A brief note on the definition of sliding block codes and the Curtis-Hedlund-Lyndon Theorem

Marcelo Sobottka
UFSC – Department of Mathematics
88040-900 Florianópolis - SC, Brazil
sobottka@mtm.ufsc.br

Daniel Gonçalves
UFSC – Department of Mathematics
88040-900 Florianópolis - SC, Brazil
daemig@gmail.com

Abstract

In this note we propose an alternative definition for sliding block codes between shift spaces. This definition coincides with the usual definition in the case that the shift space is defined on a finite alphabet, but it encompass a larger class of maps when the alphabet is infinite. In any case, the proposed definition keeps the idea that a sliding block code is a map with a local rule. Using this new definition we prove that the Curtis-Hedlund-Lyndon Theorem always holds for shift spaces over countable alphabets.

1 Introduction

Let $\mathbb{S}$ denote either the set $\mathbb{N}$ of positive integers or the set $\mathbb{Z}$ of all integers. Given a countable alphabet $A$ we consider it with the discrete topology and define $A^S$ as the set of all sequences over $A$ indexed by $S$. On $A^S$ we consider the product topology, which has a basis consisting of the collection of clopen sets (called cylinders) of the form $[a_0 \ldots a_{\ell+r}] := \{ x \in A^S : x_{i+t} = a_i, \quad i = 0, \ldots, -\ell+r \}$, where $\ell, r \in \mathbb{S}$ and $a_i \in A$ for each $i = 0, \ldots, -\ell+r$. With this topology $A^S$ is totally disconnected. Let $\sigma : A^S \to A^S$ given by $\sigma(x_i) = x_{i+1}$ be the shift map.

A shift space over the alphabet $A$ is a set $\Lambda \subset A^S$ which is closed under the topology of $A^S$ and shift invariant, that is, $\sigma(\Lambda) \subset \Lambda$. Given $n \in \mathbb{N}$, let $B_n(\Lambda) \subset A^n$ denote the set of all words of length $n$ that appear in some sequence of $\Lambda$, that is,

$B_n(\Lambda) := \{ (w_1 \ldots w_n) \in A^n : \exists x \in \Lambda \text{ s.t. } x_1 \ldots x_n = w_1 \ldots w_n \}$.

Definition 1 (Classical sliding block codes). A map $\Phi : \Lambda \subset A^S \to B^S$ is a sliding block code if there exists $\ell, r \in \mathbb{N}$ and a local rule $\phi : B_{\ell+r+1}(\Lambda) \to B$ such that

$[(\Phi(x))_i] = \phi(x_{i-\ell} \ldots x_{i+r})$, \quad \forall x \in \Lambda, \ \forall i \in \mathbb{S}.$

When $A$ is finite, the Curtis-Hedlund-Lyndon Theorem (see [4, Theorem 6.2.9]) states that a map $\Phi : \Lambda \subset A^S \to B^S$ is a sliding block code if, and only if, it is continuous and shift commuting (that is, $\Phi \circ \sigma = \sigma \circ \Phi$). However, if $A$ is infinite, then it is possible to construct continuous shift-commuting maps which do not satisfy Definition 1. For instance, suppose $A = \mathbb{N}$ and consider the map $\Phi : A^\mathbb{N} \to A^\mathbb{N}$ given by $(\Phi(x))_i = x_{i+x_i}$, for all $x \in A^\mathbb{N}$ and $i \in \mathbb{N}$. It is straightforward that $\Phi$ is continuous and shift commuting, but it is not possible to describe $\Phi$ via a local rule $\phi : B_n(A^\mathbb{N}) \to B$ as in Definition 1. This difference between the finite alphabet case and the infinite alphabet case arises from the fact that in the first case shift spaces are always compact spaces, while in the second case shift spaces are always non-compact (frequently they are not even locally compact).

In [1], the authors give a version of the Curtis-Hedlund-Lyndon Theorem for maps defined on shift spaces over infinite alphabets. More specifically, we proved that a map defined on a shift space over an infinite alphabet satisfies Definition 1 if, and only if, it is uniformly continuous and shift commuting (see [1, Theorem 6.4.1]).
In [2] a Curtis-Hedlund-Lyndon Theorem was proved for a compactification of one sided shift spaces over infinite alphabets and in [3] some weaker versions of Curtis-Hedlund-Lyndon Theorem were proved for a compactification proposed for two sided shift spaces over infinite alphabets.

In this paper, building from the ideas in [2] and [3], we propose an alternative definition for sliding block codes (see Definition 2). This new definition coincides with the classical definition when defined on shift spaces over finite alphabets, but enlarges the class of sliding block codes when dealing with shift space over infinite alphabets. More specifically, we will consider maps \( \Phi : \Lambda \to B^S \) such that, for all \( x \in \Lambda \) and \( i \in \mathbb{S} \), the symbol \( (\Phi(x))_i \) depends only on a finite number of entries \( (x_{i-\ell}, \ldots, x_{i+r}) \), for some \( \ell, r \geq 0 \) that depend only on the configuration of \( x \) around \( x_i \) (with \( \ell = 0 \) if \( \mathbb{S} = \mathbb{N} \)). Such maps also have local rules, however these local rules may not be bounded. The inspiration for considering these types of maps comes from the notion of variable length Markov processes, see [5]. Our proposed generalization of sliding block codes becomes natural once we rewrite the definition of classical sliding block codes in an equivalent way, so that we can see classical sliding block codes exactly as the maps \( \Phi : \Lambda \to B^S \) such that, for all \( i \in \mathbb{S} \), the map \( (\Phi(\cdot))_i : \Lambda \to B \) is a simple function (see Section 2).

## 2 Generalized sliding block codes

Suppose that \( A \) is finite and \( \Phi : \Lambda \to B^S \) is a sliding block code. It follows from the Curtis-Hedlund-Lyndon Theorem that \( \Phi \) is continuous and shift commuting. So, for each cylinder \( [b]_1 \) of \( B^S \), the set \( C_b := \Phi^{-1}([b]_1) \) is a clopen set of \( \Lambda \) and therefore it can be written as a (possibly empty) union of disjoint cylinders of \( \Lambda \). In particular, since \( \Lambda \) is compact, each \( C_b \) is a finite (possibly empty) union of cylinders of \( \Lambda \) and, furthermore, \( \{C_b\}_{b \in B} \) is a partition of \( \Lambda \). Now, since \( \Phi \) is shift commuting, for each \( i \in \mathbb{S} \) such that \( \sigma^i(\cdot) \in C_b \) we have that \( (\Phi(x))_{i+1} = (\Phi(\sigma^i(\cdot)))_i = b \) (recall that if \( x \in C_b \) then \( (\Phi(x))_i = b \)). In other words, the local rule \( \phi \) of \( \Phi \) is defined exactly by the words of \( \Lambda \) that define the sets \( C_b \). More precisely, the local rule is implicitly given by the simple function \( (\Phi(\cdot))_i = \sum_{b \in B} b1_{C_b} \circ \sigma^{i-1}(\cdot) \).

The above observation leads us to propose that sliding block codes should not be defined as maps whose local rules have a bound on the length of the cylinders used to define them, since this is just a consequence of the compactness of the shift space. The fundamental feature of a sliding block code is the fact that, for each \( i \in \mathbb{S} \), the map \( (\Phi(\cdot))_i : \Lambda \to B \) is a simple function which does not depend on \( i \) and such that, to decide the image of any \( x \in \Lambda \), one just needs to know a finite (yet unbounded) number of entries around \( x_i \).

### Definition 2 (Generalized sliding block codes)

A map \( \Phi : \Lambda \subset A^S \to B^S \) is a generalized sliding block code if

\[
(\Phi(x))_i = \sum_{b \in B} b1_{C_b} \circ \sigma^{i-1}(x), \quad \forall x \in \Lambda, \forall i \in \mathbb{S},
\]

where \( \{C_b\}_{b \in B} \) is a partition of \( \Lambda \) such that each nonempty \( C_b \) is a union of cylinders of \( \Lambda \), \( 1_{C_b} \) is the characteristic function of the set \( C_b \) and \( \sum \) stands for the symbolic sum.

### Example 3.

Let \( A = \mathbb{N} \) and let \( \Phi : A^\mathbb{N} \to A^\mathbb{N} \) be defined by

\[
(\Phi(x))_j = x_{j+x_j}, \quad \forall j \in \mathbb{S}.
\]

It follows that \( \Phi \) is a generalized sliding block code where for each \( b \in A \),

\[
C_b := [b]_1^2 \cup \bigcup_{a_2 \in A} [2a_2b] \cup \bigcup_{a_2, a_3 \in A} [3a_2a_3b] \cup \ldots
\]

Furthermore, it is direct that \( \Phi \) is continuous and shift commuting.

### Example 4.

Let \( A = \mathbb{N} \) and let \( \{A_k\}_{k \in \mathbb{N}} \) be a partition of \( A \) into finite sets such that at least one of them has two or more elements. Let \( \Lambda := \bigcup_{k \in \mathbb{N}} A_k^\mathbb{N} \) and \( \Phi : \Lambda \to A^\mathbb{N} \) be the map given by

\[
(\Phi(x))_j = \max_{i \geq j} x_i.
\]
It follows that $\Phi$ is not a generalized sliding block code and furthermore is not continuous (although it is shift commuting).

3 Curtis-Hedlund-Lyndon Theorem for generalized sliding block codes

In [1, Theorem 1.9.1] it was proved that, whatever the cardinality of $A$ is, a map $\Phi : \Lambda \to B^S$ is a sliding block code (according to Definition 1) if, and only if, it is uniformly continuous and shift commuting. If $A$ is a finite alphabet then $\Lambda$ is compact and it follows that the family of continuous maps coincides with the family of uniformly continuous maps. Also due to the compactness of the space, the continuity of a map implies that there exists $n = \ell + r + 1$ such that, for all $x \in \Lambda$ and $i \in S$, the symbol $(\Phi(x))_i$ depends only on the configuration $(x_{i-\ell}, \ldots, x_{i+r})$. In other words, the existence of a bounded local rule is linked to the uniform continuity of the map, which is an automatic consequence of the finiteness of the alphabet. However, for shift spaces over infinite alphabets, there exist continuous maps which are not uniformly continuous but, using the notion of generalized sliding block codes, we can state a more general version of the Curtis-Hedlund-Lyndon Theorem, whose proof follows the same outline of the proof for classical sliding block codes:

Theorem 5. Let $A$ and $B$ be two countable alphabets. A map $\Phi : \Lambda \subset A^S \to B^S$ is a generalized sliding block code if, and only if, it is continuous and shift commuting.

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