbb{P}^n = \text{Proj}(k[V_0, \ldots, V_{n+1}]/(\sum_{i=0}^{n} V_i)) \). Let \( A : V_0 \cdots V_n = 0; \infty : V_{n+1} = 0 \). The fact that \( d\gamma_{n-1} \) has log poles on the divisors \( V_i = 0, 0 \leq i \leq n \)
implies that
\[
(5.2) \quad \gamma_{n-1} \in \Gamma \left( \mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log(A + \infty))(\langle -\infty \rangle) \right)
\]
In particular, \( \gamma_{n-1} \) has log poles, so we can take the residue along components of \( A \). (The configuration \( A \) does not have normal crossings. The sheaf \( \Omega^{n-1}_{\mathbb{P}^n}(\log(A + \infty)) \) is defined to be the subsheaf of \( j_* \Omega^{n-1}_{\mathbb{P}^n - A - \infty} \) generated by forms without poles and the evident log forms with residue 1 along one hyperplane and \((-1)\) along another one. According to [1], the global sections of this naturally defined log sheaf compute de Rham cohomology.)

Lemma 5.1. We have the following residue formulae
\[
(5.3) \quad \text{Res}_{v_i=0} \gamma_n = (-1)^i \gamma_{n-1}; \quad 1 \leq i \leq n+1
\]
\[
\text{Res}_{v_0=0} \gamma_n = \gamma_{n-1}.
\]
Proof. The assertion for \( 1 \leq i \leq n+1 \) is straightforward. For \( i = 0 \), one can either compute directly or argue indirectly as follows:
\[
(5.4) \quad d\text{Res}_{v_0=0} \gamma_n = \text{Res}_{v_0=0} d\gamma_n = \text{Res}_{v_0=0} \nu_{n+1} = \nu_n = d\gamma_{n-1}.
\]
Since the global sections \((5.2)\) has dimension 1, this suffices to determine \( \text{Res}_{v_0=0} \gamma_n \). (To verify dim. 1, let \( A' \subset A \) be defined by
V_1 \cdots V_n = 0. Then \( \mathcal{A}' + \infty \) consists of \( n+1 \) hyperplanes in general position in \( \mathbb{P}^n \), so \( \Omega_{\mathbb{P}^n}^{n-1}(\log(\mathcal{A'} + \infty)) = \wedge^{n-1}\Omega_{\mathbb{P}^n}^{1}(\log(\mathcal{A'} + \infty)) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n} \).

One looks at the evident residue

\[
\Omega_{\mathbb{P}^n}^{n-1}(\log(\mathcal{A} + \infty))(-\infty) \to \Omega_{\mathbb{P}^{n-1}}^{n-2}(\log(\mathcal{A} + \infty))(-\infty).
\]

along \( V_0 = 0 \).}

\[ \square \]

**Theorem 5.2.** The assignment \( x \mapsto \text{Tr}_{k(x)/k}(\gamma(x)) \) gives an isomorphism

\[ SH^n(k, n) \cong \Omega_k^{n-1}. \]

**Proof.** Let \( X \subset Q^{n+1}_k \) be a curve in good position. For a zero-cycle \( c \) in good position on \( Q^n \) we write \( \text{Tr}\gamma_{n-1}(c) \in \Omega_k^{n-1} \) (absolute differentials) for the evident linear combination of traces from residue fields of closed points. We must show \( \text{Tr}\gamma_{n-1}(\partial X) = 0 \). Let \( \overline{X} \) denote the closure of \( X \) in \( \mathbb{P}^n \). The form \( \gamma_{n-1} \) dies when restricted to \( \infty \). We may therefore replace \( \partial X \) with \( \partial \overline{X} \).

Consider the diagram with \( D_j := \partial_j(Q^n) \cap \overline{X}, D = \sum_{j=0}^n D_j \)

\[ (5.5) \]

\[
\begin{array}{ccc}
H^0(\overline{X}, \Omega_{\overline{X}/\mathbb{Z}}(\log D)) & \xrightarrow{\sum \text{Res}_{D_j}} & \oplus_{j=0}^n \Omega_{D_j}^{n-1} \\
\downarrow & & \downarrow a \\
H^0(\overline{X} \setminus D, \Omega_{\overline{X}/\mathbb{Z}}) & \xrightarrow{\delta} & \oplus_{j=0}^n H^1_{D_j}(\overline{X}, \Omega_{\overline{X}/\mathbb{Z}}) \\
& & \xrightarrow{b} H^1(\overline{X}, \Omega_{\overline{X}/\mathbb{Z}}) \\
& & \downarrow \text{Tr} \\
& & \Omega_k^{n-1}.
\end{array}
\]

The map \( \text{Tr} \circ \delta \circ a \) is the trace map used to define our map. By Lemma \[ \ref{5.1}, \gamma_{n-1}(D_j) = (\text{Res}_{v_j = 0}\gamma_n)(D_j). \]
The desired vanishing follows, taking \( \gamma_n \in H^0(\overline{X}, \Omega_{\overline{X}/\mathbb{Z}}(\log D)) \) and using \( b \circ \delta = 0 \).

We now have

\[
\Omega_k^{n-1} \xrightarrow{\phi} SH^n(k, n) \xrightarrow{\gamma_{n-1}} \Omega_k^{n-1}.
\]

It suffices to check the composition is multiplication by \((-1)^{n+1}\). Given \( b_1, \ldots, b_{n-1} \in k \) with \( c := \sum b_i - 1 \neq 0 \), the composition is computed to be (use \[ \ref{5.1} \] and Proposition \[ \ref{4.2} \])

\[
a \cdot d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1}) \mapsto \\
- a \sum (-1)^i d\log(\frac{b_1}{ac}) \wedge \ldots \wedge d\log(\frac{b_i}{ac}) \wedge \ldots \wedge d\log(\frac{1}{ac}).
\]
Expanding the term on the right yields
\[- a \sum (-1)^i d \log \left( \frac{b_i}{c} \right) \wedge \ldots \wedge d \log \left( \frac{b_i}{c} \right) \wedge \ldots \wedge d \log \left( \frac{-1}{c} \right) +
\]
\[- a \cdot d \log (a) \wedge \left( \ldots \right),\]
and it is easy to check that the terms involving \(d \log (a)\) cancel. In this way, one reduces to the case \(a = 1\). Here
\[(5.6) \quad - \sum (-1)^i d \log \left( \frac{b_i}{c} \right) \wedge \ldots \wedge d \log \left( \frac{b_i}{c} \right) \wedge \ldots \wedge d \log \left( \frac{-1}{c} \right) =
\]
\[(-1)^n \frac{db_1}{b_1} \wedge \ldots \wedge \frac{db_{n-1}}{b_{n-1}} + \frac{dc}{c} \wedge \left( \ldots \right).\]
Again the terms involving \(d \log (c)\) cancel formally, completing the proof.

**Remark 5.3.** The isomorphism of the main theorem 5.2 depends, according to the Remark 4.3, on the scale of the coordinates.

**Challenge 5.4.** Finally, as a challenge we remark that the Kähler differentials have operations (exterior derivative, wedge product,...) which are not evident on the cycles \(SZ\). For example, one can show that the map
\[(5.7) \quad \nabla(x_0, \ldots, x_n) = (x_0, \frac{-x_1 x_0}{1 - x_0}, \ldots, \frac{-x_n x_0}{1 - x_0}, \frac{-x_0}{1 - x_0})\]
satisfies
\[(5.8) \quad \gamma_n(\nabla(x)) = (-1)^n d \gamma_{n-1}(x)\]
and hence induces the exterior derivative on the 0-cycles. The map is not uniquely determined by this property, and this particular map does not preserve good position for cycles of dimension \(> 0\). Can one find a geometric correspondence on the complex \(SZ^*\) which induces \(d\) on the 0-cycles? What about the pairings \((a, b) \mapsto a \wedge b\) or \((a, b) \mapsto a \wedge db\)?
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