Generating permutations with a given major index

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Abstract

In [S. Effler, F. Ruskey, A CAT algorithm for listing permutations with a given number of inversions, I.P.L., 86/2 (2003)] the authors give an algorithm, which appears to be CAT, for generating permutations with a given major index. In the present paper we give a new algorithm for generating a Gray code for subexcedant sequences. We show that this algorithm is CAT and derive it into a CAT generating algorithm for a Gray code for permutations with a given major index.

1 Introduction

We present the first guaranteed constant average time generating algorithm for permutations with a fixed index. First we give a co-lex order generating algorithm for bounded compositions. Changing its generating order and specializing it for particular classes of compositions we derive a generating algorithms for a Gray code for fixed weight subexcedant sequences; and after some improvements we obtain an efficient version of this last algorithm. The generated Gray code has the remarkable property that two consecutive sequences differ in at most three adjacent positions and by a bounded amount in these positions. Finally applying a bijection introduced in [7] between subexcedant sequences and permutations with a given index we derive the desired algorithm, where consecutive generated permutations differ by at most three transpositions.

Often, Gray code generating algorithms can be re-expressed simpler as algorithms with the same time complexity and generating the same class of objects, but in different (e.g. lexicographical) order. This is not the case in our construction: the Grayness of the generated subexcedant sequences is critical in the construction of the efficient algorithm generating permutations with a fixed index.

A statistic on the set \( S_n \) of length \( n \) permutations is an association of an element of \( \mathbb{N} \) to each permutation in \( S_n \). For \( \pi \in S_n \) the major index, \( \text{MAJ} \), is a statistic defined by (see, for example, [3, Section 10.6])

\[
\text{MAJ} \pi = \sum_{1 \leq i < n} i.
\]

Definition 1. For two integers \( n \) and \( k \), an \( n \)-composition of \( k \) is an \( n \)-sequence \( c = c_1c_2\ldots c_n \) of non-negative integers with \( \sum_{i=1}^{n} c_i = k \). For an \( n \)-sequence \( b = b_1b_2\ldots b_n \), \( c \) is said \( b \)-bounded if \( 0 \leq c_i \leq b_i \), for all \( i \), \( 1 \leq i \leq n \).
In this context \( b_1b_2 \ldots b_n \) is called bounding sequence and we will consider only bounding sequences with either \( b_i > 0 \) or \( b_i = b_{i-1} = \ldots = b_1 = 0 \) for all \( i, 1 \leq i \leq n \). Clearly, \( b_i = 0 \) is equivalent to fix \( c_i = 0 \). We denote by \( C(k,n) \) the set of all \( n \)-compositions of \( k \), and by \( C^b(k,n) \) the set of \( b \)-bounded \( n \)-compositions of \( k \); and if \( b_i \geq k \) for all \( i \), then \( C^b(k,n) = C(k,n) \).

**Definition 2.** A subexcedant sequence \( c = c_1c_2 \ldots c_n \) is an \( n \)-sequence with \( 0 \leq c_i \leq i-1 \), for all \( i \); and \( \sum_{i=1}^{n} c_i \) is called the weight of \( c \).

We denote by \( S(k,n) \) the set of length \( n \) and weight \( k \) subexcedant sequences, and clearly \( S(k,n) = C^b(k,n) \) with \( b = 012 \ldots (n-1) \).

## 2 Generating fixed weight subexcedant sequences

We give three generating algorithms, and the third one generates efficiently combinatorial objects in bijection with permutations having fixed index:

- **Gen\textunderscore Colex** generates the set \( C^b(k,n) \) of bounded compositions in co-lex order (defined later).
- **Gen1\textunderscore Gray** which is obtained from Gen\textunderscore Colex by:
  - changing its generating order, and
  - restricting it to the bounding sequence \( b = 01 \ldots (n-1) \).

It produces a Gray code for the set \( S(k,n) \), and it can be seen as the definition of this Gray code.

- **Gen2\textunderscore Gray** is an efficient version of Gen1\textunderscore Gray.

Finally, in Section 4 regarding the subexcedant sequences in \( S(k,n) \) as McMahon permutation codes (defined in Section 3), a constant average time generating algorithm for a Gray code for the set of permutations of length \( n \) with the major index equals \( k \) is obtained.

### 2.1 Algorithm Gen\textunderscore Colex

This algorithm generates \( C^b(k,n) \) in co-lex order, which is defined as: \( c_1c_2 \ldots c_n \) precedes \( d_1d_2 \ldots d_n \) in co-lex order if \( c_n \geq c_{n-1} \geq \ldots \geq c_1 \) precedes \( d_n \geq d_{n-1} \ldots \geq d_1 \) in lexicographical order. Its worst case time complexity is \( O(k) \) per composition.

For a set of bounded compositions \( C^b(k,n) \), an increasable position (with respect to \( C^b(k,n) \)) in a sequence \( c_1c_2 \ldots c_n \notin C^b(k,n) \) is an index \( i \) such that:

- \( c_1 = c_2 = \ldots c_{i-1} = 0 \), and

- there is a composition \( d_1d_2 \ldots d_n \in C^b(k,n) \) with \( c_i < d_i \) and \( c_{i+1} = d_{i+1}, c_{i+2} = d_{i+2}, \ldots, c_n = d_n \).
Figure 1: The path from the root 0 0 0 0 to the composition 0 1 2 3 3 ∈ \( C^{01234}(9,5) \): (a) before deleting redundant nodes (in boldface); and (b) in the generating tree induced by the call of Gen\_Colex(9,5) where redundant nodes are avoided.

For example, for \( C^{01233}(3,5) \) the increasable positions are underlined in the following sequences: 0 0 0 1 0 and 0 0 2 0 0. Indeed, the first two positions in 0 0 0 1 0 are not increasable since there is no composition in \( C^{01233}(3,5) \) with the suffix 0 1 0; and the third position in 0 0 2 0 0 is not increasable because 2 is the maximal value in this position. Clearly, if \( \ell < r \) are two increasable positions in \( c \), then each \( i, \ell < i < r \), is still an increasable position in \( c \) (unless \( b_i = 0 \)).

Here is the sketch of the co-lex order generating procedure for \( C^b(k,n) \):

- initialize \( c \) by the length \( n \) sequence 0 0 ... 0;
- for each increasable position \( i \) in \( c \), increase \( c_i \) by one and call recursively the generating procedure if the obtained sequence \( c \) is not a composition in \( C^b(k,n) \), and output it elsewhere.

The complexity of the obtained algorithm is \( O(k) \) per generated composition and so inefficient. Indeed, too many nodes in the generating tree induced by this algorithm have degree one. Algorithm Gen\_Colex in Figure 2 avoids some of these nodes. We will identify a node in a generating tree by the corresponding value of the sequence \( c \); and a redundant node in a generating tree induced by the previous sketched algorithm is a node with a unique successor and which differs in the same position from its ancestor and its successor.

For example, in Figure 1(a) redundant nodes are: 0 0 1 0, 0 0 3, 0 1 3, 0 2 3 and 0 1 3 3. These nodes occur when, for a given suffix, the smallest value allowed in an increasable position in the current sequence \( c \) is not 1, and this position is necessarily \( \ell \), the leftmost increasable one. Algorithm Gen\_Colex avoids redundant nodes by setting \( c_\ell \) to its minimal value \( e = k - \sum_{j=1}^{\ell-1} b_j \) (and \( \sum_{j=1}^{i} b_j \) can be computed for each \( i, 1 \leq i \leq n \), in a preprocessing step). For example, in Figure 1(b) there are no redundant nodes. However, in the generating tree induced by Gen\_Colex there still remain arbitrary length sequences of successive nodes with a unique successor; they are avoided in procedure Gen2\_Gray.

Algorithm Gen\_Colex is given in Figure 2 where \( \ell \) is the leftmost increasable position in the current sequence \( c \), and \( r \) the leftmost non-zero position in \( c \), and thus the rightmost increasable position in \( c \) is \( r \) if \( c_r < b_r \) and \( r - 1 \) elsewhere (\( b_1 b_2 \ldots b_n \) being the bounding sequence). The main call is Gen\_Colex(\( k,n \)) and initially \( c \) is 0 0 ... 0. (As previously, in this algorithm the function \( k \mapsto \min\{s \mid \sum_{j=1}^{s} b_j \geq k\} \) can be computed and stored in an array, in a pre-processing step.)

The induced generating tree for the call Gen\_Colex(4,5) is given in Figure 3(a).
procedure Gen_Colex(k, r)
    global n, c, b;
    if k = 0
        then print c;
    else if c[r] = b[r]
        then r := r − 1;
    end if
    ℓ := min{s | ∑ j=1 s b[j] ≥ k};
    for i := ℓ to r do
        if i = ℓ then e := k − ∑ j=1 ℓ−1 b[j];
        else e := 1;
        end if
        c[i] := c[i] + e;
        Gen_Colex(k − e, i);
        c[i] := c[i] − e;
    end do
end if
end procedure.

Figure 2: Algorithm Gen_Colex.

2.2 Algorithm Gen1_Gray

This algorithm is defined in Figure 4 and is derived from Gen_Colex: the order of recursive calls is changed according to a direction (parameter dir), and it is specialized for bounding sequences \( b = 0 1 2 \ldots (n−1) \), and so it produces subexcedant sequences. It has the same time complexity as Gen_Colex and we will show that it produces a Gray code.

The call of Gen1_Gray with \( dir = 0 \) produces, in order, a recursive call with \( dir = 0 \), then \( r − ℓ \) calls in the \( for \) statement with \( dir \) equals successively:

- 0, 1, \ldots, 0, 1, if \( r − ℓ \) is even, and
- 1, 0, \ldots, 1, 0, 1, if \( r − ℓ \) is odd.

In any case, the value of \( dir \) corresponding to the last call is 1.

The call of Gen1_Gray with \( dir = 1 \) produces the same operations as previously but in reverse order, and in each recursive call the value of \( dir \) is replaced by \( 1 − dir \). Thus, the call of Gen1_Gray with \( dir = 1 \) produces, in order, \( r − ℓ \) calls in the \( for \) statement with \( dir \) equals alternatively 0, 1, 0, \ldots, then a last call with \( dir = 1 \). See Figure 3 (b) for an example of generating tree induced by this procedure.

Let \( S(k,n) \) be the ordered list for \( S(k,n) \) generated by the call Gen1_Gray(\( k, n, 0 \)), and it is easy to see that \( S(k,n) \) is suffix partitioned, that is, sequences with the same suffix are contiguous; and Theorem 4 shows that \( S(k,n) \) is a Gray code.

For a sequence \( c \), a \( k \geq 1 \) and \( dir \in \{0, 1\} \) we denote by first(\( k; dir; c \)) and last(\( k; dir; c \)), the first and last subexcedant sequence produced by the call of Gen1_Gray(\( k, r, dir \)) if the current sequence is \( c \), and \( r \) the position of the leftmost non-zero value in \( c \). In particular, if \( c = 0 0 \ldots 0 \), then first(\( k; 0; c \)) is the first sequence in \( S(k,n) \), and last(\( k; 0; c \)) the last one.
Figure 3: (a): The tree induced by the call of $\text{Gen}_\text{Colex}(4, 5)$ with $b = 0 \ 1 \ 2 \ 3 \ 4$, and (b): that induced by $\text{Gen}_1\text{Gray}(4, 5)$. Terminal nodes are in bold-face.
Remark 1.

1. For a sequence \(c\), the list produced by the call \(\text{Gen1_gray}(k, r, 0)\) is the reverse of the list produced by the call \(\text{Gen1_gray}(k, r, 1)\), and with the previous notations we have
   \[
   \text{last}(k; \text{dir}; c) = \text{first}(k; 1 - \text{dir}; c),
   \]
   for \(\text{dir} \in \{0, 1\}\).

2. Since the bounding sequence is \(b = 01 \ldots (n - 1)\) it follows that, for \(c = 00 \ldots 0 c_i c_{i+1} \ldots c_n\), \(c_i \neq 0\), first \((k; 0; c)\)

   - if \(k \leq \sum_{j=1}^{i-1} (j - 1) = \frac{(i-1)(i-2)}{2}\), where \(a_1 a_2 \ldots a_{i-1}\) is the smallest sequence, in co-lex order, in \(S(k, i-1)\),

   - if \(k > \frac{(i-1)(i-2)}{2}\), where \(a_1 a_2 \ldots a_i\) is the smallest sequence, in co-lex order, in \(S(k + c_i, i)\).

Figure 4: Algorithm \(\text{Gen1_gray}\), the Gray code counterpart of \(\text{GenColex}\) specialized to subexcedant sequences.

Now we introduce the notion of close sequences. Roughly speaking, two sequences are close if they differ in at most three adjacent positions and by a bounded amount in these positions. Definition 3 below defines formally this notion, and Theorem 4 shows that consecutive subexcedant sequences generated by \(\text{Gen1_gray}\) are close.

Let \(s = s_1 s_2 \ldots s_n\) and \(t = t_1 t_2 \ldots t_n\) be two subexcedant sequences of same weight which differ in at most three adjacent positions, and let \(p\) be the rightmost of them (notice
that necessarily \( p \geq 3 \). The difference between \( s \) and \( t \) is the 3-tuple

\[(a_1, a_2, a_3) = (s_{p-2} - t_{p-2}, s_{p-1} - t_{p-1}, s_p - t_p).\]

Since \( s \) and \( t \) have same weight it follows that \( a_1 + a_2 + a_3 = 0 \); and we denote by \( -(a_1, a_2, a_3) \) the tuple \( (-a_1, -a_2, -a_3) \).

**Definition 3.** Two sequences \( s \) and \( t \) in \( S(k,n) \) are close if:

- \( s \) and \( t \) differ in at most three adjacent positions, and
- if \( (a_1, a_2, a_3) \) is the difference between \( s \) and \( t \), then

\[(a_1, a_2, a_3) \in \{\pm(0,1,-1), \pm(0,2,-2), \pm(1,-2,1), \pm(1,-3,2), \pm(1,1,-2), \pm(1,0,-1)\}.

Even if the second point of this definition sound somewhat arbitrary, it turns out that consecutive sequences generated by algorithm \texttt{Gen1\_Gray} are close under this definition, and our generating algorithm for permutations with a given index in Section 4 is based on it.

**Example 1.** The following sequences are close: \(01201\) and \(00301\); \(01003\) and \(01021\); \(00201\) and \(01011\); \(01132\) and \(01204\); the positions where the sequences differ are underlined.

Whereas the following sequences are not close: \(00211\) and \(01030\) (they differ in more than three positions); \(01201\) and \(01030\) (the difference 3-tuple is not a specified one).

**Remark 2.** If \( s \) and \( t \) are two close subexcedant sequences in \( S(k,n) \), then there are at most two ‘intermediate’ subexcedant sequences \( s', s'' \) in \( S(k,n) \) such that the differences between \( s \) and \( s' \), between \( s' \) and \( s'' \), and \( s'' \) and \( t \) are \( \pm(1,-1,0) \).

**Example 2.** Let \( s = 010111 \) and \( t = 002011 \) be two sequences in \( S(4,6) \). Then \( s \) and \( t \) are close since they difference is \((1,-2,1)\), and there is one ‘intermediate’ sequence \( s' = 001111 \) in \( S(4,6) \) with

- the difference between \( s \) and \( s' \) is \((1,-1,0)\),
- the difference between \( s' \) and \( t \) is \((-1,1,0)\).

A consequence of Remark 1.2 is:

**Remark 3.** If \( s \) and \( t \) are close subexcedant sequences and \( m \) is an integer such that both \( u = \text{first}(m;0; s) \) and \( v = \text{first}(m;0; t) \) exist, then \( u \) and \( v \) are also close.

**Theorem 4.** Two consecutive sequences in \( S(k,n) \) generated by the algorithm \texttt{Gen1\_Gray} are close.

**Proof.** Let \( s \) and \( t \) be two consecutive sequences generated by the call of \texttt{Gen1\_Gray}(\( k,n,0 \)). Then there is a sequence \( c = c_1c_2 \ldots c_n \) and a recursive call of \texttt{Gen1\_Gray} acting on \( c \) (referred later as the root call for \( s \) and \( t \)) which produces, in the for statement, two calls so that \( s \) is the last sequence produced by the first of them and \( t \) the first produced by the second of them.

By Remark 1.1 it is enough to prove that \( s \) and \( t \) are close when their root call has direction 0.
Let $\ell$ and $r$, $\ell \neq r$, be the leftmost and the rightmost increasable positions in $c$ (and so $c_1 = c_2 = \ldots = c_{r-1} = 0$, and possibly $c_r = 0$); and $i$ and $i + 1$ be the positions where $c$ is modified by the root call in order to produce eventually $s$ and $t$. Also we denote $m = k - \sum_{j=1}^{n}c_j$ and $e = m - \frac{\ell(\ell - 1)}{2}$.

We will give the shape of $s$ and $t$ according to the following four cases.

1. $i = \ell$ and $r - \ell$ is even,

2. $i = \ell$ and $r - \ell$ is odd,

3. $i \neq \ell$ and the call corresponding to $i$ in the for statement of the root call has direction 0 (and so that corresponding to $i + 1$ has direction 1),

4. $i \neq \ell$ and the call corresponding to $i$ in the for statement of the root call has direction 1 (and so that corresponding to $i + 1$ has direction 0).

Case 1.

\[
\begin{align*}
{s} & = \text{last}(m - e; 0; 00 \ldots ec_{\ell+1} \ldots c_n) \\
   & = \text{first}(m - e; 1; 00 \ldots ec_{\ell+1} \ldots c_n) \\
   & = \begin{cases} 
   \text{first}(m - e - (\ell - 2); 0; 00 \ldots (\ell - 2)e c_{\ell+1} \ldots c_n) & \text{if } e = \ell - 1 \\
   \text{first}(m - e - (\ell - 2); 0; 00 \ldots (\ell - 3)(e + 1)c_{\ell+1} \ldots c_n) & \text{if } e < \ell - 1,
   \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
{t} & = \text{first}(m - 1; 0; 00 \ldots (c_{\ell+1} + 1) \ldots c_n) \\
   & = \text{first}(m - e; 0; 00 \ldots (e - 1)(c_{\ell+1} + 1) \ldots c_n) \\
   & = \text{first}(m - e - (\ell - 2); 0; 00 \ldots (\ell - 2)(e - 1)(c_{\ell+1} + 1) \ldots c_n).
\end{align*}
\]

Case 2. In this case $s$ is the same as in the previous case and

\[
\begin{align*}
{t} & = \text{first}(m - 1; 1; 00 \ldots 0(c_{\ell+1} + 1) \ldots c_n) \\
   & = \begin{cases} 
   \text{first}(m - 2; 0; 00 \ldots 0(c_{\ell+1} + 2) \ldots c_n) & \text{if } c_{\ell+1} + 2 \leq \ell \\
   \text{first}(m - e; 0; 00 \ldots 0(e - 1)(c_{\ell+1} + 1) \ldots c_n) & \text{if } c_{\ell+1} + 2 > \ell
   \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
{t} & = \text{first}(m - 1; 1; 00 \ldots 0(c_{i+1} + 1) \ldots c_n) \\
   & = \begin{cases} 
   \text{first}(m - 2; 0; 00 \ldots 0(c_{i+1} + 2) \ldots c_n) & \text{if } c_{i+1} + 2 \leq i \\
   \text{first}(m - 2; 0; 00 \ldots 1(c_{i+1} + 1) \ldots c_n) & \text{if } c_{i+1} + 2 > i
   \end{cases}
\end{align*}
\]

Case 3. In this case $c_i = 0$ and

\[
\begin{align*}
{s} & = \text{last}(m - 1; 0; 00 \ldots 01c_{i+1} \ldots c_n) \\
   & = \text{last}(m - 2; 1; 00 \ldots 02c_{i+1} \ldots c_n) \\
   & = \text{first}(m - 2; 0; 00 \ldots 02c_{i+1} \ldots c_n),
\end{align*}
\]

and

\[
\begin{align*}
{t} & = \text{first}(m - 1; 1; 00 \ldots 0(c_{i+1} + 1) \ldots c_n) \\
   & = \begin{cases} 
   \text{first}(m - 2; 0; 00 \ldots 0(c_{i+1} + 2) \ldots c_n) & \text{if } c_{i+1} + 2 \leq i \\
   \text{first}(m - 2; 0; 00 \ldots 1(c_{i+1} + 1) \ldots c_n) & \text{if } c_{i+1} + 2 > i
   \end{cases}
\end{align*}
\]
Case 4. As previously, \( c_i = 0 \) and

\[
\begin{align*}
    s &= \text{last}(m - 1; 1; 00 \ldots 01c_{i+1} \ldots c_n) \\
    &= \text{first}(m - 1; 0; 00 \ldots 01c_{i+1} \ldots c_n),
\end{align*}
\]

and

\[
    t = \text{first}(m - 1; 0; 00 \ldots 00(c_{i+1} + 1) \ldots c_n).
\]

Finally, by Remark 3 it follows that in each of the four cases \( s \) and \( t \) are close, and the statement holds.

As a byproduct of the previous theorem and Remark 1.2 we have

**Remark 4.** If \( s = s_1s_2 \ldots s_n \) and \( t = t_1t_2 \ldots t_n \) are two consecutive sequences generated by \( \text{Gen1.Gray} \) and \( p \) is the rightmost position where they differ, then \( s_1s_2 \ldots s_{p-2} \) and \( t_1t_2 \ldots t_{p-2} \) are the smallest, in co-lex order, sequences in \( S(x, p-2) \) and \( S(y, p-2) \), respectively, with \( x = s_1 + s_2 + \ldots + s_{p-2} \) and \( y = t_1 + t_2 + \ldots + t_{p-2} \). Remark that \( s_1s_2 \ldots s_{p-2} = t_1t_2 \ldots t_{p-2} \), and so \( x = y \), if \( s \) and \( t \) differ in two (adjacent) positions.

### 2.3 Algorithm Gen2.Gray

Since the generating tree induced by the call of \( \text{Gen1.Gray} \) contains still arbitrary length branches of nodes of degree one it has a poor time complexity. Here we show how some of these nodes can be avoided in order to obtain the efficient generating algorithm \( \text{Gen2.Colex} \) presented in Figure 6.

A quasi-terminal node (q-terminal node for short) in the tree induced by a generating algorithm is defined recursively as: a q-terminal node is either a terminal node (node with no successor) or a node with only one successor which in turn is a q-terminal node. The q-terminal nodes occur for the calls of \( \text{Gen1.Gray} (k, r, \text{dir}) \) when \( k = r(r-1)/2 \). See Figure 5 for an example.

The key improvement made by \( \text{Gen2.Gray} \) consists in its last parameter \( p \), which gives the rightmost position where the current sequence differ from its previous one in the list \( S(k, n) \), and \( \text{Gen2.Gray} \) stops the recursive calls of more than three successive q-terminal calls. Thus, \( \text{Gen2.Gray} \) generates only suffixes of the form \( c_{p-2}c_{p-1}c_p \ldots c_n \); see Table 1 for an example. Since two consecutive sequences in the Gray code \( S(k, n) \) differ in at most three adjacent positions, these suffixes are enough to generate efficiently \( S(k, n) \), and to generate (in Section 4) a Gray code for the set of length \( n \) permutations having the major index equal to \( k \).

Now we explain how the parameter \( p \) propagates through recursive calls. A non terminal call of \( \text{Gen2.Gray} \) produces one or several calls. The first of them (corresponding
procedure Gen2 Gray(k, r, dir, p, u)
  global n, c, b;
  if k = 0 or (p - r) ≥ 3 and k = \frac{r(r-1)}{2}
    then output(p, u);
  else if c[r] = r - 1
    then r := r - 1;
    end if
  end do
end procedure.

Figure 6: Algorithm Gen2 Gray.

to a left child in the generating tree) inherits the value of the parameter p from its parent
call; in the other calls the value of this parameter is the rightmost position where the
current sequence differs from its previous generated one; this value is \(i\) if \(dir = 0\) and
\(i + 1\) if \(dir = 1\). So, each call keeps in the last parameter \(p\) the rightmost position where
the current generated sequence differs from its previous one in the list \(S(k, n)\). Procedure
Gen2 Gray prevents to produce more than three successive q-terminal calls. For
cconvenience, initially \(p = 0\).

The last two parameters \(p\) and \(u\) of procedure Gen2 Gray and output by it are used
by procedure Update Perm in Section 4 in order to generates permutations with a given
major index; \(u\) keeps the value of \(c_1 + c_2 + \ldots + c_p\), and for convenience, initially \(u = 0\).

Even we will not make use later we sketch below an algorithm for efficiently generating
the list \(S(k, n)\):

- initialize \(d\) by the first sequence in \(S(k, n)\), i.e, the the smallest sequence in \(S(k, n)\)
in co-lex order, or equivalently, the largest one in lexicographical orders, and \(c\) by
0 \ldots 0,

- run Gen2 Gray \((k, n, 0, 0, 0)\) and for each \(p\) output by it update \(d\) as: \(d[p-2] := c[p-2]\),
\(d[p-1] := c[p-1]\), \(d[p] := c[p]\).

Analyze of Gen2 Gray

For a call of Gen2 Gray \((k, r, dir, p, u)\) necessarily \(k ≤ \frac{r(r-1)}{2}\), and if \(k > 0\) and
Table 1: The subexcedant sequences generated by the call of \texttt{Gen1Gray}(4, 6, 0) and their corresponding length 6 permutations with major index equals 4, permutations descent set is either \{1, 3\} or \{4\}. The three leftmost entries \((c_{p-2}, c_{p-1}, c_p)\) updated by the call of \texttt{Gen2Gray}(4, 6, 0, 0, 0) are underlined, where \(p\) is the rightmost position where a subexcedant sequence differ from its predecessor.

| sequence | \(p\) | permutation | sequence | \(p\) | permutation |
|----------|-------|-------------|----------|-------|-------------|
| 012100   |       | 214356      | 010012   | 5     | 536124      |
| 010300   | 4     | 324156      | 001012   | 3     | 635124      |
| 001300   | 3     | 423156      | 000112   | 4     | 135624      |
| 002200   | 4     | 413256      | 000022   | 5     | 235614      |
| 011200   | 3     | 314256      | 000004   | 6     | 345612      |
| 012010   | 5     | 215346      | 000013   | 6     | 245613      |
| 011110   | 4     | 315246      | 000103   | 5     | 145623      |
| 002110   | 3     | 513246      | 010003   | 4     | 645123      |
| 000310   | 4     | 123546      | 010003   | 3     | 546123      |
| 001210   | 4     | 523146      | 010021   | 6     | 436125      |
| 010210   | 3     | 325146      | 001021   | 3     | 634125      |
| 010030   | 5     | 435126      | 000121   | 4     | 134625      |
| 001030   | 3     | 534126      | 000031   | 5     | 234615      |
| 000130   | 4     | 134526      | 000211   | 5     | 124635      |
| 000040   | 5     | 234516      | 001111   | 4     | 624135      |
| 000220   | 5     | 124536      | 010111   | 3     | 426135      |
| 001120   | 4     | 524136      | 002011   | 4     | 614235      |
| 010120   | 3     | 425136      | 011011   | 3     | 416235      |
| 002020   | 4     | 514236      | 011101   | 5     | 316245      |
| 011020   | 3     | 415236      | 002101   | 3     | 613245      |
| 011002   | 6     | 516234      | 000301   | 4     | 123645      |
| 002002   | 3     | 615234      | 001201   | 4     | 623145      |
| 000202   | 4     | 125634      | 010201   | 3     | 326145      |
| 001102   | 4     | 625134      | 012001   | 4     | 216345      |
| 010102   | 3     | 526134      |           |       |             |
• $k \leq \frac{(r-1)(r-2)}{2}$, then this call produces at least two recursive calls,

• $\frac{(r-1)(r-2)}{2} < k < \frac{r(r-1)}{2}$, then this call produces a unique recursive call (of the form $\text{Gen2}\_\text{Gray}(k',r,\cdot,\cdot,\cdot)$, with $k' = k - \frac{(r-1)(r-2)}{2}$), which in turn produce two calls,

• $k = \frac{r(r-1)}{2}$, then this call is q-terminal call.

Since the procedure $\text{Gen2}\_\text{Gray}$ stops after three successive q-terminal calls, with a slight modification of Ruskey and van Baronaigien’s ‘CAT’ principle (see also [5]) it follows that $\text{Gen2}\_\text{Gray}$ runs in constant amortized time.

3 The McMahon code of a permutation

Here we present the bijection $\psi : S(n) \rightarrow \mathfrak{S}_n$, introduced in [7], which have the following properties:

• the image through $\psi$ of $S(k,n)$ is the set of permutations in $\mathfrak{S}_n$ with major index $k$,

• $\psi$ is a ‘Gray code preserving bijection’ (see Theorem 6),

• $\tau$ is easily computed from $\sigma$ and from the difference between $s$ and $t$, the McMahon code of $\sigma$ and $\tau$, if $s$ and $t$ are close.

In the next section we apply $\psi$ in order to construct a list for the permutations in $\mathfrak{S}_n$ with a major index equals $k$ from the Gray code list $S(k,n)$.

Let permutations act on indices, i.e., for $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ and $\tau = \tau_1 \tau_2 \ldots \tau_n$ two permutations in $\mathfrak{S}_n$, $\sigma \cdot \tau = \sigma_{\tau_1} \sigma_{\tau_2} \ldots \sigma_{\tau_n}$. For a fixed integer $n$, let $k$ and $u$ be two integers, $0 \leq k < u \leq n$, and define $[[u,k]] \in \mathfrak{S}_n$ as the permutation obtained after $k$ right circular shifts of the length-$u$ prefix of the identity in $\mathfrak{S}_n$. In two line notation

$$[[u,k]] = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & u & u+1 & \cdots & n \\ u-k+1 & u-k+2 & \cdots & u & 1 & \cdots & u-k & u+1 & \cdots & n \end{pmatrix}.$$

For example, in $\mathfrak{S}_5$ we have: $[[3,1]] = 31245$, $[[3,2]] = 23145$ and $[[5,3]] = 34512$ (the rotated elements are underlined).

Let $\psi : S(n) \rightarrow \mathfrak{S}_n$ be the function defined by

$$\psi(t_1t_2\ldots t_n) = [[n,t_n]] \cdot [[n-1,t_{n-1}]] \cdot \ldots \cdot [[i,t_i]] \cdot \ldots \cdot [[2,t_2]] \cdot [[1,t_1]] = \prod_{i=n}^{1}[[i,t_i]]. \quad (1)$$

Lemma 5 ([7]).

1. The function $\psi$ defined above is a bijection.

2. For every $t = t_1t_2\ldots t_n \in S(n)$, we have $\text{maj} \prod_{i=n}^{1}[[i,t_i]] = \sum_{i=1}^{n} t_i$. 

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The first point of the previous lemma says that every permutation \( \pi \in \mathfrak{S}_n \) can be uniquely written as \( \prod_{i=1}^{\lfloor n/2 \rfloor} [i, t_i] \) for some \( t_i \)'s, and the subexcedant sequence \( t_1 t_2 \ldots t_n \) is called the **McMahon code** of \( \pi \). As a consequence of the second point of this lemma we have:

**Remark 5.** The restriction of \( \psi \) maps bijectively permutations in \( S(k, n) \) into permutations in \( \mathfrak{S}_n \) with major index equals \( k \).

**Example 3.** The permutation \( \pi = 5 2 1 6 4 3 \in \mathfrak{S}_n \) can be obtained from the identity by the following prefix rotations:

\[
\begin{align*}
1 2 3 4 5 6 & \quad \rightarrow \quad [6, 3] \cdot [5, 4] \cdot [4, 2] \cdot [3, 2] \cdot [2, 1] \cdot [1, 0] \\
4 5 6 1 2 3 & \quad \rightarrow \quad [5, 4] \cdot [4, 2] \cdot [3, 2] \cdot [2, 1] \cdot [1, 0] \\
5 6 1 2 4 3 & \quad \rightarrow \quad [4, 2] \cdot [3, 2] \cdot [2, 1] \cdot [1, 0] \\
5 6 1 2 4 3 & \quad \rightarrow \quad [3, 2] \cdot [2, 1] \cdot [1, 0] \\
5 2 1 6 4 3 & \quad \rightarrow \quad [2, 1] \cdot [1, 0] \\
5 2 1 6 4 3 & \quad \rightarrow \quad [1, 0] \\
5 2 1 6 4 3 &
\end{align*}
\]

so

\( \pi = [6, 3] \cdot [5, 4] \cdot [4, 2] \cdot [3, 2] \cdot [2, 1] \cdot [1, 0] \),

and thus

\[
\text{maj} \ \pi = 3 + 4 + 2 + 2 + 1 + 0 = 12.
\]

Theorem 6 below states that if two permutations have their McMahon code differing in two adjacent positions, and by \( 1 \) and \( -1 \) in these positions, then these permutations differ by the transposition of two entries. Before proving this theorem we need the following two propositions, where the transposition \( \langle u, v \rangle \) denote the permutation \( \pi \) (of convenient length) with \( \pi(i) = i \) for all \( i \), except \( \pi(u) = v \) and \( \pi(v) = u \).

**Proposition 1.** Let \( n, u \) and \( v \) be three integers, \( n \geq 3 \), \( 0 \leq u \leq n - 2 \), \( 1 \leq v \leq n - 2 \), and \( \sigma, \tau \in \mathfrak{S}_n \) defined by:

- \( \sigma = [n, u] \cdot [n - 1, v] \), and
- \( \tau = [n, u + 1] \cdot [n - 1, v - 1] \).

Then

\[
\tau = \sigma \cdot \langle n, v \rangle.
\]

**Proof.** First, remark that:

- \([n, u + 1] \) is a right circular shift of \([n, u] \), and
- \([n - 1, v - 1] \) is a left circular shift of the first \( n - 1 \) entries of \([n - 1, v] \),

and so \( \sigma(i) = \tau(i) \) for all \( i, 1 \leq i \leq n \), except for \( i = n \) and \( i = v \).

**Example 4.** For \( n = 7, u = 4 \) and \( v = 3 \) we have

- \( \sigma = [n, u] \cdot [n - 1, v] = [7, 4] \cdot [6, 3] = 7 1 2 4 5 6 3 \),
- \( \tau = [n, u + 1] \cdot [n - 1, v - 1] = [7, 5] \cdot [6, 2] = 7 1 3 4 5 6 2 \),
- \( \langle n, v \rangle = \langle 7, 3 \rangle \),

and \( \tau = \sigma \cdot \langle n, v \rangle \).
Proposition 2. If \( \pi \in S_n \) and \( \langle u, v \rangle \) is a transposition in \( S_n \), then
\[
\pi^{-1} \cdot \langle u, v \rangle \cdot \pi = \langle \pi^{-1}(u), \pi^{-1}(v) \rangle.
\]

Proof. Indeed, \((\pi^{-1} \cdot \langle u, v \rangle \cdot \pi)(i) = i\), for all \( i \), except for \( i = \pi^{-1}(u) \) and \( i = \pi^{-1}(v) \).

Theorem 6. Let \( \sigma \) and \( \tau \) be two permutations in \( S_n \), \( n \geq 3 \), and \( s = s_1 s_2 \ldots s_n \) and \( t = t_1 t_2 \ldots t_n \) their McMahon codes. If there is a \( f, 2 \leq f \leq n - 1 \) such that \( t_i = s_i \) for all \( i \), except \( t_f = s_f - 1 \) and \( t_{f+1} = s_{f+1} + 1 \), then \( \tau \) and \( \sigma \) differ by a transposition. More precisely,
\[
\tau = \sigma \cdot \langle \alpha^{-1}(u), \alpha^{-1}(v) \rangle
\]
where
\[
\alpha = \prod_{i=f-1}^{1} [i, s_i] = \prod_{i=f-1}^{1} [i, t_i],
\]
and \( u = f + 1, v = s_f \).

Proof.

\( \tau = \prod_{i=n}^{1} [i, t_i] \), and so \( \tau \cdot \alpha^{-1} = \prod_{i=n}^{f} [i, t_i] \), and

\( \sigma = \prod_{i=n}^{1} [i, s_i] \), and \( \sigma \cdot \alpha^{-1} = \prod_{i=n}^{f} [i, s_i] \).

But, by Proposition 1
\[
\prod_{i=n}^{f} [i, t_i] = \prod_{i=n}^{f} [i, s_i] \cdot \langle f + 1, s_f \rangle
\]
or, equivalently
\[
\tau \cdot \alpha^{-1} = \sigma \cdot \alpha^{-1} \cdot \langle f + 1, s_f \rangle,
\]
and by Proposition 2 the results holds.

The previous theorem says that \( \sigma \) and \( \tau \) ‘have a small difference’ provided that their McMahon code, \( s \) and \( t \), do so. Actually, we need that \( s \) and \( t \) are consecutive sequences in the list \( S(k, n) \) and they have a more particular shape (see Remark 4). In this context, permutations having minimal McMahon code play a particular role.

It is routine to check the following proposition (see Figure 7 for an example).

Proposition 3. Let \( n \) and \( k \) be two integers, \( 0 < k \leq \frac{n(n-1)}{2} \); \( a = a_1 a_2 \ldots a_n \) be the smallest subexcedant sequence in co-lex order with \( \sum_{i=1}^{n} a_i = k \), and \( \alpha = \alpha_{n,k} = \psi(a) \) be the permutation in \( S_n \) having its McMahon code \( a \). Let \( j = \max \{ i : a_i \neq 0 \} \), that is, \( a \) has the form
\[
012 \ldots (j-3)(j-2)a_j00 \ldots 0.
\]

Then
\[
\alpha(i) = \begin{cases} 
  j - a_j - i & \text{if } 1 \leq i \leq j - (a_j + 1), \\
  2j - a_j - i & \text{if } j - (a_j + 1) < i \leq j, \\
  i & \text{if } i > j.
\end{cases}
\]
Remark 6. The permutation $\alpha$ defined in Proposition 3 is an involution, that is $\alpha^{-1} = \alpha$.

Combining Proposition 3 and Remark 6, Theorem 6 becomes in particular

**Proposition 4.** Let $\sigma$, $\tau$, $s$ and $t$ be as in Theorem 6. In addition, let suppose that there is a $j$, $0 \leq j \leq f - 1$, such that

1. $s_i = t_i = 0$ for $j < i \leq f - 1$, and
2. if $j > 0$, then
   - $s_j = t_j \neq 0$, and
   - $s_i = t_i = i - 1$ for $1 \leq i < j$.

Then

$$\tau = \sigma \cdot \langle \phi_j(f + 1), \phi_j(s_f) \rangle$$

with

$$\phi_j(i) = \begin{cases} 
  j - s_j - i & \text{if } 1 \leq i \leq j - (s_j + 1), \\
  2j - s_j - i & \text{if } j - (s_j + 1) < i \leq j, \\
  i & \text{if } i > j.
\end{cases}$$ (3)

Notice that, the conditions 1 and 2 in the previous proposition require that $s_1 s_2 \ldots s_{f-1} = t_1 t_2 \ldots t_{f-1}$ be the smallest subexcedant sequence, in co-lex order, in $S(f - 1)$ with fixed value for $\sum_{i=1}^{f-1} s_i = \sum_{i=1}^{f-1} t_i$. Also, for point 2, necessarily $j \geq 2$.

4 Generating permutations with a given major index

Let $\sigma$ and $\tau$ be two permutations with their McMahon code $s = s_1 s_2 \ldots s_n$ and $t = t_1 t_2 \ldots t_n$ belonging to $S(k, n)$, and differing in positions $f$ and $f + 1$ by 1 and $-1$ in these positions.

Let

- $v = s_f - t_f \in \{-1, 1\}$, and
\[ x = \sum_{i=1}^{f-1} s_i = \sum_{i=1}^{f-1} t_i. \]

If \( s_1 s_2 \ldots s_{f-1} \) is the smallest sequence in \( S(x, f - 1) \), in co-lex order, then applying Proposition 4 it follows that the run of the procedure \( \text{transp}(v, f, x) \) defined in Figure 8 transforms \( \sigma \) into \( \tau \) and \( s \) into \( t \).

```
procedure transp(v, f, x)
  j := min\{i : (i-1) \geq x \};
  if v = 1 then
    \( \sigma := \sigma \cdot \langle \phi_j(f + 1), \phi_j(s[f]) \rangle \);
  else
    \( \sigma := \sigma \cdot \langle \phi_j(f + 1), \phi_j(s[f] + 1) \rangle \);
  endif
  s[f] := s[f] - v;
  s[f + 1] := s[f + 1] + v;
end procedure.
```

Figure 8: Algorithm \text{transp}, where \( \phi_j \) is defined in relation (3).

Let now \( f \) be the leftmost position where two consecutive sequences \( s \) and \( t \) in the list \( S(k, n) \) differ, and \( \sigma \) and \( \tau \) be the permutations having they McMahon code \( s \) and \( t \). By Remarks 2 and 4 we have that, repeated calls of \( \text{transp} \) transform \( s \) into \( t \) and \( \sigma \) into \( \tau \). This is true for each possible 3-tuples given in Definition 3 and corresponding to two consecutive subexcedant sequences in \( S(k, n) \), and algorithm \text{Update_Perm} in Figure 9 exhausts all these 3-tuples.

For example, if \( s \) and \( t \) are the two sequences in Example 2 with they difference \((1, -2, 1)\), \( f = 2 \) and \( x = 0 \), then the calls

\[ \text{transp}(1, f, x); \]
\[ \text{transp}(-1, f + 1, x + s[f]); \]

transform \( s \) into \( t \) and \( \sigma \) into \( \tau \).

Algorithm \text{Gen2_Gray} provides \( p \), the rightmost position where the current sequence \( c \) differs from the previous generated one, and \( u = \sum_{i=1}^{p} c_i \). Algorithm \text{Update_Perm} uses \( f \), the leftmost position where \( c \) differs from the previous generated sequence, and \( x = \sum_{i=1}^{f-1} c_i \).

Now, we sketch the generating algorithm for the set of permutations in \( S_n \) having index \( k \).

- initialize \( s \) by the smallest, in co-lex order, sequence in \( S(k, n) \) and \( \sigma \) by the permutation in \( S_n \) having its McMahon code \( s \),
- run \text{Gen2_Gray}(k, n, 0, 0, 0) where output\((p, u)\) is replaced by \text{Update_Perm}(p, u).

The obtained list of permutations is the image of the Gray code \( S(k, n) \) through the bijection \( \psi \) defined in relation (1); it consists of all permutations in \( S_n \) with major index equal to \( k \), and two consecutive permutations differ by at most three transpositions. See Table 1 for the list of permutations in \( S_6 \) and with major index 4.
procedure Update_Perm(p, u)
    x := u - c[p] - c[p - 1];
    if p - 2 ≥ 1 and s[p - 2] = c[p - 2]
    then f := p - 1;
    else f := p - 2; x := x - c[f];
    endif
    (a1, a2) := (s[f] - c[f], s[f + 1] - c[f + 1]);
    if f + 2 > n then a3 := 0; else a3 := s[f + 2] - c[f + 2]; endif
    if a1 > 0 then v := 1; else v := -1; endif
    case (a1, a2, a3) of
        ±(1, -1, 0) : transp(v, f, x);
        ±(2, -2, 0) : transp(v, f, x); transp(v, f, x)
        ±(1, -2, 1) : transp(v, f, x); transp(-v, f + 1, x + s[f]);
        ±(1, -3, 2) : transp(v, f, x); transp(-v, f + 1, x + s[f]); transp(-v, f + 1, x + s[f]);
        ±(1, 1, -2) : transp(v, f + 1, x + s[f]); transp(v, f, x); transp(v, f + 1, x + s[f]);
        (1, 0, -1) : transp(1, f, x); transp(1, f + 1, x + s[f]);
        (-1, 0, 1) : transp(-1, f + 1, x + s[f]); transp(-1, f, x);
    end case
end procedure.

Figure 9: Algorithm Update_Perm.

5 Final remarks

Numerical evidences show that if we change the generating order of algorithm Gen_Colex as for Gen1_Gray, but without restricting it to subexcedant sequences, then the obtained list for bounded compositions is still a Gray code with the closeness definition slightly relaxed: two consecutive compositions differ in at most four adjacent positions. Also, T. Walsh gives in [8] an efficient generating algorithm for a Gray code for bounded compositions of an integer, and in particular for subexcedant sequences. In this Gray code two consecutive sequences differ in two positions and by 1 and -1 in these positions; but these positions can be arbitrarily far, and so the image of this Gray code through the bijection ψ defined by relation (1) in Section 3 does not give a Gray code for permutations with a fixed index.

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